DISCRETE LINEAR GROUPS CONTAINING ARITHMETIC GROUPS

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Abstract. If $H$ is a simple real algebraic subgroup of real rank at least two in a simple real algebraic group $G$, we prove, in a substantial number of cases, that a Zariski dense discrete subgroup of $G$ containing a lattice in $H$ is a lattice in $G$. For example, we show that any Zariski dense discrete subgroup of $\text{SL}_n(\mathbb{R})$ ($n \geq 4$) which contains $\text{SL}_3(\mathbb{Z})$ (in the top left hand corner) is commensurable with a conjugate of $\text{SL}_n(\mathbb{Z})$.

In contrast, when the groups $G$ and $H$ are of real rank one, there are lattices $\Delta$ in a real rank one group $H$ embedded in a larger real rank one group $G$ and that extends to a Zariski dense discrete subgroup $\Gamma$ of $G$ of infinite co-volume.

1. Introduction

In this paper, we study special cases of the following problem raised by Madhav Nori:

Problem 1 (Nori, 1983). If $H$ is a real algebraic subgroup of a real semi-simple algebraic group $G$, find sufficient conditions on $H$ and $G$ such that any Zariski dense subgroup $\Gamma$ of $G$ which intersects $H$ in a lattice in $H$, is itself a lattice in $G$.

If the smaller group $H$ is a simple Lie group and has real rank strictly greater than one (then the larger group $G$ is also of real rank at least two), we did not know any example when the larger discrete group $\Gamma$ is not a lattice until the recent work of Danciger Guéritaud and Kassel [DGK]. The goal of the present paper is to study the following question, related to Nori’s problem.

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Question 2. If a Zariski dense discrete subgroup of a simple non-compact Lie group \( G \) intersects a simple non-compact Lie subgroup \( H \) in a lattice, is the larger discrete group a lattice in the larger Lie group \( G \)?

Analogous questions have been considered before: (e.g. \([F-K]\), Bass-Lubotzky \([B-L]\), Hee Oh \([O]\), and \([V2]\)). In the present paper, we give some evidence for a positive answer to this question.

Here are two consequences that we have: consider the “top left hand corner” embedding \( \text{SL}_k \subseteq \text{SL}_n \), \((k \geq 3)\).

The embedding is as follows: an \( \text{SL}_k \) matrix \( M \) is thought of as an \( n \times n \) matrix \( M' \) such that the first \( k \times k \) entries of \( M' \) are the same as those of \( M \), the last \( (n-k) \times (n-k) \) entries of \( M' \) are those of the identity \( (n-k) \times (n-k) \) matrix, and all other entries of \( M' \) are zero.

**Theorem 1.** Suppose that \( \text{SL}_3 \) is embedded in \( \text{SL}_n \) (in the “top left hand corner”) as above. Suppose that \( \Gamma \) is a Zariski dense discrete subgroup of \( \text{SL}_n(\mathbb{R}) \) whose intersection with \( \text{SL}_3(\mathbb{R}) \) is a subgroup of \( \text{SL}_3(\mathbb{Z}) \) of finite index. Then, \( \Gamma \) is commensurate to a conjugate of \( \text{SL}_n(\mathbb{Z}) \), and is hence a lattice in \( \text{SL}_n(\mathbb{R}) \).

Here we recall that two groups are called commensurate when their intersection has finite index in each of them. We may similarly embed \( \text{Sp}_k \subseteq \text{Sp}_g \) in the “top left hand corner”, where \( \text{Sp}_g \) is the symplectic group of \( 2g \times 2g \)-matrices preserving the non-degenerate symplectic form \( J_g = \text{diag}(J_2, \ldots, J_2) \) in \( 2g \)-variables, where \( J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Denote by \( \text{Sp}_g(\mathbb{Z}) \) the integral symplectic group.

**Theorem 2.** If \( \Gamma \) is a Zariski dense discrete subgroup of \( \text{Sp}_g(\mathbb{R}) \) whose intersection with \( \text{Sp}_2(\mathbb{R}) \) is commensurate to \( \text{Sp}_2(\mathbb{Z}) \) then a conjugate of \( \Gamma \) is commensurate with \( \text{Sp}_g(\mathbb{Z}) \) and hence \( \Gamma \) is a lattice in \( \text{Sp}_g(\mathbb{R}) \).

The proof of Theorem 1 depends (as does the proof of Theorem 2) on a general super-rigidity theorem for discrete subgroups \( \Gamma \) which contain a “large” enough higher rank lattice. More precisely, our main result is the following.

**Theorem 3 (Main result).** Let \( H \) be a semi-simple subgroup of a simple Lie group \( G \) with \( \mathbb{R} - \text{rank}(H) \geq 2 \). Let \( \Gamma \) be a Zariski dense subgroup of \( G \) whose intersection with \( H \) is an irreducible lattice in \( H \). Let \( G = NAK \) be the Iwasawa decomposition of \( G \) and \( P_0 = AN \). If the
isotropy of $H$ acting on $G/P_0$ is positive dimensional and non-compact at every point of $G/P_0$, then $\Gamma$ is a super-rigid subgroup of $G$.

Here we recall that a lattice $\Delta$ in a semisimple Lie group $H$ is irreducible if for every non-central normal subgroup $N \subseteq H$, $\Delta$ is dense when projected onto $H/N$. One of the referees has pointed out that the super-rigidity of Theorem 3 is still true, if we only assume that the intersection $\Gamma \cap H$ is a super-rigid subgroup of $H$ (thus it can also apply when $H$ has real rank equal to one but is isomorphic, up to compact factors, to $Sp(n,1)$ or the real rank one form of $F_4$). Yehuda Shalom has pointed out to us that in Theorem 3, we may replace the hypothesis that $G$ is simple, by the assumption that $G$ is semi-simple:

**Theorem 4.** Let $H$ be a semi-simple subgroup of a semi-simple Lie group $G$ with $\mathbb{R} - \text{rank}(H) \geq 2$. Let $\Gamma$ be a Zariski dense subgroup of $G$ whose intersection with $H$ is an irreducible lattice in $H$. If the normal subgroup of $G$ generated by $H$ is all of $G$, and if all the isotropy subgroups of $H$ acting on $G/P_0$ are positive dimensional, then $\Gamma$ is a super-rigid subgroup of $G$.

The proof is identical to that of Theorem 3, if the additional hypotheses of Theorem 4 are taken into account.

The conditions of Theorem 3 are satisfied if the dimension of $H$ is sufficiently large (for example, if $\dim(K) < \dim(H)$). We then get the following as a corollary of Theorem 1:

**Theorem 5.** Let $\Gamma$ be a Zariski dense discrete subgroup of a simple Lie group $G$ which intersects a semi-simple subgroup $H$ of $G$ (with $\mathbb{R} - \text{rank}(H) \geq 2$) in an irreducible lattice. Let $K$ be a maximal compact subgroup of $G$ and assume that $\dim(H) > \dim(K)$. Then $\Gamma$ is a super-rigid subgroup of $G$.

Here, super-rigid is in the sense of Margulis [M]. That is, all linear representations - satisfying some mild conditions - of the group $\Gamma$ virtually extend to (i.e. coincide on a finite index subgroup of $\Gamma$ with) a linear representation of the ambient group $G$. If the answer to Question 2 is in the affirmative, then $\Gamma$ must be a lattice and by the Super-rigidity Theorem of Margulis, must satisfy the super-rigidity property; Theorem 5 is therefore evidence that the answer to Question 2 is “yes”.

It was Madhav Nori who first raised the question (to the second named author of the present paper) whether these larger discrete groups containing higher rank lattices have to be lattices themselves (i.e. arithmetic groups, in view of Margulis’ Arithmeticity Theorem). We will in
fact show several examples of pairs of groups \((H, G)\), such that \(\Gamma\) does turn out to be a lattice, confirming Nori’s guess (see Theorem 1 and Corollary 6).

For instance, as a consequence of Theorem 5 we obtain the following.

**Corollary 6.** If \(n \geq 4\) and \(H = \text{SL}_{n-1}(\mathbb{R}) \subseteq \text{SL}_n(\mathbb{R})\) and \(\Gamma\) is a Zariski dense discrete subgroup of \(\text{SL}_n(\mathbb{R})\) which intersects \(\text{SL}_{n-1}(\mathbb{R})\) in a finite index subgroup of \(\text{SL}_{n-1}(\mathbb{Z})\), then a conjugate of \(\Gamma\) in \(\text{SL}_n(\mathbb{R})\) is commensurate to \(\text{SL}_n(\mathbb{Z})\).

Analogously, we prove

**Corollary 7.** If \(g \geq 3\), then under the standard embedding \(\text{Sp}_{g-1} \subseteq \text{Sp}_g\), every Zariski dense discrete subgroup of \(\text{Sp}_g(\mathbb{R})\) which contains a finite index subgroup of \(\text{Sp}_{g-1}(\mathbb{Z})\) is commensurable to a conjugate of \(\text{Sp}_g(\mathbb{Z})\).

The proof of Theorem 5 runs as follows.

We adapt Margulis’ proof of super-rigidity to our situation; the proof of Margulis uses crucially that a lattice \(\Gamma\) acts ergodically on \(G/S\) for any non-compact subgroup \(S\) of \(G\), whereas we do not have the ergodicity available to us. We use instead the fact that the representation of the discrete group \(\Gamma\) is rational on the smaller group \(\Delta\).

