On the relation between integrability and infinite-dimensional algebras

M. D. Freeman and P. West

Department of Mathematics
King’s College London, Strand, London WC2R 2LS

Abstract

We review our work on the relation between integrability and infinite-dimensional algebras. We first consider the question of what sets of commuting charges can be constructed from the current of a $\mathfrak{U}(1)$ Kac-Moody algebra. It emerges that there exists a set $S_n$ of such charges for each positive integer $n > 1$; the corresponding value of the central charge in the Feigin-Fuchs realization of the stress tensor is $c = 13 - 6n - 6/n$. The charges in each series can be written in terms of the generators of an exceptional $\mathfrak{W}$-algebra. We show that the $\mathfrak{W}$-algebras that arise in this way are symmetries of Liouville theory for special values of the coupling. We then exhibit a relationship between the non-linear Schrödinger equation and the KP hierarchy. From this it follows that there is a relationship between the non-linear Schrödinger equation and the algebra $\mathfrak{W}_{1+\infty}$. These examples provide evidence for our conjecture that the phenomenon of integrability is intimately linked with properties of infinite dimensional algebras.

*Talk given at the 1992 John Hopkins meeting, Goteborg, Sweden
1 Introduction

A dynamical system with an infinite number of degrees of freedom is said to be integrable if it possesses an infinite number of conserved quantities in involution, i.e. for which the Poisson bracket between any pair is zero. Such systems have been studied for many years and include the KdV equation,

\[
\frac{\partial u}{\partial t} = \frac{1}{8} u''' - \frac{3}{8} uu',
\]

(1.1)

the Liouville equation,

\[
\partial \bar{\phi} - e^{-\phi} = 0,
\]

(1.2)

and the sinh-Gordon equation

\[
\partial \bar{\phi} + 2 \sinh \phi = 0.
\]

(1.3)

These systems are in fact related to the Lie algebra \(sl(2)\). Corresponding to every Lie algebra there is an integrable generalization of each of them \([1, 2]\), namely the generalized KdV equations, Toda equations and affine Toda equations respectively.

All of these equations can be written in Hamiltonian form, in which the Hamiltonian generating the time evolution has the property that it commutes with all of the conserved charges with respect to the Poisson bracket. Each equation is therefore a member of an infinite hierarchy of equations, obtained by using the conserved charges as Hamiltonians to define commuting time evolutions for infinitely many times.

In another development, which has no obvious connection with integrability, conformal field theories have been intensively studied recently. A theory that is Poincaré invariant possesses a conserved energy-momentum tensor, and if such a theory is in addition conformally invariant then the energy-momentum tensor is traceless. In two dimensions this implies that, in light-cone coordinates \((z, \bar{z})\), the component \(T \equiv T_{zz}\) satisfies

\[
\partial T = 0,
\]

(1.4)

and so any polynomial in \(T\) is conserved. Thus the integrals of such polynomials are conserved charges for the theory.

There is a remarkable connection between the integrable systems referred to above and the infinite-dimensional conformal algebra and its extensions. Of course any conformally invariant theory in two dimensions automatically possesses an infinite number of conserved quantities, as argued above, but in order to have integrability it is necessary to select from the set of all polynomials in \(T\) a subset for which the Poisson bracket between any pair of charges is zero. It turns out that the charges arrived at in this way are essentially the conserved charges of the sinh-Gordon and KdV equations, even though these systems are
not conformally invariant. A similar result holds for the generalizations of these systems corresponding to arbitrary Lie algebras. For each Lie algebra there is an extension of the conformal algebra, referred to generically as a $W$-algebra, and choosing a commuting subset of conserved quantities in the $W$-algebra corresponding to the Lie algebra $\mathfrak{g}$ gives the conserved charges for the integrable systems corresponding to $\mathfrak{g}$.

Let us spell out in more detail how this procedure works in the simplest case. The first step is to write the KdV equation in Hamiltonian form. With respect to the so-called second Poisson bracket structure for the field $u(x)$,

$$\{u(x), u(y)\} = (u(x)\partial_x + \partial_x u(x) - \partial_x^3) \delta(x - y)$$ (1.5)

the time evolution is given by the Hamiltonian

$$H_2 = \frac{1}{16} \int u^2.$$ (1.6)

It is through this Poisson bracket that the connection with the conformal algebra is made. It was recognized in [3] that the Poisson bracket for the Fourier modes of $u$ is precisely the two-dimensional conformal algebra, so that we can identify $u$ with the energy-momentum tensor $T$. The conserved quantities of the KdV equation are then integrals of polynomials in $T$ and its derivatives.

If now we take a realization of the second Poisson bracket structure of the KdV equation in terms of some field $\Phi$, so that $u$ is expressed in terms of $\Phi$ and the Poisson bracket for $u$ follows from that of $\Phi$, we can obtain an integrable hierarchy of evolution equations for $\Phi$. This hierarchy is obtained simply by taking the conserved charges for the KdV equation, expressing them in terms of $\Phi$, and then using these as Hamiltonians to generate time evolutions for $\Phi$. Every possible realization of the Virasoro algebra leads to such a hierarchy.

An example of this is the realization of the second Poisson bracket structure of the KdV equation given by the Miura transformation. The field $u(x)$ is expressed in terms of a field $v(x)$ satisfying

$$\{v(x), v(y)\} = \partial_x \delta(x - y)$$ (1.7)

through the transformation

$$u = \frac{1}{2} v^2 + v'.$$ (1.8)

The hierarchy of evolution equations obtained in this way for $v(x)$ is precisely the mKdV hierarchy. We could, in addition, take $v$ to be the derivative of a scalar field $\phi$. The Poisson bracket of $u$ with $\int dx \exp(-\phi(x))$ is then easily seen to be zero. In fact the conserved quantities of the KdV equation, when expressed in terms of $v$ through the Miura transformation, are even functions of $v$, and so these charges must also commute with
\[ \int dx \exp(\phi(x)) \] and hence with \[ \int dx \cosh(\phi(x)) \]. But \[ \int dx \cosh(\phi(x)) \] is the Hamiltonian for the sinh-Gordon equation, and so the conserved charges for the KdV equation are also conserved for the sinh-Gordon equation. Hence the KdV, mKdV and sinh-Gordon equations have essentially the same conserved quantities because their fields provide different realizations of the classical Virasoro algebra.

There are analogues of this picture for other integrable systems. In particular let us consider the \( N \)'th KdV hierarchy, for variables \( u_i, i = 1, \ldots, N \), corresponding to the Lie algebra \( sl(N + 1) \). The equations of this hierarchy can be written in Hamiltonian form with Poisson brackets that are a classical limit of the \( W(\mathfrak{N} + 1) \) algebra of Fateev and Lukyanov [4]. There is a realization of the \( u_i \) in terms of fields \( \phi_j \) with Poisson brackets \( \{ \phi_i'(x), \phi_j'(y) \} = \delta'(x - y) \) via the generalized Miura transformation

\[ \prod_i (\partial + h_i \cdot \phi) = \sum u_s \partial^{N-s}, \quad (1.9) \]

where the \( h_i \) are the weights of the fundamental representation of \( sl(N + 1) \). The evolution equations for the \( \phi_i \) generated by the Hamiltonians of the \( N \)'th KdV hierarchy are those of the \( N \)'th mKdV hierarchy, and these equations have the same conserved quantities as the affine Toda field theory based on \( sl(N + 1) \).

