FAST TRACK COMMUNICATION

\( \mathcal{PT} \)-symmetry, Cartan decompositions, Lie triple systems and Krein space-related Clifford algebras

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Abstract

Gauged \( \mathcal{PT} \) quantum mechanics (PTQM) and corresponding Krein space setups are studied. For models with constant non-Abelian gauge potentials and extended parity inversions compact and noncompact Lie group components are analyzed via Cartan decompositions. A Lie-triple structure is found and an interpretation as \( \mathcal{PT} \)-symmetrically generalized Jaynes–Cummings model is possible with close relation to recently studied cavity QED setups with transmon states in multilevel artificial atoms. For models with Abelian gauge potentials a hidden Clifford algebra structure is found and used to obtain the fundamental symmetry of Krein space-related \( J \)-self-adjoint extensions for PTQM setups with ultra-localized potentials.

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Introduction

During the last 10 years many of the basic features of quantum mechanics with \( \mathcal{PT} \)-symmetric Hamiltonians (PTQM) \([1, 2]\) have been worked out in detail and are now to a certain degree well understood. This concerns the mapping of the PTQM sector of exact \( \mathcal{PT} \)-symmetry to conventional (von-Neumann) quantum mechanics with Hermitian Hamiltonians \([3]\), the relevance of the \( \mathcal{C} \)-operator as dynamically adapted mapping \([4]\) between Krein space-related indefinite metric structures \([5]\) and positive definite metrics of usual Hilbert spaces (required for a sensible probabilistic interpretation of the related wavefunctions) as well as the understanding of \( \mathcal{PT} \)-symmetric Hamiltonians as self-adjoint operators in Krein spaces \([6–10]\).

Here, we will discuss some up to now unnoticed structural links of PTQM, and Krein space-related models in general, to Lie algebra and Lie group-related Cartan decompositions.
underlying structures will help in recognizing hidden $\mathcal{PT}$-like involutory structures in physical models which are up to now not related with $\mathcal{PT}$-symmetry and to deeper understand these models and the role of $\mathcal{PT}$-symmetry in general.

We start from the simplest $\mathcal{PT}$-symmetric Hamiltonian $H$, $[\mathcal{PT}, H] = 0$, of differential operator type:

$$H = p^2 + V(x), \quad p := -i\partial_x, \quad V(-x) = V^*(x), \quad \mathcal{P}x\mathcal{P} = -x, \quad \mathcal{P}p\mathcal{P} = -p$$

$$Ti\mathcal{T} = -i\mathcal{T}, \quad \mathcal{T}x\mathcal{T} = x, \quad Tp\mathcal{T} = -p. \quad (1)$$

In general, this Hamiltonian is a $\mathcal{P}$-self-adjoint operator in a Krein space $(\mathcal{K}, \langle \cdot, \cdot \rangle_\mathcal{P})$ (see, e.g., [19, 20]) with $[\cdot, \cdot]_\mathcal{P} := (\cdot, \cdot)\mathcal{P}$ being the $\mathcal{PT}$ inner product [2], $[H\phi, \psi]_\mathcal{P} = [\phi, H\psi]_\mathcal{P}$, i.e.

$$\mathcal{P}H = H^\dagger\mathcal{P}. \quad (2)$$

Because of $\mathcal{P}p = -p\mathcal{P} = -p^\dagger\mathcal{P}$, i.e. $[p\phi, \psi]_\mathcal{P} = [-\phi, p\psi]_\mathcal{P}$, this $\mathcal{P}$-self-adjointness is spoilt for the gauged Hamiltonian

$$H_g = (p - A)^2 + V(x), \quad A(-x) = A^*(x), \quad \mathcal{P}H_g \neq H_g^\dagger\mathcal{P}. \quad (3)$$

Instead the gauge transformation (Kummer–Liouville transformation)

$$U : H_g \mapsto H = U H_g U^{-1}, \quad U = e^{-i\int A(x) dx}$$

together with (2), $\mathcal{P} = \mathcal{P}^\dagger$ and $[U^\dagger]^{-1} = [U^{-1}]^\dagger$ leads to the pseudo-Hermiticity condition

$$\eta H_g = H_g^\dagger \eta, \quad \eta := U^\dagger \mathcal{P}U, \quad \eta = \eta^\dagger. \quad (4)$$

$\mathcal{PT}$-symmetry of the system remains preserved under the gauge transformation $U$:

$$[\mathcal{PT}, U] = 0, \quad [\mathcal{PT}, H_g] = 0, \quad [\mathcal{PT}, H] = 0. \quad (5)$$

These facts are well known and have been widely discussed for various PTQM models [21–25].

