A generalization of Cauchy-Khinchin-van Dam inequality

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Abstract: We first give an alternative proof of a theorem originally presented by E. R. van Dam. Then we show a generalization of the van Dam matrix inequality.

1. Introduction

D. de Caen (1998) gave an upper bound on the sum of squares of degrees in a graph by considering some positive semidefinite quadratic form related to the line graph of the complete graph. Following de Caen’s idea, van Dam (1998) gave a matrix inequality, which generalizes the Cauchy-Schwarz inequality for vectors, and Khinchin’s inequality for zero-one matrices. In Section 2, we first present a different proof of Theorem 1 of van Dam (1998). In Section 3, we give the main result of this paper, a generalization of the van Dam matrix inequality. Then, in Section 4, we compare with the result of Yan (2011).

2. An alternative proof of van Dam’s theorem

Theorem 1 (van Dam, 1998, Theorem 1) Let \( A = (a_{ij}) \) be a real \( m \times n \) matrix. Then

\[
m \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \right)^2 + n \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} \right)^2 \leq \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right)^2 + mn \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2.
\]

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PUBLIC INTEREST STATEMENT

D. de Caen had gave an upper bound on the sum of squares of degrees in a graph by considering some positive semidefinite quadratic form related to the line graph of the complete graph. Following de Caens idea, E. R. van Dam gave a matrix inequality, which generalizes the Cauchy-Schwarz inequality for vectors, and Khinchins inequality for zero-one matrices. In this paper, we present a different proof of van Dam’s inequality and then give a generalization. Finally, we compare with the generalization of Zizong Yan (2011). We hope that the result can be used to the investigation of quantum entanglement.

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The equality holds if and only if \( a_j = b_j + c_j \) for some real \( b_j \) and \( c_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \).

von Dam (cf. 1998) proved the theorem using the positivity of the matrix \( nI_n - J_m \), where \( I_n \) is the identity matrix of order \( n \) and \( J_n \) is the square matrix of order \( n \) with all elements are equal to 1.

It is easy to see that the matrix \( nI_n - J_n \) has eigenvalue 0 with multiplicity 1 and \( n \) with multiplicity \( n - 1 \). By \( a_n = \frac{1}{\sqrt{n}} (1, 1, \ldots, 1) \), we denote the eigenvector of \( nI_n - J_n \) associated to the eigenvalue 0, and \( \beta^1, \beta^{2n}, \ldots, \beta^{n-1} \) denote the orthonormal basis of eigenspace associated to the eigenvalue \( n \). For positive integers \( m \) and \( n \), we can obtain eigenvalues of the matrix \( (mI_m - J_m) \otimes (nI_n - J_n) \) are 0 with multiplicity \( m + n - 1 \) and \( mn \) with multiplicity \((m - 1)(n - 1)\). Moreover, the vector set \( \{ a_m \otimes a_n, a_m \otimes \beta^1, \ldots, a_m \otimes \beta^{n-1}, \beta^1 \otimes a_n, \ldots, \beta^{m-1} \otimes a_m \} \) is an orthonormal basis of eigenspace of \((mI_m - J_m) \otimes (nI_n - J_n)\) with eigenvalue 0. Given an arbitrary real \( m \times n \) matrix \( A = (a_{ij}) \), we have an \( mn \) column vector defined as

\[
\text{Vec}(A) = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn})^\top.
\]

From the positive-semidefinite property of \((nI_n - J_n) \otimes (mI_m - J_m)\) we have that

\[
\text{Vec}(A)^\top (mI_m - J_m) \otimes (nI_n - J_n) \text{Vec}(A) \geq 0.
\]

Hence

\[
mn \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \right)^2 - n \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} \right)^2 + \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right)^2.
\]

\[
= \text{Vec}(A)^\top (mnI_m \otimes I_n - mI_m \otimes J_n - nJ_m \otimes I_n + J_m \otimes J_n) \text{Vec}(A)
\]

\[
= \text{Vec}(A)^\top [(nI_n - J_n) \otimes (mI_m - J_m)] \text{Vec}(A) \geq 0.
\]

This proves (1).

Also, from the positive-semidefinite property of \((nI_n - J_n) \otimes (mI_m - J_m)\), it follows that \( \text{Vec}(A)^\top [(mI_m - J_m) \otimes (nI_n - J_n)] \text{Vec}(A) = 0 \) if and only if \( [(mI_m - J_m) \otimes (nI_n - J_n)] \text{Vec}(A) = 0 \), i.e.

