LIMITING DISTRIBUTION OF TRANSLATES OF A CLOSED ORBITS OF THE DIAGONAL GROUP ON $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$

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Abstract. Given a closed orbit of the diagonal group in $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, it naturally carries a possibly infinite Haar measure. We classify all possible limit measures of it when translated by a sequence of elements from $SL(n, \mathbb{R})$. This is a natural extension of Shapira and Zheng’s work where only divergent orbits are considered.

1. Introduction

1.1. Motivation. Observed by Duke-Rudnick-Sarnak [6], counting integer points on a homogeneous affine variety of the form $G/H$ for $G, H$ reductive linear algebraic group defined over $\mathbb{Q}$ is related to the limiting measure of translates of the $H$–invariant measure $\mu_{H\delta(\mathbb{Z})/G(\mathbb{Z})}$ in the quotient space $G(\mathbb{R})/G(\mathbb{Z})$. Assume $G$ to have no rational characters, equivalently, $G(\mathbb{R})/G(\mathbb{Z})$ has finite measure (see [17]). Then the latter problem can be approached by theorems about unipotent-invariant probability measures known as Ratner’s theorems ([18]) for the following reason:

Given a sequence $g_k$ in $G(\mathbb{R})$ and take $\nu$ to be a limit point of $(g_k)_*\mu_{H(\mathbb{R})G(\mathbb{Z})/G(\mathbb{Z})}$. Then $\nu$ is invariant under any limit points of $\exp(Ad(g_k)v_k)$ for some sequence $v_k \in \text{Lie}(H)$. Now suppose $Ad(g_k)v_k/|v_k|$ diverges to infinity but $Ad(g_k)v_k$ converges to a nonzero vector, then $v_k$ necessarily goes to 0 and all eigenvalues of $v_k$ goes to 0. But $Ad(g_k)v_k$ has the same eigenvalues as $v_k$ hence any limit of $Ad(g_k)v_k$ must have all eigenvalues vanishing, i.e. a nilpotent matrix. For this reason, the limit measure $\nu$ has to be invariant under exponential of some nontrivial nilpotent matrix, i.e. a nontrivial unipotent element. Of course, another important issue here is that $\nu$ needs to be a nonzero measure. Indeed, this approach has been carried out successfully by Eskin-Mozes-Shah in [9] and [7]. In some special cases one may even approach by mixing as in the work of Eskin-McMullen [8]. However, one assumption that has been made was that $H(\mathbb{R})/H(\mathbb{Z})$ has finite covolume. The barrier was crossed by Oh-Shah [16], where they allow $H$ to be a $Q$–split torus in $SL_2(\mathbb{Q})$. A further refinement is obtained by Kelmer-Kontorovich [10] using different methods. This was generalized later to the case $SL_n(\mathbb{Q})$ with $\mathbb{H}$ maximal $Q$–split torus by Shapira-Zheng [20] where they give a complete classification of all possible limiting measures. As a corollary, they also classify all limiting measures for $\mathbb{H}$ reductive and contains a maximal $Q$–split torus. In the present work their results are further generalized to all $\mathbb{R}$–split maximal torus defined over $Q$ in $SL_N(\mathbb{Q})$, which corresponds to the case when the diagonal group has a closed orbit. With additional (unnecessary) assumptions on the sequence which is used to translate the measure, Shah had obtained a similar result. So the emphasize here would be a complete classification.

Let’s remark that results in [16], [10] are effective. And methods from [8] can be made effective to treat the case when $H$ is a ”symmetric subgroup” with finite covolume, for instance, the case when $H$ is an anisotropic torus in $SL_2(\mathbb{Q})$. Indeed this has been carried out by Benoist-Oh in [2]. However, it seems that no such effective results is known for translation of maximal torus in $SL_n(\mathbb{Q})$ for $n$ at least 3. And the proof presented here, or the proof in [20], makes crucial use of Ratner’s theorem, which only yields non-effective results.

One should also mention the work of Richard and Zamojski [19], where they consider translation of $\Omega x$ where $\Omega$ is a bounded open in $H$ and $x \in G/T$ is arbitrary. It might be possible to use their methods as a substitute of [9], but we did not try.

1.2. Main results. Take $G = SL_N$ with the $Q$–structure given by $SL_N(\mathbb{Q})$, $\Gamma = G(\mathbb{Z})$ or any other commensurable lattices and $T$ is a maximal $Q$–torus that is $\mathbb{R}$–split. In this paper, convergence of class of

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Remark 1.7. $C$ is in the sense defined in [20], i.e. $\mu_i \to \mu$ if and for all $f_1, f_2 \in C_c(X)$,
\[
\lim i \left\{ \frac{f_1(x)\mu_i(x)}{f_2(x)\mu_i(x)} = \frac{f_1(x)\mu(x)}{f_2(x)\mu(x)} \right\}
\]
whenever $\int f_2(x)\mu(x) \neq 0$. Another criterion, which will be used here, is that $\mu_i \to \mu$ iff there is a sequence of positive numbers $\lambda_k$ such that for all $f \in C_c(X)$,
\[
\lim i \frac{1}{\lambda_i} \int f(x)\mu_i(x) = \int f(x)\mu(x)
\]
From this description one can see that when restricted to the class of probability measures, one recovers the weak-$\ast$ convergence.

For each reductive subgroup $H$ defined over $\mathbb{Q}$, there is a unique class (up to scalar) of $H$-invariant ($H = \mathbb{H}^0(\mathbb{R})$) measure supported on the closed set $HT/T$ which we denote by $\mu_{HT}$. That $HT$ is closed can be seen from the fact that $G/H$ is an affine variety defined over $\mathbb{Q}$.

Condition (*). Let’s give a sequence $g_k \in G(\mathbb{R})$. Assume for some subtorus $S_0 \subset T$ defined over $\mathbb{Q}$ the following holds:
1) $g_k \in Z_G(S_0)$;
2) $g_k$ diverges in $G/Z_G(S)$ for all $S$ torus defined over $\mathbb{Q}$ strictly containing $S_0$.

Remark 1.1. A useful fact that we shall implicitly use is that conjugacy class of a semisimple element is closed. We can verify this fact directly over $\mathbb{C}$. Hence $g_k$ diverges in $G/Z_G(S)$ is equivalent to that $g_k s g^{-1}_k$ diverges for some $s \in S$. And condition (*) above is equivalent to $g_k \in Z_G(S_0)$ and for all $s \notin S_0$, $g_k s g^{-1}_k$ diverges.

Remark 1.2. In [20], the authors work at Lie algebra level and defined $\mathcal{A}(g_k, T)$ to be the collection of vectors in $\text{Lie}(T)$ where $\text{Ad}(g_k)v$ is bounded. One can check these two approaches are essentially equivalent in the present situation and our approach is certainly inspired by theirs. However, we prefer to work at the group level in the hope that same statement might work for more general situation.

Remark 1.3. Note although there are infinitely many subtorus, there are only finitely many candidates of $S_0$. So if we start with arbitrary $g_k$, by dividing into finitely many disjoint subsequences and modifying from left by some bounded sequence, we may always assume this to be true.

Remark 1.4. One can directly see that the $S_0$ that can satisfy the above condition is unique. And even if we don’t assume it is connected to start with, it turns out to be connected, i.e. a subtorus in this case (not true in general).

Theorem 1.5. Assume the assumption above holds. Let $H := Z_G(S)$, then $\lim_k (g_k)_\ast \mu_{TT} = \mu_{HT}$.

Remark 1.6. Let $D$ be the real points of the subgroup of diagonal matrices and suppose $Dx_\Gamma$ is closed for some $x \in G(\mathbb{R})$. By the result of Tomanov and Weiss ([23], Theorem 1.1), $x^{-1}Dx$ is a maximal $\mathbb{Q}$-torus that is $\mathbb{R}$-split. Hence the study of limiting distribution of translates of $Dx_\Gamma$ reduces to the situation considered in the theorem above. This explains the title of the paper.

Remark 1.7. In the case $T$ is $\mathbb{Q}$-split, this is done in [20]. In the case $T$ is $\mathbb{Q}$-anisotropic, taking into account of [22], this is essentially done in [9].

Remark 1.8. Note in the assumption, we put everywhere ”defined over $\mathbb{Q}$”. This is a much stronger condition compared to ”defined over $\mathbb{C}$”. Hence the assumption becomes weaker. In one extremality, when the $T$ is anisotropic corresponding to a field without subextension (to be explained in section 7), then there is no nontrivial such subtorus satisfying $\ast$ and the assumption is simply $g_k$ diverges in $G/T$. Let’s give some examples to illustrate this.

Example 1. $N=3$, $G = SL_3$. $T = T_a T_s$ with $T_a = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1/a^2 \end{bmatrix} \right\}$, $T_s = \left\{ \begin{bmatrix} b & c \\ 2c & b \end{bmatrix} \right\}$. Take $g_k = \begin{bmatrix} 1 & d_k \\ 1 & e_k \end{bmatrix}$ with one of $(d_k, e_k)$ diverging, then $(g_k)_\ast \mu_{TT} = \mu_{GT}$.
Example 2. N=4, \( G = SL_4 \). \( T_s = \begin{bmatrix} a & a \\ 1/a & 1/a \end{bmatrix}, T_a = \begin{bmatrix} b & c & \vdots \\ 2c & b & \vdots \\ \vdots & \vdots & \ddots \end{bmatrix} \), \( b^2 - 2c^2 = d^2 - 3e^2 = 1 \)

Take \( g_k = \begin{bmatrix} I_2 & F_k \\ 0 & I_2 \end{bmatrix} \) with \( F_k \) diverges, then \( (g_k)_{\ast} \mu_{HT} = \mu_{GT} \).

In the process of proving a theorem below by induction we are forced to prove

**Theorem 1.9.** Let \( G' = R_{M/Q} SL_N \) for \( M \) a totally real field and \( T' \) (the real points of) a maximal \( \mathbb{R} \)-split \( \mathbb{Q} \)-torus. \( \Gamma' := G'(\mathbb{Z}) \). Given \( g_k \in G' \) that diverges in \( G'/Z_{G'}(S') \) for all \( \mathbb{Q} \)-subtorus \( S' \subset T' \), we have \( \lim_k (g_k)_{\ast} \mu_{GT'} = \mu_{G'T'} \).

Before we state the next theorem, let’s observe that any connected reductive subgroup in \( SL_N \) containing a maximal torus is of the form \( Z_G(S) \) for some (connected) torus \( S = Z(H) \). Given a sequence in \( \{ g_k \} \) and \( H_1 \) a reductive subgroup defined over \( \mathbb{Q} \) containing \( T \) with center \( S_1 \), let’s assume:

Condition (**), there exists a \( \mathbb{Q} \)-subtorus \( S_0 \) of \( S_1 \) such that:
1. \( g_k \) is contained in \( Z_G(S_0) \);
2. \( g_k \) diverges in \( Z_G(S) \) for all \( \mathbb{Q} \)-subtorus \( S \) of \( S_1 \) strictly containing \( S_0 \);

Same as above, this assumption can always be achieved by dividing the original sequence into finitely many disjoint subsequences and modifying from left by a bounded sequence.

**Theorem 1.10.** Assume condition (***) hold. Take \( H := Z_G(S_0) \). Then \( \lim_k (g_k)_{\ast} \mu_{H_1 \Gamma} = \mu_{HT} \).

Example 3. N=4, \( G = SL_4 \). \( T_s = \begin{bmatrix} a & a \\ 1/a & 1/a \end{bmatrix}, T_a = \begin{bmatrix} b & c & \vdots \\ 2c & b & \vdots \\ \vdots & \vdots & \ddots \end{bmatrix} \), \( b^2 - 2c^2 = d^2 - 3e^2 = 1 \)

Take a \( \mathbb{Q} \)-subgroup of \( G \) defined by \( \mathbb{H}(\mathbb{Q}) = \begin{bmatrix} b & c & f & h \\ 2c & b & 2h & f \\ j & l & d & e \\ 2l & j & 2e & d \end{bmatrix} \) then \( b,c,d,e,f,h,j,l \in \mathbb{Q}, \det = 1 \)

\( \mathbb{H} \) centralize a subtorus of \( T \). Take \( g_k = \begin{bmatrix} 1 & f_k & h_k \\ & 1 & 2h_k \\ & & 1 \end{bmatrix} \) with \( (f_k, h_k) \) diverges, then \( (g_k)_{\ast} \mu_{HT} = \mu_{HT} \).

Applying the method of [6] (c.f.[8]), we obtain the following counting result.

**Theorem 1.11.** There exist a constant \( c_p > 0 \) such that

\[ \lim_{n} \frac{|X_p'(|\mathbb{Z}) \cap B_R|}{c_p R^{m_0(n-1)/2} (\log R)^{a_0 + l_0 - 1}} = 1 \]

where \( M \) is a totally real number field and \( p(x) \in \mathcal{O}_M[x] \) a fixed polynomial with distinct \( \mathbb{Q} \)-roots. \( X_p'(\mathbb{Z}) \) is the collection of all \( n \)-by-\( n \) \( \mathcal{O}_M \) matrices whose characteristic polynomials are equal to \( p(x) \). \( B_R \) is the ball of radius \( R \) in the geometric embedding with respect to the Euclidean metric. \( m_0 \) is the degree of \( M \) over \( \mathbb{Q} \) and \( a_0 + l_0 \) is the number of irreducible factors of \( p(x) \) in \( M[x] \).

See section 9 for a more precise formulation.

1.3. **Strategy of the proof.** As mentioned before, our proof ultimately relies on the powerful measure classification theorem of Ratner [18] and the linearization technique developed by Dani-Margulis [5]. This part of the argument does not show explicitly here. The problem of translating a possibly infinite measure can be easily reduced to translating finite ones. Then Ratner’s theorem and linearization technique are used via the work of Eskin-Mozes-Shah [9], which reduced our problem in the “generic” setting (namely,
the setting of Theorem 1.1) to the following:
1. Prove non-divergence;
2. Eliminate the possibilities of intermediate groups.

The first part was easy for Shapira-Zheng [20], i.e. when the torus under consideration is \( \mathbb{Q} \)-split, but poses some difficulty for us (i.e. the case when it is \( \mathbb{R} \)-split but not necessarily \( \mathbb{Q} \)-split). In section 3, this is conquered by taking advantage of Galois action. To prove the second part, an important step is to show the polytope of nondivergence contains a ball of arbitrarily large radius. This was due to [20] and also work in our situation after suitable modification, see section 4. Then the rest follows from some algebraic group arguments, see section 5.

To treat the intermediate cases, one attempts by applying induction. In [20], there is no issue. Indeed one can easily see that all the centralizer of any split torus are products of \( SL \) and some split torus. In our case, however, the analysis is a bit more complicated. In section 7.3 we essentially classify all possible intermediate subgroups based on and generalizing the work of Tomanov [22]. It turns our they are products of a central torus and some groups obtained by applying restriction of scalars to \( SL \). This forced us to treat also the generic case for those groups. Luckily, most part of the proof goes the same way as before except more care is required when one applies the non-divergence result of Kleinbock-Margulis [11]. This is done in section 6.

Then in section 8, all this was applied to study the translates of closed orbits of \( \mathbb{Q} \)-reductive subgroups containing a maximal torus defined over \( \mathbb{Q} \).

In the last section, we sketch an argument (based on computations made in [20]) of deducing a counting result from the measure classifications.

Overall in this paper, we have preferred to use concrete linear algebra arguments than appealing to general facts from the theory of linear algebraic group, which are reserved for later when we come back to treat the general \( \mathbb{Q} \)-semisimple groups.

