A $T_2$ separable $g$ boundary for localizing spacetime singularities

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Abstract. We establish a relationship between the $g$ and $a$ boundary constructions. We show that the non-$T_2$ separability of the $g$ boundary is naturally resolved by defining the open sets of the reduced tangent bundle $G = TM \setminus \{0\}$ in terms of embeddings that allow one to appropriately define a desired equivalence relation on $G$. This leads to a natural and subtle modification of the $g$ boundary, denoted $\tilde{g}$, which is $T_2$ separable. We establish an explicit embedding of this new boundary into the $a$ boundary. We demonstrate that certain points of $G_I \subset G$ - a subset of $G$, consisting of points associated with incomplete geodesics - appearing in different equivalence classes of the $g$ boundary appear in the same class in the new construction, $\tilde{g}$, resolving tensions with Geroch’s original construction. We provide illustrative examples: Misner’s simplified version of the Taub-NUT spacetime, and the Schwarzschild spacetime.

Keywords: General Relativity, Singularities, Spacetime Boundaries

1. Introduction

Not long after the field equations were introduced by Einstein, solutions were found with singular points where known physical laws break down. Some of these were only apparently singular points and were actually due to the choice of coordinates. It is possible to choose a set of coordinates that maximally extended the spacetime, where the metric becomes analytic through these points (eg. the Kruskal embedding, [1]). Truly singular points are those which maximally analytic extensions could not remove. As open sets of the topology on such spacetime manifolds do not contain these singular points, interpreting such solutions, as well as the effects of the singular points on the solutions, has been a daunting task for mathematical physicists. Over the past sixty years, schemes have been developed, [2, 3, 4, 5], which “capture” these points in...
constructed boundaries. These boundaries are then attached to the spacetime manifold, a process known as manifold completion. The resulting manifold is called the manifold with boundary.

An early investigation of geodesic completeness was carried out by Szekeres, [6], using a power series expansion of the Schwarzschild solution around the coordinate singularity $r = 2m$ to obtain a transformation under which $r = 2m$ is a regular manifold point. In 1968, Geroch, [2], provided the first boundary construction for singular spacetimes, the $g$ boundary. This involves quotienting the set of incomplete geodesics in the spacetime under the equivalence that the geodesics limit to the same point. The $g$ boundary is restricted in terms of the curves considered because it assumes a definition of a singularity in terms of geodesic incompleteness. However, it is possible to have a non-singular geodesically incomplete spacetime, [3, 7, 8]. This prompted the construction of alternate boundaries.

In 1971, Schmidt, [4], constructed the $b$ boundary. This construction maps endpoints of incomplete curves in the bundle of frames of $M$, $\mathcal{L}(M)$ - generated via Cauchy completion - as additional points of $M$. It is best known for its application by Hawking and others to singularity theorems, [7], and considers a broader class of curves than the $g$ boundary. However, this boundary construction is fraught with undesirable properties. For one, a 4-dimensional manifold has a 20-dimensional frame bundle. Though it was shown, [4], that it is sufficient to consider only the bundle of orthonormal frames - which is 10-dimensional - intuition and computations are problematic. As a result, only situations with sufficient symmetries to reduce the dimensions can reasonably be approached by this scheme. Furthermore, there is the problem of the $b$ boundary being non-Hausdorff which means that a $b$ boundary point may be arbitrarily close to manifold points.

In 1972 Geroch et al., [5], constructed the $c$ boundary. This construction is made solely using the causal structure of the spacetime manifold. It attaches future (respectively, past) endpoints to every inextensible curve in the spacetime manifold. These endpoints are called ideal points, and their collection can be interpreted as the boundary at conformal infinity. The attachment is done in such a way that the ideal points only depend on the past (resp. future) of the curves. These ideal points are represented by terminal indecomposable future (resp. past) sets [9] which are the maximal future (resp. past) sets that are not the chronological future (resp. past) of any manifold point and cannot be represented as the union of two proper subsets, both of which are open future (resp. past) sets.

As with the other constructions, the $c$ boundary construction is problematic and carries a lot of complexities. The original topology constructed by Geroch et al. has topological separation problems as well as non-intuitive results for certain solutions. As such there have been several modifications of the $c$ boundary via the construction of different topologies (see references [9, 10, 11, 12, 13, 14]).

In 1994, Scott and Szekeres constructed the abstract boundary or simply the $a$ boundary, [8]. Given the existence of open embeddings $\phi_i : M \rightarrow \tilde{M}_i$, the $a$ boundary
is constructed by defining an equivalence relation on the topological boundaries \( \partial_{\phi_i} M \) with respect to the \( \hat{M}_i \). This construction stands out as it considers a much broader class of curves relative to the other constructions, as well as by virtue of being \( T_2 \) separable, which is a crucial requirement for a physically reasonable spacetime [15]. Its generality means that it is applicable in contexts in which one has only an affine connection; for example, Yang-Mills and Einstein-Cartan theories [3]. This construction also turns out to be easier to implement in general compared to the other construction schemes.

