THE CONE OF WEIGHTS OF TOTAL STABILITY FOR TYPE A QUIVERS

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ABSTRACT. Reineke posed the following problem in a 2003 paper: given any Dynkin quiver $Q$, determine if there exists a weight $\theta \in \mathbb{R}^{Q_0}$ such that all indecomposable representations of $Q$ are stable with respect to the classical slope function (a.k.a. standard linear stability condition) determined by $\theta$. This problem was recently solved for type $A$ quivers by Apruzzese-Igusa and independently by Huang-Hu. In this paper, we describe all solutions to this problem for type $A$ quivers via an explicit minimal set of inequalities defining the cone in $\mathbb{R}^{Q_0}$ of all such weights.

1. Introduction

1.1. Problem statement. Let $Q$ be a quiver and fix an algebraically closed field $k$ over which all representations are taken. While some familiarity with quiver representations is assumed in the introduction, a detailed recollection of the necessary background is found in Section 2.

For a weight $\theta \in \mathbb{R}^{Q_0}$, we define the classical slope function (i.e., standard linear stability condition) on the space of nonzero dimension vectors for $Q$ by $\mu_\theta(d) = (\theta \cdot d)/|d|$. We extend this notation to nonzero representations by $\mu_\theta(V) := \mu_\theta(\dim V)$, and a representation of $Q$ is called $\mu_\theta$-stable if $\mu_\theta(W) < \mu_\theta(V)$ for all nonzero, proper subrepresentations $0 < W < V$. Stability of quiver representations has connections with many other notions in mathematics and mathematical physics, such as moduli spaces of representations, semi-invariants, Harder-Narasimhan filtrations, and green paths and sequences. We refer the reader to [Igu] and the references therein for more detail about these connections.

We are interested in weights $\theta$ such that every indecomposable representation of $Q$ is $\mu_\theta$-stable.

Definition 1.1. A weight $\theta$ for a quiver $Q$ is a weight of total stability if every indecomposable representation of $Q$ is $\mu_\theta$-stable. The set of weights of total stability for $Q$ is denoted

$$TS(Q) = \{ \theta \in \mathbb{R}^{Q_0} \mid \mu_\theta(V) > \mu_\theta(W) \text{ for all } V \in \text{Ind}(Q) \text{ and for all } 0 < W < V \}.$$  

Since stable representations have 1-dimensional endomorphism ring, $TS(Q)$ can only be nonempty if $Q$ is of Dynkin type. Noting that for fixed $0 < W < V$ the expression $\mu_\theta(V) - \mu_\theta(W)$ is linear in $\theta$, and that there are only finitely many such expressions for $V$ indecomposable in Dynkin type, we see that $TS(Q)$ is defined by finitely many linear inequalities in $\mathbb{R}^{Q_0}$. In this terminology, we propose the following variant of Reineke’s conjecture [Rei03, Conjecture 7.1] (also see Remark 1.5).

Problem 1.3. Given $Q$ of $\mathbb{A}\mathbb{D}\mathbb{E}$ Dynkin type, find a minimal set of defining inequalities for $TS(Q)$.

In this paper we mainly consider quivers of Dynkin type $A$. This means that the underlying undirected graph is of the form

$$1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n.$$ 

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and we say the quiver is of type $A_n$ if we want to specify that it has $n$ vertices. For equioriented type $A$ quivers, meaning all arrows point in the same direction, it is easy to verify this conjecture [Rei03, Example A] and solve Problem 1.3. This due to the fact that the all indecomposable representations are uniserial in this case, and the result is that the only conditions for a weight to be in $TS(Q)$ is that its entries must decrease along the direction of the arrows. In general, however, $C^Q$ depends on the orientation of $Q$ (see Examples 1.11 and 4.1 below).

For $Q$ of type $A$ and arbitrary orientation, $TS(Q)$ was recently shown to be nonempty in independent papers of Apruzzese-Igusa [AI] and Huang-Hu [HH], using quite different methods. In [AI] it is a consequence of more general results about an extension of Reineke’s conjecture to affine type $A$, in the context of maximal green sequences. In Theorem 1.21 of this paper, we solve Problem 1.3, using methods which are independent of the above cited papers. Then in Corollary 1.24, we give an elementary proof that $TS(Q)$ is nonempty for $Q$ of Dynkin type $A$.

**Remark 1.5.** In recent work with Yari Diaz and Cody Gilbert, we have found counterexamples to [Rei03, Conjecture 7.1] (i.e. shown that $TS(Q)$ is empty) for certain quivers of Dynkin types $D_n$ for all $n \geq 9$ and types $E_7, E_8$ (and shown that there are not counterexamples in other Dynkin types). These results will be the subject of a future paper with these authors. □

### 1.2. Results.

The following notation for type $A$ quivers is useful to organize the proof of the main theorem. A running example illustrating the notation starts with Example 1.11.

**Notation 1.6.** Given a type $A$ quiver $Q$ as in (1.4), recursively define functions $x, y : Q_0 \to \mathbb{R}$ by setting $x(1) = y(1) = 0$, and then for $i > 1$:

\[
\begin{align*}
(x(i + 1) = x(i) + 1 & \text{ and } y(i + 1) = y(i) & \text{ if there is an arrow } i \to i + 1, \\
(x(i + 1) = x(i) & \text{ and } y(i + 1) = y(i) + 1 & \text{ if there is an arrow } i + 1 \to i.
\end{align*}
\]

(Visually, these give us an embedding $Q \subset \mathbb{R}^2$ by specifying the $x, y$-coordinates of the vertices and then connecting them with arrows in the simplest way; see (1.12)).

