Cartan geometries and multiplicative forms

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Abstract

In this paper we show that Cartan geometries can be studied via transitive Lie groupoids endowed with a special kind of vector-valued multiplicative 1-forms. This viewpoint leads us to a more general notion, that of Cartan bundle, which encompasses both Cartan geometries and $G$-structures.

1 Introduction

The history of Cartan geometries is well known and dates back to the XIX century, when mathematicians began a systematic study of non-Euclidean geometries. In this perspective, the idea of Felix Klein was to shift the attention from the geometric objects to their symmetries: the slogan of his so-called Erlangen program was that each "geometry" should be described by a specific group of transformations. Later, Élie Cartan took these geometries as standard models and used them to give rise to his espaces généralisés.

The research in this field then progressed on two paths. On the one hand, many authors used Cartan’s ideas to obtain important results on relevant examples, such as parabolic geometries (see [4]). On the other hand, people used Cartan’s approach to develop a general framework for studying these geometries; the standard modern reference is the famous book Differential Geometry: Cartan’s generalisation of Klein’s Erlangen program [18] by Richard Sharpe.

Our interest in these topics sparked from a different area in geometry. Recently, the concept of Pfaffian groupoid have been introduced [17] in order to understand the structure behind the jet groupoid of a Lie pseudogroup. Our original goal was to give an alternative description of the class of transitive Pfaffian groupoids, using the principal bundle canonically associated to any transitive groupoid. It has been quite an astonishing surprise to discover that, from the object we obtained, called a Cartan bundle, one could recover as a particular case the definition of a Cartan geometry. We believe that this perspective can shed more lights in this field; we are currently investigating further applications [1].

Let us give a few more details. A Cartan geometry is defined as a principal $H$-bundle $P$ together with an invariant vector valued form $\theta$, called a Cartan connection, satisfying certain properties. A Cartan bundle consists of a principal $H$-bundle $P$ with an invariant vector-valued 1-form $\theta$ with conditions less restrictive than those of a Cartan geometry. As mentioned above, there is a bijective correspondence between Cartan bundles and transitive Pfaffian groupoids with isotropy $H$. In particular, when $\ker(\theta) = 0$, one recovers the standard Cartan geometries, which correspond to a subclass of transitive Pfaffian groupoids.

Recall also that any reductive Cartan geometry defines a $H$-structure, i.e. a reduction of the structure group of the frame bundle of $M$, together with an (Ehresmann) connection on it. In the formalism of Cartan bundles $(P, \theta)$, another important particular case is when $\ker(\theta)$ coincide with the vertical bundle of $P$; this yields precisely the class of $H$-structures on $M$ without any choice of a connection.

Through this paper we will use, without recalling the basics, the theory of Lie groupoids, Lie algebroids and principal groupoid bundles. For an introduction on these topics we refer to [14, 16, 15, 8]. Multiplicative forms on Lie groupoids are also a standard notion (see [13] for a nice review) but somehow less known, especially in the case when the coefficients are not trivial. Since they constitute our main tool, we included an appendix with the definitions and the statements we use in the rest of the paper. Some of those result are not just technical lemmas but are original, and will appear in greater generality in the author’s PhD thesis [5].

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2 Cartan geometries

Let us recall the basic definitions and properties of Cartan geometries.

Definition 2.1. A Klein geometry is a pair \((G, H)\), where \(G\) is a Lie group and \(H \subseteq G\) a Lie subgroup such that the quotient manifold \(G/H\) is connected. A Klein geometry is called \textit{reductive} if there exists an \(H\)-module \(I\) which is a complement of \(\mathfrak{h} = \text{Lie}(H)\) in \(\mathfrak{g} = \text{Lie}(G)\), i.e. \(\mathfrak{g} = \mathfrak{h} \oplus I\). In particular, considering the adjoint representation on \(\mathfrak{h}, \mathfrak{g}\) can be seen as an \(H\)-module as well.

A Klein pair is a pair \((\mathfrak{g}, \mathfrak{h})\) of a Lie algebra \(\mathfrak{g}\) and a Lie subalgebra \(\mathfrak{h} \subseteq \mathfrak{g}\). A Klein pair is called \textit{reductive} if there exists an \(\mathfrak{h}\)-module \(I\) which is a complement of \(\mathfrak{h}\) in \(\mathfrak{g}\), i.e. \(\mathfrak{g} = \mathfrak{h} \oplus I\). In particular, considering the adjoint representation on \(\mathfrak{h}, \mathfrak{g}\) can be seen as an \(\mathfrak{h}\)-module as well.

A model geometry is a Klein pair \((\mathfrak{g}, \mathfrak{h})\) together with the choice of an integration \(H\) of \(\mathfrak{h}\) and of a representation \(H \to GL(\mathfrak{g})\) which extends the adjoint representation \(Ad : H \to GL(\mathfrak{h})\).

Clearly, any (reductive) Klein geometry induces a (reductive) Klein pair. Note also that, if \((G, H)\) is reductive, the induced Klein pair is automatically a model geometry.

Definition 2.2 (Definition 3.1 of [18, chapter 5]). Let \((\mathfrak{g}, \mathfrak{h})\) be a model geometry. A Cartan geometry \((P, \theta)\) modelled on \((\mathfrak{g}, \mathfrak{h})\) is a principal \(H\)-bundle \(P \to M\) together with a form \(\theta \in \Omega^1(P, \mathfrak{g})\), called a Cartan connection on \(P\), such that

- \(\theta\) is a pointwise isomorphism, i.e. \(\theta_p : T_pP \to \mathfrak{g}\) is a linear isomorphism for every \(p \in P\)
- \(\theta\) is \(H\)-equivariant, i.e. \((R_h)^*\theta = h^{-1} \cdot \theta\) for every \(h \in H\)
- \(\theta(X^R) = X\) for every \(X \in \mathfrak{h}\), with \(X^R \in \mathfrak{X}(H)\) the right-invariant vector field associated to \(X\), interpreted as a vector field on \(P\) via bundle trivialisations.

It follows by dimension counting that \(\dim(M) = \dim(\mathfrak{g}) - \dim(\mathfrak{h})\).

Example 2.3. Any model geometry is a Cartan geometry modelled on itself. It is enough to consider the principal \(H\)-bundle \(G \to G/H\); then the Maurer-Cartan form \(\omega_G \in \Omega^1(G, \mathfrak{g})\) satisfies the requirements. More general examples of Cartan geometries include Riemannian structures, affine structures, projective structures or conformal structures (see e.g. chapter 6-7-8 of [18] and chapter 4 of [4]).

