Polarization modulation in Young’s interference experiment

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Abstract. Polarization properties at the observation screen in Young’s interference experiment are examined. Several recent results on the modulation of Stokes parameters, including the minimum number of modulated parameters, are reviewed. The theory is then applied to find out the relation between the Stokes parameters at the pinholes and the Pancharatnam–Berry phase at the screen.

1. Introduction
Interference is one of the most fundamental properties of light. It gives rise to many intriguing effects related, e.g., to propagation, diffraction, and coherence properties of electromagnetic radiation. Owing to the predominant role of the scalar description of light propagation, the research of interference effects has been mainly restricted to the study of the intensity-interference, such as the direct visibility of the fringes in Young’s interference experiment.

It has been known for quite long time that even though two coherent but orthogonally polarized beams do not give rise to intensity-interference fringes, they lead to a spatially varying polarization pattern. Such a phenomenon can be exploited, for example, in the fabrication of holographic gratings in polarization-sensitive materials with orthogonally polarized fields [1–4]. This example, among the others, clearly shows that the interference phenomenon with non-uniformly polarized light fields is much more rich in nature compared to the scalar approach.

The subject of interference and polarization has been under rather intensive research also within the optical coherence theory [5–14]. For example, several candidates for the degree of coherence, each of which emphasize different aspects in the random electromagnetic field, have been put forward [15–18]. In this article, we shall get acquainted with some of the recent results on the subject, in particular those related to the polarization modulation in Young’s interference experiment.

2. Polarization contrast parameters in Young’s interference experiment
Consider classical Young’s interference experiment: A random electromagnetic field, described in the space–frequency domain by a random vector $E(r, \omega)$ is input to the aperture $A$ that has two pinholes at positions $Q_1$ and $Q_2$ in it. Depending on the correlation properties of the field at the pinholes, we may observe various interference effects at the screen $B$. The correlation at pinholes is described by the cross-spectral density matrix [8]

$$W(Q_1, Q_2, \omega) = \langle E^*(Q_1, \omega)E^T(Q_2, \omega) \rangle,$$ (1)
where the asterisk and $T$ denote the complex conjugation and the transpose operation, respectively. Moreover, the angle brackets denote the ensemble average over the statistical ensemble in the space–frequency domain.

In the analysis of polarization properties at screen $\mathcal{B}$, the Stokes representation of the field is particularly useful. In order to make use of it, we recall from Refs. [7, 19] the so-called two-point Stokes parameters, denoted by $S_j(Q_1, Q_2, \omega), j = 0 \ldots 3$, that are connected to the elements of the cross-spectral density matrix by:

$$
S_0(Q_1, Q_2, \omega) = W_{xx}(Q_1, Q_2, \omega) + W_{yy}(Q_1, Q_2, \omega),
$$

$$
S_1(Q_1, Q_2, \omega) = W_{xx}(Q_1, Q_2, \omega) - W_{yy}(Q_1, Q_2, \omega),
$$

$$
S_2(Q_1, Q_2, \omega) = W_{xy}(Q_1, Q_2, \omega) + W_{yx}(Q_1, Q_2, \omega),
$$

$$
S_3(Q_1, Q_2, \omega) = i[W_{xy}(Q_1, Q_2, \omega) - W_{yx}(Q_1, Q_2, \omega)],
$$

(2)

as well as their normalized versions, that may be called the polarization contrast parameters for the reason that shall become shortly obvious,

$$
\eta_j(Q_1, Q_2, \omega) = \frac{S_j(Q_1, Q_2, \omega)}{[S_0(Q_1, \omega)S_0(Q_2, \omega)]^{1/2}}, \quad j = 0 \ldots 3,
$$

(3)

where $S_j(r, \omega) = S_j(r, r, \omega), j = 0 \ldots 3$, are the standard (one-point) Stokes parameters.

