Moment Generating Function of the AoI in Multi-Source Systems With Packet Management

Mohammad Moltafet, Markus Leinonen, and Marian Codreanu

Abstract

We consider a status update system consisting of two independent sources and one server in which packets of each source are generated according to the Poisson process and packets are served according to an exponentially distributed service time. We derive the moment generating function (MGF) of the age of information (AoI) for each source in the system by using the stochastic hybrid systems (SHS) under two existing source-aware packet management policies which we term self-preemptive and non-preemptive policies. In the both policies, the system (i.e., the waiting queue and the server) can contain at most two packets, one packet of each source; when the server is busy and a new packet arrives, the possible packet of the same source in the waiting queue is replaced by the fresh packet. The main difference between the policies is that in the self-preemptive policy, the packet under service is replaced upon the arrival of a new packet from the same source, whereas in the non-preemptive policy, this new arriving packet is blocked and cleared. We use the derived MGF to find the first and second moments of the AoI and show the importance of higher moments.

Index Terms– Age of information (AoI), stochastic hybrid systems (SHS), moment generating function (MGF).

Mohammad Moltafet and Markus Leinonen are with the Centre for Wireless Communications–Radio Technologies, University of Oulu, 90014 Oulu, Finland (e-mail: mohammad.moltafet@oulu.fi; markus.leinonen@oulu.fi), and Marian Codreanu is with Department of Science and Technology, Linköping University, Sweden (e-mail: marian.codreanu@liu.se)
I. INTRODUCTION

In low-latency cyber-physical system applications information freshness is critical. In such systems, various sensors are assigned to generate status update packets of various real-world physical processes. These packets are transmitted through a network to a sink. Awareness of the sensors’ state needs to be as timely as possible. Recently, the age of information (AoI) was proposed to measure the information fresnens at the sink [1]–[3]. If at a time instant \( t \), the most recently received status update packet contains the time stamp (i.e., the time when status update was generated) \( U(t) \), AoI is defined as the random process \( \Delta(t) = t - U(t) \). Thus, the AoI measures for each sensor the time elapsed since the last received status update packet was generated.

The seminal work [2] introduced a powerful technique, called stochastic hybrid systems (SHS), to calculate the average AoI. In [4], the authors extended the SHS analysis to calculate the moment generating function (MGF) of the AoI. The SHS technique has been used to analyze the AoI in various queueing models [5]–[12].

The authors of [5] considered a multi-source queueing model in which sources have different priorities and derived the average AoI for two priority based packet management policies. In [6], the authors studied a single-source status update system in which the updates follow a route through a series of network nodes where each node is a last-come first-served (LCFS) queue that supports preemption in service. The work in [7] derived the average AoI in a single-source queueing model with multiple servers with preemption in service. In [8], the authors derived the average AoI in a multi-source LCFS queueing model with multiple servers that employ preemption in service. The work in [9] derived the average AoI in a multi-source system with preemption in service and packet delivery errors. The authors of [10] studied the average AoI in a simplified CSMA based system. In our work [11], we derived the average AoI under three proposed source-aware packet management policies; these were shown to result in low AoI and high fairness among the sources.

In this letter, we extend our AoI analysis in [11] by calculating the MGF of the AoI for each source under the two source-aware packet management policies proposed in [11] which we
term self-preemptive and non-preemptive policies. In both policies, the system (i.e., the waiting queue and the server) can contain at most two packets, one packet of each source; and when the server is busy and a new packet arrives, the possible packet of the same source in the waiting queue is replaced by the fresh packet. The main difference between the policies is that in the self-preemptive policy, the packet under service is replaced upon the arrival of a new packet from the same source, whereas in the non-preemptive policy, this new arriving packet is blocked and cleared. The MGF enables characterization of higher moments of the AoI which can be used for optimizing the performance of status update systems. By using the derived MGF, we derive the first and second moments of the AoI and show the importance of higher moments.

II. System Model

We consider a status update system consisting of two sources and one server. Each source observes a random process at random time instants and the measured value of the monitored process is transmitted as a status update packet. Each status update packet contains the measured value of the monitored process and a time stamp representing the time when the sample was generated. We assume that the packets of each source are generated according to the Poisson process with rates $\lambda_c$, $c \in \{1, 2\}$, and the packets are served according to an exponentially distributed service time with rate $\mu$.

Let $\rho_c = \lambda_c / \mu$, $c \in \{1, 2\}$, be the load of source $c$. Since packets of each source are generated according to the Poisson process and the sources are independent, the packet generation in the system follows the Poisson process with rate $\lambda = \lambda_1 + \lambda_2$, and the overall load in the system is $\rho = \lambda / \mu$. Let $\Delta_c$, $c \in \{1, 2\}$, be the average AoI of source $c$.

