Shape Matters: Understanding the Implicit Bias of the Noise Covariance

Jeff Z. HaoChen  
Stanford University  
jhaochen@stanford.edu

Colin Wei  
Stanford University  
colinwei@stanford.edu

Jason D. Lee  
Princeton University  
jasonlee@princeton.edu

Tengyu Ma  
Stanford University  
tengyuma@stanford.edu

Abstract
The noise in stochastic gradient descent (SGD) provides a crucial implicit regularization effect for training overparameterized models. Prior theoretical work largely focuses on spherical Gaussian noise, whereas empirical studies demonstrate the phenomenon that parameter-dependent noise — induced by mini-batches or label perturbation — is far more effective than Gaussian noise. This paper theoretically characterizes this phenomenon on a quadratically-parameterized model introduced by Vaskevicius et al. (2019) and Woodworth et al. (2020). We show that in an over-parameterized setting, SGD with label noise recovers the sparse ground-truth with an arbitrary initialization, whereas SGD with Gaussian noise or gradient descent overfits to dense solutions with large norms. Our analysis reveals that parameter-dependent noise introduces a bias towards local minima with smaller noise variance, whereas spherical Gaussian noise does not. Code for our project is publicly available.1

1 Introduction
One central mystery of deep artificial neural networks is their capability to generalize when having far more learnable parameters than training examples (Zhang et al., 2016). To add to the mystery, deep nets can also obtain reasonable performance in the absence of any explicit regularization. This has motivated recent work to study the regularization effect due to the optimization (rather than objective function), also known as implicit bias or implicit regularization (Arora et al., 2019a, Gunasekar et al., 2017, 2018a,b, Soudry et al., 2018). The implicit bias is induced by and depends on many factors, such as learning rate and batch size (Goyal et al., 2017, Hoffer et al., 2017, Keskar et al., 2016, Li et al., 2019b, Smith et al., 2017), initialization and momentum (Sutskever et al., 2013), adaptive stepsize (Kingma and Ba, 2014, Neyshabur et al., 2015, Wilson et al., 2017), batch normalization (Arora et al., 2018, Hoffer et al., 2018, Ioffe and Szegedy, 2015) and dropout (Srivastava et al., 2014, Wei et al., 2020).

Among these sources of implicit regularization, the SGD noise is believed to be a vital one (Keskar et al., 2016, LeCun et al., 2012). Previous theoretical works (e.g., (Li et al., 2019b)) have

1https://github.com/jhaochenz/noise-implicit-bias
The effect of noise covariance in neural network and quadratically-parameterized models. We demonstrate that label noise induces a stronger regularization effect than Gaussian noise. In both real and synthetic data, adding label noise to large batch (or full batch) SGD updates can recover small-batch generalization performance, whereas adding Gaussian noise with optimally-tuned variance $\sigma^2$ cannot. **Left:** Training and validation errors on CIFAR100 for VGG19. Adding Gaussian noise to large batch updates gives little improvement (around 2%), whereas adding label noise recovers the small batch baseline (around 15% improvement). **Right:** Training and validation error on a 100-dimensional quadratically-parameterized model defined in Section 2. Similarly to deep models, label noise or mini-batch noise leads to better solutions than optimally-tuned spherical Gaussian noise. Moreover, Gaussian noise causes the parameter to diverge after sufficient mixing, as suggested by our negative result for Langevin dynamics (Theorem 2.2). More details are in Section A.

studied the implicit regularization effect from the scale of the noise, which is directly influenced by learning rate and batch size. However, people have empirically observed that the shape of the noise also has a strong (if not stronger) implicit bias. For example, prior works show that mini-batch noise or label noise (label smoothing) – noise in the parameter updates from the perturbation of labels in training – is far more effective than adding spherical Gaussian noise (e.g., see (Shallue et al., 2018, Section 4.6) and (Szegedy et al., 2016, Wen et al., 2019)). We also confirm this phenomenon in Figure 1 (left). Thus, understanding the implicit bias of the noise shape is crucial. Such an understanding may also be applicable to distributed training because synthetically adding noise may help generalization if parallelism reduces the amount of mini-batch noise (Shallue et al., 2018).

In this paper, we theoretically study the effect of the shape of the noise, demonstrating that it can provably determine generalization performance at convergence. Our analysis is based on a nonlinear quadratically-parameterized model introduced by (Vaskevicius et al., 2019, Woodworth et al., 2020), which is rich enough to exhibit similar empirical phenomena as deep networks. Indeed, Figure 1 (right) empirically shows that SGD with mini-batch noise or label noise can generalize with arbitrary initialization without explicit regularization, whereas GD or SGD with Gaussian noise cannot. We aim to analyze the implicit bias of label noise and Gaussian noise in the quadratically-parametrized model and explain these empirical observations.

We choose to study label noise because it can replicate the regularization effects of minibatch noise in both real and synthetic data (Figure 1), and has been used to regularize large-batch parallel training (Shallue et al., 2018). Moreover, label noise is less sensitive to the initialization and the optimization history than mini-batch noise, which makes it more amenable to theoretical analysis. For example, in an extreme case, if we happen to reach or initialize at a solution that overfits the data exactly, then mini-batch SGD will stay there forever because both the gradient and the noise

Figure 1: The effect of noise covariance in neural network and quadratically-parameterized models. We demonstrate that label noise induces a stronger regularization effect than Gaussian noise. In both real and synthetic data, adding label noise to large batch (or full batch) SGD updates can recover small-batch generalization performance, whereas adding Gaussian noise with optimally-tuned variance $\sigma^2$ cannot. **Left:** Training and validation errors on CIFAR100 for VGG19. Adding Gaussian noise to large batch updates gives little improvement (around 2%), whereas adding label noise recovers the small batch baseline (around 15% improvement). **Right:** Training and validation error on a 100-dimensional quadratically-parameterized model defined in Section 2. Similarly to deep models, label noise or mini-batch noise leads to better solutions than optimally-tuned spherical Gaussian noise. Moreover, Gaussian noise causes the parameter to diverge after sufficient mixing, as suggested by our negative result for Langevin dynamics (Theorem 2.2). More details are in Section A.

studied the implicit regularization effect from the scale of the noise, which is directly influenced by learning rate and batch size. However, people have empirically observed that the shape of the noise also has a strong (if not stronger) implicit bias. For example, prior works show that mini-batch noise or label noise (label smoothing) – noise in the parameter updates from the perturbation of labels in training – is far more effective than adding spherical Gaussian noise (e.g., see (Shallue et al., 2018, Section 4.6) and (Szegedy et al., 2016, Wen et al., 2019)). We also confirm this phenomenon in Figure 1 (left). Thus, understanding the implicit bias of the noise shape is crucial. Such an understanding may also be applicable to distributed training because synthetically adding noise may help generalization if parallelism reduces the amount of mini-batch noise (Shallue et al., 2018).

In this paper, we theoretically study the effect of the shape of the noise, demonstrating that it can provably determine generalization performance at convergence. Our analysis is based on a nonlinear quadratically-parameterized model introduced by (Vaskevicius et al., 2019, Woodworth et al., 2020), which is rich enough to exhibit similar empirical phenomena as deep networks. Indeed, Figure 1 (right) empirically shows that SGD with mini-batch noise or label noise can generalize with arbitrary initialization without explicit regularization, whereas GD or SGD with Gaussian noise cannot. We aim to analyze the implicit bias of label noise and Gaussian noise in the quadratically-parametrized model and explain these empirical observations.

We choose to study label noise because it can replicate the regularization effects of minibatch noise in both real and synthetic data (Figure 1), and has been used to regularize large-batch parallel training (Shallue et al., 2018). Moreover, label noise is less sensitive to the initialization and the optimization history than mini-batch noise, which makes it more amenable to theoretical analysis. For example, in an extreme case, if we happen to reach or initialize at a solution that overfits the data exactly, then mini-batch SGD will stay there forever because both the gradient and the noise
vanish (Vaswani et al., 2019). In contrast, label noise will not accidentally vanish, so analysis is more tractable. Understanding label noise may lead to understanding mini-batch noise or replacing it with other more robust choices.

In our setting, we prove that with a proper learning rate schedule, SGD with label noise recovers a sparse ground-truth classifier and generalizes well, whereas SGD with spherical Gaussian noise generalizes poorly. Concretely, SGD with label noise biases the parameter towards sparse solutions and exactly recovers the sparse ground-truth, even when the initialization is arbitrarily large (Theorem 2.1). In this same regime, noise-free gradient descent quickly overfits because it trains in the NTK regime (Arora et al., 2019b, Chizat and Bach, 2018, Du et al., 2018b, Jacot et al., 2018, Oymak and Soltanolkotabi, 2020). Adding Gaussian noise is insufficient to fix this, as this algorithm would end up sampling from a Gibbs distribution with infinite partition function and fail to converge to the ground-truth (Theorem 2.2). In summary, with not too small learning rate or noise level, label noise suffices to bias the parameter towards sparse solutions without relying on a small initialization, whereas Gaussian noise cannot.

Our analysis suggests that the fundamental difference between label or mini-batch noise and Gaussian noise is that the former is parameter-dependent, and therefore introduces stronger biases than the latter. The conceptual message highlighted by our analysis is that there are two possible implicit biases induced by the noise: 1. prior work (Keskar et al., 2016) shows that by escaping sharp local minima, noisy gradient descent bias the parameter towards solutions which are more robust (i.e., solutions with low curvature, or “flat” minima), and 2. when the noise covariance varies across the parameter space, there is another (potentially stronger) implicit bias effect toward parameters where the noise covariance is smaller. Label or mini-batch noise benefit from both biases, whereas Gaussian noise is independent of the parameter, so it benefits from the first bias but not the second. For the quadratically-parameterized model, this first bias is not sufficient for finding solutions with good generalization because there is a large set of overfitting global minima of the training loss with reasonable curvature. In contrast, the covariance of label noise is proportional to the scale of the parameter, inducing a much stronger bias towards low norm solutions which generalize well.

1.1 Additional Related Works

There has been a line of work empirically studying how noise influences generalization. Keskar et al. (2016) argued that large batch training will converge to “sharp” local minima which do not generalize well. Hoffer et al. (2017) argued that large batch size doesn’t hurt generalization much if training goes on long enough and additional noise is added with a larger learning rate. Goyal et al. (2017) and Shallue et al. (2018) showed large batch training with proper learning rate and additional label noise can achieve similar generalization as small batch. Agarwal et al. (2020) disentangled the effects of update direction and scale for a variety of optimizers. Chaudhari and Soatto (2018), Wei and Schwab (2019), Yaida (2018) (heuristically) suggested that SGD may encourage solutions with smaller noise covariance. Martin and Mahoney (2018) used random matrix theory to analyze implicit regularization effects of noises. The noise induced by dropout has been shown to change the expected training objective, hence provides a regularization effect (Arora et al., 2020, Mianjy and Arora, 2019, Mianjy et al., 2018, Wei et al., 2020). Wei et al. (2020) showed that there also exists an implicit bias induced by dropout noise.

Blanc et al. (2019) and Zhu et al. (2019) also studied implicit regularization effects which arise due to shape, rather than scale, of the noise, but only considered the local effect of the noise near some local minimum of the loss. In contrast, our work analyzes the global effect of noise. For a more detailed comparison with (Blanc et al., 2019), see Section 2.2.
Langevin dynamics or the closely-related stochastic gradient descent with Gaussian noise, has been studied in previous works (Ge et al., 2015, Mou et al., 2017, Neelakantan et al., 2015, Negrea et al., 2019, Raginsky et al., 2017, Roberts et al., 1996, Teh et al., 2016, Welling and Teh, 2011, Zhang et al., 2017). In particular, Raginsky et al. (2017) and Li et al. (2019a) provided generalization bounds for SGLD using algorithmic stability.

A number of works have theoretically analyzed implicit regularization in simplified settings (Gunasekar et al., 2018b, Ji and Telgarsky, 2018a, Soudry et al., 2018). Gunasekar et al. (2017) and Li et al. (2017) showed that gradient descent finds low rank solutions in matrix completion. Gradient descent also been shown to maximize the margin in linear and homogeneous models (Gunasekar et al., 2018a, Ji and Telgarsky, 2018b, Lyu and Li, 2019, Nacson et al., 2018, 2019, Poggio et al., 2017, Soudry et al., 2018). Du et al. (2018a) showed that gradient descent implicitly balances the layers of deep homogeneous models. Other works showed that it may not be always possible to characterize implicit biases in terms of some norm (Arora et al., 2019a, Razin and Cohen, 2020). Gissin et al. (2019) showed that gradient descent dynamics exhibit different implicit biases based on depth. Hardt et al. (2015) derived stability-based generalization bounds for SGD based on training speed.

Vaskevicius et al. (2019), Woodworth et al. (2020) analyze the effect of initialization for the same model that we study, showing that a large initialization trains in the NTK regime (shown to generalize poorly (Ghorbani et al., 2019, Wei et al., 2019)) whereas small initialization does not. We show that when the initialization is large, adding noise helps avoid the NTK regime (Du et al., 2018b, Jacot et al., 2018, Li and Liang, 2018, Woodworth et al., 2020) without explicit regularization.

Recent works also suggest that explicit regularization may mitigate the lack of implicit regularization, especially in noisy or imbalanced settings. For example, Wei and Ma (2019) show that Lipschitz-ness regularization improves the performance in clean or noisy label setting when the learning rate is sub-optimal. Cao et al. (2019) show that additional regularization improves the generalization performance of rare classes. Nakkiran et al. (2020) show that explicit regularization can mitigate the double descent phenomenon in linear regression, which is caused by the fact that the implicit regularization of gradient descent with zero initialization is insufficient for the regime when the number of parameters is close to the number of datapoints.

# 2 Setup and Main Results

## 2.1 Setup and Backgrounds

**Parameterization.** We focus on the nonlinear model parametrization: \( f_v(x) \triangleq \langle v^\odot 2, x \rangle \), where \( v \in \mathbb{R}^d \) is the parameter of the model, \( x \in \mathbb{R}^d \) is the data, and \( v^\odot 2 \) denotes the element-wise square of \( v \). Prior works (Li et al., 2017, Vaskevicius et al., 2019, Woodworth et al., 2020) have studied this model because it is an interesting and informative simplification of nonlinear models. As SGD noise exhibits many of the same empirical behaviors in this simplified model as in deep networks, we use this model as a testbed to develop a mathematical understanding of various sources of implicit biases. As shown in Figure 1, both SGD with mini-batch noise and label noise generalize better than GD or SGD with spherical Gaussian noise.

**Data distribution assumptions and overparametrization.** We assume that there exists a ground-truth parameter \( v^* \in \mathbb{R}^d \) that generates the label \( y = \langle v^*^\odot 2, x \rangle \) given a data point \( x \), which is assumed to be generated from \( \mathcal{N}(0, I_{d \times d}) \). A dataset \( D = \{(x^{(i)}, y^{(i)})\}_{i=1}^n \) of \( n \) i.i.d data points are

\(^2\)In contrast, the implicit bias of noise wouldn’t show up in a simpler linear regression model.
generated from this distribution. The implicit bias is only needed in an over-parameterized regime, and therefore we assume that \( n \ll d \). To make the ground-truth vector information-theoretically recoverable, we assume that the ground-truth vector \( v^* \) is \( r \)-sparse. Here \( r \) is much smaller than \( d \), and casual readers can treat it as a constant. Because the element-wise square in the model parameterization is invariant to any sign flip, we assume \( v^* \) is non-negative without loss of generality. For simplicity, we also assume it only takes value in \( \{0, 1\} \). We use \( S \subset [d] \) with \( |S| = r \) to denote the support of \( v^* \) throughout the paper.

We remark that we can recover \( v^* \) by re-parameterizing \( u = v^* \circ 2 \) and applying LASSO (Tibshirani, 1996) in the \( n \)-space when \( n \geq \tilde{O}(r) \), which is minimax optimal (Raskutti et al., 2012). However, the main goal of the paper, similar to several prior works (Li et al., 2017, Vaskevicius et al., 2019, Woodworth et al., 2020), is to prove that the implicit biases of non-convex optimization can recover the ground truth without explicit regularization in the over-parameterized regime when \( n = \text{poly}(r) \ll d \). We also assume throughout the paper that \( n, d \) are larger than some sufficiently large universal constant.

**Loss function.** We use the mean-squared loss denoted by \( \ell(i)(v) \triangleq \frac{1}{4} (f_v(x^{(i)}) - y^{(i)})^2 \) for the \( i \)-th example. The empirical loss is written as \( \mathcal{L}(v) \triangleq \frac{1}{n} \sum_{i=1}^{n} \ell(i)(v) \).

**Initialization.** We use a large initialization of the form \( v[0] = \tau \cdot 1 \) where \( 1 \) denotes the all 1’s vector, where we allow \( \tau \) to be arbitrarily large (but polynomial in \( d \)).

**Algorithm 1 Stochastic Gradient Descent with Label Noise**

**Require:** Number of iterations \( T \), a sequence of step sizes \( \eta[0:T] \), noise level \( \delta \), initialization \( v[0] \)

1: \textbf{for} \( t = 0 \) to \( T - 1 \) \textbf{do}
2: \hspace{0.5cm} Sample index \( i_t \sim [n] \) uniformly and add noise \( s_t \sim \{\pm \delta\} \) to \( y^{(i_t)} \).
3: \hspace{0.5cm} Let \( \tilde{\ell}^{(i_t)}(v) = \frac{1}{4}(f_v(x^{(i_t)}) - y^{(i_t)} - s_t)^2 \)
4: \hspace{0.5cm} \( v[t+1] \leftarrow v[t] - \eta[t] \nabla \tilde{\ell}^{(i_t)}(v[t]) \)  \hspace{1cm} \triangleright \text{update with label noise}

**SGD with label noise.** We study SGD with label noise as shown in Algorithm 1. We sample an example, add label noise sampled from \( \{\pm \delta\} \) to the label, and apply the gradient update. Computing the gradient, we obtain the update rule written explicitly as:

\[
v[t+1] \leftarrow v[t] - \eta[t] \left( (v[t]^\circ 2 - v^* \circ 2) x^{(i_t)} \right)^\top x^{(i_t)} \odot v[t] + \eta[t] s_t x^{(i_t)} \odot v[t],
\]

(1)

**Langevin dynamics/diffusion.** We compare SGD with label noise to Langevin dynamics, which adds spherical Gaussian noise to gradient descent (Neal et al., 2011):

\[
v[t+1] \leftarrow v[t] - \eta[t] \nabla \mathcal{L}(v[t]) + \sqrt{2\eta/\lambda} \cdot \xi,
\]

(2)

where the noise \( \xi \sim \mathcal{N}(0, I_d \times d) \) and \( \lambda > 0 \) controls the scale of noise. Langevin dynamics (LD) or its more computationally-efficient variant, stochastic gradient Langevin dynamics (SGLD), is known to converge to the Gibbs distribution \( \mu(v) \propto e^{-\lambda \mathcal{L}(v)} \) under various settings with sufficiently small learning rate (Bubeck et al., 2018, Dalalany, 2017, Raginsky et al., 2017, Roberts et al., 1996). In our negative result about Langevin dynamics/diffusion, we directly analyze the Gibbs distribution in order to disentangle the convergence and the generalization.

\(^3\)Our analysis can be straightforwardly extended to \( v^* \) with other non-zero values.