Remark 2.4. In many results on Cartan geometries, it is often assumed the model geometry \((G, H)\) to be effective, i.e. that there are no proper subgroups of \(H\) which are normal in \(G\). Under this assumption, there is a correspondence between Cartan geometries and their “coordinate version”, namely Cartan atlases (see sections 5.2-5.3 of [18]). Without the effectiveness, one can only prove that a Cartan geometry induces a Cartan atlas, but not the converse (which requires some sort of “glueing” which makes forcibly use of effectiveness). Since we will not use the point of view of Cartan atlases in this paper, we do not ask any hypotheses of effectiveness.

Remark 2.5. Let \((P, \theta)\) be a Cartan geometry over \(M\) modelled on \((\mathfrak{g}, \mathfrak{h})\); then the tangent bundle of \(M\) is isomorphic to the vector bundle associated to \(P\) and the representation \(\mathfrak{g}/\mathfrak{h} \in \text{Rep}(H)\):

\[
T M \cong P[\mathfrak{g}/\mathfrak{h}] := (P \times \mathfrak{g}/\mathfrak{h})/H.
\]

This is a well known result (see e.g. Theorem 3.15 of [18, Chapter 5]), which will be relevant in the later sections. It follows by the fact that the tangent space \(T_xM\) at any point \(x = [p] \in M\) can be identified with the vector space \(\mathfrak{g}/\mathfrak{h}\). Note that such an identification depends on the choice of the representative; for each \(p \in P\) there is a canonical linear isomorphism \(\phi_p : T_xM \to \mathfrak{g}/\mathfrak{h}\), induced by the Cartan connection \(\theta_p : T_pP \to \mathfrak{g}\).

2.1 Cartan geometries and connections on \(G\)-structures

We review now the precise relation between Cartan geometries and another well known framework to study geometric structures: \(G\)-structures. The goal is to motivate the generalisation of Cartan geometries to Cartan bundles, introduced in the next section.

Let us first recall the basics of \(G\)-structures (see e.g. [7,19,12]). Let \(G \subseteq GL(n, \mathbb{R})\) be a Lie subgroup; a \(G\)-structure on an \(n\)-dimensional manifold \(M\) is a reduction of the structure group of the principal \(GL(n, \mathbb{R})\)-bundle of frames \(Fr(M) \to M\).
Definition 2.6. Let \( \pi : P \to M \) be a \( G \)-structure; its \textit{tautological form} \( \theta_{\text{taut}} \in \Omega^1(P, \mathbb{R}^n) \) is defined as

\[
(\theta_{\text{taut}})_p(v) = p^{-1}(d\pi(v)),
\]

where we interpret the frame \( p \in P \) as a linear isomorphism \( p : \mathbb{R}^n \to T_{\pi(p)}M \).

The form \( \theta_{\text{taut}} \) has many properties: among the most important ones, it is \( G \)-invariant, pointwise surjective, and satisfies \( \ker(\theta_{\text{taut}}) = \ker(d\pi) \). The following fundamental statement appeared first as Theorem 2 in [11], and is discussed also in Appendix A.2 of [18] and Section 1.3 of [4].

Proposition 2.7. Let \( H \subseteq GL(n, \mathbb{R}) \) be a Lie subgroup and \( M \) an \( n \)-dimensional manifold. Then there is a bijective correspondence

\[
\begin{align*}
\{ \text{(isomorphism classes of) Cartan geometries over } M \text{ modelled on } (H \ltimes \mathbb{R}^n, H) \} & \leftrightarrow \{ \text{(isomorphism classes of) } H \text{-structures over } M \text{ together with a compatible connection} \}.
\end{align*}
\]

The correspondence is given as follows. Given a Cartan geometry \((P, \theta)\) as above, note that it is automatically reductive. Indeed, the Lie algebra of \( G = H \ltimes \mathbb{R}^n \) splits as \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l} \), with \( \mathfrak{l} = \mathbb{R}^n \) and the standard \( H \)-action on \( \mathfrak{l} \) given by matrix multiplication. Accordingly, we can decompose the Cartan connection \( \theta \in \Omega^1(P, \mathfrak{g}) \) into \( \theta_h \in \Omega^1(P, \mathfrak{h}) \) and \( \theta_l \in \Omega^1(P, \mathfrak{l}) \). Then \( \theta_l \) is an Ehresmann connection on \( P \), while \( \theta_h \) can be interpreted as the tautological form of a \( G \)-structure as follows.

Fixing a basis \((e_1, \ldots, e_n)\) of \( \mathfrak{g}/\mathfrak{h} \), for any \( p \in P \) we can consider the linear isomorphism \( \phi_p : T_xM \to \mathfrak{g}/\mathfrak{h} \) from Remark 2.5, so that \( (\phi_p^{-1}(e_1), \ldots, \phi_p^{-1}(e_n)) \) is a basis of \( T_{\pi(p)}M \). Denoting by \( Q \subseteq Fr(M) \) the set of all frames of the form \( \phi_p^{-1}(e_1), \ldots, \phi_p^{-1}(e_n) \), for any \( p \in P \), one checks easily that \( Q \) is a \( H \)-structure and \( P \to Q \) an isomorphism of principal bundles. Then \( P \) can be seen as a \( H \)-structure, and \( \theta_l \) as its tautological form, identifying the vector space \( \mathfrak{l} \) with \( \mathfrak{g}/\mathfrak{h} \).

Conversely, given a \( H \)-structure \( P \subseteq Fr(M) \) and a connection \( \gamma \in \Omega^1(P, \mathfrak{h}) \), we define a Cartan connection on \( P \) as the sum \( \theta = \gamma + \theta_{\text{taut}} \in \Omega^1(P, \mathfrak{g}) \), where \( \theta_{\text{taut}} \in \Omega^1(P, \mathbb{R}^n) \) is the tautological form of \( P \).

