Localizing Vector Optimization Problems with Application to Welfare Economics

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Abstract

In the present paper, the Polyak’s principle, concerning convexity of the images of small balls through $C^{1,1}$ mappings, is employed in the study of vector optimization problems. This leads to extend to such a context achievements of local programming, an approach to nonlinear optimization, due to B.T. Polyak, which consists in exploiting the benefits of the convex local behaviour of certain nonconvex problems. In doing so, solution existence and optimality conditions are established for localizations of vector optimization problems, whose data satisfy proper assumptions. Such results are subsequently applied in the analysis of welfare economics, in the case of an exchange economy model with infinite-dimensional commodity space. In such a setting, the localization of an economy yields existence of Pareto optimal allocations, which, under certain additional assumptions, lead to competitive equilibria.

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Key words: modulus of convexity; Polyak’s convexity principle; openness at a linear rate; Lagrangian function; vector optimization; $\epsilon$-localization of a problem; exchange economy; regular feasible allocation; Pareto optimality; competitive equilibrium.

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1 Introduction

In a series of papers appeared at the beginnings of the current millennium (see [20, 21, 22]), the term “local programming” was used to denote the theory emerging in connection with a special class of nonlinear optimization problems. This class includes mathematical programming problems, with equality and inequality constraints, that, even in the absence of convexity assumptions on their data, surprisingly do exhibit a local behaviour, which is typical of convex optimization problems. The doubtless advantages arising when one handles problems with convex data should underline the importance of local programming. The unexpected appearance of a local convex behaviour within the “ocean of nonlinear optimization” has a deep reason, resting upon the Polyak’s convexity principle. This crucial achievement of modern nonlinear analysis states, in its original formulation, that a mapping between Hilbert spaces, which is \( C^{1,1} \) around a regular point, carries balls centered at that point to convex sets, provided that the radius of the balls is small enough. In many questions related to optimization, it is already the convexity of images of sets, not only that of the involved functions, which does the trick, with a lot of proficuous consequences. Then reason why \( C^{1,1} \) smoothness of a mapping along with its regularity should imply convexity of the image of balls is even deeper, having to do with profound geometric properties of the underlying space and with the preservation of convexity through linear approximations.

In the present paper, such a ultimate reason is left at that. Instead, the main theme is the extension of the local programming approach to vector optimization. In fact, also in such context, a class of nonlinear problems can be singled out, whose local convex behaviour bear interesting consequences. The study of them is carried out in a general setting. Constrained vector optimization problems will be supposed to be defined in a proper subclass of reflexive Banach spaces. Nonetheless, some of the findings that are going to be presented here seem to be novel even for finite-dimensional problems.

The material exposed in the paper is organized as follows. In Section 2 key concepts and results from nonlinear analysis, essentially employed in subsequent investigations, are recalled, along with the most part of the notation in use throughout the paper. Section 3 contains the main result of the paper, describing the effect of localizing problems in nonlinear vector optimization. It deals, in particular, with existence of solutions and optimality conditions for detecting them. Section 4 is reserved for an application of the main result to a topic of mathematical economics, known as welfare theory. More precisely, a model of (pure) exchange economy, with an infinite-dimensional commodity space and finitely many consumers, is considered. In such model, the existence of Pareto optimal allocations, which, under an adequate qualification, turn out to be also equilibria, for proper localizations of the original economy is obtained.

2 Mathematical preliminaries

Throughout the paper, whenever \((X, \| \cdot \|)\) denotes a Banach space, \( B(x; r) \) denotes the ball with centre at \( x \in X \) and radius \( r \geq 0 \). The null vector of a Banach space is marked by \( 0 \). The unit ball, i.e. the set \( B(0; 1) \), is simply denoted by \( \mathbb{B} \), whereas the unit sphere by \( S \). If \( S \) is a subset of a Banach space, \( \text{int} S \), \( \text{bd} S \) and \( \text{cl} S \) denote the interior, the boundary and the (topological) closure of \( S \), respectively. Fixed \( x \in S \), \( N(S; x) \) denotes the normal cone to \( S \) at \( x \) in the sense of convex analysis.

For the purposes of the present analysis, general Banach spaces are a setting too wide. In fact, the main result presented in this paper and its application essentially rely on certain geometrical features of a specific class of Banach spaces, features that are related to the rotundity of the balls. The rotundity property of a Banach space \((X, \| \cdot \|)\) can be quantitatively described by means of the function \( \delta_X : [0, 2] \to [0, 1] \), defined by

\[
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{x_1 + x_2}{2} : x_1, x_2 \in \mathbb{B}, \|x_1 - x_2\| \geq \epsilon \right\},
\]

which is called the modulus of convexity of \((X, \| \cdot \|)\). Notice that \( \delta_X \) is not invariant under equivalent renormings.
of $X$. Such notion allows one to define a special class of Banach spaces, whose introduction is due to J.A. Clarkson (see, for instance, [5, 9, 14]).

**Definition 2.1.** A Banach space $(X, \| \cdot \|)$ is called *uniformly convex* (or, *uniformly rotund*) if it is $\delta_X(\epsilon) > 0$ for every $\epsilon \in (0, 2]$.

In what follows, the modulus of convexity of a (uniformly convex) Banach space is said to fulfil the *quadratic growth condition* if there exists $\kappa > 0$ such that

$$\delta_X(\epsilon) \geq \kappa \epsilon^2, \quad \forall \epsilon \in [0, 2].$$

The class of uniformly convex Banach spaces, with modulus of convexity fulfilling the quadratic growth condition, reveals to be the proper setting, in which to develop the analysis of the question under consideration.

**Example 2.1.** (i) Since the modulus of convexity of a Hilbert space $H$ can be easily calculated to be

$$\delta_H(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}, \quad \forall \epsilon \in [0, 2],$$

it is clear that every Hilbert space is uniformly convex, with a modulus of convexity fulfilling the quadratic growth condition with $0 < \kappa \leq 1/8$.

(ii) More generally, such Banach spaces as $l^p$, $L^p$, and $W^p_m$, with $1 < p < 2$, are known to have a modulus of convexity satisfying the relation

$$\delta_{l^p}(\epsilon) = \delta_{L^p}(\epsilon) = \delta_{W^p_m}(\epsilon) > p - \frac{1}{8} \epsilon^2, \quad \forall \epsilon \in (0, 2].$$

Therefore, they also are an example of uniformly convex space with a modulus of convexity satisfying the quadratic growth condition (see, for instance, [9]).

(iii) Concerning the notion of uniform convexity, a caveat is due: even finite-dimensional Banach spaces may fail to be uniformly convex. Consider, for instance, $\mathbb{R}^2$ equipped with the Banach space structure given by the norm $\| \cdot \|_{\infty}$.

**Remark 2.1.** (i) In [19] it was proved that the modulus of convexity $\delta_X$ of every real Banach space, having dimension greater than 1, admits the following estimate from above

$$\delta_X(\epsilon) \leq 1 - \sqrt{1 - \frac{\epsilon^2}{4}}, \quad \forall \epsilon \in [0, 2].$$

This implies that the quadratic growth is a maximal one.

