EXPANSIONS OF THE GROUP OF INTEGERS BY
BEATTY SEQUENCES

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Abstract. We study the model theoretic structure \((\mathbb{Z}, +, P_r)\) where \(r > 1\) is an irrational number and the elements of \(P_r\) are of the form \([nr]\) for some \(n \in \mathbb{Z} \setminus \{0\}\). We axiomatize of this structure and prove a quantifier elimination result. As a consequence we get that definable subsets are not sparse unless they are finite.

1. Introduction

The structure \((\mathbb{Z}, +, 0)\) is very easy from the model theoretic perspective. Adding the ordering makes it slightly harder, but it is still quite benign. On the other hand, the ring of integers \((\mathbb{Z}, +, 0, 1)\) is as wild as it gets. In this paper, we study expansions of \((\mathbb{Z}, +, 0)\) that are well-behaving and yet not stable. Namely, we study the structure \((\mathbb{Z}, +, 0, P_r)\) where \(r > 1\) is an irrational number and

\[ P_r := \{[nr] : n \in \mathbb{Z} \setminus \{0\}\}. \]

The number 0 is taken out for some technical reasons and of course it has no effect on our results. In contrast, if we have considered \(P_r^+ := P_r \cap \mathbb{N}\) in the place of \(P_r\), then there is a considerable difference. Because \((\mathbb{Z}, +, 0, P_r^+)\) defines the ordering of \(\mathbb{Z}\), whereas we show in Corollary 6.5 that \((\mathbb{Z}, +, 0, P_r)\) does not define the ordering. We also know that \(P_r\) is not definable in \((\mathbb{Z}, +, 0, <)\), since it is shown in [3] that the only reduct of \((\mathbb{Z}, +, 0, <)\) defining addition is \((\mathbb{Z}, +, 0)\).

There have been some work on expansions of \((\mathbb{Z}, +, 0)\) by a predicate by Poizat, Palacín-Sklínos, and Lambotte-Point ([8, 7, 6]). In all these, the predicate is sparse, with certain different but similar meanings of the word sparse. Later, the paper [2] by Conant generalized those results, where it is proven that the expansion of \((\mathbb{Z}, +, 0)\) by an infinite subset \(A\) of a submonoid of \((\mathbb{N}_+, \cdot)\) is superstable of \(U\)-rank \(\omega\). Hence in that setting, sparsity comes from the multiplicative structure.

Our study of the structure \((\mathbb{Z}, +, 0, P_r)\) is complimentary to work mentioned above, as the set \(P_r\) is certainly not sparse: If \(k \in P_r\), then the next element in \(P_r\) is either \(k + [r]\) or \(k + [r] + 1\). (This is also the reason of \((\mathbb{Z}, +, 0, P_r^+)\) defining the ordering.) Actually, we prove that no infinite subset of \(\mathbb{Z}\) definable in \((\mathbb{Z}, +, 0, P_r)\) is sparse in the following sense:
Theorem 1.1. Let $X \subseteq \mathbb{Z}$ be infinite and definable in $(\mathbb{Z}, +, 0, P_r)$. Then there is $N = N(X) \in \mathbb{N}_+$ such that for any $x \in X$, one of the integers $x + 1, x + 2, \ldots, x + N$ is in $X$.

The proof of this result goes through a quantifier elimination result. In order to state that, let $L_\star$ be the extension of the language $\{+, -, 0, 1\}$ of abelian groups with a distinguished element by unary predicate symbols $D_{m,+}$ and $D_{m,-}$ for each $m \geq 1$. We interpret the new symbols in $\mathbb{Z}$ as follows:

$$D_{m,+}^\mathbb{Z} := \{x \in \mathbb{Z} : x = my \text{ for some } y \in P_r\},$$

and

$$D_{m,-}^\mathbb{Z} := \{x \in \mathbb{Z} : x = my \text{ for some } y \notin P_r\}.$$ 

Note that $D_{1,+}^\mathbb{Z} = P_r$ and that $D_{m,+}^\mathbb{Z} \cup D_{m,-}^\mathbb{Z} = m\mathbb{Z}$.

Theorem 1.2. The structure $(\mathbb{Z}, +, 0, 1, (D_{m,+}^\mathbb{Z})_{m \geq 1}, (D_{m,-}^\mathbb{Z})_{m \geq 1})$ has quantifier elimination.

A particular formula with quantifiers is

$$\exists y \left( \bigwedge_{i \in I} x_i + k_i y \in P \land \bigwedge_{j \notin I} x_j + k_j y \notin P \right),$$

where $k_1, \ldots, k_n \in \mathbb{Z}$ and $I \subseteq \{1, \ldots, n\}$. So we have a quantifier-free $L_\star$-formula $\psi_{k,I}(\vec{x})$ such that

$$(\forall x_1 \cdots \forall x_n) \left( \exists y \left( \bigwedge_{i \in I} x_i + k_i y \in P \land \bigwedge_{j \notin I} x_j + k_j y \notin P \right) \iff \psi_{k,I}(\vec{x}) \right).$$

holds in $(\mathbb{Z}, +, 0, 1, (D_{m,+}^\mathbb{Z})_{m \geq 1}, (D_{m,-}^\mathbb{Z})_{m \geq 1})$. As a matter of fact, $\psi_{k,I}$ is a formula in the language $L_P = \{+, -, 0, 1, P\}$.

With this notation at hand, we have the following axiomatization.

Theorem 1.3. Let $\mathcal{M} = (M, +, -, 0, 1, P^\mathcal{M})$ be an $L_P$-structure. Then $\mathcal{M}$ is elementarily equivalent to $(\mathbb{Z}, +, -, 0, 1, P_r)$ if and only if the following hold

1. $(M, +, -, 0, 1) \equiv (\mathbb{Z}, +, -, 0, 1),$
2. for every $k_1, \ldots, k_n \in \mathbb{Z}$ and $I \subseteq \{1, \ldots, n\}$ the sentence $(\star)$ holds in $\mathcal{M},$
3. $k \in P_r$ if and only if $k \in P^\mathcal{M}$ for every $k \in \mathbb{Z}$.

The technical parts of the proofs are done in an isomorphic structure: Let

$$\Gamma_r := \{ \exp(\frac{n2\pi \sqrt{-1}}{r}) \in \mathbb{C} : n \in \mathbb{Z} \}.$$
So $\Gamma_r = h(\mathbb{Z})$ where $h$ is the group isomorphism sending $n$ to $\exp\left(\frac{n2\pi\sqrt{-1}}{r}\right)$. Being a subgroup of the unit circle, $\Gamma_r$ has orientation on it; the precise definition is given in the next section. Then the image of $P_r$ under $h$ becomes an orientation interval; see Lemma 3.1. This makes it easier to work in $\Gamma_r$ and the notations get simpler. For this reason, in Section 2 we recall some facts about the circle and its subgroups.

In Section 3, we introduce Beatty Sequences and prove a few results about them to be used in the model theoretic arguments.

