WEYL FAMILIES OF ESSENTIALLY UNITARY PAIRS

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Abstract. It is known that the Weyl families corresponding to unitary boundary pairs \((\mathcal{H}, \Gamma)\) belong to the class \(\tilde{\mathcal{R}}(\mathcal{H})\) of Nevanlinna families. Here we extend the theorem to the case of essentially unitary pairs by showing that the closures of members of the Weyl families belong to the class \(\tilde{\mathcal{R}}(\mathcal{H})\). Thus bounded Weyl functions of essentially unitary pairs are of class \(\mathcal{R}(\mathcal{H})\).

1. Introduction

Throughout \(\mathcal{H}\) and \(\mathcal{H}\) denote Hilbert spaces. Let \(\Gamma \subseteq \mathcal{H}_2 \times \mathcal{H}_2\) be a linear relation from a \(J_{\mathcal{H}}\)-space to a \(J_{\mathcal{H}}\)-space \([AI89, Section 1]\), where the canonical symmetry \(J_{\mathcal{H}}\) acts on \(\mathcal{H}_2\) as the multiplication operator by the second Pauli matrix \((0 \ -i \ i \ 0)\).

Let \(\Gamma^*[\ast]\) denote the Krein space adjoint of \(\Gamma\) \([DHM17, Equation (2.6)]\), \([DHMdS12, Section 7.2]\). Then \(\Gamma\) is said to be \((J_{\mathcal{H}}, J_{\mathcal{H}}^\ast)\)-isometric if \(\Gamma^{-1} \subseteq \Gamma^*[\ast]\) and \((J_{\mathcal{H}}, J_{\mathcal{H}}^\ast)\)-unitary if \(\Gamma^{-1} = \Gamma^*[\ast]\) \([DHM17, Definition 2.2]\), and essentially \((J_{\mathcal{H}}, J_{\mathcal{H}}^\ast)\)-unitary if \(\overline{\Gamma}^{-1} = \Gamma^*[\ast]\) \([DHMdS06]\), where the overbar denotes the closure. For notational convenience, \(\Gamma\) is simply referred to as either isometric or (essentially) unitary.

Let \(A\) be a closed symmetric linear relation in \(\mathfrak{H}\) and let \(\Gamma\) be an isometric linear relation. We assume that \(A^* := \text{dom} \Gamma\) is dense in \(A^\ast\) with respect to the topology on \(\mathfrak{H}_2\). We put \(A := A^* = \text{mul} \Gamma^*[\ast]\), so that the above assumptions are always satisfied by default. Likewise, putting \(A_* := \text{dom} \overline{\Gamma}\) and \(A := A_* \equiv (A_*)^\ast\), and using that \(\overline{\Gamma}\) is isometric, one finds that the linear relation \(A\) is closed and symmetric in \(\mathfrak{H}\), and \(A^*\) is dense in \(A^\ast\). Note that \(A = \overline{A}\); if in addition \(\Gamma\) is unitary, then \(A = \overline{A} = S\), where \(S := \ker \Gamma\).

By the above assumptions, the pair \((H, \Gamma)\) is an isometric/unitary boundary pair (for \(A^\ast\)) if \(\Gamma\) is isometric/unitary \([DHM17, Definition 3.1]\). In the terminology of \([DHMdS06, Definition 3.1]\) \(\Gamma\) is a boundary relation (for \(S^\ast\)) iff it is unitary; see also \([DHMdS06, Proposition 3.2]\). If \(\Gamma\) is essentially unitary, we also say that the pair \((H, \Gamma)\) is an essentially unitary pair (for \(A^\ast\)).

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The first part of [DHMdS06, Theorem 3.9] states that the Weyl family \( M_\Gamma(z), z \in \mathbb{C}_* := \mathbb{C} \setminus \mathbb{R} \), corresponding to a boundary relation \( \Gamma \) is a Nevanlinna family, that is, it belongs to the class \( \mathcal{N}(\mathcal{H}) \) (Definition 4.1). Here we prove an analogue of this statement for an essentially unitary \( \Gamma \).

**Theorem 1.1.** Let \( M_\Gamma(z), z \in \mathbb{C}_* \), be the Weyl family corresponding to an essentially unitary pair \((\mathcal{H}, \Gamma)\). Then the closure \( \overline{M_\Gamma(z)} = M_\overline{\Gamma}(z) \) belongs to a Nevanlinna family.

Here \( \{M_\Gamma(z)\} \) is the Weyl family corresponding to the unitary boundary pair \((\mathcal{H}, \overline{\Gamma})\). For \( \Gamma \) unitary (hence closed), the theorem clearly reduces to the first part of [DHMdS06, Theorem 3.9]. By assuming additionally that the Weyl function \( M_\Gamma(z) \) is bounded (hence closed), one deduces another corollary.

