Heroes in oriented complete multipartite graphs

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Abstract
The dichromatic number of a digraph is the minimum size of a partition of its vertices into acyclic induced subgraphs. Given a class of digraphs $\mathcal{C}$, a digraph $H$ is a hero in $\mathcal{C}$ if $H$-free digraphs of $\mathcal{C}$ have bounded dichromatic number. In a seminal paper, Berger et al. give a simple characterisation of all heroes in tournaments. In this paper, we give a simple proof that heroes in quasitransitive oriented graphs (that are digraphs with no induced directed path on three vertices) are the same as heroes in tournaments. We also prove that it is not the case in the class of oriented multipartite graphs, disproving a conjecture of Aboulker, Charbit and Naserasr, and give a characterisation of heroes in oriented complete multipartite graphs up to the status of a single tournament on six vertices.

KEYWORDS
dichromatic number, heroes, multipartite graphs

1 | INTRODUCTION

1.1 | Definitions and notations

In this paper, we only consider directed graphs (digraphs in short) with no digons (a cycle on two vertices), loops nor multiarcs. Let $G$ be a digraph. We denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For a vertex $x$ of $G$, we denote by $x^+$ (resp., $x^-$) the set of its out-neighbours (resp., in-neighbours) and by $x^o$ the set of its nonneighbours with the convention that $x \not\in x^o$. For a given set of vertices $X \subseteq V$, we denote by $G[X]$ the subgraph of $G$ induced by $X$. 
Given two disjoint sets of vertices $X, Y$ of a digraph $D$, we write $X \Rightarrow Y$ to say that for every $x \in X$ and for every $y \in Y, xy \in A(G)$, and we write $X \rightarrow Y$ to say that every arc with one end in $X$ and the other one in $Y$ is oriented from $X$ to $Y$ (but some vertices of $X$ might be nonadjacent to some vertices of $Y$). When $X = \{x\}$ we write $x \Rightarrow Y$ and $x \rightarrow Y$.

We also use the symbol $\Rightarrow$ to denote a composition operation on digraphs: for two digraphs $D_1$ and $D_2$, $D_1 \Rightarrow D_2$ is the digraph obtained from the disjoint union of $D_1$ and $D_2$ by adding all arcs from $V(D_1)$ to $V(D_2)$.

A tournament is an orientation of a complete graph. A transitive tournament is an acyclic tournament and we denote by $TT_n$ the unique acyclic tournament on $n$ vertices. Given two tournaments $H_1$ and $H_2$, we denote by $\Delta(1, H_1, H_2)$ the tournament obtained from pairwise disjoint copies of $H_1$ and $H_2$ plus a vertex $x$, and all arcs from $x$ to the copy of $H_1$, all arcs from the copy of $H_1$ to the copy of $H_2$, and all arcs from the copy of $H_2$ to $x$. When $\ell$ and $k$ are integers, we write $\Delta(1, k, H)$ for $\Delta(1, TT_k, H)$ and $\Delta(1, \ell, k)$ for $\Delta(1, TT_\ell, TT_k)$. The tournament $\Delta(1, 1, 1)$ is also denoted by $C_3$ and called a directed triangle.

A k-dicolouring of $G$ is a partition of $V(G)$ into $k$ sets $V_1, \ldots, V_k$ such that $G[V_i]$ is acyclic for $i = 1, \ldots, k$. The dichromatic number of $G$, denoted by $\chi^*(G)$ and introduced by Neuman-Lara [13] is the minimum integer $k$ such that $G$ admits a $k$-dicolouring. We will sometimes extend $\chi^*$ to subsets of vertices, using $\chi^*(X)$ to mean $\chi^*(G[X])$ where $X \subseteq V$.

Given a set of digraphs $\mathcal{H}$, we say that a digraph $G$ is $\mathcal{H}$-free if it contains no member of $\mathcal{H}$ as an induced subgraph. We denote by $Forb_{\text{ind}}(\mathcal{H})$ the class of $\mathcal{H}$-free digraphs. We write $Forb_{\text{ind}}(F_1, \ldots, F_k)$ instead of $Forb_{\text{ind}}(\{F_1, \ldots, F_k\})$ for simplicity. Given a class of digraphs $\mathcal{C}$, a digraph $H$ is a hero in $\mathcal{C}$ if every $H$-free digraph in $\mathcal{C}$ has a bounded dichromatic number.

We denote by $P_3$ the directed path on three vertices. An oriented complete multipartite graph is an orientation of a complete multipartite graph. Given two digraphs $G_1$ and $G_2$, $G_1 + G_2$ is the disjoint union of $G_1$ and $G_2$. We denote by $K_1$ the unique digraph on one vertex. Observe that oriented complete multipartite graphs are precisely the digraphs in $Forb_{\text{ind}}(K_1 + TT_2)$.

The main goal of this paper is to identify heroes in oriented complete multipartite graphs.

1.2 Context and results

In a seminal paper, Berger et al. [7] characterised heroes in tournaments:

**Theorem 1.1** (Berger et al. [7]). A digraph $H$ is a hero in tournaments if and only if:

- $H = K_1$, or
- $H = H_1 \Rightarrow H_2$, where $H_1$ and $H_2$ are heroes in tournaments, or
- $H = \Delta(1, k, H_i)$ or $H = \Delta(1, H_i, k)$, where $k \geq 1$ and $H_i$ is a hero in tournaments.

Observe that if a class of digraphs $\mathcal{C}$ contains all tournaments, then a hero in $\mathcal{C}$ must be a hero in tournaments. In [3], it is conjectured that heroes in oriented complete multipartite graphs are the same as heroes in tournaments (actually a wider conjecture is proposed, see Section 5). We disprove this conjecture by showing the following:

**Theorem 1.2.** The digraphs $\Delta(1, 2, C_3), \Delta(1, C_3, 2), \Delta(1, 2, 3)$ and $\Delta(1, 3, 2)$ are not heroes in oriented complete multipartite graphs.
On the positive side, we prove that:

**Theorem 1.3.** A digraph $H$ is a hero in oriented complete multipartite graphs if:

- $H = K_1$,
- $H = H_1 \Rightarrow H_2$, where $H_1$ and $H_2$ are heroes in oriented complete multipartite graphs, or
- $H = \Delta(1, 1, H_1)$ where $H_1$ is a hero in oriented complete multipartite graphs.

Observe that the second bullet of the theorem above implies that a digraph is a hero in oriented complete multipartite graphs if and only if each of its strong connected components is. Indeed, the only if part of the assertion holds because an induced subgraph of a hero in any class is a hero in this class.

Since a hero in oriented complete multipartite graphs must be a hero in tournaments, Theorems 1.1, 1.2 and 1.3 imply that, to get a full characterisation of heroes in oriented complete multipartite graphs, it suffices to decide whether $\Delta(1, 2, 2)$ is a hero in oriented complete multipartite graphs or not. If it is not, then heroes in oriented complete multipartite graphs are precisely the ones described in Theorem 1.3. If it is, then a digraph $H$ is a hero in oriented complete multipartite graphs if and only if:

- $H = K_1$ or $H = \Delta(1, 2, 2)$,
- $H = H_1 \Rightarrow H_2$, where $H_1$ and $H_2$ are heroes in oriented complete multipartite graphs, or
- $H = \Delta(1, 1, H_1)$ where $H_1$ is a hero in oriented complete multipartite graphs.

**Question 1.4.** Is $\Delta(1, 2, 2)$ a hero in oriented complete multipartite graphs?

**Remark 1.5.** Between the submission of this paper and its acceptation, Bartosz Walczak proved that noninterlaced ordered graphs (see Section 4 for the definition) have an unbounded chromatic number, which, together with Theorem 4.2, implies that $\Delta(1, 2, 2)$ is not a hero. According to the discussion above, this result settles the question of characterising the heroes in oriented complete multipartite graphs. We believe in the proof but since it is not yet officially reviewed and published, we preferred to not yet claim the complete theorem.

