Dualities for Ising networks

Yu-tin Huang,1,2, Chia-Kai Kuo,1, Congkao Wen,3

1 Department of Physics and Astronomy, National Taiwan University, Taipei 10617, Taiwan
2 National Tsing-Hua University, No.101, Section 2, Kuang-Fu Road, Hsinchu, Taiwan and
Centre for Research in String Theory, School of Physics and Astronomy Queen Mary University of London,
Mile End Road, London E1 4NS, United Kingdom

In this note, we study the equivalence between planar Ising networks and cells in the positive orthogonal Grassmannian. We present a microscopic construction which establishes the correspondence for any Ising network. Since the number of cells of a positive orthogonal Grassmannian is finite, this implies that Ising networks belonging to the same cell are dual. We establish local duality transformations that relate these Ising networks. Sequential applications of these duality transformations allow us to convert non-reduced Ising networks to reduced ones with calculable effective couplings. When applied to self-repeating lattices dual to the same cell, the transformation rules become the exact Renormalization group equations, and fixed points correspond to phase transitions. We use the Ising lattice on the Sierpinski triangle with general anisotropic couplings as an example, and show that for this case there are no phase transitions at finite temperatures.

INTRODUCTION

Recent years there has been a fascinating interplay between the physics of observables in quantum field theories and geometries in mathematics. Consistency conditions of the observables, arising from fundamental principles of unitarity, locality and symmetries, are often connected to the defining properties of certain mathematical objects. For instance, scattering amplitudes of gauge theories are connected to the positive Grassmannian [1][2] and further into the Amplituhedron [3], couplings of higher-dimensional operators in effective field theories, or four-point functions of a conformal field theory, are bounded by cyclic polytopes [4]. In these examples, the mathematical object of interest has an intrinsic definition that does not make any direct reference to physics. Thus the principles that determine these physical observables are in effect equivalent to the mathematical properties that define the object. Put in another way, the physical principle becomes emergent.

Recently a fascinating new connection was revealed by Galashin and Pylyavskyy [5]. There, the observable in question is the correlation functions of 2D planar Ising network, which were shown to be equivalent to cells in the positive Orthogonal Grassmannian (OG). The latter also describes scattering amplitudes of supersymmetric Chern-Simon matter theories [2][6].

In the examples mentioned in the beginning, it is the factorization properties of the physical observables that ensure their identification with the mathematical objects. In this note, we identify the corresponding physical property for Ising correlators: under the amalgamation, where two of the external spin sites are identified, the correlation function of the new network can be written as a function of the former. The map is non-linear. However, when cast into the OG, the map is linearised and manifestly preserves positivity. Since any planar network can be built through amalgamation, this ensures that all Ising networks are dual to cells in the positive OG.

Importantly, for fixed boundary sites the cells in OG are finite, whereas there are infinite possible Ising networks. This implies that there are duality transformations that translate between the networks of the same equivalence class. We identify the duality transforms which act locally on the Ising network, and express the effective couplings of the new network in terms of the original one. More precisely, the correlation functions for the original network, is equivalent to that computed in the dual network using the effective coupling. We use the Ising network on the Sierpinski triangle as an example. Note that since this is a self-replicating network, the duality transform becomes the exact Renormalization group (RG) equation for the couplings, and fixed points correspond to phase transitions. We use this correspondence to demonstrate the lack of finite temperature phase transition for general anisotropic couplings.

MAP ISING NETWORK TO $OG_{\geq 0}(n,2n)$

We begin by considering two-point correlation functions of a planar Ising network with $n$ boundary spin sites. Higher-point functions can be written as products of two-point functions, and thus the correspondence generalizes easily [5]. The two point function is defined as

$$
\langle \sigma_i \sigma_j \rangle = \frac{\sum_{\sigma \in \{\pm 1\}} \sigma_i \sigma_j P(J_{ab})}{\sum_{\sigma \in \{\pm 1\}} P(J_{ab})}, \quad P(J_{ab}) = \prod_{a,b \in \{E\}} e^{J_{ab} \sigma_a \sigma_b}
$$

