ON STAR COLORING OF SPLITTING GRAPHS

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Abstract. In this paper, we consider the problem of a star coloring. In general case the problems in NP-complete. We establish the star chromatic number for splitting graph of complete and complete bipartite graphs, as well of paths and cycles. Our proofs are constructive, so they lead to appropriate star colorings of graphs under consideration.

1. Introduction

We consider only finite, undirected, loopless graphs without multiple edges.

The notion of star chromatic number was introduced by Branko Grünbaum in 1973. A star coloring of a graph $G$ is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two color classes has connected components that are star graphs. The star graph is a tree with at most one vertex with degree larger than 1. Star coloring is a strengthening of acyclic coloring, i.e. proper coloring in which every two color classes induce a forest. The star chromatic number $\chi_s(G)$ of $G$ is the least number of colors needed to star coloring of $G$.

Guillaume Fertin et al. gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, $d$-dimensional grids ($d \geq 3$), $d$-dimensional tori ($d \geq 2$), graphs with bounded treewidth, and cubic graphs.

In this paper we consider star coloring of some splitting graphs. For a given graph $G$ the splitting graph $S(G)$ of graph $G$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$, where $N(x)$ is the neighborhood of vertex $x$. For example, the splitting graph of $K_{2,3}$ is given in Fig. 1.

Albertson et al. showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when $G$ is a graph that is both planar and bipartite. Coleman and Moré proved that finding an optimal star coloring is NP-hard and remain so even for bipartite graphs.

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One can ask whether there is a subclass of planar graphs such that admits optimal star coloring in polynomial time. Sampathkumar and Walikar [8] posed an open problem: full characterization of graphs whose splitting graphs are planar. In this situation considering star coloring of splitting graphs seems to be desirable. Moreover, star coloring problem has application in combinatorial scientific computing. In particular, it has been employed since 1980’s to efficiently compute sparse Jacobian and Hessian matrices using either finite differences or automatic differentiation [4].

Additional graph theory terminology used in this paper can be found in [2,6].

For the completeness of the reasoning given in this paper, we recall some known results.

**Theorem 1.1** ([5]). If $C_n$ is a cycle on $n \geq 3$ vertices, then

$$\chi_s(C_n) = \begin{cases} 4 & \text{when } n = 5 \\ 3 & \text{otherwise.} \end{cases}$$

In this paper we prove results concerning the star chromatic number of splitting graph of complete graphs, paths, complete bipartite graphs and cycles.

2. **Main Results**

2.1. **Star Coloring of Splitting graph of complete graphs.**

**Theorem 2.1.** Let $K_n$ be a complete graph on $n \geq 2$ vertices. Then

$$\chi_s(S(K_n)) = n + 1.$$  

**Proof.** Let $V(S(K_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\}$. We define star $(n + 1)$-coloring $\sigma$ of $S(K_n)$ in the following way:

$$\sigma(v_i) = c_i : 1 \leq i \leq n; \sigma(v'_i) = c_{n+1} : 1 \leq i \leq n.$$
Clearly the vertices of complete graph $K_n$ needs $n$ colors for a proper coloring and hence for star coloring. Thus, $\chi_s(S(K_n)) \geq n$. In the further part we will show that $n$ colors are insufficient for star coloring of $S(K_n)$.

By definition of splitting graph, for $1 \leq j \leq n$, the vertex $v_j$ is adjacent to all $v_i$ except for $i = j$. Hence $\sigma(v_j) \neq \sigma(v^i_j)$ for $i \neq j$. If $\sigma(v^i_j) = \sigma(v^i_j)$, then there exist bicoloried paths $v^i_jv^i_{j+1}v^i_{j+1}$ for $1 \leq i \leq n$. A contradiction to proper star coloring. Thus, $\chi_s(S(K_n)) = n + 1$. \hfill \Box

2.2. Star coloring of splitting graph of paths.

**Theorem 2.2.** Let $P_n$ be a path on $n \geq 4$ vertices. Then

$$\chi_s(S(P_n)) = 4.$$ 

**Proof.** Let $V(S(P_n)) = \{v_1, v_2, \ldots, v_n, v'_1, \ldots, v'_n\}$.

It is clear that vertices of $P_n$ need three colors for proper star coloring. Thus $\chi_s(S(P_n)) \geq 3$.

