Research Article

Spin Splitting Spectroscopy of Heavy Quark and Antiquarks Systems

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Phenomenological potentials describe the quarkonium systems like $c\bar{c}$, $b\bar{b}$, and $Bc$ where they give a good accuracy for the mass spectra. In the present work, we extend one of our previous works in the central case by adding spin-dependent terms to allow for relativistic corrections. By using such terms, we get better accuracy than previous theoretical calculations. In the present work, the mass spectra of the bound states of heavy quarks $c\bar{c}$, $b\bar{b}$, and $Bc$ mesons are studied within the framework of the nonrelativistic Schrödinger equation. First, we solve Schrödinger’s equation by the Nikiforov-Uvarov (NU) method, which gives asymptotic expressions for the eigenfunctions and eigenvalues of the Schrödinger equation.

1. Introduction

In the twentieth century, quarkonium systems have been discovered. Theorists have been trying to explain some aspects of those systems like mass spectra and decay mode properties. [1–5]. Some of them used lattice quantum chromodynamics view [6–12], effective field theory [13], relativistic potential models [14, 15], semirelativistic potential models [16], and nonrelativistic potential models [17–19] which have shared in common the Coulomb and linear potentials. There are other groups which use confinement power potential $r^n$ [20–22], the Bethe-Salpeter approach [23–25]. In the present work, we use mixed potential: nonrelativistic potential models (Coulomb+linear) and confinement power potentials plus spin-dependent splitting terms as a correction. Schrödinger’s equation is solved by the Nikiforov-Uvarov (NU) method [26–34], which gives asymptotic expressions for the eigenfunctions and eigenvalues of the Schrödinger equation.

2. Methodology

In the quarkonium system which deals with quark and antiquark interaction in the center of mass frame, the masses of the quark and antiquark are bigger than chromodynamics scaling, i.e., $M_{q\bar{q}} \gg \Lambda_{QCD}$. So, this allows for nonrelativistic treatment and is considered as heavy bound systems. Using Schrödinger’s equation of the two-body system in a spherical symmetric potential one obtains the following:

$$\frac{d^2Q(r)}{dr^2} + \left[ \frac{2\mu}{R^2} (E - V_{tot}(r)) - \frac{l(l+1)}{r^2} \right] Q(r) = 0, \quad (1)$$

where $\mu$ is reduced mass, $E$ is energy eigenvalue, $l$ is orbital quantum number, $V_{tot}(r)$ is total potential of the system, $Q(r) = rR(r)$, and $R(r)$ is a radial wave function.
solution of Schrödinger’s equation. Our radial potential is taken as follows:

\[ V(r) = -\frac{b}{r} + ar + dr^2 + pr^4. \] (2)

Also, we use spin-dependent splitting terms: spin-spin interaction, spin-orbital interaction, and tensor interaction, respectively \[35–43\].

\[ V_{s-T}(r) = V_{s-T}(r), \] (3)

where

\[ V_{s-T}(r) = \frac{2}{3m_q m_q} V^2 V(r) \left[ \tilde{S}_y \cdot \frac{\bar{S}'}{r} \right], \]

\[ V_{s-T}(r) = \frac{1}{2m_q m_q} \left[ 3\frac{dV(r)}{dr} - \frac{dV_s(r)}{dr} \right] \left[ \tilde{L} \cdot \bar{S} \right], \]

\[ V_T(r) = \frac{1}{12m_q m_q} \left[ \frac{dV(r)}{dr} - \frac{d^2 V(r)}{dr^2} \right] \left[ \frac{6}{r} (\tilde{S}_y \cdot \frac{\bar{S}'}{r}) - 2\tilde{S}_y \cdot \bar{S} \right]. \] (4)

\[ V_V \] is a vector potential term, and \( V_s \) is a scalar potential term.

So, the total potential becomes

\[ V_{tot}(r) = V(r) + V_{s-T}(r), \]

\[ V_{tot}(r) = \frac{4a_F}{3m_q m_q} \nu(ss) + \frac{\nu(is)}{m_q m_q} \left[ \frac{3a_F - a_S}{2r} + \frac{3b}{2r^3} - d - 2pr^2 \right] \]

\[ + \frac{\nu(T)}{12m_q m_q} \left[ \frac{a_F}{r} + \frac{3b}{r^3} \right] - b \frac{ar + dr^2 + pr^4}{r}, \] (5)

where \( \nu(ss) = \left[ \tilde{S}_y \cdot \bar{S} \right], \nu(is) = \left[ L \cdot \bar{S} \right], \nu(T) = -2 \tilde{S}_y \cdot \bar{S} + 3 \tilde{(S}_y \cdot \bar{S})/r \) \( a_S + a_F = a \).

