Levinson theorem for discrete Schrödinger operators on the line with matrix potentials having a first moment

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Abstract

This paper proves new results on spectral and scattering theory for matrix-valued Schrödinger operators on the discrete line with non-compactly supported perturbations whose first moments are assumed to exist. In particular, a Levinson theorem is proved, in which a relation between scattering data and spectral properties (bound and half bound states) of the corresponding Hamiltonians is derived. The proof is based on stationary scattering theory with prominent use of Jost solutions at complex energies that are controlled by Volterra-type integral equations.

1 Introduction

This paper proves a Levinson theorem for a matrix-valued Schrödinger operator of the form

\[ H = H_0 + V \]

on the Hilbert space \( \ell^2(\mathbb{Z}, \mathbb{C}^L) \) with \( L \in \mathbb{N} \), where the free operator \( H_0 \) is the discrete Laplacian, up to an additive constant, given by

\[ H_0 u(n) := u(n + 1) + u(n - 1), \quad u \in \ell^2(\mathbb{Z}, \mathbb{C}^L), \]

where \( V(n) \in \mathbb{C}^{L \times L} \) is a self-adjoint \( L \times L \) matrix for every \( n \in \mathbb{N} \). The main assumption is that its first moment exists, i.e.

\[ \sum_n \|nV(n)\| < \infty. \]

Levinson theorem, see Theorem 4 below, relates scattering data to the number of bound and half-bound states of \( H \). The objects of scattering theory are formulated in terms of
Jost solutions which are formal eigenvectors of $H$ with prescribed asymptotic behavior. More precisely, we will identify $H$, $H_0$ and $V$ with extended operators acting on sequences $u \in (C^L)^Z$ or $u \in M^Z$ where $M = C^{L \times L}$, given by the same formulas, and then consider the formal (or generalized) eigenvalue equation

$$Hu = Eu,$$

where $E \in \mathbb{C}$ is parameterized in the following form

$$E = z + 1/z, \quad z \in \mathbb{C}.\quad (4)$$

Then the Jost solutions $u^z_\pm \in M^Z$ are specified by

$$u^z_\pm(n) = z^n(1 + o(1)), \quad n \to \pm \infty,\quad (5)$$

where $1$ is the identity matrix in $M$. As $n \mapsto z^n1$ are the solutions of the free equation $H_0u = Eu$, one can alternatively state that the Jost solutions $u^z_\pm$ are those solutions of \eqref{eq:3} that behave as the free solutions asymptotically at $\pm \infty$. In Section 2 we prove their existence and several of their properties. Our previous work \cite{9} only addresses the case $0 < |z| \leq 1$, which is hence here extended to cover $|z| > 1$. The full picture, that is $z \in \mathbb{C} \setminus \{0\}$, and furthermore a discussion of the case $z = 0$ are necessary for a proof of Levinson’s theorem. The following elementary remark allows to introduce the scattering matrix and will be referred to at several reprises.

**Remark 1.** For each $z \in \mathbb{C} \setminus \{1, 0, -1\}$, due to the asymptotic behavior \eqref{eq:5} of the Jost solutions, the columns of the matrix $(u^z_+, u^1/z)$ are linearly independent for $z \in \mathbb{C} \setminus \{0, -1, 1\}$ and, therefore, they form a basis of solutions of the generalized eigenvalue equation \eqref{eq:3}. $\diamond$

**Definition 2.** For $z \in \mathbb{C} \setminus \{1, 0, -1\}$, we denote by $M^z_\pm$ and $N^z_\pm$ the $L \times L$ matrices satisfying

$$u^z_+ = u^-_+M^z_+ + u^-_+N^z_+, \quad u^z_- = u^1/z M^z_- + u^1/z N^z_-.\quad (6)$$

The matrices $M^z_\pm$ and $N^z_\pm$ are the key ingredients in stationary scattering theory. The scattering matrix is built from them in the following way:

**Definition 3.** For $z \in \mathbb{C} \setminus \{1, 0, -1\}$ with $|z| \leq 1$ such that $M^z_\pm$ is invertible, the scattering matrix is given by

$$S^z = \begin{pmatrix} (M^z_+)^{-1} & -N^z_+(M^z_+)^{-1} \\ -N^z_-(M^z_-)^{-1} & (M^z_-)^{-1} \end{pmatrix}.\quad (7)$$

The entries of $S^z$ also define the transmission and reflection coefficients matrices by

$$S^z = \begin{pmatrix} T^z_+ & R^z_- \\ R^z_+ & T^z_- \end{pmatrix}.\quad$$

Proposition 14 below implies that $S^z$ is well-defined and unitary for $z \in S^1 \setminus \{1, -1\}$. Moreover, Proposition 13 combined with Proposition 22 implies that the function $z \in S^1 \setminus \{1, -1\}$
\{-1, 1\} \mapsto \det(S^z) \text{ is differentiable. Proposition} \ref{prop16} \text{ and Remark} \ref{rem17} \text{ below show that } M^\pm_\epsilon \text{ are invertible in a neighborhood of } 1 \text{ in } \mathbb{D} \setminus \{-1, 1\}, \text{ and similarly in a neighborhood of } -1.

As a final preparation for the statement of the main result, let us introduce the path } \Gamma^\epsilon_+ \text{ for } \epsilon > 0 \text{ as the truncated upper semicircle parameterized by } \gamma^\epsilon_+: [0, 1] \to S^1 \text{ given by } 

\gamma^\epsilon_+(t) = e^{i\pi((1-t)\epsilon + t(1-\epsilon))}.

Theorem 4 (Levinson Theorem). The Hamiltonian } H \text{ has only a finite number } J_h \text{ of eigenvalues } E_1, \ldots, E_{J_h} \in \mathbb{R} \text{ (listed with their multiplicity) and they are outside of } [-2, 2]. \text{ Moreover, at the thresholds } E = \pm 2, \text{ there are } J_{h+}^\pm \leq L \text{ linearly independent bounded solutions of } Hu = \pm 2u \text{ which are called half-bound states. With } J_h = J_h^- + J_h^+, \text{ one has }

2\pi i (J_b + \frac{1}{2} J_h - L) = -\lim_{\epsilon \to 0} \int_{\Gamma^\epsilon_+} \det(S^z)^{-1} \frac{d}{dz} \det(S^z)dz.

This article does not assume that the potential is compactly supported, as we did in our previous work \cite{8}. Non assuming compactly supported potentials requires different techniques: the compactly supported case \cite{8} is prominently based on transfer matrices. The existence and differentiability of the Jost solutions in \cite{8} pends on transfer matrix techniques that, to our best knowledge, do not transpose to the case with non-compact support. This implies that the proofs in this manuscript differ substantially from \cite{8}. As already stated above, the Jost solutions are here studied as solutions of integral equations, similarly as in \cite{9}. To avoid overlaps with the earlier works, we state needed results from \cite{8, 9} without proofs and focus on the innovative aspects of the arguments.

Finally, let us give a brief account of earlier related works on scattering for one-dimensional discrete Schrödinger operators. Foundations and inverse scattering theory for the scalar case are laid out in \cite{11, 15, 16, 17, 5, 24}. Levinson’s theorem for one-dimensional discrete operators in the scalar case is proved in \cite{18} and more recently in \cite{12, 19, 22}. Scattering in a periodic background is treated in \cite{14}. Works on the scattering theory for the matrix-valued case are scarce \cite{23, 6, 7}, but the latter two also construct Jost solutions and a scattering matrix under a moment condition similar as is done below. What is missing in \cite{7}, however, is the fine analysis of the analytic behavior of the Jost solutions and the scattering matrix at the band edges so that the authors could not conclude that there is a finite number of bound states nor analyze half-bound states nor prove a Levinson theorem. For scattering theory for continuous one-dimensional Schrödinger operators with a matrix-valued potential, there is also abundant literature, most of which is cited in the recent monograph by Aktosun and Weder \cite{1}. A Levinson theorem in that framework is proved in \cite{3, 2}, and an index-theoretic perspective is given in \cite{20} (in the scalar case, but this readily transposes to the matrix-valued case, e.g. \cite{10, 19}). Complementary references for scattering theory on matrix Schrödinger operators and inverse scattering can be found in \cite{21, 4}.

This paper is organized as follows. Section 2 constructs Jost solutions and proves some key regularity properties and estimates for them. Section 3 then deduces analytical properties of the scattering matrix. Sections 4 and 5 address respectively the half-bound and bound states appearing in Levinson’s theorem. Section 6 gives a formula for the time delay that was
already used in [8] and for which a new simplified proof is provided here. In Section 7, we prove our main result (Levinson Theorem) using the results and constructions of the other sections.

2 Jost solutions

2.1 Existence

In this section, we construct fundamental solutions to Eq. (3).

Definition 5. For every \( z \in \mathbb{C} \setminus \{0\} \), we denote by \( s^z \) the scalar solutions \( s^z \in \mathbb{C}^Z \) of \( H_0 u = (z + 1/z)u \) such that \( s^z(0) = 0 \), \( s^z(1) = 1 \).