Remark 2.8. From the correspondence above, one gets further relations between other relevant objects. For instance, to any Cartan geometry \((P, \theta)\) one associates its \textbf{curvature} via the classical Maurer-Cartan formula:

\[
\Omega := d\theta + \frac{1}{2} [\theta, \theta] \in \Omega^2(P, \mathfrak{g}),
\]

and its \textbf{torsion} by taking the component in \( \mathfrak{l} \):

\[
\Omega_l := d\theta_l + \frac{1}{2} [\theta_l, \theta_l]_l \in \Omega^2(P, \mathfrak{l}).
\]

Then one can easily write the precise relations between the torsion of the Cartan geometry \((P, \theta)\) and that of the connection \( \theta_h \); similarly for their respective curvatures (see e.g. Theorem 3 of [11]). Moreover,

- if the homogeneous space \( G/H \) is symmetric, i.e. \( \mathfrak{l} \cong \mathfrak{g}/\mathfrak{h} \) is a Lie algebra satisfying \( [\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{h} \), then the torsion of \((P, \theta)\) coincides with the torsion of the connection \( \theta_h \),

- if, furthermore, \([\mathfrak{l}, \mathfrak{l}] = 0\) (i.e. \( \mathfrak{l} \) is an abelian Lie algebra), then also the curvature of \((P, \theta)\) coincides with the curvature of the connection \( \theta_h \).

A Cartan geometry is called flat or torsion-free if, respectively, its curvature or its torsion vanishes. For instance, if \( H = GL(n, \mathbb{R}) \), one has the affine space \( A^n = H \ltimes \mathbb{R}^n \); a Cartan geometry modelled on \( (A^n, H) \) is an affine geometry. Since \( \mathbb{R}^n = A^n/H \) is an abelian Lie algebra, such a Cartan geometry is flat and torsion-free precisely when the corresponding connection on \( Fr(M) \) is flat and torsion-free, recovering the standard notion of \textit{affine structure} on a manifold.

The correspondence from Proposition 2.7 gives therefore a compact framework to investigate \( G \)-structures with connections, which is a topic extensively studied in the literature. However, much of the theory of \( G \)-structure can be carried out without the choice of a connection; this motivates the generalisation to Cartan bundles, described in the next sections.
3 Cartan geometries and Lie groupoids

In order to investigate Cartan geometries from the point of view of Lie groupoids, let us recall the following object.

**Definition 3.1.** Given a principal $G$-bundle $P \rightarrow M$, its **gauge groupoid** $\text{Gauge}(P)$ is the quotient of the product $P \times P$ with respect to the diagonal action of $G$, i.e

$$(P \times P)/G \equiv P/G \equiv M.$$  

The arrows $[p, q] \in \text{Gauge}(P)$ have source $[q]$ and target $[p]$, the multiplication is $[p, q][q, r] = [p, r]$, the unit $1_{[p]} = [p, p]$ and the inverse $[p, q]^{-1} = [q, p]$; its isotropy groups are all isomorphic to $G$. Moreover, $\text{Gauge}(P)$ is trivially transitive: for any two points $[q], [p] \in M$ there exists an arrow $[p, q] \in \text{Gauge}(P)$ sending one to the other. -Actually, gauge groupoids exhaust all transitive Lie groupoids:

**Proposition 3.2.** Given a transitive Lie groupoid $\mathcal{G} \rightarrow M$, fixing a point $x \in M$, the s-fibre $P = s^{-1}(x)$ is a principal bundle over $M$ with structure group the isotropy group $G = \mathcal{G}_x$, and

$$\text{Gauge}(P) \rightarrow \mathcal{G}, \quad [g, h] \mapsto g \cdot h^{-1}$$

becomes an isomorphism. This induces a bijective correspondence:

$$\begin{cases}
\text{(isomorphism classes of)} \\
\text{transitive Lie groupoids over } M
\end{cases} \leftrightarrow \begin{cases}
\text{(isomorphism classes of)} \\
\text{principal bundles over } M
\end{cases}.$$  

Our theorem restricts the correspondence above by considering on the right-hand side the class of Cartan geometries; in the proof, we will use results from the Appendix.

**Theorem 3.3.** Let $(P, \theta)$ be a Cartan geometry over $M$ modelled on $(g, h)$, with $\theta \in \Omega^1(P, g)$, and consider the gauge groupoid $\mathcal{G}$ associated to principal $H$-bundle $P \rightarrow M$ and the representation $E = P[g] \in \text{Rep}(\mathcal{G})$ induced from $g \in \text{Rep}(H)$. Then $\mathcal{G}$ is endowed with a form $\omega \in \Omega^1(\mathcal{G}, t^*E)$ such that

- $\omega$ is multiplicative (Definition A.2)
- $\omega$ is pointwise surjective
- $\text{ker}(ds) \cap \text{ker}(\omega) = 0$.

Conversely, any transitive Lie groupoid, endowed with such a form $\omega$, arises from a Cartan geometry.

**Proof.** We apply Proposition A.10 to the principal $H$-bundle $P$. Here we consider the Cartan connection $\theta$ on $P$ and the zero form $0 \in \Omega^1(H, g)$ on $H$; since $\theta$ is $H$-invariant, the $H$-action is multiplicative (Example A.5). Accordingly, the following differential form $\omega \in \Omega^1(\mathcal{G}, t^*E)$:

$$\omega_{[p, q]}([v, w]) = \theta_g(v) - [p, q]^{-1} \cdot \theta_g(w).$$

is well defined and multiplicative. Moreover, $\omega$ is pointwise surjective since $\theta$ is so. Last, applying Proposition A.11 we conclude that

$$\pi^*(\text{ker}(ds) \cap \text{ker}(\omega)) = TP \cap \text{ker}(\theta) = \text{ker}(\theta) = 0,$$

since $\theta$ is pointwise injective.

Conversely, given $(\mathcal{G}, \omega)$, with $\omega$ taking values in $E \in \text{Rep}(\mathcal{G})$, fix any $x \in M$. We are going to show that the principal $H$-bundle $P := s^{-1}(x) \rightarrow M$ is a Cartan geometry with the representation $g := E_x \in \text{Rep}(\mathcal{G}_x)$ and the differential form $\theta \in \Omega^1(P, g)$, $\theta_g(v) := g^{-1} \cdot \omega_g(v)$.

