Gauge Theory for Finite-Dimensional Dynamical Systems

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Abstract

Gauge theory is a well-established concept in quantum physics, electrodynamics, and cosmology. This theory has recently proliferated into new areas, such as mechanics and astrodynamics. In this paper, we discuss a few applications of gauge theory in finite-dimensional dynamical systems with implications to numerical integration of differential equations. We distinguish between rescriptive and descriptive gauge symmetry. Rescriptive gauge symmetry is, in essence, re-scaling of the independent variable, while descriptive gauge symmetry is a Yang-Mills-like transformation of the velocity vector field, adapted to finite-dimensional systems. We show that a simple gauge transformation of multiple harmonic oscillators driven by chaotic processes can render an apparently “disordered” flow into a regular dynamical process, and that there exists a remarkable connection between gauge transformations and reduction theory of ordinary differential equations. Throughout the discussion, we demonstrate the main ideas by considering examples from diverse engineering and scientific fields, including quantum mechanics, chemistry, rigid-body dynamics and information theory.

1
1 Introduction

In modern physics, gauge theories are probably among the most powerful methods for understanding interactions among fields. The importance of gauge theories for physics stems from the tremendous success of the mathematical formalism in providing a unified framework to describe the quantum field theories of electromagnetism, the weak force and the strong force. Modern theories like string theory, as well as some formulations of general relativity, are, in some sense, gauge theories. The Yang-Mills theory, the standard approach to quantum field theory, is a particular example of gauge theories with non-Abelian symmetry groups. Gauge symmetries are the core mathematical mechanism of gauge theory, reflecting a redundancy in a description of a system.

The earliest physical theory which had a gauge symmetry was Maxwell’s electrodynamics. However, the importance of this symmetry remained unnoticed in the earliest formulations. After Einstein’s development of general relativity, Hermann Weyl, in an attempt to unify general relativity and electromagnetism, conjectured that “Eich-invarianz” or invariance under the change of scale (or “gauge”) might also be a local symmetry of the theory of general relativity. Weyl coined the use of gauge symmetry in modern physics \[40, 41, 42\], saying that “symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection”.

The question to be raised at this point is: Can the gauge-theoretical approach be applied on finite-dimensional systems, and to what degree of success? Keeping in mind that Lie symmetry has been a major tool in the study of ordinary differential equations (ODEs) \[24\], the answer must be positive; however, we have not established yet a clear-cut connection between Lie symmetry and gauge symmetry. We plan do so in the sequel, and show that, indeed, Lie point-symmetry is closely connected to gauge symmetry, albeit this connection is not always straightforward. In order to facilitate the establishment of such a connection, we use gauge symmetry in a more general context, a context of a symmetry defined by diffeomorphisms. This will ulti-
mately allow us to combine various manifestations of gauge symmetry in finite-dimensional dynamical systems under a single mathematical realm.

The idea to apply gauge transformations on ODEs is not new; it was suggested by Kunin [20], and has been recently revived by Efroimsky [11, 12, 13, 32, 10, 17], who developed a gauge-generalized astrodynamical theory for modeling the effect of orbital perturbations using non-osculating orbital elements. Efroimsky’s gauge theory, however, does not deal with scale transformations, while Kunin’s gauge theory is mostly concerned with local gauge groups and discrete symmetries. Thus far, there has not been a unified gauge theory which is able to support both Yang-Mills-like gauge theories and scaling theories in finite-dimensional systems.

In the current paper, we shall attempt to fill this gap by developing a gauge theory for finite-dimensional dynamical systems through two gauge symmetry mechanisms; the first symmetry mechanism will be called rescriptive gauge symmetry, evoked by carrying out a rescriptive gauge transformation.

Rescriptive gauge symmetry succumbs to the fundamental notion of gauge transformations, namely, a change of scale, and is also intimately connected to bilinearity. To show rescriptive gauge symmetry, we shall carry out an infinitesimal transformation of the independent variable – which in the bulk of our subsequent discussion will be time – into a different scale. The manifestation of gauge symmetry in this case will be reflected in the ability to obtain equivalence between the direction fields of the original and gauge-transformed systems. In many practical applications, this implies that the system can be reduced to a form amenable for quadrature (e.g. linear ODEs). We shall formalize this observation by establishing a mechanism for reduction through rescriptive gauge symmetry.

To illustrate the concept of rescriptive gauge symmetry, we will present a myriad of physical examples taken from diverse scientific and engineering fields, including rigid-body dynamics, finite-dimensional quantum mechanical systems, chemistry, and information theory.

An instrumental constituent of our new theory is the gauged pendulum. Generally speaking, a gauged pendulum is a physical system
with a quadratic integral of motion, whose behavior in the time domain can be arbitrary, although its phase space structure remains invariant under a change of scale. This implies that after a suitable scale transformation, harmonic oscillations will emerge. We show that many physical systems can be either re-formulated to match the formalism of the gauged pendulum, or are natural gauge pendulums per se; a classical example for a natural gauged pendulum is the Euler-Poinsot system, to be subsequently analyzed.

We ultimately utilize the notion of a gauged pendulum to question some common engineering misconceptions of chaotic and stochastic phenomena, and show that seemingly “disordered” (deterministic) or “random” (stochastic) behaviors can be “ordered”, or, put differently, evoke simple patterns [28, 29] using an infinitesimal transformation of the time scale. This brings into play the notion of observation and observables; we show that temporal observations may be misleading when used for chaos detection.

The second symmetry mechanism, reminiscent of the gauge symmetry arising in Maxwell’s equations [31] and its generalization into the Yang-Mills field theory, will be referred to as descriptive gauge symmetry. The concomitant gauge transformation will be called a descriptive gauge transformation. Descriptive gauge symmetry naturally arises in Newtonian mechanical systems, and can be thought of as an invariance of some configuration space under a gauge transformation of the covariant derivative. In fact, descriptive gauge symmetry may be best understood by relating it to the method of variations-of-parameters (VOP), which is an analytical formalism for solving inhomogeneous (forced) differential equations.

Euler invented the VOP method [14, 15] for treating highly nonlinear problems emerging in celestial mechanics. However, it was Lagrange who employed this method for deriving his system of equations describing the evolution of the orbital elements [21, 22, 23], known as Lagrange’s Planetary Equations. The relation between the VOP method and descriptive gauge symmetry can be explained as follows. According to the VOP method, the integration constants of the homogeneous solution of a given ODE are endowed with a time variation due to the presence of an external force. However, the transforma-
tion from the state variables of the original problem’s phase space to the new state variables defined as the time-varying constants involves an inherent freedom, which, in practical calculations, can be removed by means of a user-defined constraint. The constraint may be essentially arbitrary insofar as it does not come into contradiction with the equations of motion written for the variable “constants.” The internal freedom emerges under the following circumstances: First, one should perturb some $n$-dimensional differential equation, and solve it by the VOP method (i.e., using the unperturbed generic solution $q(t, x_1 \ldots x_n)$ as an ansatz, and making its constants $x_i$ time-varying); second, the number of “constants” promoted to variables must exceed $n$. Thus, when the said equations are written as equations for the new state variables $x_i$, the number of these variables will exceed that of equations; hence the internal freedom. Mathematically, this freedom is analogous to the gauge symmetry in electrodynamics, while the removal of this freedom by imposing an arbitrary constraint is analogous to fixing of a gauge in the Maxwell theory.

From a practical standpoint, we show that descriptive gauge symmetry may be used to considerably mitigate the numerical truncation error of numerical integrations, and even “symplectify” non-symplectic integrators. We also show how a given system of ODEs can undergo a reduction under a descriptive gauge symmetry transformation.

2 Preliminaries and Definitions

Consider a finite-dimensional dynamical system whose dynamics are modeled using first-order vector differential equations of the form

$$\frac{dx}{dt} = f(x)$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Assume that this system is subjected to structural modifications resulting from some re-formulation of the phase space, perturbations, control inputs, exogenous disturbances or modeling uncertainties. We shall generalize these modifications under a single mathematical umbrella which we call a rescription. A rescription operator in the time domain, $\mathcal{R}^t$, acts upon the vector field

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The vector field $\mathbf{u}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, $m \leq n$ will be called a rescriptor. The rescriptor may be either static or dynamic. In the former case, one may write $\mathbf{u} = K(x,t)$, with $K: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, while in the latter case the rescriptor constitutes a dynamical system of the general form

$$\frac{d\mathbf{u}}{dt} = h(x,dx/dt,y,u)$$  \hspace{1cm} (3a) \\
$$\frac{dy}{dt} = h_1(y,u).$$  \hspace{1cm} (3b) \\

where $y \in \mathcal{M} \subset \mathbb{R}^q$, $\mathcal{M}$ being some compact differentiable manifold, and $h_1 : \mathcal{M} \times \mathbb{R}^m \to \mathbb{R}^q$ is a vector field. By restricting the dynamic rescriptor from dependance on $\mathbf{u}$, we can unify the dynamic and static rescription, viz. a static rescription can be defined as a special case of a dynamic rescription.

In most cases, the rescription modifies - sometimes intentionally, such as in the case of control inputs - the fundamental properties of the original system. These “fundamental properties” may be, for instance, integrability, symmetry and structure/volume-preserving measures. The new properties of the rescribed system can be investigated both in the time domain and in the phase space.

However, in some cases the rescription is merely an illusion; that is, the rescription does not change the phase space and the fundamental properties of the original system, although it could modify the flow $\varphi(x_i(t = t_0), t)$. A classical example is the action of the rotation group $G_0 = SO(n)$ on Lagrangians of the form $\mathcal{L} = k\dot{x}^2$, which remain left-invariant under the transformation $x \mapsto Tx$, $T \in SO(n)$. We shall exclude these trivial occurrences from our discussion, and will explore a more general setting. In this general setting, the system is invariant under the action of some (possibly time-varying) finite-dimensional gauge group, $\mathcal{G}$. 

f in the following manner:

$$\mathbf{f} \circ \mathbf{f} = g(x, u(x,t)) = \frac{dx}{dt}$$  \hspace{1cm} (2)
3 Rescriptive Gauge Symmetry

We ask whether the system can be “de-rescribed” by finding new independent variables, $\tau_j$, possibly different for each rescriptor component $u_i$, satisfying

$$d\tau_j = G_i(x, u_i(x, t), dx, dt)$$  \hspace{1cm} (4)

for which

$$\mathfrak{F}_\tau \circ g = f(x) = x',$$  \hspace{1cm} (5)

where the operator $(\cdot)'$ denotes differentiation of each $x_i$ with respect to some $\tau_j$,

$$x' = \frac{dx_i}{d\tau_j}, \quad i = 1, \ldots, n, \quad j \in [1, \ldots, n].$$  \hspace{1cm} (6)

If $\exists d\tau_j, j \in [1, \ldots, n]$ satisfying (4) such that (5) holds, then we shall say that system (2) exhibits full rescriptive gauge symmetry$^1$ under the rescriptive gauge transformation (4). In this case $u$ becomes either a static or a dynamic rescriptive gauge function.

