THE STRONG ASYMPTOTIC ANALYSIS OF THE FIRST KIND
ORTHOGONAL TRIGONOMETRIC POLYNOMIAL

HAN HUILI, HUA LIU, AND WANG YUFENG

ABSTRACT. In this paper we study the asymptotic analysis of the orthogonal trigonometric
polynomials by the Riemann-Hilbert problem for the periodic analytic functions.

1. ORTHOGONAL TRIGONOMETRIC POLYNOMIAL

If a nonnegative locally-integrable function $w(x)$ defined on the real axis $\mathbb{R}$ satisfying
\[
 w(x + 2\pi) = w(x) \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad \int_0^{2\pi} w(x) dx > 0,
\]
such a function $w$ is called a $2\pi$—periodic weight. And the real inner product is defined by
\[
 \langle f, g \rangle = \int_0^{2\pi} f(x) g(x) w(x) dx,
\]
which induces the norm
\[
 \| f \|_2 = \int_0^{2\pi} |f(x)|^2 w(x) dx.
\]

By use of the inner product (1.2), the Gram-Schmidt orthogonalization of the following
ordered trigonometric monomials
\[
 1, \cos t, \sin t, \cdots, \cos nt, \sin nt, \cdots
\]
leads to the system of orthonormal trigonometric polynomials $\{\omega_n, n = 0, 1, 2, \cdots \}$, where
\[
 \omega_{2n}(x) = \alpha_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_0 \quad \text{with} \quad \alpha_n > 0
\]
and
\[
 \omega_{2n+1}(x) = \beta_n \sin nx + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_0 \quad \text{with} \quad \beta_n > 0.
\]
The trigonometric polynomial $\omega_{2n}$ defined by (1.5) is usually called the first kind orthogonal
trigonometric polynomial (OTP) and $\alpha_n$ is said to be its leading coefficient. And $\omega_{2n+1}$
defined by (1.6) is similarly called the second kind OTP and $\beta_n$ is the corresponding leading
coefficient.

2000 Mathematics Subject Classification. Primary: 42B20, 43A65, 44A15; Secondary: 22D30, 30E25,
30G35, 32A35.

Key words and phrases. periodically Riemann-Hilbert problem, trigonometric function, asymptotic
behaviour.
For convenience, we define two real vector spaces according to the number of the base

\[ T_{2n}(\mathbb{R}) = \left\{ a_n \cos nx + \sum_{k=0}^{n-1} (a_k \cos kx + b_k \sin kx) + a_0, a_j, b_j \in \mathbb{R} \right\}, \quad (1.7) \]

\[ T_{2n+1}(\mathbb{R}) = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) + a_0, a_j, b_j \in \mathbb{R} \right\}. \quad (1.8) \]

Therefore, one has

\[ \begin{cases} 
\langle \omega_{2n}, \cos kx \rangle = 0, & k = 0, 1, 2, \ldots, n-1, \\
\langle \omega_{2n}, \sin kx \rangle = 0, & k = 1, 2, \ldots, n-1, \\
\alpha_n^2 = \langle \omega_{2n}, \omega_{2n} \rangle = 1,
\end{cases} \quad (1.9) \]

where \( \omega_{2n} \) is the first kind OTP defined by (1.5). Conversely, if \( \omega_{2n} \) satisfies (1.9), one easily knows that it is just the first kind OTP defined by (1.5). Further, the first two equalities in (1.9) are equivalent to

\[ \omega_{2n} \perp T_{2n-1}(\mathbb{R}), \quad (1.10) \]

where \( T_{2n-1}(\mathbb{R}) \) is defined by (1.8).

Further, we define the first kind monic OTP

\[ \omega_{2n}(x) = \frac{\omega_{2n}(x)}{\alpha_n}, \quad (1.11) \]

where \( \alpha_n \) is the corresponding leading coefficient.

A \( 2\pi \)-periodic weight \( w \) is called strictly-positive analytic periodic weight if \( w \in A(\mathbb{R}) \) and \( w(x) > 0, x \in \mathbb{R} \). In this section, we always assume that \( w \) is strictly-positive analytic periodic weight. Thus, there exists \( \rho > 0 \) such that

\[ \begin{cases} 
w \in A(\mathbb{R}), \\
w(z) \neq 0, & z \in \mathbb{R},
\end{cases} \quad (1.12) \]

where \( A(\mathbb{R}) \) is the set of analytic functions on the rectangle domain defined by

\[ \square_{a,b} = \left\{ z; \text{Re}z \in (0, 2\pi), \text{Im}z \in (a, b) \right\} \quad \text{for} \ a < b. \quad (1.13) \]

2. Characterization of the First Kind OTP

In this section, we will give the Fokas-Its-Kitaev characterization of the first kind OTP. Now, we state the homogeneous periodic Riemann-Hilbert problem: Find a sectionally-analytic \( 2\pi \)-periodic function \( Y(z) \) satisfying the following conditions

\[ \begin{cases} 
Y^+(x) = Y^-(x) \begin{pmatrix} 1 & e^{-inx}w(x) \\ 0 & 1 \end{pmatrix}, & x \in [0, 2\pi], \\
Y(z)\Xi_1(z) \to I, & z \to +\infty i, \\
Y(z)\Xi_2(z) \to I, & z \to -\infty i,
\end{cases} \quad (2.1) \]
where $I$ is the $2 \times 2$ identity matrix and

$$
\Xi_1(z) = \begin{pmatrix}
\frac{1}{\cos n z} & 0 \\
0 & e^{iz}
\end{pmatrix}, \quad \Xi_2(z) = \begin{pmatrix}
\frac{1}{\cos n z} & 0 \\
0 & e^{i(2n-1)z}
\end{pmatrix}.
$$