Now, we will show that three colors are insufficient for star coloring of $S(P_n), n \geq 4$. We start from any star 3-coloring of $P_n$. we will denote it by the coloring $c$. Let $v_i, v_{i+1}, v_{i+2}, v_{i+3}, 1 \leq i \leq n-3$, be any four consecutive vertices of $P_n$. It is clear that there exists at least one pair of vertices of length two among these four ones with different colors assigned. Without lost of generality we may assume that $c(v_i) \neq c(v_{i+2})$. Then color that may be used to color vertex $v'_{i+1}$ is determined to $c(v_{i+1})$. Let us try to assign appropriate color to $v'_{i+2}$. We have two possibilities - we can use color $c(v_i)$ or $c(v_{i+2})$. But any of these two choices leads to 2-chromatic $P_4$. Thus $\chi_s(S(P_n)) \geq 4$.

The star 4-coloring $\sigma$ of $S(P_n)$ is defined as follows:

$$\sigma(v_i) = \begin{cases} c_1 & \text{if } i \equiv 1 \text{ mod 3} \\ c_2 & \text{if } i \equiv 2 \text{ mod 3} \\ c_3 & \text{if } i \equiv 0 \text{ mod 3} \end{cases}$$

and

$$\sigma(v'_i) = c_4$$

for $1 \leq i \leq n$. It is easy to verify that any two color classes in coloring $\sigma$ induce a forest whose components are $K_{1,2}$ and $K_2$, hence by definition $\sigma$ is a proper star 4-coloring. Hence, $\chi_s(S(P_n)) = 4$. \hfill \Box

It is easy to verify that $\chi_s(S(P_2)) = \chi_s(S(P_3)) = 3$.

2.3. Star coloring of splitting graph of complete bipartite graphs.

**Theorem 2.3.** Let $K_{m,n}, m \leq n$ be complete bipartite graph. Then

$$\chi_s(S(K_{m,n})) = 2m + 1.$$ 

**Proof.** Let $V(K_{m,n}) = X \cup Y = \{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_n\}$. Then we have $V(S(K_{m,n})) = \{X, X', Y, Y'\} = \{v_1, v_2, \ldots, v_m; v'_1, v'_2, \ldots, v'_m; u_1, u_2, \ldots, u_n; u'_1, u'_2, \ldots, u'_n\}$.
Note that in any star \(k\)-coloring of complete graph \(K_{m,n}\), \(k < m+n\), there is only one bipartite partition set, let say \(A\), including at least two vertices from the same color class. Vertices in the second bipartite partition set, let say \(B\), must be colored with different \(|B|\) colors and these colors cannot be assigned to vertices in set \(A\). We will named such multicolored bipartite partition set \(B\) as **rainbow**.

This implies that at least two out of four partition sets \(X,X',Y,Y'\) of \(S(K_{m,n})\) must be rainbow: \(X,Y\), \(X,X'\), \(Y,Y'\), or \(X',Y'\).

**Case 1:** \(X\) and \(Y\) are rainbow.

Note that such coloring of \(X\) and \(Y\) uses different \((m+n)\) colors and it may be extended to proper star \((m+n+1)\)-coloring of \(S(K_{m,n})\).

**Case 2:** \(X\) and \(X'\) are rainbow.

Note that such coloring of \(X\) and \(X'\) may use \(m\) colors, but then it cannot be extended to any star \(k\)-coloring of \(S(K_{m,n})\) with \(k < m+n+1\). We may extend it only to star \(k\)-coloring with \(k \geq m+n+1\). Indeed, \(Y\) must be also raibow in this case, otherwise 2-chromatic \(P_4\) arises.

If coloring of \(X\) and \(X'\) uses \(2m\) colors, then it may be extended to star \((2m+1)\)-coloring. Notice that \(2m+1 \leq m+n+1\) for \(m \leq n\).