To date, there have been no successful results relating the various boundary constructions, although there has been some work on relating \( b \)-completeness and \( g \)-completeness, [7]. Moreover, within different boundary constructions, there have been several modifications to deal with some of the problems faced by these constructions, [5,9,11,12,13,14]. As Ashley observed, [10], there does not seem to be a “natural” and general way to map between different topologies that may be placed upon a particular boundary construction.

The aim of this paper is to explore the relationship between the \( g \) and \( a \) boundaries. This was identified as an important open problem in the original paper constructing the \( a \) boundary. By the construction of the \( g \) boundary, open sets containing incomplete geodesics are “very close” (in a sense to be made precise in section 2) to the identified endpoints. However, we also know that this concept is related to the attachment used in the construction of the attached point topology on the \( a \) boundary [15]. As such, modifying Geroch’s definition of the open sets on the reduced tangent bundle to capture the attachment relation, given the existence of envelopments, seems a natural way to attempt to relate the two constructions and generate a \( T_2 \) “Geroch-like” boundary. This also would naturally provide a platform to identify the source of the non-\( T_2 \) separability of Geroch’s original construction.

In section 2 we give an overview of the \( g \) and \( a \) boundaries. Section 3 presents our new construction which modifies that of Geroch by using notions from the construction of the attached point topology on the \( a \) boundary. In section 4 we give examples to show the equivalence of certain related points that would otherwise appear in different equivalence classes as per the Geroch’s construction. In section 5 we summarize our results and anticipate future avenues of research.

2. Some notes on the \( g \) and \( a \) boundaries

In this section we provide some background on the \( g \) and \( a \) boundary constructions, as well as the topologies that can be placed on these constructions. We follow the standard references [2, 3, 7, 10, 15].

2.1. The \( g \) boundary

Let \( M \) be a geodesically complete spacetime manifold, and let \( \Sigma \) be a co-dimension 1 submanifold of \( M \) such that \( M = M_1 \sqcup \Sigma \sqcup M_2 \). In other words, \( \Sigma \) divides \( M \) into
disjoint subsets $M_1$ and $M_2$. Suppose one is given one of the subsets, say $M_1$. A natural question arises as to how much information about $\Sigma$ can be obtained from what we know about $M_1$. The idea is to use the information about the incomplete geodesics in $M_1$ - those in $M_1$ which, when extended in $M$, pass through $\Sigma$ - to recover “parts” of $\Sigma$. One then groups these geodesics as follows: Let $\gamma$ be an incomplete geodesic in $M_1$ and generate a family of geodesics by allowing small variations in the initial conditions - a point $p \in \gamma$ and a tangent vector at $p$ - of $\gamma$. This family of geodesics traces out a 4-dimensional tube called a “thickening” of $\gamma$. Another incomplete geodesic $\gamma'$ is said to be related to $\gamma$ if $\gamma'$ enters and remains in every thickening of $\gamma$. However, since finding $\Sigma$ may not always suffice, the above construction must be generalized [2].

Let $G = TM \setminus \{0\}$ be the reduced tangent bundle on a spacetime manifold $M$, made up of the non-zero vectors in $TM$. The set $G \subset TM$ is an 8-dimensional manifold whose elements are the pairs $(p, \xi^\alpha)$. Each $(p, \xi^\alpha) \in G$ uniquely determines the geodesic $\gamma$ for which $\gamma(0) = p$, and $\gamma'(0) = \xi^\alpha$. By geodesic we mean a parametrized curve which

- has one specified end point;
- has been extended as far as possible in some direction from the end point;
- has a given affine parameter; and
- for which the affine parameter vanishes at the end point and is positive elsewhere on the curve.

The $g$ boundary is constructed from the subset $G_1 \subset G$, where $G_1$ is obtained via the construction of a field $\varphi$ that identifies points of $G$ with the total affine length of the associated geodesics uniquely determined by those points. Elements of $G_1$ are then those points $(p, \xi^\alpha) \in G$ such that $\varphi(p, \xi^\alpha)$ is finite.

One constructs a 9-dimensional manifold $H = G \times (0, \infty)$, and writes $H$ as the disjoint union of the sets $H_0$ and $H_+$, where $H_0$ and $H_+$ are defined as

\begin{align*}
H_0 &= \{(p, \xi^\alpha, d) \mid \varphi(p, \xi^\alpha) = d\}, \\
H_+ &= \{(p, \xi^\alpha, d) \mid \varphi(p, \xi^\alpha) > d\}.
\end{align*}

Then there is a natural map $\Psi : H_+ \to M$, that assigns to each point $(p, \xi^\alpha, d) \in H_+$ a point $p' \in M$ obtained by moving a distance $d$ along the geodesic uniquely associated to $(p, \xi^\alpha)$. One defines a topology on $G_1$ as follows: Let $O$ be an open set of $M$, and define open subsets $S(O)$ as consisting of those points $(p, \xi^\alpha) \in G_1$ such that there exists an open set $U \subseteq H$ with $(p, \xi^\alpha, \varphi(p, \xi^\alpha)) \in U$ and $\Psi(U \cap H_+) \subseteq O$.