Let

$$
Y(z) = \begin{pmatrix}
Y_{1,1}(z) \\
Y_{2,1}(z)
\end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R},
$$

and the periodic matrix Riemann-Hilbert problem (2.1) is equivalent to the system of the following four scalar Riemann-Hilbert problems

\begin{align*}
\begin{cases}
Y_{1,1}^+(x) = Y_{1,1}^-(x), & x \in [0, 2\pi], \\
Y_{1,1}(z) & \rightarrow 1, & z \rightarrow +\infty i,
\end{cases} \\
\begin{cases}
Y_{1,2}^+(x) = Y_{1,2}^-(x) + e^{-inx}w(x)Y_{1,1}^-(x), & x \in [0, 2\pi], \\
e^{iz}Y_{1,2}(z) & \rightarrow 0, & z \rightarrow +\infty i,
\end{cases} \\
\begin{cases}
Y_{2,1}^+(x) = Y_{2,1}^-(x), & x \in [0, 2\pi], \\
Y_{2,1}(z) & \rightarrow 0, & z \rightarrow +\infty i,
\end{cases} \\
\begin{cases}
Y_{2,2}^+(x) = Y_{2,2}^-(x) + e^{-inx}w(x)Y_{2,1}^-(x), & x \in [0, 2\pi], \\
e^{iz}Y_{2,2}(z) & \rightarrow 1, & z \rightarrow +\infty i,
\end{cases}
\end{align*}

and

\begin{align*}
\begin{cases}
Y_{2,1}(z) & \rightarrow 1, & z \rightarrow -\infty i,
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
Y_{2,2}^+(x) = Y_{2,2}^-(x) + e^{-inx}w(x)Y_{2,1}^-(x), & x \in [0, 2\pi], \\
e^{i(2n-1)z}Y_{2,2}(z) & \rightarrow 1, & z \rightarrow -\infty i.
\end{cases}
\end{align*}

In order to deal with these periodic Riemann-Hilbert problems, we need introduce the following periodic Cauchy-type integral operator.

$$
C_\nu[f](z) = \frac{1}{4\pi i} \int_0^{2\pi} f(t) \cot \frac{t - z}{2} w(t) dt, \quad z \in \mathbb{C} \setminus \mathbb{R}
$$

with $f \in H([0, 2\pi])$. For convenience, we similarly define two complex vector spaces of trigonometric polynomials

$$
T_{2n}(\mathbb{C}) = \left\{ a_n \cos nx + \sum_{k=0}^{n-1} (a_k \cos kx + b_k \sin kx) + a_0, a_j, b_j \in \mathbb{C} \right\},
$$

$$
T_{2n+1}(\mathbb{C}) = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) + a_0, a_j, b_j \in \mathbb{C} \right\}.
$$

Clearly, $T_{2n}(\mathbb{R}) \subset T_{2n}(\mathbb{C})$ and $T_{2n+1}(\mathbb{R}) \subset T_{2n+1}(\mathbb{C})$. 
Theorem 2.1  The matrix Riemann-Hilbert problem (2.1) has the unique solution expressed by

\[
Y(z) = \begin{pmatrix} \varpi_{2n}(z) & e^{-inz} C_w[\varpi_{2n}](z) \\ a_n \varpi_{2n-1}(z) & a_n e^{-inz} C_w[\varpi_{2n-1}](z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R}
\]

(2.11)

with

\[
a_n = \frac{2\pi i}{\beta_{n-1}^2},
\]

(2.12)

where \(\varpi_{2n}, \varpi_{2n-1}\) are the monic OLPs, and \(\beta_{n-1}\) is the leading coefficients of OTPs defined by (1.6).

Proof: First, we will solve scalar Riemann-Hilbert problems (2.4) and (2.5). By Theorem 4.1 in [6], the solution of Riemann-Hilbert problems (2.4) can be expressed by

\[
Y_{1,1}(z) = \sum_{k=0}^{n} (a_k \cos k\tau + b_k \sin k\tau)
\]

(2.13)

with \(a_j, b_j \in \mathbb{C}\) for \(j = 0, 1, \ldots, n\). This leads to

\[
\lim_{z \to \pm \infty} \frac{Y_{1,1}(z)}{\cos nz} = a_n \pm b_n i,
\]

(2.14)

by the growth conditions in (2.4). Therefore, one has

\[
Y_{1,1}(z) = t_{2n}(z) \in T_{2n}(\mathbb{C}),
\]

(2.15)

and the leading coefficient of \(t_{2n}(z)\) is 1.

Let \(\tilde{Y}_{1,2}(z)\) satisfy

\[
\begin{aligned}
\tilde{Y}_{1,2}^+(x) &= \tilde{Y}_{1,2}(x) + t_{2n}(x)w(x), \quad x \in [0, 2\pi], \\
\tilde{Y}_{1,2}(\pm \infty i) &= 0.
\end{aligned}
\]

(2.16)

By Theorem 4.1 in [6],

\[
\tilde{Y}_{1,2}(z) = C_w[t_{2n}](z) = \frac{1}{4\pi i} \int_{0}^{2\pi} t_{2n}(\tau) \cot \frac{\tau - z}{2} w(\tau) d\tau, \quad z \in \mathbb{C} \setminus \mathbb{R}
\]

(2.17)

is the unique solution of (2.16) if it is solvable. Therefore, if (2.5) is solvable, one has

\[
Y_{1,2}(z) = e^{-inz} \tilde{Y}_{1,2}(z) = \frac{e^{-inz}}{4\pi i} \int_{0}^{2\pi} t_{n}(\tau) \cot \frac{\tau - z}{2} w(\tau) d\tau, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

(2.18)

Observe

\[
\cot \frac{\tau - z}{2} = \begin{cases} 
  i \left( 1 + 2 \sum_{k=1}^{\infty} e^{ik\tau} e^{-i\tau k} \right), & \text{Im} z > 0 \\
  -i \left( 1 + 2 \sum_{k=1}^{\infty} e^{-ik\tau} e^{i\tau k} \right), & \text{Im} z < 0
\end{cases}
\]

for \(\tau \in [0, 2\pi]\).
Now, inserting (2.19) into (2.18), one has the following Fourier expansion

\[
Y_{1,2}(z) = \begin{cases} \\
\frac{1}{4\pi i} \int_{0}^{2\pi} t_{2n}(\tau)w(\tau) d\tau e^{-inz} + \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{0}^{2\pi} t_{2n}(\tau)e^{-ik\tau}w(\tau)d\tau e^{-i(n-k)z}, & \text{Im} z > 0, \\
-\frac{1}{4\pi i} \int_{0}^{2\pi} t_{2n}(\tau)w(\tau) d\tau e^{-inz} - \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{0}^{2\pi} t_{2n}(\tau)e^{ik\tau}w(\tau)d\tau e^{-i(n+k)z}, & \text{Im} z < 0.
\end{cases}
\] (2.20)

