The Prelle-Singer method and Painlevé hierarchies

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Abstract

We consider systems of ordinary differential equations (ODEs) of the form $BK = 0$, where $B$ is a Hamiltonian operator of a completely integrable partial differential equation (PDE) hierarchy, and $K = (K, L)^T$. Such systems, whilst of quite low order and linear in the components of $K$, may represent higher-order nonlinear systems if we make a choice of $K$ in terms of the coefficient functions of $B$. Indeed, our original motivation for the study of such systems was their appearance in the study of Painlevé hierarchies, where the question of the reduction of order is of great importance. However, here we do not consider such particular cases; instead we study such systems for arbitrary $K$, where they may represent both integrable and nonintegrable systems of ordinary differential equations. We consider the application of the Prelle-Singer (PS) method — a method used to find first integrals — to such systems in order to reduce their order. We consider the cases of coupled second order ODEs and coupled third order ODEs, as well as the special case of a scalar third order ODE; for the case of coupled third order ODEs, the development of the PS method presented here is new. We apply the PS method to examples of such systems, based on dispersive water wave, Ito and Korteweg-de Vries Hamiltonian structures, and show that first integrals can be obtained. It is important to remember that the equations in question may represent sequences of systems of increasing order. We thus see that the PS method is a further technique which we expect to be useful in our future work.
I. INTRODUCTION

In the last fifteen years or so, there has been a surge of interest in Painlevé hierarchies, that is, hierarchies of ordinary differential equations (ODEs) having the Painlevé property, as well as other properties (underlying linear problems, Bäcklund transformations etc.) possessed by the Painlevé equations. Such hierarchies are often derived using associated hierarchies of completely integrable partial differential equations (PDEs), either as similarity reductions\(^1\text{–}^4\) or using non-isospectral extensions thereof\(^5,^6\).

However, when one reduces a sequence of PDEs to a sequence of ODEs, one is faced with the question of whether or not the order of every member of that sequence of ODEs can be reduced. When this is possible, we can then usually talk of a resulting Painlevé hierarchy. For example, the dispersive water wave hierarchy results in a sequence of ODEs the order of every member of which can be reduced by two, and such that the first member of the sequence then yields the fourth Painlevé equation: we thus talk of a fourth Painlevé hierarchy\(^7\text{–}^9\). From the dispersive water wave hierarchy we can also derive a second Painlevé hierarchy (a hierarchy of ODEs based on the second Painlevé equation)\(^7,^9\).

It is often the case that some of the equations resulting from the reduction of an integrable PDE hierarchy are of the form

\[ \mathcal{B} K = 0, \]  

for some particular choice of \(K\), and where \(\mathcal{B}\) is one of the Hamiltonian operators of the PDE hierarchy. For example, one case of reduction from the dispersive water wave hierarchy gives \(^8,^{11}\) with

\[ \mathcal{B} = \frac{1}{2} \begin{pmatrix} 2\partial & \partial u - \partial^2 \\ u\partial + \partial^2 & v\partial + \partial v \end{pmatrix}, \]  

where \(\partial = \frac{\partial}{\partial x}\), \(u = u(x)\) and \(v = v(x)\), for the choice

\[ K = \begin{pmatrix} K \\ L \end{pmatrix} = L_n + \sum_{i=1}^{n-1} c_i L_i + g_n \begin{pmatrix} 0 \\ x \end{pmatrix}. \]  

Here \(g_n\) and all \(c_i\) are constants, and the quantities \(L_i\) (variational derivatives of a corresponding sequence of Hamiltonian densities of the dispersive water wave hierarchy) are
defined by the recursion relation \( C L_{i+1} = B L_i \), \( i = 0, 1, 2, \ldots \), where

\[
C = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}
\]  \hspace{1cm} (4)

is another of the three Hamiltonian operators of the dispersive water wave hierarchy and \( L_0 = (0, 2)^T \). It is by reducing the order of the system (1) that we then obtain a fourth Painlevé hierarchy.

We considered the question of how to reduce the order of a system which can be put in the form (1) in Refs. [10,12]. In Refs. [10,12] we developed a method based on the factorization of the Hamiltonian operator \( B \) under a Miura map. A discrete version of this approach was given in Ref. [13]. In the present paper we consider an alternative approach to the reduction of order of a system (1), namely, the Prelle-Singer (PS) method.