A rescriptive gauge symmetry of order $p$ or simply partial rescriptive gauge symmetry comes about when the rescriptive gauge transformation de-rescribes only $p$ state variables, $p < n$, viz.

$$\mathfrak{F}_\tau \circ g_i = f_i(x) = x_i', \quad i \in \mathbb{N}^p.$$  \hspace{1cm} (7)

In this case, if $t \in [0, t_f], t_f \leq \infty$ and $\tau_j \in \mathbb{R}, j \in [1, \ldots, p]$, then $\exists \tau_{j0}, t_0, x_i(t_0), x_i(\tau_{j0})$ such that the flow satisfies the gauge homeomorphism

$$\varphi(x_i(t_0), t) = \varphi(x_i(\tau_{j0}), \tau_j)$$  \hspace{1cm} (8)

for $t \cap \mathbb{R}$, where the flow is interpreted as the one-parameter group of transformations

$$G_t : x_i(t_0) \to x_i(t), \quad G_\tau : x_i(\tau_0) \to x_i(\tau).$$  \hspace{1cm} (9)

The notion of rescriptive gauge symmetry has far-reaching applica-

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$^1$The definition of rescriptive is given in Webster’s Revised Unabridged Dictionary (1913): “Pertaining to, or answering the purpose of, a rescript; hence, deciding; settling; determining”.

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tions in “ordering” seemingly “disordered” phenomena, solving ordinary differential equations (ODEs) and improving numerical integration thereof. We shall illustrate these ideas by discussing a few examples of practical interest. We embark on our quest by presenting the notion of a gauged pendulum, dwelt upon in the following subsection.

### 3.1 The Gauged Pendulum

Finite-dimensional systems can often be modeled by Hamiltonian vector fields induced by a nominal Hamiltonian, \( \mathcal{H} \), and a perturbing Hamiltonian \( \Delta \mathcal{H} \). Moreover, in ubiquitous fields of science and engineering, \( \mathcal{H} \) is comprised of \( n \) uncoupled harmonic oscillators \(2\), namely

\[
\mathcal{H}[q(t), p(t)] = \frac{1}{2} (p^T p + q^T \Omega q) = \frac{1}{2} \sum_{i=1}^{n} (p_i^2 + \omega_i^2 q_i^2) = \frac{1}{2} \sum_{i=1}^{n} H_i[q_i(t), p_i(t)]
\]

(10)

where \( \Omega = \text{diag}(\omega_1^2, \ldots, \omega_n^2) \),

\[
q = [q_1, \ldots, q_n]^T, \quad p = [p_1, \ldots, p_n]^T,
\]

(11)

are the generalized coordinates and conjugate momenta, respectively, so that \( (q, p) \in T^*Q \), where \( Q \) is the configuration space, \( T^*Q \) is the cotangent bundle of \( Q \), and \( \dim T^*Q = 2n \) is the dimension of the phase space.

Hamilton’s equations for \( i = 1, \ldots, n \) are then

\[
\dot{q}_i = p_i \quad (12a) \\
\dot{p}_i = -\omega_i^2 q_i. \quad (12b)
\]

Carrying out the point transformation into action-angle variables, given by

\[
q_i = \sqrt{\frac{\Phi_i}{\omega_i}} \sin \phi_i, \quad p_i = \sqrt{\Phi_i \omega_j} \cos \phi_i,
\]

(13)
simplifies the Hamiltonian (10) even further, into

\[ H[\omega(t), \Phi(t)] = \frac{1}{2} \omega^T \Phi = \frac{1}{2} \sum_{i=1}^{n} \omega_i \Phi_i, \quad (14) \]

where

\[ \omega = [\omega_1, \ldots, \omega_n]^T, \quad \Phi = [\Phi_1, \ldots, \Phi_n]^T. \quad (15) \]

We note that from the topological standpoint, in both (10) and (14) \( H \in S^n \subset \mathbb{R}^{n+1} \) is always homeomorphic to an \( n \)-ellipsoid, and \( H_i \in S^1 \) is an integral. Moreover, the transformation \((q, p) \mapsto (\phi, \Phi)\) is a (universal) covering map of \( S^n \).

It is clear that the Hamiltonian (10) is left-invariant under the action of the rotation group \( G_0 = SO(n) \) on \( q \) and \( p \) if \( \omega_i = \omega_0 \). However, we shall seek a broader invariance of \( H \) with respect to the gauge group, \( G \), which does not necessarily adhere to the \( SO(n) \) symmetry.

To that end, let us choose an arbitrary (not necessarily smooth) scalar field \( u_i(q, p) : T^*Q \to \mathbb{R} \) to serve as our rescriptor, coupling the dynamics of the \( n \) pendulums, and re-write (12) into the strictly bilinear form\(^2\) in \([q, p]\) and \( u \):

\begin{align*}
\dot{q}_i &= p_i u_i(q, p) \quad (16a) \\
\dot{p}_i &= -\omega_i^2 q_i u_i(q, p). \quad (16b)
\end{align*}

Obviously, a constant of motion for each of the pairs \((q_i, p_i)\) would be

\[ C_i = \frac{1}{2} \left( p_i^2 + \omega_i^2 q_i^2 \right), \quad i = 1, \ldots, n \quad (17) \]

although \( C_i \) is no longer the Hamiltonian. Nevertheless, system (16) remains integrable regardless of the particular form of \( u_i \), since there are \( n \) integrals for \( n \) degrees-of-freedom. This can readily observed by performing the (affine in \( dt \)) rescriptive gauge transformation

\[ d\tau_i = u_i(p, q) dt, \quad (18) \]

\(^2\)A strictly bilinear system with respect to \( x \) and \( u \) has the structure \( \dot{x} = Mxu, M \in \mathbb{R}^{n \times n} \). See [5] for details.
which, on one hand, extends \((16)\) into the state-space model
\[
\begin{align*}
\dot{q}_i &= p_i u_i(p, q) \\
\dot{p}_i &= -\omega_i^2 q_i u_i(p, q) \\
\dot{\tau}_i &= u_i(p, q),
\end{align*}
\]
but, on the other hand, transforms \((16)\) back into the simple harmonic oscillator form in the independent variables \(\tau_i\), assuming the symplectic structure
\[
\begin{align*}
q'_i &= p_i \\
p'_i &= -\omega_i^2 q_i.
\end{align*}
\]
Thus, \(u_i\) is a rescriptive gauge function \(\forall i\), and \(C_i\) can be interpreted as the Hamiltonian again, that is,
\[
\mathcal{H}[q(\tau_i), p(\tau_i)] = \frac{1}{2} \sum_{i=1}^{n} C_i = \frac{1}{2} \sum_{i=1}^{n} (p_i^2 + \omega_i^2 q_i^2).
\]
In this example the transformation \(t \mapsto \tau_i, i = 1, \ldots, n\) is therefore a static rescriptive gauge transformation. This means that \(u_i\) may be used to control the flow of \(p_i\) and \(q_i\) in the time domain, but the persistence of the integrability under the transformation \((18)\) forces the system to exhibit the same behavior as the harmonic oscillator in the modified times \(\tau_i\) for each degree-of-freedom.

In order to generalize this concept and illustrate how rescriptive gauge functions emerge in common physical systems, we must allow \(u_i\) to be an output of a dynamical system, giving rise to dynamic rescription, defined in \((2\) In this case, system \((16)\), written for each degree-of-freedom, \(i = 1, \ldots, n\), becomes
\[
\begin{align*}
\dot{q}_i &= p_i u_i(p, q) \\
\dot{p}_i &= -\omega_i^2 q_i u_i(p, q) \\
\dot{u}_i &= h_i(p, q, u_i, y) \\
\dot{y} &= h_1(u, y).
\end{align*}
\]
Carrying out the rescriptive gauge transformation \((18)\) reveals a par-
tial rescriptive gauge symmetry:

\[ q'_i = p_i \]  \hspace{1cm} (23a)  \\
\[ p'_i = -\omega^2_i q_i \] \hspace{1cm} (23b)  \\
\[ u'_i = \frac{1}{u_i} h_i(p, q, u, y) \] \hspace{1cm} (23c)

Thus, independently of the particular characteristics of the dynamic (or static) rescriptive gauge function, \( u_i \), the system re-assumes the harmonic oscillator structure for \((q_i, p_i)\). This situation can therefore be viewed as a generalization of the pendulum model. The persistence of the harmonic oscillations under the rescriptive gauge transformation gives rise to the concept of a *gauged pendulum*. The gauged pendulum is a dynamical system whose flow becomes periodic under the rescriptive gauge transformation, although the flow of the original system may exhibit arbitrary behavior in the time domain. Such systems arise in ubiquitous fields of science and engineering. For example, the following model arises in the study of quantum mechanical phenomenon (assuming a zero decoherence coefficient) [18]:

\[ \dot{r}_1 = -u_1(r_1, r_2)u_2(r_1, r_2) r_2 \] \hspace{1cm} (24)  \\
\[ \dot{r}_2 = u_1(r_1, r_2)u_2(r_1, r_2) r_1. \] \hspace{1cm} (25)

This is obviously a gauged pendulum with the static rescriptor \( u = u_1 u_2 \). we shall subsequently dwell upon additional physical examples.

An alternative formulation of systems exhibiting partial rescriptive gauge symmetry with a dynamic rescriptive gauge function may written as

\[ \dot{q}_i = p_i u_i(p, q) \] \hspace{1cm} (26a)  \\
\[ \dot{p}_i = -\omega^2_i q_i u_i(p, q) \] \hspace{1cm} (26b)  \\
\[ \dot{u}_i = h_i(p, q, u) \] \hspace{1cm} (26c)
which becomes

\[ \begin{align*}
q'_i &= p_i \quad (27a) \\
p'_i &= -\omega_i^2 q_i \quad (27b) \\
u'_i &= \frac{1}{u_i} h_i(p, q, u_i) \quad (27c)
\end{align*} \]

after de-rescription using our standard rescriptive gauge transformation. Here the rescriptor \( u_i \) still constitutes a dynamic rescriptive gauge, albeit it is not an output of an auxiliary dynamical system anymore. In fact, if we relieve \( h_i \) from direct dependance upon \( u_i \), viz. \( u_i = h_i(p, q) \), then we uncover additional integrals of the motion, \( K_i \), defined by the quadrature

\[ K_i = \frac{1}{2} u_i^2 - \int h_i(q(\tau_i), p(\tau_i)) d\tau_i. \quad (28) \]

These new constants posses a clear meaning, revealed by writing

\[ \begin{align*}
\dot{u}_i &= -\partial K_i \partial \tau_i = h_i \quad (29a) \\
\dot{\tau}_i &= \partial K_i \partial u_i = u_i. \quad (29b)
\end{align*} \]

