Reconstruction formula for differential systems with a singularity.

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Abstract. Our studies concern some aspects of scattering theory of the singular differential systems $y’ - x^{-1}Ay - q(x)y = \rho By$, $x > 0$ with $n \times n$ matrices $A, B, q(x), x \in (0, \infty)$, where $A, B$ are constant and $\rho$ is a spectral parameter. We concentrate on the important special case when $q(\cdot)$ is smooth and $q(0) = 0$ and derive a formula that express such $q(\cdot)$ in the form of some special contour integral, where the kernel can be written in terms of the Weyl-type solutions of the considered differential system. Formulas of such a type play an important role in constructive solution of inverse scattering problems: use of such formulas, where the terms in their right-hand sides are previously found from the so-called main equation, provides a final step of the solution procedure. In order to obtain the above-mentioned reconstruction formula we establish first the asymptotical expansions for the Weyl-type solutions as $\rho \to \infty$ with $o(\rho^{-1})$ rate remainder estimate.

1 Introduction

Our studies concern some aspects of scattering theory of the differential systems

$$y’ - x^{-1}Ay - q(x)y = \rho By, \quad x > 0$$

with $n \times n$ matrices $A, B, q(x), x \in (0, \infty)$, where $A, B$ are constant and $\rho$ is a spectral parameter.

Differential equations with coefficients having non-integrable singularities at the end or inside the interval often appear in various areas of natural sciences and engineering. For $n = 2$, there exists an extensive literature devoted to different aspects of spectral theory of the radial Dirac operators, see, for instance [1], [2], [3], [4], [5].

Systems of the form (1) with $n > 2$ and arbitrary complex eigenvalues of the matrix $B$ appear to be considerably more difficult for investigation even in the ”regular” case $A = 0$ [6]. Some difficulties of principal matter also appear due to the presence of the singularity. Whereas the ”regular” case $A = 0$ has been studied fairly completely to date [6], [7], [8], for system (1) with $A \neq 0$ there are no similar general results.

In this paper, we consider the important special case when $q(\cdot)$ is smooth and $q(0) = 0$ and, provided also that the discrete spectrum is empty, derive a formula that express such $q(\cdot)$ in the form of some special contour integral, where the kernel can be written in terms of the Weyl-type solutions of system (1). Formulas of such a type play an important role in constructive solution of inverse scattering problems: use of such formulas, where the terms in their right-hand sides are previously found from the so-called main equation (see, for instance, [9], [10]), provides a final step of the solution procedure. In order to obtain the above-mentioned reconstruction formula we establish first the asymptotical expansions for the Weyl-type solutions as $\rho \to \infty$ with $o(\rho^{-1})$ rate remainder estimate.

2 Preliminary remarks

Consider first the following unperturbed system:

$$y’ - x^{-1}Ay = \rho By$$

and its particular case corresponding to the value $\rho = 1$ of the spectral parameter

$$y’ - x^{-1}Ay = By$$
but to complex (in general) values of \( x \).

**Assumption 1.** Matrix \( A \) is off-diagonal. The eigenvalues \( \{ \mu_j \}_{j=1}^n \) of the matrix \( A \) are distinct and such that \( \mu_j - \mu_k \not\in \mathbb{Z} \) for \( j \neq k \), moreover, \( \text{Re} \mu_1 < \text{Re} \mu_2 < \cdots < \text{Re} \mu_n \), \( \text{Re} \mu_k \neq 0 \), \( k = 1, n \).

**Assumption 2.** \( B = \text{diag}(b_1, \ldots, b_n) \), the entries \( b_1, \ldots, b_n \) are nonzero distinct points on complex plane such that \( \sum_{j=1}^n b_j = 0 \) and such that any 3 points are noncollinear.

Under Assumption 1 system \([3]\) has the fundamental matrix \( c(x) = (c_1(x), \ldots, c_n(x)) \), where
\[
c_k(x) = x^{\mu_k} \hat{c}_k(x),
\]
det \( c(x) \equiv 1 \) and all \( \hat{c}_k(\cdot) \) are entire functions, \( \hat{c}_k(0) = h_k \), \( h_k \) is an eigenvector of the matrix \( A \) corresponding to the eigenvalue \( \mu_k \). We define \( C_k(x, \rho) := c_k(\rho x) \), \( x \in (0, \infty) \), \( \rho \in \mathbb{C} \). We note that the matrix \( C(x, \rho) \) is a solution of unperturbed system \([2]\) (with respect to \( x \) for given spectral parameter \( \rho \)).

