Multiplicative asset exchange with arbitrary return distributions

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Abstract. The conservative wealth exchange process derived from trade interactions is modeled as a multiplicative stochastic transference of value, where each interaction multiplies the wealth of the poorest of the two intervening agents by a random gain $\eta = 1 + \kappa$, with $\kappa$ a random return. Analyzing the kinetic equation for the wealth distribution $P(w,t)$, general properties are derived for arbitrary return distributions $\pi(\kappa)$. If the geometrical average of the gain is larger than one, i.e. if $\langle \ln \eta \rangle_\pi > 0$, in the long time limit a nontrivial equilibrium wealth distribution $P(w)$ is attained. Whenever $\langle \ln \eta \rangle_\pi < 0$, on the other hand, wealth condensation occurs, meaning that a single agent gets the whole wealth in the long run. This concentration phenomenon happens even if the average return $\langle \kappa \rangle_\pi$ of the poor agent is positive. In the stable phase, $P(w)$ behaves as $w^{(T-1)}$ for $w \to 0$, and we find $T$ exactly. This exponent is nonzero in the stable phase but goes to zero on approach to the condensation interface. The exact wealth distribution can be obtained analytically for the particular case of Kelly betting, and it turns out to be an exponential $P(w) = e^{-w}$. We show, however, that our model is never reversible, no matter what $\pi(\kappa)$ is. In the condensing phase, the wealth of an agent with relative rank $x$ is found to be $w(x,t) \sim e^{xT(\ln \eta)_x}$ for finite times $t$. The wealth distribution is consequently $P(w) \sim 1/w$ for finite times, while all wealth ends up in the hands of the richest agent for large times. Numerical simulations are carried out and found to satisfactorily compare with the above-mentioned analytical results.

Keywords: critical phenomena of socio-economic systems, interacting agent models, socio-economic networks, stochastic processes
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## 1. Multiplicative trade model

The pervasive existence of inequalities in the distribution of wealth in human societies has long puzzled observers, but has only recently become a focus of research by physicists [1]. The observation of a power-law distribution for wealth and income in capitalist societies, originally made by Pareto [2], has been confirmed and perfected by analyzing extensive sets of data that are nowadays available. These show that the upper 5–10% of the richest individuals follow a power-law, while the middle to low income sector of a population follows a Gibbs or lognormal law [3,1,4]. The range of typical wealth variation may be orders of magnitude larger than the natural variability in individual abilities and capacities, in case one would attempt to resort to these in order to explain the former. Clearly, it is of interest to understand what mechanisms may drive the appearance of such startling differences. Apart from the fact that wealthier individuals or entities have, on average, more political power to influence their social environment to their advantage, thus producing a self-reinforcing inequality cascade, it is also valid to ask oneself whether the microscopic mechanisms of wealth production and redistribution carry within themselves the property of spontaneously producing inequality.

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In a pioneering work [5], Angle proposed the use of conservative wealth exchange models in order to explain wealth inequalities. He envisaged the wealth exchange process as a stochastic transfer of ‘surplus’, in which the loser transfers a random fraction $\kappa$ of its wealth to the winner. This is nowadays called the ‘loser scheme’ since it is a fraction of the loser’s wealth that is at stake [6]. If both interacting agents are assumed to win with the same probability, then such a process avoids wealth concentration by favoring the poorest agent, because he risks to lose less than he can win. Aiming to understand the observed wealth concentration, Angle then argues that the richer agent, because of the competitive advantage allowed by his larger wealth, usually has a larger probability of winning in each encounter. Therefore, in the context of Angle’s initial proposal, wealth concentration is a consequence of an explicit advantage, or edge favoring the richer agent$^1$.

Although being richer does provide an advantage in the context of certain wealth appropriation processes, it is now recognized [7,8] that, from a statistical point of view, an explicit advantage favoring the rich is not a necessary ingredient for wealth concentration to happen. This is a remarkable result that only recently has been stressed in the econophysics literature. For certain realistic wealth exchange rules to be discussed in this work, in the long run all wealth may end up in the hands of just one agent, even if the poor agent has an edge over the rich one. The key ingredient for this rather counterintuitive phenomenon is the fact that the amount at stake in each transaction is a fraction of the poorest agent’s wealth [6]–[10], not of the loser’s wealth. This apparently minor difference in the rules of these so-called ‘poorest scheme’ models profoundly alters the outcome, as now the poor agent has to be given an explicit advantage in order to avoid a catastrophic concentration of wealth called wealth condensation. In other words, multiplicative stochastic transfer whose scale is dictated by the wealth of the poorest intervening agent implies a ‘hidden’ bias in favor of the rich. This is one of the statistical factors driving wealth concentration.

