q-Bernoulli polynomials and q-umbral calculus

by
Dae San Kim and Taekyun Kim

Abstract
In this paper, we investigate some properties of q-Bernoulli polynomials arising from q-umbral calculus. Finally, we derive some interesting identities of q-Bernoulli polynomials from our investigation.

1 Introduction and preliminaries
Throughout this paper we will assume q to be a fixed number between 0 and 1. We denote by $D_q$ the q-derivative of a function

$$ (D_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad \text{(see [8, 10])}. $$

The Jackson definite q-integral of the function $f$ is defined by

$$ \int_0^x f(t) d_q t = (1 - q) \sum_{a=0}^{\infty} f(q^a x) x q^a, \quad \text{(see [8, 12, 13])}. $$

From (1) and (2), we note that

$$ D_q \int_0^x f(t) d_q t = f(x), \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. $$

In this paper, we use the following notations:

$$ [x]_q = \frac{1 - q^x}{1 - q}, \quad (a + b)_q^n = \prod_{i=0}^{n-1} (a + q^i b), \quad (n \in \mathbb{Z}_+) $$

and

$$ (1 + a)_q = \prod_{j=0}^{\infty} (1 + q^j a), \quad [n]_q! = [n]_q [n - 1]_q \cdots [2]_q [1]_q. $$

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The $q$-analogue of exponential function is defined by

$$e_q(t) = \frac{1}{(1 - (1 - q)t)^\infty} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \quad (\text{see } [5, 6, 8, 10]).$$

(5)

In [10], the $q$-analogues of Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e_q(t) - 1} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!}, \quad (\text{see } [8 - 14]).$$

(6)

In the special case, $x = 0$, $B_{n,q}(0) = B_n,q$ is called the $n$-th $q$-Bernoulli number.

From (6), we can derive the following equation:

$$B_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l}_q x^{n-l} B_{l,q} = \sum_{l=0}^{n} B_{n-l,q} x^l \binom{n}{l}_q,$$

(7)

where $\binom{n}{l}_q = \frac{[n]_q!}{[l]_q! [n-l]_q!}$.

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k \bigg| a_k \in \mathbb{C}\right\}.$$

(8)

Let $\mathbb{P} = \mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. Now we denote by $\langle L | p(x) \rangle$ the action of the linear functional $L$ on the polynomial $p(x)$. We remind that the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle, \quad \langle cL | p(x) \rangle = c \langle L | p(x) \rangle,$$

where $c$ is any constant in $\mathbb{C}$ (see [15, 16]).

For $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k \in \mathcal{F}$, we define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t) | x^n \rangle = a_n \quad \text{for all } n \geq 0.$$

(9)

Thus, by (8) and (9), we note that

$$\langle t^k | x^n \rangle = [n]_q! \delta_{n,k}, \quad (n, k \geq 0),$$

(10)

where $\delta_{n,k}$ is the Kronecker’s symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{[k]_q!} t^k$. Then, by (8) and (9), we see that $\langle f_L(t) | x^n \rangle = \sum_{k=0}^{n} \frac{\langle L | x^k \rangle}{[k]_q!} a_k$.
\( \langle L|x^n \rangle \) and so as linear functionals \( L = f_L(t) \). It is easy to show that the map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) is thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the \( q \)-umbral algebra. The \( q \)-umbral calculus is the study of \( q \)-umbral algebra. By (5) and (10), we easily see that \( \langle e_q(yt)|x^n \rangle = y^n \) and so \( \langle e_q(yt)|p(x) \rangle = p(y) \).

Notice that for all \( f(t) \) in \( \mathcal{F} \)

\[
f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k \rangle}{[k]_q!} t^k, \tag{11}
\]

and for all polynomials \( p(x) \)

\[
p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x) \rangle}{[k]_q!} x^k, \tag{12}
\]

For \( f_1(t), f_2(t), \ldots, f_n(t) \in \mathcal{F} \), we have

\[
\langle f_1(t) \cdots f_m(t)|x^n \rangle = \sum_{i_1 + \cdots + i_m = n} \binom{n}{i_1, \ldots, i_m}_q \langle f_1(t)|x^{i_1}\rangle \cdots \langle f_m(t)|x^{i_m}\rangle, \tag{13}
\]

where \( \binom{n}{i_1, \ldots, i_m}_q = \frac{[n]_q}{[i_1]_q \cdots [i_m]_q} \).