(ii) In the sequel, the fact will be used that every uniformly convex Banach space is reflexive. In the Banach space theory, such result is known under the name of Milman-Pettis theorem (see [14]). It is worth mentioning that uniform convexity is not characterized by reflexivity. Indeed, in [6] a large class of reflexive (separable and strictly convex) Banach spaces is exhibited, which are not isomorphic to uniformly convex spaces.

(iii) Let $(X, \| \cdot \|)$ be a uniformly convex Banach space, with a modulus of convexity $\delta_X$ fulfilling the quadratic growth condition, and $n \in \mathbb{N}$. As a consequence of Theorem 5.2.25 in [14], the space $(X^n, \| \cdot \|_2)$, where the direct sum space $X^n = X \oplus \ldots \oplus X$ is normed with the $2$-norm $\| \cdot \|_2 : X^n \to [0, +\infty)$

$$\|(x_1, \ldots, x_n)\|_2 = \left(\sum_{i=1}^{n} \|x_i\|_2^2\right)^{1/2},$$

is a uniformly convex space.
Another key concept, playing a crucial role in this paper, is openness at a linear rate, a property for mappings, which postulates a certain quantitative surjectivity behaviour. More precisely, a mapping \( f : X \to Y \) between Banach spaces is said to be \emph{open at a linear rate} around \((x_0, f(x_0))\), with \(x_0 \in X\), if there exist positive \(\delta, \zeta\) and \(\sigma\) such that

\[
f(B(x; r)) \supseteq B(f(x); \sigma r) \cap B(f(x_0); \zeta), \quad \forall x \in B(x_0; \delta), \; \forall r \in [0, \delta). \tag{1}
\]

Clearly, inclusion (1) has crucial consequences on the local solvability of the equation \( f(x) = y \) as well as on the Lipschitz behaviour of its solution set \( f^{-1}(y) \) near \(x_0\). So, it comes not surprising that many efforts have been directed to find out criteria able to detect the occurrence of such a property. The following result, known in nonlinear analysis as Lyusternik-Graves theorem, provides a characterization of openness at a linear rate for strictly differentiable mappings (see, for instance, Theorem 1.57 in [17]). Throughout the paper, the Fréchet derivative at \(x \in X\) of a mapping \(f : X \to Y\) between Banach spaces is denoted by \(Df(x)\).

**Theorem 2.1.** Let \(f : X \to Y\) be a mapping between Banach spaces. Suppose \(f\) to be strictly differentiable at \(x_0 \in X\). Then \(f\) is open at a linear rate around \((x_0, f(x_0))\) iff \(Df(x_0)\) is onto.

For the analysis conducted in the present paper, strict differentiability will be not enough. Instead, in the main result, mappings will be supposed to be \(C^{1,1}\). Recall that a mapping \(f : X \to Y\) between Banach spaces is said to be \(C^{1,1}(\Omega)\), with \(\Omega\) being an open subset of \(X\), if it admits Fréchet derivative at \(x\), for every \(x \in \Omega\), and the mapping \(Df : \Omega \to \mathcal{L}(X, Y)\) is Lipschitz continuous on \(\Omega\), where \(\mathcal{L}(X, Y)\) stands for the Banach space of all linear bounded mappings between \(X\) and \(Y\), equipped with the operator norm. In the special case \(Y = \mathbb{R}\), the symbol \(\mathcal{L}(X, Y)\) is replaced by \(X^*\). Given \(x^* \in X^*\), its kernel is denoted by \(\ker x^*\).

The main tool of analysis in the subsequent section will be the Polyak’s convexity principle. It states that \(C^{1,1}\) mappings, which are open at a linear rate around a given point, carry small balls centered at that point to convex sets. This important result was originally established for mappings between Hilbert spaces (see [20] [21] [22] and, later on, it was extended to mappings defined on uniformly convex Banach spaces, with modulus of convexity fulfilling the quadratic growth condition (see [23]). In order to give the present analysis a proper level of generality, motivated by applications to models of welfare economics exposed in the last section, the Polyak’s convexity principle is below formulated in its most recent version.

**Theorem 2.2.** Let \(f : X \to Y\) be a mapping between Banach spaces, let \(\Omega\) be an open subset of \(X\), let \(x_0 \in \Omega\), and \(r > 0\) such that \(B(x_0; r) \subseteq \Omega\). Suppose that:

(i) \((X, \| \cdot \|)\) is uniformly convex with modulus \(\delta_X\) satisfying the quadratic growth condition;

(ii) \(f \in C^{1,1}(\Omega)\) and \(Df(x_0) \in \mathcal{L}(X, Y)\) is onto.

Then, there exists \(\epsilon_0 \in (0, r)\) such that \(f(B(x_0; \epsilon))\) is convex, for every \(\epsilon \in (0, \epsilon_0]\).

**Remark 2.2.** The following complement of Theorem 2.2, already remarked in [20], will be exploited in the sequel. From hypothesis (ii) and Theorem 2.1, one has that \(f(\text{int} B(x_0; \epsilon)) \subseteq \text{int} f(B(x_0; \epsilon)) \neq \emptyset\), for every \(\epsilon \in (0, \epsilon_0]\). Therefore, it holds

\[
f^{-1}(\text{bd} f(B(x_0; \epsilon))) \subseteq \text{bd} B(x_0; \epsilon).
\]

### 3 A localization property in vector optimization

Consider a vector optimization problem of the following form:

\[
(VOP) \quad K-\max_{x \in X} h(x) \quad \text{subject to} \quad g(x) \in C,
\]

where \(h : X \to \mathbb{W}\) and \(g : X \to Y\) are given mappings, \(C\) is a nonempty subset of \(Y\) and \(K \subseteq \mathbb{W}\) is a closed, convex and pointed cone (with apex at the null vector of \(\mathbb{W}\)). Here \((\mathbb{X}, \| \cdot \|), (\mathbb{W}, \| \cdot \|)\) and \((\mathbb{Y}, \| \cdot \|)\) are real
which shows how openness at a linear rate of introduction may change substantially the geometry of the problem.

In other terms, $K$ can be regarded as the positive cone with respect to a partial ordering $\leq_K$ defined on $W$. By $K^\oplus = \{ w^* \in W^* : \langle w^*, w \rangle \geq 0, \ \forall w \in K \}$ the positive dual cone of $K$ is denoted. The feasible region associated with $(VOP)$ is indicated by

$$R = \{ x \in X : g(x) \in C \} = g^{-1}(C).$$

Recall that $\bar{x} \in R$ is said to be locally $K$-optimal for (or a local solution to) $(VOP)$ if there exists $r > 0$ such that

$$h(R \cap \mathcal{B}(\bar{x}; r)) \cap (h(\bar{x}) + K) = \{ h(\bar{x}) \}.$$

Of course, if in the above equality $\mathcal{B}(\bar{x}; r)$ can be replaced by $X$, $\bar{x}$ is also globally $K$-optimal for $(VOP)$.

According to a longstanding approach, in order to investigate optimization problems of the form $(VOP)$ near certain reference points of its feasible region. The approach here proposed leads to introduce the concept of problem localization. Let $\sigma r > 0$. By means of such mapping, letting

$$x \in \mathcal{B}(\bar{x} ; r) \cap (h(\bar{x}) + K)$$

defined by

$$I^{\text{VOP}}_x(x) = (h(x) - h(\bar{x}), g(x)). \quad (2)$$

By means of such mapping, letting

$$Q = (K \setminus \{ 0 \}) \times C,$$

one is in a position to formulate the following set characterization of local $K$-optimality.

