FINITE ELEMENTS FOR DIVDIV-CONFORMING SYMMETRIC TENSORS IN THREE DIMENSIONS

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ABSTRACT. Two types of finite element spaces on a tetrahedron are constructed for divdiv conforming symmetric tensors in three dimensions. Besides the normal-normal component, another trace involving combination of first order derivatives of stress should be continuous across the face. Due to the rigid of polynomials, the symmetric stress tensor element is continuous at vertices, and on the plane orthogonal to each edge. Hilbert complex and polynomial complexes are presented and several decomposition of polynomial vector and tensors spaces are revealed from the complexes. The constructed divdiv conforming elements are exploited to discretize the mixed formulation of the biharmonic equation. Optimal order and superconvergence error analysis is provided. Hybridization is given for the ease of implementation.

1. INTRODUCTION

Recently we have constructed finite elements for divdiv conforming symmetric tensors in two dimensions [5]. In this paper, we shall continue our study of space

$\mathcal{H}(\text{div div}, \Omega; \mathbb{S}) := \{\tau \in L^2(\Omega; \mathbb{S}) : \text{div div} \tau \in L^2(\Omega)\}$

and construct corresponding finite element spaces in three dimensions.

It turns out the construction in three dimensions is much harder than that in two dimensions. One obvious reason is the complication of interplay of differential operators (grad, curl, and div) and matrix operators (sym, dev, and mspn etc) in three dimensions. The essential difficulty arises from the Hilbert complex

$$
\begin{array}{ccccccc}
RT & \subset & H^1(\Omega; \mathbb{R}^3) & \xrightarrow{\text{dev grad}} & H(\text{sym curl}, \Omega; \mathbb{T}) & \xrightarrow{\text{sym curl}} & H(\text{div div}, \Omega; \mathbb{S}) & \xrightarrow{\text{div div}} & L^2(\Omega) & \rightarrow 0.
\end{array}
$$

In the divdiv complex in three dimensions, the Sobolev space before $H(\text{div div}, \Omega; \mathbb{S})$ consists of tensor functions rather than vector functions in two dimensions. The subspace of polynomial space with vanishing boundary degree of freedoms is not clear as no finite element spaces for $H(\text{sym curl}, \Omega; \mathbb{T})$ is known. In two dimensions, we have $\ker(\text{div div}) \cap H(\text{div div}, \Omega; \mathbb{S}) = \text{sym curl}(H^1(\Omega; \mathbb{R}^2))$ and finite element spaces for $H^1(\Omega; \mathbb{R}^2)$ are relatively mature. By analogy, an $H^{-1}(\text{div div}, \Omega; \mathbb{S})$ nonconforming finite element in two dimensions was advanced for discretizing the mixed formulation of the biharmonic equation in [8, 9, 11] in 1960s, while the corresponding $H^{-1}(\text{div div}, \Omega; \mathbb{S})$ nonconforming finite element in three dimensions was constructed to solve a mixed formulation of the linear elasticity until 2011 [14], rather than the biharmonic equation.

\[2010 \text{ Mathematics Subject Classification. 65N30; 65N12; 65N22;}
\]

The first author was supported by NSF DMS-1913080.

The second author was supported by the National Natural Science Foundation of China Project 11771398 and the Fundamental Research Funds for the Central Universities 2019110066.

1
To attack the main difficulty, we present two polynomial complexes and reveal several decomposition of polynomial vector and tensors spaces from the complexes. With further help from the Green’s identity and characterization of the trace, we are able to construct two types of finite element spaces on a tetrahedron. Here we present the BDM-type space below. Let $K$ be a tetrahedron and let $k \geq 3$ be a positive integer. The shape function space is simply $P^k(K;S)$. The set of edges of $K$ is denoted by $\mathcal{E}(K)$, the faces by $\mathcal{F}(K)$, and the vertices by $\mathcal{V}(K)$. For each edge, we chose two normal vectors $n_1$ and $n_2$. The degrees of freedom are given by

1. $\tau(\delta) \quad \forall \delta \in \mathcal{V}(K)$,
2. $(n^T \tau n_j, q)_e \quad \forall q \in P_{k-2}(e), e \in \mathcal{E}(K), i, j = 1, 2$,
3. $(n^T \tau n, q)_F \quad \forall q \in P_{k-3}(F), F \in \mathcal{F}(K)$,
4. $(2 \text{div}_F(\tau n) + \partial_n(n^T \tau n), q)_F \quad \forall q \in P_{k-1}(F), F \in \mathcal{F}(K)$,
5. $(\tau, \zeta)_K \quad \forall \zeta \in \nabla^2 P_{k-2}(K) \oplus \text{sym}(x \times P_{k-2}(K;T))$,
6. $(\tau n, n \times x q)_{F_1} \quad \forall q \in P_{k-2}(F_1)$,

where $F_1 \in \mathcal{F}(K)$ is an arbitrary but fixed face. The degrees of freedom (6) will be regarded as interior degrees of freedom to the tetrahedron $K$, that is the degrees of freedom (6) are double-valued on each face $F \in \mathcal{F}_k^h$ when defining the global finite element space. RT-type space can be obtained by further reducing the index of degree of freedoms by 1 except the moment with $\nabla^2 P_{k-2}(K)$.

The boundary degree of freedoms (1)-(4) are motivated by the Green’s formulae and the characterization of the trace of $H(\text{div div}, \Omega; \mathcal{S})$. The interior moments of $\nabla^2 P_{k-2}(K)$ is to determine $\text{div} \text{div} \tau$. Together with $\text{sym}(x \times P_{k-2}(K;T))$, the volume moments can determine the polynomial of degree up to $k-1$.

We then use the vanished trace and the symmetry of the tensor. Similarly as the RT and BDM elements [2], the vanishing normal-normal trace (34) implies the normal-normal part of $\tau$ is zero. To determine the normal-tangential terms, further degrees of freedoms are needed. Due to the symmetry of $\tau$, it is sufficient to provide additional degrees of freedoms on one face, which are inspired by the RT and BDM elements in two dimensions.

It is arduous to figure out the explicit basis functions dual to the degrees of freedom (1)-(6). Hybridization is thus provided for the ease of implementation. The basis functions of the standard Lagrange element can be used to implement the hybridized mixed finite element methods. The constructed divdiv conforming elements are exploited to discretize the mixed formulation of the biharmonic equation. Optimal order and superconvergence error analysis is provided.

The rest of this paper is organized as follows. In Section 2, we present some operations for vectors and tensors. Two polynomial complexes related to the divdiv complex, and direct sum decompositions of polynomial spaces are shown in Section 3. We derive the Green’s identity and characterize the trace of $H(\text{div div}, \Omega; \mathcal{S})$ on polyhedrons in Section 4, and then construct the conforming finite elements for $H(\text{div div}, \Omega; \mathcal{S})$ in three dimensions. Mixed finite element methods for the biharmonic equation are developed in Section 5.

2. Matrix and vector operations

In this section, we shall survey operations for vectors and tensors. Some of them are standard but some are not unified in the literature. In particular, we shall
introduce operators appending to the right side of a matrix for operations applied to columns of a matrix. We will mix the usage of row and column vectors which will be clear in the context.

Given a plane \( F \) with normal vector \( n \), for a vector \( v \in \mathbb{R}^3 \), we have the orthogonal decomposition
\[
v = \Pi_n v + \Pi_F v := (v \cdot n)n + (n \times v) \times n.
\]
The vector \( \Pi_F v := n \times v \) is also on the plane \( F \) and is a rotation of \( \Pi_F v \) by 90° counter-clockwise with respect to \( n \). Therefore
\[
\Pi_F u \cdot \Pi_F v = ((n \times u) \times n) \cdot ((n \times v) \times n) = (n \times u) \cdot (n \times v) = \Pi_F u \cdot \Pi_F v.
\]
We treat Hamilton operator \( \nabla = (\partial_1, \partial_2, \partial_3)^\top \) as a column vector and define
\[
\nabla_F^\perp := n \times \nabla, \quad \nabla_F := \Pi_F \nabla = (n \times \nabla) \times n.
\]
For a scalar function \( v \),
\[
\nabla_F v = \Pi_F (\nabla v), \quad \nabla_F^\perp v = n \times \nabla v,
\]
are the surface gradient of \( v \) and surface curl, respectively. For a vector function \( v \), \( \nabla_F \cdot v \) is the surface divergence and by definition
\[
\nabla_F \cdot v = \nabla_F \cdot (\Pi_F v) = (n \times \nabla) \cdot (n \times v) = \nabla_F^\perp \cdot (n \times v).
\]
By the cyclic invariance of the mix product, the surface rot operator is
\[
\nabla_F^\perp \cdot v = (n \times \nabla) \cdot v = n \cdot (\nabla \times v).
\]
In particular, for \( n = (0, 0, 1) \), \( F \) is the \( x-y \) plane. Then \( \nabla_F^\perp \) is the two dimensional rot operator which is a rotation of the two dimensional divergence operator \( \nabla_F^\top \).

The matrix-vector product \( Ab \) can be interpret as the inner product of \( b \) with the row vectors of \( A \). We thus define the dot operator
\[
b \cdot A := Ab.
\]
Namely the vector inner product is applied row-wise to the matrix. Similarly we can define the cross product row-wise from the left \( b \times A \). Here rigorously speaking when a column vector \( b \) is treat as a row vector, \( b^\top \) should be used. In most places, however, we will scarify this precision for the ease of notation.

