The Intersection Graph of Subgroups of the Dihedral Group of Order 2pq

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Abstract
For a finite group G, the intersection graph \( \Gamma_G \) of G is the graph whose vertex set is the set of all proper non-trivial subgroups of G, where two distinct vertices are adjacent if their intersection is a non-trivial subgroup of G. In this article, we investigate the detour index, eccentric connectivity, and total eccentricity polynomials of the intersection graph \( \Gamma_G \) of subgroups of the dihedral group \( G = D_{2pq} \) for distinct primes \( p < q \). We also find the mean distance of the graph \( \Gamma_G \).

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1. Introduction

The concept of intersection graph of subgroups of a finite group was defined and studied by Csákány and Pollák in 1969 [1]. They found the clique number and degree of vertices of an intersection graph of subgroups of a dihedral group, quaternion group, and quasi-dihedral group.

Let $G$ be a finite non-abelian group. The intersection graph $\Gamma_G$ of $G$ is an undirected simple (without loops and multiple edges) graph whose vertex-set consists of all nontrivial proper subgroups of $G$, for which two distinct vertices $H$ and $K$ of $\Gamma_G$ are adjacent if $H \cap K$ is a non-trivial subgroup of $G$. This kind of graph has been studied by researchers; we refer the reader to see [2-6].

Let $\Gamma$ be any graph. The set of vertices and the set of edges of $\Gamma$ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. If there is an edge between vertices $u$ and $v$, then we write $uv \in E(\Gamma)$. The cardinality of $V(\Gamma)$, denoted by $|V(\Gamma)|$, is called the order of $\Gamma$, while the cardinality of $E(\Gamma)$, denoted by $|E(\Gamma)|$, is called the size of $\Gamma$. For any vertex $v$ in $\Gamma$, the number of edges incident to $v$ is called the degree of $v$ and denoted by $deg\Gamma v$ [7]. The chromatic number of a graph $\Gamma$ is $\chi(\Gamma)$, which is the smallest number of colors for $V(\Gamma)$ such that adjacent vertices have different colors.

A $u - v$ path is a walk with no two vertices repeated, for any two distinct vertices $u$ and $v$ in $\Gamma$. The shortest $u - v$ path in $\Gamma$ is called the distance between $u$ and $v$, denoted by $d(u, v)$, and the longest $u - v$ path in $\Gamma$ is called the detour distance between $u$ and $v$, denoted by $D(u, v)$. The eccentricity of a vertex $v \in V(\Gamma)$, denoted by $ecc(\Gamma)$, is the longest distance between $v$ and all other vertices of $\Gamma$. The diameter of a graph $\Gamma$, denoted by $diam(\Gamma)$, is defined as $diam(\Gamma) = max\{ecc(v) \mid v \in V(\Gamma)\}$ [8]. The detour index, eccentric connectivity and total eccentricity polynomials are defined by $D(\Gamma, x) = \sum_{u,v \in V(\Gamma)} x^{D(u,v)}$ [9], $\zeta(\Gamma, x) = \sum_{u \in V(\Gamma)} deg(\Gamma) x^{ecc(\Gamma)}$ and $\theta(\Gamma, x) = \sum_{u \in V(\Gamma)} x^{ecc(\Gamma)}$ [10], respectively. The detour index $dd(\Gamma)$, the eccentric connectivity index and the total eccentricity $\zeta(\Gamma)$ of a graph $\Gamma$ are the first derivatives of their corresponding polynomials at $x = 1$, respectively. The transmission of a vertex $v$ in $\Gamma$ is $\tau(\Gamma, v) = \sum_{u \in V(\Gamma)} d(\Gamma, v)$.

In this paper, we consider the graph $\Gamma_{D_{2pq}}$ of the dihedral group $D_{2pq}$ where $p$ and $q$ are distinct primes. Some properties of the connected graph $\Gamma_{D_{2pq}}$ will be presented. The dihedral group $D_{2pq}$ of order $2pq$ is defined by $D_{2pq} = \langle r, s \rangle: r^{2p} = s^2 = 1, srs = r^{-1} \rangle$ for prime numbers $p < q$.

