Parity linkage and the Erdős-Pósa property of odd cycles through prescribed vertices in highly connected graphs

Felix Joos

Abstract

We show the following for every sufficiently connected graph $G$, any vertex subset $S$ of $G$, and given integer $k$: there are $k$ disjoint odd cycles in $G$ containing each a vertex of $S$ or there is set $X$ of at most $3k - 3$ vertices such that $G - X$ does not contain any odd cycle that contains a vertex of $S$. We prove this via an extension of Kawarabayashi and Reed’s result about parity-$k$-linked graphs (Combinatorica 29, 215-225). From this result it is easy to deduce several other well known results about the Erdős-Pósa property of odd cycles in highly connected graphs. This strengthens results due to Thomassen (Combinatorica 21, 321-333), and Rautenbach and Reed (Combinatorica 21, 267-278), respectively.

Keywords: cycles, packing, covering
AMS subject classification: 05C70,
this bound to 24k, and Kawarabayashi and Wollan [12] improved this further to 144k.

More than 50 years ago, Dirac [5] showed that in every $k$-connected graph $G$, there is a cycle containing any prescribed set of $k$ vertices. Later, Bondy and Lovász [2] extended Dirac’s result and proved among other results along this line that for every $k$-connected non-bipartite graph $G$, there is an odd cycle containing any prescribed set of $k - 1$ vertices.

If one asks for many disjoint cycles through a prescribed set $S$ of vertices it is natural to start with disjoint cycles each containing at least one element of $S$. We call such cycles $S$-cycles. Pontecorvi and Wollan [14] showed that the class of $S$-cycles has the Erdős-Pósa property with $f(k) = O(k \log k)$, which improved the quadratic bound from [10]. Bruhn et al. [3] proved that the class of all $S$-cycles of length at least $\ell$ have the Erdős-Pósa property with $f(k, \ell) = O(k \log k)$. For $S = V(G)$, these results equal the Erdős-Pósa property for cycles and cycles of length at least $\ell$, respectively.

Although, the Erdős-Pósa property does not hold for odd cycles, it is proved in [5] that a half-integral version for the Erdős-Pósa property of odd $S$-cycles holds. This generalizes a result of Reed [16], who proved the case $S = V(G)$.

In this paper we continue the study of $S$-cycles by showing the following theorem. We say a set of vertices $X$ is an odd cycle cover of $G$ if $G - X$ is bipartite. As mentioned above, the results in [11, 12, 15] show that linear connectivity ensures that a graph has $k$ vertex disjoint odd cycles or an odd cycle cover of size $2k - 2$. We show that a sufficiently connected graph has $k$ vertex disjoint odd $S$-cycles for any prescribed vertex set $S$ of at least $k$ vertices or has an odd cycle cover of size $3k - 3$.

Moreover, the bound of $3k - 3$ is tight for any connectivity and this can be seen as follows. Let $G$ arise from a large complete bipartite graph with bipartition $(A, B)$ by adding the edges of a clique on $2k - 1$ vertices to $A$ and the edges of a clique on $k$ vertices to $B$. Let $S$ be the set of $k$ vertices in $B$ containing the $k$-clique. There do not exist $k$ disjoint odd $S$-cycles, nor an odd cycle cover of size $3k - 4$ in $G$.

**Theorem 1.** For any integer $k$, any $50k$-connected graph $G$, and any subset $S$ of at least $k$ vertices of $G$, at least one of the following statements hold:

1. $G$ contains $k$ odd $S$-cycles.
2. There is a set $X$ with $|X| \leq 3k - 3$ such that $G - X$ is bipartite.

In fact, we prove more detailed results than Theorem 1. These results in turn imply the already known result that every $50k$-connected graph $G$ that fails to have $k$ disjoint odd cycles contains an odd cycle cover of size $2k - 2$, which corresponds to the case $S = V(G)$ in [11, 12, 15].

It is not difficult to see that there are arbitrarily highly connected graphs that contain $k$ disjoint odd cycles and an odd cycle cover of size $2k - 2$. In this paper, we present an equivalent condition for $50k$-connected graphs for having $k$ disjoint odd cycles and deduce the known Erdős-Pósa-type result from this result.

A graph is $k$-linked if for every set of distinct vertices $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ there are disjoint paths $P_1, \ldots, P_k$ such that $P_i$ connects $s_i$ and $t_i$. Moreover, a graph $G$ is parity-$k$-linked if it is $k$-linked and we can additionally specify whether the length of each $P_i$ should be odd or even for every $1 \leq i \leq k$. 

