POSITIVE SPEED SELF-AVOIDING WALKS ON GRAPHS WITH MORE THAN ONE END

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Abstract. A self-avoiding walk (SAW) is a path on a graph that visits each vertex at most once. The mean square displacement of an $n$-step SAW is the expected value of the square of the distance between the ending point and the starting point of an $n$-step SAW, where the expectation is taken with respect to the uniform measure on $n$-step SAWs starting from a fixed vertex. It is conjectured that the mean square displacement of an $n$-step SAW is asymptotically $n^{2\nu}$, where $\nu$ is a constant. Computing the exact values of the exponent $\nu$ on various graphs has been a challenging problem in mathematical and scientific research for long.

In this paper we show that on any locally finite Cayley graph of an infinite, finitely-generated group with more than two ends, the number of SAWs whose end-to-end distances are linear in lengths has the same exponential growth rate as the number of all the SAWs. We also prove that for any infinite, finitely-generated group with more than one end, there exists a locally finite Cayley graph on which SAWs have positive speed - this implies that the mean square displacement exponent $\nu = 1$ on such graphs.

These results are obtained by proving more general theorems for SAWs on quasi-transitive graphs with more than one end, which make use of a variation of Kesten’s pattern theorem in a surprising way, as well as the Stallings’ splitting theorem. Applications include proving that SAWs have positive speed on the square grid in an infinite cylinder, and on the infinite free product graph of two connected, quasi-transitive graphs.

1. Introduction

Self-avoiding walks, which are paths on graphs visiting no vertex more than once, were first introduced as a model for long-chain polymers in chemistry ([7], see also [21]). Despite its simple definition, SAWs have been notoriously difficult to enumerate and to study the geometric properties due to the fact that SAWs are, in general, non-Markovian.

In this paper, we consider SAWs on quasi-transitive graphs. Let $G = (V, E)$ be an infinite, connected graph, and let Aut($G$) be the automorphism group for $G$. We say that $G$ is quasi-transitive, if there exists a subgroup $\Gamma$ of Aut($G$) acting quasi-transitively on $G$, i.e. there exists a finite set of vertices $W \subset V$, $|W| < \infty$, such
that for any $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $w = \gamma v$. The set $W$ is called a fundamental domain.

The connective constant for SAWs on a quasi-transitive graph is the exponential growth rate of the number of $n$-step SAWs starting from a fixed vertex. More precisely, let $c_n(v)$ be the number of $n$-step SAWs starting from a fixed vertex $v \in V$, the connective constant $\mu$ is defined to be

$$\mu := \lim_{n \to \infty} \sup_{v \in V} c_n(v)^{\frac{1}{n}}$$

(1.1)

The limit on the right hand side of (1.1) is known to exist by a sub-additivity argument. It is proved in [17] that the connective constant $\mu$ defined in (1.1), can be expressed as follows

$$\mu = \lim_{n \to \infty} c_n(v)^{\frac{1}{n}}, \quad \forall v \in V.$$

Although the definition of the SAW is quite simple, a lot of fundamental questions concerning SAWs remain unknown. For example, it is still an open problem to compute the exact value of the connective constant for the 2-dimensional square grid. A recent breakthrough is a proof of the fact that the connective constant of the hexagonal lattice is $\sqrt{2 + \sqrt{2}}$; see [5]. See [14, 13, 16, 10, 11] for results concerning bounds of connective constants on quasi-transitive graphs; and see [12, 15] for results concerning the dependence of connective constants on local structures of graphs.

The number of ends of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph.

Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with more than one end. Let $\Gamma \subseteq \text{Aut}(G)$ be a subgroup of the automorphism group of $G$ acting quasi-transitively on $G$. We may make the following assumptions on the graph $G$.  

**Assumption 1.1.** There exist a finite set of vertices $S, S \subset V$ and $|S| < \infty$, such that

1. $S$ is connected;
2. $G \setminus S$ (the graph obtained from $G$ by removing all the vertices in $S$ and their incident edges) has at least two infinite components;
3. for any component $A$ of $G \setminus S$, let $\partial_A S$ be the set consisting of all the vertices in $S$ incident to a vertex in $A$. There exists an infinite component $B$ of $G \setminus S$ and a graph automorphism $\gamma \in \Gamma$, such that $\gamma A \subseteq B$; for any $v \in \partial_A S$ $\gamma v \in \partial_B S \cup B$, $v$ and $\gamma v$ are joined by a path in $G \setminus (A \cup \gamma A)$, whose length is bounded above by a constant $N$ independent of $A, v$. Denote $\gamma$ by $\phi(S, A) := \gamma$.

**Assumption 1.2.** There exist a finite set of vertices $S, S \subset V$ and $|S| < \infty$ satisfying Assumption 1.1. Moreover, assume that
there exist a finite set of vertices $S'$, such that $S \subseteq S'$. Let $\partial S'$ be the set consisting of all the vertices in $S'$ incident to a vertex in $G \setminus S'$. For any two distinct vertices $u, v \in \partial S'$, there exists an SAW $l_{uv}$ joining $u$ and $v$ and visiting every vertex in $S$.

Here are the main results of the paper.

**Theorem 1.3.** Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with more than one end. Let $\mu$ be the connective constant of $G$. Let $\pi_n^v$ be an $n$-step SAW on $G$ starting from a fixed vertex $v$, and let

$$\|\pi_n^v\| = \text{dist}_G(\pi(n), \pi(0)),$$

where $\text{dist}_G(\cdot, \cdot)$ is the graph distance on $G$.

A. If $G$ satisfies Assumption 1.1, then there exists $a \in (0, 1]$ such that

$$\limsup_{n \to \infty} \sup_{v \in V} |\{\pi_n^v : \|\pi_n^v\| \geq an\}|^{\frac{1}{n}} = \mu.$$

B. If $G$ satisfies Assumption 1.2, then $\pi_n^v$ has positive speed, i.e., there exist constants $C, \alpha, \beta > 0$, such that

$$P_n(\|\pi_n^v\| \leq \alpha n) \leq Ce^{-n\beta}.$$

where $P_n$ is the uniform measure on the set of $n$-step SAWs on $G$ starting from a fixed vertex.

The approach to prove Theorem 1.3 is to consider a finite “cut set” $S$, such that SAWs, once crossing this “cut set”, will move to another component of $G \setminus S$ and most of them may never come back again. The analysis involves arguments and technical details inspired by the pattern theorem ([19]), see also ([21, 4, 13]). Applications of Theorem 1.3 include a proof that SAWs on an infinite cylindrical square grid have positive speed, and that SAWs on an infinite free product graph of two quasi-transitive, connected graphs have positive speed.

**Example 1.4.** (Cylinder) Consider the quotient graph of the square grid $\mathbb{Z}^2$, $\mathbb{Z} \times \mathbb{Z}_l$, where $l$ is a positive integer. This is a graph with two ends. We can choose $S = \{0\} \times \mathbb{Z}_l$ and $S' = \{-1, 0, 1\} \times \mathbb{Z}$. Then Assumption 1.2 is satisfied and SAWs have positive speed. See also [8] for discussions about SAWs on a cylinder.

**Definition 1.5.** (Free product of graphs) Let $G_1 = (V_1, E_1, o_1)$, $G_2 = (V_2, E_2, o_2)$ be two connected and quasi-transitive, rooted graphs with vertex sets $V_1$, $V_2$; edge sets $E_1$, $E_2$ and roots $o_1 \in V_1$, $o_2 \in V_2$, respectively. Assume that $|V_i| \geq 2$, for $i = 1, 2$. Let $V_i^x = V_i \setminus \{o_i\}$ for $i = 1, 2$; and let $I(x) = i$ if $x \in V_i^x$. Define

$$V := V_1 * V_2 = \{x_1x_2 \ldots x_n | n \in \mathbb{N}, x_k \in V_1^x \cup V_2^x, I(x_k) \neq I(x_{k+1})\} \cup \{o\}$$
We define an edge set $E$ for the vertex set $V$ as follows: if $i \in \{1, 2\}$ and $x, y \in V_i$, and $(x, y) \in E_i$, then $(wx, wy) \in E$ for all $w \in V$. See [9] for discussions of SAWs on free product graphs of quasi-transitive graphs.

**Theorem 1.6.** Let $G = (V, E)$ be the free product graph of two connected and quasi-transitive, rooted graphs $G_1 = (V_1, E_1, o_1)$ and $G_2 = (V_2, E_2, o_2)$ with $|V_i| \geq 2$, for $i = 1, 2$, as defined in 1.5. Then SAWs on $G$ have positive speed.

Theorem 1.3 also has the following corollaries.

**Theorem 1.7.** Let $\Gamma$ be an infinite, finitely-generated group with more than two ends. Let $G = (V, E)$ be a locally finite Cayley graph of $\Gamma$. For $v \in V$ Let $\pi_n^v$ be an $n$-step SAW on $G$ starting from $v$. Then

$$\limsup_{n \to \infty} \left|\{\pi_n^v : \|\pi_n^v\| \geq an\}\right|^\frac{1}{n} = \mu.$$ 

**Theorem 1.8.** Let $\Gamma$ be an infinite, finitely-generated group with more than one end. There exists a locally finite Cayley graph $G = (V, E)$ of $\Gamma$, such that SAWs on $G$ have positive speed.

For a graph satisfying Assumption 1.2, Theorem 1.3 implies that the mean square displacement of SAWs on the graph is of the order $n^2$, i.e.

$$\langle \|\pi\|^2 \rangle_n \sim n^2,$$

where $\langle \cdot \rangle_n$ is the expectation taken with respect to the uniform measure for $n$-step SAWs on $G$ starting from a fixed vertex; and “$\sim$” means that there exist constants $C_1, C_2 > 0$, independent of $n$, such that $C_1 n^2 \leq \langle \|\pi\|^2 \rangle_n \leq C_2 n^2$.

The mean square displacement exponent $\nu$ for SAWs, defined by

$$\langle \|\pi\|^2 \rangle_n \sim n^{2\nu},$$

has been an interesting topic to mathematicians and scientists for long. It is conjectured that $\nu = \frac{4}{d}$ for SAWs on $\mathbb{Z}^2$, $\nu = 0.5$ for SAWs on $\mathbb{Z}^d$ with $d \geq 4$, and that $\nu = 1$ for SAWs on a non-amenable graph with bounded vertex degree.

The conjecture that $\nu = 1$ when $d \geq 5$ was proved in [3, 18]. See [1] for related results when $d = 4$, and [4] for related results for $d \geq 2$.

It is proved in [24] that if a non-amenable Cayley graph satisfies

$$(d - 1) \rho \mu^{-1} < 1, \quad \text{(1.3)}$$

then SAWs have positive speed. Here $d$ is the vertex degree, $\rho$ is the spectral radius for the transition matrix of the simple random walk on the graph, and $\mu$ is the connective constant as defined in (1.1). Combining with the results in [23, 2, 25], it is known that for any finitely generated non-amenable group, there exists a locally finite Cayley graph on which SAWs have positive speed.
It is proved in [22] that SAWs have positive speed for certain regular tilings of the hyperbolic plane. An upper bound of the spectral radius for a planar graph with given maximal degree is proved in [6], which, combining with (1.3), can be used to show that SAWs have positive speed on a large class of planar graphs.

The organization of the paper is as follows. In Section 2, we prove Theorem 1.3 A. In Section 3, we prove Theorem 1.3 B. Theorems 1.7 and 1.8 are proved in Section 4. In Section 5, we prove Theorem 1.6.

2. Proof of Theorem 1.3 A.

This section is devoted to prove Theorem 1.3 A. Let \( G = (V, E) \) be a graph satisfying the assumption of Theorem 1.3. Let \( S \) be a finite set of vertices satisfying Assumption 1.1. Recall that \( \Gamma \) is a subset of \( \text{Aut}(G) \) acting quasi-transitively on \( G \). Let \( \Gamma S \) be the set of images of \( S \) under \( \Gamma \). By quasi-transitivity of \( G \), for each \( \gamma \in \Gamma \), \( \gamma S \) still satisfies Assumption 1.1.

Let \( \pi \) be an \( n \)-step SAW on \( G \). We say that \( E^* \) occurs at the \( j \)th step of \( \pi \) if there exists \( \gamma S \in \Gamma S \) such that \( \pi(j) \in \gamma S \), and all the vertices of \( \gamma S \) are visited by \( \pi \). For \( k \geq 1 \), we say that \( E_k \) occurs at the \( j \)th step of \( \pi \), if there exists \( \gamma S \in \Gamma S \), such that \( \pi(j) \in \gamma S \), and at least \( k \) vertices of \( \gamma S \) are visited by \( \pi \). We say that \( \tilde{E}_k \) occurs at the \( j \)th step of \( \pi \) if \( E^* \) or \( E_k \) (or both) occur there.

