Five-Brane Effective Field Theory on Calabi-Yau Threefolds

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Abstract

We consider the compactification of the dual form of $N = 1$ $D = 10$ supergravity on a six-dimensional Calabi-Yau manifold. An $N = 1$ off-shell supergravity effective Lagrangian in four dimensions can be constructed in a dual version of the gravitational sector (new-minimal supergravity form). Superspace duality has a simple interpretation in terms of Poincaré duality of two-form cohomology.

The resulting 4D Lagrangian may describe the low-energy point-field limit of a five-brane theory, dual to string theory, provided Calabi-Yau spaces are consistent vacua of such dual theory.
1 Introduction

In recent years it has been shown that five-branes arise as “soliton” solutions \([1]\) of the effective theory describing 10\(D \) heterotic strings \([2]\), i. e. \(N = 1\) ten dimensional supergravity with a 2-form coupled to anomaly-free Yang-Mills multiplets \([3]\). In this respect, five-branes solutions are the analog of the monopole solutions of 4\(D \) gauge theories and they are non-perturbative in nature \([4, 5]\).

This allowed to speculate that if a fundamental theory of five-branes could be found, this theory would probably exchange weak and strong coupling regimes of string theory \([6, 7]\), by allowing us to get non-perturbative results of string dynamics by perturbative calculations of five-brane interactions.

In the case of a five-brane, the point-field theory limit in \(D = 10\) is \(N = 1\) supergravity in a dual formulation, i. e. with a six-form antisymmetric tensor \([8-11]\).

The aim of this paper is to present the most general \(N = 1\) off-shell supergravity action of a five-brane in \(D = 4\) with six dimensions compactified on a Calabi-Yau manifold \([12]\). Five-branes compactified on a six-dimensional torus have already been investigated and there \([13]\), the powerful constraints of \(N = 4\) supersymmetry have been extensively used to discuss the symmetry properties of the massive excitations \([13, 14]\).

Here we consider the effective theory in a case of \(N = 1\) unbroken supergravity in \(D = 4\) where it is also possible to present an off-shell formulation of the theory.

The main results of this investigation are:

1) The classical low-energy effective theory of 10\(D \) five-brane compactified on a Calabi-Yau manifold can not be written off-shell in standard supergravity form but a “dual” version of auxiliary supergravity fields must be used \([15]\).

2) Linear multiplets \([16]\), describing the zero modes of the Calabi-Yau manifold are naturally associated to the \(H(2, 2)\) cohomology which is dual to the \(H(1, 1)\) cohomology. This is encoded in a change of coordinates in the moduli space that has an obvious superspace interpretation.

3) Mirror symmetry can not be imposed any longer on the tree-level Lagrangian and can only arise if the space-time manifold \(M_4\) is equipped with a non-trivial topology \([17]\). This statement supports the folklore that tree-level results of string theory should arise as non-perturbative effects in the dual theory \([4, 5, 18]\). It also confirms some evidence that the \(\sigma\)-model and loop expansions are “dual” in the five-brane formulation of the theory.

4) The dilaton chiral multiplet is no longer protected by a perturbative Peccei-Quinn symmetry and its axion may have a discrete shift symmetry.
2 Four-dimensional reduction of ten dimensional supergravity with a 6-form and the zero mode analysis.

We would like to consider in this section some properties of the dual 10D supergravity theory when a six-form (rather than a two-form) is introduced in the theory [8-11].

Let us confine our analysis to the bosonic massless degrees of freedom of the gravitational sector, including the antisymmetric tensor as well as the dilaton. Let us first recall the zero mode analysis for the heterotic superstring theory. The bosonic massless fields of heterotic string theory are: the graviton $G_{\hat{\mu}\hat{\nu}}$, the 2-form $B_{\hat{\mu}\hat{\nu}}$, $\hat{\mu}, \hat{\nu} = 1, \ldots, 10$ and the dilaton $\Phi$. After compactification of six dimensions on a Calabi-Yau manifold they give rise to the following massless fields in four dimensions

$$G_{\hat{\mu}\hat{\nu}} \rightarrow (g_{\mu\nu}, g_{i\bar{j}}, g_{ij}, g_{i\bar{j}})$$

where $g_{\mu\nu}$ is the graviton field, and the internal components of the metric give

