The motion of a rigid body in a quadratic potential: an integrable discretization

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Abstract

The motion of a rigid body in a quadratic potential is an important example of an integrable Hamiltonian system on a dual to a semidirect product Lie algebra $\mathfrak{so}(n) \ltimes \text{Symm}(n)$. We give a Lagrangian derivation of the corresponding equations of motion, and introduce a discrete time analog of this system. The construction is based on the discrete time Lagrangian mechanics on Lie groups, accompanied with the discrete time Lagrangian reduction. The resulting multi–valued map (correspondence) on the dual to $\mathfrak{so}(n) \ltimes \text{Symm}(n)$ is Poisson with respect to the Lie–Poisson bracket, and is also completely integrable. We find a Lax representation based on matrix factorisations, in the spirit of Veselov–Moser.
1 Introduction

The rigid body dynamics are rich with problems interesting from the mathematical point of view, in particular, with integrable problems. Certainly, the most famous ones are the three integrable cases, named after Euler, Lagrange, and Kovalevskaya, of the rotation of a heavy rigid body around a fixed point in a homogeneous gravity field. They were discovered in the 18th and the 19th century, and can be called classical. However, the list of integrable problems of the rigid body dynamics is by far not exhausted by these ones.

In the present paper we turn our attention to the rotation of a rigid body around its fixed center of mass in an arbitrary quadratic potential. The integrability of this problem is a much more recent observation due to Reyman [R] and Bogoyavlensky [B]. (However, some particular case of this result was given already by Brun [Br]; the equations of motion in this case are identical with those describing the integrable case of the motion of a rigid body in an ideal fluid, due to Clebsch [C]).

A problem we solve in the present paper, is a construction of an integrable discretization of the above mentioned integrable mechanical system. Our construction is close in spirit to the work by Moser and Veselov [MV], where they used an approach based on the discrete time Lagrangian mechanics. Several important integrable mechanical problems were discretized in [MV], including the Euler top and the Neumann system. The general theory of discrete time Lagrangian mechanics on Lie groups was developed further in [BS1], where the list of Moser and Veselov was extended by a discrete time Lagrange top. A further development of the general theory was undertaken in [BS2], where general discrete time Euler–Poisson equations on semidirect product Lie algebras were obtained as a result of a reduction procedure, applied to discrete time Lagrangian systems on Lie groups. A discretization of a top in a quadratic potential, achieved in the present paper, serves as a spectacular illustration to the abstract constructions in [BS2].

We recall the general theory of the Lagrangian reduction in the continuous time and in the discrete time contexts, respectively, in Sect. 2 and 3. Further, we give in Sect. 4 a Lagrangian derivation of the equations of motion of a rigid body in a quadratic potential. In Sect. 5 we introduce the discrete time analogs of these constructions. Finally, in Sect. 6 the specialization of these results for the Clebsch case is given. Conclusions are contained in Sect. 7.

2 Lagrangian mechanics and Lagrangian reduction on $TG$

Recall that a continuous time Lagrangian system is defined by a smooth function $L(g, \dot{g}) : TG \mapsto \mathbb{R}$ on the tangent bundle of a smooth manifold $G$. The function $L$ is called the Lagrange function. We will be dealing here only with the case when $G$ carries an additional structure of a Lie group. For an arbitrary function $g(t) : [t_0, t_1] \mapsto G$ one can consider the action functional

$$S = \int_{t_0}^{t_1} L(g(t), \dot{g}(t))dt .$$  (2.1)

A standard argument shows that the functions $g(t)$ yielding extrema of this functional (in the class of variations preserving $g(t_0)$ and $g(t_1)$), satisfy with necessity the Euler–Lagrange equations. In
local coordinates \( \{ g^i \} \) on \( G \) they read:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{g}^i} \right) = \frac{\partial L}{\partial g^i} .
\] (2.2)

The action functional \( S \) is independent of the choice of local coordinates, and thus the Euler–Lagrange equations are actually coordinate independent as well. For a coordinate–free description in the language of differential geometry, see [A], [MR].

Introducing the quantities

\[
\Pi = \nabla \dot{g} L \in T_g^* G ,
\] (2.3)

one defines the Legendre transformation:

\[
(g, \dot{g}) \in TG \mapsto (g, \Pi) \in T^* G .
\] (2.4)

If it is invertible, i.e. if \( \dot{g} \) can be expressed through \( (g, \Pi) \), then the Legendre transformation of the Euler–Lagrange equations (2.2) yield a Hamiltonian system on \( T^* G \) with respect to the standard symplectic structure on \( T^* G \) and with the Hamilton function

\[
H(g, \Pi) = \langle \Pi, \dot{g} \rangle - L(g, \dot{g}) ,
\] (2.5)

(where, of course, \( \dot{g} \) has to be expressed through \( (g, \Pi) \)).

