Fredholm Properties and $L^p$-Spectra of Localized Rotating Waves in Parabolic Systems

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W.-J. Beyn, D. Otten. Fredholm Properties and $L^p$-Spectra of Localized Rotating waves in Parabolic Systems. Preprint to appear, 2016.

W.-J. Beyn, D. Otten. Spatial Decay of Rotating Waves in Reaction Diffusion Systems. Dyn. Partial Differ. Equ., 13(3):191-240, 2016.

D. Otten. Spatial decay and spectral properties of rotating waves in parabolic systems. PhD thesis, Bielefeld University, Shaker Verlag, 2014.
Outline

1. Rotating patterns in $\mathbb{R}^d$
2. Spatial decay of rotating waves
3. Eigenvalue problem for rotating waves and some basic definitions
4. Fredholm properties of linearization in $L^p$
5. Essential $L^p$-spectrum and dispersion relation
6. Point $L^p$-spectrum and shape of eigenfunctions
7. Cubic-quintic complex Ginzburg-Landau equation
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1. Rotating patterns in $\mathbb{R}^d$

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7. Cubic-quintic complex Ginzburg-Landau equation
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

$$
\begin{align*}
  u_t(x, t) &= A \triangle u(x, t) + f(u(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
  u(x, 0) &= u_0(x), \quad \ t = 0, \ x \in \mathbb{R}^d.
\end{align*}
$$

where $u : \mathbb{R}^d \times [0, \infty] \to \mathbb{R}^m$, $A \in \mathbb{R}^{m, m}$, $f : \mathbb{R}^m \to \mathbb{R}^m$, $u_0 : \mathbb{R}^d \to \mathbb{R}^m$.

Assume a rotating wave solution $u_* : \mathbb{R}^d \times [0, \infty] \to \mathbb{R}^m$ of (1)

$$
u_*(x, t) = v_*(e^{-tS}x)
$$

$v_* : \mathbb{R}^d \to \mathbb{R}^m$ profile (pattern), $0 \neq S \in \mathbb{R}^{d, d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS}x, t)$ solves

$$
\begin{align*}
  v_t(x, t) &= A \triangle v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
  v(x, 0) &= u_0(x), \quad \ t = 0, \ x \in \mathbb{R}^d.
\end{align*}
$$

$$
\begin{align*}
\langle Sx, \nabla v(x) \rangle &= Dv(x)Sx = \sum_{i=1}^{d} \sum_{j=1}^{d} S_{ij} x_j D_i v(x) = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij} (x_j D_i - x_i D_j) v(x).
\end{align*}
$$

(Drift term) (Rotational term)
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

$$u_t(x, t) = A \triangle u(x, t) + f(u(x, t)), \ t > 0, \ x \in \mathbb{R}^d, \ d \geq 2,$$

$$u(x, 0) = u_0(x), \quad , \ t = 0, \ x \in \mathbb{R}^d.$$  \hspace{1cm} (1)

where $u : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$, $f : \mathbb{R}^m \to \mathbb{R}^m$, $u_0 : \mathbb{R}^d \to \mathbb{R}^m$.

Assume a rotating wave solution $u_\star : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^m$ of (1)

$$u_\star(x, t) = v_\star(e^{-tS}x)$$

$v_\star : \mathbb{R}^d \to \mathbb{R}^m$ profile (pattern), $0 \neq S \in \mathbb{R}^{d \times d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS}x, t)$ solves

$$v_t(x, t) = A \triangle v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \ t > 0, \ x \in \mathbb{R}^d, \ d \geq 2,$$

$$v(x, 0) = u_0(x), \quad , \ t = 0, \ x \in \mathbb{R}^d.$$  \hspace{1cm} (2)

$$\langle Sx, \nabla v(x) \rangle = Dv(x)Sx = \sum_{i=1}^{d} \sum_{j=1}^{d} S_{ij} x_j D_i v(x) = ^{S \top} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij} (x_j D_i - x_i D_j) v(x)$$

(drift term) \hspace{1cm} (rotational term)
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(drift term)  \hspace{1cm} (rotational term)
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Consider a reaction diffusion system

$$u_t(x,t) = A \triangle u(x,t) + f(u(x,t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2,$$

$$u(x,0) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^d. \quad (1)$$

where $u : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{R}^m, \ A \in \mathbb{R}^{m,m}, \ f : \mathbb{R}^m \rightarrow \mathbb{R}^m, \ u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^m$.

Assume a rotating wave solution $u_\star : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{R}^m$ of (1)

$$u_\star(x,t) = v_\star(e^{-tS}x)$$

$v_\star : \mathbb{R}^d \rightarrow \mathbb{R}^m$ profile (pattern), $0 \neq S \in \mathbb{R}^{d,d}$ skew-symmetric. 

**Transformation (into a co-rotating frame):** $v(x,t) = u(e^{tS}x, t)$ solves

$$v_t(x,t) = A \triangle v(x,t) + \langle Sx, \nabla v(x,t) \rangle + f(v(x,t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2,$$

$$v(x,0) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^d. \quad (2)$$

Note: $v_\star$ is a stationary solution of (2), i.e. $v_\star$ solves the rotating wave equation

$$A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2.$$
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

$$
\begin{align*}
  u_t(x, t) &= A \Delta u(x, t) + f(u(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
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\end{align*}
$$

(1)

where $u : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^m$, $A \in \mathbb{R}^{m,m}$, $f : \mathbb{R}^m \to \mathbb{R}^m$, $u_0 : \mathbb{R}^d \to \mathbb{R}^m$.

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\end{align*}
$$

(2)

Questions and Ingredients: I1: exp. decay of $v_*$, I2: spectral properties

Q1: Nonlinear stability of rotating waves on $\mathbb{R}^d$? (Tools: I1+I2)

Q2: Truncations of rotating waves to bounded domains? (Tools: I1+...)

Q3: Spatial approximation (e.g. with finite element method)? (open problem)

Q4: Temporal approximation (e.g. with Euler or BDF)? (open problem)
Examples for rotating waves

**Cubic-quintic complex Ginzburg-Landau equation:** (spinning solitons)

$$u_t = \alpha \triangle u + u \left( \delta + \beta |u|^2 + \gamma |u|^4 \right)$$

$$u(x, t) \in \mathbb{C}, \; x \in \mathbb{R}^d, \; t \geq 0, \; \alpha, \beta, \gamma \in \mathbb{C}, \; \text{Re} \alpha > 0, \; \delta \in \mathbb{R}, \; d \in \{2, 3\}.$$

**λ-ω system:** (spiral waves, scroll waves)

$$u_t = \alpha \triangle u + \left( \lambda (|u|^2) + i \omega (|u|^2) \right) u$$

$$u(x, t) \in \mathbb{C}, \; x \in \mathbb{R}^d, \; t \geq 0, \; \lambda, \omega : [0, \infty] \rightarrow \mathbb{R}, \; \alpha \in \mathbb{C}, \; \text{Re} \alpha > 0, \; d \in \{2, 3\}.$$

**Barkley model:** (spiral waves, also scroll waves)

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle u + \left( \frac{1}{\varepsilon} u_1 (1 - u_1)(u_1 - \frac{u_2 + b}{a}) \right) u_1 - u_2$$

$$u(x, t) \in \mathbb{R}^2, \; x \in \mathbb{R}^d, \; t \geq 0, \; 0 \leq D \ll 1, \; \varepsilon, a, b > 0, \; d \in \{2, 3\}.$$
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Spatial decay of rotating waves

**Theorem 1: (Exponential decay of profile \( v_\star \))**

Let \( f \in C^2 \) \((\mathbb{R}^m, \mathbb{R}^m)\), \( v_\infty \in \mathbb{R}^m \), \( f(v_\infty) = 0 \), \( Df(v_\infty) \leq -\beta_\infty I_m < 0 \), assume (A1)-(A3) for some \( 1 < p < \infty \), and let \( \theta(x) = \exp \left( \mu \sqrt{|x|^2 + 1} \right) \) be a weight function for \( \mu \in \mathbb{R} \).

