Abstract

The Lottery Ticket Hypothesis is a conjecture that every large neural network contains a subnetwork that, when trained in isolation, achieves comparable performance to the large network. An even stronger conjecture has been proven recently: Every sufficiently overparameterized network contains a subnetwork that, even without training, achieves comparable accuracy to the trained large network. This theorem, however, relies on a number of strong assumptions and guarantees a polynomial factor on the size of the large network compared to the target function. In this work, we remove the most limiting assumptions of this previous work while providing significantly tighter bounds: the overparameterized network only needs a logarithmic factor (in all variables but depth) number of neurons per weight of the target subnetwork.

1 Introduction

The recent success of neural network (NN) models in a variety of tasks, ranging from vision [Khan et al., 2020] to speech synthesis [van den Oord et al., 2016] to playing games [Schrittwieser et al., 2019, Ebendt and Drechsler, 2009], has sparked a number of works aiming to understand how and why they work so well. Proving theoretical properties for neural networks is quite a difficult task, with challenges due to the intricate composition of the functions they implement and the high-dimensional regimes of their training dynamics. The field is vibrant but still in its infancy, many theoretical tools are yet to be built to provide guarantees on what and how NNs can learn. A lot of progress has been made towards understanding the convergence properties of NNs (see e.g., Allen-Zhu et al. [2019], Zou and Gu [2019] and references therein). The fact remains that training and deploying deep NNs has a large cost [Livni et al., 2014], which is problematic. To avoid this problem, one could stick to a small network size. However, it is becoming evident that there are benefits to using oversized networks, as the literature on overparametrized models [Ma et al., 2018] points out. Another solution, commonly used in practice, is to prune a trained network to reduce the size and hence the cost of prediction/deployment. While missing theoretical guarantees, experimental works show that pruning can considerably reduce the network size without sacrificing accuracy.

The influential work of Frankle and Carbin [2019] has pointed out the following observation: a) train a large network for long enough and observe its performance on a dataset, b) prune it substantially to reveal a much smaller subnetwork with good (or better) performance, c) reset the weights of the subnetwork to their original values and remove the rest, d) retrain the subnetwork in isolation; then the subnetwork reaches the same test performance as the large network, and trains faster. Frankle and Carbin [2019] thus conjecture that every successfully trained network
contains a much smaller subnetwork that, when trained in isolation, has comparable performance
to the large network, without sacrificing computing time. They name this phenomenon the
Lottery Ticket Hypothesis, and a ‘winning ticket’ is the subnetwork in question.

Ramanujan et al. [2019] went even further by observing that if the network architecture is
large enough, then it contains a smaller network that, even without any training, has comparable
accuracy to the trained large network. They support their claim with empirical results using
a new pruning algorithm, and even provide a simple asymptotic justification that we rephrase
here: Starting from the inputs and progressing toward the outputs, for any neuron of the target
network, sample as many neurons as required until one calculates a function within small error of
the target neuron; then, after pruning the unnecessary neurons, the newly generated network will
be within some small error of the target network. Interestingly, Ulyanov et al. [2018] pointed
out that randomly initialized but untrained ConvNets already encode a great deal of the image
statistics required for restoration tasks such as de-noising and inpainting, and the only prior
information needed to do them well seems to be contained in the network structure itself, since
no part of the network was learned from data.

Very recently, building upon the work of Ramanujan et al. [2019], Malach et al. [2020] proved
a significantly stronger version of the “pruning is all you need” conjecture, moving away from
asymptotic results to non-asymptotic ones: With high probability, any target network of ℓ layers
and n neurons per layer can be approximated within ε accuracy by pruning a larger network
whose size is polynomial in the size of the target network. To prove their bounds, Malach et al.
[2020] make assumptions about the norms of the inputs and of the weights. This polynomial
bound already tells us that unpruned networks contain many ‘winning tickets’ even without
training. Then could the most important task of gradient descent be pruning?

Building on top of these previous works, we aim at providing stronger theoretical guarantees
still based on the motto that “pruning is all you need” but hoping to provide further insights into
how ‘winning tickets’ may be found. In this work we relax the aforementioned assumptions while
greatly strengthening the theoretical guarantees by improving from polynomial to logarithmic
order in all variables except the depth, for the number of samples required to approximate one
target weight.

How this paper is organized. After some notation (Section 2) and the description of the
problem (Section 3), we provide a general approximation propagation lemma (Section 4), which
shows the effect of the different variables on the required accuracy. Next, we show how to
construct the large, fully-connected ReLU network in Section 5 identical to Malach et al. [2020],
except that weights are sampled from a hyperbolic weight distribution instead of a uniform one.
We then give our theoretical results in Section 6, showing that only \( O(\log(\ell n_{\text{max}}/\varepsilon)) \) neurons
per target weight are required under some similar conditions as the previous work (with ℓ layers,
n_{\text{max}} neurons per layer and ε accuracy) or \( O(\ell \log(n_{\text{max}}/\varepsilon)) \) (with some other dependencies inside
the log) if these conditions are relaxed. For completeness, the most important technical result is
included in Section 7; other technical results, a table of notation, and further ideas can be found
in Appendix C.

2 Notation and definitions

A network architecture \( A(\ell, n, \sigma) \) is described by a positive integer \( \ell \) corresponding to the
number of fully connected feed-forward layers, and a list of positive integers \( n = (n_0, n_1, \ldots, n_\ell) \)
corresponding to the profile of widths, where \( n_i \) is the number of neurons in layer \( i \in [\ell] = \{1, \ldots, \ell\} \)
and \( n_0 \) is the input dimension, and a list of activation functions \( \sigma = (\sigma_1, \ldots, \sigma_\ell) \)—all neurons
in layer $i$ use the activation function $\sigma_i$. Networks from the architecture $A(\ell, n, \sigma)$ implement functions from $\mathbb{R}^{n_0}$ to $\mathbb{R}^{n_1}$ that are obtained by successive compositions: $\mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_1} \rightarrow \cdots \rightarrow \mathbb{R}^{n_{\ell}}$.

Let $F$ be a target network from architecture $A(\ell, n, \sigma)$. The composition of such $F$ is as follows: Each layer $i \in [\ell]$ has a matrix $W_i^* \in [-u_{\text{max}}, u_{\text{max}}]^{n_{i-1} \times n_i}$ of connection weights, and an activation function $\sigma_i$, such as tanh, the logistic sigmoid, ReLU, Heaviside, etc. The network takes as input a vector $x \in \mathcal{X} \subset \mathbb{R}^{n_0}$ where for example $\mathcal{X} = \{-1, 1\}^{n_0}$ or $\mathcal{X} = [0, x_{\text{max}}]^{n_0}$, etc. In layer $i$, the neuron $j$ with in-coming weights $W_{i,j}^*$ calculates $f_{i,j}(y) = \sigma_i(W_{i,j}^* y)$, where $y \in \mathbb{R}^{n_{i-1}}$ is usually the output of the previous layer. Note that $W_{i,j}^*$ is the $j$-th row of the matrix $W_i^*$. The vector $f_i(y) = [f_{i,1}(y), \ldots, f_{i,n_i}(y)]^\top \in \mathbb{R}^{n_i}$ denotes the output of the whole layer $i$ when it receives $y \in \mathbb{R}^{n_{i-1}}$ from the previous layer. Furthermore, we recursively define $F_i(x)$ for a given network input $x \in \mathcal{X}$ by setting $F_0(x) = x$, and for $i \in [\ell]$ then $F_i(x) = f_i(F_{i-1}(x))$. The output of neuron $j \in [n_i]$ in layer $i$ given network input $x$ is $F_{i,j}(x) = f_{i,j}(F_{i-1}(x))$. The network output is $F(x) = F_{\ell}(x)$.

For an activation function $\sigma(.)$, let $\lambda$ be its Lipschitz factor (when it exists), that is, $\lambda$ is the smallest real number such that $|\sigma(x) - \sigma(y)| \leq \lambda |x - y|$ for all $(x, y) \in \mathbb{R}^2$. For ReLU and tanh we have $\lambda = 1$, and for the logistic sigmoid, $\lambda = 1/4$. Let $\lambda_i$ be the $\lambda$ corresponding to the activation function $\sigma_i$ of all the neurons in layer $i$, and let $\lambda_{\text{max}} = \max_{\ell \in [\ell]} \lambda_i$.