When the vector is on the right of the matrix, the operation is defined column-wise. That is
\[
A \cdot b := b^\top A = (A^\top b)^\top = (b \cdot A^\top)^\top, \quad \quad A \times b := -(b \times A^\top)^\top.
\]
By moving the column operation to the right, it is consistent with the transpose operator. For the transpose of product of two objects, we take transpose of each one, switch their order, and add a negative sign if it is the cross product.

For two vectors \( b, c \) and matrix \( A \), we define the following products
\[
b \cdot A \cdot c := b^\top A^\top c = (b \cdot A) \cdot c = b \cdot (A \cdot c), \quad \quad b \cdot A \times c := (Ab) \times c = b \cdot (A \times c), \quad \quad b \times A \times c := (b \times A) \times c = b \times (A \times c).
\]
By moving the column operation to the right, these operations are associative. That is the ordering of performing the products does not matter.
Apply these matrix-vector operations to the Hamilton operator $\nabla$, we get row-wise differentiation
\[ \nabla \cdot A, \quad \nabla \times A, \]
and column-wise differentiation
\[ A \cdot \nabla, \quad A \times \nabla. \]
And we can write the divdiv operator applied to a matrix as
\[ (7) \quad \text{div div} A = \nabla \cdot (A \cdot \nabla). \]

Denote the space of all $3 \times 3$ matrix by $\mathbb{M}$, all symmetric $3 \times 3$ matrix by $\mathbb{S}$, all skew-symmetric $3 \times 3$ matrix by $\mathbb{K}$, and all trace-free $3 \times 3$ matrix by $\mathbb{T}$. For any matrix $B \in \mathbb{M}$, we can decompose it into symmetric and skew-symmetric part as
\[ B = \text{sym}(B) + \text{skw}(B) := \frac{1}{2} (B + B^\top) + \frac{1}{2} (B - B^\top). \]
We can also decompose it into a direct sum of a trace free matrix and a diagonal matrix as
\[ \text{dev} B = \frac{1}{3} \text{tr}(B) I := \frac{1}{3} (B - \text{tr}(B) I) + \frac{1}{3} \text{tr}(B) I. \]

For a vector function $u = (u_1, u_2, u_3)^\top$, $\text{curl} u = \nabla \times u$ and $\text{div} u = \nabla \cdot u$ are standard differential operations. The gradient $\nabla u := u \otimes \nabla = u \nabla^\top$ is a matrix
\[ \begin{pmatrix} \nabla u_1 \\ \nabla u_2 \\ \nabla u_3 \end{pmatrix}. \]
Its symmetric part is defined as
\[ \text{def} u := \text{sym} \nabla u = \frac{1}{2} (\nabla u + (\nabla u)^\top) = \frac{1}{2} (u \nabla^\top + \nabla u^\top). \]
In the last identity notation $u \nabla^\top$ is used to emphasize the symmetry form. Similarly we can define $\text{curl}$ operator for a matrix $A$
\[ \text{sym} \text{curl} A = \frac{1}{2} (\nabla \times A + (\nabla \times A)^\top) = \frac{1}{2} (\nabla \times A - A^\top \times \nabla). \]

We define an isomorphism of $\mathbb{R}^3$ and the space of skew-symmetric matrices as follows: for a vector $\omega = (\omega_1, \omega_2, \omega_3)^\top \in \mathbb{R}^3$,
\[ \text{mspn} \omega := [\omega]_\times := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \]
For two vectors $u, v$, one can easily verify that
\[ u \times v = [u]_\times v = (\text{mspn} u)v. \]
We will use the following identities which can be verified by direct calculation.
\[ \text{skw}(\nabla u) = \frac{1}{2} (\text{mspn} \nabla \times u), \]
\[ \text{skw}(\nabla \times A) = \frac{1}{2} \text{mspn} \left[ (A \cdot \nabla)^\top - \nabla (\text{tr}(A)) \right], \]
\[ \text{div} \text{mspn} u = - \text{curl} u, \]
\[ \text{curl}(u I) = - \text{mspn} \text{grad}(u). \]
More identities involving the matrix operation and vector differentiation are summarized in [1].
3. DIVDIV COMPLEX AND POLYNOMIAL COMPLEXES

In this section, we shall consider the divdiv complex and establish two related polynomial complexes. We assume \( \Omega \subset \mathbb{R}^3 \) is a bounded and strong Lipschitz domain which is topologically trivial in the sense that it is homeomorphic to a ball. Without loss of generality, we also assume \((0,0,0) \in \Omega\).

3.1. The div div complex. The div div complex in three dimensions reads as \([1, 13]\)

\[
\begin{array}{cccc}
RT & \subset & H^1(\Omega; \mathbb{R}^3) & \overset{\text{dev grad}}{\longrightarrow} H(\text{sym curl}, \Omega; T) & \overset{\text{sym curl}}{\longrightarrow} H(\text{div div}, \Omega; S) & \overset{\text{div div}}{\longrightarrow} L^2(\Omega) & \longrightarrow & 0.
\end{array}
\]

For completeness, we prove the exactness of the complex (11) following [13].

**Theorem 3.1.** Assume \( \Omega \) is a bounded and topologically trivial strong Lipschitz domain in \( \mathbb{R}^3 \). Then (11) is a complex and exact sequence.

**Proof.** Any skew-symmetric \( \tau \) can be written as \( \tau = \text{mspn} v \). Assume \( v \in C^2(\Omega; \mathbb{R}^3) \), it follows from (9) that

\[
\text{div div} \tau = \text{div div} \text{mspn} v = -\text{div(curl} v) = 0.
\]

Since \( \text{div div} \tau = 0 \) for any smooth skew-symmetric tensor field \( \tau \), we obtain

\[
\text{div div} H(\text{div div}, \Omega; S) = \text{div div} H(\text{div div}, \Omega; M) = L^2(\Omega).
\]

For any \( \tau \in C^3(\Omega; T) \), it follows from (7) that

\[
\text{div div} \text{sym curl} \tau = \frac{1}{2} \nabla \cdot (\nabla \times A - A^T \times \nabla) \cdot \nabla = 0.
\]

Hence \( \text{div div} \text{sym curl} H(\text{sym curl}, \Omega; T) = 0 \). For any \( v \in C^2(\Omega; \mathbb{R}^3) \), it holds from (10) that

\[
\text{sym curl dev grad} v = \text{sym curl} \left( \text{grad} v - \frac{1}{3} (\text{div} v) I \right) = -\frac{1}{3} \text{sym curl}((\text{div} v) I) = \frac{1}{3} \text{sym mspn}(\text{grad}(\text{div} v)) = 0.
\]

We get \( \text{sym curl dev grad} H^1(\Omega; \mathbb{R}^3) = 0 \). As a result, (11) is a complex.

For any \( \sigma \in H(\text{div div}, \Omega; S) \cap \ker(\text{div div}) \), there exists \( v \in L^2(\Omega; \mathbb{R}^3) \) such that

\[
\text{div} \sigma = \text{curl} v = -\text{div} (\text{mspn} v).
\]

Hence here exists \( \tau \in H^1(\Omega; M) \) such that

\[
\sigma = -\text{mspn} v + \text{curl} \tau.
\]

By the symmetry of \( \sigma \), we have \( \sigma = \text{sym curl} \tau \). Noting that

\[
\text{sym curl}((\text{tr} \tau) I) = -\text{sym mspn}(\text{grad}(\text{tr} \tau)) = 0,
\]

it follows \( \sigma = \text{sym curl dev} \tau \). Thus

\[
H(\text{div div}, \Omega; S) \cap \ker(\text{div div}) = \text{sym curl} H(\text{sym curl}, \Omega; T).
\]

For any \( \tau \in H(\text{sym curl}, \Omega; T) \cap \ker(\text{sym curl}) \), by the fact that \( \text{tr} \tau = 0 \), we have from (8) that

\[
\text{curl} \tau = \text{skw curl} \tau = \frac{1}{2} \text{mspn}(\text{div} \tau^T - \text{grad}(\text{tr} \tau)) = \frac{1}{2} \text{mspn}(\text{div} \tau^T).
\]

Then

\[
\text{curl}(\text{div} \tau^T) = -\text{div}(\text{mspn} \text{div} \tau^T) = -2\text{div}(\text{curl} \tau) = 0.
\]
Thus there exists \( u \in L^2_0(\Omega) \) satisfying \( \text{div} \mathbf{T}^T = 2 \text{grad} w \), which implies
\[
\text{curl} \ \mathbf{T} = \text{mspn} \ \text{grad} w = -\text{curl}(wI).
\]
Hence there exists \( v \in H^1(\Omega; \mathbb{R}^3) \) such that \( \mathbf{T} = -wI + \text{grad} \ v \). Since \( \mathbf{T} \) is trace-free, we achieve
\[
\mathbf{T} = \text{dev} \ \mathbf{T} = \text{dev} \ \text{grad} \ v,
\]
which means \( H(\text{sym} \ \text{curl}, \Omega; T) \cap \ker(\text{sym} \ \text{curl}) = \text{dev} \ \text{grad} \ H^1(\Omega; \mathbb{R}^3) \). Therefore the complex (11) is exact. \( \square \)

3.2. Polynomial complexes. Given a bounded domain \( G \subset \mathbb{R}^3 \) and a non-negative integer \( m \), let \( P_m(G) \) stand for the set of all polynomials in \( G \) with the total degree no more than \( m \), and \( P_m(G; \mathbb{R}^3) \) denote the tensor or vector version.