**Proposition 3.1.** An element $\bar{x} \in R$ is a local solution to $(VOP)$ iff there exists $r > 0$ such that

$$I^{\text{VOP}}_x(B(\bar{x}; r)) \cap Q = \emptyset.$$

**Proof.** The proof stems directly from the definition of $I^{\text{VOP}}_x$ and from the aforementioned notion of local $K$-optimality. $\square$

Within the context of vector optimization, the issue addressed in this section deals with the local behaviour of $(VOP)$ near certain reference points of its feasible region. The approach here proposed leads to introduce the concept of problem localization. Let $x_0 \in R$ and $\epsilon > 0$. By $\epsilon$-localization of $(VOP)$ around $x_0$ the following problem is meant

$$(VOP_{x_0, \epsilon}) \quad K^- \max_{x \in \mathcal{B}(x_0, \epsilon)} h(x) \ \text{subject to} \ g(x) \in C$$

The reader should notice that, because $B(x_0; \epsilon)$ is closed, $(VOP_{x_0, \epsilon})$ actually contains a further constraint. Its introduction may change substantially the geometry of the problem.

In order to investigate the effect of localizing vector optimization, the next general proposition is needed, which shows how openness at a linear rate of $I^{\text{VOP}}_{x_0}$ can not be consistent with the $K$-optimality of a feasible element $x_0 \in R$.

**Proposition 3.2.** With reference to a problem $(VOP)$, let $x_0 \in R$. If mapping $I^{\text{VOP}}_{x_0}$ is open at a linear rate near $(x_0, (0, g(x_0)))$, then $x_0$ fails to be a solution to $(VOP_{x_0, \epsilon})$, for every $\epsilon > 0$.

**Proof.** Fix an arbitrary $\epsilon > 0$. By hypothesis, according to (1) there exist positive $\delta$, $\zeta$, $\sigma$ such that

$$I^{\text{VOP}}_{x_0}(\mathcal{B}(x; r)) \supseteq \mathcal{B}(I^{\text{VOP}}_{x_0}(x), \sigma r) \cap \mathcal{(0, g(x_0))}; \zeta, \ \forall x \in \mathcal{B}(x_0, \delta), \ \forall r \in [0, \delta). \quad (3)$$
Thus, by taking \( x = x_0 \) and \( r \) such that
\[
0 < r < \min \left\{ \epsilon, \frac{\zeta}{\sigma} \right\},
\]
one finds
\[
\mathcal{I}_{x_0}^{\mathcal{VOP}}(B(x_0; r)) \supseteq B((0, g(x_0)); \sigma r).
\]
Since \( r < \epsilon \) and \( B((0, g(x_0)); \sigma r) \cap Q \neq \emptyset \), the last inclusion entails
\[
\mathcal{I}_{x_0}^{\mathcal{VOP}}(B(x_0; r)) \cap Q \neq \emptyset,
\]
what excludes that \( x_0 \) is \( K \)-optimal for \((\mathcal{VOP}_{x_0, \epsilon})\), according to Proposition 3.1.

Now, the analysis is focussed on the subclass of those elements \( x_0 \in \mathbb{R} \), such that \( \mathcal{I}_{x_0}^{\mathcal{VOP}} \) is open at a linear rate near \((x_0, (0, g(x_0)))\). If assuming the data \( h \) and \( g \) to be at least strictly differentiable at \( x_0 \), by virtue of Theorem 2.1 the surjectivity condition on \( D(h, g)(x_0) \) singles out points at which \( \mathcal{I}_{x_0}^{\mathcal{VOP}} \) is open at a linear rate. These points, while being not solution to \((\mathcal{VOP})\), nevertheless turn out to enjoy an interesting property. Indeed, whenever the Polyak’s convexity principle can be invoked, the \( \epsilon \)-localization of \((\mathcal{VOP})\) around them, for \( \epsilon \) sufficiently small, reveal to do admit a solution, which can be detected by a method proper of convex optimization, i.e. via an optimality condition stating the maximality of such solution for the Lagrangian function. Here, by Lagrangian function associated with \((\mathcal{VOP})\), the classical function \( L : \mathbb{W}^* \times \mathbb{Y}^* \times \mathbb{X} \longrightarrow \mathbb{R} \), defined by
\[
L(w^*, y^*; x) = \langle w^*, h(x) \rangle + \langle y^*, g(x) \rangle,
\]
is meant. This localization property is formulated in the next result.

**Theorem 3.1.** With reference to \((\mathcal{VOP})\), let \( \Omega \subseteq \mathbb{X} \) open, \( C \subseteq \mathbb{Y} \) a nonempty, closed and convex set, and \( x_0 \in \Omega \cap \mathbb{R} \). Suppose that:
(i) \((\mathbb{X}, \| \cdot \|)\) is uniformly convex with modulus \( \delta_\epsilon \) satisfying the quadratic growth condition;
(ii) \((\mathbb{W}, \| \cdot \|)\) and \((\mathbb{Y}, \| \cdot \|)\) are reflexive Banach spaces;
(iii) \( h, g \in C^1,1(\Omega) \) and \( D(h, g)(x_0) \in L(\mathbb{X}, \mathbb{W} \times \mathbb{Y}) \) is onto.

Then, there exists \( \epsilon_0 > 0 \) such that for every \( \epsilon \in (0, \epsilon_0] \) there are \( x_\epsilon \in \partial B(x_0; \epsilon) \) and \( (w^*_\epsilon, y^*_\epsilon) \in (\mathbb{W}^* \times \mathbb{Y}^*) \setminus \{(0^*, 0^*)\} \) such that
\[
x_\epsilon \text{ is a global solution to } (\mathcal{VOP}_{x_0, \epsilon});
\]
\[
w^*_\epsilon \in K^\oplus \setminus \{0^*\}, \quad -y^*_\epsilon \in N(g(x_\epsilon); C);
\]
\[
L(w^*_\epsilon, y^*_\epsilon; x_\epsilon) \geq L(w^*_\epsilon, y^*_\epsilon; x), \quad \forall x \in B(x_0; \epsilon).
\]

