Energetics of correlations in interacting systems

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A fundamental connection between thermodynamics and information theory arises from the fact that correlations exhibit an inherent work value. For noninteracting systems this translates to a work cost for establishing correlations. Here we investigate the relationship between work and correlations in the presence of interactions that cannot be controlled or removed. For such naturally coupled systems, which are correlated even in thermal equilibrium, we determine general strategies that can reduce the work cost of correlations, and illustrate these for a selection of exemplary physical systems.

I. INTRODUCTION

Quantum information (QI) and quantum thermodynamics (QT) can both be framed as resource theories [1]. Based on the fundamental laws of quantum physics, these theories describe the (minimal) resources needed to perform certain tasks of interest. In order to identify the relevant resources, one first determines which states and operations are freely available, taking into account practical limitations on physical operations. Within QT, thermal systems and energy preserving operations are considered to be “for free”, whereas systems out of equilibrium and operations that require external energy constitute resources [2, 3]. In QI, on the other hand, the paradigmatic task is efficient communication. In this context one assumes local operations and classical communication (LOCC) to be free, whereas entangled quantum systems are resources that enable tasks beyond the restrictions of LOCC [4, 5].

Both of these resource theories can be considered to be simplifications of a more general physical framework: In either case only the restrictions of one area are taken into account. However, especially in quantum systems limitations from both thermodynamics and information theory present themselves simultaneously, which has greatly stimulated investigations of the connection between QI and QT (see Refs. [6–8] for recent reviews). For instance, locality restrictions on the allowed operations can limit the efficiency of thermodynamic processes [9–14], while correlations can enhance the performance of thermodynamic tasks [10, 15–21], and may even change the natural direction of the heat flow [22, 23]. Conversely, a nonzero ambient temperature induces a nonzero entropy, which limits the capacity for establishing (quantum) correlations [24, 25]. To overcome the constraints of LOCC, QI tasks hence require a supply of thermodynamic resources in the form of free energy.

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Here, we aim to study the exchange between energy and correlations in the particularly transparent setting considered in Refs. [24, 25]: Given a collection of uncorrelated thermal states at the same temperature T, one is interested in determining the minimal energy W that is needed to create (quantum) correlations. For a bipartite system with access to an auxiliary thermal bath at temperature T, one finds the relation [15, 25–27]

\[ W \geq T \Delta S_{S_1S_2}, \tag{1} \]

where \( \Delta S_{S_1S_2} \) is the gain of correlations between the subsystems \( S_1 \) and \( S_2 \) as measured by the mutual information, and we note that we work in units such that \( \hbar = k_B = 1 \) throughout this paper. The expression in Eq. (1) represents a fundamental bound on the exchange between energy and correlations, if the subsystems \( S_1 \) and \( S_2 \) are not interacting, that is, if the systems Hamiltonian is of the form \( H_S = H_{S_1} + H_{S_2} \). In this work we relax this assumption, and explore how the relation in Eq. (1) is modified for interacting systems.

An important difference to previous results lies in the fact that thermal states of interacting Hamiltonians are generally already correlated, and may potentially even be entangled [28]. This naturally raises the question of whether the presence of interactions provides advantages for the generation of (additional) correlations. We answer this question affirmatively, by constructing explicit strategies to achieve \( W < T \Delta S_{S_1S_2} \) for some energy range in any finite-dimensional system with arbitrary interacting Hamiltonian. While these procedures can improve on the best protocols for non-interacting systems, they are not necessarily optimal in the sense that other protocols may exist that generate more correlations at the same energy cost. To complement this approach we therefore develop optimal strategies for two physically relevant cases: two interacting, fermionic or bosonic modes.

This paper is structured as follows. We first provide a short summary of the framework for this investigation in Sec. II. We then approach the problem of generating correlations in interacting systems in Sec. III, where we develop general strategies to use the energy...
contained in the interactions to improve upon the bound in Eq. (1), and explicitly demonstrate their applicability in a system of two qubits. In Sec. IV, we then turn to another finite-dimensional example, two interacting, fermionic modes. This system, restricted by superselection rules, is amenable to a numerical approach that we use to determine the optimal conversion of energy into correlations. Finally, in Sec. V, we study the generation of correlations in the infinite-dimensional system of two interacting, bosonic modes.

II. FRAMEWORK

We consider a bipartite system $S$, made up of subsystems $S_1$ and $S_2$, initially at thermal equilibrium at ambient temperature $T = 1/\beta$, described by a thermal state

$$\tau(\beta) = Z^{-1}(\beta) e^{-\beta H_S},$$

(2)

where $Z(\beta) = \text{Tr}(e^{-\beta H_S})$ is the partition function, and $H_S$ is the system Hamiltonian. We further assume the presence of an auxiliary heat bath $B$, that is, an arbitrarily large system in thermal equilibrium with $S$. The total Hamiltonian is $H = H_S + H_B$, and the initial state can be written as $\tau_{SB}(\beta) = \tau(\beta) \otimes \tau_B(\beta)$. In order to transform this equilibrium state, we consider arbitrary unitary operations $U_{SB}$ on $SB$. Since the joint systems $SB$ is closed, these unitaries correspond to the most general operations permissible in this situation. The average work cost of transforming the state of $S$ from $\tau(\beta)$ to a final state $\rho = \text{Tr}_B(U_{SB} \tau_{SB}(\beta) U_{SB}^\dagger)$ is given by,

$$W = \text{Tr} \left( H [U_{SB} \tau_{SB}(\beta) U_{SB}^\dagger - \tau_{SB}(\beta)] \right),$$

(3)

which corresponds to the total external energy input (see, e.g., Refs. [29, 30]). In Ref. [27] (see also Refs. [31–33] for the same result in related frameworks), it is shown that $W$ can be bounded by the (nonequilibrium) free energy difference,

$$W(\tau \to \rho) \geq \Delta F_S = F(\rho) - F(\tau),$$

(4)

where the free energy with respect to the reservoir at temperature $T$ is

$$F(\rho) = E(\rho) - TS(\rho).$$

(5)

Here, $E(\rho) = \text{Tr}(H_S \rho)$ is the average energy, and $S(\rho) = -\text{Tr}(\rho \ln(\rho))$ is the von Neumann entropy. Note that $F(\rho)$ depends only on the state of $S$ and the temperature of $B$. Equality in (4) can be obtained in a quasistatic process [31–33], in which case the work cost becomes minimal.

