Bipartite entanglement and the arrow of time

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Abstract

We provide a new perspective on the close relationship between entanglement and time. Our main focus is on bipartite entanglement, where this connection is foreshadowed both in the positive partial transpose criterion due to Peres [A. Peres, Phys. Rev. Lett., 77, 1413 (1996)] and in the classification of quantum within more general non-signalling bipartite correlations [M. Frembs and A. Döring, http://arxiv.org/abs/2204.11471]. Extracting the relevant common features, we identify a necessary and sufficient condition for bipartite entanglement in terms of a compatibility condition with respect to time orientations in local observable algebras, which express the dynamics in the respective subsystems. We discuss the relevance of the latter in the broader context of von Neumann algebras and the thermodynamical notion of time naturally arising within the latter.
I. INTRODUCTION

The connection between quantum entanglement and the arrow of time has been the subject of numerous research enterprises, some recent ones include [35, 41, 46, 50, 60]. Here, we mainly focus on bipartite entanglement. It has been surmised that the operation of partial transposition in the positive partial transpose criterion for bipartite entanglement due to Peres [47] (henceforth referred to as the ‘PPT criterion’\(^1\)) is related to time reversal [32]. Yet, this relationship seems not to have been made precise before.

A related area of research, where time unexpectedly enters the picture, is the classification of quantum from non-signalling bipartite correlations. More precisely, the present author and Andreas Döring have recently shown that quantum states are characterised by a compatibility condition with respect to time orientations—roughly, the unitary evolution—in local observable algebras [20].

We review the basics of and extract some key insights from these results in the following subsections. Building on those, in Sec. II A we prove a necessary and sufficient criterion for bipartite entanglement. In Sec. II B we show that this, too, can be recast as a compatibility condition with respect to time orientations. Our work opens up various directions for future research, including practical considerations of our entanglement criterion as well as its generalisation to von Neumann algebras [14] (see Sec. II C and Sec. III).

A. The PPT criterion

Peres noted that the operation of partial transposition transforms separable states into separable states. In turn, any bipartite quantum state \( \rho = \sum_{ij} c_{ij} \rho_{A,i} \otimes \rho_{B,j} \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B}) \) with \( c_{ij} \in \mathbb{C}, \rho_{A,i} \in \mathcal{S}(\mathcal{A}), \rho_{B,j} \in \mathcal{S}(\mathcal{B}) \), whose partial transpose \( \rho^{T_A} := \sum_{ij} c_{ij} \rho_{A,i}^{T} \otimes \rho_{B,j} \) has at least one negative eigenvalue, is necessarily entangled [47]. Throughout, we write \( \mathcal{A} = \mathcal{L}(\mathcal{H}_A) \) for \( \dim(\mathcal{H}_A) \) finite. The partial-transpose criterion is necessary and sufficient in low dimensions, \( \dim(\mathcal{H}_A) = 2 \) and \( \dim(\mathcal{H}_B) = 2, 3 \), but is merely sufficient in higher dimensions [30]. Driven mainly by practical considerations the result has been sharpened in various ways (see [32] and references therein). This development, while rich and still active, has overshadowed the physical significance of Peres’ insight. In contrast, here we will only be concerned with the

\(^1\) Sometimes the criterion is also referred to as the Peres-Horodecki criterion [30, 47].

\(^2\) We will identify states \( \sigma \in \mathcal{S}(\mathcal{A}) \) on \( \mathcal{A} \) with their respective density matrices via \( \sigma(a) = \text{tr}[\rho a] \) for all \( a \in \mathcal{A} \).
conceptual importance, leaving the practical value of our work for future study.

**Pure and purified mixed states.** We recall the following simple fact.

**Proposition 1.** The PPT criterion is necessary and sufficient for pure bipartite states.

**Proof.** Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and consider the Schmidt decomposition $|\psi\rangle = \sum_i \alpha_i |i\rangle$, $\alpha_i \in \mathbb{R}_+$, with density matrix $\rho_\psi = |\psi\rangle\langle \psi| = \sum_{ij} \alpha_i \alpha_j |i\rangle \otimes |j\rangle \in \mathcal{S}(A \otimes B)$. Note that $\rho_\psi$ is separable if and only if the sum in the Schmidt decomposition collapses to a single term. Applying partial transposition on system $A$ we obtain $\rho_\psi^{TA} = \sum_{ij} \alpha_i \alpha_j |j\rangle \otimes |i\rangle$. It is easy to see that this operator has a negative eigenvalue for every pair of non-zero coefficients $\alpha_i, \alpha_j \neq 0$. 

This is of course well known. The following observation is equally straightforward: we can apply Prop. 1 to any bipartite state by considering purifications. What is more, a version of the Schrödinger-HJW theorem assures independence of the choice of purification [33, 53] (see also [37]). The PPT criterion thus becomes necessary and sufficient with respect to purifications. Of course, for pure states there are easier ways to check whether a state is entangled or separable. Nevertheless, as we will see below the criterion works because it works on the level of purifications. This is best expressed in terms of channels.

**Transposition vs Hermitian adjoint.** Note that transposition reverses the order of matrix multiplication: let $a \in M_{k \times l}(\mathbb{C})$, $b \in M_{l \times m}(\mathbb{C})$, then

$$(ab)_{ki}^T = (ab)_{ik} = \sum_{j=1}^l a_{ij} b_{jk} = \sum_{j=1}^l (b_{kj})^T (a_{ji})^T = (b^T a^T)_{ki} .$$

This fact is somewhat left implicit from the perspective of bipartite states $\rho \in \mathcal{S}(A \otimes B)$. In order to make it explicit, we identify a bipartite state $\rho$ with its quantum channel $\phi_\rho : A \rightarrow B$ under the Choi-Jamiołkowski isomorphism [9, 34] (see also, [19]): recall that every quantum channel $\phi : A \rightarrow B$, i.e., every completely positive linear map, determines a bipartite state $\rho_\phi$ (up to normalisation) by

$$\rho_\phi = \sum_{ij} E_{ij} \otimes \phi(E_{ij}) .$$

where $E_{ij}$ is the matrix with 1 in the entry $(i,j)$ and 0 elsewhere.

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3 For instance, note that we have used the Schmidt rank in Prop.

4 Sometimes this is written as $\rho_\phi = (\text{id} \otimes \phi)(|\Phi\rangle\langle \Phi|)$, where $|\Phi\rangle = \sum_i |i\rangle \otimes |i\rangle$ is a maximally mixed state.
Conversely, every bipartite state $\rho \in S(A \otimes B)$ corresponds to the quantum channel

$$\phi_\rho(a) = \text{tr}_{\mathcal{H}_A}[\rho(a^T \otimes 1_B)] \quad (2)$$

Clearly, with respect to the choice of basis in Eq. (2) we have $(E_{ij})^T = E_{ij}^*$. This allows us to replace transposition with the (Hermitian) adjoint.

**Lemma 1.** Let $\rho \in S(A \otimes B)$, let $\phi_\rho$ be the map under the linear isomorphism in Eq. (2), and let $(\Phi_\rho, v, K)$ be a Stinespring dilation of $\phi_\rho$, i.e., $\phi_\rho = v^* \Phi_\rho v$ with $v : \mathcal{H}_B \to K$ linear and $\Phi_\rho : A \to \mathcal{B}(K)$ a $C^*$-homomorphism [56]. Then $\phi_{\rho^T_A} = \phi_\rho^* = v^* \Phi_\rho^* v$.

