On the Exchange Interactions in Holographic $p$-adic CFT

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Abstract

There is a renewed interest in conformal field theories (CFT) on ultrametric spaces ($p$-adic field and its algebraic extensions) in view of their natural adaptability in the holographic setting. We compute the contributions from the exchange interactions to the four-point correlator of the CFT using Witten diagrams with three-scalar interaction vertex. Together with the contributions from the bulk four-point interaction, the contact term, these provide a complete answer. We remark on the singularity structure in Mellin space, and argue that all these models are analogues of $\text{adS}_2/\text{CFT}_1$.

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1 Introduction

Conformally invariant field theories (CFT) are distinguished in the space of quantum field theories, and are of fundamental importance in understanding their dynamics. The holographic duality [1–3] offers a different approach to analysing CFTs. It is of interest to study CFTs based on different spaces to have a better handle on their underlying mathematical structure. This has been the motivation to explore CFTs based on ultrametric fields ($p$-adic fields $\mathbb{Q}_p$, for a fixed prime $p$, and its algebraic extensions $\mathbb{K}$ of $\mathbb{Q}_p$ [4–7]). Recently, Gubser et al [8] (see also [9]) initiated a study of CFTs on ultrametric spaces as holographic dual to bulk theories on uniform lattices (Bruhat-Tits trees). In the simplest setting, the bulk theory is that of a scalar field [10,11], defined by a lattice action [12]. The correlators of the boundary CFT are computed by evaluating Witten diagrams in the bulk theory. Specifically, the two-, three- and four-point correlation functions in the ultrametric CFT were calculated and shown to have structures very similar to the usual (Archimedean) CFTs. There are some key differences, however, arising from the absence of descendent fields, which in turn may be attributed to the vanishing of local derivatives along the ultrametric boundary.

Specifically, the four-point correlation function in Ref. [8] was computed with a four-point contact interaction in the bulk. In this paper, we compute the contribution to the same correlator from the exchange diagrams resulting from a three-point interaction vertex in the bulk\(^1\).

We also argue that all such ultrametric CFTs (on $\mathbb{Q}_p$ or its algebraic extensions) are really like one-dimensional CFTs, that provide discrete analogues of the holographic duality $\text{adS}_2/\text{CFT}_1$. Finally, we analyze the Mellin space representation of the four-point function.

The projective $p$-adic line $\mathbb{Q}_p\mathbb{P}^1$ is the asymptotic boundary of an infinite lattice, the Bruhat-Tits tree $\mathcal{T}_p$ (more familiar to physicists as Bethe lattice with coordination number $p + 1$) that has been used as the ‘worldsheet’ of open $p$-adic string [12,14]. This is exactly analogous to the real case. The group action $\text{GL}(2,\mathbb{Q}_p)$ naturally leads to the symmetric space $\mathcal{T}_p$ obtained by a quotient of its maximal compact subgroup $\text{GL}(2,\mathbb{Z}_p)$, in which the entries are $p$-adic integers. Ultrametric spaces that are finite algebraic extensions of $\mathbb{Q}_p$ have a similar structure (possibly with different coordination number). Moreover, one may think of a family of so-called ramified extensions of $\mathbb{Q}_p$ as a continuum limit [15].

\(^1\)In fact, these contributions have been considered recently in Ref. [13]. However, the authors reduce these to the ‘geodesic bulk diagrams’ with the objective to investigate the conformal blocks. We have done a direct computation. The sum of various configurations evaluated in the following should contribute to a conformal block.
2 A brief review of ultrametric holography

The set up we consider is that of Ref. [8], therefore, we shall be brief. A $p$-adic number $x \in \mathbb{Q}_p$ (for a fixed prime $p$) has a Laurent series expansion\(^2\) in $p$

$$x = p^N \left( a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots \right), \quad a_0 \neq 0, \quad a_n = \{0, 1, 2, \ldots, p-1\} \quad \text{and} \quad N \in \mathbb{Z}.$$ 

Its ($p$-adic) norm is $|x|_p = p^{-N}$. The above may be used to construct the bulk, a Bethe lattice with coordination number $p + 1$ as shown in Fig. 1. It is also the equivalence classes of lattices in $\mathbb{Q}_p^2$. An algebraic extension $\mathbb{K}$ of it is a finite dimensional vector space over $\mathbb{Q}_p$. It is also an ultrametric space that share all qualitative features of $\mathbb{Q}_p$. In particular, there is an associated Bruhat-Tits tree (which may be constructed as a symmetric space from the group action on $\mathbb{K}$, with its coordination number depending on $p$ and the ramification index of the extension). What is important, however, is the fact that it is still a tree, an infinite graph with no closed loop. Therefore, the path between any two points, either in the bulk or on its asymptotic boundary, is unique (discounting backtracking).

