LOWER BOUNDS FOR THE DEPTH OF SECOND POWER OF
EDGE IDEALS

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ABSTRACT. Assume that $G$ is a graph with edge ideal $I(G)$. We provide sharp
lower bounds for the depth of $I(G)^2$ in terms of the star packing number of $G$.

1. Introduction

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in $n$
variables over $\mathbb{K}$. Computing and finding bounds for the depth (or equivalently, projective
dimension) of homogenous ideals of $S$ and their powers have been studied in several papers (see
e.g., [3, 4, 5, 8, 9, 11, 12, 13, 15, 17, 18, 19, and 21]).

In [6], Fouli, Hà and Morey introduced the notion of initially regular sequence and
used it to provide a method for estimating the depth of a homogenous ideal. Using
this method, in [7], the same authors determined a combinatorial lower bound for the
depth of edge ideals of graphs. Indeed, they proved that for every graph $G$ with edge
ideal $I(G)$, we have

$$\text{depth}_S I(G) \geq \alpha_2(G) + 1,$$

where $\alpha_2(G)$ denotes the so-called star packing number of $G$ (see Section 2 for the
definition of star packing number). Let $I(G)^{(2)}$ denote the second symbolic power of
$I(G)$. In [20], we showed that for any graph $G$,

$$\text{depth}_S I(G)^{(2)} \geq \alpha_2(G).$$

It is natural to ask whether the same inequality is true if one replaces the symbolic
power by ordinary. The answer is negative as we will see in Examples 3.7. However,
we prove in Theorem 3.6 that for any graph $G$, we have

$$\text{depth}_S I(G)^2 \geq \alpha_2(G) - 2.$$  

Moreover, if $G$ is a $W(K_3)$-free graph (i.e., has no induced subgraph isomorphic to a
whiskered triangle), then

$$\text{depth}_S I(G)^2 \geq \alpha_2(G) - 1.$$  

Also, for any triangle-free graph $G$, we have

$$\text{depth}_S I(G)^2 \geq \alpha_2(G).$$

Furthermore, we provide examples showing that the above inequalities are sharp.

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2. Preliminaries and known results

In this section, we provide the definitions and the known results which will be used in the next section.

Let $G$ be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. So, we identify the vertices of $G$ with the variables of $S$. Also, by abusing the notation, every edge of $G$ will be written by the product of its vertices. For a vertex $x_i$, the neighbor set of $x_i$ is $N_G(x_i) = \{x_j \mid x_ix_j \in E(G)\}$. We set $N_G[x_i] := N_G(x_i) \cup \{x_i\}$. The cardinality of $N_G(x_i)$ is the degree of $x_i$. A vertex of degree one is called a leaf of $G$ and the unique edge incident to a leaf is a pendant edge. For every subset $U \subset V(G)$, the graph $G \setminus U$ has vertex set $V(G \setminus U) = V(G) \setminus U$ and edge set $E(G \setminus U) = \{e \in E(G) \mid e \cap U = \emptyset\}$. A subgraph $K$ of $G$ is called induced provided that two vertices of $K$ are adjacent if and only if they are adjacent in $G$. For any graph $H$, We say that $G$ is a $H$-free graph if it has no induced subgraph isomorphic to $H$. We denote a triangle by $K_3$. A whiskered triangle, denoted by $W(K_3)$ is the graph obtained from $K_3$ by attaching a pendant edge to each of its vertices.

The edge ideal of a graph $G$ is the monomial ideal generated by quadratic squarefree monomials corresponding to the edges of $G$. In other words,

$$I(G) = \langle x_ix_j \mid x_ix_j \in E(G) \rangle \subset S.$$

Let $G$ be a graph and $x$ be a vertex of $G$. The subgraph $St(x)$ of $G$ with vertex set $N_G[x]$ and edge set $\{xy \mid y \in N_G(x)\}$ is called a star with center $x$. A star packing of $G$ is a family $\mathcal{X}$ of stars in $G$ which are pairwise disjoint, i.e., $V(St(x)) \cap V(St(x')) = \emptyset$, for $St(x), St(x') \in \mathcal{X}$ with $x \neq x'$. The quantity

$$\max \{ |\mathcal{X}| \mid \mathcal{X} \text{ is a star packing of } G \}$$

is called the star packing number of $G$. Following [7], we denote the star packing number of $G$ by $\alpha_2(G)$.

As it was mentioned in introduction, Fouli, Hà and Morey [8] determined a lower bound for the depth of $I(G)$ in terms of the star packing number of $G$.

**Proposition 2.1.** Let $G$ be graph with edge ideal $I(G)$. Then

$$\text{depth}_S I(G) \geq \alpha_2(G) + 1.$$

We close this section by recalling the concept of polarization.