We now wish to invest some work $W$ to increase the correlations within the bipartite state of $S$ as much as possible. (Note that $B$ is only an auxiliary system and we do not wish to create correlations between $S$ and $B$.) In Refs. [24, 25] this problem was considered for noninteracting Hamiltonians, $H_S = H_{S_1} + H_{S_2}$. Here, we want to depart from this paradigm and consider an interacting Hamiltonian of the form

$$H_S = H_{S_1} + H_{S_2} + H_I.$$  

(6)

As discussed above, the work cost of transforming $\tau(\beta)$ to a final state $\rho$ satisfies, $W(\tau \to \rho) \geq \Delta F_S = F(\rho) - F(\tau)$, and equality can be achieved in a quasi-static process and with a sufficiently large bath [31–33]. The task is then to maximize the correlations of $\rho$ under the constraint $F(\rho) - F(\tau) \leq W$, where $W$ is the amount of available work.

Before continuing, note that the main ingredients of the investigations in Refs. [24, 25] are preserved:

(i) The initial state is in thermal equilibrium, and therefore, the energy cost of transforming $\tau(\beta)$ to any final state $\rho$ is nonnegative, $W \geq 0$.

(ii) We assume arbitrary (in particular, unitary) operations can be performed on $S$ and the auxiliary thermal bath, which allows obtaining fundamental bounds on the work cost of correlations.

We quantify the amount of correlations between the subsystems by the mutual information,

$$I_{S_1,S_2}(\rho) = S(\rho_{S_1}) + S(\rho_{S_2}) - S(\rho),$$

(7)

where $\rho_{S_1(S_2)} = \text{Tr}_{S_2(S_1)}(\rho)$ are the reduced states of the subsystems. The main quantity of interest throughout this paper will be the correlations gain,

$$\Delta I_{S_1,S_2} = I_{S_1,S_2}(\rho) - I_{S_1,S_2}(\tau).$$

(8)

That is, we take a point of view inspired by Landauer’s principle, and ask how many units of correlations $\Delta I_{S_1,S_2}$ can be newly generated (on top of the preexisting correlations) at the expense of one unit of energy. Note that, in the noninteracting case, $\Delta I_{S_1,S_2}$ and $I_{S_1,S_2}$ coincide, as the initial thermal state of the Hamiltonian $H_{S_1} + H_{S_2}$ is an uncorrelated product state. In the interacting case, $\Delta I_{S_1,S_2}$ and $I_{S_1,S_2}$ still arise from the same optimization procedure, but $I_{S_1,S_2} \geq \Delta I_{S_1,S_2}$. Since $\Delta I_{S_1,S_2}$ quantifies the amount of correlations generated through the investment of $W$, we first focus on $\Delta I_{S_1,S_2}$, establishing strategies to achieve $W < T \Delta I_{S_1,S_2}$ in Sec. III.

III. GENERAL CONSIDERATIONS

A. Work cost of generating correlations

Let us now relate the correlation gain $\Delta I_{S_1,S_2}$ to the minimal work cost $\Delta F_S$. Inserting the Hamiltonian from...
respectively. The density operators for such systems are

\[ \rho_{s_2} = \frac{1}{d_1} \left( I_{s_1} + \sum_{m=1}^{d_2-1} a_m \sigma_m^{s_1} \right) + \frac{1}{d_2} \left( I_{s_2} + \sum_{n=1}^{d_2-1} b_n \sigma_n^{s_2} \right), \]

and \( t_{mn} \) are subject to constraints arising from the positivity of \( \rho \). The reduced states are then immediately obtained as

\[ \rho_{s_1} = \frac{1}{d_1} \left( I_{s_1} + \sum_{m=1}^{d_2-1} a_m \sigma_m^{s_1} \right), \quad \rho_{s_2} = \frac{1}{d_2} \left( I_{s_2} + \sum_{n=1}^{d_2-1} b_n \sigma_n^{s_2} \right), \]

with real coefficients \( c_{mn} \). Any terms of the form \( I_{s_1} \otimes \sigma_m^{s_2} \) and \( \sigma_m^{s_1} \otimes I_{s_2} \) that may appear in such a decomposition of \( H \) can be absorbed into the local Hamiltonians \( H_{s_i} \).

Returning to the relation of Eq. (10), notice that the interactions allow surpassing the bound in Eq. (12) whenever \( H_{s_i} = 0 \), respectively, whereas the interaction term \( \text{Tr}(H_{s_i} \rho - \tau) \) depends only on the correlation tensor \( c_{mn} \). With this we can formulate two complementary strategies. However, it is important to keep in mind that any choice of the coefficients \( c_{mn} \) and \( a_m, b_n \) is subject to the positivity constraint \( \rho \geq 0 \).

