Axiomatic classical (prequantum) field theory. Jet formalism

G. Sardanashvily
Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Abstract. In contrast with QFT, classical field theory can be formulated in a strict mathematical way if one defines even classical fields as sections of smooth fiber bundles. Formalism of jet manifolds provides the conventional language of dynamic systems (nonlinear differential equations and operators) on fiber bundles. Lagrangian theory on fiber bundles is algebraically formulated in terms of the variational bicomplex of exterior forms on jet manifolds where the Euler–Lagrange operator is present as a coboundary operator. This formulation is generalized to Lagrangian theory of even and odd fields on graded manifolds. Cohomology of the variational bicomplex provides a solution of the global inverse problem of the calculus of variations, states the first variational formula and Noether’s first theorem in a very general setting of supersymmetries depending on higher-order derivatives of fields. A theorem on the Koszul–Tate complex of reducible Noether identities and Noether’s inverse second theorem extend an original field theory to prequantum field-antifield BRST theory. Particular field models, jet techniques and some quantum outcomes are discussed.

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I. Introduction

Our final purpose is QFT, whose existent mathematical formulation meets many problems. Note that, from the physical viewpoint, it seems more reasonable to study dequantization of quantum fields. However, we start with classical fields. Firstly, a generating functional of Green functions in perturbative QFT is depends on an action functional of classical fields. Secondly, it may happen that there exist non-quantizable classical fields, e.g., a Higgs field. Thirdly, classical field theory can be formulated in a strict mathematical way if one defines even classical fields as sections of smooth fiber bundles (Axiom 1). Jet formalism [85, 125, 129, 181, 197] provides the conventional language of classical field theory as dynamic theory on fiber bundles [3, 25, 37, 52, 63, 81, 85, 99, 102, 104, 112, 129, 130, 144, 158, 161, 202]. We agree to call it axiomatic classical field theory (henceforth ACFT). Section II gives its brief exposition. In Sections III and IV, particular field models, different jet techniques and some quantum outcomes are discussed.

Bearing in mind quantization, we treat ACFT as Lagrangian theory (Axiom 2) (see item 22 for covariant Hamiltonian field theory). We are not concerned with solutions of field equations, but develop ACFT as prequantum field theory. Lagrangian theory on fiber bundles is algebraically formulated in terms of the variational bicomplex of exterior forms on jet manifolds [6, 7, 25, 88, 144, 157, 180, 203, 205]. This formulation is generalized to Lagrangian theory of even and odd variables on graded manifolds (Axiom 3) [12, 18, 20, 93].

Theorem 1 on cohomology of the variational bicomplex provides a solution of the global inverse problem of the calculus of variations (Theorem 2), states the first variational formula (Theorem 3) and, as a consequence, leads to Noether’s first theorem in a very general setting of supersymmetries depending on higher-order derivatives of fields (Theorem 4).

Quantization of a Lagrangian field theory essentially depends on its degeneracy [22, 23, 100]. Its Euler–Lagrange operator generally obeys Noether identities which need not be independent, but satisfy first-stage Noether identities, and so on. Theorem 5 on the Koszul–Tate complex of reducible Noether identities, Noether’s inverse second theorem (Theorem 6), and Theorem 7 on solutions of the master equation extend ACFT to prequantum field-antifield BRST theory [10, 12, 21, 33, 80]. Its Lagrangian depends on antifields and ghosts, associated to Noether identities and gauge symmetries of an original Lagrangian, and obeys the classical master equation. This prequantum BRST theory can be quantized in the framework of perturbative QFT in functional integral terms [23, 80, 100]. A problem is that functional integrals are not expressed into jets of fields [149]. However, there is a certain relation between the algebras of jets of classical fields and the algebras of quantum fields such that, in particular, any variational symmetry of a classical Lagrangian yields the identities which Euclidean Green functions of quantum fields satisfy [194, 195].
II. ACFT. The general framework

1 The main postulate

Generalizing the geometric formulations of classical gauge theory and gravitation theory in fiber bundle terms, let us postulate the following.

**Axiom 1.** Even classical fields are sections of smooth fiber bundles.

By virtue of Axiom 1, ACFT is represented as dynamic theory on fiber bundles and, therefore, is conventionally formulated in terms of jets of sections of these fiber bundles [37, 85, 129, 158, 161]. Note that we throughout are in the category of finite-dimensional smooth real manifolds, which are Hausdorff, second-countable and, consequently, paracompact. The paracompactness of manifolds is very essential for our consideration because of the abstract de Rham theorem on the sheaf cohomology (see item 4). In particular, analytic manifolds are also treated as the smooth ones since a paracompact analytic manifold need not admit the partition of unity by analytic functions.

2 Jet manifolds

Given a smooth fiber bundle $Y \rightarrow X$, a $k$-order jet $j^k$ s at a point $x \in X$ is defined as an equivalence class of sections $s$ of $Y \rightarrow X$ identified by $k + 1$ terms of their Taylor series at $x$. A key point is that a set $J^k Y$ of all $k$-order jets is a finite-dimensional smooth manifold coordinated by $(x^\lambda, y^i, y^i_\lambda, \ldots, y^i_{\lambda_1 \ldots \lambda_k})$, where $(x^\lambda, y^i)$ are bundle coordinates on $Y \rightarrow X$ and $y^i_{\lambda_1 \ldots \lambda_k}$ are coordinates of derivatives, i.e., $y^i_{\lambda_1 \ldots \lambda_k} \circ s = \partial_{\lambda_1} \ldots \partial_{\lambda_k} s(x)$ [85, 181, 197]. Accordingly, the infinite order jets are defined as equivalence classes of sections of a fiber bundle $Y \rightarrow X$ identified by their Taylor series. Infinite order jets form a paracompact Fréchet (not smooth) manifold $J^\infty Y$ [7, 85, 129, 181, 203]. It coincides with the projective limit of the inverse system of finite order jet manifolds

$$X \leftarrow Y \leftarrow J^1 Y \leftarrow \cdots J^{r-1} Y \leftarrow J^r Y \leftarrow \cdots.$$  \hspace{1cm} (1)

The main advantage of jet formalism is that it enables us to deal with finite-dimensional jet manifolds instead of infinite-dimensional spaces of fields. In the framework of jet formalism, a $k$-order differential equation on a fiber bundle $Y \rightarrow X$ is defined as a closed subbundle $\mathcal{E}$ of the jet bundle $J^k Y \rightarrow X$. Its solution is a section $s$ of $Y \rightarrow X$ whose jet prolongation $J^k s$ lives in $\mathcal{E}$. A necessary condition of the existence of a solution of a differential equation $\mathcal{E}$ is so called formal integrability of $\mathcal{E}$ [85, 129, 161]. A $k$-order differential operator on $Y \rightarrow X$ is defined as a morphism of the jet bundle $J^k Y \rightarrow X$ to some vector bundle $E \rightarrow X$. However, the kernel of a differential operator (e.g., an Euler–Lagrange operator) need not be a differential equation in a strict sense.

Note that there are different notions of jets. Jets of sections are particular jets of maps [125, 163] (see item 20) and jets of submanifolds [85, 129] (see item 24). Let us also mention the jets of modules over a commutative ring [129, 144] which are representative objects of
differential operators on modules [1, 106, 129]. In particular, given a smooth manifold $X$, jets of a projective $C^\infty(X)$-module $P$ of finite rank are exactly jets of sections of the vector bundle over $X$ whose module of sections is $P$ in accordance with the Serre–Swan theorem. The notion of jets is extended to modules over graded commutative rings [94] and modules over algebras of operadic type [155]. Jets of modules over a noncommutative ring however are not defined [94, 187]. A definition of higher-order differential operators in noncommutative geometry also meets a problem [94, 142, 185].

3 Jets and connections

Jet manifolds provides the language of modern differential geometry. Due to the canonical bundle monomorphism $J^1Y \to T^*X \otimes VY$ over $Y$, any connection $\Gamma$ on a fiber bundle $Y \to X$ is represented by a global section

$$\Gamma = dx^\lambda \otimes (\partial_\alpha + \Gamma^i_\lambda(x^\mu, y^j)\partial_i) = dx^\lambda \otimes (\partial_\alpha + y^j_\lambda\partial_i) \circ \Gamma$$

of the jet bundle $J^1Y \to X$ and vice versa [85, 144, 181, 197]. Accordingly, we have the $T^*X \otimes VY$-valued first order differential operator

$$D = dx^\lambda \otimes (y^i_\alpha - \Gamma^i_\lambda(x^\mu, y^j)\partial_i)$$

on $Y$. It is called the covariant differential.

Classical field theory and time-dependent mechanics, developed as particular field theory on bundles over $X = \mathbb{R}$ (see item 23), involve the concept of a connection in many aspects [143, 144]. Quantum theory appeals to an algebraic notion of a connection on modules and sheaves [94, 128, 144, 188]. Jets of modules underlie the notion of a connection on modules over commutative rings. This notion is equivalent to that of a connection on vector bundles $Y \to X$ in the case of $C^\infty(X)$-modules of their sections. In contrast with jets, connections on modules over a noncommutative ring are also well defined [56, 94, 136].

4 Lagrangian theory of even fields

We restrict our consideration to Lagrangian field theory, i.e., field equations are Euler–Lagrange equations. Note that, if a field model is characterized by a nonvariational operator, the Koszul–Tate complex of its Noether identities can be constructed [191], and this field model can be extended to the BRST one.

**Axiom 2.** ACFT is Lagrangian theory.

There is the extensive literature on the calculus of variations and Lagrangian formalism on fiber bundles in terms of jet manifolds [3, 25, 52, 81, 85, 99, 102, 104, 112, 130, 202, 207]. We formulate Lagrangian theory of even fields in algebraic terms of the variational bicomplex [6, 7, 25, 88, 144, 157, 180, 203, 205]. Namely, one associates to a fiber bundle $Y \to X$ the following graded differential algebra (henceforth GDA) $\mathcal{O}_\infty^*Y$. 

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The inverse system (1) of jet manifolds yields the direct system

\[ O^*X \rightarrow O^*Y \rightarrow O^*_1Y \rightarrow \cdots O^*_{r-1}Y \rightarrow O^*_rY \rightarrow \cdots \]  

(2)
of GDAs \( O^*_\infty Y \) of exterior forms on jet manifolds \( J^rY \). Its direct limit is the above mentioned GDA \( O^*_\infty Y \) of all exterior forms on finite order jet manifolds (local forms in the terminology of [10, 12, 33]). This GDA is locally generated by horizontal forms \( dx^\lambda \) and contact forms \( \theta^\Lambda_A = dy^i_A - y^i_{\Lambda+A}dx^\lambda \), where \( \Lambda = (\lambda_k...\lambda_1) \) denotes a symmetric multi-index, and \( \lambda + \Lambda = (\lambda\lambda_k...\lambda_1) \). There is the canonical decomposition of \( O^*_\infty Y \) into the modules \( O^{k,m}_\infty Y \) of \( k \)-contact and \( m \)-horizontal forms \( (m \leq n = \dim X) \). Accordingly, the exterior differential on \( O^*_\infty Y \) falls into the sum \( d = d_Y + d_H \) of the vertical differential \( d_Y : O^{k,*}_\infty Y \rightarrow O^{k+1,*}_\infty Y \) and the total one \( d_H : O^{*,m}_\infty Y \rightarrow O^{*,m+1}_\infty Y \). One also introduces the projector \( \varrho \) on \( O^{*,0}_\infty Y \) such that \( \varrho \circ d_H = 0 \) and the variational operator \( \delta = \varrho \circ d \) on \( O^{*,n}_\infty Y \) such that \( \delta \circ d_H = 0 \), \( \delta \circ \delta = 0 \). All these operators split the GDA \( O^*_\infty Y \) into the variational bicomplex. We consider its subcomplexes

\[ 0 \rightarrow \mathbb{R} \rightarrow O^0_\infty Y \xrightarrow{d_H} O^{0,1}_\infty Y \rightarrow \cdots \xrightarrow{E_1} \delta \xrightarrow{E_2} \cdots, \]  

(3)

\[ 0 \rightarrow O^{1,0}_\infty Y \xrightarrow{d_H} O^{1,1}_\infty Y \rightarrow \cdots \xrightarrow{E_1} \delta \rightarrow E_2 \rightarrow 0, \quad (4) \]

Their elements \( L \in O^{0,n}_\infty Y \) and \( \delta L \in E_1 \) are a finite order Lagrangian on a fiber bundle \( Y \rightarrow X \) and its Euler–Lagrange operators, respectively.

The algebraic Poincaré lemma [157, 205] states that the variational bicomplex \( O^*_\infty Y \) is locally exact. In order to obtain its cohomology, one therefore can use the abstract de Rham theorem on sheaf cohomology [115] and the fact that \( Y \) is a strong deformation retract of \( J^{\infty}Y \), i.e., sheaf cohomology of \( J^{\infty}Y \) equals that of \( Y \) [7, 88]. A problem is that the paracompact space \( J^{\infty}Y \) admits the partition of unity by functions which do not belong to \( O^{\infty}_\infty Y \). Therefore, one considers the variational bicomplex \( Q^*_\infty Y \supset O^*_\infty Y \) whose elements are locally exterior forms on finite order jet manifolds, and obtains its cohomology [6, 203]. Afterwards, the \( d_H \)- and \( \delta \)-cohomology of \( O^*_\infty Y \) is proved to be isomorphic to that of \( Q^*_\infty Y \) [87, 88, 180]. In particular, cohomology of the variational complex (3) equals the de Rham cohomology of \( Y \), while the complex (4) is exact.

The exactness of the complex (4) at the last term states the global first variational formula which, firstly, shows that an Euler–Lagrange operator \( \delta L \) is really a variational operator of the calculus of variations and, secondly, leads to Noether’s first theorem. Cohomology of the variational complex (3) at the term \( O^{0,n}_\infty Y \) provides a solution of the global inverse problem of the calculus of variations on fiber bundles. It is the cohomology of variationally trivial Lagrangians, which are locally \( d_H \)-exact. Note that this cohomology has been also derived from cohomology of variational sequences of finite order [6, 131, 208], and in a different way in [207].

Noether’s first theorem is stated in a general case of variational symmetries depending on higher-order derivatives of fields. Noether’s second theorem is also formulated in jet terms in a general setting [17, 62, 78]. In the case of reducible degenerate Lagrangian systems, one however meets a problem of definition of higher-stage Noether identities. This problem
is solved by constructing their Koszul–Tate complex [18, 20], but its construction involves odd antifields and leads to Grassmann-graded extension of original even field theory.

5 Odd fields

The algebraic formulation of Lagrangian theory of even fields in terms of the variational bicomplex is generalized to odd fields [10, 12, 18, 20, 93]. Note that odd fields in ACFT need not satisfy the standard spin-statistic connection. These are odd bosonic ghosts and antifields, though there exist odd Klein transformations bringing them into even fields [29].

In many field models (e.g., SUSY gauge theory), odd fields unlike even ones have no geometric feature. ACFT overcomes this inconsistence. There are different geometric descriptions of odd fields in terms both of supermanifolds [49, 73] and graded manifolds [42, 43, 151, 152, 178]. Note that graded manifolds [27, 127, 145] are not supermanifolds [13, 14, 36, 166], though every graded manifold can be associated to a DeWitt $H^\infty$-supermanifold, and vice versa [13, 24, 54]. Both graded manifolds and supermanifolds are described in terms of sheaves of graded commutative algebras [13, 144]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. Lie supergroups, vector and principal superbundles are defined both in the category of graded manifolds [5, 32, 113, 199] and that of supermanifolds [13, 50, 144, 196]. Let us mention a different definition of a super Lie group as a Harish–Chandra pair of a Lie group and a super Lie algebra [44, 53].

In ACFT, odd and even fields are described on the same level due to an appropriate extension of the GDA $O^\ast Y$. Since QFT deals with linear spaces of fields, let a bundle $Y \to X$ of classical fields be a vector bundle. Then all jet bundles $J_kY \to X$ are also vector bundles. Let us consider a subalgebra $P^\ast_{\infty}Y \subset O^\ast_{\infty}Y$ of exterior forms whose coefficients are polynomial in fiber coordinates $y^i, y^i_\Lambda$ on these bundles. In particular, the commutative ring $P^0_{\infty}Y$ consists of polynomials of coordinates $y^i, y^i_\Lambda$ with coefficients in the ring $C^\infty(X)$. One can associate to such a polynomial a section of the symmetric tensor product $\bigotimes^m(J^kY)^*$ of the dual of some jet bundle $J^kY \to X$, and vice versa. Moreover, one can show that any element of $P^\ast_{\infty}Y$ is an element of the Chevalley–Eilenberg differential calculus over $P^0_{\infty}Y$. This construction is extended to the case of odd fields if, given a vector bundle $F \to X$ and jet bundles $J^kF \to X$, one considers their exterior products, whose sections form a graded commutative ring (see item 6). The result is a Grassmann-graded GDA $S^\ast_{\infty}[F;Y] \supset O^\ast_{\infty}Y$ which is split into the variational bicomplex (see item 7) and, thus, describes Lagrangian theory both of even and odd fields. We therefore postulate the following.