\(^4\)We also remark that it’s common to obtain only sub-optimal sample complexity guarantees in the sparsity parameters with non-convex optimization methods (Chi et al., 2019, Ge et al., 2016, Li et al., 2017, Vaskevicius et al., 2019) due to technical limitations.
This paper equates discrete time Langevin dynamics (equation (2)) with gradient descent with Gaussian noise, because LD with learning rate \( \eta \) and temperature parameter \( \lambda \) is exactly equivalent to gradient descent with learning rate \( \eta \) and spherical Gaussian noise with standard deviation \( \sigma = \sqrt{2/(\lambda \eta)} \). Thus technically the negative result for the Gibbs distribution (Theorem 2.2) applies to gradient descent with \( \sigma \)-Gaussian noise when keeping \( \eta \sigma^2 \) fixed (to be any number) and letting \( \eta \) be sufficiently small.\(^5\)

**Notations.** Unless otherwise specified, we use \( O(\cdot), \Omega(\cdot), \Theta(\cdot) \) to hide absolute multiplicative factors and \( \tilde{O}(\cdot), \tilde{\Theta}(\cdot), \tilde{\Omega}(\cdot) \) to hide poly-logarithmic factors in problem parameters such as \( d \) and \( \tau \). For example, every occurrence of \( \tilde{O}(x) \) is a placeholder for a quantity \( f(x) \) that satisfies that for some absolute constants \( c_1, c_2 > 0, \forall x, |f(x)| \leq c_1 |x| \cdot \log^c_2(d \tau) \).

### 2.2 Main Results

Our main result can be summarized by the following theorem, which suggests that stochastic gradient descent with label noise can converge to the ground truth despite a potentially large initialization.

**Theorem 2.1.** In the setting of Section 2.1, given a target error \( \epsilon > 0 \). Suppose we have \( n \geq \tilde{\Theta}(\tau^2) \) samples. For any label noise level \( \delta \geq \tilde{\Theta}(\tau^2 d^2) \), we run SGD with label noise (Algorithm 1) with the following learning rate schedule:

1. learning rate \( \eta_0 = \tilde{\Theta}(1/\delta) \) for \( T_0 = \tilde{\Theta}(1) \) iterations,
2. learning rate \( \eta_1 = \tilde{\Theta}(1/\delta^2) \) for \( T_1 = \tilde{\Theta}(1/\eta_1) \) iterations,
3. learning rate \( \eta_2 = \tilde{\Theta}(\epsilon^2/\delta^2) \) for \( T_2 = \tilde{\Theta}(1/\eta_2) \) iterations.

Then, with probability at least 0.9, the final iterate \( v[T] \) at time \( T = T_0 + T_1 + T_2 \) satisfies

\[
\|v[T] - v^*\|_\infty \leq \epsilon.
\]

Here \( \tilde{\Theta}(\cdot) \) omits poly-logarithmic dependencies on \( 1/\epsilon, d \) and \( \tau \).

In other words, with arbitrarily large initialization scale \( \tau \), we can choose large label noise level and the learning rate schedule so that SGD with label noise succeeds in recovering the ground truth. In contrast, when \( \tau \) is large, gradient flow without noise trains in the “kernel” regime as shown by (Chizat and Bach, 2018, Woodworth et al., 2020). The solution in this kernel regime minimizes the RKHS distance to initialization, and in our setting equates to finding a zero-error solution with minimum \( \|v[0] - v[0]\|_2 \). Such a solution could be arbitrarily far away when initialization scale \( \tau \) is large and therefore have poor generalization. Figure 1 (right) confirms GD performs poorly with large initialization whereas SGD with minibatch or label noise works. We outline the analysis of Theorem 2.1 in Section 3.

Blanc et al. (2019) also study the implicit bias of the label noise. For our setting, their result implies that when the iterate is near a global minimum for sufficient time, the iterates will locally move to the direction that reduces the \( l_2 \)-norm of \( v \) by a small distance (that is larger than random fluctuation). However, it does not imply the global convergence to a solution with good generalization with large (or any) initialization, which is what we prove in Theorem 2.1.\(^6\) Moreover, our

\(^5\)We also note that when \( \eta \sigma^2 \) also tends to zero, the effect of the noise will vanish and very likely gradient descent with Gaussian noise perform similarly to gradient descent.

\(^6\)It also appears difficult to generalize the local analysis directly to a global analysis, because once the iterate leaves the local minimum, all the local tools do not apply anymore, and it’s unclear whether the iterate will converge to a new local minimum or getting stuck at some region.
analysis captures the effect of the large noise or large learning rate – we require the ratio between the noise and the gradient, which is captured by the value $\eta \delta^2$, to be sufficiently large. This is consistent with empirical observation that good generalization requires sufficiently large learning rate or small batch (Goyal et al., 2017).

On the other hand, the following negative result for Langevin dynamics demonstrates that adding Gaussian noise fails to recover the ground truth even when $v^* = 0$. This suggests that spherical Gaussian noise does not induce a strong enough implicit bias towards low-norm solutions.

**Theorem 2.2.** Assume in addition to the setting in Section 2.1 that the ground truth $v^* = 0$. When $n \leq d/3$, with probability at least 0.9 over the randomness of the data, for any $\lambda > 0$, the Gibbs distribution is not well-defined because the partition function explodes:

$$\int_{\mathbb{R}^d} e^{-\lambda \mathcal{L}(v)} dv = \infty.$$

As a consequence, Langevin diffusion does not converge to a proper stationary distribution.

Theorem 2.2 helps explain the behavior in Figure 1, where adding Gaussian noise generalizes poorly for both synthetic and real data. In particular, in Figure 1 (right) adding Gaussian noise causes the parameter to diverge for synthetic data, and Theorem 2.2 explains this observation. A priori, the intuition regarding Langevin dynamics is as follows: as $\lambda \to +\infty$, the Gibbs distribution (if it exists) should concentrate on the manifold of global minima with zero loss. The measure on the manifold of global minima should be decided by the geometry of $\mathcal{L}(\cdot)$, and in particular, the curvature around the global minimum. As $\lambda \to +\infty$, the mass should likely concentrate at the flattest global minimum (according to some measure of flatness), which intuitively is $v^* = 0$ in this case.

However, our main intuition is that when $n < d$, even though the global minimum at $v^*$ is the flattest, there are also many bad global minima with only slightly sharper curvatures. The vast volume of bad global minima dominate the flatness of the global minimum at $v^* = 0$ for any $\lambda$, and hence the partition function blows up and the Gibbs distribution doesn’t exist. More details in Section 4.

### 3 Analysis Overview of SGD with Label Noise (Theorem 2.1)

#### 3.1 Warm-up: Updates with Only Parameter-dependent Noise

Towards building intuition and tools for analyzing the parameter-dependent noise, in this subsection we start by studying an extremely simplified random walk in one dimensional space. The random walk is purely driven by mean-zero noisy updates and does not involve any gradient updates:

$$v \leftarrow v + \eta \xi \cdot v,$$

where $\xi \sim \{\pm 1\}$. (3)

Indeed, attentive readers can verify that when dimension $d = 1$, sample size $n = 1$, and $v^* = 0$, equation (2) degenerates to the above random walk if we omit the gradient update term (second to last term in equation (2)). We compare it with the standard Brownian motion (which is the analog of gradient descent with spherical Gaussian noise under this extreme simplification)

$$v \leftarrow v + \eta \xi, \text{ where } \xi \sim \mathcal{N}(0, 1).$$

---

7In fact, one can show that if this phenomenon happens for some $\lambda > 0$, then it happens for all other $\lambda$. 
We initialize at $v = 1$. We observe that both random walks have mean-zero updates, so the mean is preserved: $E[v] = 1$. The variances of the two random walks are also both growing because any mean-zero update increases the variance. Moreover, the Brownian motion diverges because it has a Gaussian marginal with variance growing linearly in $t$, and there is no limiting stationary distribution.

However, the parameter-dependent random walk $(3)$ has dramatically different behavior when $\eta < 1$: the random variable $v$ will eventually converge to $v = 0$ with high probability (though the variance grows and the mean remains at $1$). This is because the variance of the noise depends on the scale of $v$. The smaller $v$ is, the smaller the noise variance is, and so the random walk tends to get “trapped” around $0$. In fact, this claim has the following informal but simple proof that does not strongly rely on the exact form of the noise and can be extended to more general high-dimensional cases.

Consider an increasing concave potential function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\phi'' < 0$ (e.g., $\phi(v) = \sqrt{v}$ works). Note that when $\eta < 1$, the random variable $v$ stays nonnegative. We can show that the expected potential function decreases after any update

$$E[\phi(v + \eta \xi v)] \approx E[\phi(v) + \phi'(v)\eta \xi v + \phi''(v)\eta^2 \xi^2 v^2]$$

(by Taylor expansion)

$$= E[\phi(v)] + E[\phi''(v)\eta^2 \xi^2 v^2] < E[\phi(v)]$$

(by $\phi''(v) < 0$ and $E[\xi] = 0$.)

With more detailed analysis, we can formalize the Taylor expansion and control the decrease of the potential function, and conclude that $E[\phi(v)]$ converges to zero. Then, by Markov’s inequality, with high probability, $\phi(v)$ is tiny and so is $v$.

**From the 1-D case to the high-dimensional case.** In one dimension, it may appear that the varying scale of noise or norm of the covariance introduces the bias. However, in the high dimensional case, the shape of the covariance also matters. For example, if we generalize the random walk $(3)$ to high-dimensions by running $d$ of the random walks in parallel, then we will observe the same phenomenon, but the noise variances in different dimensions are *not identical* — they depend on the current scales of the coordinates. (Precisely, the noise variance for dimension $k$ is $\eta^2 v^2_k$.) However, suppose we instead add noise of the *same variance* to all dimensions. Even if this variance depends on the norm of $v$ (say, $\eta^2 \|v\|_2^2$), the implicit bias will be diminished, as the smaller coordinates will have relatively outsized noise and the larger coordinates will have relatively insufficient noise.

**Outline of the rest of the subsections.** We will give a proof sketch of Theorem 2.1 that consists of three stages. We first show in the initial stage of the training that label noise effectively decreases the parameter on all dimensions, bringing the training from large initialization to a small initialization regime, where better generalization is possible (Section 3.2). Then, we show in Section 3.3 that when the parameter is decently small, with label noise and a *decayed learning rate*, the algorithm will increase the magnitude of those dimensions in support set of $v^*$, while keep decreasing the norm of the rest of dimensions. Finally, with one more decay, the algorithm can recover the ground truth.

### 3.2 Stage 0: Label Noise with Large Learning Rate Reduces the Parameter Norm

We first analyze the initial phase where we use a relatively large learning rate. When the initialization is of a decent size, GD quickly *overfits* to a bad global minimum nearest to the initialization. In contrast, we prove that SGD with label noise biases towards the small norm region, for a similar reason as the random walk example with parameter-dependent noise in Section 3.1.

---

8The same proof strategy fails for the Brownian motion because $v$ is not always nonnegative, and there is no concave potential function over the real that can be bounded from below.
Theorem 3.1. In the setting of Theorem 2.1, recall that we initialize with \( v^{[0]} = \tau \cdot 1 \). Assume \( n \geq \Theta(\log d) \). Suppose we run SGD with label noise with noise level \( \delta \geq \Theta(\tau^2 d^2 / \delta^2) \) and learning rate \( \eta_0 \in [\Theta(\tau^2 d^2 / \delta^2), \Theta(1/\delta)] \) for \( T_0 = \Theta(1/(\eta^2 \delta^3)) \) iterations. Then, with probability at least 0.99 over the randomness of the algorithm,
\[
\|v^{[T]}\|_{\infty} \leq 1/d.
\]
Moreover, the minimum entry of \( v^{[T]} \) is bounded below by \( \exp(-\tilde{O}(1/\delta)) \).

We remark that our requirement of \( \eta \) being large is consistent with the empirical observation that large initial learning rate helps generalization (Goyal et al., 2017, Li et al., 2019b). We provide intuitions and a proof sketch of the theorem in the rest of the subsection and defer the full proof to Section B. Our proof is based on the construction of a concave potential function \( \Phi \) similar to Section 3.1. We will show that, at every step, the noise has a second order effect on the potential function and decrease the potential function by a quantity on the order of \( \eta^2 \delta^2 \) (omitting the dependency). On the other hand, the gradient step may increase the potential by a quantity at most on the order of \( \eta \) (omitting \( d \) dependency again). Therefore, when \( \eta^2 \delta^2 \gtrsim \eta \), we expect the algorithm to decrease the potential and the parameter norm.

In particular, we define \( \Phi(v) \triangleq \sum_{k=1}^{d} \phi(v_k) = \sum_{k=1}^{d} \sqrt{v_k} \). By the update rule (1), the update for a coordinate \( k \in [d] \) can be written as
\[
v_{k}^{[t+1]} \leftarrow v_{k}^{[t]} - \eta s_t x_k(i_t) v_{k}^{[t]} - \eta \left( (v_{k}^{[t]} \odot 2 - v^* \odot 2)^{\top} x(i_t) \right) x_k(i_t) v_{k}^{[t]},
\]
where \( s_t \) is sampled from \( \{-\delta, \delta\} \) and \( i_t \) is sampled from \([n]\). Let \( g_k^{(i_t)} \triangleq \left( (v_{k}^{[t]} \odot 2 - v^* \odot 2)^{\top} x(i_t) \right) x_k(i_t) \) be the component coming from the stochastic gradient. Using the fact that \( \phi(ab) = \phi(a)\phi(b) \) for any \( a, b > 0 \), we can evaluate the potential function at time \( t+1 \),
\[
\mathbb{E} \left[ \phi(v_{k}^{[t+1]}) \right] = \mathbb{E} \left[ \phi(v_{k}^{[t]}) \phi(1 - \eta s_t x_k(i_t) - \eta g_k^{(i_t)}) \right] = \phi(v_{k}^{[t]}) \mathbb{E} \left[ \phi(1 - \eta s_t x_k(i_t) - \eta g_k^{(i_t)}) \right]. \tag{4}
\]

Here the expectation is over \( s_t \) and \( i_t \). We perform Taylor-expansion on the term \( \phi(1 - \eta s_t x_k(i_t) - \eta g_k^{(i_t)}) \) to deal with the non-linearity and use the fact that \( \eta s_t x_k(i_t) \) is mean-zero:
\[
\mathbb{E} \left[ \phi(1 - \eta s_t x_k(i_t) - \eta g_k^{(i_t)}) \right] \approx \phi(1) - \phi'(1) \eta \mathbb{E} \left[ g_k^{(i_t)} \right] + \frac{1}{2} \phi''(1) \mathbb{E} \left[ \left( \eta s_t x_k(i_t) - \eta g_k^{(i_t)} \right)^2 \right] \\
\leq \phi(1) - \phi'(1) \eta \mathbb{E} \left[ g_k^{(i_t)} \right] + \frac{1}{2} \phi''(1) \mathbb{E} \left[ \left( \eta s_t x_k(i_t) \right)^2 \right] - \Omega(\eta^2 \delta^2). \tag{5}
\]
In the second line we used \( \phi''(1) < 0 \) from the concavity and \( \mathbb{E}[\eta s_t x_k(i_t)] = 0 \), and the third line uses the fact that \( s_t \sim \{\pm \delta\} \) and \( \mathbb{E}[x(i_t)^2] \approx 1 \) (by the data assumption). The rest of the proof consists of bounding the second term in equation (5) from above to show the potential function is contracting.

We first note for every \( i_t \), it holds that \( |g_k^{(i_t)}| \leq \|v_{k}^{[t]} \odot 2 - v^* \odot 2\|_1 \|x(i_t)\|_{\infty} \leq (\|v_{k}^{[t]}\|_2 + r) \|x(i_t)\|_2^2 \). Furthermore, we can bound the \( \ell_2 \) norm of \( v^{[t]} \) with the following lemma:

Lemma 3.2. In the setting of Theorem 3.1, for some failure probability \( \rho > 0 \), let \( b_0 \triangleq 6 \tau d / \rho \). Then, with probability at least \( 1 - \rho / 3 \), we have that \( \|v_{k}^{[t]}\|_2 \leq b_0 \) for any \( t \leq T_0 \).

\[\text{In general, any mean-zero noise has a second order effect on any potential function. Therefore, when the noise level is fixed, as } \eta \to 0, \text{ the effect of the noise diminishes. This is why a lower bound on the learning rate is necessary for the noise to play a role.}\]

\[\text{In the formal proof we will use a slightly different version of potential function (see Definition B.1).}\]
Note that \( v^{[0]} \) has \( \ell_2 \) norm \( \tau \sqrt{d} \), and here we prove that the norm does not exceed \( \tau d \) with high probability. At the first glance, the lemma appears to be mostly auxiliary, but we note that it distinguishes label noise from Gaussian noise, which empirically causes the parameter to blow up as shown in Figure 1. The formal proof is deferred to Section B.

By Lemma 3.2 and the bound on \( |g_k^{[i]}| \) in terms of \( \|v^{[i]}\|_2 \), we have \( |g_k^{[i]}| \leq (b_0^2 + r)\|x^{(i)}\|_\infty \leq \tilde{O}(b_0^2 + r) \) with \( b_0 \) defined in Lemma 3.2 (up to logarithmic factors). Here we use again that each entry of the data is from \( \mathcal{N}(0, 1) \). Plugging these into equation (5) we obtain

\[
\mathbb{E}\left[ \phi(1 - \eta s_i x_k^{(i)} - \eta g_k^{[i]}) \right] \leq 1 + \eta \tilde{O}(b_0^2 + r) - \Omega(\eta^2 \delta^2) < 1 - \Omega(\eta^2 \delta^2)
\]

where in the last inequality we use the lower bound on \( \eta \) to conclude \( \eta^2 \delta^2 \gtrsim \eta \tilde{O}(b_0^2 + r) \). Therefore, summing equation (4) over all the dimensions shows that the potential function decreases exponentially fast: \( \mathbb{E}[\Phi(v^{[i+1]})] < (1 - \Omega(\eta^2 \delta^2))\Phi(v^{[i]}) \). After \( T \approx \log(d)/(\eta^2 \delta^2) \) iterations, \( v^{[T]} \) will already converge to a position such that \( \mathbb{E}[\Phi(v^{[T]})] \leq \sqrt{1/d} \), which implies \( \|v^{[T]}\|_\infty \leq 1/d \) with probability at least \( 1 - \rho \) and finishes the proof.

### 3.3 Stage 1: Getting Closer to \( v^* \) with Annealed Learning Rate

Theorem 3.1 shows that the noise decreases the \( \infty \)-norm of \( v \) to \( 1/d \). This means that \( \ell_1 \) or \( \ell_2 \)-norm of \( v \) is similar to or smaller than that of \( v^* \) if \( r \) is constant, and we are in a small-norm region where overfitting is less likely to happen. In the next stage, we anneal the learning rate to slightly reduce the bias of the label noise and increase the contribution of the signal. Recall that \( v^* \) is a sparse vector with support \( S \subset [d] \). The following theorem shows that, after annealing the learning rate (from the order of \( 1/\delta^2 \) to \( 1/\delta \)), SGD with label noise increases entries in \( v_S \) and decreases entries in \( v_{\bar{S}} \) simultaneously, provided that the initialization has \( \ell_\infty \)-norm bounded by \( 1/d \). (For simplicity and self-containedness of the statement, we reset the time step to 0.)

**Theorem 3.3.** In the setting of Section 2.1, given a target error bound \( \epsilon_1 > 0 \), we assume that \( n \geq \tilde{\Theta}(r^2 \log^2(1/\epsilon_1)) \). We run SGD with label noise (Algorithm 1) with an initialization \( v^{[0]} \) whose entries are all in \( [\epsilon_\min, 1/d] \), where \( \epsilon_\min \geq \exp(-\tilde{O}(1)) \). Let noise level \( \delta \geq \tilde{\Theta}(\log(1/\epsilon_1)) \) and learning rate \( \eta = \tilde{\Theta}(1/\delta^2) \), and number of iterations \( T = \tilde{\Theta}(\log(1/\epsilon_1)/\eta) \). Then, with probability at least 0.99, after \( T \) iterations, we have

\[
\|v_S^{[T]} - v_S^*\|_\infty \leq 0.1 \text{ and } \|v_{\bar{S}}^{[T]} - v_{\bar{S}}^*\|_1 \leq \epsilon_1.
\]

We remark that even though the initialization is relatively small in this stage, the label noise still helps alleviate the reliance on small initialization. Li et al. (2017), Vaskevicius et al. (2019) showed that GD converges to the ground truth with sufficiently small initialization, which is required to be smaller than target error \( \epsilon_1 \). In contrast, our result shows that with label noise, the initialization does not need to depend on the target error, but only need to have an \( \ell_\infty \)-norm bound on the order of \( 1/d \). In other words, \( v \) gets closer to \( v^* \) on both \( S \) and \( \bar{S} \) in our case, whereas in (Li et al., 2017, Vaskevicius et al., 2019) the \( v_S \) grows slowly.