A back-and-forth system constructed in Section 4 is used to prove the theorems mentioned above in Sections 5 and 6.

Notations and Conventions. The set $\mathbb{N}$ of natural numbers contains 0 and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. We let the letters $m, n, k, l$ vary in $\mathbb{Z}$, and if $m$ is in $\mathbb{N}$ (or $\mathbb{N}_+$), we simply write $m \geq 0$ (or $m > 0$).

For $n > 0$, we denote the set $\{1, \ldots, n\}$ as $[n]$.

For a real number $a$, the notation $\lfloor a \rfloor$ denotes the largest integer smaller than or equal to $a$ and $\{a\}$ denotes the difference $a - \lfloor a \rfloor$. (There will not be any occasions where this could be confused with the singleton containing $a$.)

2. The Circle and Its Subgroups

In the next section, we work with an infinite subgroup of the unit circle $S := \{\beta \in \mathbb{C} : |\beta| = 1\}$. One may study such a group in the generality of oriented abelian groups as defined in [4], however there is no need to do so for our purposes.

Here we recall some generalities about $S$ and at the end we say a few words about its subgroups. Let

$$e : \mathbb{R} \to S, \quad e(x) := \exp(2\pi \sqrt{-1}x).$$

This is a surjective group homomorphism with kernel $\mathbb{Z}$.

We equip $S$ with counter-clockwise orientation: Given $\alpha, \beta, \gamma \in S$, the relation $O(\alpha, \beta, \gamma)$ holds if and only if there are $x, y, z \in \mathbb{R}$ such that $\alpha = e(x), \beta = e(y), \gamma = e(z), x < y < z$, and $z - x < 1$.

If we fix $\alpha \in S$, then we get a linear ordering $O_\alpha(\cdot, \cdot) := O(\alpha, \cdot, \cdot)$ on $S \setminus \{\alpha\}$. For $\alpha = 1$, we denote the ordering of $S \setminus \{1\}$ by the usual ordering sign:

$$\beta < \gamma \iff O(1, \beta, \gamma).$$

Clearly, if $\alpha < \beta$ and $\beta < \gamma$, then $O(\alpha, \beta, \gamma)$. We extend the definition of $<$ to all of $S$ by setting $1 < \alpha$ for $\alpha \neq 1$.

The group operation respects this orientation:

$$O(\alpha, \beta, \gamma) \iff O(\alpha \delta, \beta \delta, \gamma \delta)$$
for every $\alpha, \beta, \gamma, \delta \in S$. The group inversion reverses it:

$$O(\alpha, \beta, \gamma) \iff O(\gamma^{-1}, \beta^{-1}, \alpha^{-1})$$

for every $\alpha, \beta, \gamma \in S$.

The circle $S$ has the topology induced from the Euclidean topology on $\mathbb{C}$ and a basis for this topology consists of orientation intervals: Given $\alpha, \beta \in S$ we define the orientation interval determined by $\alpha$ and $\beta$ to be

$$(\alpha, \beta) := \{ \gamma \in S : O(\alpha, \gamma, \beta) \}.$$ We do not assume $\alpha < \beta$ for this definition. So both $(\alpha, \beta)$ and $(\beta, \alpha)$ are orientation intervals and they are distinct as long as they are non-empty. They are empty only when $\alpha = \beta$.

We define the length of an orientation interval $(\alpha, \beta)$ to be $\beta \alpha$. So if $\alpha = e(a)$ and $\beta = e(b)$ with $0 \leq b - a < 1$, then the length of $(\alpha, \beta)$ is $e(b - a)$. In particular, the length of an interval is 1 if and only if it is empty.

Below the word interval always means orientation interval, and we use notations such as $[\alpha, \beta)$ with the obvious meanings.

**Proposition 2.1.** Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in S$ be such that $\alpha_i \neq \beta_i$ for each $i$. Then the following conditions are equivalent:

1. $\bigcap_{i=1}^{m} (\alpha_i, \beta_i) \neq \emptyset$.

2. There is $i_0 \in [m]$ such that $\alpha_{i_0} \in (\alpha_i, \beta_i)$ for every $i \neq i_0$.

3. There is $j_0 \in [m]$ such that $\beta_{j_0} \in (\alpha_i, \beta_i)$ for every $i \neq j_0$.

Moreover, if one of these conditions hold, then

$$\bigcap_{i=1}^{m} (\alpha_i, \beta_i) = (\alpha_{i_0}, \beta_{j_0}).$$

Proof. Suppose $\gamma \in (\alpha_i, \beta_i)$ for every $i \in [m]$. Then $O_n(\beta_i, \alpha_i)$ for each $i \in [m]$. Choose $i_0$ such that $\alpha_{i_0}$ is maximal among $\alpha_i$ with respect to $O_n$. Clearly, $\alpha_{i_0} \in (\alpha_i, \beta_i)$ for each $i \neq i_0$. Similarly, taking $\beta_{j_0}$ is minimum among $\beta_i$ with respect to $O_n$ we see that $\beta_{j_0} \in (\alpha_i, \beta_i)$ for each $i \neq j_0$. So the first condition implies the others.

Conversely, assume that $\alpha_{i_0} \in (\alpha_i, \beta_i)$ for each $i \neq i_0$. Then

$$(\alpha_{i_0}, \beta_{j_0}) \subseteq \bigcap_{i=1}^{m} (\alpha_i, \beta_i),$$

where $\beta_{j_0}$ is minimum among $\beta_i$ with respect to $O_{\alpha_{i_0}}$. 
If $\beta_{j_0} \in (\alpha_i, \beta_i)$ for each $i \neq i_0$. Then we take $\alpha_{i_0}$ to be maximum among $\alpha_i$ with respect to $O_{\beta_{j_0}}$ to once again get

$$(\alpha_{i_0}, \beta_{j_0}) \subseteq \bigcap_{i=1}^{m}(\alpha_i, \beta_i).$$

The last sentence of the proposition follows from the rest of the proof above.

For $k \in \mathbb{N}_+$, let $\zeta_k$ denote the primitive $k^{\text{th}}$ root of unity $e\left(\frac{1}{k}\right)$.

**Proposition 2.2.** Let $\alpha, \beta, \gamma \in \mathbb{S}$ and $k \in \mathbb{N}_+$. Suppose that the length of $(\alpha, \gamma)$ is less than $\zeta_k$. Then $\beta^k \in (\alpha^k, \gamma^k)$ if and only if $\beta \in (\zeta_k^s \alpha, \zeta_k^s \gamma)$ for some $s \in \mathbb{Z}$.

**Proof.** Note that if $\alpha = \gamma$, then the result is clear; so we assume $\alpha \neq \gamma$. We may also assume that $\alpha < \gamma$ by multiplying $\alpha$ and $\gamma$ with an appropriate power of $\zeta_k$. So let $0 \leq a < c < 1$ be such that $\alpha = e(a)$ and $\gamma = e(c)$. Note that $c - a < \frac{1}{k}$ by assumption. Also let $\beta = e(b)$ for some $0 \leq b < 1$.

