Phonon–induced superconducting state on a square lattice

K. Szewczyk, M. W. Jarosik, A. P. Durajski, and R. Szczęśniak

1 Division of Theoretical Physics, Institute of Physics, Jan Długosz University in Częstochowa, Ave. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
2 Institute of Physics, Częstochowa University of Technology, Ave. Armii Krajowej 19, 42-200 Częstochowa, Poland

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The tendency to induce the superconducting state on a square lattice (the Eliashberg formalism) is characterised in the work. We took into account the pairing mechanism based on a linear electron–phonon interaction. We proved that the superconducting phase cannot exist if the value of the coupling constant in the diagonal channel of the effective energy \( \lambda_D \) is equal to the value of the coupling constant in the non–diagonal channel \( \lambda_{ND} \). The linear electron–phonon interaction induces, however, the superconducting state for values of the unbalance parameter \( \gamma = \lambda_D / \lambda_{ND} \) less than \( \gamma_C = 0.42 \). The reason for the decrease in the critical temperature value with the growth of the \( \gamma \) parameter value is the anomalous increase in the electron effective mass. The results were achieved by the fully self–consistent numerical analysis.

Physical properties of the phonon–induced superconducting state on a square lattice were determined in the present work within the Eliashberg formalism. The initial results, obtained in the limit of momentum approximation (which neglects the self–consistency with respect to Matsubara frequency), were already presented by us elsewhere [1]. There we showed that the balanced superconducting state on a square lattice cannot arise in the case of the constant value of the electron–phonon coupling function \( g_\mathbf{q} \sim g \). By the balanced superconducting state we mean such a state, for which the coupling constant \( \lambda_D \) in the diagonal channel of the matrix self–energy \( M_\mathbf{k}(\omega_n) \) is equal to the coupling constant in the non–diagonal channel \( \lambda_{ND} \). Please notice that the diagonal elements of the self–energy determine, \textit{inter alia}, the influence of the electron–phonon interaction on the value of the effective electron mass. The off–diagonal elements of the \( M_\mathbf{k}(\omega_n) \) are directly related to the order parameter of the phase transition between the superconducting and the normal state.

At present, we know many physical mechanisms which can lead to the unbalanced phonon–induced superconducting state. Particular attention should be paid to the spin–fluctuation (paramagnon) scattering, which contributes to the effective coupling constant in the diagonal channel in the opposite way than does in the antidiagonal channel (\( s \)-wave symmetry) [2, 3, 4, 5, 6, 7]. The unbalance of the superconducting state can also be generated by the non–conventional terms of the electron–phonon interaction, e.g. the electron–electron–phonon [8] or, similarly, the hole–hole–phonon interaction [9]. Additionally, they induce the asymmetry of the electron density of states, the pseudo–gap in the electron density of states, or the anomalous dependence of the energy gap on doping [10]. These effects can be of particular importance for the correct description of the superconducting state in cuprates [11, 12, 13], especially since they are observed experimentally [14, 15, 16]. At the level of the conventional Eliashberg formalism, the unbalanced superconducting state arises also when the non–phonon contribution from the \( d \)-wave type symmetry is taken into account in the \( a_0^2 F(\omega) \) Eliashberg function modelling the electron–phonon interaction [17, 18, 19]. Nevertheless the considered model is too simple for the satisfactory explanation of even some of the anomalous properties of the superconducting state in cuprates.

The purpose of our investigation, the results of which are gathered in the present work, was to find information, as precise as possible, on the physical properties of the phonon–induced superconducting state on a square lattice. Therefore we solved the Eliashberg equations [20] with respect both to the electron wave vector and to the Matsubara frequency in the fully self–consistent way. We took into consideration the electron–phonon coupling function explicitly dependent on the phonon transfer of momentum between electron states [21]. It is worth noticing that the Eliashberg equations on a square lattice are for the first time analysed in such a precise manner.

The obtained results allowed also to comment on the problem of the significance of the electron–phonon interaction in the context of determination of the pairing mechanism in cuprates [8, 13]. This issue is discussed at greater length in the concluding part of the work.

