Abstract
e-Valuate is a game on arithmetic expressions. The players have contrasting roles of maximizing and minimizing the given expression. The maximizer proposes values and the minimizer substitutes them for variables of his choice. When the expression is fully instantiated, its value is compared with a certain minimax value that would result if the players played to their optimal strategies. The winner is declared based on this comparison.

We use a game tree to represent the state of the game and show how the minimax value can be computed efficiently using backward induction and alpha-beta pruning. The efficacy of alpha-beta pruning depends on the order in which the nodes are evaluated. Further improvements can be obtained by using transposition tables to prevent reevaluation of the same nodes. We propose a heuristic for node ordering. We show how the use of the heuristic and transposition tables lead to improved performance by comparing the number of nodes pruned by each method.

Keywords: Arithmetic expressions, game trees, alpha-beta pruning

1 Introduction

Given an arithmetic expression $E$ involving variables and the standard operators (+, −, *, and /), players Amogha and Dhruva evaluate $E$ with contrasting goals; Amogha would like to maximize $E$ while Dhruva would like to minimize $E$. Towards this end, they take turns to instantiate the variables. Amogha starts and, at each move, proposes a value (digit 0–9) and Dhruva substitutes the value for a variable of his choice. When the expression is fully instantiated, it is evaluated and compared with a certain minimax value that would result if the players played to their optimal strategies. Let $\text{val}(E)$ be the value of $E$ at the
end of the game and $\text{minimax}(E)$ be the minimax value. The winner is then determined in the following way.

- If $\text{val}(E) > \text{minimax}(E)$ then Amogha is declared the winner.
- If $\text{val}(E) < \text{minimax}(E)$ then Dhruva is declared the winner.
- If $\text{val}(E) = \text{minimax}(E)$, then the game is a draw.

For example, if $E = X \ast (Y - Z)$, a possible sequence of moves is

1. Amogha chooses 5 and Dhruva replaces $X$ with 5 so that $E = 5 \ast (Y - Z)$.
2. Next Amogha chooses 3 that Dhruva substitutes 3 for $Z$ leading to $E = 5 \ast (Y - 3)$.
3. Finally Amogha chooses 9 which Dhruva substitutes for the remaining variable $Y$ and the final value for the expression is $5(9 - 3) = 30$.

With more strategic play from either player, the expression is evaluated differently. For instance, with the same moves from Amogha and optimal play from Dhruva, the substitutions would be $5 \rightarrow Y, 3 \rightarrow X$ and $9 \rightarrow Z$ and the expression evaluates to $-12$. With optimal play from both players, a possible sequence of moves is $6 \rightarrow Y, 3 \rightarrow X$ and $0 \rightarrow Z$ with $E$ evaluating to the minimax value 18.

We will refer to this version of the game as e-Valuate. Specific instances of the game have appeared in books on mathematical puzzles. For example, in [9], the expression is a difference of two four digit numbers and the reader is asked to find the minimax value.

Some possible variations on this form of the game are the following.

- The expression as well as the domain can be generalized. For example, other mathematical operators can be introduced in the expression and the domain can include other values over which the expression can be evaluated.

- An alternate way of playing the game is for the players to switch roles at the end of the game and reevaluate the expression. If the expression evaluates to a larger value in one of the games then the maximizer in that game is the winner. This version could be applicable when the number of variables is large enough that computing the minimax value is infeasible.

Another variant is for the first player to take on the role of the minimizer and the second player that of the maximizer. This is however equivalent to the original version since $\min(E) = -\max(-E)$ and $\max(E) = -\min(-E)$ where the minimum and maximum are carried out over the domain of the variables. Thus, the final value under optimal play from both players is $-\text{minimax}(-E)$. 
Minimax is a more general term and applies to any two player zero-sum game. By using a game tree to represent the states of the game and the moves of the players, the minimax algorithm can be used to determine the best move at each position in the game in the following manner. First values are assigned to the leaf nodes using an evaluation function. Next, the players MAX and MIN attempt to maximize and minimize the value of the nodes corresponding to their turn of play. For an intermediate node that corresponds to MAX’s turn to play, the value of the node is the maximum of the values of its children. Similarly, for an intermediate node that corresponds to MIN’s turn to play, the value of the node is the minimum of the values of its children. The value at the root is the minimax value of the game. For example, if a game is designed such that under optimal play, MIN has a winning strategy, and the leaf nodes are assigned a value of +1 or −1 according to whether the corresponding position is a win for MAX or MIN, then the minimax value will be −1.

