HOLOMORPHIC FOLIATIONS OF DEGREE TWO AND ARBITRARY DIMENSION

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Abstract. Let $\mathcal{F}$ be a holomorphic foliation of degree 2 on $\mathbb{P}^n$ with dimension $k \geq 2$. We prove that either $\mathcal{F}$ is algebraically integrable, or $\mathcal{F}$ is the linear pull-back of a degree 2 foliation by curves on $\mathbb{P}^{n-k+1}$, or $\mathcal{F}$ is a logarithmic foliation of type $(1^{n-k+1}, 2)$, or $\mathcal{F}$ is the linear pull-back of a degree-2 foliation of dimension 2 on $\mathbb{P}^{n-k+2}$ tangent to an action of the Lie algebra $\mathfrak{aff}(\mathbb{C})$. As a byproduct, we describe the geometry of Poisson structures on $\mathbb{P}^n$ with generic rank two.

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1. Introduction

One of the central problems in the global theory of holomorphic foliations is the description and classification of the irreducible components of the space of foliations on the projective space $\mathbb{P}^n$, $n \geq 3$, at least for low degrees. A complete classification is known for degrees 0 and 1 in any dimension, and for degree 2 only in dimension $n-1$. The space $\text{Fol}(d, k, n)$ of degree-$d$ dimension-$k$ foliations on $\mathbb{P}^n$ is a projective subvariety of $\mathcal{P} H^0(\Omega_{\mathbb{P}^n}^d(n - k + d + 1))$ whose general element is an integrable $(n-k)$-form without zeros of codimension one. A classical result of Darboux implies that every degree-0 foliation is given by the fibers of a linear rational map $\mathbb{P}^n \rightarrow \mathbb{P}^{n-p}$, thus $\text{Fol}(0, k, n)$ is isomorphic to the Grassmannian $G(k, n + 1)$ in its Plücker embedding, see [15, Théorème 3.8]. In degree $d = 1$, Loray, Pereira, and Touzet [23], generalizing upon Jouanolou [22], proved that $\text{Fol}(1, k, n)$ has two irreducible components: one parameterizing linear pullbacks of degree-one foliations by curves on $\mathbb{P}^{n-k+1}$; the other parameterizing rational foliations of type $(1^{n-k-1}, 2)$, given by the fibers of a rational map $\mathbb{P}^n \rightarrow \mathbb{P}(1^{n-k-1}, 2)$. For $d = 2$ and $p = 1$, Cerveau and Lins Neto [5] proved that $\text{Fol}(2, n-1, n)$ has six irreducible components and described their general element. Since then, several works investigated the classification in dimension $n-1$ and higher degrees [6, 17, 25]; most remarkably [13] where the authors present an almost complete classification for degree 3. In a different direction, the present work focuses on degree-2 foliations of dimension $k \leq n-2$. The main result of this work is the following.

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Theorem A. Let $\mathcal{F}$ be a degree-$2$ foliation on $\mathbb{P}^n$ of dimension $k \geq 2$. Then one of the following holds:

1. $\mathcal{F}$ is algebraically integrable and $T\mathcal{F}$ is (semi)stable for $k > 2$ (resp. $k = 2$);
2. $\mathcal{F} \in \text{Log}(1^{n-k+1}; 2; k, n)$ is a logarithmic foliation of type $(1^{n-k+1}, 2)$;
3. $\mathcal{F} \in \text{Log}(1^{n+1}; k, n)$ is a logarithmic foliation of type $(1^{n+1})$;
4. $\mathcal{F} \in \text{Exc}(r_1, \ldots, r_m, k, n)$ is an exceptional foliation; or
5. $\mathcal{F} \in \text{LPB}(2, k, n)$ is a linear pullback of a foliation by curves in $\mathbb{P}^{n-k+1}$.

The notation is explained in §2. Theorem A gives a precise description of the irreducible components of $\text{Fol}(2, k, n)$ whose general element is not algebraically integrable with either $k > 2$ and semistable tangent sheaf, or $k = 2$ and stable tangent sheaf. Suppose $\mathcal{F}$ is an algebraically integrable foliation of degree $2$ defined by a rational $(n-k)$-form without zeros in codimension one.

The exceptional foliations in item (4) of the Theorem come from actions of the Lie algebra of affine transformations of the line $\text{aff}(\mathbb{C})$, and these correspond to integer partitions $r_1 + \cdots + r_m = n - k + 3$. Thus, the number of irreducible components of $\text{Fol}(2, k, n)$ grows with the number $p(n - k + 3)$ of partitions of $n - k + 3$:

$$\# \{\text{irreducible components of } \text{Fol}(2, k, n) \} \geq p(n - k + 3) + 1$$

For a discussion on foliations associated with actions of $\text{aff}(\mathbb{C})$ and the number of irreducible components, we refer to [29].

Along our way to proving Theorem A we proved the following result, see Proposition 3.3. We observe that this result was also obtained by da Costa [12] with independent techniques.

Proposition B. Let $\mathcal{F}$ be a $2$-dimensional foliation of degree $d$ on $\mathbb{P}^n$ such that $h^0(T\mathcal{F}) > 0$. Then, either $T\mathcal{F} \cong O_{\mathbb{P}^n}(a) \oplus O_{\mathbb{P}^n}(1 + a - d)$, $a \in \{0, 1\}$, or $T\mathcal{F}$ is not locally free.

Related to holomorphic foliations is the study of Poisson structures, see §5 for a definition. A Poisson structure of generic rank $2$ on $\mathbb{P}^n$ corresponds to a foliation of dimension $2$ and degree at most $2$. In this framework, Polishchuk found in [30] a partial classification of foliations of degree $2$ on $\mathbb{P}^3$. Coming from the foliations side, Loray, Pereira, and Touzet [23] classified Poisson structures on Fano $3$-folds of Picard rank one. In [26, section 5.3], Matviichuk, Pym, and Schedler classified Poisson structures on even-dimensional projective spaces which are log symplectic along the divisor $H$ given by the union of the coordinate hyperplanes, i.e., generically symplectic Poisson structures whose the degeneracy locus is the Poisson divisor $H$.

Let $(\mathbb{P}^n, \sigma)$ be a holomorphic Poisson structure such that $\sigma \wedge \sigma = 0$. Suppose that the associated symplectic foliation $\mathcal{F}_\sigma$ has no divisorial zeros. It follows from Theorem A that the tangent sheaf $T\mathcal{F}_\sigma$ of the associated symplectic foliation is either split, semi-stable and logarithmic, or stable with algebraic leaves. This means that either the bivector $\sigma$ is globally decomposable or non-decomposable with some Poisson divisor $D$. In general, we will say that a non-decomposable bivector generic rank $2$ is of the normal crossing type, if so is $D$.

Due to our study of foliations, we prove the following result about the geometry of Poisson structures with generic rank $2$.

Corollary C. Let $(\mathbb{P}^n, \sigma)$ be a general holomorphic Poisson structure such that $\sigma \wedge \sigma = 0$. Suppose that the associated symplectic foliation $\mathcal{F}_\sigma$ has no divisorial zeros. If $\sigma$ is decomposable, so either
• \( \sigma = v_0 \wedge v_1 \in \wedge^2 \mathfrak{h} \), where \( \mathfrak{h} = H^0(\mathbb{P}^n, TP^n(-\log(D))) \) is the Abelian Lie algebra of vector fields tangent to an arrangement of Poisson planes \( D = \cup_{i=1}^{n+1} H_i \), and the vector fields \( v_i \) induce on each symplectic leaf of \( \sigma \) a transversely Euclidean holomorphic foliation.

• \( \sigma = v_0 \wedge v \), where \( v_0 \) is a constant vector field, \( v \) is a homogeneous polynomial vector field of degree 2, \( [v_0, v] = 0 \) and \( H \) is the Poisson hyperplane determined by \( v_0 \in \mathbb{P} H^0(\mathbb{P}^n, TP^n(-1)) \cong \mathbb{P}^n \).

• \( \sigma = v_0 \wedge v_1 \in \wedge^2 \mathfrak{h} \), where \( \mathfrak{h} = H^0(\mathbb{P}^n, TP^n) \) and \( v_0 \) and \( v_1 \) are linear vector fields such \( [v_0, v_1] = v_0 \), and the vector field \( v_0 \) induces on each symplectic leaf of \( \sigma \) a transversely affine holomorphic foliation. Moreover, Given a partition \( r_1 + \cdots + r_m = n + 1 \), with \( r_1 \geq r_2 \geq \cdots \geq r_m \), there is a linear change of coordinates

\[
(x_0^1 \cdots x_{r_1-1}^1 \cdots x_0^k \cdots x_{r_m-1}^k).
\]

such that \( v_0 = v_{(r_1, \ldots, r_m)} = \delta_{r_1} + \cdots + \delta_{r_m} \) and \( v_1 = w_{(r_1, \ldots, r_m)} = \rho_{r_1} + \cdots + \rho_{r_j} \), where

\[
\delta_{r_j} = \sum_{i=0}^{r_j-1} x_i^j \frac{\partial}{\partial x_{i+1}^j}, \quad \text{and} \quad \rho_{r_j} = \sum_{i=0}^{r_j-1} (r_j - 1 - \epsilon_j)x_i^j \frac{\partial}{\partial x_i^j}, \quad \epsilon_1 + \cdots + \epsilon_k = 0 \in \mathbb{C}.
\]

If \( \sigma \) is not decomposable and of normal crossing type, so either

• \( \sigma \in H^0(\mathbb{P}^n, \wedge^2 TP^n(-\log(D))) \), where \( D = H_1 \cup \cdots \cup H_{n-2} \cup V \) is a Poisson divisor such that \( H_i \) are hyperplanes and \( V \) is a cubic, or

• \( \sigma \in H^0(\mathbb{P}^n, \wedge^2 TP^n(-\log(D))) \), where \( D = H_1 \cup \cdots \cup H_{n-3} \cup Q_1 \cup Q_2 \) is a Poisson divisor such that \( H_i \) are hyperplanes and \( Q_1 \) are quadrics.

• \( \sigma \in H^0(\mathbb{P}^n, \wedge^2 TP^n(-\log(D))) \), where \( D = H_1 \cup \cdots \cup H_{n-1} \cup Q \) is a Poisson divisor such that \( H_i \) are hyperplanes and \( Q \) is a quadric and there is \( v \in H^0(\mathbb{P}^n, TP^n(-\log(D))) \) which is Hamiltonian with respect to \( \sigma \).

