Chiral Symmetry in an Extended Constituent Quark Potential Model

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Abstract

The chiral symmetry is applied to an extended constituent quark potential model. With random phase approximation (RPA), a small component effect is added to the constituent quark model. To obtain the pseudoscalar $\pi$ meson as a Goldstone boson, the quark effective potentials are modified in the model to account for the dynamical breaking of chiral symmetry. Also the vector $\rho$ meson is calculated and the KSRF relation about $\pi$ and $\rho$ meson decay constants is derived in the model.

Key words: chiral symmetry, quark potential model, random phase approximation

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1 Introduction

Due to the complication of a non-Abelian $SU(3)$ gauge theory, quantum chromodynamics (QCD), which describes the strong interaction, has many non-perturbative features such as the dynamical breaking of chiral symmetry and quark confinement. In the study of hadron structure at low energy scale, two kinds of models of QCD are often used, each incorporating some important QCD features. The constituent quark potential model which incorporates the QCD quark confinement has been impressively successful in hadron spectroscopy and decays [1], except for the pseudo-scalar $\pi$ meson which is a Goldstone boson and has very low mass. It was shown recently that the RPA $y$-component must be considered in the treatment of pions [2]. On the other hand, the Nambu-Jona-Lasinio (NJL) model [3] and several of its extensions [4,5,6] which incorporate the chiral symmetry can describe the $\pi$ meson very

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well as a massless Goldstone boson of dynamical breaking of chiral symmetry. However, the NJL model lacks the important QCD property of quark confinement.

In Ref. [7], we proposed an extension to the constituent quark model to comprise the RPA $y$-component quark excitations. As in NJL model, we start from an effective quark Hamiltonian. For meson structure, this leads to a Hamiltonian with two channels — the ordinary valence quark $x$-channel and a new $y$-channel, which is an extension of the quark potential model. This extended quark potential model has a new coupling potential connecting the ordinary valence quark $x$-channel and the small component $y$-channel. The coupling potential can be related to the ordinary valence quark potential in the model.

In this paper, we implement the chiral symmetry in the extended constituent quark potential model. First we introduce the extended quark potential model. In Sec. 3, we analyze $\pi$ and $\rho$ mesons using this framework. The restriction from chiral symmetry is considered. With the meson wave functions containing $y$-channel quark excitations, KSRF relation [8,9] about $\pi$ and $\rho$ meson decay constants is derived. Finally, we summarize our discussion.

## 2 Extending Quark Potential Model with RPA

QCD perturbation theory is very successful when applied in the high energy processes. In the low energy regime, Hamiltonian approach can provide an alternative method to understand the non-perturbative features of QCD, such as the quark confinement. Many efforts were made to obtain an effective Hamiltonian from the exact QCD Lagrangian [10,11,12]. Here we start from the effective quark Hamiltonian, assuming the quark fields have been separated approximately from the gluon fields

\[
H = H_2 + H_4, \quad (1)
\]

\[
H_2 = \int d^3x \bar{\Psi}(x)(-i\alpha \cdot \nabla + \beta m_0)\Psi(x), \quad (2)
\]

\[
H_4 = \frac{1}{2} \sum \epsilon \int d^3x d^3y K_\epsilon(x - y)\bar{\Psi}(x)\Gamma_\epsilon\Psi(x)\bar{\Psi}(y)\Gamma_\epsilon\Psi(y), \quad (3)
\]

where $H_2$ is the free current quark Hamiltonian and $m_0$ is the current quark mass. $H_4$ includes all effective quark interactions. $K_\alpha(x)$ is the kernel function for coupling vertex $\Gamma_\alpha$. 

Apart from a constant, the Hamiltonian (1) can be written in normal order as

\[
H = H_0 + : H_4 : \quad (4)
\]
where $H_0$ represents the free particle energy of constituent quark and its general form is

$$H_0 = \int d^3x \Psi^\dagger(x) \left[ -iA(-\nabla^2) \alpha \cdot \nabla + \beta B(-\nabla^2) \right] \Psi(x), \quad (5)$$

where the dependence on the three-vector momentum arises from the instantaneous approximation of quark interactions.

