Fermionic Bound States and Pseudoscalar Exchange

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We discuss the possibility that fermions bind due to Higgs or pseudoscalar exchange. It is reasonable to believe on qualitative grounds that this can occur for fermions with a mass larger than 800-900 GeV. An exchange of a pseudoscalar boson leads in the nonrelativistic limit to an unacceptable potential which behaves like $1/r^3$ at the origin. We show that this singular behaviour is smeared out when relativistic effects are included.

I. INTRODUCTION

In this note, we obtain an estimate for the mass of a fermion which is heavy enough to bind to its anti-particle via Higgs exchange. We then try to repeat the argument for pseudoscalar exchange, since pseudoscalars arise naturally in some extensions of the standard model. We will find eventually that the situation is similar to that in Higgs exchange, but the analysis is now much more complex as the exchange of a non-relativistic pseudoscalar seems to lead to a potential which is singular at the origin. We shall show that if the pseudoscalar exchange is treated in a relativistic framework this singularity vanishes.

We begin by considering Higgs exchange between the usual quarks taken to be of mass $m$. This leads to a weak attraction with coupling constant (see for example Reference [1]):

$$g = \sqrt{2} \left( \frac{m}{v} \right) \ll 1; \quad v = \left( G_F \sqrt{2} \right)^{-1/2} = 245 \text{ GeV}$$

(1)

Even for the $t$-quark with $m_t \simeq 180$ GeV, $g \approx 1$. (This does not necessarily invalidate a perturbative approach since the typical parameter of the perturbative expansion is $g^2/8\pi^2 \ll 1$). Even this coupling, however, is not strong enough to create bound states of $tt$ or $tt$. In principle, however, heavier fermions $f$ for which the coupling $g$ is greater than one can be found where bound states $ff$ are bound by Higgs exchange.

At a qualitative level it is easy to estimate how heavy fermions should be for them to be bound by means of Higgs exchange. Obviously for non-relativistic particles, the kinetic energy $p^2/m$ (we use the reduced mass $m/2$ of the two particles with mass $m$) should be smaller than the potential energy which is

$$|S| = \frac{g^2}{4\pi} \frac{1}{r} e^{-m_H r}$$

(2)

where $m_H$ is the Higgs mass. If the composite state has the radius $a$, then $p^2/m \sim 1/ma^2$ and we get the condition:

$$\frac{1}{ma^2} < \frac{g^2}{4\pi a} e^{-m_H a}$$

(3)

This equation can be rewritten in the form

$$\frac{g^2 m}{m_H} > \frac{4\pi e^{m_H a}}{m_H a}$$

(4)

Whatever the value of $a$ the right hand side of Eq (4) is always larger than $4\pi e$. Therefore

$$\frac{g^2 m}{m_H} > 4\pi e = 34.3$$

(5)

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We use Eq (1) to estimate $g$. So

$$\frac{m^3}{v^2 m_H} > 17$$  \hfill (6)

For $m = m_t = 180$ GeV this leads to $m_H < 5.7$ GeV, which is unacceptable. However if we take $m_H = 400$ GeV (for example), Eq. (6) gives:

$$m > 800 \text{ GeV}$$  \hfill (7)

This would be a very heavy fermion, but fermions of mass in the TeV region arise in many extensions of the standard model.

We will now try to extend this simple analysis to pseudoscalar exchange. Though in the SM there is no pseudoscalar Higgs they occur in many extensions of the SM, for example in supersymmetric extensions. We show in the next Section that the exchange of a pseudoscalar meson $A$ leads in the nonrelativistic approximation to the potential

$$P(r) = -\frac{1}{4\pi v_A^2} e^{-m_A r} \left[ \frac{1}{r^3} (\bar{\sigma}_1 \cdot \bar{n})(\bar{\sigma}_2 \cdot \bar{n}) (1 + m_A r) \\
- \frac{m_A^2}{r} (\bar{\sigma}_1 \cdot \bar{n})(\bar{\sigma}_2 \cdot \bar{n}) \right],$$  \hfill (8)

where $\bar{\sigma}_1, \bar{\sigma}_2$ are the Pauli matrices for the two fermions, $\bar{n} = \vec{r} / r$ and $m_A, v_A$ are the mass of the $A$-boson and the scale, analogous to $v = 246$ GeV in the standard model. The potential of Eq (8) depends on the spins of the fermions and in certain spin states can lead to an attraction between fermions. In this case the analogue of the inequality (3) is

$$\frac{1}{m a^2} < \frac{1}{4\pi v_A^2} \frac{1}{a^3} e^{-m_A a}$$  \hfill (9)

Clearly for a small enough radius $a$ this condition can be satisfied. But this corresponds to the well-known fact that an attractive potential like $1/r^3$ which is more singular at the origin than the centrifugal term leads to the “collapse” of the particle to the origin and hence to a problem which is not well-defined [2]. We shall demonstrate, however, that while the expression (8) holds for nonrelativistic fermions and leads to collapse, a relativistic treatment will lessen the singularity at $r = 0$ so that there is no collapse to the centre.