On the other hand, it can be generally argued that stochastic multiplicative ‘poorest scheme’ transfer rules constitute an appropriate simple model of the wealth exchange process occurring during commercial interaction, or trade [7,10]. Wealth transfer occurs in a trade operation not because money changes hands, which is not necessarily always the case, but as a consequence of the difference in values between the items swapped. This confers on the interaction a clear stochastic character, as none of the agents is perfectly aware of the true values of the items being interchanged. Furthermore, it can be argued that the amount at stake must be proportional to the wealth of the poorest agent, since an interaction in which the richest agent has the possibility to lose orders of magnitude more than he can win cannot be considered realistic if one wishes to reproduce a consensual trade process [7,10].

In this work, an analytic and numerical study is presented of wealth exchange models in which the transference is stochastic, multiplicative and proportional to the poorest agent’s wealth. In each interaction, the poor agent risks a fraction $\kappa$ of its wealth, where $\kappa$ is a random variable called the return. Scafetta et al [10] have numerically studied a model of this type in which the return distribution depends on the wealths of both intervening agents. In order to simplify the derivation of analytic results, we restrict ourselves to the case in which the distribution of returns $\pi(\kappa)$ is the same for all pairs of agents.

$^1$ An agent is said to have an edge when the expectation value of a single trade favors him.
Section 2 presents the model and a fast heuristic determination of its condensation interface. In section 3, the kinetic equation for $P(w)$ is introduced and analytic results are drawn from it. These are compared with numerical results in section 4. Section 5 presents a discussion of our results.

2. Trading with an arbitrary distribution of returns

2.1. The model

Trade interactions are modeled as a process in which wealth is stochastically transferred between a pair of agents, according to the following rules. In each transaction a pair of agents is chosen at random and the poorest one, initially with wealth $w_{\text{poor}}$, receives a gain $\kappa w_{\text{poor}}$, where $-1 < \kappa < 1$ is a random return with distribution $\pi(\kappa)$. The richest agent’s wealth changes by $-\kappa w_{\text{poor}}$. The transaction is thus conservative, and given by

$$
\begin{align*}
  w_{\text{poor}}^{t+1} &= w_{\text{poor}}^t + \kappa_t w_{\text{poor}}^t, \\
  w_{\text{rich}}^{t+1} &= w_{\text{rich}}^t - \kappa_t w_{\text{poor}}^t.
\end{align*}
$$

(1)

The condition $|\kappa| < 1$ ensures that both agents have positive wealth after the trade.

Yard-Sale [7, 11, 9, 8] is a particular case of this process that can be described as a ‘bet’ for a fraction $f$ of the wealth of the poorest agent, and where the poorest agent has a probability $p$ of winning. Therefore $\kappa = +f$ with probability $p$ and $\kappa = -f$ with probability $q = 1 - p$, so $\pi_{YS}(\kappa) = p\delta(\kappa - f) + q\delta(\kappa + f)$.

2.2. Wealth condensation

Depending on $\pi(\kappa)$, long term evolution under rules (1) may give rise either to a stable wealth distribution $P(w)$ or to wealth condensation [8]. The surface consisting of distributions $\pi(\kappa)$ separating these two cases is called the condensation interface. We now derive the location of this interface as follows: consider an agent who has become so poor that, in most subsequent trades he will be the poorest. His own wealth will thus almost always evolve according to

$$
\begin{align*}
  w_{t+1} &= w_t(1 + \kappa_t),
\end{align*}
$$

(2)

i.e. it will undergo a random multiplicative process [12] with multiplier $\eta_t = (1 + \kappa_t)$ at each timestep. After a large number $N$ of timesteps, the appropriate central tendency estimator for its wealth is therefore not the arithmetic average

$$
\langle w_N \rangle_\pi = w_0 (1 + \kappa)^N, 
$$

(3)

but the geometric average

$$
e^{\langle \ln w_N \rangle_\pi} = w_0 e^{N \langle \ln(1 + \kappa) \rangle_\pi}.
$$

(4)

Clearly the wealth of a poor agent will diminish steadily if $\langle \ln(1 + \kappa) \rangle_\pi < 0$, in which case there is a sustained transference of wealth from poorer to richer agents, the system is in a condensing phase, and the whole wealth ends up in the hands of one agent in the long run [8]. This catastrophic collapse of the wealth distribution is called wealth condensation. By the heuristic arguments above, the condensation interface is therefore defined by

$$
\langle \ln(1 + \kappa) \rangle = 0.
$$

(5)

This result will be rederived later in section 3.3 by means of a rigorous analysis of the kinetic equation for this process.
3. Kinetic equation analysis

3.1. Kinetic equation in the stationary limit

In appendix A we show that the equilibrium wealth distribution $P(w)$ satisfies

$$P(w) = \left\langle \frac{P(w/(1+\kappa))\mathcal{P}_>(w/(1+\kappa))}{(1+\kappa)} + \int_0^{w/(1-\kappa)} dv \, P(v)P(w+\kappa v) \right\rangle_\pi,$$

(6)

where $\mathcal{P}_>(w) = \int_w^\infty P(v) \, dv$ is the fraction of agents with wealth above $w$, and $\langle \rangle_\pi$ indicates the expectation value with respect to the return distribution $\pi(\kappa)$. The first and second terms on the right-hand side of (6) represent the contributions of exchanges with agents that have a wealth respectively larger and smaller than $w$. This equation can be solved exactly only for special cases that we discuss later in section 3.4. In the general case, however, useful exact results can still be extracted from it, as described next.