Indeed, from Lemma A.3 (based on the multiplicativity of $\omega$), it follows that $\theta$ is $H$-equivariant:

$$(\langle R_h \rangle^* \theta)_g(v) = \theta_{gh}(dR_h(v)) = (gh)^{-1} \cdot \omega_{gh}(dR_h(v)) = h^{-1} g^{-1} \cdot ((\langle R_h \rangle^* \omega)_g(v) = h^{-1} \cdot g^{-1} \cdot \omega_g(v) = h^{-1} \cdot \theta_g(v).$$

Moreover, by Proposition A.11 $\text{ker}(\theta)$ can be computed as

$$\text{ker}(\theta) = \pi^*(\text{ker}(\omega) \cap TP) = \pi^*(\text{ker}(\omega) \cap \text{ker}(ds)) = 0,$$

which proves that $\theta$ is pointwise injective. Since $\omega$ is pointwise surjective, $\theta$ is pointwise surjective as well, so $(P, \theta)$ is a Cartan geometry. Q.E.D.
The pair \((\mathcal{G}, \omega)\) we have just described is an instance of the following object:

**Definition 3.4.** A Pfaffian groupoid \((\mathcal{G}, \omega)\) over \(M\) consists of a Lie groupoid \(\mathcal{G} \rightrightarrows M\) together with a representation \(E \to M\) of \(\mathcal{G}\) and a differential form \(\omega \in \Omega^1(\mathcal{G}, t^*E)\) such that

1. \(\omega\) is multiplicative (Definition A.2)
2. \(\omega\) is of constant rank
3. The subbundle
   \[
   g(\omega) := (\ker(\omega) \cap \ker(ds))|_M \subseteq \text{Lie}(\mathcal{G})
   \]
   is a Lie subalgebroid of \(\text{Lie}(\mathcal{G})\).

We call \(g(\omega)\) the **symbol space** of \((\mathcal{G}, \omega)\). Moreover, a Pfaffian groupoid \((\mathcal{G}, \omega)\) is called

- **full** if the form \(\omega\) is pointwise surjective.
- **Lie-Pfaffian**, or of Lie type, if it satisfies the additional condition
  \[
  \ker(\omega) \cap \ker(dt) = \ker(\omega) \cap \ker(ds).
  \]

Theorem 3.3 can be rephrased as follows:

\[
\left\{ \begin{array}{c}
\text{(isomorphism classes of)} \\
\text{transitive full Pfaffian groupoids over } M
\end{array} \right\} 
\overset{\sim}{\longleftrightarrow} 
\left\{ \begin{array}{c}
\text{(isomorphism classes of)} \\
\text{Cartan geometries over } M
\end{array} \right\}.
\]

The notion of Pfaffian groupoid was first introduced in [17] in order to understand the structure behind the jet groupoids of Lie pseudogroups; see also [20] for a revisitation of Cartan’s original work on pseudogroups in this framework. Equivalently, a Pfaffian groupoid can be interpreted as the Lie theoretic version of a Pfaffian bundle, the notion encoding the essential properties of PDEs on jet bundles together with their Cartan forms (see [6]).

**Remark 3.5** (relations with previous works). Our approach on Cartan geometries fit in some recent reformulations using the language of Lie groupoids and algebroids. Blaom introduced in [2] the notion of Cartan algebroid, i.e. a Lie algebroid together with a compatible connection. When such algebroid is transitive, e.g. it is the Atiyah algebroid \(A = TP/H\) associated to a principal bundle \(P\), then it describes the infinitesimal counterpart of a Cartan connection \(\theta\) on \(P\). Since \(A\) is the Lie algebroid of the gauge groupoid \(\text{Gauge}(P)\), the way to recover his result from our formalism is via the correspondence between multiplicative 1-forms on Lie groupoids and Spencer operators on Lie algebroids described in [9].

Blaom described in [3] also the global counterpart of a transitive Cartan algebroid in term of distributions on the gauge groupoid of \(P\) which are compatible with the groupoid multiplication. In particular, our Theorem 3.3 resembles Blaom’s [3, theorem 1.1]: given our result, one can prove Blaom’s by considering the distribution \(\ker(\theta)\). We believe that our proof is more natural since it follows directly from the general properties of multiplicative forms on Lie groupoids.

Last, we also mention the recent book [10] by Crampin and Saunders. They proposed a revised approach to Cartan geometries, introducing a notion of infinitesimal Cartan connection on a Lie algebroid, which generalises further Blaom’s Cartan algebroids. However, little focus is given on the global counterpart of these objects.

**3.1 Cartan bundles**

Given the discussions in the previous section, we present now a generalisation of Cartan geometries which arises from transitive Lie-Pfaffian groupoid with non-trivial symbol.

**Definition 3.6.** A Cartan bundle \((P, \theta)\) is a principal \(H\)-bundle \(P \rightrightarrows M\), for \(H\) a Lie group, together with a representation \(V \in \text{Rep}(H)\) and a differential form \(\theta \in \Omega^1(P, V)\) such that

- \(\ker(\theta) \subseteq \ker(d\pi)\) and it is an involutive distribution
- \(\theta\) is \(H\)-equivariant, i.e. \((R_h)^*\theta = h^{-1} \cdot \theta\) for every \(h \in H\).
As anticipated, this general definition has the following two extreme cases, when \( \ker(\theta) \) is the largest or the smallest possible distribution.

**Example 3.7.** A Cartan geometry \((P, \theta)\) modelled on a reductive Klein pair \((g, h)\) is a Cartan bundle with \(V = g\) and the \(H\)-representation extending the adjoint representation of \(H\). In particular, the form \(\theta \in \Omega^1(P, g)\) satisfies \(\ker(\theta) = 0\), hence \(\ker(\theta)\) is trivially an involutive distribution inside the vertical bundle.

**Example 3.8.** Let \(H \subseteq GL(n, \mathbb{R})\) be a Lie subgroup; then a \(H\)-structure \(\pi : P \to M\) is a Cartan bundle with \(V = \mathbb{R}^n\) the natural representation of \(H \subseteq GL(n, \mathbb{R})\) and \(\theta\) the tautological form of \(P\). In particular, \(\ker(\theta) = \ker(d\pi)\), hence involutivity comes for free.

**Remark 3.9.** Recall from Proposition 2.7 that a Cartan geometry can be viewed as a \(H\)-principal bundle together with a given connection; in the framework of Cartan bundles, we have decoupled the \(H\)-structure from the connection. Note also that a principal bundle together with a connection \(\theta\) is of course not a Cartan bundle: it does not satisfy the condition \(\ker(\theta) \subseteq \ker(d\pi)\) and \(\ker(\theta)\) is involutive only if it the connection is flat.