The PS method has been successfully extended and used to derive first integrals of a wide variety of systems of ODEs. However here our emphasis is somewhat different. We are interested in systems (1), where \( B \) is a Hamiltonian operator and \( K \) is left unspecified: we recall that the reductions of order we have carried out thus far in order to obtain Painlevé hierarchies can in fact be achieved independently of the form of \( K \) (the Painlevé hierarchies then arise on specifying \( K \)). Here we are therefore concerned with systems (1), independently of whether they have a connection with Painlevé hierarchies or not. Thus in the present paper we are interested in systems (1) which represent a wide variety of ODE systems, both integrable and nonintegrable (although our original motivation was the integrable case).

We then seek first integrals of systems (1), treating them as systems in the components of \( K \) with coefficients the functions appearing in the Hamiltonian operator \( B \); these systems represent higher-order systems in these coefficient functions once the components of \( K \) have been specified in terms of these last, e.g., as in (3). Thus we have an application of the reduction of order of quite low-order systems (1), which are in fact linear in the components of \( K \), to higher-order nonlinear systems.

The layout of the paper is as follows. In Section II we recall the PS method for coupled second order ODEs. As an example we then consider its application to the system (1) with \( B \) given by (2). Thus we recover our previous results for the dispersive water wave case. In Section III we extend the PS method to the case of coupled third order ODEs; this represents a new extension of this method. As an application we consider an example of Ito type. We
also consider as a special case of coupled third order ODEs an example of the case of a single third order ODE. The final section is devoted to conclusions.

II. PS METHOD FOR COUPLED SECOND ORDER ODES

Sometime ago Prelle and Singer [18] have proposed a procedure for solving first order ODEs that presents the solution in terms of elementary functions if such a solution exists. The attractiveness of the PS method is that if the given system of first order ODEs has a solution in terms of elementary functions then the method guarantees that this solution will be found. Very recently Duarte et al [19] have modified the technique developed by Prelle and Singer [18] and applied it to second order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of $t$, $x$ and $\dot{x}$. For a class of systems these authors have deduced first integrals and in some cases for the first time through their procedure [19]. Recently Chandrasekar et al have generalized the theory given in [19] and pointed out a procedure to obtain all the integrals of motion/general solution and solved a class of nonlinear equations. Here we adopt the above said procedure for coupled second and third order ODEs.

Let us consider the following two coupled second order ODEs

\[
\begin{align*}
L_{xx} &= \Phi_1(x, L, K, L_x, K_x), \\
K_{xx} &= \Phi_2(x, L, K, L_x, K_x).
\end{align*}
\]

Let us suppose that the system (5) admits a first integral of the form $I(x, L, K, L_x, K_x) = C$, with $C$ constant on the solutions so that the total differential gives

\[
dI = I_x dx + I_L dL + I_K dK + I_{L_x} dL_x + I_{K_x} dK_x = 0.
\]

Here a subscript denotes partial differentiation with respect to the corresponding variable and, by a common abuse of notation, we also use subscripts (for example in equation (5) where $K = K(x)$ and $L = L(x)$) to denote differentiation with respect to a unique independent variable. Rewriting (5) in the form

\[
\begin{align*}
\Phi_1 dx - dL_x &= 0, \\
\Phi_2 dx - dK_x &= 0
\end{align*}
\]
and adding null terms \( s_1(x, L, K, L_x, K_x)L_xdx - s_1(x, L, K, L_x, K_x)dl \) and
\( s_2(x, L, K, L_x, K_x)K_xdx - s_2(x, L, K, L_x, K_x)dK \) with the first equation in (7), and
\( u_1(x, L, K, L_x, K_x)L_xdx - u_1(x, L, K, L_x, K_x)dl \) and \( u_2(x, L, K, L_x, K_x)K_xdx - u_2(x, L, K, L_x, K_x)dK \) with the second equation in (7), respectively, we obtain that, on the solutions, the 1-forms

\[
(\Phi_1 + s_1L_x + s_2K_x)dx - s_1dl - s_2dK - dL_x = 0, \quad (8a)
\]

\[
(\Phi_2 + u_1L_x + u_2K_x)dx - u_1dl - u_2dK - dK_x = 0. \quad (8b)
\]

