Too good to be true: when overwhelming evidence fails to convince

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Is it possible for a large sequence of measurements or observations, which support a hypothesis, to counterintuitively decrease our confidence? Can unanimous support be too good to be true? The assumption of independence is often made in good faith, however rarely is consideration given to whether a systemic failure has occurred.

Taking this into account can cause certainty in a hypothesis to decrease as the evidence for it becomes apparently stronger. We perform a probabilistic Bayesian analysis of this effect with examples based on (i) archaeological evidence, (ii) weighing of legal evidence, and (iii) cryptographic primality testing.

We find that even with surprisingly low systemic failure rates high confidence is very difficult to achieve and in particular we find that certain analyses of cryptographically-important numerical tests are highly optimistic, underestimating their false-negative rate by as much as a factor of $2^{80}$.

INTRODUCTION

In a number of branches of science, it is now well-known that deleterious effects can conspire to produce a benefit or desired positive outcome. A key example where this manifests is in the field of *stochastic resonance* [1,3] where a small amount of random noise can surprisingly improve system performance, provided some aspect of the system is nonlinear. Another celebrated example is that of *Parrondo’s Paradox* where individually losing strategies combine to provide a winning outcome [1,5].

Loosely speaking, a small amount of ‘bad’ can produce a ‘good’ outcome. But is the converse possible? Can too much ‘good’ produce a ‘bad’ outcome? In other words, can we have too much of a good thing?

The answer is affirmative—when improvements are made that result in a worse overall outcome this situation is known as *Verschlimmbesserung* [4] or disimprovement. Whilst this converse paradigm is less well known in the literature, a key example is the *Braess Paradox* where an attempt to improve traffic flow by adding bypass routes can counterintuitively result in worse traffic congestion [7-9]. Another example is the *truel*, where three gunmen fight to the death—it turns out that under certain conditions the weakest gunman surprisingly reduces his chances of survival by firing a shot at either of his opponents [10]. These phenomena can be broadly considered to fall under the class of anti-Parrondo effects [11,12], where the inclusion of winning strategies fail.

In this paper, for the first time, we perform a Bayesian mathematical analysis to explore the question of multiple confirmatory measurements or observations for showing when they can—surprisingly—disimprove confidence in the final outcome. We choose the striking example that increasing confirmatory identifications in a police line-up or identity parade can, under certain conditions, reduce our confidence that a perpetrator has been correctly identified.

Imagine that as a court case drags on, witness after witness is called. Let us suppose thirteen witnesses have testified to having seen the defendant commit the crime. Witnesses may be notoriously unreliable, but the sheer magnitude of the testimony is apparently overwhelming. Anyone can make a misidentification but intuition tells us that, with each additional witness in agreement, the chance of them all being incorrect will approach zero. Thus one might naïvely believe that the weight of as many as thirteen unanimous confirmations leaves us beyond reasonable doubt.

However, this is not necessarily the case and more confirmations can surprisingly disimprove our confidence that the defendant has been correctly identified as the perpetrator. This type of possibility was recognised intuitively in ancient times. Under ancient Jewish law [13], one could not be unanimously convicted of a capital crime—it was held that the absence of even one dissenting opinion among the judges indicated that there must remain some form of undiscovered exculpatory evidence.

Such approaches are greatly at odds with standard practice in engineering, where measurements are often taken to be independent. When this is so, each new measurement tends to lend support to the outcome with which it most concords. An important question, then, is to distinguish between the two types of decision problem; those where additional measurements truly lend support, and those for which increasingly consistent evidence ei-
A HYPOTHETICAL ROMAN POT

Let us begin with a simple scenario, the identification of the origin of a clay pot that has been dug from British soil. Its design identifies it as being from the Roman era, and all that remains is to determine whether it was made in Roman-occupied Britain or whether it was brought from Italy by travelling merchants. Suppose that we are fortunate and that a test is available to distinguish between the clay from the two regions; clay from one area—let us suppose that it is Britain—contains a trace element which can be detected by laboratory tests with an error rate $p_e = 0.3$. This is clearly excessive, and so we run the test several times. After $k$ tests have been made on the pot, the number of errors will be binomially-distributed $E \sim \text{Bin}(k, p_e)$. If the two origins, Britain and Italy, are a priori equally likely, then the most probable origin is the one suggested by the greatest number of samples.

