Non relativistic SUSY in variants of the planar Lévy-Leblond equation

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Abstract

An \(N = 2\) SUSY extension of the Schrödinger symmetry is shown to exist in the solution space of the free planar Lévy-Leblond equation, an \(N = 1\) part of which survives for the gauged version of the equation and also when it is coupled to Chern-Simons theory.

1 Introduction

A natural way to obtain models with non relativistic SUSY is to consider the non relativistic limit of relativistic, supersymmetric theories [1]. Here we are interested rather in extending the combination of Galilean and non relativistic conformal symmetries [2] (called Schrödinger symmetry) by anticommuting generators in models/equations containing no apparent supersymmetry or superpartners. SUSY extensions of the Schrödinger symmetry were considered previously [3] [4] [5] [6] [7]. In [6] it is shown that in \(d > 2\) space dimensions Schrödinger symmetry admits a unique \(N = 1\) extension, while in \(d = 2\) there are two such extensions that combine into an \(N = 2\) extension. In [7] a physical realization of this special extension is described on the example of a spin \(1/2\) particle moving in the field of a magnetic vortex.

In this paper we look for realization of supersymmetry in the solution space of certain variants of the planar Lévy-Leblond equation namely in the free and in the gauged equations and when it is coupled to Chern-Simons theory. The \(3 + 1\) dimensional Lévy-Leblond equation [8] may be thought of as a non relativistic Dirac equation: it is a first order differential equation for a spin \(1/2\) particle, the square of which gives the Pauli equation. For the \(3 + 1\) dimensional free Lévy-Leblond equation an \(N = 1\) extension of the Schrödinger symmetry is described in [9].

Since the Schrödinger symmetry for these three variants of the planar Lévy-Leblond equation is shown in [10] we look here for anticommuting generators (that also anticommute with the Lévy-Leblond differential operator) that extend the symmetry.

The motivation to study these planar systems comes not only from mathematics but also from physics as \(2 + 1\) dimensional Chern Simons electrodynamics is generally thought to provide a viable alternative to describe interesting physical phenomena like high \(T_c\) superconductivity [11] or the quantized Hall effect [12].

We investigate this possible non relativistic supersymmetry in variants of the planar Lévy-Leblond equation (LLE) in a Kaluza-Klein type framework. The main idea is that non relativistic \(2 + 1\) dimensional space time \(R\) may be viewed as the quotient of a \(3 + 1\) dimensional Lorentzian manifold \((M,g)\) by the integral curves of a covariantly constant light-like vector \(\xi\) (such a manifold is called a Bargman space) [13]. In this framework the non relativistic symmetries are the higher dimensional ones leaving \(\xi\) invariant. An adapted coordinate system on \(M\) is given by \((t,x^j,s)\) \((j = 1,2)\), where \(\xi \equiv \partial_s\), and \((t,x^j)\) are coordinates on \(R\), such that \((x^1,x^2)\) give the positions and \(t\) is non relativistic absolute time. In this paper we consider the case when \(M\) is flat Minkowski space with metric \(ds^2 = \sum (dx^i)^2 + 2dtds\).

The paper is organized as follows: In sect.2 we derive the \(N = 2\) extension of the Schrödinger symmetry for the free planar LLE, sect.3 deals with the gauged LLE, while in sect.4 we investigate the case when the LLE is coupled to Chern-Simons theory. In each of these three sections we first review what is known about the Schrödinger symmetry in that particular case before we look into the supersymmetric extension. We make our conclusions in sect.5, which is followed by three appendices.

2 Search for non relativistic SUSY in the planar Lévy-Leblond equations

2.1 Free LLE and its bosonic symmetries

First we consider the free LLE; we obtain them from the free massless Dirac equation on 4d Minkowski space

\[ \nabla \psi = 0 \text{.} \]
Using the Dirac matrices (satisfying \{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu})

\[
\begin{align*}
\gamma^t &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
\gamma^i &= \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}, \quad i = 1, 2, \\
\gamma^s &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix},
\end{align*}
\]

(1)

the equivariance condition \(\nabla_\xi \psi = im\psi\), \((\xi \equiv \partial_\alpha)\) and the Ansatz \(\psi = e^{ims} \begin{pmatrix} \Phi \\ \chi \end{pmatrix}\) (where \(\Phi\) and \(\chi\) depend on \(t\) and \(x^j\) only) in the Dirac equation we find

\[
\begin{pmatrix} -i\sigma^j \partial_j & -2im \\ i\sigma^j \partial_j \end{pmatrix} \begin{pmatrix} \Phi \\ \chi \end{pmatrix} = 0.
\]

(2)

The 4d chirality matrix

\[
\Gamma = -\frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma,
\]

\[
\Gamma = \begin{pmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}
\]

splits eq.\(2\) into two uncoupled equations for the chiral components \(\Gamma \psi_\epsilon = -ie\psi_\epsilon, \epsilon = \pm\). As discussed in \[10\] the independent (two component) equations for \(\psi_\pm\)

\[
\psi_+ = e^{ims} \begin{pmatrix} \phi_+ \\ 0 \\ \chi_+ \end{pmatrix}, \quad \psi_- = e^{ims} \begin{pmatrix} 0 \\ \phi_- \\ \chi_- \end{pmatrix},
\]

\[
\begin{pmatrix} -i(\partial_1 + i\partial_2) & -2im \\ i(\partial_1 + i\partial_2) \end{pmatrix} \begin{pmatrix} \phi_+ \\ \chi_+ \end{pmatrix} = 0,
\]

\[
\begin{pmatrix} -i(\partial_1 - i\partial_2) & -2im \\ i(\partial_1 - i\partial_2) \end{pmatrix} \begin{pmatrix} \phi_- \\ \chi_- \end{pmatrix} = 0,
\]

are the two possible (free) LL equations in two spatial dimensions. (The existence of two LLEs is a special property of two spatial dimensions). Nevertheless, for reasons becoming clear below, (and to describe them simultaneously) we keep the 4d matrix form of the LLE even for these chiral components. Note that the two component \(\Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}\) and \(\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}\) are not chiral, but are composed in a particular way of the components of \(\psi_\pm\). Since \(\nabla^{\mu} \nabla^{\nu} = (-2im\partial_t - \partial_k \partial_k)1_4\), the “square” of the LLE can be written as

\[
i\partial_t \begin{pmatrix} \Phi \\ \chi \end{pmatrix} = H \begin{pmatrix} \Phi \\ \chi \end{pmatrix}, \quad H = -\frac{1}{2m} \partial_k \partial_k 1_4.
\]

(3)

This implies, that every component of every solution of any of the free LLEs automatically solves the free Schrödinger eq.

In \[10\] we showed that the \((\equiv \partial_\alpha)\) preserving conformal transformations (besides preserving the equivariance and chirality conditions) are symmetries of the free LLE. This means that the operators \((B)\) implementing the infinitesimal transformations satisfy

\[
[B, \nabla] = \Sigma_B \nabla,
\]

(4)

where \(\Sigma_B = -\frac{1}{4} \nabla_\mu X_B^\mu\) with \(X_B^\mu \partial_\mu\) being the vector field describing the conformal transformation on Minkowski space. For later reference the explicit forms of dilatation \((d)\), Galilean boost \((b^j)\), expansion \((K)\) and rotation \((J)\) are listed here as they follow from eq.\(4.7)\) and \((4.1)\) in \[10\]:

\[
d = \begin{pmatrix} 2t\partial_t + x^k \partial_k + 1 \\ 0 \end{pmatrix}, & \quad d = \begin{pmatrix} 2t\partial_t + x^k \partial_k + 1 \\ 0 \end{pmatrix}, \quad b^j = \begin{pmatrix} t\partial_j - imx^j/2 \\ i\sigma^j \partial_j \end{pmatrix},
\]

\[
K = \begin{pmatrix} t^2\partial_t + tx^k \partial_k + t - imr^2/2 \\ \sigma^jx_j \end{pmatrix} & \quad K = \begin{pmatrix} t^2\partial_t + tx^k \partial_k + t - imr^2/2 \\ \sigma^jx_j \end{pmatrix} - v^2 = x^k x^k,
\]

\[
J = -\begin{pmatrix} x_1\partial_2 - x_2\partial_1 + im^3/2 \\ x_1\partial_2 - x_2\partial_1 + im^3/2 \end{pmatrix}, & \quad J = -\begin{pmatrix} x_1\partial_2 - x_2\partial_1 + im^3/2 \\ x_1\partial_2 - x_2\partial_1 + im^3/2 \end{pmatrix}.
\]

