INSERTION OF A CONTRA-$\gamma$-CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE REAL-VALUED FUNCTIONS

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Abstract A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a contra-$\gamma$-continuous function between two comparable real-valued functions.

1 Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [5]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term preopen, was used for the first time by A.S. Mashhour, M.E. Abi El-Monsef and S.N. El-Deeb [21], while the concept of a locally dense, set was introduced by H.H. Corson and E. Michael [5].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [18]. A subset $A$ of a topological space $(X, \tau)$ is called semi-open if $A \subseteq \text{Cl}(\text{Int}(A))$. A set $A$ is called semi-closed if its complement is semi-open or equivalently if $\text{Int}(\text{Cl}(A)) \subseteq A$.

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$-continuous [24] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$-continuity. However, for unknown concepts the reader may refer to [6, 12]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [7] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 9, 10, 11, 13, 14, 23].

Hence, a real-valued function $f$ defined on a topological space $X$ is called contra-$\gamma$-continuous (resp. contra-semi-continuous , contra-precontinuous) if the preimage of every open subset of $\mathbb{R}$ is $\gamma$-closed (resp. semi-closed , preclosed) in $X$[7].

Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient conditions for the insertion of a contra-$\gamma$-continuous function between two comparable real-valued functions.

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ (resp. $g < f$) in
case \(g(x) \leq f(x)\) (resp. \(g(x) < f(x)\)) for all \(x\) in \(X\).

The following definitions are modifications of conditions considered in [17].

A property \(P\) defined relative to a real-valued function on a topological space is a \(c_\gamma\)–property provided that any constant function has property \(P\) and provided that the sum of a function with property \(P\) and any contra-\(c_\gamma\)–continuous function also has property \(P\). If \(P_1\) and \(P_2\) are \(c_\gamma\)–property, the following terminology is used: (i) A space \(X\) has the weak \(c_\gamma\)–insertion property for \((P_1, P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g \leq f, g\) has property \(P_1\) and \(f\) has property \(P_2\), then there exists a contra-\(c_\gamma\)–continuous function \(h\) such that \(g \leq h \leq f\). (ii) A space \(X\) has the \(c_\gamma\)–insertion property for \((P_1, P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g < f, g\) has property \(P_1\) and \(f\) has property \(P_2\), then there exists a contra-\(c_\gamma\)–continuous function \(h\) such that \(g < h < f\). (iii) A space \(X\) has the weakly \(c_\gamma\)–insertion property for \((P_1, P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g < f, g\) has property \(P_1\), \(f\) has property \(P_2\) and \(f - g\) has property \(P_2\), then there exists a contra-\(c_\gamma\)–continuous function \(h\) such that \(g < h < f\).

In this paper, is given a sufficient condition for the weak \(c_\gamma\)–insertion property. Also for a space with the weak \(c_\gamma\)–insertion property, we give a necessary and sufficient condition for the space to have the \(c_\gamma\)–insertion property. Several insertion theorems are obtained as corollaries of these results.

### 2 The Main Result

Before giving a sufficient condition for insertability of a contra-\(c_\gamma\)–continuous function, the necessary definitions and terminology are stated.

Let \((X, \tau)\) be a topological space, the family of all \(\gamma\)–open, \(\gamma\)–closed, semi-open, semi-closed, preopen and preclosed will be denoted by \(\gamma O(X, \tau), \gamma C(X, \tau), sO(X, \tau), sC(X, \tau), pO(X, \tau)\) and \(pC(X, \tau)\), respectively.

**Definition 2.1.** Let \(A\) be a subset of a topological space \((X, \tau)\). We define the subsets \(A^A\) and \(A^V\) as follows:

\[
A^A = \cap\{O : O \supseteq A, O \in (X, \tau)\} \quad \text{and} \quad A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}.
\]

In [8, 19, 22], \(A^A\) is called the kernel of \(A\).