In the following, we will use \( E \) to denote any of \( E^* \), \( E_k \) or \( \tilde{E}_k \). If \( m \) is a positive integer, we say that \( E(m) \) occurs at the \( j \)th step of \( \pi \) if \( E \) occurs at the \( m \)th step of the \( 2m \)-step subwalk \( (\pi(j-m), \ldots, \pi(j+m)) \). (If \( j-m < 0 \) or \( j+m > n \), then an obvious modification must be made in this definition: for \( j-m < 0 \), it means that \( E \) occurs at the \( j \)th step of \( \pi(0), \ldots, \pi(j+m) \); for \( j+m > n \), it means that \( E \) occurs at the \( m \)th step of \( \pi(j-m), \ldots, \pi(n) \)). In particular, if \( E(m) \) occurs at the \( j \)th step of \( \pi \), then \( E \) occurs at the \( j \)th step of \( \pi \).

Let \( c_n(v) \) be the number of \( n \)-step SAWs on \( G \) starting from a fixed vertex \( v \). For \( r \geq 0 \), let \( c^v_n(r, E) \) (resp. \( c^v_n(r, E(m)) \)) be the number of \( n \)-step SAWs starting from \( v \) for which \( E \) (resp. \( E(m) \)) occurs at no more than \( r \) different steps.

Let

\[
\begin{align*}
c_n &= \sup_{v \in V} c_n(v); \\
c_n(r, E) &= \sup_{v \in V} c^v_n(r, E); \\
\lambda(E) &= \limsup_{n \to \infty} c_n(0, E) \frac{1}{n}.
\end{align*}
\]

Let \( \mu \) be the connective constant of \( G \) as defined in (1.1). We have that \( \lambda(E) < \mu \) if and only if

(2.1) there exist \( \epsilon > 0, M \in \mathbb{N} \), such that \( c_m(0, E) < [\mu(1-\epsilon)]^m \), for \( m \geq M \).
Lemma 2.1. Suppose that
\[ \lambda(E) < \mu. \]
Let \( \epsilon, M \) satisfy (2.1), and let \( m \geq M \) satisfy
\[ c_m \leq [\mu(1 + \epsilon)]^m. \]
Then there exist \( a = a(\epsilon, m) \) and \( R = R(\epsilon, m) \in (0, 1) \), such that
\[ \limsup_{n \to \infty} c_n(an, E(m))^{1/n} < R \mu. \]

Proof. Assume that (2.2) holds. Let \( \epsilon, M \) satisfy (2.3). Since \( c_m(0, E(m)) = c_m(0, E(m)) \),
we have
\[ c_m(0, E(m)) \leq [\mu(1 - \epsilon)]^m. \]
Let \( \pi \) be an \( n \)-step SAW on \( G \) and \( N = \left\lfloor \frac{n}{m} \right\rfloor \). If \( E(m) \) occurs in no more than \( r \) steps of \( \pi \), then \( E(m) \) occurs at no more than \( r \) of the \( m \)-step subwalks
\( (\pi((j - 1)m), \pi((j - 1)m + 1), \ldots, \pi(jm)), \quad 1 \leq j \leq N. \)

Counting the number of ways in which \( r \) or fewer of these subwalks can contain an occurrence of \( E(m) \), we have that
\[ c_n(r, E(m)) \leq \sum_{i=1}^{r} \binom{N}{i} c_m^i [c_m(0, E(m))]^{N-i} c_{n-Nm} \]
\[ \leq \mu^{Nm} c_{n-Nm} \sum_{i=1}^{r} \binom{N}{i} (1 + \epsilon)^{im}(1 - \epsilon)^{(N-i)m} \]
For \( \xi \) small and positive, we have
\[ \sum_{i=0}^{\xi N} \binom{N}{i} (1 + \epsilon)^{im}(1 - \epsilon)^{(N-i)m} \]
\[ \leq (\xi N + 1) \binom{N}{\xi N} \left( \frac{1 + \epsilon}{1 - \epsilon} \right)^{\xi N m} (1 - \epsilon)^{Nm} \]
The \( N \)th root of the right hand side converges as \( N \to \infty \) to
\[ f(\xi) = \frac{1}{\xi^\xi(1 - \xi)^{1-\xi}} \left( \frac{1 + \epsilon}{1 - \epsilon} \right)^{\xi m} (1 - \epsilon)^m, \]
which is strictly less than 1 for \( 0 < \xi < \xi_0 \), and some \( \xi_0 = \xi_0(\epsilon, m) > 0 \). Therefore when \( 0 < a < \frac{\xi}{m} \), and \( R = f(\xi)^{\frac{1}{n}} \), we have
\[ c_n(an, E(m))^{1/n} < R \mu, \]
and the proof is complete. \( \square \)
Lemma 2.2. Let $G_1 = (V(G_1), E(G_1))$ be a component of $G \setminus \Gamma S$. There exists a subgroup $\Gamma_1$ of $\Gamma$ acting quasi-transitively on $G_1$.

Proof. Since $\Gamma$ acts on $G$ quasi-transitively, $V$ has finitely many orbits under the action of $\Gamma$. Let $W_1$ be the subset of $V(G_1)$ consisting of one representative in each orbit of $V$ under the action of $\Gamma$, such that the intersection of the orbit with $V(G_1)$ is nonempty, then $|W_1| < \infty$.

Let

$$\Gamma_1 = \{ \gamma \in \Gamma : \forall w \in W_1, \gamma w \in V(G_1) \}.$$ 

Then it is straightforward to check that $\Gamma_1$ is a subgroup of $\Gamma$, and that $\Gamma_1$ acts on $G_1$ quasi-transitively. \qed

Lemma 2.3. There exists a component $G_1 = (V(G_1), E(G_1))$ of $G \setminus \Gamma S$, such that $\lambda(\tilde{E}_1)$ is the connective constant of $G_1$, i.e.

\begin{align}
\lambda(\tilde{E}_1) &= \lim_{n \to \infty} \sup_{v \in V(G_1)} \tilde{c}_n(v)^{\frac{1}{n}} \quad \text{(2.4)} \\
\lambda(\tilde{E}_1) &= \lim_{n \to \infty} \tilde{c}_n(v)^{\frac{1}{n}}, \forall v \in V(G_1) \quad \text{(2.5)}
\end{align}

where $\tilde{c}_n(v)$ is the number of $n$-step SAWs on $G_1$ starting from $v$.

Proof. By definition of $\lambda(\tilde{E}_1)$, we have

$$\lambda(\tilde{E}_1) = \lim_{n \to \infty} \sup_{v \in V \setminus \Gamma S} \tilde{c}_n(v)^{\frac{1}{n}},$$

where $\tilde{c}_n(v)$ is the number of $n$-step SAWs starting from $v$ on the component of $G \setminus \Gamma S$ including $v$.

By the quasi-transitivity of $G \setminus \Gamma S$ under the action of $\Gamma$, there exists $v_1 \in G \setminus \Gamma S$, such that

$$\lambda(\tilde{E}_1) = \lim_{n \to \infty} \sup_{v \in G \setminus \Gamma S} \tilde{c}_n(v_1)^{\frac{1}{n}}.$$ 

Let $G_1$ be the component of $G \setminus \Gamma S$ containing $v_1$, then

$$\lambda(\tilde{E}_1) = \lim_{n \to \infty} \sup_{v \in G \setminus \Gamma S} \tilde{c}_n(v_1)^{\frac{1}{n}} \leq \lim_{n \to \infty} \sup_{v \in V(G_1)} \tilde{c}_n(v)^{\frac{1}{n}} \leq \lim_{n \to \infty} \sup_{v \in V \setminus \Gamma S} \tilde{c}_n(v)^{\frac{1}{n}} = \lambda(\tilde{E}_1).$$

Therefore $\lambda(\tilde{E}_1)$ is the connective constant for $G_1$, and (2.4) follows. The identity (2.5) follows from Lemma 2.2 and (1.1), (1.2). \qed

Lemma 2.4. $\lambda(\tilde{E}_1) < \mu$. 
Proof. Since $\Gamma$ acts on $G$ quasi-transitively, let $W$ be a fundamental domain such that $|W| < \infty$. Let

$$N_0 = \max_{w_1, w_2 \in W} \text{dist}_G(w_1, w_2).$$

For any $v \in V$, let

$$B_G(v, N_0) = \{u \in V : \text{dist}_G(v, u) \leq N_0\}.$$

It is not hard to see that for any $v \in V$, $B_G(v, N_0)$ contains a vertex in each orbit of $V$ under the action of $\Gamma$. Therefore for any $v \in V$, there exists $\gamma S \in \Gamma S$, such that

$$B_G(v, N_0) \cap \gamma S \neq \emptyset.$$

Let $G_1$ be a component of $G \setminus \Gamma S$, such that $\lambda(\tilde{E}_1)$ is the connective constant of $G_1$. The existence of $G_1$ is guaranteed by Lemma 2.3. Let $T_n(v)$ be the set of all $n$-step SAWs on $G_1$ starting from the vertex $v$. For each $\pi \in T_n(v)$, find indices $j_1, \ldots, j_u$, such that for $i \neq l$, we have

$$\text{dist}_{G_1}(\pi(j_i), \pi(j_l)) \geq 8(N_0 + |S|) + N, \quad (2.6)$$

here $N$ is given by Assumption 1.1 (3).

We may assume that $u = \kappa n$. For each $B_G(\pi(j_i), N_0)$, there exists $\gamma_{ji} S \in \Gamma S$, such that

$$B_G(\pi(j_i), N_0) \cap \gamma_{ji} S \neq \emptyset.$$

Let $\pi(t_i)$ be a closest vertex on $\pi$, in graph distance of $G$, to $\gamma_{ji} S$. Then

$$\text{dist}_{G_1}(\pi(j_i), \pi(t_i)) \leq 2N_0 + |S|. \quad (2.7)$$

By rearrangements if necessary, we may assume that

$$t_1 < t_2 < \ldots < t_{\kappa n}.$$

We choose a subset of indices $H \subset \{t_1, t_2, \ldots, t_{\kappa n}\}$ to perform a manipulation, which will be described later. Assume that $H = \{h_1, \ldots, h_{\delta n}\}$, where $0 < \delta < \kappa$, and $h_a < h_b$ if $a < b$. For each $h_l \in H$ with $h_l = t_i$, let $S_l = \gamma_{ji} S$.

We construct a new SAW $\pi_1$ as follows

- let $\alpha_1$ be the subwalk of $\pi$ from $\pi(0)$ to $\pi(h_1)$;
- use a shortest path to join $\pi(h_1)$ and $S_1$, denoted by $\omega_1$;
- let $A_1$ be the component of $G \setminus S_1$ containing $\pi(h_1)$; use $\phi(S_1, A_1)$ to map the concatenation of the reversed $\omega_1$ and $\pi(h_1), \pi(h_1 + 1), \ldots, \pi(n)$ to another component of $G \setminus S_1$, and denote the image of the concatenation of the two subwalks under $\phi(S_1, A_1)$ by $\theta_1$ - Here $\phi(S_1, A_1)$ is given as in Assumption 1.1 (3);
• use an SAW $\beta_1$ in $G \setminus [A_1 \cup \phi(S_1, A_1)A_1]$ to join the last vertex of $\omega_1$ and its image under $\phi(S_1, A_1)$ (note that the existence of $\beta_1$ is guaranteed by Assumption 1.1);
• let $\pi_1$ be the concatenation of $\alpha_1$, $\omega_1$, $\beta_1$ and $\theta_1$.

It is not hard to check that $\pi_1$ is indeed an SAW. Note that $\alpha_1$ is a subwalk of both $\pi$ and $\pi_1$; and the subwalk $\pi \setminus \alpha_1$ is mapped, by $\phi(S_1, A_1)$, to a subwalk of $\theta_1$. Let $h_1^1 = h_1$, and let $h_i^1$ ($2 \leq i \leq \delta n$) be the step in $\pi_1$ of the image of $\pi(h_i)$ under $\phi(S_1, A_1)$.

We claim that for $2 \leq i \leq \delta n$, $\pi_1(h_i)$ is a closest vertex on $\pi_1$ to $\phi(S_1, A_1)S_i$. To see why that is true, first of all, since $\pi(h_i)$ is the image of $\pi(h_i)$ under $\phi(S_1, A_1)$, we have that for $2 \leq i \leq \delta n$, $\pi_1(h_i)$ is a closest vertex on the subwalk $\pi_1(h_i - h_2 + h_1), \ldots, \pi_1(n - h_2 + h_2)$ to $\phi(S_1, A_1)S_i$. Moreover, we have

$$\text{dist}_G(\pi_1(h_i^1), \phi(S_1, A_1)S_i) \leq N_0, \text{ for } 2 \leq i \leq \delta n.$$  

However, for any vertex $v$ on the subwalk $\pi(0), \ldots, \pi_1(h_2 - h_2 + h_1)$, we have

$$\text{dist}_G(v, \phi(S_1, A_1)S_i) > N_0, \text{ for } 2 \leq i \leq \delta n. \tag{2.8}$$

To see why (2.8) is true, note that from (2.6), (2.7), we have

$$\text{dist}_G(\pi_1(h_2^1 - h_2 + h_1), \phi(S_1, A_1)S_i) \geq 3N_0 + 6|S| + N.$$ 

Then for any $u \in \beta_1 \cup (\pi_1(h_2^1 - h_2 + h_1), \ldots, \pi_1(n - h_2 + h_2)) \cup \alpha_1 \cup \omega_1$, we have

$$\text{dist}_G(\pi_1(h_2^1 - h_2 + h_1), \phi(S_1, A_1)S_i) \geq 2N_0 + 5|S|.$$ 

This way we obtain (2.8), and the claim follows.