Hence, on the solutions, the 1-forms (8) must be proportional. Multiplying (8a) by the factor \( R_1(x, L, K, L_x, K_x) \) and (8b) by the factor \( R_2(x, L, K, L_x, K_x) \), which act as the integrating factors for (8a) and (8b), respectively, we have on the solutions that

\[
dI = R_1(\Phi_1 + SL_x)dx + R_2(\Phi_2 + UK_x)dx - R_1Sdx - R_2UdK - R_1dl - R_2dK_x = 0, \quad (9)
\]

where \( S = \frac{R_1s_1 + R_2u_1}{R_1} \) and \( U = \frac{R_1s_2 + R_2u_2}{R_2} \). Comparing equations (9) and (10) we have, on the solutions, the relations

\[
I_x = R_1(\Phi_1 + SL_x) + R_2(\Phi_2 + UK_x), \quad I_L = -R_1S, \quad I_K = -R_2U, \quad I_{L_x} = -R_1, \quad I_{K_x} = -R_2. \quad (10)
\]

The compatibility conditions between equations (10), namely \( I_xL_x = I_{L_x}, I_xL_x = I_{L_x}, I_xK = I_{K_x}, I_xK_x = I_{K_x}, I_LK = I_{KL}, I_{LL_x} = I_{L_xL}, I_{KK_x} = I_{K_xK}, I_{LK} = I_{KL}, I_{KK_x} = I_{K_xK}, I_{LxK} = I_{K_xL} \) and \( I_{L_xK} = I_{K_xL} \), provide us with the conditions,

\[
D[S] = -\Phi_1L - \frac{R_2}{R_1}\Phi_2L + \frac{R_2}{R_1}S\Phi_2L_x + S\Phi_1L_x + S^2, \quad (11)
\]

\[
D[U] = -\Phi_2K - \frac{R_1}{R_2}\Phi_1K + \frac{R_1}{R_2}U\Phi_1K_x + U\Phi_2K_x + U^2, \quad (12)
\]

\[
D[R_1] = -(R_1\Phi_1L_x + R_2\Phi_2L_x + R_1S), \quad (13)
\]

\[
D[R_2] = -(R_2\Phi_2K_x + R_1\Phi_1K_x + R_2U), \quad (14)
\]

\[
SR_{1K} = -R_1S_K + UR_2L_x + R_2U_L, \quad R_{1L} = SR_1L_x + R_1S_{L_x}, \quad (15)
\]

\[
R_{1K} = UR_2L_x + R_2U_{L_x}, \quad R_{2L} = SR_1K_x + R_1S_{K_x}, \quad (16)
\]

\[
R_{2K} = UR_2K_x + R_2U_{K_x}, \quad R_{1L} = R_2L_x. \quad (17)
\]
Here the total differential operator, $D$, is defined by

$$D = \frac{\partial}{\partial x} + L_2 \frac{\partial}{\partial L} + K_2 \frac{\partial}{\partial K} + \Phi_1 \frac{\partial}{\partial L_x} + \Phi_2 \frac{\partial}{\partial K_x}.$$  

Integrating equations (10), we obtain the integral of motion,

$$I = r_1 + r_2 + r_3 + r_4 - \int \left[ R_2 + \frac{\partial}{\partial K_x} \left( r_1 + r_2 + r_3 + r_4 \right) \right] dK_x,$$  

where

$$r_1 = \int \left( R_1(\Phi_1 + SL_x) + R_2(\Phi_2 + UK_x) \right) dx,$$

$$r_2 = -\int \left( R_1 S + \frac{\partial}{\partial L}(r_1) \right) dL,$$

$$r_3 = -\int \left( R_2 U + \frac{\partial}{\partial K}(r_1 + r_2) \right) dK,$$

$$r_4 = -\int \left[ R_1 + \frac{\partial}{\partial L_x} \left( r_1 + r_2 + r_3 \right) \right] dL_x.$$

Solving the determining equations, (11)-(17), consistently we can obtain expressions for the functions $(S, U, R_1, R_2)$. Substituting them into (18) and evaluating the integrals we can construct the associated integrals of motion.

