FANO SCHEMES FOR GENERIC SUMS OF PRODUCTS OF LINEAR FORMS

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Abstract. We study the Fano scheme of $k$-planes contained in the hypersurface cut out by a generic sum of products of linear forms. In particular, we show that under certain hypotheses, linear subspaces of sufficiently high dimension must be contained in a coordinate hyperplane. We use our results on these Fano schemes to obtain a lower bound for the product rank of a linear form. This provides a new lower bound for the product ranks of the $6 \times 6$ Pfaffian and $4 \times 4$ permanent, as well as giving a new proof that the product and tensor ranks of the $3 \times 3$ determinant equal five. Based on our results, we formulate several conjectures.

1. Introduction

Given an embedded projective variety $X \subset \mathbb{P}^n$, its Fano scheme $F_k(X)$ is the fine moduli space parametrizing projective $k$-planes contained in $X$. Such Fano schemes have been considered extensively for the case of sufficiently general hypersurfaces [AK77, BVdV79, Lan97] but less so for particular hypersurfaces [HMP98, Beh06, CI15]. In this article, we study the Fano schemes $F_k(X)$ for the special family of irreducible hypersurfaces $X = X_{r,d} = V \left( \sum_{i=1}^{r} \prod_{j=1}^{d} c_{ij} x_{ij} \right) \subset \mathbb{P}^{rd-1}$, $c_{ij} \in \mathbb{K}^*$ for any $r > 1, d > 2$. Up to projective equivalence these hypersurfaces do not depend on the choice of the $c_{ij} \in \mathbb{K}^*$. Hence, in the following we may assume that $c_{ij} = 1$ for all $1 \leq i \leq r$ and $1 \leq j \leq d$. Moreover, for $d > 2$ the hypersurfaces $X_{r,d}$ are always singular along a union of coordinate hyperplanes of codimension $2r$. We exclude the case $d = 2$ since this is a smooth quadric hypersurface with significantly different behaviour.

In [IT16, §3], Z. Teitler and the first author considered the Fano scheme $F_5(X_{4,3})$. With the help of a computer-assisted calculation, they observed the curious fact that every 5-plane $L$ of $X_{4,3}$ is either contained in a coordinate hyperplane, or there exist $1 \leq a < b \leq 4$ such that $L$ is contained in $V(x_{a_1}x_{a_2}x_{a_3} + x_{b_1}x_{b_2}x_{b_3})$. This motivates the following definition:

Definition 1.1 ($\lambda$-splitting). Consider $\lambda \in \mathbb{N}$. A $k$-plane $L$ contained in $X_{r,d}$ admits a $\lambda$-splitting if there exist $1 \leq a_1 < a_2 < \ldots < a_\lambda \leq r$ such that $L$ is contained in $V \left( \sum_{i=1}^{\lambda} \prod_{j=1}^{d} x_{a_{i,j}} \right) \subset \mathbb{P}^{rd-1}$.
We say that $F_k(X_{r,d})$ is $m$-split if every $k$-plane of $X_{r,d}$ admits a $\lambda$-splitting for some $\lambda \leq m$.

The above-mentioned observation from [IT16] can now be rephrased as the statement that $F_5(X_{4,3})$ is two-split.

We make two conjectures regarding the splitting behaviour of these Fano schemes:

**Conjecture 1.2** (One-Splitting). Assume $r \geq 2$ and $d \geq 3$. The Fano scheme $F_k(X_{r,d})$ is one-split if and only if

$$k \geq \begin{cases} \frac{r}{2} \cdot d & \text{r even} \\ \frac{r-1}{2} \cdot d + 1 & \text{r odd} \end{cases}.$$

**Conjecture 1.3** (Two-Splitting). Assume $r$ is even and $d \geq 3$. The Fano scheme $F_k(X_{r,d})$ is two-split if

$$k \geq \frac{r}{2} \cdot d - 1.$$

We show in Example 3.1 that the bound on $k$ of Conjecture 1.2 is indeed necessary for one-splitting. However, the sufficiency of the conditions of Conjectures 1.2 and 1.3 for one- and two-splitting is less obvious. Our belief in these conjectures is motivated by our Theorem 1.8 below and the connection with the property $C_d^m$ described below. It would be interesting to formulate conjectures characterizing $m$-splitting of $F_k(X_{r,d})$ in general (including the case $r$ odd and $m = 2$) but we don’t know what they would be. For our application below (Theorem 1.9), understanding the cases $m = 1$ and $m = 2$ suffices.

The following example illustrates the ideas we will use to attack Conjectures 1.2 and 1.3.

**Example 1.4** ($F_5(X_{4,3})$ is two-split). We are considering the hypersurface $X_{4,3}$ in $\mathbb{P}^{11}$, equipped with coordinates $x_{11}, \ldots, x_{43}$. Any 5-plane $L$ in $\mathbb{P}^{11}$ can be represented as the rowspan of a full rank $6 \times 12$ matrix $B = (b_{i,j})$, with rows indexed by $\alpha = 0, \ldots, 5$ and column $ij$ corresponding to the homogeneous coordinates $x_{ij}$ on $\mathbb{P}^{11}$. We define linear forms $y_{ij}$ in $\mathbb{K}[z_0, \ldots, z_5]$ by

$$y_{ij} = \sum_{\alpha} b_{\alpha,ij} z_{\alpha}$$

and note that $L \subset X_{4,3}$ if and only if the form

$$h := 4 \prod_{i=1}^{4} \prod_{j=1}^{3} y_{ij}$$

is equal to zero.

Since $B$ has full rank, we can assume that there is a submatrix $B'$ of $B$ consisting of six columns which form the identity matrix. Grouping the columns of $B$ into the 4 blocks whose indices $ij$ have the same $j$ value, we see that either:

1. There are two blocks which each contain at least two columns of $B'$. This implies $h = f_1z_0z_1 + f_2z_2z_3 - \ell_1 - \ell_2$, where $f_1, f_2$ are linear forms and $\ell_1, \ell_2$ are products of three linear forms. Or,

2. Every block contains at least one column of $B'$, and there is a block containing three columns of $B'$. This implies $h = z_0z_1z_2 + f_1z_3 + f_2z_4 + f_3z_5$. 
If \( L \subset X_{4,3} \) and hence \( h = 0 \), the second case cannot occur, since \( z_0 z_1 z_2 \) is not in the ideal generated by \( z_3, z_4, z_5 \). In the first case, the equation \( h = 0 \) translates to

\[
\ell_1 + \ell_2 = z_0 z_1 f_1 + z_2 z_3 f_2.
\]

We will see in \( \mathcal{P} \) that this is only possible if either one of \( f_1, f_2 \) vanishes (in which case \( L \) is one-split), or (after permuting indices) \( \ell_1 = z_0 z_1 f_1 \) (in which case \( L \) is two-split).

More generally, in our study of \( F_k(X_{r,d}) \) we are led to consider degree \( d > 1 \) homogeneous equations of the form

\[
(1) \quad \ell_1 + \ldots + \ell_m = \sum_{i=0}^m f_i x_i
\]

where the \( x_i \) are pairwise coprime squarefree monomials, the \( \ell_i \) are degree \( d \) products of linear forms (possibly equal to 0), and the \( f_i \) are degree \( d - \deg x_i \) products of linear forms in some polynomial ring (also possibly equal to 0). The following property will be essential in our analysis:

**Definition 1.5** (Property \( C_{d,m}^d \)). We say that \( C_{d,m}^d \) is true if, for any equation of the form (1) satisfying \( \deg x_i + \deg x_j \geq d + 2 \) for all \( i \neq j \), it follows that there is some \( i \) for which \( f_i = 0 \).

**Example 1.6.** Property \( C_1^2 \) is simply stating the obvious fact that for variables \( x_1, \ldots, x_4 \), the form \( x_1 x_2 + x_3 x_4 \) is not a product of linear forms. Property \( C_1^3 \) states the less-obvious fact that for variables \( x_1, \ldots, x_5 \) and non-zero linear form \( f \), the form \( x_1 x_2 x_3 + f x_4 x_5 \) is not a product of linear forms.

Our first main result relates the above definition to our two conjectures:

**Theorem 1.7.** Fix \( r \geq 2, d \geq 3 \) such that either

1. \( d \) is even,
2. \( r \) is even and \( d \geq r \), or
3. \( r \) is odd and \( r \leq 5 \).

Suppose that \( C_{d,m}^d \) is true for all

\[
m \leq \frac{r - 1}{2}.
\]

Then Conjectures 1.2 and 1.3 hold for this choice of \( (r, d) \).

Secondly, we use this to prove our conjectures in some special cases.

**Theorem 1.8.** Property \( C_{m}^d \) is true if \( m \leq 2 \) or if \( d = 4 \). Furthermore, Conjectures 1.2 and 1.3 hold if \( r \leq 6 \) or if \( d = 4 \).

Our analysis of Equation (1) makes use of relatively elementary methods. However, a more sophisticated approach should also be possible. Equation (1) posits that \( \sum_{i=0}^m f_i x_i \) is a point in the \((m-1)\)th secant variety of a Chow variety parametrizing degree \( d \) products of linear forms. Equations for the Chow variety are classical, going back to Brill and Gordan [GKZ08]. More recently, Y. Guan has provided some equations for secant varieties of Chow varieties [Gua15, Gua16]. It would be interesting to see if these equations shed light on the vanishing of the \( f_i \) from Equation (1).
Our motivation for studying $\mathbf{F}_k(X_{r,d})$ is twofold. Firstly, we wish to add to the body of examples of varieties $X$ for which one understands the geometry of $\mathbf{F}_k(X)$. If the Fano scheme $\mathbf{F}_k(X_{r,d})$ is $m$-split for some $m < r$, then $k$-dimensional linear subspaces of $X_{r,d}$ can be understood in terms of linear subspaces of $X_{r',d}$ for certain $r' < d$. We illustrate this by describing the irreducible components of $\mathbf{F}_k(X_{r,d})$ for $k \geq (r-2)(d-1)+1$ whenever $r \leq d+1$ or $d = 4$, see Examples [3,7] and [3,9]. We also characterize when $\mathbf{F}_k(X_{r,d})$ is connected, see Theorem 3.5.

