Perfect imaging with geodesic waveguides

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Abstract. Transformation optics is used to prove that a spherical waveguide filled with an isotropic material with radial refractive index $n = 1/r$ has radially polarized modes (i.e. the electric field is only radial) with the same perfect focusing properties as the Maxwell fish-eye (MFE) lens. An approximate version of that device, comprising a thin waveguide with a homogeneous core, paves the way to experimentally attaining perfect imaging in the MFE lens.

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1. Introduction

The Maxwell fish-eye (MFE) lens is a positive, isotropic and radial refractive-index distribution, which has been well studied in the geometrical optics framework. One of the most interesting characteristics of the refractive-index distribution of the MFE is that any point of the space has a perfect conjugate, i.e. the rays issuing from the point are perfectly focused at another point. These two perfect conjugate points are related by an inversion about the origin. Recently, in [1, 2], it was proved that this refractive-index distribution has perfect wave-optical properties as well. In particular, in [1], Leonhardt proved that the two-dimensional (2D) Helmholtz scalar waves generated by a point source are perfectly focused on an ‘infinitely well localized drain’ located at the conjugate point of the source. These conclusions were confirmed later using a different approach in [3].

Our aim in this paper is to show by means of transformation optics that one can design a spherical waveguide with the same perfect focusing properties. We will show that this spherical waveguide is easier to manufacture than the MFE gradient-index distribution and thus the experimental realization of its properties should be closer. In the final section, we extend this method to more general 2D radial refractive-index distributions and their corresponding geodesic waveguides.

This paper is organized as follows. Section 1.1 reviews the tools of transformation optics that will be employed thereafter, in section 2, to generate the transformation between the MFE and a spherically symmetric isotropic medium. In section 3, with the aid of two perfect conductor sheets, the transformation is restricted to the media within two corresponding waveguides. The extension of the method to more general geodesic waveguides is given in section 4. Finally, our conclusions are presented in section 5.

1.1. Media and coordinate transformations

Maxwell’s equations for a current-free, charge-free medium can be written as

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0. \tag{1}
\]

They are completed with the following constitutive relations wherein \( \epsilon \) and \( \mu \) are the \( 3 \times 3 \) permittivity and permeability tensors,

\[
\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu_0 \mu \mathbf{H}. \tag{2}
\]

Assume that \( x_1, x_2 \) and \( x_3 \) are the usual Cartesian spatial coordinates. A change of the variables \( x_1, x_2, x_3 \) can have two different interpretations [4]–[9]. In the first, which is referred as a media transformation, the new variables are considered to define another Cartesian space \( x'_1, x'_2, x'_3 \) (which we will call the final space, to distinguish it from the initial space). Then the functions \( x'_1(x_1, x_2, x_3), x'_2(x_1, x_2, x_3) \) and \( x'_3(x_1, x_2, x_3) \) define a mapping between the initial and final spaces. The permittivity and permeability tensors of the final space are computed from the transformation formulae and we can easily calculate electromagnetic vector fields in one space if we know the electromagnetic vector field in the other space. The second interpretation of a change to the variables \( u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3) \) and \( u_3(x_1, x_2, x_3) \) is to consider it as describing the same space with the same medium but with a different coordinate system (in general, curvilinear) \( u_1, u_2 \) and \( u_3 \). This can be referred to as a coordinate transformation.
The same Maxwell equations are formally fulfilled for this coordinate transformation provided we use ‘normalized electromagnetic fields’ $\hat{E}$, $\hat{H}$, $\hat{D}$ and $\hat{B}$ (which are actually not true electromagnetic fields) and ‘normalized tensors’ $\hat{\varepsilon}$ and $\hat{\mu}$ (which are not true permittivity and permeability tensors either). The relationship of the normalized field $\hat{E}$ with the actual electric field $E$ in the new coordinate system (i.e. $E = E_1 u_1 + E_2 u_2 + E_3 u_3$; $u_1$, $u_2$, $u_3$ are the three unit vectors of this new coordinate system) is $\hat{E} = \mathcal{H}_u E$, where $\mathcal{H}_u$ is the diagonal matrix $\text{diag}(h_1, h_2, h_3)$, and $h_i$ are the scale factors $h_i^2 = (\partial x_i / \partial u_i)^2 + (\partial x_j / \partial u_i)^2 + (\partial x_k / \partial u_i)^2$. The same relationship applies to the normalized $\hat{H}$ and actual $H$ magnetic fields.

