UNIVERSAL ALGEBRAIC EQUIVALENCES BETWEEN TAUTOLOGICAL CYCLES ON JACOBIANS OF CURVES

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Abstract. We present a collection of algebraic equivalences between tautological cycles on the Jacobian $J$ of a curve, i.e., cycles in the subring of the Chow ring of $J$ generated by the classes of certain standard subvarieties of $J$. These equivalences are universal in the sense that they hold for all curves of given genus. We show also that they are compatible with the action of the Fourier transform on tautological cycles and compute this action explicitly.

Introduction

Let $J$ be the Jacobian of a smooth projective complex curve $C$ of genus $g \geq 2$. For every $d$, $0 \leq d \leq g$, consider the morphism $\sigma_d : \text{Sym}^d C \to J : D \mapsto O_C(D - dp)$, where $p$ is a fixed point on $C$ (for $d = 0$ this is an embedding of the neutral point into $J$). It is well-known that $\sigma_d$ is birational onto its image. Let us denote by $w_d = [\sigma_{g-d}(\text{Sym}^{g-d})] \in \text{CH}^d(J)$, where $0 \leq d \leq g$, the corresponding classes in the Chow ring $\text{CH}(J)$ of $J$. Following Beauville who studied the subring in $\text{CH}(J)$ generated by these classes (see [3]) we call an element of this subring a tautological cycle on $J$. Note that $w_0 = 1$, while $w_1$ is the class of the theta divisor on $J$. The Poincaré formula states that $w_d$ is homologically equivalent to $\frac{w_d}{d!}$. However, it is known that in general this formula fails to hold modulo algebraic equivalence (although it does hold for hyperelliptic curves, see [5]).

More precisely, Ceresa in [4] has shown that if $C$ is generic of genus $g \geq 3$ then $w_d$ is not algebraically equivalent to $[-1]^*w_d$ for $1 < d \leq g - 1$, where $[-1]^*$ is the involution of the Chow ring induced by the inversion morphism $[-1] : J \to J$. This raises the problem of finding universal polynomial relations between the classes $w_d$ that hold modulo algebraic equivalence. In this paper we derive a number of such relations. We do not know whether our set of relations is complete for a generic curve. However, we show that these relations are in some sense consistent with the action of the Fourier transform on $\text{CH}(J)$.

Let us set $p_k = -N^k(w) \in \text{CH}^k(J)$ for $k \geq 1$, where $N^k(w)$ are the Newton polynomials on the classes $w_1, \ldots, w_g$:

$$N^k(w) = \frac{1}{k!} \sum_{i=1}^{g} \lambda_i^k,$$

where $\lambda_1, \ldots, \lambda_g$ are roots of the equation $\lambda^g - w_1\lambda^{g-1} + \ldots + (-1)^g w_g = 0$. For example, $p_1 = -w_1$, $p_2 = w_2 - w_1^2/2$, $p_3 = w_2w_1/2 + w_3/2 - w_1^3/6$. From the Poincaré formula it is easy to see that the classes $p_n$ for $n > 1$ are homologically trivial. Let us denote by $\text{CH}(J)_\mathbb{Q}/(\text{alg})$ the quotient of the Chow ring of $J$ with rational coefficients modulo the ideal of classes algebraically equivalent to 0. It is also convenient to have a notation

\footnotetext{Supported in part by NSF grant.}
for the divided powers of \( p_i \): \( p_i^{[d]} := p_i^d/d! \). Our main result is the following collection of relations in \( \text{CH}(J)_\mathbb{Q}/(\text{alg}) \).

**Theorem 0.1.** (i) Let us define the differential operator \( D \) acting on polynomials in infinitely many variables \( x_1, x_2, \ldots \):

\[
D = -g \partial_1 + \frac{1}{2} \sum_{m,n \geq 1} \binom{m+n}{m} x_{m+n-1} \partial_m \partial_n,
\]

where \( \partial_i = d/dx_i \). Then for every polynomial \( F \) of the form

\[
F(x_1, x_2, \ldots) = D^d(x_1^{m_1} \cdots x_k^{m_k}),
\]

where \( m_1 + 2m_2 + \ldots + km_k = g, m_1 < g \) and \( d \geq 0 \), one has

\[
F(p_1, p_2, \ldots) = 0.
\]

in \( \text{CH}^{g-d}(J)_\mathbb{Q}/(\text{alg}) \).

(ii) Here is another description of the same collection of relations. For every \( k \geq 1 \), every \( n_1, \ldots, n_k \) such that \( n_i > 1 \), and every \( d \) such that \( 0 \leq d \leq k - 1 \), one has

\[
\sum_{[1,k]=I_1 \sqcup I_2 \sqcup \ldots \sqcup I_m} \binom{m-1}{d+m-k} b(I_1) \cdots b(I_m)p_1^{[g-d-m+k-\sum_{i=1}^k n_i]}p_{d(I_1)} \cdots p_{d(I_m)} = 0
\]

(0.1)

in \( \text{CH}^{g-d}(J)_\mathbb{Q}/(\text{alg}) \), where the summation is over all partitions of the set \( [1,k] = \{1, \ldots, k\} \) into the disjoint union of nonempty subsets \( I_1, \ldots, I_m \) such that \( -d+k \leq m \leq g-d+k-\sum_{i=1}^k n_i \) (two partitions differing only by the ordering of the parts are considered to be the same); for a subset \( I = \{i_1, \ldots, i_s\} \subset [1,k] \) we denote

\[
b(I) = \frac{(n_{i_1} + \ldots + n_{i_s})!}{n_{i_1}! \cdots n_{i_s}!},
\]

\[
d(I) = n_{i_1} + \ldots + n_{i_s} - s + 1.
\]

Let us point out some corollaries of these relations.

**Corollary 0.2.** The class \( p_n \) is algebraically equivalent to 0 for \( n \geq g/2 + 1 \).

In [6] Colombo and Van Geemen proved that \( p_n \) is algebraically equivalent to 0 for \( n \geq d \), where \( d \) is the minimal degree of a nonconstant morphism from \( C \) to \( \mathbb{P}^1 \) (see [3], Proposition 4.1). For generic curve this is equivalent to the statement of the above corollary.

**Corollary 0.3.** Every tautological cycle in \( \text{CH}^{g-d}(J)_\mathbb{Q} \) is algebraically equivalent to a linear combination of the classes \( p_1^{[g-d-\sum_{i=1}^k n_i]}p_n \cdots p_k \), where \( 0 \leq k \leq d \) and \( n_i > 1 \) for all \( i = 1, \ldots, k \). More precisely, for arbitrary \( n_1, \ldots, n_k \) such that \( n_i > 1 \) and for \( d \) such that \( 0 \leq d \leq k - 1 \) one has

\[
p_1^{[g-d-\sum_{i=1}^k n_i]}p_n \cdots p_k = \sum_{j=1}^d (-1)^{k+d-1-j} \sum_{[1,k]=I_1 \sqcup I_2 \sqcup \ldots \sqcup I_j} b(I_1) \cdots b(I_j)p_1^{[g-d-j+\sum_{i=1}^k n_i]}p_{d(I_1)} \cdots p_{d(I_j)}
\]

(0.2)

in \( \text{CH}^{d-1}(J)_\mathbb{Q}/(\text{alg}) \).
For example, in the case \( d = 1 \) the above corollary states that for \( n_1, \ldots, n_k \) such that \( n_i > 1 \) and \( \sum_{i=1}^{k} n_i \leq g - 1 \) one has

\[
(-1)^n S(p_1^{[n]} p_{n_1} \cdots p_{n_k}) = \sum_{[1,k]=I_1 \sqcup I_2 \sqcup \ldots \sqcup I_m} b(I_1) \cdots b(I_m) p_1^{[g-n-m-\sum_{i=1}^{k} n_i]} p_{d(I_1)} \cdots p_{d(I_m)}
\]

in \( \text{CH}^{g-1}(J)_{Q/\text{alg}} \), where the summation is similar to the one in equation (0.1).