A. Packet Management Policies

In both packet management policies, the system (i.e., the waiting queue and the server) can contain at most two packets, one packet of each source; and when the server is busy and a new packet arrives, the possible packet of the same source in the waiting queue is replaced by the fresh packet. The main difference between the policies is that in the self-preemptive policy, the
packet under service is replaced upon the arrival of a new packet from the same source, whereas in the non-preemptive policy, this new arriving packet is blocked and cleared.

B. Summary of the Main Results

In this paper, we derive the MGF of the AoI for each source under the self-preemptive and non-preemptive policies which are summarized by the following two theorems.

Theorem 1. The MGF of the AoI of source 1 under the self-preemptive policy is given as

$$M_{\Delta_1}(s) = \frac{\rho_1}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{\rho_2^2(1 - \bar{s})^2 + 2\rho_2(1 - \bar{s})^3 + (1 - \bar{s})^4 + \sum_{k=1}^{4} \rho_1^k \gamma_k}{(\rho_1 - \bar{s})(1 - \bar{s})^2(1 + \rho_1 - \bar{s})^2(1 + \rho - \bar{s})^2} \right],$$

(1)

where $\bar{s} = \frac{s}{\mu}$ and

$$\gamma_1 = \rho_2^2(4 - 6\bar{s} + 4\bar{s}^2 - \bar{s}^3) + \rho_2(8 - 20\bar{s} + 16\bar{s}^2 - 6\bar{s}^3 + \bar{s}^4) + (1 - \bar{s})^3(4 - \bar{s}),$$
$$\gamma_2 = \rho_2^2(5 - 6\bar{s} + 2\bar{s}^2) + \rho_2(12 - 22\bar{s} + 14\bar{s}^2 - 3\bar{s}^3) + 3(1 - \bar{s})^2(2 - \bar{s}),$$
$$\gamma_3 = \rho_2^2(2 - \bar{s}) + \rho_2(8 - 10\bar{s} + 3\bar{s}^2) + 3\bar{s}^2 - 7\bar{s} + 4,$$
$$\gamma_4 = \rho_2(2 - \bar{s}) + 1 - \bar{s}.$$

Proof: The proof of Theorem 1 appears in Section IV-A.

Theorem 2. The MGF of the AoI of source 1 under the non-preemptive policy is given as

$$M_{\Delta_1}(s) = \frac{\rho_1}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{\rho_2^3(1 - \bar{s})^2 + 3\rho_2^2(1 - \bar{s})^3 + 3\rho_2(1 - \bar{s})^4 + (1 - \bar{s})^5 + \sum_{k=1}^{3} \rho_2^k \bar{\gamma}_k}{(\rho_1 - \bar{s})(1 - \bar{s})^3(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} \right],$$

(2)

where $\bar{s} = \frac{s}{\mu}$ and

$$\bar{\gamma}_1 = \rho_2^3(3 - 3\bar{s} + \bar{s}^2) + \rho_2^2(9 - 19\bar{s} + 11\bar{s}^2 - 2\bar{s}^3) +$$
$$\rho_2(9 - 28\bar{s} + 29\bar{s}^2 - 11\bar{s}^3 + \bar{s}^4) + 3(1 - \bar{s})^4,$$
$$\bar{\gamma}_2 = \rho_2^2(2 - \bar{s}) + \rho_2^2(8 - 10\bar{s} + 3\bar{s}^2) + \rho_2(9 - 19\bar{s} + 11\bar{s}^2 - 2\bar{s}^3) + (1 - \bar{s})^3,$$
$$\bar{\gamma}_3 = \rho_2^2(2 - \bar{s}) + \rho_2(3 - 3\bar{s} + \bar{s}^2) + (1 - \bar{s})^2.$$
Proof: The proof of Theorem 2 appears in Section IV-B.

III. The SHS Technique to Calculate MGF

In the following, we briefly present how to use the SHS technique for our MGF analysis in Section IV. We refer the readers to [2], [4] for more details.

The SHS technique models a queueing system through the states \((q(t), x(t))\), where \(q(t) \in Q = \{0, 1, \ldots, m\}\) is a continuous-time finite-state Markov chain that describes the occupancy of the system and \(x(t) = [x_0(t) \cdots x_n(t)] \in \mathbb{R}^{1 \times (n+1)}\) is a continuous process that describes the evolution of age-related processes in the system. Following the approach in [2], [11], we label the source of interest as source 1 and employ the continuous process \(x(t)\) to track the age of source 1 updates at the sink.

The Markov chain \(q(t)\) can be presented as a graph \((Q, L)\) where each discrete state \(q(t) \in Q\) is a node of the chain and a (directed) link \(l \in L\) from node \(q_l\) to node \(q'_l\) indicates a transition from state \(q_l \in Q\) to state \(q'_l \in Q\).