Given a representation \(\rho\) of the group \(\Gamma\) on a vector space over a local field \(k'\), we use a construction of Furstenberg to obtain a \(\Gamma\)-equivariant measurable map \(\phi\) from \(G/P_0\) into the space \(\mathcal{P}\) of probability measures on the projective space of the vector space. Using the fact that the isotropy subgroup of \(H\) at any point in \(G/P_0\) is non-compact, we deduce that on \(H\)-orbits, the map \(\phi\) is rational, and by pasting together the rationality of \(\phi\) on many such orbits, we deduce the rationality of the representation \(\rho\).

Theorem 5 implies Corollary 6 as follows: the super-rigidity of \(\Gamma\) in Corollary 6 implies, as in [M], that \(\Gamma\) is a subgroup of an arithmetic subgroup \(\Gamma_0\) of \(\text{SL}_n(\mathbb{R})\). It follows that the \(\mathbb{Q}\)-form of \(\text{SL}_n(\mathbb{R})\) associated to this arithmetic group has \(\mathbb{Q}\)-rank greater than \(n/2\). The classification of the \(\mathbb{Q}\)-forms of \(\text{SL}_n(\mathbb{R})\) then implies that the \(\mathbb{Q}\)-form must be \(\text{SL}_n(\mathbb{Q})\) and that \(\Gamma\) is, up to conjugation, commensurate to a subgroup of \(\text{SL}_n(\mathbb{Z})\). Since \(\Gamma\) is Zariski dense and virtually contains \(\text{SL}_{n-1}(\mathbb{Z})\), it follows from [V1] Corollary (3.8) that \(\Gamma\) is commensurate to \(\text{SL}_n(\mathbb{Z})\).
We now give a list (not exhaustive) of more pairs \((H, G)\) which satisfy the condition \((\dim(H) > \dim(K))\) of Theorem 5.

**Corollary 8.** If \((H, G)\) is one of the pairs

1. \(H = \text{Sp}_g \subseteq \text{SL}_2^g \) with \(g \geq 2\),
2. \(H = \text{SL}_p \times \text{SL}_p \subseteq G = \text{SL}_{2p} \) with \(p \geq 3\)
3. \(H = \text{Sp}_a \times \text{Sp}_a \subseteq G = \text{Sp}_{2a} \) with \(1 \leq a\)

then any Zariski dense discrete subgroup \(\Gamma\) of \(G(\mathbb{R})\) whose intersection with \(H(\mathbb{R})\) is an irreducible lattice, is super-rigid in \(G(\mathbb{R})\).

There are examples of pairs \((H, G)\) satisfying the condition of Theorem 1 which are not covered by Theorem 5. In the cases of the following Corollary, it is easy to check that \(\dim(H) = \dim(K)\) and that the isotropy of \(H\) at any generic point of \(G/P\) is a non-compact Cartan subgroup of \(H\). When \(G = H(\mathbb{C})\), we view \(G\) as the group of real points of a complex algebraic group, and Zariski density of a subgroup \(\Gamma \subseteq G\) is taken to mean that \(\Gamma\) is Zariski dense in \(G(\mathbb{C}) = H(\mathbb{C}) \times H(\mathbb{C})\).

**Corollary 9.** Let \(H\) be a real simple algebraic group defined over \(\mathbb{R}\) with \(\mathbb{R} - \text{rank}(H) \geq 2\) embedded in the complex group \(G = H(\mathbb{C})\). If \(H\) has no compact Cartan subgroup, then every Zariski dense discrete subgroup \(\Gamma\) of \(G\) which intersects \(H\) in a lattice is super-rigid in \(G\).

Notice that Theorem 5 and Theorem 3 allow us to deduce that if \(\Gamma\) is as in Theorem 5 or Theorem 3, then \(\Gamma\)'s is a subgroup of an arithmetic group in \(G\) (see Theorem 14), reducing Nori’s question to the following apparently simpler one:

**Question 3.** If a Zariski dense subgroup \(\Gamma\) of a lattice in a simple non-compact Lie group \(G\) contains a higher rank lattice of a smaller group, is \(\Gamma\) itself a lattice in \(G\)?

We now shortly describe the proof of Theorem 1. If Theorem 5 is to be applied directly, then the dimension of the maximal compact of \(\text{SL}_n(\mathbb{R})\) must be less than the dimension of \(\text{SL}_3(\mathbb{R})\), which can only happen if \(n = 4\); instead, what we will do, is to show that the group generated by \(\text{SL}_3(\mathbb{Z})\) (in the top left hand corner of \(\text{SL}_n(\mathbb{R})\) as in the statement of Theorem 1) and a conjugate of a unipotent root subgroup of \(\text{SL}_3(\mathbb{Z})\) by a generic element of \(\Gamma\), (modulo its radical), is a Zariski dense discrete subgroup of \(\text{SL}_4(\mathbb{R})\). By applying Corollary 5 for the pair \(\text{SL}_3(\mathbb{R})\) and \(\text{SL}_4(\mathbb{R})\), we see that the Zariski dense discrete subgroup \(\Gamma\) of \(\text{SL}_n(\mathbb{R})\) contains, virtually, a conjugate of \(\text{SL}_4(\mathbb{Z})\).
We can apply the same procedure to $\text{SL}_4$ instead of $\text{SL}_3$ and obtain $\text{SL}_5(\mathbb{Z})$ as a subgroup of $\Gamma$, ..., and finally obtain that $\Gamma$ virtually contains a conjugate of $\text{SL}_n(\mathbb{Z})$. This proves Theorem 1. The proof of Theorem 2 is similar: use Corollary 7 in place of Corollary 6.

When $H$ has real rank one, the answer to the counterpart of Question 2 is in the negative:

For $G$ of rank two or higher, work of D. Johnson and J. Millson in [J-M] produces a Zariski dense subgroup $\Gamma$ of $\text{SL}_n(\mathbb{R})$, isomorphic to a lattice in $SO(n - 1, 1)$ (hence of infinite co-volume in $\text{SL}_n(\mathbb{R})$) and intersecting a subgroup $H$ isomorphic to $SO(n - 2, 1)$ in a lattice $\Delta$ (see Remarque 1.3 of [Ben1] and Corollaire 2.10 of [Ben2]).

Even if $G$ has rank one, given a proper subgroup $H \subseteq G$ and a lattice $\Delta \subseteq H$, one can always produce a Zariski dense discrete subgroup $\Gamma$ in $G$ which is not a lattice in $G$, and whose intersection with $H$ is a subgroup of finite index in $\Delta$ (Theorem 15). The method is essentially that of Fricke and Klein [F-K] who produce, starting from Fuchsian groups, Kleinian groups of infinite co-volume, by using a “ping-pong” argument.

At the suggestion of the referee, we make some remarks on extension of these methods to groups over non-archimedean fields. The lattices in such groups are known to be co-compact and therefore do not contain unipotent elements. Therefore, the algebraic methods of of the present paper, proving that certain subgroups are lattices, by exhibiting many unipotent elements, do not work. However, if Theorem is formulated as saying that the subgroup $H$ operates on $G/P_0$ with positive dimensional isotropy groups, then it can be shown by similar methods that a Zariski dense discrete subgroup of $G$ which intersects $H$ in an irreducible lattice, is a super-rigid group.

We end with some remarks on Zariski closures. If $G$ is a connected algebraic group defined over $\mathbb{R}$, then it is known that $G(\mathbb{R})$ is Zariski dense in $G$; hence, if $\Gamma \subseteq G(\mathbb{R})$ is a subgroup, then the Zariski closure of $\Gamma$ in $G$ has the property that the smallest real algebraic group containing $\Gamma$ has finite index in the real points of the complex Zariski closure. For this reason, we abuse notation a little and refer to the real Zariski closure of $\Gamma$ the real points of the Zariski closure of the group $\Gamma$. 

2. Preliminaries on measurable maps

The aim of this section is to recall a few well known facts used in the proof of Theorem 5 and prove the following key fact. In the following Proposition, when we talk of an algebraic subgroup $J$ of $G'(k')$ where $k'$ is an archimedean local field, we mean that $J$ is an algebraic subgroup of group $R_{k'/\mathbb{R}}(G')$ obtained from $G'$ by the Weil restriction of scalars.

**Proposition 10.** Let $\tilde{H}$ be an almost $\mathbb{Q}$-simple, simply connected algebraic group with $\mathbb{R} - \text{rank}(\tilde{H}) \geq 2$ and $\Delta \subseteq \tilde{H}(\mathbb{Z})$ an arithmetic subgroup. Let $\rho: \Delta \to G'(k')$ be a representation, where $G'$ is a linear algebraic group over a local field $k'$ of characteristic zero. Suppose that $S \subset \tilde{H}(\mathbb{R})$ a closed non-compact subgroup and let $J \subset G'(k')$ be an algebraic subgroup. Let $\phi: \tilde{H}(\mathbb{R})/S \to G'(k')/J$ be a Borel measurable map which is $\Delta$-equivariant. Then the map $\phi: \tilde{H}(\mathbb{R})/S \to G'(k')/J$ is a rational map. More precisely:

If $k'$ is an archimedean local field, there exist a homomorphism $\tilde{\rho}: \tilde{H}(\mathbb{R}) \to G'(k')$ of real algebraic groups defined over $\mathbb{R}$ and a point $p \in G'(k')/J$, such that the map $\phi(h) = \tilde{\rho}(h)(p)$ for all $h \in \tilde{H}(\mathbb{R})$ (and is therefore an $\mathbb{R}$-rational map of real varieties).

If $k'$ is a non-archimedean local field, then the map $\phi$ is constant.

