\textbf{$\mathbb{B}_0$-VALUED MONOGENIC FUNCTIONS TO THE THEORY OF PLANE ANISOTROPY}

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Abstract

A solution of the elliptic type PDE of the 4th order, being a reduction of the Eqs. of stress function corresponding to any case of plane anisotropy which is not equal to isotropy (proved by S. G. Mikhlin), is described in terms of hypercomplex "analytic" functions with values in two-dimensional semisimple algebra over the field of complex numbers in case when a domain under consideration is bounded and simply-connected. A boundary value problem on finding a function which satisfies this PDE in the considered domain (bounded and simply-connected) and permits continuations (to the boundary) of operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ acted to it, is reduced to certain BVP for these hypercomplex functions.

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1 Introduction

The algebraic-analytic approaches to the investigation of elastic media in terms of “analytic” functions satisfying a system of partial differential equations (a generalization of the “Cauchy–Riemann conditions”) with values in finite-dimensional algebras were developed in [1–14] (commutative algebras and isotropic plane media), [15–17] (commutative algebras and orthotropic two-dimensional media), [19–24] (algebra of quaternions and isotropic three-dimensional media), [24, 25] (approaches of using different kinds of monogenic functions with values in the Clifford algebras for solving the equilibrium system of the isotropic three-dimensional media), [27–29] (algebras of complex (2×2) matrices and anisotropic plane media), and [30] (algebras of complex (3×3) matrices and anisotropic plane media).

The present paper is devoted to the construction of classes of “analytic” functions $\Phi$ with values in two-dimensional commutative algebras over the field of complex numbers containing bases $(e_1, e_2)$ with some algebraic properties (in what follows, we construct all mentioned bases and the corresponding algebra in the explicit form) sufficient for the real components of these functions to satisfy the following equations for fixed $p > 0, p \neq 1$:

$$\tilde{l}_p u(x, y) := \left(\frac{\partial^4}{\partial y^4} + A_p \frac{\partial^4}{\partial x^2 \partial y^2} + B_p \frac{\partial^4}{\partial x^4}\right) u(x, y) = 0,$$

where $A_p := p^2 + 1, B_p = p^2$, $u$ is a real-valued solution of (1), an argument $(x, y) \in D$, while the latter is belonging to the Cartesian plane $xOy$.

The operator $\tilde{l}_p$ can be factorized in the form:

$$\tilde{l}_p = \tilde{l}_{1,p} \circ \Delta_2 = \Delta_2 \circ \tilde{l}_{1,p}, \quad \tilde{l}_{1,p} := \frac{\partial^2}{\partial y^2} + p^2 \frac{\partial^2}{\partial x^2},$$

where $\tilde{l}_{1,p} \circ \Delta_2$ is a symbol of composition of operators $\tilde{l}_{1,p}$ and $\Delta_2$, $\Delta_2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2-D Laplasian.

Equation (1) is a special case of the generalized biharmonic equation (this term was used, e.g., in [31] and [32, p. 603]), which is extremely important in the anisotropic two-dimensional theory of elasticity (see [32–36, 39–41]) and determines (in absence of body forces) the equation for finding the stress function $u(x, y)$ (in the isotropic case a similar function is often called the Airy function and Eqs. (1) turns into the biharmonic equation).

S. G. Mikhlin proved in [33] that an equation of finding the stress function in any cases of plane anisotropy (except of isotropic case) can be reduced to Eqs. (1). From the other side, formally Eqs. (1) corresponds to orthotropic material being an equation of finding the stress function in a special case of plane anisotropy — orthotropy (cf., e.g., [35, pp. 33,34], [37]).

Note that the Eqs. (1) is considered with $A_p := 2p, B_p := 1, p > 1$ in [16,17], or $-1 < p < 1$ in [18].
2 Two-dimensional algebras over the field of complex numbers and their bases associated with Eqs. (1.1)

It is known (see [38]) that there exist (to within isomorphism) two associative algebras of the second rank with identity $e$ commuting over the field of complex numbers $\mathbb{C}$. These algebras are generated by the bases $(e, \rho)$ and $(e, \omega)$, respectively:

$$\mathbb{B} := \{c_1 e + c_2 \rho : c_k \in \mathbb{C}, k = 1, 2\}, \quad \rho^2 = 0, \quad (3)$$

$$\mathbb{B}_0 := \{c_1 e + c_2 \omega : c_k \in \mathbb{C}, k = 1, 2\}, \quad \omega^2 = e. \quad (4)$$

