AB effect and Aharonov–Susskind charge non-superselection

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Abstract
We consider a particle in a coherent superposition of states with different electric charges moving in the vicinity of a magnetic flux. Formally, it should acquire a (gauge-dependent) Aharonov–Bohm (AB) relative phase between the charge states, even for an incomplete loop. If measurable, such a geometric, rather than topological, AB phase would seem to break gauge invariance. Wick, Wightman and Wigner argued that since (global) charge-dependent phase transformations are physically unobservable, charge state superpositions are unphysical (‘charge superselection rule’). This would resolve the apparent paradox in a trivial way. However, Aharonov and Susskind disputed this superselection rule: they distinguished between such global charge-dependent transformations and transformations of the relative inter-charge phases of two particles, and showed that the latter could in principle be observable! Finally, the paradox again disappears once we consider the ‘calibration’ of the phase measured by the Aharonov–Susskind phase detectors, as well as the phase of the particle at its initial point. It turns out that such a detector can only distinguish between the relative phases of two paths if their (oriented) difference forms a loop around the flux.

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1. Introduction

The Aharonov–Bohm (AB) effect requires little introduction, and this special issue celebrating its Golden Jubilee is a vivid illustration that it continues to inspire present day research. The Aharonov–Susskind (AS) papers disputing the notion of superselection rules in non-relativistic quantum mechanics (QM), on the other hand, although themselves are over 40 years old, and despite having introduced the very important concept of quantum reference frames, have for decades remained mostly unknown to all but quantum foundations specialists. Part of
the reason, at least, may have been the fact that they apparently deal exclusively with the possibility of the existence of coherent superpositions of states with different electric charges or spin, which are admittedly esoteric and seemingly irrelevant to most of the rest of physics. Nevertheless, about a decade ago they regained a much wider audience in the context of apparent paradoxes related to the role played by the coherent superpositions of the number of states in quantum optics and the theory of Bose–Einstein condensates. Ironically, the ubiquitous and innocent-looking optical coherent state, which was one of the sources of inspiration of AS, has now been claimed to be a ‘convenient fiction’ [1], and much of the theory utilizing it was in desperate need of being reinterpreted [2]—a task accomplished by applying to it the very same framework of quantum reference frames first invoked by AS for the esoteric analogues mentioned earlier.

When a particle with an electric charge $Q$ travels (slowly) along a curve $C$ (i.e. its wavepacket is assumed to be small compared to the length of the curve, and its centroid to move along the latter) in the force-free region near a solenoid, it ostensibly picks up a phase $\phi^{[A]}_{AB}(C) \equiv Q \int_C A \cdot dr$, where $A$ is the vector potential generated by the current in the solenoid. However, this formal expression is gauge dependent (hence the superscript label), and by gauge invariance should be unobservable. This unobservability also follows trivially from the fact that this is just an overall phase of the state vector. One can get a similar relative phase if one considers the particle state to correspond to a coherent superposition of localized packets following two paths which share the same initial and final points. The relative phase between the two branches of the wavefunction can then be measured by interference at the final point. Astonishingly, this relative phase turns out to be non-zero (the AB effect, of course), and is equal to

$$\phi^{[A]}_{AB}(C) \equiv \phi^{[A]}_{AB}(C_1) - \phi^{[A]}_{AB}(C_2) = Q \oint_{C_1 - C_2} A \cdot dr,$$

(1)

where $C \equiv C_1 - C_2$ is the directed difference of the two curves—a closed loop. As we know, a simple calculation shows $\phi_{AB}(C) = nQ\Phi_1$, where $\Phi_1$ is the magnetic flux threading the solenoid, and $n$ is the (signed) number of times $C$ winds around it (a purely topological property). The rhs is manifestly gauge invariant, which justifies the dropping of the superscript on the lhs.

What if you could have a coherent superposition of states with different charges, say $Q$ and 0? Suppose such a packet moved along $C$, starting with a null relative phase. Then the same expression we had before, $\phi^{[A]}_{AB}(C) = Q \int_C A \cdot dr$, would correspond to the relative phase between the two components after reaching the end of the curve! Are relative phases between different charge states physically observable? Wick, Wightman and Wigner [3] argued that they never are (for other reasons), and should be viewed as notational redundancies of the formalism, on a par with the overall phase. If the inter-charge phases at either end of $C$ are unobservable, as claimed, then so is their difference, resolving our apparent paradox trivially. Ironically, Aharonov and Susskind [4] argued that such phases are in principle observable! In the tradition of Yakir Aharonov [5], my mentor, I will try to use this apparent paradox to gain a better understanding of the charge superselection rule: to what extent it can be violated without violating gauge invariance.

