Fractional Noether’s Theorem with Classical and Caputo Derivatives: constants of motion for non-conservative systems

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Abstract Since the seminal work of Emmy Noether it is well known that all conservation laws in physics, e.g., conservation of energy or conservation of momentum, are directly related to the invariance of the action under a family of transformations. However, the classical Noether’s theorem can not yields informations about constants of motion for non-conservative systems since it is not possible to formulate physically meaningful Lagragians for this kind of systems in classical calculus of variation. On the other hand, in recent years the fractional calculus of variation within Lagrangians depending on fractional derivatives has emerged as an elegant alternative to study non-conservative systems. In the present work, we obtained a generalization of the Noether’s theorem for Lagrangians depending on mixed classical and Caputo derivatives that can be used to obtain constants of motion for dissipative systems. In addition, we also obtained Noether’s conditions for the fractional optimal control problem.

Keywords Noether’s Theorem · Caputo Derivatives · Fractional Calculus of Variation and Optimal Control

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1 Introduction

The fractional calculus with derivatives and integrals of non-integer order started more than three centuries ago, with l'Hôpital and Leibniz, when the derivative of order $1/2$ was suggested (see [16,33,50,57] for the history of fractional calculus). This subject was then considered by several mathematicians like Euler, Fourier, Liouville, Grunwald, Letnikov, Riemann, and many others up to nowadays. Although the fractional calculus is almost as old as the usual integer order calculus, only in the last three decades it has gained more attention due to its many applications in various fields of science, engineering, economics, biomechanics, etc. (see [24,26,30,38,52] for a review). Actually, there are several definitions for fractional derivatives, being the Riemann-Liouville and the Caputo the most popular definitions. In special, the Caputo fractional derivative was introduced by Caputo and Mainardi in a seminal work [11] to model dissipation phenomenons. Fractional derivatives are generally nonlocal operators and are historically applied to study nonlocal or time dependent processes. In special, the first and well established application of fractional calculus in Physics was in the framework of anomalous diffusion observed in many physical systems (e.g. in dispersive transport in amorphous semiconductor, liquid crystals, polymers, proteins, etc [31,23,40]). Recently, the study of nonlocal quantum phenomena through fractional calculus began a fast development, where the nonlocal effects are due to either long-range interactions or time-dependent processes with many scales [26,27,32,33,43,54,60]. Relativistic quantum mechanics [25,29,42,50,62] and field theories [6,10,34,55,59] has been also recently considered in the context of fractional calculus.

The fractional calculus of variation was introduced in the context of classical mechanics. Riewe [51] showed that a Lagrangian involving fractional time derivatives leads to an equation of motion with non-conservative forces such as friction. It is a remarkable result since frictional and non-conservative forces are beyond the usual macroscopic variational treatment [7]. Riewe generalized the usual calculus of variations for a Lagrangian depending on Riemann-Liouville fractional derivatives [51] in order to deal with linear non-conservative forces. Actually, several approaches have been developed to generalize the least action principle to include problems depending on Caputo fractional derivatives, Riemann–Liouville fractional derivatives, Riesz fractional derivatives and others [1,3,5,12,35,41,45] (see [39] for a recent review). Among these approaches, recently it was show that the action principle for dissipative systems can be generalized, fixing the mathematical inconsistencies present in the original Riewe’s formulation, by using Lagrangians depending on classical and Caputo derivatives [36]. The great importance of these results is the fact that the calculus of variation with Lagrangians depending on both classical and Caputo derivatives enable us to use all the mathematical machinery of classical mechanics to study non-conservative systems.

Among the mathematical machinery of classical mechanics, the Noether’s theorem of calculus of variation becomes one of the most important theorems for physics in the 20th century. Since the seminal work of Emmy Noether it is
well know that all conservations laws in mechanics, e.g., conservation of energy or conservation of momentum, are directly related to the invariance of the action under a family of transformations. On the other hand, non-conservative forces remove energy from the systems and, as a consequence, the standard Noether constants of motion are broken. In this context, the generalization of the Noether’s theorem for the fractional calculus of variation is fundamental to investigate the action symmetries for non-conservative systems. Recently, it was show that it is still possible to obtain Noether-type theorems for fractional calculus of variations which cover both conservative and nonconservative cases [15,16,17,18,19,20,21]. In the present work, we generalize Noether’s theorem for Lagrangians depending on mixed classical and Caputo derivatives. It is important to stress that our results are based in the classical notion of conserved quantity \( C \), that is, the classical derivative of such a quantity is equal to zero (\( dC/dt = 0 \)). It is a different approach from previous works [17,18,19,20,21] where it was introduced the notion of fractional-conserved quantity, where the classical derivative is substituted by a bilinear fractional operator \( D (D(C) = 0) \). Consequently, our present work is free from the difficulties introduced by the notion of fractional-conserved quantity (see [4] for a detailed discussion). Furthermore, the generalized Noether’s theorem we obtain enable us to investigate constants of motion for dissipative systems in the context of the action principle formulated in [30]. As an example of application to non-conservative systems, we study the problem of a particle under a frictional force. Furthermore, we also generalize the Noether-type theorems for the optimal control problem with classical and Caputo derivatives.

The paper is organized in the following way. In Section 2 we review the basic notions of Riemann-Liouville and Caputo fractional calculus, that are needed for formulating the fractional problem of the calculus of variations. The Euler-Lagrange equation and the Noether’s theorem for Lagrangians depending on mixed classical and Caputo derivatives are obtained in Section 3. An example of application of the Noether’s theorem for a particle under a frictional force is presented in Section 4. In Section 5 we generalize the Noether’s theorems for the optimal control problem with classical and Caputo derivatives. Finally, the conclusions are presented in Section 6.

2 Preliminaries on Fractional Calculus

In this section we fix notations by collecting the definitions and properties of fractional integrals and derivatives needed in the sequel [2,41,47,53].

**Definition 1** (Riemann–Liouville fractional integrals) Let \( f \) be a continuous function in the interval \([a, b]\). For \( t \in [a, b] \), the left Riemann–Liouville fractional integral \( aI^\alpha_t f(t) \) and the right Riemann–Liouville fractional integral...
\( _a I_t^\alpha f(t) \) of order \( \alpha \), are defined by
\[
_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \theta)^{\alpha - 1} f(\theta) d\theta,
\]
\[
_t I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\theta - t)^{\alpha - 1} f(\theta) d\theta,
\]
where \( \Gamma \) is the Euler gamma function and \( 0 < \alpha < 1 \).