The two functions $A(k^2)$ and $B(k^2)$ should be obtained from a self-consistent mean field calculation. However in a constituent quark potential model, the quark interactions contain the linear quark confinement which has severe infrared divergence in momentum space. On the other hand, the constituent potential quark models always assume that the constituent quarks have fixed masses. As an extension of the constituent potential quark model, here we assume that the $A$ and $B$ functions can be approximated as constants. So we treat the constituent quark mass as a parameter, $m_c = B$ (with $A = 1$). The free particle Hamiltonian becomes

$$H_0 = \int d^3x \Psi^\dagger(x) \left[ -i\alpha \cdot \nabla + \beta m_c \right] \Psi(x). \quad (6)$$

In this paper, we will only consider meson structures. In naive quark model, meson fields are approximately expressed in terms of local quark fields as

$$\phi_M(x) = \alpha_M \bar{\Psi}(x)\Gamma_M \Psi(x), \quad (7)$$

where $\Gamma_M = i\gamma_5 \tau_i$ for iso-vector pseudo-scalar $\pi$ mesons or $\Gamma_M = \gamma \cdot \epsilon_i \tau_j$ for iso-vector vector $\rho$ mesons. According to $1/N$ expansion, the above meson structures are exact in the limit $N_c \to \infty [13,14]$.

In the non-relativistic limit, as quark momentum $k \ll k^0$, one has[15]

$$\phi_M(x) = \alpha_M \sum_{s_1s_2} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{2k_1^0} \frac{1}{2k_2^0} e^{-i(k_1+k_2) \cdot x} \times \left[ \bar{u}(-k_1, s_1)\Gamma_M v(-k_2, s_2)b^\dagger(-k_1, s_1)d^\dagger(-k_2, s_2) + \bar{v}(k_1, s_1)\Gamma_M u(k_2, s_2)d(k_1, s_1)b(k_2, s_2) \right], \quad (8)$$

for $\pi$ and $\rho$ mesons. On the other hand, the field operators of $\pi$ and $\rho$ can also be expressed in the $Q$ and $Q^\dagger$ of meson creation and annihilation operators

$$\phi_M(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \left[ Q_M(k)e^{-ik \cdot x} + Q_M^\dagger(k)e^{ik \cdot x} \right]. \quad (9)$$
If the mesons are composed of creation of valence quark pairs as in the potential models, one deduces from the equivalence of eqs. (8) and (9)

\[
Q_M^+(\mathbf{k}) = \alpha_M \sum_{s_1 s_2} \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{1}{2k_1^0} \frac{1}{2k_2^0} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})
\]

\[
\times \bar{u}(\mathbf{k}_1, s_1) \Gamma_M v(\mathbf{k}_2, s_2) b^\dagger(\mathbf{k}_1, s_1) d(\mathbf{k}_2, s_2).
\]

(10)

However, generally one can only obtain

\[
Q_M^+(\mathbf{k}) = \alpha_M \sum_{s_1 s_2} \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{1}{2k_1^0} \frac{1}{2k_2^0} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})
\]

\[
\times \left[ X_M \bar{u}(\mathbf{k}_1, s_1) \Gamma_M v(\mathbf{k}_2, s_2) b^\dagger(\mathbf{k}_1, s_1) d(\mathbf{k}_2, s_2)
\right.
\]

\[
+ Y_M \bar{u}(\mathbf{k}_1, s_1) \Gamma_M u(-\mathbf{k}_2, s_2) d(-\mathbf{k}_1, s_1) b(-\mathbf{k}_2, s_2) \right] .
\]

(11)

This is just the RPA excitation operator — the meson state is a superposition of creation and annihilation of $q\bar{q}$ pairs on the vacuum. The second part, i.e. the $y$-component is simply discarded in the ordinary quark potential model as a small component.

Thus, as an extension to the potential model, we take the mesons as excitation modes of the vacuum of the RPA type:

\[
|Q\rangle = Q^+ |0\rangle.
\]

(12)

For a meson in the rest frame, after we take account of the quark interactions, $Q^+$ excitation operators are

\[
Q^+ = \sum_{s_1 s_2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2k^0} \left[ x(\mathbf{k}, s_1, s_2) b^\dagger(\mathbf{k}s_1) d(-\mathbf{k}s_2)
\right.
\]

\[
\left. + y(\mathbf{k}, s_1, s_2) d(-\mathbf{k}s_1) \bar{b}(\mathbf{k}s_2) \right] ,
\]

(13)

where the time reversals are defined as

\[
\tilde{b}(\mathbf{k}s) \equiv (-1)^{1/2-s} b(\mathbf{k} - s),
\]

(14)

\[
\tilde{d}(\mathbf{k}s) \equiv (-1)^{1/2-s} d(\mathbf{k} - s),
\]

(15)

and $x$ and $y$ are the RPA amplitudes determined from the well-known RPA equation of motion[16]

\[
\langle 0| [\delta Q, [H, Q^+]] |0\rangle = E \langle 0| [\delta Q, Q^+] |0\rangle.
\]

