Quantum Information Scrambling in Systems with Nonlocal Interactions

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How fast quantum information scrambles such that it becomes inaccessible by local probes turns out to be central to various fields. Motivated by recent works on spin systems with nonlocal interactions, we study information scrambling in different variants of the Ising model. Our work reveals operator dynamics in the presence of nonlocal interactions not precisely captured by out-of-time-order correlators (OTOCs). In particular, the operator size exhibits a slowdown in systems with generic powerlaw interactions despite a highly nonlinear lightcone. A recently proposed microscopic model for fast scrambling does not show this slowdown, which uncovers a distinct analogy between a local operator under unitary evolution and the entanglement entropy following a quantum quench. Our work gives new insights on scrambling properties of systems in reach of current quantum simulation platforms and complements results on possibly observing features of quantum gravity in the laboratory.

Introduction. — The dynamics of quantum information under unitary evolution lies at the heart of numerous ongoing questions in theoretical physics [1, 2]. Due to significant improvements on quantum simulation platforms [3–7], we are nowadays able to probe information dynamics of simple quantum lattice models in an experimental environment [8–13]. In particular, in nonintegrable many-body systems, initial local quantum information can spread under unitary evolution such that local measurements are insufficient to reconstruct it at later times. This scrambling of quantum information has received a great deal of attention lately. It is inherently related to thermalization [14, 15] and its absence [16, 17], as well as the simulability of many-body systems [18], and even quantum gravity [19].

Information Scrambling. — One particular probe of scrambling relates to the growing support and complexity of local operators under unitary evolution [20, 21], known as operator spreading. We can diagnose the spread of a local operator \( W \) via the squared commutator with an auxiliary operator \( V \) at some distant site \( r \)

\[
C_r(t) = \frac{1}{2} \left\langle [W(t), V_r]^\dagger [W(t), V_r] \right\rangle, \tag{1}
\]

where the expectation value is either evaluated in some pure state \( |\Psi\rangle \), or a thermal state \( \rho_{\text{th}} \sim e^{-\beta H} \). Once the operator \( W(t) \) has spread such that its support overlaps with \( V \), Eq. (1) begins to grow and saturates afterwards at some finite value. By varying \( r \), one can track how the operator spreads over the system’s degrees of freedom. If \( W \) and \( V \) are unitary, the nontrivial part of Eq. (1) is determined by the OTOC, i.e., \( F_r(t) = \left\langle [W(t)V_r, W(t)V_r] \right\rangle \).

In a local quantum system, an emergent lightcone constrains information propagation in accordance with the Lieb-Robinson bound [22, 23]. However, today’s experimental platforms often entail nonlocal (powerlaw) interactions \( \sim 1/r^\alpha \), leading to Hamiltonians that do not necessarily comply the assumption of locality. Seminal studies on systems with powerlaw interactions and experiments with trapped ions revealed vastly different nonequilibrium physics [24–28], e.g., the breakdown of quasilocality. That is, information can propagate faster than allowed by the Lieb-Robinson bound. Since then, many works have focused on generalized bounds for systems with powerlaw interactions [29–33], the akin process of operator spreading [34–36] and improved protocols for information processing tasks such as state transfer [31, 37, 38]. In a nutshell, powerlaw interactions can induce an emergent nonlinear ‘lightcone’, i.e., information about a local operator can spread superballistically over the system’s degrees of freedom, see Fig. 1(a).

Surprisingly, powerlaw interactions typically lead to a slowdown of entanglement growth [24, 39]. It can be logarithmically slow for \( \alpha < d \), where \( d \) is the spatial dimension [40, 41]. The entanglement entropy of a region \( A \) regarding a nonequilibrium state \( |\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle \) is given by the Von Neumann entropy of its reduced density matrix \( S_A(\rho(t)) = -\text{Tr}[\rho_A \log(\rho_A)] \). Essentially, it probes how information about \( A \) becomes inaccessible by measurements on \( A \) due to entanglement with increasingly many degrees of freedom and is thereby a complementary probe of scrambling. At first sight, operator spreading and entanglement growth seem to behave oppositely in the given scenario, and some studies argue that they are two different phenomena [42].

