Toroidal free oscillations of a viscous liquid spherical shell with mixed boundary conditions

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Abstract. The problem of toroidal eigenmodes of free oscillations of a viscous fluid spherical shell is solved under mixed boundary conditions. These conditions are a combination of free surface conditions and solid boundary conditions. Such an artificial combination is used sometimes in the numerical simulation of geodynamo problem. The scheme for calculating the eigenmodes is described. It is shown that the use of mixed conditions leads to the disappearance of the first eigenmode with the smallest eigenvalue. A possible reason for this is the physical incorrectness of the mixed boundary conditions.

1. Introduction

The spherical shell of a viscous liquid is one of the most important objects of study for mathematical geophysics. Examples of such shells are the liquid core and the mantle of the Earth. Convective flows in the first of them generate a geomagnetic field, and motions in the second determine plate tectonics. It is the motions of tectonic plates that form the global seismic process on the Earth.

When studying the motions of continuous media, special attention is always paid to free oscillations, which are basic motions from the physical and mathematical points of view. The problem of free oscillations of spherical shells in general has long been solved, for example [1]. However, the spectrum of eigenvalues and the parameters of eigenmodes significantly depend on the boundary conditions used.

For the liquid core of the Earth and the mantle, it is most natural to consider the problem with adhesion boundary conditions, since solid boundaries are taken. However, in numerical simulation in the geodynamo problem, mixed boundary conditions are sometimes considered, for example [2]. They are a combination of adhesion conditions and free surface conditions. This is due to the fact that under the conditions of adhesion, the generation of the model field occurs only in the Ekman layers, and with mixed layers, the entire liquid core “works”.

These mixed conditions look like this. For the radial velocity component, a zero condition is set in order to fix the shell boundary. In addition, zero values of the stress tensor components $\sigma_{\phi\phi}$ and $\sigma_{\phi\phi}$ are required. These are the conditions for a free surface.

This combination of conditions is artificial and physically difficult to interpret. It can be expected that the “non-physicality” of these boundary conditions should somehow manifest itself in the solution of the problem of natural oscillations.
In this paper, we consider the solution to this problem, confining ourselves to velocity toroidal eigenmodes.

2. General form of toroidal eigenmodes

The mathematical formulation of the problem of free oscillations looks like this. A nonzero solution \((v, p)\) to the spectral problem

\[
\mu v + \Delta v - \nabla p = 0, \quad \nabla v = 0
\]

(1)
is sought. Here \(\mu\) - is the eigenvalue, \(v\) - is the velocity eigenmode of the, and \(p\) - is the pressure eigenmode. Equations (1) are closed by the necessary boundary conditions.

It is clear that free oscillations of a viscous fluid must decay, so only eigenvalues with a positive real part make sense. Moreover, the operator of problem (1) is Hermitian, that is, the eigenvalues are real. Finally, we can say that we are interested in real positive \(\mu\).

It is known that the components \(\sigma_{r\theta}\) and \(\sigma_{r\phi}\) of the stress tensor in spherical coordinates have the form:

\[
\sigma_{r\theta} = \eta \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial v_{\phi}}{\partial r} - \frac{v_{\theta}}{r} \right) \quad \text{and} \quad \sigma_{r\phi} = \eta \left( \frac{1}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi} - \frac{\partial v_{r}}{\partial \phi} - \frac{v_{\phi}}{r} \right),
\]

where \(\eta\) - is the viscosity of the medium. Therefore, we will use at the boundaries \(r = r_1\) and \(r = r_0 = 1\) the following boundary conditions:

\[
v_r = 0, \quad \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial v_{\phi}}{\partial r} - \frac{v_{\theta}}{r} = 0, \quad \frac{\partial v_{r}}{\partial \phi} + \frac{1}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi} - \frac{v_{\phi}}{r} = 0.
\]

(2)

We assume, that the length scale is the shell outer radius \(r_0\) and problem (1-2) is written in dimensionless form.