Explicitly, one can verify that

\[
s^z(n) = \begin{cases} 
\frac{1}{z-z^{-1}}(z^n - z^{-n}), & z^2 \neq 1, \\
(\pm 1)^{n+1}n, & z = \pm 1.
\end{cases} \tag{8}
\]

Proposition 6 (Fundamental solution). For every \( z \in \mathbb{C} \setminus \{0\} \), there exist solutions \( u^z_\pm \in \mathcal{M}^Z \) to Eq. (3) with \( E = z + \frac{1}{z} \), such that

\[
u^z_\pm(n) = z^n(1 + o(1)), \quad n \to \pm \infty. \tag{9}\]

For \( 0 < |z| \leq 1 \), they satisfy:

\[
u^z_+(n) = z^n 1 - \sum_{j={n+1}}^{\infty} s^z(j-n)V(j)u^z_+(j), \quad n \in \mathbb{Z}, \tag{10}\]

\[
u^z_-(n) = z^{-n} 1 + \sum_{j=-\infty}^{n-1} s^{1/z}(j-n)V(j)u^{1/z}_-(j), \quad n \in \mathbb{Z}.
\]

Moreover, if one defines

\[
\hat{u}^z_\pm(n) := z^{-n}u^z_\pm(n), \quad n \in \mathbb{Z}, \ z \in \mathbb{C} \setminus \{0\},
\]

\[
\hat{u}^0_+(n) = 1, \quad \hat{u}^{1/z}_+(n)|_{z=0} = 1,
\]

then for each \( n \in \mathbb{Z} \) the functions \( z \mapsto \hat{u}^z_+(n) \) and \( z \mapsto \hat{u}^{1/z}_-(n) \) are analytic on the open unit disc

\[
\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \tag{11}
\]

and continuous on its closure \( \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\} \) with a uniform bound in \( n \) and \( z \in \overline{\mathbb{D}} \).

Proof. Existence, analyticity and continuity of the solutions \( u^z_+ \) and \( u^{1/z}_- \), for \( 0 < |z| \leq 1 \), was already proved in Lemma 7 of [9] which also contains (10). The argument is essentially based on solutions of the Volterra equation as stated in Theorem 24.
The extension of $\tilde{u}_+^z$ and $\tilde{u}_{-1}^{1/z}$ to $z = 0$ is derived using the proof of Lemma 7 in [9] setting $K^0(n, j) = 0$. We recall that $K^z(n, j) = - z^{j-n} s^z (j-n) V(j)$ in [9]. Notice that $\lim_{z \to 0} z^m s^z (m) \to 0$ for $m \geq 0$, which implies that $z \mapsto K^z(n, j)$ is analytic on $|z| < 1$. The uniform bound on $\tilde{u}_+^z(n)$, w.r.t. $n$ and $z$, follows again from Theorem 24.

Next we construct the solution $u_+^z$ for $|z| > 1$, the construction of the solution $u_{-1}^{1/z}$ is analogous. Let $z \in \mathbb{C}$ with $|z| > 1$ and take $m \in \mathbb{N}$ such that

$$\sum_{j=m}^{\infty} \|V(j)\| < \left| \frac{z^2 - 1}{2z} \right|. \tag{12}$$

Consider the Volterra-type equation for $n > m$

$$Y(n) = z^n 1 + \frac{z}{z^2 - 1} \sum_{j=n+1}^{\infty} z^{n-j} V(j) Y(j) + \frac{z}{z^2 - 1} \sum_{j=m}^{n} z^{j-n} V(j) Y(j), \tag{13}$$

which is equivalent to Eq. (3) (this is verified in (15) below, see also [1] p. 31 for the continuous setting). In order to find a solution $Y$ to Eq. (13), let us set $X(n) = z^{-n} Y(n)$ and rewrite Eq. (13) in the next way

$$X(n) = 1 + \frac{z}{z^2 - 1} \sum_{j=n+1}^{\infty} V(j) X(j) + \frac{z}{z^2 - 1} \sum_{j=m}^{n} z^{2(j-n)} V(j) X(j). \tag{14}$$

For each $n, j \geq m$, one hence defines $K(n, j) \in \mathcal{M}$ by

$$K(n, j) = \begin{cases} \frac{z}{z^2 - 1} z^{2(j-n)} V(j), & m \leq j \leq n, \\ \frac{z}{z^2 - 1} V(j), & n + 1 \leq j. \end{cases}$$

The definition implies that $\|K(n, j)\| \leq \left| \frac{z}{z^2 - 1} \right| \|V(j)\|$, for all $n, j \geq m$. Next consider the operator $T : \ell^\infty([m, \infty) \cap \mathbb{N}, \mathcal{M}) \to \ell^\infty([m, \infty) \cap \mathbb{N}, \mathcal{M})$ defined by

$$(TX)(n) = \sum_{j=m}^{\infty} K(n, j) X(j).$$

Notice that $T$ is well-defined because one has due to (12)

$$\sum_{j=m}^{\infty} \|K(n, j) X(j)\| \leq \|X\| \sum_{j=m}^{\infty} \|K(n, j)\| \leq \|X\| \left| \frac{z}{z^2 - 1} \right| \sum_{j=m}^{\infty} \|V(j)\| < \frac{1}{2} \|X\| \infty$$

for all $n \in [m, \infty) \cap \mathbb{N}$, where $\| \cdot \|_\infty$ denotes the norm in the Banach space $\ell^\infty([m, \infty) \cap \mathbb{N}, \mathcal{M})$. It follows that the operator norm satisfies

$$\|T\| < \frac{1}{2}.$$

Eq. (14) takes the form of

$$X = 1 + TX.$$
Therefore its solution is
\[ X = (1 - T)^{-1} \mathbf{1} \in \ell^\infty(\mathbb{N}, \mathcal{M}), \]
which exists because \( \|T\| \leq \frac{1}{2} \). It is easy to see from Eq. (14) that \( X(n) \to \mathbf{1} \) for \( n \to +\infty \).

Now we define for \( n \geq m \),
\[ u^\pm_+(n) := Y(n), \]
and recursively for \( n \leq m \),
\[ u^\pm_+(n - 1) := Eu^\pm_+(n) - V(n)u^\pm_+(n) - u^\pm_+(n + 1). \]

It follows from its definition that \( u^\pm_+ \) satisfies (9), and for \( n \leq m \) it satisfies (3). The fact that \( u^\pm_+ \) satisfies (3) for \( n \geq m + 1 \) follows from its definition and the next computation:

\[
Y(n + 1) + Y(n - 1) = (z^{n+1} + z^{n-1})\mathbf{1} + \frac{z}{z^2 - 1} \left( \sum_{j=0}^{\infty} z^{n+1-j}V(j)Y(j) + \sum_{j=1}^{\infty} z^{n-j}V(j)Y(j) \right)
+ \frac{z}{z^2 - 1} \left( \sum_{j=m}^{n+1} z^{j-n-1}V(j)Y(j) + \sum_{j=m}^{n} z^{j-n+1}V(j)Y(j) \right)
= E_z^n \mathbf{1} + \frac{z}{z^2 - 1} \left( \sum_{j=0}^{n+1} (z^{n+1-j} + z^{n-j})V(j)Y(j) \right)
+ \frac{z}{z^2 - 1} \left( \sum_{j=m}^{n+1} z^{j-n-1}V(j)Y(j) - V(n + 1)Y(n + 1) + z^{-1}V(n)Y(n) + V(n + 1)Y(n + 1) - zV(n)Y(n) \right)
= E_z^n \mathbf{1} + E_z^n \frac{z}{z^2 - 1} \sum_{j=0}^{n+1} z^{n-j}V(j)Y(j) + E_d \frac{z}{z^2 - 1} \sum_{j=m}^{n} z^{j-n}V(j)Y(j) - V(n)Y(n)
= EY(n) - V(n)Y(n). \tag{15}
\]

This completes the proof. \hfill \Box

### 2.2 Derivatives of Jost solutions

The previous section constructed the solutions \( u^\pm_+ \) and \( u^{1/2}_- \) and demonstrated their analyticity on open unit disc \( \mathbb{D} \) as well as their continuity on its closure \( \overline{\mathbb{D}} \). In this section, it is proved they are continuously differentiable \( \overline{\mathbb{D}} \) in the sense of the definition below. The proof is inspired by the continuous case that is presented in Deift and Trubowitz [13].

**Definition 7.** Suppose that \( U \) is a subset of \( \mathbb{C} \). A function \( g : U \to \mathbb{C} \) is said to be differentiable at \( z \in U \) with differential (or derivative) \( g'(z) \) or \( \dot{g}(z) \) if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[
0 < |h| < \delta, \ z + h \in U \implies \left| \frac{g(z + h) - g(z)}{h} - g'(z) \right| < \epsilon.
\]
Then \( g \) is said to be differentiable on \( U \) if it is differentiable at every point of \( U \), and continuously differentiable on \( U \) if its derivative is a continuous function. Likewise, these concepts are defined for vector or matrix values functions.

Notice that if \( z \) is an interior point of \( U \), this coincides with the usual definition of analyticity. For the proof of the differentiability of \( u^z_n \), we will again use the sequence \( \tilde{u}^z_n(n) = z^{-n}u^z_n(n) \). Due to (10), it satisfies

\[
\tilde{u}^z_n(n) = 1 + \sum_{j=n+1}^{\infty} z^{j-n} s^z(j-n)V(j)\tilde{u}^z_n(j), \quad z \in \mathbb{D}.
\]

Hence let us set, for \( z \in \mathbb{D} \) and \( j > n \in \mathbb{N} \),

\[
H(z, n, j) = -z^{j-n} s^z(j-n) = -\sum_{m=0}^{j-n-1} z^{2m+1},
\]

so that

\[
\tilde{u}^z_n(n) = 1 + \sum_{j=n+1}^{\infty} H(z, n, j)V(j)\tilde{u}^z_n(j).
\] (16)

From the definition, one readily checks that

\[
|H(z, n, j)| \leq j - n,
\] (17)

and

\[
\hat{H}(z, n, j) = \frac{2(j-n)z^{j-n}}{1 - z^2} + \frac{z^{2(j-n)} - 1}{1 - z^2} \cdot \frac{1 + z^2}{1 - z^2} = \frac{2(j-n)z^{j-n}}{1 - z^2} - \frac{1 + z^2}{1 - z^2} \sum_{m=0}^{j-n-1} z^{2m}.
\]

It follows that

\[
|(1 - z^2)\hat{H}(z, n, j)| \leq 4(j - n).
\] (18)

Now formally deriving (16) w.r.t. \( z \) one obtains the equation

\[
\frac{d}{dz} \tilde{u}^z_n(n) = \sum_{j=n+1}^{\infty} \hat{H}(z, n, j)V(j)\tilde{u}^z_n(j) + \sum_{j=n+1}^{\infty} H(z, n, j)V(j)\frac{d}{dz} \tilde{u}^z_n(j),
\] (19)

for \( n \in \mathbb{N} \). Let us first verify that this equation, multiplied with \( 1 - z^2 \), has a bounded solution.