Let

$$\pi_0 := \pi, \quad h_i^0 = h_i,$$

for $1 \leq i \leq \delta n$. Assume that we have constructed SAWs $\pi_1, \ldots, \pi_s$, where $1 \leq s < \delta n$. Assume that $\alpha_s = (\pi_s - 1(0), \ldots, \pi_s - 1(h_s^{s-1}))$ is a subwalk of both $\pi_{s-1}$ and $\pi_s$, and that $\pi_{s-1} \setminus \alpha_s$ is mapped by a graph automorphism $\phi_s \in \Gamma$ to a subwalk of $\pi_s$. Let $h_i^s = h_i^{s-1}$, for $1 \leq i \leq s$, and let $h_i^s (s + 1 \leq i \leq \delta n)$ be the step in $\pi_s$ of the image of $\pi_{s-1}(h_i^{s-1})$ under the graph isomorphism $\phi_s$. Assume also that $\pi_s(h_q^s)$ is the closest vertex on $\pi_s$ to $\phi_s \ldots \phi_1 S_q$ for $s + 1 \leq q \leq \delta n$.

Now we construct an SAW $\pi_{s+1}$, following the procedure below.

• let $\alpha_{s+1}$ be the subwalk of $\pi_s$ from $\pi_s(0)$ to $\pi_s(h_{s+1}^s)$;
• use a shortest path $\omega_{s+1}$ to join $\pi_s(h_{s+1}^s)$ and $\tilde{S}_{s+1} := \phi_s \ldots \phi_1 S_{s+1}$;
• let $A_{s+1}$ be the component of $G \setminus \tilde{S}_{s+1}$ containing $\pi_s(h_{s+1}^s)$; use $\phi_{s+1} := \phi(\tilde{S}_{s+1}, A_{s+1})$ to map the concatenation of the reversed $\omega_{s+1}$ and $\pi_s(h_{s+1}^s), \ldots, \pi_s(n+$
\[ h_{s+1} \to \text{another component of } G \setminus \tilde{S}_{s+1}, \text{ and denote the image of the concatenation of the two subwalks under } \phi(\tilde{S}_{s+1}, A_{s+1}) \text{ by } \theta_{s+1}; \]

- use an SAW \( \beta_{s+1} \) in \( G \setminus [A_{s+1} \cup \phi(\tilde{S}_{s+1}, A_{s+1})A_{s+1}] \) to join the last vertex of \( \omega_{s+1} \) and its image under \( \phi_{s+1} \) (note that the existence of \( \beta_{s+1} \) is guaranteed by Assumption 1.1);

- let \( \pi_{s+1} \) be the concatenation of \( \alpha_{s+1}, \omega_{s+1}, \beta_{s+1} \) and \( \theta_{s+1} \).

We can check that \( \pi_{s+1} \) is indeed an SAW, and that \( \pi_{s+1}(h_{s+1}p) \) is a closest vertex on \( \pi_{s+1} \) to \( \phi_{s+1} \ldots \phi_1 S_p \), for \( s + 2 \leq p \leq \delta n \), where \( \pi_{s+1}(h_{s+1}p) = \phi_{s+1}(\pi_s(h_p)) \).

We repeat the above induction process until we construct the SAW \( \pi_{\delta n} \). Note that \( \pi_{\delta n} \) is an SAW whose length is at least \( n \) and at most \( n + \delta n(N + 2N_0) \).

We consider the number of pairs \((\pi, H)\), and obtain

\[
|\{\pi, H\}| \geq \xi_n(v) \left( \frac{\kappa n}{\delta n} \right).
\]

Moreover,

\[
|\{\pi, H\}| \leq \left( \sum_{t=n}^{n+\delta n(N+2N_0)} c_t(v) \right) [d^{[S]}|S|(2N_0 + N)^4]^{2\delta n}
\]

where \( d \) is the maximal vertex degree of \( G \); \( d \) is obviously finite since \( G \) is quasitransitive. Taking \( n \)th root of (2.9), (4), and letting \( n \to \infty \), we have

\[
(2.11) \quad \lambda(\tilde{E}_1) \frac{\kappa^{\kappa}}{\delta^{\kappa}(\kappa - \delta)^{\kappa - \delta}} \leq \mu \left( d^{[S]}|S|(2N_0 + N)^4 \right)^{2\delta}
\]

but this is impossible when \( \lambda(\tilde{E}_1) = \mu \), and \( \frac{\delta}{\kappa} > 0 \) and is sufficiently small. Therefore we have \( \lambda(\tilde{E}_1) < \mu \). \( \square \)

To prove Theorem 1.3, we will analyze both the case \( \lambda(E^*) = \mu \) and the case \( \lambda(E^*) < \mu \).

We first assume that

\[
(2.12) \quad \lambda(E^*) = \mu.
\]

Let \( 1 \leq k \leq |S| \). We make the following observations. First, \( c_n(0, \tilde{E}_k) \) is a non-decreasing function of \( k \); secondly, if \( E^* \) does not occur on a given walk, then \( \tilde{E}_{|S|} \) cannot occur. Therefore

\[
c_n(0, E^*) \leq c_n(0, \tilde{E}_{|S|}) \leq c_n.
\]

As a result, (2.12) implies

\[
\limsup_{n \to \infty} c_n(0, \tilde{E}_{|S|})^{\frac{1}{n}} = \mu.
\]
By Lemma 2.4, \( \lambda(\tilde{E}_1) < \mu \). We may choose \( k \) with \( 1 \leq k < |S| \), such that

\[
\lambda(\tilde{E}_k) < \mu, \quad \lambda(\tilde{E}_{k+1}) = \mu \tag{2.13}
\]

Let \( \epsilon, m \) satisfy (2.1) and let \( m \geq M \) satisfy (2.3). By Lemma 2.1, there exists \( a = a(\epsilon, m) > 0 \), such that

\[
\limsup_{n \to \infty} c_n(a, \tilde{E}_k(m))^{\frac{1}{n}} < \mu. \tag{2.14}
\]

By quasi-transitivity of \( G \), \( \lambda(\tilde{E}_{k+1}) = \mu \) implies that there exists a vertex \( v_0 \) such that

\[
\limsup_{n \to \infty} c_{an}(0, \tilde{E}_{k+1})^{\frac{1}{n}} = \mu. \tag{2.15}
\]

Moreover, \( \lambda(\tilde{E}_k) < \mu \) implies that for any vertex \( v \) of \( G \),

\[
\limsup_{n \to \infty} c_{an}(an, \tilde{E}_k(m))^{\frac{1}{n}} < \lim_{n \to \infty} c_n(v)^{\frac{1}{n}} = \mu. \tag{2.16}
\]

where \( a, m \) satisfy (2.14).

Now assume that \( \lambda(E^*) < \mu \), by quasi-transitivity of \( G \), (1.2) and Lemma 2.1, for any vertex \( v \) of \( G \),

\[
\limsup_{n \to \infty} c_{an}^{v}(an, E^*(m))^{\frac{1}{n}} < \lim_{n \to \infty} c_n(v)^{\frac{1}{n}} = \mu. \tag{2.17}
\]

Let \( \pi \) be an \( n \)-step SAW on \( G \). We say that \( F \) occurs at the \( j \)th step of \( \pi \) if

(a) if \( \lambda(E^*) < \mu \), then

• \( E^*(m) \) occurs at the \( j \)th step of \( \pi \); and

• assume that \( \pi(j) \in \gamma S, \gamma \in \Gamma \) and the subwalk \((\pi(j-m), \ldots, \pi(j+m))\)
visits all the vertices of \( \gamma S \). Let \( \pi(\alpha) (\alpha \geq j - m) \) be the first vertex of \( \gamma S \) visited by \( \pi \), and let \( \pi(\beta) (\alpha < \beta \leq j + m) \) be the last vertex of \( \gamma S \) visited by \( \pi \). Then \( \pi(\alpha - 1) \) and \( \pi(\beta + 1) \) are in distinct components of \( G \setminus \gamma S \).

(b) if \( \lambda(E^*) = \mu \), let \( k \) be as in (2.13), then

• \( \tilde{E}_k(m) \) occurs at the \( j \)th step, and \( \tilde{E}_{k+1} \) does not occur at the \( j \)th step; and

• assume that \( \pi(j) \in \gamma S, \gamma \in \Gamma \) and \( \pi \) visits exactly \( k \) vertices of \( \gamma S \). Let \( \pi(\alpha) (\alpha \geq j - m) \) be the first vertex of \( \gamma S \) visited by \( \pi \), and let \( \pi(\beta) (\alpha < \beta \leq j + m) \) be the last vertex of \( \gamma S \) visited by \( \pi \). Then \( \pi(\alpha - 1) \) and \( \pi(\beta + 1) \) are in distinct components of \( G \setminus \gamma S \).

For \( r \geq 0 \), and \( v \in V \)

• if \( \lambda(E^*) < \mu \), let \( b^v_n(r, F) \) be the number of \( n \)-step SAWs on \( G \) starting from \( v \), such that \( E^*(m) \) occurs at least \( an \) times, and \( F \) occurs no more than \( r \) steps;
Lemma 2.5. \(1.1\) \(2.13\) \(3\). We construct a new SAW \(2.18\) \(2.15\) \(2.18\) and let \(\lambda \) \(\pi \) from \(a\) \(12\) ZHONGYANG LI
\[ \limsup_{n \to \infty} b^v_n(0, F)^{\frac{1}{n}} < \mu. \]

(b) If \(\lambda(E^*) = \mu\), let \(v\) be a vertex satisfying \((2.15)\), then
\[ \limsup_{n \to \infty} b^v_n(0, F)^{\frac{1}{n}} < \mu. \]

Proof. Let \(v\) be a vertex satisfying the assumptions of the lemma. Assume that
\[ \limsup_{n \to \infty} b^v_n(0, F)^{\frac{1}{n}} = \mu; \]
we will obtain a contradiction.

If \(\lambda(E^*) < \mu\), let \(U_n\) be the set consisting of all the \(n\)-step SAWs on \(G\) starting from \(v\) such that \(E^*(m)\) occurs at least \(an\) times, and \(F\) never occurs.

If \(\lambda(E^*) = \mu\), let \(k\) be given as in \((2.13)\) and let \(U_n\) be the set consisting of all the \(n\)-step SAWs on \(G\) starting from \(v\), such that \(E_k(m)\) occurs at least \(an\) times, and \(E_{k+1}\) never occurs, and \(F\) never occurs.

Let \(\pi \in U_n\).

Let \(j_1, \ldots, j_n\) be the indices of the SAW \(\pi\) where \(E^*(m)\) occurs if \(\lambda(E^*) < \mu\) (where \(E_k(m)\) occurs if \(\lambda(E^*) = \mu\)), such that
\[ \text{dist}_G(\pi(j_i), \pi(j_i)) \geq 4(2m + 1)|S|. \]

We may assume \(u = \kappa n\) such that \((2.18)\) occurs. We choose a subset
\[ H = \{t_1, \ldots, t_{\delta n}\} \subset \{j_1, \ldots, j_{\kappa n}\}, \]
where \(0 < \delta < \kappa\), \(t_i < t_j\) when \(i < j\), to perform the following inductive manipulations.

For \(1 \leq i \leq \delta n\), let \(\gamma_i S\) be the copy of \(S\) (where \(\gamma_i \in \Gamma\)) such that \(\pi(t_i) \in \gamma_i S\), and all the vertices of \(\gamma_i S\) are visited by \(\pi\) and by the subwalk \((\pi(t_i - m), \ldots, \pi(t_i + m))\) if \(\lambda(E^*) < \mu\) (exactly \(k\) vertices of \(\gamma_i S\) are visited by the subwalk \((\pi(t_i - m), \ldots, \pi(t_i + m))\) if \(\lambda(E^*) = \mu\). Let \(\pi(\alpha_i)\) \((\alpha_i \geq t_i - m)\) be the first vertex of \(\gamma_i S\) visited by \(\pi\), and let \(\pi(\beta_i)\) \((\alpha_i < \beta_i \leq t_i + m)\) be the last vertex of \(\gamma_i S\) visited by \(\pi\).

