Extending the global-direction stencil with “face-area-weighted centroid” to unstructured finite volume discretization from integral form

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Abstract

Accuracy of unstructured finite volume discretization is greatly influenced by the gradient reconstruction. For the commonly used k-exact reconstruction method, the cell centroid is always chosen as the reference point to formulate the reconstructed function. But in some practical problems, such as the boundary layer, cells in this area are always set with high aspect ratio to improve the local field resolution, and if geometric centroid is still utilized for the spatial discretization, the severe grid skewness cannot be avoided, which is adverse to the numerical performance of unstructured finite volume solver. In previous work, we explored a novel global-direction stencil and combine it with face-area-weighted centroid on unstructured finite volume methods from differential form to realize the skewness reduction and a better reflection of flow anisotropy. Note, however, that the differential form is hard to achieve higher-order accuracy, and in order to set stage for the method promotion on higher-order numerical simulation, in this research, we demonstrate that it is also feasible to extend this novel method to the unstructured finite volume discretization in integral form. Numerical examples governed by linear convective,

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Euler and Laplacian equations are utilized to examine the correctness as well as effectiveness of this extension. Compared with traditional vertex-neighbor and face-neighbor stencils based on the geometric centroid, the grid skewness is almost eliminated and computational accuracy as well as convergence rate is greatly improved by the global-direction stencil with face-area-weighted centroid. As a result, on unstructured finite volume discretization from integral form, the method also has a better numerical performance.

**Keywords:** unstructured finite volume methods, k-exact reconstruction algorithm, global-direction stencil, grid skewness, face-area-weighted centroid

**Introduction**

Compared with the block-structured grid, unstructured grid is highly automated in grid generation [1-3] and convenient for the grid adaptation [4,5], while the computational accuracy and stabilities are hard to be guaranteed [6], especially on high-aspect-ratio triangular grids [7-9]. Regarding the current difficulties, scholars are continued trying to improve the discretization algorithms to break through the bottleneck of accuracy loss and stability deterioration on highly anisotropic grids, and realize the unification of automated grid generation and accurate numerical simulation [10,11].

As for the research about unstructured finite volume discretization on high-aspect-ratio triangular grids, Diskin and Thomas [12,13] test the accuracy of gradient reconstruction, and results show that the accuracy of reconstructed gradient on such grid is determined by both solution and grid, and the existing discretization schemes cannot meet the requirements of grid quality and computational accuracy simultaneously. On this basis, Diskin and Thomas [14,15] et al., systematically compare the numerical performance of inviscid and viscous fluxes on different
node-centered and cell-centered unstructured finite volume methods. Research suggests that for high-aspect-ratio triangular grids, there are few discretization schemes able to preserve the simulation accuracy, and after adding disturbance to grid points, the magnitude of computational errors obtained by all schemes are proportional to cell aspect ratios. Similar conclusions are obtained in Ref. [16].

On this basis, we notice that although there are numerous discretization approaches for high-aspect-ratio triangular grids, studies related to the stencil selection is quite few, and within the framework of unstructured finite volume method, both inviscid and viscous fluxes are greatly influenced by the gradient reconstruction [17], where different stencils play a crucial role. Commonly used stencils are face-neighbor and vertex-neighbor stencils that consist of cells sharing faces or vertices with the central cell. Apart from these two commonly used stencils, an ingenious stencil augmentation method is proposed in Ref. [18]. For each vertex, an extra cell closest to geometric centroid is selected from vertex-adjacent cells and appended to the face-neighbor stencil. On isotropic triangular grid, the instability of face-neighbor stencil is effectively avoided by this improvement, while if the grid has high aspect ratio, the numerical performance is also unstable and results in the computing divergence [19]. Besides, Xiong [20] et al., propose a local-direction stencil selection method, where stencil cells are augmented along two local directions. In this method, characteristics of the flowfield are taken into account during the process of determining local directions. On isotropic triangles, two local directions are close to the normal and tangential directions of the wall, while on high-aspect-ratio triangular grids, stencil cells selected by this method will deviate a lot from the boundary normal, and the numerical performance on such grids will get deteriorated [21,22].
Compared with the local-direction stencil selection method, global direction stencil selection method [22,23] effectively overcomes the deviation on grids with high aspect ratios, and cells are always selected along two global directions, that is, normal and tangential directions of the wall. Hence, the flow anisotropy could be well reflected. And after verification, on high-aspect-ratio triangular grids, global-direction stencil has a better numerical performance on both computational accuracy and efficiency than commonly used face-neighbor and vertex-neighbor stencils as well as the local-direction stencil.

Although global-direction stencil preliminary exhibits a better numerical performance, the distribution of reference points is still much more skewed, especially on high-aspect-ratio triangular grids. You and Mittal [24] et al., first propose the grid skewness on such grids, and conclude that grid skewness is adverse to the computational accuracy and stabilities of CFD solvers. Regarding the grid skewness, there are various definitions of that, such as the angle between face normal and the vector pointing from face centroid to the cell centroid, the minimal internal angle of grid cell [25], ratio of the max diagonal to the minimum [26], etc. Nevertheless, from different definitions, the same conclusion could be drawn that on high-aspect-ratio triangular grids, the grid skewness is always quite evident [27].

Different from above skewness measures, Nishikawa [28] propose a novel definition of that, and it is defined at a face shared by two neighbor cells, say, A and B, as the dot product of the unit vector pointing from the centroid of cell A to that of cell B. Therefore, a non-skewed grid has the measure one and highly-skewed grid nearly zero. Specifically, on highly-skewed triangular grids, the reference point distribution is irregular and exhibits deflective phenomenon. Thus, although global-direction stencil cells mentioned above are along the normal and tangential directions of the
wall, the irregular distribution of reference points is not changed, and it is hard to say whether flowfield characteristics are well reflected or not. In order to reduce the grid skewness and optimize the reference point distribution, Nishikawa propose a novel face-area-weighted centroid [28] for the second-order unstructured finite volume discretization from differential form. After verification, the second-order accuracy is also achieved, and the convergence rate is greatly improved. Besides, compared with the traditional geometric centroid, the distribution of this novel reference point is more regular, and the connection of that is almost parallel to the boundary normal vector.

Based on this phenomenon, in previous work, we combine this novel centroid and global-direction stencil for the second-order unstructured finite volume method [23], and it is verified that the global-direction stencil with face-area-weighted centroid has a lower discretization error than stencils with the geometric centroid including face-neighbor and vertex-neighbor stencils as well as the global-direction stencil. What’s more, this novel method has a more stable numerical performance on high-mach-number flow. As a result, the current situation related to accuracy loss and stability deterioration on high-aspect-ratio triangular grids are greatly ameliorated by the employment of this improved global-direction stencil.

