The ideals of the slice Burnside $p$-biset functor

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ABSTRACT

Let $G$ be a finite group and $K$ be a field of characteristic zero. Our purpose is to investigate the ideals of the slice Burnside functor $K\Xi$. It turns out that they are the subfunctors $F$ of $K\Xi$ such that for any finite group $G$, the evaluation $F(G)$ is an ideal of the algebra $K\Xi(G)$. This allows for a determination of the full lattice of ideals of the slice Burnside $p$-biset functor $K\Xi_p$.

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1. Introduction

The biset category $C$ of finite groups has all finite groups as objects, the group of morphisms from a finite group $G$ to a finite group $H$ is the double Burnside group $B(H,G)$, i.e. the Grothendieck group of $(H,G)$-bisets. In particular the endomorphism ring of a finite group $G$ is the double Burnside ring $B(G,G)$.

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A biset functor is an additive functor over this preadditive category, with values in abelian groups, and biset functors form an abelian category \( \mathcal{F} \). More generally, one can extend the morphisms to \( RB(H,G) = R \otimes_{\mathbb{Z}} B(H,G) \), where \( R \) is a commutative ring, and consider the \( R \)-linear functors with values in the category of \( R \)-modules. Thus, one obtain an \( R \)-linear abelian category \( \mathcal{F}_R \).

A fundamental example of biset functor is the Burnside functor: it can be viewed as the representable functor \( RB(1,-) \), or the Yoneda functor corresponding to the trivial group.

In particular it is a projective object of the category \( \mathcal{F}_R \), which allows by the Yoneda–Dress construction to build enough projective objects on the category \( \mathcal{F}_R \).

Moreover, the Burnside functor has a multiplicative structure which endows it with a structure of Green biset functor, and the modules over this Green functor are the biset functors.

We have a good parametrization of the isomorphism classes of simple \( RB \)-modules by isomorphism classes of pairs \( (H,V) \), where \( H \) is a finite group and \( V \) is a simple \( R \text{Out}(H) \)-module. To each such \( (H,V) \) corresponds the isomorphism class of \( S_{H,V} \), where \( S_{H,V}(G) \) is the quotient of \( L_{H,V}(G) = RB(G,H) \otimes_{RB(H,H)} V \) by

\[
J_{H,V}(G) = \left\{ \sum_i \phi_i \otimes v_i \in L_{H,V}(G) \mid \forall \psi \in \mathbb{K}B(H,G), \sum_i (\psi \phi_i) v_i = 0 \right\}.
\]

An important property of \( H \) is that it is a minimal group for \( S_{H,V} \), and all the minimal groups for a simple biset functor are isomorphic.

Note that in general, the explicit computation of the evaluation \( S_{H,V}(G) \) of a simple functor or more generally of a simple module over a Green biset functor is not easy (cf. [9] and [5]). The study of the functor \( \mathbb{K}B \) (cf. [1]) where \( \mathbb{K} \) is a field of characteristic zero, has allowed for an explicit description of some simple biset functors by the introduction of a new class of finite groups, the \( B \)-groups.

In this paper, we consider the slice Burnside ring introduced in ([3]). It is an analogue of the classical Burnside ring constructed from the morphisms of \( G \)-sets instead the \( G \)-sets themselves, and it shares most of its properties. In particular, as already shown by Serge Bouc (see [3] for more complete description), the slice Burnside ring is a commutative ring, which is free of finite rank as a \( \mathbb{Z} \)-module, and it becomes a split semisimple \( \mathbb{Q} \)-algebra, after tensoring with \( \mathbb{Q} \). The correspondence which assigns to each finite group its slice Burnside ring has a natural biset functor structure, for which it becomes a Green biset functor.

We restrict our attention to the subcategory \( \mathcal{C}_p \) of \( \mathcal{C} \) whose objects are all the finite \( p \)-groups (for a given prime number \( p \)). To achieve the description of the ideals of the slice Burnside \( p \)-biset functor over a field of characteristic zero, we introduce the concept of \( T \)-slices instead of \( B \)-groups. In studying the ideals of the slice Burnside \( p \)-biset functor, we find another counterexample to Serge Bouc’s conjecture saying that the minimal groups for a simple module over a Green biset functor form a single isomorphism class.
The study in this direction has its origin in Nadia Romero’s paper ([8]), in which she found the first counterexample to this conjecture for the monomial Burnside ring over a cyclic group of order four.

The plan of the paper is as follows. In Section 2 we recall the basic definitions and preliminary properties of the classical Burnside functor. Section 3 recalls some properties of the slice Burnside ring and the slice Burnside functor. Section 4 is devoted to the action of biset operations on idempotents. In that section we exploit the case of the deflation and find a constant $m_{G,S,N}$. Section 5 gives some properties of the constant $m_{G,S,N}$. Section 6, exploits $m_{G,S,N}$ in the case of $p$-groups. Section 7 gives a characterization of the ideals of the slice Burnside functor. In Section 8 we further specialize the biset category to be the $p$-biset category and we consider the slice Burnside $p$-biset functor. In the final Section, we give a counterexample to Bouc’s conjecture.

2. Definitions and preliminary properties

The aim of this section is to fix some notation and recall some properties of biset functors. Throughout this paper, we fix a field $\mathbb{K}$ of characteristic zero and we identify the prime subfield of $\mathbb{K}$ with $\mathbb{Q}$. We assume that the reader is familiar with the concept of biset.

Notation 2.1. If $Y$ is an $(H,G)$-biset and $X$ is a $(K,H)$-biset then the tensor product $X \times_H Y$ is defined as the set of $H$-orbits of $X \times Y$ under the $H$-action defined by $(x,y).h := (xh^{-1}, y)$ for $h \in H$ and $(x,y) \in X \times Y$. The $H$-orbit of $(x,y)$ is denoted by $(x,Hy)$ and the set $X \times_H Y$ of elements $(x,Hy)$ is a $(K,G)$-biset under

$$k(x,Hy)g := (kx,Hyg).$$

This operation defines a bilinear map

$$- \times_H - : B(K,H) \times B(H,G) \rightarrow B(K,G), \quad ([X],[Y]) \mapsto [X \times_H Y],$$

where $B(H,G)$ is the Burnside group of $(H,G)$-bisets. If $G$ is a finite group, we denote by $\text{Id}_G$ the $(G,G)$-biset $G$.

This leads to the following definitions.

Definition 2.2. The biset category $\mathcal{C}$ of finite groups is the category defined as follows:

- The objects of $\mathcal{C}$ are finite groups.
- If $G$ and $H$ are finite groups, then $\text{Hom}_\mathcal{C}(G,H) = B(H,G)$.
- If $G$, $H$, and $K$ are finite groups, then the composition $v \circ u$ of the morphism $u \in \text{Hom}_\mathcal{C}(G,H)$ and the morphism $v \in \text{Hom}_\mathcal{C}(H,K)$ is equal to $v \times_H u$.
- For any finite group $G$, the identity morphism of $G$ in $\mathcal{C}$ is equal to $\text{Id}_G$. 
More generally, when $\mathbb{K}$ is a commutative ring, we define similarly the category $\mathbb{K}\mathcal{C}$ by extending coefficients to $\mathbb{K}$, i.e. by setting

$$\text{Hom}_{\mathbb{K}\mathcal{C}}(G, H) = \mathbb{K} \otimes \mathbb{Z} B(H, G),$$

which will be simply denoted by $\mathbb{K}B(H, G)$.

**Remark 2.3.** The category $\mathcal{C}$ is a preadditive category: the sets of morphisms in $\mathcal{C}$ are abelian groups, and the composition of morphisms is bilinear. Let $G$ and $H$ be finite groups. Then, any morphism from $G$ to $H$ in $\mathcal{C}$ is a linear combination with integral coefficients of morphisms of the form $[(G \times H)/L]$, where $L$ is some subgroup of $H \times G$.

**Notation 2.4.** Let $H$ and $G$ be finite groups.

- We indicate by $H \leq G$ that $H$ is a subgroup of $G$. We write $H < G$ if $H \leq G$ and $H \neq G$.
- We set $H^x := x^{-1}Hx$, for $x \in G$ and $H \leq G$.
- If $H \leq G$, we denote by $\text{Res}^G_H$ the set $G$, viewed as an $(H, G)$-biset for left and right multiplication, and by $\text{Ind}^G_H$ the same set viewed as a $(G, H)$-biset.
- If $N \leq G$, and $H = G/N$, we denote by $\text{Inf}^G_H$ the set $H$, viewed as a $(G, H)$-biset for the left action of $G$, and right action of $H$ by multiplication. Also we denote by $\text{Def}^G_H$ the set $H$, viewed as an $(H, G)$-biset.
- If $f : G \rightarrow H$ is a group isomorphism, we denote by $\text{Iso}^H_G$ or $\text{Iso}(f)$ the set $H$, viewed as an $(H, G)$-biset for left multiplication in $H$, and right action of $G$ given by multiplication by the image under $f$.

The bisets $\text{Ind}^G_H, \text{Res}^G_H, \text{Inf}^G_H, \text{Def}^G_H$ and $\text{Iso}^H_G$ are called the *elementary bisets*.

**Lemma 2.5.** [1.1.3 Relations [2]] *Commutation conditions:*

- (Mackey formula) If $H$ and $K$ are subgroups of $G$, then

$$\text{Res}^G_H \circ \text{Ind}^G_K \cong \sum_{x \in [H\backslash G/K]} \text{Ind}^H_{H\cap xK} \circ \text{Iso}(\gamma_x) \circ \text{Res}^K_{H\cap xK},$$

where $[H\backslash G/K]$ is a set of representatives of $(H, K)$-double cosets in $G$, and $\gamma_x$ is the group isomorphism induced by conjugation by $x$.
- If $H$ is a subgroup of $G$, and if $N$ is a normal subgroup of $G$, then

$$\text{Def}^G_{G/N} \circ \text{Ind}^G_N \cong \text{Ind}^G_{H\cap N/N} \circ \text{Iso}(\varphi) \circ \text{Def}^H_{H\cap N},$$

where $\varphi : H/H \cap N \rightarrow HN/N$ is the canonical group isomorphism.

The following is a straightforward consequence of Goursat’s Lemma on subgroups of a direct product of two groups:
Lemma 2.6. [[2], Lemma 2.3.26] Any transitive \((H, G)\)-biset is isomorphic to a composition

\[
\text{Ind}_{D}^{H} \circ \text{Inf}^{D/C}_{D/C} \circ \text{Iso}^{B/A}_{D/C} \circ \text{Def}^{B}_{B/A} \circ \text{Res}^{G}_{B},
\]

where \(A \leq B \leq G\), \(C \leq D \leq H\), and \(f : B/A \xrightarrow{\cong} D/C\) is a group isomorphism.

Remark 2.7. Let \(E\) be the set of triples \(((D, C), f, (B, A))\) where \(A \leq B \leq G\), \(C \leq D \leq H\), and \(f : B/A \xrightarrow{\cong} D/C\) is a group isomorphism. The group \(H \times G^{\text{op}}\) (where \(G^{\text{op}}\) is the opposite group of \(G\)) acts by conjugation on \(E\) and the set of all elements

\[
\text{Ind}^{H}_{D} \circ \text{Inf}^{D/C}_{D/C} \circ \text{Iso}^{B/A}_{D/C} \circ \text{Def}^{B}_{B/A} \circ \text{Res}^{G}_{B}
\]

is a \(K\)-basis of \(KB(H, G)\), where the triple \(((D, C), f, (B, A))\) runs over representatives of \((H \times G)\)-orbits in \(E\).

Definition 2.8. A \textit{biset functor} is an additive functor from \(C\) to abelian groups. A biset functor with values in \(K\)-Vect is a \(K\)-linear functor from \(KC\) to the category \(K\)-Vect of \(K\)-vector spaces.

Remark 2.9. Biset functors form an abelian category \(\mathcal{F}\) (where morphisms are natural transformations of functors).