The formulae for the transformation between two spaces with arbitrary coordinate systems $u_1, u_2, u_3$ and $u'_1, u'_2, u'_3$ can be obtained by chaining the media and coordinate transformations. When the two coordinate systems (initial and final) are orthogonal, the resulting formulae are much simpler than in the general case [10]: the permittivity and permeability tensors of the original medium in the first orthogonal system are related with the permittivity and permeability tensors of the final medium, in the second orthogonal system, by

$$
\varepsilon' = \frac{1}{\det(M)} M \varepsilon M^T, \quad \mu' = \frac{1}{\det(M)} M \mu M^T. \tag{3}
$$

Here, $M^T$ denotes the transpose of the matrix $M$, which is

$$
M = \begin{pmatrix}
    h'_1 \partial u'_1 & h'_1 \partial u'_2 & h'_1 \partial u'_3 \\
    h_1 \partial u_1 & h_1 \partial u_2 & h_1 \partial u_3 \\
    h'_2 \partial u'_1 & h'_2 \partial u'_2 & h'_2 \partial u'_3 \\
    h_2 \partial u_1 & h_2 \partial u_2 & h_2 \partial u_3 \\
    h'_3 \partial u'_1 & h'_3 \partial u'_2 & h'_3 \partial u'_3 \\
    h_3 \partial u_1 & h_3 \partial u_2 & h_3 \partial u_3
\end{pmatrix}. \tag{4}
$$

Here, $h'_i$ are the scale factors of the variables $u'_i$. The fields $E$ (and $H$) of the original medium in the first orthogonal system are related to $E'$ (and $H'$) of the transformed medium in the second orthogonal system by

$$
E = M^T E', \quad H = M^T H'. \tag{5}
$$

Note that none of the vectors and tensors in equations (3) and (5) are normalized, in the sense previously defined; that is, all of them are actual physical fields.

2. Transformation from the Maxwell fish-eye (MFE) to a spherical medium

2.1. The initial space

The 2D analysis of the MFE by Leonhardt [1] describes the propagation of linear polarized waves in an isotropic cylindrical medium in which the electric field vector $E$ is parallel to the cylinder axis, and the point sources and drains become lines. That analysis, however, also describes the propagation of linearly polarized fields in the $z$-axis for more general media,
not necessarily isotropic or cylindrical. For instance, it can be easily checked that the time-
harmonic field $E = E(x, y)e^{-i\omega t}z$, the modulus of which, $E(x, y)$, fulfils the 2D Helmholtz
equation (equation (6) of [1])

$$\Delta E + k^2 n^2 E = 0,$$

is a solution of Maxwell’s equations (1) for an anisotropic medium with permittivity and
permeability tensors of the form $\varepsilon = \text{diag}(\varepsilon_x, \varepsilon_y, \varepsilon_z)$, $\mu = \text{diag}(\mu_x, \mu_y, \mu_z)$, where

$$\varepsilon_z \text{ is independent of } z, \quad \mu_x = \mu_y = \mu_\perp = \text{constant},$$

and $k = \omega/c$ and $n^2 = \varepsilon_z \mu_\perp$. Therefore, the waves described by Leonhardt [1] also propagate in
that anisotropic medium, provided that $\varepsilon_z$ and $\mu_\perp$ also fulfill

$$\sqrt{\varepsilon_z \mu_\perp} = n(\rho) = \frac{2}{1 + \rho^2}. \tag{8}$$

Here, $\rho^2 = x^2 + y^2$. Note that the elements $\varepsilon_x, \varepsilon_y$ and $\mu_z$ may depend on $z$, showing that this
medium is not necessarily cylindrically symmetric.

We are looking for a transformation from a space with a medium of this type (that
propagates the 2D Helmholtz fields of the 2D-MFE) into another final space with a simpler
medium (with spherical symmetry). The specific selection of the initial medium will be made
later after the transformation between the two spaces is described.

