WAVE EQUATIONS WITH MASS AND DISSIPATION

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Abstract. In this paper we consider a wave model with non-effective mass and dissipation terms and provide sharp descriptions of its representation of solutions. In particular we conclude estimates for a corresponding energy and estimates of dispersive type.

1. Introduction

The study of wave models with lower order terms and their influence on energy and dispersive estimates for them has led to some interesting observations and also challenging problems. In this paper we will consider the Cauchy problem

\[ u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \]

(1.1)

for a damped Klein–Gordon equation with variable mass and dissipation and investigate the precise interplay between both coefficients and asymptotic properties of solutions as \( t \) tends to infinity. For an overview on some results in this direction see \[12\] and the monograph \[7\], we recall just some selected results to motivate the conditions imposed on the coefficient functions as well as the problems discussed later on.

One of the starting points for our considerations was the study of Reissig and Smith, \[5\], treating wave equations with bounded time-dependent speed of propagation and deriving \( L^p - L^q \) decay estimates for their solutions. A second one is the treatment of wave equations with time-dependent dissipation of the second author \[8\] and \[9\] introducing the classification of dissipation terms according to their strength and influence on the large-time behaviour of solutions. There the notion of effective and non-effective dissipation was used to distinguish between lower order terms giving asymptotically dominating or sub-ordinate contributions to the large-time behaviour. Let us assume for a moment that no mass term is present, \( m = 0 \), and that \( b \) is bounded, non-negative, sufficiently smooth and satisfies a condition of the form

\[ |\partial_t^k b(t)| \leq C_k b(t) \left( \frac{1}{1 + t} \right)^k, \quad k = 1, 2. \]

(1.2)

Then there are essentially two cases. Either

\[ \limsup_{t \to \infty} tb(t) < 1. \]

(1.3)

Then solutions behave in an asymptotic sense like free waves multiplied by a decay factor,

\[ \left( \frac{\nabla u(t, x)}{u_t(t, x)} \right) \sim \frac{1}{\lambda(t)} \left( \frac{\nabla v(t, x)}{v_t(t, x)} \right), \quad t \to \infty \]

(1.4)

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the asymptotic relation understood in an appropriate $L^p$-sense, and with $v$ a solution to the free wave equation $v_{tt} = \Delta v$ and $\lambda$ given as

$$
\lambda(t) = \exp \left( \frac{1}{2} \int_0^t b(\tau) d\tau \right).
$$

This is the above mentioned non-effective case. On the other hand, if

$$
\lim_{t \to \infty} tb(t) < \infty
$$

solutions to the damped wave equation are asymptotically related to solutions of a parabolic equation

$$
u(t, x) \sim w(t, x), \quad t \to \infty,
$$

where $b(t)w_t = \Delta w$. This is made precise in the so-called diffusion phenomenon for damped waves, see [10] or the work of Nishihara [4] for the case of constant dissipation term.

The main tool to prove such statements are asymptotic, but explicit, representations of solutions in terms of Fourier multipliers. In the present paper we will consider both mass and dissipation terms and concentrate on the non-effective case when solutions are still asymptotically hyperbolic and the derivation of $L^p-L^q$ decay estimates depending on high-frequency asymptotics of solutions.

A difference to earlier accounts is the more systematic study of the low-frequency parts based on asymptotic integration techniques. This allows to understand conditions like the constant appearing on the right-hand side of (1.3), or better, the construction of asymptotic low-frequency solutions in terms of leading order terms in

$$
b(t) = \frac{b_0}{1 + t} + o \left( \frac{1}{1 + t} \right), \quad m(t) = \frac{m_0}{1 + t} + o \left( \frac{1}{1 + t} \right).
$$

The paper is organised as follows. In Section 2 we will give precise conditions on coefficients, outline our basic strategy and give asymptotic constructions of representation of solutions in different zones of the phase space. In Section 3 we derive energy estimates, $L^p-L^q$ estimates and discuss their sharpness. Finally, the Appendix A collects some useful asymptotic integration theorems for differential equations.

## 2. Representations of solutions

We consider the following Cauchy problems for damped Klein–Gordon equations

$$
\dot{u} - \Delta u + b(t)u_t + m(t)u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $b = b(t)$ constitutes a dissipation term and $m = m(t)$ describes a mass term under the following basic assumptions.

**Hypothesis 1.** Suppose that $b, m \in C^\ell(\mathbb{R}_+)$ are real-valued and

$$
|\partial_t^k b(t)| \leq C_k \left( \frac{1}{1 + t} \right)^{k+1}, \quad \text{and} \quad |\partial_t^k m(t)| \leq C_k \left( \frac{1}{1 + t} \right)^{k+2},
$$

holds true for all $k = 0, 1, \ldots, \ell$. The number $\ell$ will be specified later on. Some statements need a higher regularity.
Hypothesis 2. Suppose that the following limits
\[ \lim_{t \to \infty} (1 + t)b(t) = b_0 \quad \text{and} \quad \lim_{t \to \infty} (1 + t)^2m(t) = m_0 \] (2.3)
exist and that
\[ \int_1^\infty |tb(t) - b_0| \frac{dt}{t} < \infty \quad \text{and} \quad \int_1^\infty |2m(t) - m_0| \frac{dt}{t} < \infty \] (2.4)
holds true with exponent \( \sigma \) satisfying
\[ (A1) \quad \sigma = 1 \quad \text{or} \quad (A2) \quad \sigma \in (1, 2]. \]

Results will depend on relations between the constants \( b_0 \) and \( m_0 \). It will not be necessary to restrict considerations to \( b_0 \geq 0 \) and \( m_0 \geq 0 \), results will however depend on the constraint \( 4m_0 > b_0(b_0 - 2) \) or additional conditions imposed on initial data. See, e.g., the statements of Theorems 3.1, 3.2 or 3.4 for further details.

We further define the auxiliary function
\[ \lambda(t) = \exp \left( \frac{1}{2} \int_0^t b(\tau)d\tau \right) \] (2.5)
related to the dissipative term \( b(t)u_t \). It will play an important role in the resulting estimates. Under part (A1) of Hypothesis 1 it follows that
\[ \lambda(t) \asymp (1 + t)^{b_0}, \quad t \to \infty. \] (2.6)

When assuming (A2) a further sub-polynomial correction term appears.

2.1. Zones and general strategy. Applying the partial Fourier transform in (2.1) with respect to the spatial variables, we obtain the ordinary differential equation
\[ \hat{u}_{tt} + |\xi|^2\hat{u} + b(t)\hat{u}_t + m(t)\hat{u} = 0, \quad \hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi) \] (2.7)
parameterised by the frequency variable \( \xi \). We are going to construct asymptotically a parameter-dependent solution to this equation. In order to do so, we divide the extended phase space \([0, \infty) \times \mathbb{R}_\xi^2\) into three zones depending on a constant \( N > 0 \),
\[ Z_{diss}(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : (1 + t)|\xi| \leq N\}, \]
\[ Z_{hyp}(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : |\xi| \leq N \leq (1 + t)|\xi|\}, \] (2.8)
\[ Z_{hyp}^s(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : |\xi| \geq N\}. \]
The constant \( N \) will be specified later on. In the zone \( Z_{hyp}^s(N) \) we consider only large frequencies and in the zones \( Z_{diss}(N) \) and \( Z_{hyp}^s(N) \) we consider small frequencies. Furthermore, the boundary between zones \( Z_{diss}(N) \) and \( Z_{hyp}^s(N) \) is given by the implicitly defined function
\[ \theta_{.|\xi|} : (0, N] \to [0, \infty), \quad (1 + \theta_{.|\xi|})|\xi| = N. \] (2.9)

We also put \( \theta_0 = \infty \), and \( \theta_{.|\xi|} = 0 \) for any \( |\xi| \geq N \). In order to localise the consideration to the three parts of the extended phase space, we further introduce a function \( \chi \in C^\infty(\mathbb{R}_+) \) such that \( \chi(t) = 1 \) for \( t \leq 1 \), \( \chi(t) = 0 \) for \( t \geq 2 \) and \( \chi'(t) \leq 0 \),

and define the cut-off functions $\varphi_{\text{diss}}$, $\varphi_{\text{hyp}}^\ell$ and $\varphi_{\text{hyp}}^s$ of the zones $Z_{\text{diss}}(N)$, $Z_{\text{hyp}}^\ell(N)$ and $Z_{\text{hyp}}^s(N)$ by

\[
\varphi_{\text{diss}}(t, \xi) = \chi(|\xi|N^{-1}) \chi \left(|(1+t)|\xi|N^{-1}\right)
\]

\[
\varphi_{\text{hyp}}^\ell(t, \xi) = \chi \left(|\xi|N^{-1}\right) (1 - \chi \left(|(1+t)|\xi|N^{-1}\right))
\]

\[
\varphi_{\text{hyp}}^s(t, \xi) = \chi \left(|\xi|N^{-1}\right) \left(1 - \chi \left(|\xi|N^{-1}\right)\right)
\]

(2.10)

such that $\varphi_{\text{diss}}(t, \xi) + \varphi_{\text{hyp}}^\ell(t, \xi) + \varphi_{\text{hyp}}^s(t, \xi) = 1$. We consider the micro-energy

\[
U(t, \xi) = (h(t, \xi)\hat{u}, D_t\hat{u})^T,
\]

where

\[
h(t, \xi) = \frac{N}{1 + t} \varphi_{\text{diss}}(t, \xi) + |\xi| \left(\varphi_{\text{hyp}}^\ell(t, \xi) + \varphi_{\text{hyp}}^s(t, \xi)\right)
\]

(2.11)

is a suitable time-dependent version of the usual Sobolev weight $(1 + |\xi|^2)^{1/2}$ and $D_t = -i\partial_t$ denotes the Fourier derivative.

In the hyperbolic zone we apply a diagonalization procedure to a first-order system corresponding to equation (2.7) in order to derive a representation for the fundamental solution. We follow some ideas of Wirth [8] and Yagdjian [13]. We will consider a system with a coefficient matrix composed of a diagonal main part and a remainder part. The goal of this diagonalization is to keep the diagonal part in every step of the diagonalization and to improve the remainder terms.

To derive the asymptotic behavior of the fundamental solution to (2.7) in the dissipative zone we will perform, for $L^1$ condition (A1), one step of diagonalization and apply the Levinson Theorem A.1 and, for $L^\sigma$ condition (A2), we will apply the Hartman–Wintner Theorem A.2. For the $L^\sigma$ condition we need one more step of diagonalization (see proof of Theorem A.2).

2.2. Treatment in the dissipative zone. In the dissipative zone the micro-energy (2.11) becomes

\[
U(t, \xi) = \left(\frac{N}{1 + t}\hat{u}, D_t\hat{u}\right).
\]

(2.13)

Therefore, equation (2.7) rewrites as system

\[
D_t U(t, \xi) = \tilde{A}(t, \xi) U(t, \xi) = \left(\frac{1 + t}{1 + t} \frac{N}{1 + t} \frac{N}{1 + t} \frac{iN}{ib(t)}\right) U(t, \xi)
\]

(2.14)

with coefficient matrix $\tilde{A}(t, \xi)$ within $Z_{\text{diss}}(N)$. In order to estimates its fundamental solution $\mathcal{E}(t, s, \xi)$ we apply Levinson’s Theorem A.1. Note, that the matrix $\tilde{A}(t, \xi)$ behaves like $(1 + t)^{-1}$ in the zone and therefore it reasonable to rewrite it as Fuchs type system

\[
(1 + t)\partial_t U(t, \xi) = (A + R(t, \xi)) U(t, \xi)
\]

(2.15)

with matrices

\[
A = \begin{pmatrix}
-1 & iN \\
0 & -b_0
\end{pmatrix}
\]

(2.16)

and

\[
R(t, \xi) = \begin{pmatrix}
0 & 0 \\
\frac{i(1+t)^2|\xi|^2 + i(1+t)^2m(t) - m_0}{N} & \frac{b_0 - (1+t)b(t)}{N}
\end{pmatrix}
\]

(2.17)
By Hypothesis 2 in the form (A1) and the definition of the zone we know that
\[
\sup_{|\xi| < N} \int_1^{|\xi|} \| R(t, \xi) \| \frac{dt}{t} < \infty
\]
(2.18)
and \( R(t, \xi) \) is a remainder term in the sense of Theorem A.1. Furthermore, as \( \operatorname{tr} A = -1 - b_0 \) and \( \det A = b_0 + m_0 \) the eigenvalues of \( A \) are given as
\[
\mu_\pm = -\frac{b_0 + 1}{2} \pm \sqrt{\frac{(b_0 - 1)^2}{4} - m_0}.
\]
(2.19)
In particular we see that
\[
4m_0 \neq (b_0 - 1)^2
\]
(2.20)
implies that the eigenvalues are distinct.