Similarly, biset functors with values in \(K\)-Vect form a \(K\)-linear abelian category \(\mathcal{F}_{K}\).

Definition 2.10. Let \(F\) be a biset functor on \(C\).

A \textit{minimal group} for \(F\) is a finite group \(H\) such that \(F(H) \neq 0\), but \(F(K) = 0\) for any finite group \(K\) with \(|K| < |H|\). The class of minimal groups for \(F\) is denoted by \(\text{Min}(F)\). Note that \(\text{Min}(F) \neq \emptyset\) if and only if \(F\) is not the zero functor.

Definition 2.11. A biset functor \(A\) is a \textit{Green biset functor} (on \(C\)) if it is endowed with bilinear products \(A(G) \times A(H) \to A(G \times H)\), denoted by \((a, b) \mapsto a \times b\), for any finite groups \(G, H\), and an element \(\epsilon_{A} \in A(1)\), satisfying the following conditions:

1. (Associativity) Let \(G, H\) and \(K\) be finite groups. If

\[
\alpha_{G,H,K} : G \times (H \times K) \to (G \times H) \times K
\]

is the canonical group isomorphism, then for any \(a \in A(G)\), \(b \in A(H)\), and \(c \in A(K)\)

\[
(a \times b) \times c = \text{Iso}(\alpha_{G,H,K})(a \times (b \times c)).
\]

2. (Identity element) Let \(G\) be a finite group. Let \(\lambda_{G} : 1 \times G \to G\) and \(\rho_{G} : G \times 1 \to G\) denote the canonical group isomorphisms. Then for any \(a \in A(G)\)

\[
a = \text{Iso}(\lambda_{G})(\epsilon_{A} \times a) = \text{Iso}(\rho_{G})(a \times \epsilon_{A}).
\]
3. (Functoriality) If $\varphi : G \to G'$ and $\psi : H \to H'$ are morphisms in $\mathbb{K}C$, then for any $a \in A(G)$ and $b \in A(H)$

$$A(\varphi \times \psi)(a \times b) = A(\varphi)(a) \times A(\psi)(b).$$

**Definition 2.12.** Let $A$ be a Green biset functor (resp. with values in $\mathbb{K}$) on $C$. A *left A-module* $M$ is an object of $\mathcal{F}$ (resp. $\mathcal{F}_\mathbb{K}$) endowed, for any finite groups $G$ and $H$, with bilinear product maps

$$A(G) \times M(H) \to M(G \times H),$$

denoted by $(a, m) \to a \times m$, fulfilling the following conditions:

- For any $a \in A(G)$, $b \in A(H)$, and $m \in M(K)$
  $$(a \times b) \times m = \text{Iso}(\alpha_{G,H,K})(a \times (b \times m)).$$

- For any $m \in M(G)$
  $$m = \text{Iso}(\lambda_G)(\epsilon_A \times m).$$

- For any $a \in A(G)$ and $m \in M(H)$
  $$M(\phi \times \psi)(a \times m) = A(\phi)(a) \times M(\psi)(m).$$

**Example 2.13.** Assigning to each finite group $G$ its Burnside ring $B(G)$ yields an example of a Green biset functor. The product is induced by the bifunctor sending a $G$-set $X$ and an $H$-set $Y$ to the $(G \times H)$-set $X \times Y$. The identity element is $1 \in B(1)$, i.e. the class of a set of cardinality one in $B(1) \cong \mathbb{Z}$.

**Definition 2.14.** Let $A$ be a Green biset functor on $C$. A *left ideal* of $A$ is an $A$-submodule of the left $A$-module $A$. In other words it is a biset subfunctor $I$ of $A$ such that

$$A(G) \times I(H) \subseteq I(G \times H),$$

for any finite groups $G$ and $H$.

One defines similarly a *right ideal* of $A$.

A *two sided ideal* of $A$ is a left ideal which is also a right ideal.

A Green biset functor $A$ is called *simple* if its only two sided ideals are 0 and $A$.

**Proposition 2.15** *(See [7])*. Let $A$ be a Green biset functor on $C$ (resp. with values in $\mathbb{K}$-Vect) and let $G$ be a finite group. Then the product
\[ : A(G) \times A(G) \to A(G) \]
\[(u, v) \mapsto u.v = \text{Iso}_{\triangle(G)}^G \text{Res}_{\triangle(G)}^G (u \times v) \]

where \( \triangle(G) \) denoted the diagonal subgroup of \( G \), endows \( A(G) \) with a ring structure (resp. a structure of \( \mathbb{K} \)-algebra).

Moreover, if \( H \) is a finite group, then
\[ a \times b = (\text{Inf}_G^{G \times H} a). (\text{Inf}_H^{G \times H} b) \]
for any \( a \in A(G) \) and \( b \in A(H) \).

**Proposition 2.16.** Let \( A \) be a Green biset functor. The following assertions are equivalent.

1. \( F \) is a subfunctor of \( A \) and the evaluation at any finite group \( G \) of \( F \) is an ideal of \( A(G) \).
2. \( F \) is an ideal of \( A \) as Green biset functor.

**Proof.** Let \( G \) be a finite group.

Assume that Assertion (2) holds for \( F \) i.e. \( F \) is an ideal of \( A \). Then for any \( u \in A(G) \) and any \( v \in F(G) \), the product \( u \times v \) is in \( F(G \times G) \).

Now the product \( (u, v) \mapsto u.v \) in \( A(G) \) can be recovered from the product \( (u, v) \mapsto u \times v \) as
\[ u.v = \text{Iso}_{\triangle(G)}^G \text{Res}_{\triangle(G)}^G (u \times v). \]

Since \( u \times v \) is in \( F(G \times G) \), we have \( \text{Res}_{\triangle(G)}^G (u \times v) \in F(\triangle(G)) \) and \( \text{Iso}_{\triangle(G)}^G \text{Res}_{\triangle(G)}^G (u \times v) \in F(G) \). Thus the product \( u.v \) is in \( F(G) \) and \( F(G) \) is an ideal of \( A(G) \).

Conversely, assume that \( F \) is a subfunctor of \( A \) such that \( F(G) \) is an ideal of \( A(G) \) for any finite group \( G \).

Let \( H \) be finite group.

Let \( a \) be an element of \( A(G) \) and let \( b \) be an element of \( F(H) \). Then \( \text{Inf}_{G \times H/1 \times H}^{G \times H} (a) \) is in \( A(G \times H) \) and \( \text{Inf}_{G \times H/1 \times H}^{G \times H} (b) \) is in \( F(G \times H) \).

Since \( F(G \times H) \) is an ideal of \( A(G \times H) \), we have that the product \( a \times b = (\text{Inf}_{G \times H}^G a). (\text{Inf}_H^{G \times H} b) \) lies in \( F(G \times H) \). Hence, the subfunctor \( F \) is an ideal of \( A \). This completes the proof. \( \square \)

3. **Review of slice Burnside ring and slice Burnside functor**

We first recall the definition and basic properties of the slice Burnside ring introduced in [3], to which we refer the reader for all statements without proof.

**Definition 3.1.** A slice \((T, S)\) is a pair of finite groups with \( S \leq T \). When \( G \) is a finite group, a slice of \( G \) is a slice \((T, S)\) with \( T \leq G \).
Notation 3.2. Let $H$ and $G$ be finite groups.

- The set of slices of $G$ is denoted by $\Pi(G)$.
- When $(T, S)$ in an element of $\Pi(G)$, denote by $G/S \rightarrow G/T$ the projection morphism.
- We say that two slices $(T, S)$ and $(V, U)$ of $G$ are conjugate, if there exists an element $g$ of $G$ such that $T = V^g$ and $S = U^g$.

We note $N_G(T, S) = N_G(T) \cap N_G(S) = \{ g \in G \mid T^g = T $ and $ S^g = S \}$.

We write $(T, S) =_G (V, U)$ if the slices $(T, S)$ and $(V, U)$ of $G$ are conjugate, and $(T, S) \neq_G (V, U)$ otherwise.

- We say that $(V, U)$ is a quotient of $(T, S)$ and we denote $(T, S) \twoheadrightarrow (V, U)$, if there exists a surjective group homomorphism $\varphi : T \rightarrow V$ such that $\varphi(S) = U$. If $\phi$ is an isomorphism, we say that $(V, U)$ and $(T, S)$ are isomorphic.

Definition 3.3. Let $G$-set be the category of finite $G$-sets. Then the category $G$-Mor of morphisms of $G$-set has as objects the morphisms of $G$-set, and a morphism from $f : A \rightarrow B$ to $g : A' \rightarrow B'$ is a pair of morphisms of $G$-sets $h : A \rightarrow A'$ and $k : B \rightarrow B'$ making the following diagram commute

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow h & & \downarrow k \\
A' & \longrightarrow & B'
\end{array}
\]

Note that the category $G$-Mor admits products (induced by the direct product of $G$-sets) and coproducts (induced by the disjoint union of $G$-sets).

Definition 3.4. Let $G$ be a finite group. The slice Burnside group $\Xi(G)$ of $G$ is the Grothendieck group of the category of morphisms of finite $G$-sets defined as the quotient of the free abelian group on the set of isomorphism classes $[X \xrightarrow{f} Y]$ of morphisms of finite $G$-sets, by the subgroup generated by elements of the form

\[
[ X_1 \sqcup X_2 \xrightarrow{f_1\sqcup f_2} Y ] - [ X_1 \xrightarrow{f_1} f(X_1) ] - [ X_2 \xrightarrow{f_2} f(X_2) ],
\]

whenever $X \xrightarrow{f} Y$ is a morphism of finite $G$-sets with a decomposition $X = X_1 \sqcup X_2$ as a disjoint union of $G$-sets, where $f_1 = f|_{X_1}$ and $f_2 = f|_{X_2}$.

The product of morphisms induces a commutative unital ring structure on $\Xi(G)$. The identity element for multiplication is the image of the class $[\bullet \rightarrow \bullet]$, where $\bullet$ denotes a $G$-set of cardinality 1. For a morphism of $G$-sets $f : X \rightarrow Y$, let $\pi(f)$ denote the image in $\Xi(G)$ of the isomorphism class of $f$. 

Theorem 3.5. [[3], Theorem 3.9] Let \( G \) and \( H \) be finite groups, and let \( U \) be a finite \((H,G)\)-biset. Then the correspondence

\[
(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)
\]

is a functor from \( G\text{-Mor} \) to \( H\text{-Mor} \) and induces a group homomorphism \( \Xi(U) : \Xi(G) \rightarrow \Xi(H) \).

Proof. Since \( f : X \rightarrow Y \) is a morphism of \( G \)-sets, the map given by

\[
U \times_G f : U \times_G X \rightarrow U \times_G Y, \quad (u, g x) \mapsto (u, g f(x))
\]

is a well defined morphism of \( H \)-sets. Let \( f' : X' \rightarrow Y' \) be a morphism of \( G \)-sets and let \((\alpha, \beta)\) be a morphism from \( f \) to \( f' \) i.e. a pair of morphisms of \( G \)-sets such that \( \beta \circ f = f' \circ \alpha \). Then \( U \times_G f, U \times_G f', U \times_G \alpha \) and \( U \times_G \beta \) are morphisms of \( H \)-sets and they satisfy \((U \times_G \beta) \circ (U \times_G f) = (U \times_G f') \circ (U \times_G \alpha)\). Indeed, for any \( x \in X \) and \( u \in U \), we have

\[
(U \times_G \beta) \circ (U \times_G f)(u, g x) = \left( u, g \beta(f(x)) \right) = \left( u, g f'(\alpha(x)) \right)
\]

\[
= (U \times_G f') \circ (U \times_G \alpha)(u, g x).
\]

Thus \((U \times_G \alpha, U \times_G \beta)\) is a morphism from \( U \times_G f \) to \( U \times_G f' \) and it follows that the correspondence

\[
(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)
\]

define a functor from \( G\text{-Mor} \) to \( H\text{-Mor} \).