2.2. The transformation

Luneburg [11] realized that the 2D-MFE can be derived by a conformal transformation from
a spherical surface. This particular conformal transformation is the stereographic projection.
In consequence, any ray path in the MFE can be mapped by the stereographic projection on a
godesic of the sphere, i.e. on great circles. With this new point of view, the surprising properties
of the MFE as an absolute optical instrument became obvious. We will use this conformal
transformation between surfaces for building our transformation between spaces. Consider the
cylindrical coordinates $\rho, \psi, z$ for the initial medium and the spherical coordinates $r, \theta, \phi$ for the
final medium (we have dropped the primes of the $r, \theta, \phi$ variables to simplify our nomenclature).
These three functions define the transformation

$$\rho = \tan \frac{\theta}{2}, \quad \psi = \phi, \quad z = f(r). \tag{9}$$

The first two equations of equation (9) define the stereographic projection (see figure 1),
mapping points of the plane $z = \text{constant}$ onto points of the sphere $r = \text{constant}$. This
stereographic projection is not exactly the same as that used in [1], where North was
the projection pole. The present stereographic projection simplifies some of the following
equations. The transformation equations of the North Pole stereographic projection are the same
as equation (9) if $\theta$ is substituted by $\pi - \theta$.

Equations (3) and (5) are particularly simple when the transformation is such that it can be
written as three functions of a single variable, which is the case for equation (9). Taking into
account that the scale factors of the cylindrical and spherical coordinates are \( h_\rho = 1, h_\psi = \rho, h_z = 1 \) and \( h_r = 1, h_\theta = r, h_\phi = r \sin \theta \), the matrix \( M \) is

\[
M = \begin{pmatrix}
    h_r \frac{\partial}{\partial r} & h_r \frac{\partial}{\partial r} & h_r \frac{\partial}{\partial r} \\
    h_\rho \frac{\partial}{\partial \rho} & h_\psi \frac{\partial}{\partial \psi} & h_z \frac{\partial}{\partial z} \\
    h_\rho \frac{\partial}{\partial \rho} & h_\theta \frac{\partial}{\partial \theta} & h_\phi \frac{\partial}{\partial \phi} \\
    h_\rho \frac{\partial}{\partial \rho} & h_\psi \frac{\partial}{\partial \psi} & h_z \frac{\partial}{\partial z}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \frac{1}{f'(r)} \\
r(1 + \cos \theta) & 0 & 0 \\
0 & r(1 + \cos \theta) & 0
\end{pmatrix}.
\]

(10)

2.3. Transformed fields

Let us consider first the case in which we choose \( f(r) = \ln(r) \). This selection makes a bijective mapping from \(-\infty < z < \infty\) to \( r > 0 \), covering the whole initial and final spaces. According to equation (5), the electric field in the final medium \( \mathbf{E}' \), as a function of the electric field in the initial medium of \( \mathbf{E} (\mathbf{E} = E_\rho \rho + E_\psi \psi + E_z z) \), is given by

\[
\mathbf{E}' = E'_r r + E'_\theta \theta + E'_\phi \phi = \frac{E_z}{r} r + \frac{E_\rho \theta + E_\psi \phi}{r(1 + \cos \theta)}.
\]

(11)

Since we are interested in the \( z \)-polarized waves, i.e. \( \mathbf{E} \) has only a \( z \)-component (\( E_\rho = 0, E_\psi = 0 \)), we obtain that the transformed field in spherical coordinates \( \mathbf{E}' \) has only its \( r \)-component,

\[
\mathbf{E}' = \frac{E_z}{r} r.
\]

(12)

Also, its magnitude is proportional to that of the \( z \)-polarized wave in the initial medium.
2.4. Permittivity and permeability tensors of the final medium

Consider the initial medium described in section 2.1, i.e. with \( \varepsilon = \text{diag}(\varepsilon_\rho, \varepsilon_\psi, \varepsilon_z) \), \( \mu = \text{diag}(\mu_\perp, \mu_\perp, \mu_z) \) and with \( \varepsilon_z \) and \( \mu_\perp \) fulfilling equations (7) and (8). From equation (3) it can be proved that the permittivity and permeability of the final medium in spherical coordinates are, respectively, \( \text{diag}(\varepsilon_r, \varepsilon_\theta, \varepsilon_\phi) \) and \( \text{diag}(\mu_r, \mu_\theta, \mu_\phi) \), where

\[
\begin{align*}
\varepsilon_r &= \frac{\varepsilon_z}{r (1 + \cos \theta)^2}, & \varepsilon_\theta &= \frac{\varepsilon_\rho}{r}, & \varepsilon_\phi &= \frac{\varepsilon_\psi}{r}, \\
\mu_r &= \frac{\mu_z}{r (1 + \cos \theta)^2}, & \mu_\theta &= \frac{\mu_\perp}{r}, & \mu_\phi &= \frac{\mu_\perp}{r}.
\end{align*}
\]

