Hydrodynamic limit for perturbation of a hyperbolic equilibrium point in two-component systems

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Abstract

We consider one-dimensional, locally finite interacting particle systems with two conservation laws. The models have a family of stationary measures with product structure and we assume the existence of a uniform bound on the inverse of the spectral gap which is quadratic in the size of the system. Under Eulerian scaling the hydrodynamic limit for the macroscopic density profiles leads to a two-component system of conservation laws. The resulting pde is hyperbolic inside the physical domain of the macroscopic densities, with possible loss of hyperbolicity at the boundary.

We investigate the propagation of small perturbations around a hyperbolic equilibrium point. We prove that the perturbations essentially evolve according to two decoupled Burgers equations. The scaling is not Eulerian: if the lattice constant is \( n^{-1} \), the perturbations are of order \( n^{-\beta} \) then time is speeded up by \( n^{1+\beta} \). Our derivation holds for \( 0 < \beta < \frac{1}{5} \). The proof relies on Yau’s relative entropy method, thus it applies only in the regime of smooth solutions.

This result is an extension of [10] and [11] where the analogue result was proved for systems with one conservation law. It also complements [12] where it was shown that perturbations around a non-hyperbolic boundary equilibrium point are driven by a universal two-by-two system of conservation laws.

1 Introduction

There are several results dealing with the perturbation analysis of hydrodynamic limits for interacting particle systems. In the landmark paper [4] the authors prove that for the asymmetric simple exclusion, in dimensions higher than 2, perturbations of order \( n^{-1} \) of a constant profile evolve according to a certain parabolic equation under diffusive scaling (time rescaled by \( n^2 \), space by \( n \)). It is well-known, that under Eulerian scaling (time rescaled by \( n \), space by \( n \)) the hydrodynamic limit leads to a hyperbolic conservation law (the Burgers equation), the perturbation limit gives the same equation with the Navier-Stokes correction. (For a survey on the microscopic interpretations of the Navier-Stokes equations see the end of Chapter 7 of [6].)

Motivated by [4] T. Seppäläinen investigated a similar problem in one dimension for the so-called totally asymmetric stick process. In [10] he proves that an \( \mathcal{O}(n^{-\beta}) \) perturbation of

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the constant profile is governed by the Burgers equation (even after the appearance of shocks) if time is rescaled by \( n^{1+\beta} \) and space by \( n \), where \( \beta \in (0, \frac{1}{2}) \) is a fixed constant. Independently, in [11] the authors partially extend this result by proving that one gets universally the Burgers equation in the hydrodynamic limit for similar perturbations of equilibrium for a wide class of one-dimensional interacting particle systems with one conservation law. The models are not reversible and not necessarily attractive. The proof relies on H. T. Yau’s relative entropy method, it only applies in the smooth regime of solutions and it only works for \( \beta \in (0, \frac{1}{5}) \). It is conjectured that the result should hold for all \( \beta \in (0, \frac{1}{2}) \) even without the smoothness condition as in the result of [10].

This universal result may be explained by the following arguments. Under Eulerian scaling these systems admit in the hydrodynamic limit a hyperbolic conservation law of the form

\[
\partial_t u + \partial_x J(u) = 0. \tag{1}
\]

Taking a point \( u_0 \) with \( J''(u_0) \neq 0 \) simple (although formal) calculations yield that solutions of (1), with initial conditions which are small perturbations \( u_0 \), are governed by the Burgers equation. See [11] for the 'more precise' formulation.

In the present paper we give an extension of the results of [10, 11] for systems with 2 conserved quantities. In [12] a general one-dimensional family of lattice-models was introduced. The models are locally finite interacting particle systems with two conservation laws which possess a family of stationary measures with product structure. In that paper it is shown (in the regime of smooth solutions) that in Eulerian scaling we get a hydrodynamic limit of the form

\[
\begin{align*}
\partial_t u + \partial_x \Phi(u, v) &= 0, \\
\partial_t v + \partial_x \Psi(u, v) &= 0, \tag{2}
\end{align*}
\]

where \((u, v) \in \mathcal{D}\) and \(\mathcal{D}\) is a convex compact polygon, the the physical domain (see [11] for the definition). We also note, that [8] gives the first major result about the Eulerian hydrodynamic limit of multi-component hyperbolic systems. In [12] it was also shown that an Onsager-type symmetry relation holds for the macroscopic flux functions \( \Phi, \Psi \) (see Lemma 1). One of the consequences of this relation is that inside the physical domain \(\mathcal{D}\) the pde is (weakly) hyperbolic, i.e. the Jacobian can be diagonalized in the real sense. Experience shows, that the limiting pde is strongly hyperbolic (the Jacobian has two distinct real eigenvalues) in the whole physical domain except some special points on the boundary \(\partial \mathcal{D}\).

We consider perturbations of order \( n^{-\beta} \) around a constant equilibrium point \((u_0, v_0) \in \mathcal{D}\), which is strictly hyperbolic. We prove that rescaling time by \( n^{1+\beta} \) and space by \( n \) the evolution of the perturbations are governed by two decoupled equations. (These are 'usually' Burgers equations, see the remark at the end of subsection 3.1.) This result agrees with the formal perturbation of the pde (2) e.g. with the method of weakly nonlinear geometric optics (see [3, 5]).
The reason for the decoupling of the resulting pde system is the strict hyperbolicity, basically, the two different eigenvalues (sound speeds) cause the equations to separate. In the paper [13] perturbation around a special non-hyperbolic point was considered in a similar setting, it was proved that in that case in the limit the evolution obeys a two-by-two system of conservation laws which cannot be decoupled. The treatment of that problem needs more complex tools than our proofs, sophisticated pde methods are used besides Yau’s method.

Our proof follows the relative entropy method using similar steps as [11] (thus it only applies in the regime of smooth solutions), but it also heavily relies on the Onsager-type symmetry relation proved in [12]. We assume the existence of a uniform bound on the inverse of the spectral gap, quadratic in system size, to be able to prove the so-called one block estimate. We do not deal with the proof of the spectral gap bound, but we remark that with the techniques of [7] one can get the desired gap estimates for a large class of systems. Our result holds for \( \beta \in (0, \frac{1}{5}) \). Assuming the stronger (but harder to prove) logarithmic-Sobolev bound we could get the result for \( \beta \in (0, \frac{1}{3}) \).

2 Microscopic models

We consider the family of microscopic models investigated in [12]. We go over the definitions and the important properties, for the details we refer the reader to the original paper. There are several concrete examples introduced in [12], we do not list them here.

2.1 State space, conserved quantities, generator

Throughout this paper we denote by \( T^n \) the discrete tori \( \mathbb{Z}/n\mathbb{Z} \), \( n \in \mathbb{N} \), and by \( T \) the continuous torus \( \mathbb{R}/\mathbb{Z} \). We will denote the local spin state by \( \Omega \), we only consider the case when \( \Omega \) is finite. The state space of the interacting particle system is

\[ \Omega^n := \Omega^{T^n}. \]

Configurations will be denoted

\[ \omega := (\omega_j)_{j \in T^n} \in \Omega^n, \]

The two conserved quantities are denoted by

\[ \zeta : \Omega \to \mathbb{Z}, \quad \eta : \Omega \to \mathbb{Z}, \]

we also use the notations \( \zeta_j = \zeta(\omega_j), \eta_j = \eta(\omega_j) \). We assume that the conserved quantities are different and non-trivial, i.e. the functions \( \zeta, \eta \) and the constant function 1 on \( \Omega \) are linearly independent.

We consider the rate function \( r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+ \). The dynamics of the system consists of elementary jumps effecting nearest neighbor spins, \((\omega_j, \omega_{j+1}) \to (\omega'_j, \omega'_{j+1})\), performed with rate \( r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1}) \).

We require that the rate function \( r \) satisfy the following conditions:
(A) If \( r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0 \) then
\[
\zeta(\omega_1) + \zeta(\omega_2) = \zeta(\omega'_1) + \zeta(\omega'_2),
\]
\[
\eta(\omega_1) + \eta(\omega_2) = \eta(\omega'_1) + \eta(\omega'_2).
\]
This means that \( \zeta \) and \( \eta \) are indeed conserved quantities.

(B) For every \( Z \in [n \min \zeta, n \max \zeta] \cap \mathbb{Z}, N \in [n \min \eta, n \max \eta] \cap \mathbb{Z} \) the set
\[
\Omega^\mathbb{Z}_{Z,N} := \left\{ \omega \in \Omega^n : \sum_{j \in \mathbb{T}^n} \zeta_j = Z, \sum_{j \in \mathbb{T}^n} \eta_j = N \right\}
\]
is an irreducible component of \( \Omega^n \), i.e. if \( \omega, \omega' \in \Omega^\mathbb{Z}_{Z,N} \) then there exists a series of elementary jumps with positive rates transforming \( \omega \) into \( \omega' \). This ensures that there are no hidden conservation laws.

