ASL STRUCTURES OF SOME QUADRICS

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ABSTRACT. Let $K$ be a field and $X, Y$ denote matrices such that, the entries of $X$ are either indeterminates over $K$ or $0$ and the entries of $Y$ are indeterminates over $K$ which are different from those appearing in $X$. We consider ideals of the form $I_1(XY)$, which is the ideal generated by the $1 \times 1$ minors of the matrix $XY$. We prove that the quotient ring $K[X, Y]/I_1(XY)$ admits an ASL structure for certain $X$ and $Y$.

INTRODUCTION

Let $K$ be a field and $\{x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}, \{y_j; 1 \leq j \leq n\}$ be indeterminates over $K$. Let $R = K[x_{ij}]$ and $S = K[x_{ij}, y_j]$ denote the polynomial algebras over $K$. Let $X$ denote an $m \times n$ matrix such that its entries belong to the ideal $\langle \{x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \rangle$ and $Y = (y_j)_{n \times 1}$ the generic $n \times 1$ column matrix. Let $I_1(XY)$ denote the ideal generated by the $1 \times 1$ minors or the entries of the $m \times 1$ matrix $XY$. We assume that $m = n$ and write $I = I_1(XY) = \langle g_1, g_2, \ldots, g_n \rangle$. The ideal $I_1(XY)$ is a special case of the defining ideal of a variety of complexes, see [1]. These ideals also feature in [5], in the study of the structure of a universal ring of a universal pair. Tchernev has proved that the set of standard monomials form a free basis for the universal ring.

Under the assumption that $X$ is generic (respectively generic symmetric) and with respect to any monomial order satisfying $x_{11} > x_{22} > \cdots > x_{nn}; x_{ij}, y_j < x_{nn}$ for every $1 \leq i \neq j \leq n$ (respectively $1 \leq i < j \leq n$), it is true that the set $\{g_1, g_2, \ldots, g_n\}$ forms a Gröbner basis for the ideal $I_1(XY)$; see [3]. Another Gröbner basis exists and that also appears in [3], which has been used to prove normality in [3] and compute primary decomposition of these ideals in [4]. In this paper we will show that the knowledge of Gröbner basis actually leads us to the fact that $K[X, Y]/I_1(XY)$ admits an ASL structure.

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1. ALGEBRA WITH STRAIGHTENING LAW

ASL or the Algebra with Straightening Law is a special structure on an algebra $A$, over a partially ordered subset of $A$. We list the definition and some basic facts below. We refer to [2] for definitions and pertinent results.

**Definition 1.** Let $A$ be a commutative ring with 1 and $H$ a subset of $A$. Suppose that $H$ is a partially ordered set (poset). A standard monomial is a product of the form $m_1 \cdots m_k$, such that $m_1 \leq \cdots \leq m_k$.

**Definition 2.** Let $B$ be a commutative ring with 1 and $A$ an algebra over the ring $B$. Let $H$ be a finite partially ordered subset of $A$, which generates $A$ as a $B$ algebra. Then $A$ is an algebra with straightening law (on $H$, over $B$) if the following conditions are satisfied:

1. The algebra $A$ is a free $B$ module whose basis is the set of standard monomials.
2. If $\alpha$ and $\beta$ in $H$ are incomparable and if 
   \[ \alpha \beta = \sum r_i m_1 m_2 \cdots m_k, \]
   where $r_i \neq 0$ is in $B$ and $m_1 \leq m_2 \leq \cdots \leq m_k$ is the unique expression for $\alpha \beta$ in $A$ as a linear combination of standard monomials, then $m_i \leq \alpha, \beta$ for every $i$.

**Theorem 1.1.** Let us consider the $K$ algebra $S/I_1(\{X,Y\})$, where $X = (x_{ij})$ is generic $n \times n$ matrix of indeterminates $x_{ij}$ and $Y$ is generic $n \times 1$ matrix of indeterminates $y_i$. Then $S/I_1(\{X,Y\})$ is an algebra with straightening law on the partially ordered set $H = \{x_{ij} + I_1(\{X,Y\}), y_i + I_1(\{X,Y\}) \mid 1 \leq i, j \leq n\}$ over $K$. The partial order $\preceq$ on $H$ is given by following chains:

1. $\bar{x}_{12} \preceq \cdots \preceq \bar{x}_{1n} \preceq \bar{x}_{21} \preceq \bar{x}_{23} \preceq \cdots \preceq \bar{x}_{2, n} \preceq \cdots \preceq \bar{x}_{n, (n-1)} \preceq \bar{x}_{nn} \preceq \bar{x}_{(n-1), (n-1)} \preceq \cdots \preceq \bar{x}_{11},$
2. $\bar{x}_{n, (n-1)} \preceq \bar{y}_{n} \preceq \cdots \preceq \bar{y}_{1},$
3. $\bar{x}_{22} \preceq \bar{y}_{1},$
4. $\bar{y}_{n} \preceq \bar{x}_{(n-1), (n-1)},$
5. $\bar{x}_{(i+1), (i+1)} \bar{y}_{i} \preceq \bar{x}_{(i-1), (i-1)}$, for $2 \leq i \leq n - 1$.

Here $\bar{\cdot}$ denotes the residue modulo $I_1(\{X,Y\})$.

**Proof.** We fix monomial order on $S$ as in the theorem for the Gröbner basis for the generic case:

1. $x_{11} > x_{22} > \cdots > x_{nn};$
2. $x_{ij}, y_j < x_{nn}$ for every $1 \leq i \neq j \leq n$.

Let $\mathcal{I} = I_1(\{X,Y\}) = \langle g_1, g_2, \cdots, g_n \rangle$, where $g_i = \sum_{j=1}^{n} x_{ij} y_j$. Then, the set $\{g_1, \cdots, g_n\}$ forms a Gröbner basis for the ideal $\mathcal{I}$ with respect to the
monomial order written above. Therefore, \( \text{in}(\mathcal{I}) = \langle \{x_{ii}y_i \mid 1 \leq i \leq n\} \rangle \) and the set \( L = \{ m \mid m \text{ is a monomial such that } m \notin \text{in}(\mathcal{I}) \} \) forms a basis of \( K \) algebra \( S/\mathcal{I} \). Since only \( x_{ii} \) and \( y_i \) are incomparable in \( H \), for all \( 1 \leq i \leq n \), then it is obvious that \( L \) is set of standard monomials in \( S/\mathcal{I} \) with respect to the given partial order on \( H \). Therefore the first condition in ASL holds. Now we have the expression

\[
\bar{x}_{ii} \bar{y}_i = -\left( \sum_{j=1, j \neq i}^{n} \bar{x}_{ij} \bar{y}_j \right).
\]

Here, for each \( 1 \leq i \leq n \), we have \( \bar{x}_{ij} \preceq \bar{y}_j \) and \( \bar{x}_{ij} \preceq \bar{y}_i \), \( \bar{x}_{ij} \preceq \bar{x}_{ii} \), for all \( j \neq i \). \[\square\]

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