When working with the tangent bundle of a Lie group, it is convenient to trivialize it, translating all vectors to the group unit by left or right multiplication. We consider only the left trivialization here:

\[
(g, \dot{g}) \in TG \mapsto (g, \Omega) \in G \times g ,
\] (2.6)

where

\[
\Omega = L_{g^{-1}} \dot{g} \iff \dot{g} = L_g \Omega .
\] (2.7)

Denote the Lagrange function pushed through the above map, by \( L^{(l)}(g, \Omega) : G \times g \mapsto \mathbb{R} \), so that

\[
L^{(l)}(g, \Omega) = L(g, \dot{g}) ,
\] (2.8)

The trivialization (2.6) of the tangent bundle \( TG \) induces the following trivialization of the cotangent bundle \( T^* G \):

\[
(g, \Pi) \in T^* G \mapsto (g, M) \in G \times g^* ,
\] (2.9)

where

\[
M = L_g^* \Pi \iff \Pi = L_{g^{-1}}^* M .
\] (2.10)
We now consider the Lagrangian reduction procedure, in the case when the Lagrange function is symmetric with respect to the left action of a certain subgroup of $G$, namely an isotropy subgroup of some element in the representation space of $G$. So, the next ingredient of our construction is a representation $\Phi : G \times V \mapsto V$ of a Lie group $G$ in a linear space $V$; we denote it by

$$\Phi(g) \cdot v \quad \text{for} \quad g \in G, \ v \in V .$$

We denote also by $\phi$ the corresponding representation of the Lie algebra $\mathfrak{g}$ in $V$:

$$\phi(\xi) \cdot v = \frac{d}{d\epsilon} \left( \Phi(e^{\epsilon \xi}) \cdot v \right) \bigg|_{\epsilon=0} \quad \text{for} \quad \xi \in \mathfrak{g}, \ v \in V . \quad (2.11)$$

The map $\phi^* : \mathfrak{g} \times V^* \mapsto V^*$ defined by

$$\langle \phi^*(\xi) \cdot y, v \rangle = \langle y, \phi(\xi) \cdot v \rangle \quad \forall v \in V, \ y \in V^* , \ \xi \in \mathfrak{g} , \quad (2.12)$$

is an anti–representation of the Lie algebra $\mathfrak{g}$ in $V^*$. We shall use also the bilinear operation $\diamond : V^* \times V \mapsto \mathfrak{g}^*$ introduced in [HMR, CHMR] and defined as follows: let $v \in V, \ y \in V^*$, then

$$\langle y \diamond v, \xi \rangle = -\langle y, \phi(\xi) \cdot v \rangle \quad \forall \xi \in \mathfrak{g} . \quad (2.13)$$

(Notice that the pairings on the left–hand side and on the right–hand side of the latter equation are defined on different spaces).

Fix an element $a \in V$, and consider the isotropy subgroup $G[a]$ of $a$, i.e.

$$G[a] = \{ h : \Phi(h) \cdot a = a \} \subset G . \quad (2.14)$$

Suppose that the Lagrange function $L(g, \dot{g})$ is invariant under the action of $G[a]$ on $TG$ induced by left translations on $G$:

$$L(hg, L_{h*} \dot{g}) = L(g, \dot{g}) , \quad h \in G[a] . \quad (2.15)$$

The corresponding invariance property of $L^{(l)}(g, \Omega)$ is expressed as:

$$L^{(l)}(hg, \Omega) = L^{(l)}(g, \Omega) , \quad h \in G[a] . \quad (2.16)$$

We want to reduce the Euler–Lagrange equations with respect to this left action. As a section $(G \times \mathfrak{g})/G[a]$ we choose the set $\mathfrak{g} \times O_a$, where $O_a$ is the orbit of $a$ under the action $\Phi$:

$$O_a = \{ \Phi(g) \cdot a , \ g \in G \} \subset V . \quad (2.17)$$

The reduction map is

$$(g, \Omega) \in G \times \mathfrak{g} \mapsto (\Omega, P) \in \mathfrak{g} \times O_a , \quad \text{where} \quad P = \Phi(g^{-1}) \cdot a , \quad (2.18)$$
so that the reduced Lagrange function \( L^\circ : g \times O_a \mapsto \mathbb{R} \) is defined as
\[
L^\circ(\Omega, P) = L^\circ(g, \Omega), \quad \text{where} \quad P = \Phi(g^{-1}) \cdot a.
\] (2.19)

The reduced Lagrangian \( L^\circ(\Omega, P) \) is well defined, because from
\[
P = \Phi(g^{-1}) \cdot a = \Phi(g_1^{-1}) \cdot a = \Phi(g_2^{-1}) \cdot a
\]
there follows \( \Phi(g_1^{-1}) \cdot a = a \), so that \( g_2^{-1} \in G^{[a]} \), and \( L^\circ(g_1, \Omega) = L^\circ(g_2, \Omega) \).