$$g_{i\bar{j}} \rightarrow (0; (1, 1)) - \text{form} \rightarrow h_{(1,1)} \text{ real scalars}$$

$$g_{ij} \rightarrow (0; (2, 1)) - \text{form} \rightarrow h_{(2,1)} \text{ complex scalars}$$

where we have denoted in brackets the degree of forms in $M_4 \times K_6$. For the ten-dimensional 2-form we get

$$B_{\hat{\mu}\hat{\nu}} \rightarrow (b_{\mu\nu}, b_{i\bar{j}})$$

where $b_{\mu\nu}$ is a $(2; (0, 0))$-form, dual to a pseudoscalar in space-time, while the internal components $b_{ij}$ give rise to space-time scalars:

$$b_{i\bar{j}} \rightarrow (0; (1, 1)) - \text{form} \rightarrow h_{(1,1)} \text{ real scalars}$$

Finally $\Phi$ gives rise to a $(0; (0, 0))$-form, i.e. the 4-dimensional dilaton:

$$\Phi \rightarrow \text{dilaton (real scalar)}$$

The supersymmetric multiplets are: the gravity multiplet ($g_{\mu\nu} \oplus \text{gravitino } \psi_{\mu}$); the linear multiplet [19-22] ($\Phi, b_{\mu\nu} \oplus \text{dilatino } \chi; h_{(1,1)}$) chiral...
multiplets \((g_{i\tau}, b_{i\tau} \oplus \text{fermionic partners})\); and \(h_{(2,1)}\) chiral multiplets \((g_{ij} \oplus \text{fermionic partners})\).

There is a (string) perturbative Peccei-Quinn symmetry due to the \(b_{\mu\nu}\) gauge symmetry, a perturbative \(\sigma\)-model Peccei-Quinn symmetry on the \(b_{i\tau}\), the last being broken by world-sheet non-perturbative effects \([23, 24]\), that is when

\[
\int b_{i\tau} dx^i d\tau^\tau \neq 0 \tag{2.7}
\]

corresponding to the presence of world-sheet instantons.

The \(b_{\mu\nu}\) gauge symmetry can be broken when \([17]\)

\[
\int H_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho \neq 0 \tag{2.8}
\]

i.e. if \(H^3(M_4, \mathbf{Z}) \neq 0\) and \(b_{\mu\nu}\) becomes a singular gauge field.

In the dual 10D supergravity theory, \(B_{\hat{\mu}\hat{\nu}}\) is replaced by \(A_{\hat{\mu}_1...\hat{\mu}_6}\). There are the same number of massless modes but now (2.5-2.7) are replaced by

\[
A_{\hat{\mu}_1...\hat{\mu}_6} \rightarrow (b \epsilon_{ijk} \tau_{\tau jk}, b_{\mu_\nu \tau jk}) \tag{2.9}
\]

where

\[
b\epsilon_{ijk} \tau_{\tau jk} \rightarrow (0; (3, 3)) - \text{form} \rightarrow 1 \text{ pseudoscalar} \tag{2.10}
\]

\[
b_{\mu_\nu \tau jk} \rightarrow (2; (2, 2)) - \text{form} \rightarrow h_{(1,1)} \text{ antisymmetric tensors}. \tag{2.11}
\]

So in the supergravity sector the multiplets are now: the dilaton chiral multiplet \((\Phi, b \oplus \text{fermionic partners})\) and \(h_{(1,1)}\) linear multiplets \((g_{ij}, b_{\mu_\nu \tau jk}, \oplus \text{fermionic partners})\). The other multiplets remain unchanged.

We observe that the linear multiplets are given in terms of \((1, 1)\) and \((2, 2)\) forms so it is natural to replace \(g_{i\tau}\) with its dual form defined as follows:

\[
*J = g_{i\tau k} dx^i d\tau^\tau dx^j d\tau^\tau \tag{2.12}
\]

where \(*J\) is the dual of the Kähler form \(J = ig_{i\tau} dx^i d\tau^\tau\).