**Corollary 1.2.** Let \((\mathcal{H}, \Gamma)\) be an essentially unitary pair and \( M_\Gamma(z) \in B(\mathcal{H}) \) \( \forall z \in \mathbb{C}_* \). Then the Weyl function \( M_\Gamma(z) = M_\overline{\Gamma}(z) \) belongs to the subclass \( \mathcal{N}[\mathcal{H}] \) of Nevanlinna functions. \( \square \)

According to [DHMdS06, Proposition 5.9] a Nevanlinna function of class \( \mathcal{N}[\mathcal{H}] \) can be realized as the Weyl function of a \( B \)-generalized boundary pair \((\mathcal{H}, \Gamma)\) [DHM17, Definition 3.5]. Let us recall that ordinary, \( B \)-generalized, \( S \)-generalized, \( ES \)-generalized boundary pairs are all unitary boundary pairs; see [DHM17] for more details. Yet Corollary 1.2 shows that one can find a non-unitary boundary pair with the same Weyl function.

Assuming the hypotheses in Corollary 1.2 and \( \text{ran} \overline{\Gamma} = \mathcal{H}^2 \), one concludes that the Weyl function \( M_\Gamma(z) = M_\overline{\Gamma}(z) \) belongs to the subclass \( \mathcal{N}^u[\mathcal{H}] \) of uniformly strict Nevanlinna functions. The single-valued linear relation \( \Gamma \) with such properties arises, for example, in the study of triplet extensions associated to the scales of Hilbert spaces of self-adjoint operators [Jur18, Section 7.5]; see also Section 5.

The proof of the main theorem is organized as follows: In Section 2 we list some preparatory results. In Section 3 we compute the adjoint \( M_\Gamma(z)^* \) for an isometric pair \((\mathcal{H}, \Gamma)\); it follows that \( M_\Gamma(z)^* = M_{\overline{\Gamma}(z)} \) for \( \Gamma \) essentially unitary. Since \( \{M_{\overline{\Gamma}(z)}\} \) is a Nevanlinna family for \( \overline{\Gamma} \) unitary—the fact that we actually show without referring to [DHMdS06, Theorem 3.9]—this implies Theorem 1.1; see Section 4.

Throughout we use the standard symbols \( \text{dom}, \text{ran}, \text{mul}, \text{ker} \) to denote the domain, the range, the multivalued part, and the kernel of a linear relation. For more details related to the theory of linear relations and Nevanlinna families the reader may consult the papers in [BBM+18, DM17, BMN15, BHdS+13, dSWW11, DHMdS09, HdSS09, BHdS08, HSdSS07, HdS96, DM91] and also an extensive list of references therein.
2. Preliminaries

Here and elsewhere below a linear relation $\Gamma \subseteq \mathcal{H}^2 \times \mathcal{H}^2$ from a $J_{\mathcal{B}}$-space to a $J_{\mathcal{H}}$-space is assumed to be isometric, unless explicitly stated otherwise. Then the Green identity holds:

$$[\hat{f}, \hat{g}]_{\mathcal{B}} = [\hat{h}, \hat{k}]_{\mathcal{H}}$$

for $(\hat{f}, \hat{h}) \in \Gamma$, $(\hat{g}, \hat{k}) \in \Gamma$. The $J_{\mathcal{B}}$-metric $[\cdot, \cdot]_{\mathcal{B}}$ is written in terms of the $\mathcal{H}^2$-scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ according to

$$[\hat{f}, \hat{g}]_{\mathcal{B}} := \langle \hat{f}, J_{\mathcal{B}}\hat{g} \rangle_{\mathcal{B}}^2 = -i(\langle \hat{f}, \hat{g} \rangle_{\mathcal{B}} - \langle \hat{f}', \hat{g} \rangle_{\mathcal{B}})$$

for $\hat{f} = (f, f') \in \mathcal{H}^2$, $\hat{g} = (g, g') \in \mathcal{H}^2$, provided that the $\mathcal{B}$-scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ is conjugate-linear in the first argument. The same applies to the $J_{\mathcal{H}}$-metric $[\cdot, \cdot]_{\mathcal{H}}$.

The Krein space adjoint $\Gamma^* = \{ (\hat{k}, \hat{g}) \in \mathcal{H}^2 \times \mathcal{H}^2 \mid (\forall (\hat{f}, \hat{h}) \in \Gamma) [\hat{f}, \hat{g}]_{\mathcal{B}} = [\hat{h}, \hat{k}]_{\mathcal{H}} \}$.

Thus $\Gamma^{-1} \subseteq \Gamma^*$, and the equality holds iff $\Gamma$ is unitary.

