Bounded Point Evaluations For Certain Polynomial And Rational Modules

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Abstract

Let $K$ be a compact subset of the complex plane $\mathbb{C}$. Let $P(K)$ and $R(K)$ be the closures in $C(K)$ of analytic polynomials and rational functions with poles off $K$, respectively. Let $A(K) \subseteq C(K)$ be the algebra of functions that are analytic in the interior of $K$. For $1 \leq t < \infty$, let $P^t(1, \phi_1, \ldots, \phi_N, K)$ be the closure of $P(1, \phi_1, \ldots, \phi_N, K) = P(K) + P(K)\phi_1 + \ldots + P(K)\phi_N$ in $L^t(dA|_K)$, where $dA|_K$ is the area measure restricted to $K$ and $\phi_1, \ldots, \phi_N \in L^t(dA|_K)$. Let $HP(\phi_1, \ldots, \phi_N, K)$ be the closure of $P(1, \phi_1, \ldots, \phi_N, K)$ in $C(K)$, where $\phi_1, \ldots, \phi_N \in C(K)$. In this paper, we prove if $R(K) \neq C(K)$, then there exists an analytic bounded point evaluation for both $P^t(1, \phi_1, \ldots, \phi_N, K)$ and $HP(\phi_1, \ldots, \phi_N, K)$ for certain smooth functions $\phi_1, \ldots, \phi_N$, in particular, for $\bar{z}, \bar{z}^2, \ldots, \bar{z}^N$. We show that $A(K) \subseteq HP(\bar{z}, \bar{z}^2, \ldots, \bar{z}^N, K)$ if and only if $R(K) = A(K)$. In particular, $C(K) \neq HP(\bar{z}, \bar{z}^2, \ldots, \bar{z}^N, K)$ unless $R(K) = C(K)$. We also give an example of $K$ showing the results are not valid if we replace $\bar{z}^N$ by certain $\phi_n$, that is, there exist $K$ and a function $\phi \in A(K)$ such that $R(K) \neq A(K)$, but $A(K) = HP(\phi, K)$.

1 Introduction

Let $P$ denote the set of polynomials in the complex variable $z$. For a compact subset $K$ of the complex plane $\mathbb{C}$, let $Rat(K)$ be the set of all rational functions with poles off $K$ and let $C(K)$ denote the Banach algebra of complex-valued continuous functions on $K$ with customary norm $\| \cdot \|_K$. Let $P(K)$ and $R(K)$ denote the closures in $C(K)$ of $P$ and $Rat(K)$, respectively. Let $A(K) \subseteq C(K)$ be the algebra of functions that are analytic in the interior of $K$. For $\phi_1, \ldots, \phi_N \in C(K)$, let $HP(\phi_1, \ldots, \phi_N, K)$ denote the closure of $P(\phi_1, \ldots, \phi_N, K)$ in $C(K)$. For $1 \leq t < \infty$, let $L^t(K) = L^t(dA|_K)$, where $dA|_K$ is the area measure restricted to $K$. For $\phi_1, \ldots, \phi_N \in L^t(K)$, let $P^t(1, \phi_1, \ldots, \phi_N, K)$ be the closure of $P(\phi_1, \ldots, \phi_N, K)$ in $L^t(K)$. For a subset $A \subseteq \mathbb{C}$, we set $Int(A)$ for its interior, $\bar{A}$ for its complement, and $\chi_A$ for its characteristic function.

For a subspace $A$ of $C(K)$ and a function $f \in C(K)$, we define the distance from $f$ to $A$ by

$$dist(f, A) = \inf_{g \in A} \| f - g \|_K.$$ 

For a subspace $B$ of $L^t(K)$ and a function $f \in L^t(K)$, we define the distance from $f$ to $B$ by

$$dist(f, B) = \inf_{g \in B} \| f - g \|_{L^t(K)}.$$ 

Set

$$B(\lambda_0, \delta) = \{ z : |z - \lambda_0| < \delta \}.$$ 

The open unit disk is denoted by $D = B(0, 1)$. The constants used in the paper such as $C, C_0, C_1, C_N, \delta_0, \delta_1, \delta_N, \epsilon_0, \epsilon_1, \epsilon_N, \ldots$ may change from one step to the next.

We denote the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{ \infty \}$. For a compact subset $E \subseteq \mathbb{C}$, we define the analytic capacity of $E$ by

$$\gamma(E) = \sup_{f \in A(E)} |f'(\infty)|,$$ 

where $A(E)$ is the algebra of analytic functions in $E$. The capacity $\gamma(E)$ is a measure of how large $E$ is in terms of the growth of analytic functions at infinity. It is a fundamental tool in potential theory and complex analysis.
where \( \mathcal{A}(E) \) consists of those functions \( f \) analytic in \( \mathbb{C}_\infty \setminus E \) for which \( f(\infty) = 0, \|f(z)\| \leq 1 \) for all \( z \in \mathbb{C}_\infty \setminus E \), and
\[
f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).
\]
The analytic capacity of a general \( E_1 \subset \mathbb{C} \) is defined to be
\[
\gamma(E_1) = \sup\{\gamma(E) : E \subset E_1, \; E \text{ compact}\}.
\]
The continuous analytic capacity for a compact subset \( E \) is defined similarly as
\[
\alpha(E) = \sup_{f \in \mathcal{A}(E)} |f'(\infty)|,
\]
where \( \mathcal{A}(E) = \mathcal{A}(E) \cap C(\mathbb{C}_\infty) \). For a general \( E_1 \subset \mathbb{C} \),
\[
\alpha(E_1) = \sup\{\alpha(E) : E \subset E_1, \; E \text{ compact}\}.
\]
(see Gamelin [1969] and Conway [1991] for basic information of rational approximation and analytic capacity).

Let \( \nu \) be a compactly supported finite measure on \( \mathbb{C} \). The Cauchy transform of \( \nu \) is defined by
\[
C\nu(z) = \int \frac{1}{w-z}d\nu(w)
\]
for all \( z \in \mathbb{C} \) for which \( \int \frac{d|\nu(w)|}{|w-z|} < \infty \). A standard application of Fubini’s Theorem shows that \( C\nu \in L^s_{\text{loc}}(\mathbb{C}) \) for \( 0 < s < 2 \), in particular that it is defined for \( \text{Area} \) almost all \( z \), and clearly \( C\nu \) is analytic in \( \mathbb{C}_\infty \setminus \text{sp}\mu \).

We denote the map
\[
E^t(\lambda) : p_0 + \sum_{i=1}^N p_i \phi_i \to \begin{bmatrix} p_0(\lambda) \\ p_1(\lambda) \\ \vdots \\ p_N(\lambda) \end{bmatrix}, \quad (1-1)
\]
where \( p_0, p_1, \ldots, p_N \in \mathcal{P} \). If \( E(\lambda) \) is bounded from \( P^t(1, \phi_1, \ldots, \phi_N, K) \) to \( C^{N+1}[\|x\|_{N+1}] \), where \( \|x\|_{N+1} = \sum_{i=0}^N |x_i| \) for \( x \in \mathbb{C}^{N+1} \), then every component in the right hand side extends to a bounded linear functional on \( P^t(1, \phi_1, \ldots, \phi_N, K) \) and we will call \( \lambda \) a bounded point evaluation for \( P^t(1, \phi_1, \ldots, \phi_N, K) \).

A bounded point evaluation \( \lambda_0 \) is called an analytic bounded point evaluation for \( P^t(1, \phi_1, \ldots, \phi_N, K) \) if there is a neighborhood \( B(\lambda_0, \delta) \) of \( \lambda_0 \) such that every \( \lambda \in B(\lambda_0, \delta) \) is a bounded point evaluation and \( E^t(\lambda) \) is analytic as a function of \( \lambda \) on \( B(\lambda_0, \delta) \) (equivalently \( 1-1 \) is uniformly bounded for \( \lambda \in B(\lambda_0, \delta) \)). Similarly, we can define a bounded point evaluation (or analytic bounded point evaluation) \( \lambda \) for \( HP(\phi_1, \ldots, \phi_N, K) \) by replacing \( 1-1 \) with the following map:
\[
E(\lambda) : r + \sum_{i=1}^N p_i \phi_i \to \begin{bmatrix} p_1(\lambda) \\ p_2(\lambda) \\ \vdots \\ p_N(\lambda) \end{bmatrix}, \quad (1-2)
\]
where \( p_1, p_2, \ldots, p_N \in \mathcal{P} \) and \( r \in \text{Rat}(K) \). Notice that the rational function \( r \in \text{Rat}(K) \) does not appear on the right hand side of definition (1-2).

For an arbitrary finite compactly supported positive measure \( \mu \), Thomson [1991] describes completely the structure of \( P^t(\mu) \), the closed subspace of \( L^t(\mu) \) spanned by \( \mathcal{P} \). Conway and Elias [1993] extends some results of Thomson’s Theorem to the space \( R^t(K, \mu) \), the closure of \( \text{Rat}(K) \) in \( L^t(\mu) \), while Brennan [2008] expresses \( R^t(K, \mu) \) as a direct sum that includes both Thomson’s theorem and results of Conway and Elias [1993]. For a compactly supported complex Borel measure \( \nu \) of \( \mathbb{C} \), by estimating analytic capacity of the set \( \{\lambda : |C\nu(\lambda)| \geq c\} \), Brennan [2006, English], Aleman et al [2005], and Aleman et al [2010] provide interesting alternative proofs of Thomson’s theorem. Both their proofs rely on X. Tolsa’s deep results on analytic capacity. Yang [2018] extends some results to a rationally multicentral subnormal operator (restriction of a normal operator on a separable Hilbert space to an invariant subspace).