where $\sigma_i$ represents spin sites taking value $\pm 1$, $E$ is the set of edges in the Ising network and $J_{ab}$ is the coupling of the edge connecting sites $a$ and $b$, taking any positive value. Intuitively, since we are considering ferromagnetic couplings we expect that the correlator to be non-negative. However, as it is given by a summation over
The resulting symmetric matrix can be embedded in a $\text{OG}(n, n)$ in a $n \times n$ unit symmetric matrix with the following map:

$$i \neq j : \quad m_{i,2j-1} = -m_{i,2j} = \text{Sign}[i-j](-)^{i+j} \langle \sigma_i \sigma_j \rangle,$$

$$i = j : \quad m_{i,2i-1} = m_{i,2i} = 1.$$  

The resulting $n \times 2n$ matrix contains rows that are $2n$-dimensional mutually null vectors, an Orthogonal Grassmannian $\text{OG}(n, 2n)$: the moduli space of a null $n$-plane in a $2n$-dimensional space. The correlation functions can then be recovered by the inverse map, written as

$$\langle \sigma_i \sigma_j \rangle = \sum_{I \in \mathcal{S} \in \{1, 2\}} \frac{\Delta_I}{\sum_{I \in \mathcal{S}} \Delta_I},$$

where $\Delta_I$ denotes the $n \times n$ minors of $\text{OG}(n, 2n)$ with columns $I = \{i_1, i_2, \cdots, i_n\}$. Note that for $\text{OG}(n, 2n)$, $\Delta_I = \Delta_J$ where $\bar{I}$ is the complement of $I$. The notation $\varepsilon(S)$ represents all $n$-element subsets of $2n$, such that for each $i$, $I \cap (2n-1, 2i)$ even times if and only if $i \in S$. For example, for $n = 2$

$$\sum_{I \in \{1, 2\}} \Delta_I = \Delta_{12} + \Delta_{34} = 2\Delta_{12},$$

$$\sum_{I \in \emptyset} \Delta_I = \Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24} = 2(\Delta_{13} + \Delta_{14}).$$

Note that the correlation function is given by ratios of the minors and hence $\text{GL}(n)$ invariant. As the simplest example where two boundary spin sites are connected by a single edge, the corresponding $\text{OG}(2, 4)$ is

$$\begin{pmatrix}
1 & s(J) & 0 & -c(J) \\
0 & c(J) & 1 & s(J)
\end{pmatrix},$$

where

$$s(J) = \frac{2}{e^{2J} + e^{-2J}}, \quad c(J) = \frac{e^{2J} - e^{-2J}}{e^{2J} + e^{-2J}}.$$

With $\Delta_{12} = c(J), \Delta_{13} = 1, \Delta_{14} = s(J)$, using eq.

$$\langle \sigma_i \sigma_j \rangle_{\text{amal}} = \frac{\langle \sigma_i \sigma_j \rangle + \langle \sigma_i \sigma_{n-1} \sigma_j \sigma_1 \rangle}{1 + \langle \sigma_n \sigma_{n-1} \rangle}.$$  

The first move is somewhat trivial, as one simply declares one of the external spin sites to be internal. Let us then focus on the second operation which corresponds to identifying two boundary spin sites, $n$ and $n-1$. In terms of correlation functions, there is a simple relation between correlators before and after the "amalgamation":

$$\langle \sigma_i \sigma_j \rangle_{\text{amal}} = \frac{\langle \sigma_i \sigma_j \rangle + \langle \sigma_i \sigma_{n-1} \sigma_j \sigma_1 \rangle}{1 + \langle \sigma_n \sigma_{n-1} \rangle}.$$  

The four-point function in the formula can be further recast as sum of products of two-point functions:

$$\langle \sigma_n \sigma_{n-1} \sigma_i \sigma_j \rangle = \langle \sigma_n \sigma_{n-1} \rangle \langle \sigma_i \sigma_j \rangle - \langle \sigma_n \sigma_i \rangle \langle \sigma_j \sigma_{n-1} \rangle + \langle \sigma_{n-1} \sigma_i \rangle \langle \sigma_n \sigma_j \rangle.$$  