Secondly, we may use our results to obtain lower bounds on the product rank of certain linear forms. Recall that the product rank (also known as Chow rank) of a degree $d$ form $f$ is the smallest number $r$ such that we can write

$$f = \ell_1 + \ldots + \ell_r$$

where the $\ell_i$ are products of $d$ linear forms. We denote the product rank of $f$ by $\text{pr}(f)$. Note that product rank may be used to give a lower bound on tensor rank, see [IT16, §1.3] for details.

A form $f$ in $n+1$ variables is concise if it cannot be written as a form in fewer variables after a linear change of coordinates. The hypersurface $V(f)$ of a concise degree $d$ form of product rank at most $r$ is isomorphic to the intersection of $X_{r,d} \subset \mathbb{P}^{rd-1}$ with an $n$-dimensional linear subspace. This allows us to relate properties concerning linear subspaces contained in $V(f)$ to product rank. Generalizing [IT16, Theorem 3.1], we prove the following:

**Theorem 1.9.** Let $f$ be a concise irreducible degree $d > 1$ form in $n+1$ variables such that $V(f) \subset \mathbb{P}^n$ is covered by $k$-planes, and let $r \in \mathbb{N}$.

1. If $\mathbf{F}_k(X_{r,d})$ is one-split, then $\text{pr}(f) \neq r$.
2. If $r$ is even, $k > n-r$, and

$$\mathbf{F}_k(X_{r,d}), \mathbf{F}_{k-d}(X_{r-2,d}), \ldots, \mathbf{F}_{k-d(r-d)}(X_{4,d})$$

are two-split, then $\text{pr}(f) \neq r$.

Applying this to the $3 \times 3$ determinant of a generic matrix, we recover that its product and tensor ranks are five [IT16]. Note that we have replaced the computer-aided computation of $\mathbf{F}_5(X_{3,4})$ with a conceptual proof. We may also apply Theorem 1.9 to the $4 \times 4$ determinant $\text{det}_4$ of a generic matrix to obtain $\text{pr}(\text{det}_4) \geq 7$; this is equal to the lower bound one obtains from Derksen and Teitler’s lower bound on the Waring rank [DT15]. In Example 4.4 we apply the theorem to the Pfaffian $f$ of a generic $6 \times 6$ skew-symmetric matrix to obtain $\text{pr}(f) \geq 7$, beating the previous lower bound of 6. Finally, in Example 4.5 we use a slightly different argument to obtain that the product rank (and tensor rank) of the $4 \times 4$ permanent is at least 6, beating the previous lower bound of 5.

The rest of the paper is organized as follows. In §2 we study equations of the form (1). We use this in §3 to show our splitting results for the Fano schemes $\mathbf{F}_k(X_{r,d})$, as well as studying several cases in more detail. Finally, we prove Theorem 1.9 in §4 and apply our results to a number of examples including the $6 \times 6$ Pfaffian and $4 \times 4$ permanent.

For simplicity, we will be working over an arbitrary algebraically closed field $\mathbb{K}$. Note however that all our main results clearly hold for arbitrary fields simply by restricting from $\mathbb{K}$ to any subfield.
2. Special Sums of Products of Linear Forms

2.1. Preliminaries. In this section we will prove that property \( C^d_m \) holds for \( m \leq 2 \) or \( d \leq 4 \). We will obtain this result by using induction arguments. These arguments involve a refined version of the \( C^d_m \) property. Consider an equation of the form

\[
\ell_1 + \ldots + \ell_m = \sum_{i=1}^{k+n} f_i x_i,
\]

with \( n > m \). As before, the \( x_i \) are pairwise coprime squarefree monomials and the \( \ell_i \) are degree \( d \) products of linear forms, possibly equal to zero. In fact, whenever we say that some polynomial \( g \) is homogeneous of degree \( d \), we include the possibility that \( g = 0 \).

We now assume simply that \( f_i \) are degree \( (d - \deg x_i) \) forms (or zero), no longer requiring that they be products of linear forms. It will be convenient to order the summands on the right hand side so that

\[
\deg x_1 \leq \deg x_2 \leq \ldots \leq \deg x_{k+n}.
\]

We will maintain this ordering convention throughout all of §2. Similar to the property \( C^d_m \) we make the following definition.

**Definition 2.1 (Property \( C^d_{k,m,n} \)).** We say that \( C^d_{k,m,n} \) is true, if for any equation of the form (2) satisfying

\[
\deg x_i + \deg x_j \geq d + 1 \quad \text{for } i > k
\]

\[
\deg x_i + \deg x_j \geq d + 2 \quad \text{for } i, j > k.
\]

it follows that there are \( i_1, \ldots, i_{n-m} > k \) for which \( f_{i_j} = 0 \).

Note that by definition \( C^d_{0,m,m+1} \) implies \( C^d_m \).

**Lemma 2.2.** Fix \( d \) and \( m \) and assume that \( C^d_{k,m,m+1} \) holds for every \( k \geq 0 \). Then \( C^d_{k,m,n} \) holds for every \( n > m \).

**Proof.** We may argue by induction on \( n \). Obviously, the hypotheses for \( C^d_{k,m,n} \) imply those for \( C^d_{k+1,n+1} \). Hence, by the induction hypothesis we have the vanishing of \( (n-m-1) \) of the \( f_i \), with \( i > k + 1 \). Using this we end up with the hypotheses for \( C^d_{k,m,n-(n-m)-1} = C^d_{k,m,m+1} \) being fulfilled, and we see that another of the \( f_i \) with \( i > k \) has to vanish. \( \square \)

2.2. Property \( C^d_1 \). We will first analyze the case \( m = 1 \). Therefore, we consider an equation of the form

\[
\ell = \sum_{i=1}^{r} f_i x_i
\]

with \( f_i \) forms of degree \( (d - \deg x_i) \) (possibly equal to zero), and \( \ell \) a product of linear forms, also possibly equal to zero. As before, we order indices such that \( \deg x_1 \leq \deg x_2 \leq \ldots \leq \deg x_r \).

**Remark 2.3 (Cancellation).** Assume we are given a variable \( x \) which divides \( \ell \), one monomial \( x_i \), and all \( f_j \) for \( j \neq i \). Setting

\[
\ell' = \ell/x \quad x'_j = \begin{cases} x_j/x & i = j \\ x_j & i \neq j \end{cases} \quad f'_j = \begin{cases} f_j/x & i \neq j \\ f_j & i = j \end{cases}
\]
leads to
\[ \ell' = \sum_{i=1}^{r} f'_i x'_i \]
where we have reduced from forms of degree \( d \) to degree \( d - 1 \). We call this the cancellation of \((5)\) by \( x \).

**Lemma 2.4.** Let \( l \) be a linear form dividing \( \sum f_i x_i \), where the \( f_i \) are forms of degree \((d - \deg x_i)\).

1. If \( \deg x_1 + \deg x_2 \geq d + 2 \), then for all \( i \), \( l \) divides \( x_i \) or \( f_i \).
2. If \( \deg x_1 + \deg x_2 \geq d + 1 \) and \( l \) is a monomial, then for all \( i \), \( l \) divides \( x_i \) or \( f_i \).

**Proof.** We first prove the second statement. We have
\[ \sum f_i x_i = xg \]
for some variable \( x \) and form \( g \). Expanding the left hand side as a sum of monomials, we see that the degree condition ensures that no terms from \( f_i x_i \) cancel with \( f_j x_j \) for \( i \neq j \). But every monomial on the right hand side is divisible by \( x \), hence also on the left hand side. The claim follows.

For the first statement, we reduce to the second by performing a change of coordinates taking \( l \) to a monomial. This can be achieved while preserving all variables in the \( x_1, \ldots, x_r \) with at most one exception, say in \( x_i \). After factoring out this one linear form from \( x_i \), the pairwise of sum degrees is still at least \( d + 1 \) and we may apply the second claim. \( \square \)

**Lemma 2.5.** Suppose \( \ell = \sum f_i x_i \) for \( r \geq 2 \), \( f_i \) forms of degree \((d - \deg x_i)\). Assume \( \ell \neq 0 \), and let \( \lambda \) be the number of distinct factors of \( \ell \). If
\[ \left\lceil \prod \frac{\deg x_i}{\lambda} \right\rceil > \prod \frac{\deg x_i}{\deg x_1 \cdot \deg x_2} \]
then there is a variable \( x \) dividing both \( \ell \) and one of the \( x_i \). This is true if \( \deg x_1 \cdot \deg x_2 > \lambda \), in particular if \( \deg x_1 \cdot \deg x_2 > d \).

**Proof.** For each \( x_i \), choose some variable \( x_i \) dividing it. Setting \( x_1 = x_2 = \ldots = x_r = 0 \) will result in the equality \( \ell = 0 \), hence there exists one factor of \( \ell \) depending only on \( x_1, \ldots, x_r \). There are \( \prod \frac{\deg x_i}{\deg x_1 \cdot \deg x_2} \) possible ways to choose the \( x_i \), and \( \lambda \) factors of \( \ell \), so there must be one factor of \( \ell \) which depends only on the \( x_1, \ldots, x_r \) for
\[ \left\lceil \prod \frac{\deg x_i}{\lambda} \right\rceil \]
different choices. On the other hand, the intersection of more than
\[ \prod \frac{\deg x_i}{\deg x_1 \cdot \deg x_2} \]
choices of the \( x_1, \ldots, x_r \) contains at most one variable. Hence, if the above inequality is satisfied, the claim follows. \( \square \)

**Remark 2.6.** If \( \ell = 0 \), the conclusion of the above lemma is trivial since every variable divides \( \ell \).
Proposition 2.7. Suppose
\[ \ell = f_1 x_1 + f_2 x_2 \]
with \( f_i \) forms of degree \( (d - \deg x_i) \).

1. If \( \deg x_1 + \deg x_2 \geq d + 2 \), then either \( f_1 \) or \( f_2 \) vanishes.
2. If \( \ell \) is not squarefree and \( \deg x_1 + \deg x_2 \geq d + 1 \), then either \( f_1 \) or \( f_2 \) vanishes.