However, above method is designed for the differential finite volume solver, where both solution and source term vectors are evaluated as point values rather than the cell-averaged, and it is not directly applicable to the integral form. Besides, for this special finite volume discretization, the flux divergence is approximated by the cell integral average, and therefore, the error is already committed in governing equations, which cannot be eliminated no matter how accurately the flux integral is discretized, unless the flux is a linear function or a constant. In this situation, the
unstructured finite volume discretization from differential form is only second-order accurate, and if we want to extend it to higher-order accuracy, we can only reconstruct the primitive function \([29]\) of flux to ensure the governing equation is analytically accurate. However, again, there is no detailed research related to the high-order unstructured finite volume discretization from differential form, and although the primitive function of flux could be reconstructed, the unique value at the face integral point and the corresponding boundary condition implementation still need to be further discussed.

From above analysis, we wonder whether the reference point could also be moved in unstructured finite volume method from integral form to set stage for the skewness reduction. In this research, we first verify the feasibility of moving reference point, and derive the \(k\)-exact reconstruction process \([30-36]\) based on any local origins. In addition, we extend the global-direction stencil with face-area-weighted centroid to the integral form, and set four representative numerical examples to verify the effectiveness of this extension.

The paper is organized as follows. In Section 2, governing equations and spatial discretization from both differential and integral forms are given at first. And then, we theoretically derive the \(k\)-exact reconstruction process based on any local origins. Different stencil selection methods and global-direction stencil with face-area-weighted centroid are presented in Section 3. In Section 4, numerical examples governed by linear convective, Euler and Laplacian equations are employed to verify the effectiveness, feasibilities as well as superiorities of the method on integral unstructured finite volume solver. Concluding remarks and future work will be summarized in Section 5.
2. Unstructured finite volume method

In this section, we give unstructured finite volume discretizations from both differential and integral forms at first. In addition, the $k$-exact reconstruction method based on any local origins is also derived in this section.

2.1 Finite volume discretization from differential form

Usually, the differential inviscid governing equation could be formulated as,

$$\partial_t u_j + \text{div} F^c_j = s_j,$$  \hspace{1cm} (1)

where, $u_j$ is a solution vector, $F^c_j$ is the convective flux and $s_j$ is a source (or forcing) term vector. In differential unstructured finite volume method, both solution and source term vectors are evaluated as point values at the geometric centroid, rather than cell-averaged values.

Approximating the flux divergence by the integral average, we obtain

$$\partial_t u_j + \frac{1}{V_j} \oint_{\partial V_j} f^c_j \, dS = s_j,$$  \hspace{1cm} (2)

where, $V_j$ and $\partial V_j$ are the volume and boundaries of cell $j$, and $f^c_j$ is convective flux along the face normal vector. This equation is not accurate in general since, by definition,

$$\text{div} F^c_j = \lim_{r \to 0} \frac{1}{V_j} \oint_{\partial V_j} f^c_j \, dS \approx \frac{1}{V_j} \oint_{\partial V_j} f^c_j \, dS.$$  \hspace{1cm} (3)

Therefore, the governing equation itself has already committed the lower order errors in approximation, and it could not be eliminated no matter how precisely the flux integral is discretized. Hence, currently, the differential form is only second-order accurate.

Note, however, that higher-order accurate discretization from the differential form is also possible to realize. In Ref. [29], where Masatsuka provide an idea by reconstructing the primitive function of flux to guarantee the governing equation is analytically accurate. However, there is no
relevant research about extending the discretization to higher-order accuracy. Therefore, in order

to set stage for the method promotion, this research is concentrated on the integral form.

2.2 Finite volume discretization from integral form

Compared with the unstructured finite volume discretization from differential form, the governing equation in integral form could be written as,

\[ \int_{V_j} \partial_t \mathbf{u}_j \, dV + \int_{\partial V_j} \text{div} \mathbf{F}_j^c \, dS = \int_{V_j} \mathbf{s}_j \, dV. \]  

(4)

According to the divergence theorem, Eq. (4) could be transformed as,

\[ \int_{V_j} \partial_t \mathbf{u}_j \, dV + \oint_{\partial V_j} \mathbf{f}_j^e \cdot \mathbf{n} \, dS = \int_{V_j} \mathbf{s}_j \, dV. \]  

(5)

On this basis, Eq. (5) could be discretized as follows,

\[ \partial_t \mathbf{u}_j + \frac{1}{V_j} \sum_{k \in \{k_j\}} \Phi_{jk} A_{jk} = \frac{1}{V_j} \int_{\partial V_j} \mathbf{s}_j \, dS, \]  

(6)

where, \( \mathbf{u}_j \) is the cell-averaged solution vector and \( \{k_j\} \) are faces of cell \( j \). Besides, \( \Phi_{jk} \) is the numerical flux along the normal vector at the \( k-th \) face, and \( A_{jk} \) is the \( k-th \) face area or length (in 2D). The diagram of unstructured finite volume discretization is shown as follows.

As shown in Fig. 1, where \( \mathbf{u}_L \) and \( \mathbf{u}_R \) are two state vectors obtained by owner and neighbor cells of the common face. \( r_j \) and \( r_k \) are two vectors pointing from the cell centroid of owner and neighbor stencil cells to integral point respectively. Two state vectors could be computed by

\[ \mathbf{u}_L = \mathbf{u}_j + \nabla \mathbf{u}_j \cdot r_j, \quad \mathbf{u}_R = \mathbf{u}_k + \nabla \mathbf{u}_k \cdot r_k, \]  

(7)
where, $\mathbf{u}_j$ and $\mathbf{u}_k$ are point values of owner and neighbor cells. Actually, these two point values are always evaluated by the cell-averaged for the second-order unstructured finite volume methods, and in Section 2.3, we will discuss it in detail. Besides, $\nabla \mathbf{u}_j$ and $\nabla \mathbf{u}_k$ are solution gradients of two cells. On this basis, the numerical flux along the face normal vector could be formulated as,

$$
\Phi_{jk}(\mathbf{u}_k, \mathbf{u}_k) = \begin{pmatrix}
\rho u_n \\
\rho u_n v + P n_j \\
\rho u_n H
\end{pmatrix},
$$

where $u_n = \mathbf{v} \cdot \mathbf{n}_j$ is computed by the Roe flux [37], and $H = \left(\gamma p_p / (\gamma - 1) + \rho v^2 / 2\right) / \rho$ is the specific total enthalpy.

2.3 **k-exact reconstruction based on any local origins**

Least Square (LSQR) gradient reconstruction [38-40] is commonly used in the second-order unstructured finite volume methods [13,14,18,41]. Note, however, that for integral finite volume discretization, the real point value utilized in reconstruction and flux evaluation process is always given by the cell-averaged. This approximation is second-order accurate only if the reference point is located in the geometric centroid, and if the reference point is chosen anywhere, the point value is unable to be replaced by the cell-averaged, unless the solution is a constant. As a result, provided the reference point is moved, the LSQR gradient reconstruction method is inapplicable, because the point value within grid cell cannot be obtained.