An important ingredient in the proof is the following easy generalisation of the Ergodicity Theorem of Moore (see [Z], Theorem (2.2.6)). We give the proof for sake of completeness.

**Lemma 1.** Let $\tilde{H}$ be an almost $\mathbb{Q}$-simple, simply connected algebraic group with $\mathbb{R} - \text{rank}(\tilde{H}) \geq 1$ and $\Delta \subseteq \tilde{H}(\mathbb{Z})$ an arithmetic subgroup. For any closed non-compact subgroup $S \subset \tilde{H}(\mathbb{R})$, the group $\Delta$ acts ergodically on the quotient $\tilde{H}(\mathbb{R})/S$.

The almost $\mathbb{Q}$-simplicity of $\tilde{H}$ implies that there exists a connected absolutely almost simple simply connected group $H_0$ defined over a number field $K$, such that $\tilde{H} = R_{K/\mathbb{Q}}(H_0)$, where $R_{K/\mathbb{Q}}(H_0)$ is the Weil restriction of $H_0$ from $K$ to $\mathbb{Q}$. Consequently,

$$\tilde{H}(\mathbb{R}) = H_0(K \otimes \mathbb{R}) = \prod_{\alpha \in \infty} H_0(K_{\alpha}),$$

where $\infty$ denote the set of equivalence classes of archimedean embeddings of $K$. Let $A \subseteq \infty$ denote the archimedean embeddings $\alpha$ such that $H_0(K_{\alpha})$ is non-compact. Then, $\tilde{H} = H_u \times H^*$ where $H_u := \prod_{\alpha \in \infty \setminus A} H_0(K_{\alpha})$ is a compact group, and $H^* := \prod_{\alpha \in A} H_0(K_{\alpha})$.
Lemma 2. Let $\tilde{H}$ be an almost $\mathbb{Q}$-simple, simply connected algebraic group with $\mathbb{R} - \text{rank}(\tilde{H}) \geq 1$ and $\Delta \subseteq \tilde{H}(\mathbb{Z})$ an arithmetic subgroup. Suppose that $\pi$ is a unitary representation of $\tilde{H}(\mathbb{R})$ on a Hilbert space, such that for any simple factor $H_\alpha$ of $H^*$, the space $\pi^{H_\alpha}$ of vectors of $\pi$ invariant under the subgroup $H_\alpha$ is zero. Then, the space $\pi^S$ of vectors in $\pi$ invariant under the non-compact subgroup $S \subseteq \tilde{H}(\mathbb{R})$ is also zero.

Proof. By the Howe-Moore theorem, (see [Z], Theorem (2.2.20)), there exists a proper function $\Xi : H^* \rightarrow [0, +\infty)$ (where $[0, +\infty)$ is the closed open interval with end points 0 and $+\infty$) with the following property. Let $\pi$ be a unitary representation of $H^*$ on a Hilbert space $V$ such that for any non-compact simple factor $H_\alpha := H_0(K_\alpha)$ the space of invariants $V^{H_\alpha}$ is zero. For $v, w \in V$, denote their inner product by $\langle v, w \rangle$ and define $|v|$ by the formula $|v|^2 = \langle v, v \rangle$. Then, for all $v, w \in V$, and all $g \in H^*$ we have

$$|\langle \pi(g)v, w \rangle| \leq \frac{|v||w|}{\Xi(g)}.$$

Suppose now that $\pi$ is a unitary representation of $\tilde{H}(\mathbb{R})$ on a Hilbert space such that for any non-compact simple factor $H_\alpha$ of $H^*$, the space of $H_\alpha$ invariants in $\pi$ is zero. Let $g \in S \subseteq \tilde{H}(\mathbb{R}) = H^* \times H_u$; we write $g = (g^*, g_u)$ accordingly. Then, by the estimate of the preceding paragraph applied to $g^* \in H^*$ and the vectors $\pi(g_u)v$ in place of $v$, and $v$ in place of $w$, we get:

$$|\langle \pi(g)v, v \rangle| \leq \frac{|v||v|}{\Xi(g^*)}.$$

If $\pi$ has non-zero $S$-invariant vectors $v$ (say, of norm 1), then the foregoing estimate shows that $\Xi(g^*)$ is bounded on $S$, which by the properness of $\Xi$ implies that $S$ is compact; since this is false by assumption, we have proved that $\pi$ does not have any non-zero vectors invariant under $S$.  

We are now ready to complete the proof of Lemma 1.

Proof of Lemma 1. Let $V_0 = L^2[\Delta \setminus \tilde{H}(\mathbb{R})]$ be the space of square integrable functions on $\Delta \setminus \tilde{H}(\mathbb{R})$; the latter space has finite volume (Theorem 1 of [BHC]), and hence contains the space of constant functions. For any simple factor $H_\alpha$ of $H^* \subseteq \tilde{H}(\mathbb{R})$, the space $V_0^{H_\alpha}$ is just the space of constants, by strong approximation (see [M], Chapter (II), Theorem (6.7); in the notation of [M], we may take $B = \{\alpha\}$ to be
a singleton). Consequently, if \( \pi \) denotes the space of functions in \( V_0 \) orthogonal to the constant functions, then \( \pi \) satisfies the assumptions of Lemma 2. Therefore, by Lemma 2, \( \pi^S = 0 \), and hence the only functions on \( \Delta \backslash \tilde{H}(\mathbb{R}) \) invariant under the non-compact group \( S \) are constants. This proves Lemma 1. \( \square \)

We now record a statement which will be used in the proof of Proposition 10. We thank the referee for pointing out the simple proof and the correct formulation of the following lemma.

**Lemma 3.** Let \( \tilde{H} \) be a \( \mathbb{Q} \)-simple, simply connected algebraic group with \( \mathbb{R} \)-rank(\( \tilde{H} \)) \( \geq 1 \) and \( \Delta \subseteq \tilde{H}(\mathbb{Z}) \) an arithmetic subgroup. Suppose that \( s \in \tilde{H}(\mathbb{R}) \) generates an infinite discrete subgroup and let \( \tau : s^{\mathbb{Z}} \rightarrow \mathbb{Z} \) be an isomorphism. Then, there is no \( s^{\mathbb{Z}} \)-equivariant Borel measurable map

\[
\phi^* : \Delta \backslash \tilde{H}(\mathbb{R}) \rightarrow \mathbb{Z}.
\]

**Proof.** The image (push-out) of the Haar measure on \( \tilde{H}(\mathbb{R})/\Delta \) under an \( s^{\mathbb{Z}} \)-equivariant Borel measurable map \( \phi^* \) gives a finite \( \mathbb{Z} \) invariant measure on \( \mathbb{Z} \), which cannot exist. \( \square \)

The following is a version of the Margulis super-rigidity theorem, except that the Zariski closure of the image \( \rho(\Delta) \) is not assumed to be an absolutely simple group and that \( \rho(\Delta) \) is not assumed to have non-compact closure in \( G'(k') \) if \( k' \) is archimedean.

**Theorem 11** (Margulis). Let \( \tilde{H} \) be a \( \mathbb{Q} \)-simple simply connected algebraic group defined over \( \mathbb{Q} \) of \( \mathbb{R} \)-rank(\( \tilde{H} \)) \( \geq 2 \), \( \Delta \subseteq \tilde{H}(\mathbb{Z}) \) a subgroup of finite index and \( \rho : \Delta \rightarrow G'(k') \) a homomorphism into a linear algebraic group over a local field \( k' \) of characteristic zero.

1. If \( k' \) is archimedean, then the map \( \rho(\Delta) \) coincides, on a subgroup of finite index, with a representation \( \tilde{\rho} : \tilde{H}(\mathbb{R}) \rightarrow G'(k') \).
2. If the local field \( k' \) is non-archimedean, then \( \rho(\Delta) \) is contained in a compact subgroup of \( G'(k') \).

**Proof.** The usual statement of Margulis’ super-rigidity says that if \( \rho \) is a homomorphism of an irreducible lattice \( \Gamma \) in a real semi-simple linear Lie group \( H \) without compact factors, then representations from \( \Gamma \) into \( G'(k') \) with Zariski dense image, where \( G' \) is an absolutely simple group, extends to a smooth representation of \( H \) into \( G'(k') \) (see [M], Chapter VIII, Theorem (C)). However, if \( \tilde{H} \) is the group of real points of a \( \mathbb{Q} \)-simple simply connected algebraic group, then \( \tilde{H}(\mathbb{R}) \) may have compact factors and hence \( \tilde{H}(\mathbb{Z}) \) (or a finite index subgroup of \( \tilde{H}(\mathbb{Z}) \)) may have representations whose Zariski closures have compact factors. Even so,
representations of $\tilde{H}(\mathbb{Z})$ do extend to $\tilde{H}(\mathbb{R})$ under the hypotheses of Theorem 11.

\[ \square \]

We can now prove the main result (Proposition 10) of this section.

**Proof of Proposition 10.** Suppose first that $k'$ is archimedean. Let $\Delta' \subseteq \Delta$ be a subgroup of finite index such that there exists (according to Theorem 11 quoted above) a representation $\tilde{\rho} : \tilde{H}(\mathbb{R}) \to G'(k')$ which coincides with $\rho$ on $\Delta'$. Consider the map $\phi^*(h) = \tilde{\rho}(h)^{-1}(\phi(h))$ from $\tilde{H}(\mathbb{R})$ into the quotient $G'(k')/J$. Then, for all $\delta \in \Delta'$, $h \in \tilde{H}(\mathbb{R})$, $s \in S$, we have $\phi^*(\delta h) = \phi^*(h)$ and $\phi^*(hs) = \tilde{\rho}(s)^{-1}(\phi^*(h))$. That is, the map $\phi^*$ is $\Delta'$ invariant and $S$-equivariant for the action of $\tilde{H}(\mathbb{R})$ on $G'(k')/J$ via the representation $\tilde{\rho}$.