Combining the growth conditions in (2.5) with (2.20), one easily knows, if and only if

\[
\begin{cases}
\int_{0}^{2\pi} t_{2n}(\tau)e^{-ik\tau}w(\tau)d\tau = 0, \\
\int_{0}^{2\pi} t_{2n}(\tau)e^{ik\tau}w(\tau)d\tau = 0,
\end{cases}
\] (2.21)

the unique solution of (2.5) can be expressed by (2.18) or (2.20). Obviously, (2.21) is equivalent to

\[
\begin{aligned}
\langle t_{2n}, \cos k\tau \rangle &= 0, \\
\langle t_{2n}, \sin k\tau \rangle &= 0,
\end{aligned}
\] (2.22)

which in turn implies

\[Y_{1,1}(z) = t_{2n}(z) = \omega_{2n}(z),\] (2.23)

which is the first kind monic OLPs defined by (1.5). Then the solution of Riemann-Hilbert problem (2.5) must be rewritten as

\[Y_{1,2}(z) = \frac{e^{-inz}}{4\pi i} \int_{0}^{2\pi} \omega_{2n}(\tau) \cot \frac{\tau - z}{2} w(\tau) d\tau, \ z \in \mathbb{C} \setminus \mathbb{R},\] (2.24)

or say

\[Y_{1,2}(z) = \begin{cases} \\
\sum_{k=n}^{\infty} \frac{1}{2\pi i} \int_{0}^{2\pi} t_{2n}(\tau)e^{-ik\tau}w(\tau)d\tau e^{-i(n-k)z}, & \text{Im} z > 0, \\
-\sum_{k=n}^{\infty} \frac{1}{2\pi i} \int_{0}^{2\pi} t_{2n}(\tau)e^{ik\tau}w(\tau)d\tau e^{-i(n+k)z}, & \text{Im} z < 0.
\end{cases}
\] (2.25)

On the contrary, reversing from (2.20) to (2.16), we easily know that (2.24) is exactly the unique solution of (2.5).

Secondly, we will solve Riemann-Hilbert problems (2.6) and (2.7). Similarly to the preceding discussion, the solution of Riemann-Hilbert problem (2.6) can be explicitly expressed by

\[Y_{2,1}(z) = t_{2n-1}(z) \in T_{2n-1}(\mathbb{C}).\] (2.26)

If (2.7) is solvable, let \(\tilde{Y}_{2,2}(z) = e^{inz}Y_{2,2}(z)\). Then we have

\[
\begin{cases}
\tilde{Y}_{2,2}^+(x) = \tilde{Y}_{2,2}^-(x) + t_{2n-1}(x)w(x), & x \in [0, 2\pi], \\
\tilde{Y}_{2,2}(\pm \infty i) = 0.
\end{cases}
\] (2.27)

The unique solution of Riemann-Hilbert problem (2.27) can be written as

\[
\tilde{Y}_{2,2}(z) = C_w[t_{2n-1}](z) = \frac{1}{4\pi i} \int_{0}^{2\pi} t_{2n-1}(\tau) \cot \frac{\tau - z}{2} w(\tau) d\tau, \ z \in \mathbb{C} \setminus \mathbb{R},
\] (2.28)
which in turn implies
\[
Y_{2,2}(z) = \frac{e^{-inz}}{4\pi i} \int_{0}^{2\pi} t_{2n-1}(\tau) \cot \frac{\tau - z}{2} w(\tau) d\tau, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\] (2.29)

Further, putting (2.19) into (2.29), one has
\[
Y_{2,2}(z) = \begin{cases} 
\frac{1}{4\pi} \int_{0}^{2\pi} t_{2n-1}(\tau) w(\tau) d\tau e^{-inz} + \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} t_{2n-1}(\tau) e^{-ik\tau} w(\tau) d\tau e^{-i(n-k)z}, & \text{Im} z > 0, \\
-\frac{1}{4\pi} \int_{0}^{2\pi} t_{2n-1}(\tau) w(\tau) d\tau e^{-inz} - \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} t_{2n-1}(\tau) e^{ik\tau} w(\tau) d\tau e^{-i(n+k)z}, & \text{Im} z < 0.
\end{cases}
\] (2.30)

Thus, considering two growth conditions in (2.7), one easily knows, if and only if
\[
\begin{cases} 
\int_{0}^{2\pi} t_{2n-1}(\tau) e^{-ik\tau} w(\tau) d\tau = 0, \quad k = 0, 1, 2 \cdots, n - 2, \\
\frac{1}{2\pi} \int_{0}^{2\pi} t_{2n-1}(\tau) e^{-i(n-1)\tau} w(\tau) d\tau = 1
\end{cases}
\] (2.31)

and
\[
\begin{cases} 
\int_{0}^{2\pi} t_{2n-1}(\tau) e^{ik\tau} w(\tau) d\tau = 0, \quad k = 1, 2 \cdots, n - 2, \\
-\frac{1}{2\pi} \int_{0}^{2\pi} t_{2n-1}(\tau) e^{i(n-1)\tau} w(\tau) d\tau = 1,
\end{cases}
\] (2.32)

the unique solution of (2.7) can be expressed by (2.29) or (2.30). Further, (2.31) and (2.32) are equivalent to
\[
\begin{cases} 
\int_{0}^{2\pi} t_{2n-1}(\tau) \cos k\tau w(\tau) d\tau = 0, \quad k = 0, 1, 2 \cdots, n - 1, \\
\int_{0}^{2\pi} t_{2n-1}(\tau) \sin k\tau w(\tau) d\tau = 0, \quad k = 1, 2 \cdots, n - 2, \\
\int_{0}^{2\pi} t_{2n-1}(\tau) \sin(n-1)\tau w(\tau) d\tau = 2\pi i,
\end{cases}
\] (2.33)

which leads to
\[
Y_{2,1}(z) = t_{2n-1}(z) = a_n \varpi_{2n-1}(z),
\] (2.34)

where \(a_n\) is defined by (2.12).

