False-Name Manipulation in Weighted Voting Games is Hard for Probabilistic Polynomial Time

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Abstract

False-name manipulation refers to the question of whether a player in a weighted voting game can increase her power by splitting into several players and distributing her weight among these false identities. Analogously to this splitting problem, the beneficial merging problem asks whether a coalition of players can increase their power in a weighted voting game by merging their weights. Aziz et al. [ABEPT11] analyze the problem of whether merging or splitting players in weighted voting games is beneficial in terms of the Shapley–Shubik and the normalized Banzhaf index, and so do Rey and Rothe [RR10] for the probabilistic Banzhaf index. All these results provide merely NP-hardness lower bounds for these problems, leaving the question about their exact complexity open. For the Shapley–Shubik and the probabilistic Banzhaf index, we raise these lower bounds to hardness for PP, “probabilistic polynomial time,” and provide matching upper bounds for beneficial merging and, whenever the number of false identities is fixed, also for beneficial splitting, thus resolving previous conjectures in the affirmative. It follows from our results that beneficial merging and splitting for these two power indices cannot be solved in NP, unless the polynomial hierarchy collapses, which is considered highly unlikely.

1 Introduction

Weighted voting games are an important class of succinctly representable, simple games. They can be used to model cooperation among players in scenarios where each player is assigned a weight, and a coalition of players wins if and only if their joint weight meets or exceeds a given quota. Typical real-world applications of weighted voting games include decision-making in legislative bodies (e.g., parliamentary voting) and shareholder voting (see the book by Chalkiadakis et al. [CEW11] for further concrete applications and literature pointers). In particular, the algorithmic and complexity-theoretic properties of problems related to weighted voting have been studied in depth, see, e.g., the work of Elkind et al. [ECJ08, EGGW09], Bachrach et al. [BEM+09], Zuckerman et al. [ZFB08], and [CEW11] for an overview.
Bachrach and Elkind [BE08] were the first to study false-name manipulation in weighted voting games: Is it possible for a player to increase her power by splitting into several players and distributing her weight among these false identities? Relatedly, is it possible for two or more players to increase their power in a weighted voting game by merging their weights? The most prominent measures of a player’s power, or influence, in a weighted voting game are the Shapley–Shubik and Banzhaf power indices. Merging and extending the results of [BE08] and [AP09], Aziz et al. [ABEP11] in particular study the problem of whether merging or splitting players in weighted voting games is beneficial in terms of the Shapley–Shubik index [Sha53, SS54] and the normalized Banzhaf index [Ban65] (see Section 2 for formal definitions). Rey and Rothe [RR10] extend this study for the probabilistic Banzhaf index proposed by Dubey and Shapley [DS79]. All these results, however, provide merely NP-hardness lower bounds. Aziz et al. [ABEP11, Remark 13 on p. 72] note that “it is quite possible that our problems are not in NP” (and thus are not NP-complete). Faliszewski and Hemaspaandra [FH09] provide the best known upper bound for the beneficial merging problem with respect to the Shapley–Shubik index: It is contained in the class PP, “probabilistic polynomial time,” which is considered to be by far a larger class than NP, and they conjecture that this problem is PP-complete. Rey and Rothe [RR10] observe that the same arguments give a PP upper bound also for beneficial merging in terms of the probabilistic Banzhaf index, and they conjecture PP-completeness as well.

We resolve these conjectures in the affirmative by proving that beneficial merging and splitting (for any fixed number of false identities) are PP-complete problems both for the Shapley–Shubik and the probabilistic Banzhaf index. Beneficial splitting in general (i.e., for an unbounded number of false identities) belongs to NP^{PP} and is PP-hard for the same two indices. Thus, none of these six problems can be in NP, unless the polynomial hierarchy collapses to its first level, which is considered highly unlikely.

2 Preliminaries

We will need the following concepts from cooperative game theory (see, e.g., the textbook by Chalkiadakis et al. [CEW11]). A coalitional game with transferable utilities, $G = (N, v)$, consists of a set $N = \{1, \ldots, n\}$ of players (or, synonymously, agents) and a coalitional function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, where $2^N$ denotes the power set of $N$. $G$ is monotonic if $v(B) \leq v(C)$ whenever $B \subseteq C$ for coalitions $B, C \subseteq N$, and it is simple if it is monotonic and $v : 2^N \to \{0, 1\}$, that is, $v$ maps each coalition $C \subseteq N$ to a value that indicates whether $C$ wins (i.e., $v(C) = 1$) or loses (i.e., $v(C) = 0$), where we require that the grand coalition $N$ is always winning. The probabilistic Banzhaf power index of a player $i \in N$ in a simple game $G$ (see [DS79]) is defined by

$$Banzhaf(G, i) = \frac{1}{2^{n-1}} \sum_{C \subseteq N \setminus \{i\}} (v(C \cup \{i\}) - v(C)).$$

Intuitively, this index measures the power of player $i$ in terms of the probability such that $i$ turns a losing coalition $C \subseteq N \setminus \{i\}$ into a winning coalition by joining it, and therefore is pivotal for the normalized Banzhaf index.