Given that the correlation functions can be embedded in $\text{OG}(n, 2n)$, let us now see what is the image of the amalgamation operation in the Grassmannian. Since we are identifying two external sites, this operation correspond to reducing $\text{OG}(n, 2n)$ to $\text{OG}(n-1, 2n-2)$. We would like to find an expression for the minors of $\text{OG}(n-1, 2n-2)$, which computes $\langle \sigma_i \sigma_j \rangle_{\text{amal}}$, in terms of that of $\text{OG}(n, 2n)$, which computes the correlator of the

1 The metric in $2n$ dimensions has an alternating signature.
pre-amalgamated network. Let us take an explicit example, from \( \text{OG}(3,6) \) to \( \text{OG}(2,4) \). Begin with the matrix
\[
\begin{pmatrix}
c_1 & c_2 & c_3 & c_4 \\
1 & 1 & -c_4 & c_3 \\
1 & 1 & -c_3 & c_2 \\
-\langle \sigma_1 \sigma_2 \rangle & -\langle \sigma_1 \sigma_2 \rangle & -\langle \sigma_1 \sigma_3 \rangle & -\langle \sigma_1 \sigma_3 \rangle \\
\langle \sigma_1 \sigma_3 \rangle & \langle \sigma_1 \sigma_3 \rangle & \langle \sigma_2 \sigma_3 \rangle & \langle \sigma_2 \sigma_3 \rangle \\
\end{pmatrix},
\]
where we label the columns that will be removed by the amalgamation as \( a, b \).\(^2\) From eq.\(^5\) we see that the two-point function for the amalgamated network is simply
\[
\langle \sigma_1 \sigma_2 \rangle^{\text{amal}} = \frac{\Delta_{\text{OG}(3,6)}^{\text{OG}(2,4)}}{\Delta_{\text{OG}(2,4)} + \Delta_{\text{OG}(2,4)}},
\]
(11)
where the minors above are that of \( \text{OG}(2,4) \). On the other hand the RHS of eq.\(^8\) tells us that the same two-point function can also be written as the two-point function of the pre-amalgamated Ising network which resides in \( \text{OG}(3,6) \). Equating the two leads us to the following identification:
\[
\Delta_{ij}^{\text{OG}(2,4)} = \Delta_{ij}^{\text{OG}(3,6)} + \Delta_{ij}^{\text{OG}(3,6)},
\]
(12)
One can straight forwardly check that this generalizes to higher points, where the image of eq.\(^8\) in the \( \text{OG} \) is
\[
\Delta_{Ij}^{\text{OG}(n-1,2n-2)} = \Delta_{Ij}^{\text{OG}(n,2n)} + \Delta_{Ij}^{\text{OG}(n,2n)}.
\]
(13)
This is precisely the amalgamation operation of Grassmannian.\(^1\) Note the same is also true for the first case of eq.\(^7\) with the position of columns \( a, b \) shifted.\(^3\)

Importantly, since the minors of the amalgamated Ising network is simply a positive sum of the pre-amalgamated network, the positivity of the Grassmannian is preserved! Thus as we iteratively build up more and more complicated Ising network, when mapped in to the Grassmannian via eq.\(^3\), it will always reside in \( \text{OG}_{\geq 0}(n,2n) \).

For higher-order correlators, the translation between the amalgamated and pre-amalgamated is simply modified to:
\[
\langle \sigma_A \rangle^{\text{amal}} = \langle \sigma_A \rangle + \langle \sigma_n \sigma_{n-1} \sigma_A \rangle \frac{1}{1 + \langle \sigma_n \sigma_{n-1} \rangle},
\]
(14)
where \( \{ A \} \) labels the set of spin sites defining the correlator. Using the map given in eq.\(^5\):
\[
\langle \sigma_A \rangle = \sum_{I \in \epsilon(\{ A \})} \Delta_I \sum_{I \in \epsilon(\emptyset)} \Delta_I,
\]
(15)
we see that it again translates to the amalgamation of the minors.

