Multi-particle processes in $\kappa$-Poincaré inspired by 2+1D gravity

Jerzy Kowalski-Glikman$^1$ and Giacomo Rosati$^1$

$^1$Institute for Theoretical Physics, University of Wroclaw, Pl. Maksy Borna 9, PL-50-204 Wroclaw, Poland

Inspired by a Chern-Simons description of 2+1D gravity coupled to point particles we propose a new Lagrangian of a multiparticle system living in $\kappa$-Minkowski/$\kappa$-Poincaré spacetime. We derive the dynamics of interacting particles with $\kappa$-momentum space, alternative to the one proposed in the “principle of relative locality” literature. In this construction the locality of particle processes is naturally implemented, even for distant observers. In particular each particle process is characterized by a local deformed energy-momentum conservation law. On the other hand, the relation between non-causally-connected events still reflects the effects of deformed kinematics and relativity of locality.

I. INTRODUCTION

Gravity in 2+1 dimensions [1], [2] is a remarkable theory. It is described by a topological field theory and therefore it does not possess any dynamical degrees of freedom, so that the gravitational waves and Newtonian interactions of particles are not present. In spite of its apparent dullness, this theory contains rich physics when coupled to point particles or fields. The reason is the very simplicity of 2+1D gravity. In the case of the gravitating particle(s), one can find explicitly the form of the gravitational field and then substitute it back to the action to obtain the effective particle action that includes exactly the gravitational back-reaction [3], [4], [5]. In the case of the quantum theory of a field coupled to gravity, in the path-integral formalism one can exactly integrate out all the gravitational degrees of freedom, obtaining an effective field theory [6], [7]. In both cases the resulting, effective theories share a couple of important properties, their relativistic symmetries are deformed, spacetimes are non-commutative, and momentum spaces possess a nontrivial geometry. All these effects are characterized by a single mass scale $\kappa$, which is inverse proportional to the 2+1 dimensional Newton’s constant, and disappear when the gravitational coupling goes to zero. The above are the features of a class of theories belonging to the general relative locality framework [8] and therefore gravity in 2+1 dimensions coupled to particles and/or fields serves as a basic explicit example of this class of theories.

It has been shown [9, 10] that in a relativistic theory (in the sense of DSR [11–13]), the introduction of the invariant scale $\kappa$, with dimension of momentum, characterizing the momentum space geometry, makes it necessary to relax the notion of locality, which becomes a relative, observer-dependent notion. In this respect a particle process is described as local only in the coordinates of an observer sitting “at” the process. A distant observer, in her coordinatization, describes the process as “non-local”. Technically, these relative locality effects are encoded in the non-linearity of the relativistic transformation laws connecting the coordinates of inertial observers.

A general feature of theories with (DSR) deformed relativistic symmetries, like are relative locality theories, is that a deformed composition rule of particles momenta is needed to implement energy/momentum conservation in a particle process. For instance, for a process involving two incoming particles with momenta $p(1), p(2)$, and two outgoing particles with momenta $p(3), p(4)$, we can interpret the composite system of particles characterized by the total momentum $p^{(\text{in})\text{tot}} = p(1) \oplus p(2)$, as the in-state of a particle process, and the composite system of particles characterized by the total momentum $p^{(\text{out})\text{tot}} = p(3) \oplus p(4)$, as the outgoing state, where the $\oplus$ encodes the deformed summation rule. The conservation of energy/momentum of the process is then implemented by the constraint

$$ p^{(\text{in})\text{tot}} = p^{(\text{out})\text{tot}}. \quad (1) $$

In [8, 14] the deformed summation law is associated to the nontrivial geometry of momentum space. This deformed momentum summation law characterizing particle processes is implemented through a suitable boundary term, strictly related to the translational symmetries of the theory, which was a subject of the studies presented in [15, 16]. It was shown that the requirement for the theory to be relativistic restricts the class of possible boundary terms. One of the outcomes of these studies was that the relative locality emerges from the description of the spacetime position of the event of interaction.

jerzy.kowalski-glikman@ift.uni.wroc.pl

giacomo.rosati@ift.uni.wroc.pl
The basic property of the models investigated in the framework of relative locality (see [17] for a recent review) was that, apart from the vertex, the particles’ kinematics is governed by a free, albeit in general deformed, action. Such construction is based on the intuition stemming from the Feynman diagrams construction, where we have to do with free particles on the diagram lines interacting at a final number of vertices. However, the Feynman diagrams are aimed to describe local processes and it is not clear if they are really the best point of departure in the case nonlocal theories. In particular, it is well established in the context of the theory of particles coupled to 2+1 gravity that, after solving out the gravity degrees of freedom, the effective multi-particles action is not a sum of their free actions, and that one has to do with a nontrivial, nonlocal topological interactions between the particles [1].

In this paper we generalize this construction to the case of a two-particle system in 3+1 spacetime dimensions, and choose as illustrative example the case in which the momentum space of the particle is the $AN(3)$ group i.e., (a part of) de Sitter space. It is well known [14, 18] that in the free particle case this example corresponds to the $\kappa$-Poincaré deformation [19–21]. Here we will find that in the multiparticle case the symmetries of the system are described by the $\kappa$-Poincaré Hopf algebra. Our construction differs from the models considered previously [3, 13, 16, 22]. The system we obtain has the property that under the action of total momentum the two particle coordinates translate “rigidly”, by the same amount. Noting that our model implements translational invariance in a way compatible with the description of boundary terms achieved in [15, 16], we propose a new expression for the action of two-particle processes, which combines our braided two-particle Lagrangian and the expression for the boundary terms in a natural way.