**Axiom 3.** The algebra of ACFT of even and odd fields is the GDA $S^\ast_{\infty}[F;Y]$ introduced below.
6 The algebra of even and odd fields

Treating odd fields on a smooth manifold \( X \), we follow the Serre–Swan theorem generalized to graded manifolds. It states that, if a Grassmann \( C^\infty(X) \)-algebra is the exterior algebra of some projective \( C^\infty(X) \)-module of finite rank, it is isomorphic to the algebra of graded functions on a graded manifold with a body \( X \) [20]. Note that \( X \) need not be compact [94, 164, 179]. By virtue of the Batchelor theorem [13], any graded manifold with a body \( X \) is isomorphic to a graded manifold \((X, \mathcal{A}_F)\) with the structure sheaf \( \mathcal{A}_F \) of germs of sections of the exterior bundle

\[
\wedge F^* = \mathbb{R} \oplus F^* \oplus \wedge^2 F^* \oplus \cdots,
\]

where \( F^* \) is the dual of some vector bundle \( F \to X \). In field models, Batchelor’s isomorphism is fixed from the beginning. We call \((X, \mathcal{A}_F)\) the simple graded manifold modelled over \( F \).

Then the Grassmann-graded Chevalley–Eilenberg differential calculus

\[
0 \to \mathbb{R} \to \mathcal{A}_F \xrightarrow{d} S^1[F; X] \xrightarrow{d} \cdots S^k[F; X] \xrightarrow{d} \cdots
\]

over \( \mathcal{A}_F \) can be constructed [75, 94]. One can think of its elements as being graded differential forms on \( X \). In particular, there is a monomorphism \( O^*X \to S^*[F; X] \). Following suit of an even GDA \( P^*_\infty Y \), let us consider simple graded manifolds \((X, \mathcal{A}_{J^rF})\) modelled over the vector bundles \( J^rF \to X \). We have the direct system of corresponding GDAs

\[
S^*[F; X] \longrightarrow S^*[J^1F; X] \longrightarrow \cdots S^*[J^rF; X] \longrightarrow \cdots,
\]

whose direct limit \( S^\infty_\infty[F; X] \) is the Grassmann counterpart of an even GDA \( P^*_\infty Y \).

The total algebra of even and odd fields is the graded exterior product

\[
P^*_\infty[F; Y] = P^*_\infty Y \wedge_{\mathcal{O}^*X_\infty} S^*_\infty[F; X]
\]

of the GDAs \( P^*_\infty Y \) and \( S^*_\infty[F; X] \) over their common subalgebra \( \mathcal{O}^*X \) [18, 93]. In particular, \( P^0_\infty[F; Y] \) is a graded commutative \( C^\infty(X) \)-ring whose even and odd generating elements are sections of \( Y \to X \) and \( F \to X \), respectively. Let \((x^\lambda, y^i, y^i_\lambda)\) be bundle coordinates on jet bundles \( J^kY \to X \) and \((x^\lambda, c^a, c^a_\lambda)\) those on \( J^rF \to X \). For simplicity, let these symbols also stand for local sections \( s \) of these bundles such that \( s^i_\lambda(x) = y^i_\lambda \) and \( s^a_\lambda(x) = c^a_\lambda \). Then the GDA \( P^*_\infty[F; Y] \) (5) is locally generated by elements \((y^i, y^i_\lambda, c^a, c^a_\lambda, dx^\lambda, dy^i, dy^i_\lambda, dc^a, dc^a_\lambda)\). By analogy with \((y^i, y^i_\lambda)\), one can think of odd generating elements \((c^a, c^a_\lambda)\) as being (local) odd fields and their jets.

Note that this definition of jets of odd fields differs both from the above mentioned notion of jets of modules over a graded commutative ring [94] and the definition of jets of graded fiber bundles [113, 152]. However, it enables us to consider even and odd fields on the same level, and reproduces the heuristic notion of jets of odd ghosts in Lagrangian
BRST theory [12, 35]. Moreover, one can say that sections of vector bundles \( Y \rightarrow X \) and \( F \rightarrow X \) seen as generating elements of the ring \( \mathcal{P}_\infty^0[F; Y] \) are sui generis prequantum fields.

In a general setting, if \( Y \rightarrow X \) is not a vector bundle, we consider graded manifolds \( (J^rY, \mathfrak{A}_r) \) whose bodies are jet manifolds \( J^rY \), and \( F_r = J^rY \times J^rF \) is the pull-back onto \( J^rY \) of the jet bundle \( J^rF \rightarrow X \) [20, 21]. As a result, we obtain the direct system of GDAs
\[
\mathcal{S}^*[Y \times F; Y] \rightarrow \mathcal{S}^*[F_1; J^1Y] \rightarrow \cdots \mathcal{S}^*[F_r; J^rY] \rightarrow \cdots,
\]
whose direct limit is the GDA \( \mathcal{S}^\infty_\infty[F; Y] \) in Axiom 3. It is a differential calculus over the ring \( \mathcal{S}^\infty_\infty[F; Y] \) of graded functions. The monomorphisms \( \mathcal{O}^*_rY \rightarrow \mathcal{S}^*[F_r; J^rY] \) yield a monomorphism of the direct system (2) to that (6) and, consequently, the monomorphism \( \mathcal{O}^*_\infty Y \rightarrow \mathcal{S}^\infty_\infty[F; Y] \) of their direct limits. Moreover, \( \mathcal{S}^\infty_\infty[F; Y] \) is a \( \mathcal{O}^0_\infty \)-algebra. It contains the \( \mathcal{C}^\infty(X) \)-subalgebra \( \mathcal{P}^*_\infty[F; Y] \) if a fiber bundle \( Y \rightarrow X \) is affine. The \( \mathcal{O}^0_\infty \)-algebra \( \mathcal{S}^\infty_\infty[F; Y] \) is locally generated by elements \( (c^\alpha, c^\alpha_A, dx^\lambda, dy^i, dy^i_\lambda, dc^\alpha, dc^\alpha_A) \) with coefficient functions depending on coordinates \( (x^\lambda, y^i, y^i_\lambda) \). One calls \( (y^i, c^\alpha) \) the local basis for the GDA \( \mathcal{S}^\infty_\infty[F; Y] \). We further use the collective symbol \( s^A \) for its elements. Accordingly, \( s^A \) denote jets of \( s^A \), \( \theta^A_A = ds^A_A - s^A_A d^\lambda \) are contact forms, and \( d^\lambda \) are graded derivations of the \( \mathbb{R} \)-ring \( \mathcal{S}^0_\infty[F; Y] \) such that \( \delta d^\lambda = \delta^\lambda d^\lambda \). The symbol \( [A] = [s^A] = [s^A] \) stands for the Grassmann parity.

7 Lagrangian theory of even and odd fields

There is the canonical decomposition of the GDA \( \mathcal{S}^\infty_\infty[F; Y] \) into modules \( \mathcal{S}^{k,m}_\infty[F; Y] \) of \( k \)-contact and \( m \)-horizontal graded forms. Accordingly, the graded exterior differential on \( \mathcal{S}^\infty_\infty[F; Y] \) falls into the sum \( d = d_V + d_H \) of the vertical differential \( d_V \) and the total differential
\[
d_H(\phi) = dx^\lambda \wedge d\lambda(\phi), \quad d\lambda = \partial\lambda + \sum_{0 \leq |A|} s^A_A d^\lambda, \quad \phi \in \mathcal{S}^\infty_\infty[F; Y],
\]
\[
d_H \circ h_0 = h_0 \circ d, \quad h_0: \mathcal{S}^\infty_\infty[F; Y] \rightarrow \mathcal{S}^{0,*}_\infty[F; Y].
\]
We also have the graded projection endomorphism \( \varrho \) of \( \mathcal{S}^{0,n}_\infty[F; Y] \) such that \( \varrho \circ d_H = 0 \) and the graded variational operator \( \delta = \varrho \circ d \) such that \( \delta \circ d_H = 0, \delta \circ \delta = 0. \) With these operators the GDA \( \mathcal{S}^\infty_\infty[F; Y] \) is split into the Grassmann-graded variational bicomplex. It contains the subcomplexes
\[
0 \rightarrow \mathbb{R} \rightarrow \mathcal{S}^0_\infty[F; Y] \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty[F; Y] \xrightarrow{d_H} \mathcal{S}^{0,n}_\infty[F; Y] \xrightarrow{\delta} \mathbf{E}_1 = \varrho(\mathcal{S}^{1,n}_\infty[F; Y]), \quad (7)
\]
\[
0 \rightarrow \mathcal{S}^{1,0}_\infty[F; Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty[F; Y] \xrightarrow{d_H} \mathcal{S}^{1,n}_\infty[F; Y] \xrightarrow{\varrho} \mathbf{E}_1 \rightarrow 0. \quad (8)
\]
One can think of their even elements
\[
L = L \omega \in \mathcal{S}^{0,n}_\infty[F; Y], \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad (9)
\]
\[
\delta L = \theta^A \wedge \mathcal{E}_A \omega = \sum_{0 \leq |A|} (-1)^{|A|} \theta^A \wedge d^A_A L \omega \in \mathbf{E}_1, \quad (10)
\]
as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

The algebraic Poincaré lemma states that the complexes (7) and (8) are locally exact at all the terms, except $R_{12, 55, 93}$. Then one can obtain cohomology of these complexes in the same manner as that of the complexes (3) and (4) [21, 93, 190].

**Theorem 1.** Cohomology of the variational complex (7) equals the de Rham cohomology $H^*(Y)$ of $Y$. The complex (8) is exact.

Cohomology of the complex (7) at the term $S_{1,n}^{1,n}[F; Y]$ provides the following solution of the global inverse problem of the calculus of variation for graded Lagrangians.

**Theorem 2.** A $\delta$-closed (i.e., variationally trivial) graded density reads

$$L_0 = h_0 \psi + \delta H \xi, \quad \xi \in S_{0}^{1,n-1}[F; Y],$$

where $\psi$ is a non-exact $n$-form on $Y$. In particular, a $\delta$-closed odd density is $d_H$-exact.

Exactness of the complex (8) at the last term implies that any Lagrangian $L$ admits the decomposition

$$dL = \delta L - d_H \Xi, \quad \Xi \in S_{1,n}^{1,n-1}[F; Y],$$

where $L + \Xi$ is a Lepagean equivalent of $L$ [93]. This decomposition leads to the first variational formula (Theorem 3) and Noether’s first theorem (Theorem 4).

**8 Noether’s first theorem in a general setting**

Infinitesimal supersymmetries of ACFT, described by the GDA $S_\infty^*[F; Y]$, are defined as contact graded derivations of the $\mathbb{R}$-ring $S_\infty^0[F; Y]$ [18, 93]. Its graded derivation $\vartheta$ is called contact if the Lie derivative $L_\vartheta$ of the GDA $S_\infty^*[F; Y]$ preserves the ideal of contact graded forms. Contact graded derivations take the form

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda \partial_\lambda + \left( v^A \partial_A + \sum_{|A|>0} d_A v^A \partial_A^A \right), \quad v^A = \vartheta^A - s_\mu^A \partial_\mu,$$

where $\vartheta^\lambda$, $\vartheta^A$ are local graded functions. They constitute the most general class of so called generalized (depending on derivatives) symmetries. Generalized symmetries of differential equations [8, 37, 116, 129, 157] and Lagrangian systems [37, 62, 93, 157] have been intensively investigated. In Lagrangian field theory, generalized symmetries are exemplified by BRST transformations [12, 18, 23, 93, 100].

**Theorem 3.** It follows from the decomposition (11) that the Lie derivative $L_{\vartheta}L$ of a Lagrangian $L$ (9) with respect to an arbitrary supersymmetry $\vartheta$ (12) fulfills the first variational formula

$$L_{\vartheta}L = \vartheta_V |\delta L + d_H (h_0 (\vartheta |\Xi_L)) + d_V (\vartheta_H |\omega) L.$$

In particular, let $\vartheta$ be a vertical supersymmetry treated as an infinitesimal variation of dynamic variables. Then the first variational formula (13) shows that the Euler–Lagrange equations $\delta L = 0$ are variational equations.
A supersymmetry \( \vartheta \) (12) is called a variational symmetry of a Lagrangian \( L \) if the Lie derivative \( L_{\vartheta} L \) of \( L \) is \( d_H \)-exact. One can show that \( \vartheta \) is a variational symmetry iff its vertical part \( \nu_V \) (12) is well. Therefore, we further restrict our consideration to vertical supersymmetries

\[
\vartheta = (\nu^A \partial_A + \sum_{|A| > 0} d_A \nu^A \partial_A^A).
\tag{14}
\]

A glance at the expression (14) shows that a vertical supersymmetry is an infinite jet prolongation of its first summand \( \nu = \nu^A \partial_A \), called the generalized vector field. Substituting \( \vartheta \) (14) into the first variational formula (13), we come to Noether’s first theorem.

**Theorem 4.** If \( \vartheta \) (14) is a variational symmetry of a Lagrangian \( L \) (i.e., \( L_{\vartheta} L = d_H \sigma, \sigma \in S_{0,n-1}^\infty \)), the weak conservation law

\[
0 \approx d_H (h_0(\vartheta)] \Xi_L) - \sigma
\]

the Noether current \( J_\vartheta = h_0(\vartheta)] \Xi_L) \) holds on the shell \( \delta L = 0 \).

A vertical supersymmetry \( \vartheta \) (14) is called nilpotent if \( L_{\vartheta}(L_{\vartheta} \phi) = 0 \) for any horizontal graded form \( \phi \in S_{0,\ast}^\infty[F;Y] \). An even supersymmetry is never nilpotent.

For the sake of simplicity, the common symbol further stands for a generalized vector field \( \nu \), the contact graded derivation \( \vartheta \) (14) determined by \( \nu \) and the Lie derivative \( L_\vartheta \).

We agree to call all these operators a graded derivation of the GDA \( S_{0,n-1}^\infty[F;Y] \).

9 The Koszul–Tate complex of Noether identities

As was mentioned above, quantization of a Lagrangian field theory essentially depends on its degeneracy. The Euler–Lagrange operator (10) generally obeys non-trivial Noether identities, which need not be independent, but satisfy first-stage Noether identities, and so on. Thus, there is a hierarchy of reducible Noether identities. Note that any Euler–Lagrange operator obeys trivial Noether identities which are defined as boundaries of a certain chain complex [20, 21, 191]. A problem is that trivial higher-stage Noether identities need not be boundaries, unless a certain condition holds.

The notion of reducible Noether identities came from that of reducible constraints. By analogy with constraints, the Koszul–Tate complex of reducible Noether identities has been invented under rather restrictive regularity condition that Noether identities of arbitrary stage can be locally separated into independent and dependent ones [67, 68]. This condition has also come from the case of constraints locally given by a finite set of functions which the inverse mapping theorem is applied to. A problem is that, in contrast with constraints, Noether identities of any stage are differential operators. They are locally given by a set of functions and their jet prolongations on an infinite order jet manifold. Since the latter is a Fréchet, but not Banach manifold, the inverse mapping theorem fails to be valid.

We show that, if non-trivial Noether identities of any stage are finitely generated and if they obey a certain homology regularity condition, one can associate to the Euler–Lagrange operator of a degenerate Lagrangian system the exact Koszul–Tate complex whose boundary operator provides all the Noether identities (Theorem 5) [20, 21, 96]. This complex is an
extension of the original GDA $S^*_\infty[F;Y]$ by means of antifields whose spaces are density-dual to the modules of Noether identities.

Let us introduce the following notation. The density dual of a vector bundle $E \rightarrow X$ is $\overline{E}^* = E^* \otimes \overset{n}{\wedge} T^*X$. Given vector bundles $E \rightarrow X$ and $V \rightarrow X$, let $S^*_\infty[V \times F;Y \times E]$ be the extension of the GDA $S^*_\infty[F;Y]$ whose additional even and odd generators are sections of $E \rightarrow X$ and $V \rightarrow X$, respectively. We consider its subalgebra $P^*_\infty[V,F;Y,E]$ with coefficients polynomial in these new generators. Let us also assume that the vertical tangent bundle $VY$ of $Y$ admits the splitting $VY = Y \times W$, where $W \rightarrow X$ is a vector bundle. In this case, there no fiber bundles under consideration whose transition functions vanish on the shell $\delta L = 0$. Let $\overline{Y}^*$ denote the density-dual of $W$ in this splitting.