In the five-brane theory, the perturbative and non-perturbative symmetries are best seen by looking to the \(p\)-brane sheet Wess-Zumino term:

\[
(1 - \text{brane} \equiv \text{String}) \rightarrow \int B_{\mu\nu} dx^\mu dx^\nu \tag{2.13}
\]

\[
(5 - \text{brane}) \rightarrow \int A_{\mu_1...\mu_6} dx^{\mu_1}...dx^{\mu_6} \tag{2.14}
\]

As a consequence of (2.10) the background fields which contribute to eq. (2.15) are
\[ \int b V_6 \]  \hspace{1cm} (2.15)

\[ \int b_{\mu_1 \mu_2 \cdots \mu_6} dx^\mu dx^\nu dx^j dx^l dx^k \]  \hspace{1cm} (2.16)

where \( V_6 \) denotes the volume form on the Calabi-Yau manifold.

The Peccei-Quinn symmetry on \( b \) is broken by classical effects since we may have \[ \int b V_6 \neq 0 \]  \hspace{1cm} (2.17)

while the Peccei-Quinn symmetry for the \((2,2)\) forms can be broken if the gauge invariance of the \( b_{\mu_\nu A} \), \( A = 1, \ldots, h_{(1,1)} \), 2-forms is broken. This can only happen if space-time has a non-trivial topology as was the case for the dilaton multiplet (see eq. (2.9)) in the dual string theory \[ \text{[7]} \].

Comparing eqs. (2.16) and (2.17) to eqs. (2.7) and (2.8) we see that the roles of the dilaton and moduli multiplets have been interchanged, supporting the idea that what is perturbative in strings is non-perturbative in five-branes and vice versa \[ \text{[1, 5, 13, 14, 25]} \].

### 3 Supersymmetry and duality transformations.

The string-effective theory for the \((1,1)\) and \((2,1)\) moduli and dilaton sector is given by the following superfield expression for the Lagrangian density \[ \text{[25-30]} \]

\[ \mathcal{L} = \Phi(T^A + \bar{T}^A, \psi_\alpha, S) S_0 \mathcal{S}_0 |_D. \]  \hspace{1cm} (3.1)

Here \( T^A, \psi_\alpha \) are the \((1,1)\) and \((2,1)\) moduli, \( S \) is the (dual) dilaton multiplet and \( \Phi \) is defined by

\[ \Phi = \left[ d_{ABC} (T + \bar{T})^A (T + \bar{T})^B (T + \bar{T})^C \right]^{1/3} (S + \bar{S})^{1/3} e^{-K_2(\psi)/3}. \]  \hspace{1cm} (3.2)

\( d_{ABC} \) are the intersection numbers of the Calabi-Yau manifold and the Kähler potential \( K \), which is additive in the three sectors of fields \( T^A, \psi \) and \( S \) is given by \[ \text{[32]} \]

\[ K = -3 ln \Phi \]  \hspace{1cm} (3.3)

\( S_0 \) is the chiral compensator of conformal supergravity \[ \text{[31-33]} \], which after superconformal gauge fixing, gives the off-shell standard supergravity multiplet.
To go to the five-brane formulation, one should perform a duality transformation \cite{34} between the $T^A$ multiplets and the $h_{(1,1)}$ linear multiplets $L_A$. However, this is impossible for $\Phi$ homogenous of degree 1 in the $T^A$ fields. Indeed, the duality transformation, which amounts to replace $\text{Re} T^A$ by the real linear multiplet $L_A$ via a superspace Legendre transformation \cite{34}, would give

$$\Phi_A = L_A$$

where $\Phi_A = \frac{\partial \Phi}{\partial T^A} = \frac{\partial \Phi}{\partial (T^A + \overline{T}^A)}$. This implies

$$\frac{\partial \Phi_A}{\partial L_B} = \frac{\partial (T + \overline{T})^C}{\partial L_B} \Phi_{AC} = \delta_B^A$$

which gives

$$\frac{\partial (T + \overline{T})^C}{\partial L_B} \Phi_{AC}(T + \overline{T})^A = (T + \overline{T})^B$$

and this is impossible since $\Phi_{AC}(T + \overline{T})^A = 0$. The lack of invertibility of this relation was noticed in \cite{35} in the context of no-scale supergravity models \cite{36}. Recently it has been discussed in \cite{25} for an orbifold compactification of the 10D five-brane.

This difficulty means that linear multiplets with particular couplings cannot be coupled to standard supergravity.