By substituting in equation (1), we get

\[ \frac{d^2 Q}{dr^2} + \left[ \epsilon - \frac{B}{r} - \frac{L(l + 1)}{r^2} - \frac{C}{r} - Ar - Fr^2 - Pr^4 \right] Q = 0, \] (6)

where in natural units,

\[ \epsilon = 2\mu E + \frac{2\mu \nu(is)}{m_q m_q}, \] (7)

\[ B = \frac{\mu \nu(is) + \nu(t)}{2m_q m_q}. \] (8)

\[ A = 2\mu a_F = \frac{2\mu m_q}{m_q m_q} \left[ -2\nu(is) + dm_q m_q \right], \] (9)

\[ C = \frac{\mu [16a_F \nu(is) + 6(3a_F - a_S) \nu(is) + a_F \nu(T) - 12m_q m_q b]}{6m_q m_q}, \]

\[ p = 2\mu p. \] (10)

Let \( x = 1/r \), and by substituting in equation (6), we obtain

\[ \frac{d^2 Q}{dx^2} + 2 \frac{dQ}{dx} + \left[ \frac{\epsilon - C}{x^2} - l(l + 1) - A \frac{x^2}{x^2} - F \frac{x^2}{x^2} \right] Q = 0. \] (11)

In equation (11), one can use the Nikiforov–Uvarov method (NU) to get eigenvalue and eigenfunction equations.

Due to the singularity point in equation (11), put \( x = 1/\delta = 1/r \), and using the Taylor series to expand to second order terms, one obtains

\[ \frac{d^2 Q}{dx^2} + 2 \frac{dQ}{dx} + \frac{1}{x^2} \left[ -q + wx - xz \right] Q = 0, \] (12)

where

\[ -\frac{6e}{\delta^3} + \frac{3C}{\delta^2} + l(l + 1) + \frac{10A}{\delta^3} + \frac{15F}{\delta^4} + \frac{28p}{\delta^5} = q, \] (13)

\[ -\frac{8e}{\delta^5} - B + \frac{3C}{\delta^4} + \frac{15A}{\delta^5} + \frac{44F}{\delta^6} + \frac{48p}{\delta^7} = w, \] (14)

\[ \frac{3e}{\delta^6} + \frac{C}{\delta^5} + \frac{6A}{\delta^6} + \frac{10F}{\delta^7} + \frac{21p}{\delta^8} = z. \] (15)

We get

\[ z = \left[ \frac{w}{2n + 1 + 2\sqrt{q + 1/4}} \right]^2. \] (16)

By substituting equations (13)–(15) in equation (16) and arrange it, we get

\[ \epsilon = \frac{C\delta^6}{\delta^3} + \frac{2A}{\delta^3} + \frac{10F}{\delta^4} + \frac{7p}{\delta^5} + \frac{1}{\delta^7} \]

\[ \left[ \frac{3C + 15A/\delta^2 + 44F/\delta^3 + 48p/\delta^4 - 8e/\delta^5 - B/\delta^6}{2n + 1 + 2\sqrt{(l+1)/2} + 3C/\delta - 6e/\delta^2 + 10A/\delta^3 + 15F/\delta^4 + 28p/\delta^5} \right]^2. \] (17)

We substitute equations (8)–(10) into equation (17) to obtain the energy eigenvalue equation.

\[ E = \xi - \frac{\mu}{6} \left[ \frac{\xi}{2n + 1 + 2\sqrt{(l+1)/2} + \mu} \right]^2, \] (18)

where \( \delta = 1/r_0, \xi = \left[ (16a_F \nu(is) + 6(3a_F - a_S) \nu(is) + a_F \nu(T) - 12m_q m_q b)/36m_q m_q r_0 \right] + 2a_F + \left( 10/3m_q m_q \right) \left[ -2\nu(is) + dm_q m_q \right] m_q r_0^2 + 7pr_0^4 - dv(is)/m_q m_q \).
Process (1): after we deduced the energy eigenvalues and substituted the mass parameters in the different states using the quantum numbers \( n \) and \( L \). We gather a set of equations to be compared with the experimental data and save them in MATLAB file like equations.m. In this file one has five undefined parameters, we should have five equations synchronized with them.

Process (2): we will initialize the undefined parameters and use a built-in function "fsolve" to get the suggested values of the coefficients of the potential \( a, b, d, \) and \( p \) and the characteristic radius \( r_0 \) with the condition that they must be real.

Process (3): substitute the suggested values of the coefficients of the potential in the energy eigenvalue equation then get the theoretical data and calculate the error between the experimental data and the theoretical one.