Then for every \( 0 < \varepsilon < 1 \) there exists \( K_1 = K_1(\varepsilon) > 0 \) with the following property:

Every classical solution \( v_\star \in C^2 \) \((\mathbb{R}^d, \mathbb{R}^m)\) of

\[
A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,
\]

such that

\[
\sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0
\]

satisfies

\[
v_\star - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^m)
\]

for every exponential decay rate

\[
0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.
\]

\[
\begin{pmatrix}
a_{\max} &=& \rho(A) \\
-a_0 &=& s(-A) \\
-b_0 &=& s(Df(v_\infty))
\end{pmatrix}
\]

: spectral radius of \( A \)

: spectral bound of \(-A\)

: spectral bound of \( Df(v_\infty) \)
Spatial decay of rotating waves

**Theorem 1: (Exponential decay of profile $v_\star$)**

Let $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp \left( \mu \sqrt{|x|^2 + 1} \right)$ be a weight function for $\mu \in \mathbb{R}$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ of

$$(RWE) \quad A \Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,$$

such that

$$(TC) \quad \sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1$$

for some $R_0 > 0$ satisfies

$$v_\star - v_\infty \in W^{2,p}_\theta(\mathbb{R}^d, \mathbb{R}^m)$$

for every exponential decay rate

$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0b_0}}{a_{\max}p}.$$ 

\[
\begin{pmatrix}
 a_{\max} & = & \rho(A) \\
 -a_0 & = & s(-A) \\
 -b_0 & = & s(Df(v_\infty))
\end{pmatrix} : \text{spectral radius of } A \\
: \text{spectral bound of } -A \\
: \text{spectral bound of } Df(v_\infty)
\]
Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$: higher regularity)

Let $f \in C^{\max\{2, k-1\}}(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 3$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of

\[(RWE) \quad A\triangledown v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d, \]

such that

\[(TC) \quad \sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0 \]

satisfies

$$v_\star - v_\infty \in W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^m)$$

for every exponential decay rate

$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.$$
Theorem 1: (Exponential decay of profile $v_\star$: pointwise estimates)

Let $f \in C^{\max\{2, k-1\}}(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 3$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of

\[(RWE) \quad A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,\]

such that

\[\text{(TC) \quad } \sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0\]

satisfies

\[v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^m), \quad |D^\alpha(v_\star(x) - v_\infty)| \leq C \exp\left(-\mu \sqrt{|x|^2 + 1}\right) \forall x \in \mathbb{R}^d\]

for every exponential decay rate

\[0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p} \quad \left(\begin{array}{c}
a_{\max} = \rho(A) \\
-a_0 = s(-A) \\
-b_0 = s(Df(v_\infty))
\end{array}\right) : \begin{array}{c}\text{spectral radius of } A \\
\text{spectral bound of } -A \\
\text{spectral bound of } Df(v_\infty)
\end{array}\]

and for every multiindex $\alpha \in \mathbb{N}^d$ satisfying $d < (k - |\alpha|)p$. 
Theorem 2: (Exponential decay of eigenfunctions \( \nu \))

Let \( f \in C^{\max\{2,k\}}(\mathbb{R}^m, \mathbb{R}^m) \), \( \nu_\infty \in \mathbb{R}^m \), \( f(\nu_\infty) = 0 \), \( Df(\nu_\infty) \leq -\beta_\infty I_m < 0 \), assume (A1)-(A3) for some \( 1 < p < \infty \), and let \( \theta_j(x) = \exp \left( \mu_j \sqrt{|x|^2 + 1} \right) \) be a weight function for \( \mu_j \in \mathbb{R} \), \( j = 1, 2 \), \( k \in \mathbb{N} \), \( p \geq \frac{d}{2} \) (if \( k \geq 2 \)). Then for every \( 0 < \varepsilon < 1 \) there exists \( K_1 = K_1(\varepsilon) > 0 \) such that for every classical solution \( \nu_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m) \) of (RWE) satisfying (TC) the following property holds: Every classical solution \( \nu \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m) \) of

\[
A \triangle \nu(x) + \langle Sx, \nabla \nu(x) \rangle + Df(\nu_\star(x))\nu(x) = \lambda \nu(x), \quad x \in \mathbb{R}^d,
\]

with \( \lambda \in \mathbb{C} \), \( \Re \lambda \geq -(1 - \varepsilon)\beta_\infty \), such that

\[
\nu \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m) \quad \text{for some exp. growth rate} \quad -\sqrt{\frac{\varepsilon \gamma A\beta_\infty}{2d|A|^2}} \leq \mu_1 < 0
\]
satisfies

\[
\nu \in W^{k,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m) \quad \text{for every exp. decay rate} \quad 0 \leq \mu_2 \leq \varepsilon \sqrt{a_0b_0 \max p}
\]
and

\[
|D^\alpha \nu(x)| \leq C \exp \left( -\mu_2 \sqrt{|x|^2 + 1} \right) \quad \forall x \in \mathbb{R}^d
\]

for every multiindex \( \alpha \in \mathbb{N}_0^d \) satisfying \( d < (k - |\alpha|)p \).
Exponentially weighted Sobolev spaces and assumptions

**Exponentially weighted Sobolev spaces:** For $K \in \{\mathbb{R}, \mathbb{C}\}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$, and weight function $\theta(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right)$ with $\mu \in \mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d, K^m) := \{ v \in L^1_{\text{loc}}(\mathbb{R}^d, K^m) | \| \theta v \|_{L^p} < \infty \},$$

$$W^{k,p}_\theta(\mathbb{R}^d, K^m) := \{ v \in L^p_\theta(\mathbb{R}^d, K^m) | D^\beta u \in L^p_\theta(\mathbb{R}^d, K^m) \forall |\beta| \leq k \}.$$

**Assumptions:**

(A1) (**$L^p$-dissipativity condition**): For $A \in \mathbb{R}^{m,m}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \Re \langle w, Aw \rangle + (p - 2) \Re \langle w, z \rangle \Re \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \forall z, w \in \mathbb{R}^m$$

(A2) (**System condition**): $A, Df(v_\infty) \in \mathbb{R}^{m,m}$ simultaneously diagonalizable over $\mathbb{C}$

(A3) (**Rotational condition**): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^\top$

**Note:** Assumption (A1) is equivalent with

(A1') (**$L^p$-antieigenvalue condition**): $A \in \mathbb{R}^{m,m}$ is invertible and

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{R}^m \\
w \neq 0 \\
Aw \neq 0}} \frac{\Re \langle w, Aw \rangle}{|w||Aw|} > \frac{|p - 2|}{p} \text{ for some } 1 < p < \infty$$

($\mu_1(A) :$ first antieigenvalue of $A$)
Exponentially weighted Sobolev spaces and assumptions

**Exponentially weighted Sobolev spaces:** For $K \in \{\mathbb{R}, \mathbb{C}\}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$, and weight function $\theta(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right)$ with $\mu \in \mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d, K^m) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^d, K^m) \mid \| \theta v \|_{L^p} < \infty \right\},$$

$$W^{k,p}_\theta(\mathbb{R}^d, K^m) := \left\{ v \in L^p_\theta(\mathbb{R}^d, K^m) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, K^m) \forall |\beta| \leq k \right\}.$$

**Assumptions:**

(A1) *(L^p\text{-dissipativity condition})*: For $A \in \mathbb{R}^{m,m}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \text{Re} \langle w, Aw \rangle + (p - 2) \text{Re} \langle w, z \rangle \text{Re} \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \ \forall \ z, w \in \mathbb{R}^m$$

(A2) *(System condition)*: $A, Df(v_\infty) \in \mathbb{R}^{m,m}$ simultaneously diagonalizable over $\mathbb{C}$

(A3) *(Rotational condition)*: $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^\top$

Additionally:

(A4) *(L^q\text{-dissipativity condition})*: For $A \in \mathbb{R}^{m,m}$, $q = \frac{p}{p-1}$, there is $\delta_A > 0$ with

$$|z|^2 \text{Re} \langle w, A^H w \rangle + (q - 2) \text{Re} \langle w, z \rangle \text{Re} \langle z, A^H w \rangle \geq \delta_A |z|^2 |w|^2 \ \forall \ z, w \in \mathbb{R}^m$$
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

**Exponential Decay:** To show exponential decay for the solution $v_\star$ of

$$A \Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d,$$

investigate the linear system ($w_\star(x) := v_\star(x) - v_\infty$)

$$A \Delta w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$

**Operators:** Study the following operators

$$\mathcal{L}_c v := A \Delta v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v + Q_c v, \quad \text{(exp. decay)}$$

$$\mathcal{L}_s v := A \Delta v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v, \quad \text{(exp. decay)}$$

$$\mathcal{L}_\infty v := A \Delta v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v, \quad \text{(far-field operator)} \quad \text{(exp. decay)}$$

$$\mathcal{L}_0 v := A \Delta v + \langle S \cdot, \nabla v \rangle. \quad \text{(Ornstein-Uhlenbeck operator)} \quad \text{(max. domain)}$$

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D. Otten.

Exponentially weighted resolvent estimates for complex Ornstein-Uhlenbeck systems, 2015.
The identification problem for complex-valued Ornstein-Uhlenbeck operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, 2016.
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W.-J. Beyn, D. Otten.

Spatial Decay of Rotating Waves in Reaction Diffusion Systems, 2016.
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

**Exponential Decay:** To show exponential decay for the solution $v_\star$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d,$$

investigate the linear system ($w_\star(x) := v_\star(x) - v_\infty$)

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + \left(Df(v_\infty) + Q_s(x) + Q_c(x)\right) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$

**Operators:** Study the following operators

- $\mathcal{L}_c v := A\triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty)v + Q_s v + Q_c v$, (exp. decay)
- $\mathcal{L}_s v := A\triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty)v + Q_s v$, (exp. decay)
- $\mathcal{L}_\infty v := A\triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty)v$, (far-field operator) (exp. decay)
- $\mathcal{L}_0 v := A\triangle v + \langle S \cdot, \nabla v \rangle$. (Ornstein-Uhlenbeck operator) (max. domain)

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**Exponential Decay:** To show exponential decay for the solution \( v_\star \) of

\[
A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d,
\]

investigate the linear system (\( w_\star(x) := v_\star(x) - v_\infty \))

\[
A \triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.
\]

**Operators:** Study the following operators

\[
L_c v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v + Q_c v, \quad \text{(exp. decay)}
\]

\[
L_s v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v, \quad \text{(exp. decay)}
\]

\[
L_\infty v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v, \quad \text{(far-field operator)} \quad \text{(exp. decay)}
\]

\[
L_0 v := A \triangle v + \langle S \cdot, \nabla v \rangle. \quad \text{(Ornstein-Uhlenbeck operator)} \quad \text{(max. domain)}
\]