**Proof.** Consider the mapping \( \mathcal{I}_{x_0}^{\mathcal{VOP}} \) as defined in [2], with \( \bar{x} = x_0 \). Under the assumptions made, it is \( \mathcal{I}_{x_0}^{\mathcal{VOP}} \in C^1,1(\Omega) \) and \( D\mathcal{I}_{x_0}^{\mathcal{VOP}}(x_0) = D(h, g)(x_0) \) is onto. By virtue of hypothesis (i) it is possible to invoke the Polyak’s convexity principle (Theorem 2.2). According to it, there exists \( \epsilon_0 > 0 \) such that \( \mathcal{I}_{x_0}^{\mathcal{VOP}}(B(x_0; \epsilon)) \) is a convex closed set with nonempty interior, for every \( \epsilon \in [0, \epsilon_0] \). Notice that, since \( \mathcal{I}_{x_0}^{\mathcal{VOP}} \) is continuous at \( x_0 \), then without loss of generality one can assume the set \( \mathcal{I}_{x_0}^{\mathcal{VOP}}(B(x_0; \epsilon)) \) to be also bounded. Now, fix any \( \epsilon \in (0, \epsilon_0] \). Let us denote by \( \Pi_\mathbb{W} : \mathbb{W} \times \mathbb{Y} \longrightarrow \mathbb{W} \) the projection operator on the space \( \mathbb{W} \). Define \( \bar{w} \) as one of the \( K \)-minimal element of the set
\[
\Pi_\mathbb{W}(\mathcal{I}_{x_0}^{\mathcal{VOP}}(B(x_0; \epsilon) \cap (K \times C)) \).
Let us show that such definition makes sense, that is \( \hat{w} \) does actually exist. Since \( D\mathcal{I}^{\text{VOP}}_{x_0}(x_0) \) is onto, \( x_0 \) cannot be a local solution to \( (\text{VOP}) \). According to Proposition 5.1 it must be
\[
\mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap Q \neq \emptyset
\]
and hence
\[
\mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C) \neq \emptyset.
\]
Observe that, \( \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \) being convex and closed, it is also weakly closed. As the space \( \mathcal{W} \times \mathcal{Y} \) is reflexive by hypothesis (ii), \( \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \), which is also bounded, turns out to be weakly compact. Since \( K \times C \) is convex and closed, it is also weakly closed. Thus, it is possible to deduce that the nonempty set \( (\mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C)) \) is weakly compact. On the other hand, the projection mapping \( \Pi_\mathcal{W} \) is weakly continuous on \( \mathcal{W} \times \mathcal{Y} \). This enables one to conclude that the image of \( \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C) \) through \( \Pi_\mathcal{W} \) is weakly compact. Then, by virtue of a well-known existence result for vector optimization problems (see, for instance, Theorem 6.5(a) in [12]), there must exists \( \hat{w} \in \Pi_\mathcal{W}(\mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C)) \), which is \( K \)-maximal. Now, take any element \( \hat{w}, \hat{y} \in \Pi_\mathcal{W}^{-1}(\hat{w}) \).

Corresponding to such a \( (\hat{w}, \hat{y}) \), there exists \( x_\epsilon \in B(x_0; \epsilon) \) with the property
\[
\hat{w} = h(x_\epsilon) - h(x_0), \quad \hat{y} = g(x_\epsilon).
\]
Notice that, being \( (\hat{w}, \hat{y}) \in \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C) \), it is \( x_\epsilon \in B(x_0; \epsilon) \cap R \). Let us prove that \( x_\epsilon \) verifies the first assertion in the thesis. Ab absurdo, assume that \( \hat{x} \in B(x_0; \epsilon) \cap R \) can be found such that
\[
h(\hat{x}) \in (h(x_\epsilon) + K) \setminus \{h(x_\epsilon)\}.
\]
This means that for some \( \hat{k} \in K \setminus \{0\} \) is has to be
\[
h(\hat{x}) = h(x_\epsilon) + \hat{k},
\]
whence
\[
h(\hat{x}) - h(x_\epsilon) \in K \setminus \{0\}.
\]
Thus, one obtains
\[
h(\hat{x}) - h(x_0) = h(\hat{x}) - h(x_\epsilon) + h(x_\epsilon) - h(x_0) \in \hat{w} + K,
\]
with \( h(\hat{x}) - h(x_0) \neq \hat{w} \). It follows from (7) that the \( K \)-maximality of \( \hat{w} \) is violated. Indeed, it is
\[
(h(\hat{x}) - h(x_0), g(\hat{x})) \in \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C),
\]
because \( h(\hat{x}) - h(x_0) \in \hat{w} + K \subseteq K \) and \( g(\hat{x}) \in C \).

Observe that the \( K \)-maximality of \( \hat{w} \) entails that \( \hat{w} \in \text{bd} \Pi_\mathcal{W}(\mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C)) \) and this fact, in turn, entails that \( (\hat{w}, \hat{y}) \in \text{bd} (\mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \cap (K \times C)) \). Therefore, \( (\hat{w}, \hat{y}) \) must belong to the boundary of at least one of the two subsets, \( \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \) or \( K \times C \). If it were \( (\hat{w}, \hat{y}) \in \text{int} \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \), it would exist \( \hat{k} \in K \setminus \{0\} \) such that \( (\hat{w} + \hat{k}, \hat{y}) \in \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \). Since \( (\hat{w} + \hat{k}, \hat{y}) \in K \times C \), this would be inconsistent with the \( K \)-maximality of \( \hat{w} \). So one can conclude that \( (\hat{w}, \hat{y}) \in \text{bd} \mathcal{I}^{\text{VOP}}_{x_0}(B(x_0; \epsilon)) \). Then, according to what noticed in Remark 2.2, \( x_\epsilon \) must belong to \( \text{bd} B(x_0; \epsilon) \). By using again the characterization of \( K \)-optimality for problem \( (\text{VOP})_{x_0,x} \), one obtains
\[
\mathcal{I}^{\text{VOP}}_{x_\epsilon}(B(x_0; \epsilon)) \cap Q = \emptyset.
\]
From the definition of mapping \( \mathcal{I}^{\text{VOP}}_{x_\epsilon} \) one can readily see that it holds
\[
\mathcal{I}^{\text{VOP}}_{x_\epsilon}(x) = \mathcal{I}^{\text{VOP}}_{x_0}(x) + (h(x_0) - h(x_\epsilon), 0), \quad \forall x \in X.
\]
Therefore, as a mere translation of a convex, closed set with nonempty interior (remember Remark 2.2), also \( \mathcal{I}^{\mathbb{C}}(B(x_0; \epsilon)) \) has such properties. Being disjoint from \( \mathcal{Q} \), it can be linearly separated from \( \text{cl} \mathcal{Q} = K \times C \), by virtue of the Eidelheit’s theorem. This means that there exists \((w^*_\epsilon, y^*_\epsilon) \in (W^* \times Y^*) \setminus \{(0^*, 0^*)\}\) and \( \alpha \in \mathbb{R} \) such that

\[
\langle w^*_\epsilon, h(x) - h(x_\epsilon) \rangle + \langle y^*_\epsilon, g(x) \rangle \leq \alpha, \quad \forall x \in B(x_0; \epsilon),
\]

and

\[
\langle w^*_\epsilon, w \rangle + \langle y^*_\epsilon, y \rangle \geq \alpha, \quad \forall (w, y) \in K \times C.
\]

If \( x = x_\epsilon \), from inequality (8) one gets

\[
\langle y^*_\epsilon, g(x_\epsilon) \rangle \leq \alpha.
\]

On the other hand, being \((0, g(x_\epsilon)) \in K \times C\), from inequality (9) one has

\[
\langle y^*_\epsilon, g(x_\epsilon) \rangle \geq \alpha,
\]

whence

\[
\langle y^*_\epsilon, g(x_\epsilon) \rangle = \alpha.
\]

As for every \( y \in C \) it is \((0, y) \in K \times C\), one has

\[
\langle y^*_\epsilon, y \rangle \geq \alpha,
\]

whence it results in

\[
\langle y^*_\epsilon, y - g(x_\epsilon) \rangle \geq 0, \quad \forall y \in C.
\]