2. Basic Notations

2.1. Groups. Unless otherwise specified, we will use a ”blackboard” letter to denote an ”abstract” linear algebraic group. The corresponding normal letter will mean the connected component of the real points of that group with respect to Euclidean topology. For instance, we will let \( G := SL_N \) and \( \bar{G} = SL_N(\mathbb{R}) \).

For a vector space \( V \) defined over some field \( M \), \( V(M) \) denotes its \( M \) points and \( V \) denotes its \( \mathbb{R} \) points.

For a subset \( C \) in a linear algebraic group, \( \overline{C} \) denotes its Zariski closure.

\( \Gamma := SL_N(\mathbb{Z}) \) is a lattice, a discrete subgroup of finite covolume, in \( G \). Let \( M \) be a totally real number field. \( \mathbb{T} \) is a maximal torus of \( G \) defined over \( M \). We assume \( \mathbb{T} \) to be \( \mathbb{R} \)-split. Let \( \mathbb{D} \) be the subgroup consisting of diagonal elements in \( G \).

Now we want to make \( \mathbb{T} \) more explicit. \( \mathbb{T} \) can be written as an almost direct product of \( \mathbb{T}_a \mathbb{T}_s \), its \( M \)-anisotropic part and \( M \)-split part. We shall assume there exist \( l_0, ..., l_{a_0} \in \mathbb{N}^+ \) with \( l_0 + l := l_0 + \sum_{i=1}^{a_0} l_i = N \) such that \( \mathbb{T}_a \) and \( \mathbb{T}_s \) take the following form:

\[
T_a = \begin{bmatrix}
  I_{l_0} & T_{a_1}^1 & \cdots & T_{a_1}^{a_1} \\
  & I_{l} & \cdots & I_{l} \\
  & & \ddots & \vdots \\
  & & & I_{l}
\end{bmatrix}
\quad \text{and} \quad
T_s = \begin{bmatrix}
  x_1 & x_{l_0} & \cdots & x_{l_0} \\
  & y_1 I_{l_1} & \cdots & y_1 I_{l_1} \\
  & & \ddots & \vdots \\
  & & & y_{a_0} I_{l_{a_0}}
\end{bmatrix}
\]

where each \( T_{a_i}^i (i \geq 1) \) is a maximal \( M \)-anisotropic torus in \( SL_{l_i} \). Each block as above corresponds to a unique copy of \( SL_{l_i} \) embedded in \( G \), which we denote by \( \mathbb{G}_i \).

Now we make \( \mathbb{T}_a \) be more explicit.

Take \( L_1, ..., L_{a_0} \) totally real number fields containing \( M \) and \( deg(L_i/M) = l_i \). Take \( L \) to be a Galois number field containing all \( L_i \). For each \( i = 1, ..., a_0 \), choose a basis \( v_1^i, ..., v_{l_i}^i \) in \( O_{L_i} \) for \( L_i \) as a \( M \)-vector space. \( L_i/M \) may not be Galois. We let \( \mathbb{Q}_i \) be a fixed algebraic closure of \( \mathbb{Q} \) which is assumed to contain \( L \). And we will abbreviate \( Gal := Gal(\mathbb{Q}/\mathbb{Q}) \). The coset space \( Gal(\mathbb{Q}/\mathbb{Q})/Gal(\mathbb{Q}_i/\mathbb{Q}) \) has cardinality \( l_i \) and we will choose \( \{\sigma_1^i, ..., \sigma_{l_i}^i\} \) to be a full set of representatives in \( Gal(\mathbb{Q}/\mathbb{Q})/Gal(\mathbb{Q}_i/\mathbb{Q}) \).

For \( i = 1, ..., a_0 \), define \( (A_i)_{\xi, \xi} = (\sigma_\xi^i v_\xi^i) \), an \( l_i \)-by-\( l_i \) matrix. \( A_0 := diag(l_{a_0}, A_1, ..., A_{a_0}) \). We define \( \mathbb{T}_a \) to be \( (A_i)^{-1} \mathbb{D} A_i \) and \( \mathbb{T}_a \subset (A_0)^{-1} \mathbb{D} A_0 \).
Lemma 2.1 ([13] or [22]). Any $\mathbb{R}$–split maximal $\mathbb{Q}$–torus of $SL_N(\mathbb{Q})$ is conjugate to $\mathbb{T}$ constructed above in $SL_N(\mathbb{Q})$.

In view of this lemma, we may and will work with this explicit $\mathbb{T}$.

Finally, let’s also fix a choice of precompact nonempty open neighborhood $\Omega_a(\Omega_s$ resp.) of 0 in $Lie(T_o)(Lie(T_s)$ resp.). And $\Omega := \Omega_a + \Omega_s$, let $\Omega^A := Ad(A_0)\Omega$ and similarly we define $\Omega^a$ and $\Omega^A$.

2.2. Standard representation. $G$ acts on $V := M^N$ via the standard representation. According to the decomposition of $T_a$ into blocks, we write $M^N = M^{l_0} \oplus \ldots \oplus M^{l_{a_0}}$. For $i = 0, \ldots, a_0$, we let $\{e_j^i\}_{j=1}^{l_i}$ be the set of standard basis in the copy of $T$ in $M^N$. This set of basis will give $V$ and its exterior powers a $\Omega_M$–structure and hence it makes sense to talk about $\bigwedge^k V(\mathcal{O}_M)$. Also let’s endow this vector space with a Euclidean metric such that they form an orthonormal basis.

For $i = 0, \ldots, a_0$ and $j = 1, \ldots, l_i$, $f_j^i := A_0 e_j^i$. So for $i = 0, f_j^i = e_j^i$.

$[l_i] := \{1, \ldots, l_i\}$. For $0 \leq d_i \leq l_i$, $[l_i]_{d_i}$ is the collection of subsets of $[l_i]$ of cardinality equal to $d_i$. $\mathcal{A}_0 := 2^{[a_0]} \times \{\varnothing, [l_1]\} \times \ldots \times \{\varnothing, [l_{a_0}]\}$.

For $i \in \{0, \ldots, a_0\}$, $j = 0, \ldots, l_i$, we let $\chi^i_j$ be characters of $Lie(D)$ such that $\forall t \in Lie(D)$, exp$(t)e_j^i = \exp(\chi^i_j(t))e_j^i$. Let $B \in G$. For a subset $\xi \in [l_i]$, $[l_i]_\xi := \bigwedge_{\eta \in \xi} e_j^1$ and $\chi^\xi_{1, t}$ is defined by $e_j^1 \mapsto \exp(t)\chi^\xi_{1, t} e_j^1$ for all $t \in Lie(D)$, and $B\chi^\xi := B \bigwedge_{\eta \in \xi} e_j^1$.

For a subset $\xi = (\xi_0, \ldots, \xi_{a_0}) \in \bigwedge_{i=0}^{a_0} \xi_i \mapsto \bigwedge_{i=0}^{a_0} e_j^i$ and $\chi^\xi := \bigwedge_{i=0}^{a_0} \chi^i_{1, t}$ is defined by $e_j^i \mapsto \exp(t)\bigwedge_{i=0}^{a_0} \chi^i_{1, t} e_j^i$ for all $t \in Lie(D)$. Let’s remark here that $\{e_j^i\}_{\xi \in \mathcal{A}_0}$ is exactly the set of common rational eigenvectors with respect to $T$–action in all the exterior powers of the standard representation.

For $i \in \{0, \ldots, a_0\}$, $e_i := \bigwedge_{j=1}^{l_i} e_j^i = e_{[l_i]}$ and $\chi^i$ is defined by $\chi^i(t) \bigwedge_{j=1}^{l_i} e_j^i := \exp(t)\bigwedge_{j=1}^{l_i} e_j^i$ for all $t \in Lie(D)$. $B \chi^i := B \bigwedge_{j=1}^{l_i} e_j^i$. For $i \in \{1, \ldots, a_0\}$, $\xi = (\xi_1, \ldots, \xi_{a_0})$, $\xi = (\xi_1, \ldots, \xi_{a_0}) \in \bigwedge_{i=1}^{a_0} [l_i]_{d_i}$, we define $\beta^\xi_{\xi_i}$ by

$$f^i_{\xi_i} = \sum_{\xi \in [l_{a_0}]} \beta_{\xi_i}^\xi e_j^i$$

And $\beta^\xi_{\xi_i} = \prod_j \beta_{\xi_i}^j$. Note they all belong to $\mathcal{O}_L$.

2.3. Others. We let $\mathbb{H} := Z_G(T_s)$, so $H = \begin{bmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{l_0} & \\ & & & H_{l_0} \end{bmatrix}$ with each $H_i$ an $l_i$–by-$l_i$ matrix.

Also $\mathbb{H} = \mathbb{H}_s T_s$ where $\mathbb{H}_s$ denotes the semisimple part of $\mathbb{H}$.

For a cocharacter $\lambda : G_m \to G$, $\mathbb{P}(\lambda)(\mathbb{C}) := \{x \in G(\mathbb{C}) \mid \text{lim}_{a \to 0} \lambda(a)x\lambda(a)^{-1} \text{ exists}\}$. If $\lambda$ is defined over $M$, so is $\mathbb{P}(\lambda)$. We may and will choose a cocharacter $\lambda_0 : G_m \to T_s$ regular such that $\mathbb{P}(\lambda_0)$ contains the full upper triangular unipotent matrices. $\mathbb{P}_T := \mathbb{P}(\lambda_0)$.

Now $P_T = R_u(\mathbb{P}_T)H = R_u(\mathbb{P}_T)\mathbb{H}_s T_s$.

Let’s remark that $G/P$ is compact and $R_u(\mathbb{P}_T)(M)\mathbb{H}_s (M)T_s(M)$ is dense with respect to Euclidean topology in $\mathbb{P}_T(\mathbb{R})$.

Similarly for other subtor $S \subset T$ we can defined $P_S$ and its reductive(Levi) part will be denoted $H_S$. If $S$ is defined over $M$ then all the associated groups are also defined over $M$.

We fix a right invariant Riemannian metric induced from Killing form on $G$ such that Vol$(G/T) = 1$. Note this volume form descends to $G/T$ and is automatically $G$–invariant. With respect to this metric, for a reductive subgroup $\mathbb{H} \subset G$, we write $\mu_{HT}$ for the measure induced from the Riemannian metric on $HT/T$. $\mu_{HT}$ is $H$–invariant. When we have a finite measure $\mu$, $\tilde{\mu}$ will mean the unique probability measure obtained from $\mu$ by suitable scaling.
3. The polytope of non-divergence

When \( M = \mathbb{Q} \), the orbit \( TT/T \subset G/T \) is closed and has a divergent part if \( T \neq \{1\} \). When translated by \( B \in G \), it is expected that more and more part of this orbit will be brought back to a fixed compact part of the space. We want to have a simple description for which \( T \subset \) the part of the space. We want to have a simple description for which one in \( [0,1] \). We want to have a simple description for which one in \( [0,1] \) by taking logarithm on both sides, hence the name "polytope" is justified.

Remark 3.2. For \( \xi \in \mathfrak{a}_0 \), \( A_0\xi \in \mathfrak{a}_0 \). We nevertheless keep \( A_0 \) in the definition to emphasize the integral structure we are concerned with.

Remark 3.3. \( \|B(\exp t)A_0\xi\| = \prod_{i(\xi_i \neq 0)} \det A_i \|B(\exp t)\xi\| = \prod_i \det A_i \exp(\chi_i(t))\|B\xi\| \), hence by taking logarithm on both sides,

\[
\Omega_{B,\varepsilon} = \{ t \in \text{Lie}(T) \: \chi(t) \geq \log(\varepsilon/\prod_i \det A_i) - \log\|B\xi\|, \forall \xi \in \mathfrak{a}_0 \}
\]

The above defining inequalities make the boundary consisting of hyperplanes and hence the name "polytope" is justified.

Remark 3.4. The polytope is invariant if one replaces \( B \) by \( Bt_a \) for any \( t_a \in T_a \). It is uniquely determined by this property: if we start with arbitrary \( T \) defined over \( M \), then \( t_1 \) will be defined as the set of common eigenvectors (up to scaling) of \( T \) defined over \( M \) in all the exterior products of the standard representation. Since \( T_a \) is anisotropic over \( M \), \( T_a \) has to fix all these vectors.

Recall we fixed a precompact nonempty open neighborhood \( \Omega = \Omega_a + \Omega_o \) of 0 in \( \text{Lie}(T) \). Also recall \( \Omega^4 := Ad(A_0)(\Omega) \) is a neighborhood of 0 in \( \text{Lie}(D) \).

Proposition 3.5. Let \( M = \mathbb{Q} \), then there exists a constant \( \varepsilon' = \varepsilon'(\varepsilon, \Omega) \) with \( \varepsilon' \to \infty \) as \( \varepsilon \to \infty \). For any \( B \in G \), \( t \in \Omega_{B,\varepsilon} \), \( k \in \{1, ..., \ell \} \) and \( v \in A_0(\wedge^k V)(\mathbb{Z}) \),

\[
\sup_{t' \in \Omega^4} \|B\exp(t + t')v\| \geq \varepsilon'
\]

Remark 3.6. In the case of \( T_a = T \), there is no constraints on \( t \) and this proposition is a special case of one in [7]. And our proof differs from theirs in this case.

This proposition will follow from

Proposition 3.7. There exists a constant \( \kappa_1 = \kappa_1(\Omega) \). For any \( k = 1, ..., \ell \) and nonzero

\[
v = \sum_{I \in \{[l_1], ..., [l_{\ell_o}]\}} \alpha_I e_I \in \bigwedge^k V(\mathcal{O}_M)
\]

there exists

1. \( (d_1, ..., d_{\ell_o}) \in \{0,1, ..., l_1\} \times ... \times \{0,1, ..., l_{\ell_o}\} \);
2. \( J_0 \subset [l_0] \);
3. \( \mathcal{B}_0 \subset [l_1]d_1 \cup ... \cup [l_{\ell_o}]d_{\ell_o} \) and \( \xi_0 \in \mathcal{B}_0 \) (so \( \mathcal{B}_0 \) is nonempty);
4. \( \theta \in \text{Perm}\{1, ..., a_0\} \);
5. \( \sum_{\xi \in \{[l_1], ..., [l_{\ell_o}]\}} \alpha_{\xi} \beta_{\xi}^{\ell_0} \neq 0 \)

such that for all \( B \in G \), \( t \in \text{Lie}(T) \),

\[
\sup_{t' \in \Omega^4} \|B\exp(t + t')A_0v\| \geq \kappa_1 |\mathcal{N}_{L/M}(\sum_\zeta \alpha_{\zeta} \beta_{\zeta}^{\ell_0})| \prod_{j=0}^{\ell_o} \left( \exp \left( \sum_{i=0}^{j} \chi^{(i)}(t) \right) \|B_{J_0} \wedge B^{\theta_1} \wedge ... \wedge B^{\theta_{j+1}}\| \right) \cdot |\mathcal{B}_0(\{d_{j}/d_{0,j} - d_{0,j+1}/d_{0,j+1}\})|
\]
Here the summation is over \( \zeta \in [t_1]_{d_1} \sqcup \ldots \sqcup [l_{a_0} d_{a_0}} \) and \( \chi^0 := \chi_{J_0} \). \( \text{Gal}_0 \) is a subgroup of \( \text{Gal}(L/\mathbb{Q}) \) and \( d_0/\mathbb{Q} := 1, \ d_{(a_0+1)}/\mathbb{Q} := 0. \)

Remark 3.8. We allow \( M \) other than \( \mathbb{Q} \) because this proposition will be invoked again in later sections when dealing with the group \( R_{M/\mathbb{Q}} \text{SL}_N. \)

Proof of Proposition 3.5 assuming Proposition 3.7. Let \( M = \mathbb{Q} \).

