Thermal conductivity in a mixed state of a superconductor at low magnetic fields

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We evaluate accurate low-field/low-temperature asymptotics of the thermal conductivity perpendicular to magnetic field for one-band and two-band s-wave superconductors using Keldysh-Usadel formalism. We show that heat transport in this regime is limited by tunneling of quasiparticles between adjacent vortices across a number of local points and therefore widely-used approximation of averaging over the circular unit cell is not valid. In the single-band case, we obtain parameter-free analytical solution which provides theoretical lower limit for heat transport in the mixed state. In the two-band case, we show that the heat transport is controlled by the ratio of gaps and diffusion constants in different bands. Presence of a weaker second band strongly enhances the thermal conductivity at low fields.

I. INTRODUCTION

The thermal conductivity of a metal in the superconducting state is markedly different from its value in the normal state. The physical reason is that the Cooper pairs give no contribution to the transport of heat. Therefore heat transport occurs solely due to the quasiparticle excitations controlled by the energy gap. Due to this reason, measurements of the thermal conductivity were extensively used during last decades as a tool to probe the energy gap symmetry in various new superconducting materials.

Application of a magnetic field $H$ provides a way to generate quasiparticles in a type-II superconductor. In a s-wave superconductor at $T \ll T_c$, the quasiparticles are localized near the vortex cores and thermal conduction perpendicular to the magnetic field is due to tunneling between adjacent vortices. Thermal conductivity in the mixed state of superconductors was extensively studied in the past experimentally\cite{1,2,3} and theoretically\cite{4,5}. Early theoretical work\cite{6} addressed superconductors in diffusive regime (dirty limit) in the mixed state with a temperature gradient applied transverse to the magnetic flux. The electronic thermal conductivity was calculated using the quasiclassical time-dependent superconductivity theory, assuming homogeneous temperature gradient, using the circular-cell approximation, and averaging over the unit cell. This approach was later extended to study heat transport in various unconventional superconductors. Theory for d-wave superconductors was developed in Ref. \textsuperscript{6} for zero magnetic field and in Refs. \textsuperscript{7} and \textsuperscript{8} for the mixed state. More recently, with the discovery of magnesium diboride (MgB$_2$), theory was extended to the case of multiband superconductivity\textsuperscript{9}. Discovery of pnictides, with possibly unconventional pairing mechanism due to spin fluctuations leading to the so-called $s_\pm$ pairing state, motivated extension of heat transport theory\textsuperscript{10} taking into account multiband superconductivity and resonant interband impurity scattering. Many recent experimental studies reconsider old superconductors, such as NbSe$_2$\textsuperscript{12}, and addressed novel superconductors, such as borocarbides (LuNi$_2$B$_2$C)\textsuperscript{11}, Sr$_2$RuO$_4$\textsuperscript{13}, $C_6$Yb\textsuperscript{14}, heavy-fermion compounds CeIn$_3$, CeCoIn$_5$ and UPt$_3$\textsuperscript{15}, MgB$_2$\textsuperscript{16}–\textsuperscript{18} and iron-silicides\textsuperscript{19}. Being very sensitive to the gap structure, thermal transport at low magnetic fields varies considerably for different compounds.

In this paper we reconsider more accurately the problem of the thermal conductivity across the magnetic fields for s-wave superconductors. The case of s-wave superconductor is a standard reference, all other situations are compared with this case. Surprisingly, an accurate result for the low-field asymptotics of the thermal conductivity was never derived. The widely-used circular unit cell approximation does not give correct result in low magnetic fields because it misses essential physics. Namely, as the local thermal conductivity is strongly inhomogeneous, the heat transport is limited by tunneling between adjacent vortices across certain local points in the vortex lattice unit cell (bottlenecks). This leads to general low-field asymptotics of the electronic thermal conductivity, $\kappa \propto \exp(-\beta \sqrt{B_{c2}/B})$, where $B_{c2}$ is the upper critical field\textsuperscript{2}. Surprisingly, the theoretical value of the numerical constant $\beta$ is not available neither for clean nor for dirty s-wave superconductors. For clean case, we provide estimate for this numerical constant using asymptotics of the Bogolyubov wave functions of the localized states at zero energy and microscopic value of the upper critical field. In the dirty case we were able to perform more quantitative analysis using the Keldysh-Usadel formalism. We calculate the thermal conductance at low temperature and low magnetic field for single- and two-band superconductors in the dirty limit. In this regime we obtain parameter-free analytical solution in a single-band case which provides theoretical lower limit for heat transport in the mixed state. We find that in dirty case the low-field thermal conductivity is drastically suppressed in comparison with clean case. Further, we generalize the developed formalism to a two-band superconductor, taking MgB$_2$ as an example.
II. TUNNELING OF QUASIPARTICLES BETWEEN VORTEX CORES IN CLEAN ISOTROPIC SUPERCONDUCTOR