Next we assume, for simplicity, a purely real coordinate dependence $x \in \Omega \subseteq \mathbb{R}$ with $\Omega$ any $\mathcal{P}$-symmetric interval. Then splitting $A(x) = A_+(x) + i A_-(x)$ into even and odd components, $[\mathcal{P}A_\pm(x) = A_\pm(-x) = \pm A_\pm(x)$, leads to a factorization of $U$ into unitary and Hermitian $\mathcal{P}$-self-adjoint factors

$$U = U_u U_h, \quad U_u = e^{-i\int_{\Omega} A_+(x) dx}, \quad U_h = e^{i\int_{\Omega} A_-(x) dx} \quad (7)$$

$$U_u^\dagger = U_u^{-1}, \quad U_h^\dagger = U_h, \quad \mathcal{P}U = U^\dagger \mathcal{P}, \quad \mathcal{P}U_u = U_u^\dagger \mathcal{P}, \quad \mathcal{P}U_h = U_h. \quad (8)$$

This is just the simplest (Abelian) version of a polar decomposition which here is naturally associated with the corresponding decomposition of the metric $\eta = J|\eta|$ into the modulus $|\eta| := \sqrt{\eta^2} = U_h^\dagger \eta U_h$ and involution $J := J|\eta|^{-1} = U_u^{-1} \mathcal{P}U_u = J^\dagger = J^{-1}$. It shows that $H_g$ is $J$-self-adjoint in the weighted ($|\eta|$-deformed) Hilbert space $L_2(|\eta| dx)$ with the inner product $(\phi, \psi)|\eta| := \int_{\Omega} \psi(x)\phi^*(x) e^{2i\int_{\Omega} A_-(x) dx} \, dx$

$$\langle H_g \phi, J \psi \rangle|\eta| = (\phi, J H_g \psi)|\eta|. \quad (9)$$

Obviously, the unitary component $U_u$ of the gauge transformation $U(x)$ rotates the original involution (Krein space metric) $\mathcal{P}$ into the new involution $J = U_u^{-1} \mathcal{P}U_u$ whereas the Hermitian component $U_h$ induces the new integration weight $|\eta|$, i.e. we have a Krein space mapping $U : (\mathcal{K}_\mathcal{P}, 1, \cdot, \cdot) \mapsto (\mathcal{K}_J, 1, \cdot, \cdot|\eta|)$.

A further mapping $\rho$ will be needed to pass from $L_2(|\eta| dx)$ in (9) to a Hilbert space $\mathcal{H}$ where a Hamiltonian $H_g$ with a real spectrum (exact $\mathcal{PT}$-symmetry) will be not only $J$-self-adjoint but self-adjoint [3, 26]. This $\rho$ will strongly depend on the concrete form of the $\mathcal{PT}$-symmetric potentials $A(x)$, $V(x)$ and, in general, it will be highly nonlocal [2, 27].
Subsequently, we mainly concentrate on the symmetry structures inherent in the model and we will not focus on the nonlocalities as the latter are typical, e.g., for the construction of \( C \) operators for Hamiltonians built over differential operators [28].

The above decomposition (7) indicates on two ways of possible model generalizations based (i) on a generalization of the Abelian gauge potential to a non-Abelian one or, via slightly different structures, (ii) on the direct use of a hidden Clifford algebra.

Non-Abelian gauge potentials, Cartan decompositions and Lie triple systems

First we note that the decomposition (7) of the gauge transformation \( U \) into unitary and Hermitian components can be regarded as the trivial Abelian version of a Cartan decomposition of a Lie group into a compact subgroup and a noncompact homogeneous coset space. Subsequently we demonstrate the interrelation of \( PT \)-symmetry and Cartan decompositions of Lie groups (and Lie algebras) on the simplest example of a matrix Hamiltonian with non-Abelian but constant\(^3\) gauge potential \( A \). The parity inversion \( P \) is assumed to be of tensor product type, i.e. we set

\[
H_g = (p-A)^2 + V(x), \quad A \in \mathbb{C}^{m \times m}, \quad V(x) \in \mathbb{C}^{m \times m} \otimes L_1(\mathbb{R})
\]

\[
[PT, H_g] = 0, \quad P = \Theta \otimes T, \quad \Theta \in \mathbb{R}^{m \times m}, \quad \Theta^2 = I_m, \quad P^2 = I_m \otimes I.
\]