The order \( O(f(t)) \) of the power series \( f(t) \) is the smallest integer \( k \) for which \( a_k \) does not vanish. If \( O(f(t)) = 0 \), then \( f(t) \) is called an invertible series. If \( O(f(t)) = 1 \), then \( f(t) \) is called a delta series.

Let \( p^{(k)}(x) = D_q^k p(x) \). Then, by (12), we get

\[
p^{(k)}(x) = \sum_{l=k}^{\infty} \frac{\langle t^l|p(x) \rangle}{[l]_q!} [l]_q [l-1]_q \cdots [l-k+1]_q x^{l-k}. \tag{14}
\]

From (14), we have

\[
p^{(k)}(0) = \langle t^k|p(x) \rangle \text{ and } \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0). \tag{15}
\]

By (15), we get

\[
t^k p(x) = p^{(k)}(x) = D_q^k p(x). \tag{16}
\]

Let \( f(t), g(t) \in \mathcal{F} \) with \( O(f(t)) = 1 \) and \( O(g(t)) = 0 \). Then there exists a unique sequence \( s_n(x) \) (\( \deg s_n(x) = n \)) of polynomials such that \( \langle g(t)f(t)^k|s_n(x) \rangle = \)
From (19), we note that
\[
B_{n,q}(x) \sim \left( \frac{e_q(t) - 1}{t}, t \right).
\] (19)

By (19), we get
\[
B_{n,q}(x) = \left( \frac{t}{e_q(t) - 1} \right) x^n, \quad (n \geq 0).
\] (20)

From (7) and (16), we note that
\[
tB_{n,q}(x) = D_q B_{n,q}(x) = [n]_q B_{n-1,q}(x).
\] (21)

By (1) and (10), we easily see that
\[
\left\langle \frac{e_q(t) - 1}{t} \right| x^n \right\rangle = \frac{1}{[n+1]_q} \left\langle \frac{e_q(t) - 1}{t} \right| t x^{n+1} \right\rangle
\]
\[
= \frac{1}{[n+1]_q} \left\langle e_q(t) - 1 \right| x^{n+1} \right\rangle = \frac{1}{[n+1]_q}
\]
\[
= \int_0^1 x^n d_q x.
\] (22)
Thus, from (22), we have
\[
\left\langle \frac{e_q(t) - 1}{t} \right| p(x) \right\rangle = \int_0^1 p(x)d_qx, \text{ for } p(x) \in \mathbb{P}.
\]
(23)

In particular, if we take \( p(x) = B_{n,q}(x) \), then
\[
\int_0^1 B_{n,q}(x)d_qx = \left\langle \frac{e_q(t) - 1}{t} \right| B_{n,q}(x) \right\rangle = \left\langle 1 \right| \frac{e_q(t) - 1}{t} B_{n,q}(x) \right\rangle
\]
\[= \langle t^0 \mid x^n \rangle = [n]_q! \delta_{n,0}. \] (24)

From (7), we can derive
\[
\int_0^1 B_{n,q}(x)d_qx = \sum_{k=0}^n B_{n-k,q} \binom{n}{k} \int_0^1 x^k d_qx \int_0^1 \]
\[= \sum_{k=0}^n \frac{B_{n-k,q}}{[k+1]_q} \binom{n}{k} \] (25)

Therefore, by (24) and (25), we obtain the following proposition.

