Variational Quantum Eigensolver in Compressed Space for Nearest-Neighbour Quadratic Fermionic Hamiltonians

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Variational quantum eigensolvers (VQE) are one of the possible tasks that can be performed in noisy intermediate-scale quantum (NISQ) computers. While one of the limitations of NISQ platforms is their restricted number of available qubits, the theory of compressable matchgate circuits can be used to circumvent this limitation for certain types of Hamiltonians. We show how VQE algorithms can be used to find the ground state of quadratic fermionic Hamiltonians, providing an expressible ansatz in a logarithmic number of qubits. In particular, for systems of \(n\) orbitals encoded to 2-local qubit models with nearest neighbour interactions, the ground state energy can be evaluated with \(O(\log n)\) sets of measurements. This result is invariant of the dimensions in which the \(n\) sites are arranged.

\section{I. INTRODUCTION}

The simulation of quantum physical systems has been one of the most anticipated applications of quantum computation since Feynman’s seminal paper \([1]\). Current devices, so-called noisy intermediate-scale quantum (NISQ) computers, suffer from practical limitations that restrict the number of qubits and their coherence. Despite these limitations, several algorithms have been proposed that overcome this resource scarcity. Amongst these, the variational quantum eigensolver (VQE) \([2]\) has found applications in the simulation of many-body physics, quantum chemistry or combinatorial problems (see \([3]\) for a review of NISQ algorithms and their applications). In a VQE algorithm, a parameterized circuit \(U(\theta)\) is minimized with respect to an objective function, usually the expectation value of the energy of a Hamiltonian \(H\),

\[
\min_{\theta} \langle 0 \mid U(\theta) H U(\theta) \mid 0 \rangle, \tag{1}
\]

with the expectation value computed in a quantum device and the minimization procedure performed classically.

A particularly interesting class of many-body quantum systems are those that can be mapped to non-interacting fermions, i.e., their Hamiltonians are quadratic in the fermionic operators. Their eigenstates in the fermionic picture are single-mode non-interacting states and, in many cases, the modes can be found analytically. Despite their simplicity, quadratic Hamiltonians appear in a variety of settings. They arise as mean field approximations to higher order Hamiltonians such as the BCS model of superconductivity \([4]\) or the Fermi-Hubbard model \([5]\). They also contain the simplest examples of topological superconductivity \([4]\) or the Fermi-Hubbard model \([5]\).

In this paper we study the compressed simulation of free fermionic systems using the matchgate formalism in VQEs. In section II we review the main ideas of matchgate circuits and their compression. In section III we show that the expectation value of a compressed quadratic operator with nearest-neighbor interactions can be computed in a logarithmic number of sets of measurements. In section IV we build an ansatz with a quadratic number of parameters in the system size, which we conjecture that can be reduced to a linear number of parameters. Finally, in section V we discuss the results.

\section{II. COMPRESSED FERMIONIC GAUSSIAN STATES}

Let us first recall the main ideas in the compression of fermionic Gaussian states, to illustrate how to run a VQE algorithm in compressed space. We refer the reader to Jozsa et al. \([19, 20]\) and Boyajian \([21]\) for details.

A quadratic fermionic Hamiltonian on \(n\) fermions can be written as

\[
H = i \sum_{j \neq k=0}^{2n-1} \hbar_{j,k} \gamma_j \gamma_k, \tag{2}
\]

where the \(\gamma\)'s are Majorana operators and \(\hbar\) is a real, antisymmetric matrix. The unitaries generated by these
Hamiltonians are called Gaussian. Majorana operators transform under conjugation by Gaussian operators $U$ as
\[ U^\dagger \gamma_j U = \sum_{k=0}^{2n-1} R_{jk}^{\dagger} \gamma_k , \] (3)
with $R = e^{i\theta} \in SO(2n)$. A Gaussian operator on $n$ qubits is expressible as a circuit of $O(n^2)$ nearest-neighbour matchgates (i.e., acting on consecutive qubits). A matchgate is a quantum gate of the form
\[ G(A, B) = \begin{pmatrix} p & 0 & 0 & q \\ 0 & wx & 0 & 0 \\ 0 & yz & 0 & 0 \\ r & 0 & 0 & s \end{pmatrix}, \]
with $A$ and $B$ unitary matrices satisfying $\det(A) = \det(B)$. A nearest-neighbour matchgate circuit on $n$ qubits can either be simulated efficiently in a classical computer \cite{19, 22}, or in a quantum computer in $\log n + 1$ qubits \cite{20}, provided that we measure a quadratic operator.