1They also note that the same arguments cannot be transferred immediately to the corresponding problem for the normalized Banzhaf index.
success of $C$. (For comparison, the normalized Banzhaf index of $i$ in $\mathcal{G}$ defined by Banzhaf [Ban65], who rediscovered a notion originally introduced by Penrose [Pen46], is obtained by dividing the raw Banzhaf index of $i$ in $\mathcal{G}$, which is the term $\sum_{C \subseteq N, \{i\}} (v(C \cup \{i\}) - v(C))$ in (1), not by $2^{n-1}$, but by the sum of the raw Banzhaf indices of all players in $\mathcal{G}$; see [DS79, FM05, RR10] for a discussion of the differences between these two power indices.)

Unlike the Banzhaf indices, the Shapley–Shubik index of $i$ in $\mathcal{G}$ takes the order into account in which players enter coalitions and is defined by

$$\text{Shapley–Shubik}(\mathcal{G}, i) = \frac{1}{n!} \sum_{C \subseteq N \setminus \{i\}} ||C||! \cdot (n - ||C||)! \cdot (v(C \cup \{i\}) - v(C)).$$

Since the number of coalitions is exponential in the number of players, specifying coalitional games by listing all values of their coalitional function would require exponential space. For algorithmic purposes, however, it is important that these games can be represented succinctly. Simple games can be compactly represented by weighted voting games. A weighted voting game (WVG) $\mathcal{G} = (w_1, \ldots, w_n; q)$ consists of nonnegative integer weights $w_i$, $1 \leq i \leq n$, and a quota $q$, where $w_i$ is the $i$th player’s weight. For each coalition $C \subseteq N$, letting $w(C)$ denote $\sum_{i \in C} w_i$, $C$ wins if $w(C) \geq q$, and it loses otherwise. Requiring the quota to satisfy $0 < q \leq w(N)$ ensures that the empty coalition loses and the grand coalition wins. Weighted voting games have been intensely studied from a computational complexity point of view (see, e.g., [ECJ08, EGGW09, BEM09, ZFBE08] and [CEW11] Chapter 4 for an overview).

Aziz et al. [ABEP11] introduce the merging and splitting operations for WVGs. We use the following notation. Given a WVG $\mathcal{G} = (w_1, \ldots, w_n; q)$ and a nonempty coalition $S \subseteq \{1, \ldots, n\}$, let $\mathcal{G}_{S,i} = (w(S), w_{j_1}, \ldots, w_{j_{n-||S||}}; q)$ with $\{j_1, \ldots, j_{n-||S||}\} = N \setminus S$ denote the new WVG in which the players in $S$ have been merged into one new player of weight $w(S)$. Similarly, given a WVG $\mathcal{G} = (w_1, \ldots, w_n; q)$, a player $i$, and an integer $m \geq 2$, define the set of WVGs

$$\mathcal{G}_{i:m} = (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_m, w_{i+1}, \ldots, w_n; q)$$

in which $i$ with weight $w_i$ is split into $m$ new players $i_1, \ldots, i_m$ with weights $w_{i_1}, \ldots, w_{i_m}$ such that $\sum_{j=1}^m w_{i_j} = w_i$. (Note that there is a set of such WVGs $\mathcal{G}_{i:m}$, since there might be several possibilities of distributing $i$’s weight $w_i$ to the new players $i_1, \ldots, i_m$ satisfying $\sum_{j=1}^m w_{i_j} = w_i$.) For a power index $\text{PI}$, the beneficial merging and splitting problems are defined as follows.

| PI-BENEFICIALMERGE |
|---------------------|
| **Given:** A WVG $\mathcal{G} = (w_1, \ldots, w_n; q)$ and a nonempty coalition $S \subseteq \{1, \ldots, n\}$. |
| **Question:** Is it true that $\text{PI}(\mathcal{G}_{S,i}, 1) > \sum_{i \in S} \text{PI}(\mathcal{G}, i)$? |

We distinguish between two splitting problems: In the first problem, the number $m$ of false identities some player splits into is not part of the given problem instance (rather, the problem itself is parameterized by $m$), whereas $m$ is given in the instance for the second problem. (This distinction wouldn’t make sense for beneficial merging.)

\footnote{We omit the empty coalition, since this would slightly change the idea of the problem.}
The goal of this paper is to classify these problems in terms of their complexity for both the Shapley–Shubik and the probabilistic Banzhaf index. We assume that the reader is familiar with the basic complexity-theoretic concepts such as the complexity classes P and NP, the polynomial-time many-one reducibility, denoted by \( \leq_p \), and the notions of hardness and completeness with respect to \( \leq_p \) (see, e.g., the textbook by Papadimitriou [Pap95]). Valiant [Val79] introduced \#P as the class of functions that give the number of solutions of the instances of NP problems. For a decision problem \( A \in \text{NP} \), we denote this function by \#A. For example, if SAT is the satisfiability problem from propositional logic, then \#SAT denotes the function mapping any boolean formula \( \phi \) to the number of truth assignments satisfying \( \phi \). There are various notions of reducibility between functional problems in \#P (see [FH09] for an overview, literature pointers, and discussion). Here, we need only the most restrictive one: We say a function \( f \) parsimoniously reduces to a function \( g \) if there exists a polynomial-time computable function \( h \) such that for each input \( x \), \( f(x) = g(h(x)) \). That is, for functional problems \( f,g \in \text{#P} \), a parsimonious reduction \( h \) from \( f \) to \( g \) transfers each instance \( x \) of \( f \) into an instance \( h(x) \) of \( g \) such that \( f(x) \) and \( g(h(x)) \) have the same number of solutions. We say that \( g \) is \#P-parsimonious-hard if every \( f \in \text{#P} \) parsimoniously reduces to \( g \). We say that \( g \) is \#P-parsimonious-complete if \( g \) is in \#P and \#P-parsimonious-hard. It is known that, given a WVG \( \mathcal{G} \) and a player \( i \), computing the raw Banzhaf index is \#P-parsimonious-complete [PK90], whereas computing the raw Shapley–Shubik index is not [FH09], although it, of course, is in \#P as well.