We define the subsets \(\gamma(A^A), \gamma(A^V), p(A^A), p(A^V), s(A^A)\) and \(s(A^V)\) as follows:

\[
\gamma(A^A) = \cap\{O : O \supseteq A, O \in \gamma O(X, \tau)\},
\]

\[
\gamma(A^V) = \cup\{F : F \subseteq A, F \in \gamma C(X, \tau)\},
\]

\[
p(A^A) = \cap\{O : O \supseteq A, O \in pO(X, \tau)\},
\]

\[
p(A^V) = \cup\{F : F \subseteq A, F \in pC(X, \tau)\},
\]

\[
s(A^A) = \cap\{O : O \supseteq A, O \in sO(X, \tau)\} \quad \text{and}
\]

\[
s(A^V) = \cup\{F : F \subseteq A, F \in sC(X, \tau)\}.
\]

\(\gamma(A^A)\) (resp. \(p(A^A), s(A^A)\)) is called the \(\gamma\)–kernel (resp. \(\gamma\)–prekernel, \(\gamma\)–semi–kernel) of \(A\).

The following first two definitions are modifications of conditions considered in [15, 16].

**Definition 2.2.** If \(\rho\) is a binary relation in a set \(S\) then \(\bar{\rho}\) is defined as follows: \(x \bar{\rho} y\) if and only if \(y \rho u\) implies \(x \rho u\) and \(u \rho y\) for any \(u\) and \(v\) in \(S\).

**Definition 2.3.** A binary relation \(\rho\) in the power set \(P(X)\) of a topological space \(X\) is called a strong binary relation in \(P(X)\) in case \(\rho\) satisfies each of the following conditions:

1. If \(A_1, \rho B_j\) for any \(i \in \{1, \ldots, m\}\) and for any \(j \in \{1, \ldots, n\}\), then there exists a set \(C\) in \(P(X)\) such that \(A_i \rho C\) and \(C \rho B_j\) for any \(i \in \{1, \ldots, m\}\) and any \(j \in \{1, \ldots, n\}\).
2. If \(A \subseteq B\), then \(A \bar{\rho} B\).
3. If \(A \rho B\), then \(\gamma(A^A) \subseteq B\) and \(A \subseteq \gamma(B^V)\).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:
Definition 2.4. If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq \{x \in X : f(x) < \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is called a lower indefinite cut set in the domain of $f$ at the level $\ell$.

We now give the following main result:

Theorem 2.1. Let $g$ and $f$ be real-valued functions on the topological space $X$, in which $\gamma$-kernel sets are $\gamma$-open, with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, \ell)$ and $A(g, \ell)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, there then exists a contra-$\gamma$-continuous function $h$ defined on $X$ such that $g \leq h \leq f$.

Proof. Let $g$ and $f$ be real-valued functions defined on the $X$ such that $g \leq f$. By hypothesis there exists a strong binary relation $\rho$ on the power set of $X$ and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions $F$ and $G$ mapping the rational numbers $Q$ into the power set of $X$ by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2), G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [16] it follows that there exists a function $H$ mapping $Q$ into the power set of $X$ such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any $x$ in $X$, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If $x$ is in $H(t)$ then $x$ is in $G(t')$ for any $t' > t$; since $x$ is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If $x$ is not in $H(t)$, then $x$ is not in $F(t')$ for any $t' < t$; since $x$ is not in $F(t') = A(f, t')$ implies that $f(x) > t'$. Hence $h \leq f$.

Also, for any rational numbers $t_1$ and $t_2$ with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \gamma(H(t_2)^{\circ}) \setminus \gamma(H(t_1)^{\circ})$. Hence $h^{-1}(t_1, t_2)$ is $\gamma$-closed in $X$, i.e., $h$ is a contra-$\gamma$-continuous function on $X$.

The above proof used the technique of theorem 1 in [15].