The above considerations have all been classical, and it is natural to consider whether the same picture holds after quantization. The quantum KdV, quantum mKdV and quantum sinh-Gordon equations were considered in [3], and the first few commuting conserved quantities were constructed. It was indeed found that for these quantities the connection spelt out above held also at the quantum level. Thus the charges for the quantum KdV equation, when expressed in terms of a scalar field through the Feigin-Fuchs construction, were precisely the charges found for the quantum mKdV and sinh-Gordon equations. The first few conserved charges for these systems have also been found from the perspective of perturbations of conformal field theories [3, 8, 8]. The existence of conserved commuting charges at all expected levels has argued in a free-field realization in [9], but the methods that are used classically to construct the conserved charges and establish their properties have not so far been carried over to the quantum theories. The difficulties in doing this can be traced to the necessity of normal-ordering the quantum conserved currents and their products.

One approach to establishing the integrability of quantum theories would be to attempt to exploit the remarkable connection spelt out above at the classical level between infinite-dimensional algebras and integrability. Indeed one might conjecture that this pattern is the general case, and that every integrable system is associated with the Virasoro algebra, or an extension of it, in this way. Then different realizations of a given algebra will lead to different integrable systems, but all such systems originating from a given algebra will have the same conserved quantities.
Clearly, given an infinite set of commuting quantities, we can define an integrable system by taking them as Hamiltonians. In this sense one can regard the infinite set of commuting quantities as the primary object. From this point of view the conjecture is that any such set of quantities is associated with the Virasoro algebra or a $W$-algebra. One can also go further and conjecture that the commuting quantities can be seen as a Cartan subalgebra of an even larger algebra.

In recent papers we have attempted to shed some light on these issues, and here we review these developments. In section 2 we examine commuting charges that can be constructed from the generators of a $U(1)$ Kac-Moody algebra. It was found that there exists an infinite number of series of commuting charges, with each series itself containing an infinite number of charges. In terms of the free field realization of the Virasoro algebra each series exists only for a particular value of $c$. The $n$'th series contains even spin currents for every even value of the spin, but in addition contains odd spin currents at spins $m(2n - 2) + 1$, for $m$ a positive integer. The lowest such non-trivial odd spin is $2n - 1$, and the corresponding current can be taken to be a primary field. Furthermore this field and the identity operator generate a $W$-algebra, denoted by $W(2n - 1)$, which plays an important role in determining the structure of the charges. We were able to prove analytically that these charges did commute, and furthermore we found that the $W$-algebra and an extension of it provide a natural algebraic framework in which this result can be understood.

As explained above, given an infinite set of commuting currents we can use them as Hamiltonians to specify a dynamical system which has the commuting quantities as conserved quantities. We explain at the end of section 2 how this identification allows us to conclude that the above sets of commuting currents are the conserved quantities for Sine-gordon theory for particular values of the coupling.

A given commuting set does not, however, generate the exceptional $W$-algebra associated with it, since it contains only particular moments of the generators of this algebra. It is interesting to ask what theory does possess the exceptional algebras as symmetries. Liouville theory is well known to be conformally invariant for any value of the coupling, and in section 3 we show that for specific values of the coupling this theory admits an extended symmetry, namely one of the exceptional $W$-algebras. This is a new result which is not contained in our earlier papers.

In section 4, we review the work of reference [10], which asked whether any integrable system can be related to some infinite-dimensional algebra. To this end we examined the non-linear Schrödinger equation. We will show that there is a relation between this equation and the KP hierarchy which is analogous to that which exists between the mKdV and KdV hierarchies. The essence of the connection between the non-linear Schrödinger equation and the KP hierarchy is that the first Poisson bracket structure [11] of the KP
hierarchy is isomorphic \([12, 13]\) to the algebra \(W_{1+\infty}\), which is a linear extension of the Virasoro algebra containing a single spin-\(i\) quasiprimary field for each spin \(i \geq 1\) \([14]\), and this algebra has a realization in terms of a complex scalar field \(\psi, \psi^*\). The quantum KP hierarchy is then defined using the quantum \(W_{1+\infty}\) algebra, and it is shown to possess conserved quantities at least at the first few levels. The connection between the classical non-linear Schrödinger equation and the KP hierarchy spelt out above is shown to extend to the quantum analogues.

2 Commuting quantities and \(W\)-algebras

In this section we attempt to gain some insight into the nature of integrable systems by examining whether we can construct sets of commuting currents from the simplest possible algebra namely a \(U(1)\) Kac-Moody algebra. We will not assume in our search that the currents should contain any other algebra. The \(U(1)\) Kac-Moody algebra \(\hat{U}(1)\),

\[
[J_n, J_m] = n\delta_{n+m,0},
\]

has generators \(J_n = \oint dz z^n J(z)\) with a corresponding operator product expansion

\[
J(z)J(w) = \frac{1}{(z-w)^2} + \ldots.
\]

To be precise we wish to address the question of what sets of mutually commuting quantities can be constructed as integrals of polynomials in \(J\) and its derivatives. To put this another way, we want to find sets of mutually commuting operators in the enveloping algebra of \(\hat{U}(1)\). We know that it is possible to write a stress energy tensor in terms of \(J\),

\[
T = \frac{1}{2} : J^2 : + \alpha J',
\]

and so we certainly expect to find the commuting charges that correspond to the KdV equation. We shall therefore look for sets of mutually commuting charges that include operators not contained in this series of charges.

Given currents \(A\) and \(B\) with charges

\[
Q_A = \oint dz A(z), \quad Q_B = \oint dz B(z),
\]

the commutator of \(Q_A\) with \(Q_B\) is given by

\[
[Q_A, Q_B] = \oint_0 dw \oint w dz A(z) B(w).
\]
The only contribution to this commutator comes from the single pole term in the OPE of $A$ and $B$, so $Q_A$ and $Q_B$ will commute precisely when this single pole term is a derivative.

It is straightforward to see that

$$Q_1 \equiv \oint dz J(z) \tag{2.6}$$

commutes with any charge constructed from $J$ and its derivatives. It is also true that

$$Q_2 \equiv \oint dz :J^2(z): \tag{2.7}$$

commutes with all other charges, since if $P$ is an arbitrary differential polynomial in $J$, the single pole term in $:J^2(z): P(w)$ comes from the single contraction term, namely

$$2 \sum_{n=0}^{\infty} \frac{(n+1)!}{(z-w)^{n+2}} :J(z) \frac{\partial P(w)}{\partial J^{(n)}(w)} :\tag{2.8}$$

and the coefficient of the single pole in this expression is just

$$2 \sum_{n=0}^{\infty} :J^{(n+1)} \frac{\partial P}{\partial J^{(n)}} : = 2P' \tag{2.9}$$

This is also apparent from the Feigin-Fuchs representation—the integral of $1/2 :J^2:$ is just $L_{-1}$, the generator of translations.

