BOUNDARY-VALUE PROBLEMS FOR WEAKLY SINGULAR INTEGRAL EQUATIONS

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Abstract. We consider a perturbed linear boundary-value problem for a weakly singular integral equation. Assume that the generating boundary-value problem is unsolvable for arbitrary inhomogeneities. Efficient conditions for the coefficients guaranteeing the appearance of the family of solutions of the perturbed linear boundary-value problem in the form of Laurent series in powers of a small parameter $\varepsilon$ with singularity at the point $\varepsilon = 0$ are established.

1. Introduction. Boundary-value problems for integral equations are mathematical models of various processes in physics, chemistry, biology, economics, etc. In particular, numerous phenomena are described by using weakly singular integral equations (see, e.g., [1, 9]), which are also known as equations with weak singularity or equations with weakly polar kernel (see [13, 20, 25]). The differential properties of the solutions of weakly singular integral equations and the approximate methods for their solution were studied in the works by I. G. Graham [14], E. Vainikko and G. Vainikko [26], C. Huang, T. Tang and Z. Zhang [17], J. Shen, C. Sheng and Z. Wang [24], and other researchers [21, 23, 10, 12]. The conditions for the existence of solutions to the Fredholm boundary-value problems of nonzero index for these equations and the structure of these solutions were studied in [3]. In the present paper, we continue our investigations originated in [3] and obtain conditions for the bifurcation of solutions of the perturbed linear boundary-value problem for a weakly singular integral equation under the condition that the generating boundary-value problem does not have solutions.

In the space $L^2[a, b]$, we consider a perturbed linear boundary-value problem for a weakly singular integral equation

$$x(t) - \int_{a}^{b} K(t, s)x(s)ds = f(t) + \varepsilon \int_{a}^{b} K(t, s)x(s)ds,$$

$$lx(\cdot) = \alpha + \varepsilon Jx(\cdot).$$

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We establish conditions for the bifurcation of solutions \( x \in L^2[a, b] \) of the boundary-value problem (1), (2) and determine the structure of these solutions under the condition that the generating boundary-value problem

\[
x(t) - \int_a^b K(t, s)x(s)ds = f(t),
\]

\[
lx(\cdot) = \alpha
\]

is unsolvable.

Here, \( K(t, s) = \frac{H(t, s)}{|t - s|^{\gamma}} \) and \( \bar{K}(t, s) = \frac{\bar{H}(t, s)}{|t - s|^{\beta}} \), where \( H(t, s), \bar{H}(t, s) \) are functions bounded in the domain \([a, b] \times [a, b], 0 < \gamma < 1, 0 < \beta < 1, f \in L^2[a, b], \)

\( l = \text{col} (l_1, l_2, \ldots, l_p) : L^2[a, b] \rightarrow \mathbb{R}^p, \)

\( J = \text{col} (J_1, J_2, \ldots, J_p) : L^2[a, b] \rightarrow \mathbb{R}^p \)

are bounded linear functionals, \( l_\nu, J_\nu : L^2[a, b] \rightarrow \mathbb{R}, \nu = 1, p, \alpha = \text{col} (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p, \)

and \( \varepsilon \ll 1 \) is a small parameter.

The perturbed boundary-value problems of the form (1), (2) with kernels \( K(t, s) \) and \( \bar{K}(t, s) \) of different types were studied in the works of numerous authors. In particular, the boundary-value problems for integral and integro-differential equations with degenerate kernels were investigated in [28, 11] by using the theory of generalized inverse operators (see [2, 7]). At the same time, the boundary-value problems for integral equations with square summable kernels were studied in [5].

In the present paper, unlike the cited publications, we consider a boundary-value problem (1), (2) with unbounded kernels \( K(t, s) \) and \( \bar{K}(t, s) \). For this problem, the issue of bifurcation of its solutions has not been studied earlier.

2. Reduction of the integral equation (1) to the Fredholm equation. In [3], we show that the boundary-value problem for the integral equation with unbounded kernel (3), (4) can be reduced to a boundary-value problem for the Fredholm integral equation and, in view of [2, 7, 5, 6], establish the conditions of solvability and determine the general form of solutions to the generating problem (3), (4). By using the results obtained in [3], we show that the study of the problem of appearance of solutions of the boundary-value problem (1), (2) reduces to the corresponding task for the perturbed boundary-value problem for the Fredholm integral equation.

To substantiate the indicated transition, we first present some facts from the theory of weakly singular integral operators. It is known (see [13, p. 34], [20, p. 59], [25, p. 61]) that if two integral operators \( H_1(t, s) \) and \( H_2(t, s) \) with weakly singular kernels \( \frac{H_1(t, s)}{|t - s|^{\gamma_1}} \) and \( \frac{H_2(t, s)}{|t - s|^{\gamma_2}} \), respectively, are given, then the product \( H_1H_2 \) of these operators has a kernel of the form

\[
F(t, s) = \int_a^b \frac{H_1(t, \xi)H_2(\xi, s)}{|t - \xi|^{\gamma_1}|\xi - s|^{\gamma_2}} d\xi,
\]

which has the same structure and its index is a number that does not exceed \( \gamma_1 + \gamma_2 - 1 \). If the condition

\[
\gamma_1 + \gamma_2 - 1 < \frac{1}{2}
\]
is satisfied, then the kernel \( F(t, s) \) is square summable. Moreover, if the condition 
\[
\gamma_1 + \gamma_2 - 1 < 0
\]
is satisfied, then the kernel \( F(t, s) \) is bounded.

Consider integral operators 
\[
(Kw)(t) = \int_a^b K(t, s)w(s)ds \quad \text{and} \quad (Kw)(t) = \int_a^b K(t, s)w(s)ds
\]
with unbounded kernels \( K(t, s) = \frac{H(t, s)}{|t-s|^{\gamma}} \) and \( \overline{K}(t, s) = \frac{H(t, s)}{|t-s|^{\beta}} \) and iterated kernels \( K_n(t, s), \overline{K}_n(t, s), n \in \mathbb{N} \), given by the recurrence relations 
\[
K_{n+1}(t, s) = \int_a^b K(t, \xi)K_n(\xi, s)d\xi, \quad \overline{K}_{n+1}(t, s) = \int_a^b \overline{K}(t, \xi)\overline{K}_n(\xi, s)d\xi,
\]
and 
\[
K_1(t, s) = K(t, s), \quad \overline{K}_1(t, s) = \overline{K}(t, s).
\]

According to (5), the iterated kernels \( K_n(t, s) \) and \( \overline{K}_n(t, s) \) have the same structure as weakly singular kernels \( K(t, s) \) and \( \overline{K}(t, s) \) but the numbers \( \gamma \) and \( \beta \) are replaced with the numbers \( 1 - n(1 - \gamma) \) and \( 1 - n(1 - \beta) \), respectively, which are negative for sufficiently large \( n \). Therefore (see [13, p. 34], [20, p. 61], [25, p. 63]), for all \( n \) such that conditions 
\[
n > \frac{1}{2(1 - \gamma)}, \quad n > \frac{1}{2(1 - \beta)}
\]
are satisfied, the kernels \( K_n(t, s), \overline{K}_n(t, s) \) are square summable. Moreover, if the conditions 
\[
n > \frac{1}{1 - \gamma}, \quad n > \frac{1}{1 - \beta}
\]
are satisfied, then the kernels \( K_n(t, s) \) and \( \overline{K}_n(t, s) \) are bounded.