A digraph $G$ is *quasitransitive* if for every triple of vertices $x, y, z$, if $xy, yz \in A(G)$, then $xz \in A(G)$ or $zx \in A(G)$. Observe that the class of quasitransitive digraphs is precisely $\text{Forb}_{\text{ind}}(\vec{P}_3)$. Our last result is:

**Theorem 1.6.** Heroes in quasitransitive digraphs are the same as heroes in tournaments.

**Organisation of the paper:** We prove in Section 2 that $\Delta(1, 2, C_3), \Delta(1, C_3, 2), \Delta(1, 2, 3), \Delta(1, 3, 2)$ are not heroes in oriented complete multipartite graphs. We prove in Section 3.1 that if $H_1$ and $H_2$ are heroes in oriented complete multipartite graphs, then so is $H_1 \Rightarrow H_2$ and in Section 3.2 that if $H$ is a hero in oriented complete multipartite graphs, then so is $\Delta(1, 1, H)$. We give some insight about whether $\Delta(1, 2, 2)$ should be a hero or not in oriented complete multipartite graphs in Section 4 and finally, we prove Theorem 1.6, detail related results and propose some leads for further works in Section 5.
The goal of this section is to prove that $\Delta(1, 2, C_3)$, $\Delta(1, C_3, 2)$, $\Delta(1, 2, 3)$ and $\Delta(1, 3, 2)$ are not heroes in oriented complete multipartite graphs. Since reversing all arcs of a $\Delta(1, 2, C_3)$-free oriented complete multipartite graph results in a $\Delta(1, C_3, 2)$-free oriented complete multipartite graph and does not change the dichromatic number, if $\Delta(1, 2, C_3)$ is not a hero in oriented complete multipartite graphs then $\Delta(1, C_3, 2)$ is not either. Similarly, if $\Delta(1, 2, 3)$ is not a hero in oriented complete multipartite graphs then $\Delta(1, 3, 2)$ is not either. Hence, it is enough to prove that $\Delta(1, 2, C_3)$ nor $\Delta(1, 2, 3)$ are heroes in oriented complete multipartite graphs. This is implied by the existence of $[\Delta(1, 2, C_3),\Delta(1, 2, 3)]$-free oriented complete multipartite graphs with arbitrarily large dichromatic number. The rest of this section is dedicated to the description of such digraphs.

A feedback arc set of a given digraph $G$ is a set of arcs $F$ of $G$ such that their deletion from $G$ yields an acyclic digraph. The idea of the construction comes from the fact that a feedback arc set of $\Delta(1, 2, C_3)$ or of $\Delta(1, 2, 3)$ must induce a digraph with at least one vertex of in- or out-degree at least 2. We then describe an oriented complete multipartite graph with a large dichromatic number in which every subtournament has a feedback arc set inducing disjoint directed paths, implying that it does not contain $\Delta(1, 2, C_3)$ nor $\Delta(1, 2, 3)$ by the fact above.

Given an undirected graph $H$, a $k$-colouring of $H$ is a partition of $V(G)$ into $k$ independent sets. The chromatic number of $H$ is the minimum $k$ such that $H$ is $k$-colourable. Let $G$ be a digraph. We denote by $\chi(G)$ the chromatic number of the underlying graph of $G$. The (undirected) line graph of $G$ is denoted by $LG$ and defined as follows: its vertex set is $A(G)$, and two of its vertices $ab, cd \in A(G)$ are adjacent if and only if $b = c$.

Be aware that the next lemma deals with chromatic number and not dichromatic number. We think it appears for the first time in [9].

**Lemma 2.1** (Erdős and Hajnal [9]). For every digraph $G$, we have $\chi(L(G)) \geq \log(\chi(G))$.

**Proof.** Let $G$ be a digraph and assume $L(G)$ admits a $k$-colouring. Observe that a colouring of $L(G)$ is the same as a colouring of the arcs of $G$ in such a way that no $\overrightarrow{P}_3$ is monochromatic. Consider the following colouring of $G$: for each $v \in V(G)$, colour $v$ with the set of colours received to be the arcs entering in $v$. This is a $2^k$-colouring of $G$ because the colouring of $A(G)$ does not have monochromatic $\overrightarrow{P}_3$. \qed

Let $s \geq 3$ be an integer and let us describe the graph $L(L(TT_s))$. Assuming the vertices of $TT_s$ are numbered $v_1, \ldots, v_s$ in the topological ordering (i.e., for all $1 \leq i < j \leq s$, we have $v_i v_j \in A(T)$), for any $i < j < k$, $\{v_i, v_j, v_k\}$ induces a $\overrightarrow{P}_3$ in $TT_s$. This way, we get a natural name for the vertices of $L(L(TT_s))$, namely, $V(L(L(TT_s))) = \{(v_i, v_j, v_k) : \text{for every } i < j < k\}$. Moreover, edges of $L(L(TT_s))$ are of the form $(v_i, v_j, v_k)(v_j, v_k, v_\ell)$ for every $i < j < k < \ell$. For $2 \leq j \leq s - 1$, set $V_j = \{(v_i, v_j, v_k) : i < j < k\}$. So $V_j$’s partition the vertices of $L(L(TT_s))$ into stable sets.

We now define the digraph $D_s$ from $L(L(TT_s))$ as follows. The vertices of $D_s$ are the same as the vertices of $L(L(TT_s))$ and $D_s$ is an oriented complete multipartite graph with parts ($V_2, V_3, \ldots, V_{s-1}$) and we orient the arcs as follows: given $j < k$, the edges of $L(L(TT_s))$ are oriented from $V_j$ to $V_k$ and all the other arcs are oriented from $V_k$ to $V_j$. This completes the description of $D_s$. 
The arcs \( v_i v_j \) such that \( i < j \) are called the forward arcs of \( D_s \), and the other arcs are the backward arcs of \( D_s \). Observe that the underlying graph induced by the forward arcs of \( D_s \) is \( L(L(TT_3)) \).

The following remark is the crucial feature of \( D_s \).

**Remark 2.2.** Given a vertex \((v_i, v_j, v_k)\) of \( D_s \), the out-neighbours of \((v_i, v_j, v_k)\) are all in \( V_k \) and the in-neighbours of \((v_i, v_j, v_k)\) are all in \( V_i \).

Observe that a digraph that does not contain \( P_3 \) as a subgraph is bipartite: all its vertices have in-degree 0 or out-degree 0, and the set of vertices with in-degree 0 (resp., with out-degree 0) form a stable set.

**Lemma 2.3.** For every integer \( s \geq 1 \), \( D_s \) has a feedback arc set formed by a disjoint union of directed paths.

**Proof.** Let \( V_2, \ldots, V_{s-1} \) be the partition of \( D_s \) as in the definition. Recall that \( V(D_s) = \{(v_i, v_j, v_k) : 1 \leq i < j < k \leq s\} \). Denote by \( F_s \) the digraph induced by the forward arcs of \( D_s \). So the underlying graph of \( F_s \) is \( L(L(TT_3)) \) and by Lemma 2.1, \( \chi(F_s) \geq \log \log(s) \).

Let \( R \) be an acyclic induced subgraph of \( D_s \). Observe that a directed path on three vertices in \( D_s \) using only arcs in \( F_s \) must be of the form \((v_{i_1}, v_{i_2}, v_{i_3}) \rightarrow (v_{i_2}, v_{i_3}, v_{i_4}) \rightarrow (v_{i_3}, v_{i_4}, v_{i_5})\), where \( 1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq s \) and is thus contained in a directed triangle of \( D_s \) (because \((v_{i_1}, v_{i_2}, v_{i_3})(v_{i_2}, v_{i_3}, v_{i_4})\) is not an edge of \( L(L(TT_3)) \), and thus is not an arc of \( F_s \), and thus \((v_{i_3}, v_{i_4}, v_{i_5})(v_{i_1}, v_{i_2}, v_{i_3})\) is an arc of \( D_s \)). Hence, the digraph with arcs \( A(R) \cap A(F_s) \) does not contain \( P_3 \) as a subgraph and is thus bipartite. Hence, a \( t \)-dicolouring of \( D_s \) implies a \( 2t \)-(undirected) colouring of \( F_s \). As we have that \( \chi(F_s) \geq \log \log(s) \), the result follows.

**Lemma 2.4.** If \( T \) is a tournament contained in \( D_s \), then \( T \) has a feedback arc set formed by a disjoint union of directed paths.

**Proof.** Let \( T \) be a subgraph of \( D_s \) inducing a tournament. Then each vertex of \( T \) belongs to a distinct \( V_i \) and thus, by Remark 2.2, the forward arcs of \( D_s \) that are in \( T \) induce a disjoint union of directed paths and clearly form a feedback arc set of \( T \).

**Lemma 2.5.** For every \( s \geq 1 \), \( D_s \) does not contain \( \Delta(1, 2, C_3) \) nor \( \Delta(1, 2, 3) \).

**Proof.** Observe that the two digraphs \( \Delta(1, 2, C_3) \) and \( \Delta(1, 2, 3) \) only differ on the orientation of one arc: reversing an arc of the copy of \( C_3 \) in \( \Delta(1, 2, C_3) \) leads to \( \Delta(1, 2, 3) \) and reversing an arc of the copy of \( TT_3 \) in \( \Delta(1, 2, 3) \) leads to \( \Delta(1, 2, C_3) \). Our argument does not make any use of the orientations between the vertices inside this oriented \( K_3 \).

