Instability of Supersonic Cold Streams Feeding Galaxies I: Linear Kelvin-Helmholtz Instability with Body Modes

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ABSTRACT
Massive galaxies at high redshift are predicted to be fed from the cosmic web by narrow, dense streams of cold gas that penetrate through the hot medium encompassed by a stable shock near the virial radius of the dark-matter halo. Our long-term goal is to explore the heating and dissipation rate of the streams and their fragmentation and possible breakup, in order to understand how galaxies are fed, and how this affects their star-formation rate and morphology. We present here the first step, where we analyze the linear Kelvin-Helmholtz instability (KHI) of a cold, dense slab or cylinder in 3D flowing supersonically through a hot, dilute medium. The current analysis is limited to the adiabatic case with no gravity. By analytically solving the linear dispersion relation, we find a transition from a dominance of the familiar rapidly growing \textit{surface} modes in the subsonic regime to more slowly growing \textit{body} modes in the supersonic regime. The system is parametrized by three parameters: the density contrast between stream and medium, the Mach number of stream velocity with respect to the medium, and the stream width with respect to the halo virial radius. A realistic choice for these parameters places the streams near the mode transition, with the KHI exponential-growth time in the range 0.01-10 virial crossing times for a perturbation wavelength comparable to the stream width. We confirm our analytic predictions with idealized hydrodynamical simulations. Our linear estimates thus indicate that KHI may be effective in the evolution of streams before they reach the galaxy. More definite conclusions await the extension of the analysis to the nonlinear regime and the inclusion of cooling, thermal conduction, the halo potential well, self-gravity and magnetic fields.

Key words: cosmology — galaxies: evolution — galaxies: formation — hydrodynamics — instabilities

1 INTRODUCTION

According to the standard $\Lambda$CDM model of cosmology, the most massive haloes at any epoch lie at the nodes of the cosmic web, and are penetrated by cosmic filaments of dark matter (e.g. Bond, Kofman & Pogosyan 1996; Springel \textit{et al}. 2005 [Dekel \textit{et al}. 2009]. These represent high-sigma peaks in the density fluctuation field, much more massive than the Press-Schechter mass, $M_\star$, of typical haloes at that time (Press & Schechter 1974).

At redshift $z = 1 - 4$, when star-formation is at its peak and most of the mass is assembled into galaxies (Madau, Pozzetti & Dickinson 1998; Hopkins & Beacom 2006; Madau & Dickinson 2014), such haloes have virial masses of $M_\star \sim 10^{12} M_\odot$ and above, larger than the critical mass for shock heating $M_{\text{shock}} \lesssim 10^{12} M_\odot$ [Birnboim & Dekel 2003; Dekel & Birnboim 2006]. They thus contain hot gas at the virial temperature, $T_v \sim 10^6$ K. However, for such high-sigma peaks the filaments that feed the halo are significantly narrower than the virial radius, and the gas residing in them is much denser than the halo gas. Therefore, the radiative cooling time of the stream gas is
shorter than the local compression time, preventing the formation of a stable virial shock within the streams. The streams are thus expected to remain cold, with temperatures of $T_e \gtrsim 10^4 \text{K}$, allowing them to penetrate efficiently through the hot halo circumgalactic medium (CGM) onto the central galaxy (Dekel & Birnboim 2006).

The above theoretical picture is supported by cosmological simulations (Keres et al. 2005; Ocvirk, Pichon & Tevssier 2008; Dekel et al. 2009; Ceverino, Dekel & Bournaud 2010; Faucher-Giguère, Keres & Ma 2011; van de Voort et al. 2011). In these simulations, cold streams with widths of a few to ten percent of the virial radius penetrate deep into the halo. This gas supply allows the high star-formation rates (SFRs) of $\sim 20 - 200 M_\odot \text{yr}^{-1}$ observed in massive star-forming galaxies (SFGs) with baryonic masses of $\sim 10^{11} M_\odot$ at $z \sim 2$ (Genzel et al. 2006; Förster Schreiber et al. 2006; Elmegreen et al. 2007; Genzel et al. 2008; Stark et al. 2008). These high SFR values are only a factor of $\lesssim 2$ lower than the accretion rate at the virial radius, implying that at least half the gas mass flux brought into the halo by the streams must reach the central galaxy (Dekel et al. 2009), irrespective of what happens to the stream structure and thermal properties along the way. The streams also play a key role in the buildup of angular momentum in disk galaxies (Pichon et al. 2011; Kimm et al. 2011; Stewart et al. 2011, 2013; Codis et al. 2012; Danovich et al. 2012, 2013).

Cosmological simulations indicate that the streams maintain roughly constant inflow velocities as they travel from the outer halo to the central galaxy (Dekel et al. 2009; Goerdt & Ceverino 2013). The constant velocity, as opposed to the expected gravitational acceleration, indicates energy loss into radiation which may be observed as Lyman-α cooling emission (Dijkstra & Loeb 2009; Goerdt et al. 2010; Faucher-Giguère et al. 2010), though the dissipation process has not been explored yet. Based on radiative transfer models, the total luminosity and the spatial structure of the emitted radiation appear similar to Lyman-α “blobs” observed at $z > 2$ (Steidel et al. 2000; Matsuda et al. 2006, 2011). The models find that roughly half the radiation comes from the dissipation of gravitational energy while the other half is due to heating from the UV background (Goerdt et al. 2010, though see also Faucher-Giguère et al. 2010 who found somewhat lower luminosities in their simulations). Radiative transfer models also show that a central quasar can power the emission by supplying seed photons which scatter inelastically within the filaments, producing Lyman-α cooling emission that extends to several hundred kpc and appears similar to observed structures (Cantalupo et al. 2014). Recent observations using the MUSE integral-field instrument suggest that such extended Lyman-α emitting nebulae are ubiquitous around the brightest quasars at $z \sim 3.5$ (Borisova et al. 2010). In addition to emission, the cold streams consisting of mostly neutral Hydrogen should also be visible in absorption, and can account for observed Lyman-limit systems (LLSs) and damped Lyman-α systems (DLAs) (Fumagalli et al. 2011; Goerdt et al. 2012; van de Voort et al. 2012).

Observations using absorption features along quasar sight-lines to probe the CGM of massive SFGs at $z \sim 1 - 2$ reveal low-metallicity, co-planar, co-rotating accreting material (Bouché et al. 2013, 2016; Prochaska, Lau, & Hennawi 2014), providing further observational support for the cold-stream paradigm. Strong Lyman-α absorption has also been detected in the CGM of massive sub-millimeter galaxies (SMGs) at $z \sim 2$ (Fu et al. 2014).

Despite the growing evidence from simulations and observations that cold streams are a fundamental part of galaxy formation at high redshift, several important questions remain regarding their evolution. How much of the stream energy is dissipated as they travel through the CGM? What are the implications on the emitted radiation and the mass inflow rate onto the galaxy? How do the streams join the galaxy, in terms of coherency versus fragmentation/clumpiness, temperature, and velocity? How does this affect the growth of angular momentum and the SFR in the disk?

While there is some preliminary observational evidence for the fragmentation of cold streams (Cantalupo et al. 2014), most attempts to address these questions have used cosmological simulations. Unfortunately, in current cosmological simulations the resolution within the streams is never better than a few hundred pc, and is often on the order of a kpc, comparable to the stream width itself. They thus cannot resolve the detailed physical processes associated with stream instabilities necessary to properly address these questions. Grid-based adaptive-mesh-refinement (AMR) codes show streams that remain cold and coherent outside of $\sim 0.3 R_{\text{v}}$, inside of which a messy interaction region is seen where the streams collide, fragment, and experience strong torques before settling onto the disc (Ceverino, Dekel & Bournaud 2010; Danovich et al. 2012). These simulations exhibit high gas mass accretion rates onto the central galaxy, roughly half the virial accretion rate (Dekel et al. 2013). Simulations using the moving mesh code AREPO (Springel 2010; Vogelsberger et al. 2012) suggest that the streams heat up at $\sim 0.25 - 0.5 R_{\text{v}}$, with most of the accreted gas heating to roughly the virial temperature before falling onto the galaxy (Nelson et al. 2013). Nevertheless, the mass inflow rate onto the central galaxy ends up very similar to the virial accretion rate, likely because the dense stream gas in the inner halo rapidly cools after heating. It is unclear whether this gas is ever in hydrostatic equilibrium within the halo. The same study argued that previous reports of streams remaining cold and coherent in Smooth-Particle-Hydrodynamic (SPH) simulations were due to numerical inaccuracies associated with the standard formulation of SPH.

Since current cosmological simulations are far from being able to properly resolve the streams, a more fundamental analytical and numerical approach is warranted. The physical problem of the evolution
of a supersonic, cold, dense, gas stream in a hot, dilute medium has not been addressed thus far in the literature, even at its simplest hydrodynamic level of Kelvin-Helmholtz instability (KHI)\(^1\). This is the first in a series of papers where we study this issue, in the context of cold streams feeding massive galaxies at high redshift, using analytic models of increasing complexity together with idealized simulations, and concluding with full-scale cosmological simulations with tailored mesh refinement in the streams.

In this paper we take the first step and address KHI under fully compressible conditions\(^2\). We derive the dispersion relation for the growth of linear instabilities in a confined planar-slab and cylinder, and ask whether such instabilities grow to non-linear amplitudes in a virial crossing time. In a forthcoming paper (Padnos et al., in preparation), we will address in detail, analytically and using idealized simulations, the non-linear evolution of these instabilities. In future work we will add one-by-one thermal conduction, cooling, the external potential of the host halo, self gravity and magnetic fields. In the final phase, we will study cosmological simulations with forced mesh refinement in the streams, to explore the effect of stream instability on galaxy formation at the halo centre and the effects of feedback on the streams.

This paper is organized as follows: In \(\S 2\) we summarize the derivation of the linear dispersion relation for compressible KHI in different idealized geometries. Mathematical details of the derivations are provided in several appendices, which may be of interest to the mathematically inclined reader. In \(\S 3\) we use numerical simulations to test the analytic predictions of the preceding section. In \(\S 4\) we apply the analytic formalism to the case of cosmic cold streams and estimate the number of e-foldings of growth experienced by initially small perturbations within a virial crossing time. In \(\S 5\) we speculate as to the effects of additional physics not included in our analysis, presenting an outline for future work. We discuss our results and summarise our conclusions in \(\S 6\).

## 2 COMPRESSIBLE KHI

In this section we derive the dispersion relations for compressible KHI in several different geometries. For simplicity and analytic tractability, we begin by deriving the relation in planar geometry, first discussing a “two-zone instability”, or a sheet, where two semi-infinite fluids are separated by a single planar interface, and then a “three-zone instability”, or a slab, where one fluid is confined to a planar slab of finite thickness and surrounded by a second (background) fluid from both sides. KHI in a slab is qualitatively different than in a sheet due to the appearance of body modes, unstable perturbations caused by waves reverberating back and forth between the slab boundaries, that dominate the instability at high Mach numbers (\(\S 2.3.3\)). We then derive the dispersion relation for a cylindrical stream, and show that the behaviour of linear perturbations with wavelengths comparable to or smaller than the stream radius is effectively identical to that of perturbations in a slab.

Several previous studies have addressed linear stability of astrophysical jets to KHI in both planar and cylindrical geometries, both analytically and numerically (e.g. Ferrari, Trussoni & Zaninetti 1978; Birkinhshaw 1984, 1990; Payne & Cohn 1985; Hardee 1987; Hardee & Norman 1988; Norman & Hardee 1988; Bodo et al. 1994; Perucho et al. 2004). These studies focus primarily on hot, dilute jets travelling in cold, dense media. As we will see below, the main difference between such a scenario and that studied here, of cold streams in a hot medium, is the ratio of the stream sound crossing time to the KHI exponential-growth time. While this can be important for the overall stability of the stream (see \(\S 4\)), it does not fundamentally alter the linear dispersion relation, and our derivation is similar to those presented in Payne & Cohn (1985) (for the cylinder) and Hardee & Norman (1988) (for the slab). However, there are certain differences in our approaches and conclusions which we highlight in the text, and we find our analysis to be more complete, addressing a sheet, a slab, and a cylinder in a self-contained and consistent way.

### 2.1 General KHI in Planar Coordinates

We begin with the basic equations of hydrodynamics, which represent conservation of mass (the continuity equation), momentum (the Euler equation) and energy:

\[
\frac{\partial \rho}{\partial t} + \left( \vec{v} \cdot \vec{\nabla} \right) \rho + \rho \vec{\nabla} \cdot \vec{v} = 0, \tag{1}
\]

\[
\rho \left[ \frac{\partial \vec{v}}{\partial t} + \left( \vec{v} \cdot \vec{\nabla} \right) \vec{v} \right] + \vec{\nabla} P = 0, \tag{2}
\]

\[
\frac{\partial P}{\partial t} + \left( \vec{v} \cdot \vec{\nabla} \right) P - c^2 \left[ \frac{\partial \rho}{\partial t} + \left( \vec{v} \cdot \vec{\nabla} \right) \rho \right] = 0. \tag{3}
\]

Above, \(\rho\) is the fluid’s density, \(\vec{v}\) its velocity and \(P\) the pressure in the fluid. We assume an ideal equation of state, so \(c = (\gamma P/\rho)^{1/2}\) is the sound speed, where \(\gamma\) is the adiabatic index of the fluid, \(\gamma = (\partial \ln P/\partial \ln \rho)_s\).

We consider a time-independent flow in the \(\hat{z}\) direction, where the flow velocity and the fluid density are arbitrary functions of \(x\) (in Cartesian coordinates):

\[
\rho_0(x, y, z) = \rho_0(x), \quad \vec{v}_0(x, y, z) = v_0(x) \hat{z}. \tag{4}
\]
In this case, with no external forces, eq. (2) dictates that the pressure is uniform, \( P_0(x, y, z) = P_0 \). On top of this equilibrium flow, we impose small perturbations in the fluid variables, \( \rho = \rho_0 + \rho_1, \vec{u} = \vec{v}_0 + \vec{u} \) and \( P = P_0 + P_1 \), where the perturbation in each variable \( f \) obeys \( f_1 << f_0 \). To study the growth of instabilities, we decompose each of the perturbed quantities into Fourier modes of the form

\[
f_1(x, y, z, t) = f_1(x)e^{i(k_y y + k_z z - \omega t)},
\]

In other words, the perturbations are travelling waves in the \( yz \) plane with wave vector \( \vec{k} = k_y \hat{y} + k_z \hat{z} \) and frequency \( \omega \), and an arbitrary \( x \) dependence.

By inserting these perturbations into eqs. (1)-(3) and linearizing, we can derive algebraic relations between the \( x \) dependent amplitudes of the pressure perturbation and its derivatives with respect to \( x \) to those of all other perturbations. Using the conventions \( \partial f/\partial x = f' \), we obtain:

\[
\rho_1 = -\frac{1}{k^2 (v_k - \omega/\vec{k})^2} \left[ P_1'' - \frac{2v_k'}{v_k - \omega/\vec{k}} P_1' - k^2 P_1 \right],
\]

\[
u_z = -\frac{\cos(\varphi)}{\rho_0 (v_k - \omega/\vec{k})} \left( \frac{v_k'}{k^2 \cos^2(\varphi)} \right) (v_k - \omega/\vec{k}) P_1' + P_1,
\]

\[
u_y = -\frac{\sin(\varphi)}{\rho_0 (v_k - \omega/\vec{k})} P_1,
\]

\[
u_x = \frac{i}{\rho_0 k} \left( \frac{v_k'}{v_k - \omega/\vec{k}} \right) P_1'.
\]

Note that in eqs. (6)-(9) all fluid variables are functions of \( x \). In addition, we are left with a second order ordinary differential equation for \( P_1(x) \):

\[
P_1'' - \left( \frac{2v_k'}{v_k - \omega/\vec{k}} + \frac{\rho_1'}{\rho_0} \right) P_1' - k^2 \left[ 1 - \left( \frac{v_k - \omega/\vec{k}}{c} \right)^2 \right] P_1 = 0.
\]

Eq. (10) is an eigenvalue equation. Given profiles along \( x \) for the unperturbed density and velocity and boundary conditions for \( P_1 \), solutions exist only for certain combinations of \( \omega \) and \( k \), which define the dispersion relation, \( \omega(k) \). Since eq. (10) depends only on the component of the velocity parallel to the perturbation wave vector, \( v_k \), with no explicit dependence on the propagation angle \( \varphi \), we restrict our analysis to perturbations where \( k_y = 0 \) so that \( v_k = v_0 \).

2.1.1 Temporal vs Spatial Stability Analysis

There are in general two types of stability analyses, temporal and spatial. In the former, the wavenumber \( k \) is real while the frequency \( \omega \) is complex. Physically, this represents seeding the entire system with a spatially-oscillating perturbation and studying its temporal growth. In the latter, \( \omega \) is real while \( k \) is complex. This represents seeding a temporally-oscillating perturbation at the stream origin and studying its downstream spatial growth. This distinction is particularly important when performing numerical simulations, as they change the required boundary and initial conditions. Studies of the stability of jets whose source has intrinsic variability (such as an AGN or a gamma ray burst) often employ spatial stability analyses. However, the cosmic web streams we are studying do not have a well defined variable source outside the halo, but rather experience perturbations from the halo throughout their extent. Therefore, we perform a temporal stability analysis, envisioning a stationary stream suffering some perturbation across its extent, and asking how much the perturbation will grow in a virial crossing time.

2.2 The Planar Sheet

We first consider the classic case of two fluids separated at \( x = 0 \). For consistency with later sections when we discuss a confined stream with finite thickness in a background, we label the two fluids with subscripts ‘b’ and ‘s’, for \( x > 0 \) and \( x < 0 \) respectively. We assume each fluid to have initially uniform density and velocity. This problem is often referred to as the “vortex sheet” and was first addressed for two compressible fluids by Landau (1944). In this case, the second term in eq. (10) vanishes for all \( x \neq 0 \). The equation must be solved separately in the regions \( x > 0 \) and \( x < 0 \), subject to the boundary conditions that the pressure perturbation vanishes at infinity and is continuous across the boundary at \( x = 0 \). The solution is

\[
P_1 = \begin{cases} 
Ae^{-q_{b,s}x} & x > 0 \\
Ae^{q_{b,s}x} & x < 0
\end{cases},
\]

where \( A \) is a constant of integration that depends on the initial conditions, and we have defined the generalized wavenumbers \( q_b \) and \( q_s \) by

\[
q_{b,s} = k \left[ 1 - \left( \frac{\omega - k v_{b,s}}{k v_{b,s}} \right)^2 \right]^{1/2}.
\]

Since \( \omega \) is in general complex, \( q_{b,s} \) is the square root of a complex number, forcing us to choose a branch cut in the complex plane. We have chosen to define \( \mathrm{Re}(q_{b,s}) > 0 \), which ensures that the amplitude of perturbations decays exponentially away from the interface between the two fluids. These are therefore known as “surface modes”. A somewhat technical discussion of the meaning and justification of this branch cut can be found in appendix A.