However, even for \( \mu = dA|_K \), it is difficult to obtain necessary and sufficient conditions under which \( P^t(1, K) = L^t(K) \) or \( P^t(1, K) \) has a bounded point evaluation. Brennan and Militzer [2011] proves if \( R(K) \neq C(K) \), then \( P^t(1, K) \) has a bounded point evaluation. Yang [2016] shows that there exists a compact subset \( K \subset \mathbb{C} \) with \( R(K) = C(K) \), but \( P^t(1, K) \) still has bounded point evaluations. The first part of this is to extend the above result of Brennan and Militzer [2011] to \( P^t(1, \phi_1, \ldots, \phi_N, K) \) and
Theorem 1. There exist absolute constants $\epsilon_N, C_N > 0$ that only depend on $N$. If 
\[
\gamma(B(\lambda_0, \delta) \setminus K) < \epsilon_N \delta,
\]
then
\[
|p_N(\lambda)| \leq \frac{C_N}{\delta^{N+2}} \int_{K \setminus \bar{B}(\lambda_0, \delta)} \left| \sum_{k=0}^{N} p_k \bar{z}^k \right| dA, \ \lambda \in \bar{B} \left( \lambda_0, \frac{1}{2} \delta \right),
\]
(1)
\[
|p_N(\lambda)| \leq \frac{C_N}{\delta^{N+1}} \left\| \sum_{k=1}^{N} p_k \bar{z}^k + r \right\|_{K \setminus \bar{B}(\lambda_0, \delta)}, \ \lambda \in \bar{B} \left( \lambda_0, \frac{1}{2} \delta \right),
\]
(2)
where $r \in \text{Rat}(K \cap \bar{B}(\lambda_0, \delta))$ and $p_k \in \mathcal{P}$.

The proof of Theorem 1 depends on a careful modification of Thomson’s coloring scheme on dyadic squares. Thomson’s coloring scheme, for a point $a \in \mathbb{C}$ and a positive integer $m$, starts with a dyadic square of side length $2^{-m}$ containing $a$ and either terminates at some finite stage or produces an infinite sequence of annuli surrounding $a$. These annuli are made up of dyadic squares colored red (heavy square). When the scheme terminates at some finite stage, one can find a path consisting of many dyadic squares colored green (light square). The definition of a light square in Thomson (1993) (see also page 168 of Thomson (1993) or page 461 of Aleman et al. (2009)) only works for $P^t(1, K)$ and $HP(\bar{z}, K)$. Our definition of a light square (2-2) allows us to recursively extend (1-4) and (1-5) for $N > 1$.

Define 
\[
BA(\lambda_0, \delta) = \{ f \in C(C_{\infty}) : \exists m, f \in C^{(m)}(\bar{B}(\lambda_0, \delta)), \det A(\lambda_0, \delta) = 0 \},
\]
where $\bar{A}$ is the Cauchy-Riemann operator. We now state our first main result.

Theorem 2. Let $\lambda_0 \in K$ be a nonpeak point for $R(K)$ and $1 \leq t < \infty$. If there exist $\delta > 0$ and $F_1, F_2, \ldots, F_N \in BA(\lambda_0, \delta)$ such that
\[
\det \left( \partial^n F_i(\lambda_0) \right)_{n \times n} = 0,
\]
then
(1) $\lambda_0$ is an analytic bounded point evaluation for $P^t(1, F_1, \ldots, F_N, K)$. In particular, $\lambda_0$ is an analytic bounded point evaluation for $P^t(1, \bar{z}, \ldots, \bar{z}^N, K)$.

(2) $\lambda_0$ is an analytic bounded point evaluation for $HP(F_1, \ldots, F_N, K)$. In particular, $\lambda_0$ is an analytic bounded point evaluation for $HP(\bar{z}, \ldots, \bar{z}^N, K)$.

Let $\Lambda$ be a constant coefficient elliptic differential operator in $\mathbb{R}^2$. For a compact $K \subset \mathbb{C}$, let $H(K, \Lambda)$ and $h(K, \Lambda)$ denote the uniform closures in $C(K)$ of the set 
\[
\{ f \in K : \Lambda f = 0 \text{ in some neighborhood of } K \}
\]
and the set 
\[
C(K) \cap \{ f \in K : \Lambda f = 0 \text{ in the interior of } K \}
\]
respectively. Notice that, for $\Lambda = \bar{\partial}$, the space $H(K, \bar{\partial}) = R(K)$ and $h(K, \bar{\partial}) = A(K)$. For $\Lambda = \bar{\partial}^{n}$, the $n$th power of Cauchy-Riemann operator, the space
\[
H(K, \bar{\partial}^n) = \text{clo} (R(K) + \bar{z}R(K) + \ldots + \bar{z}^{n-1}R(K))
\]
and
\[
h(K, \bar{\partial}^n) = \text{clo} (A(K) + zA(K) + \ldots + z^{n-1}A(K)).
\]
One of uniform approximation problems is the following:

Problem 1. Find necessary and sufficient conditions for $K$ so that $H(K, \Lambda) = h(K, \Lambda)$.

A complete solution for $\Lambda = \Delta$ was obtained by Deny (1949) and Keldysh (1966) using a duality argument relying on classical potential theory. Let $\text{Cap}$ denote the Wiener capacity in potential theory. Deny and Keldysh show that the identity $H(K, \Delta) = h(K, \Delta)$ occurs if and only if for each open ball $B$ one has $\text{Cap}(B \setminus \text{int} K) = \text{Cap}(B \setminus K)$. 

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Using a constructive scheme for uniform approximation (based on a localization operator), Vitushkin (1967) proves that the identity \( H(K, \partial) = h(K, \partial) \) occurs if and only if for each open disc \( O \) one has \( \alpha(O \setminus \text{int}K) = \alpha(O \setminus K) \).

The inner boundary of \( K \), denoted by \( \partial K \), is the set of boundary points which do not belong to the boundary of any connected component of \( C \setminus K \). The remarkable paper Tolsa (2004) proves that the continuous analytic capacity is semiadditive. The result implies an affirmative answer to the so called inner boundary conjecture (see Vitushkin and Melnikov (1984), Conjecture 2). That is, if \( \alpha(\partial K) = 0 \), then \( R(K) = A(K) \).

For \( A = \partial^2 \), Trent and Wang (1981) show if \( K \) is a compact subset without interior, then \( H(K, \partial^2) = h(K, \partial^2) = C(K) \). Verdera (1993) proves that each Dincontinuous function in \( h(K, \partial^2) \) belongs to \( H(K, \partial^2) \). Finally, the excellent paper Mazalov (2004) completely solved the problem by proving \( H(K, \partial^2) = h(K, \partial^2) \) for any compact subset \( K \).

In Baranova et al. (2016), the authors consider an interesting analogous problem: find necessary and sufficient conditions so that \( P(K) + P(K)\bar{z}^n \) is dense in \( A(K) + A(K)\bar{z}^n \). The paper obtained some results for a Caratheodory compact set \( K \) with \( n \geq 2 \) (see Baranova et al. (2016), Theorem 1).

We define
\[
\phi(z, K) = \text{clos} \left( \sum_{i=1}^{N} P(K)\phi_i + A(K) \right).
\]

As analogous to Problem 1 we are interested in the following problem:

**Problem 2.** Find necessary and sufficient conditions so that
\[
\phi(z, K) = hP(\phi_1, \phi_2, ..., \phi_N, K) = hP(\phi_1, \phi_2, ..., \phi_N, K).
\]

For \( N = 1 \), the problem was studied by several authors. In Thomson (1992), Thomson proves if \( R(K) \neq C(K) \), then \( hP(z, K) \) is not equal to \( C(K) \). Yang (1992) and Yang (1994) study the generalized space \( hP(g, K) \) and prove that for a smooth function \( g \) with \( \partial g \neq 0 \), then \( hP(g, K) = hP(g, K) \) if and only if \( A(K) = R(K) \).

In the second part of this paper, we will study Problem 2 when \( N > 1 \). Our second main theorem is the following:

**Theorem 3.** Let \( q_1(z, \bar{z}), q_2(z, \bar{z}), ..., q_N(z, \bar{z}) \) be polynomials in two variables \( z \) and \( \bar{z} \). If
\[
A(K) \subset hP(q_1, ..., q_N, K),
\]
then \( A(K) = R(K) \). In particular, if \( R(K) \neq C(K) \), then \( hP(q_1, ..., q_N, K) \neq C(K) \). Notice that an important special case is that \( q_1(z, \bar{z}) = \bar{z}, q_2(z, \bar{z}) = \bar{z}^2, ..., q_N(z, \bar{z}) = \bar{z}^N \).

By Stone-Weierstrass theorem, \( \sum_{k=1}^{\infty} P(K)\bar{z}^k + R(K) \) is dense in \( C(K) \). So it is critical here to assume that \( N \) is a finite integer.

In Baranova et al. (2016), the authors are interested in the question of finding necessary and sufficient conditions so that
\[
\text{clos} \left( A(K) + \sum_{i=1}^{N} A(K)\bar{z}^{d_i} \right) = \text{clos} \left( P(K) + \sum_{i=1}^{N} P(K)\bar{z}^{d_i} \right),
\]
where \( d_1, ..., d_N \) are positive integers. Our theorem implies if (1-7) holds, then \( A(K) = R(K) \). So (1-7) is equivalent to
\[
\text{clos} \left( R(K) + \sum_{i=1}^{N} R(K)\bar{z}^{d_i} \right) = \text{clos} \left( P(K) + \sum_{i=1}^{N} P(K)\bar{z}^{d_i} \right).
\]

The following result shows that Theorem 3 will not hold if we replace \( q_n \) by certain functions.