The electronic transverse thermal conductivity in mixed state at low temperatures and fields is determined by the probability quasiparticle tunneling between the cores of neighboring vortices. In this section we evaluate this quantity with exponential accuracy for clean isotropic superconductors. Even though this estimate is very straightforward and could be done long time ago, to our great surprise, we did not succeed to find it in the literature.

The Bogolyubov wave function of the localized state in the vortex core at $E = 0$ decays as\cite{19}

$$\Psi(r) \propto \exp(-r/\xi)$$

(1)

where $\xi$ is the superconducting gap and $v_F$ is the Fermi velocity. Therefore, the probability of tunneling between the vortex cores separated by distance $d = \sqrt{2\Phi_0/\sqrt{3}B}$ can be estimated as

$$P \propto |\Psi(a)|^2 \approx \exp\left(-2a/\xi\right)$$

(2)

The upper critical field $B_{c2}$ for a clean isotropic superconductor at $T = 0$ is given by\cite{20,21}

$$B_{c2} = \frac{e^2}{4\pi\xi_0}$$

(3)

with $\xi_0 = \frac{v_F}{2\pi T_c}$, $T_c = \gamma 0.771$ being the Euler constant. Using also the BCS relation $\pi T_c = \gamma \Delta$, we evaluate

$$2\sqrt{\frac{2\Phi_0}{\sqrt{3}B_{c2}\xi_0}} = \sqrt{\frac{16\pi}{\sqrt{3}\gamma e^2}} = \approx 1.486.$$  

This allows us to represent the tunneling probability \cite{19} as

$$P \propto \exp\left(-\frac{16\pi B_{c2}}{B}\right) \approx \exp\left(-1.486 \frac{B_{c2}}{B}\right).$$

(4)

This result also determines the low-field asymptotics of the electronic thermal conductivity of clean isotropic superconductor. The constant in the exponent is obviously sensitive to anisotropy of Fermi surface. A more quantitative analysis which would include also evaluation of the preexponential factor requires much more complicated microscopic kinetic theory for clean limit. In the following sections we make quantitative calculations of thermal transport at low fields for dirty superconductor.

III. THE FORMALISM: THERMAL TRANSPORT IN DIRTY SUPERCONDUCTORS WITHIN QUASICLASSICAL KELDYSH-USADEL MODEL

Our study is based on the quasiclassical Keldysh-Ussadel formalism\cite{21,22}, which was developed to describe nonequilibrium properties of dirty superconductors. Below we will reproduce the main relations of this formalism needed for our derivations. Within this formalism a superconductor is described by the Green’s function

$$G = \begin{pmatrix} \hat{G}_R^I & \hat{G}_K^I \\ \hat{G}_R^K & \hat{G}_K^K \end{pmatrix},$$

(5)

where in dirty limit the retarded (advanced) Green’s functions $G^{R(A)}$ satisfy the Usadel equation\cite{23}

$$-iD\nabla [\hat{G}^{R(A)}] = \begin{pmatrix} iE\hat{\gamma}_3 + \hat{\Delta} \hat{G}^{R(A)} \end{pmatrix}.$$  

(6)

Here $E$ is quasiparticle energy, $D$ is the electronic diffusion coefficient, $\Delta$ is the pair potential

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix},$$

(7)

$$\hat{G}^K = \hat{G}^R \hat{f} - \hat{f} \hat{G}^A,$$

(8)

where the distribution function, $f = 1 + f_F$, where $f$ and $f_T$ are odd and even in energy components of the distribution function, one can rewrite the thermal current in the form\cite{24}

$$J_{th} = \frac{N_0 D}{4} \int E \langle \hat{\gamma}_3 \hat{G}^R \nabla \hat{G}^K + \hat{G}^K \nabla \hat{G}^A \rangle dE,$$  

(9)

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$$\nabla \cdot (D_\alpha \nabla f_\alpha) + \text{Im} \text{Im} J_{th,\alpha} = 2 R_{ij} f_I,$$

