COX RINGS OF SURFACES AND THE ANTICANONICAL IITAKA DIMENSION

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Abstract. In this paper we investigate the relation between the finite generation of the Cox ring $R(X)$ of a smooth projective surface $X$ and its anticanonical Iitaka dimension $\kappa(-K_X)$.

Introduction

This paper discusses the problem of deciding which smooth projective surfaces over the complex numbers have finitely generated Cox ring. More precisely, if the Picard group Pic($X$) of such a surface $X$ is finitely generated, or equivalently $q(X) = 0$, then the Cox ring of $X$ is:

$$R(X) := \bigoplus_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

The Cox ring of a surface $X$ is known to be a polynomial ring if and only if $X$ is toric [7, 12]. In [22, Cor. 5.1] Totaro proved that the Cox ring of a klt Calabi-Yau pair of dimension two over $\mathbb{C}$ is finitely generated if and only if its effective cone is rational polyhedral. The surfaces in this class, in particular, have numerically effective anticanonical divisor. In the recent paper [20] by Testa, Várilly-Alvarado and Velasco, the authors prove that if $X$ is a smooth rational surface with $-K_X$ big, i.e. such that $\dim \varphi_{-nK_X}(X) = 2$ for $n$ big enough, then $R(X)$ admits a finite number of generators. These results motivate the research for a relation between the anticanonical Iitaka dimension of $X$ (see [14, Def. 2.1.3]):

$$\kappa(-K_X) := \max\{\dim \varphi_{-nK_X}(X) : n \in \mathbb{N}\},$$

whose values can be $2, 1, 0$ and $-\infty$, and the finite generation of $R(X)$. Our first result is the following.

Theorem. Let $X$ be a smooth rational surface with $\kappa(-K_X) = 1$. Then the following are equivalent:

(i) the effective cone $\text{Eff}(X)$ is rational polyhedral;
(ii) the Cox ring $R(X)$ is finitely generated;
(iii) $X$ contains finitely many $(-1)$-curves.

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Moreover we prove that, if \( \kappa(-K_X) = 1 \), then the effective cone of \( X \) is rational polyhedral if and only if the same is true for its relative minimal model. This allows us to show (Corollary 4.9) that there exist surfaces with finitely generated Cox ring and Iitaka anticanonical dimension 1 of any Picard number \( \geq 9 \).

In case \(-K_X\) is nef we prove the following result, which relies on Nikulin’s description of surfaces with rational polyhedral effective cone [16, Ex. 1.4.1].

**Theorem.** Let \( X \) be a smooth projective surface with \( q(X) = 0 \) and \(-K_X\) nef. Then \( R(X) \) is finitely generated if and only if one of the following holds:

(i) \( X \) is the minimal resolution of singularities of a Del Pezzo surface with Du Val singularities;

(ii) \( \varphi_{[-mK_X]} \) is an elliptic fibration for some \( m > 0 \) and the Mordell-Weil group of the Jacobian fibration of \( \varphi_{[-mK_X]} \) is finite.

(iii) \( X \) is either a K3-surface or an Enriques surface with finite automorphism group \( \text{Aut}(X) \).

Surfaces of type (i) are classically known by [15], surfaces in (ii) can be classified by means of [9, 6] (for \( m = 1 \)) and Ogg-Shafarevich theory [17, 19], while surfaces of type (iii) have been classified in a series of papers by Nikulin and Kondō (see [16, Ex. 1.4.1] for precise references).

The paper is structured as follows. In Section 1 we introduce four cones in \( \text{Pic}(X) \otimes \mathbb{R} \): the effective cone, the closed light cone, the nef cone and the semiample cone. Section 2 deals with the structure of the effective cone of rational surfaces with \( \kappa(-K_X) \geq 0 \). Our main result here is the following.

**Theorem.** Let \( X \) be a smooth rational surface with \( \kappa(-K_X) \geq 0 \) and \( \rho(X) \geq 3 \), then \( \text{Eff}(X) = E(X) \), where \( E(X) \) is the cone generated by classes of integral curves of \( X \) with negative self-intersection.

In Section 3 we prove that on a smooth rational surface \( X \) with \( \kappa(-K_X) = 1 \) every nef divisor is semiample. The problem of finite generation of the Cox ring of such surfaces is considered in Section 4. Section 5 is devoted to the problem of finite generation of \( R(X) \) under the hypothesis \(-K_X\) nef. Finally, Section 6 shows an example of non-rational surface \( X \) with \( \rho(X) = 2 \) and rational polyhedral effective cone whose Cox ring does not admit a finite set of generators.

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## 1. Basic setup

In what follows \( X \) will denote a smooth projective surface defined over the complex numbers. Given a divisor \( D \) of \( X \) we will adopt the short notation \( H^i(D) \) for the cohomology group \( H^i(X, \mathcal{O}_X(D)) \) and we will denote its dimension by \( h^i(D) \). Also, we will denote by \( \sim \) the linear equivalence between divisors, by \([D]\) the class of \( D \) in \( \text{Pic}(X) \) and by \(|D|\) the complete linear series associated to \( D \). We recall that the **effective cone** of an algebraic surface \( X \) is defined as:

\[
\text{Eff}(X) := \left\{ \sum_i a_i [D_i] : D_i \text{ is an effective divisor}, a_i \in \mathbb{R}_{\geq 0} \right\}.
\]

The **closed light cone** of \( X \) is the cone of classes with non-negative self-intersection:

\[
L(X) := \{ [D] \in \text{Pic}(X) : D^2 \geq 0 \}.
\]
We define \( L_a(X) \) to be the half-cone of \( L(X) \) which contains an ample class. We start proving the following (see also [16, §1]).

**Proposition 1.1.** Let \( X \) be a smooth projective surface such that \( \rho(X) \geq 3 \) and \( \text{Eff}(X) \) is polyhedral. Then

\[
\text{Eff}(X) = \sum_{[E] \in \text{Exc}(X)} \mathbb{R}_+ \cdot [E]
\]

where \( \text{Exc}(X) \) is the set of classes of integral curves \( E \) of \( X \) with \( E^2 < 0 \).