Gill [Gil77] introduced the class PP (“probabilistic polynomial time”) that contains all decision problems \( X \) for which there exist a function \( f \in \text{#P} \) and a polynomial \( p \) such that for all instances \( x \), \( x \in X \) if and only if \( f(x) \geq 2^{p(|x|)} \). It is easy to see that \( \text{NP} \subseteq \text{PP} \); in fact, PP is considered to be by far a larger class than NP, due to Toda’s theorem [Tod91]: PP is at least as hard (in terms of polynomial-time Turing reductions) as any problem in the polynomial hierarchy (i.e., \( \text{PH} \subseteq \text{PP} \)). \( \text{NP}^{\text{PP}} \), the second level of Wagner’s counting hierarchy [Wag86], is the class of problems solvable by an NP machine with access to a PP oracle; Mundhenk et al. [MGLA00] identified \( \text{NP}^{\text{PP}} \)-complete problems related to finite-horizon Markov decision processes.

3 Beneficial Merging and Splitting is PP-Hard

In this section we prove that beneficial merging and splitting is PP-hard, and we provide matching upper bounds for beneficial merging and splitting (for any fixed number of false identities) both for
the Shapley–Shubik and the probabilistic Banzhaf index. We start with the latter.

3.1 The Probabilistic Banzhaf Power Index

We will use the following result due to Faliszewski and Hemaspaandra [FH09 Lemma 2.3].

**Lemma 3.1 (Faliszewski and Hemaspaandra [FH09])** Let $F$ be a $\#P$-parsimonious-complete function. The problem $\text{COMPARE-}F = \{(x,y) \mid F(x) > F(y)\}$ is $\#P$-complete.

The well-known NP-complete problem $\text{SUBSETSUM}$ (which is a special variant of the Knapsack problem) asks, given a sequence $(a_1, \ldots, a_n)$ of positive integers and a positive integer $q$, do there exist $x_1, \ldots, x_n \in \{0,1\}$ such that $\sum_{i=1}^n x_i a_i = q$? It is known that $\#\text{SUBSETSUM}$ is $\#P$-parsimonious-complete (see, e.g., the textbook by [Pap95] for parsimonious reductions from $\#3\text{-SAT}$ via $\#\text{EXACTCOVERBY3-SETS}$ to $\#\text{SUBSETSUM}$). Hence, by Lemma 3.1 we have the following.

**Corollary 3.2** $\text{COMPARE-}\#\text{SUBSETSUM}$ is $\#P$-complete.

Our goal is to $\leq^p_\#$-reduce $\text{COMPARE-}\#\text{SUBSETSUM}$ to Banzhaf-BeneficialMerge. However, to make this reduction work, it will be useful to consider two restricted variants of $\text{COMPARE-}\#\text{SUBSETSUM}$, which we denote by $\text{COMPARE-}\#\text{SUBSETSUM-}R$ and $\text{COMPARE-}\#\text{SUBSETSUM-}RR$, show their PP-hardness, and then reduce $\text{COMPARE-}\#\text{SUBSETSUM-}RR$ to the problem Banzhaf-BeneficialMerge. This will be done in Lemmas 3.3 and 3.4 and in Theorem 3.5. In all restricted variants of $\text{COMPARE-}\#\text{SUBSETSUM}$ we may assume, without loss of generality, that the target value $q$ in the related $\#\text{SUBSETSUM}$ instances $((a_1, \ldots, a_n), q)$ satisfies $1 \leq q \leq \alpha - 1$, where $\alpha = \sum_{i=1}^n a_i$, such that $\#\text{SUBSETSUM}$ remains $\#P$-parsimonious-complete.

| **COMPARE-\#SUBSETSUM-\text{R}** |
|---|
| **Given:** | A sequence $A = (a_1, \ldots, a_n)$ of positive integers and two positive integers $q_1$ and $q_2$ with $1 \leq q_1, q_2 \leq \alpha - 1$, where $\alpha = \sum_{i=1}^n a_i$. |
| **Question:** | Is the number of subsequences of $A$ summing up to $q_1$ greater than the number of subsequences of $A$ summing up to $q_2$, that is, does it hold that $\#\text{SUBSETSUM}((a_1, \ldots, a_n), q_1) > \#\text{SUBSETSUM}((a_1, \ldots, a_n), q_2)$? |

**Lemma 3.3** $\text{COMPARE-}\#\text{SUBSETSUM} \leq^p_\# \text{COMPARE-}\#\text{SUBSETSUM-\text{R}}$.

**Proof.** Given an instance $(X, Y)$ of $\text{COMPARE-}\#\text{SUBSETSUM}$, $X = ((x_1, \ldots, x_m), q_x)$ and $Y = ((y_1, \ldots, y_n), q_y)$, construct a $\text{COMPARE-}\#\text{SUBSETSUM-}R$ instance $(A, q_1, q_2)$ as follows. Let $\alpha = \sum_{i=1}^m x_i$ and define $A = (x_1, \ldots, x_m, 2\alpha y_1, \ldots, 2\alpha y_n)$, $q_1 = q_x$, and $q_2 = 2\alpha q_y$. This construction can obviously be achieved in polynomial time.

