Anisotropic induced gravity and inflationary universe

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Existence and stability analysis of the Kantowski-Sachs type universe in a higher derivative induced gravity theory is studied in details. Existence of one stable mode and one unstable mode is shown to be in favor of the inflationary universe. As a result, the de Sitter background can be made to be stable against anisotropic perturbations with proper constraints imposed on the coupling constants of the induced gravity model.

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I. INTRODUCTION

Inflationary theory provides a natural resolution for the the flatness, monopole, and horizon problems of our present universe described by the standard big bang cosmology [1]. In particular, our universe is homogeneous and isotropic to a very high degree of precision [2, 3]. Such an universe can be described by the well known Friedmann-Robertson-Walker (FRW) metric [4].

Moreover, gravitational physics could be different from the standard Einstein models near the Planck scale [5, 6]. For example, quantum gravity or string corrections could lead to some interesting cosmological applications [5]. In particular, investigations have been conducted on the possibility of deriving inflation from higher order gravitational corrections [7, 8, 9, 10].

For example, a general analysis on the stability conditions of gravity theories could be useful in screening physical models compatible with our physical universe. In particular, the stability condition for a variety of pure gravity theories as a potential candidate of inflationary universe in the flat Friedmann-Robertson-Walker (FRW) space is derived in Ref. [10, 12, 13].

In addition, the highly isotropic universe should evolve from some initially anisotropic state before it becomes isotropic to such a high degree of precision. Nonetheless, it is interesting by itself to study the stability analysis of the anisotropic space during the post-inflationary epoch even anisotropy can be smoothed out by the proposed inflationary process. One would like to know whether our universe can evolve from certain anisotropic universe to a stable and isotropic final state. In particular, it is known that such inflationary solution exists for an NS-NS model with a metric, a dilaton, and an axion field [11]. This inflationary solution is also shown to be stable against small perturbations [14]. Note that similar stability analysis has also been studied for a various of interesting models [15, 16].

The importance of higher derivative models have been known in many aspects. In particular, higher derivative terms are known to be important for the Planck scale physics [10, 13]. For example, higher order corrections from quantum gravity or string theory have been considered as possible inflationary models [17]. In addition, higher derivative terms also arise as the quantum corrections of the matter fields [17]. Therefore, it is important to study the implications of the stability analysis of all possible higher derivative models.

Recently, there are also growing interests in the study of Kantowski-Sachs (KS) type anisotropic spaces [18, 19, 20]. We will hence try to study the problem of existence and stability conditions of an inflationary de Sitter final state for some higher derivative model in Kantowski-Sachs spaces. In particular, a large class of pure gravity models with inflationary KS/FRW solutions was presented in Ref. [11]. Any KS type solution leading itself to an asymptotic FRW final state will be referred to as the KS/FRW solution in this paper for convenience.

It has been shown that the existence of a stable de Sitter background is closely related to the choices of the coupling constants. Indeed, a pure gravity model given below:

\[
\mathcal{L} = -R - \alpha R^2 - \beta R_{\mu
u}R_{\mu\nu} + \gamma R_{\mu\nu}R_{\mu\nu} + \beta\gamma R_{\mu\nu}R_{\mu\nu} + \gamma R^{\beta\gamma}_{\sigma\rho}R^{\beta\gamma}_{\sigma\rho}\]  

(1)

admits an inflationary solution with a constant Hubble parameter determined by \(H_0^4 = 1/4\gamma\) if \(\gamma > 0\). Here \(\alpha, \beta,\) and \(\gamma\) are coupling constants. This shows that (a) the \(\gamma R^2\) cubic term determines the scale of the inflation characterized by the Hubble parameter \(H_0\) and (b) quadratic terms are irrelevant to the scale \(H_0\) in the de Sitter phase. The quadratic terms are, however, important to the stability of the de Sitter phase.

Indeed, perturbing the KS type metric with \(H_i \rightarrow H_0 + \delta H_i\), one can show that

\[
\delta H_i = c_i \exp[-3H_0 t/2] (1 + \delta_1) + d_i \exp[-3H_0 t/2] (1 - \delta_1) \]  

(2)
for
\[ \delta_1 = \sqrt{1 + 8/[27 - 9(6a + 2\beta)H_0^2]} \]
and some arbitrary constants \( c_i, d_i \) to be determined by the initial perturbations. Here \( H_i \equiv \dot{a}_i/a_i \) with \( a_i(t) \) the scale factor in \( i \)-direction. We will describe the notation shortly in section II. It is easy to see that any small perturbation \( \delta H_i \) will be stable against the de Sitter background if both modes characterized by the exponents
\[ \Delta_\pm = -[3H_0\delta/2][1 \pm \delta_1] \]
are all negative. This will happen if \( \delta_1 < 1 \). In such case, the inflationary de Sitter space will remain a stable background as the universe evolves. More specifically, the constraint \( \beta H_0^2 > 35/18 \) is required for both modes to be stable.