Involving property \( \Theta^2 = I_m \) and reality \( \Theta \in \mathbb{R}^{m \times m} \) imply diagonalizability and symmetry of the matrix \( \Theta = \Theta^T \). This means that without loss of generality, i.e. modulo a global SO(\( m, \mathbb{R} \)) rotation, we may fix henceforth \( \Theta = I_{p,q} = \text{diag}(I_p,-I_q), \quad p + q = m \). Furthermore, we assume for simplicity that \( T \) acts as the same complex conjugation as for the scalar Hamiltonian (3), i.e., \( T \cong I_m \otimes T \) so that involution commutativity concerning the extended parity inversion \( P \) is fulfilled trivially\(^4\) [\( P, T \) = 0]. In this case \( PT \)-symmetry, \( [PT, H_g] = 0 \), implies

\[
\Theta A^* \Theta = A, \quad \Theta V^*(-x) \Theta = V(x)
\]

whereas \( P \)-self-adjointness \( PHP^* = H \) of the globally re-gauged Hamiltonian

\[
H = UH_gU^{-1} = p^2 + e^{-iAx} V(x) e^{iAx}, \quad U = e^{-iAx}
\]

leads to the additional conditions

\[
\Theta A^i \Theta = -A_i, \quad \Theta V^i(-x) \Theta = V(x).
\]

Together (12) and (14) give \( A = -A^T, \quad V = V^T \), and they fix via (13) the Lie group structure of the gauge transformation \( U \). Denote the set of corresponding Lie group elements by \( G_\Theta \supseteq U \) and the vector space of its Lie algebra elements by \( g_\Theta \). Then for the elements \( a \in g_\Theta \), because of \( a := -iA \), it holds

\[
a = -a^T, \quad \Theta a^i \Theta = a.
\]

\(^3\) In case of non-Abelian local (coordinate-dependent) gauge potentials in theories over a spacetime manifold \( M \) (e.g. over usual Minkowski space) finite gauge transformation operators \( U \) will have the form of path-ordered exponentials. For simplicity we restrict our consideration here to constant gauge transformations only.

\(^4\) In general, the time involution \( T \) may be extended nontrivially to any anti-linear involution \( T = \mu \otimes T \) with \( \mu^2 = I_m, \mu \in \mathbb{C}^{m \times m} \). In the simplest case of \( \mu \in \mathbb{R}^{m \times m} \), involution commutativity \( [P, T] = 0 \) together with fixed \( \Theta = I_{p,q} \) implies a block-diagonal \( \mu = \text{diag}(\mu_p,\mu_q) = S L_s S^{-1}, \quad S \in SO(m,\mathbb{R}) \) with a possibly different signature \( (r,s) \neq (p,q) \). Moreover, even involution commutativity may be violated, \( [P, T] \neq 0 \) as, e.g., for the spinor-representations [29] of the Dirac equation. We leave corresponding considerations to future research and restrict our attention here to the simplest ansatz \( T = I_m \otimes T \) only.
Hence, \( g_\Theta \) is constituted by the \( \Theta\)-Hermitian elements of \( so(m, \mathbb{C}) \). In order to understand the role of this \( \Theta\)-Hermiticity condition we first note that the compact subgroup of the special complex orthogonal group \( SO(m, \mathbb{C}) \) is the real orthogonal group \( SO(m, \mathbb{R}) \), whereas the (homogeneous) coset space \( SO(m, \mathbb{C})/SO(m, \mathbb{R}) \) parameterizes the noncompact (‘boost’-type) transformations. This is well known (see, e.g. [11], chapter 9, section II) and follows trivially from the Cartan decomposition of general \( GL(m, \mathbb{C}) \) matrices into unitary compact components and Hermitian noncompact components (i.e. from their polar decomposition). In fact, the corresponding Cartan involution \( \tau \) for the Lie algebra \( gl(m, \mathbb{C}) \) of the Lie algebra \( gl(m, \mathbb{C}) \) can be decomposed as \( gl(m, \mathbb{C}) = \mathfrak{t} \oplus \mathfrak{p} \) with \( \tau \mathfrak{t} = \mathfrak{t} \), \( \tau \mathfrak{p} = -\mathfrak{p} \) for compact subalgebra \( \mathfrak{t} \) and the set of noncompact coset elements \( \mathfrak{p} \), respectively. Imposing the additional antisymmetry restriction \( a = -a^T \) for \( so(m, \mathbb{C}) \) elements the Cartan involution reduces to complex conjugation \( \tau(a) = -a^T = a^* = T a \). Accordingly, \( T \) splits \( so(m, \mathbb{C}) \) just into real and purely imaginary components

\[
so(m, \mathbb{C}) = \mathfrak{t} \oplus \mathfrak{p}, \quad \mathfrak{t} = so(m, \mathbb{R}), \quad \mathfrak{p} = \{ b \in so(m, \mathbb{C}) | b = i f, f \in so(m, \mathbb{R}) \} \quad (16)
\]

\[
\mathcal{T} \mathfrak{t} = \mathfrak{t}, \quad \mathcal{T} \mathfrak{p} = -\mathfrak{p}. \quad (17)
\]