To proceed, we require a fourth boundary
condition. This is achieved by realizing that the velocity perpendicular to the interface between the fluids causes a spatial displacement in the interface position, from \( x = 0 \) to \( x = h \), and this displacement must be the same when approaching the interface from either side. This is often called “the Landau condition”. Expanding \( h \) in the same Fourier modes as the other perturbed quantities, this results in the first-order equation

\[
u_x |_{x=0} = \frac{\partial h}{\partial t} + (\vec{v} \cdot \nabla) h = i k \left( v_0 |_{x=0} - \frac{\omega}{k} \right) h. \tag{13}\]

Inserting eq. (19) and eq. (14) into eq. (13) from both sides of the interface and then dividing out \( h \) yields the dispersion relation

\[
\frac{(\omega - kv_b)^2}{(\omega - kv_s)^2} = -\frac{\rho_s q_b}{\rho_b q_s}. \tag{14}\]

In the incompressible limit, the speed of sound goes to infinity in both media, and therefore from eq. (15), becomes

\[
\omega = \frac{\rho_s q_b + \rho_b q_s}{\rho_b + \rho_s}k \pm i \sqrt{\rho_s \rho_b |v_s - v_b|} / k. \tag{15}\]

Since the growth rate cannot depend on the frame of reference, we analyse the general case in the frame where the background is static, \( v_0 = 0 \), and the stream is moving with velocity \( v_s = V \). Furthermore, we define unitless variables

\[
\varpi = \frac{\omega}{kV}, \quad \delta = \frac{\rho_s}{\rho_b}, \quad M_{b,s} = \frac{V}{c_{b,s}}. \tag{16}\]

Here, \( \varpi \) is the phase velocity in units of the stream velocity, \( \delta \) is the density contrast between the stream and the background, and \( M_{b,s} \) are the Mach number of the stream velocity with respect to the background and the stream itself, respectively. Since pressure equilibrium is assumed, \( M_b = \sqrt{\delta} M_s \). In this notation, the dispersion relation for the incompressible sheet, eq. (15), becomes

\[
\varpi = \frac{\delta}{1 + \delta} \pm \frac{\delta^{1/2}}{1 + \delta}. \tag{17}\]

By further defining

\[
Z = \frac{1}{\delta} \left( \frac{\varpi}{\varpi - 1} \right)^2 \left( \frac{1 - \delta M_s^2 \varpi}{1 - M_b^2 \varpi^2} \right)^{1/2}, \tag{18}\]

the dispersion relation for the compressible sheet, eq. (14), becomes

\[
Z = 1, \tag{19}\]

an algebraic equation for the unknown \( \varpi \). We learn from the equation that \( \varpi \) depends only on \( \delta \) and \( M_b \), with no dependence on \( k \). This implies that \( \omega \propto V k \), which could have been predicted from dimensional analysis, since the only length scale in the problem is the perturbation wavelength.

By squaring both sides of eq. (19), inserting eq. (18)and rearranging, we get a sixth degree polynomial equation in \( \varpi \) that can be factored as the product of a quadratic with a quartic

\[
\left[ \delta (\varpi - 1)^2 - \varpi^2 \right] \left[ \delta (\varpi - 1)^2 - M_b^2 \varpi^2 - 1 - \varpi^2 \right] = 0. \tag{20}\]

A detailed analysis of this equation is presented in [13]. We summarize the main points below. The two roots of the quadratic part and two of the four roots of the quartic part are always real, and do not solve eq. (19). Rather they are solutions to the equation \( Z = -1 \), arising from the fact that we squared eq. (19). At low Mach numbers, the two remaining roots of the quartic part are complex conjugates, representing a growing unstable mode and a decaying mode. Both of these are indeed solutions to eq. (19), with \( Z = 1 \). Eq. (19) thus admits only two solutions, a growing and a decaying mode, as in the incompressible limit (eq. 17). However, above a critical Mach number these two complex roots become real as well (while still solving \( Z = 1 \)), and the dispersion relation admits only stable solutions. For \( M_b \gg 1 \) the solutions converge to

\[
\varpi_\infty = M_b^{-1}, \quad 1 - M_s^{-1}, \tag{21}\]

which represent waves with phase velocities \( \omega / V = c_{b,s}, V - c_s \).

The critical Mach number for stability is given by

\[
M_{\text{crit}} = \left( 1 + \delta^{-1/3} \right)^{3/2}. \tag{22}\]

This generalizes the result of Landau (1944), who showed that for identical fluids, with \( \delta = 1 \), the flow is stable above \( M_{\text{crit}} = \sqrt{8} \approx 2.83 \). For \( \delta = 10 \) and 100, \( M_{\text{crit}} \approx 1.77 \) and 1.34 respectively.

The analytic expression for the growing mode
solution as a function of $M_b$ and $\delta$ can be found by finding the roots of the quartic polynomial in eq. (20) and picking the complex root with the positive imaginary part. However, the full expression is very long and intractable. We show the growth rate of the imaginary part. However, raising $M_b$ from 0 to a relatively small value causes the growth rate to become larger, meaning the system becomes more unstable. However, raising $M_b$ further to larger values causes the growth rate to decline, until it reaches zero at $\delta_0$, shown by the white curve.

Some intuition as to why the sheet becomes stable to linear perturbations at high Mach numbers can be gained by considering what happens when the initially flat interface is perturbed with a sinusoidal displacement. Upstream of each “crest”, the fluids are set to collide, creating a high pressure area, while downstream the fluids are set to separate, creating a low pressure area (see Fig. 2). The flow that develops in response to this pressure perturbation tends to increase the perturbation amplitude. The typical timescale for this process to occur is the sonic time across a perturbation wavelength, $\lambda/c$. However, if the flow is sufficiently fast with a high Mach number, this becomes very long compared to the relevant timescale for the steady state flow, $\lambda/V$. In this case, the fluid upstream does not have time to react to the displacement of the interface, colliding with the crests rather than flowing around them and suppressing the instability.

By inserting the growing mode solution into eq. (12) and then into eq. (11) we obtain the spatial form of the pressure perturbation. This is shown in Fig. 2 for the case $M_b = 1.0$ and $\delta = 100$. We have normalized the perturbation by its maximum amplitude $A$, so that it is unity at $x = 0$. For reference we also show the expected form of the perturbed interface, with an amplitude $h = 0.025\lambda$ where $\lambda$ is the perturbation wavelength. The pressure perturbation decays rapidly with distance from the interface because $q_{b,s}$ are nearly real. This is a general feature of surface modes. The differences in penetration depth and propagation angle between the two fluids are caused by differences in the real and imaginary parts of $q$ respectively between the two fluids. In the language of acoustic waves, this is caused by a change in the acoustic impedance of the two fluids.

2.3 The Planar Slab

We now consider a three zone problem, which we refer to as the slab. The slab is confined to the region $|x| < R_s$ and is infinite in the $y$ and $z$ directions. We refer to the fluid at $|x| > R_s$ as the background, and assume it to be the same fluid on either side of the slab. As before, we assume each unperturbed medium to have uniform density and velocity. Following the same procedure as in [22] we solve eq. (10) in each region subject to the boundary conditions that $P_1$ vanishes at infinity and is continuous across both slab interfaces. The result is

$$P_1 = \begin{cases} 
A e^{-q_b(x-R_s)} & x > R_s \\
\frac{A \sinh (q_b |x + R_s|) - D \sinh (q_b |x - R_s|)}{\sinh (2q_b R_s)} & |x| < R_s \\
D e^{q_b(x+R_s)} & x < -R_s
\end{cases}$$

(23)

where $A$ and $D$ are two constants of integration.

By applying the Landau condition (eq. 13) at the interfaces, $x = \pm R_s$, we learn that they are not independent. A self consistent solution where both $A$ and $D$ are non-zero requires $A = \pm D$, which in turn gives a relationship between the displacement of the two interfaces from equilibrium $h_{R_+} = h_{R_−}$. The case $A = D$, $h_{R_+} = -h_{R_−}$ corresponds to a symmetric perturbation of the pressure and is called the Pinch mode, hereafter P mode. The other case corresponds to an anti-symmetric perturbation of the pressure and

Figure 2. Pressure perturbation in a sheet with $M_b = 1.0$ and $\delta = 100$, from eq. (11), with the dense fluid on the bottom. The white line represents the perturbation in the interface height, which has an amplitude of $h = 0.025\lambda$. This is a surface mode, which decays exponentially with distance from the interface, because $q_{b,s}$ are nearly real. The wave penetrates deeper into the fluid with lower density, and the angle of wave propagation breaks at the interface between the fluids.

Figure 3. Schematic representation of the two instability modes in the planar slab. The left panel represents the anti-symmetric Sinusoidal (S) mode while the right panel represents the symmetric Pinch (P) mode.
is called the Sinusoidal mode, hereafter S mode. These are shown schematically in Fig. 5.

Inserting these two solutions into eq. (23) gives for the pressure perturbation within the slab

\[ P_{1,s}(x) = A \frac{S(qx)}{S(qx/R_s)}, \]

where \( S(x) = \sin(x) \) or \( \cosh(x) \) for S modes or P modes respectively. The corresponding dispersion relations are

\[ \frac{(\omega - kv_s)^2}{(\omega - kv_b)^2} = -\frac{\rho_b}{\rho_s} q_b T(q_b R_s), \]

where \( T(x) = \tanh(x) \) or \( \coth(x) \) for S modes or P modes respectively.

To simplify eq. (25), we again move into the frame where the background is static and the slab velocity is \( v_s = V \), and rewrite the equation in unitless form, using eq. (16) and

\[ K = kR_s. \]

The result is

\[ Z = T \left( [1 - \delta M_b^2 (\omega - 1)^2]^{0.5} K \right), \]

where \( Z \) is defined in eq. (13). This should be compared to the dispersion relation for the sheet, \( Z = 1 \).

The dispersion relations for S and P-modes can be written as a single equation by inverting eq. (27) and writing \( K \) as a function of \( \omega \)

\[ K = 0.5 \left( [1 - \delta M_b^2 (\omega - 1)^2]^{0.5} \right) (\alpha + i \beta) \]

\[ \alpha = \ln(|1 + Z|) - \ln(|1 - Z|) \]

\[ \beta = \arg(1 + Z) - \arg(1 - Z) + n\pi \]

where \( n \) is any whole number, odd for P-modes and even for S-modes, and \( \arg(1 \pm Z) \) is between \( -\pi \) and \( \pi \) due to our chosen branch cut (see §4).

Eqs. (28a), (28b) and (28c) can be used interchangeably as the dispersion relation for the compressible slab. From eqs. (28), we learn that the slab solutions exhibit a qualitatively different behaviour than the sheet, for two reasons. Firstly, in the sheet, \( \varpi \) was independent of \( k \) which resulted in the scaling \( \varpi \propto k \). On the other hand, in the slab, \( \varpi \) depends explicitly on \( K \). This is due to the additional length scale in the problem, the slab width, and will lead to a non-trivial dependence of the growth rate on wavenumber. Secondly, while eq. (14) admitted only one solution for the growing mode \( \omega(k) \), in slab geometry there can be an infinite number of modes for a fixed wavenumber \( k \), each corresponding to a different value of \( n \) in eqs. (28), arising from the periodicity of tanh for complex arguments. We will discuss this in detail in the following sections, where we begin by examining various limits of the dispersion relation.

### 2.3.1 Incompressible Limit

In the incompressible limit, when in eq. (12), \( q_b \rightarrow k \), the slab dispersion relation, eq. (25), reduces to

\[ \omega_{\pm} = \rho_b \nu_b + T(K) \rho_s v_s \]

\[ \frac{K \pm i \sqrt{T(K) \rho_s |v_s - v_b|}}{\rho_b + T(K) \rho_s}, \]

(29)

It is straightforward to see that this converges to eq. (14) for short wavelengths, \( K >> 1 \). In practice, convergence is achieved for wavelengths \( \lambda \lesssim 3R_s \). At long wavelengths, \( K << 1 \), both modes have \( \omega \propto k^{1/5} \), meaning that the growth rate for the slab decays more rapidly than for the sheet as \( k \rightarrow 0 \). The dashed lines in Fig. 4 show the growth rates (Im(\( \omega \)), left) and oscillation frequencies (Re(\( \omega \)), right) as a function of \( K \) for the incompressible slab with \( \delta = 100 \), in comparison to the compressible slab discussed below.

### 2.3.2 Long Wavelength Limit

In the long wavelength limit, as \( K \rightarrow 0 \), we show in appendix C that \( q_b \rightarrow 0 \) as well. Therefore, \( \tanh(q_b R_s) \approx 1/\coth(q_b R_s) \approx q_b R_s \). This can be used to simplify eq. (27) and expand \( \varpi \) in a power series in \( K \). The result is that both S and P-modes are unstable at long wavelengths for any \( M_b \) and \( \delta \). For this reason, the long wavelength modes are referred to as fundamental modes. To leading order in \( K \), the dispersion relations for the S and P-modes are (see C for the derivation)

\[ \varpi_{S,i} \sim \delta K \pm i(\delta K)^{1/2} \]

(30a)

\[ \varpi_{P,i} \sim 1 \pm i\delta^{-1/2}(1 - M_b^2)^{-1/4}K^{1/2} \]

(30b)

As \( K \rightarrow 0 \), the fundamental S and P-modes approach \( \varpi = 0 \) and \( 1 \) respectively, which result in \( (1 + Z)/(1 - Z) = 1 \) and \( -1 \). These modes thus correspond to \( n = 0 \) and \( n = -1 \) in eq. (28).

It is instructive to compare these solutions to the long-wavelength limit of the incompressible slab (eq. 29). To leading order in \( K \), the fundamental S-mode is identical to the incompressible case. Corrections dependent on Mach number are all higher order in \( K \). On the other hand, the growth rate of the fundamental P-mode is multiplied by a factor \( (1 - M_b^2)^{-1/4} \) compared to the incompressible case. For \( M_b << 1 \), the compressible growth rate is enhanced by a factor \( \sim (1 + 0.25M_b^2) \), while for \( M_b >> 1 \) it is suppressed by a factor \( \sim \sqrt{2M_b^2} \). Thus, for sufficiently high Mach numbers the instability is suppressed, in qualitative similarity to the compressible sheet.

### 2.3.3 Short Wavelength Limit

At short wavelengths, \( K >> 1 \), the slab solution converges to the sheet solution, but it does so in...
different ways depending on the Mach number. We summarize the main points below, providing more details in [14]. We begin by searching for solutions to eqs. (28) where \( K \rightarrow \infty \) and the right-hand-side of eq. (28a) is real, since \( K \) is real by definition in the temporal stability analysis we are performing.

At low Mach numbers, \( M_b \ll 1 \), when the sheet is unstable, we have \( \text{Im}(q_n) \ll \text{Re}(q_n) \), and the asymptotic solution is given by \( Z = 1 \), which is the dispersion relation for the sheet (eq. 18). These are surface modes, decaying exponentially with depth in the slab. It is unsurprising that such modes resemble the sheet, since in the limit \( \lambda << R_s \) we expect the perturbations not to be affected by the slab geometry.

At high Mach numbers, \( M_b > M_n \text{\textsubscript{crit}} \), when the sheet is stable, we have \( \text{Im}(q_n) >> \text{Re}(q_n) \), and the asymptotic solution is given by \( 1 - \delta M_n^2 (1 - \pi)^2 = 0 \). This leads to \( \pi = 1 - M_n^{-1} = \pi_\infty \), which is the high-Mach number (stable) limit of the growing mode in the slab (eq. 21). Physically, modes with \( \text{Im}(q_n) >> \text{Re}(q_n) \) are body modes, which traverse the width of the slab without decaying, and bring the two interfaces into causal contact. As \( \pi \rightarrow \pi_\infty \), \( Z \) goes to zero (eq. 18), so eqs. (28) can be expanded to derive an expression for the asymptotic transverse wavenumber within the slab

\[
b_n R_s = [1 - \delta M_n^2 (\pi - 1)^2]^{1/2} K \simeq \frac{(n + 2)\pi}{2}. 
\]

Note there is an extra \( 2\pi \) here compared to eqs. (28), because as \( K \) is increased from 0 to \( \infty \), \( (1 + Z)/(1 - Z) \) completes a full revolution about the origin in the complex plane, while its argument is defined in the range \( (-\pi, \pi) \) (14). These represent standing waves within the slab, with wavelengths \( \lambda_{\perp} = 4R_s/(n + 2) \). So the \( n \)-th mode has \( n + 1 \) nodes across the slab width of \( 2R_s \), which can be seen qualitatively in Fig. 4. The slab acts as a waveguide for these modes, each of which have phase velocity \( \omega/k = V - c_s \). This is a qualitatively new phenomenon compared to the sheet. In 2.3.3, we show that each of these modes is unstable at finite wavelengths, and characterize the instability. Therefore, while each individual mode (each individual \( n \)) converges to the vortex sheet solution at short wavelengths for all Mach numbers, the appearance of higher order unstable modes at shorter and shorter wavelengths renders the slab unstable at all Mach numbers, unlike the sheet which is stable at high \( M_b \).

2.3.4 Unstable Body Modes

At long wavelengths, solutions to the dispersion relation (eqs. 28) exist only for \( n = -1, 0 \). In the incompressible limit, \( M_b \ll 1 \), these are the only two solutions at any wavelength. However, at high Mach numbers, there are an infinite number of body modes solutions at short wavelengths. The questions we need to address are when do these modes appear, whether they are unstable, and what their growth rate is. These are answered in detail in [14] and we summarize the main results below.