3.2. Small-wealth limit for $P(w)$

The small-wealth behavior of the wealth distribution $P(w)$ in the stable phase can be derived as follows. Assume $P(w) \sim w^{(T-1)}$ for $w \to 0$. Plug this expression into the stationary kinetic equation (6), approximate $P_>(w) \approx 1$ for small $w$, and notice that the last integral only contributes higher order terms, to find

$$\left\langle \frac{1}{(1+\kappa)^T} \right\rangle_\pi = 1.$$

(7)

This result can be rationalized by referring to Kesten processes [13]–[15], as discussed in section 5. Numerical results to be presented later in section 4 support the validity of equation (7) in the limit of small wealth.

Numerical simulation (section 4) shows that, using the value of $T$ resulting from (7), the entire wealth distribution can be approximated by a gamma function

$$P(w) = aw^{(T-1)}e^{-w/b},$$

(8)

where the normalization conditions on the zeroth and first moments of $P(w)$ fix $a = \Gamma(T+1)/\Gamma(T)^{T+1}$, and $b = \Gamma(T)/\Gamma(T+1)$. Notice, however, that the wealth distribution (see figures 1 and 2) is not exactly given by (8), except in special cases.

3.3. Condensation interface

Equation (7) allows us to determine the location of the condensation interface by the following rigorous argument. Given that $P(w) \sim w^{(T-1)}$ for $w \to 0$, the fraction $\mathcal{P}_<(\epsilon) = \int_0^\epsilon P(v) \, dv$ of agents whose wealth is below an arbitrarily small but finite level $\epsilon$ is finite for all $T > 0$, but diverges as $T \to 0^+$. The divergence of $\mathcal{P}_<(\epsilon)$ indicates that most agents impoverish absolutely, and equivalently that all wealth becomes concentrated in the hands of a few. Therefore, the condensation interface is defined by the condition $T \to 0^+$. Now rewrite (7) as $\langle e^{-T \ln(1+\kappa)} \rangle_\pi = 1$ and expand it in powers of $T$ to obtain

$$\sum_{r=1}^{\infty} (-1)^r T^r \langle \ln^r(1+\kappa) \rangle_\pi / r = 0.$$

(9)
Figure 1. Wealth distribution for multiplicative exchange with binary return distribution (Yard-Sale), with $f$ as indicated in the figures. (a) $p(f) = (f + 1)/2$, which corresponds to $T = 1$ according to (25). The full line is $e^{-w}$. (b) $p(f)$ satisfies (26), for which $T = 2$ is expected. The full line is a gamma function (8) with $T = 2$.

Figure 2. Wealth distribution for multiplicative exchange with flat return distribution between $a$ and $b$. (a) $a = \log(z)/(z - 1) - 1$ and $b = z \log(z)/(z - 1) - 1$, with $z$ as indicated in the figure. This corresponds to $T = 1$, according to equation (29) (see text). The full line is $e^{-w}$. (b) Here $a = -b/(1 + b)$, with $b$ as indicated in the figure. This corresponds to $T = 2$, according to equation (30). The full line is a Gamma function (8) with $T = 2$.

After eliminating the trivial solution $T = 0$, we are left with

$$\langle \ln(1 + \kappa) \rangle = T \left( \sum_{r=0}^{\infty} \frac{(-1)^r T^r \langle \ln^{r+2}(1 + \kappa) \rangle}{r + 2} \right).$$

Therefore, to lowest order,

$$T \approx 2 \langle \ln(1 + \kappa) \rangle / \langle \ln^2(1 + \kappa) \rangle.$$  

We thus find that the condensation condition $T \to 0$ amounts to $\langle \ln(1 + \kappa) \rangle \pi = 0$, which is the same as equation (5), derived previously by analyzing the typical behavior of a poor agent’s wealth.
For the case of Yard-Sale, where $\kappa = \pm f$ with probabilities $p$ and $q = 1 - p$, the condensation interface (5) is given by $p_c = \ln(1/(1 - f))/\ln((1 + f)/(1 - f))$, a result that has been verified numerically [8]. Notice that $p_c > 1/2$, i.e. the poor agent has to be given a significant explicit advantage, or edge, in order for condensation not to occur.