Several optimizations to this method of computing the minimax value have been studied. Some well known techniques are

- **Alpha-beta pruning**: This is a windowing procedure that starts with an interval of $(−\infty, +\infty)$ for the minimax value. As nodes are evaluated, the window shrinks and any node that evaluates to a value outside this window is pruned along with the subtree rooted at that node.

- **Negascout**: Negascout works by assuming that for each node, the first child will be in the principal variation (the sequence of moves leading to the minimax value). It uses a null search window for the remaining children and on failure, uses a full search window. Thus this method is most effective when there is a good ordering for evaluating the nodes.

- **Transposition tables**: This is a memoization technique where the values of nodes that are evaluated are stored and retrieved when another node that corresponds to the same game position has to be evaluated. This effectively prunes the subtree rooted at that node.

The computational challenge in e-Valuate is an efficient way of determining the minimax value in order to identify the winner. We show how these techniques lead to more efficient ways of determining $\minimax(E)$.

In the next section, we introduce the game tree for e-Valuate and show how the minimax value can be computed using backward induction. In Section we show how improved performance can be obtained by combining the minimax algorithm with alpha-beta pruning. We describe these methods in the context of our game. The efficacy of alpha-beta pruning methods depends on the order in which the the children of each node are evaluated. We describe a heuristic for determining this order. Further improvements can be obtained by avoiding
repeated reevaluation of the same game position through the use of transposition tables. In Section 4, we provide implementation details and compare the number of nodes pruned by the two methods, alpha-beta and alpha-beta with node ordering and transposition tables, for different arithmetic expressions. We conclude with some unanswered questions related to this game.

We fix some notations. For an arithmetic expression $E$, let $n$ be the number of variables in $E$. $E(i \rightarrow X)$ denotes the expression $E$ with variable $X$ replaced by $i$. The players MAX and MIN will denote the maximizer and minimizer respectively.

For general aspects of game theory, see [5]; [3] is a useful online resource for lectures, glossary of terms and articles related to game theory. The game algorithms we have outlined above are well documented in books on artificial intelligence (e.g. [4], [8]).

2 The Game Tree for e-Valuate

In the framework of game theory, e-Valuate can be classified as a finite, sequential, two person game with perfect information. It is finite as the game ends after a finite number of moves, sequential since the players take turns in making their moves (rather than move simultaneously as in the rocks, paper and scissors game) and it’s a game of perfect information as each player is aware of the other’s moves at any point in the game.

Sequential games with perfect information can be represented using a game tree. The root of the tree corresponds to the initial configuration of the game (in our case, the expression $E$) and the edges represent possible moves that the players make. Each node in the tree represents a position in the game. The root and the leaf nodes are MAX nodes and the nodes at intermediate levels are alternately MAX and MIN nodes and represent positions where the maximizer or minimizer has to make a move. Thus each MAX node has 10 children corresponding to 10 possible moves (choosing any digit). A MIN node at a height $d$ has $(d+1)/2$ children that correspond to $(d+1)/2$ uninstantiated variables. The height of the tree is $2n$. We will denote by $tree(E)$, the game tree corresponding to $E$.