2
There are several results of the form if $G$ is $g_1(k)$-connected, then $G$ is $k$-linked. The best result is due to Thomas and Wollan [13] who proved that $g_1(k) = 10k$ suffices. They even proved the following stronger result.

**Theorem 2 ([13]).** Every $2k$-connected graph $G$ with at least $5k|V(G)|$ edges is $k$-linked.

Analogously, there are results of the form if $G$ is $g_2(k)$-connected and without an odd cycle cover of size $4k - 4$, then $G$ is parity-$k$-linked. In particular, Kawarabayashi and Reed [11] proved the following.

**Theorem 3 ([11]).** Every $50k$-connected graph without an odd cycle cover of size $4k - 4$ is parity-$k$-linked.

The condition of having no small odd cycle cover is necessary and best possible — there are graphs of arbitrary high connectivity and with an odd cycle cover of size $4k - 4$ that are not parity-$k$-linked. For example, consider a large complete bipartite graph $G$ with bipartition $(A, B)$ where we add to $A$ the edges of a clique on $2k - 1$ vertices and we add to $B$ the edges of a clique on $2k$ vertices minus a perfect matching.

One can apply Theorem 3 almost directly to obtain that every $50k$-connected graph $G$ without an odd cycle cover of size $4k - 4$ has $k$ disjoint odd $S$-cycles for any set $S$ of at least $k$ vertices. However, the bound on the size of the odd cycle cover is not optimal. In this paper we prove a stronger version of Theorem 3 and reprove the Erdős-Pósa property for odd cycles for $50k$-connected graphs, and as the main result of this paper, we prove Theorem 1.

In addition, we prove several results on the way that may be of independent interest.

The paper is organized as follows. In Section 2 we deal with the results concerning the parity-$k$-linkage and in Section 3 we prove the results about the Erdős-Pósa property for odd $S$-cycles.

## 2 Highly parity linked graphs

In the next theorem we explicitly characterize the obstruction for a $50k$-connected graph and a set $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ of $2k$ distinct vertices for not having $k$ disjoint $P_1, \ldots, P_k$ paths of prescribed length parity where $P_i$ connects $s_i$ and $t_i$. Before we state the theorem, we introduce some definitions. A partition $(A, B)$ of $G$ is partition of the vertex set into two sets $A$ and $B$. For a partition $(A, B)$ of $G$, we denote by $G_{A,B}$ the graph $G[A] \cup G[B]$. A partition $(A, B)$ of $G$ is nice if there is a minimum odd cycle cover $X$ of $G$ for which $(A \setminus X, B \setminus X)$ is a bipartition of $G - X$ such that a vertex of $X$ is in $A$ if and only if it has more neighbors in $B \setminus X$ than in $A \setminus X$. We say that a minimum odd cycle cover $X$ induces a nice partition $(A, B)$ of $G$ if $(A \setminus X, B \setminus X)$ is a bipartition such that a vertex of $X$ is in $A$ if and only if it has more neighbors in $B \setminus X$ than in $A \setminus X$.

Let $(A, B)$ be nice partition of $G$ and let $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ be a set of $2k$ distinct vertices. A parity breaking matching for $S$ (with respect to the partition $(A, B)$) is a matching $M$ such that $M \subseteq E(G_{A,B})$ and there is no edge $rr' \in M$ with $r \in \{s_i, t_i\}$ and $r' \in \{s_j, t_j\}$ for $i \neq j$.

3
Theorem 4. Let $k \in \mathbb{N}$. Let $G$ be a $50k$-connected graph and let $S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ be a set of $2k$ distinct vertices. Exactly one of the following two statements holds.

1. $G$ contains $k$ disjoint paths $P_1, \ldots, P_k$ of any prescribed parity such that $P_i$ connects $s_i$ and $t_i$.

2. For all nice partitions of $G$, there is no parity breaking matching for $S$ of size $k$.

There are plenty of consequences of Theorem 4. Firstly, it is easy to see that it implies Theorem 5.

Theorem 5. Let $k \in \mathbb{N}$ and let $G$ be a $50k$-connected graph. Exactly one of the following two statements holds.

1. $G$ is $k$-parity linked.

2. There is a set $\{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subset V(G)$ such that for all nice partitions of $G$, there is no parity breaking matching of size $k$.