(The time translation \((\sim \partial_t)\) and spatial translation \((\sim \partial_j)\) operators are trivial). All \(\Sigma_B\) vanish with the exception of \(\Sigma_d = -1_4\) and \(\Sigma_K = -t1_4\). The structure of the symmetry algebra (called planar Schrödinger algebra \(sch(2)\)) is the following: translations and boosts form a Heisenberg algebra \(h(2)\) [\([i\partial_j, b^k] = m\delta^{jk}1_4\)], \((i\partial_j, d, K)\) form the \(sl_2\) algebra of nonrelativistic conformal symmetry \[2\].

\[
\begin{align*}
[d, i\partial_t] &= -2i\partial_t, & [i\partial_t, K] &= id, & [d, K] &= 2K,
\end{align*}
\]

(5)

\[\text{As shown in} [10\text{]} \text{the} \psi_\pm \text{describe the spin} 1/2 \text{ (spin} -1/2\text{) representations of the} 2 + 1 \text{ dimensional Schrödinger group.}\]
$J$ forms the $so(2)$ of planar rotations and they combine into $sch(2)$ as $sch(2) = (sl_2 \oplus so(2)) \circledast \mathbb{H}(2)$. We emphasize, that all generators commute with $\Gamma$, indicating that $\psi_+$ and $\psi_-$ span two different representations.

Anticipating the forthcoming question of the $sl_2$ subalgebra containing $H$, $d$ and $K$ we compute

$$[d, H] = -2H. \quad (6)$$

### 2.2 Supercharge candidates and the concept of weak identification

Now we look for the “fermionic” extensions of these bosonic symmetries, i.e. for $\mathcal{F}$-s satisfying

$$\{\mathcal{F}, \mathcal{Y}\} = \Sigma_{\mathcal{F}} \mathcal{Y}, \quad (7)$$

where $\Sigma_{\mathcal{F}}$ may depend on the coordinates, but cannot contain derivatives [9]. We found the following trivial solutions:

$$\mathcal{F} = 1_4, \quad \Sigma_{14} = 21_4; \quad \mathcal{F} = \Gamma, \quad \Sigma_{\Gamma} = 0;$$

together with two non-trivial ones:

$$\mathcal{F} = \tilde{Q} = \frac{1}{\sqrt{2m}} \left( \begin{array}{cc} -i\epsilon_{kl}\sigma^k\partial_l & 0 \\ 0 & i\epsilon_{kl}\sigma^k\partial_l \end{array} \right) = \frac{1}{\sqrt{2m}} \gamma^k\epsilon_{kl}\partial_l, \quad \Sigma_{\tilde{Q}} = 0,$$

and

$$\mathcal{F} = \Lambda = \left( \begin{array}{cc} 0 & \beta \\ \alpha \partial_l & 0 \end{array} \right) = \alpha \gamma^l\partial_l - \frac{\beta}{2} \gamma^l, \quad \Sigma_{\Lambda} = 0, \quad \text{if} \quad \beta - 2ima = 0. \quad (8)$$

The first two transformations generate a (continuous) chiral rotation $\exp(\alpha \Gamma)$ that multiplies $\psi_{\pm}$ by $\exp(\pm i\alpha)$ and are thus not very interesting.

The interesting solutions are third one (which is the 4d version of the “twisted” $Q_{2d}$ [11, 12] and which is normalized such that $\tilde{Q}\tilde{Q} = H$, and the fourth one with square $\Lambda\Lambda = \alpha\beta \partial_l 1_4$ which is proportional to $\partial_l$, just like in the $3+1$ dimensional case investigated in [9]. Both $\tilde{Q}$ and $\Lambda$ commute with $\xi \equiv \partial_t$, $([\tilde{Q}, \xi] = 0 = [\Lambda, \xi])$, thus the symmetry they generate descends to the planar equations. A direct computation shows, that

$$\{\Gamma, \tilde{Q}\} = 0, \quad \{\Gamma, \Lambda\} = 0, \quad \{\Lambda, \tilde{Q}\} = 0. \quad (9)$$

Therefore both $\tilde{Q}\psi_\pm$ and $\Lambda\psi_\pm$ have opposite chirality to $\psi_\pm$. (In 2+1 dimensions this means that the 1/2 and −1/2 spin representations are changed into each other by $\Lambda$ and $\tilde{Q}$). Furthermore it also means, that if $\psi_+$ solves its LL equation then $\tilde{Q}\psi_+$ (or $\Lambda\psi_+$) solves the equation for $\psi_-$ rather than the one for $\psi_+$. Thus, strictly speaking, neither $\tilde{Q}$ nor $\Lambda$ is a symmetry of either the $\psi_+$ or the $\psi_-$ equations, only of the system in (2), and to represent these operators we need both the spin 1/2 and −1/2 spinors.

$\tilde{Q}$ and $\Lambda$ commute with translations and are scalar under rotation

$$[i\partial_j, \Lambda] = 0, \quad [J, \Lambda] = 0, \quad [i\partial_j, \tilde{Q}] = 0, \quad [J, \tilde{Q}] = 0,$$

and have the same dimensions

$$[d, \Lambda] = -\Lambda, \quad [d, \tilde{Q}] = -\tilde{Q}. \quad (10)$$

Adding to this the fact that $\tilde{Q}\tilde{Q} = H$, and $\Lambda\Lambda = \alpha\beta \partial_l 1_4$ makes one wonder whether one can use them as supercharge candidates in the sought of fermionic extension. For a supercharge its square should be (proportional to) a bosonic generator; this condition is met for $\Lambda$, but not - at least naively - for $\tilde{Q}$; as the bosonic generators listed above contain only first order derivatives. Thus it seems that $\tilde{Q}$ is eliminated.

However, if we choose $\alpha\beta = i$, which implies

$$\alpha = \frac{1}{\sqrt{2m}}, \quad \beta = i\sqrt{2m}; \quad \text{and} \quad \Lambda = \left( \begin{array}{cc} 0 & i\sqrt{2m} \\ \partial_l & 0 \end{array} \right); \quad (10)$$

then, for solutions of LLE, we can write

$$\Lambda\Lambda = i\partial_l 1_4 = H = \tilde{Q}\tilde{Q}, \quad (11)$$

in light of the Schrödinger eq., [3]. Putting it differently, for solutions of LLE (“weakly”), we identify $i\partial_t$ and $-\frac{1}{2am}\partial_t \partial_\theta$. Of course we have to check whether in the bosonic algebra one can consistently make this identification. Since $H$ trivially commutes with translations and rotation just like $i\partial_t$ and a direct algebraic computation gives

$$[i\partial_t, b^j] = i\partial_j, \quad [H, b^j] = i\partial_j,$$
we look whether \((H, d, K)\) also form (only “weakly” of course) an \(sl_2\) algebra. In the light of eq.(14) we must compute \([H, K]\) to check this. We find algebraically
\[
[H, K] = \begin{pmatrix} 2t(-\frac{1}{2m}\partial_k\partial_k) + ix^k\partial_k + i & 0 \\ -\frac{1}{2m}\sigma^k\partial_k & 2t(-\frac{1}{2m}\partial_k\partial_k) + ix^k\partial_k + i \end{pmatrix}.
\]
Using the identification (1) this can be written as
\[
[H, K] = id + \frac{1}{2m} \begin{pmatrix} 0 & 0 \\ -i\sigma^k\partial_k & -2mi \end{pmatrix} = id + \frac{1}{2m} \gamma^k \gamma^l.
\]
However, for solutions of the LLE, eq.(2), (i.e. “weakly”) the second term vanishes. Therefore we can say that on the solution manifold of LLE \((H, d, K)\) also form an \(sl_2\) algebra, thus the identification in (1) works. One may wonder whether the weakly vanishing term appearing in \([H, K]\) preserves weakly also the Jacobi identity, i.e. generates only further weakly vanishing terms when one forms the commutators \([B, [H, K]]\) for any \(B\) in the Jacobi identity. We investigate this question in Appendix A in a somewhat wider context, when we consider also the various weakly vanishing terms coming from the (anti)commutators of the forthcoming fermionic operators.