Define $n_{\text{max}} = \max_{\ell \in [\ell]} n_{\ell}$ to be the maximum number of neurons per layer. The total number of connection weights in the architecture $A(\ell, n, \sigma)$ is denoted $N^*$, and we have $N^* \leq n_{\text{max}}^2$. For all $x \in \mathcal{X}$, let $F_{\text{max}}(x) = \max_{\ell \in [\ell]} \max_{j \in [n_{\ell - 1}]} |F_{\ell - 1,j}(x)|$ be the maximum activation at any layer of a target network $F$, including the network inputs but excluding the network outputs. We also write $F_{\text{max}}(\mathcal{X}) = \sup_{x \in \mathcal{X}} F_{\text{max}}(x)$; when $\mathcal{X}$ is restricted to the set of inputs of interest (not necessarily the set of all possible inputs) such as a particular dataset, we expect $F_{\text{max}}(\mathcal{X})$ to be bounded by a small constant in most if not all cases. For example, $F_{\text{max}}(\mathcal{X}) \leq 1$ for a neural network with only sigmoid activations and inputs in $[-1, 1]^{n_0}$. For ReLU activations, $F_{\text{max}}(\mathcal{X})$ can in principle grow as fast as $(n_{\text{max}} u_{\text{max}})^\ell$, but since networks with sigmoid activations are universal approximators, we expect that for all functions that can be approximated with a sigmoid network there is a ReLU network calculating the same function with $F_{\text{max}}(\mathcal{X}) = O(1)$.

The large network $G$ has an architecture $A(\ell', n', \sigma')$, possibly wider and deeper than the target network $F$. The pruned network $\hat{G}$ is obtained by pruning (setting to 0) many weights of the large network $G$. For each layer $i \in [\ell']$, and each pair of neurons $j_1 \in [n_i]$, and $j_2 \in [n_{i - 1}]$, for the weight $w_{i,j_1,j_2}$ of the large network $G$ there is a corresponding mask $b_{i,j_1,j_2} \in \{0, 1\}$ such that the weight of the pruned network $\hat{G}$ is $w_{i,j_1,j_2}' = b_{i,j_1,j_2} w_{i,j_1,j_2}$. The pruned network $\hat{G}$ will have a different architecture from $F$, but at a higher level (by grouping some neurons together) it will have the same ‘virtual’ architecture, with virtual weights $\hat{W}$. As in previous theoretical work, we consider an ‘oracle’ pruning procedure, as our objective is to understand the limitations of even the best pruning procedures.

For a matrix $M \in [-c, c]^{n \times m}$, we denote by $\|M\|_2$ its spectral norm, equal to its largest singular value, and its max-norm is $\|M\|_{\text{max}} = \max_{i,j} |M_{i,j}|$. In particular, for a vector $v$, we have $\|v\|_2 \leq \|M\|_2 \|v\|_2$ and $\|M\|_{\text{max}} \leq \|M\|_2 \leq \sqrt{mn} \|M\|_{\text{max}}$ and also $\|M\|_{\text{max}} \leq c$. This means for example that $\|M\|_2 \leq 1$ is a stronger condition than $\|M\|_{\text{max}} \leq 1$.

3 Objective

Objective: Given an architecture $A(\ell, n, \sigma)$ and accuracy $\epsilon > 0$, construct a network $G$ from some larger architecture $A(\ell', n', \sigma')$, such that if the weights of $G$ are randomly initialized (no training), then for any target network $F$ from $A(\ell, n, \sigma)$, setting some of the weights of $G$ to 0
(pruning) reveals a subnetwork $\hat{G}$ such that with high probability,
\[
\sup_{x \in \mathcal{X}} \|F(x) - \hat{G}(x)\|_2 \leq \varepsilon
\]

**Question:** How large must $G$ be to contain all such $\hat{G}$? More precisely, how many more neurons or how many more weights must $G$ have compared to $F$?

Ramanujan et al. [2019] were the first to provide a formal asymptotic argument proving that such a $G$ can indeed exist at all. Malach et al. [2020] went substantially further by providing the first polynomial bound on the size of $G$ compared to the size of the target network $F$. To do so, they make the following assumptions on the target network: (i) the inputs $x \in \mathcal{X}$ must satisfy $\|x\|_2 \leq 1$, and at all layers $i \in [\ell]$: (ii) the weights must be bounded in $[-1/\sqrt{n_{\text{max}}}, 1/\sqrt{n_{\text{max}}}]$, (iii) they must satisfy $\|W_i^*\|_2 \leq 1$ at all layers $i$, and (iv) the number of non-zero weights at layer $i$ must be less than $n_{\text{max}}$: $\|W_i^*\|_0 \leq n_{\text{max}}$. Note that these constraints imply that $F_{\text{max}}(\mathcal{X}) \leq 1$.

Then under these conditions, they prove that any ReLU network with $\ell$ layers and $n_{\text{max}}$ neurons per layer can be approximated\(^1\) within $\varepsilon$ accuracy with probability $1 - \delta$ by pruning a network $G$ with $2\ell$ ReLU layers and each added intermediate layer has $n_{\text{max}}^2 \frac{64\ell n_{\text{max}}^3 \log 2n_{\text{max}}^2}{\left(\frac{2}{3}\right)}$ neurons. These assumptions are rather strong, as in general this forces the activation signal to decrease quickly with the depth. Relaxing these assumptions while using the same proof steps would make the bounds exponential in the number of layers. Our work builds upon these first theoretical results and re-uses some of their techniques to provide sharper bounds while removing these assumptions.

### 4 Approximation Propagation

In this section, we analyze how the approximation error between two networks with the same architecture propagates through the layers. The following lemma is a generalization of the (end of the) proof of Malach et al. [2020, Theorem A.6] that removes their aforementioned assumptions and provides better insight into the impact of the different variables on the required accuracy, but is not sufficient in itself to obtain better bounds. For two given networks with the same architecture, it determines what accuracy is needed on each individual weight so the outputs of the two neural networks differ by at most $\varepsilon$ on any input. Note that no randomization appears at this stage.

**Lemma 1** (Approximation propagation). Consider two networks $F$ and $\hat{G}$ with the same architecture $A(\ell, n, \sigma)$ with respective weight matrices $W^*$ and $\hat{W}$, each weight being in $[-w_{\text{max}}, w_{\text{max}}]$. Given $\varepsilon > 0$, if for each weight $w^*$ of $F$ the corresponding weight $\hat{w}$ of $\hat{G}$ we have $|w^* - \hat{w}| \leq \varepsilon_w$, and if
\[
\varepsilon_w \leq \varepsilon / \left( \ell \lambda_{\text{max}} n_{\text{max}}^{3/2} F_{\text{max}}(\mathcal{X}) \prod_{i=1}^{\ell} \max\{1, \lambda_i\|\hat{W}_i\|_2\} \right)
\]
then
\[
\sup_{x \in \mathcal{X}} \|F(x) - \hat{G}(x)\|_2 \leq \varepsilon.
\]

The proof is given in Appendix C.

**Example 2.** Consider an architecture with only ReLU activation function ($\lambda = 1$), weights in $[-1,1]$ and assume that $F_{\text{max}}(\mathcal{X}) = 1$ and take the worst case $\|\hat{W}_i\|_2 \leq w_{\text{max}} n_{\text{max}} = n_{\text{max}}$, then Lemma 1 tells us that the approximation error on each individual weight must be at most $\varepsilon_w \leq \varepsilon / (\ell n_{\text{max}}^{3/2})$ so as to guarantee that the approximation error between the two networks is

\(^1\)Note that even though their bounds are stated in the 1-norm, this is because they consider a single output—for multiple outputs their result holds in the 2-norm, which is what their proof uses.
We now explain how to construct the large network $G$ given only the architecture $A(\ell, n, \sigma)$, the accuracy $\varepsilon$, and the domain $[-w_{\max}, w_{\max}]$ of the weights. Apart from this, the target network $F$ is unknown. In this section all activation functions are ReLU $\sigma(x) = \max\{0, x\}$, and thus $\lambda = 1$.

We use a similar construction of the large network $G$ as Malach et al. [2020]: both the target network $F$ and the large network $G$ consist of fully connected ReLU layers, but $G$ may be wider and deeper. The weights of $F$ are in $[-w_{\max}, w_{\max}]$. The weights for $G$ (at all layers) are all sampled from the same distribution, the only difference with the previous work is the distribution of the weights: we use a hyperbolic distribution instead of a uniform one.