We now introduce a kernel 
\[
R(t, s, \varepsilon) = K(t, s) + \varepsilon\overline{K}(t, s)
\]
and rewrite equation (1) in the form 
\[
x(t) = f(t) + \int_a^b R(t, s, \varepsilon)x(s)ds.
\]

According to [13, p. 33], [20, p. 62], [25, p. 64], equation (11), and hence, equation (1), can be reduced to an equation with integral operators with square summable kernels.

Consider the iterated kernels \( R_n(t, s, \varepsilon) \), \( n \in \mathbb{N} \), given by the following recurrence relations:
\[
R_{n+1}(t, s, \varepsilon) = \int_a^b R(t, \xi, \varepsilon)R_n(\xi, s, \varepsilon)d\xi, \quad R_1(t, s, \varepsilon) = R(t, s, \varepsilon).
\]

Thus, according to (12), (10), (7), the iterated kernels \( R_2(t, s, \varepsilon) \) and \( R_3(t, s, \varepsilon) \) have the form 
\[
R_2(t, s, \varepsilon) = R_0^2(t, s) + \varepsilon R_1^2(t, s) + \varepsilon^2 R_2^2(t, s),
\]
\[
R_3(t, s, \varepsilon) = R_0^3(t, s) + \varepsilon R_1^3(t, s) + \varepsilon^2 R_2^3(t, s) + \varepsilon^3 R_3^3(t, s),
\]
where

\[ R_0^1(t, s) = K_2(t, s), \quad R_1^1(t, s) = M_{11}(t, s) + L_{11}(t, s), \quad R_2^1(t, s) = \overline{K}_2(t, s), \]

\[ R_0^2(t, s) = K_3(t, s), \quad R_1^2(t, s) = M_{21}(t, s) + \int_a^b K(t, \xi)L_{11}(\xi, s)d\xi + L_{12}(t, s), \]

\[ R_2^2(t, s) = M_{12}(t, s) + \int_a^b \overline{K}(t, \xi)M_{11}(\xi, s)d\xi + L_{21}(t, s), \quad R_3^2(t, s) = \overline{K}_3(t, s), \]

\[ M_{ij}(t, s) = \int_a^b K_i(t, \xi)\overline{K}_j(\xi, s)d\xi, \quad L_{ij}(t, s) = \int_a^b \overline{K}_i(t, \xi)K_j(\xi, s)d\xi. \]

Applying the iterative procedure (12) \( n - 1 \) times, we conclude that the kernel \( R_n(t, s, \varepsilon) \) has the form

\[ R_n(t, s, \varepsilon) = \sum_{k=0}^{n} \varepsilon^k R_n^k(t, s), \tag{13} \]

where \( R_n^k(t, s), k = 0, n, \) are the sums of \( C_n^k \) kernels of all possible products of \( n - k \) integral operators \( K \) and \( k \) integral operators \( \overline{K} \) (6)

\[ R_0^n(t, s) = K_n(t, s), \quad \ldots, R_n^n(t, s) = \overline{K}_n(t, s). \]

By using induction, we show that if conditions (8) are satisfied, then the kernel \( R_n(t, s, \varepsilon) \) is square summable and if conditions (9) are satisfied, then the kernel \( R_n(t, s, \varepsilon) \) is bounded.

First, we consider the kernel

\[ R_2(t, s, \varepsilon) = \sum_{k=0}^{2} \varepsilon^k R_2^k(t, s). \]

The kernels \( R_2^k(t, s) \) have the form (5) and their indices do not exceed \( 2\gamma - 1, \gamma + \beta - 1, \) and \( 2\beta - 1, \) respectively. If the conditions

\[ 2\gamma - 1 < \frac{1}{2}, \quad \gamma + \beta - 1 < \frac{1}{2}, \quad 2\beta - 1 < \frac{1}{2} \] \tag{14}

are satisfied, then the kernel \( R_2(t, s, \varepsilon) \) is square summable. At the same time, if the conditions

\[ 2\gamma - 1 < 0, \quad \gamma + \beta - 1 < 0, \quad 2\beta - 1 < 0 \] \tag{15}

are satisfied, then the kernel \( R_2(t, s, \varepsilon) \) is bounded. However, adding the first and third inequalities (14) and dividing both sides of the resulting inequality by two, we conclude that the second condition in (14) is unnecessary. Similarly, the second condition in (15) follows from the first and third conditions in (15). Hence, if the conditions

\[ 2\gamma - 1 < \frac{1}{2}, \quad 2\beta - 1 < \frac{1}{2} \]

are satisfied, then the kernel \( R_2(t, s, \varepsilon) \) is square summable. At the same time, if the conditions

\[ 2\gamma - 1 < 0, \quad 2\beta - 1 < 0 \]

are satisfied, then the kernel \( R_2(t, s, \varepsilon) \) is bounded.
We consider a kernel
\[ R_{n-1}(t, s, \varepsilon) = \sum_{k=0}^{n-1} \varepsilon^k R^k_{n-1}(t, s). \]  

Assume that the indices of the kernels \( R^k_{n-1}(t, s) \) do not exceed \((n-1)\gamma - (n-2), (n-2)\gamma + \beta - (n-2), \ldots, (n-1-k)\gamma + k\beta - (n-2), \ldots, \gamma + (n-2)\beta - (n-2)\) and \((n-1)\beta - (n-2)\), respectively. If the conditions
\[ (n-1)\gamma - (n-2) < \frac{1}{2}, \quad (n-1)\beta - (n-2) < \frac{1}{2} \]
are satisfied, then the kernel \( R_{n-1}(t, s, \varepsilon) \) is square summable. At the same time, if the conditions
\[ (n-1)\gamma - (n-2) < 0, \quad (n-1)\beta - (n-2) < 0 \]
are satisfied, then the kernel \( R_{n-1}(t, s, \varepsilon) \) is bounded.