Lemma 3.2. The polynomial complex
\[
(13) \quad \mathbf{RT} \subset \xrightarrow{\text{dev} \ \text{grad}} P_{k+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{sym} \ \text{curl}} P_{k+1}(\Omega; \mathbb{T}) \xrightarrow{\text{div} \ \text{grad}} P_k(\Omega; S) \xrightarrow{\text{div} \ \text{div}} P_{k-2}(\Omega) \xrightarrow{0}
\]
is exact.

Proof. It follows from (12) that
\[
\text{div} \ \text{div} \ P_k(\Omega; S) = \text{div} \ \text{div} \ P_k(\Omega; M) = P_{k-2}(\Omega).
\]
For any \( q \in P_{k+2}(\Omega; \mathbb{R}^3) \cap \ker(\text{grad}) \), we have \( \nabla q = \frac{1}{2}((\text{div} q) I) \). Hence
\[
-\text{mspn}(\nabla \text{div} q) = \text{curl}((\text{div} q) I) = 3 \text{curl}(\nabla q) = 0,
\]
which means \( \nabla q \) is the identity matrix multiplied by a constant.

For any \( \mathbf{T} \in P_{k+1}(\Omega; T) \cap \ker(\text{sym} \ \text{curl}) \), there exists \( v \in H^1(\Omega; \mathbb{R}^3) \) satisfying \( \mathbf{T} = \text{dev} \ \text{grad} \ v \), i.e., \( \mathbf{T} = \nabla v - \frac{1}{3}(\text{div} v) I \). Then
\[
\text{mspn}(\nabla \text{div} v) = -\text{curl}((\text{div} v) I) = 3 \text{curl}(\mathbf{T} - \nabla v) = 3 \text{curl} \mathbf{T} \in P_k(\Omega; K),
\]
from which we get \( \text{div} v \in P_{k+1}(\Omega) \), and thus \( \nabla v \in P_{k+1}(\Omega; M) \). As a result \( v \in P_{k+2}(\Omega; \mathbb{R}^3) \). And we also have
\[
\dim \text{sym} \ \text{curl} \ P_{k+1}(\Omega; T) = \dim P_{k+1}(\Omega; T) - \dim \text{dev} \ \text{grad} \ P_{k+2}(\Omega; \mathbb{R}^3)
\]
\[
= \frac{1}{6}(5k^3 + 36k^2 + 67k + 36),
\]
(14)
\[
\dim P_k(\Omega; S) \cap \ker(\text{div} \ \text{div}) = \frac{1}{6}(5k^3 + 36k^2 + 67k + 36).
\]
(15)
Finally we conclude \( P_k(\Omega; S) \cap \ker(\text{div} \ \text{div}) = \text{sym} \ \text{curl} \ P_{k+1}(\Omega; T) \) from (14) and (15). Therefore the complex (13) is exact. \( \square \)

Define operator \( \pi_{\mathbf{RT}} : C^1(\Omega; \mathbb{R}^3) \rightarrow \mathbf{RT} \) as
\[
\pi_{\mathbf{RT}} \mathbf{v} := \mathbf{v}(0, 0, 0) + \frac{1}{3}(\text{div} \ \mathbf{v})(0, 0, 0) \mathbf{x}.
\]
The following complex is a generalization of the Koszul complex for vector functions.

Lemma 3.3. The polynomial complex
\[
(16) \quad 0 \subset \xrightarrow{\mathbf{x} \mathbf{x}^T} P_{k-2}(\Omega) \xrightarrow{\mathbf{x} \times} P_k(\Omega; S) \xrightarrow{\mathbf{x} \times} P_{k+1}(\Omega; T) \xrightarrow{\mathbf{x} \times} P_{k+2}(\Omega; \mathbb{R}^3) \xrightarrow{\pi_{\mathbf{RT}}} \mathbf{RT} \xrightarrow{0}
\]
is exact.
\textbf{Proof.} Since
\[ \text{tr}(x \times \tau) = 2x^\top v \text{spn}(\text{skw} \tau) \quad \forall \ \tau \in L^2(\Omega; \mathbb{M}), \]
thus (16) is a complex.

For any \( v \in P_{k+2}(\Omega; \mathbb{R}^3) \) satisfying \( \pi_{RT} v = 0 \), since \( v(0,0,0) = 0 \), there exist \( \tau_1 \in P_{k+1}(\Omega; \mathbb{T}) \) and \( q \in P_{k+1}(\Omega) \) such that \( v = \tau_1 x + qx \). Noting that
\[ \text{div}(\tau_1 x) = x^\top \text{div}(\tau_1^\top) + \text{tr}(\tau_1) = x^\top \text{div}(\tau_1^\top), \]
we have
\[ \pi_{RT}(q)x = \pi_{RT} v - \pi_{RT}(\tau_1 x) = 0, \]
which indicates \( (\text{div}(q)x))(0,0,0) = 0 \) and thus \( q(0,0,0) = 0 \). Hence there exists \( q_1 \in P_k(\Omega; \mathbb{R}^3) \) such that \( q = q_1^\top x \). Taking \( \tau = \tau_1 + \frac{2}{3}q_1^\top x - \frac{1}{6}q_1^\top xI \in P_{k+1}(\Omega; \mathbb{T}) \),
we get
\[ \tau x = \tau_1 x + xq_1^\top x = \tau_1 x + qx = v. \]
Hence \( P_{k+2}(\Omega; \mathbb{R}^3) \cap \ker(\pi_{RT}) = P_{k+1}(\Omega; \mathbb{T})x \) holds.

It holds
\begin{equation}
\pi_{RT} v = v \quad \forall \ v \in RT.
\end{equation}
Namely \( \pi_{RT} \) is a projector. Consequently, the operator \( \pi_{RT} : P_{k+1}(\Omega; \mathbb{R}^3) \to RT \)
is surjective. And we have
\[ \dim(P_{k+1}(\Omega; \mathbb{T}) \cap \ker(x \cdot)) = \dim(P_{k+1}(\Omega; \mathbb{T}) \cap \ker(x \times)) = \dim(P_{k+1}(\Omega; \mathbb{T}) \cap \ker(x \times)) = \dim(P_{k+1}(\Omega; \mathbb{T}) \cap \ker(x \cdot)) = 1/6(5k^3 + 36k^2 + 67k + 36). \]

For any \( \tau \in P_k(\Omega; \mathbb{S}) \) satisfying \( x \times \tau = 0 \), there exists \( v \in P_{k-1}(\Omega; \mathbb{R}^3) \) such that \( \tau = vx^\top \). By the symmetry of \( \tau \), it follows
\[ x \times (xv^\top) = x \times (vx^\top) = x \times \tau^\top = 0, \]
which indicates \( x \times v = 0 \). Then there exists \( q \in P_{k-2}(\Omega) \) satisfying \( v = qx \). Hence \( \tau = qxv^\top \). Therefore \( P_k(\Omega; \mathbb{S}) \cap \ker(x \times) = P_{k-2}(\Omega)xv^\top \). And we have
\[ \dim(x \times P_k(\Omega; \mathbb{S})) = \dim(P_k(\Omega; \mathbb{S}) \cap \ker(x \times)) = \dim(P_k(\Omega; \mathbb{S}) \cap \ker(x \times)) = \dim(P_k(\Omega; \mathbb{S}) \cap \ker(x \times)) = 1/6(5k^3 + 36k^2 + 67k + 36), \]
which together with (18) implies \( P_{k+1}(\Omega; \mathbb{T}) \cap \ker(x \cdot) = x \times P_k(\Omega; \mathbb{S}) \). Therefore the complex (16) is exact. \hfill \square

Those two complexes (13) and (16) are connected as
\begin{equation}
RT \xrightarrow{\text{dev grad} \atop x} P_{k+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{sym curl} \atop x \times} P_{k+1}(\Omega; \mathbb{T}) \xrightarrow{\text{div div} \atop x^\top} P_k(\Omega; \mathbb{S}) \xrightarrow{\text{div} \atop x^\top} P_{k-2}(\Omega) \xrightarrow{\text{dev grad} \atop x} 0.
\end{equation}

Unlike the Koszul complex for vectors functions, we do not have the identity property applied to homogenous polynomials. Fortunately decomposition of polynomial spaces using Koszul and differential operators still holds.

Let \( H_k(\Omega) := P_k(\Omega)/P_{k-1}(\Omega) \) be the space of homogenous polynomials of degree \( k \). Then by Euler’s formula
\begin{equation}
\forall q \in H_k(\Omega), \quad x \cdot \nabla q = kq.
\end{equation}
Due to (20), for any \( q \in P_k(\Omega) \) satisfying \( x \cdot \nabla q + q = 0 \), we have \( q = 0 \). And
\[
\mathbb{P}_k(\Omega) \cap \ker(x \cdot \nabla) = \mathbb{P}_0(\Omega),
\]
for any positive integer \( \ell \).