**Proposition 1.** There exist a compact subset \( K \subset \mathbb{C} \) and a function \( \phi \in A(K) \) such that \( R(K) \neq A(K) \), but \( A(K) = hP(\phi, K) \).

The proposition raises the following question:

**Problem 3.** For a compact subset \( K \) of \( \mathbb{C} \), is there a function \( \phi \in A(K) \) such that \( A(K) = hP(\phi, K) \)?
2 Bounded Point Evaluations

In this section, we will prove Theorem 1 and Theorem 2 [Tolsa 2004] proves the astounding result about analytic capacity $\gamma$ that implies the semiadditivity of analytic capacity. That is,

$$\gamma\left(\bigcup_{i=1}^{m} E_i\right) \leq C_T \sum_{i=1}^{m} \gamma(E_i) \tag{2-1}$$

where $C_T$ is an absolute constant.

For a square $S$ (also denoted by $S(cS, dS)$), whose edges are parallel to x-axis and y-axis, let $cS$ denote the center and $dS$ denote the side length. For $a > 0$, $aS$ is a square with the same center of $S$ ($c_{aS} = c_S$) and the side length $d_{aS} = ad_S$. For a given $\epsilon > 0$, a closed square $S$ is defined to be light $\epsilon$ if

$$\gamma(\text{Int}(S \setminus K)) > \epsilon d_S. \tag{2-2}$$

A square is called heavy $\epsilon$ if it is not light $\epsilon$. Let $R = \{z : -1/2 < \text{Re}(z), \text{Im}(z) < 1/2\}$ and $Q = \bar{D} \setminus R$.

We now sketch our version of Thomson’s coloring scheme for $S$. For each integer $k > \log \frac{1}{\epsilon}$ let $\{S_{k,j}\}$ be an enumeration of the closed squares contained in $C$ with edges of length $2^{-k}$ parallel to the coordinate axes, and corners at the points whose coordinates are both integral multiples of $2^{-k}$ (except the starting square $S_{m1}$, see (3) below). In fact, Thomson’s coloring scheme is just needed to be modified slightly as the following:

1. Use our definition of a light $\epsilon$ square (2-2).
2. A path to $\infty$ means a path to any point that is outside of $Q$ (replacing the polynomially convex hull of $\Phi$ by $Q$).
3. The starting yellow square $S_{m1}$ in the $m$-th generation is $R$. Notice that the length of $S_{m1}$ in $m$-th generation is 1 (not $2^{-m}$).

We will borrow notations that are used in Thomson’s coloring scheme such as $\{\gamma_n\}_{n \geq m}$ and $\{\Gamma_n\}_{n \geq m}$, etc. We denote

$$\text{YellowBuffer}_m = \sum_{k=m+1}^{\infty} k^2 2^{-k}.$$  

Two things can happen (depending on $m$):

Case I. The scheme terminates. In our setup, this means Thomson’s coloring scheme reaches a square $S$ in $n$-th generation that is not contained in $Q$. One can construct a polygonal path $P$, which connects the centers of adjacent squares, from the center of a square (contained in $Q$) adjacent to $S$ to the center of a square adjacent to $R$ so that the orange (not green in Thomson’s coloring scheme) part of length is no more than $\text{YellowBuffer}_m$. Let $GP = \cup S_j$, where $\{S_j\}$ are all light $\epsilon$ squares with $P \cap S_j \neq \emptyset$. By Tolsa’s Theorem (2-1), we see

$$\gamma(P) \leq C_T(\gamma(\text{Int}(GP)) + \text{YellowBuffer}_m).$$

Since $P$ is a connected set, $\gamma(P) \geq \frac{1}{100}$ (Theorem 2.1 on page 199 of Gamelin 1969). We can choose $m$ to be large enough so that

$$\gamma(\text{Int}(GP)) \geq \frac{1}{400C_T} - \text{YellowBuffer}_m = \epsilon_m > 0. \tag{2-3}$$

Case II. The scheme does not terminate. In this case, one can construct a sequence of heavy $\epsilon$ barriers inside $Q$, that is, $\{\gamma_n\}_{n \geq m}$ and $\{\Gamma_n\}_{n \geq m}$ are infinite.

For simplicity, we will use scheme($Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m$) to stand for our version of Thomson’s coloring scheme.

If a function $f$ is analytic at $\infty$, then $f$ can be represented by its Laurent series

$$f(z) = f(\infty) + a_1(z - z_0)^{-1} + a_2(z - z_0)^{-2} + ...$$

in a neighborhood of infinite. We define $f'(\infty)$ to be $a_1$ and $\beta(f, z_0)$ to be $a_2$. The number $f'(\infty)$ does not depend on $z_0$, but $\beta(f, z_0)$ does depend on the choice of $z_0$.

Lemma 1. For a square $T$, if

$$\gamma(\text{Int}(T) \setminus K) \geq \epsilon_1 d_T,$$

then for two complex numbers $|\alpha| \leq 1$ and $|\beta| \leq 1$, there exists a function $f$ in $C(\mathbb{C}_\infty)$ such that the following hold:
(1) \(\|f\| \leq \frac{34}{\epsilon_1}\);  
(2) \(f \in R(\mathbb{C}_\infty \setminus (\text{Int}(T) \setminus K))\);  
(3) \(f(\infty) = 0\);  
(4) \(f'(\infty) = adT\);  
(5) \(\beta(f, c\gamma) = \beta d_j^2\).

Proof: There exists a function \(f_1\) in \(C(\mathbb{C}_\infty)\) such that \(\|f_1\| \leq 1\), \(f_1\) is analytic off a compact subset of \(\text{Int}(T) \setminus K\), \(f_1(\infty) = 0\), and \(f_1'(\infty) > \epsilon_1 d_T/2\). Set \(f_2 = d_T f_1/f_1'(\infty)\). Then by Theorem 2.5 of Gamelin (1969) on page 201, we get

\[|\beta(f_2, 0)| \leq \frac{12}{\epsilon_1 d_j^2}.\]

Let

\[f = \alpha \left( f_2 - \frac{\beta(f_2, 0)}{d_j^2} \right) + \beta f_2^2,\]

then \(f \in R(\mathbb{C}_\infty \setminus (\text{Int}(T) \setminus K))\) and satisfies the conditions (1)-(5).

Let \(\varphi\) be a smooth function with compact support. The localization operator \(T_\varphi\) is defined by

\[(T_\varphi f)(\lambda) = \frac{1}{\pi} \int \frac{f(z) - f(\lambda)}{z - \lambda} \partial \varphi(z) dA(z),\]

where \(f\) is a continuous function on \(\mathbb{C}_\infty\). One can easily prove the following norm estimation for \(T_\varphi\):

\[\|T_\varphi f\| \leq 4\|f\| \text{diameter(supp } \varphi\text{)} \|\partial \varphi\|.\]

Lemma 2. Suppose Case I of scheme \((Q, \epsilon, m, \gamma, n, n \geq m)\) is true, then

\[\gamma(D \setminus K) \geq \epsilon_1,\]

where

\[\epsilon_1 = 10^{-8} \epsilon_m\]

and \(\epsilon_m\) is from (2-3).

Proof: we will follow the second part of proof of Lemma B in Aleman et al. (2009) on pages 464-465 with slight modifications. Let \(GP = \cup S_j\), where \(\{S_j\}\) are light \(\epsilon\) squares discussed above, so that \(\gamma(\text{Int}(GP)) \geq \epsilon_m\). For each \(j\) let \(\dd_j\) be the center of \(S_j\), \(d_j\) be the edge length of \(S_j\), \(Q_j\), \(R_j\) be the closed squares with center \(\dd_j\) and sides parallel to the coordinate axes of lengths \(\frac{1}{3}d_j\), \(\frac{2}{3}d_j\), respectively.

The collection \(\{S_j\}\) has the following properties (see (2.16)-(2.18) on page 464 of Aleman et al. (2009)):

(a) No point lies in more than four \(Q_j\)'s.

(b) There are \(C^\infty\) functions \(\phi_j\) with \(0 \leq \phi_j \leq 1\), \(\text{spt} (\phi_j) \subset \text{Int}(Q_j)\), \(\|\partial \phi_j\| \leq \frac{50}{d_j}\), \(\phi_j = 1\) on \(R_j\), and \(\sum \phi_j = 1\) on \(GP\).