As \( \sum \alpha \zeta \beta^0_\zeta \neq 0 \in \mathcal{O}_L \), we have \( N_{\mathbb{Q}/M} \sum \alpha \zeta \beta^0_\zeta \neq 0 \in \mathbb{Z} \). So \( |N_{\mathbb{Q}/M} (\sum \alpha \zeta \beta^0_\zeta)| = 1. \)

On the other hand, \( t \in \Omega_{B, \varepsilon} \) implies for all \( j, \)

\[
\varepsilon \leq |B \exp (t) A_0 \varepsilon^{J_0} \wedge \varepsilon^{\theta_1} \wedge \ldots \wedge \varepsilon^{\theta_j} |
\]

\[
\leq |B \exp (t) e_{J_0} \wedge \varepsilon^{\theta_1} \wedge \ldots \wedge \varepsilon^{\theta_j} |
\]

\[
\leq \exp \left( \sum_{i=0}^j \theta^{\delta(i)} (t) \right) |B_{J_0} \wedge B^{\theta_1} \wedge \ldots \wedge B^{\theta_j} |
\]

Hence by Proposition above,

\[
sup_{t' \in \Omega} |B \exp (t + t') A_0 v| \geq \kappa_1 \cdot 1 \cdot \prod_{j=0}^{a_0} |B_0| (d_{e_j} - d_{e_{j+1}}) \exp \left( d_{e_{j+1}} - d_{e_j} \right)
\]

\[
\geq \kappa_1 |B_0|
\]

Now, taking \( \varepsilon' := \kappa_1 \min \{ \varepsilon, \varepsilon^{2j} \} \) completes the proof.

To prove Proposition 3.7, the following easy lemma proves to be useful (see [7]). Proof is omitted here.

Lemma 3.9. (1) (i) The collection of functions \( \{ t \to \exp \chi_{e(t)} \}_{\chi \subseteq [l_0] \times \ldots \times [l_{a_0}}} \) from \( \Omega \to \mathbb{R} \) are linearly independent.

(ii) The subcollection \( \{ t \to \exp \chi_{e(t)} \}_{\chi \subseteq \mathcal{A}} \) are still linearly independent when restricted to \( \Omega_{\mathcal{A}} \).

(2) There exists \( \kappa_2 = \kappa_2(\Omega) \) such that for any \( \mathcal{A} \subset [l_0] \times \ldots \times [l_{a_0}] \) nonempty, for any \( \{ v_\chi \}_{\chi \subseteq \mathcal{A}} \) collection of vectors in some normed vector space,

(i) if \( \{ \chi_{|\text{Lie}(T_s)} \}_{\chi \subseteq \mathcal{A}} \) are linearly independent,

\[
\sup_{t \in \Omega} \left\| \sum_{\chi \subseteq \mathcal{A}} \exp \chi_{e(t)} \right\| \geq \kappa_2 \sup_{\chi \subseteq \mathcal{A}} \left\| v_\chi \right\|
\]

(ii) if \( \{ \chi_{|\text{Ad}(A_0)\text{Lie}(T_s)} \}_{\chi \subseteq \mathcal{A}} \) are linearly independent,

\[
\sup_{t \in \Omega} \left\| \sum_{\chi \subseteq \mathcal{A}} \exp \chi_{e(t)} \right\| \geq \kappa_2 \sup_{\chi \subseteq \mathcal{A}} \left\| v_\chi \right\|
\]

Proof of Proposition 3.7. Take such a vector \( A_0 v = A_0 \sum_{f \in \mathcal{A}} \alpha f e_{I_f} \). We may rewrite as:

\[
A_0 v = A_0 \sum_{d=J, \ldots, J_{a_0}} e_J \wedge v_d = \sum_{d=J, \ldots, J_{a_0}} e_J \wedge A_0 v_d
\]

with \( v_d \in \bigwedge^{d_1} \mathcal{O}_{M_1} \wedge \ldots \wedge \bigwedge^{d_{a_0}} \mathcal{O}_{M_{a_0}} \).

As \( A_0 \) preserves each copy of \( \mathcal{O}_{M_1} \), \( A_0 v_d \) belongs to \( \bigwedge^{d_1} \mathcal{O}_{L_1} \wedge \ldots \wedge \bigwedge^{d_{a_0}} \mathcal{O}_{L_{a_0}} \).

Note that for \( w \in \bigwedge^{d_1} L_1 \wedge \ldots \wedge \bigwedge^{d_{a_0}} L_{a_0}, e_J \wedge w \) is a common eigenvector for \( T_s \) and if we denote by \( \chi_{J,d} \) the corresponding character, \( \{ \chi_{J,d} \}_{J,d} \) are linearly independent. By Lemma 3.9, we see

\[
\sup_{t' \in \Omega} \left\| B \exp (t + t' + t') A_0 v \right\| \geq \kappa_2 \sup_{d,J} \left\| B \exp (t + t') e_J \wedge A_0 v_d \right\|
\]

Take \( d = (d_1, \ldots, d_{a_0}), J_0 \) that achieves the supreme on the right hand side. We write

\[
v_d = \sum_{\zeta, \xi, \ldots, \xi \subseteq [l_1] \ldots [l_{a_0}]} \alpha \xi^{1} \wedge \ldots \wedge \xi^{a_0}
\]
, then
\[
e_{J_0} \wedge A_0 v_d = \sum_{\zeta = \xi_1 \cup \ldots \cup \xi_n \subseteq [1]_{d_1} \cup \ldots \cup [1]_{d_n}} \alpha_\zeta e_{J_0} \wedge \xi_1^1 \wedge \ldots \wedge \xi_n^1
\]
\[
= \sum_{\zeta} \alpha_\zeta e_{J_0} \wedge \left( \sum_{\xi_1 \subseteq [1]_{d_1}} \beta_{\xi_1}^1 e_{\xi_1}^1 \right) \wedge \ldots \wedge \left( \sum_{\xi_n \subseteq [1]_{d_n}} \beta_{\xi_n}^1 e_{\xi_n}^1 \right)
\]
\[
= \sum_{\zeta} \alpha_\zeta \sum_{\xi} \beta_{\xi} e_{J_0} \wedge e_{\xi_1}^1 \wedge \ldots \wedge e_{\xi_n}^1
\]
\[
= \sum_{\zeta} \left( \sum_{\xi} \alpha_\zeta \beta_{\xi} e_{J_0} \wedge e_{\xi_1}^1 \wedge \ldots \wedge e_{\xi_n}^1 \right).
\]
(12)

And now we can apply Lemma 3.9 again to see

\[
\kappa_2 \sup_{\ell \in \Omega_a} \|B \exp(t + t'_1)e_{J_0} \wedge A_0 v_d\| \geq \kappa_2^2 \sup_{\zeta} \left\| \sum_{\zeta} \alpha_\zeta \beta_{\xi}^1 \|B \exp(t)e_{J_0} \wedge e_{\xi_1}^1 \wedge \ldots \wedge e_{\xi_n}^1 \| \right\|
\]

Take $\xi_0$ such that $\sum_{\zeta} \alpha_\zeta \beta_{\xi}^1 \neq 0$.

Before we proceed, let’s say a few words about the induced action of Galois group $Gal = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Gal naturally acts on $Gal(\overline{\mathbb{Q}}/\mathbb{Q})/Gal(L_i/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$ which can be identified with $[1]_d$ by $\sigma_j \mapsto j$. Hence $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $[1]_{d_1} \cup [1]_{d_2} \cup \ldots \cup [1]_{d_n}$. Also, for $\sigma \in Gal$, $\sigma(\beta_{\xi}) = sgn(\sigma, \xi) \beta_{\xi}^\sigma$ for some function $sgn = sgn(\sigma, \xi)$ (independent of $\xi$) taking value in $\{-1, 1\}$.

Now let’s define $B_0 := Gal : \xi_0 \subseteq [1]_{d_1} \cup \ldots \cup [1]_{d_n}$ and let $Gal_0$ be the restriction of stabilizer of $\xi_0$ to $Gal(L/\mathbb{Q})$ (Note Gal action factors through that of $Gal(L/\mathbb{Q})$). Recall $L$ is defined to be a Galois number field that contains all $L_i$.

\[
| \prod_{\xi \in \mathcal{B}_0} \sum_{\zeta} \alpha_\zeta \beta_{\xi}^1 | \prod_{\xi \in \mathcal{B}_0} \|B \exp(t)e_{J_0} \wedge e_{\xi}\|
\]
\[
= \prod_{\sigma \in Gal(L_i/\mathbb{Q})/Gal_0} \sigma(\sum_{\zeta} \alpha_\zeta \beta_{\xi}^1) \prod_{\xi \in \mathcal{B}_0} (\exp\chi_{J_0 \cup \xi}(t)) \|B e_{J_0} \wedge e_{\xi}\|
\]
\[
= |Nm_{L/\mathbb{Q}}(\sum_{\zeta} \alpha_\zeta \beta_{\xi}^1)|^{1/|Gal_0|} \exp(\sum_{\xi \in \mathcal{B}_0} \chi_{J_0 \cup \xi}(t)) \prod_{\xi \in \mathcal{B}_0} \|B e_{J_0} \wedge e_{\xi}\|
\]

Now it remains to estimate $\exp(\sum_{\xi \in \mathcal{B}_0} \chi_{J_0 \cup \xi}(t)) \prod_{\xi \in \mathcal{B}_0} \|B e_{J_0} \wedge e_{\xi}\|$.

Because of the transitive Galois action on each $[1]_d$, we can find a non-negative constant $c_i$ for $i = 1, \ldots, a_0$ such that $c_i = \{\xi \in \mathcal{B}_0 | j \in \xi\}$ for all $j \in [1]_d$. One can see that

\[
\sum_{\xi \in \mathcal{B}_0} \chi_{J_0 \cup \xi} = |\mathcal{B}_0| \chi_{J_0} + \sum_{i=1}^{a_0} c_i \chi^i
\]

Applying both sides to $\sum_{j \in [1]_d} E_{i,j}^{j'} (E_{i,j}^{j'})^T$ is a $N$-by-$N$ matrix such that $E_{i,j}^{j'} e_{j'} = e_{i}$ and when $(i', j') \neq (i, j)$, $E_{i,j}^{j'} e_{j'} = 0$ we see $d_i|\mathcal{B}_0| = c_i l_i \implies c_i = |\mathcal{B}_0| d_i/l_i$.

Take $\theta \in \text{Perm}\{1, \ldots, a_0\}$ such that $1 \geq d_1|l_1| \geq d_2|l_2| \geq \ldots \geq d_{a_0}|l_{a_0}| \geq 0$. Also define $d_0/l_0 = d_{a_0}/l_{a_0} = 1$ and $d_{a_0+1}/l_{a_0+1} = d_{\theta(a_0+1)}/l_{\theta(a_0+1)} = 0$. To lighten the notation we shall omit $\theta$ in the following. Set $\chi^0 := \chi_{J_0}$. With these notations we have

\[
\chi_{J_0}^0 + \sum_{j=1}^{a_0} c_j |\mathcal{B}_0| \chi^j = \chi_{J_0} + \sum_{j=1}^{a_0} \frac{d_j}{l_j} \chi^j = \sum_{j=0}^{a_0} \frac{d_j}{l_j} \chi^j = \sum_{j=0}^{a_0} (d_j/l_j - d_{j+1}/l_{j+1})(\sum_{i=0}^{j} \chi^i)
\]

Hence

\[
\exp\left( \sum_{\xi \in \mathcal{B}_0} \chi_{J_0 \cup \xi}(t) \right) \prod_{\xi \in \mathcal{B}_0} \|B e_{J_0} \wedge e_{\xi}\| = \exp(\sum_{j=0}^{a_0} (d_j/l_j - d_{j+1}/l_{j+1})(\sum_{i=0}^{j} \chi^i(t))) \prod_{\xi \in \mathcal{B}_0} \|B e_{J_0} \wedge B_{\xi}\|
\]
For each ξ = ξ₁ \sqcup ... \sqcup ξₙ₀ ∈ ℜ₀ we write ξᵢ = \{kᵢ₁ < ... < kᵢ_dᵢ\}. For I ∈ [l₀] \sqcup ... \sqcup [lₙ₀], define V_I := the subspace represented by Bₑ[I]. Observe that if I ⊂ J, then V_I ⊂ V_J and hence dist(V_I, v) ≥ dist(V_J, v) for any vector v (distance is computed with respect to the Euclidean metric). For \(i ∈ \{1, ..., a₀\}\), \(k ∈ [l_i]\), define \(k^< := J_0 \sqcup [l_1] \sqcup ... \sqcup [l_{i-1}] \sqcup \{1, ..., k - 1\}\).

Now we compute:

\[
\prod_{ξ ∈ ℜ₀} \|B_{J₀} \land B_ξ\| \\
= \prod_{ξ ∈ ℜ₀} \text{covol}(\oplus_{ι ∈ J₀} ξ \mathbb{Z}B_{ι}) \\
= \prod_{ξ ∈ ℜ₀} \text{covol}(\oplus_{ι ∈ J₀} ξ \mathbb{Z}B_{ι}) \text{dist}(B_{k₁}, V_{J₀}) \text{dist}(B_{k₂}, V_{J₀ \cup \{k_1\}}) \cdots \text{dist}(B_{k^{a₀}_{J₀}}, V_{J₀ \cup \{k_1, ..., k^{a₀}_{J₀} - 1\}}) \\
≥ \prod_{ξ ∈ ℜ₀} \|B_{J₀}\| \text{dist}(B_{k₁}, V_{k}) \text{dist}(B_{k₂}, V_{k}) \cdots \text{dist}(B_{k^{a₀}_{J₀}}, V_{k^{a₀}_{J₀}}) \\
= \|B_{J₀}\| \prod_{j = 1}^{a₀} \prod_{k ∈ [l_j]} d(B_{k}, V_{k}) |\mathbb{R}_0| d_j/l_j
\]

Here "covol" is computed from the Euclidean metric.

The right hand side can be transformed into:

\[
\|B_{J₀}\| |\mathbb{R}_0| \prod_{j = 1}^{a₀} \prod_{k ∈ [l_j]} d(B_{k}, V_{k}) |\mathbb{R}_0| d_j/l_j \\
= \|B_{J₀}\| |\mathbb{R}_0| \sum_{a ≥ j} d_a/l_a - d_{a+1}/l_{a+1} \\
\]

(19)

\[
= \prod_{j = 1}^{a₀} \|B_{J₀}\| |\mathbb{R}_0| (d_j/l_j - d_{j+1}/l_{j+1}) \prod_{k ∈ [l_i] \cup ... \cup [l_j]} d(B_{k}, V_{k}) |\mathbb{R}_0| (d_j/l_j - d_{j+1}/l_{j+1}) \\
= \prod_{j = 0}^{a₀} \|B_{J₀} \land B_1 \land ... \land B_j\| |\mathbb{R}_0| (d_j/l_j - d_{j+1}/l_{j+1})
\]

Combining equations 11, 13, 14, 17, 18 and 19 completes the proof.

Recall for a finite measure \(μ, \hat{μ}\) denotes the probability measure obtained by rescaling.

Now we can invoke Theorem 5.2 from [11] (see also [12]) to conclude:

**Corollary 3.10.** For any δ > 0 there exists ε'' = ε''(δ, ε, Ω) such that for all B ∈ G, t ∈ Lie(T) with (Ω + t) ∩ Ω_{B, ε} ≠ ∅, we have

\[(20) \quad \text{Leb}\{t' ∈ Ω | B \exp(t + t') \mathbb{Z}^N \notin \mathcal{L}_{ε''}\} ≤ δ\]

Here \(\mathcal{L}_{ε''} := \{G | \inf_{v ∈ \mathbb{Z}^N} ||g \mathbb{Z}^N|| ≥ ε''\}\).

