Supersymmetric Construction of W-Algebras from Super Toda and WZNW Theories

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ABSTRACT

A systematic construction of super W-algebras in terms of the WZNW model based on a super Lie algebra is presented. These are shown to be the symmetry structure of the super Toda models, which can be obtained from the WZNW theory by Hamiltonian reduction. A classification, according to the conformal spin defined by an improved energy-momentum tensor, is discussed in general terms for all super Lie algebras whose simple roots are fermionic. A detailed discussion employing the Dirac bracket structure and an explicit construction of W-algebras for the cases of \( OSP(1, 2) \), \( OSP(2, 2) \), \( OSP(3, 2) \) and \( D(2, 1|\alpha) \) are given. The N=1 and N=2 super conformal algebras are discussed in the pertinent cases.

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1 Introduction

There has been an increasing demand in the study of the symmetry structures of two-dimensional conformal field theories. One of the interesting developments in this direction was the introduction of extensions of the Virasoro algebra by higher spin generators constituting the so called W-algebra [1]. These are not Lie algebras and have been shown to be the symmetries of a variety of models [2, 3, 4]. The geometrical and algebraic meaning of such transformations are not well understood yet, but some results have already been obtained in this direction [3, 5].

The W-algebras have been extensively studied in the context of the Toda models. These can be cast in three different classes according to the algebraic structure. There are the Conformal Toda models (CT) associated to finite Lie algebras which are conformally invariant and which the simplest example is the Liouville model. The Affine Toda models (AT) associated to the Kac-Moody algebras with vanishing central extension (loop algebras) and which the simplest example is the Sinh-Gordon model. These are not conformally invariant but have been shown to possess infinite number of conserved charges in involution [7]. Finally there are the recently proposed Conformal Affine Toda models (CAT) [8, 9] which are associated to the full Kac-Moody algebra and constitute a conformal extension of the AT models.

The W structure appears in a very elegant way in the CT models through a Hamiltonian reduction procedure. These models have been shown to be constrained WZNW theories [10]. In order to preserve the conformal invariance the energy momentum tensor has to be modified to commute with the constraints. As a consequence, the conformal spins of the currents are changed giving rise to higher spin generators. The remaining Kac-Moody currents under the reduction, which are the symmetries of the CT models, become the generators of the W-algebra. Another framework for constructing the W-generators for the CT models, involving differential operators, is based on the work of Drinfeld-Sokolov [11, 2]. The CAT models have also been shown to be obtained by a Hamiltonian reduction procedure from a WZNW type model associated to a two loop Kac-Moody algebra [3, 4]. The W structure of such models is not described just by the remaining Kac-Moody currents like in [10]. There is an infinite number of W-generators and a method for generating them was proposed in [13]. The AT models, being not conformally invariant, were not studied along these lines. But an interesting point to be explored in this context is the connection of the W symmetries of the CAT models and the integrability structures of the AT models via a breakdown of the conformal symmetry (see for instance [14]).

The supersymmetric version of the CT models have also received a lot of attention. They are superconformal models and constitute a natural place to study the role of supersymmetry in the structures discussed above. In fact, several aspects of the super W-algebras for such models have been studied [14]. A construction of $N = 1$ and $N = 2$ super conformal algebra in terms of the fields of super conformal Toda models (SCT) was proposed in [15]. Their algebra was derived from the Poisson brackets obtained from the SCT action. Recently, the SCT models have been shown to be constrained super WZNW models associated to super Lie algebras whose simple roots can be chosen to be all fermionic [17, 18, 19]. Such result paves the way to explore the symmetries of SCT models on the lines of ref. [10]. Indeed,
the spin of the W generators of those models were calculated in ref. \[19\] using the methods of ref. \[10\].

In this paper we propose a systematic construction of the super W-algebra for the super conformal Toda models. Our starting point is a conformally invariant WZNW model underlined by a super Lie algebra \( G \) whose simple roots are all fermionic. Such theory, however is not supersymmetric because the number of bosonic and fermionic generators in \( G \) do not match. In section 2 we review the Hamiltonian reduction procedure which constrains our model in a conformally invariant manner in order to obtain the SCT model, which is supersymmetric. In section 3 we give a general method to find the remaining super Kac-Moody currents and their corresponding conformal spins. It is shown that both, the bosonic and fermionic currents decompose each into rank \( G \) multiplets of a special \( SL(2) \) subalgebra of \( G \). It is also argued in general terms that after constraining and gauge fixing there is a single remaining current associated to the highest weight of each \( SL(2) \) multiplet. That constitute a generalization to super Lie algebra of the analysis of ref. \[10\]. Next in section 4 we analyze all cases where the simple roots are fermionic and determine the conformal spin of each remaining current. In section 5 we present the super W-algebra for some examples by calculating explicitly the Dirac brackets for the remaining currents. The examples discussed are: a) \( OSP(1,2) \) which possesses a \( N = 1 \) superconformal algebra; b) \( OSP(2,2) \) which presents \( N = 2 \) superconformal algebra; c) \( OSP(3,2) \) with an extension by a spin \( 5/2 \) of the \( N = 1 \) superconformal algebra and finally d) the case of the exceptional super Lie algebra \( D(2,1|\alpha) \) (\( \alpha \neq 0, -1 \)) which presents three non-commuting Virasoro subalgebras plus two generators of spin \( 3/2 \) and one of spin \( 5/2 \). It is shown that such super W-algebra possesses two non commuting \( N = 1 \) superconformal subalgebras.

2 The reduction of super WZNW theory

We consider a WZNW theory based on a field \( g(x) \) which takes value on a connected real Lie supergroup \( G \). Such theory possesses many of the properties of the ordinary WZNW like conformal invariance and left and right Kac-Moody symmetries. The equations of motion are given by

\[
\partial_- J_R = 0 ; \quad \partial_+ J_L = 0
\]

where \( J_R \) and \( J_L \) are respectively the left and right Kac-Moody currents

\[
J_R = kg^{-1}\partial_+ g; \quad J_L = -k\partial_- gg^{-1}
\]

where \( k \) is the central term of the KM algebra. In order to obtain the Super-Toda theories as reduced models from the above WZNW models, the superalgebra \( G \) of \( G \) must be a Basic Lie superalgebra which possess purely odd simple root system, \[17, 18\]. These are \( sl(n + 1|n) \), \( OSP(2n - 1|2n) \), \( OSP(2n|2n) \), \( OSP(2n + 1|2n) \), \( OSP(2n + 2|2n) \) and \( D(2,1|\alpha) \) with \( n \geq 1 \) and \( \alpha \neq 0, -1 \). The supercommutation relations for \( G \), in the Chevalley basis, can be written as

\[
[H_a, H_b] = 0 ; \quad [H_a, E_{\alpha_b}] = K_{ab}E_{\alpha_b} \quad [H_a, E_{-\alpha_b}] = \delta_{ab}H_a ; \quad [H_a, E_{-\alpha_b}] = -K_{ab}E_{-\alpha_b}
\]

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where $H_a$ are the generators of the Cartan subalgebra, $E_{a_a}$ and $E_{-a_a}$ are the odd operators associated to the positive and negative simple roots respectively, $K_{ab}$ is the Cartan matrix of $G$ (which can be chosen real and symmetric), $\alpha_a$ are the simple roots of $G$ (all assumed to be odd roots) and $a, b = 1, 2, \ldots, r$, where $r$ is the rank of $G$. The superalgebra $G$ possesses an invariant nondegenerate supersymmetric bilinear form given by

$$STr(H_a H_b) = K_{ab}; \quad STr(E_{a_a} E_{-a_a}) = \delta_{ab}$$

$$STr(H_a E_{a_b}) = STr(H_a E_{-a_b}) = STr(E_{a_a} E_{a_b}) = STr(E_{-a_a} E_{-a_b}) = 0$$  \hspace{1cm} (4)

The bilinear form for the remaining step operators $E_{\pm \alpha}$ is such that $STr(H_a E_{\pm a}) = 0$, and $STr(E_{a} E_{\beta}) = 0$ for $\alpha + \beta \neq 0$. We introduce the components of the KM currents

$$J_R(T) \equiv k STr(T g^{-1} \partial_T g) ; \quad J_L(T) \equiv -k STr(T \partial_T g g^{-1})$$  \hspace{1cm} (5)

Under the Poisson bracket each chiral component generates a copy of the (super) KM algebra, and currents of different chiralities commute among themselves. The energy-momentum tensor is of the Sugawara form,

$$T(x) = \frac{1}{2k} \eta^{ij} J_i(x) J_j(x)$$  \hspace{1cm} (6)

where $\eta^{ij}$ is the inverse of the Killing form, $\eta_{ij} = STr(T_i T_j)$, where $T_i$'s constitute a basis of $G$ and $J$ stand for either $J_L$ or $J_R$. All currents have conformal spin one with respect to $T(x)$, i.e.,

$$[T(x), J_i(y)] = J_i(y) \delta'(x-y) - J_i(y) \delta(x-y)$$  \hspace{1cm} (7)

The Super-Toda theories are obtained from the WZNW model by a Hamiltonian reduction procedure where the following constraints are imposed on the KM currents [17]

$$J_L(E_{a_a}) = 0 ; \quad J_R(E_{a_a}) = 0$$

$$J_L(E_{a_b}) = \mu_{(ab)} (1 + \delta_{ab}) K_{ab} ; \quad J_R(E_{-a_b}) = -\mu_{(ab)}^r (1 + \delta_{ab}) K_{ab}$$

$$J_L(E_{0}) = 0 ; \quad J_R(E_0) = 0$$  \hspace{1cm} (8)

where $E_{a_b}$ and $E_{-a_b}$ are even simple root step operators defined by

$$E_{a_b} \equiv [E_{a_a}, E_{a_b}]; \quad E_{-a_b} \equiv [E_{-a_a}, E_{-a_b}]$$

and $E_0$ and $E_{-0}$ are respectively the positive and negative remaining step operators of $G$ in the Chevalley basis. Using the super Jacobi identity and the invariance of the bilinear form (4) one gets

$$STr(E_{a_b} E_{-a_c}) = -(\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) K_{cd}$$  \hspace{1cm} (10)

The roots $\alpha_{ab} = \alpha_a + \alpha_b$, associated to the step operators $E_{a_b}$, are the simple roots of the even subalgebra $G_0$, where $G = G_0 + G_1$ is the $\mathbb{Z}_2$ grading of $G$. The reason is that $\alpha_{ab}$ can not be written as the sum of two positive even roots of $G$. Notice that $G_0$ may not be a simple ordinary Lie algebra, and its set of simple roots is the union of the sets of simple roots of each one of its simple components.
The equations of motion of the Super-Toda theories are obtained from (1) by parametrizing the group elements close to the identity by a Gauss type decomposition

\[ g = \mathcal{N}\mathcal{A}\mathcal{M} \]  

where

\[ \mathcal{N}(x_+, x_-) = \exp \mathcal{F}_+(x_+, x_-); \quad \mathcal{M}(x_+, x_-) = \exp \mathcal{F}_-(x_+, x_-); \quad \mathcal{A}(x_+, x_-) = \exp(\phi_a(x_+, x_-)H_a) \]  

and

\[ \mathcal{F}_+ = \sum_{a=1}^{\text{rank}\mathcal{G}} \zeta_a(x_+, x_-)E_{\alpha_a} + \tilde{\mathcal{F}}_+; \quad \mathcal{F}_- = \sum_{a=1}^{\text{rank}\mathcal{G}} \xi_a(x_+, x_-)E_{-\alpha_a} + \tilde{\mathcal{F}}_- \]  

where \( \tilde{\mathcal{F}}_+ \) and \( \tilde{\mathcal{F}}_- \) are real linear combinations of the positive and negative non simple roots of \( \mathcal{G} \) respectively, \( \phi_a \) are bosonic fields and \( \zeta_a \) and \( \xi_a \) are grassmannian fields.