The \( \Theta\)-Hermiticity condition in (15) refines this decomposition by an additional \( \Theta\)-related block structure. Explicitly \( \Theta a^T \Theta = a \) implies

\[
a := \begin{pmatrix}
ui & v \\
-v^T & iw
\end{pmatrix}, \quad u \in \mathbb{R}^{p \times p}, \quad v \in \mathbb{R}^{p \times q}, \quad w \in \mathbb{R}^{q \times q} \quad (18)
\]

\[
\mathfrak{t}_\Theta = \left\{ b \in so(m, \mathbb{R}) | b = \begin{pmatrix} 0 & v \\
-v^T & 0 \end{pmatrix} \right\}, \quad (19)
\]

\[
\mathfrak{p}_\Theta = \left\{ c \in so(m, \mathbb{C}) | c = i f = \begin{pmatrix} u & 0 \\
0 & iu \end{pmatrix}, \right\} \quad f \in so(p, \mathbb{R}) \oplus so(q, \mathbb{R}) \quad (20)
\]

\[
b^\dagger = -b, \quad b \in \mathfrak{t}_\Theta, \quad c^\dagger = c, \quad c \in \mathfrak{p}_\Theta. \quad (21)
\]

Denoting the Cartan decomposition of \( su(p, q) \) by\(^5\)

\[
\mathfrak{t}_\Theta = su(p, q) = \mathfrak{l} \oplus \mathfrak{q}, \quad \mathfrak{l} = s(u(p) \oplus u(q)), \quad \mathfrak{q} = su(p, q) \ominus \mathfrak{l} \quad (22)
\]

we see from \( a = -i A \) with \( A = -A^T \) and \( \Theta A^T \Theta = -A \), i.e. \( A \in so(m, \mathbb{C}) \cap su(p, q) \), that \( g_\Theta = \{ a \in so(m, \mathbb{C}) | a = i f, f \in so(m, \mathbb{C}) \cap su(p, q) \} = \mathfrak{t}_\Theta \oplus \mathfrak{p}_\Theta \)

\[
\mathfrak{t}_\Theta = so(m, \mathbb{C}) \cap iq, \quad \mathfrak{p}_\Theta = so(m, \mathbb{C}) \cap i\mathfrak{l}. \quad (23)
\]

This means that \( g_\Theta \) can be considered as a ‘Wick rotated’ \( so(m, \mathbb{C}) \cap su(p, q) \), an \( so(m, \mathbb{C}) \cap su(p, q) \) with Weyl unitary trick applied not only to the noncompact component \( \mathfrak{q} \) but to the algebra as a whole. Correspondingly the roles of compact and noncompact components in \( su(p, q) \cap so(m, \mathbb{C}) \) and \( g_\Theta \) are interchanged \( l, q \Rightarrow \mathfrak{p}_\Theta, \mathfrak{t}_\Theta \). The latter fact explains the block-diagonal decomposition of the noncompact \( \mathfrak{p}_\Theta \) in (19) and the off-diagonal block form of \( \mathfrak{t}_\Theta \).

Next we note that the intersection set \( g_\Theta \) is not a Lie algebra itself. Rather this Lie algebra subspace \( g_\Theta \) forms a Lie triple system (LTS) (see, e.g., [14], section 1.1; [17], section 10). To see this we follow standard techniques [12–16] and denote by \( \kappa \) the Lie algebra involution

\[
\kappa(a) := -\Theta a^T \Theta. \quad (24)
\]

\(^5\) Recall that the compact subgroup of \( SU(p, q) \) is \( S(U(p) \times U(q)) \) (see, e.g., [11]).
Then the $\Theta$-Hermiticity condition in (15) defines $g_\Theta$ as $\kappa$-odd subspace in $so(m, \mathbb{C})$

$$g_\Theta = \{a \in so(m, \mathbb{C})| \kappa(a) = -a\},$$

(25)

whereas the commutator $[g_\Theta, g_\Theta]$ is $\kappa$-even $\kappa([g_\Theta, g_\Theta]) = [g_\Theta, g_\Theta]$, i.e. $g_\Theta$ does not close under the Lie bracket $[g_\Theta, g_\Theta] \not\subset g_\Theta$. It only closes under the ternary composition

$$a, b, c \in g_\Theta : \quad [a, [b, c]] \in g_\Theta$$

(26)

so that $g_\Theta$ is indeed a Lie triple system (LTS) $[[g_\Theta, g_\Theta], g_\Theta] \subset g_\Theta$.