This determines two sequences of subsets of $Q_0$, which are pairwise disjoint within each sequence:

\[
(1.8) \quad X_k^Q = \{z \in Q_0 \mid x(z) = k\}, \quad Y_k^Q = \{z \in Q_0 \mid y(z) = k\}, \quad \text{for } k \in \mathbb{Z}_{\geq 1}.
\]

We furthermore define

\[
(1.9) \quad \widetilde{X}_i^Q := \bigcup_{k=i}^{x(n)} X_k^Q \quad \text{ and } \quad \widetilde{Y}_i^Q := \bigcup_{k=i}^{y(n)} Y_k^Q
\]

to get chains of subsets of $Q_0$:

\[
(1.10) \quad \widetilde{X}_{x(n)}^Q \supset \widetilde{X}_{x(n) - 1}^Q \supset \cdots \supset \widetilde{X}_2^Q \supset \widetilde{X}_1^Q \supset \widetilde{X}_0^Q = Q_0 = \widetilde{Y}_0^Q \supset \widetilde{Y}_1^Q \supset \widetilde{Y}_2^Q \supset \cdots \supset \widetilde{Y}_{y(n) - 1}^Q \supset \widetilde{Y}_{y(n)}^Q.
\]
Example 1.11. The quiver below shows the orientation of a type A quiver embedded in $R^2$ as described in Notation 1.6.

\[
Q = \begin{array}{ccc}
    & 7 & \\
1 & \rightarrow & 8 \\
& 4 & \rightarrow & 5 & \rightarrow & 6 \\
& 1 & \rightarrow & 2 & \rightarrow & 3
\end{array}
\]

The corresponding partitions of $Q_0$ come from vertically and horizontally aligned subsets of $Q_0$:

1. $X_0 = \{1\}$, $X_1 = \{2\}$, $X_2 = \{3,4\}$, $X_3 = \{5\}$, $X_4 = \{6,7\}$, $X_4 = \{8\}$
2. $Y_0 = \{1,2,3\}$, $Y_1 = \{4,5,6\}$, $Y_2 = \{7,8\}$.

The chains in (1.10) come from filtering the vertices by $x$-coordinate and $y$-coordinate respectively:

\[
\begin{align*}
8 & \subset \{6,7,8\} \subset \{5,6,7,8\} \subset \{3,4,\ldots,8\} \subset \{2,\ldots,8\} \subset Q_0 \\
Q_0 & \supset \{4,5,6,7,8\} \supset \{7,8\}.
\end{align*}
\]

We use the following shorthand for inequalities defining $TS(Q)$.

Notation 1.17. For a proper nonzero subrepresentation $W < V$ of $Q$, define the linear function of $\theta \in R^{Q_0}$:

\[
I_{W < V}^Q(\theta) = \mu_\theta(V) - \mu_\theta(W).
\]

Given a nonempty subset $S \subseteq Q_0$, we define the function of $\theta$:

\[
\text{Avg}(\theta; S) := \text{Avg}\{\theta_i \mid i \in S\}.
\]

For thin representations $W < V$ with $S = \text{Supp} \, W$, $T = \text{Supp} \, V$, we have the equivalence

\[
I_{W < V}^Q(\theta) > 0 \iff \text{Avg}(\theta; S) < \text{Avg}(\theta; T).
\]

The main result of the paper is the description of $TS(Q)$ below.

Theorem 1.21. Let $Q$ be a quiver of Dynkin type $A_n$ as in (1.4) and recall Notations 1.6 and 1.17. A weight $\theta \in R^{Q_0}$ is in the cone $TS(Q)$ of weights of total stability for $Q$ if and only if the $n - 1$ inequalities below hold good:

\[
\begin{align*}
\text{Avg}(\theta; X_0^Q) > & \text{Avg}(\theta; X_1^Q) > \cdots > \text{Avg}(\theta; X_{x(n)}^Q), \\
\text{Avg}(\theta; Y_0^Q) < & \text{Avg}(\theta; Y_1^Q) < \cdots < \text{Avg}(\theta; Y_{y(n)}^Q).
\end{align*}
\]

Furthermore, the inequalities above are the minimal set of inequalities defining this cone.

The proof of this theorem, which uses entirely elementary methods, will be given in Section 3. One notices however that Theorem 1.21 does not provide immediate insight on Reineke’s original conjecture that $TS(Q)$ is nonempty, since each $\theta$ variable appears twice in the sequence of inequalities and on potentially conflicting sides of inequalities (see Example 4.1). But with a little more work, we obtain the following corollary (as was known from [AI, HH]). Its proof is in Section 3 as well.

Corollary 1.24. For any quiver of Dynkin type $A$, the cone $TS(Q)$ is nonempty.
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2. Background

In this section we establish our notation and make some initial reductions for the proof of the main theorem. More detailed background can be found in textbooks such as [Sch14, DW17] and the survey [Rei08].

2.1. Quiver representations. We write \( Q_0 \) for the set of vertices of a quiver \( Q \), and \( Q_1 \) for its set of arrows, while \( t\alpha \) and \( h\alpha \) denote the tail and head of an arrow \( \alpha \). A representation \( V \) of \( Q \) assigns a finite-dimensional vector space \( V(z) \) to each \( z \in Q_0 \), and to each \( \alpha \in Q_1 \) a choice of linear map \( V(\alpha) : V(t\alpha) \to V(h\alpha) \). A subrepresentation \( W \subseteq V \) is a collection of subspaces \( (W(z) \subseteq V(z))_{z \in Q_0} \) such that \( V(\alpha)(W(t\alpha)) \subseteq W(h\alpha) \) for all \( \alpha \in Q_1 \). The support of a representation \( V \), written \( \text{Supp} V \), is the set of vertices \( z \in Q_0 \) such that \( V(z) \neq 0 \). A representation \( V \) is thin if \( \dim V(z) \leq 1 \) for all \( z \in Q_0 \). Definitions of standard notions such as morphisms, direct sum, and indecomposability can be found in the references above.