**Theorem 2.1.** Assume Hypothesis 2 with \( \sigma = 1 \) together with (2.20). Then the matrix-valued fundamental solution of the system (2.15) satisfies
\[
\| \mathcal{E}(t, s, \xi) \| \lesssim \left( \frac{1 + t}{1 + s} \right)^{\Re \mu_+}
\]
(2.21)
uniformly in \( 0 \leq s \leq t \) and \((t, \xi) \in Z_{\text{diss}}(N)\).

**Proof.** This follows from Theorem A.1 applied to (2.15) with \( R(t, \xi) \) extended by zero outside \( Z_{\text{diss}}(N) \). To simplify notation, we denote by \( e_\pm \) the two normalised eigenvectors corresponding to \( \mu_\pm \). From \( \mu_+ \neq \mu_- \) we conclude that there exist two linearly independent solutions to (2.15) of the form
\[
U_\pm(t, \xi) = (e_\pm + o(1))(1 + t)^{\mu_\pm}, \quad t \to \infty
\]
(2.22)
within \( Z_{\text{diss}}(N) \) and uniformly in \( \xi \). Constructing the fundamental matrix as in Remark A.1 we see that
\[
\mathcal{E}(t, 0, \xi) = (U_-(t, \xi)|U_+(t, \xi))(U_-(0, \xi)|U_+(0, \xi))^{-1},
\]
(2.23)
and hence, we obtain
\[
\| \mathcal{E}(t, 0, \xi) \| \lesssim (1 + t)^{\Re \mu_+}
\]
(2.24)
for any \((t, \xi) \in Z_{\text{diss}}(N)\). Using the scaling from Remark A.2 (taking into account the shift in time) we obtain (2.21) uniformly in \( 0 \leq s \leq t \leq \theta|\xi| \). \( \square \)

In order to treat the form (A2) of Hypothesis 2 by the Hartmann–Wintner Theorem A.2 we need to ensure that \( \Re \mu_+ \neq \Re \mu_- \). This happens if both are real and distinct. The latter is equivalent to
\[
4m_0 < (b_0 - 1)^2.
\]
(2.25)

**Theorem 2.2.** Assume Hypothesis 2 with \( \sigma \in (1, 2] \) together with (2.25). Let further \( \sigma' \) be the dual Lebesgue index to \( \sigma \). Then the fundamental solution of the system (2.15) satisfies
\[
\| \mathcal{E}(t, s, \xi) \| \lesssim \left( \frac{1 + t}{1 + s} \right)^{\mu_+} e^{(\ln (\frac{1 + t}{1 + s}))^{\frac{1}{\sigma'}}}
\]
(2.26)
uniformly in \( 0 \leq s \leq t \) and \((t, \xi) \in Z_{\text{diss}}(N)\).
Proof. As in the previous we extend $R(t, \xi)$ by zero outside $Z_{\text{diss}}(N)$ and denote by $e_{\pm}$ normalized eigenvectors of $A$ corresponding to $\mu_{\pm}$. Forming the unitary matrix $P = (e_{-} | e_{+})$ with these eigenvectors as columns and defining $\tilde{R}(t, \xi) = P^{-1}R(t, \xi)P$ allows to rewrite (2.15) in the new unknown $\tilde{U}(t, \xi) = PU(t, \xi)$ as

\[
(1 + t)\partial_{t}\tilde{U}(t, \xi) = \left( \text{diag}(\mu_{-}, \mu_{+}) + \tilde{R}(t, \xi) \right) \tilde{U}(t, \xi)
\]  

(2.27)

We apply Theorem A.2 to this system. As $\mu_{\pm}$ are real and distinct, they clearly satisfy (A.28). Furthermore, the matrix $\tilde{R}(t, \xi)$ contains combinations of $(1 + t)b(t) - b_{0}$ and $(1 + t)^{2}m(t) - m_{0}$ controlled by (A2) and terms of the form $(1 + t)^{2}|\xi|^{2}$ which are uniformly bounded and integrable with respect to $dt/t$ by the definition of the zone. Hence, Hypothesis 2 in the form (A2) implies (A.27) with $\sigma \in (1, 2]$. Therefore, Theorem A.2 applies and gives us a matrix $N(t, \xi) \in L^{\nu}(\mathbb{R}, dt/t)$ transforming (2.15) for $t \geq t_{0}$ into Levinson form

\[
(1 + t)\partial_{t}V(t, \xi) = \left( \text{diag}(\mu_{-} + \tilde{\tau}_{-}, \mu_{+} + \tilde{\tau}_{+}) + \tilde{R}_{1}(t, \xi) \right) V(t, \xi)
\]  

(2.28)

in the new unknown $V(t, \xi) = (I + N(t, \xi))^{-1}\tilde{U}(t, \xi)$ and with the new remainder $\tilde{R}_{1} \in L^{1}(\mathbb{R}, dt/t)$. By $\tilde{\tau}_{-}(t, \xi)$ and $\tilde{\tau}_{+}(t, \xi)$ we denote the diagonal entries of $\tilde{R}(t, \xi)$. The new diagonal part satisfies the dichotomy condition (A.3), the additional diagonal entries satisfy by Hölder’s inequality

\[
\int_{s}^{t} |\tilde{\tau}_{++}(\tau, \xi)| \frac{d\tau}{1 + \tau} \leq C \left( \ln \frac{1 + t}{1 + s} \right)^{\frac{\sigma'}{1 + \sigma}}
\]  

(2.29)

with $\sigma'$ the dual index and are thus small compared to

\[
\int_{s}^{t} (\mu_{+} - \mu_{-}) \frac{d\tau}{1 + \tau} = (\mu_{+} - \mu_{-}) \left( \ln \frac{1 + t}{1 + s} \right).
\]  

(2.30)

Hence, Levinson’s theorem A.1 yields a fundamental system of solutions together with the estimate

\[
\|\mathcal{E}_{V}(t, t_{0}, \xi)\| \leq (1 + t)^{\mu_{+}} \exp \left( C \left( \ln \left( 1 + t \right) \right)^{\frac{1}{1 + \sigma}} \right), \quad t \geq t_{0},
\]  

(2.31)

for the matrix-valued fundamental solution to the transformed system. The scaling argument from Remark A.2 extends this estimate to variable starting times $t_{0} \leq s \leq t \leq b_{0}|\xi|$ as

\[
\|\mathcal{E}_{V}(t, s, \xi)\| \lesssim \left( \frac{1 + t}{1 + s} \right)^{\mu_{+}} \exp \left( C \left( \ln \left( 1 + t \right) \right)^{\frac{1}{1 + \sigma}} \right).
\]  

(2.32)

Transforming back to the original system combined with compactness of the remaining bit of $Z_{\text{diss}}(N)$ where the transform was not defined yields the desired statement. The theorem is proved. \hfill \Box

Remark 2.1. If $2 \text{Re} \mu_{+} < -b_{0}$, i.e., if

\[
b_{0}(b_{0} - 2) < 4m_{0},
\]  

(2.33)

then Theorems 2.1 and 2.2 imply

\[
\|\mathcal{E}(t, s, \xi)\| \lesssim \frac{\lambda(s)}{\lambda(t)}.
\]  

(2.34)
for all $0 \leq s \leq t$ and $(t, \xi) \in Z_{\text{diss}}(N)$. In the first case this is obvious, while in the second case we observe that for all $\varepsilon > 0$ there exists a constant $c_\varepsilon$ such that

$$\exp \left( C \left( \ln \frac{1+t}{1+s} \right)^\frac{1}{\varepsilon} \right) \leq c_\varepsilon \left( \frac{1+t}{1+s} \right)^\varepsilon.$$  

(2.35)

Therefore

$$\left( \frac{1+t}{1+s} \right)^{\mu^+} \exp \left( C \left( \ln \frac{1+t}{1+s} \right)^\frac{1}{\varepsilon} \right) \lesssim \left( \frac{1+t}{1+s} \right)^{\mu^++\varepsilon} \lesssim \left( \frac{1+t}{1+s} \right)^{-\frac{b_0}{2} - \varepsilon} \lesssim \frac{\lambda(s)}{\lambda(t)}$$  

(2.36)

uniformly in $0 \leq s \leq t$.

### 2.3. The zone-boundary.

In order to combine the estimates from the dissipative zone with the treatment in the hyperbolic zone, we need one further estimate. It is conditional in the sense that it is entirely based on the final estimate from the dissipative zone and not on the precise assumptions used to prove it. It is also the first statement using Hypothesis 1.

**Lemma 2.3.** Assume Hypothesis 1 and Hypothesis 2 in combination with (2.33). Then for $|\xi| \leq N$ the symbol-like estimates

$$\| D_\theta^\alpha \xi(\theta|\xi|, 0, \xi) \| \leq C_\alpha \frac{1}{\lambda(\theta|\xi|)} |\xi|^{-|\alpha|}$$  

(2.37)

are valid for all $|\alpha| \leq \ell$.

**Proof.** To prove this fact we use Duhamel’s formula for $\xi-$derivatives of (2.14). Let first $|\alpha| = 1$. Then $D_\xi D_\xi^\alpha \xi = \left( D_\xi^2 \tilde{A} \right) \xi + \tilde{A} \left( D_\xi^2 \xi \right)$ and thus using $D_\xi^2 \xi(0, 0, \xi) = 0$ we obtain the representation

$$D_\xi^\alpha \xi(t, 0, \xi) = i \int_0^t \xi(t, \tau, \xi) \left( D_\xi^2 \tilde{A}(\tau, \xi) \right) \xi(\tau, 0, \xi) \mathrm{d}\tau.$$  

(2.38)

Since $\| D_\xi^2 \tilde{A}(t, \xi) \| \lesssim 1$ this implies from (2.34) the estimate

$$\| D_\xi^\alpha \xi(t, 0, \xi) \| \lesssim \frac{t}{\lambda(t)} \lesssim \frac{1}{\lambda(t)} |\xi|^{-1}$$  

(2.39)

uniformly on $Z_{\text{diss}}(N)$.

For $|\alpha| = \ell > 1$ we use Leibniz formula to represent $D_\xi^\ell \xi(t, 0, \xi)$ by a corresponding Duhamel integral using a sum of terms $D_\xi^{\alpha_1} \tilde{A}(t, \xi) D_\xi^{\alpha_2} \xi(t, 0, \xi)$ for $|\alpha_1| + |\alpha_2| = \ell$, $\alpha_2 < \alpha$ and apply induction over $\ell$ to obtain

$$\| D_\xi^\ell \xi(t, 0, \xi) \| \lesssim \frac{1}{\lambda(t)} |\xi|^{-|\alpha|}$$  

(2.40)

uniformly within $Z_{\text{diss}}(N)$.