It follows from (17) and the complex (16) that
\[
\mathbb{P}_{k+2}(\Omega; \mathbb{R}^3) = x \cdot \mathbb{P}_{k+1}(\Omega; \mathbb{T}) \oplus RT.
\]

We then move to the space \( \mathbb{P}_{k+1}(\Omega; \mathbb{T}) \).

**Lemma 3.4.** We have the decomposition
\[
\mathbb{P}_{k+1}(\Omega; \mathbb{T}) = x \times \mathbb{P}_k(\Omega; \mathbb{S}) \oplus \text{dev grad } \mathbb{P}_{k+2}(\Omega; \mathbb{R}^3).
\]

**Proof.** Since the dimension of space in the left hand side is the summation of the dimension of the two spaces in the right hand side in (23), we only need to prove that the sum in (23) is the direct sum. For any \( q \in \mathbb{P}_{k+2}(\Omega; \mathbb{R}^3) \) satisfying \( q(0, 0, 0) = 0 \) and \( \text{dev grad } q \in x \times \mathbb{P}_k(\Omega; \mathbb{S}) \), we have \( \text{dev grad } q = 0 \). This ends the proof. \( \Box \)

Finally we present a decomposition of space \( \mathbb{P}_k(\Omega; \mathbb{S}) \). Let
\[
\mathbb{C}_k(\Omega; \mathbb{S}) := \text{sym curl } P_{k+1}(\Omega; \mathbb{T}), \quad \mathbb{C}_k^0(\Omega; \mathbb{S}) := x x^\top P_{k-2}(\Omega).
\]

Their dimensions are
\[
\dim \mathbb{C}_k(\Omega; \mathbb{S}) = \frac{1}{6}(5k^3 + 36k^2 + 67k + 36), \quad \dim \mathbb{C}_k^0(\Omega; \mathbb{S}) = \frac{1}{6}(k^3 - k).
\]

**Lemma 3.5.** We have

(i) \( \text{div } \text{div } (x x^\top q) = (k + 4)(k + 3)q \) for any \( q \in \mathbb{H}_k(\Omega) \).

(ii) \( \text{div } : \mathbb{C}_k^0(\Omega; \mathbb{S}) \to \mathbb{P}_{k-2}(\Omega; \mathbb{R}^3) \) is a bijection.

(iii) \( \mathbb{P}_k(\Omega; \mathbb{S}) = \mathbb{C}_k(\Omega; \mathbb{S}) \oplus \mathbb{C}_k^0(\Omega; \mathbb{S}) \).

**Proof.** Since \( \text{div } x x^\top q = (\text{div } x x^\top q) x \) and \( \text{div } x q = (x \cdot \nabla) q + 3q \), we get
\[
\text{div } \text{div } (x x^\top q) = \text{div } ((x \cdot \nabla + 4)q x) = (x \cdot \nabla + 3)(x \cdot \nabla + 4)q.
\]
Hence property (i) follows from (20). Property (ii) is obtained by writing \( \mathbb{P}_{k-2}(\Omega; \mathbb{R}^3) = \bigoplus_{i=0}^{k-3} \mathbb{H}_i(D; \mathbb{R}^3) \). Now we prove property (iii). First the dimension of space in the left hand side is the summation of the dimension of the two spaces in the right hand side in (iii). Assume \( q \in \mathbb{P}_{k-2}(\Omega) \) satisfies \( x x^\top q \in \mathbb{C}_k(\Omega; \mathbb{S}) \), which means
\[
\text{div } \text{div } (x x^\top q) = 0.
\]
Thus property (iii) holds from (24) and (22). \( \Box \)
For the simplification of the degree of freedoms, we need another decomposition of the symmetric tensor polynomial space which can be derived from the dual complex of polynomial divdiv complexes.

Lemma 3.6. It holds

\[ \mathbb{P}_k(\Omega; \mathbb{S}) = \nabla^2 \mathbb{P}_{k+2}(\Omega) \oplus \text{sym}(x \times \mathbb{P}_{k-1}(\Omega; \mathbb{T))). \]

Proof. Noting that the dimension of space in the left hand side is the summation of the dimension of the two spaces in the right hand side in (25), we only need to prove the direct sum.

For any \( \tau = \nabla^2 q \) with \( q \in \mathbb{P}_{k+2}(\Omega) \) satisfying \( \tau \in \text{sym}(x \times \mathbb{P}_{k-1}(\Omega; \mathbb{T})) \), it follows \( (x \cdot \nabla)(x \cdot \nabla)q - q = x^\top(\nabla^2 q)x = 0 \). Applying (21) and (20), we get \( q \in \mathbb{P}_1(\Omega) \) and \( \nabla^2 q = 0 \). Thus the decomposition (25) holds. \( \square \)

4. Divdiv Conforming finite element spaces

We first present a Green’s identity based on which we can characterize the trace of \( H(\text{div} \text{div}, \Omega; \mathbb{S}) \) on polyhedrons and give a sufficient continuity condition for a piecewise smooth function to be in \( H(\text{div} \text{div}, \Omega; \mathbb{S}) \). Then we construct the finite element space and prove the unisolvence.

4.1. Notation. Let \( \{T_h\}_{h>0} \) be a regular family of polyhedral meshes of \( \Omega \). Our finite element spaces are constructed for tetrahedrons but some results, e.g., traces and Green’s formulae etc, hold for general polyhedrons. For each element \( K \in T_h \), denote by \( n_K \) the unit outward normal vector to \( \partial K \), which will be abbreviated as \( n \) for simplicity. Let \( F_h, \mathcal{F}_h, \mathcal{E}_h, \mathcal{V}_h \) and \( \mathcal{V}_h^i \) be the union of all faces, interior faces, all edges, interior edges, vertices and interior vertices of the partition \( T_h \), respectively. For any \( F \in \mathcal{F}_h \), fix a unit normal vector \( n_F \) and two unit tangent vectors \( t_{F,1} \) and \( t_{F,2} \), which will be abbreviated as \( t_1 \) and \( t_2 \) without causing any confusions. For any \( e \in \mathcal{E}_h \), fix a unit tangent vector \( t_e \) and two unit normal vectors \( n_{e,1} \) and \( n_{e,2} \), which will be abbreviated as \( n_1 \) and \( n_2 \) without causing any confusions. For \( K \) being a polyhedron, denote by \( \mathcal{F}(K), \mathcal{E}(K) \) and \( \mathcal{V}(K) \) the set of all faces, edges and vertices of \( K \), respectively. For any \( F \in \mathcal{F}_h \), let \( \mathcal{E}(F) \) be the set of all edges of \( F \). And for each \( e \in \mathcal{E}(F) \), denote by \( n_{F,e} \) the unit vector being parallel to \( F \) and outward normal to \( \partial F \). Furthermore, set

\[ \mathcal{F}^i(K) := \mathcal{F}(K) \cap \mathcal{F}_h^i, \quad \mathcal{E}^i(F) := \mathcal{E}(F) \cap \mathcal{E}_h^i. \]

4.2. Green’s identity. We start from the Green’s identity for smooth functions on polyhedrons.

Lemma 4.1 (Green’s identity). Let \( K \) be a polyhedron, and let \( \tau \in C^2(K; \mathbb{S}) \) and \( v \in H^2(K) \). Then we have

\[
(\text{div} \text{div} \tau, v)_K = (\tau, \nabla^2 v)_K - \sum_{F \in \mathcal{F}(K)} \sum_{e \in \mathcal{E}(F)} (n_{F,e}^\top \tau n, v)_e
- \sum_{F \in \mathcal{F}(K)} \left[ (n_{F}^\top \tau n, \partial_n v)_F - (2 \text{div}_F(\tau n) + \partial_n(n_{F}^\top \tau n), v)_F \right].
\]
Proof. We start from the standard integration by parts
\[
(\text{div} \tau, v)_K = - (\text{div} \tau, \nabla v)_K + \sum_{F \in \mathcal{F}(K)} (n^T \text{div} \tau, v)_F
\]
\[
= (\tau, \nabla^2 v)_K - \sum_{F \in \mathcal{F}(K)} (\tau n, \nabla v)_F + \sum_{F \in \mathcal{F}(K)} (n^T \text{div} \tau, v)_F.
\]
We then decompose \(\nabla v = \partial_n v n + \nabla_F v\) and apply the Stokes theorem to get
\[
(\tau n, \nabla v)_F = (\tau n, \partial_n v n + \nabla_F v)_F
\]
\[
= (n^T \tau n, \partial_n v)_F - (\text{div}_F (\tau n), v)_F + \sum_{e \in \mathcal{E}(F)} (n^T_{F,e} \tau n, v)_e.
\]
Now we rewrite the term
\[
(n^T \text{div} \tau, v)_F = (\text{div} (\tau \cdot n), v)_F = (\text{div}_F (\tau n), v)_F + (\partial_n (n^T \tau n), v)_F.
\]
Thus the Green’s identity (26) follows by merging all terms. \(\square\)

When the domain is smooth in the sense that \(\mathcal{E}(K)\) is an empty set, the term
\[
\sum_{F \in \mathcal{F}(K)} \sum_{e \in \mathcal{E}(F)} (n^T_{F,e} \tau n, v)_e
\]
does not appear. When \(v\) is continuous on edge \(e\), this term will define a jump of the tensor.