Finally, inserting (2.34) into (2.29), one gets
\[
Y_{2,1}(z) = a_n e^{-inz} C_w[\varpi_{2n-1}](z).
\] (2.35)

Similarly, reversing step by step, we obtain that (2.35) is just the unique solution of (2.7). This completes the proof.
ORTHOGONAL TRIGONOMETRIC POLYNOMIAL

3. THE STEEPEST DESCENT ANALYSIS

In this section, we will carry out an array of transforms

\[ Y \mapsto F \mapsto S \mapsto R, \quad (3.1) \]

and the model Riemann-Hilbert problem is obtained. And the steepest descent analysis bases on those transforms.

3.1. the first transform \( Y \mapsto F \)

Let

\[ \Gamma(z) = \frac{1}{4\pi i} \int_{0}^{2\pi} \ln w(\tau) \cot \frac{\tau - z}{2} d\tau, \quad z \notin \mathbb{R}. \quad (3.2) \]

Clearly,

\[ \Gamma(\pm \infty) = \pm \frac{1}{4\pi} \int_{0}^{2\pi} \ln w(\tau) d\tau. \quad (3.3) \]

We define

\[ \begin{align*}
D^+(z) &= e^{\Gamma(z)-C}, \quad \text{Im} z > 0, \\
D^-(z) &= e^{-\Gamma(z)-C}, \quad \text{Im} z < 0
\end{align*} \quad (3.4) \]

with

\[ C = \frac{1}{4\pi} \int_{0}^{2\pi} \ln w(\tau) d\tau. \quad (3.5) \]

Obviously,

\[ \begin{align*}
D^+ &\in A(\square_{a,+\infty}), \quad D^- \in A(\square_{-\infty,a}), \\
D^+(z + 2\pi) &= D^+(z), \quad D^-(z + 2\pi) = D^-(z), \\
D^+(x)D^-(x) &= e^{-2C} w(x), \quad x \in [0, 2\pi], \\
D^\pm(\pm \infty i) &= 1,
\end{align*} \quad (3.6) \]

where

\[ \begin{align*}
\square_{a,+\infty} &= \{ z, \text{Re} z \in (0, 2\pi), \text{Im} z \in (a, +\infty) \} \quad \text{with} \ a \in \mathbb{R} \\
\square_{-\infty,a} &= \{ z, \text{Re} z \in (0, 2\pi), \text{Im} z \in ( -\infty, a) \}
\end{align*} \quad (3.7) \]

are infinite rectangle domains.

Further, we also define

\[ \mathfrak{D}^+(z) = \begin{cases} 
D^+(z), & \text{Im} z \geq 0, \\
e^{-2C} w(z)[D^-(z)]^{-1}, & \text{Im} z \in (-\rho, 0),
\end{cases} \quad (3.8) \]

\[ \mathfrak{D}^-(z) = \begin{cases} 
D^-(z), & \text{Im} z \leq 0, \\
e^{-2C} w(z)[D^+(z)]^{-1}, & \text{Im} z \in [0, \rho),
\end{cases} \quad (3.9) \]
3.2. The second transform

is the solution of Riemann-Hilbert problem (2.1). And hence, by (1.12) and (3.6),

\[
\begin{aligned}
&\mathcal{D}^+ \in \mathcal{A}(\Box_{-\rho}^{-\infty}), \quad \mathcal{D}^- \in \mathcal{A}(\Box_{-\infty}^{-\rho}), \\
&\mathcal{D}^+(z + 2\pi) = \mathcal{D}^+(z), \quad \mathcal{D}^-(z + 2\pi) = \mathcal{D}^-(z), \\
&\mathcal{D}^+(x) \mathcal{D}^-(x) = e^{-2C} w(x), \quad x \in [0, 2\pi], \\
&\mathcal{D}^+(\pm \infty i) = 1.
\end{aligned}
\] (3.10)

Now, set

\[
\mathbf{U}(z) = \begin{cases}
\left( \begin{array}{cc}
e^{i\pi \mathcal{D}^+(z)} & 0 \\
0 & \frac{1}{\mathcal{D}^+(z)} \\
e^{-i\pi \mathcal{D}^-(z)} & 0 \\
0 & \frac{1}{\mathcal{D}^-(z)}
\end{array} \right), & z \in \Box_{0, \infty}, \\
\left( \begin{array}{cc}
e^{i\pi \mathcal{D}^+(z)} & 0 \\
0 & \frac{1}{\mathcal{D}^+(z)} \\
e^{-i\pi \mathcal{D}^-(z)} & 0 \\
0 & \frac{1}{\mathcal{D}^-(z)}
\end{array} \right), & z \in \Box_{-\infty, 0},
\end{cases}
\] (3.11)

and we define the first transform

\[
\mathbf{F}(z) = \mathbf{Y}(z) \mathbf{U}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\] (3.12)

**Lemma 3.1** If \(\mathbf{Y}\) is the solution of Riemann-Hilbert problem (2.1), then \(\mathbf{F}(z)\) defined by (3.12) is the solution of the following Riemann-Hilbert problem

\[
\begin{aligned}
&\mathbf{F}^+(x) = \mathbf{F}^-(x) \left( \begin{array}{cc}
e^{i2n_x \mathcal{D}^+(x)^2} & 2e^{ix} \\
0 & e^{-i2(n-1)x \mathcal{D}^-(x)^2} \end{array} \right) \quad e^{2C}, \quad x \in [0, 2\pi], \\
&\mathbf{F}(z) = \mathbf{I} + \mathbf{o}(1), \quad z \to +\infty i, \\
&\mathbf{F}(z) = \mathbf{I} + \mathbf{o}(1), \quad z \to -\infty i.
\end{aligned}
\] (3.13)

Conversely, if \(\mathbf{F}(z)\) is the solution of the Riemann-Hilbert problem (3.13), then

\[
\mathbf{Y}(z) = \mathbf{F}(z) \mathbf{U}^{-1}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}
\] (3.14)
is the solution of Riemann-Hilbert problem (2.1).