In its most advanced form, the analysis of the phonon–induced superconducting state is performed within the Eliashberg formalism, which is based on the self–consistent calculation of the matrix elements of the thermodynamic Green’s function: \( G_\mathbf{k}(\omega_n) = \langle \langle \Psi_\mathbf{k}^\dagger \Psi_\mathbf{k} \rangle \rangle_{\omega_n} \). The symbol \( \Psi_\mathbf{k}^\dagger \) denotes the Nambu spinor with Hermitian coupling: \( \Psi_\mathbf{k}^\dagger = \left( c_{\mathbf{k} \uparrow}, c_{\mathbf{k} \downarrow}^\dagger \right) \), and \( \omega_n = \frac{\pi}{\beta} (2n - 1) \) stands for the fermionic Matsubara frequency (\( \beta = 1/k_B T \)). The quantity \( c_{\mathbf{k} \sigma}^\dagger \) is the creation operator of the electron state with momentum \( \mathbf{k} \) and spin \( \sigma \in \{\uparrow, \downarrow\} \). The explicit form

*Electronic address: kamila.szewczyk@ajd.czest.pl
of the self-energy \( M_k (i\omega_n) \) is determined as a rule by application of the Hamiltonian which models the linear coupling between the electron and the phonon subsystems [22]: 

\[
H = \sum_k \varepsilon_k \Psi_k \tau_3 \Psi_k + \sum_q \omega_q b_q^\dagger b_q + \frac{1}{\sqrt{N}} \sum_{kq} g_{kq} \Psi_{k+q} \tau_3 \Psi_k \phi_q,
\]

where \( \varepsilon_k \) is the electronic dispersion relation and \( \phi_q = b_q + b_q^\dagger \). The quantity \( b_q^\dagger \) denotes the creation operator of the phonon state with momentum \( q \). The symbol \( \omega_q \) represents phonon dispersion relation (\( \omega_q = \omega_n \)). The relatively easy quantum operator calculations bring about the exact result: 

\[
M_k (i\omega_n) = \frac{1}{\pi} \sum_q g_{kq} \tau_3 \left( \left. \langle \Psi_{k+q} \phi_q \Psi_k^\dagger \phi_q^\dagger \rangle \right|_{i\omega_n} \right).
\]

Further analysis of the problem is based on the Migdal approximation and on the Wick’s theorem [22], by virtue of which the thermodynamic Green’s function \( \left( \left. \langle \Psi_{k+q} \phi_q \Psi_k^\dagger \phi_q^\dagger \rangle \right|_{i\omega_n} \right) \) is approximated by the product of the \( G_k (i\omega_n) \) function and the phonon propagator for the non-interacting phonons. The procedure of reaching self-consistency allows to obtain the non-linear Eliashberg equations, which determine the thermodynamics of the superconducting phase on the quantitative level:

\[
\varphi_k (i\omega_n) = \frac{1}{\beta N} \sum_{m_q} K_q (\omega_n - \omega_m) \frac{\varphi_{k-q} (i\omega_m)}{D_{k-q} (i\omega_m)},
\]

\[
Z_k (i\omega_n) = 1 + \frac{\gamma}{\beta N} \sum_{m_q} \frac{\omega_m}{\omega_n} K_q (\omega_n - \omega_m) \frac{Z_{k-q} (i\omega_m)}{D_{k-q} (i\omega_m)}.
\]

The symbol \( \varphi_k (i\omega_n) \) denotes the order parameter function, the order parameter being defined by the \( \Delta_k (i\omega_n) = \varphi_k (i\omega_n) / Z_k (i\omega_n) \) ratio, where \( Z_k (i\omega_n) \) represents the wave function renormalisation factor. The pairing kernel of the electron–phonon interaction is given by the formula: 

\[
K_q (\omega_n - \omega_m) = 2g_{kq}^2 \frac{\omega_q}{(\omega_n - \omega_m)^2 + \omega_q^2}.
\]

Additionally, \( D_k (i\omega_n) = (\omega_n Z_k (i\omega_n))^2 + \varepsilon_k^2 + \varphi_k^2 (i\omega_n) \). Usually the Eliashberg equations are solved in the isotropic approximation: 

\[
\varphi_k (i\omega_n) \sim \varphi (i\omega_n) \quad \text{and} \quad Z_k (i\omega_n) \sim Z (i\omega_n),
\]

while the pairing kernel is transformed in the following way: 

\[
K_q (\omega_n - \omega_m) \sim 2 \int_0^{\omega_D} d\omega' \frac{\alpha^2 F (\omega')}{(\omega_n - \omega_m)^2 + \omega'^2},
\]

where \( \omega_D \) is the Debye phonon frequency. The spectral function \( \alpha^2 F (\omega') \) for a specific physical system can be determined by means of the DFT method. This approach works perfectly e.g., for the recently discovered electron–phonon hydrogen–containing superconductors [28, 30, 31, 32, 33, 34, 35], characterised by the extremely high values of the critical temperature: \( \text{H}_2\text{S} \) \( (T_C = 150 \text{ K for } p = 150 \text{ GPa}) \), \( \text{H}_3\text{S} \) \( (T_C = 203 \text{ K for } p = 150 \text{ GPa}) \) [36, 37], and \( \text{LaH}_3 \) \( (T_C = 215 \text{ K for } p = 150 \text{ GPa}) \) [38], \( (T_C = 260 \text{ K for } p = 150 \text{ GPa}) \) [39].
\( p \in (180 - 200) \text{ GPa} \) \[39\].