It can be shown that the stability equation for the anisotropic KS space and the stability equation for the isotropic FRW space in the presence of the same inflationary de Sitter background turns out to be identical \[10\, 12\, 13\]. Therefore, the stability of isotropic perturbations also ensures the stability of the anisotropic perturbations. The stability of the isotropic perturbations for the FRW space is important for any physical models. Unfortunately, inflationary models that are stable against any isotropic perturbations will have problem with the graceful exit process. Therefore, the pure gravity model may have troubles dealing with the stability and exit mechanism all together.

Instead of the pure gravity theory, a slow rollover scalar field may help resolving this problem. An inflationary de Sitter solution in a scalar-tensor model is expected to have one stable mode (against the perturbation in \( \delta H \) direction) and one unstable mode (against the perturbation in \( \delta \phi \) direction). As a result, the inflationary era will come to an end once the unstable mode takes over after a brief period of inflationary expansion. Therefore, we propose to study the effect of such theory.

In particular, we will show in this paper that the roles played by the higher derivative terms are dramatically different in the inflationary phase of our physical universe in both pure gravity theory and scalar-tensor theory. First of all, third order term will be shown to determine the expansion rate \( H_0 \) for the inflationary de Sitter space. The quadratic terms will be shown to have nothing to do with the expansion rate of the background de Sitter space. They will however affect the stability condition of the de Sitter phase. Their roles played in the existence and stability condition of the evolution of the de Sitter space are dramatically different.

II. NON-REDUNDANT FIELD EQUATION AND BIANCHI IDENTITY IN KS SPACE

Given the metric of the following form:
\[ ds^2 = -dt^2 + c^2(t)dr^2 + a^2(t)(d\theta^2 + f^2(\theta)d\varphi^2) \]
with \( f(\theta) = (\theta, \sin \theta, \sin \theta) \) denoting the flat, open and close anisotropic space known as Kantowski-Sachs type anisotropic spaces. More specifically, Bianchi I (BI), III(BIII), and Kantowski-Sachs (KS) space corresponds to the flat, open and closed model respectively. This metric can be rewritten as
\[ ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1 - kr^2} + r^2d\theta^2) + a_z^2(t)dz^2 \]
with \( r, \theta, \) and \( z \) read as the polar coordinates and \( z \) coordinate for convenience and for easier comparison with the FRW metric. Note that \( k = 0, 1, -1 \) stands for the flat, open and closed universes similar to the FRW space.

Writing \( H_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu} \), Einstein equation can be written as \( D_\mu H^{\mu\nu} = 0 \) incorporating the Bianchi identity \( D_\mu G^{\mu\nu} = 0 \) and the energy momentum conservation \( D_\mu T^{\mu\nu} = 0 \). Here \( G^{\mu\nu} \) and \( T^{\mu\nu} \) represent the Einstein tensor and the energy momentum tensor coupled to the system respectively. With the metric \[8\], it can be shown that the \( r \) component of the equation \( D_\mu H^{\mu\nu} = 0 \) implies that
\[ H_r = H_0. \]
This result also says that any matter coupled to the system has the symmetric property \( T^r_r = T^0_0 \). In addition, the equations \( D_\mu H^{\mu\theta} = 0 \) and \( D_\mu H^{\muz} = 0 \) both vanish identically for all kinds of energy momentum tensors. More interesting information comes from the \( t \) component of this equation. It says:
\[ (\partial_t + 3H)H_t^t = 2H_1H_r^t + H_z^z, \]
This equation implies that (i) $H^i_1 = 0$ implies that $H^{i}_r = H^{i}_z = 0$ and (ii) $H^{i}_r = H^{i}_z = 0$ only implies $(\partial t + 3H)H^i_1 = 0$ instead of $H^i_1 = 0$. Case (ii) can be solved to give $H^i_1 = \text{constant} \times \exp[-a^2a_z]$ which approaches zero when $a^2a_z \to \infty$.