However we can cure this difficulty by going to the dual formulation of supergravity by “dualizing” \cite{34} the compensator $S_0$ into a linear multiplet compensator $L_0$.

This is done by replacing $S_0 \overline{S}_0 \rightarrow e^U$ and by imposing the condition $U = \Sigma + \Sigma^* (S_0 = e^\Sigma)$ by adding the Lagrange multiplier term $-UL_0$ to the action (3.1).

A trivial calculation gives for the dual form of eq. (3.1)

$$\tilde{\mathcal{L}} = -L_0 \ln L_0 + L_0 \ln \Phi = L_0 \ln \Phi / L_0$$

which describes a matter system coupled to “new-minimal” supergravity with a “dual” gravity multiplet given by \cite{13, 37}

$$(e_{a\mu}, \psi_\mu, A_\mu, a_{\mu\nu})$$

Since $\ln \Phi$ is additive in the different multiplets we may now dualize separately the term in $\Phi$ containing the $T^A$ scalars:

$$\frac{1}{3} L_0 \ln \left[ d_{ABC}(T + \overline{T})^A(T + \overline{T})^B(T + \overline{T})^C \right]$$

Since $\ln \Phi$ is additive in the different multiplets we may now dualize separately the term in $\Phi$ containing the $T^A$ scalars:
which must be replaced by

$$\frac{1}{3} L_0 \ln(d_{ABC} U^A U^B U^C) - U^A L_A$$  \hspace{1cm} (3.11)

It is immediate to see that eq. (3.11) gives rise to the following term in the modified Lagrangian of $\tilde{\mathcal{L}}$

$$\frac{1}{3} L_0 \ln \left[ d_{ABC} (T + \overline{T})^A (L/L_0) (T + \overline{T})^B (L/L_0) (T + \overline{T})^C (L/L_0) \right]$$  \hspace{1cm} (3.12)

where $(T + \overline{T})^A (L/L_0)$ is the solution of the equations

$$d_{ABC}(T + \overline{T})^B (T + \overline{T})^C \frac{d_{DEF}(T + \overline{T})^D (T + \overline{T})^E (T + \overline{T})^F}{d_{ABC}(T + \overline{T})^A (L/L_0)} = \frac{L_A}{L_0}$$  \hspace{1cm} (3.13)

Eq. (3.13) can be inverted. Indeed, if one takes

$$G_{AB} = -\partial_A \partial_B \ln(\Phi^3) \quad \left( \partial_A = \frac{\partial}{\partial T^A} \right)$$  \hspace{1cm} (3.14)

one clearly realizes that expression (3.13) can be rewritten as

$$\frac{1}{3} G_{AB} (T + \overline{T})^B = \frac{L_A}{L_0}$$  \hspace{1cm} (3.15)

with $G_{AB}$ invertible.

The final Lagrangian is therefore

$$\mathcal{L}_{TOT,AL} = -L_0 \ln L_0 + \tilde{\mathcal{L}}$$  \hspace{1cm} (3.16)

where

$$\tilde{\mathcal{L}} = L_0 \left[ \mathcal{F} \left( \frac{L_A}{L_0} \right) + \frac{1}{3} \ln(S + \overline{S}) - \frac{1}{3} K_2(\psi) \right] \bigg|_D$$  \hspace{1cm} (3.17)

with $\mathcal{F} (L_A/L_0)$ given by (3.12) without the factor $L_0$.

To make the dependence in $\mathcal{F}$ explicit it is convenient to change variables in eq.(3.11)

$$U^A = \hat{U}^A R \quad R = \left( d_{ABC} U^A U^B U^C \right)^{1/3}$$  \hspace{1cm} (3.18)

Then eq.(3.13) becomes

$$R = \frac{L_0}{L_A \hat{U}^A} \quad \left( d_{ABC} \hat{U}^A \hat{U}^B \hat{U}^C = 1 \right)$$  \hspace{1cm} (3.19)

with $\hat{U}^A$ homogeneous (of degree 0) in the $L_A$’s given by
The function $\mathcal{F}$ becomes

$$\mathcal{F}(L_A/L_0) = \ln R = \ln \frac{L_0}{L_A U^A}$$

(3.21)

The linear multiplet contribution in $\mathcal{L}_{TOTAL}$ is therefore given by

$$- L_0 \ln L_A U^A(L)$$

(3.22)

where

$$L_A \frac{\partial}{\partial L_A} U^B(L) = 0.$$  

(3.23)

Hence the Lagrangian (3.16) cannot be written in standard supergravity form because the inverse duality transformation $L_0 \to S_0$ can not be performed due to the fact that $L_0$ appears linearly in eq. (3.16).