The proof of this theorem balances the contribution of the gradient against that of the noise on \( S \) and \( \bar{S} \). On \( S \), the gradient provides a stronger signal than label noise, whereas on \( \bar{S} \), the implicit bias of the noise, similarly to the effect in Section 3.2, outweighs the gradient and reduces the entries to zero. The analysis is more involved than that of Theorem 3.1, and we defer the full proof to Section C.
3.4 Stage 2: Convergence to the ground-truth $v^*$

The conclusion of Theorem 3.3 still allows constant error in the support, namely, $\|v_S - v^*_S\|_\infty \leq 0.1$. The following theorem shows that further annealing the learning rate will let the algorithm fully converge to $v^*$ with any target error $\epsilon$.

**Theorem 3.4.** [informal version of Theorem D.1] Assume initialization $v^{[0]}$ satisfies $\|v^{[0]}_S - v^*_S\|_\infty \leq 0.1$. Suppose we run SGD with label noise with any noise level $\delta \geq 0$ and small enough learning rate $\eta$ for $T = \Theta(1/\eta)$ iterations. Then, with high probability over the randomness of the algorithm and data, there is $\|v^{[T]}_S - v^*_S\|_\infty \leq 2\|v^{[0]}_S - v^*_S\|_\infty/10$.

The formal version of Theorem 3.4 and its proof can be found in Section D.

**Proof of Theorem 2.1.** In Section E of Appendix, we combine Theorem 3.1, Theorem 3.3, and Theorem D.1 to prove our main Theorem 2.1.

4 Analysis Overview of Langevin Dynamics (Theorem 2.2)

To prove Theorem 2.2, recall that we would like to show that $\int_{\mathbb{R}^d} e^{-\lambda \mathcal{L}(v)} du$ is infinite. Our approach will be to change variables to $v = v^{\otimes 2}$ and compute this integral over $u$. First, we note that all such $u$ must lie in the convex cone where each coordinate is positive. Second, we observe that the loss $\mathcal{L}(u)$ is invariant in the affine space $u + \mathbb{R}^d$, where $\mathbb{R}^d$ is the orthogonal subspace to the data. Thus, for some fixed $u'$, we have

$$
\int_{v > 0} e^{-\lambda \mathcal{L}(v)} dv > \int_{u > 0, u \in u' + \mathbb{R}^d} e^{-\lambda \mathcal{L}(v')} \left| \frac{\partial v}{\partial u} \right| du.
$$

Thus, the aim is to show that with high probability over the data, for any choice of $u'$, the integral $\int_{u > 0, u \in u' + \mathbb{R}^d} \left| \frac{\partial v}{\partial u} \right| du$ is infinite. To this end, we will perform another change of variables $u = u' + A\tilde{u}$ where $\tilde{u} \in \mathbb{R}^{d-n}$, and $A = [a^{(1)}, \ldots, a^{(d-n)}] \in \mathbb{R}^{d \times d-n}$ is a specially constructed matrix whose columns form an orthogonal basis for $\mathbb{R}^d$. We will select $a^{(1)} = \mu$ where $\mu \in \mathbb{R}^d$ and $\mu$ is positive in every dimension. The existence of such $\mu$ is guaranteed with high probability, as shown in Section F. Now by construction of $A$, we will always have $u' + A\tilde{u} > 0$ if $\tilde{u}$ is sufficiently large. Thus, there exists a convex cone $\{ |\tilde{u}| \leq c\tilde{u}_1, \forall 2 \leq i \leq d - n \}$, such that every $\tilde{u}$ in this cone satisfies $u = u' + A\tilde{u} > 0$. Integrating (6) over this cone is similar to integrating a polynomial with degree $-d/2$ for $d > 2n$. The full proof is in Section F.

5 Conclusion

In this work, we study the implicit bias effect induced by noise. For a quadratically-parameterized model, we theoretically show that the parameter-dependent noise has a strong implicit bias, which can help recover the sparse ground-truth from limited data. In comparison, our negative result shows that such a bias cannot be induced by spherical Gaussian noise. Our result provides an explanation for the empirical observation that replacing mini-batch noise or label noise with Gaussian noise usually leads to degradation in the generalization performance of deep models.
Acknowledgements

JZH acknowledges support from the Enlight Foundation Graduate Fellowship. CW acknowledges support from an NSF Graduate Research Fellowship. JDL acknowledges support of the ARO under MURI Award W911NF-11-1-0303, the Sloan Research Fellowship, and NSF CCF 2002272. TM acknowledges support of Google Faculty Award. The work is also partially supported by SDSI and SAIL at Stanford.

References

Naman Agarwal, Rohan Anil, Elad Hazan, Tomer Koren, and Cyril Zhang. Disentangling adaptive gradient methods from learning rates. *arXiv preprint arXiv:2002.11803*, 2020.

Dennis Amelunxen, Martin Lotz, Michael B McCoy, and Joel A Tropp. Living on the edge: Phase transitions in convex programs with random data. *Information and Inference: A Journal of the IMA*, 3(3):224–294, 2014.

Raman Arora, Peter Bartlett, Poorya Mianjy, and Nathan Srebro. Dropout: Explicit forms and capacity control. *arXiv preprint arXiv:2003.03397*, 2020.

Sanjeev Arora, Zhiyuan Li, and Kaifeng Lyu. Theoretical analysis of auto rate-tuning by batch normalization. *arXiv preprint arXiv:1812.03981*, 2018.

Sanjeev Arora, Nadav Cohen, Wei Hu, and Yuping Luo. Implicit regularization in deep matrix factorization. In *Advances in Neural Information Processing Systems*, pages 7411–7422, 2019a.

Sanjeev Arora, Simon S Du, Wei Hu, Zhiyuan Li, and Ruosong Wang. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. *arXiv preprint arXiv:1901.08584*, 2019b.

Guy Blanc, Neha Gupta, Gregory Valiant, and Paul Valiant. Implicit regularization for deep neural networks driven by an ornstein-uhlenbeck like process. *arXiv preprint arXiv:1904.09080*, 2019.

Sébastien Bubeck, Ronen Eldan, and Joseph Lehec. Sampling from a log-concave distribution with projected langevin monte carlo. *Discrete & Computational Geometry*, 59(4):757–783, 2018.

Kaidi Cao, Colin Wei, Adrien Gaidon, Nikos Arechiga, and Tengyu Ma. Learning imbalanced datasets with label-distribution-aware margin loss. In *Advances in Neural Information Processing Systems*, pages 1565–1576, 2019.

Pratik Chaudhari and Stefano Soatto. Stochastic gradient descent performs variational inference, converges to limit cycles for deep networks. In *2018 Information Theory and Applications Workshop (ITA)*, pages 1–10. IEEE, 2018.

Yuejie Chi, Yue M Lu, and Yuxin Chen. Nonconvex optimization meets low-rank matrix factorization: An overview. *IEEE Transactions on Signal Processing*, 67(20):5239–5269, 2019.

Lenaic Chizat and Francis Bach. A note on lazy training in supervised differentiable programming. *arXiv preprint arXiv:1812.07956*, 8, 2018.

Fan Chung and Linyuan Lu. Concentration inequalities and martingale inequalities: a survey. *Internet Mathematics*, 3(1):79–127, 2006.
Arnak S Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79(3):651–676, 2017.

Simon S Du, Wei Hu, and Jason D Lee. Algorithmic regularization in learning deep homogeneous models: Layers are automatically balanced. In *Advances in Neural Information Processing Systems*, pages 384–395, 2018a.

Simon S Du, Jason D Lee, Haochuan Li, Liwei Wang, and Xiyu Zhai. Gradient descent finds global minima of deep neural networks. *arXiv preprint arXiv:1811.03804*, 2018b.

Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points—online stochastic gradient for tensor decomposition. In *Conference on Learning Theory*, pages 797–842, 2015.

Rong Ge, Jason D Lee, and Tengyu Ma. Matrix completion has no spurious local minimum. In *Advances in Neural Information Processing Systems*, pages 2973–2981, 2016.

Behrooz Ghorbani, Song Mei, Theodor Misiakiewicz, and Andrea Montanari. Limitations of lazy training of two-layers neural network. In *Advances in Neural Information Processing Systems*, pages 9108–9118, 2019.

Daniel Gissin, Shai Shalev-Shwartz, and Amit Daniely. The implicit bias of depth: How incremental learning drives generalization. *arXiv preprint arXiv:1909.12051*, 2019.

Priya Goyal, Piotr Dollár, Ross Girshick, Pieter Noordhuis, Lukasz Wesolowski, Aapo Kyrola, Andrew Tulloch, Yangqing Jia, and Kaiming He. Accurate, large minibatch sgd: Training imagenet in 1 hour. *arXiv preprint arXiv:1706.02677*, 2017.

Suriya Gunasekar, Blake E Woodworth, Srinadh Bhojanapalli, Behnam Neyshabur, and Nati Srebro. Implicit regularization in matrix factorization. In *Advances in Neural Information Processing Systems*, pages 6151–6159, 2017.

Suriya Gunasekar, Jason Lee, Daniel Soudry, and Nathan Srebro. Characterizing implicit bias in terms of optimization geometry. *arXiv preprint arXiv:1802.08246*, 2018a.

Suriya Gunasekar, Jason D Lee, Daniel Soudry, and Nati Srebro. Implicit bias of gradient descent on linear convolutional networks. In *Advances in Neural Information Processing Systems*, pages 9461–9471, 2018b.

Moritz Hardt, Benjamin Recht, and Yoram Singer. Train faster, generalize better: Stability of stochastic gradient descent. *arXiv preprint arXiv:1509.01240*, 2015.

Elad Hoffer, Itay Hubara, and Daniel Soudry. Train longer, generalize better: closing the generalization gap in large batch training of neural networks. In *Advances in Neural Information Processing Systems*, pages 1731–1741, 2017.

Elad Hoffer, Ron Banner, Itay Golan, and Daniel Soudry. Norm matters: efficient and accurate normalization schemes in deep networks. In *Advances in Neural Information Processing Systems*, pages 2160–2170, 2018.

Sergey Ioffe and Christian Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. *arXiv preprint arXiv:1502.03167*, 2015.
Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks. In Advances in neural information processing systems, pages 8571–8580, 2018.

Ziwei Ji and Matus Telgarsky. Gradient descent aligns the layers of deep linear networks. arXiv preprint arXiv:1810.02032, 2018a.

Ziwei Ji and Matus Telgarsky. Risk and parameter convergence of logistic regression. arXiv preprint arXiv:1803.07300, 2018b.

Nitish Shirish Keskar, Dheevatsa Mudigere, Jorge Nocedal, Mikhail Smelyanskiy, and Ping Tak Peter Tang. On large-batch training for deep learning: Generalization gap and sharp minima. arXiv preprint arXiv:1609.04836, 2016.

Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

Yann A LeCun, Léon Bottou, Genevieve B Orr, and Klaus-Robert Müller. Efficient backprop. In Neural networks: Tricks of the trade, pages 9–48. Springer, 2012.

Jian Li, Xuanyuan Luo, and Mingda Qiao. On generalization error bounds of noisy gradient methods for non-convex learning. arXiv preprint arXiv:1902.00621, 2019a.

Yuanzhi Li and Yingyu Liang. Learning overparameterized neural networks via stochastic gradient descent on structured data. In Advances in Neural Information Processing Systems, pages 8157–8166, 2018.

Yuanzhi Li, Tengyu Ma, and Hongyang Zhang. Algorithmic regularization in over-parameterized matrix sensing and neural networks with quadratic activations. arXiv preprint arXiv:1712.09203, 2017.

Yuanzhi Li, Colin Wei, and Tengyu Ma. Towards explaining the regularization effect of initial large learning rate in training neural networks. In Advances in Neural Information Processing Systems, pages 11669–11680, 2019b.

Kaifeng Lyu and Jian Li. Gradient descent maximizes the margin of homogeneous neural networks. arXiv preprint arXiv:1906.05890, 2019.

Charles H Martin and Michael W Mahoney. Implicit self-regularization in deep neural networks: Evidence from random matrix theory and implications for learning. arXiv preprint arXiv:1810.01075, 2018.

Poorya Mianjy and Raman Arora. On dropout and nuclear norm regularization. arXiv preprint arXiv:1905.11887, 2019.

Poorya Mianjy, Raman Arora, and Rene Vidal. On the implicit bias of dropout. arXiv preprint arXiv:1806.09777, 2018.

Wenlong Mou, Liwei Wang, Xiyu Zhai, and Kai Zheng. Generalization bounds of sgld for non-convex learning: Two theoretical viewpoints. arXiv preprint arXiv:1707.05947, 2017.

Mor Shpigel Nacson, Jason Lee, Suriya Gunasekar, Pedro HP Savarese, Nathan Srebro, and Daniel Soudry. Convergence of gradient descent on separable data. arXiv preprint arXiv:1803.01905, 2018.
Mor Shpigel Nacson, Suriya Gunasekar, Jason D Lee, Nathan Srebro, and Daniel Soudry. Lexicographic and depth-sensitive margins in homogeneous and non-homogeneous deep models. \textit{arXiv preprint arXiv:1905.07325}, 2019.

Preetum Nakkiran, Prayaag Venkat, Sham Kakade, and Tengyu Ma. Optimal regularization can mitigate double descent. \textit{arXiv preprint arXiv:2003.01897}, 2020.

Radford M Neal et al. Mcmc using hamiltonian dynamics. \textit{Handbook of markov chain monte carlo}, 2 (11):2, 2011.

Arvind Neelakantan, Luke Vilnis, Quoc V Le, Ilya Sutskever, Lukasz Kaiser, Karol Kurach, and James Martens. Adding gradient noise improves learning for very deep networks. \textit{arXiv preprint arXiv:1511.06807}, 2015.

Jeffrey Negrea, Mahdi Haghifam, Gintare Karolina Dziugaite, Ashish Khisti, and Daniel M Roy. Information-theoretic generalization bounds for sgd via data-dependent estimates. In \textit{Advances in Neural Information Processing Systems}, pages 11013–11023, 2019.

Behnam Neyshabur, Russ R Salakhutdinov, and Nati Srebro. Path-sgd: Path-normalized optimization in deep neural networks. In \textit{Advances in Neural Information Processing Systems}, pages 2422–2430, 2015.

Samet Oymak and Mahdi Soltanolkotabi. Towards moderate overparameterization: global convergence guarantees for training shallow neural networks. \textit{IEEE Journal on Selected Areas in Information Theory}, 2020.

Tomaso Poggio, Kenji Kawaguchi, Qianli Liao, Brando Miranda, Lorenzo Rosasco, Xavier Boix, Jack Hidary, and Hrushikesh Mhaskar. Theory of deep learning iii: explaining the non-overfitting puzzle. \textit{arXiv preprint arXiv:1801.00173}, 2017.

Maxim Raginsky, Alexander Rakhlin, and Matus Telgarsky. Non-convex learning via stochastic gradient langevin dynamics: a nonasymptotic analysis. \textit{arXiv preprint arXiv:1702.03849}, 2017.

Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Minimax-optimal rates for sparse additive models over kernel classes via convex programming. \textit{Journal of Machine Learning Research}, 13 (Feb):389–427, 2012.

Noam Razin and Nadav Cohen. Implicit regularization in deep learning may not be explainable by norms. \textit{arXiv preprint arXiv:2005.06398}, 2020.

Gareth O Roberts, Richard L Tweedie, et al. Exponential convergence of langevin distributions and their discrete approximations. \textit{Bernoulli}, 2(4):341–363, 1996.

Christopher J Shallue, Jaehoon Lee, Joseph Antognini, Jascha Sohl-Dickstein, Roy Frostig, and George E Dahl. Measuring the effects of data parallelism on neural network training. \textit{arXiv preprint arXiv:1811.03600}, 2018.

Karen Simonyan and Andrew Zisserman. Very deep convolutional networks for large-scale image recognition. \textit{arXiv preprint arXiv:1409.1556}, 2014.

Samuel L Smith, Pieter-Jan Kindermans, Chris Ying, and Quoc V Le. Don’t decay the learning rate, increase the batch size. \textit{arXiv preprint arXiv:1711.00489}, 2017.
Daniel Soudry, Elad Hoffer, Mor Shpigel Nacson, Suriya Gunasekar, and Nathan Srebro. The implicit bias of gradient descent on separable data. *The Journal of Machine Learning Research*, 19(1): 2822–2878, 2018.

Nitish Srivastava, Geoffrey Hinton, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov. Dropout: a simple way to prevent neural networks from overfitting. *The journal of machine learning research*, 15(1):1929–1958, 2014.

Ilya Sutskever, James Martens, George Dahl, and Geoffrey Hinton. On the importance of initialization and momentum in deep learning. In *International conference on machine learning*, pages 1139–1147, 2013.

Christian Szegedy, Vincent Vanhoucke, Sergey Ioffe, Jon Shlens, and Zbigniew Wojna. Rethinking the inception architecture for computer vision. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 2818–2826, 2016.

Yee Whye Teh, Alexandre H Thiery, and Sebastian J Vollmer. Consistency and fluctuations for stochastic gradient langevin dynamics. *The Journal of Machine Learning Research*, 17(1):193–225, 2016.

Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological)*, 58(1):267–288, 1996.

Tomas Vaskevicius, Varun Kanade, and Patrick Rebeschini. Implicit regularization for optimal sparse recovery. In *Advances in Neural Information Processing Systems*, pages 2968–2979, 2019.

Sharan Vaswani, Francis Bach, and Mark Schmidt. Fast and faster convergence of sgd for overparameterized models and an accelerated perceptron. In The 22nd International Conference on Artificial Intelligence and Statistics, pages 1195–1204, 2019.

Colin Wei and Tengyu Ma. Data-dependent sample complexity of deep neural networks via lipschitz augmentation. In *Advances in Neural Information Processing Systems*, pages 9722–9733, 2019.

Colin Wei, Jason D Lee, Qiang Liu, and Tengyu Ma. Regularization matters: Generalization and optimization of neural nets vs their induced kernel. In *Advances in Neural Information Processing Systems*, pages 9709–9721, 2019.

Colin Wei, Sham Kakade, and Tengyu Ma. The implicit and explicit regularization effects of dropout. *arXiv preprint arXiv:2002.12915*, 2020.

Mingwei Wei and David J Schwab. How noise affects the hessian spectrum in overparameterized neural networks. *arXiv preprint arXiv:1910.00195*, 2019.

Max Welling and Yee W Teh. Bayesian learning via stochastic gradient langevin dynamics. In *Proceedings of the 28th international conference on machine learning (ICML-11)*, pages 681–688, 2011.

Yeming Wen, Kevin Luk, Maxime Gazeau, Guodong Zhang, Harris Chan, and Jimmy Ba. Interplay between optimization and generalization of stochastic gradient descent with covariance noise. *arXiv preprint arXiv:1902.08234*, 2019.

Ashia C Wilson, Rebecca Roelofs, Mitchell Stern, Nati Srebro, and Benjamin Recht. The marginal value of adaptive gradient methods in machine learning. In *Advances in Neural Information Processing Systems*, pages 4148–4158, 2017.
Blake Woodworth, Suriya Gunasekar, Jason D Lee, Edward Moroshko, Pedro Savarese, Itay Golan, Daniel Soudry, and Nathan Srebro. Kernel and rich regimes in overparametrized models. *arXiv preprint arXiv:2002.09277*, 2020.

Sho Yaida. Fluctuation-dissipation relations for stochastic gradient descent. *arXiv preprint arXiv:1810.00004*, 2018.

Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals. Understanding deep learning requires rethinking generalization. *arXiv preprint arXiv:1611.03530*, 2016.

Yuchen Zhang, Percy Liang, and Moses Charikar. A hitting time analysis of stochastic gradient langevin dynamics. *arXiv preprint arXiv:1702.05575*, 2017.