Process (4): put processes (2),(3) inside a loop to repeat the operation continuously considering the previous suggested values of the coefficients which were initialization point in every process in the loop.

Figure 1: This is a flow chart about how one can calculate the theoretical data. It consists of three stages to do that. We found that after some iteration processes in the loops, the previous suggested values of the coefficients are the same in all last processes, as there is no change in the data. So, we used another program which gives a small change in the parameter values which was Origin lab program. In the third stage, after we got the values of the coefficients of the potential with the smallest chi-square values from the Origin program, we will substitute them in the energy eigenvalue equation and calculate the theoretical data (present work) for the different \( n \) and \( L \) states by using "fsolve" built in function in MATLAB like first flow chart.
### Table 1: Parameter values of each system.

| Variables | Systems | $m_q$ | $m_{ar{q}}$ | $r_0$ | $a_s$ | $a_{sv}$ | $b$ | $d$ | $p$ |
|-----------|---------|-------|--------------|-------|-------|--------|-----|-----|-----|
| Units     | cc system | GeV   | GeV          | GeV$^{-1}$ | GeV$^2$ | GeV$^2$ | –   | GeV$^2$ | GeV$^5$ |
|           | 1$^3$S$_0$ | 3.033 | 2.93 | 2.981 | 2.984 | 2.989 | 2.979 | 2.980 | 2.980 | 2.982 | 3.088 | 2.979 | 2.984 (49) |
|           | 1$^3$S$_1$ | 3.126 | 3.11 | 3.096 | 3.097 | 3.094 | 3.097 | 3.097 | 3.097 | 3.090 | 3.168 | 3.096 | 3.097 (29) |
|           | 2$^3$S$_0$ | 3.666 | 3.68 | 3.635 | 3.637 | 3.602 | 3.623 | 3.597 | 3.633 | 3.630 | 3.669 | 3.600 | 3.639 (27) |
|           | 2$^3$S$_1$ | 3.701 | 3.68 | 3.685 | 3.679 | 3.681 | 3.673 | 3.685 | 3.690 | 3.707 | 3.707 | 3.680 | 3.686 (15) |
|           | 3$^3$S$_0$ | 4.158 | – | 3.989 | 4.004 | 4.058 | 3.991 | 4.014 | 3.992 | 4.043 | 4.067 | 4.011 | – |
|           | 3$^3$S$_1$ | 4.055 | 3.80 | 4.039 | 4.030 | 4.129 | 4.022 | 4.095 | 4.030 | 4.072 | 4.094 | 4.077 | 4.039 (16) |
|           | 4$^3$S$_0$ | 4.415 | – | 4.401 | 4.264 | 4.448 | 4.250 | 4.433 | 4.244 | 4.384 | 4.398 | 4.397 | – |
|           | 4$^3$S$_1$ | 4.415 | – | 4.427 | 4.281 | 4.514 | 4.273 | 4.477 | 4.273 | 4.406 | 4.420 | 4.454 | 4.421 (6) |
|           | 5$^3$S$_0$ | 4.607 | – | 4.811 | 4.459 | 4.799 | 4.446 | – | 4.440 | – | – | – | – |
|           | 5$^3$S$_1$ | 4.585 | – | 4.837 | 4.472 | 4.863 | 4.463 | – | 4.464 | – | – | – | – |
|           | 6$^3$S$_0$ | 4.754 | – | 5.155 | – | 5.124 | 4.595 | – | 4.601 | – | – | – | – |
|           | 6$^3$S$_1$ | 4.733 | – | 5.167 | – | 5.185 | 4.608 | – | 4.621 | – | – | – | – |
| Level     | 1$^3$P$_0$ | 3.407 | 3.32 | 3.413 | 3.415 | 3.428 | 3.433 | 3.416 | 3.392 | 3.424 | 3.448 | 3.488 | 3.415 (8) |
|           | 1$^3$P$_1$ | 3.487 | 3.49 | 3.511 | 3.521 | 3.468 | 3.510 | 3.508 | 3.491 | 3.505 | 3.520 | 3.514 | 3.511 (24) |
|           | 1$^3$P$_2$ | 3.502 | 3.43 | 3.525 | 3.526 | 3.470 | 3.519 | 3.527 | 3.524 | 3.524 | 3.516 | 3.516 | 3.536 (23) |
|           | 1$^3$P$_3$ | 3.522 | 3.55 | 3.555 | 3.553 | 3.480 | 3.556 | 3.558 | 3.570 | 3.556 | 3.564 | 3.565 | 3.556 (33) |
|           | 2$^3$P$_0$ | 3.899 | 3.83 | 3.870 | 3.848 | 3.897 | 3.842 | 3.844 | 3.845 | 3.852 | 3.870 | 3.947 | 3.918 (19) |
|           | 2$^3$P$_1$ | 3.786 | 3.67 | 3.906 | 3.914 | 3.938 | 3.901 | 3.940 | 3.902 | 3.925 | 3.934 | 3.972 | – |
|           | 2$^3$P$_2$ | 3.821 | 3.75 | 3.926 | 3.916 | 3.943 | 3.908 | 3.960 | 3.922 | 3.934 | 3.950 | 3.996 | – |
|           | 2$^3$P$_3$ | 3.905 | – | 3.949 | 3.937 | 3.955 | 3.937 | 3.994 | 3.949 | 3.972 | 3.976 | 4.021 | 3.927 (21) |
|           | 3$^3$P$_0$ | 4.120 | – | 4.301 | 4.146 | 4.296 | 4.131 | – | 4.192 | 4.202 | 4.214 | – | – |
|           | 3$^3$P$_1$ | 4.123 | 3.91 | 4.319 | 4.192 | 4.338 | 4.178 | – | 4.178 | 4.271 | 4.275 | – | – |
|           | 3$^3$P$_2$ | 4.164 | – | 4.337 | 4.193 | 4.344 | 4.184 | – | 4.137 | 4.279 | 4.291 | – | – |
|           | 3$^3$P$_3$ | 4.144 | – | 4.354 | 4.211 | 4.358 | 4.208 | – | 4.212 | 4.317 | 4.316 | – | – |
|           | 4$^3$P$_0$ | 4.362 | – | 4.698 | – | 4.653 | – | – | – | – | – | – |
|           | 4$^3$P$_1$ | 4.373 | – | 4.728 | – | 4.696 | – | – | – | – | – | – |
|           | 4$^3$P$_2$ | 4.42 | – | 4.744 | – | 4.704 | – | – | – | – | – | – |
|           | 4$^3$P$_3$ | 4.411 | – | 4.763 | – | 4.718 | – | – | – | – | – | – |
|           | 5$^3$P$_0$ | 4.543 | – | – | – | 4.983 | – | – | – | – | – | – |
|           | 5$^3$P$_1$ | 4.561 | – | – | – | 5.026 | – | – | – | – | – | – |
|           | 5$^3$P$_2$ | 4.611 | – | – | – | 5.034 | – | – | – | – | – | – |
Table 3: Charmonia mass spectrum of $D$ and $F$ waves in GeV.