\(^2\) At this moment the choice of the positions of columns \( a \) and \( b \) is simply that it is required by eq.\(^5\), but it becomes evident if we map Ising networks fig.\(^7\) to on-shell diagrams using the rules presented in the next section.

\(^3\) We thank Pavel Galashin for pointing this out.

FIG. 1: Examples of cells in \( \text{OG}_{\geq 0}(3,6) \). (a): here we can see from the paths each column vector is spanned by three other vectors. For example the path shown in the graph indicates that column 5 is spanned by (2,3,4). Since each column is three-dimensional that implies that all \( \Delta_i \neq 0 \) and this is a top cell. (b): here we see that 1 is spanned by (2,3) indicating that \( \Delta_{1,2,3} = 0 \), and thus a co-dimension one cell.

**THE STRUCTURE OF \( \text{OG}_{\geq 0}(n,2n) \)**

The space of \( \text{OG}_{\geq 0}(n,2n) \) consists of cells defined by the set of vanishing minors \( \Delta_I \). This forms a stratification: starting from the top cell with all \( \Delta_I \)s non-zero, one has the co-dimension one boundaries where one of the consecutive minors vanishes, and etc. As shown in \( \text{fig.\(^8\)} \), the combinatorics of the cell structure for \( \text{OG}_{\geq 0}(n,2n) \) is topologically a ball. Each cell can be represented by an on-shell diagram constructed by quartic vertices, which encodes the information of which \( \Delta_I \) vanishes. For each diagram with \( 2n \)-boundary sites, one can associate it with permutation paths that are determined by the rule that one never turns when passing through a vertex. If a path connects boundary sites \( i, j \), then the column vector \( j \) in the \( n \times 2n \) matrix is spanned by the columns of \( i \) and those between \( i \) and \( j \). For example, cells of \( \text{OG}_{\geq 0}(3,6) \) are shown in fig.\(^8\). The total number of vertices represent a dimension of the cell. For more detailed discussion, please see \( \text{fig.\(^9\)} \).

Since each Ising network should correspond to a cell in \( \text{OG}_{\geq 0}(n,2n) \), it must correspond to an on-shell diagram. Indeed starting with an Ising network, one can identify the on-shell diagram by adding a vertex to each Ising edge and two vertices on the two sides of the boundary spin site \( i \), labeled by \((2i-1,2i)\). If the vertices sit on edges that intersect, they are connected by a line. If an edge extends to the boundary, then the vertex on the edge must connect to the neighboring vertex on the boundary. See examples in fig.\(^\text{fig.\(^10\)}\). It is straightforward to see that the resulting graph will be quartic in nature, as required for \( \text{OG}_{\geq 0}(n,2n) \).

Armed with the on-shell diagrams, we can interpret the operation of amalgamation as adding an extra edge and taking the coupling to infinity to identify the spin sites. Take a simple example where one has trivial Ising
network with $n = 4$ and obtain a new network with $n = 3$:

$$J \rightarrow \infty$$

Since we are adding a new edge, we are introducing an extra degree of freedom for our $OG(n, 2n)$. In terms of on-shell diagram in $OG_{\geq 0}(n, 2n)$ this correspond to adding a new vertex and taking its boundary:

$$\hat{a} = \alpha a + \beta b, \quad \hat{b} = \beta a + \alpha b$$

where $\alpha^2 - \beta^2 = 1$ to maintain the orthogonal condition of the new cell. Going to the boundary then correspond to $\alpha, \beta \rightarrow \infty$, which sets $(\hat{a}, \hat{b})$ collinear. This produces the on-shell diagram for the amalgamated network.

**LOCAL DUALITY TRANSFORMATIONS**

Now since a given $OG(n, 2n)$ has finite number of cells, while there are infinite possibilities for planar Ising networks, this implies that the different networks are secretly dual to each other and fall into equivalence classes. In other words there exist duality transformations that relate different Ising networks belonging to the same cell!