Suppose $\beta^k \in (\alpha^k, \gamma^k)$ and take $b'$ such that $\beta^k = e(b')$ and $ka < b' < kc$. Then $s := kb - b' \in \mathbb{Z}$ and we get

$$ka + s < kb < kc + s.$$ 

Dividing by $k$, we have

$$a + \frac{s}{k} < b < c + \frac{s}{k}.$$ 

Applying $e$, we get that $\beta \in (\zeta_k^s \alpha, \zeta_k^s \gamma)$.

Conversely, let $\beta \in (\zeta_k^s \alpha, \zeta_k^s \gamma)$ for some $s \in \mathbb{Z}$. In other words, $\beta \zeta_k^{-s} \in (\alpha, \gamma)$. Take $b''$ such that $\beta \zeta_k^{-s} = e(b'')$ and $a < b'' < c$. Then $t := b - b'' - \frac{s}{k} \in \mathbb{Z}$ and

$$a + t < b - \frac{s}{k} < c + t.$$ 

Now multiplying by $k$ and applying $e$ we get $\beta^k \in (\alpha^k, \beta^k)$. \qed

Note that the $s$ in this proposition can be chosen among $0, 1, \ldots, k - 1$.

Given $\alpha = e(x)$ with $0 \leq x < 1$ and $k \in \mathbb{N}_+$, we let $\alpha^{1/k}$ denote $e\left(\frac{x}{k}\right)$; so $\alpha^{1/k}$ is the $k^{\text{th}}$ root of $\alpha$ with the smallest argument. Note that $1^{1/k} = 1$ and that for $x \in \mathbb{R}$ we have $e(x)^{1/k} = e\left(\frac{x}{k}\right) \zeta_k^{-\lfloor x \rfloor}$. In particular, if $\alpha = e(a)$, then $(\alpha^k)^{1/k} = \alpha^{1/k} \zeta_k^{-\lfloor ak \rfloor}$. However, we always have $(\alpha^{1/k})^k = \alpha$.

Using this new notation, the previous proposition has the following consequences.
Corollary 2.3. Let $\alpha, \beta, \gamma \in \mathbb{S}$ and $k \in \mathbb{N}_+$. Suppose that $\gamma \neq \alpha$. Then $\beta^k \in (\alpha, \gamma)$ if and only if there is $s \in \{0, 1, \ldots, k - 1\}$ such that $\beta \in \left(\zeta_k^{s} \alpha^{1/k}, \zeta_k^{s+1} \gamma^{1/k}\right)$.

Corollary 2.4. Let $\alpha, \beta, \gamma \in \mathbb{S}$ and $k \in \mathbb{N}_+$. Suppose that $\gamma < \alpha$. Then $\beta^k \in (\alpha, \gamma)$ if and only if there is $s \in \{0, 1, \ldots, k - 1\}$ such that $\beta \in \left(\zeta_k^{s} \alpha^{1/k}, \zeta_k^{s+1} \gamma^{1/k}\right)$.

Regularly Dense Groups. Let $\Gamma \leq \mathbb{S}$. Then $\Gamma$ is either finite or dense in $\mathbb{S}$. When it is finite it consists of $m^{th}$ roots of unity for some $m > 0$. When $\Gamma$ is dense, it is indeed regularly dense in the following sense.

Proposition 2.5. Let $\Gamma \leq \mathbb{S}$ be infinite. Then for any distinct $\alpha, \beta \in \Gamma$ and prime $p$, there is $\gamma \in \Gamma$ such that $\gamma^p \in (\alpha, \beta)$.

For the proof of this, we refer the reader to Definition 8.1.7 in [4] and the remark succeeding it. Note that the conclusion of the proposition above is slightly different than the original definition of regularly dense, but it is easy see that they are indeed equivalent. It follows that for any $n > 0$ and distinct $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $\gamma^n \in (\alpha, \beta)$.

3. Beatty Sequences

Let $r > 1$ be an irrational number. The Beatty Sequence generated by $r$ is $B_r = (\lfloor mr \rfloor)_{m > 0}$; we put $b_m = \lfloor mr \rfloor$. This is an increasing sequence and we let $P^+_r$ denote the set whose elements are the terms of $B_r$.

A related sequence is $S_r = \left(\left\lfloor \frac{n+1}{r} \right\rfloor - \left\lfloor \frac{n}{r} \right\rfloor \right)_{n > 0}$; we put $s_n = \left\lfloor \frac{n+1}{r} \right\rfloor - \left\lfloor \frac{n}{r} \right\rfloor$. Note that $s_n \in \{0, 1\}$ for each $n > 0$. Actually, it is better to think of $S_r$ as an infinite word in the alphabet $\{0, 1\}$. As such, it is called the Characteristic Sturmian Word of Slope $\frac{1}{r}$. It has the property that for every $m$, it has exactly $m + 1$ many different subwords of length $m$.

Both Beatty Sequences and Sturmian Words have rich theories that we prefer not to get into in this paper and we refer the interested reader to [1]. We only need the following connection between $B_r$ and $S_r$ which is Lemma 9.1.3 of [1], but we include a proof for completeness.

Lemma 3.1. Let $n \in \mathbb{N}_+$. Then $n \in P^+_r$ if and only if $s_n = 1$.

Proof. Let $n \in P^+_r$. Then $n = \lfloor kr \rfloor$ for some $k \in \mathbb{N}_+$. So $kr - 1 < n < kr$ and after diving by $r$ we have

$$k - \frac{1}{r} < \frac{n}{r} < k.$$

Therefore $\lfloor \frac{n}{r} \rfloor = k - 1$ and $\lfloor \frac{n+1}{r} \rfloor = k$. Thus $s_n = \left\lfloor \frac{n+1}{r} \right\rfloor - \left\lfloor \frac{n}{r} \right\rfloor = 1$. As all the implications are reversible we get the desired result. $\square$
We would like to consider the negative elements as well; so we define

\[ P_r = \{ \lfloor nr \rfloor : n \in \mathbb{Z} \setminus \{0\} \}. \]

For \( m > 0 \), we have \(-m \in P_r\) if and only if \( m - 1 \in P_r^+\). So

\[ P_r = P_r^+ \cup (-P_r^+ - 1). \]

We also extend the definitions of \( b_n \) and \( s_n \) to all integers \( n \).

