On the Fine-grained Complexity of One-Dimensional Dynamic Programming

Marvin Künemann  Ramamohan Paturi  Stefan Schneider
University of California, San Diego

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Abstract

In this paper, we investigate the complexity of one-dimensional dynamic programming, or more specifically, of the Least-Weight Subsequence (LWS) problem: Given a sequence of \( n \) data items together with weights for every pair of the items, the task is to determine a subsequence \( S \) minimizing the total weight of the pairs adjacent in \( S \). A large number of natural problems can be formulated as LWS problems, yielding obvious \( O(n^2) \)-time solutions.

In many interesting instances, the \( O(n^2) \)-many weights can be succinctly represented. Yet except for near-linear time algorithms for some specific special cases, little is known about when an LWS instantiation admits a subquadratic-time algorithm and when it does not. In particular, no lower bounds for LWS instantiations have been known before. In an attempt to remedy this situation, we provide a general approach to study the fine-grained complexity of succinct instantiations of the LWS problem. In particular, given an LWS instantiation we identify a highly parallel core problem that is subquadratically equivalent. This provides either an explanation for the apparent hardness of the problem or an avenue to find improved algorithms as the case may be.

More specifically, we prove subquadratic equivalences between the following pairs (an LWS instantiation and the corresponding core problem) of problems: a low-rank version of LWS and minimum inner product, finding the longest chain of nested boxes and vector domination, and a coin change problem which is closely related to the knapsack problem and \((\min,+)\)-CONVOLUTION. Using these equivalences and known \textsc{SETH}-hardness results for some of the core problems, we deduce tight conditional lower bounds for the corresponding LWS instantiations. We also establish the \((\min,+)\)-CONVOLUTION-hardness of the knapsack problem. Furthermore, we revisit some of the LWS instantiations which are known to be solvable in near-linear time and explain their easiness in terms of the easiness of the corresponding core problems.

1 Introduction

Dynamic programming (DP) is one of the most fundamental paradigms for designing algorithms and a standard topic in textbooks on algorithms. Scientists from various disciplines have developed DP formulations for basic problems encountered in their applications. However, it is not clear...
whether the existing (often simple and straightforward) DP formulations are in fact optimal or nearly optimal. Our lack of understanding of the optimality of the DP formulations is particularly unsatisfactory since many of these problems are computational primitives.

Interestingly, there have been recent developments regarding the optimality of standard DP formulations for some specific problems, namely, conditional lower bounds assuming the Strong Exponential Time Hypothesis (SETH) [27]. The longest common subsequence (LCS) problem is one such problem for which almost tight conditional lower bounds have been obtained recently. The LCS problem is defined as follows: Given two strings $x$ and $y$ of length at most $n$, compute the length of the longest string $z$ that is a subsequence of both $x$ and $y$. The standard DP formulation for the LCS problem involves computing a two-dimensional table requiring $O(n^2)$ steps. This algorithm is only slower than the fastest known algorithm due to Masek and Paterson [34] by a polylogarithmic factor. However, there has been no progress in finding more efficient algorithms for this problem since the 1980s, which prompted attempts as early as in 1976 [6] to understand the barriers for efficient algorithms and to prove lower bounds. Unfortunately, there have not been any nontrivial unconditional lower bounds for this or any other problem in general models of computation. This state of affairs prompted researchers to consider conditional lower bounds based on conjectures such as 3-Sum conjecture [19] and more recently based on ETH [28] and SETH [27]. Researchers have found ETH and SETH to be useful to explain the exact complexity of several NP-complete problems (see the survey paper [33]). Surprisingly, Ryan Williams [39] has found a simple reduction from the CNF-SAT problem to the orthogonal vectors problem which under SETH leads to a matching quadratic lower bound for the orthogonal vectors problem. This in turn led to a number of conditional lower bound results for problems in P (including LCS and related problems) under SETH [7, 11, 23]. Also see [37] for a recent survey.

The DP formulation of the LCS problem is perhaps the conceptually simplest example of a two-dimensional DP formulation. In the standard formulation, each entry of an $n \times n$ table is computed in constant time. The LCS problem belongs to the class of alignment problems which, for example, are used to model similarity between gene or protein sequences. Conditional lower bounds have recently been extended to a number of alignment problems [9, 7, 11, 3].

In contrast, there are many problems for which natural quadratic-time DP formulations compute a one-dimensional table of length $n$ by spending $O(n)$-time per entry. In this work, we investigate the optimality of such DP formulations and obtain new (conditional) lower bounds which match the complexity of the standard DP formulations.

1-dimensional DP: The Least-Weight Subsequence (LWS) Problem. In this paper, we investigate the optimality of the standard DP formulation of the LWS problem. A classic example of an LWS problem is airplane refueling [25]: Given airport locations on a line, and a preferred distance per hop $k$ (in miles), we define the penalty for flying $k'$ miles as $(k - k')^2$. The goal is then to find a sequence of airports terminating at the last airport that minimizes the sum of the penalties. We now define the LWS problem formally.

**Problem 1.1 (LWS).** We are given a sequence of $n + 1$ data items $x_0, \ldots, x_n$, weights $w_{i,j} \in \{-W, \ldots, W\} \cup \{x\}$ for every pair $i < j$ of indices where the weights may also be functions of the values of data items $x_i$, and an arbitrary function $g: \mathbb{Z} \to \mathbb{Z}$. The LWS problem is to determine $T[n]$ which is defined by the following DP formulation.

$$
\begin{align*}
T[0] &= 0, \\
T[j] &= \min_{0 \leq i < j} g(T[i]) + w_{i,j} \\
& \quad \text{for } j = 1, \ldots, n. 
\end{align*}
$$

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To formulate airplane refueling as an LWS problem, we let $x_i$ be the location of the $i$’th airport, $g$ be the identity function, and $w_{i,j} = (x_j - x_i - k)^2$.

In the definition of the LWS problem, we did not specify the encoding of the problem (in particular, the type of data items and the representation of the weights $w_{i,j}$) so we can capture a larger variety of problems: it not only encompasses classical problems such as the pretty printing problem due to Knuth and Plass [31], the airplane refueling problem [23] and the longest increasing subsequence (LIS) [18], but also the unbounded subset sum problem [36 [10], a more general coin change problem that is effectively equivalent to the unbounded knapsack problem, 1-dimensional $k$-means clustering problem [24], finding longest $R$-chains (for an arbitrary binary relation $R$), and many others (for a more complete list of problems definitions, see Section 2).

Under mild assumptions on the encoding of the data items and weights, any instantiation of the LWS problems can be solved in time $O(n^2)$ using (1) for determining the values $T[j]$, $j = 1, \ldots, n$ in time $O(n)$ each. However, the best known algorithms for the LWS problems differ quite significantly in their time complexity. Some problems including the pretty printing, airline refueling and LIS turn out to be solvable in near-linear time, while no subquadratic algorithms are known for the unbounded knapsack problem or for finding the longest $R$-chain.

The main goal of the paper is to investigate the optimality of the LWS DP formulation for various problems by proving conditional lower bounds.

**Succinct LWS instantiations.** In the extremely long presentation of an LWS problem, the weights $w_{i,j}$ are given explicitly. This is however not a very interesting case from a computational point of view, as the standard DP formulation takes linear time (in the size of the input) to compute $T[n]$. In the example of the airplane refueling problem the size of the input is only $O(n)$ assuming that the values of the data items are bounded by some polynomial in $n$. For such succinct representations, we ask if the quadratic-time algorithm based on the standard LWS DP formulation is optimal. Our approach is to study several natural succinct versions of the LWS problem (by specifying the type of data items and the weight function) and determine their complexity. We refer to Section 2 for examples of succinct instantiations of the LWS problem.

**Our Contributions and Results.** The main contributions of our paper include a general framework for reducing succinct LWS instantiations to what we call the core problems and proving subquadratic equivalences between them. The subquadratic equivalences are interesting for two reasons. First, they allow us to conclude conditional lower bounds for certain LWS instantiations, where previously no lower bounds are known. Second, subquadratic (or more general fine-grained) equivalences are more useful since they let us translate hardness as well as easiness results.

Our results include tight (up to subpolynomial factors) conditional lower bounds for several LWS instantiations with succinct representations. These instantiations include the coin change problem, low rank versions of the LWS problem, and the longest subchain problems. Our results are somewhat more general. We propose a factorization of the LWS problem into a core problem and a fine-grained reduction from the LWS problem to the core problem. The idea is that core problems (which are often well-known problems) capture the hardness of the LWS problem and act as a potential barrier for more efficient algorithms. While we do not formally define the notion of a core problem, we identify several core problems which share several interesting properties. For example, they do not admit natural DP formulations and are easy to parallelize. In contrast, the quadratic-time DP formulation of LWS problems requires the entries $T[i]$ to be computed in order, suggesting that the general problem might be inherently sequential.

\[ \text{In all our applications, the function } g \text{ is the trivial identity function.} \]
The reductions between LWS problems and core problems involve a natural intermediate problem, which we call the Static-LWS problem. We first reduce the LWS problem to the Static-LWS problem in a general way and then reduce the Static-LWS problem to a core problem. The first reduction is divide-and-conquer in nature and is inherently sequential. The latter reduction is specific to the instantiation of the LWS problem. The Static-LWS problem is easy to parallelize and does not have a natural DP formulation. However, the problem is not necessarily a natural problem. The Static-LWS problem can be thought of as a generic core problem, but it is output-intensive.

In the other direction, we show that many of the core problems can be reduced to the corresponding LWS instantiations thus establishing an equivalency between LWS instantiations and their core problems. This equivalence enables us to translate both the hardness and easiness results (i.e., the subquadratic-time algorithms) for the core problems to the corresponding LWS instantiations.

The first natural succinct representation of the LWS problem we consider is the low rank LWS problem, where the weight matrix $W = (w_{i,j})$ is of low rank and thus representable as $W = L \cdot R$ where $L$ and $R^T$ are $(n \times n^{o(1)})$-matrices. For this low rank LWS problem, we identify the minimum inner product problem ($\text{MinInnProd}$) as a suitable core problem. It is only natural and not particularly surprising that $\text{MinInnProd}$ can be reduced to the low-rank LWS problem which shows the SETH-hardness of the low-rank LWS problem. The other direction is more surprising: Inspired by an elegant trick of Vassilevska Williams and Williams [40], we are able to show a subquadratic-time reduction from the (highly sequential) low-rank LWS problem to the (highly parallel) $\text{MinInnProd}$ problem. Thus, the very compact problem $\text{MinInnProd}$ problem captures exactly the complexity of the low-rank LWS problem (under subquadratic reductions).