Let \( \Sigma \) be the following union of lines through the origin in \( \mathbb{C} \):
\[
\Sigma = \bigcup_{(k,j): j \neq k} \{ z : \text{Re}(zb_j) = \text{Re}(zb_k) \}.
\]
By virtue of Assumption 2 for any \( z \in \mathbb{C} \setminus \Sigma \) there exists the ordering \( R_1, \ldots, R_n \) of the numbers \( b_1, \ldots, b_n \) such that \( \text{Re}(R_1 z) < \text{Re}(R_2 z) < \cdots < \text{Re}(R_n z) \). Let \( S \) be a sector \( \{ z = r \exp(i \gamma), r \in (0, \infty), \gamma \in (\gamma_1, \gamma_2) \} \) lying in \( \mathbb{C} \setminus \Sigma \). Then \([11]\) system \([3]\) has the fundamental matrix \( \psi(x) = (\psi_1(x), \ldots, \psi_n(x)) \) which is analytic in \( S \), continuous in \( \overline{S} \setminus \{0\} \) and admits the asymptotics:
\[
\psi_k(x) = e^{xR_k(f_k + x^{-1} \eta_k(x))}, \quad \eta_k(x) = O(1), \quad x \to \infty, \quad x \in \overline{S},
\]
where \( (f_1, \ldots, f_n) = f \) is a permutation matrix such that \( (R_1, \ldots, R_n) = (b_1, \ldots, b_n)f \). We define \( E(x, \rho) := e(\rho x) \).

Everywhere below we assume that the following additional condition is satisfied.

**Condition 1.** For all \( k = 2, \ldots, n \) the numbers
\[
\Delta_{0k} := \text{det}(e_1(x), \ldots, e_{k-1}(x), c_k(x), \ldots, c_n(x))
\]
are not equal to 0.

Under Condition 1 system \([3]\) has the fundamental matrix \( \psi_0(x) = (\psi_01(x), \ldots, \psi_0n(x)) \) which is analytic in \( S \), continuous in \( \overline{S} \setminus \{0\} \) and admits the asymptotics:
\[
\psi_{0k}(xt) = \exp(xtR_k(f_k + o(1))), \quad t \to \infty, \quad x \in S, \quad \psi_{0k}(x) = O(x^{\mu_k}), \quad x \to 0.
\]
We define \( \Psi_0(x, \rho) := \psi_0(\rho x) \). As above, we note that the matrices \( E(x, \rho), \Psi_0(x, \rho) \) solve \([2]\).

In the sequel we use the following notations:
- \( \{ e_k \}_{k=1}^n \) is the standard basis in \( \mathbb{C}^n \);
- \( A_m \) is the set of all ordered multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_m) \), \( \alpha_1 < \alpha_2 < \cdots < \alpha_m \), \( \alpha_j \in \{1, 2, \ldots, n\} \);
- for a sequence \( \{ u_j \} \) of vectors and a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \) we define \( u_\alpha := u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m} \);
- for a numerical sequence \( \{ a_j \} \) and a multi-index \( \alpha \) we define
\[
a_\alpha := \sum_{j \in \alpha} a_j, \quad a^\alpha := \prod_{j \in \alpha} a_j;
\]
for a multi-index $\alpha$ the symbol $\alpha'$ denotes the ordered multi-index that complements $\alpha$ to $(1, 2, \ldots, n)$;

- for $k = 1, n$ we denote

$$\vec{a}_k := \sum_{j=1}^{k} a_j, \quad \vec{a}_k := \sum_{j=k}^{n} a_j, \quad \vec{a}_k := \prod_{j=1}^{k} a_j, \quad \vec{a}_k := \prod_{j=k}^{n} a_j.$$  

We note that Assumptions 1, 2 imply, in particular, $\sum_{k=1}^{n} \mu_k = \sum_{k=1}^{n} R_k = 0$ and therefore for any multi-index $\alpha$ one has $R_{\alpha'} = -R_{\alpha}$ and $\mu_{\alpha'} = -\mu_{\alpha}$.

- the symbol $V^{(m)}$, where $V$ is $n \times n$ matrix, denotes the operator acting in $\wedge^m \mathbb{C}^n$ so that for any vectors $u_1, \ldots, u_m$ the following identity holds:

$$V^{(m)}(u_1 \wedge u_2 \wedge \cdots \wedge u_m) = \sum_{j=1}^{m} u_1 \wedge u_2 \wedge \cdots \wedge u_{j-1} \wedge V u_j \wedge u_{j+1} \wedge \cdots \wedge u_m;$$

- if $h \in \wedge^n \mathbb{C}^n$ then $|h|$ is a number such that $h = |h| e_1 \wedge e_2 \wedge \cdots \wedge e_n$;

- for $h \in \wedge^m \mathbb{C}^n$ we set: $\|h\| := \sum_{\alpha \in A_m} |h_\alpha|$, where $\{h_\alpha\}$ are the coefficients from the expansion $h = \sum_{\alpha \in A_m} h_\alpha e_\alpha$.