As usual, the eigenspaces of $A^*$ are denoted by

$$\mathcal{N}_z(A^*) := \ker(A^* - z), \quad \hat{\mathcal{N}}_z(A^*) := \{ \hat{f}_z = (f_z, zf_z) \mid f_z \in \mathcal{N}_z(A^*) \}$$

for $z \in \mathbb{C}$, and similarly for other linear relations. Since $A^*$ is closed in $\mathcal{B}$, its eigenspace is also closed, and one attains the orthogonal decomposition $\mathcal{B} = \mathcal{N}_{\mathcal{B}}(A^*) \oplus \mathcal{N}_{\mathcal{B}}(A^*)^\perp$, where $\mathcal{N}_{\mathcal{B}}(A^*)^\perp = \text{ran}(A - z)$. Then the $J_{\mathcal{B}}$-orthogonal complement [AI89, Definition 1.11] $\hat{\mathcal{N}}_z(A_s)(\perp)$ of $\hat{\mathcal{N}}_z(A_s)$ can be written as

$$\hat{\mathcal{N}}_z(A_s)(\perp) = \hat{\mathcal{N}}_{\mathcal{H}}(A^*) \perp \mathcal{D}_z, \quad \mathcal{D}_z := \cap \{ \mathcal{N}_z(A_s)(\perp) \times \mathcal{N}_z(A_s)(\perp) \}$$

where $\perp$ denotes the componentwise sum [HdSS09, Section 2.4] and $I_{\mathcal{B}\mathcal{H}}(A^*)$ denotes (the graph of) the identity operator restricted to $\mathcal{N}_{\mathcal{B}}(A^*)^\perp$.

**Remark 2.4.** Let $\mathcal{D} := \cap \{ \mathcal{D}_z \mid z \in \mathbb{C}_+ \}$; then $\mathcal{D} = \{ 0 \} \times \mathcal{M}$ where

$$\mathcal{M} := \bigcap \{ \mathcal{N}_z(A_s)(\perp) \mid z \in \mathbb{C}_+ \}.$$ 

Clearly $\mathcal{D} \subseteq S$ iff $\mathcal{M} \subseteq \text{mul } S$. Since $A_s \subseteq A^*$ densely, $\mathcal{M} = \{ 0 \}$ iff $A$ is simple [LT77], in which case a closed symmetric linear relation $A$ (and hence $S \subseteq A$) is an operator. The equality $\mathcal{M} = \{ 0 \}$ also shows that the closed linear span

$$\mathcal{H}_s := \bigvee \{ \mathcal{N}_z(A_s) \mid z \in \mathbb{C}_+ \}$$

coincides with $\mathcal{H}$. When $A = S$ is simple, a unitary $\Gamma$ is minimal [DHMdS06, Definition 3.4], and vice versa.
3. Weyl families

The Weyl family of $A$ corresponding to an isometric pair $(\mathcal{H}, \Gamma)$ is defined by [DHM17, Definition 3.2] $M_{\Gamma}(z) := \Gamma \hat{\mathcal{N}}_z(A_\ast)$ for $z \in \mathbb{C}_\ast$. Put

$$\Gamma_z := \Gamma |_{\hat{\mathcal{N}}_z(A_\ast)} := \Gamma \cap (\hat{\mathcal{N}}_z(A_\ast) \times \mathcal{H}^2)$$

then the linear relation $M_{\Gamma}(z)$ and its adjoint can be described by

$$M_{\Gamma}(z) = \text{ran} \Gamma_z, \quad M_{\Gamma}(z)^* = \ker \Gamma_z$$

where $\Gamma_z^{[s]}$ denotes the Krein space adjoint of $\Gamma_z$. Since $\Gamma$ is isometric and $\Gamma_z \subseteq \Gamma$ for $z \in \mathbb{C}$, it is evident that

$$\Gamma_z^{-1} \subseteq \Gamma^{-1} \subseteq \Gamma^{[s]} \subseteq \Gamma_z^{[s]},$$

that is, $\Gamma_z$ is also isometric. Moreover, $\Gamma_w^{-1} \subseteq \Gamma_z^{[s]}$ for all $z, w \in \mathbb{C}$.

**Lemma 3.2.** $\mathcal{O}_z \subseteq \text{mul} \Gamma_z^{[s]}$ for $z \in \mathbb{C}$.

**Proof.** Let $\hat{g} \in \mathcal{O}_z$; then by (2.3) $\hat{g} = (g, zg + f)$, $g \in \mathcal{N}_z(A^\ast)^\perp$, $f \in \mathcal{N}_z(A_\ast)^\perp$. Then by (2.2) $(\forall (\hat{f}, \hat{h}) \in \Gamma_z)$

$$[\hat{f}, \hat{g}]_\mathcal{H} = -i \langle \hat{f}, f \rangle_\mathcal{H} = 0 = [\hat{h}, (0, 0)]_\mathcal{H}$$

hence $((0, 0), \hat{g}) \in \Gamma_z^{[s]}$. \hfill $\square$

**Theorem 3.3.** The adjoint is given by

$$M_{\Gamma}(z)^* = (\Gamma_z^{[s]})^{-1} \hat{\mathcal{N}}_z(A^\ast)$$

for $z \in \mathbb{C}_\ast$.