Now by Mahler’s criterion (see for instance [1]) we have:

**Corollary 3.11.** For any sequence \(g_k\) in \(G\),

1. all weak limit of \((g_k) \ast \mu_{\Omega_{g_k, \cdot}}\) are probability measures on \(G/Γ\).
2. Assume \((t_k + Ω) ∩ Ω_{g_k, \cdot} ≠ ∅\), all weak limits of \((g_k \exp t_k) \ast \mu_{\exp Ω}\) are probability measures on \(G/Γ\).

4. Split part, graph and convex polytope

In this section \(M = \mathbb{Q}\) and we are given a sequence \(\{g_k\} ⊂ G\) such that \(g_k \mathbb{Z}_G(S)\) diverges in \(G/\mathbb{Z}_G(S)\) for any subtorus \(S\) of \(T_s\). The main statement we want to establish here is
Proposition 4.1. There exists a sequence of real numbers $\omega_k \to +\infty$. If we define $\Omega_{g_k,\varepsilon} := \Omega_{g_k,\varepsilon+\omega_k}$, its volume remains asymptotically the same as $\Omega_{g_k,\varepsilon}$, i.e.
\[
\lim_{k} \frac{\text{Vol}(\Omega_{g_k,\varepsilon})}{\text{Vol}(\Omega_{g_k,\varepsilon+\omega_k})} = 1
\]  

The proof will go in the same way as that in [20] except that a few definitions need to be modified. For this reason we only make the necessary definitions and state the main steps but will omit the proofs and refer the reader to [20].

First as $G/P_T$ is compact, we may assume $g_k \in P_T$. Also $P_T = R_s(P_T)H_sT_s$, so we may write $g_k = u_k h_k t_k$. Since $t_k$ will only translate the polytope and hence preserve the volume, we may assume $g_k = u_k h_k$. Now recall our definition of $\Omega_{g_k,\varepsilon}$ and that $h_k e_t = e_t$ for all $I \in \mathcal{I}_0$, we have $\Omega_{u_k h_k,\varepsilon} = \Omega_{u_k,\varepsilon}$. One also sees that when we replace $g_k$ by $u_k$ this way, $u_k Z_G(S)$ still diverges for all subtorus $S$ of $T_s$. In summary, to prove the above proposition we may assume $g_k = u_k$.

However, to define the graph correctly, we need to further:

1. divide $\{u_k\}$ into union of finitely many disjoint subsequences and replace $u_k$ by any one of them and
2. modify $\{u_k\}$ by a bounded sequence in $G$ from left such that for each $i, j \in [0] \cup \{1, \ldots, a_0\}$ with $i < j$ (elements from $[0]$ should always be smaller than those from $\{1, \ldots, a_0\}$), either $u_k^{(i,j)}$ diverges or remains 0. Recall that we have divided $u_k$ into blocks similar to that of $T$. Let’s remark here that such a modification is necessary for Proposition 4.2 below to be true.

Now we can define a graph $G(u_k) = (\mathcal{V}, \mathcal{E})$. $\mathcal{V} := [0] \cup \{1, \ldots, a_0\}$ and $\{i < j\} \subset \mathcal{V}$ is in $\mathcal{E}$ iff $u_k^{(i,j)}$ diverges.

Proposition 4.2. $G(u_k)$ is connected.

Recall the definition of UDS from [20]:

Definition 4.3. A subset $I \subset \mathcal{V}$ is called UDS iff for any $j \in I$, any $i \in \mathcal{V}$ that is smaller than $j$, $\{i, j\} \in \mathcal{E}$ implies $i \in I$.

Proposition 4.4. Let $I \subset [0]$ and $J \subset \{1, \ldots, a_0\}$.
1. $u_k e_0^I \cap e_J$ either diverges or remains bounded;
2. $u_k e_0^I \cap e_J$ diverges iff $I \sqcup J$ is UDS.

Arguing the same way as was done in [20] (see 5.8, 6.1, 6.2 therein), we obtain the following important output of the above graph argument.

Proposition 4.5. There exists $R_k \to +\infty$ such that $\Omega_{u_k,\varepsilon}$ contains a ball of radius $R_k$.

Then by using some interesting properties of convex polytopes we can conclude the proof of Proposition 4.1 (see section 4 of [20]).

5. LIMITING MEASURES IN THE GENERIC CASE

If $M = \mathbb{Q}$, and we are given $\{g_k\}$ diverging in $G/Z_G(S)$ for all $S \subset T$ subtorus defined over $\mathbb{Q}$. Let’s take $t_k \in \text{Lie}(T_s)$ satisfying $(t_k + \Omega_s) \cap \Omega_{g_k,\varepsilon} \neq \emptyset$. By Corollary 3.11 from section 3, $(g_k t_k)_* \mu_{\Omega^T}$ is nondivergent. Passing to a subsequence we assume $\lim_k (g_k t_k)_* \mu_{\Omega^T} = \nu \in \text{Prob}(G/\Gamma)$.

Theorem 5.1. $\nu = \bar{\mu}_{G/\Gamma}$.

Corollary 5.2. $\lim_k (g_k)_* \mu_{T S/\Gamma} = \bar{\mu}_{G/\Gamma}$.

Again by Corollary 3.11 or nondivergence, we may find a bounded sequence $\delta_k$ in $G$ and $\gamma_k \in \mathcal{G}(\mathbb{Z})$ such that $g_k t_k = \delta_k \gamma_k$. By Theorem 2.1 from [9], it suffices to prove

Proposition 5.3. $\bigcup_k (\delta_k^{-1} \gamma_k T \gamma_k^{-1}) = \mathbb{G}$.

Remark 5.4. Actually one hypothesis of Theorem 2.1 in [9] is not satisfied here. Namely, our ”$H$" definitely is allowed to have a lot of characters. But still exactly the same argument as the proof of Theorem 2.1 carries through in our situation. Let’s also emphasize here that to deduce Theorem 5.1 from Proposition 5.3 is highly nontrivial. This was done in [9] by applying Ratner’s theorem [18] and Linearization [5] methods. Arguing like that in [20], one may choose to apply these two directly and avoid the use of the proposition. But we feel it is cleaner to separate the argument.
Proposition 5.5. Given \( \{b_k\}_{k \in \mathbb{N}} \subset G(M) \) that diverges in \( G/Z_G(S) \) for all \( S \subset T \) torus defined over \( M \). Let \( L := \langle \bigcup_k b_k T b_k^{-1} \rangle \). Assume \( L \neq G \) and let \( P \) be a maximal connected \( M \)-subgroup containing \( L \). Then \( P \) is parabolic.

To prove this, we need two lemmas which should be known but are included here for the lack of reference.

Lemma 5.6. There is no proper semisimple subgroup of \( G \) containing a maximal torus.

Remark 5.7. This is definitely not true for all simple groups, say \( SO(n,n) \subset SO(n,n+1) \).

Proof. Let \( F \) be such a group and \( F := F(\mathbb{C}) \), without loss of generality we assume \( F \) contains \( D \). Hence for \( i < j \), \( E_{i,j} \in \text{Lie}(F) \) iff there exists \( A \in \text{Lie}(F) \) with \( A_{ij} \neq 0 \). Now look at the graph \( F^+ = (V, E) \) with \( V = \{1, \ldots, N\} \) and \( i < j \) \( \in V \) iff \( E_{ij} \in \text{Lie}(F) \). This graph has to be connected for otherwise we can write \( V = I_1 \cup I_2 \), both nonempty and are not connected to each other. Consider \( a \in F \) defined to be exp \((1/|1|) \sum_{i \in I_1} E_{ii} - 1/|I_2| \sum_{i \in I_2} E_{ii} \), it centralizes the full upper triangular group in \( M \) and hence centralize the full \( F \), hence a contradiction. Now \( F^+ \) is connected. For any \( i < j \), find \( i = i_0, i_1, \ldots, i_k = j \) such that \( \{i_0, i_1, i_2\} \subset \mathcal{E} \), then \( \pm E_{ij} = [-[E_{i_0 j}, E_{i_1 j}], E_{i_2 j}], \ldots, E_{i_k - i_1 j}] \) is in \( \text{Lie}(F) \). Hence all upper triangular nilpotent matrices are in \( \text{Lie}(F) \). Similarly one can prove this for the lower triangular ones.

Hence \( F = G(\mathbb{C}) \).

For two subgroups \( A, B \) of \( G \) defined over a subfield of \( \mathbb{R} \), Let \( Z(A,B) := \{g \in G, g^{-1}Ag \subset B\} \).

Lemma 5.8. For a maximal \( \mathbb{R} \)-split torus \( T \) and a subtorus \( S \) which is automatically \( \mathbb{R} \)-split, \( Z(S,T) = Z_G(S)N_G(T) \).

Proof. Wlog, we may assume \( T = \mathbb{D} \). Take a regular element \( s \in S \) and \( g \in Z(S,T) \). Hence \( g^{-1}sg \) is diagonalized, which is differed from \( s \) by some permutation. But as \( \mathbb{D} \) is \( \mathbb{R} \)-split, \( N_G(T)/Z_G(T) \) equals to the Weyl group, and in particular, it contains all permutations. Hence we may find \( \sigma \in N_G(T) \) s.t. \( g^{-1}sg = \sigma^{-1}s\sigma \) which implies \( \sigma g^{-1}sg \sigma^{-1} = s \). So \( \sigma g^{-1} \in Z_G(s) = Z_G(S) \), done.

Remark 5.9. For the two lemmas above, the same proof works for finite product of \( SL_n \)'s.

Proof of Proposition 5.5. By [14] or [3], \( P \) is either parabolic or reductive. Indeed, in our case this can be seen directly by assuming it is not reductive and take the normalizer of its unipotent radical. Let’s assume it is reductive and search for a contradiction.

By Lemma 5.6 above, \( F \) has a nontrivial center, call it \( S_0 \). \( S_0 \) is defined over \( M \). Let’s define \( T_k := b_k T b_k^{-1} \) and \( b_k := b_k b_k^{-1} \). Then

1. \( P \) contains \( T_k \);
2. \( b_k \) diverges in all \( G/Z_G(S) \) for all \( S \subset T_1 \), subtorus defined over \( M \).

(1) implies \( b_k T_k b_k^{-1} \supset S_0 \) for all \( k \). Hence \( b_k \in Z(S_0, T_1) \). By Lemma 5.8 above, we may write \( b_k = c_k d_k \) for \( c_k \in Z_G(S_0) \) and \( d_k \in N_G(T) \). Since \( N_G(T)/Z_G(T) \) is finite, passing to a subsequence, we may assume \( d_k = d_k d_0 \) with \( d_k \in Z_G(T) \) and \( d_0 \in N_G(T) \). Now \( b_k^{-1} S_0 b_k = d_k^{-1} S_0 d_k \) is defined over \( M \). Hence by assumption \( b_k = c_k d_k d_0 \) diverges in \( G/Z_G(d_0^{-1} S_0 d_0) = G/d_0^{-1} Z_G(S_0) d_0 \). So \( c_k d_k \) diverges in \( G/Z_G(S_0) \). But \( c_k, d_k \in Z_G(S_0) \), contradiction.

Definition 5.10. \( V_1 \subset M^N \) is said to be adapted to \( \mathcal{A_0} \) if \( V_1 \) is spanned by \( \{e_i\}_{i \in I} \) for some \( I \in \mathcal{A_0} \).

Proposition 5.11. Any \( M \)-subspace of \( M^N \) invariant under \( T \) is adapted to \( \mathcal{A_0} \).

Lemma 5.12. A maximal \( M \)-anisotropic torus \( S_2 \) of \( SL_n \) acts irreducibly on \( M^n \).

Proof. We know the torus is reductive in \( SL_n \), hence if its action were not irreducible, then \( M^n = V_1 \oplus V_2 \). But \( S_2 \) has no \( M \)-character, so this gives an embedding of \( S_2 \) into \( SL(V_1) \oplus SL(V_2) \), which is a contradiction.

Proof of Proposition 5.11. With respect to the \( T \)-action, \( M^N = M_1 \oplus \ldots \oplus M_l \oplus M_1^{\prime} \oplus M_2^{\prime} \oplus \ldots \oplus M_k^{\prime} \), where each \( M_i \). Write \( \pi_1, \ldots, \pi_l, \pi_1^{\prime}, \ldots, \pi_k^{\prime} \) for the associate projection. Let \( V_1 \) to be a subspace defined over \( M \) and invariant under \( T \). If \( \pi(V_1) \) is nonzero, then we can find \( v \in V_1 \) s.t. \( v' = \pi(v) \neq 0 \). By the previous lemma, \( T^1 \) acts irreducible on \( M^l \). Hence \( M \)-span \( \{T^1 v \} = M^m \). Also \( T^1 \) acts as identity map on the complementary subspace. Hence \( M \)-span \( \{T^1 v \} \) minus \( M \)-span \( \{T^1 v \} \) contains \( M^l \). Hence
Let $M^i \subset V_i$. In a similar fashion, one can show if $\pi_i(V_i)$ is nonzero, then $V_i$ contains the corresponding copy of $M^i$. In other words, $V_i$ is spanned by several $M^i$ and $M_j$’s. This finishes the proof.

Now we can prove Proposition 5.3, so we assume $M = \mathbb{Q}$ here.

**Proof of Proposition 5.3.** Assume this proposition is false. So we can find a proper maximal connected $\mathbb{Q}$-subgroup $\mathbb{P}$ containing $\bigcup_k \gamma_k T \gamma_k^{-1}$. By Proposition 5.5, $\mathbb{P}$ is parabolic. Now $\gamma_1^{-1} \mathbb{P} \gamma_1$ will stabilize a minimal flag $\{\{0\} \subset V \subset \mathbb{Q}^N\}$.

As $\mathbb{T} \subset \gamma_1^{-1} \mathbb{P} \gamma_1$, we can apply Proposition 5.11 to conclude $V$ is adapted to $\mathcal{A}_0$. Hence $V$ is spanned by $\{e_i\}_{i \in I}$ for some $I \in \mathcal{A}_0$. In other words, for all $k$, $\gamma_1^{-1} \gamma_k T (\gamma_1^{-1} \gamma_k)^{-1} e_I \subset \mathbb{R} e_I$. Now if $\mathbb{T}$ is $\mathbb{Q}$-anisotropic, we are already done. So assume it is not.

Hence $T$ preserves the $\mathbb{Q}$-subspace corresponding to $\gamma_1^{-1} \gamma_k^{-1} e_I$ which implies $\gamma_k^{-1} \gamma_k (\gamma_1^{-1} \gamma_k)^{-1} e_I = \lambda_k e_I$. For some $J_k \in \mathcal{A}_0$, $\lambda_k \in \mathbb{Q}$. But $\mathcal{A}_0$ is finite. Passing to a subsequence we may assume $J_k = J_0$ for all $k$. $(\gamma_1^{-1} \gamma_k)^{-1} e_I = \lambda_k e_I$. Recall all $\gamma$’s are integral matrices, so $\lambda_k \in \mathbb{Z}$. By looking at $(1/\lambda_k) e_I = \gamma_1^{-1} \gamma_k e_I$ we see $1/\lambda_k \in \mathbb{Z}$. Hence $\lambda_k = \pm 1$.