Given two open sets $O_1, O_2$ in $M$, it can be shown that $S(O_1) \cap S(O_2) = S(O_1 \cap O_2)$ [2]. These open sets therefore form the basis of a topology on $G_1$. Equivalence classes of elements of $G_1$ can now be constructed by requiring that two elements $\alpha, \beta \in G_1$ are equivalent if every open set $S(O) \ni \alpha$ also contains $\beta$, and every open set $S'(O) \ni \beta$ also contains $\alpha$. The collection of such equivalence classes forms the $g$ boundary, where the induced topology on the $g$ boundary is the quotient topology, and is $T_0$ separable [2][16]. However, the ideal separation property would be the $T_2$ separation property. Geroch did provide a recipe for constructing a $g$ boundary that is $T_2$ separable, but
this requires the construction of all $T_2$ separable equivalence relations on $G_I$, which in general would not be feasible.

A new manifold (called the spacetime with $g$ boundary) can now be constructed from the disjoint union of the spacetime manifold and the $g$ boundary: $M^* \equiv M \sqcup g$. Subsets of $M^*$ are of the form $(O, U)$, where $O$ is an open set of $M$ and $U$ is an open set of $g$. A subset $(O, U)$ will be called open in $M^*$ if $S(O) \supset U$. These open sets form a basis for a topology on $M^*$. Full open sets of $M^*$ are those $(O, U)$ for which $U = S(O)$. For more details and discussions see reference [2].

Geroch’s original construction only considered the set $G_I$ of incomplete geodesics, and only allows for the identification of singular boundary points. Since we are interested in relating the $g$ and $a$ boundaries, in what follows we will consider the set $G$ which allows one to treat a wider class of curves, and, hence, of boundary points. This allows for a modification of the $g$ boundary, which we shall exploit in order to establish a relationship with the $a$ boundary (see section 3.2).

2.2. The $a$ boundary

The construction of the $a$ boundary relies on the existence of open embeddings into manifolds of the same dimension, or envelopments [3]. An advantage of the $a$ boundary construction is that it can be applied to any manifold $M$ and it is independent of both the affine connection on $M$ and the chosen family of curves in $M$. If we specify a family of curves $C$ in $M$, satisfying the bounded parameter property (to be discussed in section 3) then the $a$ boundary points can be classified as regular, points at infinity, unapproachable points, or singularities.

Let $M$ be a spacetime manifold, and let $\phi : M \rightarrow \hat{M}$ be an envelopment of $M$ into $\hat{M}$ where $\text{dim} (M) = \text{dim} (\hat{M})$.

Definition 2.1 A boundary point of $M$ is a point $p \in \partial_\phi (M)$ in the topological boundary of $\phi (M) \subseteq \hat{M}$. A boundary set is a non-empty subset $B$ of $\partial_\phi M$, comprised of boundary points.

Let $\phi' : M \rightarrow \hat{M}'$ be a second envelopment of $M$ into $\hat{M}'$, and let $B' \subseteq \partial_{\phi'} M$. We define a covering relation as follows:

Definition 2.2 A boundary set $B \subseteq \partial_\phi M$ in $\hat{M}$ covers a boundary set $B' \subseteq \partial_{\phi'} M$ in $\hat{M}'$ if for every open neighborhood $U$ of $B$ in $\hat{M}$, there exists an open neighborhood $U'$ of $B'$ in $\hat{M}'$ such that

$$\phi \circ \phi'^{-1} (U' \cap \phi' (M)) \subseteq U.$$  (3)

Definition 2.3 A boundary set $B$ is equivalent to a boundary set $B'$ if $B$ covers $B'$ and $B'$ covers $B$.

The covering relation defines an equivalence relation on the set of all boundary sets induced by all possible envelopments of $M$. An equivalence class $[B]$ of boundary sets is called an abstract boundary set.
Definition 2.4 An abstract boundary point is an abstract boundary set that has a singleton $p$ as a representative element. The set of all abstract boundary points is called the abstract boundary or simply the a boundary.

Let $O$ be an open set in $M$ and let $B$ be a boundary set of an envelopment $\phi$. A boundary point $p \in B$ (resp. a boundary set $B$) is attached to the open set $O$ of $M$ if for every open neighborhood $U$ of $p$ (resp. of $B$), we have that $U \cap \phi(O) \neq \emptyset$. Thus, for a boundary set to be attached to the open set $O$, we require at least one boundary point $p \in B$ to be attached to $O$. An abstract boundary point $[p]$ is attached to an open set $O \subseteq M$ if the boundary point $p$ is attached to $O$.