We also show that the coin change problem is subquadratically equivalent to the $(\text{min,+})$-CONVOLUTION problem. In the coin change problem, the weight matrix $W$ is succinctly given as a Toeplitz matrix. At this point, the conditional hardness of the $(\text{min,+})$-CONVOLUTION problem is unknown. The quadratic-time hardness of the $(\text{min,+})$-CONVOLUTION problem would be very interesting, since it is known that the $(\text{min,+})$-CONVOLUTION problem is reducible to the 3-sum problem and the APSP problem. However, recent results give surprising subquadratic-time algorithms for special cases of $(\text{min,+})$-CONVOLUTION [13]. If these subquadratic-time algorithms extend to the general $(\text{min,+})$-CONVOLUTION problem, our equivalence result also provides a subquadratic-time algorithm for the coin change problem and the closely related unbounded knapsack problem. As a corollary, our reductions also give a quadratic-time $(\text{min,+})$-CONVOLUTION-based lower bound for the bounded case of knapsack.

We next consider the problem of finding longest chains: here, we search for the longest subsequence (chain) in the input sequence such that all adjacent pairs in the subsequence are contained in some binary relation $R$. We show that for any binary relation $R$ satisfying certain conditions the chaining problem is subquadratically equivalent to a corresponding (highly parallel) selection problem. As corollaries, we get equivalences between finding the longest chain of nested boxes ($\text{NestedBoxes}$) and $\text{VectorDomination}$ as well as between finding the longest subset chain ($\text{SubsetChain}$) and the orthogonal vectors (OV) problem. Interestingly, these results have algorithmic implications: known algorithms for low-dimensional vector domination and low-dimensional orthogonal vectors translate to faster algorithms for low-dimensional $\text{NestedBoxes}$ and $\text{SubsetChain}$ for small universe size.

Table 1 lists the LWS succinct instantiations (as discussed above) and their corresponding core problems. All LWS instantiations and core problems considered in this paper are formally defined in Section 2.

Finally, we revisit classic problems including the longest increasing subsequence problem, the
unbounded subset sum problem and the concave LWS problem and analyze the Static-LWS instantiations to immediately infer that the corresponding core problem can be solved in near-linear time. Table 2 gives an overview of some of the problems we look at in this context.

**Related Work.** LWS has been introduced by Hirschberg and Lamore [25]. If the weight function satisfies the *quadrangle inequality*, formalized by Yao [41], one obtains the *concave LWS* problem, for which they give an $\mathcal{O}(n \log n)$-time algorithm. Subsequently, improved algorithms solving concave LWS in time $\mathcal{O}(n)$ were given [38, 21]. This yields a fairly large class of weight functions (including, e.g., the pretty printing and airplane refueling problems) for which linear-time solutions exist. To generalize this class of problems, further works address convex weight functions [20, 35, 30] as well as certain combinations of convex and concave weight functions [16] and provide near-linear time algorithms. For a more comprehensive overview over these algorithms and further applications of the LWS problem, we refer the reader to Eppstein’s PhD thesis [17].

Apart from these notions of concavity and convexity, results on the succinct LWS problems are typically more scattered and problem-specific (see, e.g., [18, 31, 10, 24]; furthermore, a closely related recurrence to (1) pops up when solving bitonic TSP [15]). An exception to this rule is a study of the parallel complexity of LWS [22].

**Organization.** Section 3 contains the result on low-rank LWS. This is also where we formally

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### Table 1: Summary of our results

| Name          | Weights                                      | Equivalent Core       | Reference |
|---------------|----------------------------------------------|-----------------------|-----------|
| Coin Change   | Toeplitz matrix: $w_{i,j} = w_{j-i}$         | (min, +)-CONVOLUTION  | Theorem 4.8 |
| Remark:       | Subquadratically equivalent to UNBOUNDEDKNAPSACK |                       |           |
| LowRankLWS    | Low rank representation: $w_{i,j} = \langle \sigma_i, \mu_j \rangle$ | MININNPROD           | Theorem 3.9 |
| R-chains      | matrix induced by $R$: $w_{i,j} = w_j$ if $R(x_i, x_j)$ and $\infty o/w$ | SELECTION(R)         | Theorem 5.3 |
| Remark:       | Result below are corollaries.                |                       |           |
| NestedBoxes   | $w_{i,j} = -1$ if $B_j$ contains $B_i$       | VECTORDOMINATION      |           |
| SubsetChain   | $w_{i,j} = -1$ if $S_i \subseteq S_j$       | ORTHOGONALVECTORS     |           |

### Table 2: Near-linear time algorithms following from the proposed framework.

| Name                  | Weights                                      | $\mathcal{O}(n)$-time reducible to   | Reference |
|-----------------------|----------------------------------------------|--------------------------------------|-----------|
| Longest Increasing    | matrix induced by $R_{<i}$:                  | SORTING                              | [18],     |
| Subsequence           | $w_{i,j} = -1$ if $x_i < x_j$                |                                      |           |
| Unbounded Subset      | Toeplitz $\{0, \infty\}$ matrix:            | CONVOLUTION                          | [10],     |
| Sum                   | $w_{i,j} = w_{j-i} \in \{0, \infty\}$       |                                      | Observation 3 |
| Concave 1-dim. DP     | concave matrix:                              | SMAWK problem                        | [25, 21, 38], |
|                       | $w_{i,j} + w_{i',j'} \leq w_{i'-j} + w_{i,j'}$ |                                      |            |
|                       | for $i \leq i' \leq j \leq j'$              |                                      | Observation 4 |

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2See Section 2 for definitions.
3A weight function is convex if it satisfies the inverse of the quadrangle inequality.
introduce Static-LWS. Section 4 proves the subquadratic equivalence of the coin change problem and (min, +)-CONVOLUTION, while Section 5 discusses chaining problems and their corresponding selection (core) problem. Our results on near-linear time algorithms are given in Section 6.

2 Preliminaries

In this section, we state our notational conventions and list the main problems considered in this work.

Problem $A$ subquadratically reduces to problem $B$, denoted $A \preceq_2 B$, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that an algorithm for $B$ with time $O(n^{2-\varepsilon})$ implies an algorithm for $A$ with time $O(n^{2-\delta})$. We call the two problems subquadratically equivalent, denoted $A \equiv_2 B$, if there are subquadratic reductions both ways.

We let $[n] := \{1, \ldots, n\}$. When stating running time, we use the notation $\tilde{O}(\cdot)$ to hide poly-logarithmic factors. For a problem $P$, we write $T_P$ for its time complexity. We generally assume the word-RAM model of computation with word size $w = \Theta(\log n)$. For most problems defined in this paper, we consider inputs to be integers in the range $(-W, \ldots, W)$ where $W$ fits in a constant number of words. For vectors, we use $d$ for the dimension and generally assume $d = n^{o(1)}$.

Core Problems and Hypotheses. One of the most popular problems in the field of quadratic-time conditional hardness is the following problem.

Problem 2.1 (Orthogonal Vectors (OV)). Given $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{0, 1\}^d$, determine if there is a pair $i, j$ satisfying $\langle a_i, b_j \rangle = 0$.

Recall that for OV (and the related problems below) we assume $d = n^{o(1)}$. Thus the naive algorithm solves OV in time $O(n^2 \cdot d) = O(n^{2+o(1)})$.

One of the reasons for the popularity of OV is its surprising connection to the Strong Exponential Time Hypothesis (SETH) [27]. It states that for every $\varepsilon > 0$ there is a $k$, such that the $k$-SAT problem requires time $\Omega(2^{(1-\varepsilon)k})$. By an elegant reduction due to Williams [39], OV is quadratic-time SETH-hard, i.e., there is no algorithm with running time $O(n^{2-\varepsilon})$ for any $\varepsilon > 0$ unless SETH is false.

We consider the following generalizations of OV.

Problem 2.2 (MININNPROD). Given $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{-W, \ldots, W\}^d$ and a natural number $r \in \mathbb{Z}$, determine if there is a pair $i, j$ satisfying $\langle a_i, b_j \rangle \leq r$.

Problem 2.3 (ALLINNPROD). Given $a_1, \ldots, a_n \in \{-W, \ldots, W\}^d$ and $b_1, \ldots, b_n \in \{-W, \ldots, W\}^d$, determine for all $j \in [n]$, the value $\min_{i \in [n]} \langle a_i, b_j \rangle$.

Problem 2.4 (VECTORDOMINATION). Given $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{-W, \ldots, W\}^d$ determine if there is a pair $i, j$ such that $a_i \leq b_j$ component-wise.

Problem 2.5 (SETCONTAINMENT). Given sets $a_1, \ldots, a_n, b_1, \ldots, b_n \subseteq [d]$ given as vectors in $\{0, 1\}^d$ determine if there is a pair $i, j$ such that $a_i \subseteq b_j$.

Note that SETCONTAINMENT is a special case of VECTORDOMINATION and computationally equivalent to OV, as $\langle a, b \rangle = 0$ if and only if $a \subseteq \overline{b}$ (in this slight misuse of notation we think of the Boolean vectors $a, b$ as sets and let $\overline{b}$ denote the complement of $b$).

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4For the purposes of our reductions, even values up to $W = 2^{2n^{o(1)}}$ would be fine.
Since subquadratic solutions to any of these problems trivially give a subquadratic solution to OV, these problems are also quadratic-time SETH-hard. However, the converse does not necessarily hold. In particular, the strongest currently known upper bounds differ: while for OV and SetContainment for small dimension \( d = c \cdot \log(n) \), an \( n^{2-1/O(\log c)} \)-time algorithm is known [4], for VectorDomination the best known algorithm runs only in time \( n^{2-1/O(c \log^2 c)} \) [20] [13].

Another fundamental quadratic-time problem is \((\min, +)\)-CONVOLUTION, defined below.

**Problem 2.6** \((\min, +)\)-CONVOLUTION. Given vectors \( a = (a_0, \ldots, a_{n-1}) \), \( b = (b_0, \ldots, b_{n-1}) \in \{-W, \ldots, W\}^n \), determine its \((\min, +)\)-CONVOLUTION \( a \ast b \) defined by

\[
(a \ast b)_k = \min_{0 \leq i, j < n, i+j=k} a_i + b_j \quad \text{for all } 0 \leq k \leq 2n - 2.
\]

As opposed to the classical convolution, which we denote as \( a \odot b \), solvable in time \( O(n \log n) \) using FFT, no strongly subquadratic algorithm for \((\min, +)\)-CONVOLUTION is known. Compared to OV, we have less support for believing that no \( O(n^{2-\varepsilon}) \)-time algorithm for \((\min, +)\)-CONVOLUTION exists. In particular, interesting special cases can be solved in subquadratic-time [14] and there are subquadratic-time co-nondeterministic and nondeterministic algorithms [8] [12]. At the same time, breaking this long-standing quadratic-time barrier is a prerequisite for progress on refuting the 3SUM and APSP conjectures. This makes it an interesting target particularly for proving subquadratic equivalences, since both positive and negative resolutions of this open question appear to be reasonable possibilities.