Suppose now that

$$m_A a \ll 1$$  \hfill (10)

Then (10) means that

$$a < \frac{m}{4\pi v_A^2}$$  \hfill (11)

The condition that the fermions should be nonrelativistic $p^2/m < m$ at $p \approx 1/a$ together with eq. (11) can be written in the form:

$$\frac{1}{m} < a < \frac{m}{4\pi v_A^2}$$  \hfill (12)

It is thus required that

$$m > \sqrt{4\pi v_A}$$  \hfill (13)

For $v_A \sim v = 245$ GeV this leads to

$$m > 870 \text{ GeV}$$  \hfill (14)

This restriction is not very different from (7) for the scalar case. Note that for $a \sim (870 \text{ GeV})^{-1}$ the condition (10) is probably satisfied since we can expect the pseudoscalar $A$ to be lighter than the Higgs (the present limit is only $m_A > 24.3$ GeV) [3].

Note an interesting feature of bound states mediated by scalar and pseudoscalar exchange: unlike the situation with vector exchange, states of the same mass can appear for both $f \bar{f}$ and $ff$ systems. This is a consequence of the fact that scalar and pseudoscalar interactions are even under charge conjugation.
II. PSEUDOSCALAR EXCHANGE (NON-RELATIVISTIC CASE)

We now look in more detail at pseudoscalar exchange. Start with the interaction of the pseudo-scalar boson \( A \) with a fermion via the coupling \( i(2m/v_A) (\bar{\psi} \gamma_5 \psi) A \). The Feynman amplitude corresponding to the exchange of \( A \) can then be represented in the form

\[
M = \frac{4m^2}{v_A} (\phi_2^0 \bar{\sigma} \sigma_2 \phi_2) (\phi_1^0 \bar{\sigma} \sigma_1 \phi_1) \frac{1}{m_A + q^r},
\]  

(15)

where \( q = p_1' - p_1 = p_2' - p_2 \) is the three-momentum transfer and we have taken the non-relativistic limit so that the four-component spinors \( \psi \) are replaced by the two-component spinors \( \phi \).

The Born amplitude \( A \) is related to \( M \) by

\[
A(q) = \frac{1}{8\pi W} M
\]  

(16)

where \( W \) is the total centre of mass energy, and the potential \( P(\vec{r}) \) is the Fourier transform of \( A(q) \):

\[
P(\vec{r}) = \frac{4\pi}{m} \int e^{i\vec{q}\cdot\vec{r}} A(q) \frac{d^3q}{(2\pi)^3},
\]  

(17)

From the last three equations we can easily calculate \( P(\vec{r}) \) (we use the nonrelativistic value \( W \simeq 2m \)):

\[
P(\vec{r}) = \frac{1}{v_A^2} (\phi_2^0 \bar{\sigma} \sigma_2 \phi_2) (\sigma_1^0 \bar{\sigma} \sigma_1 \phi_1) \frac{1}{4\pi r} \left[ \frac{1}{4m_A^r} e^{-m_A r} \right] =
\]  

\[
= \frac{1}{4\pi v_A^2} e^{-m_A r} \left[ \frac{1}{r^3} \left( \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{2} - 3 \left( \frac{\vec{\sigma}_1 \cdot \vec{n}}{2} \right) \left( \frac{\vec{\sigma}_2 \cdot \vec{n}}{2} \right) \right) (1 + m_A r) \right]
\]  

(18)

In the last formulae we have omitted the spinors \( \phi_1, \phi_1' \ldots \) but have labelled the \( \sigma \)'s by the indices showing to which fermion they refer.

Following the discussion of the Introduction we shall consider only fermions which are heavy enough to bind. The radius of the bound states is of order \( a \) which must be small enough to ensure the validity of the inequality:

\[
m_{Aa} \ll 1
\]  

(19)

So the potential \( (18) \) would be of the form:

\[
P(\vec{r}) = -\frac{1}{4\pi v_A^2} e^{-m_A r} \frac{1}{r^3} \left[ \vec{\sigma}_1 \cdot \vec{\sigma}_2 - 3 \left( \frac{\vec{\sigma}_1 \cdot \vec{n}}{2} \right) \left( \frac{\vec{\sigma}_2 \cdot \vec{n}}{2} \right) \right].
\]  

(20)

To compare our nonrelativistic approach with relativistic equations we shall use the technique developed by Królikowski in handling the spin dependence of the wave function. For the Schrödinger equation

\[
\frac{-1}{m} \nabla^2 \psi + P(r) \psi = E\psi
\]  

(21)

we decompose \( \psi \) into spin-zero and spin-one parts:

\[
\phi = P_0 \psi, \quad \chi = \frac{1}{2} (\vec{\sigma}_1 - \vec{\sigma}_2) P_1 \psi,
\]  

(22)

where the projection operators are:

\[
P_0 = \frac{1}{4} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2), \quad P_1 = \frac{1}{4} (3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2)
\]  