Lemma 3.1 is actually correct for all \( n \in \mathbb{Z} \):

\[ n \in P_r \iff s_n = 1. \tag{3.1} \]

It is easy to see that \( s_n = 1 \) if and only if \( \{r\frac{n}{a}\} > 1 - \frac{1}{r} \). Putting this together with (3.1), for every \( n \in \mathbb{Z} \), we get

\[ n \in P_r \iff \{r\frac{n}{a}\} > 1 - \frac{1}{r}. \tag{3.2} \]

Since \( r \) is irrational, the image of \( \mathbb{Z}\frac{1}{r} \) under \( e \) is not finite, hence it is a dense subgroup of \( S \). Let \( \Gamma_r \) be that subgroup, and let \( h \) denote the map \( n \mapsto e(n) \). So we have an isomorphism of abelian groups with a distinguished element:

\[ h : (\mathbb{Z}, +, -, 0, 1) \cong (\Gamma_r, \cdot, -1, 1, h(1)). \]

By (3.2), the image of \( P_r \) under \( h \) is the orientation interval \( (h(-1), 1) \) in \( \Gamma_r \). Therefore expanding \( \mathbb{Z} \) by \( P_r \) is the same as expanding \( \Gamma_r \) by \( (h(-1), 1) \). We will come back to this in the next section.

We let \( P_r \) denote also the interval \( (h(-1), 1) \).

Using Proposition 2.2 and its corollaries, we give a criterion for certain linear combinations of integers being in \( P_r \) in terms of intervals in \( \mathbb{S} \).

**Proposition 3.2.** Let \( k \in \mathbb{N}_+ \) and \( a, c \in \mathbb{Z} \). Then \( a + kc \in P_r \) if and only if there is \( s \in \mathbb{Z} \) such that

\[ h(c) \in (h(-a - 1)^{1/k} \zeta_k^s, h(-a)^{1/k} \zeta_k^{s+s_n}). \]

**Proof.** First, note that \( a \in P_r \) if and only if \( h(-a) < h(-a - 1) \).

By (3.2) we have

\[ a + kc \in P_r \iff h(a) h(c)^k \in (h(-1), 1) \iff h(c)^k \in (h(-a - 1), h(-a)). \]

Now combining Corollaries 2.3 and 2.4 and using the first sentence of this proof, we get that

\[ h(c)^k \in (h(-a - 1), h(-a)) \]

if and only if there is \( s \in \mathbb{Z} \) with

\[ h(c) \in (h(-a - 1)^{1/k} \zeta_k^s, h(-a)^{1/k} \zeta_k^{s+s_n}). \]

This gives the desired equivalence. \( \square \)
Corollary 3.3. Let \( k \in \mathbb{N}_+ \) and \( a, c \in \mathbb{Z} \). Then \( a + kc \notin P_r \) if and only if there is \( s \in \mathbb{Z} \) such that
\[
h(c) \in \left[ h(-a)^{1/k} \zeta_k^{s+s_a}, h(-a - 1)^{1/k} \zeta_k^{s+1} \right].
\]

Proof. Clear from the previous proposition. \( \square \)

Next result will be useful in handling the cases when \( k \) is negative.

Lemma 3.4. Let \( a, c \in \mathbb{Z} \) and \( k < 0 \). Then \( a + kc \in P_r \) if and only if \( -a - 1 - kc \in P_r \).

Proof. Clear from the fact that \([ -x ] = -[ x ] - 1 \) for \( x \notin \mathbb{Z} \). \( \square \)

Definition 3.5. For \( a \in \mathbb{Z}, k \in \mathbb{N}_+ \), and \( s \in \{0, 1, \ldots, k - 1\} \), let
\[
U_{a,k,s} := (h(-a - 1)^{1/k} \zeta_k^s, h(-a)^{1/k} \zeta_k^{s+s_a}),
\]
\[
V_{a,k,s} := [h(-a)^{1/k} \zeta_k^{s+s_a}, h(-a - 1)^{1/k} \zeta_k^{s+1}].
\]

Also let
\[
U_{a,k} := \bigcup_{s=0}^{k-1} U_{a,k,s} \quad \text{and} \quad V_{a,k} := \bigcup_{s=0}^{k-1} V_{a,k,s}.
\]

We extend the definitions to \( k = 0 \) as follows:
\[
U_{a,0} := \begin{cases} \mathbb{S} & \text{if } a \in P_r \\ \emptyset & \text{if } a \notin P_r \end{cases} \quad \text{and} \quad V_{a,0} := \begin{cases} \emptyset & \text{if } a \in P_r \\ \mathbb{S} & \text{if } a \notin P_r \end{cases}
\]

Finally, we let \( \tilde{V}_{a,k} \) denote the interior of \( V_{a,k} \).

With this notation in hand, Proposition 3.2 and Corollary 3.3 translate as follows: Given \( k \in \mathbb{N} \) and \( a, c \in \mathbb{Z} \) we have
\[
a + kc \in P_r \iff h(c) \in U_{a,k},
\]
and
\[
a + kc \notin P_r \iff h(c) \in V_{a,k}.
\]

Lemma 3.6. Let \( a, b \in \mathbb{Z} \) and \( k, l \in \mathbb{N}_+ \). Suppose \( g = \gcd(k, l) \) and write \( k = gk' \) and \( l = gl' \). Then the following hold.

1. Suppose \( \zeta_{l'} < h(1) \). Then there is \( s \in \mathbb{Z} \) such that \( h(b)^{1/l} \zeta_{l'}^s \in U_{a,k} \).

2. Suppose \( h(1) < \zeta_{l'} \). Then there is \( s \in \mathbb{Z} \) such that \( h(b)^{1/l} \zeta_{l'}^s \in U_{a,k} \) if and only if
\[
h(l'a + k'b + l') \in \left( 1, h(k') \right).
\]

Proof. Using Corollaries 2.3 and 2.4, \( h(b)^{1/l} \zeta_{l'}^s \in U_{a,k} \) if and only if
\[
h(k'b)^{1/l'} \zeta_{l'}^{sk'} \in \left( h(-a - 1), h(-a) \right).
\]

Since \( \gcd(k', l') = 1 \), there is \( s \in \mathbb{Z} \) with \( h(b)^{1/l} \zeta_{l'}^s \in U_{a,k} \) if and only if there is \( t \in \mathbb{Z} \) with \( h(k'b)^{1/l'} \zeta_{l'}^t \in \left( h(-a - 1), h(-a) \right) \).
If $\zeta_r < h(1)$, then there is such a $t$.  
If $h(1) < \zeta_r$, then we may use Proposition \ref{prop:sufficient} to conclude that there is $s \in \mathbb{Z}$ with $h(b)^{1/l} \zeta^s \in U_{a,k}$ if and only if  
$$
    h(k'b) \in (h(-l'a - l'), h(-l'a)).
$$
After simplification, this means $h(l'a + k'b + l') \in (1, h(l'))$.

This proof can be modified to prove the next analogous result.

**Lemma 3.7.** Let $a, b \in \mathbb{Z}$ and $k, l \in \mathbb{N}_+$. Suppose $g = \gcd(k, l)$ and write $k = gl'$ and $l = gl'$. Then the following hold.