Consider the space of Poisson structures of generic rank 2

\[
\text{Poisson}(X)_2 := \left\{ \sigma \in \mathbb{P} H^0(X, \wedge^2 TX) \left| [\sigma, \sigma] = 0, \sigma \wedge \sigma = 0 \right. \right\} \subset \text{Poisson}(X).
\]

As a consequence of the study of foliations of degree 2 and dimension 2 we obtain.

**Corollary D.** The space \( \text{Poisson}(\mathbb{P}^n)_2 \) has at least \( p(n+1)+1 \) irreducible components.

This work is organized as follows. In Section 2 we establish notation and recall basic results on holomorphic foliations. We describe in §§2.2 the 2-dimensional foliations associated with an action of \( \text{aff}(\mathbb{C}) \), see Proposition 2.3. This part is done in greater generality in [29]. We also describe a rational first integral for a general foliation associated with the partition \( (n+1) \), see Proposition 2.4. In Section 3 we prove Theorem 3.1 which is Theorem A for \( k = 2 \). In this section, we also prove Proposition 3.3. In Section 4, we prove Theorem 4.1, which concludes the proof of Theorem A. Finally, in Section 5, we discuss Poisson structures and we prove Corollary C.

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### 2. Holomorphic foliations

In this section, we review the basic concepts needed throughout the work. Let \( X \) be a smooth complex projective variety; an experienced reader may find no difficulty relaxing this requirement.
A singular holomorphic distribution $\mathcal{F}$ on $X$ is the data of a saturated subsheaf $T\mathcal{F}$ of the tangent sheaf $TX$ of $X$. These sheaves give a short exact sequence

\begin{equation}
0 \rightarrow T\mathcal{F} \rightarrow TX \rightarrow N\mathcal{F} \rightarrow 0
\end{equation}

where $N\mathcal{F}$ is a torsion-free sheaf, called the normal sheaf of $\mathcal{F}$, and $T\mathcal{F}$ is a reflexive sheaf, called the tangent sheaf of $\mathcal{F}$. The dimension of $\mathcal{F}$ is the generic rank of $T\mathcal{F}$. If $T\mathcal{F}$ is involutive, i.e. $[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$, then $\mathcal{F}$ is a foliation; this means that through a general point in $X$ passes a unique connected immersed (analytic) subvariety $L \subset X$ such that $TL = T\mathcal{F}|_L$, called a leaf. A foliation $\mathcal{F}$ is algebraically integrable if its leaves are embedded algebraic varieties.

Taking exterior power on (2.1) we get a morphism $ω_\mathcal{F} : \bigwedge^p TX \rightarrow \text{det}(N\mathcal{F})$, where $p = \text{rk}N\mathcal{F}$, whose image is $\text{det}(N\mathcal{F})$ tensored by the ideal sheaf $I\mathcal{F}$ of the vanishing locus of $ω_\mathcal{F}$ seen as a global section of $Ω^n_L ⊗ \text{det}(N\mathcal{F})$. We call $Z$ the singular scheme of $\mathcal{F}$, and denote it by $\text{Sing}(\mathcal{F})$. Since $T\mathcal{F}$ is saturated, $\text{codim}(\text{Sing}(\mathcal{F})) ≥ 2$.

Given a distribution $\mathcal{F}$, the twisted $p$-form $ω_\mathcal{F} ∈ H^0(Ω^n_L ⊗ \text{det}(N\mathcal{F}))$ is rather special, for any $p ∈ X \setminus \text{Sing}(\mathcal{F})$ there exists an open neighborhood $U$ of $p$ and 1-forms $η_1, \ldots, η_p ∈ Ω^1_X(U)$ such that $ω_\mathcal{F}|_U = η_1 ∧ \cdots ∧ η_p$.

Moreover, $\mathcal{F}$ is a foliation if and only if $\text{d}η_j ∧ ω_\mathcal{F}|_U = 0$, for $j = 1, \ldots, p$.

### 2.1. Foliations on projective spaces.

A $k$-dimensional distribution $\mathcal{F}$ on $\mathbb{P}^n$ corresponds to a twisted $(n - k)$-form $ω ∈ H^0(Ω^n_{\mathbb{P}^n}^{-k}(d + n - k + 1))$, where $d ≥ 0$ is called the degree of $\mathcal{F}$. The Euler sequence allows us to interpret $ω$ as a polynomial differential form

$$ω = \sum_{i=0}^n A_i dx_i$$

where $A_i$ are homogeneous of degree $d + 1$, and the contraction with the radial vector files $\text{rad} = x_0 \frac{\partial}{\partial x_0} + \cdots + x_n \frac{\partial}{\partial x_n}$ vanishes: $t_{\text{rad}} ω = 0$. The local decomposability translates to the Plücker conditions

\begin{equation}
(i_v ω) ∧ ω = 0 \quad \text{for every } v ∈ \bigwedge^{n-k-1}(\mathbb{C}^{n+1}),
\end{equation}

and the integrability condition becomes

\begin{equation}
(i_v ω) ∧ dω = 0 \quad \text{for every } v ∈ \bigwedge^{n-k-1}(\mathbb{C}^{n+1}).
\end{equation}

We refer to [22, 11] for details.

Given integers $d$, $k$, and $n$, the conditions (2.2) and (2.3) define the following subvariety of $\mathbb{P}H^0(Ω^n_{\mathbb{P}^n}^{-k}(n - k + d + 1))$:

\begin{equation}
\text{Fol}(d, k, n) = \left\{ [ω] \mid \omega \text{ is integrable, and } \text{codim}(\text{Sing } ω) ≥ 2 \right\}^{\text{Zar}},
\end{equation}

the space of $k$-dimensional foliations of degree $d$ on $\mathbb{P}^n$. For $d ≤ 1$ the irreducible components of $\text{Fol}(d, k, n)$ are well-known. A degree-0 foliation is defined by a linear rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-k}$, see [15, Théorème 3.8]. In particular,

$$\text{Fol}(0, n - k, n) \cong G(n - k + 1, n + 1)$$

the Grassmannian of $n-k+1$-planes in $\mathbb{C}^{n+1}$. For degree one, there is the following classification.

**Theorem 2.1.** [23, Theorem 6.2] If $\mathcal{F}$ is a foliation of degree 1 and codimension $q$ on $\mathbb{P}^n$, then we are in one of the following cases:
(1) \( \mathcal{F} \) is defined by a dominant rational map \( \rho : \mathbb{P}^n \dashrightarrow \mathbb{P}(1^{(q+1)}, 2) \), with irreducible general fiber determined by \( q \) linear forms and one quadratic form; or
(2) \( \mathcal{F} \) is the linear pullback of a foliation induced by a global holomorphic vector field on \( \mathbb{P}^{n-k+1} \).

Hence, \( \text{Fol}(1, n-k, n) \) has two irreducible components. In degree 2, only the case of codimension-one foliations is known.

**Theorem 2.2.** [5, Theorem A] The space of codimension-one foliations of degree 2 in \( \mathbb{P}^n, n \geq 3 \), has 6 irreducible components:

- \( S(2, n) \) composed of linear pullbacks of degree-2 foliations on \( \mathbb{P}^2 \);
- \( R(2, 2) \) given by pencils of quadrics;
- \( R(1, 3) \) given by pencils of cubics with a triple hyperplane;
- \( L(1, 1, 1, 1) \) whose general member is defined by a logarithmic 1-form with poles in 4 hyperplanes;
- \( L(1, 1, 2) \) whose general member is defined by a logarithmic 1-form with poles in two hyperplanes and a quadric;
- \( E(n) \) the exceptional component whose general member is defined by an algebraic action of the Lie algebra of affine transformations \( \text{aff}(\mathbb{C}) \).

Below we detail each one of these components. Actually, we describe their generalizations to any codimension.

**Example 2.1** (Linear pullback foliations). Given a linear projection \( \rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-k+1} \) and a \( \mathcal{F} \) degree-\( d \) foliation by curves on \( \mathbb{P}^{n-k+1} \), then we define \( \mathcal{F} = \rho^* \mathcal{G} \) a degree-\( d \) foliation of dimension \( k \). Varying \( \rho \) and \( \mathcal{G} \), we get an irreducible component \( \text{LPB}(d, k, n) \subset \text{Fol}(d, k, n) \). For a proof, see [10, Corollary 5.1].

**Example 2.2** (Rational foliations). Given homogeneous polynomials \( F_0, \ldots, F_{n-k} \), we have a rational map \( \rho : \mathbb{P}^n \dashrightarrow \mathbb{P}(d_0, \ldots, d_{n-k}) \), where \( d_j = \deg(F_j) \). Its fibers define an algebraically integrable foliation that we call rational if \( \rho \) does not ramify in codimension one, i.e., the maximal minors of the Jacobian matrix of \( \rho \) do not share a common factor. These foliations fill an irreducible component

\[
\text{Rat}(d_0, \ldots, d_{n-k}; n) \subset \text{Fol}(d, k, n),
\]

where \( d = -n + k - 1 + \sum_{j=0}^{n-k} d_j \). For a proof, see [11].

**Remark 2.1.** In degree 2, there are two components of rational foliations. Indeed, the equation

\[
2 = -n + k - 1 + \sum_{j=0}^{n-k} d_j
\]

admits only two solutions:

\[
(d_0, \ldots, d_{n-k}) = (1^{n-k-2}, 2, 2) \text{ or } (1^{n-k-1}, 3).
\]

If a foliation \( \mathcal{F} \) of dimension \( k \) and degree 2 is tangent to a rational map \( \rho : \mathbb{P}^n \dashrightarrow \mathbb{P}(d_0, \ldots, d_{n-k}) \), such that the \( \mathcal{F} \)-invariant divisor \( \{ F_0 = \cdots = F_{n-k} = 0 \} \) is normal crossing, then by [4] we get \( d_0 + \cdots + d_{n-k} \leq n - k + 3 \). This implies that \( \mathcal{F} \) is rational.