| Level | Present work | [47] | [46] | [37] | [24] | [48] | [15] | [49] | [40] | [50] | [51] |
|-------|--------------|------|------|------|------|------|------|------|------|------|------|
| $1^3D_3$ | 3.307 | 3.755 | 3.813 | 3.808 | 3.869 | 3.799 | 3.831 | 3.844 | 3.806 | 3.809 | 3.798 |
| $1^3D_2$ | 3.376 | 3.765 | 3.807 | 3.805 | 3.739 | 3.796 | 3.824 | 3.802 | 3.802 | 3.803 | 3.796 |
| $1^3D_1$ | 3.348 | 3.772 | 3.795 | 3.807 | 3.550 | 3.798 | 3.824 | 3.788 | 3.800 | 3.804 | 3.794 |
| $1^3D_1$ | 3.374 | 3.775 | 3.783 | 3.792 | – | 3.787 | 3.804 | 3.729 | 3.785 | 3.789 | 3.792 |
| $2^3D_3$ | 3.797 | 4.176 | 4.220 | 4.112 | 3.806 | 4.103 | 4.202 | 4.132 | 4.167 | 4.167 | 4.425 |
| $2^3D_2$ | 3.836 | 4.182 | 4.196 | 4.108 | – | 4.099 | 4.191 | 4.105 | 4.105 | 4.158 | 4.158 |
| $2^3D_2$ | 3.801 | 4.188 | 4.190 | 4.109 | – | 4.100 | 4.189 | 4.095 | 4.158 | 4.158 | 4.223 |
| $2^3D_1$ | 3.800 | 4.188 | 4.105 | 4.095 | – | 4.089 | 4.164 | 4.057 | 4.142 | 4.143 | 4.222 |
| $3^3D_3$ | 4.163 | 4.549 | 4.574 | 4.340 | – | 4.331 | – | 4.351 | – | – | – |
| $3^3D_2$ | 4.176 | 4.553 | 4.549 | 4.336 | – | 4.326 | – | 4.330 | – | – | – |
| $3^3D_2$ | 4.135 | 4.557 | 4.544 | 4.337 | – | 4.327 | – | 4.322 | – | – | – |
| $3^3D_1$ | 4.113 | 4.555 | 4.507 | 4.324 | – | 4.317 | – | 4.293 | – | – | – |
| $4^3D_3$ | 4.436 | 4.890 | 4.920 | – | – | – | – | 4.526 | – | – | – |
| $4^3D_2$ | 4.429 | 4.892 | 4.898 | – | – | – | – | 4.509 | – | – | – |
| $4^3D_2$ | 4.383 | 4.896 | 4.896 | – | – | – | – | 4.504 | – | – | – |
| $4^3D_1$ | 4.345 | 4.891 | 4.857 | – | – | – | – | 4.480 | – | – | – |
| $1^3F_2$ | 3.403 | 3.990 | 4.041 | – | – | – | 4.068 | – | 4.029 | – | – |
| $1^3F_3$ | 3.375 | 4.012 | 4.068 | – | 3.999 | – | 4.070 | – | 4.029 | – | – |
| $1^3F_3$ | 3.403 | 4.017 | 4.071 | – | 4.037 | – | 4.066 | – | 4.026 | – | – |
| $1^3F_4$ | 3.315 | 4.036 | 4.093 | – | – | – | 4.062 | – | 4.021 | – | – |
| $2^3F_2$ | 3.812 | 4.378 | 4.361 | – | – | – | – | 4.351 | – | – | – |
| $2^3F_3$ | 3.823 | 4.396 | 4.400 | – | – | – | – | 3.352 | – | – | – |
| $2^3F_3$ | 3.858 | 4.400 | 4.406 | – | – | – | – | 4.350 | – | – | – |
| $2^3F_4$ | 3.814 | 4.415 | 4.434 | – | – | – | – | 4.348 | – | – | – |
| $3^3F_2$ | 4.111 | 4.730 | – | – | – | – | – | – | – | – | – |
| $3^3F_3$ | 4.152 | 4.746 | – | – | – | – | – | – | – | – | – |
| $3^3F_3$ | 4.194 | 4.749 | – | – | – | – | – | – | – | – | – |
| $3^3F_4$ | 4.186 | 4.761 | – | – | – | – | – | – | – | – | – |
Table 4: Bottomonia mass spectrum of $S$ and $P$ waves in GeV.