3.2. The second transform \(\mathbf{F}(z) \longrightarrow \mathbf{S}(z)\)

The coefficient matrix in the boundary condition in (3.13) can be decomposed as follows

\[
\begin{aligned}
&\left( \begin{array}{cc}
e^{i2n_x \mathcal{D}^+(x)^2} & 2e^{ix} \\
0 & e^{-i2(n-1)x \mathcal{D}^-(x)^2} \end{array} \right) \quad (x \in [0, 2\pi]) \\
= &\left( \begin{array}{cc}
\frac{1}{2}e^{-i(2n-1)x \mathcal{D}^-(x)^2} & 0 \\
0 & 1 \end{array} \right) \left( \begin{array}{cc}
0 & 2e^{ix} \\
-\frac{1}{2}e^{ix} & 0 \end{array} \right) \left( \begin{array}{cc}
\frac{1}{2}e^{i(2n-1)x \mathcal{D}^+(x)^2} & 0 \\
0 & 1 \end{array} \right).
\end{aligned}
\] (3.15)

And hence the boundary condition in (3.13) is changed to

\[
\mathbf{F}^+(x) \left( \begin{array}{cc}
\frac{1}{2}e^{i(2n-1)x \mathcal{D}^+(x)^2} & 0 \\
0 & 1 \end{array} \right) = \mathbf{F}^-(x) \left( \begin{array}{cc}
\frac{1}{2}e^{-i(2n-1)x \mathcal{D}^-(x)^2} & 0 \\
0 & 1 \end{array} \right) \left( \begin{array}{cc}
-\frac{1}{2}e^{ix} & 0 \\
0 & 2e^{ix} \end{array} \right) e^{2C}.
\] (3.16)
In order to construct the second transform, we introduce some symbols. For \( r \in (0, \rho) \), we define two oriented line segments and a contour

\[
L_r = [2\pi + ir, ir], \quad L_{-r} = [2\pi - ir, -ir], \quad \Gamma = L_r + [0, 2\pi] + L_{-r}.
\]

And the contour \( \Gamma \) divides the basic strip

\[
\Box_{-\infty, +\infty} = \{ z, \text{Re} z \in (0, 2\pi), \text{Im} z \in (-\infty, +\infty) \}
\]

into two domains

\[
A^+ = A_1^+ + A_2^+, \quad A^- = A_1^- + A_2^-
\]

with

\[
A_1^+ = \Box_{-\infty, -r}, \quad A_2^+ = \Box_{0, r}, \quad A_1^- = \Box_{-r, 0}, \quad A_2^- = \Box_{r, +\infty}.
\]

Let

\[
V(z) = \begin{cases}
I, & z \in A_1^+ \\
\begin{pmatrix} 1 & 0 \\
\frac{1}{2} e^{-i(2n-1)z} \frac{|D^+(z)|^2}{w(z)} & 1 \end{pmatrix}, & z \in A_1^- \\
e^{-iz} \begin{pmatrix} 1 & 0 \\
-\frac{1}{2} e^{-i(2n-1)z} \frac{|D^+(z)|^2}{w(z)} & 1 \end{pmatrix}, & z \in A_2^+ \\
e^{-iz} I & z \in A_2^- 
\end{cases}
\]

and we define the second transform

\[
S(z) = F(z)V(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

Similarly to Lemma 3.1, one has the following.

**Lemma 3.2** If \( F \) is the solution of Riemann-Hilbert problem (3.13), then \( S(z) \) defined by (3.22) is the following Riemann-Hilbert problem

\[
\begin{cases}
S^+(t) = S^-(t)\Upsilon(t), & t \in \Gamma, \\
S(z) = e^{-iz} \left[ I + o(1) \right], & z \to +\infty i, \\
S(z) = I + o(1), & z \to -\infty i
\end{cases}
\]

with

\[
\Upsilon(t) = \begin{cases}
\begin{pmatrix} 1 & 0 \\
\frac{1}{2} e^{i(2n-1)t} \frac{|D^+(t)|^2}{w(t)} & 1 \end{pmatrix}, & t \in L_r, \\
\begin{pmatrix} 2 & 0 \\
-\frac{1}{2} & 0 \end{pmatrix}^{-1} e^{-2C}, & t \in [0, 2\pi], \\
\begin{pmatrix} 1 & 0 \\
\frac{1}{2} e^{-i(2n-1)t} \frac{|D^-(t)|^2}{w(t)} & 1 \end{pmatrix}, & t \in L_{-r}.
\end{cases}
\]

Conversely, if \( S(z) \) is the solution of the Riemann-Hilbert problem (3.23), then

\[
F(z) = S(z)V^{-1}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}
\]

is the solution of Riemann-Hilbert problem (3.13).
3.3. The third transform \( S(z) \rightarrow R(z) \)

To eliminate the jump of \( S(z) \) on the interval \([0, 2\pi]\), we need to find the sectionally periodic analytic function \( M(z) \) satisfying the following conditions:

\[
\begin{cases}
M^+(x) = M^-(x) \begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix} e^{2C}, & x \in [0, 2\pi], \\
M(z) = \begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix} e^{2C} + o(1), & z \rightarrow +\infty i, \\
M(z) = I + o(1), & z \rightarrow -\infty i,
\end{cases}
\]

where \( C \) is defined by (3.5). By Liouville’s Theorem, one easily knows that

\[
M(z) = \begin{cases}
\begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix} e^{2C}, & \text{Im} z > 0, \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{Im} z < 0
\end{cases}
\]

is the unique solution of (3.26).