**Example 2.3** (Logarithmic foliations). A logarithmic foliation on \( \mathbb{P}^n \) of dimension \( k \) is defined by a closed rational \( (n-k) \)-form

\[
\omega = \sum_{0 \leq i_1 < \cdots < i_{n-k} \leq r} \lambda_{i_1, \ldots, i_{n-k}} \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_{n-k}}}{F_{i_{n-k}}}
\]

where \( F_0, \ldots, F_r \) are homogeneous polynomials of degrees \( d_0, \ldots, d_r \), with \( r \geq n-k \), and \( \lambda_{i_1, \ldots, i_{n-k}} \in \mathbb{C} \) satisfy Plücker conditions. We also assume that \( \omega \) does not vanish in codimension one. Owing
Note that \( n + 1 \) by conjugation. Each nilpotent orbit corresponds to a Jordan normal form, thus to a partition of 
\[ G \]
It is a closed subvariety of the Grassmannian \( \text{Sl}(2) \), where
\[ \langle \begin{array}{c} T \end{array} \rangle \]
To see that these are irreducible components is enough to prove they have the same dimension:
\[ \text{SLie}(2, \mathfrak{s}_n) \]
the Lie bracket. A result by Motzkin-Taussky says that the algebraic variety parametrizing pairs
\[ \text{SLie}(2, \mathfrak{s}_n) \]
dimensional logarithmic foliation of the same type. Indeed, for \( k > 2 \), \( n-k+3 < n+1 \), and, up to a linear change of coordinates, the foliation is defined by fewer than \( n+1 \) variables. For dimension \( k = 2 \), a general logarithmic foliation of type \( (1^n+1) \) is defined by a pair commuting global vector fields on \( \mathbb{P}^n \), i.e., it is associated with an action of the commutative Lie algebra \( C^2 \rightarrow H^0(T\mathbb{P}^n) \).

2.2. Foliations associated with a Lie algebra action. Given a \( k \)-dimensional Lie subalgebra \( \mathfrak{g} \subset \mathfrak{s}_n = H^0(T\mathbb{P}^n) \), it defines an element \( \omega(\mathfrak{g}) \in H^0(\Omega_{\mathbb{P}^n}^n(n+1)) \). If \( \omega(\mathfrak{g}) \neq 0 \), and does not vanish in codimension one, it defines a foliation of degree equal to its dimension, \( d = k \). We refer to [15, 10, 29] for generalities and details about these foliations. Here we will focus on \( 2 \)-dimensional Lie algebras. In [29], Pereira and dos Santos describe the scheme of Lie subalgebras and the irreducible components of the space of foliations they define. We reproduce below some of their arguments.

Consider the algebraic variety of \( 2 \)-dimensional Lie subalgebras of \( \mathfrak{s}_n = H^0(T\mathbb{P}^n) \):
\[ \text{SLie}(2, \mathfrak{s}_n) = \{ V \subset \mathfrak{s}_n | [V, V] \subset V, \dim V = 2 \}. \]

It is a closed subvariety of the Grassmannian \( G(2, \mathfrak{s}_n) \) defined by the equations imposed by the Lie bracket. A result by Motzkin-Taussky says that the algebraic variety parametrizing pairs of commuting matrices is irreducible, and it is the closure of the set of pairs of (simultaneously) diagonalizable matrices, see [27, Theorem 5] and [20]. This gives one irreducible component \( C \) of \( \text{SLie}(2, \mathfrak{s}_n) \). The other components correspond to non-Abelian subalgebras.

Up to isomorphism, there is only one \( 2 \)-dimensional noncommutative Lie algebra: \( \text{aff}(\mathbb{C}) = \{ e, f | [e, f] = c \} \). Then, computing the derived subalgebra induces a map
\[ \Phi: \text{SLie}(2, \mathfrak{s}_n) \setminus C \rightarrow \mathcal{N}; \quad \mathfrak{g} \mapsto [\mathfrak{g}, \mathfrak{g}], \]
where \( \mathcal{N} \subset \mathbb{P}(\mathfrak{s}_n) \) is the projectivization of the nullcone, the subvariety of nilpotent elements.

The celebrated Jacobson-Morosov Theorem says that any nilpotent element \( c \) of a semisimple Lie algebra extends to an \( \mathfrak{s}_2 \)-triple: \( \{ f, e, h \}, [h, e] = 2e, [h, f] = -2f, \) and \( [e, f] = h \) [8, Theorem 3.7.1]. Then, \( \Phi \) is surjective.

The variety \( \mathcal{N} \) can be stratified by the orbits of nilpotent elements under the action of \( \text{SL}_n \) by conjugation. Each nilpotent orbit corresponds to a Jordan normal form, thus to a partition of \( n+1 \). Given a partition \( r_1 + \cdots + r_m = n+1 \), we denote the corresponding orbit by \( O_{(r_1,\ldots,r_m)} \).

Note that \( \Phi^{-1}(O_{(r_1,\ldots,r_m)}) = \text{SL}_n \setminus \Phi^{-1}([v]) \), for any \( [v] \in O_{(r_1,\ldots,r_m)} \). Therefore,
\[ \mathcal{S}_{(r_1,\ldots,r_m)} := \Phi^{-1}(O_{(r_1,\ldots,r_m)}) \]
are irreducible closed subvarieties of \( \text{SLie}(2, \mathfrak{s}_n) \) whose (finite) union cover \( \text{SLie}(2, \mathfrak{s}_n) \setminus C \).

To see that these are irreducible components is enough to prove they have the same dimension:
dim sl_{n+1} - 1. Note that for [v] ∈ ℵ we have
\[ \Phi^{-1}([v]) = \{ (v, w) ∈ SLie(2, sl_{n+1}) \mid [v], w \} \cong \{ [w] ∈ Hol(sl_{n+1}) \mid [v], w \}. \]
Fix w_0 such that [v, w_0] = v. Then, any w as above can be written as w = λw_0 + w' with λ ∈ ℂ and [v, w'] = 0, i.e., w' ∈ C_{sl_{n+1}}(v), the centralizer of v. Thus, dim \Phi^{-1}([v]) = dim C_{sl_{n+1}}(v) - 1.

On the other hand, dim \mathcal{O}_{r_1, . . . , r_m} = dim sl_{n+1} - dim C_{sl_{n+1}}(v). Hence,
\[ \dim \mathcal{S}_{r_1, . . . , r_m} = \dim sl_{n+1} - 1. \]

Consider the map \mathcal{F}: SLie(2, sl_{n+1}) → Hol(2, n − 2, n) associating with a Lie subalgebra g the foliation \mathcal{F}(g) it defines, whenever it makes sense. It follows from [33, Theorem A] that \mathcal{F} is an embedding where it is defined. We saw in Remark 2.2 that \mathcal{F}(C) = Log(1^{n+1}; 2, n) the logarithmic component. For the other components, we have the following result, which is a consequence of [10, Corollary 6.1]. The following result is [29, Theorem 3.6].

**Proposition 2.3.** We have that Exc(r_1, . . . , r_m, 2, 2) := \mathcal{F}(\mathcal{S}_{r_1, . . . , r_m}) is an irreducible component of Hol(2, n − 2, n) unless (r_1, . . . , r_m) is one of the following partitions:
\[ \{(3, 1^{n−2}), (2, 2, 1^{n−3}), (2, 1^{n−1}), (1^{n+1})\}. \]
Moreover, \mathcal{F} is not defined on \mathcal{S}_{(2, 2, . . . , 2)} and \mathcal{S}_{(1^{n+1})}.

**Proof.** We may assume n ≥ 4 since the case n = 3 is well-known. Then [10, Corollary 6.1] says that a small deformation of a foliation \mathcal{F}(g) ∈ \mathcal{F}(\mathcal{S}_{r_1, . . . , r_m}) is induced by another Lie algebra g' ∈ SLie(2, sl_{n+1}) if codim Sing(\mathcal{F}(g)) ≥ 3. Therefore, it is enough to produce a foliation in \mathcal{F}(\mathcal{S}_{r_1, . . . , r_m}) with singularities in codimension at least 3.

Recall that the singular locus of a distribution given by a pair of vector fields v = \sum_{j=0}^{n} a_j \frac{∂}{∂x_j} and w = \sum_{j=0}^{n} b_j \frac{∂}{∂x_j} is defined by the homogeneous ideal I_3(A) generated by the 3 × 3 minors of the matrix
\[ A = \begin{bmatrix} a_0 & \cdots & a_n \\ b_0 & \cdots & b_n \\ x_0 & \cdots & x_n \end{bmatrix}. \]
Given any submatrix A' of A, we have I_3(A) ⊂ I_3(A'). Hence, to show that codim V(I_3(A)) ≥ 3 is enough to prove that the same holds for some submatrix A'. We will apply this strategy to our specific case.

Fix a partition r_1 ≥ r_2 ≥ ⋯ ≥ r_m and fix adapted homogeneous coordinates:
\[ (x_0^1 : \cdots : x_{r_1−1}^1 : \cdots : x_{r_1+k}^k : \cdots : x_{r_m−1}^r). \]
Then define the vector fields
\[ \delta_{r_1} = x_i^r \frac{∂}{∂x_i^r}, \quad \delta_{r_j} = \sum_{i=0}^{r_j−2} x_i^r \frac{∂}{∂x_i^{r_j+1}}, \quad \rho_{r_1} = \sum_{j=0}^{r_j−1} (r_j − 1 − 2i + \epsilon_j)x_i^r \frac{∂}{∂x_i^r}, \quad \epsilon_1 + \cdots + \epsilon_k = 0 ∈ ℂ. \]

(2.5) It follows that [v_{(r_1, . . . , r_m)}, v_{(r_1, . . . , r_m)}] = v_{(r_1, . . . , r_m)}. Let A be the matrix defined above. If either r_1 ≥ 5, or r_1 = 4 and r_2 ≥ 1, or r_1 = 3 and r_2 ≥ 2, then let A′ be the submatrix given by the first 5 columns of A. If r_1 = r_2 = r_3 = 2 then let A′ be the submatrix composed of the first 6 columns of A. Then, for \( \epsilon = (\epsilon_1, . . . , \epsilon_k) \) generic, codim V(I_3(A′)) ≥ 3. This is a lengthy but straightforward computation, and we leave it to the reader.

The partitions not covered by the above argument are: (1^{n+1}), (2, 1^{n−1}), (3, 1^{n−2}), (2, 2, 1^{n−3}), since n ≥ 4. Note that \( v_{1^{n+1}} = 0 \) and \( v_{(2, 1^{n−1})} = x_0^1 \frac{∂}{∂x_1} \) has singularities in codimension one. The other cases have singularities in codimension 2. □
Remark 2.3. The partition (4) gives the classical exceptional component on $\mathbb{P}^3$ [5]. The partition $(n+1)$ for $n \geq 3$ figures in [10, Example 6.8]. These are all rigid, i.e., the component $\text{Exc}(n+1, 2, n)$ is the orbit closure of a general foliation in it. We also remark that $\mathcal{F}(S_{(n+1, 2, 1)})$ and $\mathcal{F}(S_{(n+1, 2, 1)})$ do not fill irreducible components of $\text{Fol}(2, n-2, n)$, both fall into the logarithmic component $\text{Log}(1^{n-1}, 2, 2, n)$, see Proposition 3.6 below.