Consider the kernel \( R_n(t, s, \varepsilon) \). According to (12), (10), (16), the kernel \( R_n(t, s, \varepsilon) \) has the form
\[ R_n(t, s, \varepsilon) = \int_a^b R(t, \xi, \varepsilon) R_{n-1}(\xi, s, \varepsilon) d\xi \]
\[ = \int_a^b K(t, \xi) R_{n-1}(\xi, s, \varepsilon) d\xi + \int_a^b K(t, \xi) R_{n-1}(\xi, s, \varepsilon) d\xi \]
\[ = \sum_{k=0}^{n-1} \varepsilon^k \int_a^b K(t, \xi) R^k_{n-1}(\xi, s) d\xi + \sum_{k=0}^{n-1} \varepsilon^{k+1} \int_a^b K(t, \xi) R^{k-1}_{n-1}(\xi, s) d\xi \]
\[ = \sum_{k=0}^{n-1} \varepsilon^k R^k_n(t, s), \]
where
\[ R^0_n(t, s) = \int_a^b K(t, \xi) R^0_{n-1}(\xi, s) d\xi, \quad R^n_n(t, s) = \int_a^b K(t, \xi) R^{n-1}_{n-1}(\xi, s) d\xi, \]
\[ R^k_n(t, s) = \int_a^b K(t, \xi) R^k_{n-1}(\xi, s) d\xi + \int_a^b K(t, \xi) R^{k-1}_{n-1}(\xi, s) d\xi, \quad k = 1, n-1. \]

The kernels \( R^k_n(t, s), k = 0, n \) have the form (5) and their indices do not exceed \( n\gamma -(n-1), (n-1)\gamma + \beta - (n-1), \ldots, (n-1-k)\gamma + k\beta - (n-1), \ldots, \gamma + (n-1)\beta - (n-1) \) and \( n\beta - (n-1) \) respectively. If the conditions
\[ n\gamma - (n-1) < \frac{1}{2}, \quad (n-1)\gamma + \beta - (n-1) < \frac{1}{2}, \]
\[ \ldots, \quad (n-1-k)\gamma + k\beta - (n-1) < \frac{1}{2}, \quad \ldots, \]
\[ \gamma + (n-1)\beta - (n-1) < \frac{1}{2}, \quad n\beta - (n-1) < \frac{1}{2} \]

are satisfied, then the kernel \( R^k_n(t, s, \varepsilon) \) is bounded. If the conditions
\[ n\gamma - (n-1) < \frac{1}{2}, \quad (n-1)\gamma + \beta - (n-1) < \frac{1}{2}, \]
\[ \ldots, \quad (n-1-k)\gamma + k\beta - (n-1) < \frac{1}{2}, \quad \ldots, \]
\[ \gamma + (n-1)\beta - (n-1) < \frac{1}{2}, \quad n\beta - (n-1) < \frac{1}{2} \]

are satisfied, then the kernel \( R^k_n(t, s, \varepsilon) \) is square summable.
are satisfied, then the kernel $R_n(t,s,\varepsilon)$ is square summable. At the same time, if the conditions
\[
\begin{align*}
n\gamma - (n - 1) &< 0, \quad (n - 1)\gamma + \beta - (n - 1) < 0, \\
\ldots, \quad (n - k)\gamma + k\beta - (n - 1) &< 0, \quad \ldots, \\
\gamma + (n - 1)\beta - (n - 1) &< 0, \quad n\beta - (n - 1) < 0
\end{align*}
\]
are satisfied, then the kernel $R_n(t,s,\varepsilon)$ is bounded.

However, the conditions $(n-k)\gamma + k\beta - (n-1) < \frac{1}{2}$ and $(n-k)\gamma + k\beta - (n-1) < 0$, $k = 1, n-1$, are unnecessary. In fact, multiplying both sides of the first inequality (17) by $\frac{n}{n}$ and both sides of the last inequality (17) by $\frac{k}{n}$ and adding the resulting inequalities, we see that all conditions (17), except the first and last, conditions are unnecessary. Similarly, all conditions (18) are superfluous, except the first and last conditions.

Hence, if the conditions
\[
n\gamma - (n - 1) < \frac{1}{2}, \quad n\beta - (n - 1) < \frac{1}{2}
\]
are satisfied, then the kernel $R_n(t,s,\varepsilon)$ is square summable. At the same time, if the conditions
\[
n\gamma - (n - 1) < 0, \quad n\beta - (n - 1) < 0
\]
are satisfied, then the kernel $R_n(t,s,\varepsilon)$ is bounded.

Therefore, if, for sufficiently large $n$, conditions (8), i.e., the conditions
\[
n > \frac{1}{2(1 - \gamma)}, \quad n > \frac{1}{2(1 - \beta)}
\]
are satisfied, then the kernel $R_n(t,s,\varepsilon)$ is square summable. At the same time, if conditions (9), i.e., the conditions
\[
n > \frac{1}{1 - \gamma}, \quad n > \frac{1}{1 - \beta}
\]
average over the interval $[a,b]$, we get
\[
\int_a^b R(t,s,\varepsilon)x(s)ds = \int_a^b R(t,s,\varepsilon)f(s)ds + \int_a^b R_2(t,s,\varepsilon)x(s)ds.
\]
Continuing this process, we obtain
\[
\int_a^b R_2(t,s,\varepsilon)x(s)ds = \int_a^b R_2(t,s,\varepsilon)f(s)ds + \int_a^b R_3(t,s,\varepsilon)x(s)ds,
\]
Adding all equations obtained as a result to equation (11) and taking into account (13), we find that \( x(t) \) is a solution of the equation

\[
x(t) = f_n(t) + \sum_{k=0}^{n-1} \varepsilon^k \int_a^b R_n^k(t, s)x(s)ds,
\]

\[
f_n(t) = f(t) + \sum_{k=1}^{b} \int_a^b R_n^k(t, s)f(s)ds + \sum_{k=1}^{n-1} \varepsilon^k \sum_{m=k}^{b} \int_a^b R_n^m(t, s)f(s)ds.
\]

Therefore, according to [13, 20, 25], in view of conditions (8), after finitely many steps, we arrive at equation (19) with square summable kernel \( R_n(t, s, \varepsilon) \). Hence, any solution of equation (1) is a solution of equation (19). The converse statement is, in general, not true. However, it is possible to choose \( n \) such that conditions (9) and, hence, conditions (8) are satisfied and any solution of equation (19) is, at the same time, a solution of equation (1), i.e., equations (1) and (19) are equivalent (see [20, p. 63]). In what follows, we assume that \( n \) is chosen as indicated above. For fixed \( n \), we can pass from the analysis of the perturbed boundary-value problem for the weakly singular integral equation (1), (2) to the investigation of the perturbed boundary-value problem for the Fredholm integral equation (19), (2).