Secondly, later we deduce Theorem 3. The third consequence (Corollary 6) shows that the bound “$4k - 4$” in Theorem 3 can be strengthened to “$2k - 2$” if $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ is an independent set. Note that both bounds “$4k - 4$” and “$2k - 2$” are best possible, respectively. As a fourth consequence we prove the Erdős-Pósa property for odd $S$-cycles (Theorem 1) in Section 3.

We say that $G$ is parity-$k$-linked restricted to independent sets if for every independent set of $2k$ vertices $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$, there are disjoint paths $P_1, \ldots, P_k$ such that $P_i$ connects $s_i$ and $t_i$, and we can choose whether the length of $P_i$ is odd or even.

Corollary 6. Let $k \in \mathbb{N}$ and let $G$ be a $50k$-connected graph. At least one of the following statements holds.

1. $G$ is parity-$k$-linked restricted to independent sets.

2. There is a set $X$ of $2k - 2$ vertices such that $G - X$ is bipartite.

Next, we mention two results needed in the proof of Theorem 4. The first result is basically due to Mader and there is a slightly improved version in the textbook of Diestel on page 13 [4].

Lemma 7 (Mader [13]). If $G$ is a graph such that $|E(G)| \geq |V(G)| - k$, then $G$ contains a $(k + 1)$-connected graph $H$ such that $\frac{|E(H)|}{|V(H)|} \geq \frac{|E(G)|}{|V(G)|} - k$.

Hence, if $\delta(G) \geq 12k$, then $G$ contains a $2k$-connected subgraph $H$ such that $\frac{|E(H)|}{|V(H)|} \geq 5k$. Using Theorem 8 this implies in turn that the subgraph $H$ is $k$-linked.

Another result which is used in the proof of Theorem 4 is due to Geelen et al.. For a graph $G$ and a set of vertices $Z$, a $Z$-path is a path $P$ such that $V(P) \cap Z$ are exactly the end vertices of $P$.

Theorem 8 (Geelen et al. [8]). For any set $Z$ of vertices of a graph $G$ and any positive integer $\ell$ at least one of the following statements holds;
• there are \( \ell \) disjoint odd \( Z \)-paths or

• there is a vertex set \( X \) of order at most \( 2\ell - 2 \) such that \( G - X \) contains no odd \( Z \)-path.

We proceed with the proof of Theorem 4 that leads to the several consequences mentioned above.

**Proof of Theorem 4.** Suppose the first statement holds. Let \( (A, B) \) be some nice partition of \( G \). Let \( P_1, \ldots, P_k \) be disjoint paths where \( P_i \) is a \( s_i, t_i \)-path and we choose the parity of \( P_i \) to be even if exactly one vertex of \( \{s_i, t_i\} \) belongs to \( A \) and odd otherwise. Thus \( P_i \) contains at least one edge \( m_i \) in \( E(G_{A,B}) \).

Therefore, \( \{m_1, \ldots, m_k\} \) is a parity breaking matching for \( S \) of size \( k \).

Next, suppose the second statement does not hold; that is, there is a nice partition \( (A, B) \) of \( G \) with a parity breaking matching \( M = \{m_1, \ldots, m_k\} \). If an edge of \( M \) covers a vertex of \( \{s_i, t_i\} \), let \( m_i = x_iy_i \) be this edge and choose \( x_i, y_i \) such that \( s_i = x_i \) or \( t_i = y_i \). Let \( X \) be a minimum odd cycle cover of \( G \) that induces the nice partition \( (A, B) \).

Suppose first that \( |X| < 8k \). By the definition of a nice partition and the fact that \( G \) is \( 50k \)-connected, we know that every vertex in \( a \in A \) has at least \( 20k \) neighbors in \( B \) and at least \( 12k \) in \( B \setminus X \). The same holds vice versa for the vertices in \( B \). Therefore, we can find a set of \( 4k \) distinct vertices \( \bigcup_{i=1}^{k}\{s_i, t_i, x_i, y_i\} \subset V(G) \setminus (X \cup \bigcup_{i=1}^{k}\{s_i, t_i, x_i, y_i\}) \) such that \( z \) is a neighbor of \( z \) for \( z \in \bigcup_{i=1}^{k}\{s_i, t_i, x_i, y_i\} \) (symbolically written). Hence exactly one vertex of the set \( \{z_i, z_i'\} \) belongs to \( A \).