(c) For each \(z \in \mathbb{C}\),

\[\sum \min \left\{ 1, \frac{\delta_j^3}{|z - \dd_j|^3} \right\} \leq 10,000.\]

Let \(f \in C(\mathbb{C}_\infty)\) such that \(f\) is analytic off a compact subset of \(\text{Int}(GP)\), \(f(\infty) = 0\), \(\|f\| \leq 1\), and \(f'(\infty) > \frac{400}{d_j}\). Then from (b), we see that \(f - \sum T_{\phi_j} f\) is zero on \(GP\) and analytic off \(GP\). Hence,

\[f(z) = \sum T_{\phi_j} f(z)\]

for all \(z \in \mathbb{C}\) and

\[T_{\phi_j} f(\infty) = \sum T_{\phi_j} f \leq 400, \quad \|T_{\phi_j} f\| \leq 400d_j, \quad \|T_{\phi_j} f(\infty)\| \leq 400d_j, \quad \text{and} \quad |\beta(T_{\phi_j} f, \dd_j)| \leq 400d_j^2.\]

For \(\alpha = \frac{(T_{\phi_j} f)'(\infty)}{400d_j}\) and \(\beta = \frac{\beta(T_{\phi_j} f, \dd_j)}{400d_j^2}\), using Lemma [1] for the light \(\epsilon\) square \(S_j\), we find a function \(f_j\) in \(C(\mathbb{C}_\infty)\) such that \(\|f_j\| \leq \frac{34}{\epsilon_j}\), \(f_j \in R(\mathbb{C}_\infty \setminus (\text{Int}(S_j) \setminus K))\), and \(\frac{\beta(T_{\phi_j} f, \dd_j)}{400d_j} - f_j\) has triple zeros at \(\infty\).

Therefore, \(\frac{\beta(T_{\phi_j} f, \dd_j)}{400d_j} - f_j) (z - \dd_j)^3\) is analytic on \(\mathbb{C}_\infty \setminus Q_j\). Using the Maximum Modulus Theorem, we see if \(z \in \mathbb{C}_\infty \setminus Q_j\), then

\[\left| \frac{T_{\phi_j} f(z)}{400} - f_j(z) \right| \leq (1 + \frac{54}{\epsilon_j}) \frac{\delta_j^3}{|z - \dd_j|^3} \leq \frac{55}{\epsilon_j} \frac{\delta_j^3}{|z - \dd_j|^3}.\]
Hence, for \( z \in \mathbb{C}_\infty \),

\[
\left| \frac{T_{\psi_j} f(z)}{400} - f_j(z) \right| \leq \frac{55}{\epsilon^3} \min \left( 1, \frac{\delta^3}{|z - z_j|^3} \right).
\]

Set \( F = 400 \sum f_j \), then \( F \) is analytic off a compact subset of \( \text{Int}(GP) \setminus K \), \( F(\infty) = 0 \), \( F'(\infty) = f'(\infty) \), and

\[
\|F\|_\infty \leq \|f\|_\infty + \sum \|T_{\psi_j} f - 400 f_j\|_\infty \\
\leq 1 + \frac{2200}{\epsilon^3} \sum \min \left( 1, \frac{\delta^3}{|z - z_j|^3} \right) \\
\leq 5 \times 10^7.
\]

where the last step follows from (c). Therefore,

\[
\gamma(\mathbb{D} \setminus K) \geq \gamma(\text{Int}(GP) \setminus K) \geq \frac{F'(\infty)}{8 \times 10^7} > 10^{-8} \epsilon^3 \epsilon_m = \epsilon_1.
\]

**Lemma 3.** Suppose Case II of scheme \((Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)\) is true.

1. If there exists \( \epsilon_1 > 0 \) such that for every heavy \( \epsilon \) square \( S \) in scheme \((Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)\),

\[
\text{dist} \left( z^N, P^1(1, \bar{z}, ..., \bar{z}^{N-1}, K \cap S) \right) \geq \epsilon_1 d_S^{N+2}, \tag{2-6}
\]

then there exists a constant \( C_{m,n} \) (depends on \( m \) and \( N \)) such that

\[
|p_N(\lambda)| \leq C_{m,n} \left\| \sum_{j=0}^N p_j \bar{z}^j \right\|_{L^1(K \cap Q)}, \tag{2-7}
\]

where \( p_j \in \mathcal{P} \) and \( N = 1, 2, ..., \). For \( N = 0 \), if (2-6) is replaced by

\[
\text{Area}(K \cap S) \geq \epsilon_0 d_S^2, \tag{2-8}
\]

then (2-7) holds for \( N = 0 \).

2. If there exists \( \epsilon_1 > 0 \) such that for every heavy \( \epsilon \) square \( S \) in scheme \((Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)\),

\[
\text{dist} \left( z^N, HP(z, \bar{z}, ..., \bar{z}^{N-1}, K \cap S) \right) \geq \epsilon_1 d_S^N, \tag{2-9}
\]

then there exists a constant \( C_{m,n} \) (depends on \( m \) and \( N \)) such that

\[
|p_N(\lambda)| \leq C_{m,n} \left\| \sum_{j=1}^N p_j \bar{z}^j + r \right\|_{K \cap Q}, \tag{2-10}
\]

where \( r \in \text{Rat}(K \cap Q) \), \( p_j \in \mathcal{P} \), and \( N = 1, 2, ... \).

Proof: The proofs of section 4 in [Thomson (1991)] and [Thomson (1993)] will work if we make the following modifications:

(a) For \( w \in \gamma_n \cap S \), where \( S \) is a heavy \( \epsilon \) square, by the Hahn-Banach theorem, there exists a finite Borel measure \( \sigma_w \) supported in \( K \cap S \) and \( \|\sigma_w\| = 1 \) such that for (1), \( \sigma_w = \Phi_w dA \) with \( \Phi_w \in L^\infty(K \cap S) \),

(2-6) becomes

\[
\text{dist} \left( z^N, P^1(1, \bar{z}, ..., \bar{z}^{N-1}, K \cap S) \right) = \int z^N d\sigma_w \geq \epsilon_1 d_S^{N+2},
\]

where \( \int f d\sigma_w = 0 \) for \( f \in P^1(1, \bar{z}, ..., \bar{z}^{N-1}, K \cap S) \), and (2-8) becomes

\[
\sigma_w = \chi_{K \cap S} dA, \quad \text{Area}(K \cap S) = \int d\sigma_w \geq \epsilon_0 d_S^2;
\]

and for (2), (2-9) becomes

\[
\text{dist} \left( z^N, HP(z, \bar{z}, ..., \bar{z}^{N-1}, K \cap S) \right) = \int z^N d\sigma_w \geq \epsilon_1 d_S^N,
\]

where \( \int f d\sigma_w = 0 \) for \( f \in HP(z, \bar{z}, ..., \bar{z}^{N-1}, K \cap S) \). Define \( \tau_w = \frac{z^N d\sigma_w}{\int z^N d\sigma_w} \).
(b) Set \( L = e_\lambda (e_\lambda(f) = f(\lambda)) \) for \( \lambda \in R \). Use the same argument as in section 4 in [Thomson (1993)], one can construct \( \mu_{n+q} \) defined by linear combination of \( \tau_w \), then
\[
\|\mu_{n+q}\| \leq C_{n+q}^N \|\mu\| \leq C_{n+q} \psi \|\mu\| \leq C_{n+q} \psi \|\mu\|
\]
for (1) and
\[
\|\mu_{n+q}\| \leq C_{n+q} \psi \|\mu\| \leq C_{n+q} \psi \|\mu\|
\]
for (2). Let \( \mu = \mu_n + \mu_{n+q} + \ldots \), then \( \|\mu\| \leq C_{n,N} \), for (1)
\[
p_N(\lambda) = \int \sum_{j=0}^{N} p_j \lambda^j d\mu,
\]
and for (2)
\[
p_N(\lambda) = \int (\sum_{j=1}^{N} p_j \lambda^j + r) d\mu.
\]
Clearly, the support of \( \mu \) is outside \( R \). The proof is completed.

The idea to prove our Theorem 1 is to find sufficient small \( \epsilon \) so that Case II of scheme \((Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)\) is true. Then we use mathematical induction to show that for every heavy \( \epsilon \) square, we can find \( \epsilon_N \) so that (2-6), (2-8), and (2-9) all hold. We will demonstrate the idea by proving the following corollary, which is the case for \( N = 0 \) in (1) of Theorem 1 and Lemma B in [Aleman et al. (2009)].

**Corollary 1.** There are absolute constants \( \epsilon_0 > 0 \) and \( C_0 < \infty \) with the following property. For a compact subset \( K \subset \subset \), let \( R > 0 \) and \( \gamma(RD \setminus K) < \epsilon_0 R \). Then
\[
|p(\lambda)| \leq \frac{C_0}{R^2} \int (RD \setminus K) |p| \frac{dA}{\pi}
\]
for \( |\lambda| \leq \frac{B}{2} \) and all \( p \in P \).

Proof: Since \( \gamma(RD \setminus K) = R \gamma(D \setminus \frac{B}{2}) \), by a simple changing of variables from \( z \) to \( Rz \), we assume \( R = 1 \). Let \( \epsilon_1 = \epsilon_0 = 10^{-\epsilon} \epsilon_m \) in (2-5) with \( \epsilon < \sqrt{\frac{1}{4\pi}} \). Then from Lemma 2 we conclude that Case II of scheme \((Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)\) must be true. Let \( S \) be a heavy \( \epsilon \) square, then \( \gamma(Int(S) \setminus K) \leq \epsilon S \).

By Theorem 3.2 on page 204 of [Gamelin (1969)], we get
\[
\text{Area}(S \setminus K) \leq 4\pi \gamma(Int(S) \setminus K)^2 \leq 4\pi \epsilon^2 d_S^2.
\]
Therefore,
\[
\text{Area}(S \cap K) \geq (1 - 4\pi \epsilon^2) d_S^2.
\]
So (2-8) holds and the corollary follows from Lemma 3.