Again, one wants a topology on the union of the spacetime manifold $M$ and the abstract boundary $a$: $\bar{M} \equiv M \sqcup a$. Subsets of $\bar{M}$ are of the form $(O \cup B, C)$, where $O$ is an open set of $M$, $B$ is the set of all abstract boundary points which are attached to $O$, and $C$ is some subset of the abstract boundary. Then the sets of the form $(O \cup B, C)$ form a basis for a topology on $\bar{M}$, which was shown to be Hausdorff [15].

By construction, the identification of an abstract boundary point $[p]$ (a primary motivation for boundary constructions) is completely determined by those open sets of $M$ to which $[p]$ is attached.

3. A new construction

In this section we first briefly discuss the class of curves of interest, i.e., those satisfying the bounded parameter property. We then proceed to the construction of the modified $g$ boundary, and its relationship with the a boundary.

3.1. The bounded parameter property

The a boundary considers a broad class of curves satisfying the bounded parameter property (b.p.p) [3].

Let $\gamma : [a, b) \rightarrow M$ (with $a < b \leq \infty$) be a parametrized and regular (the tangent vector $\dot{\gamma}$ is nowhere vanishing on the interval $[a, b)$) curve in $M$, with $\gamma(a) = p$ as the starting point of $\gamma$. A curve $\gamma' : [a', b') \rightarrow M$ is a subcurve of $\gamma$ if $a \leq a' < b' \leq b$ and $\gamma' = \gamma|_{[a', b')}$. If $a = a'$ and $b > b'$ we say that $\gamma$ is an extension of $\gamma'$.

Definition 3.1 (Change of parameter) A change of parameter is a monotone increasing $C^1$ function

$$s : [a, b) \rightarrow [a', b'),$$

which maps $a$ to $a'$ and $b$ to $b'$, and $\frac{ds}{d\lambda} > 0$ for $\lambda \in [a, b)$. The curve $\gamma'$ is obtained from $\gamma$ via the change of parameter $s$ if the following diagram commutes:

$$\begin{array}{ccc}
[a, b) & \xrightarrow{\gamma} & M \\
\downarrow{s} & & | \\
[a', b').
\end{array}$$
Definition 3.2 (Bounded parameter property (b.p.p.)) A family of parametrized curves $C$ in $M$ is said to have the bounded parameter property if

- for any $p \in M$, $\exists$ at least one $\gamma \in C$ such that $\gamma(\lambda) = p$ for some $\lambda \in [a, b]$;
- if $\gamma \in C$, so is every subcurve $\gamma'$ of $\gamma$; and
- for any pair $\gamma, \gamma' \in C$ such that there exists an $s$ such that the diagram in (4) commutes, we have either the parameter on both $\gamma$ and $\gamma'$ is bounded or the parameter on both $\gamma$ and $\gamma'$ is unbounded.

3.2. A $T_2$ separable $g$ boundary

In this subsection we define open sets on $G$ making use of the attachment relation from the $a$ boundary. This allows us to define an equivalence relation on $G$, and the collection of equivalence classes defines a modified $g$ boundary which we shall denote $\tilde{g}$. This construction is $T_2$ separable. All manifolds are assumed to be smooth, Hausdorff, connected, paracompact, and time orientable (a notion of past and future can be defined).

Let $O$ be an open subset of $M$ and let $B$ be a boundary set in $\partial_a M$ (such that $B$ is in an equivalence class of the $a$ boundary) attached to $O$, where $\phi$ is an envelopment of $M$ into a spacetime manifold $\hat{M}$. We define an open set on $G$ as follows:

$$S(O) = \{(p, \xi^\alpha) \in G | \pi(p, \xi^\alpha) \in \phi^{-1}(\phi(O) \cap N_B)\}. \quad (5)$$

We note that (5) is defined for all neighborhoods $N_B$ of $B$ over all possible boundary sets $B$ attached to $O$. The map $\pi : TM \rightarrow M$ is the bundle projection. Of course, $S(O)$ is empty if no boundary set is attached to $O$. We show that finite intersections of these open sets is also open, i.e. that $S(O_1) \cap S(O_2) = S(O_1 \cap O_2)$:

$$S(O_1) \cap S(O_2)$$

$$= \{(p, \xi^\alpha) \in G | \pi(p, \xi^\alpha) \in \phi^{-1}(\phi(O_1) \cap N_B) \cap \phi^{-1}(\phi(O_2) \cap N_B')\}$$

$$= \{(p, \xi^\alpha) \in G | \pi(p, \xi^\alpha) \in \phi^{-1}(\phi(O_1) \cap N_B) \cap \phi^{-1}(\phi(O_2) \cap N_B')\}$$

$$= \{(p, \xi^\alpha) \in G | \pi(p, \xi^\alpha) \in \phi^{-1}[\phi(O_1 \cap O_2) \cap (N_B \cap N_B')]\}$$

$$= S(O_1 \cap O_2).$$

Thus, the collection of these open sets forms a basis for the open sets of a topology on $G$.