For completeness, we display the Cartan decomposition of the group elements of the set $G_\Theta = K_\Theta \Pi_\Theta$. Separately considered the compact and the noncompact subset, $K_\Theta \subset SO(m, \mathbb{R})$ and $\Pi_\Theta \subset SO(m, \mathbb{C})/SO(m, \mathbb{R})$, have parameterizations induced by the corresponding Lie algebra elements in (19), (20) (see e.g. [11], chapter 9, section IV)

$$K_\Theta = \left\{ U_\ell \in SO(m, \mathbb{R})| U_\ell = e^{\kappa_b} = \begin{pmatrix} \cos(\sqrt{\nu} \nu^T x) & \nu \sin(\sqrt{\nu} \nu^T x) \sqrt{\nu} \nu^T x \\ -\sin(\sqrt{\nu} \nu^T x) \sqrt{\nu} \nu^T x & \cos(\sqrt{\nu} \nu^T x) \end{pmatrix}, \ b \in \mathfrak{k}_\Theta \right\},$$

(27)

$$\Pi_\Theta = \{ U_p \in SO(m, \mathbb{C})/SO(m, \mathbb{R}) \} \quad U_p = e^{i\sigma} = \text{diag}(e^{i\sigma_1}, e^{i\sigma_2}), \ c \in \mathfrak{p}_\Theta \}.$$  

Furthermore, it follows from (21) that

$$U_\ell^\dagger = U_\ell^{-1}, \quad U_p^\dagger = U_p$$

(28)

as the generalization of decomposition (7) for the Abelian gauge transformation.

In the trivial case of $\Theta = I_m$ there is no compact subgroup present at all and the global gauge transformations $U$ are pure boosts

$$U = e^{i\alpha x} \in \Pi_I, \quad A = -A^T \in \mathbb{R}^{m \times m}, \quad U = U^\dagger.$$  

(29)

This fact is due to the obvious anti-Hermiticity of the gauge potential $A = -A^\dagger$ which is in clear contrast to the Hermitian gauge potentials present in the Hermitian Hamiltonians of conventional (von Neumann) quantum mechanics. For $m = 2$, e.g., it holds $i\sigma_1 = a\sigma_2$, $a \in \mathbb{R}$ with $A = i\sigma_2$ so that $U = e^{i\sigma_2 x} = \cosh(\alpha x)I_2 + \sinh(\alpha x)\sigma_2$ similar to earlier findings e.g. in [33, 34].

In contrast, for $\Theta \neq I_m, m \geq 2$ and vanishing noncompact component, we find the gauge potentials $A$ as antisymmetric Hermitian matrices $A \in i\mathfrak{k}_\Theta = \{A \in so(m, \mathbb{C})| A = ib, \ b \in so(m, \mathbb{R})\}$. In the simplest case, $m = 2$, this reduces to $\Theta = \sigma_3, \ A = a\sigma_2, \ a \in \mathbb{R}$ and $U_\ell = e^{-i\sigma_2 x} \in SO(2, \mathbb{R}) \subset U(2)$.

For general $\Theta$ the gauge potential $A$ will be composed simultaneously of anti-Hermitian as well as Hermitian components corresponding to non-compact and compact components of the Lie algebra element $\alpha$, respectively.

The global gauge transformations $U \in G_\Theta$ are $\mathbf{PT}$-symmetry preserving

$$[\mathbf{PT}, U] = 0, \quad [\mathbf{PT}, H_\ell] = 0, \quad [\mathbf{PT}, H_I] = 0,$$

(30)

in analogy to (6) for Abelian systems. In contrast, the $\mathbf{P}$-symmetry properties of the $U \in G_\Theta$ components are reversed compared to that for the Abelian $U$ in (8):

$$U \in G_\Theta : \quad \mathbf{P} U_\ell = U_\ell \mathbf{P}, \quad \mathbf{P} U_p = U_p^{-1} \mathbf{P}.$$  

(31)

This reversed behavior can be traced back to the special interplay of complex conjugation and the antisymmetry of the gauge potential as an $so(m, \mathbb{C})$ element. On its turn it implies (via $\mathbf{P}$-Hermiticity of the re-gauged Hamiltonian $H$ in (13), the relation to the original Hamiltonian

6 From the large number of recent studies on ternary and $n$-ary Lie algebras as well as metric Lie 3- and $n$-algebras we note as few examples [17, 30–32].
with a set of transmon states of a multilevel artificial atom with level energies describing a special type of Jaynes–Cummings type Hamiltonian\(^7\) with additional non-Hermitian\(^8\) For other regard to \([43, 44]\) allowing for the interaction of a single (\(d\))-particle-induced excitation process in a multi-level quantum system. Models of this type can be considered, e.g., as an Abelian gauge potential is due to the non-vanishing derivative term \(i\partial_x A(x)\) in \(H_g\). The vanishing of this term \(i\partial_x A = 0\) for the constant (global) gauge potential \(A\) removes this obstruction and leads to preserved \(P\)-self-adjointness of \(H_g\) in \(\text{(10)}\), \([H_g\phi, \psi]_P = [\phi, H_g\psi]_P\). Effectively, this results from the sign invariance of the \(P\)-term under the simultaneous action of \(\mathcal{P}\mathcal{T}\) and \(\Theta A = -A\Theta\) used for the construction of the Krein space adjoint with regard to \([\cdot, \cdot]_P\).