2. Some properties of the intersection graph of $D_{2pq}$ for prime numbers $p < q$

In order to determine the vertex set of the graph $\Gamma_{D_{2pq}}$, it is required to list all non-trivial proper subgroups of the dihedral group $D_{2pq}$ for distinct primes $p < q$. In [6], the set of all non-trivial proper subgroups of the group $D_{2n}$ are classified for all $n \geq 3$. Here, we only consider the case when $n = pq$ for distinct primes $p < q$.

Lemma 2.1[6]. The non-trivial proper subgroups of the dihedral group $D_{2pq}$ for distinct primes $p < q$ are:

1- cyclic groups $G_i = \langle s^{i}, r \rangle$ of order 2, where $i = 1, 2, \ldots, pq$.
2- dihedral groups $H_i^p = \langle r^p, s^{i/p} \rangle$ of order 2p, where $i = 1, 2, \ldots, p$ and $H_i^q = \langle r^q, s^{i/q} \rangle$ of order 2q, where $i = 1, 2, \ldots, q$. 

4924
3- cyclic groups $l_p = \langle r^p \rangle$ of order $q$, $l = \langle r \rangle$ of order $pq$, and $l_q = \langle r^q \rangle$ of order $p$.

According to the above classification of subgroups of the group $D_{2pq}$ for primes $p < q$, as given in Lemma 2.1, we can determine the structure of the set of vertices of the graph $\Gamma_{D_{2pq}}$ as the non-trivial proper subgroups by $V(\Gamma_{D_{2pq}}) = A \cup B \cup C$, where $A = \{G_1, G_2, ..., G_{pq}\}$, $B = \{H_i^p \mid 1 \leq i \leq p \}$, and $C = \{l_p, l_q\}$. So, we can distinguish subgraphs $\Gamma_A$ as complement of the complete graph $K_{pq}$, $\Gamma_{B \cup \{l\}}$ as the complete graph $K_{p+q+1}$, and $\Gamma_{C - \{l\}}$ as the complement of the complete graph $K_2$. Through this article, we fixed the sets $A$, $B$, and $C$.

In this section, some basic properties of the intersection graph of $D_{2pq}$ are investigated, such as the order and chromatic number of the graph $\Gamma_{D_{2pq}}$.

**Theorem 2.2.** The order of the graph $\Gamma_{D_{2pq}}$ is $|V(\Gamma_{D_{2pq}})| = p + q + 3$.

Proof: Since the set of vertices of $\Gamma_{D_{2pq}}$ are the non-trivial subgroups of $D_{2pq}$ which are classified in the sets $A$, $B$, and $C$, and since $|A| = pq$, $|B| = p + q$, and $|C| = 3$, then $|V(\Gamma_{D_{2pq}})| = |A| + |B| + |C| = pq + p + q + 3$.

**Theorem 2.3.** The size of the graph $\Gamma_{D_{2pq}}$ is $|E(\Gamma_{D_{2pq}})| = \frac{(p+q)^2 + 4(pq+1) + 3(p+q)}{2}$.

Proof: It is clear that each vertex of $A$ is adjacent with only two vertices of $B$. The vertices in the set $A$ are non-adjacent. Also, each vertex of $B \cup \{l\}$ is adjacent with all other vertices of $B \cup \{l\}$, that is, $B \cup \{l\}$ is a complete graph. Moreover, the vertex $l_p \in C$ is adjacent with $p$ vertices of $B$, which are $H_i^p \mid i = 1, 2, ..., p$, and $l_q \in C$ is adjacent with $q$ vertices of $B$ which are $H_i^q \mid j = 1, 2, ..., q$. Finally, the vertex $l \in C$ is adjacent with $l_p$ and $l_q$. Thus $|E(\Gamma_{D_{2pq}})| = 2pq + \frac{(p+q)^2 + 4(pq+1) + 3(p+q)}{2} + p + q + 2$.