Note that the expression in the identity (0.1) is equal to the RHS of equation (0.4) in the case \( n = -1, d = k - 1 \). More interesting observation is the formal consistency between Theorems 0.1 and 0.4 proved in part (iii) of the next theorem.

**Theorem 0.5.** For a fixed \( g \geq 2 \) let us denote by \( R^\text{Jac}_g \) the quotient-space of \( Q[x_1, x_2, \ldots] \) by the linear span of all the polynomials appearing in Theorem 0.1 together with all polynomials of degree \( > g \) where \( \deg(x_i) = i \). In other words,

\[
R^\text{Jac}_g = Q[x_1, x_2, \ldots]/I_g
\]

where the subspace \( I_g \) is spanned by all polynomials \( F \) such that \( \deg F > g \) and by polynomials of the form \( D^d(x_i^{m_1} \cdots x_k^{m_k}) \), where \( d \geq 0, m_1 + 2m_2 + \ldots + km_k = g, m_i < g \). Let us denote by \( p_i \) the image of \( x_i \) in \( R^\text{Jac}_g \). Then

(i) \( I_g = \cap_{n \geq 1} \text{im}(D^n) \), where \( \text{im}(D^n) \) denotes the image of the operator \( D^n \) acting on \( Q[x_1, x_2, \ldots] \);

(ii) \( I_g \) is an ideal in \( Q[x_1, \ldots, x_g] \), so \( R^\text{Jac}_g \) has a commutative ring structure;

(iii) the formula (0.4) gives a well-defined operator \( S \) on \( R^\text{Jac}_g \) such that

\[
S^2(p_1^n p_{n_1} \cdots p_{n_k}) = (-1)^{k+\sum_{i=1}^{k} n_i} p_1^n p_{n_1} \cdots p_{n_k},
\]

where \( n_i > 1 \);

(iv) the operators

\[
e(F) = x_1 \cdot F, \ f = -D, \ h = -g + \sum_{n \geq 1} (n + 1)x_n \partial_n
\]

on \( Q[x_1, x_2, \ldots] \) define a representation of the Lie algebra \( \mathfrak{sl}_2 \). Furthermore, they preserve the ideal \( I_g \) and therefore define a representation of \( \mathfrak{sl}_2 \) on \( R^\text{Jac}_g \). The operators \( S, e, f \) and \( h \) on \( R^\text{Jac}_g \) satisfy the standard compatibilities:

\[
SeS^{-1} = -f, \ SfS^{-1} = -e, \ ShS^{-1} = -h.
\]
Thus, the ring $R_{g}^{\text{Jac}}$ can be equipped with the same special structures as $\text{CH}(J)_{Q}/(\text{alg})$ (namely, the Fourier transform, the $sl_{2}$-action and the Pontryagin product) and the natural homomorphism $R_{g}^{\text{Jac}} \rightarrow \text{CH}(J)_{Q}/(\text{alg})$ respects these structures.

The next result points to possible connections between the structures on $R_{g}^{\text{Jac}}$ and those on the cohomology of Hilbert schemes $(\mathbb{A}^{2})^{[n]}$ of points on the plane. For example, the differential operators $D_{k}$ appearing below are very similar to the operators describing the action of the Chern character of the tautological bundle on $H^{*}((\mathbb{A}^{2})^{[n]},Q)$ (see [8],[9]).

**Theorem 0.6.** For every $k \geq 2$ let us consider the following differential operator on $Q[x_{1}, x_{2}, \ldots]$: 

$$D_{k} = \frac{1}{k!} \sum_{n_{1}, \ldots, n_{k} \geq 1} \frac{(n_{1} + \ldots + n_{k})!}{n_{1}! \ldots n_{k}!} x_{n_{1} + \ldots + n_{k} - 1} \partial_{n_{1}} \ldots \partial_{n_{k}}.$$ 

Note that the operator $D$ considered above is $D = D_{2} - g\partial_{1}$. Then for every $k \geq 3$ we have $[D_{k}, D] = 0$, so $D_{k}$ descends to an operator on $R_{g}^{\text{Jac}}$. Furthermore, for every $k \geq 2$ one has 

$$S(p_{k} \cdot S^{-1} F) = D_{k+1}(F), \quad (0.6)$$ 

where $F \in R_{g}^{\text{Jac}}$.

**Remarks.** 1. Applying the homomorphism $R_{g}^{\text{Jac}} \rightarrow \text{CH}(J)_{Q}/(\text{alg})$ we derive that the relation (0.6) also holds in $\text{CH}(J)_{Q}/(\text{alg})$ whenever $F$ is a tautological cycle. In fact, this relation can also be proved geometrically using similar arguments as for the case $k = 1$, which corresponds to equation (2.2).

2. The Lie algebra generated by operators $D_{k}$ for $k \geq 2$ and by the operators of multiplication by $x_{k}$ can be easily described: it is isomorphic to the Lie algebra of (polynomial) hamiltonian vector fields on the plane vanishing at the origin (where the plane is equipped with the standard symplectic form). Elsewhere we will show that the action of this Lie algebra on $R_{g}^{\text{Jac}}$ extends to an action on the entire Chow ring (not just tautological cycles).

We should point out that at present it is not known whether there exists a curve for which one of the classes $p_{n}$ with $n \geq 3$ is actually nonzero. Ceresa’s theorem in [4] can be interpreted as nonvanishing of $p_{2}$ for generic curve of genus $g \geq 3$. Using Theorem 0.4 we find that $S(p_{2}) = p_{1}^{g-3}p_{2}$. This implies that for generic curve of genus $g \geq 3$ one has $p_{1}^{g-3}p_{2} \neq 0$. We conjecture that for a generic curve the map 

$$R_{g}^{\text{Jac}} \rightarrow \text{CH}(J)_{Q}/(\text{alg})$$ 

is injective, i.e., the set of relations of Theorem 0.1 is complete.

**Acknowledgment.** I am grateful to Arkady Vaintrob for valuable discussions and to Alexander Postnikov for providing the references on Hurwitz identity (equation (2.6)).

**Notation.** We use the notation \(\binom{x}{n} = x(x-1) \ldots (x-n+1)/n!\), where $n \geq 0$ and $x$ is either a number or an operator. Thus, we have $\binom{a}{0} = 1$ while for $n > 0$ and for an integer $m$ we have $\binom{m}{n} = 0$ if $m \in \{0, 1, \ldots, n-1\}$. If $n < 0$ then we set $\binom{x}{n} = 0$. 

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1. Fourier transform of cycles: brief reminder

Let $X$ be an abelian variety of dimension $g$, $\hat{X}$ the dual abelian variety, $\mathcal{P}$ the Poincaré line bundle on $X \times \hat{X}$. Recall that the Fourier-Mukai transform is an equivalence $S : D^b(X) \to D^b(\hat{X})$ sending a complex of coherent sheaves $F$ to $Rp_2*(p_1^*F \otimes \mathcal{P})$, where $p_1$ and $p_2$ are projections of the product $X \times \hat{X}$ to its factors (see [10]). It induces the transform on Grothendieck groups and therefore on Chow groups with rational coefficients. Explicitly, the Fourier transform of Chow groups $S : CH(X)_{\mathbb{Q}} \to CH(\hat{X})_{\mathbb{Q}}$ is given by the formula $S(\alpha) = p_2*(p_1^*\alpha \cdot ch(\mathcal{P}))$, where $ch$ denotes the Chern character.

This operator was introduced and studied by Beauville in [1], [2]. It is clear that it preserves the ideal of cycles algebraically (resp. homologically) equivalent to $0$. In particular, it induces a well-defined map $S : CH(X)_{\mathbb{Q}}/(alg) \to CH(\hat{X})_{\mathbb{Q}}/(alg)$.

For every $n \in \mathbb{Z}$ let us denote by $[n]_*, [n]^* : CH(X) \to CH(X)$ the operators on the Chow group given by the push-forward and the pull-back with respect to the endomorphism $[n] : X \to X$. The main properties of the Fourier transform that we will use are:

$$S^2 = (-1)^g[-1]^*,$$
$$S(\alpha * \beta) = S(\alpha) \cdot S(\beta),$$
$$S \circ [n]_* = [n]^* \circ S,$$  \hspace{1cm} (1.1)

where $\alpha * \beta$ denotes the Pontryagin product of $\alpha$ and $\beta$ (the push-forward of $\alpha \times \beta$ be the group law morphism $X \times X \to X$).