Derivates of $\xi$ with respect to $t$ are estimated directly by the differential equation and $\| \tilde{A}(t, \xi) \| \lesssim (1 + t)^{-1}$ giving

$$\| \partial_t D_\xi^\ell \xi(t, 0, \xi) \| \leq \frac{1}{\lambda(t)} \left( \frac{1}{1 + t} \right)^k |\xi|^{-|\alpha|}.$$  

(2.41)
But now the statement follows from Faà di Bruno’s formula combined with
\[
|D^a_\xi \theta_\xi| \lesssim |\xi|^{-1-|\alpha|}, \quad (2.42)
\]

Remark 2.2. The result of the Lemma \[2.3\] can be reformulated in the following form. The symbol \( \lambda(\theta_\xi)E(\theta_\xi, 0, \xi) \) is an element of the homogeneous symbol class
\[
S^0_\ell = \{ m \in C^\infty(\mathbb{R}^n \setminus \{0\}) : |D^m_\xi| \leq C_\alpha |\xi|^{-|\alpha|} \text{ for all } |\alpha| \leq \ell \}
\]
of order zero and restricted smoothness \( \ell \).

2.4. Treatment in the hyperbolic zone. First, we recall the definition of the hyperbolic symbol class \( S^0_{\ell_1, \ell_2} \{m_1, m_2\} \) from \[6\] and \[7\].

Definition 1. The time-dependent amplitude function \( a = a(t, \xi) \) belongs to the hyperbolic symbol class \( S^0_{\ell_1, \ell_2} \{m_1, m_2\} \) with restricted smoothness \( \ell_1, \ell_2 \) if it satisfies the symbol estimates
\[
|D^a_\xi D^b_\xi a(t, \xi)| \leq C_{k, \alpha} |\xi|^{m_1-|\alpha|} \left( \frac{1}{1+t} \right)^{m_2+k} \quad (2.44)
\]
for all \( (t, \xi) \in \mathcal{Z}_{\text{hyp}}(N) \), all non-negative integers \( k \leq \ell_1 \) and all multi-indices \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq \ell_2 \). We will further use the notation
\[
\mathcal{H}^0_{\ell_1, \ell_2} \{k\} = \bigcap_{m_1 + m_2 = k} S^0_{\ell_1, \ell_2} \{m_1, m_2\} \quad (2.45)
\]
and \( \mathcal{S}_{\ell_1} \{m_1, m_2\} \) as short-hand for \( \mathcal{S}^0_{\ell_1} \{m_1, m_2\} \) and similarly for \( \mathcal{H}_{\ell_1} \{k\} \).

2.4.1. Diagonalisation. Within the hyperbolic zone the micro-energy \[2.3\] satisfies the system
\[
D_t U = A(t, \xi) U \quad (2.46)
\]
with
\[
A(t, \xi) = \begin{pmatrix}
0 & |\xi| \\
|\xi| + \frac{m(t)}{|\xi|} & i h(t)
\end{pmatrix} \quad \text{mod } \mathcal{H}_{\ell_1} \{1\} \quad (2.47)
\]
as consequence of \( h(t, \xi) = |\xi| \mod \mathcal{H}_{\ell_1} \{1\} \). Note, that \( A(t, \xi) \in \mathcal{S}^{0, \infty}_{\ell_1} \{1, 0\} \). We denote by \( \mathcal{E}(t, s, \xi) \) the fundamental solution to \( (2.46) \), i.e., the matrix-valued solution to
\[
D_t \mathcal{E}(t, s, \xi) = A(t, \xi) \mathcal{E}(t, s, \xi), \quad \mathcal{E}(s, s, \xi) = I \in \mathbb{C}^{2 \times 2} \quad (2.48)
\]
for \( t \geq s \) and \( (s, \xi) \in \mathcal{Z}_{\text{hyp}}(N) \). In order to estimate \( \mathcal{E}(t, s, \xi) \) we follow \[6\] and apply a diagonalisation scheme within the hyperbolic symbol classes \( \mathcal{S}^0_{\ell_1, \ell_2} \{\cdot, \cdot\} \).

Preliminary step. In the first step we use the matrix
\[
M = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}, \quad M^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}, \quad (2.49)
\]
to transform the principal part of the system. This yields for \( V(t, \xi) = M^{-1} U(t, \xi) \) the new system
\[
D_t V(t, \xi) = (D(\xi) + B(t) + C(t, \xi)) V(t, \xi) \quad (2.50)
\]
with coefficient matrices

\[ D(\xi) = \text{diag}(\|\xi\|, -|\xi|) \in \mathcal{S}_N\{1, 0\}, \quad (2.51) \]

\[ B(t) = \frac{ib(t)}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{S}^{(t, \infty)}_N\{0, 1\} \quad (2.52) \]

and

\[ C(t, \xi) = \frac{m(t)}{2|\xi|} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathcal{S}^{(t, \infty)}_N\{-1, 2\}. \quad (2.53) \]

The matrix \( D(\xi) \) gives the diagonal principal part and our aim is to apply further transformations to improve the symbolic behaviour of the remainders.

**First step.** We give the first step in detail. We want to find a matrix-valued symbol \( N^{(1)}(t, \xi) \) from \( \mathcal{S}^{(t, \infty)}_N\{-1, 1\} \) and a diagonal matrix \( F^{(0)}(t, \xi) \) from \( \mathcal{S}^{(t, \infty)}_N\{0, 1\} \) such that the operator identity

\[ B^{(1)}(t, \xi) = (D_t - D(\xi) - B(t) - C(t, \xi))(I + N^{(1)}(t, \xi)) \]

\[ - (I + N^{(1)}(t, \xi))(D_t - D(\xi) - F^{(0)}(t, \xi)) \quad (2.54) \]

defines a matrix \( B^{(1)}(t, \xi) \) from \( \mathcal{S}^{(t, -1, \infty)}_N\{-1, 2\} \). In order for this to happen, we have to satisfy the commutator identity

\[ [D(\xi), N^{(1)}(t, \xi)] + B(t) = F^{(0)}(t, \xi). \quad (2.55) \]

As \( D(\xi) \) is diagonal, the commutator on the left has zero diagonal entries. Therefore, \( F^{(0)}(t, \xi) = \text{diag} B(t) \) (where \( \text{diag} \) denotes the diagonal part of the matrix).

In particular it follows that \( F^{(0)}(t, \xi) \) is independent of \( \xi \) and is of the desired class. Similarly, the off-diagonal parts of \( N^{(1)}(t, \xi) \) are determined by the commutator equation while we are free to choose the diagonal entries. Requiring in addition that \( \text{diag} N^{(1)}(t, \xi) = 0 \), we obtain

\[ N^{(1)}(t, \xi) = \frac{ib(t)}{4|\xi|} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.56) \]

and it is clear that this matrix belongs to the symbol class \( \mathcal{S}^{(t, \infty)}_N\{-1, 1\} \).

**Iterative improvements.** We construct recursively matrices \( N^{(k)}(t, \xi) \) from the classes \( \mathcal{S}^{(t, +1, -k, \infty)}\{-k, k\} \) and diagonal matrices \( F^{(k)}(t, \xi) \) from \( \mathcal{S}^{(t, -1, -k, \infty)}_N\{1 - k, k\} \) such that for

\[ N_k(t, \xi) = I + \sum_{j=1}^{k} N^{(j)}(t, \xi), \quad F_{k-1}(t, \xi) = \sum_{j=0}^{k-1} F^{(j)}(t, \xi), \quad (2.57) \]

the operator identity

\[ B^{(k)}(t, \xi) = (D_t - D(\xi) - B(t) - C(t, \xi))N_k(t, \xi) \]

\[ - N_k(t, \xi)(D_t - D(\xi) - F_{k-1}(t, \xi)) \quad (2.58) \]

yields a remainder \( B^{(k)}(t, \xi) \in \mathcal{S}^{(t, -k, \infty)}_N\{-k, k+1\} \). This again implies a commutator identity and from that

\[ F^{(k-1)}(t, \xi) = \text{diag} B^{(k-1)}(t, \xi), \quad (2.59) \]

\[ N^{(k)}(t, \xi) = \frac{1}{2|\xi|} \begin{pmatrix} 0 & -(B^{(k-1)}(t, \xi))_{12} \\ (B^{(k-1)}(t, \xi))_{21} & 0 \end{pmatrix}, \quad (2.60) \]
where $(\cdot)_{ij}$ stands for the $ij$-entry of the matrix. By induction we also obtain that all matrices belong to the desired symbol classes.

**Proposition 2.4.** Assume Hypothesis [1] with derivatives up to order $\ell$. Then $N^{(k)} \in S^{\ell+1-k,\infty}_N\{-k,k\}$ and $B^{(k)} \in S^{\ell-k,\infty}_N\{-k,k+1\}$ holds true for all $k = 1, \ldots, \ell$. Moreover, for any such $k$ we find a zone constant $N$ such that the matrix $N_k(t,\xi)$ is invertible in $Z_{\text{hyp}}(N)$ and $N_k(t,\xi)^{-1} \in S^{\ell-k+1,\infty}_N\{0,0\}$.

**2.4.2. Fundamental solutions.** In the following we choose $N$ large enough for $N_k$ to be invertible within the zone $Z_{\text{hyp}}(N)$. If we denote $R_k(t,\xi) = -N_k(t,\xi)^{-1}B^{(k)}(t,\xi)$ and consider the new unknown $V_k(t,\xi) = N_k(t,\xi)^{-1}V(t,\xi)$ we obtain the transformed system

$$D_tV_k(t,\xi) = (D(\xi) + E_{k-1}(t,\xi) + R_k(t,\xi))V_k(t,\xi) \quad (2.61)$$

within $Z_{\text{hyp}}(N)$ with diagonal $F_{k-1}(t,\xi) \in S^{\ell+1-k,\infty}_N\{0,1\}$ and with non-diagonal remainder $R_k(t,\xi) \in S^{\ell-k,\infty}_N\{-k,k+1\}$. It remains to estimate its fundamental solution $\mathcal{E}_k(t,s,\xi)$. This follows along the lines of [6].

**Theorem 2.5.** Assume Hypothesis [1]. Then the fundamental solution $\mathcal{E}_k(t,s,\xi)$, $k \geq 1$, of the diagonalized system $(2.61)$ can be represented as

$$\mathcal{E}_k(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\mathcal{E}_0(t,s,\xi)Q_k(t,s,\xi), \quad (2.62)$$

for $t \geq s$ and $(s,\xi) \in Z_{\text{hyp}}(N)$, where

1. the function

$$\lambda(t) = \exp\left(\frac{1}{2} \int_0^t b(\tau) \, d\tau\right) \quad (2.63)$$

2. the matrix $\mathcal{E}_0(t,s,\xi)$ given by

$$\mathcal{E}_0(t,s,\xi) = \begin{pmatrix} e^{it-s(|\xi|)} & 0 \\ 0 & e^{-it-s(|\xi|)} \end{pmatrix} \quad (2.64)$$

is the fundamental solution of the free wave equation,

3. the matrix $Q_k(t,s,\xi)$ satisfies for all multi-indices $|\alpha| \leq \min\{k-1,\ell-k-1\}$ the symbol like estimates

$$\|D^\alpha_k Q_k(t,s,\xi)\| \leq C_\alpha |\xi|^{-|\alpha|} \quad (2.65)$$

uniformly in $t \geq s \geq \theta_{|\xi|}$ and

$$\|D^\alpha_k Q_k(t,\theta_{|\xi|},\xi)\| \leq C_\alpha |\xi|^{-|\alpha|} \quad (2.66)$$

uniformly in $(t,\xi) \in Z_{\text{hyp}}(N)$.