This entails that \(-y^*_\epsilon \in N(g(x_\epsilon); C)\). To complete the proof of (5), take an arbitrary \( w \in K \). Being \((w, g(x_\epsilon)) \in K \times C\), from (9) one gets

\[
\langle w^*_\epsilon, w \rangle \geq 0,
\]

that is \( w^*_\epsilon \in K^\oplus \). To show that \( w^*_\epsilon \neq 0^* \), assume instead that \( w^*_\epsilon = 0^* \). By the open covering property of \( g \) around \((x_0, g(x_0))\), which is a consequence of the surjectivity of \( Dg(x_0) \), it holds

\[
g(B(x_0; r)) \supset B(g(x_0); \sigma r)
\]

for proper positive \( \sigma \) and \( r < \epsilon \). In the light of (8), this yields

\[
\langle y^*_\epsilon, g(x_0) + \eta u \rangle \leq \alpha, \quad \forall u \in \mathbb{S}, \quad \forall \eta \in [0, \sigma r),
\]

whereas, by inequality (9), the inclusion \((0, g(x_\epsilon)) \in K \times C\) implies

\[
\langle y^*_\epsilon, g(x_0) \rangle \geq \alpha.
\]

Consequently, one finds

\[
\eta\langle y^*_\epsilon, u \rangle \leq \alpha - \langle y^*_\epsilon, g(x_0) \rangle \leq 0, \quad \forall u \in \mathbb{S},
\]

which evidently contradicts the fact that \( y^*_\epsilon \neq 0^* \) (remember that \((w^*_\epsilon, y^*_\epsilon) \in W^* \times Y^* \setminus \{(0^*, 0^*)\})).

To conclude the proof it suffices to observe that (4) is a straightforward consequence of inequality (5) and of (10). This completes the proof. \( \square \)
Remark 3.1. (i) In view of a subsequent application of Theorem 3.1 it is to be noted that, whenever set $C$ is, in particular, a cone with apex at the null vector of $\mathbb{Y}$, then the thesis of the theorem can be refined by adding that

$$y^* \in \{g(x)\}^\perp.$$  

To see this, it suffices to put $y = 2g(x_\varepsilon)$ and then $y = 0$ in inequality (11), which is valid all over $C$.

(ii) A remarkable feature of Theorem 3.1 is that the multiplier $w_\varepsilon^*$, corresponding to the solution to $(\text{VOP}_{x_\varepsilon, \varepsilon})$, does not vanish.

(iii) Theorem 3.1 extends to the context of vector optimization Theorem 4.1 in [20]. Nevertheless, as it is possible to show by means of easy counterexamples, the uniqueness of the solution to $\varepsilon$-localizations of the problem, which is valid in scalar optimization, can not be restored in such extension.

As a comment to Theorem 3.1 it is worth noting that its thesis relates to two different issues arising in the study of optimization problems. The first one has to do with the existence of solution to $\varepsilon$-localizations of the original problem. Whereas the solution existence for localizations comes out automatically in the case of finite-dimensional problems, because $h$ and $g$ are locally continuous around $x_0$ and $B(x_0; \varepsilon)$ is compact, the question becomes subtler when $X$ is infinite-dimensional. In such circumstance, indeed, $B(x_0; \varepsilon)$ turns out to be weakly compact as $X$ is reflexive, but $h$ may fail to be weakly continuous, in the absence of any convexity assumption, and, for a similar reason, $R = g^{-1}(C)$ may fail to be weakly closed. It is at that point that one appreciates the power of the Polyak’s convexity principle. The second aspect is relevant independently of the dimension of the underlying space. It deals with the necessary optimality condition, which turns out to hold for the Lagrangian function, for a proper choice of multipliers. This is done below.

Proposition 3.3. Under the same hypotheses of Theorem 3.1, in the same notations, let $\varepsilon \in (0, \varepsilon_0]$ and $z \in B(x_0; \varepsilon)$. If there exists $(w^*, y^*) \in (W^* \times X^*) \setminus \{(0^*, 0^*)\}$ fulfilling the following conditions:

$$w^* \in K^\perp, \quad \ker w^* = \{0\}, \quad \text{and} \quad -y^* \in \text{N}(g(z); C),$$

and

$$L(w^*, y^*; z) \geq L(w^*, y^*; x), \quad \forall x \in B(x_0; \varepsilon),$$

then $z$ is a solution to $(\text{VOP}_{x_0, \varepsilon})$.

Proof. Take an arbitrary $x \in R \cap B(x_0; \varepsilon)$. Being $g(x) \in C$, by virtue of the third relation in (12), one has

$$\langle y^*, g(x) - g(z) \rangle \geq 0.$$  

Therefore, from inequality (13) it follows

$$\langle w^*, h(z) \rangle \geq \langle w^*, h(x) \rangle + \langle y^*, g(x) - g(z) \rangle \geq \langle w^*, h(x) \rangle, \quad \forall x \in R \cap B(x_0; \varepsilon),$$

whence

$$\langle w^*, h(z) - h(x) \rangle \geq 0, \quad \forall x \in R \cap B(x_0; \varepsilon).$$  

(14)
Now, assume ab absurd that \( z \) fails to be a solution to \( \text{VOP}_{x_0, \epsilon} \). Then, there must exists \( \hat{x} \in R \cap B(x_0; \epsilon) \) such that

\[
h(\hat{x}) \in (h(z) + K) \setminus \{ h(z) \},
\]

and hence

\[
h(\hat{x}) - h(z) \in K \setminus \{ 0 \}.
\]

Consequently, since it is \( w^* \in K^\oplus \), one finds

\[
\langle w^*, h(\hat{x}) - h(z) \rangle \geq 0,
\]

which, along with inequality (14), implies

\[
\langle w^*, h(\hat{x}) - h(z) \rangle = 0.
\]

In the light of the condition in (12) on the triviality of ker \( w^* \), the last equality allows one to conclude that \( h(\hat{x}) = h(z) \), what contradicts inclusion (15). Thus the proof is complete. \qed

4 An application to welfare economics

4.1 The economic model

This section is concerned with a model of pure exchange economy, considering finitely many consumers. Private commodities to be consumed (or desired) by them are formalized as elements of a vector space \((\mathbb{X}, \| \cdot \|)\), which is assumed to be a real Banach space. This allows one to modelize economies with an infinite-dimensional commodity space. Motivations for considering such kind of models, widely recognized in the modern mathematical economics literature, are discussed for instance in [1] and in some references therein.

Let \( I = \{1, \ldots, n\} \) index the consumer set. Each consumer \( i \in I \) is described in the model by:

- a (nonempty) consumption set \( \Omega_i \subseteq \mathbb{X} \), representing the set of commodities, where consumer \( i \) makes her choices;
- a utility function \( u_i : \Omega_i \rightarrow \mathbb{R} \), representing preferences of the consumer \( i \) over commodities.