It holds that integers from $A$ can only sum up to $q_x < \alpha - 1$ if they do not contain multiples of $2\alpha$, thus $\#\text{SUBSETSUM}(A, q_1) = \#\text{SUBSETSUM}((x_1, \ldots, x_m), q_x)$. On the other hand, $q_2$ can only be obtained by multiples of $2\alpha$, since $\sum_{i=1}^m x_i = \alpha$ is too small. Thus, it holds that $\#\text{SUBSETSUM}(A, q_2) =$
#SubsetSum \((y_1, \ldots, y_n), q_s\). It follows that \((X, Y)\) is in Compare-#SubsetSum if and only if \((A, q_1, q_2)\) is in Compare-#SubsetSum-R.

In order to perform the next step, we need to ensure that all integers in a Compare-#SubsetSum-R instance are divisible by 8. This can easily be achieved, since for a given instance \(((a_1, \ldots, a_n), q_1, q_2)\), we can multiply each integer by 8, obtaining \(((8a_1, \ldots, 8a_n), 8q_1, 8q_2)\) without changing the number of solutions for both related SubsetSum instances. Thus, from now on, without loss of generality, we assume that for a given Compare-#SubsetSum-R instance \(((a_1, \ldots, a_n), q_1, q_2)\), it holds that \(a_i, q_j \equiv 0 \mod 8\) for \(1 \leq i \leq n\) and \(j \in \{1, 2\}\).

Now, we consider our even more restricted variant of this problem.

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**Lemma 3.4** Compare-#SubsetSum-R \(\leq^P\) Compare-#SubsetSum-RR.

**Proof.** Given an instance \((A, q_1, q_2)\) of Compare-#SubsetSum-R, where we assume that \(A = (a_1, \ldots, a_n), q_1, q_2\) satisfy \(a_i, q_j \equiv 0 \mod 8\) for \(1 \leq i \leq n\) and \(j \in \{1, 2\}\), we construct an instance \(B\) of Compare-#SubsetSum-RR as follows. (This reduction is inspired by the standard reduction from SubsetSum to Partition due to Karp [Kar72].)

Letting \(\alpha = \sum_{i=1}^{n} a_i\), define

\[
B = (a_1, \ldots, a_n, 2\alpha - q_1, 2\alpha + 1 - q_2, 2\alpha + 3 + q_1 + q_2, 3\alpha).
\]

This instance can obviously be constructed in polynomial time. Observe that

\[
T = \left(\sum_{i=1}^{n} a_i\right) + (2\alpha - q_1) + (2\alpha + 1 - q_2) + (2\alpha + 3 + q_1 + q_2) + 3\alpha = 10\alpha + 4,
\]

and therefore, \((T/2) - 2 = 5\alpha\) and \((T/2) - 1 = 5\alpha + 1\).

We show that \((A, q_1, q_2)\) is in Compare-#SubsetSum-R if and only if the constructed instance \(B\) is in Compare-#SubsetSum-RR.

First, we examine which subsequences of \(B\) sum up to \(5\alpha\). Consider the following cases: If \(3\alpha\) is added, \(2\alpha + 3 + q_1 + q_2\) cannot be added, as it would be too large. Also, \(2\alpha + 1 - q_2\) cannot be added, leading to an odd sum. So, \(2\alpha - q_1\) has to be added, as the remaining \(\alpha\) are too small. Since \(3\alpha + 2\alpha - q_1 = 5\alpha - q_1\), \(5\alpha\) can be achieved by adding some \(a_i\)'s if and only if there exists a subset \(A' \subseteq \{1, \ldots, n\}\) such that \(\sum_{i \in A'} a_i = q_1\) (i.e., \(A'\) is a solution of the SubsetSum instance \((A, q_1)\)). If \(3\alpha\) is not added, but \(2\alpha + 3 + q_1 + q_2\), an even number can only be achieved by adding \(2\alpha + 1 - q_1\) such that \(\alpha - 4 - q_1\) remain. So, \(2\alpha - q_1\) is too large, while no subsequence of \(A\) sums up to \(\alpha - 4 - q_1\), because of the assumption of divisibility by 8. If neither \(3\alpha\) nor \(2\alpha + 3 + q_1 + q_2\)}
are added, the remaining 5\(\alpha + 1 - q_1 - q_2\) are too small. Thus, the only possibility to obtain 5\(\alpha\) is to find a subsequence of \(A\) adding up to \(q_1\). Thus, \(#\text{SUBSETSUM}(A,q_1) = #\text{SUBSETSUM}(B,5\alpha)\).