We introduce

\[ K_R \equiv \mathcal{M}g^{-1}\partial_+g\mathcal{M}^{-1} = \mathcal{A}^{-1}\mathcal{N}^{-1}\partial_+\mathcal{N}\mathcal{A} + \mathcal{A}^{-1}\partial_+\mathcal{A} + \partial_+\mathcal{M}\mathcal{M}^{-1} \]
\[ K_L \equiv \mathcal{N}^{-1}\partial_-gg^{-1}\mathcal{N} = \mathcal{N}^{-1}\partial_-\mathcal{N} + \partial_-\mathcal{A}\mathcal{A}^{-1} + \mathcal{A}\partial_-\mathcal{M}\mathcal{M}^{-1}\mathcal{A}^{-1} \]  

The equations of motion (1) can then be written as

\[ \partial_-K_R = -[K_R, \partial_-\mathcal{M}\mathcal{M}^{-1}] \]
\[ \partial_+K_L = [K_L, \mathcal{N}^{-1}\partial_+\mathcal{N}] \]  

After the constraints (8) are imposed, the currents take the form

\[ J_R = kg^{-1}\partial_+g = \sum_{(ab)} \mu^+_{(ab)}E_{\alpha_{ab}} + j_R \]
\[ J_L = -kg^{-1}\partial_-g = \sum_{(ab)} \mu^-_{(ab)}E_{-\alpha_{ab}} + j_L \]  

where \( j_R \) (\( j_L \)) is a linear combination of Cartan subalgebra generators and of negative (positive) root step operators. From (8) and the invariance of the bilinear form \( STr \) it follows that

\[ [E_{\alpha_{ab}}, E_{-\alpha_c}] = -K_{ab}(\delta_{bc}E_{\alpha_a} + \delta_{ac}E_{\alpha_b}) \]
\[ [E_{-\alpha_{ab}}, E_{\alpha_c}] = K_{ab}(\delta_{bc}E_{-\alpha_a} + \delta_{ac}E_{-\alpha_b}) \]  

From (14), (17) and (18) we conclude that, after the constraints are imposed

\[ (\mathcal{A}^{-1}\mathcal{N}^{-1}\partial_+\mathcal{N}\mathcal{A})_{\text{constr.}} = \sum_{(ab)} \mu^+_{(ab)}(E_{\alpha_{ab}} + K_{ab}(\xi_bE_{\alpha_a} + \xi_aE_{\alpha_b})) \]
\[ (\mathcal{A}\partial_-\mathcal{M}\mathcal{M}^{-1}\mathcal{A}^{-1})_{\text{constr.}} = \sum_{(ab)} \mu^-_{(ab)}(E_{-\alpha_{ab}} + K_{ab}(\zeta_bE_{-\alpha_a} + \zeta_aE_{-\alpha_b})) \]
Further, after a conjugation by the abelian subgroup \( A \), we find
\[
(N^{-1} \partial_+ N)^{\text{constr.}} = \sum_{(ab)} \mu^+_{(ab)} \left( e^{(K_{ac}+K_{bc})\phi_c} E_{\alpha_{ab}} + K_{ab} \left( \xi_b e^{K_{ac}\phi_c} E_{\alpha_a} + \zeta_a e^{K_{bc}\phi_c} E_{\alpha_b} \right) \right)
\]
\[
(\partial_- M M^{-1})^{\text{constr.}} = \sum_{(ab)} \mu^-_{(ab)} \left( e^{(K_{ac}+K_{bc})\phi_c} E_{-\alpha_{ab}} + K_{ab} \left( \zeta_b e^{K_{ac}\phi_c} E_{-\alpha_a} + \xi_a e^{K_{bc}\phi_c} E_{-\alpha_b} \right) \right)
\] (21)

Now using (20) and (21) in (13) and (16), we find the equations of motion for the fields \( \phi_a \), \( \xi_a \) and \( \zeta_a \). The equation for \( \phi_a \) correspond to the coefficients of the Cartan subalgebra in either (13) or (16), whilst the coefficient of the positive and negative simple root step operator of (13) and (14), respectively, yields the equations of motion for \( \xi_a \) and \( \zeta_a \). They are
\[
\begin{align*}
\partial_+ \partial_- \phi_a &= 4 \sum_b K_{a,b} \mu^+_{ab} \mu^-_{ab} e^{(K_{ac}+K_{bc})\phi_c} - 4 \sum_{b,d} \mu^+_{ab} \mu^-_{ad} K_{ac} \zeta_a \zeta_d e^{K_{bc}\phi_c} \\
\partial_- \xi_a &= 2 \sum_b \mu^-_{ab} K_{ab} \xi_b e^{K_{ac}\phi_c} \\
\partial_+ \zeta_a &= 2 \sum_b \mu^+_{ab} K_{ab} \zeta_b e^{K_{ac}\phi_c}.
\end{align*}
\] (22)

These are the equations of motion of the super conformal Toda models.

### 3 The Super \( W \)-algebra

The symmetries of the WZNW theory are given by the left and right KM currents \( \mathfrak{g} \). The symmetries of the Super-Toda theories are described by the currents which remain after the constraints \( \mathfrak{h} \) and the corresponding gauge fixings are imposed. The algebra of these remaining currents, under the Dirac bracket, is not a subalgebra of the KM algebra. In fact, it is not even a Lie algebra. Their algebra constitute what is called a super \( W \)-algebra. We now describe a method to obtain the \( W \) generators and their algebra. It is a generalization to the case of superalgebras, of the method described in \[ \text{[14]} \text{ and [15]. Our discussion applies} \]

\[
L(x) = T(x) + 2 \partial_x J_{\tilde{\delta},H}(x)
\] (23)

where \( J_{\tilde{\delta},H}(x) = \text{STr}(J(x)\tilde{\delta}.H) \) and \( \tilde{\delta} = \frac{1}{2} \sum_{\alpha>0} \alpha \alpha \), one half of the sum of the even coroots of \( \mathfrak{g}_0 \). With respect to the improved energy-momentum tensor \( L(x) \) the conformal spins of the currents \( J_i \) are changed. In particular, \( J_R(E_{-\alpha_{ab}}) \) become scalars. Those corresponding to the Cartan subalgebra remain unchanged whilst those corresponding to positive root step operators are increased as
\[
[L(x), J(E_{\alpha}(y))] = (1 + h(\alpha))J(E_{\alpha}(y))\delta(x-y) - J^\alpha(y)\delta(x-y)
\] (24)

where \( h(\alpha) = 2\tilde{\delta}.\alpha \) is called the height of the root \( \alpha \). After the constraints \( \mathfrak{h} \) are imposed, the current \( J_R \) takes the form \( \text{[17]}. \) The constant operator
\[
I_+ = \sum_{(ab)} \mu^-_{(ab)} E_{\alpha_{ab}}
\] (25)
appearing on the r.h.s of (17) plays a crucial role in what follows. It belongs to a special $Sl(2)$ subalgebra $\mathcal{S}$ of $\mathcal{G}_0$
\begin{equation}
[T_3, I_{\pm}] = \pm I_{\pm}; \quad [I_+, I_-] = 2T_3 \tag{26}
\end{equation}
where
\begin{equation}
T_3 = 2\tilde{\delta}.H; \quad I_- = \sum_{(ab)} w_{(ab)} E_{-\alpha_{ab}} \tag{27}
\end{equation}
where $w_{(ab)}$ are determined by imposing (26). Since $T_3$ is an element of the Cartan subalgebra, the step operators of $\mathcal{G}$ are its eigenstates
\begin{equation}
[T_3, E_s] = h(s)E_s \tag{28}
\end{equation}
The simple roots $\alpha_{ab}$ of $\mathcal{G}_0$ have unit height. From the super Jacobi identity it follows that if $s$, $s'$ and $s + s'$ are roots then $h(s + s') = h(s) + h(s')$. Therefore the simple roots of $\mathcal{G}$ must satisfy $h(\alpha_a) + h(\alpha_b) = 1$ for any pair $\alpha_a, \alpha_b$. The adjoint representation of $\mathcal{G}$ can be decomposed into irreducible representations of $\mathcal{S}$. Since these are finite, it follows that the eigenvalues of $T_3$ must be integers or half integers. Therefore one concludes that
\begin{equation}
h(\alpha_a) = \frac{1}{2} \tag{29}
\end{equation}
for any simple root $\alpha_a$ of $\mathcal{G}$. The decomposition of the adjoint representation of $\mathcal{G}$ into multiplets of $\mathcal{S}$ is very useful in what follows. Since $\mathcal{S}$ contains only even generators its multiplets will be constituted of only even or only odd generators of $\mathcal{G}$. In fact the adjoint of $\mathcal{G}_0$ itself decomposes into multiplets of $\mathcal{S}$. The weights of $\mathcal{S}$ appearing in it are all integers (zero and the heights of the even roots). The generators of $\mathcal{G}_1$ define an even dimensional representation of $\mathcal{G}_0$. Indeed the odd roots of $\mathcal{G}$ are the weights of this representation. Such representation of $\mathcal{G}_0$ will also break into multiplets of $\mathcal{S}$. The weights of $\mathcal{S}$ appearing in it are all half integers. In an irreducible representation of $\mathcal{S}$ the eigenvalues of $T_3$ are not degenerate and so it follows that the Cartan subalgebra generators of $\mathcal{G}$ must belong to different multiplets. Since multiplets with integer spin necessarily contains the zero weight, the number of $\mathcal{S}$-multiplets in $\mathcal{G}_0$ is exactly the rank of $\mathcal{G}$. For the same reason the simple roots of $\mathcal{G}$ must belong to distinct multiplets. Since any positive (negative) odd root necessarily has height greater than $\frac{1}{2}$ (smaller than or equal to $-\frac{1}{2}$) they must belong to one of the multiplets where the simple root step operators are. Therefore the representation of $\mathcal{G}_0$ on $\mathcal{G}_1$ also decomposes into exactly rank $\mathcal{G} \mathcal{S}$-multiplets. In ref. [20] the adjoint representation of $\mathcal{G}$ was decomposed into super multiplets of a special $OSP(1,2)$ subalgebra.