First, we focus on the local terms \( \Delta \tilde{F}_{s_i} \). Defining the local Gibbs states as \( \gamma_{s_i} = Z_{s_i}^{-1} e^{-\beta H_{s_i}} \), which are generically different from the local initial states \( \gamma_{s_i} \), it is useful to rewrite \( \Delta \tilde{F}_{s_i} \) as,

\[ \beta \Delta \tilde{F}_{s_i} = \beta \left( F(\rho_{s_i}) - F(\gamma_{s_i}) \right) - \beta \left( F(\rho_{s_i}) - F(\gamma_{s_i}) \right) = S(\rho || \gamma_{s_i}) - S(\tau_{s_i} || \gamma_{s_i}), \]

where \( S(\rho || \tau) = -S(\rho) - \text{Tr}(\rho \ln \tau) \) is the relative entropy. Here, we have used that \( [F(\rho) - F(\tau(\beta))] = S(\rho || \tau(\beta)) \), which can easily be shown using Eqs. (2) and (5). Since \( S(\rho || \tau) \) is a measure of distance between two quantum states, the quantities \( \Delta \tilde{F}_{s_i} \) are negative whenever the final reduced states \( \rho_{s_i} \) are closer to the local Gibbs states \( \gamma_{s_i} \) than the initial state marginals \( \tau_{s_i} \). This provides a simple strategy to minimize \( \Delta \tilde{F}_{s_i} \): The Bloch coefficients \( a_m \) and \( b_n \) of the final state \( \rho \) should be chosen as close as possible to \( a_{m}^{(\gamma)} \) and \( b_{n}^{(\gamma)} \), respectively, where

\[ a_{m}^{(\gamma)} = \frac{d_2}{d_2 - 1} \text{Tr}(\gamma_{s_1} \sigma_m^{s_2}) \quad \text{and} \quad b_{n}^{(\gamma)} = \frac{d_2}{d_2 - 1} \text{Tr}(\gamma_{s_2} \sigma_n^{s_2}). \]

This strategy ensures that \( \Delta \tilde{F}_{s_i} < 0 \).

The second strategy entails the minimization of the term \( \text{Tr}(H_{s_i} \rho - \tau) \). Using Eqs. (13) and (15), we can express it in terms of the correlation tensors \( c_{mn}^{(\rho)} \) and
\( c^{(\tau)}_{mn} \) of \( \rho \) and \( \tau \), respectively, obtaining
\[
\text{Tr}(H_i[\rho - \tau]) = \sum_{m=1}^{d_i^2 - 1} \sum_{n=1}^{d_i^2 - 1} (c^{(\rho)}_{mn} - c^{(\tau)}_{mn}) \epsilon_{mn}.
\] (17)

This relation has a clear geometrical interpretation. Mapping \( \epsilon_{mn} \), \( c^{(\tau)}_{mn} \), and \( \epsilon_{mn} \) to vectors \( c^{(\rho)} \), \( c^{(\tau)} \), and \( \epsilon \) in a Euclidean vector space of dimension \((d_i^2 - 1)(d_i^2 - 1)\), the condition of Eq. (17) becomes
\[
\text{Tr}(H_i[\rho - \tau]) = (c^{(\rho)} - c^{(\tau)}) \cdot \epsilon.
\] (18)

To minimize the expression in (18) it is hence desirable to select the vector \((c^{(\rho)} - c^{(\tau)})\) to be as antiparallel as possible to \( \epsilon \).

The considerations discussed in this section hence provide two complementary strategies to obtain \( \beta W < \Delta T_{s_1,s_2} \), as desired. In general, the choices of \( a_m^{(\rho)} \), \( b_m^{(\rho)} \), and \( c_m^{(\rho)} \) are modified to \( \tau \), and \( \epsilon \) is due to the positivity constraint, \( \rho \geq 0 \) (and of course also by the amount of available work, \( W \)). In the next section we illustrate possible issues with the positivity of \( \rho \) in more detail for a particular example of two interacting qubits.

C. Improved generation of correlations for two qubits

We consider a system of two qubits, coupled by the Hamiltonian
\[
H_s = \omega (\sigma_z^{s_1} + \sigma_z^{s_2}) + \epsilon \sigma_z^{s_1} \otimes \sigma_z^{s_2},
\] (19)

where \( \omega \geq 0 \) and \( \epsilon \in \mathbb{R} \) can take either sign. In this simple example, the presence of the interaction Hamiltonian \( H_i = \epsilon \sigma_z^{s_1} \otimes \sigma_z^{s_2} \) does not change the eigenstates of \( H_s \), but the eigenvalues of the noninteracting system are modified to \( (\epsilon \pm 2\omega) \) and \(-\epsilon \) (twice degenerate). The initial thermal state \( \tau(\beta) = e^{-\beta H_s} / Z \) is hence of the form
\[
\tau(\beta) = Z^{-1} \text{diag}(e^{-\beta(\epsilon + 2\omega)}, e^{\beta\epsilon}, e^{\beta\epsilon}, e^{-\beta(\epsilon - 2\omega)})
\] (20)

with \( Z = \text{Tr}(e^{-\beta H_s}) \geq 0 \). The nonzero coefficients of the Bloch decomposition of \( \tau(\beta) \) are
\[
a_z^{(\tau)} = b_z^{(\tau)} = -2Z^{-1} e^{-\beta\epsilon} \sinh(2\beta\omega) < 0,
\] (21a)
\[
c_z^{(\tau)} = 1 - \frac{4e^{\beta\epsilon}}{Z}.
\] (21b)