(16)
We obtain the equations for RPA amplitudes

\[
\begin{align*}
(E - 2k^0) & \, x(k, s_1, s_2) \\
= & \sum_{s_3 s_4} \int \frac{d^3k'}{(2\pi)^3} \left[ \langle k, s_1, s_2 \mid U \mid k', s_3, s_4 \rangle x(k', s_3, s_4) \\
+ & \langle k, s_1, s_2 \mid V \mid k', s_3, s_4 \rangle y(k', s_3, s_4) \right], \\
(E + 2k^0) & \, y(k, s_1, s_2) \\
= - & \sum_{s_3 s_4} \int \frac{d^3k'}{(2\pi)^3} \left[ \langle k, s_1, s_2 \mid V \mid k', s_3, s_4 \rangle x(k', s_3, s_4) \\
+ & \langle k, s_1, s_2 \mid U \mid k', s_3, s_4 \rangle y(k', s_3, s_4) \right].
\end{align*}
\]

Here we have introduced two effective potentials \(U\) and \(V\). Their matrix elements are

\[
\begin{align*}
\langle k, s_1, s_2 \mid U \mid k', s_3, s_4 \rangle & \equiv -\frac{1}{2k^0 k'^0} \sum_{\epsilon} K_{\epsilon}(k - k') \bar{u}(k's_3)\Gamma_{\epsilon}u(k's_4)\bar{v}(-k's_4)\Gamma_{\epsilon}v(-k's_2), \\
\langle k, s_1, s_2 \mid V \mid k', s_3, s_4 \rangle & \equiv \frac{1}{2k^0 k'^0} \sum_{\epsilon} K'_{\epsilon}(k - k') \bar{u}(k's_3)\Gamma_{\epsilon}\bar{v}(-k's_4)\bar{u}(k's_4)\Gamma_{\epsilon}v(-k's_2).
\end{align*}
\]

Thus, with the RPA approximation, the quark potential model is extended to a coupling system with two channels. The first part, i.e. x-amplitude \(x(k, s_1, s_2)\) which we will call the x-channel wave function, is the large component which is the sole valence quark contribution in the ordinary quark potential model. The second part, i.e. the y-amplitude \(y(k, s_1, s_2)\) which we will call the y-channel wave function, is the small component which is discarded in the ordinary quark potential model. The potential \(U\) interacts only within each individual channel while the new coupling potential \(V\) couples the two channels together. If the coupling between the two channels \(V\) is small, the system decouples. The extended potential model reduces to the ordinary potential model and the potential \(U\) is just the ordinary quark potential. Here we will show that the coupling potential \(V\) is important due to chiral symmetry.

One can use the Dirac kets and bras to simplify notations. \(x(k, s_1, s_2)\) and \(y(k, s_1, s_2)\) are represented by two kets \(|x\rangle\) and \(|y\rangle\) respectively,

\[
\begin{align*}
x(k, s_1, s_2) & = \langle k, s_1, s_2 \mid x \rangle, \\
y(k, s_1, s_2) & = \langle k, s_1, s_2 \mid y \rangle.
\end{align*}
\]

Eqs. (17) and (18) can be written concisely as
\[ (E - 2k^0) |x\rangle = U |x\rangle + V |y\rangle, \]  
\[ (E + 2k^0) |y\rangle = -V |x\rangle - U |y\rangle, \]  

(23) 
(24) 

or the familiar matrix form

\[
\begin{bmatrix}
2k^0 + U & V \\
-V & -2k^0 - U
\end{bmatrix}
\begin{bmatrix}
|x\rangle \\
|y\rangle
\end{bmatrix}
= E
\begin{bmatrix}
|x\rangle \\
|y\rangle
\end{bmatrix},
\]

(25) 

where \( k^0 = \sqrt{m^2 + k^2} \).

By definitions (19) and (20), the potentials \( U \) and \( V \) will be evaluated from the same kernels in the quark interaction \( H_4 \), thus they can be related to each other. In the non-relativistic limit \( k, k' \to 0 \), we have

\[
U = S - V - A \sigma_1 \cdot \sigma_2 - T \sigma_1 \cdot \sigma_2,
\]

(26) 

\[
V = P - V \sigma_1 \cdot \sigma_2 - A - T \sigma_1 \cdot \sigma_2,
\]

(27) 

where on the right hand side of equations, the kernels are divided into five types as in ref. [17], i.e., scalar (\( S \)), pseudo-scalar (\( P \)), vector (\( V \)), axial-vector (\( A \)), and tensor (\( T \)); \( \sigma \)'s are the pauli spin matrices.