For every monomial ideal $I$, we denote the set of minimal monomial generators of $I$ by $G(I)$. Let $I$ be a monomial ideal of $S$ with $G(I) = u_1, \ldots, u_m$, where $u_j = \prod_{i=1}^n x_i^{a_{ij}}$, $1 \leq j \leq m$. For every $i$ with $1 \leq i \leq n$, let

$$a_i := \max\{a_{i,j} \mid 1 \leq j \leq m\},$$

and suppose that

$$T = \mathbb{K}[x_{1,1}, x_{1,2}, \ldots, x_{1,a_1}, x_{2,1}, x_{2,2}, \ldots, x_{2,a_2}, \ldots, x_{n,1}, x_{n,2}, \ldots, x_{n,a_n}]$$

is a polynomial ring over the field $\mathbb{K}$. Let $I^{\text{pol}}$ be the squarefree monomial ideal of $T$ with minimal generators $u_1^{\text{pol}}, \ldots, u_m^{\text{pol}}$, where $u_j^{\text{pol}} = \prod_{i=1}^n x_{i,k}^{a_{i,j}}$, $1 \leq j \leq m$. 
The monomial $u_j^{\text{pol}}$ is called the polarization of $u_j$, and the ideal $I^{\text{pol}}$ is called the polarization of $I$.

3. Main Results

In this section, we prove the main result of this paper, Theorem 3.6 which provides lower bounds for the depth of second power of an edge ideal $I(G)$ in terms of the star packing number of $G$. To do this, we need to prove some auxiliary lemmas. We first estimate the star packing number of a graph obtained from $G$ by deleting a certain subset of its vertices.

Lemma 3.1. Let $G$ be a $W(K_3)$-free graph and suppose that $x_1, x_2, x_3$ are three vertices of $G$ which form a triangle. Then

$$\alpha_2(G \setminus \bigcup_{i=1}^{3} N_G(x_i)) \geq \alpha_2(G) - 2.$$ 

Proof. To simplify the notation, set $A := N_G(x_1) \cup N_G(x_2) \cup N_G(x_3)$. Let $S$ be the set of the centers of stars in a largest star packing of $G$. In particular, $|S| = \alpha_2(G)$. Since every vertex in $A$ is adjacent to at least one of the vertices $x_1, x_2, x_3$, it follows from the definition of star packing that $|S \cap A| \leq 3$. If $|S \cap A| \leq 2$, then the stars in $G \setminus A$ centered at the vertices in $S \setminus A$ form a star packing in $G \setminus A$ of size at least $\alpha_2(G) - 2$ and the assertion follows. So, we need to consider the case $|S \cap A| = 3$.

Let $z_1, z_2, z_3$ be the vertices belonging to $S \cap A$. First, assume that

$$\{z_1, z_2, z_3\} \cap \{x_1, x_2, x_3\} \neq \emptyset.$$ 

For example, suppose that $x_1 = z_1$. Then we have either $z_2 \in N_G(x_2)$ or $z_2 \in N_G(x_3)$. In the first case, $x_2 \in N_G(z_1) \cap N_G(z_2)$ and in the second case, $x_3 \in N_G(z_1) \cap N_G(z_2)$. Both are contradictions. Hence,

$$\{z_1, z_2, z_3\} \cap \{x_1, x_2, x_3\} = \emptyset.$$ 

Without loss of generality, we may assume that $z_i \in N_G(x_i) \setminus \{x_1, x_2, x_3\}$, for each integer $i = 1, 2, 3$. Remind that for $i \neq j$, we have $N_G[z_i] \cap N_G[z_j] = \emptyset$. This yields that $z_i z_j, z_i x_j \notin E(G)$, for $i \neq j$. Consequently, the vertices $x_1, x_2, x_3, z_1, z_2, z_3$ form an induced $W(K_3)$ in $G$ which is a contradiction. The contradiction shows that $|S \cap A| \leq 2$ which completes the proof. \hfill $\square$

The next two lemmas will be used in the proof of Corollary 3.4 which provides a lower bound for the depth of a certain ideal constructed from a graph $G$.