For the anisotropic KS spaces, the metric contains two independent variables $a$ and $a_z$. The Einstein field equations have, however, three non-vanishing components: $H^i_1 = 0$, $H^{i}_r = H^{i}_z = 0$ and $H^{i}_z = 0$. The Bianchi identity implies that the $tt$ component is not redundant and will hence be retained for complete analysis. Ignoring either one of the $rr$ or $zz$ components will not affect the final result of the system. In short, the $H^i_1 = 0$ equation, known as the generalized Friedman equation, is a non-redundant field equation as compared to the $H^i_r = 0$ and $H^i_z = 0$ equations.

In addition, restoring the $g_{tt}$ component $b^2(t) = 1/B_1$ will be helpful in deriving the non-redundant field equation associated with $G_{tt}$ that will be shown shortly. More specifically, the generalized KS metric will be written as:

$$ds^2 = -b^2(t)dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2d\theta^2\right) + a_z^2(t)dz^2.$$  \hspace{1cm} (9)

In principle, the Lagrangian of the system can be reduced from a functional of the metric $g_{\mu\nu}$, $\mathcal{L}(g_{\mu\nu})$, to a simpler function of $a(t)$ and $a_z(t)$, namely $L(t) \equiv a^2a_zL(g_{\mu\nu}(a(t), a_z(t)))$. The equation of motion should be reconstructed from the variation of the reduced Lagrangian $L(t)$ with respect to the variable $a$ and $a_z$. The result is, however, incomplete because, the variation of $a$ and $a_z$ are related to the variation of $g_{rr}$ and $g_{zz}$ respectively. The field equation from varying $g_{tt}$ can not be derived without restoring the variable $b(t)$ in advance. This is the motivation to introduce the metric $[9]$ such that the reduced Lagrangian $L(t) \equiv b^2a_zL(g_{\mu\nu}(b(t), a(t), a_z(t)))$ retains the non-redundant information of the $H^i_1 = 0$ equation. Non-redundant Friedmann equation can be reproduced resetting $b = 1$ after the variation of $b(t)$ has been done.

After some algebra, all non-vanishing components of the curvature tensor can be computed: \[21\]

\begin{align*}
R^{ti}_{\phantom{ij}ij} &= \frac{1}{2}B_1H_i + B_1(\dot{H}_i + H^i_1)\delta^i_j, \\
R^{ij}_{\phantom{kl}kl} &= B_1H_iH_j \epsilon_{ijm}\epsilon_{klm} + \frac{k}{a^2}\epsilon^{ijz}\epsilon_{klz}
\end{align*}

(10)\hspace{1cm}(11)

with $H_i \equiv (\dot{a}/a, \dot{a}/a, \dot{a}_z/a_z) \equiv (H_1, H_2, H_3)$ for $r, \theta$, and $z$ component respectively.

Given a Lagrangian $L = \sqrt{g}\mathcal{L} = L(b(t), a(t), a_z(t))$, it can be shown that

$$L = \frac{a^2a_z}{\sqrt{B_1}}L(R^{ti}_{\phantom{ij}ij}, R^{ij}_{\phantom{kl}kl}) = \frac{a^2a_z}{\sqrt{B_1}}L(H_i, \dot{H}_i, a^2)$$  \hspace{1cm} (12)

The variational equations for this action can be shown to be: \[21\]

$$\mathcal{L} + H_i\frac{d}{dt}(\frac{d}{dt} + 3H)L^i = H_iL_i + \dot{H}_iL^i$$  \hspace{1cm} (13)

$$\mathcal{L} + (\frac{d}{dt} + 3H)^2L^z = (\frac{d}{dt} + 3H)L_z$$  \hspace{1cm} (14)

Here $L_i \equiv \delta\mathcal{L}/\delta H_i$, $L^i \equiv \delta\mathcal{L}/\delta \dot{H}_i$, and $3H \equiv \sum_i H_i$. For simplicity, we will write $\mathcal{L}$ as $L$ from now on in this paper. As a result, the field equations can be written in a more comprehensive form:

$$DL \equiv L + H_i\frac{d}{dt}(\frac{d}{dt} + 3H)L^i - H_iL_i - \dot{H}_iL^i = 0$$  \hspace{1cm} (15)

$$D_zL \equiv L + (\frac{d}{dt} + 3H)^2L^z - (\frac{d}{dt} + 3H)L_z = 0$$  \hspace{1cm} (16)

