Some applications of second-order differential subordination for a class of analytic function defined by the lambda operator

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Abstract: In this paper, we introduce a new class of analytic functions by using the lambda operator and obtain some subordination results.

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1. Introduction

Let \( \mathbb{C} \) be complex plane and let \( \mathbb{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} = \mathbb{U} \setminus \{0\} \) be an open unit disc in \( \mathbb{C} \). Also let \( H(\mathbb{U}) \) be a class of analytic functions in \( \mathbb{U} \). For \( n \in \mathbb{N} = \{1, 2, 3, \cdots \} \) and \( a \in \mathbb{C} \), let \( H[a, n] \) be a subclass of \( H(\mathbb{U}) \) formed by the functions of the form

\[
f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots
\]

with \( H_0 \equiv H[0,1] \) and \( H \equiv H[1,1] \). Suppose that \( A_n \) is a class of all analytic functions of the form

\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k
\]

(1)

in the open unit disk \( \mathbb{U} \) with \( A_1 = A \). A function \( f \in H(\mathbb{U}) \) is univalent if it is a one-to-one function in \( \mathbb{U} \). By \( S \), we denote a subclass of \( A \) formed by functions univalent in \( \mathbb{U} \). If a function \( f \in A \) maps \( \mathbb{U} \) onto a convex domain and \( f \) is univalent, then \( f \) is called a convex function. By

\[
K = \left\{ f \in A : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \ z \in \mathbb{U} \right\},
\]

we denote a class of all convex functions defined in \( \mathbb{U} \) and normalized by \( f(0) = 0 \) and \( f'(0) = 1 \).

Let \( f \) and \( F \) be elements of \( H(\mathbb{U}) \). A function \( f \) is said to be subordinate to \( F \), if there exists a Schwartz function \( w \) analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), \( z \in \mathbb{U} \), such that \( f(z) = F(w(z)) \). In this case, we write \( f(z) \prec F(z) \) or \( f \prec F \). Furthermore, if the function \( F \) is univalent in \( \mathbb{U} \), then we get the following equivalence [1,2]:

\[
f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathbb{U}) \prec F(\mathbb{U}).
\]

The method of differential subordinations (also known as the method of admissible functions) was first introduced by Miller and Mocanu in 1978 [3], and the development of the theory was originated in 1981 [4]. All details can be found in the book by Miller and Mocanu [2]. In recent years, numerous authors studied the properties of differential subordinations (see [5–8], etc.).

Let \( \Psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C} \) and let \( h \) be univalent in \( \mathbb{U} \). If \( p \) is analytic in \( \mathbb{U} \) and satisfies the second-order differential subordination:

\[
\Psi \left( p(z),zp'(z),zp''(z);z \right) \prec h(z),
\]

(2)
then $p$ is called the solution of differential subordination. The univalent function $q$ is called a dominant of the solution of the differential subordination or, simply, a dominant if $p \prec q$ for all $p$ satisfying (2). The dominant $q_1$ satisfying $q_1 \prec q$ for all dominants $q$ of (2) is called the best dominant of (2).

Let us recall lambda function \[9\] defined by:

$$\lambda(z,s) = \sum_{k=2}^{\infty} \frac{z^k}{(2k+1)!}$$

where $z \in \mathbb{U}, s \in \mathbb{C}$, when $|z| < 1, \Re(s) > 1$, when $|z| = 1$ and let $\lambda^{-1}(z,s)$ be defined such that

$$\lambda(z,s) \ast \lambda^{-1}(z,s) = \frac{1}{(1-z)^{\mu+1}}, \mu > -1.$$

We now define \((z\lambda^{-1}(z,s))\) as:

$$(z\lambda(z,s)) \ast (z\lambda^{-1}(z,s)) = \frac{z}{(1-z)^{\mu+1}} = z + \sum_{k=2}^{\infty} \frac{(\mu+1)_{k-1}}{(k-1)!} z^k, \mu > -1$$