2. Superselection rules and quantum reference frames

2.1. Are all Hermitian operators measurable?

In von Neumann’s original axiomatic formulations of non-relativistic quantum mechanics [6] (1932), the observables of the theory were identified with the set of Hermitian operators on the
Hilbert space. The assumption that all Hermitian operators are observable implies that all rays in Hilbert space are physically distinguishable. In other words, the only redundancy in the description of states by normalized vectors in Hilbert space was the overall phase. Twenty years later, motivated by ambiguities in state description in relativistic quantum mechanics pointed out by Yang and Tiomno [7], Wick, Wightman and Wigner [3] (WWW) restricted the class of observables in relativistic as well as non-relativistic QM. They argued that superpositions of different spin or charge eigenstates should be viewed as formal expressions in which the phases between differing eigenspaces are operationally meaningless, on a par with the aforementioned overall phase. Using the term ‘selection rule’ synonymously with ‘conservation law’, they coined the term ‘superselection rule’ for selection rules which are mirrored in a fundamental limitation on measurements as well, i.e. measurements of phases between certain subspaces (‘superselection sectors’) are impossible: ‘We shall say that a superselection rule operates between subspaces if there are neither spontaneous transitions between their state vectors (i.e. if a selection rule operates between them) and if, in addition to this, there are no measurable quantities with finite matrix elements between their state vectors’. Linear momentum is contrasted with intrinsic particle parity and with electrical charge: the first has a selection rule, but no superselection rule, the second has both and the third is postulated to also have both1.

The strongest evidence for a superselection rule, according to WWW, regards the unphysicality of the phase between the fermionic and bosonic subspaces of the whole Hilbert space. They give a formal proof hinging on the different effect the time inversion operator (introduced earlier by Wigner), \( T \), has on these two classes of states. Namely, applying \( T^2 \) to a formal superposition of the two types of states will result in a relative minus sign between them. Since \( T^2 \) should have no observable effect (two time inversions should result in a physically equivalent state), neither should the corresponding relative phase (an illuminating analysis of this argument, as well as the AS counter-argument appears in [5]). A more circumstantial argument (in their view) is given for the charge superselection rule, based on the symmetries of the Hamiltonians then in use for field theories of charged particles, which ‘in all cases [. . .] is invariant against a simultaneous multiplication of all fields by the same \( e^{i\alpha} \). This property is known to be connected with the principle of conservation of the total charge and represents a very restricted type of gauge invariance . . We are thus led to postulate that: multiplication of the state vector \( F \) by the operator \( e^{i\alpha Q} \) produces no physically observable modification of the state of a system of (mutually interacting) charged fields’.

2.2. AS relativity: all selection rules on same footing, require reference frames

In 1967 these two superselection rules were challenged for the first time. AS argued that, to begin with, the arguments used by WWW for the parity- and charge-superselection rules applied equally well to any and all selection rules, so if valid, they would constitute a no-go theorem on measurements of linear and angular momentum! They then proceeded to resolve the apparent paradox by analysing those two examples in detail. Finally, they showed how an inter-charge phase could be measured [4] in analogy to analogous measurements in quantum optics (and similarly for the phase between two different spin states [9]).

AS agreed that the relative phase induced between different charge sectors by the global operation \( e^{i\alpha Q} \) is unobservable, when considering a closed system (such as the whole universe), but argued that the operation \( e^{i\alpha L_z} \), of rotating the entire universe about the \( z \) axis, for example, is just as physically meaningless. Only angles with respect to frames defined by other physical

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1 If one adopts the point of view that quantum measurements consist of interactions describable within the theory, as is now commonplace (in defiance of Bohr’s forebodings), then the very distinction between selection and superselection rules seems to beg clarification (a possible distinction was suggested by WWW in their reply to AS [8]).
objects are meaningful, and these are unchanged when everything is rotated simultaneously. They consider a closed system consisting of an electron and two large magnets. The two magnets are assumed without loss of generality to have zero total angular momentum, and the electron to be in the $\sigma_Z = +1$ eigenstate (in the frame defined by the previous assumption).

The electron interacts with the first magnet in such a way as to transform its spin state to $\sqrt{1-r^2}|\sigma_Z = +1\rangle + r e^{i\theta}|\sigma_Z = -1\rangle$. This phase is evidently physically meaningful, and has a simple interpretation in terms of the direction of polarization of the electron. This phase can be measured using the second magnet, provided the relative orientation of the two magnets is well defined. A simple analysis shows that the two magnets can indeed be in an angular momentum eigenstate and have an approximately well-defined relative phase, i.e. the sum of the angular momenta and the difference of the angles are both well defined (this is analogous to the original EPR state being a simultaneous eigenstate of $\hat{p}_1 + \hat{p}_2$ and $\hat{x}_1 - \hat{x}_2$).