### III. HIGHER DERIVATIVE INDUCED GRAVITY MODEL

In this section, we will study the higher derivative induced gravity model:

$$L = -\frac{\epsilon}{2}\phi^2R - \alpha R^2 - \beta R_{\mu\nu}R_{\mu\nu} + \frac{\gamma}{\phi^2}\rho_{\mu\nu}R_{\mu\nu}R_{\beta\gamma}R_{\beta\gamma} + \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V(\phi) \equiv \frac{\epsilon}{2}\phi^2L_0 + L_2 + \frac{\gamma}{\phi^2}L_3 + L_{\phi}$$  \hspace{1cm} (17)

with $L_0 = -R$, $L_2 = -\alpha R^2 - \beta R_{\mu\nu}R_{\mu\nu}$, $L_3 = R_{\mu\nu}R_{\beta\gamma}R_{\beta\gamma} + \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V(\phi)$ denoting the lowest order curvature coupling, the higher order terms, and the scalar field Lagrangian respectively. Induced gravity models assume that all dimensionful parameters or coupling constants are induced by a proper choice of dynamical fields. For
example, the gravitational constant is replaced by \(8\pi G = 2/(\epsilon \phi^2)\) as a dynamical field. In addition, the cosmological constant becomes \(V(\phi)\) in this model. There is no need for any induced parameters for the quadratic terms \(R^2\) and \(R_{\mu\nu}^2\), because the coupling constants \(\alpha\) and \(\beta\) are both dimensionless by itself. This action of the system is also invariant under the global scale transformation: \(g_{\mu\nu} \to \Lambda^{-2} g_{\mu\nu}\) and \(\phi \to \Lambda \phi\) with arbitrary constant parameter \(\Lambda\).

The corresponding Lagrangian can be shown to be:

\[
L = \epsilon \phi^2 (2A + B + 2C + D) - 4\alpha \left[4A^2 + B^2 + 4C^2 + D^2 + 4AB + 8AC + 4AD + 4BC + 2BD + 4CD\right]
- 2\beta \left[3A^2 + B^2 + 3C^2 + D^2 + 2AB + 2AC + 2AD + 2BC + 2CD\right] + \frac{8}{\phi^2} \left[2A^3 + B^3 + 2C^3 + D^3\right]
\]

\[+ \frac{1}{2} \dot{\phi}^2 - V(\phi)
\]

(18)

with

\[A = \dot{H}_1 + H_1^2,
\]

(19)

\[B = H_1^2 + \frac{k}{\alpha^2},
\]

(20)

\[C = H_1 H_2,
\]

(21)

\[D = \dot{H}_2 + H_2^2.
\]

(22)

This Lagrangian can be shown to reproduce the de Sitter models when we set \(H_i \to H_0\) in the isotropic limit. The Friedmann equation reads:

\[
\frac{1}{2} \epsilon \phi^2 DL_0 + DL_2 + \frac{\gamma}{\phi^2} DL_3 + \epsilon \phi \dot{H}_1 L_0 - 2 \frac{\gamma}{\phi^2} \dot{\phi} H_1 L_3 = \frac{1}{2} \dot{\phi}^2 + V(\phi)
\]

(23)

for the induced gravity model. In addition, the scalar field equation can be shown to be:

\[
\ddot{\phi} + 3H_0 \dot{\phi} + V' = \epsilon \phi L_0 - 2 \frac{\gamma}{\phi^2} L_3.
\]

(24)

The leading order de Sitter solution with \(\phi = \phi_0\) and \(H_i = H_0\) for all directions can be shown to be:

\[
V_0 \equiv V(\phi_0) = 3\epsilon' \phi_0^2 H_0^2,
\]

(25)

\[
V'(\phi_0) = 12\epsilon' \phi_0 H_0^2
\]

(26)

under the slow roll-over assumption \(\dot{\phi} << V'\) and \(H_0 \dot{\phi} << V'\). Here \(\epsilon' \equiv \epsilon [1 - 8\gamma H_0^2/(\epsilon \phi_0^2)]\). Indeed, It can be shown that in the de Sitter inflationary phase, the ignored part of the scalar field equation evolves as \(\ddot{\phi} + 3H_0 \dot{\phi} \sim 0\). This equation leads to the approximated solution

\[
\phi \sim \phi_0 + \frac{\dot{\phi}_0}{3H_0} [1 - \exp(-3H_0 t)]
\]

(27)

during the de Sitter phase \(H_i = H_0\). This result is clearly consistent with the slow roll-over assumption we just made.