The main novelty of our model lies in the fact that, differently from the previous formulations, the total momentum of the particles system is at any time equal to the deformed sum of their individual momenta and that the locality of particle processes is preserved under translations. We comment on the interpretation of this result for the case of multiple processes, postponing a detailed analysis to a forthcoming paper. In particular, if we adopt the characterization of translational invariance proposed in [16] and discussed later in [23], we find that relative locality effects are still present when comparing causally unconnected chains of processes, so that we expect that the physical predictions of our model may not differ from those reported in there. The detailed analysis of the physical predictions of our model will be presented in a forthcoming paper.

## II. 2-PARTICLE LAGRANGIAN

Using the Chern-Simons description of 2+1D gravity coupled to point particles [3, 22] one can derive the following expression for the kinetic term of an effective Lagrangian description of the particle dynamics for a system of two particles:

$$\mathcal{L}_{1+2}^{\text{kin}} = \left\langle \dot{P}_1 P_1^{-1}, x_1(1) \right\rangle + \left\langle \dot{P}_2 P_2^{-1}, x_2(2) \right\rangle + \left\langle P_1 \dot{P}_2 P_2^{-1} P_1^{-1} - \dot{P}_2 P_2^{-1}, x(1) \right\rangle. \quad (2)$$

In this formula $P(a) \in \mathfrak{g}$ is an appropriate group element corresponding to the (group valued) momentum of the $a$’th particle, which is equal to the holonomy of the (flat) connection, representing the gravitational field, around the particle’s worldline. The dual phase space coordinates $x(a) \in \mathfrak{g}^*$, belonging to the vector space dual to the Lie algebra of the group, have a natural interpretation of the particle positions, and the brackets $\left\langle \cdot \right\rangle$ stand for the gauge invariant inner product. Finally $\tau$ plays a role of the global external time. In the framework of 2+1 gravity the models of this kind have been specifically constructed for momentum group manifold being either $\mathfrak{g} = SO(2,1)$ group (Anti de Sitter momentum space $\mathfrak{g}$, $\mathfrak{g}$ or $\mathfrak{g} = AN(2)$ group (momentum space equal to the half of de Sitter space covered by flat cosmological coordinates) [25].

The expression (2) was, strictly speaking, derived from gravity only in 2+1 dimensions, but it is well defined in any number of spacetime dimensions, equal to the dimension of the group in question. We take this Lagrangian as a starting point for our investigations in 3+1 dimensions, taking as a group $\mathfrak{g}$ the group $AN(3)$ [14]. As it is well known [14] this group serves as the basic building block for the theories with $\kappa$-Poincaré symmetries [19–21].

We introduce momentum space coordinates $p_{a(\mu)} = P_{a(\mu)}(\tau)$ by writing the group valued momentum in the form

$$P(a) = e^{\mathcal{X}^\mu p_{a(\mu)}}, \quad (3)$$

where $\mathcal{X}^\mu$ satisfy the $an(3)$ ($\kappa$-Minkowski [21, 26]) algebra commutation relations

$$[\mathcal{X}^i, \mathcal{X}^0] = \frac{1}{\kappa} \mathcal{X}^i, \quad [\mathcal{X}^i, \mathcal{X}^k] = 0. \quad (4)$$

Since the group $AN(D)$ is defined for arbitrary $D$ and has the dimension $D + 1$, the results of our investigations can be readily applied to an arbitrary spacetime dimension.
Expanding the spacetime coordinates as \( x^{(a)}(\tau) = A_\mu x^\mu_{(a)}(\tau) \), where \( A_\mu \in an^*(3) \) are the generators of the dual algebra, the inner product is defined by the pairing \( \langle \mathcal{X}^\mu, A_\nu \rangle = \delta^\mu_\nu \).

Using (4) we get useful relations

\[
\left[ e^{\mathcal{X}^j p_{(a)j}}, \mathcal{X}^0 \right] = \frac{1}{\kappa} p_{(a)j} \mathcal{X}^j e^{\mathcal{X}^j p_{(a)j}}, \quad \left[ \mathcal{X}^j, e^{\mathcal{X}^0 p_{(a)0}} \right] = (e^{P_{(a)0}/\kappa} - 1) e^{\mathcal{X}^0 p_{(a)0}} \mathcal{X}^j, \tag{5}\]

from which it follows that \( \dot{f} = df/d\tau \)

\[
\dot{p}_{(a)} = \left( \mathcal{X}^\mu \dot{p}_{(a)\mu} + \frac{1}{\kappa} \mathcal{X}^j p_{(a)j} \dot{p}_{(a)0} \right) \mathcal{P}_{(a)}. \tag{6}\]

By using Eq. (5) and (4), and solving the inner product, the kinetic term becomes

\[
\mathcal{L}^{kin}_{1D} = \left( x^0_{(1)} + \frac{1}{\kappa} x^j_{(1)} p_{(1)j} \right) \dot{p}_{(1)0} + x^j_{(1)} \dot{p}_{(1)j} + \left( x^0_{(2)} + \frac{1}{\kappa} x^j_{(2)} p_{(2)j} - \frac{1}{\kappa} x^j_{(1)} \left( (1 - e^{-P_{(1)0}/\kappa}) (p_{(2)j} - p_{(1)j}) \right) \right) \dot{p}_{(2)0} + \left( x^j_{(2)} - x^j_{(1)} (1 - e^{-P_{(1)0}/\kappa}) \right) \dot{p}_{(2)j}. \tag{7}\]