The representation $\tilde{\rho}$ is algebraic; moreover, since $k'$ is archimedean, by assumption the group $J$ is a real algebraic subgroup of $G'(k')$, and hence the action of $\tilde{H}(\mathbb{R})$ on $G'(k')/J$ is smooth. Let $S_1$ denote the Zariski closure of the image $\tilde{\rho}(S)$. The $S_1$-action on $G'(k')/J$ is smooth, hence the quotient $S_1 \backslash G'(k')/J$ is countably separated. On the other hand, by Lemma 1, the action of $S$ on $\Delta' \backslash \tilde{H}(\mathbb{R})$ is ergodic. Hence, by Proposition (2.1.11) of [Z], the image of $\phi^*$ is essentially contained in an $S_1$-orbit i.e. there exists a Borel set $E$ of measure zero in $\tilde{H}(\mathbb{R})/S$, such that the image under $\phi$ of the complement of $E$ is contained in an $S_1$-orbit.

Since $S$ is a non-compact Lie group, $S$ contains an element $s$ of infinite order which generates a discrete non-compact subgroup. We may replace $S$ by the closure of the group generated by the element $s$ and assume that $S$ is abelian. Similarly, we may replace $S_1$ by the Zariski closure of the image of $s^\mathbb{Z}$ and assume that $S_1$ is also abelian. Hence the $S_1$-orbit of the preceding paragraph is of the form $S_1/S_2$ with $S_2$ an algebraic subgroup of the abelian group $S_1$.

**Case 1:** Suppose that the inverse image $S' = S \cap \tilde{\rho}^{-1}(S_2)$ is a non-compact subgroup. By Lemma 1, the group $S'$ acts ergodically on $\Delta \backslash \tilde{H}(\mathbb{R})$; therefore, the map $\phi^*$ - being $S'$ invariant - is constant: $\phi^*(\Delta \backslash \tilde{H}(\mathbb{R})) = \{ p \}$ for some point $p \in G'(k')/J$. That is $\phi(h) = \tilde{\rho}(h)(p)$, and is rational.

**Case 2:** Suppose that $S'$ is compact. Since $s^\mathbb{Z}$ generates a discrete non-compact subgroup and $S'$ is compact, the image $s_1^\mathbb{Z} := \tilde{\rho}(s^\mathbb{Z})$ also generates a discrete non-compact subgroup in $S_1/S_2$ ($\tilde{\rho}$ being an algebraic, hence continuous, map). Hence $s_1^\mathbb{Z}$-orbits in $S_1/S_2$ are closed so that the space $s_1^\mathbb{Z} \backslash S_1/S_2$ is countably separated. By Lemma 1, $s^\mathbb{Z}$ acts ergodically on $\Delta \backslash \tilde{H}(\mathbb{R})$. By applying Proposition (2.1.11) of [Z] to the
$s_1^Z$-invariant map $\bar{\phi} : \Delta \backslash \tilde{H}(\mathbb{R}) \to s_1^Z \backslash S_1 / S_2$, we deduce that the image of $\phi^*$ is essentially contained in an orbit of $s_1^Z$ in the quotient group $S_1 / S_2$. Then, by Lemma 3, it follows that $\phi^*$ cannot exist and we are in Case 1.

If $k'$ is non-archimedean, then by Theorem 11, the image $\rho(\Delta)$ is contained in a compact group $K$ which acts smoothly on $G'(k') / J$ so that $K \backslash G'(k') / J$ is countably separated. The group $\Delta$ acts ergodically on $\tilde{H}(\mathbb{R})$. Hence, by Proposition (2.1.11) [Z], the $S$-invariant map $\bar{\phi} : \Delta \backslash \tilde{H}(\mathbb{R}) \to K \backslash G'(k') / J$ is essentially constant, and therefore the image of $\phi$ is essentially contained in an orbit of $K$.

Since $K$ is a compact subgroup of the $p - \text{adic}$ group $G'(k')$, it has a decreasing sequence of open subgroups $(K_n)_{n \geq 1}$ (of finite index in $K$) such that the intersection $\cap_{n \geq 1} K_n = \{1\}$ is trivial. Then $\Delta_n = \Delta \cap \rho^{-1}(K_n)$ is of finite index in $\Delta$. By Theorem 11 applied to $\Delta_n$, it follows that the image of $\phi$ is contained in an orbit of $K_n$ for each $n \geq 1$. But since the subgroups $K_n$'s converge to the identity subgroup, it follows that the image of $\phi$ is a singleton. That is, $\phi$ is constant on a conull subset of $\tilde{H}(\mathbb{R}) / S$.

We now mention two consequences of Fubini’s Theorem we will need in the proof of Theorem 5.

**Notation 1.** Suppose that $H$ is a locally compact Hausdorff second countable topological group with a Haar measure $\mu$ and assume that $(H, \mu)$ is $\sigma$-finite. Suppose that $(X, \nu)$ is a $\sigma$-finite measure space on which $H$ acts such that the action $H \times X \to X$ - denoted by $(h, x) \mapsto hx$ - is measurable and so that for each $h \in H$, the map $x \mapsto hx$ on $X$ preserves the measure class of $\nu$. Let $Z$ be a measure space and let $f : X \to Z$ be a measurable map. Given a measure space $(X, \nu)$, we will say that a measurable subset $X_1 \subseteq X$ is **co-null**, if the complement $X \setminus X_1$ has measure zero with respect to $\nu$.

**Lemma 4.** Under the preceding notation, suppose that there exists a co-null subset $X_1 \subseteq X$ such that for each $x \in X_1$, the map $h \mapsto f(hx)$ is constant on a co-null subset $H_x$ of $H$.

Then, given $h \in H$, there exists a measurable subset $X_h$ which is co-null in $X_1$ and such that for all $x \in X_h$

$$f(hx) = f(x).$$

**Proof.** This is essentially a consequence of Fubini’s Theorem. Define $E = \{(h, x) \in H \times X : hx \in X_1\}$. Then, for each $h \in H$, we have $E_h \overset{def}{=} \{x \in X : hx \in X_1\} = h^{-1}(X_1)$. Therefore $E_h$ is co-null and hence by Fubini’s Theorem, $E$ is co-null in $H \times X$. 

Given $h_0 \in H$, $y \in X_1$ and $h \in H$, consider the set
\[ W_y = \{ h \in H : f(h_0hy) = f(hy) \}. \]

Since $W_y \supset h_0^{-1}H_y \cap H_y$, it follows that $W_y$ is co-null in $H$. Consider the set $E_1 = \{(h, y) \in E : y \in X_1, \text{ and } f(h_0hy) = f(hy)\}$. Then $E_1$ is measurable and co-null in $E$ by Fubini, since, for each $y \in X_1$, the set $E_y^1$ is nothing but $W_y$ and is co-null. Write $X_{h_0} \overset{\text{defn}}{=} \{ hy : (h, y) \in E_1 \}$. Then, $X_{h_0} \subset X_1$ (since for every $(h, y) \in E$ we have $hy \in X_1$) and for almost all $h$, we have $(E_1)_h = \{ y \in X_1 : hy \in X_{h_0} \} = h^{-1}X_{h_0}$ is measurable and co-null. Hence $X_{h_0}$ is measurable and co-null in $X_1$.

Let $x \in X_{h_0}$. Then, by the definition of the set $X_{h_0}$, there exists $(h, y) \in E_1$ such that $x = hy$. Moreover, $f(h_0x) = f(h_0hy) = f(hy) = f(x)$ (the last but one equation holds because $(h, y) \in E_1$). This proves the lemma. \hfill \Box

**Notation 2.** Let $G$ be a locally compact Hausdorff second countable group with a $\sigma$-finite Haar measure. Equip $G \times G$ with the product measure $\mu \times \mu$. Let $Z$ be a measure space and $f : G \to Z$ a measurable map.

The following lemma is again a simple application of Fubini’s Theorem.

**Lemma 5.** With the preceding notation, suppose that given $g \in G$, there exists a measurable co-null subset $X_g \subseteq X$, such that
\[ f(gx) = f(x) \quad \forall x \in X_g. \]
Then there exists a measurable co-null subset $Y \subseteq X$ such that $y \mapsto f(y)$ is constant on $Y$.

**Proof.** Let $E = \{(g, x) \in G \times G : f(gx) = f(x)\}$. Then $E$ is measurable and for almost all $g \in G$, the set $E_g = \{ x \in G : (g, x) \in E \}$ is measurable. Moreover, by the definition of $E$, $E_g = \{ x \in G : f(gx) = f(x)\}$ and hence contains the co-null subset $X_g$. Therefore, $E_g$ is co-null for almost all $g \in G$, which implies, by the Fubini Theorem, that $E$ is co-null in $G \times G$. Hence, again by the Fubini Theorem, there exists a co-null subset $X_1$ in $G$ such that for all $x \in X_1$, we have $E^x = \{ g \in G : (g, x) \in E \}$ is co-null in $G$. But by the definition of $E$, $E^x = \{ g \in G : f(gx) = f(x)\}$. Fix $x \in X_1$. Set $Y = E^x$. Then $Y$ is co-null in $G$. Given $y \in Y$, there exists $g \in E^x$ such that $y = gx$. Therefore, $f(y) = f(gx) = f(x)$ for all $y \in Y$. This proves the lemma. \hfill \Box
Finally, we recall a well known result of Furstenberg we will need, commonly known as Furstenberg’s Lemma but due to Zimmer in the form given below.