(C) There exists a probability measure \( \pi \) on \( \Omega \) which puts positive mass on each element of \( \Omega \) and for any \( \omega_1, \omega_2, \omega'_1, \omega'_2 \in \Omega \)
\[
Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,
\]
where
\[
Q(\omega_1, \omega_2) := \sum\limits_{\omega'_1, \omega'_2 \in \Omega} \left\{ \frac{\pi(\omega'_1)\pi(\omega'_2)}{\pi(\omega_1)\pi(\omega_2)} r(\omega_j, \omega_j+1; \omega'_1, \omega'_2) - r(\omega_j, \omega_j+1; \omega_1, \omega_2) \right\}.
\]
This condition will imply that the measure \( \prod_{j \in \mathbb{T}^n} \pi \) is stationary for our process on \( \Omega^n \).

For a precise formulation of the infinitesimal generator on \( \Omega^n \) we first define the map \( \Theta^{\omega', \omega''}_j : \Omega^n \to \Omega^n \) for every \( \omega', \omega'' \in \Omega \), \( j \in \mathbb{T}^n \):
\[
(\Theta^{\omega', \omega''}_j \omega)_i = \begin{cases} 
\omega' & \text{if } i = j \\
\omega'' & \text{if } i = j + 1 \\
\omega_i & \text{if } i \neq j, j + 1.
\end{cases}
\]
The infinitesimal generator of the process defined on \( \Omega^n \) is
\[
L^n f(\omega) = \sum\limits_{j \in \mathbb{T}^n} \sum\limits_{\omega'_1, \omega'_2 \in \Omega} r(\omega_j, \omega_j+1; \omega'_1, \omega'_2)(f(\Theta^{\omega', \omega''}_j \omega) - f(\omega)).
\]
We denote by \( \chi^n_t \) the Markov process on the state space \( \Omega^n \) with infinitesimal generator \( L^n \).

2.2 Stationary measures

For every \( \theta, \tau \in \mathbb{R} \) let \( G(\theta, \tau) \) be the moment generating function defined below:
\[
G(\theta, \tau) := \log \sum_{\omega \in \Omega} e^{\theta \zeta(\omega) + \tau \eta(\omega)} \pi(\omega).
\]
We define the probability measures
\[ \pi_{\theta,\tau}(\omega) := \pi(\omega) \exp(\theta \zeta(\omega) + \tau \eta(\omega) - G(\theta, \tau)) \] (4)
on \Omega. Using condition (C), by very similar considerations as in [1], [2], [9] or [11] one can show that for any \( \theta, \tau \in \mathbb{R} \) the product measure
\[ \pi^n_{\theta,\tau} := \prod_{j \in \mathbb{T}^n} \pi_{\theta,\tau} \]
is stationary for the Markov process \( X^n_t \) on \( \Omega^n \) with infinitesimal generator \( L^n \). We will refer to these measures as the \textit{canonical} measures. Since \( \sum_j \zeta_j \) and \( \sum_j \eta_j \) are conserved, the canonical measures on \( \Omega^n \) are not ergodic. The conditioned measures defined on \( \Omega^n_{Z,N} \) by:
\[ \pi^n_{Z,N}(\omega) := \frac{\pi^n_{\theta,\tau}(\omega) \mathbb{1}\{\omega \in \Omega^n_{Z,N}\}}{\pi^n_{\theta,\tau}(\Omega^n_{Z,N})} \]
are also stationary and due to condition (B) satisfied by the rate functions they are also ergodic. We shall call these measures the \textit{microcanonical measures} of our system. It is easy to see that the measure \( \pi^n_{Z,N} \) does not depend on the values of \( \theta, \tau \).

\subsection*{2.3 Expectations, fluxes}
Expectation, variance, covariance with respect to the measures \( \pi^n_{\theta,\tau} \) will be denoted by \( E_{\theta,\tau}(.) \), \( \text{Var}_{\theta,\tau}(.) \), \( \text{Cov}_{\theta,\tau}(.) \).

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters \( \theta \) and \( \tau \):
\[ u(\theta, \tau) := E_{\theta,\tau}(\zeta) = \sum_{\omega \in \Omega} \zeta(\omega) \pi_{\theta,\tau}(\omega) = \theta G(\theta, \tau) = G_{\theta}, \]
\[ v(\theta, \tau) := E_{\theta,\tau}(\eta) = \sum_{\omega \in \Omega} \eta(\omega) \pi_{\theta,\tau}(\omega) = \tau G(\theta, \tau) = G_{\tau}. \]

We will usually note partial derivatives by using the respective subscripts, as long as it does not cause confusion. Elementary calculations show, that the matrix-valued function
\[ \begin{pmatrix} u_{\theta} & u_{\tau} \\ v_{\theta} & v_{\tau} \end{pmatrix} = \begin{pmatrix} G_{\theta\theta} & G_{\theta\tau} \\ G_{\theta\tau} & G_{\tau\tau} \end{pmatrix} =: G(\theta, \tau) \]
is equal to the covariance matrix \( \text{Cov}_{\theta,\tau}(\zeta, \eta) \) and as a consequence the function \( (\theta, \tau) \mapsto (u(\theta, \tau), v(\theta, \tau)) \) is invertible. We denote the inverse function by \( (u, v) \mapsto (\theta(u, v), \tau(u, v)) \). Denote by \( (u, v) \mapsto S(u, v) \) the convex conjugate (Legendre transform) of the strictly convex function \( (\theta, \tau) \mapsto G(\theta, \tau): \)
\[ S(u, v) := \sup_{\theta, \tau} \left( u\theta + v\tau - G(\theta, \tau) \right), \] (5)
$D := \{(u, v) \in \mathbb{R}_+ \times \mathbb{R} : S(u, v) < \infty\}$
\[(6)\]
\[= \text{co}\{(\zeta(\omega), \eta(\omega)) : \omega \in \Omega\},\]

where co stands for convex hull. In probabilistic terms: $S(u, v)$ is the rate function for joint large deviations of $(\sum_j \zeta_j, \sum_j \eta_j)$. If $(u, v)$ is inside $D$ then we have

$$\theta(u, v) = S_u(u, v), \quad \tau(u, v) = S_v(u, v).$$

With slight abuse of notation we shall denote: $\pi_\theta(u, v) = \pi_{u, v}, \pi_\tau(u, v) = \pi_{u, v}$, $E_\theta(u, v) = E_{u, v}$, etc. Clearly, $\pi_{u, v}$ can be defined naturally on the boundary of $D$, in that case $\pi_{u, v}$ does not put zero weight on some of the elements of $\Omega$.

We introduce the flux of the conserved quantities. The infinitesimal generator $L^n$ acts on the conserved quantities as follows:

$L^n \zeta_i = -\phi(\omega_i, \omega_{i+1}) + \phi(\omega_{i-1}, \omega_i) =: -\phi_i + \phi_{i-1},$

$L^n \eta_i = -\psi(\omega_i, \omega_{i+1}) + \psi(\omega_{i-1}, \omega_i) =: -\psi_i + \psi_{i-1},$

where

$$\phi(\omega_1, \omega_2) := \sum_{\omega'_1, \omega'_2 \in \Omega} r(\omega_1, \omega_2; \omega'_1, \omega'_2)(\zeta(\omega'_2) - \zeta(\omega_2)) + C_1$$

$$\psi(\omega_1, \omega_2) := \sum_{\omega'_1, \omega'_2 \in \Omega} r(\omega_1, \omega_2; \omega'_1, \omega'_2)(\eta(\omega'_2) - \eta(\omega_2)) + C_2$$

(7)

(The constants $C_1, C_2$ may be chosen arbitrarily, we will fix them later.) We shall denote the expectations of these functions with respect to the canonical measure $\pi_{u, v}^2$ by

$$\Phi(u, v) := E_{u, v}(\phi), \quad \Psi(u, v) := E_{u, v}(\psi).$$

(8)

The following lemma was proved in [12].