Theorem 2.2. Let $P_1$ and $P_2$ be $c\gamma$-property and $X$ be a space that satisfies the weak $c\gamma$-insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the $c\gamma$-insertion property for $(P_1, P_2)$ if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $(D_n)$ of subsets of $X$ with empty intersection and such that for each $n$, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by contra-$\gamma$-continuous functions.

Proof. Assume that $X$ has the weak $c\gamma$-insertion property for $(P_1, P_2)$. Let $g$ and $f$ be functions such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$. By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence $(D_n)$ such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and such that for each $n$, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by contra-$\gamma$-continuous functions. Let $k_n$ be a contra-$\gamma$-continuous function such that $k_n = 0$ on $A(f - g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function $k$ on $X$ be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the properties of contra-$\gamma$-continuous functions, the function $k$ is a contra-$\gamma$-continuous function. Since $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if $x$ is in $A(f - g, 3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If $x$ is any point in $X$, then $x \notin A(f - g, 1)$ or for some $n$,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since $P_1$ and $P_2$ are $c\gamma$-properties, then $g_1$ has property $P_1$ and $f_1$ has property $P_2$. Since $X$ has the weak $c\gamma$-insertion property for $(P_1, P_2)$, then there exists a contra-$\gamma$-continuous function $h$ such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that $X$ satisfies the $c\gamma$-insertion property for $(P_1, P_2)$. (The technique of this proof is by Katětov[15]).
Conversely, let \( g \) and \( f \) be functions on \( X \) such that \( g \) has property \( P_1 \), \( f \) has property \( P_2 \) and \( g < f \). By hypothesis, there exists a contra-\( \gamma \)-continuous function \( h \) such that \( g < h < f \). We follow an idea contained in Lane [17]. Since the constant function 0 has property \( P_1 \), since \( f - h \) has property \( P_2 \), and since \( X \) has the \( c\gamma \)-insertion property for \((P_1, P_2)\), there exists a contra-\( \gamma \)-continuous function \( k \) such that \( 0 < k < f - h \). Let \( A(f - g, 3^{-n+1}) \) be any lower cut set for \( f - g \) and let \( D_n = \{ x \in X : k(x) < 3^{-n+2} \} \). Since \( k > 0 \) it follows that \( \bigcap_{n=1}^\infty D_n = \emptyset \). Since

\[
A(f - g, 3^{-n+1}) \subseteq \{ x \in X : (f - g)(x) \leq 3^{-n+1} \} \subseteq \{ x \in X : k(x) \leq 3^{-n+1} \}
\]

and since \( \{ x \in X : k(x) \leq 3^{-n+1} \} \) and \( \{ x \in X : k(x) \geq 3^{-n+2} \} = X \setminus D_n \) are completely separated by contra-\( \gamma \)-continuous functions \( \sup \{ 3^{-n+1}, \inf \{ k, 3^{-n+2} \} \} \), it follows that for each \( n \), \( A(f - g, 3^{-n+1}) \) and \( X \setminus D_n \) are completely separated by contra-\( \gamma \)-continuous functions.

**3 Applications**

The abbreviations \( c\gamma c \), \( cpc \) and \( csc \) are used for contra-\( \gamma \)-continuous, contra-precontinuous and contra-\emph{semi}-\( \gamma \)-continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2, we suppose that \( X \) is a topological space whose \( \gamma \)-kernel sets are \( \gamma \)-open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \) of \( X \), there exist \( \gamma \)-closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then \( X \) has the weak \( c\gamma \)-insertion property for (cpc, cpc) (resp. (csc, csc)).

**Proof.** Let \( f \) and \( g \) be real-valued functions defined on \( X \), such that \( f \) and \( g \) are cpc (resp. csc), and \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( p(A^\rho) \subseteq p(B^\rho) \) (resp. \( s(A^\rho) \subseteq s(B^\rho) \)), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). Since \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then

\[
A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);
\]

since \( \{ x \in X : f(x) \leq t_1 \} \) is a preopen (resp. semi-open) set and since \( \{ x \in X : g(x) < t_2 \} \) is a preclosed (resp. semi-closed) set, it follows that \( p(A(f, t_1)^\rho) \subseteq p(A(g, t_2)^\rho) \) (resp. \( s(A(f, t_1)^\rho) \subseteq s(A(g, t_2)^\rho) \)). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 2.1.