| Level | Present work | [52] | [46] | [37] | [24] | [53] | [15] | [49] | [54] | Exp. [44] |
|-------|--------------|------|------|------|------|------|------|------|------|-----------|
| $1^3S_0$ | 9.472 | 9.402 | 9.398 | 9.390 | 9.414 | 9.389 | 9.393 | 9.392 | 9.455 | 9.398 (74) |
| $1^3S_1$ | 9.525 | 9.465 | 9.460 | 9.460 | 9.460 | 9.460 | 9.460 | 9.502 | 9.460 (65) |
| $2^3S_0$ | 10.028 | 9.976 | 9.990 | 9.990 | 9.987 | 9.987 | 9.987 | 9.991 | 9.990 | 9.999 (28) |
| $2^3S_1$ | 10.049 | 10.003 | 10.023 | 10.015 | 10.089 | 10.016 | 10.023 | 10.024 | 10.015 | 10.023 (26) |
| $3^3S_0$ | 10.36 | 10.336 | 10.329 | 10.326 | – | 10.330 | 10.345 | 10.323 | 10.330 | – |
| $3^3S_1$ | 10.371 | 10.354 | 10.355 | 10.343 | – | 10.351 | 10.364 | 10.346 | 10.349 | 10.355 (16) |
| $4^3S_0$ | 10.592 | 10.523 | 10.573 | 10.584 | – | 10.595 | 10.623 | 10.558 | – | – |
| $4^3S_1$ | 10.598 | 10.635 | 10.586 | 10.597 | – | 10.611 | 10.643 | 10.575 | 10.607 | 10.579 (19) |
| $5^3S_0$ | 10.79 | 10.869 | 10.851 | 10.800 | – | 10.817 | – | 10.741 | – | – |
| $5^3S_1$ | 10.87 | 10.878 | 10.869 | 10.811 | – | 10.831 | – | 10.755 | 10.818 | 10.876 (6) |
| $6^3S_0$ | 10.961 | 11.097 | 11.061 | 10.997 | – | 11.011 | – | 10.892 | – | – |
| $6^3S_1$ | 11.022 | 11.102 | 11.088 | 10.988 | – | 10.988 | – | 10.904 | 10.995 | 11.019 (3) |
| $1^3P_0$ | 9.84 | 9.847 | 9.859 | 9.864 | 9.815 | 9.865 | 9.861 | 9.862 | 9.855 | 9.859 (19) |
| $1^3P_1$ | 9.875 | 9.876 | 9.892 | 9.903 | 9.842 | 9.897 | 9.891 | 9.888 | 9.874 | 9.893 (18) |
| $1^3P_2$ | 9.884 | 9.882 | 9.900 | 9.909 | 9.806 | 9.903 | 9.900 | 9.896 | 9.879 | 9.899 (15) |
| $2^3P_0$ | 10.202 | 10.226 | 10.233 | 10.220 | 10.254 | 10.226 | 10.230 | 10.241 | 10.221 | 10.232 (30) |
| $2^3P_1$ | 10.229 | 10.246 | 10.255 | 10.249 | 10.120 | 10.251 | 10.255 | 10.256 | 10.236 | 10.255 (26) |
| $2^3P_2$ | 10.254 | 10.261 | 10.268 | 10.264 | – | 10.269 | 10.271 | 10.268 | 10.246 | 10.269 (15) |
| $3^3P_0$ | 10.299 | 10.552 | 10.521 | 10.490 | – | 10.502 | – | 10.511 | 10.500 | – |
| $3^3P_1$ | 10.339 | 10.538 | 10.541 | 10.515 | 10.303 | 10.524 | – | 10.507 | 10.513 | – |
| $3^3P_2$ | 10.362 | 10.541 | 10.544 | 10.519 | – | 10.529 | – | 10.497 | 10.516 | – |
| $4^3P_0$ | 10.406 | 10.550 | 10.550 | 10.528 | – | 10.540 | – | 10.516 | 10.521 | – |
| $4^3P_1$ | 10.532 | 10.775 | 10.781 | – | – | 10.732 | – | – | – | – |
| $4^3P_2$ | 10.571 | 10.788 | 10.802 | – | – | 10.753 | – | – | – | – |
| $5^3P_0$ | 10.637 | 10.798 | 10.812 | – | – | 10.767 | – | – | – | – |
| $5^3P_1$ | 10.769 | 11.014 | – | – | – | 10.951 | – | – | – | – |
| $5^3P_2$ | 10.792 | 11.016 | – | – | – | 10.955 | – | – | – | – |
Table 5: Bottomonia mass spectrum of \( D \) and \( F \) waves in GeV.