**Definition 2** (Fractional derivatives in the sense of Riemann–Liouville) Let \( f \) be a continuous function in the interval \([a, b] \). For \( t \in [a, b] \), the left Riemann–Liouville fractional derivative \( _a D_t^\alpha f(t) \) and the right Riemann–Liouville fractional derivative \( t D_b^\alpha f(t) \) of order \( \alpha \) are defined by
\[
_a D_t^\alpha f(t) = \frac{d}{dt} _a I_t^{1-\alpha} f(t)
= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t - \theta)^{-\alpha} f(\theta) d\theta
\]
and
\[
t D_b^\alpha f(t) = -\frac{d}{dt} t I_b^{1-\alpha} f(t)
= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\theta - t)^{-\alpha} f(\theta) d\theta.
\]

**Definition 3** (Fractional derivatives in the sense of Caputo) Let \( f \) be a continuously differentiable function in the interval \([a, b] \). For \( t \in [a, b] \), the left Caputo fractional derivative \( _a C D_t^\alpha f(t) \) and the right Caputo fractional derivative \( C D_b^\alpha f(t) \) of order \( \alpha \) are defined by
\[
_a C D_t^\alpha f(t) = _a I_t^{1-\alpha} \left( \frac{d}{dt} f(t) \right)
= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t - \theta)^{-\alpha} \frac{d}{d\theta} f(\theta) d\theta
\]
and
\[
C D_b^\alpha f(t) = t I_b^{1-\alpha} \left( -\frac{d}{dt} \right) f(t)
= -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\theta - t)^{-\alpha} \frac{d}{d\theta} f(\theta) d\theta.
\]

**Remark 1** If the Riemann–Liouville and the Caputo fractional derivatives exist, then they are connected by the following relations:
\[
_a C D_t^\alpha f(t) := _a D_t^\alpha \left( f - f(a) \right)(t) \quad \text{resp.} \quad C D_b^\alpha f(t) := t D_b^\alpha \left( f - f(b) \right)(t).
\]

Let us note that if \( f(a) = 0 \) (resp. \( f(b) = 0 \)), then \( _a C D_t^\alpha f(t) = _a D_t^\alpha f(t) \) (resp. \( C D_b^\alpha f(t) = t D_b^\alpha f(t) \)).
Remark 2 The Caputo fractional derivative of a constant is always equal to zero. This is not the case of the strict fractional derivative in the Riemann–Liouville sense.

Remark 3 In the classical case $\alpha = 1$, the fractional derivatives of Riemann–Liouville and Caputo both coincide with the classical derivative. Precisely, modulo a $(-1)$ term in the right case, we have $\frac{d}{dt} = -t\frac{d}{dt} = -I_b^\alpha t$. 

Theorem 1 Let $f$ and $g$ be two continuously differentiable functions on $[a, b]$. Then, for all $t \in [a, b]$, the following property holds:

$$C_a^\alpha \left( f(t) + g(t) \right) = C_a^\alpha f(t) + C_a^\alpha g(t).$$

We now present the integration by parts formula for fractional derivatives.

Lemma 1 If $f$, $g$, and the fractional derivatives $C_a^\alpha g$ and $tD_b^\alpha f$ are continuous at every point $t \in [a, b]$, then

$$\int_a^b g(t) \cdot C_a^\alpha f(t)\,dt = \int_a^b f(t) \cdot tD_b^\alpha g(t)\,dt + \left[ tI_b^{1-\alpha} g(t) \cdot f(t) \right]_{t=a}^{t=b}$$

for any $0 < \alpha < 1$. Moreover, if $f$ is a function such that $f(a) = f(b) = 0$, we have simpler formula:

$$\int_a^b g(t) \cdot C_a^\alpha f(t)\,dt = \int_a^b f(t) \cdot tD_b^\alpha g(t)\,dt.$$

Remark 4 We note that formula (1) is still valid for $\alpha = 1$ provided $f$ or $g$ are zero at $t = a$ and $t = b$.

3 Main results: Euler–Lagrange equations and Noether’s theorems for variational problems with classical and Caputo derivatives

In Section 3.1 we prove two important results for variational problems: a necessary optimality condition of Euler–Lagrange type (Theorem 3) and a Noether-type theorem (Theorem 7). The results are then extended in Section 5 to the more general setting of optimal control.

3.1 Fractional variational problems with classical and Caputo derivatives

We begin by formulating the fundamental problem in Lagrange form under investigation.
Problem 1 The fractional problem of the calculus of variations with classical and Caputo derivatives in Lagrange form consists to find the stationary functions of the functional

$$I[q(\cdot)] = \int_{a}^{b} L\left(t, q(t), \dot{q}(t), \frac{D_{t}^{\alpha} q(t)}{\alpha} \right) dt$$  \hspace{1cm} (PC)$$

subject to given appropriate boundary conditions, where \([a, b] \subseteq \mathbb{R}, a < b, 0 < \alpha < 1, \dot{q} = \frac{dq}{dt}\), and the admissible functions \(q : t \mapsto q(t)\) and the Lagrangian \(L : (t, q, v, v_{1}) \mapsto L(t, q, v, v_{1})\) are assumed to be \(C^{2}\):

\[q(\cdot) \in C^{2}([a, b]; \mathbb{R}^{n});\]
\[L(\cdot, \cdot, \cdot, \cdot) \in C^{2}([a, b] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{n}; \mathbb{R}).\]

Functional of the kind \((PC)\) with mixed integer order and Caputo fractional derivatives was previously considered in \([36,44]\). However, a Noether-type theorem for \((PC)\) is not yet considered in the literature. Furthermore, despite that the fundamental problem \((PC)\) could easily be generalized for \(\alpha > 1\), we choose \(0 < \alpha \leq 1\) for simplicity. Along the work, we denote by \(\partial_{i}L\) the partial derivative of \(L\) with respect to its \(i\)th argument, \(i = 1, \ldots, 4\).

3.1.1 Fractional Euler–Lagrange equations

The Euler–Lagrange necessary optimality condition is central in achieving the main results of this work. Our results are formulated and proved using the Euler–Lagrange equations \([3]\).

Definition 4 (Space of variations) We denote by \(S_{h}(a, b)\) the set of functions \(h(\cdot) \in C^{2}([a, b]; \mathbb{R}^{n})\) such that \(h(a) = h(b) = 0\).

Definition 5 The functional \(I[\cdot]\) is \(S_{h}(a, b)\)-differentiable on a curve \(q(\cdot) \in C^{2}([a, b]; \mathbb{R}^{n})\) if and only if its Frchet differential

$$\lim_{\epsilon \to 0} \frac{I[q + \epsilon h] - I[q]}{\epsilon}$$

exists in any direction \(h(\cdot) \in S_{h}(a, b)\), then \(DI\) is called its differential and is given by

$$DI[q](h) = \lim_{\epsilon \to 0} \frac{I[q + \epsilon h] - I[q]}{\epsilon}.$$

Definition 6 (Fractional \(S_{h}\)-extremal with classical and Caputo derivatives). We say that \(q(\cdot)\) is an \(S_{h}\)-extremal with classical and Caputo derivatives for functional \((PC)\) if for any \(h(\cdot) \in S_{h}(a, b)\)

$$DI[q](h) = 0.$$
Theorem 2  The differential of $I[\cdot]$ on $q(\cdot) \in C^2 ([a,b]; \mathbb{R}^n)$ is given by

$$DI[q](h) = \int_a^b \left[ \partial_2 L (t, q, \dot{q}, C^{\alpha}_a D^\alpha_t q) \cdot h + \partial_3 L (t, q, \dot{q}, C^{\alpha}_a D^\alpha_t q) \cdot \dot{h} 
+ \partial_4 L (t, q, \dot{q}, C^{\alpha}_a D^\alpha_t q) \cdot C^{\alpha}_a D^\alpha_t h \right] dt .$$  \hspace{1cm} (2)

Proof We obtain equation 2 by direct computations with help of a Taylor expansion.