We now discuss the choice of gauge fixing. We want the Poisson bracket of a constraint with its respective gauge fixing to be proportional to the currents $J_R(E_{-\alpha_{ab}})$ which are set to constants in (8). Therefore the gauge fixing of the constraints
\begin{equation}
\varphi_{(ab)} = J_R(E_{-\alpha_{ab}}) + \mu_{(ab)}^+(1 + \delta_{ab})K_{ab} \tag{30}
\end{equation}
can be taken to be the Cartan subalgebra generators $J_R(H_a)$. Notice that if $\mathcal{G}_0$ has $U(1)$ factors the number of even simple roots $\alpha_{ab}$ is smaller than the rank of $\mathcal{G}$. Therefore one does not have all the Cartan subalgebra generators as gauge fixing. The gauge fixing of the
constraint \( J_R(E_{-\alpha}) \), where \( \alpha \) is an even positive non simple root, can be taken to be \( J_R(E_{\beta}) \) where \( \beta \) is a positive even root such that \([E_{\beta}, E_{-\alpha}]\) is proportional to an even negative simple root step operator \( E_{-\alpha a b} \). This means that \( h(\alpha) - h(\beta) = 1 \). By doing this one can convince oneself that there will always be a remaining current (not used as gauge fixing) of height \( j \) whenever there is a \( S \)-multiplet with highest weight \( j \). Therefore the number of remaining currents with integer height is equal to the number of \( S \)-multiplets in the adjoint of \( \mathcal{G}_0 \) which is equal to rank \( \mathcal{G} \). This is in fact the result discussed in [10].

We can now apply the same procedure to choose the gauge fixing of the constraints associated to the odd roots. The gauge fixing of \( J_R(E_{-\alpha a}) \), \( a = 1, 2, \ldots \) can be taken to be themselves, since their Poisson brackets are proportional to \( J_R(E_{-\alpha a b}) \) which are set to constants in (8). The gauge fixing of the constraints \( J_R(E_{\sigma}) \), where \( \sigma \) is an odd positive non simple root, can be chosen to be \( J_R(E_{\sigma'}) \) where \( \sigma' \) is a positive odd root such that \([E_{\sigma'}, E_{-\alpha a b}]\) is proportional to a negative even simple root operator \( E_{-\alpha a b} \). Again \( h(\sigma) - h(\sigma') = 1 \). By the same reasoning as above one concludes that the number of remaining currents with half integer height is equal to the number of \( S \)-multiplets in \( \mathcal{G}_1 \) which is also equal to the rank of \( \mathcal{G} \).

Summarizing, we have shown that

1. The adjoint representation of \( \mathcal{G}_0 \) decomposes into rank \( \mathcal{G} \) multiplets of the subalgebra \( S \) (26).
2. The representation of \( \mathcal{G}_0 \) in \( \mathcal{G}_1 \) also decomposes into rank \( \mathcal{G} \) \( S \)-multiplets.
3. After the gauge fixing there is a one to one correspondence between remaining currents and \( S \)-multiplets in the adjoint of \( \mathcal{G} \).
4. The number of remaining currents with integer and half integer height is the same and equal to the rank of \( \mathcal{G} \).
5. The conformal spin, w.r.t. \( L(x) \), of the remaining current associated to the \( S \)-multiplet with highest weight \( j \) is \((j + 1)\).

We now show that the remaining currents can be written in terms of the Super-Toda fields and its derivatives only, showing that they are indeed the symmetries of the WZNW theory which survives the reduction procedure. From (14) and (20) one gets

\[
J_{R,\text{constr.}}^{\text{constr.}} = k \left( g^{-1} \partial_+ g \right)_{\text{constr.}} = \mathcal{M}^{-1} \left( I_+ + I_2 + \partial_+ \Phi \right) \mathcal{M} + \mathcal{M}^{-1} \partial_+ \mathcal{M} = I_+ + j_R \tag{31}
\]

where \( \partial_+ \Phi = \mathcal{A}^{-1} \partial_+ \mathcal{A} = \sum_{a=1}^{\text{rank} \mathcal{G}} \partial_+ \phi_a H_a \), \( I_+ \) is defined in (25), and

\[
I_2 \equiv \sum_{(ab)} \mu_{(ab)}^+ K_{ab} (\xi_b E_{\alpha a} + \xi_a E_{\alpha b}) \tag{32}
\]

The fields of the Super-Toda theory are the parameters \( \phi_a \) of the abelian subgroup \( \mathcal{A} \), \( \zeta_a \) and \( \xi_a \) (\( a = 1, 2, \ldots \) rank \( \mathcal{G} \)). The fields of the WZNW model which we want to eliminate from the remaining currents, and which appear in (31) are the parameters of the subgroup \( \mathcal{M} \) contained in \( \mathcal{F}_- \) (see (13)). We now show that after the gauge fixing described above.
the remaining currents depend only on the fields of the Super-Toda theory and therefore are symmetries of it.

The analysis is made simpler by grading the generators with eigenvalues of $T_3$ defined in (27). We write $\tilde{F}_- = \sum_s \tilde{F}_-^s$, and $(J_R)^{\text{constr.}} = I_+ + \sum_{s'} J_{R}^{-s'}$, where

$$[T_3, \tilde{F}_-^s] = -s \tilde{F}_-^s; \quad [T_3, J_{R}^{-s'}] = -s' J_{R}^{-s'}$$

We then have

$$J^1_R = I_+$$
$$J^\frac{1}{2}_R = 0$$
$$J^0_R = [I_+, \tilde{F}_-^{-1}] + \left[I_\frac{1}{2}, \xi_a E_{-\alpha_a}\right] + \partial_+ \Phi + \frac{1}{2}\left[\xi_a E_{-\alpha_a}, [\xi_b E_{-\alpha_b}, I_+]\right]$$
$$J^{-\frac{1}{2}}_R = [I_+, \tilde{F}_-^{-3/2}] + ...$$

and so on. Therefore

$$J^{-s}_R = [I_+, \tilde{F}_-^{-s-1}] + X^{-s}$$

where $X^{-s}$ involves $\tilde{F}_-^{-s'}$ for $s' \leq s + \frac{1}{2}$ only. We can then write the fields appearing in $\tilde{F}_-$ in terms of the fields of the Super-Toda theory by gauge fixing the currents recursively. For instance, we set to zero a number of components of $J^0_R$ equal to the dimension of the subspace generated by $[I_+, \tilde{F}_-^{-1}]$. That is equal to rank $G$ except for the cases where $G_0$ has $U(1)$ factors. We then have the fields in $\tilde{F}_-^{-1}$ written in terms of $\phi_a$ and $\xi_a$. Analogously we set to zero a number of components of $J^{-\frac{1}{2}}_R$ equal to the dimension of the subspace generated by $[I_+, \tilde{F}_-^{-3/2}]$ and eliminate the fields in $\tilde{F}_-^{-3/2}$. Therefore one observes that the number of remaining currents (not used as gauge fixing) of height $s$ is equal the number of components of $J^{-s}_R$ minus the dimension of the subspace spanned by $[I_+, \tilde{F}_-^{-s-1}]$. But this is exactly the number of multiplets of $S$ with highest weight $s$, showing that this is the same gauge fixing discussed above. Since at the end of the process the fields in $\tilde{F}_-$ will all be written in terms of the fields of the Super-Toda theory, so will all the remaining currents. These currents will be generators of symmetries of the Super-Toda theory. As we have shown before their number is twice the rank of $G$. Half of them have integer height and the other half have half integer height.

Notice that the remaining gauge symmetry, after the constraints (8) are imposed, is given by the subgroup generated by negative non simple roots. Indeed, the form of the constrained current (17) is left unchanged by the gauge transformation

$$J_R \rightarrow \tilde{M}^{-1}J_R \tilde{M} + \tilde{M}^{-1}\partial_+ \tilde{M}$$

where $\tilde{M}$ is an exponentiation of a real linear combination of the step operators corresponding to negative non simple roots. The negative simple root step operators can not be included because their super commutator with $E_{\alpha_{ab}}$ would produce positive simple root step operators.

8
4 Higher Spin Generators for the Super Toda Model

In this section we discuss in detail how the framework of section 3 can be applied to the cases where \( G \) contain all the simple roots of \( G \). We only deal with the fermionic sector since the decomposition of the adjoint representation of \( G_0 \) into multiplets of \( \text{Sl}(2) \) has been discussed by Balog et al [10]. The \( T_3 \) generator defined in eq.(27) is constructed in terms of the roots of \( G_0 \) and these in terms of a set of unit length vectors, \( \epsilon_i 's \) and \( \rho_a 's \), where \( \epsilon_i \cdot \epsilon_j = -\delta_{i,j} \), \( \rho_a \cdot \rho_b = \delta_{a,b} \). Let us consider case by case all possibilities.

1. \( G = OSP(2l+1, 2n) \), \( G_0 = SO(2l + 1) \otimes SP(2n) \), \( n, l = 1, 2, \ldots \)
The even and odd set of roots according to [21] are given respectively as

\[
\Delta_0 = \{ \pm \epsilon_i \pm \epsilon_j, \pm \epsilon_i, \pm \rho_a \pm \rho_b, \pm 2 \rho_a, \ i, j = 1, 2, \ldots; a, b = 1, 2, \ldots n \}
\]
\[
\Delta_1 = \{ \pm \rho_a, \pm \epsilon_i \pm \rho_a \}
\]

(37)

The height \( h(a) \) of a root \( a \) is defined in (28) and is determined in terms of \( \bar{\delta} = \frac{1}{2} \sum_{\alpha>0} \frac{\bar{\alpha}^2}{\alpha} \).