For a flat distribution of returns between two limits $a < \kappa < b$, on the other hand, the critical condition for condensation reads

$$
\langle \ln(\eta) \rangle_\pi = \ln((1 + b)/(1 + a)^{(1+a)}) - (b - a) = 0.
$$

(12)

Here again, notice that it is possible to have condensation even when the average return of the poor agent, which is $\langle \kappa \rangle_\pi = (a + b)/2$, is positive.

3.4. Exponential solution and Kelly betting

We now show that, for certain return distributions $\pi(\kappa)$, the equilibrium wealth distribution is exponential. For this we replace $P(w) = e^{-w}$ in (6), and get

$$
\langle (e^{-w(1-\kappa)/(1+\kappa)}) - e^{-w(1+\kappa)/(1-\kappa)} + 1)/(1 + \kappa) \rangle_\pi = 1,
$$

(13)

which is a sufficient condition for the stable wealth distribution to be exponential. Particularized to $w = 0$, this condition reduces to $\langle 1/(1 + \kappa) \rangle_\pi = 1$, which is just equation (7) in the case $T = 1$, as appropriate for an exponential distribution. However, notice that (13) is much more restrictive than just (7) with $T = 1$, because it has to be satisfied for all $w$. Trivially, if two return distributions $\pi_1(\kappa)$ and $\pi_2(\kappa)$ satisfy (13), so does any normalized linear combination of them. Therefore, a meaningful approach to solving (13) consists in first finding simple return distributions which satisfy it, and then building more general ones by linear combination. Proposing a binary distribution $\pi(\kappa) = p\delta(\kappa - a) + q\delta(\kappa - b)$, one finds that equation (13) is only satisfied if $-a = b = f$ and, additionally, $p = (1 + f)/2$, that is

$$
\pi(f)(\kappa) = \frac{1 + f}{2}\delta(\kappa - f) + \frac{1 - f}{2}\delta(\kappa + f).
$$

(14)

This return distribution corresponds to Yard-Sale [8] but particularized to the case of Kelly betting [16,17], which fixes $f = 2p - 1$. Possible implications of this result are explored in section 5.

3.5. More general return distributions with exponential solutions

An arbitrary superposition of return distributions of the form (14) with different values of $f$ will also admit an exponential solution for $P(w)$. In other words, for any positive $W(f)$ normalized in $[0,1]$ we have that

$$
\pi(\kappa) = \int_0^1 df W(f) \left\{ \frac{1 + f}{2}\delta(\kappa - f) + \frac{1 - f}{2}\delta(\kappa + f) \right\},
$$

(15)

which is easily integrated to give

$$
\pi(\kappa) = \frac{1 + \kappa}{2} W(|\kappa|),
$$

(16)

which satisfies (13), i.e. gives rise to $P(w) = e^{-w}$ in equilibrium.
3.6. Wealths by rank in the condensing phase

Let $w_R$ be the wealths of the $N$ agents ordered by rank $R$, so that $w_1 > w_2 > \cdots > w_N$. When an agent with rank $R$ interacts with another agent with rank $S$, we have

$$w_R^{(t+1)} = \begin{cases} w_R^{(t)} + \kappa w_R^{(t)} & \text{if } R > S \\ w_R^{(t)} - \kappa w_S^{(t)} & \text{if } R < S. \end{cases} \quad (17)$$

Notice that (17) is not valid in general, since it disregards rank changes resulting from interactions. Its validity is restricted to the case in which agents keep fairly constant ranks, i.e. there is no ‘social mobility’. This holds in the condensed phase, but not in the stable phase. In the condensed phase, furthermore, one has $w_S/w_R \ll 1$ so that the interaction with poorer agents can be neglected altogether, to write

$$\ln\left(\frac{w_R^{(t+1)}}{w_R^{(t)}}\right) \approx \begin{cases} \ln \eta & \text{if } R > S \\ 0 & \text{if } R < S. \end{cases} \quad (18)$$

Averaging over $\pi(\kappa)$, over all $(N-1)$ possible choices of $S$, and defining the relative rank $r = (R-1)/(N-1)$ so that $r = 0$ corresponds to the richest agent,

$$\langle \ln(w_R^{(t+1)}/w_R^{(t)}) \rangle_\pi = r \langle \ln \eta \rangle_\pi \Rightarrow \langle \ln w_R^{(t)} \rangle_\pi = rt \langle \ln \eta \rangle_\pi. \quad (19)$$

The typical value of $w(r,t)$ therefore satisfies

$$w(r,t) \sim e^{-rt\phi}, \quad (20)$$

where we have defined $\phi = -\langle \ln \eta \rangle_\pi > 0$. Normalization for a system of $N$ agents with a total wealth $W$ then results in

$$w(r,t) = W \frac{(1 - e^{-t\phi}/N)}{(1 - e^{-t\phi})} e^{-t\phi}. \quad (21)$$

Now since $r = P_>(w(r))$ we have that $P(w) = -1/(\partial w(r)/\partial r)$. From (20) we thus obtain

$$P(w) \sim \frac{1}{w}. \quad (22)$$

The validity of (20) and (22) is verified for uniformly distributed returns in section 4.