Let $L$ be a Lagrangian (9) and $\delta L$ its Euler–Lagrange operator (10). In order to describe Noether identities which $\delta L$ satisfies, let us enlarge the GDA $S^*_\infty[F;Y]$ to the GDA $P^*_\infty[\overline{Y}^*,F;Y,\overline{F}^*]$ with the local basis $\{s^A, \overline{s}_A\}$; $[\overline{s}_A] = ([A] + 1) \mod 2$. Its elements $\overline{s}_A$ are called antifields of antifield number Ant$[\overline{s}_A] = 1$ [12, 100]. The GDA $P^*_\infty[\overline{Y}^*,F;Y,\overline{F}^*]$ is endowed with the nilpotent right graded derivation $\overline{\delta} = \overline{\partial}^A \overline{E}_A$. We have the chain complex

$$0 \leftarrow \text{Im} \overline{\delta} \leftarrow P^0_{\infty}[\overline{Y}^*,F;Y,\overline{F}^*]_1 \leftarrow P^0_{\infty}[\overline{Y}^*,F;Y,\overline{F}^*]_2$$

(15)
of graded densities of antifield number $\leq 2$. Its one-cycles define the above mentioned Noether identities, which are trivial iff cycles are boundaries. Accordingly, elements of the first homology $H_1(\overline{\delta})$ of the complex (15) correspond to non-trivial Noether identities modulo the trivial ones [20, 21, 96, 191]. We assume that $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a projective Grassmann-graded $C^\infty(X)$-module $C_{(0)} \subset H_1(\overline{\delta})$ of finite rank with a local basis $\{\Delta_r\}$ such that any Noether identity is a corollary of the Noether identities

$$\overline{\delta}\Delta_r = \sum_{0 \leq |A|} \Delta^{A,A}_r d_A E_A = 0.$$  

(16)
The Noether identities (16) need not be independent, but obey first-stage Noether identities described as follows. By virtue of the Serre–Swan theorem, the module $C_{(0)}$ is isomorphic to a module of sections of the product $\overline{V}^* \times \overline{E}^*$, where $\overline{V}^*$ and $\overline{E}^*$ are the density-duals of some vector bundles $V \rightarrow X$ and $E \rightarrow X$. Let us enlarge the GDA $P^*_\infty[\overline{V}^*,F;Y,\overline{F}^*]$ to the GDA $P^*_\infty[\overline{V}^*,F;Y,\overline{F}^* \times \overline{V}]$ possessing the local basis $\{s^A, \overline{s}_A, \overline{\tau}_r\}$ of Grassmann parity $[\overline{\tau}_r] = ([\Delta_r] + 1) \mod 2$ and antifield number Ant$[\overline{\tau}_r] = 2$. This GDA is provided with the nilpotent right graded derivation $\overline{\delta}_0 = \overline{\delta} + \overline{\partial}^r \Delta_r$ such that its nilpotency condition is equivalent to the Noether identities (16). Then we have the chain complex

$$0 \leftarrow \text{Im} \overline{\delta}_0 \leftarrow P^0_{\infty}[\overline{V}^*,F;Y,\overline{F}^*]_1 \leftarrow P^0_{\infty}[\overline{V}^*,F;Y,\overline{F}^* \times \overline{V}]_2$$

(17)
of graded densities of antifield number $\leq 3$. It has the trivial homology $H_0(\overline{\delta}_0)$ and $H_1(\overline{\delta}_0)$. The two-cycles of this complex define the above mentioned first-stage Noether identities.
They are trivial if cycles are boundaries, but the converse need not be true, unless a certain homology condition holds \([20, 21, 96, 191]\). If the complex (17) obeys this condition, elements of its second homology \(H_2(\delta_0)\) define non-trivial first-stage Noether identities modulo the trivial ones. Let us assume that \(H_2(\delta_0)\) is finitely generated. Namely, there exists a projective Grassmann-graded \(C^\infty(X)\)-module \(\mathcal{C}_{(1)} \subset H_2(\delta_0)\) of finite rank with a local basis \(\{\Delta_{r_1}\}\) such that any first-stage Noether identity is a corollary of the equalities

\[
\sum_{0 \leq |\Lambda|} \Delta_{r_1}^{r, A} d_A \Delta_r + \overline{\delta} h_{r_1} = 0. \tag{18}
\]

The first-stage Noether identities (18) need not be independent, but satisfy the second-stage ones, and so on. Iterating the arguments, we come to the following \([20, 21, 96]\).

\textbf{Theorem 5.} One can associate to a degenerate \(N\)-stage reducible Lagrangian system the exact Koszul–Tate complex (21) with the boundary operator (20) whose nilpotency property restarts all Noether and higher-stage Noether identities \((16)\) and \((22)\) if these identities are finitely generated and iff this complex obeys the homology regularity condition.

Namely, there are vector bundles \(V_1, \ldots, V_N, E_1, \ldots, E_N\) over \(X\) and the GDA

\[
\overline{P}_\infty^* \{N\} = P_\infty^* \{E_N \times \cdots \times E_1 \times E^* \times Y^*, F; Y, F^* \times \nabla^* \times \nabla_1 \times \cdots \times \nabla_N\} \tag{19}
\]

with a local basis \(\{s^A, \tau_A, \tau_{r_1}, \ldots, \tau_{r_N}\}\) of antifield number \(\text{Ant}[\tau_{r_k}] = k + 2\). Let the indexes \(k = -1, 0\) further stand for \(\overline{\tau}_A\) and \(\tau_r\), respectively. The GDA \(\overline{P}_\infty^* \{N\}\) (19) is provided with the nilpotent right graded derivation (the Koszul–Tate differential)

\[
\delta_N = \overleftarrow{\partial} A \mathcal{E}_A + \sum_{0 \leq |\Lambda|} \overleftarrow{\partial} r \Delta_{r_1}^{r, A} \mathcal{E}_A + \sum_{1 \leq k \leq N} \overleftarrow{\partial} r_k \Delta_{r_k}, \tag{20}
\]

of antifield number \(-1\). With \(\delta_N\), we have the exact chain complex

\[
0 \leftrightarrow \text{Im} \overleftarrow{\partial} \overleftarrow{\partial} P_\infty^0 \{N\} \subset P_\infty^0 \{N - 1\} \subset \cdots
\]

\[
\delta_{N-1} \overleftarrow{\partial} \overrightarrow{P}_\infty^0 \{N - 1\} \leftrightarrow \delta_N \overleftarrow{\partial} \overrightarrow{P}_\infty^0 \{N\} \leftrightarrow \delta_N \overleftarrow{\partial} \overrightarrow{P}_\infty^0 \{N\} \leftrightarrow \cdots \tag{21}
\]

of graded densities of antifield number \(\leq N + 3\) which is assumed to satisfy the homology regularity condition. This condition states that any \(\delta_{k < N - 1}\)-cycle \(\phi \in \overrightarrow{P}_\infty^0 \{k\}_{k+3} \subset \overrightarrow{P}_\infty^0 \{k + 1\}_{k+3}\) is a \(\delta_{k + 1}\)-boundary. The nilpotency property of the boundary operator \(\delta_N\) (20) implies the Noether identities (16) and the \((k \leq N)\)-stage Noether identities

\[
\sum_{0 \leq |\Lambda|} \Delta_{r_1}^{r, A} d_A \left( \sum_{0 \leq |\Sigma|} \Delta_{r_{k-1}}^{r, \Sigma} \mathcal{E}_{\Sigma k-2} \right) + \overline{\delta} \left( \sum_{0 \leq |\Sigma|, |\Xi|} h_{r_k}^{(r_{k-2}, \Sigma)} \mathcal{E}_{\Sigma k-2} \mathcal{E}_A \right) = 0. \tag{22}
\]
Noether’s inverse second theorem

Noether’s second theorem in different variants relates the Noether and higher-stage Noether identities to the gauge and higher-stage gauge symmetries of a Lagrangian system [17, 18, 78, 96]. However, the notion of a general gauge symmetry of a Lagrangian system and, consequently, a formulation of Noether’s direct second theorem meet difficulties. In particular, it may happen that gauge symmetries are not assembled into an algebra, or they form an algebra on-shell [77, 100]. At the same time, Noether identities are well defined (Theorem 5). Therefore, one can Noether’s inverse second theorem (Theorem 6) in order to obtain gauge symmetries of a degenerate Lagrangian system. This theorem associates to the antifield Koszul–Tate complex (21) the cochain sequence (24) of ghosts, whose ascent operator (25) provides gauge and higher-stage gauge symmetries of a Lagrangian field theory.

Given the GDA $P_\infty^\ast\{N\}$ (19), let us consider the GDA

$$\mathcal{P}_\infty^\ast\{N\} = \mathcal{P}_\infty^\ast[V_N \times \cdots \times V_1 \times V, F; Y, E \times E_1 \times \cdots \times E_N]$$

possessing the local basis $\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}\}$ of Grassmann parity $[c^{r_k}] = ([c^{r_k}] + 1) \text{mod} \ 2$ and antifield number $\text{Ant}[c^{r_k}] = -(k + 1)$. Its elements $c^{r_k}, k \in \mathbb{N}$, are called the ghosts of ghost number $\text{gh}[c^{r_k}] = k + 1$ [12, 100].

**Theorem 6.** Given the Koszul–Tate complex (21), the graded commutative ring $\mathcal{P}_\infty^0\{N\}$ is split into the cochain sequence

$$0 \to S_\infty^0[F; Y] \xrightarrow{u_e} \mathcal{P}_\infty^0\{N\}_1 \xrightarrow{u_e} \mathcal{P}_\infty^0\{N\}_2 \xrightarrow{u_e} \cdots,$$

with the odd ascent operator

$$u_e = u + \sum_{1 \leq k \leq N} u_{(k)},$$

$$u = u^A \frac{\partial}{\partial s^A}, \quad u^A = \sum_{0 \leq |A|} c^r_A \eta(\Delta_{rA})^A,$$

$$u_{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}}, \quad u^{r_{k-1}} = \sum_{0 \leq |A|} c^{r_k}_A \eta(\Delta_{r_{k-1}A})^A, \quad k = 1, \ldots, N,$$

$$\eta(f)^A = \sum_{0 \leq |A| \leq k - |A|} (-1)^{|A+\Lambda|} C_{|A\Sigma\Lambda|}^{[\Sigma\Lambda]} f^{\Sigma+\Lambda}, \quad C_b^a = \frac{b!}{a!(b-a)!}.$$

The components $u$ (26), $u_{(k)}$ (27) of the ascent operator $u_e$ (25) are the above mentioned gauge and higher-stage gauge symmetries of an original Lagrangian, respectively. Indeed, let us consider the total GDA $P_\infty^*\{N\}$ generated by original fields, ghosts and antifields

$$\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}, s^A, c^r, c^{r_1}, \ldots, c^{r_N}\}.$$ 

It contains subalgebras $\mathcal{P}_\infty^\ast\{N\}$ (19) and $\mathcal{P}_\infty^\ast\{N\}$ (23), whose operators $\delta_N$ (20) and $u_e$ (25) are prolonged to $P_\infty^*\{N\}$. Let us extend an original Lagrangian $L$ to the Lagrangian

$$L_e = L_e \omega = L + L_1 = L + \sum_{0 \leq k \leq N} c^{r_k} \Delta_{r_k} \omega = L + \delta_N(\sum_{0 \leq k \leq N} c^{r_k} \bar{c}^{r_k} \omega)$$

(29)
of zero antifield number. It is readily observed that the Koszul–Tate differential \( \delta_N \) is a variational symmetry of the Lagrangian \( L_e \) (29), i.e., we have the equalities

\[
\frac{\delta (c^r \Delta r)}{\delta \bar{s}_A} \mathcal{E}_A \omega = u^A \mathcal{E}_A \omega = d_H \sigma_0, \tag{30}
\]

\[
\left[ \frac{\delta (c^r \Delta r)}{\delta \bar{s}_A} + \sum_{k<i} \frac{\delta (c^r \Delta r)}{\delta \bar{c}_r} \Delta_r^k \right] \omega = d_H \sigma_i, \quad i = 1, \ldots, N. \tag{31}
\]

A glance at the equality (30) shows that the graded derivation \( u \) (26) is a variational symmetry of an original Lagrangian \( L \). Parameterized by ghosts \( c^r \), it is a gauge symmetry of \( L \) [18, 93]. The equalities (31) are brought into the form

\[
\sum_{0 \leq |\Sigma|} d_\Sigma u^{\gamma_{\Sigma}} = \bar{\delta} (\alpha^{\gamma_{\Sigma}}), \quad \alpha^{\gamma_{\Sigma}} = - \sum_{0 \leq |\Sigma|} \eta \left( h^{(\gamma_{\Sigma})}_{(A, \Xi)} \right) \Sigma d_\Sigma (c^r \bar{s}_A). \tag{33}
\]

It follows that graded derivations \( u_{(k)} \) (27) are the \( k \)-stage gauge symmetries of a reducible Lagrangian system [17, 18, 96].

We agree to call \( u_e \) (25) the total gauge operator. In contrast with the Koszul–Tate one, this operator need not be nilpotent. However, one can say that gauge and higher-stage gauge symmetries of a Lagrangian system form an algebra (resp. an algebra on the shell) if the total gauge operator \( u_e \) can be extended to a graded derivation \( u_E \) of ghost number 1 which is nilpotent (resp. nilpotent on the shell) [19, 21, 189]. It reads

\[
u_E = u_e + \xi = u^A \partial_A + \sum_{1 \leq k \leq N} (u^{r_{k-1}} + \xi^{r_{k-1}}) \partial_{r_{k-1}} + \xi^{r_N} \partial_{r_N}, \tag{32}\]

where the coefficients \( \xi^{r_{k-1}} \) are at least quadratic in ghosts, and \( (u_E \circ u_E)(f) \) is zero (resp. \( \bar{\delta} \)-exact) on graded functions \( f \in \mathcal{P}^0_\infty \{N\} \). For instance, the total gauge operator in irreducible gauge theory is an operator of gauge transformations whose parameter functions are replaced with the ghosts. Its nilpotent extension (32) is a familiar BRST operator [19, 93].

11 BRST extended field theory

ACFT extended to ghosts and antifields exemplifies so called field-antifield Lagrangian systems of the following type [12, 100].

Given a fiber bundle \( Z \rightarrow X \) and a vector bundle \( Z' \rightarrow X \), let us consider a GDA \( \mathcal{P}^*_\infty [\mathbb{Z}, Z'; Z, Z'] \) with a local basis \( \{ z^a, \bar{z}_a \} \), where \( [\bar{z}_a] = ([z^a] + 1) \mod 2 \). One can think of its elements \( z^a \) and \( \bar{z}_a \) as being fields and antifields, respectively. Its submodule \( \mathcal{P}^{0, n}_\infty [\mathbb{Z}, Z'; Z, Z'] \) of horizontal densities is provided with the binary operation

\[
\{ \mathcal{L}_\omega, \mathcal{L}'_\omega \} = \left[ \frac{\bar{\delta} \mathcal{L}}{\delta z^a} \frac{\delta \mathcal{L}'}{\delta \bar{z}_a} + (-1)^{|z^a||\mathcal{L}'|} \frac{\bar{\delta} \mathcal{L}'}{\delta z^a} \frac{\delta \mathcal{L}}{\delta \bar{z}_a} \right] \omega, \tag{33}\]

14
called the antibracket by analogy with that in field-antifield BRST theory \[100\]. One treats this operation as \textit{sui generis} odd Poisson structure \[2, 11\]. Let us associate to a Lagrangian \( \mathcal{L} \) the odd graded derivations

\[
v_\mathcal{L} \equiv \delta \frac{\mathcal{L}}{\delta z^a} \partial_{z^a}, \quad \bar{v}_\mathcal{L} \equiv \partial \frac{\delta \mathcal{L}}{\delta z^a}.
\]  

(34)

Then the following conditions are equivalent: (i) the graded derivation \( v_\mathcal{L} \) (34) is a variational symmetry of a Lagrangian \( \mathcal{L} \), (ii) so is the graded derivation \( \bar{v}_\mathcal{L} \), (iii) the (classical) master equation

\[
\{ \mathcal{L}, \mathcal{L} \} = 2 \delta \frac{\mathcal{L}}{\delta z^a} \partial_{z^a} \omega = d_H \sigma
\]  

holds. For instance, any variationally trivial Lagrangian satisfies the master equation. We say that a solution of the master equation is not trivial if no graded derivation (34) vanishes.