To prove to second part of the Theorem the only thing to check is that the defining relations of \( \Xi(G) \) are mapped to relations in \( \Xi(H) \). But if \( X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y \) is a morphism of finite \( G \)-sets, then \( U \times_G (X_1 \sqcup X_2) \) is isomorphic to \((U \times_G X_1) \sqcup (U \times_G X_2)\).

Moreover, the image of the map \( U \times_G f_1 \) is equal to \( U \times_G f_1(X_1) \). It follows that the relation \([X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f_1(X_1)] - [X_2 \xrightarrow{f_2} f_2(X_2)]\) in \( \Xi(G) \) is mapped to the relation

\[
[(U \times_G f_1) \sqcup (U \times_G f_2)] - [U \times_G f_1] - [U \times_G f_2].
\]

\[\square\]

Proposition 3.6. The induction functor is left adjoint to the restriction functor. The adjunction means, there is a natural bijection

\[
\text{Hom}_{G\text{-Mor}}( \text{Ind}_H^G X \xrightarrow{\text{Ind}_H^G f} \text{Ind}_H^G Y, \ A \xrightarrow{\alpha} B ) \cong \text{Hom}_{H\text{-Mor}}( X \xrightarrow{f} Y, \ \text{Res}_H^G A \xrightarrow{\text{Res}_H^G \alpha} \text{Res}_H^G B )
\]
**Proof.** There is an adjunction $\text{Hom}_{\text{G-set}}(\text{Ind}_H^G X, A) \cong \text{Hom}_{\text{H-set}}(X, \text{Res}_H^G A)$ which assigns to a morphism of $H$-sets $k : X \to A$ the morphism of $G$-sets $G \times_H X \to A$, $(g, x) \mapsto gk(x)$. It induces obvious bijection between $\text{Hom}_{\text{G-Mor}}(\text{Ind}_H^G X, \text{Ind}_H^G Y$, $A \alpha \to B)$ and $\text{Hom}_{\text{H-Mor}}(X \to Y$, $\text{Res}_H^G A \to \text{Res}_H^G B)$. □

Similarly,

**Proposition 3.7.** The deflation functor is left adjoint to the inflation functor in the category of morphisms of $G$-sets.

**Proposition 3.8.** The assignment $G \mapsto \Xi(G)$ is a biset functor.

**Proof.** If $G$ and $H$ are finite groups, and if $U$ is a finite $(H, G)$-biset, then the functor

$$I_U : (X \xrightarrow{f} Y) \mapsto (U \times_G U \xrightarrow{U \times_G f} U \times_G Y)$$

induces an abelian group homomorphism $\Xi(U) : \Xi(G) \to \Xi(H)$, by Theorem 3.5. If $U'$ is an $(H, G)$-biset isomorphic to $U$, then the functors $I_U$ and $I_{U'}$ are clearly isomorphic. Hence $\Xi(U) = \Xi(U')$. And if $U$ is the disjoint union of two $(H, G)$-bisets $U_1$ and $U_2$, then the functor $I_U$ is isomorphic to the disjoint union of the functors $I_{U_1}$ and $I_{U_2}$. It follows that $\Xi(U) = \Xi(U_1) + \Xi(U_2)$. If $K$ is another finite group, and if $V$ is a finite $(K, H)$-biset, then there is a natural isomorphism of $K$-sets $V \times_H (U \times_G X) \to (V \times_H U) \times_G X$. This induces an isomorphism of functors $I_V \circ I_U \cong I_{V \times_H U}$ and so $\Xi(V) \circ \Xi(U) = \Xi(V \times_H U)$. Finally, if $U$ is the identity biset $\text{Id}_G$, then the functor $I_{\text{Id}}$ is isomorphic to the identity functor. Thus $\Xi(\text{Id}_G) = \text{Id}_{\Xi(G)}$. □

**Proposition 3.9.** [[3], Theorem 3.9] Assigning to each finite group $G$ the $\mathbb{Z}$-module $\Xi(G)$ defines a Green biset functor.

**Proof.** We have already seen that the correspondence $G \mapsto \Xi(G)$ is a biset functor. Let $f : X \to Y$ be a morphism of $G$-sets and let $g : Z \to T$ be a morphism of $H$-sets then $f \times g : X \times Z \to Y \times T$ is a morphism of $G \times H$-sets. It induces the product

$$\times : \Xi(G) \times \Xi(H) \to \Xi(G \times H)$$

$$(a, b) \mapsto a \times b.$$  

Moreover, the morphism $\bullet \mapsto \bullet$ of 1-sets is obviously an identity element for this product, up to identification $G \times 1 = G$. 
Finally if $G$, $H$, $G'$, $H'$ are finite groups, if $U$ is a finite $(H,G)$-biset, if $U'$ is a finite $(H',G')$-biset, it is clear that the morphisms $(U \times U') \times_{G \times G'} (f \times f')$ and $(U \times G f \times (U' \times G' f'))$ are isomorphic morphisms of $(H \times H')$-sets.

Thus we have shown that $\Xi(G)$ satisfies the three conditions of Definition 2.11, with $\epsilon_\Xi$ being the identity element of the product in $\Xi(1)$.

So, the correspondence $G \mapsto \Xi(G)$ is a Green biset functor. $\square$

If $S \leq T$ are subgroups of $G$, set

$$\langle T, S \rangle_G = \pi(G/S \to G/T).$$

Lemma 3.10. ([3], Lemma 3.4] Let $f : X \to Y$ be a morphism of $G$-sets. Then in the group $\Xi(G)$,

$$\pi(f) = \sum_{x \in [G \setminus X]} \langle G_{f(x)}, G_x \rangle_G,$$

where $G_\bullet$ denotes the stabilizer of $\bullet$.

Thus the group $\Xi(G)$ is, as an additive group, free with basis $\{ \langle T, S \rangle_G \mid (T, S) \in \Pi(G) \}$, where $\Pi(G)$ is a set of representatives of conjugacy classes of slices of $G$.

Remark 3.11. In this basis, the multiplication of $\Xi(G)$ can be computed as follows

$$\langle T, S \rangle_G \cdot \langle Y, X \rangle_G = \sum_{g \in [S \setminus G/X]} \langle T \cap ^g Y, S \cap ^g X \rangle_G$$

for any slices $(T, S)$ and $(Y, X)$ of $G$.

For any slice $(T, S)$ of $G$ and any morphism of $G$-sets $f : X \to Y$, we write

$$\phi_{T, S}^G : \langle X \xrightarrow{f} Y \rangle := |\text{Hom}_{G \to \text{Mor}}(G/S \to G/T, \ X \xrightarrow{f} Y)|.$$

Then $\phi_{T, S}^G$ induces a ring homomorphism $\Xi(G) \to \mathbb{Z}$, still denoted by $\phi_{T, S}^G$. Conjugate slices give the same homomorphism.

Let

$$\Phi = \prod_{(T, S) \in \Pi(G)} \phi_{T, S}^G : \Xi(G) \to \prod_{(T, S) \in \Pi(G)} \mathbb{Z}.$$

The ring homomorphism $\Phi$ is a monomorphism between free $\mathbb{Z}$-modules having the same rank. Hence tensoring with $\mathbb{Q}$ gives a $\mathbb{Q}$-algebra isomorphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \Xi(G) \to \prod_{(T, S) \in \Pi(G)} \mathbb{Q}.$$
The commutative $\mathbb{Q}$-algebra $\mathbb{Q}\Xi(G)$ ($= \mathbb{Q} \otimes_{\mathbb{Z}} \Xi(G)$) is split semisimple.

**Notation 3.12.** For a slice $(T, S)$ of the group $G$, set

$$\xi_{T, S}^G = \frac{1}{|N_G(T, S)|} \sum_{U \leq S \leq V \leq T} |U|\mu(U, S)\mu(V, T)\langle V, U \rangle_G$$

where $\mu$ is the Möbius function of the poset of subgroups of $G$ and $N_G(T, S) = N_G(T) \cap N_G(S)$.

Note that $\xi_{T, g}^G = \xi_{T^g, S^g}^G$ for any $g \in G$.

**Proposition 3.13.** [3, Theorem 5.2] Let $G$ be a finite group. Then the elements $\xi_{T, S}^G$, for $(T, S) \in [\Pi(G)]$ are the primitive idempotents of $\mathbb{Q}\Xi(G)$.

**Proof.** An easy computation shows that if $(Y, X)$ is a slice of $G$, then $\mathbb{Q}\phi_{Y, X}^G(\xi_{T, S}^G)$ is equal to 0 if $(Y, X)$ and $(T, S)$ are not $G$-conjugate, and to 1 otherwise. \qed

**Remark 3.14.** After extending the scalars from $\mathbb{Q}$ to $\mathbb{K}$, we get that the commutative algebra

$$\mathbb{K}\Xi(G) = \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{Q}\Xi(G)$$

is also split semisimple. We abuse notation and denote by $\xi_{T, S}^G$ the element $1 \otimes \xi_{T, S}^G$ of $\mathbb{K}\Xi(G)$.

**Proposition 3.15.** Let $G$ be a finite group. Let $(T, S)$ be a slice of $G$. Then for any element $v$ of $\mathbb{K}\Xi(G)$

$$v \cdot \xi_{T, S}^G = \phi_{T, S}^G(v) \xi_{T, S}^G.$$

Conversely, if $v \in \mathbb{K}\Xi(G)$ is such that $v.u$ is a scalar multiple of $u$ for any $v \in \mathbb{K}\Xi(G)$, then there exists $(T, S) \in \Pi(G)$ such that $u \in \mathbb{K}\xi_{T, S}^G$ and $v.u = \phi_{T, S}^G(v)u$.

**Proof.** The set of elements $\xi_{T, S}^G$, for $(T, S) \in [\Pi(G)]$ is a basis of $\mathbb{Q}\Xi(G)$. Thus for any $v \in \mathbb{K}\Xi(G)$, there are scalars $r_{T, S}$, for $(T, S) \in \Pi(G)$, such that $v = \sum_{(T, S) \in [\Pi(G)]} r_{T, S} \xi_{T, S}^G$.

Moreover, for a slice $(V, U) \in [\Pi(G)]$, we have

$$\phi_{V, U}^G(v) = \sum_{(T, S) \in [\Pi(G)]} r_{T, S} \phi_{V, U}^G(\xi_{T, S}^G) = r_{V, U}.$$

It follows that for all $(V, U) \in [\Pi(G)]$

$$v \cdot \xi_{V, U}^G = \phi_{V, U}^G(v) \xi_{V, U}^G.$$
Conversely, let $u$ be an element of $\mathbb{K} \Xi(G)$ satisfying $v.u = \lambda(v)u$ for any $v \in \mathbb{K} \Xi(G)$ where $\lambda(v) \in \mathbb{K}$. If $u = 0$, then there is nothing to prove. Otherwise, there exists a slice $(T, S)$ of $G$ such that $\xi_{T,S}^G u \neq 0$. Then $\xi_{T,S}^G u = \phi_{T,S}^G(\lambda(\xi_{T,S}^G)u) = \lambda(\xi_{T,S}^G)u \neq 0$, so $u$ is a scalar multiple of $\xi_{T,S}^G$. \hfill \Box

4. Effect of biset operations on idempotents

Recall that the algebra $\mathbb{K} \Xi(G)$ is commutative and split semi-simple, for any finite group $G$. Since $\mathbb{K} \Xi$ is a biset functor, by Remark 2.7, it becomes natural to look at the effect of elementary operations (Ind, Res, Inf, Def and Iso) on idempotents of the algebra $\mathbb{K} \Xi(G)$.