**Lemma 1.** Suppose we have a particle system with two conserved quantities and rates satisfying conditions (A) and (C). Then

$$\partial_\theta \Psi(u(\theta, \tau), v(\theta, \tau)) = \partial_\tau \Phi(u(\theta, \tau), v(\theta, \tau)).$$

The first derivative matrix of the fluxes $\Phi$ and $\Psi$ (with resp. to $u, v$) will be denoted by

$$D = D(u, v) := \begin{pmatrix} \Phi_u & \Phi_v \\ \Psi_u & \Psi_v \end{pmatrix}. \tag{9}$$

From Lemma 1 it follows that $D(u, v)$ is (weakly) hyperbolic, it can be diagonalized in a real sense (see [12]). We denote the two eigenvalues of $D$ by $\lambda$ and $\mu$, and the corresponding right and left eigenvectors by $r = (r_1, r_2)^\dagger, s = (s_1, s_2)^\dagger$ and $l = (l_1, l_2), m = (m_1, m_2)$:

$$Dr = \lambda r, \quad ID = \lambda l,$n
$$Ds = \mu s, \quad mD = \mu m.$$
Although we do not denote it explicitly, all of these are functions of \((u,v)\). We can assume \(|r| = |s| = 1, \quad 1 \cdot r = 1, \quad m \cdot s = 1\).

The second derivatives of the macroscopic fluxes are denoted by \(\Phi'', \Psi''\), these are symmetric two-by-two matrices depending on \((u,v)\).

### 2.4 The spectral gap condition

Let \(l\) be a positive integer and \((Z, N)\) integers with \(Z \in [l \min \zeta, l \max \zeta], \quad N \in [l \min \eta, l \max \eta]\).

Expectation with respect to the measure \(\pi^l_{Z,N}\) is denoted by \(\mathbf{E}^l_{Z,N}( \cdot )\). For \(f : \Omega^l_{Z,N} \to \mathbb{R}\) let

\[
L^l_{Z,N}f(\omega) := \sum_{j=1}^{l-1} \sum_{\omega', \omega''} r(\omega_j, \omega_{j+1}; \omega', \omega'')(f(\Theta^\omega_{j,j+1}\omega) - f(\omega)),
\]

\[
D^l_{Z,N}(f) := \frac{1}{2} \sum_{j=1}^{l-1} \sum_{\omega', \omega''} r(\omega_j, \omega_{j+1}; \omega', \omega'')(f(\Theta^\omega_{j,j+1}\omega) - f(\omega))^2.
\]

\(L^l_{Z,N}\) is the infinitesimal generator restricted to the hyperplane \(\Omega^l_{Z,N}\), and \(D^l_{Z,N}\) is the Dirichlet form associated to \(L^l_{Z,N}\) (or to its symmetric part). Note, that \(L^l_{Z,N}\) is defined with free boundary conditions.

We will assume the following additional condition on our models:

(D) There exists a positive constant \(W\) independent of \(l, Z, N\) such that for any \(f : \Omega^l_{Z,N} \to \mathbb{R}\) with \(\mathbf{E}^l_{Z,N}f = 0\)

\[
\mathbf{E}^l_{N,Z}f^2 \leq W l^2 D^l_{Z,N}(f).
\]

**Remark.** Presumably (D) is true for all (or a large class of) the models satisfying conditions [A]-[C], the techniques of [7] should be suitable to get the desired gap estimates.

We do not know about any published results covering our case.

### 3 Perturbation of the Eulerian hdl

In [12] it was proved by the application of Yau’s relative entropy method, that under Eulerian scaling the local density profiles of the conserved quantities evolve according to the following system of partial differential equations:

\[
\begin{aligned}
\partial_t u + \partial_x \Phi(u,v) &= 0, \\
\partial_t v + \partial_x \Psi(u,v) &= 0.
\end{aligned}
\]  

This pde is usually a strictly hyperbolic conservation law (i.e. \(D(u,v)\) has two distinct real eigenvalues), weak hyperbolicity follows from Lemma [1] (see [12]). Since the relative entropy
method needs smoothness conditions for the solution of the limiting equation, the previous result holds only up to a finite time, till the appearance of the first shock. We also note, that [8] gives the first major result about the Eulerian hydrodynamic limit of multi-component hyperbolic systems, also with the application of Yau’s method.

3.1 Formal perturbation

We will investigate the hydrodynamic behavior of small perturbations of an equilibrium point. For that we need to understand the asymptotics of small perturbations of a constant solution of (10). One of the perturbation techniques is the so-called method of weakly nonlinear geometric optics (see e.g. [3, 5]) which gives the following formal result.

Fix a point \((u_0, v_0)\) in \(D\) and suppose that this point is strictly hyperbolic, i.e.

\[
\lambda \neq \mu,
\]

at \((u_0, v_0)\). Suppose \((u_\varepsilon(t,x), v_\varepsilon(t,x))\) is the solution of the pde (10) with initial conditions

\[
\begin{align*}
    u_\varepsilon(0, x) &= u_0 + \varepsilon u^*(x), \\
    v_\varepsilon(0, x) &= v_0 + \varepsilon v^*(x),
\end{align*}
\]

where \(u^*(x), v^*(x)\) are fixed \(\mathbb{T} \mapsto \mathbb{R}\) smooth functions. Denote

\[
\begin{align*}
    \sigma_0(x) := l \cdot (u^*(x), v^*(x))^\dagger, & \quad c_\sigma := \int_\mathbb{T} \sigma_0(y)dy, \\
    \delta_0(x) := m \cdot (u^*(x), v^*(x))^\dagger, & \quad c_\delta := \int_\mathbb{T} \delta_0(y)dy,
\end{align*}
\]

and

\[
\begin{align*}
    a_1 := l \cdot (r_\dagger \Phi'' r, r_\dagger \Psi'' r)^\dagger, & \quad a_2 := l \cdot (r_\dagger \Phi'' s, r_\dagger \Psi'' s)^\dagger, \\
    b_1 := m \cdot (s_\dagger \Phi'' s, s_\dagger \Psi'' s)^\dagger, & \quad b_2 := m \cdot (s_\dagger \Phi'' s, s_\dagger \Psi'' s)^\dagger,
\end{align*}
\]

where \(l, m, r, s\) and \(\Phi'', \Psi''\) are the respective vector- and matrix-valued functions taken at \((u_0, v_0)\).

Then, according to the formal computations of the geometric optics method,

\[
\begin{pmatrix}
    u_\varepsilon(t,x) \\
    v_\varepsilon(t,x)
\end{pmatrix}
= 
\begin{pmatrix}
    u_0 \\
    v_0
\end{pmatrix}
+ \varepsilon \sigma(\varepsilon t, x - \lambda t) \begin{pmatrix}
    r_1 \\
    r_2
\end{pmatrix}
+ \varepsilon \delta(\varepsilon t, x - \mu t) \begin{pmatrix}
    s_1 \\
    s_2
\end{pmatrix}
+ \mathcal{O}(\varepsilon^2),
\]

as \(\varepsilon \to 0\), where \(\sigma\) and \(\delta\) are the solutions of the following Cauchy problems:

\[
\begin{align*}
    \partial_t \sigma(t,x) + \partial_x \left( a_1 \cdot \frac{1}{2} \sigma(t,x)^2 + c_\sigma a_2 \sigma(t,x) \right) &= 0, \\
    \sigma(0,x) &= \sigma_0(x),
\end{align*}
\]

(15)

and

\[
\begin{align*}
    \partial_t \delta(t,x) + \partial_x \left( b_1 \cdot \frac{1}{2} \delta(t,x)^2 + c_\delta b_2 \delta(t,x) \right) &= 0, \\
    \delta(0,x) &= \delta_0(x).
\end{align*}
\]

(16)
Remarks

1. This result means that a small perturbation of a constant solution of (10) is governed by the solutions of two decoupled equations (at least, by formal computations). If $a_1$ and $b_1$ are nonzero, then these equations are linear transforms of the Burgers equation. Otherwise the respective equations become linear transport equations. It is easy to check, that

$$a_1 \neq 0, \quad b_1 \neq 0$$

hold exactly when the point $(u_0, v_0)$ is genuinely nonlinear, i.e.

$$\nabla \lambda \cdot r \neq 0, \quad \nabla \mu \cdot s \neq 0$$

at $(u_0, v_0)$.

2. The geometric optics method is based on series expansion, thus it needs smoothness as a condition which could only be true up to a finite time in our case. Surprisingly, this formal method gives good approximation of the solutions even after the shocks. In [3] the authors prove that the equation (14) is valid, in the sense that for any $t > 0$ the $L_1$-norm of the difference of the two sides is bounded by $C t \varepsilon^2$. In fact, this result is valid for the case if we consider the pde (10) on $\mathbb{T}$ (as we do), on $\mathbb{R}$ they have even stronger bounds.

3.2 The main result

Our main theorem is a similar result on the microscopic level. We will apply Yau’s method, thus our results will hold in the regime of smooth solutions, only up to a finite time before the first appearance of shocks.