**Proposition 1.** For \( n \in \mathbb{Z}_+ \), we have
\[B_{0,q} = 1, \quad \sum_{k=1}^n \binom{n}{k} \frac{1}{[k+1]_q} B_{n-k,q} = -B_{n,q}, \quad (n > 0).\]

By (17) and (19), we get
\[
p(x) = \sum_{k=0}^\infty \frac{1}{[k]_q!} \left\langle \frac{e_q(t) - 1}{t} \right| t^k p(x) \right\rangle B_{k,q}(x)
\]
\[= \sum_{k=0}^\infty \frac{1}{[k]_q!} \left\langle \frac{e_q(t) - 1}{t} \right| t^k p(x) \right\rangle B_{k,q}(x)
\]
\[= \sum_{k=0}^\infty \frac{1}{[k]_q!} B_{k,q}(x) \int_0^1 t^k p(x)d_qx. \]

It is known that
\[ (x - 1)^n_q = (x - 1)(x - q) \cdots (x - q^{n-1}) \sim (e_q(t), t) \] (27)
From (17) and (27), we have
\[
B_{n,q}(x) = \sum_{k=0}^{n} \frac{1}{[k]_q!} \left< e_q(t) t^k \right| B_{n,q}(x) \right> (x - 1)^k
\]
\[
= \sum_{k=0}^{n} \frac{1}{[k]_q!} \left< e_q(t) \right| t^k B_{n,q}(x) \right> (x - 1)^k
\]
\[
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q B_{n-k,q}(1)(x - 1)^k.
\]

From (3), we can derive
\[
(x - 1)_q^n = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right)_q (-1)^{n-m} q^{\frac{n-m}{2}} x^m.
\]  
(29)

Thus, by (29), we get
\[
t^k(x - 1)_q^n = \sum_{m=k}^{n} \left( \begin{array}{c} n \\ m \end{array} \right)_q (-1)^{n-m} q^{\frac{n-m}{2}} \frac{[m]!}{[m-k]_q!} x^{m-k}
\]
\[
= \frac{[n]_q!}{[n-k]_q!} \sum_{m=0}^{n-k} \left( \begin{array}{c} n-k \\ m \end{array} \right)_q (-1)^{n-k-m} q^{\frac{n-k-m}{2}} x^m
\]
\[
= \frac{[n]_q!}{[n-k]_q!} (x - 1)_q^{n-k}.
\]  
(30)

By (17) and (30), we get
\[
(x - 1)_q^n = \sum_{k=0}^{n} \frac{1}{[k]_q!} \left< e_q(t) - \frac{1}{t} \right| t^k (x - 1)_q^n \right> B_{k,q}(x)
\]
\[
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q B_{k,q}(x) \left< \frac{e_q(t) - 1}{t} \right| (x - 1)_q^{n-k} \right>
\]
\[
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q B_{k,q}(x) \int_{0}^{1} (x - 1)_q^{n-k} d_q x
\]
\[
= \sum_{k=0}^{n} \sum_{m=0}^{n-k} \left( \begin{array}{c} n \\ k \end{array} \right)_q \left( \begin{array}{c} n-k \\ m \end{array} \right)_q B_{k,q}(x)(-1)^{n-k-m} q^{\frac{n-k-m}{2}} \frac{1}{[m+1]_q}.
\]}
From (6) and (10), we note that

\[
\left\langle \frac{t}{e_q(t) - 1} \right\rangle x^n = \sum_{k=0}^{\infty} \frac{B_{k,q}}{[k]_q !} \left\langle t^k \right\rangle x^n = B_{n,q}.
\]

(32)

Let \( \mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n \} \).

For \( p(x) \in \mathbb{P}_n \), let us assume that

\[
p(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}(x).
\]

(33)

By (19), we see that

\[
\left\langle \left( \frac{e_q(t) - 1}{t} \right) t^k \right\rangle B_{n,q}(x) = \left[n\right]_q ! \delta_{n,k}, \quad (n, k \geq 0).
\]

(34)

Thus, from (33) and (34), we have

\[
\left\langle \left( \frac{e_q(t) - 1}{t} \right) t^k \right\rangle p(x) = \sum_{l=0}^{n} b_{l,q} \left\langle \left( \frac{e_q(t) - 1}{t} \right) t^k \right\rangle B_{l,q}(x)
\]

\[
= \sum_{l=0}^{n} b_{l,q} \left[l\right]_q ! \delta_{l,k} = \left[k\right]_q ! b_{k,q}.
\]

(35)

From (16), (23) and (35), we have

\[
b_{k,q} = \frac{1}{[k]_q !} \left\langle \left( \frac{e_q(t) - 1}{t} \right) t^k \right\rangle p(x) = \frac{1}{[k]_q !} \left\langle \frac{e_q(t) - 1}{t} \right\rangle D_q^k p(x)
\]

\[
= \frac{1}{[k]_q !} \int_{0}^{1} p^{(k)}(x) dx, \quad \text{where } p^{(k)}(x) = D_q^k p(x).
\]

(36)

Therefore, by (33) and (36), we obtain the following theorem.