Compared with LSQR, the existing problems could be well eliminated by *k-exact* reconstruction method. By solving reconstructed equations, derivatives as well as point value are all obtained, which could be utilized for the flux evaluation, and none of discretization errors are introduced during this process. As a result, although this research is focused on the second-order
unstructured finite volume discretization, we also employ the $k$-exact reconstruction method to
solve the solution gradient and obtain the point value at the corresponding reference point.

In $k$-exact reconstruction [34,35], the geometric centroid is always used to formulate the
reconstructed function $R_j \left( x - x_j \right)$,

$$R\left( x - x_j^p \right) = u_j^p + \frac{\partial u}{\partial x} \bigg|_{x_j} (x - x_j^p) + \frac{\partial u}{\partial y} \bigg|_{y_j} (y - y_j^p),$$  \hspace{1cm} (9)

where, $x_j^p$ and $y_j^p$ are coordinates of the geometric centroid in Cartesian system. Taking the
integral average of this expansion over the control volume gives

$$\frac{1}{V_j} \int_{V_j} R_j \left( x - x_j^p \right) \, dV = u_j^p + \frac{\partial u}{\partial x} \bigg|_{x_j} x_j + \frac{\partial u}{\partial y} \bigg|_{y_j} y_j,$$  \hspace{1cm} (10)

where $x_j^p$ and $y_j^p$ is the integration of geometric quantities and is called Moments [34,35]. Besides,
Conservation of the mean requires that the average of the reconstructed function $R_j \left( x - x_j \right)$ and
original solution over control volume be the same,

$$\frac{1}{V_j} \int_{V_j} R_j \left( x - x_j \right) \, dV = \frac{1}{V_j} \int_{V_j} u(x) \, dV = \Pi_j.$$  \hspace{1cm} (11)

In other words, provided Eq. (11) is satisfied, the reconstructed function could be formulated
based on any local origins. Thus, we first rewrite the reconstructed function and integrate it over
the control volume

$$\frac{1}{V_j} \int_{V_j} R_j \left( x - x_{i}^{m} \right) \, dV = u_{j}^{m} + \frac{\partial u}{\partial x} \bigg|_{x_{j}} x_{j} + \frac{\partial u}{\partial y} \bigg|_{y_{j}} y_{j},$$  \hspace{1cm} (12)

where, $x_{i}^{m}$ and $y_{i}^{m}$ are coordinate of any local origins. Replacing $(x - x_{i}^{m})$ and $(y - y_{i}^{m})$
with $(x - x_j^p) + (x_j^p - x_{i}^{m})$ and $(y - y_j^p) + (y_j^p - y_{i}^{m})$ respectively, we obtain

$$\frac{1}{V_j} \int_{V_j} R_j \left( x - x_{i}^{m} \right) \, dV = u_{j}^{m} + \frac{\partial u}{\partial x} \bigg|_{x_{j}} x_{j} + \frac{\partial u}{\partial y} \bigg|_{y_{j}} y_{j} + \frac{\partial u}{\partial x} \bigg|_{x_{j}} \Delta x_j + \frac{\partial u}{\partial y} \bigg|_{y_{j}} \Delta y_j,$$  \hspace{1cm} (13)
where, $\Delta x_j$ and $\Delta y_j$ are abbreviations of \( (x^o_j - x_0^o) \) and \( (y^o_j - y_0^o) \), and it is easily distinguished from Eq. (10). If the reconstructed function is expanded based on any local origins, there are another two terms in its integral average. Abbreviate these two terms as $Rem_j$, and the mean constraint of Eq. (11) could be rewrote as

$$ \bar{u}_j = u_{j}^{ori} + \frac{\partial u}{\partial x_j} \bar{x}_j + \frac{\partial u}{\partial y_j} \bar{y}_j + Rem_j. $$

(14)

On this basis, if we integrate the reconstructed function over a stencil cell $k$, we obtain

$$ \bar{u}_k = u_{j}^{ori} + \frac{\partial u}{\partial x_j} \int (x - x^o_j) dV + \frac{\partial u}{\partial y_j} \int (y - y^o_j) dV. $$

(15)

Similarly replace \( (x - x^o_j) \), \( (y - y^o_j) \) by \( (X - X^o_k) + (x^o_k - x^o_j) \) and \( (Y - Y^o_k) + (y^o_k - y^o_j) \) respectively, and reconstructed equation on stencil cell $k$ could be written as,

$$ \bar{u}_k = u_{j}^{ori} + \frac{\partial u}{\partial x_j} \left( \bar{x}_k + (x^o_k - x^o_j) \right) + \frac{\partial u}{\partial y_j} \left( \bar{y}_k + (y^o_k - y^o_j) \right). $$

(16)

This equation is written for every stencil cell, of which there should be more than the number of derivatives to be solved to create an overconstrained system. If we write the novel mean constraint together with the Eq. (16) for each stencil cell, we have

\[
\begin{bmatrix}
1 & x & y & \ldots \\
\omega_{j1} & \omega_{j1} x_1 & \omega_{j1} y_1 & \ldots \\
\omega_{j2} & \omega_{j2} x_2 & \omega_{j2} y_2 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\omega_{jN} & \omega_{jN} x_N & \omega_{jN} y_N & \ldots \\
\end{bmatrix}
\begin{bmatrix}
u^{ori}_j \\
\frac{\partial u}{\partial x_j} \bigg|_{x_j} \\
\frac{\partial u}{\partial y_j} \bigg|_{y_j} \\
\vdots \\
\omega_{jN} \frac{\partial u}{\partial x_j} \bigg|_{x_N} \\
\end{bmatrix} = \begin{bmatrix}
\bar{u}_j - Rem_j \\
\omega_{j1} \bar{x}_1 \\
\omega_{j2} \bar{x}_2 \\
\vdots \\
\omega_{jN} \bar{x}_N \\
\end{bmatrix},
\]

(17)

where the first row is the mean constraint, and geometric terms could be expressed as,

\[
\bar{x}^m y^m_{j \mu} = \frac{1}{A_k} \int_{\Delta_k} \left( (x - X^o_k) + (x^o_k - x^o_j) \right)^m \left( (y - Y^o_k) + (y^o_k - y^o_j) \right)^n dA
\]

\[
= \sum_{p=0}^{m} \sum_{q=0}^{m} \frac{m!}{p!(m-p)!} \frac{n!}{q!(n-q)!} (x^o_k - x^o_j)^p (y^o_k - y^o_j)^q \bar{x}^m y^m_{k \mu},
\]

(18)

and the weights are set to emphasize the geometrically adjacent data,
where $x_j$ and $x_i$ are local origins of central and stencil cells. By solving Eq. (17), solution vectors at any local origins as well as its gradient could be obtained. Therefore, the $k$-exact reconstruction method based on any local origins is derived, and it sets stage for the employment of novel reference points on unstructured finite volume discretization from integral form.

3. Global-direction stencil based on the face-area-weighted centroid

In Section 2.3, we theoretically derive the feasibility of $k$-exact reconstruction based on any local origins. In this section, we first introduce some commonly used stencil selection methods. And then, we briefly discuss the grid skewness on high-aspect-ratio triangular grids as well as the effect of skewness reduction by the employment of face-area-weighted centroid. Finally, to reduce the grid skewness and achieve a better reflection of flow anisotropy, the global-direction stencil with this novel centroid is introduced. The main content in this section has been analyzed in Ref. [23], while considering the completeness of the article, it is also discussed here.

