The $\gamma$-Vectors of Pascal-like Triangles Defined by Riordan Arrays

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Abstract

We define and characterize the $\gamma$-matrix associated to Pascal-like matrices that are defined by ordinary and exponential Riordan arrays. We also define and characterize the $\gamma$-matrix of the reversions of these triangles, in the case of ordinary Riordan arrays. We are led to the $\gamma$-matrices of a one-parameter family of generalized Narayana triangles. Thus these matrices generalize the matrix of $\gamma$-vectors of the associahedron. The principal tools used are the bivariate generating functions of the triangles and Jacobi continued fractions.

1 Introduction

A polynomial $P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$ of degree $n$ is said to be reciprocal if

$$P_n(x) = x^n P_n(1/x).$$

Thus we have

$$[x^k]P_n(x) = a_{n,k} = [x^k]x^n P_n(1/x).$$

Now

$$[x^k]x^n P_n(1/x) = [x^{k-n}] \sum_{i=0}^{n} a_{n,i} \frac{1}{x^i} = [x^{k-n}] \sum_{i=0}^{n} a_{n,i} x^{-i} = a_{n,n-k}.$$ 

Thus $P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$ defines a family of reciprocal polynomials if and only if $a_{n,k} = a_{n,n-k}$. We shall call a lower-triangular matrix $(a_{n,k})$ Pascal-like if

1. $a_{n,k} = a_{n,n-k}$
2. $a_{n,0} = a_{n,n} = 1.$

Such a matrix will then be the coefficient array of a family of monic reciprocal polynomials. We have the following well-known result [7]

1
Proposition 1. Let $P_n(x)$ be a reciprocal polynomial of degree $n$. Then there exists a unique polynomial $\gamma_n$ of degree $\lfloor \frac{n}{2} \rfloor$ with the property

$$P_n(x) = (1 + x)^n \gamma_n \left( \frac{x}{(1 + x)^2} \right).$$

If $P_n(x)$ has integer coefficients then so does $\gamma_n(x)$.

By this means, we can associate to every Pascal-like matrix $(a_{n,k})$ a matrix $(\gamma_{n,k})$ so that for all $n$, we have

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k} x^k (1 + x)^{n-2k}.$$

We shall call this matrix the $\gamma$-matrix associated to the coefficient array $(a_{n,k})$ of the family of polynomials $P_n(x)$.

We can characterize the matrix $(a_{n,k})$ in terms of the $\gamma$-matrix $(\gamma_{n,k})$ as follows. Before we do this, we shall change our notation somewhat. In algebraic topology, it is customary to use the notation $h(x)$ for palindromic (reciprocal) polynomials [8, 14]. Thus we shall set $h_n(x) = \sum_{k=0}^{n} h_{n,k} x^k$, where $(h_{n,k})$ now denotes a Pascal-like matrix. We shall denote by $h(x, y)$ the bivariate generating function of this matrix.

Proposition 2. For a Pascal-like matrix $(h_{n,k})$ we have

$$h_{n,k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n - 2i}{k - i} \gamma_{n,i}.$$

Proof. We have

$$h_{n,k} = [x^k] \sum_{i=0}^{n} h_{n,i} x^i$$

$$= [x^k] \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} x^i (1 + x)^{n-2i}$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} [x^k] x^i (1 + x)^{n-2i}$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} [x^{k-i}] \sum_{j=0}^{n-2i} \binom{n-2i}{j} x^j$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} \binom{n - 2i}{k - i}. \qed$$
Example 3. The identity
\[
{n \choose k} = \sum_{i=0}^{\lfloor n/2 \rfloor} {n-2i \choose k-i} \delta_{i,0}
\]
shows that the matrix that begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
is the $\gamma$-matrix for the binomial matrix $B = \left( {n \choose k} \right) \text{A007318}$. Here, we have used the $\text{Aannnnnn}$ number of the On-Line Encyclopedia of Integer Sequences [12, 13] for the binomial matrix (Pascal’s triangle).

When $(\gamma_{n,k})$ is the $\gamma$-matrix for $(h_{n,k})$, we shall say the $(\gamma_{n,k})$ generates, or is the generator of, the matrix $(h_{n,k})$.

Example 4. The matrix that begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
with $\gamma_{n,0} = 1$, $\gamma_{n,\lfloor n/2 \rfloor} = 1$, and 0 otherwise, generates the matrix $(h_{n,k})$ that begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 7 & 4 & 1 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 & 0 \\
\end{pmatrix}
\].