Furthermore, $Q_k(t,s,\xi)$ is invertible and converges for $t \to \infty$ to the invertible matrix $Q_k(\infty,s,\xi)$ locally uniform with respect to $\xi \neq 0$.

**Proof.** The fundamental solution to the main diagonal part $D_t - D(\xi) - F_0(t)$ is given by

$$\frac{\lambda(s)}{\lambda(t)}\mathcal{E}_0(t,s,\xi), \quad (2.67)$$

therefore the matrix $Q_k(t,s,\xi)$ in $(2.62)$ solves

$$D_tQ_k(t,s,\xi) = R_k(t,s,\xi)Q_k(t,s,\xi), \quad Q_k(s,s,\xi) = I, \quad (2.68)$$
for $\theta_{\xi} \leq s, t$ and with
\[
R_k(t, s, \xi) = E_0(s, t, \xi)\left(F_{k-1}(t, \xi) - F_0(t, \xi) + R_k(t, \xi)\right)E_0(t, s, \xi). \tag{2.69}
\]

**Uniform bounds.** As symbols from $S^0_\alpha\{-1, 2\}$ are uniformly integrable in $t$ within the hyperbolic zone, it follows that
\[
\sup_{(s, \xi) \in \mathcal{E}_{\text{hyp}}(N)} \int_s^\infty \|R_k(t, s, \xi)\|dt = C < \infty \tag{2.70}
\]
and $Q_k(t, s, \xi)$ is uniformly bounded in $t, s \geq \theta_{\xi}$ as consequence of the representation by Peano–Baker formula
\[
Q_k(t, s, \xi) = I + \sum_{j=1}^{\infty} \int_s^t \cdots \int_s^{t_j} R_k(t_j, s, \xi) \cdots dt_2 dt_1. \tag{2.71}
\]
Furthermore, Liouville theorem yields
\[
|\det Q_k(t, s, \xi)| = \left|\exp\left(i \int_s^t \text{tr} R_k(\tau, s, \xi) d\tau\right)\right| \geq \exp(-dC) > 0 \tag{2.72}
\]
and thus uniform invertibility of $Q_k(t, s, \xi)$. The convergence for $t \to \infty$ follows from the Cauchy criterion combined with (2.71).

**Estimates for derivatives.** For estimating derivatives, we have to treat the diagonal terms and the remainder separately. Writing
\[
Q_k(t, s, \xi) = \exp\left(i \int_s^t \left(F_{k-1}(\tau, \xi) - F_0(\tau, \xi)\right) d\tau\right) \tilde{Q}_k(t, s, \xi), \tag{2.73}
\]
we obtain an equation
\[
D_t \tilde{Q}_k(t, s, \xi) = \tilde{R}_k(t, s, \xi) \tilde{Q}_k(t, s, \xi), \quad \tilde{Q}_k(s, s, \xi) = I, \tag{2.74}
\]
with improved integrability properties of $\tilde{R}_k(t, s, \xi)$. The exponential in equation (2.73) is uniformly bounded and behaves both in $(t, \xi)$ as well as in $(s, \xi)$ as symbol from $S^0_\alpha\{0, 0\}$ uniformly in the remaining variable. This follows from Proposition 2.4.

The matrix $\tilde{R}_k(t, s, \xi)$ has symbol-like estimates for large enough $k$, where the good behaviour of the remainder $\tilde{R}_k \in S^0_{\alpha-k}(-k, k+1)$ allows to compensate the badly behaving derivatives of $E_0$. This implies (in combination with the definition of the zone)
\[
\left\|D^\beta_\xi D^\alpha_\xi \tilde{R}_k(t, s, \xi)\right\| \leq \left(\frac{1}{1 + t}\right)^2 |\xi|^{-|\alpha|+|\beta|} \tag{2.75}
\]
for $|\alpha| - |\beta| \leq k - 1$ and $|\beta| \leq \ell - k - 1$. Combined with Peano–Baker formula and estimate (2.42) for derivatives of the zone-boundary the estimates
\[
\left\|D^\alpha_\xi \tilde{Q}_k(t, s, \xi)\right\| \leq C_\alpha |\xi|^{-|\alpha|}, \quad |\alpha| \leq k - 1, \tag{2.76}
\]
and
\[
\left\|D^\alpha_\xi \tilde{Q}_k(t, \xi, \xi)\right\| \leq C_\alpha |\xi|^{-|\alpha|}, \quad |\alpha| \leq \min\{k - 1, \ell - k - 1\} \tag{2.77}
\]
follow. They imply the desired statements. \hfill \Box
2.4.3. Transforming back to the original problem. After constructing the fundamental solution $E_k(t, s, \xi)$, we transform back to the original problem and get in the hyperbolic zone the representation

$$E(t, s, \xi) = \frac{\lambda(s)}{\lambda(t)} M^{-1} N_k(t, \xi)^{-1} E_0(t, s, \xi) Q_k(t, s, \xi) N_k(s, \xi) M,$$

with uniformly bounded symbols $N_k, N_k^{-1} \in \mathcal{S}^{t-k,1,\infty} N, 0, 0$, unitary $E_0$ and $Q_k$ of known properties. For large frequencies this is used with $s = 0$, while for $|\xi| \leq N$ we have to take into account the fundamental solution constructed in the dissipative zone. This leads to

$$E(t, 0, \xi) = \frac{1}{\lambda(t)} M^{-1} N_k(t, \xi)^{-1} E_0(t, \theta_{|\xi|}, \xi)$$

$$\times Q_k(t, \theta_{|\xi|}, \xi) N_k(\theta_{|\xi|}, \xi) M \lambda(\theta_{|\xi|}) E(\theta_{|\xi|}, 0, \xi),$$

for $t \geq \theta_{|\xi|}$.

2.5. Collecting the estimates. We want to collect the estimates proved so far. Estimates will be obtained as combination from results of the hyperbolic zone and from the dissipative zone. One is related to the large-time behaviour of all non-zero frequencies, while the other plays a role in estimating the exceptional frequency $\xi = 0$. 

Figure 1. Collecting the restimates
2.5.1. Estimates for the fundamental solution. High frequencies are described by the a WKB expansion of solutions giving an overall decay estimate based on the function \( \lambda(t) \). In Figure 1 this corresponds to the dashed line in the complex plane. The two dots correspond to the exponents \( \mu_{\pm} \) arising from the Levinson’s theorem. They are responsible for the small frequency behaviour and the interplay of the relation of these dots and the dashed line will be the major reason for the appearing different cases of final estimates.

The main estimates obtained so far can be seen in Tables 1 and 2. We have to distinguish between the situation of condition (A1) in Hypothesis 2 and the situation of condition (A2) in Hypothesis 2. In the latter case we can only treat mass terms satisfying \( 4m_0 < (b_0 - 1)^2 \).

2.5.2. Choice of parameters. The number of diagonalisation steps needed in the hyperbolic zone determines the zone constant \( N \) and thus the decomposition of the phase space. When proving energy estimates it will be enough to apply one non-trivial step of diagonalisation in the hyperbolic zone and for this any choice of \( N \) will be good. When proving dispersive estimates several such steps are necessary and \( N \) has to be chosen large enough.

The number \( \ell \) of derivatives required in Hypothesis 1 depends on the number of diagonalisation steps to be used and the needed symbol properties of the matrix function \( Q_{k}(t, \theta, \xi) \). When proving energy estimates, \( \ell = 1 \) is sufficient.

3. Estimates

3.1. Energy estimates and their sharpness. The representation of solutions obtained so far allow us to conclude estimates for the solution and its derivatives. This section is devoted to the study of estimates, which are directly related to our micro-energy.

**Theorem 3.1.** Assume Hypothesis 1 with \( \ell = 1 \), Hypothesis 2 with \( \sigma = 1 \) and \( b_0(b_0 - 2) \leq 4m_0 \). Then the \( L^2 \) estimate

\[
\|(1 + t)^{-\frac{1}{2}}u(t, \cdot)\|_{L^2} + \|\nabla u(t, \cdot)\|_{L^2} + \|u_{t}(t, \cdot)\|_{L^2} \lesssim \frac{1}{\lambda(t)} \left( \|u_0\|_{H^1} + \|u_1\|_{L^2} \right) \tag{3.1}
\]

holds true for any solution \( u \) of (2.1) to initial data \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \).
Proof. We first recall the stronger statement
\[ \|E(t, s, \xi)\| \lesssim \frac{\lambda(s)}{\lambda(t)} \]
for the fundamental solution \(E(t, s, \xi)\) constructed in Section 2. We start by considering the dissipative zone. Here Theorem 2.1 in combination with Remark 2.1 yields
\[ \|E(t, s, \xi)\| \lesssim \left( \frac{1 + t}{1 + s} \right) \mu_k \lesssim \frac{\lambda(s)}{\lambda(t)} \]
uniform with respect to \(0 \leq s \leq t \leq \theta |\xi|\).

Next we consider the hyperbolic zone and apply Proposition 2.4 and Theorem 2.5 with \(k = 1\). Hence, with uniformly bounded matrices \(N_1(t, \xi)\), \(N_1(t, \xi)^{-1}\) and \(Q_1(t, s, \xi)\) and the unitary matrix \(E_0(t, s, \xi)\) we obtain from (2.7)
\[ E(t, s, \xi) = \frac{\lambda(s)}{\lambda(t)} M^{-1} N_1(t, \xi)^{-1} E_0(t, s, \xi) Q_1(t, s, \xi) N_1(s, \xi) M \]
and thus the desired norm bound.

Now we show how this implies the desired energy estimate. We first observe that the definition of the micro-energy implies
\[ \|U(0, \cdot)\|_{L^2} = \|F^{-1} U(0, \cdot)\|_{L^2} \approx \|u_0\|_{H^1} + \|u_1\|_{L^2} \]
as equivalence of norms (with constants depending on the zone constant \(N\)). Furthermore,
\[ |\xi| \hat{u}(\xi) = \frac{|\xi|}{h(t, \xi)} h(t, \xi) \hat{u}(\xi) \]
with \(|\xi|/h(t, \xi)\) uniformly bounded by 1. Similarly
\[ \frac{1}{1 + t} \hat{u}(\xi) = \frac{1}{(1 + t)h(t, \xi)} h(t, \xi) \hat{u}(\xi) \]
and again by the definition of the zone \((1 + t)h(t, \xi) \geq N\). Therefore, from \(U(t, \xi) = E(t, 0, \xi) U(0, \xi)\) we conclude the desired estimates
\[ \|u(t, \cdot)\|_{L^2} \leq \|U(t, \cdot)\|_{L^2} \leq \frac{1}{\lambda(t)} \|U(0, \cdot)\|_{L^2} \leq \frac{1}{\lambda(t)} (\|u_0\|_{H^1} + \|u_1\|_{L^2}) \]
as well as
\[ \|\nabla u(t, \cdot)\|_{L^2} = \|\xi| \hat{u}(\xi)\|_{L^2} \leq \|U(t, \cdot)\|_{L^2} \leq \frac{1}{\lambda(t)} (\|u_0\|_{H^1} + \|u_1\|_{L^2}) \]
and
\[ \|(1 + t)^{-1} u(t, \cdot)\|_{L^2} \leq \|U(t, \cdot)\|_{L^2} \leq \frac{1}{\lambda(t)} (\|u_0\|_{H^1} + \|u_1\|_{L^2}) \]
by the aid of Plancherel’s theorem. This completes the proof. \qed

If we assume Hypothesis 2 with \(\sigma > 1\) we have to restrict the admissible values of \(m_0\) further. The proof goes in analogy to the above one replacing Theorem 2.1 by Theorem 2.2 for the treatment of the dissipative zone.