Now we have $\gamma_k e_{J_0} = \pm \gamma_1 e_I$ for all $k$. But $\delta_k \gamma_k = g_k t_k$ and $||g_k t_k e_{J_0}|| \rightarrow +\infty$, so $||\gamma_k e_{J_0}|| \rightarrow +\infty$, a contradiction.

\[ \square \]

6. Restriction of scalars

Recall $M$ is a totally real number field of degree $m_0$, not necessarily Galois. Let $\mathbb{T} := R_{M/\mathbb{Q}^T}$, $\mathbb{G}' := R_{M/\mathbb{Q}^T} \mathbb{G}$, $\Gamma'$ an arithmetic lattice. In this section we want to show for $g_k \in \mathbb{G}'$ which diverges in all $G'/Z_{G'} S$ for all $S \subset \mathbb{T}$ subtorus defined over $\mathbb{Q}$.

**Theorem 6.1.** $(g_k)_{\mu \in \mathbb{T}'} \rightarrow \mu_{\mathbb{G}' / \Gamma'}$.

6.1. Preparations. In this section we collect some notations and lemmas for the restriction of scalars. For a reference, see [21].

Take $\{w_1, \ldots, w_{m_0}\}$ to be a basis for $\mathcal{O}_M$ as a free $\mathbb{Z}$-Module. Let $\{\tau_1, \ldots, \tau_{m_0}\}$ be a full set of representatives in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})/Gal(\overline{\mathbb{Q}}/\mathcal{O}_M)$.

$V' := Res_{M/\mathbb{Q}} V$ is defined to be $V_{\oplus m_0}$ with an integral($\mathbb{Z}$) structure given by the following basis:

\[
\{(\tau_i (w_k e_j), \ldots, \tau_{m_0} (w_k e_j)) | k \in \{1, \ldots, m_0\}, i \in \{0, \ldots, a_0\}, j \in \{1, \ldots, l_i\}\}
\]

Actually, $V'$ is not so different from $V$ in that $V(M) \cong V'(\mathbb{Q})$, $V(\mathcal{O}_M) \cong V'(\mathbb{Z})$ where the bijection is given by $v \mapsto v' := (\tau_i v, \ldots, \tau_{m_0} v)$. Hence $V'(\mathbb{Z})$ becomes an $\mathcal{O}_M$-module via this bijection, explicitly, $w \cdot (v_1, \ldots, v_{m_0}) = (\tau_1 (w v_1), \ldots, \tau_{m_0} (w v_{m_0}))$ for $w \in M$ and $(v_1, \ldots, v_{m_0}) \in V'(\mathbb{Q})$. There is a crucial difference, however, lying in $\wedge^a V$ and $\wedge^a V'$. For instance, when $m_0 > 1$ and $a = N + 1$, the former space is reduced to $\{0\}$ while the latter is not.

For $v = v_1 \wedge \ldots \wedge v_a \in \wedge^a V(M)$ define

\[
R_{M/\mathbb{Q}}(v_i) := w_1 \cdot v_1' \wedge w_2 \cdot v_2' \wedge \ldots \wedge w_{m_0} \cdot v_{m_0}'
\]

\[
= (\tau_1 (w_1 v_1), \ldots, \tau_{m_0} (w_1 v_1)) \wedge \ldots \wedge (\tau_1 (w_{m_0} v_1), \ldots, \tau_{m_0} (w_{m_0} v_1))
\]

\[
= \det (\tau_k (w_j))_{k,j} (\tau_1 v_1, 0, \ldots, 0) \wedge \ldots \wedge (0, \ldots, 0, \tau_{m_0} v_1)
\]

and

\[
R_{M/\mathbb{Q}}(v) := R_{M/\mathbb{Q}}(v_1) \wedge \ldots \wedge R_{M/\mathbb{Q}}(v_a)
\]

\[
= \det (\tau_k (w_j))_{k,j} (\tau_1 v_1, 0, \ldots, 0) \wedge \ldots \wedge (0, \ldots, 0, \tau_{m_0} v)
\]

In other words, for $v_1, \ldots, v_a$ in $V(M)$ independent over $M$, if we take the $\mathcal{O}_M$-module spanned by them and regard it as a $\mathbb{Z}$-module sitting inside $V'(\mathbb{Z})$, then it has a set of basis given by $\{w_1 \cdot v_1', \ldots, w_{m_0} \cdot v_{m_0}'\}$ whose exterior product gives $R_{M/\mathbb{Q}}(v)$.

We equip $V'(\mathbb{R}) = V(\mathbb{R})_{\oplus m_0}$ with the product Euclidean metric under which $||(v_1, \ldots, v_{m_0})||^2 = \sum ||v_i||^2$ and $\{(v_1', 0, \ldots, 0), \ldots, (0, \ldots, 0, e_{m_0}')\}$ form an orthonormal basis. Under this metric, we can talk about covolume of a $\mathbb{Z}$-submodule in the $\mathbb{R}$-span of this lattice. Taking exterior products of this basis of metrics induces metrics on all exterior powers of $V'$. We have if $\Lambda' = \mathbb{Z} v_1' + \ldots + \mathbb{Z} v_a'$ then $\text{covol}(\Lambda') = ||v_1' \wedge \ldots \wedge v_a'||$. Also note that

\[
||R_{M/\mathbb{Q}} v|| = ||\text{det} (\tau_k (w_j))_{k,j}||^a \prod_{i=1}^{m_0} ||\tau_i v||
\]
Lemma 6.2. [Decrease of covolume] There is a constant $\kappa_5 > 0$ depending only on the field $M$ and dimension of the vector space $N$ such that for all $v \in V(O_M)$,

$$||R_{M/Q}v|| \leq \kappa_5||v'||^m_0$$

Proof. Indeed,

$$\text{LHS}^2 = \left| \det (\tau_k(w_j))_{k,j} \right|^2 \prod_i ||v_i||^2$$

$$\leq \left| \det (\tau_k(w_j))_{k,j} \right|^2 (\sum_i ||v_i||^2/m_0)^{m_0} = \text{RHS}^2$$

where $\kappa_5$ is defined by the last equation. $\square$

Remark 6.3. Ideally one may show a more general version of "decrease of covolume": for $\Lambda \subset V'(Z)$ a $Z$–sublattice and $\Lambda'$ the minimal $O_M$–module containing $\Lambda$ then $\text{covol}(\Lambda') \ll \text{covol}(\Lambda)^n$ where the implicit constant only depends on the field $M$ and dimension $N$. We don’t know whether this is true.

Define $\mathcal{P}_0 := \{\text{Primitive } Z \text{ lattices in } V'(Z)\}$, $\mathcal{P}_M := \{\Lambda \in \mathcal{P}, O_M \Lambda \subset \Lambda\}$.

Lemma 6.4. Given $\Lambda \in \mathcal{P}_0$ a primitive $Z$–module in $V'(Z)$. If there is a full rank $Z$–submodule $\Lambda' \subset \Lambda$ that is also an $O_M$–module, then $\Lambda$, i.e. $\Lambda \in \mathcal{P}_M$.

Proof. Take $\Lambda' \subset \Lambda$ as in the lemma. $O_M\Lambda' \subset \Lambda'$ implies $O_M\Lambda' \otimes Z Q \subset \Lambda' \otimes Z Q$. But $\Lambda$, being primitive, is equal to $(\Lambda' \otimes Z Q) \cap V'(Z)$. So we are done.

Lemma 6.5. Given $v = v_1 \wedge \ldots \wedge v_a \in \bigwedge^a V(O_M)$. $R_{M/Q}v$ corresponds to a lattice($Z$–module) that is also a free $O_M$–module with basis $v'_1, \ldots, v'_a$. Conversely all free $O_M$–modules in $V'(Z)$ arise this way.

Proof. This is an immediate consequence of how we identify $V(O_M) \cong V'(Z)$.

Lemma 6.6. There is a constant $\kappa_6 > 0$ depending only on the field $M$ and dimension $N$ such that any $O_M$–submodule of $V'(Z)$ has a free $O_M$–submodule of index smaller than $\kappa_6$.

Proof. Take $\Lambda$ to be such an $O_M$–submodule of rank $l + 1$. As $O_M$ is a Dedekind domain, by Theorem 1.32 in [15] and that our $O_M$–module is torsion-free, we see $\Lambda \cong O_M^\oplus I$ as an $O_M$–module for some ideal $I \subset O_M$. Hence to prove the lemma, we are reduced to the case when $\Lambda \cong I$.

Recall the class group of a number field is finite, hence we can find a finite set of ideals $\{I_1, \ldots, I_k\}$ such that any other ideal maps bijectively to one of them by multiplying by some element in $M$. Clearly this bijection would be an $O_M$–module isomorphism. So to prove the lemma, it suffices to prove it for a single $I$ allowing the constant (the index of free submodule) to depend on $I$. Take $I$ such that $I^k \cong O_M$ by finiteness of class number. So $I^k \subset I$ is a free module of index $|I/I^k| = |R/I^k|^{k-1}$ which is finite. So we are done.

Compatible with $V' = R_{M/Q}V$ we can define $G' = R_{M/Q}G$ by $G'(Z) := \{g \in G^\oplus m_0, gV'(Z) = V'(Z)\}$. Hence $G'(Q) = \{g' := (\tau_1 g, \ldots, \tau_m g), g \in G(M)\} \cong G(M)$. Then $T'_s := R_{M/Q}T$ is a maximal torus of $G'$ defined over $Q$. Moreover its split part is given by $T'_s(Q) = \{t(\Delta(t) := (t, t, \ldots, t), t \in T_s(Q)\}$. Note this is strictly smaller than $R_{M/Q}T_s$.

The following compatibility can be checked directly:

Lemma 6.7. For $g \in G(M)$ and $v \in \bigwedge^a V(M)$,

$$g' R_{M/Q}v = R_{M/Q}(gv)$$

6.2. Polytope of non-divergence. For $B = (\tau_1 B, \ldots, \tau_m B) \in G'(Q)$ and $\varepsilon > 0$,

$$\Omega_{B,\varepsilon}^{R_{M/Q}} := \{(t, \ldots, t) \in \text{Lie}(T'_s) ||B\Delta(\exp(t)) R_{M/Q}e_I|| \geq \varepsilon \forall I \in \mathcal{A}_0\}
= \{(t, \ldots, t) \in \text{Lie}(T'_s) | m_0 \chi_I(t) \geq \log \frac{\varepsilon}{|\det(\tau w_j)|} - \log \prod_{i=1,\ldots,m_0} ||\tau_I B_{1,\ldots,1}|| \forall I \in \mathcal{A}_0\}$$

Remark 6.8. This polytope can be defined for all $B = (B_1, \ldots, B_m) \in G'(R)$. But the above emphasize that we are "doing restriction of scalars".

One can see this is again a polytope. Let’s fix $\Omega'$, a neighborhood of 0 in $\text{Lie}(T')$. 

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Proposition 6.9. There exists $\varepsilon'_4 = \varepsilon'_4(\varepsilon, \Omega')$ which goes to $\infty$ as $\varepsilon$ does so. For all $B \in G'$, all $t \in \Omega_{B, \varepsilon}$, all $v = v_1 \wedge ... \wedge v_a$ with $v_i \in V(O_M)$ for some $a$, we have

$$
(30) \quad \sup_{t' \in \Omega'} \| B\Delta(\exp t') \exp t' R_{M/Q} v \| \geq \varepsilon'_4
$$

Proof. This follows from Proposition 3.7 the same as how Proposition 3.5 follows.

To deduce non-divergence from this proposition, we need the following theorem from [11] or [12].

Theorem 6.10. For $0 < k_1 \leq N_{m_0}$, $\Omega' \subset \mathbb{R}^n$ a ball of finite radius and $C, \alpha > 0$. There exists $\kappa_2 = \kappa_2(k_1, C)$. Take any continuous map $\phi : \Omega \to SL_{N_{m_0}}(\mathbb{R})$ and poset $P \subset P(\mathbb{Z}^{N_{m_0}})$ of length $k_1$ and $\eta \in (0, 1)$. Assume for any $\Lambda \in \mathcal{P}$,

1. $x \mapsto \| \phi(x)\Lambda \|$ is $(C, \alpha)$-good on $\Omega'$;
2. $\sup_{x \in \Omega'} \| \phi(x)\Lambda \| \geq \eta^{r_{k}(\Lambda)}$.

Then for any $0 < \varepsilon_2 < 1$,

$$
(31) \quad \mu_{\Omega'} \{x \in \Omega' \mid x \text{ is not } \varepsilon_2 \text{-protected relative to } \mathcal{P} \} \leq \kappa_2 \varepsilon_2^a
$$

To make use of this theorem we shall take $\mathcal{P} = \mathcal{P}_M$ and $k_1 = \text{length of } \mathcal{P}_M$ which by definition is the maximal length of flags in $\mathcal{P}_M$. $\Omega'$ is the one in Lie($\mathcal{T}_n$) we defined before. $\phi$ will be one of $\phi_{B,t}$ defined by $\phi_{B,t}(t') = B \phi(\exp t) \exp t'$. Condition (1) in the theorem holds for some $C, \alpha$ (independent of $B, t$, see [7]). For condition (2),

Lemma 6.11. Same notation as in Proposition 6.9. For all $\Lambda \in \mathcal{P}_M$,

$$
(32) \quad \sup_{t' \in \Omega'} \| \phi_{B,t}(t')\Lambda \| \geq \varepsilon'_4
$$

Proof. By Lemma 6.6, there exists a free $O_M$ submodule $A_1$ of $\Lambda$ of index smaller than $\kappa_6$. By Lemma 6.5, $A_1$ can be represented by $R_{M/Q} v$ for some $v \in \Lambda^* V(O_M)$. By Proposition 6.9,

$$
\sup_{t' \in \Omega'} \| B\Delta(\exp t) \exp t' A_1 \| \geq \varepsilon'_4
$$

Here $\| \cdot \|$ denotes the covol. Hence we are done taking into account the fact $\lambda$ has index smaller than $\kappa_6$.

We take $\eta := \min_{k = 1, \ldots, N_{m_0}} \{1/2, (\varepsilon'_4/\kappa_6)^{1/k} \}$ and condition (2) is satisfied by this lemma. $\eta$, same as $\varepsilon'_4$, depends only on $\varepsilon$ and $\Omega'$.

The following lemma relates the property $\varepsilon_2$-protected to actual non-divergence.

Lemma 6.12. Take $\varepsilon_2 \in (0, 1)$. Assume $x$ is $\varepsilon_2$-protected relative to $\mathcal{P}_M$ and $\phi = \phi_{B,t}$, then for any $v' \in V'(\mathbb{Z}) \cong \mathbb{Z}^{N_{m_0}}$, $\| \phi(x)v' \| \geq (\varepsilon_2 \eta/\kappa_5)^{1/{m_0}}$. In other words, $\phi(x)\mathbb{Z}^{N_{m_0}} \in \mathcal{L}_{N_{m_0}}((\varepsilon_2 \eta/\kappa_5)^{1/{m_0}})$.

Proof. By denseness we assume $\phi(x) \in \mathcal{G}'(\mathbb{Q})$. Take $v \in V(O_M)$ corresponding to $v' \in V'(\mathbb{Z})$.

Arguing the same way as in Proposition 3.8 of [12], one can show $\| \phi(x)R_{M/Q}(v) \| \geq \varepsilon_2 \eta$. On the other hand, by Lemma 6.2 and 6.7 $\| \phi(x)R_{M/Q}(v) \| = \| R_{M/Q}(\phi(x)v) \| \leq \kappa_5 \| \phi(x)v \|^{{m_0}}$.

Hence $\| \phi(x)v \| \geq (\varepsilon_2 \eta/\kappa_5)^{1/{m_0}}$ and we are done.