**Corollary 3.2.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \), there exist \( \gamma \)-closed sets \( F_1 \) and \( F_2 \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then every contra-precontinuous (resp. contra-\emph{semi}-\( \gamma \)-continuous) function is contra-\( \gamma \)-continuous.

**Proof.** Let \( f \) be a real-valued contra-precontinuous (resp. contra-\emph{semi}-\( \gamma \)-continuous) function defined on \( X \). Set \( g = f \), then by Corollary 3.1, there exists a contra-\( \gamma \)-continuous function \( h \) such that \( g = h = f \).

**Corollary 3.3.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \) of \( X \), there exist \( \gamma \)-closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then \( X \) has the \( c\gamma \)-insertion property for (cpc, cpc) (resp. (csc, csc)).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on \( X \), such that \( f \) and \( g \) are cpc (resp. csc), and \( g < f \). Set \( h = (f + g)/2 \), thus \( g < h < f \), and by Corollary 3.2, since \( g \) and \( f \) are contra-\( \gamma \)-continuous functions hence \( h \) is a contra-\( \gamma \)-continuous function.

**Corollary 3.4.** If for each pair of disjoint subsets \( G_1, G_2 \) of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi-open, there exist \( \gamma \)-closed subsets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \) then \( X \) have the weak \( c\gamma \)-insertion property for (cpc, cpc) and (csc, cpc).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on \( X \), such that \( g \) is cpc (resp. csc) and \( f \) is csc (resp. cpc), with \( g < f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( s(A^\rho) \subseteq s(B^\rho) \) (resp. \( p(A^\rho) \subseteq p(B^\rho) \)), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \).
If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then
\[
A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);
\]
since \( \{ x \in X : f(x) \leq t_1 \} \) is a semi-open (resp. preopen) set and since \( \{ x \in X : g(x) < t_2 \} \) is a preclosed (resp. semi-closed) set, it follows that \( s(A(f, t_1)^o) \subseteq p(A(g, t_2)^c) \) (resp. \( p(A(f, t_1)^o) \subseteq s(A(g, t_2)^c) \)). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 2.1. ■

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

**Lemma 3.1.** The following conditions on the space \( X \) are equivalent:

(i) For each pair of disjoint subsets \( G_1, G_2 \) of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi-open, there exist \( \gamma \)-closed subsets \( F_1, F_2 \) of \( X \) such that \( G_1 \subseteq F_1, G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \).

(ii) If \( G \) is a semi-open (resp. preopen) subset of \( X \) which is contained in a preclosed (resp. semi-closed) subset \( F \) of \( X \), then there exists a \( \gamma \)-closed subset \( H \) of \( X \) such that \( G \subseteq H \subseteq \gamma(H^c) \subseteq F \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( G \subseteq F \), where \( G \) and \( F \) are semi-open (resp. preopen) and preclosed (resp. semi-closed) subsets of \( X \), respectively. Hence, \( F^c \) is a preopen (resp. semi-open) and \( G \cap F^c = \emptyset \).

By (i) there exists two disjoint \( \gamma \)-closed subsets \( F_1, F_2 \) such that \( G \subseteq F_1 \) and \( F^c \subseteq F_2 \). But
\[
F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,
\]
and
\[
F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c
\]
hence
\[
G \subseteq F_1 \subseteq F_2^c \subseteq F
\]
and since \( F_2^c \) is a \( \gamma \)-open subset containing \( F_1 \), we conclude that \( \gamma(F^c_1) \subseteq F_2^c \), i.e.,
\[
G \subseteq F_1 \subseteq \gamma(F^c_1) \subseteq F.
\]
By setting \( H = F_1 \), condition (ii) holds.