Lemma 3.2. Let $G$ be a graph and suppose that $x_i x_j$ is an edge of $G$. Set $L := N_G(x_i) \cap N_G(x_j)$ and let $G'$ be the graph with $V(G') = V(G) \setminus L$ and edge set

$$E(G') = E(G \setminus L) \cup \{x_p x_q \mid x_p \in N_G[L](x_i), x_q \in N_G[L](x_j)\}.$$ 

Then

$$(I(G) : x_i) \cap (I(G) : x_j) = I(G') + (L).$$
Proof. Observe that if \( x_p \in N_{G \setminus L}(x_i) \) and \( x_q \in N_{G \setminus L}(x_j) \), then \( x_p \neq x_q \). We first prove the inclusion "\( \supseteq \)". Let \( u \) be a monomial in \( (I(G) : x_i) \). By symmetry, it is enough to prove that \( u \in (I(G) : x_i) \). If \( u \in (L) \), then \( u \) is divisible by a variable \( x_r \in L \). It follows from the definition of \( L \) that \( x_r x_i \in I(G) \). Consequently, \( u x_i \in I(G) \) which implies that \( u \in (I(G) : x_i) \). Therefore, assume that \( u \notin (L) \). Then, \( u \in I(G') \). If \( u \in I(G) \), then clearly we have \( u \notin (I(G) : x_i) \). Hence, suppose that \( u \notin I(G) \).

Then we conclude from the definition of \( G' \) that there are vertices \( x_p \in N_{G \setminus L}(x_i) \) and \( x_q \in N_{G \setminus L}(x_j) \) such that \( x_p x_q \) divides \( u \). As \( x_p \in N_G(x_i) \), we have \( x_p x_i \in I(G) \). Since \( x_p x_i \) divides \( u x_i \), we deduce that \( u x_i \in I(G) \). This means that \( u \in (I(G) : x_i) \).

To prove the reverse inclusion, let \( v \) be a monomial in \( (I(G) : x_i) \cap (I(G) : x_j) \) and suppose that \( v \notin (L) \). We must show that \( v \in I(G') \). If \( v \in I(G) \), then we are done. Hence, assume that \( v \notin I(G) \). It follows from \( v x_i \in I(G) \) that there is a vertex \( x_i \in N_G(x_i) \) which divides \( v \). As \( v \notin (L) \), we deduce that \( x_i \in N_{G \setminus L}(x_i) \). Similarly, there is a vertex \( x_s \in N_{G \setminus L}(x_j) \) which divides \( v \). We conclude from \( x_i , x_s \notin L \) that \( x_i \neq x_s \). Consequently, \( x_i x_s \) divides \( u \). It follows from the definition of \( G' \) that \( x_i x_s \in I(G') \) which implies \( v \in I(G') \).

\[ \square \]

Lemma 3.3. Let \( G \) be a graph and suppose that \( x_i x_j \) is an edge of \( G \). Let \( A \) be a subset of \( N_G(x_i) \cup N_G(x_j) \) with \( x_i , x_j \notin A \). Set

\[ J := (I(G \setminus A) : x_i) \cap (I(G \setminus A) : x_j) \]

and \( S_A := \mathbb{K}[x_k : 1 \leq k \leq n , x_k \notin A] \). Then

\[ \text{depth}_{S_A} J \geq \alpha_2(G) \]

Proof. Consider the following short exact sequence.

\[
\begin{array}{c}
0 \rightarrow S_A \rightarrow \frac{S_A}{(I(G \setminus A) : x_i) \oplus (I(G \setminus A) : x_j)} \rightarrow \frac{S_A}{(I(G \setminus A) : x_i) + (I(G \setminus A) : x_j)} \rightarrow 0
\end{array}
\]

Applying depth Lemma [2, Proposition 1.2.9] on the above exact sequence, it suffices to prove that

(a) \( \text{depth}_{S_A}(I(G \setminus A) : x_i) \geq \alpha_2(G) \),

(b) \( \text{depth}_{S_A}(I(G \setminus A) : x_j) \geq \alpha_2(G) \), and

(c) \( \text{depth}_{S_A}((I(G \setminus A) : x_i) + (I(G \setminus A) : x_j)) \geq \alpha_2(G) - 1 \).

To prove (a), note that

\[ (I(G \setminus A) : x_i) = I(G \setminus (A \cup N_{G \setminus A}[x_i])) + (\text{the ideal generated by } N_{G \setminus A}(x_i)) \]

\[ = I(G \setminus (A \cup N_G[x_i])) + (\text{the ideal generated by } N_{G \setminus A}(x_i)). \]

Hence,

\[ \text{depth}_{S_A}(I(G \setminus A) : x_i) = \text{depth}_{S_A'} I(G \setminus (A \cup N_G[x_i])), \]

where \( S_A' = \mathbb{K}[x_k : 1 \leq k \leq n , x_k \notin A \setminus \{x_i\}] \).
where $S' = \mathbb{K}[x_k : 1 \leq k \leq n, x_k \notin N_{G \setminus A}(x_i) \cup A]$. Obviously, $x_i$ is a regular element of $S'/I(G \setminus (A \cup N_G[x_i]))$. Therefore, Proposition 2.1 implies that

\[
(2) \quad \text{depth}_{S} I(G \setminus (A \cup N_G[x_i])) \geq \alpha_2(G \setminus (A \cup N_G[x_i])) + 2.
\]

It follows from [20, Lemma 3.1] that

\[
(3) \quad \alpha_2(G \setminus (A \cup N_G[x_i])) \geq \alpha_2(G) - 2.
\]

Thus, we conclude from equality (1) and inequalities (2) and (3) that

\[
\text{depth}_{S_A} (I(G \setminus A) : x_i) \geq \alpha_2(G),
\]

and this completes the proof of (a). The proof of (b) is similar to that of (a). We now prove (c).