Let \( H \) be one of \( \Delta(1, 2, C_3) \) or \( \Delta(1, 2, 2) \), and let \( x \) be the vertex in the copy of \( K_1 \), and \( y_1 \) and \( y_2 \) are the vertices in the copy of \( TT_2 \). See Figure 1.

Thanks to Lemma 2.4, it is enough to prove that in every feedback arc set of \( H \), there exists a vertex with in- or out-degree at least 2. Let \( F \) be a feedback arc set of \( H \) and assume for contradiction that it induces a disjoint union of directed paths. Then both \( xy_1 \) and \( xy_2 \) do not belong to \( F \). So we may assume without loss of generality that \( xy_1 \notin F \). But then \( F \) must intersect the three disjoint paths of length 2 that go from
to $x$, which necessarily implies that $F$ contains either two arcs coming out of $y_1$ or two arcs coming in $x$.

By Lemmas 2.3 and 2.5, $\Delta(1, 2, C_3)$ and $\Delta(1, 2, 3)$ are not heroes in oriented complete multipartite graphs.

3 | HEROES IN ORIENTED COMPLETE MULTIPARTITE GRAPHS

3.1 | Strong components

The goal of this subsection is to prove the following:

**Theorem 3.1.** If $H_1$ and $H_2$ are heroes in $\text{Forb}_{\text{ind}}(K_1 + TT_2)$, then so is $H_1 \Rightarrow H_2$.

We actually prove the following stronger result:

**Theorem 3.2.** Let $H_1$, $H_2$ and $F$ be digraphs such that $H_1 \Rightarrow H_2$ is a hero in $\text{Forb}_{\text{ind}}(F)$ and $H_1$ and $H_2$ are heroes in $\text{Forb}_{\text{ind}}(K_1 + F)$. Then $H_1 \Rightarrow H_2$ is a hero in $\text{Forb}_{\text{ind}}(K_1 + F)$.

To see that Theorem 3.2 implies Theorem 3.1, take $F = TT_2$ and observe that $\text{Forb}_{\text{ind}}(TT_2)$ is the class of digraphs with no arc and thus every digraph is a hero in $\text{Forb}_{\text{ind}}(TT_2)$. We explain why such a stronger version can be of interest to future works in Section 5.

Note also that by taking $F = K_1$, we have that $\text{Forb}_{\text{ind}}(F)$ is empty and that $\text{Forb}_{\text{ind}}(K_1 + F)$ is the class of tournaments, so Theorem 3.2 yields the result of [7] (see 3.1) stating that $H$ is a hero in tournaments if and only if all of its strong components are. Then, by induction, we get the same result for the class of digraphs with bounded independence numbers, reproving Theorem 1.4 of [11].

The rest of this subsection is devoted to the proof of Theorem 3.2, which is inspired but simpler (we got rid of the intricate notion of $r$-mountain) than the analogous result for tournaments in [7], even though our result is more general.

We start with a few definitions and notations. First, to simplify statements of the lemmas, we assume $H_1$, $H_2$ and $F$ are fixed all along the subsection and are as in the statement of Theorem 3.2. So there exists constants $c$ and $h$ such that:

![Diagram](image-url)
H₁ and H₂ have at most h vertices,

- digraphs in Forb₁₂₃(F, H₁ ⇒ H₂) have dichromatic number at most c,
- for i = 1, 2, digraphs in Forbᵢ(k + F, Hᵢ) have dichromatic number at most c.

If G is a digraph and u,v ∈ A(G), we set Cuvw = v⁺ ∩ u⁻, that is, the set of vertices that form a directed triangle with u and v. Finally, for t ∈ N, we say that a digraph K is a t-cluster if \( \overrightarrow{\chi}(K) \geq t \) and \( |V(K)| \leq f(t) \), where \( f(t) \) is the function defined recursively by \( f(1) = 1 \) and

\[
f(t) = 1 + f(t-1)(1 + f(t-1)).
\]

The structure of the proof is very simple, we prove that digraphs in Forb₁₂₃(K₁ + F, H₁ ⇒ H₂) that do not contain a t-cluster have dichromatic numbers bounded by a function of t (Lemma 3.3), and then that the ones that contain a t-cluster also have dichromatic number bounded by a function of t if t is large enough (Lemma 3.4).

**Lemma 3.3.** There exists a function \( \phi \) such that if \( t \) is an integer and \( G \) is a digraph in Forb₁₂₃(K₁ + F, H₁ ⇒ H₂) which contains no t-cluster as a subgraph, then \( \overrightarrow{\chi}(G) \leq \phi(c, h, t) \).

**Proof.** We prove this by induction on t. For \( t = 1 \) the result is trivial as a 1-cluster is simply a vertex. Assume the existence of \( \phi(c, h, t-1) \), and assume \( G \) is a digraph in Forb₁₂₃(K₁ + F, H₁ ⇒ H₂) which contains no t-cluster. Say an arc uv is heavy if \( C_{uv} \) contains a \((t-1)\)-cluster, and light otherwise. For a vertex \( u \) we define \( g(u) = \{ v ∈ V(G) | uv or vu is a heavy arc \} \).

**Claim 3.3.1.** For any vertex \( u \), \( g(u) \) contains no \((t-1)\)-cluster.

**Proof.** Assume by contradiction that \( K \) is a \((t-1)\)-cluster in \( g(u) \). By definition of \( g(u) \), for every \( v ∈ V(K) \), there exists a \((t-1)\)-cluster \( K_v \) in \( C_{uv} \) or \( C_{vu} \) (depending on which of \( uv \) or \( vu \) is an arc). Let \( K' = \{ u \} \cup V(K) \cup (\cup v ∈ K V(K_v)) \). We claim that \( K' \) is a \( t \)-cluster. First note that the number of vertices of \( K' \) is at most

\[
1 + f(t-1) + f(t-1) \cdot f(t-1) = f(t).
\]

We need to prove that \( K' \) is not \((t-1)\)-colourable, so let us consider for contradiction a \((t-1)\)-colouring of its vertices, and without loss of generality assume \( u \) gets colour 1. Because \( K \) is a \((t-1)\)-cluster, some vertex \( v \) in \( K_v \) must also receive colour 1, and since \( K_v \) is also a \((t-1)\)-cluster, some vertex \( w \) in \( K_v \) must also receive colour 1, which produces a monochromatic directed triangle. So \( K' \) is indeed a \( t \)-cluster, a contradiction. □

**Claim 3.3.2.** For any vertex \( u \), \( \min(\overrightarrow{\chi}(u⁻), \overrightarrow{\chi}(u⁺)) \leq (h+1) \cdot \phi(c, h, t-1) + c \).

**Proof.** Let \( u ∈ V(G) \). By the previous claim and the induction hypothesis, \( g(u) \) induces a digraph of dichromatic number at most \( \phi(c, h, t-1) \), so it is enough to prove that one of the sets \( u⁻ = (u⁻ \setminus g(u)) \) or \( u⁺ = (u⁺ \setminus g(u)) \) induces a digraph with dichromatic number at most \( h \cdot \phi(c, h, t-1) + c \cdot (h+1) \).

If \( u⁺ \) induces an \( H₂ \)-free digraph, then it has a dichromatic number at most \( c < h \cdot \phi(c, h, t-1) + c \cdot (h+1) \), so we can assume that there exists \( V₂ ⊆ u⁻ \) such that \( G[V₂] = H₂ \). We now cover \( u⁻ \) with three sets A, B, C, each of which will have a bounded dichromatic number.
Let \( A = u^- \cap (\bigcup_{v \in V} v^+) = u^- \cap (\bigcup_{v \in V} C_{iv}) \). For every \( v \in V_2, uv \in A(G) \) is light (because \( V_2 \subseteq u^- ) \), so \( G[C_{iv} \cap A] \) does not contain a \((t - 1)\)-cluster and is thus \( \phi(c, h, t - 1)\)-colourable by induction. Now, since \( H_2 \) contains at most \( h \) vertices, we get \( \chi'(A) \leq h \cdot \phi(c, h, t - 1) \).

Let \( B = u^- \cap (\bigcup_{v \in V} v^0) \). Since \( G \) is \((K_1 + F, H_1 \Rightarrow H_2)\)-free, for every \( v \in V_2, v^0 \) is \((F, H_1 \Rightarrow H_2)\)-free and thus \( \chi'(G[v^0]) \leq c \). Hence, \( \chi'(B) \leq c \cdot h \).