(ii) \( \Rightarrow \) (i) Suppose that \( G_1, G_2 \) are two disjoint subsets of \( X \), such that \( G_1 \) is preopen and \( G_2 \) is semi-open.

This implies that \( G_2 \subseteq G_1^c \) and \( G_1^c \) is a preclosed subset of \( X \). Hence by (ii) there exists a \( \gamma \)-closed set \( H \) such that \( G_2 \subseteq H \subseteq \gamma(H^c) \subseteq G_1^c \).

But
\[
H \subseteq \gamma(H^c) \Rightarrow H \cap \gamma((H^c)^c) = \emptyset
\]
and
\[
\gamma(H^c) \subseteq G_1^c \Rightarrow G_1 \subseteq \gamma((H^c)^c).
\]
Furthermore, \( \gamma((H^c)^c) \) is a \( \gamma \)-closed subset of \( X \). Hence \( G_2 \subseteq H, G_1 \subseteq \gamma((H^c)^c) \) and \( H \cap \gamma((H^c)^c) = \emptyset \). This means that condition (i) holds. ■

**Lemma 3.2.** Suppose that \( X \) is a topological space. If each pair of disjoint subsets \( G_1, G_2 \) of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi-open, can be separated by \( \gamma \)-closed subsets of \( X \) then there exists a contra-\( \gamma \)-continuous function \( h : X \to [0,1] \) such that \( h(G_2) = \{0\} \) and \( h(G_1) = \{1\} \).

**Proof.** Suppose \( G_1 \) and \( G_2 \) are two disjoint subsets of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi-open. Since \( G_1 \cap G_2 = \emptyset \), hence \( G_2 \subseteq G_1^c \). In particular, since \( G_1^c \) is a preclosed subset of \( X \) containing the semi-open subset \( G_2 \) of \( X \), by Lemma 3.1, there exists a \( \gamma \)-closed subset \( H_{1/2} \) such that
\[
G_2 \subseteq H_{1/2} \subseteq \gamma(H^c_{1/2}) \subseteq G_1^c.
\]
Note that \( H_{1/2} \) is also a preclosed subset of \( X \) and contains \( G_2 \), and \( G_1^c \) is a preclosed subset of \( X \) and contains the semi-open subset \( \gamma(H^c_{1/2}) \) of \( X \). Hence, by Lemma 3.1, there exists \( \gamma \)-closed subsets \( H_{1/4} \) and \( H_{3/4} \) such that
\[
G_2 \subseteq H_{1/4} \subseteq \gamma(H^c_{1/4}) \subseteq H_{1/2} \subseteq \gamma(H^c_{1/2}) \subseteq H_{3/4} \subseteq \gamma(H^c_{3/4}) \subseteq G_1^c.
\]
By continuing this method for every \( t \in D \), where \( D \subseteq [0, 1] \) is the set of rational numbers that their denominators are exponents of 2, we obtain \( \gamma \)-closed subsets \( H_t \) with the property that if \( t_1, t_2 \in D \) and \( t_1 < t_2 \), then \( H_{t_1} \subseteq H_{t_2} \). We define the function \( h \) on \( X \) by \( h(x) = \inf \{ t : x \in H_t \} \) for \( x \notin G_t \) and \( h(x) = 1 \) for \( x \in G_t \).