Suppose, that $(u_0, v_0)$ is a point in the physical domain which is strictly hyperbolic, see (11). Let $u^*(x), v^*(x)$ be smooth real functions on $\mathbb{T}$. Define $\sigma(t, x), \delta(t, x)$ according to (12), (13), (15) and (16), and suppose that they are smooth in $\mathbb{T} \times [0, T]$. Fix a small positive parameter $\beta$, and suppose that a particle system on $\Omega^n$ satisfying conditions (A)-(D) has initial distribution for which the density profiles of the two conserved quantities are 'close’ to the functions $u_0 + n^{-\beta} u^*(\cdot), v_0 + n^{-\beta} v^*(\cdot)$. I.e. the profiles are a small perturbation of the constant $(u_0, v_0)$ profile. We also assume, that $(u_0 + n^{-\beta} u^*(x), v_0 + n^{-\beta} v^*(x)) \in \mathcal{D}$ holds for every $x \in \mathbb{T}$, at least for $n > n_0$. Then, uniformly for $0 \leq t \leq T$, at time $n^{1+\beta} t$ the respective density profiles will be ‘close’ to the functions $u_0 + n^{-\beta} u^{(n)}(t, \cdot), v_0 + n^{-\beta} v^{(n)}(t, \cdot)$, where

$$\begin{pmatrix} u^{(n)}(t, x) \\ v^{(n)}(t, x) \end{pmatrix} := \sigma(t, x - \lambda n^{\beta} t) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \delta(t, x - \mu n^{\beta} t) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \quad (17)$$

For the precise formulation of the result we need to introduce some additional notations. We will denote by $\mu^n_t$ the true distribution of the system at microscopic time $n^{1+\beta} t$:

$$\mu^n_t := \mu^n_0 \exp\{n^{1+\beta} t I^n\} \quad (18)$$

We define the time-dependent reference measure $\nu^n_T$ as

$$\nu^n_T := \prod_{j \in \mathbb{T}^n} \pi_{u_0 + n^{-\beta} u^{(n)}(t, \frac{j}{n}), v_0 + n^{-\beta} v^{(n)}(t, \frac{j}{n})} \quad (19)$$
with $u^{(n)}, v^{(n)}$ defined in (17). This measure mimics on a microscopic level the macroscopic profiles $u_0 + n^{-\beta} u^{(n)}(t, \cdot), v_0 + n^{-\beta} v^{(n)}(t, \cdot)$. We also choose an absolute reference measure

$$\pi^n := \prod_{j \in T^n} \pi^n_{v_0^j, v_0^j},$$

(20)

which is a stationary measure of our Markov process on $\Omega^n$. The point $(u_0^n, v_0^n)$ is chosen in a way that it lies inside the domain $\mathcal{D}$ and

$$|u_0 - u_0^n| + |v_0 - v_0^n| < n^{-\beta}.$$  

(21)

If $(u_0, v_0)$ is inside $\mathcal{D}$, then we may choose $(u_0^n, v_0^n) = (u_0, v_0)$. By choosing $(u_0^n, v_0^n)$ inside $\mathcal{D}$ we get that any probability measure on $\Omega^n$ is absolutely continuous with respect to $\pi^n$. Condition (21) ensures that $\pi^n$ is 'close enough' to $\mu^n_t$ in entropy sense, uniformly in $t$.

**Theorem.** Let $\beta \in (0, \frac{1}{3})$ be fixed. Under the stated conditions, if

$$H(\mu^n_0 | \nu^n_0) = o(n^{1-2\beta}),$$

(22)

then

$$H(\mu^n_t | \nu^n_t) = o(n^{1-2\beta}),$$

(23)

uniformly for $0 \leq t \leq T$.

The following corollary is a simple consequence of the Theorem and the entropy inequality.

**Corollary.** Assume the conditions of Theorem 3.2. Let $g : \mathbb{T} \to \mathbb{R}$ be a test function. Then for any $t \in [0, T]$

$$n^{-1+\beta} \sum_{j \in T^n} g\left(\frac{j}{n}\right) \left(\zeta_j(n^{1+\beta}t) - u_0\right) - \int_{\mathbb{T}} g(x) \left(\sigma(t, x - \lambda n^{\beta}t) r_1 + \delta(t, x - \mu n^{\beta}t) s_1 \right) dx \to 0,$$

$$n^{-1+\beta} \sum_{j \in T^n} g\left(\frac{j}{n}\right) \left(\eta_j(n^{1+\beta}t) - v_0\right) - \int_{\mathbb{T}} g(x) \left(\sigma(t, x - \lambda n^{\beta}t) r_2 + \delta(t, x - \mu n^{\beta}t) s_2 \right) dx \to 0.$$

**Remarks.**

1. The Theorem states that if the initial distribution of the system is 'close' to $\nu^n_0$ in relative entropy sense then at time $n^{1+\beta}t$ it will be close to $\nu^n_t$. The fact, that 'close' should mean $o(n^{1-2\beta})$ can be easily explained, see e.g. [11] or [12].

2. If instead of condition (11) we assume a similar uniform bound on the logarithmic-Sobolev constant then our Theorem is valid for $\beta \in (0, \frac{1}{3})$. 

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4 Proof

We will assume, that

\[(u_0, v_0) = (0, 0), \quad \Phi_v(0, 0) = \Psi_u(0, 0) = 0.\]  \(24\)

It is easy to see, that we can always reduce the general case to get \(24\), via some suitable linear transformations on \((\zeta, \eta)\). Also, with the proper choice of the constants in the definition \(7\) we can set

\[\Phi(0, 0) = \Psi(0, 0) = 0.\]  \(25\)

Assumptions \(24\) imply, that

\[D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad l = r^\dagger = (1, 0), \quad m = s^\dagger = (0, 1),\]  \(26\)

and

\[u^{(n)}(t, x) = \sigma(t, x - \lambda n^\beta t), \quad v^{(n)}(t, x) = \delta(t, x - \mu n^\beta t).\]  \(27\)

We introduce the notations

\[\Phi'' = \begin{pmatrix} \Phi_{uu} & \Phi_{uv} \\ \Phi_{vu} & \Phi_{vv} \end{pmatrix} =: \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}, \quad \Psi'' = \begin{pmatrix} \Psi_{uu} & \Psi_{uv} \\ \Psi_{vu} & \Psi_{vv} \end{pmatrix} =: \begin{pmatrix} b_3 & b_2 \\ b_2 & b_1 \end{pmatrix}.\]  \(28\)

Clearly, these definitions agree with the definition \(13\).

We define the functions \(\sigma(t, x_1, x_2), \delta(t, x_1, x_2)\) as

\[\sigma(t, x_1, x_2) := \frac{1}{\lambda - \mu} \left( a_2 \sigma(t, x_1) \delta(t, x_2) + a_1 \sigma_x(t, x_1) \int_0^{x_2} (\delta(t, z) - c_3)dz + \frac{a_3}{2} \delta(t, x_2)^2 \right),\] \(29\)

\[\delta(t, x_1, x_2) := \frac{1}{\mu - \lambda} \left( b_2 \sigma(t, x_1) \delta(t, x_2) + b_1 \delta_x(t, x_2) \int_0^{x_1} (\sigma(t, z) - c_\sigma)dz + \frac{b_3}{2} \sigma(t, x_1)^2 \right).\]

The defining partial differential equations \(15, 16\) of the functions \(\sigma, \delta\) are conservation laws, thus for any \(0 \leq t \leq T\):

\[\int_T \sigma(z, t)dz = c_\sigma, \quad \int_T \delta(z, t)dz = c_\delta.\]

From that it follows that \(\overline{\sigma}, \overline{\delta}\) are well-defined smooth functions on \([0, T] \times \mathbb{T} \times \mathbb{T}\) (i.e. periodic in \(x_1\) and \(x_2\)) with bounded derivatives.

