Weak solutions to an initial-boundary value problem for a continuum equation of motion of grain boundaries

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Abstract. We investigate an initial-(periodic-)boundary value problem for a continuum equation, which is a model for motion of grain boundaries based on the underlying microscopic mechanisms of line defects (disconnections) and integrated the effects of a diverse range of thermodynamic driving forces. We first prove the global-in-time existence and uniqueness of weak solution to this initial-boundary value problem in the case with positive equilibrium disconnection density parameter $B$, and then investigate the asymptotic behavior of the solutions as $B$ goes to zero. The main difficulties in the proof of main theorems are due to the degeneracy of $B = 0$, a non-local term with singularity, and a non-smooth coefficient of the highest derivative associated with the gradient of the unknown. The key ingredients in the proof are the energy method, an estimate for a singular integral of the Hilbert type, and a compactness lemma.

Keywords. Motion of grain boundaries; Initial-boundary value problem; Global existence; Weak solutions; Disconnections

1 Introduction

A polycrystalline material can be regarded as a network of grain boundaries (GBs) on the mesoscale. This GB network has a great impact on a wide range of materials properties, such as strength, toughness, electrical conductivity, and its evolution is important for engineering materials \cite{27}. Grain boundaries are the interfaces between differently oriented crystalline grains, which are a kind of two-dimensional defects in materials. Grain boundary migration controls many microstructural evolution processes in materials. Since GBs are interfaces between crystals, the microscopic mechanisms by which they move are intrinsically different from other classes of interfaces, such as solid-liquid interfaces and biological cell membranes.

Recent experiments and atomistic simulations have shown that the microscopic mechanism of GB migration is associated with the motion of topological line defects, i.e., disconnections \cite{7,17,14,21,15,22,23,28}. This dependence on microscopic structures enables broad-range and deep understandings of GB migration, e.g, the stress-driven...
motion and the shear coupling effect [19, 10], which cannot be described by the classical motion by mean curvature models (driven by capillary forces) [27].

A new continuum equation for motion of grain boundaries based on the underlying disconnection mechanisms was developed by Zhang et al. [33] in 2017. This continuum model integrates the effects of a diverse range of thermodynamic driving forces including the stress-driven motion and is able to describe the shear coupling effect during the GB motion. Generalizations of this continuum model with multiple disconnection modes and GB triple junctions have been further developed [29, 30, 34].

In the present article, we will study the existence of weak solutions to the initial-boundary value problem of the continuum equation for GB motion developed in Ref. [33], which reads

$$h_t = -M_d((\sigma_i + \tau)b + \Psi H - \gamma H h_{xx})(|h_x| + B)$$

for \((t, x) \in (0, \infty) \times \Omega\), where \(\Omega = (a, d)\). The boundary and initial conditions are

$$h|_{x=a} = h|_{x=d}, \quad h_x|_{x=a} = h_x|_{x=d}, \quad (t, x) \in (0, T_e) \times \partial \Omega,$$

$$h(0, x) = h_0(x), \quad x \in \Omega,$$

where

$$\sigma_i(t, x) = \text{P.V.} \int_{-\infty}^{\infty} \frac{K\beta h_x(t, x_1)}{x-x_1} dx_1,$$  

and

$$\beta = \frac{b}{H}, \quad K = \frac{\mu}{2\pi(1-\nu)}, \quad B = \frac{2H}{a}e^{-F_d/(k_B T)}.$$

The unknown function \(h\) in Equation (1.1) is the height of grain boundary from reference line, and \(\sigma_i(x, t)\) is the stress due to the elastic interaction between disconnections based on their dislocation nature [13, 32]. The parameters \(b\) and \(H\) respectively are the Burgers vector and step height of a disconnection, \(\mu\) and \(\nu\) respectively are the shear modulus and Poisson ratio, \(\gamma\) is the GB energy, \(\tau\) is the applied stress, \(\Psi\) is energy jump across the GB, and \(M_d\) is the mobility constant. The parameter \(B\) is associated with the equilibrium density of the disconnection pairs, where \(F_d\) is the disconnection formation energy, \(a\) is the lattice constant, \(k_B\) is the Boltzmann constant, \(T\) is the temperature, and \(\frac{2H}{a}e^{-F_d/(k_B T)}\) is the equilibrium disconnection density.