Let \( G' = G - (X \cup \bigcup_{i=1}^{k}\{s_i, t_i, x_i, y_i\}) \). Thus \( G' \) is \( 38k \)-connected and bipartite. In addition, by Theorem 2, we obtain that \( G' \) is \( 2k \)-linked, which implies that we can choose up to \( 2k \) paths in \( G' \) with different endvertices to be disjoint—two for each \( i \in \{1, \ldots, k\} \).

For every \( i \) we proceed as follows. If our choice of the parity of \( P_i \) shall respect the parity naturally given by the sides of the partition \( (A, B) \), then let \( P_i' \) be a path connecting \( s_i' \) and \( t_i' \) in \( G' \) and let \( P_i \) be the conjunction of \( s_is_i' \), the path \( P_i' \), and \( t_it_i' \). Otherwise, let \( P_i' \) be a path connecting \( s_i' \) and \( x_i \) and \( P_i'' \) be a path connecting \( t_i' \) and \( y_i \). If \( \{s_i, t_i\} \cap \{x_i, y_i\} = \emptyset \), then and let \( P_i \) be the conjunction of \( s_is_i' \), the path \( P_i' \), the path \( x_iy_i \), the path \( P_i'' \), and \( t_it_i' \). If \( s_i = x_i \) and \( t_i = y_i \), then let \( P_i \) be the conjunction of \( s_is_i' \), the path \( P_i' \), and \( x_ix_it_i' \). Finally, if \( s_i = x_i \) and \( t_i = y_i \), then let \( P_i = s_is_i't_i' \).

As mentioned above, because \( G' \) is \( 2k \)-linked, we can choose the corresponding paths \( P_1', P_1'', \ldots, P_k', P_k'' \) in \( G' \) to be pairwise disjoint and hence also \( P_1, \ldots, P_k \) are pairwise disjoint.

It remains to show that if \( X \) has size at least \( 8k \), then the first statement holds. This part of the proof can basically be found in [11]. However, we change some arguments which leads to a shorter proof. Let \( G' = G - \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \) and let \( (A', B') \) be a partition of \( G' \) such that \( |E(G_{A',B'})| \) is minimized. Note that \( \delta(G') \geq 24k \). By Lemma 7 there is a \( 4k \)-connected subgraph \( H \) of \( G' \) with \( |E(H)| \geq 10k|V(H)| \). Moreover, by Theorem 2 the graph \( H \) is \( 2k \)-linked.

Theorem 8 guarantees a set \( Y \) with \( |Y| \leq 6k - 6 \) that intersects all odd \( A' \)-paths in \( G' \) or \( 3k \) disjoint odd \( A' \)-paths in \( G' \). Suppose that there is a set
Theorem 8 implies the existence of 3 disjoint paths in $G'$. Note that the length of these paths could be zero. Nevertheless, combining these two paths with one part of the cycle $C$ leads to an odd $A'$-path, which is a contradiction. This in turn implies that $S \cup Y$ is an odd cycle cover of size $8k - 6$, which is a contradiction. Thus Theorem 8 implies the existence of $3k$ odd $A'$-paths.

Let $P$ be one of these $3k$ odd $A'$-paths. There is a natural partition of $E(P)$ into $H$-paths. Because $P$ is an odd $A'$-path, there is a subpath $P'$ of $P$ such that $P'$ is an odd $H$-path and both endvertices of $P'$ lie in the same side of the bipartition of $H$ or $P'$ is an even $H$-path and exactly one endvertex of $P'$ lies in $A'$.

Therefore, there is a set $Q$ of $3k$ disjoint $H$-paths $Q_1, \ldots, Q_{3k}$ where the length of $Q_i$ is odd if both endvertices lie in the same side of the bipartition of $H$ and even otherwise.

Since $G$ is $50k$-connected, there is a set of $2k$ disjoint paths $\mathcal{P} = \{P_1, \ldots, P_{2k}\}$ connecting $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ and $H$. Choose these paths such that they intersect as few as possible paths from $Q$. Under this condition choose these paths such that their edge intersection with $Q$ is as large as possible. The latter condition implies that if $Q \in \mathcal{Q}$ has nonempty intersection with a path in $\mathcal{P}$ – let $z', z''$ be the endvertices of $Q_i$, and let $P$ be first path that intersects $Q$ seen from the direction of $z'$ – then $P$ follows the path $Q$ up to $z'$ beside the case that $P$ is the only path intersecting $Q_i$, and $P$ follows $Q$ to $z''$. Hence for every $Q \in \mathcal{Q}$ that intersects a path in $\mathcal{P}$, there is at least one path $P \in \mathcal{P}$ such that there is vertex $z$ that is an endvertex of $P$ and $Q$. Therefore, the paths in $\mathcal{P}$ intersect at most $2k$ paths in $Q$ and hence there is a collection $Q' = \{Q_1', \ldots, Q_{2k}'\} \subset Q$ of $k$ paths such that $Q \cap P = \emptyset$ for $P \in \mathcal{P}$ and $Q \in Q'$. Since $H$ is $2k$-linked, we can find the desired disjoint paths of specified parity connecting $s_i$ and $t_i$ by using the paths $\mathcal{P}$ and then either directly linking the ends in $H$ of the paths belonging to $s_i$ and $t_i$ or by using the path $Q_i'$ in between.