Hence, \( \tau_i, u_i \) can be interpreted as generalized coordinates and conjugate momenta, respectively, evolving on a \( 2n \)-dimensional symplectic manifold. \( K_i \) is then a Hamiltonian, and the dynamics of \( (u_i, \tau_i) \) is integrable. This remarkable structure implies that if the time derivative of the rescriptor does not explicitly depend upon the rescriptor, then the rescriptive gauge transformation may be viewed as the symplectomorphism

\[ \begin{bmatrix}
\dot{q}_i = \partial H(q_i(t), p_i(t)) \\
\dot{p}_i = -\partial H(q_i(t), p_i(t)) \\
\dot{u}_i = -\partial K_i \partial \tau_i \\
\dot{\tau}_i = \partial K_i \partial u_i
\end{bmatrix} \mapsto \begin{bmatrix}
\dot{q}'_i = \partial H(q_i(\tau_i), p_i(\tau_i)) \\
\dot{p}'_i = -\partial H(q_i(\tau_i), p_i(\tau_i)) \\
\dot{u}_i = -\partial K_i \partial \tau_i \\
\dot{\tau}_i = \partial K_i \partial u_i
\end{bmatrix}. \quad (30) \]
The formulation in (30) is general, and is not limited to Hamiltonians of the form (21); rather, if \( \mathcal{H}_i(p_i, q_i) = \text{const.} \) is a given Hamiltonian, then Hamilton’s equations
\[
\dot{q}_i = \frac{\partial\mathcal{H}_i(p_i, q_i)}{\partial p}, \quad \dot{p}_i = -\frac{\partial\mathcal{H}_i(p_i, q_i)}{\partial q}
\] (31)
undergoing a rescription
\[
\dot{q}_i = \frac{\partial\mathcal{H}_i(p_i, q_i)}{\partial p} u_i(p, q), \quad \dot{p}_i = -\frac{\partial\mathcal{H}_i(p_i, q_i)}{\partial q} u_i(p, q)
\] (32)
will still possess still \( \mathcal{H}_i(p_i, q_i) = \text{const.} \) as an integral, and can be de-rescribed using the rescriptive gauge transformation \( d\tau = u_i(p, q) dt \) into
\[
\dot{q}_i = \frac{\partial\mathcal{H}_i(p_i, q_i)}{\partial p}, \quad \dot{p}_i = -\frac{\partial\mathcal{H}_i(p_i, q_i)}{\partial q}. \] (33)
Finally, a slightly different formulation of the gauged pendulum with a dynamics rescriptive gauge symmetry, to be illustrated in §3.3, may be written as
\[
\dot{q}_i = k_1 p_i u_i(p, q) \quad (34a) \\
\dot{p}_i = k_2 q_i u_i(p, q) \quad (34b) \\
\dot{u}_i = h_i(p, q, u_i) \quad (34c)
\]
which becomes
\[
q_i' = k_1 p_i \quad (35a) \\
p_i' = k_2 q_i \quad (35b) \\
u_i' = \frac{1}{u_i} h_i(p, q, u_i) \quad (35c)
\]
after de-rescription using the rescriptive gauge transformation \( \tau_i = u_i dt \). Here we have
\[
C_i = \frac{1}{2} \left( k_1 p_i^2 - k_2 q_i^2 \right), \quad i = 1, \ldots, n \quad (36)
\]
as integrals. However, an important caveat is that (35a)-(35b) may be viewed as a gauged pendulum only if \( k_1 k_2 < 0 \). Otherwise, the de-rescription will yield hyperbolic motion in the variable \( \tau \).
3.2 Newtonian Systems

We shall now show that Newtonian systems can be re-written into the gauged pendulum formalism. To that end, consider the system

\[ \dot{q} = vp \tag{37a} \]
\[ \dot{p} = -vq \tag{37b} \]
\[ \dot{v} = f(x,v). \tag{37c} \]

Performing the “action-angle” transformation (cf. Eq. (13)) \( p = \cos x, q = \sin x \) and re-writing (37) yields

\[ \dot{x} = v \tag{38a} \]
\[ \dot{v} = f(x,v), \tag{38b} \]

which is a state space representation of the Newtonian system

\[ \ddot{x} = f(x,\dot{x}). \tag{39} \]

We immediately observe that our rescriptor is the velocity, \( v \). Hence, when dealing with problems in the Newtonian context, the rescriptive gauge function is simply the gauge velocity.

Our Newtonian system therefore possesses a trivial partial rescriptive gauge symmetry, found by performing the rescriptive gauge transformation \( d\tau = dx = vdt \). In other words, in the Newtonian case \( \tau = x \), and the de-rescribed system becomes

\[ q' = p \tag{40a} \]
\[ p' = -q \tag{40b} \]
\[ v' = f(x,v)/v, \tag{40c} \]

where (\( )' \) denotes differentiation with respect to \( x \). If \( f(x,v) = f(x) \), then

\[ K = \frac{v^2}{2} - \int f(x)dx \tag{41} \]

is an integral, \( K \) is the Hamiltonian for the original system \( q,p \), and \( H = (q^2 + p^2)/2 \) is the Hamiltonian of \( (q,p) \). Thus, any Newtonian system can be written in the gauged pendulum form by extending
the phase space dimension by one. Consequently, any system whose state-space model is similar to (37) is a Newtonian system in disguise.

3.3 Eulerian Systems

In a body-fixed frame, the attitude dynamics of a rigid body are usually formulated by means of the Euler-Poinsot equations. In a free-spin case, these equations look as

$$ \dot{\omega} + \omega \times I \omega = 0, $$

(42)

$I$ being the inertia tensor and $\omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{S}$ is the body angular velocity vector, where $\mathbb{S}$ is the foliation \{(I\omega_1, I\omega_2, I\omega_3)|I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = G^2}\), $G$ being the total angular momentum.

Assuming that the body axes coincide with the principal axes of inertia,

$$ I = \text{diag}(I_1, I_2, I_3), $$

(43)

the Euler-Poinsot equations are

$$ \dot{\omega}_1 = \sigma_1 \omega_2 \omega_3 $$

(44a)

$$ \dot{\omega}_2 = \sigma_2 \omega_1 \omega_3 $$

(44b)

$$ \dot{\omega}_3 = \sigma_3 \omega_1 \omega_2 $$

(44c)

where

$$ \sigma_1 = \frac{I_2 - I_3}{I_1}, \quad \sigma_2 = \frac{I_3 - I_1}{I_2}, \quad \sigma_3 = \frac{I_1 - I_2}{I_3}. $$

(45)

We shall now show that the Euler-Poinsot equations are a classical example of the gauged pendulum concept with a dynamic rescriptive gauge, exhibiting partial rescriptive gauge symmetry of order 2. To that end, define the rescriptive gauge transformation

$$ d\tau = \omega_3 dt $$

(46)
and re-write (44) into

\[ \omega'_1 = \sigma_1 \omega_2 \]  
\[ \omega'_2 = \sigma_2 \omega_1 \]  
\[ \omega'_3 = \frac{\sigma_3}{\omega_3} \omega_1 \omega_2, \]  

which adheres to the gauged pendulum model (35). Thus, in the modified scale \( \tau \), \( \omega_1 \) and \( \omega_2 \) will exhibit harmonic oscillations with frequency \( \sqrt{|\sigma_1 \sigma_2|} \) if \( \sigma_1 \sigma_2 < 0 \), given by

\[
\begin{align*}
\omega_2(\tau) &= \frac{-\sigma_2 \omega_{10} \sin(\omega_0 \tau_0) + \omega_{20} \omega_0 \cos(\omega_0 \tau_0)}{\omega_0} \cos(\omega_0 \tau) \\
&\quad + \frac{\sigma_2 \omega_{10} \cos(\omega_0 \tau_0) + \omega_{20} \omega_0 \sin(\omega_0 \tau_0)}{\omega_0} \sin(\omega_0 \tau) \\
\omega_1(\tau) &= \frac{\sigma_2 \omega_{10} \cos(\omega_0 \tau_0) + \omega_{20} \omega_0 \sin(\omega_0 \tau_0)}{\sigma_2} \cos(\omega_0 \tau) \\
&\quad - \frac{\omega_{20} \omega_0 \cos(\omega_0 \tau_0) - \sigma_2 \omega_{10} \sin(\omega_0 \tau_0)}{\sigma_2} \sin(\omega_0 \tau)
\end{align*}
\]

where \( \omega_0 = \sqrt{|\sigma_1 \sigma_2|}, \ \omega_{10} = \omega_1(\tau_0), \ \omega_2(\tau_0) = \omega_{20}. \)

The solution for the dynamic rescriptor \( \omega_3 \) can now be easily solved by quadrature. Since

\[ C = \frac{1}{2} \omega_3^2 - \sigma_3 \int \omega_1 \omega_2 d\tau \]

is an integral,

\[ \omega_3 = \sqrt{2C + A \sigma_3 \cos^2(\omega_0 \tau) + B \sigma_3 \sin(\omega_0 \tau) \cos(\omega_0 \tau)} \]

Note that here the rescriptor has units of angular velocity, while in the Newtonian case it was the velocity. We shall re-iterate on this issue in the following sections.
where

\[ A = \frac{\omega_0^2 \sigma_2}{\omega_0^2 \sigma_2} - 2 \sigma_2^2 \cos^2(\omega_0 \tau_0) \omega_0^2 \omega_20 \sin(\omega_0 \tau_0) \omega_0 \sigma_2 \]

\[ + \frac{\omega_0^2 \sigma_2}{\omega_0^2 \sigma_2} - \omega_0^2 \omega_20 + 2 \omega_0^2 \omega_20 \cos^2(\omega_0 \tau_0) + \sigma_2^2 \omega_10^2 \]

\[ B = \frac{\omega_0^2 \sigma_2}{\omega_0^2 \sigma_2} - 2 \sigma_2^2 \cos(\omega_0 \tau_0) \omega_10 \sin(\omega_0 \tau_0) + 4 \cos(\omega_0 \tau_0) \omega_10 \omega_20 \omega_0 \sigma_2 \]

\[ + \frac{\omega_0^2 \sigma_2}{\omega_0^2 \sigma_2} - 2 \omega_0 \sigma_2 \omega_10 + 2 \omega_0^2 \omega_20 \sin(\omega_0 \tau_0) \cos(\omega_0 \tau_0) \]

(52)

From (46), the new independent variable is

\[ \tau = \int \omega_3 dt. \]

(54)

To understand its physical meaning, we recall that if we let the hat map \( \widehat{\omega} : \mathbb{R}^3 \to \mathfrak{so}(3) \) denote the usual Lie algebra isomorphism that identifies \( (\mathfrak{so}(3), [, ,]) \) with \( (\mathbb{R}^3, \times) \), then

\[ \widehat{\omega} = -\dot{R}R^T \]

(55)

where

\[ \widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \]

(56)

and \( R \in SO(3) \) is a rotation matrix from inertial to body coordinates.

If we take the rotation sequence \( 3 \to 1 \to 3, \phi \to \theta \to \psi \), evaluation of (55) will entail the well-known expressions for the components of the vector of the body angular velocity \( \omega \) in terms of the Euler angles rates \( \dot{\phi}, \dot{\theta}, \dot{\psi} \):

\[ \omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \]

(57a)

\[ \omega_2 = \dot{\phi} \cos \psi \sin \theta - \dot{\theta} \sin \psi \]

(57b)

\[ \omega_3 = \dot{\psi} + \dot{\phi} \cos \theta \]

(57c)

Thus,

\[ \tau = \int \omega_3 = \psi + \int \dot{\phi} \cos \theta dt, \]

(58)
so the $\omega_i - \tau$ dynamics may be viewed as a solution for the phase space of the Eulerian system.

The rescription in the Eulerian case possesses an interesting symmetry. In the above discussion, we detected the dynamic rescriptor $\omega_3$ for the pair $(\dot{\omega}_1, \dot{\omega}_2)$, but there are two other possible rescriptions: $\omega_1$ for $(\dot{\omega}_2, \dot{\omega}_3)$ and $\omega_2$ for $(\dot{\omega}_1, \dot{\omega}_3)$.

Finally, we note that the rescriptive gauge transformation linearized the Euler-Poinsot equations; observe that (47a)-(47b) are linear, and (47c) is a simple linear quadrature thereof in the variable $z = \omega_3^2$. We shall further dwell upon this finding in §3.6.