Indeed two different looking on-shell diagrams can actually be equivalent if they are related by equivalence moves $\mathbb{1}$. For $OG_{>0}(n, 2n)$, the equivalence moves consist of tadpole reductions, bubble reductions and triangle move $\mathbb{2} \mathbb{9}$. The equivalence moves reflect the fact that the diagrams are just different, in some cases redundant, parameterizations of the same cell, and thus there exists map that translate between the two charts. When translated into Ising networks means that different Ising networks related by equivalence moves yield the same boundary correlator, with duality transformations applied on the coupling constants.

We begin with the simplest tadpole reductions. It is the case where on-shell diagrams contain a tadpole, whereas the Ising networks contain an external spin coupled to itself or an internal one coupled to the rest via only one edge:

$$c(J_{12}) = \frac{c(J_{1a})c(J_{2a})}{1 + s(J_{1a})s(J_{2a})},$$

$$s(J_{12}) = \frac{s(J_{1a}) + s(J_{2a})}{1 + s(J_{1a})s(J_{2a})}.$$
Another kind of bubble reduction is given by,

\begin{equation}
\begin{split}
\text{As evident from the graph, for this type of bubble we simply have}
J_{12} &= J_{12}^{(1)} + J_{12}^{(2)} \
\end{split}
\end{equation}

where we denote two couplings of Ising graph on LHS of fig. (21) as $J_{12}^{(1)}$ and $J_{12}^{(2)}$, and $J_{12}$ the coupling of Ising graph on RHS. On-shell diagrams are reducible if they contain bubbles (or equivalently Ising networks contain sub-diagrams as LHS of fig. (19) and fig. (21), and they can be reduced by applications of bubble reductions. Reducible sub-diagrams may actually be hidden, and can be made visible using another equivalence move as we will discuss below.

Indeed Ising networks exist the triangle move, which is to relate two triangle on-shell diagrams:

\begin{equation}
\text{The duality transformation which preserves all minors, and therefore correlators, is given by [9]}
\begin{split}
s(J_{ia}) &= \frac{s(J_{i+1,i+2})c(J_{i+1,i+2})s(J_{i+2})}{c(J_{i+1,i+2}) + c(J_{i+1})c(J_{i+2})} , \\
c(J_{i+1}) &= \frac{c(J_{i+1})c(J_{i+2})s(J_{i+2})}{s(J_{i+2}) + s(J_{ia})s(J_{i+1})},
\end{split}
\end{equation}

for $i = 1, 2, 3$ with $i + 3 := i$ is understood. As we have emphasized, Ising networks related by equivalence moves yield the same boundary correlation functions after applying the above duality transformations on the couplings. In particular, there is a unique top cell for a given $OG_{\geq 0}(n, 2n)$, and one can generate all lower cells by taking boundaries which corresponds to the limit $J \to 0$ or $J \to \infty$. We list examples of top cells of $OG_{\geq 0}(n, 2n)$ for $n = 3, 4, 5, 6$ in fig. (2).

**Exact RG and the Sierpinski triangle**

When the equivalence move is applied to a network with self-similar structure, the resulting map for the effective coupling becomes an exact RG equation. In this case we can identify phase transitions as fixed points of this map. To illustrate the idea, we apply the equivalence moves to Ising model on the Sierpinski triangle. The Sierpinski triangle can be built recursively of triangles, see fig. (3). Due to the self similar structure, the Sierpinski triangle can be brought into a single triangle by applying the following equivalence moves repeatedly:

\begin{equation}
\text{Using the duality transformations discussed previously, it is straightforward to obtain that the effective couplings of the final network in terms of the couplings of the original Ising network. Begin with the simplest case where all the couplings are identical, we find the transformation is given by}
\begin{equation}
\text{This recursion relation has appeared previously in [10, 11] derived using different methods.}
\end{equation}
\end{equation}

As mentioned previously the Sierpinski triangle can be brought into a single triangle by applying the transformation eq. (26) iteratively. In such case, eq. (26) actually
FIG. 4: The fix-point surface of anisotropic couplings, where the intersection represents a phase transition. This demonstrates that there are no finite temperature phase transitions.