From the first of the transformation equations in equation (9), we can deduce that

\[
\frac{2}{1 + \rho^2} = 1 + \cos \theta.
\]

Thus, using equation (8), we obtain

\[
\varepsilon_r = \frac{\varepsilon_z}{r (1 + \cos \theta)^2} = \frac{1}{\mu_\perp r}.
\]

Since \( \mu_\perp \) must be constant and \( \varepsilon_z \) must be independent of \( z \) (we should better say that \( \varepsilon_z \) must be independent of \( r \), because equation (15) is written in the coordinates \( r, \theta, \phi \)), \( \varepsilon_z \) should be proportional to \( (1 + \cos \theta)^2 \). If we choose \( \mu_\perp = 1 \), then \( \varepsilon_z = (1 + \cos \theta)^2 = (2/(1 + \rho^2))^2 \). Now we can freely select \( \varepsilon_\rho, \varepsilon_\psi \) and \( \mu_z \) and calculate from equation (13) the components \( \varepsilon \) and \( \mu \), without affecting the expression for the transformed field given by equation (12). This means that we can freely select \( \varepsilon_\theta, \varepsilon_\phi \) and \( \mu_r \) in the transformed medium. For instance, we can choose them to obtain an isotropic, impedance-matched final medium of refractive index \( n = 1/r \), with parameters

\[
\varepsilon_r = \varepsilon_\theta = \varepsilon_\phi = \mu_r = \mu_\theta = \mu_\phi = \frac{1}{r}.
\]

This medium has already been considered in [12] with a different purpose: by the addition of a shell with the parameters \( \varepsilon \) and \( \mu \) also proportional to \( 1/r \) but negative, it was shown to perform as the spherical symmetric equivalent of Pendry’s perfect lens.

Other remarkable solutions exist when we select the function \( f(r) = r \) in equation (9). One immediately arrives at an alternative medium (having the same perfect focusing properties) with the same refractive index \( n = 1/r \), which is also isotropic, but non-magnetic,

\[
\varepsilon_r = \varepsilon_\theta = \varepsilon_\phi = \frac{1}{r^2}, \quad \mu_r = \mu_\theta = \mu_\phi = 1.
\]

In contrast to the selection \( f(r) = \ln(r) \), in the case of \( f(r) = r \), the mapping between the initial and final spaces is bijective only when the initial space is restricted to the domain \( z > 0 \). This is not a problem, since we are specifically interested in the field solutions in limited domains of the input and output spaces, as discussed next.
3. Waveguides

The \( z \)-polarized waves of the initial medium are not perturbed by the addition of two perfect conductor plates at \( z = \) constant planes. Let us say \( z = \ln R_1 \) and \( z = \ln R_2 \) (see figure 2), since the electric field component tangent to that plane is null. These conductors create a waveguide within which the fields are isolated from the outside so that waves can propagate confined in it. Correspondingly, the \( r \)-polarized fields in the final medium given by equation (12) propagate confined within the spherical waveguide formed by two perfect conductor concentric spheres \( r = R_1 \) and \( r = R_2 \). Due to the transformation equation (9), the interior of that planar waveguide is transformed into the interior of the spherical waveguide and the flat conductors into the spherical conductors.

Let us consider the following specific time-harmonic solution of \( z \)-polarized wave in the initial medium described by Leonhardt in his equations (11) and (12) in [1],

\[
E = E_z = \frac{P_\nu(\zeta) - e^{i\nu\pi} P_\nu(-\zeta)}{4 \sin(\nu\pi)} e^{-iot} z. \tag{18}
\]

Here, \( P_\nu \) is the Legendre function of the first kind,

\[
\nu = \frac{-1 + \sqrt{1 + 4k^2}}{2} \tag{19}
\]

and

\[
\zeta = \frac{\rho^2 - 1}{\rho^2 + 1}. \tag{20}
\]

