A SEMIDEFINITE HIERARCHY FOR DISJOINTLY CONSTRAINED MULTILINEAR PROGRAMMING

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Abstract. Disjointly constrained multilinear programming concerns the problem of maximizing a multilinear function on the product of finitely many disjoint polyhedra. While maximizing a linear function on a polytope (linear programming) is known to be solvable in polynomial time, even bilinear programming is NP-hard. Based on a reformulation of the problem in terms of sum-of-squares polynomials, we study a hierarchy of semidefinite relaxations to the problem. It follows from the general theory that the sequence of optimal values converges asymptotically to the optimal value of the multilinear program. We show that the semidefinite hierarchy converges generically in finitely many steps to the optimal value of the multilinear problem.

We outline two applications of the main result. For nondegenerate bimatrix games, a Nash equilibrium can be computed by the sum of squares approach in finitely many steps. Under an additional geometric condition, the NP-complete containment problem for projections of $\mathcal{H}$-polytopes can be decided in finitely many steps.

1. Introduction

A disjointly constrained multilinear programming problem is the optimization problem of maximizing a multilinear function on the product of finitely many disjoint polyhedra. To be more precise, let $l \geq 2$ and $f(x_1, x_2, \ldots, x_l)$ be a real-valued function linear in the variable tuples $x_i = (x_{i,1}, \ldots, x_{i,d_i})$ for $i = 1, \ldots, l$. Consider nonempty polyhedra $P_i \subseteq \mathbb{R}^{d_i}$ as given by the intersection of finitely many halfspaces and an affine subspace. Then a (disjointly constrained) multilinear program has the form

$$f^* := \max f(x_1, x_2, \ldots, x_l)$$

$$\text{s.t. } (x_1, \ldots, x_l) \in P_1 \times \cdots \times P_l.$$  

(1.1)

If $l = 1$, this reduces to linear programming. Throughout the paper we assume $l \geq 2$. For $l = 2$, problem (1.1) is known as bilinear programming or indefinite quadratic optimization. This NP-hard problem lies in-between linear and quadratic optimization. Originally, bilinear programming has been considered as a generalization of bimatrix games in game theory. Today it is motivated by a various of applications, including (constrained) bimatrix games [17], quantum information theory [1], and geometric containment problems [9] [11]. Prominent approaches to solve bilinear and multilinear programs are based on vertex tracking algorithms [6] [12].

In this note we state a semidefinite hierarchy to multilinear programs based on the concept of sum of squares; see, e.g., the books of Blekherman et. al. [2], or Lasserre [14], or the survey by Laurent [15]. To that end the problem is rewritten as a program with a nonnegativity constraint. This constraint is then replaced by a weaker condition, namely
being a sum of polynomial squares which can be formulated as a semidefinite constraint. A particular question is whether the multilinear structure can be used in sum of squares certificates in order to state convergence statements. Indeed, we show finite convergence in generic cases by using an extension of Putinar’s Positivstellensatz stated by Marshall [19]. This has impact on various applications including polyhedral containment and game theory.

It is well-known in game theory that a bimatrix game (cf. Section 4.2) has a formulation as a disjointly constrained bilinear optimization problem. The sum of squares approach to (polynomial) games has been considered by Laraki and Lasserre [13], and Parrilo [22]. They did not establish finite convergence. By our main result a Nash equilibrium of a nondegenerate bimatrix game can be computed via sums of squares in finitely many steps. This can be extended to finite \( l \)-person games.

In [11] Theobald and the author formulated the problem of whether an \( \mathcal{H} \)-polytope is contained in a \( \mathcal{V} \)-polytope as a bilinear programming problem and proved finite convergence for the sum of squares hierarchy under some conditions. In Section 4.1 we extend this result to the NP-complete projective polyhedral containment problem, the problem of whether the (coordinate) projection of an \( \mathcal{H} \)-polyhedron is contained in the projection to another \( \mathcal{H} \)-polyhedron [9]. Note that the projection of an \( \mathcal{H} \)-polyhedron is again a polyhedron but a projection-free representation is not easy to achieve.

The paper is structured as follows. In Section 2 we outline some basic geometric behavior of multilinear programs. A semidefinite hierarchy based on sum of squares and a proof of its convergence in well-defined cases is given in Section 3. In Section 4 we sketch the application of our main result to polyhedral containment and game theory.