To correlate the system, we apply a two-step protocol based on the strategies discussed in Sec. IIIB. In the first phase of the protocol, step I, we aim to minimize the term \( \text{Tr}(H_s[\rho - \tau]) \). To do this, we transform the state \( \tau \) to \( \rho_1 \), such that the local Bloch vector components remain invariant, \( a_z^{(\rho_1)} = b_z^{(\rho_1)} = a_z^{(\tau)} \), while the (nonzero) correlation tensor coefficient is mapped to
\[
c_z^{(\rho_1)} = c_z^{(\tau)} - \text{sgn}(\epsilon) a_z,
\] (22)

for \( \alpha \geq 0 \). With this, one finds \( \text{Tr}(H_s[\rho_1 - \tau]) = -|\epsilon| \alpha_1 \) and from Eq. (10) we obtain
\[
W_1 = T \Delta T_{s_1,s_2} - |\epsilon| \alpha_1,
\] (23)

where we assumed that the process is quasistatic, so that \( W = \Delta F_{s_1} \). (Note that the same assumption is made also later in step II.) The correlations are hence generated at a work cost that is lower than in the noninteracting case, \( W_1 \leq T \Delta T_{s_1,s_2} \). However, it is crucial to note that the transformation in Eq. (22) is limited by the positivity constraint, \( \rho_1 \geq 0 \), requiring \( 2|a_z^{(\tau)}| - 1 \leq c_z^{(\rho_1)} \leq 1 \). Depending on the sign of the interaction term, one of these bounds is reached, when enough energy is supplied. That is, \( c_z^{(\rho_1)} \) eventually tends towards either \( c_z^{(\rho_1)} = 2|a_z^{(\tau)}| - 1 \) or \( c_z^{(\rho_1)} = 1 \) for \( \epsilon > 0 \) or \( \epsilon < 0 \), respectively.

If more energy is available than is needed to saturate the positivity constraint in step I, we may employ the complementary strategy discussed in Sec. IIIB in step II, the second phase of the protocol. Now, we keep the correlation tensor fixed while changing the local Bloch vector components to minimize \( \Delta F_{s_1} \). This entails moving the marginals closer to the states \( \gamma_{s_i} \) that are locally thermal with respect to \( H_{s_i} \). These local Gibbs states are here given by
\[
\gamma_{s_i} = \frac{e^{-\beta H_{s_i}}}{Z_{s_i}} = \frac{1}{2} (I_2 - \tanh(\beta\omega) \sigma_z^{s_i}),
\] (24)

with \( a_z^{(\gamma)} = -\tanh(\beta\omega) < 0 \). We hence map \( \rho_1 \) to the state \( \rho_\alpha \) with Bloch vector components given by
\[
a_z^{(\rho_\alpha)} = (1 - \alpha) a_z^{(\tau)} + \alpha a_z^{(\gamma)},
\] (25)

where \( 0 \leq \alpha \leq 1 \). Again, the positivity constraint \( \rho_\alpha \geq 0 \) must still be taken into account. For \( \epsilon < 0 \) we find that the full range of \( \alpha \) is compatible with the positivity of \( \rho_\alpha \). The work cost of step II is given by \( W_\alpha = T \Delta T_{s_1,s_2} + \Delta F_{s_1} + \Delta F_{s_2} \), and, as illustrated in Fig. 1, we indeed find that \( \Delta F_{s_i} \leq 0 \) for all values of \( T \geq 0, 0 \leq \alpha \leq 1 \), and \( \epsilon < 0 \).

For \( \epsilon > 0 \), on the other hand, the positivity constraints require that \( |a_z^{(\rho_\alpha)}| \leq |a - z^{(\tau)}| \). Since \( a_z^{(\tau)} < 0 \) and \( a_z^{(\gamma)} = -\tanh(\beta\omega) < 0 \), Eq. (25) yields \( |a_z^{(\rho_\alpha)}| = (1 - \alpha) |a_z^{(\tau)}| + \alpha \tanh(\beta\omega) \geq |a_z^{(\gamma)}| \). Unfortunately, since \( |a_z^{(\gamma)}| = \sinh(2\beta\omega)/\cosh(2\beta\omega) + e^{-\beta\epsilon} \leq \tanh(\beta\omega) \), one finds that \( |a_z^{(\rho_\alpha)}| \geq |a - z^{(\tau)}| \), that is, the positivity constraint does not allow for step II of the protocol to be carried out for \( \epsilon > 0 \).

In addition to the strategies discussed here, the states obtained after steps I and II may be further correlated until the maximal value of correlation is reached. How-
advantage over the noninteracting case arises when $\Delta \tilde{F}_S$, from Eq. (11) becomes negative. $\Delta \tilde{F}_S$ is plotted here against the temperature $T$ in units of $\omega$ (recall that we use units where $\hbar = k_B = 1$) for $\alpha_n = 0.5$, and the different curves correspond to values of $\epsilon$ (also in units of $\omega$) from $\epsilon = 0$ (top) to $\epsilon = -1$ (bottom) in steps of 0.1. The advantage increases with increasing coupling strength $\epsilon$, but does not monotonically decrease with the temperature. Instead, the advantage becomes maximal at a finite temperature. Although curves are only shown for a fixed value $\alpha_n = 0.5$, we have checked that other values yield analogous behaviour and the advantage increases monotonically with $\alpha_n$.

However, the work cost per newly generated unit of correlation beyond this point may be the same, or even higher than in the noninteracting case. We shall shed light on this possibility in the next sections, by studying in detail protocols to correlate bipartite systems of fermions and bosons.

IV. TWO FERMIONIC MODES

In this section, we consider the two systems $S_1$ and $S_2$, which are to be correlated, to be two fermionic modes with ladder operators $b_{\lambda}$ and $b_{\lambda}^\dagger$, respectively. Such fermionic systems have been well studied in the context of quantum information processing, see, e.g., Refs. [36–38], but we shall briefly review some key features. The annihilation and creation operators satisfy the usual anticommutation relations $\{b_m, b_{\lambda}^\dagger\} = \delta_{mn}$ and $\{b_m^\dagger, b_n\} = 0$. While the Pauli exclusion principle limits the dimension of the corresponding two-mode Fock space to 4, the anticommutation relations nonetheless imply a different subsystem structure as compared to a two-qubit Hilbert space [39]. Despite this inequivalence of qubits and fermionic modes, the marginals and correlation measures for the fermionic system are well-defined when imposing a superselection rule that forbids superpositions of even and odd numbers of fermions [40, 41].