Let \(A_i\) be the component of \(G \setminus \gamma_i S\) containing \(\pi(\alpha_i - 1)\) and \(\pi(\beta_i + 1)\), and let \(B_{\gamma}\), \(\phi(\gamma_i S, A_i)\) be as described in Assumption 1.1 \((3)\). We construct a new SAW \(\pi_1\) from \(\pi\) as follows.

- The subwalk \((\pi_1(0), \ldots, \pi_1(\alpha_1))\) is the same as the subwalk \((\pi(0), \ldots, \pi(\alpha_1))\).
We map \((\pi(\beta_1 + 1), \ldots, \pi(n))\), as an SAW in \(A_1\), to an SAW \(\theta_1\) in \(B_1\) by 
\(\phi(\gamma_1 S, A_1)\), as described in Assumption 1.1 (3).

- We use an SAW \(\omega_1\) in \(G \setminus [A_1 \cup \phi(\gamma_1 S, A_1)]\) joining \(\pi(\alpha_1)\) and \(\phi(\gamma_1 S)\pi(\beta_1)\).

This is possible since \(S\) is connected by Assumption 1.1 (1), and \(\gamma_1 S\) is an identical copy of \(S\).

- Let \(\pi_1\) be the concatenation of \((\pi_1(0), \ldots, \pi_1(\alpha_1))\), \(\omega_1\) and \(\theta_1\).

Let 
\[\pi_0 := \pi, \quad \alpha^0_\iota = \alpha_\iota, \quad \beta^0_\iota = \beta_\iota.\]

Assume that we have constructed SAWs \(\pi_1, \ldots, \pi_k\), where \(1 \leq k < \delta n\). Assume that 
\(\eta_k = (\pi_{k-1}(0), \ldots, \pi_{k-1}(\alpha_{k-1}^{k-1}))\) is a subwalk of both \(\pi_{k-1}\) and \(\pi_k\), and that \(\pi_{k-1} \setminus \eta_k\) is mapped by a graph automorphism \(\phi_k \in \Gamma\) to a subwalk of \(\pi_k\). Let 
\[
\alpha^k_\iota = \alpha^{k-1}_\iota, \quad \text{for} \ 1 \leq i \leq k;
\]
\[
\beta^k_j = \beta^{k-1}_j, \quad \text{for} \ 1 \leq j \leq k - 1.
\]

Let \(\alpha^k_i \ (k + 1 \leq i \leq \delta n)\) (resp. \(\beta^k_j \ (k \leq j \leq \delta n)\)) be the step in \(\pi_k\) of the image of 
\(\pi_{k-1}(\alpha_{k-1}^{k-1})\) (resp. \(\pi_{k-1}(\beta_{k-1}^{k-1})\)) under the graph isomorphism \(\phi_k\).

Now we construct an SAW \(\pi_{k+1}\), following the procedure below.

- Let \(\eta_{k+1}\) be the subwalk of \(\pi_k\) from \(\pi_k(0)\) to \(\pi_k(\alpha_{k+1}^k)\).
- Let \(\bar{S}_{k+1} := \phi_k \ldots \phi_1 S_{k+1}\). Let \(A_{k+1}\) be the component of \(G \setminus \bar{S}_{k+1}\) containing 
\(\pi_k(\alpha_{k+1}^k)\); use \(\phi_{k+1} := \phi(S_{k+1}, A_{k+1})\) to map the subwalk \(\pi_k(\beta_{k+1}^k), \ldots, \pi_k(n - \beta_{k+1}^k + \beta_{k+1}^k)\) to an SAW \(\theta_{k+1}\) in another component of \(G \setminus \bar{S}_{k+1}\).
- Use an SAW in \(G \setminus [A_{k+1} \cup \phi_{k+1}(S_{k+1}, A_{k+1}) A_{k+1}]\) to join \(\pi_k(\alpha_{k+1}^k)\) and 
\(\phi_{k+1}(\pi_k(\beta_{k+1}^k)))\) (note that the existence of such an SAW is guaranteed by 
Assumption 1.1);
- let \(\pi_{k+1}\) be the concatenation of \(\eta_{k+1}\), \(\omega_{k+1}\) and \(\theta_{k+1}\).

We can check that \(\pi_{k+1}\) is indeed an SAW.

We claim that
\[
(2.19) \quad [\phi_k \ldots \phi_1 S_{k+1}] \cap [\phi_{k+1} \ldots \phi_1 S_{k+2}] = \emptyset.
\]

To see why (2.19) is true, note that by (2.18), we have 
\[\text{dist}(S_{k+1}, S_{k+2}) \geq 2|S|,\]
Since \(\phi_i \ (1 \leq i \leq k)\)'s are automorphisms of \(G\), we have
\[\text{dist}(\phi_k \ldots \phi_1 S_{k+1}, \phi_k \ldots \phi_1 S_{k+2}) \geq 2|S|,\]

This implies
\[\emptyset = [\phi_k \ldots \phi_1 S_{k+1}] \cap [\phi_k \ldots \phi_1 S_{k+2}].\]
Since $\phi_{k+1}$ maps $\phi_k \ldots \phi_1 S_{k+2}$ from one component of $G \setminus [\phi_k \ldots \phi_1 S_{k+1}]$ to another component of $G \setminus [\phi_k \ldots \phi_1 S_{k+1}]$, (2.19) follows.

We continue the above construction process until we have already constructed the SAW $\pi_{\delta_n}$. Counting the number of pairs $|\langle \pi, H \rangle|$, we have

$$|\langle \pi, H \rangle| \geq |U_n| \left( \frac{\kappa n}{\delta_n} \right),$$

and

$$|\langle \pi, H \rangle| \leq \left( \sum_{i=1}^{n+\delta_n N} c_i \right) \left( \sum_{i=1}^{2m} c_i d^{|S|} S (N + d^{|S|})^3 \right)^{\delta_n}.$$

We have

$$\limsup_{n \to \infty} |U_n|^\frac{1}{n} \leq \frac{\delta^\delta (\kappa - \delta)^{\kappa - \delta}}{\kappa^\delta} \mu^{1+2N^\delta} \left[ \sum_{i=1}^{2m} c_i d^{|S|} S (N + d^{|S|})^3 \right]^\delta := g(\delta).$$

Note that

$$g(0) = \mu, \quad g'(0) = -\infty.$$

Hence for $0 < \delta \leq \delta_0$, $g(\delta) < \mu$. Therefore we have

$$\limsup_{n \to \infty} |U_n|^\frac{1}{n} < \mu,$$

and the proof is complete. \hfill \Box

**Lemma 2.6.** For $r \geq 0$, let

$$b_n(r, F) = \sup_{v \in G} b_n^v(r, F).$$

Then

$$\lim_{n \to \infty} \limsup b_n(0, F)^\frac{1}{n} < \mu.$$

**Proof.** If $\lambda(E^*) < \mu$, then (2.20) follows from Part (a) of Lemma 2.5, and the quasi-transitivity of $G$ under the action of $\Gamma$.

Now assume that $\lambda(E^*) = \mu$. Let $k$ be a positive integer satisfying (2.13). We consider the following cases

A. The starting vertex $v$ of SAWs satisfy

$$\limsup_{n \to \infty} c_n^v(0, \tilde{E}_{k+1})^\frac{1}{\pi} = \mu.$$

B. The starting vertex $v$ of SAWs satisfy

$$\limsup_{n \to \infty} c_n^v(0, \tilde{E}_{k+1})^\frac{1}{\pi} < \mu.$$
In Case A., (2.20) follows from Part (b) of Lemma 2.5. In Case B., (2.20) follows from (2.21) and the following fact
\[ b_n^v(0, F) \leq c_n^v(0, \bar{E}_{k+1}). \] □

**Lemma 2.7.** There exist \( a', R' \in (0, 1) \), such that
\[ \limsup_{n \to \infty} b_n(a'n, F)^{1/n} < R'\mu. \]

**Proof.** Let us consider the number of \( n \)-step SAWs on \( G \) starting from a fixed vertex, such that \( F \) occurs less than \( \xi n \) times, where \( \xi \) is a small positive number.

Let \( \epsilon, m \) satisfy (2.3). If \( F \) occurs in less than \( \xi n \) times, then it occurs in less than \( \xi n \) of the \( N = \left\lceil \frac{n}{m} \right\rceil \) \( m \)-step subwalks
\[ (\pi((im + 1)), \ldots, \pi((i + 1)m)), \text{ for } 0 \leq i \leq n. \]

Hence we have
\[ b_n(\xi n, F) \leq \sum_{i=1}^{\xi n} \left( \begin{array}{c} N \\ i \end{array} \right) \left[ \mu(1 - \epsilon) \right]^N \left[ \mu(1 + \epsilon) \right]^m c_{n-Nm}. \]

Note that when \( \lambda(E^*) < \mu \), \( \left[ \mu(1 - \epsilon) \right]^m \) gives an upper bound for the number of \( m \)-step SAWs starting from a fixed vertex, such that one of the followings holds
\begin{itemize}
  \item \( E^*(m) \) occurs less than \( am \) times; or
  \item \( E^*(m) \) occurs at least \( am \) times, and \( F \) never occurs,
\end{itemize}
where \( a, m \) satisfies (2.17).

When \( \lambda(E^*) = \mu \), \( \left[ \mu(1 - \epsilon) \right]^m \) gives an upper bound for the number of \( m \)-step SAWs starting from a fixed vertex such that
\begin{itemize}
  \item \( \bar{E}_k(m) \) occurs less than \( am \) times; or
  \item \( \bar{E}_k(m) \) occurs at least \( am \) times, \( E_{k+1} \) never occurs, and \( F \) never occurs.
\end{itemize}

Here \( k \) satisfies (2.13), and \( a, m \) satisfies (2.14).

When \( \xi \) is sufficiently small, we have
\[ \limsup_{n \to \infty} b_n(\xi n, F)^{1/n} < \mu, \]
and the lemma follows. □

**Proof of Theorem 1.3 A.** If \( \lambda(E^*) < \mu \), let \( d_n^v \) be the number of \( n \)-step SAWs in \( G \) starting from \( v \) such that \( E^*(m) \) occurs at least \( an \) times, and \( F \) occurs at least \( a'n \) times, where \( a \) is given by (2.17) and \( a' \) is given by Lemma 2.7. Then we have
\[ \limsup_{n \to \infty} d_n^v = \mu. \] pt1 (2.22)
Now assume that $\lambda(E^*) = \mu$. Let $v$ be a vertex of $G$ satisfying (2.15). Let $a > 0$ be given by (2.16), and $a' > 0$ be given by Lemma 2.7. Let $d_n^v$ be the number of $n$-step SAWs in $G$ such that $E_{k+1}$ never occurs, $E_k(m)$ occurs at least $an$ times, and $F$ occurs at least $a'n$ times. Then

\begin{equation}
\limsup_{n \to \infty} d_n^v = \mu.
\end{equation}

Note that in an SAW $\pi$, if $F$ occurs at least $a'n$ times, then $\|\pi\| \geq a'n$. The theorem follows from (2.22) and (2.23).

\section{Proof of Theorem 1.3 B.}

We prove Theorem 1.3 B. in this section. Let $G = (V,E)$ be a graph satisfying Assumption 1.2. Since any graph satisfying Assumption 1.2 must satisfy Assumption 1.1 as well, all the results proved in Section 2 also apply to graphs satisfying Assumption 1.2.

Let $\pi$ be an $n$-step SAW on $G$. Recall that $E^*$ occurs at the $j$th step of $\pi$ if there exists $\gamma_S \in \Gamma_S$ such that $\pi(j) \in \gamma_S$, and all the vertices of $\gamma_S$ are visited by $\pi$. For $k \geq 1$, we say that $E_k$ occurs at the $j$th step of $\pi$, if there exists $\gamma_S' \in \Gamma_S'$, such that $\pi(j) \in \gamma_S'$, and at least $k$ vertices of $\gamma_S'$ are visited by $\pi$, where $S'$ containing $S$ is given as in Assumption 1.2. We say that $\tilde{E}_k$ occurs at the $j$th step of $\pi$ if $E^*$ or $E_k$ (or both) occur there.

In the following, we will use $E$ to denote any of $E^*$, $E_k$ or $\tilde{E}_k$. If $m$ is a positive integer, we say that $E(m)$ occurs at the $j$th step of $\pi$ if $E$ occurs at the $m$th step of the $2m$-step subwalk $(\pi(j - m), \ldots, \pi(j + m))$.