Now imagine that several manufacturers of pottery deliberately introduced large quantities of this element during their production process, and that therefore it will be detected with 90% probability in their pots, which make up $p_c = 1\%$ of those found; of these, half are of British origin. We call $p_c$ the contamination rate. This is the hidden failure state to which we alluded in the introduction. Then, after the pot tests positive several times, we will become increasingly certain that it was manufactured in Britain. However, as more and more test results are returned from the laboratory, all positive, it will become more and more likely that the pot was manufactured with this unusual process, eventually causing the probability of British origin, given the evidence, to fall to 50%. This is the essential paradox of the system with hidden failure states—overwhelming evidence can itself be evidence of uncertainty, and thus be less convincing than more ambiguous data.

Formal model

Let us now proceed to formalise the problem above. Suppose we have two hypotheses, $H_0$ and $H_1$, and a series of measurements $X = (X_1, X_2, \ldots, X_n)$. We define a variable $F \in \mathbb{N}$ that determines the underlying measurement distribution, $p_{X|F,H_i}(x)$. We may then use Bayes’ law to find

$$P[H_i|X] = \frac{P[X|H_i]P[H_i]}{P[X]}$$

(1)

which can be expanded by condition with respect to $F$, yielding

$$\sum_{f} P[X|H_i, f]P[H_i, F = f] = \sum_{f, H_k} P[X|H_k, f]P[H_k, F = f].$$

(2)

In our examples there are a number of simplifying conditions—there are only two hypotheses and two measurement distributions, reducing Eqn. (2) to
Computation of these \textit{a posteriori} probabilities thus requires knowledge of two distributions: the measurement distributions \( P[X|H_k, F] \), and the state probabilities \( P[H_i, F] \). Having tabulated these, we may substitute them into Eqn. \[E3\] yielding the \textit{a posteriori} probability for each hypothesis. In this paper, the measurement distributions \( P[X|H_i, F = f] \) are all binomial, however this is not the case in general.

### Analysis of the pot origin distribution

In the case of the pot, the hypotheses and measurement distributions—the origin and contamination, respectively—are shown in Table \[I\].

Each measurement is Bernoulli-distributed, and the number of positive results is therefore described by a Binomial distribution, with the probability mass function

\[
P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}
\]

after \( N \) trials, the probability \( p \) being taken from the measurement distribution section of Table \[I\].

![FIG. 1. Probability that the pot is of British origin given \( n \) numbers of tests, all coming back positive, for a variety of contamination rates \( p_c \) and a 30% error rate. In the case of the pot above, with \( p_c = 10^{-2} \), we see a peak at \( n = 5 \), after which the level of certainty falls back to 0.5 as it becomes more likely that the pot originates at a contaminating factory. When \( p_c = 0 \), this is the standard Bayesian analysis where failure states are not considered. We see therefore that even small contamination rates can have a large effect on the global behaviour of the testing methodology.](image)

It is worth taking a moment, however, to briefly discuss the effects of weakening certain conditions; in particular, we consider two cases: that where the rate of contamination depends upon the origin of the pot, and that where the results after contamination are also origin-depandant.

Where the rate of contamination depends upon the origin, evidence of contamination provides some small evidence of where the pot came from. Thus if 80\% of contaminated pots are of British origin, then Figure \[I\] will eventually converge to 0.8 rather than 0.5.

If the probability of a positive test is dependent upon the origin even when contaminated, then the behaviour of the test protocol changes qualitatively. Supposing that
in the contaminated case the test is substantially less accurate, the probability of British origin will drop towards 0.5, as now, due to the increased likelihood of contamination; eventually, however, it will rise again towards 1.0 as sufficient data becomes available to make use of the less probative experiment that we now know to be taking place.