Zhanxing Zhu, Jingfeng Wu, Bing Yu, Lei Wu, and Jinwen Ma. The anisotropic noise in stochastic gradient descent: Its behavior of escaping from sharp minima and regularization effects. 2019.
A Experimental Details

A.1 Experimental Details for the Quadratically-Parameterized Model

In the experiment of our quadratically-parameterized model, we use a 100-dimensional model with $n = 40$ data randomly sampled from $\mathcal{N}(0, I_{100 \times 100})$. We set the first 5 dimensions of the ground-truth $v^*$ as 1, and the rest dimensions as 0. We always initialize with $v^{[0]} = 1$. We use a constant learning rate 0.01 for all the experiments except for label noise. For label noise, we start from 0.01 and then decay the learning rate by a factor of 10 after $1 \times 10^5$ and $2 \times 10^5$ iterations. For “full batch” experiment, we run full batch gradient descent without noise. For “small batch” experiment, in order to tune the scale of mini-batch noise while keeping the learning rate fixed, we add the following zero-mean noise to full gradient to simulate small batch noise with batch size 1: for each iteration, we randomly sample two data $i$ and $j$ from $[n]$, and add $\delta(\nabla \ell^{(i)}(v) - \nabla \ell^{(j)}(v))$ to the full gradient (we set $\delta = 1.0$ in our experiment). For label noise, we randomly sample $i \in [n]$ and $s \in \{\delta, -\delta\}$ (we set $\delta = 1.0$ in our experiment), and add noise $\nabla \tilde{\ell}^{(i)}(v) - \nabla \ell^{(i)}(v)$ to full gradient, where $\tilde{\ell}^{(i)}(v) \triangleq \frac{1}{2}(f_v(x^{(i)}) - y^{(i)} - s)^2$. For Gaussian noise experiments, we add noise $\xi \sim \mathcal{N}(0, \sigma^2 I_{d \times d})$ to full gradient every iteration, where the values of $\sigma$ are shown in Figure 1. For experiments except for Gaussian noises, we train a total of $3 \times 10^5$ iterations. For a more generous comparison, we run all the Gaussian noise experiments for 4 times longer (i.e., $1.2 \times 10^6$ iterations) while plotting them in the same figure after scaling the x-axis by a factor of 4. The test error is measured by the square of $\ell_2$ distance between $v^{s/2}$ and $v^{c/2}$, which is the same as the expectation of loss on a freshly randomly sampled data. The training and test error are plotted in Figure 1.

A.2 Experimental Details for Deep Neural Networks on CIFAR100

We train a VGG19 model (Simonyan and Zisserman, 2014) on CIFAR100, using a small and large batch baseline. We also experiment with adding Gaussian noise to the parameters after every gradient update as well as adding label noise in the following manner: with some probability that depends on the current iteration count, we replace the original label with a randomly chosen one.

To add additional mean-zero noise to the gradient which simulates the effect of label noise in the regression setting, we compute a noisy gradient of the cross-entropy loss $\ell_{ce}$ with respect to model output $f(x)$ as follows:

$$\nabla f \ell_{ce}(f, y) = \nabla f \ell_{ce}(f, y) + \sigma_{ln} z$$

where $z$ is a 100-dimensional vector (corresponding to each class) distributed according to $\mathcal{N}(0, I_{100 \times 100})$, and $y$ is the (possibly flipped) label. We backpropagate using this noisy gradient when we compute the gradient of loss w.r.t. parameters for the updates. After tuning, we choose the initial label-flipping probability as 0.1, and reduce it by a factor of 0.5 every time the learning rate is annealed. We choose $\sigma_{ln}$ such that $\sigma_{ln} \sqrt{\mathbb{E}[\|z\|^2]} = 0.1$, and also decrease $\sigma_{ln}$ by a factor of 0.5 every time the learning rate is annealed.

To add spherical Gaussian noise to the parameter every update, we simply set $W \leftarrow W + \sigma z$ after every gradient update, where $z$ is a mean-zero Gaussian whose coordinates are drawn independently from $\mathcal{N}(0, 1)$. We tune this $\sigma$ over the values shown in Figure 1.

We turn off weight decay and BatchNorm to isolate the regularization effects of just the noise alone. Standard data augmentation is still present in our runs. Our small batch baseline uses a

---

11The standard label smoothing (Szegedy et al., 2016) does not introduce a mean-zero noise and therefore has also a bias. Here we use the mean-zero version to isolate the effect of noise.
batch size of 26, and our large batch baseline uses a batch size of 256. In runs where we add noise, the batch size is always 256. For all runs, we use an initial learning rate of 0.004. We train for 410550 iterations (i.e., minibatches), annealing the learning rate by a factor of 0.1 at the 175950-th and 293250-th iteration. Our models take around 20 hours to train on a single NVIDIA TitanXp GPU when the batch size is 256. The final performance gap between label noise or small minibatch training v.s. large batch or Gaussian noise is around 13% accuracy.
B Proof of Stage 0 (Theorem 3.1)

In this section, we will first prove several lemmas on which the proof of Theorem 3.1 is built upon. Then we will provide a proof of Theorem 3.1.

Since the gradient descent with label noise algorithm will blow up with some very small chance, we first define a coupled version of each optimization trajectory such that it is bounded and behaves similarly to the original trajectory.

**Definition B.1.** (\(b\)-bounded coupling) Let \(v[0], v[1], \ldots, v[T]\) be a trajectory of label noise gradient descent with initialization \(v[0]\). We call the following random sequence \(\tilde{v}[t]\) a \(b\)-bounded coupling of \(v[t]\): starting from \(\tilde{v}[0] = v[0]\), for each time \(t < T\), if \(\|\tilde{v}[t]\|_1 \leq b\), we let \(\tilde{v}[t+1] = v[t+1]\); otherwise if \(\|\tilde{v}[t]\|_1 > b\) we don’t update, i.e., \(\tilde{v}[t+1] \neq v[t]\).

We first prove that the coupled trajectory \(\tilde{v}\) has bounded \(\ell_1\) norm with high probability.

**Lemma B.2.** In the setting of Theorem 3.1, assume for all \(i \in [n]\), \(\|x^{(i)}\|_\infty \leq b_x\) for some scalar \(b_x\). Let \(\eta \leq \frac{\rho}{6Tb_x^2(b_0 + r)}\), where \(b_0 = \frac{\text{Grad}}{\rho}\). Let \(\tilde{v}[t]\) be the \(b_0\)-bounded coupling of \(v[t]\). If \(\tilde{v}[t]\) is always positive on each dimension, then with probability at least \(1 - \frac{\rho}{\eta}\), there is

\[
\|\tilde{v}[T]\|_1 \leq b_0.
\]

**Proof of Lemma B.2.** Recall the update at \(t\)-th iteration is:

\[
v[t+1] = v[t] - \eta((v[t] \odot 2 - v^* \odot 2)^\top x^{(i)}) x^{(i)} \odot v[t] - \eta s_i x^{(i)} \odot v[t].
\]

We first bound the increase of \(\|\tilde{v}[t]\|_1\) in expectation. When \(\|\tilde{v}[t]\|_1 \leq b_0\), there is:

\[
\mathbb{E}[\tilde{v}[t+1]] = \tilde{v}[t] - \eta \mathbb{E}[(v[t] \odot 2 - v^* \odot 2)^\top x^{(i)} x^{(i)} \odot \tilde{v}[t] - \eta s_i x^{(i)} \odot \tilde{v}[t]]
\]

\[
\leq \tilde{v}[t] + \eta \left(\|\tilde{v}[t] \odot 2\|_1 + \|v^* \odot 2\|_1\right) b_x^2 \tilde{v}[t]
\]

\[
\leq \tilde{v}[t] + \eta (b_0^2 + r) b_x^2 \tilde{v}[t]
\]

where the first inequality is because we can separate the last term into \(v[t] \odot 2\) part and \(v^* \odot 2\) part and bound them with \(\|v[t] \odot 2\|_1\) and \(\|v^* \odot 2\|_1\), respectively, the second inequality is by \(\|v[t]\|_2 \leq \|v[t]\|_1\) and sparsity of \(v^*\). So summing over all dimensions we have \(\mathbb{E}[\|\tilde{v}[t+1]\|_1] \leq \|\tilde{v}[t]\|_1 + \eta b_0 b_x (b_0^2 + r)\). This bound is obviously also true when \(\|\tilde{v}[t]\|_1 > b_0\), in which case \(\tilde{v}[t+1] = \tilde{v}[t]\).

We then bound the probability of \(\|\tilde{v}[T]\|_1\) being too large:

\[
\Pr(\|\tilde{v}[T]\|_1 > b_0) \leq \frac{\mathbb{E}[\|\tilde{v}[T]\|_1]}{b_0}
\]

\[
\leq \frac{\tau d + T \eta b_0 b_x^2 (b_0^2 + r)}{b_0}
\]

\[
\leq \frac{\rho}{3},
\]

where the first inequality is Markov Inequality, the second is by the previous equation, and the third is by assumption of \(\eta\) and the definition of \(b_0\).

**Proof of Lemma 3.2.** Notice that when \(\|\tilde{v}[T]\|_1 \leq b_0\), there is \(v[T] = \tilde{v}[T]\), Lemma 3.2 naturally follows from Lemma B.2.
We also define the following potential function which is similar to the one introduced in Section 3.2 but is only non-zero in a bounded area.

**Definition B.3. (b-bounded potential function)** For a vector \( v \) that is positive on each dimension, we define the \( b \)-bounded potential function \( \Phi(v) \) as follows: if \( \|v\|_1 \leq b \), we let \( \Phi(v) \triangleq \sum_{k=1}^d \sqrt{v_k} \); otherwise \( \Phi(v) \triangleq 0 \).

Next, we prove that this potential function decreases to less than \( \sqrt{c_0} \) with high probability after some number of iterations.

**Lemma B.4.** In the setting of Theorem 3.1, let \( \epsilon_0 = 1/d \). Assume \( \|x(t)\|_\infty \leq b_x \) for \( i \in [n] \) with some \( b_x > 0 \), \( \mathbb{E}_i[(x_k(t))^2] \geq \frac{2}{3} \) for all \( k \in [d] \). Let \( b_0 = \frac{6rd}{\rho} \). Assume \( \eta \delta x_k + \eta (b_0^2 + r) b_x^2 \leq \frac{1}{16} \). \( \eta \delta^2 \geq 32(b_0^2 + r)b_x^2 \) and \( T = \lceil \frac{32}{\eta \rho b_x} \log(\frac{\rho \sqrt{c_0}}{3d}) \rceil \).

Let \( \tilde{v}^{[t]} \) be the \( b_0 \)-bounded coupling of \( v^{[t]} \), and \( \Phi(\cdot) \) is the \( b_0 \)-bounded potential function. If \( \tilde{v}^{[t]} \) is always positive on each dimension, then with probability at least \( 1 - \frac{\rho}{3} \), there is

\[
\Phi(\tilde{v}^{[T]}) \leq \sqrt{c_0}.
\]

**Proof of Lemma B.4.** We first show \( \Phi(\tilde{v}^{[t]}) \) decreases exponentially in expectation. If \( \|\tilde{v}^{[t]}\|_1 \leq b_0 \), we have:

\[
\mathbb{E}\left[ \Phi(\tilde{v}^{[t+1]}) \right] \leq \sum_{k=1}^d \mathbb{E}\left[ \sqrt{v_k^{[t+1]}} \right]
\]

\[
= \sum_{k=1}^d \mathbb{E}_{s_t,i_t} \left[ \sqrt{v_k^{[t]} - \eta s_t x_k^{(i_t)} \tilde{v}_k^{[t]} - \eta ((\tilde{v}^{[t]} \odot - v^{[t]})^\top x^{(i_t)}) x_k^{(i_t)} \tilde{v}_k^{[t]}} \right]
\]

\[
\leq \sum_{k=1}^d \sqrt{v_k^{[t]} \mathbb{E}_{s_t,i_t}} \left[ \sqrt{1 + \eta s_t x_k^{(i_t)} + \eta (b_0^2 + r) b_x^2} \right]
\]

where the second inequality is because \( \|\tilde{v}^{[t]}\|_2 = \|\tilde{v}^{[t]}\|_1^2 \leq b_0^2 \). Toward bounding the expectation, we notice that by Taylor expansion theorem, there is for any general function \( g(x) = \sqrt{1 + x} \), there is

\[
g(1 + x) \leq g(1) + g'(1)x + \frac{1}{2}g''(1) x^2 + \frac{M}{6} |x|^3,
\]

where \( M \) is upper bound on \( |g''(1 + x')| \) for \( x' \) in \( 0 \) to \( x \), which is less than \( 3 \) if \( |x| \leq \frac{1}{2} \). So in our theorem if \( \Delta \triangleq \eta s_t x_k^{(i_t)} + \eta (b_0^2 + r) b_x^2 \in [-\frac{1}{2}, \frac{1}{2}] \), we have

\[
\sqrt{1 + \eta s_t x_k^{(i_t)} + \eta (b_0^2 + r) b_x^2} \leq 1 + \frac{1}{2} \Delta - \frac{1}{8} \Delta^2 + \frac{1}{2} |\Delta|^3.
\]

Also since \( \mathbb{E}_{s_t,i_t}[\Delta] = \eta (b_0^2 + r) b_x^2 \), \( \mathbb{E}_{s_t,i_t}[\Delta^2] \geq \eta^2 \delta^2 \mathbb{E}_{s_t,i_t}[(x_k^{(i_t)})^2] \geq \frac{2}{7} \eta^2 \delta^2 \), we have when \( |\Delta| \leq \frac{1}{16} \) and \( \eta \delta^2 \geq 32(b_0^2 + r)b_x^2 \), we have \( \mathbb{E}_{s_t,i_t}[(\sqrt{1 + \Delta}) \leq 1 - \mathbb{E}_{s_t,i_t}[(1 - \frac{1}{16} \Delta^2)] \leq 1 - \frac{1}{32} \eta^2 \delta^2 \).

So

\[
\mathbb{E}\left[ \Phi(\tilde{v}^{[t+1]}) \right] \leq (1 - \frac{1}{32} \eta^2 \delta^2) \Phi(\tilde{v}^{[t]}).
\]

Also notice that when \( \|\tilde{v}^{[t]}\|_1 > b_0 \), there is \( \Phi(\tilde{v}^{[t+1]}) = \Phi(\tilde{v}^{[t]}) = 0 \), so obviously we have

\[
\mathbb{E}[\Phi(\tilde{v}^{[t+1]})] \leq (1 - \frac{1}{32} \eta^2 \delta^2) \Phi(\tilde{v}^{[t]}) \text{ always true}.
\]
Next we prove that $\Phi(\tilde{v}[T]) \leq \sqrt{\epsilon_0}$ with probability more than $1 - \frac{\rho}{2}$. This is because:

$$\Pr \left( \Phi(\tilde{v}[T]) > \sqrt{\epsilon_0} \right) \leq \frac{\mathbb{E} \left[ \Phi(\tilde{v}[T]) \right]}{\sqrt{\epsilon_0}}$$

$$\leq \frac{(1 - \frac{1}{32} \eta^2 \delta^3) T d \sqrt{\tau}}{\sqrt{\epsilon_0}}$$

$$\leq \frac{\rho}{3},$$

where the first inequality is if by Markov Inequality, the second inequality is by the previous inequality, and the last inequality is because $T = \lceil \frac{32}{\eta^2 \delta} \log \left( \frac{3d \sqrt{\tau}}{\rho \sqrt{\epsilon_0}} \right) \rceil$.

□

Now we are ready to prove Theorem 3.1 by combining the lemmas above.

**Proof of Theorem 3.1.** Let $\rho = 0.01$, $\epsilon_0 = 1/d$. By Lemma G.1, and Lemma G.2, when $n \geq \Theta(\log d)$, with probability at least $1 - \frac{\rho}{3}$ there is $\|x^{(i)}\|_\infty \leq b_x$ for all $i \in [n]$ with some $b_x = \Theta(\sqrt{\log(nd)})$, and $\mathbb{E}_i[(x^{(i)}_k)^2] \geq \frac{\rho}{2}$ for all $k \in [d]$.

Let $b_0 = \frac{6 \rho d}{\rho}$. We try to define $\eta$ and $\delta$ such that when $T = \lceil \frac{32}{\eta^2 \delta} \log \left( \frac{3d \sqrt{\tau}}{\rho \sqrt{\epsilon_0}} \right) \rceil$, the assumptions $\eta \leq \frac{6 \sqrt{b_0^2 (b_0^2 + r)}}{\sqrt{\rho} \sqrt{\epsilon_0}}$ and $\tilde{v}[t]$ always being positive in Lemma B.2 and assumptions $\eta \delta b_x + \eta (b_0^2 + r) b_x^2 \leq \frac{1}{16}$ and $\eta \delta^2 \geq 32 (b_0^2 + r) b_x^2$ in Lemma B.4 are satisfied.

Assume $\delta \geq 6 \times 32^2 b_x^3 (b_0^2 + r) \log \left( \frac{3d \sqrt{\tau}}{\rho \sqrt{\epsilon_0}} \right)$, then we only need $\eta \in \left[ \frac{6 \sqrt{b_0^2 (b_0^2 + r)}}{\sqrt{\rho} \sqrt{\epsilon_0}}, \frac{1}{32 b_x} \right]$, and then all the above assumptions are satisfied.

Let $\tilde{v}[t]$ be the $b_0$-bounded coupling of $v^{[t]}$. According to Lemma B.4, we know with probability at least $1 - \frac{\rho}{3}$, $\Phi(\tilde{v}[T]) \leq \sqrt{\epsilon_0}$, which means that either $\sum_{k=1}^d \sqrt{\tilde{v}^{[T]}_k} \leq \sqrt{\epsilon_0}$ or $\|\tilde{v}^{[T]}\|_1 > b_0$. According to Lemma B.2, we know with probability at most $\frac{\rho}{3}$, $\|\tilde{v}^{[T]}\|_1 > b_0$. Combining these two statements, we know with probability at least $1 - \frac{2 \rho}{3}$, $\|\tilde{v}^{[T]}\|_1 \leq b_0$ and $\sum_{k=1}^d \sqrt{\tilde{v}^{[T]}_k} \leq \sqrt{\epsilon_0}$. Notice that $\|\tilde{v}^{[T]}\|_1 \leq b_0$ implies $v^{[T]} = \tilde{v}^{[T]}$, while $\sum_{k=1}^d \sqrt{\tilde{v}^{[T]}_k} \leq \sqrt{\epsilon_0}$ implies $\tilde{v}^{[T]}_k \leq \epsilon_0$ for all dimension $k$, so we’ve finished the proof for the upper bound.

We then give a lower bound for each dimension of $\tilde{v}^{[T]}$. We can bound the decrease of any dimension $k$ at time $t$:

$$\tilde{v}^{[t+1]}_k \geq (1 - \eta \delta - \eta (b_0^2 + r)) \tilde{v}^{[t]}_k$$

$$\geq (1 - 2 \eta \delta) \tilde{v}^{[t]}_k,$$

where the first inequality is by update rule and the second is because $\delta > (b_0^2 + r)$. Putting in the value of $T$, we have

$$\tilde{v}^{[T]}_k \geq (1 - 2 \eta \delta)^T \tau$$

$$> \exp \left( - \frac{64}{\eta \delta} \log \left( \frac{3d \sqrt{\tau}}{\rho \sqrt{\epsilon_0}} \right) \right).$$

□
C Proof of Stage 1 (Theorem 3.3)

In this section, we will first prove several lemmas on which the proof of Theorem 3.3 is built upon. Then we will provide a proof of Theorem 3.3.

Similar to Section B, we first define a coupled version of each optimization trajectory such that it is bounded and behaves similarly to the original trajectory. The difference here is that since those dimensions in $S$ are expected to grow to larger than those dimensions not in $S$, we use different boundaries for these two type of dimensions.