As promised, Cartan bundles extend the correspondence from Theorem 3.3 to the more general case of transitive Lie-Pfaffian groupoids with any symbol:

**Theorem 3.10.** For any manifold \(M\) there is a 1-1 correspondence

\[
\left\{ \text{(isomorphism classes of)} \right. \quad \begin{array}{c}
\text{transitive Lie-Pfaffian groupoids on } M \\
\text{Cartan bundles on } M
\end{array}\left. \right\} \cong \left\{ \text{(isomorphism classes of)} \right. \quad \begin{array}{c}
\text{transitive Lie-Pfaffian groupoids on } M \\
\text{Cartan bundles on } M
\end{array}\left. \right\}.
\]

**Proof.** The proof goes like in Proposition 3.3. Let \((\mathcal{G}, \omega)\) be a transitive Lie-Pfaffian groupoid, with \(\omega\) taking values in \(E \in \text{Rep}(\mathcal{G})\), and fix any \(x \in M\). Then \(V := E_x\) is a representation of the isotropy group \(H := \mathcal{G}_x\) and the principal \(H\)-bundle \(P := s^{-1}(x) \searrow M\) is a Cartan bundle with the differential form

\[
\theta \in \Omega^1(P, V), \quad \theta_g(v) := g^{-1} \cdot \omega_g(v).
\]

Indeed, from Lemma A.3 (based on the multiplicativity of \(\omega\)), it follows that \(\theta\) is \(H\)-equivariant:

\[
((R_h)^*\theta)_g(v) = \theta_{gh}(dR_h(v)) = (gh)^{-1} \cdot \omega_{gh}(dR_h(v)) = h^{-1}g^{-1} \cdot ((R_h)^*\omega)_g(v) = h^{-1} \cdot g^{-1} \cdot \omega_g(v) = h^{-1} \cdot \theta_g(v).
\]

Moreover, by Proposition A.11, \(\ker(\theta)\) can be computed as

\[
\ker(\theta) = \pi^* (\ker(\omega) \cap \ker(ds)).
\]

Since \(\omega\) is \(s\)-involutive, \(\ker(\theta)\) is involutive, and since \((\mathcal{G}, \omega)\) is of Lie type, \(\ker(\theta)\) is contained in \(\ker(dt)\).

Conversely, consider a Cartan bundle \((P, \theta)\); the gauge groupoid \(\mathcal{G} := (P \times P)/H\) carries the representation \(E := P[V]\) and the following differential form \(\omega \in \Omega^1(\mathcal{G}, t^*E)\):

\[
\omega_{[p, q]}([v, w]) = \theta_p(v) - [p, q]^{-1} \cdot \theta_q(w).
\]

From the \(H\)-equivariance of \(\theta\), it follows that \(\omega\) is well defined and multiplicative by Proposition A.10. Moreover, consider the zero form \(\theta \in \Omega^0(H, V)\) on \(H\); since \(\theta\) is \(H\)-invariant, the \(H\)-action is multiplicative (Example A.5). Then we apply Proposition A.11 and we conclude that

\[
\pi^*(\ker(ds) \cap \ker(\omega)) = \ker(\theta).
\]

Since \(\ker(\theta)\) is involutive, \(\ker(ds) \cap \ker(\omega)\) is involutive as well. It follows from the definition of \(\omega\) that

\[
\ker(ds) \cap \ker(\omega) = [\ker(\theta), \text{Im}(a)] = [\text{Im}(a), \ker(\theta)] = \ker(dt) \cap \ker(\omega),
\]

hence \((\mathcal{G}, \omega)\) is of Lie type.

Q.E.D.
Proposition 3.11 (Representation associated to a Cartan bundle). Let $\mathcal{G} = \text{Gauge}(P)$ be the Pfaffian groupoid associated to a Cartan bundle $(P, \theta)$, and assume that $\theta$ is pointwise surjective. Then the fibre of the representation $E = P[V] \in \text{Rep}(\mathcal{G})$ splits as
\[ E_x = T_xM \oplus T_xH/\mathfrak{g}_1(x), \]
where $\mathfrak{g}(\omega)$ is the symbol space of $(\mathcal{G}, \omega)$ (Definition 3.4). Moreover, the linear $\mathcal{G}$-action on $E$ restricts to the following action on $T_M$:
\[ g \cdot v = d_g t(\alpha), \quad \forall g \in s^{-1}(x), v \in T_xM \quad (*) \]
where $\alpha$ is any element of $\ker(\omega_g)$ such that $v = d_g s(\alpha)$.

Proof. For any $x = \pi(p) \in M$ it is immediate to check that
\[ E_x = V = \text{Im}(\theta_p) \cong T_pP/\ker(\theta_p) \cong T_pP/\ker(d_p\pi) \oplus \ker(d_p\pi)/\ker(\theta_p) \cong \]
\[ \cong T_xM \oplus T_xP_x/\mathfrak{g}_1_x(\omega) \cong T_xM \oplus T_xH/\mathfrak{g}_1_x(\omega), \]
where we identified $l$ with $\g$. On the other hand, for a $\mathfrak{h}$-structure $(P, \theta)$ (Example 3.7), the symbol space $\mathfrak{g}(\omega)$ of the associated Pfaffian groupoid $(\mathcal{G}, \omega)$ is zero, so that the fibre of its representation is
\[ E_x = \mathfrak{h} \oplus T_xM. \]
This can also be seen directly: since the Klein pair $(\mathfrak{g}, \mathfrak{h})$ is reductive, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$, the representation $E = P[\mathfrak{l}]$ splits as
\[ E = P[\mathfrak{h}] \oplus P[\mathfrak{l}] \cong P[\mathfrak{h}] \oplus TM, \]
where we identified $\mathfrak{l}$ with $\g/\mathfrak{h}$ and used Remark 2.5.

Example 3.12. It is interesting to describe the splitting of the representation $E$ from Proposition 3.11 in the two particular cases we have examined. For a reductive Cartan geometry $(P, \theta)$ (Example 3.7), the symbol space $\mathfrak{g}(\omega)$ of the associated Pfaffian groupoid $(\mathcal{G}, \omega)$ is zero, so that the fibre of its representation is
\[ E_x = \mathfrak{h} \oplus T_xM. \]
This can also be seen directly: since the Klein pair $(\mathfrak{g}, \mathfrak{h})$ is reductive, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$, the representation $E = P[\mathfrak{l}]$ splits as
\[ E = P[\mathfrak{h}] \oplus P[\mathfrak{l}] \cong P[\mathfrak{h}] \oplus TM, \]
where we identified $\mathfrak{l}$ with $\g/\mathfrak{h}$ and used Remark 2.5.