**Proposition 4.1.** Let $(T, S)$ be a slice of $G$ and $H$ be subgroup of $G$. Then

$$\text{Res}_H^G \xi_{T,S}^G = \sum_{(T', S') \in [\Pi(H)]} \xi_{T', S'}^H.$$  

**Proof.** Let $(T, S)$ be slice of $G$.

By Proposition 3.15, the idempotent $\xi_{T,S}^G$ has the property that $\xi_{T,S}^G \pi(f) = \phi_{T,S}^G(\pi(f)) \xi_{T,S}^G$ for any $\pi(f) \in \mathbb{K} \Xi(G)$. Observe that the restriction functor $\text{Res}_H^G : G\text{-set} \to H\text{-set}$ induces a ring homomorphism $\text{Res}_H^G : \mathbb{K} \Xi(G) \to \mathbb{K} \Xi(H)$. It follows that $\text{Res}_H^G \xi_{T,S}^G$ is an idempotent of $\mathbb{K} \Xi(H)$, hence a sum of idempotents $\xi_{T', S'}^H$, for some slices $(T', S')$ of $H$. The idempotent $\xi_{V,U}^H$ appears in this sum if and only if

$$0 \neq \xi_{V,U}^H \text{Res}_H^G \xi_{T,S}^G = \phi_{V,U}^H(\text{Res}_H^G \xi_{T,S}^G) \xi_{V,U}^H = \phi_{V,U}^H(\xi_{T,S}^G) \xi_{V,U}^H$$

where the last equality follows from Proposition 3.6.

Thus $\xi_{V,U}^H$ appears in this sum if and only if $(V, U)$ is conjugate to $(T, S)$ in $G$. \hfill \Box

**Proposition 4.2.** Let $H$ be a subgroup of $G$ and $(V, U)$ be a slice of $H$. Then

$$\text{Ind}_H^G \xi_{V,U}^H = \frac{|N_G(V, U)|}{|N_H(V, U)|} \xi_{V,U}^G.$$  

**Proof.** It is straightforward from Notation 3.12 since for any element $\langle V, U \rangle_H$ of $\mathbb{K} \Xi(H)$, we have $\text{Ind}_H^G(\langle V, U \rangle_H) = \langle V, U \rangle_G$. \hfill \Box

**Proposition 4.3.** Let $N$ be a normal subgroup of $G$. Then for any slice $(T, S)$ of $G$ such that $N$ is contained in $S$, we have

$$\text{Ind}_{G/N}^G \xi_{T,S/N}^G = \sum_{(Y, X) \in [\Pi(G)]} \xi_{Y,X}^G.$$  


Proof. We observe that \( \Inf_{G/N}^{G} : \mathbb{K}\Xi(G/N) \to \mathbb{K}\Xi(G) \) is a ring homomorphism.

It follows that \( \Inf_{G/N}^{G} \xi_{G/N}^{G,N} \) is an idempotent of \( \mathbb{K}\Xi(G) \), hence a sum of idempotents \( \xi_{Y,X}^{G} \), for some slices \((Y, X) \in \Pi(G)\). The idempotent \( \xi_{Y,X}^{G} \) appears in this sum if and only if

\[
0 \neq \xi_{Y,X}^{G} \Inf_{G/N}^{G} \xi_{G/N}^{G,N} = \phi_{Y,X}^{G} (\Inf_{G/N}^{G} \xi_{G/N}^{G,N}) \xi_{Y,X}^{G} = \phi_{Y/N,N,X/N,N}^{G/N,N} (\xi_{T,N,S/N}^{G,N,N}) \xi_{Y,X}^{G}
\]

i.e. if and only if \((YN, XN)\) is conjugate to \((T, S)\) in \(G\) i.e. if and only if \((Y, X) \times (T, S)\) modulo \(G\). \(\square\)

Proposition 4.4. Let \(N\) be a normal subgroup of \(G\). Then

\[
\Def_{G/N}^{G} \xi_{G,S}^{G} = m_{G,S,N} \xi_{G/N,S/N}^{G,N}
\]

where

\[
m_{G,S,N} = \frac{|N_{G}(SN) : SN|}{|N_{G}(S)|} \sum_{U \leq S \leq V \leq G \atop V N = G, U N = SN} |U| \mu(U, S) \mu(V, G).
\]

More generally, if \((T, S)\) is some slice of \(G\), then

\[
\Def_{G/N}^{G} \xi_{T,S}^{G} = \frac{|N_{T}(S)||N_{G}(TN, SN)|}{|N_{G}(T, S)||N_{T}(SN)|} m_{T,S,T \cap N} \xi_{T,N,S/N,N/S,N}^{G/N,N}.
\]

Proof. We observe that if \(X\) is a \(G\)-set, then \(\Def_{G/N}^{G} X = (G/N) \times_{G} X \cong N \setminus X\).

So the map \(\Def_{G/N}^{G} : Q\Xi(G) \to Q\Xi(G/N)\) is such that \(\Def_{G/N}^{G}(\pi( X \underset{f}{\to} Y )) = \pi( N \setminus X \underset{f}{\to} N \setminus Y )\) for any morphism of \(G\)-sets \(f : X \to Y\).

Now if \(Z\) is a \(G/N\)-set, and \(X\) is a \(G\)-set, then there is an isomorphism of \(G/N\)-sets

\[
\alpha_{Z,X} : Z \times (N \setminus X) \to (N \setminus (\Inf_{G/N}^{G} Z) \times X).
\]

Let \(Z \to T\) is a morphism of \(G/N\)-sets, and \(X \to Y\) is a morphism of \(G\)-sets then the morphisms \(g \times (N \setminus f)\) and \(N \setminus ((\Inf_{G/N}^{G} g) \times f)\) have the same image in \(Q\Xi(G/N)\). Then

\[
v\Def_{G/N}^{G} u = \Def_{G/N}^{G} ((\Inf_{G/N}^{G} v) u),
\]

for any \(v \in Q\Xi(G/N)\) and \(u \in Q\Xi(G)\). For \(u = \xi_{G,S}^{G}\), this gives
\[ v\text{Def}_{G/N}^G \xi_{G,S}^G = \text{Def}_{G/N}^G (\{(\text{Inf}_{G/N}^G v)\xi_{G,S}^G\}) \]
\[ = \phi_{G,S}^G (\text{Inf}_{G/N}^G v) \text{Def}_{G/N}^G \xi_{G,S}^G \]
\[ = \phi_{G/N}^G \text{Def}_{G/N,SN/N}^G (v) \text{Def}_{G/N}^G \xi_{G,S}^G \]

and by Proposition 3.15 it follows that \( \text{Def}_{G/N}^G \xi_{G,S}^G \) is equal to a scalar multiple of \( \xi_{G/N,SN/N}^G \). In particular, there is a scalar \( m \) such that

\[ \text{Def}_{G/N}^G \xi_{G,S}^G = m \xi_{G/N,SN/N}^G \]

and the case \( \mathbb{K} = \mathbb{Q} \) shows that \( m \in \mathbb{Q} \). We set \( \bar{G} = G/N, \bar{S} = SN/N, \bar{Y} = Y/N \) and \( \bar{X} = X/N \).

Since \( \text{Def}_{G/N}^G (T,S)_G = (TN,SN)_G = (TN/N,SN/N)_G/N \) for any slice \((T,S)\) of \( G \), this yields

(1) \[ \text{Def}_{G/N}^G \xi_{G,S}^G = \frac{1}{|N_G(S)|} \sum_{U \leq S \leq V \leq G} |U| \mu(U,S) \mu(V,G) \langle VN/N,UN/N \rangle_{G/N} \]
\[ = m \xi_{G/N,SN/N}^G \]
\[ = \frac{m}{|N_G(S)|} \sum_{\bar{X} \leq \bar{S} \leq \bar{Y} \leq \bar{G}} |\bar{X}| \mu(\bar{X},\bar{S}) \mu(\bar{Y},\bar{G}) \langle \bar{Y}, \bar{X} \rangle_{\bar{G}} \]

(2) \[ = \frac{m}{|N_G/N(SN/N)||N|} \sum_{X \leq SN \leq Y \leq G \atop N \leq X} |X| \mu(X,SN) \mu(Y,G) \langle \bar{Y}, \bar{X} \rangle_{\bar{G}} \]

The coefficient of \( \langle G/N,SN/N \rangle_{G/N} \) is equal to \( \frac{m|SN|}{|N_G/N(SN/N)||N|} \) in (2), and equal to

\[ \frac{1}{|N_G(S)|} \sum_{U \leq S \leq V \leq G \atop \bar{V}N = \bar{G} \atop UN = gSN} |U| \mu(U,S) \mu(V,G) \]

in (1).

Since \( U \leq S \) and \( UN = gSN \) implies that \( UN = SN \), and the coefficient of \( \langle G/N,SN/N \rangle_{G/N} \) is equal to

\[ \frac{1}{|N_G(S)|} \sum_{U \leq S \leq V \leq G \atop \bar{V}N = \bar{G} \atop UN = SN} |U| \mu(U,S) \mu(V,G) \]

in (1).
It follows that
\[ m_{G,S,N} := m = \frac{|N_G(SN/N)||N|}{|N_G(S)||SN|} \sum_{U \leq S \leq V \leq G, \frac{VN}{U} = SN} |U| \mu(U, S) \mu(V, G). \]

We observe also from (2) that,
\[ \xi^G_{T,S} = \frac{|N_T(S)|}{|N_G(T, S)|} \operatorname{Ind}^G_T \xi^T_{T,S}, \]
for any slice \((T, S)\) of \(G\). Thus, by Lemma 2.5
\[
\def^G_{G/N} \xi^G_{T,S} = \frac{|N_T(S)|}{|N_G(T, S)|} \operatorname{Def}^G_{G/N} \operatorname{Ind}^G_T \xi^T_{T,S} \\
= \frac{|N_T(S)|}{|N_G(T, S)|} \operatorname{Ind}^G_{T/N} \operatorname{Iso}^T_{T/T \cap N} \operatorname{Def}^T_{T/T \cap N} \xi^T_{T,S} \\
= \frac{|N_T(S)|}{|N_G(T, S)|} m^G_{T,S,T \cap N} \operatorname{Ind}^G_{T/N} \operatorname{Iso}^T_{T/T \cap N} \xi^T_{T,T \cap N, S(T \cap N)/T \cap N} \\
= \frac{|N_T(S)|}{|N_G(T, S)|} m^G_{T,S,T \cap N} \operatorname{Ind}^G_{T/N} \xi^T_{T/N,N \cap S(T \cap N)/T \cap N} \\
= \frac{|N_T(S)||N_G(TN/N, SN/N)|}{|N_G(T, S)||N_T(SN/N)|} m^G_{T,S,T \cap N} \xi^G_{T/N,N \cap S(T \cap N)/T \cap N}.
\]
This completes the proof. \qed

**Proposition 4.5.** If \( \varphi : G \to G' \) is a group isomorphism, and \((T, S)\) is a slice of \(G\), then
\[ \operatorname{Iso}(\varphi)(\xi^G_{T,S}) = \xi^{G'}_{\varphi(T),\varphi(S)}. \]

**Proof.** This is straightforward. \qed

### 5. Properties of \(m_{G,S,N}'s\)

Let \((G, S)\) be a slice and \(N\) be a normal subgroup of \(G\).

**Proposition 5.1.** Let \(S \leq G\) be finite groups. If \(M\) and \(N\) are normal subgroups of \(G\) with \(N \leq M\), then
\[ m_{G,S,M} = m_{G,S,N} m_{G/N,SN/N,M/N}. \]
**Proof.** This follows from the transitivity property of deflation maps:

\[
\text{Def}_{G/M}^{G,N} \xi_{G,S}^G = m_{G,S} \text{Def}_{M}^{G/M} \xi_{S,M}^G = m_{G,N} \text{Def}_{N}^{G} \xi_{G,N}^{G/N} = m_{G,S,N} \text{Def}_{N}^{G} \xi_{G/N,S,N}^{G/N} = m_{G,S,N} m_{G/N,S,N,M} \xi_{G/M}^{G/M}.
\]

\[\square\]

**Remark 5.2.** If \(G\) is a finite group, and \(N\) is a normal subgroup of \(G\). We observe that the constant \(m_{G,G,N}\) is equal to the constant \(m_{G,N}\) introduced in [1] and [4].