Let

\[
\bar{\delta}_{\text{SP}(2n)} = \frac{1}{4} \sum_{a<b} (\rho_a - \rho_b) + \frac{1}{4} \sum_{a<b} (\rho_a + \rho_b) + \frac{1}{4} \sum_{a=1}^n \rho_a = \frac{1}{2} \sum_{a=1}^n (n - a + \frac{1}{2}) \rho_a
\]

(38)

and

\[
\bar{\delta}_{\text{SO}(2l+1)} = -\frac{1}{4} \sum_{i<j} (\epsilon_i - \epsilon_j) - \frac{1}{4} \sum_{i<j} (\epsilon_i + \epsilon_j) - \frac{1}{2} \sum_i \epsilon_i = -\frac{1}{2} \sum_i (n - i + 1) \epsilon_i
\]

(39)

from where we derive

\[
h(\rho_a) = \bar{\delta}_{\text{SP}(2n)} \cdot \rho_a = n - a + \frac{1}{2}
\]

(40)

\[
h(\epsilon_i) = \bar{\delta}_{\text{SO}(2l+1)} \cdot \epsilon_i = l - i + 1
\]

(41)

The number \( N(\frac{2j+1}{2}) \) of odd roots of height \( \frac{2j+1}{2} \) can be evaluated for the two cases of interest, i.e.

1.a) \( l = n - 1, \) i.e. \( G = OSP(2n - 1, 2n) \)

\[
N(\frac{2j+1}{2}) = 2n - j - 1
\]

(42)

1.b) \( l = n, \) i.e. \( G = OSP(2n + 1, 2n) \)

\[
N(\frac{2j+1}{2}) = 2n - j
\]

(43)

For both cases, when \( j = 0 \), \( N(\frac{1}{2}) \) yields the number of \( \text{Sl}(2) \) multiplets \( \Delta_1 \) is decomposed. This agrees with the general argument of the previous section to be rank \( G \). Further, since there is a remaining current associated to the highest weight of each multiplet, the number of W-generators of conformal spin \( \frac{2j+3}{2} \) is therefore found to be equal to

\[
N(\frac{2j+1}{2}) - N(\frac{2j+3}{2}) = 1
\]

(44)
2.) $\mathcal{G} = OSP(2l,2n), \quad \mathcal{G}_0 = SO(2l) \otimes SP(2n)$

$$\Delta_0 = \{ \pm \epsilon_i \pm \epsilon_j, \pm \rho_a \pm \rho_b, \pm 2 \rho_a, \quad i, j = 1, 2, \ldots l, \quad a, b = 1, 2, \ldots n \}$$

$$\Delta_1 = \{ \pm \epsilon_i \pm \rho \}$$

Again the height of the roots of $OSP(2l,2n)$ is given by eq.(28) where

$$\tilde{\delta}_{SO(2l)} = -\frac{1}{2} \sum_{i<j} (\epsilon_i - \epsilon_j) - \frac{1}{2} \sum_{i<j} (\epsilon_i + \epsilon_j) = -\frac{1}{2} \sum_{i=1}^{l}(l-i)\epsilon_i$$

It then follows

$$h(\epsilon_i) = \tilde{\delta}_{SO(2l)} \cdot \epsilon_i = l - i$$

(45)

For the cases of interest, i.e.

2.a) $l = n$, i.e. $\mathcal{G} = OSP(2n,2n), n > 1$,

$$N\left(\frac{2j+1}{2}\right) = 2n - j$$

(47)

2.b) $l = n + 1$, i.e. $\mathcal{G} = OSP(2n+2,2n)$

$$N\left(\frac{2j+1}{2}\right) = 2n - j + 1$$

(48)

In all cases discussed so far there is a single $W$-generator of conformal spin $\frac{2j+3}{2}$.

3.) $\mathcal{G} = OSP(2,2), \quad \mathcal{G}_0 = SO(2) \otimes SP(2)$

$$\Delta_0 = \{ \pm 2\rho \}$$

$$\Delta_1 = \{ \pm \epsilon \pm \rho \}$$

The $T_3$ generator is given in terms of the fundamental weight of the bosonic subalgebra $\mathcal{G}_0$,

$$T_3 = \frac{1}{2}\rho.H$$

(49)

The fermionic roots in $\Delta_1$ decomposes therefore into two doublets, namely $\epsilon \pm \rho$ and $-\epsilon \pm \rho$ yielding two remaining currents of conformal weight $\frac{3}{2}$, i.e.

$$G_+ = J_{\epsilon + \rho}, \quad G_- = J_{-\epsilon + \rho}$$

(50)

This two generators of spin $\frac{3}{2}$ give rise to a $N = 2$ superconformal theory. This fact shall be explicitly shown in the next section using Dirac brackets.

4.) $\mathcal{G} = SU(n+1,n), \quad \mathcal{G}_0 = SU(n+1) \otimes SU(n) \otimes U(1)$

$$\Delta_0 = \{ \pm (\epsilon_i - \epsilon_j), \pm (\rho_a - \rho_b), \quad i, j = 1, 2, \ldots n+1, \quad a, b = 1, 2, \ldots n \}$$
The height of the fermionic roots in $\Delta_1$ is determined from the definition of $T_3$ in eq.(27) where

$$
\tilde{\delta}_{su}(l) = \frac{1}{2} \sum_{i<j} \frac{(\sigma_i - \sigma_j)}{|\sigma_i - \sigma_j|^2} = \frac{1}{4} \sum_{i=1}^{l+1} (l - 2i + 1) \frac{\sigma_i}{|\sigma_i|^2}
$$

where $\sigma_i$ stand for either $\epsilon_i$ or $\rho_a$. Therefore

$$
h(\epsilon_i - \rho_a) = a - i + \frac{1}{2}
$$

The number of odd roots of height $\frac{2i+1}{2}$ is then given by

$$
N\left(\frac{2j+1}{2}\right) = 2n - 2j
$$

Hence the number of W-generators of conformal spin $\frac{2j+3}{2}$ is then

$$
N\left(\frac{2j+1}{2}\right) - N\left(\frac{2j+3}{2}\right) = 2.
$$

This example shows that there are always 2 $W$-generators of conformal spin $\frac{2j+1}{2}$ and hence should be associated to a $N = 2$ superconformal theory.

5.) \( G = D(2,1|\alpha), \quad G_0 = SU(2) \otimes SU(2) \otimes SU(2), \alpha \neq 0, -1. \) The bosonic and fermionic roots are, respectively

$$
\Delta_0 = \{\pm 2\epsilon_i, \quad i = 1, 2, 3\}
$$

$$
\Delta_1 = \{\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}
$$

where $\epsilon_1^2 = -\frac{\alpha+1}{2}$, $\epsilon_2^2 = \frac{1}{2}$ and $\epsilon_3^2 = \frac{\alpha}{2}$. The $T_3$ generator is then defined to be

$$
T_3 = 2\tilde{\delta} \cdot H = \left( -\frac{\epsilon_1}{\alpha + 1} + \epsilon_2 + \frac{\epsilon_3}{\alpha} \right) \cdot H
$$

The 8 odd roots of $G$ therefore decomposes into 2 doublets and one quadruplet as follows

$$
h(\epsilon_1 + \epsilon_2 + \epsilon_3) = \frac{3}{2}
$$

$$
h(\epsilon_1 + \epsilon_2 - \epsilon_3) = h(\epsilon_1 - \epsilon_2 + \epsilon_3) = h(-\epsilon_1 + \epsilon_2 + \epsilon_3) = \frac{1}{2}
$$

Although there are two $W$-generators of conformal weight $\frac{3}{2}$ this model contains two non-commuting $N = 1$ superconformal systems, whose structure is displayed in Section 5.
5 Examples

In this section we discuss in detail how the structure described in the previous sections can be illustrated to construct Super W-algebras. In particular, this presents a construction of representations of N=1 and N=2 super-conformal algebras in terms of the super Toda fields. Our notation for the super Kac-Moody algebra is as follows,

\[
\begin{align*}
[J_{H_i}(x), J_{H_j}(y)] & = kSTr(H_i H_j) \delta'(x-y) \\
[J_{H_i}(x), J_{\pm \alpha}(y)] & = \pm \alpha \cdot \epsilon_i J_{\pm \alpha}(y) \delta(x-y) \\
[J_{\alpha}(x), J_{\beta}(y)] & = \\
& \begin{cases} 
\varepsilon(\alpha, \beta) J_{\alpha+\beta}(x) \delta(x-y) & \text{if } \alpha + \beta \text{ is a root} \\
B_\alpha \alpha_i J_{H_i} \delta(x-y) + k B_\alpha \delta'(x-y) & \text{if } \alpha + \beta = 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

where \( B_\alpha = STr(E_{\alpha} E_{-\alpha}) \), \( \alpha = \sum_{i=1}^{\text{rank} G} \alpha_i \epsilon_i \) and \( \epsilon_i \)'s constitute a basis for the root space. Much of the calculation in this section can be performed using a computer program for algebraic manipulation.