**Method of finding the integrating factors $R_1$ and $R_2$**

To solve the determining equations for the functions $S$, $U$, $R_1$ and $R_2$ we follow the following procedure. Let us take a total derivative of equations (13) and (14). Doing so we get

$$D^2[R_1] = -D[R_1 \Phi_1 L_x + R_2 \Phi_2 L_x + R_1 S], \quad D^2[R_2] = -D[R_2 \Phi_2 K_x + R_1 \Phi_1 K_x + R_2 U].$$  

Using the identities

$$D[R_1 S] = -(R_1 \Phi_1 L + R_2 \Phi_2 L), \quad D[R_2 U] = -(R_1 \Phi_1 K + R_2 \Phi_2 K).$$  

equation (19) can be rewritten in a coupled form for $R_1$ and $R_2$ as

$$D^2[R_1] + D[R_1 \Phi_1 L_x + R_2 \Phi_2 L_x] = R_1 \Phi_1 L + R_2 \Phi_2 L,$$  

$$D^2[R_2] + D[R_2 \Phi_2 K_x + R_1 \Phi_1 K_x] = R_1 \Phi_1 K + R_2 \Phi_2 K.$$  

The determining equations (21) and (22) form a system of linear PDEs in $R_1$ and $R_2$. Substituting the known expressions for $\Phi_1$ and $\Phi_2$ and their derivatives into (21) and (22) and solving them one can obtain expressions for the integrating factors $R_1$ and $R_2$. To obtain the explicit form of integrating factors $R_1$ and $R_2$ one may assume a suitable ansatz for these functions and substitute them into Eqs.(21) and (22) and solving the resultant equations consistently. Once $R_1$ and $R_2$ are known then the functions $(S, U)$ can be fixed.
through the relation (13)-(14). Knowing \( S, U, R_1 \) and \( R_2 \), one has to make sure that the set \((S, U, R_1, R_2)\) also satisfies the remaining compatibility conditions (15) - (17). The set \((S, U, R_1, R_2)\) which satisfies all the equations (11) - (17) is then the acceptable solution and one can then determine the associated integral \( I \) using the relation (18). For complete integrability we require four independent compatible sets \((S_i, U_i, R_{1i}, R_{2i})\), \( i = 1, 2, 3, 4 \). We note here that the examples which we consider in this paper are linear systems. In this case, as we show below, the determining equations (21) and (22) for the integrating factors coincide with the original system of equations (5) which we consider initially. In other words any particular solution of the given equation forms an integrating factor for the given equation. So we do not discuss the method of solving the determining equations elaborately.

**Example 1**

Let us consider equation (1) with \( B \) given by (2) and \( K = (K, L)^T \), i.e.,

\[
L_{xx} = (u(x)L)_{x} + 2K_{x} = \Phi_1,
\]
\[
K_{xx} = -u(x)K_{x} - 2v(x)L_{x} - L_{x}v(x) = \Phi_2.
\]

We recall that for the particular choice (3) this system (after reduction of order by two) results in a fourth Painlevé hierarchy; we do not however, make this choice here. Substituting \( \Phi_1, \Phi_2 \) and their derivatives in (21) and (22) we get

\[
D^2[R_1] = -D[u(x)R_1 - 2R_2v(x)] + R_1u_x(x) - R_2v_x(x),
\]
\[
D^2[R_2] = -D[-u(x)R_2 + 2R_1].
\]

Rewriting

\[
D^2[R_1] + D[u(x)R_1 - 2R_2v(x)] - R_1u_x(x) + R_2v_x(x) = 0,
\]
\[
D^2[R_2] + D[-u(x)R_2 + 2R_1] = 0.
\]

If we choose the integrating factors \( R_2 = -L \) and \( R_1 = K \) then Eq. (25) becomes

\[
D^2[K] + D[u(x)K + 2Lv(x)] - Ku_x(x) - Lv_x(x) = 0,
\]
\[
- D^2[L] + D[u(x)L + 2K] = 0.
\]

The above equation is of the form

\[
K_{xx} + u(x)K_{x} + 2v(x)L_{x} + L_{x}v(x) = 0,
\]
\[
L_{xx} - (u(x)L)_{x} - 2K_{x} = 0.
\]
which is nothing but (23). Thus the integrating factors are $R_1 = K$ and $R_2 = -L$.