4.3. **Traces and continuity across the boundary.** We first recall the trace of the space \(H(\text{div} \tau, K; \mathbb{S})\) on the boundary of polyhedron \(K\) (cf. [7, Lemma 3.2] and [17, 15]). Define trace spaces
\[
H^{1/2}_{n,0}(\partial K) := \{\partial_n v|_{\partial K} : v \in H^2(K) \cap H^1_0(K)\}
\]
\[
= \{g \in L^2(\partial K) : g|_F \in H^{1/2}_{00}(F) \quad \forall F \in \mathcal{F}(K)\}
\]
with norm
\[
\|g\|_{H^{1/2}_{n,0}(\partial K)} := \inf_{v \in H^2(K) \cap H^1_0(K)} \|v\|_2,
\]
and
\[
H^{3/2}_{t,0}(\partial K) := \{v|_{\partial K} : v \in H^2(K), \partial_n v|_{\partial K} = 0, v|_e = 0 \quad \text{for each edge} \ e \in \mathcal{E}(K)\}
\]
with norm
\[
\|g\|_{H^{3/2}_{t,0}(\partial K)} := \inf_{v \in H^2(K)} \|v\|_2.
\]
Let \(H^{-1/2}_n(\partial K) := (H^{1/2}_{n,0}(\partial K))^\prime\) for the normal-normal trace, and \(H^{-3/2}_t(\partial K) := (H^{3/2}_{t,0}(\partial K))^\prime\) for the trace involving combination of derivatives.

**Lemma 4.2.** For any \(\tau \in H(\text{div} \tau, K; \mathbb{S})\), it holds
\[
\|n^T \tau n\|_{H^{-1/2}_n(\partial K)} + \|2 \text{div}_F (\tau n) + \partial_n (n^T \tau n)\|_{H^{-3/2}_t(\partial K)} \lesssim \|\tau\|_{H(\text{div} \tau)}.
\]
Conversely, for any \(g_n \in H^{-1/2}_n(\partial K)\) and \(g_t \in H^{-3/2}_t(\partial K)\), there exists some \(\tau \in H(\text{div} \tau, K; \mathbb{S})\) such that
\[
\frac{n^T \tau n}{\partial K} = g_n, \quad 2 \text{div}_F (\tau n) + \partial_n (n^T \tau n) = g_t,
\]
\[
\|\tau\|_{H(\text{div} \tau)} \lesssim \|g_n\|_{H^{-1/2}_n(\partial K)} + \|g_t\|_{H^{-3/2}_t(\partial K)}.
\]
The hidden constants depend only the shape of the domain \(K\).
Notice that the term \((n^T_e \tau n, v)_e\) in the Green’s identity (4.1) is not covered by Lemma 4.2. Indeed, the full characterization of the trace of \(H(\text{div div}, K; \mathbb{S})\) is defined by \((\text{div div}\tau, v) - (\tau, \nabla^2 v)_K\), which cannot be equivalently decoupled [7, Lemma 3.2]. But it is possible to face-wisely localize the trace if imposing additional smoothness.

We then present a sufficient continuity condition for piecewise smoothing functions to be in \(H(\text{div div}, \Omega; \mathbb{S})\).

**Lemma 4.3** (cf. Proposition 3.6 in [7]). Let \(\tau \in L^2(\Omega; \mathbb{S})\) such that

(i) \(\tau|_K \in H(\text{div div}, K; \mathbb{S})\) for each polyhedron \(K \in \mathcal{T}_h\);  
(ii) \((2 \text{div}_F(\tau n_F) + \partial_n(\mathbf{n}^T \tau n))|_F \in L^2(F)\) is single-valued for each \(F \in \mathcal{F}_h^i\);  
(iii) \((\mathbf{n}^T \tau n)|_F \in L^2(F)\) is single-valued for each \(F \in \mathcal{F}_h^i\);  
(iv) \((\mathbf{n}^T \tau n_j)|_e \in L^2(e)\) is single-valued for each \(e \in \mathcal{E}_h; i, j = 1, 2\), then \(\tau \in H(\text{div div}, \Omega; \mathbb{S})\).

**Proof.** For any \(v \in C^\infty_0(\Omega)\), we get from the Green’s identity (26) that

\[
(\tau, \nabla^2 v) = \sum_{K \in \mathcal{T}_h} (\text{div div}\tau, v)_K + \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}^i_K} \sum_{e \in \mathcal{E}^i(F)} (n^T_{F,e} \tau n, v)_e \\
+ \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}^i_K} \left[ (\mathbf{n}^T \tau n, \partial_n v)_F - (2 \text{div}_F(\tau n) + \partial_n(\mathbf{n}^T \tau n), v)_F \right].
\]

Since the terms in (ii)-(iv) are single-valued and each interior face is repeated twice in the summation with opposite orientation, it follows

\[
(\text{div div}\tau, v) = \sum_{K \in \mathcal{T}_h} (\text{div div}\tau, v)_K.
\]

Thus we have \(\tau \in H(\text{div div}, \Omega; \mathbb{S})\) by the definition of derivatives of the distribution, and \((\text{div div}\tau)|_K = \text{div}(\tau|_K)\) for each \(K \in \mathcal{T}_h\). \(\square\)

For any piecewise smooth \(\tau \in H(\text{div div}, \Omega; \mathbb{S})\), the single-valued term \((\mathbf{n}^T \tau n_j)|_e\) in (iv) in Lemma 4.3 implies that there is some compatible condition for \(\tau\) at each \(\delta \in \mathcal{V}^i_{h}\). Indeed, for any \(\delta \in \mathcal{V}^i_{h}\) and \(F \in \mathcal{F}^i_{h}\) with \(\delta\) being a vertex of \(F\), let \(n_1 = t_1 \times n_F\) and \(n_2 = t_2 \times n_F\), where \(t_1\) and \(t_2\) are the unit tangential vectors of two edges of \(F\) sharing \(\delta\). Then by (iv) we have

\[
[n^T_1 \tau n_1|_F(\delta) = [n^T_2 \tau n_2|_F(\delta) = [n^T_{1F} \tau n_F|_F(\delta) = [n^T_{2F} \tau n_F|_F(\delta) = 0,
\]

where \([\cdot]\) is the jump across \(F\). Hence this suggests the tensor value at vertex as the degree of freedom when defining the finite element.

Continuity of \((\mathbf{n}^T \tau n_j)|_e\) is a sufficient but not necessary condition for functions in \(H(\text{div div}, \Omega; \mathbb{S})\). Sufficient and necessary conditions are presented in [7, Proposition 3.6].

**4.4. Finite element spaces for symmetric tensors.** Let \(K\) be a tetrahedron. Take the space of shape functions

\[
\Sigma_{\ell, k}(K) := \mathcal{C}_\ell(K; \mathbb{S}) \oplus \mathcal{C}^\beta_k(K; \mathbb{S})
\]

with \(k \geq 3\) and \(\ell \geq \max\{k - 1, 3\}\). By Lemma 3.5, we have

\[
\mathbb{P}_{\min(\ell, k)}(K; \mathbb{S}) \subseteq \Sigma_{\ell, k}(K) \subseteq \mathbb{P}_{\max(\ell, k)}(K; \mathbb{S}) \quad \text{and} \quad \Sigma_{k, k}(K) = \mathbb{P}_k(K; \mathbb{S}).
\]
The most interesting cases are \( \ell = k - 1 \) and \( \ell = k \) which correspond to RT and BDM \( H(\text{div}) \)-conforming elements for the vector functions, respectively.

For each edge, we chose two normal vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \). The degrees of freedom are given by

\[
\begin{align*}
(27) & \quad \mathbf{\tau}(\delta) \quad \forall \delta \in \mathcal{V}(K), \\
(28) & \quad (\mathbf{n}_1^T \mathbf{\tau} \mathbf{n}_{j_e}, q)_{e} \quad \forall q \in \mathbb{P}_{\ell-2}(e), e \in \mathcal{E}(K), \quad i, j = 1, 2, \\
(29) & \quad (\mathbf{n}_1^T \mathbf{\tau} \mathbf{n}, q)_{F} \quad \forall q \in \mathbb{P}_{\ell-3}(F), F \in \mathcal{F}(K), \\
(30) & \quad (2 \text{div} \mathcal{F}(\mathbf{\tau} \mathbf{n}) + \partial_n(\mathbf{n}_1^T \mathbf{\tau} \mathbf{n}), q)_{F} \quad \forall q \in \mathbb{P}_{\ell-1}(F), F \in \mathcal{F}(K), \\
(31) & \quad (\mathbf{\tau}, \varsigma)_K \quad \forall \varsigma \in \nabla^2 \mathbb{P}_{k-2}(K) \ominus \text{sym}(\mathbf{x} \times \mathbb{P}_{\ell-2}(K; \mathbb{T})), \\
(32) & \quad (\mathbf{\tau} \mathbf{n}, \mathbf{n} \times \mathbf{x} q)_{F_1} \quad \forall q \in \mathbb{P}_{\ell-2}(F_1),
\end{align*}
\]

where \( F_1 \in \mathcal{F}(K) \) is an arbitrarily but fixed face. The degrees of freedom (32) will be regarded as interior degrees of freedom to the tetrahedron \( K \), that is the degrees of freedom (32) are double-valued on each face \( F \in \mathcal{F}_h(K) \) when defining the global finite element space.