**THE RELIABILITY OF IDENTITY PARADES**

We initially described the scenario of a court case, in which witness after witness testifies to having seen the defendant commit the crime of which he is accused. But in-court identifications are considered unreliable, and in reality if identity is in dispute then the identification is made early in the investigation under controlled conditions \[16\]. At some point, whether before or after being charged, the suspect has most likely been shown to each witness amongst a number of others, known as fillers, who are not under suspicion. Each witness is asked to identify the true perpetrator, if present, amongst the group.

This process, known as an *identity parade* or *line-up*, is an experiment intended to determine whether the suspect is in fact the same person as the perpetrator. It may be performed only once, or repeated many times with many witnesses. As human memory is inherently uncertain, the process will include random error; if the experiment is not properly carried out then there may also be systematic error, and this is the problem that concerns us in this paper.

Having seen how a unanimity of evidence can create uncertainty in the case of the unidentified pot, we now apply the same analysis to the case of an identity parade. If the perpetrator is not present—that is to say, if the suspect is innocent—then in an unbiased parade the witness should be unable to choose the suspect with a probability greater than chance. Ideally, they would decline to make a selection, however this does not always occur in practice \[16\] \[17\], and forms part of the random error of the procedure. If the parade is biased—whether intentionally or unintentionally—for example because (i) the suspect is somehow conspicuous \[18\], (ii) the staff running the parade direct the witness towards him, (iii) by chance he happens to resemble the perpetrator more closely than the fillers, or (iv) because the witness holds a bias, for example because they have previously seen the suspect \[19\], then an innocent suspect may be selected with a probability greater than chance. This is the hidden failure state that underlies this example; we assume in our analysis that this is completely binary—either the parade is completely unbiased or it is highly biased against the suspect.

In recent decades, a number of experiments \[17\] \[19\] have been carried out in order to establish the reliability of this process. Test subjects are shown the commission of a simulated crime, whether in person or on video, and asked to locate the perpetrator amongst a number of people. In some cases the perpetrator will be present, and in others not. The former allows estimation of the false-negative rate of the process—the rate that the witness fails to identify the perpetrator when present—and the latter the false-positive rate—the rate at which an innocent suspect will be mistakenly identified. Let us denote by \( p_f \) the false-negative rate; this is equal to the proportion of subjects who failed to correctly identify the perpetrator when he was present, and was found in \[17\] to be 48%.

Estimating the false positive rate is complicated by the fact that only one suspect is present in the lineup—when the suspect is innocent, an eyewitness who incorrectly identifies a filler as being the perpetrator has correctly rejected the innocent suspect as being the perpetrator, despite their error. For the purposes of our analysis, we assume that the witness selects at random in this case, and therefore divide the 80% perpetrator-absent selection rate of \[17\] by the number of participants \( L = 6 \), yielding a false-positive rate of \( p_p = 0.133 \).

Let us now suppose that there is a small probability \( p_c \) that the line-up is conducted incorrectly—for example, volunteers have been chosen who fail to adequately match the description of the perpetrator—leading to identification of the suspect 90% of the time, irrespective of his guilt. For the sake of analysis we assume that if this occurs, it will occur for all witnesses, though in practice the police might perform the procedure correctly for some witnesses and not others. The probability of the suspect being identified for each of the cases is shown in Table II.

| TABLE II. The model parameters for the hypothetical identity parade. In a similar fashion to the first example, we assume *a priori* a 50% probability of guilt. In this case, the measurement distributions are substantially assymmetric with respect to innocence and guilt, unlike Table I. |
|---|
| \( P(F, H_j) \) | \( P(Identification \mid F, H_j) \) |
| **Suspect is...** | \( H_0 \) | \( H_1 \) | \( H_0 \) | \( H_1 \) |
| **Innocent** | 0.005 | 0.005 | Y | 0.9 |
| **Guilty** | 0.495 | 0.495 | N | 0.13 |
| \( F = 0 \) | \( \% p_c \) | \( \% p_c \) | \( \% \left(1 - p_c\right) \) | \( \% \left(1 - p_c\right) \) |
| \( F = 1 \) | \( \% \left(1 - p_c\right) \) | \( \% \left(1 - p_c\right) \) | \( \% \left(1 - p_c\right) \) | \( \% \left(1 - p_c\right) \) |

If we assume a 50% prior probability of guilt, and independent witnesses, the problem is now identical to that of identifying the pot. The probability of guilt, given the unanimous parade results, is shown in Figure 2 as a function of the number of unanimous witnesses.
FIG. 2. Probability of guilt given varying numbers of unanimous line-up identifications, assuming a 50% prior probability of guilt and identification accuracies given by [17]. Of note is that for the case that we have plotted here where the witnesses are unanimous, with a failure rate $p_e = 0.01$ it is impossible to reach 95% certainty in the guilt of the suspect, no matter how many witnesses have been found.