Recall some properties of this constant \(m_{G,N}\) (see [1]).

- For any finite group \(G\) and any normal subgroup \(N\) of \(G\), we have \(m_{G,N} = m_{G/\Phi(G),N\Phi(G)/\Phi(G)}\) where \(\Phi(G)\) is the Frattini subgroup of \(G\). So, if \(G\) is a finite \(p\)-group (where \(p\) is a prime), we can assume that \(G\) is elementary abelian, in order to compute \(m_{G,N}\).
- If \(G\) is an elementary abelian \(p\)-group of rank \(n > 2\) and \(N\) is a subgroup of \(G\) of rank \(k\) with \(1 \leq k \leq n - 2\), then

\[
m_{G,N} = \prod_{i=1}^{k} (1 - p^{n-1-i}).
\]

In this case the constant \(m_{G,N}\) is equal to zero if and only if \(G\) is noncyclic and the quotient \(G/N\) is cyclic.

**Definition 5.3 (see [2], Definition 5.4.6).** A finite group \(G\) is called a \(B\)-group (over \(K\)) if \(|G| \neq 0\) in \(K\), and if for any non trivial normal subgroup \(N\) of \(G\), the constant \(m_{G,N}\) is equal to zero (in \(K\)).

**Proposition 5.4 (see [2], Section 5.6.9).** Let \(G\) be a \(p\)-group. Then \(G\) is a \(B\)-group if and only if \(G\) is trivial or isomorphic to \(C_p \times C_p\).

**Proposition 5.5.** Let \(S \leq G\) be finite groups and \(N\) be a normal subgroup of \(G\). Then

\[
m_{G,S,N} = \frac{|N_G(SN) : SN|}{|N_G(S) : S|} m_{S,S \cap N} m_{G,S,N}^{o} \mu(V,G).
\]

where \(m_{G,S,N}^{o} = \sum_{S \leq V \leq G} \mu(V,G)\).
Proof. By Dedekind’s identity we have,
\[ m_{S,S \cap N} = \frac{1}{|S|} \sum_{X(S \cap N) = S} |X| \mu(X, S) \]
\[ = \frac{1}{|S|} \sum_{X \leq S, XN = SN} |X| \mu(X, S). \]

We can now calculate the product
\[ m_{S,S \cap N} m_{G,S,N}^o = \left( \frac{1}{|S|} \sum_{X(S \cap N) = S} |X| \mu(X, S) \right) \left( \sum_{S \leq V \leq G, \forall N = G} \mu(V, G) \right) \]
\[ = \left( \frac{1}{|S|} \sum_{X \leq S, XN = SN} |X| \mu(X, S) \right) \left( \sum_{S \leq V \leq G, \forall N = G} \mu(V, G) \right) \]
\[ = \frac{1}{|S|} \sum_{X \leq S \leq V \leq G, \forall N = G} |X| \mu(X, S) \mu(V, G) \]
\[ = \frac{|N_G(S) : S|}{|N_G(SN) : SN|} m_{G,S,N}^o. \]

Proposition 5.6. Let \( G \) be a finite group.
If \( N \) is a minimal abelian normal subgroup of \( G \), then
\[ m_{G,1,N} = \frac{1}{|N|} \left( 1 - |K_G(N)| \right) = \frac{1}{|N|} m_{G,1,N}^o, \]
where \( K_G(N) \) is the set of complements of \( N \) in \( G \).

Proof. Let \( X \) be a subgroup of \( G \) such that \( XN = G \). Then \( X \cap N \) is normalized by \( X \), and by \( N \) since \( N \) is abelian. It follows that \( X \cap N \leq G \), thus \( X \cap N = N \) or \( X \cap N = 1 \) by minimality of \( N \). In the first case \( X = G \), and in the second case \( X \in K_G(N) \), so \( |X| = |G : N| \), and moreover \( X \) is a maximal subgroup of \( G \), so \( \mu_G(X, G) = -1 \). This shows that
\[ m_{G,1,N} = \frac{1}{|N|} \left( \sum_{XN = G} \mu_G(X, G) \right) = \frac{1}{|N|} \left( 1 + \sum_{XN = G, X \neq G} \mu_G(X, G) \right) \]
\[ = \frac{1}{|N|} \left( 1 - |K_G(N)| \right). \]

Lemma 5.7. Let \( S \leq G \) be finite groups and \( N \) be a normal subgroup of \( G \). Then
\[ m_{G,S,N}^o = m_{G/\Phi(G),S\Phi(G)/\Phi(G),N\Phi(G)/\Phi(G)}. \]
Proof. We set $\bar{G} = G/\Phi(G)$, $\bar{S} = S\Phi(G)/\Phi(G)$ and $\bar{N} = N\Phi(G)/\Phi(G)$. Since $\mu(V, G) = 0$ if $\Phi(G) \nmid V$, it follows that

$$m^\circ_{G,S,N} = \sum_{S \leq V \leq G \atop V_N = G} \mu(V, G) = \sum_{S \Phi(G) \leq V \leq G \atop V\Phi(G)N = G} \mu(V, G) = m^\circ_{G,S(\Phi(G)), N\Phi(G)}$$

$$= \sum_{\bar{S} \leq \bar{V} \leq \bar{G} \atop \bar{V}_N = G} \mu(\bar{V}, \bar{G}) = m^\circ_{G,\bar{S}, \bar{N}}$$

where $\bar{V}$ denotes $V/\Phi(G)$. □

6. The case of $p$-groups

Let $p$ be a prime number, and $G$ be a finite $p$-group. Let $N$ be a normal subgroup of $G$. Since,

$$m^\circ_{G,S,N} = m^\circ_{G/\Phi(G), S\Phi(G)/\Phi(G), N\Phi(G)/\Phi(G)}$$

for any subgroup $S$ of $G$, in order to compute $m^\circ_{G,S,N}$ we may assume that $G$ is an elementary abelian $p$-group without lost of generality.

Proposition 6.1. Let $G$ be an elementary abelian $p$-group of rank $n$ with $n \geq 1$, and let $N$ be subgroup of $G$ of rank $k$ with $k \geq 1$. Then

$$m^\circ_{G,1,N} = \prod_{i=1}^{k} (1 - p^{n-i}).$$

More generally, if $S$ is a subgroup of $G$, then $m^\circ_{G,S,N} = m^\circ_{G/S,1,NS/S}$.

Proof. We argue by induction on $|G|$. If $n = 1$ then by Proposition 5.6

$$m^\circ_{C_p,1,C_p} = 1 - 1 = 0.$$

One can assume that the result holds for all finite groups $H$ with $|H| < |G|$.

We assume first that $k = 1$.

Among the $(p^n - 1)/(p - 1)$ maximal subgroups of $G$, a total of $(p^{n-1} - 1)/(p - 1)$ contain $N$. Hence $N$ has $(p^n - p^{n-1})/(p - 1) = p^{n-1}$ complements in $G$, and so Proposition 5.6 gives

$$m^\circ_{G,1,N} = 1 - p^{n-1} \text{ in this case.}$$

We now assume $k > 1$. 

Let $M \leq G$ with $|M| = p$. Then the induction hypothesis applied to the group $G/M$ implies that

$$m_{G/M, 1, N/M}^o = (1 - p^{n-2}) \cdots (1 - p^{n-k}).$$

And by Proposition 5.1

$$m_{G, 1, N}^o = m_{G, 1, M}^o m_{G/M, 1, N/M}^o = (1 - p^{n-1})(1 - p^{n-2}) \cdots (1 - p^{n-k}).$$

The second part is an immediate consequence of the definition of the Möbius function and the fact that $V \mapsto V/S$ is an isomorphism from the poset of subgroups $V$ of $G$ containing $S$ such that $VN = G$ to the poset of subgroups $V/S$ of $G/S$ such that $V/S.NS/S = G/S$. Thus

$$m_{G, S, N}^o = \sum_{S \leq V \leq G \atop V/N = G} \mu(V, G) = \sum_{S \leq V \leq G \atop V/SN = G} \mu(V, G)$$

$$= \sum_{1 \leq V/S \leq G/S \atop V/S.SN/S = G/S} \mu(V/S, G/S) = m_{G/S, 1, NS/S}^o. \quad \square$$

**Remark 6.2.** The previous results show that if $G$ is a $p$-group and $S, N$ are subgroups of $G$, then: the constant $m_{G, S, N}^o$ is equal to zero if and only if $G \neq S$ and $G = SN$.

**Remark 6.3.** Let $T$ be a finite $p$-group and let $S$ be a subgroup of $T$. Then for any normal subgroup $N$ of $T$,

$$m_{T, S, N}^o = 0 \iff \begin{cases} S \text{ noncyclic} & \text{or} \\ S/S \cap N \text{ cyclic} & T \neq S \\ T = SN. \end{cases}$$

7. Characterization of the ideals of $\mathbb{K} \Xi$

In this section, we aim to give a combinatorial description of the ideals of the Green slice Burnside functor $\mathbb{K} \Xi$.

First we recall from Section 3 that for any finite group $G$, the algebra $\mathbb{K} \Xi(G)$ is a split semisimple commutative $\mathbb{K}$-algebra, whose primitive idempotents $\xi^G_{T, S}$ are indexed by a set of representatives of conjugacy classes of slices of $G$. In particular, if $F$ is an ideal of $\mathbb{K} \Xi$, we denote by $A_{F, G}$ the set of slices $(T, S)$ of $G$ such that $\xi^G_{T, S} \in F(G)$. Then $F(G)$ is the sum of one-dimensional $\mathbb{K}$-vector spaces generated by $\xi^G_{T, S}$, for $(T, S)$ in the family $A_{F, G}$. This sum is direct if the slices $(T, S)$ are chosen up to conjugation in $G$.

Our next result will be essential in our analysis of the ideals of $\mathbb{K} \Xi$.

**Theorem 7.1.** If $F$ is an ideal of the Green functor $\mathbb{K} \Xi$, then $A_{F, G}$ satisfies the four following conditions:
A Let \((T, S)\) be an element of \(A_{F,G}\) and \((T', S')\) be a slice of \(G\). If \((T, S)\) and \((T', S')\) are isomorphic, then \((T', S')\) is also an element of \(A_{F,G}\).

B The family \(A_{F,G}\) depends only on \(F\) and not on the group \(G\). So it will be denoted by \(A_F\) and \(A_F = \{(T, S) \mid \xi_{T,S}^T \in F(T)\}\).

C Let \(N\) be a normal subgroup of \(T\) and let \((T/N, S/N)\) be an element of \(A_F\). If there exist a slice \((Y, X)\) and a surjective group homomorphism \((Y, X) \to (T, S)\), then \((Y, X)\) lies in \(A_F\).

D Let \(N\) be a normal subgroup of \(T\) and let \((T, S)\) be an element of \(A_F\) such that the constant \(m_{T,S,N}\) is different from zero. Then the slice \((T/N, S/N)\) lies in \(A_F\).

Conversely, if a family \(A\) of slices is given, we set

\[
F_A(G) = \sum_{(T, S) \in A} \mathbb{K}\xi_{T,S}^G,
\]

for any finite group \(G\). If \(A\) fulfills Conditions A to D, then \(F_A\) is an ideal of the Green functor \(\mathbb{K}\Xi\).

**Proof.** Let \(F\) be an ideal of \(\mathbb{K}\Xi\). We claim that Conditions A to D are fulfilled.

Condition A is obvious by Proposition 4.5.

For each \(T \leq G\), we must have \(\text{Ind}_T^G F(T) \subseteq F(G)\). Let \((T, S)\) be an element of \(A_{F,T}\).

By Proposition 4.2, the idempotent \(\xi_{T,S}^G\) is a non zero scalar multiple of \(\text{Ind}_{T}^{G}\xi_{T,S}^{T}\).