Between layer $i - 1$ and $i$ of the target architecture, for the large network $G$ we insert an intermediate layer $i - 1/2$ of ReLU neurons. Layer $i - 1$ is fully connected to layer $i - 1/2$ which is fully connected to layer $i$. By contrast to the target network $F$, in $G$ the layers $i - 1$ and $i$ are not directly connected. The insight of Malach et al. [2020] is to use two intermediate (fully connected ReLU) neurons $z^+$ and $z^-$ of the large network $G$ to mimic one weight $w^*$ of the target network (see Fig. 1): Calling $z^+_n, z^-_n, z^+_m, z^-_m$ the input and output weights of $z^+$ and $z^-$ that match the input and output of the connection $w^*$, then in the pruned network $G$ all connections apart from these 4 are masked out by a binary mask $b$ set to 0. These two neurons together implement a ‘virtual’ weight $\hat{w}$ and calculate the function $x \mapsto \hat{w}x$ by taking advantage of the identity $x = \sigma(x) - \sigma(-x)$:

$$\hat{w} = z^+_m \sigma(z^+_m x) + z^-_m \sigma(z^-_m x)$$

Hence, if $z^+_m \approx w^* \approx -z^-_m$ and $z^+_m \approx 1 \approx -z^-_m$, the virtual weight $\hat{w}$ made of $z^+$ and $z^-$ is approximately $w^*$. Then, for each target weight $w^*$, Malach et al. [2020] sample many such intermediate neurons to ensure that two of them can be pruned so that $|w^* - \hat{w}| \leq \varepsilon w$ with high probability. This requires $\Omega(1/\varepsilon^2)$ samples and, when combined with Lemma 1 (see Example 2), makes the general bound on the whole network grow exponentially in the number of layers, unless strong constraints are imposed.

5 Construction of the ReLU Network $G$ and Main Ideas

At most $\varepsilon$. This is exponential in the number of layers. If we assume instead that $\|\hat{W}_i\|_2 \leq 1$ as in previous work then this reduces to a mild polynomial dependency: $\varepsilon w \leq \varepsilon/(e\ell n^{\max})$. △
To obtain a logarithmic dependency on $\varepsilon_w$, we use three new insights that take advantage of the composability of neural networks: 1) 'binary' decomposition of the weights, 2) product weights, and 3) batch sampling. We detail them next.

**Weight decomposition.** Our most important improvement is to build the weight $\hat{w}$ not with just two intermediate neurons, but with $O(\log \frac{1}{\varepsilon})$ of them, so as to decompose the weight into pieces of different precisions, and recombine them with the sum in the neuron at layer $i+1$ (see Fig. 1), using a suitable binary mask vector $b$ in the pruned network $\hat{G}$. Intuitively, the weight $\hat{w}$ is decomposed into its binary representation up to a precision of $k = \lceil \log_2 \frac{1}{\varepsilon} \rceil$ bits: \[ \sum_{s=1}^k b_s 2^{-s}. \] Using a uniform distribution to obtain these weights $2^{-s}$ would not work however. But, because the high precision bits are now all centered around 0, we can use a hyperbolic accuracy with the virtual weight $\hat{w}$ such that:

\[ \hat{w} x = \sum_{s=1}^k b_s \left[ z_{\text{out},s}^+ \sigma(z_{\text{in},s}^+ x) + z_{\text{out},s}^- \sigma(z_{\text{in},s}^- x) \right] \approx \sum_{s=1}^k b_s \text{sgn}(w^*) 2^{-s} x \approx w^* x \quad (1) \]

where $b_s \in \{0,1\}$ is factored out since all connections have the same mask, and where $z_{\text{out},s}^+ \approx z_{\text{out},s}^- \approx z_{\text{in},s}^+ \approx \text{sgn}(w^*) 2^{-s}$ and $z_{\text{out},s}^+ > 0$, $\text{sgn}(z_{\text{in},s}^+) = \text{sgn}(w^*)$, $z_{\text{out},s}^- < 0$ and $z_{\text{in},s}^- = -\text{sgn}(w^*)$. In fact, we do not use an actual binary decomposition but a looser 'golden-ratio decomposition' (Lemma 9 in Section 7) because of the inexactness of the sampling process.

**Product weights.** Recall that $z_{\text{out},s}^+ \sigma(z_{\text{in},s}^+ x) = z_{\text{out},s}^+ \max\{0, z_{\text{in},s}^+ x\}$. For fixed signs of $z_{\text{out},s}^+$ and $z_{\text{in},s}^+$, this function can be equivalently calculated for all possible values of these two weights such that the product $z_{\text{out},s}^+ z_{\text{in},s}^+$ remains unchanged. Hence, forcing $z_{\text{out},s}^+$ and $z_{\text{in},s}^+$ to take 2 specific values is wasteful as one can take advantage of the cumulative probability mass of all their combinations. We thus make use of the induced product distribution, which avoids squaring the number of required samples. Define the distribution $p_{w>0}$ for positive weights $w \in [\alpha, \beta]$ with $0 < \alpha < \beta$ and $p_w$, symmetric around 0, for $w \in [-\beta, -\alpha] \cup [\alpha, \beta]$:

\[ p_{w>0}(w) = \frac{1}{w \ln(\beta/\alpha)} \propto \frac{1}{w}, \quad \text{and} \quad p_w(w) = p_w(-w) = \frac{1}{2} p_{w>0}(|w|) = \frac{1}{2|w| \ln(\beta/\alpha)}. \]

Then, instead of sampling uniformly until both $z_{\text{out},s}^+ \approx 1$ and $z_{\text{in},s}^+ \approx w^*$, we sample both from $p_w$ so that $z_{\text{out},s}^+ z_{\text{in},s}^+ \approx w^*$, taking advantage of the induced product distribution $p_{w^*} \approx \frac{1}{2} p_{w>0}$ (Lemma 28).

**Batch sampling.** Sampling sufficiently many intermediate neurons so that a subset of them approximate one target weight $w^*$ with high probably and then discarding (pruning) all other intermediate neurons is wasteful. Instead, we allow these samples to be ‘recycled’ to be used for other neurons in the same layer. This is done by partitioning the neurons in different buckets (categories) and ensuring that each bucket has enough neurons (Lemma 29).

### 6 Theoretical Results

We now have all the elements to present our central theorem, which tells us how many intermediate neurons to sample to approximate a layer of the target network with high probability.
Theorem 3 (ReLU sampling bound). For a given architecture $A(\ell, n, \sigma)$ where $\sigma$ is the ReLU function, with weights in $[-w_{\max}, w_{\max}]$ and a given accuracy $\varepsilon$, the network $G$ constructed as above with weights sampled from $p_w$ with $[\alpha, \beta] = [\alpha', \beta'/q]$, $\alpha' = 2\varepsilon/w, \beta' = 2w_{\max}/3$, and $q = (\alpha'\beta')^{1/4}$, requires only to sample $M_i$ intermediate neurons for each layer $i$, where

$$M_i = \left[16k' \left(n_i n_{i-1} + \ln \frac{2\ell k'}{\delta} \right)\right]$$

and $k' = \log_{3/2} \frac{3w_{\max}}{\varepsilon_w}$ and $\varepsilon_w = \varepsilon / \left(\varepsilon \ell n_{\max}^{3/2} F_{\max}(X) \prod_{i=1}^{\ell} \max\{1, \|W_i\|_2\}\right)$

($\varepsilon_w$ is in Lemma 1 with $\lambda = 1$ for ReLU) to ensure that for any target network $F$ with the given architecture $A(\ell, n, \sigma)$, there exist binary masks $b_{i,j} = (b_{i,j,1}, \ldots, b_{i,j,n_{i-1}})$ of $G$ such that for the resulting network $\hat{G}$,

$$\sup_{x \in X} \|F(x) - \hat{G}(x)\|_2 \leq \varepsilon.$$

**Proof.** **Step 1. Sampling intermediate neurons to obtain product weights.** Consider a single target weight $w^*$. Recalling that $z_{\text{out},s}^+ > 0$ and $z_{\text{out},s}^- < 0$, we rewrite Eq. (1) as

$$\hat{w}x = \sum_{s=1}^{k} b_s z_{\text{out},s}^+ \sigma(z_{\text{in},s}^+ x) + \sum_{s=1}^{k} b_s z_{\text{out},s}^- \sigma(z_{\text{in},s}^- x)$$

$$= \sum_{s=1}^{k} b_s \sigma(z_{\text{out},s}^+ z_{\text{in},s}^+ x) + \sum_{s=1}^{k} -b_s \sigma(-z_{\text{out},s}^- z_{\text{in},s}^- x)$$

The two virtual weight $\hat{w}^+$ and $\hat{w}^-$ are obtained separately. We need both $|w^* - \hat{w}^+| \leq \varepsilon_w/2$ and $|w^* - \hat{w}^-| \leq \varepsilon_w/2$ so that $|w^* - \hat{w}| \leq \varepsilon_w$.