(23)
The following set of identities can be used:

\[ P_0 (\sigma_1 - \sigma_2) = (\sigma_1 - \sigma_2) P_1, \quad P_0 (\sigma_1 + \sigma_2) = (\sigma_1 + \sigma_2) P_0 = 0, \]

\[ P_0 (\sigma_{i1} \sigma_{2n} - \sigma_{1n} \sigma_{2i}) = i P_0 \epsilon_{int} (\sigma_{1i} - \sigma_{2i}) \]

\[ P_0 (\sigma_{i1} \sigma_{2n} + \sigma_{1n} \sigma_{2i}) = -2 \delta_{in} P_0, \]

\[ P_0 (\sigma_{1i} - \sigma_{2i}) (\sigma_{1n} + \sigma_{2n}) = 2i P_0 \epsilon_{int} (\sigma_{1i} - \sigma_{2i}) \]

\[ \rightarrow \quad P_0 (\sigma_{1i} - \sigma_{2i}) (\sigma_{1n} - \sigma_{2n}) = 4 P_0 \delta_{in} \]

Acting by \( P_0 \) on both sides of eq. (21) one sees that the tensor forces vanish in the singlet state, so that \( \phi \) obeys the free Schrödinger equation. On the other hand by applying the operator \( (\sigma_1 - \sigma_2) P_1 = P_0 (\sigma_1 - \sigma_2) \) to eq. (21) we can derive the equation for the triplet \( \chi \) function:

\[- \frac{1}{m} \nabla^2 \chi = \frac{1}{2 \pi \hbar^2} \frac{1}{r^3} (\chi - 3 \vec{n} (\chi, \vec{n})) = E \chi \]

Following [4] we now split \( \chi (\vec{r}) \) into “electric”, “longitudinal” and “magnetic” parts:

\[ \chi (\vec{r}) = \chi_e (\vec{r}) + \chi_L (\vec{r}) + \chi_M (\vec{r}), \]

where each obeys the following expansion in spherical harmonics:

\[ \chi_e (\vec{r}) = \sum_{jm} \vec{n} Y_{jm} (\vec{n}_\perp) \chi^{ jm}_e (r), \]

\[ \chi_L (\vec{r}) = - \sum_{jm} \frac{\partial}{\partial \vec{n}_\perp} \left( \frac{Y_{jm} (\vec{n}_\perp)}{j (j + 1)} \right) \chi^{ jm}_L (r), \]

\[ \chi_M (\vec{r}) = - \sum_{jm} (\vec{n} \times \frac{\partial}{\partial \vec{n}_\perp}) \frac{Y_{jm} (\vec{n}_L)}{j (j + 1)} \chi^{ jm}_M (r) \]

In eqns. (26) we introduce differentiation in the transverse direction

\[ \frac{\partial}{\partial n_{i\perp}} \equiv (\delta_{in} - n_i n_j) \frac{\partial}{\partial n_j}, \]

so that

\[ n_i \frac{\partial}{\partial n_{i\perp}} = 0. \]

On the other hand the gradient and the Laplacian are expressed in terms of \( \partial/\partial r, \partial/\partial \vec{n}_\perp \) as follows:

\[ \frac{\partial}{\partial \vec{r}} = \frac{1}{r} \frac{\partial}{\partial \vec{n}_\perp} + \vec{n} \frac{\partial}{\partial r}, \]

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial n_{\perp}^2}. \]

Note that \( \partial^2/\partial n_{\perp}^2 \) represents the angular part of the Laplacian. Note also that:
\[ \frac{\partial}{\partial \mathbf{n}_\perp} \mathbf{n}_\perp = 2, \quad \frac{\partial}{\partial \mathbf{n}_\perp} \mathbf{n}_\parallel = \mathbf{n}_\parallel \left( \mathbf{n}_\perp \frac{\partial}{\partial \mathbf{n}_\perp} \right), \quad \frac{\partial^2}{\partial \mathbf{n}_\perp^2} Y_{jm} = -j (j + 1) Y_{jm} \]  

(30)

The quantum number \( j \) corresponds to the total rather than the orbital angular momentum. Indeed let us define the total angular momentum \( \mathbf{J} = \mathbf{L} + \mathbf{S} \) acting on the vector \( \mathbf{\chi} \) of eq. (22) as:

\[ (L_i \chi_k)_i = -i \epsilon_{ipq} \frac{\partial \chi_k}{\partial q_p}, \quad (S_i \chi_k)_i = -i \epsilon_{ikl} \chi_l. \]  

(31)

It is easy to see then that the operator \( \mathbf{\tilde{J}} \) when acting on \( \mathbf{\tilde{\chi}}_e, \mathbf{\tilde{\chi}}_L \) and \( \mathbf{\tilde{\chi}}_M \) can be commuted through the vectors \( \mathbf{n}_\perp, \frac{\partial}{\partial \mathbf{n}_\perp} \) and \( (\mathbf{n}_\perp \times \frac{\partial}{\partial \mathbf{n}_\perp}) \) and that the contribution of \( \mathbf{\tilde{S}} \) in \( \mathbf{\tilde{J}} = \mathbf{\tilde{L}} + \mathbf{\tilde{S}} \) cancels out so that \( \mathbf{\tilde{J}} \) acts exactly in the same way as \( \mathbf{\tilde{L}} \) on \( Y_{jm} \). For example, taking (\( \mathbf{\tilde{\chi}}_e (\mathbf{\tilde{r}}) \)) \( k \equiv \chi_{ek} (\mathbf{\tilde{r}}) \), we get

\[ (J_{ek})_i = \sum\chi_e^{jm} (r) \left\{ -i \epsilon_{ipql} \frac{\partial}{\partial q_l} (n_k Y_{jm}) - i \epsilon_{ikl} \frac{\partial Y_{jm}}{\partial q_l} \right\} = \sum\n_k \chi_e^{jm} (r) (L_i Y_{jm}) \]  