**Succinct LWS Versions and Applications.** In the definition of LWS (Problem [13]) we did not fix the encoding of the problem (in particular, the choice of data items, as well the representation of the weights \( w_{i,j} \) and the function \( g \)). Assuming that \( g \) can be determined in \( O(1) \) and that \( W = \text{poly}(n) \), this problem can naturally be solved in time \( O(n^2) \), by evaluating the central recurrence [1] for each \( j = 1, \ldots, n \) — this takes \( O(n) \) time for each \( j \), since we take the minimum over at most \( n \) expressions that can be evaluated in time \( O(1) \) by accessing the previously computed entries \( T[0], \ldots, T[j-1] \) as well as computing \( g \). In all our applications, \( g \) will be the identity function, hence it will suffice to define the type of data items and the corresponding weight matrix. Throughout this paper, whenever we fix a representation of the weight matrix \( W = (w_{i,j})_{i,j} \), we denote the corresponding problem LWS(\( W \)).

In the remainder of this section, we list problems considered in this paper that can be expressed as an LWS instantiations. At this point, we typically give the most natural formulations of these problems — the corresponding definitions as LWS instantiations are given in the corresponding sections.

We start off with a natural succinct “low-rank” version of LWS.

**Problem 2.7** \((\text{LowRankLWS})\). \( \text{LowRankLWS} \) is the LWS problem where the weight matrix \( W \) is of rank \( d \ll n \). The input is given succinctly as two matrices \( A \) and \( B \), which are \( (n \times d) \)- and \( (d \times n) \)-matrices respectively, and \( W = A \cdot B \).

Alternatively, \( \text{LowRankLWS} \) may be interpreted in the following way: There are places \( 0, 1, \ldots, n \), each of which is equipped with an in- and an out-vector. The cost of going from place \( i \) to \( j \) is then defined as the inner product of the out-vector of \( i \) with the in-vector of \( j \), and the task is to compute the minimum-cost monotonically increasing path to reach place \( n \) starting from \( 0 \). In Section [3] we prove subquadratic equivalence to \( \text{MinInnProd} \).

We consider the following coin change problem and variations of \( \text{Knapsack} \).
Problem 2.8 (CC). We are given a weight sequence \( w = (w_1, \ldots, w_n) \) with \( w_i \in \{-W, \ldots, W\} \cup \{\infty\} \), i.e., the coin with value \( i \) has weight \( w_i \). Find the weight of the multiset of denominations \( I \) such that \( \sum_{i \in I} i = n \) and the sum of the weights \( \sum_{i \in I} w_i \) is minimized.

Problem 2.9 (UNBOUNDEDKNAPSACK). We are given a sequence of profits \( p = (p_1, \ldots, p_n) \) with \( p_i \in \{0, 1, \ldots, W\} \), i.e., the item of size \( i \) has profit \( p_i \). Find the total profit of the multiset of indices \( I \) such that \( \sum_{i \in I} i \leq n \) and the total profit \( \sum_{i \in I} p_i \) is maximized.

Note that if we replace multiset by set in the above definition, we obtain the bounded version of the problem, which we denote by KNAPSACK.

We remark that our perspective on CC and UNBOUNDEDKNAPSACK (as well as UNBOUNDEDSUBSETSUM below) using LWS is slightly different than many classical accounts of Knapsack: We define the problem size as the budget size instead of the number of items, thus our focus is on pseudo-polynomial time algorithms for the typical formulations of these problems.

Note that we state the coin change problem as allowing positive or negative weights, but UNBOUNDEDKNAPSACK only allows for positive profits. Furthermore, CC is a minimization problem, while UNBOUNDEDKNAPSACK is a maximization problem. For CC, the maximization problem is trivially equivalent as we can negate all weights. Furthermore, we can freely translate the range of the weights in the coin change problem by defining \( w'_i = i \cdot M + w_i \) for all \( i \) and sufficiently large or small \( M \). The most significant difference between CC and UNBOUNDEDKNAPSACK is that for CC the indices have to sum to exactly \( n \), while for UNBOUNDEDKNAPSACK \( n \) is only an upper bound.

We will encounter an important generalization of the two problems above, defined as follows.

Problem 2.10 (o1CC). The output-intensive version of CC is to determine, given an input to CC, the weight of the optimal multiset such that the denominations sum up to \( j \) for all \( 1 \leq j \leq n \).

It is easy to see that o1CC is at least as hard as both CC and UNBOUNDEDKNAPSACK. We will relate the above KNAPSACK variants to (min, +)-CONVOLUTION in Section 4.

In Section 6 we will revisit near-linear time algorithms for the following special case of the coin change problem.

Problem 2.11 (UNBOUNDEDSUBSETSUM). Given a subset \( S \subseteq [n] \), determine whether there is a multiset of elements of \( S \) that sums up to exactly \( n \).

We also discuss problems where the goal is to find the longest chain among data items, where the notion of a chain is defined by some binary relation \( R \). We first give the definition of the general problem which is parameterized by \( R \).

Problem 2.12 (CHAINLWS). Fix a set \( X \) of objects and a relation \( R \subseteq X \times X \). The Weighted Chain Least-Weight Subsequence Problem for \( R \), denoted CHAINLWS(\( R \)), is the following problem: Given data items \( x_0, \ldots, x_n \in X \), weights \( w_1, \ldots, w_{n-1} \in \{-W, \ldots, W\} \), find the weight of the increasing sequence \( i_0 = 0 < i_1 < i_2 < \ldots < i_k = n \) such that for all \( j \) with \( 1 \leq j \leq k \) the pair \((x_{i_{j-1}}, x_{i_{j}})\) is in the relation \( R \) and the weight \( \sum_{j=1}^{k-1} w_{i_{j}} \) is minimized.

The following problems are specializations of this problem for different relations.

Problem 2.13 (NESTEDBOXES). Given \( n \) boxes in \( d \) dimensions, given as non-negative, \( d \)-dimensional vectors \((b_1, \ldots, b_n)\), find the longest chain such that each box fits into the next (without rotation). We say box \( a \) fits into box \( b \) if for all dimensions \( 1 \leq i \leq d \), \( a_i \leq b_i \).
**Problem 2.14** (SubsetChain). Given $n$ sets from a universe $U$ of size $d$, given as Boolean, $d$-dimensional vectors $(b_1, \ldots, b_n)$, find the longest chain such that each set is a subset of the next.

Note that SubsetChain is a special case of NestedBoxes.

**Problem 2.15** (LIS). Given a sequence of $n$ integers $x_1, \ldots, x_n$, compute the length of the longest subsequence that is strictly increasing.

Finally, we will briefly discuss the following class of LWS problems that turn out to be solvable in near-linear time.

**Problem 2.16** (ConcLWS). Given an LWS instance in which the weights satisfy the quadrangle inequality

$$w_{i,j} + w_{i',j'} \leq w_{i',j} + w_{i,j'} \quad \text{for } i \leq i' \leq j \leq j',$$

solve it. The weights are not explicitly given, but each $w_{i,j}$ can be queried in constant time.

## 3 LowRankLWS

Let us first analyze the following canonical succinct representation of a low-rank weight matrix $W = (w_{i,j})_{i,j}$: If $W$ is of rank $d \ll n$, we can write it more succinctly as $W = A \cdot B$, where $A$ and $B$ are $(n \times d)$- and $(d \times n)$ matrices, respectively. We can express the resulting natural LWS problem equivalently as follows.

**Problem 3.1** (LowRankLWS). We define the following LWS instantiation $\text{LowRankLWS} = \text{LWS}(W_{\text{LowRank}})$.

**Data items:** out-vectors $\mu_0, \ldots, \mu_{n-1} \in \{-W, \ldots, W\}^d$, in-vectors $\sigma_1, \ldots, \sigma_n \in \{-W, \ldots, W\}^d$

**Weights:** $w(i,j) = \langle \mu_i, \sigma_j \rangle$ for $0 \leq i < j \leq n$

In this section, we show that this problem is equivalent, under subquadratic reductions, to the following non-sequential problem.

**Problem 3.2** (MinInnProd). Given $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{-W, \ldots, W\}^d$ and a natural number $r \in \mathbb{Z}$, determine if there is a pair $i, j$ satisfying $\langle a_i, b_j \rangle \leq r$.

We first give a simple reduction from MinInnProd that along the way proves quadratic-time SETH-hardness of LowRankLWS.

**Lemma 3.3.** It holds that $T_{\text{MinInnProd}}(n, d, W) \leq T_{\text{LowRankLWS}}(2n+1, d+2, dW) + O(nd)$.

**Proof.** Given $a_1, \ldots, a_n, b_1, \ldots, b_n \in \{-W, \ldots, W\}^d$, let $O = (0, \ldots, 0) \in \mathbb{Z}^d$ be the all-zeroes vector and define the following in- and out-vectors

$$\mu_0 = (dW, 0, O), \quad \sigma_0 = (0, 0, O),$$

$$\mu_i = (0, dW, a_i), \quad \sigma_i = (0, 0, O), \quad \text{for } i = 1, \ldots, n,$$

$$\mu_n = (0, 0, O), \quad \sigma_n = (dW, 0, b_j), \quad \text{for } j = 1, \ldots, n.$$

To prove correctness, we show that in the constructed LowRankLWS instance, we have $T[2n+1] = \min_{i,j} \langle a_i, b_j \rangle$, from which the results follows immediately. Inductively, we have $T[i] = 0$ for $i = 1, \ldots, n$, since $\langle \mu_i, \sigma_i \rangle = 0$ for all $0 \leq i' < i \leq n$. Similarly, for $j = 1, \ldots, n$ one can inductively show...
that $T[n+j] = \min_{1 \leq i \leq n, j' \leq j} \langle a_i, b_{j'} \rangle$, using that $\langle \mu_0, \sigma_{n+j} \rangle = (dW)^2 \geq \max_{i,j} \langle a_i, b_j \rangle$, $\langle \mu_1, \sigma_{n+j} \rangle = \langle a_i, b_j \rangle$ and $\langle \mu_{n+j}, \sigma_{n+j} \rangle = 0$ for all $1 \leq i, j \leq n$ and $j' \leq j$. Finally, using (1) $\langle \mu_0, \sigma_{2n+1} \rangle = (dW)^2 \geq \max_{i,j} \langle a_i, b_j \rangle$ and $T[0] = 0$, (2) $\langle \mu_1, \sigma_{2n+1} \rangle = (dW)^2 \geq \max_{i,j} \langle a_i, b_j \rangle$ and $T[i] = 0$ for $i = 1, \ldots, n$ and (3) $\langle \mu_{n+j}, \sigma_{2n+1} \rangle = 0$ and $T[n+j] = \min_{1 \leq i \leq n, 1 \leq j' \leq j} \langle a_i, b_{j'} \rangle$ for all $j = 1, \ldots, n$, we can finally determine $T[2n+1] = \min_{i,j} \langle a_i, b_j \rangle$.

To prove the other direction, we will give a quite general approach to compute the sequential LWS problem by reducing to a natural static subproblem of LWS:

**Problem 3.4 (Static-LWS(W)).** Fix an instance of LWS(W). Given intervals $I := \{a+1, \ldots, a+N\}$ and $J := \{a+N+1, \ldots, a+2N\}$, together with the correctly computed values $T[a+1], \ldots, T[a+N+1]$, the Static Least-Weight Subsequence Problem (Static-LWS) asks to determine

$$T'[j] := \min_{i \in J} T[i] + w_{i,j} \quad \text{for all } j \in J.$$

**Lemma 3.5 (LWS(W) \leq_2 Static-LWS(W)).** For any choice of $W$, if Static-LWS(W) can be solved in time $O(N^{2-\varepsilon})$ for some $\varepsilon > 0$, then LWS(W) can be solved in time $O(n^{2-\varepsilon})$.