Equation (1) is solved separately in the subspaces of toroidal and poloidal fields. It is known that the general solution, the expression for toroidal modes has the form [1]:

\[
t_{nm} = \nabla \times \left( R_n(r) \cdot Y_n^m(\theta, \phi) \cdot \mathbf{r} \right) = R_n(r) \frac{1}{\sin \theta} \frac{\partial Y_n^m}{\partial \phi} \mathbf{e}_\theta - R_n(r) \frac{\partial Y_n^m}{\partial \theta} \mathbf{e}_\phi,
\]

(3)

where \(Y_n^m(\theta, \phi)\) - spherical functions, and \(R_n(r)\) expressed in terms of spherical Bessel functions \(j_n(\cdot)\) and \(y_n(\cdot)\) by the formula:

\[
R_n(r) = A_n j_n(\sqrt{nr}) + B_n y_n(\sqrt{nr}).
\]

(4)

In what follows, we will also assume that the modes \(t_{nm}\) are normalized in the following sense [3]:

\[
\int_{r_1}^{r_0} \int_{\theta}^{2\pi} \int_{\phi}^{\pi} t_{nm}^2 \, dr = n(n + 1) \int_{r_1}^{r_0} r^2 R_n^2(r) \, dr = 1.
\]

(5)

This normalization corresponds to the unit kinetic energy of each mode.

The general form of the eigenmodes (3-4) is not related to which particular boundary conditions closes problem (1). It is clear that the specific form of the boundary conditions must determine the equation for the eigenvalues and the magnitude of the coefficients \(A_n\) and \(B_n\).

Next, we describe the calculation of these eigenvalues and coefficients for mixed conditions (2).
3. Eigenmodes parameters for mixed boundary conditions

It is clear that for all modes (3) the first of conditions (2) is fulfilled automatically and does not impose any restrictions. It is easy to see that for the second and third conditions (2) to be satisfied, it is necessary and sufficient that for \( r = r_i \) and \( r = 1 \)

\[
\frac{dR_n(r)}{dr} - \frac{R_n(r)}{r} = 0 .
\]

Then, from formulas (4), we obtain that

\[
A_n \hat{j}_n(r_i, \mu) + B_n \hat{y}_n(r_i, \mu) = 0, \quad A_n \hat{j}_n(1, \mu) + B_n \hat{y}_n(1, \mu) = 0, \quad (6)
\]

where

\[
\hat{j}_n(r, \mu) = \frac{d\phi_n(\sqrt{\mu} r)}{dr} - \frac{j_n(r)}{r}, \quad \hat{y}_n(r, \mu) = \frac{dy_n(\sqrt{\mu} r)}{dr} - \frac{\gamma_n(r)}{r} .
\]

Further, we require the degeneracy of the main matrix of system (6) and obtain the eigenvalue equation

\[
\hat{j}_n(r_i, \mu) \cdot \hat{y}_n(1, \mu) - \hat{y}_n(r_i, \mu) \cdot \hat{j}_n(1, \mu) = 0. \quad (7)
\]

This equation for each \( n \) has a countable infinite set of solutions \( \mu_{kn} \). Then double indexing will be for the coefficients \( A_n = A_{kn} \) and \( B_n = B_{kn} \), as well as for functions \( R_n = R_{kn} \). Finally, you can talk about eigenmodes \( t_{km} = \nabla \times (R_{kn}(r) \cdot Y_{mn}^{\theta \phi}(\theta, \phi, r)) \). The indices \( k = 0, 1, 2, \ldots, \quad n = 1, 2, 3, \ldots, \quad m = -n, \ldots, n \) correspond to the quantization of free oscillations for \( r \), \( \theta \) and \( \phi \) directions.

After calculation \( \mu_{kn} \), you can define \( A_{kn} \) and \( B_{kn} \) first by expressions

\[
A_{kn} = \hat{y}_n(1, \mu_{kn}), \quad B_{kn} = -\hat{j}_n(1, \mu_{kn}),
\]

and then renormalize them by formula (5).