Such a wave in the initial medium (equation (18)) is radial symmetric, generated by the line source located at the origin (\( \rho = 0 \)) and propagating outwards to infinity (\( \rho \to \infty \)), where it is drained. The line source is transformed by the first equation of equation (9) in the radial half-line \( \theta = 0 \), whereas the line drain is transformed to the opposite radial half-line \( \theta = \pi \). Also, from the first equation of equation (9), we obtain that \( \zeta = -\cos \theta \), which implies that the time-harmonic transformed field inside the spherical waveguide can be written from equations (12) and (18) as

\[
E' = \frac{1}{r} \frac{P_\nu(-\cos \theta) - e^{i\nu\pi} P_\nu(\cos \theta)}{4 \sin(\nu\pi)} e^{-iot} r, \quad R_1 \leq r \leq R_2. \tag{21}
\]
As a consequence, this field shows also an asymptotic behavior in the neighborhoods of the transformed source ($\theta = 0$) and drain ($\theta = \pi$) as the one shown by the field in the original medium (equation (18)). Since the final medium is spherical symmetric, the source and drain absolute positions are arbitrary and so the field equation (21) is also obviously a solution after a rotation centered at the origin.

3.1. Homogeneous waveguide

The electric field of equation (21) inside the spherical waveguide has a $1/r$ dependence, the same as the refractive index $n = 1/r$ of the filling medium. However, the range of variation of $r$ is limited to $R_1 \leq r \leq R_2$, so they are the field and refractive index variations. This implies that the closer $R_1$ and $R_2$ are selected, the closer the medium is to being homogeneous. Therefore, if a spherical waveguide is manufactured with $R_1$ and $R_2$ close enough and is filled with an isotropic homogeneous material of refractive index $n = 2/(R_1 + R_2)$, its behavior is expected to be equally close to the MFE. Such proximity should lead to super-resolution properties, provided the MFE has the infinite resolution claimed by Leonhardt in [1].

4. Generalization to a class of refractive-index distributions: geodesic waveguides

In section 3, we found an isotropic non-magnetic medium wherein Helmholtz scalar waves can be replicated showing perfect imaging properties. Not only is this newly discovered medium isotropic and non-magnetic but also we have found that the waves can be confined in a waveguide within which the refractive index can be made almost constant. A key point of such a result is that the transformation from cylindrical variables $\rho$, $\psi$ to spherical variables $\theta$, $\phi$ is conformal. This keeps $\mu_\theta = \mu_\phi$ in the transformation medium, which is a necessary condition for the final medium to be isotropic. The transformation between any $z = \text{constant}$ plane of the original medium and their corresponding $r = \text{constant}$ spheres is conformal. Nevertheless, we would only need the transformation between planes and spheres to be conformal within the spherical waveguide, i.e. in $R_1 \leq r \leq R_2$.

In this section, we will analyze other rotational symmetric geodesic waveguides. For some of the common refractive-index distributions in a plane (i.e. 2D) with rotational symmetry $\eta(\rho)$ (such as the Eaton and Gutman lenses in addition to the MFE), it is possible to establish a conformal mapping between points of the plane and points of a rotational symmetric surface, so the geodesic curves on this surface are mapped in ray trajectories of a medium with refractive index $\eta(\rho)$ [13]. Let us call that surface the geodesic surface corresponding to $\eta(\rho)$. Sometimes the mapping cannot be established for all points of the plane. By placing two mirrors at two parallel surfaces placed on both sides of the geodesic surface and separated by a small thickness $\tau$, it is possible to design a light guide with rays following trajectories close to the geodesic curves of the geodesic surface (by 'parallel surfaces' we mean wavefront surfaces generated from the original surface by propagating it in a medium with constant refractive index, using geometrical optics). When $\tau \to 0$, the ray trajectories in the light guide are mapped into ray trajectories of the refractive-index distribution $\eta(\rho)$. Obviously, the light guide corresponding to an MFE distribution is a spherical shell.

With the aid of transformation optics, we will next show that waves with the electric field polarized normal to the waveguide surface are transformed from waves into a planar waveguide having a $z$-polarized electric field. This means that the mapping between geodesic
waveguides and cylindrical refractive-index distributions, a mapping valid in the geometrical optics framework, can be extended to certain modes of wave optics.