**Maximal domain of** \( L_0 \) **given by**

\[
\mathcal{D}_{loc}^p(L_0) = \{ v \in W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) : L_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^m) \}, \ 1 < p < \infty
\]

satisfies \( \mathcal{D}_{loc}^p(L_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^m) \).
The operator $\mathcal{L}_0$

**Ornstein-Uhlenbeck operator**

$$[\mathcal{L}_0 \nu](x) = A \triangle \nu(x) + \langle S x, \nabla \nu(x) \rangle, \ x \in \mathbb{R}^d, \ d \geq 2.$$ 

↓

**Heat kernel**

$$H_0(x, \xi, t) = (4 \pi t A)^{-\frac{d}{2}} \exp \left( - (4tA)^{-1} \left| e^{tS} x - \xi \right|^2 \right), \ x, \xi \in \mathbb{R}^d, \ t > 0.$$ 

↓

**Semigroup** in $L^p(\mathbb{R}^d, \mathbb{C}^m), \ 1 \leq p \leq \infty$

$$[T_0(t) \nu](x) = \int_{\mathbb{R}^d} H_0(x, \xi, t) \nu(\xi) d\xi, \ t > 0.$$ 

↓

**strong ↓ continuity**

**Infinitesimal generator**

$$(A_p, \mathcal{D}(A_p)), \ 1 \leq p < \infty.$$ 

semigroup theory $\checkmark$

A-priori → exponential decay,

max. domain and max. realization, $1 < p < \infty$

identification problem

unique solv. of resolvent equ. for $A_p$, $1 \leq p < \infty$, Re$\lambda > 0$

$$(\lambda I - A_p) \nu_\star = g \in L^p.$$ 

$\nu_\star \in W^{1,p}_\theta.$

$A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\text{loc}}(\mathcal{L}_0).$
Identification problem of $L_0$

$$D^p_{\text{loc}}(L_0) := \left\{ v \in W^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) \mid L_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^m) \right\}, \ 1 < p < \infty.$$ 

**Infinitesimal generator**

$$(A_p, D(A_p)), \ 1 \leq p < \infty.$$ 

$S$ is a core for $(A_p, D(A_p))$ 

**Identification of $L_0$ maximal domain and maximal realization for $1 < p < \infty$:**

$A_p = L_0$ on $D(A_p) = D^p_{\text{loc}}(L_0)$

**Ornstein-Uhlenbeck operator**

$$[L_0 v](x) = A \triangle v(x) + \langle Sx, \nabla v(x) \rangle, \ x \in \mathbb{R}^d, \ d \geq 2.$$ 

$L^p$-resolvent estimates and unique solv. of resolvent equ. for $L_0$ in $D^p_{\text{loc}}(L_0)$, $1 < p < \infty$

**$L^p$-dissipativity condition:**

$$\exists \gamma_A > 0 \quad |z|^2 \text{Re} \langle w, Aw \rangle + (p - 2)\text{Re} \langle w, z \rangle \text{Re} \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \ \forall \ z, w \in \mathbb{K}^m$$ 

$L^p$-first antieigenvalue condition

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{K}^m \\forall w \neq 0 \\forall Aw \neq 0}} \frac{\text{Re} \langle w, Aw \rangle}{|w||Aw|} > \frac{|p - 2|}{p}, \ 1 < p < \infty$$

Denny Otten
Spectral Properties of Localized Rotating Waves
Bremen 2016
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Outline

1. Rotating patterns in $\mathbb{R}^d$

2. Spatial decay of rotating waves

3. **Eigenvalue problem for rotating waves and some basic definitions**

4. Fredholm properties of linearization in $L^p$

5. Essential $L^p$-spectrum and dispersion relation

6. Point $L^p$-spectrum and shape of eigenfunctions

7. Cubic-quintic complex Ginzburg-Landau equation
Eigenvalue problem for linearization at rotating waves

**Motivation:** Stability is determined by spectral properties of linearization $\mathcal{L}$.

**Eigenvalue problem:**

$$ (\lambda I - \mathcal{L})v(x) = 0, \ x \in \mathbb{R}^d, \ d \geq 2, \ \lambda \in \mathbb{C}. $$

$$ \mathcal{L}v(x) = A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x), \ x \in \mathbb{R}^d, \ d \geq 2. $$

**Definition 3:** (Strongly spectrally stable)

A rotating wave $u_*(x, t) = v_*(e^{-tS}x)$ is called strongly spectrally stable iff

1. $\text{Re} \sigma(\mathcal{L}) \leq 0$ (spectrally stable) and
2. $\forall \lambda \in \sigma(\mathcal{L}) \cap i\mathbb{R}: \lambda \in \sigma_{pt}(\mathcal{L})$, $\lambda$ is caused by the $\text{SE}(d)$-group action and

$$ \sum_{\lambda \in \sigma(\mathcal{L}) \cap i\mathbb{R}} \text{alg}(\lambda) = \frac{d(d + 1)}{2} = \text{dimSE}(d), \ \text{alg}(\lambda) := \text{algebraic mult. of } \lambda. $$
Recall from spectral theory

**Linearized operator** is closed and densely defined

\[ L v(x) = A \Delta v(x) + \langle S x, \nabla v(x) \rangle + D f(v_*(x)) v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2, \]

\[ D^p_{\text{loc}}(L_0) = \{ v \in W^{2,p}_{\text{loc}} \cap L^p \mid L_0 v \in L^p \}, \quad \| v \|_{L_0} := \| v \|_{L^p} + \| L_0 v \|_{L^p}. \]

**Definition 4: (Spectrum of \( L \))**

1. **Resolvent set**
   \[ \rho(L) := \{ \lambda \in \mathbb{C} \mid (\lambda I - L)^{-1} : L^p \to D^p_{\text{loc}}(L_0) \text{ exists and is bounded} \}. \]

2. **Spectrum** \( \sigma(L) := \mathbb{C} \setminus \rho(L) \). \( 0 \neq v \in D^p_{\text{loc}}(L_0) \) is an **eigenfunction** of \( L \) with eigenvalue \( \lambda \in \sigma(L) \) if \( (\lambda I - L)v = 0 \). An eigenvalue \( \lambda \in \sigma(L) \)
   - is isolated if \( \exists \varepsilon > 0 \forall \lambda_0 \in \mathbb{C} \text{ with } 0 < |\lambda - \lambda_0| < \varepsilon : \lambda_0 \in \rho(L) \).
   - has finite (algebraic) multiplicity if \( \dim(N(\lambda I - L)) < \infty \) and \( \exists n_\lambda \in \mathbb{N} \forall y \in D^p_{\text{loc}}(L_0) \text{ s.t. } y(\lambda_0) = \sum_{j=0}^{n_\lambda} (\lambda - \lambda_0)^j y_j \text{ with } y_0 \neq 0: \)
     \[ [(\lambda I - L)y]^{(\nu)}(\lambda) = 0 \text{ for } \nu = 0, \ldots, n - 1 \text{ and } [(\lambda I - L)y]^{(n)}(\lambda) \neq 0. \]

3. **Point spectrum**
   \[ \sigma_{\text{pt}}(L) := \{ \lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of finite alg. multiplicity} \}. \]

   \( \lambda \in \rho(L) \cup \sigma_{\text{pt}}(L) \) is called a **normal point** of \( L \).

4. **Essential spectrum**
   \[ \sigma_{\text{ess}}(L) := \{ \lambda \in \mathbb{C} \mid \lambda \text{ is not a normal point of } L \}. \]

Note: \( \mathbb{C} = \rho(L) \cup \sigma(L), \quad \sigma(L) = \sigma_{\text{ess}}(L) \cup \sigma_{\text{point}}(L). \)
Recall from spectral theory

**Linearized operator** is closed and densely defined

\[ \mathcal{L}v(x) = A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2, \]

\[ \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) = \{ v \in W^{2,p}_{\text{loc}} \cap L^p \mid \mathcal{L}_0 v \in L^p \}, \quad \| v \|_{\mathcal{L}_0} := \| v \|_{L^p} + \| \mathcal{L}_0 v \|_{L^p}. \]

**Definition 5: (Fredholm operator)**

The linear operator \( \lambda I - \mathcal{L} : \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \to L^p \) is called **Fredholm** iff

1. \( \lambda I - \mathcal{L} \) is closed,
2. \( \dim(\mathcal{N}(\lambda I - \mathcal{L})) < \infty \) and
3. \( \text{codim}(\mathcal{R}(\lambda I - \mathcal{L})) < \infty. \)

The **index** \( \kappa \) of the Fredholm operator \( \lambda I - \mathcal{L} \) is defined by

\[ \kappa := \dim(\mathcal{N}(\lambda I - \mathcal{L})) - \text{codim}(\mathcal{R}(\lambda I - \mathcal{L})). \]

with \( \text{codim}(\mathcal{R}(\lambda I - \mathcal{L})) := \dim(\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)/\mathcal{R}(\lambda I - \mathcal{L})). \)

**Adjoint operator:** Let \( q = \frac{p}{p-1} \) for \( 1 < p < \infty \)

\[ \mathcal{L}_0^* v(x) = A^H \triangle v(x) + \langle S^T x, \nabla v(x) \rangle + Df(v_*(x))^H v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2, \]

\[ \mathcal{D}_{\text{loc}}^q(\mathcal{L}_0^*) = \{ v \in W^{2,q}_{\text{loc}} \cap L^q \mid \mathcal{L}_0^* v \in L^q \}, \quad \| v \|_{\mathcal{L}_0^*} := \| v \|_{L^q} + \| \mathcal{L}_0^* v \|_{L^q}. \]
Outline

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7 Cubic-quintic complex Ginzburg-Landau equation
Theorem 6: (Fredholm properties of $\mathcal{L}$)

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\text{Re}\lambda \geq -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.

Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

1. **Fredholm properties**.

   $\lambda I - \mathcal{L} : (\mathcal{D}_{loc}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^N), \|\cdot\|_{L^p})$ is Fredholm of index 0.
Properties of linearization at localized rotating waves

**Theorem 6: (Fredholm properties of $\mathcal{L}$)**

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\text{Re}\lambda \geq -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.

Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

1. **(Fredholm alternative).** Let in addition to (A4) hold for $q = \frac{p}{p-1}$ and $\lambda \in \sigma_{\text{pt}}(\mathcal{L})$ with geom. mult. $1 \leq n = \dim \mathcal{N}(\lambda I - \mathcal{L}) < \infty$.

   Then, there are exactly $n$ linearly indep. eigenfunctions $v_j \in D^p_{\text{loc}}(\mathcal{L}_0)$ and adjoint eigenfunctions $\psi_j \in D^q_{\text{loc}}(\mathcal{L}_0^*)$ with
   $$(\lambda I - \mathcal{L})v_j = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^*\psi_j = 0 \quad \text{for} \quad j = 1, \ldots, n.$$  

Moreover,

(IP)  

$$(\lambda I - \mathcal{L})v = g, \quad g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$$ 

has at least one (not necessarily unique) solution $v \in D^p_{\text{loc}}(\mathcal{L}_0)$ iff

$g \in (\mathcal{N}(\lambda I - \mathcal{L})^*)^\perp$, \quad i.e. $\langle \psi_j, g \rangle_{q,p} = 0, j = 1, \ldots, n.$

In this case, one can select a solution $v \in D^p_{\text{loc}}(\mathcal{L}_0)$ of (IP) with

$$\|v\|_{\mathcal{L}_0} \leq C \|g\|_{L^p} \quad \text{and} \quad \|v\|_{W^{1,p}} \leq C \|g\|_{L^p}.$$
Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of $\mathcal{L}$)

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\Re \lambda \geq -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.

Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

1. **(Exponential decay).** Let in addition to 2:
   \[
   \theta_j(x) = \exp \left( \mu_j \sqrt{|x|^2 + 1} \right), \quad x \in \mathbb{R}^d, \quad \mu_j \in \mathbb{R}, \quad j = 1, \ldots, 4.
   \]

   Then, every classical solution $v \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ of
   \[
   (\lambda I - \mathcal{L})v = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^*\psi = 0
   \]
such that $v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in L^q_{\theta_3}(\mathbb{R}^d, \mathbb{C}^m)$ for some exp. growth rate
   \[
   -\sqrt{\varepsilon \gamma A(\beta_\infty - b_0 + \gamma)} \leq \mu_1 \leq 0 \quad \text{and} \quad -\sqrt{\varepsilon \frac{\delta A(\beta_\infty - b_0 + \gamma)}{2d|A|^2}} \leq \mu_3 \leq 0
   \]
satisfies $v \in W^{1,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in W^{1,q}_{\theta_4}(\mathbb{R}^d, \mathbb{C}^m)$ for every exp. decay rate
   \[
   0 \leq \mu_2 \leq \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max}p} \quad \text{and} \quad 0 \leq \mu_4 \leq \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max}q}.
   \]
Theorem 6: (Fredholm properties of \( L \))

Assume (A1)-(A3) for some \( 1 < p < \infty \), \( v_\infty \in \mathbb{R}^m \), \( f(v_\infty) = 0 \), \( f \in C^2(\mathbb{R}^m, \mathbb{R}^m) \) and \( \lambda \in \mathbb{C} \), \( \text{Re}\lambda \geq -b_0 + \gamma \) for some \( \gamma > 0 \) and \( -b_0 = s(Df(v_\infty)) \).

Then, for any \( 0 < \varepsilon < 1 \) there is \( K_1 = K_1(\varepsilon) > 0 \) such that for any classical solution \( v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m) \) of (RWE) satisfying (TC) the following properties hold:

4. **(Pointwise estimates for \( v \)).** Let in addition to \( \Box \):

   - \( p \geq \frac{d}{2} \), \( f \in C^k(\mathbb{R}^m, \mathbb{R}^m) \), \( v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m) \), \( v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m) \), \( 2 \leq k \in \mathbb{N} \).

   Then, \( v \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^m) \) and

   \[
   |D^{\alpha}v(x)| \leq C \exp \left( -\mu_2 \sqrt{|x|^2 + 1} \right), \quad x \in \mathbb{R}^d
   \]

   for any \( \mu_2 \in \mathbb{R} \), \( 0 \leq \mu_2 \leq \varepsilon \frac{\sqrt{a_0} \gamma}{a_{\max} p} \) and \( \alpha \in \mathbb{N}_0^d \), \( d < (k - |\alpha|)p \).

5. **(Pointwise estimates for \( \psi \)).** Let in addition to \( \Box \):

   - \( \min\{p, q\} \geq \frac{d}{2} \), \( \psi \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m) \).

   Then, \( \psi \in W_{\theta_4}^{k,q}(\mathbb{R}^d, \mathbb{C}^m) \) and

   \[
   |D^{\alpha}\psi(x)| \leq C \exp \left( -\mu_4 \sqrt{|x|^2 + 1} \right), \quad x \in \mathbb{R}^d
   \]

   for any \( \mu_4 \in \mathbb{R} \), \( 0 \leq \mu_4 \leq \varepsilon \frac{\sqrt{a_0} \gamma}{a_{\max} q} \) and \( \alpha \in \mathbb{N}_0^d \), \( d < (k - |\alpha|)q \).
Outline of proof: Theorem 6 (Fredholm properties of \( \mathcal{L} \))

\[ \mathcal{L}v = A\nabla^2 v + \langle Sx, \nabla v \rangle + Df(v_*(x))v. \]

1. Splitting off the stable part: \( Q(x) = Df(v_*(x)) - Df(v_\infty) \) implies

\[ \mathcal{L}v = A\nabla^2 v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q(x))v \]

\( v_*(x) \to v_\infty \) as \( |x| \to \infty \) \( \Rightarrow \ \sup_{|x| \geq R} |Q(x)| \to 0 \) as \( R \to \infty \)

2. Decomposition of \( Q \):

\[ \mathcal{L}v = A\nabla^2 v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x))v \]

\( Q(x) = Q_s(x) + Q_c(x) \), \( Q_s, Q_c \in L^\infty \), \( Q_s \) small w.r.t. \( \| \cdot \|_{L^\infty} \), \( Q_c \) comp. supported

3. Decomposition of \( \lambda \): \( \lambda \in \mathbb{C}, \ \text{Re} \lambda \geq -b_0 + \gamma \) for some \( \gamma > 0 \), then

\[ \lambda = \lambda_1 + \lambda_2 \quad \text{with} \quad \lambda_2 := -b_0 + \gamma, \quad \lambda_1 := \lambda - \lambda_2. \]

4. Decomposition of \( \lambda I - \mathcal{L} \):

\[ \lambda I - \mathcal{L} = (I - Q_c(\cdot)(\lambda_1 - \mathcal{L}_s)^{-1})(\lambda_1 I - \mathcal{L}_s) \]

\( \mathcal{L}_s = \mathcal{L}_s - \lambda_2 I, \quad \mathcal{L}_s v = A\nabla^2 v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q_s(x))v \)
Outline of proof: Theorem 6 (Fredholm properties of \( \mathcal{L} \))

Decomposition of \( \lambda I - \mathcal{L} \):

\[
\lambda I - \mathcal{L} = \left( I - Q_c(\cdot)(\lambda_1 - \tilde{\mathcal{L}}_s)^{-1} \right) \left( \lambda_1 I - \tilde{\mathcal{L}}_s \right)
\]

\[
\tilde{\mathcal{L}}_s = \mathcal{L}_s - \lambda_2 I, \quad \mathcal{L}_s v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q_s(x)) v
\]

5. Fredholm properties:

- \( \lambda_1 I - \tilde{\mathcal{L}}_s \) is Fredholm of index 0:
  - unique solvability of resolvent equation for \( \tilde{\mathcal{L}}_s \)

- \( I - Q_c(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_s)^{-1} \) Fredholm of index 0:
  - \( Q_c(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_s)^{-1} \) is compact
  - compact perturbation of identity
  - unique solvability of resolvent equation for \( \tilde{\mathcal{L}}_s \)
  - \( \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^m) \)

- \( \lambda I - \mathcal{L} \) Fredholm of index 0:
  - Theorem on products of Fredholm operators
Outline

1. Rotating patterns in $\mathbb{R}^d$
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4. Fredholm properties of linearization in $L^p$
5. **Essential $L^p$-spectrum and dispersion relation**
6. Point $L^p$-spectrum and shape of eigenfunctions
7. Cubic-quintic complex Ginzburg-Landau equation
Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(L)$