### 2.3 The fermionic extensions

One can obtain new fermionic symmetry generators by commuting \(\Lambda\) and \(\hat{Q}\) with the bosonic generators. We start with \(\Lambda\), and since \(\Lambda\) commutes with translations and rotation consider first
\[
[\Lambda, b^i] := Z^i, \quad j = 1, 2 \quad Z^j = \begin{pmatrix} i^\frac{j}{2} \sigma^j & 0 \\ \alpha\partial_j & -i\frac{\alpha}{\sigma} \sigma^j \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} -m\sigma^j & 0 \\ \partial_j & m\sigma^j \end{pmatrix}.
\]
If \(\Lambda\) anticommutes with \(\hat{\gamma}\), then, since \([b^i, \hat{\gamma}] = 0\), \(Z^j\) should also anticommute with \(\hat{\gamma}\), i.e. they also generate a fermionic symmetry. They have vanishing dimension and their anticommutator with \(\Lambda\) is proportional to translation:
\[
[d, Z^j] = 0, \quad \{\Lambda, Z^j\} = i\partial_j 1_4. \quad (12)
\]
The anticommutator of the (fermionic) \(Z^j\)-s contains a central element
\[
\{Z^j, Z^k\} = m\delta^{jk} 1_4, \quad (13)
\]
showing they form a fermionic Heisenberg algebra.

The second operator we introduce is the commutator of \(\Lambda\) and expansion:
\[
[\Lambda, K] := \hat{S}, \quad \hat{S} = \frac{1}{\sqrt{2m}} \begin{pmatrix} -m\sigma^j x_j & 2imt \\ (t\partial_j + x^j\partial_j + 1) & m\sigma^j x_j \end{pmatrix}.
\]
One can check, that \(\{\hat{S}, \hat{\gamma}\} = 0\), thus this operator also generates a fermionic symmetry. The dimension of \(\hat{S}\) follows from that of \(\Lambda\) and \(K\): \([d, \hat{S}] = \hat{S}\). Furthermore the “square” of this operator is proportional to expansion
\[
\hat{S}\hat{S} = iK, \quad (14)
\]
showing that \(\hat{S}\) may be thought of as a conformal supercharge. At this point the construction of new fermionic generators with the aid of \(\Lambda\) comes to an end: we can not repeat this procedure starting with \(Z^j\) or \(\hat{S}\) since
\[
[Z^j, b^k] = 0, \quad [\hat{S}, b^j]\] = 0, \quad [Z^j, K] = 0, \quad [\hat{S}, K] = 0.
\]

Next we look for additional fermionic symmetry generators constructed from \(\hat{Q}\). Using the Galilean boost we find
\[
[\hat{Q}, b^j] := \hat{Z}^j, \quad \hat{Z}^j = \frac{\epsilon_{jk}}{\sqrt{2m}} \begin{pmatrix} ma^k & 0 \\ -\partial_k & -ma^k \end{pmatrix}.
\]
It is important to note, that
\[
\hat{Z}^j = -\epsilon_{jk} Z^k. \quad (15)
\]
Then, the (anti)commutators among the \(\hat{Z}^j\)-s are obtained simply from those among the \(Z^j\)-s:
\[
\{\hat{Z}^j, \hat{Z}^k\} = m\delta^{jk} 1_4, \quad (16)
\]
Furthermore a direct computation shows
\[
\{\hat{Q}, \hat{Z}^j\} = i\partial_j 1_4. \quad (17)
\]
Using the generator of expansion one finds

\[ [\tilde{Q}, K] := \tilde{S}, \quad \tilde{S} = \frac{1}{\sqrt{2m}} \begin{pmatrix} -i\epsilon_{kl}\sigma^k(t\partial_l - imx_l) & 0 \\ -i\sigma^3 - (x_1\partial_2 - x_2\partial_1) & i\epsilon_{kl}\sigma^k(t\partial_l - imx_l) \end{pmatrix}, \]

satisfying also \{\tilde{S}, \nabla\} = 0 and \{d, \tilde{S}\} = \tilde{S}. Computing the square of this operator and exploiting the identification (11) we find

\[ \tilde{S}\tilde{S} = iK + \frac{t}{2m}\gamma^4\nabla. \]

Since the second term gives zero on solutions of LLE we can write, that weakly

\[ \tilde{S}\tilde{S} = iK \]

holds weakly in this case. One finds also

\[ [\tilde{Z}^j, b^k] = 0, \quad [\tilde{S}, b^j] = 0, \quad [\tilde{Z}^j, K] = 0, \quad [\tilde{S}, K] = 0. \]

Thus we see that both the \((\Lambda, Z^j, \tilde{S})\) set and the \((\tilde{Q}, \tilde{Z}^j, \tilde{S})\) one give a fermionic extension of the bosonic symmetry: the square of the supercharges (respectively of the conformal supercharges) gives the (appropriate form of) Hamiltonian (respectively the operator of expansion), the two sets of \(Z\)-s form two fermionic Heisenberg algebras, and the anticommutator of the supercharges with the corresponding \(Z\)-s give spatial translations. The still missing anticommutators within each set are the following:

\[ \{\tilde{S}, Z^j\} = ib^j, \quad \{\tilde{S}, \tilde{S}\} = id, \quad \{\tilde{S}, \tilde{Z}^j\} = ib^j, \quad \{\tilde{Q}, \tilde{S}\} = id + \frac{1}{2m}\gamma^4\nabla. \quad (19) \]

(In the last equality the second term vanishes again on solutions of LLE, thus “weakly” it is absent). Combining (19) with the previous anticommutators [9 [11 [12 [13 [14 [15 [16 [17 [18] one can conclude, that extending the bosonic algebra (time and space translations, rotation, Galilean boosts, dilatation and expansion) with either the \((\Lambda, Z^j, \tilde{S})\) or the \((\tilde{Q}, \tilde{Z}^j, \tilde{S})\) sets results in two closing super (i.e. \(Z_2\)-graded) algebras, the structure of which is an \(N = 1\) (super) extension of the 2dim. Schrödinger symmetry [5 [6]. The only difference between the two cases is that for the first extension the algebra closes without using the LLE, while in the second case one has to use it.

The interesting question is whether one can use the two fermionic sets simultaneously to extend the bosonic symmetry. To decide this one has to compute the various anticommutators between the elements of the different sets and check whether the algebra closes. Some of these anticommutators are easy to compute exploiting \(\{Q, \tilde{S}\}\) (which also equals \(-\{\tilde{Q}, \tilde{S}\}\)) requires a direct computation:

\[ \{\Lambda, \tilde{S}\} = iJ + Y \]

\[ Y = -\frac{i\alpha}{\sqrt{2m}} \begin{pmatrix} im\sigma^3 & 0 \\ \epsilon_{pq}\sigma^p\partial_q & im\sigma^3 \end{pmatrix} = -\frac{i}{2m} \begin{pmatrix} im\sigma^3 & 0 \\ \epsilon_{pq}\sigma^p\partial_q & im\sigma^3 \end{pmatrix} \]

The first term on the r.h.s. is proportional to rotation, thus it is there in the bosonic symmetry algebra, but the second term is a new (bosonic) one. Thus we have to check whether the symmetry algebra is closed even after including this new element. One finds, remarkably, that \(Y\) commutes with all generators of bosonic symmetry:

\[ [B, Y] = 0, \quad B = i\partial_l, \quad i\partial_j, \quad b^l, \quad d, \quad K, \quad J. \quad (20) \]

Furthermore,

\[ [\Lambda, Y] = i\tilde{Q}, \quad (21) \]

and algebraically

\[ [\tilde{Q}, Y] = \frac{-i}{\sqrt{2m}} \begin{pmatrix} -i\sigma^l\partial_l & 0 \\ \frac{1}{m}\partial_k\partial_l & i\sigma^l\partial_l \end{pmatrix} \]

However, using the identification (11) \(\partial_k\partial_l = -2mi\partial_l\) we can write

\[ [\tilde{Q}, Y] = \frac{-i}{\sqrt{2m}} \begin{pmatrix} 0 & 2mi \\ \partial_l & 0 \end{pmatrix} + \frac{-i}{\sqrt{2m}} \begin{pmatrix} -i\sigma^l\partial_l & -2mi \\ \partial_l & i\sigma^l\partial_l \end{pmatrix} = -i\Lambda - \frac{i}{\sqrt{2m}}\nabla. \]

Since the second term vanishes on solutions of LLE, we can write, that weakly

\[ [\tilde{Q}, Y] = -i\Lambda. \quad (22) \]

The commutators between \(Y\) and the additional fermionic generators are obtained from the definitions of these operators and (20) [21 [22]:

\[ [Z^j, Y] = i\tilde{Z}^j, \quad [\tilde{S}, Y] = i\tilde{S}, \quad [\tilde{S}, Y] = -i\tilde{S}. \]
Thus we conclude, that one can safely include $Y$ in the algebra; in fact its properties (commuting with all bosonic generators and transforming the two supercharges into each other) remind that bosonic $U(1)$ present in $N = 2$ superconformal algebra.