**Definition C.1.** $(b, \epsilon)$-bounded coupling) Let $v[0], v[1], \ldots, v[T]$ be a trajectory of label noise gradient descent with initialization $v[0]$. Recall $S \subset [d]$ is the support set of $v^\star$, we notate $v_S^t$ a $r$-dimensional vector composed with those dimensions in $S$ of $\tilde{v}[t]$, and $\tilde{v}_S^t$ the other $d-r$ dimensions. We call the following random sequence $\tilde{v}[t]$ a $(b, \epsilon)$-bounded coupling of $v[t]$: starting from $\tilde{v}[0] = v[0]$, for each time $t < T$, if $\|\tilde{v}[t] - S\|_1 \leq \epsilon$ and $\|\tilde{v}[t] - S\|_\infty \leq b$, we let $\tilde{v}[t+1] \triangleq v[t+1]$; otherwise $\tilde{v}[t+1] \triangleq \tilde{v}[t]$.

We first show that dimensions in $S$ don’t become much larger than the ground truth (which is 1 for these dimensions).

**Lemma C.2.** In the setting of Theorem 3.3, let $\rho \triangleq \frac{1}{10^0}$, $c_1 \triangleq \frac{1}{10^r}$, $\epsilon_1 \triangleq \frac{12}{\rho}$. Assume $\|x(i)\|_\infty \leq b_x$ for $i \in [n]$ for some $b_x > 0$, and $\mathbb{E}_s[(x_k(i))^2] \geq \frac{2}{3}$ for $k \in [d]$. Let $\bar{v}[t]$ be a $(1 + c_1, \epsilon_1)$-bounded coupling of $v[t]$. Assume $\mathbb{E}_s[(x_k(i))^2] \geq \log \frac{\nu r T^2}{\rho}$, $(\epsilon_1^2 + r)C_x b_x^2 \leq \frac{c_1}{2b_x}$ and $\delta \geq b_x(\epsilon_1^2 + r)$. Then, with probability at least $1 - \frac{\rho}{b_x^2}$, there is $\|\tilde{v}[T] - S\|_\infty \leq 1 + c_1$.

**Proof of Lemma C.2.** For any fixed $1 \leq t_1 < t_2 \leq T$ and dimension $k \in S$, we consider the event that $\tilde{v}[t_1] \in [1 + \epsilon_1, 1 + \frac{c_1}{2}]$, and at time $t_2$ it is the first time in the trajectory such that $\tilde{v}[t_2] > 1 + c_1$. We first bound the probability of this event happens, i.e., the following quantity:

$$\Pr \left( \tilde{v}[t_2] > 1 + c_1 \land \tilde{v}[t_1] \leq 1 + \frac{c_1}{2} \land \tilde{v}[t_1:t_2] \in [1 + \frac{c_1}{3}, 1 + c_1] \right),$$

where $\tilde{v}[t_1:t_2] \in [1 + \frac{c_1}{3}, 1 + c_1]$ means that for all $t$ such that $t_1 \leq t < t_2$, there is $1 + \frac{c_1}{3} \leq \tilde{v}[t] \leq 1 + c_1$.

Notice that when $\|\tilde{v}[t] - S\|_1 \leq \epsilon_1$ and $\|\tilde{v}[t] - S\|_\infty \leq 1 + c_1$ and $\tilde{v}[t_1:t_2] \in [1 + \frac{c_1}{3}, 1 + c_1]$, there is

$$\mathbb{E}[\tilde{v}[t_1] - 1] = \mathbb{E}_{s^{i}, i} \left( \left[ 1 + \eta_s (x_k(i) - x_k(i)) (x_k(i)) \right] \tilde{v}[t] - 1 \right) \leq (\epsilon_1^2 + 1) - \frac{2}{3} \eta_\epsilon (\epsilon_1^2 + 1) (\epsilon_1^2 + 1) + \eta (\epsilon_1^2 C_x + r C_x) b_x \tilde{v}[t] \leq (1 - \eta) (\epsilon_1^2 + 1).$$

where the first inequality is because $\|\tilde{v}[t] - S\|_1 \leq \epsilon_1$ and $\mathbb{E}_s[(x_k(i))^2] \geq \frac{1}{2}$, the second inequality is because $(\epsilon_1^2 + r) b_x^2 C_x \leq \frac{c_1}{2b_x}$. Also, we can bound the variance of this martingale as

$$\text{Var} \left[ \tilde{v}[t_1] - 1 \mid \tilde{v}[t] - 1 \right] = \text{Var} \left[ \eta_s (x_k(i) - \tilde{v}[t]) \right] + \text{Var} \left[ \eta (\epsilon_1^2 C_x + r C_x) b_x \tilde{v}[t] \right] \leq \left( \eta \delta b_x (1 + c_1) \right)^2 + \eta^2 (\epsilon_1^2 + r) b_x^2 (1 + c_1)^2 \leq 4 \eta^2 \delta^2 b_x^2,$$
where the first inequality is because $\eta S \langle \bar{x}_k \rangle \bar{v}_k$ is mean-zero, the second inequality is by $\|x^{(i)}\|_\infty \leq b_x$, the third inequality is by $\delta \geq b_x (\varepsilon_1^2 + r)$.

By Lemma G.3, we have

$$\Pr(\bar{v}_k^{[t_2]} - 1 > c_1) \leq e^{\frac{-t_2}{\rho}} \leq e^{\frac{-t_2}{\rho}},$$

where the first inequality is by Lemma G.3, the second inequality is by taking the sum of denominator.

Finally, we finish the proof with a union bound. Since if $\|\bar{v}_S^{[T]}\|_\infty > 1 + c_1$, the event in Equation C has to happen for some $k \in S$ and $1 \leq t_1 < t_2 \leq T$, so we have

$$\Pr \left( \|\bar{v}_S^{[T]}\|_\infty > 1 + c_1 \right) \leq \sum_{k \in S} \sum_{1 \leq t_1 < t_2 \leq T} \Pr \left( \bar{v}_k^{[t_2]} > 1 + c_1 \land \bar{v}_k^{[t_1]} \leq 1 + \frac{c_1}{2} \land \bar{v}_k^{[t_1 : t_2]} \in [1 + \frac{c_1}{3}, 1 + c_1] \right)$$

$$\leq T^2 e^{\frac{-t_2}{\rho}} \leq \frac{\rho}{6},$$

where the last inequality is by assumption.

Then, we prove that those dimensions not in $S$ don’t become much larger than ground truth (which is 0 for these dimensions).

**Lemma C.3.** In the setting of Lemma C.2, assume $(\varepsilon_1^2 + r)C_x \leq \frac{\rho}{12T^2\eta b_x}$. Then, with probability at least $1 - \frac{\rho}{6}$, there is $\|\bar{v}_S^{[T]}\|_1 \leq \tilde{\epsilon}_1$.

**Proof of Lemma C.3.** We first bound the increase of $\|\bar{v}_S^{[t]}\|_1$ in expectation. When $\|\bar{v}_S^{[t]}\|_\infty \leq 1 + c_1$ and $\|\bar{v}_S^{[t]}\|_1 \leq \tilde{\epsilon}_1$, for any $k \notin S$, there is:

$$\mathbb{E} [\bar{v}_k^{[t+1]}] = \bar{v}_k^{[t]} - \eta \mathbb{E}_i \left[ ((\bar{v}_S^{[t]} \odot 2 - v^{* \odot 2})^T x^{(i)}) x^{(i)} \bar{v}_k^{[t]} \right]$$

$$\leq \bar{v}_k^{[t]} + \eta (\varepsilon_1^2 + r) C_x b_x^2 \tilde{v}_k^{[t]}.$$

because we can bound the dimensions in $S$ and those not in $S$ respectively. So summing over all dimensions not in $S$ we have $\mathbb{E} [\|\bar{v}_S^{[t+1]}\|_1] \leq \|\bar{v}_S^{[t]}\|_1 + \eta (\varepsilon_1^2 + r) C_x b_x^2 \tilde{\epsilon}_1$. This bound is obviously also true when $\|\bar{v}_S^{[t]}\|_\infty > 1 + c_1$ and $\|\bar{v}_S^{[t]}\|_1 > \tilde{\epsilon}_1$. 

24
We then bound the probability of $\|\tilde{v}_{S}^{[T]}\|_1$ being too large:

$$
\Pr \left( \|\tilde{v}_{S}^{[T]}\|_1 > \tilde{\epsilon}_1 \right) \leq \frac{\mathbb{E} \left( \|\tilde{v}_{S}^{[T]}\|_1 \right)}{\tilde{\epsilon}_1} 
\leq \frac{1 + T \eta(\tilde{\epsilon}_1^2 + r)C_x b_x^2}{\tilde{\epsilon}_1} \leq \frac{\rho}{6}. 
$$

where the first inequality is Markov Inequality, the second is by $\|\tilde{v}_{S}^{[0]}\|_1 \leq 1$ since every dimension is less than $1/d$, the third inequality is because $\tilde{\epsilon}_1 = \frac{12}{\rho}$ and $(\tilde{\epsilon}_1^2 + r)C_x \leq \frac{1}{12T \eta \theta^2}$.

Next, we prove that suppose all the dimensions (in $S$ or not) are never much larger than the ground truth, for each dimension in $S$, there is some time such that this dimension is very close to the ground truth.

**Lemma C.4.** In the setting of Lemma C.2, assume $(\tilde{\epsilon}_1^2 + r)C_x b_x^2 < \frac{c_1}{12} - \frac{c_2^2}{4}$, $\eta \theta^2 \leq \frac{c_1}{8}$, $T \eta \geq \frac{16}{c_1} \log \frac{1}{\epsilon_{\min}}$ and $\frac{T}{\sigma^2} \geq \frac{29}{c_1} \log \frac{6r}{\rho}$. Then, for any $k \in S$, with probability at least $1 - \frac{\rho}{6r}$, either $\max_{t \leq T} \tilde{v}_k^{[t]} \geq 1 - \frac{c_1}{2}$, or $\|\tilde{v}_S^{[T]}\|_\infty > 1 + c_1$, or $\|\tilde{v}_S^{[T]}\|_1 > \tilde{\epsilon}_1$.

**Proof of Lemma C.4.** Fix $k \in S$. Let $\hat{v}^{[t]}$ be the following coupling of $\tilde{v}^{[t]}$: starting from $\hat{v}^{[0]} = \tilde{v}^{[0]}$, for each time $t < T$, if $\|\tilde{v}_S^{[t]}\|_1 \leq \tilde{\epsilon}_1$ and $\|\tilde{v}_S^{[t]}\|_\infty \leq 1 + c_1$ and $\hat{v}_k^{[t]} \leq 1 - \frac{c_1}{2}$, we let $\hat{v}_k^{[t+1]} \triangleq \hat{v}_k^{[t]}$; otherwise $\hat{v}_k^{[t+1]} \triangleq (1 + \frac{c_1}{2})\hat{v}_k^{[t]}$. Intuitively, whenever $\tilde{v}^{[t]}$ exceeds the proper range, we only times $\hat{v}^{[t]}$ by $1 + \frac{c_1}{2} \eta$ afterwards, otherwise we let it be the same as $\hat{v}^{[t]}$.

We first show that $-t \log (1 + \frac{c_1}{2} \eta) + \log \hat{v}_k^{[t]}$ is a supermartingale, i.e., $\mathbb{E}[\log \hat{v}_k^{[t+1]} | \hat{v}_S^{[t]}] \geq \log (1 + \frac{c_1}{2} \eta) + \log \hat{v}_k^{[t]}$. This is obviously true if $\|\tilde{v}_S^{[t]}\|_1 > \tilde{\epsilon}_1$ or $\|\tilde{v}_S^{[t]}\|_\infty > 1 + c_1$ or $\hat{v}_k^{[t]} > 1 - \frac{c_1}{2}$. Otherwise, there is

$$
\mathbb{E}[\log \hat{v}_k^{[t+1]} | \hat{v}_S^{[t]}] = \mathbb{E}[\log \hat{v}_k^{[t+1]} | \hat{v}_S^{[t]}] 
= \mathbb{E}_{s_t, i_t} \left[ \log \left( 1 + \eta s_t - \eta (\tilde{v}_k^{[t]} \hat{v}_S^{[t]} + \tilde{v}_x^{[t]} x_{k}^{(i)} x_{x}^{(i)}) \right) \right] + \log \hat{v}_k^{[t]} 
\geq \mathbb{E}_{s_t} \left[ \log \left( 1 + \eta s_t + \frac{2}{3} \eta (1 - (\tilde{v}_k^{[t]} \hat{v}_S^{[t]} + \tilde{v}_x^{[t]} x_{k}^{(i)} x_{x}^{(i)}) \right) \right] + \log \hat{v}_k^{[t]} 
\geq \log (1 + \frac{c_1}{4} \eta) + \log \hat{v}_k^{[t]},
$$

25
where the first inequality is by the update rule, the second inequality is because \((\tilde{c}^2 + r)C_x b_x^2 < \frac{c_1^2}{2} - \frac{c_2^2}{4}\) and \(4\eta^2 \delta^2 \leq \frac{c_2}{2}\) and \(\delta \geq \tilde{c}^2 + r\). So by Azuma inequality, we have

\[
\Pr \left( \tilde{v}_k^{[T]} < 1 - \frac{c_1}{2} \right) \leq e^{-2(T \log (1 + \frac{c_1}{2}) \eta) \log \frac{e}{\min_{m,n} - \log (1 - \frac{c_1}{2})}} = e^{-\frac{T}{2T} \log (1 + \frac{c_1}{2})} \leq e^{-\frac{T}{2T} \log (1 + \frac{c_1}{2})} \leq e^{-\frac{T}{2T}} \leq \frac{\rho}{6}\text{.}
\]

where the first inequality is because Azuma inequality and \(\Var[\log \tilde{v}_k^{[t]} | \tilde{v}_k^{[t]}] \leq (2\eta \delta)^2\), and the second inequality is because \(T \log (1 + \frac{c_1}{2}) \eta \geq 2 \log \frac{1}{\min_{m,n}}\) which is true because \(T \eta \geq \frac{16}{\eta_1} \log \frac{1}{\min_{m,n}}\), the third inequality is because \(\log (1 + \frac{c_1}{2}) \eta \geq \frac{\eta}{2}\), the last inequality is because \(\frac{T}{2T} \geq \frac{\eta}{2\rho} \log \frac{6\rho}{\rho}\).

Next, we prove that for each dimension in \(S\), whenever it gets close to ground truth, it never becomes much smaller than the ground truth.

**Lemma C.5.** In the setting of Lemma C.2, assume \((\tilde{c}^2 + r)C_x b_x^2 \leq \frac{c_1}{20}\) and \(\frac{\tilde{c}^2}{8\eta^2} \geq \log \frac{6T^2 \rho}{\rho}\). Then, for any \(k \in S\), with probability at least \(1 - \frac{\rho}{6\eta^2}\), either \(\max_{t < T} \tilde{v}_k^{[t]} < 1 - \frac{c_1}{2}\) or \(\tilde{v}_k^{[T]} \geq 1 - c_1\).

**Proof of Lemma C.5.** For any fixed \(1 \leq t_1 < t_2 \leq T\) and dimension \(k \in S\), we consider the event that \(\tilde{v}_k^{[t_1]} \in [1 - \frac{c_1}{2}, 1 - \frac{c_1}{3}]\), and at time \(t_2\) it is the first time in the trajectory such that \(\tilde{v}_k^{[t_2]} > 1 - c_1\). We first bound the probability of this event happens, i.e., the following quantity:

\[
\Pr \left( \tilde{v}_k^{[t_2]} < 1 - c_1 \land \tilde{v}_k^{[t_1]} \geq 1 - \frac{c_1}{2} \land \tilde{v}_k^{[t_1:t_2]} \in [1 - c_1, 1 - \frac{c_1}{3}] \right),
\]

where \(\tilde{v}_k^{[t_1:t_2]} \in [1 - c_1, 1 - \frac{c_1}{3}]\) means that for all \(t\) such that \(t_1 \leq t < t_2\), there is \(1 - c_1 \leq \tilde{v}_k^{[t]} \leq 1 - \frac{c_1}{3}\).

Notice that when \(\left\| \tilde{v}_k^{[t]} \right\|_1 \leq \tilde{c}_1\) and \(\left\| \tilde{v}_k^{[t]} \right\|_\infty \leq 1 + c_1\) and \(\tilde{v}_k^{[t_1:t_2]} \in [1 - c_1, 1 - \frac{c_1}{3}]\),

\[
\mathbb{E}[1 - \tilde{v}_k^{[t_2]}] = \mathbb{E}_{s, s', i, t} \left[ 1 - \left( 1 + \eta s t x_{k}^t - \eta (\tilde{v}_k^{[t]} )^2 - \mathcal{O}(\eta^2) \right) \mathcal{T} x (i) x_k^{(i)} \tilde{v}_k^{[t]} \right]
\leq (1 - \tilde{v}_k^{[t]}) - \frac{3}{3} \tilde{v}_k^{[t]} (\tilde{v}_k^{[t]} + 1) (1 - \tilde{v}_k^{[t]}) + \eta (\tilde{c}_1^2 + r) C_x b_x^2 \tilde{v}_k^{[t]}.
\]

where the first inequality is because \(\left\| \tilde{v}_k^{[t]} \right\|_2 \leq \tilde{c}_1^2\), the second inequality is because \((\tilde{c}_1^2 + r)C_x b_x^2 \leq \frac{c_1}{20}\).

Also, we can bound the variance of this martingale as

\[
\Var \left[ 1 - \tilde{v}_k^{[t]} \mid \tilde{v}_k^{[t]} \right] \leq (2\eta \delta)^2 \text{.}
\]

By Lemma G.3, we have

\[
\Pr(1 - \tilde{v}_k^{[t_2]} > c_1) \leq e^{-2 \log (1 - \frac{c_1}{2T}) \log (1 - \frac{c_1}{2})} \leq e^{-\frac{T}{2T}} \leq \frac{\rho}{6},
\]

and the last inequality is true because \(\frac{T}{2T} \geq \frac{\eta}{2\rho} \log \frac{6\rho}{\rho}\).
where the first inequality is by Lemma G.3, the second inequality is by taking the sum of denominator.

Finally, we finish the proof with a union bound. Since if \( \max_{t<T} \tilde{v}^{[t]}_k > 1 - \frac{c_1}{2} \) but \( \tilde{v}^{[T]}_k < 1 - c_1 \), the event in Equation C has to happen for some \( 1 \leq t_1 < t_2 \leq T \), so we have

\[
\Pr \left( \max_{t<T} \tilde{v}^{[t]}_k > 1 - \frac{c_1}{2} \land \tilde{v}^{[T]}_k < 1 - c_1 \right) \\
\leq \sum_{1 \leq t_1 < t_2 \leq T} \Pr \left( \tilde{v}^{[t_2]}_k < 1 - c_1 \land \tilde{v}^{[t_1]}_k \geq 1 - \frac{c_1}{2} \land \tilde{v}^{[t_1:t_2]}_k \in [1 - c_1, 1 - \frac{c_1}{3}] \right) \\
\leq T^2 e^{-\frac{c_1^2}{32\eta \delta^2}} \\
\leq \frac{\rho}{6r}.
\]

\[\square\]

Similar to Section B, we define a potential function in a bounded area. Since now we only want those dimensions not in \( S \) to decrease, the potential function is only defined on these dimensions.

**Definition C.6.** \((b, \epsilon)-bounded potential function\) For a vector \( v \) that is positive on each dimension, we define the \((b, \epsilon)\)-bounded potential function \( \Phi(v) \) as follows: if \( \|v_S\|_1 \leq \epsilon \) and \( \|v_S\|_\infty \leq b \), we let \( \Phi(v) \triangleq \sum_{k \notin S} \sqrt{v_k} \); otherwise \( \Phi(v) \triangleq 0 \).

Then we prove that the this potential function decreases to less than \( \sqrt{\epsilon_1} \) after proper number of iterations.