(10)

$$\nabla \cdot (D_\alpha \nabla f_\alpha) + \text{Im} \text{Im} J_{th,\alpha} = 0.$$

(11)

where $R = \frac{1}{4T} \left[\hat{\Delta} (\hat{G}_R + \hat{G}_A)\right]$. In thermal equilibrium $f_L = \tanh (E/2k_B T)$ and $f_T = 0$. The thermal conductivity components are defined as

$$\kappa_{\alpha} = \langle J_{th,\alpha} \rangle / \langle \nabla \alpha T \rangle,$$  

(12)
the form homogeneous within the unit cells and only changes in unit cell. In such situation the temperature is mostly \( \propto \) regions. This means that the local thermal resistance is mostly determined by the diffusion direction perpendicular to the field. The local thermal resistance in the bottleneck region marked by rectangle in the left picture.

\[
\begin{align*}
D (\nabla^2 \theta - p^2 \cos \theta \sin \theta) + 2 \Delta \cos \theta + 2 i E \sin \theta &= 0 \quad (13)
\end{align*}
\]

where \( p^2 = \nabla^2 \theta - p^2 \cos \theta \sin \theta \) is the average temperature gradient and \( \alpha = (x, y, z) \).

In the following, we shall use the standard \( \theta \)-parametrization, \( G^{\theta}(r) = \tau_3 \cos[\theta(r)] + \tau_1 \sin[\theta(r)] \) in which the Usadel equation has the following form:

\[
D (\nabla^2 \theta - p^2 \cos \theta \sin \theta) + 2 \Delta \cos \theta + 2 i E \sin \theta = 0
\]

where \( p = (\nabla \phi - (2\pi/\Phi_0) A) \) is superconducting momentum (within circular cell approximation \( p = 1/r - r/r^2, r_s = \sqrt{\Phi_0/\pi H} \)). The selfconsistency equation has the form

\[
\Delta \ln \frac{T}{T_c} = \frac{T}{T_c} \sum \omega \left( \sin \theta - \Delta \frac{\omega}{\omega} \right)
\]

The diffusion coefficients in the \( \theta \)-parametrization are given by the expressions \( D_T(E, r) = D \cosh^2 (\text{Im}[\theta(E, r)]) \), \( D_L(E, r) = D \cos^2 (\text{Re}[\theta(E, r)]) \).

### IV. THERMAL TRANSPORT IN THE VORTEX STATE AT LOW FIELDS

We consider a superconductor in low magnetic field, \( B \ll B_c \). We assume an ideal triangular vortex lattice of vortex lines and study thermal transport in the direction perpendicular to the field. The local thermal conductivity is mostly determined by the diffusion constant \( D_L(E, r) \) at low energies. This quantity is very inhomogeneous in the vortex state. It has maxima at the vortex cores and rapidly drops away from the cores reflecting localization of quasiparticles in the core regions. This means that the local thermal resistance \( \propto 1/D_L(0, r) \) is maximal at the boundaries of the lattice unit cell. In such situation the temperature is mostly homogeneous within the unit cells and only changes in the boundary regions between the cells, see Fig. 1. This means that thermal transport occurs via “bottlenecks”, saddle points of \( D_L(E, r) \). Our purpose is to evaluate average thermal conductivity limited by these bottleneck regions. Since in the vicinity of bottleneck, the spectral supercurrent \( \text{Im} J \) vanishes by symmetry, the expression for local energy current simplifies and has the form:

\[
J_{\text{loc}}(r) \approx N_0 \int dE D_L(E) \nabla J(E).
\]

We start with evaluation of the diffusion constant \( D_L(E, r) \) which is determined by the real part of the Green’s function \( \theta(E, r) \). For an isolated vortex, at distances \( r \gg \xi \) from its core we can present \( \Delta \) and \( \theta \) as \( \theta(r) = \theta_0 + \tilde{\theta}(r), \Delta(r) = \Delta_0 + \tilde{\Delta}(r) \) where \( \Delta_0 \) and \( \theta_0 \) are the equilibrium values at zero magnetic field, \( \tan \theta_0 = i\Delta_0/E \) and \( \theta(r) = \tilde{\theta}(r) + i\tilde{\Delta}(r) \) is small correction. For \( E \ll \Delta_0 \), \( \theta_0 \approx \pi/2 + i E/\Delta_0 \), therefore the energy-diffusion constant in this region \( D_L(E, r) \approx D \tilde{\theta}^2(r) \).