1. Suppose $\zeta_r < h(-1)$. Then there is $s \in \mathbb{Z}$ such that $h(b)^{1/l} \zeta^s \in V_{a,k}$.
2. Suppose $h(-1) < \zeta_r$. Then there is $s \in \mathbb{Z}$ such that $h(b)^{1/l} \zeta^s \in V_{a,k}$ if and only if  
$$
    h(l'a + k'b' + l') \in [h(k'), 1].
$$

Let $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{N}_+^n$, $I \subseteq [n]$ and $J \subseteq [n] \setminus I$ be given. We define  
$$
    V_{\vec{a}, \vec{k}, J} = \bigcap_{j \in J} (V_{a_j, k_j} \setminus \tilde{V}_{a_j, k_j}) \cap \bigcap_{j \in [n] \setminus (I \cup J)} \tilde{V}_{a_j, k_j}.
$$
Note that $V_{a_j, k_j} \setminus \tilde{V}_{a_j, k_j}$ has $2k_j$ many points. Therefore $V_{\vec{a}, \vec{k}, J}$ is finite for $J \neq \emptyset$ and $V_{\vec{a}, \vec{k}, \emptyset}$ is an open subset of $\mathbb{S}$.

We record the following without proof.

**Lemma 3.8.** Let $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{N}_+^n$, and $\emptyset \neq J \subseteq [n]$. Then $h(c) \in V_{\vec{a}, \vec{k}, J} \cap \Gamma_r$ if and only if there is a subset $J'$ of $J$ such that $c = \frac{-a_j}{k_j}$ for every $j \in J'$, $c = \frac{-a_j - 1}{k_j}$ for every $j \in J \setminus J'$, and $h(c) \in \bigcap_{j \in [n] \setminus J} \tilde{V}_{a_j, k_j}$.

**Definition 3.9.** Let $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $I \subseteq [n]$. We say that $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ realize ($\vec{k}, I$)-patterns if there is $c \in \mathbb{Z}$ such that  
$$
    a_i + k_i c \in P_r \iff i \in I.
$$
For $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let $|\vec{k}| = (|k_1|, \ldots, |k_n|)$. Then using Lemma \ref{lem:pattern}, $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ realizes the ($\vec{k}, I$)-pattern if $(a'_1, \ldots, a'_n)$ realizes the ($|\vec{k}|, I$)-pattern where $a'_i = a_i$ for $k_i \geq 0$ and $a'_i = -a_i - 1$ for $k_i < 0$. Therefore, we may focus on the case that $\vec{k} \in \mathbb{N}_+^n$.

By \ref{def:pattern} and \ref{lem:pattern}, if $\vec{k} \in \mathbb{N}_+^n$, then $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ realize ($\vec{k}, I$)-patterns if and only if  
$$
    \bigcap_{i \in I} U_{a_i, k_i} \cap \bigcap_{j \not\in I} V_{a_j, k_j} \cap \Gamma_r \neq \emptyset.
$$
We may decompose the intersection above as
\[
\left( \bigcap_{i \in I} U_{a_i, k_i} \cap V_{\vec{a}, \vec{k}, 0} \cap \Gamma_r \right) \cup \left( \bigcap_{i \in I} U_{a_i, k_i} \cap \bigcup_{\emptyset \neq J \subseteq [n] \setminus I} V_{\vec{a}, \vec{k}, j} \cap \Gamma_r \right).
\]
Using Lemma 3.8 the finite component is under control.

Let’s focus on \( \bigcap_{i \in I} U_{a_i, k_i} \cap V_{\vec{a}, \vec{k}, 0} \cap \Gamma_r \). Since \( \Gamma_r \) is dense in \( S \), this set is nonempty if and only if
\[
\bigcap_{i \in I} U_{a_i, k_i} \cap V_{\vec{a}, \vec{k}, 0} \neq \emptyset.
\]
By Proposition 2.1 this intersection is nonempty if and only if one of the following hold:

1. there are \( i_0 \in I \) and \( s \in \mathbb{Z} \) such that
   \[
h(-a_{i_0} - 1)^{1/k_0} \zeta_{k_0}^s \in \bigcap_{i \in I, i \neq i_0} U_{a_i, k_i} \cap \bigcap_{j \notin I} \tilde{V}_{a_j, k_j},
   \]
2. there are \( j_0 \in [n] \setminus I \) and \( s \in \mathbb{Z} \) such that
   \[
h(-a_{j_0})^{1/k_0} \zeta_{k_0}^s \in \bigcap_{i \in I} U_{a_i, k_i} \cap \bigcap_{j \notin I, j \neq j_0} \tilde{V}_{a_j, k_j}.
   \]

In order to summarize these observations, we make the following definitions: let \( k, l \in \mathbb{N} \) with \( g = \gcd(k, l) \) and \( k' := k/g, l' := l/g \)

\[
A_{k, l} := \begin{cases} \mathbb{Z} \times \mathbb{Z} & : \text{if } \zeta_{k'} < h(1) \\ \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : h(l'a + k'b + l') \in (1, h(k'))\} & : \text{if } h(1) < \zeta_{k'} \end{cases}
\]

\[
B_{k, l} := \begin{cases} \mathbb{Z} \times \mathbb{Z} & : \text{if } \zeta_{k'} < h(-1) \\ \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : h(l'a + k'b + l') \in (h(k'), 1)\} & : \text{if } h(-1) < \zeta_{k'} \end{cases}
\]

\[
C_{k, l} := \begin{cases} \mathbb{Z} \times \mathbb{Z} & : \text{if } \zeta_{k'} < h(1) \\ \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : h(l'a + k'b) \in (1, h(k'))\} & : \text{if } h(1) < \zeta_{k'} \end{cases}
\]

\[
D_{k, l} := \begin{cases} \mathbb{Z} \times \mathbb{Z} & : \text{if } \zeta_{k'} < h(-1) \\ \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : h(l'a + k'b) \in (h(k'), 1)\} & : \text{if } h(-1) < \zeta_{k'} \end{cases}
\]

Combining Lemmas 3.6 and 3.7 with the observations above, we get.

**Lemma 3.10.** Let \( \vec{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n \) and \( I \subseteq [n] \). Then \( \vec{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) realizes the \( (\vec{k}, I) \)-pattern if and only if one of the following conditions hold:

1. \( \bigcap_{i \in I} U_{a_i, k_i} \cap \bigcup_{\emptyset \neq J \subseteq [n] \setminus I} V_{\vec{a}, \vec{k}, j} \cap \Gamma_r \neq \emptyset. \)
The final result of this section expresses the interval \( (h(k), 1) \) in terms of \( P_r \) when \( \zeta_k > h(1) \).

**Lemma 3.11.** Let \( k > 0 \) be such that \( h(1) < \zeta_k \). Then \( \alpha \in (1, h(k)) \) if and only if

\[
\alpha^{-1} \in \bigcup_{i=0}^{k-1} h(-i)P_r \cup \{h(-1), h(-2), \ldots, h(-(k-1))\}.
\]

(Here \( h(i)P_r \) is short for the interval \( (h(-i - 1), h(-i)) \).)