Let us return to an original Lagrangian system \( L \) and its extension \( L_e \) (29) to antifields and ghosts (28), together with the odd graded derivations (34) which read

\[
v_\omega \equiv \delta \frac{\mathcal{L}_1}{\delta s^A} \partial_{s^A} + \sum_{0 \leq k \leq N} \delta \frac{\mathcal{L}_1}{\delta \bar{c}^r_k} \partial_{\bar{c}^r_k}, \quad \bar{v}_\omega \equiv \partial \frac{\delta \mathcal{L}_1}{\delta s^A} \partial_{s^A} + \sum_{0 \leq k \leq N} \bar{v}_\omega \delta \mathcal{L}_1 \partial_{\bar{c}^r_k} \partial_{\bar{c}^r_k}.
\]

An original Lagrangian \( L \) provides a trivial solution of the master equation. A goal is to extend it to a nontrivial solution

\[
L + L_1 + L_2 + \cdots = L_e + L' \quad (36)
\]

of the master equation by means of terms \( L_i \) of polynomial degree \( i > 1 \) in ghosts and zero antifield number. Such an extension need not exists. However, one can show the following \[21\].

**Theorem 7.** (i) A solution (36) of the master equation exists only if the graded derivation \( u_e \) (25) is extended to a graded derivation nilpotent on the shell. (ii) If the total gauge operator \( u_e \) (25) admits a nilpotent extension \( u_E \) (32) independent of antifields, then the Lagrangian

\[
L_E = L_e + \sum_{1 \leq k \leq N} \xi^{r_k-1} \bar{c}_{r_{k-1}} \omega = L + u_E ( \sum_{0 \leq k \leq N} c^{r_k-1} \bar{c} r_{k-1} ) \omega + d_H \sigma \quad (37)
\]

satisfies the master equation \( \{ L_E, L_E \} = 0 \), and \( v_e = u_E \) called the BRST operator.

Let ACFT with a Lagrangian \( L \) be extended to antifields and ghosts (28) which come from Theorems 5 and 6, and let its Lagrangian \( L \) admit an extension to a solution of the master equation (35). One can think of this BRST extended system as being prequantum field theory, which is quantized both in the framework of the Batalin–Vilkoviski (BV) quantization \[23, 100, 80\] and in a different way (see Section IV). For instance, this is the case of Yang–Mills gauge and SUSY gauge theories (see item 16).
12 Relative BRST cohomology

A solution of the classical master equation (35) is not unique. At least, it is defined up to variationally trivial Lagrangians. In particular, let a bundle $Y \to X$ of even classical fields be affine. Its de Rham cohomology equals that of $X$. Then by virtue of Theorem 2, any variationally trivial Lagrangian reads $L_0 = \phi + d_H \xi$ where $\phi$ is a non-exact $n$-form on $X$.

Note that the generating functional in perturbative QFT depends on the action functional and, thus, is defined with accuracy to variationally trivial Lagrangians, which are $d_H$-exact in QFT on $X = \mathbb{R}^n$. This fact motivates us to treat all Lagrangians in field-antifield BRST theory up to $d_H$-exact ones. They are defined as elements of the cohomology at the last term of the cochain complex

$$0 \to \mathbb{R} \to P^0_\infty \{N\} \xrightarrow{d_H} P^0_{1,\infty} \{N\} \cdots \xrightarrow{d_H} P^0_{0,n} \{N\} \to 0.$$ 

Equivalently, one considers so called local functionals $\int L \, d^n x$ which, in the case of even fields, are evaluated for the jet prolongations of sections of $Y \to X$ of compact support [2, 11, 12, 33].

In particular, any vertical graded derivation $\vartheta$ (14) obeys the relation $\vartheta \circ d_H = d_H \circ \vartheta$. If $\vartheta$ is nilpotent, we therefore have a complex of complexes of horizontal graded forms $P^0_* \{N\}$ with respect to the nilpotent operator $\vartheta$ and the total differential $d_H$. For instance, let $\vartheta$ be a BRST operator. Then one studies $d_H$-relative (local in the terminology of [12]) and iterated BRST cohomology [10, 12, 33, 87, 93]. Relative and iterated cohomology of graded densities coincide with each other. For instance, a glance at the formula (37) shows that an original Lagrangian $L$ and its BRST extension $L_E$ are of the same relative BRST cohomology class.

III. Particular models

13 Gauge theory of principal connections

Let us consider gauge theory of principal connections on a principal bundle $P \to X$ with a structure Lie group $G$. Principal connections are $G$-equivariant connections on $P \to X$ and, therefore, they are represented by sections of the quotient bundle

$$C = J^1 P / G \to X$$

[85, 144]. This is an affine bundle coordinated by $(x^\lambda, a^\lambda_\mu)$ such that, given a section $A$ of $C \to X$, its components $A^\lambda_\mu = a^\lambda_\mu \circ A$ are coefficients of the familiar local connection form [123] (i.e., gauge potentials). Therefore, one calls $C$ (38) the bundle of principal connections.

A key point is that its first order jet manifold $J^1 C$ admits the canonical splitting over $C$ given by the coordinate expression

$$a^{\mu \lambda}_\mu = \frac{1}{2} R^{\mu \lambda}_\mu + \frac{1}{2} S^{\mu \lambda}_\mu = \frac{1}{2} (a^{\mu \lambda}_\mu + a^{\mu \lambda}_{\mu \lambda} - c^p_{pq} a^p_\lambda a^q_\mu) + \frac{1}{2} (a^{\mu \lambda}_\mu - a^{\mu \lambda}_{\mu \lambda} + c^p_{pq} a^p_\lambda a^q_\mu),$$

(39)
where \( c^{r}_{pq} \) are the structure constants of the Lie algebra \( \mathfrak{g} \) of \( G \), and \( F^{r}_{\lambda \mu} = \mathcal{F}^{r}_{\lambda \mu} \circ J^1 A \) is the curvature (the strength (42)) of a principal connection \( A \).

There is a unique (Yang–Mills) quadratic gauge invariant Lagrangian \( L_{YM} \) on \( J^1 C \) which factorizes through the component \( \mathcal{F}^{r}_{\lambda \mu} \) of the splitting (39). It obeys the irreducible Noether identities

\[
\alpha_{ji}^{a \lambda} a^{a \lambda} + \partial_{\lambda} \xi^r \partial^r = 0.
\]

The corresponding gauge symmetries are \( G \)-invariant vertical vector fields on \( P \). They are given by sections \( \xi^r = \xi^r e^r \) of the Lie algebra bundle \( V_{G \mathcal{P}} = VP/G \), and define vector fields

\[
\xi = (- \alpha_{ij}^{a \lambda} a^{a \lambda} + \partial_{\lambda} \xi^r) \partial^r
\]

on the bundle of principal connections \( C \) such that \( L_{J^1 \xi} L_{YM} = 0 \). As a consequence, the basis \( (a^{a \lambda}, c^r, \pi^r, \tau_r) \) for the BRST extended gauge theory consists of gauge potentials \( a^{a \lambda} \), ghosts \( c^r \) of ghost number 1, and antifields \( \pi^r, \tau_r \) of antifield numbers 1 and 2, respectively. Replacing gauge parameters \( \xi^r \) in \( \xi (40) \) with odd ghost \( c^r \), we obtain the total gauge operator \( u_e (25) \), whose nilpotent extension is the well known BRST operator

\[
u_E = (- \alpha_{ij}^{a \lambda} a^{a \lambda} + \partial_{\lambda} \xi^r) \partial^r - \frac{1}{2} \alpha_{ij}^{a \lambda} c^j \partial^r + \frac{1}{2} \alpha_{ij}^{a \lambda} c^j \partial^r.
\]

Hence, the Yang–Mills Lagrangian is extended to a solution of the master equation

\[
L_{E} = L_{YM} + (- \alpha_{ij}^{a \lambda} a^{a \lambda} + \partial_{\lambda} \xi^r) \partial^r - \frac{1}{2} \alpha_{ij}^{a \lambda} c^j \partial^r + \frac{1}{2} \alpha_{ij}^{a \lambda} c^j \partial^r.
\]

14 Topological Chern–Simons theory

Vector fields \( \xi (40) \) are variational symmetries of the Lagrangian of topological Chern–Simons theory. One usually considers Chern–Simons theory whose Lagrangian is the local Chern–Simons form derived from the local transgression formula for the second Chern characteristic form. The global Chern–Simons Lagrangian is well defined, but depends on a background gauge potential [30, 31, 65, 90].

The fiber bundle \( J^1 P \to C \) is a trivial \( G \)-principal bundle canonically isomorphic to \( C \times P \to C \). It admits the canonical principal connection

\[
\mathcal{A} = dx^\lambda \otimes (\partial_\lambda + a^p_{\lambda} \varepsilon_p) + da^r_\lambda \otimes \partial^r
\]

[82, 144]. Its curvature defines the canonical \( V_G P \)-valued 2-form

\[
\mathcal{F} = (da^r_\mu \wedge dx^\mu + \frac{1}{2} c^r_{pq} a^p_{\mu} a^q_{\mu} dx^\lambda \wedge dx^\mu) \otimes e_r
\]

(41) on \( \mathcal{C} \). Given a section \( A \) of \( C \to X \), the pull-back

\[
F_A = A^* \mathcal{F} = \frac{1}{2} F_{\lambda \mu} dx^\lambda \wedge dx^\mu \otimes e_r
\]

(42)
of \( \mathfrak{g} \) onto \( X \) is the strength form of a gauge potential \( A \).

Let \( I_k(e) = b_{r_1 \ldots r_k} e^{r_1} \cdots e^{r_k} \) be a \( G \)-invariant polynomial of degree \( k > 1 \) on the Lie algebra \( \mathfrak{g} \). With \( \mathfrak{g} \) (41), one can associate to \( I_k \) the closed gauge-invariant 2\( k \)-form

\[
P_{2k}(\mathfrak{g}) = b_{r_1 \ldots r_k} e^{r_1} \wedge \cdots \wedge e^{r_k}
\]
on \( C \). Given a section \( B \) of \( C \to X \), the pull-back \( P_{2k}(F_B) = B^* P_{2k}(\mathfrak{g}) \) of \( P_{2k}(\mathfrak{g}) \) is a closed characteristic form on \( X \). Let the same symbol stand for its pull-back onto \( C \). Since \( C \to X \) is an affine bundle and the de Rham cohomology of \( C \) equals that of \( X \), the forms \( P_{2k}(F_B) \) and \( P_{2k}(F_B) \) possess the same cohomology class \( [P_{2k}(\mathfrak{g})] = [P_{2k}(F_B)] \) for any principal connection \( B \). Thus, \( I_k(e) \mapsto \int [P_{2k}(F_B)] \in H^*(X) \) is the familiar Weil homomorphism. Furthermore, we obtain the transgression formula

\[
P_{2k}(\mathfrak{g}) - P_{2k}(F_B) = dS_{2k-1}(B), \quad S_{2k-1}(B) = k \int_0^1 \mathcal{P}_{2k}(t, B) dt,
\]
on \( C \). Its pull-back by means of a section \( A \) of \( C \to X \) gives the transgression formula

\[
P_{2k}(F_A) - P_{2k}(F_B) = dS_{2k-1}(A, B)
\]
on \( X \). For instance, if \( P_{2k}(F_A) \) is the characteristic Chern 2\( k \)-form, then \( S_{2k-1}(A, B) \) is the familiar Chern–Simons \((2k - 1)\)-form. Therefore, we agree to call \( S_{2k-1}(B) \) (43) the Chern–Simons form on the bundle \( C \). Let us consider the pull-back of this form onto the jet manifold \( J^1 C \) denoted by the same symbol \( S_{2k-1}(B) \). Then \( L_{CS} = h_0 S_{2k-1}(B) \) is the global Lagrangian of topological Chern–Simons theory. One can show that its Lie derivative with respect to any vector field \( \xi \) (40) is \( d_H \)-exact [90].

### 15 Topological BF theory

Let us consider topological BF theory of two exterior forms \( A \) and \( B \) of form degree \( |A| + |B| = n - 1 \) on a smooth manifold \( X \) [28]. It is a reducible \((n - 3)\)-stage degenerate Lagrangian theory [17]. Since the verification of the homology regularity condition in a general case is rather complicated, we here restrict our consideration to the simplest example of the topological BF theory when \( A \) is a function [20, 21, 96].

Let us consider the fiber bundle \( Y = \mathbb{R} \times \times X^{n-1} T^* X \), coordinatied by \((x^\lambda, A, B_{\mu_1 \ldots \mu_{n-1}})\) and provided with the canonical \((n - 1)\)-form

\[
B = \frac{1}{(n - 1)!} B_{\mu_1 \ldots \mu_{n-1}} dx'^{\mu_1} \wedge \cdots \wedge dx'^{\mu_{n-1}}.
\]

The Lagrangian of topological BF theory reads

\[
L_{BF} = \frac{1}{n} Ad_H B.
\]
Let us extend the original GDA $O^*_\infty Y$ of BF theory to the GDA $P^*_\infty \{Y^*, Y\}$ possessing the local basis $\{A, B_{\mu_1\ldots\mu_{n-1}}, s, \overline{s}^{\mu_1\ldots\mu_{n-1}}\}$, where $s, \overline{s}^{\mu_1\ldots\mu_{n-1}}$ are odd antifields of antifield number 1. It is provided with the nilpotent Koszul–Tate differential

$$\delta = \frac{\partial}{\partial s} \mathcal{E} + \frac{\partial}{\partial \overline{s}^{\mu_1\ldots\mu_{n-1}}} \mathcal{E}^{\mu_1\ldots\mu_{n-1}}.$$ 

Then one can show that the Noether identities (16) read

$$\overline{\delta} \Delta^{\mu_2\ldots\mu_{n-1}} = d_{\mu_1} \mathcal{E}^{\mu_1\mu_2\ldots\mu_{n-1}} = 0, \quad \Delta^{\mu_2\ldots\mu_{n-1}} = d_{\mu_1} \overline{s}^{\mu_1\mu_2\ldots\mu_{n-1}}. \quad (45)$$

The graded densities $\Delta^{\nu_2\ldots\nu_{n-1}}$ (45) form a local basis for a projective $O^\infty(X)$-module of finite rank which, by virtue of the Serre–Swan theorem, is isomorphic to the module of sections of the vector bundle

$$\mathcal{V}^* = \bigwedge^n T X \otimes \bigwedge^\infty T^* X, \quad \mathcal{V} = \bigwedge^n T^* X.$$ 

Therefore, let us extend the GDA $P^*_\infty \{0\}$ to the BGDA $P^*_\infty \{0\} = P^*_\infty \{Y^*, Y, V\}$ possessing the local basis $\{A, B_{\mu_1\ldots\mu_{n-1}}, s, \overline{s}^{\mu_1\ldots\mu_{n-1}}, e, \overline{e}^{\mu_1\ldots\mu_{n-1}}, \ldots, \overline{e}^{\mu_{n-1}}\}$, where $\overline{e}^{\mu_2\ldots\mu_{n-1}}$ are even antifields of antifield number 2. We have the nilpotent graded derivation

$$\delta_0 = \overline{\delta} + \frac{\partial}{\partial \overline{e}^{\mu_2\ldots\mu_{n-1}}} \Delta^{\mu_2\ldots\mu_{n-1}}$$

of $P^*_\{0\}$. Its nilpotency is equivalent to the Noether identities (45).