In summary, the leading order equations give us a few constraints on the field parameters:

\[
4V_0 = \phi_0 \frac{\partial V}{\partial \phi}(\phi = \phi_0) = 12\epsilon' \phi_0^2 H_0^2.
\]

(28)

An appropriate effective spontaneously symmetry breaking potential \(V\) of the following form

\[
V(\phi) = \frac{\lambda}{4} (\phi^2 - \phi_0^2)^2 + 6\epsilon' H_0^2 (\phi^2 - \phi_0^2) + 3\epsilon' H_0^2 \phi_0^2
\]

(29)

with arbitrary coupling constant \(\lambda\) can be shown to be a good candidate satisfying all the scaling conditions \([28]\). The value of \(H_0\) can be chosen to induce enough inflation for a brief moment as long as the slow rollover scalar field remains close to the initial state \(\phi = \phi_0\). The de Sitter phase will hence remain valid and drive the inflationary process for a brief moment determined by the decaying speed of the scalar field. The stability conditions associated with this effective potential implied by the field equations will be studied in the following section.
Note that the local extremum of this effective potential can be shown to be $\phi = 0$ (local maximum) and $\phi^2 = \phi^2_m = \phi^2_0 - 12\epsilon H^2_0 / \lambda < \phi^2_0$ (local minimum). In addition the minimum value of the effective potential can be shown to be

$$V_m = V_0 - 36\epsilon^2 H^4_0 / \lambda < V_0.$$  

The constraint $V_m > 0$ implies that $\lambda \phi^2_0 > 12\epsilon H^2_0$. Or equivalently, it implies that $\phi^2_m > 0$. In addition, we will set $\epsilon \phi^2_m / 2 = 1/(8\pi G) = 1$ in Planck unit later in this paper.

When the scalar field settles down to the local minimum $\phi_m$ of the effective potential at large time in the post inflationary era, it will oscillate around the local minimum and kick off the reheating process. The scalar field will eventually become a constant background field with a small cosmological constant $V_m = V(\phi_m)$.

\section*{IV. STABILITY OF HIGHER DERIVATIVE INFLATIONARY SOLUTION}

Our universe could start out anisotropic and evolves to the present highly isotropic state in the post inflationary era. Therefore, a stable KS/FRW solution is necessary for any physical model of our universe.

Given an effective action of the sort described by Eq. (29), the scaling constraints (28) is required for the existence of the de Sitter solution $H_i = H_0$ and the static condition $\phi = \phi_0$. In addition to these constraints, small perturbations, $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta \phi$, against the background de Sitter solution $(H_0, \phi_0)$ may also put a few more constraint to the stability requirement of the system. This perturbation will enable one to understand whether the background solution is stable or not. In particular, it is interesting to learn whether a KS $\rightarrow$ FRW (KS/FRW) type evolutionary solution is stable or not.

It can be shown that the perturbation equation for the Bianchi models are identical to the perturbation equation for the FRW models. Therefore, any inflationary solutions with a stable mode and an unstable mode will provide a natural way for the inflationary universe to exit the inflationary phase. Such models will, however, also be unstable against the anisotropic perturbations. Therefore, any inflationary solutions with a stable mode and an unstable mode is also a negative result to our search for a stable and isotropic inflationary model. As a result, such solution will be harmful for an anisotropic space to reach an isotropic FRW space once the de Sitter phase start to collapse. It will be shown shortly that the higher derivative induced gravity theory could hopefully resolve this problem all together.

In practice, perturbing the background de Sitter solution along the $\delta H_i$ direction should be stable for at least a brief moment of the order $\Delta T \sim 60 H_0^{-1}$ for a physical model. As a result around 60 e-fold inflation can be induced before the de Sitter phase collapses. And the resulting universe is stable against isotropic and anisotropic perturbations. In addition, the scalar field is expected to roll slowly from the initial state $\phi = \phi_0$. Therefore, the perturbation along the $\delta \phi$ direction is expected to be unstable. This will favor the system for a natural mechanism for graceful exit. Hence, we will try to study the stability equations of the system for small perturbations against the de Sitter background solutions.