**Lemma C.7.** In the setting of Lemma C.2, assume \( \frac{2}{3} \eta \delta^2 > 32 (\epsilon_1^2 + r)C_x b_x^2 \) and \( T \eta^2 \delta^2 \geq 16 \log \left( \frac{6\sqrt{7}}{b \sqrt{\epsilon_1}} \right) \). Then, with probability at least \( 1 - \frac{\rho}{6} \), there is \( \Phi(\tilde{v}^{[T]}_k) \leq \sqrt{\epsilon_1} \).

**Proof of Lemma C.7.** We first show \( \Phi(\tilde{v}^{[t]}_k) \) decreases exponentially in expectation. For any \( 0 \leq t \leq T \), if \( \|\tilde{v}^{[t]}_S\|_\infty \leq 1 + c_1 \) and \( \|\tilde{v}^{[t]}_S\|_1 \leq \epsilon_1 \), we have:

\[
\mathbb{E} \left[ \Phi(\tilde{v}^{[t+1]}_k) \right] \leq \sum_{k \notin S} \mathbb{E} \left[ \sqrt{\tilde{v}^{[t+1]}_k} \right] \\
= \sum_{k \notin S} \mathbb{E} [s_{t,i} \left( \sqrt{\tilde{v}^{[t]}_k + \eta s_t x_k^{(i)} \tilde{v}^{[t]}_k} - \eta (\tilde{v}^{[t]}_k \odot 2 - v \odot 2) x_k^{(i)} \tilde{v}^{[t]}_k \right) \right] \\
\leq \sum_{k \notin S} \mathbb{E} [s_{t,i} \left( \left( 1 + \eta s_t x_k^{(i)} \right) + \eta (\epsilon_1^2 + r) C_x b_x^2 \right) \right] \\
\leq (1 - \frac{1}{16} \eta^2 \delta^2) \Phi(\tilde{v}^{[t]}_k),
\]

where the second inequality is because \( \|\tilde{v}^{[t]}_k\|_2 \leq \epsilon_1^2 \), the last inequality is by Taylor expansion and \( \frac{2}{3} \eta \delta^2 > 32 (\epsilon_1^2 + r)C_x b_x^2 \). Also notice that when \( \|\tilde{v}^{[t]}_S\|_\infty > 1 + c_1 \) or \( \|\tilde{v}^{[t]}_S\|_1 > \epsilon_1 \), there is \( \Phi(\tilde{v}^{[t+1]}_k) = p(\tilde{v}^{[t]}_k) = 0 \), so obviously we have \( \mathbb{E} [\Phi(\tilde{v}^{[t+1]}_k)] \leq (1 - \frac{1}{16} \eta^2 \delta^2) \Phi(\tilde{v}^{[t]}_k) \) always true.
Next we bound the probability of $\tilde{\Phi}(\tilde{v}^T) \leq \sqrt{\epsilon_1}$:

$$\Pr(\tilde{\Phi}(\tilde{v}^T) > \sqrt{\epsilon_1}) \leq \frac{\mathbb{E}[\tilde{\Phi}(\tilde{v}^T)\sqrt{\epsilon_1}]}{\sqrt{\epsilon_1}} \leq \sqrt{\epsilon_1} \leq e^{-\frac{1}{16}T\eta^2\delta^2} \sqrt{d} \sqrt{\epsilon_1} \leq e^{-\frac{1}{16}T\eta^2\delta^2}$$

where the first inequality if by Markov Inequality, the second inequality is by the previous inequality and initially $\tilde{\Phi}(v^{[0]}) \leq \sqrt{d}$, the third is by $1 - x \leq e^{-x}$ for any $x \in \mathbb{R}$, and the last inequality is by $T\eta^2\delta^2 \geq 16 \log \left( \frac{6\sqrt{d}}{\rho\sqrt{\epsilon_1}} \right)$.

Now we are ready to prove Theorem 3.3 by combining the lemmas above.

**Proof of Theorem 3.3.** Let $\rho = 0.01$, $c_1 = 0.1$, $\tilde{c}_1 \triangleq \frac{12}{\rho}$, $C_x \triangleq \max_{j \neq k} |E_i[x_j(i)]|$. Let $b_x = \sqrt{2 \log \frac{30d\rho}{\rho}} = \tilde{\Theta}(1)$. According to Lemma G.1, when $n \leq d$, there is with probability at least $1 - \frac{\rho}{15}$, we have $\|x^{(i)}\|_\infty \leq b_x$ for $i \in [d]$.

Assume $\delta$ be positive number such that $\frac{16}{\delta} \log \frac{6\sqrt{d}}{\rho \epsilon_{\min} \sqrt{\epsilon_1}} \leq 1$ and $\delta \geq b_x(\tilde{c}_1^2 + r)$. (since $\epsilon_{\min} \geq \exp(-\tilde{O}(1))$ this means $\delta \geq \tilde{\Theta}(r + \log(1/\epsilon_1))$.) Let $P = \frac{c_1^2}{32\delta^2 b_x^2 \log \frac{Q}{\rho}}$, $Q = 2 \log \frac{1}{P}$, $\eta = \min \{ P, \frac{16}{\delta^2} \} = \tilde{\Theta}(\frac{1}{\delta})$, $T = \frac{16}{\delta^2} \log \frac{6\sqrt{d}}{\rho \epsilon_{\min} \sqrt{\epsilon_1} \eta} = \tilde{\Theta}(\log(1/\epsilon_1)/\eta)$. Assume $C_x b_x^2 (\tilde{c}_1^2 + r) \leq \min \{ \frac{32\delta^2}{48}, \frac{\rho}{12T \eta} \} = \tilde{\Theta}(\rho/\log(1/\epsilon_1))$. (this means $C_x \leq \tilde{\Theta}(\frac{\rho}{\log(1/\epsilon_1)})$.)

We show the assumptions in the previous lemmas are all satisfied. The assumption $\frac{c_1^2}{8\delta^2 b_x^2} \geq \log \frac{6rT^2}{\rho}$ in Lemma C.2 is satisfied by

$$\frac{c_1^2}{8\delta^2 b_x^2} \geq \log \frac{6rT^2}{\rho} \iff \frac{c_1^2}{8\delta^2 b_x^2} \geq \log \frac{6r}{\rho} + 4 \log \frac{1}{\eta} \iff \eta \log \frac{1}{\eta} \leq \frac{c_1^2}{32\delta^2 b_x^2 \log \frac{Q}{\rho}} = P,$$

where the first is by $T \leq \frac{1}{\eta^2}$, the second is by $\log \frac{6r}{\rho} + 4 \log \frac{1}{\eta} \leq 4 \log \frac{6r}{\rho} \log \frac{1}{\eta}$, and the last line is true because

$$\eta \log \frac{1}{\eta} \leq P \frac{\log Q}{Q} \leq P \left( \frac{\log Q}{Q} + \frac{\log 1/P}{Q} \right) \leq P.$$
The assumption $\delta \geq b_x(\epsilon_1^2 + r)$ in Lemma C.2 is satisfied by definition of $\delta$. The assumption $(\epsilon_1^2 + r)C_xb_x^2 \leq \frac{\rho}{12n\eta}$ in Lemma C.2 is satisfied by

$$C_xb_x^2(\epsilon_1^2 + r) \leq \frac{\eta\delta^2}{48} \leq \frac{c_1^2}{96} \leq \frac{c_1}{20},$$

where we use

$$\eta\delta^2 \leq \frac{\delta^2 P}{Q} \leq \frac{c_1^2}{2}.$$

The assumption $(\epsilon_1^2 + r)C_xb_x^2 \leq \frac{\rho}{12T\eta}$ in Lemma C.3 is satisfied by assumption of $C_x$.

The assumption $\frac{T}{\delta^2} \geq \frac{\eta\delta^2}{c_1\log \frac{6r}{\rho}}$ in Lemma C.4 is satisfied by

$$\frac{T}{\delta^2} \geq \frac{16}{\eta^2\delta^4} \log \frac{6r}{\rho} \geq \frac{26}{c_1^2} \log \frac{6r}{\rho} \geq \frac{9}{c_1} \log \frac{6r}{\rho}.$$

The other two assumptions $(\epsilon_1^2 + r)C_xb_x^2 < \frac{c_1}{18} - \frac{c_1^2}{T}$ and $\eta\delta^2 \leq \frac{c_1}{2}$ in Lemma C.4 follows from $C_xb_x^2(\epsilon_1^2 + r) < \frac{\eta\delta^2}{48}$ and $\eta\delta^2 \leq \frac{c_1}{2}$. The assumption $T\eta \geq \frac{16}{c_1} \log \frac{1}{\epsilon_{min}}$ in Lemma C.4 is satisfied by the definition of $T$.

The assumptions $(\epsilon_1^2 + r)C_x \leq \frac{c_1}{18}$ and $\frac{c_1^2}{8n\eta \delta^2} \geq \log \frac{6rT^2}{\rho}$ in Lemma C.5 are satisfied by the same reason as that of Lemma C.2. The assumption $T\eta^2 \delta^2 \geq 16 \log \left( \frac{6\sqrt{3}}{\rho \sqrt{\epsilon_1}} \right)$ in Lemma C.7 is satisfied by the definition of $T$, the assumption $\frac{2}{3}\eta\delta^2 > 32(\epsilon_1^2 + r)C_xb_x^2$ in Lemma C.7 is satisfied by the definition of $C_x$.

Since data are randomly from $\mathcal{N}(0, I)$, with $n \geq \tilde{\Theta}(\frac{r(\log(1/\epsilon_1)^2)}{\rho})$ data, there is with probability at least $1 - \frac{\rho}{16}$ there is $C_xb_x^2(\epsilon_1^2 + r) \leq \min \left\{ \frac{\eta\delta^2}{48}, \frac{\rho}{12T\eta} \right\} = \tilde{\Theta}(\log(1/\epsilon_1)/\eta)$. Meanwhile, according to Lemma G.2, with $n \geq \tilde{\Theta}(1)$ data with probability at least $1 - \frac{\rho}{16}$ there is $\mathbb{E}_i[(x_i^{(k)})^2] \geq \frac{2}{3}$ for all $k \in [d]$. According to definition of $b_x$, we know when $n \leq d$, with probability at least $1 - \frac{\rho}{16}$ there is also $\|x^{(i)}\|_\infty \leq b_x$ for all $i \in [n]$. In summary, with $d \geq n \geq \tilde{\Theta}(\frac{r(\log(1/\epsilon_1)^2)}{\rho})$ data, with probability at least $1 - \frac{\rho}{6}$ there is $C_xb_x^2(\epsilon_1^2 + r) \leq \min \left\{ \frac{\eta\delta^2}{48}, \frac{\rho}{12T\eta} \right\}$ and $\mathbb{E}_i[(x_i^{(k)})^2] \geq \frac{2}{3}$ for all $k \in [d]$ and $\|x^{(i)}\|_\infty \leq b_x$ for all $i \in [n]$.

Now we use these lemmas to finish the proof of the theorem. Let $\tilde{v}^{[t]}$ be a $(1 + c_1, \tilde{\epsilon}_1)$-bounded coupling of $v^{[t]}$, we only need to prove with probability at least $1 - \rho$, there is $\|\tilde{v}^{[T]} - 1\|_\infty \leq c_1$ and $\|\tilde{v}^{[T]}_S\|_1 \leq \epsilon_1$, which follows from a union bound of the previous propositions. In particular, Lemma C.2 and Lemma C.3 tell us that probability of $\|\tilde{v}^{[T]}\|_\infty > 1 + c_1$ or $\|\tilde{v}^{[T]}_S\|_1 > \tilde{\epsilon}_1$ is at most $\frac{\rho}{6}$. Lemma C.4 and Lemma C.5 tell us for any $k \in S$, probability of $\tilde{v}^{[T]}_k < 1 - c_1$ and $\|\tilde{v}^{[T]}_S\|_1 \leq 1 + c_1$ and $\|\tilde{v}^{[T]}_S\|_1 \leq \tilde{\epsilon}_1$ is at most $\frac{\rho}{36}$. Lemma C.7 tells us the probability of $\|\tilde{v}^{[T]}\|_1 > \epsilon_1$ and $\|\tilde{v}^{[T]}_S\|_\infty \leq 1 + c_1$ and $\|\tilde{v}^{[T]}_S\|_1 \leq \tilde{\epsilon}_1$ is at most $\frac{\rho}{6}$. Combining them together tells us that probability of $\|\tilde{v}^{[T]} - 1\|_\infty > c_1$ or $\|\tilde{v}^{[T]}_S\|_1 > \epsilon_1$ is at most $\rho$. □
D Proof of Stage 2 (Theorem 3.4)

The conclusion of Theorem 3.3 still allows constant error in the support, namely, \( \|v_S - v_S^*\|_\infty \leq 0.1 \).

To prove that further annealing the learning rate will let the algorithm fully converge to \( v^* \), we leverage a “bootstrapping” type of proof, where we first prove that whenever the support dimensions of the iterate is already somewhat close to \( v^* \), it can always become even closer (by a factor of 10) to ground truth, while at the same time the other dimensions don’t increase by too much. By repeatedly using this analysis, we can prove that eventually the iterates will be arbitrarily close to the ground truth. Formally, we index the number of rounds that we use this analysis to be \( s = 2, 3, \cdots \), and assume initially the iterate’s distance to \( v^* \) at the end of Theorem 3.3 is \( \|v_S - v_S^*\|_\infty \leq c_1 \triangleq 0.1 \) and \( \|v_S - v_S^*\|_1 \leq \epsilon_1 \). We prove the following theorem:

**Theorem D.1.** Let \( s \geq 2 \) be the index of the current round of bootstrapping. Let constant \( c_0 = 1/10 \). In the setting of Section 2.1, assume \( v^{[0]} \) is an initial parameter satisfying \( \|v_S^{[0]} - v_S^*\|_\infty \leq c_{s-1} \) and \( \|v_S^{[0]} - v_S^*\|_1 \leq \epsilon_{s-1} \), where \( 0 < \epsilon_{s-1} \leq c_{s-1} \leq c_0 \). Given a failure rate \( \rho > 0 \). Assume \( n \geq \Theta(r^2) \). Suppose we run SGD with label noise with noise level \( \delta \geq 0 \) and learning rate \( \eta \leq \tilde{\Theta}(c_0^2/((\delta^2 + r^2))) \) for \( T = \log(4/c_0)/\eta \) iterations. Then, with probability at least \( 1 - \rho \) over the randomness of the algorithm and data, there is \( \|v_S^{[T]} - v_S^*\|_\infty \leq c_s \triangleq c_{s-1}c_0 \) and \( \|v_S^{[T]} - v_S^*\|_1 \leq \epsilon_s \triangleq (4/c_0)^{2c_{s-1}} \epsilon_{s-1} \). Here \( \tilde{\Theta}(\cdot) \) omits poly logarithmic dependency on \( \rho \).

In the rest of this section, we will first prove several lemmas on which the proof of Theorem D.1 is built upon. Then we will provide a proof of Theorem D.1.

To begin with, we define the following coupled version of trajectories that are bounded to a region close to the ground truth.

**Definition D.2.** \( ((b, \epsilon)-to-v^* \text{-coupling}) \) Let \( v^{[0]}, v^{[1]}, \cdots, v^{[T]} \) be a trajectory of label noise gradient descent with initialization \( v^{[0]} \). Recall \( S \subset [d] \) is the support set of \( v^* \). We call the following random sequence \( \tilde{v}[t] \) a \((b, \epsilon)-to-v^* \text{-coupling of } v^{[t]} \): starting from \( \tilde{v}^{[0]} = v^{[0]} \), for each time \( t < T \), if \( \|\tilde{v}_S^{[t]}\|_1 \leq \epsilon \) and \( \|\tilde{v}_S^{[t]} - v_S^*\|_\infty \leq b \), we let \( \tilde{v}^{[t+1]} = v^{[t+1]} \); otherwise \( \tilde{v}^{[t+1]} = \tilde{v}^{[t]} \).

First, we show that with high probability, those dimensions in \( S \) don’t get too far away from ground truth (which is 1 for these dimensions).

**Lemma D.3.** In the setting of Theorem D.1, let \( C_x \triangleq \max_{j \in [n]} \mathbb{E}_{i}[x_j^{(i)}x_k^{(i)}] \). Assume \( \|x^{(i)}\|_\infty \leq b_x \) for \( i \in [n] \) for some \( b_x > 0 \), and \( \mathbb{E}_{i}[x_k^{(i)}] \geq \frac{2}{3} \) for \( k \in [d] \). Let \( \tilde{v}^{[t]} \) be a \((2c_{s-1}, \epsilon_s)-to-v^* \text{-coupling of } v^{[t]} \).

Assume \( \frac{c_0}{2\sqrt{b_x}(\delta^2 + b_x(\epsilon_x^2 + r^2))} \geq \log \frac{10rT^2}{\rho} \) and \( (\epsilon_s^2 + 4rc_{s-1})C_xb_x^2 \leq c_{s-1}/10 \). Then, with probability at least \( 1 - \frac{\rho}{T} \), there is \( \|\tilde{v}_S^{[T]} - 1\|_\infty \leq 2c_{s-1} \).

**Proof of Lemma D.3.** For any fixed \( 1 \leq t_1 < t_2 \leq T \) and dimension \( k \in S \), we consider the event that \( \tilde{v}^{[t_1]} \in [1 + \frac{2}{3}c_{s-1}, 1 + c_{s-1}] \), and at time \( t_2 \) it is the first time in the trajectory such that \( \tilde{v}_k^{[t_2]} > 1 + 2c_{s-1} \). We first bound the probability of this event happens, i.e., the following quantity:

\[
\Pr \left( \tilde{v}_k^{[t_2]} - 1 > 2c_{s-1} \land \tilde{v}_k^{[t_1]} - 1 \leq c_{s-1} \land \tilde{v}_k^{[t_1:t_2]} \in [1 + \frac{2}{3}c_{s-1}, 1 + 2c_{s-1}] \right),
\]

where \( \tilde{v}_k^{[t_1:t_2]} \in [1 + \frac{2}{3}c_{s-1}, 1 + 2c_{s-1}] \) means that for all \( t \) such that \( t_1 \leq t < t_2 \), there is \( 1 + \frac{2}{3}c_{s-1} \leq \tilde{v}_k^{[t]} \leq 1 + 2c_{s-1} \).
Notice that when \( \| \tilde{v}[t] \|_1 \leq \epsilon_s \) and \( \| \tilde{v}[t] - 1 \|_\infty \leq 2c_{s-1} \) and \( \tilde{v}[t; t+1] \in [1 + \frac{2}{3}c_{s-1}, 1 + 2c_{s-1}] \), there is
\[
\mathbb{E}[\tilde{v}[t; t+1] - 1] = \mathbb{E}_{a, i, \theta}[1 + \eta s_t x^i_{k} - \eta (\tilde{v}[t]^{\circ 2} - v^{\circ 2})^\top x(i) x(i)]v_k - 1 \\
\leq (\tilde{v}[t] - 1) - \frac{2}{3}\eta \tilde{v}[t](v_k[2] + 1)(\tilde{v}[t] - 1) + \eta (\epsilon_s^2 + 4rc_{s-1})C_x b_x^2 \tilde{v}[t] \\
\leq (1 - \eta)(\tilde{v}[t] - 1).
\]
where the first inequality is because \( \| \tilde{v}[t] \|^2 \leq \epsilon_s^2 \) and properties of the data, the second inequality is because \( (\epsilon_s^2 + 4rc_{s-1})b_x^2 C_x \leq \frac{c_{s-1}}{10} \). Also, we can bound the variance of this martingale as
\[
\text{Var}[\tilde{v}[t; t+1] - 1 | \tilde{v}[t] - 1] = \text{Var}[\eta s_t x^i_{k} \tilde{v}[t]] + \text{Var}[\eta ((\tilde{v}[t]^{\circ 2} - v^{\circ 2})^\top x(i)) x(i) \tilde{v}[t]] \\
\leq (\eta b_s (1 + 2c_{s-1}))^2 + \eta^2 (\epsilon_s^2 + r)^2 b_x^2 (1 + 2c_{s-1})^2 \\
\leq 4\eta^2 b_x^2 (\delta^2 + b_x^2 (\epsilon_s^2 + r)^2),
\]
By Lemma G.3, we have
\[
\text{Pr}(\tilde{v}[t; t] > 2c_{s-1} \land \tilde{v}[t] - 1 \leq c_{s-1} \land \tilde{v}[t; t+1] \in [1 + \frac{2}{3}c_{s-1}, 1 + 2c_{s-1}]) \\
\leq e^{2\eta^2 b_x^2 (\delta^2 + b_x^2 (\epsilon_s^2 + r)^2) \sum_{i=0}^{t-1}(1 - \eta)^{2i}} \\
\leq e^{2\eta^2 b_x^2 (\delta^2 + b_x^2 (\epsilon_s^2 + r)^2)}
\]
where the first inequality is by Lemma G.3, the second inequality is by taking the sum of denominator.