Then, the set

\[
\Omega = \prod_{i \in I} \Omega_i \subseteq \mathbb{X}^n
\]

defines the social consumption set, where \( \mathbb{X}^n \) is the \( n \) times Cartesian product of \( \mathbb{X} \). Its elements are consumption bundles, denoted by \( x = (x_1, \ldots, x_n) \), with \( x_i \in \mathbb{X} \), for every \( i \in I \). \( \mathbb{X}^n \) will be structured as a direct sum, equipped with the 2-norm \( \| \cdot \|_2 \). Notice that in this model the happiness of each consumer is affected only by those commodities that she may consume, not by commodities considered by other consumers. Such a circumstance is labelled by stating that the consumers have separable utilities. As it is classical in general equilibrium theory, this excludes strategical interactions between consumers (instead typical in game theory), focussing on how agents in the economy respond to price systems stimulations.

A vector \( \omega \in \mathbb{X} \) denotes the aggregate initial endowment of the model, whereas \( \Theta \subseteq \mathbb{X} \) represents the net demand constraint. In this setting, a bundle \( x \in \mathbb{X}^n \) is said to be a feasible (or attainable) allocation if

\[
x \in \Omega \quad \text{and} \quad \sum_{i \in I} x_i - \omega \in \Theta.
\]
Notice that, in the case in which $X$ is a partially ordered vector space and it is $\Theta = -X_+$, the feasibility condition for allocations becomes

$$\sum_{i \in I} x_i \leq_{X_+} \omega.$$ 

Nevertheless, such an order structure on the commodity space will be not required in the present approach.

The feasibility constraint is expressed by means of the mapping $c : \Omega \rightarrow X$

$$c(x) = \sum_{i \in I} x_i - \omega.$$ 

The set $A = \{x \in \Omega : c(x) = \sum_{i \in I} x_i - \omega \in \Theta\}$ collects all feasible allocations.

The commodity-price duality associated with the model is indicated by $\langle X^*, X \rangle$. This means that the elements of the dual space $X^*$ have to be interpreted as prices, so that the value of a commodity $x \in X$ at a price $p \in X^*$ is denoted by $\langle p, x \rangle$.

The resulting economy is therefore defined by

$$\mathcal{E} = (I, (X^*, X), (\Omega_i, u_i)_{i \in I}, \omega, \Theta).$$

After the pioneering work of L. Walras, given an economy $\mathcal{E}$, a great amount of quantitative studies on the principles ruling its mechanism are focussed on general equilibrium theory (historical commentaries can be found, for instance, in [1, 3, 18]). In this theory, the concept of Pareto optimal allocation and the notion of equilibrium play a crucial role.

With reference to an exchange economy $\mathcal{E}$, a feasible allocation $\bar{x} \in A$ is said to be Pareto optimal if it is $R^n_+$-optimal for the vector optimization problem

$$(\mathcal{POP}) \quad R^n_+ \max u(x) \text{ subject to } x \in A,$$

where $u : \Omega \rightarrow R^n$ is the multiobjective mapping that arrays the utility functions of all the consumers, i.e.

$$u(x) = (u_1(x_1), \ldots, u_n(x_n)).$$

Notice that, this being the case, the space $R^n$ is partially ordered by the natural componentwise order relation.

Roughly speaking, Pareto optimality for $(\mathcal{POP})$ denotes any feasible allocation, which can not increase the happiness of any consumer without decreasing that of another one. In this sense, mapping $u$ quantifies the social efficiency of a given consumption bundle.

The notion of equilibrium is more involved. In the model under consideration (in fact, in many others), it cannot be disjoined from the notion of supporting price, which lies at the very core of the decentralization mechanism. A price system, in an equilibrium situation, should be able to summarize the informations on relative scarcities in the given economy; consequently, it can be imagined to induce a distribution among the consumers of the aggregate initial endowment, according to which each consumer maximizes her utility function over her budget set, as the latter results from the endowment distribution. More precisely, given a price $p \in X^* \setminus \{0^*\}$, a distribution induced by $p$ of the aggregate endowment $\omega$ among the consumers is any bundle $(\omega_i)_{i \in I} \in X^n$ such that

$$\langle p, \sum_{i=1}^n \omega_i \rangle = \langle p, \omega \rangle.$$ 

It is worth noting that endowment distributions are not uniquely defined by a price system and by the aggregate initial endowment. The notion of equilibrium can be therefore formalized as follows.

**Definition 4.1.** With reference to an exchange economy $\mathcal{E}$, a feasible allocation $\bar{x} \in A$ is called a (competitive) equilibrium if there exists a price system $\bar{p} \in X^* \setminus \{0^*\}$ that supports $\bar{x}$, in the sense that all the following conditions are fulfilled:
(i) $\bar{p} \in \mathbb{N} \left( \sum_{i \in I} \bar{x}_i - \omega; \Theta \right)$ (price positivity);
(ii) $\langle \bar{p}, \sum_{i \in I} \bar{x}_i \rangle = \langle \bar{p}, \omega \rangle$ (market clear condition);
(iii) $\bar{p}$ induces an endowment distribution $(\omega_i)_{i \in I}$ of $\omega$, according to which
$$u_i(\bar{x}_i) = \max_{x_i \in \Omega_i} u_i(x_i) \text{ subject to } \langle \bar{p}, x_i \rangle \leq \langle \bar{p}, \omega_i \rangle, \quad \forall i \in I,$$
(individual optimality)
with $\langle \bar{p}, \bar{x}_i \rangle = \langle \bar{p}, \omega_i \rangle$.

In the study of welfare economics, the above two notions appear to be strictly intertwined by two classical fundamental results, known as first and second welfare theorem. Roughly speaking, under appropriate assumptions, the first welfare theorem states that every equilibrium is Pareto optimal, whereas the second one is concerned with the opposite implication (for their first formulation in a modern setting the reader is referred to [2] and [7]). A critical feature of the original theory is that such achievements can be obtained by making an essential use of convexity. In the more recent literature devoted to welfare economics, an active research line revolves around the extension of the second welfare theorem to models of nonconvex economies (see, among the others, [1, 4, 10, 11, 13, 15, 16, 18]). The reason justifying such an interest has to do with the fact that, as well recognized in the economic literature, the relevance of convexity assumptions is doubtful, when even not contradicted in concrete models. A detailed discussion of such difficulty and of various attempts to overcome it can be found in the references cited above.

In this paper, starting with the same problem, a different perspective on the issue is considered. Trying to interpret the spirit of local programming, it is shown that, even in the very absence of convexity assumptions, if properly localized, an exchange economy admits feasible Pareto optimal allocations near a special class of commodity bundles, here termed regular. If some additional conditions are satisfied, these Pareto optimal allocations reveal to be equilibria.

### 4.2 Model assumptions

In this subsection all assumptions, upon which the result next presented in the paper is established, are listed and discussed.

(A1) The commodity space $(X, \| \cdot \|)$ is supposed to be a uniformly convex real Banach space, whose modulus of convexity satisfies the quadratic growth condition, and such that $(X^n, \| \cdot \|_2)$ fulfils the same property (remember Remark 2.1(iii)).

(A2) Each consumption set has nonempty interior, i.e.
$$\text{int } \Omega_i \neq \emptyset, \quad \forall i \in I.$$ 

This implies the existence of commodities in the consumption set, whose small perturbations in any direction still belong to the consumption set. Technically, such an assumption is connected with the next one.