Theorem 3.2. Assume Hypothesis 1 with \(\ell = 1\), Hypothesis 2 with \(\sigma \in (1, 2]\) and \(b_0(b_0 - 2) \leq 4m_0 < (b_0 - 1)^2\). Then the \(L^2 - L^2\) estimate
\[ \|(1 + t)^{-1} u(t, \cdot)\|_{L^2} + \|\nabla u(t, \cdot)\|_{L^2} + \|u_t(t, \cdot)\|_{L^2} \lesssim \frac{1}{\lambda(t)} (\|u_0\|_{H^1} + \|u_1\|_{L^2}) \]
holds true for any solution \(u\) of (2.1) to initial data \(u_0 \in H^1(\mathbb{R}^n)\) and \(u_1 \in L^2(\mathbb{R}^n)\).
We will give some examples to show the applicability of the previous results.

**Example 3.1.** Let us consider for $b_0, m_0 \in \mathbb{R}$

\[
    b(t) = \frac{b_0}{1 + t} + \frac{h_1(t)}{1 + t} \\
    m(t) = \frac{m_0}{(1 + t)^2} + \frac{h_2(t)}{(1 + t)^2}
\]

(3.12)

(3.13)

with uniformly bounded $h_j(t)$, $j = 1, 2$ and uniformly bounded $t\partial_t h_j(t)$ and with the integrability condition

\[
    \int_0^\infty |h_j(t)| \frac{dt}{1 + t} < \infty, \quad j = 1, 2.
\]

(3.14)

Then Hypothesis 1 is satisfied with $\ell = 1$ and Hypothesis 2 is satisfied with $\sigma = 1$. If we further suppose that $b_0(b_0 - 2) < 4m_0$, then the energy estimate

\[
    \| (1 + t)^{-1}u(t, \cdot), \frac{\partial}{\partial x}u(t, \cdot) \|_2 \lesssim (1 + t)^{-\frac{1}{2}} \left( \| u_0 \|_{H^1} + \| u_1 \|_2 \right)
\]

(3.15)

holds true. The decay is independent of $m_0$ and related to the decay for non-effective wave damped models treated in [8].

**Example 3.2.** We consider the same situation as in the previous example, but replace (3.14) by

\[
    \int_0^\infty |h_j(t)|^\sigma \frac{dt}{1 + t} < \infty, \quad j = 1, 2,
\]

(3.16)

then under the more restrictive condition $b_0(b_0 - 2) < 4m_0 < (b_0 - 1)^2$ on the numbers $m_0$ and $b_0$ the same estimate (3.15) holds true. To be more specific, this allows to treat

\[
    b(t) = \frac{b_0}{1 + t} + \frac{b_1}{(e + t)(\ln(e + t))^{\gamma}}, \\
    m(t) = \frac{m_0}{(1 + t)^2} + \frac{m_1}{(e + t)^2(\ln(e + t))^{\gamma}}
\]

(3.17)

(3.18)

with arbitrary $b_1$, $m_1$ and $\gamma \in (1/2, 1]$. It satisfies (3.16) with $\sigma \in (\gamma^{-1}, 2]$.

**Remark 3.3.** The Examples 3.1 and 3.2 show us that small mass terms have no influence on our estimates of the energy of solution to problem 2.1.

The energy estimates of the above theorems are sharp. For this we refer to Theorem 2.5 and the uniform invertibility of $Q(t, s, \xi)$ within the hyperbolic zone. We give only a preliminary statement, but it implies the sharpness.

**Lemma 3.3.** Assume the initial data $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ are Schwartz functions such that supp $u_0$ and supp $u_1$ are compact and contained in the set $\{ \xi : |\xi| > N \}$. Then the limit

\[
    \lim_{t \to \infty} \lambda(t)^2 (\| \nabla u(t, \cdot) \|_2^2 + \| u(t, \cdot) \|_2^2) > 0
\]

exists and is different from zero.

**Proof.** As the Fourier support is preserved under evolution, we conclude from Plancherel identity that $\| \hat{U}(t, \cdot) \|_2^2 = \| \nabla u(t, \cdot) \|_2^2 + \| u(t, \cdot) \|_2^2$. Therefore, by Theorem 2.5

\[
    \lambda(t)\| U(t, \xi) \| = \| N_1(t, \xi)^{-1} \mathcal{E}_0(t, 0, \xi) Q_1(t, 0, \xi) N_1(0, \xi) U(0, \xi) \|
\]

(3.19)
and \( \| N_1(t, \xi) - I \| \lesssim (1 + t)^{-1} \) uniformly in \( \xi \in \text{supp} \ U(0, \cdot) \). This implies by using

a Neumann series \( \| N_1(t, \xi)^{-1} - I \| \lesssim (1 + t)^{-1} \) and, hence, using the fact that \( \mathcal{E}_0 \) is unitary,

\[
\lambda(t) \| U(t, \xi) \| = \| Q_1(t, 0, \xi) \| N_1(0, \xi) U(0, \xi) \| + C(1 + t)^{-1}. \quad (3.21)
\]

The constant \( C \) depends on the solution. The first term tends to a limit and thus

\[
\lim_{t \to \infty} \lambda(t) \| U(t, \xi) \| = \| Q_1(\infty, 0, \xi) \| N_1(0, \xi) U(0, \xi) \| \geq \| Q_1(\infty, 0, \xi)^{-1} \| \| N_1(0, \xi)^{-1} \| \| U(0, \cdot) \|. \quad (3.22)
\]

Integrating the square of this inequality with respect to \( \xi \) proves the statement. \( \Box \)

### 3.2. Large dissipation and improvements by moment conditions.

If the dissipation term is large, \( 4m_0 < b_0(b_0 - 2) \), then the exponent \( \mu_+ \) from the dissipative zone is larger than \( -b_0/2 \). Therefore, small frequencies yield slower decay than the hyperbolic zone. This phenomenon characterises the transition from non-effective to effective dissipation.

Assuming conditions on the initial data we can control this behaviour of small frequencies and still obtain the same decay rates as before. This is obvious, if \( 0 \notin \text{supp} \ U(0, \cdot) \). We want to do better and use moment conditions. We denote by \( L^\infty_n(\mathbb{R}^n) \) the weighted Lebesgue space

\[
L^\infty_n(\mathbb{R}^n) = \{ f : \int |f(x)| (1 + x^2)^{\kappa/2} \, dx < \infty \} \quad (3.23)
\]

and assume data have finite energy and belong to such a space.

**Theorem 3.4.** Assume Hypothesis \([1]\) with \( \ell = 1 \), Hypothesis \([2]\) with \( \sigma \geq 1 \) and \( 4m_0 < b_0(b_0 - 2) \). Let further

\[
2\kappa = 1 + \sqrt{(b_0 - 1)^2 - 4m_0} \quad (\sigma = 1) \quad (3.24)
\]

\[
2\kappa > 1 + \sqrt{(b_0 - 1)^2 - 4m_0} \quad (\sigma > 1). \quad (3.25)
\]

If the initial data belong to \( u_0 \in H^1(\mathbb{R}^n) \cap L^\infty_n(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \cap L^\infty_n(\mathbb{R}^n) \) with \( |\alpha'| > \kappa - \frac{\sigma}{2} \) and satisfy the moment conditions

\[
\int x^n u_0(x) \, dx = \int x^n u_1(x) \, dx = 0 \quad (3.26)
\]

for all multi-indices \( |\alpha| \leq \kappa' \) then the solution satisfies

\[
\|(1 + t)^{-1} u(t, \cdot) \|_{L^2} + \| \nabla u(t, \cdot) \|_{L^2} + \| u_t(t, \cdot) \|_{L^2} \lesssim \frac{1}{\lambda(t)}. \quad (3.27)
\]

**Proof.** Before giving the details, we outline the main strategy. The \( L^1 \)-condition imposed on the initial data implies continuity in the Fourier image, such that moment conditions determine the order of zero in \( \xi = 0 \). If we can show that \( |\xi|^{-\kappa} U(0, \xi) \) is locally square-integrable near \( \xi = 0 \), it suffices to prove the estimate

\[
|\xi|^\kappa \| \mathcal{E}(t, 0, \xi) \| \lesssim \frac{1}{\lambda(t)} \quad (3.28)
\]

uniformly within \( |\xi| \leq N \). Then \( (1 + t)^{-1} \lesssim h(t, x) \) and \( |\xi| \lesssim h(t, \xi) \) imply \( 3.27 \).

**Step 1. Proof of estimate (3.28).** First, we consider the case \( \sigma = 1 \). Assume, \((t, \xi) \in Z_{\text{diss}}(N) \). Then Theorem \([2.1]\) yields

\[
\| \mathcal{E}(t, 0, \xi) \| \lesssim (1 + t)^{\mu+} \quad (3.29)
\]
with \( \mu_+ \) given in (2.19). If \( \kappa \) satisfies (3.24), then

\[
\kappa > 0 \quad \text{and} \quad \mu_+ - \kappa = -\frac{b_0}{2}
\]  

(3.30)

and the definition of the zone implies

\[
|\xi|\kappa (1 + t)^{\mu_+} = |\xi|\kappa (1 + t)^{\kappa} (1 + t)^{-b_0/2} \leq N^\kappa (1 + t)^{-b_0/2}.
\]  

(3.31)

By (2.6) we know \((1 + t)^{b_0/2} \simeq \lambda(t)\) and Remark 2.1 yields (3.28) for all \((t, \xi) \in Z_{\text{diss}}(N)\). Furthermore, Theorem 2.5 with \(k = 1\) extends this estimate to all \((t, \xi)\) with \(|\xi| \leq N\).

The treatment of \( \sigma > 1 \) is similar. Theorem 2.2 yields for arbitrary \( \varepsilon > 0 \)

\[
\|E(t, 0, \xi)\| \lesssim (1 + t)^{\mu_+ + \varepsilon},
\]

(3.32)

such that for sufficiently small \( \varepsilon \) again \(|\xi|\kappa (1 + t)^{\mu_+} \leq N^\kappa (1 + t)^{-b_0/2 - \varepsilon/2} \lesssim 1/\lambda(t)\).

This implies (3.27) within \(Z_{\text{diss}}(N)\) and Theorem 2.5 extends this to \(|\xi| \leq N\).

Step 2. Preparation of initial data. The assumption \(u_j \in L^1_{\text{loc}}(\mathbb{R}^n)\) implies that \(\hat{u}_j \in C[0, 1](\mathbb{R}^n)\). Furthermore, by the moment condition we know that

\[
\partial^\alpha_x \hat{u}_j(\xi)|_{\xi=0} = 0, \quad |\alpha| \leq [\kappa]
\]

(3.33)

and therefore for \(|\xi| \leq N\)

\[
|\xi|^{-[\kappa]}\|U(0, \xi)\| \lesssim 1.
\]

(3.34)

As \(|\xi|^{-[\kappa]} - \kappa\) is locally square integrable, the same is true for the function \(|\xi|^{-\kappa}U(0, \xi)\) and the desired statement follows. \(\square\)

3.3. Modified scattering result. Now we discuss the sharpness of energy estimates again and formulate a more precise statement. In fact, there is a relation between solutions to the Cauchy problem with mass and dissipation

\[
u_{tt} - \Delta v = 0, \quad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x),
\]

(3.36)

with appropriate related data. We follow some ideas of Wirth [13] and gives (in combination with the energy conservation for free waves) a very precise description of sharpness of the above energy estimates.