As a result, contrary to what one may expect from the dimension $d$ of (an unramified extension) $\mathbb{K}$ as a $\mathbb{Q}_p$-vector space, which may lead one believe that the boundary CFT on $\mathbb{K}$ is $d$-dimensional, it is still only an analogue of a one-dimensional CFT. This fact is consistent with what we shall find for the boundary $n$-point functions: they depend on $(n-3)$ cross ratios, as in one-dimensional Archimedean CFTs [18]. There is further support from the fact that $\text{GL}(2, \mathbb{K})$, the group of symmetries, of the CFT is rather akin to $\text{SL}(2, \mathbb{R})$. After some confusion in the early days of ultrametric string theory, those based on algebraic extensions of $\mathbb{Q}_p$ were also realised to be analogues of open string theory for which the boundary of the worldsheet is one-dimensional [12].

It would suffice, therefore, to restrict to the simplest case on $\mathbb{Q}_p$. Nevertheless to facilitate comparison with the existing literature, we shall present the contributions from the bulk three-vertex to the boundary CFT correlators for the general case of an $n$-dimensional (unramified) extension $\mathbb{K}$ of $\mathbb{Q}_p$. Essentially, this amounts to replacing $p$ by $q \equiv p^n$ in some of the formulas [8].

To get back to $p$-adic CFT one sets $n = 1$.

![Figure 1: A part of the Bruhat-Tits tree corresponding to the 3-adic field $\mathbb{Q}_3$ at its asymptotic boundary. The dotted line connects 0 to $\infty$. The label on the node $C_N$ corresponds to the leading power in the Laurent expansion of $x$, and the branching is decided by the coefficients $a_n$.](image)

\(^2\)We shall not recount any aspect of ultrametric analysis since succinct accounts of the facts relevant to us exist in many physics articles — see, e.g., Appendix A of [16] or Sec. 2 of [8], or [17] for a more comprehensive account.
The bulk theory is a scalar field theory defined by the discrete lattice action

\[ S[\phi] = \frac{1}{2} \sum_{E_{(ab)} \in \mathcal{T}_p} (\phi_a - \phi_b)^2 + \sum_{a \in \mathcal{T}_p} \left( \frac{1}{2} m^2 \phi_a^2 + \frac{1}{3!} g_3 \phi_a^3 + \frac{1}{4!} g_4 \phi_a^4 \right) \]

where the first sum is over all edges \( E_{(ab)} \) and the second over all nodes of the tree. Using the fact that the tree is the equivalence classes of two-dimensional lattices, Ref. [8] labelled a bulk point \( a \in \mathcal{T}_p \) by a pair of \( p \)-adic numbers \( a = (r, x) \), where \( x \in \mathbb{Q}_p \) and \( r = p^m \in \mathbb{Q}_p, (m \in \mathbb{Z}) \). This denotes an equivalence class of the form \( x + r \mathbb{Z}_p \). One may think of \( r \) as the ‘radial’ coordinate related to the notion of a ‘depth’ from the ‘trunk’ joining 0 to \( \infty \).

In the following, we shall compute the contribution of the exchange diagrams resulting from \( \phi^3 \) interaction to the boundary four-point correlation function. Together with the contact term contribution from the \( \phi^4 \) interaction, computed in [8], this accounts for all the contributions to the bulk four-point amplitude. The corresponding Witten diagrams in the usual Archimedean adS/CFT\(_1\) are shown in Fig. 2.

\[ \begin{array}{c}
\mathbb{RP}^1 \\
\text{adS}_2
\end{array} \]

\[ \begin{array}{c}
\mathbb{RP}^1 \\
\text{adS}_2
\end{array} \]

Figure 2: Representative Witten diagrams contributing to the four-point correlators in Archimedean adS\(_2\)/CFT\(_1\) from bulk three- and four-point interaction vertices. The correlators involve integrating over the bulk points \( a_1, a_2 \) and \( a \).