2 Pascal-like matrices defined by Riordan arrays

We now wish to characterize the $\gamma$-matrices that are generators for the family of Pascal-like matrices that are determined by the one-parameter family of Riordan arrays
\[
\left( \frac{1}{1-x}, \frac{x(1+rx)}{1-x} \right).
\]
We shall also determine the (generalized) \( \gamma \)-matrices associated to the reversion of these triangles. We recall that an ordinary Riordan array \((g(x), f(x))\) is defined \([1, 9, 10]\) by two power series

\[
\begin{align*}
g(x) &= 1 + g_1 x + g_2 x^2 + \ldots, \\
f(x) &= x + f_2 x^2 + f_3 x^3 + \ldots,
\end{align*}
\]

where the \((n, k)\)-th element of the resulting lower-triangular matrix is given by

\[
a_{n,k} = [x^n] g(x)f(x)^k.
\]

Such matrices are invertible. When they have integer entries, the inverse again is an integer matrix (note that we have \(a_{n,n} = 1\) in our case because \(g_0 = 1\) and \(f_1 = 1\)). The bivariate generating function of the Riordan array \((g, f)\) is given by

\[
\frac{g(x)}{1 - yf(x)}.
\]

Matrices defined in a similar manner but with \(f(x)\) replaced by \(\phi(x) = x^2 + \phi_3 x^3 + \ldots\) are called “stretched” Riordan arrays \([5]\). They are not invertible but they do possess left inverses.

**Example 5.** The stretched Riordan array \((\frac{1}{1-x}, x^2)\) begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

It is the \(\gamma\)-matrix for the Pascal-like triangle that begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 4 & 1 & 0 & 0 \\
1 & 5 & 9 & 5 & 1 & 0 \\
1 & 6 & 14 & 14 & 6 & 1 \\
1 & 7 & 20 & 29 & 20 & 7 \\
\end{pmatrix}
\]

**Example 6.** The matrix \(\binom{n-k}{k}\) is the stretched Riordan array \((\frac{1}{1-x}, \frac{x^2}{1-x})\) that begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 3 & 0 & 0 & 0 \\
1 & 5 & 6 & 1 & 0 & 0 \\
\end{pmatrix}
\]
It generates the Pascal-like matrix that begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 5 & 5 & 1 & 0 & 0 & 0 \\
1 & 7 & 13 & 7 & 1 & 0 & 0 \\
1 & 9 & 25 & 25 & 9 & 1 & 0 \\
1 & 11 & 41 & 63 & 41 & 11 & 1
\end{pmatrix}.
\]

We shall see that this is the Riordan array \(\left(\frac{1}{1-x}, \frac{x(x+1)}{1-x}\right)\), which is A008288, the triangle of Delannoy numbers.

The bivariate generating function of the stretched Riordan array \((g(x), \phi(x))\) is given by
\[
g(x) = \frac{1}{1 - y\phi(x)}.
\]

We have the following proposition [4].

**Proposition 7.** The Riordan array \(\left(\frac{1}{1-x}, \frac{x(x+1)}{1-x}\right)\) is Pascal-like (for any \(r \in \mathbb{Z}\)).

This is clear since in this case we have
\[
h_{n,k} = \sum_{j=0}^{k} \binom{k}{j} (n - j)(n - k - j) r^{j} = \sum_{j=0}^{k} \binom{k}{j} \binom{n - k}{n - k - j} (r + 1)^{j}.
\]

We can now characterize the \(\gamma\)-matrices that generate these Pascal-like matrices.

**Proposition 8.** The \(\gamma\)-matrices that generate the Pascal-like matrices \(\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)\) defined by ordinary Riordan arrays are given by the stretched Riordan arrays
\[
\left(\frac{1}{1-x}, \frac{rx^{2}}{1-x}\right),
\]
with \((n, k)\)-th term
\[
\gamma_{n,k} = \binom{n - k}{k} r^{k}.
\]

**Proof.** The generating function of the Pascal-like matrix \(\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)\) is given by
\[
h(x, y) = \frac{1}{1 - x} \frac{1}{1 - y^{x(1+rx)/(1-x)}} = \frac{1}{1 - (1+y)x - rx^{2}y}.
\]

Similarly, the generating function of the matrix \(\binom{n-k}{k} r^{k}\) is given by
\[
\gamma(x, y) = \frac{1}{1 - x} \frac{1}{1 - y^{r x^{2}/(1-x)}} = \frac{1}{1 - x - r x^{2} y}.
\]
We now have
\[ h(x, y) = \gamma \left( (1 + y)x, \frac{y}{(1 + y)^2} \right). \]

We recall that for a generating function \( f(x) \), its \textsc{invert}(\(\alpha\)) transform is the generating function
\[ \frac{f(x)}{1 + \alpha xf(x)}. \]

Note that
\[ \frac{v}{1 + \alpha x v} = v, \]
and thus the inverse of the \textsc{invert}(\(\alpha\)) transform is the \textsc{invert}(\(-\alpha\)) transform.