**Proof.** We split the proof into three steps.

**Step 1.** We show that $M_{\Gamma}(z)^* = (\Gamma_z^{[s]})^{-1} \hat{\mathcal{N}}_z(A_\ast)^{[\perp]}$. Consider $\hat{k} \in M_{\Gamma}(z)^* = M_{\Gamma}(z)^{[\perp]}$; then $(\forall (\hat{f}, \hat{h}) \in \Gamma_z \Leftrightarrow \forall \hat{h} \in M_{\Gamma}(z))$ $(\forall \hat{g} \in \hat{\mathcal{N}}_z(A_\ast)^{[\perp]})$

$$[\hat{f}, \hat{g}]_\mathcal{H} = 0 = [\hat{h}, \hat{k}]_\mathcal{H}$$

and so $\hat{k} \in (\Gamma_z^{[s]})^{-1} \hat{\mathcal{N}}_z(A_\ast)^{[\perp]}$. Conversely, consider $\hat{k} \in (\Gamma_z^{[s]})^{-1} \hat{\mathcal{N}}_z(A_\ast)^{[\perp]}$; then $(\exists \hat{g} \in \hat{\mathcal{N}}_z(A_\ast)^{[\perp]}) (\hat{k}, \hat{g}) \in \Gamma_z^{[s]}$, and so $(\forall (\hat{f}, \hat{h}) \in \Gamma_z)$ equation (3.4) holds; hence $\hat{k} \in M_{\Gamma}(z)^{[\perp]}$.

**Step 2.** We show that $M_{\Gamma}(z)^* = (\Gamma_z^{[s]})^{-1} \hat{\mathcal{N}}_z(A^\ast)$. By using (2.3) and $M_{\Gamma}(z)^*$ obtained in the first step, $M_{\Gamma}(z)^*$ contains $\hat{h} \in \mathcal{H}^2$ such that $(\exists \hat{f} \in \hat{\mathcal{N}}_z(A^\ast)) (\exists \hat{g} \in \mathcal{O}_z) (\hat{h}, \hat{f} + \hat{g}) \in \Gamma_z^{[s]}$. By Lemma 3.2, on the other hand, $((0, 0), \hat{g}) \in \Gamma_z^{[s]}$, which shows that $(\hat{h}, \hat{f}) \in \Gamma_z^{[s]}$ by linearity of a subspace $\Gamma_z^{[s]}$.

**Step 3.** Here we derive the final formula of $M_{\Gamma}(z)^*$. Let

$$X_z := (\Gamma_z^{[s]})^{-1} |_{\hat{\mathcal{N}}_z(A^\ast)}, \quad Y_z := (\Gamma_z^{[s]})^{-1} |_{\hat{\mathcal{N}}_z(A^\ast)}$$
and consider \( \hat{k} \in \mathcal{H}^2 \) and \( \hat{g}_{\pi} = (g_{\pi}, \bar{z}g_{\pi}) \in \hat{\mathfrak{M}}_{\pi}(A^*) \) such that \((\hat{g}_{\pi}, \hat{k}) \in X_\pi \setminus Y_\pi\). Then it follows that \((\hat{k}, \hat{g}_{\pi}) \in \Gamma_\pi \setminus \Gamma_\pi^r\), and hence \((\hat{g}_{\pi}, \hat{k}) \notin \Gamma (\subseteq (\Gamma_\pi^r)^{-1})\). Since \(A_\star \subseteq A^*\) densely, there exists a sequence \((g_{\pi,n}) \subseteq \mathfrak{M}_{\pi}(A_\star)\) such that \(\delta_{\pi,n} := g_{\pi} - g_{\pi,n} \to 0\) in \(\mathfrak{S}\) as \(n \to \infty\). Putting \(\hat{g}_{\pi,n} = (g_{\pi,n}, \bar{z}g_{\pi,n})\) one has that \(\hat{g}_{\pi} = \hat{g}_{\pi,n} + \delta_{\pi,n, \bar{z}}\), where the sequence \((\hat{g}_{\pi,n}) \subseteq \hat{\mathfrak{M}}_{\pi}(A_\star)\). By the above it thus follows that \((\hat{g}_{\pi,n}, \hat{k}) \notin \Gamma\) for \(n\) sufficiently large. This shows that \((\hat{g}_{\pi,n}, \hat{k}) \notin \Gamma\) for \(n\) large, or equivalently \(\hat{k} \notin M_\Gamma(z) \subseteq M_\Gamma(z)^*\). On the other hand, since \((\hat{k}, \hat{g}_{\pi}) \in \Gamma_\pi\), it follows from the second step that \(\hat{k} \in M_\Gamma(z)^*\). Thus, using the decomposition \(X_\pi = Y_\pi \cup (X_\pi \setminus Y_\pi)\) one deduces that

\[
M_\Gamma(z)^* = M_\Gamma^*(z) \cup \Delta_\Gamma(z)
\]

where

\[
M_\Gamma^*(z) := (\Gamma_\pi^r)^{-1}\hat{\mathfrak{M}}_{\pi}(A^*), \quad \Delta_\Gamma(z) := M_\Gamma(z)^* \setminus M_\Gamma(z).
\]

It remains to point out that \(\Delta_\Gamma(z) \subseteq M_\Gamma^*(z)\).