(A3) The utility function of each consumer is a $C^{1,1}$ function, i.e.
$$u_i \in C^{1,1}(\text{int } \Omega_i), \quad \forall i \in I.$$ 

As a comment to such assumption, note that, in the model under consideration, it is implicitly supposed that each consumer’s observed preferences agree with the behaviour axioms, under which the existence of an utility function can be derived. The latter being not a primitive concept, one should complement the analysis of the behavioural axioms, justifying the specific property $C^{1,1}$ requested on $u_i$. In this regard, take into account that the assumption on $u_i$ to be $C^2$, often made when dealing with smooth utility functions, entails in particular (A3). This stronger assumption is discussed in [8].
(A_4) The net demand constraint set Θ is a (nonempty) closed, convex, cone (with apex at 0). Its introduction allows one to provide a unifying framework for different situations arising in economic models. For example, Θ may reduce to \{0\}, when the market clear condition is forced by the model over all feasible allocations. Otherwise, Θ may coincide with \(-X_+\), in the presence of implicit free disposal of commodities. Again, it may describe situations in which information is incomplete or/and uncertainty enters the economic model.

(A_5) The following local qualification condition for the endowment distribution induced by a price system is supposed to hold: for every \( x \in A \) and for every \( p \in N \left( \sum_{i \in I} x_i - \omega; \Theta \right) \), \( p \) induces an endowment distribution \( (\omega_i)_{i \in I} \) among consumers, such that for every \( \epsilon > 0 \) and for \( i \in I \) there exists \( z_i \in B(x_i; \epsilon) \) with the property

\[ \langle p, z_i \rangle < \langle p, \omega_i \rangle. \]

The above condition says that, near the \( i \)-th component of each feasible allocation, any price system yields a budget set for the consumer \( i \), which contains, among others, commodities not exhausting the endowment share distributed to \( i \).

(A_6) Each consumer \( i \in I \) is supposed to be locally non-satiated with respect to subsets of \( \Omega_i \). This amounts to say that, for every \( i \in I \), one has

\[ \forall x \in \Omega_i, \forall \epsilon > 0, \forall S \subseteq \Omega_i, \text{ with } B(x; \epsilon) \cap S \neq \emptyset, \exists z \in B(x; \epsilon) \cap S \text{ such that } u_i(z) > u_i(x). \]

This last assumption is an enforced version of a well-known condition, usually appearing in model of welfare economics.

### 4.3 Regular feasible allocations

In the setting under consideration, the localization approach to the analysis of welfare economics leads to single out the following class of feasible allocations, to which the next result applies.

**Definition 4.2.** With reference to an exchange economy \( \mathcal{E} \), whose elements satisfy assumptions \((A_1) - (A_3)\), a feasible allocation \( x_0 \in A \) is said to be regular if \( x_{0,i} \in \text{int} \Omega_i \), for every \( i \in I \), and \( D(u, c)(x_0) \) is onto.

**Remark 4.1.** As an immediate consequence of Proposition \ref{def:uniform} one has that if \( x_0 \in A \) is a regular allocation for \( \mathcal{E} \), then it can not be a Pareto optimal one for any \( \epsilon \)-localization around \( x_0 \) of problem \((POP)\).

When dealing with equilibria of an economy, the notion of problem localization must be somehow adapted. Namely, given a feasible allocation \( x_0 \) and \( n \) positive reals \( \epsilon_1, \ldots, \epsilon_n \), a \( (\epsilon_1, \ldots, \epsilon_n) \)-localization of an economy \( \mathcal{E} \) around \( x_0 \) is the exchange economy defined by

\[ \mathcal{E}_{x_0, \epsilon_1, \ldots, \epsilon_n} = (I, \langle X, X^* \rangle, (B(x_0,i; \epsilon_i), u_i)_{i \in I}, \omega, \Theta). \quad (16) \]

Having done that, one is in a position to formulate the following result.

**Theorem 4.1.** Let \( \mathcal{E} \) be an exchange economy satisfying assumptions \((A_1) - (A_4)\) and let \( x_0 \in A \) be a regular feasible allocation for \( \mathcal{E} \). Then there exists \( \epsilon_0 > 0 \) such that, for every \( \epsilon \in (0, \epsilon_0] \) there is \( x_\epsilon \in A \cap \text{bd } B(x_0, \epsilon) \), which is Pareto optimal for \((POP_{x_0, \epsilon})\). If, in addition, assumption \((A_5) - (A_6)\) hold true and, letting \( \eta_i = \|x_{\epsilon,i} - x_{0,i}\| \), it results in

\[ \min \{\eta_1, \ldots, \eta_n\} > 0, \]

then such a \( x_\epsilon \) is an equilibrium of the localized economy \( \mathcal{E}_{x_0, \eta_1, \ldots, \eta_n} \).
Proof. Since \( x_0 \) is regular, it belongs to \( \prod_{i \in I} \text{int} \Omega_i \) and \( D(u, c)(x_0) \) is onto. Thus, under the assumptions (A1) — (A4), \( \mathbb{R}^n \) and \( X \) being reflexive spaces, it is possible to apply Theorem 3.1 with \( h, g, \Omega \) and \( C \) replaced by \( u, c, \prod_{i \in I} \text{int} \Omega_i \) and \( \Theta \), respectively. As a consequence, one gets the existence of \( \epsilon_0 > 0 \) such that, for every \( \epsilon \in (0, \epsilon_0) \), there exists \( x_\epsilon \in \partial B(x_0; \epsilon) \cap A \) solving the localized problem

\[
\mathbb{R}_+^n - \max_{x \in B(x_0; \epsilon)} u(x) \quad \text{subject to} \quad x \in A.
\]

The reader should notice that by (A2), without loss of generality, up to a reduction of \( \epsilon_0 \), one can assume that \( B(x_0; \epsilon_0) \subseteq \prod_{i \in I} \Omega_i \), and hence \( B(x_0; \epsilon) \subseteq \Omega_i \), for every \( i \in I \). This prove the first assertion in the theorem. As to the second one, fix \( \epsilon \in (0, \epsilon_0) \) and note that, along with the existence of \( x_\epsilon \), Theorem 3.1 enables one to get the existence of \( (\mu_\epsilon, x^*_\epsilon) \in (\mathbb{R}^n \times X^*) \setminus \{(0, 0^*)\} \), such that

\[
\mu_\epsilon \in \mathbb{R}_+^n \setminus \{0\}, \quad -x^*_\epsilon \in N \left( \sum_{i \in I} x_{\epsilon, i} - \omega; \Theta \right),
\]

and

\[
\sum_{i \in I} \mu_{\epsilon, i} u_i(x_{\epsilon, i}) + \left\langle x^*_\epsilon, \sum_{i \in I} x_{\epsilon, i} - \omega \right\rangle \geq \sum_{i \in I} \mu_{\epsilon, i} u_i(x_i) + \left\langle x^*_\epsilon, \sum_{i \in I} x_i - \omega \right\rangle, \quad \forall x \in B(x_0; \epsilon).
\]