Once $R_1$ and $R_2$ are known then the null forms $S$ and $U$ can be determined using the ideas given in the theory. They turn out to be

$$S = -u(x) - 2v(x)\frac{L}{K} - \frac{K_x}{K},$$
$$U = u(x) + 2\frac{K}{L} - \frac{L_x}{L}. \quad (28)$$

Substituting the forms $R_2$, $R_1$, $S$ and $U$ into (18) and evaluating the integrals we find

$$r_1 = LKu(x) + L^2v(x), \quad r_2 = LK_x, \quad r_3 = K^2 - KL_x, \quad r_4 = 0 \quad (29)$$

and

$$I_1 = LKu(x) + L^2v(x) + LK_x + K^2 - KL_x. \quad (30)$$

The first equation of (23) can be integrated straightforwardly and yields the second integral

$$I_2 = L_x - 2K - u(x)L. \quad (31)$$

The respective integrating factors are $R_1 = 1$ and $R_2 = 0$.

### III. PS METHOD FOR COUPLED THIRD ORDER ODES

We now develop the PS method for coupled third order ODEs; this has not appeared previously in the literature. We consider a system of the form

$$L_{xxx} = \Phi_1(x, L, K, L_x, K_x, L_{xx}, K_{xx}),$$
$$K_{xxx} = \Phi_2(x, L, K, L_x, K_x, L_{xx}, K_{xx}). \quad (32)$$

Let us assume that the above system admits a first integral of the form $I(x, L, K, L_x, K_x, L_{xx}, K_{xx}) = C$, which is constant on the solutions. Then the total differentiation gives

$$dI = I_xdx + I_LdL + I_KdK + I_{L_x}dL_x + I_{K_x}dK_x + I_{L_{xx}}dL_{xx} + I_{K_{xx}}dK_{xx} = 0. \quad (33)$$

We rewrite (32) in the following form

$$\Phi_1dx - dL_{xx} = 0, \quad \Phi_2dx - dK_{xx} = 0. \quad (34)$$
Adding null terms in the above equation, finally we obtain the following equation

\[
\begin{align*}
(\Phi_1 + S_1 L_x + S_2 K_x + M_1 L_{xx} + M_2 K_{xx})dx - S_1 dL - S_2 dK - M_1 dL_x \\
- M_2 dK_x - dL_{xx} = 0, \\
(\Phi_2 + U_1 L_x + U_2 K_x + N_1 L_{xx} + N_2 K_{xx})dx - U_1 dL - U_2 dK - N_1 dL_x \\
- N_2 dK_x - dK_{xx} = 0.
\end{align*}
\]

Multiplying \((35)\) by the integrating factor \(R_1(x, L, K, L_x, K_x, L_{xx}, K_{xx})\) and \((36)\) by \(R_2(x, L, K, L_x, K_x, L_{xx}, K_{xx})\), we obtain the equation

\[
dI = R_1(\Phi_1 + SL_x + ML_{xx})dx + R_2(\Phi_2 + UK_x + NK_{xx})dx - R_1 S dL \\
- R_2 U dK - R_1 M dL_x - R_2 N dK_x - R_1 dL_{xx} - R_2 dK_{xx} = 0
\]

where \(S = \frac{R_1 S_1 + R_2 U_1}{R_1}, U = \frac{R_1 S_2 + R_2 U_2}{R_2}, M = \frac{R_1 M_1 + R_2 N_1}{R_1}\) and \(N = \frac{R_1 M_2 + R_2 N_2}{R_2}\).

Comparing the above equation with \((33)\), we obtain the following equations:

\[
\begin{align*}
I_x &= R_1(\Phi_1 + SL_x + ML_{xx}) + R_2(\Phi_2 + UK_x + NK_{xx}), \\
I_L &= -R_1 S, \quad I_K = -R_2 U, \quad I_{Lx} = -R_1 M, \\
I_{Kx} &= -R_2 N, \quad I_{L_{xx}} = -R_1, \quad I_{K_{xx}} = -R_2.
\end{align*}
\]
Using the compatibility conditions, we obtain the following determining equations

