Hierarchical Matrices in the See-Saw Mechanism, large Neutrino Mixing and Leptogenesis

Werner Rodejohann*

Scuola Internazionale Superiore di Studi Avanzati, I-34014 Trieste, Italy
and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34014 Trieste, Italy

Abstract

We consider the see-saw mechanism for hierarchical Dirac and Majorana neutrino mass matrices $m_D$ and $M_R$, including the $CP$ violating phases. Simple arguments about the structure of the neutrino mass matrix and the requirement of successful leptogenesis lead to the situation that one of the right-handed Majorana neutrinos is much heavier than the other two, which in turn display a rather mild hierarchy. It is investigated how for the neutrino mixing one small and two large angles are generated. The mixing matrix element $|U_{e3}|^2$ is larger than $10^{-3}$ and a characteristic ratio between the branching ratios of lepton flavor violating charged lepton decays $\ell_j \rightarrow \ell_i \gamma$ is found. Successful leptogenesis implies sizable $CP$ violation in oscillation experiments. As in the original minimal see-saw model, the signs of the baryon asymmetry of the universe and of the $CP$ asymmetry in neutrino oscillations are equal and there is no connection between the leptogenesis phase and the effective mass as measurable in neutrinoless double beta decay.

*Email: werner@sissa.it
1 Introduction

The fact that two mixing angles in the neutrino mixing matrix are large \cite{1} is a severe difference with respect to the quark sector. In the latter, hierarchical mass matrices are the most natural explanation for small mixing angles. Thus, it is natural to assume that in a GUT framework also the Dirac mass matrix $m_D$ and the Majorana mass matrix $M_R$, both appearing in the see–saw mechanism \cite{2}, are of hierarchical structure, i.e., of close to diagonal form. In the see–saw mechanism the neutrino mass matrix $m_\nu$ is a matrix product containing $m_D$ and $M_R$. Consequently, it is possible that $m_\nu$ does not display a close to diagonal structure\footnote{The names “see–saw enhancement” or “correlated hierarchy” are sometimes given to this phenomenon.}, in contrast to the fundamental matrices $m_D$ and $M_R$ \cite{3}. Accordingly, the observed neutrino mixing can take the characteristic form with two large angles and one small one. The purpose of the present note is to readdress this point including effects of the $CP$ phases and investigate its consequences for leptogenesis and for the branching ratios of lepton flavor violating (LFV) charged lepton decays like $\mu \to e \gamma$. In order to reach a hierarchical mass spectrum, the 23 block of $m_\nu$ has to be approximately degenerate with entries larger than the remaining elements \cite{4,5,6}. Working within useful parameterizations of $m_D$ and $M_R$, these requirements lead to the possibility that one of the right–handed Majorana neutrinos is much heavier than the other two. Successful leptogenesis then implies a rather mild hierarchy between the latter. In this simple framework one can obtain neutrino mixing phenomenology in accordance with data, predicts $|U_{e3}|^2 \gtrsim 10^{-3}$ and finds a characteristic ratio of the branching ratios of the LFV charged lepton decays. The baryon asymmetry of the universe and the $CP$ asymmetry measurable in neutrino oscillations are directly connected, since they depend in the same way on the same phase. No connection between the leptogenesis phase and the effective mass as testable in neutrinoless double beta decay is present. The model under study is in this sense very similar to the minimal see–saw model \cite{7}, which contains only two heavy Majorana neutrino and two zeros in the Dirac mass matrix.

In Section 2 we will shortly review the formalism of neutrino mixing and leptogenesis. We investigate how hierarchical Dirac and Majorana mass matrices lead to large neutrino mixing in a simplified $2 \times 2$ case in Section 3. The realistic $3 \times 3$ case is treated in Section 4 where also the predictions for leptogenesis and low energy observables are investigated. We conclude in Section 5.

2 Framework

The neutrino mass matrix is given by the see–saw mechanism \cite{2} as

$$m_\nu \simeq -m_D M_R^{-1} m_D^T ,$$

where $m_D$ is a Dirac mass matrix and $M_R$ the mass matrix of the right–handed Majorana neutrinos. We shall work in a basis in which both the charged lepton mass matrix and $M_R$
are real and diagonal, i.e., $M_R = \text{diag}(M_1, M_2, M_3)$ with real $M_3 > M_2 > M_1$. The largest mass $M_3$ is expected to lie around or below the unification scale $M_{\text{GUT}} \simeq 10^{16}$ GeV. The matrix $m_\nu$ is observable in terms of

$$m_\nu = U^\dagger m_\nu^\text{diag} U^* .$$

Here $m_\nu^\text{diag}$ is a diagonal matrix containing the light neutrino mass eigenstates $m_i$ and $U$ is the unitary Pontecorvo–Maki–Nagakawa–Sakata lepton mixing matrix, which can be parametrized as

$$U = O_{23} O_{13}^\delta O_{12} P .$$

$O_{ij}$ are rotation matrices, e.g.,

$$O_{13}^\delta = \begin{pmatrix} c_{13} & 0 & s_{13} e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{-i\delta} & 0 & c_{13} \end{pmatrix} ,$$

where $c_{13} = \cos \theta_{13}$, $s_{13} = \sin \theta_{13}$ and $\delta$ is the “Dirac phase” measurable in neutrino oscillations. The matrices $O_{12}$ and $O_{23}$ are real and $P$ is a diagonal phase matrix containing the two additional Majorana phases. In total,

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} e^{i\delta} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} e^{i\delta} \end{pmatrix} \text{diag}(1, e^{i\alpha}, e^{i\beta}) .$$

Observation from previous experiments as well as inclusion of the recent SNO salt phase data implies the following values of the oscillation parameters, given at 3σ:

$$\tan 2\theta_{12} \simeq 1.5 \ldots 4.4 ,$$
$$\tan 2\theta_{13} \lesssim 0.45 ,$$
$$|\tan 2\theta_{23}| \gtrsim 2 ,$$
$$\Delta m_{\odot}^2 \simeq (5.4 \ldots 9.5) \cdot 10^{-5} \text{eV}^2 ,$$
$$\Delta m_A^2 \simeq (1.4 \ldots 3.7) \cdot 10^{-3} \text{eV}^2 .$$

Typical best–fit points are $\tan^2 \theta_{12} = 0.45$ and $\theta_{23} = \pi/4$, corresponding to $\tan 2\theta_{12} \simeq 2.4$ and $\tan 2\theta_{23} \gg 1$. We have therefore two large and one small mixing angle, in sharp contrast to the situation present in quark mixing.