The real and imaginary parts of \( \tilde{\theta} \) obey the following equations:

\[
\begin{align*}
D \nabla^2 \tilde{\theta}_r - D \frac{E^2 + \Delta_0^2}{E^2 - \Delta_0^2} p^2 \tilde{\theta}_r - 2 \sqrt{\Delta_0^2 - E^2} \tilde{\theta}_r &= 0, \\
D \nabla^2 \tilde{\theta}_i - D \frac{E^2 + \Delta_0^2}{E^2 - \Delta_0^2} p^2 \tilde{\theta}_i - 2 \sqrt{\Delta_0^2 - E^2} \tilde{\theta}_i &= 0,
\end{align*}
\]

where \( p = 1/r \) is the gauge-invariant phase gradient. For an isolated vortex \( \tilde{\theta}_r \) and \( \tilde{\theta}_i \) have qualitatively different behavior at large distances for \( E < \Delta_0 \): \( \tilde{\theta}_i \) decays as \( p^2 \propto 1/r^2 \) and \( \tilde{\theta}_r \) decays exponentially

\[
\tilde{\theta}_r(r) = C_r \frac{\exp(-k_\xi r)}{\sqrt{k_\xi r}},
\]

\[
k_\xi(E) = \sqrt{2 \left( \Delta_0^2 - E^2 \right)^{1/4} / \sqrt{D}}.
\]

Evaluation of numerical constant requires solution of full nonlinear problem. Numerical solution by the method described provides \( C_r \approx 4.2 \). In the vortex lattice away from the core regions \( \tilde{\theta}_r \) can be represented as a sum of contributions from individual vortices

\[
\tilde{\theta}_{\text{lat}}(r) \approx \sum_{\mathbf{R}} C_{\mathbf{R}} \left( \frac{\exp(-k_\xi |\mathbf{R}|)}{\sqrt{k_\xi |\mathbf{R}|}} \right),
\]

where \( \mathbf{R} \) are the vortex coordinates. In particular, for triangular lattice \( \mathbf{R} = (n/2 + ma + na/2, n\sqrt{3}/2) \) where \( a = \sqrt{2\Phi_0/\sqrt{3}B} \) is the lattice constant and \( m \) and \( n \) are integers.

At small field thermal transport is determined by energy flow via bottlenecks, saddlepoints of \( D_L(E, r) \) at boundaries of the lattice unit cell. Near the bottleneck point, \((x, y) = (0, 0)\), we can keep only contribution from two neighboring vortices located at \((\pm a/2, 0)\) which gives

\[
\tilde{\theta}_{\text{lat}}(r) \approx \frac{2C_r}{\sqrt{k_\xi a/2}} \exp(-k_\xi a/2) \cosh(k_\xi x) \exp(-k_\xi y^2/a)
\]
and
\[ D_L(E, r) \approx \frac{8C^2}{k_\xi a} \exp(-k_\xi y/a) \cosh^2(k_\xi x) \exp(-2k_\xi y^2/a). \] (20)

Using also quasi-equilibrium approximation for the gradient of \( f_L(E, r) \), \( \nabla_x f_L(E, r) \approx -\cosh^{-2}(E/2k_B T)(E/2k_B T^2)\nabla_E T(r) \), we obtain the heat flow near the bottleneck \( J_{th}(r) \approx -k_\xi(r)\nabla_E T(r) \), where the local thermal conductivity \( k_\xi(r) \) is given by

\[ k_\xi(r) = 16\pi^2 3C^2 k_B^2 N_0 D\exp(-k_\xi y/a) \cosh^2(k_\xi x) \exp\left(\frac{-2k_\xi y^2}{a}\right) \frac{E/2k_B T^2}{\cosh^3(E/2k_B T)}. \] (21)

At low temperatures \( T < \Delta_0 \xi/a \) the main contribution to the energy integral comes from the region \( E \lesssim T \). This allows us to neglect the energy dependence of \( k_\xi(E) \) and replace \( k_\xi(E) \rightarrow k_\xi \) = \( \sqrt{2\Delta_0/D} = 1/\xi_\Delta \). In this case, using \( \int_{-\infty}^{\infty} x^2 \cosh^{-2} x \, dx = \pi^2/6 \), we obtain

\[ k_\xi(r) = \frac{16\pi^2}{3} C^2 k_B^2 N_0 D \exp(-k_\xi y/a) \cosh^2(k_\xi x) \exp\left(\frac{-2k_\xi y^2}{a}\right) \frac{E/2k_B T^2}{\cosh^3(E/2k_B T)}. \] (21)