Iterating the arguments, we come to the GDA $P^*_\{n-2\}$ possessing the local basis

$$\{A, B_{\mu_1\ldots\mu_{n-1}}, c_{\mu_2\ldots\mu_{n-1}}, \ldots, c_{\mu_{n-1}}, e, \overline{e}^{\mu_1\ldots\mu_{n-1}}, \overline{e}^{\mu_2\ldots\mu_{n-1}}, \ldots, \overline{e}^{\mu_{n-1}}\},$$

where $\overline{e}^{\mu_2\ldots\mu_{n-1}}$, $k = 0, \ldots, n-3$, are antifields of Grassmann parity $(k+1)\text{mod} 2$ and antifield number $k + 3$, $\overline{e}$ is an antifield of Grassmann parity $(n-1)\text{mod} 2$ and antifield number $n + 1$, and $c_{\mu_2\ldots\mu_{n-1}}, \ldots, c_{\mu_{n-1}}, c$ are the corresponding ghosts. The GDA $P^*_\{n-2\}$ is provided with the Koszul–Tate differential

$$\delta_{n-2} = \delta_0 + \sum_{1 \leq k \leq n-3} \frac{\partial}{\partial \overline{e}^{\mu_2\ldots\mu_{n-1}}} \Delta^{\mu_2\ldots\mu_{n-1}} + \frac{\partial}{\partial e} \Delta,$n-2

$$\Delta^{\mu_2\ldots\mu_{n-1}} = d_{\mu_{k+1}} \overline{e}^{\mu_{k+1}\mu_2\ldots\mu_{n-1}}, \quad \Delta = d_{\mu_{n-1}} \overline{e}^{\mu_{n-1}}.$$ 

Its nilpotency results in the Noether identities (45) and the $k$-stage Noether identities

$$d_{\mu_{k+2}} \Delta^{\mu_2\ldots\mu_{n-1}} = 0, \quad k = 0, \ldots, n-3.$$ 

By virtue of Noether’s inverse second theorem, the total gauge operator reads

$$u_e = -d_{\mu_1} c_{\mu_2\ldots\mu_{n-1}} \frac{\partial}{\partial B_{\mu_1\ldots\mu_{n-1}}} - \sum_{1 \leq k \leq n-3} d_{\mu_1} c_{\mu_{k+2}\ldots\mu_{n-1}} \frac{\partial}{\partial c_{\mu_{k+1}\ldots\mu_{n-1}}} - d_{\mu} c \frac{\partial}{\partial c_{\mu}}$$

It is nilpotent and, thus, is the BRST operator. Accordingly, the Lagrangian $L_{BF}$ (44) is extended to a solution of the master equation

$$L_E = L_{BF} + [c_{\mu_2\ldots\mu_{n-1}} d_{\mu_1} \overline{e}^{\mu_1\mu_2\ldots\mu_{n-1}} + \sum_{1 \leq k \leq n-3} c_{\mu_{k+2}\ldots\mu_{n-1}} d_{\mu_{k+1}} \overline{e}^{\mu_{k+1}\mu_{k+2}\ldots\mu_{n-1}} + c d_{\mu_{n-1}} \overline{e}^{\mu_{n-1}}] \omega.$$
16 SUSY gauge theory

SUSY gauge theory is mainly developed as Yang–Mills type theory [156, 209, 210]. However, its geometric formulation meets difficulty because formalism of principal bundles in the categories of graded manifolds and supermanifolds is rather sophisticated [13, 144, 196, 199]. ACFT overcomes this difficulty [194, 195].

Let \( G = G_0 \oplus G_1 \) be a finite-dimensional real Lie superalgebra with a basis \( \{ e_r \} \), \( r = 1, \ldots, m \), and real structure constants \( c^r_{ij} \). Recall that

\[
\begin{align*}
c^r_{ij} &= \frac{1}{2}(\mathcal{F}^r_{\lambda \mu} - \mathcal{S}^r_{\lambda \mu}) = \frac{1}{2}(a^r_{\lambda \mu} - a^r_{\mu \lambda} + c^r_{ij} a^i_\lambda a^j_\mu) + \frac{1}{2}(a^r_{\lambda \mu} + a^r_{\mu \lambda} - c^r_{ij} a^i_\lambda a^j_\mu).
\end{align*}
\]

Then the Yang–Mills graded Lagrangian takes the form

\[
L_{YM} = \frac{1}{4} h^{ij} a^r_{\lambda \mu} a^s_{\nu \lambda} \mathcal{F}^{i}_{\lambda \nu} \mathcal{F}^{j}_{\mu \nu},
\]

where \( h^{ij} \) is an Euclidean metric on \( \mathbb{R}^n \). Its variational derivatives \( \mathcal{E}^r_\lambda \) obey the irreducible Noether identities

\[
c^r_{ij} a^j_\lambda \mathcal{E}^r_\lambda + d_\lambda \mathcal{E}^r_\lambda = 0.
\]

Therefore, we enlarge the GDA \( \mathcal{P}^*[Q, Y] \) to the GDA \( \mathcal{P}^*\{0\} \) whose basis

\[
\{ a^r_\lambda, c^r, \pi^r_\lambda, \tau_r \}, \quad [c^r] = ([r] + 1) \text{mod } 2, \quad [\pi^r_\lambda] = [\tau_r] = [r],
\]

consists of gauge potentials \( a^r_\lambda \), ghosts \( c^r \) of ghost number 1, and antifields \( \pi^r_\lambda, \tau_r \) of antifield numbers 1 and 2, respectively. Then the total gauge operator \( u_e \) (25) reads

\[
u_e = u^r_\lambda \frac{\partial}{\partial a^r_{\lambda}} = (-c^r_{ij} a^i_\lambda + c^r_\lambda a^j_\lambda) \frac{\partial}{\partial a^r_{\lambda}}.
\]
It admits the nilpotent BRST extension

$$u_E = u_e + \xi = (-c^r_{ij}c^j a^i_\lambda + c^r_\lambda) \frac{\partial}{\partial a^i_\lambda} - \frac{1}{2} c^r_{ij} c^i c^j \frac{\partial}{\partial c^r},$$

where $\xi$ are the modified structure constants (46). Then the Yang–Mills graded Lagrangian is extended to a solution of the master equation

$$L_E = L_{YM} + (-c^r_{ij} c^j a^i_\lambda + c^r_\lambda) \partial_a \lambda d^m x - \frac{1}{2} c^r_{ij} c^i c^j \omega.$$

### 17 Field theory on composite bundles

Let us consider a composite fiber bundle

$$Y \to \Sigma \to X,$$  \hspace{1cm} (47)

where $\pi_{YS} : Y \to \Sigma$ and $\pi_{SX} : \Sigma \to X$ are fiber bundles. It is provided with fibered coordinates $(x^\lambda, \sigma^m, y^i)$, where $(x^\mu, \sigma^m)$ are bundle coordinates on $\Sigma \to X$, i.e., the transition functions of coordinates $\sigma^m$ are independent of coordinates $y^i$. The following facts make composite bundles useful for physical applications [85, 143, 173, 197].

Given a composite bundle (47), let $h$ be a global section of $\Sigma \to X$. Then the restriction $Y_h = h^* Y$ (48) of the fiber bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$ is a subbundle of the fiber bundle $Y \to X$.

Every section $s$ of the fiber bundle $Y \to X$ is a composition of the section $h = \pi_{YS} \circ s$ of the fiber bundle $\Sigma \to X$ and some section of the fiber bundle $Y \to \Sigma$ over $h(X) \subset \Sigma$.

Let $J^1 \Sigma$, $J^1 Y$, and $J^1 Y$ be jet manifolds of the fiber bundles $\Sigma \to X$, $Y \to \Sigma$ and $Y \to X$, respectively. They are provided with the adapted coordinates $(x^\lambda, \sigma^m, \sigma^m, y^i)$ and $(x^\lambda, \sigma^m, y^i, \tilde{y}^i_\lambda, y^m_\lambda)$ and $(x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda)$. There is the canonical map

$$\varrho : J^1 \Sigma \times J^1 \Sigma \to J^1 Y, \hspace{1cm} y^i_\lambda \circ \varrho = y^i_\lambda \sigma^m_\lambda + \tilde{y}^i_\lambda.$$

Due to this map, any pair of connections

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i),$$

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m)$$

on fiber bundles $Y \to \Sigma$ and $\Sigma \to X$, respectively, yields the composite connection

$$\gamma = A_\Sigma \circ \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_\lambda + A^i_m \Gamma^m_\lambda) \partial_i)$$

on the fiber bundle $Y \to X$. For instance, let us consider a vector field $\tau$ on the base $X$, its horizontal lift $\Gamma \tau$ onto $\Sigma$ by means of the connection $\Gamma$ and, in turn, the horizontal lift
$A_\Sigma(\Gamma\tau)$ of $\Gamma\tau$ onto $Y$ by means of the connection $A_\Sigma$. Then $A_\Sigma(\Gamma\tau)$ is the horizontal lift of $\tau$ onto $Y$ by means of the composite connection $\gamma$ (50).

Given a composite bundle $Y$ (47), there is the exact sequence of bundles

$$0 \to V_\Sigma Y \to VY \to Y \times V\Sigma \to 0,$$

where $V_\Sigma Y$ is the vertical tangent bundle of the fiber bundle $Y \to \Sigma$. Every connection $A$ (49) on the fiber bundle $Y \to \Sigma$ yields the splitting

$$y^i \partial_i + \dot{\sigma}^m \partial_m = (y^i - A^i_m \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A^i_m \partial_i)$$

of the exact sequences (51). This splitting defines the first order differential operator

$$\tilde{D} = dx^\lambda \otimes (y^i - A^i_\lambda - A^i_m \sigma^m_\lambda) \partial_i$$

on the composite bundle $Y \to X$. This operator, called the vertical covariant differential, possesses the following important property. Let $h$ be a section of the fiber bundle $\Sigma \to X$ and $Y_h$ the subbundle (48) of the composite bundle $Y \to X$. Then the restriction of the vertical covariant differential $\tilde{D}$ (52) to $\mathcal{J}^1 Y_h \subset \mathcal{J}^1 Y$ coincides with the familiar covariant differential relative to the pull-back connection

$$A_h = h^* A_\Sigma = dx^\lambda \otimes [\partial_\lambda + ((A^i_m \circ h) \partial_\lambda h^m + (A \circ h)^i_\lambda) \partial_i]$$

on $Y_h \to X$ [85, 144, 173].

The peculiarity of field theory on a composite bundle (47) is that its Lagrangian depends on a connection on $Y \to \Sigma$, but not $Y \to X$, and it factorizes through the vertical covariant differential (52). This is the case of field theories with broken symmetries, spinor fields, gauge gravitation theory [144, 176, 182, 192, 193] and mechanical models with parameters [89, 91, 94, 143, 177].

18 Symmetry breaking and Higgs fields

In gauge theory on a principal bundle $P \to X$, a symmetry breaking is defined as reduction of the structure Lie group $G$ of this principal bundle to a closed (consequently, Lie) subgroup $H$ of exact symmetries [45, 85, 121, 154, 168, 193]. From the mathematical viewpoint, one speaks on the Klein–Chern geometry or a reduced $G$-structure [101, 124, 211].

By virtue of the well-known theorem [123, 200], reduction of the structure group of a principal bundle takes place iff there exists a global section $h$ of the quotient bundle $P/H \to X$. This section is treated as a Higgs field. Thus, we have the composite bundle

$$P \to P/H \to X,$$

where $P \to P/H$ is a principal bundle with the structure group $H$ and $\Sigma = P/H \to X$ is a $P$-associated fiber bundle with the typical fiber $G/H$. Moreover, there is one-to-one correspondence between the global sections $h$ of $\Sigma \to X$ and reduced $H$-principal subbundles $P^h = \pi_{P\Sigma}^{-1}(h(X))$ of $P$. 

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Let $Y \to \Sigma$ be a vector bundle associated to the $H$-principal bundle $P \to \Sigma$. Then sections of the composite bundle $Y \to \Sigma \to X$ describe matter fields with the exact symmetry group $H$ in the presence of Higgs fields. Given bundle coordinates $(x^\lambda, \sigma^m, y^i)$ on $Y$, these sections are locally represented by pairs $(\sigma^m(x), y^i(x))$. Given a global section $h$ of $\Sigma \to X$, sections of the vector bundle $Y_h$ describe matter fields in the presence of the background Higgs field $h$. Moreover, for different Higgs fields $h$ and $h'$, the fiber bundles $Y_h$ and $Y_{h'}$ need not be equivalent [85, 168, 193].

Note that $Y \to X$ fails to be associated to a principal bundle $P \to X$ with the structure group $G$ and, consequently, it need not admit a principal connection. Therefore, one should consider a principal connection (49) on the fiber bundle $Y \to \Sigma$, and a Lagrangian on $J^1Y$ factorizes through the vertical covariant differential $\tilde{D}$ (52). In the presence of a background Higgs field $h$, the restriction of $\tilde{D}$ to $J^1Y_h$ coincides with the covariant differential relative to the pull-back connection (53) on $Y_h \to X$.

Riemannian and pseudo-Riemannian metrics on a manifold $X$ exemplify classical Higgs fields. Let $X$ be an oriented four-dimensional smooth manifold and $LX$ the fiber bundle of linear frames in the tangent spaces to $X$. It is a principal bundle with the structure group $GL_4 = GL^+(4, \mathbb{R})$. By virtue of the well known theorem [200], this structure group is always reducible to its maximal compact subgroup $SO(4)$. The corresponding global sections of the quotient bundle $LX/SO(4)$ are Riemannian metrics on $X$. However, the reduction of the structure group $GL_4$ of $LX$ to its Lorentz subgroup $SO(1,3)$ need not exist, unless $X$ satisfies certain topological conditions. The quotient bundle

$$\Sigma_{\text{PR}} = LX/SO(1,3) \to X,$$

is a natural bundle (see item 20), associated to $LX$. Its global section $h$, called a tetrad field, defines a principal Lorentz subbundle $L^hX$ of $LX$. Therefore, $h$ can be represented by a family of local sections $\{h_a\}_i$ of $LX$ on trivialization domains $U_i$ which take values in $L^hX$ and possess Lorentz transition functions. One calls $\{h_a\}$ the tetrad functions or vielbeins. They define an atlas $\Psi^h = \{(\{h_a\}_i, U_i)\}$ of $LX$ and associated bundles with Lorentz transition functions. There is the canonical imbedding of the bundle $\Sigma_{\text{PR}}$ (55) onto an open subbundle of the tensor bundle $\overset{2}{\wedge} T^*X$ such that its global section $h = g$ is a pseudo-Riemannian metric $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ on $X$. This fact motivates us to treat a metric (or tetrad) gravitational field as a Higgs field [117, 169, 182, 192].

Note that, if $G = GL_4$ and $H = SO(1,3)$, we are in the case of so called reductive $G$-structure [97] when the Lie algebra $\mathfrak{g}$ of $G$ is the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

of the Lie algebra $\mathfrak{h}$ of $H$ and a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $ad(g)(\mathfrak{m}) \subset \mathfrak{m}$, $g \in H$. In this case, the pull-back of the $\mathfrak{h}$-valued component of any principal connection on $P$ onto a reduced subbundle $P^h$ is a principal connection on $P^h$. 

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19 Dirac spinor fields

Dirac spinors as like as other ones are described in the Clifford algebra terms [74, 137]. The Dirac spinor structure on a four-dimensional manifold $X$ is defined as a pair $(P^h, z_h)$ of a principal bundle $P^h \to X$ with the structure spin group $L_s = SL(2, \mathbb{C})$ and its bundle morphism $z_h : P^h \to LX$ to the frame bundle $LX$ [9, 137]. Any such morphism factorizes

$$P^h \to L^h X \to LX$$

through some reduced principal subbundle $L^h X \subset LX$ with the structure proper Lorentz group $\Sigma = SO^+(1, 3)$, whose universal two-fold covering is $L_s$. The corresponding quotient bundle $\Sigma_T = LX/L$ is a two-fold covering of the bundle $\Sigma_{PR}$ (55). Its global sections are $L$-valued tetrad fields $h$. Thus, any Dirac spinor structure is associated to a Lorentz reduced structure, but the converse need not be true. There is the well-known topological obstruction to the existence of a Dirac spinor structure. For instance, a Dirac spinor structure on a non-compact manifold $X$ exists iff $X$ is parallelizable.

Given a Dirac spinor structure (57), the associated Dirac spinor bundle $S^h$ can be seen as a subbundle of the bundle of Clifford algebras generated by the Lorentz frames $\{t_a\} \in L^h X$ [26, 137]. This fact enables one to define the Clifford representation

$$\gamma_h(dx^\mu) = h^\mu_a \gamma^a$$

of coframes $dx^\mu$ in the cotangent bundle $T^*X$ by Dirac’s matrices, and introduce the Dirac operator on $S^h$ with respect to a principal connection on $P^h$. Then sections of a spinor bundle $S^h$ describe Dirac spinor fields in the presence of a tetrad field $h$. However, the representations (58) for different tetrad fields fail to be equivalent. Therefore, one meets a problem of describing Dirac spinor fields in the presence of different tetrad fields and under general covariant transformations.

In order to solve this problem, let us consider the universal two-fold covering $\tilde{GL}_4$ of the group $GL_4$ and the $\tilde{GL}_4$-principal bundle $\tilde{LX} \to X$ which is the two-fold covering bundle of the frame bundle $LX$ [51, 137, 201]. Then we have the commutative diagram

$$\begin{array}{ccc}
\tilde{LX} & \xrightarrow{c} & LX \\
\downarrow & & \downarrow \\
P^h & \to & L^h X
\end{array}$$

for any Dirac spinor structure (57) [76, 176, 182]. As a consequence, $\tilde{LX}/L_s = LX/L = \Sigma_T$. Since $\tilde{LX} \to \Sigma_T$ is an $L_s$-principal bundle, one can consider the associated spinor bundle $S \to \Sigma_T$ whose typical fiber is a Dirac spinor space $V_s$ [144, 176, 182]. We agree to call it the universal spinor bundle because, given a tetrad field $h$, the pull-back $S^h = h^*S \to X$ of $S$ onto $X$ is a spinor bundle on $X$ which is associated to the $L_s$-principal bundle $P^h$. The universal spinor bundle $S$ is endowed with bundle coordinates $(x^\lambda, \sigma^m_a, y^A)$, where $(x^\lambda, \sigma^m_a)$ are bundle coordinates on $\Sigma_T$ and $y^A$ are coordinates on the spinor space $V_s$. The universal
spinor bundle $S \to \Sigma_T$ is a subbundle of the bundle of Clifford algebras which is generated by the bundle of Minkowski spaces associated to the $L$-principal bundle $LX \to \Sigma_T$. As a consequence, there is the Clifford representation

$$
\gamma_\Sigma : T^*X \otimes S \to S, \quad \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a,
$$

whose restriction to the subbundle $S^h \subset S$ restarts the representation (58).