To complete the construction of the Lagrangian for the system of two particles we have to impose the mass-shell constraints on each particle. The mass-shell relation is given by \( \kappa \)-Poincaré mass Casimir

\[
\mathcal{C}_{(a)} = 4 \kappa^2 \sinh \left( \frac{p_{(a)0}}{2 \kappa} \right)^2 - e^{P_{(a)0}/\kappa} P_{(a)}^2. \tag{8}\]

Thus the two-particle Lagrangian is

\[
\mathcal{L}_{1D} = \mathcal{L}^{kin}_{1D} + \lambda_{(1)} \left( \mathcal{C}_{(1)} - m_{(1)}^2 \right) + \lambda_{(2)} \left( \mathcal{C}_{(2)} - m_{(2)}^2 \right). \tag{9}\]

The equations of motion following from the variation of positions \( x^\mu_{(a)} \) are the usual momentum conservation conditions

\[
\dot{p}_{(1)\mu} = \dot{p}_{(2)\mu} = 0. \tag{10}\]

The equations resulting from the variations of momenta \( p_{(a)\mu} \) are then

\[
\dot{x}^0_{(1)} = \lambda_{(1)} \left( \frac{\partial \mathcal{C}_{(1)}}{\partial p_{(1)0}} - \frac{1}{\kappa} p_{(1)}, \frac{\partial \mathcal{C}_{(1)}}{\partial p_{(1)j}} \right), \quad \dot{x}^j_{(1)} = \lambda_{(1)} \frac{\partial \mathcal{C}_{(1)}}{\partial p_{(1)j}}, \tag{11}\]

\[
\dot{x}^0_{(2)} = \lambda_{(2)} \left( \frac{\partial \mathcal{C}_{(2)}}{\partial p_{(2)0}} - \frac{1}{\kappa} p_{(2)}, \frac{\partial \mathcal{C}_{(2)}}{\partial p_{(2)j}} \right) - \lambda_{(1)} \frac{1}{\kappa} p_{(1)} \frac{\partial \mathcal{C}_{(1)}}{\partial p_{(1)j}} + \lambda_{(1)} \left( 1 - e^{-P_{(1)0}/\kappa} \right) \frac{\partial \mathcal{C}_{(1)}}{\partial p_{(1)j}}, \quad \dot{x}^j_{(2)} = \lambda_{(2)} \frac{\partial \mathcal{C}_{(2)}}{\partial p_{(2)j}} + \lambda_{(1)} \left( 1 - e^{-P_{(1)0}/\kappa} \right) \frac{\partial \mathcal{C}_{(1)}}{\partial p_{(1)j}},
\]

which take the explicit form

\[
\dot{x}^0_{(1)} = 2 \lambda_{(1)} \left( \kappa \sinh \left( \frac{P_{(1)0}}{\kappa} \right) + \frac{1}{2 \kappa} e^{P_{(1)0}/\kappa} P_{(1)}^2 \right), \quad \dot{x}^j_{(1)} = -2 \lambda_{(1)} e^{P_{(1)0}/\kappa} p_{(1)j}, \tag{12}\]

\[
\dot{x}^0_{(2)} = 2 \lambda_{(2)} \left( \kappa \sinh \left( \frac{P_{(2)0}}{\kappa} \right) + \frac{1}{2 \kappa} e^{P_{(2)0}/\kappa} P_{(2)}^2 \right) + 2 \lambda_{(1)} \frac{1}{\kappa} e^{P_{(1)0}/\kappa} P_{(1)j}, \quad \dot{x}^j_{(2)} = -2 \lambda_{(2)} e^{P_{(2)0}/\kappa} p_{(2)j} - \lambda_{(1)} \left( e^{P_{(1)0}/\kappa} - 1 \right) p_{(1)j}.
\]

These equations express the (deformed) relations between velocities and momenta.

---

2 Notice that this can be associated with the geodesic length \( \mu \) in deSitter-momentum-space with the cosmological constant \( \kappa^2 \) by setting \( m_{(a)}^2 = 2 \kappa^2 \left( \cosh \left( \mu_{(a)}/\kappa \right) - 1 \right) \).
We want the coordinate variation to be \( \delta \) to \( \xi \). The Poisson brackets then are \( \frac{\partial L}{\partial \dot{\pi}} \) to recover the standard translation in the limit \( \frac{1}{\kappa} \rightarrow 0 \). To generate “rigid translations” for the 2-particle system, i.e. to be such that the coordinates of both particles translate simultaneously, one can verify that Eqs. (10) and (12) can be rewritten as

\[
\begin{align*}
\{ \pi_0, \dot{x}_j \} &= 1, \\
\{ \pi_0, \dot{x}_j \} &= 0, \\
\{ \pi_j, \dot{x}_0 \} &= -\frac{1}{\kappa} \pi_j, \\
\{ \pi_j, \dot{x}_k \} &= \delta_{jk}, \\
\{ \pi_0, \dot{x}_j \} &= 0, \\
\{ \pi_0, x_j \} &= -\frac{1}{\kappa} \pi_j, \\
\{ \pi_j, x_0 \} &= \delta_{jk}, \\
\{ x_0, x_k \} &= 0.
\end{align*}
\]