In order to evaluate the contribution of a Witten diagram \( W \), one needs the propagators. The (unnormalized) bulk-to-bulk propagator \( G(a, b) \ (a, b \in \mathcal{T}_p) \) is the Green’s function of the free massive field

\[ G(a_1, a_2) = p^{-d(a_1, a_2)\Delta}, \tag{1} \]

where \( d(a_1, a_2) \) is the number of edges connecting the nodes and \( \Delta \) is related to the mass by the local zeta function at the prime \( p \), \( \zeta_p(s) = 1/(1 - p^{-s}) \) through \( m^{-2} = -\zeta_p(\Delta - n)\zeta_p(-\Delta) \). The bulk-to-boundary propagator \( K(a, x) \) was obtained as a regularised limit of sending one of the points in the unnormalized bulk-to-bulk propagator to the boundary, and is expressed as

\[ K(a, x) \equiv K((r_y, y), x) = \frac{|r_y|_p^\Delta}{\left( \sup \left\{ |r_y|_p, |y - x|_q \right\} \right)^2 \Delta}. \tag{2} \]

These propagators are similar to those in Archimedean adS/CFT.

Let us consider the four-point correlation function of the boundary operator \( \mathcal{O} \) of dimension \( \Delta \), dual to the field \( \phi \). Using the \( \text{GL}(2, \mathbb{K}) \) symmetry, the positions of three insertions in \( \mathcal{A}_4(x_1, x_2, x_3, x_4) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle \) can be sent to three predetermined points. Hence, modulo dependence fixed by scaling, it is a function of the cross ratio

\[ u = \left| \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)} \right|_q = p^{-d(c_1, c_2)}. \tag{3} \]
The second expression for the cross ratio involves the distance between the bulk points $c_1$ and $c_2$, which are the two ends of the segment along which the path from $x_1$ to $x_3$ and $x_2$ to $x_4$ overlap. It is important that these paths are unique, and all paths (without any backtracking) on the tree are geodesics. The four point function in a $d$-dimensional CFT depends on two independent cross ratios $u$ and $v$ for $d \geq 2$ [18]. The $p$-adic $\text{adS}/\text{CFT}$, however, is analogous to the case $d = 1$, where there is only one independent cross ratio. It is clear that once the three insertions are fixed, say at $x_1 = 0$, $x_3 = 1$ and $x_4 \to \infty$, inserting the fourth one anywhere leads to only one independent cross ratio.

The contribution to $A_4^{(4)}$ from the contact interaction, computed in [8], is

$$A_4^{(4)} = g_4 u^{2\Delta} \left[ \log_u \left( \frac{\zeta_p(2\Delta)}{\zeta_p(4\Delta)} - 1 \right) + 3 \right] \times \frac{\zeta_p(4\Delta - n)}{|x_{12}| q |x_{34}| q} \frac{1}{2\Delta} \quad (4)$$

Some of the Witten diagrams on the tree, dubbed subway diagrams in [8], are shown in Fig. 3. In spite of their deceptive appearance, these are contact diagrams.

Figure 3: Examples of Witten (subway) diagrams with bulk 4-point contact interaction that contribute to the CFT four-point correlator. The position of the interaction point $a$ is integrated over the tree $T_p$.

3 Boundary correlators with bulk 3-point interaction

We shall now evaluate the contribution to the boundary correlators from the 3-point interaction vertex in the bulk. (The contact interaction will not be of importance in what follows, therefore, henceforth we may as well set $g_4 = 0$.) Let the bulk vertices be at the nodes $a_1$ and $a_2$ of the tree $T_p$, where $a_1$ is connected to $x_1$ and $x_2$, and $a_2$ to $x_3$ and $x_4$, by bulk-to-boundary propagators Eq. (2). In addition, $a_1$ and $a_2$ are connected by a bulk-to-bulk propagator Eq. (1). Consider the (unique) paths in $T_p$ that join the boundary points. The interaction points $a_1, a_2$ can be joined to these paths at the points $b_1$ and $b_2$ respectively. The path from $x_1$ to $x_3$ overlap that from $x_2$ to $x_4$ along the segment $\langle c_1, c_2 \rangle$ in the bulk. Depending on the locations of $a_1$ and $a_2$, which are to be summed over all the nodes, we get different configurations, representatives of which are shown in Figs. 4 and 5. The bulk-to-boundary propagators can now be decomposed in terms of products of bulk-to-boundary and bulk-to-bulk propagators in order to evaluate and simplify the amplitudes.