Now, the contour

\[
\Gamma_z = L_r + L_{-r}
\]

divides the basic strip \( \Box_{-\infty, +\infty} \) into two domains

\[
\mathbb{B}^+ = \Box_{-r, r} \quad \text{and} \quad \mathbb{B}^- = \mathbb{B}_1^- + \mathbb{B}_2^-
\]

with

\[
\mathbb{B}_1^- = \Box_{-\infty, -r}, \quad \mathbb{B}_2^- = \Box_{r, +\infty}.
\]

We define the third transform

\[
R(z) = S(z)M^{-1}(z), \quad z \in \mathbb{B}^+ \cup \mathbb{B}^-,
\]

where \( M^{-1}(z) \) is the inverse of \( M(z) \) defined by (3.27). By the boundary conditions in (3.23) and (3.26), one has

\[
S^+(x)[M^+(x)]^{-1} = S^-(x)[M^-(x)]^{-1}, \quad x \in [0, 2\pi],
\]

which implies that \( R(z) \) can be analytically extended across \( \mathbb{R} \). And hence, we always assume that \( R(z) \) is analytic on \( \mathbb{R} \) in what follows.
Further, by a simple calculation, one easily gets
\[
\begin{align*}
\mathbf{R}^-(t) &= \mathbf{S}^-(t)\mathbf{M}^{-1}(t) = \mathbf{S}^+(t) \left( -\frac{1}{2} e^{i(2n-1)t} \frac{|\mathcal{D}^+(t)|^2}{w(t)} \right) \mathbf{M}^{-1}(t) \\
&= \mathbf{S}^+(t)\mathbf{M}^{-1}(t)\mathbf{M}(t) \left( -\frac{1}{2} e^{i(2n-1)t} \frac{|\mathcal{D}^+(t)|^2}{w(t)} \right) \mathbf{M}^{-1}(t) \\
&= \mathbf{R}^+(t) \left( \begin{array}{cc}
0 & 2 \\
-\frac{1}{2} & 0 
\end{array} \right) e^{2C} \left( -\frac{1}{2} e^{i(2n-1)t} \frac{|\mathcal{D}^+(t)|^2}{w(t)} \right) \mathbf{M}^{-1}(t) \\
&= \mathbf{R}^+(t) \left( \begin{array}{cc}
1 & 2e^{i(2n-1)t} \frac{|\mathcal{D}^+(t)|^2}{w(t)} \\
0 & 1 
\end{array} \right), \quad t \in L_r
\end{align*}
\] (3.33)

and
\[
\begin{align*}
\mathbf{R}^+(t) &= \mathbf{S}^+(t)\mathbf{M}^{-1}(t) \\
&= \mathbf{S}^-(t) \left( \frac{1}{2} e^{-i(2n-1)t} \frac{|\mathcal{D}^-(t)|^2}{w(t)} \right) \mathbf{M}^{-1}(t) \\
&= \mathbf{S}^-(t)\mathbf{M}(t)\mathbf{M}^{-1}(t) \left( \frac{1}{2} e^{-i(2n-1)t} \frac{|\mathcal{D}^-(t)|^2}{w(t)} \right) \mathbf{M}^{-1}(t) \\
&= \mathbf{R}^-(t) \left( \frac{1}{2} e^{-i(2n-1)t} \frac{|\mathcal{D}^-(t)|^2}{w(t)} \right) \mathbf{M}^{-1}(t) \\
&= \mathbf{R}^-(t) \left( \begin{array}{cc}
1 & 0 \\
-\frac{1}{2} e^{-i(2n-1)t} \frac{|\mathcal{D}^-(t)|^2}{w(t)} & 1 
\end{array} \right), \quad t \in L_{-r}.
\end{align*}
\] (3.34)

And then, the following lemma is obtained.

**Lemma 3.3** If \( \mathbf{S} \) is the solution of Riemann-Hilbert problem (3.23), then \( \mathbf{R}(z) \) defined by (3.31) is the solution of the model Riemann-Hilbert problem
\[
\begin{align*}
\mathbf{R}^+(t) &= \mathbf{R}^-(t)\mathbf{G}(t), \quad t \in \Gamma_2 = L_r + L_{-r}, \\
\mathbf{R}(z) &= e^{-2C_{-2}z} \left[ \left( \begin{array}{cc}
0 & -2 \\
\frac{1}{2} & 0 
\end{array} \right) + o(1) \right], \quad z \to +\infty i, \\
\mathbf{R}(z) &= \mathbf{I} + o(1), \quad z \to -\infty i.
\end{align*}
\] (3.35)

with
\[
\mathbf{G}(t) = \begin{cases}
\left( \begin{array}{cc}
1 & -2e^{i(2n-1)t} \frac{|\mathcal{D}^+(t)|^2}{w(t)} \\
0 & 1 
\end{array} \right), & t \in L_r, \\
\left( \begin{array}{cc}
\frac{1}{2} e^{-i(2n-1)t} \frac{|\mathcal{D}^-(t)|^2}{w(t)} & 0 \\
-\frac{1}{2} e^{-i(2n-1)t} \frac{|\mathcal{D}^-(t)|^2}{w(t)} & 1 
\end{array} \right), & t \in L_{-r}.
\end{cases}
\] (3.36)

Conversely, if \( \mathbf{R}(z) \) is the solution of the model Riemann-Hilbert problem (3.35), then
\[
\mathbf{S}(z) = \mathbf{R}(z)\mathbf{M}(z), \quad z \in \mathbb{C} \setminus \Gamma_2
\] (3.37)
is the solution of Riemann-Hilbert problem (3.23).
4. Strong Asymptotic analysis of OTP

First, we set up a lemma needed in the sequel.

**Lemma 4.1.** Let $R^-$ be the negative boundary value of $R$ which is the solution for the model Riemann-Hilbert problem (3.35). Then

$$k = \frac{1}{4\pi} \int_{\Gamma_z} R^-(\tau)(G(\tau) - I)d\tau = -\frac{1}{2}I,$$

(4.1)

where $G$ is given by (3.36) and $I$ is the identity matrix.