For $r \geq 0$ and $v \in G$, let $c_n^v(r,E)$ (resp. $c_n^v(r,E(m))$) be the number of $n$-step SAWs starting from $v$ for which $E$ (resp. $E(m)$) occurs at no more than $r$ different steps. Let

\begin{align*}
c_n(r,E) &= \sup_{v \in V} c_n^v(r,E) \\
c_n(r,E(m)) &= \sup_{v \in V} c_n^v(r,E(m))
\end{align*}

Let

\begin{equation*}
\lambda(E) = \limsup_{n \to \infty} c_n(0,E)^\frac{1}{n}.
\end{equation*}

Let $\mu$ be the connective constant of $G$. We have that $\lambda(E) < \mu$ if and only if there exist $\epsilon > 0$, $M \in \mathbb{N}$, such that $c_m(0,E) < [\mu(1-\epsilon)]^m$, for $m \geq M$.

\begin{lemma}
Let $k$ satisfy $1 \leq k \leq |S|$, and
\begin{equation}
\lambda(E) < \mu.
\end{equation}
\end{lemma}
Let $\epsilon, M$ satisfy (3.1), and let $m \geq M$ satisfy
\begin{equation}
\tag{3.3}
c_m \leq [\mu(1 + \epsilon)]^m.
\end{equation}
Then there exist $a = a(\epsilon, m)$ and $R = R(\epsilon, m) \in (0, 1)$, such that
\begin{equation}
\limsup_{n \to \infty} c_n(an, \mathcal{E}(m))^{\frac{1}{n}} < R\mu.
\end{equation}
Proof. Same as the proof of Lemma 2.1. \qed

**Lemma 3.2.** $\lambda(\tilde{E}_1) < \mu$.

Proof. The lemma follows from Lemma 2.4 and the fact that $\lambda(\tilde{E}_1) < \lambda(\tilde{E}_1)$. \qed

**Lemma 3.3.** $\lambda(E^*) < \mu$.

Proof. Assume that
\begin{equation}
\tag{3.4}
\lambda(E^*) = \mu;
\end{equation}
we will obtain a contradiction.

We make the following observations. First, $c_n(0, \tilde{E}_k)$ is a non-decreasing function of $k$; secondly, if $E^*$ does not occur on a given walk, then $\mathcal{E}_{|S'_b|}$ cannot occur. Therefore
\begin{equation}
c_n(0, \mathcal{E}^*) \leq c_n(0, \tilde{E}_{|S'_b|}) \leq c_n.
\end{equation}
As a result, (3.4) implies
\begin{equation}
\limsup_{n \to \infty} c_n(0, \tilde{E}_{|S'_b|})^{\frac{1}{n}} = \mu.
\end{equation}

By Lemma 3.2, $\lambda(\tilde{E}_1) < \mu$. We may choose $k$ with $1 \leq k < |S|$, such that
\begin{equation}
\tag{3.5}
\lambda(\tilde{E}_k) < \mu, \quad \lambda(\tilde{E}_{k+1}) = \mu.
\end{equation}

Let $\epsilon, m$ satisfy (3.1) and let $m \geq M$ satisfy (3.3). By Lemma 3.1, there exists $a = a(\epsilon, m) > 0$, such that
\begin{equation}
\limsup_{n \to \infty} c_n(an, \tilde{E}_k(m))^{\frac{1}{n}} < \mu.
\end{equation}

By (3.5) and (3.6) and the quasi-transitivity of $G$, there exists a vertex $v_0$ of $G$, such that
\begin{equation}
\tag{3.7}
\limsup_{n \to \infty} c_n(0, \tilde{E}_{k+1})^{\frac{1}{n}} = \mu;
\end{equation}
and for any $v \in V$,
\begin{equation}
\tag{3.8}
\limsup_{n \to \infty} c_n^v(an, \tilde{E}_k(m))^{\frac{1}{n}} < \mu.
\end{equation}
Let \( T_n \) be the set consisting of all the \( n \)-step SAWs starting from \( v_0 \) for which \( \mathcal{E}_{k+1} \) never occurs, but \( \mathcal{E}_k(m) \) occurs at least \( an \) times. We have that
\[
|T_n| \geq c_n^m(0, \mathcal{E}_{k+1}) - c_n^m(an, \mathcal{E}_k(m)).
\]

By (3.7) and (3.8), we have
\[
\limsup_{n \to \infty} |T_n|^\frac{1}{n} = \mu.
\]

The rest of the proof is devoted to show the existence of \( t < 1 \), such that
\[
\limsup_{n \to \infty} |T_n|^\frac{1}{n} < t\mu.
\]

This contradicts (3.9), and the lemma follows.

Let \( d \) be the maximal vertex degree of \( G \). Let \( \pi \in T_n \), so that \( \pi \) contains at least \( an \) occurrences of \( \mathcal{E}_k(m) \). We can find \( j_1 < \ldots < j_u \) with \( u = \lfloor \kappa n \rfloor - 2 \), where
\[
\kappa = \frac{a}{(2m + 2)d[4S']}.
\]

such that
\( \mathcal{E}_k(m) \) occurs at the \( j_1 \)th, \( j_2 \)th, \ldots, \( j_u \)th steps of \( \pi \),

(and perhaps other steps as well), and in addition,

- \( 0 < j_1 - m, j_u + m < n \),
- \( j_t + m < j_{t+1} - m \);
- for any \( \gamma_1, \gamma_2 \in \Gamma \), such that \( \pi(j_i) \in \gamma_1 S' \) and \( \pi(j_l) \in \gamma_2 S' \), \( i \neq l \), \( \gamma_1 S' \) and \( \gamma_2 S' \) are disjoint.

Such \( j_i \)'s may be found by the following iterative construction. First \( j_1 \) is the smallest \( j > m \) such that \( \mathcal{E}_k(m) \) occurs at the \( j \)th step of \( \pi \). Having found \( j_1, j_2, \ldots, j_r \), let \( j_{r+1} \) be the smallest \( j \) such that
\[
(a) \ j_r + m < j_{r+1} - m;
(b) \ \mathcal{E}_k(m) \) occurs at the \( j \)th step of \( \pi \);
(c) for any \( \gamma_1, \gamma_2 \in \Gamma \), such that \( \pi(j_i) \in \gamma_1 S' \) (\( i \leq r \)), and \( \pi(j_{r+1}) \in \gamma_2 S' \), we have \( \gamma_1 S' \cap \gamma_2 S' = \emptyset \).

Condition (a) gives rise to the factor \((2m + 2)\) in the denominator of (3.10), and Condition (c) gives rise to the factor \( d[4S'] \).

Let \( t \in \{1, 2, \ldots, u\} \). Since \( \mathcal{E}_k(m) \) but not \( \mathcal{E}_{k+1} \) occurs at the \( j_t \)th step, \( \pi \) visits at most \( k \) vertices in each \( \gamma S' \) containing \( \pi(j_t) \). Let \( \Psi_t \) be the set consisting of all the copies \( \gamma S' \) of \( S' \) in \( \Gamma S' \), such that \( \pi(j_t) \in \gamma S' \), \( \pi \) visits exactly \( k \) vertices of \( \gamma S' \), and these \( k \) vertices lie between \( \pi(j_t - m) \) and \( \pi(j_t + m) \) on \( \pi \). Choose \( \gamma_t S' \in \Psi_t \). For \( t = 1, 2, \ldots, u \), let
\[
\alpha_t = \min\{i : \pi(i) \in \gamma_t S'\}, \quad \beta_t = \max\{i : \pi(i) \in \gamma_t S'\},
\]
so that

\[ j_t - m \leq \alpha_t \leq j_t \leq \beta_t \leq j_t + m. \]

We next describe the strategy for the replacement of the subwalk \((\pi(\alpha_t), \pi(\alpha_t + 1), \ldots, \pi(\beta_t))\). Starting from \(\pi(\alpha_t)\), the walk follows an SAW in \(\gamma_t S\) joining \(\pi(\alpha_t)\) and \(\pi(\beta_t)\), and visits every vertex in \(\gamma_t S\). Such an SAW exists by Assumption 1.2.

Let \(\delta\) satisfy \(0 < \delta < \kappa\), to be chosen later, and set \(s = \delta n\). Let \(H = (h_1, \ldots, h_s)\) be an oriented subset of \(\{j_1, \ldots, j_u\}\). We shall make an appropriate substitution in the neighborhood of each \(\pi(h_t)\) to obtain an SAW \(\pi^* = \pi, H\).

We estimate the number of pairs \((\pi, H)\) as follows. First, the number \(|(\pi, H)|\) is at least the cardinality of \(T_n\) multiplied by the minimum number of possible choices of \(H\) as \(\pi\) ranges over \(T_n\). Any subset of \(\{j_1, \ldots, j_u\}\) with cardinality \(s = \delta n\) may be chosen for \(H\), whence

\[ |(\pi, H)| \geq |T_n| \left( \frac{\kappa n - 2}{\delta n} \right) \]

We bound \((\pi, H)\) above by counting the number of SAWs \(\pi^*\) of \(G\) with length not exceeding \(n + |S'|\delta n\), and multiplying by an upper bound for the number of pairs \((\pi, H)\) giving rise to a particular \(\pi^*\).

The number of possible choices of \(\pi^*_s\) is no greater than \(\sum_{i=0}^{n+|S'|\delta n} c_i\). A given \(\pi^*_s\) contains \(|H| = \delta n\) occurrences of visits to all the vertices of some copy of \(S'\) under \(\Gamma\). At the \(t\)th such occurrence, \(\pi(h_t)\) is a point on \(\gamma_t S'\) and there are no more than \(|S'|\) different choices of \(\pi(h_t)\). For given \(\pi^*_s\) and \((\pi(h_t) : t = 1, 2, \ldots, s)\), there are at most \(\left[\sum_{i=1}^{2m} c_i\right]^\delta\) corresponding SAWs \(\pi\) of \(G\). Therefore

\[ |(\pi, H)| \leq \left( d^{|S'|}|S'|^2 \sum_{i=1}^{2m} c_i \right)^{\delta n} \left( \sum_{i=0}^{n+|S'|\delta n} c_i \right) \]

We combine (3.11) and (3.12), take \(\sqrt[n]{\text{nth root}}\) and the limit as \(n \to \infty\), we obtain, by the fact that \(c^\frac{1}{n} \to \mu\),

\[ \mu^{\frac{\kappa^\kappa}{\delta(\kappa - \delta)^{\kappa - \delta}}} \leq \left( d^{|S'|}|S'|^2 \sum_{i=1}^{2m} c_i \right)^{\delta} \mu^{1+|S'|\delta}. \]

There exists \(Z = Z(\epsilon, m, d, |S'|)\), such that

\[ d^{|S'|}|S'|^2 \mu^{|S'|} \sum_{i=1}^{2m} c_i \leq Z, \]

therefore

\[ \mu \leq f(\eta)^{\kappa} \mu, \]
where \( f(\eta) = Z^\eta \eta^\eta (1 - \eta)^{1-\eta} \), and \( \eta = \frac{\delta}{\kappa} \). Since
\[
\lim_{\eta \to 0} f(\eta) = 1, \quad \lim_{\eta \to 0} f'(\eta) = -\infty,
\]
we have \( f(\eta) < 1 \) for sufficiently small \( \eta = \eta(Z) > 0 \). The contradiction implies the lemma.

**Proof of Theorem 1.3 B.** By Lemma 3.3, \( \lambda(E^*) < \mu \). Let \( \sigma_n \) be the number of \( n \)-step SAWs on \( G \) starting from a fixed vertex such that one of the followings holds
- \( E^*(m) \) occurs less than \( an \) times; or
- \( E^*(m) \) occurs at least \( an \) times, and \( F \) occurs less than \( a'n \) times
where \( a \) satisfies (2.17) and \( a' \) satisfies Lemma 2.7. We have
\[
\limsup_{n \to \infty} \sigma_n < \mu.
\]
For any \( n \)-step SAW \( \pi \) on \( G \) starting from a fixed vertex not counted in \( \sigma_n \), \( F \) occurs at least \( a'n \) times, and therefore \( \|\pi\| \geq a'n \). This implies that SAWs on \( G \) have positive speed. \( \square \)

4. **Groups with more than one end**

In this section, we prove Theorem 1.7. The proof is based on the stalling’s splitting theorem, and an explicit construction of the set \( S \) satisfying Assumption 1.1.

**Lemma 4.1.** Let \( G = (V, E) \) be an infinite, connected, locally finite graph with maximal vertex degree \( d \). Let \( A, B \subseteq V \) be two finite set of vertices of \( G \) satisfying \( A \subseteq B \). Let \( G \setminus A \) (resp. \( G \setminus B \)) be the subgraph of \( G \) obtained from \( G \) by removing all the vertices in \( A \) (resp. \( B \)) as well as their incident edges. If \( G \setminus A \) has at least two infinite components, then \( G \setminus B \) has at least two infinite components.

**Proof.** Let \( R_1 \) and \( R_2 \) be two infinite components of \( G \setminus A \). We will show that each one of \( R_1 \setminus B \) and \( R_2 \setminus B \) has at least one infinite component.