Thus, looking at the (anti)commutators $[9] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22]$, we see that the generators of the Schrödinger symmetry accompanied by $Y$ form the bosonic generators

$$B = i\partial_t, \ i\partial_j, \ b^j, \ d, \ K, \ J, \ Y,$$

while the $(\Lambda, Z^i, \dot{S})$ and the $(\bar{Q}, \bar{Z}^i, \bar{S})$ sets the fermionic ones

$$\mathcal{F} = \Lambda, \ Z^i, \ \dot{S}, \ \bar{Q}, \ \bar{Z}^i, \ \bar{S}$$

of an $N = 2$ superalgebra. Taking into account the central element and the relation between $Z^j$ and $\bar{Z}^k$, we conclude, that this algebra is the special two dimensional one of $[6] [7]$. We note that all bosonic generators commute with $\Gamma$, but the fermionic ones anticommute with it

$$[B, \Gamma] = 0, \ \{\mathcal{F}, \Gamma\} = 0,$$

thus we need both chirality spinors to represent this algebra. Also it is important to emphasize that to show the closure of this algebra one has to use the LL equations, i.e. the algebra closes “weakly”, on the solution space of LLEs. This manifests itself in the appearance of weakly vanishing terms in various (anti)commutators, and we show in Appendix A that these terms preserve weakly the generalized (“graded”) Jacobi identity.

### 2.4 Remarks, discussion

Perhaps it is enlightening to point out that although this $N = 2$ extension of the Schrödinger symmetry is the same as the one found with respect of the planar Pauli equation $[7]$, in fact it is generated (in part at least) by operators which are related to the ones in the Pauli equation problem in a rather tricky way. What I mean is that, obviously, $\bar{Q}$ is the 4d version of the “twisted” $Q_{2d}$ used in the Pauli problem, however $\Lambda$ seems to be related to $Q_{2d}$ in a surprising way. In fact the straightforward 4d generalization of $Q_{2d}$

$$Q = \frac{1}{\sqrt{2m}} \begin{pmatrix} -i\sigma^k \partial_k & 0 \\ 0 & -i\sigma^k \partial_k \end{pmatrix}$$

(that also squares to $H$: $QQ = H$) commutes (rather than anticommutes) with the Dirac operator, $[Q, \gamma] = 0$. Therefore, although it generates a symmetry of the LLE, the symmetry it generates is a bosonic one, and most likely $Q$ itself is (the light-like reduction of) one of the many bosonic symmetries found in $[13]$ for the free massless Dirac equation in Minkowski space. On the other hand algebraically

$$Q = \Lambda + \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma,$$

which means, that weakly, on the solution space of the LLE, the two operators coincide. (I was unable to find a 4d generalization of $Q_{2d}$ that would anticommute with the Dirac operator).

It would be interesting to see whether the $N = 2$ extension found here can be given some additional structure somewhat similarly to $[9], [15], [16]$.

To emphasize that our findings depend crucially on working in $2 + 1$ dimensions we note that in $d$ (space) $+ 1$ (time) dimensions there is just one LLE when $d$ is odd, while the number of LLE-s is two if $d$ is even. This can be seen in our "light-like" Kaluza Klein framework in the following way $[17]$; this time we start with the free massless Dirac equation on $D = d + 2$ dimensional Minkowski space. The $2^{d/2}$ dimensional (Dirac) spinor representation is an irreducible one when $D$ is odd (and in this case there is no chirality matrix), while for even $D$ there is a chirality matrix and it splits the Dirac spinor representation into two $(2^{d/2})$ dimensional Weyl spinor representations. Therefore when $D$ (d) is odd there is just one LLE after the Kaluza Klein reduction, while when $D$ (d) is even, there are two LLE-s, since the chirality matrix preserves $\xi$, and it splits also the reduced Dirac equation into two independent equations for the two Weyl spinors. The appropriate generalization of $\Lambda$ works for any $d$ independently whether $d$ is even or odd, and the corresponding $N = 1$ extension of the Schrödinger symmetry exists. However in the even case $\Lambda$ anticommutes with the chirality matrix indicating it maps the two Weyl spinors into each other, therefore we need their direct sum to represent the extension, just like for $d = 2$. To have "more" than $N = 1$ SUSY we would need more supercharges (like in $d = 2$), but from the epsilon tensor, the gamma matrices and first derivatives one can make a scalar only in two dimensions. Thus the $N = 2$ extension exists only in the planar case, in accord with $[10]$.
3 Search for SUSY in the gauged Lévy-Leblond equations

3.1 The gauged Lévy-Leblond equations and its conformal symmetry

The gauged LL equations are obtained by “light-like” reduction from the gauged massless Dirac equation \( \bar{\psi} \gamma_0 \psi = 0 \), where \( D_\mu = \nabla_\mu - ie a_\mu \), with \( a_\mu(x) \) being a \( U(1) \) gauge field on \( M_4 \). Its field strength \( f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \) satisfies the Bianchi identity \( \partial_\mu f_{\nu \rho} = 0 \), (i.e. \( 2f = f_{\mu \nu} dx^\mu \wedge dx^\nu \) is a closed two form), but at this point we assume no dynamical equation for it and treat the gauge field as an “external” one. Since we are concerned here with the light like reduction of the 4d Dirac equation we have to impose some condition on \( f_{\mu \nu} (a_\mu) \) that guarantees the possibility of this. A useful condition is

\[
f_{\mu \nu} \xi^\nu = 0,
\]

since through the Bianchi identity it guarantees, that \( f_{\mu \nu} \) is a lift of a closed 2 form \( F_{\alpha \beta} \) (\( \alpha, \beta = t, 1, 2 \)) defined on \( 2 + 1 \) dimensional non relativistic space time. Therefore without loss of generality \( a_\mu \) can be chosen to be the lift of a vector potential \( A_\alpha = (A_t, A_\perp) \) for \( F_{\alpha \beta} \). Thus effectively we get the gauged LLE from [2] by the substitution \( \partial_j \rightarrow D_j \equiv \partial_j - ie A_j; \partial_t \rightarrow D_t \equiv \partial_j - ie A_t \):

\[
\left( -i\sigma^j D_j + 2 \frac{im}{2m} \right) \Phi = 0.
\]

The crucial difference to the free case is that while the ordinary derivatives commute, the covariant ones do not, thus several new terms may appear. These terms show up already in the “square” of (24):

\[
\left( -D_j^2 - e\sigma^j \epsilon_{kl} \partial_k A_l - 2imD_t \right) \left( -D_j^2 - e\sigma^j \epsilon_{kl} \partial_k A_l - 2imD_t \right) \Phi = 0.
\]

If we restrict our attention to static, purely magnetic gauge fields then \( F_{\perp \perp} \equiv 0 \), and \( D_t = \partial_t \), and this equation can be written:

\[
i\partial_t \left( \begin{array}{c} \Phi \\ \chi \end{array} \right) = H_e \left( \begin{array}{c} \Phi \\ \chi \end{array} \right), \quad H_e = -\frac{1}{2m} \left( D_j^2 + e\sigma^j \epsilon_{kl} \partial_k A_l \right) \left( 0 \begin{array}{c} 0 \\ D_j^2 + e\sigma^j \epsilon_{kl} \partial_k A_l \end{array} \right).
\]