3.1 Stencil selection methods

Commonly used stencils are vertex-neighbor and face-neighbor stencils. As Fig. 2 exhibits, face-neighbor stencil includes entire neighbor cells that share faces with the central cell, and vertex-neighbor stencil is similarly constructed by cells that share vertices with the central cell. Besides, both of them are topological-dependent, and therefore are limited by the original mesh topology. Furtherly, the stencil size of them is hard to accurately control, especially for vertex-neighbor stencil, and characteristics of flowfield cannot be well reflected.
Fig. 2 Vertex-neighbor and face-neighbor stencils, where different numbers represent stencil layers (e.g., for face-neighbor stencil, the first layer stencil is composed of all cells that share faces with the central cell, and the second layer stencil consists of cells that share faces with the first layer stencil.)

Apart from two commonly used stencils, in 2018, Xiong et al. put forward the local-direction stencil selection method, by which selected stencil cells are along two local directions. As shown in Fig. 3(a), on isotropic grid, two local directions are close to the normal and tangential directions of the wall, while on high-aspect-ratio triangular grids, as Fig. 3(b) displays, one of the local directions has severely deviated from the normal direction of the wall, and flow anisotropy is not well reflected. Besides, it is verified that on this grid type, accuracy loss and stability deterioration cannot be avoided, and the implementation process of this stencil selection method is quite complicated.

Fig. 3 Local-direction stencil cells on minor and high-aspect-ratio triangular grids.

In previous work, based on the existing problems on local-direction stencil, a novel global-direction stencil selection method was proposed. Compared with the local-direction
stencil, problems mentioned above are well solved by this novel stencil. Specifically, for this
method, two global directions, that is normal and tangential directions of the wall, are determined
at first. And then, for each central cell, two lines which are parallel to global directions
respectively and pass the cell centroid are generated. Finally, cells in a given set, such as the
vertex-adjacent cells that intersect with these two lines are selected to construct the new stencil,
and the stencil size is governed by layer of vertex-adjacent cells.

![Global-direction stencil cells on minor and high-aspect-ratio triangular grids.](image)

As Fig. 4 demonstrates, stencil cells selected by this method are always along two global
directions no matter the grid with high aspect ratio or not. Therefore, flow anisotropy can be well
reflected, and in corresponding numerical examples [23], global-direction stencil has a better
numerical performance than commonly used vertex-neighbor and face-neighbor stencils as well as
local-direction stencil. Besides, on cases with simple shapes, two global directions could be easily
determined, while for complex surface, there is no analytical expressions to obtain the normal and
tangential directions of the wall. Hence, in this situation, we can refer to the method of computing
wall distance to get the corresponding normal vector [42], and construct the global-direction
stencil.

But after analysis, although global-direction stencil cells are always along the normal and
tangential directions of the wall, no matter on grid with high aspect ratio or not, the only data
obtained by $k$-exact reconstruction and required for the flux evaluation are solution vectors stored at the reference point rather than stencil cells themselves, and it is hard to guarantee whether flow anisotropy is well reflected or not. In the next section, we will focus on different locations of reference points.

3.2 Face-area-weighted centroid and global-direction stencil

In this section, we give a brief analysis about grid skewness and face-area-weighted centroid on triangular mesh, and explain the reason as well as the idea of the combination of global-direction stencil and this novel reference point in detail.

3.2.1 Grid skewness measure and face-area-weighted centroid

In introduction, we have analyzed that although there are various definitions of grid skewness, the same conclusion could be drawn that on high-aspect-ratio triangles, the grid skewness is always evident. Here, we consider a typical skewness measure.

As shown in Fig. 5, the grid skewness is defined at common face shared by two neighbor cells, where $\hat{e}_j$ is a unit vector pointing from centroid of cell $j$ to that of cell $k$, $\hat{n}_{jk}$ is the unit outward normal vector of common face. The grid skewness is measured by $|\hat{e}_j \cdot \hat{n}_{jk}|$, and from Fig. 5, it could be easily found that the non-skewed grid has the measure one, and with the increase of cell aspect ratio, the highly-skewed grid is close to zero. On this basis, the most direct intuition is using isotropic or minor-aspect-ratio triangles to ensure the computing process is carried out on non-skewed grids, while for some typical flows, such as boundary-layer-type flow, solutions in boundary layer are changed dramatically, particularly along the normal direction of
the wall. Therefore, to enhance the resolution in this local field, highly-anisotropic grids cannot be
avoided, and we can only rely on a novel local origin to reduce the grid skewness and improve the
numerical performance.

For high-aspect-ratio triangular grids, a novel face-area-weighted centroid was proposed by
Nishiakwa [28] on the second-order differential unstructured finite volume solver. By the
employment of this reference point, the grid skewness is almost eliminated. In the following
analysis, we will briefly summarize the face-area-weighted formula and the effect about skewness
reduction,

A typical choice for the local origin is the geometric centroid that could be written for a
triangle by the arithmetic average of face midpoints,

\[
\left( x_j, y_j \right) = \frac{1}{3} \sum_{k=1}^{3} (x_{mk}, y_{mk}),
\]

(20)

where, \((x_{mk}, y_{mk})\) is coordinate of the \(k\)-th face centroid. Besides, we generalize Eq. (20) to the
face-area-weighted centroid formula:

\[
\left( x_j, y_j \right) = \frac{\sum_{k=1}^{3} A_j^{\hat{k}} (x_{mk}, y_{mk})}{\sum_{k=1}^{3} A_j^{\hat{k}}}, \quad \hat{A}_j = \frac{A_j}{\max_{k=1,2,3} A_j},
\]

(21)

where, \(A_j^{\hat{k}}\) is the area or length (in 2D) of the face shared by two neighbor cells, and \(p > 0\) is a
real value that controls the skewness degree. Note when \(p = 0\), face-area-weighted centroid is just
consistent with the geometric centroid.

As shown in Fig. 6, where \(h\) is the grid spacing in \(y\)-direction and \(R\) is cell aspect ratio. For a
typical triangular grid in Cartesian-coordinate system, skewness measure along the \(x\)-direction
approaches one, while in \(y\)-direction is nearly zero. Thus, in the following analysis, we will focus
on faces shared by cell 1 and cell 2 as well as cell 2 and cell 3.
Fig. 6 A typical high-aspect-ratio triangular grid in Cartesian-coordinate system.