4.1 Changing the time-dependent reference measure

The usual way to prove a result like Theorem \(3.2\) is to get a Grönwall-type estimate on \(H(\mu^n_t | \nu^n_t)\):

\[H(\mu^n_t | \nu^n_t) - H(\mu^n_0 | \nu^n_0) \leq C \int_0^t H(\mu^n_s | \nu^n_s) + \sigma(n^{1-2\beta}),\]
via bounding the derivative $\partial_t H(\mu^n_t | \nu^n_t)$. We will use a slightly different approach, by proving a similar estimate for $H(\mu^n_t | \nu^n_t)$:

$$H(\mu^n_t | \nu^n_t) - H(\mu^n_0 | \nu^n_0) \leq C \int_0^t H(\mu^n_s | \nu^n_s) + o(n^{1-2\beta}).$$

(30)

Here

$$\tilde{\nu}^n_t := \prod_{j \in \mathbb{T}^n} \pi_{n-\beta \tilde{u}^{(n)}(t, \frac{j}{n}), n-\beta \tilde{v}^{(n)}(t, \frac{j}{n})}$$

(31)

and $\tilde{u}^{(n)}, \tilde{v}^{(n)}$ are smooth functions defined as

$$\tilde{u}^{(n)}(x, t) := u^{(n)}(x, t) + n^{-\beta} \sigma(t, x - \lambda n^\beta t, x - \mu n^\beta t)$$

$$= \sigma(t, x - \lambda n^\beta t) + n^{-\beta} \sigma(t, x - \lambda n^\beta t, x - \mu n^\beta t),$$

$$\tilde{v}^{(n)}(x, t) := v^{(n)}(x, t) + n^{-\beta} \sigma(t, x - \lambda n^\beta t, x - \mu n^\beta t),$$

(32)

$$= \delta(t, x - \mu n^\beta t) + n^{-\beta} \sigma(t, x - \lambda n^\beta t, x - \mu n^\beta t).$$

Because of Lemma 2 and condition (22) we have $H(\mu^n_0 | \tilde{\nu}^n_0) = o(n^{1-2\beta})$ and therefore from (30)

$$H(\mu^n_t | \tilde{\nu}^n_t) = o(n^{1-2\beta})$$

will follow uniformly for $0 \leq t \leq T$. Using Lemma 2 again we get Theorem 3.2.

**Lemma 2.** Let $\mu^n_t, \nu^n_t, \tilde{\nu}^n_t$ be the measures defined as before, with $t \in [0, T]$. Then

$$H(\mu^n_t | \nu^n_t) = o(n^{1-2\beta}) \iff H(\mu^n_t | \tilde{\nu}^n_t) = o(n^{1-2\beta}).$$

**Proof.** We start with

$$H(\mu^n_t | \nu^n_t) - H(\mu^n_t | \tilde{\nu}^n_t) = - \int_{\Omega^n} \log \frac{d\nu^n_t}{d\mu^n_t} d\mu^n_t.$$  

(33)

By subsections 22 and 23 we can calculate that

$$\log \frac{d\nu^n_t}{d\mu^n_t}(\omega) = \sum_{j \in \mathbb{T}^n} \left\{ \left( \theta(\sigma^{(n)} u, \nu^{(n)}) - \theta(\sigma^{(n)} \tilde{u}, \nu^{(n)}) \right) \zeta_j 
+ \left( \tau(\sigma^{(n)} u, \nu^{(n)}) - \tau(\sigma^{(n)} \tilde{u}, \nu^{(n)}) \right) \eta_j 
- G \left( \theta(\sigma^{(n)} u, \nu^{(n)}), \tau(\sigma^{(n)} u, \nu^{(n)}) \right) 
+ G \left( \theta(\sigma^{(n)} \tilde{u}, \nu^{(n)}), \tau(\sigma^{(n)} \tilde{u}, \nu^{(n)}) \right) \right\},$$

where, for typographical reasons, we omitted the arguments $(t, \frac{j}{n})$ from the functions $u^{(n)}, v^{(n)}, \tilde{u}^{(n)}, \tilde{v}^{(n)}$. From the previous expression via power-series expansion:

$$\log \frac{d\nu^n_t}{d\mu^n_t}(\omega) \leq O(n^{-3\beta}) + C n^{-2\beta} \sum_{j \in \mathbb{T}^n} \left| \zeta_j - u^{(n)}(t, \frac{j}{n}) \right| + \left| \eta_j - v^{(n)}(t, \frac{j}{n}) \right|,$$

$$= O(n^{-3\beta}) + C n^{-2\beta} \sum_{j \in \mathbb{T}^n} \left( \left| \zeta_j - \tilde{u}^{(n)}(t, \frac{j}{n}) \right| + \left| \eta_j - \tilde{v}^{(n)}(t, \frac{j}{n}) \right| \right),$$

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with uniform error terms. Using this with (33) and the entropy inequality with respect to \( \nu^n_t \) and \( \tilde{\nu}^n_t \) the lemma follows.

We also note, that applying the same arguments as in the proof of Lemma 2 we get that from the condition (22)

\[
H(\mu^n_0|\pi^n) = O(n^{1-2\beta})
\]

follows. Since \( \pi^n \) is a stationary measure,

\[
H(\mu^n_t|\pi^n) \leq H(\mu^n_0|\pi^n) = O(n^{1-2\beta}) \tag{34}
\]

for all \( t \geq 0 \).

The proof of the following lemma is a simple application of the entropy inequality with the entropy bound (34). Mind that because of (24) and (25)

\[
E_{\pi^n}\zeta = E_{\pi^n}\eta = E_{\pi^n}\phi = E_{\pi^n}\psi = 0.
\]

Lemma 3. Suppose \( b_1, b_2, \ldots \) are real numbers with \( |b_j| \leq 1 \) and \( \xi_j \) stands for either of \( \eta_j, \zeta_j, \psi_j \) or \( \phi_j \). Then

\[
\int \frac{1}{n} \sum_{j \in \mathbb{T}^n} b_j \xi_j d\mu^n_t \leq C n^{-\beta}.
\]

with an absolute constant \( C \).

In the rest of the paper we prove inequality (30).

4.2 Preparatory computations

We define

\[
\tilde{\vartheta}^{(n)}(t, x) := n^\beta (n^{-\beta} a^{(n)}(t, x), n^{-\beta} \tilde{v}^{(n)}(t, x)), \\
\tilde{\tau}^{(n)}(t, x) := n^\beta (n^{-\beta} \bar{u}^{(n)}(t, x), n^{-\beta} \bar{v}^{(n)}(t, x)), \\
\theta^n_0 := n^\beta (n^{-\beta} u^n_0, n^{-\beta} v^n_0), \\
\tau^n_0 := n^\beta (n^{-\beta} u^n_0, n^{-\beta} v^n_0). \tag{35}
\]

It is easy to check, that the partial derivatives \( \partial_x \tilde{\vartheta}^{(n)}(t, x), \partial_x \tilde{\tau}^{(n)}(t, x), \partial_x^2 \tilde{\vartheta}^{(n)}(t, x), \partial_x^2 \tilde{\tau}^{(n)}(t, x) \) are uniformly bounded in \([0, T] \times \mathbb{T}\). From subsection 2.2 we have

\[
\bar{f}^n_t := \frac{d\tilde{\vartheta}^n_t}{d\pi^n_t} = \exp \sum_{j \in \mathbb{T}^n} \left\{ n^{-\beta} \left( \tilde{\vartheta}^{(n)}(t, \frac{j}{n}) - \theta^n_0 \right) \xi_j + n^{-\beta} \left( \tilde{\tau}^{(n)}(t, \frac{j}{n}) - \tau^n_0 \right) \eta_j \right. \\
- G \left( n^{-\beta} \tilde{\vartheta}^{(n)}(t, \frac{j}{n}), n^{-\beta} \tilde{\tau}^{(n)}(t, \frac{j}{n}) \right) + G \left( n^{-\beta} \theta^n_0, n^{-\beta} \tau^n_0 \right) \right\} \tag{36}.
\]
Differentiating the identity
\[ H(\mu^n_t|\nu^n_t) - H(\mu^n_0|\pi^n) = -\int_{\Omega^n} \log \tilde{f}^n_t \mu^n_t \]
and noting that \( \partial_t H(\mu^n_t|\nu^n_t) \leq 0 \) we get the following bound on \( \partial_t H(\mu^n_t|\nu^n_t) \):
\[ n^{2\beta - 1} \partial_t H(\mu^n_t|\nu^n_t) \leq -\int_{\Omega^n} \left( n^{3\beta} L^n \log \tilde{f}^n_t + n^{-1+2\beta} \partial_t \log \tilde{f}^n_t \right) d\mu^n_t. \]
Integrating with respect to the time:
\[ n^{2\beta - 1} (H(\mu^n_t|\nu^n_t) - H(\mu^n_0|\nu^n_0)) \leq -\int_0^t \int_{\Omega^n} \left( n^{3\beta} L^n \log \tilde{f}^n_s + n^{-1+2\beta} \partial_t \log \tilde{f}^n_s \right) d\mu^n_s dt. \tag{37} \]
We estimate the two terms on the right-hand side separately in the next two subsections.