Before we prove the unisolvence, we give characterization of the space of shape functions restricted to edges and faces, and derive some consequence of vanishing degree of freedoms.

**Lemma 4.4.** For any \( \mathbf{\tau} \in \Sigma_{\ell,k}(K) \), we have

\[
\begin{align*}
n_1^T \mathbf{\tau} n_{j_e}|_e & \in \mathbb{P}_{\ell}(e), \quad n_1^T \mathbf{\tau} n|_F \in \mathbb{P}_{\ell}(F), \\
2 \text{div} \mathcal{F}(\mathbf{\tau} \mathbf{n}) + \partial_n(\mathbf{n}_1^T \mathbf{\tau} \mathbf{n})|_F & \in \mathbb{P}_{\ell-1}(F)
\end{align*}
\]

for each edge \( e \in \mathcal{E}(K) \), each face \( F \in \mathcal{F}(K) \) and \( i, j = 1, 2 \).

**Proof.** Take any \( \mathbf{\tau} = \mathbf{x} \mathbf{x}^T q \in \mathbb{C}^\oplus_{\ell}(K; \mathbb{S}) \) with \( q \in \mathbb{P}_{k-2}(K) \). Since \( n_1^T x \) is constant on each edge of \( K \) and \( n_1^T x \) is constant on each face of \( K \),

\[
n_1^T \mathbf{\tau} n_{j_e}|_e = (n_1^T x)(n_1^T x) q \in \mathbb{P}_{k-2}(e), \quad n_1^T \mathbf{\tau} n|_F = (n_1^T x)^2 q \in \mathbb{P}_{k-2}(F),
\]

And

\[
2 \text{div} \mathcal{F}(\mathbf{\tau} \mathbf{n}) + \partial_n(\mathbf{n}_1^T \mathbf{\tau} \mathbf{n}) = (\text{div} \mathcal{F}(\mathbf{\tau} \mathbf{n}) + n_1^T \text{div} \mathbf{\tau})|_F = n_1^T x (\text{div} \mathcal{F}(\mathbf{x} \mathbf{q}) + \text{div}(\mathbf{x} \mathbf{q}) + q) \in \mathbb{P}_{k-2}(F).
\]

Thus we conclude the results from the requirement \( \ell \geq k - 1 \). \( \square \)

**Lemma 4.5.** For any \( \mathbf{\tau} \in \Sigma_{\ell,k}(K) \) with the degrees of freedom (27)-(31) vanishing, we have

\[
\begin{align*}
(33) & \quad n_1^T \mathbf{\tau} n_{j_e}|_e = 0 \quad \forall e \in \mathcal{E}(K), \quad i, j = 1, 2, \\
(34) & \quad n_1^T \mathbf{\tau} n|_F = 0 \quad \forall F \in \mathcal{F}(K), \\
(35) & \quad (2 \text{div} \mathcal{F}(\mathbf{\tau} \mathbf{n}) + \partial_n(\mathbf{n}_1^T \mathbf{\tau} \mathbf{n})|_F = 0 \quad \forall F \in \mathcal{F}(K), \\
(36) & \quad \text{div} \text{div} \mathbf{\tau} = 0, \\
(37) & \quad (\mathbf{\tau}, \varsigma)_K = 0 \quad \forall \varsigma \in \mathbb{P}_{\ell-1}(K; \mathbb{S}).
\end{align*}
\]

**Proof.** According to Lemma 4.4, we acquire (33)-(35) from the vanishing degrees of freedom (27)-(30) directly. The scalar function \( n_1^T \mathbf{\tau} n|_F \) is the standard Lagrange element and the vanishing function value \( \mathbf{\tau}(\delta) \) at vertices are used to ensure (34).
Noting that \( \text{div} \tau \in P_{k-2}(K) \), we get from the Green’s identity (26), (33)-(35) and the vanishing degrees of freedom (31) that \( \text{div} \tau = 0 \). Applying the Green’s identity (26) and (33)-(35), it follows

\[
(\tau, \nabla^2 v)_K = 0 \quad \forall \, v \in H^2(K),
\]

which together with (31) and the decomposition (25) yields (37).

With previous preparations, we prove the unisolvence as follows. For any \( \tau \in \Sigma_{\ell,k}(K) \) satisfying \( \text{div} \tau = 0 \), we have \( \tau \in P_\ell(K) \) as no contribution from \( C^0_k(K;\mathcal{S}) \). By (37) the volume moments can only determine the polynomial of degree up to \( \ell - 1 \).

We then use the vanished trace. Similarly as the RT and BDM elements [2], the vanishing normal-normal trace (34) implies the normal-normal part of \( \tau \) is zero. To determine the normal-tangential terms, further degrees of freedoms are needed. Due to the symmetry of \( \tau \), it is sufficient to provide additional degrees of freedoms on one face, which are inspired by the RT and BDM elements in two dimensions.

Unlike the traditional approach by transforming back to the reference element, we will chose an intrinsic coordinate. For ease of presentation, denote four faces in \( \mathcal{F}(K) \) by \( F_i \), which is opposite to the \( i \)th vertex of \( K \), and by \( n_i \) the outward unit normal vector of \( F_i \) for \( i = 1, 2, 3, 4 \). Let \( t_i \) be the unit tangential vector of the edge from vertex 4 to vertex \( i \); see Fig. 1. The set of three vectors \( \{t_1, t_2, t_3\} \) forms a basis of \( \mathbb{R}^3 \) although they may not be orthogonal in general. Consequently \( \{t_i \otimes t_j\}_{i,j=1}^3 \) forms a basis of the second order tensor and \( (t_i, n_i) \neq 0 \) for \( i = 1, 2, 3 \).

![Figure 1. Local coordinate formed by three edge vectors.](image_url)

**Lemma 4.6.** The degrees of freedom (27)-(32) are unisolvent for \( \Sigma_{\ell,k}(K) \).

**Proof.** We first count the number of the degrees of freedom (27)-(32) and the dimension of the space, i.e., \( \text{dim} \Sigma_{\ell,k}(K) \). The number of the degrees of freedom (27)-(32)
constructed in the last section to solve the biharmonic equation \( \Sigma \) which is the same as \( \dim \Sigma_{\ell,k}(K) \).

Take any \( \tau \in \Sigma_{\ell,k}(K) \) and suppose all the degrees of freedom (27)-(32) vanish. We are going to prove the function \( \tau = 0 \). Using the local coordinate sketched in Fig. 1, we have

\[
\tau = \sum_{i,j=1}^{3} \tau_{ij} \mathbf{t}_i \otimes \mathbf{t}_j \quad \text{with} \quad \tau_{ij} = \frac{n_i^j \tau n_j}{(t^j_i n_i)(t^i_j n_j)}.
\]

As \( \tau \) is symmetric, \( \tau_{ij} = \tau_{ji} \). By (34), it follows

\[
\tau_{ii}|_{F_1} = n_i^j \tau n_j|_{F_1} = 0, \quad i = 1, 2, 3.
\]

Thus there exists \( q_{\ell-1} \in \mathbb{P}_{\ell-1}(K) \) satisfying \( \tau_{ii}|_{F_1} = \lambda_i q_i \), where \( \lambda_i(x) \) is the \( \ell \)

barycentric coordinate with respect to the tetrahedron \( K \). Taking \( \varsigma = q_{\ell-1} n_i \otimes n_i \) in (37) produces \( \tau_{ii} = 0 \).

On the other hand, from (33) and (35) we have \( \tau n_1 = \Pi_{F_1}(\tau n_1) \in H_0(\text{div}_{F_1}, F_1) \) and

\[
2 \text{div}_{F_1}(\tau n_1)|_{F_1} = 0.
\]

Hence there exists \( q_{\ell-2} \in \mathbb{P}_{\ell-2}(F_1) \) such that \( n_1 \times (\tau n_1) = \nabla_{F_1}(b_{F_1} q_{\ell-2}) \), where \( b_{F_1} \) is the cubic bubble function on face \( F_1 \). Together with (32) and the fact \( \text{div}_{F_1}(\varsigma \mathbb{P}_{\ell-2}(F_1)) = \mathbb{P}_{\ell-2}(F_1) \), we get \( (\tau n_1)|_{F_1} = 0 \). Then there exists \( q_{\ell-1} \in \mathbb{P}_{\ell-1}(K; \mathbb{R}^3) \) such that \( \tau n_1 = \lambda_1 q_{\ell-1} \), combined with (37) yields \( \tau n_1 = 0 \). That is \( \tau_{11} = \tau_{12} = \tau_{13} = 0 \).