We now obtain the fractional Euler–Lagrange necessary optimality condition.

Theorem 3 (Fractional least-action principle). If $q(\cdot)$ is a $S_h$-extremal to Problem 1, then it satisfies the following Euler–Lagrange equation with classical and Caputo derivatives:

$$\partial_2 L (t, q(t), \dot{q}(t), C^{\alpha}_a D^\alpha_t q(t)) - \frac{d}{dt} \partial_3 L (t, q(t), \dot{q}(t), C^{\alpha}_a D^\alpha_t q(t)) 
+ \tau D^\alpha_t \partial_4 L (t, q(t), \dot{q}(t), C^{\alpha}_a D^\alpha_t q(t)) = 0 , \quad t \in [a,b] .$$ \hspace{1cm} (3)

Remark 5 If $\alpha = 1$, Problem 1 is reduced to the classical problem of the calculus of variations,

$$I[q(\cdot)] = \int_a^b F (t, q(t), \dot{q}(t)) \rightarrow \min$$ \hspace{1cm} (4)

with $F (t, q(t), \dot{q}(t)) := L (t, q(t), \dot{q}(t), \ddot{q}(t))$, and one obtains from Theorem 3 the standard Euler–Lagrange equations 37:

$$\partial_2 F (t, q, \dot{q}) = \frac{d}{dt} \partial_3 F (t, q, \dot{q}) .$$ \hspace{1cm} (5)

Remark 6 Our variational Problem 1 only involves Caputo fractional derivatives but both Caputo and Riemann-Liouville fractional derivatives appear in the necessary optimality condition given by Theorem 3. This is different from [1,18] where the necessary conditions only involve the same type of derivatives (Riemann-Liouville) as those in the definition of the fractional variational problem.

Proof (of Theorem 3) According with Definition 6 a necessary condition for $q$ to be a $S_h$-extremal is given by

$$\int_a^b \left[ \partial_2 L (t, q, \dot{q}, C^{\alpha}_a D^\alpha_t q) \cdot h + \partial_3 L (t, q, \dot{q}, C^{\alpha}_a D^\alpha_t q) \cdot \dot{h} 
+ \partial_4 L (t, q, \dot{q}, C^{\alpha}_a D^\alpha_t q) \cdot C^{\alpha}_a D^\alpha_t h \right] dt = 0 .$$ \hspace{1cm} (6)
Using the fact that \( h \in S_h(a, b) \), and the classical and Caputo integration by parts formulas in the second and third terms of the integrand of (1), respectively, we obtain

\[
\int_a^b \left[ \partial_2 L \left( t, q, \dot{q}, \frac{C}{a} D_t^\alpha q \right) - \frac{d}{dt} \partial_3 L \left( t, q, \dot{q}, \frac{C}{a} D_t^\alpha q \right) + \frac{1}{C} \partial_4 L \left( t, q, \dot{q}, \frac{C}{a} D_t^\alpha q \right) \right] h dt = 0.
\]

Equality (3) follows from the application of the fundamental lemma of the calculus of variations (see, e.g., [22]).

### 3.1.2 Fractional Noether’s theorem

A classical result of Emmy Noether provides a relation between groups of symmetries of a given equation and constants of motion, i.e. first integrals. Precisely, if a Lagrangian system is invariant under a group of symmetries then it admits an explicit conservation law.

The symmetries are defined via the action of one parameter group of diffeomorphisms as follows

**Definition 7** (Group of symmetries) For any real \( \varepsilon \), let \( \psi(\varepsilon, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism. Then \( \Psi = \{ \psi(\varepsilon, \cdot) \}_{\varepsilon \in \mathbb{R}} \) is a one parameter group of diffeomorphisms of \( \mathbb{R}^n \) if it satisfies:

1. \( \psi(0, \cdot) = i_{\mathbb{R}^n} \);
2. \( \forall \varepsilon, \varepsilon' \in \mathbb{R}, \psi(\varepsilon, \cdot) \circ \psi(\varepsilon', \cdot) = \psi(\varepsilon + \varepsilon', \cdot) \);
3. \( \psi(\cdot, \cdot) \) is of class \( C^2 \) with respect to \( \varepsilon \).

Usual examples of one parameter groups of diffeomorphisms are given by translations in a given directions \( v \)

\[
\psi : q \mapsto q + \varepsilon v, \quad q \in \mathbb{R}^n
\]

and rotations of angle \( \omega \)

\[
\psi : q \mapsto q e^{i\varepsilon \omega}, \quad q \in \mathbb{C}.
\]

In [16,17] the authors use the related notion of one parameter family of infinitesimal transformations, instead of group of diffeomorphisms. They are obtained using a Taylor expansion of \( y_t(\varepsilon) = \psi(\varepsilon, q(t)) \) in a neighborhood of 0. We obtain

\[
y_t(\varepsilon) = \psi(0, q(t)) + \varepsilon \frac{\partial \psi}{\partial \varepsilon}(0, q(t)) + o(\varepsilon).
\]

Having in mind that \( \psi(0, \cdot) = i_{\mathbb{R}^n} \), we deduce that an infinitesimal transformation is of the form

\[
q(t) \mapsto q(t) + \varepsilon \xi(t, q(t)) + o(\varepsilon)
\]

where \( \frac{\partial \psi}{\partial \varepsilon}(0, q(t)) = \xi(t, q(t)) \).
In order to prove a fractional Noether’s theorem for Problem 1 we adopt a technique used in [18,28]. The proof is done in two steps: we begin by proving a Noether’s theorem without transformation of the time (without transformation of the independent variable); then, using a technique of time-reparametrization, we obtain Noether’s theorem in its general form.

The action of one parameter group of diffeomorphisms on a Lagrangian allows to define the notion of a symmetry for a fractional functional \( \mathcal{P}_\varepsilon \).

**Definition 8** (Invariance without transforming the time). Functional \( \mathcal{P}_\varepsilon \) is said to be \( \varepsilon \)-invariant under the action of one parameter group of diffeomorphisms \( \Psi_2 = \{ \psi_2(\varepsilon, \cdot) \}_{\varepsilon \in \mathbb{R}} \) of \( \mathbb{R}^n \) if it satisfies for any solution \( q(\cdot) \) of (3)

\[
\int_{t_a}^{t_b} L \left( t, \psi(t), q(t) \right) \frac{\partial \psi_2}{\partial \varepsilon} (0, q(t)) \, dt = \int_{t_a}^{t_b} L \left( t, \psi_2(\varepsilon, q(t)), \frac{d\psi_2}{dt}(\varepsilon, q(t)), C_a \mathcal{D}_r^\varepsilon \psi_2(\varepsilon, q(t)) \right) \, dt
\]  

for any subinterval \( [t_a, t_b] \subseteq [a, b] \).