The presence of heavy right–handed Majorana neutrinos in the see–saw mechanism means that the possibility of leptogenesis is included. Thus, the see–saw mechanism gains a large amount of attractiveness. Leptogenesis explains the baryon asymmetry of the universe
through the $CP$ asymmetric out–of–equilibrium decay of heavy right–handed Majorana neutrinos occurring much before the electroweak phase transitions. It is governed by the decay asymmetry $^{11,12}$

$$\varepsilon_1 \simeq \frac{1}{8 \pi v^2} \frac{1}{(m_D^2 m_D)_{11}} \sum_{j \neq i} \text{Im}(m_D^\dagger m_D)_{ij}^2 f(M_j^2/M_1^2), \quad (7)$$

where $f(x)$ is a function whose limit for $x \gg 1$, i.e., hierarchical neutrinos$^2$, is $-3/\sqrt{x}$. Values of $|\varepsilon_1| \gtrsim 10^{-7}$ and $M_1 \gtrsim 10^9$ GeV are required in order to produce a sufficient baryon asymmetry $^{12,14}$. There is a tendency of this lower mass limit to be in conflict with bounds on the reheating temperature, which stem from the requirement that the decay products of the gravitino do not spoil Big Bang Nucleosynthesis predictions. From this condition one finds upper limits of less than $M_1 \lesssim 10^9 \ldots 10^{10}$ GeV $^{15}$. The baryon asymmetry is positive when $\varepsilon_1$ is negative, because it holds $Y_B \propto c \varepsilon_1$ $^{12}$, where $c$ is a negative constant stemming from the conversion of the lepton asymmetry into a baryon asymmetry.

3 2 × 2 Case

We shall analyze the generation of large mixing in $m_\nu$ from hierarchical $m_D$ and $M_R$ first in a simplified 2 × 2 framework. Consider a complex symmetric matrix

$$m = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad (8)$$

which is diagonalized by a unitary matrix $U$ through

$$m^{\text{diag}} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = U^\dagger m U, \quad \text{where} \quad U = \begin{pmatrix} \cos \theta & \sin \theta e^{i\phi} \\ -\sin \theta e^{-i\phi} & \cos \theta \end{pmatrix}. \quad (9)$$

In general, a symmetric matrix 2 × 2 is diagonalized by $UP$, where $U$ is given above and $P$ is a diagonal phase matrix. By redefining the charged lepton fields, these two additional phases can be absorbed. The eigenvalues $m_1$ and $m_2$ with $m_2 > m_1$ are trivial to obtain. The mixing angle $\theta$ is given by the equation

$$\tan 2\theta = \frac{2b}{d e^{-i\phi} - a e^{i\phi}}. \quad (10)$$

The phase $\phi$ is defined by the requirement of the angle $\theta$ being real, i.e.,

$$\text{arg}(b) = \text{arg}(d e^{-i\phi} - a e^{i\phi}). \quad (11)$$

$^2$We shall not discuss the possibility of degenerate Majorana neutrinos, whose decay asymmetry is resonantly enhanced $^{13}$. 

4
Now consider in a simple $2 \times 2$ case hierarchical Dirac and Majorana mass matrices, i.e.,

$$m_D = m \begin{pmatrix} \epsilon_D^2 & A \epsilon_D \\ B \epsilon_D & 1 \end{pmatrix} \quad \text{and} \quad M_R = M \begin{pmatrix} \epsilon_M & 0 \\ 0 & 1 \end{pmatrix}, \quad (12)$$

with $\epsilon_D, \epsilon_M \ll 1$ but an unspecified hierarchy between $\epsilon_D$ and $\epsilon_M$. The complex coefficients $A = a e^{i\alpha}$ and $B = b e^{i\beta}$ with real $a$ and $b$ have absolute values of order one. Inserting the matrices in the see–saw formula (1) yields

$$m_\nu = -m^2 M \begin{pmatrix} \frac{\epsilon_D^4}{\epsilon_M^2} + A^2 \epsilon_D^2 & A \epsilon_D + B \frac{\epsilon_D^3}{\epsilon_M^2} \\ 1 + B^2 \epsilon_D^2 \epsilon_M^2 \end{pmatrix}, \quad (13)$$

where we defined the characteristic quantity $\eta \equiv \frac{\epsilon_D^2}{\epsilon_M}$. The magnitude of the mixing angle is therefore governed by the ratio of the hierarchies of the Dirac and Majorana masses. Namely:

$$\tan 2\theta = 2 \epsilon_D \frac{A + \eta B}{1 + \eta (B^2 - e^{2i\phi} A^2 \epsilon_M - e^{2i\phi} \epsilon_D^2)} e^{i\phi}. \quad (14)$$

From Eq. (14) one encounters several interesting special cases, some of which are discussed in the following:

1) $\eta \approx 1$ but $\epsilon_{M,D} \ll 1$: similar hierarchy in $m_D$ and $M_R$

Then, we find for the mass matrix and the mixing angle

$$m_\nu \approx -\frac{m^2}{M} \begin{pmatrix} 0 & \epsilon_D (A + B) \\ \cdot & 1 + B^2 \end{pmatrix} \Rightarrow \tan 2\theta \approx 2 \epsilon_D \sqrt{\frac{a^2 + b^2 + 2ab \cos 2\beta}{1 + b^4 + 2b^2 \epsilon_D^2}} \epsilon_D . \quad (15)$$

Values of $\beta \simeq \pi/2$ and $b \simeq 1$ can thus lead to (close-to-)maximal mixing as observed in the atmospheric neutrino oscillation experiments. In this case, $\phi \simeq -\arg(A + i)$. Also, relaxing the conditions for $b$ and $\beta$ a bit can lead to the observed large but not maximal mixing in solar neutrino oscillation experiments.

2) $\eta \ll 1$: stronger hierarchy in $m_D$

The mass matrix and mixing are now given by

$$m_\nu \approx -\frac{m^2}{M} \begin{pmatrix} 0 & A \epsilon_D \\ \cdot & 1 \end{pmatrix} \Rightarrow \tan 2\theta \approx 2 \epsilon_D a, \quad (16)$$

which, for large but still reasonable choices of $\epsilon_D \simeq \sin \theta_C \simeq 0.22$ and $a \gtrsim 4$ yields $\tan 2\theta \gtrsim \sqrt{3}$, i.e., $\theta \gtrsim \pi/6$, as implied by the observed non–maximal large mixing in the solar neutrino oscillation experiments. More naturally, smaller values of $\epsilon_D$ and $a$ can easily reproduce the small mixing parameter as implied by the CHOOZ and Palo Verde reactor neutrino oscillation experiments. For the phase holds $\phi \simeq -\alpha$. 