Before we turn to the discussion of Clifford algebra-related structures in the \(\mathcal{P}\mathcal{T}\)-symmetric scalar Schrödinger equation, we note that the \(\mathcal{P}\mathcal{T}\)-symmetric matrix Hamiltonian \(H_g\) in \(\text{(10)}\) with the constant gauge potential \(A\) and appropriately chosen \(V(x)\) can be related to a Jaynes–Cummings type Hamiltonian\(^7\) with additional non-Hermitian \(\mathcal{P}\mathcal{T}\)-symmetric degrees of freedom. To see this we introduce creation and annihilation operators \(d^\dagger := (-ip + x)/\sqrt{2}, d := (ip + x)/\sqrt{2}\) and split the Lie algebra element \(a\) (see equation \((18)\)) in strictly upper and lower triangular (nilpotent) components

\[
a = c - c^T, \quad c := \begin{pmatrix} i\hat{u} & v \\ 0 & i\hat{w} \end{pmatrix}, \quad c^m = 0
\]

with \(\hat{u}, \hat{w}\) the strictly upper triangular components of \(u, w\). For \(V(x) = (x^2 - 1)\lambda_0 + 2(c + c^T)x + a^2 + 2\omega, \quad \omega = \text{diag}[(\omega_1, \ldots, \omega_m)], \quad \omega_j \in \mathbb{R}\) (34) and particle number operator \(N = d^\dagger d\) this yields, e.g.,

\[
\frac{1}{2}H_g = N + \sqrt{2}(cd + c^T d^\dagger) + \omega
\]

describing a special type of \(\mathcal{P}\mathcal{T}\)-symmetry preserving (gain-loss-balanced\(^8\)) \(d\)-particle-induced excitation process in a multi-level quantum system. Models of this type can be considered, e.g., as \(\mathcal{P}\mathcal{T}\)-symmetric generalization of the recently studied circuit and cavity QED setups [43, 44] allowing for the interaction of a single \((d)\)-mode of the cavity electromagnetic field with a set of transmon states of a multilevel artificial atom with level energies \(\omega_j\).

**Krein space-related hidden Clifford algebra**

The analysis of the scalar \(\mathcal{P}\mathcal{T}\)-symmetric Hamiltonian \((3)\) with the local Abelian gauge potential \(A(x)\) can be pursued in another direction by concentrating on the symmetry properties of the unitary factor \(U_u = e^{-i\mathcal{Q}}\), \(\mathcal{Q} := \int_0^1 A_s(s) ds\) in \(\text{(7)}\) which was responsible for the rotation of the involution as \(U_u: \mathcal{P} \mapsto J = U_u^{-1}\mathcal{P}U_u\). Representing \(\mathcal{Q}\) as

\[
\mathcal{Q} = \mathcal{R}q, \quad \mathcal{R} := \text{sign}(\mathcal{Q}), \quad q := |\mathcal{Q}|
\]

we see that the essential structure underlying the \(\mathcal{P}\)-Hermiticity condition \(\mathcal{P}U = U^\dagger \mathcal{P}\) together with \(\mathcal{P}\mathcal{Q} = -\mathcal{Q}\mathcal{P}\) and \(\mathcal{P}q = q\mathcal{P}\) is the anticommutation of space reflection operator \(\mathcal{P}\) and sign operator \(\mathcal{R}\):

\[
\mathcal{PR} = -\mathcal{R}\mathcal{P}.
\]
From the fact that $\mathcal{R}$ and $\mathcal{P}$ are involutions, $\mathcal{R}^2 = \mathcal{P}^2 = \mathcal{I}$, we find that they can be interpreted as basis (generating) elements of the real Clifford algebra

$$R_{2,0} = \text{span}_{\mathbb{R}}\{I, \mathcal{P}, \mathcal{R}, \mathcal{PR}\}$$

or its complex extension

$$C_{2} = \text{span}_{\mathbb{C}}\{I, \mathcal{P}, \mathcal{R}, \mathcal{PR}\}.$$ (38)