The next theorem establishes a necessary condition of invariance.

**Theorem 4** (Necessary condition of invariance). If functional \( \mathcal{P}_\varepsilon \) is invariant, in the sense of Definition 8, then

\[
\frac{\partial \psi_2}{\partial \varepsilon} (0, q(t)) \cdot \frac{d}{dt} \partial_3 L \left( t, q(t), \dot{q}(t) \right) + C_a \mathcal{D}_r^\varepsilon q(t) \cdot \frac{d}{dt} \frac{\partial \psi_2}{\partial \varepsilon} (0, q(t)) - \frac{\partial \psi_2}{\partial \varepsilon} (0, q(t)) \cdot C_a \mathcal{D}_r^\varepsilon L \left( t, q(t), \dot{q}(t) \right) = 0.
\]  

**Proof** As condition (7) is valid for any subinterval \( [t_a, t_b] \subseteq [a, b] \), we can get rid of the integral signs in (7). Differentiating this condition with respect to \( \varepsilon \), substituting \( \varepsilon = 0 \), the usual chain rule for the classical derivatives implies

\[
0 = \partial_2 L \left( t, q, \dot{q}, C_a \mathcal{D}_r^\varepsilon q \right) \cdot \frac{\partial \psi_2}{\partial \varepsilon} (0, q) + \partial_3 L \left( t, q, \dot{q}, C_a \mathcal{D}_r^\varepsilon q \right) \cdot \frac{d}{dt} \frac{\partial \psi_2}{\partial \varepsilon} (\varepsilon, q) \bigg|_{\varepsilon=0}
+ \partial_4 L \left( t, q, \dot{q}, C_a \mathcal{D}_r^\varepsilon q \right) \cdot \frac{\partial}{\partial \varepsilon} \left[ C_a \mathcal{D}_r^\varepsilon \psi_2(\varepsilon, q) \right] \bigg|_{\varepsilon=0}.
\]  

Using the definitions and properties of the Caputo fractional derivatives given in Section 2 and the fact \( d/dt \) and \( C_a \mathcal{D}_r^\varepsilon \) act on variable \( t \) and \( \partial/\partial \varepsilon \) on variable \( \varepsilon \), and \( \psi_2(\varepsilon, q) \in C^2 \) with respect to \( \varepsilon \) (see Definition 7), we deduce that

\[
\frac{\partial}{\partial \varepsilon} \left[ \frac{d\psi_2}{dt} (\varepsilon, q) \right] \bigg|_{\varepsilon=0} = \frac{d}{dt} \frac{\partial \psi_2}{\partial \varepsilon} (0, q).
\]
and
\[ \frac{\partial}{\partial \epsilon} \left[ C_a D_t^\alpha \psi_2(\epsilon, q) \right] \bigg|_{\epsilon=0} = C_a D_t^\alpha \frac{\partial \psi_2}{\partial \epsilon}(0, q). \] (11)

Substituting the quantities (10) and (11) into (9), and using the Euler–Lagrange equation (3), the necessary condition of invariance (9) is equivalent to
\[
\frac{\partial \psi_2}{\partial \epsilon}(0, q(t)) \frac{d}{dt} \left[ C_a D_t^\alpha \psi_2(t, q(t)) \right] + \partial_\alpha L \left( t, q(t), \dot{q}(t), C_a D_t^\alpha q(t) \right) \cdot C_a D_t^\alpha \frac{\partial \psi_2}{\partial \epsilon}(0, q(t))
\]
\[- \partial \psi_2 \frac{d}{dt} \left( 0, q(t) \right) \cdot t D_b^\alpha \psi_2 \frac{d}{dt} \left( 0, q(t) \right) = 0.
\]

The proof is completed.

**Theorem 5** (Transfer formula [8]).

Consider functions \( f, g \in C^\infty([a, b]; \mathbb{R}^n) \) and assume the following condition (C): the sequences \( (g^{(k)}(t, q(t)), C_a D_t^\alpha q(t))_{k \in \mathbb{N}_0} \) and \( (f^{(k)}(t, q(t)), C_a D_t^\alpha g(t))_{k \in \mathbb{N}_0} \) converge uniformly to 0 on \([a, b]\). Then, the following equality holds:
\[
g \cdot C_a D_t^\alpha f - f \cdot t D_b^\alpha g = \frac{d}{dt} \left[ \sum_{r=0}^{\infty} \left( (-1)^r g^{(r)} \cdot a I_t^{r+1-\alpha}(f - f(a)) + f^{(r)} \cdot t I_b^{r+1-\alpha}g \right) \right].
\]

**Theorem 6** (Fractional Noether’s theorem without transformation of time).

If functional \( \mathcal{P}_C \) is invariant in the sense of Definition 8 and functions \( \frac{\partial \psi_2}{\partial \epsilon}(0, q) \) and \( \partial \alpha L \) satisfy condition (C) of Theorem 5 then
\[
\frac{d}{dt} \left[ f_2 \cdot \partial \alpha L + \sum_{r=0}^{\infty} \left( (-1)^r \partial_r L \cdot a I_t^{r+1-\alpha}(f_2 - f_2(a)) \right. \right.
\]
\[+ \left. f_2^{(r)} \cdot t I_b^{r+1-\alpha} \partial \alpha L \right] = 0 \quad (12)
\]
along any fractional \( S_b \)-extremal with classical and Caputo derivatives \( q(t), t \in [a, b] \) (Definition 8). In (12) \( f_2 \) denotes \( \frac{\partial \psi_2}{\partial \epsilon}(0, q) \).

**Proof** We combine equation (8) and Theorem 5.

The next definition gives a more general notion of invariance for the integral functional \( \mathcal{P}_C \). The main result of this section, the Theorem 7, is formulated with the help of this definition.
Definition 9 (Invariance of \( P_C \)). Functional \( P_C \) is said to be \( \varepsilon \)-invariant under the action of one parameter group of diffeomorphisms \( \Psi_{i=1,2} = \{ \psi_i(\varepsilon, \cdot) \}_{\varepsilon \in \mathbb{R}} \) if it satisfies for any solution \( q(\cdot) \) of (3)

\[
\int_{t_a}^{t_b} L \left( t, q(t), \dot{q}(t), \frac{C}{\alpha} D^\alpha_q q(t) \right) dt \\
= \int_{\psi_1(\varepsilon, t_a)}^{\psi_1(\varepsilon, t_b)} L \left( \psi_1(\varepsilon, t), \psi_2(\varepsilon, q(t)), \frac{\dot{\psi}_2(\varepsilon, q(t))}{\psi_1(\varepsilon, t)}, \frac{C}{\alpha} D^\alpha_q \psi_2(\varepsilon, q(t)) \right) \dot{\psi}_1(\varepsilon, q(t)) dt
\]

(13)

for any subinterval \([t_a, t_b] \subseteq [a, b] \), where \( \dot{\psi}_i = \frac{d\psi_i}{dt}, i = 1, 2, \bar{a} = \psi_1(\varepsilon, t_a) \) and \( \bar{t} = \psi_1(\varepsilon, t) \).