Consider $\hat{w}^+$ (the case $\hat{w}^-$ is similar). We now sample $m$ intermediate neurons, fully connected to the previous and next layers, but only keeping the connection between the same input and output neurons as $w^*$ (the other weights are zeroed out by the mask $b$). For a single sampled intermediate neuron $\hat{z}$, all its weights, including $z_{\text{in}}^+$ and $z_{\text{out}}^+$, are sampled from $p_w$, thus the product $|z_{\text{out}}^+ z_{\text{in}}^+|$ is sampled from the induced product distribution $p_{w \times}$, and, a quarter of the time, $z_{\text{out}}^+$ and $z_{\text{in}}^+$ have the correct signs (recall we need $z_{\text{out}}^+ > 0$ and $\text{sgn}(z_{\text{in}}^+) = \text{sgn}(w^*)$). Define

$$p^+(z_{\text{out}}^+ z_{\text{in}}^+) = P(w = z_{\text{out}}^+ z_{\text{in}}^+ \land z_{\text{out}}^+ \sim p_w \land z_{\text{out}}^+ > 0$$

$$\land z_{\text{in}}^+ \sim p_w \land \text{sgn}(z_{\text{in}}^+) = \text{sgn}(w^*)$$

then with $p^+(z_{\text{out}}^+ z_{\text{in}}^+) \geq p_{w \times}(|z_{\text{out}}^+ z_{\text{in}}^+|)/4 \geq p_{w \geq s}(|z_{\text{out}}^+ z_{\text{in}}^+|)/8$ where the last inequality follows from Lemma 28 for $|z_{\text{out}}^+ z_{\text{in}}^+| \in [\alpha', \beta']$, $z_{\text{out}}^+ \in [\alpha, \beta]$ and $z_{\text{in}}^+ \in [\alpha, \beta]$, and similarly for $z_{\text{out}}^-$ with $p^-(z_{\text{out}}^- z_{\text{in}}^-) \geq p_{w \geq s}(|z_{\text{out}}^- z_{\text{in}}^-|)/8$.

Note that because $\text{sgn}(z_{\text{out}}^+) = -\text{sgn}(z_{\text{out}}^-)$ and $\text{sgn}(z_{\text{in}}^+) = -\text{sgn}(z_{\text{in}}^-)$, the samples for $\hat{w}^+$ and $\hat{w}^-$ are mutually exclusive which will save us a factor 2 later.

**Step 2. ‘Binary’ decomposition/recomposition.** Consider a target weight $w^* \in [-w_{\max}, w_{\max}]$. Noting that Corollary 10 equally applies for negative weights by first negating them, we obtain $\hat{w}^+$ and $\hat{w}^-$ by two separate applications of Corollary 10 where we substitute $P_{\varepsilon} \rightsquigarrow P_{\varepsilon}/8 = p_{w \geq s}/8$ and $P_{\varepsilon/2} \rightarrow \delta_w$. Substituting $P_{\varepsilon}$ with $p_{w \geq s}/8$ in Eq. (2) shows that this leads to a factor 8 on $m$. Therefore, by sampling $m = 8[k' \ln \frac{k'}{\delta_w}]$ weights from $p_{w \times}$ in $[\alpha', \beta'] = [2\varepsilon/w, 2w_{\max}/3]$ with $k' = \log_{3/2} \frac{3w_{\max}}{\varepsilon_w}$ ensures that there exists a binary mask $b$ of
size at most $k'$ such that $|w^* - \hat{w}^*| \leq \varepsilon_w/2$ with probability at least $1 - \delta_w$. We proceed similarly for $w^-$. Note that Corollary 10 guarantees $|\hat{w}| \leq |w^*| \leq w_{\text{max}}$, even though the large network $G$ may have individual weights larger than $w_{\text{max}}$.

**Step 2'. Batch sampling.** Take $k := \lceil \log_{3/2} \frac{w_{\text{max}}}{\varepsilon_w} \rceil \leq k'$ to be the number of ‘bits’ required to decompose a weight with Corollary 10 (via Lemma 9). Sampling $m$ different intermediate neurons for each target weight and discarding $m - k$ samples is wasteful: Since there are $n_i n_{i-1}$ target weights at layer $i$, we would need $n_i n_{i-1}$ intermediate neurons, when in fact most of the discarded neurons could be recycled for other target weights.

Instead, we sample all the weights of layer $i$ at the same time, requiring that we have at least $n_i n_{i-1}$ samples for each of the $k$ intervals of the ‘binary’ decompositions of $\hat{w}^*$ and $\hat{w}^-$. Then we use Lemma 29 with $2k$ categories: The first $k$ categories are for the decomposition of $\hat{w}^*$ and the next $k$ ones are for $\hat{w}^-$. Note that these categories are indeed mutually exclusive as explained in Step 1. and, adapting Eq. (2), each has probability at least \( \frac{1}{2} \int_{|w| < \sqrt{\gamma}} \rho_{w}(w) dw \geq \frac{1}{4} (8 \log_{3/2} (3 w_{\text{max}} / \varepsilon_w)) = 1/(8k') \) (for any $w$). Hence, using Lemma 29 where we take $n \rightarrow n_i n_{i-1}$ and $\delta \rightarrow \delta_i$, we only need to sample \( \lceil 16 k' (n_i n_{i-1} + \ln \frac{1}{\delta_i}) \rceil \leq \lceil 16 k' (n_i n_{i-1} + \ln \frac{2k'}{\delta_i}) \rceil = \ell_i \) intermediate neurons to ensure that with probability at least $1 - \delta_i$ each $\hat{w}^*$ and $\hat{w}^-$ can be decomposed into $k$ product weights in each of the intervals of Lemma 9.

**Step 3. Network approximation.** Using a union bound, we need $\delta_i = \delta / \ell$ for the claim to hold simultaneously for all $\ell$ layers. Finally, when considering only the virtual weights $\hat{w}$ (constructed from $\hat{w}^*$ and $\hat{w}^-$), $G$ and $F$ now have the same architecture, hence choosing $\varepsilon_w$ as in Lemma 1 ensures that with probability at least $1 - \delta$, sup \( x \in \mathcal{X} \left\| F(x) - G(x) \right\| \leq \varepsilon$. □

**Remark 4.** Consider $n_i = n_{\text{max}}$ for all $i$ and assume $\|W_i\|_2 \leq 1$, $w_{\text{max}} = 1$ and $F_{\text{max}}(\mathcal{X}) \leq 1$. Then $\varepsilon_w \geq \varepsilon/(\ell n_{\text{max}}^2)$ and $k' \leq \log_{3/2} (3\ell n_{\text{max}}^2 / \varepsilon)$. Then we can interpret Theorem 3 as follows: When sampling the weights of a ReLU architecture from the hyperbolic distribution, we only need to sample $M_i/n_{\text{max}}^2 \leq 16k' + \ln(2k' / \delta)/n_{\text{max}}^2 = \tilde{O}(\log(\ell n_{\text{max}}^2 / \varepsilon))$ neurons per target weight (assuming $n_{\text{max}}^2 > \log(\ell k' / \delta)$). Compare with the bound of Malach et al. [2020, Theorem A.6] where, under the further constraints that $w_{\text{max}} \leq 1/\sqrt{\log n_{\text{max}}}$ and $\max_i \|W_i^*\|_0 \leq n_{\text{max}}$ and with uniform sampling in $[-1,1]$, $M_i/n_{\text{max}}^2 = [64\ell^2 n_{\text{max}}^3 \log(2N/\delta)/\varepsilon^2]$ sampled neurons per target weight are needed.