\[ D[R_1] = -(R_1 \Phi_1 L_{xx} + R_2 \Phi_2 L_{xx} + R_1 M) \],
\[ D[R_2] = -(R_1 \Phi_1 K_{xx} + R_2 \Phi_2 K_{xx} + R_2 N) \],
\[ D[S] = S \Phi_1 L_{xx} + \left( \frac{SR_2}{R_1} \right) \Phi_2 L_{xx} + MS - \Phi_1 L - \left( \frac{R_2}{R_1} \right) \Phi_2 L, \]
\[ D[U] = U \Phi_2 K_{xx} + \left( \frac{R_1 U}{R_2} \right) \Phi_1 K_{xx} + NU - \Phi_2 K - \left( \frac{R_1}{R_2} \right) \Phi_1 K, \]
\[ D[M] = M \Phi_1 L_{xx} + \left( \frac{MR_2}{R_1} \right) \Phi_1 K_{xx} + M^2 - \Phi_1 L - \left( \frac{R_2}{R_1} \right) \Phi_2 L - S, \]
\[ D[N] = N \Phi_2 K_{xx} + \left( \frac{R_1 N}{R_2} \right) \Phi_1 K_{xx} + N^2 - \Phi_2 K - \left( \frac{R_1}{R_2} \right) \Phi_1 K - U, \]
\[ R_{1K_x} M + R_1 K_{M_x} = R_{2L_x} N + R_2 N_{L_x}, \quad R_{1L} = R_{1L_x} S + R_1 S_{L_x}, \]
\[ R_{1K_x} S + R_1 S_{K_x} = R_{2L} N + R_2 N_L, \quad R_{1K} = R_{2L_x} U + R_2 U_{L_x}, \]
\[ R_{2K_x} U + R_2 U_{K_x} = R_{2K} N + R_2 N_K, \quad R_{1L_x} = R_{1L_x} M + R_1 M_{L_x}, \]
\[ R_{1K_x} S + R_1 S_K = R_{2L} U + R_2 U_L, \quad R_{2K} = R_{2K_x} U + R_2 U_{K_x}, \]
\[ R_{2L_x} U + R_2 U_{L_x} = R_{1M} K + M R_{1K}, \quad R_{2L} = R_{1K_x} S + R_1 S_{K_x}, \]
\[ R_{2L_x} = R_{1K_x} M + R_1 M_{K_x}, \quad R_{2K_x} = R_{2K_x} N + R_2 N_{K_x}, \]
\[ R_{1K_x} = R_{2L_x}. \]

Once the compatible solution, \( R_1, R_2, S, U, M \) and \( N \) are determined then the integral can be readily constructed by substituting all the expressions in \( (38) \) and integrating the resulting equation, that is

\[ I = r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - \int (R_2 + \frac{\partial}{\partial K_{xx}} (r_1 + r_2 + r_3 + r_4 + r_5 + r_6))dK_{xx} \]

where

\[ r_1 = \int [R_1(\Phi_1 + SL_x + ML_{xx}) + R_2(\Phi_2 + UK_x + NK_{xx})]dx, \]
\[ r_2 = -\int (R_1 S + \frac{\partial}{\partial L}(r_1))dL, \]
\[ r_3 = -\int (R_2 U + \frac{\partial}{\partial K}(r_1 + r_2))dK, \]
\[ r_4 = -\int (R_1 M + \frac{\partial}{\partial L_x}(r_1 + r_2 + r_3))dL_x, \]
As we did in the second order case to determine the integrating factors \( R_1 \) and \( R_2 \) we rewrite the determining equations (39)-(44) as two equations.

\[
\begin{align*}
D^3[R_1] + D^2[R_1\Phi_{1Lxx} + R_2\Phi_{2Lxx}] - D[R_1\Phi_{1Lx} + R_2\Phi_{2Lx}] + R_1\Phi_{1L} + R_2\Phi_{2L} &= 0, \quad (54) \\
D^3[R_2] + D^2[R_1\Phi_{1Kxx} + R_2\Phi_{2Kxx}] - D[R_1\Phi_{1Kx} + R_2\Phi_{2Kx}] + R_1\Phi_{1K} + R_2\Phi_{2K} &= 0, \quad (55)
\end{align*}
\]

where \( D = \frac{\partial}{\partial x} + L_{x\Phi_{1L}} + K_{x\Phi_{1K}} + L_{xx\Phi_{1L}} + K_{xx\Phi_{1K}} + \Phi_1 \frac{\partial}{\partial Lx} + \Phi_2 \frac{\partial}{\partial Kx} \).