**Theorem 1** [HMR, CHMR]

a) Under the left trivialization \((g, \dot{g}) \mapsto (g, \Omega)\) and the subsequent reduction \((g, \Omega) \mapsto (\Omega, P)\), the Euler–Lagrange equations (2.2) become the following **Euler–Poincaré equations**:
\[
\begin{align*}
\dot{M} &= \text{ad}^* \Omega \cdot M + \nabla_P L^\circ \circ P, \\
\dot{P} &= -\phi(\Omega) \cdot P,
\end{align*}
\] (2.20)

where
\[
M = L_g^* \Pi = \nabla_\Omega L^\circ.
\] (2.21)

b) If the “Legendre transformation”
\[
(\Omega, P) \in g \times O_a \mapsto (M, P) \in \mathfrak{g}^* \times O_a,
\] (2.22)
is invertible, then (2.20) is a Hamiltonian system of \( \mathfrak{g}^* \times O_a \) with the Hamilton function
\[
H(M, P) = \langle M, \Omega \rangle - L^\circ(\Omega, P),
\]
with respect to the Poisson bracket given by
\[
\{F_1, F_2\} = \langle M, [\nabla_M F_1, \nabla_M F_2] \rangle + \langle \nabla_P F_1, \phi(\nabla_M F_2) \cdot P \rangle - \langle \nabla_P F_2, \phi(\nabla_M F_1) \cdot P \rangle
\] (2.23)

for two arbitrary functions \( F_1, F_2(M, P) : \mathfrak{g}^* \times O_a \mapsto \mathbb{R} \).

**Remark 1.** The formula (2.23) defines a Poisson bracket not only on \( \mathfrak{g}^* \times O_a \), but on all of \( \mathfrak{g}^* \times V \). Rewriting this formula as
\[
\{F_1, F_2\} = \langle M, [\nabla_M F_1, \nabla_M F_2] \rangle + \langle P, \phi^*(\nabla_M F_2) \cdot \nabla_P F_1 - \phi^*(\nabla_M F_1) \cdot \nabla_P F_2 \rangle
\] (2.24)
one immediately identifies this bracket with the Lie–Poisson bracket of the semiproduct Lie algebra \( \mathfrak{g} \ltimes V^* \) corresponding to the representation \(-\phi^*\) of \( \mathfrak{g} \) in \( V^* \).

**Remark 2.** In an important particular case of constructions of this section, the vector space is chosen as the Lie algebra of our basic Lie group: \( V = \mathfrak{g} \), the group representation is the adjoint one: \( \Phi(g) \cdot v = \text{Ad} g \cdot v \), so that \( \phi(\xi) \cdot v = \text{ad} \xi \cdot v = [\xi, v] \), and the bilinear operation \( \circ \) is nothing but the coadjoint action of \( \mathfrak{g} \) on \( \mathfrak{g}^* \): \( y \circ v = \text{ad}^* v \cdot y \). This is the framework, e.g., for the heavy top mechanics.
3 Lagrangian mechanics and Lagrangian reduction on $G \times G$

We now turn to the discrete time analog of these constructions, Our presentation of the general discrete time Lagrangian mechanics is an adaptation of the Moser–Veselov construction [V], [MV] for the case when the basic manifold is a Lie group. The presentation of the discrete time Lagrangian reduction follows [BS1], [BS2]. Almost all constructions and results of the continuous time Lagrangian mechanics have their discrete time analogs. The only exception is the existence of the “energy” integral (2.5).

Let $L(g, \hat{g}) : G \times G \mapsto \mathbb{R}$ be a smooth function, called the (discrete time) Lagrange function.

For an arbitrary sequence $\{g_k \in G, k = k_0, k_0 + 1, \ldots, k_1\}$ one can consider the action functional

$$S = \sum_{k=k_0}^{k_1-1} L(g_k, g_{k+1}).$$

(3.1)

Obviously, the sequences $\{g_k\}$ delivering extrema of this functional (in the class of variations preserving $g_{k_0}$ and $g_{k_1}$), satisfy with necessity the discrete Euler–Lagrange equations:

$$\nabla_1 L(g_k, g_{k+1}) + \nabla_2 L(g_{k-1}, g_k) = 0.$$

(3.2)

Here $\nabla_1 L(g, \hat{g})$ ($\nabla_2 L(g, \hat{g})$) denotes the gradient of $L(g, \hat{g})$ with respect to the first argument $g$ (resp. the second argument $\hat{g}$). So, in our case, when $G$ is a Lie group and not just a general smooth manifold, the equation (3.2) is written in a coordinate free form, using the intrinsic notions of the Lie theory. As pointed out above, an invariant formulation of the Euler–Lagrange equations in the continuous time case is more sophisticated. This seems to underline the fundamental character of discrete Euler–Lagrange equations.