and obtain the linear operator $I^*_p f(z) = (z\lambda^{-1}(z,s)) \ast f(z)$, where $f \in A, z \in \mathbb{U}$ and $(z\lambda^{-1}(z,s)) = z + \sum_{k=2}^{\infty} \frac{(\mu+1)_{k-1}(2k-1)^s}{(k-1)!} z^k$. A simple computation gives us

$$I^*_p f(z) = z + \sum_{k=2}^{\infty} L(k, \mu, s) a_k z^k,$$

where

$$L(k, \mu, s) = \frac{(\mu+1)_{k-1}(2k-1)^s}{(k-1)!},$$

where $(\mu)_k$ is the Pochhammer symbol defined in terms of the Gamma function by:

$$(\mu)_k = \frac{\Gamma(\mu+k)}{\Gamma(\mu)} = \begin{cases} 1, & \text{if } k = 0; \\ \mu(\mu+1) \cdots (\mu+k-1), & \text{if } k \in \mathbb{N}. \end{cases}$$

**Definition 1.** Let $\mathcal{L}_{p,a}(q)$ be a class of function $f \in A$ satisfying the inequality

$$\Re(I^*_p f(z)) \geq q,$$

where $z \in \mathbb{U}$, $0 \leq q < 1$ and $I^*_p f(z)$ is the Lambda operator.

**Lemma 1.** Let $h$ be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ be a complex number with $\Re\{\gamma\} \geq 0$. If $p \in H[a,n]$ and

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z),$$

then $p(z) \prec q(z) \prec h(z)$, where

$$q(z) = \frac{\gamma}{\pi n^2} \int_0^{2\pi} \frac{z}{1 - \gamma^2} h(t) dt, \ z \in \mathbb{U}.$$ The function $q$ is convex and is the best dominant for subordination (5).

**Lemma 2.** \[10\] Let $\Re\{\gamma\} > 0$, $n \in \mathbb{N}$ and $w = \frac{n^2 + |\mu|^2 - |\mu|^2}{4n \Re(\gamma)}$. Also, let $h$ be an analytic function in $\mathbb{U}$ with $h(0) = 1$. Suppose that

$$\Re\left\{1 + \frac{z^\mu}{\Gamma(\gamma)}\right\} > -w.$$ If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ and

$$p(z) + \frac{1}{\mu} z p'(z) \prec h(z),$$

then $p(z) \prec q(z) \prec h(z)$, where $q(z)$ is defined as above.
then \( p(z) \prec q(z) \), where \( q \) is a solution of the differential equation \( q(z) + \frac{n}{z} q'(z) = h(z), \quad q(0) = 1 \), given by

\[
q(z) = \frac{\mu}{m!} \int_0^z t^{m-1} h(t) dt, \quad z \in U.
\]

Moreover, \( q \) is the best dominant for the differential subordination (6).

**Lemma 3.** [11] Let \( r \) be a convex function in \( U \) and let \( h(z) = r(z) + n q z r'(z), \quad z \in U, \) where \( q > 0 \) and \( n \in \mathbb{N} \). If \( p(z) = r(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad z \in U, \) is holomorphic in \( U \) and \( p(z) + q z p'(z) \prec h(z), \quad z \in U, \) then \( p(z) \prec r(z) \) and this result is sharp.

In the present paper, we use the subordination results from [10] to prove our main results.

2. Main results

**Theorem 1.** The set \( \mathcal{L}_{h,s}(q) \) is convex.

**Proof.** Let \( f_j(z) = z + \sum_{k=2}^{\infty} a_{kj} z^k, \quad z \in U, \) \( j = 1, \cdots, m \) be in the class \( \mathcal{L}_{h,s}(q) \). Then, by Definition 1, we get

\[
\Re \left\{ (I^s_{\mu} f_j(z))' \right\} = \Re \left\{ 1 + \sum_{k=2}^{\infty} \lambda(k, \mu, s) a_{kj} k z^{k-1} \right\} > \varphi.
\]  

(7)