The Poisson brackets then are

\[
\begin{align*}
\{ \pi_0, \dot{x}_j \} &= 1, \\
\{ \pi_0, \dot{x}_j \} &= 0, \\
\{ \pi_j, \dot{x}_0 \} &= -\frac{1}{\kappa} \pi_j, \\
\{ \pi_j, x_k \} &= \delta_{jk}, \\
\{ \pi_0, x_j \} &= 0, \\
\{ x_0, x_k \} &= 0.
\end{align*}
\]

Notice that as a consequence of the Lagrangian (7), the two-particle phase space gets mixed.

By Eq. (9), the Hamiltonian is

\[
\mathcal{H} = \mathcal{L} - \sum \dot{x}_i \pi_i = \sum \lambda_i (\mathcal{L}_i - m_i^2).
\]

This Hamiltonian generates the evolution in terms of time \( \tau \). Indeed, using the Poisson brackets defined in this section one can verify that Eqs. (10) and (12) can be rewritten as

\[
\begin{align*}
\dot{p}_0 &= \{ \mathcal{H}, p_0 \} = 0, \\
\dot{x}_i &= \{ \mathcal{H}, x_i \}.
\end{align*}
\]

IV. RIGID TRANSLATIONS, THE TOTAL MOMENTUM OF THE SYSTEM

The fact that the momenta are conserved, Eq. (10), implies that also every (even non-linear) combination of momenta is conserved. Then, the requirement for the total momentum to be conserved does not single out a unique generator for translations. In principle, the conserved charges, and hence the generators of symmetries, associated with translations, can be chosen arbitrarily. There must be therefore some other property, besides conservation, constraining the form of the translations generators. We will show now that the requirement for the total momentum to be conserved does not single out a unique generator for translations by the same amount, suffices to single out an expression for the total momentum.

To see how this argument works let us revisit the Lagrangian (7) and express it in terms of the relative position of the particles \( x_\mu^- \) and the average position \( x_\mu^+ \). The average position \( x_\mu^+ \) is \( x_\mu^+ = \frac{1}{2} x_\mu^+ \). It follows from our discussion above that under rigid translations, with translation parameter \( \xi = (\xi^0, \xi^j) \), such that \( x_\mu^+(\xi) = x_\mu^+(0) + \xi \), the relative position does not change: \( \delta \xi x_\mu^- = 0 \). Therefore the variation of the Lagrangian under infinitesimal translations will be proportional to \( \delta \xi x_\mu^+ \). Notice first that, after rearranging the coordinates, the kinetic term varies as

\[
\delta \mathcal{L}_{\xi}^{kin} = \delta \xi x_\mu^+ \left( \dot{\pi}_1 + \dot{\pi}_2 \right)
\]

We want the coordinate variation to be \( \delta \xi x_\mu^+ = -\xi + O(1/\kappa) \), with coordinate-independent parameters \( \xi^\mu \), in order to recover the standard translation in the limit \( \frac{1}{\kappa} \rightarrow 0 \). Let us first consider terms in the variation proportional to \( \xi^0 \). It is straightforward from (10), considering that the term proportional to \( \delta \xi x_\mu^+ \) is a total derivative, that imposing

\[
\delta \xi^0 x_\mu^+ = -\xi^0, \quad \delta \xi^0 x_\mu^+ = 0,
\]

imposing
it follows
\[ \delta \xi \mathcal{L}_{1 \oplus 2}^{\text{kin}} = -\xi^0 \frac{d}{d\tau} (p_{(1)0} + p_{(2)0}), \]
so that the zeroth component of the conserved total momentum is
\[ p_0^{\text{tot}} = p_{(1)0} + p_{(2)0}. \]

It is far less trivial to find the spatial component on the conserved total momentum. From (19) one can notice that the terms proportional to \( \delta x^j_{(+)} \) do not to add up to a total derivative. However one can verify that by setting the variation parametrized by \( \xi \) to be
\[ \delta \xi x^0_{(+)} = \frac{1}{\kappa} \xi^j \left( p_{(1)j} + e^{-p_{(1)0}/\kappa} p_{(2)j} \right), \quad \delta \xi x^j_{(+)} = -\xi^j, \]
we get
\[ \delta \xi \mathcal{L}_{1 \oplus 2}^{\text{kin}} = -\xi^j \frac{d}{d\tau} \left( p_{(1)j} + e^{-p_{(1)0}/\kappa} p_{(2)j} \right), \]
so that the spatial component of the total momentum is
\[ p_j^{\text{tot}} = p_{(1)j} + e^{-p_{(1)0}/\kappa} p_{(2)j}. \]

One can check by (14)-(15), that the total momentum \( p_{\mu}^{\text{tot}} \), generates the translations (20) by Poisson brackets, its action on the single particle coordinates being
\[ \delta \xi x_{(a)}^{\mu} = -\left\{ \xi^\nu p_{\nu}^{\text{tot}}, x_{(a)}^{\mu} \right\}, \]
\[ \delta \xi x^0_{(1)} = \delta \xi x^0_{(2)} = -\xi^0 + \frac{1}{\kappa} \xi \cdot \mathbf{p}^{\text{tot}}, \quad \delta \xi x^j_{(1)} = \delta \xi x^j_{(2)} = -\xi^j. \]