**Lemma 6** (Furstenberg, [Z] Corollary (4.3.7) and Proposition (4.3.9)). *Suppose that $\Gamma$ is a closed subgroup of a locally compact topological group $G$ and that $P_0$ is a closed amenable subgroup of $G$. Let $X$ be a compact metric $\Gamma$-space. Then there exists a Borel measurable $\Gamma$-equivariant map from a conull subset of $G/P_0$ to $\mathcal{P}(X)$, the space of probability measures on $X$."

3. **Proof of the super-rigidity result (Theorem 1)**

We will now proceed to the proof of Theorem 3. In this section, we suppose that $H$ is a semi-simple Lie subgroup of a *simple* Lie group $G$. Let $P$ be a minimal real parabolic subgroup of $G$ and $S$ and $N$ denote respectively a maximal real split torus of $P$ and the unipotent radical of $P$. Let $A$ be the connected component of identity in $S$. Denote by $P_0$ the subgroup $AN$ of $P$ (if $G$ is split, then $P_0 = P$; in general we have replaced the minimal parabolic subgroup $P$ by a subgroup $P_0$ which has no compact factors such that $P/P_0$ is compact). Let $K$ be a maximal compact subgroup of $G$. We have the Iwasawa decomposition $G = P_0K = ANK$.

We will treat the archimedean and non-archimedean cases separately.

We start by mentioning the following easy observation.

**Lemma 7.** *Let $H$ be a semi-simple subgroup of simple group $G$ and $K$ a maximal compact subgroup of $G$. Assume that $\dim(H) > \dim(K)$. Let $G = NAK$ be an Iwasawa decomposition of $G$ and $P_0 = NA$. Then the isotropy subgroup of $H$ at any point in $G/P_0$ is a non-compact subgroup of $H$."

*Proof. Since $G/P_0 = K$, we have $\dim(G/P_0) = \dim(K)$, and since $\dim(H) > \dim(K)$, at any point $p \in G/P_0$, the isotropy of $H$ is a positive dimensional subgroup, which is conjugate to a subgroup of $P_0$; the latter has no compact subgroups, hence the isotropy of $H$ at $p$ is a non-compact subgroup. $\square$

**Theorem 12.** *Suppose that $\Gamma$ is a Zariski dense discrete subgroup of a simple Lie group $G$ which intersects a semi-simple Lie subgroup $H$ (of real rank at least two) of $G$ in an irreducible lattice $\Delta$. Suppose that $H$ acts with non-compact isotropy at any point of $G/P_0$ (or that $\dim(H) > \dim(K)$ for a maximal compact subgroup $K$ of $G$). Then, the group $\Gamma$ is non-archimedean super-rigid."
Proof. Suppose that \( \rho : \Gamma \to G' \) is a representation of \( \Gamma \) into an absolutely simple algebraic group \( G' \) over a non-archimedean local field \( k' \) of characteristic zero with Zariski dense image. By a construction of Furstenberg (Lemma 6), there exists a \( \Gamma \) equivariant measurable map \( \phi \) from \( G/P_0 \) into \( \mathcal{P}(G'(k)/P'(k)) \) where \( \mathcal{P} \) refers to the space of probability measures on the relevant space. Being a closed subgroup of \( P \), the group \( P_0 \) is amenable. In particular, the map \( \phi \) is \( \Delta \)-equivariant.

By assumption (or by Lemma 7), at any point \( p \in G/P_0 \), the map \( \phi \) is \( \Delta \)-equivariant. Therefore, by Proposition 10, the map \( \phi : G/P_0 \to \mathcal{P}(G'(k)/P'(k)) \) is (by [Z]) an extension of an algebraic subgroup \( G' \) of \( G' \) by a compact subgroup of \( \Gamma \).

By Theorem (3.2.17) of [Z], the group \( G' \) acts smoothly on \( \mathcal{P}(G'(k)/P'(k)) \) and hence the quotient space \( \mathcal{P}(G'(k)/P'(k))/G'(k') \) is countably separated. By Lemma 11, \( \Delta \) acts ergodically on \( H(\mathbb{R})/S \). Therefore, the map

\[
\phi_p : H/S \to \mathcal{P}(G'(k)/P'(k)), h \mapsto \phi(hp)
\]

is essentially contained in a \( G'(k') \)-orbit (Proposition (2.1.11) of [Z]). Write this orbit as \( G'(k')/J \) for some closed subgroup \( J \) of \( G'(k') \). Then, by Proposition 10, the map \( \phi : H/S \to G'(k')/J \) is constant (we are therefore using the non-archimedean Margulis super-rigidity Theorem to conclude that the image of the lattice in \( H \) is contained in a compact subgroup in \( G'(k') \)). Therefore, by Lemma 4, given \( h \in H \), there exists a co-null set \( X_1 \) in \( G/P_0 \), such that \( \phi(hp) = \phi(p) \) for all \( p \in X_1 \).

Since \( G \) is simple and \( \Gamma \) is Zariski dense, there exist finitely many elements \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma \) such that the span of the Lie algebras of the conjugates \( \gamma^i(H) = \gamma_i H \gamma_i^{-1} \) is the Lie algebra of \( G \). The existence of such a set follows from the simplicity of \( G \) and the Zariski density of \( \Gamma \) in \( G \). Then, we apply the conclusion of the preceding paragraph successively to the groups \( \gamma^i(H) \) to conclude that if \( h_i \in \gamma^i(H) \) \( (1 \leq i \leq k) \), then the map \( x \mapsto \phi(h_1 h_2 \cdots h_k x) \) does not depend on the \( h_i \), on a co-null set \( X_1 \). Since the product set \( \gamma_1(H) \cup \cdots \cup \gamma_k(H) \) contains a Zariski open set in \( G \) by the choice of the \( \gamma_i \), by replacing the set \( \{\gamma_i\} \) with a larger set if necessary, we may assume that the product set \( \gamma_1(H) \cdots \gamma_k(H) = G \). It follows from Lemma 5 that the map \( \phi \) is constant, and hence \( \rho(\Gamma) \) fixes a probability measure on \( G'(k')/P'(k') \).

But the isotropy subgroup in \( G'(k') \) of a probability measure \( \mu \) on \( G'(k')/P'(k') \) is (by [Z]) an extension of an algebraic subgroup \( G'_\mu \) of \( G' \) by a compact subgroup of \( G'(k') \). Since \( \rho(\Gamma) \) is Zariski dense in \( G'(k') \), it follows that \( G'_\mu \) is normalised by \( G' \). Since \( G' \) is absolutely simple, either \( G'_\mu = G' \) or else \( G'_\mu = 1 \). If \( G'_\mu = G' \), this means that \( G'(k') \) fixes a probability measure on \( G'(k')/P' \) which is impossible; therefore, \( G'_\mu = 1 \), and \( \rho(\Gamma) \) is contained in a compact subgroup of \( G'(k') \).
This means that $\Gamma$ is non-archimedean super-rigid in $G$, and proves Theorem 12.

To treat the archimedean case, we need the following preliminary result.

**Lemma 8.** If $f : X \to Y$ is a surjective map of irreducible affine varieties over $\mathbb{R}$, such that $X(\mathbb{R})$ (resp. $Y(\mathbb{R})$) is Zariski dense in $X$ (resp. $Y$), and $\phi : Y(\mathbb{R}) \to \mathbb{R}$ is a set theoretic map such that the composite $\phi \circ f$ is a rational map on $X(\mathbb{R})$, then $\phi$ is a rational map on $Y(\mathbb{R})$.

**Proof.** We may assume - by replacing $X$ by an open set - that $X$ is a finite cover $\tilde{U}$ of a open set $U$ of a product $Y \times Z$ of $Y$ with an affine variety $Z$; and that the map $f$ is the composite of the covering map $p : \tilde{U} \to U$ with the first projection $\pi_1 : Y \times Z \to Y$.

The assumptions imply that the composite $p \circ \pi_1 \circ \phi$ is rational on $\tilde{U}$. Since $p$ is a finite cover, it follows that $\pi_1 \circ \phi$ is rational on $U$ (a rational map on a finite cover which descends to a function on the base is a rational function on the base), and hence rational on $Y \times Z$. Since this map is independent of the point $z \in Z$, it follows that the map $\phi$ is rational on $Y$ (a rational function in two variables which depends only on the first variable is a rational function in the first variable).

**Theorem 13.** Let $H$ be a semi-simple Lie subgroup (of real rank at least two) of a simple Lie group $G$ which acts with non-compact isotropies on $G/P_0$ (or, which satisfies the stronger condition $\dim(H) > \dim(K)$ for a maximal compact subgroup $K$ of $G$). Let $\Gamma \subseteq G$ be a Zariski dense discrete subgroup which intersects $H$ in an irreducible lattice. Let $\rho : \Gamma \to G'(k')$ be a homomorphism of $\Gamma$, with $k'$ an archimedean local field, and $G'$ an absolutely simple algebraic group over $k'$. If $\rho(\Gamma)$ is not relatively compact in $G'(k')$ and is Zariski dense in $G'$, then $\rho$ extends to an algebraic homomorphism of $G$ into $R_{k'/\mathbb{R}}G'$ defined over $\mathbb{R}$.

**Proof.** Here $G(k)$ is viewed as the group of real points of a real algebraic group $R_{k'/\mathbb{R}}(G')$ where $R$ is the Weil restriction of scalars. We view the semi-simple linear group $G$ also as the group of real points of a real algebraic group.