A transition occurs when a packet arrives or departs in the system. Since the time elapsed between departures and arrivals is exponentially distributed, transition \(l \in L\) from state \(q_l\) to state \(q'_l\) occurs with the exponential rate \(\lambda^{(l)} \delta_{q_l, q(l)}\), where the Kronecker delta function \(\delta_{q_l, q(l)}\) ensures that the transition \(l\) occurs only when the discrete state \(q(t)\) is equal to \(q_l\). When a transition \(l\) occurs, the discrete state \(q_l\) changes to state \(q'_l\), and the continuous state \(x\) is reset to \(x'\) according to a binary reset map matrix \(A_l \in \mathbb{B}^{(n+1) \times (n+1)}\) as \(x' = xA_l\). In addition, as long as discrete state \(q(t)\) is unchanged we have \(\dot{x}(t) = \frac{\partial x(t)}{\partial t} = 1\), where \(1\) is the row vector \([1 \cdots 1] \in \mathbb{R}^{1 \times (n+1)}\).

Note that unlike in a typical continuous-time Markov chain, a transition from a state to itself (i.e., a self-transition) is possible in \(q(t) \in Q\). In the case of a self-transition, a reset of the continuous state \(x\) takes place, but the discrete state remains the same. In addition, for a given pair of states \(\bar{q}, \hat{q} \in Q\), there may be multiple transitions \(l\) and \(l'\) so that the discrete state changes from \(\bar{q}\) to \(\hat{q}\) but the transition reset maps \(A_l\) and \(A_{l'}\) are different (for more details, see [2, Section III]).

To calculate the MGF of the AoI using the SHS technique, the state probabilities of the Markov chain, the correlation vector between the discrete state \(q(t)\) and the continuous state \(x(t)\), and
the correlation vector between the discrete state \( q(t) \) and the exponential function \( e^{sx(t)} \), \( s \in \mathbb{R} \), need to be defined. Let \( \pi_q(t) \) denote the probability of being in state \( q \) of the Markov chain. Let \( v_q(t) = [v_{q0}(t) \cdots v_{qn}(t)] \in \mathbb{R}^{1 \times (n+1)} \) denote the correlation vector between the discrete state \( q(t) \) and the continuous state \( x(t) \). Let \( v^s_q(t) = [v^s_{q0}(t) \cdots v^s_{qn}(t)] \in \mathbb{R}^{1 \times (n+1)} \) denote the correlation vector between the discrete state \( q(t) \) and the exponential function \( e^{sx(t)} \). Accordingly, we have

\[
\pi_q(t) = \Pr(q(t) = q) = \mathbb{E}[\delta_{q,q(t)}], \quad \forall q \in Q, \quad (3)
\]

\[
v_q(t) = [v_{q0}(t) \cdots v_{qn}(t)] = \mathbb{E}[x(t)\delta_{q,q(t)}], \quad \forall q \in Q, \quad (4)
\]

\[
v^s_q(t) = [v^s_{q0}(t) \cdots v^s_{qn}(t)] = \mathbb{E}[e^{sx(t)}\delta_{q,q(t)}], \quad \forall q \in Q. \quad (5)
\]

Let \( L'_q \) denote the set of incoming transitions and \( L_q \) denote the set of outgoing transitions for state \( q \), defined as

\[
L'_q = \{ l \in L : q'_l = q \}, \quad L_q = \{ l \in L : q_l = q \}, \quad \forall q \in Q.
\]

Following the ergodicity assumption of the Markov chain \( q(t) \) in the AoI analysis \[2\], \[4\], the state probability vector \( \pi(t) = [\pi_0(t) \cdots \pi_m(t)] \) converges uniquely to the stationary vector \( \bar{\pi} = [\bar{\pi}_0 \cdots \bar{\pi}_m] \) satisfying

\[
\bar{\pi}_q \sum_{l \in L_q} \lambda^{(l)} = \sum_{l \in L'_q} \lambda^{(l)} \bar{\pi}_q, \quad \forall q \in Q, \quad \sum_{q \in Q} \bar{\pi}_q = 1.
\]