By definition \(\Delta_\Gamma(z)\) contains \(\hat{k} \in \mathcal{H}^2\) such that \(\hat{k}[\perp]M_\Gamma(z)\) and \(\hat{k} \notin M_\Gamma(z)\). On the other hand, \(M_\Gamma^*(z)\) is the set of \(\hat{k} \in \mathcal{H}^2\) such that \((\hat{k}, \hat{g}_{\pi}) \in \Gamma_\pi \subseteq \Gamma_\pi^r\) for some \(\hat{g}_{\pi} \in \hat{\mathfrak{M}}_{\pi}(A^*)\); hence \(\hat{k}[\perp]M_\Gamma(z)\). Since also \(M_\Gamma^*(z) \supseteq M_\Gamma(z)\), one has \(\Delta_\Gamma(z) \subseteq M_\Gamma^*(z)\).

**Remark 3.5.** It follows from (3.1) that the intersection \(M_\Gamma(z) \cap M_\Gamma(z)^*\) is a subset of the set of neutral vectors [AI89, Definition 1.3] of a \(J_{\mathcal{H}}\)-space. Thus, by applying the Green identity (2.1) for \((\hat{f}_\pi, \hat{h}) \in \Gamma_\pi\) such that \([\hat{h}, \hat{h}]_{\mathcal{H}} = 0\), one concludes that \(M_\Gamma(z) \cap M_\Gamma(z)^* = \text{mul } \Gamma_\pi \cap \text{mul } \Gamma\). This result also follows from Theorem 3.3 by noting that

\[
\Gamma\hat{\mathfrak{M}}_\pi(A_\star) \cap (\Gamma_\pi^r)^{-1}\hat{\mathfrak{M}}_{\pi}(A^*) = \Gamma(\hat{\mathfrak{M}}_\pi(A_\star) \cap \hat{\mathfrak{M}}_{\pi}(A^*)),
\]

that \(\hat{\mathfrak{M}}_\pi(A_\star)\) is non-degenerate [AI89, Definition 1.14] for \(\exists z \neq 0\), and that \(\Gamma\{0,0\} = \text{mul } \Gamma\). One therefore has yet another proof of the relation \(M_\Gamma(z) \cap M_\Gamma(z)^* = \text{mul } \Gamma\) \((z \in \mathbb{C}_\star)\), which is stated without the proof in [DHM17, Lemma 3.6(i)], [DHMds12, Lemma 7.52(i)], and which is shown in [DHMds06, Lemma 4.1(i)] for a unitary pair \((\mathcal{H}, \Gamma)\). By using Theorem 3.3 one finds other invariance results for \(M_\Gamma(\cdot)\).

**Corollary 3.6.** Let \((\mathcal{H}, \Gamma)\) be an essentially unitary pair. Then \(M_\Gamma(z)^* = M_\Gamma(z)\) for \(z \in \mathbb{C}_\star\).

**Proof.** Since \(\Gamma^{-1} = \Gamma_\pi^r\), one has by Theorem 3.3 \(M_\Gamma(z)^* = \Gamma\hat{\mathfrak{M}}_\pi(A^*)\), that is, \(M_\Gamma(z)^*\) is the set of \(\hat{h} \in \mathcal{H}^2\) such that \((\exists \hat{f}_\pi \in \hat{\mathfrak{M}}_{\pi}(A^*)) (\hat{f}_\pi, \hat{h}) \in \Gamma\). But also, it must hold \(\hat{f}_\pi \in \text{dom } \Gamma =: \tilde{A}_\star\). Using that \(A^* \supseteq \tilde{A}_\star\) one concludes that \(\Gamma\hat{\mathfrak{M}}_{\pi}(A^*) = \Gamma\hat{\mathfrak{M}}_{\pi}(\tilde{A}_\star)\). □

If in addition \(\Gamma\) is unitary, Corollary 3.6 shows that \(M_\Gamma(z)^* = M_\Gamma(z)\) for \(z \in \mathbb{C}_\star\).
4. Nevanlinna families

The following definition of a Nevanlinna family is due to [DHMdS12, Definition 9.12], [BHdS08, Definition 2.1], [DHMdS06, Section 2.6].

Definition 4.1. A family $M(z), z \in \mathbb{C}_s$, of linear relations in $\mathcal{H}$ belongs to the class $\tilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families, or is said to be a Nevanlinna family, if:

(a) For $\Im z > 0/\Im z < 0$, the relation $M(z)$ is maximal dissipative/accumulative, and the operator family $(M(z) + w)^{-1} \in \mathcal{B}(\mathcal{H}), w \in \mathbb{C}_+ / \mathbb{C}_-$, is analytic;

(b) $M(z)^* = M(\overline{z})$.