The number of nodes in the tree, $T(n)$, depends only on $n$ and satisfies the recursion

$$T(n) = 11 + 10nT(n - 1)$$

which follows from observing that the root node has 10 children each of which has $n$ children that correspond to game trees on expressions with $(n - 1)$ variables.
We can use this to bound $T(n)$ by

$$2n!10^n \leq T(n) \leq 2n!10^n e^{1/10}$$

from the following argument. Let $N = n!10^n$ be the number of leaves of $\text{tree}(E)$. Starting from the bottom and counting the number of nodes at each level we get

$$T(n) = N + N/1 + N/(1 \cdot 10) + N/(1 \cdot 10 \cdot 2) + \cdots + N/(n!10^n)$$

$$= \sum_{i=0}^{n} N/(i!10^i) + \sum_{i=1}^{n} N/(i!10^{i-1}) \leq 2 \sum_{i=0}^{n} N/(i!10^i) \leq 2e^{1/10}N$$

as desired.

We identify each node in a tree by

- a sequence of instantiations of the variables and possibly an additional digit (for a MIN node). For example if $E = (10 - X) \cdot Y$, then a MAX node in $\text{tree}(E)$ is $\{1 \to Y\}$ and a MIN node is $\{1 \to Y, 3\}$. Thus MAX nodes correspond to partially instantiated expressions and MIN nodes to (expression, digit) pairs.

- a value which is the minimax value of the partially instantiated expression for a MAX node and the minimum of the minimax values of the children for a MIN node. This is the value $E$ would evaluate to under optimal play starting from the position given by the node. This is also referred to as the score of the position given by the node [4].

The game tree $\text{tree}((10 - X) \cdot Y)$ is shown partially in Figure 2. The edges are labelled by the moves corresponding to the players.

The minimax value is computed by the method of backward induction applied to $\text{tree}(E)$. This procedure works by reasoning backwards from the end of the game and computing the optimal move for the players at each position. At a terminal (MAX) node, the expression is a constant, and the value of the node is this constant. Working up, each MIN node has as its value, the minimum of the values of its valid children and each MAX node, the maximum of the values of its valid children. The value of the root is $\text{minimax}(E)$.

### 3 Alpha-beta Pruning and Node Ordering

To determine $\text{minimax}(E)$, it’s not necessary to evaluate every node in the tree. Suppose alpha is the current maximum (over the children evaluated so far) for

\[\text{An internal node is deemed valid if it has a valid child. A terminal node is valid if it evaluates to a finite value}\]
Figure 1: A partial game tree for $E = (10 - X) \ast Y$

a MAX node and beta the current minimum for a MIN node. For a MAX node, if its alpha value is at least the beta value of its parent, then there is no reason to explore the node further as the final value of its parent will be smaller than alpha. Each pruning of a subtree of a MAX node in this manner is referred to as a beta cutoff. Similarly, for a MIN node, if its beta value is at most the alpha value of its parent then the remaining subtrees of this node can be pruned as the final value of its parent will be larger than beta. These prunings are alpha cutoffs.

A subtree at height $d$ that is pruned by an alpha cutoff is rooted at a MAX node and prunes $T(d/2)$ nodes. A subtree at height $d$ that is pruned by a beta cutoff is rooted at a MIN node and prunes $(T(\lceil d/2 \rceil) - 1)/10$ nodes.

For example, suppose $E = (10 - X) \ast Y$. Then $\text{minimax}(E) = 45$ and a terminal node that achieves this value is $\{5 \rightarrow X, 9 \rightarrow Y\}$. To compute $\text{minimax}(E)$, we start at the root node and evaluate the MIN nodes $\{0\}, \{1\}, \ldots, \{5\}$ in succession which return the values $0, 10, 20, 30, 40$ and $45$ respectively. At this point, the alpha value at the root is $\max(0, 10, 20, 30, 40, 45) = 45$. When node $\{6\}$ is explored, the MIN node computes the value of the MAX node $\{6 \rightarrow X\}$ which returns $36$ as its minimax value. Thus the beta value of $\{6\}$ is $36$ which is smaller than $45$, the alpha value of its parent. As a result, the node $\{6 \rightarrow Y\}$ is not evaluated. Similarly, the nodes $\{7 \rightarrow Y\}, \{8 \rightarrow Y\}$ and $\{9 \rightarrow Y\}$ are not
evaluated leading to 4 alpha cutoffs.

