ON BOOTH LEMNISCATE OF STARLIKE FUNCTIONS

RAHIM KARGAR, ALI EBADIAN AND JANUSZ SOKÓŁ

Abstract. Assume that ∆ is the open unit disk in the complex plane and A is the class of normalized analytic functions in Δ. In this paper we introduce and study the class
$$BS(\alpha) := \left\{ f \in A : \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{z}{1 - \alpha z^2}, z \in \Delta \right\},$$
where $0 \leq \alpha \leq 1$ and $\prec$ is the subordination relation. Some properties of this class like differential subordination, coefficients estimates and Fekete-Szegő inequality associated with the $k$-th root transform are considered.

1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form:
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
which are analytic and normalized in the open unit disk $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. The subclass of $A$ consisting of all univalent functions $f(z)$ in $\Delta$ is denoted by $S$. A function $f \in S$ is called starlike (with respect to 0), denoted by $f \in S^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. Robertson introduced in [8], the class $S^*(\gamma)$ of starlike functions of order $\gamma \leq 1$, which is defined by
$$S^*(\gamma) := \left\{ f \in A : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, z \in \Delta \right\}.$$
If $\gamma \in [0, 1)$, then a function in $S^*(\gamma)$ is univalent. In particular we put $S^*(0) \equiv S^*$. We denote by $B$ the class of analytic functions $w(z)$ in $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \Delta)$. If $f$ and $g$ are two of the functions in $A$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$, if there exists a $w \in B$ such that $f(z) = g(w(z))$, for all $z \in \Delta$.

Furthermore, if the function $g$ is univalent in $\Delta$, then we have the following equivalence:
$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta)).$$

We now recall from [7], a one-parameter family of functions as follows:
$$F_{\alpha}(z) := \frac{z}{1 - \alpha z^2} = z + \sum_{n=1}^{\infty} \alpha^n z^{2n+1} \quad (z \in \Delta, \ 0 \leq \alpha \leq 1).$$

The function $F_{\alpha}(z)$ is starlike univalent for $0 \leq \alpha < 1$. We have also $F_{\alpha}(\Delta) = D(\alpha)$, where
$$D(\alpha) = \left\{ x + iy \in \mathbb{C} : (x^2 + y^2)^2 - \frac{x^2}{(1 - \alpha)^2} - \frac{y^2}{(1 + \alpha)^2} < 0 \right\},$$

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when $0 \leq \alpha < 1$ and
\[ D(1) = \{ (x + iy) \in \mathbb{C} : (\forall t \in (-\infty, -i/2] \cup [i/2, \infty)) (x + iy \neq it) \}. \]

Figure 1. The boundary curve of $D(1/2)$

The Persian curve (cf. [10]) is a plane algebraic curve of order four that is the line of intersection between the surface of a torus and a plane parallel to its axis. The equation in rectangular coordinates is
\[
(x^2 + y^2 + p^2 + d^2 - r^2)^2 = 4d^2 (x^2 + p^2),
\]
where $r$ is the radius of the circle describing the torus, $d$ is the distance from the origin to its center and $p$ is the distance from the axis of the torus to the plane. We remark that a curve described by
\[
(x^2 + y^2)^2 - (n^4 + 2m^2)x^2 - (n^4 - 2m^2)y^2 = 0 \quad (x, y) \neq (0, 0),
\]
is a special case of Persian curve that studied by Booth and is called the Booth lemniscate [2]. The Booth lemniscate is called elliptic if $n^4 > 2m^2$ while, for $n^4 < 2m^2$, it is termed hyperbolic. Thus it is clear that the curve
\[
(x^2 + y^2)^2 - \frac{x^2}{(1 - \alpha)^2} - \frac{y^2}{(1 + \alpha)^2} = 0 \quad (x, y) \neq (0, 0),
\]
is the Booth lemniscate of elliptic type (see figure 1). Two other special case of Persian curve are Cassini oval and Bernoulli lemniscate.