From eqs. (18) and (28), we see that when \( \pi = 0 \) the wavenumber of the \( n \)-th mode is \( K_{n,0} = n\pi/(2\sqrt{M_n^2 - 1}) \). Since \( K \) must be real, such solutions are only possible for \( n \geq 1 \) if \( M_n \geq 1 \). This was incorrectly identified by previous authors (e.g. Payne & Cohn 1983; Hardee \& Norman 1988) as the condition for unstable body modes. However, for every \( n \geq 1 \) the solution \( (\pi, K) = (0, K_{n,0}) \) is stable, meaning that solutions to the dispersion relation with \( K \geq K_{n,0} \) have real \( \pi \). A necessary and sufficient condition for body modes to be unstable is not \( M_n > 1 \), but rather

\[
M_{tot} = \frac{V}{c_s + c_b} = \frac{\sqrt{\delta}}{1 + \sqrt{\delta}} M_b > 1. \quad (32)
\]

If \( M_{tot} < 1 \), only the fundamental modes with \( n = -1, 0 \) are unstable, and these modes are surface modes.[5]

When \( M_{tot} > 1 \), the smallest unstable wavenumber (corresponding to the longest unstable wavelength, and hereafter referred to as marginal stability) for the \( n \)-th body mode is well approximated by

\[
K_n \simeq \frac{n\pi}{2\sqrt{\delta(M_n^2 - 1)^2 - 1}}. \quad (33)
\]

Defining \( \kappa \equiv K - K_{n} \), the growth rate of the \( n \)-th body mode near marginal stability scales as

\[
\text{Im}(\pi_n) \propto \delta^{1/4} M_n^3 n^{-3/2} \kappa^{1/2}. \quad (34)
\]

This growth rate diverges strongly with Mach number, which is in contrast to the fundamental modes. Recall that near marginal stability at \( K = 0 \), the growth rate of the fundamental S-mode was independent of \( M_b \), while the growth rate of the fundamental P-mode scaled as \( M_b^{-1/2} \) (eqs. 30).

Since the growth rate of each mode goes to zero as \( K \rightarrow \infty \) (2.3.3), it must reach a maximum at some intermediate \( K \), hereafter referred to as mode resonance. The resonant wavenumber is well approximated by

\[
K_{n, \text{res}} \simeq \frac{n\pi}{2M_{tot}}. \quad (35)
\]

At a given wavelength, the effective growth rate of the slab is determined by the mode with the largest growth rate at that wavelength (see Fig. 4). This growth rate, \( \omega_i = \text{Im}(\omega_i) \), can be written as the limit of an infinite sequence of functions (11)

\[
\omega_{i,1} = t_{sc}^{-1} \ln \left( \frac{4M_{tot} \sqrt{\delta}}{1 + \sqrt{\delta}} K \right) \quad (36a)
\]

\[
\omega_{i,j} = \omega_{i,1} - t_{sc}^{-1} \ln \left( t_{sc} \omega_{i,j-1} \right), \quad (36b)
\]

where \( \omega_i = \lim_{j \rightarrow \infty} \omega_{i,j} \) and

\[
t_{sc} = 2R_s/c_s \quad (37)
\]

is the slab sound crossing time. In practice, the sequence converges by \( j = 3 \) even for relatively low values of \( K \) (Fig. 11). In the asymptotic limit \( K \rightarrow \infty \),

5 Note that when \( \delta << 1 \), which was the regime studied by Payne \& Cohn (1983); Hardee \& Norman (1988), \( M_{tot} \sim M_b \).
\[ \omega \propto \ln(k) \]

Hence, the effective growth rate for KHI in a compressible slab diverges logarithmically with wavenumber. In contrast, the growth rate for an incompressible sheet or slab diverges linearly with wavenumber, \( \omega_1 \propto k \), while the compressible sheet becomes stable at high Mach numbers, \( \omega_1 = 0 \). The scaling of \( \omega_1 \propto \ln(k) \) for the effective growth rate of the compressible slab is in some sense a compromise between these two extremes, though recall that each individual mode does stabilize as \( k \to \infty \).

At resonance, \( q_0 \approx q_s \) (41). This means that the penetration depth of the perturbation, \( \Delta_{b,s} = 1/\text{Re}(q_{b,s}) \), is comparable in both the background and the slab. Furthermore, the propagation angle of the perturbation wave with respect to the normal to the slab, given by \( \cot(\theta_{b,s}) = \text{Im}(q_{b,s})/k \), is the same in both media (see Fig. 5 panels marked \( n = 1 \) and \( n = 2 \)). At resonance we have \( \sin(\theta_{b,s}) \approx M_{\text{tot}}^{-1} \), commonly referred to as the Mach angle.

The physical origin of body modes in a slab can be understood in the following way. When a perturbation is excited, the two interfaces between the slab and the background initially behave as independent sheets, only coming into causal contact once a slab sound crossing time has elapsed. At high Mach numbers, surface modes are stable and the perturbation does not grow in amplitude, but rather results in acoustic waves propagating between the two interfaces, being reflected off of and transmitted through them. The pressure perturbation within the slab (eq. 24) can be written as the sum of an incident and a reflected wave, with wavenumber \( q_s \), while the pressure perturbation in the background can be thought of as a transmitted wave, with wavenumber \( q_0 \). At certain critical incident angles, there is constructive interference between waves emanating from different points along the slab, assumed to have infinite extent. The pressure perturbation is

\[ 2 \pi t_{\text{sc}} = \frac{2 \pi R_s}{\lambda} \]

\[ \omega_R \cdot 2R_s/c_s = 2\pi t_{\text{sc}}/t_{\text{period}} \]

Figure 4. Numerical solution to the slab dispersion relation, eq. (24), for \( M_b = 1.5 \) and \( \delta = 100 \). For this choice of \( M_b \) and \( \delta \), the sheet is stable (Fig. 4), while the slab is unstable through body modes. The left-hand panel shows the growth rate, \( \omega_1 \), normalized by the inverse of the sound crossing time in the slab, \( t_{\text{sc}}^{-1} = c_s/(2R_s) \), i.e. the ratio of the sound crossing time to the Kelvin-Helmholtz time. The right-hand panel shows the oscillation frequency of the wave, \( \omega_R \), normalized by \( t_{\text{sc}}^{-1} \). The solid black and beige lines show the fundamental (compressible) S and P-modes respectively, while the dashed lines show the corresponding solutions for the incompressible slab (eq. 29). The coloured lines show the \( n = 1 - 40 \) modes (\( n = 1 \) in green, \( n = 2 \) in blue, and so on, with even/odd \( n \) representing S/P-modes respectively). At long wavelengths, the two fundamental modes are similar to their incompressible counterparts. However, at \( K > 1 \) the growth rates for the fundamental modes decay while the incompressible modes diverge as \( \omega_1 \propto k \). Modes with \( n \geq 1 \) are excited at finite wavenumbers that scale linearly with \( \lambda \), reach a maximum growth rate at resonance, and then decay at large \( K \). The vertical dotted lines mark, from left to right, wavelengths of \( \lambda = 2R_s \), \( R_s \) and \( 0.5R_s \), where the dominant modes are \( n = 2 \), 4 and 9 respectively. The ridge line connecting the maximal growth rates of each mode diverges logarithmically, and is well fit by \( \omega_{1.3} \) from eq. (29), shown by the thick, dash-dotted line. As each mode stabilizes, its phase velocity converges to \( \omega_{1.3}/k = v - c_s \) (right panel).
Figure 5. Pressure perturbation in a slab with $M_b = 1.5$ and $\delta = 100$ normalized by its maximal value at the interfaces, $A$ (eq. 23), for the first 6 unstable modes. The longitudinal wavelength of the perturbation (along $z$) is equal to the slab diameter, $\lambda = 2R_s$, and the amplitude of fluid displacement at the interfaces is $h = 0.025\lambda$, which is shown by the white curves. The top row shows the first three P-modes: $n = -1$ (the fundamental mode, left), $n = 1$ (centre) and $n = 3$ (right). The bottom row shows the first three S-modes: $n = 0$ (the fundamental mode, left), $n = 2$ (centre) and $n = 4$ (right). All unstable modes for this case are body modes, that penetrate to large depths in both the slab and the background. The number of transverse nodes within the slab (along $x$) is $(n + 1)$, creating a more complex standing wave pattern as $n$ increases.

2.3.5 Numerical Solution

We here summarize, in Fig. 4 and Fig. 5, all the features of unstable body modes in the slab derived above. Figure 4 shows a numerical solution to the slab dispersion relation, eq. (27), for $\delta = 100$ and $M_b = 1.5$. We show as a function of wavenumber, $K = kR_s$, the growth rate of the perturbation, $\omega_I = \text{Im}(\omega) = t_{KH}^{-1}$ (left), and the oscillation frequency of the wave, $\omega_r = \text{Re}(\omega) = 2\pi t_{\text{period}}^{-1}$ (right). In both cases, we normalise the frequency by the inverse sound crossing time in the slab, $t_{sc}^{-1}$. We show the two fundamental modes $(n = -1, 0)$ and the modes $n = 1 - 40$ (odd/even for P/S modes). We also show the solutions for the incompressible slab for comparison. Note that for this choice of $M_b$ and $\delta$, the sheet is stable (Fig. 1), while the slab is unstable due to body modes.

At long wavelengths, $K \ll 1$, only the fundamental modes are unstable, and their behaviour is similar to the corresponding incompressible solutions. Higher order unstable modes are gradually excited at shorter and shorter wavelengths, according to eq. (33). Each mode reaches a maximal growth rate at a resonance wavelength (eq. 35), and these dominate over the fundamental modes at intermediate and short wavelengths. At short wavelengths, each mode stabilizes as $\omega \rightarrow \omega_\infty = 1 - M_b^{-1}$, so that $\omega \rightarrow (V - c_s)k$. However, the ridge line formed by the peak resonant growth rates acts as an effective growth rate for the slab, which is always unstable. The effective growth rate of this ridge line is well fit by $\omega_I \propto t_{sc}^{-1}\ln(K)$ (eq. 36).

Figure 5 shows the spatial structure of the pressure perturbation in the $xz$ plane, $P_1$, for the first 6 unstable modes with $\delta = 100$ and $M_b = 1.5$. The longitudinal wavelength is equal to the slab diameter, $\lambda = 2R_s$, and the displacement amplitude of the fluid interfaces is $h = 0.025\lambda$ (shown in white). To compute the transverse wavenumbers, $q_{bs}$, we insert the numerical solutions to the dispersion relation (Fig. 4) into eq. (12). The top row shows the first three P-modes: $n = -1$ (the fundamental mode), $n = 1$ and $n = 3$ from left to right. The bottom row shows the first three S-modes: $n = 0$ (the fundamental mode), $n = 2$ and $n = 4$ from left to right. Since sheet is stable for these values of $M_b$ and $\delta$, surface modes such as shown in Fig. 2 are stable. All unstable modes, including the fundamentals, are body modes, whose exponential decay length in the transverse direction is comparable to or larger than the slab width. The $n$-th mode has $n + 1$
nodes across the slab width, creating a more complex standing wave pattern as \( n \) increases. For \( \lambda = 2R_s \), the \( n = 1 \) and \( n = 2 \) modes are near resonance, so the angle of wave propagation is nearly the same in the slab and the background. For the other modes, the pattern breaks at the slab interfaces.

### 2.4 The Cylindrical stream

We now consider a flow with cylindrical, rather than planar symmetry. Using the standard cylindrical coordinates, \((r, \varphi, z)\), we assume an equilibrium configuration where the density and flow velocity depend only on \( r \), the flow is in the \( \hat{z} \) direction, and the pressure is constant: \( \rho_0(r), v_0(r) \hat{z}, P_0 \). By rewriting the hydrodynamic equations (eqs. 1 - 5) in cylindrical coordinates, inserting perturbations of the form \( f(r) \exp[i(kz + m \varphi - \omega t)] \) where \( m \) is an integer, and linearising, we obtain analogous expressions to eqs. (6) - (9) that relate the perturbations in density and velocity to the pressure perturbation. We also obtain a second order differential equation for the pressure perturbation, analogous to eq. (10):

\[
P_1'' - \left[ \frac{2\nu'}{v - \omega/k} + \frac{\rho_0'}{\rho_0} - \frac{1}{r} \right] P_1' - k^2 \left[ 1 - \left( \frac{v - \omega/k}{c} \right)^2 + \left( \frac{m}{kr} \right)^2 \right] P_1 = 0
\]

where \( f' = \partial f/\partial r \). This is identical to eq. (10), except for the geometrical terms \( 1/r \) and \( m/(kr) \).

We consider an infinitely long cylindrical stream of radius \( r = R_s \), centred on the \( z \) axis, with density and velocity \( \rho = \rho_0 \) and \( \vec{v} = \vec{V} \hat{z} \). The background, at \( r > R_s \), has \( \rho = \rho_0 \) and \( \vec{v} = 0 \). Eq. (12) reduces to two modified Bessel equations, for \( r < R_s \) and \( r > R_s \). Using the boundary conditions that the pressure perturbation is finite at \( r = 0 \) and \( r \to \infty \) and is continuous at \( r = R_s \),

we obtain the solution

\[
P_1(r) = \begin{cases} A \frac{I_m(q_r)}{I_m(q_s R_s)} & r < R_s, \\ A \frac{K_m(q_r)}{K_m(q_s R_s)} & r > R_s. \end{cases}
\]

\( I_m \) and \( K_m \) are the \( m \)-th order modified Bessel functions of the first and second kind respectively and \( A \) is a constant of integration. By applying the Landau condition at the stream boundary, in analogy to eq. (13), we obtain the dispersion relation

\[
Z = -\frac{I_m \left( \sqrt{1 - \delta M_s^2} - 1 \right) K_m}{I_m' \left( \sqrt{1 - \delta M_s^2} - 1 \right) K_m'}
\]

(40)

Comparing eqs. (39) and (40) to the corresponding equations for the slab, eqs. (23) and (27), we see one qualitative difference between the two configurations. While the slab admitted only two symmetry modes, the symmetric P-modes and the antisymmetric S-modes (Fig. 3), the cylinder admits infinitely many symmetry modes, represented by the index \( m \). Through the Landau condition, which introduces the perturbation to the cylinder surface, we learn that \( m \) is the number of azimuthal modes on this surface. This is shown schematically in Fig. 6 where we show a slice through the \( z = 0 \) plane for the first six symmetry modes, \( m = 0 - 5 \). The \( m = 0 \) modes are axisymmetric pinch modes, analogous to the P-modes in the slab. The \( m = 1 \) modes are antisymmetric helical modes, analogous to S-modes in the slab. Modes with \( m > 1 \) are fluting modes with no direct analogue in the slab, but as we shall see they do not qualitatively change the growth of instabilities at short wavelengths.

#### 2.4.1 Long Wavelength Behaviour

At long wavelengths, \( K \to 0 \), we use the asymptotic form of the modified Bessel functions for \( \xi \ll 1 \) (Abramowitz & Stegun 1963)

\[
I_m(\xi) \approx \begin{cases} 1 - 0.25\xi^2 & m = 0 \\ \frac{1}{(2m!)^{-1}} \xi^m & m \geq 1, \end{cases}
\]

\[
K_m(\xi) \approx \begin{cases} 1.27 - \ln(\xi) & m = 0 \\ \frac{2m-1}{(m-1)!} \xi^{-m} & m \geq 1. \end{cases}
\]

(41)

(42)

Inserting these into eq. (40) results in the leading order dispersion relations

\[
\omega \approx \begin{cases} 1 + i \frac{1}{\sqrt{28}} \sqrt{\ln(\frac{1}{\sqrt{|M_s^2 - 1|} K})} & m = 0 \\ \frac{\delta}{1 + \delta} + i \frac{\sqrt{\delta}}{1 + \delta} & m \geq 1. \end{cases}
\]

(43)

These modes are unstable at all wavenumbers and for all values of \( \delta \) and \( M_s \). They thus represent the fundamental modes for the cylinder. As in the
slab, each symmetry mode has one fundamental mode. Modes with \( m \geq 1 \) all have the same growth rate which is independent of Mach number, similar to the fundamental S-mode in the slab (eq. \( 30a \)). It is fascinating to note that this is exactly the dispersion relation for the incompressible sheet (eq. \( 15 \)). The fundamental \( m = 0 \) mode is suppressed at large Mach numbers for a given \( K \), qualitatively similar to the fundamental P-mode in the slab (eq. \( 30 \)).

### 2.4.2 Short Wavelength Behaviour

At short wavelengths, when \( K \gg m + 1 \), we use the asymptotic form of the modified Bessel functions for \( |\xi| \gg m + 1 \) (Abramowitz & Stegun [1965]):

\[
\mathcal{T}_m(\xi) \propto \begin{cases} 
\xi^{-1/2} e^{\xi} & \text{Re}(\xi) > 0 \\
|\xi|^{-1/2} \cos \left( |\xi| - \frac{(2m + 1)\pi}{4} \right) & |\xi| = i|\xi| \\
\xi^{-1/2} e^{-\xi} & \text{Re}(\xi) < 0 
\end{cases} 
\tag{44}
\]

\[
K_m(\xi) \propto \xi^{-1/2} e^{-\xi}. 
\tag{45}
\]

Note that these approximations become valid at shorter wavelengths for larger \( m \). Inserting these into eq. \( 40 \), we obtain the asymptotic form of the dispersion relation:

\[
Z \approx \begin{cases} 
1 & \text{Re}(q_s) \gg \text{Im}(q_s) \\
\tanh \left( q_s R_b - i \frac{(2m - 1)\pi}{4} \right) & \text{Re}(q_s) \ll \text{Im}(q_s) 
\end{cases} 
\tag{46}
\]

As for slab and sheet geometries, \( \text{Re}(q_s) \gg \text{Im}(q_s) \) at low Mach numbers, when surface modes are unstable. In this case, the dispersion relations for the sheet, the slab and the cylinder all converge to \( Z = 1 \). At higher Mach numbers when surface modes are stable, \( \text{Re}(q_s) \ll \text{Im}(q_s) \), and the dispersion relation for the cylinder becomes very similar to that of the slab. Except for the extra \( -i\pi/4 \) in the argument of the \( \tanh \) function in eq. \( 46 \), with \( m = 0 \) and \( m = 1 \) corresponds exactly to eq. \( 24 \) for P and S modes respectively. We conclude that for any given \( m \), at short enough wavelengths, \( \lambda < 2\pi R_b/(m + 1) \), shorter than any features on the cylinder surface, we are not sensitive to the geometry (planar or cylindrical) and the dispersion relation for the cylinder converges to that of the slab.

Given the similarity of eq. \( 40 \) and eq. \( 24 \) for the short wavelength behaviour of body modes, we can apply our analysis of the slab \( (2.3.4 \text{ and } 19 - 20) \) to the cylinder as well. It is straightforward to see that all our results for the marginally stable wavenumbers (eq. \( 33 \)), the resonant wavenumbers (eq. \( 35 \)), and the asymptotic transverse wavelength across the stream (eq. \( 34 \)), can all be applied to the cylinder under the transformation \( \nu_{\text{slab}} \rightarrow 2\nu_{\text{cyl}} + m - 1/2 \). As in the slab case, the mode number \( n \) represents the number of nodes of the perturbation along the stream width (Fig. \( 5 \)). Most importantly, the effective growth rate of instabilities in the stream, given by the maximal growth rate at each wavenumber (eqs. \( 36 \)), is the same in both the slab and the cylinder.