**Proof.** The assumption \( h(1) < \zeta_k \) gives \( h(i - 1) < h(i) \) for every \( i \in [k] \).

So we have the decomposition

\[
\begin{align*}
(1, h(k)) &= (1, h(1)) \cup (h(1), h(2)) \cup \cdots \cup (h(k - 1), h(k)) \\
&= (1, h(1)) \cup \cdots \cup (h(k - 1), h(k)) \cup \{h(1), \ldots, h(k - 1)\} \\
&= \bigcup_{i=0}^{k-1} h(i) (1, h(1)) \cup \{h(1), \ldots, h(k - 1)\}
\end{align*}
\]

This finishes the proof, since \( \beta \in (1, h(1)) \) if and only if \( \beta^{-1} \in P_r \). \( \square \)

4. **Expanding the Group of Integers**

We would like to consider the model theoretic structure obtained by expanding the abelian group of integers by the subset \( P_r \). We have seen above that \( \mathbb{Z} \) is isomorphic as an abelian group with a subgroup \( \Gamma_r \) of \( \mathbb{S} \) that happens to be dense in \( \mathbb{S} \). The work in the previous section was mostly done in \( \Gamma_r \), but it is straightforward to pull those results back to \( \mathbb{Z} \) via the map \( h \).

Let \( L := \{+, -, 0, c\} \) be the language of abelian groups with a distinguished element \( c \). Let \( T \) be the theory of the \( L \)-structure \( (\mathbb{Z}, +, -, 0, 1) \).

We extend \( L \) to \( L_P := L \cup \{P\} \) where \( P \) is a unary relation symbol. Our main objective is to study the \( L_P \)-structure

\[
\mathfrak{Z} := \left( \mathbb{Z}, +, -, 0, 1, P_r \right).
\]

For \( \vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n \) and \( I \subseteq [n] \), we define \( \phi_{\vec{k}, I}(x_1, \ldots, x_n) \) be the following \( L_P \)-formula:

\[
\exists y \left( \bigwedge_{i \in I} x_i + k_i y \in P \land \bigwedge_{j \notin I} x_j + k_j y \notin P \right).
\]
Therefore for \((a_1, \ldots, a_n) \in \mathbb{Z}^n\), we have \(\mathfrak{Z} \models \phi_{\vec{k}, I}(a_1, \ldots, a_n)\) if and only if \((a_1, \ldots, a_n)\) realizes the \((\vec{k}, I)\)-pattern. Then using Lemmas 3.10 and 3.11 there is a quantifier-free \(L_P\)-formula \(\psi_{\vec{k}, I}(x_1, \ldots, x_n)\) such that

\[
\mathfrak{Z} \models \forall x_1 \cdots \forall x_n (\phi_{\vec{k}, I}(x_1, \ldots, x_n) \leftrightarrow \psi_{\vec{k}, I}(x_1, \ldots, x_n))
\]

Let \(T_P\) be the \(L_P\)-theory extending \(T\) by the condition above; namely for every \(\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n\) and \(I \subseteq [n]\), we add the following sentence as an axiom:

\[
\forall x_1 \cdots \forall x_n (\phi_{\vec{k}, I}(x_1, \ldots, x_n) \leftrightarrow \psi_{\vec{k}, I}(x_1, \ldots, x_n))
\]

We shall construct a back-and-forth system between certain substructures of models of \(T_P\).

Let \(\mathcal{M}\) and \(\mathcal{N}\) be \(\aleph_0\)-saturated models of \(T_P\). Let \(S_{\mathcal{M}}\) be the collection of countable \(L_P\)-substructures \(\mathcal{M}'\) of \(\mathcal{M}\) such that \(\mathcal{M}'\) is a pure subgroup of \(\mathcal{M}\). We define \(S_{\mathcal{N}}\) in a similar way.

Note that the group \(\mathbb{Z}\) has a copy in each member of \(S_{\mathcal{M}}\) and \(S_{\mathcal{N}}\) as the subgroup generated by the constant \(c\). However, that copy may not be an \(L_P\)-substructure of \(\mathcal{M}\) or \(\mathcal{N}\).

Let \(\mathfrak{B}(\mathcal{M}, \mathcal{N})\) be the collection of \(L_P\)-isomorphisms \(f : \mathcal{M}' \to \mathcal{N}'\), where \(\mathcal{M}' \in S_{\mathcal{M}}\) and \(\mathcal{N}' \in S_{\mathcal{N}}\).

**Proposition 4.1.** For \(\aleph_0\)-saturated models \(\mathcal{M}\) and \(\mathcal{N}\) of \(T_P\), the collection \(\mathfrak{B}(\mathcal{M}, \mathcal{N})\) is a back-and-forth system.

**Proof.** Let \(f : \mathcal{M}' \to \mathcal{N}'\) be in \(\mathfrak{B}(\mathcal{M}, \mathcal{N})\) and \(\alpha \in \mathcal{M} \setminus \mathcal{M}'\). By symmetry, it suffices to extend \(f\) to an element of \(\mathfrak{B}(\mathcal{M}, \mathcal{N})\) that contains \(\alpha\) in its domain.

Let \(\mathcal{M}''\) be the pure subgroup of \(\mathcal{M}\) generated by \(\mathcal{M}'\) and \(\alpha\); namely:

\[
\mathcal{M}'' = \langle \mathcal{M}' \cup \{\alpha\} \rangle_M := \{ \gamma \in \mathcal{M} : m\gamma \in \mathcal{M}' \oplus \mathbb{Z}\alpha \text{ for some } m > 0 \}
\]

Also let \(\mathcal{N}''\) be the \(L_P\)-substructure of \(\mathcal{M}\) with the underlying set \(\mathcal{M}''\). We would like to extend \(f\) to \(\mathcal{M}'\). That amounts to finding \(\beta \in \mathcal{N}\) with the following property:

\((*)\) For every \(a \in \mathcal{M}'\), \(k \in \mathbb{Z}\), \(n > 0\), and \(\gamma \in \mathcal{M}\) if \(a + k\alpha = n\gamma\), then there is \(\delta \in \mathcal{N}\) such that \(f(a) + k\beta = n\delta\), and

\[
\gamma \in P \iff \delta \in P.
\]

This condition without the last part just means that \(\langle \mathcal{M}' \cup \{\alpha\} \rangle_M\) and \(\langle \mathcal{N}' \cup \{\beta\} \rangle_N\) are isomorphic as groups. Since the reducts of \(\mathcal{M}\) and \(\mathcal{N}\) to \(L\) are models of \(T\), there is certainly such an element \(\beta\) in \(\mathcal{N}\). So the point is to find \(\beta\) in a way that that isomorphism of groups is indeed an \(L_P\)-isomorphism.
By saturation, it suffices to find $\beta \in N$ satisfying a given finite fragment of $(\ast)$. So let $a_1, \ldots, a_m \in M'$, $k_1, \ldots, k_m \in \mathbb{Z}$, $n_1, \ldots, n_m \in N$, and $\gamma_1, \ldots, \gamma_m \in M$ be such that $a_i + k_i \alpha = n_i \gamma_i$ for each $i$. Then we need to find $\beta, \delta_1, \ldots, \delta_m \in N$ such that $f(a_i) + k_i \beta = n_i \delta_i$ and $\delta_i \in P$ if and only if $\gamma_i \in P$ for every $i$.