To normalize the model wave functions, we apply the standard boson commutation relation

\[
[Q(P), Q^\dagger(P')] = (2\pi)^3 2P^0 \delta^3(P - P')
\]

(28) 

to rest mesons. We obtain

\[
\langle Q | Q \rangle = (2\pi)^3 2E\delta^3(0).
\]

(29) 

Then we find the normalization relation of RPA type

\[
2E = \sum_{s_1, s_2} \int \frac{d^3k}{(2\pi)^3} \left\{ |x(k, s_1, s_2)|^2 - |y(k, s_1, s_2)|^2 \right\}.
\]

(30) 

Again with Dirac kets and bras, its form is

\[
2E = \langle x | x \rangle - \langle y | y \rangle.
\]

(31) 

With the normalized meson wave functions, one can calculate the static prop-
erties of mesons. The \( \pi \) weak decay constant is defined as
\[
if_\pi q^\mu \delta_{ij} = \langle 0 \mid \bar{\Psi}(0) \gamma^\mu \gamma_5 \frac{1}{2} \tau_i \Psi(0) \mid \pi_j(q) \rangle. \tag{32}
\]
For pions at rest, the above equation becomes
\[
if_\pi m_\pi \delta_{ij} = \langle 0 \mid \bar{\Psi}(0) \gamma^0 \gamma_5 \frac{1}{2} \tau_i \Psi(0) \mid Q_{\pi j} \rangle. \tag{33}
\]
In the non-relativistic limit, we obtain
\[
if_\pi m_\pi = \langle \mathbf{r} = 0, S = 0, M_S = 0 \mid x \rangle + \langle \mathbf{r} = 0, S = 0, M_S = 0 \mid y \rangle, \tag{34}
\]
where \( \langle \mathbf{r}, S = 0, M_S = 0 \mid x \rangle \) and \( \langle \mathbf{r}, S = 0, M_S = 0 \mid y \rangle \) are the coordinate wave functions in \( x \)- and \( y \)- channel of the \( \pi \) meson with its total spin \( S = 0, M_S = 0 \). The \( \rho \) electro-magnetic decay constant is defined as
\[
\frac{m_\rho^2}{f_\rho} \epsilon_k^\mu \delta_{ij} = \langle 0 \mid \bar{\Psi}(0) \gamma^\mu \frac{1}{2} \tau_i \Psi(0) \mid \rho_{jk}(q) \rangle, \tag{35}
\]
where \( i, j \) are the isospin indices, and \( k \) is the \( \rho \) meson polarization index. For static \( \rho \) mesons
\[
\frac{m_\rho^2}{f_\rho} \delta_{ij} = \langle 0 \mid \bar{\Psi}(0) \gamma^k \frac{1}{2} \tau_i \Psi(0) \mid Q_{\rho jk} \rangle. \tag{36}
\]
In the non-relativistic limit, one obtains
\[
\frac{m_\rho^2}{f_\rho} = \langle \mathbf{r} = 0, S = 1, M_S \mid x \rangle + \langle \mathbf{r} = 0, S = 1, M_S \mid y \rangle. \tag{37}
\]

3 \( \pi \) and \( \rho \) meson Properties

Following the quark potential model [1], we choose the effective quark interaction \( H_4 \) as a vector interaction plus a scalar confinement interaction
\[
H_4 = \frac{1}{2} \int d^3x d^3y K_s(\mathbf{x} - \mathbf{y}) \bar{\Psi}(\mathbf{x}) \psi(\mathbf{x}) \bar{\Psi}(\mathbf{y}) \bar{\Psi}(\mathbf{y}) \\
+ \frac{1}{2} \int d^3x d^3y K_v(\mathbf{x} - \mathbf{y}) \bar{\Psi}(\mathbf{x}) \gamma^\mu \frac{\lambda^c}{2} \Psi(\mathbf{x}) \bar{\Psi}(\mathbf{y}) \gamma_\mu \frac{\lambda^c}{2} \Psi(\mathbf{y}). \tag{38}
\]
According to lattice calculation, the scalar confinement kernel is linear in $r$

$$K_s(r) = br.$$  \hfill (39)

The vector kernel is taken from the one-gluon exchange plus a constant

$$K_v(r) = c + \frac{\alpha_s(r)}{r},$$  \hfill (40)

where $\alpha_s(r)$ is the QCD running coupling constant.

In our analysis, we use the simple harmonic oscillator wave functions for pions and rho mesons as a first approximation. In this way the underlying physics is emphasized.

$$x(k, s_1, s_2) = X \Psi_{000}(k) \left( \frac{1}{2} s_1 \frac{1}{2} s_2 \right) |SM_S\rangle,$$  \hfill (41)

$$y(k, s_1, s_2) = Y \Psi_{000}(k) \left( \frac{1}{2} s_1 \frac{1}{2} s_2 \right) |SM_S\rangle,$$  \hfill (42)

where $S, M_S$ are the spin quantum numbers of the meson, and $\Psi_{000}(k)$ is the ground state wave function of harmonic oscillator in momentum space

$$\Psi_{000}(k) = \frac{1}{(\sqrt{\pi}\beta)^{3/2}} e^{-k^2/2\beta^2}.$$  \hfill (43)