**Proof.** In what follows, we fix LWS as LWS(W) and Static-LWS as Static-LWS(W).

We define the subproblem $S(\{i_1, \ldots, j_1\}, \{t_1, \ldots, t_1\})$ that given an interval spanned by $1 \leq i \leq j \leq n$ and values $t_k = \min_{0 \leq k < i} T'[k] + w_{k',k}$ for each point $k \in \{i, \ldots, j\}$, computes all values $T[k]$ for $k \in \{i, \ldots, j\}$. Note that a call to $S([n], (w_{0,1}, \ldots, w_{0,n}))$ solves the LWS problem, since $T[0] = 0$ and thus the values of $t_k, k \in [n]$ are correctly initialized.

We solve $S$ using Algorithm 1. We briefly argue correctness, using the invariant that $t_k = \min_{0 \leq k < i} T'[k] + w_{k',k}$ in every call to $S$. If $S$ is called with $i = j$, then the invariant yields $t_i = \min_{0 \leq k < i} T'[k] + w_{k',i} = T[i]$, thus $T[i]$ is computed correctly. For the call in Line 5 the invariant is fulfilled by assumption, hence the values $(T[i], \ldots, T[i+m-1])$ are correctly computed. For the call in Line 9 we note that for $k = i + m, \ldots, i + 2m - 1$, we have

$$t'_k = \min \{t_k, T'[k]\} = \min \{ \min_{0 \leq k' < i} T'[k'] + w_{k',k}, \min_{i \leq k' < i+m} T'[k'] + w_{k',k} \} = \min_{0 \leq k' < i+m} T'[k'] + w_{k',k}.$$
Hence the invariant remains satisfied. Thus, the values \((T[i + m], \ldots, T[i + 2m - 1])\) are correctly computed. Finally, if \(j = i + 2m\), we compute the remaining value \(T[j]\) correctly, since \(t_j = \min_{0 \leq k < i} T[k] + w_{k,j}\) by assumption.

To analyze the running time \(T^S(n)\) of \(S\) on an interval of length \(n := j - i + 1\), note that each call results in two recursive calls of interval lengths at most \(n/2\). In each call, we need an additional overhead that is linear in \(n\) and \(T^{\text{STATIC-LWS}}(n/2)\). Solving the corresponding recursion \(T^S(n) \leq 2T^S(n/2) + T^{\text{STATIC-LWS}}(n/2) + O(n)\), we obtain that an \(O(N^{2 - \varepsilon})\)-time algorithm \(\text{STATIC-LWS}\), with \(0 < \varepsilon < 1\) yields \(T^{\text{LWS}}(n) \leq T^S(n) = O(n^{2 - \varepsilon})\). Similarly, an \(O(N \log^c N)\)-time algorithm for \(\text{STATIC-LWS}\) would result in an \(O(n \log^{c+1} n)\)-time algorithm for \(\text{LWS}\).

For the special case of \(\text{LOWRANK-LWS}\), it is straightforward to see that the static version boils down to the following natural reformulation.

**Problem 3.6 (ALLINNPROD).** Given \(a_1, \ldots, a_n \in \{-W, \ldots, W\}^d\) and \(b_1, \ldots, b_n \in \{-W, \ldots, W\}^d\), determine for all \(j \in [n]\), the value \(\min_{i \in [n]} \langle a_i, b_j \rangle\). (Again, we typically assume that \(d = n^{o(1)}\) and \(W = 2^{n^{o(1)}}\).)

**Lemma 3.7 (STATIC-LWS(\(W_{\text{LOWRANK}}\) \(\leq 2\) ALLINNPROD).** We have

\[ T^{\text{STATIC-LWS}}(W_{\text{LOWRANK}})(n, d, W) \leq T^{\text{ALLINNPROD}}(n, d + 1, nW) + O(nd). \]

**Proof.** Consider \(\text{STATIC-LWS}(W_{\text{LOWRANK}})\). Let \(I = \{a + 1, \ldots, a + N\}\), \(J = \{a + N + 1, \ldots, a + 2N\}\) and values \(T[a + 1], \ldots, T[a + N]\) be given. To determine \(T'[j] = \min_{i \in I} T[i] + w_{i,j}\) for all \(j \in J\), it is sufficient to solve \(\text{ALLINNPROD}\) on the vectors \(a_{a+1}, \ldots, a_{a+N}, b_{a+N+1}, \ldots, b_{a+2N} \in \{nW, \ldots, nW\} \times \ldots \times \{nW, \ldots, nW\}\)

\[ a_i := \langle \mu_i, T[i] \rangle \quad \text{and} \quad b_j := \langle \sigma_j, 1 \rangle, \]

for all \(i \in I, j \in J\), since then \(\langle a_i, b_j \rangle = T[i] + \langle \mu_i, \sigma_j \rangle = T[i] + w_{i,j}\). The claim immediately follows (note that \(|T[i]| \leq nW\)).

Finally, inspired by an elegant trick of [40], we reduce \(\text{ALLINNPROD}\) to \(\text{MININNPROD}\).

**Lemma 3.8 (ALLINNPROD \(\leq 2\) MININNPROD).** We have

\[ T^{\text{ALLINNPROD}}(n, d, W) \leq O(n \cdot T^{\text{MININNPROD}}(\sqrt{n}, d + 3, ndW^2) \cdot \log^2 nW). \]

**Proof.** We first observe that we can tune \(\text{MININNPROD}\) to also return a witness \((i, j)\) with \(\langle a_i, b_j \rangle \leq r\), if it exists. To do so, we replace each \(a_i\) by the \((d + 2)\)-dimensional vector \(a_i' = (a_i, n, (i-1)n, -1)\) and similarly, each \(b_j\) by the \((d + 2)\)-dimensional vector \(b_j' = (b_j \cdot n, -1, j - 1)\). Clearly, we have \(\langle a_i', b_j' \rangle = \langle a_i, b_j \rangle n^2 - (i - 1)n - (j - 1)\). Thus \(\langle a_i', b_j' \rangle \leq r n^2\) if and only if \(\langle a_i, b_j \rangle \leq r\) since \(i, j \in [n]\).

Using a binary search over \(r\), we can find \(\min_{i,j} \langle a_i', b_j' \rangle\), from whose precise value we can determine also a witness, if it exists. Thus the running time \(\text{wit}(n, d, W)\) for finding such a witness is bounded by \(O(\log nW) \cdot T^{\text{MININNPROD}}(n, d + 2, nW)\).

To solve \(\text{ALLINNPROD}\), i.e., to compute \(p_j := \min_{i \in [n]} \langle a_i, b_j \rangle\) for all \(j \in [n]\), we employ a parallel binary search. Consider in particular the following problem \(P\): Given arbitrary \(r_1, \ldots, r_n\), determine for all \(j \in [n]\) whether there exists \(i \in [n]\) such that \(\langle a_i, b_j \rangle \leq r_j\). We will show below that this problem can be solved in time \(O(n \cdot \text{wit}(\sqrt{n}, d + 1, dW^2))\). The claim then follows, since starting from feasible intervals \(\mathcal{R}_1 = \cdots = \mathcal{R}_n = \{-dW^2, \ldots, dW^2\}\) satisfying \(p_j \in \mathcal{R}_j\), we can
halve the sizes of each interval simultaneously by a single call to $\mathcal{P}$. Thus, after $O(\log(dW))$ calls, the true values $p_j$ can be determined, resulting in the time guarantee $T^{\text{ALLINNP} \text{ROD}}(n, d, w) = O(n \cdot \text{wit}(\sqrt{n}, d + 1, dW^2) \cdot \log(dW)) = O(n \cdot T^{\text{MININNP} \text{ROD}}(\sqrt{n}, d + 3, ndW^2) \log^2(nW))$, as desired.

We complete the proof of the claim by showing how to solve $\mathcal{P}$. Without loss of generality, we can assume that $r_j \leq dW^2$ for every $j$, since no larger inner product may exist. We group the vectors $a_1, \ldots, a_n$ in $g := \lceil \sqrt{n} \rceil$ groups $A_1, \ldots, A_g$ of size at most $\sqrt{n}$ each, and do the same for the vectors $b_1, \ldots, b_n$ to obtain $B_1, \ldots, B_g$. Now, we iterate over all pairs of groups $A_k, B_\ell$, $k, \ell \in [g]$: For each such choice of pairs, we do the following process. For each vector $a_i \in A_k$, we define the $(d + 1)$-dimensional vector $\tilde{a}_i := (a_i, -1)$ and for every vector $b_j \in B_\ell$, we define $\tilde{b}_j := (b_j, r_j)$. In the obtained instance $\{\tilde{a}_1\}_{a \in A_k}, \{\tilde{b}_j\}_{b \in B_\ell}$, we try to find some $i, j$ such that $\langle \tilde{a}_i, \tilde{b}_j \rangle \leq 0$, which is equivalent to $\langle a_i, b_j \rangle \leq r_j$. If we succeed in finding such a witness, we delete $b_j$ and $\tilde{b}_j$ (but remember its witness) and repeat finding witnesses (an deleting the witnessed $b_j$) until we cannot find any. The process then ends and we turn to the next pair of groups.

It is easy to see that for all $j \in [n]$, we have $\langle a_i, b_j \rangle \leq r_j$ for some $i \in [n]$ if and only if the above process finds a witness for $b_j$ at some point. To argue about the running time, we charge the running time of every call to witness finding to either (1) the pair $A_k, B_\ell$, if the call is the first call in the process for $A_k, B_\ell$, or (2) to $b_j$, if the call resulted from finding a witness for $b_j$ in the previous call. Note that every pair $A_k, B_\ell$ is charged by exactly one call and every $b_j$ is charged by at most one call (since in after a witness for $b_j$ is found, we delete $b_j$ and no further witness for $b_j$ can be found). Thus in total, we obtain a running time of at most $(g^2 + n) \cdot \text{wit}(\sqrt{n}, d + 1, dW^2) + O(n) = O(n \cdot \text{wit}(\sqrt{n}, d + 1, dW^2))$.

**Theorem 3.9.** We have $\text{LOWRANKLWS} \equiv_2 \text{MININNP} \text{ROD}$.

**Proof.** In Lemmas 3.8, 3.9, 3.11, and 3.8 we have proven

$$\text{MININNP} \text{ROD} \leq_2 \text{LOWRANKLWS} = \text{LWS}(W_{\text{LOWRANK}}) \leq_2 \text{STATICLWS}(W_{\text{LOWRANK}}) \leq_2 \text{ALLINNP} \text{ROD} \leq_2 \text{MININNP} \text{ROD},$$

proving the claim. □

4 Coin Change and Knapsack Problems

In this section, we focus on the following problem related to Knapsack: Assume we are given coins of denominations $d_1, \ldots, d_m$ with corresponding weights $w_1, \ldots, w_m$ and a target value $n$, determine a way to represent $n$ using these coins (where each coin can be used arbitrarily often) minimizing the total sum of weights of the coins used. Since without loss of generality $d_i \leq n$ for all $i$, we can assume that $m \leq n$ and think of $n$ as our problem size. In particular, we describe the input by weights $w_1, \ldots, w_n$ where $w_i$ denotes the weight of the coin of denomination $i$ (if no coin with denomination $i$ exists, we set $w_i = \infty$). It is straightforward to see that this problem is an LWS instance $LWS(W_{cc})$, where the weight matrix $W_{cc}$ is a Toeplitz matrix.