Note that

\[
(I(G \setminus A) : x_i) + (I(G \setminus A) : x_j)
\]

\[
= I(G \setminus (A \cup N_G[x_i] \cup N_G[x_j])) + \text{(the ideal generated by } N_{G \setminus A}(x_i) \cup N_{G \setminus A}(x_j))
\]

\[
= I(G \setminus (A \cup N_G[x_i] \cup N_G[x_j])) + \text{(the ideal generated by } N_{G \setminus A}(x_i) \cup N_{G \setminus A}(x_j))
\]

\[
= I(G \setminus (N_G[x_i] \cup N_G[x_j])) + \text{(the ideal generated by } N_{G \setminus A}(x_i) \cup N_{G \setminus A}(x_j)),
\]

where the last equality follows from $A \subseteq N_G(x_i) \cup N_G(x_j)$. We conclude that

\[
(4) \quad \text{depth}_{S''} ((I(G \setminus A) : x_i) + (I(G \setminus A) : x_j)) = \text{depth}_{S''} I(G \setminus (N_G[x_i] \cup N_G[x_j])),
\]

where $S'' = \mathbb{K}[x_k : 1 \leq k \leq n, x_k \notin N_{G \setminus A}(x_i) \cup N_{G \setminus A}(x_j) \cup A]$. Using Proposition 2.1 we deduce that

\[
(5) \quad \text{depth}_{S''} I(G \setminus (N_G[x_i] \cup N_G[x_j])) \geq \alpha_2(G \setminus (N_G[x_i] \cup N_G[x_j])) + 1.
\]

We also know from [20, Lemma 3.1] that

\[
(6) \quad \alpha_2(G \setminus (N_G[x_i] \cup N_G[x_j])) \geq \alpha_2(G) - 2.
\]

Consequently, the assertion of (c) follows from equality (1) and inequalities (5) and (6). \hfill \Box

The following corollary is a consequence of Lemmata 3.2 and 3.3.

**Corollary 3.4.** Assume that $G$ is a graph and $x_i x_j$ is an edge of $G$. Let $A$ be a subset of $N_G(x_i) \cup N_G(x_j)$ with $x_i, x_j \notin A$. Set $L := N_{G \setminus A}(x_i) \cap N_{G \setminus A}(x_j)$ and $S_A := \mathbb{K}[x_k : 1 \leq k \leq n, x_k \notin A]$. Suppose $G'$ is the graph with $V(G') = V(G) \setminus (A \cup L)$ and edge set

\[
E(G') = E(G \setminus (A \cup L)) \cup \{x_p x_q \mid x_p \in N_{G \setminus (A \cup L)}(x_i), x_q \in N_{G \setminus (A \cup L)}(x_j)\}.
\]

Then

\[
\text{depth}_{S_A} (I(G') + (L)) \geq \alpha_2(G).
\]

**Proof.** Let $J$ be the ideal defined in Lemma 3.3. By substituting $G$ with $G \setminus A$ in Lemma 3.2 we obtain that $J = I(G') + (L)$. The claim now follows from Lemma 3.3. \hfill \Box
The following lemma is the most technical part of the proof of Theorem 3.6.

**Lemma 3.5.** Let $G$ be a graph and suppose that $x_ix_j$ is an edge of $G$. Let $A$ be a subset of $N_G(x_i) \cup N_G(x_j)$ with $x_i, x_j \notin A$. Then
\[
\text{depth}_A(I(G \setminus A)^2 : x_ix_j) \geq \alpha_2(G) - 2.
\]
If moreover, $G$ is a $W(K_3)$-free graph, then
\[
\text{depth}_A(I(G \setminus A)^2 : x_ix_j) \geq \alpha_2(G) - 1.
\]