**Proof.** We have

$$\sum_{i,j} E_{ij} \otimes \phi_{\rho^T_A}(E_{ij}) = \rho_{T_A} = (\rho^*)_{T_A} = \sum_{i,j} E_{ij} \otimes \phi_\rho^*(E_{ij}) = \sum_{i,j} E_{ij} \otimes v^* \Phi_\rho^*(E_{ij}) v,$$

where we used Eq. (1) in the first and third step and $\rho^* = \rho$ in the second. \hfill \Box

Partial transposition therefore assumes a more natural interpretation in terms of the adjoint operation on the local system $\mathcal{B}$ (see also [19]). In Sec. II C we will see that this encodes a difference between time orientations on the system $\mathcal{B}$. The latter also play a crucial role in selecting quantum from more general non-signalling bipartite correlations [20].

**B. Quantum from non-signalling correlations**

It is instructive to view the problem of entanglement classification from the broader perspective of classifying quantum from non-signalling correlations. In general, non-signalling distributions are far from being quantum [49]. Considering product quantum observables, a Gleason-type argument restricts non-signalling bipartite correlations to normalised linear functionals that are positive on pure tensors (POPT), yet not necessarily positive [8, 21, 38, 63]. To further single out quantum correlations among the latter, various additional physical principles have been proposed, see e.g. [48]. While successful in some instances, none has been shown to recover the quantum state space in general [44].

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5 There are different versions of this isomorphism; for a detailed discussion, see [19].

6 Notably, this is different than so-called co-positive maps, i.e., maps $\phi^T := T_B \circ \phi$, where $\phi : A \to \mathcal{B}$ is a completely positive map: by a similar computation we obtain $\phi_{\rho^T_A} = \phi_\rho^T$, yet, $\phi_\rho^T \neq \phi_\rho^T v$ in general.

7 We remark that $\phi^* := * \circ \phi$ denotes the adjoint of the image of the channel $\phi$, not its (Heisenberg) dual.
Recently, a classification of quantum states from more general non-signalling bipartite correlations has been obtained: quantum states correspond with those correlations, which satisfy (i) an extension of the no-signalling principle to dilations, and (ii) a relative consistency condition between the canonical unitary evolution in the respective subsystems (see Def. 1, Thm. 2 and Def. 2, Thm. 3 in [20]). Correlations under (i) (but not necessarily (ii)) correspond with decomposable maps under the Choi-Jamiołkowski isomorphism in Eq. (2).

Recall that a linear map $\phi: A \rightarrow B(H)$ is decomposable if there exists a Hilbert space $K$, a bounded linear operator $v: H \rightarrow K$, and a Jordan $*$-homomorphism $\Phi$, i.e., $\Phi(aa' + a'a) = \Phi(a)\Phi(a') + \Phi(a')\Phi(a)$ and $\Phi^*(a) = \Phi(a^*)$ for all $a, a' \in A$ (for details, see Sec. [11B]), such that $\phi = v^*\Phi v$. Such maps are more general than quantum channels $\phi: A \rightarrow B(H)$, which are of similar form: $\phi = v^*\Phi v$ with $\Phi$ a $C^*$-homomorphism. By Stinespring’s theorem [56], the latter is equivalent to $\phi$ being completely positive: if $x_{ij} \in M_n(A)_+ = (M_n(\mathbb{C}) \otimes A)_+$, then $\phi(x_{ij}) := \text{id}_{M_n(\mathbb{C})} \otimes \phi(x_{ij}) \in M_n(B(H))_+$. Similarly, decomposable maps can be characterised by a weaker positivity condition [58]: if $x_{ij} \in M_n(A)_+$ and $x_{ji} \in M_n(A)_+$, then $\phi(x_{ij}) \in M_n(B(H))_+$. Let $S_D(A \otimes B)$ denote the class of bipartite states corresponding to decomposable maps under the Choi-Jamiołkowski isomorphism. Interestingly, (the weaker positivity condition in) $S_D(A \otimes B)$ is preserved under partial transposition (see also [19]).

**Proposition 2.** Let $\rho \in S_D(A \otimes B)$, i.e., $\rho$ corresponds to a decomposable map under the Choi-Jamiołkowski isomorphism in Eq. (1). Then $\rho^{T_A} \in S_D(A \otimes B)$.

**Proof.** By a similar argument to the one in Lm. 1, $\phi_{\rho^{T_A}} = \phi^*_\rho = v^*\Phi^*_\rho v$, where $\Phi_\rho$ is a Jordan $*$-homomorphism, hence, $\Phi^*_\rho := \ast \circ \Phi_\rho = \Phi_\rho \circ \ast$. But then so is $\Phi^*_\rho$: for all $a_1, a_2 \in A$,

$$\Phi^*_\rho(\{a_1, a_2\}) = \Phi_\rho(\{a_1, a_2\}^*) = \Phi_\rho(\{a_1^+, a_2^+\}) = \{\Phi_\rho(a_1^+), \Phi_\rho(a_2^+)\} = \{\Phi^*_\rho(a_1), \Phi^*_\rho(a_2)\}.$$ 

Consequently, $\phi_{\rho^{T_A}}$ is decomposable and $\rho^{T_A} \in S_D(A \otimes B)$. \qed

Summarising this motivational prologue, in Sec. [A] we remarked that the PPT criterion becomes necessary and sufficient when applied topurifications, and used the Choi-Jamiołkowski isomorphism to translate the criterion from bipartite states to bipartite channels. In particular, we recast partial transposition into the (Hermitian) adjoint in Lm. [1] 

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8 We remark that for $\dim(H_A) = 2$ and $\dim(H_B) = 2, 3$ every positive map $\phi: A \rightarrow B$ is decomposable [57, 65], which implies necessity of the PPT criterion in those dimensions [30] (see also Thm. 2 below).
which further suggests to lift the PPT criterion from the level of bipartite quantum channels \( \phi \) to \( C^* \)-homomorphisms \( \Phi \) in Stinespring dilations \( \phi = v^* \Phi v \).

More generally, in Sec. [I.B] we considered the PPT criterion with respect to dilations of decomposable maps. Prop. [2] shows that the partial transpose preserves the respective positivity condition of linear functionals corresponding to such maps. We deduce that the PPT criterion is sensitive precisely to the difference between decomposable and completely positive maps. In the context of classifying quantum from non-signalling bipartite correlations, this is achieved by enforcing a compatibility condition with respect to the relative time orientation between systems \( A \) and \( B \) [20]. Building on this motivation, in the next section we identify a necessary and sufficient condition for bipartite separability in terms of a compatibility condition with respect to different time orientations between \( A \) and \( B \).

II. ENTANGLEMENT AND THE ARROW OF TIME

In this main part, we combine the insights gained in previous sections. In Sec. [II.A] we identify a necessary and sufficient criterion for bipartite entanglement (Thm. [1]). In Sec. [II.B] we employ the structure theory of Jordan and (associative) \( C^* \)-algebras to translate this criterion into a compatibility condition between canonical time orientations on local subsystems (Thm. [2]). Finally, in Sec. [II.C] we interpret our results in light of the intrinsic flow of time in von Neumann algebras by means of Tomita-Takesaki theory, Connes cocyles, and the background-independent thermodynamical arrow of time [14].