### 4.3 Estimating the first term of (37)

From the definitions
\[
n^{3\beta} L^n \log \tilde{f}^n_t (\omega) = n^{-1+2\beta} \sum_{j \in \mathbb{T}^n} \left( \phi_j - \Phi(n^{-\beta} \tilde{u}^{(n)}_j, n^{-\beta} \tilde{v}^{(n)}_j) \right) \partial_x \tilde{\theta}^{(n)}(t, \frac{j}{n}) \]
\[ + \left( \psi_j - \Psi(n^{-\beta} \tilde{u}^{(n)}_j, n^{-\beta} \tilde{v}^{(n)}_j) \right) \partial_x \tilde{\tau}^{(n)}(t, \frac{j}{n}) \]
\[ + \text{Err}_1^n(t, \omega) + \text{Err}_2^n(t), \tag{38} \]
where
\[
\text{Err}_1^n(t, \omega) := n^{-1+2\beta} \sum_{j \in \mathbb{T}^n} \left\{ \phi_j \left( \nabla^n \tilde{\theta}^{(n)}(t, \frac{j}{n}) - \partial_x \tilde{\theta}^{(n)}(t, \frac{j}{n}) \right) \right. \]
\[ + \psi_j \left( \nabla^n \tilde{\tau}^{(n)}(t, \frac{j}{n}) - \partial_x \tilde{\tau}^{(n)}(t, \frac{j}{n}) \right) \}, \tag{39} \]
\[
\text{Err}_2^n(t) := n^{-1+2\beta} \sum_{j \in \mathbb{T}^n} \left\{ \Phi(n^{-\beta} \tilde{u}^{(n)}_j, n^{-\beta} \tilde{v}^{(n)}_j) \partial_x \tilde{\theta}^{(n)}(t, \frac{j}{n}) \right. \]
\[ + \Psi(n^{-\beta} \tilde{u}^{(n)}_j, n^{-\beta} \tilde{v}^{(n)}_j) \partial_x \tilde{\tau}^{(n)}(t, \frac{j}{n}) \}. \tag{40} \]
We used the (slightly abused) shorthanded notations
\[
\tilde{u}^{(n)}_j = \tilde{u}^{(n)}(t, \frac{j}{n}), \quad \tilde{v}^{(n)}_j = \tilde{v}^{(n)}(t, \frac{j}{n}),
\]
and \( \nabla^n \) denotes the discrete gradient:
\[
\nabla^n g(x) := n \left( g(x + \frac{1}{n}) - g(x) \right).
\]
Using the smoothness of \( \tilde{\theta}^{(n)} \) and \( \tilde{\tau}^{(n)} \) and Lemma 3, the expectation of the first error term can be easily estimated:
\[
\int_{\Omega^n} |\text{Err}_1^n(t, \omega)| d\mu^n_t = O(n^{-1+\beta}). \tag{41} \]
Using Lemma 1 it is easy to see that there exists a smooth function \( U(u, v) \) such that
\[
\partial_\theta U (u(\theta), v(\theta), \tau(\theta)) = \Phi (u(\theta), v(\theta), \tau(\theta)), \quad \partial_\tau U (u(\theta), v(\theta), \tau(\theta)) = \Psi (u(\theta), v(\theta), \tau(\theta)).
\]
Thus \( \text{Err}_2(t) \) takes the form:
\[
\text{Err}_2^n(t, \omega) = n^{3\beta - 1} \sum_{j \in \mathbb{T}^n} \partial_x U \left( n^{-\beta} u_j(n) (t, \frac{j}{n}), n^{-\beta} v_j(n) (t, \frac{j}{n}) \right),
\]
from which
\[
\text{Err}_2(t) = O(n^{-1 + 2\beta}), \quad (42)
\]
uniformly for \( t \in [0, T] \). From previous bounds we have
\[
\int_0^t \int_{\Omega^n} n^3 L^n \log f^n_s(\omega) d\mu^n_s ds =
\]
\[
n^{-1 + 2\beta} \int_0^t \int_{\Omega^n} \sum_{j \in \mathbb{T}^n} \left( \phi_j - \Phi (n^{-\beta} u_j(n), n^{-\beta} v_j(n)) \right) \partial_x \theta_j(n) (s, \frac{j}{n})
+ \left( \psi_j - \Psi (n^{-\beta} u_j(n), n^{-\beta} v_j(n)) \right) \partial_x \tau_j(n) (s, \frac{j}{n}) d\mu^n_s ds \quad (43)
+ O(n^{-1 + 2\beta})
\]
In the next step we introduce the block averages. We will denote the block size with \( l = l(n) \), it will be large microscopically, but small on the macroscopic scale. In the first computations we only assume \( l \gg n^{2\beta} \), the exact order of \( l \) will only be determined at the end of the proof, after collecting all the error terms. For a local function \( \kappa_j (j \in \mathbb{T}^n) \) we define its block average with
\[
\kappa^l_j := \frac{1}{l} \sum_{i=0}^{l-1} \kappa_{j+i}.
\]
By partial summation for a smooth function \( \rho(x) : \mathbb{T} \mapsto \mathbb{R} \) we have
\[
\left| \sum_{j \in \mathbb{T}^n} \kappa_j \rho (\frac{j}{n}) - \sum_{j \in \mathbb{T}^n} \kappa^l_j \rho (\frac{j}{n}) \right| \leq \| \partial_x \rho \|_{\infty} \left| \sum_{j \in \mathbb{T}^n} \kappa_j \right| \frac{l}{n}
\]
Using this with Lemma 3 we can replace \( \phi_j, \psi_j \) in (43) with the respective block averages:
\[
\int_0^t \int_{\Omega^n} n^3 L^n \log f^n_s(\omega) d\mu^n_s ds =
\]
\[
n^{-1 + 2\beta} \int_0^t \int_{\mathbb{T}^n} \sum_{j \in \mathbb{T}^n} \left( \phi^l_j - \Phi (n^{-\beta} u_j(n), n^{-\beta} v_j(n)) \right) \partial_x \theta_j(n) (s, \frac{j}{n})
+ \left( \psi^l_j - \Psi (n^{-\beta} u_j(n), n^{-\beta} v_j(n)) \right) \partial_x \tau_j(n) (s, \frac{j}{n}) \right) d\mu^n_s ds \quad (44)
+ O(n^{\beta - 1} l).
Finally, using Lemma 4 (the one-block estimate), we replace the block averages $\phi^l_j, \psi^l_j$ by their 'local equilibrium value': $\Phi(\zeta^l_j, \eta^l_j)$ and $\Psi(\zeta^l_j, \eta^l_j)$, respectively:

$$\int_0^t \int_{\Omega^n} n^{3\beta} L^n \log \tilde{f}_s^n(\omega) d\mu^n_s ds =$$

$$n^{-1+2\beta} \int_0^t \int_{\Omega^n} \sum_{j \in T^n} \left( \Phi(\zeta^l_j, \eta^l_j) - \Phi(n^{-\beta} \bar{u}^{(n)}_j, n^{-\beta} \bar{v}^{(n)}_j) \right) \partial_s \tilde{\theta}^{(n)}(s, \frac{j}{n})$$

$$+ \left( \Psi(\zeta^l_j, \eta^l_j) - \Psi(n^{-\beta} \bar{u}^{(n)}_j, n^{-\beta} \bar{v}^{(n)}_j) \right) \partial_s \tilde{\tau}^{(n)}(s, \frac{j}{n}) d\mu^n_s ds \quad (45)$$

$$+ O(n^{\beta-1} l \vee n^{-1-\beta l^3} \vee l^{-1})$$

**Lemma 4 (One block estimate).**

$$\frac{1}{n} \int_0^t \int_{\Omega^n} \sum_{j \in T^n} \left| \phi^l_j - \Phi(\zeta^l_j, \eta^l_j) \right| d\mu^n_s dt \leq C(n^{-1-2\beta l^3} + l^{-1}),$$

$$\frac{1}{n} \int_0^t \int_{\Omega^n} \sum_{j \in T^n} \left| \psi^l_j - \Psi(\zeta^l_j, \eta^l_j) \right| d\mu^n_s dt \leq C(n^{-1-3\beta l^3} + l^{-1}).$$

The proof relies on the spectral gap condition (12). It uses the Feynman-Kac formula, the Raleigh-Schrödinger perturbation technique and the 'equivalence of ensembles' (see the Appendix of [6] for all three). A detailed proof can be found in [11] for the one component case which can be easily adapted for our purposes.

**Remark.** If instead of the condition (12) we have a similar uniform bound on the logarithmic-Sobolev constant, then the previous lemma may be strengthened: it holds with the bound $C(n^{-1-3\beta l^2} + l^{-1})$.