The first order perturbation equation for $DL$, with $H_i \rightarrow H_0 + \delta H_i$, can be shown to be:

$$\delta(DL) = < H_i L^{ij} \delta \dot{H_j} > + 3H < H_i L^{ij} \delta \ddot{H_j} > + 3H < (H_i L^j_j + L^j) \delta H_j > + < H_i L^i > \delta (3H) - < H_i L^{ij} \delta H_j >$$

for any $DL$ defined by Eq. (15) with all functions of $H_i$ evaluated at some FRW background with $H_i = H_0$. The notation $< A_i B_i > = \sum_{i=1}^{z} A_i B_i$ denotes the summation over $i = 1$ and $z$ for repeated indices. Note that we have absorbed the information of $i = 2$ into $i = 1$ since they contribute equally to the field equations in the KS type spaces. In addition, $L^i_j = \delta^2 L_i H_j + \delta H_i \delta H_j$ and similarly for $L_{ij}$ and $L^{ij}$ with upper index $i$ and lower index $j$ denoting variation with respect to $\dot{H}_i$ and $H_j$ respectively for convenience. In addition, perturbing Eq. (16) can also be shown to reproduce the Eq. (31) in the de Sitter phase[1].

In addition, it can be shown that

$$< H_i L^i_{1} >= 2 < H_i L^i_{2} >$$

$$< H_i L^i_{1} > = 2 < H_i L^i_{2} >$$

$$L^1 = 2L^2,$$

$$< H_i L^i_{1} > = 2 < H_i L^i_{2} >$$

in the inflationary de Sitter background with $H_0 = \text{constant}$. Therefore, the stability equations (31) can be greatly simplified. For convenience, we will define the operator $D_L$ as

$$D_L \delta H \equiv < H_i L^{ij} > \delta \dot{H} + 3H < H_i L^{ij} > \delta \ddot{H} + 3H < H_i L^{i}_{1} + L^1 > \delta H + 2 < H_i L^i > \delta H - < H_i L^{i1} > \delta H = 0.$$
As a result, the stability equation (31) becomes
\[ \delta(DL) = D_L(\delta H_1 + \delta H_2/2) = \frac{3}{2}D_L(\delta H) = 0 \] (37)

with \( H = (2H_1 + H_2)/3 \) as the average of \( H_i \).

Hence the leading order perturbation equation in \( \delta H \) and \( \delta \phi \) for the Friedmann equation of this model can be shown to be:
\[ \frac{\gamma H_0^4}{\epsilon \phi_0^4} - 6\epsilon (1 - 24\frac{\gamma H_0^4}{\epsilon \phi_0^4})r_0H_0[\delta \phi - H_0\delta \phi] = \frac{\epsilon}{2}\phi_0^2 \delta(DL_0) + \delta(DL_2) + \frac{\gamma}{\phi^2} \delta(DL_3) = \frac{3}{4}\epsilon \phi_0^2 D_0 \delta H + \frac{3}{2}D_2 \delta H + \frac{3\gamma}{2\epsilon^2} D_3 \delta H \] (38)

with \( D_0 \delta H \equiv D_L \delta H, \ D_2 \delta H \equiv D_L \delta H \) and \( D_3 \delta H \equiv D_L \delta H \) as short-handed notations. This equation can further be shown to be:
\[ \epsilon (1 - 24\frac{\gamma H_0^4}{\epsilon \phi_0^4}) \phi_0[\delta \ddot{\phi} - H_0 \dot{\delta \phi}] = 4(3\alpha + \beta - 6\gamma \frac{H_0^2}{\phi_0^2}) (\dot{\delta H} + 3H_0 \delta \dot{H}) + (24\gamma \frac{H_0^4}{\phi_0^4} - \epsilon \phi_0^2) \delta H. \] (39)

Similarly, the leading perturbation of the scalar field equation can be shown to be:
\[ \delta \ddot{\phi} + 3H_0 \dot{\delta \phi} + (V'' - 12\epsilon \frac{H_0^3}{\phi_0}) \delta \phi = 6\epsilon \dot{\phi}_0 (\delta \dot{H} + 4H_0 \delta H) \] (40)

The variational equation of \( a_z \) can be shown explicitly to be redundant in the limit \( H_i = H_0 + \delta H_i \) and \( \phi = \phi_0 + \delta \phi \) following the Bianchi identity.