**Theorem 2.4.** The chromatic number of the graph $\Gamma_{D_{2pq}}$ is $\chi(\Gamma_{D_{2pq}}) = p + q + 1$.

Proof: From Theorem 2.2, $cl(\Gamma_{D_{2pq}}) = p + q + 1$. This means that the graph $\Gamma_{D_{2pq}}$ is at least $p + q + 1$ colorable. The vertices $G_1, G_2, ..., G_{pq}$ and $H_i^p$ can be colored with the same color as the vertex $I$, the vertices $l_p$ and $H_i^q$ can share the same color, and the vertices $l_q$ and $H_i^p$ can share the same color. Thus, the minimum number of colors that can be used to color the graph $\Gamma_{D_{2pq}}$ is $p + q + 1$.

Therefore, $\chi(\Gamma_{D_{2pq}}) = p + q + 1$.

**Theorem 2.5.** Let $\Gamma = \Gamma_{D_{2pq}}$ be the graph of the dihedral group $D_{2pq}$. Then $diam(\Gamma) = 3$.

Proof: Let $u$ and $v$ be two distinct vertices in $\Gamma$. If $u$ and $v$ are joint by an edge, then $d(u, v) = 1$. Otherwise, $u \cap v = \{e\}$. There are five cases to consider.

Case1. If $u = G_i$ and $v = G_j$, where $i \equiv j (mod p)$ or $i \equiv j (mod q)$, then there exists $v' \in B$ such that $v' = H_k^p$ or $v' = H_k^q$, for some $k$ and $k'$. If $i \equiv k (mod p)$ or $j \equiv k' (mod q)$, then $uv'$, $v'v \in E(\Gamma)$ and so $d(u, v) = 2$. Otherwise, if $i \not\equiv j (mod p)$ and $i \not\equiv j (mod q)$, take $v' = H_k^p$ and then there exists $w \in B$, where $w = H_l^p$ such that $k \not\equiv l (mod p)$ and $k \not\equiv l (mod q)$. Thus, $uv'$, $v'w$, $wv \in E(\Gamma)$ and then $d(u, v) = 3$.

Case2. If $u = G_i$ and $v = H_i^p$ or $v = H_i^q$, $i = 1, ..., p$; $k = 1, ..., q$, where $i \not\equiv j (mod p)$ and $k \not\equiv j (mod q)$, then there exists $v' \in B$ such that $v' = H_i^p$ or $v' = H_i^q$, where $j \equiv l (mod q)$ or $j \not\equiv l (mod p)$, so $uv'$, $v'v \in E(\Gamma)$ and $d(u, v) = 2$.

Case3. If $u = I_p$ and $v = I_q$, then we take $v' = I_l$ so that $uv'$, $v'v \in E(\Gamma)$ and $d(u, v) = 2$.

Case4. If $u = l_p$ and $v \in \{H_i^q \mid i = 1, ..., q\}$ (or $u = l_q$ and $v \in \{H_i^p \mid i = 1, ..., p\}$), then we take $w = I$, which implies that $uw$, $wv \in E(\Gamma)$ and so $d(u, v) = 2$. 

4925
Case 5. If \( u = G_j \) and \( v \in C \), then there are three possibilities for \( v \). If \( v = I_p \), then there exists \( v' \in \{ H_i^p \mid i = 1, \ldots, p \} \) such that \( uu', v'v \in E(\Gamma) \) if \( i \equiv j \mod p \). If \( v = I_q \), then there exists \( v' \in \{ H_i^q \mid i = 1, \ldots, q \} \) such that \( uu', v'v \in E(\Gamma) \) if \( i \equiv j \mod q \). Finally, if \( v = I \), then there exists \( v' \in B \) such that \( uu', v'v \in E(\Gamma) \). In all possibilities, \( d(u, v) = 2 \).

As a consequence from the above theorem, we state the following.