(32)

If we now repeat this operation:

\[ \mathbf{\tilde{J}}^2 \chi_{ek} = J_i (J_{ek})_i = \sum_{jm} n_k \chi_e^{jm} (r) \mathbf{\tilde{L}}^2 Y_{jm} = j (j + 1) \chi_{ek}, \]  

(33)

which proves that \( j \) is actually the total momentum. The same proof can be given for \( \chi_L \) and \( \chi_M \). We are ready now to derive the equations for the radial wave functions. For that purpose we project eq. (24) on \( \mathbf{\tilde{n}} \) and act on both sides of the equation by \( \partial / \partial \mathbf{n}_\perp \) and \( \mathbf{n}_\perp \times \partial / \partial \mathbf{n}_\perp \). Using the expansions (27) and (26) it is easy to obtain after some algebra the following equations:

\[ \frac{d^2 \chi_e}{dr^2} + \frac{2 d \chi_e}{dr} - j (j + 1) + \frac{1}{r^2} \chi_e - \frac{2 \chi_L}{r^2} + \frac{m}{\pi v_A^2 r^3} \chi_e + m E \chi_e = 0, \]

\[ \frac{d^2 \chi_L}{dr^2} + \frac{2 d \chi_L}{dr} - j (j + 1) \chi_L - \frac{2 j (j + 1)}{r^2} \chi_e - \frac{m \chi_L}{2 \pi v_A^2 r^3} + m E \chi_L = 0, \]

\[ \frac{d^2 \chi_M}{dr^2} + \frac{2 d \chi_M}{dr} - j (j + 1) \chi_M - \frac{m}{2 \pi v_A^2} \chi_M + m E \chi_M = 0, \]

where we have omitted the indices \( jm \) of the functions \( \chi_e^{jm}, \ldots \) in these equations.

We see that the functions \( \chi_e \) and \( \chi_L \) are coupled in the first two equations (34) whereas the function \( \chi_M \) is separated from \( \chi_e, \chi_L \). This is related to the fact that \( \chi_M \) corresponds to the value of the orbital momentum \( \ell = j \) while \( \chi_e \) and \( \chi_L \) are the mixtures of the states \( \chi^+ \) with \( \ell = j + 1 \) and \( \chi^- \) with \( \ell = j - 1 \). These mixtures are [4]:

\[ \chi_e = \sqrt{\frac{j}{2j + 1}} \chi^- + \sqrt{\frac{j + 1}{2j + 1}} \chi^+, \]

\[ \chi_L = \sqrt{j (j + 1)} \left[ -\sqrt{\frac{j + 1}{2j + 1}} \chi^- + \sqrt{\frac{j}{2j + 1}} \chi^+ \right] \]  

(35)

Passing in equs. (34) from \( \chi_e \) and \( \chi_L \) to \( \chi^\pm \) we obtain:

\[ \frac{d^2 \chi^-}{dr^2} + \frac{2 d \chi^-}{dr} - \frac{j (j - 1)}{r^2} \chi^- + m E \chi^- + \frac{m}{2 \pi v_A^2 r^3} \left[ \frac{j - 1}{2j + 1} \chi^- + 3 \sqrt{j (j + 1)} \chi^+ \right] = 0, \]

(36)

\[ \frac{d^2 \chi^+}{dr^2} + \frac{2 d \chi^+}{dr} - \frac{(j + 1) (j + 2)}{r^2} \chi^+ + m E \chi^+ + \frac{m}{2 \pi v_A^2 r^3} \left[ \frac{j + 2}{2j + 1} \chi^+ + 3 \sqrt{j (j + 1)} \chi^- \right] = 0. \]
We see from (36) that \( j (j - 1) \) in the first equation and \( (j + 1)(j + 2) \) in the second correspond to \( \ell (\ell + 1) \) for \( \ell = j \mp 1 \). Note that in each part of Eq. (36) the potentials are attractive and singular as \( \sim 1/r^3 \) at \( r \to 0 \). Note also that \( \chi_M \) in eqn (24) does not have this singularity. We show in the next Section that this singularity is not present at all in a fully-relativistic treatment of pseudoscalar exchange: in the Appendix we show that the apparent singularity arises only when a non-relativistic approximation is made. So Eq. (36) is valid provided that \( r \) is not too small; more precisely that \( r > a \), where the cutoff \( a \) is given by Eq. (12).