3 Asymptotics of the Weyl - type solutions

Let $\mathcal{S} \subset \mathbb{C} \setminus \Sigma$ be an open sector with vertex at the origin. For arbitrary $\rho \in \mathcal{S}$ and $k \in \{1, \ldots, n\}$ we define the $k$-th Weyl - type solution $\Psi_k(x, \rho)$ as a solution of (1) normalized with the asymptotic conditions:

$$\Psi_k(x, \rho) = O(x^{\mu_k}), \quad x \to 0, \quad \Psi_k(x, \rho) = \exp(\rho x R_k)(f_k + o(1)), \quad x \to \infty. \quad (4)$$

If $q(\cdot)$ is off-diagonal matrix function summable on the semi-axis $(0, \infty)$ then for arbitrary given $\rho \in \mathcal{S}$ $k$-th Weyl - type solution exists and is unique provided that the characteristic function:

$$\Delta_k(\rho) = |F_{k-1}(x, \rho) \wedge T_k(x, \rho)|$$

does not vanish at this $\rho$. Here $\{F_k(x, \rho)\}_{k=1}^{n}$, $\{T_k(x, \rho)\}_{k=1}^{n}$ are certain tensor-valued functions (fundamental tensors) defined as solutions of certain Volterra integral equations, see [14], [16] for details.

For arbitrary fixed arguments $x, \rho$ (where $\Delta_k(\rho) \neq 0$) the value $\Psi_k = \Psi_k(x, \rho)$ is the unique solution of the following linear system:

$$F_{k-1} \wedge \Psi_k = F_k, \quad \Psi_k \wedge T_k = 0. \quad (5)$$

This fact and also some properties of the Weyl - type solutions were established in works [14], [17], in particular, the following asymptotics for $\rho \to \infty$ was obtained:

$$\Psi_k(x, \rho) = \Psi_{0k}(x, \rho) + o(\exp(\rho x R_k)). \quad (6)$$

For our purposes we need more detailed asymptotics that can be obtained provided that the potential $q(\cdot)$ is smooth enough and vanishes as $x \to 0$. 

We denote by \( \mathcal{P}(\mathcal{S}) \) the set of functions \( F(\rho), \rho \in \mathcal{S} \) admitting the representation:

\[
F(\rho) = \sum_{\lambda \in \Lambda} f(\lambda) \exp(\lambda \rho).
\]

Here the set \( \Lambda \) (depending on \( F(\cdot) \in \mathcal{P}(\mathcal{S}) \)) is such that \( \text{Re}(\lambda \rho) < 0 \) for all \( \lambda \in \Lambda, \rho \in \mathcal{S} \). We note that the set of scalar functions belonging to \( \mathcal{P}(\mathcal{S}) \) is an algebra with respect to pointwise multiplication.

**Theorem 1.** Suppose that \( q(\cdot) \) is absolutely continuous off-diagonal matrix function such that \( q(0) = 0 \). Denote by \( \hat{q}_o(\cdot) \) the off-diagonal matrix function such that \([B, \hat{q}_o(x)] = -q(x)\) for all \( x > 0 \) (here \([\cdot, \cdot]\) denotes the matrix commutator). Define the diagonal matrix \( d(x) = \text{diag}(d_1(x), \ldots, d_n(x)) \), where

\[
d_k(x) := \int_x^\infty t^{-1} ([\hat{q}_o(t), A])_{kk} \, dt
\]

and set \( \hat{q}(x) := \hat{q}_o(x) + d(x) \).

Suppose that all the functions \( q_{ij}(\cdot), q'_{ij}(\cdot) \) and \( \hat{q}_{ij}(\cdot) \), where \( \hat{q}(x) := \hat{q}'(x) + x^{-1}[\hat{q}(x), A] \), belong to \( X_p := L_1(0, \infty) \cap L_p(0, \infty), p > 2 \).

Then for each fixed \( x > 0 \) and \( \rho \rightarrow \infty, \rho \in \mathcal{S} \) the following asymptotics holds:

\[
\rho(\Psi(x, \rho) - \Psi_0(x, \rho)) \exp(-\rho x R) = f \Gamma(x) + \hat{q}(x)f + \mathcal{E}(x, \rho) + o(1),
\]

where \( \Gamma(x) \) is some diagonal matrix, \( \mathcal{E}(x, \cdot) \in \mathcal{P}(\mathcal{S}) \).