Here $\mathbb{C}_+ / \mathbb{C}_-$ is the set of $z \in \mathbb{C}$ such that $\Im z > 0/\Im z < 0$; hence $\mathbb{C}_* = \mathbb{C}_+ \cup \mathbb{C}_-$. A linear relation $M(z)$ is dissipative (resp. accumulative) if $(\forall (h, h') \in M(z)) \Im \langle h, h' \rangle \geq 0$ (resp. $\leq 0$). We emphasize that the $\mathcal{H}$-scalar product is conjugate-linear in the first argument. A dissipative (resp. accumulative) $M(z)$ is maximal dissipative (resp. maximal accumulative) if $M(z)$ has no proper dissipative (resp. accumulative) extensions.

The Weyl family $M_\Gamma(z), z \in \mathbb{C}_s$, corresponding to an isometric pair $(\mathcal{H}, \Gamma)$ is dissipative/accumulative for $\Im z > 0/\Im z < 0$. Indeed, in view of (3.1), $\tilde{h} = (h, h') \in M_\Gamma(z)$ implies that $(\tilde{f}_z, \tilde{h}) \in \Gamma_z$ for some $\tilde{f}_z \in \tilde{\mathcal{N}}_z(A_s)$. Then, by the Green identity (2.1), $\Im \langle h, h' \rangle \mathcal{H} = (3z)\|f_z\|^2$, hence the claim. But then $(M_\Gamma(z) + w)^{-1}, w \in \mathbb{C}_+ / \mathbb{C}_-$, is an operator family by [DdS74, Theorem 3.1(i)].

If in addition $M_\Gamma(z)^* = M_\Gamma(\overline{z})$, then $M_\Gamma(\overline{z})^* = M_\Gamma(z)$, and therefore each member of the Weyl family is closed in this case: $M_\Gamma(z)^{**} = M_\Gamma(\overline{z})^* = M_\Gamma(z)$. But then the operator $(M_\Gamma(z) + w)^{-1}$ is bounded by [DdS74, Theorem 3.1(vi)], and the relation $M_\Gamma(z)$ is maximal dissipative/accumulative by [DdS74, Theorem 3.4(ii)].

It follows from the above that:

Lemma 4.2. The Weyl family $M_\Gamma(z), z \in \mathbb{C}_s$, is a Nevanlinna family iff $M_\Gamma(z)^* = M_\Gamma(\overline{z})$. \hfill \Box

By applying Corollary 3.6 and Lemma 4.2 one accomplishes the proof of Theorem 1.1.

Remark 4.3. Let us recall that the Weyl family of $A$ and its simple part coincide. Indeed, let $A_s$ be the simple part [LT77, Proposition 1.1] of $A$ and let $\Gamma_s$ be the restriction of $\Gamma$ to $\mathcal{H}_s$. Put $A_{ss} := \text{dom } \Gamma_s = A_s \cap \mathcal{H}_s$. Then $\tilde{\mathcal{N}}_z(A_{ss}) = \tilde{\mathcal{N}}_z(A_s) \cap \mathcal{H}_s = \tilde{\mathcal{N}}_z(A_s)$. Thus, since $\Gamma$ is isometric, $\Gamma_s \subseteq \Gamma$ is also isometric, and the corresponding Weyl family of $A_s$ is given by $M_{\Gamma_s}(z) = M_\Gamma(z), z \in \mathbb{C}_s$, by noting that $\Gamma_s \cap \Gamma_z = \Gamma_z$. In addition, given an isometric $\Gamma$, assume that $\Gamma_s$ is essentially unitary. Then $\Gamma$ is also essentially unitary, whose closure $\overline{\Gamma} = \overline{\Gamma_s}$. 
5. Example

Let $A$ be a densely defined, closed, symmetric operator in a Hilbert space $H$ with defect numbers $(d, d)$, for some $d \in \mathbb{N}$. Let $L$ be a self-adjoint extension of $A$ in $H$. Then by the von Neumann formula the adjoint $A^* \supseteq L$ is described by $\text{dom } A^* = \text{dom } L + \mathcal{N}_z(A^*)$, where the eigenspace $\mathcal{N}_z(A^*)$ is spanned by the deficiency elements $g_\sigma(z)$, $z \in \mathbb{C}_*$, with $\sigma$ ranging over an index set $S$ of cardinality $d$. That is, $\mathcal{N}_z(A^*) = g_z(\mathbb{C}^d)$, where one puts $g_z(c) := \sum_{\sigma \in S} c_\sigma g_\sigma(z)$ for $c = (c_\sigma) \in \mathbb{C}^d$.

Let $(\mathcal{H}_n)_{n \in \mathbb{Z}}$ be the scale of Hilbert spaces associated with $L$; hence $\mathcal{H}_2 = \text{dom } L$ and $\mathcal{H}_0 = \mathcal{H}$. Then a deficiency element $g_\sigma(z)$ can be defined in the generalized sense as $g_\sigma(z) = (L - z)^{-1} \varphi_\sigma$, for some functional $\varphi_\sigma \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$. Thus, $A$ is the symmetric restriction of $L$ to the domain of $u \in \mathcal{H}_2$ such that $\langle \varphi, u \rangle = 0$. Here one uses the vector notation $\langle \varphi, \cdot \rangle = (\langle \varphi_\sigma, \cdot \rangle)_{\mathcal{H}_0}$.