5
3) $\eta \gg 1$: stronger hierarchy in $M_R$

The mixing is found to be

$$m_\nu \simeq -\frac{m^2}{M} \begin{pmatrix} 0 & B\epsilon_D \eta \\ B^2 \eta & 0 \end{pmatrix} \Rightarrow \tan 2\theta \simeq 2\epsilon_D \frac{1}{B},$$ (17)

for which similar arguments as for the case $\eta \ll 1$ hold. The phase is given by $\phi \simeq \beta$.

To sum up, hierarchical Dirac and Majorana mass matrices reproduce for specific choices of the hierarchies and parameters all observed types of neutrino mixing, (close–to–)maximal, non–maximal large and small mixing. Exactly maximal and vanishing mixing requires some fine–tuning. Vanishing mixing would be obtained for $|A + \eta B| \simeq 0$ or equivalently $a^2 + b^2\eta^2 = -2ab\epsilon\alpha_\beta$. We show in Fig. 1 several examples of the mixing obtained with specific choices of $\epsilon_D$, $A$ and $B$. One finds from the figure and the discussion in this Section that in order to obtain (close–to–)maximal mixing there is — in the given parametrization — a crucial dependence on the hierarchies of the fundamental matrices $m_D$ and $M_R$. Also the phases play an important role. Leptogenesis in turn requires the presence of $CP$ violation and — from Eq. (7) — depends on $m_D$ and $M_R$, therefore also on the ratio of the hierarchies. We should thus analyze leptogenesis in this scenario. The decay asymmetry reads

$$\varepsilon_1 = \frac{3\epsilon_M}{4\pi} \frac{m^2}{v^2} \frac{1}{b^2 + \epsilon_D^2} \left( (a\epsilon_D^2 \cos \alpha + b \cos \beta)(b \sin \beta - a\epsilon_D^2 \sin \alpha) \right) \simeq \frac{3\epsilon_M}{8\pi} \sin 2\beta,$$ (18)

where terms of order $\epsilon_D^2$ were neglected and $m \simeq v$ was used. We can construct a very interesting special case: suppose that the mass matrix parameters take the values $b \simeq 1$, $\epsilon_D \simeq 0.1$ and $\eta \simeq 1$. Then, from Eq. (15), we see that maximal mixing is only possible for $\beta \simeq \pi/2$. For this value of the phase, however, the decay asymmetry is highly suppressed. Therefore, maximal mixing implies a too small baryon asymmetry, or in other words, requiring a non–zero baryon asymmetry implies non–maximal neutrino mixing. We shall encounter a slightly similar effect in the next Section for the $3 \times 3$ case. Stressed is here that the same $CP$ phase can affect the magnitude of neutrino mixing angles and the value of the baryon asymmetry of the universe.

4 The $3 \times 3$ case

Let us turn now to the appropriate 3 flavor case. We can parametrize the relevant mass matrices $m_D$ and $M_R$ now as

$$m_D \simeq m \begin{pmatrix} 0 & A\epsilon_D^2 & 0 \\ B\epsilon_D^2 & \epsilon_D^2 & F\epsilon_D^2 \\ 0 & g\epsilon_D^2 & 1 \end{pmatrix}, \quad M_R = M \begin{pmatrix} \epsilon_{M1} & 0 & 0 \\ 0 & \epsilon_{M2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ (19)

\footnote{Note though that in general no link between low and high energy $CP$ violation exists \cite{16,17} and any such connection will be model dependent.}
For later use we define $A = a e^{i\alpha}$, $B = b e^{i\beta}$ and $F = f e^{i\phi}$; $g$ can be chosen real. Again, the complex coefficients have absolute values of order one, so has $g$. Small entries in the 11, 13 and 31 elements of $m_D$ are neglected (see below) and it holds $\epsilon_{M1} < \epsilon_{M2}$. We choose now the following parameters describing the relative hierarchy in $m_D$ and $M_R$:

$$\eta_1 = \epsilon_D^4/\epsilon_{M1} \quad \text{and} \quad \eta_2 = \epsilon_D^4/\epsilon_{M2} \quad \text{with} \quad \eta_1 > \eta_2 . \quad (20)$$

Let us choose a typical expansion parameter in $m_D$ of $\epsilon_D \simeq 0.1$ and an overall mass scale $m \simeq v \simeq 174$ GeV. Using the see–saw formula we find for $m_\nu$:

$$m_\nu \simeq -\frac{m_D^2}{M} \begin{pmatrix}
A \epsilon_D^2 \eta_2 & A \epsilon_D \eta_2 & A g \epsilon_D \eta_2 \\
\eta_2 + B^2 \epsilon_D^2 \eta_1 + F^2 \epsilon_D^4 & F \epsilon_D^2 + g \eta_2 \\
1 + g^2 \eta_2
\end{pmatrix}. \quad (21)$$

The light neutrino mass scheme will of course be hierarchical. To have an approximately degenerate spectrum in the 23 submatrix of $m_\nu$ (with scale $\sim \sqrt{\Delta m^2}$) it is required that $g \simeq 1$ and $\eta_2 \simeq 1$ or $\eta_2 \simeq 10$. Larger values are incompatible with $m \simeq v$ and $M \lesssim 10^{16}$ GeV. Later on it will be shown that $\tan 2\theta_{12}$, where $\theta_{12}$ is the mixing angle governing the solar neutrino oscillations, is proportional to $\epsilon_D \eta_2$ and thus the larger value of $\eta_2 \simeq 10$ is implied. Thus, $\epsilon_{M2} = \epsilon_D^4/\eta_2 \simeq 10^{-5}$, i.e., the heaviest Majorana neutrino has a much larger mass than the other two.

We can gain even more insight in the hierarchy of $M_R$ by looking at the decay asymmetry of the heavy Majorana neutrinos. It reads

$$\epsilon_1 = \frac{3 m_D^2}{8 \pi v^2} \epsilon_D^4 \left( \frac{\epsilon_{M1}}{\epsilon_{M2}} \sin 2\beta + f^2 \epsilon_{M1} \sin 2(\beta - \phi) \right)$$

$$\simeq 0.1 \epsilon_D^4 \left( \frac{\epsilon_{M1}}{\epsilon_{M2}} \sin 2\beta + f^2 \epsilon_{M1} \sin 2(\beta - \phi) \right) \quad (22)$$

where we used $\epsilon_{M1} \ll 1$ and assumed again $m \simeq v$. We can identify the leptogenesis phase $\beta$. Since the decay asymmetry should be negative, we can constrain $\beta$ to lie between $\pi/2$ and $\pi$ or between $3\pi/2$ and $2\pi$. In order to reach a favorable value of $|\epsilon_1| \gtrsim 10^{-7}$, the factor $\epsilon_{M2}/\epsilon_{M1} = \eta_1/\eta_2$ should not exceed $\sim 10$. Therefore, the two lightest Majorana neutrinos display a rather mild hierarchy. The requirements for the structure of $m_\nu$ and successful leptogenesis therefore determine the hierarchy of $M_R$.