Eigenvalue problem:

\[(\lambda I - L)v = 0, \ x \in \mathbb{R}^d\]

\[Lv = A\Delta v + \langle Sx, \nabla v \rangle + Df(v_*(x))v\]

1. Splitting off the stable part: $Q(x) = Df(v_*(x)) - Df(v_\infty)$ implies

\[(\lambda I - L_Q)v = 0, \ x \in \mathbb{R}^d\]

\[L_Qv = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q(x))v = Lv\]

$v_*(x) \to v_\infty$ as $|x| \to \infty$ \ \Rightarrow \ \sup_{|x| \geq R} |Q(x)| \to 0$ as $R \to \infty$
Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Splitting off the stable part:

\[(\lambda I - \mathcal{L}_Q)v = 0, \ x \in \mathbb{R}^d\]

\[\mathcal{L}_Q v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q(x))v\]

\[Q(x) = Df(v_\star(x)) - Df(v_\infty), \ \sup_{|x| \geq R} |Q(x)| \to 0 \text{ as } R \to \infty\]

2. Orthogonal transformation: $S \in \mathbb{R}^{d,d}$, $S = -S^\top$, implies $S = P\Lambda^SP^\top$ with

\[P \in \mathbb{R}^{d,d} \text{ orth.}, \ \Lambda^S_b = \text{diag}(\Lambda^S_1, \ldots, \Lambda^S_k, 0), \ \Lambda^S_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \ \pm i\sigma_j \in \sigma(S).\]

Then, $\tilde{v}(y) = v(T_1(y))$ with $x = T_1(y) = Py$ yields

\[(\lambda I - \mathcal{L}_1)\tilde{v} = 0, \ y \in \mathbb{R}^d\]

\[\mathcal{L}_1\tilde{v} = A\Delta\tilde{v} + \langle \Lambda^S_b y, \nabla\tilde{v} \rangle + (Df(v_\infty) + Q(T_1(y)))\tilde{v}\]

\[\langle \Lambda^S_b y, \nabla\tilde{v} \rangle = \sum_{l=1}^k \sigma_l (y_{2l}\partial_{y_{2l-1}} - y_{2l-1}\partial_{y_{2l}})\tilde{v}\]
Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Orthogonal transformation:

$$(\lambda I - \mathcal{L}_1)\tilde{v} = 0, \ y \in \mathbb{R}^d$$

$$\mathcal{L}_1\tilde{v} = A\Delta\tilde{v} + \langle \Lambda^S_y, \nabla\tilde{v} \rangle + (Df(v_\infty) + Q(T_1(y)))\tilde{v}$$

$$\langle \Lambda^S_y, \nabla\tilde{v} \rangle = \sum_{l=1}^{k} \sigma_l \left( y_{2l-1}\partial_{y_{2l-1}} - y_{2l-1}\partial_{y_{2l}} \right) \tilde{v}$$

3. Several planar polar coordinates: For $\phi \in (-\pi, \pi]^k, \ r \in (0, \infty)^k$ define

$$\left( \begin{array}{c} y_{2l-1} \\ y_{2l} \end{array} \right) = T(r_l, \phi_l) := \left( \begin{array}{c} r_l \cos \phi_l \\ r_l \sin \phi_l \end{array} \right), \ l = 1, \ldots, k,$$

$$T_2(\xi) = (T(r_1, \phi_1), \ldots, T(r_k, \phi_k), \tilde{y}), \ \xi = (r_1, \phi_1, \ldots, r_k, \phi_k, \tilde{y}), \ \tilde{y} = (y_{2k+1}, \ldots, y_d).$$

Then, $\hat{v}(\xi) = \tilde{v}(T_2(\xi))$ with $y = T_2(\xi)$ and $Q(\xi) = Q(T_1(T_2(\xi))))$ yields

$$(\lambda I - \mathcal{L}_2)\hat{v} = 0, \ \xi \in \Omega$$

$$\mathcal{L}_2\hat{v} = A \left[ \sum_{l=1}^{k} \left( \partial_{r_l}^2 + \frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l}^2 \right) + \sum_{l=2k+1}^{d} \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} \hat{v} + (Df(v_\infty) + Q(\xi))\hat{v}.$$
Several planar polar coordinates: \( \Omega = ((0, \infty) \times (-\pi, \pi))^k \times \mathbb{R}^{d-2k} \)

\[
(\lambda l - \mathcal{L}_2)\hat{v} = 0, \quad \xi \in \Omega
\]

\[
\mathcal{L}_2 \hat{v} = A \left[ \sum_{l=1}^{k} \left( \frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l}^2 \right) + \sum_{l=2k+1}^{d} \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} \hat{v} + (Df(v_\infty) + Q(\xi)) \hat{v}.
\]

\[
Q(\xi) = Q(T_1(T_2(\xi)))
\]

4. Limit operator (far-field operator, simplified operator):
Let formally \( |x| \to \infty \) (i.e. \( r_l \to \infty \)) and use \( |Q(x)| \to 0 \) as \(|x| \to \infty \)

\[
(\lambda l - \mathcal{L}_\infty^{\text{sim}})\hat{v} = 0, \quad \xi \in \Omega
\]

\[
\mathcal{L}_\infty^{\text{sim}} \hat{v} = A \left[ \sum_{l=1}^{k} \partial_{r_l}^2 + \sum_{l=2k+1}^{d} \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} \hat{v} + Df(v_\infty) \hat{v}
\]
Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

**Limit operator:**
$$\Omega = ((0, \infty) \times (-\pi, \pi))^k \times \mathbb{R}^{d-2k}$$

$$(\lambda I - \mathcal{L}_{\infty}^\text{sim})\hat{\nu} = 0, \xi \in \Omega$$

$$\mathcal{L}_{\infty}^\text{sim} \hat{\nu} = A \left[ \sum_{l=1}^{k} \partial^2_{r_l} + \sum_{l=2k+1}^{d} \partial^2_{y_l} \right] \hat{\nu} - \sum_{l=1}^{k} \sigma_l \partial \phi_l \hat{\nu} + Df(\nu_\infty)\hat{\nu}$$

**5. Angular Fourier transform:**

For $n \in \mathbb{Z}^k$, $\omega \in \mathbb{R}^k$, $\rho, \tilde{y} \in \mathbb{R}^{d-2k}$, $\nu \in \mathbb{C}^m$, $|\nu| = 1$, $\phi \in (-\pi, \pi)^k$, $r \in (0, \infty)^k$.

Inserting

$$\hat{\nu}(\xi) = \exp \left( i \sum_{l=1}^{k} \omega_l r_l \right) \exp \left( i \sum_{l=1}^{k} n_l \phi_l \right) \exp \left( i \sum_{l=2k+1}^{d} \rho_l y_l \right) \nu, $$

$$= \exp(i\langle \omega, r \rangle + i\langle n, \phi \rangle + i\langle \rho, \tilde{y} \rangle)\nu$$

yields the *m-dimensional eigenvalue problem*

$$\left( \lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^{k} n_l \sigma_l I_m - Df(\nu_\infty) \right) \nu = 0.$$
Angular Fourier transform: \( \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k, \mathbf{v} \in \mathbb{C}^m, |\mathbf{v}| = 1 \)

\[
\left( \lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty) \right) \mathbf{v} = 0.
\]

6. Dispersion relation: Every \( \lambda \in \mathbb{C} \) satisfying

\[
(\text{DR}) \quad \det \left( \lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty) \right) = 0
\]

for some \( \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \) belongs to \( \sigma_{\text{ess}}(L) \).

Dispersion set:

\[
\sigma_{\text{disp}}(L) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \}.
\]
Illustration: Dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

(DR) \[ \det \left( \lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty) \right) = 0 \]

$\sigma_{\text{disp}}(\mathcal{L}) = \{ \lambda \in \mathbb{C} | \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \}$

$S \in \mathbb{R}^{d,d}, S = -S^\top, \pm i\sigma_1, \ldots, \pm i\sigma_k$ nonzero eigenvalues of $S$, $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$.

\begin{align*}
\text{Parameters for illustration: } & A = \frac{1}{2} + \frac{1}{2} i, \quad Df(v_\infty) = -\frac{1}{2}, \\
& \sigma_1 = 1.027, \quad \sigma_1 = 1, \quad \sigma_2 = 1.5, \quad \sigma_1 = 1, \quad \sigma_2 = \frac{\exp(1)}{2}
\end{align*}

$\sigma_{\text{disp}}(\mathcal{L}) \subseteq \{ \lambda \in \mathbb{C} | \Re \lambda \leq s(Df(v_\infty)) \}$ dense $\iff \exists \sigma_n, \sigma_m: \sigma_n \sigma_m^{-1} \notin \mathbb{Q}$. 

$d = 2$ or $3$ \quad $d = 4$ (not dense) \quad $d = 4$ (dense)
Essential $L^p$-spectrum of $\mathcal{L}$

$$(\text{DR}) \quad \det \left( \lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty) \right) = 0$$

$$\sigma_{\text{disp}}(\mathcal{L}) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \}$$

$S \in \mathbb{R}^{d,d}$, $S = -S^\top$, $\pm i \sigma_1, \ldots, \pm i \sigma_k$ nonzero eigenvalues of $S$, $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$.