5.1 \( G = \text{OSP}(1,2) \) and the \( N = 1 \) Super Conformal Theory

Let us consider the super Lie algebra \( \text{OSP}(1,2) \) where the even and odd roots are, respectively \( \pm 2 \rho \) and \( \pm \rho \), \( (\rho^2 = 1) \). The Lie algebra is

\[
\begin{align*}
[H, E_{\pm 2\rho}] & = \pm 2E_{\pm \rho} ; & [H, E_{\pm \rho}] & = \pm E_{\pm \rho} \\
[E_{+2\rho}, E_{-2\rho}] & = 2B_2 H ; & \{ E_{+\rho}, E_{-\rho} \} & = B_{\rho} H ; \\
\{ E_{\pm \rho}, E_{\mp \rho} \} & = \varepsilon(\pm \rho, \mp \rho) E_{\pm 2\rho} ; & [E_{\pm 2\rho}, E_{\mp \rho}] & = \varepsilon(\pm 2\rho, \mp \rho) E_{\pm \rho} \\
\{ E_{\pm 2\rho}, E_{\mp \rho} \} & = 0
\end{align*}
\]

Since we are already dealing with a conformal invariant field theory we shall only consider one chirality, say \( J_R = J \). Following the Gauss decomposition,\([11]-[13]\), we shall describe as bosonic fields are \( \phi(x), f(x) \) and \( g(x) \) and the fermionic fields as \( \zeta(x) \) and \( \xi(x) \). From \([12]\) we write,

\[
\begin{align*}
\mathcal{F}_+ & = \zeta(x) E_\rho + f(x) E_{2\rho} ; & \mathcal{F}_- & = \xi(x) E_{-\rho} + g(x) E_{-2\rho}
\end{align*}
\]

and find the components of the current defined in \([4]\) to be (normalizing \( Str H^2 = 1 \),

\[
\frac{1}{k} J_{-2\rho} = B_{2\rho} (\partial_+ f - \frac{\varepsilon(\rho, \rho)}{2} \zeta \partial_+ \xi) e^{-2\phi}
\]
\[
\frac{1}{k} J_H = \frac{2}{k} g J_{-2\rho} - B_\rho \xi \partial_+ \zeta e^{-\phi} + \partial_+ \phi
\]

\[
\frac{1}{k} J_{2\rho} = B_{2\rho} \left( -\frac{2}{k} g^2 J_{-2\rho} + 2B_\rho g \xi \partial_+ \zeta e^{-\phi} - 2g \partial_+ \phi + \partial_+ g - \frac{1}{2} \varepsilon (\rho, \rho) \xi \partial_+ \xi \right)
\]

\[
\frac{1}{k} J_\rho = B_\rho \left( \frac{2}{k} g \xi J_{-2\rho} - \varepsilon (2\rho, -\rho) g \partial_+ \zeta e^{-\phi} + \xi \partial_+ \phi - \partial_+ \xi \right)
\]

\[
\frac{1}{k} J_{-\rho} = B_\rho \left( \frac{\varepsilon (2\rho, -\rho)}{kB_{2\rho}} \xi J_{-2\rho} + \partial_+ \zeta e^{-\phi} \right)
\]

All currents have conformal weight one, with respect to the Sugawara energy momentum tensor, \((\mathbf{6})\). However, with respect to the modified energy-momentum tensor,

\[
L(x) = T(x) + \partial_+ \partial^\perp \phi
\]

(62)

the current \(J_{-2\rho}\) becomes scalar and can be set to a constant \(\lambda\) without breaking the conformal symmetry. Also following \((\mathbf{8})\) the current associated to the negative fermionic root \(J_{-\rho}\) should be set to vanish. These two constraints imply further subsidiary conditions (gauge fixings) to be imposed. In order to evaluate the Dirac brackets of the remaining currents, we need to invert the matrix

\[
\Delta_{ij}(x, y) = \{ \psi_i(x), \psi_j(y) \}_{PB|_{\text{constrained}}} ; \ i, j = 1, 2, 3.
\]

(63)

where the constraints and gauge fixings are

\[
\psi_1 = J_{-2\rho} - \lambda ; \ \psi_2 = J_H ; \ \psi_3 = J_{-\rho}
\]

(64)

The Dirac bracket is defined as

\[
\{ A(x), B(x) \}_{DB} = \{ A(x), B(y) \}_{PB} - \{ A(x), \psi_i(z) \}_{PB} \Delta^{-1}_{ij}(z, z') \{ \psi_j(z'), B(y) \}
\]

(65)

where integrations over \(z\) and \(z'\) are implicit.

We therefore find that the algebra of the remaining currents \(J_{2\rho}\) and \(J_\rho\) under Dirac bracket yields the \(N = 1\) super conformal algebra if we define

\[
L(x) = \frac{\lambda}{kB_{2\rho}} J_{2\rho} ; \ G(x) = \left( \frac{2\lambda}{kB_{2\rho} \varepsilon (\rho, \rho)} \right)^{\frac{1}{2}} J_\rho
\]

(66)

In other words, the Dirac bracket yields

\[
[L(x), L(y)]_{DB} = 2L(y)\delta'(x - y) - L'(x)\delta(x - y) - k^4 x - y
\]

\[
[L(x), G(y)]_{DB} = \frac{3}{2} G(y)\delta'(x - y) - G'(x)\delta(x - y)
\]

(67)
\[ [G(x), G(y)]_{DB} = 2L(y)\delta(x - y) - k\delta''(x - y) \]

where we have used the following relations

\[ \varepsilon(2\rho, -\rho)B_\rho = \varepsilon(\rho, \rho)B_{2\rho} \] (68)

and

\[ 2B_\rho = -\varepsilon(2\rho, -\rho)^2. \]

which were obtained from the Super Jacobi identities and the symmetry properties of the brackets.

Moreover, solving the constraints and gauge fixings, \( \psi_i(x) = 0 \), \( i = 1, 2, 3 \) for \( L(x) \) and \( G(x) \), we find an explicit realization of the algebraic structure (67) in terms of the Super Toda (Super Liouville) fields, i.e.,

\[
\begin{align*}
L(x) &= \left\{ \frac{1}{2}(\partial_+ \phi)^2 - \frac{1}{2}\partial^2_+ \phi - \frac{1}{2}\lambda\varepsilon(\rho, \rho)\xi\partial_+ \xi \right\} \\
G(x) &= kB_\rho\sqrt{\frac{2\lambda}{kB_2\varepsilon(\rho, \rho)}}\left\{ \xi\partial_+ \phi - \partial_+ \xi \right\}.
\end{align*}
\]

Such realization can be done if the Dirac bracket is replaced by the canonical Poisson bracket derived from the super Toda action, \([15]\).

5.2 \( G = \text{OSP}(2,2) \) and the \( N = 2 \) Super Conformal Theory

In this case, the even and odd roots are respectively \( \pm 2\rho \) and \( \pm \rho \pm \epsilon \), \( (\rho^2 = 1, \ \epsilon^2 = -1) \). The Lie algebra is

\[ [H_\rho, E_{\pm \alpha}] = \pm \rho \cdot \alpha E_{\pm \alpha}; \quad [H_\epsilon, E_{\pm \alpha}] = \pm \epsilon \cdot \alpha E_{\pm \alpha}; \quad \text{for} \quad \alpha = 2\rho, \ (\rho + \epsilon), \ (\rho - \epsilon); \] (71)

\[ [H_\rho, H_\epsilon] = 0; \quad [E_{\pm 2\rho}, E_{-2\rho}] = 2B_{2\rho}H_\rho; \quad \{E_{+\rho+\epsilon}, E_{-\rho-\epsilon}\} = B_{\rho+\epsilon}(H_\rho + H_\epsilon) \]

\[ \{E_{+\rho-\epsilon}, E_{-\rho+\epsilon}\} = B_{\rho-\epsilon}(H_\rho - H_\epsilon); \quad [E_{\pm 2\rho}, E_{\mp(\rho+\epsilon)}] = \varepsilon(\pm 2\rho, \mp(\rho + \epsilon))E_{\pm(\rho-\epsilon)} \]

\[ [E_{\pm 2\rho}, E_{\mp(\rho-\epsilon)}] = \varepsilon(\pm 2\rho, \mp(\rho - \epsilon))E_{\pm(\rho+\epsilon)}; \quad \{E_{\pm(\rho+\epsilon)}, E_{\pm(\rho-\epsilon)}\} = \varepsilon(\pm(\rho + \epsilon), \pm(\rho - \epsilon))E_{\pm 2\rho} \]

\[ [E_{\pm 2\rho}, E_{\pm(\rho\pm\epsilon)}] = \{E_{\pm(\rho+\epsilon)}, E_{\mp(\rho-\epsilon)}\} = \{E_{\pm(\rho-\epsilon)}, E_{\pm(\rho+\epsilon)}\} = \{E_{\pm(\rho-\epsilon)}, E_{\pm(\rho-\epsilon)}\} = 0 \]

Following Eq. (12) we define the bosonic and fermionic fields as

\[ \mathcal{F}_+ = \zeta_1(x)E_{\rho+\epsilon} + \zeta_2(x)E_{-\rho-\epsilon} + f(x)E_{2\rho}; \quad \mathcal{F}_- = \xi_1(x)E_{-\rho-\epsilon} + \xi_2(x)E_{-\rho+\epsilon} + g(x)E_{-2\rho} \] (72)
and two extra bosonic fields, $\phi_1(x)$ and $\phi_2(x)$, associated to the Cartan subalgebra. The components of the current are the following, (with the normalization $Str H^2_\rho = - Str H^2_\epsilon = 1$),

$$\frac{1}{k} J_{-2\rho} = B_{2\rho} \left\{ \partial_+ f - \frac{\varepsilon (\rho + \epsilon, \rho - \epsilon)}{2} (\zeta_1 \partial_+ \zeta_2 + \zeta_2 \partial_+ \zeta_1) \right\} e^{-2\phi_1}$$

$$\frac{1}{k} J_{H_\rho} = \partial_+ \phi_1 + \frac{2}{k} g J_{-2\rho} - B_{\rho + \epsilon} \xi_1 \partial_+ \zeta_1 e^{-\phi_1 + \phi_2} - B_{\rho - \epsilon} \xi_2 \partial_+ \zeta_2 e^{-\phi_1 - \phi_2}$$

$$\frac{1}{k} J_{H_\epsilon} = -\partial_+ \phi_2 + \frac{\varepsilon (\rho + \epsilon, \rho - \epsilon)}{k} \xi_1 \xi_2 J_{-2\rho} + B_{\rho + \epsilon} \xi_1 \partial_+ \zeta_1 e^{-\phi_1 + \phi_2} - B_{\rho - \epsilon} \xi_2 \partial_+ \zeta_2 e^{-\phi_1 - \phi_2}$$

$$\frac{1}{k} J_{j_{\rho + \epsilon}} = B_{\rho + \epsilon} \left\{ \frac{2}{k} g \xi_1 J_{-2\rho} + \partial_+ g - \frac{\varepsilon (\rho + \epsilon, \rho - \epsilon)}{2} (\xi_1 \partial_+ \zeta_2 + \zeta_2 \partial_+ \zeta_1) - 2 g \partial_+ \phi_1 + B_{\rho + \epsilon} \xi_1 \partial_+ \zeta_1 e^{-\phi_1 + \phi_2} + B_{\rho - \epsilon} g \xi_2 \partial_+ \zeta_2 e^{-\phi_1 - \phi_2} + \varepsilon (\rho + \epsilon, \rho - \epsilon) \xi_1 \xi_2 \partial_+ \phi_2 \right\}$$

$$\frac{1}{k} J_{j_{\rho - \epsilon}} = B_{\rho - \epsilon} \left\{ \frac{2}{k} g \xi_2 J_{-2\rho} - (\varepsilon (2\rho, -\rho + \epsilon) g + B_{\rho - \epsilon} \xi_1 \xi_2) \partial_+ \zeta_2 e^{-\phi_1 - \phi_2} - \partial_+ \xi_1 + \xi_1 (\partial_+ \phi_1 - \partial_+ \phi_2) \right\}$$

$$\frac{1}{k} J_{-j_{\rho + \epsilon}} = B_{\rho + \epsilon} \left\{ \frac{\varepsilon (2\rho, -\rho + \epsilon)}{kB_{2\rho}} \xi_2 J_{-2\rho} + \partial_+ \zeta_1 e^{-\phi_1 + \phi_2} \right\}$$

$$\frac{1}{k} J_{-j_{\rho - \epsilon}} = B_{\rho - \epsilon} \left\{ \frac{\varepsilon (2\rho, -\rho - \epsilon)}{kB_{2\rho}} \xi_1 J_{-2\rho} + \partial_+ \zeta_2 e^{-\phi_1 - \phi_2} \right\}$$