**Corollary 2.6.** Let \( \Gamma = \Gamma_{D_{2pq}} \) be the graph of the dihedral group \( D_{2pq} \). Then

\[
d(u, v) = \begin{cases} 
1 & \text{if } u = G_i, v = H_j^p \land i \equiv j \mod p, \quad 1 \leq i \leq p, 1 \leq j \leq p, \\
& \text{or } u = G_i, v = H_j^q \land i \equiv j \mod q, 1 \leq i \leq q, 1 \leq j \leq q, \\
2 & \text{if } u = G_i, v = G_j, (i \equiv j \mod p \text{ or } q) 1 \leq i, j \leq p, 1 \leq j \leq q, \\
& \text{or } u = G_i, v = H_j^p \land i \not\equiv j \mod p, 1 \leq i \leq p, 1 \leq j \leq p, \\
& \text{or } u = G_i, v = H_j^q \land i \not\equiv j \mod q, 1 \leq i \leq p, 1 \leq j \leq q, \\
3 & \text{if } u = G_i, v = G_j, (i \not\equiv j \mod p \land i \not\equiv j \mod q) 1 \leq i, j \leq p. 
\end{cases}
\]

**Lemma 2.7.** Let \( \Gamma = \Gamma_{D_{2pq}} \) be the intersection graph of subgroups of the dihedral group \( D_{2pq} \) with distinct primes \( p \) and \( q \). Then

\[
\deg_r(v) = \begin{cases} 
2 & \text{if } v = G_i, \text{ for } 1 \leq i \leq p, \\
p + 1 & \text{if } v = I_p, \\
q + 1 & \text{if } v = I_q, \\
p + q + 2 & \text{if } v = I, \\
p + 2q + 1 & \text{if } v = H_i^p, \text{ for } 1 \leq i \leq p, \\
2p + q + 1 & \text{if } v = H_i^q, \text{ for } 1 \leq j \leq q.
\end{cases}
\]

Proof: see [7].

### 3. Detour index, eccentric connectivity, and total eccentricity polynomials of the graph \( \Gamma_{D_{2pq}} \)

In this section, we find detour index, eccentric connectivity, and total eccentricity polynomials of the intersection graph \( \Gamma_{D_{2pq}} \) of \( D_{2pq} \).

**Theorem 3.1.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with primes \( p < q \). Then

\[
D(u, v) = \begin{cases} 
3p + q - 1 & \text{if } u = H_i^p, v = H_j^p, 1 \leq i, j \leq p \land i \not\equiv j, \\
3p + q & \text{if } u = H_i^p, v \in \{I_p, I_q, H_j^q; 1 \leq j \leq q\}, 1 \leq i \leq p, \\
3p + q + 1 & \text{if } u = H_i^p, v \in \{I_p, G_j; 1 \leq j \leq pq\}, 1 \leq i \leq p, \\
& \text{or } u = H_i^p, v \in \{I_q, I\}, 1 \leq i \leq q, \\
& \text{or } u = I, v \in \{I_p, I_q\}, \\
& \text{or } u = G_i, v = H_j^q, 1 \leq i \leq p, 1 \leq j \leq q \\
& \land uv \in E(\Gamma), \\
3p + q + 2 & \text{if } u = G_i, v \in \{I_p, I_q\}, 1 \leq i \leq p, q, \\
& \text{or } u = I_p, v \in \{I_q, H_j^q; 1 \leq i \leq q\}, \\
& \text{or } u = G_i, v = H_j^q, 1 \leq i \leq p, 1 \leq j \leq q \\
& \land uv \in E(\Gamma), \\
3p + q + 3 & \text{if } u = G_i, v \in \{I_p, G_j\}, 1 \leq i, j \leq p, 1 \leq j \leq q \land i \not\equiv j.
\end{cases}
\]