A. A necessary and sufficient criterion for bipartite entanglement

The PPT criterion translates between bipartite states and bipartite channels as follows:

\[
\begin{align*}
\rho^{T_A} \text{ positive} & \quad \text{Choi's theorem} \quad \phi \rho^{T_A} \text{ completely positive} \\
& \quad \text{Lm. [1]} \quad \phi^* \rho \text{ completely positive} \\
& \quad \text{Stinespring's theorem} \quad \phi^*_\rho = (v')^* \Phi_{v'} v', \text{ where } (v', \Phi_{v'}, K') \text{ is a Stinespring dilation}
\end{align*}
\]

Now since \( \phi^* \rho \) is completely positive, it also has a Stinespring dilation \( \phi^*_\rho = v^* \Phi_{v'} v \). We may thus strengthen the PPT criterion as follows: rather than \( \phi^*_\rho \) admitting any Stinespring dilation, we ask when \( v' = v, \Phi_{v'} = \Phi^*_\rho, \text{ i.e., when } \Phi^*_\rho \text{ is a } C^* \text{-homomorphism.} \)
Note first that this condition does not depend on the choice of Stinespring dilation.

**Lemma 2.** Let $\phi = v_1\Phi_1v_1^* = v_2\Phi_2v_2^*$ be two Stinespring dilations of $\phi$. Then $\Phi_1 : A \to \mathcal{B}(K_1)$ is a $C^*$-homomorphism if and only if $\Phi_2^* : A \to \mathcal{B}(K_2)$ is a $C^*$-homomorphism.

**Proof.** There is a partial isometry $W : K_1 \to K_2$ defined by $W\Phi_1v_1|\psi\rangle = \Phi_2v_2|\psi\rangle$ for all $|\psi\rangle \in \mathcal{H}_B$ such that $\Phi_1 = W^*\Phi_2W$. Hence, $\Phi_1 = W^*\Phi_2W$ and the claim follows.

Next, we have the following important characterisation.

**Lemma 3.** Let $\Phi : A \to \mathcal{B}(K)$ be a $C^*$-homomorphism. Then $\Phi^* : A \to \mathcal{B}(K)$ is a $C^*$-homomorphism if and only if $\Phi(A) \subset \mathcal{B}(K)$ is a commutative subalgebra, equivalently, $\Phi = \Phi|_V$ for $V \subset A$ a commutative subalgebra.

**Proof.** If $\Phi^*$ is a $C^*$-homomorphism, then for all $a_1, a_2 \in A$

$$\Phi(a_1)\Phi(a_2) = \Phi(a_1a_2) = \Phi^*((a_1a_2)^*) = \Phi^*(a_2^*a_1^*) = \Phi^*(a_2^*)\Phi^*(a_1^*) = \Phi(a_2)\Phi(a_1),$$

hence, $\Phi(A) \subset \mathcal{B}(K)$ is a commutative subalgebra. Conversely, if $\Phi(A) \subset \mathcal{B}(K)$ is a commutative subalgebra, then for all $a_1^*, a_2^* \in A$

$$\Phi^*(a_2^*a_1^*) = \Phi^*((a_1a_2)^*) = \Phi(a_1a_2) = \Phi(a_1)\Phi(a_2) = \Phi(a_2)\Phi(a_1) = \Phi^*(a_2^*)\Phi^*(a_1^*).$$

Hence, $\Phi^*$ is a $C^*$-homomorphism.

Since $\Phi(A) \subset \mathcal{B}(K)$ is a commutative subalgebra, there exists a maximal commutative subalgebra $V \subset A$ such that $\Phi(V) = \Phi(A)$. We show that $\Phi(a) = 0$ for all $a \perp V$ (with respect to the Hilbert-Schmidt inner product $(a, a') := \text{tr}[a^*a]$). Without loss of generality, we may assume that $V$ is the commutative subalgebra generated by diagonal matrices. We want to show that $\Phi(a) = 0$ for all $a \in V^\perp$. The latter implies $\text{tr}[a] = 0$, hence, $a = [b, c]$ for some $b, c \in A$ [54]. We have $\Phi(a) = \Phi([b, c]) = [\Phi(b), \Phi(c)] = 0$, since $\Phi$ is a homomorphism and $\Phi(A)$ is commutative. Consequently, $\Phi$ acts non-trivially only on $V$ and is zero otherwise.

The following key result shows that this property is equivalent to separability.

**Theorem 1.** Let $\rho \in \mathcal{S}(A \otimes B)$, let $\phi_\rho$ be the map under the isomorphism in Eq. (2) and let $\phi_\rho = v^*\Phi_\rho v$ be a Stinespring dilation of $\phi_\rho$. Then $\rho$ is separable if and only if $\phi_\rho = v^*\Phi_\rho^* v$ is a Stinespring dilation of $\phi_\rho^*$, i.e., if and only if $\Phi_\rho^*$ is a $C^*$-homomorphism.

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9 Note that for minimal Stinespring dilations $W$ is unitary.
10 It is interesting to note that Bell’s theorem holds for states over $C^*$-algebras as long as one of them is commutative [4]. We discuss the relation with Bell’s theorem and Bell nonlocality elsewhere [18].
For a general Stinespring dilation $\phi_\rho = v^* \Phi_\rho v$ to be of the following simple form: $v : \mathcal{H}_B \to \mathcal{K}$ for $\mathcal{K} = \mathcal{H}_B \otimes \mathcal{H}_A$, and $\Phi_\rho = 1_B \otimes \text{id}_{\text{supp}(\Phi_\rho)}$. In fact, if $\Phi_\rho^*$ is a $C^*$-homomorphism, then $\phi_\rho^{\tau_A} = \phi_\rho^* = v^* \Phi_\rho^* v$ is a Stinespring dilation (see Lm. 11). By Lm. 3 $\Phi_\rho(A) \in \mathcal{B}(\mathcal{K})$ is a commutative subalgebra and there exists a commutative subalgebra $V \subset A$ such that $\Phi_\rho(V) = \Phi_\rho(A) \subset \mathcal{B}(\mathcal{K})$. We thus have $\Phi_\rho(a) = \Phi_\rho|_V(a) = 1_B \otimes p_V a p_V$, where $p_V$ denotes the projection onto $V$.

Next, let $(|\xi_k\rangle)_k$ be a basis of $\mathcal{H}_A$ such that $|\xi_k\rangle\langle\xi_k| = p_k$ for all one-dimensional projections $p_k \in \mathcal{P}_1(V)$, and let $(|e_i\rangle)_i$ be an orthonormal basis of $\mathcal{H}_B$. We can decompose $v : \mathcal{H}_B \to \mathcal{K}$ by its action on basis states, $v(|e_j\rangle) := \sum_{ij} c_{ij}^k |e_i\rangle \otimes |\xi_k\rangle$ with $c_{ij}^k \in \mathbb{C}$, hence, $v = \sum_{ijk} c_{ij}^k |e_i\rangle \otimes |\xi_k\rangle = \sum_k X_k \otimes |\xi_k\rangle$ with $X_k = \sum_{ij} c_{ij}^k |e_i\rangle \otimes |\xi_k\rangle$. Consequently, for all $a = \sum_r a_r p_r \in V$

$$\phi_\rho(a) = v^* \Phi_\rho|_V(a) v = v^*(1_B \otimes a) v = \sum_{kl} X_i^* X_k \langle \xi_i| \sum_r a_r p_r |\xi_k\rangle = \sum_k E_k \text{tr}_{\mathcal{H}_A}[|\xi_k\rangle\langle\xi_k| a] \text{.}$$

Clearly, $E_k = X_k^* X_k \in \mathbb{B}_+$, hence, (after normalisation $E_k \to E_k/\text{tr}[E_k]$), $|\xi_k\rangle\langle\xi_k| \to \text{tr}[E_k]|\xi_k\rangle\langle\xi_k|)$ $\phi_\rho$ is an entanglement-breaking channel, equivalently $\rho$ is a separable state.