We see that after a certain number of unanimously positive identifications the probability of guilt diminishes. Even with only one in ten-thousand line-ups exhibiting this bias towards the suspect, the peak probability of guilt is reached with only five unanimous witnesses, completely counter to intuition—in fact, with this rate of failure, ten identifications in agreement provide less evidence of guilt than three. We see also that even with a 50% prior probability of guilt, a 1% failure rate renders it impossible to achieve 95% certainty if the witnesses are unanimous.

This tendency to be biased towards a particular member of the lineup when an error occurs has been noted [16, paragraph 4.31] prior to the more rigorous research stimulated by the advent of DNA testing, leading us to suspect that our sub-1% contamination rates are probably overly optimistic.

**ANCIENT JUDICIAL PROCEDURE**

The acknowledgement of this type of phenomenon is not entirely new; indeed, the adage “too good to be true” dates to the sixteenth century [20, good, P5.b]. Moreover, its influence on judicial procedure was visible in Jewish law even in the classical era; until the Romans ultimately removed the right of the Sanhedrin to confer death sentences, a defendant unanimously condemned by the judges would be acquitted [13, Sanhedrin 18b], the Talmud stating “If the Sanhedrin unanimously find guilty, he is acquitted. Why? — Because we have learned by tradition that sentence must be postponed till the morrow in hope of finding new points in favour of the defence”.

The value of this rule becomes apparent when we consider that the Sanhedrin was composed, for ordinary capital offenses, of 23 members [13, Sanhedrin 2a]. In our line-up model, this many unanimous witnesses would indicate a probability of guilt scarcely better than chance, suggesting that the inclusion of this rule should have a substantial effect.

We show the model parameters for the Sanhedrin decision in Table III, which we use to compute the possibility of guilt in Figure 3 for various numbers of judges condemning the defendant. We see that the probability of guilt falls as judges approach unanimity, however excluding unanimous decisions substantially reduces the probability of false conviction.

It is worth stressing that the exact shapes of the curves in Figure 3 are unlikely to be entirely correct; communication between the judges will prevent their verdicts from being entirely independent, and false-positive and false-negative rates will be very much dependent upon the evidentiary standard required to bring charges, the strength of the contamination when it does occur, and the accepted burden of proof of the day. However, it is nonetheless of qualitative interest that with reasonable parameters, this ancient law can be shown to have a sound statistical basis.

**THE RELIABILITY OF CRYPTOGRAPHIC SYSTEMS**

We now consider a different example, drawn from cryptography. An important operation in many protocols is the generation and verification of prime numbers; the se-
FIG. 3. Probability of guilt as a function of judges in agreement out of 23—the number used by the Sanhedrin for most capital crimes—for various contamination rates $p_c$. We assume as before that half of defendants are guilty, and use the estimated false-positive and false-negative rates of juries from [21, model (2)], 0.14 and 0.25 respectively. We arbitrarily assume that a ‘contaminated’ trial will result in the a positive vote 95% of the time. The panel of judges numbers 23, with conviction requiring a majority of two and at least one dissenting opinion [13, Sanhedrin]; the majority of two means that the agreement of at least 13 judges is required in order to to cast a sentence of death, to a maximum of 22 votes in order to satisfy the requirement of a dissenting opinion. These necessary conditions for a conviction by the Sanhedrin are shown as the pink region in the graph.

Security of some protocols depends upon the primality of a number that may be chosen by an adversary; in this case, one may test whether it is a prime, whether by brute-force or by using another test such as the Rabin-Miller [22, p. 176] test. As the latter is probabilistic, we repeat it until we have achieved the desired level of security—in [22], a probability $2^{-128}$ of accepting a composite as prime is considered acceptable. However, a naïve implementation cannot achieve this level of security, as we will demonstrate.