We observe that, in Geroch’s construction of the $g$ boundary, [2], elements of an equivalence class are those elements of $G$, whose images under the projection $\pi$ lie in the same open set $\phi^{-1}(\phi(O) \cap N_B)$ (see figure 1), for open subset $O$ of $M$ and fixed $N_B$ and $B$. Fix $N_B$ and $B$. Given the sets $\phi(O_1)$ and $\phi(O_2)$, such that $\phi(O_1 \cap O_2) \cap N_B \neq \emptyset$, we observe that Geroch’s original construction is not $T_2$ separable because we are not guaranteed that points in $G$ associated with curves originating in $\phi(O_1 \cup O_2) \cap N_B$ are equivalent. For example, it is possible that two curves $\gamma$ and $\gamma'$, associated with two points in the same open set $\phi(O_1 \cup O_2) \cap N_B$ might approach boundary sets (or points) that are not equivalent under the covering relations. As such, though the $g$ boundary
is $T_0$ by construction, it will not be $T_2$ separable. We avoid this problem by defining an equivalence relation on $G$.

**Definition 3.3** Let $S(O_1) \ni (p_1, \xi_1^a)$ and let $S(O_2) \ni (p_2, \xi_2^a)$. We define an equivalence relation, denoted $E_1$, as follows: $(p_1, \xi_1^a) \sim (p_2, \xi_2^a)$ iff the geodesics associated with the images of $\pi (p_1, \xi_1^a)$ and $\pi (p_2, \xi_2^a)$ under an envelopment $\phi$ approach the same abstract boundary point, and the geodesics associated with each of the $(p, \xi^a)$ are entirely contained in the intersections $\phi(O) \cap N_B$ of equation (5).

The last requirement encapsulates Geroch’s requirement that geodesics lie entirely in $O$. The proof that $E_1$ is indeed an equivalence relation follows immediately from the definition of an abstract boundary point.

**Definition 3.4** The collection of all equivalence classes in $G$ under the equivalence relation $E_1$ will be called the modified $g$ boundary, $\tilde{g}$.

The $g$ boundary constructed in this way is $T_2$ separable (the topology on $G$ induces a $T_2$ topology on $\tilde{g}$). We emphasize that our construction relies on the existence of open embeddings. Of course this is not a problem as, in general relativity, when one is presented with a spacetime metric it is usually given in terms of some coordinate system, which amounts to an envelopment. Geroch mentioned, [2], that there is no unique way of defining the $g$ boundary, and that his construction was ad hoc. However, we see that, under the condition that envelopments of the spacetime manifold exist, the a boundary provides us with a natural way of defining open sets on $G$ as well as of defining a quotient on $G$ that defines a $T_2$ separable $g$ boundary.

We can now complete the spacetime manifold $M$ by attaching $\tilde{g}$: $M^* \equiv M \sqcup \tilde{g}$, where the basis of a topology on $M^*$ is given by open subsets of the form $(O, \hat{U})$, and $\hat{U}$ are the open sets of $\tilde{g}$, induced by the open sets on $G$. As disjoint unions preserve the Hausdorff property, $M^*$ is Hausdorff.
3.3. Relation to the abstract boundary

We now present the entire construction relating the \( \tilde{g} \) and \( a \) boundaries in terms of maps. We first define an equivalence relation on \( \phi(M) \).

**Definition 3.5**: Let \( \phi(p_1), \phi(p_2) \in \phi(M) \). The **equivalence relation**, denoted \( E_2 \), is defined as follows: \( \phi(p_1) \sim \phi(p_2) \) iff

a. \( \phi(p_1) = \phi \circ \pi (p_1, \xi^a_1) \) and \( \phi(p_2) = \phi \circ \pi (p_2, \xi^a_2) \), where \( (p_i, \xi^a_i) \) are in open sets of \( G \), and

b1. \( \phi(p_1) \) and \( \phi(p_2) \) lie on the same geodesic, or

b2. \( \phi(p_1) \) and \( \phi(p_2) \) lie on geodesics approaching the same abstract boundary point.