Conversely, let $\rho$ be a separable state. Then $\phi_\rho$ is an entanglement-breaking channel, i.e., there exists states $E_k \in \mathcal{S}(\mathcal{B})$ and positive operators $F_k \geq 0$ such that

$$\phi_\rho(a) = \sum_{k=1}^K E_k \text{tr}_{\mathcal{H}_A}[F_k a] \text{.}$$

We may extend $F_k$ to a positive operator-valued measure (POVM) $(F_k)^{\mathcal{K}}$, $F_k \in \mathcal{A}_+$ by setting $F_0 := 1 - \sum_{k=1}^K F_k$ such that $\sum_{k=0}^K F_k = 1$. By Naimark’s theorem [43, 56], $F$ admits a dilation $F = \tilde{v}^* \pi \tilde{v}$, where $\tilde{v} : \mathcal{H}_A \to \tilde{\mathcal{A}}$ is a linear map and $(\pi_k)^{K_{0}} \in \mathcal{P}(\tilde{\mathcal{A}})$ a projection-valued measure (PVM). Consequently, $\phi_\rho(a) = \tilde{\phi}_\rho(a) - E_0 \text{tr}[F_0 a]$, where e.g. $E_0 \propto 1$ and

$$\tilde{\phi}_\rho(a) = \sum_{k=0}^K E_k \text{tr}_{\tilde{\mathcal{A}}}[\tilde{v}^* \pi_k \tilde{v} a] = \sum_{k=0}^K E_k \text{tr}_{\tilde{\mathcal{A}}}[\tilde{v}^* \pi_k \tilde{\phi}_\rho(a) \tilde{v}] = \sum_{k=0}^K E_k \text{tr}_{\tilde{\mathcal{A}}}[\pi_k \tilde{\phi}_\rho(a)] \text{.}$$

Here, $\tilde{\phi}_\rho : \mathcal{A} \to \tilde{\mathcal{A}} = \mathcal{B}(\tilde{\mathcal{A}})$, $\tilde{\phi}_\rho(a) := \tilde{v} a \tilde{v}^*$ is the natural embedding under the isometry $\tilde{v}$ (that is, $\tilde{v}^* \tilde{v} = 1$), and we used that $\tilde{\phi}_\rho \tilde{v} \tilde{v}^* = \tilde{\phi}_\rho$, where $\tilde{v} \tilde{v}^* \in \mathcal{P}(\tilde{\mathcal{A}})$, $(\tilde{v} \tilde{v}^*) \tilde{\mathcal{A}} \equiv \mathcal{A}$ in the last step. Note that $(a, a') := \text{tr}_{\mathcal{A}}[a^* a]$ defines an inner product on $\mathcal{A}$, It follows that we can

\begin{enumerate}
\item For a general Stinespring dilation, one needs $\mathcal{K} = \mathcal{H}_B \otimes \mathcal{H}_A \otimes \mathcal{H}_A$ (cf. 43, 56). However, as it turns out in the case that both $\Phi$ and $\Phi^*$ are both $C^*$-homomorphisms, $\mathcal{K}$ can be reduced by one factor of $\mathcal{H}_A$.
\item This representation allows to interpret a quantum channel $\phi$ as a coarse-grained unitary bipartite channel on the target and some unknown ancillary system, after tracing out the latter. In particular, generalised measurements can be understood as projective measurements on a larger system [45].
\end{enumerate}

\begin{enumerate}
\item In particular, $\rho^{\tau_A}$ is positive in this case.
\item Entanglement-breaking channels are also called measure-prepare channels or in Holevo form (see [29, 31]).
\item Clearly, $\rho^{\tau_A}$ is positive, equivalently $\phi_\rho^{\tau_A}$ is completely positive in this case.
\end{enumerate}
restrict the action of $\tilde{\Phi}_\rho$ to the pre-image of the commutative subalgebra $W := \langle \pi_k \rangle_{k=0}^K \subset \tilde{A}$, spanned by the projections $\pi_k$. More precisely, we define $\Phi_\rho : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ for $\mathcal{K} = \mathcal{H}_B \otimes \mathcal{H}_{\tilde{A}}$ by

$$\Phi_\rho(a) = \begin{cases} 1_B \otimes \tilde{\Phi}_\rho & \text{for all } a \in \tilde{\Phi}_\rho^{-1}(W) \\ 0 & \text{otherwise} \end{cases}.$$ 

Clearly, $\Phi_\rho$ is a $C^*$-homomorphism since $\tilde{\Phi}_\rho|_{\tilde{\Phi}_\rho^{-1}(W)}$ is. Moreover, $\Phi_\rho(\mathcal{A}) \subset \mathcal{B}(\mathcal{K})$ is a commutative subalgebra by construction, hence, $\Phi_\rho^*$ is a $C^*$-homomorphism by Lm. 3. Finally, we obtain Stinespring dilations $\phi_\rho = v^*\Phi_\rho v$ and $\phi_\rho^* = v^*\Phi_\rho^* v$ as before: define $v = \sum_{k=1}^K X_k \otimes |\xi_k\rangle$ with $X_k^* X_k = E_k$ and with $((|\xi_k\rangle)_k)_{k=0}^K$ the orthonormal basis of $\mathcal{H}_{\tilde{A}}$, corresponding to the commutative subalgebra $W \subset \tilde{A}$, i.e., $|\xi_k\rangle\langle \xi_k|$ = $\pi_k$.

We have used the fact that a state is separable if and only if its corresponding quantum channel in Eq. (2) is a measure-prepare channel \cite{31}. The latter requires a decomposition of $v$ as in Eq. (3). Clearly, such a decomposition exists for any commutative subalgebra $V \subset \mathcal{A}$. In turn, Thm. \ref{thm1} shows that such a decomposition exists for all of $\mathcal{A}$ if and only if $\Phi_\rho(\mathcal{A})$ is a commutative subalgebra.

Moreover, Thm. \ref{thm1} sheds new light on the reason why the PPT criterion is not necessary in general: let $\phi_\rho = v^*\Phi_\rho v$ be a Stinespring dilation and let $\phi_\rho^* = \phi_\rho|_V$ be completely positive, this does not imply that $\phi_\rho^* = v^*\Phi_\rho^* v$ is a Stinespring dilation for $\phi_\rho$.

We record the following corollary of Thm. \ref{thm1}.

**Corollary 1.** Let $\rho \in S(\mathcal{A} \otimes \mathcal{B})$, let $\phi_\rho$ the map under the isomorphism in Eq. (2), and let $\phi_\rho = v^*\Phi_\rho v$ be a Stinespring dilation. Then $\rho$ is separable if and only if $\phi_\rho = \phi_\rho|_V$ for a commutative subalgebra $V \subset \mathcal{A}$.

**Proof.** This follows immediately from Lm. \ref{lm2}, Thm. \ref{thm1} and Lm. \ref{lm3}.

We surmise that Thm. \ref{thm1}—especially in the form of Cor. \ref{cor1}—entails improvements of existing protocols for practical verification of entanglement, e.g. in the form of semi-definite linear programmes in \cite{15, 16}. We leave this as an exciting direction for future research. In the remainder, we focus on the physical content of Thm. \ref{thm1} in terms of the arrow of time.
B. Entanglement and time orientations

Comparing Prop. 2 with Thm. 1 it is natural to study the difference between Jordan $\ast$-homomorphisms and $C^\ast$-homomorphisms. To this end, we first review some basic facts about Jordan algebras and their dynamics in terms of one-parameter groups of automorphisms, before proving a reformulation of Thm. 1 in terms of local time orientations.