**Table 1.** Eigenvalues \( \mu_{kn} \) and coefficients \( A_{kn} \) and \( B_{kn} \) of eigenmodes \( t_{km} \) for mixed boundary conditions.

| \( k \) | \( n \) | \( \mu_{kn} \) | \( A_{kn} \) | \( B_{kn} \) |
|---|---|---|---|---|
| 0 | 1 | 37.495 | 6.68063 | 2.28433 |
| 1 | 1 | 109.71 | 2.9192 | 12.1948 |
| 2 | 1 | 227.005 | -16.5853 | 7.85202 |
| 3 | 1 | 390.696 | -11.6821 | -21.2548 |
| 4 | 1 | 601.013 | 26.5723 | -14.338 |
| 0 | 2 | 6.18258 | 3.69254 | -0.00811232 |
| 1 | 2 | 50.9135 | 4.49043 | -0.00256117 |
| 2 | 2 | 123.114 | 6.30113 | 3.90986 |
| 3 | 2 | 239.617 | -1.0306 | 10.6656 |
| 4 | 2 | 402.857 | -14.0291 | 1.3956 |
| 0 | 3 | 14.8878 | 3.19044 | -0.0048223 |
| 1 | 3 | 69.8819 | 3.60709 | -0.269795 |
| 2 | 3 | 143.729 | 5.3943 | 0.544838 |
| 3 | 3 | 258.95 | 4.86861 | 5.95692 |
| 4 | 3 | 421.325 | -4.80533 | 8.8434 |
| 0 | 4 | 25.9335 | 2.99077 | -0.00183279 |
| 1 | 4 | 92.9942 | 3.23314 | -0.191585 |
| 2 | 4 | 171.623 | 4.37545 | -0.448992 |
| 3 | 4 | 285.503 | 5.72524 | 2.0037 |
| 4 | 4 | 446.399 | 2.26111 | 7.55521 |
Note that the expressions themselves for the left-hand part of equations (7) and integrals (5) are very complex and their error-free compilation is not easy. Therefore, we used the Maple package both for the automatic generation of expressions and for the numerical solution of equations (7) and the calculation of integrals (5).

Let us now consider the calculation results. In the calculations we used \( r_i = 0.35 \), which corresponds to the aspect ratio for the Earth’s liquid core.

Table 1 shows the values of the parameters of some modes.

Figure 1 shows graphs of functions \( R_{kn}(r) \) for \( n = 1, 2, 3 \) and \( k = 0, 1, 2, 3, 4 \). There is a noticeable difference in the structure of the functions for \( n = 1 \) (figure 1 (a)) and \( n = 2, 3 \) (figure 1 (b, c)). For functions \( R_{k2}(r) \) and \( R_{k3}(r) \), the number of zero points between the shell boundaries is the same as the index \( k \) value. A similar structure is observed with others \( n > 1 \). Note that the same structure is possessed \( R_{kn}(r) \) for any \( k \) and \( n \) for eigenmodes when physically correct adhesion conditions are used [3,4].

But the functions in figure 1 (a) the number of zero points is one more than the index \( k \). One gets the impression that one of its eigenmodes has disappeared from the solution. In the author's opinion, the reason for this “disappearance” is in the physical incorrectness of the mixed boundary conditions.

4. Conclusion
In this paper, we considered the solution to the problem of free oscillations of a liquid spherical shell using mixed boundary conditions. These conditions artificially combine the free surface conditions for the stress tensor components \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) and the non-penetration conditions for the radial velocity component. Such artificial conditions are sometimes used in direct numerical simulation of the geodynamo problem. From a physical point of view, the conditions are incorrect, since they combine completely different types of boundaries.
The idea of the work was to see how this incorrectness manifests itself in the problem of free oscillations – basic motions, that have an obvious physical meaning. It turned out that there is a loss of the eigenmode with the smallest eigenvalue, which should decay longer than all others. The result obtained shows that the use of spectral methods in solving problems on the dynamics of shells under mixed boundary conditions can lead to severe errors.

Acknowledgements
The work was performed within the framework of the Vitus Bering Kamchatka State University research project AAAA-A19-11907290002-9 “Natural disasters in Kamchatka – earthquakes and volcanic eruptions (monitoring, forecast, study, psychological support of the population)”.

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