**Proof.** Denote:

\[
\tilde{F}_k(x, \rho) := \exp \left( -\rho x \tilde{R}_k \right) F_k(x, \rho), \quad \tilde{T}_k(x, \rho) := \exp \left( -\rho x \tilde{R}_k \right) T_k(x, \rho).
\]

By virtue of Theorem 1 \([16]\) the following asymptotics hold:

\[
\rho \tilde{F}_k(q, x, \rho) = \rho \tilde{F}_0 k(x, \rho) + \sum_{\alpha \in A_k} f_{k, \alpha}(x) f_\alpha + \mathcal{E}(x, \rho) + o(1),
\]

\[
\rho \tilde{T}_k(q, x, \rho) = \rho \tilde{T}_0 k(x, \rho) + d_{0k} \tilde{T}_{0k}(x, \rho) + \sum_{\alpha \in A_{n-k+1}} T^0_{k, \alpha^*(k)} g_{k, \alpha, \alpha^*(k)}(x) f_\alpha + \mathcal{E}(x, \rho) + o(1), \quad (7)
\]

where \( \alpha^*(k) := (k, \ldots, n), \alpha^*(k) := (1, \ldots, k) \); \( f_{k, \alpha}(x), g_{k, \alpha, \alpha^*(k)}(x) \) are some scalars that can be written explicitly in terms of \( q(\cdot) \).

For the Weyl - type solutions of the unperturbed system we have the asymptotics (following directly from their definition):

\[
\tilde{\Psi}_0 k(x, \rho) = f_k + \mathcal{E}(x, \rho) + O \left( \rho^{-1} \right), \quad (8)
\]

where \( \tilde{\Psi}_0 k(x, \rho) := \exp(-\rho x R_k) \tilde{\Psi}_0 k(x, \rho) \). Here and below we use the same symbol \( \mathcal{E}(\cdot, \cdot) \) for different functions such that \( \mathcal{E}(x, \cdot) \in \mathcal{P}(\mathcal{S}) \) for each fixed \( x \).

We rewrite relations \([5]\) in the form of the following linear system with respect to value \( \tilde{\Psi}_k = \tilde{\Psi}_0 k(x, \rho) \) of the function \( \tilde{\Psi}_k(x, \rho) := \exp(-\rho x R_k) \tilde{\Psi}_k(x, \rho) \):

\[
\tilde{F}_{k-1} \land \tilde{\Psi}_k = \tilde{F}_k, \quad \tilde{\Psi}_k \land \tilde{T}_k = 0.
\]

By making the substitution:

\[
\tilde{\Psi}_k = \tilde{\Psi}_0 k + \tilde{\Psi}_k,
\]

\[
(9)
\]
we obtain:
\[ \bar{F}_{k-1} \wedge \hat{\Psi}_k = \bar{F}_k - \bar{F}_{k-1} \wedge \bar{\Psi}_0k, \quad \hat{\Psi}_k \wedge \bar{T}_k = -\bar{\Psi}_0k \wedge \bar{T}_k. \]

The obtained relations we transform into the following system of linear algebraic equations:
\[ \sum_{j=1}^{n} m_{ij} \gamma_{jk} = u_i, \quad i = \overline{1,n} \tag{10} \]

with respect to coefficients \( \{\gamma_{jk}\} \) of the expansion:
\[ \hat{\Psi}_k(x, \rho) = \sum_{j=1}^{n} \gamma_{jk}(x, \rho) f_j. \tag{11} \]

Coefficients \( \{m_{ij}\}, \{u_i\} \) can be calculated as follows:
\[ m_{ij} = \left| \bar{F}_{k-1} \wedge f_j \wedge f_\alpha \right|, \]
\[ u_i = \left| (\bar{F}_k - \bar{F}_{k-1} \wedge \bar{\Psi}_0k) \wedge f_\alpha \right|, \quad \alpha = \alpha^*(k) \setminus i, \quad i = k,n, \]
\[ m_{ij} = \left| f_\alpha \wedge f_j \wedge \bar{T}_k \right|, \quad u_i = -\left| f_\alpha \wedge \bar{\Psi}_0k \wedge \bar{T}_k \right|, \quad \alpha = \alpha^*(k-1) \setminus i, \quad i = 1,k-1. \]