For a graph $G$, let a set of vertices $X$ of $G$ be a vertex cover of $G$ if every edge is incident to at least one vertex of $X$. Let the vertex cover number $\tau(G)$ of $G$ be the least number $k$ such that $G$ has a vertex cover $X$ with $|X| = k$. Since a vertex cover has to contain at least one vertex of every edge in a matching $M$, we have on the one hand $|M| \leq \tau(G)$ for every matching $M$ in $G$. On the other hand, we observe the following.

If $M$ is a maximal matching of $G$, then the vertices covered by $M$ form a vertex cover of $G$ and hence $\tau(G) \leq 2|M|$.

A graph $G$ is $\tau$-critical if $\tau(G - e) < \tau(G)$ and $\tau(G - v) < \tau(G)$ for every edge $e \in E(G)$ and every vertex $v \in V(G)$. A result of Erdős and Gallai [10] says if $G$ is $\tau$-critical, then $\tau(G) \geq |V(G)|/2$.

Having these definitions in mind we reprove Kawababayashi’s and Reed’s result and directly afterwards Corollary 9.

Proof of Theorem 8. Suppose that $X$ is a minimum odd cycle cover and $|X| \geq 4k - 3$. We show by induction on $k$ that $G$ contains a parity breaking matching.
Let \((A, B)\) be a nice partition induced by \(X\). Note that \(X\) is a minimum vertex cover of \(G_{A,B}\).

Suppose \(k = 1\). Since \(|X| \geq 1\), the graph \(G_{A,B}\) must contain an edge \(e\) and \(e\) is a parity breaking matching.

Hence we may assume that \(k \geq 2\). Because \(2k < 4k - 3\), there is a vertex \(r\) such that \(r \in X\setminus\{s_1, \ldots, s_k, t_1, \ldots, t_k\}\). Since \(X\) is minimum, \(r\) has a neighbor \(r'\) in \(G_{A,B}\). If \(r' \notin \{s_1, \ldots, s_k, t_1, \ldots, t_k\}\), we have \(\tau(H - \{s_1, t_1, r, r'\}) \geq 4(k-1) - 3\) and combining \(rr'\) and the induction hypothesis, we conclude that \(G\) contains a parity breaking matching \(M\). Now we may assume that, by symmetry, \(r' = s_1\).

However, using the same argument again, there is a parity breaking matching for \(S\). Applying Theorem 4 completes the proof.

Proof of Corollary 6. Suppose that \(X\) is a minimum odd cycle cover and \(|X| \geq 2k - 1\). Fix some independent set \(S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}\) of \(2k\) distinct vertices. We show by induction on \(k\) that \(G\) contains a parity breaking matching.

Let \((A, B)\) be a nice partition induced by \(X\). Note that \(X\) is a minimum vertex cover of \(G_{A,B}\).

In the following we show that there is a parity breaking matching of size \(k\).

We proceed by induction on \(k\). If \(k = 1\), then \(G_{A,B}\) contains an edge because \(|X| \geq 1\) and this edge is a parity breaking matching of size 1.

Suppose \(G_{A,B} - S\) contains no edges. This implies that \(G_{A,B}\) is bipartite with bipartition \((V(G_{A,B}) \setminus S, S)\). By König’s Theorem, the matching number of \(G_{A,B}\) equals the vertex cover number and hence \(G_{A,B}\) contains a matching \(M\) of size \(2k - 1\). Let \(N\) be the matching obtained from \(M\) by deleting one of the matching edges \(s_i p\) and \(t_i q\) if both exist in \(M\). Therefore, \(|N| \geq k\) and every set of \(k\) edges of \(N\) is a parity breaking matching of size \(k\).