**Theorem 7: (Essential $L^p$-spectrum of $\mathcal{L}$)**

Let that assumptions of Theorem 1 (pointwise estimates) be satisfied. Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property: For every classical solution $v_\ast \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) it holds

$$\sigma_{\text{disp}}(\mathcal{L}) \subseteq \sigma_{\text{ess}}(\mathcal{L}) \quad \text{in} \quad L^p(\mathbb{R}^d, \mathbb{C}^N).$$

- essential spectrum is determined by the far-field linearization
- Thm. 7 holds only for exponentially localized rotating waves, but **not** for nonlocalized rotating waves (e.g. spiral waves, scroll waves)
- essential spectrum for spiral waves much more involved (→ Floquet theory)
Outline

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6. Point $L^p$-spectrum and shape of eigenfunctions
7. Cubic-quintic complex Ginzburg-Landau equation
Rotating wave equation:

\[(RWE)\quad 0 = A\nabla v(x) + \left\langle Sx, \nabla v(x) \right\rangle + f(v(x)), \quad x \in \mathbb{R}^d\]

\textbf{SE}(d)-group action: 

\[ [a(R, \tau)v](x) = v(R^{-1}(x - \tau)), \quad x \in \mathbb{R}^d, (R, \tau) \in \text{SE}(d). \]

1. Generators of \textbf{SE}(d)-group action: Applying the generators

\[ D_l = \partial_{x_l} \quad \text{and} \quad D^{(i,j)} = x_j D_i - x_i D_j \]

to (RWE) leads to \[ \frac{d(d+1)}{2} = d + \frac{d(d-1)}{2} \] equations

\[ 0 = D_l \left( A\nabla v(x) + \left\langle Sx, \nabla v(x) \right\rangle + f(v(x)) \right) \]

\[ 0 = D^{(i,j)} \left( A\nabla v(x) + \left\langle Sx, \nabla v(x) \right\rangle + f(v(x)) \right) \]

for \( l = 1, \ldots, d, \quad i = 1, \ldots, d - 1, \quad j = i + 1, \ldots, d. \)
Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

Generators of $\mathbb{SE}(d)$-group action:

$$D_l = \partial_{x_l} \quad \text{and} \quad D^{(i,j)} = x_j D_i - x_i D_j$$

$$0 = D_l \left( A \triangle v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) \right)$$

$$0 = D^{(i,j)} \left( A \triangle v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) \right)$$

for $l = 1, \ldots, d$, $i = 1, \ldots, d - 1$, $j = i + 1, \ldots, d$.

2. Commutator relations of generators:

$$D_l D_k = D_k D_l,$$

$$D_l D^{(i,j)} = D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j,$$

$$D^{(i,j)} D^{(r,s)} = D^{(r,s)} D^{(i,j)} + \delta_{is} D^{(r,j)} - \delta_{ir} D^{(s,j)} - \delta_{js} D^{(r,i)} + \delta_{jr} D^{(s,i)},$$

$$0 = \mathcal{L}(D_l v_*) - \sum_{n=1}^{d} S_{ln} D_n v_* ,$$

$$0 = \mathcal{L}(D^{(i,j)} v_*) - \sum_{n=1}^{d} S_{jn} D^{(i,n)} v_* - \sum_{n=1}^{d} S_{in} D^{(n,j)} v_*.$$
Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

Commutator relations of generators: $l = 1, \ldots, d$, $i = 1, \ldots, d-1$, $j = i+1, \ldots, d$

\[
0 = \mathcal{L}(D_i v_\star) - \sum_{n=1}^{d} S_{ln} D_n v_\star,
\]
\[
0 = \mathcal{L}(D^{(i,j)} v_\star) - \sum_{n=1}^{d} S_{jn} D^{(i,n)} v_\star - \sum_{n=1}^{d} S_{in} D^{(n,j)} v_\star.
\]

3. Finite-dimensional eigenvalue problem: Linear combination of generators

\[
v(x) = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} C_{ij}^{\text{rot}} D^{(i,j)} v_\star(x) + \sum_{l=1}^{d} C_{l}^{\text{tra}} D_l v_\star(x) = \langle C^{\text{rot}} x + C^{\text{tra}}, \nabla v_\star(x) \rangle
\]

reduces $\mathcal{L} v = \lambda v$ to the following $\frac{d(d+1)}{2}$-dimensional eigenvalue problem

\[
\lambda C^{\text{tra}} = -SC^{\text{tra}},
\]
\[
\lambda C^{\text{rot}} = S^\top C^{\text{rot}} + C^{\text{rot}} S.
\]

- **Unknowns:** $\lambda \in \mathbb{C}$, $C^{\text{rot}} \in \mathbb{C}^{d,d}$ skew-symmetric, $C^{\text{tra}} \in \mathbb{C}^{d}$
- EVP appears in **block diagonal form** ⇒ solve EVPs separately
Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

**Finite-dimensional eigenvalue problem:** $S \in \mathbb{R}^{d,d}, \ S = -S^\top$

\begin{align*}
(1) & \quad \lambda C^{\text{tra}} = -SC^{\text{tra}}, \\
(2) & \quad \lambda C^{\text{rot}} = S^\top C^{\text{rot}} + C^{\text{rot}} S.
\end{align*}

Unknowns: $\lambda \in \mathbb{C}, \ C^{\text{rot}} \in \mathbb{C}^{d,d}$ skew-symmetric, $C^{\text{tra}} \in \mathbb{C}^d$.

4. **Solution of (1)-(2):** $S$ is unitary diagonalizable, i.e.

$\Lambda_S = U^H S U, \ U \in \mathbb{C}^{d,d}$ unitary, $\Lambda_S = \text{diag}(\lambda_1^S, \ldots, \lambda_d^S)$, $\sigma(S) = \{\lambda_1^S, \ldots, \lambda_d^S\}$

A transformation of (1)-(2) implies

\begin{align*}
\lambda &= -\lambda_i^S, \quad C^{\text{rot}} = 0, \quad C^{\text{tra}} = U e_i, \quad (d \text{ solutions}), \\
\lambda &= -(\lambda_i^S + \lambda_j^S), \quad C^{\text{rot}} = U (l_{ij} - l_{ji}) U^\top, \quad C^{\text{tra}} = 0, \quad \left(\frac{d(d-1)}{2} \text{ solutions}\right)
\end{align*}

Symmetry set:

$$
\sigma_{\text{sym}}(\mathcal{L}) = \sigma(S) \cup \{\lambda_i^S + \lambda_j^S \mid 1 \leq i < j \leq d\}$$
Illustration: Symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

$$\sigma_{\text{sym}}(\mathcal{L}) = \sigma(S) \cup \{\lambda_i^S + \lambda_j^S \mid 1 \leq i < j \leq d\} \text{ & algebraic multiplicities}$$

Number of elements $\frac{d(d+1)}{2} = d + \frac{d(d-1)}{2}$ equals $\dim \text{SE}(d)$.

| $d$ | $\text{SE}(d)$ |
|-----|----------------|
| 2   | 3              |
| 3   | 6              |
| 4   | 10             |
| 5   | 15             |
Theorem 8: (Point $L^p$-spectrum of $\mathcal{L}$)

Let the assumptions of Theorem 6 be satisfied. Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property: For every classical solution $\nu_* \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) it holds

$$\sigma_{\text{sym}}(\mathcal{L}) \subseteq \sigma_{\text{pt}}(\mathcal{L}) \quad \text{in} \quad L^p(\mathbb{R}^d, \mathbb{C}^N).$$

In particular, Theorem 6 implies exponential decay of eigenfunctions and adjoint eigenfunctions.

- **point spectrum** is determined by the group action
- Thm. 8 even holds for nonlocalized rotating waves (spiral waves, scroll waves)
- $\nu(x) = \langle Sx, \nabla \nu_*(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geq 2$
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Example

Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

\[
    u_t = \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad u = u(x,t) \in \mathbb{C}
\]

with \( u : \mathbb{R}^d \times [0, \infty[ \to \mathbb{C}, \ d \in \{2,3\} \). For the parameters

\[
    \alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2}
\]

this equation exhibits so called **spinning soliton** solutions.

Freezing method implies numerical results for profile \( v_\star \) and velocities \( S \).

\[
    \text{Re} \ v_\star(x) = \pm 0.5 \quad \text{Im} \ v_\star(x) = \pm 0.5 \quad |v_\star(x)| = 0.5
\]
Spatial decay of a spinning soliton in QCGL for \( d = 3 \): Assume
\[
\text{Re} \alpha > 0, \quad \text{Re} \delta < 0, \quad p_{\text{min}} = \frac{2|\alpha|}{|\alpha| + \text{Re} \alpha} < p < \frac{2|\alpha|}{|\alpha| - \text{Re} \alpha} = p_{\text{max}}
\]

**Decay rate of spinning soliton:**
\[
0 \leq \mu < \frac{\sqrt{-\text{Re} \alpha \text{Re} \delta}}{|\alpha| p} =: \mu^{\text{pro}}(p) < \frac{\sqrt{-\text{Re} \alpha \text{Re} \delta}}{|\alpha| \max\{p_{\text{min}}, \frac{d}{2}\}} =: \mu^{\text{pro}}_{\text{max}}.
\]

**Parameters:**
\[
\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i,
\]
\[
\mu = -\frac{1}{2}, \quad \nu_{\infty} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_0 = \text{Re} \alpha,
\]
\[
a_{\text{max}} = |\alpha|, \quad b_0 = \beta_{\infty} = -\text{Re} \delta = -\frac{1}{2},
\]