Let \( \phi \) be a smooth function supported in \( D \) such that:
\[
0 \leq \phi \leq 1, \quad \phi(z) = \phi(|z|), \quad \|\partial_N \phi\| \leq C_N, \quad \int \phi dA = 1.
\]
(2-11)

Proof of Theorem 1 (1): We only need to prove the case that \( \lambda_0 = 0 \) and \( \delta = 1 \). In fact, using the elementary properties of analytic capacity (see p. 196 of [Gamelin (1969)]), one sees that condition (1-3) is equivalent to
\[
\gamma \left(B(0,1) \setminus \frac{K - \lambda_0}{\delta}\right) < \epsilon N.
\]
The inequality (1-4)
\[
|p_N(\lambda)| \leq \frac{C_N}{\delta^{N+2}} \left\| \sum_{k=0}^{N} p_k \lambda^k \right\|_{L^1(K \cap B(\lambda_0,\delta))}\]
\[
= \frac{C_N}{\delta^2} \left\| \sum_{k=0}^{N} \frac{q_k}{\delta} \left( \frac{\lambda - \lambda_0}{\delta}\right)^k \right\|_{L^1(K \cap B(\lambda_0,\delta))},
\]
for \( \lambda \in B(\lambda_0, \frac{1}{2}\delta) \), where \( q_N = p_N, q_k (1 \leq k \leq N - 1) \) are certain linear combinations of \( p_k \), is equivalent to
\[
|p_{N}(\lambda)| \leq C_N \left\| \sum_{k=0}^{N} p_k \lambda^k \right\|_{L^1(K \cap B(0,1))}, \quad \lambda \in B \left(0, \frac{1}{2}\right),
\]
8
Hence, 

\[ \left\| \sum_{j=0}^{N} p_j \hat{z}^j \right\|_{L^1(K \cap S)} \leq 2 \operatorname{dist}\left( \hat{z}^{N+1}, P^1(1, \hat{z}, \ldots, \hat{z}^N, K \cap S) \right) < d_S^{N+3}. \]  

(2-12)

Hence,

\[ \left\| \sum_{j=0}^{N} p_j \hat{z}^j \right\|_{L^1(K \cap S)} \leq d_S^{N+3} + \left\| z^{N+1} \right\|_{L^1(K \cap S)} \leq 2 d_S^{N+3}. \]

Since \( S \) is a heavy \( \epsilon \) square, we get

\[ \gamma(B(0, \frac{d_S}{2}) \setminus K) \leq \gamma(S \setminus K) < \epsilon d_S \leq \epsilon N \frac{d_S}{2}. \]

Applying the induction assumption for \( N \), from (1-4), we have

\[ |p_N(\lambda)| \leq \frac{C_N}{d_S} \left\| \sum_{j=0}^{N} p_j \hat{z}^j \right\|_{L^1(K \cap B(0, \frac{d_S}{2}))} < 2C_N d_S, \]

for \( \lambda \in B(0, \frac{d_S}{2}) \). This implies

\[ \left\| z^{N+1} + P_N(z)z^N \right\|_{L^1(K \cap B(0, \frac{d_S}{2}))} \leq 3C_N d_S^{N+3}. \]

From (2-12), we conclude

\[ \left\| \sum_{j=0}^{N-1} p_j \hat{z}^j \right\|_{L^1(K \cap B(0, \frac{d_S}{2}))} \leq 4C_N d_S^{N+3}. \]

In general, there is an absolute constant \( C_{N+1} > 0 \) so that

\[ \left\| \sum_{j=0}^{N-k} p_j \hat{z}^j \right\|_{L^1(K \cap B(0, \frac{d_S}{2k+1}))} \leq C_{N+1} d_S^{N+3}. \]

Since \( S \) is a heavy \( \epsilon \) square, we get

\[ \gamma(B(0, \frac{d_S}{2k+1}) \setminus K) \leq \gamma(S \setminus K) < \epsilon d_S \leq \epsilon N \frac{d_S}{2k+1}. \]

Apply the induction assumption for \( N - k \), from (1-4), we get

\[ |p_{N-k}(\lambda)| \leq \frac{C_{N-k}}{d_S^{N-k+2}} \left\| \sum_{j=0}^{N-k} p_j \hat{z}^j \right\|_{L^1(K \cap B(0, \frac{d_S}{2k+1}))} \leq C_{N-k} C_{N+1} d_S^{k+1}, \]

for \( \lambda \in B(0, \frac{d_S}{2k+1}) \). So there is an absolute constant which we still use \( C_{N+1} > 0 \) such that

\[ \left\| z^{N+1} + \sum_{j=0}^{N} p_j(z) \hat{z}^j \right\| \leq C_{N+1} d_S^{N+1}. \]
Lemma 4. Let $g_{N+1} = z^{N+1} + \sum_{j=0}^{N} p_j z^j$, then $|g_{N+1}(z)| \leq C_{N+1} d_S^{N+1}$ on $z \in B(0, \frac{d_S}{2^{N+1}})$. Let $\phi_{N+1}(z) = \phi(z^{N+1}),$ where $\phi$ is in (2-11), then $spt(\phi_{N+1}) \subset B(0, \frac{d_S}{2^{N+1}})$,

$$0 \leq \phi_{N+1} \leq 1, \quad \|\partial^{N+1} \phi_{N+1}\| < C_{N+1} \phi_{N+1}/d_S^{N+1}, \quad \int \phi_{N+1} dA = \frac{d_S}{4^{N+1}}.$$

Then

$$(N+1)!d_S^{N+1} = \int \phi_{N+1} dA$$

$$\leq \int g_{N+1} \partial^{N+1} \phi_{N+1} dA$$

$$\leq \frac{C_{N+1} \phi_{N+1}}{d_S^{N+1}} \left( \|g_{N+1}\| L^1(K \cap B(0, \frac{d_S}{2^{N+1}})) + \|g_{N+1}\| L^1(B(0, \frac{d_S}{2^{N+1}}) \setminus K) \right)$$

$$\leq \frac{C_{N+1} \phi_{N+1}}{d_S^{N+1}} \|g_{N+1}\| L^1(K \cap S) + C_{N+1} \phi_{N+1} \text{Area}(B(0, \frac{d_S}{2^{N+1}}) \setminus K)$$

$$\leq 2 \frac{C_{N+1} \phi_{N+1}}{d_S^{N+1}} \text{dist} \left( z^{N+1}, P^1(1, \bar{z}, \ldots, \bar{z}, K \cap S) \right) + 4\pi C_{N+1} C_{N+1} \gamma(K \cap S) \geq \frac{(N+1)!}{2^{2N+4}C_{N+1}} d_S^{N+3}.$$

where the last step follows from (2-12) and Theorem 3.2 on page 204 of Gamelin [1969]. Now choose

$$c_1^2 = \frac{(N+1)!}{2^{2N+4}C_{N+1} C_{N+1}}.$$

then since $S$ is a heavy $\epsilon$-square, we have

$$\text{dist} \left( z^{N+1}, P^1(1, \bar{z}, \ldots, \bar{z}, K \cap S) \right) \geq \frac{(N+1)!}{2^{2N+4}C_{N+1} C_{N+1}} d_S^{N+3}.$$

So (2-6) holds and the theorem follows from Lemma 3.1.

For a smooth function $\varphi$ with compact support, $T^*_\varphi$ is a bounded linear operator on $C(K)$. Let $M$ be the space of finite complex Borel measures supported on $K$ ($= C(K)^*$), then $T^*_\varphi$ is a bounded linear operator on $M$. Moreover, for $\mu \in M$,

$$\int T^*_\varphi f d\mu = \int f dT^*_\varphi \mu$$

and

$$\|T^*_\varphi \mu\| \leq 4\text{diameter(supp \varphi)} ||\partial \varphi|| \|\mu\|.$$  

(2-13)

Consequently, $T^*_\varphi \mu \perp R(K)$ for each $\mu \perp R(K)$.

**Lemma 4.** Let $S$ be a square with center 0 and $d_S < 1$. Let $g_N = \bar{z}^N + \sum_{k=1}^{N-1} p_k \bar{z}^k$, where $p_k$ are polynomials and $\|g_N\| \leq C_N d_S^N$ ($C_N$ is an absolute constant depending on $N$) on $S$. If

$$\text{dist}(g_N, R(K \cap S)) \leq c_N d_S^N,$$

where

$$c_N = \frac{N!}{2^{2N+4}C_N C_N}$$

and $C_N^\phi$ is in (2-11). Then

$$\gamma(K \cap S) \leq \frac{4c_N}{C_N + 4c_N} d_S.$$

Proof: Let $\phi_1 = \bar{\partial}^{N+1} \phi(\frac{z}{d_S})$, where $\phi$ is in (2-11). Let $f_1 = T^*_\varphi g_N$, then

$$\|f_1\| \leq 2^{N+3} C_N^\phi C_N d_S$$

and

$$f_1(\infty) = \frac{1}{\pi} \int \partial g_N \phi_1 dA = (-1)^N \frac{N! d_S^3}{4\pi}.$$
We have the following computation:

\[ \text{dist}(f_1, R(\mathbb{C}_\infty \setminus \{\text{Int}(S) \setminus K\})) = \sup_{\mu \perp R(\mathbb{C}_\infty \setminus \{\text{Int}(S) \setminus K\})} \left| \int f_1 d\mu \right| \]

\[ = \sup_{\mu \perp R(\mathbb{C}_\infty \setminus \{\text{Int}(S) \setminus K\})} \left| \int g_N dT_{\phi_1}^\mu \right| \]

\[ \leq \sup_{\mu \perp R(\mathbb{C}_\infty \setminus \{\text{Int}(S) \setminus K\})} \|T_{\phi_1}^\mu\| \text{dist}(g_N, R(K \cap S)) \]

\[ \leq 8d_S \|\partial^N \phi_1\| \text{dist}(g_N, R(K \cap S)) \leq 2N+3C_N^2 \epsilon_N d_S, \]

where we use the fact that $T_{\phi_1}^\mu \perp R(K \cap S)$ for $\mu \perp R(\mathbb{C}_\infty \setminus \{\text{Int}(S) \setminus K\})$. Let

\[ f = \frac{f_1}{(-1)^N-2N+3C_N^2 \epsilon_N d_S}, \]

then $f$ is analytic off $S$, $\|f\| \leq 1$, $f'(\infty) = 8C_N^2 \epsilon_N d_S$, and

\[ \text{dist}(f, R(\mathbb{C}_\infty \setminus \{\text{Int}(S) \setminus K\})) \leq \frac{C_N}{C_N^2}. \]