We remark that this action at the quantum level is potentially anomalous due to the gauge invariance implied by the auxiliary field sector of the supergravity multiplet [38, 22]. If therefore a consistent quantum theory exists, a quantum anomaly cancellation should take place.

4 Results at the component level.

The most general $N = 1$ supergravity Lagrangian in new-minimal formulation containing at most two derivatives is described by 3 functions: a real function $F(S_i, S_i, L_A)$ of a set of matter (chiral and linear) multiplets, a holomorphic function $f(S_i)$ of the chiral multiplets and a Yang-Mills covariant symmetric holomorphic tensor $f_{ab}(S_i)$. The Poincaré linear multiplets $L_A$ correspond to the ratios $L_A/L_0$ of conformal multiplets of the previous section, while the Poincaré function $F$ is related to the conformal function $\mathcal{F}$ after superconformal gauge fixing by $F = -3\mathcal{F}$. The bosonic part of the Lagrangian has been given in [37]:

$$e^{-1} \mathcal{L}_{BOS} = \frac{1}{2} (1 - (nz)^j F_j) (R + 6 H_\mu H^\mu) + (2 A_\mu + i F_j D_\mu z^j - i F_\mu D_\mu z^7) H^\mu$$

$$+ \frac{1}{4} F^{AB} \partial_\mu C_A \partial^\mu C_B - \frac{1}{4} F^{AB} V_\mu V_B^\mu$$

$$+ \frac{i}{2} (F^A_j D_\mu z^j - F^A_\mu D_\mu z^7) V_A^\mu$$

$$- F_\mu D_7 z^7 D^\mu z^7 + F_{j7} h^j h^7 + f_j h^j + \bar{f}_j \bar{h}^j$$
\[- \frac{1}{4} \text{Re} f_{ab} F_{\mu\nu}^a F^{b,\mu\nu} + \frac{1}{4} \text{Im} f_{ab} F_{\mu\nu}^a F^{b,\mu\nu} + \frac{1}{2} \text{Re} f_{ab} D^a D^b + (T_\alpha z)^j F_j D^a \]

where $H^\mu$ is the field strength of the auxiliary field $a_{\mu\nu}$ in (3.9), $A_\mu = A_\mu - 3H^\mu$ and $D^-_\mu$ are covariant derivatives with a modified chiral connection (see [37]).

In eq. (4.1) upper indices on $F$ denote differentiation with respect to the linear multiplet scalars ($C_A$) and lower indices differentiation with respect to the chiral and antichiral ones ($z^i$, $\overline{z}^j$).

We start with a function $F((T + \overline{T})^A, Z^i)$ that does not contain linear multiplets but where some of the chiral ones ($T^A$) appear only in the particular combination $(T + \overline{T})^A$. These multiplets can appear in $f_{ab}$ in the same real combination (though we will not consider it here), but must be absent from the superpotential $f$. The terms of the bosonic Lagrangian (4.1) containing fields of these multiplets are

\[ e^{-1} \hat{L} = i(F_A \partial_\mu t^A - F_A \partial_\mu \overline{T}^A)H^\mu - F_{AB} \partial_\mu t^A \partial^\mu t^B - F_A \partial_\mu t^A D^-_\mu \overline{z}^j \]

\[ - F_{jA} D^-_\mu z^j \partial^\mu \overline{T}^A + F_{AB} h^A \overline{h}^B + F_A h^A \overline{h} + F_{jB} h^j \overline{h}^B \] (4.2)

Since we intend to dualize the Lagrangian in (4.2) into one containing linear multiplets instead of the chiral $T^A$, we must eliminate the auxiliary fields $h^A$ from their equations of motion:

\[ h^A = -(F^{-1})^{BA} F_{jB} h^j \]

\[ \overline{h}^A = -(F^{-1})^{AB} F_{jA} \overline{h}^j \] (4.3)

which upon replacement into $\hat{L}$ gives for the auxiliary field part

\[ e^{-1} \hat{L}_{AUX} = -h^i F_i c (F^{-1})^{CD} F_{17} \overline{h}\] (4.4)