In [10] we investigated the symmetries of the gauged LLE, (24), and showed, that the infinitesimal \( \xi \) preserving conformal transformations satisfy

\[
[B, \bar{\psi}] = -ie\gamma^\mu (L_{X_\mu} a)_\mu - 4 \nabla_\mu X^\mu E \bar{\psi},
\]

where \( (L_{X_\mu} a)_\mu \) is the Lie derivative of the gauge field with respect to \( X^\mu \partial_\mu \). Note that - because of the terms with the Lie derivative - this equation is more complicated then (3) in the case of the free LLE. Therefore if \( \psi \) solves the gauged LLE, (24), then \( B\psi \) solves rather

\[
\bar{\psi} (B\psi) - ie\gamma^\mu (L_{X_\mu} a)_\mu \psi = 0,
\]

i.e. the transformed form of the gauged LLE. For later reference we list here the explicit form of the commutators between the various \( B \)-s and \( \bar{\psi} \) as they follow from eq.(26), when the gauge field is static and purely magnetic (i.e. when \( A_t = 0, \partial_t A_j = 0 \)):

\[
[i\partial_j, \bar{\psi}] = -ie\gamma^k (i\partial_j A_k), \quad [i\partial_t, \bar{\psi}] = 0, \quad [J, \bar{\psi}] = -ie\gamma^k (\epsilon_{ij} x^j \partial_j A_k + \epsilon_{kj} A_j),
\]

\[
[d, \bar{\psi}] = -ie\gamma^k (x^j \partial_j A_k + A_k) - \bar{\psi}, \quad [\bar{\psi}, \bar{\psi}] = -ie(\gamma^k (t\partial_j A_k) + \gamma^j A_k),
\]

\[
[K, \bar{\psi}] = -ie(\gamma^k (t x^j \partial_j A_k + A_k) + \gamma^j A_k x^m) - t\bar{\psi}.
\]

3.2 The fermionic extension in case of the gauged LLE

Next we look whether the supercharges and the associated fermionic extensions found in the case of the free LLE work also for the gauged LLE. We start with \( \Lambda \), eq.(11), and find that for a static, purely magnetic gauge field it anticommutes with \( \bar{\psi} \):

\[
\{\Lambda, \bar{\psi}\} = 0, \quad \text{if} \quad A_t = 0, \quad \text{and} \quad \partial_t A_j = 0.
\]

This is a good sign, however it is not enough as in the case of the free LLE, and to progress we have to check whether the modifications in eq.(23) (28) represented by the terms with the Lie derivatives are consistent with the algebra of the extension (\( \Lambda, Z^j, S \)).

To start to investigate this we use two identities, both of which are obtained by simple algebra exploiting eq.(29): a “bosonic” one

\[
\{[\Lambda, B], \bar{\psi}\} = \{\Lambda, [B, \bar{\psi}]\}.
\]
valid for any bosonic generator B, and a “fermionic” one

\[ [\Lambda, \{ F, \tilde{\Phi} \}] = \{ [\Lambda, F], \tilde{\Phi} \}, \]  

(31)

valid for any fermionic generators.

\( \Lambda \) commutes with translations \((B = i\partial_j)\) and rotation \((B = J)\) thus when \([30]\) applied in these cases the l.h.s. vanishes. Thus we have to check whether \( \Lambda \) indeed anticommutes with the r.h.s. of the first and third expressions in \([26]\). One can prove in general, that

\[ \{ \Lambda, W \} = 0, \quad \text{for} \quad W = -ie\gamma^k W_k, \quad \text{provided} \quad \partial_i W_k = 0, \]  

(32)

and the explicit expressions of \([i\partial_j, \tilde{\Phi}]\) and \([J, \tilde{\Phi}]\) are precisely of this form. Thus we conclude that \( [\Lambda, i\partial_j] = 0 \) and \( [\Lambda, J] = 0 \) are consistent with \([26]\).

We also notice, that applying \([30]\) for \( B = d \) we get zero on the r.h.s. when using eq.\([27]\) as a consequence of \([29]\) and \([32]\). However this is consistent, since on the l.h.s. \( [\Lambda, d] = \Lambda \) and \( [\Lambda, \tilde{\Phi}] = 0 \).

Applying \([30]\) for \( B = b^j \) \((B = K)\) determines the anticommutators of \( Z^j \) \((\tilde{S})\) and \( \tilde{W} \) explicitly:

\[ \{ Z^j, \tilde{\Psi} \} = \mathcal{N}^j, \quad \mathcal{N}^j = \frac{e}{\sqrt{2m}} \left( \begin{array}{cc} 2mA_j & 0 \\ -\sigma^k A_k & 2mA_j \end{array} \right), \]  

(33)

and

\[ \{ \tilde{S}, \tilde{\Psi} \} = \mathcal{L} - \frac{1}{\sqrt{2m}} \gamma^i \tilde{\Phi}, \quad \mathcal{L} = \frac{-ie}{\sqrt{2m}} \left( \begin{array}{cc} 2miA_m x^m & 0 \\ -\sigma^k M_k & 2miA_m x^m \end{array} \right), \]  

(34)

with \( \mathcal{M}_k = x^j \partial_j A_k + A_k \). The \( \mathcal{N}^j \) and \( \mathcal{L} \) are the equivalents of the Lie derivative terms in \([24]\): if \( \psi \) solves \( \tilde{\Psi} = 0 \), then on the basis of \([30, 31]\) \( Z^j \psi \) and \( \tilde{S} \psi \) solve the transformed equations

\[ \tilde{\Psi} (Z^j \psi) - \mathcal{N}^j \psi = 0, \quad \tilde{\Psi} (\tilde{S} \psi) - \mathcal{L} \psi = 0. \]

The consistency of these terms with \([26, 28]\) is investigated in Appendix B.

We apply the fermionic identity, eq.\([31]\), first for \( F = Z^j \). The l.h.s. can be written

\[ [\Lambda, \{ Z^j, \tilde{\Phi} \}] = [\Lambda, \{ [\Lambda, [b^j, \tilde{\Phi}]] \}] = [\Lambda \Lambda, [b^j, \tilde{\Phi}]] = [i\partial_i, [b^j, \tilde{\Phi}]] = i\partial_i ([b^j, \tilde{\Phi}]) = [i\partial_i, \tilde{\Phi}], \]

where in the first equality the definition of \( Z^j \) and \([30]\) is used, the second equality is simple algebra, the third equality uses the square of \( \Lambda \) while in the last equality we used the explicit form of \([b^j, \tilde{\Phi}]\), \([27]\), and the fact that \( \partial_i A_j = 0 \). Thus we conclude that \([26, 28]\) are also consistent with \( [\Lambda, Z^j] = i\partial_j 14 \).

In a similar way applying \([31]\) for \( F = \tilde{S} \) we get

\[ [\Lambda, \{ \tilde{S}, \tilde{\Phi} \}] = [\Lambda, \{ [\Lambda, [K, \tilde{\Phi}]] \}] = [\Lambda \Lambda, [K, \tilde{\Phi}]] = [i\partial_i, [K, \tilde{\Phi}]] = i\partial_i ([K, \tilde{\Phi}]) = [i\partial_i, \tilde{\Phi}], \]

where the last equality is based on \([28]\) with \( \partial_i A_m = 0 \) and \([27]\). Thus we conclude that \([26, 28]\) are also consistent with \( [\Lambda, \tilde{S}] = id \).

Unfortunately we found no general framework to check the consistency of the rest of the fermionic anticommutation relations and eq.\([26, 28]\) thus we have to resort to a case by case analysis. We collect some of these not very illuminating computations in Appendix B and here merely state that the outcome is positive: the fermionic algebra generated by \((\Lambda, Z^j, \tilde{S})\) in case of the free LLE is consistent with eq.\([26, 28]\), thus the whole \( N = 1 \) super extension of the Schrödinger symmetry works also for the gauged LLE at least when the external gauge field is static and purely magnetic.