Now \( \tau = 2\tau_{23} \text{sym}(\mathbf{t}_2 \otimes \mathbf{t}_3) \). Multiplying \( \tau \) by \( n_4 \) from both sides and restricting to \( F_4 \), we have

\[
\tau_{23}|_{F_4} = \frac{1}{2} \frac{n_i^j \tau n_j}{(t^j_i n_i)(t^i_j n_j)}|_{F_4} = 0.
\]

The denominator is non-zero as \( \mathbf{t}_2, \mathbf{t}_3 \) are non-tangential vectors of face \( F_4 \). Again there exists \( q_{\ell-1} \in \mathbb{P}_{\ell-1}(K) \) satisfying \( \tau_{23}|_{F_4} = \lambda_4 q_{\ell-1} \). Taking \( \varsigma = \text{sym}(\mathbf{t}_2 \otimes \mathbf{t}_3) q_{\ell-1} \) in (37) gives \( \tau_{23} = 0 \). \( \square \)

Due to (30), it is arduous to figure out the explicit basis functions of \( \Sigma_{\ell,k}(K) \), which are dual to the degrees of freedom (27)-(32). Alternatively we can hybridize the degrees of freedom (30), and use the basis functions of the standard Lagrange element. We will discuss the hybridization in the next section.

5. Mixed finite element methods for biharmonic equation

In this section we will apply the \( H(\text{div \ div}) \)-conforming finite element constructed in the last section to solve the biharmonic equation

\[
\begin{cases}
\Delta^2 u = -f & \text{in } \Omega, \\
u = \partial_n u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

(38)
where \( f \in L^2(\Omega) \). The biharmonic equation in three dimensions arises in elasticity [3, Section 4.2.1], Stokes flow [16, Section 8.12.6] and phase-field models [12]. A mixed formulation of the biharmonic equation (38) is to find \( \sigma \in H(\text{div}, \Omega; \mathbb{S}) \) and \( u \in L^2(\Omega) \) such that

\[
\begin{align*}
(q, \tau) + (\text{div} \tau, u) &= 0 \quad \forall \tau \in H(\text{div}, \Omega; \mathbb{S}), \\
(\text{div} \sigma, v) &= (f, v) \quad \forall v \in L^2(\Omega).
\end{align*}
\]

Notice that the homogenous Dirichlet boundary condition becomes natural boundary condition in the mixed formulation.

5.1. **Mixed finite element methods.** From now on we assume each element in \( T_h \) is a tetrahedron. Define the global finite element spaces

\[
\Sigma_h := \{ \tau_h \in H(\text{div}, \Omega; \mathbb{S}) : \tau_h|_K \in \Sigma_{\ell,k}(K) \text{ for each } K \in T_h, \text{ all the degrees of freedom (27)-(30) are single-valued} \},
\]

\[
Q_h := \mathbb{P}_{k-2}(T_h) = \{ q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_{k-2}(K) \text{ for each } K \in T_h \}.
\]

Employing the finite element spaces \( \Sigma_h \times Q_h \) to discretize \( H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega) \), we propose the following discrete methods for the mixed formulation (39)-(40): find \( \sigma_h \in \Sigma_h \) and \( u_h \in Q_h \) such that

\[
\begin{align*}
(q, \tau_h) + (\text{div} \tau_h, u_h) &= 0 \quad \forall \tau_h \in \Sigma_h, \\
(\text{div} \sigma_h, v_h) &= (f, v_h) \quad \forall v_h \in Q_h.
\end{align*}
\]

Since the degrees of freedom (27) are not well-defined for tensors in \( H^2(\Omega; \mathbb{S}) \), we will define a quasi-interpolation operator for \( \Sigma_h \). To this end, let

\[
\bar{\Sigma}_{\ell,k}(K) := \{ \tau \in \Sigma_{\ell,k}(K) : \text{all degrees of freedom (27)-(30) vanish} \},
\]

\[
\bar{\mathbb{P}}_{k-2}(K) := \mathbb{P}_{k-2}(K)/\mathbb{P}_1(K)
\]

for each \( K \in T_h \).

**Lemma 5.1.** For each \( K \in T_h \), it holds

\[
\text{div} \text{div} \bar{\Sigma}_{\ell,k}(K) = \bar{\mathbb{P}}_{k-2}(K).
\]

**Proof.** It is obviously true that \( \text{div} \text{div} \Sigma_{\ell,k}(K) \subseteq \bar{\mathbb{P}}_{k-2}(K) \). On the other side, for any \( q \in \bar{\mathbb{P}}_{k-2}(K) \), due to the fact that \( \text{div} \text{div} H^2_0(K; \mathbb{S}) = L^2(K)/\mathbb{P}_1(K) \) [6], there exists \( \bar{\tau} \in H^2_0(K; \mathbb{S}) \) such that

\[
\text{div} \text{div} \bar{\tau} = q.
\]

Then take \( \tau \in \bar{\Sigma}_{\ell,k}(K) \) determined by

\[
\begin{align*}
(q - \bar{\tau}, \varsigma)_K &= 0 \quad \forall \varsigma \in \nabla^2 \mathbb{P}_{k-2}(K) \oplus \text{sym}(\mathbf{x} \times \mathbb{P}_{k-2}(K; \mathbb{T})), \\
((\tau - \bar{\tau}) \mathbf{n}, \mathbf{n} \times \mathbf{x} q)_{F_1} &= 0 \quad \forall q \in \mathbb{P}_{k-2}(F_1).
\end{align*}
\]

Applying the Green’s identity (26), we get

\[
(\text{div} \text{div}(\tau - \bar{\tau}), q)_K = 0 \quad \forall q \in \mathbb{P}_{k-2}(K).
\]

This implies \( \text{div} \text{div} \tau = \text{div} \text{div} \bar{\tau} = q \). \qed
Let \( Q^K : L^2(K) \rightarrow \mathbb{P}_k(K) \) be the \( L^2 \)-orthogonal projector, and its vector or tensor version is denoted by \( Q^K \). Define \( \tilde{\Pi}_h : H^2(\Omega; S) \rightarrow \Sigma_h \) as follows: for any \( \tau \in H^2(\Omega; S) \), let \( \tilde{\Pi}_h \tau \in \Sigma_h \) be determined by

\[
(\tilde{\Pi}_h \tau)(\delta) = \frac{1}{\#\partial^{-3}\delta} \sum_{K \in \partial^{-3}\delta} (Q^K \tau)(\delta),
\]

\[
(n^T(\tilde{\Pi}_h \tau)n_j, q)_e = (n^T \tau n_j, q)_e \quad \forall\ q \in \mathbb{P}_{\ell-2}(e), i, j = 1, 2,
\]

\[
(2 \text{div}_F((\tilde{\Pi}_h \tau - \tau)n) + \partial_h (n^T(\tilde{\Pi}_h \tau - \tau)n), q)_F = 0 \quad \forall q \in \mathbb{P}_{\ell-1}(F),
\]

\[
(\tilde{\Pi}_h \tau, \varsigma)_K = (\tau, \varsigma)_K \quad \forall \varsigma \in \nabla^2 \mathbb{P}_{k-2}(K) \oplus \text{sym}(x \times \mathbb{P}_{\ell-2}(K; T)),
\]

\[
((\tilde{\Pi}_h \tau)n, n \times xq)_F = (\tau n, n \times xq)_F \quad \forall q \in \mathbb{P}_{\ell-2}(F_1)
\]

for each \( \delta \in \mathcal{V}_h, e \in \mathcal{E}_h, F \in \mathcal{F}_h, K \in T_h \) and \( F_1 \in \mathcal{F}(K) \), where \( \#\partial^{-3}\delta \) is the number of elements in \( \partial^{-3}\delta \) and \( \#\partial^{-3}\delta \) is the number of elements in \( \partial^{-3}\delta \).

Since \( \ell \geq 3 \), it follows from the Green’s identity (26) that

\[
Q^K \left( Q_h \text{div} \text{div} \tau - \text{div} \text{div}(\tilde{\Pi}_h \tau) \right) = 0 \quad \forall \tau \in H^2(\Omega; S), \quad K \in T_h,
\]

where \( Q_h : L^2(\Omega) \rightarrow \mathbb{P}_{k-2}(T_h) \) is defined by \( (Q_h q)_K := Q^K_{k-2}(q|_K) \) for each \( K \in T_h \). Applying (43), there exists \( \tilde{\tau} \in \Sigma_h \) such that \( \tilde{\tau}|_K \in \tilde{\Sigma}_{\ell,k}(K) \) for each \( K \in T_h \), and

\[
Q_h \text{div} \text{div} \tau - \text{div} \text{div}(\tilde{\Pi}_h \tau) = \text{div} \text{div} \tilde{\tau}.
\]

Now define \( \Pi_h : H^2(\Omega; S) \rightarrow \Sigma_h \) as \( \Pi_h \tau := \tilde{\Pi}_h \tau + \tilde{\tau} \). Hence we have

\[
\text{div} \text{div}(\Pi_h \tau) = Q_h \text{div} \text{div} \tau \quad \forall \tau \in H^2(\Omega; S).
\]

Employing the standard techniques in analyzing the error estimate of the quasi-interpolation operator, we have

\[
h^i|\tau - \Pi_h \tau|_i \lesssim h^s|\tau|_s \quad \forall \tau \in H^s(\Omega; S), \quad i = 0, 1, 2
\]

with \( 2 \leq s \leq \min\{\ell, k\} + 1 \). According to (44) and (45), it holds the inf-sup condition

\[
\|v_h\|_0 \lesssim \sup_{\tau_h \in \Sigma_h} \frac{\text{div} \text{div} \tau_h, v_h}{\|\tau_h\|_{H(\text{div} \text{div})}} \quad \forall\ v_h \in Q_h.
\]

Hence the mixed finite element method (41)-(42) is well-posed.