| Level | Present work | \([52]\) | \([46]\) | \([37]\) | \([24]\) | \([53]\) | \([15]\) | \([49]\) | \([54]\) | Exp. \([44]\) |
|-------|--------------|--------|--------|--------|--------|--------|--------|--------|--------|---------|
| \(1^3D_3\) | 9.849 | 10.115 | 10.166 | 10.157 | 10.232 | 10.156 | 10.163 | 10.177 | 10.127 | – |
| \(1^3D_2\) | 9.767 | 10.148 | 10.163 | 10.153 | 10.194 | 10.152 | 10.158 | 10.166 | 10.123 | – |
| \(1^3D_2\) | 10.096 | 10.147 | 10.161 | 10.153 | 10.145 | 10.151 | 10.157 | 10.162 | 10.122 | 10.163 \((66)\) |
| \(1^1D_1\) | 9.666 | 10.138 | 10.154 | 10.146 | – | 10.145 | 10.149 | 10.147 | 10.117 | – |
| \(2^3D_3\) | 10.175 | 10.455 | 10.449 | 10.436 | – | 10.442 | 10.456 | 10.447 | 10.422 | – |
| \(2^3D_2\) | 10.093 | 10.450 | 10.445 | 10.432 | – | 10.439 | 10.452 | 10.440 | 10.419 | – |
| \(2^3D_2\) | 10.071 | 10.449 | 10.443 | 10.432 | – | 10.438 | 10.450 | 10.437 | 10.418 | – |
| \(2^3D_1\) | 9.996 | 10.441 | 10.435 | 10.425 | – | 10.432 | 10.443 | 10.428 | 10.414 | – |
| \(3^3D_3\) | 10.446 | 10.711 | 10.717 | – | – | 10.680 | – | 10.652 | – | – |
| \(3^3D_2\) | 10.368 | 10.706 | 10.713 | – | – | 10.677 | – | 10.646 | – | – |
| \(3^3D_2\) | 10.345 | 10.705 | 10.711 | – | – | 10.676 | – | 10.645 | – | – |
| \(3^3D_1\) | 10.272 | 10.698 | 10.704 | – | – | 10.670 | – | 10.637 | – | – |
| \(4^3D_3\) | 10.676 | 10.939 | 10.963 | – | – | 10.866 | – | 10.817 | – | – |
| \(4^3D_2\) | 10.599 | 10.935 | 10.959 | – | – | 10.883 | – | 10.813 | – | – |
| \(4^3D_2\) | 10.576 | 10.934 | 10.957 | – | – | 10.882 | – | 10.811 | – | – |
| \(4^3D_1\) | 10.504 | 10.928 | 10.949 | – | – | 10.877 | – | 10.811 | – | – |
| \(1^3F_2\) | 9.642 | 10.350 | 10.343 | 10.338 | – | – | 10.353 | – | 10.315 | – |
| \(1^3F_3\) | 9.754 | 10.355 | 10.346 | 10.340 | 10.302 | – | 10.356 | – | 10.321 | – |
| \(1^3F_3\) | 9.778 | 10.355 | 10.347 | 10.339 | 10.319 | – | 10.356 | – | 10.322 | – |
| \(1^1F_4\) | 9.896 | 10.358 | 10.349 | 10.340 | – | – | 10.357 | – | – | – |
| \(2^3F_2\) | 9.971 | 10.615 | 10.610 | – | – | – | 10.610 | – | – | – |
| \(2^3F_3\) | 10.081 | 10.619 | 10.614 | – | – | – | 10.613 | – | – | – |
| \(2^3F_3\) | 10.104 | 10.619 | 10.647 | – | – | – | 10.613 | – | – | – |
| \(2^3F_4\) | 10.219 | 10.622 | 10.617 | – | – | – | 10.615 | – | – | – |