Finally, consider \( C = u^- \cap (A \cup B) \). By definition of \( A \) and \( B \), we get \( C \Rightarrow V_2 \). Since \( G \) is \((K_1 + F, H_1 \Rightarrow H_2)\)-free, \( G[C] \) is \( H_1 \)-free, and therefore \( \chi'(C) \leq c \).

All together, we get \( \chi'(x^-) \leq h \cdot \phi(c, h, t - 1) + c \cdot (h + 1) \) as desired.

By the previous claim, we can partition the set of vertices into the two sets \( V^- \) and \( V^+ \) defined by

\[
V^- = \{ u \in V \mid \chi'(u^-) \leq (h + 1) \cdot (c + \phi(c, h, t - 1)) \},
\]
\[
V^+ = \{ u \in V \mid \chi'(u^+) \leq (h + 1) \cdot (c + \phi(c, h, t - 1)) \}.
\]

If \( G[V^-] \) is \( H_1 \)-free and \( G[V^+] \) is \( H_2 \)-free, then \( \chi'(G) \leq 2c < \phi(c, h, t) \) and we are done. Assume that there exists \( V_1 \subseteq V^- \) such that \( G[V_1] = H_1 \) (the case where \( V^+ \) contains an induced copy of \( H_2 \) is symmetrical).

We now cover \( V(G) \setminus V_1 \) with three sets of vertices depending on their relation with \( V_1 \) and prove that each of these sets induces a digraph with bounded dichromatic number.

Let \( A = \bigcup_{v \in V} v^- \). By definition of \( V^- \) and since \( V_1 \subseteq V^- \), for every \( v \in V_1, v^- \) has dichromatic number at most \((h + 1)(c + \phi(c, h, t - 1))\), and since \( H_1 \) has \( h \) vertices we get that \( \chi'(A) \leq h \cdot (h + 1) \cdot (c + \phi(c, h, t - 1)) \).

Let \( B = \bigcup_{v \in V} v^0 \). Since \( G \) is \((K_1 + F, H_1 \Rightarrow H_2)\)-free, for every \( v \in V_1, v^0 \) is \((F, H_1 \Rightarrow H_2)\)-free and thus \( \chi'(G[v^0]) \leq c \). Hence, \( \chi'(B) \leq c \cdot h \).

Finally, let \( C = V(G) \setminus (A \cup B \cup V_1) \). By definition of \( A \) and \( B \), we have \( V_1 \Rightarrow C \), hence \( C \) is \( H_2 \)-free and thus \( \chi'(C) \leq c \).

All together, we get that \( \chi'(G) \leq h + h \cdot (h + 1) \cdot (c + \phi(c, h, t - 1)) + ch + c = \phi(c, h, t) \).

The proof of the theorem will follow from the second lemma below.

**Lemma 3.4.** If \( G \in \text{Forb}_{\text{ind}}(K_1 + F, H_1 \Rightarrow H_2) \) contains a \((3c + 1)\)-cluster, then \( \chi'(G) \leq c \cdot 2^{f(3c+1)+1} \).

**Proof.** Let \( K \) be a \((3c + 1)\)-cluster in \( G \). Assume there exists a vertex \( u \in V(G) \) such that \( u^- \cap V(K) \) is \( H_1 \)-free and \( u^+ \cap V(K) \) is \( H_2 \)-free. Since \( u^0 \cap V(K) \) is by assumption \((F, H_1 \Rightarrow H_2)\)-free, we get a partition of \( V(K) \) into three sets that induce digraphs with a dichromatic number at most \( c \), a contradiction (this still holds if \( u \in K \) as we can add it to any of the sets without increasing the dichromatic number).

So, for every \( u \in V(G) \), either \( u^- \cap V(K) \) contains a copy of \( H_1 \), or \( u^+ \cap V(K) \) contains a copy of \( H_2 \). Now for every \( V_1 \subseteq V(K) \) such that \( G[V_1] \) is isomorphic to \( H_1 \), the
set of vertices $u$ such that $V_i \subseteq u^-$ is $H_2$-free and therefore has dichromatic number at most $c$. Similarly, for every $V_2 \subseteq V(K)$ such that $G[V_2]$ is isomorphic to $H_2$, the set of vertices $u$ such that $V_2 \subseteq u^+$ is $H_1$-free and therefore has a dichromatic number at most $c$. By doing this for every possible copy of $H_1$ or $H_2$ inside $V(K)$ we can cover every vertex of $V(G)$. Moreover, the number of subsets of $V(K)$ that induces a copy of $H_1$ (resp., of $H_2$) is at most $2^{f(3c+1)}$. Hence, we get that $\chi (G) \leq c \cdot 2^{f(3c+1)+1}$. □

Proof of Theorem 3.2. By Lemmas 3.3 and 3.4, we get that every digraph in \( \text{Forb}_{\text{ind}}(K_1 + F, H_1 \Rightarrow H_2) \) has a dichromatic number at most \( \max(\phi(c, h, 3c + 1), 2^{f(3c+1)+1c}) \), which proves Theorem 3.2. □

Remark 3.5. Let $K(c, h)$ be an integer such that digraphs in \( \text{Forb}_{\text{ind}}(K_1 + F, H_1 \Rightarrow H_2) \) have dichromatic numbers at most $K(c, h)$. From the proof above we can deduce that taking $K(c, h) = \max\left((2h \cdot (h + 1))^{5c+1}, 2^{2^{23c+1}+1} \cdot c\right)$ works (proving as intermediate steps that for every integer $t$, we can take $f(t) \leq 2^{2 \cdot 3^t}$ and $\phi(c, h, t) \leq (2h \cdot (h + 1))^{2c+1}$).

3.2 Growing a hero

The goal of this subsection is to prove the following theorem:

**Theorem 3.6.** If $H$ is a hero in oriented complete multipartite graphs, then so is $\Delta(1, H, 1)$.

Lemma 3.7 is proved in [7] (see 4.2) for tournaments but actually holds for every digraph.

**Lemma 3.7.** Let $G$ a digraph and let $(X_1, \ldots, X_n)$ a partition of $V(G)$. Suppose that $d$ is an integer such that:

- $\forall 1 \leq i \leq n \chi(X_i) \leq d$,
- $\forall 1 \leq i < j \leq n$, if there is an arc $uv$ with $u \in X_j$ and $v \in X_i$, then $\chi(X_{i+1} \cup X_{i+2} \cup \cdots \cup X_j) \leq d$.

Then $\chi(G) \leq 2d$.

**Proof.** Define a sequence $s_0 < s_1 < \cdots < s_t = n$ defined recursively as follows: $s_0 = 0$ and

$$s_k = \max\left\{j > s_{k-1} \mid \chi\left(\bigcup_{s_{k-1} < i \leq j} X_i\right) \leq d\right\}$$

for $k = 1, \ldots t$, and let $Y_k = \bigcup_{s_{k-1} < i \leq s_k} X_i$. By definition of the sequence $s_k$, $\chi(Y_k) \leq d$ for $k = 1, \ldots, t$ and $\chi(Y_k \cup X_{s_k+1}) > d$ for $k = 1, \ldots, t - 1$, so by the assumption of the
lemma, there cannot be an arc from $Y_j$ to $Y_i$ whenever $i \leq j - 2$. Hence, $\bigcup_{i \text{ even}} Y_i$ and $\bigcup_{i \text{ odd}} Y_i$ both have dichromatic numbers at most $d$, and thus $\overrightarrow{x}(G) \leq 2d$. □

The following is an adaptation of (4.4) in [7] with oriented complete multipartite graphs instead of tournaments (note also that their proof is concerned with $\Delta(1, k, H)$ while ours is concerned with $\Delta(1, 1, H)$).

**Lemma 3.8.** Let $G$ be a $\Delta(1, 1, H)$-free oriented complete multipartite graph given with a partition $(X_1, \ldots, X_n)$ of its vertex set $V(G)$. Suppose that $r$ is an integer such that:

- $H$-free oriented complete multipartite graphs have dichromatic numbers at most $r$,
- $\forall 1 \leq i \leq n \bigvee v \in X_i \overrightarrow{x}(v^+ \cap (X_i \cup \cdots \cup X_{i-1})) \leq r$,
- $\forall 1 \leq i \leq n \bigvee v \in X_i \overrightarrow{x}(v^- \cap (X_{i+1} \cup \cdots \cup X_n)) \leq r$.

Then $\overrightarrow{x}(G) \leq 8r + 4$.