Second, for similar reasons, a sum of 5\(\alpha + 1\) can only be achieved by adding \(3\alpha + (2\alpha + 1 - q_2)\) and a term \(\sum_{i \in A'} a_i\), where \(A'\) is a subset of \(\{1,\ldots,n\}\) such that \(\sum_{i \in A'} a_i = q_2\). Hence, \(#\text{SUBSETSUM}(A,q_2) = #\text{SUBSETSUM}(B,5\alpha + 1)\).

Thus, the relation \(#\text{SUBSETSUM}(A,q_1) > #\text{SUBSETSUM}(A,q_2)\) holds if and only if \(#\text{SUBSETSUM}(B,5\alpha) > #\text{SUBSETSUM}(B,5\alpha + 1)\), which completes the proof.

We now are ready to prove the main theorem of this section.

**Theorem 3.5** Banzhaf-BeneficialMerge is PP-complete, even if only three players of equal weight merge.

**Proof.** Membership of Banzhaf-BeneficialMerge in PP has already been observed in [RR10, Theorem 3]. It follows from the fact that the raw Banzhaf index is in \#P and that \#P is closed under addition and multiplication by two\(^3\) and, furthermore, since comparing the values of two \#P functions on two (possibly different) inputs reduces to a PP-complete problem. This technique (which was proposed by Faliszewski and Hemaspaandra [FH9] and applies their Lemma 2.10) works, since PP is closed under \(\leq^p_m\)-reducibility.

We show PP-hardness of Banzhaf-BeneficialMerge by means of a \(\leq^p_m\)-reduction from compare-\#SubsetSum-RR, which is PP-hard by Corollary [3.2] via Lemmas [3.3 and 3.4]. Our construction is inspired by the NP-hardness results by Aziz et al. [AP09] and Rey and Rothe [RR10].

Given an instance \(A = (a_1,\ldots,a_n)\) of compare-\#SubsetSum-RR, construct the following instance for Banzhaf-BeneficialMerge. Let \(\alpha = \sum_{i=1}^{n} a_i\). Define the WVG

\[ \mathcal{G} = (2a_1,\ldots,2a_n,1,1,1; \alpha), \]

and let the merging coalition be \(S = \{n+2,n+3,n+4\}\).

Letting \(N = \{1,\ldots,n\}\), it holds that

\[
\text{Banzhaf}(\mathcal{G},n+2) = \frac{1}{2^{n+3}} \sum_{C \subseteq \{1,\ldots,n+1,n+3,n+4\}} \left( \sum_{i \in C} w_i = \alpha - 1 \right) \left( \sum_{i \in A'} 2a_i = \alpha - 1 \right)
\]

\[
= \frac{1}{2^{n+3}} \left( \sum_{A' \subseteq N} 2a_i = \alpha - 1 \right) + 3 \cdot \left( \sum_{A' \subseteq N} 1 + \sum_{i \in A'} 2a_i = \alpha - 1 \right) \quad (2)
\]

\[
+ 3 \cdot \left( \sum_{A' \subseteq N} 2 + \sum_{i \in A'} 2a_i = \alpha - 1 \right) + 3 \cdot \left( \sum_{A' \subseteq N} 3 + \sum_{i \in A'} 2a_i = \alpha - 1 \right) \quad (3)
\]

\[
= \frac{1}{2^{n+3}} \left( 3 \cdot \left( \sum_{A' \subseteq N} 2a_i = \alpha - 2 \right) + \left( A' \subseteq N \left| \sum_{i \in A'} 2a_i = \alpha - 4 \right) \right) \right). \]

---

\(^3\)Again, note that this idea cannot be transferred straightforwardly to the normalized Banzhaf index, since in different games the indices have possibly different denominators, not only different by a factor of some power of two, as is the case for the probabilistic Banzhaf index.
since the $2a_i$’s can only add up to an even number. The first of the four sets in (2) and (3) refers to those coalitions that do not contain any of the players $n + 1$, $n + 3$, and $n + 4$; the second, third, and fourth set in (2) and (3) refers to those coalitions containing either one, two, or three of them, respectively. Since they all have the same weight, players $n + 3$ and $n + 4$ have the same probabilistic Banzhaf index as player $n + 2$.

Furthermore, the new game after merging is $\mathcal{G}_{\{n+2,n+3,n+4\}} = (3, 2a_1, \ldots, 2a_n, 1; \alpha)$ and, similarly to above, the Banzhaf index of the first player is calculated as follows:

$$
\text{Banzhaf}(\mathcal{G}_{\{n+2,n+3,n+4\}}, 1) = \frac{1}{2^{n+1}} \left\| \left\{ A' \subseteq \{2, \ldots, n+2\} \mid \sum_{i \in A'} 2a_i = \alpha - 1 \right\} \right\|
$$

$$
= \frac{1}{2^{n+1}} \left( \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right) + 1.
$$