Then there is a compact subset $F$ of $\text{Int}(S) \setminus K$ and a rational function $r$ with poles in $F$ such that

\[ \|f - r\|_{\mathbb{C}_\infty} < \frac{2C_N}{C_N^2}. \]

Hence, $|r(\infty)| < 2\frac{2C_N}{C_N^2}, \|r\|_{\mathbb{C}_\infty} \leq 1 + 2\frac{2C_N}{C_N^2}$, and by the maximum modulus principle,

\[ |f'(\infty) - r'(\infty)| = \lim_{z \to \infty} |z(f(z) - r(z) + r(\infty))| \leq \max_{|z|=d_S} |z(f(z) - r(z) + r(\infty))| < 4 \frac{C_N}{C_N^2} d_S. \]

Therefore,

\[ \gamma(\text{Int}(S) \setminus K) \geq \gamma(F) \geq \frac{|r'(\infty)|}{\|r\|_{\mathbb{C}_\infty} + |r(\infty)|} \geq \frac{4C_N}{C_N + 4C_N} d_S. \]

Proof of Theorem\ref{thm:main}(2): We assume $\lambda_0 = 0$ and $\delta = 1$ (same argument used in the first paragraph of the proof of Theorem\ref{thm:main}(1)).

We use mathematical induction for $N$. First we consider the case that $N = 1$. Let $\epsilon = \frac{4c_1}{C_1 + 4c_1}$ in Lemma\ref{lem:induction}. Let $\epsilon_1$ be as in (2-5). Then from assumption (1-3) and Lemma\ref{lem:caseII} we conclude that Case II of scheme$$(Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)$$is true. Let $S$ be a heavy $\epsilon$ square and $g_1 = \bar{z} - \epsilon S$, then by Lemma\ref{lem:induction} we must have

\[ \text{dist}(\bar{z}, R(S \cap K)) = \text{dist}(g_1, R(S \cap K)) \geq c_1 d_S. \]

Hence, (2-9) holds for $N = 1$, by Lemma\ref{lem:caseII} we prove (1-5) for $N = 1$.

Now we assume that (1-5) holds for $k = 1, 2, \ldots, N$. The proof is similar to that of Theorem\ref{thm:main}(1). Set

\[ \epsilon = \min\{\epsilon_0, \frac{\epsilon_N}{2}, \frac{\epsilon_{N-1}}{2^2}, \ldots, \frac{\epsilon_1}{2^N}, \frac{\epsilon_0}{2^{N+1}}\}, \]

where $\epsilon_0 > 0$ will be determined later, and $\epsilon_{N+1}$ as in (2-5). Since $\gamma(B(0,1) \setminus K) < \epsilon_{N+1}$ (assumption (1-3)), from Lemma\ref{lem:induction} we conclude that Case II of scheme$$(Q, \epsilon, m, \gamma_n, \Gamma_n, n \geq m)$$is true. Let $S$ be a heavy $\epsilon$ square. We assume

\[ \text{dist} \left( \bar{z}^{N+1}, HP(1, \bar{z}, \ldots, \bar{z}^N, K \cap S) \right) \leq \frac{1}{2} d_S^{N+1}, \]

otherwise (2-9) already holds. Without loss of generality, we assume the center of $S$ is zero. There are polynomials $p_1, \ldots, p_N$ and a rational function $r$ with poles off $K \cap S$ such that

\[ \left\| \bar{z}^{N+1} + \sum_{j=1}^N p_j \bar{z}^j + r \right\|_{K \cap S} \leq d_S^{N+1}. \]
Using the same argument of the paragraph (in the proof of Theorem 4.1) under (2-12), we get
\[
\left\|z^{N+1} + \sum_{j=1}^{N} p_j z^j\right\|_{K \cap B(0, \frac{d}{2^{N+1}})} \leq C_{N+1} d_{S}^{N+1}.
\]
Let \(g_{N+1} = z^{N+1} + \sum_{j=1}^{N} p_j z^j\), then \(\|g_{N+1}\| \leq C_{N+1} d_{S}^{N+1}\) on \(\frac{1}{2^{N+1}} S \subset B(0, \frac{d}{2^{N+1}})\). By Lemma 4 for \(\frac{1}{2^{N+1}} S\), if we choose
\[
c_0 = \frac{4c_{N+1}}{C_{N+1} + 4c_{N+1}},
\]
we must have
\[
dist\left(z^{N+1}, HP(1, z, ..., z^N, K \cap S)\right) = dist\left(g_{N+1}, HP(1, z, ..., z^N, K \cap S)\right) \geq c_{N+1} d_{S}^{N+1}.
\]
Therefore, (2-9) holds and the theorem now follows from Lemma 3.

Proof of Theorem 4. Assume that \(\lambda_0\) is not a peak point for \(R(K)\), then by Curtis’s Criterion (see Theorem 4.1 in Gamelin (1969), p. 204), we get,
\[
\limsup_{\delta \to 0} \frac{\gamma(B(\lambda_0, \delta) \setminus K)}{\delta} = 0.
\]
Then together with the assumptions for \(F_1, ..., F_N\), we can choose \(\delta > 0\) such that
\[
F_{i}(z) = \sum_{j=0}^{M} g_{ij}(z)z^j, \ z \in B(\lambda_0, \delta),
\]
and
\[
\gamma(B(\lambda_0, \delta) \setminus K) < \epsilon \delta
\]
where \(M \geq N, g_{ij} \in A(B(\lambda_0, \delta)), j = 1, ..., N, i = 0, 1, ..., M,\)
\[
\epsilon = \min\left(\epsilon_M, \frac{\epsilon_{M-1}}{2}, ..., \frac{\epsilon}{2M-1}\right),
\]
and \(\epsilon_1, ..., \epsilon_M\) are in Theorem 4. Set
\[
F_{0}(z) = \sum_{i=0}^{M} g_{i0}(z)z^i, \ z \in B(\lambda_0, \delta),
\]
where \(g_{i0} = 1\) and \(g_{io} = 0\) for \(i = 1, ..., M\). Then
\[
\begin{pmatrix}
F_{0} & F_{1} & ... & F_{N} \\
0 & \partial F_{1} & ... & \partial F_{N} \\
0 & \partial^2 F_{1} & ... & \partial^2 F_{N} \\
0 & \partial^3 F_{1} & ... & \partial^3 F_{N}
\end{pmatrix}
= \begin{pmatrix}
1 & \bar{z} & \bar{z}^2 & ... & \bar{z}^M \\
0 & 1 & 2\bar{z} & ... & M\bar{z}^{M-1} \\
0 & 0 & 2 & ... & M(M-1)\bar{z}^{M-2} \\
0 & 0 & 0 & ... & (M-N+1)\bar{z}^{M-N}
\end{pmatrix}
\begin{pmatrix}
g_{00} & g_{01} & ... & g_{0N} \\
g_{10} & g_{11} & ... & g_{1N} \\
g_{20} & g_{21} & ... & g_{2N} \\
g_{M0} & g_{M1} & ... & g_{MN}
\end{pmatrix}
\]
Since the matrix \([\partial^i F_{j}(\lambda_0)]_{1 \leq i, j \leq N}\) is invertible, we may choose above \(\delta\) small enough and \(i_0 = 1 < i_1 < i_2 < ... < i_N\) so that
\[
G(\lambda) = \begin{pmatrix}
g_{00} & g_{01} & ... & g_{0N} \\
g_{10} & g_{11} & ... & g_{1N} \\
... & ... & ... & ... \\
g_{N0} & g_{N1} & ... & g_{NN}
\end{pmatrix}
\]
is analytic and invertible on \(B(\lambda_0, \delta)\). Moreover,
\[
\|G(\lambda)^{-1}\| \leq C, \ \lambda \in B(\lambda_0, \delta).
\]
So \(G(\lambda)\) and \(G(\lambda)^{-1}\) are linear and uniform bounded operators from \((C^{N+1}, \|\cdot\|_{N+1})\) to \((C^{N+1}, \|\cdot\|_{N+1})\), where \(\|x\|_{N+1} = \sum_{i=1}^{N+1} |x_i|\). For polynomials \(p_0, p_1, ..., p_N\), we get
\[
\sum_{i=0}^{M} \sum_{j=0}^{N} p_j(\lambda) g_{ij}(\lambda) \geq \sum_{k=0}^{N} \sum_{j=0}^{N} p_j(\lambda) g_{kj}(\lambda)
\leq \|(p_0(\lambda), p_1(\lambda), ..., p_N(\lambda))G(\lambda)\|
\geq \frac{1}{C} \sum_{j=0}^{N} |p_j(\lambda)|
\]
12.
Lemma 5. \( B(0, \delta) \).