Fast Scrambling. — Recently, systems with nonlocal interactions appeared in connection to the correspondence of anti-de Sitter space and conformal field theories (AdS/CFT), where information scrambling has become a central topic [43–50]. The property of fast scrambling, i.e., a system where \( C_r(t) \sim O(1) \forall r \) in a time \( t \sim \log(N) \) is believed to be characteristic of black holes [44, 45], and holographic duals to theories of quan-
tum gravity, e.g., the Sachdev-Ye-Kitaev (SYK) model [51, 52]. The highly complex structure of the latter, however, renders an experimental probe of fast scrambling challenging, and several proposals for simpler models with this property have appeared [53–55]. The proposals in Refs. [54, 55] follow a similar structure: a fine tuned combination of a local Hamiltonian and a nonlocal all-to-all interaction \( \sim 1/r^\alpha \). Noteworthy, a slowdown of entanglement growth is absent in these models [54–56].

In addition, fast scrambling has been ruled out for systems with generic powerlaw interactions if \( \alpha > d \), where lightcones are at most polynomial [57]. Whether or not this remains true for \( \alpha \leq d \) is an open question. However, from an entanglement perspective, a system with (strong) powerlaw interactions starkly differs from the proposed fast scramblers, see Fig. 1(b).

If entanglement growth and operator spreading are both probes of information scrambling, is there a quantitative difference in operator dynamics in systems with unlike entanglement dynamics? In this Letter, we shed more light on this question and provide evidence for a distinct relation between entanglement growth and operator spreading. A local operator under unitary evolution exhibits, in a sense we make more precise, a growth that relates to the entanglement entropy following a quantum quench. Consequently, operator dynamics in a fast scrambling model is in sharp contrast to systems with generic powerlaw interactions. Our results underline that a profound understanding of the relationship between entanglement and operator dynamics may improve our ability to characterize nonequilibrium quantum matter.

![FIG. 1. Quantum information scrambling for different Ising Hamiltonians. Black (circle): local Hamiltonian \( \mathcal{H}_\infty \), Orange (diamond): powerlaw Hamiltonian \( \mathcal{H}_\alpha \) with \( \alpha = 1.1 \) and \( \kappa = 1 \), Green (hexagon): fast scrambling Hamiltonian \( \mathcal{H}_{FS} \), Purple (star): powerlaw Hamiltonian \( \mathcal{H}_\alpha \) with \( \alpha = 0.4 \) and \( \kappa = 1 \). (a) Operator spreading: Spacetime contour \( t_0(r) \) determined by \( C_e(t_0) = 0.5 \). Squared commutator is evaluated in the initial state \( |Y^+\rangle \), \( \mathcal{W} = \mathcal{V} = \mathcal{Y} \) and \( N = 32 \) (TDVP). Note that different thresholds lead to similar contour shapes. (b) Entanglement growth: Half-chain entanglement entropy (normalized by the Page value) following a quench from \( |Y^+\rangle \) for \( N = 28 \) (EXPM).](image)

Models.— In particular, let us consider the one-dimensional mixed field Ising model of \( N \) qubits with open boundary conditions

\[
\mathcal{H}_\alpha = - \sum_{m<n} J_{mn}^\alpha Z_m Z_n - h_x \sum_m X_m - h_z \sum_m Z_m, \tag{2}
\]

where \( X_m, Z_m \) are Pauli X, Z operators acting on site \( m \), and \( h_x = -1.05, h_z = 0.5 \) throughout this work. Interactions among the qubits decay as a powerlaw \( J_{mn}^\alpha = 1/\kappa \cdot J/|m-n|^{\alpha} \), where \( \kappa \) is chosen to be either unity, or the Kac normalization [58], i.e., \( \kappa = \frac{1}{N-1} \sum_{m<n} J_{mn} \) [59]. Moreover, we include the fast scrambling proposal from Ref. [54] whose Hamiltonian is given by

\[
\mathcal{H}_{FS} = \mathcal{H}_\infty - \frac{1}{\sqrt{N}} \sum_{m<n} Z_m Z_n. \tag{3}
\]

The local version of Eq. (2), i.e., \( \mathcal{H}_\infty \) has been intensively studied in the context of information scrambling [46, 47, 49, 60, 61] as it is maximally chaotic and exhibits strong thermalization [62, 63]. Moreover, in the particular quench scenario we consider here, the entropy of any region at long times is given by the value expected in a random state drawn from the Haar measure, known as the Page value \( S_P \) [64].

We simulate the dynamics generated by Eq. (2) and (3) numerically [65]. Our focus is on early to intermediate times as the considered Hamiltonians display substantially different dynamics there, while the dynamics at long times turns out to be uniform [66].