**Lemma 8.** For \( z \in \mathbb{D} \) there exists a solution \( h^z \in \ell^\infty(\mathbb{N}, \mathcal{M}) \) of the integral equation

\[
h^z(n) = \sum_{j=n+1}^{\infty} (1 - z^2)\hat{H}(z, n, j)V(j)\tilde{u}^z_n(j) + \sum_{j=n+1}^{\infty} H(z, n, j)V(j)h^z(j),
\]

which is analytic on \( \mathbb{D} \) and continuous on \( \overline{\mathbb{D}} \).
Proof. By Proposition 8, \( \tilde{u}_+^z \in \ell^\infty(\mathbb{N}, \mathcal{M}) \) with \( \|\tilde{u}_+^z\|_\infty \) uniformly bounded for \( z \in \overline{\mathbb{D}} \). For \( n \in \mathbb{N}, \) (18) implies that
\[
\sum_{j=n+1}^{\infty} \| (1 - z^2) \dot{H}(z, n, j)V(j)\tilde{u}_+^z(j) \| \leq \sum_{j=n+1}^{\infty} 4j\|V(j)\|\|\tilde{u}_+^z(j)\| < 4\|\tilde{u}_+^z\|_\infty \sum_{j=0}^{\infty} j\|V(j)\|.
\]
This implies that the function
\[
g^z(n) := \sum_{j=n+1}^{\infty} (1 - z^2) \dot{H}(z, n, j)V(j)\tilde{u}_+^z(j)
\]
belongs to \( \ell^\infty(\mathbb{N}, \mathcal{M}) \), is analytic in \( z \in \mathbb{D} \) and uniformly bounded on \( \overline{\mathbb{D}} \):
\[
\|g^z\|_\infty \leq 4 \sup_{z \in \overline{\mathbb{D}}} \|\tilde{u}_+^z\|_\infty \sum_{j=0}^{\infty} j\|V(j)\|.
\]
Then the result follows from the Theorem 24 with \( g^z \) defined as above and \( K^z(n, j) = H(z, n, j)V(j), \) \( M(j) = j\|V(j)\| \).

Proposition 9. For each \( n \in \mathbb{N}, \) the function \( z \mapsto \tilde{u}_+^z(n) \) is continuously differentiable on \( \overline{\mathbb{D}} \setminus \{1, -1\} \). Moreover, the derivative \( \frac{d}{dz}\tilde{u}_+^z \in \ell^\infty(\mathbb{N}, \mathcal{M}) \) and it satisfies (19).

Proof. For \( z \in \overline{\mathbb{D}} \setminus \{1, -1\}, \) let us set \( f^z := (1 - z^2)^{-1}h^z \) where \( h^z \) is defined in Lemma 8. Lemma 8 implies that \( z \mapsto f^z(n) \) is continuous on \( \overline{\mathbb{D}} \setminus \{1, -1\} \) and satisfies the following equation
\[
f^z(n) = \sum_{j=n+1}^{\infty} \dot{H}(z, n, j)V(j)\tilde{u}_+^z(j) + \sum_{j=n+1}^{\infty} H(z, n, j)V(j)f^z(j), \quad n \in \mathbb{N}.
\]
Take \( n \in \mathbb{N} \) and \( z \in \overline{\mathbb{D}} \setminus \{1, -1\} \). We prove that \( \frac{d}{dz}\tilde{u}_+^z(n) = f^z(n) \), then the result follows from the above. Using (16) and (21) one has
\[
\frac{\tilde{u}_+^{z+h}(n) - \tilde{u}_+^z(n)}{h} - f^z(n) = G(h) + \sum_{j=n+1}^{\infty} H(z, n, j)V(j) \left( \frac{\tilde{u}_+^{z+h}(j) - \tilde{u}_+^z(j)}{h} - f^z(j) \right),
\]
where
\[
G(h) = \sum_{j=n+1}^{\infty} \left( \frac{H(z + h, n, j) - H(z, n, j)}{h} V(j)\tilde{u}_+^{z+h}(j) - \dot{H}(z, n, j)V(j)\tilde{u}_+^z(j) \right).
\]
Eq. (18) implies that
\[
\|\dot{H}(z, n, j)V(j)\tilde{u}_+^z(j)\| \leq 4|(1 - z^2)^{-1}|j\|V(j)\|,
\]
which is hence summable in \( j \) by the main assumption \(^2\). On the other hand, the estimate

\[
\left\| \frac{H(z + h, n, j) - H(z, n, j)}{h} \right\| \leq \int_0^1 \|H(z + th, n, j)\| dt
\]

leads to

\[
\left\| \frac{H(z + h, n, j) - H(z, n, j)}{h} V(j) \tilde{u}_+^{z+h}(j) \right\| \leq 4 \left( \int_0^1 |1 - (z + th)^2|^{-1} dt \right) j \|V(j)\|.
\]

which is thus also summable in \( j \). Therefore, the Lebesgue dominated convergence theorem implies that \( G(h) \to 0 \) as \( h \to 0 \). Now by the Gronwall lemma (Lemma \(^{25}\)) and Eq. \(^{17}\), one has

\[
\left\| \tilde{u}_+^{z+h}(n) - \tilde{u}_+^z(n) \right\| - f^z(n) \leq |G(h)| \exp \left( \sum_{j=n+1}^{\infty} j \|V(j)\| \right) \to 0, \quad \text{as } h \to 0.
\]

This implies the desired result. \( \square \)

**Remark 10.** Recall that (see Proposition \(^8\)) by definition \( u^z_+(n) = z^n \tilde{u}_+^z(n) \), so that Proposition \(^9\) implies that the map \( z \mapsto u^z_+(n) \) is continuously differentiable on \( \mathbb{D} \setminus \{1, -1\} \) for \( n \in \mathbb{N} \). Moreover, Eq. \(^3\) implies that

\[
u^z_+(n-1) = (z + 1/z - V(n)) u^z_+(n) - u^z_+(n+1),
\]

which along with the above allows to prove that \( z \mapsto u^z_+(n) \) is continuously differentiable on \( \mathbb{D} \setminus \{1, -1\} \) for all \( n \in \mathbb{Z} \). In a similar way one proves that the map \( z \mapsto u_+^{1/z}(n) \) is continuously differentiable on \( \mathbb{D} \setminus \{-1, 0, 1\} \) for all \( n \in \mathbb{Z} \). The above results imply that the map \( z \mapsto u_+^{1/z}(n) \) is differentiable for \( z \in (\mathbb{C} \setminus \mathbb{D}) \setminus \{-1, 1\} \), and therefore it is differentiable on \( \mathbb{S}^1 \setminus \{-1, 1\} \). The same holds true for the function \( z \mapsto u_-^z(n) \). \( \diamond \)

### 3 Scattering matrix

All formulas in this section are identical to those in \(^8\), but their existence in the present more general context depends on the results of the previous sections. Remark \(^1\) allows us to define the scattering coefficients \( M_+^z \) and \( N_+^z \) for \( z \in \mathbb{C} \setminus \{1, 0, -1\} \) (see Definition \(^2\)). These matrices have representations in terms of the Wronskian which for two functions \( u, v \in \mathcal{M}^\mathbb{Z} \) is defined by

\[
W(u, v)(n) = \imath(u(n + 1)^* v(n) - u(n)^* v(n + 1)).
\]

Using the eigenvalue Eq. \(^3\), an elementary calculation implies that \( W(u_+^z, u_+^{1/z})(n) \) and \( W(u_+^{1/z}, u_+^z)(n) \) do not depend on \( n \) (for every \( z \)). Then, using the asymptotic behavior of Jost solutions, one concludes that for \( 0 < |z| \leq 1 \)

\[
W(u_+^z, u_+^z) = 0 = W(u_+^{1/z}, u_+^{1/z}),
\]

(23)
and for \( z \in \mathbb{C} \setminus \{0\} \)

\[
W(u_\pm^{1/z}, u_\pm^z) = (\nu^z)^{-1}1. \tag{24}
\]

where we set

\[
\nu^z = \frac{1}{z - z^{-1}}. \tag{25}
\]

**Proposition 11.** For every \( 0 < |z| \leq 1 \) with \( z^2 \neq 1 \), the following expressions hold true:

\[
M_+^z = \nu^z W(u_-^{1/z}, u_+^z), \tag{26}
\]

\[
M_-^z = -\nu^z W(u_+^{1/z}, u_-^z). \tag{27}
\]

Moreover, for every \( |z| \geq 1 \) with \( z^2 \neq 1 \),

\[
N_+^z = -\nu^z W(u_-^{1/z}, u_+^z), \tag{28}
\]

\[
N_-^z = \nu^z W(u_+^{1/z}, u_-^z). \tag{29}
\]

**Proof.** Let us start computing \( W(u_\pm^{1/z}, u_\pm^z) \) for \( 0 < |z| \leq 1 \) using equations (6), (23) and (24):

\[
W(u_\pm^{1/z}, u_\pm^z) = W(u_\pm^{1/z}, u_-^z M_+^z + u_-^{1/z} N_+^z) = W(u_\pm^{1/z}, u_-^z) M_+^z + W(u_\pm^{1/z}, u_-^{1/z}) N_+^z = (\nu^z)^{-1} M_+^z.
\]

In a similar fashion using that \( \nu^{1/z} = -\nu^z \) one has:

\[
W(u_\pm^{1/z}, u_\pm^{1/z}) = W(u_\pm^{1/z}, u_+^{1/z} M_-^z + u_+^{1/z} N_-^z) = W(u_\pm^{1/z}, u_+^{1/z}) M_-^z + W(u_\pm^{1/z}, u_+^{1/z}) N_-^z = -(\nu^z)^{-1} M_-^z.
\]

The rest of the proof is derived in a similar way. \( \square \)

Proposition [I] implies

\[
(M_+^z)^* = M_-^z \quad \text{for} \quad z \in \overline{\mathbb{D}} \setminus \{0\}, \quad (N_+^z)^* = -N_-^z \quad \text{for} \quad z \in \mathbb{S} \setminus \{1, -1\}. \tag{30}
\]

Next further properties of these coefficients are proved.