Beauville proved in [2] that there is a direct sum decomposition of the Chow groups

$$CH^d(X)_{\mathbb{Q}} = \oplus CH^d_s(X),$$

where $CH^d_s(X) = \{ x \in CH^d(X)_{\mathbb{Q}} : [n]^*x = n^{2d-s}x \text{ for all } n \in \mathbb{Z} \}$. This decomposition is compatible with various operations on the Chow groups in the following way:

$$CH^p_s \cdot CH^q_t \subset CH^{p+q}_{s+t},$$
$$CH^p_s \cdot CH^q_t \subset CH^{p+q-g}_{s+t-g},$$
$$S(CH^p_s) = CH^{q-p+s}_s.$$  \hspace{1cm} (1.2)

Remark. If $X$ is principally polarized then identifying $X$ with $\hat{X}$ we can view the Fourier transform $S$ as an automorphism of $CH(X)_{\mathbb{Q}}$. Let $c_1(L) \in CH^1(X)$ be the class of the principal polarization. Then there is a natural action of the Lie algebra $\mathfrak{sl}_2$ on $CH(X)_{\mathbb{Q}}$ such that $e(x) = c_1(L)x$, $f(x) = -S(c_1(L)S^{-1}x)$, $h(x) = (2p - g - s)x$ for $x \in CH^p_s$. Indeed, this can be deduced from the explicit formulae for the algebraic action of the group $SL_2$ on $CH(X)_{\mathbb{Q}}$ considered in [11]. The corresponding action of $\mathfrak{sl}_2$ on tautological cycles in the Jacobian will play a crucial role below.
2. Computations in the Chow ring

2.1. Let $C$ be a (connected) smooth projective curve of genus $g \geq 2$, $J$ be its Jacobian. We can identify the $J$ with its dual $\hat{J}$ and consider the Fourier transform $S$ (resp., $\hat{S}$) as an autoequivalence of $D^b(J)$ (resp., as an automorphism of $\text{CH}(J)$). More precisely, to define this transform we use the line bundle $L = \mathcal{O}_{J \times J}(p_1^{-1}(\Theta) + p_2^{-1}(\Theta) - m^{-1}(\Theta))$ as a kernel on $J \times J$, where $\Theta \subset J$ is a theta divisor, $p_i$ are projections of $J \times J$ to $J$, $m : J \times J \to J$ is the group law. Hence, $S$ induces a $\mathbb{Q}$-linear operator on $\text{CH}(J)_{\mathbb{Q}}/(\text{alg})$. The assertion of the following important lemma is proved in Proposition 2.3 and Corollary 2.4 of [3].

Lemma 2.1. 

\[ S(w_{g-1}) = \sum_{n=1}^{g-1} p_n \]

in $\text{CH}(J)_{\mathbb{Q}}/(\text{alg})$. Moreover, for every $n$ one has $p_n \in \text{CH}^n_{n-1}/(\text{alg})$.

For every collection of integers $n_1, \ldots, n_k$ we set

\[ w(n_1, \ldots, n_k) := [n_1]_* w_{g-1} \ast \ldots \ast [n_k]_* w_{g-1}. \]

The following lemma is essentially contained in section (3.3) of [3].

Lemma 2.2. One has

\[ w_1 \cdot w(n_1, \ldots, n_k) = \sum_{i=1}^{k} [gn_i^2 + n_i \sum_{j \neq i} n_j]w(n_1, \ldots, \hat{n}_i, \ldots, n_k) - \sum_{i<j} n_i n_j w(n_i + n_j, n_1, \ldots, \hat{n}_i, \ldots, \hat{n}_j, \ldots, n_k) \]

where a hat over a symbol means that it should be omitted.

Proof. By definition $w(n_1, \ldots, n_k)$ is the push-forward of the fundamental class under the composite map

\[ u : C^k \xrightarrow{(\sigma)^k} J^k \xrightarrow{n} J^k \xrightarrow{m} J, \]

where $\sigma : C \to J$ is the standard embedding, $n = [n_1] \times \ldots \times [n_k] : J^k \to J^k$, $m$ is the addition morphism. Therefore,

\[ w_1 \cdot w(n_1, \ldots, n_k) = u_* u^* w_1. \]

Using theorem of the cube one can easily show (see section (3.3) of [3]) that

\[ u^* w_1 = \sum_i n_i^2 q_i^* \sigma^* w_1 - \sum_{i<j} n_i n_j (\delta_{ij} - q_i^*[p] - q_j^*[p]) \]

modulo algebraic equivalence, where $p \in C$ is a point, $q_i$ are projections of $C^k$ to $C$, $\delta_{ij} \subset C^k$ is the class of the partial diagonal divisor given by the equation $x_i = x_j$. Since $\sigma^* w_1$ has degree $g$, it is algebraically equivalent to $g[p]$. Hence, we have

\[ u^* w_1 = \sum_i (gn_i^2 + n_i \sum_{j \neq i} n_j) q_i^*[p] - \sum_{i<j} n_i n_j \delta_{ij} \]
modulo algebraic equivalence, which immediately implies the assertion.

2.2. The idea of the proof of Theorem 0.4 is to use the following identity proved in [3], (1.7):
\[ S[-1]^*x = \exp(-p_1) \cdot (\exp(p_1) \ast [\exp(-p_1) \cdot x]). \] (2.1)
Thus, if we want to find explicitly the action of the Fourier transform on polynomials in \( p_i \), it suffices to find the formula for the action of the operator \( x \mapsto \exp(p_1) \ast x \) on such polynomials. The only geometric ingredients needed for this are Lemmata 2.1 and 2.2, the rest of the computation consists of formal manipulations. Along the way we will derive the relations of Theorem 0.1.

**Lemma 2.3.** One has
\[ S(w(n_1, \ldots, n_k)) = n_1 \ldots n_k P(n_1) \cdot \ldots \cdot P(n_k), \]
where \( P(t) = \sum_{i=1}^{g-1} p_i t^i \).

**Proof.** This follows immediately from Lemma 2.1 combined with (1.1).

Let us define the operator \( U \) on the space \( CH(J)_{\mathbb{Q}/(\text{alg})} \) by the formula
\[ U(x) = S(p_1 \cdot S^{-1}(x)). \]
Since \( p_1 \) is invariant with respect to \( [-1]^* \) we also have
\[ U(x) = S^{-1}(p_1 \cdot S(x)). \]
Let us denote \( U^{[n]} = U^n/n! \) for \( n \geq 0 \). The plan of the computations below is the following. First, we will compute the action of \( U \) on a monomial in \( p_i \)'s. Next, we will compute the action of the operators \( U^{[n]} \) on such a monomial for all \( n \geq 0 \). As a result, we will get an explicit formula for the operator \( x \mapsto \exp(p_1) \ast x \) on the tautological ring. Finally, using equation (2.1) we will find the formula for the Fourier transform.

**Lemma 2.4.** For every \( n \geq 0 \) one has
\[ U^{[n]}(x) = (-1)^{g-n} p_1^{[g-n]} \ast x. \]

**Proof.** It is well known that \( S(\exp(-p_1)) = \exp(p_1) \). Therefore, \( (-1)^{g-n} S(p_1^{[g-n]}) = p_1^{[n]} \).