Sections of the composite bundle $S \to \Sigma_T \to X$ describe Dirac spinor fields in the presence of different tetrad fields as follows [176, 182]. Due to the splitting (56), any general linear connection $K$ on $X$ (i.e., a principal connection on $LX$) yields the connection

$$
A_\Sigma = dx^\lambda \otimes (\partial_\lambda - \frac{1}{4}(\eta^{kb}_c \sigma^a_\mu - \eta^{ka}_c \sigma^b_\mu)\sigma^\mu_k L_{ab} A_{By} ^B \partial_A) +
$$

$$
\dot{d}\sigma^\mu_k \otimes (\partial_\mu + \frac{1}{4}(\eta^{kb}_c \sigma^a_\mu - \eta^{ka}_c \sigma^b_\mu)L_{ab} A_{By} ^B \partial_A)
$$
on the universal spinor bundle $S \to \Sigma_T$. Its restriction to $S^h$ is the familiar spin connection

$$
K_h = dx^\lambda \otimes [\partial_\lambda + \frac{1}{4}(\eta^{kb}_c \sigma^a_\mu - \eta^{ka}_c \sigma^b_\mu)(\partial_\mu h^\mu_k - h^\mu_k K_{\lambda \mu} \nu) L_{ab} A_{By} ^B \partial_A],
$$
defined by $K$ [162, 174]. The connection (60) yields the vertical covariant differential

$$
\tilde{D} = dx^\lambda \otimes [y_\lambda^A - \frac{1}{4}(\eta^{kb}_c \sigma^a_\mu - \eta^{ka}_c \sigma^b_\mu)(\sigma^\mu_k - \sigma^\mu_k K_{\lambda \mu} \nu) L_{ab} A_{By} ^B \partial_A],
$$
on the fiber bundle $S \to X$. Its restriction to $J^1S^h \subset J^1S$ recovers the familiar covariant differential on the spinor bundle $S^h \to X$ relative to the spin connection (61). Combining (59) and (62) gives the first order differential operator

$$
\mathcal{D} = \sigma^\lambda_a \gamma^a_{AB} [y_\lambda^A - \frac{1}{4}(\eta^{kb}_c \sigma^a_\mu - \eta^{ka}_c \sigma^b_\mu)(\sigma^\mu_k - \sigma^\mu_k K_{\lambda \mu} \nu) L_{ab} A_{By} ^B],
$$
on the fiber bundle $S \to X$. Its restriction to $J^1S^h \subset J^1S$ is the familiar Dirac operator on the spinor bundle $S^h$ in the presence of a background tetrad field $h$ and a general linear connection $K$.

### 20 Natural and gauge-natural bundles

A connection $\Gamma$ on a fiber bundle $Y \to X$ defines the horizontal lift $\Gamma \tau$ onto $Y$ of any vector field $\tau$ on $X$. There is the category of natural bundles [125, 204] which admit the functorial lift $\tilde{\tau}$ onto $T$ of any vector field $\tau$ on $X$ such that $\tau \mapsto \tilde{\tau}$ is a monomorphism of the Lie algebra of vector field on $X$ to that on $T$. One can think of the lift $\tilde{\tau}$ as being an infinitesimal generator of a local one-parameter group of general covariant transformations of $T$. The corresponding Noether current $\mathcal{J}_\tilde{\tau}$ is the energy-momentum flow along $\tau$ [84, 85, 174, 176].
Natural bundles are exemplified by tensor bundles over \( X \). Moreover, all bundles associated to the principal frame bundle \( L_X \) are natural bundles. The bundle
\[
C_K = J^1 L_X/GL_4
\]
of principal connections on \( L_X \) is not associated to \( L_X \), but it is also a natural bundle [85, 144]. As is well known, a spinor bundle \( S^h \) associated to the spinor structure (57) fails to be a natural bundle. There exists the lift of any vector field on \( X \) onto \( S^h \). It is called Kosmann’s Lie derivative [61, 98, 126]. Such a lift is a property of any reductive \( G \)-structure [97], but it is not a generator of general covariant transformations. At the same time, the universal spinor bundle \( S \rightarrow X \) associated to the two-fold covering \( \tilde{L}_X \) of \( L_X \) is a natural bundle. Therefore, there exists the functorial lift onto \( S \) of any vector field on \( X \). Its restriction to a spinor bundle \( S^h \) coincides with Kosmann’s Lie derivative [174, 176, 182].

In a more general setting, higher order natural bundles and gauge-natural bundles are called into play [60, 63, 125, 204]. Note that the linear frame bundle \( L_X \) over a manifold \( X \) is the set of first order jets of local diffeomorphisms of \( \mathbb{R}^n \) to \( X \), \( n = \dim X \), at the origin of \( \mathbb{R}^n \). Accordingly, one considers \( r \)-order frame bundles \( L^r X \) of \( r \)-order jets of local diffeomorphisms of \( \mathbb{R}^n \) to \( X \). Furthermore, given a principal bundle \( P \rightarrow X \) with a structure group \( G \), the \( r \)-order jet bundle \( J^1 P \rightarrow X \) of its sections fails to be a principal bundle. However, the product \( W^r P = L^r X \times J^r P \) is a principal bundle with the structure group \( W^r_n G \) which is a semi direct product of the group \( G^r_n \) of invertible \( r \)-order jets of maps \( \mathbb{R}^n \) to itself at its origin (e.g., \( G^1_n = GL(n, \mathbb{R}) \)) and the group \( T^r_n G \) of \( r \)-order jets of morphisms \( \mathbb{R}^n \rightarrow G \) at the origin of \( \mathbb{R}^n \). Moreover, if \( Y \rightarrow X \) is a fiber bundle associated to \( P \), the jet bundle \( J^r Y \rightarrow X \) is a vector bundle associated to the principal bundle \( W^r P \). It exemplifies gauge natural bundles, which can described as fiber bundles associated to principal bundles \( W^r P \). Natural bundles are gauge natural bundles for a trivial \( G = 1 \). The bundle of principal connections \( C \) (38) is a first order gauge natural bundle. This fact motivates somebody to develop generalized gauge theory on gauge natural bundles [60, 63, 72].

21 Gauge gravitation theory

Gauge gravitation theory (see [110, 117, 169, 192] for a survey) is described as a field theory on natural bundles over an oriented four-dimensional manifold \( X \) whose dynamic variables are linear connections and pseudo-Riemannian metrics on \( X \) [19, 64, 85, 174, 182, 206].

Linear connections on \( X \) (henceforth world connection) are principal connections on the linear frame bundle \( L_X \) of \( X \). They are represented by sections of the bundle of linear connections \( C_K \) (63). This is provided with bundle coordinates \((x^\lambda, k^{\nu}_{\alpha})\) such that components \( k^{\nu}_{\alpha} \circ K = K^{\nu}_{\alpha} \) of a section \( K \) of \( C_K \rightarrow X \) are coefficient of the linear connection
\[
K = dx^\lambda \otimes (\partial_\lambda + K^{\mu}_{\nu} x^\nu \partial_\mu)
\]
on $TX$ with respect to the holonomic bundle coordinates $(x^\lambda, \dot{x}^\lambda)$. The first order jet manifold $J^1C_K$ of $C_K$ admits the canonical decomposition taking the coordinate form

$$k_{\lambda\mu}^{\alpha\beta} = \frac{1}{2}(R_{\lambda\mu}^{\alpha\beta} + S_{\lambda\mu}^{\alpha\beta}) = \frac{1}{2}(k_{\lambda\mu}^{\alpha\beta} - k_{\mu\lambda}^{\alpha\beta} + k_{\mu}^{\alpha}k_{\lambda}^{\varepsilon\beta} - k_{\lambda}^{\alpha}k_{\mu}^{\varepsilon\beta}) + \frac{1}{2}(k_{\lambda\mu}^{\alpha\beta} + k_{\mu\lambda}^{\alpha\beta} - k_{\mu}^{\alpha}k_{\lambda}^{\varepsilon\beta} + k_{\lambda}^{\alpha}k_{\mu}^{\varepsilon\beta}).$$

If $K$ is a section of $C_K \rightarrow X$, then $R \circ K$ is the curvature of a world connection $K$.

In order to describe gravity, let us assume that the linear frame bundle $LX$ admits a Lorentz structure, i.e., reduced principal subbundles with the structure Lorentz group. Sections of the corresponding quotient bundle $\Sigma_{PR}$ (55) are pseudo-Riemannian (henceforth world) metrics on $X$. Note that the physical underlying reasons for the existence of a Lorentz structure and, consequently, a world metric are both the geometric equivalence principle and the existence of Dirac fermion fields [117, 169, 182].

The total configuration space of gauge gravitation theory in the absence of matter fields is the bundle product $\Sigma_{PR} \times C_K$ coordinated by $(\nu^\lambda, \sigma^{\alpha\beta}, k_{\mu}^{\alpha\beta})$. This is a natural bundle admitting the functorial lift

$$\tau_K = \tau^\mu \partial_\mu + (\sigma^{\nu\beta} \partial_{\nu} \tau^\alpha + \sigma^{\alpha\nu} \partial_{\nu} \tau^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (\partial_\nu \tau^\alpha k_{\mu}^{\nu\beta} - \partial_\beta \tau^\nu k_{\mu}^{\alpha\nu} - \partial_\mu \tau^\nu k_{\nu}^{\alpha\beta} + \partial_{\mu\beta} \tau^\alpha) \frac{\partial}{\partial k_{\mu}^{\alpha\beta}}$$

of vector fields $\tau$ on $X$ [19, 144]. These lifts are generators of one-dimensional groups of general covariant transformations, whose gauge parameters are vector fields on $X$.

We do not specify a gravitation Lagrangian $L_G$ on the jet manifold $J^1(\Sigma_{PR} \times C_K)$, but assume that vector fields (64) exhaust its gauge symmetries. Then the Euler–Lagrange operator $(E_{\alpha\beta} d\sigma^{\alpha\beta} + \epsilon^{\mu\alpha\beta} dk_{\mu}^{\alpha\beta}) \wedge \omega$ of this Lagrangian obeys irreducible Noether identities

$$-(\sigma^{\lambda\beta} + 2\sigma^{\nu\beta} \delta^\alpha_\nu) E_{\alpha\beta} = -2\sigma^{\nu\beta} d_\nu E_{\lambda\beta} + (-k_{\lambda}^{\mu\beta} - k_{\nu}^{\mu\beta} \delta^{\alpha}_\nu + k_{\beta}^{\mu\lambda} + k_{\lambda}^{\mu\beta}) E^{\mu\alpha\beta} + (-k_{\mu}^{\nu\beta} \delta^{\alpha}_\lambda + k_{\nu}^{\mu\alpha} \delta^{\nu}_\beta + k_{\lambda}^{\mu\beta} \delta^{\nu}_\nu) d_\nu E^{\mu\alpha\beta} + d_{\mu\beta} E^{\mu\alpha\beta} = 0$$

[19]. Taking the vertical part of vector fields $\tilde{\tau}_K$ and replacing gauge parameters $c^\lambda$ with ghosts $\lambda$, we obtain the total gauge operator and its nilpotent BRST prolongation

$$u_E = u^{\alpha\beta} \frac{\partial}{\partial \sigma^{\alpha\beta}} + u_\mu^{\alpha\beta} \frac{\partial}{\partial k_{\mu}^{\alpha\beta}} + u^\lambda \frac{\partial}{\partial c^\lambda} = (\sigma^{\mu\nu} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta - \lambda^\alpha c_\nu^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (c_\nu^\alpha k_{\mu}^{\nu\beta} - c_\nu^\nu k_{\mu}^{\alpha\beta} - c_\mu^{\nu\beta} k_{\nu}^{\alpha\beta} + c_\mu^{\alpha\beta} - \lambda^\beta k_{\mu}^{\alpha\beta}) \frac{\partial}{\partial k_{\mu}^{\alpha\beta}} + \lambda^\beta c_\nu^\nu \frac{\partial}{\partial c^\lambda},$$

but this differs from that in [105]. Accordingly, an original Lagrangian $L_G$ is extended to a solution of the master equation

$$L_E = L_G + u^{\alpha\beta} \sigma_{\alpha\beta} \omega + u^{\alpha\beta} E_\alpha^\beta \omega + u^\lambda \bar{c}_\lambda \omega,$$

where $\sigma_{\alpha\beta}, E_\alpha^\beta$ and $\bar{c}_\lambda$ are corresponding antifields.
22 Covariant Hamiltonian field theory

As is well-known, the familiar symplectic Hamiltonian technique applied to field theory leads to instantaneous Hamiltonian formalism on an infinite-dimensional phase space coordinated by field functions at some instant of time \([103]\). The true Hamiltonian counterpart of classical first order Lagrangian field theory on a fiber bundle \(Y \to X\) is covariant Hamiltonian formalism, where canonical momenta \(p^\mu_i\) correspond to derivatives \(y^i_\mu\) of field variables \(y^i\) with respect to all world coordinates \(x^\mu\). This formalism has been rigorously developed since the 1970s in the multisymplectic, polysymplectic and Hamilton–De Donder variants (see \([58, 59, 70, 85, 86, 102, 111, 134, 139, 140, 141, 146, 159, 165]\) and references therein).

The multisymplectic phase space is the homogeneous Legendre bundle

\[ Z_Y = T^*Y \wedge \left( \wedge^{n-1} T^*X \right), \tag{65} \]

coordinated by \((x^\lambda, y^i, p^\lambda_i, p)\). It is endowed with the canonical exterior form

\[ \Xi_Y = p\omega + p^\lambda_i dy^i \wedge \omega_\lambda, \]

whose exterior differential \(d\Xi_Y\) is the multisymplectic form \([38, 147]\). Given a first order Lagrangian \(L = \mathcal{L}\omega\) on \(J^1Y\), the associated Poincaré–Cartan form

\[ H_L = L + \pi^\lambda_\mu \theta^i \wedge \omega_\lambda, \quad \pi^\lambda_i = \partial^\lambda_i \mathcal{L}, \quad \omega_\lambda = \partial_\lambda | \omega. \tag{66} \]

is a Lepagean equivalent both of \(L\) and the Lagrangian

\[ \mathcal{L} = \hat{h}_0(H_L) = (\mathcal{L} + (\hat{y}^i_\lambda - y^i_\lambda) \pi^\lambda_i)\omega, \quad \hat{h}_0(dy^i) = \hat{g}^i_\lambda dx^\lambda, \tag{67} \]

on the repeated jet manifold \(J^1J^1Y\), whose Euler–Lagrange equations are the Cartan ones

\[ \partial^\mu_i \pi^\mu_\lambda (\hat{y}^\mu_\mu - y^\mu_\mu) = 0, \quad \partial_i \mathcal{L} - \partial_\lambda \pi^\lambda_i + (\hat{g}^i_\lambda - y^i_\lambda) \partial_i \pi^\lambda_\lambda = 0. \tag{68} \]

The Poincaré–Cartan form (66) yields the Legendre morphism

\[ \hat{H}_L : J^1Y \to Z_Y, \quad (p^\mu_i, p) \circ \hat{H}_L = (\pi^\mu_i, \mathcal{L} - \pi^\mu_i y^i_\mu), \]

of \(J^1Y\) to the homogeneous Legendre bundle \(Z_Y\). If its image \(Z_L = \hat{H}_L(J^1Y)\) is an imbedded subbundle \(i_L : Z_L \to Z_Y\) of \(Z_Y \to Y\), it is provided with the pull-back De Donder form \(\Xi_L = i_L^* \Xi_Y\). The Hamilton–De Donder equations for sections \(\mathfrak{s}\) of \(Z_L \to X\) read

\[ \mathfrak{s}^!(u|d\Xi_L) = 0, \tag{69} \]

where \(u\) is an arbitrary vertical vector field on \(Z_L \to X\). If the Legendre morphism \(\hat{H}_L\) is a submersion, one can show that a section \(\mathfrak{s}\) of \(J^1Y \to X\) obeys the Cartan equations (68) iff \(\hat{H}_L \circ \mathfrak{s}\) satisfies the Hamilton–De Donder ones (69) \([86, 102]\). In a general setting,
one studies different Lepagean forms in order to develop Hamilton – De Donder formalism
[134, 135].