Figure 6 shows a numerical solution to the cylindrical dispersion relation, eq. \( 10 \), for \( \delta = 100 \) and \( M_b = 1.5 \). We show as a function of wavenumber, \( K = k R_b \), the growth rate of the perturbation, \( \omega = \text{Im}(\omega) = t^{-1}_K \) normalised by the inverse sound crossing time in the stream, \( t^{-1}_s \). This can be directly compared to the corresponding solution for the slab case, shown in the left-hand panel of Fig. 3. The three panels address different azimuthal modes, \( m = 0, 1 \) and 5 as marked. For each \( m \) we show the fundamental mode, \( n = 0 \), and the modes \( n = 1 - 20 \) together with the corresponding incompressible solution. At long wavelengths, \( K < m + 1 \), only the fundamental modes are unstable, and their behaviour is similar to the corresponding incompressible solutions. Higher order unstable modes are gradually excited at shorter and shorter wavelengths, and their overall behaviour is similar to the slab case. Most importantly, the effective growth rate for the cylinder at short wavelengths, defined by the peak line of peak growth rates of each \( n \geq 1 \) mode, is well fit by the same formula as for the slab, namely \( \omega_s \propto t^{-1}_s \ln(K) \) (eq. \( 30 \)).

### 2.4.3 Surface Modes vs. Body Modes

In planar geometry, surface modes become stable when \( M_b > M_{\text{crit}} \) (eq. \( 22 \)), while body modes in the slab become unstable when \( M_{\text{tot}} > 1 \) (eq. \( 32 \)). However, recall that the Mach number was defined using only the component of the velocity parallel to the perturbation wavevector, \( v_k = \hat{v} \cdot \hat{k} \) (eq. \( 14 \)). On the surface of the cylinder, the wavevector is \( \vec{k} = k \hat{z} + (m/R)\hat{\varphi} \), resulting in \( v_k = V(1 + (m/K)^2)^{-1/2} \). Therefore, the effective value of \( M_b \), which is relevant for determining whether surface modes are stable, is reduced by a factor \( [1 + (m/K)^2]^{-1/2} \), which depends both on the azimuthal wavenumber \( m \), and on the perturbation wavelength through \( K \). As a result, at a given wavenumber surface modes will be unstable for azimuthal modes \( m > K[(M_b/M_{\text{crit}})^2 - 1]^{1/2} \). This means that surface modes are formally always unstable for the cylinder, whatever the value of \( M_b \), for large enough \( m \). While this may seem fundamentally different from the slab, it is actually very similar. Recall that we limited our analysis in 2.2.1 to perturbations where \( \vec{k}||\vec{v} \), so that \( v_k = V \). In principle, perturbations in a slab can assume any angle \( \varphi \) with respect to the flow velocity, and surface modes will be unstable so long as \( \cos(\varphi) < M_{\text{crit}}/M_b \).

However, surface tension can stabilize modes with \( m \geq 2 \). where the stream surface is highly perturbed with many small scale features and the surface to volume ratio is high (see Fig. 6). Previous studies have found that the inclusion of magnetic fields parallel to the flow, which act as a form of surface tension, stabilizes surface modes with \( m \geq 2 \) (e.g. Ferrari, Trussoni & Zaninetti [1981]; Birkinshaw [1990]). Furthermore, perturbations with \( m = 0 - 2 \) are likely...
the dominant modes in cold streams in galactic haloes, plausibly seeded by gravitational tidal interactions with satellite galaxies located either inside or outside the streams. We will therefore focus hereafter on low-$m$ modes, where body modes dominate the instability for wavelengths $\lambda \lesssim R_s$.

3 SIMULATION RESULTS

In this section we use numerical simulations to study the growth of perturbations due to KHI in the linear regime. Guided by [2] we do this in two stages. First, we study the evolution of eigenmode perturbations, whose initial spatial structure obeys the linearized equations of hydrodynamics as derived in [2]. This corresponds to a pressure perturbation obeying eq. (11) for a sheet, eqs. (20) and (24) for a slab, or eq. (28) for a cylinder, together with perturbations in the density and velocity obeying eqs. (9) - (9) in planar geometry, or the corresponding equations in cylindrical geometry. In the second stage, we study the evolution of arbitrary non-eigenmode perturbations.

3.1 Numerical Method

We use the Eulerian code RAMSES ([Teyssier 2002]), with a piecewise-linear reconstruction using the MonCen slope limiter ([van Leer 1977]) and a HLLC approximate Riemann solver ([Toro, Spruce & Speares 1994]). Since perturbations with wavelengths $\lambda \lesssim R_s$ in a slab and a cylinder should behave similarly in the linear regime, which is what interests us here, we limit our current analysis to 2D slab simulations. This allows us to achieve higher resolution than would be possible in 3D simulations of cylindrical streams. As highlighted below, the slab simulations also allow us to test our predictions for the sheet.

The simulation domain is a square of side $L = 1$, representing the $xz$ plane, extending from 0 to 1 in the $z$ direction and from $-0.5$ to 0.5 in the $x$ direction. The slab is centred at $x = 0$ with a radius of $R_s = 1/160$, and extends the full domain in the $z$ direction. We use periodic boundary conditions at $z = 0$ and 1, and outflow boundary conditions at $x = \pm 0.5$ (such that gas crossing the boundary is lost from the simulation domain). The slab and the background are both ideal gasses with adiabatic index $\gamma = 5/3$, and initial uniform pressure $P_0 = 1$. The background, at $|x| > R_s$, is initialized with density $\rho_b = 1$ and velocity $\vec{v}_{b,0} = 0$. The slab, at $|x| < R_s$, is initialized with $\rho_s = \delta$ and $\vec{v}_{s,0} = M_b c_b \hat{z}$, where $c_b$ is the sound speed in the background, $(5/3)^{1/2}$ in simulation units. We simulate several different combinations of $\delta$ and $M_b$ (Table 1).

A characteristic time common to all our simulations is the sound crossing time in the background, $t_{box} = L/c_b \approx 0.775$ in our simulation units. The characteristic time for growth of perturbations in the linear regime is the Kelvin-Helmholtz time, $t_{KH}$. This is the inverse of the imaginary part of the frequency, $t_{KH} = \omega_i^{-1} = (K \delta R_s)^{-1}$, and can be expressed as

$$t_{KH} = \frac{2K \delta^{1/2}M_b \omega_i}{c_b} t_{sc}, \quad (47)$$

with $t_{sc} = 2R_s/c_b$ the sound crossing time in the slab (eq. 57). We run each simulation for at least $5t_{KH}$ with 20 outputs per $t_{KH}$. Our use of a thin slab with $R_s << L$ ensures that $t_{KH} << t_{box}$ in all of our simulations.

![Figure 7. Growth rates for the cylinder. Shown are numerical solutions to the cylinder dispersion relation, eq. (40), for $M_b = 1.5$ and $\delta = 100$, the same case shown in Fig. 5 for the slab. The y axis shows the growth rate, $\omega_i$, normalized by the inverse of the sound crossing time in the stream, $t_{sc} = c_s/(2R_s)$, as in the left-hand panel of Fig. 5. The x axis shows the normalized wavenumber $K = kR_s$. We show solutions for $m = 0$ (left), $m = 1$ (centre), and $m = 5$ (right). Dashed black lines show the incompressible solutions. Solid black lines show the fundamental modes, $n = 0$ (one mode for each $m$). Solid red lines show the $n = 1 - 20$ modes for each $m$. Surface modes, where $\omega_i \propto K$, are unstable at wavenumbers $K \gtrsim m + 1$ and the corresponding growth rates are very similar to the incompressible case. At shorter wavelengths, body modes dominate the instability, and the effective growth rate due to the ridge line connecting the mode $m = 100$, the same case shown in Fig. 4 for the slab. The $\delta$ = 100 case was shown in Fig. 4 for the slab. The $\delta$ = 100 case was shown in Fig. 4 for the slab. The $\delta$ = 100 case was shown in Fig. 4 for the slab. The $\delta$ = 100 case was shown in Fig. 4 for the slab. The $\delta$ = 100 case was shown in Fig. 4 for the slab.
is resolved if rapidly in the denser fluid (Fig. 2) the eigenmode perturbation. Since the perturbation decays more means resolving the exponential decay length of the discontinuity to be valid, the eigenmode structure of 2 every 5 grid. The region (AMR) capabilities, we instead use a statically refined smaller than the shearing layer (Robertson et al. 2010).

In order to effectively suppress the grid noise we find that we require $\sigma > 2\Delta$, where $\Delta$ is the cell size.

While RAMSES has adaptive mesh refinement (AMR) capabilities, we instead use a statically refined grid. The region $|x| < 5R_s$ has the highest resolution, with cell size $\Delta$, and the cell size increases by a factor of 2 every $5R_s$ in the $x$ direction until a maximal cell size. In all our simulations, $\Delta = 2^{-15} \approx R_s/205$, and the maximal cell size was $2^{-9}$.

In order for our analytical solution of a sharp discontinuity to be valid, the eigenmode structure must be well resolved. For surface modes, this means resolving the exponential decay length of the perturbation. Since the perturbation decays more rapidly in the denser fluid (Fig. 2) the eigenmode is resolved if $\sigma << 1/\text{Re}(q_s)$, which for $\delta \sim 10 - 100$ corresponds to $\sigma << 0.1 - 0.3 \lambda$. Thus, properly resolving the eigenmodes while at the same time suppressing artificial perturbations requires $\Delta < 0.5\sigma << 0.1\lambda$. In practice, we find that our results are well converged for $\lambda > 30\sigma > 60\Delta$. Body modes are easier to resolve. The smallest length scale we must resolve is the transverse wavelength within the slab, $2\pi/\text{Im}(q_s) \sim 4R_s/(n + 2)$ (eq. 51). For $\lambda \lesssim R_s$, $\delta \sim 10 - 100$ and $M_0 \lesssim 2$, typical of cold streams in haloes (41), the fastest growing mode has $n \lesssim 10$ (eq. 54), yielding the requirement $\sigma << 0.3 \lambda$. Table 1 compares $\Delta$, $\sigma$, and $\lambda$ for each simulation.

7 One source of noise comes from trying to capture a sinusoidal shape of the slab boundary using a finite Cartesian grid, which leads to inaccuracies on the grid scale.

3.2 Eigenmode Simulations

Eigenmodes are simultaneous perturbations of all the fluid variables that self-consistently solve the linearised equations of hydrodynamics. To find the true eigenmodes of the problem, we would have to insert the smoothed density and velocity profiles given by eq. (48) into eq. (49), solve the ODE to find the form of the pressure perturbation, and then insert this into eqs. (1) - (3) to find the corresponding perturbations in density and velocity. However, there is no analytic solution to eq. (10) with the profiles given by eq. (48), and this would not offer a direct test of the growth rates derived in §2. Instead, we approximate the eigenmodes of the smoothed profile as smoothed versions of the eigenmodes corresponding to the “step-function” profile. More precisely, we take the pressure perturbation given by eqs. (23) - (24) and insert this into eqs. (10) - (13) assuming constant density and velocity within each region (slab and background) to find the corresponding density and velocity perturbations. As these are also discontinuous at $x = \pm R_s$, we smooth them using eq. (48) as well. This approximation for the eigenmodes, and subsequently the general discussion of a step-function slab, will be judged by how well the simulations match the predicted evolution of eigenmodes, namely that they grow exponentially in amplitude from $t = 0$ while maintaining their spatial structure.

Based on the analysis in §2 there are three regimes of instability for a slab, depending on the values of $M_{\text{tot}}$ and $\delta$. To see this, it is useful to define

$$M_{\text{tot, crit}} \equiv M_{\text{crit}} \frac{\delta^{1/2}}{1 + \delta^{1/2}} = \frac{(1 + \delta^{1/3})^{3/2}}{1 + \delta^{1/2}}, \quad (49)$$

where $M_{\text{crit}}$ is the critical value of $M_0$ above which surface modes become stable (eq. 22). $M_{\text{tot, crit}}$ is the corresponding critical value of $M_{\text{tot}}$. It is straightforward to show that $M_{\text{tot, crit}} > 1$ for any finite $\delta$. The three regimes of instability are thus:

(i) If $M_{\text{tot}} < 1$, then surface modes are unstable, while body modes are stable. The only unstable modes are the two fundamental modes, $n = -1, 0$.

(ii) If $1 < M_{\text{tot}} < M_{\text{tot, crit}}$, then both surface modes and body modes are unstable. The fundamental modes correspond to unstable surface modes while modes with $n \geq 1$ correspond to unstable body modes.

(iii) If $M_{\text{tot}} > M_{\text{tot, crit}}$, then surface modes are stable while body modes are unstable. All modes including the fundamentals correspond to unstable body modes.

To explore the three regimes, we ran a total of five eigenmode simulations. Relevant parameters of these simulations are listed in Table 1. The case $(\delta, M_0) = (1, 1.5)$ represents the first regime where only surface modes are unstable. The case $(\delta, M_0) = (10, 1.5)$ represents the second regime where both surface and body modes are unstable, and we simulate one of each: the $n = 0$ surface mode and the $n = 2$ body mode. The cases $(\delta, M_0) = (100, 1.5)$ and $(\delta, M_0) = (1, 5.0)$
Table 1. The top five entries correspond to the eigenmode runs of Fig. 8, the bottom three to the non-eigenmode runs of Fig. 8. For each simulation we list the values of $\delta$, $M_b$, and $M_{tot}$, whether it corresponds to a surface (S) or body (B) mode; the mode number, $n$; the ratio of the perturbation wavelength, $\lambda$, to the slab radius, $R_s$, the smoothing scale, $\sigma$, and the smallest cell size, $\Delta$; and the KH time, $t_{KH}$ (eq. 27), in units of the box sound-crossing time, $T_{box}$. For eigenmodes, $\lambda$ and $\lambda/\Delta$ correspond to the seeded mode. For the non-eigenmode runs, they correspond to the fastest growing mode for the given $\delta$, $M_b$, $\lambda$ and mode symmetry (they are all symmetric P-modes).

| $\delta$ | $M_b$ | $M_{tot}$ | $S/B$ | $n$ | $\lambda/R_s$ | $\lambda/\sigma$ | $\lambda/\Delta$ | $t_{KH}/T_{box}$ |
|----------|-------|-----------|-------|----|---------------|-----------------|-----------------|-----------------|
| 1        | 1.5   | 0.75      | $S$   | 0  | 2             | 102             | 410             | 0.004           |
| 10       | 1.5   | 1.13      | $S$   | 0  | 2             | 102             | 410             | 0.009           |
| 10       | 1.5   | 1.13      | $B$   | 2  | 2             | 25              | 410             | 0.038           |
| 100      | 1.5   | 1.36      | $B$   | 2  | 2             | 102             | 410             | 0.053           |
| 1        | 5.0   | 2.50      | $B$   | 4  | 2             | 102             | 410             | 0.006           |

Figure 8 shows the perturbation amplitude as a function of time, normalized by the respective KH time from Table 1 for the five eigenmode simulations. The analytical prediction is that eigenmode perturbations grow exponentially from time $t = 0$, their amplitude scaling as $\exp(\delta t/\lambda)$, which is shown by the dashed line. All simulated modes match the analytically predicted growth rate to within ~ 20% for a period of between $\sim 1.5 - 3t_{KH}$, after which the growth rate saturates due to non-linear effects.

The predicted complex frequency corresponding to each simulated mode, $\omega$, is found by numerically solving the dispersion relation, eq. (24). For surface modes, these are nearly identical to the corresponding values in the sheet given by eq. (26). The Kelvin-Helmholz time, $t_{KH}$, is then calculated using eq. (17), and listed in Table 1. Each simulation was run until time $t = 5t_{KH}$, which corresponds to a different time for each mode. Table 1 also shows the ratio of the wavelength to the smoothing scale, $\sigma$ from eq. (18). In most cases we use relatively narrow smoothing, $\sigma = \lambda/102$, in order not to deviate too far from the step-function slab. However, for the $\delta = 10$ body mode we require a larger smoothing scale, $\sigma = \lambda/25$, to suppress artificial surface modes which have much faster growth rates than the body mode. This is less of an issue in the third regime where surface modes are intrinsically stable.

In each snapshot, we estimate the perturbation amplitude by calculating the average of $|P_1| = |P - P_0| = |P - 1|$ in the high resolution region, $|x| < 5R_s$. We experimented with varying the region within which we average the perturbation between $|x| < R_s$ and $|x| < 5R_s$, calculating the root-mean-squared value of $P_1$ rather than the averaged absolute value, and using the transverse velocity or the displacement of the slab interface rather than the pressure to estimate the perturbation amplitude. These variations change our growth rates by less than ~ 10%.

The first of these two cases has $M_b$ very close to $M_{crit}$ (see Fig. 8) while the second is deep within the third regime.

This was calculated by using a passive scalar, called “color”, to differentiate the slab material from the background.
and found no noticeable effect on the results. We also reran the simulations with the same resolution while varying $\sigma/\Delta$ between 1 and 16. Larger values of $\sigma$ result in slower growth rates compared to the predicted values, especially for surface modes where eigenmodes are harder to resolve (Fig. 1). Smaller values of $\sigma$ bring the simulated growth rates into better agreement with the predictions at early times, but lead to artificial noise dominating the perturbation amplitude and hence the growth rate before the non-linear saturation of the initial seeded mode.

### 3.3 Non-Eigenmode Simulations

We now study the evolution of general perturbations, that are not eigenmodes of the problem. We initialize pressure perturbations which are harmonic along the slab axis and decaying perpendicular to it:

$$P_1 = A \cos(kz) \times \left[ \exp\left(-\frac{(x-R_h)^2}{2\Sigma^2}\right) + \exp\left(-\frac{(x+R_h)^2}{2\Sigma^2}\right) \right], \tag{50}$$

with $A = 0.05$ and $\Sigma = 5\sigma$. The precise value of $\Sigma$, the width of the perturbation, is not important as long as it is larger than $\sigma$, the width of the smoothing layer. We also tried initializing perturbations in the transverse velocity component, and found no qualitative difference in our results. We prefer to focus here on the pressure perturbations, because all simulations, eigenmode and non-eigenmode, have initially uniform pressure in the entire domain, $P_0 = 1$, whereas the slab velocity and sound speed vary between different simulations.