The constraints together with their respective gauge fixings are the following,

$$\psi_1 = J_{-2\rho} - \lambda ; \; \psi_2 = J_{H_\rho} ; \; \psi_3 = J_{-j_{\rho + \epsilon}} ; \; \psi_4 = J_{-j_{\rho - \epsilon}}.$$  

We now take the remaining currents $J_{2\rho}$, $J_{H_\epsilon}$, $J_{\rho + \epsilon}$ and $J_{\rho - \epsilon}$ and make the following combination

$$L(x) = \frac{\lambda}{kB_{2\rho}} J_{2\rho} - \frac{1}{2k} J^2_{H_\epsilon} ; \; T(x) = J_{H_\epsilon}$$

$$G_+(x) = \sqrt{\frac{2\lambda}{kB_{2\rho}(\rho + \epsilon, \rho - \epsilon)}} J_{\rho - \epsilon} ; \; G_-(x) = \sqrt{\frac{2\lambda}{kB_{2\rho}(\rho + \epsilon, \rho - \epsilon)}} J_{\rho + \epsilon}$$

15
The Dirac brackets evaluated with these operators give us the interesting case of N=2 super conformal algebra, (see [22]) i.e.,

\[ [L(x), L(y)]_{DB} = 2L(y)\delta'(x - y) - L'(x)\delta(x - y) - \frac{k}{4}\delta''(x - y) \]

\[ [L(x), G_{\pm}(y)]_{DB} = \frac{3}{2}G_{\pm}(y)\delta'(x - y) - G'_{\pm}(x)\delta(x - y) \] (79)

\[ [L(x), T(y)]_{DB} = T(y)\delta'(x - y) - T'(x)\delta(x - y) \] (80)

\[ [T(x), T(y)]_{DB} = -k\delta'(x - y) \] (81)

\[ [T(x), G_{+}(y)]_{DB} = G_{+}(y)\delta(x - y) \] (82)

\[ [T(x), G_{-}(y)]_{DB} = -G_{-}(y)\delta(x - y) \] (83)

\[ [G_{-}(x), G_{+}(y)]_{DB} = 2L(y)\delta(x - y) + 2T(y)\delta'(x - y) - T'(x)\delta(x - y) - k\delta''(x - y) \] (84)

\[ [G_{+}(x), G_{+}(y)]_{DB} = [G_{-}(x), G_{-}(y)]_{DB} = 0 \] (85)

where we have used the following relations

\[ \varepsilon(2\rho, -\rho + \epsilon)B_{\rho+\epsilon} = \varepsilon(\rho + \epsilon, \rho - \epsilon)B_{2\rho} = \varepsilon(2\rho, -\rho - \epsilon)B_{\rho-\epsilon} \] (86)

and

\[ \varepsilon(2\rho, -\rho - \epsilon)\varepsilon(2\rho, -\rho + \epsilon) = -2B_{2\rho} \]

which were obtained from the Super Jacobi identities.

Again, solving the constraints and gauge fixings for the remaining currents we finally obtain the realization of the generators of the \( N = 2 \) super conformal algebra in terms of the Toda fields, given by

\[ L(x) = \left\{ \frac{1}{2}(\partial_+ \phi_1)^2 - \frac{1}{2}(\partial_+ \phi_2)^2 - \frac{1}{2}\varepsilon(\rho + \epsilon, \rho - \epsilon)\lambda(\xi_1 \partial_+ \xi_2 + \xi_2 \partial_+ \xi_1) \right\} \]

\[ T(x) = k \left\{ -\partial_+ \phi_2 - \varepsilon(\rho + \epsilon, \rho - \epsilon)\lambda\xi_1 \xi_2 \right\} \] (87)

\[ G_{-}(x) = k \sqrt{\frac{2\lambda}{k\varepsilon(\rho + \epsilon, \rho - \epsilon)B_{2\rho}}} \left\{ -\partial_+ \xi_1 + \xi_1(\partial_+ \phi_2 - \partial_+ \phi_1) \right\} \]

\[ G_{+}(x) = k \sqrt{\frac{2\lambda}{k\varepsilon(\rho + \epsilon, \rho - \epsilon)B_{\rho+\epsilon}}} \left\{ -\partial_+ \phi_2 + \xi_2(\partial_+ \phi_1 + \partial_+ \phi_2) \right\} \]

where the bracket in equations (79) is replaced by the Poisson brackets, derived from the action of the model, [15].
5.3 $G = \text{OSP}(3,2)$ and the Super W-Algebra

The novelty of this example is the existence of an operator of conformal spin $5/2$ leading to a super W-algebra as an extension of the $N = 1$ super conformal structure already discussed in detail in sections 5.1. As described in section 4, the even and the odd roots are respectively,

$$\Delta_0 = \{\pm \epsilon, \pm 2\rho\}$$

$$\Delta_1 = \{\pm \rho, \pm \rho \pm \epsilon\}. \quad (88)$$

The constraints and gauge fixings for this case are chosen to be,

$$\psi_1 = J_{-2\rho} - \lambda; \quad \psi_2 = J_{H_{\rho}}; \quad \psi_3 = J_{-\epsilon} - \mu; \quad \psi_4 = J_{H_{\epsilon}};$$

$$\psi_5 = J_{-\rho}; \quad \psi_6 = J_{\rho-\epsilon}; \quad \psi_7 = J_{-\rho-\epsilon}; \quad \psi_8 = J_{-\rho+\epsilon}. \quad (89)$$

It was also argued that there are two bosonic remaining currents of conformal spin 2 namely $J_{\epsilon}(x)$ and $J_{2\rho}(x)$ and two fermionic of conformal spin $3/2$ and $5/2$ corresponding to $J_{\rho}(x)$ and $J_{\rho+\epsilon}(x)$, respectively.

We now take the remaining currents $J_{2\rho}, J_{\epsilon}, J_{\rho}, J_{\rho+\epsilon}$ in the following combinations

$$L(x) = \frac{\lambda}{kB_{2\rho}} J_{2\rho} + \frac{\mu}{kB_{\rho}} J_{\epsilon} \quad (90)$$

$$R(x) = \frac{\lambda}{kB_{2\rho}} J_{2\rho} - \frac{\mu}{kB_{\rho}} J_{\epsilon} \quad (91)$$

$$G(x) = \sqrt{\left(\frac{2\lambda}{kB_{2\rho}\epsilon(\rho, \rho)}\right)} \, J_{\rho} \quad (92)$$

$$W_{2^m} = \frac{\mu}{2k\epsilon(\rho + \epsilon, -\epsilon)} \sqrt{\left(\frac{2\lambda}{kB_{2\rho}\epsilon(\rho, \rho)}\right)} \, J_{\rho+\epsilon} \quad (93)$$

and evaluate the Dirac brackets. The resulting algebra is given by

$$[L(x), L(y)]_{DB} = 2L(y)\delta'(x-y) - L'(x)\delta(x-y) + \frac{3k}{4}\delta''(x-y) \quad (94)$$

$$[L(x), G(y)]_{DB} = \frac{3}{2}G(y)\delta'(x-y) - G'(x)\delta(x-y) \quad (95)$$

$$\{G(x), G(y)\}_{DB} = 2L(y)\delta(x-y) + 3k\delta''(x-y) \quad (96)$$

Equations (94)-(96) provide the realization of $N = 1$ super conformal algebra. The second spin 2 and the spin $5/2$ currents transform, under the Virasoro generator, as

$$[L(x), R(y)]_{DB} = 2R(y)\delta'(x-y) - R'(x)\delta(x-y) - \frac{5k}{4}\delta''(x-y) \quad (97)$$
\[ [L(x), W_{\frac{3}{2}}(y)]_{DB} = \frac{5}{2}W_{\frac{3}{2}}(y)\delta'(x - y) - W'_{\frac{3}{2}}(y)\delta(x - y) - \frac{1}{2}G(y)\delta''(x - y) \] (98)

Notice that the spin 5/2 current \( W_{\frac{3}{2}}(x) \) does not transform as a primary field. However, the linear combination \( W_{\frac{3}{2}}(x) - \frac{4}{3}G'(x) \) does. The remaining Dirac brackets we obtain are

\[ [R(x), R(y)]_{DB} = 2L(y)\delta'(x - y) - L'(x)\delta(x - y) + \frac{3k}{4}\delta'''(x - y) \] (99)

\[ [R(x), G(y)]_{DB} = \frac{3}{2}G(y)\delta'(x - y) - G'(x)\delta(x - y) - 4W_{\frac{3}{2}}(x)\delta(x - y) \] (100)

\[ [R(x), W_{\frac{3}{2}}(y)]_{DB} = \frac{1}{2}W_{\frac{3}{2}}(y)\delta'(x - y) - W'_{\frac{3}{2}}(y)\delta(x - y) - \frac{\mu}{k}G(y)(L(y) + R(y))\delta(x - y) + \left(G(x) + \frac{1}{2}G(y)\right)\delta''(x - y) \] (101)

\[ \{G(x), W_{\frac{3}{2}}(y)\}_{DB} = (L(y) + R(y))\delta'(x - y) + \frac{1}{4}(L'(x) - R'(x))\delta(x - y) - k\delta'''(x - y) \] (102)

\[ \{W_{\frac{3}{2}}(x), W_{\frac{3}{2}}(y)\}_{DB} = -\frac{1}{2k} \left(L^2(x) + L(x)R(x)\right)\delta(x - y) + \frac{3}{8k}(L(y) + L(x))\delta''(x - y) + \frac{1}{8k}(R(y) + R(x))\delta'''(x - y) + \frac{1}{4k}G'(x)G(x)\delta(x - y) + \frac{1}{k}W_{\frac{3}{2}}(x)G(x)\delta(x - y) - \frac{k}{4}\delta'''(x - y). \] (103)