**Problem 4.1 (CC).** We define the following LWS instantiation $CC = LWS(W_{cc})$.

**Data items:** weight sequence $w = (w_1, \ldots, w_n)$ with $w_i \in \{-W, \ldots, W\} \cup \{\infty\}$

**Weights:** $w_{i,j} = w_{j-i}$ for $0 \leq i < j \leq n$
We now define a \textit{Proof}. We first do a translation of the input. Note that for any scalars \(p_1, \ldots, p_n\) with \(p_i \in \{0, 1, \ldots, W\}\), i.e., the item of size \(i\) has profit \(p_i\). Find the total profit of the multiset of indices \(I\) such that \(\sum_{i \in I} i \leq n\) and the total profit \(\sum_{i \in I} p_i\) is maximized.

The purpose of this section is to show that both \textit{CC} and \textit{UnboundedKnapsack} are subquadratically equivalent to the \textit{(min, +)-Convolution} problem. Along the way, we also prove quadratic-time \textit{(min, +)-Convolution}-hardness of \textit{Knapsack}. Recall the definition of \textit{(min, +)-Convolution}.

\textbf{Problem 4.2 (UnboundedKnapsack).} We are given a sequence of profits \(p = (p_1, \ldots, p_n)\) with \(p_i \in \{0, 1, \ldots, W\}\), i.e., the item of size \(i\) has profit \(p_i\). Find the total profit of the multiset of indices \(I\) such that \(\sum_{i \in I} i \leq n\) and the total profit \(\sum_{i \in I} p_i\) is maximized.

\textbf{Problem 4.3 ((min, +)-Convolution).} Given vectors \(a = (a_0, \ldots, a_{n-1})\), \(b = (b_0, \ldots, b_{n-1}) \in \{-W, \ldots, W\}^n\), determine its \textit{(min, +)-Convolution} \(a \ast b\) defined by

\[
(a \ast b)_k = \min_{0 \leq i, j < n, i+j = k} a_i + b_j \quad \text{for all } 0 \leq k \leq 2n - 2.
\]

As opposed to the classical convolution, which we denote as \(a \circ b\), solvable in time \(O(n \log n)\) using FFT, no strongly subquadratic algorithm for \textit{(min, +)-Convolution} is known. Compared to the popular orthogonal vectors problem, we have less support for believing that no \(O(n^{2-\varepsilon})\)-time algorithm for \textit{(min, +)-Convolution} exists. In particular, interesting special cases can be solved in subquadratic time [14] and there are subquadratic-time co-nondeterministic and nondeterministic algorithms [8][12]. At the same time, breaking this long-standing quadratic-time barrier is a prerequisite for progress on refuting the \textit{3SUM} and \textit{APSP} conjectures. This makes it an interesting target particularly for proving subquadratic \textit{equivalences}, since both positive and negative resolutions of this open question appear to be reasonable possibilities.

To obtain our result, we address two issues: (1) We show an equivalence between the problem of determining only the value \(T[n]\), i.e., the best way to give change only for the target value \(n\), and to determine \textit{all values} \(T[1], \ldots, T[n]\), which we call the \textit{output-intensive version}. (2) We show that the output-intensive version is subquadratic equivalent to \textit{(min, +)-Convolution}.

\textbf{Problem 4.4 (oiCC).} The output-intensive version of \textit{CC} is to determine, given an input to \textit{CC}, all values \(T[1], \ldots, T[n]\).

We first consider issue (2) and provide a \textit{(min, +)-Convolution}-based lower bound for \textit{oiCC}.

\textbf{Lemma 4.5 ((min, +)Conv \(\leq_2\) oiCC).} We have \(T^{\text{min, +Conv}}(n, W) \leq T^{\text{oiCC}}(6n, 4(2W + 1)) + O(n)\).

\textit{Proof.} We first do a translation of the input. Note that for any scalars \(\alpha, \beta\), we have \((a + \alpha) \ast (b + \beta) = (a \ast b) + \alpha + \beta\). Let \(M := 2W + 1\). Without loss of generality, we may assume that

\[
2M \leq a_i \leq 3M \quad \text{for all } i = 0, \ldots, n - 1,
\]

\[
0 \leq b_j \leq M \quad \text{for all } j = 0, \ldots, n - 1.
\]

We now define a \textit{CC} instance with a problem size \(n' = 6n\) and \(W' = 4M\) by defining

\[
w = (4M)^n \circ (a_{n-1}, \ldots, a_0) \circ (4M)^n \circ (b_{n-1}, \ldots, b_0) \circ (4M)^{2n}.
\]
We now claim that $T[4n+i] = (a \ast b)_{2n-i}$ for $i = 1, \ldots, 2n$, which immediately yields the lemma.

To do so, we will prove the following sequence of identities.

\[
T[i] = 4M \quad \text{for } i \in [n], \tag{2}
\]
\[
T[n+i] = a_{n-i} \quad \text{for } i \in [n], \tag{3}
\]
\[
T[2n+i] = 4M \quad \text{for } i \in [n], \tag{4}
\]
\[
T[3n+i] = b_{n-i} \quad \text{for } i \in [n], \tag{5}
\]
\[
T[4n+i] = (a \ast b)_{2n-i} \quad \text{for } i \in [2n]. \tag{6}
\]

In the last line, we define, for our convenience, $(a \ast b)_{2n-i} = 4M$ (note that before, we defined only the entries $(a \ast b)_k$ with $k \leq 2n - 2$).

For later convenience, observe that $0 \leq w_i \leq 4M$ for all $i \in [n']$. It is easy to see that this implies $0 \leq T[i] \leq 4M$ for $i \in [n']$.

The identities in (2) are obvious.

To prove the identities in (3) inductively over $i$, recall that $T[n+i] = \min_{j=1, \ldots, n+i} T[n+i-j] + w_j$. Observe that $T[n+i-j] + w_j < 4M$ can only occur if $j \geq n+1$ (since otherwise $w_j = 4M$), which implies $n+i-j \leq n$ and $T[n+i-j] = 4M$ except for the case $j = n+i$. In this case, we have $T[n+i-j] + w_j = T[0] + w_{n+i} = a_{n-i} \leq 4M$.

To prove the identities in (4), observe that for $1 \leq j \leq 3n$, we have $w_j \geq 2M$ by assumption $\min a_i \geq 2M$. Similarly, we have already argued that $T[i'] \geq 2M$ for $1 \leq i' \leq 2n$. Thus, we can inductively show that $T[2n+i] = \min_{j=1, \ldots, 2n+i-1} T[2n+i-j] + w_j = 4M$ using $w_{2n+i} = 4M$ and that every sum in the inner minimum expression is at least $4M$.

To prove the identities in (5), note that for $T[3n+i-j] + w_j < 4M$ to hold, we must have $n+1 \leq j \leq 2n$ or $3n+1 \leq j \leq 3n+i$, since otherwise $w_j = 4M$. We observe that for $n+1 \leq j \leq 2n$, we have $w_j \geq \min a_i \geq 2M$ and $T[3n+i-j] \geq \min a_i \geq 2M$. Thus, we may assume that $3n+1 \leq j \leq 3n+i$. Note that in this case, we have $T[3n+i-j] = 4M$ except for the case $j = 3n+i$, where we have $T[3n+i-j] + w_j = T[0] + w_{3n+i} = b_{n-i} \leq 4M$.

Finally, for the identities in (6), we might have $T[4n+i] + w_j < 4M$ only if $n+1 \leq j \leq 2n$ or $3n+1 \leq j \leq 4n$. First consider the case that $i = 1$. We have

\[
T[4n+1] = \min \{ w_{4n+i}, \min_{n+1 \leq j \leq 2n} T[4n+1-j] + w_j, \min_{3n+1 \leq j \leq 4n} T[4n+1-j] + w_j \} = 4M.
\]

Inductively over $1 < i \leq 2n$, we will prove $T[4n+i] = (a \ast b)_{2n-i}$. By definition,

\[
T[4n+i] = \min \{ w_{4n+i}, \min_{n+1 \leq j \leq 2n} T[4n+i-j] + w_j, \min_{3n+1 \leq j \leq 4n} T[4n+i-j] + w_j \} = \min \{ w_{4n+i}, \min_{1 \leq j' \leq n} T[3n+i-j'] + a_{n-j'}, \min_{1 \leq j' \leq n} T[n+i-j'] + b_{n-j'} \} \tag{7}
\]

Note that

\[
\min_{1 \leq j' \leq n} T[n+i-j'] + b_{n-j'} = \min_{\max(1,i-n) \leq j' \leq \min(i-1,n)} a_{n-(i-j')} + b_{n-j'} = (a \ast b)_{2n-i}
\]

where the last equation follows from noting that the choice of $j'$ lets $n-j'$ and $n-(i-j')$ range over all admissible pairs of values in $\{0, \ldots, n-1\}$ summing up to $2n-i$. Similarly, we inductively prove that

\[
\min_{1 \leq j' \leq n} T[3n+i-j'] + a_{n-j'} = \min_{\max(1,i-n) \leq j' \leq \min(i-1,n)} a_{n-(i-j')} + b_{n-j'} = (a \ast b)_{2n-i},
\]
since \(a_{n-j'} \geq 2M\) and \(T[3n+i-j'] \geq 2M\) whenever \(j' \geq i\) or \(j' < i-n\) (where the last regime uses \(T[4n+i'] = (a*b)_{2n-i'} \geq 2M\) inductively for \(i' < i\)). Finally, since \((a*b)_{2n-i} \leq (\max_i a_i) + (\max_j b_j) \leq 4M\), we can simplify (7) to \(T[4n+i] = (a*b)_{2n-i}\).

Using the notion of Static-LWS, the other direction is straight-forward.

**Lemma 4.6.** We have \(oICC \leq_2 \text{STATIC-LWS}(W_{cc}) \leq_2 (\text{min},+)-\text{CONV}.

**Proof.** In Lemma 3.5, we have in fact reduced the output-intensive version of \(LWS(W)\) to our static problem \(\text{STATIC-LWS}(W)\), thus specialized to the coin change problem, we only need to show that \(\text{STATIC-LWS}(W_{cc})\) subquadratically reduces to \((\text{min},+)-\text{CONVOLUTION}\). Consider an input instance to Static-LWS given by \(I = \{a_1, \ldots, a+N\}, J = \{a+N+1, \ldots, a+2N\}\) and values \(T[i], i \in I\). Defining \(M := 2W + 1\) and the vectors

\[
\begin{align*}
    u &:= (nM, T[a+1], \ldots, T[a+N], nM, \ldots, nM), \\
    v &:= (nM, w_1, \ldots, w_{2N}),
\end{align*}
\]

we have \((u*v)_{N+k} = \min_{i=1, \ldots, N} T[a+i] + w_{N+k-i} = T^d[a+N+k]\) for all \(k = 1, \ldots, N\), thus a \((\text{min},+)-\text{CONVOLUTION}\) of two \((2n+1)\)-dimensional vectors solves \(\text{STATIC-LWS}(W_{cc})\), yielding the claim.