The homogeneous Legendre bundle $Z_Y$ is the trivial one-dimensional bundle $\zeta : Z_Y \to \Pi$ over the Legendre bundle

$$\Pi = \frac{n}{Y} T^* X \otimes V^* Y \otimes T X = V^* Y \wedge (\frac{n-1}{Y} T^* X), \quad (70)$$

coordinated by $(x^\lambda, y^i, p^\mu_i)$. Being provided with the canonical polysymplectic form

$$\Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda,$$

the Legendre bundle $\Pi$ is the momentum phase space of polysymplectic Hamiltonian formalism [40, 85, 86, 107, 122, 170, 171]. A Hamiltonian $H$ on $\Pi$ is defined as a section $p = -H$ of the bundle $\zeta$. The pull-back of $\Xi_Y$ onto $\Pi$ by a $H$ is a Hamiltonian form

$$H = H^* \Xi_Y = p^\lambda_i dy^i \wedge \omega - H \omega. \quad (71)$$

For every Hamiltonian form $H$ (71), there exists a connection $\gamma$ on $\Pi \to X$ such that $\gamma \Omega = dH$. This connection yields the first order Hamilton equations

$$y^i_\lambda = \partial^i_\lambda H, \quad p^\lambda_{\lambda i} = -\partial_i H \quad (72)$$

on $\Pi$ which are exactly the Euler–Lagrange equations for the first-order Lagrangian

$$L_H = h_0(H) = (p^\lambda_i y^i_\lambda - H)\omega \quad (73)$$

on $J^1 \Pi$. Let $i_N : N \to \Pi$ be a closed imbedded subbundle of the Legendre bundle $\Pi \to Y$ which is regarded as a constraint space. Let $H_N = i_N^* H$ be the pull-back of the Hamiltonian form $H$ (71) onto $N$. This form defines the constrained Lagrangian

$$L_N = h_0(H_N) = (J^1 i_N)^* L_H \quad (74)$$

on the jet manifold $J^1 N_L$. In fact, this Lagrangian is the restriction of $L_H$ to $N \times J^1 Y$. Its Euler–Lagrange equations are called the constrained Hamilton equations. One can show that any solution of the Hamilton equations (72) which lives in the constraint manifold $N$ is also a solution of the constrained Hamilton equations on $N$ [85, 86].

Lagrangian and covariant Hamiltonian formalisms are not equivalent, unless Lagrangians are hyperregular. The key point is that a non-regular Lagrangian admits different associated Hamiltonians, if any. At the same time, there is a comprehensive relation between these formalisms in the case of almost-regular Lagrangians [85, 86, 170].

Any first order Lagrangian $L$ yields the Legendre map

$$\hat{L} : J^1 Y \longrightarrow \Pi, \quad p^\lambda_i \circ \hat{L} = \partial^i_\lambda L,$$
whose image \( N_L = \hat{L}(J^1Y) \) is called the Lagrangian constraint space. Conversely, any Hamiltonian \( \mathcal{H} \) defines the Hamiltonian map
\[
\hat{H} : \Pi \to J^1Y, \quad y^i_\lambda \circ \hat{H} = \partial^i_\lambda \mathcal{H}.
\]
A Hamiltonian \( \mathcal{H} \) on \( \Pi \) is said to be associated to a Lagrangian \( L \) if it satisfies the relations
\[
p^\mu_i = \partial^\mu_i \mathcal{L}(x^\mu, y^j, \partial^j \mathcal{H}), \quad p^\mu_i \partial^i \mathcal{H} - \mathcal{H} = \mathcal{L}(x^\mu, y^j, \partial^j \mathcal{H}).
\]
A Lagrangian \( L \) is called almost-regular if the Lagrangian constraint space \( N_L \) is a closed imbedded subbundle of the Legendre bundle \( \Pi \to Y \), and \( \hat{L} : J^1Y \to N_L \) is a fibered manifold with connected fibers. In this case, the Poincaré–Cartan form (66) is the pull-back \( H_L = \hat{L}^*H \) of the Hamiltonian form \( H \) (71) for any associated Hamiltonian \( \mathcal{H} \). If an almost-regular Lagrangian admits associated Hamiltonians \( \mathcal{H} \), they define a unique constrained Lagrangian \( L_N = h_0(H_N) \) (74) on the jet manifold \( J^1N_L \) of the fiber bundle \( N_L \to X \). Then one can show that a section \( \pi \) of the jet bundle \( J^1Y \to Y \) is a solution of the Cartan equations for \( L \) iff \( \hat{L} \circ \pi \) is a solution of the constrained Hamilton equations.

For instance, the comprehensive description of systems with almost-regular quadratic Lagrangians can be obtained [85, 86, 92]. In this case, the jet bundle \( J^1Y \to Y \) admits a splitting similar to that (39) in gauge theory. As a consequence, such a Lagrangian is brought into the Yang–Mills type form, and can be accordingly quantized [15, 16, 92].

In order to quantize covariant Hamiltonian field theory, one often try to construct the multisymplectic generalization of a Poisson bracket [46, 69, 71, 114, 118, 119, 120, 183]. In a different way, we quantize covariant (polysymplectic) Hamiltonian field theory as a particular Lagrangian system with the Lagrangian \( L_\mathcal{H} \) in path integral terms [15, 16, 172].

There are attempts to generalize covariant Hamiltonian formalism (e.g., its Hamilton–De Donder variant) to higher order Lagrangian systems [4, 132, 135, 198]. However, a problem is to define the Legendre map \( \hat{L} \) for a higher order Lagrangian \( L \).

23 Time-dependent mechanics

Non-relativistic time-dependent mechanics (see item 25 for relativistic one) can be formulated as particular field theory on fiber bundles \( Q \to \mathbb{R} \) over a time axis \( \mathbb{R} \) [83, 133, 138, 143, 148, 175, 177]. In this case, polysymplectic and multisymplectic Hamiltonian formalisms provide Hamiltonian and homogeneous Hamiltonian formulations of time-dependent mechanics, whose momentum and homogeneous momentum phase spaces are the vertical cotangent bundle \( V^*Q \) of \( Q \to \mathbb{R} \) and the cotangent bundle \( T^*Q \), respectively [143, 175, 177].

At the same time, there is the essential difference between field theory and time-dependent mechanics. In contrast with gauge potentials in field theory, connections on a configuration bundle \( Q \to \mathbb{R} \) of time-dependent mechanics fail to be dynamic variables since their curvature vanishes. There is one-to-one correspondence between these connections and the trivializations \( Q \approx \mathbb{R} \times M \) of a configuration space, i.e., reference frames
[143, 144, 175]. If a reference frame holds fixed, time-dependent mechanical systems are familiarly described on the products $Q \approx \mathbb{R} \times M$, $J^1Q \approx \mathbb{R} \times TM$, $V^*Q \approx \mathbb{R} \times T^*M$, which are not subject to time-dependent transformations [39, 41, 48, 57, 108, 153].

### 24 Jets of submanifolds

Jets of sections of fiber bundles are particular jets of submanifolds. Namely, a space of jets of submanifolds admits a cover by charts of jets of sections [85, 95, 129, 150]. Three-velocities in relativistic mechanics exemplify first order jets of submanifolds (see item 25). A problem is that differential forms on jets of submanifolds do not constitute a variational bicomplex because horizontal forms (e.g., Lagrangians) are not preserved under coordinate transformations. However, one can associate to jets of $n$-dimensional submanifolds of an $m$-dimensional manifold $Z$ the jets of sections of a trivial fiber bundle

$$Z_Q = Q \times Z \to Q,$$

where $Q$ is some $n$-dimensional manifold. This relation fails to be one-to-one correspondence. The ambiguity contains, e.g., diffeomorphisms of $Q$. Lagrangian formalism on a fiber bundle (75) is developed in a standard way, but Lagrangians are required to be variationally invariant under the above mentioned diffeomorphisms of $Q$ (see item 26) [95].

Given an $m$-dimensional smooth real manifold $Z$, a $k$-order jet of $n$-dimensional submanifolds of $Z$ at a point $z \in Z$ is defined as the equivalence class $j_x^kS$ of $n$-dimensional imbedded submanifolds of $Z$ through $z$ which are tangent to each other at $z$ with order $k$. The set $J^k_nZ$ of this jets is a finite-dimensional real smooth manifold. Let $Y \to X$ be an $m$-dimensional fiber bundle over an $n$-dimensional base $X$ and $J^kY$ the $k$-order jet manifold of sections of $Y \to X$. Given an imbedding $Y \to Z$, there is the natural injection $J^kY \to J^k_nZ$ which defines a chart on $J^k_nZ$. These charts provide a manifold atlas of $J^k_nZ$.

In particular, there is obvious one-to-one correspondence between the jets $j_x^kS$ at a point $z \in Z$ and the $n$-dimensional vector subspaces of the tangent space $T_zZ$ of $Z$ at $z$. It follows that $J^1_nZ$ is a fiber bundle $\rho : J^1_nZ \to Z$ in Grassmann manifolds. It possesses the following coordinate atlas. Let $\{(U; z^a)\}$ be a coordinate atlas of $Z$. Putting $J^0_nZ = Z$, let us provide $J^0_nZ$ with the atlas obtained by replacing every chart $(U, z^A)$ of $Z$ with the $m!/(n!(m-n)!)$ charts on $U$ which correspond to different partitions of $(z^A)$ in collections of $n$ and $m-n$ coordinates $(x^a, y^i)$, $a = 1, \ldots, n$, $i = 1, \ldots, m-n$. Accordingly, the first order jet manifold $J^1_nZ$ is endowed with the coordinates $(x^a, y^i, y^i_a)$ possessing transition functions

$$x^a = x^a(x^b, y^k), \quad y^i = y^i(x^b, y^k), \quad y^i_a = (\frac{\partial y^i}{\partial y^k} y^k_b + \frac{\partial y^i}{\partial x^a} ((\frac{\partial x^b}{\partial y^a} y^i_a + \frac{\partial x^b}{\partial x^a})).$$

In particular, if coordinate transition functions $x^a$ are independent of coordinates $y^k$, the transformation law (76) comes to the familiar transformations of jets of sections.

Given a coordinate chart $(\rho^{-1}(U); x^a, y^i, y^i_a)$ of $J^1_nZ$, one can regard $\rho^{-1}(U)$ as the first order jet manifold $J^1U$ of sections of the fiber bundle $U \ni (x^a, y^i) \to (x^a) \in U_X$. The graded
Any regular element \((q^\mu, x^a, y^i, y^i_\mu)\) possessing transition functions
\[
q^\mu = q^\mu(q^\nu), \quad x^a = x^a(x^b, y^k), \quad y^i = y^i(x^b, y^k),
\]
\[
x^a_\mu = \left(\frac{\partial x^a}{\partial y^k} y^k + \frac{\partial x^a}{\partial x^b} x^b_\nu\right) \frac{\partial q^\nu}{\partial q^\mu}, \quad y^i_\mu = \left(\frac{\partial y^i}{\partial y^k} y^k + \frac{\partial y^i}{\partial x^b} x^b_\nu\right) \frac{\partial q^\nu}{\partial q^\mu}.
\]  
An element \((q^\mu, x^a, y^i, y^i_\mu)\) is called regular if an \(m \times n\) matrix with the entries \((x^a_\mu, y^i_\mu)\) of maximal rank \(n\). This property is preserved under the coordinate transformations (77). Obviously, any regular elements of \(J^1 Z_Q\) defines some jet of \(n\)-dimensional subbundles of the manifold \(\{q\} \times Z\) through a point \((x^a, y^i) \in Z\). Moreover, one can state the following relations between the elements of \(J^1_n Z\) and the regular elements of \(J^1 Z_Q\) [95].

Any jet of submanifolds \((x^a, y^i, y^i_\mu)\) through a point \(z \in Z\) defines some (but not unique) jet \((q^\mu, x^a, y^i, x^a_\mu, y^i_\mu)\) of sections of the fiber bundle \(Z_Q\) (75) through a point \(q \times z\) for any \(q \in Q\) if the jet coordinates obey the equalities
\[
y^i_a x^a_\mu = y^i_\mu.
\]  
Any regular element \((q^\mu, x^a, y^i, x^a_\mu, y^i_\mu)\) of \(J^1 Z_Q\) defines a unique element \((x^a, y^i, y^i_a)\) of the jet manifold \(J^1_n Z\) by means of the equalities
\[
y^i_a = y^i_\mu(x^{-1})^\mu_a.
\]  
The equalities (78) and (79) are maintained under coordinate transformations (76) – (77).

Note that there is a certain ambiguity between elements of \(J^1_n Z\) and \(J^1 Z_Q\). Non-regular elements of \(J^1 Z_Q\) can correspond to different jets of submanifolds. Two regular elements \((q^\mu, z^A, z^\mu_A)\) and \((q^\nu, z^A, z^\nu_A)\) of \(J^1 Z_Q\) define the same jet of submanifolds if \(z^\nu_A = M^\nu_\mu z^\mu_A\), where \(M\) is some matrix. For instance, \(M\) comes from a diffeomorphism of \(Q\).

Basing on this result, one can describe the dynamics of \(n\)-dimensional submanifolds of a manifold \(Z\) as that of sections of the fiber bundle (75) (see item 26).

## 25 Relativistic mechanics

Given an \(m\)-dimensional manifold \(Z\) coordinated by \((z^A)\), let us consider the jet manifold \(J^1_n Z\) of its one-dimensional submanifolds. Let us provide \(Z = J^0_n Z\) with coordinates \((x^0 = z^0, y^i = z^i)\). Then the jet manifold \(J^1_n Z\) is endowed with coordinates \((z^0, z^i, z^0_0)\) possessing transition functions (76) which read
\[
z^0 = z^0(z^0, z^k), \quad z^0 = z^0(z^0, z^k), \quad z^\mu_0 = \left(\frac{\partial z^\mu}{\partial z^0} z^0_0 + \frac{\partial z^\mu}{\partial z^0} z^0_0\right) \left(\frac{\partial z^0}{\partial z^0} z^0_0 + \frac{\partial z^0}{\partial z^0}\right)^{-1}.
\]
A glance at this expression shows that \( J^1 Z \rightarrow Z \) is a fiber bundle in projective spaces. For instance, put \( Z = \mathbb{R}^4 \) whose Cartesian coordinates are subject to Lorentz transformations

\[
z^0 = z^0 \cosh \alpha - z^1 \sinh \alpha, \quad z^r = -z^0 \sinh \alpha + z^1 \cosh \alpha, \quad z^{2,3} = z^{2,3}. \tag{81}
\]

Then \( z^\mu (80) \) are exactly the Lorentz transformations

\[
z^1 = \frac{z^1 \cosh \alpha - \sinh \alpha}{z^0 \sinh \alpha + \cosh \alpha}, \quad z^{2,3} = \frac{z^{2,3}}{-z^0 \sinh \alpha + \cosh \alpha}
\]

of three-velocities in relativistic mechanics [95, 143, 175, 186].

Let us consider a one-dimensional manifold \( Q = \mathbb{R} \) and the product \( Z_Q = \mathbb{R} \times Z \). Let \( \mathbb{R} \) be provided with a Cartesian coordinate \( \tau \) possessing transition function \( \tau' = \tau + \text{const.} \)

Then the jet manifold \( J^1 Z_Q \) of the fiber bundle \( \mathbb{R} \times Z \rightarrow \mathbb{R} \) is endowed with the coordinates \((\tau, z^0, z^i, z^0_\tau, z^i_\tau)\) with the transition functions

\[
z^0_\tau = \frac{\partial z^0}{\partial z^k} z^k_\tau + \frac{\partial z^0}{\partial z^0} z^0_\tau, \quad z^i_\tau = \frac{\partial z^i}{\partial z^k} z^k_\tau + \frac{\partial z^i}{\partial z^0} z^0_\tau.
\]

In the case of Lorentz transformations (81), these transition functions are transformations of four-velocities in relativistic mechanics where \( \tau \) is a proper time.

Let us consider coordinate charts \((U'; \tau, z^0, z^i, z^0_\tau)\) and \((U''; \tau, z^0, z^i, z^0_\tau, z^i_\tau)\) of the manifolds \( \mathbb{R} \times J^1 Z \) and \( J^1 Z_Q \) over the same chart \((U; \tau, z^0, z^i)\) of \( Z_Q \). Then one can associate to each element \((\tau, z^0, z^i, z^0_\tau)\) of \( U' \) the elements of \( U'' \) which obey the relations (78)–(79):

\[
z^i = z^i_\tau, \quad z^0 = \frac{z^0_\tau}{z^0_\tau} = 0. \tag{82}
\]

Given a point \((\tau, z) \in \mathbb{R} \times Z\), the relations (82) are exactly the correspondence between elements of a one-dimensional vector subspace of the tangent space \( T_{\tau} Z \) and the corresponding element of the projective space of these subspaces. In relativistic mechanics, the relations (82) are familiar equalities between three- and four-velocities, and one avoids the ambiguity between them by means of the nonholonomic constraint \((z^0_\tau)^2 - \sum_i (z^i_\tau)^2 = 1\).