Combining Lemma 6.11, 6.12 and Theorem 6.10 the following corollary follows:

Corollary 6.13. (1) For all $\delta, \varepsilon > 0$, there exists $\varepsilon'_2 = \varepsilon'_2(\varepsilon, \Omega, \delta) > 0$ such that for all $B \in G'$, $t \in \Omega'_{B, \varepsilon}$,

$$
(33) \quad \mu_{\Omega'} \{t' \in \Omega' \mid B\Delta(\exp t) \exp t' A_0 \Gamma \in \mathcal{L}_{N_{m_0}}(\varepsilon'_2) \} \geq 1 - \delta
$$

(2) For all sequences $g_k \in G'$ and $t_k \in \Omega_{B, \varepsilon}$, all limits of $(g_k \exp t_k) \mu_{\Omega'}$ are probability measures.

6.3. Graph and convex polytope. For the same reason as in section 4, we only make the necessary modifications on definitions here.

Given $g_k \in \mathcal{G}'(\mathbb{Q}) = \mathcal{G}(M)$ as before we may replace $g_k$ by $u_k \in R_u \mathbb{P}(M)$. Modify $u_k$ such that for all $(i, j)$, $\prod_{\xi} \| T_{\xi} u_{k}(i, j) \|$ is either 0 or goes to infinity. The vertices of the graph $\mathcal{G}$ remains the same and $\{i < j \} \subset \mathcal{V}$ is in $\mathcal{E}$ iff $\prod_{\xi} \| T_{\xi} u_{k}(i, j) \|$ diverges. Also, in proposition 5 one should replace $e_{j}^I \wedge e_{j}^I$ by $R_{M/Q}(e_{j}^I \wedge e_{j}^I)$. The rest remains the same. Let's only state the final consequence:
Proposition 6.14. Assume $g_k \mathcal{Z}_{G'}(S')$ diverges in $G'/\mathcal{Z}_{G'}(S')$ for all $S' \subset \mathcal{T}'$ subtorus. There exists a sequence of real numbers $\omega_k \to +\infty$ such that if we define $\Omega_{g_k,\varepsilon}^k := \Omega_{G',\varepsilon}^k + \omega_k$, its volume remains asymptotically the same as $\Omega_{g_k,\varepsilon}^k$, i.e.

$$ \lim_{k \to +\infty} \frac{\text{Vol} \left( \Omega_{g_k,\varepsilon}^k \right)}{\text{Vol} \left( \Omega_{g_k,\varepsilon}^k \right)} = 1 $$

6.4. Limiting measures. Let’s given $\{g_k\}$ diverging in $G'/\mathcal{Z}_{G'}(S')$ for all $S' \subset \mathcal{T}'$ subtorus defined over $\mathbb{Q}$. Take $t_k \in \text{Lie}(\mathcal{S}_k)$ satisfying $(t_k + \Omega') \cap \Omega_{g_k,\varepsilon} \neq \emptyset$. By Corollary 6.13, $(g_k \exp t_k)_\ast \tilde{\mu}_{(15.2.5)}$ is nondivergent. Passing to a subsequence we may assume $\lim (g_k \exp t_k)_\ast \tilde{\mu}_{(15.2.5)} = \nu \in \text{Prob}(G'/\Gamma')$. Also by nondivergence, we may find $\{\gamma_k\} \subset \Gamma'$ and a bounded sequence $\{\delta_k\} \subset G'$ such that $g_k \exp t_k = \delta_k \gamma_k$. Arguing as in [9], to prove Theorem 6.1, it suffices to show

Proposition 6.15. $\bigcup_k g_k \mathcal{T}' \gamma_k^{-1} = G'$.

We’ll proceed in the same way as in section 5.

Proposition 6.16. Given $\{b_k\} \subset \mathcal{G}'(\mathbb{Q}) = \mathcal{G}(M)$ diverging in $G'/\mathcal{Z}_{G'}(S')$ for all $S' \subset \mathcal{T}'$ subtorus defined over $\mathbb{Q}$, let $\mathcal{L}' := \bigcup_k b_k \mathcal{T}' b_k^{-1}$. Assume $\mathcal{L}' \neq \mathcal{G}'$ and let $\mathcal{P}'$ be a maximal connected $\mathbb{Q}$-subgroup containing $\mathcal{L}'$, then $\mathcal{P}'$ is parabolic.

Proof. Same as before, we know $\mathcal{P}'$ is either parabolic or reductive. So let’s assume it is reductive and find a contradiction.

There is a $\mathbb{Q}$-torus $S_0$, $\mathcal{P}' = \mathcal{Z}_{G'}(S_0)$. Take $s \neq 0 \in S_0(\mathbb{Q})$, by maximality $\mathcal{P}' = \mathcal{Z}_{G'}(s)$. Now $s \in \mathcal{G}'(\mathbb{Q}) = \mathcal{G}(M)$. Take $\mathcal{P} := \mathcal{Z}_{G'}(s)$ in $\mathcal{G}$, a subgroup defined over $M$. Then $\mathcal{P}(M) \supset \mathcal{P}'(\mathbb{Q})$ and hence $R_{M/\mathcal{Q}}(\mathcal{P})$ contains $\mathcal{P}'$. On the other hand $\mathcal{P} \neq \mathcal{G}$ so by maximality of $\mathcal{P}'$, $R_{M/\mathcal{Q}}(\mathcal{P}) = \mathcal{P}'$ and $\mathcal{P}$ is a maximal $M$-subgroup containing $\mathcal{T}$. Also $\mathcal{P}(M) = \mathcal{P}'(\mathbb{Q})$ contains $\mathcal{L}'(\mathbb{Q}) \supset b_k \mathcal{T}'(\mathbb{Q}) b_k^{-1} = b_k \mathcal{T}(\mathbb{Q}) b_k^{-1}$, so $\mathcal{P}$ contains the Zariski closure of $\bigcup_k b_k \mathcal{T}' b_k^{-1}$.

Also for all $\mathcal{S} \subset \mathcal{T}$ subtorus defined over $M$ let $\mathcal{S}' = R_{M/\mathcal{Q}}(\mathcal{S})$. Then $b_k$ diverges in $\mathcal{G}'(\mathbb{R})/\mathcal{Z}_{G'}(\mathcal{B})(\mathbb{R}) = \mathcal{G}(\mathbb{R})/\mathcal{Z}_{G}(\mathcal{B})(\mathbb{R})$ embeds in $\mathcal{H}$. Hence $b_k$ diverges in $\mathcal{G}(\mathbb{R})/\mathcal{Z}_{G}(\mathcal{B})(\mathbb{R})$. Now we can apply Proposition 5.5 to conclude. □

Lemma 6.17. (1) If $\mathcal{P}'$ is a parabolic subgroup of $\mathcal{G}'$ defined over $\mathbb{Q}$, then there exists $\mathcal{P}$, a parabolic subgroup of $\mathcal{G}$ defined over $M$ such that $R_{M/\mathcal{Q}}(\mathcal{P}) = \mathcal{P}'$.

(2) If $\mathcal{T}'$ is a maximal torus of $\mathcal{G}'$ defined over $\mathbb{Q}$, then there exists $\mathcal{T}$, a maximal torus or $\mathcal{G}$ defined over $M$ such that $R_{M/\mathcal{Q}}(\mathcal{T}) = \mathcal{T}'$.

Proof. (1) $\mathcal{P}'$ has to contain a maximal $\mathbb{Q}$-split torus of $\mathcal{G}'$. On the other hand $\Delta \mathcal{D}$ is a maximal $\mathbb{Q}$-split torus. So by Theorem 15.2.5 from [21], there exists a $b \in \mathcal{G}'(\mathbb{Q}) = \mathcal{G}(M)$ s.t. $b \mathcal{P}' b^{-1}$ contains $\Delta \mathcal{D}$. Now by proof of 15.1.2 (ii) from [21], there exists a cocharacter defined over $\mathbb{Q}$, $\lambda' : \mathbb{G}_m \to \Delta \mathcal{D}$ such that $b \mathcal{P}' b^{-1} = \mathcal{P}'(\lambda')$. Let $\mathcal{L}' : \mathbb{G}_m \to \mathcal{D}$ be such that $\lambda' = \Delta \circ \lambda$. So $\lambda$ is defined over $M$. Then $\mathcal{P}' = b^{-1} \mathcal{P}'(\lambda') b = R_{M/\mathcal{Q}}(b^{-1} \mathcal{P}'(\lambda) b)$.

(2) Take the Zariski closure of $\mathcal{T}'(\mathbb{Q}) \subset \mathcal{G}'(\mathbb{Q}) = \mathcal{G}(M)$ in $\mathcal{G}$ and call it $\mathcal{T}$. It is a subgroup of $\mathcal{G}$ defined over $M$. It is also commutative by continuity. Then $R_{M/\mathcal{Q}}(\mathcal{T})$ is a torus of $\mathcal{G}'$ defined over $\mathbb{Q}$ containing $\mathcal{T}'$ hence they are equal. □

Proof of Proposition 6.15. If the proposition were not true, we can find $\mathcal{P}'$ proper maximal connected subgroup defined over $\mathbb{Q}$ containing the LHS. By Proposition 6.16 above, $\mathcal{P}'$ is parabolic. Find $\mathcal{P}$ maximal parabolic $M$-subgroup of $\mathcal{G}$ such that $\gamma_1^{-1} \mathcal{P}' \gamma_1 = R_{M/\mathcal{Q}}(\mathcal{P})$.

Let $\mathcal{P}'$ correspond to a maximal $M$-flag $\{\{0\} \subset V_1 \subset V\}$. Now $\mathcal{P}(M) = \gamma_1^{-1} \mathcal{P}(\mathbb{Q}) \gamma_1 \supset \mathcal{T}(\mathbb{Q}) = \mathcal{T}(M)$, so $V_1$ is $\mathcal{T}(M)$-invariant and hence adapted to $\mathcal{S}_0$. So $\gamma_1^{-1} \mathcal{G}_k \mathcal{T}(\gamma_1^{-1} \mathcal{G}_k)^{-1} e_I \in \mathcal{R} e_J$ for some nonempty proper $I \subset \mathcal{S}_0$. This implies the subspace corresponding to $\gamma_1^{-1} \mathcal{G}_k^{-1} e_I$ is $\mathcal{T}$-invariant and hence we can find $c_k \in \mathbb{R}$ and $J_k \subset \mathcal{S}_0$ such that $\gamma_1^{-1} \mathcal{G}_k^{-1} e_I = c_k e_{J_k}$. As $J_k$ only has finitely many possibilities we may assume $J_k = J_1$ by passing to a subsequence.

Now let’s apply $R_{M/\mathcal{Q}}$ on both sides to get $(\gamma_1^{-1} \mathcal{G}_k^{-1}) e_I = N_{M/\mathcal{Q}} e_{J_1} = c_k R_{M/\mathcal{Q}} e_{J_1}$. As $\gamma_1$’s belong to $\mathcal{G}'(\mathbb{Z})$, $c_k \in \mathbb{Z}$. Now we can argue as in Proposition 5.3 to finish. □
7. Intermediate Cases

In this section \( M = \mathbb{Q} \) for the notation introduced in section 2. And we shall use \( M \) for a different unrelated field corresponding to a subtorus (to be defined later).

We want to treat the intermediate case, i.e., we are given a sequence \( \{g_k\} \subset G \) and a rational subtorus \( S_0 \subset \mathbb{T} \) such that:
1. \( \{g_k\} \subset \mathbb{Z}S(0) \);
2. \( g_k \) diverges in \( G/\mathbb{Z}S(S) \) for all \( S_0 \not\subset S \subset \mathbb{T} \) defined over \( \mathbb{Q} \).

Then

**Theorem 7.1.** \( \lim_k (g_k)_* \mu_T = \mu \mathbb{Z}S(0)_\Gamma \).

When \( S_0 = \{0\} \), it reduces to Theorem 5.1.

7.1. Preparations. For \( \theta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), let \( \phi^i(\theta) \in \text{Perm}[l_i] \) be defined by

\[
\theta \sigma_j^i = \sigma_{\phi^i(\theta)(j)}^i \mod \text{Gal}(\overline{\mathbb{Q}}/L_i)
\]

Putting together all \( \phi^i \)'s we obtain a permutation \( \phi(\theta) \) of \( \{l_0| \cdots \}|a_n \) acting trivially on \( \{l_0| \cdots \}|a_n \), preserving each \( [l_i] (i = 1, \ldots, a_0) \) and reducing to \( \phi^i(\theta) \) when restricted to \( [l_i] (i = 1, \ldots, a_0) \).

For each permutation \( \phi \in S_n \), we will use \( A_\phi \) to denote the \( n \) by \( n \) matrix such that \( A_\phi(x_0, \ldots, x_n)^r = (x_{\phi(1)}, \ldots, x_{\phi(n)})^r \), i.e. \( A_\phi(i,\phi(i)) = 1 \) and \( A_\phi(i,j) = 0 \) otherwise. Then \( A_\phi^{-1} \text{diag}\{x_1, \ldots, x_n\} A_\phi = \text{diag}\{x_{\phi^{-1}(1)}, \ldots, x_{\phi^{-1}(n)}\} \).

Now we take \( \theta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), \( A_i \) same as before. One can compute

\[
\theta(A_i) = A_{\phi^i(\theta)}A_i
\]

Let’s define \( i_0 \) depending on \( i \) such that \( \sigma_{i_0} = \text{id} \) for all \( i = 1, \ldots, a_0 \). Recall \( G_i \) is the copy of \( SL_{l_i} \) embedded in \( G = SL_N \).

**Lemma 7.2.** For \( x^i = \text{diag}(x_1, \ldots, x_{l_i}) \in SL_{l_i}(\overline{\mathbb{Q}}) \), \( A_i^{-1} \text{diag}(x_1, \ldots, x_{l_i}) A_i \) is contained in \( G_i(\overline{\mathbb{Q}}) \) iff there is some \( y \in L_i \) s.t. \( x_j = \sigma_j^iy \).

**Proof.** By computation above, we have for \( \theta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)

\[
\theta(A_i^{-1} \text{diag}(x_1, \ldots, x_{l_i}) A_i) = A_i^{-1}A_{\phi^i(\theta)}^{-1} \text{diag}(\theta x_1, \ldots, \theta x_{l_i}) A_{\phi^i(\theta)}A_i
\]

So \( x^i \in G_i(\overline{\mathbb{Q}}) \) iff \( \theta x_{(\phi^i(\theta))}^{-1} = x_j \) for all \( j = 1, \ldots, l_i \) and all \( \theta \in \text{Gal} \).

Let’s assume there is some \( y \in L_i \) s.t. \( x_j = \sigma_j^iy \). Then we need to check \( \theta \sigma_{(\phi^i(\theta))}^{-1} = y = \sigma_j^iy \). It suffices to show \( \theta \sigma_{(\phi^i(\theta))}^{-1} \equiv \sigma_j^i \mod \text{Gal}(\overline{\mathbb{Q}}/L_i) \) from right which, by the definition of \( \phi^i(\theta) \), is true.

Let’s assume \( x^i \in G(\overline{\mathbb{Q}}) \). So \( x_{(\phi^i(\theta))}^{-1} = \theta x_j \) or \( \theta x_{(\phi^i(\theta))}^{-1} = x_j \). Recall \( \sigma_{i_0} \equiv \text{id} \mod \text{Gal}(\overline{\mathbb{Q}}/L_i) \), \( \theta \sigma_{i_0} = \theta \equiv \text{id} \mod \text{Gal}(\overline{\mathbb{Q}}/L_i) \).