## 2. Multilinear Programming

Note that a multilinear function \( f = f(x_1, x_2, \ldots, x_l) \) can be written as

\[
f = \sum_{\emptyset \neq L \subseteq [l]} Q^{(L)}(x_i : i \in L) = \sum_{j_1 = 1}^{d_1} \cdots \sum_{j_l = 1}^{d_l} Q^{(1,\ldots,l)}_{j_1,\ldots,j_l} x_{1,j_1} \cdots x_{l,j_l} + \cdots + \sum_{i=1}^{l} \sum_{j_1 = 1}^{d_i} Q^{(i)}_{j_1} x_{i,j_1},
\]

for tensors \( Q^{(L)} \in \mathbb{R}^{\times \{d_i \mid i \in L\}} \) with \( \emptyset \neq L \subseteq [l] := \{1,\ldots,l\} \) and \( Q^{[l]} \neq 0 \). For \( i \in [l] \) let \( a^{(i)} \in \mathbb{R}^{m_i}, A^{(i)} \in \mathbb{R}^{m_i \times d_i}, b^{(i)} \in \mathbb{R}^{m_i}, B^{(i)} \in \mathbb{R}^{m_i \times d_i} \). Define the polytopes

\[
P_i = \{ x_i \in \mathbb{R}^{d_i} \mid a^{(i)} \geq A^{(i)} x_i, \ b^{(i)} = B^{(i)} x_i \}.
\]

We set \( P_1,\ldots,l := P_1 \times \ldots \times P_l \) and often use \( P \) if there is no risk of confusion.

Throughout the paper we make the following assumptions on the representations of the polytopes \( P_i \).

**Assumption 2.1.** For \( i \in [l] : 

1. The equality constraints are linearly independent, i.e., the dimension of the affine subspace \( B_i := \{ x \in \mathbb{R}^{d_i} \mid b^{(i)} = B^{(i)} x_i \} \) equals \( d_i - n_i \) and \( d_i - n_i > 0 \).
2. \( \{ x \in \mathbb{R}^{d_i} \mid a^{(i)} \geq A^{(i)} x \} \) is full-dimensional in the affine subspace \( B_i \), i.e., \( \dim \{ x \in B \mid a^{(i)} \geq A^{(i)} x \} = d_i - n_i \).
The first assumption can be achieved by Gaussian elimination. The second assumption particularly implies nonemptiness of $P_1, \ldots, P_l$. It can be tested (and achieved) by linear programming methods in polynomial time. We understand relative interior of a polytope $P_i$ as with respect to the affine subspace $B_i$.

Given a polytope $P_i$, the outer normal cone $C_i$ of a boundary point $v$ of $P_i$ is the cone in $\text{span}\{(B^{(i)})^T\}$ generated by the outer normals of the constraints that are active at that point, i.e., $C_i = \text{pos}\{(A_j^{(i)})^T | a_j^{(i)} = A_j^{(i)}v, j \in [m_i]\} \subseteq \text{span}\{(B^{(i)})^T\}$. Note that the definition of a normal cone here differs slightly from the usual one as we restrict it to the subspace $\text{span}\{(B^{(i)})^T\}$. The usual normal cone is given by $C_i + \text{span}\{(B^{(i)})^T\}$.

The boundary of the product polytope $\partial(P) = \partial(P_1 \times \ldots \times P_l)$ equals the product of the boundaries $\partial P_1 \times \ldots \times \partial P_l$. Moreover, $F$ is a face of $P_1 \times \ldots \times P_l$ if and only if $F = F_1 \times \ldots \times F_l$ for faces $F_i$ of $P_i$. In particular, any vertex of $P$ is given by a tuple of vertices of $P_1, \ldots, P_l$.

We state a basic result on multilinear programming. For a proof we refer to Konno [12]; see also Drenick [4].

**Proposition 2.2.** Let $P_1, \ldots, P_l$ be nonempty polytopes. Then the set of optimal solutions to (1.1) is part of the boundary and the maximum is attained at a tuple of vertices of $P_1, \ldots, P_l$.

We state two corollaries that will be used in the next section.

**Corollary 2.3.** The set of optimal solutions to (1.1) is an union of proper faces $F = F_1 \times \ldots \times F_l$ of $P$.

**Proof.** Let $(\bar{x}_1, \ldots, \bar{x}_l)$ be an optimal solution. By Proposition 2.2 $(\bar{x}_1, \ldots, \bar{x}_l) \in \partial(P_1 \times \ldots \times P_l)$. Let $F_{\bar{x}_i}$ be the minimal face containing $\bar{x}_i$. Then for any $(x_1, \ldots, x_l) \in F_{\bar{x}_1} \times \ldots \times F_{\bar{x}_l}$, we have $f(x_1, \ldots, x_l) = f(x_1, \ldots, \bar{x}_i, \ldots, x_l)$ by the linearity of $f$ in every $x_i = (x_{i,1}, \ldots, x_{i,d_i})$, $i \in [l]$. □

The second corollary follows directly from the latter one.