In particular, property \( C^d \) holds for every \( d > 0 \).

Proof. For the first case, the hypothesis \( \deg x_1 + \deg x_2 \geq d + 2 \) implies in particular that \( \deg x_1 \geq 2 \) and \( \deg x_1 \cdot \deg x_2 \geq d \). Hence, Lemma 2.3 guarantees that \( x \) divides both \( \ell \) and one of the \( x_i \). But then \( f_j x_j \) is divisible by \( x \) for \( j \neq i \), hence \( x \) divides \( f_j \). Cancelling by \( x \), we may proceed by induction on the degree \( d \).

For the second case, we proceed with a similar argument. The inequality
\[ \deg x_1 + \deg x_2 \geq d + 1 \]
implies that \( \deg x_1 \cdot \deg x_2 > d - 1 \), which is larger than or equal to the number of distinct factors of \( \ell \). Thus, we again find a variable \( x \) dividing both \( \ell \) and one of the \( x_i \). If in fact \( x^2 \) divides \( \ell \), then after factoring out one power of \( x \) from \( x_i \) and \( f_j \), Lemma 2.3 guarantees that \( x \) divides \( f_i \) as well. Dividing \( \ell, f_i, \) and \( f_j \) by \( x \), we reduce to the first case.

If \( x^2 \) does not divide \( \ell \), we may cancel by \( x \) as in the first case, maintaining that \( \ell/x \) is still not squarefree. To finish, we again proceed by induction on the degree \( d \).

Remark 2.8. It is clear that the degree bounds in Proposition 2.7 cannot be improved upon. If \( \deg x_1 + \deg x_2 = d + 1 \) and \( x_1, x_2 \) are variables dividing \( x_1, x_2 \) respectively, then setting \( f_i = x_i/x \) gives
\[ f_1 x_1 + f_2 x_2 = (x_1 + x_2) \frac{x_1 x_2}{x_1 x_2}. \]
Likewise, if \( \deg x_1 + \deg x_2 \leq d \) there are non-trivial degree \( d \) syzygies between \( x_1 \) and \( x_2 \), so we cannot expect the second claim to hold.

We next prove a stronger version of \( C^d \).

Proposition 2.9. Suppose for \( r \geq 3, d \geq 2 \)
\[ \ell = \sum_{i=1}^{r} f_i x_i \]
with \( f_i \) being forms of degree \( (d - \deg x_i) \). Then if \( \deg x_1 + \deg x_{r-1} \geq d + 1 \), some \( f_i \) with \( \deg x_i \geq \deg x_{r-1} \) must vanish.

In particular, \( C_{k,1,2}^d \) holds for every \( d > 0, k \geq 0 \).

Proof. Set \( \alpha = \deg x_1 \) and \( \beta = \deg x_{r-1} \). It suffices to prove the proposition in the case that \( \deg x_1 = \ldots = \deg x_{r-2} = \alpha \) and \( \deg x_{r-1} = \deg x_r = \beta \). Indeed, we may absorb variables from \( x_2, \ldots, x_{r-2}, x_r \) into the corresponding \( f_i \) to reduce to this case. Henceforth we will assume we are in such a situation.

We begin by proving the claim when \( r = 3 \) and \( \alpha = 1 \), that is, \( x_1 \) is a single variable \( x \). By using Proposition 2.7, we see that modulo \( x \), either \( f_2 \) or \( f_3 \) must vanish. But since \( \alpha + \beta \geq d + 1 \), \( f_2 \) and \( f_3 \) are both just constants, hence one must vanish outright.
Next, we consider the case when \( r = 3 \) and \( \alpha > 1 \). First, we show that some \( f_i \) must vanish, with no restriction on its degree. We apply Lemma 2.5 to find a variable \( x \) dividing some monomial \( x_i \) and \( \ell \). Applying Lemma 2.4 we may conclude that \( x \) divides \( f_j \) for \( j \neq i \). In particular, if \( d = 2 \), this implies that \( f_j = 0 \) for \( j \neq i \). For \( d > 2 \), we may cancel by \( x \) to reduce the degree by one and conclude by induction on degree that \( f_i = 0 \) for some \( i \). Now we show that we can impose the desired degree restriction on \( f_i \). Indeed, if \( i = 2, 3 \), or \( i = 1 \) and \( \alpha = \beta \), this is automatic. If instead \( i = 1 \) and \( \alpha < \beta \), then we have \( \ell = f_2x_2 + f_3x_3 \) satisfying the hypotheses of Proposition 2.7(1), from which the claim follows.

It remains to consider the cases when \( r \geq 4 \). We will now induct on \( r \).

First assume that \( \alpha < \beta \). By setting any variable \( x \) in \( x_i \) equal to zero, for \( i \leq r - 2 \), we reduce to an equation of the form (6) with one fewer summand on the right hand side, yet \( \alpha < \beta \), and \( d \) the same. Hence, by induction, \( x \) divides \( f_{r-1} \) or \( f_r \). Now, there are \( \alpha \cdot (r - 2) \) variables appearing in the \( x_i \) for \( i \leq r - 2 \), yet

\[
\deg f_{r-1} + \deg f_r = 2(d - \beta) \leq 2(\alpha - 1).
\]

Thus, if \( r > 3 \) then either \( f_{r-1} \) or \( f_r \) must vanish.

If instead \( r \geq 4 \) and \( \alpha = \beta \), we may again apply Lemma 2.5 followed by Lemma 2.4 to find a variable \( x \) dividing some \( x_i \) and \( f_j \) for \( j \neq i \). We may reorder the monomials such that \( i = 1 \), since all have the same degree \( \alpha \). Cancelling by \( x \), we again find ourselves in the situation of (6), but now with \( \alpha < \beta \), so by the above, \( xf_{r-1} \) or \( xf_r \) vanishes, thus \( f_{r-1} \) or \( f_r \) does as well.

**Remark 2.10.** Proposition 2.9 is sharp in the following sense. Suppose that in (6), we have \( \deg x_1 + \deg x_{r-1} \leq d \). Then none of the \( f_i \) need vanish. Indeed, for \( i = 2, \ldots, r-1 \) we can take \( f_i = g_i x_1 \) for any forms \( g_i \) of degree \((d - \deg x_1 - \deg x_i)\), and

\[
 f_1 = \sum_{i=2}^{r-1} -g_i x_i.
\]

Then

\[
 \sum_{i=1}^r f_i x_i = f_r x_r
\]

so if \( f_r \) is a product of linear forms, then so is the whole sum, yet for appropriate choice of \( g_i \), none of the \( f_i \) will vanish.

2.3. Property \( C_d^2 \). We now move to the case of \( C_d^2 \).

**Proposition 2.11.** Suppose

\[
 \ell_1 + \ell_2 = f_1 x_1 + f_2 x_2 + f_3 x_3
\]

with \( f_i \) forms of degree \((d - \deg x_i)\). If \( \deg x_1 + \deg x_2 \geq d + 2 \), then either \( f_1 \), \( f_2 \), or \( f_3 \) vanishes.

In particular, \( C_d^2 \) holds for every \( d > 0 \).

**Proof.** We will prove the statement by induction on the degree \( d \). For \( d = 1 \) there is nothing to prove. If we can show that \( \ell_1, \ell_2 \) have a common factor \( l \), then we are done. Indeed, by Lemma 2.4 \( l \) must divide either each \( x_i \) or \( f_i \). Pulling \( l \) out of each \( f_i \), where we can, and out of \( x_i \) in at most one position, allows us to “cancel by \( l \)” in a fashion similar to Remark 2.3. We thus reduce the degree and the claim follows by induction.
In the following, we will assume that no common factor \( l \) of \( \ell_1 \) and \( \ell_2 \) exists, and that all \( f_i \) are non-zero. For simplicity, we may assume that \( \deg x_2 = \deg x_3 \), since this case implies the more general one. We denote \( \deg x_1 \) by \( \alpha \), and \( \deg x_2 \) by \( \beta \). Our hypothesis on degrees is now simply \( \alpha + \beta \geq d + 2 \).

Consider any factor \( l \) of \( \ell_1 \) or \( \ell_2 \). Setting \( l = 0 \), we reduce to the case of Proposition 2.7(1) (if \( l \) divides some \( x_i \)) or Proposition 2.9 (by absorbing into some \( f_i \) a variable of \( x_i \) appearing in \( l \)). In either case, we see that modulo \( l \), some \( f_i \) must vanish, that is, \( l \) is a factor of \( f_i \). We may proceed to do this for all distinct divisors of \( \ell_1 \) and \( \ell_2 \). But since

\[
\deg f_1 + \deg f_2 + \deg f_3 \leq 2(d - \beta) + (d - \alpha),
\]

we conclude that together \( \ell_1 \) and \( \ell_2 \) have at most

\[
2(d - \beta) + (d - \alpha) \leq 2d - \beta - 2
\]
different factors. It follows that either both \( \ell_1 \) and \( \ell_2 \) contain a square, or else that the non-squarefree product has at most \( d - \beta - 2 \) distinct factors.

Assume first that \( \ell_1 \) is squarefree, and fix some factor \( l \). We now argue in a similar fashion to the proof of Lemma 2.5. For each \( x_1, x_2, x_3 \), fix a variable \( y_i \) so that \( l \) and all remaining variables are linearly independent. For each \( x_i \), choose some variable \( x_i \neq y_i \) dividing \( x_i \). Setting \( x_1 = x_2 = x_3 = 0 \) will result in the equality \( \ell_1 = -\ell_2 \), hence one factor of \( \ell_2 \) is zero modulo \( x_1, \ldots, x_m, l \). There are \( (\alpha - 1)(\beta - 1)^2 \) possible ways to choose the \( x_i \), and at most \( d - \beta - 2 \) factors of \( \ell_2 \), so there must be one fixed factor of \( \ell_2 \) which is zero mod \( x_1, \ldots, x_m, l \) for

\[
\left\lfloor \frac{(\alpha - 1)(\beta - 1)^2}{d - \beta - 2} \right\rfloor
\]
different choices. On the other hand, the intersection of more than \( \beta^2 \) choices of the \( x_1, \ldots, x_m \) contains no variable. Hence, since \( d - \beta - 2 < \alpha - 1 \) it follows that there is a factor of \( \ell_2 \) which is zero modulo \( l \), that is, agrees with it.