As an example, let us consider in detail the Witten diagram $W_1$ in Fig. 4, as this gives interesting new dependence on the cross ratio. Its contribution is

$$A_4^{(3)}(W_1) = g_3^2 \sum_{a_1, a_2 \in T_p} K(x_1, a_1) K(x_2, a_1) K(x_3, a_2) K(x_4, a_2) G(a_1, a_2) \quad (5)$$

We can decompose the bulk-to-boundary propagators as $K(x_1, a_1) = K(x_1, c_1) G(c_1, b_1) G(b_1, a_1)$, and similarly for the others. Likewise, the bulk-to-bulk propagator between the interaction vertices can be split as $G(a_1, a_2) = G(a_1, b_1) G(b_1, b_2) G(b_2, a_2)$. Notice that in this configuration,
where we have factored out the bulk-to-boundary propagators from the boundary points to the extreme points \( c_1 \) and \( c_2 \) of the common segment of the geodesics that join them.

We now have to sum over the bulk points \( a_1 \) and \( a_2 \), in which sum over the different possibilities for \( b_1 \) and \( b_2 \) is implicit\(^3\). The reduced amplitude is

\[
\hat{\mathcal{A}}_4^{(3)}(W_1) = (G(a_1,b_1)G(a_2,b_2))^3 (G(c_1,b_1)G(c_2,b_2))^2 G(b_1,b_2).
\]

Let us denote the lengths of the different segments as follows \( l_1 = d(a_1,b_1) \), \( l_2 = d(b_2,a_2) \), \( m_1 = d(c_1,b_1) \) and \( m_2 = d(c_2,b_2) \). The following possibilities lead to different contributions in \( \hat{\mathcal{A}}_4^{(3)}(W_1) \) as enumerated below. (In the following the first case occurs only if the coordination number of the lattice is at least 4.)

- \( m_1 = 0 \) and \( m_2 = d(c_1,c_2) \) or vice versa. This means that \( b_1 \) coincides with \( b_2 \), which in turn coincides with either \( c_1 \) or \( c_2 \), but the vertices \( a_1 \) and \( a_2 \) lie on different branches emanating from \( b_1 = b_2 = b \).

\[
2u^{2\Delta} \left[ 2 (p^n - 2) \left( \sum_{l=1}^{\infty} p^{n(l-1)} p^{-3\Delta l} \right) + 2 \left( \sum_{l=1}^{\infty} p^{n(l-1)} p^{-3\Delta l} \right)^2 \right] = 2u^{2\Delta} (p^n - 2)p^{-3\Delta} \zeta_p (3\Delta - n) \left[ 2 + (p^n - 3)p^{-3\Delta} \zeta_p (3\Delta - n) \right]
\]

- \( m_1 = 0 \) and \( m_2 = 0 \), means that \( b_1 \) coincides \( c_1 \) and \( b_2 \) with \( c_2 \).

\[
u^{\Delta} \left[ 1 + (p^n - 2) \left( \sum_{l=1}^{\infty} p^{n(l-1)} p^{-3\Delta l} \right) \right]^2 = u^{\Delta} \left[ (1 - 2p^{-3\Delta}) \zeta_p (3\Delta - n) \right]^2
\]

- \( m_1, m_2 \neq 0 \) but \( m_1 + m_2 = d(c_1,c_2) \). This means that \( b_1 \) coincides with \( b_2 \), but neither of these coincides with either \( c_1 \) or \( c_2 \).

\[
u^{2\Delta} (d - 1) \left[ 2 (p^n - 1) \left( \sum_{l=1}^{\infty} p^{n(l-1)} p^{-3\Delta l} \right) + 2 \left( \sum_{l=1}^{\infty} p^{n(l-1)} p^{-3\Delta l} \right)^2 \right] = -u^{2\Delta} (\log p u + 1) (p^n - 1)p^{-3\Delta} \zeta_p (3\Delta - n) \left[ 2 + (p^n - 2)p^{-3\Delta} \zeta_p (3\Delta - n) \right]
\]

\(^3\)One should take care to avoid double counting that may arise from degenerate cases of different configurations.
• $m_1 = 0$ but $m_2 \neq 0, d(c_1, c_2)$, which means $b_1$ coincides with $c_1$ but $b_2$ varies (or $b_2$ coincides with $c_2$ but $b_1$ varies).

$$2u^\Delta \left( \sum_{m_2=1}^{d(c_1, c_2)-1} p^{-m_2\Delta} \right) \left( 1 + \sum_{l_1=1}^\infty (p^n - 1) p^{n(l_1-1)} p^{-3\Delta l_1} \right) \left( 1 + \sum_{l_2=1}^\infty (p^n - 2) p^{n(l_2-1)} p^{-3\Delta l_2} \right)$$

$$= 2 \left( p^{-\Delta} u^\Delta - u^{2\Delta} \right) \left( 1 - 2p^{-3\Delta} \right) c_p^2(3\Delta - n) \left( \frac{\bar{\zeta}(\Delta)}{\zeta_p(3\Delta)} \right)$$