**Remark 1.** If, for some fixed \( n \) for which conditions (8) are satisfied, equation (19) possesses a unique solution, then equations (1) and (19) are equivalent. In particular, equations (1) and (19) are equivalent if equation (1), instead of the Fredholm operators (6), contains the Volterra operators (see [13, p. 35]).

3. Relationship between the boundary-value problem (19), (2) and a countably measurable system of linear algebraic equations with perturbations. We apply the approach described in [5, 6] to the study of the boundary-value problem (19), (2) and show that it can be reduced to a countable system of linear algebraic equations with perturbations. To do this, we use the results of the theory of linear integral equations of the second kind proposed by Hilbert (see [16]). Let \( \{ \varphi_i(t) \}_{i=1}^{\infty} \) be a complete orthonormal system of functions in \( L_2[a, b] \). We introduce the notation

\[
x_i = \int_a^b x(t)\varphi_i(t)dt, \quad F_i = f_i + \sum_{k=1}^{n-1} \varepsilon^k f_k^i, \quad A_{ij} = a_{ij} + \sum_{k=1}^{n} \varepsilon^k a_{ij}^k,
\]

\[
a_{ij} = \int_a^b K_n(t, s)\varphi_i(t)\varphi_j(s)ds, \quad a_{ij}^k = \int_a^b \int_a^b R_n^k(t, s)\varphi_i(t)\varphi_j(s)ds dt, \quad k = \overline{1, n},
\]

\[
f_i = \int_a^b f(t)\varphi_i(t)dt + \sum_{k=1}^{n-1} \int_a^b R_n^k(t, s)f(s)\varphi_i(t)ds dt,
\]

\[
f_i^k = \sum_{m=k}^{n} \int_a^b R_n^m(t, s)f(s)\varphi_i(t)ds dt, \quad k = \overline{1, n-1}.
\]
Finding the solution of the integral equation (19) with square summable kernel, according to [16], is equivalent to finding the solution of a countable system of linear algebraic equations

\[ x_i = F_i + \sum_{j=1}^{\infty} A_{ij} x_j, \quad i = 1, \ldots, \infty \]

or, in the expanded form,

\[ x_i - \sum_{j=1}^{\infty} a_{ij} x_j = f_i + \varepsilon^k f_i^k + \sum_{k=1}^{n} \sum_{j=1}^{\infty} \varepsilon^k a_{ij}^k x_j, \quad i = 1, \ldots, \infty. \]  

(20)

System (20) is considered in the Hilbert space \( \ell_2 \), i.e., for solutions satisfy the condition

\[ \sum_{j=1}^{\infty} |x_j|^2 < +\infty. \]  

(21)

If system (20) possesses at least one solution, which satisfies condition (21), then, according to the Riesz–Fischer theorem, one can find an element \( x \in L_2[a, b] \) such that the quantities \( x_i, i = 1, \ldots, \infty \) are the Fourier coefficients of this element. Thus, the following representation is true:

\[ x(t) = \sum_{j=1}^{\infty} x_j \varphi_j(t). \]  

(22)

As in [16, p. 266], we conclude that the element \( x(t) \) given by relations (22) is a solution of the integral equation (19). Substituting (22) in the boundary condition (2), we obtain \( p \) equations

\[ \sum_{j=1}^{\infty} l_\nu \varphi_j(\cdot) x_j = \alpha_\nu + \varepsilon \sum_{j=1}^{\infty} J_\nu \varphi_j(\cdot) x_j, \quad \nu = 1, \ldots, p. \]  

(23)

Thus, we have shown that the study of the boundary-value problem (19), (2) is equivalent to the study of a countable system of linear algebraic equations with perturbations (20), (23).

We rewrite system (20), (23) in the vector form as follows:

\[ \begin{bmatrix} \Lambda & W \end{bmatrix} z = \begin{bmatrix} g \\ \alpha \end{bmatrix} + n^{-1} \sum_{k=1}^{n} \varepsilon^k \begin{bmatrix} g_k \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} \Lambda_1 & W_1 \end{bmatrix} z + \sum_{k=2}^{n} \varepsilon^k \begin{bmatrix} \Lambda_k & 0 \end{bmatrix} z, \]  

(24)

where the vectors \( z, g, g_k, k = 1, n-1 \) and the matrices \( W, W_1, \Lambda, \Lambda_k, k = 1, n \) have the form

\[ z = \text{col} \left( x_1, x_2, \ldots, x_i, \ldots \right), \quad g = \text{col} \left( f_1, f_2, \ldots, f_i, \ldots \right), \]

\[ g_k = \text{col} \left( f_1^k, f_2^k, \ldots, f_i^k, \ldots \right), \quad W = l\Phi(\cdot), \quad W_1 = J\Phi(\cdot), \]

\[ \begin{bmatrix} a_{11} - a_{12} \ldots - a_{1i} \ldots \\ -a_{21} 1 - a_{22} \ldots - a_{2i} \ldots \\ \ldots \ldots \ldots \ldots \ldots \\ -a_{i1} - a_{i2} \ldots 1 - a_{ii} \ldots \end{bmatrix}, \quad \Lambda_k = \begin{bmatrix} a_{11}^k & a_{12}^k \ldots a_{1i}^k \ldots \\ a_{21}^k & a_{22}^k \ldots a_{2i}^k \ldots \\ \ldots \ldots \ldots \ldots \ldots \\ a_{i1}^k & a_{i2}^k \ldots a_{ii}^k \ldots \end{bmatrix}, \]

\[ \Phi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_i(t), \ldots). \]  

(25)
For the sake of convenience, we rewrite equation (24) as follows:

\[ Uz = q + \sum_{k=1}^{n-1} \varepsilon^k q_k + \sum_{k=1}^{n} \varepsilon^k U_k z, \quad (26) \]

where

\[ U = \begin{bmatrix} \Lambda \\ W \end{bmatrix}, \quad U_1 = \begin{bmatrix} \Lambda_1 \\ W_1 \end{bmatrix}, \quad U_k = \begin{bmatrix} \Lambda_k \\ 0 \end{bmatrix}, \quad k = 2, n, \]

\[ q = \begin{bmatrix} g \\ \alpha \end{bmatrix}, \quad q_k = \begin{bmatrix} g_k \\ 0 \end{bmatrix}, \quad k = 1, n-1. \]