Remark 2.4. Due to [10, Corollary 5.1], if $k \geq 3$ and $r_1 + \cdots + r_m = n - k + 3$, there are irreducible components $\text{Exc}(r_1, \ldots, r_m, k, n) \subset \text{Fol}(2, k, n)$ whose general element is a linear pullback of a foliation in $\text{Exc}(r_1, \ldots, r_m, 2, n - k + 2)$. We call these the exceptional components.

2.2.1. First integrals for exceptional foliations. The vector field $\delta_n$, as defined in (2.5), is sometimes called the basic Weitzenböck derivation. In general, a Weitzenböck derivation is a linear nilpotent derivation. They play an important role in Invariant Theory, and the name comes from Weitzenböck’s Theorem (see [32]) which asserts that any linear action of the additive group $\mathbb{G}_a$ on a polynomial ring has a finitely generated ring of invariants. That is, the ring

$$\ker \delta = \{ f \in k[y_1, \ldots, y_n] \mid \delta(f) = 0 \}$$

is finitely generated (over $k$) for $\delta$ a Weitzenböck derivation. This fails for polynomial locally nilpotent derivations of higher degree, see [28, Example 6.3.1].

In [14], the authors give explicit generators of the field of fractions of the constant ring of the basic Weitzenböck derivation. Using this description, we can show that an exceptional foliation in $\mathcal{F}(S_{(n+1, 1)})$ is algebraically integrable.

Proposition 2.4. A general exceptional foliation in $\text{Exc}(n+1, 2, n)$ is algebraically integrable.

Proof. It is enough to prove that the foliation $\mathcal{F}$ defined by $\delta_{n+1}$ and $\rho_{n+1}$ in (2.5) is algebraically integrable. Note that we only need to prove that there exist $n - 2$ algebraically independent functions $f_i/g_i \in \mathbb{C}(x_0, \ldots, x_n)$ such that $f_i$ and $g_i$ are homogeneous of the same degree, and $v(f_i/g_i) = w(f_i/g_i) = 0$.

According to [14, §5], the field of fractions of $\ker v$ is generated by algebraically independent elements $z_1, \ldots, z_n$ given by:

$$z_1 = x_0,$$

$$z_{2i} = \sum_{j=0}^{2i} (-1)^j x_j x_{2i-j},$$

$$z_{2i+1} = \frac{x_0}{2n-4i} \left( \sum_{j=0}^{2i} (-1)^j (j+1)(n-j)x_j x_{2i-j} \right) + \sum_{j=0}^{2i} (-1)^j (2i+1-j)(n-2i+j)x_j x_{2i+1-j} - x_1 \sum_{j=0}^{2i} (-1)^j x_j x_{2i-j},$$

for $j \geq 1$.

One may verify that $w(z_1) = \frac{n}{2} z_1$, $w(z_{2i}) = n - 2i$, and $w(z_{2i+1}) = \frac{n}{2} - 2i - 1$ for $j \geq 1$. A first integral for $\mathcal{F}$ is then a rational function $f = z_1^{a_1} \cdots z_n^{a_n}$ such that $w(f) = 0$, hence, the exponents $a_j$ satisfy

$$\begin{cases}
    a_1 + \sum_{i=1}^{\lfloor n/2 \rfloor} (2a_{2i} + 3a_{2i+1}) = 0 \\
    \frac{n}{2} a_1 + \sum_{i=1}^{\lfloor n/2 \rfloor} \left( (n-2i)a_{2j} + \left( \frac{3n}{2} - 2i - 1 \right) a_{2j+1} \right) = 0
\end{cases}$$
The $2 \times n$ matrix $A$ associated with this system is of rank 2. Then we can find $n - 2$ integer-valued solutions, linearly independent over $\mathbb{Q}$, which provide $n - 2$ algebraically independent rational first integrals for $\mathcal{F}$. Explicitly, an independent set of first integrals is:

$$\left\{ \frac{z_2^2 + z_1^{4/3}}{z_2^2 + z_1}, i = 1, \ldots, n - 2 \right\},$$

for $i$ even one can take a square root.

From the above proof, we also conclude that $\mathcal{F}$ is given by the $(n - 2)$-form

$$\omega = \frac{1}{z_1^{n-2}} \text{tr} d\omega^2 \wedge dz_3 \wedge d(z_1^2 z_4) \wedge d(z_1^2 z_5) \wedge \cdots \wedge d(z_1^{2\lceil \frac{n-2}{2} \rceil} z_\lambda).$$

3. Foliations of dimension 2 and degree 2

In this section, we prove the following theorem.

**Theorem 3.1.** Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n$ of dimension and degree two, $k = d = 2$. Then one of the following holds:

1. $\mathcal{T} \mathcal{F}$ is stable and $\mathcal{F}$ is algebraically integrable with uniruled leaves;
2. $\mathcal{T} \mathcal{F}$ is strictly semistable and $\mathcal{F}$ belongs to either
   - an exceptional component $\text{Exc}(d_1, \ldots, d_r; 2, n)$;
   - one of the rational components $\text{Rat}(1^{n-2}, 2, 2; n)$ or $\text{Rat}(1^{n-1}, 3; n)$; or
   - one of the logarithmic components $\text{Log}(1^{n-1}, 2; 2, n)$ or $\text{Log}(1^{n+1}, 2, n)$.
3. $\mathcal{T} \mathcal{F}$ is not semistable and $\mathcal{F}$ is a linear pullback of a degree-2 foliation by curves on $\mathbb{P}^{n-1}$.

Given a foliation $\mathcal{F}$ as in the statement of the theorem, its tangent sheaf $T \mathcal{F}$ is reflexive of rank two with zero first Chern class. Then $T \mathcal{F}$ is semistable if and only if $h^0(T \mathcal{F}(-1)) = 0$, and stable if and only if $h^0(T \mathcal{F}) = 0$. If $T \mathcal{F}$ is not semistable, then $T \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$ and $\mathcal{F}$ is a linear pullback of a degree-2 foliation by curves on $\mathbb{P}^{n-1}$. On the other hand, if $T \mathcal{F}$ is stable then $c_2(T \mathcal{F}) > 0$, see [19, Lemma 2.5]. Since $\mathbb{P}^n$ is smooth and simply connected, it follows from Druel’s results [16, Theorem 6.1 and Proposition 8.4] that $\mathcal{F}$ is algebraically integrable. Due to [24, Theorem 5.13], $\mathcal{F}$ has uniruled leaves.

It remains to understand the case where $T \mathcal{F}$ is strictly semistable. If $T \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}$ then $\mathcal{F}$ is associated with a Lie algebra action, and thus it falls in either $\text{Log}(1^{n+1}, 2, n)$ or an exceptional component from Proposition 2.3. The proof of Theorem 3.1 will be completed by the propositions 3.6 and 3.7 below.

3.1. Foliations with non-split tangent sheaf with a global section.

**Lemma 3.2.** Let $\mathcal{F}$ be a foliation of dimension 2 and degree $d$. Suppose that $h^0(T \mathcal{F}) > 0$ and $T \mathcal{F} \not\cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(2-d)$. For any $s \in H^0(T \mathcal{F}) \setminus \{0\}$, the induced foliation by curves has singularities in codimension two.

**Proof.** Denote by $\mathcal{G}$ the foliation induced by $s \in H^0(T \mathcal{F}) \hookrightarrow H^0(\mathbb{P}^n)$. Let $W := (s = 0)$ the vanishing locus of $s$. Using the inclusions $\mathcal{O}_{\mathbb{P}^n} \hookrightarrow T \mathcal{F}$ and $T \mathcal{F} \hookrightarrow T \mathbb{P}^n$ and the snake lemma,
we build the following commutative diagram:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
\mathcal{O}_{\mathbb{P}^n} & \mathcal{O}_{\mathbb{P}^n} & \mathcal{O}_{\mathbb{P}^n} & \\
\downarrow & & & \\
0 & \mathcal{T}\mathcal{F} & T\mathbb{P}^n & N\mathcal{F} & 0 \\
\downarrow & & & \downarrow & \\
0 & \mathcal{I}_{W}(2-d) & N\mathcal{G} & N\mathcal{F} & 0 \\
\downarrow & & & \\
0 & 0 & 0 & \\
\end{array}
\]

(3.1)

We have that either \( W = \emptyset \) and \( T\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(2-d) \) or \( W \) is a subscheme of pure codimension two; by hypothesis, we are in the latter case. Dualizing the bottom row of (3.1) we get

\[ \cdots \rightarrow \mathcal{E}xt^1(N\mathcal{G}, \mathcal{O}_{\mathbb{P}^n}) \rightarrow \mathcal{E}xt^1(\mathcal{I}_{W}(2-d), \mathcal{O}_{\mathbb{P}^n}) \rightarrow \mathcal{E}xt^2(N\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}) \rightarrow 0 \]

Since \( N\mathcal{F} \) is torsion-free, \( \mathcal{E}xt^2(N\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}) \) has support in codimension at least 3. On the other hand, \( \mathcal{E}xt^1(\mathcal{I}_{W}(2-d), \mathcal{O}_{\mathbb{P}^n}) = \omega_W(d+2) \) has support on \( W \). Therefore, the support of \( \mathcal{E}xt^1(N\mathcal{G}, \mathcal{O}_{\mathbb{P}^n}) \) contains \( |W| \) and \( \mathcal{G} \) has singularities in codimension two.

\[ \square \]

**Remark 3.1.** Raphael Constant da Costa recently proved that foliations on \( \mathbb{P}^3 \) with locally free tangent sheaf having a global section must split as a direct sum of line bundles, see [12, Theorem C]. Using Lemma 3.2 we can extend this result to 2-dimensional foliations on \( \mathbb{P}^n, n \geq 3 \).

**Proposition 3.3.** Let \( \mathcal{F} \) be a 2-dimensional foliation of degree \( d \) on \( \mathbb{P}^n \) such that \( h^0(T\mathcal{F}) > 0 \). Then, either \( T\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(a) \oplus \mathcal{O}_{\mathbb{P}^n}(1+a-d), a \in \{0, 1\}, \) or \( T\mathcal{F} \) is not locally free.