The ground states of quadratic fermionic Hamiltonians are called fermionic Gaussian states. They can be written as $\rho = Ke^{-i\mathbf{x}^T \mathbf{G} \mathbf{x}}$, where $\mathbf{x}$ is a vector of Majorana operators, $\mathbf{G}$ is a real, antisymmetric matrix and $K$ is a normalization factor. We are interested in the compression of this set of states and the Gaussian operators. A fermionic Gaussian state $\rho$ is completely characterized by its second moments, encoded in the covariance matrix
\[ \Gamma(\rho)_{kl} = \frac{i}{2} \text{Tr}(\rho[\gamma_k, \gamma_l]) . \]
(5)
All information about the state can be obtained from $\Gamma$ using Wick’s theorem. The covariance matrix can be used to define a density matrix $\sigma$ on $\log n + 1$ qubits as
\[ \sigma(\rho) = \frac{1}{2^n} (\mathbb{I} + i\Gamma(\rho)) . \]
(6)
Fermionic Gaussian states are closed under the action of Gaussian operators $U$. By Eq. 3 the corresponding action on $G$, $\Gamma$ and $\sigma$ is the conjugation by the rotation matrix $R$, which takes the form of a Bogoliubov transformation. Finally, the expectation value of an operator $H$ can be computed in compressed space as
\[ \langle H \rangle = n \text{Tr} \left[ R \sigma R^T A \right] \]
(7)
where
\[ A = i \sum_{j \neq k=0}^{2n-1} h_{jk} |j \rangle \langle k | \]
(8)

We have now all the necessary steps to run a VQE algorithm in compressed space: start with any compressed fermionic Gaussian state, parametrize an $SO(2n)$ rotation to explore the whole set of compressed fermionic Gaussian states and measure the compressed operator $A$ to get the the uncompressed operator’s density.

Ideally, we would like to grow the uncompressed number of orbitals exponentially and simulate them in $O(\log n)$ qubits and steps. In the rest of the paper we identify a class of Hamiltonians for which the compressed operator can be efficiently computed, and we explain how to implement an expressible, although quadratic, ansatz.

III. COMPRESSED OPERATORS

The cost of measuring an arbitrary compressed Hamiltonian when the number of constituents $n$ grows exponentially can be extremely high, because a compressed operator has up to $n(2n - 1)$ words in its Pauli string. In this section we first show how to group the Pauli string words of any compressed operator in sets of mutually commuting operators. Then, we show that nearest-neighbour interactions reduce the number of groups exponentially, even if we stack the fermions in any number of dimensions. From now on, we use $n$ to denote the number of orbitals in the system and $m = \log n + 1$ for the number of qubits in the compressed simulation.

A compressed operator (Eq. 8) of $n$ uncompressed qubits is structured in $n^2$ blocks of dimensions $2 \times 2$ as
\[ A = i \frac{1}{2} \begin{pmatrix} a_{0,0} & \cdots & a_{0,n} \\ \vdots & \ddots & \vdots \\ a_{n,0} & \cdots & a_{n,n} \end{pmatrix} \]
(9)
The lower-left elements are defined by the antisymmetry of $h$. The inner blocks are
\[ a_{i,j} = \begin{pmatrix} h_{2i,2j} & h_{2i,2j+1} \\ h_{2i+1,2j} & h_{2i+1,2j+1} \end{pmatrix} , \]
(10)
with $h_{k,k} = 0$ and $h_{j,i} = -h_{j,i}$.