\[
\text{Vec}(A) = k_0 a_m \otimes a_n + k_1 a_m \otimes \beta^1 + \cdots + k_{n-1} a_m \otimes \beta^{n-1} + l_0 \beta^1 \otimes a_n + \cdots + l_{m-1} \beta^{m-1} \otimes a_n,
\]

for some \( k_0, k_1, \ldots, k_{n-1}, l_0, \ldots, l_{m-1} \). Thus

\[
\text{Vec}(A)^\top [(mI_m - J_m) \otimes (nI_n - J_n)] \text{Vec}(A) = 0
\]

if and only if

\[
a_j = k_0 + k_1 \beta^1 + \cdots + k_{n-1} \beta^{n-1} + l_0 \beta^1 + \cdots + l_{m-1} \beta^{m-1},
\]

where

\[
b_j = l_1 \beta^1 + \cdots + l_{m-1} \beta^{m-1}, \quad c_j = k_0 + k_1 \beta^1 + \cdots + k_{n-1} \beta^{n-1}.
\]
3. Main result

We fix some notations which will be used in the following:

\[ [s] = \{1, 2, \ldots, s\}, \]
\[ \Gamma_{k,n} = \{\alpha = (i_1, \ldots, i_k), 1 \leq i_1 < i_2 < \cdots < i_k \leq n, i = 1, 2, \ldots, k\}, \]
\[ n_{a_i} = n_{i_1} \cdots n_{i_k} \quad \text{for} \quad \{i_1, i_2, \ldots, i_k\} = \alpha \in \Gamma_{k,n}, \quad \text{if} \quad k = 0, \quad \text{fix} \quad n_{a_i} = 1. \]

Since \( (n_1 I_{n_1} - J_{n_1}) \otimes (n_2 I_{n_2} - J_{n_2}) \otimes \cdots \otimes (n_s I_{n_s} - J_{n_s}) \), \( s \in \mathbb{Z}^+ \), \( n_i \in \mathbb{Z}^+ \), \( i = 1, 2, \ldots, s \), is positive, we have

**Theorem 2** Let \( A = (a_{i_1,\ldots,i_s}) \) be a real \( n_1 \times n_2 \times \cdots \times n_s \) matrix. Then

\[
\sum_{i=0}^s \sum_{a_{i_1,\ldots,i_s}} (-1)^{n_{a_i}} \left( \sum_{i,j,k,l} a_{i_1,i_2} \right)^2 \geq 0. \quad (2)
\]

The equality holds if and only if \( \text{Vec}(A) \) is in the kernel of \((n_1 I_{n_1} - J_{n_1}) \otimes (n_2 I_{n_2} - J_{n_2}) \otimes \cdots \otimes (n_s I_{n_s} - J_{n_s}) \).

Here, \( a_{i_1,\ldots,i_s} \) is the \( i_s \) component of \( \text{Vec}(A) \).

**Notation:** We have an orthonormal basis for the eigenspace with eigenvalue 0 (i.e. kernel) of \((n_1 I_{n_1} - J_{n_1}) \otimes (n_2 I_{n_2} - J_{n_2}) \otimes \cdots \otimes (n_s I_{n_s} - J_{n_s}) \)

\[ \mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \cdots \otimes \mathcal{Y}_s, \]

where \( \mathcal{Y}_i \in \{\alpha_i\} \cup \{\beta^{n_1}, \beta^{2n_1}, \ldots, \beta^{n_i-1} n_i\} \) and at least one \( \mathcal{Y}_i \in \{\alpha_i\} \).

For example, when \( s = 3 \), we have

\[
\Omega = mn l \sum_{i,j,k} a_{i,j,k}^2 + m \sum_{i} \left( \sum_{j} a_{i,j} \right)^2 + n \sum_{j} \left( \sum_{i} a_{i,j} \right)^2 + l \sum_{k} \left( \sum_{j} a_{j,k} \right)^2
- mn \sum_{j} \left( \sum_{j} a_{j,k} \right)^2 - ml \sum_{i} \left( \sum_{j} a_{i,j} \right)^2 - nl \sum_{j} \left( \sum_{i} a_{i,j} \right)^2 - \left( \sum_{i,j,k} a_{i,j,k} \right)^2 \geq 0, \quad (3)
\]

where \( n_1 = m, n_2 = n, n_3 = l. \)

4. The case of \( s = 3 \)

For the case of \( s = 3 \), Yan (2011) present another inequality.

**Theorem 3** (Yan, 2011, Theorem 1.1) Let \( A = (a_{ijk}) \) be a real \( m \times n \times l \) and \( \alpha, \beta, \gamma \) are real numbers. Then

\[
\Pi = mn l \sum_{i,j,k} a_{i,j,k}^2 + 2\beta \gamma m \sum_{j} \left( \sum_{j} a_{i,j} \right)^2 + 2\gamma m \sum_{i} \left( \sum_{j} a_{i,j} \right)^2
+ 2\alpha \gamma \sum_{k} \left( \sum_{j} a_{j,k} \right)^2
- (2\beta - \gamma^2)m \sum_{i} \left( \sum_{j} a_{i,j} \right)^2 - nl(2\alpha - \beta) \sum_{j} \left( \sum_{i} a_{i,j} \right)^2
- (\alpha + \beta + \gamma - 1)^2 \sum_{i,j,k} a_{i,j,k}^2 \geq 0. \quad (4)
\]