For numerical estimates of the obtained quantities we shall use in the following the representative values $\epsilon_{M1} = 10^{-6}$, $\epsilon_{M2} = 10^{-5}$ and $\epsilon_D = 0.1$. These choices basically eliminate the parameter $F = f e^{i\phi}$ from the problem. The ratios of the branching ratios of the LFV violating charged lepton decays in Eq. (27) remain however somewhat sensitive to this parameter. Looking with the given parameter set for $\epsilon_D$, $\epsilon_{M1}$ and $\epsilon_{M2}$ at Eq. (21), one notes that the terms including $A$ and thus $\alpha$ are subleading. One can therefore expect the phase $\beta$ to play the major role in the observables under study. We shall see that this is
indeed the case.

For thermal leptogenesis the important effective mass parameter is given by

$$\tilde{m}_1 = \frac{(m_D^\dagger m_D)_{11}}{M_1} \simeq \frac{m^2}{M} b^2 \eta_1 \epsilon^2_D , \quad (23)$$

being of the order of the entries in $m_\nu$ and thereby guaranteeing for the baryon asymmetry a not too strong wash–out factor $\kappa$ (stemming from lepton number violating scattering processes) of $\kappa \sim 0.1 - 10^{-3}$ [14].

We can get a lower limit on the heavy neutrino masses by comparing our formula for $\epsilon_1$ with its analytical upper limit, which reads [18]

$$|\epsilon_1| \lesssim \frac{3}{8 \pi v^2} M_1 \sqrt{\Delta m^2_\Lambda} . \quad (24)$$

With $\Delta m^2_\Lambda \gtrsim 10^{-3}$ eV$^2$ one finds

$$M_1 \gtrsim \epsilon_D^4 \frac{\epsilon_{M1}}{\epsilon_{M2}} 10^{15} \text{ GeV} . \quad (25)$$

Therefore, for our chosen parameters of $\epsilon_D \simeq 0.1$ and $\epsilon_{M1}/\epsilon_{M2} \simeq 0.1$, it holds $M_1 \gtrsim 10^{10}$ GeV.

We can now take a closer look at the rates of the LFV violating charged lepton decays. Assumption of universality of the slepton mass matrices at the GUT scale leads via radiative corrections to non–diagonal entries at low scale, which give rise to LFV violating charged lepton decays such as $\mu \to e + \gamma$, $\tau \to \mu + \gamma$ and $\tau \to e + \gamma$ [19]. The branching ratios for the decay $\ell_j \to \ell_i \gamma$ with $\ell_{(3,2,1)} = \tau, \mu, e$ are approximately proportional to $|(m_D m^\dagger_D)_{ji}|^2$.

In our case, their magnitude is governed by

$$BR(\mu \to e \gamma) \propto \left|(m_D m^\dagger_D)_{21}\right|^2 \simeq a^4 m^4 \epsilon^9_D \quad (26)$$

and their ratios are predicted to be

$$BR(\mu \to e \gamma) \simeq \frac{1}{g^2} BR(\tau \to e \gamma) \simeq \frac{a^2}{f^2} \epsilon^6_D BR(\tau \to \mu \gamma) . \quad (27)$$

This relation gets modified by the presence of small entries in $m_D$, see Section 4.3.

### 4.1 Diagonalization

As seen, our simple arguments lead to the situation in which one of the right–handed Majorana mass is much heavier than the other two, which in turn display a mild hierarchy. In order to compare our framework with the neutrino data, we shall next diagonalize the resulting mass matrix $m_\nu$, leaving the definitions and details to the Appendix. We did not
consider the renormalization of the mass matrix since the corrections to neutrino masses and mixings are subleading in the case of a hierarchical mass spectrum [20], which we are considering.

Observation requires large mixing in the 23 sector of the matrix \( m_\nu \) in Eq. (21), which is given by

\[
m_{23}^2 \simeq -\frac{m^2}{M} \left( \eta_2 + B \epsilon_D^2 \eta_1, \frac{g \eta_2}{1 + g^2 \eta_2} \right) \simeq -\frac{m^2}{M} \eta_2 \left( \frac{1}{g}, g^2 \right)
\]  

and diagonalized by the mixing angle

\[
\tan 2\theta_{23} \simeq \frac{2g}{g^2 - 1}.
\]  

Note that the hierarchy chosen in this analysis renders the 23 submatrix quasi real, thereby simplifying the diagonalization procedure, see the Appendix for details. In order to guarantee a large solar mixing, the determinant of \( m_{23}^2 \) should be small [4, 5], which leads from Eq. (28) to \( |1 + b^2 g^2 \epsilon_D \eta_1 e^{2i\beta}| \approx 1 \).

The deviation from maximal mixing is of order

\[
1 - \sin^2 2\theta_{23} \approx \left( \frac{1 - g^2}{1 + g^2} \right)^2.
\]  

The largest eigenvalue of \( m_{23}^2 \) is

\[
m_3' \simeq -\frac{m^2}{M} \eta_2 (1 + g^2).
\]  

Note that \( m_3' \) will not be changed significantly by the following two rotations, \( m_3' \simeq m_3 \), and can therefore already be confronted with \( \sqrt{\Delta m^2_{A}} \approx 0.05 \text{ eV} \). Values of \( m \approx v \) and \( M \approx 10^{16} \text{ GeV} \) lead to the desired value if \( g \approx 1 \) and \( \eta_2 \approx 10 \).

It is now straightforward to extend the diagonalization procedure from Section 3 in order to obtain the remaining mass and mixing parameters. See the Appendix for details. One finds for the angle \( \theta_{13} \) that

\[
\tan 2\theta_{13} \simeq \sqrt{2} a \epsilon_D \frac{1 + g}{1 + g^2},
\]  

while the solar neutrino oscillations are triggered by

\[
\tan 2\theta_{12} \simeq \frac{\sqrt{2} a \epsilon_D \eta_2 (1 - g) (1 + g^2)}{\sqrt{1 + b^2 g^2 \epsilon_D^2 \eta_1 (b^2 g^2 \epsilon_D^2 \eta_1 + 2 c_{23})}}.
\]  

One notes that \( \theta_{13} \) is naturally small, \( \tan 2\theta_{13} \propto \epsilon_D \), while \( \tan 2\theta_{12} \) is larger than \( \tan 2\theta_{13} \) by approximately a factor of \( \sim \eta_2 \). We therefore observe a hierarchy in the mixing angles of the form

\[
\tan 2\theta_{23} \propto \frac{1}{1 - g^2} > \tan 2\theta_{12} \propto \epsilon_D \eta_2 > \tan 2\theta_{13} \propto \epsilon_D,
\]  