Notation 2.1. For a subset \( S \subseteq Q_0 \), we let \([U]\) be the representation \( Q \) such that

\[
[S](z) = \begin{cases} 
  k & z \in S \\
  0 & z \notin S 
\end{cases} \quad \text{and} \quad [S](\alpha) = \begin{cases} 
  id_k & t\alpha, s\alpha \in S \\
  0 & \text{otherwise.}
\end{cases}
\]

For a quiver of type \( A_n \), it can be seen from repeated use of Gaussian elimination that as \( S \) varies over all intervals \( \{i, \ldots, j\} \) for \( 1 \leq i \leq j \leq n \), the representations \([S]\) trace out all isomorphism classes of indecomposables for type \( A \) quivers (a special case of Gabriel’s theorem [Gab72]).

2.2. Stability. A weight on a quiver \( Q \) is an element \( \theta \in \mathbb{R}^{Q_0} \), where we write \( \theta_z \in \mathbb{R} \) for the value in coordinate \( z \in Q_0 \), and we write \( |\theta| := \sum_{z \in Q_0} \theta_z \). A dimension vector for \( Q \) is a weight such that each \( \theta_z \) is a nonnegative integer. The dimension vector of a representation \( V \) of \( Q \) is \( (\dim V(z))_{z \in Q_0} \), and we usually use bold Roman letters such as \( \mathbf{d}, \mathbf{e} \) for dimension vectors. Given a weight \( \theta \) and representation \( V \) of \( Q \), we write \( \theta(V) := \theta \cdot \dim V \), where \( \cdot \) is the standard dot product on \( \mathbb{R}^{Q_0} \). The (standard linear) stability condition, or classical slope function, determined by a weight \( \theta \) is the function

\[
\mu_\theta : \mathbb{Z}^{Q_0} \to \mathbb{R}, \quad \mu_\theta(\mathbf{d}) = \frac{\theta \cdot \mathbf{d}}{|\mathbf{d}|}.
\]

The slope of a nonzero representation \( V \) is \( \mu_\theta(V) := \mu_\theta(\dim V) \). We say \( V \) is \( \theta \)-stable if \( \mu_\theta(W) < \mu_\theta(V) \) for all proper, nonzero subrepresentations \( W < V \).

Remark 2.4. Another notion of stability which is prevalent in quiver literature is the following [Kin94]. A representation \( V \) of \( Q \) is \( \theta \)-stable if \( \theta \cdot V = 0 \) and \( \theta \cdot W < 0 \) for all proper, nonzero
subrepresentations \( W < V \). Clearly a representation which is \( \theta \)-stable is \( \mu_\theta \)-stable as well, but the converse does not hold. However, for a fixed \( V \) of dimension vector \( d \), we can define
\begin{equation}
\theta' := |d| \theta - (\theta \cdot V)(1, \ldots, 1)
\end{equation}
and we have that \( V \) is \( \theta \)-stable if and only if \( V \) is \( \mu_{\theta'} \)-stable.

The \( \theta \)-stable representations for fixed \( \theta \) are the simple objects of the full, abelian subcategory of \( \theta \)-semistable representations inside the category of all finite-dimensional representations of \( Q \). Thus we can never have all indecomposable representations \( \theta \)-stable in the above sense if \( Q \) has a nonempty arrow set (by Schur’s lemma).

2.3. Initial reductions. The results of this section are valid for all quivers, not just type \( A \). Presumably all of these lemmas have been observed elsewhere, but we include proofs of everything for completeness. The first reduction shows that we only need to consider the slopes of indecomposable subrepresentations to determine if a representation is stable.

**Lemma 2.6.** Let \( Q \) be an arbitrary quiver. A representation \( V \) of \( Q \) is \( \mu_\theta \)-stable if and only if \( \mu_\theta(W) < \mu_\theta(V) \) for all proper nonzero indecomposable subrepresentations \( W < V \).

**Proof.** One implication follows from the definition. For the converse, assume \( \mu_\theta(W) < \mu_\theta(V) \) for all proper indecomposable subrepresentations \( W < V \), and let \( Y < V \) be an arbitrary proper nonzero subrepresentation. Taking \( W \leq Y \) to be an indecomposable direct summand of \( Y \) of maximal slope, we have \( \mu_\theta(Y) \leq \mu_\theta(W) \) by [Rei08, Lemma 4.1(3)], and the result follows.

Recall that from a quiver \( Q \) we obtain its **opposite quiver** \( Q^{\text{op}} \) by reversing the orientations of all arrows of \( Q \). Note that a weight on \( Q \) is also a weight on \( Q^{\text{op}} \). Taking the vector space dual at each vertex, and dual map over each arrow, gives an equivalence between the categories of representations of \( Q \) and \( Q^{\text{op}} \). The following lemma gives a helpful connection between stability in these categories.

**Lemma 2.7.** Let \( Q \) be an arbitrary quiver and \( \theta \) a weight for \( Q \). Let \( W < V \) be a proper nonzero subrepresentation and \( (V/W)^* < V^* \) the corresponding dual subrepresentation. Then \( I_{W < V}(\theta) > 0 \) if and only if \( I_{(V/W)^* < V^*}(\theta^*) < 0 \).

**Proof.** Note that \( \mu_\theta(X) = \mu_\theta(X^*) \) for any representation \( X \) of \( Q \). So this lemma is immediate from [Rei08, Lemma 4.1(2)], which says that \( \mu_\theta(W) < \mu_\theta(V) \) if and only if \( \mu_\theta(V) < \mu_\theta(W) \).