Our next result gives a general form of Noether’s theorem for fractional problems of the calculus of variations with classical and Caputo derivatives.

Theorem 7 (Fractional Noether’s theorem with classical and Caputo derivatives). If functional \( P_C \) is invariant, in the sense of Definition 9, and functions \( \frac{\partial \psi_2}{\partial \varepsilon}(0, q) \) and \( \frac{\partial}{\partial t} L \) satisfy condition \( (C) \) of Theorem 5 then

\[
\frac{d}{dt} \left[ f_2 \cdot \partial_\lambda L + \sum_{r=0}^{\infty} (-1)^r \partial_\lambda L^{(r)} \cdot \partial_\lambda f_2(r+1-\alpha) \right] \\
+ \frac{f_2^{(r)}}{t_i} \cdot I^{r+1-\alpha}_t \partial_\lambda \left( L - \dot{q} \cdot \partial_\lambda L - \alpha \partial_\lambda L \cdot \frac{C}{\alpha} D^\alpha_q q \right) = 0
\]

(14)

along any fractional \( S_\lambda \)-extremal with classical and Caputo derivatives \( q(\cdot), t \in [a, b] \). Here and the sequel \( f_2 \) and \( \tau \) denote \( \frac{d\psi_2}{dt}(0, q) \) and \( \frac{d\psi_1}{dt}(0, t) \), respectively.

Proof Our proof is an extension of the method used in [28]. For that we reparametrize the time (the independent variable \( t \)) by the Lipschitz transformation

\[
[a, b] \ni t \mapsto f(\lambda) \in [\sigma_a, \sigma_b]
\]

that satisfies

\[
t_\sigma' = \frac{dt(\sigma)}{d\sigma} = f(\lambda) = 1 \text{ if } \lambda = 0.
\]

(15)

Functional \( P_C \) is reduced, in this way, to an autonomous functional:

\[
\bar{I}[t(\cdot), q(t(\cdot))] \\
= \int_{\sigma_a}^{\sigma_b} L \left( t(\sigma), q(t(\sigma)), \dot{q}(t(\sigma)), \frac{C}{\alpha} D^\alpha_{t(\sigma)} q(t(\sigma)) \right) t_\sigma' d\sigma
\]

(16)
where \( t(\sigma_a) = a \) and \( t(\sigma_b) = b \). Using the definitions and properties of fractional derivatives given in Section 2, we get successively that

\[
\frac{C}{\sigma_a} D_t^\alpha q(t(\sigma)) = \frac{1}{\Gamma(1-\alpha)} \int_{\sigma_a}^{\sigma_b} (\sigma f(\lambda) - \theta)^{-\alpha} \frac{d}{d\theta} q(\theta f^{-1}(\lambda)) d\theta
\]

\[
= \frac{(t'_\sigma)^{-\alpha}}{\Gamma(1-\alpha)} \int_{\sigma_a}^{\sigma_b} (\sigma - s)^{-\alpha} \frac{d}{ds} q(s) ds
\]

\[
= (t'_\sigma)^{-\alpha} \frac{C}{\chi} D_\sigma^\alpha q(\sigma), \quad \left( \chi = \frac{a}{(t'_\sigma)^2} \right).
\]

We then have

\[
\bar{I}[t(\cdot), q(t(\cdot))]
\]

\[
= \int_{\sigma_a}^{\sigma_b} L\left( t(\sigma), q(t(\sigma)), \frac{q'}{t'_\sigma}, (t'_\sigma)^{-\alpha} \frac{C}{\chi} D_\sigma^\alpha q(\sigma) \right) t'_\sigma d\sigma
\]

\[
= \int_{\sigma_a}^{\sigma_b} \tilde{L}_f \left( t(\sigma), q(t(\sigma)), q'_\sigma, t'_\sigma, \frac{C}{\chi} D_\sigma^\alpha q(t(\sigma)) \right) d\sigma
\]

\[
= \int_a^b L \left( t, q(t), q(t), \frac{C}{\chi} D_t^\alpha q(t) \right) dt
\]

\[
= I[q(\cdot)].
\]

If the integral functional \( L_{C,\tau} \) is invariant in the sense of Definition 9, then the integral functional \( I^{(15)} \) is invariant in the sense of Definition 8. It follows from Theorem 6 that

\[
\frac{d}{dt} \left[ f_2 \cdot \partial_3 \bar{L}_f + \tau \frac{\partial}{\partial t'_\sigma} \bar{L}_f + \sum_{r=0}^{\infty} (-1)^r \partial_3 \bar{L}_f^{(r)} \cdot \partial_t^{r+1-\alpha} (f_2 - f_2(a)) \right]
\]

\[
+ \bar{L}_f^{(r)} \cdot t \partial_t^{r+1-\alpha} \partial_3 \bar{L}_f \right) = 0. \quad (17)
\]

For \( \lambda = 0 \), the condition \( 15 \) allow us to write that

\[
\frac{C}{\chi} D_\sigma^\alpha q(\sigma) = \frac{C}{a} D_t^\alpha q(t)
\]

and, therefore, we get

\[
\begin{cases}
\partial_3 \bar{L}_f = \partial_3 L, \\
\partial_3 \bar{L}_f = \partial_3 L
\end{cases}
\]
and
\[
\frac{\partial}{\partial t} L_f = L + \partial_3 L_f \cdot \frac{q}{\tau} \partial_3 \frac{\cdot q}{\tau} + \partial_4 L_f
\]
\[
\times \frac{\partial}{\partial \tau} \left[ \frac{(t')^\alpha}{\Gamma(1-\alpha)} \int_{(t') \sigma}^\sigma (\sigma - s)^{-\alpha} \frac{ds}{d\sigma} \right] \frac{q}{\tau}
\]
\[
= -\dot{q} \cdot \partial_3 L - \alpha \partial_4 L \cdot \frac{\cdot q}{\tau} + L.
\]

We obtain (14) substituting (18) and (19) into equation (17).

Theorem 7 gives a new and interesting result for autonomous fractional variational problems. Let us consider an autonomous fractional variational problem i.e., the case when function \( L \) of \((P.C)\) do not depends explicitly on the independent variable \( t \):
\[
I[q(\cdot)] = \int_a^b L \left( q(t), \dot{q}(t), C_{\alpha} D_t^\alpha q(t) \right) dt \rightarrow \min.
\]

**Corollary 1** For the autonomous fractional problem (20) one has
\[
\frac{d}{dt} \left( L - \dot{q} \cdot \partial_3 L - \alpha \partial_4 L \cdot C_{\alpha} D_t^\alpha q \right) = 0
\]
along any fractional \( S_h \)-extremal with classical and Caputo derivatives \( q(\cdot) \), \( t \in [a,b] \).