**Corollary 2.4.** Problem (1.1) has finitely many optimal solutions if and only if every optimal solution of the problem is a vertex of $P$.

### 3. A Semidefinite hierarchy based on sum of squares

In this section, we apply sum of squares techniques to multilinear programming problems. Our main goal is to show that the outcoming hierarchy of semidefinite programs converges after finitely many steps in generic cases (in a well-defined sense); see Theorem 3.1.  

The multilinear problem (1.1) can equivalently be stated as a nonnegativity question

$$f^* = \max \{ f(x_1, \ldots, x_l) | x_i \in P_i, i \in [l] \}$$

$$= \min \{ \mu | \mu - f(x_1, \ldots, x_l) \geq 0 \text{ for } (x_1, \ldots, x_l) \in P \}.$$  

Replacing the nonnegativity condition by sum of squares is a common method in polynomial optimization. A polynomial $p(x) \in \mathbb{R}[x]$ is a sum of squares (sos) if and only if there
exists a positive semidefinite matrix $M$ of appropriate size such that $p(x) = [x]_t^T M [x]_t$, where $[x]_t$ is the vector of monomials up to degree $t$ for some nonnegative integer $t$. We denote the set of all sos-polynomials in the variables $x$ by $\Sigma[x] \subseteq \mathbb{R}[x]$. While every sos-polynomial is nonnegative, the converse is not true in general. Thus replacing nonnegativity by being a sum of squares in the above problem (where the equality constraints have to be treated separately) yields an upper bound for $f^*$,

$$f^* \leq \min \{ \mu \mid \mu - f(x_1, \ldots, x_l) \in QM + I \},$$

where

$$QM = \left\{ \sigma_0 + \sum_{i=1}^l \sum_{j=1}^{m_i} \sigma_{ij} (a^{(i)} - A^{(i)} x_i)_j \mid \sigma_{ij} \in \Sigma[x] \right\}$$

is the quadratic module generated by the linear inequality constraints and

$$I = \left\{ \sum_{i=1}^l \sum_{j=1}^{n_i} \tau_{ij} (b^{(i)} - B^{(i)} x_i)_j \mid \tau_{i,j} \in \mathbb{R}[x] \right\}$$

is the ideal generated by the linear equality constraints.

The idea is to test membership of $\mu - f(x_1, \ldots, x_l)$ in truncations of the quadratic module and the ideal. Testing membership in these sets can be done by deciding feasibility of a certain semidefinite inequality system, which is solvable in polynomial time if the degrees are fixed. To be precise, define the $t$-th truncation as

$$QM_t = \left\{ \sigma_0 + \sum_{i=1}^l \sum_{j=1}^{m_i} \sigma_{ij} (a^{(i)} - A^{(i)} x_i)_j \mid \sigma_{ij} \in \Sigma_{2t-2}[x] \right\}$$

and

$$I_t = \left\{ \sum_{i=1}^l \sum_{j=1}^{n_i} \tau_{ij} (b^{(i)} - B^{(i)} x_i)_j \mid \tau_{i,j} \in \mathbb{R}_{2t-1}[x] \right\}.$$

The $t$-th sos program is then

$$(3.1) \quad f_t := \inf \{ \mu \mid \mu - f(x_1, \ldots, x_l) \in QM_t + I_t \}$$

with $f^* \leq \ldots \leq f_{t+1} \leq f_t$. Note that the first relaxation order making sense is $t = l$ as the degree of $f$ is $l$.

We state our main result.

**Theorem 3.1.** Let $P_1, P_2, \ldots, P_l$ be polytopes with nonempty relative interior satisfying Assumption [2.7].

1. The sequence $(f_t)_t$ of optimal values (3.1) converges asymptotically from above to the optimal value $f^*$.

2. If the multilinear programming problem (1.1) has only finitely many optimal solutions, then $f^* - f \in QM + I$. Thus the sequence of the optimal values $(f_t)_t$ converges in finitely many steps to the optimal value $f^*$. 


The proof of Theorem 3.1 extends the one for the containment problem of an $H$-polytope in a $V$-polytope in [11]. We start with the proof of part (1) of the statement as it follows directly from the general theory.

**Proposition 3.2** (Putinar’s Positivstellensatz [23]). Let $S = \{x \in \mathbb{R}^d \mid g(x) \geq 0 \ \forall g \in G, \ h(x) = 0 \ \forall h \in H\}$ for some finite subsets $G, H \subseteq \mathbb{R}[x]$. If the quadratic module $QM(G) = \{\sigma_0 + \sum_{g \in G} \sigma_g g \mid \sigma_0, \sigma_g \in \Sigma[x]\}$ is Archimedean, then the sum of the quadratic module with the ideal generated by $H$, $QM(G) + I(H)$, contains every polynomial $p \in \mathbb{R}[x]$ positive on $S$.