Next, we will have a comparison between these two results.
\[ \Pi - \Omega = (2\beta\gamma - 1)m \sum_i \left( \sum_j a_{ijk} \right)^2 + (2\alpha\gamma - 1)n \sum_j \left( \sum_k a_{ijk} \right)^2 \\
+ (2\alpha\beta - 1)n \sum_k \left( \sum_i a_{ijk} \right)^2 + (1 - \gamma)^2 mn \sum_k \left( \sum_i a_{ijk} \right)^2 \\
+ (1 - \beta)^2 mn \sum_i \left( \sum_k a_{ijk} \right)^2 + (1 - \alpha)^2 nl \sum_j \left( \sum_k a_{ijk} \right)^2 \\
- (\alpha + \beta + \gamma)(\alpha + \beta + \gamma - 2) \left( \sum_{i,j,k} a_{ijk} \right)^2. \]

Due to

\[ (\alpha + \beta + \gamma)(\alpha + \beta + \gamma - 2) = (1 - \alpha)^2 + (1 - \beta)^2 + (1 - \gamma)^2 + (2\alpha\beta - 1) + (2\alpha\gamma - 1) + (2\beta\gamma - 1), \]

we have that

\[ \Pi - \Omega = (2\beta\gamma - 1) \left[ m \sum_i \left( \sum_j a_{ijk} \right)^2 - \left( \sum_{i,j,k} a_{ijk} \right)^2 \right] + (2\alpha\gamma - 1) \left[ n \sum_j \left( \sum_k a_{ijk} \right)^2 - \left( \sum_{i,j,k} a_{ijk} \right)^2 \right] \\
+ (2\alpha\beta - 1) \left[ l \sum_k \left( \sum_i a_{ijk} \right)^2 - \left( \sum_{i,j,k} a_{ijk} \right)^2 \right] + (1 - \gamma)^2 mn \left[ \sum_k \left( \sum_i a_{ijk} \right)^2 - \left( \sum_{i,j,k} a_{ijk} \right)^2 \right] \\
+ (1 - \beta)^2 mn \left[ \sum_i \left( \sum_k a_{ijk} \right)^2 - \left( \sum_{i,j,k} a_{ijk} \right)^2 \right] + (1 - \alpha)^2 nl \left[ \sum_j \left( \sum_k a_{ijk} \right)^2 - \left( \sum_{i,j,k} a_{ijk} \right)^2 \right] \\
= \text{Vec}(A)^T \text{Vec}(A), \]
where

\[ \text{Vec}(A)^t = (a_{111}, \ldots, a_{11l}, a_{121}, \ldots, a_{12l}, \ldots, a_{ml1}, \ldots, a_{mln}) \]
and

\[ T = (2\beta\gamma - 1)(mI_m \otimes J_n \otimes J_l - J_m \otimes J_n \otimes J_l) + (2\alpha\gamma - 1)(nJ_n \otimes I_l \otimes J_m \otimes J_l) \\
+ (2\alpha\beta - 1)(l \otimes J_m \otimes I_n \otimes J_l) + (1 - \gamma)^2 (mnI_m \otimes I_l \otimes J_m \otimes J_l) \\
+ (1 - \beta)^2 (mlJ_m \otimes J_n \otimes J_l - J_m \otimes J_l \otimes J_m) + (1 - \alpha)^2 (mlI_m \otimes I_n \otimes I_l \otimes J_m \otimes J_l). \]

For the matrix \( T \), we have \( mnl \) orthogonal eigenvectors as

\[ \gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_s, \]
where \( \gamma_i \in \{ a_{ijk} \} \cup \{ \beta^{1n_1}, \beta^{2n_2}, \ldots, \beta^{n_{-1}n} \} \). By a direct calculation we can find all different eigenvalues for \( T \) are 0, \( (1 - \alpha)^2 mnI \), \( (1 - \beta)^2 mnl \), \( (1 - \gamma)^2 mnl \), \( (\alpha + \beta - 1)^2 mnl \), \( (\alpha + \gamma - 1)^2 \), \( (\beta + \gamma - 1)^2 \). So we can get that \( T \) is semipositive, i.e. \( \Pi - \Omega \geq 0 \) for all \( A = (a_{ijk}) \in \mathcal{R}^{mn \times ml} \) and \( \alpha, \beta, \gamma \in \mathcal{R} \). Then from (3) we can obtain the result (4). Or else, there is an advantage that (3) can be easily generalized.

Another comparison in the inequality (4) is that if \( \alpha = \beta = l = 1, \gamma = 0 \), then the inequality (1) can be obtained. But if \( l = 1 \) in (3), we can obtain an equality.
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Author’s contributions
Sun and Zhao involved in drafting the manuscript. Wang revised it critically and gave final approval of the version to be published.

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