**Proof** As the Lagrangian \( L \) does not depend explicitly on the independent variable \( t \), we can easily see that (20) is invariant under translation of the time variable: the condition of invariance (13) is satisfied with \( \psi_1(\varepsilon,t) = t + \varepsilon \) and \( \psi_2(\varepsilon,q(t)) = q(t) \). Indeed, given that \( \frac{d}{dt}(\varepsilon,q(t)) = 1 \), \( \tau = 1 \) and \( f_2 = 0 \), the invariance condition (13) is verified if \( C_{\alpha} D_t^\alpha \psi_2(\varepsilon,q(t)) = C_{\alpha} D_t^\alpha q(t) \). This is true because
\[
C_{\alpha} D_t^\alpha \psi_2(\varepsilon,q(t)) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\theta)^{-\alpha} \frac{d}{d\theta} \psi_2(\varepsilon,q(\theta)) d\theta
\]
\[
= \frac{1}{\Gamma(1-\alpha)} \int_{a+\varepsilon}^{t+\varepsilon} (t+\varepsilon-\theta)^{-\alpha} \frac{d}{d\theta} \psi_2(\varepsilon,q(\theta)) d\theta
\]
\[
= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \frac{d}{ds} \psi_2(\varepsilon,q(s+\varepsilon)) ds
\]
\[
= C_{\alpha} D_t^\alpha \psi_2(\varepsilon,q(t + \varepsilon)) = C_{\alpha} D_t^\alpha \psi_2(\varepsilon,q(t))
\]
\[
= C_{\alpha} D_t^\alpha q(t).
\]

**Remark 7** If \( \alpha = 1 \) Problem (20) is reduced to the classical problem of the calculus of variations,
\[
I[q(\cdot)] = \int_a^b F(q(t), \dot{q}(t)) \rightarrow \min
\]
with \( F(q(t), \dot{q}(t)) := L(q(t), \dot{q}(t), \ddot{q}(t)) \), and one obtains from Corollary 1 the famous conservation of energy in classical mechanics:

\[
F - \dot{q} \frac{\partial F}{\partial \dot{q}} = \text{constant}
\]

along any solutions of the equations (5).

4 Noether’s theorem for the linear friction problem

In order to formulate an action principle for dissipative systems free from the problems found in the original Riewe’s approach, in a recent work [36] it was proposed that the equation of motion for dissipative systems can be obtained by taking the limit \( a \to b \) with \( t = a + (b - a)/2 = (a + b)/2 \) in the extremal of the action

\[
I[q(\cdot)] = \int_a^b L(q(t), \dot{q}(t), \frac{C}{a} D^\frac{1}{2} q(t)) \, dt. \tag{21}
\]

Furthermore, it was proposed a quadratic Lagrangian for a particle under a frictional force proportional to the velocity as [36]

\[
L(q(t), \dot{q}(t), \frac{C}{a} D^\frac{1}{2} q(t)) = \frac{1}{2} m (\dot{q}(t))^2 - U(q(t)) + \frac{\gamma}{2} \left( \frac{C}{a} D^\frac{1}{2} q(t) \right)^2, \tag{22}
\]

where the three terms in (22) represent the kinetic energy, potential energy, and the fractional linear friction energy, respectively. Since the equation of motion is obtained in the limit \( a \to b \), if we consider the last term in (22) up to first order in \( \Delta t = b - a \) we get:

\[
\frac{\gamma}{2} \left( \frac{C}{a} D^\frac{1}{2} q \right)^2 \approx \frac{\gamma}{2} \left( \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \right)^2 (\dot{q})^2 \Delta t \approx \frac{2}{\pi} \gamma \dot{q} \Delta q, \tag{23}
\]

that coincide, apart from the multiplicative constant \( 2/\pi \), with the work from the frictional force \( \gamma \dot{q} \) in the displacement \( \Delta q \approx \dot{q} \Delta t \). This additional constant is a consequence of the use of fractional derivatives in the Lagrangian and do not appears in the equation of motion after we apply the action principle [36]. Furthermore, the Lagrangian (22) is physical in the sense it provide us with physically meaningful relations for the momentum and the canonical Hamiltonian [36]

\[
H = p \dot{q} + p_\frac{\gamma}{2} \frac{C}{a} D^\frac{1}{2} q - L = \frac{1}{2} m (\dot{q})^2 + U(q) + \frac{\gamma}{2} \left( \frac{C}{a} D^\frac{1}{2} q \right)^2, \tag{24}
\]

where

\[
p = \frac{\partial L}{\partial \dot{q}} = m \dot{q}, \quad p_\frac{\gamma}{2} = \frac{\partial L}{\partial \frac{C}{a} D^\frac{1}{2} q} = \gamma \frac{C}{a} D^\frac{1}{2} q. \tag{25}
\]

From (23) and (24) we can see that the Lagrangian (22) is physical in the sense it provides us a correct relation for the momentum \( p_\frac{\gamma}{2} = m \dot{q} \), and a physically meaningful Hamiltonian (it is the sum of all energies). Furthermore,
the additional fractional momentum $p_{\mathcal{S}} = \gamma C D_t^{\frac{\alpha}{2}} q$ goes to zero when we take the limit $a \to b$ [36].

Finally, the equation of motion for the particle is obtained by inserting our Lagrangian \( (22) \) into the Euler-Lagrange equation \( (3) \),

$$m \ddot{q} - \gamma _t D_t^{\frac{\alpha}{2}} q = F (q),$$  \hspace{1cm} (26)

where $F (q) = - \frac{d}{dq} U (q)$ is the external force. By taking the limit $a \to b$ with $t = (a + b)/2$ and using the approximation $C D_t^{\frac{\alpha}{2}} q \approx - \gamma D_t^{\frac{\alpha}{2}} q$ [36] we obtain the equation of motion for a particle under a linear friction force

$$m \ddot{q} + \gamma \dot{q} = F (q).$$  \hspace{1cm} (27)

Finally, Noether's invariant theorems states that if an action remains invariant with respect to a group of transformations, such transformations leads to a corresponding conservation law. Since for the Lagrangian \( (22) \) the linear friction is an autonomous fractional problem, Corollary [11] gives us

$$\frac{d}{dt} \left( L - p_{\mathcal{S}} - \frac{1}{2} p_{\mathcal{S}} C D_t^{\frac{\alpha}{2}} q \right) = \frac{d}{dt} \left( \frac{1}{2} p_{\mathcal{S}} C D_t^{\frac{\alpha}{2}} q - H \right) = \frac{d}{dt} \left( \frac{\gamma}{2} C D_t^{\frac{\alpha}{2}} q \right)^2 - H = 0.$$  \hspace{1cm} (28)

From \( (28) \) it is easy to see that the Hamiltonian for a particle under frictional forces is not a conserved quantity, as expected. The Hamiltonian and consequently the total energy of the system is only time locally conserved, when we consider only very short time intervals by taking the limit $a \to b$. In this last case we have $C D_t^{\frac{\alpha}{2}} q \to 0$ and \( (28) \) reduces to $\frac{dH}{dt} = 0$.