4. Numerical results

4.1. Stable phase

We first consider the case of Yard-Sale exchange [8], i.e. $\pi(\kappa) = p\delta(\kappa-f) + (1-p)\delta(\kappa+f)$. Equation (7) amounts in this case to

$$\frac{p}{(1+f)^T} + \frac{q}{(1-f)^T} = 1. \quad (23)$$
Some particular cases of interest are

\[ T = 0 \Rightarrow p = p_c = -\ln(1 - f) / \ln \left( \frac{1 + f}{1 - f} \right) \]  
\[ (24) \]

\[ T = 1 \Rightarrow p = \frac{1 + f}{2} \]  
\[ (25) \]

\[ T = 2 \Rightarrow p = \frac{1}{2} + \frac{3f - f^3}{4}. \]  
\[ (26) \]

Equation (24) defines the condensation interface. Its accuracy has been numerically verified in previous work [8]. Figure 1(a) shows wealth distributions for three values of \( f \), where \( p(f) \) is given by (25), and therefore corresponds to \( T = 1 \), i.e. \( P(w) \) should be constant for \( w \to 0 \). Notice that all values of \( f \) in this figure satisfy \( P(w) = e^{-w} \). This is consistent with the results derived in section 3.4, namely that the wealth distribution is exponential whenever \( f = 2p - 1 \) is satisfied. So in this case, for any pair \((p, f)\) satisfying (25) the wealth distribution is the same. For \( T = 2 \), the wealth distribution should approach the origin as \( P(w) \sim w \). This is verified by considering the data shown in figure 1(b), obtained with \( p(f) \) given by (26). However, in this case, notice that the wealth distribution does depend on \( f \), i.e. the asymptotic exponent \( T \) does not determine the whole distribution. A similar observation holds for the rest of the \((f, p)\) plane: the equilibrium distribution depends on \( f \) on all lines of constant \( T \), except on the line \( f = 2p - 1 \), where \( T = 1 \) and \( P(w) = e^{-w} \).

Let us now consider a flat return distribution \( \pi(\kappa) = 1/(b - a) \) for \( a \leq \kappa \leq b \), where \( |a|, |b| \leq 1 \). This last condition ensures that the gain \( \eta = (1 + \kappa) \) lies between zero and two, and therefore that both agents can always pay. Equation (7) can be worked out exactly also in this case, and the result is

\[ \frac{(1 + b)^{(1-T)} - (1 + a)^{(1-T)}}{(b - a)(1 - T)} = 1. \]  
\[ (27) \]

We consider the following particular cases

\[ T = 0 \Rightarrow \ln((1 + b)^{(1+b)}/(1 + a)^{(1+a)}) = b - a \]  
\[ (28) \]

\[ T = 1 \Rightarrow \ln((1 + b)/(1 + a)) = (b - a) \]  
\[ (29) \]

\[ T = 2 \Rightarrow (1 + b)/(1 + a) = 1. \]  
\[ (30) \]

The condensation interface has been obtained numerically (not shown) and found to be in accordance with (28). In the case of \( T = 1 \), equation (29) can be shown to be equivalent to writing \( a = \log z/(z - 1) - 1 \) and \( b = z \log z/(z - 1) - 1 \), where \( z = e^{(b-a)} \) is a free parameter restricted to \( 1 \leq z \leq 4.9215 \). We have simulated flat return distributions with \( a \) and \( b \) given by the expressions above with several values of \( z \) (figure 2(a)), and found wealth distributions consistent with \( T = 1 \) in all cases, i.e. reaching \( w = 0 \) as a constant. Notice that only in the limit \( z \to 1 \), in which case \( a, b \to 0 \), does the distribution approach an exponential. When \( T = 2 \), equation (30) amounts to letting \( a = -b/(1 + b) \), within the limits given by \( 0 \leq b \leq 1 \). Examples are shown in figure 2(b), where again good accordance with analytic predictions for the small-wealth exponent is found.

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4.2. Condensed phase

We now verify the approximate expression (20), derived in section 3.6, for the wealth of an agent with relative rank \( x \) in the condensing state. Figure 3(a) shows \( w(x, t) \) for returns \( \kappa \) distributed uniformly between \(-0.1\) and \(+0.1\). Therefore \( \langle \ln \eta \rangle_\pi = -1.67 \times 10^{-3} \), and the system is in the condensing state. The analytic prediction given by (20) is seen in this case to be acceptable, except perhaps for the very poorest agents. Accordingly, the wealth distribution \( P(w) \) obtained in this condensing state is, as shown in figure 3(b), consistent with \( 1/w \) as derived in section 3.6.