The partial polarization properties of the field at the screen can be obtained from the generalized interference law [20, 21]

$$
S_j(r, \omega) = S_j^{(1)}(r, \omega) + S_j^{(2)}(r, \omega)
$$

$$
+ 2\sqrt{S_0^{(1)}(r, \omega)S_0^{(2)}(r, \omega)}|\eta_j(Q_1, Q_2, \omega)|\cos[\alpha_j(Q_1, Q_2, \omega) + k(R_2 - R_1)],
$$

(4)

where

$$
\alpha_j(Q_1, Q_2, \omega) = \arg[\eta_j(Q_1, Q_2, \omega)],
$$

(5)

and $S_j^{(q)}(r, \omega), q = (1, 2)$, denote the Stokes parameters at the screen if only pinhole at $Q_q$ is open. It follows from Eq. (4) that, not only the spectral density $S_0(r, \omega)$, but also the other three Stokes parameters are generally modulated at the observation screen [21]. The strength of modulations, compared to the extrema of the spectral density, are described by the generalized visibility parameters

$$
V_j(\omega) = \frac{S_{j,\text{max}}(r, \omega) - S_{j,\text{min}}(r, \omega)}{S_{0,\text{max}}(r, \omega) + S_{0,\text{min}}(r, \omega)} = \frac{2\sqrt{S_0^{(1)}(r, \omega)S_0^{(2)}(r, \omega)}}{S_0^{(1)}(r, \omega) + S_0^{(2)}(r, \omega)}|\eta_j(Q_1, Q_2, \omega)|.
$$

(6)

In particular, if the spectral densities at the pinholes are equal, we have

$$
V_j(\omega) = |\eta_j(Q_1, Q_2, \omega)|.
$$

(7)

Thus the generalized visibility parameters can be seen as the electromagnetic extensions of the classical (scalar) degree of coherence.

Consider next an example, in which the spectral densities are equal at the pinholes, and in which the field is $x$-polarized at the first pinhole and $y$-polarized at the second one. The (normalized) polarization matrices and the cross-spectral density matrix at the pinholes are thus given by

$$
J(Q_1, \omega) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad J(Q_2, \omega) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad W(Q_1, Q_2, \omega) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix},
$$

(8)
where \( a \) is a complex number that describes the correlation between the components at the pinholes. It is obvious that it must satisfy the constraint \(|a| \leq 1\). We assume that \( \arg(a) = 0 \), since the phase is not of special interest. It now follows from Eqs. (2) and (3) that the Stokes parameters in the region around the \( z \) axis at the observation screen are of the forms

\[
S_0(\mathbf{r}, \omega) = 2C,
S_1(\mathbf{r}, \omega) = 0,
S_2(\mathbf{r}, \omega) = 2C|a|\cos[k(R_2 - R_1)],
S_3(\mathbf{r}, \omega) = 2C|a|\sin[k(R_2 - R_1)],
\] (9)

where \( C \) is a constant that depends on the geometry. The degree of polarization at the observation screen is thus given by

\[
P(\mathbf{r}, \omega) = \frac{1}{S_0(\mathbf{r}, \omega)} \left[ |S_1(\mathbf{r}, \omega)|^2 + |S_2(\mathbf{r}, \omega)|^2 + |S_3(\mathbf{r}, \omega)|^2 \right]^{1/2} = |a|.
\] (10)

Since \( S_0(\mathbf{r}, \omega) \) is constant, no intensity-interference fringes are observed at the screen, regardless on the correlation of the field at the pinholes. However, the modulation of the parameters \( S_2(\mathbf{r}, \omega) \) and \( S_3(\mathbf{r}, \omega) \) implies that the polarization at the screen is modulated periodically, provided that \( a > 0 \).