The fermionic system is hence of interest for the following reasons: First, the anticommutation relations and the restrictions of the superselection rule provide a qualitative difference to the qubit case, which makes for an interesting comparison. Second, the low-dimensional Hilbert space is amenable to a numerical treatment, allowing a rather general approach to optimal protocols for the generation of correlations. Third, fermionic fields form a conceptually fundamental ingredient in the current view of matter in the universe in terms of relativistic quantum field theory.

A. Hamiltonian and initial thermal state

Let us now turn to the specific system Hamiltonian that we consider in this section, $H = H_{S_1} + H_{S_2} + H_I$. For the noninteracting part, we consider the standard Hamiltonian for two modes of the same frequency $\omega$, i.e.,

$$H_{S_1} + H_{S_2} = \omega \left( b_1^\dagger b_1 + b_2^\dagger b_2 \right).$$

(26)

For the interaction between the modes, we will employ the most general two-mode coupling term that is quadratic in the mode operators, given by

$$H_I = H_{\text{even}} + H_{\text{odd}}$$

$$= \epsilon_{\text{even}} (b_1 b_2 + b_2^\dagger b_1^\dagger) + \epsilon_{\text{odd}} (b_1^\dagger b_2 + b_2 b_1),$$

(27)

where $H_{\text{even}}$ couples only the states $\| 0 \rangle$ and $\| 1_1 \rangle \| 1_2 \rangle = b_1^\dagger b_1^\dagger \| 0 \rangle \| 0 \rangle$ in the even subspace, while $H_{\text{odd}}$ acts in the odd subspace spanned by $\| 1_1 \rangle = b_1^\dagger \| 0 \rangle$ and $\| 1_2 \rangle = b_2^\dagger \| 0 \rangle$. Here $\| 0 \rangle$ is the vacuum state satisfying $b_i^\dagger \| 0 \rangle = 0 \forall i$, and the double-lined ket notation indicates the antisymmetrized tensor product for the excited states, i.e., $\| 1_1 \rangle \| 1_2 \rangle = \| 1_1 \rangle \\wedge \| 1_2 \rangle = -\| 1_2 \rangle \| 1_1 \rangle$ (see, e.g., Ref. [39] for more information).

The thermal state $\tau(\beta)$ of $H$ can be computed straightforwardly. The eigenvalues of $H$ read,

$$\lambda_{1,4} = \omega \pm \sqrt{\omega^2 + \epsilon_{\text{even}}^2},$$

(28a)

$$\lambda_{2,3} = \omega \pm \epsilon_{\text{odd}},$$

(28b)

where the labels 3, 4 refer to the negative relative sign.
and the corresponding eigenstates are given by
\[
\| \lambda_{1,4} \rangle \rangle = \frac{1}{\sqrt{\epsilon_{\text{even}} + \lambda_{1,4}^2}} \left( \epsilon_{\text{even}} \| 0 \rangle \rangle - \lambda_{1,4} \| 1_2 \rangle \rangle \right),
\]
\[
\| \lambda_{2,3} \rangle \rangle = \frac{1}{\sqrt{2}} \left( \| 1_2 \rangle \rangle \pm \| 1_1 \rangle \rangle \right).
\]
Then, \( \tau(\beta) \) can be written as,
\[
\tau(\beta) = Z^{-1}(\beta) \sum_i e^{-\beta \lambda_i} \| \lambda_i \rangle \rangle \langle\langle \lambda_i \|,
\]
where the partition function is \( Z(\beta) = \sum_i e^{-\beta \lambda_i} \). It is important to note that \( \tau(\beta) \) already contains correlations, which are computed in detail in the Appendix.

### B. Generation of Correlations

We now consider the task of correlating \( \tau(\beta) \) further. The simple structure of the system (an only four-dimensional Hilbert space that is further restricted by superselection rules) allows us to consider the most general protocols beyond the strategies discussed in Sec. III B. That is, given some available work \( W \), we consider the possibility to transform \( \tau \) to any state \( \rho \), provided \( \Delta F_\rho(\tau \to \rho) \leq W \) is satisfied. In order to maximize the created correlations, \( \Delta I_{s_1,s_2} \), we conveniently parametrize the final state \( \rho \), and numerically optimize its mutual information \( I_{s_1,s_2}(\rho) \) under the constraint of a maximally available free energy.