Proof: For \( D(u, v) = 3p + q - 1 \), the longest path from \( H_i^p \) to \( H_j^p \) where \( 1 \leq i, j \leq p \) and \( i \not\equiv j \) is the path that starts from \( H_i^p \), passing alternatively through \( 2p - 3 \) elements of...
A, p + q − 1 elements of B, and \(I_p, I\) and \(I_q\) vertices of B, and ending at \(H_i^p\). So, the path has length 
\[(2p - 3) + (p + q - 1) + 3 = 3p + q - 1.\] 
Hence \(D(H_i^p, H_j^p) = 3p + q - 1.\) For \(D(u, v) = 3p + q\), there are two cases. Case1, the longest path, that starts from \(H_i^p\) for some \(1 \leq i \leq p\) to \(S \in \{I, I_q\}\), is the path passing alternatively through \(2p - 1\) of vertices of A, \(p + q - 3\) elements of B, and \(I_q\) vertices of B, and ending at \(S \in \{I, I_q\}\). So, the length of this path is
\[\lfloor 1 + (2p - 1) + (p + q - 3) + (1 + 2) \rfloor = 3p + q.\]
Thus, \(D(H_i^p, X) = 3p + q\), for all \(1 \leq i \leq p\) and \(X \in \{I, I_q\}\).

Case2, the longest path, that starts from \(H_i^p\) for some \(1 \leq i \leq p\) to \(H_j^q\), for some \(1 \leq j \leq q\), is the path passing alternatively through \(2p - 1\) of vertices of A, \(p + q - 3\) elements of B, and \(I_q\) and \(I\) element of B, and ending at \(H_j^q\), for some \(1 \leq j \leq q\). So, the length of this path is
\[\lfloor 1 + (2p - 1) + (p + q - 3) + (2 + 1) + 1 \rfloor - 1 = 3p + q.\]
Thus, \(D(H_i^p, H_j^q) = 3p + q\), for all \(1 \leq i \leq p\) and \(1 \leq j \leq q\).

For \(D(u, v) = 3p + q + 1\), the longest path, that starts from \(G_i\) to \(H_j^q\) for some \(1 \leq j \leq p\) for some \(1 \leq i \leq p q\), is the path passing alternatively through \(p + q - 1\) of vertices of B, \(2p - 1\) vertices of A, and \(I\) and \(I_p\) vertices of C, and ending at \(H_j^q\) for some \(1 \leq j \leq p\). So, the length of the path is
\[\lfloor 1 + (p + q - 1) + (2p - 1) + 2 + 1 \rfloor - 1 = 3p + q + 1.\]
Thus, \(D(G_i, H_j^p) = 3p + q + 1\), for all \(1 \leq i \leq p q\) and \(1 \leq j \leq p\).

The longest path, that starts from \(H_i^p\) to \(I_p\) for some \(1 \leq i \leq p\), is the path passing alternatively through \(2p - 1\) of vertices of A and \(p + q - 3\) elements of B with \(I\), and ending at \(I_p\). So the length of the path is
\[\lfloor 1 + (2p - 1) + (p + q - 3) + (2 + 1) + 1 \rfloor - 1 = 3p + q + 1.\]
Hence, \(D(H_i^p, I_p) = 3p + q + 1\), for all \(1 \leq i \leq p\).

The longest path, that starts from the vertex \(H_i^q\) to \(I\) for some \(1 \leq i \leq q\), is the path passing alternatively through \(2p\) vertices of A, \(p + q - 1\) vertices of B, and the vertex \(I_q\) of C, and ending at \(I\). So the length of the path is
\[\lfloor 1 + (2p) + (p + q - 1) + 1 + 1 \rfloor - 1 = 3p + q + 1.\]
Hence, \(D(H_i^q, I) = 3p + q + 1\), for all \(1 \leq i \leq q\).

In a similar way, we can prove the detour distance between all other vertices in the graph \(D_{2pq}\).