Near the bottleneck region the local thermal conductivity has a form \( \kappa(x, y) = k_0 F_x(x) F_y(y) \), where the function \( F_x(x) = \cosh^2(k_\xi x) \) has minimum at \( x = 0 \) and \( F_y(y) = \exp(-2k_\xi y^2/a) \) has maximum at \( y = 0 \). The total flow per unit length along the field through the bottleneck is given by

\[ I_{th} \approx k_0 F_x(x) \left[ \int_{-\infty}^{\infty} F_y(y) dy \right] \nabla_x T. \] (22)

Energy conservation requires that \( I_{th} \) has to be \( x \)-independent. Therefore, the temperature drop \( \Delta T \) across the bottleneck can be evaluated as

\[ \Delta T \approx \frac{\int_{-\infty}^{\infty} F_x^{-1}(x) dx}{k_0 \int_{-\infty}^{\infty} F_y(y) dy} I_{th} \] (23)

meaning that the total thermal conductance through this region \( K = I_{th}/\Delta T \) can be evaluated as

\[ K = \frac{k_0 \int_{-\infty}^{\infty} F_y(y) dy}{\int_{-\infty}^{\infty} F_x^{-1}(x) dx} = \frac{8\pi^{5/2}C^2}{3\sqrt{2}} k_B^2 N_0 D \frac{\exp(-k_\xi y/a)}{\sqrt{a}k_\xi}. \] (24)

Evaluating the total average energy flow density

\[ J_{th} = \frac{I_{th} + I_{th}/4}{a\sqrt{3}/2} = \frac{5}{2\sqrt{3}} K \frac{\Delta T}{a}, \] (25)

we obtain the final result for the low-field/low-temperature limit for the thermal conductivity in the vortex-lattice state

\[ \frac{\kappa}{\kappa_N} \approx \frac{10\sqrt{2}\pi^{5/2} C^2}{3\sqrt{3}} k_B^2 N_0 D \frac{\exp(-k_\xi y/a)}{\sqrt{a}k_\xi} \frac{E/2k_B T^2}{\cosh^3(E/2k_B T)}. \] (26)

where we used relations \( B_{c2} = \Phi_0 k_\xi^2 / 4\pi \) and \( k_\xi y/a = \sqrt{(8\pi/\sqrt{3})B_{c2}/B} \).

Introducing the thermal conductivity in the normal state \( \kappa_N = \frac{27}{4} k_B^2 N_0 DT \), we rewrite the final result in the form

\[ \frac{\kappa}{\kappa_N} \approx 10\sqrt{2}\pi^{5/2} C^2 \frac{\exp(-\sqrt{(8\pi/\sqrt{3})B_{c2}/B})}{(8\pi/\sqrt{3})B_{c2}/B} \frac{E/2k_B T^2}{\cosh^3(E/2k_B T)}. \] (27)

This parameter-free analytical result provides theoretical lower limit for the heat transport in the mixed state in an isotropic dirty s-wave superconductor at low field. The constant 3.81 in the exponent is significantly higher than the constant 1.486 which we evaluated for the clean case [33] meaning that the scattering drastically suppresses the quasiparticle thermal conductivity at low fields.

V. DISCUSSION OF EXPERIMENT

Rather limited set of experimental data is available on electronic contribution to the thermal conductivity at low temperatures and low magnetic fields, since phonon contribution should be accurately subtracted in this regime. To our knowledge, the only experimental data on electronic thermal conductivity which clearly demonstrate low-field/low-temperature behavior expected for an s-wave superconductor are reported for Nb in Refs. [1] and [2] and for V$_3$Si in Ref. [12]. In these experiments the samples were in the clean limit. In Ref. [2] the electronic thermal conductivity was fitted by the expression
\( \kappa \propto \exp(-\beta \sqrt{B_{c2}/B}) \) for values of \( B \) up to about \( B_{c2}/3 \), with \( \beta = 1.66 \). This number is slightly higher than our estimate of \( \beta = 1.486 \) in Eq. (3). This difference can be explained by the influence of impurity scattering. The shape of field dependence of thermal conductivity for \( V_3Si \) reported in Ref. 12 is in qualitative agreement with the predicted exponential dependence, however the quantitative analysis was not made and the value of \( \beta \) was not explicitly extracted. In a field \( B = B_{c2}/20 \) the value of \( \kappa \approx 2.5 \cdot 10^{-5}k_N \) provided in Ref. 12 exceeds our dirty-limit estimate, Eq. (27), by about two orders of magnitude. This discrepancy can be naturally attributed to the fact that measured \( V_3Si \) samples were in the clean limit. The magnitude of thermal conductivity at not very small magnetic fields is dominated by phonons and separating electronic contribution is a challenging task.