\textbf{Corollary 9.} The generating function \( h(x, y) \) of the Pascal-like matrix \( \begin{pmatrix} 1 & 1 + x \\ -x & -x(1 + rx) \end{pmatrix} \) is the \textsc{invert}(\(y\)) transform of the generating function \( \gamma(x, y) \) of the corresponding \( \gamma \)-matrix.

\textbf{Proof.} A direct calculation shows that for \( \gamma(x, y) = \frac{1}{1 - yx \gamma(x, y)} \) we have
\[ \frac{\gamma(x, y)}{1 - yx \gamma(x, y)} = \frac{1}{1 - (y + 1)x - r x^2 y} = h(x, y). \]

Equivalently, we can say that the generating function of the \( \gamma \)-matrix is the \textsc{invert}(\(-y\)) transform of the generating function of the corresponding Pascal-like matrix.

We make the following observation, which will be relevant when we discuss a family of generalized Narayana triangles. The \( \gamma \)-matrix corresponding to the signed Pascal-like matrix
\[ \left( \frac{1}{1 + x}, -x(1 + rx) \right) \]
has generating function
\[ \frac{1}{1 + x + r x^2 y}. \]

This is the matrix with general term \((-1)^{n-k} r^k \binom{n}{k}\). By a signed Pascal-like matrix in this case we mean that \( a_{n,k} = a_{n,n-k} \) but we now have \( a_{n,0} = a_{n,n} = (-1)^n \).

We close this section by recalling the formula
\[ \gamma_n = (1 + x)^n \gamma_n \left( \frac{x}{(1 + x)^2} \right). \]

We now note that the inverse of the Riordan array
\[ \begin{pmatrix} 1, \frac{x}{(1 + x)^2} \end{pmatrix} \]
is given by
\[(1, xc(x)^2),\]
where
\[c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}\]
is the generating function of the Catalan numbers \(C_n = \frac{1}{n+1} \binom{2n}{n}\) \(A000108\). In fact, we have the following result [8].

**Proposition 10. (Zeilberger’s Lemma).**

\[
\gamma_{n,k} = [x^k] \frac{h_n(xc(x)^2)}{c(x)^n}.
\]

We can use this result to find an explicit formula for \(\gamma_{n,k}\) in terms of \(h_{n,k}\). We let \(\alpha_{n,k}\) be the general \((n, k)\)-th element of the Riordan array \((1, xc(x)^2)\). We have
\[
\alpha_{n,k} = \binom{2n-1}{n-k} \frac{2k + 0^{n+k}}{n + k + 0^{n+k}}.
\]

We let \(\beta_{n,k}\) be the general \((n, k)\)-th term of the Riordan array \((1, \frac{x}{c(x)})\). We have \(\beta_{n,n} = 1\), and
\[
\beta_{n,k} = \sum_{j=0}^{n-k} \frac{(-1)^j j}{n-k} \binom{k+j-1}{j} \binom{2(n-k)}{n-k-j},
\]
otherwise. This is essentially \(A271875\). Then we have the following result.

**Corollary 11.** We have
\[
\gamma_{n,k} = \sum_{i=0}^{k} \left( \sum_{j=0}^{n} h_{n,j} \alpha_{i,j} \right) \beta_{n+k-i,n}.
\]

**Proof.** We have
\[
[x^k] \frac{h_n(xc(x)^2)}{c(x)^n} = \sum_{i=0}^{n} [x^i] \sum_{j=0}^{n} h_{n,j} (xc(x)^2)^j [x^{k-i}] \frac{1}{c(x)^n} = \sum_{i=0}^{k} \left( \sum_{j=0}^{n} h_{n,j} [x^i] (xc(x)^2)^j \right) [x^{k-i+n}] \frac{x^n}{c(x)^n} = \sum_{i=0}^{k} \left( \sum_{j=0}^{n} h_{n,j} \alpha_{i,j} \right) \beta_{n+k-i,n}.
\]