Hence,
\[ (-1)^{g-n} S(p_1^{[g-n]} \ast x) = p_1^{[n]} \cdot S(x) = SU^{[n]}(x). \]

**Lemma 2.5.** One has the following identity of polynomials in \( t_1, \ldots, t_k \):
\[ U(P(t_1) \ldots P(t_k)) = - \sum_{i=1}^{k} [gt_i + \sum_{j \neq i} t_j] P(t_1) \ldots \widehat{P(t_i)} \ldots P(t_k) + \]
\[ \sum_{i < j} (t_i + t_j) P(t_i + t_j) P(t_1) \ldots \widehat{P(t_i)} \ldots, \widehat{P(t_j)} \ldots P(t_k), \]
where hats denote omitted symbols.
Proof. Since both sides of the identity are polynomials in $t_1, \ldots, t_k$ it suffices to prove it for $t_i = n_i \in \mathbb{Z}$. Using Lemma 2.3 we get

$$U(P(n_1) \ldots P(n_k)) = S(p_1 \cdot \frac{w(n_1, \ldots, n_k)}{n_1 \ldots n_k}).$$

It remains to apply the formula of Lemma 2.2 to compute the expression under Fourier transform and then use Lemma 2.3 again. □

Thus, we arrive to the following formula for the action of the operator $U$ on monomials in classes $p_1, \ldots, p_{g-1}$.

**Proposition 2.6.** For $n \geq 0$ and $n_1, \ldots, n_k$ such that $n_i > 1$ one has

$$U(p_1^{[n]} \ldots p_{nk}) = (-g + n - 1 + k + \sum_{i=1}^{k} n_i) p_1^{[n-1]} p_{n_1} \ldots p_{n_k} + \sum_{i<j} \binom{n_i + n_j}{n_i} p_1^{[n_i+n_j-1]} p_{n_i} \ldots p_{n_j} \ldots p_{n_k}.$$

Proof. The expression in the LHS is equal to $1/n!$ times the coefficient with $t_1^{n_1} \ldots t_k^{n_k}$ in $U(P(t_1) \ldots P(t_k) P(t)^n)$. Now the assertion is an easy consequence of Lemma 2.5. □

We can immediately recognize in the formula of the above proposition the action of the differential operator $D$ appearing in Theorem 0.1, so that for every polynomial $F(x_1, x_2, \ldots)$ we have

$$U(F(p_1, p_2, \ldots)) = DF(p_1, p_2, \ldots). \quad (2.2)$$

It is convenient to separate in $D$ the part that does not contain $\partial_1$. Namely, let us set

$$\Delta = \frac{1}{2} \sum_{m,n \geq 2} \binom{m+n}{m} x_{m+n-1} \partial_m \partial_n.$$

Then we have

$$D = \partial_1 H + \Delta \quad (2.3)$$

where

$$H = -g - 1 + x_1 \partial_1 + \sum_{n \geq 2} (n+1) x_n \partial_n,$$

so that

$$H(x_1^{n_1} x_{n_1} \ldots x_{n_k}) = (-g + n - 1 + k + \sum_{i=1}^{k} n_i) x_1^{n_1} x_{n_1} \ldots x_{n_k},$$

where $n_i > 1$ for $i = 1, \ldots, k$.

Let us set $\Delta^{[n]} = \Delta^n/n!$ for $n \geq 0$.

**Lemma 2.7.** For every polynomial $F(x_1, x_2, \ldots)$ and every $n \geq 0$ one has

$$U^{[n]}(F(p_1, p_2, \ldots)) = \sum_{i=0}^{n} \partial_1^{n-i} \Delta^{[i]} \binom{H-i}{n-i} F(p_1, p_2, \ldots).$$

in CH(J)$_Q/(\text{alg})$. 

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Proof. This can be easily deduced from (2.2) and (2.3) using the following commutation relations for the operators $\Delta$, $\partial_1$ and $H$:

$$[\partial_1, \Delta] = 0, \ [H, \partial_1] = -\partial_1, \ [H, \Delta] = -2\Delta.$$  

The powers of the operator $\Delta$ are computed in the following lemma.

**Lemma 2.8.** For $m \geq 0$, $n \geq 0$ and $n_1, \ldots, n_k$ such that $n_i > 1$ one has

$$\Delta^m[(x_1^n x_{n_1} \ldots x_{n_k})] = \sum_{[1,k]=I\sqcup J} b(I)b(J)(d(I) + d(J)) = 2(k - 1)b([1,k])$$

where we use the notation of Theorem 0.1, the summation is over all unordered partitions of the set $[1,k] = \{1, \ldots, k\}$ into the disjoint union of $k - m$ nonempty subsets.

Proof. For $m = 0$ the assertion is clear, so we can use induction in $m$. Using the definition of $\Delta$ we can immediately reduce the induction step to the following identity:

$$\sum_{[1,k]=I\sqcup J} b(I)b(J)(d(I) + d(J)) = 2(k - 1)b([1,k])$$

of polynomials in $n_1, \ldots, n_k$, where in the LHS we consider partitions of $[1,k]$ into the *ordered* disjoint union of nonempty subsets $I$ and $J$. Equivalently, we have to check that

$$\sum_{[1,k]=I\sqcup J} P_{[I]-1}(\sum_{i\in I} u_i)P_{[J]-1}(\sum_{j\in J} u_j) = 2(k - 1)P_{k-2}(\sum_{i=1}^k u_i)$$

(2.4)

where $u_1, \ldots, u_k$ are formal commuting variables; the sum is of the same kind as before; $P_n(u) := u(u - 1) \ldots (u - n + 1)$ for $n \geq 0$ ($P_0 = 1$). Note that the formal power series $P_t(u) = \sum_{n \geq 0} P_n(u)t^n/n! = (1 + t)^u$ satisfies $P_t(u + v) = P_t(u)P_t(v)$. Hence,

$$P_n(u + v) = \sum_{i=0}^n \binom{n}{i} P_i(u)P_{n-i}(v).$$

(2.5)

Using this property we can express both parts of the identity (2.4) as linear combinations of the products of the form $P_{i_1}(u_1) \ldots P_{i_k}(u_k)$, where $i_1 + \ldots + i_k = k - 2$. Thus, (2.4) is equivalent to a sequence of identities obtained by equating the coefficients with each such product. Note that the polynomials $\tilde{P}_n(u) = u^n$ also satisfy (2.5). Hence, (2.4) is equivalent to a similar identity with $P_n(u)$ replaced by $u^n$:

$$\sum_{[1,k]=I\sqcup J} (\sum_{i\in I} u_i)^{|I|-1}(\sum_{j\in J} u_j)^{|J|-1} = 2(k - 1)(\sum_{i=1}^k u_i)^{k-2}.$$  

(2.6)

This is the so called Hurwitz identity proved in [7] (see also Exer. 5.31b of [13] for a more recent treatment).

Lemmata 2.7 and 2.8 give an explicit formula for the action of all the operators $U^{[m]}$ on monomials in $p_i$’s. Using this formula we can easily prove our first main theorem.
2.3. Proof of Theorem 0.1. (i) We start with the obvious vanishing
\[ p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k} = 0 \]
in \( \text{CH}(J)_{Q/\text{alg}} \), where \( m_1 + 2m_2 + \ldots + km_k = g \) and \( m_1 < g \). Indeed, this is a class of a cycle of dimension 0 that is homologically equivalent to zero, hence, it is also algebraically equivalent to zero. Therefore, for every \( d \geq 0 \) we have
\[ U^d(p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}) = 0 \]
in \( \text{CH}(J)_{Q/\text{alg}} \). It remains to use (2.2).

(ii) Note that for \( \sum_{i=1}^{k} n_i > g \) the relation becomes trivial, so we can assume that \( \sum_{i=1}^{k} n_i \leq g \). The relations described in (i) have form
\[ U^d[p_1^{g - \sum_{i=1}^{k} n_i}][n_1 \ldots n_k] = 0. \]
Applying Lemma 2.7 we can rewrite this as
\[ \sum_{j=0}^{d} \binom{k - 1 - j}{d - j} p_1^{[g+j-d-\sum_{i=1}^{k} n_i]} \Delta[j](p_1 \ldots p_k) = 0, \quad (2.7) \]
where we use the convention \( p_1^{[n]} = 0 \) for \( n < 0 \). Now Lemma 2.8 shows that the obtained identity is equivalent to the assertion of the theorem.