Hence \((T, S) \in A_{F,G}\).

We must also have \(\text{Res}_T^G F(G) \subseteq F(T)\). Let \((T, S)\) be an element of \(A_{F,G}\).

Then by Proposition 4.1, the idempotent \(\text{Res}_T^G\xi_{T,S}^G\) is a sum of idempotents \(\xi_{T',S'}^T\) (with non zero coefficient), for slices \((T', S')\) of \(T\) such that \((T', S') = G (T, S)\).

Since \(\xi_{T,S}^T\) appears in this sum with non zero coefficient, we have that \((T, S) \in A_{F,T}\).

Thus, the slice \((T, S) \in A_{F,G}\) if and only if it lies in \(A_{F,T}\). In other words the family \(A_{F,G}\) depends only on \(F\) and \(A_{F,G} = A_F = \{(T, S) \mid \xi_{T,S}^T \in F(T)\}\).

Condition C and Condition D follow obviously from Proposition 4.3 and Proposition 4.4.

Conversely, let \(A\) be a set of slices which satisfies Conditions A, B, C and D. In order to prove that \(F_A\) is an ideal of \(\mathbb{K}\Xi\), all we have to show is that it is a biset subfunctor of \(\mathbb{K}\Xi\), since for any \(G\), the vector space \(F_A(G)\) is an ideal of \(\mathbb{K}\Xi(G)\).

As the elementary bisets generate the biset category, it is enough to check that \(F_A\) is stable under transport by isomorphism, induction, restriction, inflation and deflation.

The first case is clear.

Let \(H\) be a subgroup of \(G\) and \((V, U)\) be a slice of \(H\) such that \(\xi_{H,U}^V \in F_A(H)\). Then \(\xi_{V,U}^V \in F_A(V)\) and \(\xi_{V,U}^V \in F_A(G)\), by Condition B.

Since the idempotent \(\xi_{V,U}^V\) is a non zero scalar multiple of \(\text{Ind}_{H}^G\xi_{V,U}^V\), we observe that \(\text{Ind}_{H}^G\xi_{V,U}^V \in F_A(G)\) and so \(\text{Ind}_{H}^GF_A(H) \subseteq F_A(G)\).

Now, let \((T, S)\) be a slice of \(G\) such that \(\xi_{T,S}^G \in F_A(G)\) and \(H\) be a subgroup of \(G\).

By the Mackey formula the restriction of \(\xi_{T,S}^G\) to \(H\) is equal to
\[
\text{Res}_H^G \xi_{T,S} = \frac{|N_H(T,S)|}{|N_G(T,S)|} \text{Res}_H^G \text{Ind}_T^H \xi_{T,S},
\]

\[
= \frac{|N_H(T,S)|}{|N_G(T,S)|} \sum_{x \in [H \setminus G/T]} \text{Ind}_{H \cap xT}^H \circ \text{Iso}(\gamma_x) \circ \text{Res}_{H \cap xT}^T \xi_{T,S}^T
\]

where \([H \setminus G/T]\) is a set of representatives of \((H,T)\)-double cosets in \(G\), and \(\gamma_x\) is the group isomorphism induced by conjugation by \(x\).

We observe by Condition B, that \(\xi_{T,S}^T \in F_A(T)\). Since the restriction \(\text{Res}_{H \cap xT}^T \xi_{T,S}^T\) is either zero or \(\xi_{T,S}^T\), then \(\text{Res}_{H \cap xT}^T \xi_{T,S}^T\) lies in \(F_A(T)\).

By Condition A, the element \(\text{Iso}(\gamma_x) \circ \text{Res}_{H \cap xT}^T \xi_{T,S}^T\) lies in \(F_A(H \cap xT)\) and from the above proof, the element

\[
\text{Ind}_{H \cap xT}^H \circ \text{Iso}(\gamma_x) \circ \text{Res}_{H \cap xT}^T \xi_{T,S}^T
\]

is an element of \(F_A(H)\).

Thus \(\text{Res}_H^G F_A(G) \subseteq F_A(H)\), for any subgroup \(H\) of \(G\).

Next we show that \(F_A\) is stable under inflation. Let \(N\) be a normal subgroup of \(G\) and let \((T,S)\) be a slice of \(G\) such that \(N\) is contained in \(S\). By Proposition 4.3, we have

\[
\text{Inf}_{G/N}^G \xi_{T/N,S/N}^{G/N} = \sum_{(Y,X) \in [\Pi(G)]} \xi_{(Y,X)} G/N.
\]

Assume that \(\xi_{T/N,S/N}^{G/N} \in F_A(G/N)\). Then by Condition C, for any slice \((Y,X)\) of \(G\) such that \((Y,X) \rightarrow (T,S)\), we obtain \(\xi_{(Y,X)} G/N \in F_A(Y)\). And by Condition B, we have \(\xi_{(Y,X)} G/N \in F_A(G)\). Thus, all these \(\xi_{(Y,X)} G/N\) lie in \(F_A(G)\). Therefore \(\text{Inf}_{G/N}^G F_A(G/N) \subseteq F_A(G)\).

Now we prove that \(F_A\) is stable under deflation. Let \(N\) be a normal subgroup of \(G\) and let \((T,S)\) be a slice of \(G\). Since

\[
\text{Def}_{G/N}^G \xi_{T,S}^G = \frac{|N_T(S)|}{|N_G(T,S)|} \text{Ind}_{T/N}^G \text{Iso}_{T/N} T/N \text{Def}_{T/N}^T \xi_{T,S}^T,
\]

it suffices to show that \(\text{Def}_{T/N}^T F_A(T) \subseteq F_A(T/T \cap N)\). Recall from Proposition 4.4 that

\[
\text{Def}_{T/N}^T F_A(T) = m_{T,S,T \cap N} \xi_{T/N}^{T/N} T \cap N.
\]

Thus, if \(\xi_{T,S}^T \in F_A(T)\), then by Condition D the idempotent \(\xi_{T/N}^{T/N} T \cap N, S(T/N)/\cap N \in F_A(T/T \cap N)\) whenever \(m_{T,S,T \cap N} \neq 0\).

Thus \(\text{Def}_{T/N}^T F_A(T) \subseteq F_A(T/T \cap N)\). This completes the proof. \(\square\)

\textbf{Notation 7.2.} For any ideal \(F\) of \(\mathbb{K} \Xi\), we denote by \(A_F\) the family of slices corresponding to \(F\) i.e. such that
\[ F(G) = \sum_{(T,S) \in \mathcal{A}_F} K_{T,S}^G \]

for any finite group \(G\).

**Corollary 7.3.** Let \(F\) be an ideal of \(K\).  
If \((T, S)\) is an element of \(\mathcal{A}_F\) and \((B, A)\) is any slice, then the slice \((B \times T, A \times S)\) lies in \(\mathcal{A}_F\).

**Proof.** The subgroup \(N = B \times 1\) of \(B \times T\) is normal and,
\[
(B \times T)/(B \times 1) \simeq T \quad \text{and} \quad ((A \times S)(B \times 1))/(B \times 1) \simeq S.
\]
Thus \((B \times T, A \times S) \rightarrow (T, S)\), and by Condition C we have \((B \times T, A \times S) \in \mathcal{A}_F\). \(\square\)

**Corollary 7.4.** Let \(F\) be an ideal of \(K\).  
If \(T\) is minimal such that there exists \(S\) with \((T, S) \in \mathcal{A}_F\), then for any non trivial normal subgroup \(N\) of \(T\), the constant \(m_{T,S,N}\) is equal to zero.

**Proof.** This follows from Condition D because, if \(1 \neq N \trianglelefteq T\) and \(m_{T,S,N} \neq 0\) then \((T/N, SN/N) \in \mathcal{A}_F\). \(\square\)

The above corollary motivates the following definition:

**Definition 7.5.** A slice \((T, S)\) of \(T\) is called a \(T\)-slice (over \(K\)) if for any non trivial normal subgroup \(N\) of \(T\), the constant \(m_{T,S,N}\) is equal to zero.

8. Ideals of the slice Burnside \(p\)-biset functor \(K\Xi_p\)

Throughout this section \(p\) denotes a prime number.

We apply the results of the previous sections to the determination of the full lattice of ideals of the slice Burnside \(p\)-biset functor. Unless otherwise specified, the groups considered in this section are \(p\)-groups.

**Definition 8.1.**

- The \(p\)-biset category \(K\mathcal{C}_p\) is the full subcategory of the biset category \(K\mathcal{C}\) whose objects are finite \(p\)-groups.
- A \(p\)-biset functor on \(\mathcal{C}_p\) with values in \(K\text{-Vect}\) is a \(K\)-linear functor from \(K\mathcal{C}_p\) to \(K\text{-Vect}\).

**Remark 8.2.** By restricting to \(p\)-groups only, the category of bisets functors becomes the category of \(p\)-biset functors. All the methods of biset functors can be easily adapted and therefore our results hold in this context. In particular:
The $p$-biset functors form an abelian category.

The correspondence $G \mapsto \mathbb{K} \Xi(G)$ is a biset functor on the full subcategory of the biset category consisting of finite $p$-groups. It is denoted $\mathbb{K} \Xi_p$ and is called slice Burnside $p$-biset functor.

As before, for any ideal $F$ of $\mathbb{K} \Xi_p$, we denote by $\mathcal{A}_F$ the family of slices corresponding to $F$.

**Proposition 8.3.** The following families of slices

1. $\mathcal{A}_{\mathfrak{A}_1} = \{(T, S) \mid T \neq S\}$,
2. $\mathcal{A}_{\mathfrak{A}_2} = \{(T, S) \mid S \text{ noncyclic}\}$,
3. $\mathcal{A}_{\mathfrak{A}_3} = \{(T, S) \mid T \neq S, S \text{ noncyclic}\}$

satisfy conditions A, B, C and D.

**Proof.** It is clear for A, B and C.

In order to prove D for the families $\mathcal{A}_{\mathfrak{A}_1}$, $\mathcal{A}_{\mathfrak{A}_2}$ and $\mathcal{A}_{\mathfrak{A}_3}$, let $(T, S)$, $(V, U)$ and $(B, A)$ be slices such that $(T, S) \in \mathcal{A}_{\mathfrak{A}_1}$, $(V, U) \in \mathcal{A}_{\mathfrak{A}_2}$ and $(B, A) \in \mathcal{A}_{\mathfrak{A}_3}$.

1. Then, if $N$ is normal subgroup of $T$ such that the constant $m_{T,S,N}^o$ is nonzero then

   \[ m_{T,S,N}^o = m_{T/\Phi(T),S\Phi(T)/\Phi(T),N\Phi(T)/\Phi(T)}^o \neq 0. \]

By Proposition 6.1, this is equivalent to saying that either

\[ T/\Phi(T) = (S\Phi(T))/\Phi(T) \text{ or } \left((S\Phi(T))/\Phi(T)\right) \left((N\Phi(T))/\Phi(T)\right) \neq T/\Phi(T). \]

Since $T \neq S$, we have $S\Phi(T) \neq T$. It follows from the previous argument that $SN\Phi(T) \neq T$, and so $SN/N \neq T/N$. Thus $(T/N, SN/N) \in \mathcal{A}_{\mathfrak{A}_1}$.

2. If $N$ is normal subgroup of $V$ such that $m_{V,U,N}^o \neq 0$ then $m_{U,N/\Phi(U)}^o \neq 0$ where we denote $\bar{U} = U\Phi(T)/\Phi(T)$ and $\bar{N} = N\Phi(T)/\Phi(T)$. Since $U$ is noncyclic, the quotient $\bar{U}$ is noncyclic and by Proposition 6.1, we have $\bar{U}\bar{N}/\bar{N}$ noncyclic.

Thus $UN/N$ is noncyclic and $(V/N, UN/N) \in \mathcal{A}_{\mathfrak{A}_2}$.