VI. EXTENSION TO A TWO-BAND SUPERCONDUCTOR

Here we extend the above formalism to a two-band superconductor. At present, the most established example of such system is MgB\(_2\), which is characterized by two electronic bands: \( \pi \)-band and \( \sigma \)-band, see, e.g., recent review Ref. 28. The quasi-two-dimensional \( \sigma \)-band is characterized by stronger superconductivity than the three-dimensional \( \pi \)-band. Heat transport in a two-band superconductor was studied theoretically in Ref. 9 assuming clean limit conditions and using the averaging over unit cell method.\(^\text{29}\). In the dirty limit, theory of density of states in the mixed state and the upper critical field for MgB\(_2\) was developed in Refs. 30 and 31. Below we extend the calculations of the heat transport presented above, to the case of a diffusive two-band superconductor, taking MgB\(_2\) as an example.

\[ B_{c2}(0) = a_{c2}B_{c2}'(0), \quad a_{c2} = \exp \left( -\frac{W_1 + W_2 - \ln r_x}{2} + \frac{(W_1 + W_2 - \ln r_x)^2}{4} + W_1 \ln r_x \right), \] (33)

where indices 1 and 2 correspond to the \( \sigma \) and \( \pi \) bands,

\[ W_{1,2} = \frac{\pm(A_{11} - A_{22})/2 + \sqrt{(A_{11} - A_{22})^2/4 + A_{12}A_{21}}}{A_{11}A_{22} - A_{12}A_{21}}, \]

\[ \Lambda_{\alpha\beta} \] is the coupling-constant matrix, \( r_x = D_\sigma/D_\pi, \) \( D_{\sigma,\pi} \) are the diffusion constants in \( \sigma \) and \( \pi \) bands,

\[ B_{c2}^\omega(0) = \frac{\Delta_{BCS}\Phi_0}{2\pi D_\sigma} \] (34)

In the presence of two electronic bands with different energy gaps, low-temperature behavior of thermal transport is determined by the band with lower gap (\( \pi \)-band). Still, the generalization from single-band to two-band case involves not simply renormalization of the energy gap, but also correction to the asymptotic behavior of Green’s function in the \( \pi \)-band and to the upper critical field.

The Usadel equation for the \( \pi \)-band reads

\[ D_\pi \left( \nabla^2 \theta_\pi - p^2 \cos \theta_\pi \sin \theta_\pi \right) + 2\Delta_\pi \cos \theta_\pi + 2iE \sin \theta_\pi = 0, \] (28)

where \( D_\pi \) and \( \Delta_\pi \) are the diffusion constant and gap for the \( \pi \)-band. Similar to a single-band case, asymptotics of the Green’s function in the \( \pi \)-band at large distance from the vortex core is given by

\[ \tilde{\theta}_{\pi,x}(r) = C_\pi \exp(-k_{\pi}r) \] (29)

with

\[ k_{\pi}(E) = \sqrt{2} \left( \Delta_{\pi0}^2 - E^2 \right)^{1/4} / \sqrt{D_\pi}. \]

The main exponential dependence of the zero-energy Green’s function at the bottleneck point is \( \propto \exp(-k_{\pi0}a) \), where \( k_{\pi0} = k_{\pi}(0) = \sqrt{2\Delta_{\pi0}/D_\pi} \), which gives

\[ k_{\pi0}a = \sqrt{B_\pi/B}, \]

\[ B_\pi = \frac{2\Delta_{\pi0}}{D_\pi} \sqrt{3}. \] (31)

Therefore, the magnetic field dependence of thermal conductivity is determined by the field scale \( B_\pi \) and can be presented in the form similar to Eq. (27).

\[ \frac{\kappa}{k_{\pi0}} \approx 10 \frac{\tau_\pi}{3} C_\pi^2 \exp \left( -\frac{\sqrt{B_{c2}/B}}{B_\pi} \right), \] (32)

where \( \kappa_{\pi0} = \frac{\pi}{\sqrt{2}B_{c2}} \Delta_{\pi} T \) is the partial \( \pi \)-band contribution to the normal-state thermal conductivity.