For these two common faces, if we use the geometric centroid, the grid skewness measure is

\[
d_{12} = \frac{2R}{R^2 + 1}, \quad d_{23} = \frac{2}{\sqrt{R^2 + 4}}.
\]  

(22)

With the increase of cell aspect ratio, both measures nearly equal to zero. Hence the grid is highly skewed. But if we employ the face-area-weighted centroid and when parameter \( p \) is equaling to 2, the skewness measure becomes

\[
d_{12}^{p=2} = 1, \quad d_{23}^{p=2} = 1 - \frac{1}{2R^2}.
\]  

(23)

As a result, for high-aspect-ratio triangular grid, the grid skewness could be eliminated, and note, moreover, that with the increase of cell aspect ratio, the grid skewness, at the face like between cell 2 and cell 3, could be further reduced. Besides, it is demonstrated in Fig. 7 when the face-area-weighted centroid is employed, line connecting the novel local origins is almost parallel to the normal direction of the wall, and the serrated phenomenon exhibited on geometric centroid is effectively avoided.

This special distribution just coincides with our original motivation of designing the global-direction stencil, and in previous work, we combine the global direction stencil and
face-area-weighted centroid on the second-order unstructured finite volume solver in differential form to capture the flow anisotropy more accurately. After verification, a better numerical performance is obtained by this novel method. In this work, we further investigate feasibilities of extension to the integral form.

### 3.2.2 Combination of global-direction stencil and face-area-weighted centroid

In Section 3.1, we have demonstrated that compared with commonly used vertex-neighbor and face-neighbor stencils, although global-direction stencil cells are along the normal and tangential directions of the wall, the only data obtained by $k$-exact reconstruction and required for the flux evaluation are solution vectors stored at cells reference points, rather than stencil cells themselves, and if we still use the geometric centroid, grid skewness is unable to eliminate. Therefore, it is hard to guarantee whether the flowfield characteristics are well captured or not. But when face-area-weighted centroid is employed, the distribution of reference points is more regular, and the line connecting them is along the normal direction of the wall.

Besides, in Section 2.3, we have derived the $k$-exact reconstruction process based on any local origins, and therefore, in this section, we give the method of combining the global-direction stencil and face-area-weighted centroid in detail to realize the unified direction of both stencil cells and local origins. Global-direction stencils with geometric and face-area-weighted centroids on grids with straight and curved boundaries are displayed in Fig. 8 and Fig. 9.
From these two figures, we can easily find although same stencil cells are selected, geometric centroids are deflective and exhibit serrated phenomenon. Particularly on high-aspect-ratio triangular grids, the mentioned phenomenon is much more evident. By comparison, when face-area-weighted centroids are employed, line connecting them is close to the normal direction of the wall no matter on grid with straight or curved boundaries, and it is consistent with one of the global directions. As a result, the flow anisotropy could be well reflected, and the grid skewness is reduced.
On unstructured finite volume method from differential form, both computational accuracy and stabilities are greatly improved by this novel method, and in the next section, four representative numerical examples are designed to verify the effectiveness and superiorities of this novel method on integral unstructured finite volume solver.

4. Numerical examples

To examine the effectiveness of global-direction stencil with face-area-weighted centroid on unstructured finite volume method from integral form, in this section, numerical examples governed by linear convective, Euler and Laplacian equations are utilized. For comparison, these numerical examples are simulated with four different stencils, including vertex-neighbor and face-neighbor stencils, as well as the global direction stencil with geometric centroid and face-area-weighted centroid. Note that from Section 3.2.1, we can find the face-area-weighted centroid could be further distinguished by different $p$ values, and when $p = 2$, skewness has been almost eliminated. Hence, the face-area-weighted centroid in this section refers to the result of $p = 2$. In addition, to simplify the presentation in the following analysis, different stencils are abbreviated as follows.

| Different stencils                        | Abbreviation                          |
|-----------------------------------------|---------------------------------------|
| Vertex-neighbor stencil                 | V-Stencil                             |
| Face-neighbor stencil                   | F-Stencil                             |
| Global-direction stencil (With cell centroid) | G-Stencil (Cell centroid)           |
| Global-direction stencil (With face-area-weighted centroid) | G-Stencil (F-a-w centroid) |

4.1 Manufactured boundary layer (Governed by Linear convective equation)

In this section, we first use the Method of Manufactured Solutions (MMS) [43-46] on linear convective equation to verify feasibilities of the employment of face-area-weighted centroid on unstructured finite volume discretization from integral form, and examine the numerical
performance of global-direction stencil with this novel reference point. The governing equation can be written as,

$$\frac{\partial u}{\partial t} + a \cdot \nabla u = 0,$$  \hspace{1cm} (24)

where, \( a = \left( \cos \frac{\pi}{16}, \sin \frac{\pi}{16} \right) \) is the constant convective velocity, and to simulate characteristics of boundary layer, manufactured solution is

$$u(x, y) = 1 - e^{\frac{-(y-y_0)}{\sqrt{c\mu(x-x_0)}}},$$  \hspace{1cm} (25)

where \( c = 0.59 \) is a constant, and parameter \( \mu \) is utilized to control the thickness of boundary layer. Flowfields corresponding to different \( \mu \) values are shown in Fig. 10, and in the following test, \( \mu \) is set as \( 10^{-6} \).

![Flowfields with different \( \mu \) values](image)

(a) \( \mu = 10^{-6} \)  \hspace{1cm} (b) \( \mu = 10^{-8} \)

Fig. 10 Flowfields with different \( \mu \) values

By bringing the manufactured solution to Eq. (24), the modified equation with source term could be written as,

$$\frac{\partial u}{\partial t} + a \cdot \nabla u = a_x (y-y_0) e^{\frac{-(y-y_0)}{\sqrt{c\mu(x-x_0)}}} - a_y (y-y_0) e^{\frac{-(y-y_0)}{\sqrt{c\mu(x-x_0)}}}.$$  \hspace{1cm} (26)

On this basis, the manufactured solution represents the analytical solution of this modified governing equation, and we can calculate both \( L_2 \) and \( L_\infty \) errors of different stencils.

$$L_2 = \sqrt{\sum_{j=1}^{N} \left( u_j - u_{analy} \right) \cdot A_j / \sum_{j=1}^{N} A_j},$$

$$L_\infty = \max_{j=1,N} \left| u_j - u_{analy} \right|,$$  \hspace{1cm} (27)
where \( \tilde{u}_j \), \( u_{\text{anal}} \), and \( A_j \) are numerical and analytical solutions and area of cell \( j \) respectively.

Besides, as shown in Fig. 11, both regular and randomly perturbed triangular grids are used in this numerical example, and two levels of grid stretching, including \( 10^3 \) and \( 10^4 \) these two wall cell aspect ratios \( (AR) \), are tested. In each level, five sets of triangular grids from the coarsest to finest are generated within \( x, y \in [0.05, 1.05] \times [0, 0.001] \).