According to the analysis in [2] growing modes are eigenmodes. However, eigenmodes do not span the full range of perturbations, and an arbitrary initial perturbation cannot be decomposed into a linear combination of eigenmodes. Each eigenmode contains perturbations in all fluid variables whose amplitudes are linearly related to one another, while different eigenmodes have different transverse wavelengths. Thus, no linear combination of eigenmodes can result in a perturbation in only one fluid variable (such as the pressure) while leaving the others unperturbed. Before an arbitrary initial perturbation in one of the fluid variables can begin to grow, corresponding perturbations in the other variables will develop. The resulting set of perturbations can be decomposed into a linear combination of eigenmodes, both growing and decaying modes, and a residual perturbation. The residual propagates as a sound wave away from the slab boundary, the decaying eigenmodes decay away, and eventually the growing eigenmodes dominate. The minimal timescale over which the fluid can “arrange” itself into eigenmodes is the wavelength sound-crossing time within the hot medium, $t_\lambda = \lambda/c_s$. At $t > t_\lambda$, after eigenmodes have developed, their amplitudes will begin to grow according to their corresponding growth rates until the fastest growing mode will eventually dominate the instability.

An additional relevant timescale is the slab sound-crossing time, $t_{sc}$. At $t < t_{sc}$, the slab is not coherent, and information regarding a perturbation on one edge will not have reached the opposite edge. In our analysis of the dispersion relation for the slab (and the cylinder), we implicitly assumed both boundaries to be in causal contact, so these solutions can only be applied at $t > t_{sc}$. At earlier times, each boundary must behave as an independent sheet. In the regime where surface modes are unstable, the slab and sheet are practically identical anyway, and perturbations begin to grow at $t \gtrsim t_\lambda$. Body modes, on the other hand, can only begin to grow once the slab is coherent, at $t \gtrsim t_{sc} > t_\lambda$. These are triggered by sound waves reverberating between the slab boundaries, which is why they have been referred to as reflected modes in the literature.

We simulated several configurations, with $(\delta, M_h) = (1,1.5), (10,1.5), (1,5.0)$, similar to our eigenmode runs. Each configuration was run once with a perturbation wavelength $\lambda = 2\pi/k = 2R_h$ (as in the eigenmode runs), and once with $\lambda = \lambda_s$. We focus mainly on the runs with $\lambda = \lambda_s$, summarized in Table 1. There we list the mode number and $t_{KH}$ of the fastest growing mode into which the perturbation can decay given $M_h, \delta$ and $\lambda/\lambda_s$ (surface modes for the first two and a body mode for the third). Since the pressure perturbation we are initiating (eq. 50) is symmetric, it can only decay into P-modes.

![Figure 9. Growth rate of non-eigenmode perturbations in the simulations with wavelengths $\lambda = R_s$, represented by the three solid curves. The x axis shows time normalized by the KH time of the predicted fastest growing mode given $M_h, \delta$, and $K$, according to Table 1. For the two cases shown with $M_h = 1.5$ (green and cyan curves) the fastest growing mode is a surface mode, while for the case with $M_h = 5$ (red line) it is a body mode. The y axis shows the perturbation amplitude normalized by its value at $t = 0$, marked by the dotted line. The slope of the dashed line marks the expected exponential growth (the zero-point has been shifted for clarity). Squares mark the wavelength sound crossing time within the background, $t_k = \lambda/c_s$, after which eigenmodes develop and unstable surface modes begin to grow exponentially. Stars mark the slab sound crossing time, $t_{sc} = 2R_s/c_s$, after which the two sides of the slab come into causal contact and unstable body modes begin to grow exponentially.](image-url)
Figure 10. Transition of a general initial perturbation into the fastest growing eigenmode. The left and centre panels are taken from a simulation of a slab with $M_b = 5.0$, $\delta = 1$, and $\lambda = 2R_s$, showing the pressure perturbation, $P_b$, at time $t = 0$ (left) and $t \sim 1.63t_{sc}$ (centre), after the slab has become coherent. In both panels we list the time in units of the Kelvin-Helmholtz time of the fastest-growing mode, $t_{KH}$, the slab sound-crossing time, $t_{sc}$, and the box sound-crossing time, $T_{box}$. The color scale has been normalized by $P$, the maximal value of the perturbation on the slab interface at $t = 0$. The right-hand panel shows the analytic form of the fastest-growing mode, the $n = 5$ P-mode, with the amplitude normalized to unity at the slab interface. The resemblance of the simulated perturbation to the analytic mode is striking.

Figure 9 shows the perturbation amplitude as a function of time, similar to Fig. 8. The time has been normalized by the $t_{KH}$ corresponding to the fastest growing mode (Table 1). As in Fig. 8 we estimate the perturbation amplitude by the average of $|P_t|$, though we calculate the average in a smaller region, $|x| < 2R_s$, since the initial perturbations were localized on the slab boundaries. However, our results are not strongly dependent on the size of this region, or on whether we use pressure or transverse velocity to estimate the amplitude. For each configuration we have marked the corresponding $t_\lambda$ with squares and $t_{sc}$ with stars. As expected, the two surface modes begin to grow in amplitude at $t \sim t_\lambda$ while the body mode does not grow until $t \sim t_{sc}$. During the growth phase, the growth rates match the predicted growth rate of the fastest growing mode to within $\sim 20\%$, similar to the eigenmode runs.

We also simulated the case $(\delta, M_b) = (100, 1.5)$. Despite surface modes being formally stable for this configuration (Fig. 4), it lies very close to the $M_{crit}$ boundary and small numerical errors seed artificial surface modes. Due to the high density contrast, $t_{sc} >> t_\lambda$ and these artificial modes dominate the instability before the body mode has a chance to grow.

Figure 10 shows the pressure perturbation for the $(\delta, M_b, \lambda) = (1, 5.0, 2R_s)$ simulation at $t = 0$ and at $t \sim 1.6t_{sc}$, shortly after the perturbation begins to grow. For comparison, we show the analytic form of the corresponding fastest growing P-mode, $n = 5$, normalized to the same amplitude. The fastest growing mode can be found from eq. (35) with $K = kR_s = \pi$ and $M_{tot} = 2.5$. The resemblance of the simulation result and the analytic prediction is striking, illustrating that the initial perturbation has evolved into eigenmodes and the fastest growing mode, $n = 5$, dominates.

4 LINEAR STABILITY OF COLD FLOWS IN HOT HALOES

In this section we evaluate the potential importance of KHI in the evolution of cold streams that feed massive galaxies at high redshift. This depends on the ratio of the total time a perturbation can grow before the stream joins the central galaxy, $t_{growth}$, and the Kelvin-Helmholtz time, $t_{KH}$. In the linear regime, the amplitude of perturbations grows as $\exp(t/t_{KH})$, so the number of e-foldings in the perturbation growth is

$$N_{e\text{-}\text{folding}} \equiv t_{growth}/t_{KH} \quad (51)$$

If $N_{e\text{-}\text{folding}}$ = 1, 3, or 10, a small perturbation will grow to roughly 2.7, 20, or $2.2 \times 10^4$ times its initial amplitude. However, as we saw in Fig. 3 the exponential growth does not continue indefinitely. Once the perturbation becomes quasi-linear, the amplitude saturates before continuing to grow linearly with time in a self-similar way (Padnos et al., in preparation). For our purposes here, we loosely refer to cases where $N_{e\text{-}\text{folding}} \gtrsim 3$ as being quasi-linear and to cases where $N_{e\text{-}\text{folding}} \gtrsim 10$ as being non-linear. However, without knowledge of the initial perturbation amplitudes, these thresholds are somewhat arbitrary. All we can say is that if the perturbation begins small, significant growth in the linear regime is a necessary condition for significant growth overall.

We begin by evaluating $t_{growth}$. For haloes of $M_v \sim 10^{12}M_\odot$ at $z \sim 2$, the virial shock radius is roughly the halo virial radius, $R_v$ (e.g. Dekel & Birnboim 2006, Dekel et al. 2009), though we note that in more massive clusters the shocked region can extend to several times $R_v$ (Zinger et al., in preparation). Cosmological simulations indicate that the stream velocity is roughly constant during infall (Dekel et al. 2009, Goerdt et al. 2010, Goerdt & Ceverino 2013), comparable to the halo...
virial velocity,

\[ V \approx V_\nu = \sqrt{\frac{GM_\nu}{R_\nu}}. \] (52)

The travel time of the stream through the shock-heated medium is thus roughly the virial crossing time

\[ t_{\text{infall}} \approx \frac{R_\nu}{V_\nu}. \] (53)

In the EdS regime, valid at \( z \gtrsim 1 \), this is a constant fraction of the cosmological time, \( t_\nu \approx 0.14t_{\text{Hubble}} \) (e.g. Dekel et al. 2013).

As we saw in \( \S 4 \) unstable surface modes begin to grow after the initial perturbation has decayed into eigenmodes, roughly at time \( t_\lambda = \lambda/c_b \). We assume \( \lambda \ll R_\nu \ll R_\nu \) and \( c_b \sim V_\nu \) (see below), so \( t_\lambda \ll t_\nu \) and we can assume that surface modes begin to grow instantaneously, \( t_{\text{growth}} \sim t_\nu \). On the other hand, unstable body modes can only grow after the stream becomes coherent, at time \( t_{\text{sc}} \). Since \( c_b \) can be much slower than \( c_b \sim V_\nu \) for a dense stream, \( t_{\text{sc}} \) can be a significant fraction of \( t_\nu \), and \( t_{\text{growth}} \sim t_\nu - t_{\text{sc}} \). The relation between \( t_{\text{growth}} \) and \( t_\nu \) thus depends on whether surface modes or body modes dominate the instability.

Combining our estimate of \( t_{\text{growth}} \) for surface/body modes with eq. \( \S 4 \) for \( t_{\text{KH}} \), we obtain for the number of e-folding times

\[ N_\nu = 2K_\nu^2/M_\nu \times \max \left[ \frac{t_\nu}{T_{\text{sc}}}, \frac{t_\nu}{t_{\text{sc}}}, \right]. \] (54)

This depends on four parameters: \( M_\nu \), \( \delta \), \( t_\nu/t_{\text{sc}} \) and \( K = \lambda/c_b \). We estimate each of these in turn below.

To estimate \( M_\nu \) for typical cold streams, we make the simplifying assumption that the halo CGM is isothermal. The sound speed in the halo is thus approximately

\[ c_b = \sqrt{\frac{\gamma K_B T_\nu}{\mu m_p}}, \] (55)

where \( K_B \) is the Boltzmann constant, \( T_\nu \) is the virial temperature, \( \mu m_p \) is the mean particle mass, and \( \gamma = 5/3 \) is the adiabatic index of the gas. The temperature can be found from virial equilibrium

\[ \frac{3}{2} K_B T_\nu \simeq \frac{1}{2} \frac{G M_\nu}{R_\nu} \mu m_p = \frac{1}{2} \mu m_p V_\nu^2. \] (56)

Inserting eq. \( \S 3 \) into eq. \( \S 4 \) gives for the Mach number of the stream relative to the background \( M_\nu \simeq V_\nu/c_b \approx 1.34 \). In practice, we assume values in the range \( M_\nu \approx 0.75 - 2.25 \).

To estimate \( \delta \), we again make the simplifying assumption that both the halo CGM and the stream are isothermal. The temperature in the halo is \( T = T_\nu \) (eq. \( \S 4 \)), which for a \( \sim 10^{12} M_\odot \) halo is a few times \( 10^6 \)K. However, due to efficient cooling in the high density streams, they do not support a stable shock at the virial radius (Dekel & Birnboim 2006), and their temperature is set by the cooling curve for low metallicity gas, at a few times \( 10^4 \)K. Assuming pressure equilibrium between the streams and the hot CGM at any given halo-centric radius, this leads to a density contrast of \( \sim 10 - 100 \), consistent with cosmological simulations (e.g. Dcuvirk, Pichon & Teyssier 2008; Dekel et al. 2009; Goerdt et al. 2010).

The ratio \( t_\nu/t_{\text{sc}} \), can be related to the ratio of the stream width to the virial radius,

\[ t_{\text{sc}}/t_\nu = M_\nu^2 2R_\nu/R_\nu. \] (57)

A narrower stream has a shorter sound crossing time, leading to more rapid growth of body modes. Based on cosmological simulations, the cold streams we are discussing have characteristic widths of roughly \( 2R_\nu/R_\nu \sim 0.01 - 0.1 \).

The final ingredient is the ratio of the perturbation wavelength to the stream width, \( \lambda/(2R_\nu) \). Since shorter wavelengths grow faster for both surface and body modes, the instability will be dominated by the shortest wavelengths that can grow. In the idealized problem we studied in \( \S 2 \) perturbations with arbitrarily small wavelengths can be excited and their growth rate diverges. However, in a realistic stream, there is a finite transition layer at the interface between the stream and the background, similar to the smoothing layer introduced in our simulations (\S 3). Perturbations with wavelengths comparable to or smaller than this layer will be damped. The width of this layer is determined by processes such as thermal conduction and viscosity which are not incorporated in cosmological simulations. This is crudely addressed in \( \S 5 \) while a more rigorous study will be the topic of future work. For our purposes, we crudely consider a range of \( \lambda/(2R_\nu) \sim 0.1 - 1 \), corresponding to wavenumbers \( K \sim 3 - 30 \). As we have seen, the growth rate scales linearly with wavenumber for surface modes, \( \omega_i \sim K \), but only logarithmically for body modes, \( \omega_i \propto \ln(K) \).

Figure \( \S 11 \) shows \( N_\nu \) as a function of \( M_\nu \) and \( \delta \) for different values of \( R_\nu/R_\nu \) and \( \lambda/(2R_\nu) \). We show a wide stream with \( 2R_\nu = 0.1R_\nu \) and a stream 10 times narrower. We show a long wavelength perturbation, \( \lambda = 2R_\nu \), and one ten times smaller. When surface modes dominate the instability, we calculate the growth rates, \( \omega_i \), from the sheet dispersion relation (eq. \( \S 20 \)). These are practically identical to the growth rates of surface modes in the slab calculated numerically from eq. \( \S 27 \). When body modes dominate the instability, we use the approximation for the fastest growing mode in a slab or a cylinder given by eqs. \( \S 38 \).

The allowed range for the stream parameters places them near the transition between surface modes and body modes. When surface modes dominate, the stream is highly unstable to KHI in a virial time, and even very small perturbations will become highly non-linear. However, only a very minor change in velocity pushes the stream into the region where body modes dominate

\[ 10 \] In realistic haloes both the stream and the background become denser at smaller radii, making the stream of a conical, rather than cylindrical shape. This is discussed further in \( \S 5 \).
and the growth rate is much slower. Wide streams are marginally unstable, with $N_{\rm e-folding} < 3$. However, narrow streams can be highly unstable even in this regime, with $N_{\rm e-folding} \lesssim 30$. Furthermore, recall that if $m > 1$ in a cylinder, the effective Mach number for determining whether surface modes are still unstable is $M_{h}[1 + (m/K)^2]^{-1/2}$ (eq. 22). Thus for high $m$ modes, unstable surface modes will occupy a larger region of the parameter space, making Fig. 11 only a lower limit on $N_{\rm e-folding}$. However, as stated in (22-3), modes with $m > 1$ are likely to be suppressed in realistic streams, making our estimates of $N_{\rm e-folding}$ reasonable. Overall, we estimate that for the relevant range of parameters, $N_{\rm e-folding} \sim 0.1 - 100$, indicating that KHI can in principle be important for the evolution of cold streams.

### 5 ADDITIONAL PHYSICS

While the analysis presented here has been thorough and accurate, with interesting new understanding of the body modes in the compressible regime, it has also been very simplistic. We have only studied the linear regime of adiabatic, purely hydrodynamic instabilities. This is an important first step, which allowed us to obtain a definitive result. However, as it stands, further, more detailed analysis is necessary. In a forthcoming paper (Padnos et al., in preparation), we will present a detailed study of the non-linear evolution of idealized KHI, without cooling or gravity. Since we have seen that streams can find themselves in the non-linear regime for a wide range of plausible parameters, this is necessary to complete our study of the effect of KHI in the evolution of cold streams. Additionally, we are planning a series of papers in which we will account for additional physical processes one by one, both analytically as in this work, and numerically. Our ultimate goal is to build a comprehensive understanding of the evolution of cold streams, from the bottom up. Below, we outline the additional processes that we intend to address in future work, and their possible effects.

(i) **Cooling:** The importance of cooling in the linear regime of KHI depends on the relation between $t_{\rm KH}$ and the cooling times in the hot and cold media, $t_h$ and $t_c$, respectively. We will always have $t_c < t_h$. In the case of cold streams in hot haloes, the cooling time in the hot halo is longer than the virial time, $t_v < t_h$, which is what allows the presence of a stable virial shock to begin with. Since we are only interested in cases where $t_{\rm KH} < t_v$, we ignore the case $t_h < t_{\rm KH}$.

If $t_{\rm KH} < t_v$, then cooling should not make any difference in the linear evolution of KHI. However, if $t_c < t_{\rm KH} < t_h$, then the cold gas returns to an equilibrium temperature fast compared to the hydrodynamic growth timescale. This is exactly as if the gas were isothermal, and thus all of our analysis stays the same, except that $\gamma = 1$ in the cold medium while $\gamma = 5/3$ in the hot medium. While our analysis implicitly assumed that both media had the same value of $\gamma$, this should only affect the ratio of sound speeds and it seems unlikely that this change will dramatically alter our results for the linear evolution. In the non-linear regime, cooling

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11 This could come up in certain physical circumstances, e.g., near an X-ray source that is capable of keeping the hot gas hot.
can enhance the instability by introducing thermal instabilities in addition to hydrodynamic ones. Once KHI sets in, regions of the stream will become denser and shocks may develop. The cooling rate in these overdense regions may increase causing them to become even denser and creating a runaway process. However, cooling can also weaken the non-linear evolution of the instability. If the instability is dominated by surface modes, then the non-linear evolution occurs via a shearing layer that forms between the hot and cold media, where the two fluids mix. If the cooling time in the cold stream is shorter than the sound crossing time, this region will be confined close to the stream boundary and will not consume the stream interior. This effect has been seen in simulations of cold gas clouds surrounded by a hotter confining medium within the ISM (Vietri, Ferrara & Miniati 1997).