The determining equations (54) and (55) form a system of linear PDEs in \( R_1 \) and \( R_2 \). Substituting the known expressions for \( \Phi_1 \) and \( \Phi_2 \) and their derivatives into (54) and (55) and solving them one can obtain expressions for the integrating factors \( R_1 \) and \( R_2 \). Once \( R_1 \) and \( R_2 \) are known then the functions \( (S, U, M, N) \) can be fixed through the relations (41)-(44). Knowing \( S, U, M, N, R_1 \) and \( R_2 \), one has to make sure that the set \( (S, U, M, N, R_1, R_2) \) also satisfies the remaining compatibility conditions (45)-(52). The set \( (S, U, M, N, R_1, R_2) \) which satisfies all the equations (39)-(52) is then the acceptable solution and one can then determine the associated integral \( I \) using the relation (53).

If we choose \( K = 0 \) and \( \Phi_2 = 0 \) in the above procedure, we get the Prelle-Singer procedure for scalar third order ODEs. We also present an example for this case.

**Example 2**

Let us consider a system of Ito type, i.e., of the form (1) with\(^2\)

\[
\mathcal{B} = \begin{pmatrix}
\frac{1}{4} \partial^3 + u_1 \partial + \frac{1}{2} u_{1x} & u_0 \partial + \frac{1}{2} u_{0x} - \partial \\
u_0 \partial + \frac{1}{2} u_{0x} - \partial & \frac{1}{4} \partial^3 + u_1 \partial + \frac{1}{2} u_{1x}
\end{pmatrix},
\]

where \( u_0 = u_0(x) \), \( u_1 = u_1(x) \) and \( K = (K, L)^T \). This system we write as

\[
\begin{align*}
L_{xxx} &= -4(\frac{1}{2} u_{1L} L + u_0 K - K_x), \\
K_{xxx} &= -4(\frac{1}{2} u_{1K} K + u_0 L - L_x).
\end{align*}
\]

We are interested in showing that we can use the PS method to construct first integrals of this system in the general case, i.e., independently of whether for a particular choice of \( K \)
the system is integrable or nonintegrable, or of whether there is a connection with Painlevé hierarchies. Substituting $\Phi_1, \Phi_2$ and their derivatives in (54) and (55) we find

$$D^3[R_1] - D[R_1(-4u_1(x)) + R_2(-4u_0(x) + 4)] + R_1(-2u_{1x}(x)) + R_2(-2u_{0x}(x)) = 0,$$

$$D^3[R_2] - D[R_1(-4u_0(x) + 4) + R_2(-4u_1(x))] + R_1(-2u_{0x}(x)) + R_2(-2u_{1x}(x)) = 0. \quad (58)$$

As we noted earlier for linear ODEs, the determining equations for the integrating factors coincide with the original ODEs. If we choose the integrating factors $R_2 = L$ and $R_1 = K$ then Eq.(58) exactly coincides with (57). Thus the integrating factors are $R_2 = L$ and $R_1 = K$. Once $R_1$ and $R_2$ are known the null forms $S, U, M$ and $N$ can be determined using the ideas given in the previous section. They turn out to be

$$M = -\frac{K_x}{K}, \quad S = 4u_1(x) + \frac{4Lu_0(x) - 4L + K_{xx}}{K},$$

$$N = -\frac{L_x}{L}, \quad U = 4u_1(x) + \frac{4Ku_0(x) - 4K + L_{xx}}{L}. \quad (59)$$

Substituting the above equations in (53) we find

$$r_1 = -4LKu_1(x) - 2u_0(x)(L^2 + K^2), \quad r_2 = 2L^2 - K_{xx}L,$$

$$r_3 = 2K^2 - L_{xx}K, \quad r_4 = L_xK_x, \quad r_5 = 0, \quad r_6 = 0. \quad (60)$$

and the first integral is of the form

$$I_1 = -4 \left( LKu_1(x) + \frac{1}{2}u_0(x)(L^2 + K^2) - \frac{1}{2}L^2 + \frac{1}{4}K_{xx}L - \frac{1}{2}K^2 + \frac{1}{4}L_{xx}K - \frac{1}{4}L_xK_x \right). \quad (61)$$

