A REMARK ON THE GEOMETRIC JACQUET FUNCTOR

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Abstract. We give an action of \(N\) on the geometric Jacquet functor defined by Emerton-Nadler-Vilonen.

Let \(G_\mathbb{R}\) be a reductive linear algebraic group over \(\mathbb{R}\), \(G_\mathbb{R} = K_\mathbb{R}A_\mathbb{R}N_\mathbb{R}\) an Iwasawa decomposition, and \(M_\mathbb{R}\) the centralizer of \(A_\mathbb{R}\) in \(K_\mathbb{R}\). Then \(P_\mathbb{R} = M_\mathbb{R}A_\mathbb{R}N_\mathbb{R}\) is a Langlands decomposition of a minimal parabolic subgroup. We use lower-case fraktur letters to denote the corresponding Lie algebras and omit the subscript “\(\mathbb{R}\)” to denote complexifications. Fix a Cartan involution \(\theta\) such that \(K = \{g \in G \mid \theta(g) = g\}\). For a \((g,K)\)-module \(V\), the Jacquet module \(J(V)\) of \(V\) is defined by the space of \(n\)-finite vectors in \(\lim_{\leftarrow k} V/\theta(n)^k V\) [Cas80].

For simplicity, assume that \(V\) has the same infinitesimal character as the trivial representation. Denote the category of Harish-Chandra modules with the same infinitesimal characters as the trivial representation by \(\text{HC}_\rho\). Let \(X\) be the flag variety of \(G\), \(\text{Perv}_K(X)\) the category of \(K\)-equivariant perverse sheaves on \(X\). By the Beilinson-Bernstein correspondence and the Riemann-Hilbert correspondence, we have the localization functor \(\Delta: \text{HC}_\rho \to \text{Perv}_K(X)\). Emerton-Nadler-Vilonen gave a geometric description of \(J(V)\) by the following way [ENV04]. Fix a cocharacter \(\nu: \mathbb{G}_m \to A\) which is positive on the roots in \(\mathfrak{n}\). Define \(a: \mathbb{G}_m \times X \to X\) by \(a(t,x) = \nu(t)x\). Consider the following diagram,

\[
X \simeq \{0\} \times X \to \mathbb{A}^1 \times X \leftarrow \mathbb{G}_m \times X \xrightarrow{a} X.
\]

Let \(R\psi\) be the nearby cycle functor with respect to \(\mathbb{A}^1 \times X \to \mathbb{A}^1\). Then the geometric Jacquet functor \(\Psi\) is defined by

\[
\Psi(F) = R\psi(a^*(\mathcal{F})).
\]

Theorem 1 (Emerton-Nadler-Vilonen [ENV04, Theorem 1.1]). We have \(\Delta \circ J \simeq \Psi \circ \Delta: \text{HC}_\rho \to \text{Perv}_K(X)\).

It is easy to see that \(J(V)\) is a \((\mathfrak{g},N)\)-module for a \((\mathfrak{g},K)\)-module \(V\). Hence \(\Psi(F)\) is \(N\)-equivariant for \(F \in \text{Perv}_K(X)\). (See also [ENV04, Remark 1.3].)

In this paper, we give the action of \(N\) on \(\Psi(F)\) in a geometric way. Roughly speaking, this action is given by the “limit” of the action of \(K\).

We use the following lemma.

Lemma 2. Let \(X\) be a scheme of finite type over \(\mathbb{A}^1\), \(\mathcal{X}^0\) (resp. \(X_0\)) the inverse image of \(\mathbb{G}_m\) (resp. \(\{0\}\)), and \(\mathcal{G}\) a smooth group scheme over \(\mathbb{A}^1\). If \(\mathcal{F}^0\) is a \(\mathcal{G}^0\)-equivariant perverse sheaf on \(\mathcal{X}^0\), then \(R\psi(\mathcal{F}^0)\) is \(\mathcal{G}_0\)-equivariant.