(ii) Thermal Conduction: This can suppress the growth of short wavelength perturbations, by forming a finite transition region between the stream and the background, similar to the smoothing region in our simulations. Thermal Conduction can be treated as a diffusive flow with a diffusion coefficient $D(T, \rho) = \kappa(T)/(\rho c_s)$, where $c_s$ is the specific heat capacity and $\kappa(T)$ is the thermal conductivity. Using Spitzer (1956) to evaluate $\kappa(T)$ under the simplifying assumption that both fluids are pure hydrogen, we obtain $D \sim 6 \times 10^{28} \text{cm}^2 \text{s}^{-1} T_h^{5/2} n_i^{-1}$, where $T_h$ is the gas temperature in units of $10^6 \text{K}$ and $n_i$ is the gas density in units of $10^{-4} \text{cm}^{-3}$. The characteristic time for diffusion to “sweep out” features with a characteristic lengthscale $L$ is $t_{\text{dif}} \sim L^2/D$. To estimate the effect this will have on the growth of surface modes, we compare $t_{\text{dif}}$ over the perturbation wavelength, $L = \lambda$, to the e-folding time for perturbation growth, $t_{\text{KH}}$, as surface modes begin to grow immediately (§4). For short wavelengths, $\lambda < \lambda_c = D/(2\pi M_c \omega_i)$, we have $t_{\text{dif}} < t_{\text{KH}}$ and we thus expect the instability to be suppressed. For $\delta \sim 100$ and $M_6 \sim 1$, we have $\omega_i \sim 0.15$ (Fig. 1). For a virial temperature of $T_v \simeq 10^6 \text{K}$, we have $c_s \sim 1.2 \times 10^5 \text{cm} \text{s}^{-1}$ and the critical wavelength becomes $\lambda_c \sim 1.7 \text{kpc} T_h^{5/2} n_i^{-1}$. In the cold streams we estimate $T_h \sim 0.01$ and $n_i \sim 100$, yielding $\lambda_c \sim 1.7 \times 10^{-4} \text{pc}$. For the shock-heated halo gas, we have $T_h \sim n_i \sim 1$, yielding $\lambda_c \sim 1.7 \text{kpc}$, comparable to the stream radius. However, in the hot gas the effective diffusion coefficient is likely to be reduced from the Spitzer (1956) estimate, since the maximum rate of conductive energy transport cannot exceed $\sim \nu_c K B T$, where $\nu_c$ is the electron thermal velocity, i.e. conduction can’t move thermal energy any faster than a population of free-streaming electrons. This can decrease the critical wavelength by at least an order of magnitude. Regarding body modes, since these are triggered by sound waves reverberating within the stream, the relevant timescale is the stream sound-crossing time. Comparing $t_{\text{sc}}$ to $t_{\text{dif}}$ with $L = 2R_\delta$, $T_h \sim 0.01$ and $n_i \sim 100$, we find heat conduction can suppress the excitement of unstable body modes only if the stream radius is $R_\delta \lesssim 0.1 \text{pc}$. While more accurate conclusions await the the explicit inclusion of heat conduction in the analysis of KHI, it seems that in cold streams only wavelengths much smaller than the stream width will be suppressed.

(iii) External Gravity: We refer here to the underlying gravitational potential of the dark matter halo, which will create a density gradient, and hence a pressure gradient in the background gas halo. As the external pressure increases towards smaller radii, the stream will become narrower towards the halo centre, assuming a conical rather than cylindrical shape. However, we expect our linear analysis as presented in §2 to remain valid for perturbations with wavelengths smaller than the length-scale over which the stream radius changes by order unity. If we make the simplifying assumptions that the halo is an isothermal sphere, with $\rho_h(r) \propto r^{-2}$ and $T_h(r) = \text{const}$, and that the background and stream are in pressure equilibrium locally at each radius $r$, than the stream is also isothermal and hence $c_s$ is constant. Therefore, the conical shape causes the stream sound crossing time to decrease towards the halo centre leading to more rapid growth of instabilities, as $t_{\text{KH}} \propto t_{\text{sc}} \propto R_\delta$. Local pressure equilibrium and the isothermality of the stream imply that the density profile within the stream scales as $\rho_h(r) \propto r^{-2}$, which in turn implies that the stream width scales as $R_\delta \propto r$, i.e. a perfect cone with constant opening angle. If the stream velocity is roughly constant during infall, as indicated by cosmological simulations, this implies that $t_{\text{KH}}(r) \sim R_\delta(r) \propto r \propto t_{\text{infall}}(r)$. So at every radius, the number of e-foldings a perturbation can grow from its current state before the stream reaches the galaxy is constant. Integrated over the lifetime of the stream, this can lead to significantly more growth than predicted in §2. In future work, we will address the dispersion relation of KHI in a conical stream, and perform simulations of streams in an external spherical potential.

(iv) Self Gravity: Self gravity within the stream will have two effects. Firstly, it will result in gradients in density, pressure and perhaps velocity within the stream, as a function of distance from the stream axis. While these gradients will be much weaker than those in the transition zone between the stream and the background, it may still alter the growth rates of perturbations in an appreciable way. Secondly, self-gravity may enhance stream instability by causing overdense regions within the stream to become gravitationally unstable. It is worth noting that a crude estimate of the Jeans scale in the stream is $\lambda_J \sim 2 \text{kpc} c_{10} n_{0.1}^{-1/2}$, where $c_{10}$ is the sound speed in units of $10 \text{km s}^{-1}$, and $n_{0.1}$ is the gas density in units of $0.1 \text{cm}^{-3}$. For $T \sim 10^4$ and $n \sim 0.1$, relevant in the streams near the galaxy, the Jeans length is comparable to the stream width, and could become smaller due to overdensities caused by hydrodynamical and thermal instabilities. We may also witness a fragmentation into sub-filaments, similar to the multi-scale filamentary structure observed in Galactic molecular clouds.
(v) **Magnetic Fields:** While the intra-cluster medium is known to be mildly magnetized (Churazov et al. 2008), little is known about the magnetization state of gas in galactic haloes. Dubois & Teyssier (2010) argued that initial magnetic fields in the ICM originate from winds outflowing from dwarf galaxies, whose IGM contain magnetic fields well below $1 \mu G$ in amplitude. This indicated that the ratio of thermal to magnetic pressure (the plasma $\beta$ parameter) is $\sim 50 - 10^3$. Furthermore, we expect this to be only an upper-limit on the magnetization of the filament gas, which is in general more “pristine” and has not been affected as much by galactic processes. However, some measurements indicate magnetic fields as high as $0.3 \mu G$ in cosmic filaments feeding galaxy clusters (Bagchi et al. 2002). The KHI stability of mildly magnetized gas can be altered from its pure hydrodynamic analogue in three ways. The first is the modifications to the equation of state of the gas, though for a sub-dominant component, this effect is expected to be small. The second effect is surface tension that could possibly arise from the shearing of magnetized gas at the boundaries of the instability. The shearing layer is comprised of gas that has been entrained from the filament gas and from the ambient halo, and could be magnetized even when the filament gas is completely unmagnetized, if the halo is not. Through magnetic draping, the magnetic field lines at the shear layer will become aligned with the flow direction, and the amplitude of the magnetic fields there will increase. This process will stabilize the boundary and could strongly effect the results in the linear phases of the instability. We leave for future work a more robust comparison of the growth rate of magnetic fields in the shear layers compared to the growth rate of the instability and their dependence on the initial hydrodynamic perturbation and initial magnetic fields. The third aspect of magnetic fields is its effect on conduction perpendicular to the shearing layer and magnetic draping that is expected there, which could drastically reduce the coefficients of thermal conductivity and viscosity discussed above.

(vi) **Galaxy Formation:** Even after accounting for all the effects mentioned above, the analysis will still be idealized. In order to address the stability of realistic cold streams feeding galaxies from the cosmic-web, we must account for the presence of additional merging galaxies within the streams, interaction between several streams within the same halo, interaction between streams and feedback induced outflows from the central galaxy, and possible star-formation and feedback within the streams themselves. The best way to account for all these effects is with a fully cosmological simulation. In parallel to the methodical work outlined above, we are experimenting with running cosmological zoom-in simulations using **RAMSES**, with refinement based on gradients in density and velocity, rather than the density based quasi-Lagrangian strategy typically used (Roca-Fabrega et al., in preparation). This will allow us to resolve the streams better than any existing cosmological simulation. As described in §5.1 to properly resolve the the instabilities in cold streams, the cell size must be at least $\sim 60$ times smaller than the perturbation wavelength. For $\lambda \sim R_s \sim 1 \text{kpc}$, this corresponds to a resolution of $\lesssim 15 \text{pc}$ within the cold streams, which will be challenging to achieve. Such simulations will also allow us to gauge the effect of initially large instabilities on the streams, such as caused by the collision of a stream with a satellite galaxy in the halo. An idealized study of such a collision suggests that while the stream is initially destroyed, it reforms within $\lesssim 0.3$ virial crossing times (Wang et al. 2014), after which our linear analysis is again relevant.

### 6 SUMMARY AND CONCLUSIONS

We have presented a detailed analysis of linear Kelvin-Helmholtz instabilities for fully compressible fluids in three different geometries: a sheet, a slab and a cylindrical stream, confirming our analytical predictions using numerical simulations with **RAMSES**. We then applied our results to the problem of cold streams that feed massive SFGs at high redshift, showing that KHI can be important in their evolution. For a large region of allowed parameters, the number of $e$-foldings of growth experienced by a linear perturbation is between 10-100. However, the estimated range of parameters overlaps the phase transition between rapidly growing surface modes and more slowly growing body modes. As a result, perturbations may still remain linear by the time the stream reaches the central galaxy, with less than one $e$-folding of growth in a virial crossing time. The linear analysis of KHI in the adiabatic limit thus indicates that it can be relevant for the evolution of cold streams, but it cannot definitely assess its importance. Our main results can be summarised as follows:

(i) For a sheet, KHI is suppressed at high Mach numbers, $M_b > M_{\text{crit}} = (1 + \delta^{1/3})^{3/2}$, with $M_b = V/c_b$, the fluid velocity normalized by the sound speed in the hot medium, and $\delta = \rho_h/\rho_b$ the density contrast between the fluids. At lower Mach numbers the growth rate for instabilities scales linearly with the wavenumber, $\omega_i \propto k$, and the perturbations themselves decay exponentially with distance from the interface between the fluids.

(ii) At low Mach numbers, when the sheet is unstable, surface modes dominating the instability in a slab, which behaves similarly to a sheet for wavelengths comparable to or shorter than the slab width.

(iii) At high Mach numbers, $M_{\text{tot}} = V/(c_s + c_b) > 1$, the slab remains unstable despite surface modes having stabilized due to the appearance of body modes at shorter and shorter wavelengths. These modes are triggered by waves being reflected off the slab boundaries and are qualitatively different than the surface modes present at low Mach numbers. They penetrate the width of the slab, and resemble standing waves propagating through a waveguide. There is an infinite set of these body modes, characterized by a mode number, $n = 1, 2, 3, \ldots$, representing the number of nodes in the transverse direction. Each mode becomes
unstable at a finite wavelength, its growth rate increases towards a maximum at a resonant wavelength, and then decays to zero as $\lambda \to 0$. The effective growth rate of instabilities in the slab is given by the sequence of resonant growth rates for each mode. This is inversely proportional to the slab sound crossing time and scales logarithmically with wavenumber, $\omega_i \propto t_{\text{sc}}^{-1}\ln(kR_s)$.

(iv) The only qualitative difference between a slab and a cylinder is that a cylinder has an infinite sequence of symmetry modes, characterised by an azimuthal and a cylindrical geometry. When the slab is perturbed with an eigenmode of the problem, the fastest growing mode for the given $\delta$, $M_b$ and $\lambda/R_s$ dominates the instability.

(vi) The allowed range of parameters for cold streams in massive galaxies at high-$z$ is near the transition between surface modes and body modes. Thus, even minor variations in the stream parameters within their realistic ranges can have large effects on the growth rate of instabilities. For a realistic range of stream parameters, the number of $e$-foldings of growth within a virial crossing time can range from roughly 0.1 to 100. This implies that KHI could in principle have an important role in the evolution of cold streams, and a study of the non-linear phases of the instability and the effects of additional physical processes is well motivated.

(vii) When non-linear effects become important, the perturbation amplitude saturates at first, before continuing to grow linearly with time via the mergers of vortices. In an upcoming paper (Padno et al., in preparation) we will address this process in detail, both analytically and using simulations.

(viii) Rough, order of magnitude estimates suggest that heat conduction and magnetic fields should not drastically alter the linear analysis presented here. On the other hand, the gravitational potential of the halo may enhance the instability by causing the stream to become narrower closer to the halo centre, decreasing the sound crossing time and thus increasing the growth rate of KHI. Cooling is unlikely to drastically affect the linear evolution, but can have significant effects in the non-linear regime. In future work we will address all these effects one by one in more detail, while in parallel we will study the effects of galaxy formation on cold streams using cosmological simulations with tailored refinement in the streams.

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APPENDIX A: BRANCH CUTS OF THE SOLUTIONS

Since the generalized wavenumbers \( q_{b,s} \) (eq. 12) are given by the square root of a complex number, we must define a branch cut in the complex plane. As mentioned in (2.2), we have chosen to have \( \text{Re}(q_{b,s}) > 0 \). This is equivalent to defining arguments of complex numbers between \( -\pi \) and \( \pi \). If the argument of \( q_{b,s}^2 \) is \( 2\alpha \in (-\pi, \pi) \), then the argument of \( q_{b,s} \) is \( \alpha \in (-\pi/2, \pi/2) \) and we have \( \text{Re}(q_{b,s}) > 0 \). This ensures that the amplitude of the perturbation decays, rather than grows, away from the interface between the fluids. However, in order to understand how waves generated by the instability propagate away from the interface, we must also determine the sign of \( \text{Im}(q_{b,s}) \). We limit our discussion here to forward travelling growing modes, i.e. modes with \( k \), \( \text{Re}(\omega) \), \( \text{Im}(\omega) > 0 \).

We define \( R \exp(i\theta) = (\omega - kv)/c \). If \( \text{Re}(\omega) < kv \), then \( \pi/2 < \theta < \pi \) and \( q^2 = (k^2 - R^2\cos(2\theta)) - i (R^2\sin(2\theta)) \) has a positive imaginary part. Therefore, if the argument of \( q \) is \( \alpha \), we have \( 0 < 2\alpha < \pi \Rightarrow 0 < \alpha < \pi/2 \), which gives \( \text{Im}(q) > 0 \). On the other hand, if \( \text{Re}(\omega) > kv \), similar arguments yield \( -\pi/2 < \alpha < 0 \), so that \( \text{Im}(q) < 0 \).

In all cases presented in the text, we assumed that \( v_b = 0 \) and \( v_s = V \), and found \( 0 < \text{Re}(\omega) < kV \). This means that \( \text{Im}(q_b) < 0 \) and \( \text{Im}(q_s) > 0 \).
APPENDIX B: SOLVING THE DISPERSION RELATION OF THE COMPRESSIBLE SHEET

In this appendix we examine the 6 solutions to eq. (20) and explain why only 2 of them are solutions to the dispersion relation of the compressible sheet (eq. 19).

The two solutions to the quadratic part of the equation are $\omega^2 = \delta(1 - \omega)^2$. Both of these result in $q_0 = q_s$ and in $Z = -1$. Therefore, neither of these are solutions to the dispersion relation, but only come about because eq. (19) was squared.

Attempting to find tractable analytic expressions for the four solutions to the quartic part of eq. (20) were unsuccessful. However, analytic insight can still be gained by examining its discriminant. Given the general quartic equation

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e,$$  \hspace{1cm} (B1)

the discriminant is given by

$$\Delta_0 = 256a^3e^3 - 192a^2bde^2 - 128a(b^2e^2 - 27a^2) + 144a^2c^2 - 6ab^2d^2 - 80ab^2de + 18ab^2c^2 + 16ac^2e - 4ac^3d^2 - 27b^2e^2 + 18b^3de - 4b^3d^3 - 4b^3e^2 + b^2c^2d^2$$

When $\Delta_0 < 0$, the quartic has two real roots and two complex conjugate roots (Rees 1922). For the quartic part of eq. (20) we have

$$\Delta_0 = 16M_b^2\delta^4 \cdot [\delta^3 M_b^6 - 3\delta^2 (1 + \delta) M_b^4 - 3\delta (\delta - 1) M_b^2 - (\delta + 1)^3].$$  \hspace{1cm} (B3)

It is straightforward to show that $\Delta_0 < 0$ if and only if $M_b < M_{crit}$, where $M_{crit}$ is given by eq. (22). Since the solution must converge to that of the incompressible sheet (eq. 17) for $M_b << 1$, and since this admits only two complex conjugate solutions, we deduce that the two real solutions to the quartic part of eq. (20) at $M_b < M_{crit}$ are not solutions to the dispersion relation, $Z = 1$. Rather they must also be solutions to $Z = -1$, similar to the two roots to the quadratic part. We discuss this further below.

For $M_b > M_{crit}$ we obtain $\Delta_0 > 0$ and the nature of roots depends on the signs of two additional parameters,

$$\Delta_1 = 8ac - 3b^2,$$  \hspace{1cm} (B4)

$$\Delta_2 = 64a^3e - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4.$$  \hspace{1cm} (B5)

If both $\Delta_1 < 0$ and $\Delta_2 < 0$ when $\Delta_0 > 0$, then all four roots are real and distinct. Otherwise the four roots are two pairs of complex conjugates (Rees 1922). For the quartic part of eq. (20) we have

$$\Delta_1 = -\frac{4M_b^6 (2 + \delta (2 + M_b^2))}{\delta^2} < 0,$$  \hspace{1cm} (B6)

$$\Delta_2 = -\frac{16M_b^6 (1 + \delta)(1 + \delta + 26M_b^2)}{\delta^4} < 0.$$  \hspace{1cm} (B7)

Since $\Delta_1$ and $\Delta_2$ are always negative, all four roots are real when $M_b > M_{crit}$, and the sheet is stable.

When $M_b >> 1$, asymptotic solutions can be found for the quartic part of eq. (20),

$$\omega \simeq \pm \frac{1}{M_b}, 1 \pm \frac{1}{M_s},$$  \hspace{1cm} (B8)

where $M_s = \delta^{1/2} M_b$. Note that all four of these solutions result in $q_b$ and $q_s$ that are purely imaginary.