Both particles are translated by the same amount, consistently with our assumptions of rigid translations. Notice now that the total momentum generating rigid translations can be re-expressed as a deformed summation law for the single particle momenta
\[ p_{\mu}^{\text{tot}} = \left( p_{(1)} + p_{(2)} \right)_{\mu}, \]
which in turn, by means of (3), can be associated with the product of the two particle group valued momenta \( \mathcal{P} \) as
\[ \mathcal{P}^{\text{tot}} \left( p_{(1)} + p_{(2)} \right) = \mathcal{P}_{(1)} \left( p_{(1)} \right) \mathcal{P}_{(2)} \left( p_{(2)} \right). \]

V. NEW ACTION PROPOSAL FOR TWO-PARTICLE PROCESSES

We can draw some considerations from the results of the previous section.

- The rigid translations are generated by the total momentum obtained respectively by the group elements \( \mathcal{P}^{\text{tot}} = \mathcal{P}_{(1)} \mathcal{P}_{(2)} \). This reflects the structure of the kinetic terms (2), where the momenta appear in the same combination. Notice indeed that the kinetic term (2) can be re-expressed as
  \[ \left\langle \left[ \frac{d}{d\tau} \left( \mathcal{P}_{(1)} \mathcal{P}_{(2)} \right) \right] \left( \mathcal{P}_{(1)} \mathcal{P}_{(2)} \right)^{-1} \left( \frac{d}{d\tau} \mathcal{P}_{(2)} \right) \mathcal{P}_{(2)}^{-1}, x_{(2)} \right\rangle + \left\langle \left( \frac{d}{d\tau} \mathcal{P}_{(2)} \right) \mathcal{P}_{(2)}^{-1}, \mathcal{P}_{(2)} \right\rangle. \]

- The change in the coordinates due to rigid translations can be rewritten in terms of the total momentum as (20), i.e. the particle coordinates translate rigidly in function of the total momentum.

Consider now a process involving two incoming particles (1), (2) and two outgoing particles (3), (4). Taking into account of the above considerations and of the results reported in [14] (see also [15]), and later in [23], where the analysis of translational invariant formulations of relative locality frameworks is developed, and considering also that the two-particle system evolution is parametrized by the external time \( \tau \), we propose the following action:
\[ S = \int_{-\infty}^{\tau} d\tau \left[ \mathcal{L}_{1 \oplus 2} + \frac{d}{d\tau} \left( \zeta^\mu \left( p_{(1)} + p_{(2)} \right)_{\mu} \right) \right] + \int_{\tau}^{\infty} d\tau \left[ \mathcal{L}_{3 \oplus 4} + \frac{d}{d\tau} \left( \zeta^\mu \left( p_{(3)} + p_{(4)} \right)_{\mu} \right) \right]. \]
Here the process is taken to happen at the time $\tau = \bar{\tau}$. The interaction is described by the two boundary terms, where $\zeta^\mu$ is a function of $\tau$, $\zeta^\mu = \zeta^\mu(\tau)$. Notice that for a single process, the interaction term is analogous to the one reported\(^3\) in \cite{16} or in \cite{15}. Indeed varying the action with respect to $\zeta^\mu$, we get\(^4\)

$$
\delta \zeta^\mu (\bar{\tau}) \left( (p(1) \oplus p(2))_\mu - (p(3) \oplus p(4))_\mu \right) = 0
$$

i.e. we recover the constraint equation

$$
p^{\text{tot}}_{\text{(in)}\mu} = (p(1) \oplus p(2))_\mu = (p(3) \oplus p(4))_\mu = p^{\text{tot}}_{\text{(out)}\mu}.
$$

At the same time, varying the action with respect to the particles momenta $p(a)_\mu$, the interaction terms produce the boundary conditions

$$
x^\mu_a (\bar{\tau}) = -\zeta^\nu (\bar{\tau}) \left\{ p^{\text{tot}}_{\text{(in)}\nu}, x^\mu_a \right\} (\bar{\tau}), \quad a = 1, 2
$$

$$
x^\mu_a (\bar{\tau}) = \zeta^\nu (\bar{\tau}) \left\{ p^{\text{tot}}_{\text{(out)}\nu}, x^\mu_a \right\} (\bar{\tau}), \quad a = 3, 4.
$$

where we have taken into account of the Poisson brackets defined in Sec. \cite{11} By calling $\delta \zeta^\mu = \xi^\mu$ the change of $\zeta^\mu$ due to a translation, it follows that the particle coordinates are translated by the action of the total momentum associated to their composed system, as described in Sec. \cite{11}

The action proposed in this section can be generalized to an arbitrary number of particles and interaction vertices. The discussion of such a generalization will be presented in a forthcoming paper. We just make a remark for the case of multiple processes. If one adopts a prescription for the boundary terms analogous to that of \cite{16} (later developed in \cite{23}), then the $\zeta^\mu$ of all the processes are translated by the same amount $\xi^\mu$. While, for each time $\tau$, the translation of the particle coordinates is generated by the non-linear sum of the momenta of all the particles involved in the set of causally connected interactions, present at the time $\tau$ (cf. \cite{16}). In this case our action is constructed by dividing the system in slices bounded by the times $\tau_I$ of every vertex $I$, and taking, for each slice, a Lagrangian combining all the particle coordinates and the interaction term.