**Proof.** If \( h^0(T\mathcal{F}(-1)) > 0 \) then \( \mathcal{F} \) is a linear pullback and \( T\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1-d) \). Suppose that \( h^0(T\mathcal{F}(-1)) > 0 \) and \( T\mathcal{F} \not\cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(2-d) \). Since \( \text{rk} T\mathcal{F} = 2 \), it is enough to prove that \( c_3(T\mathcal{F}) \neq 0 \). Consider the exact sequence from the first column of (3.1):

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow T\mathcal{F} \rightarrow \mathcal{I}_{W}(2-d) \rightarrow 0 \]

By Lemma 3.2 \( W \) has pure codimension 2. Furthermore, if \( n > 3 \), then \( W \cong \mathbb{P}^{n-2} \) is a linear subspace; and if \( n = 3 \), then \( W \) is either one line or a (possibly degenerate) disjoint union of two lines, see [18, §2.3] and Lemma 3.4 below.

If \( n \geq 4 \), we take a general \( L \cong \mathbb{P}^3 \) linearly embedded in \( \mathbb{P}^n \) transverse to \( W \) then the above sequence restricts to \( L \) as

\[ 0 \rightarrow \mathcal{O}_L \rightarrow T\mathcal{F}|_L \rightarrow \mathcal{I}_{W\cap L}(2-d) \rightarrow 0, \]

\( W \cap L \) is a line, and \( c_i(T\mathcal{F}|_L) = c_i(T\mathcal{F}) \) for \( i = 1, 2, 3 \). Therefore, we may assume that \( n = 3 \).

By [21, Theorem 4.1], \( c_1(T\mathcal{F}) = 2 - d, c_2(T\mathcal{F}) = \text{deg}(W) \) and \( c_3(T\mathcal{F}) = 2p_a(W) - 2 + \text{deg}(W)(d + 2) \). If \( W \) is a line then \( \text{deg}(W) = 1 \) and \( p_a(W) = 0 \), hence \( c_3(T\mathcal{F}) = d \geq 1 \). If \( W \) is (possibly the limit of) a disjoint union of two lines, \( \text{deg}(W) = 2 \) and \( p_a(W) = -1 \), hence \( c_3(T\mathcal{F}) = 2d \geq 2 \).

We classify the possible subfoliations using Jordan normal forms.

**Lemma 3.4.** Let \( v \in H^0(T\mathbb{P}^n) \) such that \( v \) has zeros in codimension two. Then, up to a linear change of coordinates, either

1. \( v = x_0 \frac{\partial}{\partial x_0} + \lambda x_1 \frac{\partial}{\partial x_1} \) with \( \lambda \in \mathbb{C}^*; \)
Both cases follow the same argument, we only prove one. In case (3), tangent to an algebraically integrable foliation defined by a global vector field of types $F_\lambda \not\in F$.

Then $\text{locus between } F \text{ and } T \text{ is not algebraic. Then, due to Lemma } 3.2, \mathcal{G} \text{ is defined by either } v = x_0 \frac{\partial}{\partial x_0} + \lambda x_1 \frac{\partial}{\partial x_1} \text{ with } \lambda \notin \mathbb{Q} \text{ or } v = (x_0 + x_1) \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1}$. Then consider $\overline{\mathcal{G}}$ the foliation obtained by taking the Zariski closure of orbits of $v$, see [3]. One readily sees that, in either case, $\overline{\mathcal{G}}$ is given by the rational map $\mathbb{P}^n \to \mathbb{P}^{n-2}, (x_0 : \cdots : x_n) \mapsto (x_2 : \cdots : x_n)$.

On the other hand, let $E \cong \mathbb{P}^2 \subset \mathbb{P}^n$ be the Zariski closure of a general leaf of $\mathcal{G}$. The tangency locus between $E$ and $\mathcal{F}$ is closed and Zariski-dense in $E$ hence $E$ is $\mathcal{F}$-invariant. Therefore, $\mathcal{F} = \overline{\mathcal{G}}$ but we get a contradiction since $\overline{\mathcal{G}}$ has degree 0 and $\mathcal{F}$ has degree $d \geq 1$.

Summarizing the above discussion, a foliation $\mathcal{F}$ of degree and dimension equal to 2 must be tangent to an algebraically integrable foliation defined by a global vector field of types (1), (3) or (4) of Lemma 3.4. The vector fields in cases (3) and (4) are nilpotent; hence they induce actions of the additive group $(\mathbb{C}, +)$. This allows us to conclude the following result.

**Proposition 3.5.** Let $\mathcal{F}$ be a foliation of dimension 2 and degree $d \geq 1$. Suppose that $h^0(T, \mathcal{F}) > 0$ and $T, \mathcal{F} \not\in O_{\mathbb{P}^n} \oplus O_{\mathbb{P}^n}(2 - d)$. For any $s \in H^0(T, \mathcal{F}) \setminus \{0\}$, the induced foliation by curves is algebraically integrable in codimension two.

**Proof.** Let $\mathcal{G}$ be the foliation induced by a nonzero section $s \in H^0(T, \mathcal{F})$. By Lemma 3.2, we know that $\mathcal{G}$ has singularities in codimension two. Suppose, aiming at a contradiction, that the general leaf of $\mathcal{G}$ is not algebraic. Then, due to Lemma 3.4, $\mathcal{G}$ is defined by either $v = x_0 \frac{\partial}{\partial x_0} + \lambda x_1 \frac{\partial}{\partial x_1}$ with $\lambda \notin \mathbb{Q}$ or $v = (x_0 + x_1) \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1}$. Then consider $\overline{\mathcal{G}}$ the foliation obtained by taking the Zariski closure of orbits of $v$, see [3]. One readily sees that, in either case, $\overline{\mathcal{G}}$ is given by the rational map $\mathbb{P}^n \to \mathbb{P}^{n-2}, (x_0 : \cdots : x_n) \mapsto (x_2 : \cdots : x_n)$.

On the other hand, let $E \cong \mathbb{P}^2 \subset \mathbb{P}^n$ be the Zariski closure of a general leaf of $\mathcal{G}$. The tangency locus between $E$ and $\mathcal{F}$ is closed and Zariski-dense in $E$ hence $E$ is $\mathcal{F}$-invariant. Therefore, $\mathcal{F} = \overline{\mathcal{G}}$ but we get a contradiction since $\overline{\mathcal{G}}$ has degree 0 and $\mathcal{F}$ has degree $d \geq 1$.

Summarizing the above discussion, a foliation $\mathcal{F}$ of degree and dimension equal to 2 must be tangent to an algebraically integrable foliation defined by a global vector field of types (1), (3) or (4) of Lemma 3.4. The vector fields in cases (3) and (4) are nilpotent; hence they induce actions of the additive group $(\mathbb{C}, +)$. This allows us to conclude the following result.

**Proposition 3.6.** Let $\mathcal{F}$ be a foliation of dimension 2 and degree 2 on $\mathbb{P}^n$ tangent to a global vector field $v \in H^0(T, \mathbb{P}^n)$ from either (3) or (4) in Lemma 3.4. Then $\mathcal{F} \in \text{Log}(1^{n-2}, 2; 2, n)$.

**Proof.** Both cases follow the same argument, we only prove one. In case (3), $v = x_1 \frac{\partial}{\partial x_0} + x_3 \frac{\partial}{\partial x_2}$. Then it has a rational first integral $\rho : \mathbb{P}^n \to \mathbb{P}(1^{n-2}, 2)$ given by

$$\rho(x_0 : \cdots : x_n) = (x_1 : x_3 : x_4 : \cdots : x_n : x_0 x_3 - x_1 x_2).$$

Then $\mathcal{F} = \rho^* \mathcal{H}$ for $\mathcal{H}$ a foliation by curves on $\mathbb{P}(1^{n-2}, 2)$. Note that $\rho$ does not ramify in codimension one, hence

$$O_{\mathbb{P}^n}(n + 1) = \text{det}(N, \mathcal{F}) = \rho^* \text{det}(N, \mathcal{H}) = O_{\mathbb{P}^n}(n + \deg \mathcal{H}).$$

Thus $\deg(\mathcal{H}) = 1$ and $\mathcal{H}$ is defined by a global vector field. As we observed in Remark 2.2, $\mathcal{F} \in \text{Log}(1^{n-2}, 2; 2, n)$.

Summarizing the above discussion, a foliation $\mathcal{F}$ of degree and dimension equal to 2 must be tangent to an algebraically integrable foliation defined by a global vector field of types (1), (3) or (4) of Lemma 3.4. The vector fields in cases (3) and (4) are nilpotent; hence they induce actions of the additive group $(\mathbb{C}, +)$. This allows us to conclude the following result.

**Proposition 3.7.** Let $\omega \in H^0(\Omega_{\mathbb{P}^n}^{n-2}(n+1))$ not vanishing in codimension one and let $v = x_0 \frac{\partial}{\partial x_0} + \mu x_1 \frac{\partial}{\partial x_1} \in H^0(T, \mathbb{P}^n)$ be such that $i_\omega \omega = 0$ and $i_\omega d\omega = a \omega$ for $a \in \mathbb{C}$ a constant. Then $\omega$ defines a foliation $\mathcal{F}$ of one of the following types:

1. $a = \mu + 1$, $\mu \not\in \mathbb{Z}$ and $\mathcal{F} \in \text{Log}(1^{n+1}; 2, n)$;
2. $a = 0$, $\mu = -1$ and $\mathcal{F} \in \text{Log}(1^{n-1}; 2; 2, n)$;
3. $a = 0$, $\mu = -2$ and $\mathcal{F} \in \text{Rat}(1^{n-2}; 3; n)$;
4. $\mu \in \{1, 2\}$ and $a = \mu + 2$, and $\mathcal{F}$ is induced by an action of $\text{aff}(\mathbb{C})$. 