Since \( R_1 \setminus B \) can be obtained from \( R_1 \) by removing finitely many vertices and edges, if \( R_1 \setminus B \) has no infinite components, then \( R_1 \setminus B \) has infinitely many finite components. Moreover, since each vertex of \( G \) is incident to at most \( d \) edges, for any connected subgraph of \( G \), removing one vertex of the subgraph as well as all its incident edges can split the subgraph into at most \( d \) connected components. Since \( |B| < \infty \), it is not possible that \( R_1 \setminus B \) has infinitely many finite components. As a result, \( R_1 \setminus B \) has at least one infinite component. The fact that \( R_2 \setminus B \) has at least one infinite component can be proved similarly. \( \square \)

Now we prove Theorem 1.7. By Stalling’s splitting theorem ([26, 27]), a group \( \Gamma \) has more than one end if and only if one of the followings holds.
(1) the group $\Gamma$ is an amalgated free product, i.e.

$$\Gamma = H \ast_C K,$$

where $H, K$ are groups, and $C$ is a finite group such that $C \neq H$ and $C \neq K$.

(2) There exists a group $H$, two finite subgroups $C_1, C_2$ of $H$, and a group isomorphism $\phi : C_1 \to C_2$, such that $\Gamma$ is the following HNN extension

$$\Gamma = \langle H, t | t^{-1}c_1t = \phi(c_1), \forall c_1 \in C_1 \rangle$$

We will prove Theorem 1.7 for Case (1) and (2), in the subsections below.

4.1. **Proof of Theorem 1.7 when $\Gamma$ is an amalgamated free product.** In this section, we prove Theorem 1.7 when the group $\Gamma$ is a free product with amalgamation, as described in Case (1). We start with the following standard result concerning members in an amalgamated free product.

**Lemma 4.2.** *(Normal form for amalgamated free product [20])* Every element in $H \ast_C K$ which is not in the image of $C$ can be written in the normal form

$$v_1 \cdot \ldots \cdot v_n$$

where the terms $v_i$ lie in $H \setminus C$ or $K \setminus C$ and alternate between two sets. The length $n$ is uniquely determined and two such expressions $v_1 \cdot \ldots \cdot v_n$ give the same element in $H \ast_C K$ if and only if there are elements $c_1, \ldots, c_n \in C$, so that

$$w_k = c_{k-1}v_kc_k^{-1},$$

where $c_0 = c_n = 1$.

Assume that $\Gamma$ is a free product with amalgamation as described in Case (1). Let $G_H$ (resp. $G_K$) be a locally finite Cayley graph for $H$ (resp. $K$) with respect to a finite set of generators $T_H$ (resp. $T_K$). Let $G_0$ be a locally finite Cayley graph of $\Gamma$ constructed from $G_H$ and $G_K$ as follows.

(a) Construct the free product graph $G_H \ast_G K$ of $G_H$ and $G_K$. In other words $G_H \ast_G K$ is the Cayley graph for the free product $H \ast K$ with respect to the generator set $T_H \cup T_K$.

(b) Glue vertices $u \in G_H \ast_G K$ and $w \in G_H \ast_G K$ if there exists a vertex $v \in G_H \ast_G K$ such that $v^{-1}u = v^{-1}w \in C$.

Let $1_\Gamma$ be the identity element of the group $\Gamma$. Let $G$ be a locally finite Cayley graph of $\Gamma$ with respect to the generator set $T$ such that $|T| < \infty$, $T = T^{-1}$ and $1_\Gamma \notin T$. Let

$$D_0 = \max_{t \in T, v \in \Gamma} \text{dist}_{G_0}(v, vt),$$

where $\text{dist}_{G_0}(\cdot, \cdot)$ is the graph distance in $G_0$. 

(4.1)
Lemma 4.3. Let $u \in \Gamma$ have normal form starting from a term in $H \setminus C$; and let $v \in \Gamma$ have normal form starting from a term in $K \setminus C$, then any path in $G_0$ joining $1_\Gamma$ and $u^{-1}v$ must visit a point in $u^{-1}C$.

Proof. Since $u \in \Gamma$ has a normal form starting from a term in $H \setminus C$, and $v \in \Gamma$ has a normal form starting from a term in $K \setminus C$, then the concatenation of normal forms of $u^{-1}$ and $v$ gives us a normal form of $u^{-1}v$, denoted by $z_1 \cdots z_n$. This normal form gives rise to a path $\tau_{u^{-1}v}$ in $G_0$ joining $1_\Gamma$ and $u^{-1}v$. More precisely, $\tau_{u^{-1}v}$ is the concatenation of $n$ paths $\tau_i$, $1 \leq i \leq n$, such that $\tau_i$ is the shortest path in $z_1 \cdots z_{i-1}H$ (resp. $z_1 \cdots z_{i-1}K$) joining $z_1 \cdots z_{i-1}$ and $z_1 \cdots z_i$, if $z_i \in H \setminus C$ (resp. $z_i \in K \setminus C$). We can see that the path $\tau_{u^{-1}v}$ visits the vertex $u^{-1}$.

Assume that $l_{u^{-1}v}$ is a path in $G_0$ joining $1_\Gamma$ and $u^{-1}v$. Then $l_{u^{-1}v}$ gives rise to a sequence $x_1, x_2, \ldots, x_m$, such that

(a) for $1 \leq i \leq m$, $l_{u^{-1}v}$ visits every vertex in $\{x_1 \cdots x_i\}_{1 \leq i \leq m}$;
(b) for $1 \leq i \leq m$, $x_i \in H \setminus C$ or $x_i \in K \setminus C$;
(c) for $1 \leq i \leq m-1$, if $x_i \in H \setminus C$, then $x_{i+1} \in K \setminus C$;
(d) $u^{-1}v = x_1 \cdots x_m c$, where $c \in C$.

Indeed $x_1, x_2, \ldots, x_m$ can be found as follows. Let $\{y_1, \ldots, y_l\}$ be all the vertices incident to an edge in $H \setminus C$ along $l_{u^{-1}v}$ and an edge in $K \setminus C$ along $l_{u^{-1}v}$, and assume that starting from $1_\Gamma$ and traversing along $l_{u^{-1}v}$, one visits $y_1, \ldots, y_l$ in order. Let $y_0 = 1_\Gamma$. From $\{y_1, \ldots, y_l\}$, we perform the following manipulations.

A. Remove all the $y_i$’s such that $y_{i-1}^{-1}y_i \in C$; let $\{y_1^{(1)}, \ldots, y_{l_1}^{(1)}\}$ be the remaining set of vertices;

B. Remove all the $y_i^{(1)}$’s such that either both $[y_{i-1}^{(1)}]^{-1}y_i^{(1)}$ and $[y_i^{(1)}]^{-1}y_{i+1}^{(1)}$ are in $H$ or both $[y_{i-1}^{(1)}]^{-1}y_i^{(1)}$ and $[y_i^{(1)}]^{-1}y_{i+1}^{(1)}$ are in $K$; let $\{y_1^{(2)}, \ldots, y_{l_2}^{(2)}\}$ be the remaining set of vertices.

Once we have constructed $\{y_1^{(2j)}, \ldots, y_{l_2}^{(2j)}\}$, for $j \geq 1$, we perform the following manipulations.

A. Remove all the $y_i$’s such that $[y_{i-1}^{(2j)}]^{-1}y_i^{(2j)} \in C$; let $\{y_1^{(2j+1)}, \ldots, y_{l_1}^{(2j+1)}\}$ be the remaining set of vertices;
Lemma 4.4. Let $u \in \Gamma$ have normal form starting from a term in $H \backslash C$; and let $v \in \Gamma$ have normal form starting from a term in $K \backslash C$. If the lengths of normal forms of $u$ and $v$ are sufficiently large, $u$ and $v$ are in two distinct infinite components of $G \backslash C_1$.

In particular, $G \backslash C_1$ has at least two distinct infinite components.

Proof. By Lemma 4.3, any path in $G_0$ joining $1_{\Gamma}$ and $u^{-1}v$ must visit a point in $u^{-1}C$.

Assume that there exists a path $l_{uv}$ joining $u$ and $v$ in $G \backslash C_1$. Then for any edge $\langle p, q \rangle$ along $l_{uv}$, $p$ and $q$ can be joined by a path in $G_0$ which does not pass through any vertex in $C$. That is because $\text{dist}_{G_0}(p, q) \leq D_0$ by (4.1), and $\text{dist}_{G_0}(p, C) \geq D_0 + 2$ and $\text{dist}_{G_0}(q, C) \geq D_0 + 2$, by (4.2) and the fact that $p, q \in G \backslash C_1$. By replacing each edge along $l_{uv}$ with a path in $G_0$ joining $p$ and $q$ that does not pass through any vertex in $C$, we obtain that the shortest path in $G_0$ joining $u$ and $v$ does not pass through any vertex in $C$. This implies that there exists a path in $G_0$ joining $1_{\Gamma}$ and $u^{-1}v$ which does not visit any vertex of $u^{-1}C$, by the vertex-transitivity of $G_0$. This is a contradiction.

As a result, any path in $G$ joining $u$ and $v$ must visit a vertex in $C_1$. Since $C_1$ is finite, the lengths of normal forms for vertices in $C_1$ have a maximum $M_0$. If the lengths of normal forms for $u$ and $v$ exceed $M_0$, then $u \in G \backslash C_1$ and $v \in G \backslash C_1$. This
means that \( u \) and \( v \) are in two distinct components of \( G \setminus C_1 \). Moreover, we can find a singly-infinite path \( l_u \) (resp. \( l_v \)) on \( G \) starting from \( u \) (resp. \( v \)), such that moving along the path from \( u \), the length of normal forms along the path is non-decreasing. Therefore all the vertices along \( l_u \) (resp. \( l_v \)) are in \( G \setminus C_1 \). As a result, \( u \) and \( v \) are in two distinct infinite component of \( G \setminus C_1 \).

Let \( \Omega_n \) be the set of \( n \)-step SAWs on \( G \) starting from the identity vertex \( 1_\Gamma \). Let \( r_1 \in \Gamma \) have a normal form starting from a term in \( H \setminus C \), and ending in a term in \( H \setminus C \). Let \( r_2 \Gamma \) have a normal form starting from a term in \( K \setminus C \) and ending at a term in \( K \setminus C \). Assume that the lengths of normal forms of \( r_1 \) and \( r_2 \) are strictly greater than the maximal length of normal forms of elements in \( C_2 \). Also assume that

\[
d_{r_1}(4.3) \quad \operatorname{dist}_G(r_1, C_2) \geq |C_2| + 1;
\]

\[
d_{r_2}(4.4) \quad \operatorname{dist}_G(r_2, C_2) \geq |C_2| + 1.
\]

Let \( A \) (resp. \( B \)) be the collection of elements in \( \Gamma \) with a normal form starting from an element in \( H \setminus C \) (resp. \( K \setminus C \)). Let

\[
S = C_2,
\]

then by the construction of \( C_2 \) it is obvious that \( S \) is connected. The graph \( G \setminus S \) has at least two distinct infinite components by Lemma 4.4, \( C_1 \subset C_2 \) and Lemma 4.1. Let \( A_1 \subset A \) be a component of \( G \setminus S \), define \( \phi(S, A_1) = r_2 \). Let \( B_1 \subset B \) be a component of \( G \setminus S \), define \( \phi(S, B_1) = r_1 \).

**Lemma 4.5.** Assumption 1.1 (3) holds with the choice of \( S \) as given by (4.5).

**Proof.** For each component of \( G \setminus S \), either all the elements in the component have a normal form starting from a term in \( H \setminus C \), or all the elements in the component have a normal form starting from a term in \( K \setminus C \), by Lemma 4.4.

Let \( A_1 \) be a component of \( G \setminus S \) such that all the elements in \( A_1 \) have a normal form starting from a term in \( H \setminus C \). For each \( v \in A_1 \), since \( r_2 \) has a normal form ending in a term in \( K \setminus C \), and \( v \) has a normal form starting from a term in \( H \setminus C \), the concatenation of normal forms of \( r_2 \) and \( v \) gives us a normal form of \( r_2v \). Moreover \( r_2v \) has a normal form starting from a term in \( K \setminus C \) since \( r_2 \) has a normal form starting from a term in \( K \setminus C \). Therefore, \( r_2v \notin A_1 \). Given the assumption that the length of the normal form of \( r_2 \) is strictly greater than the maximal length of normal forms of elements in \( S \), the length of the normal form of \( r_2v \) is also strictly greater than the maximal length of normal forms of elements in \( S \), we have \( r_2v \notin S \). Hence \( r_2v \) is in a component of \( G \setminus S \) different from \( A_1 \). For any two vertices \( u, v \in A_1 \), by the connectivity of \( A_1 \), there exists a path \( l_{uv} \) joining \( u \) and \( v \) and consisting of vertices in \( A_1 \). Then \( r_2l_{uv} \) is a path joining \( r_2u \) and \( r_2v \) which does not intersect \( S \),
since any vertex along $r_2 l w$ is in $r_2 A_1$, and $r_2 A_1 \cap S = \emptyset$. Therefore all the vertices in $r_2 A_1$ are in the same component $B_1$ of $G \setminus S$ such that $B_1 \cap A_1 = \emptyset$.