In the non-relativistic limit, the two potentials $U$ and $V$ are

$$U = br + \left( c + \frac{\alpha_s}{r} \right) F_1 \cdot F_2,$$  \hfill (44)

$$V = c \sigma_1 \cdot \sigma_2 F_1 \cdot F_2,$$  \hfill (45)

where

$$F_i = \begin{cases} \frac{\lambda_i}{2} & \text{for quarks,} \\ \frac{\lambda_i'}{2} = -\frac{\lambda_i}{2} & \text{for antiquarks.} \end{cases}$$  \hfill (46)

Here, one must be careful with the one-gluon exchange interaction $\frac{\alpha_s}{r}$. It arises from the covariant form $\frac{4\pi\alpha_s}{Q^2}$ in momentum space, where $Q = k - k'$ is the 4-momentum of the exchanged gluon. For potential $U$, $k = (k^0, \bm{k})$ and $k' = (k'^0, \bm{k'})$ are the 4-momenta of incoming and outgoing quarks respectively. Thus in non-relativistic limit, $\frac{\alpha_s}{r} \approx \frac{4\pi}{\beta^2}$ which is a Coulomb potential in coordinate
space. However for coupling potential $V$, $k = (k^0, \mathbf{k})$ and $-k' = (k^0, -\mathbf{k}')$ are the 4-momenta of $q\bar{q}$ quark pair which were created or annihilated on the vacuum. So in non-relativistic limit, $\frac{4\pi}{Q^2} \approx \frac{4\pi}{(2mc)^2} \approx 0$. The term $\frac{\alpha_s}{r}$ is suppressed in $V$.

It is well known in NJL model, the chiral symmetry can be broken dynamically with a nonzero vacuum condensate coming from quark interaction. The current quarks gain dynamical masses and become constituent quarks which are the valence quarks in potential model. The Goldstone boson pions are still massless. To account for this dynamical chiral symmetry breaking in our model, the coupling potential $V$ must be large. This is why we introduce a constant term into the vector kernel. In this way, our model differs from the ordinary potential model (see ref. [1]) which had the constant term in the scalar interaction. In an ordinary potential model, this change will cause no difference (the potential $U$ is not changed). But in this extended potential model, the coupling potential $V$ can be made strong enough to give $\pi$ mesons very low masses.

Inserting the simple model wave functions (41) and (42) into Eq. (25) with the potentials given by (44) and (45), we obtain an eigen equation of a $2 \times 2$ matrix

$$
\begin{pmatrix}
A & B \\
-B & -A
\end{pmatrix}
\begin{pmatrix}
X \\
Y
\end{pmatrix}
= E
\begin{pmatrix}
X \\
Y
\end{pmatrix},
$$

(47)

where

$$
A = 2\langle \sqrt{\mathbf{k}^2 + m_c^2} \rangle + \frac{2b}{\sqrt{\pi} \beta} - \frac{4c}{3} - \frac{4}{3} \left\langle \frac{\alpha_s(r)}{r} \right\rangle,
$$

(48)

$$
B = -\frac{4c}{3} \langle \mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2 \rangle.
$$

(49)

The energy $E$ can be easily obtained

$$
E = \sqrt{A^2 - B^2}.
$$

(50)

The wave function parameter $\beta$ is determined by the minimum of the energy

$$
\frac{\partial E}{\partial \beta} = 0.
$$

(51)
Fig. 1. Comparison of the integral \( \int_0^\infty t^2 dt \sqrt{t^2 + x^2} \exp(-t^2) \) (solid line) in the average quark kinetic energy with \( \frac{1}{2} \sqrt{1 + \frac{\pi}{4} x^2} \) (dashed line) used in Eq. (53).

Since \( B \) is a constant, we get

\[
\frac{\partial A}{\partial \beta} = 0. \tag{52}
\]

We approximate the average quark kinetic energy by the following expression.

\[
\langle K \rangle_{mc} = \langle \sqrt{k^2 + m^2} \rangle \approx \sqrt{\frac{4}{\pi} \beta^2 + m^2}. \tag{53}
\]

Numerically the deviation is very small and less than 5% (See Fig. 1).