• $m_1, m_2 \neq 0$ and $2 \leq m_1 + m_2 < d(c_1, c_2)$. This means that both $b_1$ and $b_2$ are strictly inside the segment $(c_1, c_2)$, but $b_1$ is always to the left of $b_2$.

$$u^\Delta \left( \sum_{m_2=1}^{d(c_1, c_2)-2} p^{-m_2\Delta} \left( \sum_{m_1=1}^{d(c_1, c_2)-m_2-1} p^{-m_1\Delta} \right) \right) \left[ 1 + \sum_{l=1}^\infty (p^n - 1) p^{n(l-1)} p^{-3\Delta l} \right]$$

$$= \left( u^{2\Delta} \left( \log_p u + 2 - \zeta_p(\Delta) \right) + p^{-2\Delta} \zeta_p(\Delta) u^\Delta \right) \left( \frac{\bar{\zeta}(\Delta - n)}{\zeta_p(3\Delta)} \right)^2 \zeta_p(\Delta)$$

The sums being geometric series could be performed easily.

There are additional diagrams with the positions of $a_1, b_1$ and $a_2, b_2$ interchanged, however, some of the degenerate cases of these are indistinguishable. Exchanging the labels 1 and 2 in $W_1$ in Fig. 4, we get $\tilde{W}_1$, which contributes

$$\tilde{A}_4^{(3)}(\tilde{W}_1) = \left\{ u^{5\Delta} \left( 1 - 2p^{-3\Delta} \right)^2 + 2 \left( p^{-3\Delta} u^{2\Delta} - u^{5\Delta} \right) \left( 1 - 2p^{-3\Delta} \right) \right.$$  

$$- u^{2\Delta} \left( \log_p u + 2 - \frac{p^{-3\Delta}}{\zeta_p(3\Delta)} + p^{-6\Delta} \right) + u^{5\Delta} \right\} c_p^2(3\Delta - n)$$  

$$= \left\{ -u^{2\Delta} \left( \log_p u + 4 - \frac{p^{-3\Delta}}{\zeta_p(3\Delta)} - p^{-6\Delta} \right) + 4p^{-6\Delta} u^{5\Delta} \right\} c_p^2(3\Delta - n)$$  

(7)

where we have taken care to avoid double counting.

We draw attention to the fact that the functional dependence on the (single) cross ratio $u$ of the CFT four-point correlation function, with contributions from the bulk 3-vertex, involves terms with $u^{2\Delta}$ and $u^{5\Delta}$, familiar from the contribution from the contact interaction. However, there are additional powers $u^\Delta$ and $u^{5\Delta}$, although these do not accompany terms with $\log_p u$. We shall come back to this.

![Figure 5: Other Witten (subway) diagrams with bulk 3-point interactions that give non-trivial dependence on the cross ratio.](image)

Many more Witten diagrams contribute to the boundary four-point amplitude. These are shown in Figs. 5 and 6 without labels, for which there are more than one possibilities for each
diagram. A ‘twig’ can be attached to an external in many different ways, however, we have not shown all these diagrams explicitly. In case an exchange of the vertices 1 and 2 lead to different configurations, we denote it with a tilde.

When a twig is attached to an external leg, say, as in the diagram $W_3$ in the configuration in which the twig $\langle b_1, a_1 \rangle$ is attached to the external leg to $x_1$, it involves $K(x_1, a_1) = K(x_1, b_1)G(b_1, a_1)$. In order to perform the sum over the interaction point, it is simpler to use the fact that the bulk-to-boundary propagator is a (regularised) limit of a bulk-to-bulk propagator: $K(x_1, b_1) = \lim_{z_1 \to x_1} G(z_1, b_1)$. The same limiting procedure is used in calculating the contributions from $\tilde{W}_3$ and those in Fig. 6.