It is useful to understand the compressed operator structure in terms of Pauli operators. We choose the Jordan-Wigner representation for Majorana operators on $n$ qubits in terms of Pauli matrices,
\[ \gamma_{2i} = \bigotimes_{j=0}^{i-1} \sigma_z \sigma_x , \]
(11)
\[ \gamma_{2i+1} = \bigotimes_{j=0}^{i-1} \sigma_z \sigma_y \]
(12)
for $i \in \{0, \ldots, n - 1\}$. With this representation, a quadratic operator is
\[ \begin{pmatrix} i\sigma_z^j, & \cdots & j = 2l, k = 2l + 1 \\ -i\sigma_y^j \sigma_z^{j+1} \cdots \sigma_z^{k-1} \sigma_z^k, & \cdots & j = 2l, k = 2m \\ -i\sigma_y^j \sigma_z^{j+1} \cdots \sigma_z^{k-1} \sigma_z^k, & \cdots & j = 2l, k = 2m + 1 \\ i\sigma_z^j \sigma_z^{j+1} \cdots \sigma_z^{k-1} \sigma_z^k, & \cdots & j = 2l + 1, k = 2m \\ i\sigma_z^j \sigma_z^{j+1} \cdots \sigma_z^{k-1} \sigma_z^k, & \cdots & j = 2l + 1, k = 2m + 1 \end{pmatrix} \]
(13)
with $j < k$ and where the superscript labels the site. One can see that the diagonal blocks $a_{i,i}$ encode $n$ single qubit $\sigma_z$ operators. The non-diagonal blocks $a_{i,j}$ encode the four possible interactions between fermions $i$ and $j$. 
A. Sets of mutually commuting operators of an antisymmetric operator

An antisymmetric \( n \times n \) matrix has \( n(n-1)/2 \) free parameters, and its Pauli string will have at most the same number of words. Only antisymmetric tensor products (i.e., with an odd number of \( \sigma_y \)'s), can be present in the Pauli string of an antisymmetric matrix. Let \( A = \{ \sigma_x, \sigma_y \} \) be the set of diagonal Pauli matrices, and \( D = \{ 1, \sigma_z \} \). Let \( \sigma_M \) be a member of each set, labeled by \( M_i \in \{ D, A \} \). Then, the set of tensor products \( \sigma_M \sigma_M \ldots \sigma_M \), with at least one \( M_i = A \), is a set of mutually commuting observables. There are \( n - 1 \) such groups, and \( n/2 \) elements in each group, for a total of \( n(n-1)/2 \) parameters. We show in the Supplemental Material that each of these groups form a set of mutually commuting observables.

With these groups we can compute any compressed operator’s expectation value in \( n - 1 \) sets of measurements. To find classes of Hamiltonians where a logarithmic number of sets of measurements is needed we must relate each tensor product to the matrix elements it can populate. The non-zero elements of (omitting the tensor product symbol), \( \sigma_M \sigma_M \ldots \sigma_M \), can be tracked by starting at the center of the matrix and selecting diagonal or antidiagonal subblocks according to the label \( M_i \). For example, for a \( 4 \times 4 \) matrix, the \( \sigma_A \sigma_D \) set will have the non-zero elements

\[
A = \begin{pmatrix}
0_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\
0_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
0_{0,0} & a_{0,2} & a_{0,3} \\
0_{1,2} & a_{1,3} \\
a_{2,0} & a_{2,1} \\
a_{3,0} & a_{3,1}
\end{pmatrix}
\]

Similarly, we can trace back the set of tensor products from a matrix element by following the path backwards.

B. Compressed operators with nearest-neighbour interactions in arbitrary dimensions

While the grouping of Pauli strings in sets of commuting operators reduces the number of measurements, there is still an exponential growth with the number of compressed qubits. We now consider Pauli strings of operators with nearest-neighbour interactions, including many well-known systems such as the tight-binding models or the Kitaev wire. The structure of the compressed operator is such that the number of sets of measurements to be made in a quantum computer is reduced exponentially.