A quadratic module is called Archimedean if $N - x^T x \in QM$ for some positive integer $N$. Archimedeaness of the quadratic module implies compactness of the set $S$, but the converse is generally not true. In our specific situation with only linear constraint polynomials defining a polytope, this condition is always satisfied [18, Theorem 7.1.3].

**Proof of Theorem 3.1 (1).** By Proposition 3.2, $f^* - f + \varepsilon \in QM + I$ for every $\varepsilon > 0$. The claim follows by letting $\varepsilon$ tend to zero. $\square$

Proving part (2) is more involved. Our goal is to apply Marshall’s Nichtnegativstellensatz, an extension of Putinar’s result. To that end, we introduce some necessary notation. See Nie [21] for a similar application of Marshall’s result to general polynomial optimization problems.

Given a variety $V \subseteq \mathbb{R}^n$ of dimension $d$, denote by $I(V)$ the vanishing ideal of $V$ and by $R[V] = \mathbb{R}[x]/I(V)$ the coordinate ring of $V$.

**Proposition 3.3** (Marshall’s Nichtnegativstellensatz [18, Theorem 9.5.3]). Let $V \subseteq \mathbb{R}^n$ be a variety of dimension $d$ and let $f, g_1, \ldots, g_m \in \mathbb{R}[V]$ with $S = \{x \in V \mid g_i(x) \geq 0, \ i \in [m]\}$. Assume that the quadratic module $M$ generated by $g_1, \ldots, g_m$ is Archimedean in $\mathbb{R}[V]$. Further suppose that for any global maximizer $\bar{x}$ of $f$ on $S$ the following holds:

1. $\bar{x}$ is a nonsingular point of $V$,
2. there exist local parameters $t_1, \ldots, t_d$ and an index set $I$ of size $1 \leq k \leq d$ such that $t_i = g_i$ for $i \in I$,
3. the linear part of $f$ in the localizing parameters equals $\sum_{i \in I} c_i t_i$ for some negative coefficients $c_i$, $i \in I$, and
4. the quadratic part of $f$ in the localizing parameters is negative definite on the $t_i = 0$ for $i \in I$.

Then $f_{\text{max}} - f \in M$, where $f_{\text{max}}$ denotes the global maximum of $f$ on $S$.

The last two conditions in Proposition 3.3 are called boundary Hessian condition (BHC); see [18, 19]. To prove Theorem 3.1 we need one more lemma.

**Lemma 3.4.** Let $P_1, \ldots, P_l$ be nonempty polytopes and $c_0 - c^T x_i \geq 0$ be a redundant inequality in the representation of some $P_i$. Then it is also redundant in the hierarchy (3.1).

The lemma follows by a simple application of the affine version of Farkas’ Lemma; see e.g. [11].
Proof of Theorem 3.1 (2). Let \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_l) \in \mathbb{P} = P_1 \times \ldots \times P_l \) be an arbitrary but fixed optimal solution to (1.1). As by Assumption 2.1 the linear equality constraints are linearly independent for each \( P_i \), the rank of the gradient equals the number of equations for every point in the variety. Thus every point is nonsingular (cf. [3, Theorem 9]).

Since \( f \) is linear in \( x_i \) for any \( i \in [l] \), its derivative w.r.t. \( x_i \) has the form
\[
\frac{\partial}{\partial x_i} f(x_1, \ldots, x_l) = \sum_{\emptyset \neq L \subseteq [l], i \in L} Q^{(L)}(x_j : j \in L \setminus \{i\}),
\]
a \( d_i \)-dimensional vector with entries in the polynomial ring \( \mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l] \). Every component is a multilinear function in \( (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l) \).