**Numerical vs. theoretical decay rate:** \( (p = 2) \)
\[
\text{NDR} \approx 0.5387, \quad \text{TDR} = \mu^{\text{pro}}_{\text{max}} = \frac{\sqrt{2}}{4} \approx 0.4714.
\]
**Spectrum** of QCGL for a spinning soliton with $d = 3$: (numerical vs. analytical)

**Point spectrum** on $i\mathbb{R}$ and **essential spectrum** by dispersion relation:

$$
\sigma_{\text{disp}}(\mathcal{L}) = \{ \lambda = -\omega^2 \alpha_1 + \delta_1 + i(\mp \omega^2 \alpha_2 \pm \delta_2 - n\sigma_1) : \omega \in \mathbb{R}, n \in \mathbb{Z} \},
$$

$$
\sigma_{\text{sym}}(\mathcal{L}) = \{ 0, \pm i\sigma_1 \}, \quad \sigma_1 = 0.6888
$$

for parameters $\alpha = \frac{1}{2} + \frac{1}{2}i$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{1}{10}i$, $\mu = -\frac{1}{2}$. 
Eigenfunctions of QCGL for a spinning soliton with $d = 3$: $\text{Re}(x) = \pm 0.8$
Spatial decay of eigenfunctions of QCGL at a spinning soliton for $d = 3$: Note

$$\text{Re}\lambda \geq -(1 - \varepsilon)\beta_\infty = -(1 - \varepsilon)(-\text{Re}\delta) \iff \varepsilon \leq \frac{\text{Re}\lambda - \text{Re}\delta}{-\text{Re}\delta} =: \varepsilon(\lambda).$$

Decay rate of eigenfunctions:

$$0 \leq \mu \leq \frac{\varepsilon(\lambda)\sqrt{-\text{Re}\alpha\text{Re}\delta}}{|\alpha|p} =: \mu^\text{eig}(p, \lambda) < \frac{\varepsilon(\lambda)\sqrt{-\text{Re}\alpha\text{Re}\delta}}{|\alpha|\max\{p_{\text{min}}, \frac{d}{2}\}} =: \mu^\text{eig}_{\text{max}}(\lambda).$$

| eigenvalue                  | NDR       | TDR       |
|----------------------------|-----------|-----------|
| $8.999 \cdot 10^{-15}$     | 0.5387    | 0.4714    |
| $-5.6162 \cdot 10^{-4}$    | 0.5478    | 0.4714    |
| $0.00110 \pm 0.68827i$     | 0.5507    | 0.4714    |
| $0.00248 \pm 0.6874i$      | 0.5398    | 0.4714    |
| $-0.06622 \pm 1.0112i$     | 0.4899    | 0.4090    |
| $-0.07747 \pm 1.5274i$     | 0.5355    | 0.3984    |
| $-0.22334 \pm 1.1593i$     | 0.4756    | 0.2608    |
| $-0.26467 \pm 0.1193i$     | 0.4785    | 0.2219    |
| $-0.30232 \pm 1.9457i$     | 0.4649    | 0.1864    |
| $-0.43957 \pm 2.3248i$     | 0.3595    | 0.0570    |
| $-0.44063 \pm 1.5128i$     | 0.3310    | 0.0560    |
| $-0.47366 \pm 1.3552i$     | 0.4781    | 0.0248    |
| $-0.48294 \pm 0.9163i$     | 0.4145    | 0.0161    |
| $-0.48506 \pm 0.0991i$     | 0.2126    | 0.0141    |
| $-0.49015 \pm 0.2535i$     | 0.3307    | 0.0093    |
| $-0.55519 \pm 1.1222i$     | 0.3581    | —         |
Eigenfunctions vs. adjoint eigenfunctions of QCGL for a spinning soliton with $d = 3$:

Eigenfunctions (above) and adjoint eigenfunctions (bottom) for $\lambda \in \sigma_{\text{sym}}(\mathcal{L})$
**Eigenfunction** \( \langle Sx, \nabla v_\star(x) \rangle \) of QCGL for a spinning soliton with \( d = 3 \):
Conclusion:

Theoretical results:

1. spatial decay of rotating waves
2. spectral properties of linearization at localized rotating waves
   - Fredholm properties in $L^p$
   - symmetry set, point $L^p$-spectrum, shape of eigenfunctions and spatial decay of eigenfunctions and adjoint eigenfunctions
   - dispersion set, essential $L^p$-spectrum

Numerical results:

3. approximation of rotating waves, spectra, eigenfunctions and adjoint eigenfunctions of QCGL (computation: COMSOL, postprocessing: MATLAB)
Open problems and work in progress

- **Fredholm properties and \( L^p \)-spectra of localized rotating waves**
  (joint work with: W.-J. Beyn)

- **Fourier-Bessel method on \( \mathbb{R}^d \) and on circular domains**
  (joint work with: W.-J. Beyn, C. Döding)

- **Nonlinear stability of relative equilibria in evolution equations**
  (joint work with: W.-J. Beyn, C. Döding)

- **Freezing traveling waves in incompressible Navier-Stokes equations**
  (joint work with: W.-J. Beyn, C. Döding)

- **Nonlinear stability of rotating waves for \( d \geq 3 \)**
  (joint work with: W.-J. Beyn)

- **Approximation theorem for rotating waves**
Outline

Outline of proof: Theorem 1

Outline of proof: Theorem 2

Outline of proof: Theorem 7
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \; x \in \mathbb{R}^d, \; d \geq 2.$$  

1. Far-Field Linearization: $f \in C^1$, Taylor’s theorem, $f(v_\infty) = 0$

$$a(x) := \int_0^1 Df(v_\infty + tw_\star(x)) dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + a(x)w_\star(x) = 0, \; x \in \mathbb{R}^d.$$
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Consider the nonlinear problem

$$A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2.$$ 

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$ 

$$A \triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$ 

![Graph showing exponential decay and boundedness of $Q(x)$](image-url)
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Consider the nonlinear problem

$$A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2.$$ 

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A \triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$ 

3. Decomposition of $Q$:

$$Q(x) = Q_s(x) + Q_c(x),$$

$Q$, $Q_s$, $Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$,

$Q_s$ small, i.e. $\|Q_s\|_{L^\infty} < K_1$,

$Q_c$ compactly supported.
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Consider the nonlinear problem

$$A\nabla v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2.$$ 

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty) dt, \ w_\star(x) := v_\star(x) - v_\infty$$

$$A\nabla w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$ 

3. Decomposition of $Q$:

$$Q(x) = Q_s(x) + Q_c(x),$$

$Q, Q_s, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$, $Q_s$ small, i.e. $\|Q_s\|_{L^\infty} < K_1$, $Q_c$ compactly supported.
Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Consider the nonlinear problem

\[ A \Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d, \ d \geq 2. \]

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

\[ Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty) dt, \quad w_\star(x) := v_\star(x) - v_\infty \]

\[ A \Delta w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \quad x \in \mathbb{R}^d. \]

3. Decomposition of $Q$:

\[ Q(x) = Q_s(x) + Q_c(x), \]

$Q, Q_s, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$,

$Q_s$ small, i.e. $\|Q_s\|_{L^\infty} < K_1$,

$Q_c$ compactly supported.
Outline

Outline of proof: Theorem 1

Outline of proof: Theorem 2

Outline of proof: Theorem 7
Outline of proof: Theorem 2 (Decay of eigenfunctions)

Consider

\[ A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\ast(x))v(x) = \lambda v(x), \ x \in \mathbb{R}^d. \]

1. Splitting off the stable part:

\[ Df(v_\ast(x)) = Df(v_\infty) + (Df(v_\ast(x)) - Df(v_\infty)) =: Df(v_\infty) + Q(x), \ x \in \mathbb{R}^d, \]

leads to

\[ [L_0 v](x) + (Df(v_\infty) + Q(x))v(x) = \lambda v(x), \ x \in \mathbb{R}^d. \]

2. Decomposition of (the variable coefficient) \( Q \):

\[ Q(x) = Q_\varepsilon(x) + Q_c(x), \ Q_\varepsilon \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ small w.r.t. } ||\cdot||_{C_b}, \]

\[ Q_c \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d, \]

leads to

\[ [L_0 v](x) + (Df(v_\infty) + Q_\varepsilon(x) + Q_c(x))v(x) = \lambda v(x), \ x \in \mathbb{R}^d. \]

(\( \rightarrow \) inhomogeneous Cauchy problem for \( L_c \))
Outline

Outline of proof: Theorem 1

Outline of proof: Theorem 2

Outline of proof: Theorem 7
Outline of proof: Theorem 7 (Essential $L^p$-spectrum of $\mathcal{L}$)

Choose $R \geq 2$ large and cut-off function $\chi_R \in C^2_b$ (bounded indep. on $R$)

$$\chi_R : [0, \infty) \to [0, 1], \quad \chi_R(r) = \begin{cases} 0, & r \in l_1 \cup l_5, \\ \in [0, 1], & r \in l_2 \cup l_4, \\ 1, & r \in l_3, \end{cases}$$

$l_1 = [0, R - 1], l_2 = [R - 1, R], l_3 = [R, 2R], l_4 = [2R, 2R + 1], l_5 = [2R + 1, \infty)$.