Further, it has been shown in \[4\, Theorem 1\] that under the ergodicity assumption of the Markov chain \( q(t) \), if we can find a non-negative limit \( \bar{v}_q = [\bar{v}_{q0} \cdots \bar{v}_{qn}] \), \( \forall q \in Q \), for the correlation vector \( v_q(t) \) satisfying

\[
\bar{v}_q \sum_{l \in L_q} \lambda^{(l)} = \bar{\pi}_q 1 + \sum_{l \in L'_q} \lambda^{(l)} \bar{v}_q A_l, \quad \forall q \in Q, \quad (6)
\]

there exists \( s_0 > 0 \) such that for all \( s < s_0 \), \( v^s_q(t), \forall q \in Q \), converges to \( \bar{v}^s_q \) that satisfies

\[
\bar{v}^s_q \sum_{l \in L_q} \lambda^{(l)} = s \bar{v}^s_q + \sum_{l \in L'_q} \lambda^{(l)} [\bar{v}^s_q A_l + \bar{\pi}_q 1 \hat{A}_l], \quad \forall q \in Q, \quad (7)
\]
where $\hat{A}_t \in \mathbb{B}^{(n+1)\times(n+1)}$ is a binary matrix whose $k,j$th element, $\hat{A}_t(k,j)$, is given as

$$\hat{A}_t(k,j) = \begin{cases} 1, & k=j, \text{ and } j\text{th column of } A_t \text{ is a zero vector}, \\ 0, & \text{otherwise}. \end{cases}$$

Finally, the MGF of the continuous state $x(t)$, which is calculated by $\mathbb{E}[e^{sx(t)}]$, converges to the stationary vector [4, Theorem 1]

$$\mathbb{E}[e^{sx}] = \sum_{q \in Q} \bar{v}^s_q. \quad (8)$$

As (8) implies, if the first element of continuous state $x(t)$ represents the AoI of source 1 at the sink, the MGF of the AoI of source 1 at the sink converges to

$$M_{\Delta_1}(s) = \sum_{q \in Q} \bar{v}^s_{q0}. \quad (9)$$

From (9), the main challenge in calculating the MGF of the AoI of source 1 using the SHS technique reduces to deriving the first elements of correlation vectors $\bar{v}^s_q, \forall q \in Q$.

IV. AOI ANALYSIS USING THE SHS TECHNIQUE

In this section, we use the SHS technique to calculate the MGF of the AoI of source 1 in (9) for each source under the self-preemptive and non-preemptive policies.

The discrete state space is $Q = \{00, 01, 02, 21, 12\}$, where state $a_1a_2$ indicates that a packet of source $a_2$ is under service when $a_2 \neq 0$ and a packet of source $a_1$ is in the waiting queue when $a_1 \neq 0$. Note that $a_1 = 0$ (resp. $a_2 = 0$) indicates that the waiting queue (resp. the server) is empty.

The continuous process is $x(t) = [x_0(t) \ x_1(t) \ x_2(t)]$, where $x_0(t)$ is the current AoI of source 1 at time instant $t$, $\Delta_1(t)$; $x_1(t)$ encodes what the AoI of source 1 would become if the packet that is under service is delivered to the sink at time instant $t$; $x_2(t)$ encodes what the AoI of source 1 would become if the packet in the waiting queue is delivered to the sink at time instant $t$.

Recall that to calculate the MGF of the AoI of source 1 in (9) we need to find $\bar{v}^s_{q0}, \forall q \in Q$. 

which are the solution of the system of linear equations (7) with variables $\bar{v}_q^s, \forall q \in Q$. To form the system of linear equations (7), we need to determine $\bar{\pi}_q, A_l$, and $\hat{A}_l$ for each state $\forall q \in Q$, and transition $l \in L'_q$. Next, we derive these under the self-preemptive and non-preemptive policies in Sections [IV-A] and [IV-B] respectively.

A. MGF of the AoI Under the Self-Preemptive Policy

The Markov chain for the discrete state $q(t)$ is shown in Fig. 1. The transitions between the discrete states $q_l \rightarrow q'_l, \forall l \in L$, and their effects on the continuous state $x(t)$ are summarized in Table I. The explanations of the transitions can be found in [11, Section IV.B].

As it has been shown in [11, Section IV.B], the stationary probabilities are given as

$$\bar{\pi} = \frac{1}{2\rho_1\rho_2 + \rho + 1} \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_1\rho_2 & \rho_1\rho_2 \end{bmatrix}.$$ (10)

Recall from Section III that to calculate the MGF of the AoI, first, we need to make sure whether we can find non-negative vectors $\bar{v}_q = [\bar{v}_{q0} \cdots \bar{v}_{qn}], \forall q \in Q$, satisfying (6). As it is shown in [11, Eq. (13)], the system of linear equations in (6) has a non-negative solution. Thus, the MGF of the AoI can be calculated by solving the system of linear equations in (7). We form (7) by substituting the values of $A_l$ and $\hat{A}_l$ presented in Table I and the vector $\bar{\pi}$ in (10). By solving the formed system of linear equations, the values of $\bar{v}^s_{q0}, \forall q \in Q$, under the self-preemptive policy are calculated as presented in Appendix A.