Now let us prove (1). From (2-14) and Theorem 1 (1), we have
\[
|q_M(\lambda)| \leq \frac{C_M}{\delta^{M+1}} \left\| \sum_{i=0}^{M} q_i \bar{z}^i \right\|_{L^1(K \cap \bar{B}(\lambda_0, \frac{\delta}{2}))}
\]
on \( B(\lambda_0, \frac{\delta}{2}) \) and \( q_0, q_1, ..., q_M \) are polynomials. Using (2-14) and Theorem 1 (1) again, we have the following calculation
\[
|q_{M-1}(\lambda)| \leq \frac{C_{M-1}}{(\frac{\delta}{2})^{M+1}} \left\| \sum_{i=0}^{M-1} q_i \bar{z}^i \right\|_{L^1(K \cap \bar{B}(\lambda_0, \frac{\delta}{2}))}
\]
on \( B(\lambda_0, \frac{\delta}{2}) \). Therefore, there is a constant \( C_N > 0 \) such that
\[
\sum_{i=0}^{M} |q_i(\lambda)| \leq C_N \sum_{i=0}^{M} q_i \bar{z}^i \left\|_{L^1(K \cap \bar{B}(\lambda_0, \delta))} \right\|
\]
on \( B(\lambda_0, \frac{\delta}{2}) \). From (2-15) and (2-16), we see that
\[
\sum_{j=0}^{N} |p_j(\lambda)| \leq C \sum_{i=0}^{M} \sum_{j=0}^{N} p_j(\lambda) g_{ij}(\lambda) \leq CC_N \sum_{j=0}^{N} p_j F_j \left\|_{L^1(K \cap \bar{B}(\lambda_0, \delta))} \right\|
\]
on \( B(\lambda_0, \frac{\delta}{2}) \). So \( \lambda_0 \) is an analytic bounded point evaluation for \( P^t(1, F_1, ..., F_N, K \cap \bar{B}(\lambda_0, \delta)) \).

The proof of (2) is the same. (2-16) becomes
\[
\sum_{i=1}^{M} |q_i(\lambda)| \leq C_N \sum_{i=1}^{M} q_i \bar{z}^i + r \left\|_{K \cap \bar{B}(\lambda_0, \delta)} \right\|
\]
where \( r \) is a rational function with poles off \( K \cap \bar{B}(\lambda_0, \delta) \). (2-17) becomes
\[
\sum_{j=1}^{N} |p_j(\lambda)| \leq C \sum_{i=1}^{M} \sum_{j=1}^{N} p_j(\lambda) g_{ij}(\lambda) \leq CC_N \sum_{j=1}^{N} p_j F_j + r \left\|_{K \cap \bar{B}(\lambda_0, \delta)} \right\|
\]
on \( B(\lambda_0, \frac{\delta}{2}) \). So \( \lambda_0 \) is an analytic bounded point evaluation for \( HP(F_1, ..., F_N, K \cap \bar{B}(\lambda_0, \delta)) \).

3 Uniform Rational Approximation

In this section, we will prove Theorem 3 and Proposition 1. To prove Theorem 3 we need to prove several Lemmas.

Lemma 5. If \( R(K) \neq C(K) \), then \( HP(\bar{z}, \bar{z}^2, ..., \bar{z}^N, K) \neq C(K) \).

Proof: Notice, by Lemma 1 (2-10),
\[
|p_N(\lambda)| \leq \frac{CN}{\delta^N} \sum_{k=1}^{N} p_k \bar{z}^k + r \left\|_{K \cap \bar{B}(\lambda_0, \delta) \setminus B(\lambda_0, \frac{\delta}{2})} \right\|, \lambda \in \bar{B} \left( \lambda_0, \frac{1}{2^2} \right).
\]
Then there exists a finite Borel measure \( \mu \) supported on \( K \cap \bar{B}(\lambda_0, \delta) \setminus B(\lambda_0, \frac{\delta}{2}) \) such that
\[
p_N(\lambda_0) = \int \left( \sum_{k=1}^{N} p_k \bar{z}^k + r \right) d\mu,
\]
where \( r \in \text{Rat}(K \cap \bar{B}(\lambda_0, \delta) \setminus B(\lambda_0, \frac{\delta}{2})) \) and \( p_k \in \mathcal{P} \). Therefore, the non zero measure \( (z - \lambda_0) \mu \perp HP(\bar{z}, \bar{z}^2, ..., \bar{z}^N, K) \) and \( HP(\bar{z}, \bar{z}^2, ..., \bar{z}^N, K) \neq C(K) \).
Lemma 6. Let $T$ be a closed square and $\gamma(B(cT, \sqrt{2d}T) \setminus K) < \epsilon_N \sqrt{2d}$. If $A(K) \subset HP(z, z^2, ..., z^N, K)$, $\phi$ is a smooth function supported inside $T$, and $f \in A(K)$, then $T_\phi f \in R(K)$.

Proof: Case 1: Suppose $Int(K \cap T) = \emptyset$. Then $A(K \cap T) = C(K \cap T) \subset HP(z, z^2, ..., z^N, K \cap T)$ since each smooth function with support in $K \cap T$ belongs to $A(K)$. From Lemma 5 we get $C(K \cap T) = R(K \cap T)$. Hence, $T_\phi f \in R(K)$.

Case 2: $Int(K \cap T) \neq \emptyset$. There are sequences of $\{p_{ij}\}, 1 \leq i \leq N, 1 \leq j < \infty \subset P$ and $\{r_j\} \subset Rat(K)$ such that

$$\lim_{j \to \infty} \left( \sum_{i=1}^{N} p_{ij}z^i + r_j \right) = T_\phi f$$

uniformly on $K$ since $T_\phi A(K) \subset A(K)$. By Theorem 1(2), for each $i$, the sequence $\{p_{ij}\}, 1 \leq j < \infty$ converges to an analytic function $f_i$ uniformly on $T \subset B(cT, \sqrt{2d}T / 2)$. Hence $\{r_j\}$ converges to $r$ uniformly on $K \cap T$ that is analytic on $Int(K \cap T)$ and $T_\phi f(z) = \sum_{i=1}^{N} f_i(z)\bar{z}^i + r(z)$ on $Int(K \cap T)$. This implies $f_i(z) = 0$ on $T$ and $T_\phi f \in R(K \cap T)$. The Lemma is proved.

Lemma 7. If $A(K) \subset HP(z, z^2, ..., z^N, K)$, then $A(K) = R(K)$.

Proof: We use standard Vitushkin approximation scheme (see Gamelin 1969 for example). Let $\{\psi_n, S_n\}$ be a smooth partition of unity, where the length of $S_n$ is $\delta$, the support of $\psi_n$ is in $2S_n$, $||\partial \psi_n|| \leq C/\delta$, $\sum \psi_n = 1$, and $\cup_{n=1}^\infty S_n = \mathbb{C}$ with $Int(S_n) \cap Int(S_m) = \emptyset$. For a function $f \in A(K)$,

$$f = \sum_{n=1}^\infty T_{\psi_n} f.$$ 

For a fixed $n$, let $T = 2S_n$, if $\gamma(B(cT, \sqrt{2d}T) \setminus K) < \epsilon_N \sqrt{2d}$, then by Lemma 3 $h_n = T_{\psi_n} f \in R(K)$. If $\gamma(B(cT, \sqrt{2d}T) \setminus K) \geq \epsilon_N \sqrt{2d}$, then $\gamma(Int(2T) \setminus K) \geq \epsilon_N \sqrt{2d}$. Since

$$\left| \int f \partial \psi_n dA \right| \leq C_1 dT \omega(f, \delta), \quad \left| \int (z - ct) f \partial \psi_n dA \right| \leq C_2 dT \omega(f, \delta),$$

where

$$\omega(f, \delta) = \sup_{z, w \in B(cT, \sqrt{2d})} |f(z) - f(w)|,$$

we set

$$\alpha = \frac{\int f \partial \psi_n dA}{C_1 dT \omega(f, \delta)}, \quad \beta = \frac{\int (z - ct) f \partial \psi_n dA}{C_2 dT \omega(f, \delta)}.$$ 

Using Lemma 1 we can find a function $g \in R(\mathbb{C}_\infty \setminus \{Int(2T) \setminus K\}) \subset R(K)$ satisfying (1) to (5) of Lemma 1. Now let $h_n = C_{\alpha}(\beta, g)$, then $h_n \in R(K)$, $h_n$ is analytic off $2T$, $||h_n|| \leq C\omega(f, \delta)$, and $h_n - T_{\psi_n} f$ has triple zeros at $\infty$. So $\sum_{n=1}^\infty h_n$ goes to $f$ uniformly when $\delta$ tends to zero. This completes the proof of the lemma.

Proof of Theorem 5. Let $M$ be the largest power of $\bar{z}$ among all terms of $q_1(z, \bar{z}), q_2(z, \bar{z}), ..., q_N(z, \bar{z})$, then

$$A(K) \subset HP(q_1, ..., q_N, K) \subset HP((\bar{z}, \bar{z}^2, ..., \bar{z}^M, K)).$$

By Lemma 4 we conclude $A(K) = R(K)$.

Can Theorem 3 work for more general functions? For $N = 1$ and a smooth function $g$ with $\partial g \neq 0$, it is proved in Yang (1994) that $A(K) \subset HP(g, K)$ implies $A(K) = R(K)$. We think this may still hold for $N > 1$. However, in this section, we provide an example for a single function as stated in Proposition 1.