(34)
which is exactly the situation implied by neutrino phenomenology. It is seen that, for $\epsilon_D \simeq 0.1$, a value $\eta_2 \sim 10$ is required in order to reproduce the large solar neutrino mixing angle, which justifies our choice for $\eta_2$ as discussed above. Note that the dominator in Eq. (33) should be smaller than one. In fact, the denominator can be identified with $|1 + b^2 g^2 \epsilon_D^2 \eta_2 \epsilon^{2i\beta}|$, and the condition that this quantity is smaller than one was exactly the condition to make the determinant of the 23 submatrix of $m_\nu$ small. With our assumptions about the hierarchy parameters we can make the denominator very small for $b \simeq 1$ and $\beta \simeq \pi/2$. This value of $\beta$, however, leads via Eq. (22) to a too small baryon asymmetry. We have therefore an interplay between the baryon asymmetry of the universe and the non–maximality of $\theta_{12}$, which resembles the situation mentioned for the $2 \times 2$ case and discussed at the end of Section 3.

Regarding $\theta_{13}$, useful estimates can be performed. First of all, one can expect $\theta_{13}$ to be non–zero, because $a = 0$ will lead to a too small solar neutrino mixing. More precisely, we have for $g \simeq 1$ the estimate

$$|U_{e3}|^2 \simeq \frac{a^2 \epsilon_D^2}{2} \sim (10^{-3} - 10^{-2}),$$

where we assumed $a$ between 0.5 and 3 and $\epsilon_D = 0.1$. These values can be tested in the not too far future [21]. The magnitude of $U_{e3}$ is a crucial prediction for neutrino mass models, see, e.g., [22]. Fig. 2 shows for $\epsilon_D = 0.1$, $\epsilon_{M1} = 10^{-6}$ and $\epsilon_{M2} = 10^{-5}$ the mixing parameter $\tan^2 \theta_{12}$ as obtained from Eq. (33) for specific choices of $a$, $b$ and $g$ as a function of the leptogenesis phase $\beta$. The values of $\theta_{23}$ are close to maximal and of $\sin^2 \theta_{13}$ close to $10^{-2}$ for all cases plotted, confirming our quantitative statements from above. Also shown is — when negative — the decay asymmetry $\epsilon_1$ from Eq. (22) multiplied with $-10^6$. Its value is of the required magnitude for the solar neutrino mixing angle inside its experimental range, the angle $\theta_{13}$ below its upper limit and atmospheric mixing sufficiently large. Note that too large $\tan^2 \theta_{12}$ can lead to a too small decay asymmetry.

The two remaining mass eigenvalues are complicated functions of the parameters $\eta_1$, $\eta_2$, $\epsilon_D$, $a$, $b$, $g$, $\alpha$ and $\beta$. We saw above that for $\eta_2 \simeq 10$ and $M \simeq 10^{16}$ GeV the favorable value of $m_3 \simeq \sqrt{\Delta m^2_{\odot}}$ is achieved. With this choice for $M$, the common factor of $m_{1,2}$ is $m^2/M \simeq 3 \cdot 10^{-3}$ eV, which, when multiplied with a sum and difference of two terms of order one, can, admittedly involving some tuning, result in the required values of $|m_2|^2 - |m_1|^2 = \Delta m^2_{\odot}$. For later use we define that $\Delta m^2_{\odot} = m^4/M^2 \tilde{s}$, where $\tilde{s}$ is a function of the hierarchy parameters $\epsilon_D$, $\eta_{1,2}$ and the mass matrix parameters $a$, $b$, $g$, $\alpha$ and $\beta$. Its value is for $m \simeq v$ and $M \simeq 10^{16}$ GeV located around 10.

### 4.2 CP Violation in Neutrino Oscillation experiments and Neutrinoless Double Beta Decay

We shall investigate now the predictions of the scenario for the $CP$ asymmetries in neutrino oscillation experiments and for neutrinoless double beta decay and its connection to
leptogenesis. The interplay between these low and high energy parameters has recently been analyzed in a number of publications \[17, 23, 4, 25, 24\]. Instead of trying to identify the low energy Dirac and Majorana phases and express them in terms of the available high energy phases in Eq. (19), we shall work as convention–independent as possible.

We can calculate the rephasing invariant \( CP \) observable \( J_{CP} \), which can be written as \[24\]

\[
J_{CP} = -\frac{\text{Im}(h_{12} h_{23} h_{31})}{\Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32}},
\]

where \( h = m_{\nu} m_{\nu}^\dagger \).

(36)

With the help of \( m_{\nu} \) given in Eq. (21) we find with the choice of \( \epsilon_D^2 \eta_1 \approx 1 \) and \( \eta_1 \approx 10 \eta_2 \) that the leading term is given by

\[
-\text{Im}(h_{12} h_{23} h_{31}) \approx \frac{m_{12}^4 \epsilon_D^2 \eta_1 \eta_2^4 a^2 b^2 g^2 (1 + g^2) \sin 2\beta}{M^6} \approx \frac{2 m_{12}^4 \epsilon_D^2 \eta_1 \eta_2^4 a^2 b^2 \sin 2\beta}{M^6}.
\]

(37)

With the help of \( \Delta m^2_{31} \approx \Delta m^2_{32} \approx m_3^2 \approx (2 \eta_2 m^2 / M)^2 \) we find with our definition for \( \Delta m^2_3 \) that in leading order

\[
J_{CP} \approx \frac{1}{8} \epsilon_D^2 \eta_1 a^2 b^2 \tilde{s} \sin 2\beta.
\]

(38)

For our representative values we find that \( J_{CP} \sim 10^{-2} a^2 b^2 \sin 2\beta \). Recall that for, e.g., \( \tan^2 \theta_{12} = 0.45, \sin^2 2\theta_{23} = 1 \) and \( \sin^2 \theta_{13} = 0.01 \) the invariant \( J_{CP} \) is given by

\[
J_{CP} = \text{Im} \left\{ U_{e1} U_{\mu1}^* U_{e2}^* U_{\mu2} \right\} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta \approx 0.02 \sin \delta .
\]

(39)

Thus, it is confirmed that \( \theta_{13} \) is sizable in the framework under study. Since \( \Delta m^2_3 = |m_2|^2 - |m_1|^2 \) depends on \( \eta_1, \eta_2, \epsilon_D, a, b, g, \alpha \) and \( \beta \), whereas the decay asymmetry is proportional to \( \sin 2\beta \), there is no simple connection between the size of \( J_{CP} \) and \( Y_B \). It is seen, however, that — due to the same dependence on \( \beta \) — vanishing \( J_{CP} \) is incompatible with successful leptogenesis and that \( J_{CP} \) has the same sign as the baryon asymmetry. The case \( \epsilon_D = 0 \), i.e., the presence of only one Dirac mass, corresponds to an effective 2 flavor system in which \( J_{CP} \) has to vanish, as confirmed by Eq. (38).