Note that for every \( x \in X, 0 \leq h(x) \leq 1 \), i.e., \( h \) maps \( X \) into \([0,1]\). Also, we note that for any \( t \in D, G_2 \subseteq H_t \); hence \( h(G_2) = \{ 0 \} \). Furthermore, by definition, \( h(G_1) = \{ 1 \} \). It remains only to prove that \( h \) is a contra-\( \gamma \)-continuous function on \( X \). For every \( \alpha \in \mathbb{R} \), we have if \( \alpha \leq 0 \) then \( \{ x \in X : h(x) < \alpha \} = \emptyset \) and if \( 0 < \alpha \) then \( \{ x \in X : h(x) < \alpha \} = \bigcup \{ H_t : t < \alpha \} \); hence, they are \( \gamma \)-closed subsets of \( X \). Similarly, if \( \alpha < 0 \) then \( \{ x \in X : h(x) > \alpha \} = X \) and if \( \alpha \leq 0 \) then \( \{ x \in X : h(x) \geq \alpha \} = \bigcup \{ (H^c_t) : t \geq \alpha \} \); hence, every of them is a \( \gamma \)-closed subset. Consequently \( h \) is a contra-\( \gamma \)-continuous function. ■

**Lemma 3.3.** Suppose that \( X \) is a topological space such that every two disjoint \( \text{semi-} \)open and \( \text{preopen} \) subsets of \( X \) can be separated by \( \gamma \)-closed subsets of \( X \). The following conditions are equivalent:

(i) Every countable covering of \( \text{semi-} \)-closed (resp. \( \text{preclosed} \)) subsets of \( X \) has a refinement consisting of \( \text{semi-} \)closed (resp. \( \text{semi-} \)-closed) subsets of \( X \) such that for every \( x \in X \), there exists a \( \gamma \)-closed subset of \( X \) containing \( x \) such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence \( \{ G_n \} \) of \( \text{semi-} \)-open (resp. \( \text{preopen} \)) subsets of \( X \) with empty intersection there exists a decreasing sequence \( \{ F_n \} \) of \( \text{semi-} \)-closed (resp. \( \text{semi-} \)-closed) subsets of \( X \) such that \( \bigcap_{n=1}^{\infty} F_n = \emptyset \) and for every \( n \in \mathbb{N} \), \( G_n \subseteq F_n \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( \{ G_n \} \) is a decreasing sequence of \( \text{semi-} \)-open (resp. \( \text{preopen} \)) subsets of \( X \) with empty intersection. Then \( \{ G_n : n \in \mathbb{N} \} \) is a countable covering of \( \text{semi-} \)-closed (resp. \( \text{semi-} \)-closed) subsets of \( X \). By hypothesis (i) and Lemma 3.1, this covering has a refinement \( \{ V_n : n \in \mathbb{N} \} \) such that every \( V_n \) is a \( \gamma \)-closed subset of \( X \) and \( \gamma(V_n^c) \subseteq G_n^c \). By setting \( F_n = \gamma((V_n^c)^c) \), we obtain a decreasing sequence of \( \gamma \)-closed subsets of \( X \) with the required properties.

(ii) \( \Rightarrow \) (i) Now if \( \{ H_n : n \in \mathbb{N} \} \) is a countable covering of \( \text{semi-} \)-closed (resp. \( \text{preclosed} \)) subsets of \( X \), we set for every \( n \in \mathbb{N} \), \( G_n = (\bigcup_{i=1}^{n} H_i)^c \). Then \( \{ G_n \} \) is a decreasing sequence of \( \text{semi-} \)-open (resp. \( \text{preopen} \)) subsets of \( X \) with empty intersection. By (ii) there exists a decreasing sequence \( \{ F_n \} \) consisting of \( \text{preclosed} \) (resp. \( \text{semi-} \)-closed) subsets of \( X \) such that \( \bigcap_{n=1}^{\infty} F_n = \emptyset \) and for every \( n \in \mathbb{N} \), \( G_n \subseteq F_n \). Now we define the subsets \( W_n \) of \( X \) in the following manner:

\[ W_1 \text{ is a } \gamma \text{-closed subset of } X \text{ such that } F_1 \subseteq W_1 \text{ and } \gamma(W_1^c) \cap G_1 = \emptyset. \]

\[ W_2 \text{ is a } \gamma \text{-closed subset of } X \text{ such that } \gamma(W_1^c) \cup F_2 \subseteq W_2 \text{ and } \gamma(W_2^c) \cap G_2 = \emptyset, \text{ and so on.} \]

(By Lemma 3.1, \( W_n \) exists).