When $\omega = -M_b^{-1}$ we have $\omega < 0$ and when $\omega = 1 + M_b^{-1}$ we have $\omega > kV$. Therefore, following the discussion in (24) $q_b$ and $q_s$ will have the same sign, so $q_b/q_s > 0$ and $Z < 0$. Therefore, these cannot be solutions to $Z = 1$, but must yield $Z = -1$. We conclude that these are the high-Mach number limits of the two roots that were real even for $M_b < M_{crit}$, which we already determined were not true solutions to the dispersion relation. The other two solutions, $\omega = M_b^{-1}, 1 - M_b^{-1}$ both result in $0 < \omega < kV$ and therefore $q_b/q_s < 0$ and $Z > 0$, meaning that $Z = 1$. Thus, these are the high-Mach number limits of the true solutions to the dispersion relation.

In Fig. B1 we show numerical solutions to the quartic part of eq. (20) as a function of Mach number, $M_b$, for different values of the density contrast, $\delta$. For the two complex conjugates at low Mach numbers, representing the growing and decaying unstable modes, we show only the real part which is the same for both solutions (the complex part is shown in Fig. 1). At $M_b > M_{crit}$, these modes stabilize and become two independent travelling waves. This can be seen as the bifurcation point between the red and black lines in the figure. The other two solutions are always real, and can be thought of as a forward travelling wave with $\omega > 0$ and a backward travelling wave with $\omega < 0$. However, as discussed above, these are not solutions to the dispersion relation. At high Mach numbers, the four solutions to the quartic are well approximated by the asymptotic expressions in eq. (28), shown with dashed lines. These represent stable sound waves with phase velocities $\omega/k = v \pm c_s, \pm c_b$.

APPENDIX C: LONG WAVELENGTH LIMIT OF THE COMPRESSIBLE SLAB

In this section we derive the growth rate of the compressible slab in the long wavelength limit, where $K << 1$. The first step is to verify that in this limit, $q_b R_s << 1$ as well, as this is what is needed to simplify the dispersion relation in eq. (27). We accomplish this by showing that as $k \to 0$, $\omega \to 0$ as well. Otherwise, if $k = 0$ and $\omega \neq 0$ eq. (27) reduces to $q_{b,s} = \pm i\omega/c_{b,s}$. On the one hand, we are only interested in growing modes where $\text{Im}(\omega) > 0$, and on the other hand we require $\text{Re}(q_{b,s}) > 0$. Therefore we take $q_{b,s} = -i\omega/c_{b,s}$. Inserting this into eq. (27) yields

$$T \left(-\frac{\omega R_s}{c_b} \right) = -\delta^{-1/2}.$$  \hspace{1cm} (C1)

Solving for $\omega$ results in

$$\omega = \frac{c_s}{2R_s} \left[-n\pi + i\ln \left(\frac{\sqrt{\delta} - 1}{\sqrt{\delta} + 1}\right)\right],$$  \hspace{1cm} (C2)

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where \( n \) is any whole number, even for S-modes and odd for P-modes. For any \( \delta > 0 \), eq. (C2) results in \( \text{Im}(\omega) < 0 \), which is in contradiction to our original assumption. We conclude that there are no solutions with \( \omega(k = 0) \neq 0 \). Therefore, \( q_{b,s} \to 0 \) as \( k \to 0 \), and \( \text{tanh}(q_{b}R_{s}) \simeq 1/\text{coth}(q_{b}R_{s}) \simeq q_{b}R_{s} \). Inserting this into eq. (27) yields

\[
K = -\frac{\omega^{2}}{\delta(\omega - 1)^{2}(1 - M_{b}^{2}\omega^{2})^{1/2}} \quad \text{(S)} \tag{C3a}
\]

\[
K = -\frac{\delta(\omega - 1)^{2}(1 - M_{b}^{2}\omega^{2})^{1/2}}{\omega^{2}(1 - \delta M_{b}^{2}(\omega - 1)^{2})} \quad \text{(P)} \tag{C3b}
\]

**S-modes:** The solution to eq. (C3a) for \( K = 0 \) is \( \omega_{0} = 0 \). For \( K << 1 \), the lowest order correction is \( \omega_{1} \sim \pm i\delta^{1/2}K^{1/2} \). To find the next order correction, we insert \( \omega = \omega_{1} + \omega_{2} \) with \( \omega_{2} << \omega_{1} << 1 \). This results in \( \omega_{2} \sim \delta K \), which is the leading order term in \( \text{Re}(\omega) \). Thus, at long wavelengths S-modes are unstable with the approximate dispersion relation

\[
\omega_{S,f} \simeq -\frac{V}{R_{s}} \left[ \delta K^{2} \pm i\delta^{1/2}K^{3/2} \right]. \tag{C4}
\]

**P-modes:** When \( K = 0 \), there are two solutions to eq. (C3b): \( \omega_{0} = 1 \) and \( \omega_{0} = M_{b}^{-1} \). In the vicinity of \( \omega_{0} = 1 \) the system is unstable. Inserting \( \omega = 1 + \omega_{1} \) with \( \omega_{1} << 1 \) into eq. (C3b) yields \( \omega_{1} \sim \pm iK^{1/2}\delta^{-1/2}(1 - M_{b}^{-2})^{-1/4} \). So at long wavelengths, P-modes are unstable with the approximate dispersion relation

\[
\omega_{P,f} \simeq -\frac{V}{R_{s}} \left[ K \pm i\delta^{-1/2}(1 - M_{b}^{-2})^{-1/4}K^{3/2} \right]. \tag{C5}
\]

For the special case of \( M_{b} = 1 \), one gets instead \( \omega_{1}^{5/2} \sim (iK)/(\sqrt{2}\delta) \) and the dispersion relation becomes, to lowest order in \( K \),

\[
\omega_{P,f} \simeq -\frac{V}{R_{s}} \left[ K \pm (-2\delta^{2})^{-1/5}K^{7/5} \right]. \tag{C6}
\]

When \( M_{b} \neq 1 \), the second solution to eq. (C3b), \( \omega_{0} = M_{b}^{-1} \), is stable. Inserting \( \omega = M_{b}^{-1}(1 + \omega_{1}) \) with \( \omega_{1} << 1 \) into eq. (C3b) yields

\[
\omega_{1} = -\frac{1}{2} \left[ 1 - \frac{1}{\delta(M_{b} - 1)^{2}} \right] K^{2} \tag{C7}
\]

**APPENDIX D: SHORT WAVELENGTH LIMIT OF THE COMPRESSIBLE SLAB**

In this section we discuss in detail the short wavelength limit of the slab, where \( K \gg 1 \). There are two solutions to the dispersion relation, eqs. (D1), for \( K \to \infty \). These are \( Z = 1 \) and \( 1 - \delta M_{b}^{2}(\omega - 1)^{2} = 0 \). However, care must be taken to ensure that \( K \) remain real, which must be the case in the temporal analysis we are discussing. By examining eqs. (D1), we see that there are two different regimes, depending on the parameter

\[
\gamma \equiv \frac{\text{Im}\left(\sqrt{1 - \delta M_{b}^{2}(\omega - 1)^{2}}\right)}{\text{Re}\left(\sqrt{1 - \delta M_{b}^{2}(\omega - 1)^{2}}\right)} = \frac{\text{Im}(q_{s})}{\text{Re}(q_{s})}. \tag{D1}
\]

If \( \gamma << 1 \), then \( K \) will remain real as \( Z \to 1 \), in other words the slab dispersion relation converges to that of the sheet at short wavelengths, as expected. In this case, \( q_{s} \) is very nearly real which means that these solutions represent surface modes, similar to the sheet. These decay exponentially with depth in the slab and prevent the two surfaces from coming into causal contact. We examine the regime of validity of
results in complex $\omega$, which must correspond to an extremal point of the function $K(\omega)$ where $\omega$ is real. To see this, consider a solution with $\omega = 0$, $K(0)$, and imagine increasing $K$ (decreasing the wavelength $\lambda$). So long as $K(\omega)$ is monotonic, there is still a solution with real $\omega$, and therefore the solution remains stable. But when $K(\omega)$ reaches an extremal point, decreasing the wavelength further requires extending $\omega$ to the complex plane, indicating an instability (Fig. E1 discussed below).

There are two main branches of stable solutions, depending on whether $q_s$ is real or imaginary. We will address each of these separately, as they define different families of solutions, for low and high Mach numbers.

\begin{align}
q_s \in \mathbb{R} \Rightarrow 1 - \delta M_b^2 (1 - \omega^2) > 0
\end{align}

Such solutions represent surface waves that decay exponentially within the slab, with a penetration depth of $q_s^{-1}$. Such modes behave similarly to the sheet at short wavelengths since the two sides of the slab are not in contact with each other ($\delta = 0$). For $K$ to be real we require

\begin{align}
K &= \text{Re}(K) = \frac{\ln (|1 + Z|) - \ln (|1 - Z|)}{2\sqrt{1 - \delta M_b^2 (\omega - 1)^2}}, \\
\text{Im}(K) &= \frac{\arg(1 + Z) - \arg(1 - Z) + n\pi}{2\sqrt{1 - \delta M_b^2 (\omega - 1)^2}} = 0.
\end{align}

If $q_s = \mathbb{R}$, then $1 - M_b^2 \omega^2 \geq 0$ and $Z = -|Z|$ is on the negative real axis. If $|Z| \lesssim 1$, then $\arg(1 + Z) = 0$ and $\arg(1 - Z) = \pi$ and eq. (E1b) can be satisfied with $n = 1$. If $|Z| > 1$, then $\arg(1 + Z) = \pi$ and eq. (E1a) can be satisfied with $n = -1$. Either way, eq. (E1a) results in $K \leq 0$. Physically, we require $K = |k| R_s \geq 0$, so the only relevant solutions in this case are those with $K = 0$. If $n = 0$ then $-1 \leq Z \leq 0$, so $K = 0$ is only possible for $Z = 0$. Since we have assumed here $q_s > 0$, this in turn requires $\omega = 0$, which corresponds to the fundamental S-mode (C). If $n = -1$ then $Z < -1$, so $K = 0$ is only possible for $Z = -\infty$. This in turn requires $\omega = 1$ (or $M_b^{-1}$), which corresponds to the fundamental P-mode (C). Recall that at $K = 0$, the system is unstable for $\omega = 0$ and $\omega = 1$, but stable for $\omega = M_b^{-1}$ (C).

Note that our assumptions above, whereby $q_s > 0$ and $q_b \geq 0$, limit the range of $\omega$ to $1 - M_b^{-1} < \omega < 1 + M_b^{-1}$ and $-M_b^{-1} < \omega < M_b^{-1}$. The solutions $\omega = 0$, 1, $M_b^{-1}$ are only included in this range at low Mach numbers: $M_b < 1$, $M_b \leq 1$ and $1 - 1/\sqrt{5} < M_b < 1 + 1/\sqrt{5}$ respectively.

On the other hand, if $q_s \in \mathbb{I}$ so that $1 - M_b^2 \omega^2 < 0$, then $q_s = -i|q_s|$ and $Z = -i|Z|$ is on the negative imaginary axis. Therefore $-\pi/2 \leq \arg(1 \pm Z) \leq \pi/2$ and eq. (E1b) can be satisfied with $n = 0$ only if $Z = 0$, which would require $\omega = 0$, in contradiction to the assumption that $1 - M_b^2 \omega^2 < 0$. A solution with $n = -1$ is possible if $|Z| = \infty$ so that $(1 + Z)/(1 - Z) = -1$. In this case, provided $M_b > 1$, the solution is $K = 0$ and $\omega = 1$, corresponding to the unstable fundamental (P).

\[\text{Figure D1. The value of } \gamma = \text{Im}(q_s)/\text{Re}(q_s) \text{ for the sheet solution. When } \gamma < < 1, \text{ the dispersion relation of the slab converges to that of the sheet, } Z = 1, \text{ at short wavelengths. This happens only for low Mach numbers, } M_b < < 1. \text{ When } \gamma \gg 1, \text{ which happens when the sheet is stable, the slab converges to the solution } \omega = 1 - M_b^{-1}, \text{ which is the high-Mach number stable limit of the growing mode solution in the sheet.} \]
mode. The solution $\varpi = M_b^{-1}$ is in contradiction to the assumption that $1 - M_b^2 \varpi^2 < 0$.

**E2** $q_b \in \mathbb{I} \Rightarrow 1 - \delta M_b^2 (1 - \varpi)^2 < 0$

Such solutions represent waves that do not decay spatially within the slab. They travel from one interface to the other and are reflected off of and transmitted through the slab boundaries. These are body modes, or reflected modes. In this case, since the two sides of the slab can interact with each other through the reflection of waves, the slab will differ greatly from the sheet. In this regime $q_b = i |q_b| (1\mathbb{A})$ and $\varpi$ must obey either $1 + M_s^{-1} < \varpi$ or $\varpi < 1 - M_s^{-1}$. For $K$ to be real we require

$$K = \text{Re}(K) = \frac{\arg(1 + Z) - \arg(1 - Z) + n\pi}{2 \sqrt{\delta M_b^2 (\varpi - 1)^2 - 1}}, \quad (E2a)$$

$$\text{Im}(K) = -\frac{\ln |1 + Z| - \ln |1 - Z|}{2 \sqrt{\delta M_b^2 (\varpi - 1)^2 - 1}} = 0. \quad (E2b)$$

If $q_b \in \mathbb{I}$ so that $1 - M_b^2 \varpi^2 < 0$, then $q_b = -i |q_b| (1\mathbb{A})$ and $Z = |Z|$ is on the positive real axis. Eq. (E2b) can thus only be satisfied if $Z = 0$, which requires $\varpi = 0$, which is in contradiction to the assumption that $1 - M_b^2 \varpi^2 < 0$. We therefore conclude that no such solution exists.

On the other hand, if $q_b \in \mathbb{R}$ so that $1 - M_b^2 \varpi^2 \geq 0$, then $Z = -i |Z|$ is on the negative imaginary axis. In this case $1 - Z = 1 + Z$ and Eq. (E2b) is always satisfied. Furthermore, eq. (E2a) can be rewritten as

$$K = \frac{-\arctan(|Z|) + (n/2)\pi}{\sqrt{\delta M_b^2 (\varpi - 1)^2 - 1}}. \quad (E3)$$

For $n < 0$ eq. (E3) always results in $K < 0$, which is not relevant for our discussion. For $n = 0$, eq. (E3) results in $K \leq 0$ so the only relevant solution is $K = 0$, which requires $|Z| = 0$, which in turn requires $\varpi = 0$. Provided $M_s > 1$, this corresponds to the fundamental S-mode.

For $n \geq 1$, eq. (E3) results in $K \geq 0$ for all $\varpi$. However, our assumptions of $q_b^2 < 0$ and $q_b^2 \geq 0$ limit the range of allowed $\varpi$ values to $- M_b^{-1} \leq \varpi \leq M_b^{-1}$ and either $1 + M_s^{-1} < \varpi$ or $\varpi < 1 - M_s^{-1}$. For the $n$-th mode, $\varpi = 0$ when $K = K_{n,0} = n\pi/\left(2 \sqrt{M_b^2 - 1}\right)$, provided $M_s > 1$.

In Figure E1. we show $K$ as a function of $(M_b, \varpi)$ from eq. (E3), for $n = 1 - 10$. The three panels are representative of three different regimes of Mach number. The left panel represents low Mach numbers, $M_b < 1 - 1/\sqrt{\delta}$, corresponding to flow velocities $V < c_b - c_s$. In this case, both branches of $\varpi$ are accessible. $K(\varpi)$ increases monotonically from $\varpi = - M_b^{-1}$ until $\varpi = 1 - M_s^{-1}$ where $K \to \infty$. It then decreases monotonically from $\varpi = 1 + M_s^{-1}$, where again $K \to \infty$, until $\varpi = M_b^{-1}$. The center panel represents intermediate Mach numbers, $1 - 1/\sqrt{\delta} < M_b < 1 + 1/\sqrt{\delta}$, corresponding to flow velocities $c_b - c_s < V < c_b + c_s$. In this case, only $\varpi < 1 - M_s^{-1}$ is accessible, but the qualitative behaviour of $K(\varpi)$ in this regime is the same as in the previous case, shown in the left panel. In both these cases, there are no marginally stable points with $K > 0$. Solutions with $n \geq 1$ are stable at all wavelengths. Once the $n$-th mode has been excited, at $K = K_{n,0}$, we can increase $K$ (decrease the wavelength) continuously and there will always be a solution to the mode with real $\varpi$. Since we never require complex $\varpi$, no instability is ever triggered.

The right hand panel of Fig. E1 represents large Mach numbers, $1 + 1/\sqrt{\delta} < M_b$, corresponding to flow...
velocities which are super-sonic with respect to the sum of the two sound speeds: \( V > c_b + c_a \Rightarrow M_{\text{tot}} > 1 \). In this case, the \( K(\vec{w}) \) curve corresponding to each mode has exactly one maximum point in the range \( 0 < \vec{w} < M_b^{-1} \), as shown in the figure. We refer to these maxima as \( K_n \) and \( \vec{w}_n \). These are marginally stable points.

**APPENDIX F: MARGINALLY STABLE POINTS FOR LARGE \( N \)**

The marginally stable points are given by \( dK/d\vec{w} = 0 \) with \( K(\vec{w}) \) given by eq. (F3). This equation cannot be solved analytically in general, but in this section we derive an analytical approximation for large \( n \).

Based on the discussion in the previous section, when \( M_{\text{tot}} > 1 \) eq. (F3) is continuous in the range \(-M_b^{-1} \leq \vec{w} \leq M_b^{-1} \). Since \( 0 \leq \arctan(|Z|) \leq \pi/2 \), for \( n >> 1 \) eq. (F3) can be approximated as \( K \simeq (n\pi/2) \left( \delta M_b^2 (\vec{w}^2 - 1)^{1/2} \right) \), which is a monotonically increasing function of \( \vec{w} \). We conclude that for \( n >> 1 \), \( \vec{w}_n \) converges to \( M_b^{-1} \) (see the right-hand panel of Fig. (F1)). When \( \vec{w} = M_b^{-1} \) we get \(|Z| = \infty\), so for \( \vec{w}_n = M_b^{-1}(1-W) \), with \( 0 < W << 1 \), we have \(|Z| >> 1\) and \( \arctan(|Z|) = \pi/2 - |Z|^{-1} + O(|Z|^{-3}) \).