Altogether, it holds that

$$
\text{Banzhaf}(\mathcal{G}_{\{n+2,n+3,n+4\}}, 1) - \sum_{i \in \{n+2,n+3,n+4\}} \text{Banzhaf}(\mathcal{G}, i)
$$

$$
= \frac{1}{2^{n+1}} \left( 2 \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right)
$$

$$
- \frac{3}{2^{n+3}} \left( 3 \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right)
$$

$$
= \left( \frac{1}{2^{n+1}} \cdot 2 - \frac{3}{2^{n+3}} \cdot 3 \right) \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\|
$$

$$
+ \left( \frac{1}{2^{n+1}} - \frac{3}{2^{n+3}} \right) \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\|
$$

$$
= -\frac{1}{2^{n+3}} \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} a_i = \alpha - 2 \right\} \right\| + \frac{1}{2^{n+3}} \left\| \left\{ A' \subseteq N \mid \sum_{i \in A'} a_i = \alpha - 2 \right\} \right\|
$$

which is greater than zero if and only if $||\{A' \subseteq N \mid \sum_{i \in A'} a_i = (\alpha/2) - 2\}||$ is greater than $||\{A' \subseteq N \mid \sum_{i \in A'} a_i = (\alpha/2) - 1\}||$, which in turn is the case if and only if the original instance $A$ is in \textsc{Compare-#SubsetSum-RR}. \hfill \Box

It is known (see [RR10]) that both the beneficial merging problem for a coalition $S$ of size 2 and the beneficial splitting problem for $m = 2$ false identities can trivially be decided in polynomial time.
for the probabilistic Banzhaf index, since the sum of power (in terms of this index) of two players is always equal to the power of the player that is obtained by merging these two players. Analogously to the proof of Theorem 3.5, it can be shown that the beneficial splitting problem for a fixed number of at least three false identities is PP-complete.

Since we allow players with zero weight, we need another simple fact for the analysis of the beneficial splitting problem to be used in the proofs of Theorems 3.7 and 3.10.

**Lemma 3.6** For both the probabilistic Banzhaf index and the Shapley–Shubik index, given a weighted voting game, adding a player with weight zero does not change the original players’ power indices, and the new player’s power index is zero.

The proof of Lemma 3.6 is straightforward and therefore omitted. We are now ready to prove Theorem 3.7 which states that Banzhaf-$m$-BENEFICIAL_SPLIT is PP-complete for each $m \geq 3$.

**Theorem 3.7** Banzhaf-$m$-BENEFICIAL_SPLIT is PP-complete for each $m \geq 3$.

**Proof.** As already mentioned in the proof of Theorem 3.5, comparing the values of two #P functions on two (possibly different) inputs reduces to a PP-complete problem and thus is in PP. In particular, this is true for the problem of comparing (sums of) probabilistic Banzhaf indices in possibly different WVGs, such as testing whether the sum of the new players’ raw Banzhaf indices is greater than $2^{m-1}$ times the raw Banzhaf index of the original player $i$ (which is equivalent to “$\sum_{j=1}^{m} \text{Banzhaf}(\mathcal{G}_{i,m}, i_j) > \text{Banzhaf}(\mathcal{G}, i)$” from the definition of Banzhaf-$m$-BENEFICIAL_SPLIT), where $i$ is split into $m$ new players $i_1, \ldots, i_m$.

The main difference between the beneficial merging and splitting problems is that before comparing the two #P functions associated with beneficial splitting, one has to choose a right way of distributing $i$’s weight among the $m$ false identities of $i$. Since $m$ is fixed, there are only polynomially many (specifically, some number in $O(w_i^m)$) ways of doing so, i.e., of finding nonnegative integers $w_{i_1}, \ldots, w_{i_m}$ satisfying $\sum_{j=1}^{m} w_{i_j} = w_i$. Thus, this comparison can be done in PP for each such weight distribution. As PP is closed under union, Banzhaf-$m$-BENEFICIAL_SPLIT is in PP.

In order to show PP-hardness for Banzhaf-$3$-BENEFICIAL_SPLIT, we use the same techniques as in Theorem 3.5 appropriately modified.

First, we slightly change the definition of \textsc{Compare-SubsetSum-RR} by switching $(\alpha/2) - 2$ and $(\alpha/2) - 1$. The problem (call it \textsc{Compare-SubsetSum-YA}) of whether the number of subqueries of a given sequence $A$ of positive integers summing up to $(\alpha/2) - 1$ is greater than the number of subsequences of $A$ summing up to $(\alpha/2) - 2$, is PP-hard by the sameproof as in Lemma 3.4 with the roles of $q_1$ and $q_2$ exchanged.

Now, we reduce this problem to Banzhaf-$3$-BENEFICIAL_SPLIT by constructing the following instance of the beneficial splitting problem from an instance $A = (a_1, \ldots, a_n)$ of the problem \textsc{Compare-SubsetSum-YA}. Let $\mathcal{G} = (2a_1, \ldots, 2a_n, 1, 3; \alpha)$, where $\alpha = \sum_{j=1}^{n} a_j$, and let $i = n+2$ be the player to be split. $\mathcal{G}$ is (apart from the order of players) equivalent to the game obtained by merging in the proof of Theorem 3.5. Thus, letting $N = \{1, \ldots, n\}$, Banzhaf$(\mathcal{G}, n+2)$ equals