Finally we can analyze the prediction of the scenario for neutrinoless double beta decay. From Eq. (21) and our usual assumptions of the parameters we find that the absolute values of the \( ee \) element of \( m_{\nu} \) is

\[
\langle m \rangle \equiv |(m_{\nu})_{ee}| \approx \frac{m^2}{M} \epsilon_D^2 \eta_2 \approx 3 a \cdot 10^{-4} \text{ eV}.
\]

(40)

Neutrinoless double beta decay triggered by values of \( \langle m \rangle \) smaller than \( 10^{-3} \) eV will probably be unobservable \[26\]. With Eqs. (31) and (35) we can however write an interesting correlation of parameters, namely:

\[
\langle m \rangle \approx \sqrt{\Delta m^2_3} |U_{e3}|^2.
\]

(41)

In summary, the same phase governs the \( CP \) asymmetry in neutrino oscillations and the decay asymmetry, whereas there is no correlation of the leptogenesis phase with the effective
mass in neutrinoless double beta decay. The very same features have been found for the minimal see–saw model [7], which is defined as having only 2 heavy Majorana neutrinos and 2 zeros in the Dirac mass matrix. Given the presence of two zeros (or very small entries) in our $m_D$ (see Eq. (19)) and the fact that $M_3 \gg M_{2,1}$, it is very interesting that we encounter the same situation. Note however that different variations of the model, which have been discussed lately in the literature [25], do not necessarily display the mentioned correlations of the phases.

4.3 Effects of entries of order $\epsilon_D^4$ in $m_D$

The question arises if it is valid to neglect terms of order $\epsilon_D^4$ in the 11, 13 and 31 entries of $m_D$ in Eq. (19). We therefore repeat the calculation with terms of this order. One finds that new contributions to $m_\nu$ are suppressed by one or two orders of $\epsilon_D$. Regarding the LFV violating decays, one observes that the term $| (m_D m_D^\dagger)_{31} |^2$ now has the leading contribution proportional to $\epsilon_D^4 h_1$, where $h_1$ is the absolute value of the 13 element of $m_D$. The other terms acquire subleading new contributions stemming from the new entries in $m_D$. Thus, Eq. (27) is modified to

$$BR(\mu \to e \gamma) \simeq \frac{a^2}{h_1^2} \epsilon_D^2 BR(\tau \to e \gamma) \simeq \frac{a^2}{f^2} \epsilon_D^6 BR(\tau \to \mu \gamma) \, ,$$

or, numerically:

$$BR(\mu \to e \gamma) \sim 10^{-2} BR(\tau \to e \gamma) \sim 10^{-6} BR(\tau \to \mu \gamma) \, .$$

Note the analogy of these ratios with the ones presented in [17], where also a hierarchical $m_D$ was assumed. One sees that the small entries of order $\epsilon_D^4$ change the ratio between $BR(\mu \to e \gamma)$ and $BR(\tau \to e \gamma)$ by a factor of $\epsilon_D^2 \simeq 10^{-2}$. The decay asymmetry $\varepsilon_1$ is also slightly altered. It reads now

$$\varepsilon_1 = \frac{3 m^2}{8 \pi^2 v^2} \epsilon_D^2 \left( \frac{\epsilon_D^2}{\epsilon_{M1}} \frac{\epsilon_{M2}}{\epsilon_{M1}} \sin 2\beta + \frac{h_2^2}{b^2} \epsilon_{M1} \sin 2\delta_2 \right) \, ,$$

where $h_2$ and $\delta_2$ are the absolute value and phase of the 31 entry of $m_D$. For $\epsilon_{M1} \ll \epsilon_D^2$, the situation we are interested in, we recover the form given in Eq. (22). Thus, small entries in $m_D$, which were neglected in Eq. (19), have in our framework some influence on the ratios of the LFV violating decay branching ratios but only little influence on $m_\nu$ and $\varepsilon_1$.

5 Conclusions

The see–saw mechanism with hierarchical Dirac and Majorana neutrino masses was reanalyzed in the presence of $CP$ phases. A consistent and appealing framework of neutrino mixing phenomenology and leptogenesis was found, in which one of the heavy Majorana neutrinos is much heavier than the other two, which in turn display a mild hierarchy. It
was investigated how large neutrino mixing can be generated starting from hierarchical mass matrices in the see–saw mechanism. Ratios for the branching ratios of LFV charged lepton decays are predicted, which are sensitive to small entries in $m_D$. A natural hierarchy of the mixing angles in accordance with observation is found and it holds $|U_{e3}|^2 \gtrsim 10^{-3}$, which is observable in the not so far future. There can be an interplay between too large solar neutrino mixing and a too small baryon asymmetry. The $CP$ asymmetry in neutrino oscillations has the same sign as the baryon asymmetry of the universe and successful leptogenesis implies non–zero and measurable $J_{CP}$. Neutrinoless double beta is not linked with the leptogenesis phase and will probably not be observable. The framework under study resembles in this respect very much the minimal see–saw model.

Acknowledgments

I thank S. Pascoli and S.T. Petcov for helpful comments and discussions. The hospitality of the Max–Planck Institut für Physik, München, where part of this study was performed, is gratefully acknowledged. This work was supported in part by the EC network HPRN-CT-2000-00152.

References

[1] See the recent reviews M.C. Gonzalez-Garcia, Y. Nir, Rev. Mod. Phys. 75, 345 (2003); S. Pakvasa, and J.W.F. Valle, hep-ph/0301061; V. Barger, D. Marfatia and K. Whisnant, hep-ph/0308123.

[2] M. Gell–Mann, P. Ramond, and R. Slansky in Supergravity, p. 315, edited by F. Nieuwenhuizen and D. Friedman, North Holland, Amsterdam, 1979; T. Yanagida, Proc. of the Workshop on Unified Theories and the Baryon Number of the Universe, edited by O. Sawada and A. Sugamoto, KEK, Japan 1979; R.N. Mohapatra, G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).

[3] A.Yu. Smirnov, Phys. Rev. D 48, 3264 (1993); M. Tanimoto, Phys. Lett. B 345, 477 (1995); G. Altarelli, F. Ferruglio, and I. Masina, Phys. Lett. B 472, 382 (2000); E.K. Akhmedov, G.C. Branco, and M.N. Rebelo, Phys. Lett. B 478, 215 (2000); A. Datta, F.-S. Ling, and P. Ramond, hep-ph/0306002 see also S. Lavignac, I. Masina and C. A. Savoy, Nucl. Phys. B 633, 139 (2002).