5 Fractional optimal control problems with classical and Caputo derivatives

We now adopt the Hamiltonian formalism in order to generalize the Noether type results found in [14, 19, 58] for the more general context of fractional optimal control problems with classical and Caputo derivatives. For this, we make use of our Noether's Theorem [7] and the standard Lagrange multiplier technique (cf. [14]). The fractional optimal control problem with classical and Caputo derivatives is introduced, without loss of generality, in Lagrange form as in [4, 48]:

$$I [q (\cdot), u (\cdot), \mu (\cdot)] = \int_a^b L (t, q (t), u (t), \mu (t)) \, dt \longrightarrow \min$$  \hspace{1cm} (29)

subject to the differential system

$$\dot{q} (t) = \varphi (t, q (t), u (t)),$$  \hspace{1cm} (30)

$$C D_t^{\alpha} q (t) = \rho (t, q (t), \mu (t))$$  \hspace{1cm} (31)
and initial condition
\[ q(a) = q_a. \]  

The Lagrangian \( L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R} \), the velocity vector \( \varphi : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and the fractional velocity vector \( \rho : [a, b] \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n \) are assumed to be functions of class \( C^1 \) with respect to all their arguments. We also assume, without loss of generality, that \( 0 < \alpha < 1 \). In conformity with the calculus of variations, we are considering that the control functions \( u(\cdot) \) and \( \mu(\cdot) \) take values on an open set of \( \mathbb{R}^m \) and \( \mathbb{R}^d \), respectively.

**Remark 8** The fractional functional of the calculus of variations with classical and Caputo derivatives \( \{PC\} \) is obtained from (29)–(31) choosing \( \varphi(t, q, u) = u \) and \( \rho(t, q, \mu) = \mu \).

### 5.0.3 Fractional Pontryagin Maximum Principle

In the fifties of the twentieth century, L.S. Pontryagin and his collaborators proved the main necessary optimality condition for optimal control problems: the famous Pontryagin Maximum Principle [49].

In this subsection we prove a fractional maximum principle with the help of optimality conditions (3).

**Definition 10** (Process with classical and Caputo derivatives). An admissible triplet \((q(\cdot), u(\cdot), \mu(\cdot))\) that satisfies the control system (30)–(31) of the optimal control problem (29)–(32), \( t \in [a, b] \), is said to be a process with classical and Caputo derivatives.

For convenience of notation, we introduce the following operator:
\[
[q, u, \mu, p, p_\alpha](t) = (t, q(t), u(t), \mu(t), p(t), p_\alpha(t)).
\]

**Remark 9** In mechanics, \( p(\cdot) \) and \( p_\alpha(\cdot) \) correspond to the generalized momentum related to \( \dot{q}(\cdot) \) and \( \mathcal{C}^a D_t^\alpha q(\cdot) \), respectively. In the language of optimal control \( p(\cdot) \) and \( p_\alpha(\cdot) \) are called the adjoint variables.

**Theorem 8** (Fractional Pontryagin Maximum Principle). If \((q(\cdot), u(\cdot), \mu(\cdot))\) is a process for problem (29)–(32), in the sense of Definition, then there exists co-vector functions \( p(\cdot) \in PC^1([a, b]; \mathbb{R}^n) \) and \( p_\alpha(\cdot) \in PC^1([a, b]; \mathbb{R}^n) \) such that for all \( t \in [a, b] \) the quadruple \((q(\cdot), u(\cdot), p(\cdot), p_\alpha(\cdot))\) satisfies the following conditions:

\begin{align*}
& \text{– the Hamiltonian system} \\
& \partial_2 H[q, u, \mu, p, p_\alpha](t) = \dot{q}(t), \\
& \partial_5 H[q, u, \mu, p, p_\alpha](t) = \mathcal{C}^a D_t^\alpha q(t), \\
& \partial_3 H[q, u, \mu, p, p_\alpha](t) = -\dot{p}(t) + t D_t^\alpha p_\alpha(t) ;
\end{align*}
– the stationary conditions

\[
\begin{aligned}
\{ & \partial_t \mathcal{H}[q, u, \mu, p, p_\alpha](t) = 0, \\
& \partial_t \mathcal{H}[q, u, \mu, p, p_\alpha](t) = 0; \\
& \partial_t \mathcal{H}[q, u, \mu, p, p_\alpha](t) = 0; \\
& \partial_t \mathcal{H}[q, u, \mu, p, p_\alpha](t) = 0,
\end{aligned}
\]  

(34)

where the Hamiltonian \( \mathcal{H} \) is given by

\[
\mathcal{H}[t, q, u, \mu, p, p_\alpha] = L(t, q(t), u(t), \mu(t)) + p(t) \cdot \varphi(t, q(t), u(t)) + p_\alpha(t) \cdot \rho(t, q(t), \mu(t)).
\]

(35)

Proof Minimizing (29) subject to (30)–(31) is equivalent, by the Lagrange multiplier rule, to minimize the augmented functional \( J[q(\cdot), u(\cdot), \mu(\cdot), p(\cdot), p_\alpha(\cdot)] \) defined by

\[
J[q(\cdot), u(\cdot), \mu(\cdot), p(\cdot), p_\alpha(\cdot)] = \int_a^b \left[ \mathcal{H}[q, u, \mu, p, p_\alpha](t) \\
- p(t) \cdot \dot{q}(t) - p_\alpha(t) \cdot C^\alpha D_t^\alpha q(t) \right] dt
\]

(36)

with \( \mathcal{H} \) given by (35).

Theorem 8 is proved applying the necessary optimality condition (3) to the augmented functional (36): we only prove the one of optimality equations of Theorem 8 (the reasoning is similar for the other equations)

\[
\partial_2 L[q, u, \mu, p, p_\alpha](t) - d \frac{\partial L}{\partial \dot{q}}[q, u, \mu, p, p_\alpha](t) \\
+ t D_0^\alpha \frac{\partial L}{\partial C^\alpha D_t^\alpha q}[q, u, \mu, p, p_\alpha](t) = 0
\]

\[
\Leftrightarrow \partial_2 \mathcal{H}[q, u, \mu, p, p_\alpha](t) = -\dot{\rho} + t D_0^\alpha p_\alpha
\]

where

\[
L[q, u, \mu, p, p_\alpha](t) = \mathcal{H}[q, u, \mu, p, p_\alpha](t) - p(t) \cdot \dot{q}(t) - p_\alpha(t) \cdot C^\alpha D_t^\alpha q(t).
\]

Definition 11 (Pontryagin \( S_\alpha \)-extremal with classical and fractional derivatives). A tuple \( (q(\cdot), u(\cdot), \mu(\cdot), p(\cdot), p_\alpha(\cdot)) \) satisfying Theorem 8 is called a Pontryagin \( S_\alpha \)-extremal with classical and Caputo derivatives.