**Case 3:** \(Y\) and \(Y'\) are rainbow.

This case may be considered analogously to Case 2. We have two possibilities: star \((n+m+1)\)- or \((2n+1)\)-coloring of \(S(K_{m,n})\).

**Case 4:** \(X'\) and \(Y'\) are rainbow.

In this cases at least one out of \(X\) and \(Y\) must be also rainbow partition set and we may look for optimal star coloring of \(S(K_{m,n})\) due to Case 2 or Case 3.

Summarizing, star coloring of \(S(K_{m,n})\) with the smallest number of colors is mentioned in Case 2. In details, we define star \((2m+1)\)-coloring \(\sigma\) of \(S(K_{m,n})\) in the following way.

\[
\sigma(v_i) = c_i, 1 \leq i \leq m \\
\sigma(u_j) = c_{m+1}, 1 \leq j \leq n \\
\sigma(u'_j) = c_{m+1}, 1 \leq j \leq n \\
\sigma(v'_i) = c_{m+1+i}, 1 \leq i \leq m.
\]

\[\square\]

### 2.4. Star coloring of splitting graph of cycles.

**Theorem 2.4.** Let \(C_n\) be a cycle graph on \(n \geq 3\) vertices. Then

\[
\chi_S(S(C_n)) = \begin{cases} 
4 & \text{if } n \not\equiv 1 \mod 3 \text{ and } n \neq 5 \\
\leq 5 & \text{otherwise}
\end{cases}
\]
Proof. Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and \( V(S(C_n)) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \). By Theorem 1.1, \( \chi_s(C_n) = 3 \) and thus \( \chi_s(S(C_n)) \geq 3 \), for \( n \geq 6 \). We claim that 3 colors are insufficient for star coloring of \( S(C_n) \). First, we assign colors \( \{c_1, c_2, c_3\} \) to vertices of \( C_n \) to obtain proper star 3-coloring \( c \) of \( C_n \). If we want to extend this \( c \) coloring into whole \( S(C_n) \) without adding a new color, we may have at most three possibilities. Let \( v, v_{i+1}, v_{i+2}, v_{i+3} \) be any four consecutive vertices of \( C_n \), \( 1 \leq i \leq n - 3 \). Since the cycle is star 3-colored, then exactly two out of four vertices \( v, v_{i+1}, v_{i+2}, v_{i+3} \) are assigned the same color:

1. \( c(v) = c(v_{i+3}) \)

Then the coloring of \( v'_{i+1} \) and \( v'_{i+2} \) is determined. Vertices \( v'_{i+1} \) and \( v'_{i+2} \) must obtain the same colors as \( v_{i+1} \) and \( v_{i+2} \), respectively. But then 2-colored \( P_4 \) arises: \( v'_{i+2}, v_{i+1}, v_{i+2}, v'_{i+1} \). We need fourth color.

2. \( c(v) = c(v_{i+2}) \)

It is clear that \( c(v_{i+1}) \neq c(v_{i+3}) \) and the coloring of \( v'_{i+2} \) is determined while vertex \( v'_{i+1} \) can be assigned two colors: \( c(v_{i+1}) \) or \( c(v_{i+3}) \). If we assign color \( c(v_{i+1}) \) to vertex \( v'_{i+1} \), then we get 2-chromatic \( P_4: v_i, v'_i, v_{i+1}, v_{i+2}, v_{i+1} \). In the second choice, also 2-chromatic \( P_4 \) arises: \( v_i, v'_i, v_{i+1}, v_{i+2}, v_{i+3} \). Also in this case, we get additional color.

3. \( c(v_{i+1}) = c(v_{i+3}) \)

This case is analogous to case with \( c(v) = c(v_{i+2}) \).

Summarizing, \( \chi_s(S(C_n)) \geq 4 \). We will consider three cases depending on the value of \( n \mod 3, n \geq 3 \).