To leading order in \( \vec{W} \),

\[
|Z|^{-1} \simeq \sqrt{2\delta(M_b - 1)^2} \frac{n\pi}{2\sqrt{\delta(M_b - 1)^2 - 1}} \vec{W}^{1/2}. \tag{F1}
\]

In order to find \( K_n \), the maximum of eq. (F3), we expand \( K \) to the second leading order in \( \vec{W} \):

\[
K \simeq K_0 + K_1 W^{1/2} + K_2 W, \tag{F2a}
\]

\[
K_0 = \frac{n\pi}{2\sqrt{\delta(M_b - 1)^2 - 1}}, \tag{F2b}
\]

\[
K_1 = \frac{\sqrt{2\delta}(M_b - 1)^2}{\delta(M_b - 1)^2 - 1}, \tag{F2c}
\]

\[
K_2 = \frac{n\pi\delta(M_b - 1)}{2(\delta(M_b - 1)^2 - 1)^{3/2}}. \tag{F2d}
\]

It is now straightforward to find the maximum of eqs. (F2), which approximates the marginally stable point of the \( n \)-th mode

\[
W_n \simeq \frac{2(M_b - 1)^2}{n^2\pi^2} \left[ \delta (M_b - 1)^2 - 1 \right], \tag{F3a}
\]

\[
K_n \simeq \frac{n\pi}{2\sqrt{\delta(M_b - 1)^2 - 1}}. \tag{F3b}
\]

\( \vec{w}_n \) thus converges to \( M_b^{-1} \) as \( n^{-2} \) while \( K_n \) grows linearly with \( n \).

To ensure \( \vec{w}_n \leq M_b^{-1} \) we require \( W_n \geq 0 \). From eq. (F3a), this leads to

\[
M_b^{-1} + M_a^{-1} \leq v \geq c_b + c_a. \tag{F4}
\]

This supports what was inferred from Fig. (F1) in \( \text{E} \) namely that marginally stable points exist only for flows that are super-sonic with respect to the sum of the two sound speeds. All we assumed in deriving eqs. (F3) was that \( K(\vec{w}) \) from eq. (F3) was continuous for \( |\vec{w}| \leq M_b^{-1} \), which requires only \( v \geq c_b - c_a \). Such a flow is not necessarily supersonic at all, while the existence of unstable modes with \( n \geq 1 \) is a purely supersonic effect.

Our approximation for \( W_n \) is only valid if \(|Z|^{-1} < \pi/2\), with \(|Z|^{-1} \) given in eq. (F1). It is straightforward to verify that this condition also guarantees \( W_n << 1 \) in eq. (F3a). The value of \( n \) for which \(|Z|^{-1} \simeq 1\) is

\[
n_1 \simeq \delta (M_b - 1)^3 \tag{F5}
\]

We expect eqs. (F3) to converge for \( n >> n_1 \).

In Fig. (F1) we compare eqs. (F3) to exact numerical solutions for the maxima of eq. (F3), for different values of \( M_b \) and \( \delta \). The left-hand panel shows \( K_n \) and the right-hand panel shows \( W_n \). The approximation for \( K_n \) improves with increasing \( n \) and increasing \( M_{\text{tot}} \). For \( M_{\text{tot}} \approx 2 \), the error is less than 10% already for \( n \approx 2 \), while for lower \( M_{\text{tot}} \) this error is achieved for \( n \approx 10 \). On the other hand, the approximation for the marginally stable frequency, eq. (F5), only converges for \( n >> n_1 \) as expected (eq. (F5)). Unfortunately, for even moderately high Mach numbers and density contrasts, \( n_1 \) can be quite high. For example, for \( M_0 = 5 \) and \( \delta = 1 \), \( n_1 \approx 65 \).

Modes with \( n \geq 1 \) begin stable at \( K = K_{n,0} = 0 \), pass through marginal stability at \( K = K_n \), \( \vec{w} \approx M_b^{-1} \) and end at \( K = \infty \), \( \vec{w} = 1 - M_b^{-1} \). As a result, \(|Z|\) begins near 0, reaches large values \(|Z| >> 1\) and ends near 0 again. Therefore, \((1 + Z)/(1 - Z)\) goes from \( \sim 1 \) to \( \sim -1 \) and back to \( \sim 1 \), completing a full revolution about the origin in the complex plane. This is the reason for the extra \( 2\pi \) in eq. (F1) compared to eqs. (F3).

**APPENDIX G: GROWTH RATES NEAR MARGINAL STABILITY**

Using our approximation for the behaviour of \( K(\vec{w}) \) near marginal stability (eqs. (F2)), we can derive the growth rate of the \( n \)-th mode. We assume \( K = K_n + \kappa \) and \( \vec{w} = M_b^{-1}(1 - W_n + \xi) \), with \( \kappa << K_n \) and \(|\xi| \ll W_n \ll 1 \). By inserting this into eqs. (F2) we obtain to leading order in \( \kappa \)

\[
\xi \simeq \pm \left[ 8M_b^{3/2} \frac{\delta (M_b - 1)^2 - 1}{\sqrt{2\delta (M_b - 1)^2}} W_n^{3/2} \right]^{1/2} \kappa^{1/2}. \tag{G1}
\]

This shows that the system is indeed unstable near \( K_n \), and that this instability has a growing mode. Using our approximation for \( W_n \) from eq. (F3a), eq. (G1) becomes

\[
\xi \simeq \pm \frac{8\sqrt{|M_b - 1|^3}}{\sqrt{n\pi}^{3/2}} \left[ \delta (M_b - 1)^2 - 1 \right]^{5/4} \kappa^{1/2}. \tag{G2}
\]

The growth rate near marginal stability is

\[
\text{Im}(\omega) = kv \text{ Im}(\vec{w}) = kv c_b |\xi|. \tag{G3}
\]
\( \xi \) can be computed either from eq. (G1) using the exact (numerical) solution for \( \mathcal{W}_n \), or from eq. (G2) using our analytic approximation, which we saw to be a good approximation for \( n > n_1 \) (eq. [F5]).

The analytic approximation for \( \xi \) is compared to numerical solutions of the dispersion relation in Fig. G1 for different values of \( \delta \) and \( M_b \). The coloured lines show the result of a full numerical solution to the dispersion relation (eq. 27) for the modes \( n = 1 - 40 \). The dashed black line is the analytic approximation. We show \( \xi = \text{Im}(\omega) \), normalized by the \( n \)-dependent prefactor of \( \kappa^{1/2} \) in eq. (G1), as a function of \( \kappa \). When computing \( \xi \) and \( \kappa \) from eq. (G1), we used the numerical results for \( \mathcal{W}_n \) and \( K_n \), obtained by finding the maximum of \( K(\omega) \) in eq. (G2). Though not fully analytic, this is still easier to evaluate numerically than a full solution to the dispersion relation, and is thus still useful. In all cases examined, the scaling of \( \xi \propto \kappa^{1/2} \) is captured at \( \kappa \lesssim 0.1 \) for all \( n \), even if the normalization has not yet converged. The normalization converges rapidly as well for low Mach numbers (left panel), though more slowly for high Mach numbers, qualitatively similar to the convergence of eq. (F3a) for \( \mathcal{W}_n \) (Fig. F1).

For high Mach numbers, \( M_b \gg 1 \), eq. (G2) yields \( \xi \propto \delta^{3/4} M_b^{5/4} n^{-3/2} \) (for \( n > n_1 \)), while \( k \propto K_n \propto \delta^{-1/2} M_b^{-1} n^{-1} \). The growth rate near marginal stability thus scales as

\[
\text{Im}(\omega) \propto \delta^{1/4} M_b^{5/4} n^{-1/2}.
\]

The scaling of \( M_b^5 \) is very different than the growth rate near marginal stability for the fundamental modes. For the S-mode this is independent of \( M_b \) (eq. C4), and for the P-mode it scales as \( M_b^{-1/2} \) (eq. C5).

**APPENDIX H: FASTEST GROWING MODE IN THE SLAB**

For each mode, the growth rate is zero at marginal stability, grows larger as \( K \) is increased, and then goes to zero again at \( K \to \infty \) (13). Each mode thus has a maximal growth rate at some intermediate \( K \), hereafter the resonance of the mode. It is of particular interest to estimate this maximal growth rate, and how it scales with mode number \( n \), or alternatively with wavenumber \( K \). This will tell us which mode is dominant for perturbations of a particular wavelength, and will give an estimate of the fastest growth rate. Below, we derive an analytic approximation for the maximal growth rate which is valid for very supersonic flows, \( M_{\text{tot}} = V/(c_s + c_b) \gg 1 \) (though in practice, it is a good approximation for \( M_{\text{tot}} \gtrsim 1.5 \)).

We make two additional assumptions. The first is that \( \Im \) is sufficiently far from both 0 and 1 at resonance, so that \( \delta M_b (\Im - 1)^2 \gg 1 \) and \( M_b \Im^2 \gg 1 \). The second is that the growth rate at resonance is much smaller than the angular frequency. In other words, if \( \omega = \omega_b + i \Im \), then \( \omega_b \ll \Im \) at resonance. We empirically show these two assumptions to be valid by numerically solving the dispersion relation (see Fig. 4). However, they can both be justified analytically as well. When \( M_{\text{tot}} \gg 1 \), \( \Im \) grows from \( \gtrsim M_b^{-1} \sim 0 \) at marginal stability to \( 1 - M_b^{-1} \sim 1 \) as \( K \to \infty \). Since in both of these limits the growth rate goes to zero, it is reasonable to assume that resonance occurs far from these points, so that \( \Im \) is sufficiently far from both 0 and 1. The second assumption can be justified by realizing that while both \( \omega_b \) and \( \Im \) start very small at marginal stability, \( \omega_b \) increases monotonically to an asymptotic
value of $\sim 1$, while $\varpi_i$ flattens, reaches a maximum at resonance and then decays to zero. It is therefore reasonable to assume that $\varpi_i << \varpi_R$ at resonance, and therefore that $\omega_i << \omega_d$.

Using these assumptions, and the fact that in our chosen branch cut $\text{Re}(q_{b,s}) > 0$, $\text{Im}(q_b) < 0$, $\text{Im}(q_s) > 0$ \cite{4}, we have $q_b = \sqrt{1 - M_b^2 \omega^2} \simeq -i M_b \varpi$ and $q_s = \sqrt{1 - \delta M_b^2 (1 - \varpi)^2} \simeq i \sqrt{\delta} M_b (1 - \varpi)$. Inserting this into eq. (18) yields

$$Z \simeq \varpi \sqrt{\delta (1 - \varpi)}.$$  \hspace{1cm} (H1)

Writing $\varpi = \varpi_R + i \varpi_i$ and $(1 + Z)/(1 - Z) = A e^{i\theta}$, we obtain from eq. (18)

$$\varpi_R \simeq 1 - \frac{\theta + n \pi}{2 \sqrt{\delta} M_b K}.$$  \hspace{1cm} (H2a)

$$\varpi_i \simeq \frac{1}{2 \sqrt{\delta} M_b K} \ln (A) \Rightarrow \omega_i \simeq \frac{c_s}{2 R_s} \ln (A).$$  \hspace{1cm} (H2b)

We thus conclude that resonance occurs at the maximum of $A$. After some algebra, it is straightforward to show from eq. (H1) that

$$A^2 = \left[ \frac{1 + Z}{1 - Z} \right]^2 \simeq \left[ \varpi_R (\sqrt{\delta} - 1) - \sqrt{\delta} \right]^2 + \left[ \varpi_i (\sqrt{\delta} - 1) \right]^2 \left[ \varpi_R (\sqrt{\delta} + 1) - \sqrt{\delta} \right]^2 + \left[ \varpi_i (\sqrt{\delta} + 1) \right]^2.$$  \hspace{1cm} (H3)

Neglecting terms of order $(\varpi_i/\varpi_R)^2 << 1$, the maximum of $A$ occurs when

$$\varpi_{R, \text{res}} \simeq \frac{\sqrt{\delta}}{\sqrt{\delta} + 1} = \frac{c_b}{c_b + c_s},$$  \hspace{1cm} (H4a)

$$A_{\text{res}} \simeq \frac{2 \sqrt{\delta}}{\varpi_{1, \text{res}} \left( \sqrt{\delta} + 1 \right)^2}.$$  \hspace{1cm} (H4b)

Neglecting terms of order $\varpi_i/\varpi_R$, we have $|q_b| \simeq |q_s|$ at resonance. This means that the perturbation penetrates the same depth into the slab as into the background. It also means that the angle of propagation relative to the normal to the slab, given by

$$\cot (\Psi_{b,s}) = \frac{|\text{Im}(q_{b,s})|}{k},$$  \hspace{1cm} (H5)

is the same within the slab and the background. At resonance, the angle is

$$\sin (\Psi_{\text{res}}) \simeq \frac{\sqrt{\delta} + 1}{\sqrt{\delta} M_b} = \frac{1}{M_{\text{tot}}},$$  \hspace{1cm} (H6)

where in the last equality we have used $M_{\text{tot}}^{-1} = M_b^{-1} + M_c^{-1}$. This is commonly referred to as the Mach angle.

Inserting eq. (H4b) into eq. (H2b) gives an estimate for the growth rate at resonance

$$t_{sc} \omega_{i, \text{res}} + \ln \left( t_{sc} \omega_{i, \text{res}} \right) \simeq \ln \left( 4 M_{\text{tot}} \sqrt{\delta} K_{\text{res}} \right),$$  \hspace{1cm} (H7)

where $t_{sc} = 2 R_s/c_s$ is the sound crossing time in the slab. The solution to this equation can be expressed as an infinite sequence of functions

$$t_{sc} \omega_{i,1} = \ln \left( 4 M_{\text{tot}} \frac{\sqrt{\delta}}{1 + \sqrt{\delta}} K_{\text{res}} \right),$$  \hspace{1cm} (H8a)

$$t_{sc} \omega_{i,1,j} = t_{sc} \omega_{i,1} - \ln \left( t_{sc} \omega_{i,1,j-1} \right).$$  \hspace{1cm} (H8b)

As $K \to \infty$, the sequence converges to $\omega_{i,1}$ and in practice, it convergences to $\omega_{i,3}$ even for small $K$.  

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The growth rate at resonance diverges logarithmically with wavenumber, \( \omega_i \propto \ln(k) \). As shown in Fig. 4, this is the effective growth rate for the slab, since at each wavenumber the growth rate will be dominated by the mode closest to resonance. This should be compared to the case of the incompressible slab and the compressible/incompressible sheet. In the incompressible cases, the system is always unstable and the growth rate diverges linearly with wavenumber, \( \omega_i \propto k \). In the compressible sheet, the same scaling applies at low Mach numbers, while at high Mach numbers the system is stable, \( \omega_i = 0 \). The scaling of \( \omega_i \propto \ln(k) \) for the effective growth rate in the compressible slab can be seen as a “compromise” between these two extremes, diverging at short wavelengths, but only logarithmically. Also, recall that it is only the effective growth rate, comprised of the fastest growing modes at each wavelength, which diverges. Each individual mode stabilizes as \( k \to \infty \), as discussed in §D.

By inserting eq. (H4a) into eq. (H2a) we can estimate the wavenumber at resonance. Neglecting terms of order \( \theta = O(\pi / \theta_R) \)

\[
K_{\text{res}} \simeq \frac{n\pi}{2M_{\text{tot}}}
\]  

(H9)

The resonant wavenumber increase linearly with \( n \), similar to the marginally stable wavenumber (eq. F3b). When \( M_{\text{tot}} \gg 1 \), \( K_{\text{res}} \) is very nearly continuous.

Figure H1 and Fig. H2 compare our analytic approximations for the resonant frequency and wavelength to numerical solutions of the dispersion relation, for the first 40 modes, \( n = 1 - 40 \), for different values of \( \delta \) and \( M_b \). The left-hand panel of Fig. H1 shows the resonant wavenumber, \( K_{\text{res}} \) (eq. H9), the right-hand panel shows the phase velocity at resonance, \( \varpi_{R,\text{res}} \) (eq. H1a), and Fig. H2 shows the growth rate at resonance, \( \omega_{i,\text{res}} \) (eq. H8). In the left-hand panel of Fig. H2 we focus on one example, \( M_b = 1.5 \) and \( \delta = 100 \), and examine the convergence of the sequence given in eq. (H8), which is shown to converge by \( \omega_{i,3} \). In the right-hand panel, we compare \( \omega_{i,3} \) to numerical solutions for the same values of \( M_b \) and \( \delta \) as in Fig. H1.

The fit to \( \varpi_{R,\text{res}} \) at resonance is good in all cases, with an error of less than \( \lesssim 10\% \). On the other hand, the approximations for \( K_{\text{res}} \) and \( \omega_{i,\text{res}} \) are quite poor at low Mach numbers, \( M_{\text{tot}} \lesssim 1.3 \), but rapidly improve as \( M_{\text{tot}} \) is increased. The error in \( \omega_{i,\text{res}} \) reaches \( \lesssim 10\% \) for \( M_{\text{tot}} \gtrsim 1.3 \), and for \( K_{\text{res}} \) a similar error is achieved for \( M_{\text{tot}} \gtrsim 2 \). This is expected, since the approximations we made are strictly valid for very high Mach numbers only. We note that in the cases where the approximation is particularly poor, \( M_b = 1.5, \delta = 10 \) and \( M_b = 5.0, \delta = 0.1 \), the sheet is still unstable since \( M_b < M_{\text{crit}} \), so the slab instability is dominated by surface modes rather than body modes.
Figure H1. Comparison of our analytic estimate for the resonant wavenumber, \(K_{res}\) (eq. H9, left), and the real part of the resonant phase velocity, \(\omega_{\omega, res}\) (eq. H4a, right), to a numerical solution of the dispersion relation. We compare the first 40 modes, \(n = 1 - 40\), for the same values of \(M_b\) and \(\delta\) as in Fig. F1. The approximation for \(K_{res}\) is poor when \(M_{tot} \lesssim 1.3\), but for \(M_{tot} \gtrsim 2\) the error is less than \(\sim 10\%\) for \(n > 4\). On the other hand, the approximation for \(\omega_{\omega, res}\) is very good even at low Mach numbers.

Figure H2. Analytic estimate vs. numerical solutions for the fastest growing mode in the slab. Left: Comparison of the growth rate at resonance, \(\omega_{\omega, res}\), computed numerically for the first 40 modes (blue points) and calculated analytically using the first 4 terms in the sequence in eq. (H8) (solid lines), for \(M_b = 1.5\) and \(\delta = 100\). The analytic series converges by \(\omega_{\omega, 3}\), though in this case it converges to a slightly lower value than the numerical solution. Right: The ratio of \(\omega_{\omega, 3}\) from eq. (H8) to the numerical solution for the same values of \(M_b\) and \(\delta\) as in Fig. H1. Note that the wavenumber corresponding to \(n = 40\) is different in each case, which is why different curves end at different \(K\) values. The fit is quite poor for \(M_{tot} \lesssim 1.3\), but for higher Mach numbers the error is less than 10\%.