**Jordan algebras.** We recall that an abstract Jordan algebra $J$ is an algebra over a field with a product that satisfies $a \circ b = b \circ a$ and $(ab)(aa) = (a(b(aa))$ for all $a, b \in J$.\footnote{For an extensive study of Jordan algebras, see [42].} Given an associative algebra $A$ one obtains a Jordan algebra $J(A)$ by symmetrisation. If $J = J(A)$ for an associative algebra $A$, then the Jordan algebra is called special\footnote{The prototypical exceptional Jordan algebra is the so-called Albert algebra $H_3(O)$\footnote{Recall that a $C^\ast$-algebra is an involutive Banach algebra (closed in norm) satisfying the defining $C^\ast$-property, $\|x^\ast x\| = \|x\|^2$. A von Neumann algebra is a $C^\ast$-algebra closed in the weak operator topology.}. In particular, every $C^\ast$- (and von Neumann) algebra defines a JB(W) algebra: a JB(W) algebra is a (weakly closed) Jordan algebra that is also a Banach space ($\|a \circ b\| \leq \|a\| \cdot \|b\|$) such that $\|a^2\| = \|a\|^2 \leq \|a^2 + b^2\|$. For simplicity, here we only consider matrix algebras over the complex numbers, $A = M_n(C)$, $n \in \mathbb{N}$. In this case, the set of Hermitian matrices $H_n(C)$ under the anti-commutator $\{a, b\} := ab + ba$ defines a real Jordan algebra $J(A)_{sa} := (H_n(C), \{\cdot, \cdot\})$. We denote its complexification by $J(A) = J(A)_{sa} + iJ(A)_{sa}$.

Crucially, Jordan products are commutative. As such the Jordan algebra $J(A)_{sa}$ is the same as the Jordan algebra $J(A^{op})_{sa}$ of the opposite algebra $A^{op}$, i.e., the algebra obtained from $A$ by reversing the order of composition (matrix multiplication),

$$A := \{a \in A \mid \forall a_1, a_2 \in A : a_1 \cdot_+ a_2 = \frac{1}{2}\{a_1, a_2\} + \frac{1}{2}[a_1, a_2]\},$$

$$A^{op} := \{a \in A \mid \forall a_1, a_2 \in A : a_1 \cdot_- a_2 = \frac{1}{2}\{a_1, a_2\} - \frac{1}{2}[a_1, a_2]\}.$$ \hfill (4)

The difference between the associative algebras $A$ and $A^{op}$ is the anti-symmetric part or commutator. In order to extract from this a notion of time directionality, we relate commutators to (infinitesimal) symmetries of $J(A)_{sa}$.

**Time orientations.** Dynamics is naturally expressed in terms of one-parameter groups of Jordan automorphisms $\mathbb{R} \ni t \mapsto \Aut(J(A)_{sa})$. Recall that for $A = B(H_A)$ (in particular,
for $\mathcal{A} = M_n(\mathbb{C})$) every such one-parameter group is given by conjugation with a unitary or anti-unitary operator by Wigner’s theorem \[7, 64\]. In fact, Wigner’s theorem holds on the level of Jordan algebras \[18, 40\]. In $\mathcal{A}$, we obtain one-parameter groups of the form

$$e^{t \text{ad}(ia_1)}(a_2) = e^{ita_1} a_2 e^{-ita_1} \quad \forall t \in \mathbb{R}, a_1, a_2 \in \mathcal{J}(\mathcal{A})_{\text{sa}}.$$ (5)

If we interpret $a_1$ as the Hamiltonian of the system, then Eq. (5) is just the standard expression for unitary evolution, in which $t$ plays the role of a time parameter. More generally, Eq. (5) defines a one-parameter group for every element $a \in \mathcal{J}(\mathcal{A})_{\text{sa}}$. In particular, note that for every $a \in \mathcal{J}(\mathcal{A})_{\text{sa}}$ and $\lambda \in \mathbb{R}^+$ also $\lambda a \in \mathcal{J}(\mathcal{A})_{\text{sa}}$. Hence, we cannot give physical meaning to the absolute value of $t$ without first specifying a Hamiltonian $a$.

Nevertheless, the sign of $t$ carries physical meaning independent of the choice of $a \in \mathcal{J}(\mathcal{A})_{\text{sa}}$. To see this, we remark that inherent in Eq. (5) is the canonical identification between self-adjoint operators (observables) and generators of Jordan automorphisms (symmetry generators), $a \mapsto \text{ad}(ia)$ for all $a \in \mathcal{J}(\mathcal{A})_{\text{sa}}$ \[3, 6, 22\]. Moreover, note that changing this identification to $a \mapsto \text{ad}(-ia)$ results in a sign change for the parameter $t$ in Eq. (5), equivalently to a change in the commutator and thus to a change in the order of composition from $\mathcal{A}$ to $\mathcal{A}^{\text{op}}$ in Eq. (4) (see also Lm. 4 below).

In contrast, in $\mathcal{J}(\mathcal{A})$ there is no canonical identification between self-adjoint operators and generators of Jordan automorphisms \[3, 27\]. Consequently, in $\mathcal{J}(\mathcal{A})$ we cannot interpret the sign of the parameter $t$ in the corresponding one-parameter groups independently of the choice of Hamiltonian $a \in \mathcal{J}(\mathcal{A})_{\text{sa}}$. By comparison, lifting $\mathcal{J}(\mathcal{A})$ to $\mathcal{A}$ thus equips the latter with an intrinsic direction of time, mediated by the identification $a \mapsto \text{ad}(ia)$ \[19\]. To emphasise this distinction, we define the canonical time orientation $\Psi_\mathcal{A}$ on $\mathcal{J}(\mathcal{A})$ by \[19\]

$$\Psi_\mathcal{A} := \text{Ad} : \mathbb{R} \times \mathcal{J}(\mathcal{A})_{\text{sa}} \ni (t, a) \mapsto e^{t \text{ad}(ia)},$$ (6)

and call $\mathcal{A}_+ := (\mathcal{J}(\mathcal{A}), \Psi_\mathcal{A})$ the observable Jordan algebra together with its canonical time orientation. Similarly, we define the reverse time orientation by

$$\Psi^*_\mathcal{A} := * \circ \Psi_\mathcal{A} : \mathbb{R} \times \mathcal{J}(\mathcal{A})_{\text{sa}} \ni (t, a) \mapsto e^{-t \text{ad}(ia)},$$ (7)

and set $\mathcal{A}_- := (\mathcal{J}(\mathcal{A}), \Psi^*_\mathcal{A})$ \[21\].

19 Generalising the mapping $a \mapsto \text{ad}(ia)$, \[2\] characterise those maps which lift $\text{JB}(W)$ to $C^*$ (von Neumann) algebras. By their physical interpretation, such maps are called dynamical correspondences.

20 The notion of time orientation was introduced in \[14, 20\].

21 By Thm. 23 in \[2\], these are the only time orientations on $\mathcal{J}(\mathcal{A})$, deriving from $\mathcal{A}$ and $\mathcal{A}^{\text{op}}$, respectively.
Entanglement and time orientation. Returning to Thm. 1 we are interested in the difference between Jordan $\ast$-homomorphism and $C^*$-homomorphism. Recall that a Jordan $\ast$-homomorphism $\Phi : \mathcal{J}(A) \to \mathcal{J}(B)$ is a linear map preserving the Hermitian adjoint, $\ast \circ \Phi = \Phi \circ \ast$, equivalently, $\Phi|_{\mathcal{J}(A)_{sa}} : \mathcal{J}(A)_{sa} \to \mathcal{J}(B)_{sa}$, and the Jordan product, i.e., $\Phi([a_1, a_2]) = \{\Phi(a_1), \Phi(a_2)\}$ for all $a_1, a_2 \in \mathcal{J}(A)_{sa}$. Consequently, $\Phi : \mathcal{J}(A) \to \mathcal{J}(B)$ lifts to a $C^*$-homomorphism $\Phi : A \to B$ if and only if it preserves commutators, $\Phi([a_1, a_2]) = [\Phi(a_1), \Phi(a_2)]$. Using Eq. (8), we re-express this condition in terms of one-parameter groups of Jordan automorphisms.