We now instead assume that both \( \ell_1 \) and \( \ell_2 \) contain factors with multiplicity at least two. Consider any factor \( l \) of \( \ell_1 \) or \( \ell_2 \). As long as \( l \) is not a variable in \( x_2 \) or \( x_3 \), we may set \( l = 0 \) and conclude that \( l \) divides \( f_2 \) or \( f_3 \). Indeed, if \( l \) divides \( x_1 \) this follows from Proposition 2.7. Otherwise we may absorb into some \( f_1 \) a variable of \( x_1 \) made linear dependent modulo \( l \), and then apply Proposition 2.9 followed by Proposition 2.7(2) to conclude that two of \( f_1, f_2, \) and \( f_3 \) vanish modulo \( l \).

If at most one factor \( l \) of \( \ell_1, \ell_2 \) divides \( x_2 \) or \( x_3 \) but not \( f_2 \) or \( f_3 \), we thus obtain that \( \ell_1, \ell_2 \) have at most \( 1 + 2(d - \beta) \) distinct factors. But then either \( \ell_1 \) or \( \ell_2 \) has at most \( d - \beta \) distinct factors, so an argument similar to the previous case where \( \ell_1 \) squarefree above shows that \( \ell_1 \) and \( \ell_2 \) would have to possess a common factor.

So we now finally consider the case that at least two distinct factors \( x, y \) of \( \ell_1, \ell_2 \) are variables found in \( x_2 \) and \( x_3 \), neither dividing \( f_2 \) or \( f_3 \). It follows by Proposition 2.7 that each such factor must divide \( f_1 \). Without loss of generality, we assume that \( x \) divides \( x_3 \) and \( \ell_1 \). We obtain

\[
\ell_2 \equiv f_2 x_2 \mod x,
\]
so \( \ell_2 \) has \( \beta \) factors which only depend on \( x \) and a single variable of \( x_2 \).

If \( y \) divides \( x_3 \) and \( \ell_2 \), then setting \( x = y = 0 \), we obtain \( f_2 x_2 \equiv 0 \mod x, y \), a contradiction. If instead \( y \) divides \( x_3 \) and \( \ell_1 \) we obtain

\[
\ell_2 \equiv f_2 x_2 \mod y
\]
and \( \ell_2 \) has \( \beta \) factors which only depend on \( y \) and a single variable of \( x_2 \). Since \( \ell_2 \) has \( d < 2\beta \) factors, one must also just be a variable \( w \) of \( x_2 \). So in this case, we conclude that a variable \( w \) of \( x_2 \) divides \( \ell_2 \). If instead \( y \) divides \( x_2 \), we see by setting \( x = 0 \) that \( y \) must divide \( \ell_2 \), so we can take \( w = y \) to produce \( w \) as above.

We thus may assume that we are in the situation of variables \( x, w \) with \( x \) dividing \( x_3 \) and \( \ell_1 \), and \( w \) dividing \( x_2 \). By Proposition 2.7, \( w \) divides \( f_j \) for \( j = 1 \) or \( j = 3 \). Now let \( k \in \{1, 3\} \) be such that \( i \neq j \). We thus obtain

\[
\ell_1 \equiv f_k x_k \mod w,
\]

hence \( \ell_1 \) has \( \deg x_k \) factors which depend on \( w \) and a single variable of \( x_k \).

The right hand side of Equation (7) clearly contains monomials divisible by \( x_j \). But the left hand side cannot: while each monomial of \( \ell_1 \) has degree at least \( \deg x_k \) in the variables of \( \sum \) and \( \ell_1 \) and \( w \), and each monomial of \( \ell_2 \) has degree at least \( \beta = \deg x_2 \) in the variables of \( x_2 \) and \( x \), the part of \( x_j \) relatively prime to \( x \) has degree at least \( \deg x_j - 1 \). The inequality

\[
\deg x_j - 1 + \deg x_k \geq d + 1
\]

then shows that this impossible. We conclude that in fact some \( f_i \) must equal zero. □

Remark 2.12. Proposition 2.11 is optimal. Indeed, suppose that \( \deg x_1 + \deg x_2 \leq d + 1 \). Then by Remark 2.8 for appropriate non-vanishing choices of \( f_1, f_2, f_1 x_1 + f_2 x_2 \) is a product of linear forms, so \( f_1 x_1 + f_2 x_2 + f_3 x_3 \) is a sum of two products of linear forms for any choice of \( f_3 \).

2.4. Property \( C_{d}^{m} \) for \( d \leq 4 \). We now prove a lemma that will help us with the degree four case:

Lemma 2.13. Fix \( d \geq 2 \), \( k > 0 \), and \( m > 0 \) and let \( n = m + 1 \). Assume that (3) and (4) hold and that \( C_{k,m,n}^{d} \) holds whenever \( m' < m \), or whenever \( m' = m \) and \( k' < k \). In Equation (2), consider any linear form \( l \) dividing \( p \) of the summands \( \ell_i \) on the left hand side. Then \( l \) must also divide \( p \) of the \( f_i \) with \( i > k \).

Proof. Assume \( l \) divides some factor \( x_j \) of \( x_j \). Setting \( l = 0 \), we now have that the hypothesis for \( C_{k,m-p,m}^{d} \) is fulfilled. Since we have assumed that \( C_{k,m-p,m}^{d} \) is true, \( p \) of the \( f_i \) with \( i > k \) must vanish modulo \( l \).

Even if \( l \) does not divide any \( x_j \), we still may set \( l = 0 \), modifying the right hand side of the equation \( \ell_1 + \ldots + \ell_m = \sum_{j=1}^{k+n} f_j x_j \) to replace one factor of some \( x_j \) by a linear form \( f \) which is no longer a monomial. Now we have to distinguish two cases. Let us assume first that \( j > k \). Then, since the degree of \( x_j \) drops by one, we are in the situation of \( C_{d}^{k+1,m-p,m} \). As before by our assumption, \( p \) of the \( f_i \) with \( i > k \) must vanish modulo \( l \).

If \( j \leq k \) then the fact that the degree of \( x_j \) drops may violated condition (4). However, we may bring the summand \( f_j x_j \) to the left hand side of the equation. This leaves us in the situation of \( C_{k+1,m-p+1,m+1}^{d} \) and our assumption again provides the vanishing of \( p \) of the \( f_i \) with \( i > k \). □

We now use this lemma to show \( C_{m}^{d} \) for arbitrary \( m \) and \( d \leq 4 \):

Proposition 2.14. If \( d \leq 4 \), property \( C_{k,m,n}^{d} \) holds for arbitrary \( k > 0 \) and \( n > m > 0 \). In particular, \( C_{m}^{d} \) holds for \( m > 0 \).
Proof. By Lemma 2.2 it is enough to show that \( C^{d}_{k,m,m+1} \) holds for all \( m, k > 0 \). Now, we prove \( C^{d}_{k,m,m+1} \) by induction on \( m \) and \( k \). Note that, for \( k \) arbitrary, \( C^{d}_{k,1,2} \) follows from Proposition 2.9. Moreover, Lemma 2.2 then provides \( C^{d}_{k,1,n} \) for arbitrary \( n > 1 \).

Assume we have proven property \( C^{d}_{k,m,m'+1} \) is true whenever \( m' < m \), or whenever \( m' = m \) and \( k' < k \). For \( d \leq 4 \) we have \( \sum_{i>k} \deg f_i \leq m + 1 \). Either one of the \( f_i \) has to vanish, which would prove our claim, or by Lemma 2.13 all the linear factors of the \( \ell_i \) occur as one of the (at most) \((m + 1)\) linear factors of the \( f_i \) for \( i > k \). Here, by a linear factor we mean an equivalence class of linear forms, where two linear forms are equivalent if one is a non-zero scalar multiple of the other.

By the above the linear factors of the \( f_i \) form a multiset \( L \) of cardinality at most \( m + 1 \) and every \( \ell_j \) is divisible by one of the elements of \( L \). On the other hand, we have seen by Lemma 2.13 that every \( l \in L \) may divide at most \( m(l) \) of the \( \ell_j \), where \( m(l) \) denotes the multiplicity of \( l \) in \( L \). Since \( \#L \leq (m + 1) \), there can be at most one \( \ell_j \) which is divisible by more than one linear factor. This implies, we have \( \ell_i = t_i^d \) for all but one of the summands on the left hand side.

For \( m > 2 \) this implies that we can write the left hand side as the sum of only \((m - 1)\) products, since we can write \( t_i^d + t_j^d \) as a product of linear forms. Then using the induction hypothesis for \( C^{d}_{k,m-1,m} \) concludes the proof for the case \( m > 2 \).

To conclude, we consider the case \( m = 2 \). If none of the \( f_i \) (with \( i > k \)) vanishes, we have seen that at most three linear factors \( l_1, l_2 \) and \( l_3 \) can occur on the left hand side. We choose \( \lambda \in \mathbb{K} \) and set \( l_2 = \lambda l_3 \). Now, the left hand side depends only on two linear forms. In this situation the left hand side is actually a product of linear forms (since \( \mathbb{K} \) is algebraically closed). In the same way as in the proof of Lemma 2.13 (when setting one of the \( l_i \) to zero) we see by the induction hypothesis that one of the \( f_i \) has to be divisible by \( (l_2 - \lambda l_3) \). We have only finitely many choices for the linear factors of the \( f_i \), but we have infinitely many choices for \( \lambda \in \mathbb{K} \). Hence, there are \( \lambda, \lambda' \in \mathbb{K} \) with \((l_2 - \lambda l_3) \) dividing \((l_2 - \lambda' l_3) \), which implies either \( l_2 = 0 \) or \( l_3 = 0 \). In any case, one of the summands \( \ell_1 \) or \( \ell_2 \) has to vanish, and we are in the case of \( C^{d}_{k,1,3} \).