**Proof.** This is a consequence of the following observation: by Riemann-Roch theorem the interior of \( L_a(X) \) is contained in \( \text{Eff}(X) \). Since the effective cone is polyhedral, then it is closed, so that \( L_a(X) \subset \text{Eff}(X) \). Since \( \rho = \rho(X) \geq 3 \), then \( \partial L_a(X) \) is circular because the intersection form is hyperbolic with signature \((1, \rho - 1)\) by Hodge index theorem. Thus an element of \( \partial L_a(X) \) can not be an extremal ray of \( \text{Eff}(X) \), since otherwise \( \text{Eff}(X) \) would not be polyhedral in a neighbourhood of that ray, giving a contradiction. \( \square \)

Another important cone associated to \( X \) is the cone of numerically effective divisors, or simply the *nef cone*:

\[
\text{Nef}(X) := \{ [D] \in \text{Pic}(X) : D \cdot E \geq 0 \text{ for any } [E] \in \text{Eff}(X) \}.
\]

This cone is the dual of the effective cone with respect to the intersection form on the surface \( X \). Finally, we define the *semiample cone* \( \text{SAmp}(X) \) to be the cone spanned by the classes of semiample divisors, where \( D \) is semiample if \( |nD| \) is base point free for some \( n > 0 \) (see [14, Def. 1.1.10, 2.1.26]).

**Proposition 1.2.** Let \( X \) be a smooth projective surface. We have the following inclusions:

\[
\text{SAmp}(X) \subset \text{Nef}(X) \subset \text{Eff}(X).
\]

**Proof.** The second inclusion is due to the fact that the nef cone is the closure of the ample cone [14, Thm. 1.4.23]. For the first inclusion observe that, if \( D \) is semiample, then \( \varphi_{[nD]} : X \to \mathbb{P}^r \) is a morphism for \( n \) big enough. Thus, if \( E \) is an effective divisor, then \( nD \cdot E = \deg(\varphi_{[nD]}^*\mathcal{O}(1)_E) \geq 0 \), so that \( D \) is nef. \( \square \)

The following will be useful in the next sections.

**Proposition 1.3.** Let \( X \) be a smooth projective surface with \( q(X) = 0 \) and let \( M \) be an effective divisor of \( X \) such that \( |M| \) does not contain fixed components and \( M^2 = 0 \). Then \( M \sim aD \) with \( D \) smooth and integral, \( h^0(D) = 2 \) and \( H^0(M) \cong \text{Sym}^a H^0(D) \).

**Proof.** The linear series \( |M| \) is base point free, since otherwise two of its distinct elements would intersect at the base points giving \( M^2 > 0 \), which is a contradiction.

Let \( \varphi_{|M|} : X \to B \subset \mathbb{P}^n \) be the morphism defined by \( |M| \). From \( 0 = q(X) \geq p_a(B) \) we deduce that \( B \) is smooth and rational. Consider the Stein factorization of \( \varphi_{|M|} \):

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_{|M|}} & \mathbb{P}^n \\
\downarrow{f} & & \downarrow{\nu} \\
\mathbb{P}^1 & \xrightarrow{} & \\
\end{array}
\]
where $f$ is a morphism with connected fibers and $\nu$ is a finite map. If $a := \deg(\nu) \deg(\nu(\mathbb{P}^1))$, then $M \sim aD$, where $\mathcal{O}_X(D) = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Since $h^0(D) \geq 2$, then $n + 1 = h^0(M) = h^0(aD) \geq a + 1$. On the other hand $a \geq \deg(\nu(\mathbb{P}^1)) \geq n$ since the curve $\nu(\mathbb{P}^1)$ is non-degenerate. Thus $n = a$, $h^0(D) = 2$ and the map $\nu$ is the $a$-Veronese embedding of $\mathbb{P}^1$. Since $f$ has connected fibers, then $D$ is connected so that, by Bertini’s second theorem [13], the general element of $|D|$ is smooth. □

2. The structure of the effective cone

Let $X$ be a projective surface with $q(X) = 0$ and $\kappa(-K_X) \geq 0$. Observe that in this case, either $K_X$ is numerically trivial, or $X$ is rational by Castelnuovo’s rationality criterion [3, Thm. 3.4, VI]. We consider the problem of determining under which hypothesis the effective cone of $X$ is rational polyhedral.

Let $L_a(X)$ be the component of the closed light cone which contains the ample cone, as in the previous section, and let $E(X)$ be the convex cone generated by the classes of curves in $X$ with negative self-intersection. We will adopt the following notation:

$$\overline{\text{Eff}}(X)_{\geq 0} = \{D \in \text{Eff}(X) : D \cdot K_X \geq 0\}, \quad L_a(X)_{\geq 0} = L_a(X) \cap \overline{\text{Eff}}(X)_{\geq 0}.$$

**Theorem 2.1.** If $X$ is a smooth rational surface with $\kappa(-K_X) \geq 0$ and $\rho(X) \geq 3$, then

$$\overline{\text{Eff}}(X) = E(X).$$

**Proof.** We divide the proof in three steps.

**Step 1.** We prove that

$$\text{Chull}(E(X), L_a(X)) = \overline{\text{Eff}}(X).$$

If the sets are distinct then, since both are closed, there exists a class $[D] \in \text{Eff}(X) \setminus \text{Chull}(E(X), L_a(X))$. In particular $D^2 < 0$, so that $|D|$ contains at least a negative curve as a fixed component. Thus $D \sim D_1 + D_2$, where $D_1, D_2$ are effective and $D_1$ consists of all the negative curves contained in the fixed part of $|D|$. Since $[D_1] \in E(X)$, then $[D_2] \notin \text{Chull}(E(X), L_a(X))$, so that $D_2^2 < 0$. Then there is still a negative curve in the fixed part of $|D_2|$ and thus in the fixed part of $|D|$, giving a contradiction.