The next step is to perform the duality transformation on the kinetic part. For that [13] we write

\[ t^A = \text{Re} t^A + i \text{Im} t^A \] (4.5)

then make the replacement $\partial_\mu \text{Im} t^A \rightarrow L^A_\mu$ and add the necessary Lagrange multipliers:

\[ e^{-1} \hat{L}'_{KIN} = -F_{AB} \partial_\mu \text{Re} t^A \partial^\mu \text{Re} t^B - (F_{jA} D^-_\mu z^j + F_{jA} D^-_\mu \overline{z}^j) \partial^\mu \text{Re} t^A \]

\[ - F_{AB} L^A_\mu L^B_\mu - 2F A L^A_\mu \overline{A}_\mu H^\mu - i(F_{17} D^-_\mu \overline{z}^j - F_{jA} D^-_\mu z^j) L^A_\mu \]

\[ - \frac{1}{2} e^{-1} \epsilon^{\mu\nu\rho\sigma} b_{\mu A} \partial_\rho L^A_\sigma \] (4.6)

The equation of motion for $L^A_\mu$ is then
\[ L^{\mu B} = (F^{-1})^{BC} \left[ -F_C H^\mu + \frac{1}{2} v_\mu^C + \frac{i}{2} \left( F_J C D^{-\mu} z^j - F_C D^{-\mu} \mathcal{Z}^j \right) \right] \]  

(4.7)

with

\[ v_\mu^\rho = -\frac{1}{2} e^{-1} e^{\mu \nu \rho \sigma} \partial_\nu b_{\rho \sigma} \]  

(4.8)

Replacing (4.7) in (4.6) we get finally

\[ e^{-1} \hat{\mathcal{L}}'_{KIN} = -F_{AB} \partial_\mu R e t^A \partial^\mu R e t^B - (F_J A D^{-\mu} z^j + F_{A^7} D^{-\mu} \mathcal{Z}^j) \partial^\mu R e t^A \]

\[ + \frac{1}{4} (F^{-1})^{AB} \left[ v_\mu^A - 2 F_A H^\mu + i (F_J A D^{-\mu} z^j - F_{A^7} D^{-\mu} \mathcal{Z}^j) \right] \]

\[ \times \left[ v_B^\rho - 2 F_B H_\rho + i (F_J B D^{-\rho} z^j - F_{B^7} D^{-\rho} \mathcal{Z}^j) \right] \]

(4.9)

This expression for \( \hat{\mathcal{L}}'_{KIN} \) seems to imply (compare with eq. (4.1)) that the scalar partner of \( b_{\mu \nu} \) is

\[ C_A = F_A \]  

(4.10)

Indeed, the kinetic term for the \( b_{\mu \nu} \) field dictates the presence of the following scalar kinetic term:

\[ -\frac{1}{4} (F^{-1})^{AB} \partial_\mu F_A \partial^\mu F_B = \]

\[ -F_{AB} \partial_\mu R e t^A \partial^\mu R e t^B - (F_J A D^{-\mu} z^j + F_{A^7} D^{-\mu} \mathcal{Z}^j) \partial^\mu R e t^A \]

\[ - \frac{1}{4} (F^{-1})^{AB} (F_J A D^{-\mu} z^j - F_{A^7} D^{-\mu} \mathcal{Z}^j) (F_J B D^{-\rho} z^j - F_{B^7} D^{-\rho} \mathcal{Z}^j) \]

\[ + (F^{-1})^{AB} F_A F_{B^7} D^{-\mu} z^j D^{-\rho} \mathcal{Z}^j \]  

(4.11)

where we have used the chain rule as well as the properties that \( F \) must satisfy [37].