The last two lemmas resolve issue (2). We proceed to issue (1) and show that the output-intensive version is subquadratically equivalent to both CC and \(\text{UNBOUNDKDNPACK}\) that only ask to determine a single output number. We introduce the following notation for our convenience: Recall that weight \(w_i\) denotes the weight of a coin of denomination \(i\). For a multiset \(S \subseteq \{n\}\), we let \(d(S) := \sum_{i \in S} i\) denote its total denomination, i.e., sum of the denomination of the coins in \(S\) (where multiples uses of the same coin is allowed, since \(S\) is a multiset). We let \(w(S) := \sum_{i \in S} w_i\) denote the weight of the multiset. Analogously, when considering a Knapsack instance, \(p(S) = \sum_{i \in S} p_i\) denotes the total profit of the item (multi)set \(S\).

It is trivial to see that \(\text{UNBOUNDKDNPACK} \leq_2 oICC\). Furthermore, we can give the following simple reduction from CC to \(\text{UNBOUNDKDNPACK}\).

**Observation 1 (CC \leq_2 \text{UNBOUNDKDNPACK} \leq_2 oICC).** We have \(T^{CC}(n, W) \leq T^{\text{UNBOUNDKDNPACK}}(n, nW) + \mathcal{O}(n)\) and \(T^{\text{UNBOUNDKDNPACK}}(n, W) \leq T^{oICC}(n, W) + \mathcal{O}(n)\).

**Proof.** Given a CC instance, for every weight \(w_i < \infty\), we create an item of size \(i\) and profit \(p_i := i \cdot M - w_i\) in our resulting \(\text{UNBOUNDKDNPACK}\) instance for a sufficiently large constant \(M \geq nW\). This way, all profits are positive and every multiset \(S\) whose sizes sum up to \(B\) has a profit of \(p(S) = B \cdot M - w(S)\). Since \(M \geq nW \geq \max_{S, d(S) \leq n} |w(S)|\), this ensures that the maximum-profit multiset of total size / denomination at most \(n\) has a total size/ denomination of exactly \(n\). Thus, the optimal multiset \(S^*\) has profit \(p(s^*) = n \cdot M - \min_{S, d(S) = n} w(S) = n \cdot M - T[n]\), from which we can derive \(T[n]\), as desired.

Given an \(\text{UNBOUNDKDNPACK}\) instance, we define for every item of size \(i\) and profit \(p_i\), the corresponding weight \(w_i = -p_i\) in a corresponding CC instance. It remains to compute all \(T[1], \ldots, T[n]\) in this instance and determining their minimum, concluding the reduction.
The remaining part is similar in spirit to Lemma $\text{3.8}$. Somewhat surprisingly, the same general approach works despite the much more sequential nature of the Knapsack/CoinChange problem – this sequentiality can be taken care of by a more careful treatment of appropriate subproblems that involves solving them in a particular order and feeding them with information gained during the process.

In what follows, to clarify which instance is currently considered, we let $T^I$ denote the $T$-table of the (0/1)CC LWS problem (see Problem $\text{1.1}$) corresponding to instance $I$. Dropping the superscript always refers to $T^I$.

**Lemma 4.7** (0/1CC $\leq_2$ CC). We have that $T^{\text{0/1CC}}(n, W) \leq O(\log(nW) \cdot n \cdot T^{CC}(24\sqrt{n}, 3n^2W))$.

**Proof.** Let $I$ be an 0/1CC instance. To define our subproblems, we set $N := \lceil \sqrt{n} \rceil$ and define $N$ ranges $W_1 := \{1, \ldots, N\}, \ldots, W_N := \{(N-1)N + 1, \ldots, N^2\}$. To determine all $T[i] = \min_{S \subseteq \{1, \ldots, N\}} \max(w(S), \text{weight}(S))$, we will compute $T[i]$ for all $i \in W_j$ successively over all $j = 1, \ldots, N$. The case of $j = 1$ and $j = 2$ can be computed by the naive algorithm in time $O(N^2) = O(n)$. Consider now any fixed $j \geq 3$ and assume that all values $T[i]$ for $i \in W_j$ with $j' < j$ have already been computed. We employ a parallel binary search. For every $i \in W_j$, we set up a feasible range $R_i$ initialized to $\{nW, \ldots, nW\}$. We will maintain the invariant that $T[i] \in R_i$ and will halve the size of all feasible ranges $R_i$, $i \in W_j$ simultaneously using a small number of calls to the following problem $P(M, W)$: Given an instance $J$ for CC specified by the weights $\tilde{w}_1, \ldots, \tilde{w}_M$, as well as values $\tilde{r}_1, \ldots, \tilde{r}_M \in \{-\infty, \ldots, \infty\}$, determine whether there exists an $i \in [M]$ with $T^J[i] \leq \tilde{r}_i$, and if so, also return a witness $i$. We will later prove that this problem can be solved in time $T^P(M, W) = O(T^{CC}(2M, 3M^2W))$. Clearly, after $O(\log(nW))$ rounds of this parallel binary search, the feasible ranges consist of single values, thus determining the values of all $T[i]$ for $i \in W_j$.

Since we will show that halving all feasible ranges for range $W_j$ takes $O(N)$ calls to $P(12N, nW)$, and we need to determine at most $N$ ranges $W_3, \ldots, W_N$, the total time for this process amounts to $O(\log(nW)N^2 \cdot T^P(12N, nW)) = O(\log(nW)N^2 \cdot T^{CC}(24N, 3n^2W))$.

We now describe how to use $P$ to halve the size of all feasible ranges $R_i$, $i \in W_j$: we set $r_i$ to the median of $R_i$ and aim to determine, for all $i \in W_j$, whether $T[i] \leq r_i$, i.e., whether some multiset $S$ with $d(S) = i$ and $w(S) \leq r_i$ exists. We achieve this by the following process: For every $k = 1, \ldots, j$, we consider only two ranges, namely $W_k := \{(k-1)N + 1, \ldots, kN\}$ and $W_{j-k} \cup W_{j-k+1} = \{(j-k-1)N + 1, \ldots, (j-k+1)N\}$. Let us first consider the case $k \geq 2$. Here, we can define the $2N$-dimensional vectors $a, b$ with

$$a_\ell = \begin{cases} w((k-1)N + \ell) & \text{for } \ell \in [N], \\ \infty & \text{for } \ell > N, \end{cases}$$

$$b_\ell = T[(j-k-1)N + \ell] \quad \text{for } \ell \in [2N].$$

(Note that all $T[i], i \in W_{j-k} \cup W_{j-k+1}$ for $k \geq 2$ have already been computed by assumption.) We are interested in all those values of the $(\min, +)$-CONVOLUTION $a \ast b$ of these vectors that correspond to summing up some $w((k-1)N + \ell)$ with some $T[(j-k-1)N + \ell]$ such that $(j-2)N + \ell$ is in $W_j$. More specifically, we aim to determine whether there is some $\ell$ with $(a \ast b)_{N+\ell} \leq r_{(j-1)N+\ell}$. To do so, we use the reduction from $(\min, +)$-CONVOLUTION to 0/1CC given in Lemma $\text{4.3}$ to create an 0/1CC instance $J$. From this instance of problem size $12N$ we can read off the values of $a \ast b$ as a certain interval in the corresponding $T^J$-table. Thus, we can test whether $(a \ast b)_{N+\ell} \leq r_{(j-1)N+\ell}$ for some $\ell$ using $P(12N, nW)$: for every $\ell$, we let $i$ be the unique index in the $T^J$-table representing...
the entry \((a \ast b)_{N+\ell}\) and set \(\tilde{r}_i := r_{(i-1)N+\ell}\). For all other \(i'\), we set \(\tilde{r}_{i'} = -\infty\), thus enforces that those indices will never be reported.

For the special case \(k = 1\), we proceed slightly differently: Here, we define the 2N-dimensional vectors \(a, b\) with

\[
\begin{align*}
a_\ell &= T[\ell] & & \text{for } \ell \in [2N] \\
b_\ell &= \begin{cases} T[(j-2)N+\ell] & \text{for } \ell \in [N] \\ -\infty & \text{for } \ell > N. \end{cases}
\end{align*}
\]

(Note that all necessary \(T[i], i \in W_1 \cup W_2\) and \(T[i], i \in W_{j-1}\) have already been computed by assumption.) Analogously to above, we use \(P(12N, nW)\) to test whether \((a \ast b)_{N+\ell} \leq \tilde{r}_{(j-1)N+\ell}\) using the reduction from \((\min, +)\)-CONVOLUTION to \(\text{otcc}\) given in Lemma 4.5.

Once an \(i \in W_j\) has been reported to satisfy \(T[i] \leq r_i\) for some witnessing subproblem given by the ranges \(W_k\) and \(W_{j-k} \cup W_{j-k+1}\) for some \(k\), we set \(r_i := -\infty\) and repeat on the same subproblem \(k\) (analogously to the approach of Lemma 3.8). Note that for every \(j\), we have \(j \leq N\) subproblems and at most \(N\) many indices \(i \in W_j\) that can be reported. Thus, we use at most \(O(N)\) many calls to the subproblem \(P\).

To briefly argue correctness, note that by construction, we only determine some \(i\) with \(T[i] \leq r_i\) if we have found a witness. For the converse, let \(k\) be the largest index such that the optimal multiset for \(i\) includes a coin in \(W_k\). Then the subproblem given by the ranges \(W_k\) and \(W_{j-k} \cup W_{j-k+1}\) will give a witness. This is obvious for \(k \geq 2\). For \(k = 1\), note that no weight in \(W_{k'}\) with \(k' > 1\) is used in an optimal multiset for \(T[i] \in W_j\). In particular, the optimal multiset \(S\) can be represented as \(S = S' \cup S''\), where \(S'\) is a multiset of total denomination \(i' \in W_{j-1}\) and \(S''\) is a multiset of total denomination \(i - i' \in W_1 \cup W_2\). Thus, in the instance constructed from \(a, b\), we will find the witness \(T[i] \leq T[i'] + T[i - i'] \leq r_i\).