In order to go beyond these somewhat trivial commuting charges, we must specify some rules for the game; we will look for sets of charges commuting with the integral $Q_4$ of the spin-4 current

$$p_4 = :J^4: + g :J'^2: \tag{2.10}$$

for some value of the coupling $g$, with the same value of $g$ for each current in a given set. Moreover, we require that each set should contain a current of odd spin. We found that there were a number of different sets of such charges:

There exists an infinite number of sets of mutually commuting operators constructed from the Kac-Moody generators $J(z)$. The series $S_n$ has a unique current at every even spin and unique odd spin currents at spins $1 + m(h - 1)$ for $h = 2n - 1$ and $m = 0, 1, 2, \ldots$.

Thus the first series $S_2$ has $h = 3$ and has a unique current at every odd spin as well as at every even spin. The second series $S_3$ has unique odd spin currents at spins $5, 9, 13, \ldots$ as well as currents at every even spin.

We found such sets of odd spin currents for every odd integer $h$ larger than 1. In fact each set was uniquely fixed by the spin of the lowest non-trivial odd spin current. The charges
constructed from these currents commute with $Q_4$ provided that $g$ is given in terms of $h$ by

$$
g = \frac{1}{h + 1} (h^2 - 4h - 1) \quad (2.11)
$$

Given a set of charges commuting with a given spin-3 charge $Q_4$, it is natural to ask whether these charges commute with each other. We have verified that this is indeed the case.

The series of currents described above were found initially by using Mathematica [13]. It is a remarkable fact that there is a relatively simple explicit formula that gives all the currents in all the series, at least up to the dimension to which we calculated, namely spin 13. We refer the reader to our paper [16] for details of this formula, since we will not need them here.

We can now ask whether we can find some underlying algebraic structure that would explain the above set of currents. The first step in this process is the verification that the even spin currents can be expressed in terms of the Feigin-Fuchs energy-momentum tensor given in eqn (2.3), which has background charge $\alpha$ and central charge $c = 1 - 12\alpha^2$. Evidently $Q_2 = \oint :J^2: = 2L_{-1}$. The charge $Q_4$ can also be written in terms of $T(z)$, provided we relate the coupling $g$ and the central charge $c$ by

$$
g = -\frac{1}{3} (c + 5); \quad (2.12)
$$

for this value of $g$ we have

$$
Q_4 \equiv \oint :J^4: + g :J^2: + J^2:
\quad = 4 \oint :T^2: . \quad (2.13)
$$

Here $T^2$ is normal-ordered according to the standard prescription in conformal field theory, which takes the normal product of two operators $A(z)$ and $B(w)$ to be the term of order $(z - w)^0$ in their OPE [17],

$$
(AB)(Z) \equiv \int dw \frac{A(w)B(z)}{w - z}. \quad (2.14)
$$

To reconcile the two expressions for $Q_4$ in (2.13), we must relate the normal ordering in terms of $J$, which is the usual Wick ordering, to that in terms of $T$. Using the formula [17]

$$
((AB)C) = A(BC) + [(AB),C] - A[B,C] - [A,C]B \quad (2.15)
$$

with $A = B = J$ and $C = :J^2:$, and also making use of

$$
:J :J^2: \equiv - :J^2: J: = - J'', \quad (2.16)
$$
it is straightforward to recover eqn (2.13).

In this way it is possible to express the even spin currents in terms of $T$ alone, provided we use the freedom to add and subtract total derivatives, but the odd spin currents cannot be expressed only in terms of $T$. To summarize the result:

The even spin currents can be built in terms of the energy-momentum tensor $T = \frac{1}{2} J^2 + 3 \frac{(h-1)}{\sqrt{h+1}} J'$ where the central charge is $c = \frac{1}{h+1}(-3h^2 + 7h - 2)$.

The value $c_n$ of the central charge corresponding to the series $S_n$ can be written in the form

$$c_n = 13 - 6n - \frac{6}{n}. \tag{2.17}$$

We now consider if some even larger algebra can play a role. It is known for $n = 2, 3$ and $4$ that a W-algebra $W(2n - 1)$ exists [18, 19], where $W(2n - 1)$ is an algebra generated by the identity operator together with a single primary field of spin $2n - 1$. For $n = 2$ this algebra is the $W(3)$-algebra of Zamolodchikov [20], which exists for any value of $c$, while for $n = 3$ the algebra $W(5)$ exists for only five values of $c$, namely $6/7, -350/11, -7, 134 \pm 60/\sqrt{5}$, and for $n = 4 W(7)$ exists for the single value $-25/2$ of $c$. There is an argument to suggest that an algebra $W(2n - 1)$ exists for any $n$ at the value of $c$ given in eqn (2.17). This can be seen by examining the primary field algebra for the conformal field theory with $c$ given by (2.17). In this model the field $\phi^{(3,1)}$ has dimension $2n - 1$ and is expected to have an OPE with itself of the form [21]

$$[\phi^{(3,1)}][\phi^{(3,1)}] = [1] + [\phi^{(3,1)}] + [\phi^{(5,1)}]. \tag{2.18}$$

The dimension of $[\phi^{(5,1)}]$ is $6n - 2$, however, which is too high to appear in the OPE of a spin $2n - 1$ field with itself, and so we expect the identity operator together with $[\phi^{(3,1)}]$ to form a closed operator algebra. This observation has also been made previously by Kausch [22]. We note in addition that for the values of $c$ we are considering the field $[\phi^{(3,1)}]$ has odd spin, and as a consequence cannot appear on the right-hand side of the OPE (2.18). Thus the OPE of $[\phi^{(3,1)}]$ with itself gives only the identity operator in this case.

It is natural to suppose that the series $S_n$ is associated in some way with the algebra $W(2n - 1)$. This is confirmed by the realization that the first non-trivial field of odd spin in $S_n$ can be chosen to be a primary field and that its OPE with itself is just that for $W(2n - 1)$.

We have amassed sufficient evidence to conclude that:

The series $S_n$ can be associated with the algebra $W(2n - 1)$, the current $p_{2n-1}$
being a primary field which is the generator of $W(2n-1)$. All odd spin currents in $S_n$ are descendants of $p_{2n-1}$.

All the above deliberations were based on computer results using the Mathematica programme of Thielemans \[23\]. The calculations involved currents of spin 13 and less.