It is clear that the algebra $\mathbb{B}_0$ is semisimple (for the definition, see, e.g., [42, p. 33]) and contains a basis with orthogonal idempotents $(I_1, I_2)$, where

$$I_1 = \frac{1}{2} (e + \omega), \quad I_2 = \frac{1}{2} (e - \omega), \quad I_1 I_2 = 0. \quad (5)$$

It is clear that

$$I_1 + I_2 = e, \quad I_1 - I_2 = \omega. \quad (6)$$

In the works of different researchers, several names are used for the algebra $\mathbb{B}_0$. Thus, in [43], it is called unipodal. Moreover, it determines the simplest case of complex Clifford algebra (cf., e.g., [43, 44]). Algebra (3) is a complexification of the algebra of hyperbolic or double numbers $\mathbb{P}$ over the field of real numbers $\mathbb{R}$:

$$\mathbb{B}_0 = \mathbb{P} \oplus i \mathbb{P}, \quad \mathbb{P} := \{xe + hy : x, y \in \mathbb{R}\}, \quad h := \omega,$$

where $i$ is the imaginary unit.

The element $w = c_1 I_1 + c_2 I_2$ from $\mathbb{B}_0$ is invertible if and only if $c_k \neq 0, k = 1, 2$. In this case, the inverse element is given by the equality (see [44, p. 38])

$$w^{-1} = \frac{1}{c_1} I_1 + \frac{1}{c_2} I_2. \quad (7)$$

Since the algebra $\mathbb{B}$ contains a nonzero radical $\{c\rho : c \in \mathbb{C}\}$ (see [4]), the algebra $\{c\rho : c \in \mathbb{C}\}$ is not semisimple. An element $a = c_1 e + c_2 \rho$ from $\mathbb{B}$ is invertible if and only if $c_1 \neq 0$. In this case, the equality $a^{-1} = \frac{1}{c_1} e - \frac{c_2}{(c_1)^2} \rho$ is true (see [45]).

For any complex number $s$ we introduce the notation

$$\widehat{l}_p(s) := s^4 + (p^2 + 1) s^2 + p^2. \quad (8)$$

The equation $\widehat{l}_p(s) = 0$ is the characteristic equation of the $(1)$, its set of roots is

$$\{s_1, s_2, \bar{s}_1, \bar{s}_2\} =: \ker \widehat{l}_p, \quad s_1 = i, \quad s_2 = ip. \quad (9)$$

where $x + iy := x - iy, \quad x, y \in \mathbb{R}.$
Now we are looking for an associative, commutative algebra of the second rank with unity $e$ over the field of complex numbers $\mathbb{C}$ and containing a basis $(e_1, e_2)$ that satisfies the condition

$$\mathcal{L}_p(e_1, e_2) := e_2^4 + G_p e_1^2 e_2^2 + H_p e_1^4 = 0. \quad (10)$$

where $G_p := (p^2 + 1), \ H_p := p^2$.

Note that a similar problem had been considered for the equation of the type like $(10)$ with $G_p := 2p, \ H_p := 1, \ p > 1$ in [16] and $-1 < p < 1$ in [18].

Doing in analogous way as in the proof of similar Theorem in [16] one can obtain the following theorem.

**Theorem 1.** The algebra $\mathbb{B}$ does not contain any basis $(e_1, e_2)$ satisfying condition $(10)$.

There exists a set of cardinality continuum of bases $(e_1, e_2)$ in $\mathbb{B}_0$ satisfying condition $(10)$:

$$e_1 = \alpha I_1 + \beta I_2, \ e_2 = \alpha \tilde{s}_1 I_1 + \beta \tilde{s}_2 I_2 \ \forall \alpha, \beta \in \mathbb{C} \setminus \{0\}, \quad (11)$$

where $\tilde{s}_k \in \ker \tilde{l}_p, \ k = 1, 2, \ \tilde{s}_1 \neq \tilde{s}_2$

Let us restrict our attention on the case $\alpha = \beta \equiv 1, \ \tilde{s}_k = s_k, \ k = 1, 2$, in $(11)$. Therefore, we have

$$e_1 = I_1 + I_2 \equiv e, \ e_2 = i(I_1 + pI_2) \equiv i (e_1 + (p - 1)I_2). \quad (12)$$