The following lemma is used to prove the minimality part of the main theorem.

**Lemma 2.8.** Let \( Q \) be a connected quiver such that \( TS(Q) \) is nonempty. Then the only subspace of \( \mathbb{R}^{Q_0} \) which has a translate contained in \( TS(Q) \) is \( \mathbb{R}(1, \ldots, 1) \).

**Proof.** For any weight \( \theta \) and \( c \in \mathbb{R} \), it can be directly checked that \( \mu_{\theta+c(1, \ldots, 1)}(V) = \mu_\theta(V) + c \), and thus \( \theta \in TS(Q) \) if and only if \( \theta + c(1, \ldots, 1) \in TS(Q) \). So \( \mathbb{R}(1, \ldots, 1) \) has a translate contained in \( TS(Q) \).

Suppose for contradiction that there exists \( \theta \in TS(Q) \) and \( \eta \in \mathbb{R}^{Q_0} \), where \( \mathbb{R} \eta \neq \mathbb{R}(1, \ldots, 1) \), such that \( \theta + \mathbb{R} \eta \subset TS(Q) \). Since \( Q \) is connected, \( \mathbb{R} \eta \neq \mathbb{R}(1, \ldots, 1) \) implies that there exists \( \alpha \in Q_1 \) such that \( \eta_{h\alpha} \neq \eta_{h\alpha} \). Let \( V \) be the thin indecomposable representation whose support is exactly \( \{t\alpha, h\alpha\} \), and \( W < V \) the simple subrepresentation supported at \( \{h\alpha\} \). Then we compute
\begin{equation}
\mu_\eta(V) - \mu_\eta(W) = \frac{\eta_{h\alpha} + \eta_{h\alpha}}{2} - \eta_{h\alpha} = \frac{\eta_{h\alpha} - \eta_{h\alpha}}{2} \neq 0.
\end{equation}
Since \( \theta + \mathbb{R} \eta \subset \mathcal{T} \mathcal{S}(Q) \), for any \( c \in \mathbb{R} \) we have
\[
\mu_{\theta+c\eta}(V) - \mu_{\theta+c\eta}(W) = (\mu_\theta(V) - \mu_\theta(W)) + c(\mu_\eta(V) - \mu_\eta(W)) > 0,
\]
a contradiction since \( c \) is arbitrary and the other values in the middle expression are fixed. \( \square \)

3. Proof of the main theorem

We begin with some elementary lemmas that are used repeatedly throughout the proof of the main theorem; their proofs are omitted.

**Lemma 3.1.** Let \( S, T \subseteq Q_0 \) be nonempty subsets such that \( S \cap T = \emptyset \). Then we have:
\[
\text{Avg}(\theta; S) < \text{Avg}(\theta; T)
\]
\[
\Leftrightarrow \text{Avg}(\theta; S) < \text{Avg}(\theta; S \cup T)
\]
\[
\Leftrightarrow \text{Avg}(\theta; S \cup T) < \text{Avg}(\theta; T).
\]

The previous lemma can be repeatedly applied up the chains (1.10) to obtain the following.

**Lemma 3.3.** A point \( \theta \in \mathbb{R}^{Q_0} \) satisfies the sequence of inequalities (1.22) if and only if it satisfies
\[
\text{Avg}(\theta; \tilde{X}_0^Q) > \text{Avg}(\theta; \tilde{X}_1^Q) > \cdots > \text{Avg}(\theta; \tilde{X}_{x(n)}^Q),
\]
Similarly, \( \theta \) satisfies the sequence of inequalities (1.23) if and only if it satisfies
\[
\text{Avg}(\theta; \tilde{Y}_0^Q) < \text{Avg}(\theta; \tilde{Y}_1^Q) < \cdots < \text{Avg}(\theta; \tilde{Y}_{y(n)}^Q).
\]

Since \( \tilde{X}_0^Q = Q_0 = \tilde{Y}_0^Q \), we may concatenate the chains (3.4) and (3.5) to obtain:
\[
\text{Avg} \left( \theta; \tilde{X}_{x(n)}^Q \right) < \cdots < \text{Avg} \left( \theta; \tilde{X}_1^Q \right) < \text{Avg}(\theta; Q_0) < \text{Avg} \left( \theta; \tilde{Y}_1^Q \right) < \cdots < \text{Avg} \left( \theta; \tilde{Y}_{y(n)}^Q \right).
\]

We make one final observation about supports of certain indecomposable representations of \( Q \).

**Lemma 3.7.** Let \( Q \) be a type \( \mathbb{A}_n \) quiver, and \( V \) an indecomposable representation of \( Q \) with \( n \in \text{Supp} V \). Then there exists \( k \) such that either \( \text{Supp} V = \tilde{X}_k^Q \) or \( \text{Supp} V = \tilde{Y}_k^Q \).

Proof of “if and only if” statement of Theorem 1.21. The \( \Rightarrow \) direction of the proof is just from the definitions: consider the subrepresentations \( [\tilde{X}_k^Q] \subset [\tilde{X}_{k-1}^Q] \) for \( 1 \leq k \leq x(n) \). For \( \theta \in \mathcal{T} \mathcal{S}(Q) \), the observation in (1.20) and Lemma 3.3 show that the series of inequalities (1.22) must hold. Similarly, the representations \( [\tilde{Y}_k^Q] \) are used to show the inequalities (1.23) also hold for \( \theta \in \mathcal{T} \mathcal{S}(Q) \).