Regimes.— A Hamiltonian like Eq. (2) will generally possess a regime of \( \alpha \) with effectively local dynamics [31, 33, 67], which we find to hold at least for \( \alpha > 2 \). In this regime, the respective velocities of entanglement growth and operator spreading show a similar dependence on the exponent \( \alpha \) [66]. For smaller exponents, the dynamics becomes nonlocal. To stress the effect on information scrambling, we compare the local Hamiltonian \( \mathcal{H}_\infty \) against \( \mathcal{H}_\alpha \) with \( \alpha = 1.1 \). Moreover, since fast scrambling might be possible if \( \alpha \leq 1 \) and in Ref. [35] it was argued for a logarithmic lightcone if \( \alpha \leq 1/2 \), we compare the dynamics of \( \mathcal{H}_{FS} \) against \( \mathcal{H}_\alpha \) with \( \alpha = 0.4 \), where we set \( \kappa = 1 \) in this case. In Fig. 1(a) we display a spacetime contour of the squared commutator (1), evaluated in the state \( |\Psi_0\rangle = |Y^+\rangle \), where we choose \( \mathcal{W} = \mathcal{V} = \mathcal{Y} \). Figure 1(b) shows the entanglement entropy following a quench from the initial state \( |\Psi_0\rangle \).

As discussed earlier, both the system with powerlaw interactions and the fast scrambler induce a highly nonlinear lightcone as opposed to the linear one associated with the local system (Fig. 1(a)). Nevertheless, the slowdown of entropy growth that one observes for systems with powerlaw interactions is absent for the fast scrambler (Fig. 1(b)), elucidating very different entanglement dynamics.
Operator State. — To unveil the interplay between entanglement growth and operator spreading, it is constructive to consider states of the form

$$|\Phi(t)\rangle := \mathcal{W}(t)|\Psi_0\rangle.$$  \hspace{1cm} (4)

If $|\Psi_0\rangle$ is a product state, the entropy $S_A(|\Phi(t)\rangle)$ of a region $A$ that contains the initial position of $\mathcal{W}$ will vanish as long as the support of $\mathcal{W}(t)$ is confined to $A$, see Fig. 2(a). Once the operator has spread beyond $A$, entropy will grow as information about the operator is leaking out, see Fig. 2(b). Entanglement growth of Eq. (4) is therefore in direct correspondence to the spread of $\mathcal{W}(t)$.

In Fig. 3, we compare the entanglement entropy $S_A(|\Phi(t)\rangle)$ of the left block $A$ with the squared commutator $C_r(t)$, where $r$ is chosen as either the leftmost or the rightmost site of the right block $B$, see Fig. 2(c) for an illustration. For the local model (Fig. 3(a)), entanglement growth agrees with the spatiotemporal structure of the squared commutator. That is, the entropy of the left block $A$ begins to grow, once the squared commutator diagnoses that the support of $\mathcal{W}(t)$ overlaps with the the right block $B$. Shortly after the support has reached the rightmost site of $B$, entropy saturates in line with the squared commutator. The fast scrambler exhibits very different dynamics, which is highly nonlocal as both entanglement entropy and squared commutator begin to grow immediately. However, both capture the same operator dynamics and behave similarly up to saturation, see Fig. 3(b).

On the contrary, this does not hold for powerlaw interactions. Although initial entropy growth agrees with the squared commutator, we observe a slowdown at intermediate times similar to the ordinary quench scenario, see Fig. 3(c), (d). While the squared commutator has reached its saturation value $C_r(t) \simeq 1$ (up to oscillations around it), the entropy $S_A(|\Phi(t)\rangle)$ is still growing.

Thus, information about $\mathcal{W}(t)$ is still leaking out of the left block $A$, although its support extends over the entire system for some time.

Noteworthy, entropies of $|\Phi(t)\rangle$ approach the Page value at long times for $\mathcal{W} = \mathcal{V}$, therefore, $\mathcal{V}(\infty)$ acts (approximately) like a Haar random unitary. Generally, if $\mathcal{V}_r|\Psi_0\rangle \sim |\Psi_0\rangle$, the squared commutator evaluated in the initial state $|\Psi_0\rangle \sim |\Phi(t)\rangle |\mathcal{V}_r|\Phi(t)\rangle$. Thus, if $|\Phi(t)\rangle$ is Haar random and $\mathcal{V}$ a traceless local operator, it follows that $C_r(t) \simeq 1 \forall r$ [68].