This gives us the following formula.
\[
\gamma_{n,k} = \sum_{i=0}^{k} \sum_{j=0}^{n} h_{n,j} \binom{2i - 1}{i - j} \frac{2j + 0^{i+j}}{i + j + 0^{i+j}} \text{If } (k = i, 1, \sum_{m=0}^{k-i} \frac{m(-1)^m}{k-i} \binom{n - 1 + m}{m} \binom{2(k-i)}{k-i-m}).
\]
Example 12. If we take $(h_{n,k})$ to be the triangle of Eulerian numbers A008292 that begins

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 & 0 & 0 \\
1 & 11 & 11 & 1 & 0 & 0 & 0 \\
1 & 26 & 66 & 26 & 1 & 0 & 0 \\
1 & 57 & 302 & 302 & 57 & 1 & 0 \\
1 & 120 & 1191 & 2416 & 1191 & 120 & 1
\end{pmatrix}
$$

we find that the $\gamma$-matrix $(\gamma_{n,k})$ is the triangle A101280 that begins

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 8 & 0 & 0 & 0 & 0 & 0 \\
1 & 22 & 16 & 0 & 0 & 0 & 0 \\
1 & 52 & 136 & 0 & 0 & 0 & 0 \\
1 & 114 & 720 & 272 & 0 & 0 & 0
\end{pmatrix}
$$

This is the triangle of $\gamma$-vectors for the permutahedra (of type A). It also gives the number of permutations of $n$ objects with $k$ descents such that every descent is a peak [11].

Example 13. We consider the Pascal-like matrix $(h_{n,k}) = \left( \frac{1}{1-x}, x \right)$ that begins

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

We note that the row elements are constant. We have that

$$
\gamma_{n,k} = \sum_{i=0}^{k} \sum_{j=0}^{n} \frac{(2i-1)}{i-j} \frac{2j+0^{i+j}}{i+j+0^{i+j}} \text{If } k = i, 1, \sum_{m=0}^{k-i} \frac{m(-1)^m}{m} \frac{(n-1+m)}{k-i} \left( \frac{2(k-i)}{k-i-m} \right).
$$

We find that the $\gamma$-matrix in this case begins

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 0 & 0 \\
1 & -4 & 3 & 0 & 0 & 0 & 0 \\
1 & -5 & 6 & -1 & 0 & 0 & 0
\end{pmatrix}
$$
This is the matrix \( \binom{n-k}{k}(-1)^k \). Thus

\[
\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i}{k-i} \binom{n-2i}{i}(-1)^{i} = \text{If}[k \leq n, 1, 0].
\]

### 3 Stretched Riordan arrays as \( \gamma \)-matrices

Every stretched Riordan array of the form

\[
\left( \frac{1}{1-x}, x^2g(x) \right),
\]

where

\[
g(x) = 1 + g_1 x + g_2 x^2 + \cdots
\]

can be used to generate a Pascal-like matrix. Thus to each power series \( g(x) \) above we can associate a Pascal-like matrix whose \( \gamma \)-matrix is given by this stretched Riordan array.

In this section, we shall concentrate on the case when \( g(x) = \frac{1+rx}{1-x} \).

**Example 14.** For \( r = 1 \), we obtain the \( \gamma \)-matrix that begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 5 & 1 & 0 & 0 & 0 & 0 \\
1 & 7 & 5 & 0 & 0 & 0 & 0 \\
1 & 9 & 13 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]

The corresponding Pascal-like matrix then begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 6 & 6 & 1 & 0 & 0 & 0 \\
1 & 9 & 17 & 9 & 1 & 0 & 0 \\
1 & 12 & 36 & 36 & 12 & 1 & 0 \\
1 & 15 & 64 & 101 & 64 & 15 & 1
\end{pmatrix}.
\]

The row sums of this matrix, which begin

\[1, 2, 5, 14, 37, 98, 261, \ldots\]

give \text{A077938}, with generating function

\[
\frac{1}{1-2x-x^2-2x^3}.
\]
The diagonal sums, which begin

$$1, 1, 2, 4, 8, 16, 31, \ldots$$

are the Pentanacci numbers \[A001591\] with generating function

$$\frac{1}{1-x-x^2-x^3-x^4-x^5}.$$

We have the following proposition.