Lemma 4. Let $\Phi : \mathcal{J}(A) \to \mathcal{J}(B)$ be a Jordan $\ast$-homomorphism. Then $\Phi : A \to B$ lifts to a $C^*$-homomorphism if and only if it preserves the canonical time orientations $\Psi_A$ and $\Psi_B$,

$$\forall t \in \mathbb{R}, a \in \mathcal{J}(A)_{sa} : \Phi \circ \Psi_A(t, a) = \Psi_B(t, \Phi(a)) \circ \Phi.$$  \hspace{1cm} (8)

Proof. Clearly, a $C^*$-homomorphism $\Phi$ preserves Eq. (8). Conversely, by differentiation,

$$\frac{d}{dt} \bigg|_{t=0} (\Phi \circ \Psi_A(t, a_1))(a_2) = \frac{d}{dt} \bigg|_{t=0} (\Psi_B(t, \Phi(a_1)) \circ \Phi)(a_2)$$

$$\iff \Phi \left( \frac{d}{dt} \bigg|_{t=0} e^{it a_1} a_2 e^{-it a_1} \right) = \frac{d}{dt} \bigg|_{t=0} e^{it \Phi(a_1)} \Phi(a_2) e^{-it \Phi(a_1)}$$

$$\iff \Phi([a_1, a_2]) = [\Phi(a_1), \Phi(a_2)].$$

for all $a_1, a_2 \in \mathcal{J}(A)_{sa}$. $\Phi$ thus preserves commutators, hence, is a $C^*$-homomorphism. 

Assume $\Phi : A \to B(K)$ in Eq. (8) is part of a Stinespring dilation $\phi_\rho = v^* \Phi v$ for the image of a bipartite state $\rho \in \mathcal{S}(A \otimes B)$ under the isomorphism in Eq. (2). Since $B$ arises from $B(K)$ by restriction under $v$, the time orientation $\Psi_B$ on $B$ uniquely lifts to a time-orientation $\Psi'_B$ on $B(K)$.

This motivates the following definition, which first appeared in the context of classifying quantum states from non-signalling bipartite correlations [20] (see also Sec. 1B).

Definition 1. Let $\rho \in \mathcal{S}(A \otimes B)$. $\rho$ is called time-oriented with respect to $A_+ = (\mathcal{J}(A), \Psi_A^\ast)$ and $B_+ = (\mathcal{J}(B), \Psi_B)$ if and only if $\Phi_\rho : A \to B(K)$ in $\phi_\rho = v^* \Phi_\rho v$ preserves time orientations $\Psi_A^\ast = \ast \circ \Psi_A$ and $\Psi'_B$,

$$\forall t \in \mathbb{R}, a \in \mathcal{J}(A)_{sa} : \Phi_\rho \circ \Psi_A^\ast(t, a) = \Psi'_B(t, \Phi(a)) \circ \Phi_\rho.$$
We remark that the appearance of the reverse time orientation $\Psi^*_A$ in Def. 1 is a consequence of the identification of bipartite quantum states and quantum channels via Choi’s theorem (for more details, see [19]). Def. 1 is the missing piece of physical data to identify bipartite non-signalling distributions with quantum states [20]. Together, this shows that quantum states encode information about the relative time orientation between subsystems.

This is a genuine quantum effect. What is more, it is intimately related with entanglement: in fact, Def. 1 allows us to reformulate the separability criterion in Thm. 1 in terms of time orientations.

**Theorem 2.** A bipartite state $\rho \in S(A \otimes B)$ is separable if and only if it is time-oriented with respect to $A_-(\mathcal{J}(A), \Psi^*_A)$ and $B_+(\mathcal{J}(B), \Psi^*_B)$ as well as $A_-$ and $B_-$.

**Proof.** By Thm. 1 $\rho$ is separable if and only if $\Phi_\rho$ and $\Phi^*_\rho$ are $C^*$-homomorphisms for any Stinespring dilation $\phi_\rho = v^*\Phi_\rho v$. Since $C^*$-homomorphisms preserve time orientations by Lm. 1 $\rho$ is time-oriented with respect to both $A_-$ and $B_+$ as well as $A_-$ and $B_-$.22

Conversely, $\rho$ is time-oriented with respect to $A_-$ and $B_+$ by Thm. 3 in [20], i.e.,

$$\forall t \in \mathbb{R}, a \in \mathcal{J}(A)_{sa} : \Phi_\rho \circ \Psi^*_A(t,a) = \Phi_\rho \circ \Psi_A(-t,a) = \Psi^*_B(t,\Phi(a)) \circ \Phi_\rho . \quad (9)$$

where $\Psi^*_A(t,a) = \ast \circ \Psi_A(t,a) = \Psi_A(-t,a)$ by Eq. (7). In particular, $\Phi_\rho$ in $\phi_\rho = v^*\Phi_\rho v$ is a $C^*$-homomorphism [50]. If $\rho$ is also time-oriented with respect to $A_-$ and $B_+$, then by Def. 1

$$\forall t \in \mathbb{R}, a \in \mathcal{J}(A)_{sa} : \Phi_\rho \circ \Psi_A^*(t,a) = \Psi^*_B(t,\Phi(a)) \circ \Phi_\rho \quad (10)$$

Differentiating Eq. (9) and Eq. (10) yields $[\Phi_\rho(a_1), \Phi_\rho(a_2)] = -[\Phi^*_\rho(a_1), \Phi^*_\rho(a_2)] = 0$ for all $a_1, a_2 \in \mathcal{J}(A)_{sa}$ (cf. Lm. 1). It follows that $\Phi_\rho(A) \subset B$ is a commutative subalgebra, by Lm. 3 $\Phi^*_\rho$ is therefore a $C^*$-algebra homomorphism and by Thm. 1 $\rho$ is separable. \hfill \square

**C. Time orientations and the arrow of time**

In this section, we embed the classification of bipartite entanglement in terms of compatibility with time orientations in local observable algebras (Thm. 2) into a wider context.

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22 Note that since $\Phi_\rho : A \to B(K)$ is Hermiticity-preserving, i.e., $\Phi^*_\rho(a) = \Phi_\rho(a^*)$ for all $a \in A$, $\Phi_\rho$ preserves time orientations $\Psi_A$ and $\Psi_B$ if and only if it preserves time orientations $\Psi^*_A$ and $\Psi^*_B$; similarly $\Phi_\rho$ preserves time orientations $\Psi^*_A$ and $\Psi_B$ if and only if it preserves time orientations $\Psi_A$ and $\Psi^*_B$.\footnote{Note that since $\Phi_\rho : A \to B(K)$ is Hermiticity-preserving, i.e., $\Phi^*_\rho(a) = \Phi_\rho(a^*)$ for all $a \in A$, $\Phi_\rho$ preserves time orientations $\Psi_A$ and $\Psi_B$ if and only if it preserves time orientations $\Psi^*_A$ and $\Psi^*_B$; similarly $\Phi_\rho$ preserves time orientations $\Psi^*_A$ and $\Psi_B$ if and only if it preserves time orientations $\Psi_A$ and $\Psi^*_B$.}
We especially focus on time orientations as a complex structure on $\mathcal{J}(\mathcal{A})$, as well as their role within the intrinsic, thermodynamic arrow of time in von Neumann algebras.