**Theorem 2.** For \( p(x) \in \mathbb{P}_n \), let \( p(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}(x) \). Then we have

\[
b_{k,q} = \frac{1}{[k]_q !} \left\langle \frac{e_q(t) - 1}{t} \right\rangle p^{(k)}(x) = \frac{1}{[k]_q !} \int_{0}^{1} p^{(k)}(x) d_q x,
\]

where \( p^{(k)}(x) = D_q^k p(x) \).
Let us consider the $q$-Bernoulli polynomials of order $r$ as follows:

$$
\left( \frac{t}{e_q(t) - 1} \right)^r e_q(x) = \left( \frac{t}{e_q(t) - 1} \right) \times \cdots \times \left( \frac{t}{e_q(t) - 1} \right) e_q(x) \tag{37}
$$

$$
= \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{[n]_q!}.
$$

In the special case, $x = 0$, $B_{n,q}^{(r)}(0) = B_{n,q}^{(r)}$ is called the $n$-th $q$-Bernoulli number of order $r$. It is easy to show that

$$
\langle \left( \frac{t}{e_q(t) - 1} \right)^r \mid x^n \rangle = \sum_{k=0}^{\infty} \frac{B_{k,q}^{(r)}}{[k]_q!} \langle t^k \mid x^n \rangle = B_{n,q}^{(r)}. \tag{38}
$$

From (37), (32) and (38), we note that

$$
B_{n,q}^{(r)} = \langle \left( \frac{t}{e_q(t) - 1} \right)^r \mid x^n \rangle \tag{39}
$$

$$
= \sum_{i_1 + \cdots + i_r = n} \binom{n}{i_1, \ldots, i_r}_q \langle \frac{t}{e_q(t) - 1} \mid x^{i_1} \rangle \cdots \langle \frac{t}{e_q(t) - 1} \mid x^{i_r} \rangle
$$

$$
= \sum_{i_1 + \cdots + i_r = n} \binom{n}{i_1, \ldots, i_r}_q B_{i_1,q} \cdots B_{i_r,q}.
$$

Therefore, by (39), we have the following lemma.

**Lemma 3.** For $n \geq 0$, we have

$$
B_{n,q}^{(r)} = \sum_{i_1 + \cdots + i_r = n} \binom{n}{i_1, \ldots, i_r}_q B_{i_1,q} \cdots B_{i_r,q}.
$$

By (37), we easily get

$$
B_{n,q}^{(r)}(x) \sim \left( \frac{t}{e_q(t) - 1} \right)^r, \tag{40}
$$

and

$$
B_{n,q}^{(r)}(x) = \left( \frac{t}{e_q(t) - 1} \right)^r x^n, \text{ where } n, r \in \mathbb{Z}_+.
$$

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Let us take $p(x) = B_{n,q}^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} q B_{n-k,q}^{(r)} x^k \in \mathbb{P}_n$. Then we may write

$$p(x) = B_{n,q}^{(r)}(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}(x).$$  \hspace{1cm} (42)

From (42), we have

$$p^{(k)}(x) = D_q^{k} B_{n,q}^{(r)}(x) = [n]_q [n-1]_q \cdots [n-k+1]_q B_{n-k,q}^{(r)}(x) \hspace{1cm} (43)$$

By (36) and (43), we get

$$b_{k,q} = \frac{1}{[k]_q!} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \left| p(x) \right| \right\rangle = \frac{1}{[k]_q!} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r D_q^{k} p(x) \right\rangle \hspace{1cm} (44)$$

Therefore, by Theorem 2 and (42), we obtain the following theorem.