We finally describe how to solve \(P(M, \tilde{W})\) in time \(T^{CC}(2M, 3M^2\tilde{W})\). First consider the problem without finding a witnessing \(i\). Let \(\tilde{w}_1, \ldots, \tilde{w}_M, \tilde{r}_1, \ldots, \tilde{r}_M\) be an instance \(J\) of \(P(M, \tilde{W})\). We define a CC instance \(K\) of problem size \(2M\) by giving the weights

\[
\begin{align*}
w'_i &:= \tilde{w}_i & & \text{for all } i \in [M], \\
w'_{2M-i} &:= -3M\tilde{W} - \tilde{r}_i & & \text{for all } i \in [M].
\end{align*}
\]

We claim that \(T^K[2M] \leq -3M\tilde{W}\) iff the input instance to \(P\) is a yes instance: First observe that \(T^K[1] = T^J[1], \ldots, T^K[M] = T^J[M]\) since the first \(M\) weights agree for both \(J\) and \(K\). Consider the case that there is some \(i \in [M]\) with \(T^J[i] \leq \tilde{r}_i\). Then we have \(T^K[2M] \leq T^K[i] + w_{2M-i} = (T^J[i] - \tilde{r}_i) - 3M\tilde{W} \leq -3M\tilde{W}\), as desired. Conversely, assume that all \(T^J[i] > \tilde{r}_i\). We distinguish the cases whether the optimal subsequence \(S\) uses only weights among \(\tilde{w}_1, \ldots, \tilde{w}_M\) or not. In the first case, since \(|\tilde{w}_i| \leq \tilde{W}\) for \(i \in [M]\), we have that \(w(S) \geq 2M \cdot \min_{i \in [n]} |\tilde{w}_i| \geq -2M\tilde{W} > -3M\tilde{W}\). Otherwise, \(S\) uses exactly one weight among \(\tilde{w}_{M+1}, \ldots, \tilde{w}_{2M}\). Let this weight be \(\tilde{w}_{2M-i}\). Then \(w(S) = T^K[i] + \tilde{w}_{2M-i} = (T^J[i] - \tilde{r}_i) - 3M\tilde{W} > -3M\tilde{W}\) since \(T^J[i] > \tilde{r}_i\), yielding the claim.

Very similar to Lemma 3.5, we can now tune the above reduction to also produce a witness \(i\) such that \(T^J[i] \leq \tilde{r}_i\). For this, we scale all weights \(w'_i, i \in [2M]\) by a factor of \(M\) and subtract a value of \(i - 1\) for every \(w'_i, i \in [M]\). It is easy to see that a yes instance \(K\) attains some value \(T^K[2M] = -\kappa \cdot M - i\) for some integers \(\kappa \geq 3\) and \(0 \leq i < n\), where \(i + 1\) is a witness for \(T^J[i + 1] \leq \tilde{r}_{i+1}\), thus computing \(T^K[2M]\) lets us derive a witness as well. Thus, problem \(P\) can be solved by a single call to \(T^{CC}(2M, 3M^2\tilde{W})\).
The results above prove the following theorem.

**Theorem 4.8.** We have \((\min,+)\text{conv} \equiv_2 \text{CC} \equiv_2 \text{UNBOUNDEDKnapsack}.\) Furthermore, the bounded version of Knapsack admits no strongly subquadratic-time algorithm unless \((\min,+)\text{-convolution}\) can be solved in strongly subquadratic time.

**Proof.** Lemmas 4.5 and 4.6 prove \((\min,+)\text{conv} \equiv_2 \text{oPCC},\) while Observation 1 and Lemma 4.7 establish \(\text{oPCC} \equiv_2 \text{CC} \equiv_2 \text{UNBOUNDEDKnapsack},\) yielding the first claim.

The second claim follows from inspecting the proofs of Lemma 4.5, Lemma 4.7 and the first claim of Observation 1 and observing that we only reduce to \(\text{CC}/\text{Knapsack}\) instances in which the optimal multiset (for each total size) is always a set, i.e., uses each element at most once. \(\square\)

## 5 Chain LWS

In this section we consider a special case of of Least-Weight Subsequence problems called the Chain Least-Weight Subsequence. This captures problems in which edge weights are given implicitly by a relation \(R\) that determines which pairs of data items we are allowed to chain – the aim is to find the longest chain.

An example of a Chain Least-Weight Subsequence problem is the NestedBoxes problem. Given \(n\) boxes in \(d\) dimensions, given as non-negative, \(d\)-dimensional vectors \(b_1, \ldots, b_n\), find the longest chain such that each box fits into the next (without rotation). We say box that box \(a\) fits into box \(b\) if for all dimensions \(1 \leq i \leq d, a_i \leq b_i\).

NestedBoxes is not immediately a least-weight subsequence problem, as for least weight subsequence problems we are given a sequence of data items, and require any sequence to start at the first item and end at the last. We can easily convert NestedBoxes into a LWS problem by sorting the vectors by the sum of the entries and introducing two special boxes, one very small box \(K\) such that \(K\) fits into any box \(b_i\) and one very large box \(J\) such that any \(b_i\) fits into \(J\).

We define the chain least-weight subsequence problem with respect to any relation \(R\) and consider a weighted version where data items are given weights. To make the definition consistent with the definition of LWS the output is the weight of the sequence that minimizes the sum of the weights.

**Problem 5.1 (ChainLWS).** Fix a set of objects \(X\) and a relation \(R \subseteq X \times X\). We define the following LWS instantiation \(\text{ChainLWS}(R) = \text{LWS}(W_{\text{ChainLWS}(R)})\).

**Data items:** sequence of objects \(x_0, \ldots, x_n \in X\) with weights \(w_1, \ldots, w_n \in [-W, \ldots, W]\).

**Weights:** \(w_{i,j} = \begin{cases} w_j & \text{if } (x_i, x_j) \in R, \\ \infty & \text{otherwise,} \end{cases}\) for \(0 \leq i < j \leq n.\)

The input to the (weighted) chain least-weight subsequence problem is a sequence of data items, and not a set. Finding the longest chain in a set of data items is \(\text{NP}\)-complete in general. For example, consider the box overlap problem: The input is a set of boxes in two dimensions, given by the top left corner and the bottom right corner, and the relation consists of all pairs such that the two boxes overlap. This problem is a generalization of the Hamiltonian path problem on induced subgraphs of the two-dimensional grid, which is an \(\text{NP}\)-complete problem [29].

We relate \(\text{ChainLWS}(R)\) to the class of selection problems with respect to the same relation \(R\).
Problem 5.2 (Selection Problem). Given data items $a_1, \ldots, a_n, b_1, \ldots, b_n$ and a relation $R(a_i, b_j)$, determine if there is a pair $i, j$ satisfying $R(a_i, b_j)$. We denote this selection problem with respect to a relation $R$ by $\text{Selection}(R)$.

The class of selection problems includes several well studied problems including $\text{MinInnProd}$, $\text{OV}$ [39][4] and $\text{VectorDomination}$ [26].

We will use the selection problems in the search variant, where we find a pair satisfying the $R$ if such a pair exists. To reduce the the search variant to the decision variants in a fine-grained way, we can use a simple, binary search type reduction from the decision problem to the search problem:

We give a subquadratic reduction from $\text{ChainLWS}(R)$ to $\text{Selection}(R)$ that is independent of $R$.

Theorem 5.3. For all relations $R$ such that $R$ can be computed in time subpolynomial in the number of data items $n$, $\text{ChainLWS}(R) \leq_{2} \text{Selection}(R)$.

The proof is again based on $\text{Static-LWS}$ and a variation on a trick of [40].

As an intermediate step, we define $\text{Static-ChainLWS}$ as the equivalent of $\text{Static-LWS}$ in the special case for chains.

Problem 5.4 (Static-ChainLWS). Fix an instance of $\text{ChainLWS}(R)$. Given intervals $I := \{a + 1, \ldots, a + N\}$ and $J := \{a + N + 1, \ldots, a + 2N\}$ for some $a$ and $N$, together with the correctly computed values $T[a + 1], \ldots, T[a + N]$, the Static Chain Least-Weight Subsequence Problem ($\text{Static-ChainLWS}$) asks to determine

$$T'[j] := \min_{i \in I : R(i, j)} T[i] + w_j$$

for all $j \in J$.

Similar to the definition of $\text{ChainLWS}$, $\text{Static-ChainLWS}$ is the special case of $\text{Static-LWS}$ where the the weights $w_{i,j}$ are restricted to be either $w_j$ or $\infty$, depending on $R$. As a result, Lemma 35 applies directly.

Corollary 5.5 ($\text{ChainLWS}(R) \leq_{2} \text{Static-LWS}(R)$). For any $R$, if $\text{Static-ChainLWS}(R)$ can be solved in time $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$, then $\text{ChainLWS}(R)$ can be solved in time $O(n^{2-\varepsilon})$.

We now reduce $\text{Static-ChainLWS}(R)$ to $\text{Selection}(R)$ with a variation on the trick by [40].

Lemma 5.6 ($\text{Static-ChainLWS}(R) \leq_{2} \text{Selection}(R)$). For all relations $R$ such that $R$ can be computed in time subpolynomial in the number of data items $n$, $\text{Static-ChainLWS}(R) \leq_{2} \text{Selection}(R)$.

Proof. As a first step, we sort the data items $a_i, i \in I = \{a + 1, \ldots, a + N\}$ by $T[i]$ in increasing order and we will assume for the remainder of the proof that for all $a + 1 \leq i < a + N$ we have $T[i] \leq T[i + 1]$. We then split the set $a_{a+1}, \ldots, a_{a+N}$ into $g := \lfloor \sqrt{N} \rfloor$ groups $A_1, \ldots, A_g$ with $A_i = \{a_{i-1}\lceil N/g\rceil, \ldots, a_{i\lfloor N/g\rfloor-1}\}$. We split the set $b_{a+N+1}, \ldots, b_{a+2N}$ into $B_1, \ldots, B_g$ in a similar fashion. We then iterate over all pairs $A_k, B_l$ with $k, l \in [g]$ in lexicographic order, and for each pair we do the following. Call the oracle for $\text{Selection}(R)$ on the input $A_k, B_l$ to find a pair $a_i, b_j$ such that the relation $R$ is satisfied on the pair. If there is no such pair, move to the next pair $A_{k'}, B_{l'}$ of sets of data items. If there is such a pair, find the first element $a_{i*} \in A_g$ such that $R(a_{i*}, b_j)$ using a simple linear scan. As we first sorted $A$ and iterate over sets $A_k, B_l$ in lexicographic order, we have $T'[j] = T[i^*] + w_j$. We then remove $b_j$ from $B_l$ and repeat.
For the runtime analysis, we observe, that the oracle can find a pair of elements at most \( \mathcal{O}(N) \) times, as each time we find a pair we remove an element from the input. In the case where we do find a pair of elements we do a linear scan that takes \( \mathcal{O}(N/g) \) time. Furthermore, each pair of sets \( A_k, B_l \) can fail to find a pair at most once. Hence, if \( T_{\text{Selection}} \) is the time to solve the selection problem and using \( g = \sqrt{N} \) we get a time of

\[
T(N) = NT_{\text{Selection}}(\sqrt{N}) + N(T_{\text{Selection}}(\sqrt{N}) + \sqrt{N}) = NT_{\text{Selection}}(\sqrt{N})
\]

which is subquadratic if \( T_{\text{Selection}}(N) \) is subquadratic.