2.4. Next we compute \( \exp(p_1) \ast x \), where \( x \) is a monomial in \( p_i \)'s.

Proposition 2.9. For \( n \geq 0 \) and \( n_1, \ldots, n_k \) such that \( n_i > 1 \) one has
\[ (-1)^g \exp(p_1) \ast (p_1^{[n]} p_1 \ldots p_k) = \sum_{0 \leq j \leq l \leq g} (-1)^l \binom{-j + n - g - 1 + k + \sum_{i=1}^{k} n_i}{l - j} p_1^{[n-l+j]} \Delta[j](p_1 \ldots p_k) \]
Proof. By Lemma 2.4 we have
\[ (-1)^g \exp(p_1) \ast x = \sum_{l=0}^{g} (-1)^l U[l](x). \]
Hence, if \( x \) is a polynomial in \( p_i \)’s then using Lemma 2.7 we get
\[ (-1)^g \exp(p_1) \ast x = \sum_{0 \leq j \leq l \leq g} (-1)^l \partial_1^{l-j} \Delta[j] \binom{H-j}{l-j} x \]
which immediately gives the required formula.

We are going to use Proposition 2.9 to compute \( S(p_{n_1} \ldots p_{n_k}) \), where \( n_i > 1 \). Then we will deduce the general case of Theorem 0.4 from this using our formula for the operators \( U[n] \).
Lemma 2.10. For $n_1, \ldots, n_k$ such that $n_i > 1$ one has

$$(-1)^g \exp(p_1) \ast (\exp(-p_1) \cdot p_{n_1} \ldots p_{n_k}) = \sum_{j \geq 0, m \geq 0} (-1)^{j+m} \left( k - j + \sum_{i=1}^{k} n_i \right) p_1^{[m]} \Delta [j] (p_{n_1} \ldots p_{n_k}).$$

Proof. We have

$$(-1)^g \exp(p_1) \ast (\exp(-p_1) \cdot p_{n_1} \ldots p_{n_k}) = (-1)^g \sum_{n \geq 0} (-1)^{n} \exp(p_1) \ast (p_1^{[n]} p_{n_1} \ldots p_{n_k}).$$

Using Proposition 2.9 this expression can be rewritten as follows:

$$\sum_{n \geq 0} \sum_{0 \leq j \leq l \leq g} (-1)^{n+l} \left(-j + n - g - 1 + k + \sum_{i=1}^{k} n_i \right) p_1^{[n-l+j]} \Delta [j] (p_{n_1} \ldots p_{n_k}) =$$

$$\sum_{j \geq 0} \Delta [j] (p_{n_1} \ldots p_{n_k}) \sum_{m \geq 0} (-1)^{j+m} p_1^{[m]} \sum_{m \leq n \leq m+g-j} \left(-j + n - g - 1 + k + \sum_{i=1}^{k} n_i \right).$$

It remains to use the elementary identity

$$\sum_{i=0}^{N} \binom{x+i}{i} = \frac{(x + N + 1)}{N}$$

to simplify the above sum. \qed

Proposition 2.11. For $n_1, \ldots, n_k$ such that $n_i > 1$ one has

$$S(p_{n_1} \ldots p_{n_k}) = \sum_{j \geq 0} p_1^{[j+g-k-\sum_{i=1}^{k} n_i]} \Delta [j] (p_{n_1} \ldots p_{n_k}),$$

where we use the notation $p^{[m]} = 0$ for $m < 0$.

Proof. Applying formula (2.1) for $x = p_{n_1} \ldots p_{n_k}$ and using Lemma 2.10 we get

$$(-1)^g S[-1]^*(p_{n_1} \ldots p_{n_k}) = \sum_{j \geq 0, m \geq 0} (-1)^{j+m} \left( k - j + \sum_{i=1}^{k} n_i \right) \exp(-p_1) p_1^{[m]} \Delta [j] (p_{n_1} \ldots p_{n_k}).$$

(2.8)

Now we observe that the LHS belongs to $\text{CH}^{g-k}(J)_{\mathbb{Q}}/(\text{alg})$. Indeed, by Lemma 2.1 we have $p_{n_1} \ldots p_{n_k} \in \text{CH}^{\sum_{i=1}^{k} n_i}_{-k+\sum_{i=1}^{k} n_i}/(\text{alg})$, so this follows from (1.2). Note also that in the RHS of (2.8) we can restrict the summation to $(j, m)$ satisfying the inequality

$$g - m \leq k - j + \sum_{i=1}^{k} n_i$$

(otherwise the binomial coefficient vanishes). For such $(j, m)$ we have

$$p_1^{[m]} \Delta [j] (p_{n_1} \ldots p_{n_k}) \in \text{CH}^{d}(J)_{\mathbb{Q}}/(\text{alg})$$
where \( d = \sum_{i=1}^{k} n_i - j + m \geq g - k \). Thus, the only terms in the RHS belonging to \( \text{CH}^{g-k}(J)_{Q/\text{alg}} \) correspond to \( m = j + g - k - \sum_{i=1}^{k} n_i \). Therefore, (2.8) implies

\[
(-1)^g S[-1]^t(p_{n_1} \ldots p_{n_k}) = (-1)^{g+k+\sum_{i=1}^{k} n_i} \sum_{j \geq 0} p_1^{[j+g-k-\sum_{i=1}^{k} n_i]} \Delta[j](p_{n_1} \ldots p_{n_k}).
\]

It remains to use the formula \([-1]^{*}p_n = (-1)^{n+1}p_n\) \(\square\).

### 2.5. Proof of Theorem 0.4

We have

\[
S(p_1^{[n]} p_{n_1} \ldots p_{n_k}) = U^{[n]}(S(p_{n_1} \ldots p_{n_k})).
\]

Hence, using Proposition 2.11 and Lemma 2.7 we get

\[
S(p_1^{[n]} p_{n_1} \ldots p_{n_k}) = \sum_{j \geq 0} \sum_{l=0}^{n} \partial_1^{l-n} \Delta[l] \left( \frac{H - l}{n - l} \right) p_1^{[j+N]} \Delta[j](p_{n_1} \ldots p_{n_k}),
\]

where \( N = g - k - \sum_{i=1}^{k} n_i \). Applying the definition of \( H \) and \( \partial_1 \) we can rewrite this as

\[
\sum_{j \geq 0} \sum_{l=0}^{n} \left( \frac{-j - 1 - l}{n - l} \right) p_1^{[j+l-n+N]} \Delta[l] \Delta[j](p_{n_1} \ldots p_{n_k}) =
\sum_{j \geq 0} \sum_{l=0}^{n} \left( \frac{-j - 1 - l}{n - l} \right) \Delta[j+l](p_{n_1} \ldots p_{n_k}) =
\sum_{m \geq 0} \sum_{l=0}^{n} \left( \frac{-m - 1 - l}{n - l} \right) \Delta[m](p_{n_1} \ldots p_{n_k}).
\]

But one has

\[
\sum_{l=0}^{n} \left( \frac{-m - 1 - l}{n - l} \right) \left( \frac{m}{l} \right) = (-1)^n
\]

(this can be proved by looking at coefficients with \( t^n \) in the identity \((1 + t)^{-m-1}(1 + t)^m = (1 + t)^{-1}\)). Hence,

\[
S(p_1^{[n]} p_{n_1} \ldots p_{n_k}) = (-1)^n \sum_{m \geq 0} p_1^{[m-n+g-k-\sum_{i=1}^{k} n_i]} \Delta[m](p_{n_1} \ldots p_{n_k}). \tag{2.9}
\]

In view of Lemma 2.8 the obtained identity is equivalent to Theorem 0.4 \(\square\).