5. Discussion

Multiplicative ‘poorest scheme’ asset exchange models with an arbitrary return distribution \( \pi(\kappa) \) were studied, analyzing the kinetic equation (6). It was shown that the whole system’s wealth ‘condenses’ onto one agent whenever \( \langle \ln(1 + \kappa) \rangle_\pi < 0 \). Given that \( e^{\langle \ln(1 + \kappa) \rangle_\pi} \leq e^{\ln\langle (1 + \kappa) \rangle_\pi} = \langle (1 + \kappa) \rangle_\pi \), it is possible to have \( \langle \ln(1 + \kappa) \rangle_\pi < 0 \), and therefore wealth condensation, even in cases in which the average return \( \langle \kappa \rangle_\pi \) is positive. But having a positive average return means, according to (3), that the expectation value of a poor agent’s wealth grows exponentially in time. The apparent paradox suggested by the fact that poor agents lose wealth steadily despite their average return being positive is solved by recognizing that the expectation value is not an appropriate central tendency estimator when considering multiplicative processes [12]. In other words, while the exponential growth indicated by (3) would only be realized after averaging over an enormously large number of repetitions of the multiplicative process, the typical outcome of one realization follows the geometric average (4), which, in the condensing phase, decreases exponentially fast in time.

Analyzing the kinetic equation, the small-wealth exponent \( T \) of \( P(w) \), was found to be given by equation (7). This result can be understood in the context of Kesten
processes [13]–[15], as follows. In the stable phase, where \( \langle \ln(1 + \kappa) \rangle_\pi > 0 \), the inverse wealth \( z = w^{-1} \) of a poor agent undergoes a contractive multiplicative process \( z \to z\xi = z/(1 + \kappa) \) due to its interaction with richer agents, with a small additive noise term given by its almost negligible interaction with poorer agents, as \( z \to z + \delta z \). Because in the stable phase \( \langle \kappa \rangle_\pi > 0 \), it follows that \( \langle \delta z \rangle_\pi > 0 \). The theory of Kesten processes then ensures that \( P(z) \) has a power-law right tail of the form \( z^{-(1+T)} \) with \( T \) satisfying \( \langle \xi^T \rangle_\pi = 1 \). By a simple change of variables, our result (7) then follows for \( P(w) \).

In section 3.4 it was shown that the wealth distribution is exactly exponential when \( \pi(\kappa) \) corresponds to Kelly betting [16,17]. Kelly betting is a gambling strategy devised to maximize the long time rate of growth of a bettor’s capital when faced with a set of risky choices. In its simplest inception, one considers a gambler who is given a choice between a single risky asset (a bet) and a riskless asset (e.g. deciding not to bet). Assuming a bet that pays double or nothing, the gambler doubles its stake or loses it altogether, respectively with probabilities \( p \) and \( q \). The bettor can decide the (fixed) fraction \( f \) of his wealth that will be risked at each time step, while the rest \((1-f)w \) is kept in the riskless asset (e.g. cash). His total wealth thus evolves as \( w \to ((1-f)w + 2fw) = (1+f)w \) if the bet is won, and as \( w \to (1-f)w \) if the bet is lost. In other words, the gambler’s gain is \( \eta = (1+f) \) with probability \( p \), and \( \eta = (1-f) \) with probability \( q \). This is exactly the way in which the wealth of the poorest agent evolves in Yard-Sale. The only difference is the fact that, in the betting optimization problem, \( f \) and \( p \) are not independent parameters, since the gambler wishes to find the most profitable \( f \) for a given \( p \).

The average gain is \( \langle \eta \rangle_\pi = (1+f)p + (1-f)q = 1 + (2p-1)f \), which is larger than one whenever \( p > 1/2 \), i.e. whenever the edge \((2p-1) \) is positive. If the bettor were to maximize \( \langle \eta \rangle_\pi \), the recommended strategy would then be choosing \( f \) as large as possible. However, this approach is doomed to fail in the long run, since sooner or later a losing bet would produce his absolute ruin. Kelly then proposes that the most profitable strategy in the long run consists in using the value of \( f \) maximizing the average growth rate \( G \) of the gambler’s wealth, defined by \( w(t) \sim e^{Gt} \). The average growth rate is then given by the average logarithmic gain \( \langle \ln(\eta) \rangle_\pi \), since a random multiplicative process typically follows its geometric average. Maximization of \( \langle \ln(\eta) \rangle_\pi = p\ln(1+f) + q\ln(1-f) \) with respect to \( f \) then results in \( f^* = (2p-1) \), which is the recommendation of Kelly theory for this problem. Equivalently, \( p = (1+f)/2 \) and \( q = (1-f)/2 \), from which we can recognize that (14) describes Yard-Sale exchange in the case of Kelly betting.