As in Assumption 1.1 (3), let $\partial A_1, S$ be the set consisting of all the vertices in $S$ incident to a vertex in $A_1$. Let $u \in A_1$, $w \in \partial A_1, S$, and $e$ be the edge of $G$ with endpoints $u$ and $w$. Since $r_2 u \in B_1$, we have $r_2 w \in B_1 \cup \partial B_1, S$, since $r_2 u$ and $r_2 w$ are adjacent vertices in $G$. Let $l_{w,r_2w}$ be a path in $G$ joining $w$ and $r_2 w$, starting from $w$ and ending in $r_2 w$. Let $p$ be the last vertex of $S$ visited by $l_{w,r_2w}$, and let $q$ be the first vertex of $r_2 S$ visited by $l_{w,r_2w}$. By the connectivity of $S$, there exists a path $L_{wp}$ joining $w$ and $p$ and consisting of vertices of $S$; also, there exists a path $L_{q,r_2w}$ joining $q$ and $r_2 w$ and consisting of vertices of $r_2 S$. By (4.4), $r_2 S \cap S = \emptyset$. Since $r_2 \in r_2 S$ and $r_2 \notin A_1$, the connectivity of $r_2 S$ implies that $r_2 S$ is in a component of $G \setminus S$ different from $A_1$; in particular $r_2 S \cap A_1 = \emptyset$. Moreover, since $r_2 w \in r_2 S$, $r_2 w \in B_1 \cup \partial B_1, S$, we have $r_2 S \subset B_1$. Let $l_{pq}$ be the portion of $l_{w,r_2w}$ between $p$ and $q$. All the vertices along $l_{pq}$ except $p$ are outside $S$, hence they are in the same component of $G \setminus S$. Since $q \in r_2 S \subset B_1$, all the vertices along $l_{pq}$ except $p$ are in $B_1$. Similarly, all the vertices along $l_{pq}$ except $q$ are outside $r_2 S$, hence they are in the same component of $G \setminus r_2 S$. Under the assumption that the length of the normal form of $r_2$ is strictly greater than the maximal length of normal forms of elements in $C_2(=S)$, we have $r_2 A_1 \cap S = \emptyset$. Since $p \in S$, $p \notin r_2 A_1$. Therefore no vertices in $l_{pq}$ except $p$ and $q$ are in $A_1 \cup r_2 A_1$. It is straightforward to check that no vertices in $L_{wp} \cup L_{q,r_2w}$ are in $A_1 \cup r_2 A_1$. Let $L_{w,r_2w} = L_{wp} \cup l_{pq} \cup L_{q,r_2w}$; then $L_{w,r_2w}$ is a path joining $w$ and $r_2 w$ in $G \setminus [A_1 \cup r_2 A_1]$. □

4.2. Proof of Theorem 1.8 when $\Gamma$ is an amalgamated free product. In this section, we prove Theorem 1.8 when $\Gamma$ is an amalgamated free product. Let $\Gamma$ be a finitely generated, infinite group which is a free product with amalgamation as described in (1). It suffices to construct a locally finite Cayley graph $G$ of $\Gamma$ such that SAWs on $G$ have positive speed.

Choose a Cayley graph $G_H$ (resp. $G_K$) for $H$ (resp. $K$) such that any two vertices in $C$ (resp. $C$) are joined by an edge. Let $G$ be the graph obtained from the free product graph $G_H \ast G_K$ by gluing the vertices $u \in G_H \ast G_K$ and $w \in G_H \ast G_K$ satisfying the condition that there exists a vertex $v \in G_H \ast G_K$ such that $v^{-1} u = v^{-1} w \in C$.

Let

$$S' = S = C$$

It is not hard to check that for the locally finite Cayley graph $G$ of $\Gamma$ constructed above with the finite set of vertices $S$, Assumption 1.2 is satisfied and SAWs on $G$ have positive speed.
4.3. HNN extension. In this section, we prove Theorems 1.7 and 1.8 when \( \Gamma \) is an HNN extension as described by (2).

Choose a set of representatives of the right cosets of \( C_1 \) in \( G \), and a set of representatives of the right cosets of \( C_2 \) in \( G \). We shall assume that 1 is the representatives of both \( C_1 \) and \( C_2 \). The choice of coset representatives is to be fixed to the rest of the discussion.

**Definition 4.6.** (Normal form for HNN extension [20]) Let \( \Gamma \) be the HNN extension with a presentation

\[
\Gamma = \langle H, t | t^{-1}c_1t = \phi(c_1), \forall c_1 \in C_1 \rangle.
\]

where \( H \) is a group, \( C_1, C_2 \) are two finite subgroups of \( H \), and \( \phi : C_1 \to C_2 \) is a group isomorphism. A normal form is a sequence \( g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n (n \geq 0) \) where

- \( g_0 \) is an arbitrary element of \( \Gamma \);
- if \( \epsilon_i = -1 \), then \( g_i \) is a representative of a coset of \( C_1 \) in \( \Gamma \);
- if \( \epsilon_i = 1 \), then \( g_i \) is a representative of a coset of \( C_2 \) in \( \Gamma \); and
- there is no consecutive subsequence \( t^{\epsilon_i}, 1, t^{-\epsilon} \).

**Theorem 4.7.** (Uniqueness of normal form [20]) Every element \( w \) of \( \Gamma \) with a presentation as in (2) has a unique representation as

\[
w = g_0 t^{\epsilon_1} \cdots t^{\epsilon_n} g_n,
\]

where \( g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n \) is a normal form. Let \( n \) be the length of the normal form of \( w \).

4.3.1. Proof of Theorem 1.7 when \( \Gamma \) is an HNN extension. Let \( \Gamma \) be an infinite, finitely generated group, which is an HNN extension as described in (2). Let \( G_H \) be a locally finite Cayley graph for \( H \) with respect to a finite set of generators \( T_H \) satisfying \( |T_H| < \infty \), \( T_H = T_H^{-1} \) and \( 1 \notin T_H \).

Let \( G_0 \) be the Cayley graph of the HNN extension \( \Gamma \) with respect to the generator set \( T_H \cup \{t, t^{-1}\} \). Let \( G \) be a locally finite Cayley graph of \( \Gamma \) with respect to a finite generator set \( T \) satisfying \( T = T^{-1} \), \( |T| < \infty \) and \( 1 \notin T \).

Let

\[
D_0 = \max_{v \in \Gamma} \text{dist}_{G_0}(v, vs);
\]

\[
S_0 = \left\{ w \in \Gamma, \text{dist}_{G_0}(w, C_1 \cup C_2) \leq \frac{D_0}{2} \right\}
\]

Obviously \( C \) is finite since both \( C_1 \) and \( C_2 \) are finite.

**Lemma 4.8.** Let \( u, v \in \Gamma \) such that one of the followings hold

(a) \( v \) has a normal form with \( \epsilon_1 = 1 \), \( g_0 \in C_1 \); and \( u \) has a normal form with \( \epsilon_1 = -1 \); or
(b) \( v \) has a normal form with \( \epsilon_1 = -1, g_0 \in C_2 \); and \( u \) has a normal form with \( \epsilon_1 = 1 \); or
(c) \( v \) has a normal form with \( \epsilon_1 = 1, g_0 \in C_1 \); and \( u \) has a normal form with \( \epsilon_1 = 1, g_0 \in H \setminus C_1 \); or
(d) \( v \) has a normal form with \( \epsilon_1 = -1, g_0 \in C_2 \); and \( u \) has a normal form with \( \epsilon_1 = -1, g_0 \in H \setminus C_2 \);

Then any path in \( G_0 \) joining \( 1_\Gamma \) and \( u^{-1}v \) must visit a vertex in \( u^{-1}[C_1 \cup C_2] \).

**Proof.** Let \( l_{u^{-1}v} \) be a path in \( G_0 \) joining \( 1_\Gamma \) and \( u^{-1}v \). Let \( y_1, \ldots, y_n \) be all the vertices along \( l_{u^{-1}v} \) such that one of the followings holds

A. both edges incident to \( y_i \) along \( l_{u^{-1}v} \) are \( t \) or \( t^{-1} \); or
B. one edge incident to \( y_i \) along \( l_{u^{-1}v} \) is \( t \) or \( t^{-1} \); the other edge incident to \( y_i \) along \( l_{u^{-1}v} \) is an edge of \( G_H \).

Count each vertex in Case A. twice in \( \{y_1, \ldots, y_n\} \), and count each vertex in Case B. once in \( \{y_1, \ldots, y_n\} \). We perform the following manipulations on the set \( \{y_1, \ldots, y_n\} \).

1. For all \( i \)'s satisfying \( y_{i-1}^{-1}y_i = t^{-1}, y_{i+1}^{-1}y_i \in C_1 \), and \( y_{i+1}^{-1}y_{i+2} = t \), remove \( y_{i-1}, y_i, y_{i+1}, y_{i+2} \) from \( \{y_1, \ldots, y_n\} \), and let \( \{y_1^{(1)}, \ldots, y_n^{(1)}\} \) be the new sequence;
2. For all \( i \)'s satisfying \( y_{i-1}^{-1}y_i = t, [y_i^{-1}y_{i+1}]^{-1}y_i \in C_2 \), and \( [y_{i+1}^{-1}y_{i+2}]^{-1}y_i = t^{-1} \), remove \( y_{i-1}^{(1)}, y_i^{(1)}, y_{i+1}^{(1)}, y_{i+2}^{(1)} \) from \( \{y_1^{(1)}, \ldots, y_n^{(1)}\} \), and let \( \{y_1^{(2)}, \ldots, y_n^{(2)}\} \) be the new sequence;

Assume we have obtained \( \{y_1^{(2j)}, \ldots, y_n^{(2j)}\} \), then we perform the following inductive manipulations

1. For all \( i \)'s satisfying \( [y_i^{(2j)}]^{-1}y_i^{(2j+1)} = t^{-1}, [y_i^{(2j+1)}]^{-1}y_i^{(2j)} \in C_1 \), and \( [y_i^{(2j+1)}]^{-1}y_i^{(2j+2)} = t \), remove \( y_{i-1}^{(2j)}, y_i^{(2j)}, y_i^{(2j+1)}, y_i^{(2j+2)} \) from \( \{y_1^{(2j)}, \ldots, y_n^{(2j)}\} \), and let \( \{y_1^{(2j+1)}, \ldots, y_n^{(2j+1)}\} \) be the new sequence;
2. For all \( i \)'s satisfying \( [y_i^{(2j+1)}]^{-1}y_i^{(2j+2)} = t, [y_i^{(2j+2)}]^{-1}y_i^{(2j+1)} \in C_2 \), and \( [y_i^{(2j+2)}]^{-1}y_i^{(2j+1)} = t^{-1} \), remove \( y_{i-1}^{(2j+1)}, y_i^{(2j+1)}, y_i^{(2j+2)}, y_i^{(2j+3)} \) from \( \{y_1^{(2j+1)}, \ldots, y_n^{(2j+1)}\} \), and let \( \{y_1^{(2j+2)}, \ldots, y_n^{(2j+2)}\} \) be the new sequence.

We continue the above process until we obtain \( \{y_1^{(2k)}, \ldots, y_n^{(2k)}\} \) such that

1. There are no \( i \)'s satisfying \( [y_i^{(2k)}]^{-1}y_i^{(2k)} = t^{-1}, [y_i^{(2k)}]^{-1}y_i^{(2k)} \in C_1 \), and \( [y_i^{(2k)}]^{-1}y_i^{(2k)} = t \); and
2. There are no \( i \)'s satisfying \( [y_i^{(2k)}]^{-1}y_i^{(2k)} = t, [y_i^{(2k)}]^{-1}y_i^{(2k)} \in C_2 \), and \( [y_i^{(2k)}]^{-1}y_i^{(2k)} = t^{-1} \).
Lemma 4.9. Let $G \setminus S_0$ be the subgraph obtained from $G$ by removing all the vertices in $S_0$ as well as their incident edges. For any $u, v \in G \setminus S_0$ one of (a) (b) (c) (d) in Lemma 4.8 holds, $u$ and $v$ are in two distinct components of $G \setminus S_0$. Moreover, $G \setminus S_0$ has at least two infinite components.

Proof. To show that $u$ and $v$ are in two distinct components of $G \setminus S_0$, it suffices to show that any path in $G$ joining $u$ and $v$ must visit a vertex in $S_0$.

Let $l_{uv}$ be a path in $G$ joining $u$ and $v$. Assume that $l_{uv}$ visits no vertices in $S_0$. For any edge $e = (p, q) \in l_{uv}$, $p$ and $q$ can be joined by a path in $G_0$ whose length does not exceed $D_0$. Then there exists a path in $G_0$ joining $u$ and $v$ and visiting no vertices in $C_1 \cup C_2$ by the definition of $S_0$ in (4.6). This is equivalent to the condition that there exists a path in $G_0$ joining $1_t$ and $u^{-1}v$, which visits no vertices in $u^{-1}[C_1 \cup C_2]$. But this is a contradiction to Lemma 4.8.