**Proposition 15.** The Pascal-like triangle that begins

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & r+5 & r+5 & 1 & 0 & 0 & 0 \\
1 & 2r+7 & 4r+13 & 2r+7 & 1 & 0 & 0 \\
1 & 3r+9 & 11r+25 & 11r+25 & 3r+9 & 1 & 0 \\
1 & 4r+11 & r^2+22r+41 & 2r^2+36r+63 & r^2+22r+41 & 4r+11 & 1
\end{pmatrix}
$$

with γ-matrix given by the stretched Riordan array \(\left(\frac{1}{1-x}, \frac{x^2(1+rx)}{1-x}\right)\), has row sums with generating function

$$\frac{1}{1-2x-x^2-2rx^3},$$

and diagonal sums given by the generalized Pentanacci numbers with generating function

$$\frac{1}{1-x-x^2-x^3-rx^4-rx^5}.$$

**4 Reverting triangles**

Let \(h(x, y)\) be the generating function of the lower-triangular matrix \(h_{n,k}\), with \(h_{0,0} = 1\). By the reversion of this triangle, we shall mean the triangle whose generating function \(h^*(x, y)\) is given by

$$h^*(x, y) = \frac{1}{x} \text{Rev}_x(xh(x, y)).$$

Procedurally, this means that we solve the equation

$$uh(u, y) = x$$

and then we divide the solution \(u(x, y)\) that satisfies \(u(0, y) = 0\) by \(x\).

**Proposition 16.** The generating function of the reversion of the Pascal-like matrix defined by the Riordan array \(\left(\frac{1}{1-x}, \frac{x^2(1+rx)}{1-x}\right)\) is given by

$$h^*(x, y) = \frac{1}{1 + x(y + 1)c} \left(\frac{-rx^2y}{(1 + x(y + 1))^2}\right),$$
where
\[ c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \]
is the generating function of the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \). (A000108).

**Proof.** Solving the equation
\[ \frac{u}{1 - u(y + 1) - ru^2y} = x \]
gives us
\[ h^*(x, y) = \frac{-1 - x(y + 1) + \sqrt{1 + 2x(y + 1) + x^2(1 + 2y(2r + 1) + y^2)}}{2rx^2y}. \]
Thus
\[ h^*(x, y) = \frac{1}{1 + x(y + 1)} c \left( \frac{-rx^2y}{(1 + x(y + 1))^2} \right). \]

We note that we can now calculate an expression for the terms of the reverted triangle, since, using the language of Riordan arrays, we have
\[ h^*(x, y) = \left( \frac{1}{1 + y(x + 1)}, \frac{-rx^2y}{(1 + x(y + 1))^2} \right) \cdot c(x). \]

**Proposition 17.** We have
\[ [x^n][y^i]h^*(x, y) = h^*_{n,i} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n (-r)^k \binom{n}{2k} C_k \binom{n - 2k}{i - k}. \]

The \( \gamma \)-matrix of the reverted triangle \( (h^*_{n,k}) \) is given by
\[ \gamma^*_{n,k} = (-1)^n (-r)^k \binom{n}{2k} C_k. \]
The \( \gamma \)-matrix \( (\gamma^*_{n,k}) \) of the reverted triangle \( (h^*_{n,k}) \) is the reversion of the triangle \( \gamma_{n,k} \).

**Proof.** The expression for \( h^*_{n,k} \) results from a direct calculation. Reverting the expression
\[ \gamma(x, y) = \frac{1}{1 - x - rx^2y} \]
in the sense above gives us
\[ \gamma^*(x, y) = \frac{1}{1 + x} c \left( \frac{-rx^2y}{(1 + x)^2} \right), \]
from which we deduce the other statements. \( \square \)
Example 18. For $r = -1, 0, 1$, the triangles $(h_{n,k})$ begin, respectively,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{pmatrix},
$$

and

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 5 & 5 & 1 & 0 & 0 & 0 \\
1 & 7 & 13 & 7 & 1 & 0 & 0 \\
1 & 9 & 25 & 25 & 9 & 1 & 0 \\
1 & 11 & 41 & 63 & 41 & 11 & 1
\end{pmatrix}.
$$

The corresponding reverted triangles $(h_{n,k}^*)$ are, respectively,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 10 & 20 & 10 & 1 & 0 & 0 \\
1 & 15 & 50 & -50 & -15 & -1 & 0 \\
1 & 21 & 105 & 175 & 105 & 21 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{pmatrix}.
$$

and

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Note that for $r = -1$, the reverted triangle is $(-1)^n$ times the Narayana triangle A001263.