A plane algebraic curve of order four whose equation in Cartesian coordinates has the form:
\[
(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4.
\]
The Cassini oval is the set of points such that the product of the distances from each point to two given points $p_2 = (-c, 0)$ and $p_1 = (c, 0)$ (the foci) is constant. When $a \geq c\sqrt{2}$ the Cassini oval is a convex curve; when $c < a < c\sqrt{2}$ it is a curve with "waists" (concave parts); when $a = c$ it is a Bernoulli lemniscate; and when $a < c$ it consists of two components. Cassini ovals are related to lemniscates. Cassini ovals were studied by G. Cassini (17th century) in his attempts to determine the Earth’s orbit.

The Bernoulli lemniscate plane algebraic curve of order four, the equation of which in orthogonal Cartesian coordinates is
\[
(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0,
\]
and in polar coordinates
\[
\rho^2 = 2a^2 \cos 2\phi.
\]
The Bernoulli lemniscate is symmetric about the coordinate origin, which is a node with tangents $y = \pm x$ and the point of inflection. The product of the distances of any point $M$ to the two given points $p_1 = (-a, 0)$ and $p_2 = (a, 0)$ is equal to the
square of the distance between the points \( p_1 \) and \( p_2 \). The Bernoulli lemniscate is a special case of the Cassini ovals, the lemniscates, and the sinusoidal spirals. The Bernoulli spiral was named after Jakob Bernoulli, who gave its equation in 1694.

In [3], the authors introduced and studied the class \( M(\delta) \) as follows:

**Definition A.** Let \( \pi/2 \leq \delta < \pi \). Then the function \( f \in A \) belongs to the class \( M(\delta) \) if \( f \) satisfies:

\[
1 + \frac{\delta - \pi}{2\sin \delta} < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\delta}{2\sin \delta} \quad (z \in \Delta).
\]

By definition of subordination and by (1.3), we have that \( f \in M(\delta) \) if and only if

\[
\left( \frac{zf'(z)}{f(z)} - 1 \right) \prec B_\alpha(z) := \frac{1 - z}{2i\sin \delta} \log \left( \frac{1 - z}{1 - ze^{-i\delta}} \right) \quad (z \in \Delta),
\]

where \( \pi/2 \leq \delta < \pi \). The above function \( B_\alpha(z) \) is convex univalent in \( \Delta \) and maps \( \Delta \) onto \( \Gamma_\delta = \{ w : (\delta - \pi)/(2\sin \delta) < \Re \{ w \} < \delta/(2\sin \delta) \} \), or onto the convex hull of three points (one of which may be that point at infinity) on the boundary of \( \Gamma_\delta \). In other words, the image of \( \Delta \) may be a vertical strip when \( \pi/2 \leq \delta < \pi \), while in other cases, a half strip, a trapezium, or a triangle.

It was proved in [6], that for \( \alpha < 1 < \beta \), the following function \( P_{\alpha,\beta} : \Delta \to \mathbb{C} \), defined by

\[
P_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left( \frac{1 - e^{2\pi i(1-\alpha) z}}{1 - z} \right) \quad (z \in \Delta),
\]

maps \( \Delta \) onto a convex domain

\[
P_{\alpha,\beta}(\Delta) = \{ w \in \mathbb{C} : \alpha < \Re \{ w \} < \beta \},
\]

conformally. Therefore, the function \( P_{\alpha,\beta}(z) \) defined by (1.4) is convex univalent in \( \Delta \) and has the form:

\[
P_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,
\]

where

\[
B_n = \frac{\beta - \alpha}{n\pi} i \left( 1 - e^{2n\pi i(1-\alpha)} \right) \quad (n = 1, 2, \ldots).
\]

The present authors (see [3]) introduced the class \( V(\alpha,\beta) \) as follows:

**Definition B.** Let \( \alpha < 1 \) and \( \beta > 1 \). Then the function \( f \in A \) belongs to the class \( V(\alpha,\beta) \) if \( f \) satisfies:

\[
\alpha < \Re \left\{ \left( \frac{z}{f(z)} \right)^2 f'(z) \right\} < \beta \quad (z \in \Delta).
\]

Therefore, by definition of subordination, we have that \( f \in V(\alpha,\beta) \) if and only if

\[
\left( \frac{z}{f(z)} \right)^2 f'(z) \prec P_{\alpha,\beta}(z) \quad (z \in \Delta).
\]

Motivated by Definition A, Definition B and using \( F_\alpha \), we introduce a new class.