**Theorem 3.2.** Let \(D_{2pq}\) be the intersection graph of \(D_{pq}\) with distinct primes \(p < q\). Then
\[
D(D_{2pq}, X) = \left(\frac{pq-1}{2}\right)^{3p+q+3} + \left(p^2 + q + 1\right)\frac{4q(p+1)+2(p+2)+q(q-1)}{2}\frac{3p+q+1}{3p+q+1} + \frac{p(q+2)x^{3p+q}}{2} + \frac{p(p-1)x^{3p+q}}{2},
\]
Proof: 
\[
D(D_{2pq}, X) = \sum_{u, v \in V(T)} x^{D(u,v)} = \left(\frac{pq-1}{2}\right)^{3p+q+3} + (pq)\left(p^2 + q + 1\right)\frac{4q(p+1)+2(p+2)+q(q-1)}{2}\frac{3p+q+1}{3p+q+1} + \frac{p(q+2)x^{3p+q}}{2} + \frac{p(p-1)x^{3p+q}}{2},
\]
It follows from Theorem3.1 that
\[
D(D_{2pq}, X) = \sum_{u, v \in V(T)} x^{D(u,v)} = \left(\frac{pq-1}{2}\right)^{3p+q+3} + (pq)\left(p^2 + q + 1\right)\frac{4q(p+1)+2(p+2)+q(q-1)}{2}\frac{3p+q+1}{3p+q+1} + \frac{p(q+2)x^{3p+q}}{2} + \frac{p(p-1)x^{3p+q}}{2},
\]
\[ p^x D(H^q J) + q^x D(H^q J) + x^D(l_p J) + x^D(l_p J_q) + x^D(l_j J) \] where \( \frac{(pq - 1)}{2} = \frac{(pq - 1)(pq - 2)}{2} \), \( \frac{p}{2} \).

Therefore,
\[ D \left( I_{2pq}, x \right) = \frac{(pq - 1)(pq - 2)}{2} x^{3p + q + 3} + \left[ p^2 (q - 1) + q + 1 \right] x^{3p + q + 2} + \left[ 2pq + p + \frac{q(q - 1)}{2} + 2q \right] x^{3p + q + 1} + p(q + 2) x^{3p + q} + \frac{p(p - 1)}{2} x^{3p + q - 1} \]

**Corollary 3.3.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with distinct primes \( p < q \). Then
\[ dd(I_{D_{2pq}}) = 3p^3 q(q + 1) + q^3 \left( \frac{p^2}{2} + \frac{1}{2} \right) - \frac{3}{2} p^3 + \frac{3}{2} p^2 q + \frac{3}{2} q^2 p + 4p^2 q^2 + 5p^2 + 3q^2 + 3pq + \frac{33}{2} p + \frac{17}{2} q + 10. \]

**Proof:** The result follows directly by taking the first derivative of \( D \left( I_{D_{2pq}}, x \right) \) at \( x = 1 \).

**Theorem 3.4.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with distinct primes \( p < q \). Then
\[ ecc(v) = \begin{cases} 2 & \text{if } v \in B \cup \{l\}, \\ 3 & \text{if } v \in A \cup C - \{l\}. \end{cases} \]

**Proof:** The proof follows directly from Corollary 2.6.

**Theorem 3.5.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with distinct primes \( p < q \). Then
\[ \zeta \left( I_{D_{2pq}}, x \right) = (2pq + p + q + 2) x^3 + \left[ (p + q)^2 + 2(pq + p + q + 1) \right] x^2. \]

**Proof:** It follows from Lemma 2.7 and Theorem 3.4 that
\[ \zeta \left( I_{D_{2pq}}, x \right) = \sum_{u \in V(I_{D_{2pq}})} \text{deg}(u) x^{ecc(u)} = 2pq x^3 + (q + 2 + p + q - 1) p x^2 + (p + 2 + p + q - 1) q x^2 + (p + 1) x^3 + (p + q + 2) x^2 + (q + 1) x^3. \]

**Theorem 3.6.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with distinct primes \( p < q \). Then
\[ \theta \left( I_{D_{2pq}}, x \right) = (pq + 2) x^3 + (p + q + 1) x^2. \]

**Proof:** It follows from Theorem 3.4 that
\[ \theta \left( I_{D_{2pq}}, x \right) = \sum_{u \in V(I_{D_{2pq}})} x^{ecc(u)} = pq x^3 + px^2 + qx^2 + px^2 + x^2 + x^2 + x^2 = (pq + 2) x^3 + (p + q + 1) x^2. \]

**Theorem 3.7.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with distinct primes \( p < q \). Then
\[ \xi \left( I_{D_{2pq}} \right) = 2(p^2 + q^2) + 7(2pq + p + q) + 10. \]

**Proof:** From Theorem 3.5, one can see that
\[ \frac{d}{dx} \zeta \left( I_{D_{2pq}}, x \right) |_{x=1} = 3(2pq + p + q + 2) + 2[(p + q)^2 + 2(pq + p + q + 1)]. \] The result follows.