**Example 5.** Under the same assumptions as Remark 4, for $n_{\text{max}} = 100$, $\ell = 10$, $\varepsilon = 0.01$, $\delta = 0.01$, the bound above for Malach et al. [2020] gives $M_i/n_{\text{max}}^2 \leq 2 \cdot 10^{-5}$, while our bound in Theorem 3 gives $M_i/n_{\text{max}}^2 \leq 630$. △

**Example 6.** Under the same conditions as Example 5, if we remove the assumption that $\|W_i\|_2 \leq 1$, then Theorem 3 gives $M_i/n_{\text{max}}^2 = \tilde{O}(\ell \log(n_{\text{max}}^2 / \varepsilon))$ and numerically $M_i/n_{\text{max}}^2 \leq 2450$. △

We can now state our final result.

**Corollary 7 (Weight count ratio).** Under the same conditions as Theorem 3, Let $N^*$ be the number of weights in the fully connected architecture $A(\ell, n, \sigma)$ and $N_G$ the number of weights of the large network $G$, then the weight count ratio is $N_G/N^* \leq 32n_{\text{max}} k' + \tilde{O}(\log(k' / \delta))$.

**Proof.** We have $N^* = \sum_{i=1}^{\ell} n_{i-1} n_i$, and the total number of weights in the network $G$ if layers are fully connected is at most $N_G = \sum_{i=1}^{\ell} (n_i n_{i-1} + n_i^2) M_i$, where $M_i = 16k' n_{i-1} n_i + O(\log(k' / \delta))$. Hence the weight count ratio is $N_G/N^* \leq 32n_{\text{max}} k' + \tilde{O}(\log(k' / \delta))$. □

**Remark 8.** Since in the pruned network $\hat{G}$ each target weight requires $k'$ neurons, the large network has at most a constant factor more neurons than the pruned network.
7 Technical lemma: Random weights

The following lemma shows that if \( m \) weights are sampled from a hyperbolic distribution, we can construct a ‘goldary’ (as opposed to ‘binary’) representation of the weight with only \( \tilde{O}(\ln \frac{2}{\delta}) \) samples. Because of the randomness of the process, we use a “base” \( 3/2 \) instead of a base 2 for logarithms, so that the different ‘bits’ have overlapping intervals. As the proof clarifies, the optimal base is \( 1/\gamma = \frac{3}{2} (\sqrt{5} + 1) \pm 1.62 \). The base \( 1/\gamma = 3/2 \) is convenient.

**Lemma 9** (Golden-ratio decomposition). For any given \( \varepsilon > 0 \) and \( 1/\varphi \leq \gamma < 1 \), where \( \varphi := \frac{1}{2}(\sqrt{5} + 1) \) is the golden ratio, define the distribution \( P_\varphi(v) := \frac{c'}{v} \) for \( v \in [\varepsilon \gamma, \gamma] \) with normalization \( c' := [\ln \frac{1}{\gamma}]^{-1} \). For any \( \delta \in (0, 1) \), if \( m = [k' \ln k'/\delta] = \tilde{O}(\ln \delta \cdot \ln \delta) \) with \( k' := \log_\gamma(\gamma \varepsilon) \), then with probability at least \( 1 - \delta \) over the random sampling of \( m \) ‘weights’ \( v_s \sim P_\varphi \) for \( s = 1, \ldots, m \), the following holds: For every ‘target weight’ \( w \in [0, 1] \), there exists a mask \( b \in \{0, 1\}^m \) with \( |b| \leq k' \) such that \( \hat{w} := b_1 v_1 + \ldots + b_m v_m \) is \( \varepsilon \)-close to \( w \), indeed \( w - \varepsilon \leq \hat{w} \leq w \).

**Proof.** Let \( k = [\log_\gamma \varepsilon] \leq 1 + \log_\gamma \varepsilon = k' \). First, consider a sequence \((v_i)_{i \in [k]}\) such each \( v_i \) is in the interval \( I_i := (\gamma^{i+1}, \gamma^i) \) for \( i = 1, \ldots, k \). We construct an approximating \( \hat{w} \) for any weight \( w_0 := w \in [0, 1] \) by successively subtracting \( v_i \) when possible. Formally

\[
\text{for}(i = 1, \ldots, k) \{ \text{if } w_{i-1} \geq \gamma^i \text{ then } w_i := w_{i-1} - v_i; \ b_i = 1 \} \text{ else } \{ w_i := w_{i-1}; \ b_i = 0 \}
\]

By induction we can show that \( 0 \leq w_i \leq \gamma^i \). This holds for \( w_0 \). Assume \( 0 \leq w_{i-1} \leq \gamma^{i-1} \); If \( w_{i-1} \geq \gamma^i \) then \( w_i \leq w_{i-1} < \gamma^i \).

If \( w_{i-1} \geq \gamma^i \) then \( w_i = w_{i-1} - v_i \leq \gamma^{i-1} - \gamma^{i+1} = (\gamma^{-1} - \gamma) \gamma^i \leq \gamma^i \).

The last inequality is true for \( \gamma \geq \frac{1}{2}(\sqrt{5} - 1) \), which is satisfied due to the restriction \( 1/\varphi \leq \gamma < 1 \). Hence the error \( 0 \leq w - \hat{w} = w_k \leq \gamma^k \leq \varepsilon \leq \gamma^{k-1} \) for \( k = [\log_\gamma \varepsilon] \geq 1 \).

Now consider a random sequence \((v_i)_{i \in [m]}\) where we sample \( v_s \sim P \) over the interval \([\varepsilon \gamma, \gamma]\) for \( s = 1, \ldots, m > k \). In the event that there is at least one sample in each interval \( I_i \), we can use the construction above with a subsequence \( \tilde{v} \) of \( v \) such that \( \tilde{v}_i \in I_i \) and \( \sum_{i \in [k]} b_i \tilde{v}_i = w_k \) as in the construction above. Next we lower bound the probability \( p \) that each interval \( I_i \) contains at least one sample. Let \( E_i \) be the event “no sample is in \( I_i \)” and let \( c = \min_{i \in [k]} P[v \in I_i] \). Then

\[
P[E_i] = (1 - P[v \in I_i])^m \leq (1 - c)^m, \text{ hence}
\]

\[
p = 1 - P[E_1 \lor \ldots \lor E_k] \geq 1 - \sum_{i=1}^k P[E_i] \geq 1 - k(1-c)^m \geq 1 - k \exp(-cm)
\]

and thus choosing \( m \geq c \left[ \frac{1}{\gamma} \ln(k/\delta) \right] \) ensures that \( p \geq 1 - \delta \). Finally,

\[
c = \min_{i \in [k]} P[v \in I_i] = \min_{i} P[\gamma^{i+1} < v \leq \gamma^{i}] = \min_{i} \int_{\gamma^{i+1}}^{\gamma^{i}} \frac{c'}{v} dv = c' \ln \frac{1}{\gamma} = 1/\log_\gamma(\gamma \varepsilon) = \frac{1}{k'}
\]

and so we can take \( m = \left[ k' \ln \frac{k'}{\delta} \right] \). \( \square \)

**Corollary 10** (Golden-ratio decomposition for weights in \([0, \text{w}_{\text{max}}]\)). For any given \( \varepsilon > 0 \), define the distribution \( P_{\varphi}(v) := \frac{c'}{v} \) for \( v \in \left[ \frac{4}{5} \varepsilon, \frac{2}{3} \text{w}_{\text{max}} \right] \) with normalization \( c' := 1/\ln \frac{3 \text{w}_{\text{max}}}{2 \varepsilon} \). Let
$k' := \log_{3/2} \frac{3w_{\text{max}}}{2\epsilon}$, For any $\delta \in (0, 1)$, if $m = \lceil k' \ln \frac{k'}{\delta} \rceil = \tilde{\Omega}(\ln \frac{1}{\epsilon} \cdot \ln \frac{1}{\delta})$, then with probability at least $1 - \delta$ over the random sampling of $m$ ‘weights’ $v_s \sim P_\epsilon$ ($s = 1, ..., m$) the following holds: For every target ‘weight’ $w \in [0, w_{\text{max}}]$, there exists a mask $b \in \{0, 1\}^m$ with $|b| \leq k'$ such that $\hat{w} := b_1 v_1 + ... + b_m v_m$ is $\varepsilon$-close to $w$, indeed $w - \varepsilon \leq \hat{w} \leq w$.

Proof. Follows from Lemma 9 with $\gamma = 2/3$ and a simple rescaling argument: First rescale $w' = w/w_{\text{max}}$ and apply Lemma 9 with $w'$ and accuracy $\varepsilon/w_{\text{max}}$. Then the constructed $\hat{w}'$ satisfies $w' - \varepsilon/w_{\text{max}} \leq \hat{w}' \leq w'$ and multiplying by $w_{\text{max}}$ gives the required accuracy. Also note that the density $P_\epsilon(v) \propto 1/v$ is scale-invariant. 