Similarly, we bound
\[
\text{Pr}(1 - \tilde{v}[t] > 2c_{s-1} \land 1 - \tilde{v}[t] \leq c_{s-1} \land \tilde{v}[t; t+1] \in [1 - 2c_{s-1}, 1 - \frac{2}{3}c_{s-1}]),
\]
Notice that when \( \| \tilde{v}[t] \|_1 \leq \epsilon_s \) and \( \| \tilde{v}[t] - 1 \|_\infty \leq 2c_{s-1} \) and \( \tilde{v}[t; t+1] \in [1 - 2c_{s-1}, 1 - \frac{2}{3}c_{s-1}] \), there is
\[
\mathbb{E}[1 - \tilde{v}[t; t+1]] = \mathbb{E}_{a, i, \theta}[1 - (1 + \eta s_t x^i_{k} - \eta (\tilde{v}[t]^{\circ 2} - v^{\circ 2})^\top x(i)) x(i)]v_k - 1 \\
\leq (1 - \tilde{v}[t]) - \frac{2}{3}\eta \tilde{v}[t](v_k[2] + 1)(1 - \tilde{v}[t]) + \eta (\epsilon_s^2 + 4rc_{s-1})C_x b_x^2 \tilde{v}[t] \\
\leq (1 - \eta)(1 - \tilde{v}[t]).
\]
where the first inequality is because \( \| \tilde{v}[t] \|^2 \leq \epsilon_s^2 \) and the properties of data, the second inequality is because \( (\epsilon_s^2 + 4rc_{s-1})b_x^2 C_x \leq \frac{c_{s-1}}{10} \). So
\[
\text{Pr}(1 - \tilde{v}[t] > 2c_{s-1} \land 1 - \tilde{v}[t] \leq c_{s-1} \land \tilde{v}[t; t+1] \in [1 - 2c_{s-1}, 1 - \frac{2}{3}c_{s-1}]) \\
\leq e^{2\eta^2 b_x^2 (\delta^2 + b_x^2 (\epsilon_s^2 + r)^2) \sum_{i=0}^{t-1}(1 - \eta)^{2i}} \\
\leq e^{2\eta^2 b_x^2 (\delta^2 + b_x^2 (\epsilon_s^2 + r)^2)}
\]
31
where the first inequality is by Lemma G.3, the second inequality is by taking the sum of denominator.

Finally, we finish the proof with a union bound. Since if \( \| \tilde{v}^{[T]}_S \|_\infty > 2c_{s-1} \), either event in Equation D or in Equation D has to happen for some \( k \in S \) and \( 1 \leq t_1 < t_2 \leq T \), so we have

\[
\Pr \left( \| \tilde{v}^{[T]}_S - 1 \|_\infty > 2c_{s-1} \right) \leq \sum_{k \in S} \sum_{1 \leq t_1 < t_2 \leq T} \Pr \left( \tilde{v}^{[t_1]}_k - 1 > 2c_{s-1} - \tilde{v}^{[t_2]}_k \right) \leq 2c_{s-1} \right) \leq \sum_{k \in S} \sum_{1 \leq t_1 < t_2 \leq T} \Pr \left( 1 - \tilde{v}^{[t_2]}_k > 2c_{s-1} - 1 - \tilde{v}^{[t_1]}_k \right) \leq (1 - 2c_{s-1}, 1 - \tilde{v}^{[t_2]}_k) \epsilon \leq 2T^2 e^{\frac{t_2^2}{2T^2(r_2^2 + r_2^2 + r_2^2)}} \leq \frac{p}{5},
\]

where the first inequality is by union bound, the second inequality is by previous results, the third inequality is by assumption of this lemma.

The next step is to show that those dimensions not in \( S \) remain close to ground truth 0.

Lemma D.4. In the setting of Lemma D.3, assume \( (\epsilon^2 + 4c_{s-1}^r)C_x b_x^2 \leq c_{s-1}, \epsilon_s > (1 + \eta c_{s-1}) T \epsilon_{s-1} \) and \( \frac{(1 + \eta c_{s-1})^T \epsilon_s - \epsilon_{s-1}^r}{2T\eta b_x^2 (r_2^2 + r_2^2 + r_2^2)} \geq \log \frac{5}{\tilde{p}} \). Then, with probability at least \( 1 - \tilde{p} \), there is \( \| \tilde{v}^{[T]}_S \|_1 \leq \epsilon_s \).

Proof of Lemma D.4. When \( \| \tilde{v}^{[t]}_S \|_\infty \leq 2c_{s-1} \) and \( \| \tilde{v}^{[t]}_S \|_1 \leq \epsilon_s \), for any \( k \notin S \), there is:

\[
\mathbb{E} \left[ \tilde{v}_k^{[t+1]} \right] = \tilde{v}_k^{[t]} - \eta \mathbb{E} \left[ \left( (\tilde{v}^{[t]}_S \otimes 2 - \bar{v}^{[t]}_S \otimes 2) x^{(i)}(\bar{v}^{[t]}_S \otimes 2) \right) \right] \leq \tilde{v}_k^{[t]} + \eta \left( \| \tilde{v}^{[t]}_S \|_1 + 4c_{s-1}r \right) C_x b_x^2 \tilde{v}_k^{[t]} \leq \tilde{v}_k^{[t]} + \eta \left( \epsilon_s^2 + 4c_{s-1}^r \right) C_x b_x^2 \tilde{v}_k^{[t]} \leq (1 + \eta c_{s-1}) \tilde{v}_k^{[t]} .
\]

where the first inequality is because we can bound the dimensions in \( S \) and those not in \( S \) with \( \| \tilde{v}^{[t]}_S \|_1 \leq C_x b_x^2 \) and \( 4c_{s-1}r C_x b_x^2 \) respectively, the second inequality is by \( \| \tilde{v}^{[t]}_S \|_2 \leq \| \tilde{v}^{[t]}_S \|_1 \), the third is because \( \epsilon_s^2 + 4c_{s-1}^r \leq c_{s-1} \). Summing over all \( k \notin S \) we have \( \mathbb{E} \left[ \| \tilde{v}^{[t+1]}_S \|_1 \right] \leq (1 + \eta c_{s-1}) \| \tilde{v}^{[t]}_S \|_1 \). This bound is obviously also true when \( \| \tilde{v}^{[t]}_S \|_\infty > 2c_{s-1} \) or \( \| \tilde{v}^{[t]}_S \|_1 > \epsilon_s \), in which case \( \tilde{v}^{[t+1]}_S = \tilde{v}^{[t]}_S \).

Therefore we know \( (1 + \eta c_{s-1}) T \| \tilde{v}^{[t]}_S \|_1 \) is a supermartingale. Also notice \( \| \tilde{v}^{[t+1]}_S \|_1 - \mathbb{E} \| \tilde{v}^{[t+1]}_S \|_1 \leq \eta \delta \epsilon_s \), By Azuma Inequality,

\[
\Pr \left( \| \tilde{v}^{[T]}_S \|_1 > \epsilon_s \right) \leq \mathbb{E} \left[ \| \tilde{v}^{[T]}_S \|_1 \right] - \mathbb{E} \left[ \| \tilde{v}^{[T]}_S \|_1 \right] \leq \eta \delta \epsilon_s \leq \frac{p}{4} .
\]

here we are using \( \epsilon_s > (1 + \eta c_{s-1}) T \epsilon_{s-1} \) by assumption and the last step is by assumption. \( \square \)
Then we prove that when every dimension (in $S$ or not) remains close to the ground truth, each dimension in $S$ will become even closer ($c_s/2$-close) to ground truth at some time.

**Lemma D.5.** In the setting of Lemma D.3, assume $(1 - \eta)^T 2c_{s-1} < \frac{c_s}{2}, \frac{c_s^2 - 1}{2c_{s-1}^2} \geq \log \frac{5r}{\rho}$, and $(\epsilon_s^2 + 4c_{s-1}r)C_xb_x^2 \leq \frac{1}{10}$. Then, for any $k \in S$, with probability at least $1 - \frac{\rho}{5r}$, either $\min_{t \leq T} |\tilde{v}_k^{[t]} - 1| \geq \frac{c_s}{2}$, or $\|\tilde{v}_S^{[T]} - 1\|_{\infty} > 2c_{s-1}$, or $\|\tilde{v}_S^{[T]}\|_1 > \epsilon_s$.

**Proof of Lemma D.5.** We first consider when $\tilde{v}_k^{[t]} \in [1 + \frac{c_s}{2}, 1 + 2c_{s-1}]$. For some $t < T_2$, if $\|\tilde{v}_S^{[t]} - 1\|_{\infty} \leq c_{s-1}$ and $\|\tilde{v}_S^{[t]}\|_1 \leq \epsilon_s$, there is

$$E[\tilde{v}_k^{[t+1]} - 1] = \tilde{v}_k^{[t]} - \eta E_t [((\tilde{v}_k^{[t]} - \epsilon_{j+2})^\top \epsilon_{j+2}) \epsilon_{j+2}]^\top \epsilon_{j+2} - 1$$

$$\leq (s_k^{[t]} - 1) - \frac{2}{3} \tilde{v}_k^{[t]} (s_k^{[t]} + 1)(\tilde{v}_k^{[t]} - 1) + \eta (\epsilon_s^2 + 4c_{s-1}r)C_xb_x^2$$

$$\leq (1 - \eta) (s_k^{[t]} - 1).$$

Here the first inequality is by assumption, the second inequality is because of $(\epsilon_s^2 + 4c_{s-1}r)C_xb_x^2 \leq \frac{1}{10}c_s$ and $c_{s-1} \leq \frac{1}{10}$. We define the event $E_t$ as $\|\tilde{v}_S^{[t]} - 1\|_{\infty} > 2c_{s-1}$ or $\|\tilde{v}_S^{[t]}\|_1 > \epsilon_s$. Since $(1 - \eta)T_2 2c_{s-1} < \frac{c_s}{2}$ by assumption, if $\tilde{v}_k^{[0]} \in [1 + \frac{c_s}{2}, 1 + 2c_{s-1}]$, by Lemma G.4 we know:

$$\Pr \left( \min_{t \leq T} \tilde{v}_k^{[t]} > 1 + c_{s-1} \wedge \|\tilde{v}_S^{[T]} - 1\|_{\infty} \leq 2c_{s-1} \wedge \|\tilde{v}_S^{[T]}\|_1 \leq \epsilon_s \right)$$

$$\leq e^{-\frac{(2c_{s-1})(1 - \eta)(1 - c_{s-1})}{2\eta c_{s-1} + 2\eta c_{s-1} + c_{s-1}^2}}$$

$$\leq e^{-\frac{\epsilon_{s-1}^2}{2\eta c_{s-1} + 2\eta c_{s-1} + c_{s-1}^2}}$$

$$\leq \frac{\rho}{5r},$$

where the second inequality is because of assumption.

Similarly, when $\tilde{v}_k^{[0]} \in [1 - 2c_{s-1}, 1 - \frac{c_s}{2}]$, there is

$$\Pr \left( \max_{t \leq T} \tilde{v}_k^{[t]} < 1 - c_{s-1} \wedge \|\tilde{v}_S^{[T]} - 1\|_{\infty} \leq 2c_{s-1} \wedge \|\tilde{v}_S^{[T]}\|_1 \leq \epsilon_s \right)$$

$$\leq e^{-\frac{\epsilon_{s-1}^2}{2\eta c_{s-1} + 2\eta c_{s-1} + c_{s-1}^2}}$$

$$\leq \frac{\rho}{5r}.$$
Lemma D.6. In the setting of Lemma D.3, assume \( \frac{s m^2 (\alpha^2 + b_s^2 (c_s^2 + r))}{\rho} \geq \log \frac{n T^2}{\rho} \) and \( (c_s^2 + 2rc_s)C_x b_x^2 \leq \frac{c_s}{20} \).
Then, for any dimension \( k \in S \), with probability at most \( \frac{\rho}{n} \), there is \( \min_{t \leq T} |\tilde{v}_k[t] - 1| \leq \frac{1}{2} c_s \) and \( |\tilde{v}_S[t] - 1|_\infty \leq 2c_{s-1} \) and \( |\tilde{v}_S[T] - 1|_1 \leq \epsilon_s \).

Proof of Lemma D.6. For any fixed \( 1 \leq t_1 < t_2 \leq T \), we consider the event that \( \tilde{v}_k[t_1] \in [1 + \frac{1}{3} c_s, 1 + \frac{1}{3} c_s] \), and at time \( t_2 \) it is the first time in the trajectory such that \( \tilde{v}_k[t_2] > 1 + c_s \). We first bound the probability of this event happens, i.e., the following quantity:

\[
\Pr \left( \tilde{v}_k[t_2] - 1 > c_s \wedge \tilde{v}_k[t_1] - 1 \leq \frac{1}{2} c_s \wedge \tilde{v}_k[t_1:t_2] \in [1 + \frac{1}{3} c_s, 1 + c_s] \right).
\]

Notice that when \( \|\tilde{v}_S[t]\|_1 \leq \epsilon_s \) and \( \|\tilde{v}_S[t]\|_\infty \leq 2c_{s-1} \) and \( \tilde{v}_k[t_1:t_2] \in [1 + \frac{1}{3} c_s, 1 + c_s] \), there is

\[
\mathbb{E}[\tilde{v}_k[t_2] - 1] = \mathbb{E}_{s, t_1} \left[ (1 + \eta s_t x_k^{t_i} - \eta \mathbb{E}_{t_1}(\tilde{v}_S[t] \circ 2 - v^\top x^{t_i})x_k^{t_i})\tilde{v}_k[t] - 1 \right] \\
\leq (\tilde{v}_k[t] - 1) - \frac{2}{3} \eta \tilde{v}_k[t] (\tilde{v}_k[t] + 1) (\tilde{v}_k[t] - 1) + \eta (c_s^2 + 2rc_s)C_x b_x^2 \tilde{v}_k[t] \\
\leq (1 - \eta)(\tilde{v}_k[t] - 1).
\]

where the first inequality is because \( \|\tilde{v}_S[t]\|_2^2 \leq \epsilon_s^2 \), the second inequality is because \( (c_s^2 + 2rc_s)C_x b_x^2 \leq \frac{c_s}{20} \). Also, we can bound the variance of this martingale as

\[
\mathbb{E} \left[ \left( \tilde{v}_k[t_2] - 1 \right) \left| \tilde{v}_k[t_1] - 1 \right. \right] = \mathbb{E} \left[ \eta (\tilde{v}_S[t] \circ 2 - v^\top x^{t_i})x_k^{t_i} \right] + \mathbb{E} \left[ (\tilde{v}_k[t] \circ 2 - v^\top x^{t_i})x_k^{t_i} \right] \\
\leq (\eta \delta b_x (1 + c_s))^2 + \eta^2 (c_s^2 + r)^2 b_x^4 (1 + c_s)^2 \\
\leq 4 \eta^2 b_x^2 (\delta^2 + b_x^2 (c_s^2 + r)^2),
\]

By Lemma G.3, we have

\[
\Pr(\tilde{v}_k[t_2] - 1 > c_s \wedge \tilde{v}_k[t_1] - 1 \leq \frac{1}{2} c_s \wedge \tilde{v}_k[t_1:t_2] \in [1 + \frac{1}{3} c_s, 1 + c_s]) \\
\leq e^{\frac{-2}{\alpha^2 + b_s^2 (c_s^2 + r)^2}} \sum_{t=0}^{T-1} (1 - \eta^2)^t \\
\leq e^{\frac{-2}{\alpha^2 + b_s^2 (c_s^2 + r)^2}},
\]

where the first inequality is by Lemma G.3, the second inequality is by taking the sum of denominator.

Similarly, we bound

\[
\Pr(1 - \tilde{v}_k[t_2] > c_s \wedge 1 - \tilde{v}_k[t_1] \leq \frac{1}{2} c_s \wedge \tilde{v}_k[t_1:t_2] \in [1 - c_s, 1 - \frac{1}{3} c_s]) \\
\leq e^{\frac{-2}{\alpha^2 + b_s^2 (c_s^2 + r)^2}}.
\]
Finally, we finish the proof with a union bound:

\[
\Pr \left( \min_{t \leq T} |\hat{v}_k^{[t]} - 1| \leq \frac{1}{2} c_s \land |\hat{v}_k^{[T]} - 1| > c_s \land \|\hat{v}_S^{[T]} - 1\|_\infty \leq 2c_{s-1} \land \|\hat{v}_S^{[T]}\|_1 \leq \epsilon_s \right)
\]
\[
\leq \sum_{1 \leq t_1 < t_2 \leq T} \Pr(\hat{v}_k^{[t_2]} - 1 > c_s \land \hat{v}_k^{[t_1]} - 1 \leq \frac{1}{2} c_s \land \hat{v}_k^{[t_1, t_2]} \in [1 + \frac{1}{3} c_s, 1 + c_s])
\]
\[
+ \sum_{1 \leq t_1 < t_2 \leq T} \Pr(1 - \hat{v}_k^{[t_2]} > c_s \land 1 - \hat{v}_k^{[t_1]} \leq \frac{1}{2} c_s \land \hat{v}_k^{[t_1, t_2]} \in [1 - c_s, 1 - \frac{1}{3} c_s])
\]
\[
\leq T^2 e^{-\eta_0(\frac{c_s^2}{b_x^2(\epsilon_s^2 + 2\epsilon_s^2 r)^2})} \leq \frac{\rho}{5r},
\]

where the first inequality is by union bound, the second inequality is by previous results, the third inequality is by assumption of this lemma. \(\square\)

Now we are ready to combine these lemmas to prove Theorem D.1.

**Proof of Theorem D.1.** Let \(C_x \triangleq \max_{j \neq k} \|\mathbb{E}[x_j^{(i)} x_k^{(i)}]\|, b_x = \sqrt{2 \log \frac{30d^2}{\rho}} = \tilde{\Theta}(1)\). According to Lemma G.1, when \(n \leq d\), there is with probability at least \(1 - \frac{\rho}{15}\) we have \(\|x^{(i)}\|_\infty \leq b_x\) for \(i \in [d]\).

Set \(\eta\) small enough such that \(\frac{c_s^2}{80b_x^2(\epsilon_s^2 + 2\epsilon_s^2 r)^2} \geq \log \frac{10T^2}{\rho}\). Obviously we only need \(\eta \leq \tilde{\Theta}(\frac{c_s^2}{\delta^2 + \rho^2})\), where \(\tilde{\Theta}(\cdot)\) omits poly logarithmic dependency on \(d\) and \(\rho\). Assume \((c_s^2 + 4c_{s-1}r)C_x b_x^2 \leq \tilde{c}_0\), which can be represented as \(C_x \leq \tilde{\Theta}(\frac{1}{\eta})\). Recall \(T = \frac{1}{\eta} \log \frac{4}{c_0}\), \(\epsilon_s = e^{2c_{s-1} \log \frac{1}{c_0} \epsilon_{s-1}}\).