In the following we may assume that \(G_{A,B} - S\) contains edges. Suppose \(s_j\) is an isolated vertex in \(G_{A,B}\). Let \(e = uv\) be an edge incident to \(t_i\) if such an edge exists otherwise let \(e\) be an edge in \(G_{A,B} - S\). By induction, \(\tau(G_{A,B} - \{u, v, s_j, t_i\}) \geq 2k - 3\) and thus a parity breaking matching \(M\) of size \(k - 1\). Combining \(M\) and \(e\) leads to the desired matching.

Therefore, we may assume that every vertex in \(S\) has a neighbor in \(G_{A,B}\). Induction can also be applied if \(|X| \geq 2k\) by deleting \(s_1, t_1\), and a neighbor of \(s_1\) from \(G_{A,B}\). Thus we may assume \(|X| = 2k - 1\).

Let \(G'_{A,B}\) be the induced subgraph of \(G_{A,B}\) which is obtained from \(G_{A,B}\) by deleting all isolated vertices from \(G_{A,B}\). We may assume that the \(\tau(G'_{A,B} - e) < \tau(G'_{A,B})\) for every \(e \in E(G'_{A,B})\) and \(\tau(G'_{A,B} - r) < \tau(G'_{A,B})\) for every \(r \in V(G'_{A,B})\setminus S\). Moreover, if \(\tau(G'_{A,B} - s_i) = \tau(G'_{A,B})\), then let \(r\) be a neighbor of \(t_i\) and the statement follows by induction because \(\tau(G'_{A,B} - \{s_i, t_i, r\}) \geq \tau(G'_{A,B}) - 2\). This implies that \(G'_{A,B}\) is a \(\tau\)-critical graph.

Since \(S\) is an independent set, the complement of an independent set is a vertex cover, and \(|X| = 2k - 1\), we conclude that \(\tau(G'_{A,B}) < n(G_{A,B})/2\).

However, this contradicts a theorem of Erdős and Gallai mentioned before.

3 Odd cycles through prescribed vertices

In this section we present several results concerning the Erdős-Pósa property of odd \(S\)-cycles in highly connected graphs. This extends the results concerning the Erdős-Pósa property of odd cycles in highly connected graphs. Furthermore,
assuming a slightly higher connectivity, we show how known results follow easily from Theorem 4.

**Lemma 9.** Let $k \in \mathbb{N}$ and let $G$ be a $50k$-connected graph. Let $S$ be a set of $k$ vertices. Exactly one of the following statements holds.

1. $G$ contains $k$ disjoint odd $S$-cycles.

2. For all nice partitions $(A, B)$ of $G$, there is no matching $M$ of size $k$ in $G_{A, B}$ where an edge in $M$ covers at most one vertex of $S$.

**Proof.** Suppose first that $G$ contains $k$ disjoint $S$-cycles and let $(A, B)$ be a partition of $G$. Since every odd cycle contains at least one edge of $E(G_{A, B})$ and by picking exactly one of these edges from every odd $S$-cycle, we get a matching of size $k$ in $G_{A, B}$ where one edge does not cover two vertices of $S$, which implies that the second statement does not hold.

Next, we suppose that the second statement does not hold and let $(A, B)$ be a nice partition such that there is a matching $M$ of size $k$ in $G_{A, B}$ and one edge in $M$ does not cover two vertices of $S$.

Let $T = \{t_1, \ldots, t_k\}$ be a set of vertices distinct from $S$ and distinct from the vertices covered by $M$. By Theorem 4 there are disjoint paths $P_1, \ldots, P_k$ with prescribed parity such that $P_i$ connects $s_i$ and $t_i$ because $M$ is a parity breaking matching for $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$.

Let $(\bar{A}, \bar{B})$ any partition of $G$. By suitable choosing the parity of $P_i$, every path uses at least one edge in $\bar{G}_{\bar{A}, \bar{B}}$ and hence there is a matching in $\bar{G}_{\bar{A}, \bar{B}}$ of size at least $k$ such that each edge covers at most one vertex of $S$.

For every $s_i$, add to $G$ a vertex $s'_i$ such that $N(s_i) = N(s'_i)$ and denote this new graph by $G'$. Note that $G'$ is $50k$-connected. Let $(A', B')$ be nice partition of $G'$. As mentioned above, in the subgraph $G$ of $G'$ for every partition $(\bar{A}, \bar{B})$, the graph $\bar{G}_{\bar{A}, \bar{B}}$ contains a matching of size $k$ such that each edge covers at most one vertex of $S$, in particular, this holds for the partition induced by $(A', B')$.