From the discussion above, \( f^* \) is the fraction at stake that produces the fastest growth in the wealth of a poor agent, for a given \( p \). However, notice that this is not the value of \( f \) that produces the least density of poor agents in Yard-Sale for a given \( p \). The density of poor agents in Yard-Sale in equilibrium is minimum in the limit \( f \to 0 \), for any fixed \( p \). This conclusion may be reached by noticing that the fraction of poor agents is smaller the larger \( T \) is and analyzing (7) particularized to Yard-Sale, in the limit \( T \to \infty \).

The link between Kelly betting and an exponential wealth distribution in these exchange models is intriguing. The Kelly strategy can be restated in the language of information theory as a way to maximize the rate of transfer of information over a noisy channel [16]. On the other hand, the exponential distribution \( P(w) = e^{-w} \) maximizes the entropy \( S = -\int P(w) \ln(P(w)) \, dw \) subject to the constraints of constant total wealth.

2 In this context, \( f > 1 \) may be acceptable and would mean that the gambler lends money from the riskless asset for gambling.
\[ \int wP(w)\,dw = 1 \text{ and number of players } \int P(w)\,dw = 1. \] Of course, there is in principle no logical relation between extremization of an entropy transfer rate, and maximizing the total entropy in equilibrium, however the connection seems worth analyzing.

Because the dynamics is conservative, if the microscopic exchange rules are reversible it can be shown \cite{18} that the equilibrium wealth distribution has to be exponential. However, the converse is of course not true, i.e. the existence of a stable exponential solution does not imply reversibility. In fact, multiplicative exchange rules of the type discussed here usually violate reversibility, since the role of both agents is clearly different. Nevertheless, it could in principle be the case that, for some specific return distributions, the general exchange rules considered here were reversible. This in turn would provide a clearcut explanation for the appearance of an exponential solution. However, in appendix B it is shown that this is not the case, i.e. reversibility is not satisfied for exchange rules of the general type (1), no matter what the return distribution \( \pi(\kappa) \) is. Therefore, the wealth distribution in equilibrium is exponential for return distributions of the form (14), not because of reversibility, but because of accidental cancelation of asymmetries in the transition rules.

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**Appendix A. Kinetic equation**

Consider an interaction between two agents \( Y \) and \( Z \) with initial wealths \( y \) and \( z \), assuming without loss of generality \( y < z \). The interaction processes contributing to \( \dot{P}(x) \) are those in which \( x \) is the wealth of one of the two interacting agents, either before or after the interaction. Adopting the shorthand notation \( D\kappa = \pi(\kappa)d\kappa \), \( D\gamma = P(\gamma)\,d\gamma \), and \( Dz = P(z)\,dz \), one has:

(i) The poorest agent \( Y \) has a return \( \kappa \), its wealth thus becoming \( y(1 + \kappa) = x \). This contributes with \( D\kappa D\gamma Dz \theta(z - y)\delta(x - y(1 + \kappa)) \).

(ii) The richest agent \( Z \) takes part in an exchange where the poorest agent \( Y \) has return \( \kappa \). \( Z \)'s wealth becomes \( z - \kappa y = x \). The contribution of this process is \( D\kappa D\gamma Dz \theta(z - y)\delta(x - z + y\kappa) \).

(iii) Either agent has wealth \( x \) before the interaction, but not after it, resulting in a contribution \( -D\kappa D\gamma Dz \theta(z - y)(\delta(x - y) + \delta(x - z)) \).

If \( r \) is the probability per unit time of a trade, the time derivative of \( P(x) \) is then given by

\[
r^{-1}\dot{P}(x) = \int D\kappa D\gamma Dz \theta(z - y)\{\delta(x - y(1 + \kappa)) + \delta(x - z + y\kappa)\}
- \delta(x - z) - \delta(x - y) \}.
\]

(A.1)
Letting $\mathcal{P}_\rho(x) = \int_x^\infty P(z) \, dz$, we are left with
\[
\frac{1}{r} \dot{P}(x) = -P(x) + \int_{-1}^1 \mathcal{D} \kappa \left\{ \frac{P(x/(1+\kappa)) \mathcal{P}_\rho(x/(1+\kappa))}{1+\kappa} + \int_0^{x/(1-\kappa)} \, dy \, P(y) P(x+y\kappa) \right\}.
\]

It can be seen that this equation conserves the zeroth- and first moments of $P(w)$, i.e. number of agents and total wealth are conserved.