2.5. Consequences. The following lemma derives a consequence of the property \( C^{d}_{m} \) which will be used later.

Lemma 2.15. Consider an equation of the form

\[
\ell_1 + \ldots + \ell_m = \sum_{i=1}^{m} x_i
\]

where the \( \ell_i \) are degree \( d \geq 3 \) products of linear forms, and the \( x_i \) are pairwise relatively prime squarefree monomials of degree \( d \). If property \( C^{d}_{m-1} \) is true, then there is a permutation \( \sigma \in S_m \) such that \( \ell_i = x_{\sigma(i)} \) for all \( i \).

Proof. Consider any factor \( l_i \) of some \( \ell_i \). If \( l_i \) does not divide any \( x_j \), we may set \( l_i = 0 \), modifying the right hand side of Equation 8 to replace one factor of some \( x_j \) by a linear form \( f \) which is no longer a monomial. But this equation still satisfies the hypotheses necessary for \( C^{d}_{m-1} \), as long as \( d \geq 3 \), so in fact, \( l_i \) must have divided one of the \( x_j \) all along.

We thus see that every factor of each \( \ell_i \) is just a variable, up to scaling. By comparing the monomials on both sides of 8, we find the desired permutation. □
Remark 2.16. We may interpret the above lemma geometrically as saying that, if $C_d^{-1}$ is true, then the subgroup of $PGL(rd-1)$ taking $X_{r,d}$ to itself is generated by the semidirect product of the torus

$$T = \{x_{11} \cdots x_{1d} = x_{21} \cdots x_{2d} = \cdots = x_{r1} \cdots x_{rd}\}$$

with the copy of the symmetric group $S_r$ permuting the indices $i$ of $x_{ij}$, and the $r$ copies of $S_d$ permuting the indices $j$ of $x_{ij}$ for some fixed $1 \leq i \leq r$.

3. Fano Schemes and Splitting

3.1. Main results. In this section, we will prove Theorems 1.7 and 1.8. For $n = rd - 1$, consider projective space $\mathbb{P}^n$ with coordinates $x_{ij}$, $1 \leq i \leq r$ and $1 \leq j \leq d$. Let $L$ be a $k$-dimensional linear subspace of $\mathbb{P}^n$. We may represent $L$ as the rowspan of a full rank $(k + 1) \times rd$ matrix $B = (b_{\alpha,ij})$, with rows indexed by $\alpha = 0, \ldots, k$ and column $ij$ corresponding to the homogeneous coordinates $x_{ij}$ on $\mathbb{P}^n$. We define linear forms $y_{ij}$ in $S = \mathbb{K}[z_0, \ldots, z_k]$ by

$$y_{ij} = \sum_\alpha b_{\alpha,ij} z_\alpha,$$

along with degree $d$ forms

$$y_i = \prod_{j=1}^d y_{ij}.$$

The condition that $L$ is contained in $X_{r,d}$ is equivalent to the condition

$$(9) \quad \sum_i y_i = 0.$$

The condition that $L$ is one-split is equivalent to the condition that some $y_{ij}$ vanishes, and also to the condition that some $y_i$ vanishes. The condition that $L$ is two-split is equivalent to the existence of $a \leq a_1 < a_2 \leq r$ such that $y_{a_1} + y_{a_2} = 0$.

Example 3.1 ($k$-planes which are not one-split). For $r = 2m$ and $k = md - 1$, let $L$ be any $k$-plane with $y_{ij}$ all linearly independent for $i \leq m$, $y_{(i+m)1} = -y_{i1}$, and $y_{(i+m)j} = y_{ij}$ for $j > 1$. Then clearly $L$ is contained in $X_{r,d}$, but is not one-split (although it is two-split).

For $r = 2m + 1$ and $k = md$, consider forms $y_{ij}$ satisfying $\{y_{ij}\}_{i \leq m}$ and $y_{r1}$ all linearly independent, and

$$y_{(i+m)1} = -y_{i1} \quad \text{for } i < m,$$

$$y_{(i+m)j} = y_{ij} \quad \text{for } i < m, j > 1,$$

$$y_{rj} = y_{(2m)j} = y_{mj} \quad \text{for } j > 1,$$

$$y_{(2m)1} = -y_{m1} - y_{r1}.$$

Let $L$ be the corresponding $md$-plane. Clearly $L$ is contained in $X_{r,d}$, but is not one-split.

We thus see that the bound on $k$ in Conjecture 1.2 is sharp.

We henceforth assume that $L \subset X_{r,d}$, that is, that $\sum_i y_i = 0$, and that none of the $y_i$ vanish, that is, $L$ is not one-split. Without loss of generality, we may inductively reorder the forms $y_i$ as follows: given $y_1, \ldots, y_s$, we take $y_{s+1}$ to be any form such that the dimension of the vector space spanned by the $\{y_{ij}\}_{i \leq s+1}$ is maximal.
After this re-ordering, we may define integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ inductively by requiring that the dimension of the vector space spanned by the $\{y_{ij}\}_{i \leq s}$ is equal to $\sum_{i \leq s} \lambda_i$. By the way we have ordered the forms $y_i$, this implies that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. Indeed, by our choice of $y_s$ we must have $\sum_{i \leq s} \lambda_i \geq \lambda_{r+1} + \sum_{i \leq s-1} \lambda_i$ and, hence, $\lambda_s \geq \lambda_{s+1}$ for every $1 \leq s \leq r$.

We may then choose a new basis
\[ z_{ij}, \quad 1 \leq i \leq r \quad 1 \leq j \leq \lambda_i; \]
for the degree one piece of $S$ with the property that each $z_{ij}$ is a factor of $y_i$, and each factor of $y_i$ is in the span of
\[ \{z_{ij}\}_{1 \leq h \leq i, 1 \leq j \leq \lambda_h}. \]

We now will assume that
\[ k \geq \begin{cases} \frac{r}{2} \cdot d - 1 & \text{r even} \\ \frac{r+1}{2} \cdot d + 1 & \text{r odd} \end{cases}. \]

**Lemma 3.2.** For $s \geq 0$, suppose that $\lambda_{r-s} = 0$. If $r$ is odd, then $s \leq \frac{r-3}{2}$. If $r$ is even, then $s \leq \frac{r-1}{2}$ In particular, we always have $s + 1 \leq r - s - 1$.

**Proof.** We have
\[ k + 1 = \sum_{i=1}^{r} \lambda_i = \sum_{i=1}^{r-s-1} \lambda_i \leq (r-s-1)d. \]

For $r$ odd, our assumptions on $k$ imply $\frac{r+1}{2}d + 2 \leq (r-s-1)d$. Hence, we have $s + \frac{2}{r} \leq \frac{r+1}{2} \frac{r-s-1}{2}$ which implies $s \leq \frac{r-3}{2}$. Since $s$ is an integer we obtain $s \leq \frac{r-3}{2}$.

For $r$ even, (10) implies $\frac{r}{2}d \leq (r-s-1)d$, which directly implies the claim. □

**Lemma 3.3.** For $s \geq 0$, suppose that $\lambda_{r-s} = 0$. Assume further that either
\begin{enumerate}
  \item $d$ is even,
  \item $r$ is even and $d \geq r - 2s$, or
  \item $r$ is odd and $r - 2s \leq 6$.
\end{enumerate}
Then $\lambda_{s+1} + \lambda_{s+2} \geq d + 2$.

**Proof.** We have that
\[ k + 1 = \sum_{i=1}^{r} \lambda_i = \sum_{i=1}^{r-s-1} \lambda_i \leq sd + \sum_{i=s+1}^{r-s-1} \lambda_i. \]

Note that by Lemma 3.2 we have $s + 1 \leq r - s - 1$, so the summation on the right hand side makes sense. Using our assumption on $k$ we thus have
\[ \sum_{i=s+1}^{r-s-1} \lambda_i \geq \begin{cases} \frac{r-2s}{2} \cdot d & \text{r even} \\ \frac{r-2s-1}{2} \cdot d + 2 & \text{r odd} \end{cases}. \]

Suppose that $\lambda_{s+1} + \lambda_{s+2} \leq d + 1$. If $d$ is even, then $\lambda_{s+2} \leq \frac{d}{2}$, and thus $\lambda_i \leq \frac{d}{2}$ for all $i \geq s + 2$. But then
\[ \sum_{i=s+1}^{r-s-1} \lambda_i \leq (d + 1) + (r - 2s - 3) \frac{d}{2} = \frac{r - 2s - 1}{2} \cdot d + 1 \]
contradicting (11), since $d \geq 3$. The proof is complete.
Assume instead that $d$ is odd. Then $\lambda_i \leq \frac{d+1}{2}$ for all $i \geq s + 2$, so

$$\sum_{i=s+1}^{r-s-1} \lambda_i \leq (r - 2s - 1) \frac{d+1}{2}.$$  

But this contradicts (11) if $r$ is even and $d \geq r - 2s$, or if $r$ is odd and $r - 2s \leq 6$. □

**Proof of Theorem 1.7.** First note that $\lambda_r = 0$. Indeed, if not, then $y_r$ contains a factor which is not in the span of the factors of the $y_i$ for $i < r$, so it is impossible to satisfy Equation (9). Suppose that we have inductively shown that $\lambda_{r-s} = 0$ for some $s \leq \frac{d-1}{2} - 1$. Then by Lemma 3.3 we have that $\lambda_{s+1} + \lambda_{s+2} \geq d + 2$. If $\lambda_{r-s-1} \neq 0$, we set $z_i = 0$ for $i = s + 3, \ldots, r - s - 1$ and use property $C_{s+1}$ applied to

$$- \sum_{i=r-s}^{r} y_i = \sum_{i=1}^{s+2} y_i \mod \{z_i\}_{s+3 \leq i \leq r-s-1}$$

to conclude that some $y_i$ for $i \leq s + 2$ vanishes modulo $\{z_i\}_{s+3 \leq i \leq r-s-1}$. But by our construction of the $y_i$, this is impossible, and we conclude that $\lambda_{r-s-1} = 0$.