The equation (3.2) is an implicit equation for $g_{k+1}$. In general, it has more than one solution, and therefore defines a correspondence (multi–valued map) $(g_{k-1}, g_k) \mapsto (g_k, g_{k+1})$. To discuss symplectic properties of this correspondence, one defines:

$$\Pi_k = \nabla_2 L(g_{k-1}, g_k) \in T^*_g G.$$

(3.3)

Then (3.2) may be rewritten as the following system:

$$\begin{cases}
\Pi_k = -\nabla_1 L(g_k, g_{k+1}) \\
\Pi_{k+1} = \nabla_2 L(g_k, g_{k+1})
\end{cases}$$

(3.4)

\footnote{For the notations from the Lie groups theory used in this and subsequent sections see, e.g., [BS1]. In particular, for an arbitrary smooth function $f : G \mapsto \mathbb{R}$ its right Lie derivative $d^f g$ and left Lie derivative $df$ are functions from $G$ into $\mathfrak{g}^*$ defined via the formulas

$$\langle df(g), \eta \rangle = \left. \frac{d}{dx} f(e^{\xi \eta} g) \right|_{\xi=0}, \quad \langle d^f g, \eta \rangle = \left. \frac{d}{dx} f(g e^{\xi \eta}) \right|_{\xi=0}, \quad \forall \eta \in \mathfrak{g},$$

and the gradient $\nabla f(g) \in T^*_g G$ is defined as

$$\nabla f(g) = R_{g^{-1}}^* df(g) = L_{g^{-1}}^* d^f g.$$}
This system defines a (multivalued) map \((g_k, \Pi_k) \mapsto (g_{k+1}, \Pi_{k+1})\) of \(T^*G\) into itself. More precisely, the first equation in (3.4) is an implicit equation for \(g_{k+1}\), while the second one allows for the explicit and unique calculation of \(\Pi_{k+1}\), knowing \(g_k\) and \(g_{k+1}\). As demonstrated in [V], [MV], this map \(T^*G \mapsto T^*G\) is symplectic with respect to the standard symplectic structure on \(T^*G\).

The tangent bundle \(TG\) does not appear in the discrete time context at all. On the contrary, the cotangent bundle \(T^*G\) still plays an important role in the discrete time theory, as the phase space with the canonical invariant symplectic structure. The left trivialization of \(T^*G\) is same as in the continuous time case:

\[
(g_k, \Pi_k) \in T^*G \mapsto (g_k, M_k) \in G \times g^* ,
\]

where

\[
M_k = L^* g_k \Pi_k \iff \Pi_k = L^* g_{k-1} M_k .
\]

Consider also the map

\[
(g_k, g_{k+1}) \in G \times G \mapsto (g_k, W_k) \in G \times G ,
\]

where

\[
W_k = g_k^{-1} g_{k+1} \iff g_{k+1} = g_k W_k .
\]

Denote the Lagrange function pushed through (3.7) by

\[
\mathbb{L}^{(l)}(g_k, W_k) = \mathbb{L}(g_k, g_{k+1}) .
\]

Suppose that the Lagrange function \(\mathbb{L}(g, \tilde{g})\) is invariant under the action of \(G^{[a]}\) on \(G \times G\) induced by left translations on \(G\):

\[
\mathbb{L}(hg, h\tilde{g}) = \mathbb{L}(g, \tilde{g}) , \quad h \in G^{[a]} .
\]

The corresponding invariance property of \(\mathbb{L}^{(l)}(g, W)\) is expressed as:

\[
\mathbb{L}^{(l)}(hg, W) = \mathbb{L}^{(l)}(g, W) , \quad h \in G^{[a]} .
\]

We want to reduce the Euler–Lagrange equations with respect to this left action. As a section \((G \times G)/G^{[a]}\) we choose the set \(G \times O_a\). The reduction map is

\[
(g, W) \in G \times G \mapsto (W, P) \in G \times O_a , \quad \text{where} \quad P = \Phi(g^{-1}) \cdot a ,
\]

so that the reduced Lagrange function \(\Lambda^{(l)} : G \times O_a \mapsto \mathbb{R}\) is defined as

\[
\Lambda^{(l)}(W, P) = \mathbb{L}^{(l)}(g, W) , \quad \text{where} \quad P = \Phi(g^{-1}) \cdot a .
\]
Theorem 2 [BS1], [BS2]

a) Under the left trivialization \((g, \hat{g}) \mapsto (g, W)\) and the subsequent reduction \((g, W) \mapsto (W, P)\), the Euler–Lagrange equations (3.2) become the following discrete Euler–Poincaré equations:

\[
\begin{cases}
\text{Ad}^* W_k^{-1} \cdot M_{k+1} = M_k + \nabla_P \Lambda^{(l)}(W_k, P_k) \circ P_k , \\
P_{k+1} = \Phi(W_k^{-1}) \cdot P_k ,
\end{cases}
\tag{3.14}
\]

where

\[M_k = d_W \Lambda^{(l)}(W_{k-1}, P_{k-1}) \in g^* . \tag{3.15}\]

b) If the “Legendre transformation”

\[(W_{k-1}, P_{k-1}) \in G \times O_a \mapsto (M_k, P_k) \in g^* \times O_a , \tag{3.16}\]

where \(P_k = \Phi(W_{k-1}^{-1}) \cdot P_{k-1} ,\) is invertible, then (3.14) define a map \((M_k, P_k) \mapsto (M_{k+1}, P_{k+1})\) of \(g^* \times O_a\) which is Poisson with respect to the Poisson bracket (2.23).

The relation between the continuous time and the discrete time equations is established, if we set

\[g_k = g , \quad g_{k+1} = g + \varepsilon \hat{g} + O(\varepsilon^2) , \quad L(g_k, g_{k+1}) = \varepsilon L(g, \hat{g}) + O(\varepsilon^2) ; \]
\[P_k = P , \quad W_k = 1 + \varepsilon \Omega + O(\varepsilon^2) , \quad \Lambda^{(l)}(W_k, P_k) = \varepsilon L^{(l)}(\Omega, P) + O(\varepsilon^2) . \]

4 A rigid body in a quadratic potential

The basic Lie group relevant for our main example is

\[G = \text{SO}(n) , \quad \text{so}(n) \]

(the “physical” rigid body corresponds to \(n = 3\)). The scalar product on \(g\) is defined as

\[\langle \xi, \eta \rangle = -\frac{1}{2} \text{tr}(\xi \eta) , \quad \xi, \eta \in g .\]

This scalar product is used also to identify \(g^*\) with \(g\), so that the previous formula can be considered also a pairing between the elements \(\xi \in g\) and \(\eta \in g^*\).

The group \(G\) is a natural configuration space for problems related to the rotation of a rigid body. Indeed, if \(E(t) = \left(e_1(t), \ldots, e_n(t)\right)\) stands for the time evolution of a certain orthonormal frame firmly attached to the rigid body (so that all \(e_k \in \mathbb{R}^n\)), then

\[E(t) = g^{-1}(t) E(0) \iff e_k(t) = g^{-1}(t) e_k(0) \quad (1 \leq k \leq n) ,\]
with some $g(t) \in G$. The Lagrange function of an arbitrary rigid body rotating about a fixed point $0 \in \mathbb{R}^n$ in a field with a quadratic potential

$$
\varphi(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j ,
$$

is equal to (cf. [B], [RSTS]):

$$
L(g, \dot{g}) = -\frac{1}{2} \text{tr}(\Omega J \Omega) + \frac{1}{2} \text{tr}(g J g^T A) ,
$$

where

- $\Omega = g^{-1} \dot{g} = g^T \dot{g}$ is the angular velocity of the rigid body in the body frame $E(t)$;
- $J$ is a symmetric matrix (tensor of inertia of the rigid body); choosing the frame $E(0)$ properly, we can assure this matrix to be diagonal, $J = \text{diag}(J_1, \ldots, J_n)$, which will be supposed from now on;
- $A = (a_{ij})_{i,j=1}^{n}$ is a symmetric matrix of coefficients of the quadratic form $\varphi(x)$.

To include this Lagrange function in the framework of Sect. 2, we make the following identifications:

- $V = \text{Symm}(n)$, the linear space of all $n \times n$ symmetric matrices; we identify $V^*$ with $V$ via the following scalar product on $V$:

$$
\langle v_1, v_2 \rangle = \frac{1}{2} \text{tr}(v_1 v_2) , \quad v_1, v_2 \in V .
$$

- The representation $\Phi$ of $G$ in $V$ is defined as

$$
\Phi(g) \cdot v = gvg^{-1} = gvg^T \text{ for } g \in G, \ v \in V .
$$

- Therefore the representation $\phi$ of $\mathfrak{g}$ in $V$ is given by

$$
\phi(\xi) \cdot v = [\xi, v] \text{ for } \xi \in \mathfrak{g}, \ v \in V ,
$$

while the anti-representation $\phi^*$ of $\mathfrak{g}$ in $V^*$ is given by

$$
\phi^*(\xi) \cdot y = -[\xi, y] \text{ for } \xi \in \mathfrak{g}, \ y \in V^* ,
$$