Fix \( i \in [l] \). Assume \( \frac{\partial}{\partial x_i} f(\bar{x}) \) lies in the sum of the outer normal cone (in \( \text{span}\{ (B^{(i)})^T \} \)) of an at least one-dimensional face \( F_i \) of \( P_i \) and the linear subspace \( \text{span}\{ (B^{(i)})^T \} \). By Corollary 2.3, we have
\[
\left( \frac{\partial}{\partial x_i} f(\bar{x}) \right)(\hat{x}_i) = \max \left\{ \left( \frac{\partial}{\partial x_i} f(\bar{x}) \right)(x_i) \mid x_i \in P_i \right\} = \left( \frac{\partial}{\partial x_i} f(\bar{x}) \right)(\bar{x}_i)
\]
for every \( \hat{x}_i \in F_i \) and thus
\[
f(\bar{x}) = \sum_{\emptyset \neq L \subseteq [l], i \in L} Q^{(L)}(\bar{x}_j : j \in L) + \sum_{\emptyset \neq L \subseteq [l], i \notin L} Q^{(L)}(\bar{x}_j : j \in L)
= \left( \frac{\partial}{\partial x_i} f(\bar{x}) \right)(\bar{x}_i) + \sum_{\emptyset \neq L \subseteq [l], i \notin L} Q^{(L)}(\bar{x}_j : j \in L)
= \left( \frac{\partial}{\partial x_i} f(\bar{x}) \right)(\bar{x}_i) + \sum_{\emptyset \neq L \subseteq [l], i \notin L} Q^{(L)}(\bar{x}_j : j \in L)
= f(\bar{x}_1, \ldots, \hat{x}_i, \ldots, \bar{x}_l),
\]
in contradiction to Corollary 2.3 and the assumption in part (2) of the theorem. Hence \( \frac{\partial}{\partial x_i} f(\bar{x}) \) lies in the sum of the outer normal cone \( C_i \) of \( \bar{x}_i \) and the subspace \( \text{span}\{ (B^{(i)})^T \} \).

As \( \bar{x}_i \) is a vertex of \( P_i \), \( C_i \) is full-dimensional in the \((d_i - n_i)\)-dimensional linear subspace \( \text{span}\{ (B^{(i)})^T \} \). Thus there exist \( d_i - n_i \) points in the outer normal cone such that \( \frac{\partial}{\partial x_i} f(\bar{x}) \) is a (strictly) positive combination of these points. The points correspond to redundant inequalities in the representation of \( P_i \) that are active in \( \bar{x}_i \). As by Lemma 3.4 redundant constraints do not affect (existence and degree of) the sum of squares certificates, we can assume that the representation of \( P_i \) is given in a form such that there exists an index set \( I_i \subseteq [m_i] \) of cardinality \( |I_i| = d_i - n_i \) corresponding to linearly independent, active constraints and there exist \( \alpha^{(i)} \in \mathbb{R}^{m_i} \) and \( \beta^{(i)} \in \mathbb{R}^{n_i} \) with
\[
(3.2) \quad \frac{\partial}{\partial x_i} f(\bar{x}) = (A^{(i)})^T \alpha^{(i)} + (B^{(i)})^T \beta^{(i)} \quad \alpha_j^{(i)} > 0 \text{ for } j \in I_i \text{ and } \alpha_j^{(i)} = 0 \text{ for } j \notin I_i.
\]
Let $I_1, \ldots, I_l$ be index sets as constructed in the previous paragraph. W.l.o.g. set $I_i = [d_i - n_i]$ for $i \in [l]$. Consider the affine variable transformation

$$
\phi : \mathbb{R}^{d_1 + \cdots + d_l} \to \mathbb{R}^{d_1 + \cdots + d_l}, \quad (x_1, \ldots, x_l) \mapsto \begin{bmatrix}
(a^{(1)} - A^{(1)} x_1)_{I_1} \\
b^{(1)} - B^{(1)} x_1 \\
\vdots \\
(a^{(l)} - A^{(l)} x_l)_{I_l} \\
b^{(l)} - B^{(l)} x_l
\end{bmatrix} =: \begin{pmatrix}s_1 \\
\vdots \\
s_l
\end{pmatrix}.
$$

The new variables $(s_1, \ldots, s_l) := (s_{11}, \ldots, s_{1d_1}, \ldots, s_{l1}, \ldots, s_{ld_l})$ serve as localizing parameters on the variety defined by $s_{ij} = (b^{(i)} - B^{(i)} x_i)_j = 0, \; j \in [d_i] \setminus I_i, \; i \in [l]$ as in Proposition [3.3]. The inverse of $\phi$ is given by

$$
\phi^{-1} : \mathbb{R}^{d_1 + \cdots + d_l} \to \mathbb{R}^{d_1 + \cdots + d_l}, \quad (s_1, \ldots, s_l) \mapsto \begin{bmatrix}A^{(1)}_{I_1} \\
B^{(1)} \\
\vdots \\
A^{(l)}_{I_l} \\
B^{(l)}\end{bmatrix}^{-1} \cdot \begin{pmatrix}(A^{(1)}_{I_1})^{-1} (\bar{a}^{(1)} - s_1) \\
\vdots \\
(A^{(l)}_{I_l})^{-1} (\bar{a}^{(l)} - s_l) \end{pmatrix}.
$$