The sharp angular momentum implies that the angle $\theta$ is completely undefined with respect to a reference frame external to the whole system, as are all angular orientations of parts of the system. However, relative angles are perfectly consistent with it. Similar arguments hold for measurements of relative positions within a closed system, the whole of which possesses well-defined momentum.

To show how a charge-superposed nucleon state can be created, and how the relative phase can be measured, AS describe a setup analogous to Ramsey interferometry of quantum optics. In Ramsey interferometry, ‘two-level atoms’ are prepared in superpositions of ground and excited states by passing through a cavity containing a coherent state of photons, and the phase can be measured by passing them through a second such cavity and measuring the final probability to be in the excited state. The analogy is effected by replacing atoms by nucleons (which can be in a superposition of isospin eigenstates: a proton $|P\rangle$, and a neutron $|N\rangle$), and photons by charged mesons (pions). Using the coherent state formalism introduced in quantum optics by Glauber [10] a few years earlier, they define coherent states of a charged meson field with mean charge $Q((Q, \theta) \hat{Q} (Q, \theta) = Q)$ and phase $\theta$:

$$|\alpha = \sqrt{Q} e^{i\theta} = |Q, \theta\rangle = \sum_n \frac{Q^n}{\sqrt{n!}} e^{i\theta} |n\rangle. \quad (2)$$

(The Fock state $|n\rangle$ has $n$ units of charge: $\hat{Q}|n\rangle = n|n\rangle$.) A nucleon passing through a cavity containing such a state will experience an effective interaction of the Jaynes–Cummings form:

$$H = g(t)(\sigma^+ a^- + \sigma^- a^+), \quad (3)$$

where $g(t)$ is constant with value $g$ during time interval $[0, T]$ corresponding to the passage of the nucleon through the cavity (during which the action of the free Hamiltonian can be neglected), and zero outside it.

A proton entering the cavity $C1$ exits in a superposition state, which in the approximation $Q \gg 1$, is given by

$$|P\rangle|Q, \theta\rangle \mapsto \cos(gT Q^{1/2})|P\rangle + i e^{i\theta} \sin(gT Q^{1/2})|N\rangle. \quad (4)$$

The absolute values of the coefficients of the charge eigenstates clearly have operational meaning as probability amplitudes for the respective states, but how can we measure the relative phase? Passing the nucleon through a second cavity, $C2$, containing the mesonic coherent state $|Q', \theta'\rangle$, we get a similar expression, but now the absolute value of each coefficient depends on $\theta - \theta'$. Thus, the probability of the nucleon to exit the second cavity as a proton depends on the relative phase of the fields in the two cavities. In other words, measuring this probability is tantamount to measuring the phase between the proton and neutron in the intermediate state, relative to the reference frame defined by $C2$. 4
So far, so good, but we have resorted to using coherent mesonic states, which themselves contain coherences between different charge eigenstates! So far then, the logic seems to be circular (an analogous objection regarding optical coherent states would only be raised decades later, as we shall note in the next section). However, AS note that just as in the angular momentum case, we do not really need well-defined phases for each cavity separately, only a well-defined relative phase $\Delta \theta \equiv \theta - \theta'$, which is consistent with a well-defined total charge $Q_1 + Q_2$ for the two cavities. They note that such a state is approximately given by

$$|i\rangle = \int |Q, \theta_1\rangle |Q', \theta_1 + \Delta \theta\rangle e^{(Q+Q')\theta_1}$$

(where the integration should be understood to be over $\theta_1$). That this is indeed an eigenvector of $Q_1 + Q_2$ is readily verified by a direct calculation, but there is a slight subtlety with the interpretation of this as having a well-defined relative phase. Not only is there no phase operator (which is why we need to assume the $Q$s are big), but also the values of the integrand evaluated at different values of $\theta_1$ are not mutually orthogonal (the coherent states, far from forming an orthonormal basis, are actually overcomplete). This leaves a gap in the proof due to the possibility that interference between different terms would change the conclusion. AS also suggest a Gedankenexperiment where a relative phase between the two cavities is established by populating them jointly by passing $Q$ mesons through an appropriate beam-splitter, such that the transmitted amplitude goes in one cavity and the reflected one in the other. The relative phase would be determined by the complex reflection and transmission coefficients (and the state would also be an exact eigenstate of $Q_1 + Q_2$). The optical analogue of this setup in this very context was rigorously analysed much later by Mølmer [1].