**Lemma 12.** For every \( z \in \mathbb{S} \setminus \{-1, 1\} \), the following identities hold true:

\[
(M_+^z)^* M_-^z = 1 + (N_+^z)^* N_-^z, \tag{31}
\]

\[
M_+^z N_-^z = -N_+^{1/z} M_-^z, \tag{32}
\]

\[
(M_+^z)^* M_-^z = 1 + (N_+^z)^* N_-^z, \tag{33}
\]

\[
M_+^z N_-^z = -N_+^{1/z} M_-^z. \tag{34}
\]

**Proof.** Equations (6) and (24) imply that

\[
(\nu^z)^{-1}1 = W(u_+^z, u_+^z) = W(u_-^z M_+^z + u_-^{1/z} N_+^z, u_+^z M_-^z + u_-^{1/z} N_+^z). \tag{35}
\]
Expanding the r.h.s. of (33) and using Equations (23), (24), one gets
\[(\nu^2)^{-1}1 = (\nu^2)^{-1}(M_+^z)^*M_+^z - (\nu^2)^{-1}(N_+^z)^*N_+^z,\]
where \(\nu^{1/z} = -\nu^z\) was used. This implies (33). Eq. (31) is obtained in similar manner by expanding \(W(u_+^z, u_-^z)\). Now let us prove (34). It follows from Equations (23) and (6) that
\[0 = W(u_+^{1/z}, u_-^{1/z}) = W(u_+^{1/z}M_+^{1/z} + u_-^{1/z}N_+^{1/z}, u_-^{1/z}M_+^{1/z} + u_+^{1/z}N_+^{1/z}).\] (36)
Expanding the r.h.s. of (36) and using Equations (23), (24), we get
\[0 = -(\nu^z)^{-1}(M_+^{1/z})^*N_+^{1/z} + (\nu^z)^{-1}(N_+^{1/z})^*M_+^{1/z} = -(\nu^z)^{-1}M_+^{1/z}N_+^{1/z} - (\nu^z)^{-1}N_+^{1/z}M_+^{1/z},\]
where the last equality follows from (30). Eq. (32) is obtained in similar manner expanding \(W(u_+^{1/z}, u_-^{1/z})\).

The next proposition allows to extend \(M_\pm^z\) to 0.

**Proposition 13.** The functions \(z \mapsto M_\pm^z\) and \(z \mapsto M_\pm^{1/z}\) are differentiable on \(\mathbb{D} \setminus \{-1, 0, 1\}\) and \(\mathbb{S}^1 \setminus \{-1, 1\}\), respectively. Moreover,
\[\lim_{z \to 0} M_\pm^z = 1.\]
Therefore the functions \(z \mapsto M_\pm^z\) are analytic on \(|z| < 1\).

**Proof.** By Remark 10 the functions \(z \mapsto (u_\pm^z(n))^*, z \mapsto u_\pm^z(n)\) are differentiable on \(\mathbb{D} \setminus \{-1, 1\}\). By (26) and (27) this implies the first claim. The second part follows from the following computation using Eq. (27)
\[M_\pm^z = -\nu^zW(u_\pm^z, u_\pm^{1/z})\]
\[= \frac{z}{z^2 - 1}(u_\pm^z(n + 1)^*u_\pm^{1/z}(n) - u_\pm^z(n)^*u_\pm^{1/z}(n + 1))\]
\[= \frac{z^2}{z^2 - 1}(\bar{z}^{-n+1}u_\pm^z(n + 1)^*z^n u_\pm^{1/z}(n) - \frac{1}{z^2 - 1}(\bar{z}^{-n}u_\pm^z(n))^*z^{n+1}u_\pm^{1/z}(n + 1))\]
\[= \frac{z^2}{z^2 - 1}(u_\pm^z(n + 1)^*\bar{u}_\pm^{1/z}(n) - \frac{1}{z^2 - 1}(\bar{u}_\pm^z(n))^*\bar{u}_\pm^{1/z}(n + 1) \to 1 \quad \text{as} \quad z \to 0,\]
because \(\bar{u}_\pm^z(n), \bar{u}_\pm^{1/z}(n) \to 1\) as \(z \to 0\) (see Proposition 6). The other limit can be computed in the same fashion. The last claim follows from the removable singularity theorem. 

Next recall Definition 3 introducing the scattering matrix \(S^z\) for \(z \in \mathbb{D}\) by (7), provided that \(M_\pm^z\) are invertible. This is the case on the unit circle:

**Proposition 14.** For \(z \in \mathbb{S}^1 \setminus \{-1, 1\}\), the matrices \(M_\pm^z\) are invertible and the scattering matrix \(S^z\) is unitary.
Proof. Eqs. (31) and (33) imply that $M^\pm_\pm$ are invertible (using that $\langle A^*A\phi, \phi \rangle = \|A\phi\|^2$ one checks injectivity and therefore surjectivity because they are finite dimensional operators). For the second part, the off-diagonal terms of $(S^z)^*S^z$ are (see Definition 3)

$$-((M^+_\pm)^{-1})^*N^z_-(M^-_\pm)^{-1} - ((M^-_\pm)^{-1})^*(N^z_+)(M^+_\pm)^{-1},$$

$$-((M^-_\pm)^{-1})^*(N^z_+)(M^+_\pm)^{-1} - ((M^-_\pm)^{-1})^*N^z_-(M^-_\pm)^{-1}$$

and they vanish by (30). The diagonal terms are

$$(M^+_\pm)^{-1}*(1 + (N^z_+)*N^z_\pm)(M^+_\pm)^{-1},$$

$$(M^-_\pm)^{-1}*(1 + (N^z_+)*N^z_\pm)(M^-_\pm)^{-1},$$

and they are both equal to 1, see (33) and (31). This proves the unitary of $S^z$. \hfill \Box

4 Half-bound states

This section analyzes the behavior of the function $z \mapsto \det(M^\pm_\pm)$ when $z \to \pm 1$. All the results in this section are presented for $z \to 1$, but they are also true for $z \to -1$ and the corresponding proofs are basically the same. Throughout this section we will denote $J^+_h = \dim \ker(W(u_1^-, u_1^+))$. Notice that this seems to differ from Theorem 4 however, these two definitions coincide as will be verified in the proof of Theorem 4. Let us start by stating a result from [9] (see Proposition 24 and Eq. (102) therein). From the definition of $u_1^\pm$, one knows that $u_1^\pm(j)$ tends to 1 as $j$ tends to infinity. Then, for large enough $j$, $u_1^\pm(j)$ is invertible. In order to simplify notations, we assume that $u_1^\pm(1)$ is already invertible (this is needed in order to apply Proposition 24 in [9]). This does not imply any restriction because one can always translate the origin.

Proposition 15. There exist invertible matrices $P, Q \in \mathcal{M}_{L \times L}$ and matrix valued functions $A(z), B(z), C(z), D(z)$, for $z \in \overline{D} \setminus \{1\}$ (recall (11)) sufficiently close to 1, such that

$$P W(u_{-1/2}^+, u_{+1}^+)u_{+1}^*(1)Q = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

(40)

where

$$A(z) = o(1 - z)A + o(|1 - z|), \quad B(z) = o(1), \quad C(z) = O(|1 - z|), \quad D(z) = D + o(1).$$

(41)

In the previous equations, $A$ is a matrix of size $J^+_h \times J^+_h$ (and this determines the dimensions of the other matrices involved). Moreover, $A$ and $D$ are invertible matrices. The invertibility of $u_1^\pm(1)$ for $z$ close to 1 follows from the invertibility of $u_1^\pm(1)$ and the continuity of Jost solutions.

In [9], Proposition 15 was used to show that the limits $T_1^\pm := \lim_{z \to 1} T_\pm^z$ exist. It also implies the next result which is a preparation for the proof of Levinson’s theorem.
Proposition 16. There is a constant \( c \in \mathbb{C} \setminus \{0\} \) such that
\[
\det(M_z^+) = (z - 1)^{d_L} \cdot (c + o(1)), \quad z \to 1, \quad z \in \mathbb{D}.
\]

Proof. Using Eqs. (26) and (40) one has for \( 0 < |z| \leq 1 \) that
\[
\det(M_z^+) = (v^z)^L \det \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} (a + o(1)),
\]
where \( a = \det(PQ)^{-1} \neq 0 \), and the continuity of the function \( z \mapsto u_+(1) \) was used. Using Schur formula for the determinant (see Proposition 28) and Eq. (41), it follows that
\[
\det \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \det(D + o(1)) \det(v(1-z)A + o(|1-z|)) = (z - 1)^{d_L} (b + o(1)),
\]
where \( b \) is a non-zero constant. Using Eqs. (42), (43) and (25), the required result follows. \( \square \)

Remark 17. Using Eq. (30) and Proposition 16 one gets a similar result for \( M_z^- \) in a neighborhood of \( z = 1 \) in \( \mathbb{D} \):
\[
\det(M_z^-) = (z - 1)^{d_L} (\bar{c} + o(1)).
\]
The corresponding result in a neighborhood of \( z = -1 \) in \( \mathbb{D} \) reads as:
\[
\det(M_z^-) = (z + 1)^{d_L} (d + o(1)),
\]
where \( d \in \mathbb{C} \setminus \{0\} \).