### 2.6. Proof of Corollary 0.2

For \( d = k - 1 \) and \( \sum_{i=1}^{k} n_i = g \) the relation (0.1) gives \( p_{g-k+1} = 0 \). It remains to observe that for every \( m \) such that \( g/2 + 1 \leq m \leq g \) we can choose \((n_1, \ldots, n_{g-m+1})\) with \( n_i \geq 2 \) and \( \sum_i n_i = g \). \(\square\)

**Proof of Corollary 0.3.** We want to represent an element

\[
p_1^{[g-d-\sum_{i=1}^{k} n_i]} p_{n_1} \ldots p_{n_k},
\]
where \( n_i > 1 \) and \( k > d \), as a linear combination of classes of the form

\[
p_{[g-d-\sum_{i=1}^{k'} n_i']} p_{n_1} \cdots p_{n_{k'}}
\]

with \( k' \leq d \) (and \( n_i' > 1 \)). For this we can use a system of relations

\[
D[1]\left(p_{[m+g-\sum_{i=1}^{k'} n_i]} \Delta^m(p_{n_1} \cdots p_{n_k})\right) =
\sum_{j=0}^{d} \left(\frac{k - 1 - m - j}{d - j}\right) \cdot \frac{1}{j!} p_{[g+m+j-d-\sum_{i=1}^{k'} n_i]} \Delta^{m+j}(p_{n_1} \cdots p_{n_k}) = 0
\]

for \( m = 0, \ldots, k - 2 \). For \( j \leq k - 1 \) let us denote

\[
a_j = p_{[g+k-1-j-d-\sum_{i=1}^{k'} n_i]} \Delta^{k-1-j}(p_{n_1} \cdots p_{n_k}),
\]

so that \( a_j = 0 \) for \( j < 0 \). Note that \( a_0, \ldots, a_{d-1} \) are linear combinations of classes of the required form. On the other hand, the above relations can be rewritten as

\[
\sum_{j=0}^{d} \left(\frac{N - j}{d - j}\right) \cdot \frac{1}{j!} a_{N-j} = 0
\]

(2.10)

for \( N \leq k - 1 \). It is clear that these relations allow to express \( a_{k-1} \) in terms of the classes \( a_0, \ldots, a_{d-1} \). To find the explicit formula we observe that we can define \( a_j \) for all \( j \in \mathbb{Z} \) by imposing the recursive relation (2.10) also for \( N \geq k \). Then the generating function

\[
F(t) = \sum_{j=0}^{d} a_j t^j
\]

satisfies the differential equation

\[
\sum_{j=0}^{d} \left(\frac{d}{j}\right) F^{(j)}(t) = 0.
\]

The initial terms \( a_0, \ldots, a_{d-1} \) determine the solution uniquely:

\[
F(t) = \sum_{j=0}^{d-1} a_j t^j \cdot \left(\sum_{m=0}^{d-1-j} \frac{t^m}{m!}\right) \cdot \exp(-t).
\]

Hence, we obtain

\[
a_N = \sum_{j=0}^{d-1} a_j \cdot \sum_{m=0}^{d-1-j} \frac{(-1)^{N-j-m}}{m!(N-j-m)!}.
\]

Using the elementary identity

\[
\sum_{m=0}^{d-1-j} (-1)^m \binom{N-j}{m} = (-1)^{d-1-j} \binom{N-1-j}{d-1-j}
\]

we derive that

\[
a_N = (-1)^{N+d-1} \sum_{j=0}^{d-1} \frac{a_j}{(N-j)!} \binom{N-1-j}{d-1-j}.
\]
Hence, for $N = k - 1$ we get
\[
p_1^{[g - d - \sum_i n_i]} p_{n_1} \cdots p_{n_k} = a_{k-1} =
\]

\[
(-1)^{k+d} \sum_{j=0}^{d-1} \binom{k-2-j}{d-1-j} p_1^{[g+k-1-j - d - \sum_i n_i]} \Delta^{[k-1-j]} (p_{n_1} \cdots p_{n_k}),
\]

which is equivalent to (0.2).

Now let us consider the relations (0.1) for small genera. Recall that to produce a nontrivial relation the parameters $(d, k, n_1, \ldots, n_k)$ should satisfy the inequalities $\sum_{i=1}^k n_i \leq g$, $d \leq k - 1$. In particular, we should have $d \leq k - 1 \leq g/2 - 1$. Thus, for $g = 4$ the only interesting relation is $p_3 = 0$ (corresponding to $d = 1$, $k = 2$, $n_1 = n_2 = 2$). Here is the complete list of relations equivalent to (0.1) for $5 \leq g \leq 10$ (where we omit the relations in codimension $g$):

$g = 5$:

\[
p_4 = 0, \quad p_2^2 = -6p_1p_3;
\]

$g = 6$:

\[
d = 2 : \quad p_4 = 0,
d = 1 : \quad p_5 = 0, \quad p_2p_3 = 0, \quad p_1p_4 = 0, \quad p_1p_2^2 = -3p_1^2p_3;
\]

$g = 7$:

\[
d = 2 : \quad p_5 = 0, \quad p_2p_3 = -5p_1p_4,
d = 1 : \quad p_6 = 0, \quad p_2p_4 = 0, \quad p_2^3 = 0, \quad p_2^3 = 45p_1^2p_4, \quad p_1p_5 = 0, \quad p_1p_2p_3 = -5p_1^2p_4,
p_1^2p_2^2 = -2p_1^3p_3;
\]

$g = 8$:

\[
d = 3 : \quad p_5 = 0,
d = 2 : \quad p_6 = 0, \quad 3p_3^2 = -10p_2p_4, \quad p_2^3 + 18p_1p_2p_3 + 45p_1^2p_4 = 0, \quad p_1p_5 = 0,
d = 1 : \quad p_7 = 0, \quad p_2p_5 = 0, \quad p_3p_4 = 0, \quad p_2^3p_4 = 0, \quad p_1p_6 = 0, \quad p_1p_2p_4 = 0, \quad p_1p_3^2 = 0,
p_1p_2^3 = 15p_1^3p_4, \quad p_1^2p_5 = 0, \quad 3p_1^2p_2p_3 = -10p_1^3p_4, \quad 2p_1^3p_2^2 = -3p_1^4p_3;
\]

$g = 9$:

\[
d = 3 : \quad p_6 = 0, \quad 3p_3^2 + 10p_2p_4 + 70p_1p_5 = 0,
d = 2 : \quad p_7 = 0, \quad p_2p_5 = 0, \quad p_3p_4 = 0, \quad 2p_2^3p_3 + 10p_1p_2p_4 + 3p_1p_3^2 = 0, \quad p_1p_6 = 0,
3p_1p_2^3 + 10p_1p_2p_4 + 70p_1^2p_5 = 0, \quad p_1p_2^3 + 9p_1^2p_2p_3 + 15p_1^3p_4 = 0,
d = 1 : \quad p_8 = 0, \quad p_2p_6 = 0, \quad p_3p_5 = 0, \quad p_1^3 = 0, \quad p_2^2p_4 = 0, \quad p_2p_3^2 = 0, \quad p_1^2 = -420p_1^3p_5,
p_1p_7 = 0, \quad p_1p_2p_5 = 0, \quad p_1p_3p_4 = 0, \quad p_1p_2^3p_3 = 35p_1^4p_5, \quad p_1^2p_2p_4 = -5p_1^3p_5,
3p_1^2p_2^2 = -20p_1^3p_5, \quad 2p_1^2p_2^2 = 15p_1^4p_4, \quad 2p_1^2p_2p_3 = -5p_1^4p_4, \quad 5p_1^4p_2^2 = -6p_1^5p_3;
\]
3. Formal consequences of relations

Throughout this section $x_1, x_2, \ldots$ are formal commuting variables. We equip the ring of polynomials $\mathbb{Q}[x_1, x_2, \ldots]$ with grading by setting $\deg(x_i) = i$.

3.1. The following lemma will allow us to express the relations in $R^l_g$ in the form asserted in Theorem 0.4.(i).

**Lemma 3.1.** Let $V_n \subset V = \mathbb{Q}[x_1, x_2, \ldots]$ denote the component of degree $n$. The linear map $D : V_{n+1} \to V_n$ induced by the differential operator $D$ is surjective for $n > g$. The image of the map $D : V_{g+1} \to V_g$ is the codimension 1 subspace spanned by all monomials of degree $g$ except for $x^g$.