Finally, substituting the values of $\bar{v}^s_{q0}, \forall q \in Q$, into (9), we obtain the MGF of the AoI of
source 1 under the self-preemptive policy, as given in Theorem 1.

| $l$ | $q_l \rightarrow q_l'$ | $\lambda_l$ | $x\mathbf{A}_l$ | $\mathbf{A}_l$ | $\hat{\mathbf{A}}_l$ |
|-----|-----------------|-------------|-----------------|--------------|-----------------|
| 1   | 00 $\rightarrow$ 01 | $\lambda_1$ | $[x_0 \ 0 \ x_2]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 2   | 00 $\rightarrow$ 02 | $\lambda_2$ | $[x_0 \ x_0 \ x_2]$ | $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 3   | 01 $\rightarrow$ 01 | $\lambda_1$ | $[x_0 \ 0 \ x_2]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 4   | 01 $\rightarrow$ 21 | $\lambda_2$ | $[x_0 \ x_1 \ x_1]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 5   | 02 $\rightarrow$ 12 | $\lambda_1$ | $[x_0 \ x_0 \ 0]$ | $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 6   | 21 $\rightarrow$ 21 | $\lambda_1$ | $[x_0 \ 0 \ 0]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 7   | 12 $\rightarrow$ 12 | $\lambda_1$ | $[x_0 \ x_0 \ 0]$ | $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 8   | 01 $\rightarrow$ 00 | $\mu$ | $[x_1 \ x_1 \ x_2]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 9   | 02 $\rightarrow$ 00 | $\mu$ | $[x_0 \ x_1 \ x_2]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 10  | 21 $\rightarrow$ 02 | $\mu$ | $[x_1 \ x_1 \ x_2]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| 11  | 12 $\rightarrow$ 01 | $\mu$ | $[x_0 \ x_2 \ x_2]$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |

### B. MGF of the AoI Under the Non-Preemptive Policy

The Markov chain of the non-preemptive policy is the same as that for the self-preemptive policy. Thus, the stationary probability vector $\mathbf{\pi}$ of the non-preemptive policy is given in (10). The transitions between the discrete states $q_l \rightarrow q'_l$, and their effects on the continuous state $x(t)$ for $l \in \{1, 2, 4, 5, 7, 8, 9, 10, 11\}$ are same as those for the self-preemptive policy. The transitions $l \in \{3, 6\}$ and their effects on the continuous state $x(t)$ are summarized in Table II. The explanations of the transitions can be found in [11, Section IV.C].

As it is shown in [11, Eq. (14)], the system of linear equations in (6) has a non-negative solution. Thus, the MGF of the AoI can be calculated by solving the system of linear equations in (7). We form (7) by substituting the values of $\mathbf{A}_l$ and $\hat{\mathbf{A}}_l$ presented in Tables I and II and the vector $\mathbf{\pi}$ in (10). By solving the formed system of linear equations, the values of $\tilde{v}^{\ast}_q$, $\forall q \in \mathcal{Q}$,
under the non-preemptive policy are calculated as presented in Appendix [B].

Finally, substituting the values of $\bar{v}_{q0}$, $\forall q \in Q$, into (9) results in the MGF of the AoI of source 1 under the non-preemptive policy, given in Theorem 2.

The following remark presents how we can derive different moments of the AoI by using the MGF.

**Remark 1.** The $m$th moment of the AoI of source 1, $\Delta_1^{(m)}$, is calculated as

$$\Delta_1^{(m)} = \frac{d^m(M\Delta_1(s))}{ds^m}\bigg|_{s=0}. \quad (11)$$

**V. FIRST AND SECOND MOMENTS OF THE AOI**

Having derived the MGF of the AoI presented in Theorems 1 and 2, we apply Remark 1 to derive the first and second moments of the AoI of source 1.

The first moment of the AoI of source 1 under the self-preemptive policy is given as

$$\Delta_1 = \frac{(\rho_2 + 1)^2 + \sum_{k=1}^{5}\rho_1^k \eta_k}{\mu \rho_1 (1+\rho_1)^2 (\rho_1^2(2\rho_2+1) + (\rho_2+1)^2(2\rho_1+1))}, \quad (12)$$

where

$$\eta_1 = 6\rho_2^2 + 11\rho_2 + 5, \quad \eta_2 = 13\rho_2^2 + 24\rho_2 + 10,$$

$$\eta_3 = 10\rho_2^2 + 27\rho_2 + 10, \quad \eta_4 = 3\rho_2^2 + 14\rho_2 + 5, \quad \eta_5 = 3\rho_2 + 1. \quad (13)$$

The first moment of the AoI of source 1 under the non-preemptive policy is given as

$$\Delta_1 = \frac{(\rho_2 + 1)^3 + \sum_{k=1}^{4}\rho_1^k \hat{\eta}_k}{\mu \rho_1 (1+\rho_1) (1+\rho_2) \left(\rho_1^2(2\rho_2+1) + (\rho_2+1)^2(2\rho_1+1)\right)}, \quad (14)$$
where
\[
\hat{\eta}_1 = 5\rho_2^3 + 14\rho_2^2 + 13\rho_2 + 4, \quad \hat{\eta}_2 = 10\rho_2^3 + 28\rho_2^2 + 25\rho_2 + 7, \quad \hat{\eta}_3 = 5\rho_2^3 + 22\rho_2^2 + 23\rho_2 + 6, \quad \hat{\eta}_4 = 5\rho_2^2 + 8\rho_2 + 2.
\]  
(15)

It is worth to note that (12) and (14) coincide with the results in [11, Theorems 2 and 3], as expected.