For the \( \Leftarrow \) direction, we need to show that if \( \theta \in \mathbb{R}^{Q_0} \) satisfies the inequalities (1.22) and (1.23), and thus the chain (3.6) as well, then \( \theta \in \mathcal{T} \mathcal{S}(Q) \). This means that \( I_{W<V}(\theta) > 0 \) for all \( V \) indecomposable and \( 0 < W < V \), and we are reduced to the case that \( W \) is also indecomposable by Lemma 2.6.

We use induction on the number of vertices of \( Q \), with the statement being vacuously true in the base case \( n = 1 \) (the unique indecomposable is stable with respect to any \( \theta = (\theta_1) \), and there are no inequalities to satisfy). Let \( Q \) be a type \( \mathbb{A}_n \) quiver and assume the theorem is true for type \( \mathbb{A}_{n-1} \) quivers. The primary challenge in the induction is that the collection of inequalities (1.22), (1.23) for \( Q \) does not simply restrict to the corresponding collection of inequalities for smaller quivers, so we cannot easily apply the induction hypothesis.
Consider the arrow \( n - 1 \to n \) in \( Q \): we can assume \( n \) is a sink without loss of generality because Lemma 2.7 gives us the \( n - 1 \leftarrow n \) case from this by reversing the directions of all inequalities, noting that the sets \( X_k \) and \( Y_k \) are interchanged when switching between \( Q \) and \( Q^{\text{op}} \). Thus we have
\[
x(n) = x(n-1) + 1 \quad \text{and} \quad y(n) = y(n-1).
\]
Let \( \overline{Q} \) be the quiver obtained by removing vertex \( n \) and the arrow connected to it, and \( \bar{\theta} \in \mathbb{R}^\overline{Q}_0 \) the restriction of \( \theta \) to \( \overline{Q}_0 \).

To apply the induction hypothesis to \( \overline{Q} \), we need to show that \( \bar{\theta} \) satisfies the sequences of inequalities in (1.22) and (1.23) associated to \( \overline{Q} \), namely:
\[
\begin{align*}
(3.8) \quad & \quad \text{Avg}(\bar{\theta}; X_{\overline{Q}}^0) > \text{Avg}(\bar{\theta}; X_{\overline{Q}}^1) > \cdots > \text{Avg}(\bar{\theta}; X_{\overline{Q}}^{x(n-1)}), \\
(3.9) \quad & \quad \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^0) < \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^1) < \cdots < \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n-1)}).
\end{align*}
\]
Whenever \( n \notin S \subseteq \overline{Q}_0 \), the function \( \text{Avg}(\theta; S) \) is independent of \( \theta_n \), and can thus be identified with the function \( \text{Avg}(\bar{\theta}; S) \) on \( \mathbb{R}^\overline{Q}_0 \). Since \( X_k^Q = X_k^Q \) for \( 0 \leq k \leq x(n-1) \) and \( Y_k^Q = Y_k^Q \) for \( 0 \leq k \leq y(n-1) - 1 \), we know \( \bar{\theta} \) satisfies all the inequalities in (3.8) and (3.9), except perhaps the far right inequality of (3.9), where we must deal with the fact that \( Y_{\overline{Q}}^{y(n-1)} = Y_{\overline{Q}}^{y(n-1)} \setminus \{n\} \).

Thus to apply the induction hypothesis, it remains to show that
\[
(3.10) \quad \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n-1)}) < \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n)}),
\]
where we use \( y(n-1) = y(n) \) to simplify the notation here and below. Recalling that \( X_{\overline{Q}}^{x(n)} = \{n\} \) since \( n \) is a sink, from (3.6) we can extract
\[
(3.11) \quad \theta_n = \text{Avg}(\bar{\theta}; \overline{X}_{\overline{Q}}^{x(n)}) < \text{Avg}(\bar{\theta}; \overline{Y}_{\overline{Q}}^{y(n-1)}).
\]
We also have by definition \( n \in Y_{\overline{Q}}^{y(n)} \subseteq \overline{Y}_{\overline{Q}}^{y(n)-1} \), so (3.11) and (3.2) imply that
\[
(3.12) \quad \text{Avg}(\bar{\theta}; \overline{Y}_{\overline{Q}}^{y(n)-1}) < \text{Avg}(\bar{\theta}; \overline{Y}_{\overline{Q}}^{y(n)-1} \setminus \{n\}).
\]
Furthermore, from (1.23) and (3.2) with \( \overline{Y}_{\overline{Q}}^{y(n)-1} = Y_{\overline{Q}}^{y(n)-1} \prod Y_{\overline{Q}}^{y(n)} \), we get
\[
(3.13) \quad \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n)-1}) < \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n)}) \quad \Rightarrow \quad \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n)-1}) < \text{Avg}(\bar{\theta}; \overline{Y}_{\overline{Q}}^{y(n)-1}),
\]
so (3.12), (3.13), and (3.2) with \( \overline{Y}_{\overline{Q}}^{y(n)-1} \setminus \{n\} = (Y_{\overline{Q}}^{y(n)} \setminus \{n\}) \prod Y_{\overline{Q}}^{y(n)-1} \) give us
\[
(3.14) \quad \text{Avg}(\bar{\theta}; Y_{\overline{Q}}^{y(n)-1}) < \text{Avg}(\bar{\theta}; \overline{Y}_{\overline{Q}}^{y(n)-1} \setminus \{n\}).
\]
This is exactly the inequality (3.10) we set out to show in this paragraph.

Now by the induction hypothesis, \( \bar{\theta} \) satisfies all inequalities \( I_{W <V}(\bar{\theta}) \) for \( V \) an indecomposable representation of \( \overline{Q} \). This means \( \theta \) satisfies all such inequalities when \( n \notin \text{Supp} \, V \). So it remains to consider the inequalities \( I_{W <V}(\theta) \) where \( n \in \text{Supp} \, V \).