3.4 Other Common Systems Exhibiting Rescriptive Gauge Symmetry

The gauged pendulum is a particular case of systems exhibiting rescriptive gauge symmetry. However, there are systems that exhibit rescriptive gauge symmetry, which cannot be rendered periodic after a rescriptive gauge transformation. Generally speaking, such systems cannot be conveniently described using the Hamiltonian formalism, although they do possess integrals. Consider, for illustration, the dynamical equations of two chemical reactants, $A$ and $B$, whose concentrations evolve according to the bilinear rate law [9]

$$\begin{align*}
\frac{d[A]}{dt} &= k_1[A][B], \quad (59a) \\
\frac{d[B]}{dt} &= k_2[A][B]. \quad (59b)
\end{align*}$$

These can be de-rescribed using e. g. $d\tau = [A]dt$, yielding the linear equations

$$\begin{align*}
\frac{d[A]}{d\tau} &= k_1[B], \quad (60a) \\
\frac{d[B]}{d\tau} &= k_2[B], \quad (60b)
\end{align*}$$

so that

$$[B(\tau)] = [B(\tau_0)]e^{k_2\tau}, \quad [A(\tau)] = [B(\tau_0)]\frac{k_1}{k_2}e^{k_2\tau} - 1 + [A(\tau_0)]. \quad (61)$$
An integral for system (60) is $C = [A] - k_1/k_2[B]$, albeit this is not the Hamiltonian. Consequently, an additional class of systems exhibiting descriptive gauge symmetry may be written as

\begin{align*}
\dot{q}_i &= q_i u_i(p, q) \\
\dot{p}_i &= q_i u_i(p, q) \\
\dot{u}_i &= h_i(p, q, u_i, y) \\
\dot{y} &= h_1(u, y).
\end{align*}

(62a) (62b) (62c) (62d)

3.5 The One-Parameter Lie Symmetry Group

Thus far we have not explicitly spelled out a relationship between the descriptive gauge transformation and Lie point-symmetry transformations. This is the purpose of the following discussion.

To keep things simple, assume a 1-DOF gauged pendulum model with a static rescriptor, $u(p, q)$:

\begin{align*}
\dot{q} &= pu(p, q) \\
\dot{p} &= -qu(p, q).
\end{align*}

(63a) (63b)

This set of equations can be analyzed by means of one-parameter groups based upon infinitesimal transformations. We demand the equation to be invariant under infinitesimal changes of the independent variable $t$, but without a simultaneous infinitesimal changes of the dependent variables. This leads to the Lie point-symmetry transformation

\begin{align*}
p &\rightarrow p \\
q &\rightarrow q \\
t &\rightarrow \tau = t + \epsilon \zeta(p, q).
\end{align*}

(64a) (64b) (64c)

We now apply (64) on (63) by following these stages: First, we write

\[ \frac{dq}{d\tau} = \frac{dq}{dt} + \epsilon \left( \frac{\partial \zeta}{\partial q} dq + \frac{\partial \zeta}{\partial p} dp \right) + O(\epsilon^2) \] 

(65)
\[ \frac{dp}{d\tau} = \frac{dp}{dt} + \epsilon \left( \frac{\partial \zeta}{\partial q} dq + \frac{\partial \zeta}{\partial p} dp \right) + O(\epsilon^2). \] (66)

Expanding (65) and (66) into a Taylor series with \( \epsilon \) as a first-order small parameter we get

\[ \frac{dq}{d\tau} = \frac{dq}{dt} - \epsilon \frac{dq}{dt} \left( \frac{\partial \zeta}{\partial q} dq + \frac{\partial \zeta}{\partial p} dp \right) + O(\epsilon^2), \] (67)

\[ \frac{dp}{d\tau} = \frac{dp}{dt} - \epsilon \frac{dp}{dt} \left( \frac{\partial \zeta}{\partial q} dq + \frac{\partial \zeta}{\partial p} dp \right) + O(\epsilon^2). \] (68)

These yield a partial differential equation (PDE) for \( \zeta(p, q) \),

\[ u^2(p, q)\epsilon \left( p \frac{\partial \zeta(p, q)}{\partial q} - q \frac{\partial \zeta(p, q)}{\partial p} \right) - u(p, q) + 1 = 0, \] (69)

the solution thereof is

\[ \zeta(p, q) = -\int_0^p \frac{u(\eta, \sqrt{c - \eta^2}) - 1}{\epsilon \sqrt{c - \eta^2} u^2(\eta, \sqrt{c - \eta^2})} d\eta + c_0 c \] (70)

where \( c = p^2 + q^2 \) and \( c_0 \) is an integration constant. To relate (65) and (66) to the generators of the infinitesimal transformation, we write

\[ \tau = t + \epsilon \zeta(p, q) + \ldots = t + \epsilon X t + \ldots \] (71)

where the operator \( X \) is given by

\[ X = \zeta(p, q) \frac{\partial}{\partial t} \] (72)

In addition, due to the fact that (63) is autonomous, it will also exhibit Lie point-symmetry with generator

\[ X_1 = \frac{\partial}{\partial t}. \] (73)

Symmetries (72) and (73) form an Abelian Lie algebra \( \mathcal{X} \) with the Lie bracket \( [X, X_1] = 0 \).
In essence, this symmetry implies that the direction field is
\[
\frac{dp}{dq} = -\frac{q}{p},
\]
and is therefore homogenous, that is, invariant under all dilations \((p, q) \mapsto (e^\lambda p, e^\lambda q), \lambda \in \mathbb{R}\), which holds true for any dynamic or static rescriptive gauge \(u(p, q)\). The connection to the rescriptive gauge symmetry can now be easily obtained via Arnold’s theorem \(\text{[1]}\), stating that if a one-parameter group of symmetries of a direction field is known, the equation \(dp/dq = f(p, q)\) can be integrated explicitly. This is obvious for the direction field (74) of the gauged pendulum.

3.6 Reduction using Rescriptive Gauge Symmetry

It is a well-known fact in dynamical system theory that under certain conditions, systems that exhibit symmetry are also reducible \(\text{[30]}\). We shall discuss reduction in the context of rescriptive gauge theory by following a few fundamental steps; ultimately, we will show that rescriptive gauge symmetry allows to reduce classes of nonlinear system into linear ODEs, solved by simple quadratures.

We begin our quest for the manifestation of reduction in the realm of rescriptive gauges by asking how a rescriptor for a given ODE can be found. We shall then show that the answer to this question is related to a more profound problem - that of exact linearization of ODEs, or, as we shall call it for clarity - global linearization. We shall dwell upon the latter issue shortly, and will first address the more basic query.

Finding a rescriptive gauge transformation for a given ODE is important, since it may allow quadrature in the modified time scale by reduction into linear forms. Consider, for illustration, the 1-DOF gauged pendulum model
\[
\begin{align*}
\dot{q} &= pu(p, q) \quad \text{(75a)} \\
\dot{p} &= -qu(p, q), \quad \text{(75b)}
\end{align*}
\]
which is readily transformed into the ODE
\[ \ddot{q} - \frac{\partial u(p, q)}{\partial q} \dot{p} q + u(p, q) q \left[ \frac{\partial u(p, q)}{\partial p} + u(p, q) \right]. \] (76)

Thus, any ODE that is written in the form (76) can be transformed into the de-rescribed gauged pendulum \( q'' + q = 0 \) using the rescriptive gauge transformation \( d\tau = u dt \). However, usually the rescriptor, \( u \), cannot be easily found. Consider, for instance, the nonlinear ODE
\[ \ddot{q} - \dot{q}^2 \cot q + q \sin^2 q = 0, \] (77)
for which the rescriptive gauge transformation
\[ d\tau = \sin q dt, \] (78)
reveals that (77) is no more than a harmonic oscillator in disguise, viz. \( q'' + q = 0 \). However, one cannot determine that \( u = \sin q \) by observation. This calls for a more rigorous methodology for finding the rescriptor.

To that end, consider a second-order ODE of the form
\[ \ddot{q} + f(q) \dot{q}^2 + b_1 u(q) \dot{q} + \psi(q) = 0. \] (79)
When can this ODE be transformed into the linear form
\[ q'' + b_1 q' + b_0 q + c = 0 \] (80)
by a rescriptive gauge transformation
\[ d\tau = u(q) dt \] (81)
only? The answer lies in the theory of exact linearization [4], which seeks a transformation rendering a nonlinear ODE amenable for quadrature. We shall prefer the term global linearization, emphasizing that this method is conceptually different from the common point linearization. We shall ultimately use global linearization theory to help us track down the rescriptor of a given ODE.

The theory of global linearization suggests that ODEs of the form
can be globally linearized by a transformation of the form

\[ z = \beta \int u \exp \left( \int f dq \right) dq, \quad d\tau = u(q)dt \]  

(82)

where \( \beta = \text{const} \), if and only if \( \beta \) can be written in the form

\[
\ddot{q} + f(q)\dot{q}^2 + b_1 u\dot{q} + u \exp \left( -\int f(q) dq \right) 
\cdot \left[ b_0 \int u \exp \left( \int f(q) dq \right) dq + \frac{c}{\beta} \right] = 0,
\]  

(83)

This fundamental result can be adapted to the case in question. In particular, since we are probing the case of rescriptive gauge transformations, we must require that \( z = q \), or, in other words, that

\[ \beta = 1, \quad u = u(q), \quad f = -\frac{1}{u dq}. \]  

(84)

In our discussion we allowed \( u \) to be a function of both \( q \) and \( p \), while (82) permit a \( u \) which is a function of \( q \) only. Thus, we must take \( u = u(q) \), as written in (84). Relations (84) modify (83) into

\[
\ddot{q} - u \frac{du}{dq} \dot{q}^2 + b_1 u \dot{q} + u \exp \left( -\int f(q) dq \right) \left[ b_0 q + \frac{c}{\beta} \right] = 0
\]  

(85)

\[
\ddot{q} - u \frac{du}{dq} \dot{q}^2 + b_1 u \dot{q} + u \exp \left( -\int \frac{1}{u dq} \right) \left[ b_0 q + \frac{c}{\beta} \right] = 0
\]  

(86)

\[
\ddot{q} - \frac{du}{dq} \dot{q}^2 + b_1 u \dot{q} + u^2 b_0 q + \frac{c}{\beta} = 0
\]  

(87)

Thus, we have proven that a second-order ODE can be transformed into a linear ODE using a rescriptive gauge transformation (assuming that the rescriptor is a function of the coordinate only) if and only if this ODE can be written as

\[ \ddot{q} - \frac{1}{u(q)} \frac{du(q)}{dq} \dot{q}^2 + b_1 u(q)\dot{q} + u(q)^2 b_0 q + \frac{c}{\beta} = 0. \]  

(88)

Eq. (88) immediately yields the rescriptor: It is the square root of the coefficient of the coordinate, \( q \), divided by \( \sqrt{b_0} \).

Returning to example (77), we see that it succumbs to the general
from (88) by substituting

\[ b_1 = 0, \quad b_0 = 1, \quad c = 0, \quad (89) \]

which yields

\[ \ddot{q} - \frac{1}{u} \frac{du}{dq} \dot{q}^2 + u^2 q = 0 \quad (90) \]

and it is immediately apparent that the rescriptor is \( u = \sin q \), which agrees with (88). As a simple verification, we also note that

\[ \frac{1}{u} \frac{du}{dq} = \cot q. \quad (91) \]

Similarly, the chemical rate equations (59), written as the single ODE

\[ \frac{d^2[A]}{dt^2} - \frac{1}{[A]} \left( \frac{d[A]}{dt} \right)^2 - k_2[A] \frac{d[A]}{dt} = 0 \quad (92) \]

may be linearized using the transformation \( d\tau = [A]dt \), as was done in §3.4.

The above process can be repeated for higher-order ODEs as well. The bottom line is that the theory of global linearization is a convenient method for finding a rescriptor of a given ODE, or, in other words, reduce it into a linear ODE using a rescriptive gauge transformation.