**Proof.** Let us consider the operator $Dx_1 : V_n \to V_n$. Then for every monomial $F = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $V_n$, where $n_i > 1$, one has

$$Dx_1 F \equiv (n + k - g)F \mod (x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}) \cap V_n,$$

where $(x_1^{n_1})$ denotes the ideal generated by $x_1^{n_1}$. This immediately implies that $Dx_1$ surjects onto $V_n$ for $n > g$ and that the image of $Dx_1|_{V_g}$ is the subspace spanned by all monomials of degree $g$ except for $x^g$. It remains to observe that for an arbitrary monomial $F$ of degree $g + 1$, $D(F)$ lies in this subspace. \qed

**Lemma 3.2.** The operators $e(F) = x_1 \cdot F$, $f = -D$ and $h = -g + \sum_{n \geq 1} (n + 1) x_n \partial_n$, define a representation of $\mathfrak{sl}_2$ on $\mathbb{Q}[x_1, x_2, \ldots]$.

The proof is straightforward and is left for the reader.

**Lemma 3.3.** Let $V$ be an $\mathfrak{sl}_2$-module. Then the subspace $\cap_{n \geq 0} f^n(V) \subset V$ is $\mathfrak{sl}_2$-invariant.

**Proof.** We have $h f^n(V) \subset f^n(V)$. Now an easy induction in $n$ shows that $e f^n(V) \subset f^{n-1}(V)$. Hence, $\cap_{n \geq 0} f^n(V)$ is invariant with respect to $e$. \qed

Let $S$ be the operator on $\mathbb{Q}[x_1, x_2, \ldots]$ defined by

$$S(x_1^{[n]} x_{n_1} \cdots x_{n_k}) = (-1)^n \sum_{m \geq 0} x_1^{[m-n+g-k-\sum_{i=1}^k n_i]} \Delta^{[m]}(x_{n_1} \cdots x_{n_k}),$$

(3.1)
where \(n_i > 1\). Of course, this is just the formula (2.9) with \(p_i\) replaced by \(x_i\) (recall that we use the convention \(x_1^{[d]} = 0\) for \(d < 0\)).

**Lemma 3.4.** For arbitrary \(n \geq 0, n_1, \ldots, n_k\) such that \(n_i > 1\) and \(n + k + \sum_{i=1}^{k} n_i \leq g\) one has

\[
S^2(x_1^{[n]} x_{n_1} \ldots x_{n_k}) = (-1)^{g+k+\sum n_i} x_1^{[n]} x_{n_1} \ldots x_{n_k}.
\]

**Proof.** The condition \(n + k + \sum n_i \leq g\) implies that in the sum defining \(S(x_1^{[n]} x_{n_1} \ldots x_{n_k})\) all terms contain nonnegative powers of \(x_1\), i.e., no terms are omitted due to our convention that \(x_1^{[d]} = 0\) for \(d < 0\). Therefore, we can apply the formula for \(S\) again and find that

\[
(-1)^{g+k+\sum n_i} S^2(x_1^{[n]} x_{n_1} \ldots x_{n_k}) = \sum_{m \geq 0, l \geq 0} (-1)^m \binom{m+l}{m} x_1^{[l+m+n]} \Delta^{[l+m]}(x_{n_1} \ldots x_{n_k}) = \sum_{d \geq 0} \sum_{m+l=d} (-1)^m \binom{d}{m} x_1^{[d+n]} \Delta^{[d]}(x_{n_1} \ldots x_{n_k}).
\]

It remains to use the fact that \(\sum_{m=0}^{d} (-1)^m \binom{d}{m} = 0\) for \(d \geq 1\). \(\Box\)

### 3.2. Proof of Theorem 0.5(i),(iii),(iv)

Part (i) is an immediate consequence of Lemma 3.1.

By Lemma 3.2 we have an action of \(\mathfrak{sl}_2\) on \(V = \mathbb{Q}[x_1, x_2, \ldots]\). Applying Lemma 3.3 and part (i) we derive that the subspace \(I_g = I_g = \cap_{n \geq 0} f^n(V) \subset V\) is \(\mathfrak{sl}_2\)-invariant.

Now let us consider the operator \(S : V \to V\) defined by (3.1). It is easy to check that

\[
Sc = -fS. \tag{3.2}
\]

Indeed, the formula (3.1) was constructed in such a way that

\[
S(x_1^{[n]} x_{n_1} \ldots x_{n_k}) = D^{[n]} S(x_{n_1} \ldots x_{n_k}),
\]

where \(n_i > 1\), Since \(f = -D\) this implies the above relation. It is also straightforward to check that

\[
Sh = -hS. \tag{3.3}
\]

Next we claim that

\[
Sf(F) + eS(F) \in I \tag{3.4}
\]

for every \(F \in V\). Using the relation (3.2) one can easily check that

\[
(Sf + eS)e = -f(Sf + eS).
\]

Hence, the claim would follow if we could prove that

\[
(Sf + eS)(x_{n_1} \ldots x_{n_k}) \in I,
\]

but this is immediate from part (i).
where \( n_i > 1 \). We have

\[
(Sf + eS)(x_{n_1} \ldots x_{n_k}) = x_1 S(x_{n_1} \ldots x_{n_k}) - S \Delta(x_{n_1} \ldots x_{n_k}) =
\sum_{m \geq 0} (g + m + 1 - k - \sum n_i) x_1^{[g + m + 1 - k - \sum n_i]} \Delta^m(x_{n_1} \ldots x_{n_k}) -
\sum_{m \geq 0} m x_1^{[g + m + 1 - k - \sum n_i]} \Delta^m(x_{n_1} \ldots x_{n_k}) =
(g + 1 - k - \sum n_i) \sum_{m \geq 0} x_1^{[g + m + 1 - k - \sum n_i]} \Delta^m(x_{n_1} \ldots x_{n_k}).
\]

If \( \sum n_i > g \) then the latter sum is zero since nonzero terms correspond to \( m \leq k - 1 \) and \( g + m + 1 - k - \sum n_i \geq 0 \). Otherwise, it is proportional to \( D_1^{[k-1]}(x_1^{g - \sum n_i} x_{n_1} \ldots x_{n_k}) \) (see the proof of Theorem 0.1(ii)). Hence, it belongs to \( I \) which proves our claim.

Next, we observe that if \( n + \sum_{i=1}^n n_i > g \) (where \( n_i > 1 \)) then

\[
S(x_1^n x_{n_1} \ldots x_{n_k}) = (-1)^n \sum_{m > k} x_1^{[m-n-g-k-\sum_{i=1}^n n_i]} \Delta^m(x_{n_1} \ldots x_{n_k}) = 0
\]

since \( \Delta^m(x_{n_1} \ldots x_{n_k}) = 0 \) for \( m > k \). Hence \( S(V_m) = 0 \) for \( m > g \), where \( V_m \) is the component of degree \( m \) in \( V \). Using (3.4) and Lemma 3.1 we conclude that \( S(I) \subseteq I \).

Finally, we want to prove that

\[
S^2(F) = (-1)^g [-1]^* (F) \text{ mod } I
\] (3.5)

for every \( F \in V \), where \([-1]^* x_1^n x_{n_1} \ldots x_{n_k} = (-1)^{k+\sum n_i} x_1^n x_{n_1} \ldots x_{n_k} \). The relations (3.2), (3.3) and (3.4) imply that the operator \( S^2 - (-1)^g [-1]^* \) induces an endomorphism of the finite-dimensional \( \mathfrak{s}_\mathfrak{l}_2 \)-module \( V/I \). Therefore, it is enough to prove that (3.5) holds for every \( F \in V \) such that \( hF = mF \) with \( m \leq 0 \). Thus, it suffices to prove that

\[
S^2(x_1^n x_{n_1} \ldots x_{n_k}) = (-1)^{g+k+\sum n_i} x_1^n x_{n_1} \ldots x_{n_k},
\]

where \( n_i > 1 \), provided that \( -g + 2n + k + \sum n_i \leq 0 \). But in this case we also have \( -g + n + k + \sum n_i \leq 0 \), so we are done by Lemma 3.4.

\[\square\]