We first consider inequalities \( I_{W <V}(\theta) > 0 \) when \( n \in \text{Supp} \, W \). Let \( S = \text{Supp} \, W \) and \( T = \text{Supp} \, V \), and write \( \bar{S} = S \setminus \{n\} \) and \( \bar{T} = T \setminus \{n\} \). Then \( I_{W <V}(\theta) > 0 \) is equivalent to \( \text{Avg}(\theta; S) < \text{Avg}(\theta; T \setminus S) \). To show this holds, it is enough to show both:
\[
(3.15) \quad (i) \quad \text{Avg}(\theta; \bar{S}) < \text{Avg}(\theta; T \setminus S) \quad \text{and} \quad (ii) \quad \theta_n < \text{Avg}(\theta; T \setminus S).
\]
But (i) is immediate from the induction hypothesis since \([\bar{S}]\) is a subrepresentation of \([\bar{T}]\) and \(\bar{T} \setminus \bar{S} = T \setminus S\). For (ii), we consider two cases. First, if \(S = \{n\}\), then (ii) follows from Lemma 3.3: indeed, the least term of the sequence (3.6) is \(\theta_n\), and \(\text{Avg}(\bar{\theta}; T)\) must appear in this chain by Lemma 3.7. This gives \(\theta_n < \text{Avg}(\bar{\theta}; T)\) and thus (ii) holds in this case. Otherwise \(S \supset \{n\}\), and we have \(X^Q_{x(n)-1} \subseteq S \subseteq T\) because \(W\) is a proper subrepresentation of \(V\). Then we have 
\[
X^Q_{x(n)-1} = X^\bar{Q}_{\bar{x}(n)-1} \subseteq \bar{S} \subseteq \bar{T}
\]
as well, so from the far right inequality of (1.22) and the induction hypothesis we get 
\[
\theta_n < \text{Avg}(\bar{\theta}; X^Q_{x(n)-1}) = \text{Avg}(\bar{\theta}; X^\bar{Q}_{\bar{x}(n)-1}) \leq \text{Avg}(\bar{\theta}; S) < \text{Avg}(\bar{\theta}; \bar{T}).
\]

Then (3.2) to the rightmost equality gives \(\text{Avg}(\bar{\theta}; \bar{S}) < \text{Avg}(\bar{\theta}; \bar{T} \setminus \bar{S})\), and then 
\[
\theta_n < \text{Avg}(\bar{\theta}; \bar{S}) < \text{Avg}(\bar{\theta}; T \setminus \bar{S}) = \text{Avg}(\theta; T \setminus S).
\]

This shows (ii) holds in this case.

We now consider inequalities \(I_{W \subsetneq V}(\theta) > 0\) when \(n \notin \text{Supp} W\) but \(n \in \text{Supp} V\). Fix such a \(V\) and set \(T = \text{Supp} V\), noting \(T \supset Y^Q_{y(n)}\) in order for \(V\) to contain a subrepresentation without \(n\) in its support. Then \(V\) has a unique maximal subrepresentation \(W\) not supported at \(n\), namely \(W_0 = [T \setminus Y^Q_{y(n)}]\). The induction hypothesis then implies that \(\mu_{\bar{\theta}}(W) = \mu_{\theta}(W)\) is maximized at \(W_0\), so it is enough to show that \(\mu_{\theta}(W_0) = \text{Avg}(\bar{\theta}; T \setminus Y^Q_{y(n)}) < \text{Avg}(\theta; T) = \mu_{\theta}(V)\). By (3.2) this is equivalent to \(\text{Avg}(\theta; T) < \text{Avg}(\theta; Y^Q_{y(n)})\), which follows from (3.6) and Lemma 3.7 (recalling \(Y^Q_{y(n)} = \bar{Y}^Q_{y(n)}\)).

Having shown that \(I_{W \subsetneq V}(\theta) > 0\) for all \(W, V\) indecomposable and \(0 < W < V\), the proof of the “if and only if” part is completed.

**Proof of minimality.** To prove minimality, we need to use that the cone is nonempty (or the inequalities would obviously not be minimal). This is proven independently to the minimality claim in Corollary 1.24 below, so let us assume it for now. For a quiver of type \(\mathbb{A}_n\), we have \(n - 1\) inequalities \(\{Y_{\alpha}(\theta) > 0\}_{\alpha \in Q_1}\) on \(\mathbb{R}^n\). If any of them could be omitted, then \(T \mathcal{S}(Q)\) could be represented as the intersection of \(n - 2\) or fewer half spaces. But then \(T \mathcal{S}(Q)\) would contain a translate of a two-dimensional subspace of \(\mathbb{R}^n\), contradicting Lemma 2.8.