Let us start with a 1D system of size \( n \). In this case, only the \( a_{i,j} \) with \( j = i \) or \( j = i + 1 \) will be non-zero. Local operators are encoded in the \( a_{i,i} \) terms, and are generated by the set \( \sigma_D \ldots \sigma_D \sigma_A \). The \( a_{i,i+1} \) encode nearest-neighbor interactions. We can collect them in \( 2(m-1) \) groups of mutually commuting observables. Let \( l = 1, \ldots, m-1 \) and, for each \( l, k = 0, \ldots, n/2l-1 \). The \( a_{i,j} \) blocks with \( i = 2^l k + 2^l -1, j = i + 1 \) are generated by the sets of strings (using subscripts to label the qubits) \( \sigma_D, \ldots \sigma_D, \sigma_A, \ldots, \sigma_A, \sigma_M \). The last label differentiates between the two possible sets of interactions in a two-qubit interaction block. Since \( l \) completely labels the sets of tensor products, there are \( m-1 \) groups in total. Periodic boundary conditions can be simulated with the same groups, because they involve the \( a_{0,n-1} \) block, whose set of tensor products is \( \sigma_A \ldots \sigma_A \sigma_M \), which is also the one for the central block \( a_{n/2-1,n/2} \).

Let us turn the system two-dimensional. We can start with the 1D system and break it in two equal parts, between positions \( n/2 - 1 \) and \( n/2 \). The compressed matrix when there is no interaction between both halves of a chain is block diagonal, with two \( n/2 \times n/2 \) blocks. Compared to the connected 1D chain, the Pauli string will lack the \( \sigma_A \ldots \sigma_A \sigma_M \) set. When connecting both halves to make a 2D system, this set will now be replaced by the interactions between positions \( i \) and \( n/2 + i \) for \( i = 0, \ldots, n/2 - 1 \). All these interactions are generated by the set \( \sigma_A \sigma_D, \ldots, \sigma_D \sigma_M \). Therefore, folding a chain by the middle position does not increase the number of sets of measurements.

In general, if we want to fold a 1D chain into a 2D system of \( p \) rows and \( q \) columns, we can start by breaking the chain in half, eliminating one set of Pauli strings, then breaking the resulting two chains in half again, and keep going until we have \( p \) chains of length \( n/p \), at which point we have eliminated \( \log p \) sets. Then, we can fold the smallest chains in pair by the second dimension, introducing a single set at each step, until we have introduced \( \log p \) sets again. The same reasoning can be made to fold a chain in more than 2 dimensions.

This shows that any nearest-neighbor quadratic Hamiltonian can be measured in compressed space in \( O(\log n) \) sets of measurements. Still, there is an exponential number of Pauli words in each of these groups. In the next section we show how to compute the expectation values without knowledge of each Pauli words’ coefficients.

C. Expectation values of quadratic operators

Wick’s theorem relates expectation values of products of Majorana operators to elements of the covariance matrix. For quadratic operators, the expectation value is

\[
\langle \gamma_k \gamma_l \rangle = -i \Gamma_{k,l} = -i \text{Tr}[\sigma (|l\rangle \langle k| - |k\rangle \langle l|)] , \quad (14)
\]

where \( \sigma \) is the compressed state. The \( |l\rangle \langle k| - |k\rangle \langle l| \) operator is a linear combination of the Pauli words in the
set of mutually commuting operators which are non-zero at position \( k, l \). With the appropriate diagonalization, we can compute \( \langle \gamma_k \gamma_l \rangle \) directly from the measurement results.