**Step 2.** We now prove that

$$L_a(X)_{\geq 0} \subset \text{Chull}(E(X), L_a(X))_{\leq 0}.$$

If the interior of $L_a(X)_{\geq 0}$ is empty, then clearly $L_a(X)_{\geq 0} \subset L_a(X)_{\leq 0}$, so we get the thesis by Step 1. Otherwise, let $[D]$ be a class in the interior of $L_a(X)_{\geq 0}$, i.e. $D^2 > 0$ and $D \cdot K_X > 0$. Observe that for some positive integers $n, m$, the multiples $nD, -mK_X$ are effective since $D^2 > 0$ and $\kappa(-K_X) \geq 0$. Thus, since $nD \cdot (-mK_X) < 0$, then $|nD|$ contains at least a negative curve in its fixed locus. Let $D_1$ be given by all curves with negative self-intersection in the fixed locus of $|nD|$. Then the divisor $D_2 = D - D_1$ is nef, so that $D_2^2 \geq 0$ and $D_2 \cdot (-K_X) \geq 0$. Thus $D_2 \in L_a(X)_{\leq 0}$. Together with Step 1, this gives:

$$\text{Chull}(E(X), L_a(X)_{\leq 0}) = \overline{\text{Eff}}(X).$$

**Step 3.** Let $l_+ \in \partial L_a(X)_{\geq 0}$, so that $l_+^2 = 0$ and $l_+ \cdot K_X > 0$. By Step 2 we have $l_+ = e + l_-$, where $e \in E(X)$ and $l_- \in L_a(X)_{\leq 0}$. Assume that $l_+$ is an extremal ray of $\overline{\text{Eff}}(X)$, then $l_+ = e$ is an extremal ray of $E(X)$. Observe that $E(X)_{\geq 0}$ and
Lemma 6.2] we have: is rational and $X$ of an effective multiple of $-K_X$. Then $l_+$ is an extremal ray of $E(X)_{>0}$, so that $l_+^2 < 0$, which is a contradiction.

Assume now that $z_- \in \partial L_a(X)_{<0}$, so that $z_-^2 = 0$ and $z_- \cdot K_X = -\epsilon < 0$. Since $X$ is rational and $\rho(X) \geq 3$, then by the Cone Theorem [14, Thm. 1.5.33] and [8, Lemma 6.2] we have:

$$\text{Eff}(X) = \text{Eff}(X)_{\geq 0} + \sum_i \mathbb{R}_+ \cdot [D_i],$$

for countably many $[D_i] \in E(X)$ which can have accumulation points only on the hyperplane $K_X^\perp$. This implies that $z_- = z_+ + e'$, where $z_+ \in \text{Eff}(X)_{\geq 0}$ and $e' \in E(X)_{\leq 0}$. If $z_-$ is an extremal ray of $\text{Eff}(X)$, then $z_- = e'$ is an extremal ray of $E(X)_{< -\epsilon/2} = E(X)_{< -\epsilon/2}$ by the Cone theorem. Then, since $z_-$ is an extremal ray, we get $z_-^2 < 0$, which is a contradiction.

We proved that neither $\partial L_a(X)_{<0}$ nor $\partial L_a(X)_{>0}$ can be at the boundary of the effective cone, thus $L_a(X) \subset \overline{E(X)}$.

\begin{corollary}
Let $X$ be a smooth rational surface with $\kappa(-K_X) \geq 0$, then the following are equivalent

(i) $\text{Eff}(X)$ is rational polyhedral,

(ii) $X$ contains finitely many $(-1)$ and $(2)$-curves.

\end{corollary}

\begin{proof}
If $\rho(X) \leq 2$, then both the conditions are obviously satisfied. So now we assume that $\rho(X) \geq 3$. The implication $(i) \Rightarrow (ii)$ is given by Proposition 1.1. To prove the converse, by Theorem 2.1 it is enough to prove that $E(X)$ is rational polyhedral, or equivalently that $X$ contains finitely many classes of integral curves with negative self-intersection. The set of such classes in $E(X)_{>0}$ is finite, since the corresponding curves belong to the base locus of an effective multiple of $-K_X$. By the Cone Theorem and [8, Lemma 6.2] the curves with negative self-intersection with classes in $E(X)_{<0}$ are rational, thus they are $(-1)$-curves by adjunction formula. Finally, a curve with negative self-intersection and orthogonal to $K_X$ is a $(2)$-curve. This concludes the proof.

\end{proof}

\begin{remark}
If $\kappa(-K_X) = 2$, then $\text{Eff}(X)$ is rational polyhedral [15, Prop. 3.3]. The effective cone of a smooth rational surface $X$ with $\kappa(-K_X) = 1$ is not necessarily polyhedral. Consider as an example the blow-up $X$ of $\mathbb{P}^2$ at the nine intersection points $\{p_1, \ldots, p_9\} = C_1 \cap C_2$ of two general plane cubics. Due to the generality assumption on the $C_i$, there are no reducible elements in the linear series $| -K_X|$, so that all the fibers of $\varphi_{| -K_X|} : X \to \mathbb{P}^1$ are integral. Hence the class $-K_X$ is an extremal ray of the effective cone. By Proposition 1.1 and the fact that $K_X^2 = 0$, we deduce that $\text{Eff}(X)$ is not polyhedral (see also [21, Cor. 3.2]).

\end{remark}

3. The nef and the semiample cones

In this section we prove a necessary and sufficient condition for $\text{Nef}(X) = \text{SAmp}(X)$ on a smooth rational surface with $\kappa(-K_X) \geq 0$. In what follows we will make use of the Zariski decomposition of a pseudoeffective divisor (see [14, Thm. 2.3.19]): a pseudoeffective divisor $D$ can be written uniquely as a sum $D = N + P$, where

$E(X)_{>0}$ have the same extremal rays, since the last convex set contains a finite number of extremal rays, which are classes of curves contained in the base locus of an effective multiple of $-K_X$. Then $l_+$ is an extremal ray of $E(X)_{>0}$, so that $l_+^2 < 0$, which is a contradiction.
where \( N, P \) are \( \mathbb{Q} \)-divisors such that \( P \) is nef, \( N \) is effective and the intersection matrix of its components is negative definite, \( P \) is orthogonal to each component of \( N \). The divisors \( P \) and \( N \) are called the positive and negative part of \( D \) respectively.

**Lemma 3.1.** Let \( X \) be a smooth rational surface with \( \kappa(-K_X) \geq 1 \). If \( E \) is an effective divisor of \( X \), such that the intersection matrix on its integral components is negative definite, then \( p_a(E) \leq 0 \).

**Proof.** We begin proving that \( h^0(K_X + E) = 0 \). Observe that \( \kappa(E) = 0 \) since the intersection form on its components is negative definite. If \( Z = K_X + E \) is an effective divisor, then \( 0 = \kappa(E) = \kappa(-K_X + Z) \geq 1 \), which is a contradiction.

Consider now the exact sequence of sheaves:

\[
0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0.
\]

Taking cohomology and using the fact that \( h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0 \) because \( X \) is rational, we get \( h^1(\mathcal{O}_E) = h^2(-E) \). The last is equal to \( h^0(K_X + E) \) by Serre’s duality theorem. Now, from what proved before, we deduce that \( h^1(\mathcal{O}_E) = 0 \) so that \( p_a(E) \leq 0 \), which proves the thesis. \( \square \)

**Lemma 3.2.** Let \( X \) be a smooth rational surface with \( \kappa(-K_X) \geq 1 \) and let \( L \) be a nef divisor with \( \kappa(L) = 2 \). Then \( L \) is semiample.