| \(3^3F_2\) | 10.246 | 10.850 | – | – | – | – | – | – | – | – |
| \(3^3F_3\) | 10.353 | 10.853 | – | – | – | – | – | – | – | – |
| \(3^3F_3\) | 10.376 | 10.853 | – | – | – | – | – | – | – | – |
| \(3^3F_4\) | 10.489 | 10.856 | – | – | – | – | – | – | – | – |
| Level  | Present work | \[47\] | \[55\] | \[46\] | \[56\] | \[57\] | Exp.\[44\] |
|--------|--------------|--------|--------|--------|--------|--------|----------|
| \(1^1S_0\) | 6.276 | 6.272 | 6.278 | 6.272 | 6.271 | 6.275 | (0) |
| \(1^3S_1\) | 6.313 | 6.321 | 6.331 | 6.333 | 6.338 | 6.314 | – |
| \(2^3S_0\) | 6.841 | 6.864 | 6.863 | 6.842 | 6.855 | 6.838 | 6.842 (1) |
| \(2^3S_1\) | 6.867 | 6.900 | 6.873 | 6.882 | 6.887 | 6.850 | – |
| \(3^1S_0\) | 7.281 | 7.306 | 7.244 | 7.226 | 7.250 | – | – |
| \(3^3S_1\) | 7.308 | 7.338 | 7.249 | 7.258 | 7.272 | – | – |
| \(4^3S_0\) | 7.634 | 7.684 | 7.564 | 7.585 | – | – | – |
| \(4^3S_1\) | 7.66 | 7.714 | 7.568 | 7.609 | – | – | – |
| \(5^1S_0\) | 7.917 | 8.025 | 7.852 | 7.928 | – | – | – |
| \(5^1S_1\) | 7.941 | 8.054 | 7.855 | 7.947 | – | – | – |
| \(6^1S_0\) | 8.144 | 8.340 | 8.120 | – | – | – | – |
| \(6^1S_1\) | 8.168 | 8.368 | 8.122 | – | – | – | – |
| \(1^3P_0\) | 6.223 | 6.686 | 6.748 | 6.699 | 6.706 | 6.672 | – |
| \(1^3P_1\) | 6.281 | 6.705 | 6.767 | 6.750 | 6.741 | 6.766 | – |
| \(1^1P_1\) | 6.290 | 6.706 | 6.769 | 6.743 | 6.750 | 6.828 | – |
| \(1^3P_2\) | 6.366 | 6.712 | 6.775 | 6.761 | 6.768 | 6.776 | – |
| \(2^3P_0\) | 6.782 | 7.146 | 7.139 | 7.094 | 7.122 | 6.914 | – |
| \(2^3P_1\) | 6.836 | 7.165 | 7.155 | 7.134 | 7.145 | 7.259 | – |
| \(2^3P_2\) | 6.846 | 7.168 | 7.156 | 7.094 | 7.150 | 7.322 | – |
| \(3^3P_0\) | 6.917 | 7.173 | 7.162 | 7.157 | 7.164 | 7.232 | – |
| \(3^3P_1\) | 7.227 | 7.536 | 7.463 | 7.474 | – | – | – |
| \(3^3P_2\) | 7.278 | 7.555 | 7.479 | 7.510 | – | – | – |
| \(3^1P_1\) | 7.287 | 7.559 | 7.479 | 7.500 | – | – | – |
| \(3^3P_3\) | 7.355 | 7.565 | 7.485 | 7.524 | – | – | – |
| \(4^3P_0\) | 7.583 | 7.885 | – | 7.817 | – | – | – |
| \(4^3P_1\) | 7.631 | 7.905 | – | 7.853 | – | – | – |
| \(4^3P_2\) | 7.640 | 7.908 | – | 7.844 | – | – | – |
| \(4^3P_3\) | 7.704 | 7.915 | – | 7.867 | – | – | – |
| \(5^3P_0\) | 7.867 | 8.207 | – | – | – | – | – |
| \(5^3P_1\) | 7.913 | 8.226 | – | – | – | – | – |
| \(5^3P_2\) | 7.922 | 8.230 | – | – | – | – | – |
The equation is given by