**Proof.** We are going to prove that $G$ satisfies the hypothesis of Lemma 3.7 with $d = 4r + 2$, which implies the result. Let $uv$ be an arc such that $u \in X_j$ and $v \in X_i$ where $1 \leq i < j \leq n$. We want to prove that $\overrightarrow{x}(X_{i+1} \cup X_{i+2} \cup \cdots \cup X_j) \leq 4r + 2$. Let $W = X_{i+1} \cup \cdots \cup X_{j-1}$. Let $Q = v^+ \cap u^- \cap W$. If $Q$ contains a copy of $H$, then together with $u$ and $v$ it forms a $\Delta(1, H, 1)$, a contradiction. So $Q$ is $H$-free and thus is $r$-colourable. Now, each vertex in $W \setminus Q$ is in $u^+ \cup v^- \cup u^o \cup v^o$. By hypothesis, $u^+ \cap W$ and $v^- \cap W$ are both $r$-colourable, and since $G$ is an oriented complete multipartite graph, $u^o$ and $v^o$ are stable sets. Finally, by hypothesis, $\overrightarrow{x}(X_j) \leq r$. All together, we get that $\overrightarrow{x}(X_{i+1} \cup \cdots \cup X_j) \leq 4r + 2$ as announced. □

**Proof of Theorem 3.6.** Let $H$ be a hero in oriented complete multipartite graphs and let $h = |V(H)|$. By applying Theorem 3.1 with $H_1 = H_2 = H$, we get that $H \Rightarrow H$ is a hero in oriented complete multipartite graphs. Applying it again with $H_1 = H_2 = H \Rightarrow H$, we get that $(H \Rightarrow H) \Rightarrow (H \Rightarrow H)$ is a hero in oriented complete multipartite graphs. So there exists a constant $c$ such that every $((H \Rightarrow H) \Rightarrow (H \Rightarrow H))$-free oriented complete multipartite graph has a dichromatic number at most $c$. Note that it also implies that every $H$-free oriented complete multipartite graph has a dichromatic number at most $c$.

Let $G$ be a $\Delta(1, 1, H)$-free oriented complete multipartite graph. We are going to prove that $\overrightarrow{x}(G) \leq 8r + 4$ for some $r$, using

Lemma 3.8 We say that $J \subseteq V(G)$ is an $H$-jewel if $G[J]$ is isomorphic to $H \Rightarrow H$. The important feature about an $H$-jewel $J$ in an oriented complete multipartite graph is that, for any vertex $x$ not in $J$, either $x^+ \cap J$ or $x^- \cap J$ contains a copy of $H$, or $x$ has both an in- and an out-neighbour in $J$. An $H$-jewel-chain of length $n$ is a sequence $(J_1, \ldots, J_n)$ of pairwise disjoint $H$-jewels such that for $i = 1, \ldots, n - 1$, $J_i \Rightarrow J_{i+1}$, and for every $1 \leq i < j \leq n$, $J_i \nrightarrow J_j$. Both notions of $H$-jewel and $H$-jewel-chain exist in [7], the ones we give here are slightly different, but are morally similar.
Consider an $H$-jewel-chain $(J_1, \ldots, J_n)$ of maximum length $n$. Set $J = J_1 \cup \cdots \cup J_n$ and $W = V(G) - J$. To simplify statements, we also consider sets $J_i$ for $i \leq 0$ and $i \geq n + 1$, that are assumed to be empty.

The easy but key properties of an $H$-jewel-chain are stated in the following claim.

**Claim 3.8.1.** For every $w \in W$ and $1 \leq j \leq n - 1$:

- $w^+ \cap J_j \neq \emptyset \Rightarrow w^+ \cap J_{j+1} \neq \emptyset$,
- $w^- \cap J_{j+1} \neq \emptyset \Rightarrow w^- \cap J_j \neq \emptyset$.

**Proof.** Assume $w^+ \cap J_j \neq \emptyset$. Then since $J_j \Rightarrow J_{j+1}$, it is not possible that $G[w^- \cap J_{j+1}]$ contains a copy of $H$ for it would create a $\Delta(1, H, 1)$. Since $G[J_{j+1}]$ is isomorphic to $H \Rightarrow H$, and since $w$ cannot have a nonneighbour in both copies of $H$ (because $G$ is an oriented complete multipartite graph), this implies that $w$ has at least one out-neighbour in $J_{j+1}$. The proof of the second item is identical up to the reversal of the arcs. \hfill $\square$

For every $w \in W$, let $g(w)$ be the smallest integer $j$ such that $w^+ \cap J_j \neq \emptyset$ if such an integer exists, and $g(w) = n + 1$ if no such integer exists. For $j = 1, \ldots, n + 1$, set $W_j = \{w : g(w) = j\}$ and $X_j = J_j \cup W_j$. Note that, by definition of the $W_j$'s, if $w \in W_j$, then $J_j \rightarrow w$ for every $i \leq j - 1$.

**Claim 3.8.2.** $\overrightarrow{\chi}(X_j) \leq 4c \cdot h^2 + c + 6h$ for $j = 1, \ldots, n + 1$.

**Proof.** Let $1 \leq j \leq n + 1$. We have $\overrightarrow{\chi}(J_j) \leq |J_j| \leq 2h$.

For each pair of vertices $a \in J_j$ and $b \in J_{j+1}$, set $A_{ab} = \{w \in W_j : bw, wa \in A(G)\}$. Since $ab \in A(G)$ (because $J_j \Rightarrow J_{j+1}$), and $G$ is $\Delta(1, H, 1)$-free, $A_{ab}$ must be $H$-free and thus is $c$-colourable for every choice of $a$ and $b$. Setting $A = \bigcup_{(a,b) \in J_j \times J_{j+1}} A_{ab}$, we get that $\overrightarrow{\chi}(A) \leq 4h^2 \cdot c$. Moreover, since every vertex in $W_j$ has an out-neighbour in $J_j$, we have $A = \{w \in W_j : w^- \cap J_{j+1} \neq \emptyset\}$.

Let $B = \{w \in W_j : w^0 \cap J_{j-1} \neq \emptyset \text{ or } w^0 \cap J_{j+1} \neq \emptyset\}$, in other words $B$ is the set of vertices in $W_j$ with at least one nonneighbour in $J_{j-1}$ or $J_{j+1}$. Since $G$ is an oriented complete multipartite graph, we have $\overrightarrow{\chi}(B) \leq |J_{j-1}| + |J_{j+1}| \leq 4h$.

Let $C = W_j \setminus (A \cup B)$. By definition of $W_j$, for every $i \leq j - 1$, $J_i \rightarrow C$. Since $C$ is disjoint from $A$, we have $C \rightarrow J_{j+1}$, and thus, by Claim 3.8.1 (second bullet), we have $C \rightarrow J_k$ for every $k \geq j + 1$. Finally, since $C$ is disjoint from $B$, we have furthermore $J_{j-1} \Rightarrow C$ and $C \Rightarrow J_{j+1}$. Now, if $C$ contains an $H$-jewel-chain $(J'_1, J'_2)$ of length 2, then $(J_1, \ldots, J_{j-1}, J'_1, J'_2, J_{j+1}, \ldots, J_n)$ is an $H$-jewel-chain of size $n + 1$, contradicting the maximality of $n$. Hence, $C$ does not contain a jewel-chain of size 2 and thus $\overrightarrow{\chi}(C) \leq c$.

All together, we get that $\overrightarrow{\chi}(X_j) \leq 4c \cdot h^2 + c + 6h$. \hfill $\square$

**Claim 3.8.3.** For $j = 1, \ldots, n$ and for every $u \in J_j$,

- $\overrightarrow{\chi}(u^+ \cap (X_1 \cup \cdots \cup X_{j-1})) \leq 4c \cdot h^2 + 2c \cdot h + c + 6h$, and
- $u^- \cap (X_{j+1} \cup \cdots \cup X_{n+1}) = \emptyset$.\hfill $\square$
Proof. Let \( 1 \leq j \leq n \) and let \( u \in J_j \). We first prove the first bullet. By definition of an \( H \)-jewel-chain, \( u \) has no out-neighbour in any \( J_i \) for \( i \leq j - 1 \) and by Claim 3.8.2, \( \overrightarrow{\chi}(X_{j-1}) \leq 4c \cdot h^2 + c + 6h \). So it is enough to prove that \( A = u^+ \cap (W_1 \cup \cdots \cup W_{j-2}) \) has dichromatic number at most \( 2c \cdot h \). By Claim 3.8.1, every vertex of \( W_1 \cup \cdots \cup W_{j-2} \) has an out-neighbour in \( J_{j-1} \). Moreover, for every \( v \in J_{j-1} \), we have \( vu \in A(G) \) (because \( J_{j-1} \rightarrow J_j \)) and \( v^- \cap A \) is \( H \)-free, for otherwise a copy of \( H \) in \( v^- \cap A \) would form, together with \( v \) and \( u \), a \( \Delta(1, H, 1) \). So \( \overrightarrow{\chi}(A) \leq |J_{j-1}| \cdot c = 2c \cdot h \) as needed.