Setting $\bar{A}^{(i)} = [A^{(i)}_{I_i}, B^{(i)}]_T, \; \bar{a}^{(i)} = (a^{(i)}_{I_i}, b^{(i)})^T$, the objective function $f$ has the form

$$
f \circ \Phi^{-1}(s_1, \ldots, s_l) = \sum_{\emptyset \neq L \subseteq [l]} Q^{(L)} \left( (\bar{A}^{(i)})^{-1} (\bar{a}^{(i)} - s_i) : i \in L \right)
$$

$$
= \sum_{\emptyset \neq L \subseteq [l]} \sum_{M \subseteq L} Q^{(L)} \left( \{-(\bar{A}^{(i)})^{-1} s_i : i \in M\} \cup \{(\bar{A}^{(i)})^{-1} \bar{a}^{(i)} : i \in L \setminus M\} \right)
$$

in the parameterization space. The homogeneous degree-1 part of $f$ in $(s_1, \ldots, s_l)$ is

$$
f_1 := \sum_{\emptyset \neq L \subseteq [l]} \sum_{i \in L} Q^{(L)} \left( \{-(\bar{A}^{(i)})^{-1} s_i \cup \{(\bar{A}^{(j)})^{-1} \bar{a}^{(j)} : j \neq i\} \right).
$$

In order to show that $f$ satisfies the conditions in Proposition [3.3], we show that the gradient of $f_1$ w.r.t $s_1, \ldots, s_l$ equals $(-\alpha^{(1)}_{I_1} - \beta^{(1)}), \ldots, -\alpha^{(l)}_{I_l} - \beta^{(l)})$. Then $f_1$ has only negative coefficients on the variety defined by the equality constraints, i.e.,

$$
f_1 = (\alpha^{(1)}_{I_1})^T(s_1)_{I_1} + \cdots + (\alpha^{(l)}_{I_l})^T(s_l)_{I_l} \quad \text{on} \quad (s_i)_{[d_i] \setminus I_i} = 0, \; i \in [l],
$$

and thus the third condition of Proposition [3.3] is satisfied. Since $|I_1| + \cdots + |I_l| = d_1 - n_1 + \cdots + d_l - n_l$ equals the dimension of the variety defined by the equality constraints in $P_1 \times \cdots \times P_l$, the last condition is obsolete. Marshall’s Nichtnegativstellensatz then implies $f^* - f(x_1, \ldots, x_l) \in \text{QM + I}$.

It remains to show the desired equality for the gradient of $f_1$. To that end note that

$$
(\bar{x}_1, \ldots, \bar{x}_l) = \phi^{-1}(0) = (\bar{A}^{(1)})^{-1} \bar{a}^{(1)}, \ldots, (\bar{A}^{(l)})^{-1} \bar{a}^{(l)}.
$$
We have
\[
\frac{\partial}{\partial s_i} f_1(0) = - \sum_{\emptyset \neq L \subseteq [i], i \in L} ((\bar{A}^{(i)})^{-1})^T Q^L ((\bar{A}^{(i)})^{-1} \bar{a}^{(i)} : j \in L \setminus \{i\})
\]
\[
= -((\bar{A}^{(i)})^{-1})^T \frac{\partial}{\partial x_i} f(\bar{x})
\]
\[
= -\alpha^{(i)}_1 - \beta^{(i)},
\]
where the last equation follows from (3.2). This completes the proof. \(\square\)

4. Applications

4.1. Geometric containment problems. As outlined in the introduction, this note is motivated by geometric containment problems. More precisely, Theobald and the author studied the classical problem (cf. [5, 7, 8]) to decide whether a given \(\mathcal{H}\)-polytope is contained in a given \(\mathcal{V}\)-polytope. While the converse containment problem, \(\mathcal{V}\)-in-\(\mathcal{H}\), is solvable in polynomial time, the \(\mathcal{H}\)-in-\(\mathcal{V}\) problem is co-NP-complete. The main result in [11] is the formulation as a certain bilinear programming problem and the proof of finite convergence in some cases. Theorem 3.1 extends this results to disjointly constrained multilinear programs.

In [9] the author gave a bilinear formulation of the projective polyhedral containment problem: For \(a \in \mathbb{R}^k\) and \(b \in \mathbb{R}^l\) let
\[
P = \{ (x, y) \in \mathbb{R}^{d+m} \mid a \geq Ax + A'y \} \quad \text{and} \quad Q = \{ (x, y') \in \mathbb{R}^{d+n} \mid b \geq Bx + B'y' \}
\]
be nonempty polyhedra and consider their projection to \(\mathbb{R}^d\), denoted by \(\pi(P)\) and \(\pi(Q)\). The projective polyhedral containment problem asks whether \(\pi(P) \subseteq \pi(Q)\). Note that the projection of a polyhedron is again a polyhedron but a projection-free description is not easy to achieve. Indeed, while deciding containment of \(P\) in \(Q\) is solvable in polynomial time, it is shown in [10, Theorem 3.2.4] that deciding containment of \(\pi(P)\) in \(\pi(Q)\) is co-NP-complete. This hardness statement holds true for \(m = 0\).