We recall that a real Clifford algebra $R_{m,n}$ with generating elements $\{e_k\}_{k=1}^{m+n}$

$$\{e_i, e_k\} := e_i e_k + e_k e_i = 0 \quad \forall i \neq k$$

$$\varepsilon_i^2 = \mathcal{I} \quad \forall i = 1, \ldots, m, \quad \varepsilon_i^2 = -\mathcal{I} \quad \forall i = m + 1, \ldots, m + n$$

is naturally related to an indefinite form $B(x, y) = \sum_{k=1}^{m} x_k y_k - \sum_{k=m+1}^{m+n} x_k y_k$ over $\mathbb{R}^{m+n}$ with fixed value $m + n$. For $C_{m,n}$ it suffices to work with basis elements of positive type $\varepsilon_i^2 = \mathcal{I}, \forall k = 1, \ldots, m + n$ so that the concrete interpretation as (38) or (39) depends only on whether one works with an $\mathbb{R}$- or a $\mathbb{C}$-span.

For a gauged scalar Hamiltonian $H_s$ the Clifford algebra structures become especially clearly pronounced, e.g. when the potentials $A(x)$ and $V(x)$ in (3) under appropriate regularization are shrunken to an ultra-local support of delta-function type (see e.g. [45, 46]). Below we demonstrate this fact on a Hamiltonian with general regularized zero-range potential at the point $x = 0$ as studied, e.g., in [45, 46]:

$$H_{\text{reg}} = p^2 + t_{11}\{\delta, \cdot\}\delta + t_{12}\{\delta', \cdot\}\delta + t_{21}\{\delta, \cdot\}\delta' + t_{22}\{\delta', \cdot\}\delta'.$$ (41)

The concrete operator realization $H_T(T = \|t_{ij}\|) \in L_2(\mathbb{R})$ can be defined by setting

$$H_T = H_{\text{reg}} \mid \mathcal{D}(H_T), \quad \mathcal{D}(H_T) = \{f \in W^2_2(\mathbb{R}\setminus\{0\}) : H_{\text{reg}} f \in L_2(\mathbb{R})\},$$ (42)

where the derivative $p^2 = -\partial_x^2$ acts on $W^2_2(\mathbb{R}\setminus\{0\})$ in the distributional sense and the regularized delta-function $\delta$ and its derivative $\delta'$ (with support at 0) are defined on the piecewise continuous functions $f \in W^2_2(\mathbb{R}\setminus\{0\})$ as (for more details see, e.g., [46])

$$\langle \delta, f \rangle = [f(0) + f(-0)]/2, \quad \langle \delta', f \rangle = [-f'(0) + f'(-0)]/2.$$ (43)

Denoting the set of $\mathcal{PT}$-symmetric operators $H_T$, $[\mathcal{PT}, H_T] = 0$, by $\mathcal{N}_{\mathcal{PT}}$ one immediately verifies that $H_T \in \mathcal{N}_{\mathcal{PT}} \iff t_{11}, t_{22} \in \mathbb{R}, \quad t_{12}, t_{21} \in i\mathbb{R}$. $\mathcal{N}_{\mathcal{PT}}$ contains the subset of $\mathcal{P}$-self-adjoint Hamiltonians which are determined by the condition $t_{12} = t_{21}$. For their $\mathcal{PT}$-symmetric potentials $V = t_{11}\{\delta, \cdot\}\delta + t_{12}\{\delta', \cdot\}\delta + t_{21}\{\delta, \cdot\}\delta' + t_{22}\{\delta', \cdot\}\delta'$ it additionally holds

$$\mathcal{PV} = V\mathcal{P}, \quad \langle Vu, v \rangle = \langle u, V^\dagger v \rangle, \quad u, v \in W^2_2(\mathbb{R}\setminus\{0\})$$ (43)

In analogy to the gauged Hamiltonians (3), this $\mathcal{P}$-self-adjointness can be modified toward a $\mathcal{P}_\phi$-self-adjointness with Clifford-rotated involution

$$\mathcal{P}_\phi = \mathcal{P} e^{i\phi/2} = e^{-i\phi R/2} \mathcal{P} e^{i\phi R/2}, \quad \mathcal{R} f(x) := \text{sign}(x) f(x)$$ (44)

so that an appropriate Krein space involution can be constructed for any parameter combination $t_{12} \neq t_{21}$ as well. The Clifford rotation angle $\phi$ is fixed by the parameters of the matrix $T$ and can be defined from the relation

$$i \sin(\phi) [\det(T) + 4] = 2 \cos(\phi)(t_{12} - t_{21}).$$ (45)
The derivation of this relation is based on the interpretation of the $\mathcal{P}\mathcal{T}$-symmetric operators $H_T$ as extensions of the symmetric operator

$$H_{\text{sym}} = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(H_{\text{sym}}) = \{ u(x) \in W^2_2(\mathbb{R}\setminus\{0\}) \mid u(0) = u'(0) = 0 \}. \quad (46)$$