The current quark mass \( m_0 \) can be obtained from the partial conservation of axial-vector current (PCAC). Consider the axial-vector current

\[
A^\mu_{ud}(x) = \bar{u}(x) \gamma^\mu \gamma_5 d(x). \tag{54}
\]

According to PCAC, its divergence is

\[
\partial_\mu A^\mu_{ud}(x) = (m_u + m_d) \bar{u}(x) i \gamma_5 d(x), \tag{55}
\]
where $m_u$ and $m_d$ are current masses of up and down quarks. Let us calculate the matrix element $\langle 0 \mid \partial_\mu A_{\mu ud}(x) \mid \pi^+ \rangle$ for $\pi^+$ meson with momentum $q_\pi$. One obtains

$$
\langle 0 \mid \partial_\mu A_{\mu ud}(x) \mid \pi^+, q_\pi \rangle = -iq_\pi^\mu \langle 0 \mid A_{\mu ud}(x) \mid \pi^+, q_\pi \rangle.
$$

From right hand side of PCAC relation (55), one also has

$$
\langle 0 \mid \partial_\mu A_{\mu ud}(x) \mid \pi^+ \rangle = (m_u + m_d) \langle 0 \mid \bar{u}(x)i\gamma_5 d(x) \mid \pi^+ \rangle.
$$

Both of the matrix elements can be easily calculated using the model wave functions for a static $\pi^+$ meson. We obtain

$$
2m_0 \equiv m_u + m_d = \frac{X_\pi + Y_\pi}{X_\pi - Y_\pi} m_\pi.
$$

Before the numerical calculation, one can make some qualitative analysis of the properties of $\pi$ and $\rho$ mesons in the model. Let

$$
u = X + Y, \quad \nu = X - Y.
$$

The matrix equation (47) can be written for $u, v$

$$
(A - B)v = Eu,
$$

$$
(A + B)u = Ev.
$$

Also the normalization relation (30) becomes

$$
X^2 - Y^2 = uv = 2E.
$$

First, let us consider the $\pi$ meson which is a Goldstone boson. We have

$$
\left(2A_0 - \frac{16c}{3}\right) v_\pi = m_\pi u_\pi,
$$

$$
\left(2A_0 + \frac{8c}{3}\right) u_\pi = m_\pi v_\pi,
$$

where

$$
A_0 = \sqrt{\frac{4}{\pi} \beta^2 + m_c^2 + \frac{b}{\sqrt{\pi} \beta} - \frac{2}{3} \left\langle \frac{\alpha_s(r)}{r} \right\rangle}.
$$
If the current quark mass $m_0 = 0$, chiral symmetry is a strict QCD symmetry. From the Goldstone theorem, $\pi$ meson should be massless. As an effective theory of QCD, the $\pi$ meson mass in our model should also be zero. Assuming $m_c$ is the constituent quark mass in the limit of $m_0 = 0$, we have

$$2 A_0|_{m_c} - \frac{16c}{3} = 0.$$  \tag{67}$$

In the real world, current quarks have small nonzero masses. So does the $\pi$ meson. As the average mass of current quarks (u and d) slightly changes to $m_0 \neq 0$, the constituent quark mass will also change a little bit to $m_c + \Delta m_c$ with $\Delta m_c \ll m_c$. So we have

$$A_0|_{m_c+\Delta m_c} \approx A_0|_{m_c} \equiv A_0$$  \tag{68}$$

and

$$A_0|_{m_c+\Delta m_c} - A_0|_{m_c} \approx \frac{\partial \langle K \rangle_{m_c}}{\partial m_c} \cdot \Delta m_c = \frac{m_c}{\langle K \rangle_{m_c}} \cdot \Delta m_c.$$  \tag{69}$$

Eqs. (64) and (65) become

$$2 \frac{\Delta m_c m_c}{\langle K \rangle_{m_c}} v_\pi = m_\pi u_\pi,$$  \tag{70}$$
$$3 A_0 u_\pi = m_\pi v_\pi.$$  \tag{71}$$

We obtain

$$m_\pi = \sqrt{\frac{6 \Delta m_c m_c A_0}{\langle K \rangle_{m_c}}},$$  \tag{72}$$
$$u_\pi = \frac{m_\pi}{3 A_0},$$  \tag{73}$$

v_\pi = \frac{m_\pi}{3 A_0}.$$

With the normalization $2m_\pi = u_\pi v_\pi$, we obtain

$$u_\pi = \sqrt{\frac{2}{3} \frac{m_\pi}{\sqrt{A_0}}},$$  \tag{74}$$
$$v_\pi = \sqrt{\frac{6 A_0}{}}.$$  \tag{75}$$

From PCAC (eq. (58)), the mass of current quark is related to $\Delta m_c$ as

$$m_0 = \frac{m_c}{\langle K \rangle_{m_c}} \Delta m_c.$$  \tag{76}$$
Next we can calculate the weak decay constant of $\pi$ meson to obtain

$$f_\pi = \sqrt{\frac{2N_c}{3A_0}} \left( \frac{\beta}{\sqrt{\pi}} \right)^3. \quad (77)$$

Eq. (73) means $u_\pi/v_\pi \approx 0$ and $Y_\pi \approx -X_\pi$, which shows the importance of the contribution to the $\pi$ meson from the small component of $y$-channel.