- Finally, the bilinear operation $\diamond : V^* \times V \mapsto \mathfrak{g}^*$ is given by

$$
y \diamond v = -[y, v] \text{ for } y \in V^* , \ v \in V .
$$
Denoting now $P = g^TAg = \Phi(g^{-1}) \cdot A$, we represent (4.2) in the form

$$L(g, \dot{g}) = \mathcal{L}^{(l)}(\Omega, P) = -\frac{1}{2} \text{tr}(\Omega J \Omega) + \frac{1}{2} \text{tr}(JP) = -\frac{1}{2} \text{tr}(\Omega J \Omega) + \langle J, P \rangle,$$

(4.3)

which is manifestly invariant under the left action of the isotropy subgroup $G^{(A)}$. Now Theorem 1 is applicable, which delivers the following equations of motion:

$$\begin{cases}
\dot{M} = [M, \Omega] + [P, J], \\
\dot{P} = [P, \Omega],
\end{cases}$$

(4.4)

where

$$M = \nabla_\Omega \mathcal{L}^{(l)} = J\Omega + \Omega J \quad \Leftrightarrow \quad M_{jk} = (J_j + J_k)\Omega_{jk}.$$  (4.5)

According to the general theory, the system (4.4) is Hamiltonian on the dual of the semidirect product Lie algebra $g \ltimes V^*$, with the Hamilton function

$$H(M, P) = \frac{1}{2} \langle M, \Omega \rangle - \langle J, P \rangle = \frac{1}{2} \sum_{j<k} \frac{M_{jk}^2}{J_j + J_k} - \frac{1}{2} \sum_{k=1}^{n} J_k P_{kk}.$$  (4.6)

Generic orbits in this Poisson phase space have dimension $n^2 - n$ (the dimension of $g \ltimes V^*$ is equal to $n(n - 1)/2 + n(n + 1)/2 = n^2$; the $n$ spectral invariants of $P \in V$ are Casimir functions of the Poisson bracket). In particular, for the “physical” case $n = 3$ the dimension of the generic orbit is equal to 6. Therefore the number of independent involutive integrals of motion necessary for complete integrability is equal to $n(n - 1)/2$ (equal to 3 for $n = 3$).

A key observation of [R], [B] consists in the following Lax representation of the above system:

$$\dot{L}(\lambda) = [L(\lambda), B(\lambda)],$$

where

$$L(\lambda) = P + \lambda M + \lambda^2 J^2, \quad B(\lambda) = \Omega + \lambda J.$$  

(The key point in the straightforward verification of this statement is the identity

$$[M, J] + [J^2, \Omega] = 0,$$

which follows directly from (4.3)). The spectral invariants of the matrix $L(\lambda)$ provide us with the necessary number of independent integrals of motion [B], and their involutivity follows from the general $r$–matrix theory (cf. [R], [RSTS]).
5 A discrete time analog of a top in a quadratic potential

To find a discrete analog of the Lagrange function (4.2), we rewrite the latter once more as
\[ L(g, \dot{g}) = \frac{1}{2} \text{tr}(\dot{g}J\dot{g}^T) + \frac{1}{2} \text{tr}(gJg^T A) \].

Let us introduce the following discrete analog:
\[ L(g^k, g^{k+1}) = \frac{1}{2\epsilon} \text{tr}\left( (g^{k+1} - g^k)J(g^{k+1} - g^k)^T \right) + \frac{\epsilon}{2} \text{tr}(g^{k+1}Jg^{k+1}^T A) \].

The powers of \( \epsilon \) are introduced in a way assuring the correct asymptotics of the discrete time Lagrange function, namely \( L(g^k, g^{k+1}) \approx \epsilon L(g, \dot{g}) \), as \( g^k = g \) and \( g^{k+1} \approx g + \epsilon \dot{g} \) (see the end of Sect. 3). Up to a constant, the function (5.2) may be rewritten as
\[ L(g^k, g^{k+1}) = -\frac{1}{\epsilon} \text{tr}(W_k J) + \frac{\epsilon}{2} \text{tr}(W_k P_k A) \].

This is representable also in terms of \( W_k = g_k^T g_{k+1} \in G \) and \( P_k = g_k^T A g_k \in O_A \subset V \):
\[ L(g^k, g^{k+1}) = \Lambda^{(l)}(W_k, P_k) = -\frac{1}{\epsilon} \text{tr}(W_k J) + \frac{\epsilon}{2} \text{tr}(W_k J P_k) \].