Since expressions of idempotents $I_k, \ k = 1, 2$, via elements of bases $(12)$ are

$$I_1 = -\frac{1}{1 - p} (pe_1 + ie_2), \ I_2 = \frac{1}{1 - p} (e_1 + ie_2), \quad (13)$$

we obtain the multiplication table for the bases $(12)$:

$$e_1 e_2 = e_2, \ e_2^2 = pe_1 + i(p + 1)e_2. \quad (14)$$

### 3 \ $\mathbb{B}_0$-valued monogenic functions and Eqs. (1)

Consider $\mu_{e_1, e_2} = \{\zeta = xe_1 + ye_2 : x, y \in \mathbb{R}\}$ which is a linear span of the elements $e_1, e_2$ of the basis $(12)$ over the field of real numbers $\mathbb{R}$. With a domain $D$ of the Cartesian plane $xOy$ we associate the congruent domain $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\} \subset \mu_{e_1, e_2}$, and corresponding domains in the complex plane $\mathbb{C}$: $D_z := \{z = x + iy : (x, y) \in D\}$, $D_{z_p} := \{z_p = x + ipy : (x, y) \in D\}$.

Let $D_*$ be a domain in $xOy$ or in $\mu_{e_1, e_2}$. Denote by $\partial D_*$ a boundary of a domain $D_*$, $\text{cl} D_*$ means a closure of a domain $D_*$.

In what follows, $(x, y) \in D, \ z = xe_1 + ye_2 \in D_\zeta, \ z = x + iy \in D_z, \ z_p = x + ipy \in D_{z_p}$.

Inasmuch as divisors of zero don’t belong to $\mu_{e_1, e_2}$, one can define the derivative $\Phi'(\zeta)$ of function $\Phi : D_\zeta \to \mathbb{B}_0$ in the same way as in the complex plane:

$$\Phi'(\zeta) := \lim_{h \to 0, \ h \in \mu_{e_1, e_2}} (\Phi(\zeta + h) - \Phi(\zeta)) h^{-1}. \quad (15)$$
We say that a function \( \Phi: D_\zeta \rightarrow \mathbb{B} \) is \textit{monogenic} in a domain \( D_\zeta \) if and denote as \( \Phi \in \mathcal{M}_{\mathbb{B}_0}(D_\zeta) \), if the derivative \( \Phi'(\zeta) \) exists in every point \( \zeta \in D_\zeta \).

Every function \( \Phi: D_\zeta \rightarrow \mathbb{B}_0 \) has a form
\[
\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) ie_1 + U_3(x, y) e_2 + U_4(x, y) ie_2, \tag{15}
\]
where \( \zeta = xe_1 + ye_2, U_k: D \rightarrow \mathbb{R}, k = \overline{1, 4} \).

Let denote every real component \( U_k, k = \overline{1, 4} \), in expansion (15) by \( U_k[\Phi] \), i.e., for arbitrary fixed \( k \in \{1, \ldots, 4\} \):
\[
U_k[\Phi(\zeta)] := U_k(x, y) \forall \zeta = xe_1 + ye_2 \in D_\zeta.
\]

We establish the following theorem similar to analogous theorems in [6, 16].

**Theorem 2.** A function \( \Phi: D_\zeta \rightarrow \mathbb{B}_0 \) is monogenic in the \( D_\zeta \) if and only if its components \( U_k: D_\zeta \rightarrow \mathbb{R}, k = \overline{1, 4} \), in decomposition (15) are differentiable in the domain \( D \) and the following analog of the Cauchy–Riemann conditions is true:
\[
\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2. \tag{16}
\]

In an extended form the condition (16) for the monogenic function (15) is equivalent to the system of four equations (cf., e.g., [4, 6]) with respect to components \( U_k = U_k[\Phi] \), \( k = \overline{1, 4} \), in (15):
\[
\begin{align*}
\frac{\partial U_1(x, y)}{\partial y} &= p \frac{\partial U_3(x, y)}{\partial x}, \tag{17} \\
\frac{\partial U_2(x, y)}{\partial y} &= p \frac{\partial U_4(x, y)}{\partial x}, \tag{18} \\
\frac{\partial U_3(x, y)}{\partial y} &= \frac{\partial U_1(x, y)}{\partial x} - (p + 1) \frac{\partial U_4(x, y)}{\partial x}, \tag{19} \\
\frac{\partial U_4(x, y)}{\partial y} &= \frac{\partial U_2(x, y)}{\partial x} + (p + 1) \frac{\partial U_3(x, y)}{\partial x}. \tag{20}
\end{align*}
\]