To prove the second bullet, observe that for every \( k \geq j + 1 \), since \( J \) is a jewel-chain, \( u \) has no in-neighbour in \( J_k \) and by definition of \( W_k \), \( u \) has no in-neighbour in \( W_k \). \( \square \)

Claim 3.8.4. For \( j = 1, \ldots, n + 1 \) and for every \( w \in W_j \),

1. \( \overrightarrow{\chi}(w^+ \cap (X_1 \cup \cdots \cup X_{j-1})) \leq 8c \cdot h^2 + 2c \cdot h + 2c + 12h \), and
2. \( \overrightarrow{\chi}(w^- \cap (X_{j+1} \cup \cdots \cup X_{n+1})) \leq 8c \cdot h^2 + 2c + 12h \).

Proof. Let \( 1 \leq j \leq n + 1 \) and let \( w \in W_j \).

We first prove the first bullet. By definition of \( W_j \), \( w \) has no out-neighbour in any of the \( J_i \) for \( i \leq j - 1 \) and by Claim 3.8.2 \( \overrightarrow{\chi}(W_{j-2} \cup W_{j-1}) \leq 8c \cdot h^2 + 2c + 12h \). So it is enough to prove that \( A = w^+ \cap (W_1 \cup \cdots \cup W_{j-2}) \) has dichromatic number at most \( 2c \cdot h \). Again by definition of \( W_j \) we have \( J_{j-2} \rightarrow w \) and \( J_{j-1} \rightarrow w \), and since \( J_{j-2} \cup J_{j-1} \) induces a tournament and \( G \) is \( (K_1 + T_{2j}) \)-free, \( w \) has at most one nonneighbour in \( J_{j-2} \cup J_{j-1} \). So there exists \( s \in (j - 2, j - 1) \) such that \( J_s \Rightarrow w \). For every \( v \in J_s \), if \( v^- \cap A \) contains a copy of \( H \), then it would form, together with \( v \) and \( w \), a \( \Delta(1, 1, H) \), a contradiction. So, for every \( v \in J_s \), \( v^- \cap A \) is \( H \)-free and is thus \( c \)-colourable. Finally, by Claim 3.8.1 every vertex in \( A \) has an out-neighbour in \( J_s \). So we get that \( \overrightarrow{\chi}(A) \leq 2c \cdot h \).

We now prove the second bullet. If \( j \geq n - 1 \), then by Claim 3.8.2 \( \overrightarrow{\chi}(X_n \cup X_{n+1}) \leq 8c \cdot h^2 + 2c + 12h \) and we are done. So we may assume that \( j \leq n - 2 \). By Claim 3.8.2, \( \overrightarrow{\chi}(X_{j+1}) \leq 4c \cdot h^2 + 6h + c \). Set \( B = w^- \cap (X_{j+2} \cup \cdots \cup X_{n+1}) \). By Claim 3.8.1, \( w \) has an out-neighbour \( v \in J_{j+1} \). For \( i \geq j + 2 \), by definition of an \( H \)-jewel-chain, \( v \rightarrow J_i \) and by definition of \( W_i \), \( v \rightarrow W_i \). So \( v \rightarrow B \) and since \( G \) is an oriented complete multipartite graph \( B \setminus (v^+ \cap B) \) is a stable set. Now, \( v^+ \cap B \) is \( H \)-free, as otherwise \( G \) would contain a \( \Delta(1, 1, H) \). So \( v^+ \cap B \) is \( c \)-colourable and thus \( \overrightarrow{\chi}(B) \leq c + 1 \) and thus \( \overrightarrow{\chi}(w^- \cap (X_{j+1} \cup \cdots \cup X_{n+1})) \leq \overrightarrow{\chi}(X_{j+1}) + c + 1 \leq 4c \cdot h^2 + 2c + 6h + 1 \) by Claim 3.8.2. \( \square \)

By Claims 3.8.2, 3.8.3 and 3.8.4, we can apply Lemma 3.8 with \( r = 12c \cdot h^2 + 4c \cdot h + 3c + 18h \) to get \( \overrightarrow{\chi}(G) \leq 8r + 4 \). \( \square \)

4 SOME INSIGHTS ABOUT \( \Delta(1, 2, 2) \)-FREE ORIENTED COMPLETE MULTIPARTITE GRAPHS

In [4] Axenovich et al. tried to characterise patterns that must appear in every ordering of the vertices of graphs with a large chromatic number. An (undirected) graph \( G \) is (what we call) noninterlaced if there exists an ordering \( (x_1, \ldots, x_n) \) on its vertices such that for every \( i_1 < i_2 < i_3 < i_4 < i_5 \), \( \{x_{i_1}x_{i_3}, x_{i_2}x_{i_4}, x_{i_3}x_{i_5}\} \not\subseteq E(G) \). See Figure 2. They left as an open question
whether noninterlaced graphs have bounded chromatic numbers or not. The goal of this section is to show that if $\Delta(1, 2, 2)$ is a hero in oriented complete multipartite graphs, then noninterlaced graphs have bounded chromatic numbers. See Theorem 4.2.

Given an oriented complete multipartite graph $D$ together with an ordering $(V_1, ..., V_n)$ on its parts, the arcs going from $V_i$ to $V_j$ are called forward arcs if $i < j$, and backward arcs otherwise. Moreover, given $i < j$, we say that $u < v$ for every $u \in V_i$ and every $v \in V_j$. Finally, we say that an oriented complete multipartite graph $D$ is flat if it admits an ordering $(V_1, ..., V_n)$ on its parts such that for every vertex $v$ of $D$, the set of vertices $\{xvx \text{ is a backward arc}\}$ is included in a single part of $D$, and the set of vertices $\{xvx \text{ is a backward arc}\}$ is also included in a single part of $D$.

**Lemma 4.1.** Let $D$ be an oriented complete multipartite graph with parts $V_1, ..., V_n$ where $(V_1, ..., V_n)$ is a flat ordering. If $D$ contains a copy of $\Delta(1, 2, 2)$, naming its vertices as in Figure 3, we must have $v_1 < v_2 < v_3 < v_4 < v_5$.

**Proof.** Suppose that $D$ contains a copy of $\Delta(1, 2, 2)$ and name its vertices as in Figure 3. Since $\Delta(1, 2, 2)$ is a tournament, $v_i$’s are contained in pairwise distinct parts of $D$, and thus are totally ordered. Since $(V_1, ..., V_n)$ is a flat ordering, the smallest vertex among $\{v_1, v_2, v_3, v_4, v_5\}$ must have in-degree at most 1 in $\Delta(1, 2, 2)$, and hence must be $v_1$. Similarly, since $v_5$ is the only vertex with out-degree 1 in $\Delta(1, 2, 2)$, $v_5$ must be the largest of the $v_i$. If $v_3 < v_2$, then $v_3 < v_2 < v_5$ and the arcs $v_2v_3$ and $v_5v_3$ contradicts the fact that $(V_1, ..., V_n)$ is a flat ordering, so $v_2 < v_3$. Similarly, if $v_4 < v_3$, then $v_4 < v_3 < v_5$ and the arcs $v_3v_4$ and $v_5v_3$ contradicts the fact that $(V_1, ..., V_n)$ is a flat ordering, so $v_3 < v_4$ and thus $v_1 < v_2 < v_3 < v_4 < v_5$. □

**Theorem 4.2.** If $\Delta(1, 2, 2)$ is a hero in oriented complete multipartite graphs, then every noninterlaced graph has a bounded chromatic number.

![Figure 2](image-url) A graph is noninterlaced if there is an ordering of its vertices that avoids the above pattern as a subgraph.