Theorem 3 The discrete time Euler–Lagrange equations for the Lagrange function (5.4) are equivalent to the following system:
\[
\begin{cases}
M_k = \frac{1}{\epsilon}(W_k J - JW_k^T) - \frac{\epsilon}{2}(P_k W_k J - JW_k^T P_k) \\
M_{k+1} = \frac{1}{\epsilon}(JW_k - W_k^T J) - \frac{\epsilon}{2}(JP_k W_k - W_k^T P_k J) \\
P_{k+1} = W_k^T P_k W_k
\end{cases}
\]

The multi-valued map (correspondence) \((M_k, P_k) \mapsto (M_{k+1}, P_{k+1})\) described by (5.5) is Poisson with respect to the Lie–Poisson bracket of the semidirect product Lie algebra \( \mathfrak{g} \ltimes V^* \), where \( \mathfrak{g} = \text{so}(n) \), \( V^* = \text{Symm}(n) \), and the representation \(-\phi^*\) of \( \mathfrak{g} \) in \( V^* \) is defined as \(-\phi^*(\xi) \cdot y = [\xi, y] \) for \( \xi \in \mathfrak{g} \), \( y \in V^* \).

Proof. We are in a position to apply Theorem 2. To this end we first calculate
\[ \nabla P \Lambda^{(l)}(W_k, P_k) = \frac{\epsilon}{2}(W_k J + JW_k^T) \],
so that the equations of motion read:
\[
\begin{cases}
W_k M_{k+1} W_k^T = M_k + \frac{\epsilon}{2} [P_k, W_k J + JW_k^T] \\
P_{k+1} = W_k^T P_k W_k
\end{cases}
\]
These equations are, obviously, a discrete time approximation of (4.4). They have to be supplemented by a discrete version of (4.5), which reads:

\[ M_{k+1} = d'_W \Lambda^{(l)}(W_k, P_k) = \frac{1}{\varepsilon} (JW_k - W_k^T J) - \frac{\varepsilon}{2} (JP_k W_k - W_k^T P_k J) . \]  

(5.7)

Plugging this into the first equation of the system (5.6), we put this system into the form (5.5).

The definition of the above correspondence (5.5) crucially depends on the solvability of the first equation in (5.5) for \( W_k \in G \). The best approach to this problem, as well as to the integrability of the correspondence, is through the matrix factorizations. The following argument is a generalization of the Moser–Veselov approach [MV] to the discrete time motion of the free \( n \)-dimensional rigid body.

Let us introduce the following matrices, depending on the spectral parameter \( \lambda \):

\[ U_k(\lambda) = \left( I - \frac{\varepsilon^2}{2} P_k \right) W_k - \varepsilon \lambda J , \]

so that

\[ U_k^T (\lambda) = W_k^T \left( I - \frac{\varepsilon^2}{2} P_k \right) + \varepsilon \lambda J . \]

(5.9)

Denote also

\[ L_k(\lambda) = \left( I - \frac{\varepsilon^2}{2} P_k \right)^2 - \varepsilon^2 \lambda M_k - \varepsilon^2 \lambda^2 J^2 = I - \varepsilon^2 \left( P_k + \lambda M_k + \lambda^2 J^2 \right) + \varepsilon^4 \frac{1}{4} P_k^2 . \]

(5.10)

Then a direct calculation allows us to verify the following statement.

**Theorem 4** The equations of motion (5.5) are equivalent to the following matrix factorizations:

\[ \left\{ \begin{array}{l}
L_k(\lambda) = U_k(\lambda) U_k^T (\lambda) , \\
L_{k+1}(\lambda) = U_k^T (\lambda) U_k(\lambda) .
\end{array} \right. \]

(5.11)

In particular, the matrix \( L_k(\lambda) \) remains isospectral in the discrete time evolution described by the equations (5.5).

More precisely, the first equation in (5.5) is equivalent to the first equation in (5.11) (under the assumption that \( W_k \in G = \text{SO}(n) \)), while the second and the third equations in (5.5) are equivalent to the second equation in (5.11). So, the problem of solvability of the first equation in (5.5) for \( W_k \) is equivalent to the matrix factorization problem expressed by the first equation in (5.11). For a treatment of a closely related factorization problem, we refer the reader to [MV].

From Theorem 4 there follows also the complete integrability of our discrete time Lagrangian map. Notice that its integrals of motion do not coincide with the integrals of motion of the continuous–time problem (with the only exception of the free rigid body motion considered in [MV]). To be more concrete, the integrals of motion of our map are obtained from the integrals of the continuous time problem by replacing \( P \) through \( P - \frac{1}{4} \varepsilon^2 P^2 \). As for the actual integration of our map in terms of theta–functions, we leave it as an open problem for want of a better occasion (cf. [MV], [B], [RSTS]).
6 A particular case: the Clebsch problem