4.1. The transformation

The original medium is described in the cylindrical coordinates $\rho, \psi, z$ and the final medium in a rotationally symmetric coordinate system $s, \sigma, \phi$. Similarly to equation (9), the three functions defining the transformation are

$$\rho = \rho(\sigma), \quad \psi = \phi, \quad z = s.$$  \hspace{1cm} (22)

For the surface defined by $s = 0$, the coordinate $\sigma$ of a point $P'$ in this surface is defined as the length of a meridian cross section curve from the axis to the point $P'$ (see figure 3). The azimuthal angle is preserved in the mapping, i.e. $\psi = \phi$. All of the points in a straight line normal to $s = 0$ at $P'$, such as $Q'$, have the same value of $\sigma$ as $P'$. The coordinate $s$ of a general point $Q'$ is the length along these straight lines from the point to the intersection of the straight line with the surface $s = 0$. We will assume that the Jacobian of the transformation defined by equation (22) is not singular in a region of the space around the plane $z = 0$ containing $-\tau/2 \leq z \leq \tau/2$ (or in its corresponding region around the surface $s = 0$). The transformation is well defined in a region around the surface $s = 0$, with $|s|$ smaller than the minimum absolute radius of curvature of the lines of curvature of the surface $s = 0$. We will assume that this minimum absolute radius of curvature, designated $\tau/2$, is greater than zero. The transformation is limited to the range $-\tau/2 \leq s \leq \tau/2$. Note that the different $s = $ constant surfaces coincide with the geometrical optics propagation in vacuum of the surface $s = 0$ interpreted as a wavefront. Thus the transformation is well defined in the volumetric region where the propagation of this wavefront has no caustics.

Note that with the above definition of coordinates, the scale factors $h_\rho$ and $h_\phi$ are $h_\rho = 1, h_\phi = (x'^2 + y'^2)^{1/2}$. The scale factor of $\sigma$ is $h_\sigma = 1$ for the points of the surface $s = 0$. Taking
into account that the scale factors of cylindrical coordinates are \( h_\rho = 1, \ h_\psi = \rho, \ h_z = 1 \), the matrix \( M \) is

\[
M = \begin{pmatrix}
    h_s \partial_s & h_s \partial_s & h_s \partial_s \\
    h_\rho \partial_\rho & h_\psi \partial_\psi & h_z \partial_z \\
    h_\rho \partial_\rho & h_\psi \partial_\psi & h_\sigma \partial_\sigma \\
    h_\rho \partial_\rho & h_\psi \partial_\psi & h_\phi \partial_\phi
\end{pmatrix} = \begin{pmatrix}
    0 & 0 & 1 \\
    h_\sigma \partial_\sigma & 0 & 0 \\
    0 & \sqrt{\frac{x'^2 + y'^2}{\rho}} & 0
\end{pmatrix}.
\] (23)

If the surface \( s=0 \) is a geodesic surface corresponding to the refractive-index distribution \( \eta(\rho) \), as explained before, then it fulfils (see for instance [13])

\[
\sqrt{x'^2 + y'^2} = \rho \eta(\rho) \ d\sigma = \eta(\rho) \ d\rho.
\] (24)

Then the matrix \( M \) is

\[
M = \begin{pmatrix}
    0 & 0 & 1 \\
    h_\sigma \eta(\rho) & 0 & 0 \\
    0 & \eta(\rho) & 0
\end{pmatrix}.
\] (25)

4.2. Transformed fields

According to equation (5), the electric field in the final medium is \( \mathbf{E}' \), as a function of the electric field in the initial medium \( \mathbf{E}(\mathbf{E} = E_\rho \rho + E_\psi \psi + E_z z) \). It is given by

\[
\mathbf{E}' = E'_s s + E'_\sigma \sigma + E'_\phi \phi = E_s s + \frac{E_\rho}{h_\sigma \eta} \sigma + \frac{E_\psi}{\eta} \phi.
\] (26)

Since we are interested in the \( z \)-polarized waves \( (E_\rho = 0, E_\psi = 0) \), we determine that the transformed field \( \mathbf{E}' \) has only an \( s \)-component

\[
\mathbf{E}' = E_z s.
\] (27)