However a major difference with the previous formulations of relative locality actions is that in our formulation the change in coordinates due to translations, which are generated by the total momentum of the system, is rigid. The position of the endpoints of the particle worldlines participating to the same vertex coincide for all the translated observers: the locality of a particle process is preserved by translations. Suppose that an observer describes a particle process locally to her origin, so that all the worldlines involved in the process end at the interaction point (i.e. at her origin). Then in our formulation, a distant observer (at relative rest), due to the rigidness of translations, will still describe the worldline endpoints to coincide.

We could conclude that in our formulation there is no relative locality for translations. However one has to keep in mind that, adopting the vertex prescription analogous to the one in \cite{16} and \cite{22}, this property holds only for causally connected processes. Indeed one can show\(^5\) that in comparing causally unconnected chains of processes translational relative-locality effects still arise.

### VI. BOOSTS AND ROTATIONS

We have discussed in the previous sections the details of translational symmetries. We now complete the analysis of spacetime symmetries by considering boosts and rotations.

---

\(^3\) The two formulations differ only when one considers more than one process. In that case in \cite{16} the translational invariance is implemented in such a way that the change of the interaction coordinate $\zeta^\mu$ under translation is the same for every process, $\delta \zeta^\mu = \xi^\mu$. In \cite{15} $\delta \zeta^\mu$ changes differently for every process, the changes being functions of the momenta of all the considered processes. The implications of this are discussed in \cite{23}.

\(^4\) We assume $\delta \zeta^\mu (-\infty) = \delta \zeta^\mu (\infty) = 0$.

\(^5\) We will discuss this point in a forthcoming paper. We will show how this implies also that the predictions for the physical effects, in our new formulation, coincide with the ones present in the literature, obtained within the formulation \cite{16}.
A. Algebra of symmetries for the two-particle system

One finds that for the two-particle system defined in the previous sections the (single-particle) boost and rotation charges are respectively

\[ N_{(1)j} = -p_{(1)j} x_j^0 + \left( \frac{\kappa}{2} \left( 1 - e^{-2p_{(1)0}/\kappa} \right) + \frac{1}{2\kappa} p_{(1)}^2 \right) x_j^1, \]
\[ N_{(2)j} = -p_{(2)j} x_j^0 + \left( \frac{\kappa}{2} \left( 1 - e^{-2p_{(2)0}/\kappa} \right) + \frac{1}{2\kappa} p_{(2)}^2 \right) x_j^1 - \frac{1}{\kappa} p_{(2)j} p_{(1)} \cdot x_1, \]

(33)

One can verify that they are conserved,

\[ \dot{N}_{(a)j} = \{ \mathcal{H}, N_{(a)j} \} = 0, \quad \dot{R}_{(a)j} = \{ \mathcal{H}, R_{(a)j} \} = 0. \]

(35)

Moreover they satisfy the \( \kappa \)-Poincaré algebra [21]:

\[
\begin{align*}
\{ N_{(a)j}, p_{(a)0} \} &= p_{(a)j}, \\
\{ N_{(a)j}, p_{(a)k} \} &= \delta_{jk} \left( \frac{\kappa}{2} \left( 1 - e^{-2p_{(a)0}/\kappa} \right) + \frac{1}{2\kappa} p_{(a)}^2 \right) - \frac{1}{\kappa} p_{(a)j} p_{(a)k}, \\
\{ R_{(a)j}, p_{(a)0} \} &= 0, \\
\{ R_{(a)j}, p_{(a)k} \} &= \epsilon_{jkl} p_{(a)l}, \\
\{ R_{(a)j}, R_{(a)k} \} &= \epsilon_{jkl} R_{(a)l}, \\
\{ N_{(a)j}, N_{(a)k} \} &= \epsilon_{jkl} N_{(a)l}, \\
\{ N_{(a)j}, R_{(a)k} \} &= \epsilon_{jkl} N_{(a)l} - \epsilon_{jkl} R_{(a)l}, \\
\{ R_{(a)j}, N_{(a)k} \} &= \epsilon_{jkl} R_{(a)l}.
\end{align*}
\]

(36)

To derive the expression for the total boost and rotation generators, we notice that the two particles system is seen by an observer with not sufficient "resolution power" as a single system carrying the momentum \( p_{(\text{tot})} \). It follows that the total momentum should transform, with respect to the symmetries generated by total boost \( N_{(\text{tot})j} \) and rotation \( R_{(\text{tot})j} \), in exactly the same way as the momenta \( p_{(a)\mu} \) transform under the symmetries generated by the single particle boost \( N_{(a)j} \) and rotation \( R_{(a)j} \), Eq. (30). This property, in turn, ensures the covariance of the energy-momentum conservation law [31] under the action of total boost and rotation (cf. [27]). Then, by direct calculation we check that the total boost and rotation generators are

\[ N_{(\text{tot})j} = N_{(1)j} + e^{-p_{(1)0}/\kappa} N_{(2)j} + \frac{1}{\kappa} \epsilon_{jkl} p_{(1)k} R_{(2)l}, \quad R_{(\text{tot})j} = R_{(1)j} + R_{(2)j}, \]

(37)

which is exactly the expression that follows from the \( \kappa \)-Poincaré coproducts [21]. We see therefore that form of the generators of total momentum, boost and rotation deduced in our classical model on the basic physical premises, reproduce the coproduct structure of the \( \kappa \)-Poincaré Hopf algebra.