![Figure 3](image-url) Two drawings of $\Delta(1, 2, 2)$ (A) $\Delta(1, 2, 2)$ and (B) a drawing of $\Delta(1, 2, 2)$ where the backward arcs (coloured in red) induce the forbidden pattern of noninterlaced graphs.
Proof. Assume that $\Delta(1, 2, 2)$ is a hero in oriented complete multipartite graphs. Let $\mathcal{F}$ be the class of flat $\Delta(1, 2, 2)$-free oriented complete multipartite graphs. Since $\Delta(1, 2, 2)$ is a hero in oriented complete multipartite graphs, there exists a constant $r$ such that every digraph in $\mathcal{F}$ has a dichromatic number at most $r$. Let $R \in \mathcal{F}$ such that $\overrightarrow{\chi}(R) = r$ and recall that $R$ has a flat ordering. We are going to prove that every noninterlaced graph has a chromatic number at most $2^r$.

Let $G$ be a noninterlaced (undirected) graph and $(x_1, \ldots, x_n)$ the ordering on $V(G)$ given by the definition of noninterlaced graphs (i.e., an ordering that avoids the pattern in Figure 2). We construct an oriented complete multipartite graph $D'(G)$ as follows. For each $x_i$, we create a stable set $V_i$ in $D'(G)$ of size $n^2$ and we assume the vertices of $V_i$ are organised as an $n \times n$ matrix. The parts of $D'(G)$ are $V_1, \ldots, V_n$. Let us now explain how we orient the arcs. Given $i < j$, if $x_i x_j \in E(G)$, we orient the arcs from each vertex of the $i$th line of $V_j$ to each vertex of the $j$th column of $V_i$. Every other arc is oriented from $V_i$ to $V_j$. This completes the construction of $D'(G)$.

Let us now prove that the ordering $(V_1, \ldots, V_n)$ of $D'(G)$ is flat. Let $v$ be any vertex of $D'(G)$ and assume $v \in V_j$ and is in the $i$th line and the $k$th column. By definition of $D'(G)$, if $v$ is the tail of some backward arcs $vw$, then $w$ belongs to the $j$th column of $V_i$ (in particular $i < j$). So all such $w$ belong to the same part. Similarly, if $uv$ is a backward arc, then $u$ belongs to the $j$th-line of $V_k$ ($j < k$). This proves that $(V_1, \ldots, V_n)$ is a flat ordering of $D'(G)$.

We now construct another oriented complete multipartite graph $D(G)$ from $D'(G)$ by introducing, for $j = 1, \ldots, n-1$, a copy $R_j$ of $R$ between $V_j$ and $V_{j+1}$ such that $\cup_{i \leq j} V_i \Rightarrow V(R_j), V(R_j) \Rightarrow \cup_{k \geq j+1} V_k$, and $V(R_j) \Rightarrow \cup_{i \geq j+1} V(R_i)$. This completes the construction of $D(G)$.

It is clear that $D(G)$ is an oriented complete multipartite graph and by inserting the flat ordering of each copy of $R$ between each consecutive $V_j$, we get a natural ordering of the parts of $D(G)$. In the rest of the proof, we speak about backward and forward arcs of $D(G)$ with respect to this ordering.

We are going to prove that $D(G) \in \mathcal{F}$ (so $\overrightarrow{\chi}(D(G)) \leq r$) and that $\chi(G) \leq 2^{2^\mathcal{F}(D(G))}$, which together imply the result.

To help in our analysis, we will say that the vertices of $D(G)$ that comes from $D'(G)$ are green.

The following claim is straightforward by construction.

Claim 4.2.1. If $uv$ is a backward arc of $D(G)$, then either both $u$ and $v$ are green, or $u$ and $v$ are both contained in one of the copies of $R$.

Claim 4.2.2. If $v_1, v_2, v_3, v_4, v_5$ are vertices of $D'(G)$ such that $v_1 < v_2 < v_3 < v_4 < v_5$, then $\{v_3 v_1, v_5 v_3, v_4 v_2\} \subseteq A(D'(G))$.

Proof. For otherwise $\{x_1 x_3, x_3 x_5, x_5 x_4\} \subseteq E(G)$, a contradiction. \qed

Let us first prove that $D(G) \in \mathcal{F}$. By Claim 4.2.1, $D(G)$ is flat and the ordering we consider is a flat ordering. Assume that $D(G)$ contains a copy of $\Delta(1, 2, 2)$ and name its vertices as in Figure 3. By Lemma 4.1, we have that the $v_i$ are in pairwise distinct parts of $D(G)$ and $v_1 < v_2 < v_3 < v_4 < v_5$. If $v_3$ is in a copy of $R$, since $v_3 v_1$ and $v_5 v_3$ are backward arcs of $D(G)$, we get by Claim 4.2.1 that $v_1$ and $v_5$ are in the same copy of $R$ as $v_3$. By
construction, since \( v_1 < v_2 < v_3 < v_4 < v_5 \), we get that \( v_2 \) and \( v_4 \) are also in this same copy of \( R \), a contradiction with the fact that \( R \) is \( \Delta(1, 2, 2) \)-free. So we may assume that \( v_3 \) is green, and so are \( v_1 \) and \( v_5 \) by Claim 4.2.1. Now, if \( v_2 \) is in a copy of \( R \), then by Claim 4.2.1 \( v_4 \) is in the same copy of \( R \), and since \( v_2 < v_3 < v_4 \), \( v_3 \) must be in that same copy of \( R \), a contradiction with the fact that \( v_3 \) is green. Hence, \( v_2 \) is green and by Claim 4.2.1 so is \( v_4 \). Thus, every \( v_i \) is green, a contradiction to Claim 4.2.2. This proves that \( \emptyset \in DG() \).

Since \( DG() \) contains copies of \( R \), it has a dichromatic number at least \( r \), and since \( \emptyset \in DG() \), we get that \( \chi DG() = r \). Consider a dicolouring \( \phi \) of \( DG() \) with \( r \) colours. We define a colouring \( \phi \) of \( V(G) \) from \( \phi \) as follows: for \( i \) in \( \phi \), \( v_i \) is the set of sets of colours used by each line of \( V_i \). This gives us a colouring of \( V(G) \) with at most \( 2^r \) colours. Let us prove that it is a proper colouring of \( G \), that is, each colour class is an independent set.

Assume for contradiction that there exists \( x, v \in E(G) \) such that \( \phi(x) = \phi(v) \) and assume without loss of generality that \( i < j \). Let us first prove that \( D(G) \) has a monochromatic backward arc. Consider the set of colours used in the \( i \)th line of \( V_j \). The same set of colours is used by the vertices of some line of \( V_i \), say the \( k \)th. Now, there is an arc from each vertex of the \( i \)th line of \( V_j \) to the \( j \)th vertex of the \( k \)th line of \( V_i \), which implies the existence of a monochromatic backward arc as announced. Let \( uv \) be this monochromatic backward arc, with \( v \in V_i \) and \( u \in V_j \). Since \( i < j \), there is a copy of \( R \) between \( V_i \) and \( V_j \). Since \( \chi(R) = r \), one of the vertices \( x \) of \( R \) is coloured with \( \phi(u) \). By construction of \( D(G) \), \( ux \) and \( xv \) are arcs of \( D(G) \) and thus \{\( u, x, v \)\} induces a monochromatic directed triangle, a contradiction. \( \square \)

5 RELATED AND FURTHER WORKS

Heroes in orientations of chordal graphs were recently fully characterised in [2].

A star is an undirected tree with at most one nonleaf vertex. An oriented forest (resp., oriented star) is an orientation of a forest (resp., of a star). In [3], the authors initiated a systematic study of heroes in \( Forb_{ind}(F) \) for a fixed digraph \( F \). We now summarise the known results in this direction and explain how our results fit in the big picture.

First observe that \( K_1 \) and \( TT_2 \) are heroes in every class of digraphs. A result in [12] implies that no digraph except for \( K_1 \) and \( TT_2 \) is a hero in \( Forb_{ind}(F) \) whenever the underlying graph of \( F \) contains a cycle. We now distinguish cases depending on whether \( F \) is an oriented forest, an oriented star or a disjoint union of at least two oriented stars.

5.1 Heroes in \( Forb_{ind}(F) \) when \( F \) is an oriented forest

It is proved in [3] that if \( F \) is not a disjoint union of oriented stars, then the only possible heroes in \( Forb_{ind}(F) \) are transitive tournaments. In the same paper the authors venture to conjecture the following (which can be seen as an oriented analogue of the well-known Gyárfás–Sumner conjecture [10, 15]):

**Conjecture 5.1** (Aboulker et al. [3]). For every oriented forest \( F \), every transitive tournament is a hero in \( Forb_{ind}(F) \).
In [14] it is proved that it is enough to prove the conjecture for trees, the conjecture has been proved to be true for oriented stars [8].