To conclude this section, we shall show that there are well-known ODEs that can be transformed into the reducible form (88) using an additional auxiliary variable transformation. This observation is inspired by §3.3, where we have shown that the Euler-Poinsot equations are transformed into a linear form in the independent variable \( \tau \) using the rescriptive gauge transformation \( d\tau = \omega_3 dt \) and the auxiliary transformation \( z = \omega_3^2 \). For example, consider the ODE:

\[ \ddot{q} + q\dot{q} + kq^3 = 0, \quad k = \text{const}. \quad (93) \]

This ODE arises in a few practical problems [25]. To render it globally linearizable using a rescriptive gauge transformation, perform the auxiliary variable transformation \( z = q^2 \), so the modified system reads
In this form, (94) adheres to ansatz (88), with the rescriptor $u = \sqrt{z} = q$ and $k = b_0$, $b_1 = 1$, $c = 0$. The rescriptive gauge transformation $d\tau = \sqrt{z}dt$ transforms (94) into

$$z'' + z' + 2kz = 0. \quad (95)$$

### 3.7 Illustrative Examples

We shall now illustrate the rescriptive gauge transformation formalism and the resulting gauged pendulum concept using a few numerical examples.

**Example 1 (A damped pendulum is a gauged pendulum)**

Consider the model \[38\]:

$$\dot{q} = up \quad (96a)$$

$$\dot{p} = -uq \quad (96b)$$

$$\dot{u} = -\omega_0^2 q - au. \quad (96c)$$

By carrying out the transformation $p = \cos \phi$, $q = \sin \phi$, these equations are immediately recognized as a state-space model for a damped nonlinear pendulum,

$$\ddot{\phi} + \omega_0^2 \sin \phi + a\dot{\phi} = 0. \quad (97)$$

System \[38\] complies with the gauged pendulum formalism \[26\]; it can be therefore viewed as a rescribed harmonic oscillator, revealed by the rescriptive gauge transformation $d\phi = udt$, so that $\tau = \phi$:

$$q' = p, \quad p' = -q. \quad (98)$$

Obviously, the rescriptor, or gauge velocity, is simply the angular velocity, i.e. $u = \dot{\phi}$. The scalar differential equation for this dynamic rescriptive gauge function, Eq. \[36c\], assumes the nonautonomous form
\[ u' = -\omega_0^2 \sin(\phi)/u - a. \] (99)

For \( a = 0 \), the rescriptive gauge function does not explicitly depend upon the rescriptor itself, and (99) is easily solved by quadrature:

\[ u(\phi) = \sqrt{2\omega_0^2(\cos \phi - \cos \phi_0) + u^2(\phi_0)}. \] (100)

It is interesting to note that under the rescriptive gauge symmetry, the harmonic oscillator and the damped nonlinear pendulum are represented by the same mathematical formalism - although for different independent variables - whereas the time flow of these models is completely different. The harmonic oscillator, which is a conservative system, does not have an attractor, since the motion is periodic. The damped pendulum, on the other hand, is a dissipative dynamical systems, in which volumes shrink exponentially, so its attractor has 0 volume in phase space. This alleged paradox stems from the fact that the dissipative time flow of the damped pendulum becomes periodic under a change of the independent variable. Thus, an observer measuring the “time”, \( \phi \), is bound to observe periodic behavior, while an observer measuring the “true” time, \( t \), will observe exponential decay.

These observations are demonstrated and validated by means of a numerical integration, comparing the flows of (96) and (98). Figure 1 compares between \( q(t) \) (Fig. 1a) and \( q(\phi) \) (Fig. 1b), and between \( p(t) \) (Fig. 1c) and \( p(\phi) \) (Fig. 1d), for \( a = 0.1, q_0 = 0.5, p_0 = 1, u_0 = 5, \phi_0 = \sin^{-1} q_0 = 0.5236 \).
Example 2 (A glimpse of order in the realm of chaos)

Consider the dynamical system

\[
\begin{align*}
\dot{q} &= yp \\
\dot{p} &= -yq \\
\dot{x} &= \sigma(y - x) \\
\dot{y} &= (r - z)x - y \\
\dot{z} &= xy - bz
\end{align*}
\]

where \( \sigma, r, b \) are constants. Eqs. (101a)-(101e) are recognized as the Lorenz system, and the entire system (101) complies with the gauged pendulum formalism of Eqs. (23). It shall be thus referred to as the
Lorenz-fed gauged pendulum.

For certain parameter values and initial conditions, the Lorenz system is known to exhibit chaos. For instance, choosing the parameter values $\sigma = 10$, $r = 28$, $b = 8/3$, the initial conditions $x(0) = 10$, $y(0) = 10$, $z(0) = 10$, and simulating for $t_f = 50$ time units, yields the trajectory depicted by Fig. 2.

Figure 2: The Lorenz strange attractor feeding the gauged pendulum.

Let us now examine the time history of $p$ and $q$, shown in Fig. 3 and ask: Do $q$ and $p$ exhibit chaotic behavior? To answer this seemingly trivial question (without using a comprehensive mapping of the phase space using Poincarè sections), we shall resort to the common
“engineering” interpretation of chaos, although more mathematically-rigorous definitions, related to the destruction of KAM tori [3] or the Kolmogorov-Sinai entropy [16], do exist. As Strogatz says in reference [36], “no definition of the term chaos is universally accepted yet, but almost everyone would agree on the three ingredients used in the following working definition”. These three ingredients are:

1. Aperiodicity: Chaos is aperiodic long-term behavior in a deterministic system. Aperiodic long-term behavior means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$.  

2. Sensitive dependence on initial conditions: Nearby trajectories separate exponentially fast, i.e., the system has a positive Lyapunov characteristic exponent (LCE).

Strogatz notes that he favors additional constraints on the aperiodic long-term behavior, but leaves open what form they may take. He suggests two alternatives to fulfill this:

3. Requiring that there exists an open set of initial conditions having aperiodic trajectories, or

4. If one picks a random initial condition $x(t_0) = x_0$ then there must be a nonzero chance of the associated trajectory $x(t, x_0)$ being aperiodic.

Returning to Fig. 3, we see that items 1, 3 and 4 in Strogatz’s list are satisfied: $p$ and $q$ exhibit aperiodic behavior, the open set of initial conditions guaranteeing aperiodic trajectories for $\sigma = 10$, $r = 28$, $b = 8/3$ are $x_0, y_0, z_0, p_0, q_0 \in \mathbb{R}\{0\}$, and hence for randomly selected initial conditions $p$ and $q$ will be aperiodic. The only remaining test is to calculate the LCEs, denoted by $\lambda_i$, $i = 1, \ldots, n$. However, as shall be illustrated shortly, calculation of the LCEs may be problematic for system (101).

First, we should note that some authors endorse the calculation of the maximal Lyapunov exponent in order to establish the presence
of chaos. For example, Ref. [34] states that “it is well-known that the ordered or the chaotic property of an orbit is characterized by the largest Lyapunov characteristic exponent”. This approach, however, is misleading for system (101). To illustrate this fact, we have calculated the maximal Lyapunov exponent for (101) using the standard method developed by [33] and [37]. The result is depicted by Fig. 4 for an integration period of 12,000 time units. It is seen that the maximal LCE satisfies $\max_i \lambda_i \approx 0.9$, which is the well-known maximal LCE of the Lorenz system. Hence, according to the rationale of [34], system (101) is chaotic - or is it?
Figure 4: The maximal Lyapunov characteristic exponent for a Lorenz-fed gauged pendulum system.

For a more rigorous analysis, the entire spectrum of LCEs should be examined. Since the LCE spectrum of the Lorenz system is well-known (the phase space contraction satisfies the relation $\sum_i \lambda_i = \nabla \cdot [\dot{x}, \dot{y}, \dot{z}] = -(\sigma + b + 1) = -13.667$), let us concentrate on the additional LCEs contributed by $p$ and $q$. A magnified view of these LCEs is shown in Fig. 5. One of these LCEs is smaller than zero, while the other one assumes the value of $4 \cdot 10^{-5}$, which allegedly indicates that the additional states are also chaotic.

However, this is a mere illusion resulting from the fact that the calculation process of the LCEs is affected by the truncation and round-off errors of the numerical integration routine used to simultaneously integrate the extended phase space of the original and linearized systems.\footnote{This causes the Lyapunov exponents themselves to exhibit a chaotic behavior; most high-order integrators are chaotic maps, as pointed out in \cite{7}. This may be viewed a manifestation of the uncertainty principle.} One may view this phenomenon as pseudochaos \cite{27}; the truth
regarding “chaos” in system (101) can be plainly revealed by realizing that (101) complies with the gauged-pendulum formalism, and can hence be subjected to a rescriptive gauge transformation of the form $d\tau = y dt$. This transformation will transform (101a), (101b) into $q' = p$, $p' = -q$, which is an integrable system and hence cannot exhibit chaos. This observation is illustrated in Fig. 6, showing plots of $q$ and $p$ as a function of $\tau$. Thus, in contrast to the prediction of the common engineering interpreting of chaos and the chaos detection tools thereof, the rescriptive gauge transformation shows that the temporal behavior of signals cannot always be used to predict the presence of chaos. This observation calls into being the concept of partial chaos [6], meaning that in a given system, both chaotic and regular signals may co-exist, even if the chaotic states overshadow the regular behavior of the other states.

Another important conclusion concerns the system observables. Observables, or outputs, is a subset of state variables, $z$, $\dim z = l \leq \dim x = n$, determined by the output map, $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^l$, such that $z = \mathcal{O}(x)$, and an observation scale, $\mathcal{T} \in \mathbb{R}$, such that $z : \mathbb{R} \to \mathbb{R}^l$. If
\( T = t \), the observation process is \textit{temporal} and the observable scale is merely the time. Our simple example shows that temporal observations may be misleading when used to detect chaos, even when using a seemingly rigorous test such as the LCE spectrum. A fictitious observer using \( T = \tau \) as the scale would have not suspected that the Lorenz-fed gauged pendulum is a chaotic process.

We further conclude that rescriptive gauge transformations may be used to isolate self-similarities of a dynamical systems. In our example, the Lorenz system remains scale-invariant; i.e. its Hausdorff dimension does not depend on the scale. However, the Lorenz-fed pendulum is \textit{not} scale invariant, and hence is a regular process in disguise.

Example 3 (Stochastic signals, coding, and Kolmogorov complexity)

The preceding example illustrated the fact that the gauged pendulum concept may be used to order pseudochaotic behavior. This is, in fact, only an understatement of the potential of rescriptive gauge theory; this theory can be used not only for ordering pseudochaotic signals, but moreover, transform seemingly stochastic signals into deterministic ones.

Our final example is therefore concerned with illustrating how rescriptive gauge symmetry, and in particular a simple gauged pendulum, may be used to establish some key ideas in modern information and coding theory through the well-known notion of Kolmogorov complexity.

The Kolmogorov complexity (also known as Kolmogorov-Chaitin complexity, stochastic complexity, and algorithmic entropy) of an object is a measure of the computational resources needed to specify the object \([19, 20, 8]\). In other words, the complexity of a string is the length of the string’s shortest description in some fixed description language. It can be shown that the Kolmogorov complexity of any string cannot be too much larger than the length of the string itself. Strings whose Kolmogorov complexity is small relative to the string’s size are not considered to be complex. The sensitivity of complexity
Figure 6: The seemingly irregular behavior of the Lorenz-fed gauged pendulum, shown in Fig. 3, can be regularized into harmonic oscillations by a rescriptive gauge transformation.

relative to the choice of description “language” is what the current example is about. To that end, consider the gauge pendulum

\[ \dot{q} = wp, \quad \dot{p} = -wq \]  

(102)

where here the rescriptor \( w \) is a band-limited white noise, that is, a white noise going through a zero-order hold with some sampling frequency \( T_w \) and power spectral density \( W \). Model (102) can be de-rescribed by \( d\tau = wd\tau \).