The following circuit diagonalizes the \( \sigma_A \sigma_A \ldots \sigma_A \) set:

\[
\begin{array}{c}
\text{Y} \\
\text{S} \\
\text{H} \\
\text{X}
\end{array}
\]

More general tensor products are diagonalized with this circuit applied to all \( \sigma_A \) positions. In terms of the matrix elements, the diagonalization is equivalent to moving element \( k, l \) to position \( k, k \) with an \( i \) factor. Let \( G \) be the diagonalizing circuit operator, then

\[
\Gamma_{k,l} = -in \text{Tr} (G \sigma G \Gamma (\langle k | \langle l | - | k \rangle \rangle G \Gamma) = n \text{Tr} (G \sigma G \Gamma (\langle k | \langle l | - | k \rangle \rangle G) = n (P(k) - P(l)).
\]

where \( P(k) \) and \( P(l) \) are the probabilities of measuring bitstrings \( k \) and \( l \) in the \( G \sigma G \Gamma \) density matrix. With the diagonalizing circuits, we can compute expectation values of quadratic operators without needing to explicitly compute the Pauli string of the compressed operator. Notice that this is more efficient even for a 1D chain, whose central element Pauli string will have \( O(n) \) terms. In general, the actual number of measurements depend on the desired accuracy \( \epsilon \). In VQE settings for molecular Hamiltonians, the number of measurements scales as \( O(1/\epsilon^2) \) [24], but can be reduced to \( O(1/\epsilon) \) with quantum phase estimation [24].

**IV. ANSÄTZE FOR COMPRESSED FERMIONIC GAUSSIAN STATES**

Quadratic Hamiltonians have fermionic Gaussian states as ground states. These states are closed under the action of Bogoliubov transformations, or, in compressed space, special orthogonal rotations. Same parity states are closed under the action of proper Bogoliubov transformations (having determinant 1), while different parity sectors are connected by an improper transformation, with determinant -1. Hence, an ansatz that implements a special orthogonal rotation will be able to explore the set of even or odd compressed Fermionic Gaussian states, if the initial state is itself a member of that set.

Here we explain how to generate the compressed version of the \( |0 \rangle \otimes n \) state. Its covariance matrix is defined by

\[
\Gamma \left( |0 \rangle \otimes n \right)_{j,k} = \begin{cases} 
-1 & \text{if } (j,k) = (2l, 2l + 1) \\
1 & \text{if } (j,k) = (2l + 1, 2l) \\
0 & \text{otherwise}
\end{cases}
\]

with \( l \in \mathbb{Z}, l \geq 0 \). The associated \( n \)-qubit density matrix is

\[
\rho = \frac{1}{n} (I \otimes |+ \rangle \langle +|)
\]

In general, compressed fermionic Gaussian states are mixed. We can purify the state in Eq. [17] in \( 2m - 1 \) qubits as

\[
|\psi\rangle = \frac{1}{\sqrt{2n}} \sum_{k=0}^{2^{m-1}-1} |j\rangle_L |0\rangle_R + i |j\rangle_L |1\rangle_R.
\]

Tracing out subsystem \( R \) generates the density matrix in Eq. [17]. The previous state can be prepared in a quantum computer with the following \( 2m - 1 \) qubit circuit,

\[
\begin{array}{c}
|0\rangle \\
\vdots \\
|0\rangle \\
0 \\
0 \\
|0\rangle \\
|0\rangle \\
\end{array}
\]

with the \( S \) gate in the \( m \)-th qubit.

With a suitable initial state like this one, we can rotate the relevant degrees of freedom using a special orthogonal rotation. Any \( SO(n) \) element can be decomposed in \( n(n-1)/2 \) Givens rotations, which are rotations around a single axis and take the form

\[
G_{i,j} (\theta) = \begin{pmatrix}
1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \cos \theta & \cdots & -\sin \theta & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \sin \theta & \cdots & \cos \theta & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix},
\]