**Proof.** We follow the proof of [20, Lemma 2.6]. Let \( \Delta \) be the union of all integral curves orthogonal to \( L \). Since \( L^2 > 0 \) then, by Hodge index theorem, the restriction to \( \Delta \) of the intersection form of \( X \) is negative definite. Moreover, by Lemma 3.1, we have that \( p_a(E) \leq 0 \) for any effective divisor supported on \( \Delta \). Thus we can apply Artin’s contractability criterion \([2, \text{Thm. 2.3}]\) to \( \Delta \). So, there exists a normal projective surface \( Y \) and a birational morphism \( \psi : X \to Y \) which contracts only the connected components of \( \Delta \). Hence, by \([2, \text{Cor. 2.6}]\), \( L \) is linearly equivalent to a divisor \( L' \) whose support is disjoint from \( \Delta \), and hence \( L \) is the pullback of a Cartier divisor on \( Y \). By Nakai-Moishezon criterion \( L' \) is ample, so that \( L \) is semiample. \( \square \)

**Lemma 3.3.** Let \( X \) be a smooth algebraic surface and let \( L \) be a nef divisor with \( \kappa(L) = 1 \). Then \( L \) is semiample and \( L^2 = 0 \).

**Proof.** First of all observe that since \( L \) is nef with \( \kappa(L) = 1 \), then \( L^2 = 0 \). Let \( nL \) be a \( \mathbb{Z} \)-divisor with \( h^0(nL) > 1 \). If \( B \) is the base locus of \( nL \), then \( \kappa(B) \leq 1 \) so that \( B^2 \leq 0 \). Since \( nL - B \) is nef with \( \kappa(nL - B) = 1 \), then \( (nL - B)^2 = 0 \). This implies that \( B^2 = B \cdot L = 0 \). The restriction of the intersection form to the space spanned by \( [B] \) and \( [L] \) is null. Thus by Hodge index theorem this implies that \( [B] = [mL] \) for some \( m > 0 \). \( \square \)

We are finally ready to prove the main theorem of this section.

**Theorem 3.4.** Let \( X \) be a smooth rational surface with \( \kappa(-K_X) \geq 1 \). Then \( \text{Nef}(X) = \text{SAample}(X) \).

**Proof.** If \( L \) is a nef divisor of \( X \), then \( L^2 \geq 0 \). If \( L^2 > 0 \) or \( K_X \cdot L < 0 \), then \( h^0(L) \geq 2 \) by Riemann-Roch formula so that \( \kappa(L) \geq 1 \) and we conclude by Lemma 3.2 and 3.3. We assume now that \( L^2 = K_X \cdot L = 0 \) and let \( -K_X \sim N + P \) be the Zariski decomposition of \( -K_X \). Then \( P \cdot L = 0 \) because \( P, N \) are effective and \( L \) is nef. The restriction of the intersection form of \( \text{Pic}(X) \) to the space spanned by
Proposition 1.3 we have $P \sim aC$ for some $a \in \mathbb{Q}_{>0}$, where $C$ is a smooth elliptic curve with $C^2 = 0$ and $h^0(C) = 2$.

Proof. Since $P$ is nef and $\kappa(P) = 1$, then $P$ is semiample by Lemma 3.3. By Proposition 1.3 we have $P \sim aC$ for some smooth integral curve $C$ with $C^2 = 0$ and $h^0(C) = 2$. By the genus formula and $-K_X \cdot P = 0$ we get $2g(C) - 2 = C \cdot K_X = 0$.

We consider the problem of the finite generation of Cox rings of smooth rational surfaces with $\kappa(-K_X) = 1$.

**Theorem 4.2.** Let $X$ be a smooth rational surface with $\kappa(-K_X) = 1$. Then the following are equivalent:

(i) the effective cone $\text{Eff}(X)$ is rational polyhedral;
(ii) the Cox ring $R(X)$ is finitely generated;
(iii) $X$ contains finitely many $(-1)$-curves.

Proof. (i)$\Rightarrow$(ii): by [1, Cor. 2.6] it is enough to prove that the nef and semiample cone of $X$ coincide and this is proved in Theorem 3.4.

(ii)$\Rightarrow$(iii): if $C$ is a $(-1)$-curve and $x \in H^0(C)$, write $x = \sum_i m_i$, where $m_i$ are monomials in the generators of $R(X)$. If $D_i$ is the zero locus of $m_i$, then $C \sim D_i$. Since $C$ is integral with $C^2 < 0$, then $D_i = C$ so that $m_i = \alpha_i x$, with $\alpha_i \in \mathbb{C}$. Hence $x$ appears in any set of homogeneous generators of $R(X)$. Since $R(X)$ is finitely generated, this implies that there are finitely many $(-1)$-curves.

(iii)$\Rightarrow$(i): We can assume that $\rho(X) \geq 9$ since otherwise $\kappa(-K_X) = 2$. By Corollary 2.2 it is enough to prove that $X$ contains finitely many $(-2)$-curves.

Let $-K_X \sim N + aC$ be the decomposition given in Proposition 4.1 and let $N_{\text{red}}$ be the reduced divisor associated to a positive multiple $rN$ with $|rN| \in \text{Pic}(X)$. If $E$ is a $(-2)$-curve, then $-K_X \cdot E = 0$, so that either $E$ is a component of $N_{\text{red}}$ or $E \cdot C = 0$. In the last case $E$ is contracted by the morphism $\varphi_{|C|} : X \to \mathbb{P}^1$, which is a fibration by Proposition 4.1. Since $N_{\text{red}}$ has a finite number of components and $\varphi_{|C|}$ has finitely many reducible fibers then $X$ contains finitely many $(-2)$-curves.
Let $X$ be a smooth rational surface with $\kappa(-K_X) = 1$ and let $-K_X \sim P + N$ be the Zariski decomposition. By Proposition 4.1 a multiple of $P$ defines an elliptic fibration. We have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_{|rP|}} & \mathbb{P}^1 \\
\pi \downarrow & & \downarrow \varphi_{|mK_Y|} \\
Y & & \\
\end{array}
\]

where $\pi$ is the blow-down map of all the $(-1)$-curves contained in the fibers of $\varphi_{|rP|}$ and $m$ is the smallest positive integer such that $h^0(-mK_Y) = 2$. The surface $Y$ is the relative minimal model and its anticanonical divisor $-K_Y$ is nef. If $m = 1$, then $Y$ is called a jacobian elliptic surface.