For the $\rho$ meson, with the help of Eq. (67), the equations are

$$2A_0v_\rho = m_\rho u_\rho, \quad (78)$$
$$A_0u_\rho = m_\rho v_\rho. \quad (79)$$

One simply obtains

$$m_\rho = \sqrt{2}A_0, \quad (80)$$
$$\frac{u_\rho}{v_\rho} = \sqrt{2}. \quad (81)$$

With the normalization $2m_\rho = u_\rho v_\rho$,

$$u_\rho = 2\sqrt{A_0}, \quad (82)$$
$$v_\rho = \sqrt{2A_0}. \quad (83)$$

Next the electromagnetic decay constant of $\rho$ meson can be calculated as

$$f_\rho^{-1} = \frac{1}{m_\rho} \sqrt{\frac{2N_c}{A_0}} \left( \frac{\beta}{\sqrt{\pi}} \right)^3. \quad (84)$$

A relation among masses of $\pi$, $\rho$ mesons and current quark mass $m_0$ can be easily obtained

$$m_\pi^2 = 6m_0A_0 = 3\sqrt{2}m_0m_\rho. \quad (85)$$

From the experimental $\rho$ meson mass $m_\rho = 770$ MeV, one can estimate that $A_0 \approx m_\rho/\sqrt{2} \approx 540$ MeV. Then we can estimate the current quark mass to be $m_0 \approx 6$ MeV from the physical pion mass $m_\pi = 140$ MeV.
Another relation about \(\pi\) and \(\rho\) decay constants can also be easily obtained from Eqs. (77) and (84),

\[
f_\pi f_\rho = \frac{1}{\sqrt{3}} m_\rho.
\]  

(86)

The KSRF relation [8,9] gives

\[
f_\pi f_\rho = \frac{1}{\sqrt{2}} m_\rho,
\]

(87)

while the experiment value is \(m_\rho/(f_\pi f_\rho) = 1.62\). In our approach \(Y_\rho/X_\rho = 1/(3 + \sqrt{2}) = 0.17\) from Eq. (81). This means the contribution to the rho meson from y-channel is rather small. If the small component is completely neglected, i.e., \(Y_\rho = 0\), we may obtain \(m_\rho = 3/2 A_0\) and \(X_\rho = \sqrt{2} m_\rho\). Then

\[
f_\rho^{-1} = \frac{1}{m_\rho} \left[ \frac{4N_c}{3A_0} \left( \frac{\beta}{\sqrt{\pi}} \right)^3 \right],
\]

(88)

thus

\[
f_\pi f_\rho = \frac{1}{\sqrt{2}} m_\rho,
\]

(89)

which is exactly the KSRF relation. The experimental value is between our result and KSRF relation.

In numerical calculation we parametrize the running coupling constant as

\[
\alpha_s(Q^2) = \sum_k \alpha_k \exp(-Q^2/4\gamma_k^2)
\]

\[
= 0.25 \exp(-Q^2) + 0.15 \exp(-Q^2/10) + 0.20 \exp(-Q^2/1000)
\]

(90)

as in Ref. [1] (with \(Q\) in GeV). Also the constituent quark mass \(m_c\) is fixed to be \(m_c = 220\) MeV. The results are listed in Table 1. First (Set 1), we choose the confinement parameter \(b = 0.18\)GeV\(^2\) according to lattice calculation. The \(\rho\) meson mass is somehow larger. Next (Set 2), we adjust the confinement parameter slightly to \(b = 0.15\)GeV\(^2\). Both of pion and rho meson masses agree with the experimental values well. The current and constituent quark masses are also reasonable in the ranges of theoretical and experimental estimations.

The decay constants \(f_\pi\) and \(f_\rho\) deviate from the experimental data by a factor 2. This is not very surprising. In the derivation of the effective Hamiltonian
Table 1
Numerical results for π and ρ mesons. Experimental data are taken from ref. [18].

|   | b (GeV$^2$) | c (MeV) | β (MeV) | $m_0$ (MeV) | $m_π$ (MeV) | $m_ρ$ (MeV) |
|---|-------------|---------|---------|-------------|-------------|-------------|
| Exp. |             |         |         |             |             |             |
| Set 1 | 0.18 | 220 | 391 | 6.5 | 152 | 843 |
| Set 2 | 0.15 | 204 | 364 | 5.6 | 136 | 779 |

|   | $\frac{\gamma_π}{X_π}$ | $\frac{\gamma_ρ}{X_ρ}$ | $f_π$ (MeV) | $f_ρ$ (MeV) | $Z$ |
|---|-----------------|-----------------|-------------|-------------|-----|
| Exp. |             |         |             |             |     |
| Set 1 | −0.842 | 0.169 | 190(93) | 2.6(5.3) | 0.70 |
| Set 2 | −0.847 | 0.169 | 178(89) | 2.5(5.1) | 0.71 |