We proceed in this fashion until we obtain $\lambda_t = 0$ for

$$t = \left\lceil \frac{r}{2} \right\rceil + 1,$$

since for

$$s = \left\lfloor \frac{r-1}{2} \right\rfloor - 1,$$

we have

$$t \geq r - s - 1.$$

If $r$ is odd, we conclude again by Lemma 3.3 that $\lambda_{t-2} + \lambda_{t-1} \geq d + 2$, and an appropriate application of property $C_{(r-1)/2}$ shows that some $y_i$ must vanish, a contradiction. If $r$ is even, we must have $\lambda_1 = \ldots = \lambda_{r/2} = d$. This is impossible if $k$ satisfies the bound of Conjecture 1.2 completing the claim regarding one-splitting. For the claim regarding two-splitting, we may apply Lemma 2.15 to conclude that $y_1 = -y_j$ for some $j > r/2$. But this implies two-splitting. □

**Proof of Theorem 1.8.** The first part of the Theorem is simply Propositions 2.7, 2.11 and 2.13. The statement regarding Conjectures 1.2 and 1.3 following immediately from Theorem 1.7 except in the cases $(r = 4, d = 3)$, $(r = 6, d = 3)$, and $(r = 6, d = 5)$. The obstruction in all these cases comes about that in the proof of Theorem 1.7 we cannot use Lemma 3.3 to conclude that $\lambda_1 + \lambda_2 \geq d + 2$. However, we may use Proposition 2.9 to compensate.

Consider for example the case $r = 6$, $d = 3$. If $\lambda_1 + \lambda_2 \leq d + 1 = 4$, then we must in fact have $\lambda_1 = \lambda_2 = \ldots = \lambda_4 = 2$, and $\lambda_5 = 1$. Setting $z_5 = 0$, we may apply Proposition 2.9 to reach a contradiction. Thus, $\lambda_5 = \lambda_6 = 0$. A similar argument shows that $\lambda_3 = 0$ as well, and we conclude as in the proof of Theorem 1.7. The other two cases are similar, and left to the reader. □

**Remark 3.4.** Consider the Fano scheme $F_k(X_{r,d})$. If we only know that $C^d_m$ is true for all $m \leq M$ for some $M$ strictly less than $(r - 1)/2$, we may still use the above arguments to conclude that $F_k(X_{r,d})$ is one-split if $k$ is sufficiently large.

For example, we know that $C^d_m$ is always true for $m = 1, 2$. For $r \leq 6$, we already know by Theorem 1.8 exactly when $F_k(X_{r,d})$ is one-split, so assume that $r \geq 7$. 


We claim that if \( k \geq d(r - 3) \), then \( F_k(X_{r,d}) \) must be one-split. Indeed, if \( d \) is even, Lemma 3.3 applies and arguing as in the proof of Theorem 1.8 shows that if \( L \) is not one-split, then \( \lambda_r = \lambda_{r-1} = \lambda_{r-2} = 0 \). But this contradicts \( k \geq d(r - 3) \). For \( d \) odd, slightly more care is needed. Assume some \( k \)-plane \( L \) is not one-split. The arguments from Lemma 3.3 will apply if

\[
k \geq \frac{(r - 1)(d + 1)}{2},
\]

in which case we are done as above. But this inequality is satisfied except for the case \( r = 7, d = 3 \). As always, \( \lambda_7 = 0 \). But certainly \( \lambda_2 + \lambda_3 \geq d + 1 = 4 \), so using Proposition 2.9 in place of \( C_5^3 \) we conclude that \( \lambda_6 = 0 \). But then one easily verifies that \( \lambda_2 + \lambda_3 \geq d + 2 = 5 \), so \( \lambda_5 = 0 \), which is impossible.

3.2. Consequences and examples. We now want to use our results on splitting to study the geometry of \( F_k(X_{r,d}) \). We first note the following result:

**Theorem 3.5.** The Fano scheme \( F_k(X_{r,d}) \) is non-empty if and only if \( k < r(d - 1) \). Such a Fano scheme is connected if and only if \( k < r(d - 1) - 1 \).

**Proof.** Consider the subtorus \( T \) of \((\mathbb{K}^*)^d\) cut out by

\[
x_{11}x_{12} \cdots x_{1d} = x_{21}x_{22} \cdots x_{2d} = \cdots = x_{r1}x_{r1} \cdots x_{rd}.
\]

This torus acts naturally on \( \mathbb{P}^{rd-1} \). Since it fixes \( X_{r,d} \), this action induces an action on \( X_{r,d} \), and hence also on \( F_k(X_{r,d}) \). It is straightforward to check that the only \( k \)-planes of \( \mathbb{P}^{rd-1} \) fixed by \( T \) are intersections of coordinate hyperplanes. Thus, any torus fixed point of \( F_k(X_{r,d}) \) corresponds to a \( k \)-plane \( L \) whose associated non-zero forms \( y_{ij} \) of \( \mathbb{K} \) are all linearly independent.

Recall that such a \( k \)-plane \( L \) is contained in \( F_k(X_{r,d}) \) if and only if Equation (9) is satisfied. But since by assumption the non-zero \( y_{ij} \) are linearly independent, this is equivalent to requiring that for each \( i \), there is some \( j \) such that \( y_{ij} = 0 \). Since every component of \( F_k(X_{r,d}) \) must contain a torus fixed point, it follows immediately that if \( k \geq r(d - 1) \), \( F_k(X_{r,d}) \) must be empty. The non-emptiness of \( F_k(X_{r,d}) \) for \( k < r(d - 1) \) is also clear.

Assume now that \( k = r(d - 1) - 1 \). By Remark 3.4, it inductively follows that any \( k \)-plane of \( X_{r,d} \) must be torus fixed. But there are \( r^d \) such fixed \( k \)-planes, so \( F_k(X_{r,d}) \) is not connected.

Suppose finally that \( k < r(d - 1) - 1 \), and let \( L \) be a torus fixed \( k \)-plane contained in \( F_k(X_{r,d}) \). We prove that \( F_k(X_{r,d}) \) is connected by deforming \( L \) to a \( k \)-plane satisfying

\[
y_{11} = y_{21} = \ldots = y_{r1} = 0.
\]

Since the set of all such \( k \)-planes forms a connected subscheme of \( F_k(X_{r,d}) \) isomorphic to the Grassmannian \( G(k + 1, r(d - 1)) \), and every irreducible component of the Fano scheme contains a torus fixed point, it follows that \( F_k(X_{r,d}) \) is connected.

To see that we can deform \( L \) to a \( k \)-plane of the desired type, let \( j_1, \ldots, j_r \) be such that \( y_{j_1}, \ldots, y_{j_r} \), all vanish; these must exist since \( L \) is torus fixed and contained in \( X_{r,d} \). Let \( i \) be the smallest index for which \( j_i \neq 1 \). The set of all \( k \)-planes satisfying \( y_{1j_1} = \ldots = y_{rj_r} = 0 \) forms a closed subscheme of \( F_k(X_{r,d}) \) isomorphic to \( G(k + 1, r(d - 1)) \). Since \( k < r(d - 1) - 1 \), this set contains a \( k \)-plane \( L' \) satisfying \( y_{11} = 0 \) along with \( y_{1j_1} = \ldots = y_{rj_r} = 0 \), and \( L \) deforms to \( L' \). Replacing \( L \) with \( L' \) we can continue this procedure until we arrive at a \( k \)-plane satisfying \( y_{11} = y_{21} = \ldots = y_{r1} = 0 \) as desired. \( \square \)
Remark 3.6. Theorem 3.5 is in stark contrast to the situation for the Fano scheme $F_k(X)$ for a general degree $d > 2$ hypersurface $X \subset \mathbb{P}^n$. Such a Fano scheme $F_k(X)$ is non-empty if and only if $\phi(n, k, d) \geq 0$, and connected if $\phi(n, k, d) \geq 1$, where

$$\phi(n, k, d) = (k + 1)(n - 3) - \binom{k + d}{k},$$

see [1497].

We now illustrate on several examples how our results help determine the irreducible component structure of $F_k(X_{r,d})$.

Example 3.7 ($F_k(X_{r,d})$) for $k \geq (r - 2)(d - 1) + 2$. For $k \geq (r - 2)(d - 1) + 2$ and $r \geq 3$, Conjecture 3.2 would imply that $F_k(X_{r,d})$ is one-split. Assume this to be true. Considering any $k$-plane $L$ contained in $X_{r,d}$, we know that some $x_{ij}$ must vanish. Intersecting $L$ with $x_{i1} = x_{i2} = \ldots = x_{id} = 0$, we obtain a linear subspace $L'$ in $X_{r-1,d}$ of dimension $k'$, where $k' \geq k - (d - 1) \geq (r - 3)(d - 1) + 2$. Hence, $L'$ is also (conjecturally) one-split, as long as $r - 1 \geq 3$. We may proceed in this fashion until we obtain a linear subspace $L''$ in $X_{2,d}$ of dimension $k''$, where $k'' \geq k - (r - 2)(d - 1) \geq 2$. If $L''$ is also one-split, then $L$ is contained in an $(r(d - 1) - 1)$-plane of the form

$$x_{1j_1} = x_{2j_2} = \ldots = x_{rj_r} = 0$$

for some choice of $j_1, \ldots, j_r$. The $k$-planes in this fixed $(r(d - 1) - 1)$-plane are parametrized by the Grassmannian $G(k + 1, r(d - 1))$. This leads to $d^r$ irreducible components of $F_k(X_{r,d})$, each isomorphic in its reduced structure to $G(k + 1, r(d - 1))$.

If on the other hand $L''$ is not one-split, then Equation (3) implies that after some permutation in the $j$ indices, $y_{1j}$ and $y_{2j}$ are linearly dependent for all $j$. In particular, $L''$ is contained in a $(d-1)$-plane of $X_{2,d}$, appearing in a $d-1$-dimensional family. Thus, the plane $L$ is contained in an $(r-1)(d-1)$-plane of $X_{r,d}$, which is moving in a $(d-1)$-dimensional family. This only can occur if $k \leq (r-1)(d-1)$. In such cases, it follows that the corresponding irreducible component of $F_k(X_{r,d})$ has dimension $(d-1) + (k+1)((r-1)(d-1) - k)$, and there are

$$\binom{r}{2}d^{r-2} \cdot (d!)$$

such components.