On the other hand, we set \( \theta \in \text{Gal}(\overline{\mathbb{Q}}/L_i) \) and \( j = i_0 \). First, we know \( \theta \sigma_{i_0} = \theta \equiv \text{id} \mod \text{Gal}(\overline{\mathbb{Q}}/L_i) \). But \( \theta \sigma_{i_0} = \sigma_{(\phi^i(\theta))}^{-1} \) so \( \phi^i(\theta) = i_0 \). Then we see \( x_{i_0} = \theta x_{(\phi^i(\theta))}^{-1} = \theta x_{i_0} \) for all \( \theta \in \text{Gal}(\overline{\mathbb{Q}}/L_i) \), thus \( x_{i_0} \in L_i \). Taking \( y := x_{i_0} \) proves the other implication.

For \( \xi \) partition of \( \{l_0| \cdots \}|a_n \), we write \( \xi = (\xi_1, \ldots, \xi_k) \) where each \( \xi_i \) denotes one part of this partition. Each partition define an equivalence relation on \( \{l_0| \cdots \}|a_n \) denoted by \( x \sim \xi y \). \( \xi \) with upper index, \( \xi^i \), will denote the restriction of \( \xi \) to \( [l_i] \). Also we let \( |\xi| = k_0, k_i := |\xi^i| \) be the number of parts for each partition.

\( D_\xi \) denotes the subtorus of \( D \) adapted to this partition, namely, \( \text{diag}(x_1, \ldots, x_N) \in D_\xi(\overline{\mathbb{Q}}) \) iff \( \prod_i x_{i} = 1 \) and \( x_i = x_j \) if \( i \sim \xi j \). Let’s observe that for a subtorus \( S \) of \( D \), \( \text{Z}(S) = S \) if \( S \subset D_\xi \) for some partition \( \xi \). We will concentrate on this class of torus. For \( x \in [l_0] \), define \( [x] := \{x\} \) and \( \phi^x = \text{id} \). This notation hopefully will not cause much confusion since later we will focus on the case when \( l_0 = 0 \) and \( [l_0] = \emptyset \).
Proposition 7.3. Let $\mathbb{D}_\xi \subset \mathbb{D}$ be such that $A_0^{-1} \mathbb{D}_\xi A_0$ is defined over $\mathbb{Q}$. For $i_1, i_2 \in [l_0] \cup \{1, ..., a_0\}$ and $j_1 \in [l_{i_1}], j_2 \in [l_{i_2}].$

1) For any $\theta$, we have $j_1 \sim_\xi j_2 \iff \phi^{i_1}(\theta)(j_1) \sim_\xi \phi^{i_2}(\theta)(j_2).

2) All $\xi^i$'s are equi-partitioned, i.e. all parts of the partition have the same cardinality.

3) For all $j \in [l_{i_1}]$ there is $j' \in [l_{i_2}]$ with $j \sim_\xi j'$ if for some $j$ this is true.

Proof.

(1): Take a generic element $A_0^{-1}yA_0 \in A_0^{-1} \mathbb{D}_\xi A_0(\mathbb{Q})$. Write $y = \text{diag}(y_1^0, ..., y_a^0)$. Now take any pair $(i_1, j_1)$ and $(i_2, j_2)$ with $j_1 \in [l_{i_1}]$ and $j_2 \in [l_{i_2}]$. By genericity, $y_{i_1} = y_{i_2}$ if $j_1 \sim_\xi j_2$. (We think of $j_1 \in [l_{i_1}]$ being different from $j_2 \in [l_{i_2}]$ if $i_1, i_2$ are different). As $y$ represents the conjugate by $A_0$ of a rational points we know by the proposition above for each $i$ we have some $y^i \in L_i$ such that $y^i = \sigma_i y^i$.

(2), (3): They are consequences of (1) and the transitivity of $\phi^i(\text{Gal})$ action.

Now for a torus as in the above proposition, we can define an equivalence condition on $[l_0] \cup \{1, ..., a_0\}$. $x \sim_\xi y$ iff there exists $i \in [l_x]$ that is equivalent to some $i' \in [l_y]$.

Proposition 7.4. Let $\xi$ be a partition of $[l_0] \cup \ldots \cup [l_{a_0}]$. Assume $A_0^{-1} \mathbb{D}_\xi A_0$ is defined over $\mathbb{Q}$.

1) Then $\dim \mathbb{D}_\xi \cap T_s = 0$ iff there is only one equivalence class in $[l_0] \cup \{1, ..., a_0\}$.

Furthermore, if this condition holds and $\dim \mathbb{D}_\xi \neq 0$ then

2) $l_0 = 0$.

3) $\xi$ is equi-partitioned and all $\xi^i$'s and $\xi$ have the same number of parts, i.e. $k_1 = \ldots = k_{a_0} = k_0$.

Proof.

(1):

$\dim \mathbb{D}_\xi \cap T_s = 0$ is the same as $\dim A_0^{-1} \mathbb{D}_\xi A_0 \cap T_s = 0$.

First assume $\dim \mathbb{D}_\xi \cap T_s = 0$. If $[l_0] \cup \{1, ..., a_0\} = C_1 \cup C_2$ is decomposed as two nonempty disjoint sets, we may easily create non-identity one-parameter subgroup $x(t) = \text{diag}(x_1^0, ..., x_{a_0}^0)$ whose entries from $C_1$ and $C_2$ are always the same. And such an element is in both $\mathbb{D}_\xi$ and $T_s$ by definition. So we have a contradiction.

Conversely, assume there is some non-identity one-parameter group $x(t)$ in $\mathbb{D}_\xi \cap T_s$ now. Take $C_1 \cup C_2 = [l_0] \cup \{1, ..., a_0\}$ such that the entries corresponding to $C_i$ ($i = 1$ or $2$) are the same. Then element from $C_1$ can not be equivalent to those from $C_2$. We are done.

(2):

Under present condition, $A_0^{-1} \mathbb{D}_\xi A_0$ has to have a nontrivial $\mathbb{Q}$-anisotropic part, equivalently, $a_0 \neq 0$. Let’s look at $A_0 T(\mathbb{Q}) A_0^{-1}$. If $l_0 \neq 0$, then by (1) there exists $j_0 \in [l_0], j' \in [l_i](i \neq 0)$ and $j_0 \sim_\xi j'$.

So $x \in \mathbb{D}_\xi(\mathbb{Q})$ implies $x' \in G_i(\mathbb{Q})$ and therefore is a scalar matrix by the Lemma 7.2. But there is only one equivalence class in $[l_0] \cup \{1, ..., a_0\}$, so the full matrix $x$ is a scalar matrix. Thus $\dim D_\xi = 0$, contradiction.

But element from $[l_0]$ can never be equivalent to those from $[l_i]$ for $i$ different from $0$ since at those entries one has element in $L_i \setminus \mathbb{Q}$. This shows $[l_0] = \emptyset$, or $l_0 = 0$.

(3): This is a consequence of (1) and Proposition 7.3 above.

7.2. A reduction. Theorem 7.1 can be reduced to the following situation:

Proposition 7.5. If we further assume $S_0$ is $\mathbb{Q}$-anisotropic, then Theorem 7.1 is true.

Proof of Theorem 7.1 assuming Proposition 7.5. Decompose $S_0 = S_0^a S_0^0$ as its $\mathbb{Q}$-split and $\mathbb{Q}$-anisotropic part. $Z(\mathbb{Z}_S S_0^a) \subset Z(\mathbb{Z}_S S_0^a) \cap Z(\mathbb{Z}_S T_s)$ because $S_0^a \subset S_0$ and $T_s$. Hence $Z(\mathbb{Z}_S S_0^a) = S_0 \cap T_s S_0^0$. So $S_0 = \mathbb{D}_\xi$ for some partition $\zeta = (\zeta_1, ..., \zeta_{a_0})$ and $\mathbb{Z}_S S_0^0$ is isomorphic to $SL(\zeta_1) \times \ldots \times SL(\zeta_{a_0}) \times S_0^0$. 


7.3. Classification of intermediate groups. In this section we work under the further assumption that $S_0$ is $\mathbb{Q}$-anisotropic and classify all possible $S_0$ and $Z_G(S_0)$. In the case when $T$ is $\mathbb{Q}$-anisotropic, this was done in [22] which shall make use of here.

As $Z_G(S_0) = S_0$, $S_0 = A_{\xi}^{-1}D_{\xi}A_0$ for some partition $\xi$ of $\{1, ..., N\}$. In view of Proposition 7.4, reordering $\xi$ and $\{\sigma_i\}$ by $\phi_0 = (\phi_{0i}, ..., \phi_{0N}) \in \text{Perm}(1, ..., N) \subset \text{Perm}(\{1\} \cup \ldots \cup \{N\})$ simultaneously we may assume elements in $D_{\xi}(\mathbb{Q})$ can be written as $\text{diag}(B_0, ..., B_0) \in D_{\xi}(\mathbb{Q})$ with $B_0 = \text{diag}(x_1, ..., x_0)$ a $k_0$-by-$k_0$ diagonal matrix with $x_i \in \mathbb{Q}$ and $\prod\xi x_i = 1$.

Recall $G_i$ is the copy of $SL_{l_i}$ embedded in $G$. Then the first $l_1/k_0$ copies of $B_0$ belong to $G_1$ and so on. Let $\pi_i$ be the projection map from $T \subset SL_{l_1} \times \ldots \times SL_{l_n} \subset G$ to $G_i$, $S_i := \pi_iS_0$. 

**Proposition 7.6.** $A_iS_iA_i^{-1} = (D_i)_{\xi_i}$. In particular $Z_{G_i}(S_i) = S_i$. 

**Proof.** "$\subset$" is true by definition. Let's take $x_i \in \mathbb{Q}$ point of RHS. It can be uniquely extended to $x \in \mathbb{Q}^{\mathbb{Q}}$ under the requirement that $x_i^2 = x_{i2}$ whenever $j_i \sim \xi_2$. One can check this $x$ is indeed in $G(\mathbb{Q})$, i.e. has determinant 1. Then $x \in D_{\xi}(\mathbb{Q})$ and $x_i = \pi_i(A_i^{-1}x A_i) \in S_i(\mathbb{Q})$. 

Because its centralizer equals to itself, we can apply Proposition 3.3 from [22] to $S_i$, implying there exists a subfield $M_i \subset L_i$ such that $S_i(\mathbb{Q}) = \{A_i^{-1}\text{diag}(\sigma_i^1(x), ..., \sigma_i^1(x))A_i | x \in M_i, \text{Nm}_i(x) = 1\}$. Hence $k_0$ defined before is actually the degree of $M_i/\mathbb{Q}$ for all $i$ because $S_i = A_{\xi_i}^{-1}(D_i)^{\xi_i}A_i$ implies there are exactly $k_0$ different embeddings of $M_i$ into $\mathbb{Q}$. Also $\sigma_i^1(M_i) = \ldots = \sigma_i^{k_0}(M_{k_0}) := M_i$. So $S_i(\mathbb{Q}) = \{A_i^{-1}\text{diag}(\sigma_i^{1i}(1), ..., \sigma_i^{1i}(1))A_i | x \in M_i, \text{Nm}_i(x) = 1\}$.

We can extract more information from the above proposition. In fact, for all $i, j \in \{1, ..., a_0\}$, $j_1, j_2 \in \{0, ..., l_i/k_0 - 1\}$, $k \in \{1, ..., k_0\}$, $\sigma_{j_1}^{1i} \cdot \sigma_{j_2}^{1i} \cdot k \cdot (\sigma_1^{1i})^{-1}x = \sigma_{j_2}^{1i} \cdot x$ for all $x$ in $M_i$ their relative positions in "$B_0" are the same. In other words, $\sigma_{j_1}^{1i} \cdot \sigma_{j_2}^{1i} \cdot k \cdot (\sigma_1^{1i})^{-1} \equiv \sigma_{j_2}^{1i} \cdot x$ mod $Gal(\mathbb{Q}/M)$ from right. Also $\{\sigma_{j_1}^{1i} \cdot \sigma_{j_2}^{1i} \cdot k \cdot (\sigma_1^{1i})^{-1} \}_{k=1,...,k_0}$ form a complete set of representatives of $Gal(\mathbb{Q}/M)$.

So for $x \in M_i$, if we define $B(x) := \text{diag}(\sigma_{j_2}^{1i} \cdot x, ..., \sigma_{j_2}^{1i} \cdot x)$, a $k_0$-by-$k_0$ matrix, the definition will not depend on the choice of $i, j$.

Take $o_1, ..., o_{k_0}$ in $O_M$ to be a basis of $M$ as $\mathbb{Q}$ vector space. Then $\sigma_i^e := \sigma_i^1 o_1, ..., \sigma_i^e o_{k_0}$ in $O_M$ is a basis of $M_i$ as $\mathbb{Q}$-vector space. For $i \in \{1, ..., a_0\}$ and $j \in \{0, ..., l_i/k_0 - 1\}$, define a $k_0$-by-$k_0$ matrix $A_M$ by requiring the $(a, b)$-th entry $(A_M)_{a,b} := \sigma_i^{1i} o_j$. By our observation above, this definition does not depend on $i, j$.

Now if $Z$ denotes the center of $GL_{l/k_0}$, the inclusion $R_{M/\mathbb{Q}}^{1i}(Z)(\mathbb{Q}) \subset R_{M/\mathbb{Q}}^{1i}(GL_{l/k_0})(\mathbb{Q})$ may be realized as

$$(37) \quad \{(A_M, ..., A_M)^{-1}B(x, ..., B(x))(A_M, ..., A_M) | x \in M, \text{Nm}_M/\mathbb{Q}(x) = 1\}$$
$$\subset \{(A_M, ..., A_M)^{-1}(B(x_{a,b})))(A_{a,b})(A_M, ..., A_M) \cap SL_N(\mathbb{Q}) | x_{a,b} \in M\}$$ 

Let's also observe that $R_{M/\mathbb{Q}}^{1i}(Z)(\mathbb{Q})$ is again the center of $R_{M/\mathbb{Q}}^{1i}(GL_{l/k_0})(\mathbb{Q})$ and the latter is the centralizer of the former in $G$. 


Proposition 7.7. Over $GL_N(Q)$, $S_0(Q) \subset Z_G(S_0)(Q)$ is conjugate to $R^{(1)}_{M/Q}(Z)(Q) \subset R^{(1)}_{M/Q}(GL_{l/k_0})(Q)$. Moreover, under the same conjugation, $T$ is equal to $R_{M/Q}(T')(SL_{l/k_0})$ for some maximal $M$-torus $T'$ in $SL_{l/k_0}$.

Proof. We want to make a special choice of $\{v_i^j\}$'s. Different choices amount to conjugating by a matrix in $GL_{l}(Q) \times \ldots \times GL_{l_0}(Q)$.

First, for $i \in \{1, \ldots, a_0\}$, choose $u_1^i, \ldots, u_{l_i/k_0}^i$ in $O_{L_i}$ to be a basis of $L_i$ as an $M_i$ vector space. Then take $\{v_j^i\}_{j=1, \ldots, l_i} = \{u_1^i', u_1^i, u_2^i, \ldots, u_{l_i/k_0}^i, u_{l_i/k_0}^i\}$. For $i \in \{1, \ldots, a_0\}$, $j \in \{1, \ldots, l_i/k_0\}$, and $x \in L_i$, define $B_j^i(x)$, a $k_0$-by-$k_0$ diagonal matrix, by $\text{diag}(\sigma_{j-1}^{-1}((x), \ldots, \sigma_{j-1}^{-1}((x) + k_0(x)))$.