The reason is that despite it being proven that each iteration of the Rabin-Miller test will reject a composite number with probability at least 0.75, a real computer may fail at any time. The chance of this occurring is small, however it turns out that the probability of a stray cosmic ray flipping a bit in the machine code, causing the test to accept composite numbers, is substantially greater than $2^{-128}$.

**Code changes caused by memory errors**

Data provided by Google [23] suggests that a given memory module has approximately an 8% probability of suffering an error in any given year, independent of capacity. Assuming a 4 GB module, this results in approximately a $\lambda = 10^{-19}$ probability that any given bit will be flipped in any given second. We will make the assumption that, in the machine code for the primality-testing routine, there exists at least one bit that, if flipped, will cause all composite numbers—or some class of composite numbers known to the adversary—to be accepted as prime. As an example of how this could happen, consider the function shown in Figure 4 that implements a brute-force factoring test. Assuming that the input is odd, the function will reach one of two return statements, returning zero or one. The C compiler GCC compiles these two return statements to

```c
int trialdivision (long to_test)
{
    long i;
    long threshold;
    if( to_test % 2 == 0)
    {
        return 1;
    }
    threshold = (long)sqrt(to_test);
    for(i = 3; i <= threshold; i += 2)
    {
        if( to_test % i == 0)
        {
            return 1;
        }
    }
    return 0;
}
```

FIG. 4. A function that tests for primality by attempting to factorise its input by brute force.

respectively. That is to say, it stores the return value as an immediate into the EAX register and then jumps to the cleanup section of the function, labelled .L3. The store instructions on lines 45 and 56 have machine-code values B801000000 and B800000000 for return values of one and zero respectively. These differ by only one bit, and therefore can be transformed into one another by a single bit-error. If the first instruction is turned into the second, this will cause the function to return zero for any odd input, thus always indicating that the input is prime.
TABLE IV. Model parameters for the Rabin-Miller test on random 2000-bit numbers. However, we have no choice but to assume the lower bound on the composite-number rejection rate, and so this model is inappropriate. Furthermore, in an adversarial setting the attacker may intentionally choose a difficult-to-detect composite number, rendering the prior distribution optimistic.

The effect of memory errors on confidence

At cryptographically-interesting sizes—on the order of $2^{2000}$—roughly one in a thousand numbers is prime [22, p. 173]. We might calculate the model parameters as before—for interest’s sake, we have done so in Table IV—and calculate the confidence in a number’s primality after a given number of tests. However, this is not particularly useful, for two reasons: first, the rejection probability of 75% is a lower bound, and for randomly chosen numbers is a substantial underestimate; second, we do not always choose numbers at random, but rather may need to test those provided by an adversary. In this case, we must assume that they have tried to deceive us by providing a composite number, and would instead like to know the probability that they will be successful. The Bayesian estimator in this case would provide only a tautology of the type: ‘given the data and the fact that the number is composite, the number is composite’.

Let us suppose that the machine containing the code is rebooted every month, and the Rabin-Miller code remains in memory for the duration of this period; then, neglecting other potential errors that could affect the test, at the time of the reboot the probability that the bit has flipped is now $p_f = 2.6 \times 10^{-13}$; this event we denote $A_F$. Let $k$ be the number of iterations performed; the probability of accepting a composite number is at most $4^{-k}$, and we assume that the adversary has chosen a composite number such that this is the true probability of acceptance. We denote the event that the prime is accepted by the correctly-operating algorithm $A_R$.

When hardware errors are taken into account, the probability of accepting a composite number is no longer $A_R$ accepted by the correctly-operating algorithm $A_R$.

\begin{align*}
4^{-k}, & \\
p_{fa} = P[A_F \cup A_R] & = P[A_F] + P[A_R] - P[A_F, A_R]. \quad (4) \\
& = 4^{-k}(1 - p_f) + p_f. \quad (5) \\
\text{Since } A_F \text{ and } A_R \text{ are independent,} & \\
& = P[A_F] + P[A_R] - P[A_F]P[A_R] \quad (6) \\
& \geq p_f. \quad (7)
\end{align*}

No matter how many iterations $k$ of the algorithm are performed, this is substantially greater than the $2^{-128}$ security level that is predicted by probabilistic analysis of the algorithm alone, thus demonstrating that algorithmic analyses that do not take into account the reliability of the underlying hardware can be highly optimistic. The false acceptance rate as a function of the number of test iterations and time in memory is shown in Figure 5.