**Proof of Corollary 1.24.** This proof is due to Hugh Thomas. We begin by setting 
\[
x_i := |X_i^Q|, \quad y_i := |Y_i^Q|, \quad \bar{x}_i := \sum_{k=1}^i x_k, \quad \bar{y}_i := \sum_{k=1}^i y_k,
\]
noting that \(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{y}_1, \bar{y}_2, \ldots\} = \{1, 2, \ldots, n\}\). We consider the linear functions of \(\theta\) defined by 
\[
f_i(\theta) := \text{Avg}(\theta; X_i^Q) - \text{Avg}(\theta; X_{i+1}^Q), \quad 1 \leq i \leq M := \max\{i : X_i^Q \neq \emptyset\},
\]
\[
g_j(\theta) := \text{Avg}(\theta; Y_j^Q) - \text{Avg}(\theta; Y_{j+1}^Q), \quad 1 \leq j \leq N := \max\{j : Y_j^Q \neq \emptyset\}.
\]
Our main theorem says that a weight is in \(T \mathcal{S}(Q)\) if and only if these functions are all strictly positive on the weight. We can assume that both \(M, N \geq 1\), since otherwise the quiver is equioriented and the corollary is immediate [Rei03, Example A].
If \(TS(Q)\) were empty, there would exist a linear combination with nonnegative coefficients

\[
0 = \sum_{i=1}^{M} a_i f_i(\theta) + \sum_{j=1}^{N} b_j g_j(\theta), \quad a_i, b_j \in \mathbb{R}_{\geq 0},
\]

where some \(a_i \neq 0\) for \(1 \leq i \leq M\) or some \(b_j \neq 0\) for \(1 \leq j \leq N\). Assume for contradiction that we have such an expression, and take one for which \(Q\) has a minimal number of vertices. We will successively consider the coefficients of \(\theta_1, \theta_2, \theta_3, \ldots\) and show that (up to a scalar multiple) the vanishing of these coefficients forces \(a_i = \bar{x}_i\) and \(b_j = \bar{y}_j\) up to a point, and then yields a contradiction when considering the coefficient of \(\theta_t\) when either \(x(t)\) or \(y(t)\) is maximal (i.e., in Notation 1.6, when we reach a vertex in the furthest right column of vertices or furthest up row of vertices).

First consider the coefficient of \(\theta_1\). Assume \(1 \to 2\) in \(Q\) (without loss of generality by the same application of Lemma 2.7 used in the proof of the main theorem). This variable appears only in \(f_1(\theta)\) and \(g_1(\theta)\), and the coefficient of \(\theta_1\) in (3.20) is \(a_1 - b_1 \frac{1}{y_1}\). Up to a scalar, we are forced to take \(a_1 = 1 = \bar{x}_1\) and \(b_1 = y_1 = \bar{y}_1\).

Proceeding inductively up the indices for \(\theta\), we next consider the coefficient of \(\theta_t\) for \(1 < t < n\) but \(y(t) = 1\) still (i.e., we have a path \(1 \to 2 \to \cdots \to t\) in \(Q\)). The coefficient of \(\theta_t\) in (3.20) receives contributions from (at most) \(f_{t-1}(\theta), f_t(\theta), g_t(\theta)\). If \(x(t)\) is not maximal, then \(t \leq M\) and for (3.20) to hold we need

\[
0 = a_{t-1} \frac{-1}{x_t} + a_t \frac{1}{x_t} + b_1 \frac{-1}{y_1}.
\]

By induction we already have \(a_{t-1} = \bar{x}_{t-1}(= t - 1)\) and \(b_1 = \bar{y}_1\), so a direct substitution into the above expression yields

\[
\frac{1}{x_t}(a_t - \bar{x}_{t-1}) - 1 = 0,
\]

forcing \(a_t = \bar{x}_{t-1} + x_t = \bar{x}_t\). However, if \(x(t)\) is maximal, then \(M = t - 1\) so for (3.20) to hold we need

\[
a_{t-1} \frac{-1}{x_t} + b_1 \frac{-1}{y_1} = 0,
\]

which is a contradiction since both terms of the left hand side are negative.

Continuing up the indices, consider the general situation of \(t \in Q_0\) such that both \(k := x(t) > 1\) and \(l := y(t) > 1\) and neither is maximal among vertices of \(Q\). The coefficient of \(\theta_t\) in (3.20) receives contributions from \(f_{k-1}(\theta), f_k(\theta), g_{l-1}(\theta), g_l(\theta)\), and (3.20) implies

\[
ak_{k-1} \frac{-1}{x_k} + ak \frac{1}{x_k} + b_{l-1} \frac{1}{y_l} + b_l \frac{-1}{y_l} = 0.
\]

By induction on \(t\) we already have \(a_{k-1} = \bar{x}_{k-1}\) and \(b_{l-1} = \bar{y}_{l-1}\), and either \(a_k = \bar{x}_k\) (if \(t - 1 \leftarrow t\) in \(Q\)) or \(b_l = \bar{y}_l\) (if \(t - 1 \to t\) in \(Q\)), so the remaining coefficient is determined in (3.24). In the case that \(t - 1 \leftarrow t\) in \(Q\), direct substitution into the above expression yields

\[
1 + \frac{1}{y_l}(\bar{y}_{l-1} - b_l) = 0,
\]

forcing \(b_l = \bar{y}_{l-1} + y_l = \bar{y}_l\). The case that \(t - 1 \to t\) in \(Q\) is similar.
At some point we arrive at $t \in Q_0$ such that either $k$ or $l$ is maximal, say $k$ (again the other case is similar). Then the arrows of $Q$ are oriented like $t - 1 \to t \leftarrow \cdots \leftarrow n$. The coefficient of $\theta_t$ has one fewer term and is by induction equal to
\begin{equation}
\label{eqn:3.26}
a_{k-1} \frac{-1}{x_k} + b_{l-1} \frac{1}{y_l} + b_l \frac{-1}{y_l} = \bar{x}_{k-1} \frac{-1}{x_k} - 1 < 0,
\end{equation}
thus nonvanishing. This is the desired contradiction and the corollary is proven. \hfill \Box

4. Example and open problems

We illustrate the main theorem by continuing our running example, and pose a follow up problem.