5 Bound states

This section is about the behavior of the function \( z \mapsto \det(M_z^\pm) \) when \( z \to r \), where \( r \) is such that \( E = r + 1/r \) is an eigenvalue of \( H \). The main result is (see Proposition 21) that the number of zeros of the function \( z \mapsto \det(M_z^\pm) \) on \( \mathbb{D} \) (counted with multiplicity) equals the number of eigenvalues of \( H \) (counted with multiplicity).

Proposition 18. For \( z \in \mathbb{C}, \ 0 < |z| < 1, \) and \( E = z + 1/z \), the following identity holds true:
\[
\dim(\text{Ker}(H - E)) = \dim(\text{Ker}(M_z^\pm)).
\]

Moreover, \( N_z^- \) restricted to \( \text{Ker}(M_z^-) \) is a bijection between \( \text{Ker}(M_z^-) \) and \( \text{Ker}(M_z^+) \).

Proof. Let us prove (46) for \( M_z^+ \) and the result for \( M_z^- \) is obtained in a similar fashion. Set
\[
S_+ = \{ \phi \in \mathbb{C}^L : \ u_+^\dagger \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L) \}, \quad S_- = \{ \phi \in \mathbb{C}^L : \ u_-^\dagger \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L) \},
\]
then the function \( T : S_+ \to \text{Ker}(H - E) \) defined by \( T(\phi) = u_+^\dagger \phi \) is linear and injective (because the columns of \( u_+^\dagger \) are linearly independent by Remark 1 and these columns are precisely solutions to the eigenvalue problem). Let \( u \in \text{Ker}(H - E) \), then there exist \( \phi \in \mathbb{C}^L \) such that
\[ u(n) = u^z_+(n) \phi \] (write \( u = u^z_+ + u^{1/z} \psi \) again by Remark 1 and notice that \( u^{1/z}_+(n) \psi \neq 0 \) implies that \( \lim_{n \to +\infty} \| u^{1/z}_+(n) \psi \| = \infty \) - see Eq. (9)). Thus \( T \) is surjective, and it is consequently an isomorphism.

Next we prove that \( S_+ = \operatorname{Ker}(M^z_+) \) which implies (46) (similarly, one proves that \( S_- = \operatorname{Ker}(M^z_-) \)). Let us take \( \phi \in \operatorname{Ker}(M^z_+) \) and multiply (6) by \( \phi \) so that

\[ u^z_+ \phi = u^{1/z}_+ N^z_+ \phi. \]  

(47)

This implies that \( u^z_+ \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L) \) and therefore \( \phi \in S_+ \), which implies that \( \operatorname{Ker}(M^z_+) \subset S_+ \).

The other contention is proved by taking \( \phi \in S_+ \) and multiplying (6) by \( \phi \). Then

\[ u^z_+ \phi = u^z_+ M^z_+ \phi + u^{1/z}_+ N^z_+ \phi. \]

Since \( u^z_+ \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L) \), it follows (using the asymptotic behavior of Jost solutions to compute the second term on the right of the next equation) that

\[ \lim_{n \to -\infty} u^z_+(n)M^z_+ \phi = \lim_{n \to -\infty} u^z_+(n)\phi - u^{1/z}_+(n)N^z_+ \phi = 0. \]

The asymptotic behavior of Jost solutions implies that \( M^z_+ \phi = 0 \) (since otherwise one would have \( \lim_{n \to -\infty} \| u^z_+(n)M^z_+ \phi \| = \infty \)). The arguments above imply the first part of the statement.

Next, let us prove that \( N^z_+|_{\operatorname{Ker}(M^z_+)} \) is a bijection between \( \operatorname{Ker}(M^z_+) \) and \( \operatorname{Ker}(M^z_+) \). Take \( \phi \in \operatorname{Ker}(M^z_-) \). Eq. (6) implies that

\[ u^{1/z}_- \phi = u^z_- N^z_- \phi, \]

and using the asymptotic behavior of Jost solutions (see Eq. (9)) one concludes that

\[ u^z_- N^z_- \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L). \]

With help of Eq. (6) for \( u^z_- \) (i.e. \( u^z_- N^z_- \phi = u^z_- M^z_- N^z_- \phi + u^{1/z}_- N^z_- N^z_- \phi \)), one deduces as before (using a blow up argument) that \( N^z_- \phi \in \operatorname{Ker}(M^z_+) \). This implies that \( N^z_- \) maps \( \operatorname{Ker}(M^z_-) \) into \( \operatorname{Ker}(M^z_+) \). Moreover, Eq. (5) and the above equations imply that

\[ u^{1/z}_- \phi = u^z_- N^z_- \phi = u^{1/z}_+ N^z_+ N^z_- \phi. \]

Taking the limit \( n \to -\infty \) in this identity (see also Eq. (9)), it follows that

\[ \phi = N^z_+ N^z_- \phi. \]

In a similar fashion, one proves that if \( \phi \in \operatorname{Ker}(M^z_+) \) then \( N^z_+ \phi \in \operatorname{Ker}(M^z_-) \) and

\[ \phi = N^z_- N^z_+ \phi. \]

Then the restriction of \( N^z_+ \) to \( \operatorname{Ker}(M^z_+) \) is the inverse of \( N^z_- \) restricted to \( \operatorname{Ker}(M^z_-) \), concluding the proof.\( \square \)
Proposition 19. The set of eigenvalues of $H$ is finite and every eigenvalue $E$ can be expressed in the form

$$E = z + 1/z,$$

for some $z \in (-1, 0) \cap (0, 1)$.

Proof. Let us first state some properties of the function

$$E : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad E(z) := z + 1/z :$$

Solving for $z$ gives

$$z = E/2 + \sqrt{E^2/4 - 1},$$

which implies that the map $E$ is surjective. The presence of the square root implies that the solutions are given by a Riemann surface with two branches. Then, for every complex number $E$, there are only two solutions $z_{in}, z_{ext}$ for the equation

$$E(z) = E.$$

Since $E(z) = E(1/z)$, one obtains that the restriction of $E$ to the disc $D$ is injective and its restriction to $\overline{D}$ is surjective (and therefore the analysis can be restricted to the case $|z| \leq 1$). Moreover, an elementary calculation yields that the equation

$$z + 1/z = E,$$

for $E$ in the real numbers and $|z| \leq 1$, is solvable if and only if $|z| = 1$ or $z \in (-1, 1)$.

Since $H$ a self-adjoint, its spectrum is contained in the real line. All eigenvalues, parameterized in the form $E(z)$, $|z| \leq 1$, must satisfy that $z \in S^1 \cup (-1, 1)$. Next let us argue that, furthermore, if $z \in S^1 \setminus \{1, -1\}$, then $E = z + 1/z$ is not an eigenvalue. Suppose that $u$ is an eigenvector of $H$ corresponding to $E(z)$, i.e. $Hu = E(z)u$, with $z \in S^1 \setminus \{-1, 1\}$. Remark 1 implies that $u$ can be written in the form

$$u = u_+^z \alpha + u_+^{1/z} \beta,$$

for some $\alpha, \beta \in C^L$. As $u$ is square integrable, one has

$$\lim_{n \to \infty} u(n) = 0.$$

Eq. (9) yields that

$$\lim_{n \to \infty} z^n \alpha + (1/z)^n \beta = 0,$$

which is only possible when $\alpha = \beta = 0$ and hence $u = 0$. Consequently all eigenvalues must lie on $[-1, 1]$. It remains to rule out the points $\{-1, 1\}$. We analyze only $z = 1$, since the analysis for $z = -1$ is the same. The proof in this case is similar, but Remark 1 is not valid anymore because $u_+^z = u_+^{1/z}$ for $z = 1$. The columns of $u_+^z$ do not generate all solutions. Nevertheless, in Definition 1 in [9] we introduce another solution $u_+^1$ such that the columns of $[u_+^1, u_+^1]$ generate all solutions. Now, following the line-of-argument for the case $z \in S^1 \setminus \{-1, 1\}$, one concludes that 1 is not an eigenvalue.
Next let us check that there are neighborhoods of 0 and ±1 in \( \mathbb{D} \) such that for \( z \) in these neighborhoods \( E(z) \) is not an eigenvalue of \( H \). Proposition 18 implies that a number \( E = z + 1/z \in \mathbb{C} \) with \( 0 < |z| < 1 \) is an eigenvalue of \( H \) if and only if the function \( z' \mapsto \det(M_{\pm}^{z'}) \) has a zero at \( z \). Now \( M_{\pm}^{z'} \) is invertible in a neighborhood of \( \pm 1 \) (by Proposition 16 and Remark 17) and also in a neighborhood of 0 by Proposition 13. This implies the claim.

Finally let us recall that the essential spectrum of \( H \) is \([-2, 2]\) which is precisely the image of \( S^1 \) under the map \( E \). The above arguments imply that the eigenvalues of \( H \) take the form \( E(z) \), for \( z \) in a compact subset \( K \) of \((0, 1) \setminus \{0\} \). Then, all eigenvalues of \( H \) must belong to the compact set \( E(K) \). This set does not intersect the essential spectrum of \( H \). Since all spectral points of \( H \) not belonging to the essential spectrum are isolated eigenvalues with finite multiplicity, we conclude that there is only a finite number of them (and they can be parametrized in the form \( E(z) \) for a finite number of \( z \)’s in \((-1, 0) \cup (0, 1)) \). \( \square \)

Proposition 18 claims that the number of zeros, counted without multiplicity, of the function \( z \mapsto \det(M_{\pm}^{z}) \) on \( \mathbb{D} \) is equal to the number of eigenvalues of \( H \), counted without multiplicity. Proposition 21 below proves that they are also the same if counted with multiplicity. For the proof, the following technical statement is needed which is a discrete version of a result from [8] that was already used in \([9]\).