**Proof.** Set $r := |(N_G(x_i) \cup N_G(x_j)) \setminus \{x_i, x_j\}|$. Then $|A| \leq r$. We proceed by backward induction on $|A|$. If $|A| = r$, then $A = (N_G(x_i) \cup N_G(x_j)) \setminus \{x_i, x_j\}$. Hence, $G \setminus A$ is the disjoint union of the edge $x_ix_j$ with the graph $G \setminus (N_G(x_i) \cup N_G(x_j))$. Thus, we conclude from [20] Lemma 2.10 that
\[
(I(G \setminus A)^2 : x_ix_j) = I(G \setminus A).
\]
Consequently, we deduce from Proposition 2.11 that
\[
\text{depth}_A(I(G \setminus A)^2 : x_ix_j) \geq \alpha_2(G \setminus A) + 1.
\]
On the other hand, we know from [20] Lemma 3.1 that $\alpha_2(G \setminus A) \geq \alpha_2(G) - 2$. This together with the above inequality implies that
\[
\text{depth}_A(I(G \setminus A)^2 : x_ix_j) \geq \alpha_2(G) - 1.
\]
Therefore, assume that $|A| \leq r - 1$.

Set $L := N_G \setminus A(x_i) \cap N_G \setminus A(x_j)$ and let $G'$ be the graph introduced in Corollary 3.4. We know from [11] Theorems 6.5 and 6.7 that
\[
(I(G \setminus A)^2 : x_ix_j) = \left(I(G \setminus A) + (x_px_q : x_p \in N_G \setminus A(x_i), x_q \in N_G \setminus A(x_j))\right) = I(G \setminus A) + (x_px_q : x_p \in N_G \setminus A(x_i), x_q \in N_G \setminus A(x_j), x_p \neq x_q) + (x_k^2 : x_k \in L).
\]
If $L = \emptyset$, then using the above equalities, we have
\[
(I(G \setminus A)^2 : x_ix_j) = I(G') = I(G') + (L).
\]
Thus, in this case the assertion follows from Corollary 3.4. Hence, suppose that $L \neq \emptyset$. Without loss of generality, assume that $L = \{x_1, \ldots, x_t\}$, for some integer $t \geq 1$. Let $H$ be the graph with
\[
I(H) = \left(I(G \setminus A)^2 : x_ix_j\right)^{\text{pol}}.
\]
In other words, $H$ is the graph with vertex set $V(H) = V(G \setminus A) \cup \{y_1, \ldots, y_t\}$ and edge set $E(H) = E(G \setminus A) \cup \{x_px_q : x_p \in N_G \setminus A(x_i), x_q \in N_G \setminus A(x_j), x_p \neq x_q\} \cup \{x_1y_1, \ldots, x_ty_t\}$. Let $T$ be the polynomial ring over $\mathbb{K}$ with variables corresponding to the vertices of $H$. It follows from [10] Corollary 1.6.3 that
\[
\text{depth}_A(I(G \setminus A)^2 : x_ix_j) = \text{depth}_T I(H) - t.
\]
Consider the short exact sequence
\[
0 \to \frac{T}{(I(H) : x_1)} \to \frac{T}{I(H)} \to \frac{T}{I(H) + (x_1)} \to 0.
\]
It follows from depth Lemma [2, Proposition 1.2.9] that
\[
\text{depth}_T I(H) \geq \min \{ \text{depth}_T(I(H) : x_1), \text{depth}_T(I(H), x_1) \}.
\] (8)
Therefore, using equality (7) and inequality (8) it is enough to prove the following statements.

(i) $\text{depth}_T(I(H) : x_1)$ is at least $\alpha_2(G) + t - 2$, and if $G$ is a $W(K_3)$-free graph, then $\text{depth}_T(I(H) : x_1) \geq \alpha_2(G) + t - 1$.

(ii) $\text{depth}_T(I(H), x_1)$ is at least $\alpha_2(G) + t - 2$, and if $G$ is a $W(K_3)$-free graph, then $\text{depth}_T(I(H), x_1) \geq \alpha_2(G) + t - 1$.