Let $C$ be an integral divisor of a smooth projective surface $X$ such that $\varphi_{|C|} : X \to \mathbb{P}^1$ is a fibration. Let $L \sim aC$ with $a \in \mathbb{Q}$ and let $C_p \in |C|$ be the unique curve through a point $p \in X$.

**Definition 4.3.** The multiplicity of $L$ at $p \in X$ is:

$$\mu(L, p) := a\mu(C_p, p).$$

**Lemma 4.4.** Let $\pi : \tilde{X} \to X$ be the blow-up at a point $p$ of a smooth rational surface with $\kappa(-K_X) = 1$. Then $\kappa(-K_{\tilde{X}}) = 1$ if and only if $\mu(P, p) > 1$, where $P$ is the positive part of the Zariski decomposition of $-K_X$. In this case we have the Zariski decomposition $-K_{\tilde{X}} \sim \tilde{P} + \tilde{N}$ with

$$\tilde{P} \sim (1 - \frac{1}{\mu})\pi^*P.$$

**Proof.** Observe that, by definition of the multiplicity $\mu = \mu(P, p)$, a multiple of the divisor $B := \pi^*P - \mu E$ is effective. Consider the decomposition:

$$-K_{\tilde{X}} \sim \pi^*(P + N) - E \sim (1 - \frac{1}{\mu})\pi^*P + \frac{1}{\mu}B + \pi^*N = \tilde{P} + \tilde{N}.$$

Since $\tilde{P} \cdot B = 0$, then an effective multiple of $B$ is strictly contained into a fiber of the fibration $\varphi_{|\pi^*C|} : \tilde{X} \to \mathbb{P}^1$, where $P \sim aC$ as in Proposition 4.1. Thus the intersection form on the integral components of $B$ is negative definite by [3, Lemma 8.2, III]. This implies that the intersection form on the integral components of $\tilde{N}$ is negative definite, since $B \cdot \pi^*N = 0$ and it is negative definite on the integral components of $\pi^*N$. From this and the fact that $\tilde{P} \cdot \tilde{N} = 0$ we deduce $\kappa(\tilde{P} + \tilde{N}) = \kappa(P)$, which proves the thesis.

**Proposition 4.5.** Let $n \geq 10$ be an integer. There exists a smooth rational surface $X$ with $\kappa(-K_X) = 1$ and $\rho(X) = n$.

**Proof.** Let $X_0$ be a smooth rational surface such that $\varphi_{|\kappa X_0|} : X_0 \to \mathbb{P}^1$ is an elliptic fibration which admits a fiber $P_0$ with a triple point. For example, $X_0$ can be the blow-up of $\mathbb{P}^2$ at the nine base points of the pencil $xy(x + y) + t(x^3 + y^3 + z^3) = 0$. We construct a sequence of blow-ups $\pi_i : X_i \to X_{i-1}$, where $\pi_i$ is the blow-up of $X_{i-1}$ at a point $p_{i-1}$ chosen in the following way. Let $p_0$ be the triple point of $P_0$ and let $p_i$ lie on the intersection of the exceptional divisor $E_{i-1} = \pi_{i-1}^{-1}(p_{i-1})$ and the strict transform $E_{i-2}'$ of $E_{i-2}$ (see Figure 1 below). Let $-K_{X_i} \sim P_i + N_i$ be
the Zariski decomposition. By Lemma 4.4 we have \( P_i \sim (1 - \frac{1}{\mu_{i-1}})\pi_i^*(P_{i-1}) \) where \( \mu_{i-1} = \mu(P_{i-1}, p_{i-1}) \). This gives the formula:

\[
P_i \sim \prod_{k=0}^{i-1} \left(1 - \frac{1}{\mu_k}\right) \phi_i^* P_0,
\]

where \( \phi_i = \pi_1 \circ \cdots \circ \pi_i : X_i \to X_0 \) is the blow-down map. In this way we can recursively calculate \( \mu_i \) obtaining:

\[
\mu_i = \prod_{k=0}^{i-1} \left(1 - \frac{1}{\mu_k}\right) \mu(\phi_i^* P_0, p_i).
\]

Observe that \( \mu_0 = 3 \) since \( p_0 \) is a triple point of a fiber of the elliptic fibration \( \varphi|_{P_0} \). Let \( a_i := \mu(\phi_i^* P_0, p_i) \), then \( a_0 = 3, a_1 = 4 \) and

\[
a_i = a_{i-1} + a_{i-2}.
\]

To see this observe that \( \phi_i^* P_0 \) contains \( E_{i-1} \) with multiplicity \( a_{i-1} \) and the strict transform of \( E_{i-2} \) with multiplicity \( a_{i-2} \). If we denote by \( b_i := a_i/a_{i-1} \), then an easy calculation based on (1) gives

\[
\mu_i = (\mu_{i-1} - 1)b_i.
\]

Observe that the \( a_i \)’s satisfy a Fibonacci type recursion and the \( b_i \)’s are rational approximations in the continued fraction expansion of the number \( \frac{1}{2}(1 + \sqrt{5}) \). In particular we claim that \( b_i > 8/5 \) for \( i > 4 \), which can be easily proved by induction using the fact that \( b_0 = 7/3 \) and \( b_1 = 1 + 1/b_0 \). We prove now by induction that \( \mu_i > 8/3 \) for each \( i \). We have \( \mu_0 = 3 \) and \( \mu_{i-1} > 8/3 = 1 + 1/(8/3 - 1) > 1 + 1/(b_i - 1) \) so that \( \mu_i = (\mu_{i-1} - 1)b_i > \mu_{i-1} > 8/3 \). Since \( \mu_i > 1 \), then \( \kappa(-K_{X_i}) = 1 \) by Lemma 4.4. Thus the surface \( X_{n-9} \) has the required properties.

\[\square\]

![Figure 1. The sequence of blow-ups in Proposition 4.5](image)

**Remark 4.6.** Observe that if we start with a node \( p_0 \) of a singular fiber of \( \varphi|_{P_0} \) and construct a sequence of blow-ups as in the proof of Proposition 4.5, then we get \( \mu_0 = 2, \mu_1 = 3/2 \) and \( \mu_2 = 5/6 < 1 \) so that \( \kappa(-K_{X_2}) < 1 \).
We now want to relate the structure of the effective cone $\text{Eff}(X)$ of a smooth rational surface $X$ with $\kappa(-K_X) = 1$ with that of its relative minimal model $Y$. Recall that the birational morphism $\pi : X \to Y$ induces an injective linear map $\pi^* : \text{Pic}(Y)_\mathbb{R} \to \text{Pic}(X)_\mathbb{R}$ which maps $\text{Eff}(Y)$ into a linear section of $\text{Eff}(X)$. Thus if $\text{Eff}(X)$ is rational polyhedral, then the same is true for $\text{Eff}(Y)$. We will show that the converse statement also holds.