Then since \( \{ F_n : n \in \mathbb{N} \} \) is a covering for \( X \), hence \( \{ W_n : n \in \mathbb{N} \} \) is a covering for \( X \) consisting of \( \gamma \)-closed sets. Moreover, we have

(i) \( \gamma(W_n^c) \subseteq W_{n+1} \)

(ii) \( F_n \subseteq W_n \)

(iii) \( W_n \subseteq \bigcup_{i=1}^{n} H_i \).

Now setting \( S_1 = W_1 \) and for \( n \geq 2 \), we set \( S_n = W_{n+1} \setminus \gamma(W_{n+1}^c) \).

Then since \( \gamma(W_{n+1}^c) \subseteq W_n \) and \( S_n \supseteq W_{n+1} \setminus W_n \), it follows that \( \{ S_n : n \in \mathbb{N} \} \) consists of \( \gamma \)-closed sets and covers \( X \). Furthermore, \( S_i \cap S_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). Finally, consider the following sets:

\[
S_1 \cap H_1, \quad S_1 \cap H_2 \\
S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3 \\
S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4 \\
\vdots \\
S_i \cap H_1, \quad S_i \cap H_2, \quad S_i \cap H_3, \quad S_i \cap H_4, \quad \ldots, \quad S_i \cap H_{i+1} \\
\vdots
\]
These sets are $\gamma$–closed sets, cover $X$ and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $\gamma$–closed set containing $x$ that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i + 1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $\gamma$–closed sets, and for every point in $X$ we can find a $\gamma$–closed set containing the point that intersects only finitely many elements of that refinement.$\blacksquare$

**Corollary 3.5.** If every two disjoint $\text{semi}$–open and preopen subsets of $X$ can be separated by $\gamma$–closed subsets of $X$, and in addition, every countable covering of $\text{semi}$–closed (resp. preclosed) subsets of $X$ has a refinement that consists of preclosed (resp. $\text{semi}$–closed) subsets of $X$ such that for every point of $X$ we can find a $\gamma$–closed subset containing that point such that it intersects only a finite number of refining members then $X$ has the weakly $c\gamma$–insertion property for $(\text{cpc, csc})$ (resp. $(\text{csc, cpc})$).

**Proof.** Since every two disjoint $\text{semi}$–open and preopen sets can be separated by $\gamma$–closed subsets of $X$, therefore by Corollary 3.4, $X$ has the weak $c\gamma$–insertion property for $(\text{cpc, csc})$ and $(\text{csc, cpc})$. Now suppose that $f$ and $g$ are real-valued functions on $X$ with $g < f$, such that $g$ is cpc (resp. csc), $f$ is csc (resp. cpc) and $f - g$ is csc (resp. cpc). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{ x \in X : (f - g)(x) \leq 3^{-n+1} \}.$$ 

Since $f - g$ is csc (resp. cpe), hence $A(f - g, 3^{-n+1})$ is a $\text{semi}$–open (resp. preopen) subset of $X$. Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of $\text{semi}$–open (resp. preopen) subsets of $X$ and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preclosed (resp. $\text{semi}$–closed) subsets of $X$ such that $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But by Lemma 3.2, the pair $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of $\text{semi}$–open (resp. preopen) and preopen (resp. $\text{semi}$–open) subsets of $X$ can be completely separated by contra-$\gamma$–continuous functions. Hence by Theorem 2.2, there exists a contra-$\gamma$–continuous function $h$ defined on $X$ such that $g < h < f$, i.e., $X$ has the weakly $c\gamma$–insertion property for $(\text{cpc, csc})$ (resp. $(\text{csc, cpc})$).$\blacksquare$

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