Taking into account the results reported in [1], we supplemented the Eq. [2] with the \( \gamma \) parameter, which determines the degree of unbalance of the electron–phonon coupling constants in the diagonal and the non–diagonal channel of self–energy.

We performed the numerical calculations on the \( N \times N \) momentum lattice, where \( N = 100 \), and we took into account 200 Matsubara frequencies. Therefore two million of equations had to be solved. To minimize the hardware requirements, the Eliashberg equations were written in the form which explicitly expressed all symmetries of solutions with respect to the wave vector and to the Matsubara frequency. Assuming that the auxiliary symbol \( f_{k}(i\omega_{n}) = f_{n}(k_{x}, k_{y}) \) should be understood as \( \varphi \) or \( Z \), one can write that: \( f_{n}(k_{x}, k_{y}) = f_{n}(-k_{x}, -k_{y}) = f_{n}(k_{x}, -k_{y}) = f_{n}(-k_{x}, k_{y}) \), and \( f_{n}(k_{x}, k_{y}) = f_{n+1}(k_{x}, k_{y}) \). The Eliashberg equations were solved by means of the AMCIA program, which generalises the numerical procedures used in the work [19]. The AMCIA numerical environment requires introduction of the initial values of the \( \varphi_{k}(i\omega_{n}) \) and \( Z_{k}(i\omega_{n}) \) functions. As far as the wave function renormalising factor is concerned, we took into account the first 25 terms of the series:

\[
Z_{m_{0}}(q_{0}) = 1 + \sum_{L=1}^{+\infty} \left( \frac{1}{\beta N} \right)^{L} \prod_{q_{1},q_{L}=0}^{\pi} \prod_{m_{1}=1}^{M} \frac{\omega_{m_{L}}}{\omega_{m_{0}}} \prod_{j=1}^{L} J_{m_{j-1} m_{j}}(q_{j-1}, q_{j}) d_{m_{j}}^{-1}(q_{j}), \tag{3}
\]

where: \( q_{j} = (q_{x}^{j}, q_{y}^{j}) \) and \( J_{m_{j-1} m_{j}}(q_{j-1}, q_{j}) = J_{m_{j-1} m_{j}}(q_{x}^{j-1}, q_{y}^{j-1}, q_{x}^{j}, q_{y}^{j}) = K_{m_{j-1} m_{j}}(q_{x}^{j-1} + q_{x}^{j}, q_{y}^{j-1} + q_{y}^{j}) \) + \( K'_{m_{j-1} m_{j}}(q_{x}^{j-1} + q_{x}^{j}, q_{y}^{j-1} - q_{y}^{j}) \), \( K_{m_{j-1} m_{j}}(q_{x}^{j-1} - q_{x}^{j}, q_{y}^{j-1} + q_{y}^{j}) \) + \( K'_{m_{j-1} m_{j}}(q_{x}^{j-1} - q_{x}^{j}, q_{y}^{j-1} - q_{y}^{j}) \), while \( K'_{m_{j-1} m_{j}}(q_{x}, q_{y}) = K_{q}(\omega_{m_{j-1}} - \omega_{m_{j}}) = K_{q}(\omega_{m_{j-1}} + \omega_{m_{j}}) \).

Additionally: \( d_{m_{j}}^{-1}(q_{j}) = \omega_{m_{j}}^{2} + \varepsilon_{q}^{2} \). We choose positive non-zero numbers as initial values for the order parameter function.