The relations from the Super Jacobi identities we have used are

\[ \varepsilon(2\rho, -\rho + \epsilon)B_{\rho+\epsilon} = \varepsilon(\rho + \epsilon, \rho - \epsilon)B_{2\rho} = \varepsilon(2\rho, -\rho - \epsilon)B_{\rho-\epsilon} \]

\[ \varepsilon(\rho + \epsilon, -\rho)B_{\epsilon} = -\varepsilon(\rho + \epsilon, -\epsilon)B_{\rho} = \varepsilon(\epsilon, \rho)B_{\rho+\epsilon} \]

\[ \varepsilon(\rho - \epsilon, -\rho)B_{\epsilon} = \varepsilon(-\rho, \epsilon)B_{\rho+\epsilon} = -\varepsilon(\rho - \epsilon, \epsilon)B_{\rho} \] (104)

\[ \varepsilon(2\rho, -\rho - \epsilon)\varepsilon(2\rho, -\rho + \epsilon) = -\varepsilon(2\rho, -\rho)^2 = -2B_{2\rho} \]

\[ \varepsilon(\rho, \rho)B_{2\rho} = \varepsilon(2\rho, -\rho)B_{\rho} ; \quad \varepsilon(\rho - \epsilon, -\rho)\varepsilon(\epsilon, -\rho) = B_{\rho}. \]

At this point we can use the Gauss decomposition to obtain the components of the current. Again, solving the appropriate constraints and gauge fixings for the remaining currents we find, in particular, another representation for the \( N = 1 \) super conformal algebra.
5.4 \( G = D(2, 1|\alpha) \) and the Super W-Algebra

This example presents three non commuting Virasoro generators, two fields of conformal spin \( 3/2 \) and one of spin \( 5/2 \). We show that they lead to two non commuting \( N = 1 \) super W-algebras.

The even and the odd roots are respectively,

\[
\Delta_0 = \{ \pm 2\epsilon_i \}, i = 1, 2, 3 \\
\Delta_1 = \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \} 
\]

The dimension of this group is 17. The simple (fermionic) roots are

\[
\alpha_1 = -\epsilon_1 + \epsilon_2 + \epsilon_3; \quad \alpha_2 = \epsilon_1 - \epsilon_2 + \epsilon_3; \quad \alpha_3 = \epsilon_1 + \epsilon_2 - \epsilon_3
\]

The other positive roots are

\[
\alpha_4 = \alpha_1 + \alpha_2 = 2\epsilon_3; \quad \alpha_5 = \alpha_1 + \alpha_3 = 2\epsilon_2; \\
\alpha_6 = \alpha_2 + \alpha_3 = 2\epsilon_1; \quad \alpha_7 = \alpha_1 + \alpha_2 + \alpha_3 = \epsilon_1 + \epsilon_2 + \epsilon_3
\]

The constraints and corresponding gauge fixings are

\[
\psi_1 = J_{-\alpha_1}; \quad \psi_2 = J_{-\alpha_2}; \quad \psi_3 = J_{-\alpha_3} \\
\psi_4 = J_{-\alpha_4} - \lambda_4; \quad \psi_5 = J_{H_4} \\
\psi_6 = J_{-\alpha_5} - \lambda_5; \quad \psi_7 = J_{H_5} \\
\psi_8 = J_{-\alpha_6} - \lambda_6; \quad \psi_9 = J_{H_6} \\
\psi_{10} = J_{-\alpha_7}; \quad \psi_{11} = J_{\alpha_3}
\]

After those are imposed we are left with six remaining currents which will generate the super W-algebra. We have chosen the following normalization for them

\[
L_1(x) = \frac{1}{k} \frac{\lambda_4}{B_{\alpha_4}} J_{\alpha_4}(x); \quad L_2(x) = \frac{1}{k} \frac{\lambda_5}{B_{\alpha_5}} J_{\alpha_5}(x); \quad L_3(x) = \frac{1}{k} \frac{\lambda_6}{B_{\alpha_6}} J_{\alpha_6}(x)
\]

\[
W^{(1)}_{\beta}(x) = \left( \alpha_1 \cdot \alpha_3 \right) \frac{\lambda_4 \lambda_5 \varepsilon(\alpha_1, \alpha_2)}{k \lambda_6} \frac{B_{\alpha_3}}{\varepsilon(\alpha_1, \alpha_3) \varepsilon(\alpha_2, \alpha_3) B_{\alpha_1} B_{\alpha_5}} J_{\alpha_1}(x) \\
W^{(2)}_{\beta}(x) = \left( \alpha_2 \cdot \alpha_3 \right) \frac{\lambda_4 \lambda_6 \varepsilon(\alpha_1, \alpha_2)}{k \lambda_5} \frac{B_{\alpha_3}}{\varepsilon(\alpha_1, \alpha_3) \varepsilon(\alpha_2, \alpha_3) B_{\alpha_2} B_{\alpha_6}} J_{\alpha_2}(x) \\
W_{\beta}(x) = \left( \alpha_1 \cdot \alpha_3 \right) \left( \alpha_2 \cdot \alpha_3 \right) \frac{\lambda_4 \lambda_5 \lambda_6}{k} \frac{\varepsilon(\alpha_1, \alpha_2)}{\varepsilon(\alpha_1, \alpha_3) \varepsilon(\alpha_2, \alpha_3) B_{\alpha_5} B_{\alpha_6} B_{\alpha_7}} J_{\alpha_1}(x)
\]

where \( B_{\alpha} \equiv \text{Str}(E_{\alpha} E_{-\alpha}) \).

The structure constants \( \varepsilon(\beta, \gamma) \) are determined from the root system and the super Jacobi identities. However there are some arbitrariness in the choice of the signs of such structure constants. We denote \( \varepsilon(\beta, \gamma) = \eta(\beta, \gamma) | \varepsilon(\beta, \gamma) | \), where \( \eta(\beta, \gamma) = \pm 1 \). We have chosen

\[
\eta(\alpha_1, \alpha_2) = \eta(\alpha_1, \alpha_3) = \eta(\alpha_2, \alpha_3) = \eta(\alpha_1, \alpha_6) = 1
\]
This choice completely fixes, through the super Jacobi identities, the signs of all other structure constants \( \varepsilon(\beta, \gamma) \).

We parametrize the scalar product on the root space as

\[
\epsilon_1^2 = -\frac{1}{2}(1 + \alpha); \quad \epsilon_2^2 = \frac{1}{2}; \quad \epsilon_3^2 = \frac{\alpha}{2}; \quad \epsilon_i \cdot \epsilon_j = 0, \quad i \neq j \tag{114}
\]

where \( \alpha \) is a parameter and \( \alpha \neq 0, -1 \). (From now on the \( \alpha \)'s appearing in the formulae are parameters and not roots).

The super W-algebra is given by the Dirac bracket of the remaining currents \( \{109, 112\} \). It has a very interesting structure as we now discuss. The currents \( L_i(x), i = 1, 2, 3 \), given in \( \{109\} \) generate three non commuting Virasoro algebras

\[
[L_i(x), L_j(y)]_{DB} = \delta_{i,j} \{2L_i(y)\delta'(x-y) - \partial_y L_i(y)\delta(x-y) - c_i\delta''(x-y)\}
\quad - \frac{1}{2k} \left( \sum_{k=1}^{3} \epsilon_{ijk} \right) W_{\frac{1}{2}}^{(1)}(y) W_{\frac{1}{2}}^{(2)}(y) \delta(x-y) \tag{115}
\]

where \( \epsilon_{ijk} \) is the completely antisymmetric symbol \( (\epsilon_{123} = 1) \), and the central terms are given by

\[
c_1 = \frac{k \text{Str}(H_3 H_3)}{2 \alpha}; \quad c_2 = \frac{k}{2} \text{Str}(H_2 H_2); \quad c_3 = -\frac{k \text{Str}(H_1 H_1)}{2 (1 + \alpha)} \tag{116}
\]

Under these Virasoro generators, the spin 3/2 currents transform as

\[
[L_2(x), W_{\frac{1}{2}}^{(1)}(y)]_{DB} = \frac{3}{2} W_{\frac{1}{2}}^{(1)}(y) \delta'(x-y) - \partial_y W_{\frac{1}{2}}^{(1)}(y) \delta(x-y) \tag{117}
\]

\[
[L_1(x) + L_3(x), W_{\frac{1}{2}}^{(1)}(y)]_{DB} = 0 \tag{118}
\]

\[
[L_1(x) - L_3(x), W_{\frac{1}{2}}^{(1)}(y)]_{DB} = W_{\frac{1}{2}}^{(1)}(y) \delta'(x-y) + \frac{2}{k^2} W_{\frac{1}{2}}^{(1)}(y) \delta(x-y)
\quad - 2 \left( W_{\frac{1}{2}}^{(2)}(y) \delta'(x-y) - \partial_y W_{\frac{1}{2}}^{(2)}(y) \delta(x-y) \right) \tag{119}
\]

and

\[
[L_3(x), W_{\frac{1}{2}}^{(2)}(y)]_{DB} = \frac{3}{2} W_{\frac{1}{2}}^{(2)}(y) \delta'(x-y) - \partial_y W_{\frac{1}{2}}^{(2)}(y) \delta(x-y) \tag{120}
\]

\[
[L_1(x) + L_2(x), W_{\frac{1}{2}}^{(2)}(y)]_{DB} = 0 \tag{121}
\]

\[
[L_1(x) - L_2(x), W_{\frac{1}{2}}^{(2)}(y)]_{DB} = W_{\frac{1}{2}}^{(2)}(y) \delta'(x-y) + \frac{2}{k^2} W_{\frac{1}{2}}^{(2)}(y) \delta(x-y)
\quad - 2 \left( W_{\frac{1}{2}}^{(1)}(y) \delta'(x-y) - \partial_y W_{\frac{1}{2}}^{(1)}(y) \delta(x-y) \right) \tag{122}
\]

The spin 5/2 current transforms as

\[
[L_1(x), W_{\frac{5}{2}}(y)]_{DB} = \frac{3}{2} W_{\frac{5}{2}}(y) \delta'(x-y) - \partial_y W_{\frac{5}{2}}(y) \delta(x-y)
\quad + \frac{1}{2} \{\alpha L_1(y) - L_2(y) - (1 + \alpha) L_3(y)\} W_{\frac{1}{2}}^{(1)}(y) \delta(x-y)
\]