For any positive numbers \( \xi_1, \xi_2, \xi_3, \cdots, \xi_m \) such that \( \sum_{j=1}^{m} \xi_j = 1 \), it is necessary to show that the function

\[
h(z) = \sum_{j=1}^{m} \xi_j f_j(z)
\]

is an element of \( \mathcal{L}_{h,s}(q) \), i.e.,

\[
\Re \left\{ (I^s_{\mu} h(z))' \right\} > \varphi.
\]  

(8)

Thus, we have

\[
(I^s_{\mu} h(z))' = 1 + \sum_{k=2}^{\infty} k \lambda(k, \mu, s) \left\{ \sum_{j=1}^{m} \xi_j a_{kj} \right\} z^{k-1}.
\]  

(9)

If we differentiate (9) with respect to \( z \), then we obtain

\[
(I^s_{\mu} h(z))' = 1 + \sum_{k=2}^{\infty} k \lambda(k, \mu, s) \left\{ \sum_{j=1}^{m} \xi_j a_{kj} \right\} z^{k-1}.
\]

Thus by using (8), we have

\[
\Re \left\{ (I^s_{\mu} h(z))' \right\} = 1 + \sum_{j=1}^{m} \xi_j \Re \left\{ \sum_{k=2}^{\infty} k \lambda(k, \mu, s) a_{kj} z^{k-1} \right\} > 1 + \sum_{j=1}^{m} \xi_j (q - 1) = \varphi.
\]

Hence, inequality (7) is true and we arrive at the desired result. \( \Box \)

**Theorem 2.** Let \( q \) be a convex function in \( U \) with \( q(0) = 1 \) and \( h(z) = q(z) + \frac{1}{\gamma + 1} z q'(z), \quad z \in U, \) where \( \gamma \) is a complex number with \( \Re \{ \gamma \} > -1 \). If \( f \in \mathcal{L}_{h,s}(q) \) and \( \mathcal{R} = \mathcal{Y}_\gamma f \), where

\[
\mathcal{R}(z) = \mathcal{Y}_\gamma f(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt,
\]  

(10)

then

\[
(I^s_{\mu} f(z))' \prec h(z)
\]  

(11)

implies that \( (I^s_{\mu} \mathcal{R}(z))' \prec q(z) \) and this result is sharp.
Proof. In view of equality (10), we can write

\[ z^\gamma q(z) = (\gamma + 1) \int_0^z t^{\gamma-1} f(t) dt. \] (12)

Differentiating (12) with respect to \( z \), we obtain \((\gamma)q(z) + zq'(z) = (\gamma + 1)f(z)\). Further, by applying the operator \( T_\mu^s \) to the last equation, we get

\[ (\gamma)T_\mu^s q(z) + z(T_\mu^s q(z))' = (\gamma + 1)T_\mu^s f(z). \] (13)

If we differentiate (13) with respect to \( z \), then we find

\[ (T_\mu^s q(z))' + \frac{1}{\gamma + 1}z(T_\mu^s f(z))'' = (T_\mu^s f(z))'. \] (14)

By using the differential subordination given by (11) in equality (14), we obtain

\[ (T_\mu^s q(z))' + \frac{1}{\gamma + 1}z(T_\mu^s f(z))'' < h(z). \] (15)

We define

\[ p(z) = (T_\mu^s q(z))'. \] (16)

Hence, as a result of simple computations, we get

\[ p(z) = \left\{ z + \sum_{k=0}^\infty L(k, \mu, s) \frac{\gamma + 1}{\gamma + k} \varrho^k \right\}' = 1 + p_1z + p_2z^2 + \cdots, \quad p \in H[1, 1]. \]

By using (16) in subordination (15), we obtain

\[ p(z) + \frac{1}{\gamma + 1}zp'(z) < h(z) = q(z) + \frac{1}{\gamma + 1}q'q(z), \quad z \in U. \]

If we use Lemma 2, then we write \( p(z) < q(z) \). Thus, we obtained the desired result and \( q \) is the best dominant.