Using (12), an element \( \zeta = xe_1 + ye_2 \in \mu_{e_1, e_2} \) turns of by the formula
\[
\zeta = z I_1 + z_p I_2 \forall \zeta \in \mu_{e_1, e_2}. \tag{21}
\]

A function \( \Phi \in \mathcal{M}_{\mathbb{B}_0}(D_\zeta) \) can be expressed in terms of two holomorphic functions of the complex variable \( z \) and \( z_p \), respectively. The following theorem obtained with use of (21) similar to analogous theorem in [16].

**Theorem 3.** The function \( \Phi: D_\zeta \rightarrow \mathbb{B}_0 \) is monogenic in the domain \( D_\zeta \) if and only if the following equality is true:
\[
\Phi(\zeta) = F_1(z) I_1 + F_2(z_p) I_2 \forall \zeta \in D_\zeta, \tag{22}
\]
where \( F_1(z) \), \( F_2(z_p) \) is a holomorphic function of its complex variable \( z \in D_z, z_p \in D_{z_p}, \) respectively.
It follows from (22) and Theorem 2 that every derivative \( \Phi^{(n)}(\zeta) \), \( n = 1, 2, \ldots \), is monogenic. Thus, we have
\[
\Phi^{(4)}(\zeta) \equiv 0.
\]
Therefore, we deduce that every component \( U_k = U_k[\Phi] \), \( k = 1, 4 \), satisfies Eqs. (1).

With use of (13) we rewrite (22) to the form
\[
\Phi(\zeta) = \frac{1}{1 - p} \left( (-p F_1(z) + F_2(z_p)) e_1 + \left( F_2(z_p) - \frac{1}{p} F_1(z) \right) e_2 \right) \quad \forall \zeta \in D_\zeta.
\] (23)

After that, substituting with loss of generality \( c_k F_k \) to \( F_k \), \( k = 1, 2 \), where \( c_1 := -\frac{p}{1 - p} \), \( c_2 := \frac{1}{1 - p} \), we obtain the following representation of the monogenic function \( \Phi \) in the bases (12) for every \( \zeta \in D_\zeta \):
\[
\Phi(\zeta) = (F_1(z) + F_2(z_p)) e_1 + i \left( F_2(z_p) - \frac{1}{p} F_1(z) \right) e_2.
\] (24)

Since now we assume that \( D \) is a bounded and simply-connected domain.

Then by solving a system (17) – (20) with \( U_1 \equiv 0 \) it is easy to deliver that a function \( \Phi_{1,0} \in \mathcal{M}_{B_0}(D_\zeta) \), such that \( U_1[\Phi_{1,0}] \equiv 0 \), has a form
\[
\Phi_{1,0}(\zeta) = ai \left( ye_1 + \left( \frac{zp}{p} + \frac{i}{p} y \right) e_2 \right) + bie_1 + ce_2 + die_2 \quad \forall \zeta \in D_\zeta,
\] (24)

where \( a, b, c, d \) are arbitrary real numbers.

We shall prove that for every fixed solution \( u \) of the equation (1) in a bounded simply connected domain \( D \subset xOy \) exists a function \( \Phi_u \in \mathcal{M}_{B_0}(D_\zeta) \) such that \( U_1[\Phi_u] \equiv u \).

There is a well-known fact (cf., e.g., [35, §20, p. 136] or [36]), that every solution \( u \) of the equation (1) is expressed in the form:
\[
u(\xi, \eta) = \text{Re} \left( F_1(z) + F_2(z_p) \right) \quad \forall (\xi, \eta) \in D,
\] (25)

where \( F_1: D_z \rightarrow \mathbb{C} \) and \( F_2: D_{z_p} \rightarrow \mathbb{C} \) are analytic functions of their variables.

By use of (23) with \( \Phi_u := \Phi \) and \( F_k \) the same as in (25), \( k = 1, 2 \), we rewrite the equality (25) in the form
\[
u(\xi, \eta) = U_1[\Phi_u(\zeta)] \quad \forall \zeta \in D_\zeta,
\] (26)

where \( \Phi_u \in \mathcal{M}_{B_0}(D_\zeta) \).

It follows now from (24) and (25) the following theorem being an analog of the classical fact that any harmonic function (in the bounded simply-connected domain of the real plane) is a real part of an analytic function of the complex variable, moreover, this representation is unique up to the imaginary constant as an addend.