We first show that the additional assumptions in the previous lemmas are satisfied. There is

\[
(1 + \eta c_{s-1})^T = (1 + \eta c_{s-1})^{\frac{1}{\eta} \log \frac{4}{c_0}} \leq e^{c_{s-1} \log \frac{4}{c_0}} \triangleq P.
\]

The assumption \(\epsilon_s > (1 + \eta c_{s-1})^T \epsilon_{s-1}\) in Lemma D.4 is therefore satisfied by definition of \(\epsilon_s\). The assumption \(\frac{(1 + \eta c_{s-1})^{-T} \epsilon_s - \epsilon_{s-1})^2}{2T\eta b_x^2(\delta^2 + b_x^2(\epsilon_s^2 + r)^2)\epsilon_s^2} \geq \frac{c_2^2}{\rho} \) in Lemma D.4 is satisfied because:

\[
\frac{(1 + \eta c_{s-1})^{-T} \epsilon_s - \epsilon_{s-1})^2}{2T\eta b_x^2(\delta^2 + b_x^2(\epsilon_s^2 + r)^2)\epsilon_s^2} \geq \frac{c_2^2(P^2 - P^{-2})^2}{2T\eta b_x^2(\delta^2 + b_x^2(\epsilon_s^2 + r)^2)\epsilon_s^2}
\]
\[
\geq \frac{c_2^2}{2T\eta b_x^2(\delta^2 + b_x^2(\epsilon_s^2 + r)^2)},
\]

which is larger than \(\log \frac{5}{\rho}\) by the definition of \(\eta\). The assumption \((1 - \eta)^T 2c_{s-1} \leq \frac{\tilde{c}_0}{2}\) in Lemma D.5 is satisfied because \((1 - \eta)^T 2c_{s-1} \leq (\frac{1}{2})^{\log \frac{4}{c_0}} 2c_{s-1} = \frac{\tilde{c}_0}{2}\). All the other assumptions in Lemma D.3, Lemma D.4, Lemma D.5 and Lemma D.6 naturally follows from the definition of \(\eta\) and the requirement of \(C_x\).

Since data are randomly from \(\mathcal{N}(0, I)\), with \(n \geq \tilde{\Theta}(r^2)\) data, there is with probability at least \(1 - \frac{\rho}{15}\) there is \((c_s^2 + 4c_{s-1}r)C_x b_x^2 \leq \tilde{c}_0\). Meanwhile, according to Lemma G.2 with \(n \geq \tilde{\Theta}(1)\) data with probability at least \(1 - \frac{\rho}{15}\) there is \(\mathbb{E}_i[(x_k^{(i)})^2] \geq \frac{2}{3}\) for all \(k \in [d]\). According to definition of \(b_x\),

35
we know when $n \leq d$, with probability at least $1 - \frac{\rho}{10}$ there is also $\|x^{(i)}\|_\infty \leq b_x$ for all $i \in [n]$. In summary, with $d \geq n \geq \tilde{\Theta}(r^2)$ data, with probability at least $1 - \frac{\rho}{5}$ there is $(\epsilon^2 + 4c_{s-1}r)C_xb_s^2 \leq \frac{c_s}{10}$ and $\mathbb{E}_i[(x^{(i)}_k)^2] \geq \frac{3}{2}$ for all $k \in [d]$ and $\|x^{(i)}\|_\infty \leq b_x$ for all $i \in [n]$.

Now we finish the proof with the above lemmas. Lemma D.3 and Lemma D.4 together tell us that with probability at least $1 - \frac{2\rho}{5}$, there is $\|\tilde{v}[T] - 1\|_\infty \leq 2c_{s-1}$ and $\|\tilde{v}[T]\|_1 \leq \epsilon_s$, in which case there is also $v^{[T]} = \tilde{v}[T]$ by the definition of $\tilde{v}[T]$. Lemma D.5 and Lemma D.6 together tell us the probability of $\|\tilde{v}[T]_S - 1\|_\infty \leq 2c_{s-1}$ and $\|\tilde{v}[T]_S\|_1 \leq \epsilon_s$ and $\|\tilde{v}[T]_S - 1\|_\infty > c_s$ is no more than $\frac{2\rho}{5}$. So together we know with probability at least $1 - \rho$, there is $\|v^{[T]}_S - 1\|_\infty \leq c_s$ and $\|v^{[T]}_S\|_1 \leq \epsilon_s$. \qed
E Proof of Theorem 2.1

Proof of Theorem 2.1. Starting from initialization \( \tau \cdot 1 \), by Theorem 3.1, running SGD with label noise with noise level \( \delta > \tilde{O}(\frac{d^2}{\rho^2}) \) and \( \eta_0 = \tilde{O}(\frac{1}{\delta}) \) for \( T_0 = \tilde{O}(1) \) iterations gives us that with probability at least 0.99, \( \epsilon_{\min} \leq v_k^{[T_0]} \leq \frac{1}{d} \) where \( \epsilon_{\min} = \exp(-\tilde{O}(1)) \). Now \( v_k^{[T_0]} \) satisfies the initial condition of Theorem 3.3.

Recall the final target precision is \( \epsilon \), set \( \epsilon_1 = \frac{1}{40^3} \epsilon \). By Theorem 3.3, with \( n = \tilde{O}(r^2 \log^2(1/\epsilon)) \) data, after running SGD with label noise with learning rate \( \eta_1 = \tilde{O}(\frac{1}{\delta^2}) \) for \( T_1 = \tilde{O}(\frac{\log(1/\epsilon)}{\eta_1}) \) iterations, with probability at least 0.99, there is
\[
\|v_{\bar{S}}^{[T_0+T_1]} - v^*_S\|_1 \leq \epsilon_1.
\]

The total failure rate of above three stages is 0.03, so with probability at least 0.97, there is
\[
\|v_{\bar{S}}^{[T_0+T_1+T_2]} - v^*_S\|_1 \leq \epsilon,
\]
which finishes the proof. \(\square\)
F Proof of Theorem 2.2

We first prove that with high probability, there is always some element-wise positive vector that is orthogonal to the subspace spanned by data.

Lemma F.1. Assume \( n \leq \frac{d}{2} - 9\sqrt{d} \). Let \( C \subset \mathbb{R}^d \) be the convex cone where each coordinate is positive, \( K \) be a random subspace of dimension \( d - n \). Then with probability at least 0.999, there is \( K \cap C \neq \{0\} \).

Proof of Lemma F.1. By Theorem 1 of (Amelunxen et al., 2014), we only need to prove
\[
\delta(C) + \delta(K) \geq d + 9\sqrt{d},
\]
where \( \delta(\cdot) \) is the statistical dimension of a set. By equation (2.1) of (Amelunxen et al., 2014), there is
\[
\delta(K) = d - n.
\]
To calculate \( \delta(C) \), we use Proposition 2.4 from (Amelunxen et al., 2014),
\[
\delta(C) = \mathbb{E}[\|\Pi_C(g)\|^2] = \frac{d}{2}.
\]
Therefore we have
\[
\delta(C) + \delta(K) = \frac{3}{2}d - n \geq d + 9\sqrt{d}.
\]

Now we use this lemma to prove Theorem 2.2.

Proof of Theorem 2.2. Let \( X^\perp \) be the subspace that is orthogonal to the subspace \( X \) spanned by data. Since data is random, with probability 1 the random subspace \( X \) is of \( n \) dimension. Therefore, according to the previous lemma, with probability at least 0.999, there is \( X^\perp \cap C \neq \{0\} \), where \( C \) is the coordinate-wise positive cone. Let \( \mu \in X^\perp \) be such a vector such that \( \mu_i > 0 \) for \( \forall i \in [d] \), and we scale it such that \( \|\mu\|_2 = 1 \). We can construct the following orthonormal matrix
\[
A = [a^1, \cdots, a^d] \in \mathbb{R}^{d \times d},
\]
such that \( \text{span}\{a^1, \cdots, a^n\} = X \) and \( a^{n+1} = \mu \). Consider the following transformation
\[
A\tilde{u} = u = v^{\otimes 2},
\]
since only the projection of \( u \) to the span of data influences \( L(v) \), we can write \( L(v) = \tilde{L}(\tilde{u}_{1:n}) \) as a function of the first \( n \) dimensions of \( \tilde{u} \).

We can lower bound the partition function with
\[
\int_{v \in \mathbb{R}^d} e^{-\lambda L(v)} dv \geq \int_{v > 0} e^{-\lambda L(v)} dv = \int_{A\tilde{u} > 0} e^{-\lambda \tilde{L}(\tilde{u}_{1:n})} \det \frac{\partial v}{\partial \tilde{u}} \det \frac{\partial u}{\partial \tilde{u}} d\tilde{u} = \frac{1}{2^d} \int e^{-\lambda \tilde{L}(\tilde{u}_{1:n})} \left( \int_{A\tilde{u} > 0} \prod_{i=1}^d \frac{1}{\sqrt{u_i}} d\tilde{u}_{n+1:d} \right) d\tilde{u}_{1:n}.
\]
Here the inner loop is integrating over the last \( d - n \) dimensions of \( \tilde{u} \) in the set such that \( A\tilde{u} \) is coordinate-wise positive. Now we prove that for each \( \tilde{u}_{1:n} \) such that \( S = \{ \tilde{u}_{n+1:d} | A\tilde{u} > 0 \} \) is not empty set, the inner loop integral is always \(+\infty\).

Fix \( \tilde{u}_{1:n} \), let \( \tilde{u}_{n+1:d}^* \) be one possible solution such that \( u^* = A\tilde{u} > 0 \). Define constant
\[
c = \min_{i \leq 1:d} \max_{j \in [n+2:d]} \frac{a_i^{n+1}}{(d - n - 1)|a_j|},
\]
we can define the following set
\[
S' = \{ \tilde{u}_{n+1:d} | \tilde{u}_{n+1} \geq \tilde{u}_{n+1}^* \land |\tilde{u}_j - \tilde{u}_j^*| \leq c(\tilde{u}_{n+1} - \tilde{u}_{n+1}^*), \forall j \in [n + 2, d] \}
\]
In other words, this is a convex cone where constraint of \( \tilde{u}_j \) is linear in \( \tilde{u}_{n+1} \) for \( j \in [n + 2 : d] \). By definition of \( c \), it is easy to verify that \( S' \) is a subset of \( S \). Also, for every \( \tilde{u}_{n+1:d} \in S' \), \( u_i \) is upper bounded by
\[
\left( A \begin{bmatrix} \tilde{u}_{1:n} \\ \tilde{u}_{n+1:d} \end{bmatrix} \right)_i = u_i^* + a_i^{n+1}(\tilde{u}_{n+1} - \tilde{u}_{n+1}^*) + \sum_{j=n+2}^{d} a_i^j(\tilde{u}_j - \tilde{u}_j^*) 
\leq u_i^* + 2a_i^{n+1}(\tilde{u}_{n+1} - \tilde{u}_{n+1}^*)
\]
Here the inequality is because of the definition of \( c \).

Let \( z = \tilde{u}_{n+1} - \tilde{u}_{n+1}^* \) we have
\[
\int_{\tilde{u}_{n+1:d} \in S'} \prod_{i=1}^{d} \frac{1}{u_i} d\tilde{u}_{n+1:d} 
\geq \int_{z \geq 0} (2cz)^{d-n-1} \prod_{i=1}^{d} \frac{1}{u_i^* + 2a_i^{n+1}z} dz 
= +\infty.
\]
Here the last step is because \( n < d/2 \), so the integrand is essentially a polynomial of \( z \) with degree larger than \(-1\), so integrating it over all positive \( z \) has to be \(+\infty\). So we finish the proof that \( \int_{v \in \mathbb{R}^d} e^{-\lambda L(v)} dv = +\infty \). \( \square \)
G Helper Lemmas

Lemma G.1. Suppose $n$ random data points are sampled i.i.d.: $\forall i \in [n], x^{(i)} \sim N(0, I_{d \times d})$. Then, with probability at least $1 - \rho$, for every $i \in [n]$ there is

$$
\|x^{(i)}\|_\infty \leq \sqrt{2 \log \frac{2nd}{\rho}}
$$

Proof. By Gaussian tail bound, there is $\Pr\left(\|x^{(i)}\| > b_x \right) \leq 2e^{-\frac{b_x^2}{2}}$. So by union bound we have $\Pr\left(\max_{i,k} |x^{(i)}_k| > b_x \right) \leq 2n de^{-\frac{b_x^2}{2}}$. Let $b_x = \sqrt{2 \log \frac{2nd}{\rho}}$ we complete the proof. \qed

Lemma G.2. Suppose $n$ random data points are sampled i.i.d.: $\forall i \in [n], x^{(i)} \sim N(0, I_{d \times d})$. Then, when $n > 24 \log \frac{d}{\rho}$, with probability at least $1 - \rho$, for every $k \in [d]$ there is

$$
\E_i[x^{(i)}_k^2] \geq \frac{2}{3}
$$

Proof. Since $\E_x[x_k^2] = 1$, $\E_x[x_k^4] = 3$, by standard concentration result for $\chi^2$ distribution (Chung and Lu, 2006), we have

$$
\Pr\left(\frac{1}{n} \sum_{i=1}^{n} x^{(i)}_k^2 < \frac{2}{3}\right) \leq e^{-\frac{n}{24}}.
$$

Therefore, when $n \geq 24 \log \frac{d}{\rho}$, by union bound we finish complete the proof. \qed

Lemma G.3. Let $c > 0, 1 > \gamma > 0$ be real constants. Let $A^{[0]}, A^{[1]}, \ldots, A^{[T]}$, be a series of random variables, such that given $A^{[0]}, \ldots, A^{[t]}$ for some $t < T$ with $A^{[t]} \in [\frac{c}{3}, c]$, there is either $A^{[t]} = A^{[t+1]} = \ldots = A^{[T]}$, or $E[A^{[t+1]}] \leq (1 - \gamma)A^{[t]}$ with variance $\text{Var}[A^{[t+1]} | A^{[0]}, \ldots, A^{[t]}] \leq a$. Then there is

$$
\Pr\left(A^{[T]} > c \land A^{[0]} \leq \frac{c}{2} \land A^{[0:T]} \in \left[\frac{c}{3}, c\right]\right) \leq e^{2\sum_{i=1}^{T} \frac{a^2}{(1-\gamma)^{2i}}},
$$

where $A^{[0:T]} \in [\frac{c}{3}, c]$ means for any $0 \leq t < T$, $A^{[t]} \in [\frac{c}{3}, c]$.

Proof of Lemma G.3. We only need to consider when $A^{[0]} \leq \frac{c}{2}$. Let $\hat{A}^{[t]}$ be the following coupling of $A^{[t]}$: starting from $\hat{A}^{[0]} = A^{[0]}$, for each time $t < T$, if $A^{[t]} = A^{[t+1]} = \ldots = A^{[T]}$ or there is $t' \leq t$ such that $A^{[t']} \notin [\frac{c}{3}, c]$, we let $\hat{A}^{[t+1]} = (1 - \gamma)\hat{A}^{[t]}$; otherwise $\hat{A}^{[t+1]} = A^{[t+1]}$. Intuitively, whenever $A^{[t]}$ stops updating or exceeds proper range, we only times $\hat{A}^{[t]}$ by $1 - \gamma$ afterwards, otherwise we let it be the same as $A^{[t]}$. Notice that if the event in Equation G.3 happens, there has to be $\hat{A}^{[T]} = A^{[T]}$ (otherwise $A^{[t]}$ stops updating or exceeds range at some time, contradicting the event). So we only need to bound $\Pr\left(\hat{A}^{[T]} > c\right)$.

We notice that $(1 - \gamma)^{-t} \hat{A}^{[t]}$ for $t = 0 \cdots T$ is a supermartingale, i.e., given history there is $E[\hat{A}^{[t+1]} | \hat{A}^{[t]}] \leq (1 - \gamma)\hat{A}^{[t]}$. This is obviously true when $\hat{A}^{[t+1]} = (1 - \gamma)\hat{A}^{[t]}$, and also true otherwise.
by assumption of the lemma. So we have
\[
\Pr(\hat{A}^T > c) \\
= \Pr((1 - \gamma)^{-T} \hat{A}^T > (1 - \gamma)^{-T} c) \\
\leq e^{-\frac{2\gamma(1-\gamma)^{-T}A[0]}{2\gamma - a}} \\
\leq e^{-\frac{2\sum_{t=0}^{T} (1-\gamma)^{-2T} a}{2\gamma}} \\
= e^{-\frac{2\gamma}{2\gamma - a} c^2}
\]
where the first inequality is because of Azuma Inequality and \( \text{Var} \left[(1 - \gamma)^{-t} \hat{A}^{t+1} | \hat{A}^t \right] \leq (1 - \gamma)^{-2(t+1)a} \), the second inequality is because \( \hat{A}^0 \leq \frac{1}{2} \). Since the event in Equation G.3 only happens when \( \hat{A}^T > c \), we’ve finished the proof. 

\[\square\]

**Lemma G.4.** Let \( 0 < c_1 < c_2 \) be real constants. Let \( A[0], A[1], \ldots, A[T] \), be a series of random variables, such that given \( A[0], \ldots, A[t] \) for some \( t < T \) with \( A[t] \in [c_1, c_2] \), there is either event \( E_t \) happens, or \( E[A[t+1]] \leq (1 - \gamma)A[t] \) with variance \( \text{Var}[A[t+1] | A[0], \ldots, A[t]] \leq a \). Then when \( A[0] \in [c_1, c_2] \) and \((1 - \gamma)^T c_2 < c_1 \) there is
\[
\Pr \left( \min_{t \leq T} A[t] > c_1 \wedge \max_{t \leq T} A[t] \leq c_2 \wedge -E_{[0,T]} \right) \leq e^{-\frac{2(\gamma - (1-\gamma)^T - A[0])^2}{\frac{\gamma}{a}}},
\]
where \(-E_{[0,T]} \) means for any \( 0 \leq t < T \), \( E_t \) doesn't happen.

**Proof of Lemma G.4.** Let \( \hat{A}[t] \) be the following coupling of \( A[t] \): starting from \( \hat{A}[0] = A[0] \), for each time \( t < T \), if exists \( t' \leq t \) such that \( E_{t'} \) happens or \( A[t'] \notin [c_1, c_2] \), we let \( \hat{A}[t+1] = (1 - \gamma)\hat{A}[t+1] \); otherwise \( \hat{A}[t+1] = A[t+1] \). Intuitively, whenever \( A[t] \) exceeds proper range, we only times \( \hat{A}[t] \) by \( 1 - \gamma \) afterwards, otherwise we let it be the same as \( A[t] \). Notice that if the event in Equation G.4 happens, there has to be \( \hat{A}[T] = A[T] \) (otherwise \( E_t \) happens sometimes or \( A[t] \notin [c_1, c_2] \), contradicting the event). So we only need to bound \( \Pr \left( \hat{A}[T] > c_1 \right) \).

We notice that \((1 - \gamma)^{-t} \hat{A}[t] \) for \( t = 0 \cdots T \) is a supermartingale, i.e., given history there is \( \mathbb{E}[\hat{A}[T+1] | \hat{A}[t]] \leq (1 - \gamma)\hat{A}[t] \). This is obviously true when \( \hat{A}[t+1] = (1 - \gamma)\hat{A}[t] \), and also true otherwise by assumption of the lemma. So we have
\[
\Pr(\hat{A}[T] > c_1) \\
= \Pr((1 - \gamma)^{-T} \hat{A}[T] > (1 - \gamma)^{-T} c_1) \\
\leq e^{-\frac{2(\gamma - (1-\gamma)^T - A[0])^2}{2\gamma - a}} \\
\leq e^{-\frac{2(\gamma - (1-\gamma)^T - c_2)^2}{\frac{\gamma}{a}}},
\]
where the first inequality is because of Azuma Inequality and \( \text{Var} \left[(1 - \gamma)^{-t} \hat{A}^{t+1} | \hat{A}^t \right] \leq (1 - \gamma)^{-2(t+1)a} \) and \( \hat{A}^0 \leq c_2 \leq e^{-\frac{2(\gamma - (1-\gamma)^T - c_2)^2}{\sum_{t=0}^{T} (1-\gamma)^{-2Ta}}} \). Since the event in Equation G.4 only happens when \( \hat{A}[T] > c_1 \), we’ve finished the proof. 

\[\square\]