III. THE RELATIVISTIC EQUATIONS

To obtain a relativistic treatment of the problem and in particular, to understand the smoothing out of the singularity \( 1/r^3 \) we begin with the two-body Dirac equation (the Breit equation)

\[
\left[ E - \gamma_0^{(1)} \left( \gamma_1 \vec{p} + m_1 \right) - \gamma_0^{(2)} \left( -\gamma_2 \vec{p} + m_2 \right) - V_{\text{int}} (r) \right] \psi = 0
\]

(37)
describing a system of two spin-1/2 particles of masses \( m_1 \) and \( m_2 \) in the centre of mass frame, interacting with each other through a potential of the form

\[
V_{\text{int}} = V_s (r) + V_p (r) + V_v (r)
\]

\[
V_s = \left( \gamma_0^{(1)} \times \gamma_0^{(2)} \right) S (r)
\]

\[
V_p = \left( \left( \gamma_0^{(1)}, \gamma_5 \right) \times \left( \gamma_0^{(2)}, \gamma_5 \right) \right) P (r)
\]

\[
V_v = \left[ \left( \gamma_0^{(1)} \gamma_5 \right) \times \left( \gamma_0^{(2)} \gamma_\mu \right) \right] V (r)
\]

(38)

where \( S (r), P (r) \), and \( V (r) \) result from the exchange of scalar, pseudoscalar and vector particles. We use the Dirac-Pauli representation for the \( \gamma \)-matrices:

\[
\gamma_0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_\sigma = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3
\]

(39)

In this paper we shall be interested mainly in \( S \) and \( P \) exchange. Of course, a static potential \( V_{\text{int}} \) which depends only on the variable \( r = |\vec{r}_1 - \vec{r}_2| \) is not a relativistically invariant quantity. Furthermore the concept of a potential can only be an approximation at best to a proper field-theoretic description of a system. These are well-known difficulties of the generalisation of the Dirac equation involving a potential to of a two-body system which we do not wish to pursue here. We shall adopt the viewpoint that Eq. (37) provides a starting point for a calculation of the relativistic corrections to the non-relativistic two-fermion system in the same way that the treatment of the Coulomb potential in the Dirac equation leads to relativistic corrections in the spectrum of the hydrogen atom of higher order in \( \alpha \) compared with the Schrodinger solution. An analysis of the constraints imposed by the relativistic invariance of a two-body fermionic system shows that provided the coordinates are chosen appropriately (which in practice means using the centre of mass frame for the equal mass case of \( m_1 = m_2 \)), then a potential \( V_{\text{int}} (r) \) can be used in a relativistic two-body equation without violating relativistic invariance. Further details can be found in the papers by Mourad and Sazdjian [5] and Crater and Long [6], which also discuss the possible forms for the relativistic wave functions which result from this analysis.

We now follow Tsibidis’ [7] treatment of the Breit equation, which itself is based on the work of Krolikowski [4]. The spinor \( \psi (\vec{r}) \) is a sixteen component wave function which can be represented as a \( 4 \times 4 \) matrix

\[
\psi (\vec{r}) = \psi_{\gamma_0^{(1)}, \gamma_0^{(2)}} = \begin{pmatrix} \psi_{++} \psi_{+-} \\ \psi_{-+} \psi_{--} \end{pmatrix}
\]

(40)

where the indices \(+, -\) refer to the eigenvalues of \( \gamma_0^{(1)} \) and \( \gamma_0^{(2)} \). Equation (37) is reduced to a set of 4 equations for \( \psi_{++}, \psi_{+-} \ldots \). Following the technique of ref. [3], which has been already used to analyse the non-relativistic case, we introduce the components

\[
\psi_{\phi_0} = \frac{i}{\sqrt{2}} \psi_{++} + \psi_{--}
\]

(41)
From Eqs. (37), (40) and (41) we now derive the following set of equations using the identities above for the projection operators $P_0$ and $P_1$:

\[
\begin{align*}
\phi^0 & = \frac{1}{2} \left( \sigma^{(1)} - \sigma^{(2)} \right) P_1 \frac{1}{\sqrt{2}} (\psi_+ \pm \psi_-) \\
\chi^0 & = P_0 \frac{i}{\sqrt{2}} (\psi_+ \mp \psi_-) \\
\bar{\phi}^0 & = \frac{1}{2} \left( \sigma^{(1)} - \sigma^{(2)} \right) P_1 \frac{1}{\sqrt{2}} (\psi_+ \pm \psi_-)
\end{align*}
\]

where $\phi$ and $\chi$ correspond to the spin zero states while $\bar{\phi}$ and $\bar{\chi}$ correspond to $S = 1$ states (compare with eq. (22)). From Eqs. (37), (40) and (41), we now derive the following set of equations using the identities above for the projection operators $P_0$ and $P_1$:

\[
\begin{align*}
\frac{1}{2} (E - S - P - V) \phi^0 - \frac{m_1 + m_2}{2} \phi - i \vec{p} \vec{\phi} & = 0 \\
\frac{1}{2} (E - S + P - V) \phi - \frac{m_1 + m_2}{2} \phi^0 & = 0 \\
\frac{1}{2} (E + S + P - V) \chi^0 - \frac{m_1 - m_2}{2} \chi - i \vec{p} \chi & = 0 \\
\frac{1}{2} (E + S - P - V) \chi - \frac{m_1 - m_2}{2} \chi^0 & = 0 \\
\frac{1}{2} (E - S - P - V) \bar{\chi} - \frac{m_1 + m_2}{2} \bar{\chi}^0 + i \vec{p} \bar{\chi}^0 & = 0 \\
\frac{1}{2} (E - S + P - V) \bar{\chi}^0 - \frac{m_1 + m_2}{2} \bar{\chi} + i \vec{p} \bar{\chi} & = 0 \\
\frac{1}{2} (E + S + P - V) \bar{\phi} - \frac{m_1 - m_2}{2} \bar{\phi}^0 + i \vec{p} \bar{\phi}^0 & = 0 \\
\frac{1}{2} (E + S - P - V) \bar{\phi}^0 - \frac{m_1 - m_2}{2} \bar{\phi} + i \vec{p} \bar{\phi} & = 0
\end{align*}
\]