5.2 | Heroes in Forb_{ind}(F) when F is an oriented star

When F is an oriented star, it is still possible that heroes in Forb_{ind}(F) are the same as heroes in tournaments. As said in Section 5.1, it is proved in [8] that for every oriented star F, all transitive tournaments are heroes in Forb_{ind}(F). The only other known result so far is concerned with $\vec{K}_{1,2}$ (the oriented star on three vertices, with one vertex of out-degree 2 and two vertices of in-degree 1): it is proved in [1, 14] that $\vec{K}_{1,2}$ is a hero in Forb_{ind}(\vec{K}_{1,2}). Note that $\vec{P}_3$ is an oriented star. We now give an easy proof that all heroes in tournaments are heroes in Forb_{ind}(\vec{P}_3).

Recall that a digraph G is quasi transitive if for every triple of vertices x, y, z, if $xy, yz \in A(G)$, then $xz \in A(G)$ or $zx \in A(G)$ and observe that the class of quasi transitive digraphs is precisely Forb_{ind}(\vec{P}_3).

Given two digraphs $G_1$ and $H_1$ with disjoint vertex sets, a vertex $u \in G_1$, and a digraph G, we say that G is obtained by substituting $H_1$ for u in $G_1$, and write $\leftarrow G_{uH_1}$ to denote G, provided that the following hold:

- $V(G) = (V(G_1) \setminus u) \cup V(H_1)$,
- $G[V(G_1) \setminus u] = G_1 \setminus u$,
- $G[V(H_1)] = H_1$,
- for all $v \in V(G_1) \setminus u$ if $uv \in A(G_1)$ (resp., $uv \in A(G_1)$, respectively, $u$ and $v$ are nonadjacent in $G_1$), then $V(H_1) \ni v$ (resp., $v \ni V(H_1)$, respectively, $V(H_1) \subseteq v^\delta$) in G.

Let $T$ be the class of tournaments and $\mathcal{A}$ the class of acyclic digraphs. Let $(\mathcal{A} \cup T)^*$ be the closure of $\mathcal{A} \cup T$ undertaking substitution, that is to say digraphs in $(\mathcal{A} \cup T)^*$ are the digraphs obtained from a vertex by repeatedly substituting vertices with digraphs in $\mathcal{A} \cup T$. A classic result of Bang-Jensen and Huang [6] (see also Proposition 8.3.5 in [5]), implies that quasi transitive digraphs are all in $(\mathcal{A} \cup T)^*$.

Theorem 5.2. Heroes in $(\mathcal{A} \cup T)^*$ are the same as heroes in tournaments. In particular, heroes in Forb_{ind}(\vec{P}_3) are the same as heroes in tournaments.

Proof. Let H be a hero in tournaments and c be the maximum dichromatic number of an H-free tournament. We prove by induction on the number of vertices that H-free digraphs in $(\mathcal{A} \cup T)^*$ are also c-dicolourable. Let $G \in (\mathcal{A} \cup T)^*$ on $n \geq 2$ vertices and assume that all digraphs in $(\mathcal{A} \cup T)^*$ on at most $n - 1$ vertices are c-dicolourable.

There exist $G_1, \ldots, G_s, H_1, \ldots, H_{s-1}$ and vertices $v_1, \ldots, v_{s-1}$ such that the $G_i$’s and the $H_i$’s are digraphs of $\mathcal{A} \cup T$ with at least two vertices, $G_1 = K_1$, $G_s = G$, $v_1 \in V(G_i)$ and for $i = 1, \ldots, s - 1$, $G_{i+1} = G_i(v_i \leftarrow H_i)$.

If all $H_i$ are tournaments, then $G$ is a tournament and is thus c-dicolourable. So we may assume that there exists $1 \leq i \leq s - 1$ such that $H_i$ is an acyclic digraph. Let $x_1, \ldots, x_t$ be the vertices of $H_i$. There exist $t$ digraphs $X_1, \ldots, X_t$ in $(\mathcal{A} \cup T)^*$ such that $G$ is obtained
from $G_{i+1}$ by substituting $x_i$ by $X_1, x_2$ by $X_2, ..., x_t$ by $X_t$ and some vertices of $V(G_{i+1})\setminus\{x_1, ..., x_t\}$ by digraphs in $(A \cup T)^*$. Note that the order in which these substitutions are performed does not matter.

Let $X = \cup_{1 \leq i \leq t} V(X_i)$. So $V(G) \setminus X$ can be partitioned into 3 sets $S^+, S^-, S^0$ such that for every $v \in X$, $S^+ \subseteq v^+$, $S^- \subseteq v^-$ and $S^0 \subseteq v^0$.

For $i = 1, ..., t$, let $D_i = G[G_i \setminus (X \setminus X_i)]$. By induction, the $D_i$'s are $c$-dicolourable. For $i = 1, ..., t$, let $\phi_i$ be a $c$-dicolouring of $D_i$. Assume without loss of generality that $|\phi_i(X_i)| \geq |\phi_i(X)|$ for $1 \leq i \leq t$. In particular $\chi'(X_i) \leq |\phi_i(X_i)|$ for $i = 1, ..., t$. Extend $\phi_i$ to a $c$-dicolouring of $D$ by dicolouring each $X_i$ (independently) with colours from $\phi_i(X_i)$. We claim that this gives a $c$-dicolouring of $G$.

Let $C$ be an induced directed cycle of $G$. If $C$ is included in $X$ or $V(G) \setminus X$, then $C$ is not monochromatic. So we may assume that $C$ intersects both $V(G) \setminus X$ and $X$. Since vertices in $X$ share the same neighbourhood outside $X$ and $C$ is induced, $C$ must intersect $X$ on exactly one vertex, and this vertex can be chosen to be any vertex of $X$. In particular we may assume that it is in $X_i$. Hence $C$ is not monochromatic.

Note that the proof of the previous theorem actually works for the following stronger statement:

**Theorem 5.3.** Let $C$ be a class of digraphs closed under taking substitution and let $(A \cup C)^*$ be the closure of $A \cup C$ under taking substitution. Then heroes in $(A \cup C)^*$ are the same as heroes in $C$.

### 5.3 Heroes in $\text{Forb}_{\text{ind}}(F)$ when $F$ is a disjoint union of at least two oriented stars

When $F$ is a disjoint union of stars, the authors of [3] conjectured that heroes in $\text{Forb}_{\text{ind}}(F)$ were the same as heroes in tournaments, and Theorem 1.2 disproves this conjecture (recall that $\text{Forb}_{\text{ind}}(K_1 + TT_2)$ is the class of oriented complete multipartite graphs).

Since $\text{Forb}_{\text{ind}}(F_1) \subseteq \text{Forb}_{\text{ind}}(F_2)$ whenever $F_1$ is an induced subgraph of $F_2$, and given our knowledge of heroes in $\text{Forb}_{\text{ind}}(F)$ when $F$ is an oriented star, let us focus on disjoint union of stars where each connected component is $K_i, TT_2$ or $P_3$.

We denote by $\overline{K}_i$ the digraph on $i$ vertices with no arc (this is a disjoint union of stars, where each connected component is $K_i$). Observe that $\text{Forb}_{\text{ind}}(\overline{K}_i)$ is the class of tournaments. In [11], it is proved that heroes in $\text{Forb}_{\text{ind}}(\overline{K}_i)$ are the same as heroes in tournaments. The proof of this result is quite hard, and shows that knowing heroes in $\text{Forb}_{\text{ind}}(F)$ does not necessarily help in understanding heroes in $\text{Forb}_{\text{ind}}(K_1 + F)$. Even worse, it is clear that every digraph is a hero in $\text{Forb}_{\text{ind}}(K_i)$ and in $\text{Forb}_{\text{ind}}(TT_2)$, while our result shows that only very few digraphs are heroes in $\text{Forb}_{\text{ind}}(K_i + TT_2)$. Theorem 3.2 suggests that the heroes in $\text{Forb}_{\text{ind}}(K_1 + TT_2)$ could be the same as heroes in $\text{Forb}_{\text{ind}}(F)$ where $F = \overline{K}_i + TT_2$ or $F = \overline{K}_i + P_3$. To prove it (up to the status of $\Delta(1, 2, 2)$), it would be enough to answer by the affirmative to the following question:

**Question 5.4.** Let $H$ and $F$ be digraphs such that $\Delta(1, 1, H)$ is a hero in $\text{Forb}_{\text{ind}}(F)$ and $H$ is a hero in $\text{Forb}_{\text{ind}}(K_1 + F)$. Then $\Delta(1, 1, H)$ is a hero in $\text{Forb}_{\text{ind}}(K_1 + F)$.
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