Introducing

$$v_R(\xi) : = \left[ \prod_{l=1}^{k} \chi_R(r_l) \right] \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R : = \frac{v_R}{\|v_R\|_{L^p}},$$

we want show that $w_R \in D^p_{loc}(\mathcal{L}_0)$ and

$$\|(\lambda I - \mathcal{L})w_R\|^p_{L^p} = \frac{\|(\lambda I - \mathcal{L})v_R\|^p_{L^p}}{\|v_R\|^p_{L^p}} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d} = \frac{C}{R} + \eta_R \to 0 \text{ as } R \to \infty.$$ 

Then, $\lambda \notin \rho(\mathcal{L})$ (by continuity of resolvent), i.e. $\lambda \in \sigma(\mathcal{L})$. But $\lambda \notin \sigma_{pt}(\mathcal{L})$ (since varying $\omega$ or $\rho$ shows that $\lambda$ is not isolated), hence $\lambda \in \sigma_{ess}(\mathcal{L})$.
Outline of proof: Theorem 7 (Essential $L^p$-spectrum of $\mathcal{L}$)

$$\chi_R(r) = \begin{cases} 0 & , r \in l_1 \cup l_5, \\ \in [0, 1] & , r \in l_2 \cup l_4, \\ 1 & , r \in l_3, \end{cases}$$
$$v_R(\xi) := \left[ \prod_{l=1}^{k} \chi_R(r_l) \right] \chi_R(\tilde{y}) \hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}}$$

$l_1 = [0, R - 1], \ l_2 = [R - 1, R], \ l_3 = [R, 2R], \ l_4 = [2R, 2R + 1], \ l_5 = [2R + 1, \infty)$. 

**Aim:**
$$\frac{\| (\lambda I - \mathcal{L}) v_R \|^p_{L^p}}{\| v_R \|^p_{L^p}} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d} \quad \text{and} \quad w_R \in D^{p}_{loc}(\mathcal{L})$$

**Show:**

1. $\| v_R \|^p_{L^p} \geq CR^d$
2. $\| (\lambda I - \mathcal{L}) v_R \|^p_{L^p} \leq CR^{d-1} + CR^d \eta_R$
3. $| (\lambda I - \mathcal{L}_2) v_R(\xi) | = 0$, if $|\tilde{y}| \in l_1 \cup l_5$ or $r_l \in l_1 \cup l_5$ for some $1 \leq l \leq k$,
   $| (\lambda I - \mathcal{L}_2) v_R(\xi) | \leq C \ \forall |\tilde{y}|, r_l \in l_2 \cup l_3 \cup l_4$ for some $1 \leq l \leq k$,
   $| (\lambda I - \mathcal{L}_2) v_R(\xi) | \leq \left( \sum_{l=1}^{k} \frac{c_l}{r_l} + \eta_R \right)^{\frac{1}{p}} \ \forall |\tilde{y}|, r_l \in l_3$ for all $1 \leq l \leq k$,
4. $\| (\lambda I - \mathcal{L}^{\text{sim}}_{\infty}) v_R \|^p_{L^p} \leq CR^{d-1}$
5. $(\lambda I - \mathcal{L}^{\text{sim}}_{\infty}) v_R(\xi) = 0$
Outline of proof: Theorem 7 (Essential $L^p$-spectrum of $\mathcal{L}$)

$$\chi_R(r) = \begin{cases} 0, & r \in I_1 \cup I_5, \\ \in [0, 1], & r \in I_2 \cup I_4, \\ 1, & r \in I_3, \end{cases}$$

$$v_R(\xi) := \left[ \prod_{l=1}^{k} \chi_R(r_l) \right] \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R := \frac{v_R}{\| v_R \|_{L^p}}$$

$I_1 = [0, R - 1], I_2 = [R - 1, R], I_3 = [R, 2R], I_4 = [2R, 2R + 1], I_5 = [2R + 1, \infty)$.

Aim: \[ \frac{\| (\lambda I - \mathcal{L}) v_R \|_{L^p}^p}{\| v_R \|_{L^p}^p} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d} \quad \text{and} \quad w_R \in D_{\text{loc}}^p(\mathcal{L}_0) \]

Show:

1. \[ \| v_R \|_{L^p}^p \geq CR^d \]
2. \[ \| (\lambda I - \mathcal{L}) v_R \|_{L^p}^p \leq CR^{d-1} + CR^d \eta_R \]
3. \[ (\lambda I - \mathcal{L}_2) v_R(\xi) = 0, \text{ if } |\tilde{y}| \in I_1 \cup I_5 \text{ or } r_l \in I_1 \cup I_5 \text{ for some } 1 \leq l \leq k, \]
4. \[ (\lambda I - \mathcal{L}_2) v_R(\xi) \leq C \\forall |\tilde{y}|, r_l \in I_2 \cup I_3 \cup I_4 \text{ for some } 1 \leq l \leq k, \]
5. \[ (\lambda I - \mathcal{L}_2) v_R(\xi) \leq \left( \sum_{l=1}^{k} \frac{C_l}{r_l} + \eta_R \right)^{\frac{1}{p}} \\forall |\tilde{y}|, r_l \in I_3 \text{ for all } 1 \leq l \leq k, \]
6. \[ \| (\lambda I - \mathcal{L}_3) v_R \|_{L^p}^p \leq CR^{d-1} \]
7. \[ (\lambda I - \mathcal{L}_3) v_R(\xi) = 0 \]
Outline of proof: Theorem 7 (Essential $L^p$-spectrum of $\mathcal{L}$)

$$\chi_R(r) = \begin{cases} 
0 & , \ r \in I_1 \cup I_5, \\
\in [0, 1] & , \ r \in I_2 \cup I_4, \\
1 & , \ r \in I_3,
\end{cases}$$

$$v_R(\xi) := \prod_{l=1}^k \chi_R(r_l) \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}}$$

$I_1 = [0, R - 1], \ I_2 = [R - 1, R], \ I_3 = [R, 2R], \ I_4 = [2R, 2R + 1], \ I_5 = [2R + 1, \infty).$

Aim: \[ \frac{\| (\lambda I - \mathcal{L}) v_R \|_{L^p}^p}{\| v_R \|_{L^p}^p} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d} \quad \text{and} \quad w_R \in D_{\text{loc}}^p(\mathcal{L}_0) \]

Show:

1. $\|v_R\|_{L^p}^p \geq CR^d$
2. $\| (\lambda I - \mathcal{L}) v_R \|_{L^p}^p \leq CR^{d-1} + CR^d \eta_R$
3. $| (\lambda I - \mathcal{L}_2) v_R(\xi) | = 0$, if $|\tilde{y}| \in I_1 \cup I_5$ or $r_l \in I_1 \cup I_5$ for some $1 \leq l \leq k$, $| (\lambda I - \mathcal{L}_2) v_R(\xi) | \leq C \ \forall |\tilde{y}|, \ r_l \in I_2 \cup I_3 \cup I_4$ for some $1 \leq l \leq k$, $| (\lambda I - \mathcal{L}_2) v_R(\xi) | \leq \left( \sum_{l=1}^k \frac{C_l}{r_l} + \eta_R \right)^\frac{1}{p} \ \forall |\tilde{y}|, \ r_l \in I_3$ for all $1 \leq l \leq k$,
4. $\| (\lambda I - \mathcal{L}_{\infty}^{\text{sim}}) v_R \|_{L^p}^p \leq CR^{d-1}$
5. $| (\lambda I - \mathcal{L}_{\infty}^{\text{sim}}) v_R(\xi) | = 0$
Outline of proof: Theorem 7 (Essential $L^p$-spectrum of $\mathcal{L}$)

\[ \chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0, 1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases} \quad v_R(\xi) := \left[ \prod_{l=1}^{k} \chi_R(r_l) \right] \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}} \]

\[ I_1 = [0, R - 1], \quad I_2 = [R - 1, R], \quad I_3 = [R, 2R], \quad I_4 = [2R, 2R + 1], \quad I_5 = [2R + 1, \infty). \]

Aim: \[ \frac{\| (\lambda I - \mathcal{L}) v_R \|_{L^p}^p}{\| v_R \|_{L^p}^p} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d} \] and \[ w_R \in D_{\text{loc}}^p(\mathcal{L}_0) \]

Show:

1. $\| v_R \|_{L^p}^p \geq CR^d$
2. $\| (\lambda I - \mathcal{L}) v_R \|_{L^p}^p \leq CR^{d-1} + CR^d \eta_R$
3. $| (\lambda I - \mathcal{L}_2) v_R(\xi) | = 0$, if $|\tilde{y}| \in I_1 \cup I_5$ or $r_l \in I_1 \cup I_5$ for some $1 \leq l \leq k$,
   $| (\lambda I - \mathcal{L}_2) v_R(\xi) | \leq C \ \forall |\tilde{y}|, \ r_l \in I_2 \cup I_3 \cup I_4$ for some $1 \leq l \leq k$,
   $| (\lambda I - \mathcal{L}_2) v_R(\xi) | \leq (\sum_{l=1}^{k} \frac{C_l}{r_l} + \eta_R)^{\frac{1}{p}} \ \forall |\tilde{y}|, \ r_l \in I_3$ for all $1 \leq l \leq k$,
4. $\| (\lambda I - \mathcal{L}_{\text{sim}}^\infty) v_R \|_{L^p}^p \leq CR^{d-1}$
5. $| (\lambda I - \mathcal{L}_{\text{sim}}^\infty) v_R(\xi) | = 0$