### 4.4 Estimating the second term of (37)

Performing the time-derivation we obtain:

$$n^{-1+2\beta} \partial_t \log \tilde{f}_t^n = \frac{1}{n} \sum_{j \in T^n} \left\{ \left( n^\beta \zeta_j - \bar{u}_j^{(n)} \right) \partial_t \tilde{\theta}^{(n)}(t, \frac{j}{n}) + \left( n^\beta \eta_j - \bar{v}_j^{(n)} \right) \partial_t \tilde{\tau}^{(n)}(t, \frac{j}{n}) \right\} \quad (46)$$

By the definitions of $\bar{u}^{(n)}_j, \bar{v}^{(n)}_j$ and Taylor-expansion we readily get that

$$\partial_t \bar{u}^{(n)}(t, x) = \sigma_t(t, x - \lambda n^\beta t) - \lambda n^\beta \sigma_x(t, x - \lambda n^\beta t)$$

$$- \lambda \bar{\sigma}_x(t, x - \lambda n^\beta t, x - \mu n^\beta t) - \mu \bar{\sigma}_x(t, x - \lambda n^\beta t, x - \mu n^\beta t)$$

$$+ O(n^{-\beta}),$$

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and
\[ n^{2\beta} \partial_x \Phi \left( n^{-\beta} \tilde{u}^{(n)}(x,t), n^{-\beta} \tilde{v}^{(n)}(x,t) \right) \]
\[ = \lambda n^\beta \sigma_x (t,x - \lambda n^\beta t) \]
\[ + \lambda \overline{\sigma}_x (t,x - \lambda n^\beta t, x - \mu n^\beta t) + \lambda \overline{\sigma}_x (t,x - \lambda n^\beta t, x - \mu n^\beta t) \]
\[ + \partial_x \left( \frac{1}{2} a_1 \sigma (t,x - \lambda n^\beta t)^2 + c_3 a_2 \sigma (t,x - \lambda n^\beta t) \right) \]
\[ + \partial_x \left( a_2 \sigma (t,x - \lambda n^\beta t) (\delta(t,x - \mu n^\beta t) - c_3) + \frac{1}{2} a_3 \delta(t,x - \mu n^\beta t)^2 \right) \]
\[ + \mathcal{O}(n^{-\beta}), \]

with uniform error terms. (\( \overline{\sigma}_x \) and \( \overline{\sigma}_x \) are the partial derivatives of \( \overline{\sigma}(t,x_1,x_2) \) with respect to the second and third variable.) Adding up these equations and checking the definitions for \( \sigma, \delta, \overline{\sigma} \) we see that all the significant terms on the right hand side cancel to give:
\[ \partial_x \tilde{u}^{(n)} + \partial_x \left( n^{2\beta} \Phi (n^{-\beta} \tilde{u}^{(n)}, n^{-\beta} \tilde{v}^{(n)}) \right) = \mathcal{O}(n^{-\beta}). \] (47)

Similarly,
\[ \partial_x \tilde{v}^{(n)} + \partial_x \left( n^{2\beta} \Psi (n^{-\beta} \tilde{u}^{(n)}, n^{-\beta} \tilde{v}^{(n)}) \right) = \mathcal{O}(n^{-\beta}). \] (48)

From (47) and (48):
\[ \partial_t \tilde{\theta}^{(n)} = \partial_u \theta \tilde{u}^{(n)} + \theta_v \tilde{v}^{(n)} \]
\[ = -n^{2\beta} \partial_u \partial_x \left( \Phi (n^{-\beta} \tilde{u}^{(n)}, n^{-\beta} \tilde{v}^{(n)}) \right) - n^{2\beta} \theta_v \partial_x \left( \Psi (n^{-\beta} \tilde{u}^{(n)}, n^{-\beta} \tilde{v}^{(n)}) \right) + \mathcal{O}(n^{-\beta}) \]
\[ = -n^\beta (\theta_u \Phi_u + \theta_v \Psi_v) \partial_x \tilde{u}^{(n)} - n^\beta (\theta_u \Phi_v + \theta_v \Psi_u) \partial_x \tilde{v}^{(n)} + \mathcal{O}(n^{-\beta}) \]
\[ = -n^\beta (\theta_u \Phi_u + \tau_u \Psi_u) \partial_x \tilde{u}^{(n)} - n^\beta (\theta_v \Phi_v + \tau_v \Psi_v) \partial_x \tilde{v}^{(n)} + \mathcal{O}(n^{-\beta}) \]
\[ = -n^\beta \Phi_u \partial_x \tilde{\theta}^{(n)} - n^\beta \Psi_v \partial_x \tilde{\tau}^{(n)} + \mathcal{O}(n^{-\beta}) \] (49)

In the fourth line we used \( \tau_u = \theta_v \) and Lemma 11. To simplify notations, we omitted the arguments \( (t,x) \) from the functions \( \tilde{\theta}^{(n)}, \tilde{\tau}^{(n)}, \tilde{u}^{(n)}, \tilde{v}^{(n)} \), and the arguments \( (n^{-\beta} \tilde{u}^{(n)}, n^{-\beta} \tilde{v}^{(n)}) \) from all the partial derivatives of \( \theta, \tau, \Phi, \Psi \) with respect to \( u, v \). Similarly,
\[ \partial_t \tilde{\tau}^{(n)} = -n^\beta \Phi_v \partial_x \tilde{\theta}^{(n)} - n^\beta \Psi_u \partial_x \tilde{\tau}^{(n)} + \mathcal{O}(n^{-\beta}). \] (50)
Hence from (46):

\[
\begin{align*}
&\quad n^{-1+2\beta} \partial_t \log \frac{f^n_t}{\mu^n_t} \\
&= -n^{-1+2\beta} \sum_{j \in \mathbb{T}^n} \left\{ \left( \zeta_j - n^{-\beta} u_{j}^{(n)} \right) \Phi_u \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\theta}^{(n)}(t, \frac{j}{n}) \\
&\quad + \left( \zeta_j - n^{-\beta} u_{j}^{(n)} \right) \Psi_u \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\tau}^{(n)}(t, \frac{j}{n}) \\
&\quad + \left( \eta_j - n^{-\beta} v_{j}^{(n)} \right) \Phi_v \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\theta}^{(n)}(t, \frac{j}{n}) \\
&\quad + \left( \eta_j - n^{-\beta} v_{j}^{(n)} \right) \Psi_v \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\tau}^{(n)}(t, \frac{j}{n}) \right\} \\
&\quad + \text{Err}_3(t, \omega),
\end{align*}
\]

where

\[
\text{Err}_3(t, \omega) = \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left( \zeta_j - n^{-\beta} u_{j}^{(n)} \right) b_j(t) + \left( \eta_j - n^{-\beta} v_{j}^{(n)} \right) c_j(t) \quad (52)
\]

and \(b_j(t)\) and \(c_j(t)\) are uniformly bounded constants. Using Lemma 9 we get that

\[
\int_{\Omega^n} |\text{Err}_3(t, \omega)| \, d\mu^n_t = O(n^{-\beta}). \quad (53)
\]

We can exchange \(\zeta_j, \eta_j\) with their block-averages \(\zeta^l_j, \eta^l_j\) (as in the previous subsection) which (after the time-integration) gives the following estimate:

\[
\begin{align*}
&\quad \int_0^t \int_{\Omega^n} n^{-1+2\beta} \partial_t \log \frac{f^n_s}{\mu^n_s} \, d\mu^n_s \\
&= -\int_0^t \int_{\Omega^n} n^{-1+2\beta} \sum_{j \in \mathbb{T}^n} \left\{ \left( \zeta^l_j - n^{-\beta} u_{j}^{(n)} \right) \Phi_u \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\theta}^{(n)}(s, \frac{j}{n}) \\
&\quad + \left( \zeta^l_j - n^{-\beta} u_{j}^{(n)} \right) \Psi_u \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\tau}^{(n)}(s, \frac{j}{n}) \\
&\quad + \left( \eta^l_j - n^{-\beta} v_{j}^{(n)} \right) \Phi_v \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\theta}^{(n)}(s, \frac{j}{n}) \\
&\quad + \left( \eta^l_j - n^{-\beta} v_{j}^{(n)} \right) \Psi_v \left( n^{-\beta} u_{j}^{(n)}, n^{-\beta} v_{j}^{(n)} \right) \partial_x \tilde{\tau}^{(n)}(s, \frac{j}{n}) \right\} \\
&\quad + O(n^{-\beta} \lor n^{\beta-1}), \quad (54)
\end{align*}
\]