4.3. Permittivity and permeability tensors of the final medium

Using equation (3) with \( \varepsilon = \text{diag}(\varepsilon_\rho, \varepsilon_\psi, \varepsilon_z), \ \mu = \text{diag}(\mu_\perp, \mu_\perp, \mu_z) \), we determine that the permittivity and permeability of the final medium in the \( s, \sigma, \phi \) coordinates are, respectively, \( \text{diag}(\varepsilon_s, \varepsilon_\sigma, \varepsilon_\phi) \) and \( \text{diag}(\mu_s, \mu_\sigma, \mu_\phi) \), where

\[
\varepsilon_s = \frac{\varepsilon_z}{h_\sigma \eta^2}, \quad \varepsilon_\sigma = h_\sigma \varepsilon_\rho, \quad \varepsilon_\phi = \frac{\varepsilon_\psi}{h_\sigma},
\]

\[
\mu_s = \frac{\mu_z}{h_\sigma \eta^2}, \quad \mu_\sigma = h_\sigma \mu_\perp, \quad \mu_\phi = \frac{\mu_\perp}{h_\sigma}.
\] (28)

Choosing again \( \mu_\perp = 1 \) and \( \varepsilon_z = \eta^2(\rho) \) (which fulfill the conditions of equation (7) of section 2.1), we can calculate \( \varepsilon_s, \mu_\sigma, \mu_\phi \) from equation (28), and we can freely choose the remaining parameters \( \varepsilon_\sigma, \varepsilon_\phi, \mu_s \), since \( \varepsilon_\rho, \varepsilon_\psi, \mu_z \) do not intervene in the expression of the \( z \)-polarized field \( \mathbf{E} \) in the original medium. This means that the transformed field \( \mathbf{E}' \)
(equation (27)) is insensitive to the selection of $\varepsilon_{\sigma}$, $\varepsilon_{\phi}$, $\mu_{s}$. For instance we can choose

$$\varepsilon_{\sigma} = \varepsilon_{\phi} = \frac{1}{h_{\sigma}}, \quad \mu_{s} = 1,$$

and obtain the following $\varepsilon$ and $\mu$ in the transformed medium,

$$\varepsilon_{s} = \varepsilon_{\sigma} = \varepsilon_{\phi} = \frac{1}{h_{\sigma}}, \quad \mu_{s} = \mu_{\sigma} = \frac{1}{h_{\sigma}}. \tag{29}$$

### 4.4. Homogeneous waveguide

Let us design a waveguide with conductor surfaces $z = -\tau/2$ and $z = \tau/2$ (initial space) or $s = -\tau/2$ and $s = \tau/2$ (final space). In order for the fields to be well defined, the Jacobian of the transformation equation (22) needs to be non-singular and single valued inside the waveguide, i.e. between surfaces $s = -\tau/2$ and $s = \tau/2$. As discussed before, assume that the surface $s = 0$ is smooth enough so this condition is fulfilled for a certain positive $\tau$.

Since $h_{\sigma} = 1$ at the points of the surface $s = 0$, the tensors $\varepsilon$ and $\mu$ at this surface are

$$\varepsilon_{s} = \varepsilon_{\sigma} = \varepsilon_{\phi} = \mu_{s} = \mu_{\sigma} = \mu_{\phi} = 1. \tag{30}$$

This corresponds to a vacuum. Then, a sufficiently small $\tau$ makes the transformed medium nearly homogeneous, isotropic and non-magnetic inside the waveguide. This means that when $\tau \rightarrow 0$, the $z$-polarized electric field mode in the initial medium can be reproduced in the waveguide filled with an isotropic, homogeneous, non-magnetic material, which seems much easier to manufacture.

The use of transformation optics seems to be a powerful tool to analyze waves in a curved waveguide giving much simpler results than the direct analysis [14].

Unlike the waveguides of this paper, which have constant thickness and are not close to flat surfaces, a tapered waveguide is essentially a flat one, with thickness changing adiabatically. A tapered waveguide can emulate 2D isotropic refractive-index distributions for some modes [13]. They can also emulate 2D metamaterial devices with anisotropic dielectric permittivity and magnetic permeability. In particular, the idea has been applied to emulate cloaking structures in the visible range [15], as well as the MFE and the ‘inverted Eaton lens’ [16]. (This other refractive-index distribution is also an absolute optical instrument, found in [17], and related to that of Eaton lens, but despite its name in [16] it is not the result of an inversion of the Eaton lens.)