We shall skip the details of each individual cases and present only the results. The relevant expressions corresponding to Figs. 5 and 6 are

$$\hat{A}_4^{(3)} (W_2) = -p^{-7\Delta+n} \left( (p^n-1) \log u - (p^n-3) \right) (2 + (p^n-1)p^{-3\Delta} \zeta_p(3\Delta - n))$$

$$\hat{A}_4^{(3)} (W_3) = 4p^{-\Delta} \frac{\zeta_p(\Delta) \zeta_p^2(3\Delta - n)}{\zeta_p(3\Delta)} \left\{ \frac{\zeta_p(\Delta)}{\zeta_p(3\Delta)} \left( p^{-\Delta} u^\Delta - u^{2\Delta} \right) + (1 - 2p^{-3\Delta}) \left( u^\Delta + u^{2\Delta} \right) \right\}$$

$$\hat{A}_4^{(3)} (\tilde{W}_3) = 4p^{-3\Delta} \frac{\zeta_p(3\Delta - n)}{\zeta_p(3\Delta)} \left\{ u^{2\Delta} + 2p^{-3\Delta} \zeta_p(3\Delta) u^{5\Delta} \right\}$$

$$\hat{A}_4^{(3)} (W_4 + \tilde{W}_4) = 4p^{-2\Delta} \frac{\zeta_p^2(3\Delta - n)}{\zeta_p(3\Delta)} \left\{ \zeta_p^2(\Delta) u^\Delta + p^{-4\Delta} \zeta_p^2(3\Delta) u^{5\Delta} \right\}$$

$$\hat{A}_4^{(3)} (W_5) = 4p^{4\Delta} \frac{\zeta_p(2\Delta) \zeta_p^2(3\Delta - n)}{\zeta_p(3\Delta)} \left( \zeta_p(\Delta) + p^{-2\Delta} \zeta_p(3\Delta) \right) u^{2\Delta}$$

$$\hat{A}_4^{(3)} (W_6 + \tilde{W}_6) = 4p^{-3\Delta} \frac{\zeta_p(2\Delta) \zeta_p^2(3\Delta - n)}{\zeta_p(3\Delta)} \left( \zeta_p(\Delta) + p^{-2\Delta} \zeta_p(3\Delta) \right) u^{2\Delta}$$

$$\hat{A}_4^{(3)} (W_7) = 4p^{-5\Delta} (p^n-1) \zeta_p(2\Delta) \zeta_p(3\Delta - n) \left\{ (2 + (p^n-2)p^{-3\Delta} \zeta_p(3\Delta - n)) \right.$$}

$$+ p^{-4\Delta+n} \zeta_p(4\Delta - n) \left( 2 + (p^n-1)p^{-3\Delta} \zeta_p(3\Delta - n) \right) \left\} u^{2\Delta}. \quad (8)$$

The boundary four-point correlator, computed from the bulk theory with three-scalar interaction vertex, is obtained by adding all the contributions discussed above. Combined with the result Eq. (4) from the bulk four-point contact term in [8], these exhaust the possible bulk contributions to the boundary four-point amplitude. It may be possible to simplify the result further, but it is unlikely to be very useful.

We see that some of the diagrams contribute terms with $u^{5\Delta}$, which would signify the presence of an operator of dimensions $5\Delta$ in the intermediate channel. However, this is not expected. Indeed, when we collect the terms in Eqs. (7) and (8), the coefficient of $u^{5\Delta}$ vanishes.

## 4 Correlators in Mellin space

Mellin space is the natural setting for conformally invariant field theories [19, 20] (see [18] for a review). The isomorphism between the additive group (\( \mathbb{R}, + \)) and the multiplicative group (\( \mathbb{R}^+, \times \)), via the exponential map, is at the heart of this idea — radial quantization maps the translation invariance of a field theory to scale invariance of CFT, and takes Fourier transform to Mellin transform. The generic branch point singularities of a CFT correlator takes the form of (isolated) poles in terms of the Mellin variables. Many authors have explored CFTs from a Mellin
Figure 6: Other Witten (subway) diagrams with bulk 3-point interactions that give trivial dependence on the cross ratio.

space perspective in recent times. For a partial list of references dealing with different aspects, see [21–27].

Let us consider the results obtained in Section 3. In the s-channel, the cross ratio defined in Eq. (3), in the four point correlation function of the boundary CFT, evaluated from the bulk with both the three- and four-point interaction terms, is $u < 1$. The amplitude has the following form

$$A_4(s) = \left( \prod_{i=1,2} K(x_i, c_1) \prod_{j=3,4} K(x_j, c_2) \right) \tilde{A}_4(s)$$

$$\tilde{A}_4(s) = \alpha_{(1)} u^\Delta + (\alpha_{(2)} + \beta_{(2)} \log_p u) u^{2\Delta},$$

where the coefficients $\alpha_{(1)}$, $\alpha_{(2)}$ and $\beta_{(2)}$, which depend only on $p$, $\Delta$, $g_3$ and $g_4$, can be determined from the results in Sections 2 and 3.