The situation of the other supercharge, \( \tilde{Q} \), and the associated extension \((\tilde{Q}, \tilde{Z}^j, \tilde{S})\) is different. \( \tilde{Q} \) has several properties, that make it a potential supercharge for the free LLE; e.g. it anticommutes with \( \tilde{\Psi} \) and its square is \( \tilde{H} \). None of these survive for the gauged LLE, no matter what kind of gauge field we have: \( [\tilde{Q}, \tilde{\Phi}] \neq 0 \), and \( \tilde{Q}\tilde{Q} = H \neq H_c \). Interestingly one can define an “external field” version of \( \tilde{Q} \)

\[ \tilde{Q}_e = \frac{1}{\sqrt{2m}} \left( \begin{array}{cc} -i\epsilon_k \sigma^k D_l & 0 \\ 0 & i\epsilon_k \sigma^k D_l \end{array} \right), \]

that squares to \( H_c \): \( H_c = \tilde{Q}_e \tilde{Q}_e \). Furthermore it anticommutes with \( \tilde{\Phi} \) when the gauge field is static and purely magnetic

\[ [\tilde{Q}_e, \tilde{\Phi}] = 0, \quad \text{if} \quad A_t = 0, \quad \text{and} \quad \partial_i A_j = 0, \]

thus in this case it generates a symmetry of the gauged LLE, \([24]\). However it cannot play the role of a supercharge in a fermionic extension of the Schrödinger symmetry, since it does not commute with translations

\[ [i\partial_j, \tilde{Q}_e] = \frac{e}{\sqrt{2m}} \gamma^k \epsilon_k \partial_j A_t \neq 0, \]

and the algebra it generates would depend on the external field.
4 SUSY in the coupled Lévy-Leblond and Chern-Simons equations

4.1 The gauged Lévy-Leblond equations coupled to Chern-Simons theory

Next we investigate the fermionic extension of the Schrödinger symmetry when the dynamics of the gauge field appearing in [24] is determined by the Chern-Simons (CS) field equations.

In [13] it is shown that on a general Bargman space the 4d form of the CS equations is the field current identity
\[ f_{\mu
u} = \frac{e}{\kappa} \sqrt{-g} \epsilon_{\mu
u\rho\sigma} \xi^\rho j^\sigma, \]
where \( \kappa \) is the CS coupling and \( j^\mu \) is some 4d current. Please note, that \[ f_{\mu
u} \] implies \[ \kappa \] thus \( f_{\mu
u} \) is a lift of a closed \( F_{\alpha\beta} \). Furthermore in [13] we showed that \( j^\mu \) projects to a 3 current \( J^\alpha = (\rho, J^k) \) (\( \alpha = t, 1, 2 \)) and \[ f_{\mu
u} \] descends in the lightlike reduction as
\[ F_{\alpha\beta} = -\frac{e}{\kappa} \sqrt{-g} \epsilon_{\alpha\beta\gamma} J^\gamma. \]

On our Minkowski space with metric \( ds^2 = \sum(dx^i)^2 + dt ds \) this can be written as
\[ B \equiv \epsilon_{ij} \partial_i A_j = -\frac{e}{\kappa} \rho, \quad \text{and} \quad E^j \equiv F_{j t} = \frac{e}{\kappa} \epsilon_{jk} J^k, \]
which are indeed the CS field equations in [19]. Since \( F_{\alpha\beta} \) is closed, \( \partial_t B + \epsilon_{ij} \partial_i E^j = 0 \), \( \rho \equiv J^t \) and \( J^j \) must satisfy the reduced conservation equation
\[ \partial_t J^t + \partial_j J^j = 0. \]

Now we couple the CS equations to the LLE by identifying the CS current in (35) with the natural (conserved) Dirac adjoint current associated to the gauged LLE. This current is made of the spinor fields and to construct it we need the
Dirac matrices (1). With our Dirac matrices \( G \) turns out to be \( G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and as a consequence of the particular form of \( \gamma^t \)
\[ \rho \equiv J^t = |\Phi|^2 = |\phi_+|^2 + |\phi_-|^2. \]

The spatial components of the current are
\[ J^j = i(\Phi^\dagger \sigma^j \chi - \chi^\dagger \sigma^j \Phi). \]

Note that the reduced conservation equation (39) for this \( \rho \) and \( J^j \) follows from (35) since \( \bar{\psi} \gamma^s \psi \sim \chi^\dagger \chi \) is independent of \( s \).

Thus the system of coupled LL and CS equations is given by [24], [35] and [35] [10]. In [10] we showed that the \( \xi \) preserving conformal transformations act as symmetries on the solutions of this system. Here we are interested whether the \( N = 1 \) superextension of these symmetries, represented by \( (\Lambda, \ Z^j, \ S) \), is also a symmetry of the coupled system.

4.2 Fermionic extension for the coupled LL and CS equations

In the previous section we showed that the algebra generated by \( (\Lambda, \ Z^j, \ S) \) is a symmetry of the gauged LLE if the gauge field satisfies
\[ A_t \equiv 0, \quad \text{and} \quad \partial_t A_j = 0. \]

While for the gauged LLE alone these conditions are relatively harmless, here, in the coupled system, they impose some non trivial restrictions, and - as we argue below - require that we consider only static solutions of [24] with definite chirality spinors.

These conditions imply that \( F_{j t} \) vanishes, thus, because of (35),
\[ J^j = 0, \quad \text{and also} \quad \partial_t \rho = 0, \]
must also hold. In light of (35) and (10) these requirements are satisfied if we look for solutions of (24) with\(^2\)
\[ \chi \equiv 0, \quad \partial_t \Phi = 0, \]

\(^2\)The other generic solution of (11), when \( \Phi \equiv 0 \), leads to vanishing \( \chi \) and pure gauge \( A_j \), thus is not interesting.
i.e. if \((24)\) simplifies to
\[-i\sigma^j D_j \Phi = 0, \quad \partial_t \Phi = 0.\]

In terms of the spinors with definite chirality these equations mean, that \(\chi_+ = 0, \chi_- = 0\) and their nonvanishing components are static and satisfy
\[(D_1 + iD_2)\phi_+ = 0, \quad (D_1 - iD_2)\phi_- = 0,\]
respectively. Although the equations for \(\phi_+\) and \(\phi_-\) look independent, they contain the same gauge field \(A_j\) and the first eq. in \((\text{40})\) couples them as
\[\epsilon_{ij} \partial_i A_j = -\frac{e}{\kappa}(|\phi_+|^2 + |\phi_-|^2).\]

One can show that \((\text{43-44})\) admit normalizable solutions when only one of the \(\phi_{\pm}\)-s is different from zero. If \(\kappa < 0\) then the normalizable solution exists for \(\phi_+\) while for \(\kappa > 0\) it exists for \(\phi_-\). (See Appendix C).

If under an infinitesimal fermionic transformation the spinors change as
\[\Phi = (\Phi, \chi) \rightarrow \begin{pmatrix} \tilde{\Phi} \\ \tilde{\chi} \end{pmatrix} + \epsilon \begin{pmatrix} \tilde{\Phi} \\ \tilde{\chi} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\Phi} \\ \tilde{\chi} \end{pmatrix} = \mathcal{F} \begin{pmatrix} \Phi \\ \chi \end{pmatrix}, \quad \mathcal{F} = \Lambda, Z^j, \hat{S},\]
then, in general, \(\rho\) and \(J^j\) also change
\[\rho \rightarrow \rho + \delta\rho = \rho + \epsilon^s \Phi^\dagger \Phi + \Phi \Phi^\dagger, \quad J^j \rightarrow J^j + \delta J^j = J^j + i(\epsilon \Phi^\dagger \sigma^j \tilde{\Phi} + \epsilon \Phi \sigma^j \tilde{\Phi} - \epsilon^s \tilde{\Phi}^\dagger \sigma^j \Phi - \epsilon^s \tilde{\Phi} \sigma^j \Phi).\]

Note that for solutions of our interest, i.e. when \((\text{42})\) holds, \(\delta J^j\) simplifies to
\[\delta J^j = i(\epsilon \Phi^\dagger \sigma^j \tilde{\Phi} - \epsilon^s \tilde{\Phi}^\dagger \sigma^j \Phi).\]

The coupled LL and CS equations admit the fermionic symmetries, since for any \(\mathcal{F}\) the \(\delta \rho\) and \(\delta J^j\) in \((\text{43-44})\) vanish (thus the CS equations preserve their form). Fortunately to show this there is no need to determine \(\Phi\) and \(\tilde{\chi}\) explicitly for the various \(\mathcal{F}\)-s. Indeed recalling that \(\{\Gamma, \mathcal{F}\} = 0\) for all \(\mathcal{F}\), we see that when we start with a positive chirality solution then its fermionic transform is of negative chirality
\[\begin{pmatrix} \Phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_+ \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\Phi} \\ \tilde{\chi} \end{pmatrix} = \mathcal{F} \begin{pmatrix} \Phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\phi}_- \\ \tilde{\chi}_- \end{pmatrix},\]
(for some \(\tilde{\phi}_-\), \(\tilde{\chi}_-\)) and vice versa
\[\mathcal{F} \begin{pmatrix} \phi_- \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_+ \\ 0 \\ \tilde{\chi}_+ \end{pmatrix}.\]