**Theorem 3.5.** Assume Hypothesis 1 with \( \ell = 1 \) and Hypothesis 2 with

\[
\sigma = 1 \quad \text{and} \quad b_0(b_0 - 2) \leq 4m_0
\]

(3.37)

or with

\[
\sigma \in (1, 2) \quad \text{and} \quad b_0(b_0 - 2) \leq 4m_0 < (b_0 - 1)^2.
\]

(3.38)

Then there exists a bounded operator

\[
W_+: H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)
\]

(3.39)

such that for Cauchy data \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) of (3.36) and associated data \((v_0, v_1) = W_+(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) to (3.36) the corresponding solution \(u = u(t, x)\) and \(v = v(t, x)\) satisfy

\[
\|\lambda(t)u(t, \cdot) - v(t, \cdot)\|_2 \to 0,
\]

(3.40)

\[
\|\lambda(t)\nabla u(t, \cdot) - \nabla v(t, \cdot)\|_2 \to 0,
\]

(3.41)

as \(t \to \infty\).
Proof. We sketch the major steps of the proof. First let for \( \varepsilon > 0 \)
\[
F_\varepsilon = \left\{ U_0 \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \hat{U}_0(\xi) = 0 \text{ for any } |\xi| \leq \varepsilon \right\}.
\] (3.42)
Then the union \( \mathcal{L} = \bigcup_{\varepsilon > 0} F_\varepsilon \) dense in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). We construct the wave operator \( W_+ \) as pointwise limit on \( \mathcal{L} \) and apply Banach–Steinhaus theorem to show strong convergence on the energy space.

We start by introducing notation. Let \( \mathcal{E}_0(t,s,\xi) \) be the free propagator from (2.64). Then for any solution \( v \) to the free wave equation (3.36) the classical micro-energy \( V(t,\xi) = (|\xi|\hat{v}, D_t \hat{v})^T \) satisfies \( V(t,\xi) = \mathcal{E}_v(t,s,\xi)V(s,\xi) \) for
\[
\mathcal{E}_v(t,s,\xi) = M^{-1}\mathcal{E}_0(t,s,\xi)M.
\] (3.43)
We compare this to the propagator \( \mathcal{E}(t,s,\xi) \) constructed in Section 2 for the micro-energy \( U(t,\xi) = (h(t,\xi)\hat{u}, D_t \hat{u})^T \) associated to solutions \( u \) of (3.35). By (2.78) and (2.79) with \( k = 1 \) this propagator is a product of known matrix functions \( \mathcal{E}_0(t,s,\xi), Q_1(t,s,\xi), N_1(t,\xi) \) and the function \( \lambda(t) \) with known behaviour. We recall from Theorem 2.5 that
\[
\lim_{t \to \infty} Q_1(t,s,\xi) = Q_1(\infty,s,\xi)
\] (3.44)
holds true locally uniform in \( s \) and \( \xi \). Furthermore, \( N_1 - I \in \mathcal{S}^{L,\infty}\{1,1\} \). Our first aim is to show that the limit
\[
W_+(\xi) = \lim_{t \to \infty} \lambda(t)\mathcal{E}_v(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi)
\] (3.45)
uniform in \( |\xi| > \varepsilon \) and thus as norm-limit on \( F_\varepsilon \). Indeed, by (2.78) and (2.79)
\[
\lambda(t)\mathcal{E}_v(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) = \lambda(\theta_{|\xi|})M^{-1}\mathcal{E}_0(0,t,\xi)N_1(t,\xi)^{-1}\mathcal{E}_0(t,\theta_{|\xi|},\xi) \\
\times Q_1(t,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)ME(\theta_{|\xi|},0,\xi)
\] (3.46)
such that using
\[
\mathcal{E}_0(0,t,\xi)N_1(t,\xi)^{-1}\mathcal{E}_0(t,\theta_{|\xi|},\xi) \\
= \mathcal{E}_0(0,\theta_{|\xi|},\xi) + \mathcal{E}_0(0,t,\xi)(N_1(t,\xi)^{-1} - I)\mathcal{E}_0(t,\theta_{|\xi|},\xi)
\] (3.47)
as well as \( N_k(t,\xi) \to I \) uniformly on \( |\xi| \geq \varepsilon \) implies
\[
\lim_{t \to \infty} \lambda(t)\mathcal{E}_v(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \\
= \lambda(\theta_{|\xi|})M^{-1}\mathcal{E}_0(0,\theta_{|\xi|},\xi)Q_1(\infty,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)ME(\theta_{|\xi|},0,\xi)
\] (3.48)
uniformly on \( |\xi| > \varepsilon \). We denote this limit as \( W_+(\xi) \). Due to the energy estimates of Theorem 3.1 we know that \( \lambda(t)\mathcal{E}_v(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \) is uniformly bounded in \( t \) and \( \xi \). Therefore, the matrix \( W_+(\xi) \) is uniformly bounded in \( \xi \). Furthermore, by Banach–Steinhaus theorem we know that
\[
W_+(D) = \lim_{t \to \infty} \lambda(t)\mathcal{E}_v(t,0,D)^{-1}\mathcal{E}(t,0,D)
\] (3.49)
exists as strong limit in \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \).

Now we relate the initial data by \( V_0 = W_+(\xi)U_0 \). Then the difference of the corresponding micro-energies satisfies
\[
\lambda(t)U(t,\xi) - V(t,\xi) = \mathcal{E}_v(t,0,\xi)\left(\lambda(t)\mathcal{E}_v(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) - W_+(\xi)\right)U(0,\xi).
\] (3.50)
such that by the strong convergence (3.39) combined with the fact that the free propagator is unitary the limit behaviour

\[ \| \lambda(t) \nabla u(t, \cdot) - \nabla v(t, \cdot) \|_2 \to 0, \]
\[ \| \lambda(t) u(t, \cdot) - v(t, \cdot) \|_2 \to 0 \]

follows for all initial data from \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). This completes the proof. \( \square \)

**Remark 3.4.** The modified scattering result involves only the hyperbolic energy terms \( \nabla u(t, \cdot) \) and \( u(t, \cdot) \). If we are interested in results containing also the solution \( u(t, \cdot) \) itself, we can not hope for the same kind of (unweighted) result. Note for this, that the estimate \( \| v(t, \cdot) \|_2 \leq t(\| v_0 \|_2 + \| v_1 \|_{H^{-1}}) \) is in general sharp for solutions to the Cauchy problem for the free wave equation, nevertheless there are no initial data with this precise rate. We only have \( \| v(t, \cdot) \|_2 = o(t) \) as \( t \to \infty \) for each (fixed) solution. Similarly one obtains for solutions to (3.35) to initial data from \( L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n) \)

\[ \lim_{t \to \infty} \frac{\lambda(t)}{1 + t} \| u(t, \cdot) \|_2 = 0. \]

This rate is sharp for general data and can only by improved by further assumptions on initial data. We omit the proof.

### 3.4. \( L^p-L^q \) estimates.

Finally we want to give dispersive type estimates for solutions. These are \( L^p-L^q \) estimates for conjugate Lebesgue indices. The estimate is again independent of \( m_0 \), but the range of admissible \( b_0 \) depends on \( m_0 \). For this statement we need to use the representations to Section 2 with \( k > 1 \) and therefore we also need higher regularity of the coefficient functions compared to the energy estimates given before.

**Theorem 3.6.** Assume Hypothesis 4 with \( \ell = n + 1 \), Hypothesis 2 with

\[ \sigma = 1 \quad \text{and} \quad b_0(b_0 - 2) \leq 4m_0 \]

or with

\[ \sigma \in (1, 2] \quad \text{and} \quad b_0(b_0 - 2) \leq 4m_0 < (b_0 - 1)^2. \]

Then the \( L^p-L^q \) estimate

\[ \| (1 + t)^{-1} u(t, \cdot) \|_{L^p} + \| \nabla u(t, \cdot) \|_{L^q} + \| u(t, \cdot) \|_{L^q} \]
\[ \leq C_{p, q} \frac{1}{\lambda(t)} (1 + t)^{-\frac{4 \sigma^2}{p + q}} \left( \| u_0 \|_{W^{\sigma+1, p}} + \| u_1 \|_{W^{\sigma, p}} \right) \]

holds true for \( p \in (1, 2] \), \( pq = p + q \) and with Sobolev regularity \( r = n \left( \frac{1}{p} - \frac{1}{q} \right) \).

**Proof.** The proof is divided into two steps, we give estimates separately for the dissipative and the hyperbolic zone of the phase space. In the dissipative zone the estimate follows by a simple argument using Hölder inequality, while for the hyperbolic zone we have to employ the stationary phase method. The latter is done by reducing the estimate to the well-known estimate for the free wave equation.

**Step 1. Considerations in the dissipative zone.** We first recall the estimate

\[ \| E(t, 0, \xi) \varphi_{\text{diss}}(t, \xi) \| \lesssim \frac{1}{\lambda(t)} \]

(3.57)
obtained in Section 2. If the initial data belong to Sobolev spaces over \( L^p \) the initial micro-energy satisfies \( U_0 \in L^q \{ |\xi| \leq N \} \) and therefore

\[
\| \mathcal{F}^{-1} (\mathcal{E}(t, 0, \cdot) \varphi_{\text{diss}}(t, \cdot) U_0) \|_q \leq \| \mathcal{E}(t, 0, \cdot) \|_p \| \varphi_{\text{diss}}(t, \cdot) \|_p \| U_0 \|_q \\
\leq \| \mathcal{E}(t, 0, \cdot) \|_\infty \| \varphi_{\text{diss}}(t, \cdot) \|_p \| U_0 \|_q \\
\lesssim \frac{1}{\lambda(t)} (1 + t)^{-\alpha(p - \frac{r}{4})} \| U_0 \|_q ,
\]

(3.58)

based on

\[
\| \varphi_{\text{diss}}(t, \cdot) \|_p \leq \left( \int_{|\xi| \leq N(1+t)^{-\frac{1}{4}}} \frac{d\xi}{(1 + t)^{-\frac{r}{4}}} \right)^{\frac{1}{p}} \lesssim (1 + t)^{-\alpha(\frac{1}{2} - \frac{r}{4})},
\]

(3.59)

which is a better decay compare to the statement of the theorem.

**Step 2. Considerations in the hyperbolic zone.** For large frequencies we use the representation of \( \varphi_{\text{diss}} \) to split the propagator into several parts and estimate each of them separately. For this we choose \( \ell \) such that

\[
\ell = 2(k - 1) \quad \text{and} \quad k - 1 \geq \left\lceil \frac{n}{2} \right\rceil.
\]

(3.60)

We use the short-hand notation \( p, r \to p, r \) to denote operators acting between the Sobolev / Bessel potential spaces \( H^{r-p} \to H^{r-p} \) of regularity \( r \) over \( L^p \). Then \( \mathcal{E}(t, 0, D)\varphi_{\text{hyp}}(D) \) equals

\[
\frac{1}{\lambda(t)} M^{-1} N_k^{-1}(t, D) \mathcal{E}_0(t, 0, D) Q_k(t, 0, D) N_k(0, D) M \varphi_{\text{hyp}}(D)
\]

(3.61)

and we estimate each factor as indicated. To estimate operators for fixed \( p \) (and \( r \), we apply the Hörmander–Mikhlin Theorem. Indeed, by Proposition 2.4 we know that \( M^{-1} N_k^{-1}(t, \xi) \in \mathcal{S}^0_{\ell-k+1} \{ 0, 0 \} \). Therefore, \( M^{-1} N_k^{-1}(t, \xi) \in \mathcal{S}^0_{\ell-k+1} \) uniformly in \( t \) and by Hörmander–Mikhlin Theorem we conclude that

\[
\| M^{-1} N_k^{-1}(t, D) \|_{q \to q} \leq C
\]

(3.62)

uniformly in \( t \). Here it is essential that \( \ell - k + 1 \geq \left\lceil \frac{n}{2} \right\rceil \).