Let us compare the representation of the “strings” \( q \) and \( p \) using the “languages” \( t \), time, and \( \tau \), a random walk obtained by integrating \( w \) (i.e., a stochastic signal in its own right). This comparison is depicted
in Fig. 7 for $T_w = 0.1$ time units and $W = 0.1$. Fig. 7a shows the signal $q(t)$, which should be compared to the signal $q(\tau)$, shown in Fig. 7b. Similarly, compare $p(t)$, Fig. 7c, to $p(\tau)$, Fig. 7d.

![Figure 7](image_url)

Figure 7: “Stochastic” signals transformed into harmonic oscillations by a rescriptive gauge transformation.

Although $q(t)$ and $p(t)$ seem stochastic and therefore Kolmogorov-complex in the “language” $t$, their alleged complexity vanishes when the “language” $\tau$ is used, and the stormy stochasticity vanishes into harmonic oscillations, implying much reduced Kolmogorov complexity. This phenomenon has practical value in terms of coding theory: Signals may be coded using the “code” $t$ and de-coded using the “key” $\tau$.

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4 Descriptive Gauge Symmetry

A more “benign” gauge symmetry can be detected by applying the rationale of the Yang-Mills gauge theory of infinite-dimensional systems on finite-dimensional dynamics. In this case, the problem can be defined as follows: Given a rescribed dynamical system, of the form

\[ \tilde{f}_t \circ f = g(x, u(x, t)) = \frac{dx}{dt}, \]

find a descriptive gauge function \( \Psi_j \), possibly different for each rescriptor component \( u_i \), satisfying

\[ d\Psi_j = H_i(x, dx, \dot{x}, u_i(x, t), t, dt), \quad (103) \]

such that the gauge automorphism

\[ \tilde{f}_\Psi \circ \varphi(\xi_i(x_0, t_0), t) = \varphi(\xi_i(x_0, t_0), t) \quad (104) \]

holds for some \( \xi_i(x, t), i \in [1, \ldots, n] \), where \( \mathcal{F} \) is some abstract configuration manifold embedded in \( \mathbb{R}^n \). If \( \exists \Psi, j \in [1, \ldots, n] \) satisfying (103) such that (104) holds, then we shall say that system (2) exhibits descriptive gauge symmetry under the descriptive gauge transformation (103). In this case each \( \Psi_j \) - and not the rescriptor, as in rescriptive gauge symmetry - becomes either a static or a dynamic descriptive gauge function.

A descriptive gauge symmetry of order \( k \) or simply partial descriptive gauge symmetry comes about when the descriptive gauge transformation does not affect \( k \) state variables, \( k < n \), viz.

\[ \tilde{f}_\Psi \circ \varphi(\xi_i(x, t_0), t) = \varphi(\xi_i(x, t_0), t), \quad i \in \mathbb{N}_p. \quad (105) \]

4.1 Newtonian Systems Revisited

Consider the Newtonian system

\[ \ddot{q}(t) + f[q(t)] = u[q(t), \dot{q}(t), t] \quad (106) \]
with the configuration manifold $Q$, $\mathbf{q} \in Q$, and the tangent bundle $TQ = \mathbb{R}^{n/2} \times \mathbb{R}^{n/2}$, so that $(\mathbf{q}, \dot{\mathbf{q}}) \in TQ$, $\mathbf{f} : Q \to \mathbb{R}^{n/2}$, and $\mathbf{u} : Q \times \mathbb{R} \to \mathbb{R}^{n/2}$ is the rescriptor. Let

$$\mathbf{q} = \gamma[\mathbf{x}(t), t] \quad (107)$$

be the solution of (106), where $\mathbf{x} : \mathbb{R} \to \mathcal{M} \subseteq \mathbb{R}^n$ are the variational coordinates. The velocity vector field is then given by the Lagrangian derivative

$$\dot{\mathbf{q}} = \beta[\mathbf{x}(t), \dot{\mathbf{x}}(t), t] = \frac{\partial \gamma[\mathbf{x}(t), t]}{\partial t} + \frac{\partial \gamma[\mathbf{x}(t), t]}{\partial \mathbf{x}} \dot{\mathbf{x}}, \quad (108)$$

or, stated in terms of field theory, the gauge covariant derivative of the configuration vector field,

$$\mathbf{D}_t \gamma = \partial_t \gamma + \nabla_{\mathbf{x}} \gamma \cdot \mathbf{D}_t \mathbf{x}, \quad (109)$$

and the velocity vector field,

$$\mathbf{D}_t \beta = \partial_t \beta + \nabla_{\mathbf{x}} \beta \cdot \mathbf{D}_t \mathbf{x}. \quad (110)$$

Denote the convective (sometimes also called advective) term by

$$\Psi = \frac{\partial \gamma[\mathbf{x}(t), t]}{\partial \mathbf{x}} \dot{\mathbf{x}}. \quad (111)$$

Substituting (107) and (108) into (106) transforms (106) into the Gauss-Poisson equations

$$\frac{d\mathbf{x}(t)}{dt} = P^T(\mathbf{x}) \left\{ \left[ \frac{\partial \gamma}{\partial \mathbf{x}} \right]^T (\mathbf{u} - \frac{d\Psi}{dt}) - \left[ \frac{\partial \beta}{\partial \mathbf{x}} \right]^T \Psi \right\} \quad (112)$$

where $P$ is the $n \times n$ skew-symmetric Poisson matrix, whose entries are the Poisson brackets, $\{x_i, x_j\}$.

We note that $\mathbf{q}$ – the physical trajectory on the configuration manifold – remains invariant under any selection of $\Psi$. Thus, we are in the liberty of choosing a descriptive gauge function vector of the form

A derivative taken with respect to a moving coordinate system. Alternatively, this operation is sometime referred to as the substantive derivative or Stokes derivative. Fluid dynamicists prefer the notation $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$, where $\mathbf{v}$ is the velocity vector field.
\[ \Psi = \begin{cases} \mathbf{W}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t), \dot{\mathbf{x}}, t), & \mathbf{u}(\mathbf{x}, t) \neq 0 \\ 0, & \mathbf{u}(\mathbf{x}, t) = 0, \end{cases} \]  
(113)

inducing an affine descriptive gauge transformation of the form

\[ d\Psi = \mathbf{H}(\mathbf{x}, d\mathbf{x}, \dot{\mathbf{x}}, d\dot{\mathbf{x}}, t)dt, \]  
(114)

with

\[ H(\mathbf{x}, d\mathbf{x}, \dot{\mathbf{x}}, d\dot{\mathbf{x}}, t) = \partial_t \Psi + \nabla_\mathbf{x} \Psi \cdot \mathbf{x} + \nabla_{\dot{\mathbf{x}}} \Psi \cdot \dot{\mathbf{x}}, \]  
(115)

such that the gauge automorphism

\[ \mathcal{F}_\Psi \circ \varphi(\mathbf{q}(\mathbf{x}_0, t_0), t) = \varphi(\mathbf{q}(\mathbf{x}_0, t_0), t) \]  
(116)

holds. We note that the gauge \( \Psi \) in the Newtonian context has dimensions of velocity, and hence can be referred to as the \textit{gauge velocity}. Recall that we have made a similar observation regarding repressive gauge symmetry in Newtonian systems (cf. \( \S 3.2 \)).

We see that Newtonian systems exhibit partial descriptive gauge symmetry, so that trajectories in the configuration space remain \textit{invariant} under a selection of a particular descriptive gauge function. Stated more eloquently, \( \mathbf{q} = \gamma[\mathbf{x}(t), t] \) remains invariant under the symmetry transformation

\[ \partial_t \gamma \mapsto \mathbf{D}_t \gamma = \partial_t \gamma + \Psi. \]  
(117)

The gauge group \( \mathcal{G} \) therefore consists of real valued functions on \( \mathbb{R}^{n/2} \), with the group operation being addition. An element \( \Psi \) acts on the velocity vector field according to the rule (117).

Eq. (112) is not necessarily integrable, and may possess no “classical” integrals whatsoever. However, regardless of the particular properties of the original system (106), the variational system (112) must \textit{always} satisfy the constraint (118). If we choose \( \Psi \equiv 0 \), then (118) becomes

\[ \frac{\partial \gamma[\mathbf{x}(t), t]}{\partial \mathbf{x}} \dot{\mathbf{x}} = 0, \]  
(118)

which may be viewed as a \textit{hidden integral} emanating from the descriptive gauge symmetry.
We emphasize that our hidden symmetry is not confined to systems in which the homogenous solution of $\ddot{\mathbf{q}} + f(\mathbf{q}) = 0$ can be found. Although there are important realms of science in which a solution to this system does exist - the most notable being the case where $f = \nabla R$, where $R$ is an inverse square gravitational potential emerging in Keplerian orbital mechanics [10] - in many other instances $\gamma$ cannot be found in closed form. This stems from the fact that the distinction between $f$ and $\mathbf{u}$, the rescriptor, is really an artificial one; we can always take $\chi = \mathbf{u} - f$, so that now

$$\ddot{\mathbf{q}}(t) = \chi[\mathbf{q}(t), \dot{\mathbf{q}}(t), t].$$

(119)

Letting $\mathbf{x} = [x_1^T \ x_2^T]^T$ entails

$$\gamma[\mathbf{x}(t), t] = x_1 t + x_2.$$

(120)

In this case the Gauss-Poisson equations are simply

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0_{n/2 \times n/2} & I \\ I & -tI \end{bmatrix} \begin{bmatrix} \Psi \\ \chi(x_1, x_2, \Psi, t) - \dot{\Psi} \end{bmatrix},$$

(121)

where $I$ is an $n/2 \times n/2$ identity matrix and $0_{n/2 \times n/2}$ is an $n/2 \times n/2$ zero matrix.

For $\Psi \equiv 0$, our hidden integral re-emerges, assuming the particular simple form

$$\dot{x}_1 t + \dot{x}_2 = 0,$$

(122)

for which

$$\dot{x}_1 = \chi(x_1, x_2, t)$$

(123a)

$$\dot{x}_2 = -t\chi(x_1, x_2, t).$$

(123b)

Eqs. (123) are particularly amenable for numerical integration, because in this form constraint (122) should be satisfied; however, due to numerical round off errors, this constraint is violated. An improved numerical integration may be achieved if the integration scheme itself is forced to satisfy this constraint during the integration process.

We shall illustrate these observations in §4.3, discussing a few nu-
numerical examples showing how descriptive gauge symmetry may be
used to improve numerical integration of ordinary differential equa-
tions. Our last section before dwelling upon actual examples deals
with reduction in descriptive gauge theory.