The energy of electron states was modeled by using the tight–binding description on the ions’ equilibrium positions assuming the nearest–neighbor (\( t \)) and next-nearest neighbor (\( t' \)) hopping [13]: \( \varepsilon_{k} = -2t(\cos(k_{x}) + \cos(k_{y})) + 4t' \cos(k_{x}) \cos(k_{y}) \), where the lattice constant is taken as unity. In the present work we assumed that energy is expressed in units of hopping integral \( t \), and \( t' = 0.1t \). We took into account the acoustic

FIG. 3: (a) The influence of temperature on the averaged value of the order parameter \( \langle \Delta(i\omega_{n=1}) \rangle \). Numerical results obtained for the unbalance parameter equal to 0.25\( \gamma_{C} \), 0.56\( \gamma_{C} \), and 0.75\( \gamma_{C} \). The continuous curves were found by means of the formula: \( \Delta(T) = \Delta(0) \sqrt{1 - (T/T_{c})^{2}} \), where \( \Gamma \in \{23.9, 22.5, 21.1 \} \) for \( \gamma \in \{0.25\gamma_{C}, 0.56\gamma_{C}, 0.75\gamma_{C} \} \), respectively. Additionally \( \Delta(0) = \Delta(T_{0}) \). For the BCS-type curves (dashed lines), it should be assumed that \( \Gamma = 3.0 \) and \( 2\Delta(0)/k_{B}T = 3.53 \) [25] [26] [27]. (b) The temperature dependence of the effective electron mass.
phonons stemming from the nearest-neighbor ($\kappa = 0.3t$) and
the next-nearest neighbor ($\kappa^* = 0.01t$) springs. We
added the mass-term ($\kappa'' = 0.001t$) that removed the
modes with zero energy [21]:

$$\omega_q = 2|\kappa| \left( \sin^2 \left( \frac{q_x}{2} \right) + \sin^2 \left( \frac{q_y}{2} \right) \right)$$

$$+ \kappa' \left( 1 - \cos (q_x) \cos (q_y) \right)$$

$$+ 2 \sqrt{\frac{\kappa^2}{4} \left( \cos (q_x) - \cos (q_y) \right)^2 + \kappa'^2 \sin^2 (q_x) \sin^2 (q_y)}.$$  

In the case of a square lattice, the electron–phonon ma-
trix elements are given by [21]: $g_q = D|q|\sqrt{1/\omega_q}$, and
$D = 0.3t$. Let us notice that the function $g_q$ reproduces
the results reported in [11][12][43], and has the structure predicted by Bloch [44].

The dependence between the averaged value of the or-
der parameter $\langle \Delta (i\omega_{n=1}) \rangle = \frac{1}{N} \sum_k \Delta_k(i\omega_{n=1})$ and
the value of the unbalance parameter is plotted in Fig.1 (a).
It can be easily noticed that the superconducting phase
cases to exist for $\gamma \geq \gamma_C = 0.42$. This means that the
balanced superconducting state cannot originate on the
square lattice. Let us remember that within the momentum
approximation with a constant value of the electron–
phonon coupling function ($g_q \sim q$ we got $\gamma_C = 0.94$ [1].
A similar value, $\gamma_C = 0.93$, is obtained from calculations
in which the momentum transfer is explicitly taken into
account in the electron–phonon coupling function (see
Fig.1 (b)).

The dependence of the electron effective mass ($m_e^* )$ on
the value of the unbalance parameter $\gamma$ is presented in
Fig. 1 (c), wherein $m_e^* = \langle Z(i\omega_{n=1}) \rangle m_e$, and
the symbol $m_e$ denotes the band mass of an electron. The
obtained results prove that the superconducting phase on
a square lattice ceases due to the fact that the electron
effective mass takes anomalously high values [$m_e^* \gamma \gamma_C =
13.42m_e$, what results in the decrease in the critical tem-
perature: $T_C \sim \exp (-m_e^*)$ [15][46]. The values of the
electron effective mass for the momentum approximation are
significantly less (Fig.1 (d)). This outcome corre-
lates well with the result achieved for the momentum approxima-
tion ($m_e^* \gamma \gamma_C \sim 4.73m_e$) under the additional
assumption $g_q \sim q$ [14]. It should be noticed that a char-
acteristic deflection occurs in the profile of the effective
electron mass versus the unbalance parameter curve for
$\gamma = \gamma_C$. It results from the disappearance of the super-
conducting state.

As far as the results achieved within the isotropic approxima-
tion ($\Delta_k(i\omega_{n=1}) \sim \Delta(i\omega_{n=1})$ and
$Z_k(i\omega_{n=1}) \sim Z(i\omega_{n=1})$) are concerned, the unbalanced
Eliashberg equations were solved by Cappelluti and Um-
marino in [17]. For the case of 3D system, the existence of
the balanced superconducting state was confirmed, and
even the unbalanced superconducting state of the $\gamma$ par-
ter value significantly exceeding unity was stated. The
calculations which we performed for the isotropic approxima-
tion on a square lattice also predict induction of either the balanced or the unbalanced
superconducting state for $\gamma > 1$ (see Appendix A). The
above results mean that the isotropic approximation, at
least with respect to the square lattice, is incorrect.