The converse of this statement seems to be false, which we observe for systems with powerlaw interactions. Although the spatiotemporal structure of the squared commutator is already homogeneous, i.e., $C_r(t) \simeq 1 \forall r$, the operator $\mathcal{V}(t)$ is still evolving towards being Haar random as reflected by sub-Page value of $S_A(|\Phi(t)\rangle)$. This indicates that some part of the operator dynamics is not properly captured by the squared commutator. Moreover, it suggests a slowdown of operator dynamics in the presence of (strong) powerlaw interactions similar to the entanglement entropy following a quantum quench.

Beyond the Quench. — So far, our focus was on the quench scenario, which is biased towards the (highly excited) initial state $|\Psi_0\rangle$. This begs the question of how much of these insights are due to this choice. For a more general treatment, let us recall that any operator can be expanded in terms of a complete orthonormal operator basis, i.e.,

$$\mathcal{W}(t) = \sum_{\lambda} c_{\lambda}(t) S_{\lambda},$$  \hspace{1cm} (5)

where $S_{\lambda} = \bigotimes_{\lambda \in A} P_{\lambda}$ are Pauli strings, with $P =$
\[ \{1, x, y, z\}, \text{ and } \text{Tr} \left( S_A S_P \right) / 2^N = \delta_{MN}. \] Considering an operator \( W \) initially supported on the leftmost site of the system, a useful measure based on the expansion (5) is the \textit{operator density} \[ p_\ell (t) = \sum_{|\Lambda| = \ell} |c_\Lambda (t)|^2, \tag{6} \]

where the sum runs over all strings whose rightmost non-identity site is \( \ell \). Note that \( \sum_\ell p_\ell (t) = 1 \forall t \). Thus, Eq. (6) measures how much weight of the operator is in strings whose support ranges from the first to the \( \ell \)-th site, i.e., strings of size \( \ell \). The operator density is related to the squared commutator evaluated in the infinite temperature ensemble \[ [\hat{A}, \hat{B}]^2 \text{ evaluated in the infinite temperature ensemble}. \]

With use of Eq. (6), one can define the \textit{operator size} as \[ L [\mathcal{W} (t)] = \sum_\ell \ell p_\ell (t). \tag{7} \]

Generally, we expect the operator size to grow monotonically and saturate at some value \( \sim N \) at long times. If \( \mathcal{W} (\infty) \) is Haar random, the coefficients in (5) should be uniformly distributed (excluding the identity) \[ [73]. \] Therefore, the operator density \( p_\ell \) is determined by the number of strings with size \( \ell \), i.e., \( p_\ell (\infty) \approx 3 \cdot 4^{\ell-1} / (4^N - 1) \). The operator size at long times then becomes

\[ L_{\text{Haar}} = N \left( 1 + \frac{1}{4^N - 1} \right) - \frac{1}{3} \approx N - \frac{1}{3}. \tag{8} \]

Equation (8) is the average operator size of a random unitary of \( N \) qubits drawn from the Haar measure.

To evaluate Eq. (6) we have to resort to exact diagonalization and compute the dynamics of \( \mathcal{W} \) in the Heisenberg picture. Analog to the saturation of entropy at the Page value following a quench from \( |\Psi_0\rangle = |Y+\rangle \), the operator size of the initial operator \( \mathcal{W} = \mathcal{Y} \) approaches Eq. (8) at long times. Moreover, the intermediate time dynamics of the operator size shows a similar behavior as the entanglement entropy \( S_A (|\Psi (t)\rangle) \), which further supports our previous results, see Fig. 4(a),(b).

This establishes a remarkable connection between the dynamics of a local operator under unitary evolution and the entanglement growth following a quantum quench. Note that the operator size is a quantity of the operator itself, independent of any initial state. Crucially, it emphasizes a sharp contrast of operator dynamics in a system with powerlaw interactions compared to a fast scrambler. At both cases the support of a local operator quickly spreads over the entire system, our analysis shows that strong powerlaw interactions lead Pauli strings of small to intermediate size to have high weight for a much longer time, see Fig. 4(c),(d).