Let $\nu = \text{lcm}(n_1, \ldots, n_m)$, and let $d \in \{0, \ldots, \nu - 1\}$ and $\alpha' \in M$ be such that $\alpha = d + \nu \alpha'$. Then $\gamma_i = a_i' + k_i' \alpha'$, where $n_i a_i' = a_i + k_i d$ and $k_i' = \frac{k_i}{\nu}$. Since $M'$ is pure in $M$, it contains $a_i'$. Therefore it suffices to find $\beta' \in N$ such that for every $i$:

$$a_i' + k_i' \alpha' \in P \iff f(a_i') + k_i' \beta' \in P$$

Taking $\bar{k} = (k_1', \ldots, k_m')$ and $I = \{i : a_i' + k_i' \alpha' \in P\}$, we have

$$\mathcal{M} \models \psi_{\bar{k}, I}(a_1', \ldots, a_m').$$

Hence

$$\mathcal{M} \models \psi_{\bar{k}, I}(a_1', \ldots, a_n') \text{ and } \mathcal{M}' \models \psi_{\bar{k}, I}(a_1', \ldots, a_n').$$

Thus

$$\mathcal{N}' \models \psi_{\bar{k}, I}(f(a_1'), \ldots, f(a_m')) \text{ and } \mathcal{N} \models \psi_{\bar{k}, I}(f(a_1'), \ldots, f(a_n')).$$

As a result $\mathcal{N} \models \phi_{\bar{k}, I}(f(a_1'), \ldots, f(a_n'))$ and hence there is $\beta' \in N$ with the desired property:

$$f(a_i') + k_i' \beta' \in P \iff i \in I \iff a_i' + k_i' \alpha' \in P.$$

\[\square\]

5. Quantifier Elimination and Axiomatization

The theory $T_P$ does not have quantifier elimination for the obvious reason that for any $n > 1$, the definable subgroup consisting of elements divisible by $n$ is not quantifier-free definable. However, we do not get quantifier elimination even after adding predicate symbols to represent those subgroups, because we also need to know whether the element obtained by dividing by $n$ is in $P$ or not. So for every $n \geq 1$ we add two new predicate symbols $D_{n,+}$ and $D_{n,-}$ to the language $L_P$ to obtain $L_\pm$ and let $T_\pm$ be the definitional extension of $T_P$ to an $L_P$-theory by adding the following for each $n \geq 1$:

$$\forall x (D_{n,+}(x) \leftrightarrow \exists y (x = ny \wedge y \in P))$$

$$\forall x (D_{n,-}(x) \leftrightarrow \exists y (x = ny \wedge y \notin P))$$

Therefore, every model $\mathcal{M}$ of $T_P$ expands to a model of $T_\pm$; we still denote this extension by $\mathcal{M}$. Note that for a model $\mathcal{M}$ of $T_\pm$, we have $D_{n,+}(\mathcal{M}) \cup D_{n,-}(\mathcal{M}) = n\mathcal{M}$ for every $n \geq 1$ and $D_{1,+}(\mathcal{M}) = P(\mathcal{M})$.

Now we are ready to prove Theorem 1.2 in a stronger form.

**Theorem 5.1.** The theory $T_\pm$ has quantifier elimination.
Proof. It suffices to prove the following:

(†) Let \( M \) and \( N \) be models of \( T \) and \( A \) a common finitely generated \( L \)-substructure of \( M \) and \( N \). Then \( M \equiv_A N \). (This means that \( M \) and \( N \) are elementarily equivalent as \( L \)-structures)

(For why this is enough, see, for instance, Proposition 18.2 of [5].)

We may assume that \( M \) and \( N \) are \( \aleph_0 \)-saturated. Let \( M' = \langle A \rangle_M \) and \( N' = \langle A \rangle_N \).

Clearly, \( M' \) and \( N' \) are isomorphic as abelian groups via a map extending the identity map on \( A \). If \( n\alpha = a \) for some \( a \in A \) and \( \alpha \in M \), then there is \( \beta \in N \) with \( n\beta = a \). Then the isomorphism sends \( \alpha \) to \( \beta \). But we also have \( \alpha \in P \) if and only if \( \beta \in P \), since either both \( M \) and \( N \) satisfy \( D_{m,+}(a) \) or they both satisfy \( D_{m,-}(a) \). Therefore \( M' \) and \( N' \) are underlying sets of \( L \)-substructures \( M' \) and \( N' \) of \( M \) and \( N \) respectively, and they are isomorphic. Since \( M' \) and \( N' \) are countable and pure in \( M \) and \( N \), that isomorphism is in \( \mathfrak{B}(M,N) \). It follows that \( M \equiv_M N \) and in particular \( M \equiv_A N \). \( \square \)

Given \( \aleph_0 \)-saturated models \( M \) and \( N \) of \( T_P \), we may still have that \( \mathfrak{B}(M,N) = \emptyset \). So in order to get completeness we extend \( T_P \) to \( T_Z \) by adding \( Z \)-axioms: Given \( k \in \mathbb{N}_+ \) if \( k \in P(\mathbb{Z}) \), then we add the axiom \( k \in P \), otherwise we add the axiom \( k \notin P \). (Recall that 1 is in the language, so \( k = 1 + \cdots + 1 \).)

With this extension, we get Theorem [13].

Theorem 5.2. The theory \( T_Z \) is complete.

Proof. Clearly, \( \mathfrak{Z} \) is an algebraically prime model of \( T_Z \). Since \( T_Z \) has quantifier elimination, we get that \( T_Z \) is complete. \( \square \)

Question. According to this theorem each \( T_Z \) is a completion of \( T_P \). Is it correct that each completion of \( T_P \) is given by an expansion of the group of integers by a Beatty Sequence?

6. Definable Sets

The quantifier elimination result in the previous section gives a very simple characterization of definable sets in models of \( T_Z \): They are Boolean combinations of sets defined by formulas of the form

\[ a + \vec{k} \vec{x} = 0, D_{m,+}(a + \vec{k} \vec{x}), \text{and} \ D_{m,-}(a + \vec{k} \vec{x}) \]

where \( \vec{x} = (x_1, \ldots, x_n) \) is a tuple of variables, \( a \in M \) and \( \vec{k} \in \mathbb{Z}^n \).

It is simpler in the sense that the negations of formulas \( D_{m,+}(a + \vec{k} \vec{x}) \) and \( D_{m,-}(a + \vec{k} \vec{x}) \) are finite disjunctions of the same kind of formulas.