Example 1. If we choose \( \gamma = i + 1 \) and \( q(z) = \frac{1+i+z}{1+i+z^2} \), in Theorem 2, then we get \( h(z) = \frac{(1+i)(1+i+z+2z^2)}{(1+i+z+2z^2)} \). If \( f \in \mathcal{L}_{\mu,s}(q) \) and \( q \) is given as \( q(z) = Y(f)z \), then, by virtue of Theorem 2, we find \( (T_\mu^s f(z))' < h(z) = \frac{1+i(1+i+z+2z^2)}{(1+i+z+2z^2)}, \) implies \( (T_\mu^s f(z))' < \frac{1+i+z}{1+i-z} \).

Theorem 3. Let \( \Re\{\gamma\} > -1 \) and \( w = \frac{1+|\gamma+1|^2-|\gamma|^2+2\gamma}{4\Re(\gamma+1)} \). Suppose that \( h \) is an analytic function in \( U \) with \( h(0) = 1 \) and that \( \Re\left\{ 1 + \frac{\Re(q(z))}{\Re(q(z))} \right\} > -w \). If \( f \in \mathcal{L}_{\mu,s}(q) \) and \( q = Y_f(z) \), where \( q \) is defined by (10), then

\[ (T_\mu^s f(z))' < h(z) \] (17)

implies that \( (T_\mu^s q(z))' < q(z) \), where \( q \) is the solution of the differential equation \( h(z) = q(z) + \frac{1}{\gamma+1}q'q(z), \quad q(0) = 1, \) given by \( q(z) = \frac{\gamma+1}{\gamma+1} \int_0^z f(t) dt \). Moreover, \( q \) is the best dominant for subordination (17).

Proof. If we choose \( n = 1 \) and \( \mu = \gamma + 1 \) in Lemma 1, then the proof is obtained by means of the proof of Theorem 3.

Theorem 4. Let

\[ h(z) = \frac{1+(2\varrho-1)z}{1+z}, \quad 0 \leq \varrho < 1 \] (18)
be convex in \(U\) with \(h(0) = 1\). If \(f \in A\) and verifies the differential subordination \((I^\gamma_\mu f(z))' \prec h(z)\), then \((I^\gamma_\mu h(z))' \prec q(z) = (2q - 1) + \frac{2(1-q)(\gamma+1)}{z^{\gamma+1}}\tau(\gamma)\), where \(\tau\) is given by the formula

\[
\tau(\gamma) = \int_0^\gamma \frac{t^\gamma}{t+1} dt
\]

and \(\Re\) is given by equation (10). The function \(q\) is convex and is the best dominant.

\textbf{Proof.} If \(h(z) = \frac{1+(2q-1)}{1+z}, 0 \leq q < 1\), then \(h\) is convex and, in view of Theorem 3, we can write \((I^\gamma_\mu h(z))' \prec q(z)\). Now, by using Lemma 1, we get

\[
q(z) = \frac{\gamma + 1}{z^{\gamma+1}} \int_0^z t^\gamma h(t) dt = \frac{1+1}{z^{\gamma+1}} \int_0^z \left\{ 1 + \frac{(2q-1)t}{1+t} \right\} dt = (2q - 1) + \frac{2(1-q)(\gamma+1)}{z^{\gamma+1}}\tau(\gamma),
\]

where \(\tau\) is given by (19). Hence, we obtain

\[
(I^\gamma_\mu h(z))' \prec q(z) = (2q - 1) + \frac{2(1-q)(\gamma+1)}{z^{\gamma+1}}\tau(\gamma).
\]

The function \(q\) is convex. Moreover, it is the best dominant. Hence the theorem is proved. 

\textbf{Theorem 5.} If \(0 \leq q < 1\), \(0 \leq \mu < 1\), \(\delta \geq 0\), \(\Re\{\gamma\} > -1\), and \(h = Y_\gamma f\) is defined by (10), then \(Y_\gamma(\mathcal{E}_{\mu,s}(\rho)) \subset \mathcal{E}_{\mu,s}(\rho)\), where

\[
\rho = \min_{|z|=1} \Re\{q(z)\} = \rho(\gamma, q) = (2q - 1) + 2(1-q)(\gamma+1)\tau(\gamma)
\]

and \(\tau\) is given by (19).