Proof. Define \(m: \mathcal{G} \times_{\mathbb{A}^1} X \to X\) by \(m(g,x) = gx\). Then \(m\) is a smooth morphism. Let \(m^0: \mathcal{G}^0 \times_{\mathbb{G}_m} X^0 \to X^0\) and \(m_0: \mathcal{G}_0 \times X_0 \to X_0\) be its restrictions. Since \(\mathcal{F}^0\) is \(\mathcal{G}^0\)-equivariant, we have an isomorphism \((m^0)^*(\mathcal{F}^0) \simeq \mathcal{F}^0\). Hence we have

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Let \( K^0 = \{(t, k) \in \mathbb{G}_m \times G \mid k \in \text{Ad}(\nu(t)^{-1})(K)\} \subset \mathbb{A}^1 \times G. \)

Let \( \mathcal{K} \) be the closure of \( K^0 \) in \( \mathbb{A}^1 \times G. \) It is a closed sub-group scheme of \( \mathbb{A}^1 \times G. \)

Then \( \mathcal{K} \) is flat over \( \mathbb{A}^1. \) Since each fiber of \( \mathcal{K} \rightarrow \mathbb{A}^1 \) is a group scheme of finite type over \( \mathbb{C} \), it is reducible [Oor66]. Hence it is smooth. Therefore, \( \mathcal{K} \) is smooth over \( \mathbb{A}^1. \)

Let \( \Sigma \) be the restricted root system of \((g, a), \Sigma^+ \) the positive system corresponding to \( n \) the restricted root space for \( \alpha \in \Sigma. \) Then \( \mathfrak{k} \) is spanned by \( \mathfrak{m} \) and \( \{X + \theta(X) \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma^+\}. \) Since

\[
\text{Ad}(\nu(t)^{-1})(X + \theta(X)) = t^{-\langle \nu, \alpha \rangle}(X + t^{2\langle \nu, \alpha \rangle}\theta(X))
\]

for \( X \in \mathfrak{g}_\alpha, \) the Lie algebra of \( \text{Ad}(\nu(t)^{-1})(K) \) is spanned by \( \mathfrak{m} \) and \( \{X + t^{2\langle \nu, \alpha \rangle}\theta(X) \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma^+\}. \)

Hence the neutral component of \( K_0 \) is \( M^o N \) where \( M^o \) is the neutral component of \( M. \) Since \( MK^o = K \) and \( \text{Ad}(\nu(t)^{-1})(M) = M, \) we have \( K_0 = MMM^o N = MN. \)

Define \( \tilde{a} : K^0 \times_{\mathbb{G}_m} (G_m \times X) \rightarrow K \times X \) (resp. \( \tilde{m} : K^0 \times_{\mathbb{G}_m} (G_m \times X) \rightarrow \mathbb{G}_m \times X, \)

\( m : K \times X \rightarrow X \) by \( \tilde{a}((t, k), (t, x)) = (\text{Ad}(\nu(t))k, \nu(t)x) \) (resp. \( \tilde{m}((t, k), (t, x)) = (t, kx), m(k, x) = kx) \).

Then we have the following commutative diagrams

\[
\begin{array}{c}
\mathcal{K}^0 \times_{\mathbb{G}_m} (G_m \times X) \xrightarrow{\tilde{m}} \mathbb{G}_m \times X \quad \quad \quad \mathcal{K}^0 \times_{\mathbb{G}_m} (G_m \times X) \xrightarrow{\text{pr}_2} \mathbb{G}_m \times X \\
\downarrow \tilde{a} \quad \quad \quad \downarrow \tilde{a} \\
K \times X \xrightarrow{m} X, \quad K \times X \xrightarrow{\text{pr}_2} X.
\end{array}
\]

Let \( \mathcal{F} \in \text{Perv}_K(X). \) Then \( m^* \mathcal{F} \simeq \text{pr}_2^* \mathcal{F}. \) Hence we get \( \tilde{a}^* m^* \mathcal{F} \simeq \tilde{a}^* \text{pr}_2^* \mathcal{F}. \) By the above diagrams, \( \tilde{m}^* a^* \mathcal{F} \simeq \text{pr}_2^* \tilde{a}^* \mathcal{F}. \) Therefore \( a^*(\mathcal{F}) \) is \( K^0 \)-equivariant. By Lemma 2, \( \Psi(\mathcal{F}) = R\psi(a^*(\mathcal{F})) \) is \( K_0 = MN \)-equivariant.

**Remark 3.** Let \( \Gamma \) be the quasi-inverse functor of \( \Delta. \) Then \( N \) acts on \( \Gamma(\Psi(\mathcal{F})). \) Moreover, this becomes a \((g, N)\)-module [Kas89, 9.1.1], namely, the infinitesimal action of \( N \) coincides with the action of \( n \subset g. \) We also have that \( J(\Gamma(\mathcal{F})) \) is a \((g, N)\)-module. Hence both \( N \)-actions have the same infinitesimal actions. Since the action of \( N \) is determined by its infinitesimal action, the \( N \)-action we defined above coincides with the \( N \)-action on \( J(\Gamma(\mathcal{F})). \)

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