Using (7), (8) and taking into account that:
\[ \delta^k \]
\[ \delta^k \]
we obtain:
\[ F_1^0(\bar{\Psi}_0k \wedge \bar{T}_k) = 0, \]
we obtain the following asymptotics for the coefficients of SLAE (10) as \( \rho \to \infty \):
\[ m_{ij}(x, \rho) = O(\rho^{-1}) \quad j \neq i, \quad m_{ii}(x, \rho) = m_{ii}^0 + O(\rho^{-1}), \quad m_{ii}^0 = (-1)^{k-i}|f|, \quad i = k,n, \tag{12} \]
and
\[ m_{ij}(x, \rho) = m_{ij}^0 + O(\rho^{-1}), \quad i = \overline{1,k-1}, \quad j = \overline{1,k-1}, \]
\[ m_{ij}(x, \rho) = m_{ij}^0 + \mathcal{E}(x, \rho) + O(\rho^{-1}), \quad i = \overline{1,k-1}, \quad j = k,n, \]
where:
\[ m_{ij}^0 = T_{k,\alpha^*(k)}^0|f_\alpha \wedge f_j \wedge f_\alpha^*(k)|, \quad \alpha = \alpha^*(k-1) \setminus i, \]
and therefore:
\[ m_{ij}(x, \rho) = O(\rho^{-1}) \quad j \neq i, \quad m_{ij}(x, \rho) = \mathcal{E}(x, \rho) + O(\rho^{-1}), \quad j = k,n, \tag{13} \]
\[ m_{ii}(x, \rho) = m_{ii}^0 + O(\rho^{-1}), \quad m_{ii}^0 = (-1)^{k-i}|f| T_{k,\alpha^*(k)}^0 \tag{14} \]
for \( i = \overline{1,k-1} \).

Proceeding in a similar way we obtain:
\[ \rho u_i(x, \rho) = u_i^1(x) + \mathcal{E}(x, \rho) + o(1), \tag{15} \]
\[ u_i^1(x) = (-1)^{k-i}|f| f_{k,\alpha}(x) - \delta_{i,k}|f| f_{(k-1),\alpha^*(k-1)}(x), \quad \alpha = \alpha^*(k-1) \cup \{i\}, \quad i = k,n, \tag{16} \]
where \( \delta_{i,k} \) is a Kronecker delta,
\[ u_i^1(x) = -(-1)^{k-i}|f| T_{k,\alpha^*(k)}^0 g_{k,\beta^*(k)}(x), \quad \beta = \alpha^* \setminus k, \quad \alpha = \alpha^*(k-1) \setminus i, \quad i = \overline{1,k-1}. \tag{17} \]

Using the obtained asymptotics we obtain from (10) the auxiliary estimate \( \gamma_{ik}(x, \rho) = O(\rho^{-1}) \).
Then, using in (10) the substitution $\gamma_{ik}(x, \rho) = \rho^{-1}\gamma_{ik}(x, \rho)$ (where, as it was shown above, $\hat{\gamma}_{ik}(x, \rho) = O(1)$) we obtain for $i = k, n$:

$$m_{ii}(x, \rho)\hat{\gamma}_{ik}(x, \rho) = u_i^1(x) + \mathcal{E}(x, \rho) - \sum_{j \neq i} m_{ij}(x, \rho)\hat{\gamma}_{jk}(x, \rho) + o(1).$$

In view of (12), (15) this yields:

$$\hat{\gamma}_{ik}(x, \rho) = \gamma_{ik}^1(x) + \mathcal{E}(x, \rho) + o(1), \quad \gamma_{ik}^1 = \frac{u_i^1(x)}{m_{ii}^0}, \quad i = k, n.$$  

Similarly, for $i < k$ we have:

$$m_{ii}(x, \rho)\hat{\gamma}_{ik}(x, \rho) = u_i^1(x) + \mathcal{E}(x, \rho) - \sum_{j \geq k} m_{ij}(x, \rho)\hat{\gamma}_{jk}(x, \rho) - \sum_{j < k, j \neq i} m_{ij}(x, \rho)\hat{\gamma}_{jk}(x, \rho) + o(1).$$

Using (13) the obtained relation can be transformed as follows:

$$m_{ii}^0\hat{\gamma}_{ik}(x, \rho) = u_i^1(x) + \mathcal{E}(x, \rho) - \sum_{j \geq k} m_{ij}(x, \rho)\hat{\gamma}_{jk}(x, \rho) + o(1).$$

Now, using in the right hand side of the obtained formula (13) for $m_{ij}(x, \rho)$ and (18) for $\hat{\gamma}_{jk}(x, \rho)$ with $j = k, n$ we conclude that formulas (18) are true for $i < k$ as well.

In our further calculations we use particular form of the coefficients $f_{k,\alpha}(x)$ and $g_{k,\alpha,\beta}(x)$ given by Theorem 1 [16].