Since the various terms determining \(\delta \rho\) and \(\delta J^j\) in \((\text{43-44})\) couple only same chirality spinors
\[\Phi^\dagger \Phi = \phi_+^\dagger \phi_+ + \phi_-^\dagger \phi_- , \quad \Phi^\dagger \sigma^j \tilde{\Phi} = \phi_+^\dagger \tilde{\chi}_+ + \phi_-^\dagger \tilde{\chi}_-, \quad \Phi^\dagger \sigma^2 \tilde{\Phi} = i(\phi_+^\dagger \tilde{\chi}_+ - \phi_-^\dagger \tilde{\chi}_-),\]
it is obvious, that they give zero, when evaluated for the definite chirality solutions and their fermionic transforms.

### 5 Conclusions

In the central part of this paper we present an \(N = 2\) extension of Schrödinger symmetry \(\text{schr}(2)\) \([\text{6}, \text{7}]\) for the free planar LLE. This extension is built in terms of operators that anticommute with the LL differential operator. The construction is based on several special properties of the planar problem e.g. on the existence of two LLE-s, the solutions of which span two representations of \(\text{schr}(2)\), describing non relativistic spin \((1/2)\) and spin \((-1/2)\) particles respectively \([\text{10}]\). It is also a special property of two spatial dimensions, that we can find two supercharges; one, \((\Lambda)\), with square \(i\partial_t\), and another one, \((\hat{Q})\), which squares to the free Pauli Hamiltonian. Both of these operators map the two non relativistic spinor representations into each other, thus we need both representations to construct the \(N = 2\) extension. The \(N = 2\) extension requires the equality of \(\Lambda\Lambda\) and \(\hat{Q}\hat{Q}\), we achieve this by identifying - on the solution manifold of the free LLE - time translation and the Pauli Hamiltonian. (This identification is made possible by the fact that all solutions of any of the two LLE-s satisfy the Pauli equation). As a consequence of this identification some of the (anti)commutators of the extended \(N = 2\) algebra contain weakly vanishing terms, i.e. terms, which are non zero algebraically, but vanish on solutions of the LLE. Thus we can say that the \(N = 2\) algebra...
closes weakly, on solutions of LLE. We also show that the weakly vanishing terms do not spoil the generalized Jacobi identity of the algebra.

Next we show that when the LLE is coupled to an external gauge field, the \( N = 1 \) part of the previous extension generated by \( A \) persists as symmetry, at least when the external gauge field is static and purely magnetic. Since \( sch(2) \) acts on the gauged LLE in a more complicated way than on the free one, the major task here is to show that the terms with the Lie derivatives of the gauge field (which form that complicating difference to the free case) are consistent with the \( N = 1 \) extension.

Finally we show that the same \( N = 1 \) extension persists as symmetry when the dynamics of the gauge field is described by the Chern Simons field equations, i.e. when we couple LLE to Chern Simons theory. This conclusion is based on the observation that through the Chern Simons equations a static and purely magnetic gauge field leads to static solutions of the LLE with definite chirality spinor s only.

Acknowledgments

I intended to present the results of this paper on the conference celebrating the retirement of prof. Peter Horvathy, but this conference has been postponed because of the pandemic. I thank Peter for his enthusiastic interest in these matters and for his remarks.

A Check of the generalized Jacobi identity

In this appendix we show that the various weakly vanishing terms appearing in some of the (anti)commutators of the \( N = 2 \) algebra keep the weak form of the generalized Jacobi identity as they generate weakly vanishing terms only.

We have the following types of (anti)commutators containing weakly vanishing terms

\[
[H, K] = id + \frac{1}{2m} \gamma^t \nabla, \quad \{\tilde{\Sigma}, \tilde{S}\} = 2iK + \frac{t}{m} \gamma^t \nabla,
\]

\[
[\tilde{Q}, Y] = -i\Lambda - \frac{i}{\sqrt{2m}} \nabla, \quad \{\tilde{Q}, \tilde{S}\} = id + \frac{1}{2m} \gamma^t \nabla.
\]

The commutator on the l.h.s. of the third expression is a fermionic operator, thus in the generalized Jacobi identity only their commutators appear with both the \( \Sigma \) and \( \gamma \) generators.

We have the following types of (anti)commutators containing weakly vanishing terms

\[
[\mathcal{B}, [\tilde{Q}, Y]] = -i[\mathcal{B}, \Lambda] - \frac{i}{\sqrt{2m}} [\mathcal{B}, \nabla], \quad \{\mathcal{F}, [\tilde{Q}, Y]] = -i\{\mathcal{F}, \Lambda] - \frac{i}{\sqrt{2m}} \{\mathcal{F}, \nabla\},
\]

since the symmetry equations \((4) \ (7)\) with the known \( \Sigma_B \) and \( \Sigma_F = 0 \) show that the terms generated are indeed weakly vanishing.

Next we consider the (anti)commutators containing the weakly vanishing term \( \frac{1}{2m} \gamma^t \nabla \). In the above mentioned commutator with bosonic generators this term generates

\[
[\mathcal{B}, \frac{1}{2m} \gamma^t \nabla] = \frac{1}{2m} ([\mathcal{B}, \gamma^t] \nabla + \gamma^t [\mathcal{B}, \nabla]) = \frac{1}{2m} ([\mathcal{B}, \gamma^t] + \gamma^t \Sigma_B) \nabla.
\]

We are not yet ready, since we have to show that the term multiplying \( \nabla \) contains no derivatives. However since \( [\mathcal{B}, \gamma^t] + \gamma^t \Sigma_B \) vanishes trivially for \( \mathcal{B} = id_j, i\partial_t, J_i^j, Y \) (since both \( [\mathcal{B}, \gamma^t] \) and \( \Sigma_B \) vanish) and non trivially for \( \mathcal{B} = d \) \((d, \gamma^t) = \gamma^t \) and \( \Sigma_d = -1 \) and \( \mathcal{B} = K \) \((K, \gamma^t) = t\gamma^t \) and \( \Sigma_K = -t \) we see that all the terms generated vanish identically. In the commutator with fermionic generators this term produces

\[
[\mathcal{F}, \frac{1}{2m} \gamma^t \nabla] = \frac{1}{2m} ([\mathcal{F}, \gamma^t] \nabla + \gamma^t [\mathcal{F}, \nabla]) = \frac{1}{2m} ([\mathcal{F}, \gamma^t] + 2\gamma^t \mathcal{F}) \nabla,
\]

where we used that \( \{\mathcal{F}, \nabla\} = 0 \). The fermionic generators \( \mathcal{F} = \tilde{Q}, \tilde{S}, Z^i, \tilde{Z}^i \) have the generic form

\[
\mathcal{F} = \begin{pmatrix} A & 0 \\ B & -A \end{pmatrix}
\]

for some \( 2 \times 2 \) \( A \) and \( B \), and for them

\[
[\mathcal{F}, \gamma^t] = \begin{pmatrix} 0 & 0 \\ -2A & 0 \end{pmatrix}, \quad 2\gamma^t \mathcal{F} = 2 \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad \text{thus} \quad [\mathcal{F}, \gamma^t] + 2\gamma^t \mathcal{F} = 0.
\]
This argument does not apply for \( \Lambda \) and \( \hat{S} \), but a direct computation gives

\[
[\Lambda, \gamma^t] + 2\gamma^t\Lambda = i\sqrt{2m}1_4, \quad [\hat{S}, \gamma^t] + 2\gamma^t\hat{S} = \sqrt{2m}\begin{pmatrix} it & 0 \\ -2\sigma^jx_j & it \end{pmatrix},
\]

and, since they contain no derivatives, we see that the terms generated for them are indeed weakly vanishing.