Each of the vector functions $\vec{\phi}, \vec{\chi}, \vec{\phi}^0, \vec{\chi}^0$, entering this equations, can be split into “electric”, “longitudinal” and “magnetic” parts according to Eqs. (25) and (26) and expanded into spherical harmonics. We then obtain for each value of the total angular momentum 12 radial wave functions for the vector components: $\phi^0_j(r), \phi^1_j(r), \phi^2_j(r), \chi^0_j(r), \chi^1_j(r), \chi^2_j(r)$ etc, together with the four radial wave functions for the scalar components: $\phi^0_j(r), \phi^1_j(r), \chi^0_j(r), \chi^1_j(r)$. We write down the sixteen equations for these functions according to the following classification.

It is easy to see that $\phi, \phi^0, \phi^1, \phi^2, \phi^3, \chi, \chi^0, \chi^1, \chi^2, \chi^3$ have parity $P = \eta (-1)^j$ where $\eta$ is the intrinsic parity (+1 for two fermions and -1 for fermion-antifermion system) and $j$ is the total momentum (we omit index $j$ at the radial components of the wave functions). the remaining 8 functions namely $\chi, \chi^0, \chi^1, \chi^2, \chi^3$ have $P = -\eta (-1)^j$ and therefore the equations for the first 8 functions do not mix with the equations for the latter 8 components. The former case may be called a pseudoscalar meson trajectory (PMT) and its spectroscopic signature is $1_j$, the latter may be called a vector meson trajectory (VMT): it has the spectroscopic signature $3_j$. A fermion-antifermion system which conserves charge conservation C will have 8 independent components so the 16 spinor components will need to reduce to 8 dynamical equations and 8 constraint equations, or 4 dynamical equations and 4 constraint equations for PMT and the same again for VMT. We demonstrate this below.

Thus we obtain after some algebra:
(i) PMT $^3j_3$, $P = \eta(-1)^j$

\[
\frac{1}{2} (E - S - P - V) \phi^0 - \frac{m_1 + m_2}{2} \phi - \left( \frac{d}{dr} + \frac{2}{r} \right) \phi_e - \frac{1}{r} \phi_L = 0
\]

\[
\frac{1}{2} (E - S + P - V) \phi - \frac{m_1 + m_2}{2} \phi^0 = 0
\]

\[
\frac{1}{2} (E - S - P - V) \chi_M - \frac{m_2 + m_2}{2} \chi_M^0 = 0
\]

\[
\frac{1}{2} (E - S + P - V) \chi_M^0 - \frac{m_1 + m_2}{2} \chi_M + \frac{j(j + 1)}{r} \phi^0 + \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_L^0 = 0
\]

\[
\frac{1}{2} (E + S + P - V) \phi_e - \frac{m_1 - m_2}{2} \phi^0_e + \frac{d\phi^0}{dr} = 0
\]

\[
\frac{1}{2} (E + S + P - V) \phi_L - \frac{m_1 - m_2}{2} \phi^0_L - \frac{j(j + 1)}{r} \phi^0 = 0
\]

\[
\frac{1}{2} (E + S - P - V) \phi_e^0 - \frac{m_1 - m_2}{2} \phi_e^0 + \frac{1}{r} \chi_M^0 = 0
\]

\[
\frac{1}{2} (E + S - P - V) \phi_L^0 - \frac{m_1 - m_2}{2} \phi_L^0 - \left( \frac{d}{dr} + \frac{1}{r} \right) \chi_M^0 = 0
\]

(ii) VMT $^3(j \pm 1)_j$, $P = -\eta(-1)^j$

\[
\frac{1}{2} (E + S + P - V) \chi^0 - \frac{m_1 + m_2}{2} \chi - \left( \frac{d}{dr} + \frac{2}{r} \right) \chi_e - \frac{1}{r} \chi_L = 0
\]

\[
\frac{1}{2} (E + S - P - V) \chi - \frac{m_1 - m_2}{2} \chi^0 = 0
\]

\[
\frac{1}{2} (E - S - P - V) \chi_e - \frac{m_1 + m_2}{2} \chi_e^0 + \frac{d\chi^0_e}{dr} = 0
\]

\[
\frac{1}{2} (E - S - P - V) \chi_L - \frac{m_1 + m_2}{2} \chi_L^0 - \frac{j(j + 1)}{r} \chi^0 = 0
\]

\[
\frac{1}{2} (E + S + P - V) \chi_e^0 - \frac{m_1 + m_2}{2} \chi_e^0 + \frac{1}{r} \phi_M^0 = 0
\]

\[
\frac{1}{2} (E + S + P - V) \chi_L^0 - \frac{m_1 + m_2}{2} \chi_L^0 - \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_M^0 = 0
\]

\[
\frac{1}{2} (E + S - P - V) \phi_M^0 - \frac{m_1 + m_2}{2} \phi_M^0 + \frac{j(j + 1)}{r} \chi_e^0 - \left( \frac{d}{dr} + \frac{1}{r} \right) \chi_L^0 = 0
\]

\[
\frac{1}{2} (E + S + P - V) \phi_L^0 - \frac{m_1 - m_2}{2} \phi_L^0 = 0
\]