Having acquired the explicit solutions of the Eliash-
berg equations, one can determine the regions of the first
Brillouin zone, for which the function $m_e^* (k)$ contributes
particularly greatly to the averaged value of the effective
electron mass. We present diagrams showing the values
of the wave function renormalisation factor versus the
wave vector in Fig. 2 (left column). It can be seen that
an increase in the effective electron mass is generated
mainly by increase in the value of $m_e^* (k)$ function in the
whole Brillouin zone. Although at the boundaries of the
Brillouin zone it is particularly high. The order parameter
function $\varphi_k(i\omega_{n=1})$, which is plotted in Fig. 2 (right
column), has a similar form.

Our results prove that only the unbalanced supercon-
ducting state of $\gamma < \gamma_C = 0.42$ can be induced on
a square lattice. We selected three values of the unbal-
ance parameter ($0.25\gamma_C, 0.56\gamma_C$, and $0.75\gamma_C$) in order to
analyse thermodynamical properties of this state. The
achieved results are plotted in Fig. 3 for both the order
parameter and the wave function renormalising factor.
It can be seen that in every case the dependence of the
order parameter on temperature deviates considerably
from the one predicted by the BCS theory [25][26]. Taking
into account the ratio $R_\Delta = 2\Delta(0)/k_BT_C$ we get:
$R_\Delta^{0.25\gamma_C} = 5.10$, $R_\Delta^{0.56\gamma_C} = 4.56$, and
$R_\Delta^{0.75\gamma_C} = 4.43$. Please notice that the BCS theory predicts $R_\Delta = 3.53$.

The influence of temperature on the value of the effective
electron mass is presented in Fig. 3 (b). The depen-
dence of $m_e^*$ on $T$ is weak, as it is both in the isotropic
case and in the momentum approximation [1][45].

The thermodynamics of the superconducting state is
fully determined by the values of both the order parameter
and the wave function renormalising
factor. Considering the averaged values of the discussed functions, we calculated the difference in free energy between the superconducting and the normal state \[49\]: 
\[
\Delta F/\rho(0) = -(2\pi/\beta) \sum_{n=1}^{M} \left( \tilde{\omega}_{n}^{2} + (\Delta_{n})^{2} - |\omega_{n}| \langle Z_{n}^{(S)} \rangle - \langle Z_{n}^{(N)} \rangle \frac{|\omega_{n}|}{\sqrt{\omega_{n}^{2} + (\Delta_{n})^{2}}} \right).
\]
Symbols \(Z_{n}^{(S)}\) and \(Z_{n}^{(N)}\) denote the values of the wave function renormalising factor in the superconducting and the normal state, respectively. The thermodynamic critical field should be determined from the formula: 
\[
H_{c} = \sqrt{-8\pi\Delta F},
\]
whereas the difference in the specific heat between the superconducting and the normal state can be found using the formula: 
\[
\Delta C = C^{S} - C^{N} = -T \frac{d\Delta F}{dT},
\]
where \(C^{N} = \gamma T\). The Sommerfeld constant is equal to: 
\[
\gamma = \frac{2}{3} \pi^{2} k_{B}^{2} \rho(0) (1 + \lambda).
\]
The achieved results we showed in Fig. 4. Then we calculated the values of the non-dimensional thermodynamic ratios: 
\[
R_{C} = \frac{[C^{S}(T_{C}) - C^{N}(T_{C})] / C^{N}(T_{C})}{C^{N}(T_{C})} \quad \text{and} \quad R_{H} = T C^{N}(T_{C}) / H_{c}^{2}(0).
\]
The considered results correspond to the universal constants in the BCS theory, the latter taking the values equal to 1.43 and 0.168, respectively [25–29]. For the superconducting state on a square lattice, we arrived at the results which differ significantly from the ones occurring in the BCS theory: 
\[
R_{C}^{25T_{C}} = 3.80, \quad R_{H}^{25T_{C}} = 0.107, \quad R_{C}^{56T_{C}} = 2.76,
\]
\[
R_{H}^{56T_{C}} = 0.121, \quad R_{C}^{75T_{C}} = 2.27, \quad R_{H}^{75T_{C}} = 0.185.
\]

To summarize, we proved in the work that the balanced phonon-induced superconducting state cannot be generated on a square lattice. On the other hand, the linear electron–phonon interaction can induce the unbalanced superconducting state in cases for which the unbalance parameter takes a value less than 0.42. The reason for a decrease in the critical temperature value with an increase in \(\gamma\) parameter value is the anomalously high increase in the effective electron mass. This effect can be observed only when the Eliashberg equations are solved in the fully self-consistent way. It should be stressed that the results presented in the work completely call into question the outcome got for the phonon–induced superconducting state on a square lattice within the isoropic approximation.