Let us recall the definition of the so-called degree of coherence for electromagnetic fields, denoted by \( \mu_\varepsilon(Q_1, Q_2, \omega) \), from Refs. [6, 15]:

\[
\mu_\varepsilon(Q_1, Q_2, \omega) = \left\{ \frac{\text{tr}[W(Q_1, Q_2, \omega)W(Q_2, Q_1, \omega)]}{S_0(Q_1, \omega)S_0(Q_2, \omega)} \right\}^{1/2} = \left[ \frac{\sum_{i,j=x,y} |W_{ij}(Q_1, Q_2, \omega)|^2}{S_0(Q_1, \omega)S_0(Q_2, \omega)} \right]^{1/2}.
\] (11)

It follows from Eqs. (2) that we have the identity

\[
3 \sum_{j=0}^3 |S_j(Q_1, Q_2, \omega)|^2 = 2 \sum_{i,j=x,y} |W_{ij}(Q_1, Q_2, \omega)|^2
\] (12)

and hence, in view of Eqs. (3) and (11), we have [22]

\[
\mu_\varepsilon^2(Q_1, Q_2, \omega) = \frac{1}{2} \sum_{j=0}^3 |\eta_j(Q_1, Q_2, \omega)|^2.
\] (13)

Thus, the \( \mu_\varepsilon \) is a direct measure of the contrasts of modulation of the standard Stokes parameters. It is thus fully analogous to the scalar degree of coherence which describes the visibility of the intensity-interference fringes at the observation screen [22].

3. Reduction of the number of modulated parameters

It is obvious that the cross-spectral density matrix does not remain invariant if (different) unitary transformations are applied to the field at the pinholes. In view of Eq. (2), the same applies to the two-point Stokes parameters and, consequently, to the polarization properties at the observation screen as well. We next investigate how the number of modulated parameters can be minimized using suitable unitary transformations.

The suitable transformations are found by performing the singular-value decomposition of the cross-spectral density matrix at the pinholes. Namely, by elementary matrix algebra, it follows that the cross-spectral density matrix has a representation

\[
W(Q_1, Q_2, \omega) = U_1^*D(Q_1, Q_2, \omega)U_2^T,
\] (14)

where \( U_1 \) and \( U_2 \) are unitary matrices and \( D(Q_1, Q_2, \omega) \) is a diagonal matrix whose diagonal elements are the singular values of the cross-spectral density matrix. It follows at once from Eq. (14) that if we
apply the adjoint transformations $U_1^\dagger$ and $U_2^\dagger$ to the field at the pinholes, the cross-spectral density matrix after the transformations, denoted by a prime, is given by

$$W'(Q_1, Q_2, \omega) = U_1^\dagger D(Q_1, Q_2, \omega) U_2^\dagger = D(Q_1, Q_2, \omega).$$

Since the transformed matrix is diagonal, it follows from Eqs. (2) and (3) that only the polarization S\parallel where therefore fully polarized [26]. Further, it follows from Eq. (4) that the polarization state at the observation it then follows that the field at the pinholes and at the observation screen is also completely coherent, and that the singular values of the cross-spectral density matrix are equal [25]. It follows at once from this result that if the field is unpolarized at the pinholes, the number of modulated Stokes parameters can be reduced to one if, and only if, the singular values of the cross-spectral density matrix are equal [25]. It follows at once from this result that if the field is unpolarized at the pinholes, the number of modulated Stokes parameters can be reduced to one if, and only if, the so-called intrinsic degrees of coherence [16, 17] are equal.

4. Pancharatnam–Berry phase at the observation screen

Assume next that the field that is input to Young’s interferometer is completely coherent in the sense of Glauber’s definition [26], i.e., that the cross-spectral density tensor is of a factorized form

$$W(r_1, r_2, \omega) = F'(r_1, \omega)F^T(r_2, \omega)$$

It then follows that the field at the pinholes and at the observation screen is also completely coherent, and therefore fully polarized [26]. Further, it follows from Eq. (4) that the polarization state at the observation screen is cyclic and hence the Poincare vector [27]

$$P(r, \omega) = \hat{s}_1 s_1(r, \omega) + \hat{s}_2 s_2(r, \omega) + \hat{s}_3 s_3(r, \omega),$$

where $s_j = S_j/S_0$ and $\hat{s}_j$, $j = (1, 2, 3)$, are three Cartesian unit vectors in the polarization space, forms a closed loop in Poincare’s sphere. Moreover, since the field is fully polarized, $||P(r, \omega)|| = 1$, i.e., the loop is located on the surface of the sphere.