For the case (1) of Lemma 3.4, $v = x_0 \frac{\partial}{\partial x_0} + \mu x_1 \frac{\partial}{\partial x_1} \in H^0(T, \mathbb{P}^n)$ defines an action of the multiplicative group $\mathbb{C}^\ast$. The next result classifies the 2-dimensional foliations invariant by such actions. Note that we don’t impose any restriction on $\mu \in \mathbb{C}^\ast$ a priori.
Proof. Since \( t_v \omega = 0 \) we may write

\[
\omega = (x_0dx_1 - \mu x_1dx_0) \wedge t_{u\text{rad}}\beta + (\mu - 1)x_0x_1\beta + \gamma,
\]

where \( \beta \) and \( \gamma \) do not depend on \( dx_0 \) nor \( dx_1 \) and \( t_{u\text{rad}}\gamma = 0 \). Then \( \beta \) and \( \gamma \) are \((n - 2)\)-forms depending on \( n - 1 \) basic differentials, which means that there exists a degree-two homogeneous polynomial \( q \) and a linear vector field \( u = \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \) such that \( \beta = t_u\Theta \) and \( \gamma = q t_{u\text{rad}}\Theta \), where \( \Theta = dx_2 \wedge \cdots \wedge dx_n \). Hence,

\[
\omega = (x_0dx_1 - \mu x_1dx_0) \wedge t_{u\text{rad}}t_u\Theta + (\mu - 1)x_0x_1t_u\Theta + q t_{u\text{rad}}\Theta.
\]

We might assume that \( u \) is not proportional to the radial in the last \( n - 1 \) variables, i.e., \( u \wedge \sum_{j=2}^{n} x_j \frac{\partial}{\partial x_j} \neq 0 \), otherwise \( \omega \) would vanish in codimension one.

By hypothesis, there exists \( a \in \mathbb{C} \) such that \( t_v \omega = a\omega \). Computing the RHS, we will see that there are few possibilities for \( a \). Start with expanding

\[
d\omega = (\mu + 1)(x_0dx_1 - \mu x_1dx_0) \wedge t_{u\text{rad}}t_u\Theta + (\mu - 1)(x_0dx_1 - \mu x_1dx_0) \wedge t_u\Theta + (\mu - 1)x_0x_1t_u\Theta + dq \wedge t_{u\text{rad}}\Theta + q t_{u\text{rad}}\Theta
\]

then,

\[
t_v \omega = (\mu + 1)(x_0dx_1 - \mu x_1dx_0) \wedge t_{u\text{rad}}t_u\Theta + (\mu - 1)(x_0dx_1 - \mu x_1dx_0) \wedge t_u\Theta + (\mu - 1)x_0x_1t_u\Theta + v(q)t_{u\text{rad}}\Theta
\]

\[
= (x_0dx_1 - \mu x_1dx_0) \wedge ((\mu + 1)t_{u\text{rad}}t_u\Theta + t_v t_{u\text{rad}}t_u\Theta) + (\mu - 1)x_0x_1((\mu + 1)t_u\Theta + t_v t_u\Theta) + v(q)t_{u\text{rad}}\Theta
\]

and by \( t_v \omega = a\omega \) imposing we get that

\[
t_v t_{u\text{rad}}t_u\Theta = (a - \mu - 1)t_{u\text{rad}}t_u\Theta,
\]

\[
t_v t_u\Theta = (a - \mu - 1)t_u\Theta + c t_{u\text{rad}}\Theta
\]

\[
v(q) = aq - (\mu - 1)c x_0x_1
\]

where \( c \in \mathbb{C} \) and the second equation follows from the first.

Notice that \( t_{u\text{rad}}t_u\Theta = \sum_{i,j} p_{ij}dx_2 \wedge \cdots \wedge d\widehat{x}_i \wedge d\widehat{x}_j \wedge \cdots \wedge dx_n \), where each \( p_{ij} \) is a quadratic polynomial not having monomials \( x_0^2, x_0x_1 \) nor \( x_1^2 \). Moreover,

\[
t_v t_{u\text{rad}}t_u\Theta = \sum_{i,j} v(p_{ij})dx_2 \wedge \cdots \wedge d\widehat{x}_i \wedge d\widehat{x}_j \wedge \cdots \wedge dx_n
\]

and we only need to compute the action of \( v \) on degree two polynomials. From \( v(x_0) = x_0, v(x_1) = \mu x_1 \) and \( v(x_j) = 0 \) for \( j \geq 2 \) we get that

\[
\begin{align*}
\frac{v(x_i x_j)}{x_i x_j} &= \begin{cases} 
2, & i = j = 0 \\
2\mu, & i = j = 1 \\
(\mu + 1), & i = 0, j = 1 \\
1, & i = 0, j \geq 2 \\
\mu, & i = 1, j \geq 2 \\
0, & i, j \geq 2 
\end{cases}
\end{align*}
\]

Then, \( t_v t_{u\text{rad}}t_u\Theta = (a - \mu - 1)t_{u\text{rad}}t_u\Theta \), we need \( a - \mu - 1 \in \{0, 1, \mu \} \).
Case 1.0: Suppose that $a - \mu - 1 = 0$ and that $\mu \notin \{-1, 1\}$. Then $\nu, du\Theta = c_{\nu}\Theta$ implies $c = 0$. Thus $v(q) = (\mu + 1)q$. From (3.2) and $\mu \notin \{-1, 1\}$, $q = c'x_0x_1$ for $c' \in \mathbb{C}$ a constant. Then

$$\omega = (x_0dx_1 - \mu x_1dx_0) \wedge \nu_{\nu}\Theta + x_0x_1((\mu - 1)\nu\Theta + c'\nu_{\nu}\Theta).$$

Consider $w = c'x_1\frac{\partial}{\partial x_1} + u$ and notice that

$$\nu_{\nu} + \nu_{\nu_{\nu}}(dx_0 \wedge \cdots \wedge dx_n) = \nu_{\nu_{\nu}}(dx_0 \wedge dx_1 \wedge \Theta) = \omega.$$

Also, since $u$ does not depend on $x_0$ nor $x_1$, $[v, w] = 0$. Thus $\omega$ defines a foliation $\mathcal{F}$ on $\mathbb{P}^n$ with trivial tangent sheaf $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$.

Case 1.1: Suppose that $a - \mu - 1 = 0$ and that $\mu = 1$. As in the previous case, $c = 0$ and $u$ does not depend on $x_0$ nor $x_1$. But now $v(q) = 2q$ implies that $q = q_0x_0^2 + 2q_1x_0x_1 + q_2x_1^2$. Then

$$\omega = (x_0dx_1 - x_1dx_0) \wedge \nu_{\nu}\Theta + (q_0x_0^2 + 2q_1x_0x_1 + q_2x_1^2)\nu_{\nu}\Theta.$$

Consider the vector field

$$w = (-q_1x_0 - q_2x_1)\frac{\partial}{\partial x_0} + (q_0x_0 + q_1x_1)\frac{\partial}{\partial x_1} + u.$$

As in the previous case, we have that $\omega = \nu_{\nu_{\nu}}(dx_0 \wedge dx_1 \wedge \Theta)$, and it defines a foliation whose tangent sheaf is trivial.

Case 1.2: Suppose that $a - \mu - 1 = 0$ and that $\mu = -1$. Again, we have $c = 0$ but now $v(q) = 0$, which implies that $q = q_0x_0x_1 + q'(x_2, \ldots, x_n)$. Then

$$\omega = d(x_0x_1) \wedge \nu_{\nu}\Theta - 2x_0x_1\nu\Theta + q\nu_{\nu}\Theta,$$

Then $\omega$ defines a foliation on $\mathbb{P}^n$ which is the pullback of the foliation on $\mathbb{P}(2, 1^{(n-1)})$ given by the vector field

$$m = (q_0y_1 + q'(y_2, \ldots, y_n))\frac{\partial}{\partial y_1} + \sum_{j=2}^n b_j(y)\frac{\partial}{\partial y_j}$$

via the map defined by $y_1 = x_0x_1$ and $y_j = x_j$ for $j \geq 2$; recall that $u = \sum_{j=2}^n b_j(x)\frac{\partial}{\partial x_j}$.

Case 2.0: Suppose that $a - \mu - 1 = 1$ and that $\mu \notin \{-2, -1, 2\}$. Then $\nu_{\nu_{\nu}}\Theta = \nu_{\nu}\Theta$ and $x_0$ must divide $\nu_{\nu}\Theta$. Hence, there exist $\xi$ and $c'$ such that $\nu\Theta = x_0\xi + c'\nu_{\nu}\Theta$. Note that $c' = -c'$. Then

$$\omega = (x_0dx_1 - \mu x_1dx_0) \wedge \nu_{\nu}\Theta + (\mu - 1)x_0^2x_1\xi + (-c(\mu - 1)x_0x_1 + q)\nu_{\nu}\Theta.$$

On the other hand, $v(q) = (\mu + 2)q - (\mu - 1)c x_0x_1$ implies that $q = c(\mu - 1)x_0x_1$. Hence, $\omega$ vanishes along $V(x_0)$ which contradicts our hypothesis.

Case 2.1: Suppose that $a - \mu - 1 = 1$ and that $\mu = -2$. Then, $v(q) = 3cx_0x_1$ which implies that $q = -3cx_0x_1 + q'(x_2, \ldots, x_n)$. Hence,

$$\omega = x_0(dx_1 + 2x_1dx_0) \wedge \nu_{\nu}\Theta - 3x_0^2x_1\xi + (3cx_0x_1 + q)\nu_{\nu}\Theta =$$

$$= x_0(dx_1 + 2x_1dx_0) \wedge \nu_{\nu}\Theta - 3x_0x_1\xi + q'\nu_{\nu}\Theta.$$

Note that $\xi$ is a constant $(n - 2)$-form on $\{dx_2, \ldots, dx_n\}$. Then, up to a linear change on the last $n - 1$ coordinates, we may assume that $\xi = dx_3 \wedge \cdots \wedge dx_n$. Then

$$\omega = \nu_{\nu}(q'dx_2 - d(x_0x_1)) \wedge dx_3 \wedge \cdots \wedge dx_n).$$

Now consider $f(x_2, \ldots, x_n)$ a degree 3 polynomial such that $\frac{\partial f}{\partial x_2} = q'$. It follows that

$$\omega = \nu_{\nu}(d(f - x_0^2x_1) \wedge dx_3 \wedge \cdots \wedge dx_n),$$
and $\omega$ defines a rational foliation of type $(3, 1^{(n-2)})$.