Since the final state needs to respect the superselection rule that forbids superpositions of even and odd numbers of fermions \cite{40}, the four-dimensional Fock space splits into two two-dimensional spaces. An arbitrary two-mode final state may therefore be written as a convex combination of two density operators, \( \rho_{\text{even}} \) and \( \rho_{\text{odd}} \), corresponding to the subspaces of even and odd fermion numbers, respectively. We hence write
\[
\rho = p \rho_{\text{even}} + (1-p) \rho_{\text{odd}},
\]
where \( 0 < p < 1 \). For each of the two subspaces, we then use a single-qubit Bloch representation, i.e.,
\[
\rho_{\text{even}} = \frac{1}{2} \begin{pmatrix} [1 + z_{\text{even}}] \| 0 \rangle \rangle \langle\langle 0 \| & [1 - z_{\text{even}}] \| 1_1 \rangle \rangle \langle\langle 1_1 \| \\
& [1 - z_{\text{even}}] \| 1_2 \rangle \rangle \langle\langle 1_2 \| & [1 + z_{\text{even}}] \| 1_1 \rangle \rangle \langle\langle 1_1 \|
\end{pmatrix} \],
\]
\[
\rho_{\text{odd}} = \frac{1}{2} \begin{pmatrix} [1 + z_{\text{odd}}] \| 1_2 \rangle \rangle \langle\langle 1_2 \| & [1 - z_{\text{odd}}] \| 1_1 \rangle \rangle \langle\langle 1_1 \| \\
& [1 + z_{\text{odd}}] \| 1_2 \rangle \rangle \langle\langle 1_2 \| & [1 - z_{\text{odd}}] \| 1_1 \rangle \rangle \langle\langle 1_1 \|
\end{pmatrix} \],
\]
where the coefficients satisfy \( |x_{\text{even,odd}}| \leq 1 \), and
\[
r_{\text{even}}^2 = x_{\text{even}}^2 + y_{\text{even}}^2 + z_{\text{even}}^2 \leq 1,
\]
\[
r_{\text{odd}}^2 = x_{\text{odd}}^2 + y_{\text{odd}}^2 + z_{\text{odd}}^2 \leq 1.
\]
In this parametrization, the entropy of the final state can easily be obtained via its eigenvalues \( \frac{1}{2}(1 + r_{\text{even}}) \) and \( \frac{1}{2}(1 + r_{\text{odd}}) \). The energy of the final state, in turn, is
\[
E(\rho) = \omega(1 - p z_{\text{even}}) - p \epsilon_{\text{even}}x_{\text{even}} + (1-p)\epsilon_{\text{odd}}x_{\text{odd}}.
\]
Lastly, the final state marginals are of the form
\[
\rho_{s_1} = \frac{1}{2} \begin{pmatrix} (1 + p z_{\text{even}} - (1-p) z_{\text{odd}}) \| 0 \rangle \rangle \langle\langle 0 \| & (1-p) (1 - p z_{\text{even}} - (1-p) z_{\text{odd}}) \| 1_1 \rangle \rangle \langle\langle 1_1 \|
\end{pmatrix},
\]
\[
\rho_{s_2} = \frac{1}{2} \begin{pmatrix} (1 + p z_{\text{even}} - (1-p) z_{\text{odd}}) \| 0 \rangle \rangle \langle\langle 0 \| & (1+p) (1-p z_{\text{even}} - (1-p) z_{\text{odd}}) \| 1_2 \rangle \rangle \langle\langle 1_2 \|
\end{pmatrix}.
\]
For the illustration of the results, it is convenient to specify the amount of available input energy in units of \( W_{\text{min}} \), the minimal free energy difference\(^2\) to a maximally correlated state. Taking into account that such a maximally correlated state must be pure, and that the free energy of the initial state is \( F(\tau) = -T \ln(Z) \), we find
\[
W_{\text{min}} = \omega - \max\{|\epsilon_{\text{even}}|, |\epsilon_{\text{odd}}|\} + T \ln(Z).
\]
With this, we may numerically evaluate the maximal amount of correlations that can be created at a fixed temperature \( T \) and a fixed energy input \( W/W_{\text{min}} \). The results of the optimization allows us to compare the energy cost of optimal protocols to generate correlations, for both interacting (\( \epsilon_{\text{even,odd}} \neq 0 \)) and noninteracting systems (\( \epsilon_{\text{even,odd}} = 0 \)). The results are shown in Fig. 2.

In agreement with our considerations in Sec. III B, one observes an initial regime where the interactions provide an advantage. However, at some point, the energy cost of \( \Delta I_{s_1,s_2} \) becomes higher for interacting systems than for noninteracting ones. This is to be expected: Since the interacting system is correlated initially, the maximal value of \( \Delta I_{s_1,s_2} \) is always lower than in the noninteracting case.

\(^2\) Note that the minimal energy for maximal correlations \( (W_{\text{min}}) \) depends on the coupling strength. Therefore, the functions \( \Delta I_{s \neq 0} \) and \( \Delta I_{s=0} \), whose difference is plotted in Fig. 2, would be multiplied by different values when converting the plots of Fig. 2 to absolute energy costs. This would result in shifted intersections with the horizontal axes. Nonetheless, since \( W_{\text{min}} \) is maximal in the absence of interactions, the intersections would all shift to the left, leaving the conclusion unchanged, that the presence of interactions may make the creation of new correlations more expensive, even if the overall amount of correlations is larger in the end.
FIG. 2. Fermionic newly generated correlation cost: The difference in correlations that can be newly generated for an available energy $W$ (in units of $W_{\text{min}}$) in the presence ($\Delta I_{\epsilon \neq 0}$) and absence ($\Delta I_{\epsilon = 0}$) of interactions is shown for temperatures $T = 0.1, \ldots, 1$ (in units of $\hbar \omega/k_B$) in steps of 0.1 (blue to red, top to bottom) for the ratios $\epsilon_{\text{even}}/\epsilon_{\text{odd}} = 2, 1,$ and 0.5 in (a),(b), and (c), respectively.

For a complete picture of the situation, it is also enlightening to study the behaviour of the total correlations $I_{S_1 S_2}(\rho)$ for interacting and noninteracting systems, as shown in Fig. 3. In all cases that we have considered, the presence of the interactions leads to a larger amount of final state correlations $I_{S_1 S_2}(\rho)$, irrespective of the (relative) size and sign of the coupling constants.

FIG. 3. Fermionic correlation cost: The maximal correlation of the final state that is achievable for a fixed input energy $W$ [in units of $W_{\text{min}}$ from Eq. (36)] is shown for temperatures $T = 0.1, \ldots, 1$ (in units of $\hbar \omega/k_B$) in steps of 0.1 (blue to red, top to bottom) for the ratios $\epsilon_{\text{even}}/\epsilon_{\text{odd}} = 2, 1,$ and 0.5 in (a),(b), and (c), respectively. In all cases, the achievable final correlation is larger in the presence of interactions (solid lines) than in their absence (dashed lines). In (a) this can be seen from the inset plot, where the horizontal axis is not scaled with $W_{\text{min}}$. For (b) and (c) one may deduce this directly from the plots, since the solid lines are strictly above their corresponding dashed lines, and $W_{\text{min}}$ is maximal when $\epsilon_{\text{even}} = \epsilon_{\text{odd}} = 0$.