The corresponding $\gamma$-matrices $(\gamma_{n,k})$ are given by, respectively,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 0 \\
1 & -4 & 3 & 0 & 0 & 0 \\
1 & -5 & 6 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$
and
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 3 & 0 & 0 & 0 \\
1 & 5 & 6 & 1 & 0 & 0
\end{pmatrix}.
\]

The corresponding reverted $\gamma$-matrices ($\gamma^*_{n,k}$) are then, respectively,
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -3 & 0 & 0 & 0 & 0 \\
1 & 6 & 2 & 0 & 0 & 0 \\
-1 & -10 & -10 & 0 & 0 & 0 \\
1 & 15 & 30 & 5 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 \\
1 & -6 & 2 & 0 & 0 & 0 \\
-1 & -10 & -10 & 0 & 0 & 0 \\
1 & -15 & 30 & -5 & 0 & 0
\end{pmatrix},
\]

and
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 3 & 0 & 0 & 0 & 0 \\
1 & -6 & 2 & 0 & 0 & 0 \\
-1 & 10 & -10 & 0 & 0 & 0 \\
1 & -15 & 30 & -5 & 0 & 0
\end{pmatrix}.
\]

It is interesting to represent the generating functions of the ($\gamma^*_{n,k}$) and the ($h^*_{n,k}$) triangles as Jacobi continued fractions. We have

**Proposition 19.** The generating function $h^*(x, y)$ can be expressed as the Jacobi continued fraction
\[
\mathcal{J}(-(y + 1), -(y + 1), -(y + 1), \ldots; -ry, -ry, -ry, \ldots).
\]
The generating function $\gamma^*(x, y)$ can be expressed as the Jacobi continued fraction
\[
\mathcal{J}(-1, -1, -1, \ldots; -ry, -ry, -ry, \ldots).
\]

**Proof.** We solve the continued fraction equation
\[
u = \frac{1}{1 + (y + 1)x + rx^2u}
\]
to retrieve the generating function $h^*(x, y)$. Similarly, we solve the continued fraction equation
\[
u = \frac{1}{1 + x + rx^2u}
\]
to retrieve the generating function $\gamma^*(x, y)$.
\[\square\]
Note that we have used the notation \( \mathcal{J}(a, b, c, \ldots; r, s, t, \ldots) \) to denote the Jacobi continued fraction \([2, 15]\)

\[
\frac{1}{1 - ax - \frac{r x^2}{1 - bx - \frac{s x^2}{1 - cx - \frac{t x^2}{\ddots}}}}.
\]

We can now express the relationship between the generating functions \( h^*(x, y) \) and \( \gamma^*(x, y) \) in terms of repeated binomial transforms.

**Corollary 20.** The generating function \( h^*(x, y) \) is the \((-y)\)-th binomial transform of the \( \gamma \) generating function \( \gamma^*(x, y) \):

\[
h^*(x, y) = \frac{1}{1 + xy} \gamma^* \left( \frac{x}{1 + xy}, y \right).
\]

Equivalently, the \( \gamma \) generating function \( \gamma^*(x, y) \) is the \( y \)-th binomial transform of the generating function \( h^*(x, y) \):

\[
\gamma^*(x, y) = \frac{1}{1 - xy} h^* \left( \frac{x}{1 - xy}, y \right).
\]

This reflects the general assertion that the reversion of an INVERT transform is a binomial transform.

### 5 The \( \gamma \)-vectors of generalized Narayana numbers

The Riordan array \( \left( \frac{1}{1 + x}, \frac{-x(1 + rx)}{1 + x} \right) \), with bivariate generating function

\[
\frac{1}{1 + x(y + 1) + rx^2y},
\]

has a \( \gamma \)-matrix with generating function

\[
\frac{1}{1 + x + rx^2y}.
\]

We shall call elements of the reversions of the Riordan array \( \left( \frac{1}{1 + x}, \frac{-x(1 + rx)}{1 + x} \right) \) \( r \)-Narayana numbers. The Narayana numbers \( N_{n,k} = \frac{1}{k+1} \binom{n+1}{k} \binom{n}{k} \) are then the 1-Narayana numbers. The bivariate generating function for the \( r \)-Narayana numbers is given by

\[
\frac{1}{1 - x(y + 1)}^r \left( \frac{rx^2y}{(1 - x(y + 1))^2} \right).
\]