Finite rank perturbation $K$. Consider the set

$$\mathfrak{K} := \text{span}\{g_\alpha := g_\sigma(z_j) \mid \alpha = (\sigma, j) \in S \times J\}$$

where an index set $J := \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, and the points $z_j \in \mathbb{C}_*$ are such that $z_j \neq z_j'$ for $j \neq j'$. The system $\{g_\alpha\}$ is linearly independent, and so the Gram matrix

$$\mathcal{G} := (\langle g_\alpha, g_\alpha' \rangle_{\mathcal{H}}) \in B(\mathbb{C}^{md})$$

is Hermitian and positive definite.

Consider another set

$$\mathfrak{K}' := \mathcal{H}_{2m+2} + \mathcal{M}_z, \quad z \in \mathbb{C}_* \cap (\mathbb{C} \setminus \{z_j \mid j \in J\})$$

where the subset $\mathcal{M}_z \subseteq \mathcal{H}_{2m} \subseteq \mathcal{H}_2$ $(m \geq 1)$ is defined by

$$\mathcal{M}_z := \text{span}\{ \sum_{j \in J} b_j(z_j)(L - z_j)^{-1} g_\sigma(z) \mid \sigma \in S\}$$

where the multiplier

$$b_j(z_j) := \prod_{j' \in J \setminus \{j\}} (z_j - z_{j'})$$

for $m > 1$, and $b_1(z) := 1$ for $m = 1$ and $z \in \mathbb{C}$.

The rank-$md$ perturbation $K$ of $L$ is defined by

$$\text{dom } K := \mathfrak{K}' + \mathfrak{R}, \quad K(u + k) := Lu + \sum_{\alpha, \alpha' \in S \times J} C_{\alpha \alpha'} \langle g_\alpha', k \rangle_{\mathcal{H}} g_\alpha$$

for $u + k \in \mathfrak{K}' + \mathfrak{R}$, where $C_{\sigma j, \sigma' j'} := z_j [G^{-1}]_{\sigma j, \sigma' j'}$. 
Linear relation $\Gamma$. Let $\mathcal{H} := \mathbb{C}^d$ and define the single-valued linear relation $\Gamma \subseteq \mathfrak{H}^2 \times \mathcal{H}^2$ by

$$\Gamma := \{((u + k, K(u + k)), (c(k), \langle \varphi, u \rangle + \mathcal{M}d(k))) \mid u + k \in \mathfrak{H} + \mathfrak{H}\}.$$ 

Here the column-vectors

$$c(k) = (c_\sigma(k)) \in \mathbb{C}^d, \quad c_\sigma(k) := \sum_{j \in J} d_{\sigma j}(k),$$

$$d(k) = (d_\alpha(k)) \in \mathbb{C}^{md}, \quad d_\alpha(k) := \sum_{\alpha' \in S \times J} [G^{-1}]_{\alpha \alpha'} \langle g_{\alpha'}, k \rangle_{\mathfrak{H}}$$

for $k \in \mathfrak{H}$, and the matrix

$$\mathcal{M} = (\mathcal{M}_{\sigma \sigma'}) \in \mathcal{B}(\mathbb{C}^{md}, \mathbb{C}^{d}), \quad \mathcal{M}_{\sigma \sigma', j} := R_{\sigma \sigma'}(z_j')$$

for some matrix-valued Nevanlinna family $R(\cdot) = (R_{\sigma \sigma'}(\cdot))$ of class $\mathcal{R}^+ [\mathcal{H}]$. In fact, if the functional $\langle \varphi^{ex}, \cdot \rangle = (\langle \varphi^{ex}_{\sigma}, \cdot \rangle)$ extends $\langle \varphi, \cdot \rangle : \mathfrak{H} \to \mathbb{C}^d$ to dom $A^*$ according to (see also [AK00, Section 3.1.3], [HK09])

$$\langle \varphi^{ex}, u \rangle = \langle \varphi, u^\# \rangle + R(z)c, \quad u = u^\# + g_z(c), \quad u^\# \in \mathfrak{H}, \quad c \in \mathbb{C}^d$$

then the matrix $R(z) \in \mathcal{B}(\mathbb{C}^d)$ is defined by

$$R_{\sigma \sigma'}(z) := \langle \varphi^{ex}_{\sigma'}(g_{\sigma'}(z))$$

for $z \in \mathbb{C}_+$. From here one verifies that $\ker \Re R(z) = \{0\}$ indeed: The imaginary part $\Re R(z)$ is the matrix with entries $(\Re z) \langle g_\sigma(z), g_{\sigma'}(z)\rangle_{\mathfrak{H}}$. Then $\ker \Re R(z)$ is the set of $c \in \mathbb{C}^d$ such that $g_z(c) \in g_z(\mathbb{C}^d)^\perp$; hence $c = 0$.