The non-relativistic analysis of the previous section for pseudoscalar exchange $P \neq 0$ refers to the spin $S = 1$ states connected to the "large" components of the relativistic wave function $\psi_{+,-}$. This implies that $\chi_e, \chi_L$ of Eq. (34) are the non-relativistic analogues of the same functions entering Eq. (14). Remember also that the $1/r^3$ singularity of Eq. (34) only involves the functions $\chi_e$ and $\chi_L$, so it is sufficient for our purposes to concentrate on the following set of equations for the case $m_1 = m_2 = m$: 

8
\[
\frac{1}{2} (E + S + P - V) \chi^0 - \left( \frac{d}{dr} + \frac{2}{r} \right) \chi_e - \frac{1}{r} \chi_L = 0
\]

\[
\frac{1}{2} (E - S - P - V) \chi_e - m \chi_e^0 + \frac{d \chi_e^0}{dr} = 0
\]

\[
\frac{1}{2} (E - S - P - V) \chi_L - m \chi_L^0 - \frac{j (j + 1)}{r} \chi^0 = 0
\]

\[
\frac{1}{2} (E - S + P - V) \chi_e^0 - m \chi_e + \frac{1}{r} \phi_0^0 = 0
\]

\[
\frac{1}{2} (E - S + P - V) \chi_L^0 - m \chi_L - \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_0^0 = 0
\]

\[
\frac{1}{2} (E - S + P - V) \phi_0^0 + \frac{j (j + 1)}{r} \phi_0^0 + \left( \frac{d}{dr} + \frac{1}{r} \right) \chi_L^0 = 0
\]

(45)

Eventually we obtain the non-relativistic limit:

\[
\frac{1}{2} (E - S + P - V) \phi_0^0 + \frac{j (j + 1)}{r} \phi_0^0 + \left( \frac{d}{dr} + \frac{1}{r} \right) \chi_L^0 = 0
\]

First of all we should try to show that the non-relativistic limit of (45) gives Eq. (34) for \( \chi_e \) and \( \chi_L \). This is not a trivial exercise but we can proceed as follows. Add and subtract the second and the fourth equation of (44) and introduce \( \chi^0 = \frac{1}{2} (\chi_e + \chi_L^0) \). Then we can get the equation (keeping only \( P \neq 0 \)):

\[
(E - 2m) \chi^+_e + \frac{d \chi^+_e}{dr} \left( 1 + \frac{P}{E + 2m} \right) + \frac{1}{r} \phi_0^0 \left( 1 - \frac{P}{E + 2m} \right) - \frac{P^2}{E + 2m} \chi^+_e = 0
\]

(46)

From (45) and (46) we can obtain for \( \chi^+_L = \frac{1}{2} (\chi_L + \chi^+_L) \):

\[
(E - 2m) \chi^+_L - \frac{j (j + 1)}{r} \left( 1 + \frac{P}{E + 2m} \right) \chi^0 - \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_0^0 \left( 1 - \frac{P}{E + 2m} \right) - \frac{P^2}{E + 2m} \chi^+_L = 0
\]

(47)

In the nonrelativistic approximation \( \chi^0 \) and \( \phi_0^0 \) are readily expressed through \( \chi^+_e \) and \( \chi^+_L \) (this is correct only for the non-relativistic limit). We get:

\[
\chi^0 \approx \frac{1}{m} \left[ \left( \frac{d \chi^+_e}{dr} + \frac{2}{r} \chi^+_e + \frac{1}{r} \chi^+_L \right) \left( 1 - \frac{P}{4m} \right) + \frac{P'}{4m} \chi^+_e \right]
\]

(48)

\[
\phi_0^0 \approx \frac{1}{m} \left[ \left( \frac{j (j + 1)}{r} \chi^+_e + \left( \frac{d}{dr} + \frac{1}{r} \right) \chi^+_L \right) \left( 1 + \frac{P}{4m} \right) - \frac{P'}{4m} \chi^+_L \right]
\]

where \( P' = \frac{dP}{dr} \). In these equations we neglect all non-relativistic corrections and keep only linear terms in \( P \). Then substituting (45) into (46) and (47) we see that only derivatives of \( P \) remain in the final equations for \( \chi^+_e \) and \( \chi^+_L \). Eventually we obtain the non-relativistic limit:

\[
m (E - 2m) \chi^+_e + \frac{d^2 \chi^+_e}{dr^2} + \frac{2}{r} \frac{d \chi^+_e}{dr} - \frac{j (j + 1) + 2}{r^2} \chi^+_e + \frac{2}{r^2} \chi^+_L + \left( \frac{P''}{4m} - \frac{P'}{8mr} \right) \chi^+_e = 0,
\]

\[
m (E - 2m) \chi^+_L + \frac{d^2 \chi^+_L}{dr^2} + \frac{2}{r} \frac{d \chi^+_L}{dr} - \frac{j (j + 1)}{r^2} \chi^+_L - \frac{2 j (j + 1)}{r^2} \chi^+_e + \frac{P'}{4m} \chi^+_L = 0
\]