An important particular case of the rigid body in a quadratic potential appears when \( A = a a^T \) with some \( a \in \mathbb{R}^n \). For example, this is the case when the quadratic potential \( (4.1) \) represents the quadratic terms in an expansion of a potential of a single point mass; it is supposed that the distance from the rigid body to this point mass is much larger than the size of the body itself, and the ratio of this two length scales is the small parameter of the above mentioned expansion. In this case the vector \( a \) points from the point mass to the fixed center of mass of the rigid body. Then

\[
p = g^T a
\]

represents the same vector in the frame firmly attached to the rigid body. We have:

\[
P = p p^T,
\]

i.e. the orbit \( O_A \) in \( V \) consists of rank 1 matrices. This case could be considered independently, along the lines of this paper. The relevant representation space \( V \) would be then \( \mathbb{R}^n \), and the representation \( \Phi \) of \( G \) in \( V \) would be defined as \( \Phi(g) \cdot v = gv \). However, we prefer to simply use the results already obtained, replacing \( A \) with \( a a^T \), and \( P \) with \( p p^T \). So, the Lagrange function \( (4.3) \) takes the form

\[
L(g, \dot{g}) = L(l)(\Omega, p) = -\frac{1}{2} \text{tr}(\Omega J\Omega) + \frac{1}{2} \langle p, Jp \rangle,
\]

where now \( \langle \cdot, \cdot \rangle \) denote the standard scalar product in \( \mathbb{R}^n \). The equations of motion \( (4.4) \) become

\[
\begin{align*}
\dot{M} &= [M, \Omega] + p \wedge (Jp), \\
\dot{p} &= -\Omega p,
\end{align*}
\]

where the notation \( p \wedge q = pq^T - qp^T \) for \( p, q \in \mathbb{R}^n \) is used. This system is Hamiltonian on the dual of the semidirect product Lie algebra \( e(n) = so(n) \ltimes \mathbb{R}^n \), with the Hamilton function

\[
H(M, p) = \frac{1}{2} \langle M, \Omega \rangle - \frac{1}{2} \langle p, Jp \rangle = \frac{1}{2} \sum_{j<k} M_{jk}^2 J_j + J_k - \frac{1}{2} \sum_{k=1}^n J_k p_k^2.
\]

The integrable discretization of this system is given by the discrete time Lagrange function

\[
\mathcal{L}(g_k, g_{k+1}) = \Lambda(l)(W_k, p_k) = -\frac{1}{\epsilon} \text{tr}(W_k J) + \frac{\epsilon}{2} \langle p_{k+1}, Jp_k \rangle = -\frac{1}{\epsilon} \text{tr}(W_k J) + \frac{\epsilon}{2} \langle p_k, W_k Jp_k \rangle,
\]

where, as usual, \( W_k = g_k^T g_{k+1} \in G \) and \( p_k = g_k^T a \in O_a \). The equations of motion of this discretization read:

\[
\begin{align*}
M_k &= \frac{1}{\epsilon} (W_k J - JW_k^T) - \frac{\epsilon}{2} p_k \wedge (Jp_{k+1}) \\
M_{k+1} &= \frac{1}{\epsilon} (JW_k - W_k^T J) + \frac{\epsilon}{2} p_{k+1} \wedge (Jp_k) \\
p_{k+1} &= W_k^T p_k.
\end{align*}
\]
The Lax matrix and the Lax representation of this map are obtained from (5.10) and Theorem 5 by replacing $P_k$ through $p_k p_k^T$.

It remains to be noticed that originally the integrable Hamiltonian (6.5) was found (in the “physical” case $n = 3$) by Clebsch [C] in another setting, namely in the problem of a motion of a rigid body in an ideal fluid. (An $n$-dimensional generalization is due to Perelomov [P]). In this setting the system appears as a result of reduction of a Lagrangian on the group $E(n)$, left–invariant under the action of a whole group, rather than a Lagrangian on $SO(n)$, left–invariant under the action of an isotropy subgroup of $a$. Nevertheless, these two different settings lead to formally identical results.

7 Conclusion

The model introduced in the present paper serves as a further important example of the completely integrable Lagrangian systems with a discrete time à la Moser–Veselov. Actually, this is the second example (after [BS1]) where the version of the discrete Lagrangian reduction is essential, leading to systems on duals to semidirect product Lie algebras. This new application was made possible due to the theoretical development in [BS2]. Probably, the type of Lagrangians introduced in these papers is able to produce further interesting examples, important for applications. At this point, I would like to express my gratitude to A.Bobenko, the collaboration with whom in [BS1], [BS2] was crucial also for this work.

Generally, we consider the discrete time Lagrangian mechanics as an important source of symplectic and, more general, Poisson maps. From some points of view the variational (Lagrangian) structure is even more fundamental and important than the Poisson (Hamiltonian) one (cf. [HMR], [MPS], where a similar viewpoint is represented). A very intriguing and still not completely understood point is a capability of the discrete Lagrangian approach to produce completely integrable systems. It would be highly desirable to continue the search for integrable Lagrangian discretizations of the known integrable systems. Hopefully, the list of such discretizations established in [V],[MV], [BS1], and the present paper, will be further extended. Also the generalizations to the infinite dimensional case, e.g. to discretization of ideal compressible fluids motion (see [HMR]), are highly desirable.

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