In the s-channel discussed above, we have considered the topology (shown in Fig. 4) in which path from $x_1$ to $x_2$ meet for the first time at $c_1$, and that from $x_3$ to $x_4$ at $c_2$. There is another possibility in which the path from $x_1$ to $x_3$ and $x_2$ to $x_4$ meet for the first time. The amplitude for this case, i.e., for the t-channel, can be obtained by a simple relabelling of the external points. Therefore, it depends on the cross ratio

$$v = \frac{|x_{13} x_{24}|}{|x_{12} x_{34}|} = \frac{1}{u}$$

Notice that $v$ is not an independent variable, since there is only one independent cross ratio, as emphasised in the Section 2. Therefore, the t-channel amplitude is

$$A_4(t) = \left( \prod_{i=1,3} K(x_i, c_1) \prod_{j=2,4} K(x_j, c_2) \right) \tilde{A}_4(t)$$

$$\tilde{A}_4(t) = \alpha_{(1)} u^{-\Delta} + (\alpha_{(2)} - \beta_{(2)} \log_p u) u^{-2\Delta},$$

which has the same form as Eq. (9) with some flips in sign.
Let us define the Mellin amplitude

\[ \tilde{A}_4(q; u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} u^z M_4(z) \, dz, \] (11)

where the amplitude in the \(q\)-channel \((q = s, t)\) can be obtained by closing the contour on the right- or the left-half plane depending on whether \(u < 1\) or \(u > 1\). We have not included any conventional factor of \(\Gamma\)-functions in the definition above because in this situation, where there is only one independent cross ratio, and the dynamics is rather simple, this does not seem necessary.

The generic form of the Mellin amplitude is then

\[ M_4(z) = \frac{a_{(1)}}{z^2 - \Delta^2} + \frac{a_{(2)}}{z^2 - 4\Delta^2} + \frac{b_{(2)}}{(z^2 - 4\Delta^2)^2}. \] (12)

It can be seen that this leads to the terms in Eqs. (9) and (10) with the correct relative signs of the different terms. The actual coefficients can be determined by a detailed comparison. We see that the operator of dimension \(\Delta\) dual to the bulk scalar field, as well as the composite operator of dimension \(2\Delta\), mediate the CFT 4-point amplitude. We also find that the \(\log u\) term only comes with \(u^{2\Delta}\). (We find this to be true also of the terms in the five-point amplitudes that we have evaluated.) This is what one expects to find in this theory. On the external legs, we have the primary operator of dimension \(\Delta\) of the boundary CFT. Therefore, the condition [19, 27] for the appearance of a logarithm is met only when the dimension of the mediating operator is \(2\Delta\).

In summary, \(p\)-adic CFT holographically dual to a bulk scalar field theory, takes a rather simple form in Mellin space, reflecting the absence of the tower of secondary operators.

**Figure 7**: A few Witten diagrams contributing to five-point amplitude in CFT.

**5 Conclusions**

We have analyzed some aspects of CFTs on ultrametric spaces in the recently proposed holographic approach. The natural relation between the \(p\)-adic field (or its algebraic extensions) as the boundary of an infinite lattice, the so called Bruhat-Tits tree, as the bulk, is essential in this correspondence. In the simplest case, the bulk theory is described by the naturally discretized action of an interacting scalar field. Two-, three- and four-point correlators of the boundary CFT were already computed with bulk-to-boundary and bulk-to-bulk propagators, and the structural similarity of these propagators, as well as the resulting amplitudes, with the usual Archimedean CFTs were emphasised in Refs. [8, 9].

\(^4\)There is a notion of \(p\)-adic Mellin transformation that uses the map from \(Q_p\) to \(Q_p^*\). However, the scalar field is complex valued and since we are considering complex valued functions on ultrametric spaces, the Mellin transform here is the standard one.
However, in computing the four-point correlator, contributions from only the bulk four-scalar interaction was used. We have computed the exchange contributions from the bulk three-scalar interaction. Since there cannot be any other contribution to the CFT four-point amplitude, this gives the complete answer. Our results show that the fundamental operator of dimension $\Delta$ and a composite of dimension $2\Delta$ flow in the intermediate channels. The latter has an anomalous contribution to its dimension resulting in a term dependent on the logarithm of the cross ratio.