### 26 String theory

Given a manifold \( Z \), one can develop Lagrangian theory of its \( n \)-dimensional submanifolds as Lagrangian theory on the fiber bundle \( Z_Q \) (75) for an appropriate \( n \)-dimensional manifold \( Q \). If \( n = 2 \), we are in the case of classical string theory.

Let \( Z_Q \) be a fiber bundle (75) coordinated by \((q^\mu, z^A)\) and \( J^1 Z_Q \) its first order jet manifold provided with coordinates \((q^\mu, z^A, z^{\mu}_A)\), possessing transition functions

\[
q^\mu(q^\nu), \quad z^{\mu}_A(z^B), \quad z^{\mu}_A = \frac{\partial z^{\mu}_A}{\partial q^\nu} \frac{\partial q^\nu}{\partial z^{\mu}_B} z^B.
\]
Let \( L = \mathcal{L}(z^A, z^A_\mu) d^n q \) be a first order Lagrangian on \( J^1 Z_Q \) and \( \delta L = \mathcal{E}_A dz^A \wedge d^n q \) its Euler–Lagrange operator. Let us consider an arbitrary vector field \( u = u^\mu(q^\nu) \partial_\mu \) on \( Q \). It is an infinitesimal generator of a one-parameter group of local diffeomorphisms of \( Q \). Since \( Z_Q \to Q \) is a trivial bundle, this vector field gives rise to a vector field \( u = u^\mu \partial_\mu \) on \( Z_Q \), and its jet prolongation onto \( J^1 Z_Q \) reads

\[
\begin{align*}
\quad u &= u^\mu \partial_\mu - z^A_\nu \partial_\mu u^\nu \partial^A = u^\mu d_\mu + [-u^\nu z^A_\nu \partial_A - d_\mu (u^\nu z^A_\nu)] \partial^A.
\end{align*}
\]  

One can regard it as a generalized vector field depending on parameter functions \( u^\mu(q^\nu) \). In order to describe jets of submanifolds of \( Z_Q \), it seems reasonable to require that a Lagrangian \( L \) on \( J^1 Z_Q \) is independent on coordinates of \( Q \) and variationally invariant under \( u \) or, equivalently, its vertical part

\[
\quad u^V = -u^\nu z^A_\nu \partial_A - d_\mu (u^\nu z^A_\nu) \partial^\mu.
\]

Then the variational derivatives of this Lagrangian obey irreducible Noether identities

\[
\quad z^A_\nu \mathcal{E}_A = 0.
\]

For instance, let us consider Lagrangian theory of two-dimensional submanifolds (strings) [95]. Let \( Z \) be an \( m \)-dimensional locally affine manifold, i.e., a toroidal cylinder \( \mathbb{R}^{m-k} \times T^k \). Its tangent bundle \( TZ \) can be provided with a constant non-degenerate fiber metric \( \eta_{AB} \). Let \( Q \) be a two-dimensional manifold. Let us consider the \( 2 \times 2 \) matrix with the entries \( h_{\mu\nu} = \eta_{AB} z^A_\mu z^B_\nu \). Then its determinant provides a Lagrangian

\[
\quad L = (\det h)^{1/2} d^2 q = (\eta_{AB} z^A_1 z^B_1) [\eta_{AB} z^A_2 z^B_2] - (\eta_{AB} z^A_1 z^B_2)^2)^{1/2} d^2 q
\]

on the jet manifold \( J^1 Z_Q \). This is the well known Nambu–Goto Lagrangian of string theory [109, 160]. It satisfies the Noether identities (84).

**IV. Quantum outcomes**

### 27 Quantum master equation

Discussing quantization of ACFT, we restrict our consideration to the case of a vector bundle \( Y \to X \) of classical field and ACFT obeying item (ii) of Theorem 7, i.e., its BRST extension \( P_\infty^\star \{ N \} \) is characterized by a Lagrangian \( L_E \) (37) and the BRST operator \( u_E \) (32). One can quantize this BRST theory in the framework of perturbative QFT in functional integral terms. This QFT is well formulated if a field Lagrangian is non-degenerate. A problem is that the BRST extended Lagrangian \( L_E = \mathcal{L}_E \omega \) is necessarily degenerate. Indeed, it obeys the classical master equation

\[
\{ L_E, L_E \} = 2 \frac{\delta \mathcal{L}_E}{\delta \omega_a} \frac{\delta \mathcal{L}_E}{\delta z^a} \omega = 0
\]  

(85)
which is reducible Noether identities. To overcome this difficulty, one often require that a
BRST extended Lagrangian is a solution $L_h$ of the quantum master equation

$$\{ L_h, L_h \} = \hbar \frac{\delta}{\delta z^a} \frac{\delta}{\delta z^a} L_h \omega, \quad h = \text{const.},$$

[23, 79, 100, 80]. Accordingly, a quantum BRST operator is defined, and quantum BRST
cohomology are studied.

28 Gauge fixing procedure

In order to make a Lagrangian $L_E$ non-degenerate, one can replace antifields in $L_E$ with
gauge-fixing terms [23, 80, 100]. For this purpose, let us consider an odd graded density $\Psi \omega$
of antifield number 1 which depends on original fields $s^A$ and ghosts $c^r_k$, $k = 0, \ldots, N$, but
not antifields $s^A, \tau_\alpha, k = 0, \ldots, N$. In order to satisfy these conditions, new field variables
must be introduced because all the ghosts are of negative antifield numbers. Therefore, let
us enlarge the BGDA $P^*\{N\}$ to the BGDA $P^*\{N\}$, possessing the basis

$$\{ s^A, c^r, c^r_1, \ldots, c^r_N, c^*_{r_1}, \ldots, c^*_{r_N}, \xi_A, \overline{c}_r, \overline{c}_r_1, \ldots, \overline{c}_r_N \},$$

where $[c^*_{r_k}] = [c^{r_k}]$ and $\text{Ant}[c^*_{r_k}] = k + 1, k = 0, \ldots, N$ [194]. Then one can choose $\Psi \omega$ as an
element of $\mathcal{F}^B_{h,n}\{N\}$. It is traditionally called the gauge-fixing fermion.

Let us replace all the antifields in the Lagrangian $L_E$ (37) with the gauge fixing terms

$$\overline{s}_A = \frac{\delta \Psi}{\delta s^A}, \quad \overline{c}_r_k = \frac{\delta \Psi}{\delta c^{r_k}}, \quad k = 0, \ldots, N.$$

We obtain the Lagrangian

$$L_\Psi = L + [u_E \frac{\delta \Psi}{\delta s^A} + \sum_{0 \leq k \leq N} u^{r_k}_E \frac{\delta \Psi}{\delta c^{r_k}}] \omega = L + u_E(\Psi) \omega + d_H \sigma. \quad (86)$$

A glance at the equalities

$$u_E(L_\Psi) = u(L) + u_E(u_E(\Psi)) \omega + d_H \sigma = d_H \sigma'$$

shows that BRST operator $u_E$ (32) is a variational symmetry of the Lagrangian $L_\Psi$. It
however is not a gauge symmetry of $L_\Psi$ if $L_\Psi$ depends on all the ghosts $c^{r_k}, k = 0, \ldots, N,$
i.e., no ghost is a gauge parameter. Therefore, we require that

$$\frac{\delta \Psi}{\delta s^A} \neq 0, \quad \frac{\delta \Psi}{\delta c^{r_k}} \neq 0, \quad k = 0, \ldots, N - 1. \quad (87)$$

In this case, Noether identities for the Lagrangian $L_\Psi$ (86) come neither from the BRST
symmetry $u_E$ nor the equalities (85). One also put

$$\Psi = \sum_{0 \leq k \leq N} \Psi^{r_k} c^*_{r_k}.$$
Finally, let $h^r_k r'_k$ be a non-degenerate bilinear form for each $k = 0, \ldots, N - 1$ whose coefficients are either real numbers or functions on $X$. Then a desired gauge-fixing Lagrangian is written in the form

$$L_{GF} = L_\Psi + \sum_{0 \leq k \leq N} \frac{1}{2} h^r_k r'_k \Psi^r \Psi^r' d^n x. \quad (89)$$

The BRST operator $u_E$ (32) fails to be a variational symmetry of the Lagrangian (89), but it can be extended to its variational symmetry

$$\tilde{u} = u_E - \sum_{0 \leq k \leq N} \frac{\partial}{\partial C^r_k} h^r_k r'_k,$$

though it is not nilpotent. Of course, the Lagrangian $L_{GF}$ and, accordingly, the generating functional of perturbative QFT essentially depends on a choice of the gauge-fixing fermion $\Psi$. The generating functional is invariant under the variations $\Psi + \delta \Psi$ of $\Psi$ if the gauge fixing Lagrangian obeys the quantum master equation [100].

### 29 Green function identities

In order to obtain the generating functional of BRST theory, one replaces horizontal densities, depending on jets, with local functionals (see item 12) evaluated for the jet prolongations of sections [2, 11, 12, 33, 149]. Note that such functionals, in turn, define differential forms on functional spaces [47, 66]. At the same time, there is the following relation between the algebras of jets of classical fields and the algebras of quantum fields [195].

Let us consider a Lagrangian field system on $X = \mathbb{R}^n$, coordinated by $(x^\lambda)$. It is described by the GDA $P^*$ possessing a local basis

$$\{ s^a, s^a_\lambda, s^a_{\lambda_1 \lambda_2}, \ldots, s^a_{\lambda_1 \cdots \lambda_k}, \cdots \}. \quad (90)$$

Let $L \in P^{0,n}$ be a non-degenerate Lagrangian. Let us quantize this Lagrangian system in the framework of perturbative Euclidean QFT. We suppose that $L$ is a Lagrangian of Euclidean fields on $X = \mathbb{R}^n$. The key point is that the algebra of Euclidean quantum fields $B_\Phi$ as like as $P^0$ is graded commutative [167, 184, 195]. It is generated by elements $\phi_{x\Lambda}^a$, $x \in X$. For any $x \in X$, there is a homomorphism

$$\gamma_x : f^A_{a_1 \cdots a_r} s^a_{\Lambda_1} \cdots s^a_{\Lambda_r} \mapsto f^A_{a_1 \cdots a_r} (x) \phi_{x\Lambda_1}^{a_1} \cdots \phi_{x\Lambda_r}^{a_r} , \quad f^A_{a_1 \cdots a_r} \in C^\infty (X), \quad (91)$$

of the algebra $P^0$ of classical fields to the algebra $B_\Phi$ which sends the basis elements $s^a_{\Lambda} \in P^0$ to the elements $\phi_{x\Lambda}^a \in B_\Phi$, and replaces coefficient functions $f$ of elements of $P^0$ with their values $f(x)$ at a point $x$. Then a state $\langle \cdot \rangle$ of $B_\Phi$ is given by symbolic functional integrals

$$\langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \rangle = \frac{1}{N} \int \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \exp \{- \int \mathcal{L}(\phi_{x\Lambda}^a) d^n x \} \prod_x [d\phi_{x}^{a}], \quad (92)$$

$$N = \int \exp \{- \int \mathcal{L}(\phi_{x\Lambda}^a) d^n x \} \prod_x [d\phi_{x}^{a}],$$

$$\mathcal{L}(\phi_{x\Lambda}^a) = \mathcal{L}(x, \gamma_x (s^a_\Lambda)).$$

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which restart complete Euclidean Green functions in the Feynman diagram technique.

Due to homomorphisms (91), any graded derivation $\vartheta$ of $P^0$ induces the graded derivation

$$\tilde{\vartheta} : \phi^a_x \rightarrow (x, s^a_x) \rightarrow u^a_{\Lambda}(x, s^b_{\Sigma}) \rightarrow u^a_{\Lambda}(x, s^b_{\Sigma}) = \tilde{\vartheta}^a_{\Lambda}(\phi^b_x)$$

of the algebra of quantum fields $B_\Phi$ [195]. With an odd parameter $\alpha$, let us consider the automorphism

$$\hat{U} = \exp\{\alpha \tilde{\vartheta}\} = \text{Id} + \alpha \tilde{\vartheta}$$

of the algebra $B_\Phi$. This automorphism yields a new state $\langle . \rangle'$ of $B_\Phi$ given by the equalities

$$\langle \phi^a_{x_1} \cdots \phi^a_{x_k} \rangle = \langle \hat{U}(\phi^a_{x_1}) \cdots \hat{U}(\phi^a_{x_k}) \rangle' = \frac{1}{N'} \int \hat{U}(\phi^a_{x_1}) \cdots \hat{U}(\phi^a_{x_k}) \exp\{- \int \mathcal{L}(\hat{U}(\phi^a_{x\Lambda})) d^m x\} \prod_x [d\hat{U}(\phi^a_x)],$$

$$N' = \int \exp\{- \int \mathcal{L}(\hat{U}(\phi^a_{x\Lambda})) d^m x\} \prod_x [d\hat{U}(\phi^a_x)].$$

It follows from the first variational formula (13) that

$$\int \mathcal{L}(\hat{U}(\phi^a_{x\Lambda})) d^m x = \int (\mathcal{L}(\phi^a_{x\Lambda}) + \alpha \partial^a_{\Lambda} \mathcal{E}_{xa}) \omega,$$

where $\mathcal{E}_{xa} = \gamma_x(\mathcal{E}_a)$ are the variational derivatives. It is a property of symbolic functional integrals that

$$\prod_x [d\hat{U}(\phi^a_x)] = (1 + \alpha \int \frac{\partial^a_{x\Lambda} d^m x}{\partial \phi^a_x} \prod_x [d\phi^a_x] = (1 + \alpha \text{Sp}(\tilde{\vartheta})) \prod_x [d\phi^a_x].$$

Then the equalities (93) result in the identities

$$\langle \tilde{\vartheta}(\phi^a_{x_1} \cdots \phi^a_{x_k}) \rangle + \langle \phi^a_{x_1} \cdots \phi^a_{x_k} \rangle \langle \text{Sp}(\tilde{\vartheta}) - \int \tilde{\vartheta}^a_{x\Lambda} \mathcal{E}_{xa} d^m x \rangle =$$

$$\langle \phi^a_{x_1} \cdots \phi^a_{x_k} \rangle \langle \text{Sp}(\tilde{\vartheta}) - \int \tilde{\vartheta}^a_{x\Lambda} \mathcal{E}_{xa} d^m x \rangle = 0. \quad (94)$$

for complete Euclidean Green functions (92).

In particular, if $\vartheta$ is a variational symmetry of a Lagrangian $L$, the identities (94) are the Ward identities

$$\langle \tilde{\vartheta}(\phi^a_{x_1} \cdots \phi^a_{x_k}) + \phi^a_{x_1} \cdots \phi^a_{x_k} \rangle \langle \text{Sp}(\vartheta) - \int \vartheta^a_{x\Lambda} \mathcal{E}_{xa} d^m x \rangle -$$

$$\langle \phi^a_{x_1} \cdots \phi^a_{x_k} \rangle \langle \text{Sp}(\vartheta) - \int \vartheta^a_{x\Lambda} \mathcal{E}_{xa} d^m x \rangle = 0. \quad (95)$$

generalizing the Ward (Slavnov–Taylor) identities in gauge theory [34, 80, 149].

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If \( \vartheta = c^a \partial_a, \ c^a = \text{const} \), the identities (94) take the form

\[
\sum_{r=1}^{k} (-1)^{\left[ a_1 \right]} \cdots \left[ a_r \right] \left[ a_{r+1} \right] \cdots \left[ a_{k} \right] \langle \varphi_{x_1}^{a_1} \cdots \varphi_{x_{r-1}}^{a_{r-1}} \delta_{x_{r+1}}^{a_r} \cdots \varphi_{x_k}^{a_k} \rangle = \sum_{r=1}^{k} (-1)^{\left[ a_1 \right]} \cdots \left[ a_r \right] \left[ a_{r+1} \right] \cdots \left[ a_{k} \right] \langle \varphi_{x_1}^{a_1} \cdots \varphi_{x_{r-1}}^{a_{r-1}} \left( \int E_{x_1} d^n x \right) \rangle + \langle \varphi_{x_1}^{a_1} \cdots \varphi_{x_k}^{a_k} \rangle \left( \int \hat{E}_{x_0} d^n x \right) = 0. \tag{96}
\]

One can think of them as being equations for complete Euclidean Green functions. Clearly, the expressions (94) – (96) are singular, unless one follows regularization and renormalization procedures, which however can induce additional anomaly terms.

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