8 Conclusion

We have proven that large randomly initialized ReLU networks contain many more subnetworks than previously shown, which gives further weight to the idea that one important task of stochastic gradient descent (and learning in general) may be to effectively prune connections by driving their weights to 0, revealing the so-called winning tickets. One could even conjecture that the effect of pruning is to reach a vicinity of the global optimum, after which gradient descent can perform local quasi-convex optimization. Then the required precision $\varepsilon$ may not need to be very high.

Although our work assumes that the distribution of the weights is hyperbolic, we conjecture (with supporting empirical evidence in Appendix A) that a similar effect could be achieved with uniform samples. Combined with some of our proof ideas (batch sampling and product weights), it may be possible to reach the same type of bounds.

Further questions include the impact of convolutional and batch norm layers, skip-connections and LSTMs on the number of required sampled neurons to maintain a good accuracy.
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A Sub-sums of Uniform Samples

Sampling uniformly requires many samples to obtain high precision anywhere in the region of interest (Fig. 2, Top). In this work we have taken advantage the summing function in neurons, combined with pruning, so as to be able to consider all \(2^k\) sub-sums of \(k\) samples (Fig. 2, Middle). However, we conjecture that a similar effect appears with sub-sums of uniform samples (Fig. 2, Bottom, but observe the large offset): For example, it suffices that 2 among \(k\) samples \(x_1\) and \(x_2\) are within \(\varepsilon\) of each other so that for all other samples \(x_3\), \(x_3 + x_1\) and \(x_3 + x_2\) are within \(\varepsilon\) of each other too.

B Variations and Improvements

Remark 11. Sampling from \(P_\varepsilon\) is easy using the inverse CDF: To obtain a sample from \(P_\varepsilon\) in \([\alpha, \beta]\), first draw a uniform sample \(u \sim U[0, 1]\) then return \(\alpha \left( \frac{\beta}{\alpha} \right) u\).

Remark 12. The sampling procedure can be made independent of \(\varepsilon\) by sampling from \(P(v) = \ln \frac{1}{\gamma v} / (\ln v)^2\) for \(0 \leq v \leq \gamma\). It is easy to see that \(c\) in the proof becomes \(\hat{\Omega}(\ln \gamma / \ln(\varepsilon \gamma^2))^2\), leading to a slightly worse bound \(m = \hat{\Omega}(\ln \frac{1}{\varepsilon} \cdot \ln \frac{1}{\delta})\) as in Lemma 9.

Remark 13. Instead of using batch sampling and Lemma 29, we can ‘recycle’ samples in a different way, which removes the leading factor 16 at the expense of a larger second order term. See Appendix D.

Remark 14. What if \(w_{\text{max}}\) is not known? A simple trick is to take \(w_{\text{max}} = 1\) every second sample, every 4th sample we take \(w_{\text{max}} = 2\), and every \(2^{j+1}\) sample we can take \(w_{\text{max}} = 2^j\). Then the total number of samples required to obtain \(M^*_i\) samples for the correct \(w^*_\text{max}\) is at most \(4w^*_\text{max}M^*_i\). Since \(w^*_\text{max}\) is expected to be small, this is likely a mild dependency. A \(\log w_{\text{max}} \log^2 \log w_{\text{max}}\) factor instead of \(w_{\text{max}}\) can even be obtained with a little more work.

Remark 15. In practice, weights are often initialized uniformly in \([-O(\sqrt{1/n}), +O(\sqrt{1/n})]\), where \(n\) is the layer width, potentially somehow averaged over two layers, i.e. weights are initially very small. All our initializations need some large weights but only very few \((O(\log n)\) outside this interval), most weights are very small too. We could even eliminate the large weights and limit our sampling procedure to weights in this interval, but sample \(O(\sqrt{n})\) times more weights to reconstruct large weights.

Remark 16. Empirically it seems that even sampling a logarithmic number of weights uniformly from \([-1; +1]\) or \([\pm 1/\sqrt{n}]\), see previous item) or from a standard Gaussian works nearly as well hyperbolic sampling, but we were not able to prove this.

Remark 17. Extreme case: Pruning ‘Boolean’ networks The difficulty of pruning can be seen easily for Boolean network with Heaviside transition function and binary inputs, and Boolean weights everywhere. Then network \(G\) would need only twice as many weights as the target network, but can still represent exponentially many functions. It is then clear that not only “Pruning really is all you need,” but also that “Pruning is as hard as learning.”

C Technical Results

Proof of Lemma 1.
Figure 2: y-axis: Difference between two consecutive points on the x-axis. Top: 1000 uniform samples in [0, 1], x-axis is sample value (previous work); Middle: 15 hyperbolic samples in [0, 1], each point is one of $2^{15}$ possible sub-sums, x-axis is the sub-sums (this work); Bottom: Like Middle, but with uniform samples in [0, 1] (future work?).
Proof. For all \( x \in X \), for a layer \( i \):

\[
\|F(x) - \hat{G}_i(x)\|_2 \overset{(a)}{=} \|f_i(F_{i-1}(x)) - \hat{g}_i(\hat{G}_{i-1}(x))\|_2
\]

\[
= \left[ \sum_{j \in [n_i]} (\sigma_i(W_{i,j}^*F_{i-1}(x)) - \sigma_i(\hat{W}_{i,j}\hat{G}_{i-1}(x)))^2 \right]^{1/2}
\]

\[
\leq \lambda_i\|W_{i}^*F_{i-1}(x) - \hat{W}_{i}\hat{G}_{i-1}(x)\|_2
\]

\[
\leq \lambda_i\|W_{i}(F_{i-1}(x) - \hat{G}_{i-1}(x))\|_2 + \lambda_i\|(W_{i}^* - \hat{W}_{i})F_{i-1}(x)\|_2
\]

\[
\leq \lambda_i\|W_{i}\|_2\|F_{i-1}(x) - \hat{G}_{i-1}(x)\|_2 + \lambda_i\|W_{i}^* - \hat{W}_{i}\|_2\|F_{i-1}(x)\|_2
\]

\[
\leq \lambda_i\|W_{i}\|_2\|F_{i-1}(x) - \hat{G}_{i-1}(x)\|_2 + \lambda_{\max}\sqrt{\max t} \max w \sqrt{\max x} F_{\max}(x)
\]

\[
\leq \lambda_i\|W_{i}\|_2\|F_{i-1}(x) - \hat{G}_{i-1}(x)\|_2 + \varepsilon_w\lambda_{\max}\sqrt{\max x} F_{\max}(x)
\]

\[
\overset{(f)}{=} \varepsilon_w\epsilon\lambda_{\max}\sqrt{\max x} F_{\max}(x) \prod_{i=1}^T \max \{1, \lambda_i\|W_{i}\|_2\}
\]

where (a) follows from the definition of \( F_i \) and \( \hat{G}_i \), (b) follows from \( |\sigma_i(x) - \sigma_i(y)| \leq \lambda_i|x-y| \) by the definition of \( \lambda_i \), (c) follows from the Minkowski inequality, (d) follows from the definition of \( \lambda_i \), (e) is by assumption that \( |w^* - \hat{w}| \leq \varepsilon_w \) and \( \|M\|_2 \leq c\sqrt{ab} \) for any \( M \in [-c, c]^{a \times b} \), and finally (f) follows from Corollary 27, using \( \|F_0(x) - \hat{G}_0(x)\|_2 = 0 \). Therefore

\[
\|F(x) - \hat{G}(x)\|_2 = \|F(T) - \hat{G}(T)\|_2 \leq \varepsilon_w\epsilon\lambda_{\max}\sqrt{\max x} F_{\max}(x) \prod_{i=1}^T \max \{1, \lambda_i\|W_{i}\|_2\}
\]

and taking \( \varepsilon_w \) as in the theorem statement proves the result. \( \square \)

**Lemma 18** (Bound on positive sequences). Assuming \( x_0 \geq 0 \), and if, for all \( t = 0, 1, \ldots, x_t \leq a_t x_{t-1} + b_t \) with \( a_t \geq 0 \) and \( b_t \geq 0 \), then

\[
\forall \tau \text{ s.t. } \{|a_t < 1 + 1/\tau|_{t \in [T]} \leq \tau \text{ we have } x_T \leq e(x_0 + c) \prod_{t \in [T]} a_t \}
\]

with \( c = \min \left\{ \tau \max t \max b_t, \max t \frac{b_t}{a_t - 1} \right\} \).