This yields equivalence classes of geodesics with limit points being the same abstract boundary point, the collection \( \phi(M)/E_2 \), which we shall denote by \( \phi(M)_{geo} \). It is not difficult to show that \( E_2 \) is indeed an equivalence relation. Reflexivity and symmetry follow immediately from the definition. To show transitivity, let \( \phi(p_1) \sim \phi(p_2) \). Then either \( \phi(p_1) \) and \( \phi(p_2) \) lie on the same curve \( \gamma \) or \( \phi(p_1) \) and \( \phi(p_2) \) lie on curves approaching the same abstract boundary point. Suppose \( \phi(p_1) \) and \( \phi(p_2) \) both lie on \( \gamma \). If \( \phi(p_2) \) and \( \phi(p_3) \) lie on \( \gamma \), then \( \phi(p_1) \sim \phi(p_3) \). Otherwise, \( \phi(p_3) \) lies on some curve \( \tilde{\gamma} \) such that \( \tilde{\gamma} \) approaches the same abstract boundary point as \( \gamma \). Since \( \phi(p_1) \) lies on \( \gamma \), this would imply \( \phi(p_1) \sim \phi(p_3) \). Now suppose \( \phi(p_1) \) and \( \phi(p_2) \) lie on geodesics \( \gamma \) and \( \tilde{\gamma} \) respectively, both approaching the same abstract boundary point. If \( \phi(p_3) \) lies on either of \( \gamma \) or \( \tilde{\gamma} \), then \( \phi(p_3) \sim \phi(p_1) \) or \( \phi(p_3) \sim \phi(p_2) \) which would imply \( \phi(p_3) \sim \phi(p_2) \) or \( \phi(p_3) \sim \phi(p_1) \) respectively. Otherwise, \( \phi(p_3) \) lies on some curve \( \tilde{\gamma} \) (different from \( \gamma \) and \( \tilde{\gamma} \)) approaching the same abstract boundary point as \( \gamma \) and \( \tilde{\gamma} \), which would imply that \( \phi(p_3) \sim \phi(p_1) \), \( \phi(p_1) \sim \phi(p_2) \) and \( \phi(p_3) \sim \phi(p_2) \).

Let the map

\[
q : G \longrightarrow \tilde{g},
\]

be the canonical quotient map defined by

\[
(p, \xi^a) \mapsto [(p, \xi^a)]_{E_1},
\]

which sends points in \( G \) to its equivalence class under the equivalence relation \( E_1 \). Define a map

\[
k : G \longrightarrow \phi(M)_{geo},
\]

by

\[
(p, \xi^a) \mapsto [\phi(\gamma_p)]_{E_2},
\]

which maps a point of \( (p, \xi^a) \) of \( G \) to the equivalence class (under the equivalence relation \( E_2 \)) containing the image of the associated geodesic under the composition \( \phi \circ \pi \). The map \( q \) is a quotient map and thus has the natural “inverse map”, \( q^{-1} \), which sends an equivalence class to the set of its elements. Clearly \( \kappa \) is constant on the set \( q^{-1}([(p, \xi^a)_{E_1}], \) for \( [(p, \xi^a)]_{E_1} \in \tilde{g} \), since all elements in \( q^{-1}([(p, \xi^a)]_{E_1}) \) are sent to \( [\phi(\gamma_p)]_{E_2} \). We recall the following theorem, [17]:
Theorem 3.6 Let $T : X \rightarrow Y$ be a quotient map. Let $Z$ be a space and let $Q : X \rightarrow Z$ be a map that is constant on each set $T^{-1}(\{y\})$, for $y \in Y$. Then $Q$ induces a map $f : Y \rightarrow Z$ such that $f \circ T = Q$. The induced map $f$ is continuous if and only if $Q$ is continuous; $f$ is a quotient map if and only if $Q$ is a quotient map.

The map $\kappa$ thus induces a map

$$r : \tilde{g} \rightarrow \phi(M)_{geo},$$

such that $r \circ q = \kappa$. Hence, the diagram

$$\begin{array}{ccc}
G & \xrightarrow{q} & \tilde{g} \\
\downarrow{\kappa} & & \downarrow{r} \\
\phi(M)_{geo} & & \\
\end{array}$$

commutes. The map $r$ sends points of $\tilde{g}$ to the appropriate equivalence class under the equivalence relation $E_2$.

We now introduce the limit operator, [18, 19, 20], which allows us to “attach” limit points/endpoints to curves from $\tilde{g}$. Let $X$ be any set and let $S(X)$ denote the set of sequences in $X$. Let $P(X)$ denote the set of parts of $X$. One defines the limit operator as a map $l : S(X) \rightarrow P(X)$, satisfying the compatibility condition of subsequences: if $\sigma_1, \sigma_2 \in S(X)$, and $\sigma_2$ is a subsequence of $\sigma_1$, then $l(\sigma_1) \subset l(\sigma_2)$.