Example 4.1. Continuing Example 1.11, Theorem 1.21 says that the minimal set of inequalities defining $\mathcal{TS}(Q)$ is:
\begin{align}
\label{eqn:4.2}
\theta_1 & > \theta_2 > \text{Avg}(\theta_3, \theta_4) > \theta_5 > \text{Avg}(\theta_6, \theta_7) > \theta_8, \\
\label{eqn:4.3}
\text{Avg}(\theta_1, \theta_2, \theta_3) & < \text{Avg}(\theta_4, \theta_5, \theta_6) < \text{Avg}(\theta_7, \theta_8).
\end{align}

Corollary 1.24 tells us that this system of inequalities does indeed have a solution, but it does not tell us how to find any specific point satisfying (4.2) and (4.3). The following remark addresses a special case when a procedure for constructing points of $\mathcal{TS}(Q)$ is available.

Remark 4.4. When $Q$ is bipartite (every vertex is either a sink or a source), we can inductively construct $\theta \in \mathcal{TS}(Q)$ by the following procedure. We recycle the notation in the proof of Theorem 1.21 (in particular we still assume $n$ is a sink). Base cases $n = 1, 2, 3$ can easily be checked by hand. The difficulty is that not every choice of $\bar{\theta} \in \mathcal{TS}(\bar{Q})$ will extend to an element of $\mathcal{TS}(Q)$, so a naive approach to induction simply does not work. Instead, we consider additional inequalities
\begin{equation}
\label{eqn:4.5}
\begin{cases}
\theta_{n-1} > \theta_{n-3} > \theta_{n-5} > \cdots \\
\theta_n < \theta_{n-2} < \theta_{n-4} < \cdots
\end{cases}
\end{equation}
and the smaller cone
\begin{equation}
\label{eqn:4.6}
\mathcal{D}(Q) := \{ \theta \in \mathbb{R}^{Q_0} \mid \theta \in \mathcal{TS}(Q) \text{ and } \theta \text{ satisfies (4.5) for all } z \in Q_0 \}.
\end{equation}
This is compatible with the assumption that $n$ is a sink in the sense that replacing $Q$ with $Q^\text{op}$ simply replaces every $\theta$ in $\mathcal{D}(Q)$ with $-\theta$, so there is still no loss of generality in making this assumption.

Now assume $\bar{\theta} \in \mathcal{D}(\bar{Q})$. We want to see that we can choose $\theta_n \in \mathbb{R}$ such that $\theta \in \mathcal{D}(Q)$. This requires three additional inequalities to be satisfied, namely:
\begin{equation}
\label{eqn:4.7}
\theta_n < \text{Avg}(\theta_{n-1}, \theta_{n-2}), \quad \text{Avg}(\theta_{n-1}, \theta_n) > \text{Avg}(\theta_{n-3}, \theta_{n-2}), \quad \theta_n < \theta_{n-2}.
\end{equation}
Since $\theta_{n-1} > \theta_{n-2}$ by the assumption that $\bar{\theta} \in \mathcal{TS}(\bar{Q})$, we have
\begin{equation}
\label{eqn:4.8}
\theta_n < \theta_{n-2} \implies \theta_n < \text{Avg}(\theta_{n-1}, \theta_{n-2}).
\end{equation}
Furthermore, the middle condition of (4.7) is equivalent to $\theta_n > \theta_{n-3} + \theta_{n-2} - \theta_{n-1}$. So there exists $\theta_n$ satisfying the conditions (4.7) if and only if $\theta_{n-2} > \theta_{n-3} + \theta_{n-2} - \theta_{n-1}$, which holds since $\theta_{n-1} > \theta_{n-3}$ by the assumption that $\bar{\theta} \in \mathcal{D}(\bar{Q})$. Thus any $\theta_n$ satisfying
\begin{equation}
\label{eqn:4.9}
\theta_{n-3} + \theta_{n-2} - \theta_{n-1} < \theta_n < \theta_{n-2}
\end{equation}
suffices to continue the inductive construction of a point $\theta \in \mathcal{D}(Q) \subset \mathcal{TS}(Q)$. \hfill \Box
The next problem and example concern a different way to represent polyhedral cones which is more useful for constructing explicit points in the cone.

**Problem 4.10.** The closure of $\mathcal{TS}(Q)$ is a polyhedral cone in $\mathbb{R}^{Q_0}$. Our description of the cone in Theorem 1.21 gives a presentation of the closure as an intersection of half spaces (i.e. an “$H$-representation” of the closure). Give a representation-theoretic description of the extremal rays of this cone (i.e. give a “$V$-representation” of the closure of $\mathcal{TS}(Q)$ in terms of representation theory).

A $V$-representation for our running example was computed with the QPA software [Qt] (authored by Ed Green and Øyvind Solberg) in conjunction with SageMath.

**Example 4.11.** For $Q$ as in Example 1.11, we have $\theta \in \mathcal{TS}(Q)$ if and only if

$$
\theta = \begin{bmatrix}
    d & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    8 & 0 & -5 & 5 & 0 & -2 & 2 & 0 \\
    0 & 0 & 5 & 5 & 0 & -2 & 2 & 0 \\
    0 & 0 & -1 & 1 & 0 & -2 & 2 & 0 \\
    0 & 0 & -3 & 3 & 0 & -6 & 6 & -8 \\
    0 & 0 & -3 & 3 & -2 & -4 & 0 & -2 \\
    0 & 0 & -9 & 9 & 0 & -18 & 2 & -8 \\
    0 & 0 & -9 & 1 & -4 & -6 & -2 & -4 
\end{bmatrix}
$$

for some $d \in \mathbb{R}$ and some $x_1, \ldots, x_7 \in \mathbb{R}_{\geq 0}$ not all 0. Note that rows 2 through 8 are not unique, as any $\mathbb{R}$ multiple of the first row can be added to each. We have no idea at this time how to interpret this with representation theory. □

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