The terms generated by the weakly vanishing \( \frac{t}{m}\gamma^t\nabla \) require a separate consideration, as some bosonic and fermionic generators contain time derivatives, thus the previous results may not apply in this case. In case of the commutators with the bosonic generators

\[
[B, \frac{t}{m}\gamma^t\nabla] = \frac{1}{m}( [B, \gamma^t] + t\gamma^t\Sigma B ) \nabla,
\]

the term multiplying \( \nabla \) vanishes trivially for all \( B \) apart from \( B = d \) and \( B = K \). However a direct computation gives

\[
[d, \gamma^t] + t\gamma^t\Sigma_d = 2t\gamma^t - \gamma^t = \gamma^t, \quad [K, \gamma^t] + t\gamma^t\Sigma_K = 2t^2\gamma^t + t\gamma^t(-i) = i\gamma^t,
\]

showing that they are weakly vanishing. In case of the commutators with the fermionic operators

\[
[F, \frac{t}{m}\gamma^t\nabla] = \frac{1}{m}( [F, \gamma^t] + 2t\gamma^tF ) \nabla,
\]

the term multiplying \( \nabla \) vanishes for \( F = \hat{Q}, \hat{S}, Z^j, \hat{Z}^j \) in the same way as before, since they contain no time derivatives. For \( \Lambda \) and \( \hat{S} \) we find

\[
[\Lambda, \gamma^t] + 2t\gamma^t\Lambda = it\sqrt{2m}1_4, \quad [\hat{S}, \gamma^t] + 2t\gamma^t\hat{S} = t\sqrt{2m}\begin{pmatrix} it & 0 \\ -2\sigma^jx_j & it \end{pmatrix}.
\]

Thus we conclude that also \( \frac{t}{m}\gamma^t\nabla \) generates only weakly vanishing terms.

## B Consistency of the Lie derivative terms and the fermionic extension

Here we collect some of the case by case checks we carried out to prove that the terms with the Lie derivatives in \((26, 28)\) are consistent with the fermionic extension \((\Lambda, Z^j, \hat{S})\). We are concerned here mainly with the anticommutators of \(Z^j\) \((\hat{S})\) and \(\hat{\Phi}\), obtained in \((33-34)\).

The first anticommutator we check is \(\{Z^j, Z^k\}\); a simple computation gives

\[
[\{Z^j, Z^k\}, \hat{\Phi}] = [Z^j, \{Z^k, \hat{\Phi}\}] + [Z^k, \{Z^j, \hat{\Phi}\}] = [Z^j, \Sigma^k] + [Z^k, \Sigma^j].
\]

Substituting here the expression one obtains using \((33)\) and the explicit form of \(Z^j\)

\[
[Z^j, \Sigma^k] = e \begin{pmatrix} 0 & 0 \\ -\partial_jA_k - \partial_kA_j & 0 \end{pmatrix}
\]

gives \(\{Z^j, Z^k\}, \hat{\Phi}\) = 0, which is consistent with \((13)\).

Next we check the consistency of \(\hat{S}\hat{S} = iK\):

\[
[\hat{S}\hat{S}, \hat{\Phi}] = [\hat{S}, \{\hat{S}, \hat{\Phi}\}] = [\hat{S}, \mathcal{L}] + \frac{-1}{\sqrt{2m}}[\hat{S}, \gamma^t\hat{\Phi}],
\]

where we used \((34)\). One finds explicitly:

\[
[\hat{S}, \mathcal{L}] = e \begin{pmatrix} -i\sigma^k tM_k & 0 \\ 0 & i\sigma^k tM_k \end{pmatrix},
\]

and

\[
\frac{-1}{\sqrt{2m}}[\hat{S}, \gamma^t\hat{\Phi}] = -it\hat{\Phi} + e \begin{pmatrix} 0 & 0 \\ A_j x^j & 0 \end{pmatrix}.
\]

Thus

\[
[\hat{S}\hat{S}, \hat{\Phi}] = i \left( -ie \begin{pmatrix} -i\sigma^k tM_k \\ A_j x^j \end{pmatrix} - t\hat{\Phi} \right) = i[K, \hat{\Phi}],
\]

where, in the last equality, eq. \((28)\) is used.

Finally we show the consistency of \([d, Z^j]\) = 0 with the Lie derivative terms and with \((33)\), i.e. we check whether \(\{[d, Z^j], \hat{\Phi}\}\) vanishes. A simple algebra gives

\[
\{[d, Z^j], \hat{\Phi}\} = [d, \{Z^j, \hat{\Phi}\}] - \{Z^j, [d, \hat{\Phi}]\}.
\]

\[(47)\]
A direct computation using (33) yields

\[ [d, \{ Z^j, \mathcal{P} \}] = \frac{e}{\sqrt{2m}} \begin{pmatrix} 2m x^k \partial_k A_j & 0 \\ -\sigma^k \partial_j (x^m \partial_m A_k) & 2m x^k \partial_k A_j \end{pmatrix}, \]

while, on the basis of (27) one can write

\[ \{ Z^j, [d, \mathcal{P}] \} = -ie \{ Z^j, \gamma^k \mathcal{M}_k \} - \{ Z^j, \mathcal{P} \}. \]

We find explicitly:

\[ -ie \{ Z^j, \gamma^k \mathcal{M}_k \} = \frac{e}{\sqrt{2m}} \begin{pmatrix} 2m (x^l \partial_l A_j + A_j) & 0 \\ -\sigma^k \partial_j (x^m \partial_m A_k + A_k) & 2m (x^l \partial_l A_j + A_j) \end{pmatrix}, \]

and, using (33), get eventually

\[ \{ Z^j, [d, \mathcal{P}] \} = \frac{e}{\sqrt{2m}} \begin{pmatrix} 2m (x^l \partial_l A_j) & 0 \\ -\sigma^k \partial_j (x^m \partial_m A_k) & 2m (x^l \partial_l A_j) \end{pmatrix}. \]

Since this is identical to \([d, \{ Z^j, \mathcal{P} \}]\) above, we see that \({ \{ Z^j, [d, \mathcal{P}] \}, \mathcal{P} \}\) in (47) vanishes indeed.

**C Solutions of (43-44)**

In this appendix we study the solutions of (43-44). First we assume, that none of the \(\phi_\pm\)'s vanishes identically. In this case, introducing their modulus and phase \(\phi_\pm = (\rho_\pm)^{1/2} e^{i \alpha_\pm}\) (where \(\rho_\pm \geq 0\), and \(\alpha_\pm\) are real), one can express the gauge field from both equations in (43); we get

\[ A_i = \frac{1}{2e} \epsilon_{ij} \partial_j \ln \rho_+ + \frac{1}{e} \partial_i \alpha_+ \]

from the equation for \(\phi_+\), while

\[ A_i = -\frac{1}{2e} \epsilon_{ij} \partial_j \ln \rho_- + \frac{1}{e} \partial_i \alpha_- \]

from the equation for \(\phi_-\). However the two gauge fields must be the same, thus

\[ \rho_- = \frac{1}{\rho_+}, \quad \text{and} \quad \alpha_- = \alpha_+ \]

must hold. Substituting this common gauge field into (44) yields

\[ \Delta \ln \rho_+ = \frac{2e^2}{\kappa} \left( \rho_+ + \frac{1}{\rho_+} \right). \]

The r.h.s. of this equation is invariant under \(\rho_+ \to \frac{1}{\rho_+}\), while the l.h.s. changes sign. Therefore both sides of the equation must vanish:

\[ \Delta \ln \rho_+ = 0, \quad \rho_+ + \frac{1}{\rho_+} = 0. \]

Since the second condition does not allow any real solutions, we conclude that there is no solution of (43-44) with both \(\phi_\pm\) non vanishing.

On the other hand, if we assume, that only one of the \(\phi_\pm\)'s is different from zero, then there is only one relevant equation in (43) and from this - after going into the \(\phi_\epsilon = (\rho_\epsilon)^{1/2}\) gauge - we obtain one of the above expressions for the gauge field (with \(\alpha_\epsilon \equiv 0\)). Using this in (44) leads to the Liouville equation in both cases

\[ \Delta \ln \rho_\pm = \pm \frac{2e^2}{\kappa} \rho_\pm. \]

A normalizable solution is obtained for \(\phi_+\) when \(\kappa\) is negative, while for \(\phi_-\) when \(\kappa\) is positive.

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