\[
\xi = \frac{16\alpha_v(s) + 6(3a_v - a_q)v(l) + a_v(t) - 12m_qm_{\bar{b}}}{2m_qm_{\bar{b}}}
\]

- \[ + 30a_0^2 + \frac{48}{m_qm_{\bar{b}}} \left[-2\nu(l) + dm_qm_{\bar{b}}\right] \rho_0^3 \]
- \[ + 96pr_0^2 - 8r_0 \left[ 2E + \frac{dv(l)}{m_qm_{\bar{b}}} - \frac{b(6\nu(l) + v(t))}{2m_qm_{\bar{b}}r_0^2} \right]. \]

\[
\eta = \frac{16\alpha_v(s) + 6(3a_v - a_q)v(l) + a_v(t) - 12m_qm_{\bar{b}}}{2m_qm_{\bar{b}}}
\]

- \[ + 20a_0^3 + \frac{30}{m_qm_{\bar{b}}} \left[-2\nu(l) + dm_qm_{\bar{b}}\right] \rho_0^4 \]
- \[ + 56pr_0^6 - 6r_0^2 \left[ 2E + \frac{dv(l)}{m_qm_{\bar{b}}} \right]. \]

(19)

Knowing that

\[
M(q\bar{q}) = E + m_q + m_{\bar{b}} \rightarrow E = M(q\bar{q}) - (m_q + m_{\bar{b}}). \]

(20)

So, the mass spectra equation becomes

\[
M(q\bar{q}) = N - \frac{\mu}{6} \left[ \frac{\xi}{2n + 1 + 2n + 3 + \mu} \right]^2 + m_q + m_{\bar{b}}. \]

(21)

The eigenfunction equation is

\[
Q(r) = N_{nl}\rho^{1/2-\sqrt{q+1/4}}e^{-\rho/2}L_n^{q+1/4} \left( \frac{2\sqrt{z}}{r} \right). \]

(22)

where \(L_n^{q+1/4} \left( \frac{2\sqrt{z}}{r} \right)\) is the Rodrigues formula of the associated Laguerre polynomial and \(N_{nl}\) is a normalization constant.

So, the radial wave function solution of Schrödinger’s equation is given by

\[
R(r) = N_{nl}r^{-1/2+\sqrt{q+1/4}}e^{-\sqrt{q+1/4}}L_n^{q+1/4} \left( \frac{2\sqrt{z}}{r} \right). \]

(23)

The energy eigenvalue equation (18) has spin-orbital-tensor coefficients \(v(ss), v(dl), v(T)\), and those can be given from references [35–37]. Also, it has potential parameters \((a_0, a_1, b, d, p)\) and \(r_0\) due to the expansion, so we have six parameters of the eigenvalue equation which can be obtained from the experimental data [44] by best fitting as shown in Figure 1.

### 3. Numerical Results and Discussions

In Table 1, potential parameters are shown for each system. It is noticed that the values of these parameters are different for different systems, and this is due to the properties of those systems like energy scale and decay mode. We use spectroscopic notation for the levels \((n^{2S+1}L_J)\).

\(S\) is the total spin of the system, \(L\) is the orbital quantum number, \(n\) is the principal quantum number, and \(J\) is the total (orbital+spin) quantum number.

By using equation (21) and Table 1, we get the mass spectra of different quantum states as shown in Tables 2–7. Previously, we used the phenomenological potential in equation (2) without spin-dependent corrections (central-dependent potential) [45]. The results obtained were good in comparison with the experimental data.

### 4. Conclusions

The above tables show that spin-dependent terms are important factors to give a better accuracy and complete quantitative description of the quarkonium systems for the cases where experimental values are available. The theoretical work agrees with the experimental data. This shows also that the Nikiforov-Uvarov method is a good method to get the energy eigenvalues for the meson spectra. The results are even better than other previous works.

### Data Availability

I used experimental data, and this is available for all. The link https://iopscience.iop.org/article/10.1088/1674-1137/40/10/10001/meta was mentioned in the paper reference [44].

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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