V. TWO BOSONIC MODES

In this section we study the creation of correlations for two bosonic modes. We present an example for which we explicitly show that the overall final-state correlation $I_{S_1 S_2}(\rho)$ is always larger in the presence of interactions. This complements the observations made in Fig. 3, where
we come to same conclusion using a numerical approach.

Let us now consider the simple, yet versatile system of two bosonic modes with creation and annihilation operators \( a_i \) and \( a_i^\dagger \) (\( i = 1, 2 \), respectively. The mode operators satisfy the canonical commutation relations \([a_i, a_j^\dagger] = \delta_{ij} \) and \([a_i, a_j] = 0 \). Such systems of two (or more) harmonics oscillators are of fundamental importance to quantum optics and quantum field theory. The correlations between bosonic modes have been extensively studied in continuous variable quantum information (see, e.g., Refs. [42, 43]) but they are also of interest in more specialized lines of research, such as, e.g., studies of entanglement in relativistic quantum field theory, see for example Refs. [44–46]. In addition to the usual free Hamiltonian

\[
H_{s_1} + H_{s_2} = \omega (a_1^\dagger a_1 + a_2^\dagger a_2),
\]

where we have assumed that the two modes have the same frequency \( \omega \), we will consider the interaction term

\[
H_{i} = \epsilon (a_1 a_2 + a_1^\dagger a_2^\dagger),
\]

with \( \epsilon \in \mathbb{R} \). Since the system Hamiltonian \( H_S = H_{s_1} + H_{s_2} + H_i \) is quadratic in the mode operators, any thermal state \( \tau \) of \( H_S \) is a Gaussian state that is fully described by its second moments, that is, its covariance matrix \( \Gamma_S \) with components

\[
(\Gamma_S)_{mn} = \text{Tr}(\tau [X_m X_n + X_n X_m]),
\]

with the quadrature operators \( X_{2n-1} = (a_n + a_n^\dagger)/\sqrt{2} \) and \( X_{2n} = -(a_n - a_n^\dagger)/\sqrt{2} \). The first moments \( \text{Tr}(\tau X_n) \), which would normally also enter into Eq. (39), vanish for the state \( \tau \). This can easily be seen by diagonalizing \( H_S \) using the Bogoliubov transformation

\[
c_1 = \cosh(u) a_1 + \sinh(u) a_2^\dagger,
\]

\[
c_2 = \cosh(u) a_2 + \sinh(u) a_1^\dagger,
\]

where \( u = \frac{1}{2} \text{artanh}(\epsilon/\omega) \), such that \([c_1, c_1^\dagger] = \delta_{ij} \) and \([c_1, c_2] = 0 \). With this transformation, the system Hamiltonian becomes

\[
H_S = \tilde{\omega} (c_1^\dagger c_1 + c_2^\dagger c_2) - 2\tilde{\omega} \sinh^2(u),
\]

where \( \tilde{\omega} = \sqrt{\omega^2 - \epsilon^2} \). The eigenstates of \( H_S \) are therefore the eigenstates of \( c_1^\dagger c_1 \) and \( c_2^\dagger c_2 \). Expanding the thermal state \( \tau = Z^{-1} e^{-\beta H_S} \) in terms of these eigenstates one quickly obtains \( \text{Tr}(\tau c_i) = \text{Tr}(\tau c_i^\dagger) = 0 \). Since the Bogoliubov transformation relating the operators \( a_i, a_i^\dagger \) and \( c_1, c_1^\dagger \) is linear, this implies that also \( \text{Tr}(\tau a_i^\dagger) = \text{Tr}(\tau a_i) = 0 \). The first moments vanish and \( \Gamma_S \) completely describes the state \( \tau \).

To assess the properties of the initial state, we then define the covariance matrix \( \Gamma_S^{(c)} \) with respect to the operators \( c_i \), in complete analogy to Eq. (39). For the thermal state \( \tau \), this \( 4 \times 4 \) matrix is proportional to the identity, that is,

\[
\Gamma_S^{(c)} = \nu(T) I_4 = \coth(\frac{\omega T}{2}) I_4,
\]

where the identity \( I_4 \) is the covariance matrix of the pure two-mode vacuum state with respect to the operators \( c_i \). The mixedness of the state is hence captured solely by the prefactor \( \nu(T) = \coth(\frac{\omega T}{2}) \). The matrices \( \Gamma_S^{(c)} \) and \( \Gamma_S \) are related by a symplectic transformation \( S \) corresponding to the unitary Bogoliubov transformation of Eq. (40), such that

\[
\Gamma_S = S \Gamma_S^{(c)} S^T.
\]

The transformation \( S \) leaves the symplectic form \( \Omega \), with components \( \Omega_{kl} = -i [X_k, X_l] \) invariant, \( S \Omega S^T = \Omega \). Consequently, also the eigenvalues \( \nu(T) \) of \( [i \Omega \Gamma_S] \), the symplectic eigenvalues are left unchanged by the transformation \( S \). This means that, up to the prefactor \( \nu(T) \), the covariance matrix \( \Gamma_S = \nu(T) SS^T \) represents a pure two-mode state, which is hence locally equivalent to two-mode squeezed state [47]. Due to the presence of \( \nu(T) \), the overall state is nonetheless mixed and correlated, but may or may not be entangled, depending on the size of \( \nu(T) \) [25, 48]. Any available energy \( W \) may then be used to further correlate the system by a combination of cooling [i.e., reducing \( \nu(T) \)] and two-mode squeezing along the direction in phase space corresponding to the two-mode squeezed state \( SS^T \). These transformations leave the subsystems in local thermal states with respect to \( H_{s_1} \) and \( H_{s_2} \) and therefore optimally correlate\(^3\) the subsystems at any given work cost [24]. Consequently, the presence of the interaction Hamiltonian \( H_i \) is here equivalent to an increased energy supply in the noninteracting case, and the overall correlations \( I_{s_1, s_2}(\rho) \) are always larger than in the noninteracting case at a fixed work cost \( W \). The correlations that can in principle be generated in this infinite-dimensional Hilbert space are unbounded. However, the energy cost of creating additional correlations increases as the state becomes more correlated. The newly generated correlations \( \Delta I_{s_1, s_2} \) may hence be more or less expensive than in the noninteracting case, depending on the initial temperature, coupling strength \( \epsilon \), and the available energy.