**Conclusions and Outlook.**— We found a connection between entanglement growth and operator spreading that reveals distinct classes of operator dynamics in the presence of nonlocal interactions. These classes are not clearly distinguishable by the squared commutator alone, at least not for system sizes of current numerical or experimental reach. In particular, the slowdown of entanglement entropy in systems with strong powerlaw interactions manifests in a slower decay of the operator density \( p_\ell \). Thus, although a highly nonlinear lightcone bounds the operator front, the operator size exhibits a slowdown analogous to the entanglement entropy. Since \( \sum_\ell p_\ell \approx 1 \) holds, a generally slower than exponential decay of \( p_\ell \) may eventually slow down the operator front and thereby prohibit fast scrambling for large enough \( N \). This behavior is in sharp contrast to the fast scrambler from Eq. (3), which shows no slowdown in both operator size and entanglement entropy.

Furthermore, this connection indicates that fast scrambling might be associated with universal entanglement dynamics. A recent study showed that fast scrambling is prohibited in models with a generic all-to-all term with prefactor \( \sim 1/N^{\gamma} \) if \( \gamma > 1/2 \) \[ [72]. Moreover, the authors of Ref. [54] argued that for the Hamiltonian (3), fast scrambling only occurs if \( \gamma = 1/2 \). Interestingly, by fur-

![FIG. 4.](image)
ther decreasing $\gamma$ from 1/2, we observe a slowdown of entanglement growth, similar to our findings for power-law interactions. Future theoretical work may explore the relationship between entanglement growth and fast scrambling in microscopic quantum systems.

An extension of this work may consider holographic models, which obey monogamy of mutual information [74]. The latter sets further restrictions on entanglement growth and is violated in systems with strong powerlaw interactions [75]. A refined understanding of entanglement growth in the presence of nonlocal interactions may result in explicit probes for holographic quantum matter.

Our work could also provide a new perspective on prethermalization in systems with powerlaw interactions, e.g., in ion traps [76]. Generally, the precise relationship between entanglement growth and operator spreading characterizes various nonequilibrium phenomena. To the best of our knowledge, there is no example where entanglement growth does not serve as a bottleneck of information dynamics, e.g., linear entanglement growth but a superlinear lightcone. A throughout understanding of this relationship may improve our ability to probe nonequilibrium phenomena and phases of quantum matter.

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See Supplemental Material.

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Supplemental Material: Quantum Information Scrambling in Systems with Nonlocal Interactions

This document provides supplemental data and information concerning the work ‘Quantum Information Scrambling in Systems with Nonlocal Interactions’. In Sec. S1, we present additional data on long-time dynamics. Section S2 is devoted to larger powerlaw exponents $\alpha$, where the dynamics of $\mathcal{H}_\alpha$ is effectively local. More data on operator spreading can be found in Sec. S3, and Sec. S4 shows details of our TDVP computation. Details on the relationship between the operator density and the squared commutator can be found in Sec. S5.

S1. Thermalization of $\mathcal{H}_\alpha$

In the main article, we claim that the long-time behavior of the Hamiltonian $\mathcal{H}_\alpha$ is not altered by the presence of nonlocal interactions. In this section, we provide further details on this issue.

Figure S1(a) shows the total magnetization in $z$-direction for various values of the exponent $\alpha$. Except for very small exponents, one can nicely see that the expectation value approaches the infinite temperature result at long times. Similar conclusions can be made if one considers how local density matrices approach the infinite temperature ensemble. Figure S1(b) shows the trace distance of a two-qubit density matrix in the middle of the system with the maximally mixed state. Further analysis supports that the deviation one observes for small exponents $\alpha$ is due to finite site effects. Figure S2 shows the time average of the total magnetization in $z$-direction, i.e., $\langle M_z(t) \rangle = 1/t \int_0^t d\tau M_z(\tau)$ for $\alpha = 0.5$ and various system sizes. With increasing system size, the deviation to the infinite temperature result systematically decreases, see the inset of Fig. S2. We note that the analysis of entanglement entropy for different system sizes leads to similar conclusions.

Summarizing, in the considered quench scenario, the long-time behaviour is the same for the different Hamiltonians that we studied. That is, local observables are determined by the expectation value in the infinite temperature ensemble and entropies saturate at the Page value.