2.3. WWW’s reply and aftermath: symmetry breaking primordial coherences or ubiquitous reference frames?

WWW [8] replied to the AS papers by acknowledging that given a mesonic charged coherent state, and the physically acceptable meson–nucleon interaction used by AS, an inter-charge sector phase could indeed be measured. However, they claimed that the assumption of its existence only begs the question: such coherence cannot be created by the postulated interaction, and so it is required to have pre-existent coherence to measure coherence (it takes one to know one, so to speak), and in particular to create it by measurement2. Since there is no evidence for what could be called ‘primordial coherence’, there is no basis for assuming that such measurement could be performed. The lack of superselection rules corresponding to other selection rules is explained by the prior existence of such god-given coherences: ‘There is, in this regard a fundamental difference between conserved quantities, such as linear and angular momentum on the one hand, and electric (and baryonic) charge on the other. We have naturally been given states which are superpositions of states with different momenta; all more or less localized states are of this nature . . . ’. They do not address the issue of reference frames.

In an ironic turn of fate, much later, in a series of papers dealing with the status of the particle number superselection rule in Bose–Einstein condensates [11, 12] and optical systems such as the laser [1], the measurability of optical phases (and their atomic-optical analogues) has itself come under suspicion. In addressing these issues, the AS reference frame concept has been called to the rescue of (the theoretical explanation of) Ramsey interferometry itself and even the existence of the paradigmatic optical coherent state itself, which inspired it, thus coming full circle!

2 This in turn, would seem to raise the question of the distinction between selection and superselection. See the previous footnote.
3. Are reference frames for charge compatible with gauge invariance?

We are now in a position to put together the different pieces of the puzzle. Let us revisit the paradox described in the introduction, and attempt to fill in the details in careful adherence to the AS measurement procedure. AS tell us that we can, in principle, prepare an initial state of the form \[ \frac{1}{\sqrt{2}} (|P\rangle + |N\rangle) \], by letting a nucleon interact with a mesonic field in cavity \( c_1 \) at point \( A \). The phase, \( \phi_A \), is defined with respect to that of the cavity field. Now after travelling along the curve \( C \), let it encounter a second cavity \( c_2 \) at its other end, \( B \). Its phase relative to that of \( c_2 \), \( \phi_B \), is again AS measurable. Now, \( \phi_{AB} \equiv \phi_B - \phi_A = Q \int_C A \cdot dr - \Delta \theta \), \( (6) \) where \( \Delta \theta \) is the relative phase between the two cavity mesonic fields. Since preparation and measurement are closely related in quantum mechanics, this relative phase can be thought of as the calibration of the cavities, viewed as measuring devices. However, this calibration itself is gauge dependent. One way to see this is to consider two similar cavities initially at \( A \) containing identical coherent states, \( |Q, \theta\rangle \), and then letting one of them travel along the curve \( C' \) to point \( B \). Described in a particular gauge, the states of the fields in the stationary and mobile cavities at the end of this process will be given by \( |Q, \theta\rangle \) and \( e^{i\phi_{AB}(C')}|Q, \theta\rangle \), respectively. Therefore, \( \phi_{AB} = \phi_{AB}^{(A)}(C) - \phi_{AB}^{(A)}(C') \) which is equal to the topological (and gauge-independent) AB phase \( \phi_{AB}^{(A)}(C-C') \). Thus, there is some trivial freedom in calibration, but the only phase information one obtains from this measurement procedure is the topological one. In effect, the detector itself ‘closes the loop’ (either around, or outside the fluxon). The latter interpretation is analogous to a result of Vaidman and Aharonov \[13, 14\] on the measurement of the relative phase between remote packets of a single photon through its absorption by a pair of atoms and conversion into a proper (two-particle) EPR—Bohm state.

4. Summary

In the AB effect, a charged particle travelling in the force-free region outside of a solenoid, formally acquires a path- and gauge-dependent phase, in a continuous fashion. However, what saves the day for gauge invariance (or for the AB effect) is the fact that this phase is to some extent an artifact of the notation. If our particle is the only system treated as quantum, then this is a manifestly redundant overall phase, except for situations where two amplitudes corresponding to different sets of Feynman paths add up, which together can be considered as closing a loop, and thus is just the topological, non-local, situation, which is gauge invariant. If, however, one allows coherent superpositions of different charge states, then the phase picked up along an open trajectory is a local relative phase, and hence measurable! At first sight this seems to conflict with gauge invariance. In fact, the AS phase, whatever its provenance is gauge dependent, to begin with, just like the initial phase of the charged particle in the regular AB effect. Since AS tell us that we should look at phases defined relative to physical reference frames, we are led to consider such frames for the two spatially separated endpoints of the curve \( C \). There is ambiguity in the relative phase of the latter, due to gauge freedom. However, the phase difference between the initial and final states relative to these frames is gauge invariant and depends only on the topological AB effect. We conclude that a reference frame for inter-charge phase can, in principle, be established locally in the AS sense; the relative phase of two spatially separate phase standards is ambiguous, in accordance with gauge invariance.
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