VI. CONCLUSION

We have investigated the work cost of creating correlations between interacting quantum systems and compared our results to previous studies [24, 25] of the corre-

\(^3\) Note that two-mode squeezing is generally not the optimal entangling transformation and may be outperformed by non-Gaussian transformations [25].
lution cost in noninteracting systems. While the notion of isolated, noninteracting systems may appear more natural from the perspective of quantum communication scenarios, our approach here is motivated by the ubiquity of interactions present in nature. Hence, assuming that the presence of the interactions cannot be avoided or controlled, we find that the interactions can nonetheless be harnessed.

For such naturally occurring interactions we have identified general strategies for finite-dimensional systems to reduce the energy cost of creating correlations. These strategies, which apply to any finite-dimensional bipartite system with arbitrary interaction Hamiltonian, improve on previous bounds for non-interacting systems, at least in some low-energy regime. Nevertheless, the exact relation between the interactions and the correlation cost is complicated. The work cost of correlations strongly depends on the exact configuration of the interaction terms and thus on the underlying physics. To illustrate the general strategies, we therefore choose some exemplary physical systems — qubits, as well as fermionic and bosonic modes — to showcase the usefulness of the interactions.

In our examination, we have focused on the mutual information as a measure of the generated correlations, capturing both classical and genuine quantum correlations. The notoriously difficult case of characterizing the cost of entangling interacting quantum systems, which would provide further insight into the relation between the practically motivated resource theories of QI and QT, is hence left open for future investigation.

Moreover, while we have here considered arbitrary operations on the system, inevitable noise and practical design may favor operations that can be directly implemented through the natural interactions present in the underlying systems. It would hence be interesting to compare such physically motivated protocols with the optimal protocols derived here, including also noncyclic processes where the interactions can be switched on or off at will. Further open questions include the general cost and impact of interactions on single-shot information processing capabilities.

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Appendix: Correlations in a fermionic thermal state

Here we study the correlations present in the initial thermal state $\tau_\beta$ of two fermionic modes, given by Eq. (30). Whereas the thermal states of the noninteracting Hamiltonian (i.e., for $\epsilon_{\text{even,odd}} = 0$) are uncorrelated, the thermal states of interacting Hamiltonians feature some correlations. For instance, consider the ground state, that is, the limit $T \to 0$ ($\beta \to \infty$), which arises as the eigenstate of the Hamiltonian with the smallest eigenvalue. As can be seen from Eq. (28), this can be either $|\lambda_3 \rangle \langle \lambda_3 |$ or $|\lambda_4 \rangle \langle \lambda_4 |$, depending on the relative sizes of $|\epsilon_{\text{odd}}|$ and $\sqrt{\omega^2 + \epsilon_{\text{even}}^2}$. That is,

$$
\tau(\beta \to \infty) = \begin{cases} 
\rho_3 & \text{if } |\epsilon_{\text{odd}}| > \sqrt{\omega^2 + \epsilon_{\text{even}}^2}, \\
\frac{1}{2}(\rho_3 + \rho_4) & \text{if } |\epsilon_{\text{odd}}| = \sqrt{\omega^2 + \epsilon_{\text{even}}^2}, \\
\rho_4 & \text{if } |\epsilon_{\text{odd}}| < \sqrt{\omega^2 + \epsilon_{\text{even}}^2},
\end{cases}
$$

(A.1)

with $\rho_3 = |\lambda_3 \rangle \langle \lambda_3 |$ and $\rho_4 = |\lambda_4 \rangle \langle \lambda_4 |$. Both $|\lambda_3 \rangle \langle \lambda_3 |$ and $|\lambda_4 \rangle \langle \lambda_4 |$ are correlated, but only the former state is maximally correlated. It is hence expected that the relative sizes of the coupling constants strongly influence the initial amount of correlations, see Fig. 4. To evaluate the mutual information of Eq. (7), we still need to specify the reduced density operators. These are found to be diagonal, with matching matrix elements, i.e.,

$$
\tau_{S_1}(\beta) = \frac{1}{2}(1 + \tau_0) |0 \rangle \langle 0 | + \frac{1}{2}(1 - \tau_0) |1 \rangle \langle 1 |,
$$

(A.2a)

$$
\tau_{S_2}(\beta) = \frac{1}{2}(1 + \tau_0) |0 \rangle \langle 0 | + \frac{1}{2}(1 - \tau_0) |2 \rangle \langle 2 |,
$$

(A.2b)

with the coefficient $\tau_0$ given by

$$
\tau_0 = \frac{\omega \sinh(\beta \sqrt{\omega^2 + \epsilon_{\text{even}}^2})}{\sqrt{\omega^2 + \epsilon_{\text{even}}^2} \cosh(\beta \epsilon_{\text{odd}}) + \cosh(\beta \sqrt{\omega^2 + \epsilon_{\text{even}}^2})}.
$$

(A.3)

With the eigenvalues of the thermal state given by $e^{-\beta \lambda_i}$, and those of the marginals by $\frac{1}{2}(1 \pm \tau_0)$ one can then easily evaluate the entropies $S(\tau)$, $S(\tau_{S_1})$ and $S(\tau_{S_2})$, and hence the mutual information, shown in Fig. 4.