[4] F. Vissani, JHEP 11, 025 (1998); Phys. Lett. B 508, 79 (2001); J. Sato and T. Yanagida, Phys. Lett. B 493, 356 (2000).

[5] S.F. King, JHEP 0209, 011 (2002).

[6] W. Rodejohann, hep-ph/0309249
[7] P.H. Frampton, S.L. Glashow, and T. Yanagida, Phys. Lett. B 548, 119 (2002); W. l. Guo, Z.Z.Xing, hep-ph/0310326.

[8] B. Pontecorvo, Zh. Eksp. Teor. Fiz. 33, 549 (1957) and 34, 247 (1958); Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. 28, 870 (1962).

[9] N. Ahmed et al. [SNO Collaboration], nucl-ex/0309004.

[10] M. Maltoni et al., hep-ph/0309130; see also A. B. Balantekin and H. Yuksel, hep-ph/0309079, G. L. Fogli, et al., hep-ph/0309100, A. Bandyopadhyay, et al., hep-ph/0309174, P. C. de Holanda and A. Y. Smirnov, hep-ph/0309299.

[11] M. Fukugita, T. Yanagida, Phys. Lett. B 174, 45 (1986); M.A. Luty, Phys. Rev. D 45, 455 (1992); M. Flanz, E.A. Paschos, and U. Sarkar, Phys. Lett. B 345, 248 (1995); L. Covi, E. Roulet, and F. Vissani, Phys. Lett. B 384, 169 (1996); M. Flanz et al., Phys. Lett. B 389, 693 (1996); M. Plümacher, Z. Phys. C 74, 549 (1997); A. Pilaftsis, Phys. Rev. D 56, 5431 (1997); W. Buchmüller, M. Plümacher, Phys. Lett. B 431, 354 (1998).

[12] See the reviews A. Pilaftsis, Int. J. of Mod. Phys. A 14, 1811 (1999); W. Buchmüller, M. Plümacher, Int. J. of Mod. Phys. A 15, 5047 (2000).

[13] For a recent analysis, see A. Pilaftsis and T. E. Underwood, hep-ph/0309342.

[14] W. Buchmüller, P. Di Bari, and M. Plümacher, Nucl. Phys. B 643, 367 (2002); G. F. Giudice, et al., hep-ph/0310123.

[15] M.Y. Khlopov and A.D. Linde, Phys. Lett. B 138, 265 (1984); J. Ellis, et al., Phys. Lett. B 145, 1984 (181); Phys. Rev. D 67, 103521 (2003); M. Kawasaki, and T. Moroi, Prog. Theor. Phys. 93, 879 (1995); E. Holtmann et al., Phys. Rev. D 60, 023506 (1999), for the prediction in different models see K. Hamaguchi, hep-ph/0212305 and references therein.

[16] G.C. Branco, T. Morozumi, and B.M. Nobre, Nucl. Phys. B 617, 475 (2001); M.N. Rebelo, Phys. Rev. D 67, 013008 (2003).

[17] S. Pascoli, S.T. Petcov, and W. Rodejohann, hep-ph/0302054, to appear in Phys. Rev. D.

[18] S. Davidson, and A. Ibarra, Phys. Lett. B 535, 25 (2002).

[19] F. Borzumati, A. Masiero, Phys. Rev. Lett. 57, 961 (1986); for the connection to neutrino mixing, see, e.g., J.A. Casas and A. Ibarra, Nucl. Phys. B 618, 171 (2001); S. Lavignac, C.A. Savoy, and C.A. Savoy, Phys. Lett. B 520, 269 (2001).
[20] See, e.g., J.A. Casas, et al., Nucl. Phys. B 573, 652 (2000); P.H. Chankowski, S. Pokorski, Int. J. Mod. Phys. A 17 (2002) 575; S. Antusch, et al., hep-ph/0305273 and references therein; for the scenario under study see the discussion in the third reference in [3]. The stability of $J_{CP}$ has also been analyzed in C.W. Chiang, Phys. Rev. D 63, 076009 (2001).

[21] See, e.g., M. Lindner, Invited talk at XXth International Conference on Neutrino Physics and Astrophysics (Neutrino 2002), Munich, Germany, 25-30 May 2002, hep-ph/0210377 and references therein.

[22] Some recent predictions for $U_{e3}$ through radiative corrections A.S. Joshipura, Phys. Lett. B 543, 276 (2002); R.N. Mohapatra, M. K. Parida and G. Rajasekaran, hep-ph/0301234 in the type II see–saw mechanism: H.S. Goh, R.N. Mohapatra and S.P. Ng, Phys. Lett. B 570, 215 (2003); for Fritzsch type mass matrices: M. Fukugita, M. Tanimoto and T. Yanagida, Phys. Lett. B 562, 273 (2003); from physics above the GUT scale: F. Vissani, M. Narayan and V. Berezinsky, hep-ph/0305233. See also the reviews S.M. Barr and I. Dorsner, Nucl. Phys. B 585 (2000) 79; M. Tanimoto, hep-ph/0305274 and references therein.

[23] See, e.g., A.S. Joshipura, E.A. Paschos and W. Rodejohann, JHEP 08, 029 (2001); W. Buchmüller, D. Wyler, Phys. Lett. B 521, 291 (2001); G.C. Branco et al., Nucl. Phys. B 640, 202 (2002); H.B. Nielsen, Y. Takanishi, Nucl. Phys. B 636, 305 (2002); J. Ellis, M. Raidal, Nucl. Phys. B 643, 229 (2002). Z.Z. Xing, Phys. Lett. B 545, 352 (2002); S. Davidson, A. Ibarra, Nucl. Phys. B 648, 345 (2003); W. Rodejohann, Phys. Lett. B 542, 100 (2002); S.F. King, Phys. Rev. D 67, 113010 (2003); S. Kaneko, M. Katsumata, and M. Tanimoto, JHEP 0307, 025 (2003); L. Velasco-Sevilla, hep-ph/0307071.

[24] G.C. Branco et al., Phys. Rev. D 67, 073025 (2003).

[25] T. Endoh et al., Phys. Rev. Lett. 89, 231601 (2002); M. Raidal and A. Strumia, Phys. Lett. B 553, 72 (2003); V. Barger et al., hep-ph/0310278 R. Gonzalez Felipe, F.R. Joaquim, and B.M.Nobre, hep-ph/0311029.

[26] O. Cremonesi, Invited talk at the XXth Internat. Conf. on Neutrino Physics and Astrophysics (Neutrino 2002), Munich, Germany, May 25-30, 2002, hep-ex/0210007.
Figure 1: Result for the mixing angle in a $2 \times 2$ framework, Eq. (14), obtained for hierarchical Dirac and Majorana neutrino mass matrices $m_D$ and $M_R$ and different values of the relevant parameters.