To summarize, the Fano scheme has two types of irreducible components:

- **Type A**: $d^r$ components of dimension $(k+1)(r(d-1) - (k+1))$, isomorphic in their reduced structures to a Grassmannian; general $k$-planes in such components are contained in the intersection of $r$ coordinate hyperplanes.

- **Type B**: Assuming $k \leq (r-1)(d-1)$, $\binom{r}{2}d^{r-2} \cdot (d!)$ components of dimension $(d-1) + (k+1)((r-1)(d-1) - k)$; general $k$-planes in such components are contained in the intersection of $(r-2)$ coordinate hyperplanes.

This analysis relied on Conjecture 3.2. By Theorem 1.8, this holds true if $r \leq 6$ or $d = 4$, so we know our above conclusions are true as long as this is satisfied. Furthermore, by Remark 4.4 the one-splitting we need follows if $k \geq d(r-3)$. But this is always satisfied as long as $r \leq d + 2$.

The above example is somewhat elementary, since all the Fano schemes appearing in the reduction steps are one-split or two-split. However, if we understand the
structure of a Fano scheme which isn’t one-split (or even two-split), we can leverage this to an understanding of $F_k(X_{r,d})$ for larger values of $r$. We will illustrate in the next two examples.

**Example 3.8** (Special components of $F_d(X_{3,d})$). By Example 3.1, we know that $F_d(X_{3,d})$ is not one-split; since $r = 3$, it is also not two-split. Nonetheless, with a bit of work, we can completely describe these Fano schemes. In this example, we will describe a special type of irreducible component; all components will be dealt with in Example 3.9.

We begin with the case $d = 2$ (although we usually have been assuming $d > 2$). The variety $X_{3,2}$ is just a non-singular quadric fourfold; it is well-known that $F_2(X_{3,4})$ is the disjoint union of two copies of $\mathbb{P}^3$.

We now suppose that $d > 2$. Let $L$ be a $d$-plane of $X_{3,d}$ which is not one-split. After reordering indices, we may assume that $y_{11}, \ldots, y_{1\lambda_1}, y_{21}, \ldots, y_{2\lambda_2}$ are linearly independent, with $\lambda_1 + \lambda_2 = d + 1$ and $\lambda_1 \geq \lambda_2$. If $\lambda_2 = 1$, then we may replace $\lambda_1$ with $\lambda_1 - 1$ and $\lambda_2$ with $\lambda_2 + 1$, unless all $y_{2j}$ are linearly dependent. But this is easily seen to contradict Equation (9). So we may assume that $\lambda_2 \geq 2$.

For each choice $y_{1j_1}, y_{2j_2}$ with $j_i \leq \lambda_i$, by setting $y_{1j_1} = y_{2j_2} = 0$, we find that one $y_{3j}$ depends only on $y_{1j_1}, y_{2j_2}$. A simple counting argument shows that some $y_{3j}$ can only depend on some $y_{1j_1}$ or $y_{2j_2}$. But by Equation (9), this form must also divide some $y_{2j_2'}$ or respectively $y_{1j_1'}$ (for $j_i' > \lambda_i$). Factoring this out of Equation (9), we arrive at the situation of a $k'$-plane in $X_{3,d-1}$, with $k' \geq d - 1$. If $k' > d - 1$, then this plane is one-split, contradicting our assumption, so in fact $k' = d - 1$. We continue in this fashion of reducing degree until we arrive at one of the two toric components of $F_2(X_{3,2})$. The component of $F_d(X_{3,d})$ is a $(d - 2)$-fold iterated $\mathbb{P}^2$ bundle over the toric component, and hence has dimension $3 + 2(d - 2) = 2d - 1$.

There are

$$2 \cdot \binom{d}{2}^3 (d - 2)!^2$$

such components: we choose one of two toric components of $F_2(X_{3,2})$; then for each index $i$ we choose two of the $y_{ij}$ which are not getting factored out. We then match each of the remaining $y_{1j}$ with a $y_{2j'}$ and $y_{3j''}$, of which there are $(d - 2)!^2$ ways.

We now leverage the above example to lower the bound on $k$ in Example 3.7 by one:

**Example 3.9** ($F_k(X_{r,d})$ for $k = (r - 2)(d - 1) + 1$). Let $L$ be any $k$-plane of $X_{r,d}$, for $k = (r - 2)(d - 1) + 1$. Similar to in Example 3.7, Conjecture 1.2 would imply that $L$ is one-split, as long as $r \geq 5$. As in Example 3.7, we successively reduce to a $k'$-plane in $X_{4,d}$, with $k' \geq 2(d - 1) + 1 = 2d - 1$. By Theorem 1.8, $L'$ is two-split.

Suppose first that $L'$ is not one-split. Then after permuting $\{1, 2, 3, 4\}$, we may assume that $y_1 + y_2 = y_3 + y_4 = 0$. The factors of $y_1$ and $y_2$ must agree up to scaling, and similarly for $y_3$ and $y_4$. Similar to the component of type $B$ in Example 3.7, we see that $L'$ is a $2d - 1$-plane of $X_{4,d}$, moving in a $2(d - 1)$-dimensional family. Thus, the plane $L$ also is moving in a $2(d - 1)$-dimensional family. It follows that the corresponding irreducible component of $F_k(X_{r,d})$ has dimension $2(d - 1)$, and there are

$$\binom{r}{r - 4, 2, 2} d^{r-4} \cdot (d!)^2$$

such components.
If $L'$ is one-split, we may reduce further to a $k''$-plane $L''$ in $X_{3,d}$ with $k'' \geq d$. Suppose next that $L''$ is not one-split. Then $k'' = d$, and $L''$ corresponds to a point in one of the $(2d-1)$-dimensional irreducible components described in Example 3.8. It follows that the corresponding irreducible component of $F_k(X_{r,d})$ has dimension $2d - 1$, and there are

$$\binom{r}{3} d^{r-3} \cdot 2 \cdot \binom{d}{2} ((d-2)!)^2$$

such components.

Finally, if $L''$ is also one-split, then we get components of types A and B similar to those appearing in Example 3.7. To summarize, assuming that the necessary splitting conjectures are true, $F_k(X_{r,d})$ for $k = (r-2)(d-1) + 1$ has the following irreducible components:

- **Type A**: $d^r$ components of dimension $2((r-2)(d-1) + 2)(d-2)$, isomorphic in their reduced structures to a Grassmannian; general $k$-planes in such components are contained in the intersection of $r$ coordinate hyperplanes.
- **Type B**: $\binom{r}{2} d^{r-2} \cdot (dl)$ components of dimension $(d-1) + ((r-2)(d-1) + 2)(d-2)$; general $k$-planes in such components are contained in the intersection of $(r-2)$ coordinate hyperplanes.
- **Type C**: there are

$$\binom{r}{3} d^{r-3} \cdot 2 \cdot \binom{d}{2} ((d-2)!)^2$$

components of dimension $2d - 1$; general $k$-planes in such components are contained in the intersection of $(r-3)$ coordinate hyperplanes.
- **Type D**: there are

$$\binom{r}{r-4,2,2} d^{r-4} \cdot (dl)^2$$

components of dimension $2(d-1)$; general $k$-planes in such components are contained in the intersection of $(r-4)$ coordinate hyperplanes.

This analysis relied on appropriate splitting statements, which (similar to Example 3.7) hold true if $r \leq 6$, $d = 4$, or $r \leq d + 1$.

**Example 3.10 ($F_5(X_{4,3})$).** For a concrete example, consider $F_5(X_{4,3})$. By Example 3.9 we see that this Fano scheme has the following components:

| Dimension Number | Type A | 12 | 81  |
|------------------|--------|----|-----|
| Type B           | 8      | 324|     |
| Type C           | 5      | 648|     |
| Type D           | 4      | 216|     |
4. Product Rank

4.1. Bounding product rank.

Proof of Theorem 4.1. Assume that $\text{pr}(f) \leq r$. Since $f$ is concise, this implies that there is an $n$-dimensional linear space $Y \subset \mathbb{P}^{rd-1}$ such that $V(f) = X_{r,d} \cap Y$. Since we are assuming that $V(f)$ is covered by $k$-planes, there must be a positive-dimensional irreducible subvariety $S \subset \mathbb{F}_k(X_{r,d})$ such that the $k$-planes corresponding to points in $S$ are all contained in $Y$, and that the linear span of these $k$-planes is exactly $Y$.

Now, if all $k$-planes parametrized by $S$ are contained in a coordinate hyperplane of $\mathbb{P}^{rd-1}$, we can clearly write $f$ as a sum of $r - 1$ products of linear forms, that is, $\text{pr}(f) \neq r$. But this is certainly the case if $\mathbb{F}_k(X_{r,d})$ is one-split.

Assume instead that $r$ is even and the two-splitting assumption of the theorem is fulfilled. As above, if every $k$-plane parametrized $S$ is contained in a coordinate hyperplane, we are done. Otherwise, by the two-splitting assumption, we can permute the indices $i = 1, \ldots, r$ such that every $k$-plane $L$ parametrized by $S$ is contained in

$$V(x_{i1} \cdots x_{id} + x_{(i-1)1} \cdots x_{(i-1)d})$$

for $i = 2, 4, \ldots, r$. Using the notation from §4, this tells us that

$$y_{i1} \cdots y_{id} + y_{(i-1)1} \cdots y_{(i-1)d} = 0, \tag{12}$$

for $i = 2, 4, \ldots, r$. After reordering the $y_{ij}$ for each fixed $i$, we conclude (by unique factorization of polynomials) that the forms $y_{ij}$ and $y_{(i-1)j}$ are proportional for $i = 2, 4, \ldots, r$ and $j \leq d$ for all $k$-planes $L$ in $S$.

For some fixed $s = 2, 4, 6, \ldots, r$, suppose that the ratio $y_{s1}/y_{(s-1)1}$ is some constant $c_s$ as $L$ ranges over $S$. Note that these constants satisfy $\prod c_j = -1$. Then every $L$ in $S$ is contained in the linear space

$$V \left( \{ x_{s1} - c_j x_{(s-1)1} \}_{j \leq d} \right)$$

so their span $Y$ is as well. This means that after restricting to $Y$ we have

$$\sum_{i=1}^r x_{i1} \cdots x_{id} = \sum_{i \neq s, s-1} x_{i1} \cdots x_{id}$$

so the product rank of $f$ is at most $r - 2$.