3. To prove D for $\mathfrak{J}_3$, it suffices to observe that $\mathcal{A}_{\mathfrak{J}_3} = \mathcal{A}_{\mathfrak{A}_1} \cap \mathcal{A}_{\mathfrak{A}_2}$. □

**Remark 8.4.** Note that by Theorem 7.1, the correspondence sending a finite $p$-group $G$ to the $\mathbb{K}$-vector space $\mathfrak{J}_1(G) = \sum_{(T,S) \in \mathcal{A}_{\mathfrak{A}_1}} \mathbb{K} \xi_T^{G} \xi_S^{G}$ (resp. $\mathfrak{J}_2(G) = \sum_{(T,S) \in \mathcal{A}_{\mathfrak{A}_2}} \mathbb{K} \xi_T^{G} \xi_S^{G}$ or $\mathfrak{J}_3(G) = \sum_{(T,S) \in \mathcal{A}_{\mathfrak{A}_3}} \mathbb{K} \xi_T^{G} \xi_S^{G}$) is an ideal of $\mathbb{K} \Xi_p$.

**Proposition 8.5.** Let $F$ be an ideal of the Green $p$-biset functor $\mathbb{K} \Xi_p$ and $(T, S)$ be a slice. The family $\mathcal{A}_F$ has the following property:

1. If $S$ is cyclic then

   \[ (T, S) \in \mathcal{A}_F \leftrightarrow (T/\Phi(T), S\Phi(T)/\Phi(T)) \in \mathcal{A}_F. \]
2. If $E$ is an elementary abelian $p$-group, and if $S$ and $E \times S\Phi(T)/\Phi(T)$ are noncyclic, then

$$(T, S) \in \mathcal{A}_F \iff \left( E \times (T/\Phi(T)), E \times (S\Phi(T)/\Phi(T)) \right) \in \mathcal{A}_F.$$  

**Proof.** Let us point out that the first assertion holds for arbitrary finite groups.

1. Assume that $S$ is cyclic and the slice $(T, S) \in \mathcal{A}_F$.

   Since $S$ is cyclic, the constant $m_{S,S\cap N}$ is different from zero, for any normal subgroup $N$ of $T$. Now by Proposition 5.5, for any normal subgroup $N$ of $T$, we have

   $$m_{T,S,N} = \frac{|N_T (SN): SN|}{|N_T (S): S|} m_{S,S\cap N} m_{T,S,N}^0.$$  

   In particular, if $N$ is equal to the Frattini subgroup $\Phi(T)$ of $T$, then $N$ is normal and the constant

   $$m_{T,S,N}^0 = \sum_{S \leq V \leq T \atop VN = T} \mu(V,T) = \mu(T,T) = 1.$$  

   Hence $m_{T,S,N}$ is non-zero and by Condition D, the slice $\left( T/\Phi(T), S\Phi(T)/\Phi(T) \right)$ lies in $\mathcal{A}_F$.

   Conversely, assume that the slice $\left( T/\Phi(T), S\Phi(T)/\Phi(T) \right)$ lies in $\mathcal{A}_F$. Then, since there is a surjective group homomorphism $\left( T, S \right) \rightarrow \left( T/\Phi(T), S\Phi(T)/\Phi(T) \right)$, Condition C implies that the slice $(T, S)$ lies in $\mathcal{A}_F$.

2. Let $E$ be an elementary abelian group and $(T, S)$ be a slice such that the groups $S$ and $E \times (S\Phi(T)/\Phi(T))$ are noncyclic. Assume that the slice $(T, S)$ lies in $\mathcal{A}_F$. Then by Corollary 7.3 the slice $\left( E \times T, E \times S \right)$ is an element of $\mathcal{A}_F$. Let $N = \Phi(E \times T) = 1 \times \Phi(T)$. Then by definition

   $$m_{E\times T, E\times S, N}^0 = \sum_{E \times S \leq L \leq E \times T \atop L(1 \times \Phi(T)) = E \times T} \mu(L, E \times T) = \mu(E \times T, E \times T) = 1.$$  

   Since the groups

   $$E \times S \quad \text{and} \quad (E \times S)/(1 \times S \cap \Phi(T)) \simeq E \times (S\Phi(T)/\Phi(T))$$  

   are not cyclic, the constant $m_{E\times S,1\times S\cap \Phi(T)}$ is nonzero. Hence by Proposition 5.5, the constant $m_{E\times T, E\times S, N}$ is nonzero. Thus by Condition D the slice

   $$\left( E \times (T/\Phi(T)), E \times (S\Phi(T)/\Phi(T)) \right)$$  

   lies in $\mathcal{A}_F$. 

Conversely, assume that \( (E \times (T/\Phi(T)), E \times (S\Phi(T)/\Phi(T))) \in \mathcal{A}_F \). Then by Condition C, we have \((E \times T, E \times S) \in \mathcal{A}_F\). Since \((E \times S)/(E \times 1) \simeq S\) is noncyclic and

\[
m_{E \times T, E \times S, E \times 1} = \sum_{E \times S \leq L \leq E \times T \atop L/(E \times 1) = E \times T} \mu(L, E \times T) = \mu(E \times T, E \times T) = 1,
\]

the constant \(m_{E \times T, E \times S, E \times 1}\) is nonzero. Thus, Condition D implies that the slice \((T, S)\) lies in \(\mathcal{A}_F\), and this completes the proof. \(\square\)

This Proposition shows that we can work with elementary abelian \(p\)-groups in order to describe all the ideals of \(\mathbb{K}\Xi_p\).

**Theorem 8.6.** Let \(E\) be an elementary abelian \(p\)-group of rank \(n\), and let \(F\) be a subgroup of \(E\). Then the slice \((E, F)\) is a \(T\)-slice if and only if up to isomorphism, it belongs to the set

\[\{(1, 1); (C_p, 1); (C_p^2, C_p^2); (C_p^3, C_p^2)\}.\]

**Proof.** Let \((E, F)\) be a \(T\)-slice.

- If \(F\) is cyclic and \(F \neq E\).
  Then, for any non-trivial subgroup \(N\) of \(E\), we have \(m_{E, F, N} = 0\). Since \(F\) is cyclic by Proposition 5.5, this is equivalent to saying that \(FN = E\), for any subgroup \(N\) such that \(|N| = p\).
  If the order of \(F\) were \(p^k\) with \(k \geq 1\), we would have a subgroup \(M\) of \(F\) of order \(p\), so that \(M\) would verify \(FM = F = E\). This cannot occur, since by hypothesis \(F \neq E\). Thus \(F\) is the trivial group. Since \(FN = E\), for any subgroup \(N\) of order \(p\), so the only possibility for \(E\) is \(E \simeq C_p\) i.e.
  \[(E, F) \simeq (C_p, 1).\]

- If \(F\) is noncyclic and \(F \neq E\).
  Since \((E, F)\) is a \(T\)-slice,
  \((\ast)\) for any subgroup \(N\) such that \(|N| = p\), either \(F/(F \cap N)\) is cyclic or \(FN = E\).
  - The first step is to consider the possibility that the quotient \(F/(F \cap N)\) is noncyclic.
    Then \(FN = F \times N = E\). Since \(F\) is not cyclic, there exists an integer \(k > 1\) such that \(F \simeq C_p^k\).
    Thus, we may find a subgroup \(M\) of \(F\) of order \(p\), and by \((\ast)\)
    \[FM = E\] or \(F/M\) is cyclic.
  But, since \(FM = F \neq E\), we have that \(F/M\) is cyclic. Thus, there exists \(n \geq 1\) such that
Let

\[ F \simeq (F/M) \times M \simeq C_{p^n} \times C_p. \]

Since \( E \) is elementary abelian, we deduce that

\[ F \simeq C_p \times C_p \quad \text{and} \quad E \simeq C_p \times C_p \times C_p. \]

Hence \( (E,F) \simeq (C^2_p,C^2_p) \).

- Now consider the possibility that the quotient \( F/(F \cap N) = F/N \) is cyclic. Then \( F \cap N \) is non-trivial and \( N \subseteq F \). Therefore, the quotient \( F/(F \cap N) \) is cyclic. Then \( F \) is isomorphic to \( C_{p^n} \times C_p \), for some integer \( n \geq 1 \). Since \( E \) is elementary abelian, we have

\[ F \simeq C_p \times C_p. \]

On the other hand, since \( F \neq E \) and \( E \) is elementary abelian, there exists a subgroup \( M \) of \( E \) of order \( p \) such that \( M \nsubseteq F \). By the hypothesis (*), the quotient \( F/(F \cap M) \) is cyclic or \( FM = E \). But, the quotient \( F/F \cap M \) is noncyclic because \( F \cap M \) is trivial and \( F \) is not cyclic. Thus

\[ E \simeq C_p \times C_p \times C_p. \]

Hence \( (E,F) \simeq (C^2_p,C^2_p) \).

- If \( E = F \neq 1 \) then, for any subgroup \( N \) of \( E \), we have \( m_{E,E,N}^2 = 1 \). It follows that the slice \( (E,E) \) is a \( T \)-slice if and only if \( E \) is a \( B \)-group. It other words \( (E,E) \) is a \( T \)-slice if and only if \( E \simeq C_p \times C_p \) or \( E = 1 \). Thus

\[ (E,E) \simeq (C_p \times C_p) \times C_p. \]

Conversely, we shall show that the slices \((1,1), (C_p,1), (C^2_p,C^2_p)\) and \((C^2_p,C^2_p)\) are \( T \)-slices.

This follows directly from the expression of \( m_{G,S,N} \) as a product of a nonzero constant by \( m_{S,S\cap N,m_{G,S,N}} \), for any slice \((G,S)\) and any normal subgroup \( N \) of \( G \).

By Proposition 6.1, we have

\[ m_{C^1_p,C_p} = m_{C^1_p,C_p}^2 = 0. \]

Since \( C_p \) is the only non trivial normal subgroup of \( C_p \), we claim that the slice \((C_p,1)\) is a \( T \)-slice.

Let \( N \) be a non trivial normal subgroup of \( C^2_p \), then

\[ m_{C^2_p,N} = m_{C^2_p,N \cap C^2_p} = 0 \quad \text{and} \quad m_{C^2_p,C^2_p,N} = 0. \]

Let \( M \) be a non trivial normal subgroup of \( C^3_p \), then \( m_{C^3_p,C^3_p,M} \) has the following decomposition:
$m_{C_3^p,C_2^p,M} = (\star)m_{C_3^p,M \cap C_2^p}m_{C_3^p,C_2^p,M}^o$,

where $(\star)$ is a non zero constant.

If $M \cap C_2^p \neq 1$, then $m_{C_3^p,M \cap C_2^p}$ is equal to zero by Proposition 5.4. It follows that the constant $m_{C_3^p,C_2^p,M}^o$ is equal to zero.

If $M \cap C_2^p = 1$, then $C_3^p = C_2^p \times M$ and by Proposition 6.1, we deduce that

$m_{C_3^p,C_2^p,M}^o = 0$ and $m_{C_3^p,C_2^p,M} = 0$.

This shows that the slice $(C_3^p,C_2^p)$ is a $T$-slice. \square

**Notation 8.7.** Let $G$ be a $p$-group and let $(T,S)$ be a slice of $G$.

For any set $I$, let $(F_\alpha)_{\alpha \in I}$ be a family of ideal of $\mathbb{K} \Xi_p$ such that $F_\alpha(T) \ni \xi_{T,S}^T$. Then

\[ \bigcap_{\alpha \in I} F_\alpha \text{ is an ideal of } \mathbb{K} \Xi_p \text{ and } \bigcap_{\alpha \in I} (F_\alpha(T)) \ni \xi_{T,S}^T. \]

The ideal $E_{T,S}$ of the Green functor $\mathbb{K} \Xi_p$ generated by $\xi_{T,S}^T$ is by definition the intersection of all ideals $F_\alpha$ of $F$ such that $\xi_{T,S}^T \in F(T)$.