The singularities, when viewed in the Mellin variables conjugate to the cross ratios, result in simple poles at $\pm \Delta$ and $\pm 2\Delta$. Moreover, there are double poles at $\pm 2\Delta$ which accounts for the term with $u^{2\Delta} \log_p u$.

It is possible to compute higher-point amplitudes in CFT, although it gets rapidly complicated due to contributions from a large number of possible diagrams. For instance, a few of the many possible configurations of the boundary five-point correlator is shown in Fig. 7. We have chosen the boundary points $x_1, \cdots, x_5$ in such a way that the propagators from $x_1$ and $x_2$ to the interaction vertex at $c_1$ meet first at $c_1$. Similarly those joining $x_3$ and $x_4$ to $a_2$ meet at $c_2$. Finally, $x_5$ and the fields in the intermediate channel meet at the vertex $a_3$. In this process the path from $x_5$ meets that joining $c_1$ and $c_2$ at $c_3$. The five-point amplitude is a function of two cross ratios, which are given by the lengths of the paths $\langle c_1, c_3 \rangle$ and $\langle c_2, c_3 \rangle$. The decomposition of the propagators and the counting proceeds in the same way as for the four-point correlator explained in Section 3. As an example, let us consider the diagram in the left-most panel in Fig. 7, and take the $b$-nodes strictly between the vertices $c_1$ and $c_3$. In this configuration, the contribution of the Witten diagram can be decomposed as follows

$$g_3^2 \left\{ K(x_1, c_1)K(x_2, c_1)K(x_3, c_2)K(x_4, c_2)K(x_5, c_3) \right\} G^2(c_2, c_3) \times \sum_{a_1, a_2, a_3, b_1, b_2, b_3} \left( G^2(c_1, b_1)G(b_1, b_3)G^2(b_2, c_3) \right) \left( G(b_1, a_1)G(b_2, a_2)G(b_3, a_3) \right)^3. \quad (13)$$

The contribution from the sum of propagators in I does not depend on the cross ratios yielding factors depending only on $p$ and $\Delta$. This factor, for this diagram, and its equivalent for the three diagrams in Fig. 7, which are, respectively,

\begin{align*}
I(A) &= -p^{-4\Delta} \zeta_p^2(\Delta) \left( \zeta_p(\Delta) - p^{-\Delta} \zeta_p(2\Delta) \right) u_1^\Delta - \zeta_p^2(\Delta) \zeta_p(2\Delta) u_1^{3\Delta} \\
&\quad - \zeta_p^2(\Delta) \left( \log_p u_1 + p^{-\Delta} \left( p^{-\Delta} \frac{\zeta_p(2\Delta)}{\zeta_p(\Delta)} - \zeta_p(\Delta) + 3 \right) \right) u_1^{2\Delta} \\
I(B) &= -p^{-\Delta} \zeta_p(\Delta) \left( \log_p u_1 + 2 + p^{-\Delta} \zeta_p(\Delta) \right) u_1^{2\Delta} + \zeta_p^2(\Delta) u_1^{3\Delta} \\
I(C) &= p^{-\Delta} \zeta_p(\Delta) u_1^{3\Delta} - \zeta_p(\Delta) u_1^{\Delta}, \\
\end{align*}

where $u_1 = p^{-d(c_1, c_3)}$ and $u_2 = p^{-d(c_2, c_3)}$ are the two cross ratios, (but the dependence on $G^2(c_2, c_3) = u_2^{2\Delta}$ is not part of I). Operators of dimensions $\Delta$ and $2\Delta$, the latter accompanied also by terms with $\log_p u_1$, can be seen to contribute to the intermediate channel. These operators mediate interactions in CFT and give a closed operator algebra.

Finally, we have argued that there is little qualitative difference between CFTs on $p$-adic space or its finite algebraic extensions, even though the latter are finite dimensional vector spaces over $\mathbb{Q}_p$. In either case, the bulk is a Bruhat-Tits tree on which paths joining nodes are unique. The restricted ultrametric geometry implies that the $n$-point correlator of the CFT is a function of $(n-3)$ cross ratios. In a sense, all these holographic CFTs are analogues of (Archimedean) $\text{adS}_2$/CFT$_1$. This is more apparent from its structure in the Mellin space. Be that as it may, the correspondence
between the infinite tree and the ultrametric spaces on its boundary is inherently holographic. It is, therefore, a natural setting where one may address unresolved issues in holography and hope to get some interesting answers.

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