S2. Linear regime of $\mathcal{H}_\alpha$

We have focused our analysis on small values of the exponent $\alpha$ since the slowdown of entanglement entropy and operator size is most dominant in this case. As mentioned in the main article, there generally exists a regime of $\alpha$ with effectively local dynamics. Accordingly, entropy exhibits a linear growth and the spread of local operators is bounded by a linear lightcone. Although we cannot rigorously prove where the transition to this regime occurs, for the model at hand our numerical results indicate that at least for $\alpha > 2$, the dynamics is effectively local. Noteworthy, the respective velocities $v_E$ and $v_B$ are similar renormalized and decrease monotonically with decreasing $\alpha$, see Fig. S3. Hence, in this local regime a connection between entanglement growth and operator spreading can be observed already, as they both diagnose a likewise slowdown of information scrambling. Moreover, this has experimentally relevance as the local dynamics of $\mathcal{H}_\alpha$ is accessible for a broader range of $\alpha$ on
experimental platforms such as trapped ions, which are typically limited to $0 < \alpha \leq 3$. We note that finite size effects on the velocities are negligible for the considered system sizes and conclude that the calculated velocities are universal properties of the Hamiltonian $\mathcal{H}_\alpha$ for all $N$.

In Fig. S4(a), the entanglement entropy for various values of $\alpha$ within the linear regime is shown. With decreasing $\alpha$, the growth rate of entanglement entropy also decreases. The inset shows a clear collapse of the data if entropies are rescaled with the respective entanglement velocity $v_E$, which highlights the linear dynamics in this regime. Figure S4(b) shows the squared commutator for various values of $\alpha$ within the linear regime. In a similar vein, the growth of the squared commutator decreases with smaller $\alpha$. A collapse at the wavefront, i.e., $C_r(t) = \frac{1}{2}$ can be observed if time is rescaled by means of the butterfly velocity, see the inset of Fig. S4(b). Moreover, we observe an increased broadening of the front with decreasing $\alpha$, which may be a first signature of the slower decay of small to intermediate Pauli strings as observed in the main article during the analysis of the operator size.

FIG. S3. Velocities $v_E$ (blue star) and $v_B$ (red hexagon) as a function of the exponent $\alpha$, where $|\Phi_0\rangle = |Y+\rangle$, $W = Y = \mathcal{Y}$. Dashed lines indicate the values for the local Hamiltonian, $\mathcal{H}_\infty$, $\kappa = \frac{1}{N-1} \sum_{m<n} J_{mn}$ system size is $N = 24$ (EXPM).

FIG. S4. Entanglement growth and operator spreading for various values of $\alpha = \{\infty, 6.0, 5.0, 4.0, 3.0, 2.5, 2.3, 2.1\}$. (a) Entanglement entropy, inset shows the collapse of all curves by rescaling entropies with the respective entanglement velocity $v_E$. (b) Squared commutator for $r = 13$. The inset shows the collapse at the wavefront if time is rescaled according to the butterfly velocity. Darker colors indicate larger values. System size is $N = 24$ (EXPM).

This section provides additional data on operator dynamics and finite size effects regarding the Hamiltonians we analyzed in the main article. The spatiotemporal structure of the OTOC $F_r(t) = \langle \mathcal{W}(t) \mathcal{V}_r \mathcal{W}(t) \mathcal{V}_r \rangle$ and the entanglement entropy $S_{A}(\langle \Phi(t) \rangle)$ of the state $\langle \Phi(t) \rangle = \mathcal{W}(t) |\Phi_0\rangle$ are shown in Figs. S6-S9. The dynamics of these two quantities is clearly in line for the local Hamiltonian $\mathcal{H}_\infty$ and the fast scrambler $\mathcal{H}_FS$. For systems with powerlaw interactions, however, the spatiotemporal structure of $S_{A}(\langle \Phi(t) \rangle)$ diagnoses a slowdown of operator dynamics not precisely captured by the OTOC.

For small powerlaw exponents $\alpha$, we observe stronger oscillations of the squared commutator. With increasing system size, these oscillations decrease which suggests that they are due to the finite system size of our simulation, see Fig. S5. Finite size effects regarding the operator size are generally quite small for the scenarios considered. Figure S10(a) shows the operator size for various system sizes and $\alpha = 1.1$. Hence, we conclude that the observed slowdown of operator size is a general feature of systems with powerlaw interaction. Moreover, the dynamics of operator size is quite similar for different operators $\mathcal{W}$, see Fig. S10(b).

FIG. S5. Entanglement growth of $|\Phi(t)\rangle$ (dashed) compared to the squared commutator $C_r(t)$ (solid), where the operator $\mathcal{W}$ is located at the leftmost site of the system, and $\mathcal{V}$ at site $r = N$. The exponent of the powerlaw decay is $\alpha = 0.8$ and $\kappa = \frac{1}{N-1} \sum_{m<n} J_{mn}$. Different systems sizes $N = \{14, 18, 22, 26\}$ are shown. Darker colors indicate larger values.