where the trigonometric functions appear at the intersections of the \( i \)-th and \( j \)-th rows and columns. To completely parametrize an \( SO(n) \) element, we can restrict to Givens rotations with \( i < j \). Let \( k \) be the index of one of the possibly many positions in which the binary strings of \( i \) and \( j \) differ. A multicontrolled RY gate on qubit \( k \) controlled by all other other qubits being on implements a Givens rotation between states \( |1 \ldots 101 \ldots 1 \rangle \) and \( |1 \ldots 111 \ldots 1 \rangle \), where the 0 and the respective 1 are at position \( k \). If any of the other qubits are 0, we must turn them into a 1 before the RY gate and then back to a 0, which can be done in at most 4 gates per qubit. A fully-controlled RY gate in \( m + 1 \) qubits can be implemented in \( O(m) \) elementary operations [24]. The total complexity of the ansatz is \( O \left( n^2 \log(n) \right) \), with the quadratic dependence coming from the number of Givens rotations.
We provide a Python package that builds this ansatz for Qiskit [26], [27].

Odd states can be explored turning an even state into an odd one with a determinant \(-1\) unitary, such as the fully-controlled \(X\), with no effect in the ansatz complexity.

This ansatz has \(n(2n-1)\) parameters and can explore all the set of \(SO(2n)\) rotations. In practice, the ansatz is too expressive, the ground state can be found by exploring rotations affecting only specific elements in the correlation matrix. We conjecture that there exist ansätze with \(O(n)\) parameters, built with one Givens rotation for every interaction in the Hamiltonian. It remains an open question how to build it and whether there are logarithmic ansätze for physically interesting Hamiltonians.

V. CONCLUSION

We have shown how to define VQEs in \(O(\log n)\) space and quadratic ansätze to find ground states of quadratic Hamiltonians. The exponential space compression is accompanied by an equivalent expansion in the time complexity due to the mapping between quadratic operators and the Pauli strings of the new compressed equivalents. We have shown that this can be alleviated for many physically interesting Hamiltonians thanks to two strategies. First, for any nearest-neighbour Hamiltonian, the number of sets to be measured is reduced exponentially. The interactions between fermions can include non-local terms in a way that allows any rectangular arrangement and open and closed boundary conditions. Second, we have shown how to compute expectation values of quadratic operators directly from measurement results.

The compressed VQE could be used to simulate free-fermionic systems in NISQ devices. Another possible use is as a starting point for algorithms initialized with compressed ground states. There exist algorithms that start with compressed states based on other schemes (10) and algorithms that start with non-compressed fermionic Gaussian states, for example for strongly correlated systems [28] or Hartree-Fock methods [11]. The study of compressed fermionic Gaussian states can help to develop other compressed algorithms.

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[27] Note that although the ansatz is the one we propose, the measurement procedure in Qiskit’s VQE algorithms uses the Pauli strings. The package can also convert a quadratic 1D Hamiltonian into its compressed Pauli strings as an example.

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S-I. CONSTRUCTION OF SETS OF MUTUALLY COMMUTING TENSOR PRODUCTS FOR AN ANTISYMMETRIC OPERATOR

Here we show that the sets of tensor products defined in Section III are sets of mutually commuting observables. The proof is based on the following identities relating commutators and anticommutators

\[ [A \otimes B, C \otimes D] = \frac{1}{2} ([A, C] \otimes [B, D] + [A, B] \otimes [C, D]) \]  \hfill (S1)

\[ \{ A \otimes B, C \otimes D \} = \frac{1}{2} ([A, C] \otimes [B, D] + [A, B] \otimes [C, D]) \]  \hfill (S2)

\[ [A \otimes B \otimes C, D \otimes E \otimes F] = \frac{1}{4} ([A, D] \otimes [B, E] \otimes [C, F] + [A, D] \otimes [B, E] \otimes [C, F] + [A, D] \otimes [B, E] \otimes [C, F] + [A, D] \otimes [B, E] \otimes [C, F]) \]  \hfill (S3)

The last identity can be proved from the other two.