**Our principal definition is the following.**

**Definition 1.1.** Let \( f \in A \) and \( 0 \leq \alpha < 1 \). Then \( f \in BS(\alpha) \) if and only if

\[
\left( \frac{z}{f(z)} \right)^2 f'(z) \prec F_\alpha(z) \quad (z \in \Delta),
\]

where \( F_\alpha \) defined by (1.2).

In our investigation, we require the following result.
Corollary 1.1. We have that \( f \in \mathcal{BS}(\alpha) \) if and only if
\[
(1.9) \quad f(z) = z \exp \int_0^z \frac{F_\alpha(w(t)) - 1}{t} \, dt \quad (z \in \Delta),
\]
for some function \( w(z) \), analytic in \( \Delta \), with \( |w(z)| \leq |z| \) in \( \Delta \).

Proof. From (1.8) it follows that there exists a function \( w(z) \), analytic in \( \Delta \), with \( |w(z)| \leq |z| \) in \( \Delta \), such that
\[
z \left( \frac{f'(z)}{f(z)} - \frac{1}{z} \right) = F_\alpha(w(z)) \quad (z \in \Delta),
\]
or
\[
z \left( \log \frac{f(z)}{z} \right)' = F_\alpha(w(z)) \quad (z \in \Delta).
\]
This gives (1.9). On the other hand, it is a easy calculation that a function having the form (1.9) satisfies condition (1.8). □

Applying formula (1.9) for \( w(z) = z \) gives that
\[
(1.10) \quad f_0(z) = z \left( \frac{1 + \sqrt{\alpha z}}{1 - \sqrt{\alpha z}} \right)^{1/\sqrt{\pi}} \quad (z \in \Delta),
\]
is in the class \( \mathcal{BS}(\alpha) \).

Lemma 1.1. Let \( F_\alpha(z) \) be given by (1.2). Then
\[
(1.11) \quad \frac{1}{\alpha - 1} < \Re \{ F_\alpha(z) \} < \frac{1}{1 - \alpha} \quad (0 \leq \alpha < 1).
\]

Proof. If \( \alpha = 0 \), then we have \(-1 < \Re \{ F_0 \} = \Re(z) < 1\). For \( 0 < \alpha < 1 \), the function \( \{ F_\alpha \} \) does not have any poles in \( \Delta \) and is analytic in \( \Delta \), thus looking for the min\{\Re\{ F_\alpha(z) \} : |z| < 1\} it is sufficient to consider it on the boundary \( \partial F_\alpha(\Delta) = \{ F_\alpha(e^{i\varphi}) : \varphi \in [0, 2\pi] \} \). A simple calculation give us
\[
\Re \{ F_\alpha(e^{i\varphi}) \} = \frac{(1 - \alpha) \cos \varphi}{1 + \alpha^2 - 2\alpha \cos 2\varphi} \quad (\varphi \in [0, 2\pi]).
\]
So we can see that \( \Re \{ F_\alpha(z) \} \) is well defined also for \( \varphi = 0 \) and \( \varphi = 2\pi \). Define
\[
g(x) = \frac{(1 - \alpha)x}{1 + \alpha^2 - 2\alpha(2x^2 - 1)} \quad ( -1 \leq x \leq 1),
\]
then for \( 0 < \alpha < 1 \), we have \( g'(x) > 0 \). Thus for \( -1 \leq x \leq 1 \), we have
\[
\frac{1}{\alpha - 1} = g(-1) \leq g(x) \leq g(1) = \frac{1}{1 - \alpha}.
\]
This completes the proof. □