4.2 Reduction using Descriptive Gauge Symmetry

Equivalently to §3.6, we shall conceive a process for reduction using
descriptive gauge symmetry. To that end, we re-write Eq. (112) into
the following form:

\[ \dot{x}_j(t) = \sum_{i=1}^{n/2} f_{ji}(x)(u_i - \dot{\psi}_i) - \sum_{i=1}^{n/2} g_{ji}(x)\psi_i, \quad j = 1 \ldots n \quad (124) \]

This yields \( n/2 \) integrals \( x_k, k \in \mathbb{N}^{n/2} \) (there are \( n!/(n/2)!^2 \)) possible combinations of constants of motion obtained by concomitant
\( n!/(n/2)!^2 \) descriptive gauge function components) obtained by solv-
ing the \( n/2 \) first-order ODEs

\[ \sum_{i=1}^{n/2} f_{ji}(x)\dot{\psi}_i + \sum_{i=1}^{n/2} g_{ji}(x)\psi_i = \sum_{i=1}^{n/2} f_{ji}(x)u_i. \quad (125) \]

The freedom to reduce system (112) stems from the existence of the
descriptive gauge function \( \Psi \). If we fix the gauge - a straightforward
selection would be \( \Psi = 0 \), as Lagrange himself had advocated in
his memoirs [21, 22, 23] - this freedom will be lost, and hence the
possibility for reduction. This process is illustrated in the following
section.
4.3 Illustrative Examples

Example 4 (Reduction using descriptive gauge symmetry)

Our first example is a simple one, illustrating the concept of reduction using descriptive gauge symmetry. To that end, consider the one-dimensional, second-order ODE

\[ \ddot{q}(t) + q(t) = \sin(t), \quad q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0. \]  

(126)

The general solution is

\[ q(t) = x_1(t)q_1(t) + x_2(t)q_2(t), \]  

(127)

where \( q_1(t) \) and \( q_2(t) \) are the fundamental solutions

\[ q_1(t) = \cos(t), \quad q_2(t) = \sin(t). \]  

(128)

Taking \( \Psi = \Psi(t) \) to be a time-dependant descriptive gauge function, the Gauss-Poisson equations (112) assume the simple form

\[ \dot{x}_1(t) = \Psi(t)\dot{q}_2(t) - q_2(t)[u(t) - \dot{\Psi}(t)] \quad (129a) \]

\[ \dot{x}_2(t) = q_1(t)[u(t) - \dot{\Psi}(t)] - \Psi(t)\dot{q}_1(t) \quad (129b) \]

Substituting (128) into (129) yields ODEs for two possible descriptive gauge functions that will transform system (129) into a single ODE. Thus, letting \( k_1 \) and \( k_2 \) denote arbitrary integration constants, taking \( \Psi(t) = \Psi_1(t) = \left[\frac{t}{2} - \sin(2t)/4 + k_1\right]/\sin t \) will reduce (129) into

\[ x_1 = \text{const.} \]  

(130a)

\[ \dot{x}_2 = -1/2[\cos t\sin t - t - 2k_1]/\sin^2 t \]  

(130b)

and taking \( \Psi(t) = \Psi_2(t) = \left[ -\cos(2t)/4 + k_2\right]/\cos t \) will reduce (129) into

\[ x_2 = \text{const.} \]  

(131a)

\[ \dot{x}_1 = -1/4[2\cos^2 t - 1 - 4k_2]/\cos^2 t. \]  

(131b)
Both (130b) and (131b) are readily solved by quadrature. Systems (130) and (131) will of course both yield the same general solution (127); this is what descriptive gauge symmetry is all about.

Example 5 (Gauss-Poisson variables reduce Hamiltonian drift)

Consider the Hamiltonian

$$H = \frac{1}{2}(\dot{q}^2 + q^2)$$  \hspace{1cm} (132)

of the harmonic oscillator

$$\ddot{q}(t) + q(t) = 0.$$  \hspace{1cm} (133)

We shall compare the numerical integration of this equation in two cases. In the first case, the state variables are chosen in standard form: \(q_1 = q, q_2 = \dot{q}\), so that (133) becomes

$$\dot{q}_1 = q_2, \quad \dot{q}_2 = -q_1,$$  \hspace{1cm} (134)

while in the second case, (133) is re-written in the form \(\ddot{q} = -q\), and the solution is taken as \(\gamma = x_1(t)t + x_2(t)\), with gauge \(\Psi = 0\). Per (123), this yields the state-space representation

$$\dot{x}_1 = -x_1 t - x_2, \quad \dot{x}_2 = x_1 t^2 + x_2 t.$$  \hspace{1cm} (135)

We used MATLAB’s ODE45 integration routine, a 5th-order Runge-Kutta integrator with an adaptive time step, to integrate (134) and (135) with an integration tolerance of \(10^{-5}\) for 5000 time units given \(H = 2.5\).

The time history of the Hamiltonian for both cases is plotted in Fig. 8. Since the ODE45 routine is not a symplectic integrator, the standard selection of states, Eq. (134), causes the Hamiltonian to decrease with time at a rate of about \(5 \cdot 10^{-4}\) units per time unit. This introduces artificial numerical damping into the system, which causes the harmonic oscillations to slowly damp out. However, the same system integrated in form (135) keeps the Hamiltonian fixed. This implies that the Gauss-Poisson formalism may be used to “symplectify” non-
symplectic integrators by a judicious selection of the descriptive gauge function.

Figure 8: Symplectifying a non-symplectic integrator using Gauss-Poisson state variables and a zero descriptive gauge function.

**Example 6 (Descriptive gauge reduces integration errors)**

Our final example shows how descriptive gauge symmetry may be used to reduce the numerical truncation error of ODE integration. We shall ultimately show that using the Gauss-Poisson state variables with an appropriate descriptive gauge can dramatically reduce the numerical truncation errors, and show how such gauge can be found.
Consider, for example, the one-dimensional, second-order ODE

\[ \ddot{q}(t) + 2\xi\omega_n\dot{q}(t) + \omega_n^2 q(t) = u(t), \quad q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0, \] (136)

where \( \omega_n \) is the natural frequency, \( \xi \) is the damping coefficient and the rescriptor \( u(t) \) is piecewise continuous. Assuming an underdamped case (\( \xi < 1 \)),

\[ q(t) = x_1(t)q_1(t) + x_2(t)q_2(t), \] (137)

where \( q_1(t) \) and \( q_2(t) \) are the fundamental solutions

\[ q_1(t) = e^{-\xi\omega_n t} \cos(\omega_d t) \] (138a)

\[ q_2(t) = e^{-\xi\omega_n t} \sin(\omega_d t) \] (138b)

and \( \omega_d = \omega_n\sqrt{1 - \xi^2} \). Taking \( \Psi = \Psi(t) \) to be a time-dependant descriptive gauge function, the Gauss-Poisson equations (112) assume the simple form

\[ \dot{x}_1(t) = \frac{\Psi(t)\dot{q}_2(t) - q_2(t) \left[ u(t) - \dot{\Psi}(t) - 2\xi\omega_n\Psi(t) \right]}{w[q_1(t), q_2(t)]} \] (139a)

\[ \dot{x}_2(t) = \frac{q_1(t) \left[ u(t) - \dot{\Psi}(t) - 2\xi\omega_n\Psi(t) \right] - \Psi(t)\dot{q}_1(t)}{w[q_1(t), q_2(t)]} \] (139b)

where \( w[q_1(t), q_2(t)] \) is the Wronskian determinant,

\[ w[q_1(t), q_2(t)] = \begin{vmatrix} q_1(t) & q_2(t) \\ \dot{q}_1(t) & \dot{q}_2(t) \end{vmatrix}. \] (140)

The initial conditions for this system are then

\[ x_1(t_0) = \frac{-q_0\dot{q}_2(t_0) + q_2(t_0)\dot{q}_0 - q_2(t_0)\Psi(t_0)}{\dot{q}_1(t_0)q_2(t_0) - \dot{q}_2(t_0)q_1(t_0)} \]

\[ x_2(t_0) = -\frac{-q_0\dot{q}_1(t_0) + q_1(t_0)\dot{q}_0 - q_1(t_0)\Psi(t_0)}{\dot{q}_1(t_0)q_2(t_0) - \dot{q}_2(t_0)q_1(t_0)} \] (141)
From the discussion in §4.1 we know that there is only a single solution for \( q(t) \) for any given initial conditions; thus, per the descriptive gauge symmetry, \( q(t) \) must remain invariant to any selection of the gauge function \( \Psi \). However, \( \Psi \) may be used as a tuning function for mitigating the numerical integration error. To that end, we define the numerical integration error of some state variable \( \cdot \) as the difference between the true solution and the numerical solution:

\[
e_{\cdot} = (\cdot)_{\text{true}} - (\cdot)_{\text{numerical}}
\]

In this example we will demonstrate how to mitigate the numerical integration error of a (fixed-step) 4th order Runge-Kutta integrator (RK4) by several orders of magnitudes. This merit is achievable by applying gauge-optimized integration. To that end, let us re-write Eqs. (139) into

\[
\begin{align*}
\dot{x}_1(t) &= f(t, \Psi) \\
\dot{x}_2(t) &= g(t, \Psi)
\end{align*}
\]  

(143a, 143b)

Obviously, in the linear case discussed herein, the transformation into the Gauss-Poisson equations has transomed the ODE integration problem into a simple quadrature, whose accuracy can be controlled by a proper selection of a time-dependant descriptive gauge function.

The integration errors resulting from numerically integrating (143) are given by

\[
\begin{align*}
e_{x_1} &= -\frac{1}{90} h^5 f^{(4)}[\xi, \Psi(\xi)] \\
e_{x_2} &= -\frac{1}{90} h^5 g^{(4)}[\xi, \Psi(\xi)]
\end{align*}
\]  

(144a, 144b)

where \( \xi \in (t_0, t) \). The total integration error of \( q(t) \) is now calculated as follows:

\[
e_q = q_1(t)e_{x_1} + q_2(t)e_{x_2} = q_1(t)[-\frac{1}{90} h^5 f^{(4)}(\xi, \Psi)] + q_2(t)[-\frac{1}{90} h^5 g^{(4)}(\xi, \Psi)].
\]  

(145)
If some $\Psi^*$ could be found for which $e_q \equiv 0$, $\forall \xi \in (t_0, t)$, then the only remaining integration error of $q(t)$ would be the numerical round-off error. Indeed, such $\Psi^*$ can be quite straightforwardly found. For example, if the forcing term is of the form

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{K_z} a_k \cos(k\omega t) + \sum_{k=1}^{K_s} b_k \sin(k\omega t),$$  \hspace{1cm} (146)$$

then $\Psi^*$ may be chosen as a Fourier series as well:

$$\Psi^*(t) = \frac{A_0}{2} + \sum_{n=1}^{N_c} \left[ A_n \cos(n\omega t) \right] + \sum_{n=1}^{N_s} \left[ B_n \sin(n\omega t) \right]$$ \hspace{1cm} (147)

Substituting (147) into (145) and solving for the coefficients by requiring $e_q = 0$ yields, after some algebra, that

$$\Psi^* = \sum_{n=1}^{N} \frac{\rho_2(n)}{\rho(n)} \omega_n \left[ a_n \cos(nt\omega) - b_n \sin(nt\omega) \right] + \sum_{n=1}^{N} \frac{\rho_1(n)}{\rho(n)} \omega_n \xi \left[ a_n \sin(nt\omega) + b_n \cos(nt\omega) \right]$$ \hspace{1cm} (148)

where

$$\rho_1 = 4[32\omega_n^6\xi^6 + (-40\omega_n^6 + 32n^2\omega_n^4\omega_0^2)\xi^4 + (-24n^2\omega_n^2\omega_0^2 + 14\omega_n^6 + 10n^4\omega_n^2\omega_0^4)\xi^2 + 5\omega_n^6n^2 + 5n^4\omega_n^2\omega_0^4 + 7n^2\omega_n^4\omega_0^2 - \omega_n^6]$$

$$\rho_2 = -4n(16\omega_n^6\xi^6 + (16^2\omega_n^4\omega_0^2 - 16\omega_n^6)\xi^4 + (-12n^2\omega_n^2\omega_0^2 + 4\omega_n^6)\xi^2 + 5\omega_n^6n^2 + 11n^2\omega_n^4\omega_0^2 + \omega_n^6 + 15n^4\omega_n^2\omega_0^4)$$

$$\rho = (256\omega_n^8\xi^8 + (320\omega_n^6\omega_0^2n^2 - 384\omega_n^8)\xi^6 + (176\omega_n^8 + 160\omega_n^4\omega_0^4n^4 - 320\omega_n^6\omega_0^2n^2)\xi^4 + (-24\omega_n^8 - 120\omega_n^4\omega_0^4n^4 + 80\omega_n^6\omega_0^2n^2)\xi^2 + 20\omega_n^6\omega_0^4n^2 + \omega_n^8 + 110\omega_n^4\omega_0^4n^4 + 100\omega_n^2\omega_0^6n^6 + 25\omega_0^8n^8).$$  \hspace{1cm} (149)

For illustration, if we desire to integrate numerically

$$\ddot{q} + q = \sin 2t, \quad q(0) = 0, \quad \dot{q}(0) = 0,$$ \hspace{1cm} (150)
then the descriptive gauge function yielding minimum numerical truncation will be, based on (148), simply

$$\Psi^* = -\frac{40}{121}\cos(2t).$$

(151)

In Figure 9 we depict a comparison of integration errors between the gauge-optimized integration utilizing the Gauss-Poisson equations with the optimal descriptive gauge function (151) and the standard choice of state variables $q_1 = q, q_2 = \dot{q}$. As can be plainly seen, the gauge-optimized integration decreases the integration error by three orders of magnitude in the examined time interval. Moreover, the integration error using the standard state variables is diverging, while the error of the gauge-optimized integration is bounded. Therefore, for a larger time interval, the use of gauge-optimized integration, utilizing the concept of descriptive gauge symmetry, becomes increasingly important.