For $i = k, n$ from (18), (16), (12) we get:

$$\gamma_{ik}^1(x) = \delta_{i,k}\gamma_{ik}^1(x) + f_{k,\alpha}(x), \quad \alpha = \alpha_*(k - 1) \cup i. \quad (19)$$

Theorem 1 [16] yields:

$$f_{k,\alpha}(x) = \chi_\alpha \left| (\hat{q}^{(k)}(x)f_{\alpha_*(k)}) \wedge f_{\alpha'} \right|, \quad \chi_\alpha := |f_\alpha \wedge f_{\alpha'}|.$$  

Recall that any arbitrary linear operator $V$ acting in $\mathbb{C}^n$ can be expanded onto the wedge algebra $\wedge \mathbb{C}^n$ so that the identity

$$V(h_1 \wedge \cdots \wedge h_m) = (Vh_1) \wedge \cdots \wedge (Vh_m)$$

remains true for any set of vectors $h_1, \ldots, h_m$, $m \leq n$; moreover, for any $h \in \wedge^n \mathbb{C}^n$ one has $Vh = |V|h$ (here $|V|$ denotes determinant of matrix of the operator $V$ in the standard coordinate basis $\{e_1, \ldots, e_n\}$). In what follows the symbol $\hat{f}$ denotes the above mentioned expansion of the operator corresponding to the transmutation matrix $\hat{f}$. We should note also that the relation $(\hat{f}^{-1}V\hat{f})^{(k)} = \hat{f}^{-1}V^{(k)}\hat{f}$ is true for any $n \times n$ matrix $V$. Taking this into account we obtain:

$$f_{k,\alpha}(x) = \chi_\alpha \left| (\hat{q}^{(k)}(x)f_{\alpha_*(k)}) \wedge (f_{\alpha'}) \right| =$$

$$= |f_\alpha \wedge f_{\alpha'}||f| \left| (\hat{q}^{(k)}(x)f_{\alpha_*(k)}) \wedge e_{\alpha'} \right| = \left| e_\alpha \wedge e_{\alpha'} \right| \left| \left( (\hat{q}^{(k)}(x)f)^{(k)} e_{\alpha_*(k)} \right) \wedge e_{\alpha'} \right|.$$  

For the particular multi-index $\alpha = \alpha_*(k - 1) \cup i$ arising at (19) and arbitrary $n \times n$ matrix $V$ we have:

$$|e_\alpha \wedge e_{\alpha'}| \left| (V^{(k)}e_{\alpha_*(k)}) \wedge e_{\alpha'} \right| = V_{ik}.$$  

Substituting the obtained relations into (19) we arrive at:

$$\gamma_{ik}^1(x) = \delta_{i,k}\gamma_{ik}^1(x) + (f^{-1}\hat{q}(x)\hat{f})_{ik}, \quad i = k, n. \quad (20)$$
Proceeding in a similar way in the case \( i < k \), using (13), (17) we obtain:

\[
\gamma_{ik}^1(x) = g_{k,\beta,\alpha^*(k)}(x), \quad \beta = \alpha' \setminus k, \alpha = \alpha_+(k - 1) \setminus i. \tag{21}
\]

Theorem 1 [16] yields:

\[
g_{k,\alpha,\beta}(x) = \chi_\alpha \left| \left( \hat{q}^{(n-k+1)}(x)f_{\beta} \right) \wedge f_{\alpha'} \right|
\]

for \( \beta \neq \alpha \). Repeating the arguments above we obtain:

\[
g_{k,\alpha,\beta}(x) = |e_\alpha \wedge e_{\alpha'}| \left| \left( \left( f^{-1}\hat{q}(x)f \right)^{(n-k+1)} e_\beta \right) \wedge e_{\alpha'} \right|.
\]

In particular, one gets:

\[
g_{k,\beta,\alpha^*(k)}(x) = |e_\beta \wedge e_{\beta'}| \left( \left( f^{-1}\hat{q}(x)f \right)^{(n-k+1)} e_{\alpha^*(k)} \right) \wedge e_{\beta'}.
\]

If \( \beta = \alpha' \setminus k, \alpha = \alpha_+(k - 1) \setminus i, i < k \), then for arbitrary \( n \times n \) matrix \( V \) we have:

\[
|e_\beta \wedge e_{\beta'}| \left| \left( V^{(n-k+1)} e_{\alpha^*(k)} \right) \wedge e_{\beta'} \right| = V_{ik}.
\]

Substituting the obtained relations into (21) we arrive at:

\[
\gamma_{ik}^1(x) = \left( f^{-1}\hat{q}(x)f \right)_{ik}, \quad i = 1, k - 1. \tag{22}
\]