S3. OPERATOR SPREADING AND FINITE SIZE EFFECTS
FIG. S6. ‘Lightcones’ for $F_r (t)$ (left), where $W = \mathcal{Y} = \mathcal{Y}$ and $1 - S_A (|\Phi(t)\rangle) / S_F$ (right). Crosses show thresholds of $\theta = 0.01, 0.5, 0.85$ respectively. Data is for $\alpha = \infty$ and $N = 22$ (EXPM).

FIG. S7. ‘Lightcones’ for $F_r (t)$ (left), where $W = \mathcal{Y} = \mathcal{Y}$ and $1 - S_A (|\Phi(t)\rangle) / S_F$ (right). Crosses show thresholds of $\theta = 0.01, 0.5, 0.85$ respectively. Data is for $\alpha = 1.1$ and $N = 22$ (EXPM).

FIG. S8. ‘Lightcones’ for $F_r (t)$ (left), where $W = \mathcal{Y} = \mathcal{Y}$ and $1 - S_A (|\Phi(t)\rangle) / S_F$ (right). Crosses show thresholds of $\theta = 0.01, 0.5, 0.85$ respectively. Data is for $\alpha = 0.4 \ (\kappa = 1)$ and $N = 22$ (EXPM).

FIG. S9. ‘Lightcones’ for $F_r (t)$ (left), where $W = \mathcal{Y} = \mathcal{Y}$ and $1 - S_A (|\Phi(t)\rangle) / S_F$ (right). Crosses show thresholds of $\theta = 0.01, 0.5, 0.85$ respectively. Data is for $\alpha = 0.4 \ (\kappa = 1)$ and $N = 22$ (EXPM).

S4. TDVP DATA

In this section we provide additional data regarding our TDVP computations. In particular, the convergence of the quantities of interest with increasing bond dimension $\chi$. In Fig. S11, the half-chain entanglement entropy following a quench from the state $|Y+\rangle$ is shown for various bond dimensions. For the time intervals we considered, the entanglement entropy is clearly converged. Only small deviations at the end of the respective time intervals can be observed. The squared commutator seems to be more sensitive and a larger bond dimension is needed for convergence, see Fig. S12. This might be due to the forward and backward evolution required for its calculation.

FIG. S10. (a) Operator size for various system sizes $N = \{10, 12, 14, 16\}$ (dotted), where $W = \mathcal{Y}$, $\alpha = 1.1$ and $\kappa = \sum_{m<n} J_{mn}$. Solid black line shows the operator size for $\alpha = \infty$ and $N = 16$. (b) Operator size for $W = \{X, Y, Z\}$ and $N = 16$. Solid lines are for $\alpha = \infty$ and dashed lines for $\alpha = 1.1$. 
FIG. S11. Half-chain entanglement entropy for various bond dimensions $\chi = \{256,512,1024\}$ (TDVP). Time step is $\delta t = 0.1 [1/J]$. Darker colors indicate larger values. Red dotted line shows numerically exact data for $N = 28$ (EXPM). (a) Local Hamiltonian $H_{\infty}$ (b) fast scrambler $H_{FS}$ (c) power law Hamiltonian $H_\alpha$ with $\alpha = 1.1$ and $\kappa = \frac{1}{12} \sum_{m<n} J_{mn}$ (d) power law Hamiltonian $H_\alpha$ with $\alpha = 0.4$ and $\kappa = 1$.

FIG. S12. Squared commutator ($W = V = \mathcal{V}$) at a fixed time $t$ as a function of $r$ for various bond dimensions $\chi = \{256,512,1024\}$ (TDVP). Time step of simulation is $\delta t = 0.1 [1/J]$. Darker colors indicate larger values. (a) power law Hamiltonian $H_\alpha$ with $\alpha = 0.4$, $\kappa = 1$ and $t = 2.0$. (b) fast scrambler $H_{FS}$ with $t = 2.5$.