Thus (4.9) provides not only the kinetic term for \( C_A \) in (4.10) but also the extra kinetic contribution for the \( z^i \) fields that, together with (4.4), is precisely what we need in order to have the appropriate form of the Lagrangian corresponding to the new function \( \tilde{F} \) that one obtains after completing the dualization procedure. That function is

\[ \tilde{F}(L, \ldots) = F(T(L) + \mathcal{T}(L), \ldots) - L_A (T(L) + \mathcal{T}(L))^A \]  

(4.12)

where \( T(L) \) in (4.12) is given by the solution of

\[ L_A = F_A = \frac{\partial F}{\partial T^A} \]  

(4.13)

whose first component is (4.10) and whose inverse is

\[ T^A + \mathcal{T}^A = -\tilde{F}^A = -\frac{\partial \tilde{F}}{\partial L_A} \]  

(4.14)
In order to verify what we have stated above, we do not need to invert explicitly (4.10) (in other words, we do not need to find the particular form of $\tilde{F}$) which may be difficult, but it is sufficient to use the following identities that always hold

$$
\begin{align*}
\tilde{F}^A_j &= F^A_j & \tilde{F}_7 &= F_7 \\
\tilde{F}^{AB} &= -(F^{-1})^{AB} & F_{AB} &= -(\tilde{F}^{-1})_{AB} \\
\tilde{F}^A_j &= (F^{-1})^{AB} F^B_j & F^A_j &= -(F^{-1})_{AB} \tilde{F}_B^j \\
\tilde{F}^A_7 &= (F^{-1})^{AB} F_{B7} & F_{A7} &= -(\tilde{F}^{-1})_{AB} \tilde{F}^B_7 \\
\tilde{F}_{ij} &= F_{ij} - F_i A (F^{-1})^{AB} F^A_j & \tilde{F}_{ij} &= F_i j + F^A_i (\tilde{F}^{-1})_{AB} \tilde{F}^B_j
\end{align*}
$$

(4.15)

It is worth stressing that, even though it may be impossible to obtain explicitly $\tilde{F}$ from $F$, or viceversa, the hybrid formulation (as given by (4.4), (4.9) plus the unmodified part of the Lagrangian) can always be reached from either $\tilde{F}$ or $F$.

Let us also mention that, even though we have not done it explicitly here, it is a simple matter to include the case when some of the $T^A$ multiplets appear in the function $f_{ab}$, where there is an extra Chern-Simons term modifying $v^A$ in (4.7) and (4.9) [19, 37].

Next we would like to show how the fact that certain particular couplings of linear multiplets can not be written in standard supergravity form, manifests itself in component language. Clearly, the duality transformation to the old-minimal set of auxiliary fields can always be performed except when the $H^\mu H^\mu$ term in the Lagrangian (4.1) vanishes, in which case we can not solve for $H^\mu$ from its equation of motion. Requiring that the total coefficient of $H^\mu H^\mu$ vanishes provides us the following obstruction condition for the function $F(L_A, Z_i, Z^\prime_\bar{i})$ in terms of its $\theta = 0$ component

$$
F^{AB} C_A C_B + 3 \left[ 1 - (nz)^j F_j \right] + 6 F^A j C_A (nz)^j + 9 F_i j (nz)^j (nz)^{\bar{i}} = 0
$$

(4.16)

There are some important aspects to note in this equation. First, only the function $F$ appears in it; the superpotential and $f_{ab}$ are not involved. Second, these exceptional theories satisfying (4.16) constitute a very large class since, if $F_{part}(C, z, \bar{z})$ is a solution, so it will be

$$
F(C, z, \bar{z}) = F_{part}(C, z, \bar{z}) + F_{(1)}(C) + F_{(0)}(C)
$$

(4.17)

where $F_{(1)}(C)$ and $F_{(0)}(C)$ are arbitrary homogenous functions of degrees 1 and 0 respectively of the $C_A$ fields.

A particular solution to (4.16) involving only linear multiplets is easy to find

$$
F(C) = \sum_A \alpha^A \ln C_A , \quad \sum_A \alpha^A = -3
$$

(4.18)

Another example is provided by the effective Lagrangian for the superstring [39] which corresponds to the function

$$
F(L, S + \bar{S}, Z\bar{Z}) = -3 \ln \frac{(S + \bar{S})^{\frac{1}{2}}}{L} + 3 Z\bar{Z} (S + \bar{S})^{\frac{1}{2}}
$$

(4.19)
where the chiral multiplet $Z$ has chiral weight $n = 1/3$.