**Proof.** Recall that $R^+$ is analytic on $\Box_{-r,r}$. We have

$$\frac{1}{4\pi} \int_{\Gamma_z} R^-(\tau)(G(\tau) - I)d\tau = \frac{1}{4\pi} \int_{\Gamma_z} R^-(\tau)G(\tau)d\tau - \frac{1}{4\pi} \int_{\Gamma_z} R^-(\tau)d\tau$$

$$= \frac{1}{4\pi} \int_{\partial\Box_{-r,r}} R^+(\tau)d\tau \frac{1}{4\pi} \int_{\Gamma_z} R^-(\tau)d\tau$$

$$= -\frac{1}{4\pi} \int_{L_r} R^-(\tau)d\tau + \frac{1}{4\pi} \int_{L_r} R^-(\tau)d\tau,$$  

(4.2)

where $L_r, L_{-r}$ are defined in (3.17). But by (3.35) we have

$$\frac{1}{4\pi} \int_{L_{-r}} R^-(\tau)d\tau = \lim_{R \to +\infty} \frac{1}{4\pi} \int_{L_{-r}} R(z)dz$$

$$= \lim_{R \to +\infty} \frac{1}{4\pi} \int_{L_{-r}} (I + o(1))dz$$

$$= \frac{1}{2}I.$$  

(4.3)

Denote by $A(z) = R(z)e^{2C+i\tau}$ and $B(w) = A(-i\ln w)$ for $w \in \mathbb{C}$. Then $B(w)$ is analytic near $\infty$. And by (3.35) its Laurent series is

$$B(w) = \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} + \frac{1}{w}c_1 + \frac{1}{w^2}c_2 + \ldots,$$

where $c_j, j = 1, 2, \ldots$ are constant matrices. So we have

$$R(z) = e^{-2C-i\tau}\left[ \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} + c_1 e^{i\tau} + o(|e^{i\tau}|) \right].$$

(4.4)

Then we obtain

$$\frac{1}{4\pi} \int_{L_r} R^-(\tau)d\tau = \lim_{R \to +\infty} \frac{1}{4\pi} \int_{L_R} R(z)dz$$

$$= \lim_{R \to +\infty} \frac{1}{4\pi} \int_{L_R} e^{-2C-i\tau}\left[ \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} + c_1 e^{i\tau} + o(|e^{i\tau}|) \right]dz.$$  

Finally, combining (4.2), (4.5) with (4.3), we get the conclusion of the lemma.  

Now, one comes to verify one of the main results, usually called Aptekarev type theorem [2, 3]. It must be pointed that all the norms in the following have the same definition with those in [2, 3].
**Theorem 4.2.** There exist constants \( \eta > 0 \) and \( \delta > 0 \) such that, for \( \|G - I\|_{\Omega_{\epsilon}} < \delta \), we have

\[
\left\| \mathbf{R}(z) - e^{-2C - iz} \begin{pmatrix} 0 \frac{1}{2} -2 \end{pmatrix} - \frac{1}{2} \mathbf{I} \right\|_{C_\Gamma^2} < \eta\|G - I\|_{\Omega_{\epsilon}} \text{ with } \Omega_{\epsilon} = \square_{-r-\epsilon,r+\epsilon} \cup \square_{-r+\epsilon,-r-\epsilon}
\]  

where \( \epsilon > 0 \) is sufficiently small, \( G \) is given by (3.36) and \( \mathbf{R} \) is the solution for the model Riemann-Hilbert problem (3.35).

**Proof.** First, let

\[
\mathbf{W}(z) = \mathbf{R}(z) - e^{-2C - iz} \begin{pmatrix} 0 \frac{1}{2} -2 \end{pmatrix}, \quad z \in \mathbb{C} \setminus \Gamma_2.
\]

By the asymptotic conditions in (2.35), one has

\[
\mathbf{W}(z) = \begin{cases} 
\mathbf{A} + o(1), & z \to +\infty i, \\
\mathbf{I} + o(1), & z \to -\infty i.
\end{cases}
\]  

(4.6)

Denote by \( \Delta = G - I \). Again by the boundary condition in (3.35), we have

\[
\mathbf{R}^+(t) = \mathbf{R}^-(t) + \mathbf{R}^-(t)\Delta = \mathbf{R}^-(t)(\mathbf{I} + \Delta), \quad t \in \Gamma_2,
\]

which leads to

\[
\mathbf{W}^+(t) = \mathbf{W}^-(t) + \mathbf{R}^-(t)\Delta, \quad t \in \Gamma_2.
\]

(4.7)

Secondly, we set

\[
\mathbf{H}(z) = \frac{1}{4\pi i} \int_{\Gamma_2} \mathbf{R}^-(\tau)\Delta(\tau) \cot \frac{\tau - z}{2} d\tau, \quad z \notin \Gamma_2.
\]

(4.9)

By Lemma 4.1, one has

\[
\begin{cases} 
\mathbf{H}^+(t) = \mathbf{H}^-(t) + \mathbf{R}^-(t)\Delta, & t \in \Gamma_2 \\
\mathbf{H}(z) = k + o(1), & z \to +\infty i, \\
\mathbf{H}(z) = -k + o(1), & z \to -\infty i,
\end{cases}
\]

(4.10)

where \( k \) is given by (4.1). By (4.8) and (4.10), we have \( \mathbf{W}(z) - \mathbf{H}(z) \) is an entire function. And by (4.6) and (4.10), \( \mathbf{W}(z) - \mathbf{H}(z) \) is bounded on the whole complex plane. Then, by Liouville Theorem, we have

\[
\mathbf{W}(z) - \mathbf{H}(z) \equiv \mathbf{I} + k = \frac{1}{2} \mathbf{I}, \quad z \in \mathbb{C},
\]

(4.11)

which leads to

\[
\mathbf{W}(z) = \frac{1}{2} \mathbf{I} + \frac{1}{4\pi i} \int_{\Gamma_2} \mathbf{R}^-(\tau)\Delta(\tau) \cot \frac{\tau - z}{2} d\tau, \quad z \notin \Gamma_2.
\]

(4.12)

Therefore, one has

\[
\mathbf{R}(z) = e^{-2C - iz} \begin{pmatrix} 0 \frac{1}{2} -2 \end{pmatrix} + \frac{1}{2} \mathbf{I} + \frac{1}{4\pi i} \int_{\Gamma_2} \mathbf{R}^-(\tau)\Delta(\tau) \cot \frac{\tau - z}{2} d\tau, \quad z \notin \Gamma_2.
\]