We first prove (i). Note that
\[
(I(H) : x_1) = I(H \setminus N_H(x_1)) + (\text{the ideal generated by } N_H(x_1)).
\]
Consequently,
\[
\text{depth}_T(I(H) : x_1) = \text{depth}_{T'} I(H \setminus N_H(x_1)),
\]
where $T'$ is the polynomial ring which is obtained from $T$ by deleting the variables in $N_H(x_1)$. It is obvious that
\[
N_H(x_1) = N_{G \setminus A}(x_1) \cup N_{G \setminus A}(x_i) \cup N_{G \setminus A}(x_j) \cup \{ y_1 \}.
\]
In particular, the vertices $x_2, \ldots, x_t$ are contained in $N_H(x_1)$. Thus, $H \setminus N_H(x_1)$ is disjoint union of the isolated vertices $y_2, \ldots, y_t$ with the graph $H'$ defined as
\[
H' := G \setminus (A \cup N_{G \setminus A}(x_1) \cup N_{G \setminus A}(x_i) \cup N_{G \setminus A}(x_j))
\]
\[
= G \setminus (A \cup N_G(x_1) \cup N_G(x_i) \cup N_G(x_j))
\]
\[
= G \setminus (N_G(x_1) \cup N_G(x_i) \cup N_G(x_j)),
\]
where the last equality follows from $A \subseteq N_G(x_i) \cup N_G(x_j)$. Since $x_1, y_2, \ldots, y_t$ is a regular sequence on $T'/I(H \setminus N_H[x_1])$, we deduce that
\[
\text{depth}_{T'} I(H \setminus N_H[x_1]) = \text{depth}_{T''} I(H') + t,
\]
where $T''$ is the polynomial ring which is obtained from $T'$ by deleting the variables $x_1, y_2, \ldots, y_t$. Using Proposition 2.1 we have
\[
\text{depth}_{T''} I(H') \geq \alpha_2(H') + 1.
\] (11)
Also, we conclude from [20, Lemma 3.1] and Lemma 3.1 that
\[
\alpha_2(H') = \alpha_2(G \setminus (N_G(x_1) \cup N_G(x_i) \cup N_G(x_j))) \geq \alpha_2(G) - 3
\]
and if $G$ is a $W(K_3)$-free graph, then
\[
\alpha_2(H') = \alpha_2(G \setminus (N_G(x_1) \cup N_G(x_i) \cup N_G(x_j))) \geq \alpha_2(G) - 2.
\]
Together with equalities \((9)\) and \((10)\) and inequality \((11)\), we obtain the assertion of (i).

To prove (ii), note that
\[(I(H), x_1) = I(H \setminus x_1) + (x_1).\]

Therefore,
\[(12) \quad \text{depth}_T(I(H), x_1) = \text{depth}_{T_1} I(H \setminus x_1),\]
where \(T_1\) is the polynomial ring obtained from \(T\) by deleting the variable \(x_1\). Since \(y_1\) is an isolated vertex of \(H \setminus x_1\), we conclude that \(y_1\) is regular on \(T_1/I(H \setminus x_1)\). Hence, equality \((12)\) yields that
\[(13) \quad \text{depth}_T(I(H), x_1) = \text{depth}_{T_2} I(H \setminus \{x_1, y_1\}) + 1,\]
where \(T_2\) is the polynomial ring obtained from \(T_1\) by deleting the variable \(y_1\). Set \(A' := A \cup \{x_1\}\). As \(x_1 \in L\), we have \(A' \subseteq N_G(x_i) \cup N_G(x_j)\). Clearly, \(x_i\) and \(x_j\) do not belong to \(A'\). Set \(S_{A'} := \mathbb{K}[x_k : 1 \leq k \leq n, x_k \notin A']\). Since \(|A' > |A|\), it follows from the induction hypothesis that
\[(14) \quad \text{depth}_{S_{A'}} (I(G \setminus A')^2 : x_i x_j) \geq \alpha_2(G) - 2,\]
and if moreover \(G\) is a \(W(K_3)\)-free graph, then
\[(15) \quad \text{depth}_{S_{A'}} (I(G \setminus A')^2 : x_i x_j) \geq \alpha_2(G) - 1.\]

It is obvious that
\[(16) \quad (I(G \setminus A')^2 : x_i x_j)^{\text{pol}} = I(H \setminus \{x_1, y_1\}).\]

Thus, Using \((10)\) Corollary 1.6.3, we have
\[(17) \quad \text{depth}_{S_{A'}} (I(G \setminus A')^2 : x_i x_j) = \text{depth}_{T_2} I(H \setminus \{x, y\}) - (t - 1).\]

Combining equalities \((13)\) and \((16)\) and inequalities \((14)\) and \((15)\) completes the proof of (ii).

We are now ready to prove the main result of this paper.

**Theorem 3.6.**

(1) For any graph \(G\), we have \(\text{depth}_S I(G)^2 \geq \alpha_2(G) - 2\).

(2) For any \(W(K_3)\)-free graph \(G\), we have \(\text{depth}_S I(G)^2 \geq \alpha_2(G) - 1\).

(3) For any triangle-free graph \(G\), we have \(\text{depth}_S I(G)^2 \geq \alpha_2(G)\).