A real cryptographic system will include many such checks in order to make sure that an attacker has not chosen weak values for various parameters, and a failure of any of these may result in the system being broken, so our calculations are somewhat optimistic.

Error-correcting-code equipped (ECC) memory will substantially reduce the risk of this type of fault, and for regularly-accessed regions of code—multiple times per second—will approach the $2^{-128}$ level. A single parity bit, as used in at least some CPU level-one instruction...
caches [24], requires two bit-flips to induce an error. Suppose the parity is checked every $R$ seconds, then the probability of an undetected bit-flip in any given second is

$$\lambda' = \frac{(\lambda R)^2}{R} = \lambda^2 R. \quad (9)$$

For code that is accessed even moderately often, this will come much closer to $2^{-128}$. For example, if $R = 100$ ms then this results in a false-acceptance rate of $2^{-108}$ after one month, much closer to the $2^{-128}$ level of security promised by analysis of the algorithm alone. The stronger error-correction codes used by the higher-level caches and main memory will detect virtually all such errors—with two-bit detection capability, the rate of undetected bit-flips will be at most

$$\lambda' = \lambda^3 R^2, \quad (10)$$

and even with check rate of only once per 100 ms, the rate of memory errors is essentially zero, increasing the false-acceptance rate by a factor of only $10^{-14}$ above the $2^{-128}$ level that would be achieved in a perfect operating environment.

DISCUSSION

This phenomenon is interesting in that it is commonly known and applied heuristically, and trivial examples such as the estimation of coin bias [25, section 2.1] have been well-analysed—see the appendix for a brief discussion—but these rare failure states are rarely, if ever, considered when a statistical approach to decision-making is applied to an entire system. Real systems that attempt to counter failure modes producing consistent data tend to focus upon the detection of particular failures rather than the mere fact of consistency. Sometimes there is little choice—a casino that consistently ejected gamblers on a winning streak would soon find itself without a clientele—however we have demonstrated that in many cases the level of consistency needed to cast doubt on the validity of the data is surprisingly low.

If this is so, then we must reconsider the use of thresholding as a decision mechanism when there is the potential for such failure modes to exist, particularly when the consequences of an incorrect decision are large. When the decision rule takes the form of a probability threshold, it is necessary to deduce an upper threshold as well, such as was shown in Figure 3 in order to avoid capturing the region indicative of a systemic failure.

That this phenomenon was accounted for in ancient Jewish legal practice indicates a surprising level of intuitive statistical sophistication in this ancient law code; though predating by millennia the statistical tools needed to perform a rigorous analysis, our simple model of the judicial panel indicates that the requirement of a dissenting opinion would have provided a substantial increase in the probability of guilt required to secure a conviction.

Applied to cryptographic systems, we see that even the minuscule probability that one particular bit in the system's machine code will be flipped due to a memory error over the course of a month, rendering the system insecure, is approximately $2^{80}$ times larger than the risk predicted by algorithmic analysis. This demonstrates the importance of strong error correction in modern cryptographic systems that strive for a failure rate on the order of $2^{-128}$, a level of certainty that appears to be otherwise unachievable without active mitigation of the effect.

The use of naturally-occurring memory errors for DNS hijacking [26] has previously been demonstrated, and the ability of a user to disturb protected addresses by writing to adjacent cells [27] has been demonstrated, however little consideration has been given to the possibility that this type of fault might occur simply by chance, implying that security analyses which assume reliable hardware are substantially flawed when applied to consumer systems lacking error-corrected memory.