$$\frac{1}{2^{n+1}} \left( 2 - \left\| \left\{ A' \subseteq N \mid \sum_{j \in A'} 2a_j = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \mid \sum_{j \in A'} 2a_j = \alpha - 4 \right\} \right\| \right).$$
Allowing players with weight zero, there are different possibilities to split player \( n + 2 \) into three players. By Lemma 3.6 splitting \( n + 2 \) into one player with weight 3 and two others with weight 0 is not beneficial. Likewise, splitting \( n + 2 \) into two players with weights 1 and 2 and one player with weight 0 is not beneficial, by Lemma 3.6 and since splitting into two players is not beneficial (by the remark above Theorem 3.7). Thus, the only possibility left is splitting \( n + 2 \) into three players of weight 1 each. This corresponds to the original game in the proof of Theorem 3.5, \( G_i \div 3 = (2a_1, \ldots, 2a_n, 1, 1, 1, 1; \alpha) \). Therefore,

\[
\text{Banzhaf}(G_i \div 3, n + 2) = \text{Banzhaf}(G_i \div 3, n + 3) = \text{Banzhaf}(G_i \div 3, n + 4) = \frac{1}{2^{n+3}} \left( 3 \cdot \left\| \left\{ A' \subseteq \mathbb{N} \bigg| \sum_{j \in A'} 2a_j = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq \mathbb{N} \bigg| \sum_{j \in A'} 2a_j = \alpha - 4 \right\} \right\| \right).
\]

Altogether, as in the proof of Theorem 3.5, the sum of the three new players’ probabilistic Banzhaf indices minus the probabilistic Banzhaf index of the original player is greater than zero if and only if

\[
\left\| \left\{ A' \subseteq \mathbb{N} \bigg| \sum_{j \in A'} a_j = (\alpha/2) - 1 \right\} \right\| > \left\| \left\{ A' \subseteq \mathbb{N} \bigg| \sum_{j \in A'} a_j = (\alpha/2) - 2 \right\} \right\|,
\]

which is true if and only if \( A \) is in \text{COMPARE-\#SUBSETSUM-\#R}. This result can be expanded to all \( m \geq 3 \) by splitting into additional players with weight 0. More precisely, if \( m > 3 \), we consider the same game \( G \) as above and split into three players of weight 1 each and \( m - 3 \) players of weight 0 each. By Lemma 3.6 the sum of the \( m \) new players’ Banzhaf power is equal to the combined Banzhaf power of the three players. Thus, PP-hardness holds by the same arguments as above.

On the other hand, a PP upper bound for the general beneficial splitting problem cannot be shown in any straightforward way. Here, we can only show membership in \( \text{NP}^{\text{PP}} \), and we conjecture that this problem is even complete for this class.

**Theorem 3.8** Banzhaf-BENEFICIALSPLIT is PP-hard and belongs to \( \text{NP}^{\text{PP}} \).

**Proof.** With \( m \) being part of the input, there are exponentially many possibilities to distribute the split player’s weight to her false identities. Nondeterministically guessing such a distribution and then, for each distribution guessed, asking a PP oracle to check in polynomial time whether their combined Banzhaf power in the new game is greater than the original player’s Banzhaf power in the original game, shows that Banzhaf-BENEFICIALSPLIT is in \( \text{NP}^{\text{PP}} \).

Since Banzhaf-3-BENEFICIALSPLIT is a special variant of the general problem Banzhaf-BENEFICIALSPLIT, PP-hardness is implied immediately by Theorem 3.7.

\[ \Box \]
3.2 The Shapley–Shubik Power Index

In order to prove PP-hardness for the merging and splitting problems with respect to the Shapley–Shubik index, we need to take a further step back.

**ExactCoverBy3Sets** (X3C, for short) is another well-known NP-complete decision problem: Given a set $B$ of size $3k$ and a family $\mathcal{S}$ of subsets of $B$ that have size three each, does there exist a subfamily $\mathcal{S}'$ of $\mathcal{S}$ such that $B$ is exactly covered by $\mathcal{S}'$?

**Theorem 3.9** ShapleyShubik-BeneficialMerge is PP-complete, even if only two players of equal weight merge.

**Proof.** The PP upper bound, which has already been observed for two players in [FH09], can be shown analogously to the proof of Theorem 3.5.

For proving the lower bound, observe that the size of a coalition a player is pivotal for is crucial for determining the player’s Shapley–Shubik index. Pursuing the techniques of Faliszewski and Hemaspaandra [FH09], we examine the problem COMPARE-#X3C, which is PP-complete by Lemma 3.1. We will apply useful properties of X3C instances shown by Faliszewski and Hemaspaandra [FH09], Lemma 2.7: Every X3C instance $(B, \mathcal{S})$, where $|B| = 3k$ and $|\mathcal{S}| = n$, that satisfies $\frac{k}{n} = \frac{2}{3}$ without changing the number of solutions, i.e., $\#X3C(B, \mathcal{S}) = \#X3C(B', \mathcal{S}')$. Now, by the properties of the standard reduction from X3C to SubsetSum (which in particular preserves the number of solutions, i.e., $\#X3C$ parsimoniously reduces to #SubsetSum, as well as the “input size” $n$ and the “solution size” $k$), we can assume that in a given COMPARE-#SubsetSum instance each subsequence summing up to the given integer $q$ is of size $2n/3$. Following the track of the reductions from COMPARE-#SubsetSum via COMPARE-#SubsetSum-R to COMPARE-#SubsetSum-RR in Lemmas 3.3 and 3.4, a solution $A' \subseteq \{1, \ldots, n\}$ to a given instance $A = (a_1, \ldots, a_n)$ of the latter problem ($A'$ satisfying either $\sum_{i \in A'} a_i = (\alpha/2) - 2$ or $\sum_{i \in A'} a_i = (\alpha/2) - 1$, where $\alpha = \sum_{i=1}^n a_i$) can be assumed to satisfy $|A'| = k = (n+2)/3$. Under this assumption, we show PP-hardness of ShapleyShubik-BeneficialMerge via a reduction from COMPARE-#SubsetSum-RR. Given such an instance, we construct the WVG $G = (a_1, \ldots, a_n, 1; a/2)$ and consider coalition $S = \{n+1, n+2\}$. Define $X = \#SubsetSum(A, (a/2) - 1)$ and $Y = \#SubsetSum(A, (a/2) - 2)$. Letting $N = \{1, \ldots, n\}$, it holds that