The bivariate generating function for the \( \gamma \)-matrix of the \( r \)-Narayana numbers is then obtained by reverting the generating function \( \frac{1}{1 + x + rx^2y} \). We thus obtain the following result.
Proposition 21. The $\gamma$-matrix for the $r$-Narayana numbers has generating function

$$\frac{1}{1-x} c \left( \frac{rx^2y}{(1-x)^2} \right).$$

This is the matrix that begins

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & r & 0 & 0 & 0 & 0 & 0 \\
1 & 3r & 0 & 0 & 0 & 0 & 0 \\
1 & 6r & 2r^2 & 0 & 0 & 0 & 0 \\
1 & 10r & 10r^2 & 0 & 0 & 0 & 0 \\
1 & 15r & 30r^2 & 5r^3 & 0 & 0 & 0
\end{pmatrix},$$

with general term

$$\begin{pmatrix} n \\ 2k \end{pmatrix} r^k C_k.$$

For $r = -1, 0, 1$, the matrices

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
-1 & -5 & -5 & -1 & 0 & 0 & 0 \\
1 & 7 & 13 & 7 & 1 & 0 & 0 \\
-1 & -9 & -25 & -25 & -9 & -1 & 0 \\
1 & 11 & 41 & 63 & 41 & 11 & 1
\end{pmatrix},$$

begin, respectively,

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
-1 & -3 & -3 & -1 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 \\
-1 & -5 & -10 & -10 & -5 & -1 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{pmatrix},$$

and

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}. $$

The corresponding matrices of $r$-Narayana numbers are, respectively,

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -2 & -4 & -2 & 1 & 0 & 0 \\
1 & -5 & -10 & -10 & -5 & 1 & 0 \\
1 & -9 & -15 & -15 & -15 & -9 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{pmatrix}. $$
This last matrix, as expected, is the Narayana triangle \(A001263\). The corresponding \(\gamma\)-matrices for these \(r\)-Narayana triangles are, respectively,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & 0 & 0 & 0 & 0 & 0 \\
1 & -6 & 2 & 0 & 0 & 0 & 0 \\
1 & -10 & 10 & 0 & 0 & 0 & 0 \\
1 & -15 & 30 & -5 & 0 & 0 & 0 \\
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 6 & 2 & 0 & 0 & 0 & 0 \\
1 & 10 & 10 & 0 & 0 & 0 & 0 \\
1 & 15 & 30 & 5 & 0 & 0 & 0 \\
\end{pmatrix}
\]

This last matrix is \(A055151\). The rows of this triangle are the \(\gamma\)-vectors of the \(n\)-dimensional (type A) associahedra [8]. We have seen that its elements are given by

\[
\gamma_{n,k} = \sum_{i=0}^{k} \sum_{j=0}^{n} N_{n,j} \left(\frac{2i-1}{i-j}\right)^{2j+0^i+j} i+j+0^i+j \text{If} \left(k = i, 1, \sum_{m=0}^{k-i} m(-1)^m \left(\begin{pmatrix}n-1+m\end{pmatrix}\left(\begin{pmatrix}2(k-i)\end{pmatrix}\right)\right)\right),
\]

where \(N_{n,k}\) denotes the \((n, k)\)-th Narayana number \(A001263\).

The relationship between the \(\gamma\)-matrix and the \(r\)-Narayana numbers can be further clarified as follows.

**Proposition 22.** The generating function of the \(r\)-Narayana numbers can be expressed as the Jacobi continued fraction

\[
\mathcal{J}((y+1), (y+1), (y+1), \ldots; ry, ry, ry, \ldots).
\]

The generating function of the corresponding \(\gamma\)-matrix can be expressed as the Jacobi continued fraction

\[
\mathcal{J}(1, 1, 1, \ldots; ry, ry, ry, \ldots).
\]
Corollary 23. The generating function of the r-Narayana numbers is the $y$-th binomial transform of the generating function of the corresponding $\gamma$-matrix.

$$h^*(x, y) = \frac{1}{1 - xy} \gamma^* \left( \frac{x}{1 - xy}, y \right).$$

Equivalently, the $\gamma$ generating function $\gamma^*(x, y)$ is the $(-y)$-th binomial transform of the generating function $h^*(x, y)$:

$$\gamma^*(x, y) = \frac{1}{1 + xy} h^* \left( \frac{x}{1 + xy}, y \right).$$

6 Pascal-like triangles defined by exponential Riordan arrays

We recall that an exponential Riordan array $[g(x), f(x)]$ [1, 6] is defined by two exponential generating functions

$$g(x) = 1 + g_1 \frac{x}{1!} + g_2 \frac{x^2}{2!} + \cdots,$$

and

$$f(x) = \frac{x}{1!} + f_2 \frac{x^2}{2!} + \cdots,$$

with its $(n, k)$-th term $a_{n,k}$ given by

$$a_{n,k} = \frac{n!}{k!} [x^n] g(x) f(x)^k.$$

In the context of Pascal-like matrices, we have that the exponential Riordan array