**Time orientations and complex structure.** Following [17, 20], we have expressed the difference between $\mathcal{J}(\mathcal{A})$ and $\mathcal{A}$ in terms of time orientations in Def. 1. As exponentials of dynamical correspondences [2], time orientations highlight the double role played by self-adjoint operators: as observables and generators of dynamics in quantum mechanics [3, 22]. This perspective has some appeal when considering axiomatic reconstructions of quantum mechanics and possible generalisations they suggest.

For instance, note that a ‘quantum formalism’ can be defined also over the real instead of the complex numbers (see e.g. [28, 59]). More generally, several results aiming to reconstruct quantum mechanics arrive at the level of (special) Jordan algebras corresponding to associative algebras over the real, complex and quaterionic numbers (e.g. [22, 36, 39, 55]). In this context, it is interesting to note that time orientations define a complex structure on (the order derivations of) $\mathcal{J}(\mathcal{A})$ [2, 11]. Compare this with Eq. (6), where we used the complex structure of the associative algebra $\mathcal{A}$ implicitly to define the canonical time orientation $\Psi_{\mathcal{A}}$. In this way, dynamical correspondences can be seen as a justification for the prominence of complex numbers in quantum mechanics [5]. By Thm. 2 these arguments are further inherently connected with quantum entanglement.

**Outlook: intrinsic dynamics and thermal time.** One of the deepest insights into the emergence of time from purely algebraic considerations arises in infinite dimensions and the structure theory of (hyperfinite) von Neumann algebras [12, 13, 24, 25]. The latter heavily rests on the foundational insights by Tomita and Takesaki [61, 62].

Given a von Neumann algebra $\mathcal{N}$ and a faithful normal state $\omega \in \mathcal{S}(\mathcal{N})$, $\omega$ becomes a cyclic and separating vector $\Omega$ in its Gelfand-Naimark-Segal (GNS) representation. The operator defined by $S_{\omega}a\Omega := a^*\Omega$ for all $a \in \mathcal{N}$ is closable, hence, has a polar decomposition $S_{\omega} = J_{\omega}\Delta_{\omega}^{\frac{1}{2}}$, where $J_{\omega}$ is an anti-unitary involution and $\Delta_{\omega}$ is a self-adjoint, positive operator. The fundamental results of *Tomita-Takesaki theory* are summarised in the statements $J_{\omega}\mathcal{N}J_{\omega} = \mathcal{N}'$, where $\mathcal{N}'$ is the commutant of $\mathcal{N}$, and $\Delta_{\omega}^{it}\mathcal{N}\Delta_{\omega}^{-it} = \mathcal{N}$ for all $t \in \mathbb{R}$ [61]. The latter implies that every faithful normal state $\omega \in \mathcal{S}(\mathcal{N})$ defines a one-parameter group of automorphisms.

\[\text{For the intimate relationship between dynamical correspondences and Noether’s theorem, see [6].}\]
\( \sigma^\omega : \mathbb{R} \to \text{Aut}(\mathcal{N}), \sigma^\omega_t(a) \mapsto \Delta^i t a \Delta^{-i t} \), called the modular automorphism group of \( \omega \).

Crucially, \( S_\omega \) and thus \( \sigma^\omega \) are state-dependent since they are defined with respect to the support of the state \( \omega \). Despite this fact, the difference between \( \sigma^\omega_t \) and \( \sigma^\omega_{-t} \) is merely an inner automorphism \( \sigma^\omega_t(a) = u_t \sigma^\omega_{-t}(a) u_t^{-1} \) for all \( a \in \mathcal{N} \), where the unitaries \( (u_t)_{t \in \mathbb{R}} \) satisfy Connes’ cocycle condition \( \sigma^\omega_{s+t} = u_s \sigma^\omega_{s+t}(u_t) \) \( \Box \). As a consequence, \( \mathcal{N} \) carries an intrinsic, i.e., state-independent notion of dynamics, given by (the subgroup of) the automorphism group generated by the \( \sigma^\omega \). In contrast, one can also study the operators \( S_\omega \) in JBW algebras. However, without the existence of a dynamical correspondence (equivalently, time orientation), a JBW algebra cannot distinguish between the one-parameter families of automorphisms \( \sigma^\omega_t \) and \( \sigma^\omega_{-t} \) \( \Box \).

What is more, the intrinsic dynamics in von Neumann algebras is further exemplified in the study of statistical mechanics in a background-independent setting \( \Box \). Here, \( \omega \in \mathcal{S}(\mathcal{N}) \) is understood as a state in thermodynamic equilibrium. In the setting of quantum statistical mechanics such states are characterised by the KMS condition \( \Box \). It is a remarkable fact that \( \sigma^\omega \) satisfies the KMS condition for every faithful normal state \( \omega \in \mathcal{S}(\mathcal{N}) \) \( \Box \). In contrast, no analogue of this condition holds for Jordan algebras \( \Box \). In effect, time orientations in von Neumann algebras allow to interpret time from a thermodynamical standpoint, encoded in a state of thermodynamic equilibrium \( \Box \).

The crucial role played by time orientations (equivalently, dynamical correspondences) in von Neumann algebras and in Thm. \( \Box \) is hardly coincidental. In particular, it is tempting to ‘explain’ the intrinsic dynamics and (thermodynamic) origin of the arrow of time more fundamentally in terms of the entanglement structure of a given faithful normal state. To this end, one would like to generalise Thm. \( \Box \) to the setting of general von Neumann algebras. We leave this and similar considerations for future work.

III. CONCLUSION

We found a necessary and sufficient criterion for bipartite entanglement using Stinespring dilations in Thm. \( \Box \). The latter adopts a clear physical meaning in terms of a compatibility condition with respect to time orientations (Def. \( \Box \) on the respective local observable algebras in Thm. \( \Box \). Moreover, we highlighted the key role time orientations play within the broader picture of the intrinsic flow of time in von Neumann algebras.
More explicitly, our results are motivated from and bear close resemblance with the PPT criterion \[30, 47\]. As such, it would be interesting to study the practical relevance of Thm. 1. For example, it seems possible that existing results on marginal extension problems, e.g. \[15, 16\], can be strengthened using Cor. 1.

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[1] A. A. Albert, *On a certain algebra of quantum mechanics*, Annals of Mathematics, 35 (1934), pp. 65–73.

[2] E. M. Alfsen and F. W. Shultz, *On Orientation and Dynamics in Operator Algebras Part I*, Communications in Mathematical Physics, 194 (1998), pp. 87–108.

[3] E. M. Alfsen and F. W. Shultz, *Orientation in Operator Algebras*, Proc. Natl. Acad. Sci. U.S.A., 95 (1998), pp. 6596–6601.

[4] J. C. Baez, *Bell’s inequality for C∗-algebras*, Lett. Math. Phys., 13 (1987).

[5] ———, *Division algebras and quantum theory*, Foundations of Physics, 42 (2011), p. 819–855.

[6] J. C. Baez, *Getting to the bottom of Noether’s theorem*, (2020).

[7] V. Bargmann, *Note on Wigner’s theorem on symmetry operations*, Journal of Mathematical Physics, 5 (1964), pp. 862–868.

[8] H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner, *Local quantum measurement and no-signaling imply quantum correlations*, Phys. Rev. Lett., 104 (2010), p. 140401.

[9] M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra Its Appl., 10 (1975), pp. 285 – 290.

[10] A. Connes, *Une classification des facteurs de type III*, Annales scientifiques de l’École Normale Supérieure, 6 (1973), pp. 133–252.

[11] ———, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von neumann*, Ann. Inst. Fourier (Grenoble), 24 (1974), pp. 121–155.