The generating equation for the operator equation (26) has the form

\[ Uz = q. \quad (27) \]

The operator \( \Lambda : \ell_2 \to \ell_2 \) appearing on the left-hand side of the operator equation (27) has the form \( \Lambda = I - A \), where \( I : \ell_2 \to \ell_2 \) is the identity operator and \( A : \ell_2 \to \ell_2 \) is a compact operator. Hence, according to S. Krein’s classification, the operator \( \Lambda : \ell_2 \to \ell_2 \) is a Fredholm operator of index zero (dim ker \( \Lambda \) = dim ker \( \Lambda^* \) < \( \infty \)) and the operator \( U : \ell_2 \to \ell_2 \times \mathbb{R}^p \) is a Fredholm operator of nonzero index (dim ker \( U < \infty \), dim ker \( U^* < \infty \)).

The following statement is true for equation (27) (see [7]):

**Theorem 3.1.** The homogeneous equation (27) \( (q = 0) \) possesses a \( d_2 \)-parameter family of solutions \( z \in \ell_2 \),

\[ z = P_{\Lambda^*} P_Q d_2 c_{d_2} \forall c_{d_2} \in \mathbb{R}^{d_2}. \]

The inhomogeneous equation (27) is solvable if and only if the following \( r + d_1 \) linearly independent conditions are satisfied:

\[ P_{\Lambda^*} g = 0, \quad P_{Q^{*}_{d_1}(\alpha - WA^+ g)} = 0 \]

and the equation possesses a \( d_2 \)-parameter family of solutions \( z \in \ell_2 \) of the form

\[ z = P_{\Lambda^*} P_Q d_2 c_{d_2} + P_{\Lambda^*} Q^*(\alpha - WA^+ g) + \Lambda^+ g \forall c_{d_2} \in \mathbb{R}^{d_2}. \]

Here, \( Q = WP_{\Lambda^*} \) is a \( (p \times r) \)-matrix, \( P_{\Lambda^*} (P_{\Lambda^*}) \) is a matrix formed by a complete system of \( r \) linearly independent columns (rows) of the matrix projector \( P_{\Lambda^*} \), where \( P_{\Lambda^*} \) \( (P_{\Lambda^*}) \) is the projector onto the kernel (cokernel) of the matrix \( \Lambda \), and \( P_Q d_2 \) \( P_Q d_2 \) is a matrix formed by the complete system of \( d_2 \) \( (d_1) \) linearly independent columns (rows) of the matrix projector \( P_Q \), where \( P_Q \) \( (P_Q) \) is the projector onto the kernel (cokernel) of the matrix \( Q \) and \( \Lambda^+ \) \( (Q^+) \) is the pseudoinverse Moore–Penrose matrix for the matrix \( \Lambda \) \( (Q^+) \).

4. **Construction of the family of solutions of the perturbed boundary-value problem (1), (2).** We now determine the conditions required for the bifurcation of solutions of the perturbed inhomogeneous boundary-value problem (1), (2) and study the structure of these solutions under the conditions that the solution of the homogeneous generating boundary-value problem (3), (4) \( (f(t) = 0, \alpha = 0) \) is not unique, i.e. (see [3]),

\[ P_{\Lambda^*} P_Q d_2 \neq 0, \]

and that the inhomogeneous generating boundary-value problem (3), (4) is unsolvable. To do this, we first determine the conditions required for the bifurcation of the solutions of equations (26). Since, by assumption, the boundary-value problem (3), (4) is unsolvable, we conclude that equation (27) also does not have solutions.
The following question arises: Is it possible to make equation (26) solvable with the help of linear perturbations $U_k$?

It is known (see [18]) that small perturbations preserve the Fredholm property of the operator, i.e., the operator $\left( U - \sum_{k=1}^{n} \varepsilon^k U_k \right)$ is a Fredholm operator with nonzero index. This enables one to investigate equation (26) by the methods of the theory of perturbed operator boundary-value problems with Fredholm linear part (see, e.g., [4, 2, 7, 22, 8]) obtained as a generalization of the classical methods of the perturbation theory of periodic boundary-value problems in the theory of oscillations (see [19, 15]).

The analysis of the appearance of solutions of the equation (26) is closely connected with the

\[
B_0 = \begin{bmatrix}
P_{A_{r}} A_1 P_{\Lambda} & P_{Q_{d_2}} \\
P_{Q_{d_2}} (W_1 - W \Lambda^+ A_1) P_{\Lambda} & P_{Q_{d_2}}
\end{bmatrix},
\]

constructed by using the coefficients of the equation (26).

By the Vishik—Lyusternik method (see [27]), we find efficient conditions for the coefficients guaranteeing the appearance of a family of solutions of the perturbed linear boundary-value problem (26) in the form of a Laurent series in powers of the small parameter $\varepsilon$ with singularity at the point $\varepsilon = 0$.

We introduce an \(((r + d_1) \times (r + d_1))\)-matrix $P_{B_0^*}$, which is a projector onto the cokernel of the matrix $B_0$ and a matrix

\[
G = \begin{bmatrix}
-P_{A_{r}} A_1 & 0 \\
P_{Q_{d_2}} W \Lambda^+ & -P_{Q_{d_2}}
\end{bmatrix},
\]

formed by $(r + d_1)$ rows and infinitely many columns. Moreover, as the matrix $B_0$, it is completely determined by the coefficients of equation (26). Therefore, the following statement is true:

**Theorem 4.1.** Assume that equation (27), which is a generating equation for (26), is unsolvable. If conditions

\[
P_{A_{r}} A_1 P_{\Lambda} P_{Q_{d_2}} \neq 0, \quad P_{B_0^*} G = 0
\]

are satisfied, then equation (26) has a $d_2$-parameter family of solutions in the form of series with singularity at the point $\varepsilon = 0$ convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon_*)$.

**Proof of Theorem 4.1.** We seek the solutions of equation (26) in the form of a segment of power series in the small parameter $\varepsilon$:

\[
z = \sum_{k=-1}^{\infty} \varepsilon^k z_k.
\]

We substitute series (31) in equation (26) and equate the coefficients of the same powers of $\varepsilon$. For $\varepsilon^{-1}$, we get the following homogeneous equation for the determination of $z_{-1}$:

\[
U z_{-1} = 0.
\]

According to Theorem 3.1, the homogeneous equation (32) always possesses a $d_2$-parameter family of solutions

\[
z_{-1} = P_{A_{r}} P_{Q_{d_2}} c_{-1}, \quad \forall c_{-1} \in \mathbb{R}^{d_2},
\]
where the column vector $c_{-1}$ is determined from the condition of solvability of the equation for $z_0$.