If $\rho$ is an archimedean representation of $\Gamma$, then, again by Furstenberg’s construction, there exists a $\Gamma$-equivariant measurable map

$$\phi : G/P \to \mathcal{P}(G'(k')/P'(k')),$$

where $P'$ is a minimal parabolic $k'$-subgroup of $G'$ and $\mathcal{P}$ denotes the space of probability measures.
We first prove that on each $H$-orbit in a conull $H$-invariant subset of $G/P$, the map $\phi$ is essentially a rational map. More precisely, we prove that the map $\phi$ is the composite of an algebraic homomorphism of $\tilde{H}$ into $G'(k')$ with the orbit map $G'(k') \to G'(k')p$ with $p \in P$. This is exactly the statement of Proposition 10 if $H = \tilde{H}(\mathbb{R})$, where $\tilde{H}$ is simply connected. If not, we replace $H$ (resp. $\Delta$) by the image of $\tilde{H}(\mathbb{R})$ where $\tilde{H}$ is the simply connected cover of $H$ (resp. $\Delta \cap \tilde{H}(\mathbb{R})$). The conclusion of Proposition 10 is unaltered.

As in the proof of Theorem 12, we fix elements $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$ such that the Lie algebras of the conjugates $\gamma_i(H) = \gamma_iH\gamma_i^{-1}$ together span that of $G$. The existence of such a set follows from the simplicity of $G$ and the Zariski density of $\Gamma$ in $G$. Then, we apply the conclusion of the preceding paragraph successively to $\gamma_i(H)$ to conclude that if $h_i \in \gamma_i(H)$ (with $1 \leq i \leq k$), then the map $x \mapsto \phi(h_1h_2\cdots h_k x)$ is simply the map

$$\tilde{\rho}(h_1)\tilde{\rho}(h_2)\cdots \tilde{\rho}(h_k)(\phi(x)),$$

for almost all $x \in G/P_0$, and all $h_i \in H_i$. Here

$$\tilde{\rho}(h_i) = \rho(\gamma_i)\rho(\gamma_i^{-1}h_i\gamma_i)\rho(\gamma_i)^{-1}$$

is the natural homomorphism on the conjugate $\gamma_i(H)$ into $G'(k')$ obtained from $\tilde{\rho} : H \to G'(k')$.

Since the product set $\gamma_1(H)\gamma_2(H)\cdots\gamma_k(H)$ contains a Zariski open set in $G$ by the choice of the $\gamma_i$, by replacing the product by a product over a larger set of $\gamma_i$'s if necessary, we may assume that the product set $\gamma_1(H)\gamma_2(H)\cdots\gamma_k(H)$ contains all of $G$ (we may assume that $G$ is connected). Given $g \in G$, there exist elements $h_i \in \gamma_i(H)$ ($1 \leq i \leq k$) such that $g = h_1\cdots h_k$. Thus, given $g \in G$, there exists an element $R(g) = \tilde{\rho}(h_1)\cdots \tilde{\rho}(h_k) \in G'(k')$ such that for a co-null set $X_1 \subseteq G/P_0$, we have

$$\phi(gx) = R(g)\phi(x) \quad \forall x \in X_1.$$

We may assume (since $\Gamma$ is countable), that $X_1$ is $\Gamma$-stable. If $R'(g) \in G'(k')$ is another element such that the equality (11) holds with $R'(g)$ in place of $R(g)$, then, we have: $R(g)^{-1}R'(g) \in I_{\phi(x)}$, the isotropy of $G'(k')$ at the point $\phi(x)$, for each $x \in X_1$. Replacing $x$ by $\gamma x$ for a fixed $\gamma \in \Gamma$, $R(g)^{-1}R'(g)$ lies in the intersection

$$I = \cap_{\gamma \in \Gamma} \rho(\gamma)I_{\phi(x)}\rho(\gamma)^{-1}.$$

But the isotropy is an algebraic subgroup of $G'(k')$ (by [Z]). Hence the intersection $I$ is also algebraic, but is normalised by the Zariski dense subgroup $\rho(\Gamma)$ of $G'(k')$, and hence is normalised by $G'(k')$. But $G' is
absolutely simple, hence the intersection is trivial and $R(g) = R'(g)$ for all $g \in G$. In particular, $R(\gamma) = \rho(\gamma)$ for all $\gamma \in \Gamma$.

Moreover, the equation 
\[
\phi(g_1g_2x) = R(g_1)R(g_2)\phi(x) = R(g_1g_2)\phi(x)
\]
holds for a fixed $g_1, g_2 \in G$, and $x \in X'$ a co-null subset of $G/P_0$. By the uniqueness of $R(g_1g_2)$ proved in the preceding paragraph, we have 
\[
R(g_1g_2) = R(g_1)R(g_2)
\]
for any fixed pair $g_1, g_2 \in G$. Hence $R : G \to G'(k')$ is an abstract homomorphism.

The composite of the product map $f : H_1 \times \cdots \times H_k \to G$ and the abstract homomorphism $R : G \to G'(k')$ is $R \circ f(h_1, \cdots, h_k) = \tilde{\rho}(h_1) \cdots \tilde{\rho}(h_k)$, which is rational on $H_1 \times \cdots \times H_k$. Then the map $R$ is rational, by virtue of Lemma 8. Theorem 13 now follows since the rationality of $R$ implies the rationality of the representation $\rho : \Gamma \to G'(k')$.

Theorem 3 is an immediate consequence of Theorem 12 and Theorem 13. Theorem 5 is a particular case of Theorem 1, in view of Lemma 7.

4. Applications (Proof of Corollary 6)

Assume that $H = SL_k(\mathbb{R})$ and $G = SL_n(\mathbb{R})$, with $SL_k$ embedded in $SL_n$ in the top left hand corner. Under the assumptions of Corollary 6 we have $k - 1 > n/2$ and $\dim(H) = k^2 - 1 > \dim(G/P_0) = n(n-1)/2$. Therefore, if $\Gamma$ is a Zariski dense discrete subgroup of $G$ which intersects $H$ in a lattice, then by Theorem 5 $\Gamma$ is super-rigid. We now prove Corollary 6.

We now recall a result, which is a generalisation of Margulis’ observation that super-rigidity implies arithmeticity. However, Margulis needed only that the discrete subgroup was a lattice. We have not assumed that $\Gamma$ is a lattice (indeed, this is what is to be proved), and we also do not assume that $\Gamma$ is finitely generated.

Theorem 14 (V3). Let $G$ be an absolutely simple real algebraic group and let $\Gamma$ be a super-rigid discrete subgroup. Then there exists an arithmetic group $\Gamma_0$ of $G$ containing $\Gamma$.

Suppose $\Gamma_0 \subseteq G$ is arithmetic. This means that there exists a number field $F$ and a semi-simple linear algebraic $F$-group $G$ such that the group $G(\mathbb{R} \otimes F)$ is isomorphic to a product $SL_n(\mathbb{R}) \times U$ of $SL_n(\mathbb{R})$ with a compact group $U$. Under this isomorphism, the projection of $G(O_F)$ of the integral points of $G$ into $G$ is commensurable with $\Gamma_0$. The simplicity of $SL_n(\mathbb{R})$ implies that $G$ may be assumed to be absolutely simple over $F$. The group $G$ is said to be an $F$-form of $SL_n$. Moreover,
if $\Gamma_0$ contains unipotent elements, then $G$ cannot be anisotropic over $F$. Hence $F$-rank of $G$ is greater than zero. In that case, $G(F \otimes \mathbb{R})$ cannot contain compact factors (since compact groups cannot contain unipotent elements). This means that $F = \mathbb{Q}$.

We now recall the classification of $\mathbb{Q}$-forms of $\text{SL}_n$.

[1] Let $d$ be a divisor of $n$ and $D$ a central division algebra over $\mathbb{Q}$ of degree $d$. Write $n = md$. Then, the algebraic group $G = \text{SL}_m(D)$ is a $\mathbb{Q}$-form of $\text{SL}_n$. The rank of $G$ is $m - 1 = n/d - 1$. If $d \geq 2$, then $m - 1 < n/2$.

[2] Let $E/\mathbb{Q}$ be a quadratic extension and $D$ a central division algebra over $E$ with an involution of the second kind with respect to $E/\mathbb{Q}$. Let $d$ be the degree of $D$ over $E$, suppose $d$ divides $n$ and let $md = n$. Let $h : D^m \times D^m \to E$ be a Hermitian form with respect to this involution, and let $G = \text{SU}(h)$. Then, $G$ is a $\mathbb{Q}$-form of $\text{SL}_n$; its $\mathbb{Q}$-rank is not more than $m/2 = n/2d \leq n/2$.

The classification of simple algebraic groups (see [T]), implies the following.

**Lemma 9.** The only $\mathbb{Q}$-forms $G$ of $\text{SL}_n$ are as above. In particular, if $G$ is a $\mathbb{Q}$-form of $\mathbb{Q}$-rank strictly greater than $n/2$, then $G$ is $\mathbb{Q}$-isomorphic to $\text{SL}_n$.