contains the information of RG, controlling how the coupling behaves as we go through each iteration. The fixed point to the RG equation then represents the point of the phase transition. Indeed the fixed point solutions to \( \text{eq.}(26) \) are simply \( c = 0 \) and \( 1 \), trivial solutions, consistent with known results [11]. Instead of homogenous coupling, we also consider more general anisotropic couplings with \((J_1, J_2, J_3)\) on the three different sides of each triangle. This is shown as

where edges with the same colors have the same coupling. Requiring that the duality map comes back to itself yields three 2-dimensional manifold in the 3-dimensional space of coupling, and the fixed points are given as the intersection points. We show the result of this plotted in fig.(4). One sees that the manifolds only intersect at \( c(J_i) = 1, 0 \), thus ruling out finite temperature phase transitions.

CONCLUSIONS AND OUTLOOK

In this note, we explore the dualities between positive orthogonal Grassmannian and 2D planar Ising networks. The dualities are established via the amalgamation, which is a basis move to build any Ising networks and corresponding on-shell diagrams. The classification of cells in positive orthogonal Grassmannian is applied to classify Ising networks, which leads to duality transformations that relate Ising networks of the same equivalence class. As an example, the equivalence moves and duality transformations are utilized to study Ising model on the Sierpinski triangle. Clearly, the idea can be applied to Ising model on other interesting lattice shapes, and one can ask whether or not the classification in terms of cells can be the determination factor of whether an Ising network exhibit a finite temperature phase transition, which may ultimately lead to a completely geometric understanding of planar Ising networks. We will leave this as well as other aspects of the dualities as the future projects.

ACKNOWLEDGEMENTS

We thank P. Galashin and P. Pylyavskyy for bringing to our attention their fascinating work. We also like to thank Nima Arkani-Hamed for very enlightening comments. C.W. is supported by a Royal Society University Research Fellowship no. UF160350. C-k Kuo and Y-t Huang is supported by MoST Grant No. 106-2628-M-002-012-MY3.

[1] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov and J Trnka, “Grassmannian Geometry of Scattering Amplitudes,” arXiv:1212.5605 [hep-th].
[2] Y. T. Huang and C. Wen, “ABJM amplitudes and the positive orthogonal grassmannian,” JHEP 1402, 104 (2014) [arXiv:1309.3252 [hep-th]].
[3] N. Arkani-Hamed and J. Trnka, “The Amplituhedron,” JHEP 1410, 030 (2014) [arXiv:1312.2007 [hep-th]].
[4] N. Arkani-Hamed, T-z Huang, and Y-t Huang : N. Arkani-Hamed, Y-t Huang and Shu-Heng Shao, To appear.
[5] “Ising model and the positive orthogonal Grassmannian”, P. Galashin and P. Pylyavskyy, arXiv:1807.03282.
[6] S. Lee, “Yangian Invariant Scattering Amplitudes in Supersymmetric Chern-Simons Theory,” Phys. Rev. Lett. 105, 151603 (2010) [arXiv:1007.4772 [hep-th]].
[7] “The Planar Ising Model and Total Positivity” M. Lis, Journal of Statistical Physics, 166(1): 72-89, 2017 [arXiv:1606.06068 [math]]
[8] “Correlation-function identities for general planar Ising systems” J. Groeneveld and R.J. Boel and P.W. Kasteleyn, Physica A: Statistical Mechanics and its Applications, 93 (1) 138-154, 1978.
[9] Y. t. Huang, C. Wen and D. Xie, “The Positive orthogonal Grassmannian and loop amplitudes of ABJM,” J. Phys. A 47, no. 47, 474008 (2014) [arXiv:1402.1479 [hep-th]].
[10] Y. Gefen, B. B. Mandelbrot and A. Aharony “Critical Phenomena on Fractal Lattices,” Phys. Rev. Lett. 45, 855 (1980)
[11] Y. Gefen, A. Aharony, Y. Shapir and B. B. Mandelbrot, “Phase transitions on fractals. II. Sierpinski gas-
kets,” Journal of Physics A: Mathematical and General, Volume 17, Issue 2, pp. 435-444 (1984)