**Theorem 4.** For $n \geq 0$, we have

$$B_{n,q}^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} q \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r \left| B_{n-k,q}^{(r)}(x) \right| \right\rangle B_{k,q}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} q B_{n-k,q}^{(r-1)} B_{k,q}(x).$$

For $p(x) \in \mathbb{P}_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} b_{k,q}^{(r)} B_{k,q}^{(r)}(x).$$  \hspace{1cm} (45)

By (40), we easily get

$$\left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \left| B_{n,q}^{(r)}(x) \right| \right\rangle = [n]_q! \delta_{n,k}, \hspace{0.5cm} (n, k \geq 0).$$  \hspace{1cm} (46)
From (45) and (46), we have
\[
\left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \bigg| p(x) \right\rangle = \sum_{l=0}^{n} b_{l,q}^{(r)} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \bigg| B_{l,q}^{(r)}(x) \right\rangle = \sum_{l=0}^{n} b_{l,q}^{(r)} [l]_q! \delta_{l,k} = [k]_q! b_{k,q}^{(r)}.
\]

By (47), we get
\[
b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \bigg| p(x) \right\rangle.
\]

Therefore, by (45) and (48), we obtain the following theorem.

**Theorem 5.** For \( p(x) \in \mathbb{P}_n \), let \( p(x) = \sum_{k=0}^{n} b_{k,q}^{(r)} B_{k,q}^{(r)}(x) \). Then we have
\[
b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \bigg| p(x) \right\rangle.
\]

Let us take \( p(x) = B_{n,q}(x) \). Then, by Theorem 5, we get
\[
B_{n,q}(x) = p(x) = \sum_{k=0}^{n} b_{k,q}^{(r)} B_{k,q}^{(r)}(x),\quad (49)
\]

where
\[
b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \bigg| p(x) \right\rangle = \frac{1}{[k]_q!} \left\langle \left( \frac{e_q(t) - 1}{t} \right)^r t^k \bigg| B_{n,q}(x) \right\rangle.
\]

For \( k < r \), by (50), we have
\[
b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left( e_q(t) - 1 \right)^r \frac{1}{t^{r-k}} B_{n,q}(x) \bigg|_{t=1} \quad (51)
\]
\[
= \frac{1}{[k]_q!} \left( \frac{1}{[n+r-k]_q \cdots [n+1]_q} \right) \left\langle (e_q(t) - 1)^r \frac{1}{t} \bigg| t^{r-k} B_{n+r-k,q}(x) \right\rangle
\]
\[
= \left( \frac{1}{[k]_q! [r-k]_q} \right) \left( \frac{[r-k]_q!}{[n+r-k]_q \cdots [n+1]_q} \right) \left\langle (e_q(t) - 1)^r \bigg| B_{n+r-k,q}(x) \right\rangle
\]
\[
= \frac{1}{[r]_q \cdot (n+r-k)_q} \sum_{j=0}^{r} (-1)^{r-j} \left\langle (e_q(t))^j \bigg| B_{n+r-k,q}(x) \right\rangle
\]
Let us assume that $k \geq r$. Then, by (50), we get

$$b_{k,q}^{(r)} = \frac{1}{[r]_q!} \langle (e_q(t) - 1)^r | t^{k-r} B_{n,q}(x) \rangle$$

(52)

$$= \frac{1}{[k]_q!} \langle n \cdot \ldots \cdot q | (e_q(t) - 1)^r | B_{n-k+r,q}(x) \rangle$$

$$= \frac{1}{[r]_q!} \binom{n}{r} \left\langle \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \binom{e_q(t)}{j} | B_{n-k+r,q}(x) \right\rangle$$

$$= \frac{1}{[r]_q!} \binom{n}{r} \left\langle \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \binom{m}{m_1, \ldots, m_j} q \right\rangle$$

Therefore, by (49), (51) and (52), we obtain the following theorem.

**Theorem 6.** For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$B_{n,q}(x) = \sum_{k=0}^{r-1} \frac{1}{[r]_q!} \binom{n}{r-k} \left\langle \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \binom{m}{m_1, \ldots, m_j} q \right\rangle$$

$$\times \left\{ \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \binom{m}{m_1, \ldots, m_j} q \right\}$$

$$\times B_{n-k+r-m,q}(x).$$

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Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
e-mail: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
e-mail: tkkim@kw.ac.kr