One might ask how the above sets of commuting quantities compare with those that are known from classical integrable systems. For the $n$th KdV hierarchy there are conserved currents with spins $2$, $3$, $\ldots$, $n + 1$, mod $n + 1$, so that the conserved charges have spins $1$, $2$, $\ldots$, $n$ mod $n + 1$. These are the exponents of the Lie algebra $sl(n)$ repeated modulo the Coxeter number, and in fact for any Lie algebra $\mathfrak{g}$ there exists an integrable system for which the charges have spins equal to the exponents of $\mathfrak{g}$ \[2, 1\]. These do not correspond to the spins of the currents for any of the series found above. The even spins however are those found for the quantum KdV system since there exists only one set of commuting even spins for a given central charge.

We now study in more detail the first series of currents, $S_2$, containing a current $J^3$. Some features of this series will generalize to the other series, although there are aspects of this series that will have no analogues for the others. Let us first show that this series, which is defined by the existence of commuting currents of spins $3$ and $4$, corresponds to the central charge having the value $c = -2$. If we take an arbitrary current $p_r$ of spin $r$, defined modulo derivatives,

$$p_r = J^r + g(r)J^{r-4}(J')^2 + \ldots,$$  \hspace{1cm} (2.19)

it is straightforward to calculate its OPE with $J^3$. We find that, up to derivatives, the coefficient of the single pole is given by

$$\frac{r(r-1)(r-2)(r-3)}{4}J^{r-4}(J')^3 + 6g(r)J^{r-4}(J')^3 + \ldots,$$  \hspace{1cm} (2.20)

so we must have

$$g(r) = -\left(\frac{r}{4}\right).$$  \hspace{1cm} (2.21)

For $r = 4$ we find the current is $J^4 - (J')^2$, and comparing with \(2.13\) we find that $c = -2$. Hence we conclude that a spin-3 and a spin-4 current can commute only if $c = -2$.

One recognizes from the computer results of reference \[16\] that, for $c = -2$, there is a considerable simplification in the form of the currents. In fact they are consistent with the formula

$$p_r = :e^{-\phi}\partial' e^{\phi}:,$$  \hspace{1cm} (2.22)

\(^1\) We will loosely say that two currents commute if their corresponding charges commute.
up to derivatives. The generating function for this series is

\[ e^{-\phi(z) + \phi(z+\alpha)} = \sum_{r=0}^{\infty} \frac{\alpha^r}{r!} p_r(z). \]  

(2.23)

It is straightforward to show that the integrals of the currents \( p_r \) commute by considering the OPE of the above exponential with itself.

These results for the series \( S_2 \) can understood in terms of the algebra \( \mathcal{W}_\infty \) of reference [24, 25]. This is a linear algebra containing a quasiprimary field \( V^i(z) \) with spin \( i + 2 \) for \( i = 0, 1, 2, \ldots \). Examining the commutation relations of \( \mathcal{W}_\infty \), as given for example in eqn(3.2) of [24], it can be seen that the modes \( V^i_{-1} \) form an infinite set of commuting operators, which we might think of as a Cartan subalgebra. Since \( V^0(z) = T(z) \), we recognize \( V^0_{-1} \) as \( L_{-1} \), and similarly \( V^1_{-2} \) is the mode \( W_{-2} \) of the spin-3 primary field \( W(z) \).

For the case of \( c = -2 \), however, the \( \mathcal{W}_\infty \) algebra has a realization in terms of a complex fermion, and this can be bosonized to give a realization in terms of a single real scalar field. In terms of this scalar field \( \phi \), the field \( V^i \) is proportional to \( :e^{-\phi \partial^2 + 2\phi} : \), up to derivatives. We recognize this as the current given in eqn (2.22). We also note that for \( c = -2 \) the fields \( V^i \) for \( i \geq 2 \) can be written as composites of the stress tensor \( T \) and the spin-3 primary field \( W \), so that the enveloping algebra of the Zamolodchikov \( \mathcal{W}(3) \)-algebra contains \( \mathcal{W}_\infty \) as a linear subalgebra for \( c = -2 \). This thus gives a nice explanation of the presence of the infinitely many commuting quantities in the enveloping algebra of \( \mathcal{W}(3) \) in terms of a linear algebra.

Let us summarize the main results for the series of currents \( S_3 \). Starting from a \( \mathfrak{u}(1) \) Kac-Moody algebra, we demanded a set of commuting currents that included currents of spins 3 and 4. We found that we could supplement these two currents by an infinite number of other currents, one at each spin, that formed a mutually commuting set. These could be expressed in terms of a stress tensor \( T \) and a spin-3 primary field \( W \) alone, where \( T \) has a central charge \( c = -2 \). Furthermore these currents can be identified with a Cartan subalgebra of \( \mathcal{W}_\infty \), which in turn can be written in terms of \( T \) and \( W \). We therefore see that in terms of the algebra \( \mathcal{W}_\infty \) it is possible to understand very easily the existence of the commuting charges. Although we do not know of a generalization of \( \mathcal{W}_\infty \) that enables us to gain a similar understanding of all the series of commuting charges, we shall see below that, for each series, there is a relation between the odd-spin currents that allows us to give a simple proof of the commutativity of the corresponding charges.

We shall now give explicit formulae for all of the odd spin fields in each of the series \( S_n \), and we shall prove that these fields commute with each other and with all of the even spin fields in the series [24]. We shall also spell out in detail the connection of the series \( S_n \)

\footnote{Here, as before, we refer to fields commuting when we really mean that their integrals commute.}
to a $W$-algebra.

Let us consider the forms of the fields of odd spin in $S_n$. The series $S_n$ exists for $c$ having the value $13 - 6n - 6/n$, with $n$ a positive integer, and the first non-trivial field of odd spin has dimension $2n - 1$ and can be chosen to be primary. From the work of Zamolodchikov \[6\] we expect there to be only two primary fields that commute with the even spin fields built out of the stress tensor, namely the two fields with null descendants at level 3. If we write an arbitrary value of the central charge as $c = 13 - 6t - 6/t$, the two corresponding fields will have dimensions $h_{(3,1)} = 2t - 1$ and $h_{(1,3)} = 2/t - 1$. These two values of the dimension $h$ are related to $c$ by

$$c = 13 - 3(h + 1) - \frac{12}{h + 1} = \frac{1}{h + 1}(-3h^2 + 7h - 2). \quad (2.24)$$

Taking $t$ to be a positive integer $n$ we indeed expect to find a field of dimension $2n - 1$ that commutes with the commuting charges constructed from $T$. In order to write down an explicit expression for this field in terms of the currents $J$ alone, we first consider a field of this dimension that can be written as an exponential of $\phi$, namely

$$V_{-\alpha_+} \equiv e^{-\alpha_+ \phi} \quad (2.25)$$

where $\alpha_+ = \sqrt{2n}$ and $\alpha_- = -\sqrt{2/n}$. There exist one other field with the same weight, but we shall concentrate our attention on the above field in what follows.