**Theorem 5.7.** Let \( D \) be the set of possible data items. For any relation \( R \) such that

1. There is a data item \( \perp \) such that \( (\perp, d) \in R \) for all \( d \in D \).
2. There is a data item \( \top \) such that \( (d, \top) \in R \) for all \( d \in D \).
3. For any set of data items \( d_1, \ldots, d_n \) there is a sequence \( i_1, \ldots, i_n \) such that for any \( j < k \), \( (d_j, d_k) \notin R \). This ordering can be computed in time \( \mathcal{O}(n^{2-\delta}) \) for \( \delta > 0 \). We call this ordering the natural ordering.

Then \( \text{Selection}(R) \leq_2 \text{ChainLWS}(R) \).

**Proof.** We construct an unweighted ChainLWS problem with all weights set to \(-1\), so that the problem is to find the longest chain. Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) be the data items of \( \text{Selection}(R) \) and sort both sets according to the natural ordering. We claim that for the sequence of data items \( \perp, a_1, \ldots, a_n, b_1, \ldots, b_n, \top \) the weight of the least weight subsequence is \(-3\) exactly if there is a pair \((a_i, b_j) \in R\). Because of the property of the natural ordering, any valid subsequence starting at \( \perp \) and ending at \( \top \) contains at most one element \( a_i \) and at most one element \( b_j \). If there is a pair \((a_i, b_j) \in R\), then the sequence \( \perp, a_i, b_j, \top \) will have value \(-3\). If there is no such pair, any valid sequence contains at most one element other than \( \perp \) and \( \top \) and its value is therefore at least \(-2\).

The proof is in the appendix.

In the rest of the section we give some interesting instantiations of the subquadratic equivalence of \( \text{Selection} \) and ChainLWS.

**Corollary 5.8 (NestedBoxes \( \equiv_2 \) VectorDomination).** The weighted NestedBoxes problem on \( d = c \log n \) dimensions can be solved in time \( n^{2 - (1/\mathcal{O}(\log^2 c)))} \). For \( d = \omega(\log n) \), the (unweighted) NestedBoxes problem cannot be solved in time \( \mathcal{O}(n^{2-\varepsilon}) \) for any \( \varepsilon > 0 \) assuming SETH.

**Proof.** Let \( R \) be the relation that contains all pairs of non-negative, \( d \)-dimensional vectors \( a, b \) such that \( a_i \leq b_i \) for all \( i \). Now \( \text{Selection}(R) \) is VectorDomination, and \( \text{ChainLWS}(R) \) is the NestedBoxes problem.

Using the reduction from Theorem 5.3 and the algorithms for vector domination of the stated runtime \( [26, 13] \) we immediately get an algorithm for NestedBoxes.

We apply Theorem 5.7 with \( \top = W^d \) where \( W \) is the largest coordinate in all input vectors, \( \perp = 0^d \) and use the sum of the coordinates of the boxes as the natural ordering. SETH-hardness of NestedBoxes then follows from the SETH-hardness of vector domination \( [39] \).

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If we restrict NestedBoxes and VectorDomination to Boolean vectors, then we get SubsetChain and SetContainment respectively. In this case the upper bound improves to \( n^{2-1/\mathcal{O}(\log c)} \).

We would like to point out that the definition of ChainLWS requires the input to be a sequence of data items, and not a set. Consider the following definition:

**Problem 5.9 (ChainSet).** Let a set of data items \( \{x_0, \ldots, x_n\} \), weights \( w_1, \ldots, w_{n-1} \in \{-W, \ldots, W\} \) and a relation \( R(x_i, x_j) \) be given. The chain set problem for \( R \), denoted ChainSet(\( R \)) asks to find the weight sequence \( i_0, i_1, i_2, \ldots, i_k \) such that for all \( j \) with \( 1 \leq j \leq k \) the pair \( (x_{i_{j-1}}, x_{i_j}) \) is in the relation \( R \) and the weight \( \sum_{j=1}^{k} w_{i_j} \) is minimized.

While ChainLWS can always be solved in quadratic time, ChainSet is NP-complete. For example, consider the box overlap problem: The input is a set of boxes in two dimensions, given by the top left corner and the bottom right corner, and the relation consists of all pairs such that the two boxes overlap. This problem is a generalization of the Hamiltonian path problem on induced subgraphs of the two-dimensional grid, which is an NP-complete problem [29]. This is a formal barrier to a more general reduction than Theorem 5.7, as we need some mechanism to impose an ordering on the data items.

### 6 Near-linear time algorithms

In this section, we classify problems to be solvable in near-linear time using the lens of our framework. Note that in these instances, near-linear time solutions have already been known, however, our focus on the static variants of LWS provides a simple, general approach to find fast algorithms by identifying a simple “core” problem. Since in this paper, we generally ignore subpolynomial factors in the running time, we concentrate here on the reduction from some LWS variant to its corresponding core problem and disregard reductions in the other direction.

#### 6.1 Longest Increasing Subsequence

The longest increasing subsequence problem LIS has been first investigated by Fredman [18], who gave an \( \mathcal{O}(n \log n) \)-time algorithm and gave a corresponding lower bound based on Sorting. The following LWS instantiation is equivalent to LIS.

**Problem 6.1 (LIS).** We define the following LWS instantiation LIS = LWS(\( W_{LIS} \)).

Data items: integers \( x_1, \ldots, x_n \in \{1, \ldots, W\} \)

Weights: \( w_{i,j} = \begin{cases} -1 & \text{if } x_i < x_j \\ \infty & \text{ow.} \end{cases} \)

It is straightforward to verify that \( -T[a] \) yields the value of the longest increasing subsequence of \( x_1, \ldots, x_n \). Using the static variant of LWS introduced in Section 3, we observe that LIS effectively boils down to Sorting.

**Observation 2.** LIS can be solved in time \( \tilde{O}(n) \).

**Proof.** By Lemma 3.3 we can reduce LIS to the static variant Static-LWS(\( W_{LIS} \)). It is straightforward to see that the latter can be reformulated as follows: Given \((a_1, T[1]), \ldots, (a_N, T[N])\) and
6.2 Unbounded Subset Sum

UnboundedSubsetSum is a variant of the classical SubsetSum, in which repetitions of elements are allowed. While improved pseudo-polynomial-time algorithms for SubsetSum could only recently be found [32, 10], there is a simple algorithm solving UnboundedSubsetSum in time $O(n \log n)$ [10]. It can be cast into an LWS formulation as follows.

**Problem 6.2 (UnboundedSubsetSum).** We define the following LWS instantiation \( \text{LIS} = \text{LWS}(W_{\text{USS}}) \).

**Data items:** \( S \subseteq [n] \)

**Weights:** \( w_{i,j} = \begin{cases} 0 & \text{if } j - i \in S \\ \infty & \text{otherwise} \end{cases} \)

Note that in this formulation, \( T[n] = 0 \) iff there is a multiset of numbers from \( S \) that sums up to \( n \). It is a straightforward observation that the static variant of UnboundedSubsetSum can be solved by classical convolution, i.e., \((+, +)\)-convolution.

**Observation 3.** UnboundedSubsetSum can be solved in time \( O(n) \).

**Proof.** Noting that all weights \( w_{i,j} \) are either 0 or \( \infty \), it is easy to see that the static variant Static-LWS\((W_{\text{USS}})\) can be reformulated as follows: Given a subset \( X \subseteq I = \{a + 1, \ldots, a + N\} \), determine, for all \( j \in J = \{a + N + 1, \ldots, a + 2N\} \), whether there exists some \( i \in X \) such that \( j - i \in S \). To do so, we do the following: We represent \( X \) as an \( N \)-bit vector \( x = (x_1, \ldots, x_N) \in \{0, 1\}^N \) with \( x_i = 1 \) iff \( a + i \in X \). Furthermore, we represent the “relevant part” of \( S \) by defining a \( 2N \)-bit vector \( s = (s_1, \ldots, s_{2N}) \in \{0, 1\}^{2N} \) with \( s_i = 1 \) iff \( i \in S \). Then the \((+, +)\)-convolution \( r = x \oplus s \) of \( x \) and \( s \) allows us to determine \( T[a + N + j] \) for \( j = 1, \ldots, N \): this value is 0 iff \( r_{N+j} > 0 \) and \( \infty \) otherwise. Correctness follows from the observation that \( r_{N+j} > 0 \) is equivalent to the existence of some \( i \in [N] \) and \( k \in [2N] \) with \( i + k = N + j \) and \( x_i = s_k = 1 \). This in turn is equivalent to \( a + i \in X \) and \( (a + N + j) - (a + i) = N + j - i = k \in S \), as desired.

Thus Static-LWS\((W_{\text{USS}})\) can be solved by a single convolution computation, which can be performed in time \( O(N \log N) \). Thus by Lemma 3.5 this gives rise to a \( O(n \log^2 n) \)-time algorithm for UnboundedSubsetSum.

6.3 Concave LWS

The concave LWS problem is a special case of LWS in which the weights satisfy the quadrangle inequality. Since a complete description of the input instance consists of \( \Omega(n^2) \) weights, we use the standard assumption that each \( w_{i,j} \) can be queried in constant time. This allows for sublinear solutions in the input description, in particular there exist \( O(n) \)-time algorithms [38, 21].
Problem 6.3 (ConCCLS). We define the following LWS instantiation LIS = LWS(W_conc).
Weights: \( w_{i,j} \) given by oracle access, satisfying \( w_{i,j} + w_{i',j'} \leq w_{i',j} + w_{i,j'} \) for \( i \leq i' \leq j \leq j' \).

We revisit ConCCLS and its known connection to the problem of computing column (or row) minima in a totally monotone \((n \times n)\)-matrix, which we call the SMAWK problem because of its remarkable \(O(n)\)-time solution called the SMAWK algorithm \(5\).

Observation 4. ConCCLS can be solved in time \(\tilde{O}(n)\).

Proof. The static variant of ConCCLS can be formulated as follows: Given intervals \( I = \{a + 1, \ldots, a + N\} \) and \( J = \{a + N + 1, \ldots, a + 2N\} \), we define a matrix \( M := (m_{i,j})_{i \in I, j \in J} \) with \( m_{i,j} = T[i] + w_{i,j} \). It is easy to see that \( M \) is a totally monotone matrix since \( w \) satisfies the quadrangle inequality. Note that the minimum of column \( j \in J \) in \( M \) is \( \min_{i \in I} T[i] + w_{i,j} = T'[j] \) by definition. Thus, using the SMAWK algorithm we can determine all \( T'[j] \) in simultaneously in time \(O(N)\).

Thus by Lemma 3.5 we obtain an \(O(n \log n)\)-time algorithm for ConCCLS. \(\square\)

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\(^5\)A matrix \( M = (m_{i,j})_{i,j} \) is totally monotone if for all \( i < i' \) and \( j < j' \), we have that \( m_{i,j} > m_{i',j'} \) implies that \( m_{i,j'} > m_{i',j} \). For a more comprehensive treatment, we refer to \(5\).
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