Since the techniques in the error analysis for the mixed finite element method (41)-(42) are similar as [5], we will omit the proof of the error estimates. By adopting the same arguments in [5], we have the inf-sup condition for \( \ell \geq k \)

\[
|v_h|_{2,h} \lesssim \sup_{\tau_h \in \Sigma_h} \frac{\text{div} \text{div} \tau_h, v_h}{\|\tau_h\|_0} \quad \forall\ v_h \in Q_h
\]

based on the squared mesh-dependent norm

\[
|v_h|_{2,h}^2 := \sum_{K \in T_h} |v_h|_{2,K}^2 + \sum_{F \in \mathcal{F}_h} (h_F^{-3}\|v_h\|_{0,F}^2 + h_F^{-1}\|\partial_{n_F} v_h\|_{0,F}^2),
\]

where \( \|v_h\| \) and \( \|\partial_{n_F} v_h\| \) are jumps of \( v_h \) and \( \partial_{n_F} v_h \) across \( F \) for \( F \in \mathcal{F}_h \), and \( \|v_h\| = v_h \) and \( \|\partial_{n_F} v_h\| = \partial_{n_F} v_h \) for \( F \in \mathcal{F}_h \backslash \mathcal{F}_i \).
Theorem 5.2. Let $\sigma_h \in \Sigma_h$ and $u_h \in Q_h$ be the solution of the mixed finite element methods (41)-(42). Assume $\sigma \in H^{\min(\ell,k)+1}(\Omega; \mathbb{S})$, $u \in H^{k-1}(\Omega)$ and $f \in H^{k-1}(\Omega)$. Then

\begin{align}
\|\sigma - \sigma_h\|_0 + \|Q_h u - u_h\|_0 & \lesssim h^{\min(\ell,k)+1}\|\sigma\|_{\min(\ell,k)+1}, \\
\|u - u_h\|_0 & \lesssim h^{\min(\ell,k)+1}\|\sigma\|_{\min(\ell,k)+1} + h^{k-1}\|u\|_{k-1}, \\
\|\sigma - \sigma_h\|_{H(\text{div div})} & \lesssim h^{\min(\ell,k)+1}\|\sigma\|_{\min(\ell,k)+1} + h^{k-1}\|f\|_{k-1}.
\end{align}

Furthermore, when $\ell \geq k$, we have the superconvergence

\begin{align}
\|Q_h u - u_h\|_{2,h} & \lesssim h^{k+1}\|\sigma\|_{k+1}.
\end{align}

Both the estimates of $\|Q_h u - u_h\|_0$ in (46) and $\|Q_h u - u_h\|_{2,h}$ in (49) are superconvergent, where the estimate of $\|Q_h u - u_h\|_{2,h}$ is fourth order higher than the optimal one and can be used to get a high order approximation of displacement by postprocessing as follows.

When $\ell \geq k$, define $u^*_h \in P_{k+2}(T_h)$ as follows: for each $K \in T_h$,

\begin{align*}
(\nabla^2 u^*_h, \nabla^2 q)_K = -(\sigma_h, \nabla^2 q)_K & \quad \forall q \in P_{k+2}(T_h), \\
(u^*_h, q)_K = (u_h, q)_K & \quad \forall q \in P_1(T_h).
\end{align*}

Then

\begin{align}
\|u - u^*_h\|_{2,h} & \lesssim h^{k+3}\|u\|_{k+3}.
\end{align}

5.2. Hybridization. In this subsection we consider a partial hybridization of the mixed finite element methods (41)-(42) by relaxing the continuity of $(2 \text{div}_F(\tau n_F) + \partial_n (n^T \tau n))|_F$. To this end, let

\begin{align*}
\tilde{\Sigma}_h := \{\tau_h \in L^2(\Omega; \mathbb{S}) : \tau_h|_K \in \Sigma_{\ell,k}(K) \text{ for each } K \in T_h, \\
& \text{ all the degrees of freedom (27)-(29) are single-valued}\}, \\
\Lambda_h := \{\mu_h \in L^2(F_h) : \mu_h|_F \in P_{\ell-1}(F) \text{ for each } F \in F_h^i, \\
& \mu_h|_F = 0 \text{ for each } F \in F_h \setminus F_h^i\}.
\end{align*}

Lemma 5.3. Let $\sigma_h \in \Sigma_h$ and $u_h \in Q_h$ be the solution of the mixed finite element methods (41)-(42). Let $(\tilde{\sigma}_h, \tilde{u}_h, \lambda_h) \in \tilde{\Sigma}_h \times Q_h \times \Lambda_h$ satisfy the partially hybridized mixed finite methods

\begin{align}
(\tilde{\sigma}_h, \tau_h) + b_h(\tau_h, \tilde{u}_h, \lambda_h) = 0 & \quad \forall \tau_h \in \tilde{\Sigma}_h, \\
b_h(\tilde{\sigma}_h, v_h, \mu_h) = (f, v_h) & \quad \forall v_h \in Q_h, \mu_h \in \Lambda_h,
\end{align}

where

\begin{align*}
b_h(\tau_h, v_h, \mu_h) := \sum_{K \in T_h} (\text{div} \tau_h, v_h)_K - \sum_{K \in T_h} (2 \text{div}_F(\tau_h n) + \partial_n (n^T \tau_h n), \mu_h)_{\partial K}.
\end{align*}

Then $\tilde{\sigma}_h = \sigma_h$ and $\tilde{u}_h = u_h$. 
One benefit of the hybridized formulation (51)-(52) is the simplification of implementation. When \( \ell = k \), we can take the following degrees of freedom for \( \mathbf{\tau} \)

\[
\mathbf{\tau}(\delta) \quad \forall \delta \in \mathcal{V}(K),
\]

\[
(n^T_i \mathbf{\tau} n_j, q)_e \quad \forall q \in \mathbb{P}_{k-2}(e), e \in \mathcal{E}(K), \quad i, j = 1, 2,
\]

\[
(t^T \mathbf{\tau} n_j, q)_e, (t^T \mathbf{\tau} t, q)_e \quad \forall q \in \mathbb{P}_{k-2}(e), e \in \mathcal{E}(K), \quad j = 1, 2,
\]

\[
(n^T \mathbf{\tau} n, q)_F \quad \forall q \in \mathbb{P}_{k-3}(F), F \in \mathcal{F}(K),
\]

\[
(t^T_i \mathbf{\tau} n, q)_F, (t^T_i \mathbf{\tau} t_j, q)_F \quad \forall q \in \mathbb{P}_{k-3}(F), F \in \mathcal{F}(K), \quad j = 1, 2,
\]

\[
(\mathbf{\tau}, \varsigma)_K \quad \forall \varsigma \in \mathbb{P}_{k-4}(K; \mathbb{S}).
\]

Here notations \( e \in \mathcal{E}(K) \) and \( F \in \mathcal{F}(K) \) mean the corresponding degrees of freedom are the interior to the tetrahedron \( K \), i.e., \( (t^T_i \mathbf{\tau} n_j, q)_e, (t^T_i \mathbf{\tau} t, q)_e, \) and \( (t^T_i \mathbf{\tau} \xi, q)_F, \xi = t_j \) or \( n \), are not single-valued on each edge \( e \in \mathcal{E}_h \) and face \( F \in \mathcal{F}_h \), respectively.

These are exactly the tensor version of the local degrees of freedom for the Lagrange element. Therefore we can adopt the standard Lagrange element basis to implement the hybridized mixed finite element methods (51)-(52) and manage the mapping of local degree of freedoms to global one.

The hybridized mixed finite element methods (51)-(52) is a saddle point problem. Following the argument in [10], we have the inf-sup condition for \( \ell \geq k \)

\[
|(v_h, \mu_h)|_{2,h} \lesssim \sup_{\mathbf{\tau}_h \in \mathbb{S}_h} \frac{b_h(\mathbf{\tau}_h, v_h, \mu_h)}{\| \mathbf{\tau}_h \|_0} \quad \forall (v_h, \mu_h) \in \mathbb{Q}_h \times \mathbb{L}_h,
\]

where the squared mesh-dependent norm

\[
|(v_h, \mu_h)|_{2,h}^2 := \sum_{K \in \mathcal{T}_h} |v_h|_{2,K}^2 + \sum_{F \in \mathcal{F}_h} \left( h_F^{-3} \| v_h - \mu_h \|_{0,F}^2 + h_F^{-1} \| \partial_n v_h \|_{0,F}^2 \right).
\]

Based on the discrete inf-sup condition (53), we can adapt the approach in [4] to construct the approximate block-factorization preconditioner, where the Schur complement corresponds to a discontinuous Galerkin method for the biharmonic equation: find \((v_h, \mu_h) \in \mathbb{Q}_h \times \mathbb{L}_h\) such that

\[
(\nabla_h^2 v_h, \nabla_h^2 v_h) + \sum_{F \in \mathcal{F}_h} h_F^{-1} (\| \partial_n v_h \|, [\partial_n v_h])_F + \sum_{F \in \mathcal{F}_h} h_F^{-3} (u_h - \lambda_h, v_h - \mu_h)_F = g(v_h, \mu_h)
\]

for some right hand side \( g(v_h, \mu_h) \). Numerical examples and fast solvers will be investigated in our future work.

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