(which we skip in this work), the current quark field $\Psi$ is renormalized into constituent quark field. This means that the constituent quark may include the contributions from configurations other than the single quark, like $qq\bar{q}$, $qg$ etc as indicated through the deep inelastic scattering experiments. For example, we may assume the normalized constituent quark field $\Psi_N$

$$\Psi_N = Z\Psi + \text{contributions from } q\bar{q}, qg, ...$$

(91)

Here we have introduced the factor $Z$. Roughly speaking, $|Z|^2$ is the probability of finding the "pure" current quark field in a complicated constituent quark. However, from the electro-weak Lagrangian we know that what really participates in the electro-weak interactions are the current quarks. In other words, only the "pure" current quark will contribute to the pion weak decay constant and rho meson electromagnetic decay constant. In our calculation we need two real current quarks to annihilate. So there will be a difference of factor $Z^2$. Decay constants $f_π$ and $f_ρ$ can then be fit to the experimental values if we set the additional normalization factor $Z \approx 0.7$ (In Table 1 the results are listed in parentheses).

4 Summary

The chiral symmetry is studied in an extended constituent quark potential model which includes the small component effect with RPA approximation. With some modification of quark interaction in potential model, $\pi$ meson is still a Goldstone boson. The small mass of $\pi$ meson can be connected to
nonzero current quark mass from PCAC. All other mesons such as the vector $\rho$ meson can also be studied in this model like in the ordinary constituent quark models. With the small component effect of $y$-channel contribution, the well-known KSRF relation of the $\pi$ and $\rho$ meson decay constants is reestablished in the model.

This extension to constituent quark potential model aims at improving the constituent quark model to comprise the small component, i.e., the $y$– channel quark excitations which are essential to $\pi$ (and K) mesons. To thoroughly investigate the effect of this extension, we need perform a full calculation of meson spectroscopy.

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References

[1] S. Godfrey, N. Isgur, Mesons in a relativized quark model with chromodynamics, Phys. Rev. D32 (1985) 189.

[2] F. J. Llanes-Estrada, S. R. Cotanch, Phys. Rev. Lett. 84 (2000) 1102.

[3] Y. Nambu, G. Jona-Lasinio, Phys. Rev. 122 (1961) 246.

[4] V. Bernard, R. Brockmann, M. Schaden, W. Weise, E. Werner, Nucl. Phys. A412 (1984) 349.

[5] S. Klimt, M. Lutz, U. Vogl, W. Weise, Generalized $SU(3)$ nambu–jona-lasinio model, Nucl. Phys. A516 (1990) 429.

[6] K. Langfeld, C. Kettner, H. Reinhardt, Nucl. Phys. A608 (1996) 331.

[7] W. Deng, X. Chen, D. LU, L. Yang, Extended quark potential model from random phase approximation, Commun. Theor. Phys. 38 (2002) 327.

[8] K. Kawarabayashi, M. Suzuki, Partially conserved axial-vector current and the decays of vector mesons, Phys. Rev. Lett. 16 (1966) 255.

[9] Riazuddin, Fayyazuddin, Algebra of current components and decay widths of $\rho$ and $K^*$ mesons, Phys. Rev. 147 (1966) 1071.

[10] K. G. Wilson, T. S. Walhout, A. Haringdranath, W.-M. Zhang, R. J. Perry, S. D. Glazek, Phys. Rev. D49 (1994) 6720.
[11] D. Robertson, E. S. Swanson, A. P. Szczepaniak, C.-R. Ji, S. R. Cotanch, Phys. Rev. D59 (1999) 074019.

[12] F. J. Llanes-Estrada, S. R. Cotanch, Nucl. Phys. A697 (2002) 303.

[13] G. ’t Hooft, A planar diagram theory for strong interaction, Nucl. Phys. B72 (1974) 461–473.

[14] E. Witten, Baryon in the $1/N$ expansion, Nucl. Phys. B160 (1979) 57–115.

[15] W. Deng, X. Chen, D. Lu, L. Yang, Meson structures from random phase approximation, Phys. Lett. B506 (2001) 85–92.

[16] P. Ring, P. Schuck, The Nuclear Many-Body Problem, Springer, Berlin, 1980.

[17] D. Gromes, Nucl. Phys. B131 (1977) 80.

[18] Particle Data Group, Review of particle physics, Euro. Phys. J. C15 (2000) 1.