The second moment of the AoI of source 1 under the self-preemptive policy is given as
\[
\Delta_1^{(2)} = \frac{2(\rho_2 + 1)^3 + 2\sum_{k=1}^{8} \rho_1^k \xi_k}{\mu \rho_1^2 (1 + \rho_1)^3 (1 + \rho)^2 (2\rho_1\rho_2 + \rho + 1)},
\]
where
\[
\xi_1 = 7\rho_2^3 + 21\rho_2^2 + 21\rho_2 + 7, \quad \xi_2 = 22\rho_2^3 + 68\rho_2^2 + 68\rho_2 + 22, \\
\xi_3 = 40\rho_2^3 + 113\rho_2^2 + 134\rho_2 + 41, \quad \xi_4 = 36\rho_2^3 + 161\rho_2^2 + 180\rho_2 + 50, \\
\xi_5 = 18\rho_2^3 + 113\rho_2^2 + 160\rho_2 + 41, \quad \xi_6 = 4\rho_2^3 + 45\rho_2^2 + 88\rho_2 + 22, \\
\xi_7 = 8\rho_2^2 + 28\rho_2 + 7, \quad \xi_8 = 4\rho_2 + 1.
\]  
(17)

The second moment of the AoI of source 1 under the non-preemptive policy is
\[
\Delta_1^{(2)} = \frac{2(\rho_2 + 1)^5 + 2\sum_{k=1}^{7} \rho_1^k \hat{\xi}_k}{\mu \rho_1^2 (1 + \rho_1)^2 (1 + \rho_2)^2 (1 + \rho)^2 (2\rho_1\rho_2 + \rho + 1)},
\]
where
\[
\hat{\xi}_1 = 6\rho_2^5 + 30\rho_2^4 + 60\rho_2^3 + 60\rho_2^2 + 30\rho_2 + 6, \\
\hat{\xi}_2 = 18\rho_2^5 + 91\rho_2^4 + 182\rho_2^3 + 180\rho_2^2 + 88\rho_2 + 17, \\
\hat{\xi}_3 = 34\rho_2^5 + 178\rho_2^4 + 361\rho_2^3 + 355\rho_2^2 + 169\rho_2 + 31, \\
\hat{\xi}_4 = 29\rho_2^5 + 190\rho_2^4 + 439\rho_2^3 + 463\rho_2^2 + 224\rho_2 + 39, \\
\hat{\xi}_5 = 9\rho_2^5 + 97\rho_2^4 + 293\rho_2^3 + 365\rho_2^2 + 192\rho_2 + 32.
\]
Fig. 2: The average AoI of source 1 and its standard deviation as a function of $\lambda_1$ under the two packet management policies for $\mu = 1$ and $\lambda = 5$. The dashed lines visualize the standard deviation of the AoI as $\Delta_{1,+}\sigma$ and $\Delta_{1,-}\sigma$.

$$\hat{\xi}_6 = 18\rho_2^4 + 92\rho_2^3 + 151\rho_2^2 + 93\rho_2 + 15, \quad \hat{\xi}_7 = 9\rho_2^3 + 24\rho_2^2 + 19\rho_2 + 3.$$  

(19)

Fig. 2 depicts the average AoI of source 1 and its standard deviation ($\sigma$) as a function of $\lambda_1$ under the two packet management policies for $\mu = 1$ and $\lambda = 5$. This figure shows that in a status update system the standard deviation of the AoI might have a large value. Thus, to have a reliable system, in addition to optimizing the average AoI, we need to take the higher moments of the AoI into account.