The pseudocode for computing the minimax value of $E$ with alpha-beta pruning is given by Algorithm 1. The function alphabeta() takes as its parameters, the current node, its height, the current value (alpha or beta) of the parent node, the digit passed (valid for a MIN node) and the current player. Apart from the minimax value, the algorithm also returns the number of nodes pruned by alpha and beta cutoffs, which are computed using the recursion formula (1), as well as the entire principal variation. The function is called with the command

\[
\text{alpha.prunes} = \text{beta.prunes} = 0; \text{principal.var} = ''; \text{alphabeta (root, 2n, } \infty, -1, \text{MAX)}
\]

### 3.1 A Heuristic for Node Ordering

The effectiveness of alpha-beta pruning depends on the order in which each node’s children are explored. For example, for the expression $E = (10 - X) \times Y$, suppose we evaluate a MAX node by choosing the digits in sequence \{5, 4, 6, 3, 7, 2, 8, 1, 9, 0\}, and evaluate a MIN node by setting the variable sequence as $(Y, X)$ if the digit passed to it is less than 5 and as $(X, Y)$ if the digit passed to it is at least 5. Then, to calculate minimax($E$), the node \{5\} is evaluated first and returns 45. Subsequently, for each of the MIN nodes, the order in which its children are explored ensures that there is an alpha cutoff.

Let $\tilde{v}(x)$ be an estimate for the value $v(x)$ of node $x$. We propose a heuristic for determining the order in which the digits are to be chosen at a MAX node. The ordering is static in the sense that it is determined by $E$ and is the same for all nodes being evaluated. We estimate the values of the MAX nodes 2 levels below the root node. These estimates are backed up, by taking the minima, to estimate the values of their parents. If these estimates are placed in decreasing order, as $\tilde{v}(\{i_0\}) \geq \tilde{v}(\{i_1\}) \geq \cdots \geq \tilde{v}(\{i_9\})$ then the children of a MAX node are evaluated in sequence $i_0, i_1, \ldots, i_9$.

For a MAX node $x = \{i \rightarrow X\}$, our estimate for $v(x)$ is simply the maximum of $E$ over some random instantiations of the variables of $E$ while fixing $X$ at $i$. More precisely, to estimate $v(\{i \rightarrow X\})$, we fix $X$ at $i$ and randomly instantiate the other variables in $E$ with digits and compute val($E$). We do this a fixed number of times and take the maximum of the resulting values.

The performance of minimax algorithm is further enhanced by noting that several nodes in tree($E$) correspond to the same game position and thus have to be evaluated only once. An example are nodes $\{2 \rightarrow X, 1 \rightarrow Y\}$ and $\{1 \rightarrow Y, 2 \rightarrow X\}$ in tree($X \ast (Y - Z)$). We exploit this fact by storing, for each MAX node $x$ that is fully evaluated (i.e. none of its children are pruned by beta cutoffs), its value and the principal variation starting at $x$. On subsequent
Algorithm 1 Minimax value of $E$ with alpha-beta pruning

function alphabeta(node, height, parent_αβ, digit, player)
    if height = 0  ▷ terminal node
        principal_var = ""
        return the value of node
    if player = MAX  ▷ process MAX node
        maxstr = ""  ▷ principal variation from this node
        maxval = $-\infty$  ▷ current $\alpha$
        for each $i$ from 0 to 9  ▷ evaluate each child in this loop
            value = alphabeta (node, height - 1, maxval, $i$, MIN)
            if value > maxval and value $\neq +\infty$  ▷ update $\alpha$
                maxval = value
                maxstr = principal_var + 'i'
            if maxval $\geq$ parent_αβ  ▷ beta prune
                beta_prunes = beta_prunes + $(9-i) \times (T(\text{height}/2) - 1)/10$
                break
        principal_var = maxstr
        return maxval
    else
        minstr = ""
        minval = $+\infty$  ▷ current $\beta$
        $j = (\text{height} + 1)/2$  ▷ number of children left to explore
        for each uninstantiated variable $v$ in node
            $j = j - 1$
            grandchild = node (digit $\rightarrow v$)  ▷ replace $v$ by digit in node
            value = alphabeta (grandchild, height - 1, minval, $-1$, MAX)
            if value < minval and value $\neq -\infty$  ▷ update $\beta$
                minval = value
                minstr = principal_var + 'v'
            if minval $\leq$ parent_αβ  ▷ alpha prune
                alpha_prunes = alpha_prunes + $j \times T((\text{height} - 1)/2)$
                break
        principal_var = minstr
        return minval
end function
Table 1: Comparison of Alpha-beta and Alpha-beta with Node Ordering