From (22), (20), (18) we obtain:

\[
\rho_{\gamma_{ik}}(x, \rho) = \gamma_{ik}(x, \rho) = \delta_{i,k}\gamma_{ik}^1(x) + \left( f^{-1}\hat{q}(x)f \right)_{ik} + \mathcal{E}(x, \rho) + o(1).
\]

In terms of the matrix \( \gamma = (\gamma_{ik})_{i,k=1,n} \) this is equivalent to:

\[
\rho\gamma(x, \rho) = \Gamma(x) + f^{-1}\hat{q}(x)f + \mathcal{E}(x, \rho) + o(1),
\]

where the matrix \( \Gamma(x) \) is diagonal. Finally, using (11) in the form \( \hat{\Psi}(x, \rho) = f\gamma(x, \rho) \) we obtain the required relation.

\( \square \)

4 Reconstruction formula

Let \( S_\nu, \nu = 1, N \) be the open pairwise nonintersecting sectors such that \( \mathbb{C} \setminus \Sigma = \bigcup_{\nu=1}^N S_\nu \). Suppose that the sectors are enumerated in counterclockwise order. We denote by \( \Sigma_\nu \) the open ray dividing \( S_\nu \) and \( S_{\nu+1} \) (assuming \( S_{N+1} := S_1 \)). We agree that the rays \( \Sigma_\nu \) are oriented from 0 to \( \infty \). Denote by \( \Sigma_\nu^+ \) and \( \Sigma_\nu^- \) the banks of cut (along \( \Sigma_\nu \)) belonging to \( S_{\nu+1} \) and \( S_\nu \) respectively. We agree that \( \Sigma_\nu^+ \) is oriented from 0 to \( \infty \) while \( \Sigma_\nu^- \) is oriented from \( \infty \) to 0.

For a function \( f(\rho), \rho \in S_\nu \cup S_{\nu+1} \) and arbitrary \( \rho_0 \in \Sigma_\nu \) we denote by \( f^\pm(\rho_0) \) the limit values (if they exist):

\[
f^-(\rho_0) := \lim_{\rho \to \rho_0, \rho \in S_\nu} f(\rho), \quad f^+(\rho_0) := \lim_{\rho \to \rho_0, \rho \in S_{\nu+1}} f(\rho).
\]

We say that off-diagonal matrix function \( q(\cdot) \in \mathbb{C} \) belongs to the class \( G^\omega_0 \) if for any \( \nu \in \{1, \ldots, N\} \) and \( k \in \{1, \ldots, n\} \) it is true that \( \Delta_k(\rho) \neq 0 \) for all \( \rho \in \mathcal{S}_\nu \). If \( q(\cdot) \in G^\omega_0 \) then the limit values \( \Psi_k^\pm(x, \rho_0) \) exist for any \( k \in \{1, \ldots, n\}, \rho_0 \in \Sigma_\nu, \nu \in \{1, \ldots, N\} \).

We denote by \( \Psi(x, \rho) \) the matrix function \( \Psi(x, \rho) = (\Psi_1(x, \rho), \ldots, \Psi_n(x, \rho)) \) and introduce the following spectral mappings matrix:

\[
P(x, \rho) := \Psi(x, \rho)\Psi^{-1}_0(x, \rho).
\]
If \( q(\cdot) \in G^p_0 \) then the limit values \( P_k^\pm(x, \rho_0) \) exist for any \( k \in \{1, \ldots, n\} \), \( \rho_0 \in \Sigma_\nu, \nu \in \{1, \ldots, N\}. \) Denote \( \hat{P}(x, \rho) := P^+(x, \rho) - P^-(x, \rho). \) Following theorem contains main result of the paper.

**Theorem 2.** Suppose that the potential \( q(\cdot) \in G^p_0 \) satisfies the conditions of Theorem 1. Then the following relation (reconstruction formula) holds:

\[
q(x) = \frac{1}{2\pi i} \int_{\Sigma} [B, \hat{P}(x, \rho)] \, d\rho,
\]

where (as above) the brackets \([\cdot, \cdot]\) denote the matrix commutator and the integral is considered as the following limit (existing for each \( x > 0 \)):

\[
\frac{1}{2\pi i} \int_{\Sigma} [B, \hat{P}(x, \rho)] \, d\rho := \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\Sigma^r} [B, \hat{P}(x, \rho)] \, d\rho,
\]

\( \Sigma^r := \Sigma \cap \{\rho : |\rho| \leq r\} \).