VI. Conclusions

We considered a status update system consisting of two independent sources and one server in which packets of each source are generated according to the Poisson process and packets in the system are served according to an exponentially distributed service time. We derived the MGF of the AoI under two packet management policies by using the SHS technique. We derived the first and second moments of the AoI by using the MGF and demonstrated the importance of considering higher moments in the AoI optimization.
\[ \dot{v}_{00}^s = \frac{\rho_1}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{(1 - \bar{s})^2 + \rho_2}{(\rho_1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})^2} + \frac{\rho_1^3 + \rho_1^2(3 - 2\bar{s} + \rho_2) + \rho_1((2 - \bar{s})^2 + \rho_2(2 - \bar{s}) - 1)}{(\rho_1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})^2} \right]. \]

\[ \dot{v}_{10}^s = \frac{\rho_1^2}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{\rho_1^3(\rho_2 + 1 - \bar{s}) + \rho_1^2\alpha_{1,1} + \rho_1\alpha_{1,2} + (1 - \bar{s})^3}{(1 - \bar{s})(\rho_1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} + \frac{\rho_1^3 + \rho_1^2(3 - 2\bar{s} + \rho_2) + \rho_1(3(1 - \bar{s}) + \bar{s}^2 + (2 - \bar{s}))}{(\rho_1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} \right], \]

where \( \alpha_{1,1} = (\rho_2 + 2)^2 + 2(\bar{s} - 2)^2 + 3(1 - \rho_2) - 9 \), and \( \alpha_{1,2} = \rho_2^2(2 - \bar{s}) + \rho_2(5 - 6\bar{s} + 2\bar{s}^2) + 3 - 7\bar{s} + 5\bar{s}^2 - \bar{s}^3 \).

\[ \dot{v}_{20}^s = \frac{\rho_1\rho_2}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{1 - 2\bar{s} + \rho_2}{(\rho_1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} + \frac{\rho_1^3 + \rho_1^2(3 - 2\bar{s} + \rho_2) + \rho_1(3(1 - \bar{s}) + \bar{s}^2 + (2 - \bar{s}))}{(\rho_1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} \right]. \]

\[ \dot{v}_{30}^s = \frac{\rho_1^2\rho_2}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{\rho_1^3(\rho_2 + 1 - \bar{s}) + \rho_1^2\alpha_{3,1} + \rho_1\alpha_{3,2} + (\rho_2 + 1)^2}{(\rho_1 - \bar{s})(1 - \bar{s})^2(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} + \frac{(1 - \bar{s})^3 - 3\rho_2\bar{s} - 1}{(\rho_1 - \bar{s})(1 - \bar{s})^2(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} \right], \]

where \( \alpha_{3,1} = (\rho_2 + 2)^2 + 2(\bar{s} - 1)^2 - \bar{s}(3\rho_2 + 1) - 3 \), and \( \alpha_{3,2} = \rho_2^2(2 - \bar{s}) + \rho_2(5 - 6\bar{s} + 2\bar{s}^2) + 3 - 7\bar{s} + 5\bar{s}^2 - \bar{s}^3 \).

\[ \dot{v}_{40}^s = \frac{\rho_1^2\rho_2}{2\rho_1\rho_2 + \rho + 1} \left[ \frac{\rho_1^3 + \rho_1^2(3 - 2\bar{s} + \rho_2) + \rho_1(3(1 - \bar{s}) - 2\bar{s} + \rho_2)}{(\rho_1 - \bar{s})(1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho - \bar{s})} \right]. \]
APPENDIX B

VALUES OF $\bar{v}_{q0}^*$ FOR THE NON-PREEMPTIVE POLICY

\[
\bar{v}_{00}^* = \frac{\rho_1}{2\rho_1 \rho_2 + \rho + 1} \left[ \frac{\rho_1^2 (1 - \bar{s})(1 + \rho_2) + \rho_1 \bar{\alpha}_{0,1} + (1 + \rho_2)^2}{(\rho_1 - \bar{s})(1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})} \right] + \frac{\rho_2 \bar{s}(\bar{s} - 3) + (1 - \bar{s})^3 - 1}{(\rho_1 - \bar{s})(1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})},
\]

where $\bar{\alpha}_{0,1} = \rho_2(\rho_2 + 3) + \rho_2 \bar{s}(\bar{s} - 3) + 2(1 - \bar{s})$.

\[
\bar{v}_{10}^* = \frac{\rho_1^2}{2\rho_1 \rho_2 + \rho + 1} \left[ \frac{\rho_1^2 \bar{\alpha}_{1,1} + \rho_1 \bar{\alpha}_{1,2} + \rho_2^3 + \rho_2^2 (3 - 4\bar{s})}{(\rho_1 - \bar{s})(1 - \bar{s})^2(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})} \right] + \frac{\rho_2 (3 - 8\bar{s} + 6\bar{s}^2 - \bar{s}^3) + (1 - \bar{s})^4}{(\rho_1 - \bar{s})(1 - \bar{s})^2(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})},
\]

where $\bar{\alpha}_{1,1} = (\rho_2 + 1)^2 + \rho_2 \bar{s}(\bar{s} - 2) + (1 - \bar{s})^2 - 1$, and $\bar{\alpha}_{1,2} = \rho_2^3 + \rho_2^2 (4 - 3\bar{s}) + \rho_2 (5 - 9\bar{s} + 4\bar{s}^2 - \bar{s}^3) + 2(1 - \bar{s})^3$.