When $c = 13 - 6n - 6/n$, we can use the screening charge $Q_+ \equiv \oint \exp \alpha_+ \phi$ to construct another primary field of weight $2n - 1$ that is annihilated by $O_{(3,1)}$. The commutator of a Virasoro generator $L_n$ with $\exp \alpha_+ \phi$ is a total derivative, so provided this field is single-valued the commutator of $Q_+$ with a primary field will be another primary field of the same weight. In general this new primary field could be zero, but that will not be the case here. For the values of $c$ in which we are interested, $\exp \alpha_+ \phi$ is indeed local with respect to $\exp -\alpha_+ \phi$, and we can therefore construct another primary field of weight $2n - 1$ by

$$p_{2n-1} \equiv [Q_+, \exp -\alpha_+ \phi(z)] = \oint_z dw \exp(\alpha_+ \phi(w)) \exp(-\alpha_+ \phi(z)). \quad (2.26)$$

This field has vanishing background charge and so is expressible entirely in terms of the current $J$ and its derivatives, and in fact it is given explicitly by the term of order $\partial^0$ in the differential operator

$$(\partial + \alpha_+ J)^{2n-1}, \quad (2.27)$$

in which the derivatives are taken to act on everything that occurs to the right of them. We claim that this is the first non-trivial field of odd spin occurring in $S_n$. For the first
few series this can be checked explicitly against the computer results given in Section 2. To show that it is true in general, we need to check that the charge constructed from this field commutes with all of the odd spin charges obtained by integrating the even spin currents, which correspond to those that occur in the quantum KdV equation. This is a consequence of the fact that the screening charge \( \exp \alpha + \phi \) commutes with any polynomial in \( T \) and its derivatives, and in particular with the odd spin charges. Since these odd spin charges are even under \( \phi \to -\phi \), it follows that \( \oint \exp -\alpha + \phi \) also commutes with these charges, as explained in ref [5].

The above considerations immediately suggest a way to construct infinitely many odd spin currents whose integrals commute with the odd spin charges. We take \( Q_+ \), as defined earlier, to be \( \oint \exp \alpha + \phi \), and we define also \( Q_- = \oint \exp -\alpha + \phi \). These two charges have conformal weights 0 and \( 2n - 2 \) respectively, and each commutes with all of the odd spin charges. We then define an operator \( \Delta \) that acts on an arbitrary field \( \Phi \) by

\[
\Delta \Phi(z) = [Q_+, [Q_-, \Phi(z)]] = \oint_z dy \oint_{y,z} dx :e^{\alpha + \phi(x)}: :e^{-\alpha + \phi(y)}: \Phi(z). \tag{2.28}
\]

If the integral of \( \Phi \) commutes with the odd spin charges, it is clear that the integral of \( \Delta \Phi \) will also commute with these charges. Our strategy is then to apply \( \Delta \) repeatedly to the primary field constructed above. In fact it is interesting to note that the primary field \( p_{2n-1} \) can itself be written as

\[
p_{2n-1} = \Delta J, \tag{2.29}
\]

since \([Q_-, J(z)] = \exp -\alpha + \phi(z)\). If we write \( \Phi^{(m)} \) for the \( m \)th odd spin field in \( S_n \), and \( Q^{(m)} \) for the corresponding charge, we have

\[
\Phi^{(m)} = \Delta \Phi^{(m-1)}, \quad \Phi^{(0)} = J, \tag{2.30}
\]

with similar formulae holding for \( Q^{(m)} \). Thus \( \Phi^{(m)} \) is given by the multiple commutator

\[
\Phi^{(m)} = \Delta^m J = [Q_+, [Q_-, \ldots [Q_+, [Q_-, J]. \tag{2.31}
\]

Furthermore it is readily seen to be expressible as a sum of terms, with each term being a product of derivatives of exponentials of \( \phi \), with the total \( U(1) \) charge for each term in the sum being zero. Thus \( \Phi^{(m)} \) can be written in terms of \( J \) alone, with no exponentials of \( \phi \). While it is conceivable that this multiple commutator could vanish, and indeed this will be the case for some orderings of the \( Q_+ \) and \( Q_- \), we believe this will not happen in general for the ordering we have chosen. In fact for the first series, it is straightforward to show that the action of \( \Delta \) does not vanish and that it ladders one up the odd currents of the series.
Having seen how to write explicit formulae for an infinite set of charges commuting with the even spin charges, it is necessary to understand why these charges commute amongst themselves. The details are given in reference [16], and we omit them here. The basic strategy, however, is to use the crucial identities

\[
[Q_+, [Q_+, [Q_+, Q_-]]] = 0, \quad (2.32)
\]

and

\[
[Q_-, [Q_-, [Q_-, Q_+]]] = 0, \quad (2.33)
\]

which can be obtained very simply just from looking at the contour integrals that need to be done in order to evaluate these expressions. Making use of these relations one can then prove by induction that

\[
[Q^{(i)}, Q^{(j)}] = 0, \quad \text{for } i + j = n,
\]

\[
Q^{(n)} \equiv Q_+ Q_- Q^{(n-1)}
\]

\[
= [Q_- Q^{(k)}, Q_+ Q^{(l)}], \quad \text{for } k + l = n - 1,
\]

\[
Q_+^2 Q^{(n-1)} = Q_-^2 Q^{(n-1)} = 0. \quad (2.34)
\]

In particular we have that \([Q^{(i)}, Q^{(j)}] = 0\) for all \(i\) and \(j\).

It is interesting to ask to what perturbation of a conformal field theory with \(c = 13 - 6n - 6/\alpha\) the commuting charges found in this paper correspond. This amounts to finding an operator that commutes with these charges. The charges constructed from the even spin currents are differential polynomials in \(T\) and so will commute with the screening charges \(\oint \exp \alpha \pm \phi\). They also commute with \(\oint \exp -\alpha \pm \phi\), on account of being even polynomials in \(\phi\). The charges for the odd spin currents, however, are not constructed from \(T\) alone, and so will not commute with the above charges in general. Nevertheless, since they are constructed from \(\oint \exp \alpha_+ \phi\) and \(\oint \exp -\alpha_- \phi\), and since \(\alpha_+ \alpha_- = -2\) implies that both of these charges commute with \(\oint \exp \pm \alpha_\pm \phi\), all the charges we have found commute with \(\oint \exp \alpha_\pm \phi\) and \(\oint \exp -\alpha_\pm \phi\). While the former is the screening charge, the latter is the field \(\phi_{1,3}\), which has weight \(2/n - 1\). Thus the commuting charges are conserved in the presence of the \(\phi_{1,3}\) perturbation. These remarks are in agreement with those of Eguchi and Yang [27], who considered the quantum sine-Gordon theory with Hamiltonian \(\oint e^{\alpha_+ \phi} - e^{-\alpha_+ \phi}\). They observed that for particular values of the sine-Gordon coupling constant extra odd-spin conserved currents existed for spins \(2n - 1 \mod (2n - 2)\). The detailed forms of their charges are different from ours, however, and it is not possible to write them in terms of \(J\) alone. Their first charge, for example, is given in our notation by \(Q_+ - Q_-\).