The relation $\Gamma$ defines the boundary space of operator $K$. Indeed, associate with $\Gamma$ the following two single-valued linear relations

$$\Gamma_0 := \{ (\tilde{f}, h) \mid (\exists h' \in \mathcal{H}) (\tilde{f}, h') \in \Gamma; \tilde{h} = (h, h') \},$$

$$\Gamma_1 := \{ (\tilde{f}, h') \mid (\exists h \in \mathcal{H}) (\tilde{f}, h) \in \Gamma; \tilde{h} = (h, h') \}.$$ 

Then the boundary form of the operator $K$ is given by

$$\langle u, Kv \rangle_{\mathfrak{H}} - \langle Ku, v \rangle_{\mathfrak{H}} = (\Gamma_0 u, \Gamma_1 v)_{\mathbb{C}^d} - (\Gamma_1 u, \Gamma_0 v)_{\mathbb{C}^d}$$

for $u, v \in \text{dom} K$, provided that $\Gamma_0$ (resp. $\Gamma_1$) is regarded as the mapping dom $K \to \mathbb{C}^d$.

Linear relation $\Gamma_z$ and its Krein space adjoint. Let $A_\pi := \text{dom} \Gamma = K$. It is shown in [Jur18] that the adjoint $K^* = A$. One also verifies that the eigenspace $\mathcal{N}_z(A_\pi) = \mathcal{N}_z(A^*)$ for $z \in \mathbb{C}_+$. Then, the single-valued linear relation $\Gamma_z$ and its Krein space adjoint $\Gamma_z^{[*]}$ are given by

$$\Gamma_z = \{ (g_z(c), zg_z(c)), (c, R(z)c) \mid c \in \mathbb{C}^d \},$$

$$\Gamma_z^{[*]} = \{ ((c, \langle \varphi, u \rangle + R(z)c), (g, zg + (L - z)u)) \mid c \in \mathbb{C}^d; u \in \mathfrak{H}_2; g \in \mathfrak{H} \}.$$ 

It follows that $\Gamma_z \subseteq \Gamma$ and $\Gamma_w^{-1} \subseteq \Gamma_z^{[*]}$ for all $z, w \in \mathbb{C}_+$. 

Closure $\Gamma$. By [Jur18], the Krein space adjoint of $\Gamma$ is given by $\Gamma^\dagger = \Gamma^{-1}$, where the closure $\Gamma = \{(u + g_z(c), A^*(u + g_z(c))), (c, \langle \varphi, u \rangle + R(z)c) \mid c \in \mathbb{C}^d; u \in \mathfrak{H}_2\}$ with $z \in \mathbb{C}_*$. It follows that $\Gamma^{-1} \subseteq \Gamma^\dagger \subseteq \Gamma_{w}^\dagger$ for all $w \in \mathbb{C}_*$.

Since $\Gamma$ is essentially unitary and $A^* = A^*$, the pair $(\mathcal{H}, \Gamma)$ is an essentially unitary pair for $A^*$; the pair $(\mathcal{H}, \Gamma)$ is a unitary pair for $A^*$, with the associated Weyl family $M_{\Gamma}(z) = R(z)$, $z \in \mathbb{C}_*$. Note that $A = \tilde{A} = \ker \Gamma$. Note also that, since $\Gamma$ and $\Gamma$ are single-valued, the pair $(\mathcal{H}, \Gamma)$ is actually an isometric boundary triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for $A^*$ [DHM17, Definition 1.8], and the pair $(\mathcal{H}, \Gamma)$ is an ordinary boundary triple $(\mathcal{H}, (\Gamma)_0, (\Gamma)_1)$ for $A^*$; here $(\Gamma)_0$ (resp. $(\Gamma)_1$) is defined similar to $\Gamma_0$ (resp. $\Gamma_1$).

The domain
$$\text{dom } M_{\Gamma}(z) = \Gamma_0 \hat{\mathfrak{N}}_z(A_*) = \mathbb{C}^d$$
and so it follows from Corollary 1.2 that
$$M_{\Gamma}(z) = M_{\Gamma}(z) = R(z), \quad z \in \mathbb{C}_*$$
is a Nevanlinna family of class $\mathcal{R}[\mathcal{H}]$; the equality $M_{\Gamma}(z) = R(z)$ can be checked by computing $\Gamma \hat{\mathfrak{N}}_z(A_*)$ directly. Moreover, since $\text{ran } \Gamma = \mathcal{H}^2$, one concludes that actually $R(\cdot) \in \mathcal{R}[\mathcal{H}]$. Finally, the matrix-valued analytic function $R(\cdot)$ on $\mathbb{C}_*$ extends to the domain of analyticity of $(g_{\sigma}(\cdot))$, namely, the resolvent set of $L$.

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