**Case 2.2:** Suppose that $a - \mu - 1 = 1$ and that $\mu = 2$. Then, $v(q) = 4q - cx_0x_1$ which implies $q = q_0x_1^2 + cx_0x_1$. Then

$$\omega = x_0(x_0dx_1 - 2x_1dx_0) \wedge \tau_{\text{rad}}\xi + x_0^2x_1\xi + (-cx_0x_1 + q)\tau_{\text{rad}}\Theta$$

As in the previous case, we may assume $\xi = dx_3 \wedge \cdots \wedge dx_n$, hence

$$\omega = x_0(x_0dx_1 - 2x_1dx_0) \wedge \tau_{\text{rad}}\xi + x_0^2x_1\xi + q_0x_1^2\tau_{\text{rad}}\Theta$$

Note that $q_0 \neq 0$ unless $\omega$ vanishes along $V(x_0)$. Define the vector field $w = \frac{q_0}{2}x_1\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_2}$. It follows that

$$\omega = \tau_{\text{rad}}\Theta$$

Also notice that $[v, w] = w$ hence $\omega$ defines a foliation associated with an algebraic action of the affine Lie algebra.

**Case 2.3:** Suppose that $a - \mu - 1 = 1$ and that $\mu = -1$. Again $x_0$ divides $\tau_{\text{rad}}\Theta$. But now $v(q) = q + 2cx_0x_1$, which implies that $x_0$ divides $q$. Then $\omega$ vanishes along $V(x_0)$ which is absurd.

**Case 2.4:** Suppose that $a - \mu - 1 = 1$ and that $\mu = 1$. Then $v(q) = 3q$ implies $q = 0$. Then,

$$\omega = \tau_{\text{rad}}\Theta$$

On the other hand, $\tau_{\text{rad}}\Theta$ implies that $u = u' + c\sum_{j=2}^{n}x_j\frac{\partial}{\partial x_j}$. Thus, we can replace $u$ with $u'$ in the definition of $\omega$. Moreover, $[v, u'] = u'$ and $\omega$ defines a foliation tangent to an action of $\text{aff}(C)$.

**Case 3:** Suppose that $a - \mu - 1 = \mu$. We may exchange $x_0$ and $x_1$ so that we get new parameters $(\mu', a') = (1/\mu, a/\mu)$. For $a = 2\mu + 1$ we get $a' = \mu' + 2$. Then we reduce to one of the previous cases. \hfill \square

### 4. Foliations of Degree Two and Higher Dimension

In dimension $k > 2$, the positivity of $T\mathcal{F}$ simplifies the classification. Indeed, $c_1(T\mathcal{F}) = k - 2$, and $\mathcal{F}$ is a Mukai foliation, see [2]. We have the following theorem (cf. Theorem A).

**Theorem 4.1.** Let $\mathcal{F}$ be a degree-2 foliation on $\mathbb{P}^n$ of dimension $k > 2$. Then one of the following holds:

1. $\mathcal{F}$ is algebraically integrable;
2. $\mathcal{F} \in \text{Log}(1^{n-k+1}; 2; k, n)$ is a logarithmic foliation of type $(1^{n-r+1}, 2)$;
3. $\mathcal{F} \in \text{Log}(1^{n+1}; k, n)$ is a logarithmic foliation of type $(1^{n+1})$;
4. $\mathcal{F} \in \text{Exc}(r_1, \ldots, r_m, k, n)$ is an exceptional foliation; or
5. $\mathcal{F} \in \text{LPB}(2, k, n)$ is a linear pullback of a foliation by curves in $\mathbb{P}^{n-k+1}$.

The proof of this result will follow from Proposition 4.2 below. We begin assuming that $\mathcal{F}$ is not a linear pullback of a foliation on $\mathbb{P}^r$ with $r < n$. This is equivalent to assuming that $\mathcal{F}$ has no subfoliation of degree 0 or that $H^0(T\mathcal{F}(1)) = 0$. Indeed, for any foliation $\mathcal{G}$ of degree 0, we have $T\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\dim \mathcal{G}}$. 

Proposition 4.2. Let $\mathcal{F}$ a degree-2 foliation on $\mathbb{P}^n$ of dimension $k > 2$. If $H^0(T\mathcal{F}(-1)) = 0$ then either

\begin{itemize}
  \item $\mathcal{F}$ is algebraically integrable, or
  \item $\mathcal{F} \in \text{Log}(1^{n-k+1,2}; k, n)$ is a logarithmic foliation of type $(1^{n-r+1}, 2)$.
\end{itemize}

Proof. Suppose that $\mathcal{F}$ is not algebraically integrable. By [1, Proposition 7.5] there exists a subfoliation $\mathcal{G} \subset \mathcal{F}$ such that $c_1(T\mathcal{G}) \geq c_1(T\mathcal{F})$, hence,

$$\dim \mathcal{G} - \deg \mathcal{G} \geq k - 2 > \dim \mathcal{G} - 2$$

which implies that $\deg \mathcal{G} \leq 1$. Since $H^0(T\mathcal{F}(-1)) = 0$, we have $\deg \mathcal{G} = 1$ and $H^0(T\mathcal{G}(-1)) = 0$; hence, $\dim \mathcal{G} = k - 1$. By Theorem 2.1, $\mathcal{G}$ is defined by a rational map $\rho: \mathbb{P}^n \to \mathbb{P}(1^{n-k+1,2})$ given by $n - k + 1$ linear forms and one quadratic form. It follows that $\mathcal{F} = \rho^* \mathcal{H}$ for some (purely transcendental) foliation by curves $\mathcal{H}$ on $\mathbb{P}(1^{n-k+1,2})$. Since $\rho$ does not ramify in codimension one, $\det(N\mathcal{F}) = \rho^* \det(N\mathcal{H})$. Hence, $\mathcal{H}$ is given by a global vector field. This means that $\mathcal{F} \in \text{Log}(1^{n-k+1,2}; k, n)$, see Remark 2.2.

Now, we apply this proposition to prove the general case.

Proof of Theorem 4.1. Suppose that $\mathcal{F}$ is neither algebraically integrable nor logarithmic of type $(1^{n-k+1,2})$. By Proposition 4.2, we must have $h = h^0(T\mathcal{F}(-1)) > 0$. Then $\mathcal{F}$ is a linear pullback of a foliation $\mathcal{G}$ on $\mathbb{P}^{n-k}$ such that $H^0(T\mathcal{G}(-1)) = 0$. It follows that $\dim \mathcal{G} \leq 2$. Indeed, if we had $\dim \mathcal{G} \geq 3$ we would get a contradiction applying Proposition 4.2 again.

If $\dim \mathcal{G} = 1$ then $\mathcal{F} \subseteq \text{LPB}(2, k, n)$. If $\dim \mathcal{G} = 2$ then, by Theorem 3.1, $\mathcal{G}$ belongs to either $\text{Log}(1^{n-h+1}; n-h)$ or an exceptional component $\text{Exc}(r_1, \ldots, r_m, 2, n-h)$. Thus $\mathcal{F}$ belongs to either $\text{Log}(1^{n+1}; k, n)$ or $\text{Exc}(r_1, \ldots, r_m, k, n)$, see Remark 2.4. \hfill $\square$

5. Holomorphic Poisson structures

In this section, we will prove Corollary C.

A holomorphic Poisson structure on a complex manifold $X$ is a $\mathbb{C}$-linear Lie bracket

$$\{\cdot, \cdot\}: \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

which satisfies the Leibniz rule $\{f, gh\} = h\{f, g\} + f\{g, h\}$ and Jacobi identities

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

for all $f, g \in \mathcal{O}_X$. The bracket corresponds to a holomorphic bivector field $\sigma \in H^0(X, \wedge^2 T^*X)$ given by $\sigma(df \wedge dg) = \{f, g\}$, for all $f, g \in \mathcal{O}_X$. We will denote a Poisson structure on $X$ as the pair $(X, \sigma)$, where $\sigma \in H^0(X, \wedge^2 T^*X)$ is the corresponding Poisson bivector field. We say that the structure $(X, \sigma)$ has rank $2k$ if

$$\sigma \wedge \cdots \wedge \sigma \neq 0 \quad \text{and} \quad \sigma \wedge \cdots \wedge \sigma = 0,$$

$k$ times

$k+1$ times

The holomorphic bivector $\sigma$ induces a morphism

$$\sigma^\#: \Omega^1_X \to TX$$

which is called the anchor map and is defined by $\sigma^\#(\theta) = \sigma(\theta, \cdot)$, where $\theta$ is a germ of a holomorphic 1-form, and $\delta: TX \to \wedge^2 TX$ is given by $\delta(v) = L_v(\sigma)$; here $L_v$ denotes the Lie derivative. We refer to [31] for more details.

Definition 5.1. The symplectic foliation associated with $\sigma$ is the foliation given by $\mathcal{F}_\sigma := \text{Ker}(\sigma^\#)$, whose dimension is the rank of the anchor map $\sigma^\#: \Omega^1_X \to TX$. A Poisson variety $(X, \sigma)$ is called generically symplectic if the anchor map $\sigma^\#: \Omega^1_X \to TX$ is generically an isomorphism. Then, the degeneracy loci of $\sigma^\#$ is an effective anti-canonical divisor $D(\sigma) \subseteq |-K_X|$. 


Definition 5.2. Let \((X, \sigma)\) be a Poisson projective variety. A Poisson connection on a sheaf of \(\mathcal{O}_X\)-modules \(E\) is a \(\mathbb{C}\)-linear morphism of sheaves \(\nabla : E \to TX \otimes E\) satisfying the Leibniz rule
\[
\nabla(fs) = \delta(f) \otimes s + f\nabla(s),
\]
where \(f\) is a germ of a holomorphic function on \(X\) and \(s\) is a germ of a holomorphic section of \(E\). We say that \(E\) is a Poisson module if it admits a Poisson flat connection, i.e., if its curvature \(\nabla^2 : E \to \Omega_X^2 \otimes E\) vanishes. Equivalently, a Poisson connection defines a \(\mathbb{C}\)-linear bracket \(\{ , \} : \mathcal{O}_X \times E \to E\) by
\[
\{f, s\} := \nabla(s)(df),
\]
where \(f\) is a germ of a holomorphic function on \(X\) and \(s\) is a germ of a holomorphic section of \(E\).

Let \(E \cong \mathcal{O}_X^{\oplus 2}\) be a Poisson module with a trace free Poisson connection \(\nabla\), then we can write
\[
\nabla = \delta + \begin{pmatrix} v_1 & v_2 \\ v_0 & -v_1 \end{pmatrix}
\]
and the flatness condition is equivalent to
\[
\delta(v_0) = v_0 \wedge v_1, \\
\delta(v_1) = 2v_0 \wedge v_2, \\
\delta(v_2) = v_1 \wedge v_2.
\]

We refer to [9] for more details.