We note that from Lemma 1.1 and by definition of subordination, the function \( f \in \mathcal{A} \) belongs to the class \( \mathcal{BS}(\alpha) \), \( 0 \leq \alpha < 1 \), if it satisfies the condition
\[
\frac{1}{\alpha - 1} < \Re \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{1}{1 - \alpha} \quad (z \in \Delta),
\]
or equivalently
\[
(1.12) \quad \frac{\alpha}{\alpha - 1} < \Re \left( \frac{zf'(z)}{f(z)} \right) < \frac{2 - \alpha}{1 - \alpha} \quad (z \in \Delta).
\]
It is clear that \( \mathcal{BS}(0) \equiv \mathcal{S}(0, 2) \subset \mathcal{S}^* \), where the class \( \mathcal{S}(\alpha, \beta) \), \( \alpha < 1 \) and \( \beta > 1 \), was recently considered by K. Kuroki and S. Owa in [6].
Corollary 1.2. If $f \in \mathcal{BS}(\alpha)$, then
\begin{equation}
\frac{zf'(z)}{f(z)} \prec P_\alpha(z) \quad (z \in \Delta),
\end{equation}
where
\begin{equation}
P_\alpha(z) = 1 + \frac{2}{\pi(1 - \alpha)} i \log \left( \frac{1 - e^{\pi i(1 - \alpha)z}}{1 - z} \right) \quad (z \in \Delta),
\end{equation}
is convex univalent in $\Delta$.

Proof. If $f \in \mathcal{BS}(\alpha)$, then it satisfies (1.12) or $zf'(z)/f(z)$ lies in a strip of the form (1.5). Then applying the definition of subordination and the function (1.4), we obtain (1.13) and (1.14).

For the proof of our main results, we need the following Lemma.

Lemma 1.2. (See [9]) Let $q(z) = \sum_{n=1}^{\infty} C_n z^n$ be analytic and univalent in $\Delta$, and suppose that $q(z)$ maps $\Delta$ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\Delta$ and satisfies the following subordination
\begin{equation}
p(z) \prec q(z) \quad (z \in \Delta),
\end{equation}
then
\begin{equation}
|A_n| \leq |C_1| \quad n \geq 1.
\end{equation}

2. Main Results

The first main result is the following theorem.

Theorem 2.1. Let $f \in A$ and $0 \leq \alpha < 1$. If $f \in \mathcal{BS}(\alpha)$ then
\begin{equation}
\log \frac{f(z)}{z} \prec \int_0^z \frac{P_\alpha(t) - 1}{t} \, dt \quad (z \in \Delta),
\end{equation}
where
\begin{equation}
P_\alpha(z) - 1 = \frac{2}{\pi(1 - \alpha)} i \log \left( \frac{1 - e^{\pi i(1 - \alpha)z}}{1 - z} \right) \quad (z \in \Delta)
\end{equation}
and
\begin{equation}
\tilde{P}_\alpha(z) = \int_0^z \frac{P_\alpha(t) - 1}{t} \, dt \quad (z \in \Delta),
\end{equation}
are convex univalent in $\Delta$.

Proof. If $f \in \mathcal{BS}(\alpha)$, then by (1.13) it satisfies
\begin{equation}
z \left\{ \log \frac{f(z)}{z} \right\}' \prec P_\alpha(z) - 1.
\end{equation}
It is known that if $F(z)$ is convex univalent in $\Delta$, then
\begin{equation}
[f(z) \prec F(z)] \Rightarrow \left[ \int_0^z \frac{f(t)}{t} \, dt \prec \int_0^z \frac{F(t)}{t} \, dt \right]
\end{equation}
and
\begin{equation}
\tilde{F}(z) = \int_0^z \frac{F(t)}{t} \, dt,
\end{equation}
is convex univalent in $\Delta$. By Corollary 1.2, we know that $P_\alpha(z) - 1$ is convex univalent in $\Delta$. Therefore, applying (2.3) in (2.2) gives (2.1) with convex univalent $\tilde{P}_\alpha(z)$. □
Corollary 2.1. If $f \in \mathcal{BS}(\alpha)$ and $|z| = r < 1$, then
\begin{equation}
\min_{|z|=r} \left| \exp \tilde{P}_\alpha(z) \right| \leq \frac{|f(z)|}{z} \leq \max_{|z|=r} \left| \exp \tilde{P}_\alpha(z) \right|.
\end{equation}

Proof. Subordination implies
\begin{equation}
\frac{f(z)}{z} \prec \exp \tilde{P}_\alpha(z)
\end{equation}
and $\exp \tilde{P}_\alpha(z)$ is convex univalent. Then (2.5) implies (2.4). □

We now obtain coefficients estimates for functions belonging to the class $\mathcal{BS}(\alpha)$.