**Lemma 20.** Let \( r \in \mathbb{R} \), with \( 0 < |r| < 1 \) and such that \( r + 1/r = E \) is an eigenvalue of \( H \). Let \( \alpha \in \text{Ker}(M_{-}^{r}) \). The following equation holds true:

\[
(N_{r}^{-} \alpha)^{*} \frac{d}{dz} M_{-}^{z} \bigg|_{z=r} \alpha = r^{-1} \| u_{-}^{1/r} \alpha \|^2 \tag{48}
\]

**Proof.** Let \( z \in \mathbb{D} \) and recall that the Jost solution \( u_{+}^{r} \) satisfies the generalized eigenvalue equations \( Hu_{+}^{r} = Eu_{+}^{r} \) with \( E = z + 1/z \), namely

\[
u_{+}^{r}(n + 1) + u_{+}^{r}(n - 1) + V(n)u_{+}^{r}(n) = (z + 1/z)u_{+}^{r}(n) \quad \forall n \in \mathbb{Z}. \tag{49}
\]

Taking derivative w.r.t. \( r \), one obtains

\[
\dot{u}_{+}^{r}(n + 1) + \dot{u}_{+}^{r}(n - 1) + V(n)\dot{u}_{+}^{r}(n) = (r + 1/r)\dot{u}_{+}^{r}(n) + (1 - 1/r^2)u_{+}^{r}(n), \tag{50}
\]

where \( \dot{u}_{+}^{r} \) is given by \( \dot{u}_{+}^{r}(n) = \frac{d}{dz} u_{+}^{r}(n) \bigg|_{z=r} \). Taking adjoints and evaluating in \( z = r \) in (49) leads to

\[
u_{+}^{r}(n + 1)^{*} + u_{+}^{r}(n - 1)^{*} + u_{+}^{r}(n)^{*}V(n) = (r + 1/r)u_{+}^{r}(n)^{*}. \tag{51}
\]

Multiplying (50) on the left by \( u_{+}^{r}(n)^{*} \) and (51) on the right by \( \dot{u}_{+}^{r}(n) \) and subtracting the resulting equations, one obtains

\[
u_{+}^{r}(n)^{*}\dot{u}_{+}^{r}(n + 1) + u_{+}^{r}(n)^{*}\dot{u}_{+}^{r}(n - 1) - u_{+}^{r}(n + 1)^{*}\dot{u}_{+}^{r}(n) - u_{+}^{r}(n - 1)^{*}\dot{u}_{+}^{r}(n)
\]

\[
= (1 - 1/r^2)u_{+}^{r}(n)^{*}u_{+}^{r}(n). \tag{52}
\]

Recalling the definition of the Wronskian \( W(u_{+}^{r}, \dot{u}_{+}^{r})(n) = i(u_{+}^{r}(n + 1)^{*}\dot{u}_{+}^{r}(n) - u_{+}^{r}(n)^{*}\dot{u}_{+}^{r}(n + 1) + 1)) \), one can rewrite the last equation as

\[
W(u_{+}^{r}, \dot{u}_{+}^{r})(n - 1) - W(u_{+}^{r}, \dot{u}_{+}^{r})(n) = i(1 - 1/r^2)u_{+}^{r}(n)^{*}u_{+}^{r}(n). \tag{53}
\]
Multiplying this equation by $N^r_\alpha$ on the right and by $(N^r_\alpha)^*$ on the left implies that

$$(N^r_\alpha)^*(W(u^r_+, \dot{u}^r_+)(n) - W(u^r_+, \dot{u}^r_+)(n - 1)) N^r_\alpha = i(1 - 1/r^2)(u^1_{1/r}(n)\alpha)^* u^1_{1/r}(n)\alpha, \tag{53}$$

where Eq. (6) was used to exchange $u^r_+(n) N^r_\alpha$ by $u^1_{1/r}(n)\alpha$ (recall that $\alpha \in \operatorname{Ker}(M^r)$). Since $\alpha \in \operatorname{Ker}(M^r)$, one has that $u^1_{1/r}\alpha \in \ell^2(\mathbb{Z}, \mathbb{C})$ (by using that $u^1_{1/r}(n)N^r_\alpha = u^1_{1/r}(n)\alpha$ and the asymptotic properties of Jost solutions). Now take the sum in both sides of the Eq. (53) to get:

$$\sum_{n \in \mathbb{Z}} s(n - 1) - s(n) = \sum_{n \in \mathbb{Z}} i(1 - 1/r^2)(u^1_{1/r}(n)\alpha)^* u^1_{1/r}(n)\alpha = i(1 - 1/r^2)\|u^1_{1/r}\alpha\|^2,$$

where $s(n) := (N^r_\alpha)^* W(u^r_+, \dot{u}^r_+)(n) N^r_\alpha$, $n \in \mathbb{Z}$. Note that the l.h.s. of the equation is a telescoping series. Thus

$$\lim_{n \to -\infty} s(n) - \lim_{n \to +\infty} s(n) = i(1 - 1/r^2)\|u^1_{1/r}\alpha\|^2. \tag{54}$$

Calculating $\dot{u}^r_+(n) = nr^{n-1}\ddot{u}^r_+(n) + r^n d/dz \dddot{u}^r_+(n)$ and noticing that $d/dz \dddot{u}^r_+, \dddot{u}^r_+ \in \ell^\infty(\mathbb{N}, \mathcal{M})$ (see Proposition 9), one obtains that

$$u^r_+(n), \quad \dot{u}^r_+ (n) \to 0, n \to +\infty. \tag{55}$$

Thus

$$s(n) \to 0 \quad \text{as} \quad n \to +\infty.$$

Using the definition of the Wronskian (see Eq. (22)) and the general fact that $d/dz f(z) \bigg|_{z = z_0} = (\frac{d}{dz} f(z) \bigg|_{z = z_0})^*$, one obtains the following (for every $n \in \mathbb{Z}$):

$$\frac{d}{dz} W(u^r_+, u^1_{1/z}) \bigg|_{z = r} = W(\ddot{u}^r_+, u^1_{1/r})(n) + W(u^r_+, \dot{u}^1_{1/r})(n), \tag{56}$$

because $\dddot{u}^1_{1/r}$ is given by $\dddot{u}^1_{1/r}(n) = d/dz u^1_{1/z}(n) \bigg|_{z = r}$ . The following computation now uses again Eq. (6) in order to replace $u^r_+(n) N^r_\alpha$ by $u^1_{1/r}(n)\alpha$ (recall that $\alpha \in \operatorname{Ker}(M^r)$) and Eq. (56)

$$s(n) = (N^r_\alpha)^* W(u^r_+, \dot{u}^r_+)(n) N^r_\alpha = W(u^r_+, N^r_\alpha, \dot{u}^r_+)(n) N^r_\alpha = W(u^1_{1/r}\alpha, \dot{u}^r_+)(n) N^r_\alpha$$

$$= \alpha^* W(u^1_{1/r}, \dot{u}^r_+)(n) N^r_\alpha = \alpha^* \left( \frac{d}{dz} W(u^r_+, u^1_{1/z}) \bigg|_{z = r} - W(u^r_+, \dot{u}^1_{1/r})(n) \right)^* N^r_\alpha =$$

$$= \alpha^*(d/dz W(u^r_+, u^1_{1/z}) \bigg|_{z = r})^* N^r_\alpha = \alpha^* W(\ddot{u}^1_{1/r}, u^1_{1/r})(n) \alpha,$$

Arguing as in (55), one gets $\dddot{u}^1_{1/r}(n), u^1_{1/r}(n) \to 0, n \to -\infty$. Taking the limit $(n \to -\infty)$ in the last equation leads to

$$s(n)^* \to (N^r_\alpha)^* \frac{d}{dz} W(u^r_+, u^1_{1/z}) \bigg|_{z = r} \alpha, \quad n \to -\infty. \tag{57}$$
Eqs. (54) and (57) and the fact that \( s(n) \to 0, \ n \to +\infty, \) show that
\[
(N^r \alpha)^* \frac{d}{dz} W(u^-_1, u^{1/z}_-) \bigg|_{z=r} \alpha = -i(1 - 1/r^2)\|u^{1/r}_- \alpha\|^2. \tag{58}
\]

Using Eq. (27) implies
\[
\frac{d}{dz} M^- \bigg|_{z=r} = -\nu^z W(u^r_+, u^{1/r}_-) - \nu^z \frac{d}{dz} W(u^r_+, u^{1/z}_-) \bigg|_{z=r}. \tag{59}
\]

Then, due to \( \alpha \in \text{Ker}(M^r) = \text{Ker}W(u^r_+, u^{1/r}_-)\),
\[
(N^r \alpha)^* \frac{d}{dz} M^- \bigg|_{z=r} \alpha = -\nu^z (N^r \alpha)^* \frac{d}{dz} W(u^r_+, u^{1/z}_-) \bigg|_{z=r} \alpha. \tag{60}
\]

Combining Eqs. (60) and (58), the result follows.

\[\square\]

**Proposition 21.** Suppose that \( r \in \mathbb{R} \) with \( 0 < |r| < 1 \) and that \( E = r + 1/r \) is an eigenvalue of \( H \). Set \( n_r = \dim(\text{Ker}(M^r_-)) = \dim(\text{Ker}(H - E)) \). Then there exists a complex number \( c_r \in \mathbb{C} \setminus \{0\} \) such that
\[
\det(M^-) = (z - r)^{nr}(c_r + O(\|z - r\|)), \quad z \to r.
\]