3.3. For \( k \geq 0 \), \( m \geq 0 \), \( k + m \geq 2 \) consider the differential operators

\[
\Delta_{k,m} = \frac{1}{k!} \sum_{n_1, \ldots, n_k \geq 2} \frac{(n_1 + \ldots + n_k + m)!}{n_1! \ldots n_k! m!} x_{n_1 + \ldots + n_k + m-1} \partial_{n_1} \ldots \partial_{n_k}.
\]

Note that \( \Delta_{0,m} = x_{m-1} \) for \( m \geq 2 \) and that \( \Delta_{2,0} = \Delta \). These operators appear when we want to write the operators \( D_k \) considered in Theorem 0.6 as polynomials in \( \partial_1 \): for every \( k \geq 2 \) one has

\[
D_k = \sum_{m=0}^{k} \Delta_{k-m,m} \partial_1^m.
\] (3.6)

Hence, we have

\[
D = D_2 - g \partial_1 = \Delta_{2,0} + \Delta_{1,1} \partial_1 + p_1 \partial_1^2 - g \partial_1.
\]

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Lemma 3.5. (i) The following commutation relations hold:

\[ [\Delta_{k,m}, \Delta_{k',m'}] = (km' - k'm)(k + k' - 1)!(m + m' - 1)! \Delta_{k+k'-1,m+m'-1} \]

for \( m + m' \geq 1 \) and

\[ [\Delta_{k,0}, \Delta_{k',0}] = 0. \]

In particular,

\[ [\Delta, \Delta_{k,m}] = (k + 1)\Delta_{k+1,m} \]

for \( m \geq 1 \) and

\[ [\Delta, \Delta_{k,0}] = 0. \]

(ii) One has \([D_k, D_{k'}] = 0\) for \( k, k' \geq 2 \) and \([D_k, \partial_1] = 0\) for \( k \geq 3\).

We leave the proof for the reader.

Let us denote by \( V' \subset V \) the subspace spanned by all the monomials \( x_1^{n_1}x_2 \ldots x_n \), where \( n_i > 1 \), such that \( n + k + \sum_{i=1}^k n_i \leq g \). Let also \( J \subset I \) denote the span of the subspaces \( V_m \) for \( m > g \) together with all the elements of the form

\[ D^{[d]}(x_1^{[g-n_1]}x_2 \ldots x_n) = \sum_{j=0}^d \binom{k-1-j}{d-j} x_1^{[g-a-n]} \Delta^j(x_2 \ldots x_n), \]

where \( k \geq 1, n_i > 1, d \geq 0 \) and \( d + \sum n_i \leq g \).

Lemma 3.6. (i) \( V = J + V' \);

(ii) \( I \cap V' = S(I) = S(I \cap V') \);

(iii) \( I = J + S(I \cap V') \).

Proof. (i) This follows essentially from the proof of Corollary 0.3. Indeed, it suffices to show that every monomial \( x_1^{n_1}x_2 \ldots x_n \) such that \( n_i > 1, n + k + \sum n_i > g \), can be expressed modulo \( J \) in terms of monomials \( x_1^{n_1}x_2^{n_1} \ldots x_n^{n_1} \), with \( n'+k' + \sum n_i' < n + k + \sum n_i \). If \( n + \sum n_i > g \) then our monomial belongs to \( J \), so we can assume that \( d := g - n - \sum n_i \geq 0 \). Our assumption implies that \( d < k \), hence the element \( D^{[d]}(x_1^{[g-n_1]}x_2 \ldots x_n) \in J \) has a nonzero coefficient with \( x_1^{[n]}x_2 \ldots x_n \). Therefore, it gives the required expression.

(ii) The inclusion \( S(V) \subset V' \) is immediate from the definition of \( S \). Since \( S \) preserves \( I \) we obtain \( S(I \cap V') \subset S(I) \subset I \cap V' \). On the other hand, Lemma 3.4 implies that \( S^2|_{V'} = (-1)^g[-1]^g \). Hence, \( I \cap V' \subset S(I \cap V') \).

(iii) Combining (i) and (ii) we get \( I = J + I \cap V' = J + S(I \cap V') \).

Remark. In fact, it is easy to show that \( J = \ker(S) \). Indeed, the same argument as in the proof below shows that \( S(F) = 0 \) for \( F \in J \). On the other hand, \( \ker(S) \cap V' = 0 \) by Lemma 3.4. Hence, \( J = \ker(S) \). It follows that we have direct sum decompositions \( V = J \oplus V' \), \( I = J \oplus S(I \cap V') \).
3.4. **Proof of Theorem 0.5(ii) and of Theorem 0.6.** First, we observe that by Lemma 3.5(ii) the operators \( D_k \) for \( k \geq 3 \) commute with \( D = D_2 - g \partial_1 \), hence they preserve \( I_g \) and induce operators on \( R_g^{\text{Jac}} \).

Next, we claim that the identity

\[
S(x_m(S(F))) = (-1)^g D_{m+1}[-1]^*F
\]  

(3.7)

holds for every \( F \in V' \), \( m \geq 2 \). It suffices to take \( F = [n] x_{i_1} \ldots x_{i_k} \), where \( n_i > 1 \), \( n + k + \sum_i n_i \leq g \). Applying the definition of \( S \) we can write

\[
S x_m S([n] x_{i_1} \ldots x_{i_k}) = (-1)^n \sum_{i \geq 0} S x_1^{[i-n+g-k-\sum_i n_i]} \Delta^i(x_{i_1} \ldots x_{i_k}) =
\]

\[
(-1)^{g-k-\sum_i n_i} \sum_{i,j \geq 0} (-1)^i x_1^{[i+j+n-m-1]} \Delta^j x_m \Delta^i(x_{i_1} \ldots x_{i_k}),
\]

where we used the assumption \( n + k + \sum_i n_i \leq g \) to be able to apply the definition of \( S \) the second time. Now we observe that for every \( N \geq 0 \) we have the equality of operators

\[
\sum_{i+j=0}^N (-1)^i \Delta^j x_m \Delta^i = \frac{1}{N!} (\text{ad} \Delta)^N(x_m).
\]

Recall that \( x_m = \Delta_{0,m+1} \), so using Lemma 3.5(i) we find

\[
(\text{ad} \Delta)^N(x_m) = N! \Delta_{N,m+1-N}
\]

for \( N \leq m + 1 \) and \( (\text{ad} \Delta)^N(x_m) = 0 \) for \( N > m + 1 \). Therefore, we obtain

\[
S x_m S([n] x_{i_1} \ldots x_{i_k}) = (-1)^{g-k-\sum_i n_i} \sum_{N=0}^{m+1} x_1^{[N+n-m-1]} \Delta_{N,m+1-N}(x_{i_1} \ldots x_{i_k}).
\]

Comparing the result with (3.6) we observe that it is equal to

\[
(-1)^{g-k-\sum_i n_i} D_{m+1}([n] x_{i_1} \ldots x_{i_k})
\]

which proves (3.7).

It follows that \( S x_m S(I \cap V') = D_{m+1}(I \cap V') \subset I \), hence

\[
x_m S(I \cap V') \subset I.
\]

(3.8)

Now we are going to show that \( S(x_m F) = 0 \) for \( F \in J \). If \( F \) has degree \( \geq g \) then this is clear, so it is enough to check that

\[
S \left( x_m \cdot \sum_{j=0}^d \binom{k-j}{d-j} x_1^{[j+g-d-\sum_i n_i]} \Delta^j(x_{i_1} \ldots x_{i_k}) \right) = 0,
\]

where \( k \geq 1, n_i > 1, d \geq 0 \) and \( d + \sum_i n_i \leq g \).

Since \( S(x_m J) = 0 \) we derive that

\[
x_m J \subset I.
\]

(3.9)

Recall that \( I = J + S(I \cap V') \) by Lemma 3.6(iii), so the inclusions (3.8) and (3.9) imply that \( x_m I \subset I \) for every \( m \geq 2 \). Since we also know that \( I \) is invariant under \( e \), it follows that \( I \) is an ideal in \( \mathbb{Q}[x_1, x_2, \ldots] \).
Now the decomposition $V = V' + I$ implies that equation (3.7) holds for all $F \in R_g^{\text{Jac}}$. This proves (0.6).

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