5 Superspace duality and $H^{(2,2)}$ cohomology.

In the previous paragraph we have seen that the duality transformation which maps the $T^A$ chiral multiplets into linear multiplets $L_A$ (eq. (3.13)) also gives a coordinate transformation between $Re t^A$ and the $\theta = 0$ components of the linear multiplets

$$C_A = L_A |_{\theta = 0} \quad (5.1)$$

This coordinate transformation has a simple interpretation in the Calabi-Yau geometry. Indeed, the $b_{\mu\nu A}$ component of $L_A$ is an element of $H^{(2,2)}$ while the metric is an element of $H^{(1,1)}$. It is then natural to map the metric $G_{ij}$ into its dual counterpart which is an element of $H^{(2,2)}$. This map induces a reparametrization of the associated moduli scalars.

For this purpose let us recall that [26, 31], if $G_{AB}$ is the metric for the $(1,1)$ forms moduli space, then

$$G_{AB} \int J \wedge J \wedge J = \frac{3}{2} \int V_A \wedge^* V_B \quad (5.2)$$

where the Kähler form $J$ can be expanded as follows

$$J = \sum_A \lambda^A V_A \quad V_A \in H^{(1,1)} \quad (5.3)$$

where the $V_A$ form a basis of $H^{(1,1)}$.

We also have

$$\int J \wedge V_A \wedge V_B = d_{ABC} \lambda^C, \quad \int J \wedge J \wedge J = d_{ABC} \lambda^A \lambda^B \lambda^C \quad (5.4)$$

The moduli coordinates $\lambda^A$ are defined through the dual homology 2-cycles $\gamma_A$ by

$$\lambda^A = \int_{\gamma_A} J \quad (5.5)$$

and are identified with $(T + \overline{T})^A/2$.

It is now natural to introduce a “dual” basis $\Omega^A$ in $H^{(2,2)}$ such that
\[ \int V_B \wedge \Omega^A = \delta^A_B \]  

which then implies

\[ \lambda^A = \int J \wedge \Omega^A \]  

Let us next consider the dual of the Kähler form

\[ *J = \sum_{A=1}^{h_{(1,1)}} \lambda^A *V_A \]  

which we now expand in the dual basis \( \Omega^A \) defined by (5.6), (5.7):

\[ *J = \sum_{A=1}^{h_{(1,1)}} C_A \int J \wedge J \wedge J \]  

By comparing eq. (5.8) with (5.9) and using (5.2) we find

\[ *V_A = \frac{2}{3} G_{AB} \Omega^A \int J \wedge J \wedge J \]  

implying

\[ \frac{2}{3} G_{AB} \lambda^B = C_A \]  

which is nothing but eq. (3.15).

We then may reinterpret the superspace duality relation (3.15) as the change of coordinates from the basis \( V_A \) of \( H^{(1,1)} \) to the dual basis \( \Omega^A \) of \( H^{(2,2)} \) with \( \Omega^A \) defined through eq. (5.6). Note that the normalization factor in (5.9) is such that

\[ \sum_A C_A \lambda^A = \frac{1}{2} \]  

6 Conclusions.

In this note we have considered the dual form of \( N = 1 \), \( D = 10 \) supergravity and studied its compactification on a Calabi-Yau threefold, which is known to be a consistent vacuum of the dual counterpart.
We have presented an off-shell $N = 1$ supergravity version of the resulting $4D$ effective theory which turns out not to be in standard $N = 1$ supergravity form.

Several type of duality transformations have been discussed, in particular the relation between superspace duality and Poincaré duality of the internal manifold.

The extension of this Lagrangian to include effects of the finite size of the five-brane has not been discussed. The relation between the perturbative expansion versus a strong coupling regime of the dual theory is also an important question to be addressed in this context. In particular the dilaton multiplet, which is now a chiral multiplet, can have non-linear interactions and the theory may exhibit a discrete duality symmetry similar to that of the moduli fields in string theory compactification. There is some evidence that, at least for toroidal compactifications, this symmetry is $SL(2, \mathbb{Z})$ [13, 14, 18]. The study of a coordinate free formulation of the linear multiplet geometry, which is “dual” to special geometry [27-30] of heterotic strings is also an interesting aspect to be investigated [26-30].

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