**Proof.** Set \(I := I(G)\) and let \(G(I) = \{u_1, \ldots, u_m\}\) be the set of minimal monomial generators of \(I\). For every integer \(k\) with \(1 \leq k \leq m\), consider the short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & S & \longrightarrow & S \\
& & (I^2 + (u_1, \ldots, u_{k-1})) : u_k & \longrightarrow & I^2 + (u_1, \ldots, u_{k-1}) \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
& S & \longrightarrow & I^2 + (u_1, \ldots, u_k) & \longrightarrow & 0,
\end{array}
\]
where for \( k = 1 \), the ideal \((u_1, \ldots, u_{k-1})\) is the zero ideal. It follows from depth Lemma \([2, \text{Proposition 1.2.9}]\) that

\[
\text{depth}_S(I^2 + (u_1, \ldots, u_{k-1})) \
\geq \min \left\{ \text{depth}_S((I^2 + (u_1, \ldots, u_{k-1})): u_k), \text{depth}_S(I^2 + (u_1, \ldots, u_k)) \right\}.
\]

Using the above inequality inductively, we have

\[
\text{depth}_S I^2 \geq \min \left\{ \text{depth}_S(I^2 + I), \min \left\{ \text{depth}_S((I^2 + (u_1, \ldots, u_{k-1})): u_k) | 1 \leq k \leq m \right\} \right\} = \min \left\{ \text{depth}_S I, \text{depth}_S((I^2 + (u_1, \ldots, u_{k-1})): u_k) | 1 \leq k \leq m \right\} \geq \min \left\{ \alpha_2(G) + 1, \text{depth}_S((I^2 + (u_1, \ldots, u_{k-1})): u_k) | 1 \leq k \leq m \right\},
\]

where the last inequality follows from Proposition \([2, \text{Proposition 1.2.9}]\). Hence, in order to prove (1) and (2) it is enough to show that for every integer \( k \) with \( 1 \leq k \leq m \), we have

\[
\text{depth}_S((I^2 + (u_1, \ldots, u_{k-1})): u_k) \geq \alpha_2(G) - 2,
\]

and if \( G \) is a \( W(K_2) \)-free graph, then

\[
\text{depth}_S((I^2 + (u_1, \ldots, u_{k-1})): u_k) \geq \alpha_2(G) - 1.
\]

Fix an integer \( k \) with \( 1 \leq k \leq m \) and assume that \( u_k = x_ix_j \). Using \([1, \text{Theorem 4.12}]\), we may suppose that for every pair of integers \( 1 \leq s < t \leq m \), one of the following conditions holds.

(i) \((u_s : u_t) \subseteq (I^2 : u_t)\); or

(ii) there exists an integer \( \ell \leq t - 1 \) such that \((u_\ell : u_t)\) is generated by a variable, and \((u_s : u_t) \subseteq (u_\ell : u_t)\).

We conclude from (i) and (ii) above that

\[
(I^2 + (u_1, \ldots, u_{k-1})): u_k = (I^2 : u_k) + (\text{some variables}).
\]

Assume that \( A \) is the set of variables belonging to \((I^2 + (u_1, \ldots, u_{k-1})): u_k)\). Let \( x_r \) be an arbitrary variable in \( A \). This means that \( x_rx_ix_j \) belongs to the ideal \( I^2 + (u_1, \ldots, u_{k-1}) \). As \( I^2 \) is generated in degree 4, we deduce that \( x_rx_ix_j \notin I^2 \). Hence, there is an integer \( l \) with \( 1 \leq l \leq k - 1 \) such that \( u_l \) divides \( x_rx_ix_j \). Since, \( u_l \neq u_k \), it follows that either \( u_l = x_rx_i \) or \( u_l = x_rx_j \). In particular,

\[
x_r \in (N_G(x_i) \cup N_G(x_j)) \setminus \{x_i, x_j\}.
\]

Consequently, \( A \subseteq (N_G(x_i) \cup N_G(x_j)) \setminus \{x_i, x_j\} \). It follows from equality (17) that

\[
((I^2 + (u_1, \ldots, u_{k-1})): u_k) = (I^2 : u_k) + (\text{the ideal generated by } A)
\]

\[
= (I^2 +(\text{the ideal generated by } A)): u_k)
\]

\[
= (I(G \setminus A)^2 + (\text{the ideal generated by } A)) : u_k)
\]

\[
= (I(G \setminus A)^2 : u_k) + (\text{the ideal generated by } A).
\]
The above equalities imply that
\[
\text{depth}_S \left( (I^2 + (u_1, \ldots, u_{k-1})) : u_k \right) = \text{depth}_{S_A} \left( (I(G \setminus A))^2 : u_k \right),
\]
where \( S_A \) is the polynomial ring obtained from \( S \) by deleting the variables in \( A \). Using Lemma 3.5, we deduce that
\[
\text{depth}_S ((I^2 + (u_1, \ldots, u_{k-1})) : u_k) \geq \alpha_2(G) - 2,
\]
and if \( G \) is a \( W(K_3) \)-free graph, then
\[
\text{depth}_S ((I^2 + (u_1, \ldots, u_{k-1})) : u_k) \geq \alpha_2(G) - 1.
\]
This completes the proof of parts (1) and (2).