5 Summary and Conclusions

This paper described how gauge theory can be adapted for finite-dimensional dynamical systems. We have defined gauge symmetry in a very broad context, and distinguished between two fundamental manifestations of gauge symmetry:

(i) Descriptive gauge symmetry results from an action of a one-parameter Lie group, yielding an Abelian Lie algebra. A descriptive gauge symmetry transformation is then an infinitesimal change of the independent variable, which renders the system integrable via reduction.

(ii) Descriptive gauge symmetry is an invariance of some configuration space under a gauge transformation of the covariant derivative. In this case the symmetry group consists of real-valued functions on the Euclidean space, with the group operation being addition.

The gauge conversation leads to a few practical conclusions. We first note that gauge symmetry is ubiquitous in a myriad of scientific fields. Gauge theory for finite-dimensional system may be thus viewed
Figure 9: The numerical integration error of the gauge-optimized integration considerably reduces the integration error compared to a standard choice of state variables.

as a generalization of dynamical systems theory into the realm of group theory, unifying various physical phenomenon into simple generating models.

Furthermore, the gauge-theoretic tools may be used to improve our understanding of chaos, randomness and their inter-relations. We discussed a few simple examples showing how a change of scale can lead to pattern evocation in seemingly chaotic and/or stochastic systems.

Finally, gauge-theoretic tools are important for improving the accuracy of numerical integration. The gauge freedom allows re-shaping of the phase space so as to render it tractable for numerical integration.
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References

[1] Arnold, V. I., *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New-York, 1983.

[2] Arribas, M., Elipe, A., Floría, L., and Riaguas, A., “Oscillators in Resonance p:q:r”, *Chaos, Solitons and Fractals*, Vol. 27, 2006, pp. 1220-1228.

[3] Aulbach, B., and Kieninger, B., “On Three Definitions of Chaos”, *Nonlinear Dynamics and Systems Theory*, Vol. 1, No. 1, 2001, pp. 23-37.

[4] Berkovich, L. M., “Method of Exact Linearization of Nonlinear Autonomous Differential Equations of Second Order”, *Journal of Applied Mathematics and Mechanics*, Vol. 43, No. 4, 1979, pp. 629-638.

[5] Bruni, C., DiPillo, G., and Koch, G., “Bilinear Systems: An Appealing Class of ‘Nearly Linear’ Systems in Theory and Applications”, *IEEE Transactions on Automatic Control*, Vol. 19, No. 4, August 1974, pp. 334-348.

[6] Campa, A., Giansanti, A., and Tenenbaum, A., “Partial Lyapunov Exponents in Tangent Space Dynamics”, *Journal of Physics A*, Vol. 25, No. 1, 1992.

[7] Cartwright, J. H.E., and Piro, O., “The Dynamics of Runge-Kutta Methods”, *International Journal of Bifurcation and Chaos*, Vol. 2, 1992, pp. 427-449.

[8] Chaitin, G. J., *The Limits of Mathematics*, Springer-Verlag, London, 2003
[9] Clary, D. C., “Geometric Phase in Chemical Reactions”, *Science*, Vol. 309, No. 5738, pp. 1195 - 1196, August 2005.

[10] Efroimsky, M., “Gauge Freedom in Orbital Mechanics”, *Annals of the New-York Academy of Science*, Vol. 1065, 2005, pp. 346-374.

[11] Efroimsky, M., “Equations for the Orbital Elements. Hidden Symmetry,” *Preprint No 1844 of the Institute of Mathematics and its Applications, University of Minnesota*, , No. [http://www.ima.umn.edu/preprints/feb02/1844.pdf](http://www.ima.umn.edu/preprints/feb02/1844.pdf) 2002.

[12] Efroimsky, M. and Goldreich, P., “Gauge Symmetry of the N-body Problem in the Hamilton-Jacobi Approach,” *Journal of Mathematical Physics*, Vol. 44, 2003, pp. 5958 – 5977.

[13] Efroimsky, M. and Goldreich, P., “Gauge Freedom in the N-body Problem of Celestial Mechanics,” *Astronomy & Astrophysics*, Vol. in press, 2004.

[14] Euler, L., *Recherches sur la question des inegalites du mouvement de Saturne et de Jupiter, sujet propose pour le prix de l'annee*, Piece qui a remporte le prix de l'academie royale des sciences (1748).
For modern edition see: L. Euler *Op. mechanica et astronomic*. (Birkhauser-Verlag, Switzerland, 1999).

[15] Euler, L., *Theoria motus Lunae exhibens omnes ejus inaequalitates etc.*, Impensis Academiae Imperialis Scientarum Petropolitanae. St.Petersburg, Russia (1753).
For modern edition see: L. Euler *Op. mechanica et astronomic*. (Birkhauser-Verlag, Switzerland, 1999).

[16] Frigg, R., “Chaos and Randomness: An Equivalence Proof of a Generalized Version of the Shannon Entropy and the Kolmogorov-Sinai Entropy for Hamiltonian Dynamical Systems”, *Chaos, Solitons and Fractals*, Vol. 28, 2006, pp. 26-31.

[17] Gurfil, P., “Analysis of J2-Perturbed Motion using Mean Non-Osculating Orbital Elements”, *Celestial Mechanics and Dynamical Astronomy*, Vol. 90, No. 3-4, November 2004, pp. 289-306.
[18] Khaneja, N., and Glaser, S. J., “Constrained Bilinear Systems”, Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, December 2002.

[19] Kreinovich, V., and Kunin, I. A., “Kolmogorov Complexity and Chaotic Phenomena”, International Journal of Engineering Science, Vol. 41, 2003, pp. 483-493.

[20] Kunin, I. A., “Gauge Theories in Mechanics”, in Trends in Application of Pure Mathematics to Mechanics, Lecture Notes in Physics, Vol. 249, 1986, pp. 246-269.

[21] Lagrange, J. L., Sur le Problème de la détermination des orbites des comètes d’après trois observations, 1er et 2-ième mémoires., Nouveaux Mémoires de l’Académie de Berlin (1778).
Later edition: in Œuvres de Lagrange. Vol. IV, Gauthier-Villars, Paris 1869.

[22] Lagrange, J. L., Sur la théorie des variations des éléments des planètes et en particulier des variations des grunds axes de leurs orbites, Lu le 22 août 1808 à l’Institut de France (1808).
Later edition: in Œuvres de Lagrange. Vol. VI, pp. 713 - 768, Gauthier-Villars, Paris 1877.

[23] Lagrange, J. L., Second mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de la mécanique, Lu le 19 février 1810 à l’Institut de France (1810).
Later edition: In Œuvres de Lagrange. Vol. VI, pp. 809 - 816, Gauthier-Villars, Paris 1877.

[24] Leach, P. G. L., Feix, M. R., and Bouquet, S., “Analysis and Solution of a Nonlinear Second-Order Differential Equation through Rescaling and through a Dynamical Point-of-View”, Journal of Mathematical Physics, Vol. 29, No. 12, December 1989, pp. 2563-2569.

[25] Lemmer, R. L., and Leach, P. G. L., “The Painlevé Test, Hidden Symmetries and the Equation $y'' + yy' + Ky^3 = 0$”, Journal of Physics A: Mathematical and General, Vol. 26, 1993, pp. 5017-5024.
[26] Li, M., and Vitanyi, P., *An Introduction to Kolmogorov Complexity and Its Applications*, Second Edition, Springer-Verlag, New York, 1997.

[27] Lowenstein, J. H., Poggiaspalla, G., and Vivaldi, F., “Sticky Orbits in a Kicked-Oscillator Model”, *Dynamical Systems: An International Journal*, Vol. 20, No. 4, December 2005, pp. 413 - 451.

[28] Marsden, J. E., and Scheurle, J., “Pattern Evocation and Geometric Phases in Mechanical Systems with Symmetry”, *Dynamics and Stability of Systems*, Vol. 10, No. 4, 1995, pp. 315-338.

[29] Marsden, J. E., Scheurle, J., and Wendlandt, J. M., “Visualization of Orbits and Pattern Evocation for the Double Spherical Pendulum”, *Proceedings of the ICIAM Conference*, Hamburg, Germany, July 1995.

[30] Marsden, J. E., and Ratiu, T.S., “Introduction to Mechanics and Symmetry”, 2nd Edition, Springer, New York, 2002.

[31] Marsden, J. E., and Weinstein, A., “The Hamiltonian Structure of the Maxwell-Vlasov Equations”, *Physica 4D*, 1982, pp. 394-406.

[32] Newman, W. I., and Efroimsky, M., “Multiple Time Scales in Orbital Mechanics”, *Chaos*, Vol. 13, 2002, pp. 476 - 485.

[33] Shimada, I., and Nagashima, T., “A Numerical Approach to Ergodic Problem of Dissipative Dynamical Systems”, *Progress of Theoretical Physics*, Vol. 61, No. 6, June 1979, pp. 1605-1617.

[34] Sándor, Z., Érdi, B., Széll, A., and Funk, B., “The Relative Lyapunov Indicator: An Efficient Method of Chaos Determination”, *Celestial Mechanics and Dynamical Astronomy*, Vol. 90, 2004, pp. 127-138.

[35] Stoer, J., and Bulirsch, R., *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.

[36] Strogatz, S. H., *Nonlinear Dynamics and Chaos*, Westview Press, 1994.
[37] Wolf, A., Swift, J. B., Swinney, H. L., and Vastano, J. A., “Determining Lyapunov Exponents from a Time Series”, Physica D, Vol. 16, 1985, pp. 285-317.

[38] Yamrom, B., Kunin, I., Metcalfe, R., and Chernykh, G., “Discrete Systems of Controlled Pendulum Type”, International Journal of Engineering Science, Vol. 41, 2003, pp. 449-458.

[39] Yamrom, B., Kunin, I. A., Chernykh, G. A., “Centroidal Trajectories and Frames for Chaotic Dynamical Systems”, International Journal of Engineering Science, Vol. 41, 2003, pp. 465-473.

[40] Weyl, H, Gesammelte Abhandlungen, hrsg. v. K. Chandrasekharan, Springer-Verlag, Berlin, Heidelberg, New York, 1968.

[41] Weyl, H., Space, Time, Matter, trans. By H. L. Brose, Dover Publications, 1950.

[42] Weyl, H. Philosophy of Mathematics and Natural Science, Atheneum, New York, 1960.