(4.13)

Thirdly, for \( \epsilon > 0 \), let

\[
\Gamma_2^\epsilon = L_{r-\frac{1}{4}} + L_{-r-\frac{1}{4}}.
\]
By (4.13), one has
\[ R(z) = e^{-2C^{-i}z} \left( \begin{array}{cc} 0 & -2 \\ 0 & 0 \end{array} \right) + \frac{1}{2} I + \frac{1}{4\pi i} \int_{\Gamma^+} R^-(\tau) \Delta(\tau) \cot \frac{\tau - z}{2} \, d\tau, \quad z \in \mathbb{B}^+ = \square_{-r,r}, \] (4.14)
which in particular implies
\[ R^+(t) = e^{-2C^{-i}t} \left( \begin{array}{cc} 0 & -2 \\ 0 & 0 \end{array} \right) + \frac{1}{2} I + \frac{1}{4\pi i} \int_{\Gamma^+} R^-(\tau) \Delta(\tau) \cot \frac{\tau - t}{2} \, d\tau, \quad t \in \Gamma^+. \] (4.15)

Further, by (4.7) and (4.15), we get
\[ R^-(t) = \left[ e^{-2C^{-i}t} \left( \begin{array}{cc} 0 & -2 \\ 0 & 0 \end{array} \right) + \frac{1}{2} I + \frac{1}{4\pi i} \int_{\Gamma^+} R^-(\tau) \Delta(\tau) \cot \frac{\tau - t}{2} \, d\tau \right] [I + \Delta(t)]^{-1}, \quad t \in \Gamma^+. \] (4.16)

This leads to the estimate
\[ \| R^- \|_{\Gamma^+} \leq \left( d_\epsilon + K_\epsilon \| \Delta \|_{\Omega^\epsilon} \| R^- \|_{\Gamma^+} \right) \| (I + \Delta)^{-1} \|_{\Omega^\epsilon} \] (4.17)
with
\[ d_\epsilon = \left\| e^{-2C^{-i}t} \left( \begin{array}{cc} 0 & -2 \\ 0 & 0 \end{array} \right) + \frac{1}{2} I \right\|_{\Omega^\epsilon}, \quad K_\epsilon = \max_{t \in \Gamma^+, \tau \in \Gamma^+} \left| \cot \frac{\tau - t}{2} \right|. \] (4.18)

By the Maximal Principle Theorem, we have
\[ \| R^- \|_{\Gamma^+} \geq \| R^- \|_{\Gamma^+}. \] (4.19)

Notice that
\[ \| (I + \Delta)^{-1} \|_{\Omega^\epsilon} \leq \sum_{n=0}^{\infty} \| \Delta \|_{\Omega^\epsilon}^n = \frac{1}{1 - \| \Delta \|_{\Omega^\epsilon}} \] (4.20)
for \( \| \Delta \|_{\Omega^\epsilon} < 1 \). Combining (4.17), (4.19) with (4.20), one has
\[ \| R^- \|_{\Gamma^+} \leq \frac{d_\epsilon + K_\epsilon \| \Delta \|_{\Omega^\epsilon} \| R^- \|_{\Gamma^+}}{1 - \| \Delta \|_{\Omega^\epsilon}} \] (4.21)
for \( \| \Delta \|_{\Omega^\epsilon} < 1 \). Now, we pick \( \delta \in (0, \frac{1}{2}) \) sufficiently small such that
\[ \| \Delta \|_{\Omega^\epsilon} < \delta \Rightarrow \frac{K_\epsilon \| \Delta \|_{\Omega^\epsilon}}{1 - \| \Delta \|_{\Omega^\epsilon}} \leq \frac{1}{2}. \] (4.22)

Therefore, combing (4.21) with (4.22), one has
\[ \| \Delta \|_{\Omega^\epsilon} < \delta \Rightarrow \| R^- \|_{\Gamma^+} \leq \frac{d_\epsilon}{1 - \| \Delta \|_{\Omega^\epsilon}} \leq \frac{2d_\epsilon}{1 - \delta} < 4d_\epsilon. \] (4.23)

Finally, according to [5], the singular integral operator is \( H^\mu \)-bounded. And hence there exists \( M > 0 \) such that
\[ \left\| \int_{\Gamma^+} R^-(\tau) \Delta(\tau) \cot \frac{\tau - t}{2} \, d\tau \right\|_{\Gamma^+} \leq MK_\epsilon \| \Delta \|_{\Omega^\epsilon} \| R^- \|_{\Gamma^+} \] (4.24)
since the integrand is differentiable. Combining (4.23) with (4.24), one easily gets the desired estimate (4.5).
Acknowledgements This paper is supported by NSFC11701597; NSFC11471250.

REFERENCES

[1] J. Y. Du, *On the collocation methods for singular integral equations with Hilbert kernel*, Mathematics of Computation, 2009, 78 (266), 891-928.

[2] Z. H. Du and J. Y. Du, *Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on the unit circle*, Chinese Ann. Math. 27A(5) (2006) 701-718.

[3] Z. H. Du and J. Y. Du, *Orthogonal trigonometric polynomials: Riemann-Hilbert analysis and relations with OPUC*, Asymptotic Analysis 79 (2012) 87-132.

[4] J. K. Lu, *Boundary value Problems for Analytic Functions*, World Scientific, 1993.

[5] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer-Verlag, Berlin, 1986.

[6] Y. F. Wang and Y. J. Wang, *On Riemann problems for single-periodic polyanalytic functions*, Math. Nachr. 287(16)(2014) 1886-1915.

[7] Yufeng Wang, Yifeng Lu and Jinyuan Du, *Strong asymptotic analysis of OLPs on the unit circle by Riemann-Hilbert approach*, S. Rogosin, A. O. Çelebi (eds.), Analysis as a Life, Trends in Mathematics, Springer Nature Switzerland AG, 2019, 139-169.

Han Huili, Ningxia University

Liu Huadaliu@163.com, Department of Mathematics, Tianjin University of Technology and Education, Tianjin 300222, China

Wang Yufeng, Wuhan University