$$[e^x, x(1 + rx/2)],$$

with general term

$$h_{n,k} = \frac{n!}{k!} \sum_{j=0}^{k} \frac{r^j}{(n - k - j)!2^j},$$

is a Pascal-like matrix [3]. This matrix begins

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & r + 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 3r + 3 & 3r + 3 & 1 & 0 & 0 & 0 \\
1 & 6r + 4 & 3r^2 + 12r + 6 & 6r + 4 & 1 & 0 & 0 \\
1 & 10r + 5 & 15r^2 + 30r + 10 & 15r^2 + 30r + 10 & 10r + 5 & 1 & 0 \\
1 & 15r + 6 & 45r^2 + 60r + 15 & 15r^2 + 90r^2 + 90r + 20 & 45r^2 + 60r + 15 & 15r + 6 & 1
\end{pmatrix}.$$
Proposition 24. The $\gamma$-matrix of the the Pascal-like exponential Riordan array $[e^x, x(1 + rx/2)]$ is the matrix with general term

$$\binom{n}{2k} r^k (2^k - 1)!!$$

In fact, the generating function of the exponential Riordan array $[e^x, x(1 + rx/2)]$ is given by

$$J(y + 1, y + 1, y + 1, \ldots; ry, 2ry, 3ry, \ldots)$$

while that of its $\gamma$-matrix is given by

$$J(1, 1, 1, \ldots; ry, 2ry, 3ry, \ldots).$$

Proposition 25. The generating function of the $\gamma$-matrix of the Pascal-like exponential Riordan array $[e^x, x(1 + rx/2)]$ has generating function

$$e^{x(1+rx/2)}.$$

Proof. By the theory of exponential Riordan arrays, the generating function of the Riordan array $[e^x, x(1 + rx/2)]$ is given by

$$e^x e^{xy(1+rx/2)}.$$

Taking the $(-y)$-th binomial transform of this, we obtain

$$e^{x(1+rx/2)}.$$

Example 26. For $r = 1$, we get the $\gamma$-matrix that begins

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 6 & 3 & 0 & 0 & 0 & 0 \\
1 & 10 & 15 & 0 & 0 & 0 & 0 \\
1 & 15 & 45 & 15 & 0 & 0 & 0
\end{pmatrix}.$$  

This is $A100861$, the triangle of Bessel numbers that count the number of $k$-matchings of the complete graph $K(n)$. The corresponding Pascal-like matrix begins

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 6 & 6 & 1 & 0 & 0 & 0 \\
1 & 10 & 21 & 10 & 1 & 0 & 0 \\
1 & 15 & 55 & 55 & 15 & 1 & 0 \\
1 & 21 & 120 & 215 & 120 & 21 & 1
\end{pmatrix}.$$  

This is $A100862$, which counts the number of $k$-matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$. 

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Example 27. For \( r = 2 \), we obtain the \( \gamma \)-matrix that begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 6 & 0 & 0 & 0 & 0 & 0 \\
1 & 12 & 12 & 0 & 0 & 0 & 0 \\
1 & 20 & 60 & 0 & 0 & 0 & 0 \\
1 & 30 & 180 & 120 & 0 & 0 & 0
\end{pmatrix}.
\]

This is \texttt{A059344}, where row \( n \) consists of the nonzero coefficients of the expansion of \( 2^nx^n \) in terms of Hermite polynomials with decreasing subscripts. The corresponding Pascal-like matrix begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 & 0 & 0 \\
1 & 9 & 9 & 1 & 0 & 0 & 0 \\
1 & 16 & 42 & 16 & 1 & 0 & 0 \\
1 & 25 & 130 & 130 & 25 & 1 & 0 \\
1 & 36 & 315 & 680 & 315 & 36 & 1
d\end{pmatrix}.
\]

The row sums of this matrix are given by \texttt{A000898}, the number of symmetric involutions of \([2n]\) (Deutsch).

7 Conclusion

It is the case that the set of Pascal-like matrices defined by Riordan arrays is a restricted one. Nevertheless, we hope that this note indicates that they have interesting properties, including in particular their generating \( \gamma \)-matrices. In the case of Pascal-like matrices defined by ordinary Riordan arrays, we have seen that by reverting them, we find additional (signed) Pascal-like triangles, including triangles of Narayana type. The \( \gamma \)-matrices of these new triangles are again the reversions of the original triangles’ \( \gamma \)-matrices.

We have also shown that stretched Riordan arrays play a useful role, and in particular can lead to further (non-Riordan) Pascal-like matrices. We have also found it useful to use Riordan array techniques to find an explicit closed form formula for the elements \( \gamma_{n,k} \) of the \( \gamma \)-matrix in terms of \( h_{n,k} \).

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