\textbf{Proof.} Assume that \(h\) is given by equation (18), \(f \in \mathcal{E}_{\mu,s}(\rho)\), and \(h = Y_\gamma f\) is defined by (10). Then \(h\) is convex and, by Theorem 3, we deduce

\[
(I^\gamma_\mu h(z))' \prec q(z) = (2q - 1) + \frac{2(1-q)(\gamma+1)}{z^{\gamma+1}}\tau(\gamma),
\]

where \(\tau\) is given by (19). Since \(q\) is convex, \(q(U)\) is symmetric about the real axis, and \(\Re\{\gamma\} > -1\), we find

\[
\Re\left\{ (I^\gamma_\mu h(z))' \right\} \geq \min_{|z|=1} \Re\{q(z)\} = \Re\{q(1)\} = \rho(\gamma, q) = (2q - 1) + 2(1-q)(\gamma+1)(1-q)\tau(\gamma).
\]

It follows from inequality (21) that \(Y_\gamma(\mathcal{E}_{\mu,s}(\rho)) \subset \mathcal{E}_{\mu,s}(\rho)\), where \(\rho\) is given by (20). Hence the theorem is proved.

\textbf{Theorem 6.} Let \(q\) be a convex function with \(q(0) = 1\) and \(h\) be a function such that \(h(z) = q(z) + zq'(z), z \in U\). If \(f \in A\), then the subordination

\[
(I^\gamma_\mu f(z))' \prec h(z)
\]

implies that \(\frac{I^\gamma_\mu f(z)}{z} \prec q(z)\), and the result is sharp.

\textbf{Proof.} Let

\[
p(z) = \frac{I^\gamma_\mu f(z)}{z}.
\]

Differentiating (23), we find

\[
(I^\gamma_\mu f(z))' = p(z) + zp'(z)
\]

We now compute \(p(z)\). This gives

\[
p(z) = \frac{I^\gamma_\mu f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} L(k, \mu, s) a_k z^k}{z} = 1 + p_1 z + p_2 z^2 + \cdots, p \in H[1, 1].
\]
By using (24) in subordination (22), we find \( p(z) + zp'(z) \prec h(z) = q(z) + zq'(z) \). Hence, by applying Lemma 3, we conclude that \( p(z) \prec q(z) \) i.e., \( \frac{T_p f(z)}{z} \prec q(z) \). This result is sharp and \( q \) is the best dominant. Hence the theorem is proved.  

**Example 2.** If we take \( \mu = 0 \) and \( s = 1 \) in equality (4) and \( q(z) = \frac{1}{1-z} \) in Theorem 5, then \( h(z) = \frac{1}{(1-z)^2} \) and

\[
I_0^1 f(z) = z + \sum_{k=2}^{\infty} \frac{(2k-1)}{(k-1)!} a_k z^k.
\]

Differentiating (25) with respect to \( z \), we get

\[
(I_0^1 f(z))' = 1 + \sum_{k=2}^{\infty} \frac{(2k-1)}{(k-1)!} a_k z^{k-1} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1, 1].
\]

By using Theorem 5, we find \( (I_0^1 f(z))' \prec h(z) = \frac{1}{(1-z)^2} \). This yields \( \frac{I_0^1 f(z)}{z} \prec q(z) = \frac{1}{1-z} \).

**Theorem 7.** Let \( h(z) = \frac{1+(2q-1)z}{1+z}, \quad z \in \mathbb{U} \) be convex in \( \mathbb{U} \) with \( h(0) = 1 \) and \( 0 \leq q < 1 \). If \( f \in A \) satisfies the differential subordination

\[
(I_0^s f(z))' \prec h(z),
\]

then \( \frac{T_p f(z)}{z} \prec q(z) = (2q - 1) + \frac{2(1-q)ln(1+z)}{z} \). The function \( q \) is convex and, in addition, it is the best dominant.