We now turn to the connection between the series of commuting charges and certain \(W\)-algebras. We explained in the previous section that the first series of charges was related to Zamolodchikov’s \(W(3)\)-algebra with the central charge taking the value \(c = -2\), and that for this value of \(c\) the enveloping algebra of the \(W(3)\)-algebra contains a linear
subalgebra which is just \( W_\infty \). In order to make a connection between \( W \)-algebras and the higher series, there was another aspect of the algebras that played an important role in our proof of the commutativity of the charges constructed from the odd-spin currents. This was the fact that we had not just a single primary field of conformal weight \( 2n - 1 \) for \( c = 13 - 6n - 6/n \), but in fact we made use of three distinct primary fields having this dimension. In our free-field representation, these were \( \exp(-\alpha_+ \phi), Q_+ \exp(-\alpha_+ \phi) \) and \( Q_+^2 \exp(-\alpha_+ \phi) \). It is known that for the values of \( c \) that we are considering there exist \( W \)-algebras generated by the stress tensor and three primary fields each having spin \( 2n - 1 \). If we denote the primary fields by \( W_i(z) \), for \( i = 1, 2 \), and \( 3 \), they have operator product expansions of the form

\[
[W^i][W^j] = \delta^{ij}[I] + \epsilon^{ijk}[W^k],
\]

so that there is an \( SU(2) \)-like structure present. The operator \( Q_+ \) can be considered as an \( SU(2) \) raising operator. We have seen that acting repeatedly with \( Q_- \) on the multiplet of primary fields gives other spin-1 \( SU(2) \) multiplets of higher conformal weight. The fields in these higher multiplets are no longer primary, but their integrals give rise to the infinite sets of commuting charges we have found.

### 3 Louiville theory and the exceptional \( W \)-algebras

We have found that a given set of commuting currents, \( S_n \), was a symmetry of the sine-Gordon theory for a particular value of the coupling. This theory, however, does not possess the full exceptional \( W \)-symmetry since the Hamiltonian only commutes with a specific mode of the \( W \)-current and the higher currents. A natural question to ask is what theory does possess the exceptional \( W \)-algebra as a symmetry. The \( W(2n - 1) \) current \( J(z) \) is of the form

\[
j(z) = [Q_+, e^{\alpha_+ \phi(z)}].
\]

Let us consider the Hamiltonian

\[
H = \oint dz : \oint dz e^{\alpha_- \phi(z)} :.
\]

Since \( H \) is a screening charge, it commutes with the Virasoro generators \( L_n \) and also with \( Q_+ \), since \( \alpha_+ \alpha_- = -2 \) and

\[
[H, Q_+] = -\oint dz \oint dz' e^{\alpha_- \phi(z)} e^{\alpha_+ \phi(w)}
\]

\[
= -\oint dz \oint dz' e^{\alpha_- \phi(z) + \alpha_+ \phi(w)}
\]

\[
= \oint dz \alpha_+ \partial \phi e^{(\alpha_- + \alpha_+ \phi(z)} = 0
\]
Similarly, $H$ commutes with $e^{-\alpha_+\phi(z)}$ and consequently $Q_-$. It follows therefore that $H$ commutes with all the modes of the current $j(z)$, and so the exceptional $W(2n-1)$ algebra which we conclude is a symmetry of Louville theory for the above coupling.

It also follows that all the moments of the higher currents are also a symmetry of this Louville theory since

$$[Q_+, [Q_-, \ldots, [Q_+, e^{-\alpha_+\phi(z)}]\ldots]].$$

also commutes with $H$.

4 $W_{1+\infty}$ and the non-linear Schrödinger equation

We now turn our attention to the non-linear Schrödinger equation, which is well-known to be integrable, in an attempt to understand the connection of this to a $W$-algebra. We do this by first establishing a connection of the non-linear Schrödinger equation to the KP hierarchy, which is in turn known to be related to the algebra $W_{1+\infty}$. We then extend these considerations to the quantum domain.

The non-linear Schrödinger equation is given by

$$\frac{\partial \psi}{\partial t} + \psi'' + 2\kappa \psi^* \psi^2 = 0,$$

(4.1)

where $\psi, \psi^*$ is a bosonic complex scalar field and $\kappa$ is a coupling constant. This equation can be written in Hamiltonian form $\dot{\psi} = \{\psi, H\}$ provided we adopt the Poisson brackets

$$\{\psi^*(x), \psi(y)\} = \delta(x - y), \quad \{\psi(x), \psi(y)\} = \{\psi^*(x), \psi^*(y)\} = 0$$

(4.2)

and take $H = \int (\psi^* \psi'' + \kappa \psi^2 \psi^2)$.

It is useful to note that it is possible to assign weights to $\psi$ and $\psi^*$ in a manner consistent with the Poisson bracket. We take $\partial$ and $\delta(x - y)$ to have weight 1, and then eqn (4.2) implies that the product $\psi^* \psi$ must have weight 1. If we now take $\kappa$ to have weight 1, the Hamiltonian $H$ will be homogeneous of weight 2.

The non-linear Schrödinger equation possesses an infinite set of conserved quantities $Q_n$, given by the formula

$$Q_n = \frac{1}{\kappa} \int dx \psi(x) Y_n(\psi, \psi^*),$$

(4.3)

where

$$Y_1 = \kappa \psi^*, \quad Y_{n+1} = Y'_n + \psi \sum_{k=1}^{n-1} Y_k Y_{n-k} \quad \text{for } n \geq 1.$$
The first few such charges are
\begin{align*}
Q_1 &= \int dx \psi^* \psi \\
Q_2 &= -\int dx \psi^* \psi' \\
Q_3 &= \int dx (\psi^* \psi'' + \kappa \psi^* \psi^2);
\end{align*}

it can be checked that these charges commute and so can be used to generate commuting evolutions, giving a hierarchy of equations.

The KP hierarchy consists of an infinite set of differential equations which has, at first sight, no relation to the non-linear Schrödinger equation. This set of equations is defined in terms of a pseudodifferential operator $Q$ of the form
\begin{equation}
Q = D + q_0 D^{-1} + q_1 D^{-2} + \ldots.
\end{equation}

Here $D$ denotes $\partial / \partial z$ and satisfies the following generalization of the Leibniz rule:
\begin{equation}
D^n f = \sum_{r=0}^{\infty} \binom{n}{r} f^{(r)} D^{n-r}, \quad n \in \mathbb{Z}.
\end{equation}

An infinite set of commuting time evolutions for the fields $q_i$, $i = 0, 1, 2, \ldots$, is given by
\begin{equation}
\frac{\partial Q}{\partial t_p} = [(Q^p)_+, Q], \quad p = 0, 1, 2, \ldots
\end{equation}

where $(Q^p)_+$ is the part of $Q^p$ involving no negative powers of $D$. These evolution equations can be written in Hamiltonian form. There are in fact a number of ways to do this, but for our purposes it is the so-called first Hamiltonian structure that is needed. The Poisson bracket in this case is obtained from the method of coadjoint orbits applied to the Lie algebra of differential operators. We omit the details here and simply give the results \[11\]:
\begin{equation}
\{q_i(x), q_j(y)\} = \sum_{r=0}^{\infty} \binom{j}{r} \partial_x^r (q_{i+j-r}(x) \delta(x-y)) - \sum_{r=0}^{\infty} \binom{i}{r} (-1)^r q_{i+j-r} \partial_x^r \delta(x-y).
\end{equation}