We have considered only a relatively simple case, in which there are only two levels of contamination. However, in practical situations we might expect any of a wide range of failure modes varying continuously. We have described a simple case in the appendix, where a coin may be biased—or not—towards either heads or tails with any strength; were one to apply this to the case of an identity parade, for example, one would find a probability that the suspect is indeed the perpetrator, as before, but taking into account that there may well be slight biases that nudge the witnesses towards or away from the suspect, not merely catastrophic ones. The result is heavily dependent upon the distribution of the bias, and lacking sufficient data to produce such a model we have chosen to eschew the complexity of the continuous approach and focus on a simple two-level model. An example of this approach is shown in [28, p. 117].

A related concept to that which we have discussed is the Duhem-Quine hypothesis [28, p. 6]; this is the idea that an experiment inherently tests hypotheses as a group—not merely the phenomenon that we wish to examine, but also the correct function of the experimental apparatus for example, and that that only the desired independent variables are being changed. Our thesis is a related one, namely that in practical systems the failure of these auxiliary hypotheses, though unlikely, result in a significant reduction in confidence when it occurs, an effect which has traditionally been ignored.

CONCLUSION

We have analysed the behaviour of systems that are subject to systematic failure, and demonstrated that with
relatively low failure rates, large sample sizes are not required in order that unanimous results start to become indicative of systematic failure. We have investigated the effect of this phenomenon upon identity parades, and shown that even with only a 1% rate of failure, confidence begins to decrease after only three unanimous identifications, failing to reach even 95%.

We have also applied our analysis of the phenomenon to cryptographic systems, investigating the effect by which confidence in the security of a parameter fails to increase with further testing due to potential failures of the underlying hardware. Even with a minuscule failure rate of 10^{-13} per month, this effect dominates the analysis and is thus a significant determining factor in the overall level of security, increasing the probability that a maliciously-chosen parameter will be accepted by a factor of more than 2^{80}.

Hidden failure states such as these reduce confidence far more than intuition leads one to believe, and must be more carefully considered than is the case today if the lofty targets that we set for ourselves are to be achieved in practice.

**COMPETING INTERESTS**

We have no competing interests.

**AUTHOR CONTRIBUTIONS**

LJG drafted the manuscript. LJG and DA devised the concept. LJG, FC-B, MDM, BRD, AA, and DA carried out analyses and checking.

All authors contributed to proofing the manuscript. All authors gave final approval for publication.

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**Analysis of a biased coin**

It is worth adding a brief discussion of a simple and well-known problem that has some relation to what we have discussed, namely the question of whether or not a coin is biased. We follow the Bayesian approach given in [23].

They use Bayes’ law in its proportional form,

\[ P(Q|\{\text{data}\}) \propto P(\{\text{data}\}|Q)P(Q), \]

(11)

where \( Q \) is the probability that a coin-toss will yield heads. Various prior distributions \( Q[H] \) can be chosen, a matter that we will discuss momentarily.

As the coin tosses are independent, the data can be boiled down to a binomial random variable \( X \sim \text{Bin}(p, n) \), where \( n \) is the number of coin tosses made. Substituting the binomial probability mass function into Eqn. [11] they find that

\[ P(Q,X) \propto Q^X(1-Q)^{n-X}P(Q). \]

(12)

As the number of samples \( n \) increases, this becomes increasingly peaked around the value \( Q = X/n \), this ‘peaking’ effect limited by the shape of \( P(Q) \). As the number of samples increases, the \( Q^X(1-Q)^{n-X} \) part of the expression eventually comes to dominate the shape of the posterior distribution \( P(Q|X) \), and we have no choice but to believe that the coin genuinely does have a bias close to \( X/n \).

In the examples previously discussed, we have assumed that bias is very unlikely; in the coin example, this corresponds to a prior distribution \( P(Q) \) that is strongly clustered around \( Q = 0.5 \); in this case, a very large number of samples will be necessary in order to conclusively reject the hypothesis that the coin is unbiased or nearly so. However, eventually this will occur, and the posterior distribution will change; when this occurs, the system has visibly failed—a casino using the coin will decide that they are not in fact playing the game that they had planned, and must cease before their loss becomes catastrophic. This is much like in the case of the Sanhedrin—if too many judges agree, the system has failed, and should not be considered reliable.

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