From the works of Pancharatnam [28] and Berry [29] we know that such a closed loop may be connected to a geometric phase, which in this context is known as Pancharatnam–Berry phase. It equals one half of the solid angle formed by the Poincare vector in the loop on Poincare’s sphere [30]. Even though this geometric phase is usually examined in connection with the polarization changes upon propagation, it can be observed whenever cyclic polarization changes appear, like in electromagnetic Young’s experiment discussed here.

It can be shown by elementary geometry and Eq. (13) that the loop formed by the vector $P(r, \omega)$ is a circle on Poincare’s sphere. Moreover, the solid angle $\Omega$ subtended by the circle is given by

$$\Omega(\omega) = 2\pi - 2\pi \frac{|S_0(Q_1, \omega) - S_0(Q_2, \omega)|}{||S_0(Q_1, \omega)P(Q_1, \omega) - S_0(Q_2, \omega)P(Q_2, \omega)||}$$

and hence the Pancharatnam–Berry phase after one full cycle at the screen is of the form

$$|\phi_{PB}(\omega)| = \Omega(\omega)/2 = \pi - \pi \frac{|S_0(Q_1, \omega) - S_0(Q_2, \omega)|}{||S_0(Q_1, \omega)P(Q_1, \omega) - S_0(Q_2, \omega)P(Q_2, \omega)||}.$$
Let us briefly sketch how the Pancharatnam–Berry phase in Young’s experiment could be measured, at least in principle. We consider the setup illustrated in Fig. 1: The original observation screen is replaced by the second double-pinhole system, in which the pinholes are placed very close to each other. Moreover, the second double-pinhole system can be easily moved in the transverse direction. Now Pancharatnam’s definition for the phase $\psi$ between the field values at the second pinhole system is given by

$$\psi(r_1, r_2, \omega) = \arg[F^*(r_1, \omega)F(r_2, \omega)] = \arg[\eta_0(r_1, r_2, \omega)], \quad (20)$$

where $r_1$ and $r_2$ are the locations of the pinholes, and $F(r, \omega)$ has the same meaning as in Eq. (16). It is well known that the phase consists of a dynamical phase that only depends on the geometry, and the polarization phase that depends on the states of polarization. It is the latter one that accounts for the Pancharatnam–Berry phase. Applying Eq. (4) to the new system, we find that $\psi(r_1, r_2, \omega)$ is given, analogously to the scalar case [31], by the spatial location of the intensity-interference pattern at the (second) observation screen. Now moving the second double-pinhole system by small steps over one period of the (first) interference pattern, measuring $\psi(r_1, r_2, \omega)$ in each step, gives us a set of phases. After subtracting the contribution from the dynamical phase from each measurement, and then summing the individual values of $\psi(r_1, r_2, \omega)$, we obtain the value of $|\phi_{PB}(\omega)|$, as desired.

**Figure 1.** Setup proposed for measurement of Pancharatnam–Berry phase: The light is input to the aperture A with two pinholes/slits in it from which it diffracts onto the second screen B. The second screen has a second double-pinhole/slit system that can be moved in the lateral direction. The locations of the interference fringes are recorded in the plane C.

### 5. Conclusions

The observation of the modulation of the Stokes parameters at the observation screen in Young’s interference experiment has led to some interesting findings. For example, the recently introduced degree of coherence for electromagnetic fields has a clear physical meaning as the measure of, not only the visibility of the intensity-interference fringes, but also the other three Stokes parameters. This result is particularly interesting, since one is able to interchange the roles of different Stokes parameters, including intensity, using simple optical elements like retarders. In addition, we may employ suitable
unitary transformations at pinholes to reduce the number of the parameters to only two, and in some cases to only one.

In addition to the coherence analysis, the modulation of Stokes parameters implies that there is a straightforward connection between the states of polarization at the pinholes and the Pancharatnam–Berry phase at the screen. The phase can be measured with a set of successive interference experiments.

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