[12] A. Connes, *A factor not anti-isomorphic to itself*, Ann. Math., 101 (1975), pp. 536–554.
A. Connes, *Factors of type III*$_1$, *property and closure of inner automorphism*, Journal of Operator Theory, (1985), pp. 189–211.

A. Connes and C. Rovelli, *Von Neumann algebra automorphisms and time-thermodynamics relation in generally covariant quantum theories*, Classical and Quantum Gravity, 11 (1994), pp. 2899–2917.

A. C. Doherty, P. Parrilo, and F. Spedalieri, *Distinguishing separable and entangled states*, Physical Review Letters, 88 (2002), p. 187904.

A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, *Complete family of separability criteria*, Physical Review A, 69 (2004), p. 022308.

A. Döring, *Two new complete invariants of von Neumann algebras*, ArXiv e-prints, (2014).

A. Döring and M. Frembs, *Contextuality and the fundamental theorems of quantum mechanics*, (2019).

M. Frembs, *Variations on the Choi-Jamiołkowski isomorphism. forthcoming*, 2022.

M. Frembs and A. Döring, *From no-signalling to quantum states*, 2022.

E. Grgin and A. Petersen, *Duality of observables and generators in classical and quantum mechanics*, Journal of Mathematical Physics, 15 (1974), pp. 764–769.

R. Haag, N. Hugenholtz, and M. Winnink, *On the equilibrium states in quantum statistical mechanics*, Communications in Mathematical Physics, 5 (1967), p. 21.

U. Haagerup, *Connes’s bicentralizer problem and uniqueness of the injective factor of type III*$_1$, Acta Mathematica, 158 (1987), pp. 95–148.

U. Haagerup, *On the uniqueness of injective III*$_1$ factors, 2016.

U. Haagerup and H. Hanche-Olsen, *Tomita-Takesaki theory for Jordan algebras*, Journal of Operator Theory, 11 (1984), pp. 343–364.

H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Monographs and studies in mathematics, Pitman Advanced Pub. Program, 1984.

L. Hardy and W. K. Wootters, *Limited holism and real-vector-space quantum theory*, Found. Phys., 42 (2011), p. 454–473.

A. S. Holevo, *Quantum coding theorems*, Russian Mathematical Surveys, 53 (1998), p. 1295–1331.
[30] M. Horodecki, P. Horodecki, and R. Horodecki, **Separability of mixed states: necessary and sufficient conditions**, Phys. Lett. A, 223 (1996), pp. 1–8.

[31] M. Horodecki, P. W. Shor, and M. B. Ruskai, **Entanglement breaking channels**, Reviews in Mathematical Physics, 15 (2003), pp. 629–641.

[32] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, **Quantum entanglement**, Rev. Mod. Phys., 81 (2009), pp. 865–942.

[33] L. P. Hughston, R. Jozsa, and W. K. Wootters, **A complete classification of quantum ensembles having a given density matrix**, Physics Letters A, 183 (1993), pp. 14–18.

[34] A. Jamiołkowski, **Linear transformations which preserve trace and positive semidefiniteness of operators**, Rep. Math. Phys., 3 (1972), pp. 275–278.

[35] D. Jennings and T. Rudolph, **Entanglement and the thermodynamic arrow of time**, Phys. Rev. E, 81 (2010), p. 061130.

[36] P. Jordan, J. v. Neumann, and E. Wigner, **On an algebraic generalization of the quantum mechanical formalism**, Ann. Math., 35 (1934), pp. 29–64.

[37] K. A. Kirkpatrick, **The Schrödinger-HJW theorem**, Foundations of Physics Letters, 19 (2003), pp. 95–102.

[38] M. Kläy, C. Randall, and D. Foulis, **Tensor products and probability weights**, Int. J. Theor. Phys., 26 (1987), pp. 199–219.

[39] M. Koecher, A. Krieg, and S. Walcher, **The Minnesota Notes on Jordan Algebras and their Applications**, no. no. 1710 in Lecture Notes in Mathematics, Springer, 1999.

[40] K. Landsman and B. Lindenhoiuis, **Symmetries in exact Bohrification**, in Reality and Measurement in Algebraic Quantum Theory, M. Ozawa, J. Butterfield, H. Halvorson, M. Rédei, Y. Kitajima, and F. Buscemi, eds., Singapore, 2018, Springer Singapore, pp. 97–118.

[41] J. Lin, M. Marcolli, H. Ooguri, and B. Stoica, **Locality of gravitational systems from entanglement of conformal field theories**, Phys. Rev. Lett., 114 (2015), p. 221601.

[42] K. McCrimmon, **A Taste of Jordan Algebras**, Springer, 2004.

[43] M. A. Naimark, **On a representation of additive operator set functions**, C. R. (Dokl.) Acad. Sci. URSS, n. Ser., 41 (1943), pp. 359–361.

[44] M. Navascués, Y. Guryanova, M. J. Hoban, and A. Ací, **Almost quantum correlations**, Nature communications, 6 (2015), p. 7.
[45] M. Ozawa, *Quantum measuring processes of continuous observables*, Journal of Mathematical Physics, 25 (1984), pp. 79–87.

[46] D. N. Page and W. K. Wootters, *Evolution without evolution: dynamics described by stationary observables*, Phys. Rev. D, 27 (1983), pp. 2885–2892.

[47] A. Peres, *Separability criterion for density matrices*, Phys. Rev. Lett., 77 (1996), pp. 1413–1415.

[48] S. Popescu, *Nonlocality beyond quantum mechanics*, Nature Phys., 10 (2014), pp. 264–270.

[49] S. Popescu and D. Rohrlich, *Quantum nonlocality as an axiom*, Found. Phys., 24 (1994), pp. 379–385.

[50] O. Racorean, *Quantum entanglement, two-sided spacetimes and the thermodynamic arrow of time*, 2019.

[51] C. Rovelli, *Statistical mechanics of gravity and the thermodynamical origin of time*, Classical and Quantum Gravity, 10 (1993), pp. 1549–1566.

[52] ——, *The statistical state of the universe*, Classical and Quantum Gravity, 10 (1993), pp. 1567–1578.

[53] E. Schrödinger, *Probability relations between separated systems*, Mathematical Proceedings of the Cambridge Philosophical Society, 32 (1936), p. 446–452.

[54] V. K. Shoda, *Einige Sätze über Matrizen*, Japanese journal of mathematics :transactions and abstracts, 13 (1936), pp. 361–365.

[55] M. P. Solèr, *Characterization of Hilbert spaces by orthomodular spaces*, Communications in Algebra, 23 (1995), pp. 219–243.

[56] W. F. Stinespring, *Positive functions on C*-algebras*, Proc. Am. Math. Soc., 6 (1955), pp. 211–216.

[57] E. Størmer, *Positive linear maps of operator algebras*, Acta Mathematica, 110 (1963), pp. 233 – 278.

[58] ——, *Decomposable positive maps on C*-algebras*, Proc. Am. Math. Soc., 86 (1982), pp. 402–402.

[59] E. C. Stueckelberg, *Quantum theory in real Hilbert space*, Helv. Phys. Acta, 33 (1960), p. 458.

[60] L. Susskind, *Copenhagen vs Everett, teleportation, and ER=EPR*, Fortschritte der Physik, 64 (2016), p. 14.
[61] M. Takesaki, *Tomita’s Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, 1970.

[62] ——, *Theory of Operator Algebras II*, Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2002.

[63] N. Wallach, *An Unentangled Gleason’s theorem*, Contemp. Math., 305 (2000).

[64] E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, Friedrich Vieweg und Sohn, Braunschweig, 1931.

[65] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Reports on Mathematical Physics, 10 (1976), pp. 165–183.