\[
\text{ShapleyShubik}(G, n + 1) = \text{ShapleyShubik}(G, n + 2)
\]

\[
= \frac{1}{(n + 2)!} \left( \sum_{C \subseteq N \text{ such that } \sum_{i \in C} a_i = (\alpha/2) - 1} \|C\|!(n + 1 - \|C\|)! \right) + \left( \sum_{C \subseteq N \text{ such that } \sum_{i \in C} a_i = (\alpha/2) - 2} \|C\|!(n - \|C\|)! \right)
\]

\[
= \frac{1}{(n + 2)!} \left( X \cdot k!(n + 1 - k)! + Y \cdot (k + 1)!(n - k)! \right).
\]

Merging the players in $S$, we obtain $G \cup S = (2, a_1, \ldots, a_n; a/2)$. The Shapley–Shubik index of the
new player in $G_S$ is

$$\text{ShapleyShubik}(G_S, 1) = \frac{1}{(n+1)!} \sum_{C \subseteq N \text{ such that } \sum_{i \in C} a_i \in \{(\alpha/2) - 1, (\alpha/2) - 2\}} ||C||!(n - ||C||)!$$

$$= \frac{1}{(n+1)!} (X + Y) \cdot (k + 1)!(n - k)!.$$

All in all,

$$\text{ShapleyShubik}(G_S, 1) - (\text{ShapleyShubik}(G, n + 1) + \text{ShapleyShubik}(G, n + 2))$$

$$= \frac{(X + Y) \cdot (k + 1)!(n - k)!}{(n+1)!} - \frac{2(X \cdot k!(n + 1 - k) + Y \cdot (k + 1)!(n - k)!}{(n+2)!}$$

$$= \frac{k!(n - k)!}{(n + 2)!}(n - 2k)(X + Y).$$

Since we assumed that $k = (n+2)/3$ and we can also assume that $n > 4$ (because we added four integers in the construction in the proof of Lemma 3.4), it holds that

$$n - 2k = \frac{n - 4}{3} > 0.$$

Thus the term (4) is greater than zero if and only if $Y$ is greater than $X$, which is true if and only if $A$ is in $\text{COMPARE-#SUBSETSUM-RR}$. 

**Theorem 3.10** ShapleyShubik-$m$-BENEFICIAL-SPLIT is PP-complete for each $m \geq 2$.

**Proof.** PP membership can be shown analogously to the PP upper bound in the proof of Theorem 3.7. PP-hardness can also be shown analogously to the proof of Theorem 3.7, appropriately modified to use the arguments from the proof of Theorem 3.9 instead of those from the proof of Theorem 3.5.

**Theorem 3.11** ShapleyShubik-BENEFICIAL-SPLIT is PP-hard and belongs to NP$^p$.

**Proof.** The upper bound of NP$^p$ holds due to analogous arguments as in the proof of Theorem 3.8. Also, analogously to the proof of Theorem 3.8, since ShapleyShubik-2-BENEFICIAL-Split is a special variant of the general ShapleyShubik-BENEFICIAL-SPLIT problem, PP-hardness is implied immediately by Theorem 3.10. 

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4 Conclusions and Open Questions

Solving previous conjectures in the affirmative, we have pinpointed the precise complexity of the beneficial merging problem in weighted voting games for the Shapley–Shubik and the probabilistic Banzhaf index by showing that it is PP-complete. We have obtained the same result for beneficial splitting (a.k.a. false-name manipulation) whenever the number of false identities a player splits into is fixed. For an unbounded number of false identities, we raised the known lower bound from NP-hardness to PP-hardness and showed that it is contained in NP^PP. For this problem, it remains open whether it can be shown to be complete for NP^PP, a huge complexity class that by Toda’s theorem [Tod91] contains the entire polynomial hierarchy. NP^PP is an interesting class, but somewhat sparse in natural complete problems. The only (natural) NP^PP-completeness results we are aware of are due to Littman et al. [LGM98], who analyze a variant of the satisfiability problem and questions related to probabilistic planning, and due to Mundhenk et al. [MGLA00], who study problems related to finite-horizon Markov decision processes.

Another interesting open question is whether our results can be transferred also to the beneficial merging and splitting problems for the normalized Banzhaf index. Finally, it would be interesting to know to which classes of simple games, other than weighted voting games, our results can be extended.

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