We are interested in cases where all the \( A \) to \( F \) are elements of the Pauli group. In that case, any two elements acting on the same number of qubits either commute or anticommute. If either \([A, D] = 0\) or \([C, F] = 0\), then the commutator of the triple tensor products is trivially proportional to the commutator after taking out the commuting operators.

\[ [A \otimes B \otimes C, D \otimes E \otimes F] = \begin{cases} \frac{1}{2} [A, D] \otimes [B \otimes C, E \otimes F] & \text{if } [A, D] = 0 \\ \frac{1}{2} [A \otimes D, B \otimes C, E \otimes F] \otimes [C, F] & \text{if } [C, F] = 0 \end{cases} \]  \hfill (S4)

If the commuting operators are \( B \) and \( E \), the resulting operator is

\[ [A \otimes B \otimes C, D \otimes E \otimes F] = \begin{cases} \frac{1}{4} [A, D] \otimes [B, E] \otimes [C, F] & \text{if } [B, E] = 0 \text{ and } [A, D] = [C, F] = 0 \\ \frac{1}{4} [A, D] \otimes [B, E] \otimes [C, F] & \text{if } [B, E] = 0 \text{ and } [A, D] = [C, F] = 0 \\ 0 & \text{otherwise} \end{cases} \]  \hfill (S5)

In any case, when \([B, E] = 0\) the result is proportional to \([A \otimes C, D \otimes F]\) up to permutation similarity. This property lets us prove the following lemma. Let \( P \) and \( Q \) be two operators built from \( m \) tensor Pauli matrices, \( P = \bigotimes_{i=1}^{m} P_i \) and \( Q = \bigotimes_{i=1}^{m} Q_i \). Let \( \hat{P} \) and \( \hat{Q} \) be the operators resulting from taking out equal pairs in \( P \) and \( Q \), \( \hat{P} = \bigotimes_{j \in \{\{p_i \neq q_i\}\}} P_j \) and similarly for \( \hat{Q} \). Then, if \( [\hat{P}, \hat{Q}] = 0 \), \( P \) and \( Q \) commute too.

The sets of tensor products we are interested in take the form \( \sigma_{M_1} \otimes \sigma_{M_2} \ldots \sigma_{M_m} \), with \( M_i \in \{D, A\} \) and at least one \( M_i = A \). \( D \) and \( A \) are the sets of diagonal and antidiagonal Pauli matrices, \( D = \{\sigma_z, \mathbb{I}\} \) and \( A = \{\sigma_x, \sigma_y\} \). Since any two elements of \( D \) always commute, the commutativity of each of those sets is equivalent to the commutativity of \( \sigma_A \ldots \sigma_A \). The antisymmetry of the compressed operator ensures that there is always an even number of different operator pairs for any two elements of that set. Let \( P \) and \( Q \) be two such elements, i.e., two \( m \)-qubit operators consisting on tensored \( \sigma_x \) and \( \sigma_y \) with an even number of \( \sigma_y \)'s. Let \( \hat{P} \) and \( \hat{Q} \) be the operators resulting from taking out equal pairs in \( P \) and \( Q \), \( \hat{P} = \bigotimes_{i=1}^{l} \sigma_i^l, \hat{Q} = \bigotimes_{k=1}^{l} \sigma_k^l \), where \( i \in \{x, y\} \), and \( \hat{P}_m = \bigotimes_{j=m}^{l} \sigma_i^j \) and similarly for \( \hat{Q} \). Since \( \sigma_i^l \neq \sigma_i^l \) if \( j = k \), \( \sigma_i^l \sigma_i^l = 0 \). In particular, for the first position \( j = k = 1 \), Eq. S1 implies that \([\hat{P}, \hat{Q}] \propto \{\hat{P}_2, \hat{Q}_2\}\). Applying Eqs. S1 and S2

\[ [\hat{P}, \hat{Q}] \propto \{\hat{P}_l, \hat{Q}_l\} \quad \text{if } l \text{ is even} \\
\{\hat{P}_l, \hat{Q}_l\} \quad \text{if } l \text{ is odd} \]  \hfill (S6)

Since the fact that having an odd number of \( \sigma_y \)'s guarantees that \( l \) be even, \( \hat{P} \) and \( \hat{Q} \) commute, and so do \( P \) and \( Q \).