For $u$ and $v$ satisfying the condition of the theorem, assume the lengths of the normal forms of $u$ and $v$ are strictly greater than the maximal lengths of normal forms of elements in $S_0$. Assume the normal form of $u$ (resp. $v$) has length $n_1$ (resp. $n_2$). Let $\epsilon_{n_1}(u)$ (resp. $\epsilon_{n_2}(v)$) be the exponent of the $n_1$th (resp. $n_2$th) $t$ in the normal form of $u$ (resp. $v$). Then $\{u t^{\epsilon_{n_1}(u)} k\}^{\infty}_{k=1}$ and $\{v t^{\epsilon_{n_2}(v)} k\}^{\infty}_{k=1}$ are in two distinct infinite components of $G \setminus S_0$. □

Let $S_1$ be a finite, connected set of vertices in $G$ containing $S_0$. By Lemmas 4.1 and 4.9, the fact that $G \setminus S_0$ has at least two distinct infinite components implies that $G \setminus S$ has at least two distinct infinite components.
4.3.2. $C_1 = C_2 = H$. In this section, we consider the case when the group $\Gamma$ is an HNN extension, as in Definition 4.6, such that $C_1 = C_2 = H$.

Lemma 4.10. Assume that the group $\Gamma$ is a finitely generated HNN extension, as in Definition 4.6, such that $C_1 = C_2 = H$. Then any locally finite Cayley graph $G$ of $\Gamma$ has two ends.

Proof. It is a well-known fact that the number of ends of locally finite Cayley graphs of a finitely generated group do not depend on the choices of finite generating set. Therefore it suffices to prove the theorem for a specific choice of generating set. Let $G$ be the Cayley graph with respect to the generating set $H \cup \{t, t^{-1}\}$. It is straightforward to check that $G$ has two ends. \qed

4.3.3. $|C_1| = |C_2| < H$. Now we prove Theorem 1.7 when the group $\Gamma$ is a finitely generated HNN extension, as in Definition 4.6, such that $C_1$ is a proper subset of $H$. Since $C_1$ is finite, this implies that $|C_1| = |C_2| < |H|$, $C_2$ is also a proper subset of $H$.

Lemma 4.11. When the group $\Gamma$ is a finitely generated HNN extension, as in Definition 4.6, such that $C_1$ is a proper subset of $H$, then there exists $a \in (0, 1]$, such that

$$\limsup_{n \to \infty} \frac{1}{n} \left| \{\pi_n : \|\pi_n\| \geq an\} \right|^\frac{1}{n} = \mu,$$

where $\pi_n$ is an $n$-step SAW starting from $1_{\Gamma}$, and $\mu$ is the connective constant.

Proof. It suffices to check Assumption 1.1, then the lemma follows from Theorem 1.3 A.

Let $S$ be a finite set of vertices including $S_0$ such that $S$ is connected.

Let $v \in S$ be incident to $w \in G \setminus S$. Let $A_w$ be the component of $G \setminus S$ including $w$. Let $k$ be an integer which is strictly greater than the maximal length of normal forms of elements in $S$.

A. If $w$ has a normal form with $g_0 \in H \setminus [C_2 \cup C_1]$, let $\phi(S, A_w) = t^{-2k}$;

B. If $w$ has a normal form with $g_0 \in C_1$, $\epsilon_1 = 1$, let $\phi(S, A_w) = t^{-2k}r_1$, where $r_1 \in H \setminus C_1$;

C. If $w$ has a normal form with $g_0 \in C_1 \setminus C_2$, $\epsilon_1 = -1$, and $A_w$ contains no vertices satisfying Case A.; let $\phi(S, A_w) = t^{2k}$;

D. If $w$ has a normal form with $g_0 \in C_2$, $\epsilon_1 = -1$, let $\phi(S, A_w) = r_2t^{-2k}$, where $r_2 \in H \setminus C_2$;

E. If $w$ has a normal form with $g_0 \in C_2 \setminus C_1$, $\epsilon_1 = 1$, and $A_w$ contains no vertices satisfying Case A. or Case C.; let $\phi(S, A_w) = t^{-2k}$.

For Case A., $\phi(S, A_w)w$ and $w$ are in distinct components of $G \setminus S$ by (b)(d) of Lemma 4.8. Let $u$ be an arbitrary vertex in $A_w$, then the following cases might occur.
(i) \( u \) has a normal form with \( g_0 \in C_1 \setminus C_2, \epsilon_1 = -1; \)
(ii) \( u \) has a normal form with \( g_0 \in C_2 \setminus C_1, \epsilon_1 = 1; \)

If \( \phi(S, A_w)u \in S \), then \( u \in t^{2k}S \), then \( u \) has normal form with \( g_0 \in C_1, \epsilon_1 = 1 \), but this is possible in neither Case (i) nor Case (ii). Therefore \( \phi(S, A_w)A_w \) is in a component of \( G \setminus S \) different from \( A_w \).

For Case B. \( \phi(S, A_w)w \) and \( w \) are in distinct components of \( G \setminus S \) by (b) of Lemma 4.8. Note that any vertex in \( A_w \) has a normal form with \( g_0 \in C_1, \epsilon_1 = 1 \). Therefore \( \phi(S, A_w)A_w \) is in a component of \( G \setminus S \) different from \( A_w \).

For Case C. \( \phi(S, A_w)w \) and \( w \) are in distinct components of \( G \setminus S \) by (a) of Lemma 4.8. Let \( u \) be an arbitrary vertex in \( A_w \), then \( u \) may have a normal form with \( g_0 \in C_2 \setminus C_1, \epsilon_1 = 1 \). If \( \phi(S, A_w)u \in S \), then \( u \in t^{-2k}S \). But this is not possible since in this case the normal form of \( u \) satisfies \( \epsilon_1 = -1 \). Therefore \( \phi(S, A_w)A_w \) is in a component of \( G \setminus S \) different from \( A_w \).

For Case D. \( \phi(S, A_w)w \) and \( w \) are in distinct components of \( G \setminus S \) by (d) of Lemma 4.8. Note that any vertex in \( A_w \) has a normal form with \( g_0 \in C_2 \setminus C_1, \epsilon_1 = 1 \). Therefore \( \phi(S, A_w)A_w \) is in a component of \( G \setminus S \) different from \( A_w \).

For Case E. \( \phi(S, A_w)w \) and \( w \) are in distinct components of \( G \setminus S \) by (a) of Lemma 4.8. Note that any vertex in \( A_w \) has a normal form with \( g_0 \in C_2 \setminus C_1, \epsilon_1 = 1 \). Therefore \( \phi(S, A_w)A_w \) is in a component of \( G \setminus S \) different from \( A_w \).

Let \( B_w \) be the component of \( G \setminus S \) containing \( \phi(S, A_w)A_w \). By the construction above, we have

\[
\phi(S, A_w)S \cap S = \emptyset.
\]

Therefore, \( \phi(S, A_w)A_w \subset B_w \) implies

\[
\phi(S, A_w)S \subset B_w.
\]

Also it is straight forward to check from Cases A.-E. that \( A_w \) and \( \phi(S, A_w)A_w \) are in two distinct components of \( G \setminus [\phi(S, A_w)S] \). For each \( v \in \partial A_w S \), we can construct a path \( l_v \) joining \( v \) and \( \phi(S, A_w)u \) as follows

1. use a path \( l_1 \) in \( S \) to join \( v \) and a vertex \( u \) in \( \partial B_w S \) - this is possible by the connectivity of \( S \);
2. use a shortest path \( l_2 \) in \( B_w \) to join \( u \) and \( \phi(S, A_w)S \); let \( x \) be the endpoint of \( l_2 \);
3. use a path \( l_3 \) in \( \phi(S, A_w)S \) to join \( x \) and \( \phi(S, A_w)u \)

Let \( l_v \) be the concatenation of \( l_1, l_2, \) and \( l_3 \).

We can check that \( l_v \in G \setminus [A_w \cup \phi(S, A_w)A_w] \) as follows. The path \( l_1 \subset S \), \( S \cap A_w = \emptyset \) and \( S \cap \phi(S, A_w)A_w = \emptyset \). Therefore \( l_1 \in G \setminus [A_w \cup \phi(S, A_w)A_w] \). The path \( l_3 \in \phi(S, A_w)S; \phi(S, A_w)S \cap \phi(S, A_w)A_w = \emptyset \) since \( S \cap A_w = \emptyset \); \( \phi(S, A_w)S \cap A_w = \emptyset \) by (4.7) and the fact that \( B_w \cap A_w = \emptyset \). Therefore \( l_3 \in G \setminus [A_w \cup \phi(S, A_w)A_w] \). The path \( l_2 \in B_w \), hence \( l_2 \cap A_w = \emptyset \). Since \( l_1 \cup l_2 \setminus \{x\} \cap \phi(S, A_w)S = \emptyset \) and \( v \) is in
the same component of $G \setminus [\phi(S, A_w)S]$ as $A_w$, we obtain that $l_1 \cup l_2 \setminus \{x\}$ are in the same component of $G \setminus [\phi(S, A_w)S]$ as $A_w$ and $[l_1 \cup l_2 \setminus \{x\}] \cap \phi(S, A_w)A_w = \emptyset$. Therefore $l_2 \in G \setminus [A_w \cup \phi(S, A_w)A_w]$.

The length of $l_1$ and $l_3$ are bounded above by $|S|$. We can make the length of $l_2$ to be bounded above by the distance of $\partial_{B_w}S$ and $\phi(S, A_w)S$, which is bounded above by the graph distance in $G$ if $1\Gamma$ and $\phi(S, A_w)1\Gamma$. The latter is bounded by $2k + \max\{\text{dist}_G(1\Gamma, r_1), \text{dist}_G(1\Gamma, r_2)\}$. Hence if we choose

$$N = 2|S| + 2k + \max\{\text{dist}_G(1\Gamma, r_1), \text{dist}_G(1\Gamma, r_2)\},$$

then Assumption 1.1(3) is satisfied.

Therefore Theorem 1.7 in this case follows from Theorem 1.3 A. □

4.3.4. Proof of Theorem 1.8 when $\Gamma$ is an HNN extension. Let $\Gamma$ be an infinite, finitely generated graph which is an HNN extension as described by (2). It suffices to construct a locally finite Cayley graph $G$ of $\Gamma$ on which SAWs have positive speed.

First we consider the case when $C_1 = C_2 = H$. Let $G$ be the Cayley graph of of $\Gamma$ with respect to generator set $H \cup \{t, t^{-1}\}$; i.e. any elements in $H$ corresponds to an edge in $G$. Let $S' = S = H$. Note that $\phi \setminus S$ has two distinct infinite components. For any component $A$ of $G \setminus S$, let $\phi(S, A)$ be the mapping from $\Gamma$ to $\Gamma$ changing each $t$ in the normal form to $t^{-1}$ and each $t^{-1}$ in the normal form to $t$. Then Theorem 1.8 in this case follows from Theorem 1.3 B.

Now we consider the case when $C_1$ is a proper subset of $H$. Let $T_H$ be a finite generator set of $H$ satisfying $T_H = T_H^{-1}, 1 \notin T_H, |T_H| < \infty$, and $[C_1 \cup C_2] \setminus \{1\Gamma\} \subset T_H$. Let $G$ be a Cayley graph of $\Gamma$ with respect to the set of generators $T_H \cup \{t, t^{-1}\}$. Let $S$ be defined as in Section 4.3.3, and let $S' = S$. Then Theorem 1.8 in this case follows from Theorem 1.3 B.

5. Free product graph of two quasi-transitive graphs

In this section, we prove Theorem 1.6.

Proof. Obviously $G$ is an infinite, connected, quasi-transitive graph. Let $S = \{o\} \in V$. Then $G \setminus S$ has at least two infinite components. Indeed, Let $x, y \in V$ satisfy

$$x = x_1 \ldots x_n;$$

$$y = y_1 \ldots y_m;$$

where $m, n \geq 1, x_i, y_j \in V_1^x \cup V_2^x, I(x_i) \neq I(x_{i+1}), I(y_j) \neq I(y_{j+1})$ (see Definition 1.5 for notations). If $x_1 \in V_1^x$ and $y_1 \in V_2^x$, then $x$ and $y$ are in two distinct components of $G \setminus S$.

Let $A$ (resp. $B$) be a component of $G \setminus S$, such that for any $x \in A$ (resp. $y \in B$), $x$ (resp. $y$) has the form (5.1) (resp. (5.2)) with $x_1 \in V_1^x$ (resp. $y_1 \in V_2^x$). Let $u \in V_2^x$ (resp. $w \in V_2^x$), and define $\phi(S, A)x = ux$ (resp. $\phi(S, B)y = vy$). Then it
is straightforward to verify Assumption 1.2 with $S$ chosen as above. Therefore the theorem follows from Theorem 1.8. □

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