Next, the well-known dispersive estimate for the free wave equation is equivalent to

\[
\| \mathcal{E}_0(t, 0, D) \|_{p, r \to p} \leq C (1 + t)^{-\frac{n}{2}(\frac{1}{2} - \frac{r}{4})}.
\]

(3.63)

For the remaining factors we observe that a Fourier multiplier is bounded between Bessel potential spaces of order \( r \) if and only if it is bounded on the \( L^p \)-spaces. Therefore, it is again sufficient to apply the Hörmander–Mikhlin multiplier theorem. By Theorem 2.3 we know that \( Q_k(t, 0, \xi) \in \mathcal{S}^0_{k-1} \) uniformly with respect to \( t \) and therefore

\[
\| Q_k(t, 0, D) \|_{p, r \to p, r} < C
\]

(3.64)

uniformly in \( t \). Again it is essential that \( k - 1 \geq \left\lceil \frac{n}{2} \right\rceil \). Finally, \( N_k(0, \xi) \in \mathcal{S}^0_{k-1} \) by construction and \( \varphi_{\text{hyp}}^\ell \in \mathcal{S}^0_{k-1} \). Therefore, it follows that

\[
\| \mathcal{E}(t, 0, D) \varphi_{\text{hyp}}(D) \|_{p, r \to q} \leq C \frac{1}{\lambda(t)} (1 + t)^{-\frac{n}{2}(\frac{1}{2} - \frac{r}{4})}.
\]

(3.65)
For the remaining small frequencies we proceed in a similar way. We have by (2.79) that $E(t, 0, D)\varphi_{\text{hyp}}(t, D)$ equals

$$\frac{1}{\lambda(t)} M^{-1} N_k^{-1}(t, D) E_0(t, \theta[D], D) Q_k(t, \theta[D], D) N_k(\theta[D], D) M \times \lambda(\theta[D]) E(\theta[D], 0, D) \varphi_{\text{hyp}}(t, D)$$

and each of the appearing operators can be again estimated separately. The estimate (3.62) for $M^{-1} N_k^{-1}(t, D)$ follows in analogy. Furthermore, we rewrite $E_0(t, \theta[D], D) = E_0(t, 0, D) E_0(0, \theta[D], D)$. Then $E_0(0, \theta[\xi], \xi) \varphi_{\text{hyp}}(t, \xi) \in \dot{S}^0$ implies that the first factor is $L^q$-bounded and therefore the dispersive estimate (3.63) for free waves yields

$$\|E_0(t, \theta[D], D) \varphi_{\text{hyp}}(t, D)\|_{p, r \to q} \leq C(1 + t)^{-\frac{n-1}{2} \left(\frac{1}{p} - \frac{1}{q}\right)}.$$  

By Theorem 2.5 it follows that $Q_k(t, \theta[\xi], \xi) \in \dot{S}^0_{k-1}$. Furthermore, by Proposition 2.4 and the properties of $\theta[\xi]$ we know that $N_k(\theta[\xi], \xi) \in \dot{S}^0$. Therefore

$$\|Q_k(t, \theta[D], D)\|_{p, r \to p, r} < C$$

uniformly in $t$ and

$$\|N_k(\theta[D], D) M\|_{p, r \to p, r} < C.$$  

By Remark 2.2 we also know that $\lambda(\theta[\xi]) E(\theta[\xi], 0, \xi) \in \dot{S}^0_{k-1}$ such that

$$\|\lambda(\theta[D]) E(\theta[D], 0, D)\|_{p, r \to p, r} \leq C.$$  

Hence, it follows that

$$\|E(t, 0, D) \varphi_{\text{hyp}}(D)\|_{p, r \to q} \leq C \frac{1}{\lambda(t)} (1 + t)^{-\frac{n-1}{2} \left(\frac{1}{p} - \frac{1}{q}\right)}$$

and combining all three parts the statement follows from the definition of the micro-energy.

\[\square\]

4. Concluding remarks

We will give some comments on the relation of the presented results to the known treatments of Wirth [8] for non-effectively damped wave equations (i.e., $m(t) = 0$) and Böhme–Reissig [1], [2] for Klein–Gordon equations with non-effectively time-dependent mass (i.e., $b(t) = 0$).

If $m_0 = 0$ and $b_0 \in [0, 1) \cup (1, 2)$ then we are in the setting of [8] (or [11]) for the particular case $b(t) = \frac{b_0}{t^2}$ and the estimates of Theorems 3.1 and 3.6 both with $\sigma = 1$, reduce to results from these papers.

If $b_0 = 0$ we can treat arbitrary $m_0$ and obtain form Theorems 3.1 and 3.6 both with $\sigma = 1$, uniform bounds on the energy as well as the standard wave type $L^p - L^q$ decay estimates. This case was treated in [2] and [11] with similar observations.

The results based on Hypothesis 2 with $\sigma > 1$ are new for both situations.
Appendix A. Asymptotic integration lemma

In this appendix we collect some theorems on the asymptotic integration of ordinary differential equations, which are particularly useful for the treatment of the dissipative zone. We formulate them in more general form than used in the present paper. They follow Sections 1.3 and 1.4 adapted to systems of Fuchs type.

A.1. Levinson type theorems. We consider the following system of ordinary differential equations
\[ t \partial_t V(t, \nu) = \left( D(t, \nu) + R(t, \nu) \right) V(t, \nu), \quad t \geq 1, \]  
(A.1)
depending on a parameter \( \nu \in \Upsilon \). The matrix
\[ D(t, \nu) = \text{diag} \left( \mu_1(t, \nu), \ldots, \mu_d(t, \nu) \right) \]  
(A.2)
is diagonal and \( R(t, \nu) \in \mathbb{C}^{d \times d} \) denotes a remainder term.

Under a dichotomy condition imposed on \( D \) and appropriate smallness conditions on the remainder \( R \) the diagonal matrix \( D \) determines asymptotic properties of solutions to (A.1). We denote by \( e_k \) the \( k \)-th basis vector of \( \mathbb{C}^d \).

**Theorem A.1.** Assume that for \( i \neq j \)
\[ \limsup_{t \to \infty} \sup_{\nu \in \Upsilon} \Re \int_1^t \left( \mu_i(s, \nu) - \mu_j(s, \nu) \right) \frac{ds}{s} < +\infty \]
or
\[ \liminf_{t \to \infty} \inf_{\nu \in \Upsilon} \Re \int_1^t \left( \mu_i(s, \nu) - \mu_j(s, \nu) \right) \frac{ds}{s} > -\infty \]  
(A.3)

together with
\[ \sup_{\nu \in \Upsilon} \int_1^{\infty} \| R(t, \nu) \| \frac{dt}{t} < \infty. \]  
(A.4)

Then there exist solutions \( V_k(t, \nu) \) to (A.1) satisfying
\[ V_k(t, \nu) = (e_k + o(1)) \exp \left( \int_1^t \mu_k(\tau, \nu) \frac{d\tau}{\tau} \right) \]  
(A.5)
uniformly in the parameter \( \nu \in \Upsilon \).

**Proof.** This is a reformulation of Theorem 1.3.1 from [3] with the substitution \( t = e^x \). For the convenience of the reader we sketch the main idea of the proof. We may replace the dichotomy condition (A.3) by an 'either-or' statement assuming in the first case that in addition
\[ \liminf_{t \to \infty} \inf_{\nu \in \Upsilon} \Re \int_1^t \left( \mu_i(s, \nu) - \mu_j(s, \nu) \right) \frac{ds}{s} = -\infty \]  
(A.6)
holds true. This yields an ordering of the diagonal entries according to their strength and we may assume without loss of generality that for \( i < j \) the first alternative holds true. Furthermore, if we write
\[ V(t, \nu) = Z(t, \nu) \exp \left( \int_1^t \mu_k(\tau, \nu) \frac{d\tau}{\tau} \right) \]  
(A.7)
for a fixed index \( k \) then the function \( Z(t, \nu) \) satisfies the transformed equation
\[ t \partial_t Z(t, \nu) = \left( D(t, \nu) - \mu_k(t, \nu)I + R(t, \nu) \right) Z(t, \nu) \]  
(A.8)
and we have to show that there exists a solution to that equation tending to $e_k$ uniformly with respect to $\nu \in \Upsilon$. Thus it is thus sufficient to prove the original theorem for the case $\mu_k = 0$. Let $\Phi(t) = \Phi_-(t, \nu) + \Phi_+(t, \nu)$ be the fundamental solution to the diagonal part, split as

$$
\Phi_-(t, \nu) = \text{diag}(\exp\left(\int_1^t \mu_1(\tau, \nu)\frac{d\tau}{\tau}\right), \ldots, \exp\left(\int_1^t \mu_{k-1}(\tau, \nu)\frac{d\tau}{\tau}\right), 0, \ldots)
$$

and

$$
\Phi_+(t, \nu) = \text{diag}(0, \ldots, 0, 1, \exp\left(\int_1^t \mu_{k+1}(\tau, \nu)\frac{d\tau}{\tau}\right), \ldots, \exp\left(\int_1^t \mu_d(\tau, \nu)\frac{d\tau}{\tau}\right))
$$

according to the asymptotics of the entries. Then $\eqref{A.1}$ rewrites as an integral equation

$$
V(t, \nu) = e_k + \Phi_-(t, \nu) \int_{t_0}^t \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau}
$$

$$
- \Phi_+(t, \nu) \int_t^\infty \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau}. \tag{A.11}
$$

By construction we obtain $\|\Phi_-(t, \nu)\Phi(\tau, \nu)^{-1}\| \leq C_-$ uniformly on $1 \leq \tau \leq t$ and $\|\Phi_+(t, \nu)\Phi(\tau, \nu)^{-1}\| \leq C_+$ uniformly on $t \leq \tau < \infty$. Thus, this equation can be solved uniquely in $L^\infty([1, \infty))$ by the contraction mapping principle as

$$
\left| \Phi_-(t, \nu) \int_1^t \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau}
$$

$$
- \Phi_+(t, \nu) \int_t^\infty \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau} \right|
$$

$$
\leq (C_- + C_+) \int_{t_0}^\infty \|R(\tau, \nu)\|\frac{d\tau}{\tau}\|V(\cdot, \nu)\|_\infty \tag{A.12}
$$

is contractive for $t_0$ sufficiently large. Thus, solutions to $\eqref{A.1}$ are uniformly bounded. To show that they tend to $e_k$ for $t \to \infty$ uniformly with respect to $\nu \in \Upsilon$ one uses the stronger form $\eqref{A.3}$–$\eqref{A.6}$ of the dichotomy condition. Indeed, writing $\eqref{A.11}$ for $t > T$ as

$$
V(t, \nu) = e_k + \Phi_-(t, \nu) \int_{t_0}^T \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau} + \Psi(t, \nu) \tag{A.13}
$$

with

$$
\Psi(t, \nu) = \Phi_-(t, \nu) \int_T^t \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau}
$$

$$
- \Phi_+(t, \nu) \int_t^\infty \Phi^{-1}(\tau, \nu) R(\tau, \nu)V(\tau, \nu)\frac{d\tau}{\tau} \tag{A.14}
$$

we obtain

$$
\|\Psi(t, \nu)\| \leq (C_- + C_+) \int_T^\infty \|R(\tau, \nu)\|\frac{d\tau}{\tau}\|V(\cdot, \nu)\|_\infty \tag{A.15}
$$

uniformly in $t \geq T$ and $\nu \in \Upsilon$. Hence, we can choose $T$ large enough such that $\|\Psi(t, \nu)\| \leq \varepsilon$. But then the dichotomy condition implies $\Phi_-(t, \nu) \to 0$ uniformly in $\nu$ and thus

$$
\|V(t, \nu) - e_k\| \leq 2\varepsilon \tag{A.16}
$$
holds true uniformly in \( \nu \in \mathcal{T} \) and \( t > T \) sufficiently large. As \( \varepsilon \) was arbitrary, the statement is proven. \( \square \)

**Remark A.1.** We will use a special form of the previous theorem, where the diagonal matrices \( D \) are constant and independent of \( \nu \),

\[
D = \text{diag}(\mu_1, \ldots, \mu_d).
\]  
(A.17)

In this case the dichotomy condition \( (\text{A.3}) \) is trivially satisfied as the appearing integrals are all logarithmic functions in \( t \) which can’t approach both infinities. Hence, \( (\text{A.4}) \) is sufficient to conclude the existence of solutions

\[
V_k(t, \nu) = (e_k + o(1)) t^{\mu_k}
\]  
(A.18)

for all \( k \) and if in addition it is known that \( \mu_i \neq \mu_j \) for \( i \neq j \) this yields a fundamental system of solutions. If the diagonal entries coincide, one has to make further assumptions on lower order terms to get precise asymptotic properties, in particular \( (\text{A.4}) \) has to be replaced by adding logarithmic terms.