To prove (3), let \( G \) be a triangle-free graph. Then we know from \([16, \text{Lemma 3.10}]\) that \( I(G)^2 = I(G)^{(2)} \), where \( I(G)^{(2)} \) denotes the second symbolic power of \( I(G) \). The assertion now follows from \([20, \text{Theorem 4.2}]\). □

The following examples show that the inequalities obtained in Theorem 3.6 are sharp.

**Examples 3.7.**

1. Suppose \( G = W(K_3) \). Then \( \text{depth}_S I(G)^2 = 1 \) which is equal to \( \alpha_2(G) - 2 \).
2. Let \( G \) be the graph which is obtained from \( W(K_3) \) by deleting one of its leaves. Then \( \text{depth}_S I(G)^2 = 1 \) which is equal to \( \alpha_2(G) - 1 \).
3. Let \( G = P_4 \) be the path of length three. Then \( \text{depth}_S I(G)^2 = 2 \) which is equal to \( \alpha_2(G) \).

**References**

[1] A. Banerjee, The regularity of powers of edge ideals, *J. Algebraic Combin.* **41** (2015), 303–321.
[2] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, Cambridge Studies in Advanced Mathematics, **39**, Cambridge University Press, 1993.
[3] L. Burch, Codimension and analytic spread, *Proc. Cambridge Philos. Soc.* **72** (1972), 369–373.
[4] G. Caviglia, H. T. Hà, J. Herzog, M. Kummini, N. Terai, N. V. Trung, Depth and regularity modulo a principal ideal, *J. Algebraic Combin.* **49** (2019), 1–20.
[5] H. Dao, J. Schweig, Bounding the projective dimension of a squarefree monomial ideal via domination in clutters, *Proc. Amer. Math. Soc.* **143** (2015), 555–565.
[6] L. Fouli, H. T. Hà, S. Morey, Initially regular sequences and depths of ideals, *J. Algebra*, **559** (2020), 33–57.
[7] L. Fouli, H. T. Hà, S. Morey, Depth of Powers of Squarefree Monomial Ideals, Advances in Mathematical Sciences, 161–171, Assoc. Women Math. Ser., 21, Springer, Cham, 2020.
[8] L. Fouli, S. Morey, A lower bound for depths of powers of edge ideals, *J. Algebraic Combin.* **42** (2015), 829–848.
[9] H. T. Hà, N. V. Trung, T. N. Trung, Depth and regularity of powers of sums of ideals, *Math. Z.*, **282** (2016), 819–838.
[10] J. Herzog, T. Hibi, *Monomial Ideals*, Springer-Verlag, 2011.
[11] J. Herzog, T. Hibi, The depth of powers of an ideal, *J. Algebra* **291** (2005), no. 2, 534–550.
[12] L. T. Hoa, K. Kimura, N. Terai, T. N. Trung, Stability of depths of symbolic powers of Stanley-Reisner ideals, *J. Algebra* **473** (2017), 307–323.
[13] K. Kimura, N. Terai, S. Yassemi, The projective dimension of the edge ideal of a very well-covered graph, *Nagoya Math. J.* **230** (2018), 160–179.
[14] S. Morey, Depths of powers of the edge ideal of a tree, *Comm. Algebra* 38 (2010), no. 11, 4042–4055.

[15] H. D. Nguyen, N. V. Trung, Depth functions of symbolic powers of homogeneous ideals, *Invent. Math.* 218 (2019), 779–827.

[16] G. Rinaldo, N. Terai, K. Yoshida, Cohen-Macaulayness for symbolic power ideals of edge ideals, *J. Algebra* 347 (2011), 1–22.

[17] S. A. Seyed Fakhari, Depth and Stanley depth of symbolic powers of cover ideals of graphs, *J. Algebra*, 492 (2017), 402–413.

[18] S. A. Seyed Fakhari, On the depth and Stanley depth of the integral closure of powers of monomial ideals, *Collect. Math.*, 70 (2019), no. 3, 447–459.

[19] S. A. Seyed Fakhari, Stability of depth and Stanley depth of symbolic powers of squarefree monomial ideals, *Proc. Amer.Math. Soc.*, 148 (2020), 1849–1862.

[20] S. A. Seyed Fakhari, On the depth of symbolic powers of edge ideals of graphs, *Nagoya Math. J.*, to appear.

[21] T. N. Trung, Stability of depths of powers of edge ideals, *J. Algebra* 452 (2016), 157–187.

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