S5. OPERATOR DENSITY AND THE SQUARED COMMUTATOR

This section provides more details on the relationship between the operator density and the squared commutator. To this end, let us consider the squared commutator, where the operator $W$ is initially placed at the left edge of the system, and the operator $V$ at site $r$

$$C_r^V(t) = \frac{1}{2} \left\langle [W(t), V_r^\dagger] [W(t), V_r] \right\rangle .$$

(S1)

Here, we use $\langle \ldots \rangle = 2^{-N} \text{Tr} \ldots$, which is the expectation value in the infinite temperature ensemble. Furthermore, we assume $W$ and $V$ to be unitary. Let us first consider $r = N$, Eq. (S5) then reads

$$\frac{1}{2} \left\langle [W(t), V_N^\dagger] [W(t), V_N] \right\rangle = \left\langle [\mathbb{P}_N W(t), V_N] \right\rangle^2 ,$$

(S2)

where we defined $\mathbb{P}_N W(t) := \sum_{|A| = N} c_A S_A$ as the projection of $W(t)$ onto strings that act nontrivially on site $N$. The first term in Eq. S2 then reads

$$\left\langle [\mathbb{P}_N W(t), V_N] \right\rangle^2 = \sum_{|A| = |A'| = N} c_A^* c_{A'} \left\langle S_A S_{A'} \right\rangle$$

$$= \sum_{|A| = N} |c_A|^2 = p_N(t) ,$$

(S3)

which is just the operator density for $r = N$. For the second term, we obtain

$$\left\langle [\mathbb{P}_N W(t), V_N] \right\rangle^2 = \sum_{|A| = |A'| = N} c_A^* c_{A'} \left\langle S_A V N S_{A'} V N \right\rangle$$

$$= \sum_{|A| = N, A_N=M} |c_A|^2 - \sum_{|A| = N, A_N \neq M} |c_A|^2 .$$

(S4)

The second line follows from the fact that $V_N S_A V_N = \pm S_A$, where we obtain a negative sign if the string $S_A$ at site $N$ is not $V$. Combining Eq. (S3) and (S4) we obtain

$$C_N^V(t) = 2 \sum_{|A| = N, A_N \neq M} |c_A|^2 .$$

(S5)

Note that Eq. (S5) depends on the choice of $\mathcal{V}$. We can define an average square commutator as

$$\overline{C}_r(t) = \frac{1}{|\mathcal{P}|} \sum_{\mathcal{V} \in \mathcal{P}} C_r^\mathcal{V}(t) = \sum_{A_r \neq 1} |c_A|^2 ,$$

(S6)

where $\mathcal{P} = \{1, \mathcal{V}, \mathcal{V}_r, 2\}$. Hence, we can establish the following equality between the squared commutator and the operator density

$$p_N(t) = \overline{C}_N(t) .$$

(S7)

Equation (S7) is a special case. In general, the average squared commutator $\overline{C}_r(t)$ is determined by all coefficients $c_A$ that belong to strings $S_A$ that act nontrivially on site $r$. For $r = N$, this coincides with all coefficients that belong to strings of size $N$. In the general case, we obtain

$$p_r(t) = \overline{C}_r(t) - \sum_{|A| > r, A_r \neq 1} |c_A|^2 .$$

(S8)

Thus, in this general case, the operator density is bounded from above by the average squared commutator, i.e.,

$$p_r(t) \leq \overline{C}_r(t) , \quad r > 1.$$
If one has access to the Heisenberg operator $\mathcal{W}(t)$, for instance, within an ED computation, the operator density can be obtained as follows: the part of the operator whose support ranges up to a given site $\ell$ can be obtained by taking the partial trace with respect to all sites to the right of $\ell$, i.e.,

$$
\mathcal{W}_\ell(t) := \frac{1}{2^{N-\ell}} \text{Tr}_{\mathcal{Z}} [\mathcal{W}(t)] = \sum_{|\Lambda| \leq \ell} c_{\Lambda}(t) S_\Lambda , \quad (S10)
$$

where $\mathcal{Z}$ is the complement of $\{1, \ldots, \ell\}$. Note that $\mathcal{W}_\ell(t)$ is not unitary anymore. It follows then straightforwardly that

$$
\langle \mathcal{W}_\ell(t) \mathcal{W}_\ell(t) \rangle = \sum_{\ell' \leq \ell} p_{\ell'}(t) . \quad (S11)
$$

Thus, by computing Eq. (S11) for all $1 \leq \ell \leq N - 1$, one can reconstruct the operator density for all $\ell$. In particular, we have

$$
p_{\ell}(t) = \langle \mathcal{W}_\ell(t) \mathcal{W}_\ell(t) \rangle - \langle \mathcal{W}_{\ell-1}(t) \mathcal{W}_{\ell-1}(t) \rangle . \quad (S12)
$$