For $\varepsilon^0$, we arrive at the following inhomogeneous equation for the determination of $z_0$:

$$Uz_0 = q + U_1z_{-1}.$$  \hspace{1cm} (33)

By Theorem 3.1, equation (33) is solvable if and only if the following $r + d_1$ linearly independent conditions are satisfied:

$$P_{\Lambda z}(g + \Lambda_1z_{-1}) = 0, \quad P_{Q_{d_1}}(\alpha + W_1z_{-1} - W\Lambda^t(g + \Lambda_1z_{-1})) = 0.$$  

Substituting the expression for $z_{-1}$ in the indicated condition of solvability, we obtain the following algebraic system for $c_{-1} \in \mathbb{R}^{d_2}$:

$$B_0c_{-1} = b_{-1},$$  \hspace{1cm} (34)

$$b_{-1} = \left[\begin{array}{c} -P_{\Lambda z}g \\ P_{Q_{d_1}}(W\Lambda^t g - \alpha) \end{array}\right] = G \left[\begin{array}{c} g \\ \alpha \end{array}\right],$$

where the matrices $B_0$ and $G$ have the form (28), (29). Note that the $(r + d_1)$-vector $b_{-1}$ is completely determined by the coefficients of equation (26).

In order that system (34) be solvable, it is necessary and sufficient that $r + d_1$ linearly independent conditions

$$P_{B_0^*}b_{-1} = 0,$$  \hspace{1cm} (35)

be satisfied. If the $r + d_1$ conditions $P_{B_0^*}G = 0$ (30) are satisfied, then conditions (35) are true and system (34) possesses a $d_2$-parameter family of solutions $c_{-1}$

$$c_{-1} = P_{B_0}{\hat{c}} + B_0^*b_{-1} \forall \hat{c} \in \mathbb{R}^{d_2},$$

where the $(d_2 \times d_2)$-matrix $P_{B_0}$ is a projector onto the kernel of the matrix $B_0$.

Therefore, the coefficient $z_{-1}$ of series (31) takes the form

$$z_{-1} = X_{-1}\hat{c} + \bar{c}_{-1} \forall \hat{c} \in \mathbb{R}^{d_2},$$  \hspace{1cm} (36)

where

$$X_{-1} = P_{\Lambda z}P_{Q_{d_2}}P_{B_0}, \quad \bar{c}_{-1} = P_{\Lambda z}P_{Q_{d_2}}B_0^*b_{-1}.$$  

Hence, substituting (36) in (33), we get

$$Uz_0 = U_1X_{-1}\hat{c} + q + U_1\bar{c}_{-1}.$$  \hspace{1cm} (37)

Equation (37) possesses a $d_2$-parameter family of solutions

$$z_0 = P_{\Lambda z}P_{Q_{d_2}}c_0 + \bar{z}_0,$$

where $c_0 \in \mathbb{R}^{d_2}$ is a column vector determined from the condition of solvability of the equation for $z_1$ and $\bar{z}_0$ is a partial solution of the inhomogeneous equation (37)

$$\bar{z}_0 = Y_{-1}\hat{c} + h_{-1} \forall \hat{c} \in \mathbb{R}^{d_2},$$

where

$$Y_{-1} = U^+U_1X_{-1}, \quad h_{-1} = U^+(q + U_1\bar{c}_{-1}).$$

For $\varepsilon^1$, we obtain the following linear inhomogeneous equation:

$$Uz_1 = q_1 + U_1z_0 + U_2\bar{z}_{-1}.$$  \hspace{1cm} (38)

Equation (38) is solvable if and only if the conditions of solvability

$$P_{\Lambda z}(g_1 + \Lambda_1z_0 + \Lambda_2\bar{z}_{-1}) = 0, \quad P_{Q_{d_1}}(W_1z_0 - W\Lambda^t(g_1 + \Lambda_1z_0 + \Lambda_2\bar{z}_{-1})) = 0$$

are true.
We substitute the expression for $z_0$ in these conditions and arrive at an algebraic system similar to (34)

$$B_0c_0 = b_0,$$

which is solvable if and only if the following $r + d_1$ linearly independent conditions are satisfied:

$$P_{B_0}b_0 = 0.$$

Here, $b_0$ is an $(r + d_1)$-vector of the form

$$b_0 = G \begin{bmatrix} g_1 + \Lambda_1 \tilde{z}_0 + \Lambda_2 \tilde{z}_{-1} \\ W_1 \tilde{z}_0 \end{bmatrix}.$$

Then the solution of system (39) is as follows:

$$c_0 = P_{B_0} \hat{c} + B_0^* b_0 = D_0 \hat{c} + \xi_0,$$

where $\hat{c} \in \mathbb{R}^{d_2}$ is an arbitrary vector and the $(d_2 \times d_2)$-matrix $D_0$ and $d_2$-vector $\xi_0$ take the form

$$D_0 = P_{B_0} + B_0^* G \begin{bmatrix} \Lambda_1 Y_1 + \Lambda_2 X_{-1} \\ W_1 Y_{-1} \end{bmatrix}$$

and

$$\xi_0 = B_0^* G \begin{bmatrix} g_1 + \Lambda_1 h_{-1} + \Lambda_2 \tilde{c}_{-1} \\ W_1 h_{-1} \end{bmatrix},$$

respectively.