On the other hand, for a $H$-structure $(P, \theta)$ (Example 3.8), the symbol space $\mathfrak{g}(\omega)$ of the associated Pfaffian groupoid $(\mathcal{G}, \omega)$ is trivial vector bundle with fibre the Lie algebra of $H$, so that the term $T_xH/\mathfrak{g}(\omega)_{1_x}$ disappears and
\[ E = TM. \]
A Appendix

In this appendix we collect some basic definitions and results on multiplicative forms on Lie groupoids, as well as Lie groupoid actions compatible with a multiplicative form. Some results are not standard, and constitute a particular case of general statements proved in the author’s PhD thesis [5].

A.1 Multiplicative forms

Definition A.1. Let $\mathcal{G}$ be a Lie groupoid; a differential form $\omega \in \Omega^k(\mathcal{G})$ is called multiplicative if

$$m^*\omega = \text{pr}_1^*\omega + \text{pr}_2^*\omega,$$

where $m : \mathcal{G} \times \mathcal{G} \subseteq \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is the multiplication of $\mathcal{G}$ and $\text{pr}_i : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ are the projections on the $i$-th component.

Multiplicative forms arise naturally in many geometric context, e.g. to study symplectic or contact structures on Lie groupoids. In this paper we consider forms with coefficients; to make sense of the multiplicativity condition, the coefficients must be the pullback bundle $t^*E$ of a representation $E$ of $\mathcal{G}$.

Definition A.2. Let $\mathcal{G}$ be a Lie groupoid and $E$ a representation of $\mathcal{G}$; a differential form $\omega \in \Omega^k(\mathcal{G}, t^*E)$ is called multiplicative if

$$(m^*\omega)_{(g,h)} = (\text{pr}_1^*\omega)_{(g,h)} + g \cdot (\text{pr}_2^*\omega)_{(g,h)} \quad \forall (g, h) \in \mathcal{G}_2.$$



To keep the notation simple, we will often write

$$m^*\omega = \text{pr}_1^*\omega + g \cdot \text{pr}_2^*\omega.$$

Here is a simple but fundamental property of multiplicative 1-forms.

Lemma A.3. Let $\mathcal{G}$ be a Lie groupoid, $E$ a representation and $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative form. Then, for every $g \in \mathcal{G}$ from $x$ to $y$: 

- $(L_g)^*(\omega|_{t^{-1}(y)}) = g \cdot \omega|_{t^{-1}(x)}$,
- $(R_g)^*(\omega|_{s^{-1}(x)}) = \omega|_{s^{-1}(g)}$.

Proof. For any $(g, h) \in \mathcal{G}_2$ and $Y \in T_h(t^{-1}(s(g)))$, we have

$$d_h L_g(Y) = d_h m(g, \cdot)(Y) = d_h m(t^{-1}(s(g)))(0, Y),$$

where the last equality comes from a straightforward computation using tangent curves. Therefore, using the multiplicativity of $\omega$, we obtain

$$((L_g)^*\omega)_h(Y) = \omega_{g \cdot h}(d_h L_g(Y)) =$$

$$= \omega_{m(g,h)}(d_{(g,h)}m_{t^{-1}(s(g))})(0, Y) = \omega_{m(g,0)} + g \cdot \omega_h(Y).$$

With the same argument, for any $(h, g) \in \mathcal{G}_2$ and $X \in T_g(s^{-1}(t(h)))$ we have

$$d_h R_g(X) = d_h m(\cdot, g)(X) = d_{(h,g)}m_{t^{-1}(t(g)) \times s^{-1}(g)}(X, 0),$$

and we conclude that

$$((R_g)^*\omega)_h(X) = \omega_h(X) + h \cdot \omega_f(0).$$

Q.E.D.

A.2 Multiplicative groupoid actions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid acting (on the left) on the manifold $P$ along the map $\mu : P \rightarrow M$; denote by $m_P$ the action map, defined on the fibred product

$$\mathcal{G} \times_{\mu} P := \{(g, p) \in \mathcal{G} \times P \mid s(g) = \mu(p)\}.$$

Moreover, let $E$ be a representation of $\mathcal{G}$, $\alpha \in \Omega^k(\mathcal{G}, t^*E)$ a multiplicative form and $\beta \in \Omega^k(P, \mu^*E)$ a differential form; we represent this setting in the following diagram:
Definition A.4. The $\mathcal{G}$-action on $P$ is called multiplicative (with respect to $\alpha$ and $\beta$) if

$$ (m^*_P \beta)_{(g, p)} = (\text{pr}_1^* \alpha)_{(g, p)} + g \cdot (\text{pr}_2^* \beta)_{(g, p)} \quad \forall (g, p) \in \mathcal{G} \times_{\mu} P. $$

As for multiplicative forms, we will often denote this as

$$ m^*_P \beta = \text{pr}_1^* \alpha + g \cdot \text{pr}_2^* \beta. $$

Multiplicative right actions are defined analogously, with the condition

$$ g \cdot (m^*_P \beta)_{(p, g)} = (\text{pr}_1^* \alpha)_{(p, g)} + (\text{pr}_2^* \beta)_{(p, g)} \quad \forall (p, g) \in \mu \times \mathcal{G}. $$

Example A.5. Let $\theta \in \Omega^1(P, \mu^* E)$ be a 1-form on a principal $H$-bundle $P$. Then the $H$-action is multiplicative w.r.t. the form $\theta$ and the zero form $0 \in \Omega^1(H, V)$ if and only if $\theta$ is $H$-invariant, i.e.

$$ R^*_h \theta = \theta \quad \forall h \in H. $$

Proposition A.6. Let $\mathcal{G}$ be a Lie groupoid, $\omega \in \Omega^1(\mathcal{G}, t^* E)$ a multiplicative form with coefficients in a representation $E$ of $\mathcal{G}$. Assume moreover that $\mathcal{G}$ acts on $\mu : P \to M$, let $\theta \in \Omega^1(P, \mu^* E)$ and consider the infinitesimal action

$$ a : \Gamma(A) \to \mathfrak{X}(P), \quad a(\alpha)_p := d_{\mu(p)} m_P (\cdot, p)(\alpha_{\mu(p)}). $$

If the action of $(\mathcal{G}, \omega)$ on $(P, \theta)$ is multiplicative, then

$$ \theta(a(\alpha)) = \omega(\alpha). $$

Proof. It follows directly from the multiplicativity of the $(\mathcal{G}, \omega)$-action $m_P$ on $(P, \theta)$:

$$ \theta_p (a_\omega(\alpha_x)) = \theta_p (d_{(1_x, p)} m_P (\alpha_x, 0)) = (m^*_P \theta)_{(1_x, p)} (\alpha_x, 0) = \omega_1 (\alpha_x) + \theta_p (\omega). $$

Q.E.D.