Proposition 4.1. [9, Theorem 3.1] Let \(\ker(B^T) \cap \mathbb{R}^l_+ \neq \{0\}\). Then \(\pi(P) \subseteq \pi(Q)\) if and only if
\[
\min \{ z^T (b - Bx) \mid \pi(P) \times (\ker(B^T) \cap \Delta^l) \} \geq 0,
\]
where \(\Delta^l = \{ z \in \mathbb{R}^l \mid 1^T z = 1, z \geq 0 \}\) is the \(l\)-simplex.

Sufficient for condition \(\ker(B^T) \cap \mathbb{R}^l_+ \neq \{0\}\) being satisfied is boundedness of \(\pi(Q)\). If \(\ker(B^T) \cap \mathbb{R}^l_+ = \{0\}\), then the simplex \(\Delta^l\) in (4.1) has to be replaced by the nonnegative orthant \(\mathbb{R}^l_+\), and thus the feasible set of the bilinear program is unbounded; see [9, Theorem 3.1]. If \(m = 0\), we can apply Theorem 3.1.

Theorem 4.2. Let \(P \subseteq \mathbb{R}^d\) be a nonempty polytope with \(m = 0\) and \(\ker(B^T) \cap \Delta^l \neq \emptyset\). Assume that every optimal solution to (4.1) is a pair of vertices of \(P\) and \(\ker(B^T) \cap \Delta^l\). Then hierarchy (3.3) decides containment in finitely many steps.

As a special case the theorem covers \(\mathcal{H}\)-in-\(\mathcal{V}\) containment since every \(\mathcal{V}\)-polytope can be represented as a projection of an \(\mathcal{H}\)-polytope.
4.2. (Constrained) bimatrix games. A bimatrix game \((A, B)\) is given by a pair of payoff matrices \(A, B \in \mathbb{R}^{m \times n}\) and the sets of mixed strategies for player 1, \(\Delta_m = \{x \in \mathbb{R}^m \mid 1_m^T x = 1, \; x \geq 0\}\), and player 2, \(\Delta_n = \{y \in \mathbb{R}^n \mid 1_n^T y = 1, \; y \geq 0\}\). The payoffs are given by \(x^T Ay\) for player 1 and by \(x^T By\) for player 2. Both players want to maximize their payoff. A pair of mixed strategies \((\bar{x}, \bar{y})\) is called a (Nash) equilibrium if
\[
\bar{x}^T A \bar{y} \geq x^T A \bar{y} \quad \forall x \in \Delta_m \quad \text{and} \quad \bar{x}^T B \bar{y} \geq \bar{x}^T B y \quad \forall y \in \Delta_n.
\]
Mangasarian showed that the set of equilibria corresponds to a subset of the vertices of certain polyhedra and can be computed via quadratic programming; see \[16, 17\]. As adding a positive constant to every entry of the matrices \(A, B\) does not change the set of equilibria (see, e.g., \[21\]), we assume that \(A, B\) are entrywise positive. Then using a projective transformation boundedness of the polyhedra can be achieved. Define
\[
P_1 := \{x \in \mathbb{R}^m \mid B^T x \leq 1_n, \; x \geq 0\} \quad \text{and} \quad P_2 := \{y \in \mathbb{R}^n \mid Ay \leq 1_m, \; y \geq 0\}.
\]

Proposition 4.3. [17] Let \(A, B \in \mathbb{R}^{m \times n}\) have only positive entries. A pair \((\bar{x}, \bar{y})\) solves the bilinear program
\[
\max \; x^T (A + B) y - x^T \mathbb{1}_m - 1^T_n y \\
x \in P_1, \; y \in P_2
\]
if and only if \((\frac{\bar{x}}{1_m^T \bar{x}}, \frac{\bar{y}}{1_n^T \bar{y}}) \in \Delta_m \times \Delta_n\) is an equilibrium point for the bimatrix game \((A, B)\).

A bimatrix game is called nondegenerate if in any basic feasible solution to the system
\[
Ay + s = \mathbb{1}_m, \; B^T x + t = \mathbb{1}_n, \; x, y, s, t \geq 0
\]
all basic variables have positive values. (See the survey by von Stengel \[21\] for equivalent definitions of nondegeneracy.) In the case of a nondegenerate bimatrix game, the number of equilibria is finite as they only appear at vertices of \(P_1 \times P_2\); see \[24\].