We shall focus on subsets of \( \mathbb{Z} \) definable in \( \mathfrak{Z} \). We prove that if such a set \( X \) is infinite, then there is a constant \( N \) such that consecutive
elements of $X$ are at most $N$ apart. While doing that, we constantly
switch between $\mathfrak{G}$ and the isomorphic structure
$$\mathfrak{G}_r = (\Gamma_r, \cdot^{-1}, 1, \eta, P(\Gamma_r)).$$

**Definition 6.1.** Let $X \subseteq \mathbb{Z}$. We say that $X$ has the uniform gaps property if there is $N = N(X)$ such that
$$X \cap \{x + 1, \ldots, x + N\} \neq \emptyset$$
for every $x \in X$.

Clearly, $\emptyset$ has the uniform gaps property and it is the only finite set
that has the uniform gaps property. The following is also clear.

**Lemma 6.2.** If $X, Y \subseteq \mathbb{Z}$ have the uniform gaps property. Then $X \cup Y$ has the uniform gaps property.

The following is Theorem 1.1 from the Introduction.

**Theorem 6.3.** Every infinite subset of $\mathbb{Z}$ definable in $\mathfrak{G}$ has the uniform gaps property.

**Proof.** By quantifier elimination for $\mathfrak{G}$ and Lemma 6.2 it suffices to prove the theorem for intersections of sets defined by formulas of the forms
$$a + kx = 0, a + kx \neq 0, D_{m,+}(a + kx), D_{m,-}(a + kx)$$
where $a, k \in \mathbb{Z}$, $m \geq 1$. Such an intersection is empty or a singleton, if it contains a set defined by $a + kx = 0$. Also the formula $a + kx \neq 0$ avoids only one element; hence we may assume that the intersection is made up of sets defined by formulas $D_{m,+}(a + kx)$ and $D_{m,-}(a + kx)$ for various $a, k, m$. Therefore let $X$ be defined by the formula
$$\bigwedge_{i=1}^{p} D_{m_i,+}(a_i + k_i x) \land \bigwedge_{j=p+1}^{q} D_{m_j,-}(a_j + k_j x).$$
Clearly, we may assume $k_i \neq 0$ for each $i$.

Let $x \in X$ be given. So let $a_i + k_i x = m_i y_i$ for $y_1, \ldots, y_p \in P_r$ and $y_{p+1}, \ldots, y_q \notin P_r$. Then
$$y_i = f_i(y_1)$$
where
$$f_i(t) = k_i \frac{m_i t - a_1}{k_i m_i} + \frac{a_i}{m_i}$$
for each $i$. Hence as $h(y_1)$ varies in $(h(-1), 1)$, the element $h(y_i)$ varies in an interval, say $J_i$. Let $I_i = J_i \cap (h(-1), 1)$ for $i = 1, \ldots, p$ and $I_j = J_j \cap [1, h(-1)]$ for $i = p + 1, \ldots, q$. Then
$$h(y_1) \in I := \bigcap_{i=1}^{q} f_i^{-1}(I_i).$$
Note that \( I \) is a nonempty interval and for every \( y'_1 \in I \), if \( x' \in \mathbb{Z} \) satisfies
\[
a_i + k_i x' = m_i f_i(y'_1) \quad \text{for every} \ i = 1, \ldots, q
\]
then \( x' \in X \).

Let \( \alpha \) be the length of \( I \) and let \( \beta = \alpha \frac{1}{\kappa} \), where \( \kappa = \text{lcm}(k_1, \ldots, k_q) \). Also let \( \mu = \text{lcm}(m_1, \ldots, m_q) \) and take \( z' \in \mu \mathbb{N}+ \) with \( z_1 < \beta \) and \( z'' \in \mu \mathbb{N}+ \) with \( \beta^{-1} < z_2 \).

Letting \( \gamma \) denote the midpoint of \( I \), if \( \gamma < h(y_1) \), then \( h(y_1 + k_1 \frac{z'}{m_1}) \in I \), and if \( h(y_1) < \gamma \), then \( h(y_1 + k_1 \frac{z''}{m_1}) \in I \). Since the choices of \( z' \) and \( z'' \) are independent of \( y_1 \) (hence \( x \)), this finishes the proof.

\[ \Box \]

**Remark.** This proof does not work for larger models of \( T \mathbb{Z} \), because the interval \( I \) might be of infinitesimal length and hence we cannot find suitable \( z' \) and \( z'' \). Actually, one can construct an example to illustrate that definable subsets of larger models do not have the uniform gaps property.

**Corollary 6.4.** Let \( X \) be an infinite subset of \( \mathbb{Z} \) that us definable in \( \mathcal{Z} \). Then there is \( N = N(X) \in \mathbb{N}+ \) such that for every \( x \in X \), the intersection \( X \cap \{x - N, x - (N - 1), \ldots, x - 1\} \) is nonempty.

**Proof.** The set \( -X := \{y \in \mathbb{Z} : -y \in X\} \) is also definable in \( \mathcal{Z} \) and hence has the uniform gaps property. That translates to \( X \) as the conclusion of the corollary. \( \square \)

**Corollary 6.5.** Ordering of \( \mathbb{Z} \) is not definable in \( \mathcal{Z} \).

**Proof.** If the ordering were definable in \( \mathcal{Z} \), then so would be the set of positive elements. However, \( \mathbb{N}+ \) does not satisfy the conclusion of Corollary 6.4. \( \square \)

**Corollary 6.6.** Multiplication is not defined in \( \mathcal{Z} \).

**Proof.** If multiplication on \( \mathbb{Z} \) is definable, then the ordering of \( \mathbb{Z} \) is also definable using Lagrange’s four-square theorem. \( \square \)

**Unstability.** The last corollary above shows that \( \mathcal{Z} \) is not too bad with the point of view of definable sets. Now we show that it is not very good either.

**Lemma 6.7.** Let \( n > 0 \), then there is \( m > 0 \) such that \( m, 2m, \ldots, nm \in P_r \) and \( -m, -2m, \ldots, -nm \notin P_r \).

**Proof.** Let \( \alpha = \min\{h(1), h(-1)\} \) and \( \beta = \alpha \frac{1}{\kappa} \). By regular density of \( \Gamma_r \) take \( m \in \mathbb{N}+ \) such that \( \beta^{-1} < h(m) \). Clearly, this \( m \) satisfies the conclusion of the lemma. \( \square \)

**Proposition 6.8.** The theory \( T \mathbb{Z} \) is not stable.
Proof. Let $\phi(x; y)$ be the $L_p$-formula $y - x \in P$. We show that this formula is unstable. For this, it suffices to prove the following: For every $n > 0$, there are $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$3 \models \phi(a_i, a_j) \iff i < j.$$  

So let $n > 0$ be given. Take $a_i = im$ where $m = m(n)$ is as in the lemma above. Now $a_j - a_i = (j - i)m$ and hence $\phi(a_i, a_j)$ holds in $3$ if and only if $i < j$.

\[\square\]

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The place of $3$ in the stability spectrum will a part of the dissertation study of the second author.

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