A.3 Principal multiplicative groupoid actions

When a multiplicative action of a Lie groupoid $\mathcal{G}$ on $P$ is also principal, the corresponding gauge groupoid $\text{Gauge}(P)$ carries a multiplicative form and its action on $P$ is multiplicative as well. In order to prove this, we first recall a couple of results on principal Lie groupoid actions.

Lemma A.7. For any principal $\mathcal{G}$-bundle $P \to M$, the $\pi$-vertical bundle of $P$ coincides with the image of the infinitesimal action $a : \mu^* A \to TP$:

$$ \ker(d\pi) = \text{Im}(a). $$

Definition A.8. A basic form on a (left) principal $\mathcal{G}$-bundle $\pi : P \to M$ with coefficients in a representation $E$ of $\mathcal{G}$ is a differential form $\theta \in \Omega^k(P, \mu^* E)$ such that

$$ g \cdot (\text{pr}_2^* \theta)_{(g, p)} = (m^*_P \theta)_{(g, p)} \quad \forall (g, p) \in \mathcal{G} \times_{\mu} P. $$

There is another characterisation of basic forms, reminiscent of the one for Lie group actions, which can be used to prove the following result:

Proposition A.9 (Proposition 8.8.5 of [20]). Let $\mathcal{G} \rightarrow X$ be a Lie groupoid, $E \in \text{Rep}(\mathcal{G})$ a representation, $\pi : P \to M$ a principal $\mathcal{G}$-bundle and $P[E] := (P \times_X E)/\mathcal{G}$ the associated vector bundle over $M$. Then the pullback from $M$ to $P$ induces an isomorphism

$$ \Omega^k(M, P[E]) \xrightarrow{\cong} \Omega^k_{\text{bas}}(P, \pi^* E), \quad \omega \mapsto \pi^* \omega. $$

And here is the promised result.
Proposition A.10. Let \( \mathcal{G} \) be a Lie groupoid and \( \omega \in \Omega^1(\mathcal{G}, \ast E) \) a multiplicative form with coefficients in a representation \( E \) of \( \mathcal{G} \). Let also \( P \) be a left principal \( \mathcal{G} \)-bundle over \( X \), whose moment map \( \mu \) is a submersion, and \( \theta \in \Omega^1(P, \mu^*E) \) a differential form such that the \( \mathcal{G} \)-action is multiplicative.

Then \( \text{Gauge}(P) \) carries a unique multiplicative form \( \tilde{\omega} \) such that

\[ \tau^* \tilde{\omega} = pr_1^* \theta - pr_2^* \theta, \]

for \( \tau \) the projection \( P \times_{\mu} P \to P \times_{\mu} P / \mathcal{G} \). Moreover, the action of \( \text{Gauge}(P), \tilde{\omega} \) on \( (P, \theta) \) is multiplicative.

Proof. Let us represent on the following diagram the spaces and the maps we are going to use.

The proof is carried out in four steps:

1. The form \( \tilde{\theta} = s^* \theta - \tilde{t}^* \theta \in \Omega^1(P \times_{\mu} P, \tilde{\mu}^*E) \) is basic.

2. There is a unique form \( \tilde{\omega} \in \Omega^k(\text{Gauge}(P), P[E]) \) such that \( \tau^* \tilde{\omega} = \tilde{\theta} \).

3. \( \tilde{\omega} \) is multiplicative.

4. The action of \( \text{Gauge}(P) \) on \( P \) is multiplicative.

First part: we denote by \( pr \) the projections from \( \mathcal{G} \times_\mu P \) on the first and second component, and by \( \tilde{pr} \) the projections from \( \mathcal{G} \times_{\tilde{\mu}} (P \times_{\mu} P) \) to either one of the three components or two of them.

Note that the vector bundle \( \mu^*E \to P \) is a trivial representation of the groupoid \( P \times_{\mu} P \): therefore both \( \tilde{s}^* \theta \) and \( \tilde{t}^* \theta \) belong to the same fibre and the \( g \cdot \) of Definition A.2 becomes redundant, so we omit it. Using the multiplicativity of \( m_P \) we find

\[
(m_P)^* \tilde{\theta} = (m_P)^* (s^* \theta - \tilde{t}^* \theta) = (s \circ m_P)^* \theta - (\tilde{t} \circ m_P)^* \theta =
\]

\[
= (m_P \circ \tilde{pr}_{13})^* \theta - (m_P \circ \tilde{pr}_{12})^* \theta = \tilde{pr}_{13}^* (m_P^* \theta) - \tilde{pr}_{12}^* (m_P^* \theta) =
\]

\[
= \tilde{pr}_{13}^* (pr_{13} \omega) + \tilde{pr}_{13}^* (g \cdot pr_2^* \theta) - \tilde{pr}_{12}^* (pr_{12} \omega) - \tilde{pr}_{12}^* (g \cdot pr_2^* \theta) =
\]

\[
= \tilde{pr}_{13}^* \omega + g \cdot \tilde{pr}_2^* \theta - \tilde{pr}_{12}^* \omega - g \cdot \tilde{pr}_2^* \theta =
\]

\[
= g \cdot ((s \circ \tilde{pr}_{23})^* \theta - (\tilde{t} \circ \tilde{pr}_{23})^* \theta) = g \cdot \tilde{pr}_2^* (s^* \theta - \tilde{t}^* \theta) = g \cdot \tilde{pr}_2^* \tilde{\theta}.
\]

By Definition A.8 we conclude that \( \tilde{\theta} \) is basic.

Second part: it is immediate to check that \( \tilde{\theta} = s^* \theta - \tilde{t}^* \theta \). Then the claim follows from Proposition A.9. Indeed, since

\[ \Omega^1(\text{Gauge}(P), P[E]) \to \Omega^1_{bas}(P \times_{\mu} P, \tilde{\mu}^*E), \quad \omega \mapsto \tau^* \omega \]

is an isomorphism, and \( \tilde{\theta} \in \Omega^1_{bas}(P \times_{\mu} P, \tilde{\mu}^*E) \), then there exists a unique form \( \tilde{\omega} \in \Omega^k(\text{Gauge}(P), P[E]) \) such that \( \tau^* \tilde{\omega} = \tilde{\theta} \).