\[
\bar{v}_{20}^* = \frac{\rho_1 \rho_2}{2\rho_1 \rho_2 + \rho + 1} \left[ \frac{\rho_1^2 (\rho_2 + 1) + \rho_1 (\rho_2^2 + 3\rho_2 + 2 - \bar{s}(2\rho_2 + 3)) + \rho_2^3}{(\rho_1 - \bar{s})(1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})} \right] + \frac{\rho_2 (2 - 3\bar{s}) + 1 - 3\bar{s} + 2\bar{s}^2}{(\rho_1 - \bar{s})(1 - \bar{s})(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})},
\]

\[
\bar{v}_{30}^* = \frac{\rho_1^2 \rho_2}{2\rho_1 \rho_2 + \rho + 1} \left[ \frac{\rho_1^2 \bar{\alpha}_{3,1} + \rho_1 \bar{\alpha}_{3,2} + \rho_2^3 + \rho_2^2 (3 - 4\bar{s})}{(\rho_1 - \bar{s})(1 - \bar{s})^3(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})} \right] + \frac{\rho_2 (3 - 8\bar{s} + 6\bar{s}^2 - \bar{s}^3) + (1 - \bar{s})^2}{(\rho_1 - \bar{s})(1 - \bar{s})^3(1 + \rho_1 - \bar{s})(1 + \rho_2 - \bar{s})(1 + \rho - \bar{s})},
\]

where $\bar{\alpha}_{3,1} = (\rho_2 + 1)^2 + \rho_2 \bar{s}(\bar{s} - 2) + (1 - \bar{s})^2 - 1$, and $\bar{\alpha}_{3,2} = \rho_2^3 + \rho_2^2 (4 - 3\bar{s}) + \rho_2 (5 - 9\bar{s} + 4\bar{s}^2 - \bar{s}^3) + 2(1 - \bar{s})^3$.
REFERENCES

[1] S. Kaul, R. Yates, and M. Gruteser, “Real-time status: How often should one update?” in Proc. IEEE Int. Conf. on Computer Commun. (INFOCOM), Orlando, FL, USA, Mar. 25–30, 2012, pp. 2731–2735.

[2] R. D. Yates and S. K. Kaul, “The age of information: Real-time status updating by multiple sources,” IEEE Trans. Inform. Theory, vol. 65, no. 3, pp. 1807–1827, Mar. 2019.

[3] M. Moltafet, M. Leinonen, and M. Codreanu, “On the age of information in multi-source queueing models,” IEEE Trans. Commun., vol. 68, no. 8, pp. 5003–5017, May 2020.

[4] R. D. Yates, “The age of information in networks: Moments, distributions, and sampling,” IEEE Trans. Inform. Theory, vol. 66, no. 9, pp. 5712–5728, Sep. 2020.

[5] S. K. Kaul and R. D. Yates, “Age of information: Updates with priority,” in Proc. IEEE Int. Symp. Inform. Theory, Vail, CO, USA, Jun. 17–22, 2018, pp. 2644–2648.

[6] R. D. Yates, “Age of information in a network of preemptive servers,” in Proc. IEEE Int. Conf. on Computer Commun. (INFOCOM), Honolulu, HI, USA, Apr. 15–19, 2018, pp. 118–123.

[7] ——, “Status updates through networks of parallel servers,” in Proc. IEEE Int. Symp. Inform. Theory, Vail, CO, USA, Jun. 17–22, 2018, pp. 2281–2285.

[8] A. Javani, M. Zorgui, and Z. Wang, “Age of information in multiple sensing,” in Proc. IEEE Global Telecommun. Conf., Waikoloa, HI, USA, USA, Dec. 9–13, 2019.

[9] S. Farazi, A. G. Klein, and D. Richard Brown, “Average age of information in multi-source self-preemptive status update systems with packet delivery errors,” in Proc. Annual Asilomar Conf. Signals, Syst., Comp., Pacific Grove, CA, USA, USA, 2019.

[10] A. Maatouk, M. Assaad, and A. Ephremides, “On the age of information in a CSMA environment,” IEEE/ACM Trans. Net., vol. 28, no. 2, pp. 818–831, Feb. 2020.

[11] M. Moltafet, M. Leinonen, and M. Codreanu, “Average AoI in multi-source systems with source-aware packet management,” (Under revision) IEEE Trans. Commun, 2020. [Online]. Available: https://arxiv.org/pdf/2001.03959.pdf

[12] M. Moltafet, M. Leinonen, and M. Codreanu, “Average age of information for a multi-source M/M/1 queueing model with packet management,” in Proc. IEEE Int. Symp. Inform. Theory, Los Angeles, CA, USA, Jun. 21–26, 2020, pp. 1765–1769.