Let $M$ be such a matching of size $k$. Note that $M$ does not cover a vertex of the set $\{s'_1, \ldots, s'_k\}$. Thus $M$ is a parity breaking matching for $S \cup \{s'_1, \ldots, s'_k\}$ in $G'$. By Theorem 4 there are disjoint paths $P'_1, \ldots, P'_k$ of odd length where $P'_i$ joins $s_i$ and $s'_i$.

By identifying $s_i$ and $s'_i$, this in turn implies the existence of $k$ disjoint odd $S$-cycles $C_1, \ldots, C_k$ in $G$ where $C_i$ contains $s_i$. □

After having proved Lemma 9 it is not difficult to prove the Erdős-Pósa property of odd $S$-cycles in highly connected graphs.

**Theorem 10.** Let $k \in \mathbb{N}$ and let $G$ be a $50k$-connected graph. Let $S$ be a set of $k$ vertices. At least one of the following statements holds.

1. $G$ contains $k$ disjoint odd $S$-cycles.

2. There is a set $X$ with $|X| = 2k - 2 + \tau(G[S])$ such that $G - X$ is bipartite.

**Proof.** Let $X$ be a minimum odd cycle cover. We may assume that $|X| \geq 2k - 1 + \tau(G[S])$. Let $(A, B)$ be nice partition of $G$ and let $Y$ be a minimum vertex cover of $S$. Since $X$ is a minimum vertex cover of $G_{A, B}$, we conclude $\tau(G_{A, B} - Y) \geq 2k - 1$. Using 4, this in turn implies the existence of a matching $M$ of size $k$ in $G_{A, B} - Y$ such that no vertex of $M$ covers more than one vertex.
of $S$. Using Lemma 9, this implies the existence of $k$ disjoint odd $S$-cycles in $G$.

Note that $\tau(G[S]) \leq k - 1$ and thus $2k - 2 + \tau(G[S]) \leq 3k - 3$. Moreover, the bound “$2k - 2 + \tau(G[S])$” is sharp for every possible value of $\tau(G[S])$ no matter how large the connectivity of $G$ is. To see this, let $G$ arise from a large complete bipartite graph with bipartition $(A, B)$ by adding the edges of a clique on $2k - 1$ vertices to $A$ and the edges of a clique on $\tau$ vertices to $B$ for some $1 \leq \tau \leq k$. Let $S$ be a set of $k$ vertices in $B$ containing the $\tau$-clique. Hence $\tau(G[S]) = \tau - 1$, there do not exist $k$ disjoint odd $S$-cycles, and there is no set $X$ of $2k - 3 + \tau(G[S])$ vertices such that $G - X$ is bipartite.

In Theorem 10 we require that $|S| = k$. However, it is not difficult to extend Theorem 10 to arbitrary subsets of $V(G)$ as follows.

On the one hand, if $|S| < k$, then $G$ does not contain $k$ disjoint odd $S$-cycles and one can remove (the small set) $S$ from $G$ to obtain a graph without an odd $S$-cycle. Of course, one cannot hope for a small set $X$ such that $G - X$ is bipartite because $G$ could simply be a large clique.

On the other hand if $|S| \geq k$, then let $S' \subset S$ such that $|S'| = k$. In the case that $G$ does not contain $k$ disjoint odd $S'$-cycles, Theorem 10 implies the existence of a set $X$ of $3k - 3$ vertices such that $G - X$ is bipartite, in particular, $G - X$ does not contain odd $S$-cycles. In addition, this proves Theorem 1.

We complete the paper with two results concerning the case $S = V(G)$.

Corollary 11. Let $k \in \mathbb{N}$ and let $G$ be a $50k$-connected graph. Exactly one of the following statements holds.

1. $G$ contains $k$ disjoint odd cycles.
2. For every nice partition $(A, B)$, the graph $G_{A,B}$ does not contain a matching of size $k$.

Proof. If the first statement holds, then the second does clearly not hold.

Suppose that the second statement does not hold. We may assume that $G$ contains an independent set of cardinality $k$, otherwise there are $k$ disjoint triangles in $G$. This can be seen as follows: Let $x$ be any vertex in $G$. The neighborhood of $x$ contains an edge. Combining this edge with $x$ leads to a triangle. Repeating this argument $k$ times, leads to the first statement. Thus $G$ contains an independent set $S$ of cardinality $k$ and we apply Lemma 9 to obtain $k$ disjoint odd ($S$-)cycles in $G$.