We have thus arrived in the situation where for each fixed $i = 2, 4, \ldots, r$, there is some $j \leq d$ such that the ratio between $y_{ij}$ and $y_{(i-1)j}$ is non-constant over $S$. A straightforward calculation shows that the dimension of the span of two general $k$-planes $L, L'$ in $S$ must be at least $k + r$, leading to the inequality $k + r \leq n$; by assumption, this is a contradiction. □

Remark 4.1. Suppose that in the situation of part two of Theorem 4.1, we know that the family of $k$-planes $S \subset \mathbb{F}_k(V(f))$ covering $V(f)$ is $m$-dimensional. Then the hypothesis $k > n-r$ may be replaced with the condition $k > n-m-r/2$. Indeed, in the conclusion of the proof of the theorem, the assumption on the dimension of $S$ guarantees that at least $m$ of the ratios $y_{ij}/y_{(i-1)j}$ vary independently of each other. Combining Equation (12) with the fact that at least one ratio $y_{ij}/y_{(i-1)j}$ varies for each $i$ guarantees that in fact a total of at least $m + r/2$ ratios vary. As above, this shows that the dimension of the span of two general $k$-planes $L, L'$ in $S$ must be at least $k + m + r/2$, leading to the desired contradiction.
4.2. Examples of bounds on product rank.

Example 4.2 ($3 \times 3$ determinant). In [IT16], Z. Teitler and the first author prove that $\text{pr}(\det_3) > 4$ over $\mathbb{C}$, where $\det_3$ is the determinant of a generic $3 \times 3$ matrix. H. Derksen gave an expression for $\det_3$ as a sum of 5 multihomogeneous products of linear forms in [Der16], so we conclude $\text{pr}(\det_3) = 5$. This also shows that the tensor rank of $\det_3$ equals five.

The proof that $\text{pr}(\det_3)$ consisted of a computer calculation showing that $F_5(X_{4,3})$ is 2-split, and then a special case of Theorem 1.9. Our Theorem 1.8 makes this computer calculation unnecessary, and is valid in arbitrary characteristic. We conclude that the product and tensor ranks of $\det_3$ are at least five over any field.

Derksen’s identity requires 2 to be invertible, so we conclude that except in characteristic 2, $\text{pr}(\det_3) = 5$. We do not know whether $\text{pr}(\det_3)$ is 5 or 6 in characteristic 2.

Example 4.3 ($4 \times 4$ determinant). Let $\det_4$ be the $4 \times 4$ determinant; this is easily seen to be a concise form. The projective hypersurface $V(\det_4)$ is covered by 11-dimensional linear spaces, see e.g. [CI15]. By Theorem 1.8, we know that $F_{11}(X_{6,4})$ and $F_7(X_{4,4})$ are both 2-split, so we may apply Theorem 1.9 to conclude that $\text{pr}(\det_4) \neq 6$. A similar application of Theorem 1.8 shows that $\text{pr}(\det_4) \neq 4, 5$. If $\text{pr}(\det_4) \leq 3$, then the projective hypersurface $V(\det_4) \subset \mathbb{P}^{15}$ must be a cone, in which case every maximal linear subspace would contain a common line. But this is not the case, so we conclude that $\text{pr}(\det_4) \geq 7$ (in arbitrary characteristic).

This is exactly the bound on product rank in characteristic zero which follows from Z. Teitler and H. Derksen’s bound on Waring rank. They show that the Waring rank of $\det_4$ is at least 50 [DT15], from which follows that $\text{pr}(\det_4) \geq 7$ by [IT16, §1.2].

Our above argument for the product rank of $\det_4$ can be generalized to show that, for $n \geq 3$, $\text{pr}(\det_n) \geq 2n - 1$, as long as we assume that Conjecture 1.3 holds. However, for $n \geq 5$ this is much worse than the bound that follows from known lower bounds on Waring rank [DT15].

Example 4.4 ($6 \times 6$ Pfaffian). Let $f$ be the Pfaffian of a generic $6 \times 6$ skew-symmetric matrix; this is also a concise form. Derksen and Teitler show that the Waring rank of $f$ is at least 24 [DT15]. Section 1.2 of [IT16] then implies that $\text{pr}(f) \geq 6$.

We will use Theorem 1.9 to show that $\text{pr}(f) \neq 6$, and hence $\text{pr}(f) \geq 7$, a new lower bound. First note that by Theorem 1.8, $F_9(X_{6,3})$ is one-split. Secondly, we have that $V(f) \subset \mathbb{P}^{14}$ is covered by projective 9-planes. Indeed, for any $6 \times 6$ singular skew-symmetric matrix $A$ with $0 \neq v \in \mathbb{K}^6$ in its kernel, consider the linear space of all $6 \times 6$ skew-symmetric matrices $B$ satisfying

$$B \cdot v = 0.$$

This is clearly a linear space of singular skew-symmetric matrices containing $A$. There are six linear conditions cutting out this linear space, but they are linearly dependent, since

$$v^{tr} \cdot A \cdot v = 0.$$

Hence, $A$ is contained in a linear space of dimension $14 - 5 = 9$. The claim $\text{pr}(f) \neq 6$ now follows from Theorem 1.9.
For our final example, we must use a different argument than Theorem 1.9 since the permanental hypersurface is not covered by high-dimensional linear spaces:

**Example 4.5** \((4 \times 4\) permanent). Let \(\text{perm}_4\) be the permanent of a generic \(4 \times 4\) matrix. Assume that the characteristic of \(K\) is not two, in which case Example 4.3 applies.

Shahefi has shown that the Waring rank of \(\text{perm}_4\) is at least 35 \cite{Sha15}, from which follows that \(\text{pr}(\text{perm}_4) \geq 5\) by \cite{IT16} §1.2. We will show that in fact, \(\text{pr}(\text{perm}_4) \geq 6\). On the other hand, Glynn’s formula gives 8 as an upper bound for the product rank of \(\text{perm}_4\) \cite{Gly10}. Our result also gives a lower bound of 6 on the tensor rank of \(\text{perm}_4\).

To fix notation, suppose that \(\text{perm}_4\) is the permanent of the matrix

\[
M = \begin{pmatrix}
z_{11} & z_{12} & z_{13} & z_{14} \\
z_{21} & z_{22} & z_{23} & z_{24} \\
z_{31} & z_{32} & z_{33} & z_{34} \\
z_{41} & z_{42} & z_{43} & z_{44}
\end{pmatrix}.
\]

The hypersurface \(V(\text{perm}_4) \subset \mathbb{P}^{15}\) contains exactly 8 11-planes \cite{CI15}: \(H_i\) and \(V_j\) for \(i, j \leq 4\) are given respectively by the vanishing of the \(i\)th row or \(j\)th column of \(M\). Any two of the \(H_i\), or any two of the \(V_j\) span the entire space \(\mathbb{P}^{15}\).

If \(\text{pr}(\text{perm}_4) \leq 5\), then \(V(\text{perm}_4)\) is isomorphic to \(X_{5,4}\) intersected with a 15-dimensional linear space. If under the embedding of \(V(\text{perm}_4)\) in \(X_{5,4}\), two of the \(H_i\) or two of the \(V_j\) are contained in a common coordinate hyperplane, it follows that \(\text{pr}(\text{perm}_4) \leq 4\), contradicting the above bound. Thus, we may assume that this is not the case.

On the other hand, it follows from Theorem 1.8 that any 11-plane of \(X_{5,4}\) is contained in the intersection of three coordinate hyperplanes. Since no pair of \(H_i\) or \(V_j\) is contained in a common coordinate hyperplane, some pair \((H_a, V_b)\) must be contained in a common coordinate hyperplane. Now, \(H_a\) and \(V_b\) span the 14-dimensional linear space \(L = V(z_{ab}) \subset \mathbb{P}^{15}\). But since this 14-dimensional linear space is contained in a coordinate hyperplane of \(X_{5,4}\), we conclude that \(\text{pr}(\text{perm}_4') \leq 4\), where \(\text{perm}_4'\) is obtained from \(\text{perm}_4\) by setting \(z_{ab} = 0\). We will show that this cannot be.

Indeed, in such a situation we would have \(V(\text{perm}_4') \subset \mathbb{P}^{14}\) isomorphic to the intersection of \(X_{4,4}\) with a 14-dimensional linear space. Intersecting \(H_i\) and \(V_j\) with \(L\), we arrive at a set of 8 11- or 10-dimensional planes \(H_i', V_j'\) with properties similar to above. Similar to above, if a pair of \(H_i'\) or \(V_j'\) is contained in a coordinate hyperplane in \(X_{4,4}\), then we would have that \(\text{pr}(\text{perm}_4') \leq 3\). But if this is not the case, an argument similar to above shows that \(\text{pr}(\text{perm}_4'') \leq 3\), where \(\text{perm}_4''\) is obtained from \(\text{perm}_4\) by setting two variables equal to zero. The key step of the argument is here another application of Theorem 1.8, showing that any 10-plane of \(X_{4,4}\) is contained in four coordinate hyperplanes.

To arrive at the final contradiction, first note that \(\text{pr}(\text{perm}_4') \leq 3\) implies \(\text{pr}(\text{perm}_4'') \leq 3\). The latter implies in particular that one can write \(\text{perm}_4''\) as a form in 12 variables, that is \(V(\text{perm}_4'') \subset \mathbb{P}^{13}\) must be a cone. However, utilizing the natural torus action on \(V(\text{perm}_4'')\) similar to in \cite{CI15} Proposition 2.3, one easily verifies that the intersections of \(H_1, H_2, H_3, H_4\) with this \(\mathbb{P}^{13}\) are all maximal linear subspaces of \(V(\text{perm}_4'')\). But their common intersection is empty,
which contradicts $V(\text{perm}_4''')$ being a cone. We conclude that $pr(\text{perm}_4''') > 3$, which in turn implies $pr(\text{perm}_4') > 4$, which finally implies $pr(\text{perm}_4) > 5$.

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