**Proof.** First, observe that \( \tau \geq 0 \), with \( \tau = 0 \) if \( T = 0 \). Let \( \bar{a}_t = \max \{a_t, 1 + 1/\tau\} \). Then

\[
c = \min \left\{ \tau \max t \max b_t, \max t \frac{b_t}{a_t - 1} \right\} = \max t \frac{b_t}{\max \{1/\tau, a_t - 1\}} = \max t \frac{b_t}{a_t - 1}.
\]

Define \( y_t = \bar{a}_t y_{t-1} + (\bar{a}_t - 1)c \) and \( y_0 = x_0 \). Then we have \( y_t + c = \bar{a}_t (y_{t-1} + c) \) and so by recurrence \( y_T + c = (y_0 + c) \prod_{t=1}^T a_t \) and thus \( y_T \leq (y_0 + c) \prod_{t=1}^T a_t \). Now, observe that \( y_t \geq \bar{a}_t y_{t-1} + b_t \geq 0 \) and so by recurrence with base case \( y_0 = x_0 \), \( x_T \leq y_T \).

Furthermore

\[
\prod_{t \in [T]} \bar{a}_t \leq \prod_{t:a_t < 1+1/\tau} \left( 1 + 1/\tau \right) \prod_{t:a_t \geq 1+1/\tau} a_t \leq \left( 1 + 1/\tau \right)^T \prod_{t:a_t \geq 1+1/\tau} a_t,
\]

noting that \( (1 + 1/\tau)^T \leq e \). \( \square \)
Remark 19. The factor $e$ should be 1 if $\min_t a_t \geq 1 + 1/\tau$.

Remark 20. $\tau = T$ is always feasible.

Remark 21. If $\min_t a_t \geq 2$, then $\tau = 1$ is feasible.

Corollary 22 (Feasible $\tau$ for Lemma 18). In the context of Lemma 18, for all $x > 1$ taking $\tau = \max\{1/(x - 1), \{|a_t < x\}_{t \in [T]}\}$ is feasible.

Proof. Take $y = 1/(x - 1)$, so $x = 1 + 1/y$ and $\tau = \max\{y, \{|a_t < 1 + 1/y\}_{t}\}$. Thus $\{|a_t < 1 + 1/\tau\}_{t} \leq \{|a_t < 1 + 1/y\}_{t}\} \leq \tau$ as required.

Remark 23. If $1 < \min_t a_t \leq x$ then taking $\tau = 1/(x - 1)$ is feasible.

Remark 24. Taking $\tau = \max\{1, \{|a_t < 2\}_{t}\}$ is feasible.

Remark 25. For $\varphi = (1 + \sqrt{3})/2 \leq 1.62$, taking $\tau = \max\{\varphi, \{|a_t < \varphi\}_{t}\}$ is feasible.

Remark 26. If $\min_t a_t > 1$ then $\tau = 1/(\min_t a_t - 1)$ is feasible (but useful only if $\min_t a_t \geq 1 + 1/T$).

Corollary 27 (Simpler bound on positive sequences). Assuming $x_0 = 0$, and if, for all $t = 0, 1, \ldots,$ $x_t \leq a_t x_{t-1} + b_t$ with $a_t \geq 0$ and $b_t \geq 0$, then

$$x_T \leq cT \max_{t} b_t \prod_{t \in [T]} \max\{1, a_t\}.$$ 

Proof. Follows from Lemma 18 with $\tau = T$ which is always feasible and observing that $\prod_{t, a_t \geq 1 + 1/T} a_t \leq \prod_t \max\{1, a_t\}$, and that $c \leq \tau \max_t b_t = T \max_t b_t$.

Lemma 28 (Product of weights). Let probability densities $P_{v}(v) := c/v$ for $v \in [a, b]$ and $0 < a < b$ with normalization $c := 1/\ln b$. Let weight $w := v \cdot v'$ with $v$ and $v'$ both sampled from $P_v$. Then $P_w(w) \geq c/2w$ for $w \in [a', b']$, where $P_w$ is the probability density of $w$, and $a' := a\sqrt{ab}$ and $b' := b\sqrt{ab}$.

Note that $w$ may be outside of $[a', b']$, but at least half of the time is inside $[a', b']$. The lemma implies that the bound in Lemma 9 also applies to the product of two weights, only getting a factor of 2 worse. Note that the scaling ranges $\frac{a'}{a} = \frac{b'}{b}$ are the same.

Proof. $\ln v$ is uniformly distributed in $[\ln a, \ln b]$: indeed, taking $y = \ln v$, we have $P_y(y) = P_v(v)/\frac{dv}{dy} = c$. Let us scale and shift this to $t := c(2\ln v - \ln ab) \in [-1, +1]$ and similarly $t' := c(2\ln v' - \ln ab) \in [-1, +1]$. Then $P_t(t) = P_v(v)/\frac{dv}{dy} = \frac{c}{2}/\frac{dy}{du} = \frac{1}{2}$ for $t \in [-1, +1]$, and same for $t'$. Let $u := t + t' \in [-2, +2]$. The sum of two uniformly distributed random variables is triangularly distributed: $P_u(u) = \frac{1}{2}(1 - \frac{1}{2}|u|)$. Using $w = v \cdot v'$ we can write $u = t + t' = 2c(\ln w - \ln(ab))$. Then $P_w(w) = P_u(u)/\frac{du}{dw} = \frac{1}{2}(1 - \frac{1}{2}|u|)/\frac{du}{dw}$. For $|u| \leq 1$ this is $\geq c/2w$. Finally $|u| \leq 1$ iff $|\ln w - \ln(ab)| \leq 1/2c$. If $\ln w \geq \ln ab + \frac{1}{2} \ln \frac{c}{2}$ then $w \in [a', b']$.

Lemma 29 (Filling $k$ categories each with at least $n$ samples). Let $P_c$ be a categorical distribution of at least $k \in \mathbb{N}$ (mutually exclusive) categories $\{1, 2, \ldots, k, \ldots\}$ such that the first $k$ categories have probability at least $c$ and at most $1/2$, that is, if $X \sim P_c$, then $c \leq P_c(X = j) \leq 1/2$ for
all \( j \in [k] \). Let \((X_i)_{i \in [M]}\) be a sequence of \( M \) random variables sampled i.i.d. from \( P_c \). 

For all \( \delta \in (0, 1) \), for all \( n \in \mathbb{N} \), if

\[
M = \left\lceil \frac{2}{\epsilon} \left( n + \ln \frac{k}{\delta} \right) \right\rceil
\]

then with probability at least \( 1 - \delta \) each category \( j \in [k] \) contains at least \( n \) samples, i.e., \( |\{X_i = j\}_{i \leq M}| \geq n \).

**Proof.** Let \( c_j \geq c \) be the probability of category \( j \in [k] \). Using the Chernoff-Hoeffding theorem on the Bernoulli random variable \( J_{X_i = j} \), where \( J \) is the indicator function and equals 1 if \( test \) is true, 0 otherwise— with \( Mc_j - x = n \geq 0 \), that is, \( x = Mc_j - n \), for each category \( j \in [k] \) we have

\[
P \left( \sum_{i=1}^{M} (1 - [X_i = j]) > M(1 - c_j) + x \right) \leq \exp \left( -\frac{x^2}{2Mc_j(1 - c_j)} \right)
\]

\[
P \left( \sum_{i=1}^{M} [X_i = j] < Mc - x \right) \leq \exp \left( -\frac{x^2}{2Mc_j(1 - c_j)} \right)
\]

\[
P \left( \sum_{i=1}^{M} [X_i = j] < n \right) \leq \exp \left( -(Mc_j/2 - n) \right)
\]

and the condition \((1 - c_j) \geq \frac{1}{2}\) is satisfied. Name \( E_j \) the event “the category \( j \in [k] \) contains fewer than \( n \) samples,” then \( P(E_j) \leq \exp(-(Mc/2 - n)) \). Then, using a union bound, the probability that any of the \( k \) categories contain fewer than \( n \) samples is at most

\[
P(E_1 \lor E_2 \lor \ldots E_k) \leq \sum_{j=1}^{k} P(E_j) \leq k \exp(-(Mc/2 - n))
\]

and since \( M \geq \frac{2}{\epsilon} \left( n + \ln \frac{k}{\delta} \right) \)

\[
P(E_1 \lor E_2 \lor \ldots E_k) \leq \delta,
\]

\[
1 - P(E_1 \lor E_2 \lor \ldots E_k) \geq 1 - \delta,
\]

which proves the claim. 