Assuming that \( \delta H = \exp[hH_0^0] \delta H_0 \) and \( \delta \phi = \exp[pH_0^0] \delta \phi_0 \) for some constants \( h \) and \( p \), one can write above equations as:
\[ \epsilon (1 - 24\frac{\gamma H_0^4}{\epsilon \phi_0^4}) \phi_0 [p - 1] \delta \phi = 4(3\alpha + \beta - 6\gamma \frac{H_0^2}{\phi_0^2}) H_0 [h^2 + 3h + \frac{24\gamma H_0^4}{\phi_0^4} - \epsilon \phi_0^2] \delta H, \] (41)
\[ H_0 \left[ p^2 + 3p + \frac{V''}{H_0^2} - 12\epsilon - 384\frac{H_0^4}{\phi_0^2}\right] \delta \phi = 6\epsilon \phi_0 [h + 4] \delta H. \] (42)

These equations are consistent when all coefficients vanish simultaneously. This implies that \( h = -4 \) and \( p = 1 \). This set of solution \( (h,p) = (-4,1) \) hence imposes two additional constraint
\[ \epsilon - 16(3\alpha + \beta) \frac{H_0^2}{\phi_0^2} + 72\gamma \frac{H_0^4}{\phi_0^4} = 0, \] (43)
\[ \lambda = 192\gamma \frac{H_0^4}{\phi_0^4} - 2\frac{H_0^2}{\phi_0^2} \] (44)

with \( 2\lambda \phi_0^2 = V'' - 12\epsilon \phi_0^2 \).

The coupling constant \( \lambda \) has to positive in order for the effective potential \( V(\phi) \) to be free from run-away negative global minimum at \( \phi \to \infty \). As a result, the constraints \( \epsilon > 0 \) and \( \lambda > 0 \) imply that
\[ \frac{2(3\alpha + \beta) \phi_0^2}{9H_0^2} > \gamma > \frac{\phi_0^4}{96H_0^4}. \] (45)

Therefore, the inflationary phase will remain stable against small perturbation along the \( \delta H (= \exp[-4H_0^0] \delta H_0) \) direction. In addition, the inflationary phase also has an unstable mode when we perturb the system along the \( \delta \phi (= \exp[H_0^0] \delta \phi_0) \) direction. The unstable mode is in fact consistent with the slow rollover assumption. The scalar field is expected to roll slowly off the initial state \( \phi_0 \) for a brief moment during the inflationary era. This unstable mode is hence responsible for the graceful exit process.

Hence such system with a stable mode and an unstable mode is a very nice candidate for a inflationary model. The stable mode \( \delta H = \exp[-4H_0^0] \delta H_0 \) implies that the de Sitter background solution will remain stable against small isotropic perturbation. It also implies that the system is stable against any anisotropic perturbation along all \( \delta H_i \) directions. Therefore, the induced gravity model with a scalar field can indeed resolve the stability problem of the pure gravity model.
V. CONCLUSION

The existence of a stable de Sitter background is closely related to the choices of the coupling constants. The pure higher derivative gravity model with quadratic and cubic interactions $[21]$ admits an inflationary solution with a constant Hubble parameter. Proper choices of the coupling constants allow the de Sitter phase to admit one stable mode and one unstable mode for the anisotropic perturbation.

The stable mode favors a strong inflationary period and the unstable mode provides a natural mechanism for the graceful exit process. It is also found that the perturbation against the isotropic FRW background space and the perturbation against the anisotropic KS type background space obey the same perturbation equations. This is true for both pure and induced gravity models. As a result, the unstable mode in pure gravity model also means that the isotropic de Sitter background is unstable against anisotropic perturbations. Therefore, small anisotropy could be generated during the de Sitter phase for pure gravity model.

We have shown that, for induced gravity models, stable mode for perturbations along the anisotropic $\delta H_i$ directions does exist with proper constraints imposed on the coupling constants. In addition, another unstable mode for perturbation against the anisotropic KS type background space obey the same perturbation equations. This is true for both pure and induced gravity models. As a result, the unstable mode in pure gravity model also means that the

It is also found that the quadratic terms will not affect the inflationary solution characterized by the Hubble parameter $H_0$. These quadratic terms play, however, critical role in the stability of the de Sitter background. In addition, it is also interesting to find that their coupling constants $\alpha$ and $\beta$ always show up as a linear combination of $3\alpha + \beta$ in these stability equations. Implications of these constraints deserve more attention for the applications to the inflationary models.

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