**4. The mean distance of the intersection graph \( \Gamma_{D_{2pq}} \)**

In this section, we find the mean distance of the intersection graph of subgroups of \( D_{2pq} \) for distinct prime numbers \( p \) and \( q \).

**Theorem 4.1.** The transmission of the graph \( \Gamma_{D_{2pq}} \) is
\[ \sigma \left( I_{D_{2pq}} \right) = p^2 (3q + 1)(q + 1) + q^2 (3p + 1) + q(8p + 7) + 7p + 8. \]

**Proof:** From Corollary 2.6, we have
\[ \sigma(G_i) = q(2) + (pq - (q + 1))(3) + 2(1) + (p + q - 2)(2) + 2(2) + (1)(3) = 3pq + q + 2p + 2, \text{ for all } i = 1, 2, ..., pq, \]
\[ \sigma(H_i^p) = q(1) + (pq - q)(2) + (p + q - 1)(1) + 2(1) + (1)(2) = 2pq + p + 3, \text{ for all } i = 1, 2, ..., p. \]

Also, \( \sigma(H_i^q) = p(1) + (pq - q)(2) + (p + q - 1)(1) + 2(1) + (1)(2) = 2pq + q + 3, \text{ for all } i = 1, 2, ..., q. \)
Note that the vertices $I_p$ and $I_q$ are non-adjacent but the vertex $I$ is adjacent to both $I_p$ and $I_q$. So, $C = \{I_p, I_q, I\}$ induced a path subgraph of $\Gamma_{D_{2pq}}$.

Thus, $(I_p) = p(q(2) + p(1) + q(2) + (1)(1) + (1)(2) = 2pq + p + 2q + 3$,

$\sigma(I_p) = p(q(2) + (p + q)(1) + 2(1) = 2pq + p + q + 2$, and

$\sigma(I_q) = pq(2) + q(1) + p(2) + (1)(1) + (1)(2) = 2pq + 2p + q + 3$.

Now, we can find the transmission of the graph $\Gamma_{D_{2pq}}$ as

$\sigma(\Gamma_{D_{2pq}}) = \sum_{i=1}^{p} \sigma(G_i) + \sum_{i=1}^{p} \sigma(H_i^p) + \sum_{i=1}^{q} \sigma(H_i^q) + \sigma(I) + \sigma(I_p) + \sigma(I_q)$

$= pq[3pq + 2p + q + 2] + p[2pq + p + 3] + q[2pq + q + 3] + 6pq + 4(p + q) + 8$

$= p^2(3q + 1)(q + 1) + q^2(3p + 1) + q(8p + 7) + 7p + 8$.

**Theorem 4.2.** The mean distance of the graph $\Gamma_{D_{2pq}}$ is

$$\mu(\Gamma_{D_{2pq}}) = \frac{p^2(3q+1)(q+1)+q^2(3p+1)+q(8p+7)+7p+8}{(p+q)(p+q+2)}.$$  

Proof: Since the order of the graph $\Gamma_{D_{2pq}}$ is $pq + p + q + 3$ and the transmission of the graph $\Gamma_{D_{2pq}}$ is given in Theorem 4.1, we can find the mean distance of the graph $\Gamma_{D_{2pq}}$ as

$$\mu(\Gamma_{D_{2pq}}) = \frac{p^2(3q+1)(q+1)+q^2(3p+1)+q(8p+7)+7p+8}{(p+q)(p+q+3)(p+q+2)}$$, where $p < q$ are prime numbers.

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