By the assumption that all manifolds considered are Hausdorff, limits of sequences will be unique. By Urysohn’s theorem, every Hausdorff second-countable regular space is metrizable [21]. Since manifolds are regular and second-countable, and since we restrict our attention to the Hausdorff case, all manifolds considered here are metrizable, which allows curves to be represented by sequences of points. Let $t_i$ be an increasing infinite sequence of real numbers. Then a curve $\phi(\gamma_p) \in \phi(M)$ can be written as the sequence $\phi(\gamma_p(t_i))$. The limit operator

$$L : \phi(M)_{geo} \rightarrow \phi(M)^l_{geo},$$

where $\phi(M)^l_g$ consists of equivalence classes of limit points of the sequences in $\phi(M)$, takes points in $\phi(M)_{geo}$ to equivalence classes of limit points in $\partial_b M$ under $E_2$. The equivalence classes are elements of the abstract boundary. The composition

$$L \circ r : \tilde{g} \rightarrow \bar{a},$$

where $\bar{a}$ is a subset of the abstract boundary, defined by

$$[(p, \xi^a)] \mapsto B,$$

where $B$ is a boundary set that is a representative of an abstract boundary point, thus gives a bijection from the $\tilde{g}$ boundary to a subset of the $a$ boundary, and we can write (10) as

$$[(p, \xi^a)] \mapsto [\hat{p}],$$

where $\hat{p}$ is related to $B$ under the covering relation equivalence. The composition $L \circ r$ gives us a desired map from $\tilde{g}$ to $\bar{a}$. Clearly, $L \circ r$ is a bijection onto its image $L \circ r(\tilde{g})$: 
any two equivalent points in $[\hat{p}]$ are endpoints/limit points of curves associated with equivalent points in $G$ (under $E_1$). Let $\hat{p} \in \partial_\phi M$, and let $N_{\hat{p}}$ be an open neighborhood of $\hat{p}$ in $\phi (M)$. Furthermore, let $B' \in \partial_{\phi'} M$ be a boundary set in a second envelopment $\phi'$ and let $B' \sim \hat{p}$ via the mutual covering relation of the $a$ boundary. For an open neighborhood $N_{B'}$ of $B'$ we can always identify $\phi' (O) \cap N_{B'}$ with $\phi (O) \cap N_{\hat{p}}$ by virtue of the mutual covering. Now let $O$ be an open subset of $M$ to which $\hat{p}$ is attached. Then $[\hat{p}]$ is attached to $O$ by virtue of the identification of such intersections (definition 12 of reference [12]). We associate an open set of $[\hat{p}]$ with $O$. Since such $O$ gives rise to the open set $S (O)$, we can map an open neighborhood of $[\hat{p}]$ to $S (O)$. We therefore have continuity of $L \circ r$. Thus, equation (11) establishes a homeomorphism from $\tilde{g}$ to $L \circ r (\tilde{g}) = \tilde{a} \subseteq a$, i.e. $L \circ r$ embeds $\tilde{g}$ into $a$. This is an important point since there might also be unapproachable boundary points (see reference [3]) that are not approached by any curve. There might also be cases where the image $L \circ r (\tilde{g}) = \tilde{a}$ coincides with the $a$ boundary, and future work could consider under what conditions this happens. We stress that this construction can be made for any family of b.p.p. curves (not just geodesics) yielding a generalization of Geroch’s $g$ boundary.

This construction allows for the classification of all boundary points and not just approachable ones, and could prove useful for future work. We emphasize that for the purpose of this paper the focus is on maps of the form (11) since we are primarily interested in curves as they approach associated limit points.

4. Examples

In this section we discuss several examples that clarify the distinction between our construction and that of Geroch. We stress that, by our use of constructions from the $a$ boundary, our $\tilde{g}$ boundary is homeomorphic to a subset of the $a$ boundary via (11).

Example 1 We start with a simple example (from reference [3]). Consider the envelopment $\phi$ of the unit interval $(0, 1)$ into $\mathbb{R}$ via the inclusion map. The boundary set $B = \{0, 1\}$ is the boundary set of this envelopment. Now let a second envelopment, $\phi'$, embed $(0, 1)$ in the unit circle via the map $\theta = 2\pi t$. A boundary point of this envelopment is 0. Under $\phi'$, the boundary points of the first envelopment are both identified with 0 of the second envelopment and so 0 and 1 of the first envelopment are equivalent (under the equivalence relation defining the abstract boundary) to the boundary point 0 of the second envelopment. The boundary set $B$ is disconnected and so takes the discrete topology. Therefore, the singletons $\{0\}$ and $\{1\}$ are open sets containing each of the boundary points.

Suppose we are presented with just the envelopment $\phi$. The boundary points 0 and 1 may wrongly be associated with different equivalence classes. With the knowledge of the second envelopment, it becomes clear that they are equivalent (since they are both equivalent to 0 in the second envelopment). We thus have 0 and 1 as elements of an equivalence class in the $\tilde{g}$ boundary.
This is an obvious example in the sense that we know that the points 0 and 1 are identified on the circle. However, this demonstrates that two equivalent points might be placed in separate equivalence classes as per Geroch’s construction.