During the grid generation process, nodes in \( x \)-direction are equidistantly distributed, while the \( y \)-coordinates of different nodes are determined by

\[
y_{j+1} = y_j + \hat{h}_y \beta^{j+1} \quad j = 1, 2, \ldots, N_x,
\]

where \( \hat{h}_y \) is the first layer vertical spacing, and \( \beta \) is the stretching factor that could be computed by the known condition \( y_N = 10^{-3} \). Besides, distribution of five sets of background quadrilateral grids from the coarsest to finest is shown in Table 2.

| Grid name | Distribution in \( x \) and \( y \) directions |
|-----------|-----------------------------------------------|
|           | \( AR = 10^3 \)                 | \( AR = 10^4 \)                 |
| vcoa      | 30\times10                       | 15\times10                      |
| coa       | 45\times15                       | 30\times20                      |
| med       | 60\times20                       | 45\times30                      |
| fin       | 80\times30                       | 60\times40                      |
| vfin      | 120\times40                      | 90\times60                      |

On this basis, computational accuracy and discretization errors of four different stencils on integral unstructured finite volume solver are counted and given in Fig. 12 and Fig. 13.
4.1.1 Computational results on regular grids

1) $AR = 10^3$

From Fig. 12 and 13, we can easily find the second-order accuracy is achieved by all stencils we tested, for both $L_2$ and $L_\infty$ errors. Therefore, the effectiveness of employing face-area-weighted centroid on unstructured finite volume method from integral form is verified. In addition, it is also
proved that the derivation of k-exact reconstruction based on any local origins is feasible, and its correctness has been well demonstrated by computational results.

Besides, combining the specific data listed in Table 3, we find both $L_2$ and $L_\infty$ errors of G-Stencil (F-a-w centroid) are the lowest among all stencils we tested, and when $AR = 10^4$, computational accuracy of $L_\infty$ errors could be obviously improved. What's more, from Table 3, we notice that G-Stencil requires the least number of stencil cells, and the efficiency can also be improved.

| Different stencils          | $AR = 10^3$ |          | $AR = 10^4$ |          | Average stencil size |
|-----------------------------|-------------|----------|-------------|----------|----------------------|
|                             | $L_2$ errors | $L_\infty$ errors | $L_2$ errors | $L_\infty$ errors |              |
| V-Stencil                   | 7.287×10^{-5} | 4.183×10^{-4} | 1.982×10^{-4} | 9.499×10^{-4} | 11.779               |
| F-Stencil                   | 6.317×10^{-5} | 4.259×10^{-4} | 1.704×10^{-4} | 9.439×10^{-4} | 8.861                |
| G-Stencil (Cell centroid)   | 6.273×10^{-5} | 3.599×10^{-4} | 1.668×10^{-4} | 8.635×10^{-4} | 6.917                |
| G-Stencil (F-a-w centroid)  | 3.31×10^{-5}  | 2.67×10^{-5}  | 9.833×10^{-5} | 4.007×10^{-4} | 6.917                |

### 4.1.2 Computational results on randomly perturbed grids

On randomly perturbed triangular grids, we also test the case of $AR = 10^3$ and $10^4$. For simplicity, results of $AR = 10^4$ are given here.

![Fig. 14 Errors of different stencils on randomly perturbed grids with $AR = 10^4$](image-url)
From Fig. 14, we find errors on randomly perturbed grids demonstrate the similar trends to results on regular grids, and when face-area-weighted centroid is employed on the global-direction stencil, both \( L_2 \) and \( L_\infty \) errors are greatly reduced. As a result, on linear convective governing equation, the effectiveness as well as superiorities of this novel method is well verified.

### 4.2 Supersonic vortex flow (Governed by Euler equations)

To further test the effectiveness and feasibilities of this novel method on Euler equations, in this section, the supersonic vortex flow is introduced. Computational domain is two concentric circular arcs with radius \( r_i = 1 \) and \( r_0 = 1.384 \) located in the first quadrant. These two circular arcs represent the inviscid wall boundary, and the flow at both inlet and outlet are supersonic.

Analytical solution \([47]\) of this numerical example could be derived by isentropic relation and is given as follows,

\[
\begin{align*}
\rho &= \rho_i \left(1 + \frac{\gamma - 1}{2} M_i^2 \left(1 - \left(\frac{r_i}{r}ight)^2\right)\right)^{\frac{1}{\gamma - 1}}, \\
P &= \frac{P_i^\gamma}{\gamma}, \quad \rho = \frac{c_i M_i}{r},
\end{align*}
\]  

(29)

where the value of Mach number at the inner radius is \( M_i = 2.25 \) and the density \( \rho_i =1 \).

Besides, the sound speed is calculated as,

\[
c_i = \sqrt{\gamma P_i / \rho_i} = 1,
\]

(30)

Flow structure of this numerical case is shown as follows

![Flowfields of supersonic vortex flow.](image)

**Fig. 15** Flowfields of supersonic vortex flow.
For this numerical example, both regular and randomly perturbed triangular grids are utilized. As shown in Fig. 16, where the regular grid is generated by splitting the background quadrilateral grid with right diagonals, and randomly perturbed grid is generated by introducing the random node perturbation to the regular grid with topology and the number of cells unchanged. Besides, two levels of grid aspect ratios are employed, and in each level, five sets of grids from the coarsest to finest are generated.

![Regular and randomly perturbed triangular grids.](image)

Considering that the aspect ratio of grid with the curved boundary is not a fixed value, similarly, the wall cell aspect ratio is utilized, and two aspect ratios are approximately equal to 4 and 8 respectively. The distribution of background quadrilateral grids in the radical and circumferential is shown in Table 4.

| Grid name | Distribution in $x$ and $y$ directions |
|-----------|--------------------------------------|
|           | $AR \approx 4$ | $AR \approx 8$ |
| vcoa      | 10×10         | 20×10         |
| coa       | 20×20         | 40×20         |
| med       | 40×40         | 80×40         |
| fin       | 60×60         | 120×60        |
| vfin      | 80×80         | 160×80        |
4.2.1 Computational errors on regular grids

1) $AR \approx 4$

As shown in Fig. 17 and Fig. 18, for three stencils with the geometric centroid, both $L_2$ and $L_\infty$ errors of global-direction stencil are the lowest. On this basis, we employ the face-area-weighted centroid, and discretization errors are further reduced. Besides, according to the specific data listed in Table 5, higher-order accuracy on both $L_2$ and $L_\infty$ errors is achieved by G-Stencil (F-a-w centroid). Particularly, $L_2$ errors can reach 2.755 order accurate. As a result, the
employment of face-area-weighted centroid on unstructured finite volume method from integral
form will not deteriorate the designed order of accuracy but greatly improve the numerical
performance of the finite volume solver after being introduced in global-direction stencil.

Table 5 Computational errors on the finest grid and $L_2$, $L_\infty$ accuracy between the last two grids of different stencils

| Different stencils         | $AR \approx 4$ | $AR \approx 8$ | $L_2$ errors | $L_\infty$ errors |
|----------------------------|----------------|----------------|--------------|-------------------|
| V-Stencil                  | 8.706×10^{-5}  | 2.002×10^{-3}  | 2.22         | 2.076             |
| F-Stencil                  | 7.489×10^{-5}  | 1.984×10^{-4}  | 2.22         | 2.076             |
| G-Stencil (Cell centroid)  | 2.946×10^{-5}  | 5.065×10^{-5}  | 2.22         | 2.076             |
| G-Stencil (F-a-w centroid) | 7.961×10^{-6}  | 6.063×10^{-5}  | 2.755        | 2.101             |

4.2.2 Computational errors on randomly perturbed grids

In order to adequately illustrate the numerical performance of different methods, errors on
randomly perturbed grids are also counted. For simplicity, the results of $AR \approx 8$ are given here.