It can be easily seen that tapering a thin spherical waveguide (figure 2) does change both $\varepsilon_{r}$ and $\mu_{\theta}$, $\mu_{\phi}$ relative to the values of the untapered waveguide ($\varepsilon_{r}$ grows in proportion to the thickness amplification, while $\mu_{\theta}$, $\mu_{\phi}$ is inversely proportional to it). The product $\varepsilon_{r}\mu_{\theta}$ (or $\varepsilon_{r}\mu_{\phi}$), however, is not, so the solution of equation (6) remains the same. This is because the specific mode of equation (12) has the electric field normal to the waveguide, which makes it insensitive to the thickness. A similar thing happens with the more general thin waveguides of the type of figure 4 for the radially polarized mode. A combined curved and tapered waveguide seems to have enough degrees of freedom to emulate anisotropic permittivity and permeability for a wide variety of modes. However, this last statement is yet to be proven.

### 5. Conclusions

With the help of transformation optics, we have been able to prove that:

1. A spherical waveguide filled with an isotropic non-magnetic material with radial refractive index $n = 1/r$ has radially polarized modes with the same perfect focusing properties
Figure 4. The planar waveguide of the original space, which is filled with an inhomogeneous medium, is transformed into a curved waveguide filled with an isotropic, homogeneous, non-magnetic medium. The original $z$-polarized electric field is transformed into one polarized normal to the curved waveguide.

as the Maxwell fish-eye lens. Moreover, if the waveguide is thin enough, the medium within it can be approximated by a homogeneous one. These results pave the way to experimentally attaining perfect imaging in the Maxwell fish-eye lens.

2. Geodesic waveguides designed with the tools of geometrical optics, from a 2D refractive-index distribution, not only map rays in geodesic curves (as is well known from geometrical optics) but also replicate the $z$-polarized electric fields obtained in the cylindrical refractive-index distribution of the waveguides.

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References

[1] Leonhardt U 2009 Perfect imaging without negative refraction New J. Phys. 11 093040
[2] Leonhardt U and Philbin T G 2010 Perfect imaging with positive refraction in three dimensions Phys. Rev. A 81 011804
[3] Benítez P, Miñano J C and González J C 2010 Perfect focusing of scalar wave fields in three dimensions Opt. Express 18 7650–63
[4] Ward A J and Pendry J B 1996 Refraction and geometry in Maxwell’s equations J. Mod. Opt. 4 773–93
[5] Schurig D, Pendry J B and Smith D R 2006 Calculation of material properties and ray tracing in transformation media Opt. Express 21 9794–804

New Journal of Physics 12 (2010) 123023 (http://www.njp.org/)
[6] Pendrury J B, Schurig D and Smith D R 2006 Controlling electromagnetic fields *Science* **312** 1780–2
[7] Leonhardt U and Philbin T G 2006 General relativity in electrical engineering *New J. Phys.* **8** 247
[8] Min Y, Wei Y and Min Q 2009 Invisibility cloaking by coordinate transformation *Prog. Opt.* **52** 261–304
[9] Leonhardt U and Philbin T G 2009 Transformation optics and the geometry of light *Prog. Opt.* **53** 69–152
[10] Chen H 2009 Transformation optics in orthogonal coordinates *J. Opt. A: Pure Appl. Opt.* **11** 075102
[11] Luneburg R K 1964 *Mathematical Theory of Optics* (Berkeley, CA: University of California Press)
[12] Ramakrishna S A and Pendry J B 2004 Spherical perfect lens: Solutions of Maxwell’s equations for spherical geometry *Phys. Rev. B* **69** 115115
[13] Coornbleet S 1994 *Microwave and Geometrical Optics* (London: Academic)
[14] Kiselev V A 1986 Main optics equations for waveguides with a curved surface *Sov. J. Quantum Electron.* **16** 1574
[15] Smolyaninov I I, Smolyaninova V N, Kildishev A V and Shalaev V M 2009 Anisotropic metamaterials emulated by tapered waveguides: application to optical cloaking *Phys. Rev. Lett.* **102** 213901
[16] Smolyaninova V N, Smolyaninov I I, Kildishev A V and Shalaev V M 2010 Maxwell fish-eye and Eaton lenses emulated by microdroplets *Opt. Lett.* **35** 3396–8
[17] Minano J C 2006 Perfect imaging in a homogeneous three-dimensional region *Opt. Express* **14** 9627–35