The second set of integrating factors can also be easily fixed from Eq.(58) as $R_2 = K$ and $R_1 = L$. The associated null terms turn out to be

$$M = -\frac{L_x}{L}, \quad S = 4u_1(x) + \frac{4Ku_0(x) - 4K + L_{xx}}{L},$$

$$N = -\frac{K_x}{K}, \quad U = 4u_1(x) + \frac{4Lu_0(x) - 4L + K_{xx}}{K}. \quad (62)$$

Substituting the above equations in (53) we find

$$r_1 = -4LKu_0(x) - 2u_1(x)(L^2 + K^2), \quad r_2 = 4KL - L_{xx}L,$$

$$r_3 = -K_{xx}K, \quad r_4 = \frac{L_x^2}{2}, \quad r_5 = \frac{K_x^2}{2}, \quad r_6 = 0. \quad (63)$$

The second integral is found to be of the form

$$I_2 = -4 \left( \frac{1}{2}u_1(x)(L^2 + K^2) + u_0(x)KL - KL + \frac{1}{4}L_{xx}L + \frac{1}{4}K_{xx}K - \frac{1}{8}L_x^2 - \frac{1}{8}K_x^2 \right). \quad (64)$$
Whilst this example was considered in Ref. [12] using a technique based on the factorization of the Hamiltonian operator $B$ under a Miura map, first integrals of the system (57) were not explicitly given in this general case (first integrals were given for a particular case, i.e., for a particular choice of $K$).

**Example 3**

Let us consider the equation

$$BL = [\partial^3 + 4u\partial + 2u_x]L = 0$$

(65)

where $u = u(x)$, i.e.,

$$L_{xxx} = \Phi_1 = -4u(x)L_x - 2u_x(x)L.$$ (66)

For a particular choice of $L$, this equation leads us (after integration) to a thirty-fourth Painlevé hierarchy. Once again, we do not make this choice of $L$ here, as we are interested in both integrable and nonintegrable cases. Substituting Eq. (66) into (54) we get

$$D^3[R_1] + 4D[u(x)R_1] - 2u_x(x)R_1 = 0,$$ (67)

which can be written

$$D^3[R_1] + 4u(x)D[R_1] + 2u_x(x)R_1 = 0.$$ (68)

Choosing $R_1 = L$, we see that equation (68) exactly coincides with (66). As a consequence one simple solution (integrating factor) of (66) can be immediately written in the form

$$R_1 = L.$$ (69)

In other words one integrating factor for the equation (66) is $L$ itself. Once $R_1$ is known then $M$ can be determined straightforwardly from Eq. (39). Simplifying the resulting expression we find

$$M = -\frac{L_x}{L}.$$ (70)

Substituting $M$ into (41) we find that $S$ is of the form

$$S = 4u(x) + \frac{L_{xx}}{L}.$$ (71)
Plugging the expressions $R_1$, $S$ and $M$ into the expression (53) and evaluating the integrals we find

$$r_1 = -2L^2u(x), \quad r_2 = -LL_{xx}, \quad r_4 = \frac{L_x^2}{2}, \quad r_3 = r_5 = r_6 = 0,$$

and

$$I = -\left(LL_{xx} + 2L^2u(x) - \frac{L_x^2}{2}\right). \quad (72)$$

Of course, in this example, the integrating factor can be identified by inspection; our aim here is to show that the PS method is also applicable in the scalar case of third order equations which may in fact, for particular choices of $L$, represent higher order ODEs. One can easily check that $\frac{dI}{dx} = 0$.

IV. CONCLUSIONS

We have considered the application of the PS method to systems of the form (1), where $B$ is a Hamiltonian operator of a completely integrable PDE hierarchy, and $K = (K, L)^T$. The original motivation for the study of such systems was their appearance in the study of Painlevé hierarchies. However, we are also interested in the study of such systems outside of that context, where they may also represent nonintegrable systems. It is interesting that these quite low-order systems, linear in the components of $K$, may represent higher-order nonlinear systems. We have considered the cases of coupled second order ODEs and coupled third order ODEs, as well as the special case of a scalar third order ODE; for the case of coupled third order ODEs, the development of the PS method presented here is new. We have successfully applied the PS method to examples of such systems, and have succeeded in obtaining first integrals. This then represents a further technique, additional to the factorization of $B$ under a Miura map, which we expect to be of use in our future work.

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