Fig. 19 Errors of different stencils on regular grids with $AR \approx 8$

From Fig. 19, errors exhibit the similar trends to that of regular grids, and both average and
max errors of G-Stencil (F-a-w centroid) are the lowest among all stencils we tested. Therefore,
correctness of k-exact reconstruction based on any local origins as well as the effectiveness and
superiorities of the global-direction stencil with face-area-weighted centroid is verified on Euler
equations as well.
4.3 Subsonic flow over a NACA0012 airfoil (Governed by Euler equations)

In Section 3.1 we give a brief introduction about the determination of global directions based on wall distance [42], and in this section, the subsonic flow over a NACA0012 airfoil governed by Euler equations is simulated to examine the effectiveness of the mentioned method as well as the effectiveness of global-direction stencil with face-area-weighted centroid. The angle of attack is $\alpha = 0^{\circ}$, and initial condition is $Ma = 0.5$.

Regular and randomly perturbed triangular grids with O-type topology are utilized in this numerical example. For regular grid, the first layer spacing in normal direction is $10^{-3}$, and there are 201 and 71 grid points distributed in circumferential and normal directions respectively. Regular and randomly perturbed grids near the airfoil are shown in Fig. 20.

![Regular and randomly perturbed grids](image)

**Fig. 20** Regular and randomly perturbed triangular grids over a NACA0012 airfoil.

Besides, to intuitively contrast the effect of geometric centroid and face-area-weighted centroid on cells adjacent to the airfoil, we display these two reference points on randomly perturbed grid respectively.

![Geometric centroid and face-area-weighted centroid](image)

**Fig. 21** Geometric centroid and face-area-weighted centroid of perturbed triangular grid over a NACA0012 airfoil.
From Fig. 21, we can easily find that compared with the geometric centroid, the distribution of face-area-weighted centroid is more regular, and the reference points are almost spread along the normal direction of the wall, especially on grid adjacent to the airfoil surface. On this basis, we count lift and drag coefficients as well as the residual of different stencils respectively, and are shown as follows.

![Fig. 22](image1)  
**Fig. 22** $C_l$ on regular and perturbed triangular grids

![Fig. 23](image2)  
**Fig. 23** $C_d$ on regular and perturbed triangular grids.

From results of lift and drag coefficients, we can easily find from local amplifications of both regular and randomly perturbed grids that the lift coefficient of G-Stencil (F-a-w centroid) is the lowest among four different stencils, and in the same time steps, there is almost no oscillations on V-Stencil and G-Stencil (Cell centroid) as well as G-Stencil (F-a-w centroid). However, compared with these three stencils, oscillations of the F-Stencil are quite evident, and has not yet converged.

Similar phenomenon could be concluded on drad coefficient, where the result of G-Stencil (F-a-w centroid) is lower than V-Stencil and G-Stencil (Cell centroid) but slightly higher than
F-Stencil. Nevertheless, oscillations of F-Stencil are quite obvious, and the convergence property of this stencil is much poorer than another three stencils.

![Fig. 24 Residuals on regular and randomly perturbed grids.](image)

Conclusions related to lift and drag coefficients mentioned above can be better illustrated by residuals of different stencils. As exhibited in Fig. 24, we can easily find that for three stencil with geometric centroid, the convergence speed of G-Stencil is close to the commonly used V-Stencil, and in the same time steps, residuals of these two stencils are decreased by 7 orders of magnitude, while only 4 orders of magnitude is decreased on F-Stencil. On this basis, with the introduction of face-area-weighted centroid on global-direction stencil, the residual is further decreased, and therefore, the convergence property of integral unstructured finite volume solver is greatly improved by global-direction stencil with face-area-weighted centroid.

In short, feasibilities and superiorities of global-direction stencil with face-area-weighted centroid on unstructured finite volume method from integral form are demonstrated again. And according to results displayed in Section 4.1, 4.2 and 4.3, we can easily find computational accuracy, efficiency and convergence speed are all improved by this novel method.

### 4.4 Dissipative term evaluation (Governed by Laplacian equation)

From Section 4.1 to 4.3, three numerical examples govern by linear convective and Euler equations are simulated, and the correctness of $k$-exact reconstruction based on any local origins is
verified. Moreover, on unstructured finite volume method from integral form, the global-direction stencil with face-area-weighted centroid also has a better computational accuracy, efficiency as well as convergence rate for convective flux evaluation.

But, the numerical performance on dissipative term is also need to be further test to set stage for the extension to viscous problems. In this section, the MMS method is also used on Laplacian equation to examine discretization errors and convergence speed of different stencils. Laplacian model equation could be formulated as,

$$\frac{\partial \phi}{\partial t} - \nabla \cdot (\nu \nabla \phi) = 0,$$

where \( \nu \) is a constant equaling to 1. Integrating this model equation over the control volume, we obtain

$$\frac{\partial \phi_j}{\partial t} - \frac{1}{V_j} \int_{V_j} \nabla \cdot (\nu \nabla \phi_j) \, dV = 0.$$  \hspace{1cm} (32)

According to the divergence theorem, Eq. (32) could be transformed as

$$\frac{\partial \phi_j}{\partial t} + \frac{1}{V_j} \int_{\partial V_j} (\nu \nabla \phi_j)_n \, dS = 0.$$  \hspace{1cm} (33)

From Eq. (33), we can clearly find the difference between convective term and dissipative term. For dissipative term computation, what we need to compute at the face integral point is not two state vectors obtained by owner and neighbor cells, but solution gradient, which could be evaluated by the arithmetic average of left and right cells for the second-order unstructured finite volume method,

$$\nabla \phi_{int}^j = \frac{1}{2} \left( \nabla \phi_{int}^l + \nabla \phi_{int}^r \right).$$  \hspace{1cm} (34)

On this basis, in order to improve the stability and reduce the truncation errors, the average face gradient is always added a solution jump term \([28,48,49]\). Here are many different schemes for
adding the jump term, such as the edge-normal (EN) and face-tangent (FT) schemes [50]. Based on the FT scheme, a novel $\alpha$-damping scheme is proposed by Nishikawa [51-54]. The accuracy as well as robustness of second-order unstructured finite volume solver is greatly improved by this scheme, and it could be formulated as

$$\nabla \phi^{\text{int}} = \frac{1}{2} \left( \nabla \phi^{\text{int}} + \nabla \phi^{\text{int}} \right) + \alpha \frac{\phi^{\text{L}} - \phi^{\text{R}}}{L_{jk}} \hat{n}_{jk}$$