Remark 10 For problems of the calculus of variations with classical and Caputo derivatives, one has \( \varphi(t, q, u) = u \) and \( \rho(t, q, \mu) = \mu \) (Remark 8). Therefore, \( \mathcal{H} = L + p \cdot u + p_\alpha \cdot \mu \). From the Hamiltonian system (33) we get

\[
\begin{aligned}
\{ & u = \dot{q}, \\
& \mu = C^\alpha D_t^\alpha q, \\
& \partial_2 L = -\dot{\rho} + t D_0^\alpha p_\alpha
\end{aligned}
\]

(37)
and from the stationary conditions \(34\)

\[
\begin{aligned}
\partial_3 \mathcal{H} &= 0 \iff \partial_3 L = -p \Rightarrow \frac{d}{dt} \partial_3 L = -\dot{p}, \\
\partial_4 \mathcal{H} &= 0 \iff \partial_4 L = -p_\alpha \Rightarrow tD^\alpha_0 \partial_4 L = -tD^\alpha_0 p_\alpha.
\end{aligned}
\] (38)

Substituting the quantities \(38\) into \(37\), we arrive to the Euler–Lagrange equations with classical and Caputo derivatives \(3\).

5.0.4 Noether’s theorem for fractional optimal control problems

The notion of variational invariance for \(29\)–\(31\) is defined with the help of the augmented functional \(36\).

**Definition 12** (Variational invariance of \(29\)–\(31\)). The augmented functional \(36\) is said to be \(\varepsilon\)-invariant under the action of one parameter group of diffeomorphisms

\[
\Psi_{\varepsilon=1,...,6} = \{ \psi_i(\varepsilon, \cdot) \}_{\varepsilon \in \mathbb{R}} \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n
\]

if it satisfies for any Pontryagin \(S_h\)-extremal with classical and Caputo derivatives

\[
\begin{aligned}
(\mathcal{H}(\psi_1(\varepsilon, t), \psi_2(\varepsilon, q(t)), \psi_3(\varepsilon, u(t)), \psi_4(\varepsilon, \mu(t)), \psi_5(\varepsilon, p(t)), \psi_6(\varepsilon, p_\alpha(t))) \\
- \psi_3(\varepsilon, p(t)) \cdot \frac{\psi_2(\varepsilon, q(t))}{\psi_1(\varepsilon, t)} - \psi_6(\varepsilon, p_\alpha) \cdot \frac{C}{\alpha} D^\alpha_0 \psi_2(\varepsilon, q(t))) \psi_1(\varepsilon, t)
\end{aligned}
\]

\[
= \left( \mathcal{H}[q, u, \mu, p_\alpha](t) - p(t) \cdot \dot{q}(t) - p_\alpha(t) \cdot \frac{C}{\alpha} D^\alpha_0 q(t) \right)
\] (39)

for any subinterval \([t_a, t_b] \subseteq [a, b]\).

In \([8]\), the author proved the Noether’s theorem without transformation of the independent variable \(t\) for the following fractional control problem:

\[
I[q(\cdot), u(\cdot)] = \int_a^b L(t, q(t), u(t)) \, dt \rightarrow \min
\]

\[
\frac{C}{\alpha} D^\alpha_0 q(t) = \varphi(t, q(t), u(t))
\]

In this case he only obtain the conservation of momentum.

Next theorem provides an extension of Noether’s theorem in general form to the wider fractional context of optimal control problems with classical and Caputo derivatives.

**Theorem 9** (Noether’s theorem in Hamiltonian form for optimal control problems with classical and Caputo derivatives). If \(29\)–\(31\) is variationally invariant, in the sense of Definition 12 and functions \(f_2\) and \(p_\alpha\) satisfy condition
(C) of Theorem 5, then

\[
\frac{d}{dt} \left[ -f_2 \cdot p - \sum_{r=0}^{\infty} (-1)^r p_\alpha \cdot _a I_t^{r+1-\alpha}(f_2 - f_2(a)) \\
+ f_2 \cdot _t I_t^{r+1-\alpha} p_\alpha \right] + \tau \left( \mathcal{H} - (1 - \alpha) p_\alpha \cdot C_a D^\alpha t q \right) = 0 \tag{40}
\]

along any Pontryagin $S_h$-extremal with classical and Caputo derivatives (Definition 11).

**Proof** The fractional conservation law (40) is obtained by applying Theorem 7 to the equivalent functional (36).

Like Theorem 7, Theorem 9 also gives an interesting result for autonomous fractional problems. Let us consider an autonomous fractional optimal control problem, i.e., the case when functions $L, \varphi$ and $\rho$ of (29)–(31) do not depend explicitly on the independent variable $t$:

\[
I[q(\cdot),u(\cdot),\mu(\cdot)] = \int_a^b L[q(t),u(t),\mu(t)] \, dt \longrightarrow \min, \tag{41}
\]

\[
\dot{q}(t) = \varphi(q(t),u(t)), \tag{42}
\]

\[
C_a D^\alpha t q(t) = \rho(q(t),u(t)). \tag{43}
\]

**Corollary 2** For the autonomous fractional problem (41)–(43) one has

\[
\frac{d}{dt} \left[ \mathcal{H}(q(t),u(t),\mu(t),p(t),p_\alpha(t)) - (1 - \alpha) p_\alpha(t) \cdot C_a D^\alpha t q(t) \right] = 0 \tag{44}
\]

along any Pontryagin $S_h$-extremal with classical and Caputo derivatives $(q(\cdot),u(\cdot),\mu(\cdot),p(\cdot),p_\alpha(\cdot))$.

**Proof** The proof is similar of Corollary 1 to taking into account the Definition 12 applied to the Problem (41)–(43).

The Corollary 2 shows that in contrast with the classical autonomous problem of optimal control, for (41)–(43) the Hamiltonian $\mathcal{H}$ does not define a conservation law. Instead of the classical equality $\frac{d}{dt} (\mathcal{H}) = 0$, we have

\[
\frac{d}{dt} \left[ \mathcal{H} + (\alpha - 1) p_\alpha \cdot C_a D^\alpha t q \right] = 0, \tag{45}
\]

i.e., conservation of the Hamiltonian $\mathcal{H}$ plus a quantity that depends on the fractional order $\alpha$ of differentiation. This seems to be explained by violation of the homogeneity of space-time caused by the fractional derivatives, when $\alpha \neq 1$. If $\alpha = 1$, then we obtain from (45) the classical result: the Hamiltonian $\mathcal{H}$ is preserved along all the Pontryagin extremals.
6 Conclusion

In the present work, we obtained a generalization of the Noether’s theorem for Lagrangians depending on mixed classical and Caputo derivatives that can be used to obtain constants of motion for dissipative systems. The Noether’s theorem of calculus of variation is one of the most important theorems for physics in the 20th century. It is well known that all conservations laws in mechanics, e.g., conservation of energy or conservation of momentum, are directly related to the invariance of the action under a family of transformations. However, the classical Noether’s theorem can not yields informations about constants of motion for non-conservative systems since it is not possible to formulate physically meaningful Lagrangians for this kind of systems in classical calculus of variation. On the other hand, in recent years the fractional calculus of variation within Lagrangians depending on fractional derivatives has emerged as an elegant alternative to study non-conservative systems. In this context, the generalization of the Noether’s theorem for the fractional calculus of variation is fundamental to investigate the action symmetries for non-conservative systems. As an example of application to non-conservative systems, we study the problem of a particle under a frictional force. In addition, we also obtained Noether’s conditions for the fractional optimal control problem.

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