**Proof.** Let \( \{u_1, \ldots, u_n\} \) be a basis of \( \text{Ker}(M^r_-) \). Since \( N^r \big|_{\text{Ker}(M^r_-)} : \text{Ker}(M^r_-) \to \text{Ker}(M^r_+) \) is an isomorphism (see Proposition (19)), it follows that \( \{N^r_u u_1, \ldots, N^r_u u_n\} \) is a basis of \( \text{Ker}(M^r_+) = \text{Ran}(M^r_-)^\perp \), the latter due to Eq. (30). Next let \( \{v_{n_r+1}, \ldots, v_L\} \) be an orthonormal basis of \( \text{Ran}(M^r_-) \) and \( \{u_{n_r+1}, \ldots, u_L\} \) such that \( M^r u_i = v_i \). Then \( \{N^r_r u_1, \ldots, N^r_r u_{n_r}, v_{n_r+1}, \ldots, v_L\} \) and \( \{u_1, \ldots, u_{n_r}, u_{n_r+1}, \ldots, u_L\} \) are basis of \( C^L \). We denote by \( U_1, V_1 \) and \( V_1, V_2 \) the matrices such that
\[
U_1 = (u_1 \ldots u_L), \quad U_2 = (u_1 \ldots u_{n_r}), \quad V_1 = (N^r_u u_1 \ldots N^r_u u_{n_r} \ v_{n_r+1} \ldots v_L), \quad V_2 = (N^r_r u_1 \ldots N^r_r u_{n_r}).
\]

Then
\[
V_1^* M^r_+ U_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

We let \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) denote the matrices satisfying
\[
V_1^* \left. \frac{d}{dz} M^- \right|_{z=r} U_1 = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix},
\]
where \( \tilde{A} = V_2^* \left. \frac{d}{dz} M^- \right|_{z=r} U_2 \). Lemma 20 shows that \( \tilde{A} \) is invertible because for \( \phi \in \mathbb{C}^{n_r} \setminus \{0\} \)
\[
\phi^* \tilde{A} \phi = (V_2 \phi)^* \left. \frac{d}{dz} M^- \right|_{z=r} U_2 \phi = (N^r_r U_2 \phi)^* \left. \frac{d}{dz} M^- \right|_{z=r} U_2 \phi = r^{-1} \|u^{1/r} U_2 \phi\|^2 \neq 0.
\]

Note that the last identity used that the columns of \( U_2 \) are linearly independent. This implies that \( U_2 \phi \neq 0 \). The fact that \( \|u^{1/r} U_2 \phi\|^2 \neq 0 \) follows from Eq. (9), which implies that if \( x \neq 0 \)
then \( r^n u_{1/r}(n)x \to x \neq 0 \) as \( n \to -\infty \). With the help of Taylor’s theorem and analyticity, it follows that

\[
V_1^* M^z U_1 = \left( \begin{array}{ll} 0 & 0 \\ 0 & 1 \end{array} \right) + (z-r)V_1^* \frac{d}{dz} M^z \bigg|_{z=r} U_1 + \mathcal{O}((z-r)^2)
\]

\[
= \left( \begin{array}{ll} (z-r) \tilde{A} & (z-r) \tilde{B} \\ (z-r) \tilde{C} & 1 + (z-r) \tilde{D} \end{array} \right) + \mathcal{O}((z-r)^2) \quad \text{as } z \to r.
\]

Using the Schur formula (see Proposition 28) for the determinant in Eq. (61) one gets

\[
\det(V_1^* M^z U_1) = \det(1 + (z-r) \tilde{D} + \mathcal{O}((z-r)^2)) - \det(M^z) - 1 \det((M^z)^*) = \det(M^z) - 1 \det(M_1^{1/z}).
\]

Proposition 22. Let \( z \in S^1 \setminus \{1, -1\} \). The following identity holds true:

\[
\det(S^z) = \det(M_-^z)^{-1} \det((M^z_+)^*) = \det(M^z)^{-1} \det(M_1^{1/z}),
\]

Proof. Applying the Schur complement formula for the determinant (see Proposition 28) to the definition (7) of the scattering matrix leads to

\[
\det(S^z) = \det(M^z_+)^{-1} \det((M^z_-)^{-1} - N^z_- N^z_+ (M^z_+)^{-1}).
\]

Using Eqs. (30) and (33) one obtains that

\[
\det((M^z_-)^{-1} - N^z_- N^z_+ (M^z_+)^{-1}) = \det((1 - N^z_- N^z_+)(M^z_+)^{-1}) = \det((1 + (N^z_-)^* N^z_+)(M^z_+)^{-1})
\]

\[
= \det((M^z_-)^*) = \det(M_1^{1/z}).
\]

Equations (62) and (63) imply the claim. \( \square \)

Propositions 22 and 13 imply that the function \( z \mapsto \det(S^z) \) is differentiable on \( S^1 \setminus \{-1, 1\} \). This allows us to state the next result.

6 Time delay

The (total) time delay is by definition the quantity

\[
\text{Tr}\left( (S^z)^* \frac{d}{dz} S^z \right) = \det(S^z)^{-1} \frac{d}{dz} \det(S^z)
\]

for \( z \in S^1 \setminus \{-1, 1\} \) (the above identity is referred to as Jacobi’s formula). This section provides a formula for it in terms of the determinant of \( M^z \).
Corollary 23. For every \( z \in S^1 \setminus \{1, -1\} \), the following hold true:
\[
\det(S^z)^{-1} \frac{d}{dz} \det(S^z) = \det(M_{1/z}^z)^{-1} \frac{d}{dz} \det(M_{1/z}^z) - \det(M_z^z)^{-1} \frac{d}{dz} \det(M_z^z). \tag{64}
\]

**Proof.** Using Proposition 22, an explicit computation gives
\[
\frac{d}{dz} \det(S^z) = \det(M_z^z)^{-1} \frac{d}{dz} \det(M_{1/z}^z) - \det(M_z^z)^{-2} \frac{d}{dz} \det(M_z^z). \tag{65}
\]
Multiplying Eq. (65) by \( \det(S^z)^{-1} = \det(M_z^z) \det(M_{1/z}^z)^{-1} \) one gets the stated result. \( \square \)

7 Proof of Levinson’s theorem

**Proof of Theorem 14.** For each \( \epsilon > 0 \), let \( \Gamma_+^\epsilon \) and \( \Gamma_-^\epsilon \) be the truncated upper and lower semicircles parameterized by \( \gamma_+^\epsilon \), \( \gamma_-^\epsilon : [0, 1] \to S^1 \),
\[
\gamma_+^\epsilon(t) = e^{\pi(1-t)\epsilon + t(1-\epsilon)}, \quad \gamma_-^\epsilon(t) = 1/\gamma_+^\epsilon(1-t).
\]
For every \( \delta > 0 \), let us denote by \( \Omega_+^{\epsilon, \delta} \) and \( \Omega_-^{\epsilon, \delta} \) the interior arcs parameterized by \( \omega_+^{\epsilon, \delta} \), \( \omega_-^{\epsilon, \delta} : [0, 1] \to \mathbb{C} \), given by
\[
\omega_+^{\epsilon, \delta} = (1-\delta)\gamma_+^\epsilon, \quad \omega_-^{\epsilon, \delta} = (1-\delta)\gamma_-^\epsilon.
\]
We let \( l_+^{\epsilon, \delta} \) be the line segment from \( \omega_+^{\epsilon, \delta}(1) \) to \( \omega_+^{\epsilon, \delta}(0) \), and \( l_-^{\epsilon, \delta} \) the line segment that goes from \( \omega_-^{\epsilon, \delta}(1) \) to \( \omega_-^{\epsilon, \delta}(0) \). Now we define the positively-oriented closed curve \( \Omega_{\epsilon, \delta} = \Omega_+^{\epsilon, \delta} + l_+^{\epsilon, \delta} + \Omega_-^{\epsilon, \delta} + l_-^{\epsilon, \delta} \). By Propositions 13, 18, 19, 21 and the argument principle one has that
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{\Omega_{\epsilon, \delta}} \det(M_z^z)^{-1} \frac{d}{dz} \det(M_z^z)dz = 2\pi i J_0. \tag{66}
\]
On the other hand, Eq. (43) implies that
\[
\frac{d}{dz} \det(M_z^z) = (z-1)^{J_0^h} - Lg'(z) + (J_0^+ - L)(z-1)^{J_0^h} - 1g(z),
\]
where \( g(z) \to \overline{z} \neq 0, \ z \to 1. \) Then
\[
\det(M_z^z)^{-1} \frac{d}{dz} \det(M_z^z) = (J_0^+ - L) \frac{1}{z-1} + g'(z) \frac{g(z)}{g(z)}. \tag{67}
\]
Using (67) and Lemma 27 one can compute the next limit
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{l_+^{\epsilon, \delta}} \det(M_z^z)^{-1} \frac{d}{dz} \det(M_z^z)dz = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{l_+^{\epsilon, \delta}} (J_0^+ - L) \frac{1}{z-1} + g'(z) \frac{g(z)}{g(z)}dz
\]
\[
= (J_0^+ - L) \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{l_+^{\epsilon, \delta}} \frac{1}{z-1}dz = -\pi i (J_0^+ - L). \tag{68}
\]
An analogous calculation using (45), shows that
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{\Gamma_{\epsilon,\delta}} \det(M_z)^{-1} \frac{d}{dz} \det(M_z)dz = -\pi i (J_h^- - L). \tag{69}
\]
Now Lemma 26 implies that
\[
\int_{\Gamma_{\epsilon,\delta}} \det(M_z)^{-1} \frac{d}{dz} \det(M_z)dz = -\int_{\Gamma_{\epsilon,\delta}} \det(M_{1/z}^{-})^{-1} \frac{d}{dz} \det(M_{1/z}^{-})dz. \tag{70}
\]
Using the previous equation and (64), one obtains that
\[
\int_{\Gamma_{\epsilon,\delta}} \det(S_z)^{-1} \frac{d}{dz} \det(S_z)dz = \int_{\Gamma_{\epsilon,\delta}} \det(M_{1/z}^{-})^{-1} \frac{d}{dz} \det(M_{1/z}^{-})dz - \det(M_{1/z}^{-})^{-1} \frac{d}{dz} \det(M_{1/z}^{-})dz \\
= - \left( \int_{\Gamma_{\epsilon,\delta}} + \int_{\Gamma_{\epsilon,\delta}} \right) \det(M_z)^{-1} \frac{d}{dz} \det(M_z)dz \\
= - \lim_{\delta \to 0} \left( \int_{\Omega_{\epsilon,\delta}^+} + \int_{\Omega_{\epsilon,\delta}^-} \right) \det(M_z)^{-1} \frac{d}{dz} \det(M_z)dz. \tag{71}
\]
Using (66), (68), (69) and (71) one arrives at
\[
2\pi i (J_h + \frac{1}{2} J_h - L) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \left( \int_{\Omega_{\epsilon,\delta}^+} - \int_{\Gamma_{\epsilon,\delta}^+} - \int_{\Gamma_{\epsilon,\delta}^-} \right) \det(M_z)^{-1} \frac{d}{dz} \det(M_z)dz \\
= \lim_{\epsilon \to 0} \lim_{\delta \to 0} \left( \int_{\Omega_{\epsilon,\delta}^+} + \int_{\Omega_{\epsilon,\delta}^-} \right) \det(M_z)^{-1} \frac{d}{dz} \det(M_z)dz \tag{72}
\]
where \( J_h = J_h^+ + J_h^- \).