**Proposition 8.8.** Let $F$ be an ideal of $\mathbb{K} \Xi$. Then

\[ F = \sum_{(T,S) \text{ slices } \xi_{T,S}^T \in F(T)} E_{T,S}. \]

**Proof.** For any finite group $G$, the ideal $F(G)$ is the sum in $\mathbb{K} \Xi(G)$ of those one dimensional subspaces $\mathbb{K} \xi_{T,S}^G$ for which $\xi_{T,S}^G \in F(G)$. We observe by Theorem 7.1 that

$\xi_{T,S}^G \in F(G) \iff \xi_{T,S}^T \in F(T)$.

If $\xi_{T,S}^G \in F(G)$, then $\mathrm{Res}_{T}^G \xi_{T,S}^G = \displaystyle\sum_{S' \equiv G S \mod T} \xi_{T,S'}^T \in F(T)$ i.e. $E_{T,S} \subseteq F$. Conversely, if $E_{T,S} \subseteq F$, i.e. if $\xi_{T,S}^T \in F(T)$, then $\xi_{T,S}^G \in F(G)$. It follows that

$F(G) = \sum_{(T,S) \in \Pi(G) \atop \xi_{T,S}^T \in F(T)} E_{T,S}(G)$,

for any finite group $G$. Thus

\[ F = \sum_{(T,S) \text{ slices } \xi_{T,S}^T \in F(T)} E_{T,S}. \]
Theorem 8.9. The full lattice of ideals of $K\Xi_p$ has the following structure:

$$
\begin{align*}
K\Xi_p & \quad \downarrow \quad \mathcal{I}_4 \\
\mathcal{I}_1 & \quad \downarrow \quad \mathcal{I}_2 \\
\mathcal{I}_3 & \quad \downarrow \quad 0
\end{align*}
$$

where

$K\Xi_p = \mathbb{E}_{1,1}$,

$\mathcal{I}_1 = \mathbb{E}_{C_p,1}$,

$\mathcal{I}_2 = \mathbb{E}_{C_p^2,C_p^2}$,

$\mathcal{I}_3 = \mathbb{E}_{C_p^3,C_p^2} = \mathcal{I}_1 \cap \mathcal{I}_2$ and $\mathcal{I}_4 = \mathcal{I}_1 + \mathcal{I}_2$.

Moreover, the following equalities hold:

1. $\mathcal{A}_{\mathbb{E}_{C_p,1}} = \{(T,S) \mid T \neq S\}$.
2. $\mathcal{A}_{\mathbb{E}_{C_p^2,C_p^2}} = \{(T,S) \mid S \text{ noncyclic}\}$.
3. $\mathcal{A}_{\mathbb{E}_{C_p^3,C_p^2}} = \{(T,S) \mid T \neq S, \ S \text{ noncyclic}\}$.

Proof. By Proposition 8.8, any ideal $F$ of $K\Xi_p$ is equal to the sum of the ideals $\mathbb{E}_{T,S}$ it contains. Now by Proposition 8.5, for any slice $(T,S)$, there is a slice $(E,F)$, where $E$ is elementary abelian, such that $\mathbb{E}_{T,S} = \mathbb{E}_{E,F}$. Moreover, by Conditions C and D, we can assume that $(E,F)$ is a T-slice, i.e. by Theorem 8.6, one of the slices $(1,1)$, $(C_p,1)$, $(C_p^2,C_p^2)$, $(C_p^3,C_p^2)$.

- The first part of this Theorem follows easily from Conditions A, B, C, D and Corollary 7.3.

Indeed, for any slice $(V,U)$, we have the projection $(V,U) \to (V/V,UV/V) = (1,1)$. Since $(1,1) \in \mathcal{A}_{\mathbb{E}_{1,1}}$, Condition C implies that $(V,U)$ lies in $\mathcal{A}_{\mathbb{E}_{1,1}}$. Thus $\mathbb{E}_{1,1} = K\Xi_p$.

Using Corollary 7.3 and the decomposition $(C_p^3,C_p^2) = (C_p^2,C_p^2) \times (C_p,1)$ it is easy to see that

$\mathcal{I}_3 \subseteq \mathcal{I}_2$ and $\mathcal{I}_3 \subseteq \mathcal{I}_1$

as ideals.
We prove the equalities in the second part of this Theorem.
This is straightforward to see that \( \mathcal{A}_{E_{C_{p,1}}} \subseteq \mathcal{A}_{\bar{J}_1} \), \( \mathcal{A}_{E_{C_{p,1}}} \subseteq \mathcal{A}_{\bar{J}_2} \) and \( \mathcal{A}_{E_{C_{p,1}}} \subseteq \mathcal{A}_{\bar{J}_3} \).
Let us show the reverse inclusions. Set \( \tilde{T} = T/\Phi(T) \) and \( \tilde{S} = S\Phi(T)/\Phi(T) \).
1. Let \((T, S)\) be a slice such that \( T \neq S \) i.e. \((T, S) \in \mathcal{A}_{\bar{J}_1}\).
If \( S \) is cyclic then there exists a normal subgroup \( N \) of the group \( T \) of index \( p \) such that \( S \subseteq N \). Moreover,
\[
T/N \simeq C_p \text{ and } SN/N = 1.
\]
Since the slice \((T/N, SN/N)\) belongs to \( \mathcal{A}_{E_{C_{p,1}}} \), it follows from Condition \( C \) that the slice \((T, S) \in \mathcal{A}_{E_{C_{p,1}}} \).
If \( S \) is noncyclic then by Proposition 8.5, we have \((E \times \tilde{T}), E \times \tilde{S}) \in \mathcal{A}_{\bar{J}_1}\) whenever \( E \times \tilde{S} \) is noncyclic. Set \( E = C_p \times C_p \), then the slice
\[
(E \times \tilde{T}, E \times \tilde{S}) = (C_p, 1) \times (C_p \times \tilde{T}, E \times \tilde{S})
\]
lies in \( \mathcal{A}_{E_{C_{p,1}}} \). Thus the same argument show that \((T, S) \in \mathcal{A}_{E_{C_{p,1}}} \).
2. Assume that \((T, S)\) is a slice where the group \( S \) is noncyclic then for any elementary abelian group such that \( |E| \geq p^2 \), we have \((T, S) \in \mathcal{A}_{\bar{J}_2}\) if and only if \((E \times \tilde{T}, E \times \tilde{S}) \in \mathcal{A}_{\bar{J}_2}\).
In particular, we can set \( E = C_p^2 \) and Corollary 7.3 implies that
\[
(E \times \tilde{T}, E \times \tilde{S}) \in \mathcal{A}_{E_{C_{p,1}}} \times E_{C_{p,1}}.
\]
Hence \((T, S) \in \mathcal{A}_{E_{C_{p,1}}} \times E_{C_{p,1}}\).
3. Assume that \( S \) is noncyclic, \( T \neq S \) and \((T, S) \in \mathcal{A}_{\bar{J}_3}\) then \((C_p^2 \times \tilde{T}, C_p^2 \times \tilde{S}) \in \mathcal{A}_{\bar{J}_3}\).
There are integers \( n > 1 \) and \( r < n \) such that
\[
\tilde{T} = C_p^m \text{ and } \tilde{S} = C_p^r.
\]
Thus
\[
(C_p^2 \times \tilde{T}, C_p^2 \times \tilde{S}) = (C_p^3, C_p^2) \times (C_p^{m-1}, C_p^r) \in \mathcal{A}_{E_{C_{p,1}}}.
\]
by Corollary 7.3. \( \square \)

Remark 8.10. There is a unital Green biset functor homomorphism \( i : \mathbb{K}B_p \to \mathbb{K}\Xi_p \) defined at a \( p \)-group \( G \) by sending the (class of a) finite \( G \)-set \( X \) to the (class of the) identity morphism of \( X \).

It was shown moreover in [4] that \( \mathbb{K}B_p \) has a unique proper non zero subfunctor, equal to the kernel \( K \) of the linearization morphism \( \mathbb{K}B_p \to \mathbb{K}R_{Q,p} \). One can check accordingly that \( i(\mathbb{K}B_p) \cap \mathcal{I}_1 = 0 \), while \( i(\mathbb{K}B_p) \cap \mathcal{I}_2 = i(\mathbb{K}B_p) \cap \mathcal{I}_4 = i(K) \).
9. The counterexample

Let $G$ be a finite $p$-group.

Consider $\mathcal{I}_3(G)$, the ideal of $K \Xi_p(G)$ generated by elements of the form $\xi_{T,S}^G$ where $(T, S)$ is any slice of $G$ such that $S \neq T$ and $S$ is noncyclic. It follows from Theorem 8.9 that the $K \Xi_p$-submodule $\mathcal{I}_3$ of $K \Xi_p$ is simple and for any finite $p$-group $G$, the dimension of $\mathcal{I}_3(G)$ is equal to the number of conjugacy classes of slices $(T, S)$ of $G$, such that $S \neq T$ and $S$ noncyclic.

Assume that $\mathcal{I}_3(G) \neq 0$. Then there exists a slice $(T, S)$ of $G$ such that $S \neq T$ and $S$ noncyclic.

In this case, it follows easily that $S$ has at least $p^2$ elements. Since $S$ is a proper subgroup of $T$, it follows that

\[(*) \quad |T| \geq p^3 \text{ and } |G| \geq p^3.\]

Now we assume that $|G| = p^3$, then $T = G$ and $S \simeq C_p \times C_p$.

If $p \neq 2$, then $G$ is isomorphic to one of the groups $C_p^3$, $C_{p^2} \times C_p$, $M_{p^3} = C_{p^2} \rtimes C_p$ or $X_{p^3} = Syl_p(\text{GL}(3, \mathbb{F}_p))$.

The groups $M_{p^3} = C_{p^2} \rtimes C_p$ and $X_{p^3} = Syl_p(\text{GL}(3, \mathbb{F}_p))$ are the two extraspecial $p$-groups of order $p^3$ (see [6]).

The lattices of subgroups of $M_{p^3}$ and $X_{p^3}$ for $p = 3$ are respectively:

\[M_{p^3} \text{ of exponent } p^2\]

\[X_{p^3} \text{ of exponent } p\]

An horizontal dotted link between two vertices means that the corresponding subgroups are conjugate. The vertex marked with a • is the centre of the group.
The lattice of subgroups of the $C_{p^2} \times C_p$ for $p = 3$ is:

![Diagram of subgroups](image)

We have therefore established that the slices $(C_{p^3}, C_p \times C_p)$, $(C_{p^2} \times C_p, C_p \times C_p)$, $(C_{p^2} \times C_p, C_p \times C_p)$ and $(X_{p^3}, C_p \times C_p)$ belong to $A_{53}$.

By (*) if $K$ is a finite $p$-group such that $|K| < p^3$ then $I_3(K) = 0$. Thus,

$$\{C_{p^3}, C_{p^2} \times C_p, M_{p^3}, X_{p^3}\} \subseteq \text{Min}(I_3)$$

and

| $G$               | $C_{p^3}$ | $C_{p^2} \times C_p$ | $M_{p^3}$ | $X_{p^3}$ |
|-------------------|-----------|----------------------|-----------|-----------|
| $\dim I_3(G)$     | $p^2 + p + 1$ | $1$                  | $1$       | $p + 1$   |

If $p = 2$, then $G$ is isomorphic to one of the following groups: $C_4 \times C_2$, $C_3^2$, $D_8$. The lattices of subgroups of $C_4 \times C_2$, $C_2 \times C_2 \times C_2$ and $D_8$ are respectively:

![Diagram of subgroups](image)
The dimension of $J_3(G)$ is therefore the following:

|       | $C_4^3$ | $C_4 \times C_2$ | $D_8$ | $\dim J_3(G)$ |
|-------|--------|------------------|-------|----------------|
| $G$   | 7      | 1                | 2     | 12             |

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