Recall $(A_i)_x = \sigma_{j-1}^{-1}(x)$, $(A_M)_a = \sigma_{j+1}^{-1}(o_k^j)$ and $B(x) := \text{diag}(\sigma_{j-1}^{-1}(x), \ldots, \sigma_{j+1}^{-1}(o_k^j))$. As we noted before, the definition of the latter two does not depend on $i, j$. The key computation is, for each $i \in \{1, \ldots, a_0\}$, arbitrary element in $S_i(Q)$ can be written as (for some $x \in M$ with norm 1),

$$\begin{align*}
(A_i) &= \sigma_{j-1}(x) = (A_M)_a,
\end{align*}$$

$$\begin{align*}
(38) & \quad A_i^{-1} B(x) A_i = B(x),
(39) & \quad A_i^{-1} B_j^i(u_1^i) A_i = B_j^i(u_1^i),
(40) & \quad A_i^{-1} B_j^i(u_1^i) A_i = B_j^i(u_1^i),
(41) & \quad A_i^{-1} B_j^i(u_1^i) A_i = B_j^i(u_1^i),
(42) & \quad \in R^{(1)}_{M/Q}(Z)(Q)
\end{align*}$$

Hence $S_0(Q) = R^{(1)}_{M/Q}(Z)(Q)$ (Above we showed “⊂” and by going backwards “⊃” can be proved) and by taking its centralizer we see $Z_G(S_0)(Q) = R^{(1)}_{M/Q}(GL_{l/k_0})(Q)$.

Lastly, $T$ is a maximal torus of $R^{(1)}_{M/Q}(Z)(Q)$ since it is already maximal in $SL_N$. And we can conclude because all maximal torus come from restriction of scalars.

Now we can complete the proof.

Proof of Proposition 7.5. By Proposition 7.7 and that $R^{(1)}_{M/Q}(GL_{l/k_0}) = Z \cdot R_{M/Q}(SL_{l/k_0})$ it suffices to prove an equidistribution result for the group $R_{M/Q}(SL_{l/k_0})$ which has been established in Theorem 6.1.

8. Translates of reductive subgroups containing $T$

In this section we prove Theorem 1.10 from the introduction. The proof will be the same as that in [20] except that we work directly at the group level instead of Lie algebra. Recall Condition (**) says we are given a sequence $\{g_k\}$ in a reductive group $H_1$ containing $T$ and a subtorous $S_0$ of $S_1$, the center of $H_1$, such that:

1. $g_k$ is contained in $Z_G(S_0) = H_{S_0}$;
(2) $g_k$ diverges in $Z_G(S)$ for all $\mathbb{Q}$-subtorus $S$ of $S_1$ that strictly contains $S_0$.

Work inside $Z_G(S_0)$, we may write $g_k = a_k u_k h_k$ with $a_k$ bounded in $Z_G(S_0)$, $u_k \in R_u(P_{S_1}) \cap Z_G(S_0)$ and $h_k \in H_1$. Now $(g_k)_{\ast \mu_{H_1}} = (a_k u_k)_{\ast \mu_{H_1}}$ hence without loss of generality we may assume $g_k = u_k$, in other words $g_k$ is in $R_u(P_{S_1})$ in addition to satisfying conditions (**) above. Let’s also take $h_m$ satisfying condition (⋆) with $S_1$ in $H_1$ such that $\lim_m (h_m)_{\ast \mu_T} = \mu_{H_1}$.

To show $(g_n)_{\ast \mu_{H_1}} \to \mu_{H, S_0}$, it amounts to show for $f_1, f_2 \in C_c(G/\Gamma')$ and $(f_2, \mu_{H, S_0}) \neq 0$, 

$$\frac{\langle f_1, (g_n)_{\ast \mu_{H_1}} \rangle}{\langle f_2, (g_n)_{\ast \mu_{H_1}} \rangle} = \lim_{n} \frac{\langle f_1, (g_n h_m)_{\ast \mu_T} \rangle}{\langle f_2, (g_n h_m)_{\ast \mu_T} \rangle}$$

if this fails, then we can find $\varepsilon_2 > 0$ and $n_i \to \infty$ such that for all $n_i$,

$$\left| \frac{\langle f_1, (g_n h_m)_{\ast \mu_T} \rangle}{\langle f_2, (g_n h_m)_{\ast \mu_T} \rangle} - \lim_{n} \frac{\langle f_1, (g_n h_m)_{\ast \mu_T} \rangle}{\langle f_2, (g_n h_m)_{\ast \mu_T} \rangle} \right| \geq \varepsilon_2$$

Hence there exists a sequence $g_n$ in $R_u(P_{S_1})$ satisfying condition (**) and $h_n$ satisfying condition (⋆) such that \(\lim_n (g_n h_m)_{\ast \mu_T} = \mu_{H, S_0}\). This is a contradiction to the following proposition.

**Proposition 8.1.** Assume we have a sequence $g_n$ in $R_u(P_{S_1})$ satisfying condition (**) with $S_0$ and $h_n$ satisfying condition (⋆) with $S_1$, then $g_n h_n$ satisfies condition (⋆) with $S_0$. As a result, \(\lim_n (g_n h_n)_{\ast \mu_T} = \mu_{H, S_0}\).

**Proof.** Take $t \in T \setminus S_0$, we want to show $(g_n h_n) t (g_n h_n)^{-1}$ diverges.

Case I: $t \in S_1 \setminus S_0$.

$(g_n h_n) t (g_n h_n)^{-1} = (g_n) t (g_n)^{-1}$ diverges by assumption on $g_n$.

Case II: $t \in T \setminus S_1$.

Write $t = t_0 t_1$ with $t_1 \in S_1$ and $t_0 (\neq 1) \in (H_1)_{ss} \cap T$, where $(H_1)_{ss}$ denotes (real points of) the semisimple part of $H_1$. Then

$$(g_n h_n) t_0 t_1 (g_n h_n)^{-1} = g_n h_n t_0 h_n^{-1} t_1 g_n^{-1} = (g_n h_n t_0 h_n^{-1} t_1 g_n^{-1} h_n t_0 h_n^{-1} t_1^{-1})(h_n t_0 h_n^{-1} t_1)$$

The first pair of parentheses contains elements in side $[R_u(P_{S_1}), H_{S_1}] = R_u(P_{S_1})$ whereas the second pair contains elements from $H_{S_2}$. Hence it diverges iff at least one of them diverges. But the latter one indeed diverges because of our assumption on $h_n$ and $t_0 \in T \setminus S_1$.

9. A counting problem

In this section we apply what proved to a counting problem. This procedure is standard, originating from [6]. Moreover, the key computations and estimates have been done in [20], section 11. We sketch a proof below using these computations.

So we are given two totally real fields $L$, $M$ and $M \subseteq L$. Let $p(x) \in O_M[x]$ a monic polynomial of degree $N$ with distinct roots in $\overline{Q}$. In $M[x]$, factor $p(x) = p_0(x) p_1(x) \cdots p_n(x)$ into irreducible monic polynomials $\{p_i\}_{i=1}^{n}$. Let $\deg p_i = l_i$. For an $n-$by-$n$ matrix $A \in M_n(O_M)$, let $p_A(x)$ denotes its characteristic polynomial. Let’s consider

$$X_p(O_M) = \{ A \in M_n(O_M) \mid p_A(x) = p(x) \}$$

To state a meaningful counting problem, we will use the geometric embedding to identify this set with

$$X_p(O_M) = X'_p(Z) = \{ (\tau_1 A, \ldots, \tau_{ma} A) \in M_n(\mathbb{R}) \times \ldots \times M_n(\mathbb{R}) \mid A \in M_n(O_M), p_A(x) = p(x) \}$$

Similarly, let

$$X'_p(\mathbb{R}) = \{ (A_1, \ldots, A_{ma}) \in M_n(\mathbb{R}) \times \ldots \times M_n(\mathbb{R}) \mid p_{A_i}(x) = \tau_0 p(x) \}$$

And $B_R$ denotes the ball of radius $R$ in $X'_p(\mathbb{R})$, i.e.

$$B_R = \{ (A_1, \ldots, A_{ma}) \in M_n(\mathbb{R}) \times \ldots \times M_n(\mathbb{R}) \mid p_{A_i}(x) = \tau_0 p(x), \sum_{k,i,j} (A_k)^2_{ij} \leq R \}$$

With these notations we can state
Theorem 9.1. There exist a constant $c_p > 0$ such that

$$
\lim_{R} \frac{|X_p'(\mathbb{Z}) \cap B_R|}{c_p R^{m\alpha(n-1)/2} (\log R)^{a_0 + l_0 - 1}} = 1
$$

First fix $x_0 \in X_p'(\mathbb{Z})$ and let $\mathcal{T}'$ in $G' = Res_{\mathbb{M}/\mathbb{Q}} SL_n$ be the centralizer of $x_0$. Then $\mathcal{T}'$ is a maximal torus conjugate over $\mathbb{Q}$ to one we considered before. Consider the map $g \to g \cdot x_0 := gx_0g^{-1}$ from $G'(\mathbb{R})/\mathcal{T}'$ to $X_p'(\mathbb{R})$. One can check this defines a homeomorphism. By a result of Borel and Harish-Chandra we know $X_p'(\mathbb{Z})$ is decomposed into finitely many disjoint orbits of $\Gamma' := G'(\mathbb{Z})$ action. Hence it suffices to count $|\Gamma' \cdot x_1 \cap B_R|$ for $x_1 \in X_p'(\mathbb{Z})$. One can verify that (say, by using the volume asymptotic established after)

Lemma 9.2. The collection of sets $\{B_R\}_R$ is well-rounded.

Thus to prove the theorem, it suffices to show

Proposition 9.3. Let $F_R(g\Gamma) := \sum_{\gamma \in \mathcal{T}' \cap \Gamma'} \chi_{BR}(g\gamma \cdot x_1)$, a positive function on $G'/\Gamma'$. There exists a constant $c_p = c_2 > 0$ such that for any $\phi \in C_c(G'/\Gamma')$,

$$
\lim_{R} \frac{F_R}{c_2 R^{m\alpha(n-1)/2} (\log R)^{a_0 + l_0 - 1}} (\phi) = (1, \phi)
$$

The integration is with respect to $\mu_{G'/\Gamma'}$.

For the definition of well-roundedness and the deduction of the theorem from proposition, see [8].

Before going to the proof, we make some preparations.

Take $g_0 \in G'$ such that $g_0 \cdot x_1 = x_2$ is a (tuple of) diagonal matrix and $(T_2)_s := g_0T_2g_0^{-1}$ is of the form we described before, i.e. first $l_0$ diagonal entries are free, the next $l_1$ entries form a scalar matrix and so on. So the matrix $x_2$ has its diagonal entries equal to (Galois conjugate of) roots of our fixed polynomial $p(x)$.

Lemma 9.4. (1) The map $\Phi_1 : K \times N \to X_p'(\mathbb{R})$ defined by $(k, n) \to kn g_0 \cdot x_1 = kn \cdot x_2$ is a homeomorphism and the push-forward of $\mu_K \times \mu_N$ is equal to $\mu_{G'/\mathcal{T}'}$.

(2) The action $K$ on $X_p'(\mathbb{R})$ preserves each $B_R$. The map from $N \to X_p'(\mathbb{R})$ defined by $n \to n \cdot x_2 = n \cdot g_0 x_1$ is an injection pushing $\mu_N$ forward to a positive multiple of $\{dx_{ij}^k\}_{k, i < j}$.

The first follows from the Iwasawa decomposition and the second is from ([20], Lemma 11.2).

Proposition 9.5. There is a constant $c_3 > 0$, the volume of unit ball in $\mathbb{R}^{m\alpha(n-1)/2}$ such that

$$
\lim_{R} \frac{\mu_{G'/\mathcal{T}'}(B_R)}{c_3 R^{m\alpha(n-1)/2}} = 1
$$

This follows from the above lemma and a direct computation.

Next we turn to the computation of nondivergence of polytope. By our assumption on $x_2$, we may write it as

$$
\Omega_{B, \varepsilon} = \{t' \in \text{Lie}(T')_s ||B \exp t g_0^{-1} R_M/\mathbb{Q} \varepsilon I|| \geq \varepsilon, \forall I \subset \mathcal{A}_0\}
$$

Applying $Ad(g_0)$ to this set we arrive at

$$
\Omega_{B g_0^{-1}, \varepsilon} = \{t' \in Ad(g_0) \text{Lie}(T')_s ||B g_0^{-1} \exp t R_M/\mathbb{Q} \varepsilon I|| \geq \varepsilon, \forall I \subset \mathcal{A}_0\}
$$

Under this correspondence the volume form will be differed by a fixed constant $c_4 > 0$. Also recall that modifying $B g_0^{-1}$ on the right by $T_2$ or on the left by $K$ leave the volume of the set invariant and hence it reduces to looking at elements from $N$.

Defined $N_R := \{N || N \cdot x_2 \leq R\}$ and for $\varepsilon_3 > 0$, $N_{R, \varepsilon_3} := \{N \in N_R ||x_{i+1}^k| < \varepsilon_3, \exists k, i\}$ by Proposition 11.8 from [20] we have

Proposition 9.6. There is a constant $c_5 > 0$, for any $\varepsilon_3 > 0$, for all $u_R \in N_R \setminus N_{R, \varepsilon_3}$, for all $k \in K$, we have

$$
\lim_{R} \frac{\text{Vol} (\Omega_{B u_R, \varepsilon})}{c_5 (\log R)^{a_0 + l_0 - 1}} = 1
$$
Proof of Theorem 9.1.
\[ c_2 R^{m_{\Omega}(n-1)/2} (\log R)^{a_0+10^{-1}} \cdot \text{LHS} \]
\[ = \int_{G'/T'} \sum_{\Gamma'/\Gamma' \subset T'} \chi_{B_R}(g \gamma \cdot x_1) \phi(g \Gamma') \mu_{\Gamma'}/\Gamma'(g \Gamma') \]
\[ = \int_{G'/T'} \int_{T'/T' \subset \Gamma'} \chi_{B_R}(g \cdot x_1) \phi(g \Gamma' \cap T') \mu_{G'/T' \cap T'}(g \Gamma' \cap T') \]
\[ = \int_{B_R} \int_{T'/T' \subset \Gamma'} \phi(k n_{\gamma} t T') \mu_{T'/T' \cap \Gamma'}(t T' \cap \Gamma') \mu_k(k) \mu_N(n) \]
Now by propositions above, for any \( \varepsilon > 0 \), \( c_3 R^{m_{\Omega}(n-1)/2} \sim \mu(B_R) \) and for any \( k \in K \), \( n \in N_R \setminus N_{R,\varepsilon} \), \( c_4 c_5 (\log R)^{a_0+10^{-1}} \sim \text{Vol}(\Omega'_{k n_{\gamma} t T'}) \). Also note that for \( n_R \in N_R \), the sequence \( k_R n_R \) is generic for any choice of \( k_R \in K \). If we define \( c_5 = c_3 c_4 c_5 \) then
\[ \text{LHS} \approx \frac{1}{\mu_{G'/T'}(B_R)} \int_{B_R, \varepsilon} \int_{G'/T'} \phi(t \Gamma') \left( kn_{\gamma} t T' \right) \frac{\mu_{T'/T'}(t T') \mu_k(k) \mu_N(n)}{\text{Vol}(\Omega'_{k n_{\gamma} t T'})} \]
\[ \rightarrow (\phi, 1) \]
And the "\( \approx \)" tends to an equality as \( \varepsilon \to 0 \). So we are done.

\[ \square \]

Remark 9.7. One might wish to compute those constants explicitly. For \( M = \mathbb{Q} \) and in some low-dimensional examples, it seems possible, but we are not able to demonstrate a general formula, especially for the number of orbits for the \( \Gamma' \) action on \( X_p(\mathbb{Z}) \).

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