Theorem 4.4. Let \((A, B)\) be a nondegenerate bimatrix game. Then an equilibrium can be computed by hierarchy \((3.1)\) in finitely many steps.

Proof. By the nondegeneracy assumption, the polytopes in \((1.2)\) are full-dimensional and all optima of the bilinear problem are vertices. Thus we can apply Theorem 3.1. □

Note that the bilinear formulation of bimatrix games can be extended to a multilinear formulation of finite \(l\)-person games; see Mills \[20\]. However, other than bimatrix games, the geometry of \(l\)-person games is not well understood. In constrained bimatrix games, the probability simplices are replaced by arbitrary polytopes. A formulation as a bilinear program is along the same lines as for the non-constrained case. In both cases, an \(l\)-person game and a constrained game, as long as the number of equilibria is known to be finite, Theorem 4.4 holds true.

References

[1] M. Berta, O. Fawzi, and V. B. Scholz. Quantum bilinear optimization. Preprint arXiv:1506.08810, 2015.

[2] G. Blekherman, P. A. Parrilo, and R. Thomas. Semidefinite Optimization and Convex Algebraic Geometry. SIAM Series on Optimization. Society for Industrial and Applied Mathematics, 2013.
[3] D. A. Cox, J. Little, and D. O'Shea. Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra. Undergraduate Texts in Mathematics. Springer New York, 2008.

[4] R. F. Drenick. Multilinear programming: Duality theories. Journal of Optimization Theory and Applications, 72(3):459–486, 1992.

[5] R. M. Freund and J. B. Orlin. On the complexity of four polyhedral set containment problems. Math. Program., 33(2):139–145, 1985.

[6] G. Gallo and A. Ülküçü. Bilinear programming: An exact algorithm. Math. Program., 12(1):173–194, 1977.

[7] P. Gritzmann and V. Klee. Computational complexity of inner and outer $j$-radii of polytopes in finite-dimensional normed spaces. Math. Program., 59(2, Ser. A):163–213, 1993.

[8] P. Gritzmann and V. Klee. On the complexity of some basic problems in computational convexity. I. Containment problems. Discrete Math., 136(1-3):129–174, 1994.

[9] K. Kellner. Containment problems for projections of polyhedra and spectrahedra. Preprint arXiv:1509.02735, 2015.

[10] K. Kellner. Positivstellensatz Certificates for Containment of Polyhedra and Spectrahedra. PhD thesis, Goethe-Universität Frankfurt am Main, 2015.

[11] K. Kellner and T. Theobald. Sum of squares certificates for containment of $H$-polytopes in $V$-polytopes. To appear in SIAM J. Disc. Math., Preprint, arXiv:1409.5008, 2016.

[12] H. Konno. A cutting plane algorithm for solving bilinear programs. Math. Program., 11(1):14–27, 1976.

[13] R. Laraki and J. B. Lasserre. Semidefinite programming for min–max problems and games. Math. Program., 131(1):305–332, 2010.

[14] J. B. Lasserre. An Introduction to Polynomial and Semi-Algebraic Optimization. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2015.

[15] M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In M. Putinar and S. Sullivant, editors, Emerging Applications of Algebraic Geometry, volume 149 of IMA Vol. Math. Appl., pages 157–270. Springer, New York, 2009.

[16] O. L. Mangasarian. Equilibrium points of bimatrix games. J. Soc. Ind. Appl. Math., 12(4):778–780, 1964.

[17] O. L. Mangasarian and H. Stone. Two-person nonzero-sum games and quadratic programming. J. Math. Anal. Appl., 9(3):348–355, 1964.

[18] M. Marshall. Positive Polynomials and Sums of Squares. Mathematical surveys and monographs. American Mathematical Society, 2008.

[19] M. Marshall. Representation of non-negative polynomials, degree bounds and applications to optimization. Canad. J. Math, 61(205-221), 2009.

[20] H. Mills. Equilibrium points in finite games. J. Soc. Ind. Appl. Math., 8(2):397–402, 1960.

[21] J. Nie. Optimality conditions and finite convergence of Lasserre’s hierarchy. Math. Program., pages 1–25, 2012.

[22] P. A. Parrilo. Polynomial games and sum of squares optimization. In Proc. of the 45th IEEE Conference on Decision and Control, pages 2855–2860. IEEE, 2006.

[23] M. Putinar. Positive polynomials on compact semi-algebraic sets. Indiana Univ. Math. J., 42(3):969–984, 1993.

[24] B. Von Stengel. Computing equilibria for two-person games. Handbook of game theory with economic applications, 3:1723–1759, 2002.

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