**Example 2** This next example was discussed in references [2] and [3]. We consider Misner’s simplified version, [8], of the Taub-NUT spacetime [22].

Let $M$ be the 2-dimensional manifold $S^1 \times \mathbb{R}$ (the infinite cylinder), with metric given by

$$ds^2 = 2dtd\psi + td\psi^2,$$

(12)

where $t \in \mathbb{R}, 0 \leq \psi \leq 2\pi$. From (12) it is clear that $\psi = c$, for some constant $c$, and $t = 0$ are null geodesics. However, there are other geodesics in both the lower and upper halves that spiral around the cylinder and toward $t = 0$. As per our construction, $\tilde{g}$ consists of just one point, $\beta$, which is the equivalence class consisting of those points in $G$ associated with all geodesics that approach $t = 0$. (There are boundary points at $t = -\infty$ and $t = +\infty$, but as these are points at infinity and we are considering singular points, we shall ignore them in this example.) We note that this spacetime is non-singular, though geodesically incomplete, as the spacetime can be compactified by identifying the two infinities in the opposite directions along $t$ (see reference [3], and references therein). This is due to the fact that a compact pseudo-Riemannian manifold is singularity free, [3]. In contrast, the $g$ boundary consists of three points, $\tilde{g}$, and two circles, $C$ and $C'$, [2], with a point on the circles representing those geodesics from both halves of the cylinder striking the circles at exactly one point. As all these geodesics approach $t = 0$, points in $C$ and $C'$ are all equivalent to points in the single equivalence class making up $\tilde{g}$, where $t = 0$ is an $a$ boundary point.

**Example 3** We next consider the well studied Schwarzschild geometry, which, in the usual $(t,r,\theta,\phi)$ coordinates, is given by the metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(13)

This is the unique vacuum spherically symmetric solution to the field equations, which describes the geometry outside a spherically symmetric body. The form of the Schwarzschild metric in [13] has two singularities: $r = 0$ and $r = 2m$. However, from the Kruskal extension, [1] - the unique analytic and locally inextendible extension of the Schwarzschild geometry - all geodesics reach the singularity $r = 0$. The “apparent” singularity at $r = 2m$ is covered by regular boundary points in the Kruskal extension. Thus, we have the $\tilde{g}$ boundary as a single point, $\alpha$, consisting of points in $G$ whose associated geodesics all approach the surface $r = 0$, which is a point in the $a$ boundary. By contrast, the $g$ boundary, the $g$ boundary is made up of two parts, each topologically $S^2 \times \mathbb{R}$, for different approaches of the geodesics to $r = 0$, [2].

In principle we could have, for example, a case in which different classes of geodesics approach different surfaces consisting of singular boundary points. As such we will have
the \( \tilde{g} \) boundary with multiple components. The construction of such examples is an ongoing project.

5. Conclusion

The aim of this paper was to establish a relationship between the \( g \) and \( a \) boundaries. This has been achieved in a natural sense. We have modified Geroch’s \( g \) boundary by exploiting properties of the \( a \) boundary. The new construction is \( T_2 \) separable in a natural manner. The realization that we can define the open sets \( S(O) \) of \( G \), \[2\], using the attachment relation between open sets of \( M \) and boundary sets of the image of \( M \) under \( \phi \), from the attached point topology on the \( a \) boundary, \[15\], was the key to our construction. In particular, this choice of the open sets, together with the desire to relate the two boundary constructions, constrains the choice of equivalence relation. It transpires that this forces \( \tilde{g} \) to be \( T_2 \) separable, a consequence of the \( T_2 \) separability of the \( a \) boundary. In the process of obtaining the definition of the equivalence relation for our construction, all other identifications we attempted were non-Hausdorff.

We have also presented several examples. The first demonstrates that if equivalent points are placed in separate equivalence classes as in the original \( g \) boundary, this is the source of the non-\( T_2 \) separability. In the second and third, we constructed the \( \tilde{g} \) boundary, which differs from the \( g \) boundary.

This paper exploits properties of the \( a \) boundary to provide insight into and rectify a major shortfall of the \( g \) boundary construction. Now that we may relate the \( \tilde{g} \) and \( a \) boundaries, there are many interesting problems that come to mind. Among these, we identify the following as exciting possible future work:

1. Can the causal, metric and differentiable structures defined on \( g \), \[2\], be extended to \( \tilde{g} \)?
2. If the answer to (1.) is yes, can our construction then provide a way to construct causal, metrical and differentiable structures on the \( a \) boundary?
3. Again, if the answer to (2.) is in the affirmative, what new information, if any, can we obtain from these structures on the \( a \) boundary?

Finally, an obvious problem to consider is the relationship between the boundaries considered in this paper to other boundary constructions.

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