**Theorem 4.** Let \( u \) be a solution of the equation (1). Then all \( \Phi \in \mathcal{M}_{B_0}(D_\zeta) \), such that
\[
u(\xi, \eta) = U_1[\Phi(\zeta)] \quad \forall \zeta \in D_\zeta,
\] (27)
are expressed by the formula
\[
\Phi(\zeta) = \Phi_u(\zeta) + \Phi_{1,0}(\zeta) \quad \forall \zeta \in D_\zeta.
\] (28)
Note, that similar Theorem is proved in with deal of the equation of the type like (1) with
\( A_p := 2p \), \( B_p := 1 \) and \( p > 1 \) or \(-1 < p < 1\) (being Eqs. of the stress function to a
certain class of orthotropic plane deformations) in \([17](p > 1)\) or \([18](−1 < p < 1)\).

4 A BVP of plane anisotropy and corresponding BVP for \( \mathbb{B}_0 \)-valued monogenic functions

Consider a boundary value problem on finding a function \( u : D \to \mathbb{R} \) satisfying conditions
\[
\begin{cases}
\tilde{l}_p u(x,y) = 0 \quad \forall (x,y) \in D, \\
\lim_{(x,y) \to (x_0,y_0) \in \partial D,(x,y) \in D} \frac{\partial u(x,y)}{\partial x} = u_1(x_0,y_0) \quad \forall (x_0,y_0) \in \partial D, \\
\lim_{(x,y) \to (x_0,y_0) \in \partial D,(x,y) \in D} \frac{\partial u(x,y)}{\partial y} = u_3(x_0,y_0) \quad \forall (x_0,y_0) \in \partial D,
\end{cases}
\tag{29}
\]
where \( u_k : \partial D \to \mathbb{B}_0, k \in \{1,3\} \), are given continuous functions.

The problem (29) has a great importance in the anisotropy theory (cf., e.g., [17]), when
\( \tilde{l}_p \) is a biharmonic operator we arrive at the isotropic case and this changed boundary
value problem (29) named as the biharmonic problem (cf., e.g., [31]). There are different
approaches to its solving (see [33, 36, 46, 47]).

Our aim is to find a new method of solving BVP (29) by use of monogenic functions. Let \( \Phi_1 \in \mathcal{M}_{\mathbb{B}_0}(D_\zeta) \) such that
\[
U_1[\Phi_1(\zeta)] = u(x,y) \quad \forall (x,y) \in D,
\tag{30}
\]
where \( u \) is a south-fought solution of Problem (29). Using the equality (17) and the
condition (16), we have
\[
\Phi'_1(\zeta) \equiv \frac{\partial \Phi_1(\zeta)}{\partial x} = \frac{\partial u(x,y)}{\partial x} e_1 + \frac{1}{p} \frac{\partial u(x,y)}{\partial x} e_2 + \frac{\partial U_2[\Phi_1(\zeta)]}{\partial x} e_1 + \frac{\partial U_4[\Phi_1(\zeta)]}{\partial x} e_2
\]
for all \( \zeta \in D_\zeta \), therefore, for all \((x,y) \in D\). Thus, BVP (29) is reduced to BVP on finding
a monogenic function \( \Phi := \Phi' \) satisfying boundary conditions
\[
\lim_{\zeta \to \zeta_0 = x_0 e_1 + y_0 e_2 \in \partial D_\zeta, \zeta \in D_\zeta} U_k[\Phi(\zeta)] = \lambda_k u_k(x_0,y_0), k \in \{1,3\}, \forall (x_0,y_0) \in \partial D,
\]
where \( \lambda_k := 1 \) if \( k = 1 \), \( \lambda_k := \frac{1}{p} \) if \( k = 3 \). Solving the latter BVP for monogenic functions,
we obtain a solution of BVP (29) in the form
\[
u(x,y) = \int_{(x_0,y_0)}^{(x,y)} (U_1[\Phi(xe_1 + ye_2)] dx + p U_3[\Phi(xe_1 + ye_2)] dy) + \text{const}_R \quad \forall (x,y) \in D,
\]
where \( \text{const}_R \) is an arbitrary real number, \((x_0,y_0) \) is a fixed point in \(D\), integration means
along any piecewise smooth curve jointing this point with a point with variable coordinates
\((x,y) \).
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