**Proof.** Consider the function:

\[
F(x, \rho) := \rho[B, P(x, \rho)] + q(x).
\]

From Theorem 1 we have the asymptotics:

\[
\hat{\Psi}(x, \rho) := (\Psi(x, \rho) - \hat{\Psi}(x, \rho)) \exp(-\rho xR) = \rho^{-1}(f\Gamma_\nu(x) + \hat{q}(x)f + \mathcal{E}_\nu(x, \rho) + o(1))
\]

as \( \rho \to \infty, \rho \in \mathcal{S}_\nu \), where \( R = \text{diag}(R_1, \ldots, R_n) \), \( \Gamma_\nu(x) \) are some diagonal matrices and \( \mathcal{E}_\nu(x, \cdot) \in \mathcal{P}(\mathcal{S}_\nu) \).

For \( \tilde{\Psi}_0(x, \rho) \) we have:

\[
\tilde{\Psi}_0(x, \rho) = f + \mathcal{E}_\nu(x, \rho) + o(1)
\]

as \( \rho \to \infty, \rho \in \mathcal{S}_\nu \) (we use the same symbol for denoting possibly different functions from \( \mathcal{P}(\mathcal{S}_\nu) \)).

Since \( |\det \Psi_0| = 1 \) the following asymptotics is also valid:

\[
\tilde{\Psi}_0^{-1}(x, \rho) = f^{-1} + \mathcal{E}_\nu(x, \rho) + o(1), \quad \rho \to \infty, \quad \rho \in \mathcal{S}_\nu.
\]

Therefore, for \( \rho \to \infty, \rho \in \mathcal{S}_\nu \) we have:

\[
P(x, \rho) = I + \hat{\Psi}(x, \rho)\tilde{\Psi}_0^{-1}(x, \rho) = I + \rho^{-1}(f\Gamma_\nu(x)f^{-1} + \hat{q}(x) + \mathcal{E}_\nu(x, \rho) + o(1)). (23)
\]

Since the matrices \( \Gamma_\nu(x) \) are diagonal the matrices \( f\Gamma_\nu(x)f^{-1} \) are diagonal as well and we have \( [B, f\Gamma_\nu(x)f^{-1}] = 0 \). Thus, from (23) we deduce:

\[
F(x, \rho) = \mathcal{E}_\nu(x, \rho) + o(1), \quad \rho \to \infty, \quad \rho \in \mathcal{S}_\nu. (24)
\]

Define:

\[
\gamma = \bigcup_{\nu=1}^{N} (\Sigma_\nu^{-} \cup \Sigma_\nu^{+}), \quad \gamma_r := \gamma \cap \{\rho : |\rho| \leq r\}, \quad \Gamma_r := \gamma_r \cup C_r,
\]

where \( C_r \) is the circle \( \{\rho : |\rho| = r\} \) with a counterclockwise orientation.

By virtue of the Jordan lemma from asymptotics (24) it follows that for any arbitrary fixed \( \rho \in \mathbb{C} \setminus \Sigma \) we have:

\[
\lim_{r \to \infty} \int_{C_r} \frac{d\zeta}{\zeta - \rho} F(x, \zeta) = 0.
\]
Therefore, the Cauchy integral formula for the closed contour $\Gamma_r$ (where $r > |\rho|$):
\[
F(x, \rho) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{d\zeta}{\zeta - \rho} F(x, \zeta)
\]
can be transformed as follows:
\[
F(x, \rho) = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\Sigma_r} \frac{d\zeta}{\zeta - \rho} (F^+(x, \zeta) - F^-(x, \zeta)).
\]
Taking into account that $F^+(x, \zeta) - F^-(x, \zeta) = \zeta[B, \hat{P}(x, \zeta)]$ we obtain:
\[
F(x, \rho) = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\Sigma_r} \frac{d\zeta}{\zeta - \rho} \zeta[B, \hat{P}(x, \zeta)]. \quad (25)
\]
On the other hand, we can proceed in a similar way applying the Cauchy formula to the function $P(x, \rho) - I$. Thus we obtain:
\[
P(x, \rho) - I = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{d\zeta}{\zeta - \rho} (P(x, \zeta) - I)
\]
and since from (24) it follows that:
\[
\lim_{r \to \infty} \int_{C_r} \frac{d\zeta}{\zeta - \rho} (P(x, \zeta) - I) = 0
\]
we arrive at the representation:
\[
P(x, \rho) = I + \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\Sigma_r} \frac{d\zeta}{\zeta - \rho} (P^+(x, \zeta) - P^-(x, \zeta)).
\]
Substituting this to the definition of the function $F(x, \rho)$ we arrive at the representation:
\[
F(x, \rho) = q(x) + \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\Sigma_r} \frac{d\zeta}{\zeta - \rho} [B, \hat{P}(x, \zeta)].
\]
Compare it with (25) we obtain the desired relation.
\[
\square
\]
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