(49)

For the exchange of a pseudoscalar particle:

\[
P = \frac{m^2}{\pi v^2_A} e^{-m_A r}
\]

(50)

So we see that the terms proportional to \( P' / r \) and \( P'' \) in Eq. (49) give the \( 1/r^3 \) singularity of eq. (34). We show in the Appendix that the relativistic equations of Eq. (44) do not give a \( 1/r^3 \) singularity. Thus this singularity is just an artefact of the non-relativistic approximation.
We conclude that heavy fermions $f$ of mass larger than 800 GeV are required if they are to bind with $\bar{f}$ by Higgs or pseudoscalar exchange. If fermions of this mass exist, a whole new region of physics will be associated with this high mass scale: the fermions themselves and their decay modes, and the $f\bar{f}$ bound states, their decays and systematics. The details of this new physics, which involves binding by an unfamiliar mechanism, cannot be predicted in advance.

We have also demonstrated that the $1/r^3$ singularity seemingly present in pseudoscalar exchange will vanish if a relativistic framework is adopted for the calculation. Pseudoscalar exchange in relativistic quantum field theory arises from renormalisable theories and hence cannot lead to collapse.

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V. APPENDIX

We now look at the relativistic case to show that there is no $1/r^3$ singularity at the origin arising from pseudoscalar exchange. It is not easy to derive relativistic equations for the two functions $\chi_e$ and $\chi_L$ of the non-relativistic approximation (49). It is easier to exclude $\chi_e, \chi_e^0, \chi_L, \chi_L^0$ from Eq. (49) and then obtain a pair of equations for $\chi^0, \phi_M^0$.

To this end we have from the second and fourth equations of (45) (keeping $P \neq 0$ and $S \neq 0$):

$$\chi_e = \frac{4m}{4m^2 - (E - S)^2 + P^2} \left[ \frac{1}{2m}(E - S + P) \frac{d\chi^0}{dr} + \frac{\phi_M^0}{r} \right]$$

$$\chi_e^0 = \frac{4m}{4m^2 - (E - S)^2 + P^2} \left[ \frac{1}{2m}(E - S - P) \phi_M^0 + \frac{d\chi^0}{dr} \right]$$

and from the third and fifth equations:

$$\chi_L = \frac{-4m}{4m^2 - (E - S)^2 + P^2} \left[ \frac{1}{2m}(E - S + P) \frac{j(j + 1)}{r^2} \chi^0 + \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_M^0 \right]$$

$$\chi_L^0 = \frac{-4m}{4m^2 - (E - S)^2 + P^2} \left[ \frac{1}{2m}(E - S - P) \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_M^0 + \frac{j(j + 1)}{r^2} \chi^0 \right]$$

Substituting these expressions into the first and the last of equations (45) we get two second order equations for $\chi^0$ and $\phi_M^0$:

$$\frac{d^2\chi^0}{dr^2} + \frac{2}{r} \frac{d\chi^0}{dr} - \frac{j(j + 1)}{r^2} \chi^0 - \frac{1}{4} \left[ 4m^2 - (E - S)^2 + P^2 \right] \frac{E + S + P}{E - S + P} \chi^0$$

$$- \frac{d}{dr} \left[ \ln \left( 4m^2 - (E - S)^2 + P^2 \right) \right] \frac{d\chi^0}{dr} - \frac{4m}{E - S + P} \frac{d}{dr} \left[ \ln \left( 4m^2 - (E - S)^2 + P^2 \right) \right] \frac{\phi_M^0}{r} = 0$$

$$\frac{d^2\phi_M^0}{dr^2} + \frac{2}{r} \frac{d\phi_M^0}{dr} - \frac{j(j + 1)}{r^2} \phi_M^0 - \frac{1}{4} \left[ 4m^2 - (E - S)^2 + P^2 \right] \frac{E + S - P}{E - S - P} \phi_M^0 -$$

$$- \frac{d}{dr} \left[ \ln \left( 4m^2 - (E - S)^2 + P^2 \right) \right] \left( \frac{d}{dr} + \frac{1}{r} \right) \phi_M^0 - \frac{4m}{E - S - P} \frac{d}{dr} \left[ \ln \left( 4m^2 - (E - S)^2 + P^2 \right) \right] \frac{j(j + 1)}{r} \chi^0 = 0$$

It is now clear what a delicate problem it is to go to the nonrelativistic limit for the pseudoscalar case. For the scalar interaction, $S$ should be retained only in $4m^2 - (E - S)^2 \approx -4m(\epsilon - S)$ where $\epsilon = E - 2m$ in the first line of Eq.
This results in the usual Schrödinger equation with potential energy $S$. For $S = 0$, $P \neq 0$ we should keep the apparently small term $P/2m$ even in the limit $m \to \infty$ term since the coupling constant in $P$ is proportional to $m^2$. We do not however keep $P^2/4m^2$ terms in order to get the nonrelativistic equations (36). The most important thing which we learn from (63) is that when $r \to 0$ there are only singularities $\sim 1/r^2$, and therefore there is no collapse. A more detailed analysis of the relativistic equation is outside the scope of this paper: Further details can be found in the papers by Mourad and Sazdjian [5] and Crater and Long [6].

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