**Case 1:** \( n \equiv 0 \mod 3 \)

We define 4-coloring \( \sigma \) of \( S(C_n) \) in the following way.

\[
\sigma(v_i) = \begin{cases} 
  c_1 & \text{if } i \equiv 1 \mod 3 \\
  c_2 & \text{if } i \equiv 2 \mod 3 \\
  c_3 & \text{if } i \equiv 0 \mod 3 
\end{cases}
\]

and \( \sigma(v'_i) = c_4, 1 \leq i \leq n \). Consider the color classes \( c_i \) and \( c_j \), \( 1 \leq i < j \leq 4 \). The components of induced subgraph of these color classes are \( K_2 \) and \( K_{1,2} \). Hence there exists no bicolored path on four vertices and thus \( \sigma \) is a proper star 4-coloring. Hence \( \chi_s(S(C_n)) = 4 \).

**Case 2:** \( n \equiv 1 \mod 3 \)

For \( n \geq 4 \), we define 5-coloring \( \sigma \) of \( S(C_n) \) in the following way.

\[
\sigma(v_i) = \begin{cases} 
  c_1 & \text{if } i \equiv 1 \mod 3 \\
  c_2 & \text{if } i \equiv 2 \mod 3 \\
  c_3 & \text{if } i \equiv 0 \mod 3 
\end{cases}
\]

for \( 1 \leq i \leq n - 1 \), and \( \sigma(v_n) = c_2 \).
\[ \sigma(v_i') = \begin{cases} 
    c_4 & \text{if } i \equiv 0 \mod 3 \\
    c_5 & \text{otherwise}
\end{cases} \]

for \( 1 \leq i \leq n \).

It is clear that every two out of three color classes corresponding to colors \( c_1, c_2, \) and \( c_3 \) avoid 2-chromatic \( P_4 \), similarly as color classes corresponding to colors \( c_4 \) and \( c_5 \). We have to only check pairs of color classes \( c_k; 1 \leq k \leq 3 \) and \( c_4 \) or \( c_5 \). It is easy to check that every such two color classes induces forest whose components are isolated vertices, \( P_2 \) or \( K_{1,2} \). Thus, \( \sigma \) is proper star 5-coloring. Hence, \( 4 \leq \chi_s(S(C_n)) \leq 5 \).

**Case 3:** \( n = 5 \)

Star 5-coloring of \( S(C_5) \) is given in Fig. 2b).

**Case 4:** \( n \equiv 2 \mod 3 \) and \( n \geq 8 \)

**Case 4.1:** \( n = 8 \)

Star 4-coloring of \( S(C_8) \) is given in Fig. 2c).

**Case 4.2:** \( n \geq 11 \)

Let \( n = 8 + 3t, t \geq 1 \). First, we color \( v_1, \ldots v_8 \) in the way given in Fig. 3.

**Figure 2.** a) \( S(C_4) \); b) \( S(C_5) \) and c) \( S(C_8) \) with their star coloring.

**Figure 3.** A part of \( S(C_n), n \geq 11 \), with its star 4-coloring.
Next, the remaining vertices of a cycle of $S(C_n)$ are colored in the following way.

$$\sigma(v_i) = \begin{cases} c_1 & \text{if } i \equiv 0 \mod 3 \\ c_2 & \text{if } i \equiv 1 \mod 3 \\ c_3 & \text{if } i \equiv 2 \mod 3 \end{cases}$$

for $9 \leq i \leq n$, and

$$\sigma(v'_i) = \begin{cases} c_3 & \text{if } c(v_i) = 3 \\ c_4 & \text{otherwise.} \end{cases}$$

Similarly as it was in Case 1 we can check that $\sigma$ is proper star 4-coloring. Hence, $\chi_s(S(C_n)) = 4$.

\[\square\]

3. Conclusion

In the paper the problem of a star coloring for some splitting graphs has been considered. As an open question we put the problem of determining exact values of star chromatic number of splitting graphs of cycles $C_n$ where $n = 5$ or $n \equiv 1 \mod 3$. Moreover, considering star coloring of degree splitting graphs, defined by Ponraj and Somasundaram \[7\], seems to be worth paying attention.

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