\( \square \)

### D Sample recycling

**Theorem 30** (ReLU sampling bound #2). Theorem 3 holds simultaneously also with

\[
M_i = \left\lceil 2k' \left( n_in_{i-1} + 4\max\{n_i, n_{i-1}\} \ln \frac{2k'N^*}{\delta} \right) \right\rceil
\]

and all other quantities are unchanged.

**Proof.** **Step 1** and **2.** Same as for Theorem 3.

**Step 2’. Sample recycling.** Let

\[
m = \left\lceil 8k' \ln \frac{k'}{\delta_w} \right\rceil, \quad k' = \log_{3/2} \frac{3w_{\max}}{\varepsilon_w}, \quad k = \left\lceil \log_{3/2} \frac{2w_{\max}}{\varepsilon_w} \right\rceil,
\]

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where $m$ is the number of neurons that need to be sampled according to Corollary 10 to $\varepsilon_w$-approximate one target weight with probability at least $1 - \delta_w$, and $k$ is an upper bound on the number of unit bits of the corresponding mask. One neuron with some pruned weights cannot be shared to approximate two target weights at the same time, which means we need at least $2kn_i n_{i-1}$ neurons ($k$ for each $\hat{w}^+$, and $k$ for each $\hat{w}^-$). For a specific target weight $w^*$, out of $m \geq 2k$ sampled neurons, that is, only $m + 2(k - 1)n_i n_{i-1}$ for the whole layer. To also cover the case $n_{i-1} > n_i$, we need to sample max{$n_i, n_{i-1}$} $m + 2(k - 1)n_i n_{i-1}$ neurons to ensure that every target weight of layer $i$ can be decomposed into $2k$ product weights, each based on $m$ independent product weight samples.

**Step 3. Network approximation.** For the guarantee to hold simultaneously over all $\hat{w}^+$ and $\hat{w}^-$, using a union bound we can take $\delta_w = \delta/(2N^*)$. Finally the claim follows from Lemma 1 and noting that $k \leq k'$.

**Remark 31.** Observe that even though the factor in front of $m$ is larger than for Theorem 3, (also $\ell \sim N^*$ in the log) we gain a constant factor 8 in front of the leading term $n_i n_{i-1} k$.

**Example 32.** Under the same conditions as Example 5, Theorem 30 gives $M_i/n_{\max}^2 \leq 144$, and under the same conditions as Example 6 we have $M_i/n_{\max}^2 \leq 574$. □

Therefore, since both Theorem 3 and Theorem 30 hold simultaneously, we can take:

$$M_i = \min \left\{ \left[ 16k' \left( n_i n_{i-1} + \ln \frac{2k'\ell}{\delta} \right) \right], \left[ 2k' \left( n_i n_{i-1} + 4 \max\{n_i, n_{i-1}\} \ln \frac{2k'N^*}{\delta} \right) \right] \right\}$$

to ensure that, with probability at least $1 - \delta$,

$$\sup_{x \in X} \|F(x) - \hat{G}(x)\|_2 \leq \varepsilon.$$
Algorithm 1 Recycling samples. We assume that $n_i \geq n_{i-1}$, otherwise the loops and the increments need to be exchanged.

1 # Sample recycling at layer $i$.  
2 for j= 1 to $n_i$: # Assumes $n_i \geq n_{i-1}$  
3  # Discard old samples and generate fresh ones.  
4 M = sample $m$ fully-connected intermediate neurons  
5 for d = 1 to $n_{i-1}$:  
6   # These indices ensure that  
7   # * all target weights are approximated,  
8   # * no input weight and no output weight is used for more  
9   #    than one target weight.  
10  idx_in = d  
11  idx_out = (d+j) % $n_i$  
12  $w^* = W^*[idx_in, idx_out]$  
13  # ‘Call’ to the golden-ratio decomposition (Corollary 10)  
14  # using the provided samples M.  
15  # It returns the set $K \subseteq M$ of sampled neurons used to decompose $w^*$.  
16  # Only uses weights at the indices $idx_{in}$ and $idx_{out}$ of the neurons in $M$  
17  # The indexes above ensure that no weight in $M$ already has $idx_{in}$ and  
18  # $idx_{out}$ zeroed out.  
19  $K^+ = GRD^+(M, idx_{in}, idx_{out}, w^*)$ # Corollary 10 for $\hat{w}^+$  
20  $K^- = GRD^-(M, idx_{in}, idx_{out}, w^*)$ # Corollary 10 for $\hat{w}^-$  
21  # These samples cannot be reused for other neurons, put them aside.  
22  M = M \ (K^+ \cup K^-)  
23  # Zero-out the input and output weights that the GRD has filtered,  
24  # as they are not independent samples anymore and cannot be reused.  
25  for n in M:  
26     n.ins[idx_in] = 0  
27     n.outs[idx_out] = 0  
28  # Fill up M to have $m$ intermediate neurons.  
29  M_new = sample $|K^+|+|K^-|$ new independent neurons # $|K^+|+|K^-| \leq 2k$  
30  M = M \cup M_new # such that $|M| = m$
# E List of Notation

| Symbol | Explanation |
|--------|-------------|
| $\mathbb{N}$ | natural numbers \{1,2,...\} |
| $\ell \in \mathbb{N}$ | number of network layers |
| $n \in \mathbb{N}^\ell$ | vector of the number of neurons |
| $n_{\text{max}} \in \mathbb{N}$ | maximum number of neurons per layer |
| $i \in [\ell]$ | layer index |
| $j \in [n_i]$ | index of $j$th neuron in layer $i$ |
| $F$ | a target network |
| $G$ | the large network to be pruned |
| $\hat{G}$ | the network $G$ after pruning |
| $F_{\text{max}}(\mathcal{X})$ | maximum absolute activation of any non-final neuron on all inputs of interest in $F$ |
| $w \in [-w_{\text{max}}, w_{\text{max}}]$ | some weight |
| $w_{\text{max}} \in \mathbb{R}^+$ | max norm of the weights |
| $w^* \in [-w_{\text{max}}, w_{\text{max}}]$ | a weight of the target network $F$ |
| $W^*$ | weights of the target network |
| $z_{\text{out}}^+, z_{\text{in}}^-, z_{\text{out}}^-$ | actual individual weights of the network $G$ |
| $\hat{w}^+, \hat{w}^-, \hat{w}$ | virtual individual weights of the network $\hat{G}$ |
| $\varepsilon > 0$ | output accuracy |
| $1 - \delta \in [0,1]$ | high probability |
| $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ | activation function |
| $\mathbf{\sigma}$ | vector of $\ell$ activation functions |
| $\lambda_i$ | Lipschitz factor of $\sigma_i$ |
| $k \in \mathbb{N}$ | number of 'bits' to represent a weight |
| $m \in \mathbb{N}$ | number of neurons sampled per target weight |
| $M \in \mathbb{N}$ | number of neurons sampled per intermediate layer |
| $x \in [-x_{\text{max}}, x_{\text{max}}]^{n_0}$ | network input |
| $x_{\text{max}} \in \mathbb{R}^+$ | max norm of the inputs |
| $p$ | probability |
| $A(\ell, n, \mathbf{\sigma})$ | architecture of a network |
| $v$ | vector |
| $b$ | binary mask vector |
| $\mathbf{b}$ | binary mask |
| $F_i$ | output of layer $i$ of the target network given network inputs |
| $G$ | the big network $i$ |
| $\hat{G}_i$ | subnetwork of the big network, approximating $F$ |
| $f_i$ | layer functions of target network given layer inputs |
| $\hat{f}_i$ | same as $f_i$ for $\hat{G}$ |
| $P_{\varepsilon}$ | $1/\nu$ distribution |
| $p_{w \geq \nu}$ | $1/\nu$ distribution |
| $p_w$ | $\pm 1/\nu$ distribution |
| $p_w \times$ | $1/\nu$ product distribution |
| $p^+$ | product distribution of $z_{\text{out}}^+$ and $z_{\text{in}}^+$ |
| $p^-$ | product distribution of $z_{\text{out}}^-$ and $z_{\text{in}}^-$ |