It will be presented in full detail in [47]. For the specific angle $\phi$ the $\mathcal{P}\mathcal{T}$-symmetric Hamiltonian $H_T$ in (42) is $\mathcal{P}_\phi$-self-adjoint, $\mathcal{P}_\phi H_T^\dagger = H_T \mathcal{P}_\phi$. Accordingly, for the $\mathcal{P}\mathcal{T}$-symmetric potential $V$ it holds (conf. (43))

$$\mathcal{P}_\phi V^\dagger = V \mathcal{P}_\phi, \quad (V u, v) = (u, V^\dagger v), \quad u, v \in W^2_2(\mathbb{R}\setminus\{0\}) \quad (47)$$

with the rotated involution $\mathcal{P}_\phi = e^{-i\phi R/2} \mathcal{P} e^{i\phi R/2}$ built from the Clifford algebra elements (involutions) $\mathcal{P}$ and $R$. In the special case of $\phi = 0$ equation (45) implies $t_{12} = t_{21}$ so that (47) indeed coincides with (43), and $\mathcal{P}_{\phi=0} = \mathcal{P}$.

Concluding remarks

- The Cartan decomposition used here for the structure analysis of the gauge potentials $A$ can also be applied to the similarity transformation\(^9\) $\rho$ which maps a spectrally diagonalizable $\mathcal{P}\mathcal{T}$-symmetric Hamiltonian $H$ with real spectrum into its equivalent Hermitian operator $h = \rho H \rho^{-1}$. Although, in general, $\rho$ is a highly nonlocal operator, as similarity transformation it can nevertheless be understood as the Lie group element. Within the framework of generalized Cartan decompositions the Hermiticity $\rho = \rho^\dagger$ and positivity $\rho > 0$ clearly indicate that $\rho$ should be an element of some noncompact coset space. For the simple finite-dimensional matrix setups of [33, 34, 37] this non-compactness of $\rho$ was clearly visible in the corresponding $SO(m, \mathbb{C})$ boost-type.

- The possible use of the generalized Jaynes–Cummings setup of [43, 44] as reliable experimental candidate for the implementation of qubit states, together with the structural links indicated here, seems to open a new and interesting playground for experimental implementations of $\mathcal{P}\mathcal{T}$-symmetric and Lie-triple setups as well.

- The symmetric operator $H_{\text{sym}}$ in (46) commutes with both generating involutions $\mathcal{P}$ and $\mathcal{R}$ from the Clifford algebra $\mathcal{Cl}_2$ in (39). It will be shown in [48] that for any involution $J$ constructed in an arbitrary way from $\mathcal{Cl}_2$-involution elements there necessarily exists a very special subclass of $J$-self-adjoint extensions of $H_{\text{sym}}$ which will have a spectrum filling the whole complex plane $\mathbb{C}$.

- It is known (see, e.g., section 1.3.5 in [18]) that a Clifford algebra $\mathcal{Cl}_m$ with $m$ basis elements $\{e_1, \ldots, e_m\}$ has a faithful representation as matrix algebra $\mathcal{Cl}_{2k} \sim \mathbb{C}^{2^k \times 2^k}$, $\mathcal{Cl}_{2k+1} \sim \mathbb{C}^{2^k \times 2^k} \oplus \mathbb{C}^{2^k \times 2^k}$. Furthermore, it is known that the $J$-self-adjoint extensions of a symmetric operator with deficiency indices $(n, n)$ are parameterized by unitary matrices $U \in U(n) \subset \mathbb{C}^{n \times n}$. Once, the extension-related Clifford elements act via a representation in this $\mathbb{C}^{n \times n}$ matrix space the maximal number $m$ of Clifford basis elements in $\mathcal{Cl}_m$ is bounded by the dimensionality of this matrix space and, hence, by $2^k \leq n$ for $m = 2k$ and $2^{k+1} \leq n$ for $m = 2k + 1$. The Hamiltonian $H_T$ in (42) is related to the symmetric operator $H_{\text{sym}}$ in (46) with deficiency indices $(2, 2)$ and parameter matrix $U \in U(2)$ [5]. This means that not more than the two Clifford basis elements $\mathcal{P}$ and $\mathcal{R}$ can be naturally associated with this operator extension.

\(^9\) We use the notations from [2, 4, 5, 27] with $\rho^2 = e^{-\mathcal{O}} = \mathcal{P}\mathcal{C}$ and the $\mathcal{C}$-operator, as usual, as dynamical symmetry $[\mathcal{C}, H] = 0$ and involution $\mathcal{C}^2 = I$. 

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