To proceed further, we have to find relation between the \( \pi \)-band field scale \( B_\pi \) and the upper critical field \( B_{c2} \) for a two-band superconductor. The upper critical field at low temperatures, \( T \ll T_c \), is given by

\[ B_{\pi}(0) = \frac{\Delta_0}{\pi D_\pi}. \] (34)
is the single-band upper critical field for the \( \sigma \)-band, and 
\[ \Delta_{BCS} = \pi e^{-\gamma} T_c \approx 1.764 T_c \]  
\( \gamma_E \approx 0.5772 \) is the Euler constant).

To make estimates for MgB\(_2\), we use the following 
coupling matrix elements,\(^{20,31} \) \( \Lambda_{11} \approx 0.81, \Lambda_{21} \approx 0.278, \) \( \Lambda_{12} \approx 0.115, \Lambda_{22} \approx 0.091, \) which gives \( W_1 \approx 0.088 \) and \( W_2 \approx 2.56. \) With such coupling matrix the two-band BCS model gives\(^{30} \) \( \Delta_{\pi 0} \approx 0.3 \Delta_{\sigma 0} \approx 0.177 \pi T_c. \) Since the parameter \( W_1 \) is small, typically the inequality 
\[ W_1 \ln r_x \ll (W_2 - \ln r_x)^2 / 4 \]  is valid. In this case one can expand Eq. \( (33) \) with respect to \( W_1 \) and obtain simple result
\[ a_{c2} \approx 1 + \frac{W_1 \ln r_x}{W_2 - \ln r_x}, \]
meaning that the upper critical field is close to \( B_{c2}(0) \) and is mostly determined by the coherence length of the \( \sigma \)-band. Using Eqs. \( (31), (33), \) and \( (34) \), we obtain the relation between \( B_\pi \) and \( B_{c2}(0) \)
\[ B_\pi = \frac{8 \pi}{\sqrt{3}a_{c2}} \frac{\Delta_{\sigma 0} D_\pi}{\Delta_{BCS} D_\pi} B_{c2}(0), \]
which presents the main result of this section. This scale 
has to be compared with the scale \((8\pi/\sqrt{3})B_{c2}(0)\) for the single-band case.

In order to determine the pre-exponential factor \( C_\pi \) in 
eqs. \( (29) \) and \( (32) \), we have to calculate the Green’s function \( \theta_{\pi \sigma}(r) \), which requires solution of the full two-band Usadel problem, as described in Ref. \( 30. \) An important parameter is the ratio of diffusion coefficients in 
two bands \( r_x \). For illustration, we consider two 
cases here: \( r_x = 1 \) and \( 0.2, \) for which the ratios of the coherence lengths in the two bands are 
\( \xi_\pi / \xi_\sigma = \sqrt{D_\pi / D_\sigma} \left( \Delta_{\pi 0} / \Delta_{\sigma 0} \right) \approx 1.83 \) and \( 4.1. \) For these two cases we compute 
\( B_\pi \approx 0.32(8\pi/\sqrt{3})B_{c2}(0), C_\pi \approx 3.6 \) for 
\( r_x = 1 \) and \( B_\pi \approx 0.065(8\pi/\sqrt{3})B_{c2}(0), C_\pi \approx 2.9 \) for 
\( r_x = 0.2. \) This gives
\[ \frac{\kappa}{\kappa_{\pi N}} \approx 130 \left( \frac{B}{B_{c2}(0)} \right)^{1/4} \exp \left( -2.15 \sqrt{\frac{B_{c2}(0)}{B}} \right), r_x = 1 \]
\[ \approx 130 \left( \frac{B}{B_{c2}(0)} \right)^{1/4} \exp \left( -0.98 \sqrt{\frac{B_{c2}(0)}{B}} \right), r_x = 0.2 \]
We can see that the field scale in the two-band case is 
strongly reduced in comparison with the single-band case leading to 
large enhancement of the thermal conductivity at low fields. This reduction is mostly caused by the 
smaller energy gap in the \( \pi \) band. Another factor which 
may contribute is possible large value of the diffusion 
constant \( D_\pi. \) The smaller field scale caused by the larger 
coherence length in the \( \pi \)-band is an established feature

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