Theorem 2.2. Assume that the function $f$ of the form (1.1) belongs to the class $\mathcal{BS}(\alpha)$ where $0 \leq \alpha \leq 3 - 2\sqrt{2}$. then
\begin{equation}
|a_n| \leq \frac{1}{n-1} \left( \frac{k}{k-1} \right) \quad (n = 3, 4, \ldots).
\end{equation}

Proof. Assume that $f \in \mathcal{BS}(\alpha)$. Then from Definition 1.1 we have
\begin{equation}
p(z) \prec 1 + F_\alpha(z) = 1 + z + \alpha z^3 + \cdots \quad (z \in \Delta),
\end{equation}
where
\begin{equation}
z f'(z) = p(z) f(z).
\end{equation}
We note that $F_\alpha$ is convex function for $0 \leq \alpha \leq 3 - 2\sqrt{2}$ (see [7, Corollary 3.3]). If we define
\begin{equation}
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,
\end{equation}
then from Lemma 1.2, we have
\begin{equation}
|p_n| \leq 1.
\end{equation}
Equating the coefficients of $z^n$ on both sides of (2.8), we find the following relation between the coefficients:
\begin{equation}
na_n = p_{n-1} + a_2 p_{n-2} + \cdots + a_{n-1} p_1 + a_n.
\end{equation}
Making use of (2.9) and (2.10), we get
\begin{equation}
|a_n| \leq \frac{1}{n-1} \sum_{k=1}^{n-1} |a_k| \quad |a_1| = 1.
\end{equation}

Obvious that, from (2.11), we have $|a_2| \leq 1$. We now need show that
\begin{equation}
\frac{1}{n-1} \sum_{k=1}^{n-1} |a_k| \leq \frac{1}{n-1} \prod_{k=2}^{n-1} \left( 1 + \frac{1}{k-1} \right) \quad (n = 3, 4, \ldots).
\end{equation}
We use induction to prove (2.12). If we take $n = 3$ in the inequality (2.12), we have $|a_2| \leq 1$, therefore the case $n = 3$ is clear. A simple calculation gives us
\begin{equation}
|a_{m+1}| \leq \frac{1}{m} \sum_{k=1}^{m} |a_k| = \frac{1}{m} \left( \sum_{k=1}^{m-1} |a_k| + |a_m| \right)
\leq \frac{1}{m} \prod_{k=2}^{m-1} \left( 1 + \frac{1}{k-1} \right) + \frac{1}{m} \times \frac{1}{m-1} \prod_{k=2}^{m-1} \left( 1 + \frac{1}{k-1} \right)
= \frac{1}{m} \prod_{k=2}^{m} \left( 1 + \frac{1}{k-1} \right),
\end{equation}
which implies that the inequality (2.12) holds for $n = m + 1$. From now (2.11) and (2.12), the desired estimate for $|a_n|$ ($n = 3, 4, \ldots$) follows, as asserted in (2.6). This completes the proof. □
The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $A$ is known as the Fekete-Szegő problem.

Recently, Ali et al. [1] considered the Fekete-Szegő functional associated with the $k$th root transform for several subclasses of univalent functions. We recall here that, for a univalent function $f(z)$ of the form (1.1), the $k$th root transform is defined by

$$F(z) = \left[ f(z^k) \right]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \quad (z \in \Delta).$$

Following, we consider the problem of finding sharp upper bounds for the Fekete-Szegő coefficient functional associated with the $k$th root transform for functions in the class $BS(\alpha)$.

In order to prove next result, we need the following lemma due to Keogh and Merkes [5]. Further we denote by $P$ the well-known class of analytic functions $p(z)$ with $p(0) = 1$ and $\Re(p(z)) > 0, z \in \Delta$.

**Lemma 2.1.** Let the function $g(z)$ given by

$$g(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be in the class $P$. Then, for any complex number $\mu$

$$|c_2 - \mu c_1^2| \leq 2 \max\{|1, |2\mu - 1|\}.$$

The result is sharp.

**Theorem 2.3.** Let $0 \leq \alpha < 1, f \in BS(\alpha)$ and $\mathfrak{F}$ is the $k$th root transform of $f$ defined by (2.13). Then, for any complex number $\mu$,

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{1}{2k} \max \left\{ 1, \left| \frac{2(\mu - 1)}{k} + 1 \right| \right\}.$$

The result is sharp.