**Proof.** Let

\[
p(z) = \frac{T_p f(z)}{z} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1, 1].
\]

Differentiating (27), we find

\[
(I_0^s f(z))' = p(z) + zp'(z).
\]

In view of (28), the differential subordination (26) becomes \( (I_0^s f(z))' \prec h(z) = \frac{1+(2q-1)z}{1+z} \), and by using Lemma 1, we deduce \( p(z) \prec q(z) = \frac{1}{2} \int h(t)dt = (2q - 1) + \frac{2(1-q)ln(1+z)}{z} \). Now, by virtue of relation (27) we obtained the desired result.  

**Corollary 1.** If \( f \in \mathcal{L}_{\mu,s}(q) \), then \( \Re \left( \frac{T_p f(z)}{z} \right) > (2q - 1) + (1-q)ln(2) \).

**Proof.** If \( f \in \mathcal{L}_{\mu,s}(q) \), then it follows from Definition 1 that \( \Re \left\{ (I_0^s f(z))' \right\} > q \), \( z \in \mathbb{U} \), which is equivalent to \( (I_0^s f(z))' \prec h(z) = \frac{1+(2q-1)z}{1+z} \). Now, by using Theorem 7, we obtain

\[
\frac{T_p f(z)}{z} \prec q(z) = (2q - 1) + \frac{2(1-q)ln(1+z)}{z}.
\]

Since \( q \) is convex and \( q(\mathbb{U}) \) is symmetric about the real axis, we conclude that

\[
\Re \left( \frac{T_p f(z)}{z} \right) > \Re(q(1)) = (2q - 1) + (1-q)ln(2).
\]

**Theorem 8.** Let \( q \) be a convex function such that \( q(0) = 1 \) and \( h \) be the function given by the formula \( h(z) = q(z) + zq'(z), \quad z \in \mathbb{U} \). If \( f \in A \) and verifies the differential subordination

\[
\left\{ \frac{zT_p f(z)}{T_p h(z)} \right\}' \prec h(z), \quad z \in \mathbb{U},
\]

then...
then \( \frac{T_\mu f(z)}{T_\mu N(z)} < q(z) \), \( z \in \mathbb{U} \), and this result is sharp.

**Proof.** For function \( f \in A \), given by Equation (1), we get

\[
T_\mu f(z) = z + \sum_{k=2}^{\infty} L(k, \mu, s) \frac{\gamma + 1}{k + \gamma} a_k b_k z^k, \quad z \in \mathbb{U}.
\]

We now consider the function

\[
p(z) = \frac{T_\mu f(z)}{T_\mu N(z)} = \frac{z + \sum_{k=2}^{\infty} L(k, \mu, s) a_k b_k z^k}{z + \sum_{k=2}^{\infty} L(k, \mu, s) \frac{\gamma + 1}{k + \gamma} a_k b_k z^k} = \frac{1 + \sum_{k=2}^{\infty} L(k, \mu, s) a_k b_k z^{k-1}}{1 + \sum_{k=2}^{\infty} \frac{L(k, \mu, s) \gamma + 1}{k + \gamma} a_k b_k z^{k-1}}.
\]

In this case, we get

\[
(p(z))' = \frac{(T_\mu f(z))'}{T_\mu N(z)} - p(z) \frac{(T_\mu N(z))'}{T_\mu N(z)}.
\]

Then

\[
p(z) + zp'(z) = \left\{ zT_\mu f(z) \right\}' \left\{ \frac{T_\mu f(z)}{T_\mu N(z)} \right\}', \quad z \in \mathbb{U}.
\]

By using relation (30) in inequality (29), we obtain \( p(z) + zp'(z) < h(z) = q(z) + zq'(z) \) and, by virtue of Lemma 3, \( p(z) < q(z) \), i.e., \( \frac{T_\mu f(z)}{T_\mu N(z)} < q(z) \). Hence the theorem is proved. \( \Box \)

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