Third part: denote by \( \tilde{m}, \tilde{pr}_1 \) and \( \tilde{pr}_2 \) the maps

\[
(P \times_{\mu} P) \times \tilde{z} (P \times_{\mu} P) \to P \times_{\mu} P
\]
therefore
\[ A.6 \]
a is an isomorphism. Therefore, we have only to show that \([\tilde{m}]\) and \([\tilde{pr}]\) the projections of those maps to the quotient \((P \times \mu P)/\Im\). With the usual arguments we get
\[
(\tau \times \tau)^*([\tilde{m}]^*\omega) = ([\tilde{m}] \circ (\tau \times \tau))^*\omega = (\tau \circ \tilde{m})^*\omega = \tilde{m}^*\tau^*\omega = \\
\tilde{pr}_1^*\tilde{\tilde{\theta}} + \tilde{pr}_2^*\tilde{\tilde{\theta}} = \tilde{pr}_1^*(\tau^*\omega) + \tilde{pr}_2^*(\tau^*\omega) = (\tau \circ \tilde{pr}_1)^*\omega + (\tau \circ \tilde{pr}_2)^*\omega = \\
= (\tilde{pr}_1 \circ (\tau \times \tau))^*\omega + (\tilde{pr}_2 \circ (\tau \times \tau))^*\omega = (\tau \times \tau)^*([\tilde{pr}_1]^*\omega + [\tilde{pr}_2]^*\omega).
\]
By the injectivity of the pullback we get the multiplicativity of \(\tilde{\omega}\).

Fourth part: we see first that the action of the fibred pair groupoid \(P \times \mu P\) on
\[
m_P : P_{id \times 1} (P \times \mu P) \to P, \quad (p, (p, q)) \mapsto q
\]
is multiplicative with respect to \(\theta\) and \(\tilde{\theta}\):
\[
(\tilde{pr}_1)^*\theta + (\tilde{pr}_2)^*\theta = ([\tilde{pr}_1]^*\theta + (\tilde{s} \circ \tilde{pr}_2)^*\theta) = \tilde{pr}_1^*\theta + \tilde{pr}_2^*\theta = \\
(\tilde{pr}_1 \circ (idP, \tau))^*\theta + (\tau \circ \tilde{pr}_2)^*\omega = ([\tilde{pr}_1] \circ (idP, \tau))^*\theta + ([\tilde{pr}_2] \circ (idP, \tau))^*\omega = \\
= (idP, \tau)^*([\tilde{pr}_1]^*\theta + [\tilde{pr}_2]^*\omega).
\]
Then, when passing to the action \([\tilde{m}_P]\) from the quotient \((P \times \mu P)/\Im\), the multiplicativity condition is preserved:
\[
(idP, \tau)^*([\tilde{m}]^*\theta) = ([\tilde{m}] \circ (idP, \tau))^*\theta = \tilde{m}^*\theta = \tilde{pr}_1^*\theta - \tilde{pr}_2^*\theta = \\
= ([\tilde{pr}_1] \circ (idP, \tau))^*\theta + (\tau \circ \tilde{pr}_2)^*\omega = ([\tilde{pr}_1] \circ (idP, \tau))^*\theta + ([\tilde{pr}_2] \circ (idP, \tau))^*\omega = \\
= (idP, \tau)^*([\tilde{pr}_1]^*\theta + [\tilde{pr}_2]^*\omega).
\]
Again, by the injectivity of the pullback we get the multiplicativity of the \(\text{Gauge}(P)\)-action on \(P\). Q.E.D.

Last, consider a Pfaffian groupoid \((\mathfrak{g}, \omega)\) (Definition 3.4) which acts multiplicatively on a manifold \(P\). If the action is principal, one has the following characterisation of the symbol space of \(\mathfrak{g}\):

**Proposition A.11.** Let \(\mathfrak{g}\) be a Lie groupoid over \(X, E\) a representation and \(\omega \in \Omega^1(\mathfrak{g}, t^*E)\) a multiplicative differential form. Moreover, let \(\pi : P \to M\) be a principal \(\mathfrak{g}\)-bundle and \(\theta \in \Omega^1(P, \mu^*E)\) a differential form such that the principal \(\mathfrak{g}\)-action is multiplicative w.r.t. \(\theta\) and \(\omega\).

Then the \(\mu\)-pullback of the symbol space \(\mathfrak{g}(\omega)\) (Definition 3.4) of the Pfaffian groupoid \((\mathfrak{g}, \omega)\) is isomorphic to the space \(\mathfrak{g}_\omega(\theta) := \ker(d\theta) \cap \ker(\theta)\):
\[
(\mu^*\mathfrak{g}(\omega))\_p \cong \mathfrak{g}_\omega(\theta)\_p \quad \forall p \in P.
\]

**Proof.** Let \(A\) be the Lie algebroid of \(\mathfrak{g}\); the isomorphism will be induced by the infinitesimal action \(\alpha : \mu^*A \to TP\). Using Lemma A.7 and the fact that infinitesimal free actions are injective, we see that
\[
a_p : A_{\mu(p)} \to \text{Im}(a_p) = \ker(d_p\pi).
\]
is an isomorphism. Therefore, we have only to show that \(a_p\) sends \(g_{\mu(p)}(\omega) = A_{\mu(p)} \cap \ker(\omega_{\mu(p)})\) to \(\mathfrak{g}_{\pi}(\theta)\_p = \ker(d_p\pi) \cap \ker(\theta_p)\). Consider \(\alpha \in g_{\mu(p)}(\omega)\); since the action is multiplicative, by Proposition A.6
\[
\theta_p(a_p(\alpha)) = \omega_{\mu(p)}(\alpha) = 0,
\]
therefore \(a_p(\alpha) \in \mathfrak{g}_{\pi}(\theta)\_p\). Conversely, if \(a_p(\alpha) \in \mathfrak{g}_{\pi}(\theta)\_p\), for some \(\alpha \in A_{\mu(p)}\), then \(\alpha \in \ker(\omega_{\mu(p)})\), hence \(\alpha \in g_{\mu(p)}(\omega)\). Q.E.D.

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