Let us construct a compact subset $K_0$ of the closed unit square with center at zero and sides parallel to coordinate axes such that $P(K_0) = C(K_0)$ and $Arclen(K_0) = a$, $0 < a < 1$. In fact, we can construct a planar Cantor set $K_0$ as the following. Given a sequence $\{\lambda_n\}$ with $0 < \lambda_n < \frac{1}{2}$, let $Q_0 = [0, 1] \times [0, 1]$. At the first step we take four closed squares inside $Q_0$, with side length $\lambda_1$, with sides parallel to the coordinate axes, and so that each square contains a vertex of $Q_0$. At the second step we apply the preceding procedure to each of the four squares obtained in the first step, but now using the proportion factor $\lambda_2$. In this way, we get 16 squares of side length $\sigma_2 = \lambda_1 \lambda_2$. Proceeding inductively, at each step we obtain $4^n$ squares $Q^n_j, j = 1, 2, ..., 4^n$ with side length $\sigma_n = \lambda_1 \lambda_2 ... \lambda_n$. Now let

$$L_n = \cup_{j=1}^{4^n} Q^n_j, \quad K_0 = \cap_{n=1}^\infty L_n,$$

and

$$\lambda_n = \frac{1}{2} a^\frac{1}{\lambda_n}.$$
Then

\[ \text{Area}(K_0) = \lim_{n \to \infty} 4^n \sigma_n^2 = a, \]

By construction, \( \mathbb{C} \setminus K_0 \) is connected, so

\[ P(K_0) = C(K_0). \]  

(3-1)

Since \( \lim_{n \to \infty} \frac{\sigma_n}{\sigma_{n+1}} = 2 \), we can choose \( n_0 \) so that for \( n \geq n_0 \), we have \( 2\sigma_{n+1} \leq \sigma_n \leq 2.1\sigma_{n+1} \). Now let \( T \) be a square with \( d_T \leq \sigma_n \), then we can find an integer \( n_1 > n_0 \) such that \( 2\sigma_{n+1} \leq \sigma_n \leq d_T \leq \sigma_{n-1} \). Suppose \( T \cap K_0 \neq \emptyset \), there exists \( Q_j^{n+1} \) with \( T \cap Q_j^{n+1} \neq \emptyset \). Therefore, \( Q_j^{n+1} \subset 2T \) and

\[ \text{Area}(2T \cap K_0) \geq \text{Area}(Q_j^{n+1} \cap K_0) = \lim_{n \to \infty} 4^n \sigma_{n+1}^2 \geq a \sigma_{n+1}^2 \geq \frac{a}{(2(2.1)^2)^2} \text{Area}(2T). \]

In this case,

\[ \gamma((2T \cap K_0)) \geq C(\text{Area}(2T)). \]  

(3-2)

Now we will construct a sequence of disjoint small open disks \( \{B_k(z_k, r_k)\}_{k=1}^{\infty} \) within \( G = B(0,1) \setminus K_0 \) satisfying the following conditions:

1. Each point in \( K_0 \) is a limit of a subsequence of the disks;
2. No point in \( G \) is a limit of a subsequence of the disks;
3. \( \sum_{k=1}^{\infty} r_k < \infty \).

Let \( \{z_k\} \subset K_0 \) be a subset that is dense in \( K_0 \). We begin with a point \( y_{11} \in G \), choose \( 0 < r_{11} < \frac{2}{3} \) with \( B(y_{11}, r_{11}) \subset G \). Let \( d_{11} = \text{dist}(y_{11}, K_0) - r_{11} \). We finish the level 1 construction. For level 2, choose \( y_{12} \in G \) so that \( \text{dist}(y_{12}, x_1) < \frac{2}{3} d_{11} \) and \( 0 < r_{11} < \min\left(\frac{d_{11}}{2}, \frac{1}{r_{11}}\right) \) with \( B(y_{11}, r_{12}) \subset G \). Define \( d_{21} = \text{dist}(y_{12}, K_0) - r_{21} \). Choose \( y_{22} \in G \) so that \( \text{dist}(y_{22}, x_2) < \frac{d_{21}}{2} \) and \( 0 < r_{22} < \min\left(\frac{d_{21}}{2}, \frac{1}{r_{22}}\right) \) with \( B(y_{22}, r_{22}) \subset G \). Define \( d_{22} = \text{dist}(y_{22}, K_0) - r_{22} \). We continue this process and get \( \{y_{ij}\}, \{r_{ij}\} \), and \( \{d_{ij}\} \) satisfying:

\[ \text{dist}(y_{ij}, K_0) = d_{ij} + r_{ij}, \sum r_{ij} < \infty; \]

and

\[ d_{i1} < \frac{d_{i+1,1}}{2}, \quad d_{ij} < \frac{d_{i+1,j}}{2}, \]

where \( 1 < j \leq i \). Let \( \{z_k\} = \{y_{ij}\} \) and \( \{r_k\} = \{r_{ij}\} \). Clearly the conditions (1)-(3) are met. Set

\[ K = \bar{B}(0,1) \setminus \left( \bigcup_{k=1}^{\infty} B(z_k, r_k) \right). \]  

(3.3)

It is easy to verify that the inner boundary \( \partial K \) equals \( K_0 \). For this \( K \), we can prove Proposition [1]

Proof of Proposition [1] Let \( \phi = C(dA|K_0) \), then \( \phi \in A(K) \). Define

\[ \mu = dz|_{\partial B(0,1)} - \sum_{k=1}^{\infty} dz|_{\partial B(z_k, r_k)}, \]

then \( \mu \) is a finite Borel measure, \( \mu \perp R(K) \), and

\[ \int \phi d\mu = \int_{K_0} \int \frac{1}{\lambda - z} d\mu(z)dA(\lambda) = -2\pi i \text{Area}(K_0) \neq 0. \]

Hence, \( R(K) \neq A(K) \). Since for a polynomial \( p \),

\[ C(pdA|K_0) - p\phi = \int_{K_0} \frac{p(w) - p(z)}{w - z} dA(w) \in R(K), \]

we see that \( C(pdA|K_0) \in HP(\phi, K) \). Using (3-1), we conclude that

\[ C(\chi_E dA) \in HP(\phi, K), \quad E \subset K_0, \]  

(3-3)

where \( \chi_E \) is the characteristic function of \( E \). Let \( T \) and \( Q_j^{n+1} \) be squares as above. There are four squares

\[ \bigcup_{i=1}^{4} Q_j^{n+1} \subset Q_j^{n+1}. \]
Choose two of them, say $Q_{k_1}^{h_1+2}$ and $Q_{k_2}^{h_1+2}$, with the same $y$ coordinates of the centers. Set $E_1 = Q_{k_1}^{h_1+2} \cap K_0$ and $E_2 = Q_{k_2}^{h_1+2} \cap K_0$. Let $\phi_1(z) = -\chi_E$, and $\phi_2(z) = -\chi_E$. Set $f_j = C(\phi_j dA)$ for $j = 1, 2$. Then from (3-3), we have $f_j \in HP(\phi, K)$ and

$$f_j(\infty) = Area(E_j)(= AE)$$

since $Area(E_1) = Area(E_2)$, and

$$\beta(f_j, ct) = \int (z - c_E)\chi_E(z) dA + (c_{E_j} - ct) Area(E_j) = (c_{E_j} - ct)AE.$$  

We Set

$$g_1 = \frac{(c_{E_2} - ct)f_1 - (c_{E_1} - ct)f_2}{(c_{E_2} - c_{E_1})AE} d_f,$$

and

$$g_2 = \frac{f_2 - f_1}{(c_{E_2} - c_{E_1})AE} d_f.$$  

Notice that $|c_{E_2} - ct| < 2d_f, |c_{E_1} - ct| < 2d_f$, and $c_{E_2} - c_{E_1}$ is comparable with $d_f$. Using the arguments before (3-2), we see that $\|g_j\| \leq C (\text{absolute constant})$, $g_j \in HP(\phi, K)$, and

$$g_1(\infty) = d_f, g_2(\infty) = 0, \beta(g_1, ct) = 0, \beta(g_2, ct) = d_f^2.$$  

Now we use standard Vitushkin approximation scheme (see Gamelin, 1963, for example). Let $\{\psi_n, S_n\}$ be a smooth partition of unity, where the length of $S_n$ is $\delta$, the support of $\psi_n$ is in $2S_n$, $\|\partial \psi_n\| \leq C/\delta$, $\sum \psi_n = 1$, and $\bigcup_{n=1}^{\infty} S_n = \mathbb{C}$ with $Int(S_n) \cap Int(S_m) = \emptyset$. We assume $\delta$ is less than $\frac{1}{2 C}$. For a function $f \in A(K)$,

$$f = \sum_{n=1}^{\infty} T_{\psi_n} f.$$  

For a fixed $n$, if $(2S_n) \cap K_0 = \emptyset$, then $h_n = T_{\psi_n} f \in R(K)$. If $(2S_n) \cap K_0 \neq \emptyset$, then let $T = 2S_n$ and

$$h_n = \frac{\int f \bar{\psi}_n dA}{\pi d_f} g_1 + \frac{\int (z - c_E) f \bar{\psi}_n dA}{\pi d_f^2} g_2,$$

then $h_n \in HP(\phi, K), h_n$ is analytic off $2T, ||h_n|| \leq C \omega(f, \delta)$, and $h_n - T_{\psi_n} f$ has triple zeros at $\infty$. So $\sum_{n=1}^{\infty} h_n$ goes to $f$ uniformly when $\delta$ tends to zero. This completes the proof of the proposition.

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