4.5 Block replacement

For a function \(\Upsilon(u, v)\) we denote

\[
R_{\Upsilon}(u_1, v_1; u_2, v_2) := \Upsilon(u_1, v_1) - \Upsilon(u_2, v_2) - \Upsilon_u(u_2, v_2)(u_1 - u_2) - \Upsilon_v(u_2, v_2)(v_1 - v_2).
\]
Collecting the estimates of the previous subsections we have
\[
H(\mu_t^n | \nu^n_0) - H(\mu_t^n | \nu^n_0)
\leq C n^{2\beta - 1} \int_0^t \int_{\Omega^n} \sum_{j \in \mathbb{T}^n} \left| R_\Phi \left( \zeta_j^l, \eta_j^l; n^{-\beta} \tilde{u}^{(n)}(s, \frac{j}{n}), n^{-\beta} \tilde{v}^{(n)}(s, \frac{j}{n}) \right) \right|
+ \left| R_\Psi \left( \zeta_j^l, \eta_j^l; n^{-\beta} \tilde{u}^{(n)}(s, \frac{j}{n}), n^{-\beta} \tilde{v}^{(n)}(s, \frac{j}{n}) \right) \right| \, d\mu^n_s \, ds
+ \mathcal{O}(n^{-\beta} \vee n^{\beta - 1} l \vee n^{-1 - \beta} l^3)
\] (55)

The second derivatives of \( \Phi \) and \( \Psi \) are bounded thus
\[
\left| R_\Phi \left( \zeta_j^l, \eta_j^l; n^{-\beta} \tilde{u}^{(n)}(s, \frac{j}{n}), n^{-\beta} \tilde{v}^{(n)}(s, \frac{j}{n}) \right) \right| + \left| R_\Psi \left( \zeta_j^l, \eta_j^l; n^{-\beta} \tilde{u}^{(n)}(s, \frac{j}{n}), n^{-\beta} \tilde{v}^{(n)}(s, \frac{j}{n}) \right) \right|
\leq C \left( (\zeta_j^l - n^{-\beta} \tilde{u}^{(n)}(s, \frac{j}{n}))^2 + (\eta_j^l - n^{-\beta} \tilde{v}^{(n)}(s, \frac{j}{n}))^2 \right),
\]
which means that it is sufficient to estimate
\[
n^{2\beta - 1} \int_{\Omega^n} \sum_{j \in \mathbb{T}^n} \left( \zeta_j^l - n^{-\beta} \tilde{u}^{(n)}(t, \frac{j}{n}) \right)^2 \, d\mu^n_t \quad \text{and} \quad n^{2\beta - 1} \int_{\Omega^n} \sum_{j \in \mathbb{T}^n} \left( \eta_j^l - n^{-\beta} \tilde{v}^{(n)}(t, \frac{j}{n}) \right)^2 \, d\mu^n_t
\]
uniformly in \( t \). We estimate the first expression, the other will follow the same way. We denote
\[
\tilde{\zeta}_j^l := \tilde{\zeta}_j^l(t, \omega) = \frac{1}{l} \sum_{i=0}^{l-1} \left( \zeta_j - n^{-\beta} \tilde{u}^{(n)}(t, \frac{j}{n}) \right).
\]

Since \( \partial_x \tilde{u}^{(n)}(t, x) \) is uniformly bounded for \( (t, x) \in [0, T] \times \mathbb{T} \), we have
\[
(\zeta_j^l)^2 - (\zeta_j^l - n^{-\beta} \tilde{u}^{(n)}(t, \frac{j}{n}))^2 = \mathcal{O}(n^{-\beta - 1} l)
\]
uniformly in \( j \in \mathbb{T}^n \), \( t \in [0, T] \) and it is enough to estimate
\[
n^{2\beta - 1} \int_{\Omega^n} \sum_{j \in \mathbb{T}^n} (\tilde{\zeta}_j^l)^2 \, d\mu^n_t.
\]

Applying the entropy inequality with respect to the time-dependent reference measure \( \nu^n_t \) and using Hölder’s inequality:
\[
n^{2\beta - 1} \int_{\Omega^n} \sum_{j \in \mathbb{T}^n} (\tilde{\zeta}_j^l)^2 \, d\mu^n_t \leq \frac{1}{\gamma} n^{2\beta - 1} H(\mu^n_t | \nu^n_0) + \frac{1}{\gamma} l n^{2\beta - 1} \sum_{j \in \mathbb{T}^n} \log \mathbb{E}_{\nu^n_t} \exp \left( \gamma l (\tilde{\zeta}_j^l)^2 \right), \quad (56)
\]
for any \( \gamma > 0 \). \( \mathcal{D} \) is compact, \( \zeta \) is bounded thus there exists a positive constant \( C \) such that
\[
\log \mathbb{E}_{u,v} \exp \left( ((\zeta - u) y) \right) \leq Cy^2
\]
for all \( (u, v) \in \mathcal{D} \) and \( y \in \mathbb{R} \). Thus as a consequence of Lemma there exists a small \( \gamma > 0 \) for which
\[
\frac{1}{n} \sum_{j \in \mathbb{T}^n} \log \mathbb{E}_{\nu^n_t} \exp \left( \gamma l (\tilde{\zeta}_j^l)^2 \right) < 1.
\]
Substituting into (56):

\[ n^{2\beta - 1} \int_{\Omega_n} \sum_{j \in \mathbb{T}_n} \left( c_i^j - n^{-\beta} \tilde{a}^{(n)}(t_i^j) \right)^2 d\mu^n_t < C n^{2\beta - 1} H(\mu^n_t|\tilde{\nu}^n_t) + \mathcal{O}(n^{2\beta - 1}). \]

Collecting all the estimates, from (55) we get

\[ n^{2\beta - 1} (H(\mu^n_t|\tilde{\nu}^n_t) - H(\mu^0_n|\tilde{\nu}^0_n)) \leq C n^{2\beta - 1} \int_0^t H(\mu^n_s|\tilde{\nu}^n_s) ds + \mathcal{O}(n^{-1} \vee n^{-1} l \vee n^{2\beta - 1}). \]

Choosing \( l \) with

\[ n^{2\beta} \ll l \ll n^{1 + \beta} \]

the error term becomes \( o(1) \) and the Theorem follows via the Grönwall inequality. (If we have the logarithmic-Sobolev condition, and thus a stronger version of Lemma 4, then \( l \) can be chosen with \( n^{2\beta} \ll l \ll n^{1 + \beta} \) to make all the error terms \( o(1) \).)

The proof of Lemma 5 can be found in [11] or [13].

**Lemma 5.** Suppose \( \xi_1, \xi_2, \ldots \) are independent random variables with \( \mathbb{E} \xi_i = 0 \) for which

\[ \log \mathbb{E} \exp(y \xi_i) \leq C y^2 \]

with a positive constant \( C \) independent of \( i \) and \( y \). Then there exists a small positive constant \( \gamma \) depending only on \( C \) such that

\[ \log \mathbb{E} \exp \left( \gamma l (\xi_i^2) \right) < 1. \]

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**References**

[1] M. Balázs: Growth fluctuations in interface models. Annales de l’Institut Henri Poincaré — Probabilités et Statistiques 39: 639-685 (2003)

[2] C. Cocozza: Processus des misanthropes. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 70: 509-523 (1985)

[3] R. J. DiPerna, A. Majda: The Validity of Nonlinear Geometric Optics for Weak Solutions of Conservation Laws. Commun. Math. Phys. 98: 313-347 (1985)
[4] R. Esposito, R. Marra, H. T. Yau: Diffusive limit of asymmetric simple exclusion. Reviews of Mathematical Physics 6: 1233-1267 (1994)

[5] J. K. Hunter, J. B. Keller: Weakly nonlinear high frequency waves. Commun Pure Appl. Math. 36, 547-569 (1983)

[6] C. Kipnis, C. Landim: Scaling Limits of Interacting Particle Systems. Springer, 1999.

[7] C. Landim, S. Sethuraman, S. R. S. Varadhan: Spectral gap for zero range dynamics. Annals of Probability 24: 1871-1902 (1986)

[8] S. Olla, S. R. S. Varadhan, and H. T. Yau: Hydrodynamical limit for Hamiltonian system with weak noise, Commun. Math. Phys. 155: 523-560 (1993)

[9] F. Rezakhanlou: Microscopic structure of shocks in one conservation laws. Annales de l'Institut Henri Poincaré — Analyse Non Lineaire 12: 119-153 (1995)

[10] T. Seppäläinen: Perturbation of the equilibrium for a totally asymmetric stick process in one dimension. Annals of Probability 29: 176-204 (2001)

[11] B. Tóth, B. Valkó: Between equilibrium fluctuations and Eulerian scaling. Perturbation of equilibrium for a class of deposition models. Journal of Statistical Physics 109: 177-205 (2002)

[12] B. Tóth, B. Valkó: Onsager relations and Eulerian hydrodynamic limit for systems with several conservation laws. Journal of Statistical Physics 112: 497-521 (2003)

[13] B. Tóth, B. Valkó: Perturbation of singular equilibria of hyperbolic two-component systems: a universal hydrodynamic limit, preprint, arxiv.org/abs/math.PR/0312256