visits to nodes that correspond to the same game position, this value is retrieved instead of being recomputed.

4 Implementation Details

We first convert $E$ to a postfix form using Dijkstra’s shunting yard algorithm [2]. During evaluation, the variables are substituted with values, and val($E$) is computed using the reverse polish notation evaluation [1] algorithm.

Table 1 compares the number of nodes pruned by alpha-beta and alpha-beta with node ordering and also shows the ordering of digits at each MAX node as determined by the heuristic. For the alpha-beta method, the number of nodes pruned is the sum of the number of nodes pruned by alpha and beta cutoffs. For alpha-beta with node ordering, the number of nodes pruned is the sum of the number of nodes pruned by alpha and beta cutoffs and the transposition tables. For expressions with five or six variables, we have observed a ten-fold speedup in the performance of the second method over the first.

We also attempted ordering the MIN nodes as well as using different orderings for MAX nodes at different heights using the same heuristic but any gains in the number of nodes pruned was offset by the computational time in determining the order. Other promising approaches such as Negascout [7] and the MTD-f [6] algorithm have not been attempted yet.
5 Conclusion

We have demonstrated the effectiveness of search algorithms for computing the minimax value of e-Valuate. Other heuristics for node values could yield more effective ordering of the nodes and thus faster algorithms.

One would also like to understand what expressions and associated domains constitute a fair game. A fair game is one where if MAX and MIN make their moves randomly, they have equal chances of winning. For example, if $E$ has only $+$ and $*$ operators or is defined on one variable, then $\text{minimax}(E) = \max(E)$ and MIN can never lose.

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References

[1] Arthur W. Burks, Don W. Warren and Jesse B. Wright, *An Analysis of a Logical Machine Using Parenthesis-Free Notation*, Mathematical Tables and Other Aids to Computation, Vol. 8, No. 46 (1954), pages 53–57

[2] E. W. Dijkstra, *Algol 60 translation: An algol 60 translator for the x1 and making a translator for algol 60*, report MR 35/61. Mathematisch Centrum, Amsterdam, 1961.

[3] Game Theory .net - Resources for Learning and Teaching Strategy for Business and Life, [http://www.gametheory.net](http://www.gametheory.net)

[4] Ian Millington and John Funge, *Artificial Intelligence and Games*, 2nd Ed., Elsevier, 2009.

[5] Martin J. Osborne and Ariel Rubinstein, *A Course in Game Theory*, The MIT Press, 1994.

[6] Aske Plaat, Jonathan Schaeffer, Wim Pijls, and Arie de Bruin, *Best-First Fixed-Depth Game-Tree Search in Practice*, Fourteenth International Joint Conference on Artificial Intelligence, IJCAI’95, Montreal, Canada, volume 1, pages 273–279.

[7] Alexander Reinefeld, *Spielbaum-Suchverfahren*, Informatik-Fachbericht 200, Springer-Verlag, 1989.

[8] Stuart Russell and Peter Norvig, *Artificial Intelligence: A Modern Approach*, 3rd Ed., Upper Saddle River, New Jersey: Pearson Education.

[9] Peter Winkler, *Mathematical Puzzles: A Connoisseur’s Collection*, A. K. Peters, 2004.