Set \( p_\epsilon = -x^*_\epsilon \), let us check that a multiple of \( p_\epsilon \) supports the allocation \( x_\epsilon \), with reference to the localized economy \( \mathcal{E}_{x_0, \eta_1, \ldots, \eta_n} \). According to the position of \( \eta_j \), it is \( x_\epsilon \in \prod_{i=1}^n B(x_0; \eta_i) \), so \( x_\epsilon \) is a feasible allocation for \( \mathcal{E}_{x_0, \eta_1, \ldots, \eta_n} \). The positivity of the price \( p_\epsilon \) is expressed by the second inclusion in (17). Since in (A4) \( \Theta \) has been assumed to be a cone, as noted in Remark 3.1(i), one has also

\[
\langle p_\epsilon, \sum_{i \in I} x_{\epsilon, i} - \omega \rangle = 0,
\]

which is exactly the market clear condition. To show that actually \( p_\epsilon \neq 0^* \), suppose to the contrary that \( p_\epsilon \) vanishes. From inequality (18), it follows

\[
\sum_{i \in I} \mu_{\epsilon, i} u_i(x_{\epsilon, i}) \geq \sum_{i \in I} \mu_{\epsilon, i} u_i(x_i), \quad \forall x \in B(x_0; \epsilon).
\]

Since it is \( \mu_\epsilon \neq 0 \), a proper \( j \in I \) can be found such that \( \mu_{\epsilon, j} > 0 \). By virtue of (A6), taking \( S = B(x_{0, j}; \eta_j) \), there exists \( z_j \in B(x_{\epsilon, j}; \epsilon) \cap B(x_{0, j}; \eta_j) \), with the property

\[
u_j(x_{\epsilon, j}) < u_j(z_j).
\]

Thus, if taking the boundle \( \tilde{x} \) defined by

\[
\tilde{x}_i = \begin{cases} x_{\epsilon, i}, & \forall i \in I \setminus \{j\}, \\ z_j, & \text{if } i = j, \end{cases}
\]

it is \( \tilde{x} \in B(x_0; \epsilon) \). Indeed, it results in

\[
\| \tilde{x} - x_0 \| = \left( \sum_{i \in I \setminus \{j\}} \| x_{\epsilon, i} - x_{0, i} \|^2 + \| z_j - x_{0, j} \|^2 \right)^{1/2} = \left( \epsilon^2 - \| x_{\epsilon, j} - x_{0, j} \|^2 + \| z_j - x_{0, j} \|^2 \right)^{1/2} = \left( \epsilon^2 - \nu_j^2 + \| z_j - x_{0, j} \|^2 \right)^{1/2} \leq \epsilon.
\]

Thus, being

\[
\sum_{i \in I} \mu_{\epsilon, j} u_i(x_{\epsilon, i}) < \sum_{i \in I} \mu_{\epsilon, j} u_i(\tilde{x}_i),
\]

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one obtains a contradiction of (20). Therefore, \( p_\epsilon \neq 0^* \). In order to complete the proof, it remains to check the individual optimality condition. To this aim, let us consider a distribution \((\omega_\epsilon,i)_{i \in I}\) of the aggregate endowment \(\omega\) among the consumers, which is induced by \(p_\epsilon\), in such a way that

\[
\langle p_\epsilon, \omega_\epsilon,i \rangle = \langle p_\epsilon, x_\epsilon,i \rangle, \quad \forall i \in I.
\]

This can be done, because it is

\[
\left( p_\epsilon, \sum_{i \in I} \omega_\epsilon,i \right) = \sum_{i \in I} \langle p_\epsilon, x_\epsilon,i \rangle = \langle p_\epsilon, \omega \rangle,
\]

according to (19). Now, fix an arbitrary \(j \in I\). For every \(z_j \in B(x_0,j; \eta_j)\), any bundle \(x\) of the form

\[
x_i = \begin{cases} x_\epsilon,i & \forall i \in I \setminus \{j\}, \\ z_j & \text{if } i = j \end{cases}
\]

still belongs to \(B(x_0; \epsilon)\), as already observed. Consequently, from (18), taking (19) into account, one obtains

\[
\mu_{\epsilon,j}u_j(x_{\epsilon,j}) \geq \mu_{\epsilon,j}u_j(z_j) - \langle p_\epsilon, \sum_{i \in I \setminus \{j\}} x_\epsilon,i + z_j - \omega \rangle = \mu_{\epsilon,j}u_j(z_j) - \langle p_\epsilon, z_j - x_{\epsilon,j} \rangle, \quad \forall z_j \in B(x_0,j; \eta_j).
\]

From the last inequality, in force of the endowment distribution, (21), it follows

\[
\mu_{\epsilon,j}u_j(x_{\epsilon,j}) \geq \mu_{\epsilon,j}u_j(z_j) - \langle p_\epsilon, z_j - \omega_{\epsilon,j} \rangle, \quad \forall z_j \in B(x_0,j; \epsilon).
\]

Notice that \(\mu_{\epsilon,j}\) must be positive. Otherwise, it would be

\[
\langle p_\epsilon, z_j - \omega_{\epsilon,j} \rangle \geq 0,
\]

so the arbitrariness of \(z_j\) would contradict the existence of \(z_j \in B(x_0,j; \epsilon) \subseteq \Omega_j\) such that

\[
\langle p_\epsilon, z_j \rangle < \langle p_\epsilon, \omega_{\epsilon,j} \rangle,
\]

which is guaranteed by virtue of the local qualification condition (A5). Thus, the positivity of \(\mu_{\epsilon,j}\) enables one to obtain from (22)

\[
u_j(x_{\epsilon,j}) \geq u_j(z_j), \quad \forall z_j \in B(x_0,j; \eta_j) \text{ such that } \left\langle \frac{p_\epsilon}{\mu_{\epsilon,j}}, z_j \right\rangle \leq \left\langle \frac{p_\epsilon}{\mu_{\epsilon,j}}, \omega_{\epsilon,j} \right\rangle,
\]

what means that \(x_{\epsilon,j}\) is optimal for the consumer \(j \in I\) over her budget set. Since the positivity condition and the market clear condition are both invariant with respect to the multiplication by positive scalars, one can take as a price system supporting \(x\), the functional \(\bar{p}_\epsilon = \frac{p_\epsilon}{\mu_{\epsilon,j}}\). This completes the proof.

It should be clear that Theorem 4.1 is not a generalization of the second welfare theorem to an exchange economy model affected by non-convexities. What it states is rather different. First of all it is an existence result. More precisely, it speaks about the behaviour of an exchange economy near its regular feasible allocations, provided that this economy is localized as in (16). Of course, any such localization modifies the geometry of the problem, with the result of yielding the existence of Pareto optimal allocations. The second part of the thesis, which is valid under additional assumptions, qualifies the above Pareto optimal allocations as equilibria.

As in many recent generalizations of the second welfare theorem, also in Theorem 4.1 some convexity assumption is dropped out: in fact, utility functions are merely supposed to be \(C^{1,1}\). The convexity of the net demand constraint set is maintained (in the cases \(\Theta = \{0\}\) and \(\Theta = -X_+\), such assumption is automatically satisfied), because the price system supporting an equilibrium is still obtained by means of the linear separation theorem. This is evidently in contrast with many of the aforementioned generalized second welfare theorems, which rely on a nonconvex separation technique (see [10] [13]) due to J.M. Borwein and A. Jofré or on the so-called extremal principle due to B.S. Mordukhovich (see [15] [10] [18]).
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