Figure 2: Result for the mixing parameter $\tan^2 \theta_{12}$, as obtained from Eq. (33), for different $a$, $b$ and $g$ as a function of the leptogenesis phase $\beta$. The range as implied by experiment is indicated. The values of $|U_{e3}|^2$ are 0.009, 0.033 and 0.027, respectively. For $g = 1.2$ (1.4) atmospheric neutrino mixing is given by $\sin^2 2\theta_{23} \simeq 0.97$ (0.90). Plotted is also the decay asymmetry $\varepsilon_1$ from Eq. (22) multiplied with $-10^5$ (dash–dotted).
A Diagonalization of a complex and hierarchical symmetric $3 \times 3$ matrix

We present for completeness our formulae for the diagonalization of a complex and hierarchical symmetric $3 \times 3$ matrix. It is a special case of the general strategy as outlined, e.g., in Ref. [5]. In the diagonalization of a $2 \times 2$ matrix three phases were present. We saw that two of them can be absorbed in the charged lepton fields. Diagonalizing a complex $3 \times 3$ matrix through three consecutive $2 \times 2$ diagonalizations will introduce 6 phases, which in principle can influence the mixing angles. In our case, however, they do not. We take advantage of the somewhat more simple structure of $m_\nu$ in the hierarchical situation we consider. It is convenient to express the results in terms of mixing angles. Regarding the phases, as stated in the text, we prefer not to identify the low energy Dirac and Majorana phases but work with convention independent quantities like $J_{CP}$. Consider a symmetrical neutrino mass matrix

$$m = \begin{pmatrix} a & b & d \\ \cdot & e & f \\ \cdot & \cdot & g \end{pmatrix}, \quad (45)$$

where the 23 block has entries larger than the other elements. The strategy outlined in [5] is to first rephase the mass matrix with $P_2 m P_2$, where $P_2$ is a diagonal phase matrix with complex entries on the 22 and 33 elements. Then, one puts zeros in the 23 and 13 elements of $m$ by diagonalizing first the 23 submatrix and then the resulting 13 submatrix. Then the matrix is again rephased by a diagonal phase matrix containing only one complex entry on the 22 element. After that, we have to diagonalize the 12 submatrix and end up in this way with a diagonal matrix. The eigenstates are however still complex. Thus, by again rephasing the diagonal matrix and absorbing these three phases in the charged leptons, we end up with the desired three real diagonal entries, three mixing angles and three phases. In our case, the 23 submatrix of Eq. (21) is effectively real, since we choose $\eta_2 \simeq 10$. Therefore, the first rephasing with $P_2$ is not necessary and there is also no phase in the 23 rotation. Thus, the 23 submatrix is diagonalized via $R_{23}^T m_23$ where

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad (46)$$

where $c_{23} = \cos \theta_{23}$ and $s_{23} = \sin \theta_{23}$. The resulting matrix $m'$ is

$$m' = \begin{pmatrix} a & b c_{23} - d s_{23} & b s_{23} + d c_{23} \\ \cdot & m'_2 & 0 \\ \cdot & \cdot & m'_3 \end{pmatrix} \equiv \begin{pmatrix} a & b' & d' \\ \cdot & m'_2 & 0 \\ \cdot & \cdot & m'_3 \end{pmatrix}, \quad (47)$$
for
\[m'_{2,3} = \frac{1}{2} \left( (e + g) \mp \sqrt{(e - g)^2 + 4 f^2} \right)\]  
and
\[\tan 2\theta_{23} = \frac{2 f}{g - e} .\]  
Now the 13 submatrix of \(m'\) is diagonalized via \(R_{13}^T m' R_{13}\) with
\[R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s^*_{13} & 0 & c_{13} \end{pmatrix},\]  
where \(c_{13} = \cos \theta_{13}\) and \(s_{13} = \sin \theta_{13} e^{i\phi_{13}}\). The resulting matrix \(m''\) reads
\[m'' = \begin{pmatrix} m''_1 & b' c_{13} & 0 \\ \cdot & m''_2 & b' s_{13} \\ \cdot & \cdot & m''_3 \end{pmatrix} \simeq \begin{pmatrix} m''_1 & b' & 0 \\ \cdot & m''_2 & 0 \\ \cdot & \cdot & m''_3 \end{pmatrix},\]  
where the last approximation takes into account the smallness of \(\theta_{13}\) as implied by the reactor experiments and the hierarchical structure of \(m\). The masses and the mixing angle are given by
\[m''_{1,3} = \frac{1}{2} \left( (a + m_3') \mp \sqrt{(a - m_3')^2 + 4 d'^2} \right)\]  
and
\[\tan 2\theta_{13} = \frac{2 d'}{m_3' e^{-i\phi_{13}} - a e^{i\phi_{13}}} \simeq \frac{2 d' e^{i\phi_{13}}}{m_3},\]  
where \(\arg(d') = \arg(m_3' e^{-i\phi_{13}} - a e^{i\phi_{13}}) \Rightarrow \phi_{13} \simeq \arg(m_3') - \arg(d')\).

From Eq. (21) we see that the 11 element of our \(m_{1\nu}\) (here called \(a\)) is much smaller than \(m_3'\) as given in Eq. (31). The phase \(\phi_{13}\) is therefore suppressed and does not influence the magnitude of \(\theta_{13}\). The eigenvalue \(m''_3 \equiv m_3\) is already the heaviest eigenvalue of the matrix \(m\). Now we rephase \(m''\) through a diagonal phase matrix \(P\) with only the 22 entry being complex, \(P = \text{diag}(1, e^{i\phi}, 1)\). Finally, the 12 submatrix of \(m''\) gets diagonalized by \(R_{12}^T m'' R_{12}\) where
\[R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s^*_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},\]  
and for the masses and mixing angle holds
\[m_{1,2} = \frac{1}{2} \left( (m''_1 + m''_2) \mp \sqrt{(m''_1 - m''_2)^2 + 4 b^2} \right)\]  
18
as well as
\[
\tan 2\theta_{12} = \frac{2b'e^{i\phi}}{m_2'e^{-i\phi_{12}} e^{2i\phi} - m_1''e^{i\phi_{12}}} , \text{ where } \arg(b'e^{i\phi}) = \arg(m_2'e^{-i\phi_{12}} e^{2i\phi} - m_1''e^{i\phi_{12}}).
\]

(56)

In our case it turns out that \(m_2' \gg m_1''\), therefore \(\phi\) and \(\phi_{12}\) do not influence the magnitude of \(\theta_{12}\). The mass states are in general still complex. Rephasing these states through a diagonal phase matrix and absorbing them in the charged lepton fields then leaves us with the correct number of three phases in \(U\).