**Appendix B. Reversibility**

We wish to determine whether the exchange rules considered in this work satisfy reversibility for some return distribution $\pi(\kappa)$. For this purpose we have to write the kinetic equation in the general form
\[
\dot{P}(x) = \int dz \, d\rho \{ P(x-\rho) P(z) W_{(x-\rho, z)} \} - P(x) P(z) W_{(x, z)} - P(x-\rho) W_{(x-\rho, z)} \}
\]
and check whether $W_{(x-\rho, z)} = W_{(x, z)}$ is satisfied. A lengthy but straightforward calculation shows that
\[
W_{(x-\rho, z)} = \theta(z-(x-\rho)) \frac{1}{x-\rho} \pi \left( \frac{\rho}{x-\rho} \right) + \theta((x-\rho)-z) \frac{1}{z} \pi \left( \frac{\rho}{z} \right).
\]
\[
W_{(x, z)} = \theta((z-\rho)-x) \frac{1}{x} \pi \left( \frac{\rho}{x} \right) + \theta(x-(z-\rho)) \frac{z}{(z-\rho)^2} \pi \left( \frac{\rho}{z-\rho} \right).
\]
If reversibility holds, the following conditions must then be met:
\[
z > x + |\rho|, \Rightarrow \frac{1}{x-\rho} \pi \left( \frac{\rho}{x-\rho} \right) = \frac{1}{x} \pi \left( \frac{\rho}{x} \right);
\]
\[
|z-x| < \rho, \Rightarrow \frac{1}{x-\rho} \pi \left( \frac{\rho}{x-\rho} \right) = \frac{z}{(z-\rho)^2} \pi \left( \frac{\rho}{z-\rho} \right);
\]
\[
\rho < |z-x|, \Rightarrow \frac{1}{z} \pi \left( \frac{\rho}{z} \right) = \frac{1}{x} \pi \left( \frac{\rho}{x} \right);
\]
\[
z < x - |\rho|, \Rightarrow \frac{1}{z} \pi \left( \frac{\rho}{z} \right) = \frac{z}{(z-\rho)^2} \pi \left( \frac{\rho}{z-\rho} \right).
\]
The first condition is independent of $z$, so it has to hold for all $x$ and $\rho$. By calling $\rho/(x-\rho) = \kappa$ and after some manipulation, this condition reads
\[
\pi(-\kappa) = \frac{1}{1-\kappa} \pi \left( \frac{\kappa}{1-\kappa} \right).
\]
Similar manipulation of the fourth case gives
\[
\pi(-\kappa) = \frac{1}{(1-\kappa)^2} \pi \left( \frac{\kappa}{1-\kappa} \right).
\]
These are only compatible with each other if $\kappa = 0$. Therefore the exchange rules discussed in this work are never reversible.

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References

[1] Yarlagadda S, Chatterjee A and Chakrabarti B K (ed), 2005 Econophysics of Wealth Distributions (Italy: Springer)
[2] Pareto V, 1896 Cours d’Economie Politique (Geneve: Droz)
[3] Chatterjee A, Chakrabarti B K and Manna S S, Money in gas-like markets: Gibbs and pareto laws, 2003 Phys. Scr. T106 36
[4] Yakovenko V M and Rosser J B, Colloquium: statistical mechanics of money, wealth, and income, 2009 Rev. Mod. Phys. 81 1703
[5] Angle J, The surplus theory of social stratification and the size distribution of personal wealth, 1986 Social Forces 2 293
[6] Ispolatov S, Krapivsky P L and Redner S, Wealth distributions in asset exchange models, 1998 Eur. Phys. J. B 2 267
[7] Hayes B, Follow the money, 2002 Am. Sci. 90 400
[8] Moukarzel C F, Goncalves S, Iglesias J R, Rodriguez-Achach M and Huerta-Quintanilla R, Wealth condensation in a multiplicative random asset exchange model, 2007 Eur. Phys. J. Spec. Top. 143 75
[9] Iglesias J R, Goncalves S, Abramson G and Vega J L, Correlation between risk aversion and wealth distribution, 2004 Physica A 342 186
[10] Scafetta N, Picozzi S and West B J, A trade-investment model for distribution of wealth, 2004 Physica D 193 338
[11] Sinha S, Stochastic maps, wealth distribution in random asset exchange models and the marginal utility of relative wealth, 2003 Phys. Scr. T106 59
[12] Redner S, Random multiplicative processes—an elementary tutorial, 1990 Am. J. Phys. 58 267
[13] Levy M and Solomon S, Power laws are logarithmic Boltzmann laws, 1996 Int. J. Mod. Phys. C 7 595
[14] Takayasu H, Sato A H and Takayasu M, Stable infinite variance fluctuations in randomly amplified Langevin systems, 1997 Phys. Rev. Lett. 79 966
[15] Sornette D, Multiplicative processes and power laws, 1998 Phys. Rev. E 57 4811
[16] Kelly J L, A new interpretation of information rate, 1956 Bell Syst. Tech. J. 35 917
[17] Rotando L M and Thorp E O, The Kelly criterion and the stock-market, 1992 Am. Math. Mon. 99 922
[18] Dragulescu A and Yakovenko V M, Statistical mechanics of money, 2000 Eur. Phys. J. B 17 723

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