**Proof.** Using (1.2), we first put

$$1 + F_\alpha(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where $B_1 = 1, B_2 = 0, B_3 = \alpha$, and etc. Since $f \in BS(\alpha)$, from Definition 1.1 and definition of subordination, there exists $w \in \mathfrak{B}$ such that

$$zf'(z)/f(z) = 1 + F_\alpha(w(z)).$$

We now define

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots.$$

Since $w \in \mathfrak{B}$, it follows that $p \in \mathcal{P}$. From (2.15) and (2.17) we have:

$$1 + F_\alpha(w(z)) = 1 + \frac{1}{2} B_1 p_1 z + \frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) z^2 + \cdots,$$

where $B_1 = 1$ and $B_2 = 0$. Equating the coefficients of $z$ and $z^2$ on both sides of (2.16) and substituting $B_1 = 1$ and $B_2 = 0$, we get

$$a_2 = \frac{1}{2} p_1,$$

and

$$a_3 = \frac{1}{8} p_1^2 + \frac{1}{4} \left( p_2 - \frac{1}{2} p_1^2 \right).$$
A computation shows that, for $f$ given by (1.1),
\begin{equation}
\hat{F}(z) = \left[ f\left(\frac{z^{1/k}}{k}\right) \right]^{1/k} = z + \frac{1}{k} a_2 z^{k+1} + \left( \frac{1}{k} a_3 - \frac{1}{2} \frac{k-1}{k^2} a_2^2 \right) z^{2k+1} + \cdots.
\end{equation}

From equations (2.13) and (2.21), we have
\begin{equation}
b_{k+1} = \frac{1}{k} a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k} a_3 - \frac{1}{2} \frac{k-1}{k^2} a_2^2.
\end{equation}

Substituting from (2.19) and (2.20) into (2.22), we obtain
\begin{equation}
b_{k+1} = \frac{1}{2k} p_1,
\end{equation}
and
\begin{equation}
b_{2k+1} = \frac{1}{4k} \left( p_2 - \frac{k-1}{k} p_1^2 \right),
\end{equation}
so that
\begin{equation}
b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{4k} \left[ p_2 - \frac{1}{2} \left( \frac{2(\mu - 1)}{k} + 2 \right) p_1^2 \right].
\end{equation}

Letting
\begin{equation}
\mu' = \frac{1}{2} \left( \frac{2(\mu - 1)}{k} + 2 \right),
\end{equation}
the inequality (2.14) now follows as an application of Lemma 2.1. It is easy to check that the result is sharp for the $k$th root transforms of the function
\begin{equation}
f(z) = z \exp \left( \int_0^z \frac{F_a(w(t))}{t} \, dt \right).
\end{equation}

Putting $k = 1$ in Theorem 2.3, we have:

**Corollary 2.2.** (Fekete-Szegő inequality) Suppose that $f \in BS(\alpha)$ and $0 \leq \alpha < 1$. Then, for any complex number $\mu$,
\begin{equation}
|a_3 - \mu a_2^2| \leq \frac{1}{2} \max \{1, |2\mu - 1|\}.
\end{equation}
The result is sharp.

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by $f^{-1}(f(z)) = z$, $z \in \Delta$ and
\begin{equation}
f(f^{-1}(w)) = w \quad (|w| < r_0; \ r_0 < 1/4),
\end{equation}
where
\begin{equation}
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.
\end{equation}

**Corollary 2.3.** Let the function $f$, given by (1.1), be in the class $BS(\alpha)$ where $0 \leq \alpha < 1$. Also let the function $f^{-1}(w) = w + \sum_{n=2} a_n w^n$ be inverse of $f$. Then
\begin{equation}
|b_2| \leq 1,
\end{equation}
and
\begin{equation}
|b_3| \leq \frac{3}{2}.
\end{equation}
Proof. Relation (2.26) give us
\[ b_2 = -a_2 \quad \text{and} \quad b_3 = 2a_2^2 - a_3. \]
Thus, we can get the estimate for \( |b_2| \) by
\[ |b_2| = |a_2| \leq 1. \]
For estimate of \( |b_3| \), it suffices in Corollary 2.2, we put \( \mu = 2 \). Hence the proof of Corollary 2.3 is completed. \( \square \)

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