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We also have

\[ \{ W^{(1)}_\frac{3}{2}, W^{(2)}_\frac{3}{2} \}_\text{DB} = \frac{1}{2} W^{(1)}_\frac{3}{2}(y) \delta'(x-y) - L_2(y) W^{(2)}_\frac{3}{2}(y) \delta(x-y) \]

\[ - \frac{k}{2} \left\{ \left( W^{(1)}_\frac{3}{2}(y) + W^{(2)}_\frac{3}{2}(y) \right) \delta''(x-y) \right\} \]

\[ - 2 \partial_y \left( W^{(1)}_\frac{3}{2}(y) + W^{(2)}_\frac{3}{2}(y) \right) \delta'(x-y) \]

\[ + \partial_y^2 \left( W^{(1)}_\frac{3}{2}(y) + W^{(2)}_\frac{3}{2}(y) \right) \delta(x-y) \] (123)

\[ [L_2(x), W^{(2)}_\frac{3}{2}(y)]_{\text{DB}} = \frac{1}{2} W^{(1)}_\frac{3}{2}(y) \delta'(x-y) - L_2(y) W^{(2)}_\frac{3}{2}(y) \delta(x-y) \]

\[ - \frac{1}{2} \left\{ \alpha L_1(y) - L_2(y) + (1 + \alpha) L_3(y) \right\} W^{(1)}_\frac{3}{2}(y) \delta(x-y) \]

\[ + \frac{k}{2} \left\{ W^{(1)}_\frac{3}{2}(y) \delta''(x-y) - 2 \partial_y W^{(1)}_\frac{3}{2}(y) \delta'(x-y) + \partial_y^2 W^{(1)}_\frac{3}{2}(y) \delta(x-y) \right\} \]

\[ + c_2 W^{(2)}_\frac{3}{2}(y) \delta''(x-y) \] (124)

\[ [L_3(x), W^{(2)}_\frac{3}{2}(y)]_{\text{DB}} = \frac{1}{2} W^{(1)}_\frac{3}{2}(y) \delta'(x-y) + (1 + \alpha) L_3(y) W^{(1)}_\frac{3}{2}(y) \delta(x-y) \]

\[ - \frac{1}{2} \left\{ \alpha L_1(y) - L_2(y) + (1 + \alpha) L_3(y) \right\} W^{(2)}_\frac{3}{2}(y) \delta(x-y) \]

\[ + \frac{k}{2} \left\{ W^{(2)}_\frac{3}{2}(y) \delta''(x-y) - 2 \partial_y W^{(2)}_\frac{3}{2}(y) \delta'(x-y) + \partial_y^2 W^{(2)}_\frac{3}{2}(y) \delta(x-y) \right\} \]

\[ - c_3 (1 + \alpha) W^{(1)}_\frac{3}{2}(y) \delta''(x-y) \] (125)

Notice that \( W^{(1)}_\frac{3}{2}(x) \) and \( W^{(2)}_\frac{3}{2}(x) \) are primary field (of spin 3/2) with respect to the diagonal Virasoro generator \( L_1(x) + L_2(x) + L_3(x) \). \( W^{(2)}_\frac{3}{2}(x) \), on the other hand, is not a primary field. This is a consequence of the gauge we are using. However one can check that

\[ V^{\frac{3}{2}}_L(x) \equiv W^{\frac{3}{2}}_L(x) + \frac{k}{3} \partial_x \left( STr(H_1H_1) W^{(1)}_\frac{3}{2}(x) + STr(H_2H_2) W^{(2)}_\frac{3}{2}(x) \right) \] (126)

is a primary field of spin 5/2 w.r.t. the diagonal Virasoro \( L_1(x) + L_2(x) + L_3(x) \).

The Dirac brackets involving spin 3/2 currents are

\[ \{ W^{(1)}_\frac{3}{2}(x), W^{(1)}_\frac{3}{2}(y) \}_\text{DB} = -2L_2(y) \delta(x-y) + 2k \delta''(x-y) \] (127)

\[ \{ W^{(2)}_\frac{3}{2}(x), W^{(2)}_\frac{3}{2}(y) \}_\text{DB} = 2(1 + \alpha) L_3(y) \delta(x-y) + 2k \delta''(x-y) \] (128)

\[ \{ W^{(1)}_\frac{3}{2}(x), W^{(2)}_\frac{3}{2}(y) \}_\text{DB} = \{ - \alpha L_1(y) + L_2(y) - (1 + \alpha) L_3(y) \} \]

\[ - \delta(x-y) - k \delta''(x-y) \] (129)

We also have

\[ \{ W^{(1)}_\frac{3}{2}(x), W^{(2)}_\frac{3}{2}(y) \}_\text{DB} = -\frac{1}{2} W^{(1)}_\frac{3}{2}(y) W^{(2)}_\frac{3}{2}(y) \delta(x-y) + k^2 \delta''(x-y) \]

\[ + k \{ (1 + \alpha) (\alpha L_1(y) - (1 + \alpha) L_3(y)) \delta'(x-y) \]
\[ \{W^{(2)}_{\frac{3}{2}}(x), W^{(2)}_{\frac{5}{2}}(y)\}_{DB} = \frac{1}{2}(1 + \alpha)W^{(1)}_{\frac{3}{2}}(y)W^{(2)}_{\frac{5}{2}}(y)\delta(x-y) + k^2 \delta''(x-y) \\
+ k \{(\alpha L_1(y) + L_2(y)) \delta'(x-y) \\
+ (1 + \alpha) (L_3(y)\delta'(x-y) - \partial_y L_3(y)\delta(x-y))\} \] (131)

and

\[ \{W_{\frac{5}{2}}(x), W_{\frac{5}{2}}(y)\}_{DB} = \left\{-(1 + \alpha)W^{(1)}_{\frac{3}{2}}(y) + W^{(2)}_{\frac{5}{2}}(y)\right\} W_{\frac{5}{2}}(y)\delta(x-y) \\
- \frac{k}{2} \{(\alpha L_1(y) - L_3(y) + 2(1 + \alpha)L_3(y))^2 \delta(x-y) \\
- 2k(1 + \alpha)L_2(y)\delta(x-y) \\
+ \frac{k^2}{2} \{(\alpha y^2) + L_1(y) - (L_2(x) + L_2(y)) \\
+ (1 + \alpha)(L_3(x) + L_3(y))\} \delta''(x-y) - \frac{k^3}{2} \delta'''(x-y) \] (132)

Notice, from (117) that \(W^{(1)}_{\frac{3}{2}}(x)\) is a spin 3/2 primary field w.r.t \(L_2(x)\). From (127) we see that the operators \(L_2(x)\) and \(W^{(1)}_{\frac{3}{2}}(x)\) generate a \(N = 1\) superconformal subalgebra of the above super \(W\)-algebra. Analogously, from (120) and (128) we see that \(L_3(x)\) and \(W^{(2)}_{\frac{5}{2}}(x)\) generate another \(N = 1\) superconformal subalgebra. However these two subalgebras do not commute.

6 Conclusions

In this paper we have achieved a systematic method of classifying and dealing with the symmetries of the supersymmetric Toda system. This was accomplished by exploiting the Hamiltonian reduction of the WZNW model associated to a super Lie algebra whose simple roots are all fermionic. This process leads to an improved energy-momentum tensor of the theory giving rise to higher spin generators. Their classification generalizes the one employed for the bosonic case (119). Each generator corresponds to a highest weight state of a representation of a special \(Sl(2)\) subalgebra of \(\mathcal{G}\) selected by the constraints.

The second main point we should highlight relies upon the Gauss decomposition formula which allows explicit constructions of the current components in terms of the WZNW fields. After imposing the constraints and gauge fixings, all degrees of freedom beyond the Super Toda fields are eliminated and the remaining currents, in particular, provides, in a systematic manner, representations of \(N = 1\) and \(N = 2\) super conformal algebras.

An interesting point we intend to pursue further concerns the construction of representations of super conformal algebras with higher supersymmetries whose algebraic structure was discussed in (22). This involves fermionic central extensions and may be related to other conformally invariant models beyond the Super Toda.
References

[1] A. B. Zamolodchikov, Theor. Math. Phys. 65 (1986) 1205; V. A. Fateev and A. B. Zamolodchikov, Nucl. Phys. B 280 (1987) 644

[2] A. Bilal, Phys. Lett. B 227 (1989) 406; A. Bilal and J. -L. Gervais, Phys. Lett. B 206 (1988) 412

[3] I. Bakas, Comm. Math. Phys. bf 134 (1990) 487

[4] C. Pope, L. Romans and X. Shen, Phys Lett. 326 (1990) 173, *ibid* B 245 (1990) 72

[5] A. Bilal, V. V. Fock and I. I. Kogan, preprint CERN-TH.5965/90

[6] J.-L. Gervais and Y. Matsuo, preprint LPTENS-91/29

[7] D. Olive and N. Turok, Nucl. Phys. [FS8] B 220 (1983) 491; *ibid* [FS14] B 257 (1985) 277

[8] O. Babelon and L. Bonora, Phys. Lett. B244 (1990) 220

[9] H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, Phys. Lett. B 254 (1991) 372

[10] J. Balog, L. Feher, P. Forgacs, L. O'Raifeartaigh and A. Wipf, Phys. Lett. B227 (1989) 214, *ibid* B 251 (1990) 361-368; Ann. Phys. 203 (1990) 76

[11] V. Drinfeld and V. Sokolov, J. Sov. Math. 30 (1984) 1975

[12] A. Schwimmer, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, preprint IFT-P.035/91, SISSA 147/91/EP, to appear in Phys. Lett. B

[13] H. Aratyn, C. P. Constantinidis, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, preprint IFT-P.023/91

[14] O. Babelon and L. Bonora, Phys. Lett. B 267 (1991) 71

[15] J. Evans and T. Hollowood, Nucl. Phys. B 352 (1991) 723

[16] H. Nohara and K. Mohri, Nucl. Phys. B 349 (1991) 253

[17] T. Inami, K. -I. Izawa, Phys. Lett. B 255 (1991), 521

[18] M. A. Olshanetsky, Commun. Math. Phys. 88 (1983) 63

[19] K. -I. Izawa, HE(TH) 91/04, preprint KUNS 1062, 1991

[20] F. Delduc, E. Ragoucy and P. Sorba, preprint ENSLAPP-L-352/91

[21] L. Frappat, A. Sciarrino and P. Sorba, Commun. Math. Phys. 121 (1989) 457

[22] A. Schwimmer and N. Seiberg, Phys. Lett. B 184 (1987), 191

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