Calling \( G_{(a)\mu} \) the generic element of the set of single-particle charge/generators (momentum, boosts, and rotations), and \( G_{(\text{tot})\mu} \) their two-particle composite version, one can verify with the help of relations (36), that the following property is satisfied:

\[ \{ G_{(\text{tot})\mu}, G_{(\text{tot})\nu} \} = \{ G_{(a)\mu}, G_{(a)\nu} \} \bigg|_{G_{(a)} \to G_{(\text{tot})}}, \]

(38)

i.e. under the the composite symmetry generators satisfy the same algebra [30] of the single-particle ones.

B. Action of boosts and rotations on a 2-particle system

Under the action of a composite boost and rotation, the spacetime coordinates change respectively as

\[ q_{(a)\mu}' = q_{(a)\mu} + \delta\chi q_{(a)\mu} = q_{(a)\mu} + \lambda \cdot \left( N_{(\text{tot})\mu} q_{(a)\mu} \right), \quad q_{(a)\mu}' = q_{(a)\mu} + \delta\theta q_{(a)\mu} = q_{(a)\mu} - \theta \cdot \left( R_{(\text{tot})\mu} q_{(a)\mu} \right), \]

(39)
\[ \lambda^j \text{ and } \theta^j \text{ being the boost and rotation parameters. The system of particle transforms under the action of the composite boost and rotation (37), as} \]

\[
\left\{ N_j^{\text{tot}}, x^0_1 \right\} = -x^j_1 - \frac{1}{\kappa} N_j^0, \quad \left\{ N_j^{\text{tot}}, x^k_1 \right\} = -\delta_{jk} x^0_1 + \frac{1}{\kappa} \epsilon_{jkl} R^l_0, \\
\left\{ N_j^{\text{tot}}, x^0_2 \right\} = -e^{-P(1)\alpha/\kappa} x^j_2 - \left( 1 - e^{-P(1)\alpha/\kappa} \right) x^j_1 - \frac{1}{\kappa} N_j^0, \quad \left\{ N_j^{\text{tot}}, x^k_2 \right\} = -\delta_{jk} \left( e^{-P(1)\alpha/\kappa} x^0_2 + \left( 1 - e^{-P(1)\alpha/\kappa} \right) x^0_1 - \frac{1}{\kappa} P(1) \cdot (x_2 - x_1) \right) + \frac{1}{\kappa} \epsilon_{jkl} R^l_0 - \frac{1}{\kappa} P(1) \cdot \left( x^k_2 - x^k_1 \right), \\
\left\{ R_j^{\text{tot}}, x^0_1 \right\} = \left\{ R_j^{\text{tot}}, x^0_2 \right\} = 0, \quad \left\{ R_j^{\text{tot}}, x^k_1 \right\} = \epsilon_{jkl} x^l_1, \quad \left\{ R_j^{\text{tot}}, x^k_2 \right\} = \epsilon_{jkl} x^l_2. \tag{40}
\]

One can verify that the action of boosts and rotations on the composite system satisfies the property

\[
\left. \left\{ N_j^{\text{tot}}, x^\mu_1 \right\} = \left\{ N_j^{\text{tot}}, x^\mu_2 \right\} \right|_{x_1=x_2} = 0, \quad \left. \left\{ R_j^{\text{tot}}, x^\mu_1 \right\} = \left\{ R_j^{\text{tot}}, x^\mu_2 \right\} \right|_{x_1=x_2} = 0. \tag{42}
\]

This property implies that, as for translations, the locality of a distant process is preserved also by boosts and rotations.

\section*{VII. SUMMARY AND DISCUSSION}

Starting from a construction of a multi-particle Lagrangian inspired by 2+1D gravity coupled to point particles, we have proposed a new expression for the action of a two-particle process with \( \kappa \)-Minkowski/\( \kappa \)-Poincaré deformed relativistic symmetries, and associated deformed composition-law of momenta. The latter is encoded in the product of group valued momenta, which in turn reflects the properties of the non-commutative spacetime symmetries, as required for a DSR description of a local particle process \cite{11}.

The peculiarity of our action is that the two-particle coordinates are “braided” together in the kinetic term, as a result of the prescription coming from the 2+1D gravity analysis. Moreover we propose a re-writing of the boundary term representing the vertex interaction which takes into account of the properties of the momentum conservation law in translational invariant formulations of “principle of relative locality” theories (cf. \cite{15, 16, 22}). In this way the action we obtain contains a single integral for each composite particle system, with its own boundary term.

The main novelty of our action, respect to previous formulations \cite{8, 15, 16, 22}, is that the locality of a particle process is preserved by the whole set of (deformed) relativistic transformations. One could conclude that relativity of locality is not present in our model. However, while we postpone a detailed study of multiple processes in our formulation to a forthcoming paper, we anticipate that when one compares causally unconnected chains of processes, relative locality effects are still present. It follows that the observable predictions of our model reproduce the results considered in the literature (cf. \cite{10, 22}).