Levinson’s theorem yields a corresponding statement for the fundamental matrix-valued solution to \( (\text{A.1}) \). This follows immediately from the following variant of Liouville theorem. We assume for simplicity that \( D \) is constant and that the entries are distinct. Then we take the solutions \( V_k \) constructed above as fundamental system. Their Wronskian satisfies

\[
W_{V_1, \ldots, V_d}(t) = \det (V_1(t, \nu)|\cdots|V_d(t, \nu)) = t^{\mu_1 + \mu_2 + \cdots + \mu_d}.
\]  
(A.19)

If we denote by \( E_V(t, 1, \nu) \) the matrix valued solution to

\[
t \partial_t E_V(t, 1, \nu) = (D + R(t, \nu)) E_V(t, 1, \nu), \quad t \geq 1,
\]  
(A.20)

combined with \( E_V(1, 1, \nu) = I \), it follows that

\[
E_V(t, 1, \nu) = (V_1(t, \nu)|\cdots|V_d(t, \nu))(V_1(1, \nu)|\cdots|V_d(1, \nu))^{-1}
\]  
(A.21)

and the norm of the inverse matrix can be estimated by Carmer’s rule combined with Hadamard’s inequality as

\[
\| (V_1(1, \nu)|\cdots|V_d(1, \nu))^{-1} \| \leq d \left( \max_{1 \leq k \leq d} \| V_k(1, \nu) \| \right)^{d-1}
\]  
(A.22)

and thus

\[
\| E_V(t, 1, \nu) \| \leq C t^{\max_j \text{Re } \mu_j}
\]  
(A.23)

uniformly in \( \nu \).

**Remark A.2.** We can use scaling properties of Fuchs type equations. If \( V(t, \nu) \) solves \( (\text{A.1}) \), then \( \tilde{V}(t, \nu) = V(\lambda t, \nu) \) solves the rescaled equation

\[
t \partial_t \tilde{V}(t, \nu) = (D(\lambda t, \nu) + R(\lambda t, \nu)) \tilde{V}(t, \nu).
\]  
(A.24)

If \( \lambda > 1 \) then

\[
\int_1^\infty \| R(\lambda t, \nu) \| \frac{dt}{t} = \int_1^\infty \| R(t, \nu) \| \frac{dt}{t} \leq \int_1^\infty \| R(t, \nu) \| \frac{dt}{t}
\]  
(A.25)

and similarly for the integrals in \( (\text{A.3}) \). Hence, the conditions of Levinson’s theorem are uniform in \( \lambda \) and thus are the constructed solutions. Therefore, any estimate of the fundamental solution given in Remark \( (\text{A.1}) \) is also uniform and therefore of the form

\[
\| E_V(\lambda t, \lambda, \nu) \| = \| E_V(t, 1, \nu) \| \leq C t^{\max_j \text{Re } \mu_j}
\]  
(A.26)

uniformly in \( \lambda > 1 \) and \( \nu \in \mathcal{T} \).
A.2. Hartman–Wintner type theorems. Now we discuss improvements of Theorem A.1 based on a diagonalisation procedure. They allow to handle remainders satisfying
\[ \int_1^\infty \|R(t,\nu)\|\sigma \frac{dt}{t} < C \]  
for some constant $1 < \sigma < \infty$. They are constructive and give precise asymptotics similar to the above theorem. We formulate it in more general form with diagonal matrix $D(t,\nu)$ with entries satisfying the stronger form of the dichotomy condition
\[ \text{Re} (\mu_i(t,\nu) - \mu_j(t,\nu)) \leq C_- \quad \text{or} \quad \text{Re} (\mu_i(t,\nu) - \mu_j(t,\nu)) \geq C_+ \]  
uniform in $t \geq t_0$ and $\nu \in \mathcal{Y}$. It implies (A.3).  

**Theorem A.2.** Assume (A.28) in combination with (A.27). Let further
\[ F(t,\nu) = \text{diag} R(t,\nu) \]  
denote the diagonal part of $R(t,\nu)$. Then we find a matrix-valued function $N(t,\nu)$ satisfying
\[ \int_1^\infty \|N(t,\nu)\|\sigma \frac{dt}{t} < C' \]  
uniformly in $\nu \in \mathcal{Y}$ such that the differential expression
\[ (t\partial_t - D(t,\nu) - R(t,\nu))(I + N(t,\nu)) \]  
\[ = (1 + N(t,\nu))(t\partial_t - D(t,\nu) - F(t,\nu)) = B(t,\nu) \]  
satisfies
\[ \int_1^\infty \|B(t,\nu)\|^{max\{\sigma/2,1\}} \frac{dt}{t} < \infty. \]  
Furthermore, $N(t,\nu) \rightarrow 0$ as $t \rightarrow \infty$ such that the matrix $I + N(t,\nu)$ is invertible for $t \geq t_0$. Hence, $\tilde{V} = (I + N(t,\nu))^{-1}V$ solves the transformed problem
\[ t\partial_t \tilde{V} = (D(t,\nu) + F(t,\nu) + R_1(t,\nu)) \tilde{V} \]  
with $R_1(t,\nu) = (I + N(t,\nu))^{-1}B(t,\nu)$ also satisfying (A.32).

**Proof.** This follows [3] Section 1.5 and is a version of the diagonalisation scheme we applied earlier on. We set $D_1(t,\nu) = D(t,\nu) + F(t,\nu)$, $F(t,\nu) = \text{diag} R(t,\nu)$ and denote $\tilde{R}(t,\nu) = R(t,\nu) - F(t,\nu)$. We construct $N(t,\nu)$ as solution to
\[ t\partial_t N(t,\nu) = D(t,\nu)N(t,\nu) - N(t,\nu)D(t,\nu) + \tilde{R}(t,\nu), \lim_{t \rightarrow \infty} N(t,\nu) = 0, \]  
such that equation (A.31) becomes
\[ B(t,\nu) = N(t,\nu)F(t,\nu) - R(t,\nu)N(t,\nu). \]  
In a first step we estimate $N(t,\nu)$. Considering individual matrix entries (A.34) reads as
\[ t\partial_t n_{jj}(t,\nu) = 0, \]  
\[ t\partial_t n_{ij}(t,\nu) = (\mu_i(t,\nu) - \mu_j(t,\nu))n_{ij}(t,\nu) + r_{ij}(t,\nu) \]  
such that the diagonal entries are given by $n_{jj}(t,\nu) = 0$. For the off-diagonal entries we formulate integral representations and use the auxiliary function
\[ \delta_{ij}(t,\nu) = \int_1^t (\mu_i(s,\nu) - \mu_j(s,\nu)) \frac{ds}{s}. \]
Then the off-diagonal entries are given by Duhamel integrals
\[ n_{ij}(t, \nu) = -e^{\delta_{ij}(t, \nu)} \int_t^{+\infty} e^{-\delta_{ij}(s, \nu)} r_{ij}(s, \nu) \frac{ds}{s} \] (A.39)
for those \( i, j \) where \( \Re(\mu_i - \mu_j) \geq C_+ > 0 \) and
\[ n_{ij}(t, \nu) = e^{\delta_{ij}(t, \nu)} \int_1^t e^{-\delta_{ij}(s, \nu)} r_{ij}(s, \nu) \frac{ds}{s} \] (A.40)
for those with \( \Re(\mu_i - \mu_j) \leq C_- < 0 \). It follows in particular that \( n_{ij}(t, \nu) \to 0 \) as \( t \to \infty \) and with \( \pm C_\pm \geq \delta > 0 \) the estimates
\[ |n_{ij}(t, \nu)| \leq \int_1^{+\infty} s^{-\delta} |r_{ij}(ts^{\pm 1}, \nu)| \frac{ds}{s}, \] (A.41)
the \( \pm \)-sign depending on the case of the Dichotomy condition. Therefore, the \( L^\sigma \)-property of \( r_{ij} \) implies by Minkowski inequality
\[ \left( \int_1^{+\infty} |n_{ij}(t, \nu)|^\sigma \frac{dt}{t} \right)^{1/\sigma} \leq \int_1^{+\infty} s^{-\delta} \left( \int_1^{+\infty} |r_{ij}(ts^{\pm 1}, \nu)|^\sigma \frac{ds}{s} \right)^{1/\sigma} \frac{ds}{s}, \] (A.42)
and thus
\[ \int_1^{+\infty} \|N(t, \nu)\|^\sigma \frac{dt}{t} < \infty. \] (A.43)
uniformly in \( \nu \in \mathcal{Y} \). Similarly, by Hölder’s inequality and with \( \sigma \sigma' = \sigma + \sigma' \).
\[ \sup_t |n_{ij}(t, \nu)| \leq \int_1^{+\infty} s^{-\delta} |r_{ij}(ts^{\pm 1}, \nu)| \frac{ds}{s} \]
\[ \leq \left( \int_1^{+\infty} s^{-\delta \sigma'} \frac{ds}{s} \right)^{1/\sigma'} \left( \int_1^{+\infty} |r_{ij}(ts^{\pm 1}, \nu)|^\sigma \frac{ds}{s} \right)^{1/\sigma} \] (A.44)
uniformly in \( \nu \in \mathcal{Y} \). Hence, the matrix \( N \) belongs to \( L^r([1, \infty), dt/t) \) for all \( \sigma \leq r \leq \infty \) uniformly in \( \nu \). If \( \sigma \geq 2 \) then equation (A.35) implies that \( B(t, \xi) \) product of two \( L^\sigma \)-functions and thus in \( L^{\sigma/2} \). If \( \sigma \in [1, 2) \), then \( \sigma' > \sigma \) and thus \( B(t, \xi) \) is product of an \( L^\sigma \)-function and an \( L^{\sigma'} \)-function and thus in \( L^1 \).
\[ \square \]

We distinguish two cases. If \( \sigma \in (1, 2] \) the transformation reduces the system to Levinson form and Theorem (A.1) applies. If \( \sigma \) is larger, than one application of the transform gives a new remainder satisfying (A.27) with \( \sigma \) replaced by \( \sigma/2 \).

In the first case the conclusion of the Theorem (A.2) is the existence of solutions
\[ V_k(t, \nu) = (\epsilon_k + o(1)) t^{\mu_k} \exp \left( \int_0^t r_{kk}(s, \nu) \frac{ds}{s} \right), \quad k = 1, \ldots, d, \] (A.45)
uniformly in the parameter, provided \( D = \text{diag}(\mu_1, \ldots, \mu_d) \) with distinct entries and \( R \in L^\sigma([1, \infty), dt/t) \).

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