The performed numerical analysis proved that the unbalanced superconducting state is described by the thermodynamic parameters, which values differ significantly from the values predicted by the BCS theory.

Please notice that our results can make the understanding of the pairing mechanism in cuprates considerably easier [8, 13]. It is generally accepted fact that electrons in cuprates form a strongly correlated system [50, 51, 52]. Numerical calculations carried out for the Hubbard model demonstrate that the value of the onsite \(U\) integral is equal to about 5 eV [22]. It should be strongly emphasised that the balanced linear electron–phonon interaction is too weak to induce the experimentally observed superconducting state (the full discussion of the considered problem at the \(ab\ initio\) level can be found in [53]). On the other hand, there exists a good deal of experimental data which indicate that the interaction between electrons and phonons in cuprates is really significant. Let us recall the results obtained by means of the ARPES method, which demonstrated the existence of a break in the energy spectrum near the phonon energy [54, 55]. Additionally, the ARPES method made possible to discover the isotope effect of the real part of the self-energy [56]. The isotope effect is also related to the critical temperature, which is particularly distinct in the strongly underdoped regions [57]. Moreover, the vibrations of the crystal lattice modify the penetration depth and the results of Raman measurements [58, 60].

Therefore an obvious suggestion comes into mind that the high–temperature superconducting state in cuprates may be induced by the strongly unbalanced electron–phonon interaction. This suggestion is quite probable for the reason that it is very difficult to prove the existence of the superconducting state of sufficiently high critical temperature value in pure electron models with \(U > 0\). Such a problem, however, does not occur in effective models with \(U < 0\).

Our hypothesis is also confirmed by the results achieved by Kim and Tesanovic [60], who showed, on the example of \(\La_{2-x}\Sr_{x}\Cu_{4}\O_{4}, \ (Y_{1-x}Pr_{x})Ba_{2}Cu_{3}O_{7-\gamma}\) and \(\YBa_{2-x}\La_{x}\Cu_{3}O_{7}\) compounds, that the strong Coulomb correlations do not suppress the phonon pairing mechanism in a wide doping range.

**Appendix A: The unbalanced superconducting state on a square lattice - the isotropic approximation**

Let us discuss the issue of the unbalanced superconducting state within the isotropic approximation. The
Eliashberg equations take the form:
\[
\varphi_n = \frac{1}{\beta} \sum_k K'(n - m) \varphi_m \sum_k p_k^{-1}(m),
\]
\[
Z_n = 1 + \frac{\gamma}{\omega_{n/2}} \sum_k K'(n - m) Z_m \omega_m \sum_k p_k^{-1}(m),
\]
where: \( p_k(m) = (Z_m \omega_m)^2 + \varepsilon_k^2 + \varphi_m^2 \). The pairing kernel of the electron–phonon interaction is given by:
\( K'(n - m) = \lambda (\nu^2 - n^2 + v^2) \), where \( \lambda \) is the electron–phonon coupling constant.

We used Kresin’s method of introducing the average phonon frequency: \( \langle \Omega \rangle \sim \omega_D \). In this case: \( \nu = \beta \omega_D / 2\pi \). The density of states per spin direction \( \rho(\varepsilon) \) is reproduced by:
\[
\rho(\varepsilon) = b_1 \ln |\varepsilon/b_2|,
\]
where \( b_1 = -0.04687 t^{-1} \) and \( b_2 = 21.17796 t \). Hence we obtain:
\[
\sum_k p_k^{-1}(m) \approx \int_{-W}^W d\varepsilon \rho(\varepsilon) p_k^{-1}(m),
\]
where \( W = 4t \) is half of the band width.

The dependence of the order parameter on \( \gamma \) is plotted in Fig. 5 (a). It can be seen that, in the case of the isotropic approximation, the superconducting state on the square lattice is induced even for high values of the unbalance parameter. The results concerning the effective electron mass are presented in Fig. 5 (b). The values of \( m^* \) in the isotropic approximation are not very high, as compared with the accurate results.

\[\text{References}\]

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