**Proof of Corollary 6.** By Theorem 5, the group $\Gamma$ is super-rigid in $G$. By Theorem 14, $\Gamma$ is contained in an arithmetic subgroup $\Gamma_0$ of $G$. Since $\Gamma_0 \supset \Gamma$ contains a finite index subgroup of $\text{SL}_k(\mathbb{Z})$ by assumption, it follows that $\Gamma_0$ contains unipotent elements. Therefore, the number field $F$ associated to $\Gamma_0$ is $\mathbb{Q}$ and there is $G$ a $\mathbb{Q}$-form of $\text{SL}_n$ such that $\Gamma_0$ is commensurate with $G(\mathbb{Z})$. Since $\Gamma \subseteq \Gamma_0$, a finite index subgroup of $\text{SL}_k(\mathbb{Z})$ is a subgroup of $G(\mathbb{Q})$ and hence its Zariski closure $\text{SL}_k$ is a $\mathbb{Q}$-subgroup of $G$. Hence the $\mathbb{Q}$-rank of $G$ is not less than the $\mathbb{Q}$-rank of $\text{SL}_k$ which is $k - 1 > n/2$ by assumption.

By Lemma 9, the $\mathbb{Q}$-form $G$ is isomorphic to $\text{SL}_n$. Hence $\Gamma_0$ is commensurable with $\text{SL}_n(\mathbb{Z})$. Moreover, the $\mathbb{Q}$-inclusion of $H = \text{SL}_k$ in $G = \text{SL}_n$ is the standard one described before the statement of Theorem 1.

Now, $\Gamma$ is Zariski dense and contains a finite index subgroup of $\text{SL}_k(\mathbb{Z})$. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{Q}^n$. Consider the change of basis which interchanges $e_k$ and $e_n$ and all other $e_i$’s are left unchanged. After this change of basis, (which leaves the diagonal torus stable), the group $\Gamma$ (or rather, a conjugate of it by the matrix effecting this change of basis) contains the highest root group and the second highest root group (in the usual notation for $\text{SL}_n$ the positive roots occur in the Lie algebra of upper triangular matrices). By Theorem
(3.5) (or Corollary (3.6)) of [V1], \( \Gamma \) must be of finite index in \( \text{SL}_n(\mathbb{Z}) \). This proves Corollary 6.

\( \square \)

Corollary 7 is proved in an analogous way.

5. Proof of Theorem 1

Notation 3. Let \( k \geq 3 \) and \( m \geq 2 \) be integers, set \( n = k + m \). Fix (real) vector spaces \( W, E \) and \( V \) of dimensions \( k, m \) and \( n \) respectively. Assume that \( V = W \oplus E \). Then there is a decomposition of the dual vector spaces \( V^* = W^* \oplus E^* \). Fix a basis \( w_1, w_2, \ldots, w_k \) of the vector space \( W \). Let \( w_1^*, w_2^*, \ldots, w_k^* \) be the dual basis in \( W^* \subseteq V^* \). Let

\[
U = \{ g \in \text{SL}(V) : g(w_i) \not\in W (1 \leq i \leq k - 1) \text{ and } g^*(w_k) \not\in E^* \cup W^* \}.
\]

Then, \( U \) is a non-empty set in \( G \) which is open in the Zariski topology on \( \text{SL}(V) \).

We will view \( \text{SL}(W) \) as a subgroup of \( \text{SL}(V) \) by letting any element of \( \text{SL}(W) \) act trivially on \( E \). Let \( U \subseteq \text{SL}(W) \) be the unipotent subgroup an element of which acts trivially on each \( w_i \) with \( 1 \leq i \leq k - 1 \) and takes \( w_k \) to \( w_k + \) a multiple of \( w_1 \). Denote by \( L(g) \) the Zariski closure in \( \text{SL}(V) \) of the subgroup generated by \( \text{SL}(W) \) and \( gUg^{-1} \).

Notice that the group \( L(g) \) is connected, as it is generated by the connected subgroups \( \text{SL}(W) \) and \( gUg^{-1} \); therefore, its Lie algebra is generated by those of \( \text{SL}(W) \) and \( gUg^{-1} \).

Lemma 10. With the above notation, (and the assumption that \( k \geq 3 \)), there exists a Zariski open set \( U \) in \( \text{SL}_n(\mathbb{R}) \) such that the quotient of the group \( L(g) \) modulo its radical is isomorphic to \( \text{SL}_{k+1}(\mathbb{R}) \), for every \( g \in U \), where \( \text{SL}_{k+1} \) is, after a conjugation, embedded in \( \text{SL}_n(\mathbb{R}) \) in the top left hand corner.

Proof. The lemma is equivalent to showing that the Lie algebra \( l(g) \) generated by \( \mathfrak{sl}(W) \) and \( gUg^{-1} \) in the Lie algebra \( \mathfrak{sl}(V) \) of the group \( \text{SL}(V) \), where \( \mathfrak{u} \) is the Lie algebra of \( U \). We first decompose the space \( \mathfrak{sl}(V) \) as a module over \( \mathfrak{sl}(W) \). Now, \( V = W \oplus \mathbb{R}^{n-k} \) where \( \mathbb{R} \) is the one dimensional trivial \( \mathfrak{sl}(W) \)-module. Hence

\[
\mathfrak{sl}(V) = \mathfrak{sl}(W) \oplus (\mathbb{C}^{n-k} \otimes W) \oplus (W^* \otimes \mathbb{C}^{n-k}) \oplus E
\]

where \( E \) is a trivial \( \mathfrak{sl}(W) \) module of the appropriate dimension and \( W^* \) is the dual of \( W \).

The Lie algebra \( \mathfrak{u} \) is generated by the linear transformation \( E_{k1} \) which takes all the elements of the standard basis of \( \mathbb{R}^n \) to zero, except the \( k \)-th basis element \( w_k \) which is taken to \( w_1 \). Therefore, the kernel of \( E_{k1} \)
has codimension 1. By the genericity assumption on $g$, the conjugate $g E_{k_1} g^{-1}$ has kernel $F(g)$ (also of codimension 1) which intersects $E$ in a subspace of codimension 1: $\dim( F(g) \cap E) = n - k - 1$.

We have the exact sequence

$$0 \to F(g) \cap E \to V \to V/(F(g) \cap E) \to 0$$

of vector spaces, which is also a sequence of $l(g)$ modules since all the terms are stable under the action of $sl(W)$ and of $gug^{-1}$. Moreover, $l(g)$ acts trivially on $F(g) \cap E$. Denote by $V$ the quotient $V/(F(g) \cap E)$, by $E$ the image of $E \subseteq V$ in $\overline{V}$. Then $E$ is one-dimensional.

Moreover, as an $sl(W)$-module, we have the decomposition $\overline{V} = W \oplus E$ and hence as an $sl(W)$-module, the decomposition

$$\text{End}(\overline{V}) = \text{End}(W) \oplus (E \otimes E) \oplus (W^* \otimes E) \oplus (E^* \otimes W).$$

Since $\dim(W) \geq 3$, the irreducible modules $W$ and $W^*$ are non-isomorphic; therefore, if a vector in $\text{sl}(\overline{V})$ projects non-trivially to $W \otimes E^*$ and to $E \otimes W^*$, then the $\text{sl}(W)$-module generated by this vector contains both spaces $W \otimes E^*$ and $E \otimes W^*$.

The genericity assumption on $g$ ensures that the projection of $g E_{k_1} g^{-1}$ viewed as an endomorphism of $\overline{V}$ projects non-trivially to $W \otimes E^*$ and to $E \otimes W^*$ in $\text{End}(\overline{V})$. Hence, by the preceding paragraph, $l(g)$ contains in its image in $\text{End}(\overline{V})$, $\text{sl}(W)$, $W \otimes E^*$ and $E \otimes W^*$; therefore, the image of $l(g)$ equals $\text{sl}(\overline{V})$. This proves Lemma 10. □

We note that in the Lemma, the embedding of $SL_{k+1}$ in $SL_n$ is in the top left hand corner.

The following Lemma was originally stated when $U$ was the vector group $\mathbb{R}^{k+1}$, the standard representation of $SL_{k+1}$. We thank the referees for pointing out that we need to prove this stronger version.

**Lemma 11.** If $\Delta \subseteq G = \mathbb{R}^{k+1} \rtimes SL_{k+1}(\mathbb{R})$ is a Zariski dense discrete subgroup of $G$ which contains $\{0\} \rtimes SL_k(\mathbb{Z})$ embedded in $SL_{k+1}(\mathbb{R})$ in the top left corner and if $k \geq 3$, then, the group $\Delta$ virtually contains a conjugate of $SL_{k+1}(\mathbb{Z})$.

**Proof.** The group $\Delta$ may be shown to be super-rigid in $G$, exactly as in the proof of Corollary 6. This means, by [V3], that the projection of $\Delta$ is contained in an arithmetic subgroup of $SL_{k+1}(\mathbb{R})$, and is therefore discrete. The Zariski density of the projection in $SL_{k+1}(\mathbb{R})$ now implies, by Corollary 6, that the projection of $\Delta$ is conjugate virtually to $SL_{k+1}(\mathbb{Z})$. Now, $\Delta$ is a subgroup of a semi-direct product and its projection to $SL_{k+1}(\mathbb{R})$ contains virtually $SL_{k+1}(\mathbb{Z})$. Hence $\Delta$ defines an element of the cohomology group $H^1(\Delta', \mathbb{R}^n)$ where $\Delta \subseteq SL_{k+1}(\mathbb{Z})$. 


is a subgroup of finite index. But the first cohomology group vanishes, by a theorem of Raghunathan (see [R2]), for \( k + 1 \geq 3 \), and hence \( \Delta \) contains, virtually, a conjugate of \( SL_{k+1}(\mathbb{Z}) \).