\section*{ACKNOWLEDGMENT}

This work was supported by funds provided by the National Science Center under the agreement DEC-2011/02/A/ST2/00294.

\begin{thebibliography}{99}

\bibitem{1} E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” Nucl. Phys. B 311 (1988) 46.
\bibitem{2} A. Achucarro and P. K. Townsend, “A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories,” Phys. Lett. B 180 (1986) 89.
\bibitem{3} P. de Sousa Gerbert, “On spin and (quantum) gravity in (2+1)-dimensions,” Nucl. Phys. B 346 (1990) 440.
\end{thebibliography}
[4] H.-J. Matschull and M. Welling, “Quantum mechanics of a point particle in (2+1)-dimensional gravity,” Class. Quant. Grav. 15, 2981 (1998) [gr-qc/9708054].
[5] C. Meusburger and B. J. Schroers, “Poisson structure and symmetry in the Chern-Simons formulation of (2+1)-dimensional gravity,” Class. Quant. Grav. 20 (2003) 2193 [arXiv:gr-qc/0301108].
[6] L. Freidel and E. R. Livine, “Ponzano-Regge model revisited III: Feynman diagrams and effective field theory,” Class. Quant. Grav. 23 (2006) 2021 [hep-th/0502106].
[7] L. Freidel and E. R. Livine, “Effective 3-D quantum gravity and non-commutative quantum field theory,” Phys. Rev. Lett. 96 (2006) 221301 [hep-th/0512113].
[8] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman and L. Smolin, “The principle of relative locality,” Phys. Rev. D 84 (2011) 084010 [arXiv:1101.0931] [hep-th].
[9] G. Amelino-Camelia, M. Matassa, F. Mercati and G. Rosati, “Taming Nonlocality in Theories with Planck-Scale Deformed Lorentz Symmetry,” Phys. Rev. Lett. 106 (2011) 071301 [arXiv:1006.2126] [gr-qc].
[10] G. Amelino-Camelia, N. Loret and G. Rosati, “Speed of particles and a relativity of locality in $\kappa$-Minkowski quantum spacetime,” Phys. Lett. B 700 (2011) 150 [arXiv:1102.4637] [hep-th].
[11] G. Amelino-Camelia, “Relativity in space-times with short distance structure governed by an observer independent (Planckian) length scale,” Int. J. Mod. Phys. D 11 (2002) 35 [gr-qc/0012051].
[12] G. Amelino-Camelia, “Testable scenario for relativity with minimum length,” Phys. Lett. B 510 (2001) 255 [hep-th/0012238].
[13] J. Kowalski-Glikman, “Observer independent quantum of mass,” Phys. Lett. A 286 (2001) 391 [hep-th/0102098].
[14] J. Kowalski-Glikman and S. Nowak, “Quantum kappa-Poincare algebra from de Sitter space of momenta,” [hep-th/0411154].
[15] J. M. Carmona, J. L. Cortes, D. Mazon and F. Mercati, “About Locality and the Relativity Principle Beyond Special Relativity,” Phys. Rev. D 84 (2011) 085010 [arXiv:1107.0939] [hep-th].
[16] G. Amelino-Camelia, M. Arzano, J. Kowalski-Glikman, G. Rosati and G. Trevisan, “Relative-locality distant observers and the phenomenology of momentum-space geometry,” Class. Quant. Grav. 29 (2012) 075007 [arXiv:1107.1724] [hep-th].
[17] J. Kowalski-Glikman, “Living in Curved Momentum Space,” Int. J. Mod. Phys. A 28 (2013) 1330014 [arXiv:1303.0196] [hep-th].
[18] J. Kowalski-Glikman and S. Nowak, “Doubly special relativity and de Sitter space,” Class. Quant. Grav. 20 (2003) 4799 [hep-th/0304101].
[19] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, “Q deformation of Poincare algebra,” Phys. Lett. B 264 (1991) 331.
[20] J. Lukierski, A. Nowicki and H. Ruegg, “New quantum Poincare algebra and k deformed field theory,” Phys. Lett. B 293 (1992) 344.
[21] S. Majid and H. Ruegg, “Bicrossproduct structure of kappa Poincare group and noncommutative geometry,” Phys. Lett. B 334 (1994) 348 [hep-th/9405107].
[22] G. Gubitosi and F. Mercati, “Relative Locality in $\kappa$-Poincaré,” Class. Quant. Grav. 30 (2013) 145002 [arXiv:1106.5710] [gr-qc].
[23] G. Amelino-Camelia, S. Bianco, F. Brighenti and R. J. Buonocore, “Causality and momentum conservation from relative locality,” [arXiv:1401.7160] [gr-qc].
[24] C. Meusburger and B. J. Schroers, “Phase space structure of Chern-Simons theory with a non-standard puncture,” Nucl. Phys. B 738 (2006) 425 [hep-th/0505143].
[25] J. Kowalski-Glikman and T. Trzesniewski, “Deformed Carroll particle from 2+1 gravity,” Phys. Lett. B 737 (2014) 267 [arXiv:1408.0154].
[26] J. Lukierski, H. Ruegg and W. J. Zakrzewski, “Classical quantum mechanics of free kappa relativistic systems,” Annals Phys. 243 (1995) 90 [hep-th/9312153].
[27] G. Amelino-Camelia, “On the fate of Lorentz symmetry in relative-locality momentum spaces,” Phys. Rev. D 85 (2012) 084034 [arXiv:1110.5081] [hep-th].