**Lemma 4.7.** Let $Y$ be a smooth rational surface with $\kappa(-K_Y) = 1$, $-K_Y$ nef and rational polyhedral $\text{Eff}(Y)$. If $a,b$ are non-negative integers with $b \neq 0$, then there is a finite number of classes $[D] \in \text{Nef}(Y)$ such that $D^2 = a$, $-K_Y \cdot D = b$.

**Proof.** Observe that $Y$ has a minimal elliptic fibration $\varphi = \varphi_{\lfloor -mK_Y \rfloor}$ for some $m > 0$ by [6, Prop. 5.6.1]. Since the effective cone of $Y$ is rational polyhedral, then there are 8 components of reducible fibers of $\varphi$ whose intersection matrix $M$ is negative definite (see Proposition 5.1(ii) below). Let $f_1, \ldots, f_8$ be the classes of such curves, $f$ be the class of a fiber of $\varphi$ and $s$ be the class of a $m$-section of $\varphi$ so that $f \cdot s = m$. Observe that the lattice $L := (f,s,f_1,\ldots,f_8)$ has rank 10, so that it has finite index $k$ in $\text{Pic}(Y)$. Let

$$[D] = \alpha f + \sum_{i=1}^{8} \alpha_i f_i + \beta s \in \text{Pic}(Y)$$

be a nef class with $-K_Y \cdot D = b$ and $D^2 = a$. Thus $b = -K_Y \cdot D = \frac{1}{m} f \cdot D = \beta$.

Since $D$ is nef and $f - f_i$ is an effective class, then $b_i = D \cdot f_i \leq D \cdot f = b$. Observe that, since $[kD] \in \text{Pic}(Y)$, then the coefficients of $[D]$ are rational with bounded denominators. Hence $b_i$ can take a finite number of non-negative rational values. For any such choice of the $b_i$’s, the coefficients $\alpha_i$ are uniquely determined since the intersection matrix $M$ of the $f_i$’s is non singular. Finally, the condition $D^2 = a$ determines $\alpha$, since $D^2 = -b^2 + (\sum \alpha_i f_i)^2 + 2b \sum \alpha_i f_i \cdot s + 2abm$. This proves that there is a finite number of classes $[D]$ as in the statement. $\square$

**Theorem 4.8.** Let $X$ be a smooth rational surface with $\kappa(-K_X) = 1$ and such that its relative minimal model $Y$ has rational polyhedral effective cone. Then $\text{Eff}(X)$ is rational polyhedral.

**Proof.** By Theorem 4.2 it is enough to prove that $X$ contains finitely many $(-1)$-curves. Let $\pi : X \to Y$ be the blow-down map. We assume that $X$ is the blow-up of $Y$ at $r$ points, eventually infinitely near, and we call $E$ the exceptional divisor of $\pi$. Observe that we can write $E = \sum_{i=1}^{r} c_i E_i$, where $c_i$ are positive integers and $E_i$ are curves (not necessarily integral) such that $E_i^2 = -1$ and $E_i \cdot E_j = 0$ for distinct $i,j$.

Let $F$ be a $(-1)$-curve of $X$ and let $-K_X \sim N + P$ be the Zariski decomposition. Observe that $P \sim \alpha \pi^*(-K_Y)$, where $\alpha$ is a rational number with $0 < \alpha < 1$ by Lemma 4.4. Since $F \cdot (-K_X) = 1$, then either $F$ is a component of $N$ or $F \cdot P = 0$ or $0 < F \cdot P \leq 1$. In the first two cases $F$ belongs to a finite set of curves, so we assume to be in the third case. Then we have

$$F \cdot E = F \cdot (\pi^*(-K_Y) + K_X) = F \cdot \pi^*(-K_Y) - 1 \leq \frac{1}{\alpha} - 1.$$

Observe that we can also assume that $F \cdot E_i \geq 0$ for each $i$, since otherwise $F$ would be a component of $E_i$ and again there is only a finite set of such components. If
Let $D = \pi(F)$, then we can write $F$ as

$$F = \pi^*(D) - \sum_{i=1}^{r} a_i E_i$$

with $a_i = F \cdot E_i \leq F \cdot E \leq \frac{1}{\alpha} - 1$ so that $D^2 \leq -1 + r(\frac{1}{\alpha} - 1)^2$ since $F^2 = -1$. Moreover

$$0 < -K_Y \cdot D = \frac{1}{\alpha} P \cdot (F + \sum_{i} a_i E_i) \leq \frac{1}{\alpha}(1 + (\frac{1}{\alpha} - 1) \sum_{i} P \cdot E_i).$$

Observe that either $D$ is a $(-1)$-curve of $Y$ or $D^2 \geq 0$ and $D$ is a nef divisor since it is integral. In the first case there are a finite number of such $D$ by Theorem 4.2. In the second case we conclude by Lemma 4.7 since both $D^2$ and $-K_Y \cdot D$ are bounded. This implies that $X$ contains finitely many $(-1)$-curves. \qed

**Corollary 4.9.** Let $n \geq 10$ be an integer. Then there exists a smooth rational surface $X$ with $\kappa(-K_X) = 1$, $\rho(X) = n$ and finitely generated Cox ring $R(X)$.