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. We have \( SL_3(\mathbb{Z}) \) is virtually contained (in the top left corner) in the Zariski dense discrete subgroup \( \Gamma \) of \( SL_n(\mathbb{R}) \). We will prove by induction, that for every \( k \geq 3 \) with \( k \leq n \), a conjugate of \( SL_k(\mathbb{Z}) \) is virtually a subgroup of \( \Gamma \). Applying this to \( k = n \) gives us Theorem 1.

Suppose that for some \( k \), \( SL_k(\mathbb{Z}) \) is virtually contained in \( \Gamma \). Let \( \gamma \in \Gamma \), and let \( \Delta = \Delta(\gamma) \) denote the subgroup of \( \Gamma \) generated by \( SL_k(\mathbb{Z}) \) and a conjugate of the unipotent element \( \gamma u \gamma^{-1} \) (where \( u \) is in a finite index subgroup of \( SL_3(\mathbb{Z}) \), as in Lemma 10). Assume further, that the element \( \gamma \) is in the open set \( U \) of Notation 3. By Lemma 10, the Zariski closure of the group \( \Delta \) maps onto \( SL_{k+1} \).

By Lemma 11 the group \( \Delta \) contains, virtually, the group \( SL_{k+1}(\mathbb{Z}) \). We have thus proved that if \( SL_k(\mathbb{Z}) \subseteq \Gamma \), then \( SL_{k+1}(\mathbb{Z}) \subseteq \Gamma \), provided \( k + 1 \leq n \). Thus the induction is completed and therefore Theorem 1 is established.

6. THE RANK ONE CASE

In this last section, we will see that the situation for Nori’s question is completely different in real rank one. More precisely, we have the following result.

Theorem 15. Let \( G \) be a real simple Lie group of real rank one and \( H \subseteq G \) a non-compact semi-simple subgroup. Suppose that \( \Delta \) is a lattice in \( H \). Then, there exists a Zariski dense discrete subgroup \( \Gamma \) in \( G \) of infinite co-volume whose intersection with \( H \) is a subgroup of finite index in the lattice \( \Delta \).

Suppose that \( H \) is a simple non-compact subgroup of a simple group \( G \) of real rank one. Let \( P \) be a minimal parabolic subgroup of \( G \) which intersects \( H \) in a minimal parabolic subgroup \( Q \). The group \( G \) acts on \( G/P \) and \( H \) leaves the open set \( U = (G/P) \setminus (H/Q) \) stable. Let \( \Delta \) be a discrete subgroup of \( H \).

Lemma 12. Given compact subsets \( \Omega_1 \) and \( \Omega_2 \) in the open set \( U \), the set \( U_H = \{ h \in H : h\Omega_1 \subseteq \Omega_2 \} \) is a compact subset of \( H \). Further, the group \( \Delta \) acts properly discontinuously on \( U \).

Proof. Choose, as one may, a maximal compact subgroup \( K \) of \( G \) such that \( K \cap H \) is maximal compact in \( H \). The set \( U \) is invariant under
and hence under $K \cap H$. We may assume that the compact sets $\Omega_1$ and $\Omega_2$ are invariant under $K \cap H$ of $H$. The Cartan decomposition of $H$ says that we may write $H = (K \cap H) A^+(K \cap H)$ with $A$ a maximal real split torus (of dimension one) and $A^+ = \{ a \in A : 0 < \alpha(h) \leq 1 \}$ where $\alpha$ is a positive root of $A$. Let $W$ denote the normaliser modulo the centraliser of $A$ in $G$. This is the relative Weyl group. Since $\mathbb{R}$-rank($G$) = 1, it follows that $W = \{1, \kappa\}$ has only two elements. Since $\mathbb{R}$-rank($H$) = 1, it follows that $\kappa \in H$. We have the Bruhat decomposition $G = P \cup U \kappa P$, where $\kappa$ is the non-trivial element of the Weyl group of $A$ in $G$ and $U$ is the unipotent radical of the minimal parabolic subgroup $P$.

If possible, let $h_m$ be a sequence in $U_H$ which tends to infinity. It follows from the previous paragraph that $h_m = k_m a_m k'_m$ with $k_m, k'_m \in K \cap H$ and $a_m \in A^+$, and $\alpha(a_m) \to 0$ as $m \to +\infty$. Let $p$ be an element of $\Omega_1 \subseteq U$. By the Bruhat decomposition in $G$ ($G$ has real rank one), it follows that $U \subseteq U \kappa P$. We may write $p = ukP$ for some $u \in U$. Since $h_mp \in \Omega_2$ and the latter is compact, we may replace $h_m$ by a sub-sequence and assume that $h_mp$ converges, say to $q \in \Omega_2$. Since $\Omega_2$ is stable under $K \cap H$, we may assume that $k_m = 1$, by replacing $q$ by a sub-sequential limit of $k_m^{-1}q$.

We compute $h_mp = a_m k'_m u \kappa P \in \Omega_2 \subseteq U$. The convergence of $k'_m$ says that $k'_m u \kappa P = u_m \kappa P$ with $u_m$ convergent (possibly after passing to a sub-sequence). We write $^a(u) \overset{def}{=} aua^{-1}$. Then, $h_mp = a_m (u_m) \kappa P$ since $\kappa$ conjugates $A$ into $P$ (in fact into $A$). Since $\alpha(a_m) \to 0$, and $u_m$ are bounded, conjugation by $a_m$ contracts $u_m$ into the identity and $h_mp$ therefore tends to $\kappa P$. But the latter is in $H/Q$ since $\kappa$ already lies in $H$. Hence the limit does not lie in $\Omega_2$, contradicting the fact that $\Omega_2$ is compact. Therefore $U_H$ is compact.

The second part of the lemma immediately follows since the intersection of $\Delta$ with $U_H$ is finite. □

**Lemma 13.** Let $\Gamma \subseteq G$ be a Zariski dense discrete subgroup. There exists an element $\gamma \in \Gamma$ such that the $\gamma$ translate of $HP/P$ does not intersect $HP/P$:

$$\gamma(HP/P) \cap HP/P = \emptyset.$$  

**Proof.** First, suppose that $V$ is a compact set contained in $G/P \setminus \{P, \kappa P\}$ (with $\kappa$ as in the proof of Lemma 12). By Bruhat decomposition, $V \subseteq U \kappa P/P$ and its $U$ part lies in a compact set. After a conjugation, we assume as we may, that $\Gamma$ contains a semisimple element $t$ in $A$ such that the positive powers of $Ad(t)$ contract elements of $U$ into identity. Moreover, this contraction is uniform on a compact
subset of $U$. Hence there exists a positive power $t^m$ of $t$ such that $t^m(V)$ lies in an arbitrarily small neighbourhood of $\kappa P \in G/P$.

The group $\Gamma$ is Zariski dense in $G$ and $HP/P$ is a Zariski closed subset of $G/P$. Therefore, there exists $g \in G$ such that $g^\pm 1P \notin HP/P$ and $g^\pm 1\kappa P \notin HP/P$. Now, $HP/P \subseteq G/P \{gP, g\kappa P\}$ is a compact set. Hence $V = g^{-1}(HP/P)$ is a compact set in $G/P \{P, \kappa P\}$. Applying a large positive power of $t \in \Gamma \cap A$ as in the preceding paragraph, we see that for some large integer $m$, $t^mV$ lies in a small neighbourhood of $\{P, \kappa P\}$. Therefore, $gt^m g^{-1}(HP/P)$ lies in a small neighbourhood of $\{gP, g\kappa P\}$.

By the choice of the element $g \in \Gamma$, the latter set does not intersect $HP/P$. Choose a small neighbourhood of $\{gP, g\kappa P\}$ which does not intersect $HP/P$; if $m$ is large, then the set $gt^m g^{-1}(HP/P)$ lies in this neighbourhood, and hence does not intersect $HP/P$. We take $\gamma = gt^m g^{-1} \in \Gamma$. This proves the Lemma. □

Given a lattice $\Delta$ in $H$, and given a compact subset $\Omega \subseteq U$, Lemma 12 shows that there exists a finite index subgroup $\Delta'$ of $\Delta$ such that non-trivial elements of $\Delta'$ drag $\Omega$ into an arbitrarily small compact neighbourhood $V$ of $H/Q = HP/P$.

By Lemma 13 there exists $g \in G - HP$ such that $g(V) \subseteq U$. Replacing $H/Q$ by $g(H/Q)$ and $\Delta$ by $g\Delta g^{-1}$, we see that all points of $H/Q$ are dragged, by non-trivial elements of $g\Delta g^{-1}$, into a small neighbourhood of $H/Q$. The ping-pong lemma then guarantees that there exists a finite index subgroup subgroup $\Delta''$ such that the group $\Gamma$ generated by $\Delta''$ and $g\Delta'' g^{-1}$ is the free product of $\Delta''$ and $g\Delta'' g^{-1}$.

We may replace $g$ by a finite set $g_1, \ldots, g_k$ such that for each pair $i, j$, the intersections $g_i(H/Q) \cap g_j(H/Q)$ and $g_i(H/Q) \cap H/Q$ are all empty. By arguments similar to the preceding paragraph, we can find a finite index subgroup $\Delta_0$ of $\Delta$ such that the group $\Gamma$ generated by $g_i\Delta_0 g_i^{-1}$ is the free product of the groups $g_i\Delta_0 g_i^{-1}$, and by choosing the $g_i$ suitably, we ensure that $\Gamma$ is Zariski dense in $G$. This proves Theorem 15 since $\Gamma$ is discrete and since it operates properly discontinuously on some open set in $G/P$, $\Gamma$ cannot be a lattice. This proves Theorem 15.

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