Definition 5.3. Let \(D\) be a reduced divisor on \(X\). The logarithmic tangent sheaf \(TX(-\log(D))\) is the subsheaf of \(TX\) consisting of germs of vector fields that preserve the ideal sheaf \(\mathcal{O}_X(-D)\). That is, \(TX(-\log(D))\) is the sheaf of vector fields tangent to \(D\). A reduced divisor \(D \subset X\) is free if \(TX(-\log(D))\) is locally free. If \(f : X \to Y\) is a rational map, we denote by \(T_{X/Y} \subset TX\) the relative tangent sheaf and by \(T_{X/Y}(-\log(D)) := T_{X/Y} \cap TX(-\log(D))\).

Definition 5.4. An analytic subspace \(Z \subset (X, \{ , \})\) is a Poisson subspace. \(Z\) is equipped with a Poisson structure \(\{ , \}_Z\) such that the inclusion \(i : Z \to X\) is a morphism of Poisson analytic spaces, i.e., it is compatible with the brackets.

Proposition 5.1. [31, Proposition 4.4.1] Let \(D \subset X\) a Poisson free divisor with respect to a holomorphic bi-vector \(\sigma \in H^0(X, \Lambda^2 TX)\). Then \(\sigma \in H^0(X, \Lambda^2 TX(-\log(D)))\).

Proof of Corollary C. Since the associated symplectic foliation \(\mathcal{F}_\sigma\) has no divisorial zeros and \(c_1(T\mathcal{F}_\sigma) = 0\), \(\mathcal{F}_\sigma\) is a foliation of dimension 2 and degree 2. It follows from Theorem 3.1 that one of the following holds:

(a) \(\mathcal{F}_\sigma\) is algebraically integrable and \(\sigma\) is not decomposable, since \(T\mathcal{F}\) is stable.

(b) \(\mathcal{F}_\sigma\) is the linear pull-back of a degree-two foliation by curves on \(\mathbb{P}^{n-1}\) with tangent sheaf split. So, \(\sigma\) is decomposable.

(c) \(T\mathcal{F}_\sigma \cong \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n}\), where \(\mathfrak{g} \subset \mathfrak{sl}(n+1, \mathbb{C})\) and either \(\mathfrak{g}\) is an Abelian Lie algebra of dimension 2 or \(\mathfrak{g} \cong \mathfrak{aff}(\mathbb{C})\). So, \(\sigma\) is decomposable.

(d) \(\mathcal{F}_\sigma\) is the pull-back by a rational map \(\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}(1^{(n-1)}, 2)\) of a non-algebraic foliation by curves induced by a global vector field on \(\mathbb{P}(1^{(n-1)}, 2)\). Then \(T\mathcal{F}\) is not split, which implies that \(\sigma\) is not decomposable.

Case (a): Since \(\sigma\) is of normal crossing type, by Remark 2.1, the symplectic foliation \(\mathcal{F}_\sigma\) is tangent to a dominant rational map \(\mathbb{P}^n \to \mathbb{P}^{n-2}\) given either by \(f_{(1, 3)} := (L_1^3, \cdots, L_{n-2}^3, P)\) or \(f_{(1, 2, 2)} := (L_1^2, \cdots, L_{n-3}^2, Q_1, Q_2)\), where \(L_i\) are linear, \(Q_i\) are quadratic, and \(P\) is cubic. Then the divisors \(D_{(1, 3)} := L_1 \cdots L_{n-2}P = 0\) and \(D_{(1, 2, 2)} := L_1 \cdots L_{n-3}Q_1Q_2 = 0\) are Poisson divisors of \(\sigma\).

If we denote by \(T_{\mathbb{P}^n/\mathbb{P}^{n-2}}\) the corresponding relative tangent sheaf either of \(f_{(1, 3)} : \mathbb{P}^n \to \mathbb{P}^{n-2}\) or of \(f_{(1, 2, 2)} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}\), then it follows from Proposition 5.1 that
• $\sigma \in H^0(\mathbb{P}^n, \wedge^2 T_{\mathbb{P}^n/p^n}(-\log(D_{1,3})))$, or
• $\sigma \in H^0(\mathbb{P}^n, \wedge^2 T_{\mathbb{P}^n/p^n}(-\log(D_{1,2,2})))$.

**Case (b):** If $\mathcal{F}_s$ is the linear pull-back of a degree two foliation by curves on $\mathbb{P}^{n-1}$, then there are $v_0$ a constant vector field, and $v$ a homogeneous polynomial vector field of degree 2 on $\mathbb{P}^{n-1}$ such that $\sigma = v_0 \wedge v$. It is clear that $[v_0, v] = 0$. Let $H$ be the Poisson hyperplane determined by $v_0 \in \mathbb{P} H^0(\mathbb{P}^n, T\mathbb{P}^n(-1)) \cong \mathbb{P}^n$. That is, $H$ is the hyperplane parameterizing lines passing through a point which are leaves of the foliation induced by $v_0$. It is clear that $H$ is a Poisson divisor.

**Case (c):** If $T\mathcal{F}_s \cong \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n}$, where $\mathfrak{g} \subset \mathfrak{sl}(n+1, \mathbb{C})$ an Abelian Lie algebra. Since $\mathcal{F}_s$ is general, then $\mathfrak{g}$ is generated by diagonal linear vector fields which are given in a suitable system of coordinates by

$$v_0 = \sum_{i=0}^{n} \lambda_{0i} z_i \frac{\partial}{\partial z_i}$$

and

$$v_1 = \sum_{j=0}^{n} \lambda_{1j} z_j \frac{\partial}{\partial z_j}.$$

Then

$$\sigma = v_0 \wedge v_1 = \sum_{i<j} (\lambda_{0i} - \lambda_{1j}) z_i z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

Since $v_r(z_i) = \lambda_r z_i$, for all $r = 0, 1$ and $i = 0, \ldots, n$, we conclude that $v_0, v_1 \in H^0(\mathbb{P}^n, T\mathbb{P}^n(-\log(D)))$ and $\sigma \in \wedge^2 \mathfrak{h}$, where $\mathfrak{h} = H^0(\mathbb{P}^n, T\mathbb{P}^n(-\log(D)))$ is the Abelian Lie algebra of vector fields tangent to the arrangement of planes $D = \{z_0 \cdots z_n = 0\}$.

Suppose that $T\mathcal{F}_s \cong \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n}$, where $\mathfrak{g} \cong \text{aff}(\mathbb{C})$. In this case, we have that the Poisson structure on $\mathbb{P}^n$ is induced by the so-called infinitesimal action

$$\sigma : \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n} \to T\mathbb{P}^n,$$

which are generated by $v_0$ and $v_1$ linear vector fields such $[v_0, v_1] = v_0$. In addition, in both case where $\sigma = v_0 \wedge v_1$, with $v_1, v_2 \in H^0(\mathbb{P}^n, T\mathbb{P}^n)$ we have that the tangent sheaf of the symplectic foliation is $T\mathcal{F}_s \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus 2}$. We have a trace-free flat Poisson connection

$$\nabla : T\mathcal{F}_s \to T\mathbb{P}^n \otimes T\mathcal{F}_s$$

given by

$$\nabla = \delta + \begin{pmatrix} v_1 & 0 \\ v_0 & -v_1 \end{pmatrix}.$$
Now, defining $(\sigma^\#)^{-1}|_{L_p}(v_i) = \omega_i$, we obtain the transversely affine structure on $L_p$ given by the pair $(\omega_0, \omega_1)$, where $\omega_0$ is the 1-form inducing the transversely affine holomorphic foliation on $L_p$. We have that

- if $\mathfrak{g}$ is Abelian, then $(\omega_0, \omega_1)$ is Euclidean,
- if $\mathfrak{g} \cong \text{aff}(\mathbb{C})$, then $(\omega_0, \omega_1)$ is affine.

If $T\mathcal{F}_\sigma \cong \mathfrak{g} \otimes \mathcal{O}_{P^n}$, where $\mathfrak{g} \cong \text{aff}(\mathbb{C})$, then the normal form of $\sigma$ follows from the proof of Proposition 2.3.

**Case (d):** The symplectic foliation $\mathcal{F}_\sigma$ is the pull-back by a rational map $\rho: \mathbb{P}^n \dashrightarrow \mathbb{P}(1^{(n-1)}, 2)$ of a foliation by curves on $\mathbb{P}(1^{(n-1)}, 2)$, and $\rho := (L_1, \ldots, L_{n-1}, Q)$, where $L_i$ are linear and $Q$ is quadratic. Then $\sigma \in H^0(\mathbb{P}^n, \wedge^2 T\mathbb{P}n(- \log(D)))$, where $D := \{L_1 \cdots L_{n-1} Q = 0\}$. Consider $v \in H^0(\mathbb{P}^n, T\mathbb{P}n(- \log(D))) \subset H^0(\mathbb{P}^n, T\mathbb{P}^n)$, the global vector field whose leaves are fibers of $\rho: \mathbb{P}^n \dashrightarrow \mathbb{P}(1^{(n-1)}, 2)$. It is clear that $\sigma \wedge v = 0$ and $dF(v) = 0$, where $F := L_1 \cdots L_{n-1} \cdot Q$. That is, $v$ is Hamiltonian with respect to $\sigma$.

\[ \square \]

**Remark 5.1.** Let $(\mathbb{P}^n, \sigma)$ be a holomorphic Poisson structure such that $\sigma \wedge \sigma = 0$. Then the associated symplectic foliation $\mathcal{F}_\sigma$ has at most degree 2. If $\sigma$ has zeros of codimension one, then by the classification of foliations of degree 0 and 1, see Section 2.1, we have that either $\sigma = Q \cdot (v_0 \wedge v_1)$, where $v_i$ are constant vector fields and $Q$ is of degree 2, or $\sigma = H_0 \cdot \tilde{\sigma}$, with $\{H_0 = 0\}$ being a Poisson hyperplane and

- $\tilde{\sigma} = v_0 \wedge v$, where $v_0$ is a constant vector field, $v$ is a linear vector field such that $[v_0, v] = 0$.
- $\tilde{\sigma} \in H^0(\mathbb{P}^n, \wedge^2 T\mathbb{P}n/\mathbb{P}^{n-2}(- \log(D)))$, where $D = H_1 \cup \cdots \cup H_{n-1} \cup Q$ is a Poisson divisor such that $H_i$ are linear and $Q$ is of degree 2.

By a similar argument given in [23, Theorem 9.1], we can deform $\sigma$ to a bivector belonging to either $\text{Log}(1^{n-1}); 2, 2, n)$ or $\text{Log}(1^{n+1}; 2, n)$.

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