**Proof.** Let $Y$ be a minimal elliptic surface with 4 fibers of Kodaira type $A_2$. Such a surface exists and its effective cone is rational polyhedral by [16, Ex. 1.4.1]. Observe that $Y$ contains a fiber with a triple point, thus proceeding as in the proof of Proposition 4.5 we construct a surface $X$ of Picard number $n$ with $\kappa(-K_X) = 1$ whose relative minimal model is $Y$. Since $\text{Eff}(Y)$ is rational polyhedral then $\text{Eff}(X)$ is rational polyhedral by Theorem 4.8. We now conclude by Theorem 4.2. \qed

### 5. Smooth Projective Surfaces with $-K_X$ nef

In this section we consider smooth projective surfaces $X$ such that $q(X) = 0$, $\text{Eff}(X)$ is rational polyhedral and $-K_X$ is nef. Observe that, since $-K_X$ is nef, then $K_X^2 \geq 0$ and the integral curves with negative self-intersection are either $(-1)$ or $(-2)$-curves. If $X$ is minimal, i.e. it does not contain $(-1)$-curves, then either $\rho(X) \leq 2$ or $K = 0$ by Proposition 1.1. In the first case, $X$ is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$ with $n = 0, 2$. In the second case, $X$ is either a K3 surface or an Enriques surface. In [1, Thm. 2.7, Thm. 2.10] it is proved that the Cox ring of these surfaces is finitely generated if and only if $\text{Eff}(X)$ is rational polyhedral. Moreover, the Picard lattices of those $X$ which admit a rational polyhedral effective cone are classified in a series of papers by Nikulin and Kondō (see [16, Ex. 1.4.1] for precise references).

If $X$ is non-minimal, then it is rational and it is either a Hirzebruch surface $\mathbb{F}_1$ or one of the surfaces described in the following (for the proof see [16, Ex. 1.4.1]).

**Proposition 5.1.** Let $X$ be a smooth rational surface with $-K_X$ nef and $\rho(X) \geq 3$. Then $\text{Eff}(X)$ is rational polyhedral if and only if one of the following holds:

(i) $K_X^2 > 0$ and $X$ is the minimal resolution of singularities of a Del Pezzo surface with Du Val singularities.

(ii) $K_X^2 = 0$ and any connected component of the set of $(-2)$-curves of $X$ is an extended Dynkin diagram of rank $r_i$ with $\sum_i r_i = 8$.

The surfaces in Proposition 5.1 (ii) have been further classified and divided in two classes.

If $\kappa(-K_X) = 1$, then $\varphi_{-mK_X}$ is an elliptic fibration for some $m > 0$. Then (ii) is equivalent to ask that the Jacobian fibration of $\varphi_{-mK_X}$ has finite Mordell-Weil
group. In case \( m = 1 \) the reducible fibers and the Mordell-Weil groups of such surfaces have been classified in \([9, 6]\).

If \( \kappa(-K_X) = 0 \), then \( |-mK_X| \) is zero-dimensional for all positive \( m \). There are exactly three such surfaces, which have been classified in \([16, \text{Ex. 1.4.1}]\).

**Remark 5.2.** If \( X \) is a smooth rational surface as in Proposition 5.1, then \( \text{Eff}(X) \) is generated by \((-1)\) and \((-2)\)-curves by Proposition 1.1. In particular, if \( \varphi = \varphi_{|K_X|} \) is an elliptic fibration, then these curves will be the sections and the components of reducible fibers of \( \varphi \) respectively.

**Theorem 5.3.** Let \( X \) be a smooth projective surface with \( q(X) = 0 \) and \( -K_X \) nef. Then \( R(X) \) is finitely generated if and only if one of the following holds:

(i) \( X \) is the minimal resolution of singularities of a Del Pezzo surface with Du Val singularities;

(ii) \( \varphi_{|mK_X|} \) is an elliptic fibration for some \( m > 0 \) and the Mordell-Weil group of the Jacobian fibration of \( \varphi_{|mK_X|} \) is finite.

(iii) \( X \) is either a K3-surface or an Enriques surface with finite automorphism group \( \text{Aut}(X) \).

**Proof.** If \( K_X \equiv 0 \), then \( X \) is either a K3 or an Enriques surface and we conclude by \([1, \text{Thm. 2.7, Thm. 2.10}]\). If \( K_X \not\equiv 0 \) and \( \rho(X) \leq 2 \), then \( X \) is either \( \mathbb{P}^2 \) or a Hirzebruch surface \( F_0, F_2 \). In all these cases \( X \) is in (i).

Assume now that \( K_X \not\equiv 0 \) and \( \rho(X) \geq 3 \). If \( K_X^2 > 0 \), then \( \kappa(-K_X) = 2 \), so that \( R(X) \) is finitely generated by \([20, \text{Thm. 2.9}]\). If \( K_X^2 = 0 \) and \( \kappa(-K_X) = 1 \) then, by Proposition 5.1 and Theorem 4.2, \( R(X) \) is finitely generated if and only if we are in (ii). If \( K_X^2 = 0 \) and \( \kappa(-K_X) = 0 \), then \( -K_X \) is nef but not semiample since \( h^0(-mK_X) = 1 \) for any \( m > 0 \). Thus \( R(X) \) is not finitely generated by \([1, \text{Corollary 2.6}]\). \( \square \)

**Corollary 5.4.** Let \( X \) be a smooth rational surface such that \( \varphi_{|K_X|} : X \to \mathbb{P}^1 \) is an elliptic fibration. Then \( R(X) \) is finitely generated if and only if the Mordell-Weil group of \( \varphi_{|K_X|} \) is finite.

**Remark 5.5.** Theorem 5.3 and \([16, \text{Ex. 1.4.1}]\) provide a negative answer to \([11, \text{Question I.3.9}]\), since they show that there are rational surfaces such that the effective cone is rational polyhedral but the Cox ring is not finitely generated.

6. An example with \( \kappa(-K_X) = -\infty \)

In this section we construct a surface \( X \) with \( \rho(X) = 2 \) such that \( \text{Eff}(X) \) is rational polyhedral but the Cox ring \( R(X) \) is not finitely generated. The surface \( X \) will be the blow-up of a smooth, very general quartic \( S \subset \mathbb{P}^3 \) at a very general point \( p \). In particular \( \kappa(-K_X) = -\infty \). If \( \pi : X \to S \) is the blow-up at \( p \) with exceptional divisor \( E \) and \( H = \pi^*O_S(1) \), we will show that:

\[
\text{Eff}(X) = \langle [E], [H - 2E] \rangle, \quad \text{Nef}(X) = \langle [H], [H - 2E] \rangle, \quad [H - 2E] \not\in \text{SAmple}(X),
\]

since \( h^0(m(H - 2E)) = 1 \) for any \( m \geq 1 \). This implies, by \([1, \text{Corollary 2.6}]\), that \( R(X) \) is not finitely generated.