This algebra is in fact precisely $W_{1+\infty}$ with zero central charge \[12, 13\]; the $q_i$ are related to the usual basis for $W_{1+\infty}$, denoted by $V^i$ \[14, 20\], by a linear transformation. It can be shown that the time evolutions \[4.3\] can be written as
\begin{equation}
\frac{\partial q_i}{\partial t_p} = \{q_i, H_{p+1}\}, \quad p = 0, 1, 2, \ldots,
\end{equation}

where
\begin{equation}
H_p = \frac{1}{p} tr(Q^p).
\end{equation}
The trace of a pseudodifferential operator is given by the integral of the coefficient of $\partial^{-1}$. Just as for the non-linear Schrödinger equation, there is an assignment of weights to the $q_i$ which is consistent with the Poisson bracket structure (4.9). If we take $\partial$ and $\delta(x - y)$ to have weight 1, then $q_i$ must have weight $i + 1$. With this choice, the operator $Q$ in eqn (4.6) does not have a definite weight, but we can correct this by working instead with

$$\tilde{Q} \equiv \kappa^{-1}D + q_0D^{-1} + q_1D^{-2} + \ldots,$$

(4.12)

where, as before, $\kappa$ is a constant of weight 1. $\tilde{Q}$ then has weight zero. Using $\tilde{Q}$ instead of $Q$ does not change the Poisson brackets between the $q_i$, but the Hamiltonians

$$\tilde{H}_p = \frac{1}{p}tr(\tilde{Q}^p)$$

(4.13)

now involve various powers of $\kappa$. The first few $\tilde{H}_p$ are given explicitly by

$$\tilde{H}_1 = \int dxq_0$$
$$\tilde{H}_2 = \kappa^{-1}\int dxq_1$$
$$\tilde{H}_3 = \kappa^{-2}\int dx(q_2 + \kappa q_0^2)$$
$$\tilde{H}_4 = \kappa^{-4}\int dx(q_3 + 3\kappa q_0q_1).$$

(4.14)

The $\tilde{H}_p$ can be viewed as a set of Poisson commuting quantities within the enveloping algebra of $\mathcal{W}_{1+\infty}$.

Having given the KP and the non-linear Schrödinger equations, we are now in a position to explain the relation between them by exploiting the fact that $\mathcal{W}_{1+\infty}$ has a realization in terms of a single complex boson. Let us take a general approach first. We consider a realization of $\mathcal{W}_{1+\infty}$ in terms of some set of fields $\phi_i$, so that the $q_i$ can be expressed in terms of the $\phi_i$ and the Poisson brackets for the $q_i$ follow from those of the $\phi_i$. If such a realization of $\mathcal{W}_{1+\infty}$ has zero central charge, we can clearly construct an infinite set of mutually commuting quantities by expressing the $H_n$’s in terms of the $\phi_i$. We can then define time evolutions for the $\phi_i$ by

$$\frac{\partial \phi_i}{\partial t_p} = \{\phi_i, H_{p+1}\}, \quad p = 0, 1, 2, \ldots,$$

(4.15)

and it follows from the Leibniz property of the Poisson bracket that this implies the KP evolutions for the $q_i$.

There is a centreless realization of $\mathcal{W}_{1+\infty}$ in terms of a complex boson with Poisson brackets given by

$$\{\psi^*(x), \psi(y)\} = \delta(x - y), \quad \{\psi(x), \psi(y)\} = \{\psi^*(x), \psi^*(y)\} = 0.$$
This is most easily seen by checking that the correct Poisson bracket (4.9) between the \( q_i \) is obtained by taking
\[
q_i = (-1)^i \psi^* \psi^{(i)}.
\]
(4.17)
The first few Hamiltonians of the KP hierarchy, when expressed in terms of \( \psi \) and \( \psi^* \), take the forms
\[
\begin{align*}
\tilde{H}_1(\psi^*, \psi) &= \int dx \psi^* \psi \\
\tilde{H}_2(\psi^*, \psi) &= -\kappa^{-1} \int dx \psi^* \psi' \\
\tilde{H}_3(\psi^*, \psi) &= \kappa^{-2} \int dx (\psi^* \psi'' + \kappa \psi^* \psi^2) \\
\tilde{H}_4(\psi^*, \psi) &= -\kappa^{-4} \int dx (\psi^* \psi''' + 3 \kappa \psi^* \psi^2 \psi').
\end{align*}
\]
(4.18)
We recognize these immediately as the first few conserved charges of the non-linear Schrödinger equation, up to overall factors. It follows from the commutativity of the KP Hamiltonians that the \( \tilde{H}_n(\psi^*, \psi) \) will also commute. However, since the charges of the non-linear Schrödinger equation are uniquely determined by the requirement that they commute with \( \tilde{H}_3 \), it follows that the charges obtained from the KP system must coincide with those of the non-linear Schrödinger equation at all levels.

Given that the classical non-linear Schrödinger equation is related to the KP hierarchy and that the quantum non-linear Schrödinger equation is known to be integrable, one might expect there to exist an integrable quantum KP hierarchy. One way to construct such a hierarchy would be to take the first Poisson bracket structure for the KP hierarchy, which is isomorphic to \( W_{1+\infty} \), and then to replace Poisson brackets by commutators. The existence of an integrable quantum KP hierarchy would then follow if we could find an infinite set of commuting Hamiltonians in the enveloping algebra of \( W_{1+\infty} \).

In this section we construct some quantum analogues of the Hamiltonians \( \tilde{H}_p \) from the generators \( V^i \) of \( W_{1+\infty} \). We use the notation of [14], where \( V^i \) is a quasi-primary field of weight \( i + 2 \). We used a computer to look for commuting charges that could be expressed as integrals of sums of products of the fields in \( W_{1+\infty} \), using the OPEdefs package of Thielemans [23]. The first two charges, namely
\[
Q_1 = \int dz V^{-1}
\]
(4.19)
and
\[
Q_2 = \int dz V^0
\]
(4.20)
are trivial, in the sense that the first of these commutes with all \( V^i(z) \) while the commutator of \( Q_2 \) with \( V^i(z) \) is just \( V'^i(z) \). For the first non-trivial charge we take
\[
Q_3 \equiv \int dz \left\{ V^1 + \kappa (V^{-1})^2 \right\};
\]
(4.21)
this is analogous to the classical Hamiltonian $H_3$ of the KP hierarchy in that it involves terms of weights 2 and 3. In this and subsequent expressions involving products of operators, we use the normal ordering standard in conformal field theory, namely we take

$$ (AB)(z) = \oint_z dw \frac{A(w)B(z)}{w-z}. $$

(4.22)