Therefore, $z_0$ has the form

$$z_0 = X_0 \hat{c} + \tilde{c}_0 \quad \forall \hat{c} \in \mathbb{R}^{d_2},$$

where

$$X_0 = P_{\Lambda}, P_{Q_{d_2}} D_0 + Y_{-1}, \quad \tilde{c}_0 = P_{\Lambda}, P_{Q_{d_2}} \tilde{z}_0 + h_{-1}.$$

Substituting (36), (40) in (38), we get

$$Uz_1 = (U_1 X_0 + U_2 X_{-1}) \hat{c} + q_1 + U_1 \tilde{c}_0 + U_2 \tilde{c}_{-1}.$$  (41)

If conditions (30) are satisfied, then equation (41) has a $d_2$-parameter family of solutions

$$z_1 = P_{\Lambda}, P_{Q_{d_2}} c_1 + \tilde{z}_1,$$

where the column vector $c_1 \in \mathbb{R}^{d_2}$ is determined from the condition of solvability of the equation for $z_2$ and $\tilde{z}_1$ is a partial solution of the inhomogeneous equation (41) of the form

$$\tilde{z}_1 = Y_0 \hat{c} + h_0 \quad \forall \hat{c} \in \mathbb{R}^{d_2},$$

where

$$Y_0 = U^+ (U_1 X_0 + U_2 X_{-1}), \quad h_0 = U^+ (q_1 + U_1 \tilde{c}_0 + U_2 \tilde{c}_{-1}).$$

We continue this process further and, for $\varepsilon^{n-1}$, arrive at the equation

$$Uz_{n-1} = q_{n-1} + \sum_{i=1}^{n} U_i z_{n-1-i}.$$  (42)

This equation is solvable if and only if the $r + d_1$ conditions

$$P_{\Lambda^*} \begin{bmatrix} g_{n-1} + \sum_{i=1}^{n} \Lambda_i z_{n-1-i} \end{bmatrix} = 0,$$

$$P_{Q_{d_1}} \left( W_1 z_{n-2} - W \Lambda^+ \left( g_{n-1} + \sum_{i=1}^{n} \Lambda_i z_{n-1-i} \right) \right) = 0$$

are satisfied.
We obtain the following system for \( c_{n-2} \in \mathbb{R}^{d_2} \):
\[
B_0 c_{n-2} = b_{n-2}.
\] (43)
This system is solvable if and only if the following \( r + d_1 \) linearly independent conditions are satisfied:
\[
P_{B_0^+} b_{n-2} = 0.
\]
Here, as earlier, \( B_0 \) is regarded as representation (28) and
\[
b_{n-2} = G \left[ g_{n-1} + \sum_{i=2}^{n} \Lambda_i \tilde{z}_{n-1-i} \right]
\]
is an \((r + d_1)\)-vector.
Thus, the solution of system (43) has the form:
\[
c_{n-2} = P_{B_0} \hat{c} + B_0^+ b_{n-2} = D_{n-2} \hat{c} + \pi_{n-2},
\]
where \( \hat{c} \in \mathbb{R}^{d_2} \) is an arbitrary vector and, moreover, the \((d_2 \times d_2)\)-matrix \( D_{n-2} \) and the \(d_2\)-vector \( \pi_{n-2} \) have the form
\[
D_{n-2} = P_{B_0} + B_0^+ G \left[ \Lambda_1 Y_{n-3} + \sum_{i=2}^{n} \Lambda_i X_{n-1-i} \right],
\]
\[
\pi_{n-2} = B_0^+ G \left[ g_{n-1} + \sum_{i=2}^{n} \Lambda_i \tilde{c}_{n-1-i} \right].
\]
The vector \( z_{n-2} \) takes the form
\[
z_{n-2} = X_{n-2} \hat{c} + \pi_{n-2} \quad \forall \hat{c} \in \mathbb{R}^{d_2},
\]
where
\[
X_{n-2} = P_{\Lambda_1} P_{Q_{d_2}} D_{n-2} + Y_{n-3}, \quad \pi_{n-2} = P_{\Lambda_1} P_{Q_{d_2}} \pi_{n-2} + h_{n-3}.
\]
Substituting \( z_i, i = -1, n - 2 \) in (42), we get
\[
U z_{n-1} = \sum_{i=1}^{n} U_i X_{n-1-i} \hat{c} + q_{n-1} + \sum_{i=1}^{n} U_i \tilde{c}_{n-1-i}.
\] (44)
If conditions (30) are satisfied, then equation (44) has a \( d_2 \)-parameter family of solutions
\[
z_{n-1} = P_{\Lambda_1} P_{Q_{d_2}} c_{n-1} + \tilde{z}_{n-1},
\]
where the column vector \( c_{n-1} \in \mathbb{R}^{d_2} \) is determined from the condition of solvability of the equation for \( z_n \) and \( \tilde{z}_{n-1} \) is a partial solution of the inhomogeneous equation (44) of the form
\[
\tilde{z}_{n-1} = Y_{n-2} \hat{c} + h_{n-2} \quad \forall \hat{c} \in \mathbb{R}^{d_2},
\]
where
\[
Y_{n-2} = U^+ \sum_{i=1}^{n} U_i X_{n-1-i}, \quad h_{n-2} = U^+ \left( q_{n-1} + \sum_{i=1}^{n} U_i \tilde{c}_{n-1-i} \right).
\]
For \( \varepsilon^n \), we get the following homogeneous equation:
\[
U z_n = \sum_{i=1}^{n} U_i z_{n-i}.
\] (45)
The conditions of solvability of this equation are as follows:
\[ \sum_{i=1}^{n} P_{A_i} A_i z_{n-i} = 0, \quad P_{Q_i} \left( W_1 z_{n-1} - \sum_{i=1}^{n} W A_i A_i z_{n-i} \right) = 0. \]
In view of (28), we arrive at the following algebraic system for \( c_{n-1} \in \mathbb{R}^{d_2} \):
\[ B_0 c_{n-1} = b_{n-1}, \quad (46) \]
where \( b_{n-1} \) is an \((r + d_1)\)-vector of the form
\[ b_{n-1} = G \left[ \Lambda_1 \tilde{z}_{n-1} + \sum_{i=2}^{n} \Lambda_i z_{n-i} \right]. \]
System (46) is solvable if and only if the following \( r + d_1 \) conditions are satisfied:
\[ P_{B_0} b_{n-1} = 0. \]
This system possesses the following \( d_2 \)-parameter family of solutions:
\[ c_{n-1} = P_{B_0} \hat{c} + B_0^+ b_{n-1} = D_{n-1} \hat{c} + \varpi_{n-1}, \quad \forall \hat{c} \in \mathbb{R}^{d_2}, \quad (47) \]
Here, the \((d_2 \times d_2)\)-matrix \( D_{n-1} \) and the \( d_2 \)-vector \( \varpi_{n-1} \) have the form
\[ D_{n-1} = P_{B_0} + B_0^+ G \left( \Lambda_1 Y_{n-2} + \sum_{i=2}^{n} \Lambda_i X_{n-i} \right), \]
\[ \varpi_{n-1} = B_0^+ G \left[ \Lambda_1 h_{n-2} + \sum_{i=2}^{n} \Lambda_i \tilde{c}_{n-i} \right]. \]
According to (47), we get the following representation for \( z_{n-1} \):
\[ z_{n-1} = X_{n-1} \hat{c} + \tilde{c}_{n-1}, \quad \forall \hat{c} \in \mathbb{R}^{d_2}, \]
where
\[ X_{n-1} = P_{A_i} P_{Q_{d_2}} D_{n-1} + Y_{n-2}, \quad \tilde{c}_{n-1} = P_{A_i} P_{Q_{d_2}} \varpi_{n-1} + h_{n-2}. \]
Substituting \( z_i, \ i = 0, n - 1 \) in (45), we get
\[ U z_n = \sum_{i=1}^{n} U_i X_{n-i} \hat{c} + \sum_{i=1}^{n} U_i \tilde{c}_{n-i}, \quad (48) \]
If conditions (30) are satisfied, then equation (48) possesses the following \( d_2 \)-parameter family of solutions:
\[ z_n = P_{A_i} P_{Q_{d_2}} c_n + \tilde{z}_n, \]
where the column vector \( c_n \in \mathbb{R}^{d_2} \) is determined from the condition of solvability of the equation for \( z_{n+1} \) and \( \tilde{z}_n \) is a partial solution of the inhomogeneous equation (48) of the form
\[ \tilde{z}_n = Y_{n-1} \hat{c} + h_{n-1}, \quad \forall \hat{c} \in \mathbb{R}^{d_2}, \]
where
\[ Y_{n-1} = U^+ \sum_{i=1}^{n} U_i X_{n-i}, \quad h_{n-1} = U^+ \sum_{i=1}^{n} U_i \tilde{c}_{n-i}. \]
By induction, conditions (30) are, at the same time, the conditions of solvability of the corresponding equation obtained in the \( k \)th step \((k > n)\) of the iterative process from which we determine the elements \( z_k \) of series (31). Substituting the
expression for \( z_k, k \geq -1 \), in series (31), we get the following \( d_2 \)-parameter family of solutions:

\[
z = \sum_{k=-1}^{\infty} \varepsilon^k (X_k \hat{c} + \tilde{c}_k) \quad \forall \hat{c} \in \mathbb{R}^{d_2},
\]

(49)

where \( X_k, k = -1, \infty \), are matrices with infinitely many rows and \( d_2 \) columns determined by the following recurrence relations:

\[
X_k = P_\Lambda, P_{Q_{d_2}} D_k + Y_{k-1}, \quad k = -1, \infty,
\]

\[
D_{-1} = P_{B_0}, \quad D_k = P_{B_0} + B_0^+ G \left[ \Lambda_1 Y_{k-1} + \sum_{i=2}^{k+2} \Lambda_i X_{k+1-i} \right], \quad k = 0, n-2,
\]

\[
D_k = P_{B_0} + B_0^+ G \left[ \Lambda_1 Y_{k-1} + \sum_{i=2}^{n} \Lambda_i X_{k+1-i} \right], \quad k = n-1, \infty,
\]

\[
Y_{-2} = 0, \quad Y_k = U^+ \sum_{i=1}^{k+2} U_i X_{k+1-i}, \quad k = -1, n-2,
\]

\[
Y_k = U^+ \sum_{i=1}^{n} U_i X_{k+1-i}, \quad k = n-1, \infty,
\]

and the infinite-dimensional vectors \( \tilde{c}_k, k = -1, \infty \) are given by the formulas:

\[
\tilde{c}_k = P_\Lambda, P_{Q_{d_2}} z_k + h_{k-1}, \quad k = -1, \infty,
\]

\[
z_{-1} = B_0^+ G \left[ \begin{array}{c} g \\ \alpha \end{array} \right],
\]

\[
z_k = B_0^+ G \left[ g_{k+1} + \Lambda_1 h_{k-1} + \sum_{i=2}^{k+2} \Lambda_i \tilde{c}_{k+1-i} \right], \quad k = 0, n-2,
\]

\[
z_k = B_0^+ G \left[ \Lambda_1 h_{k-1} + \sum_{i=2}^{n} \Lambda_i \tilde{c}_{k+1-i} \right], \quad k = n-1, \infty,
\]

\[
h_{-2} = 0, \quad h_{-1} = U^+ (q + U_1 \hat{c}_{-1}),
\]

\[
h_k = U^+ \left( q_{k+1} + \sum_{i=1}^{k+2} U_i \tilde{c}_{k+1-i} \right), \quad k = 0, n-2,
\]

\[
h_k = U^+ \sum_{i=1}^{n} U_i \tilde{c}_{k+1-i}, \quad k = n-1, \infty.
\]

By analogy with [7], we can prove that series (49) is convergent for sufficiently small fixed \( \varepsilon \in (0, \varepsilon_\ast) \) with singularity at the point \( \varepsilon = 0 \). \( \square \)

**Remark 2.** Conditions \( PB_0^+ G = 0 \) are sufficient for the existence of a \( d_2 \)-parameter family of solutions of equation (26). If conditions \( PB_0^+ G = 0 \) are not satisfied, then the family of solutions of equation (26) in the form of series (49) does not exist. However, this family of solutions may exist as a part of power series of the form (31) in the small parameter \( \varepsilon \) starting from \( k = -2, -3, \ldots \) (see [7]).
The results obtained for the perturbed equations (26) enable us to establish the conditions for the existence of a $d_2$-parameter family of solutions of the original perturbed boundary-value problem (1), (2). Indeed, if the boundary-value problem (1), (2) possesses at least one solution, then, according to the Riesz–Fischer theorem, one can find an element $x \in L^2[a,b]$ such that the quantities $x_i, i = 1, \infty$, determined from system (20), (23) are the Fourier coefficients of this element. Thus, the following representation is true:

$$x(t) = \Phi(t)z = \sum_{i=1}^{\infty} x_i \varphi_i(t),$$

(50)

where $\Phi(t)$ has the form (25).

As in [16, p. 266], we conclude that the element $x(t)$ given by relations (50) is the required $d_2$-parameter family of solutions of the original boundary-value problem (1), (2). The conditions of bifurcation and the general solution of the boundary-value problem (1), (2) are as follows:

Theorem 4.2. Suppose that the generating boundary-value problem (3), (4) is unsolvable. If conditions (30)

$$P_\Lambda P_{Q_{d_2}} \neq 0, \quad P_{B_0} G = 0,$$

are satisfied, then the boundary-value problem (1), (2) has a $d_2$-parameter family of solutions in the form of series (50) with singularity at the point $\varepsilon = 0$ convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon^*)$.

Remark 3. If the condition $P_\Lambda P_{Q_{d_2}} \neq 0$ is not satisfied, i.e., the homogeneous generating boundary-value problem (3), (4) possesses a unique solution, then the perturbed boundary-value problem (1), (2) is also uniquely solvable.

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