The following corollary is already proven by Thomassen [19] and Rautenbach and Reed [15] with a higher connectivity bound. Later Kawarabayashi and Reed [11] and Kawarabayashi and Wollan [12] improved this bound to $24k$ and $\frac{31}{2}k$, respectively.

Corollary 12. Let $k \in \mathbb{N}$ and let $G$ be a $50k$-connected graph. At least one of the following statements holds.

1. $G$ contains $k$ disjoint odd cycles.
2. $G$ contains a set $X$ of $2k - 2$ vertices such that $G - X$ is bipartite.
Proof. Suppose the second statement does not hold. Let \((A, B)\) a nice partition of \(G\) induced by a minimum odd cycle cover \(X\). Note that \(X\) is a minimum vertex cover of \(G_{A,B}\) and since \(|X| \geq 2k - 1\) by our assumption and by applying (1), the graph \(G_{A,B}\) contains a matching of size \(k\). Because this holds for every minimum odd cycle cover and so for ever nice partition of \(G\), the statement follows by the previous corollary.

Clearly, assuming 50\(k\)-connectivity in our results in not the best function in terms of \(k\) one can hope for. However, it is essentially best possible in that sense as one can easily construct graphs that show that linear connectivity in \(k\) is necessary. It would be interesting to know which connectivity is needed to ensure that our results hold. It even seems possible that the approach via a parity-\(k\)-linkage theorem cannot lead to the best connectivity bound.

References

[1] E. Birmele, J.A. Bondy, and B. Reed, The Erdös-Pósa property for long circuits, Combinatorica 27 (2007), 135–145.

[2] J.A. Bondy and L. Lovász, Cycles through specified vertices of a graph, Combinatorica 1 (1981), 117–140.

[3] H. Bruhn, F Joos, and O. Schaudt, Long cycles through prescribed vertices have the Erdös-Pósa property, manuscript.

[4] R. Diestel, Graph theory, fourth ed., Springer, Heidelberg, 2010.

[5] G.A. Dirac, In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterteilungen, Math. Nachr. 22 (1960), 61–85.

[6] P. Erdős and T. Gallai, On the minimal number of vertices representing the edges of a graph., Magyar Tud. Akad. Mat. Kutató Int. Közl. 6 (1961), 181–203.

[7] P. Erdős and L. Pósa, On the maximal number of disjoint circuits of a graph, Publ. Math. Debrecen 9 (1962), 3–12.

[8] J. Geelen, B. Gerards, B. Reed, P. Seymour, and A. Vetta, On the odd-minor variant of Hadwiger’s conjecture, J. Combin. Theory (Series B) 99 (2009), 20–29.

[9] N. Kakimura and K. Kawarabayashi, Half-integral packing of odd cycles through prescribed vertices, Combinatorica 33 (2013), 549–572.

[10] N. Kakimura, K. Kawarabayashi, and D Marx, Packing cycles through prescribed vertices, J. Combin. Theory (Series B) 101 (2011), 378–381.

[11] K. Kawarabayashi and B. Reed, Highly parity linked graphs, Combinatorica 29 (2009), 215–225.

[12] K. Kawarabayashi and P. Wollan, Non-zero disjoint cycles in highly connected group labelled graphs, J. Combin. Theory (Series B) 96 (2006), 296–301.
[13] W. Mader, *Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte*, Abh. Math. Sem. Univ. Hamburg 37 (1972), 86–97.

[14] M. Pontecorvi and P. Wollan, *Disjoint cycles intersecting a set of vertices*, J. Combin. Theory (Series B) 102 (2012), 1134–1141.

[15] D. Rautenbach and B. Reed, *The Erdős-Pósa property for odd cycles in highly connected graphs*, Combinatorica 21 (2001), 267–278.

[16] B. Reed, *Mangoes and blueberries*, Combinatorica 19 (1999), 267–296.

[17] N. Robertson and P. Seymour, *Graph minors. V. Excluding a planar graph*, J. Combin. Theory (Series B) 41 (1986), 92–114.

[18] R. Thomas and P. Wollan, *An improved linear edge bound for graph linkages*, Europ. J. Combin. 26 (2005), 309–324.

[19] C. Thomassen, *The Erdős-Pósa property for odd cycles in graphs of large connectivity*, Combinatorica 21 (2001), 321–333.