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\section{Appendix}

This appendix recollects a some technical statements that are used in the main text.

\textbf{Theorem 24} (Volterra equation, Lemma 7.8 in [24], and Theorem 26 in [9]). Let \( g \in \ell^\infty(N,M) \) and \( K(n,m) \in M \) for each \( m,n \in N \). Consider the Volterra equation
\[
f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m)f(m), \tag{73}
\]
and suppose there is a sequence \( M \in \ell^1(\mathbb{N}, \mathbb{R}) \) such that \( \|K(n,m)\| \leq M(m) \) for all \( m, n \in \mathbb{N} \). Then Eq. \((63)\) has a unique solution \( f \in \ell^\infty(\mathbb{N}, M) \). Moreover, if \( g(n) \) and \( K(n,m) \) depend continuously (resp. holomorphically) on a parameter \( z \) (for every \( n \)), \( M \) does not depend on \( z \), and \( g(n) \) is uniformly bounded w.r.t. \( n \) and \( z \), then the same is true for \( f(n) \).

**Lemma 25** (Gronwall lemma). Let \( \alpha \) a real positive number and \((w_n)_{n \in \mathbb{N}}, (u_n)_{n \in \mathbb{N}}\) real positive sequences such that
\[
\sum_{j=1}^\infty w_j < \infty, \quad u_n \leq K, \ n \in \mathbb{N},
\]
for some \( K \in \mathbb{R} \) and
\[
u_n \leq \alpha + \sum_{j=n+1}^\infty w_j u_j. \tag{74}
\]
Then for all \( n \in \mathbb{N} \), it follows that
\[
u_n \leq \alpha \exp \left( \sum_{j=n+1}^\infty w_j \right).
\]

**Proof.** Let us provide a proof as this was already stated without proof in [9]. Let us define the functions \( W, U : \mathbb{R} \to [0, \infty) \) by setting
\[
U\big|_{[-n,-n+1]} = u_n, \quad W\big|_{[-n,-n+1]} = w_n, \quad n \in \mathbb{N},
\]
and both \( U \) and \( W \) vanish on \([0, \infty)\). For every \( t \in [-n,-n+1) \), one has that
\[
U(t) = u_n \leq \alpha + \sum_{j=n+1}^\infty w_j u_j = \alpha + \int_{-\infty}^{-n} WU \leq \alpha + \int_{-\infty}^{t} WU. \tag{75}
\]
For the rest of the proof, one argues as in the proof of the Gronwall lemma for the continuous case. We provide a few lines with the key steps, for the convenience of the reader. Let us define \( V(t) = e^{-\int_{-\infty}^{t} WU} \int_{-\infty}^{t} WU \). It is clear that \( \frac{d}{dt} V(t) = e^{-\int_{-\infty}^{t} WU} \int_{-\infty}^{t} WU (U(t) - \int_{-\infty}^{t} WU) \leq \alpha e^{-\int_{-\infty}^{t} WU} \int_{-\infty}^{t} WU(t), \) for every \( t \notin -\mathbb{N} \cup \{0\} \). Integrating, one gets
\[
V(t) \leq \int_{-\infty}^{t} \alpha e^{-\int_{-\infty}^{s} WU} W(s) = \alpha (1 - e^{-\int_{-\infty}^{t} WU}).
\]
This implies that
\[
\int_{-\infty}^{t} WU \leq \alpha e^{\int_{-\infty}^{t} WU} - \alpha,
\]
which together with \((75)\) implies \( u_n = U(-n) \leq \alpha e^{\int_{-n}^{-n+1} WU} = \alpha \sum_{j=n+1}^{\infty} w_j \). \qed

**Lemma 26.** Let \( h : S^1 \setminus \{-1, 1\} \to \mathbb{C} \) a continuously differentiable function and \( \Gamma_+ \) a curve on \( S^1 \setminus \{-1, 1\} \) that is parameterized by a differentiable function \( \gamma_+ : [0, 1] \to S^1 \setminus \{-1, 1\} \). We assume that \( 0 \notin h(\Gamma_+) \). Let \( r : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) the function \( r(z) = 1/z \) and \( \Gamma_- := r(\Gamma_+) \) parameterized by \( \gamma_-(t) := 1/\gamma_+(1-t) \). The following identity holds true:
\[
\int_{\Gamma_-} \frac{h'}{h} = - \int_{\Gamma_+} \frac{(h \circ r)'}{h \circ r}.
\]
Proof.
\[
\int_{\gamma_-} \frac{h'}{h} = \int_0^1 \frac{h' \circ \gamma_-(t)}{h \circ \gamma_-(t)} (\gamma_-)'(t) dt = - \int_1^0 \frac{h' \circ \gamma_-(1 - t)}{h \circ \gamma_-(1 - t)} (\gamma_-)'(1 - t) dt
\]
\[
= \int_0^1 \frac{h'(1/\gamma_+(t))}{h(1/\gamma_+(t))} 1/(\gamma(t))^2 (\gamma_+)'(t) dt = \int_0^1 \frac{h' \circ r(\gamma_+(t))}{h \circ r(\gamma_+(t))} 1/(\gamma+)(t) (\gamma_+)'(t) dt
\]
\[
= - \int_0^1 \frac{(h \circ r)'(\gamma_+(t))}{h \circ r(\gamma_+(t))} (\gamma_+)'(t) dt = - \int_{\Gamma_+} \frac{(h \circ r)'}{h \circ r},
\]
where the penultimate equality follows from \((h \circ r)'(z) = -(1/z^2)h' \circ r(z)\). \qed

Lemma 27. Let \(r : \overline{D} \to \mathbb{C}\) a continuous function (or continuous in a closed neighborhood of 1 in \(\overline{D}\)) such that it is analytic on \(\mathbb{D}\) (or analytic in the intersection of a neighborhood of 1 with \(\mathbb{D}\)) and \(r(1) = c \neq 0\). Let \((\gamma_n)_{n \in \mathbb{N}}\) a sequence of curves satisfying
\[
\gamma_n \subset D_n \cap \mathbb{D},
\]
where \(D_n = \{z \in \mathbb{D} : |z - 1| < 1/n\}\). It follows that
\[
\lim_{n \to \infty} \int_{\gamma_n} \frac{r'(z)}{r(z)} dz = 0.
\]

Proof. Since \(c \neq 0\), we can assume w.l.o.g. that \(\Re(c) > 0\) (we multiply everything by a constant complex number). Let \(B\) an open set such that \(c \in B\) and
\[
(-\infty, 0] \cap B = \emptyset.
\]
We set
\[
\log : \mathbb{C} \setminus [-\infty, 0] \to \mathbb{C}
\]
an analytic branch of logarithm. By the continuity of \(r\), there exist \(n \in \mathbb{N}\) such that \(r(D_n \cap \mathbb{D}) \subset B\). Then, for \(m \geq n\), the function
\[
\log r : \overline{D_m} \cap \mathbb{D} \to \mathbb{C}
\]
is continuous, and analytic on \(D_m \cap \mathbb{D}\). For \(m \geq n\), we calculate
\[
\int_{\gamma_m} \frac{r'(z)}{r(z)} dz = \int_{\gamma_m} (\log r)' = \log r(\gamma_m(1)) - \log r(\gamma_m(0)).
\]
The desired result follows from the fact that \(\log r\) is continuous on \(\overline{D_m} \cap \mathbb{D}\) and, consequently,
\[
\gamma_m(1) - \gamma_m(0) \to 0, \ m \to \infty.
\]
\qed

Proposition 28 (Schur formula for the determinant). Let \(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) be a block matrix with square matrices \(A\) and \(D\). If \(D\) is invertible then
\[
\det(M) = \det(D) \det(A - BD^{-1}C).
\]
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