It is our belief that there are infinitely many further charges that commute with the charge $Q_3$, and that furthermore these charges are unique. We have found three such charges, and we have verified that all the charges we have obtained commute amongst themselves. We conjecture that there exists an infinite set of such charges, one for each spin. We give here $Q_4$ and $Q_5$:

$$ Q_4 = \int dz \left\{ V^2 + 3\kappa (V^{-1}V^0) + 3c\kappa^2/2 (V^{-1})^2 \right\} $$

$$ Q_5 = \int dz \left\{ V^3 + \kappa \left( 4(V^{-1}V^1) + 2(V^0)^2 - 5/6 (V^{-1})^2 \right) \right. $$

$$ +\kappa^2 \left( 2(V^{-1})^3 + 4c (V^{-1}V^0) \right) + 2c^2\kappa^3 (V^{-1})^2 \right\}. $$

(4.23)

There is thus strong evidence that quantization of the first Poisson bracket structure of the KP hierarchy leads to an integrable system. We now relate this to the quantum non-linear Schrödinger equation. The integrability of the quantum non-linear Schrödinger equation has been extensively studied using the quantum inverse scattering method and also from the point of view of the existence of an infinite number of commuting conserved quantities. The relation between these different approaches was discussed in [28]. In order to quantize the non-linear Schrödinger equation we adopt the following OPE for the bosonic operators $\psi$ and $\psi^*$:

$$ \psi^*(z)\psi(w) = (z-w)^{-1} + \ldots. $$

(4.24)

The time evolution is generated by

$$ H = \oint dz \left( :\psi^*\psi'' : + :\kappa\psi^*^2\psi^2 : \right), $$

(4.25)

in the sense that

$$ \frac{\partial \psi}{\partial \bar{z}} = [\psi, H], \quad \frac{\partial \psi^*}{\partial \bar{z}} = [\psi^*, H]. $$

(4.26)

We can then search for normal-ordered polynomials in $\psi, \psi^*$ and their derivatives whose integrals commute with $H$; the first few conserved currents are as follows:

$$ \psi^*\psi, \quad \psi^*\psi', \quad \psi^*\psi'' + \kappa\psi^*^2\psi^2, \quad \psi^*\psi'' + 6\kappa\psi^*\psi'' + 3\kappa\psi^*^2\psi^2. $$

(4.27)

We showed earlier that the non-linear Schrödinger equation and the KP hierarchy were related at the classical level, and we now extend that analysis to the quantum case. Just
as the Poisson algebra $W_{1+\infty}$ has a realization in terms of classical scalar fields $\psi$ and $\psi^*$, the quantum algebra $W_{1+\infty}$ has a realization in terms of operators $\psi$ and $\psi^*$ with OPE given by eqn (4.24). The quasiprimary fields $V_i$ of $W_{1+\infty}$ have the form

$$V_i = \left(\frac{2n+2}{n+1}\right)^{-1} \sum_{r=0}^{i+1} (-1)^r \binom{i+1}{r} \psi^{(r)} \psi^* \psi^{(i+1-r)}.$$  

(4.28)

On substituting these expressions into the quantum KP charges $Q_i$ of the previous section we recover the charges of the quantum non-linear Schrödinger equation given above, up to the freedom to add lower spin charges. Some care must be taken in transforming from the normal ordering in terms of the $V_i$ to the free field normal ordering used for the non-linear Schrödinger charges. Since the KP charges commute it follows that the non-linear Schrödinger charges will also commute, and because the non-linear Schrödinger and KP charges are uniquely determined by requiring that they commute with $H_3$ we conclude that we obtain all the non-linear Schrödinger charges in this way.

Acknowledgement: We are grateful to Klaus Hornfeck for discussions, and to John Schwarz who specifically encouraged us to consider the question of which theories possessed exceptional W-algebras as symmetries. M.D. Freeman is grateful to the UK Science and Engineering Research Council for financial support.

References

[1] V.G. Drinfeld and V.V. Sokolov, J. Sov. Math. 30 (1984) 1975.
[2] G. Wilson, Ergod. Th. and Dynam. Sys. 1 (1981) 361.
[3] J.-L. Gervais, Phys. Lett. B160 (1985) 277; Phys. Lett. B160 (1985) 279.
[4] V.A. Fateev and S.L. Lukyanov, Int. J. Mod. Phys. A3 (1988) 507; “Additional symmetries and exactly-soluble models in two-dimensional conformal field theory,” Moscow preprint 1988.
[5] R. Sasaki and I. Yamanaka, Adv. Studies in Pure Math. 16 (1988) 271.
[6] A.B. Zamolodchikov, “Integrable field theory from conformal field theory,” Proceedings of the Taniguchi Symposium, Kyoto, (1988); Int. J. Mod. Phys. A3 (1988) 743; Int. J. Mod. Phys. A4 (1989) 4235.
[7] T. Eguchi and S-K. Yang, Phys. Lett. B224 (1989) 373.
[8] T. Hollowood and P. Mansfield, Phys. Lett. B226 (1989) 73.
[9] B. Feigin and E. Frenkel, Phys. Lett. B276 (1992) 79.
[10] M. D. Freeman and P. West, Phys. Lett. B295 (1992) 59.

[11] Y. Watanabe, Lett. Math. Phys. 7 (1983) 99.

[12] K. Yamagishi, Phys. Lett. B259 (1991) 436.

[13] F. Yu and Y-S. Wu, Phys. Lett. B263 (1991) 220.

[14] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B242 (1990) 401.

[15] S. Wolfram, “Mathematica: a system for doing mathematics by computer,” Addison-Wesley, 1991.

[16] M.D. Freeman, K. Hornfeck and P. West, “Commuting quantities and exceptional W-algebras,” to be published in Int. J. Mod. Phys.

[17] F. Bais, P. Bouwknegt, M Surridge and K. Schoutens, Nucl. Phys. B304 (1988) 348.

[18] H.G. Kausch and G.M.T. Watts, Nucl. Phys. B354 (1991) 740.

[19] R. Blumehagen et al, Nucl. Phys. B361 (1991) 255.

[20] A.B. Zamolodchikov, Teo. Mat. Fiz 65 (1985) 347.

[21] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.

[22] H.G. Kausch, Phys. Lett. B259 (1991) 448.

[23] K. Thielemans, Int. J. Mod. Phys. C2 (1991) 787.

[24] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B236 (1990) 173;

[25] C.N. Pope, L.J. Romans and X. Shen, Nucl. Phys. B339 (1990) 191.

[26] C.N. Pope, “Lectures on W algebras and W gravity,” preprint CTP TAMU-103/91.

[27] T. Eguchi and S-K. Yang, Phys. Lett. B235 (1990) 282.

[28] M. Omote, M. Sakagami, R. Sasaki and I. Yamanaka, Phys. Rev. D35 (1987) 2423.