(35)

where, $\alpha$ is a damping coefficient, and here, a special value is $\alpha = 4/3$, which has been known to provide superior accuracy and robustness [28,48,51]. $\hat{n}_{jk}$ is the unit face normal vector, and $\hat{e}_{jk}$ is the unit vector between two cell centroids. $L_{jk}$ is the length from one centroid to another.

$$\hat{e}_{jk} = \frac{x_j - x_k}{L_{jk}}, \quad L_{jk} = |x_k - x_j|$$

(36)

Besides, $\phi^L_{jk}$ and $\phi^R_{jk}$ are solutions linearly reconstructed at integral point,

$$\phi^L_{jk} = \phi_k + \frac{1}{2} \nabla \phi_k \cdot (x_j - x_k), \quad \phi^R_{jk} = \phi_k + \frac{1}{2} \nabla \phi_k \cdot (x_j - x_k)$$

(37)

On the basis of determining the discretization method of dissipative term, the manufactured solution of Eq. (31) is

$$\phi = \cos(\omega x) \sin(10\omega y)$$

(38)

where $\omega = 2\pi$ is a constant, and the source term could be derived as,

$$s = \left(10^2 + 1\right)\omega^5 \cos(\omega x) \sin(10\omega y).$$

(39)

As shown in Fig. 25, the simulation is carried out on five sets of triangular grids generated by splitting the quadrilateral grids with regular or random diagonals, and the cell aspect ratio is $AR = 10$. The distribution of these five sets of grids is from 10×10 to 80×80, and the computational domain is $x, y \in [-0.5, 0.5] \times [-5 \times 10^{-2}, 5 \times 10^{-2}]$. 
As a result, superiorities of face-area-weighted centroid are illustrated again, and with the employment of this novel centroid, the grid skewness is almost eliminated. In order to reflect the relative position of two different centroids on grid with regular and random diagonals, the distribution of these two local origins is shown in Fig. 26.
From Fig. 26, we can easily find that face-area-weighted centroids are almost distributed along y-axis no matter on grid with regular or random diagonals, but the distribution of geometric centroid is irregular, and according to the data listed in Table 6, it has been demonstrated that the evident skewness will be introduced by this traditional centroid. Based on above analysis, we count discretization errors of four different stencils, and corresponding results are shown in Fig 27, 28 and Table 7.

Fig. 27 Errors of different stencils on grids with regular diagonals.

Fig. 28 Errors of different stencils on grids with random diagonals.
From Fig. 27, Fig. 28 and combining the specific data listed in Table 7, we find that discretization errors of G-Stencil (F-a-w centroid) are the lowest among all stencils we tested no matter on grid with regular or random diagonals, and the computational accuracy is greatly improved by this novel method.

Table 7 Computational errors on the finest regular grid and $L_2, L_\infty$ accuracy between the last two grids of different stencils

| Different stencils                                    | $L_2$ errors | $L_\infty$ errors | $L_2$ accuracy | $L_\infty$ accuracy |
|------------------------------------------------------|--------------|------------------|----------------|---------------------|
| V-Stencil                                            | 8.586×10^{-4} | 3.164×10^{-3}    | 1.991          | 2.063               |
| F-Stencil                                            | 8.386×10^{-4} | 2.626×10^{-3}    | 1.986          | 1.984               |
| G-Stencil (Cell centroid)                            | 3.617×10^{-4} | 5.405×10^{-4}    | 1.993          | 1.971               |
| G-Stencil (F-a-w centroid)                           | 2.831×10^{-4} | 4.617×10^{-4}    | 1.992          | 1.986               |

On this basis, residuals on the coarsest and finest regular grids of different stencils are shown in Fig. 29.

From Fig. 29, we can easily find that among three stencils with the geometric centroid, the G-Stencil (Cell centroid) has a faster convergence speed than two commonly used V-Stencil and F-Stencil. What’s more, when face-area-weighted centroid is employed, the convergence rate is further promoted. Hence, except for convective term, on dissipative term, a better computational accuracy and faster convergence rate are also realized by this novel method. In short, feasibilities
and superiorities of employing face-area-weighted centroid on unstructured finite volume discretization from integral form are verified again on Laplacian model equation.

5. Concluding remarks

In this paper, we have shown that the reference point used in the unstructured finite volume method from integral form does not have to be the geometric centroid, and there is a better choice that reduces the grid skewness. Besides, we combine the global-direction stencil with this novel reference point trying to improve the numerical performance of unstructured finite volume solver.

Specifically, we first illustrate in detail that the traditional LSQR reconstruction method is only applicable when the reference point is located in the geometric centroid, and that is the reason why we focus on $k$-exact reconstruction algorithm. On this basis, the $k$-exact reconstruction based on any local origins is analytically derived, and during the reconstruction process, the mean constraint can always be satisfied. Besides, we extend the global-direction stencil with novel face-area-weighted centroid from unstructured finite volume method in differential form to the integral form for the grid skewness reduction and a better reflection of flow anisotropy. More importantly, it sets stage for the promotion on higher-order accuracy.

Four representative numerical examples governed by linear convective, Euler and Laplacian equations are simulated to examine the correctness of $k$-exact derivation and the effectiveness of global-direction stencil with face-area-weighted centroid on the unstructured finite volume discretization from integral form. After verification, in numerical cases governed by linear convective and Euler equations, when the face-area-weighted centroid is utilized on finite volume discretization, the second-order accuracy also could be achieved. What’s more, the global-direction stencil with this novel reference point has the lowest discretization errors among
all stencils we tested. Besides, in subsonic flow over NACA0012 airfoil, we find this novel
method has a better numerical performance on calculating lift and drag coefficients, and has a
faster convergence speed than commonly used stencils based on the geometric centroid.

Finally, a Laplacian model equation is utilized to further test its effectiveness on dissipative
term evaluation. From the result, similar conclusions are demonstrated that errors of the
global-direction stencil with face-area-weighted centroid are still lower than another three stencils,
and the convergence rate is also greatly promoted by this novel method. As a result, this
conclusion sets stage for the further extension to viscous flows.

In short, the computational accuracy of the second-order unstructured finite volume method
from integral form will not get deteriorated by the employment of face-area-weighted centroid,
and it is feasible to extend the global-direction stencil with face-area-weighted centroid from
differential finite volume solver to the integral form. The next work will be carried out from two
aspects. Firstly, in terms of method extension, we will continue to extend this novel method to
higher-order unstructured finite volume solver. Besides, the application on viscous and
high-mach-number flows with complex surface is also necessary to further examine its numerical
performance.

Abbreviations

CFD: Computational Fluid Dynamics; 2D: Two-dimensional; LSQR: least-squares; MMS:
Method of Manufactured Solutions; V-Stencil: Vertex-neighbor stencil; F-Stencil: Face-neighbor
stencil; G-Stencil (Cell centroid): Global-direction stencil with cell centroid; G-Stencil (F-a-w
centroid): Global-direction stencil with face-area-weighted centroid; EN: Edge-normal scheme; FT:
Face-tangent scheme.
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