**Remark 6.1.** To explain why \( R(X) \) is not finitely generated, to any \( x \in H^0(D) \) we associate its degree:

\[
[x] := [D] \in \text{Pic}(X).
\]
Assume that $R(X)$ is finitely generated, so that there exists $[D] \in \text{Nef}(X)$ such that the degree $[x]$ of any generator either lives in the cone $\langle [D], [E] \rangle$ or is equal to $[H - 2E]$. A class $[D']$ as in the picture is ample, since it lives in the interior of the nef cone, so that there is a section $x' \in H^0(nD')$ which is not divisible by $z \in H^0(H - 2E)$, for $n$ big enough. Thus $[x']$ is a non-negative linear combination of the degrees of the generators of $R(X)$ distinct from $z$. This means that $[nD'] = [x']$ lives in the cone $\langle [D], [E] \rangle$, which is a contradiction.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image}
\caption{Each blue ray provides new generators of $R(X)$.}
\end{figure}

If $S \subset \mathbb{P}^3$ is a smooth quartic surface and $p \in S$ we denote by $\pi : C \to S \cap T_pS$ the normalization at $p$ of the hyperplane section $S \cap T_pS$. Observe that, for general $p$, $S \cap T_pS$ has a node at $p$ and $C$ is a smooth genus two curve.

**Lemma 6.2.** There exists a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 1$ and a point $p \in S$ such that $C$ is smooth of genus two and $K_C - q_1 - q_2$ is not a torsion point of $J(C)$, where $\{q_1, q_2\} = \pi^{-1}(p)$.

**Proof.** Let $S_0 \subset \mathbb{P}^3$ be a smooth quartic surface with $\rho(S_0) = 1$, $p \in S_0$ and $B_0 = S_0 \cap T_pS_0$. We denote by $B_1 \subset T_pS_0$ a plane quartic with just one node at $p$ such that, if $\pi_1 : C_1 \to B_1$ is its normalization, then $K_{C_1} - q_{11} - q_{12}$ is not a torsion point of $J(C_1)$, where $\{q_{11}, q_{12}\} = \pi_1^{-1}(p)$. Observe that such a plane quartic $B_1$ exists because, if $C$ is a genus two curve, then the morphism $\varphi$ associated to the linear system $|K_C + q_1 + q_2|$ is birational onto a plane quartic with just one node at $\varphi(q_1) = \varphi(q_2)$.

Let $f : B \to \mathbb{P}^1$ with $B \subset T_pS_0 \times \mathbb{P}^1$ be the pencil of plane quartics generated by $B_0 = f^{-1}(0)$ and $B_1 = f^{-1}(1)$. We denote by $\pi : C \to B$ the normalization of $B$ along the section $\{(p, t) : t \in \mathbb{P}^1\}$ and by $J(C) \to \mathbb{P}^1$ its relative jacobian. We will denote by $B_t$ and $C_t$ the fiber of $f$ and of $f \circ \pi$ respectively over $t \in \mathbb{P}^1$. Let $L_t := K_{C_t} - q_{11} - q_{12}$, where $\{q_{11}, q_{12}\} = \pi^{-1}(p, t)$. Since $L_1$ is not a torsion point of $J(C_1)$ and the set of torsion sections in $J(C)$ is countable, then $L_t$ is not a torsion point of $J(C_t)$ outside of a countable set of $t \in \mathbb{P}^1$.

Let $S := \{(S, t) \in |\mathcal{O}_{\mathbb{P}^3}(4)| \times \mathbb{P}^1 : B_t \subset S\}$ be the family of quartic surfaces which contain one $B_t$ and let $(S_t, t)$ be a curve in $S$ which maps birationally onto $\mathbb{P}^1$ and which contains $(S_0, 0)$. Observe that $B_t = S_t \cap T_pS_0$ for each $t \in \mathbb{P}^1$. Since $S_0$ is smooth with $\rho(S_0) = 1$, then the same is true for $S_t$ outside of a countable set of values (see for example [18, Thm. 1.1]). Thus there exists $t_0 \in \mathbb{P}^1$ such that $S_{t_0}$ is smooth with $\rho(S_{t_0}) = 1$ and $L_{t_0}$ is not a torsion point of $J(C_{t_0})$. \qed

**Proposition 6.3.** There exists a smooth quartic surface $S \subset \mathbb{P}^3$ with $\rho(S) = 1$ and a point $p \in S$ such that, if $\pi : X \to S$ is the blow-up at $p$, then $\text{Eff}(X)$ is rational polyhedral but $R(X)$ is not finitely generated.
Proof. Let $S$ and $p$ be as in Lemma 6.2. We start proving that $\text{Eff}(X)$ is rational polyhedral. Let $E$ be the exceptional divisor of $\pi$ and $H = \pi^*\mathcal{O}_S(1)$. The class of the strict transform $C$ of $B := S \cap T_pS$ is $[H - 2E]$. Since $C$ is integral and $(H - 2E)^2 = 0$, then $[H - 2E]$ is nef.

Let $[D] := [aC - bE]$ where $a, b$ are positive integers. If $D$ is effective, then $D \cdot C < 0$ so that $h^0(D) = h^0(D - C)$ and $(a - 1)C - bE$ is effective. Applying this reasoning $a$ times we deduce $h^0(-bE) > 0$, which is a contradiction. Hence $[D]$ is not effective. Thus $\text{Eff}(X) = ([H - 2E], [E])$ is rational polyhedral.

We will now prove that $\text{SAample}(X) \subseteq \text{Nef}(X)$, so that $R(X)$ is not finitely generated by [1, Cor. 2.6]. Since $K_X \sim E$, then by adjunction formula we have $C | C \sim K_C - E | C \sim K_C - q_1 - q_2$, where $\pi(q_1) = \pi(q_2) = p$. By Lemma 6.2 this implies that $C | C$ is not a torsion point of $J(C)$ or, equivalently, that $h^0(nC|C) = 0$ for any $n > 0$. From the exact sequence

$$0 \rightarrow H^0((n-1)C) \rightarrow H^0(nC) \rightarrow H^0(nC|C) = 0,$$

and $h^0(\mathcal{O}_X) = 1$, we get that $h^0(\mathcal{O}_X) = 1$ for any positive $n$. This implies that $H - 2E$ is not semiaffine.$\qed$

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