We will study the existence of weak solutions for both cases of \(B > 0\) and \(B = 0\). Note that the regime of \(B \to 0\) means small equilibrium disconnection density or large slope of the grain boundary profile, and when \(B = 0\), the equation (1.1) is degenerate at those points where \(h_x = 0\). Numerical results in Ref. [33] showed that sharp corners may be developed in the GB profile in the case of \(B = 0\).

The difficulties in the proofs of the existence and uniqueness theorems come from the non-local term with singularity together with a non-smooth coefficient associated with \(|h_x|\) of the highest derivative \(h_{xx}\), and the degeneracy of the equation in the case of \(B = 0\). To estimate this singular integral term, we employ a theorem in the book by Stein [26]. Regularization is performed so that the coefficient of the \(h_{xx}\) term is smooth and uniformly bounded from below, and then compactness lemmas are employed to obtain the results for the original equations. Note that dependence on non-smooth gradient terms in the coefficient of the highest derivative also appeared in the phase field
models proposed by Alber and Zhu in [1, 3] to describe the evolution of an interface driven by configurational forces, and properties of solutions have been obtained [2, 4, 5, 35, 36, 37]. Models with non-smooth gradient terms have also been investigated by Acharya, et al [6] and Hilderbrand, et al [12].

1.1 Interpretation of the formula for $\sigma_i$

This subsection is intended to give an explanation of the formula for $\sigma_i$. The material we consider is normally in a bounded domain and we assume the spatial periodic boundary conditions, which means that the unknown $h$ is defined over $\Omega$; however in formula (1.4), the integral domain is $\mathbb{R} = (-\infty, +\infty)$, which implies $h$ should be defined over $\mathbb{R}$. Letting $L = d - a$ be the smallest positive period, choosing $x \in \Omega$ we then arrive at

$$
\sigma_i(t, x) = \text{P.V.} \int_{-\infty}^{\infty} \frac{K\beta h_x(t, x_1)}{x - x_1} dx_1
$$

$$
= \sum_{k \in \mathbb{Z}} \text{P.V.} \int_{a + kL}^{a + (k+1)L} \frac{K\beta h_x(t, x_1)}{x - x_1} dx_1
$$

$$
= \sum_{k \in \mathbb{Z}} \text{P.V.} \int_{a}^{d} \frac{K\beta h_x(t, y)}{x - y + kL} dy
$$

$$
=: \sigma_{i1} + \sigma_{i2},
$$

(1.5)

where

$$
\sigma_{i1} =: \text{P.V.} \sum_{k \in \mathbb{Z}\setminus\{0\}} \int_{a}^{d} \frac{K\beta h_x(t, y)}{x - y + kL} dy,
$$

(1.6)

$$
\sigma_{i2} =: \text{P.V.} \int_{a}^{d} \frac{K\beta h_x(t, y)}{x - y} dy.
$$

(1.7)

Here $\mathbb{Z}$ denotes the set of integers. Observing that for $k \neq 0$ we see that the function $\frac{1}{x - y + kL}$ is positive and monotonically increasing in $y$ for any fixed $x$ and $L$, thus by applying the second mean value theorem of integrals, we conclude that there exists a number $\eta \in [a, d]$ such that

$$
\int_{a}^{d} \frac{K\beta h_x(t, y)}{x - y + kL} dy = \frac{1}{x - d + kL} \int_{\eta}^{d} K\beta h_x(t, y) dy,
$$

while for the case that $k = 0$, $\int_{a}^{d} \frac{K\beta h_x(t, y)}{x - y} dy$ is a singular integral for general $h$, from which one thus finds that for many kinds of $h$, the series in $\sigma_i$ may diverge.

Therefore one must understand Principal Value in the formula of $\sigma_i$ both for a series $\sigma_{i1}$ and for singular integral $\sigma_{i2}$, more precisely, the term

$$
\sigma_{i1} = \sum_{k=1}^{\infty} \int_{a}^{d} \left( \frac{K\beta h_x(t, y)}{x - y + kL} + \frac{K\beta h_x(t, y)}{x - y - kL} \right) dy,
$$

(1.8)

which implies that

$$
|\sigma_{i1}| \leq C\|h_x\|_{L^2(\Omega)} \sum_{k=1}^{\infty} \frac{1}{(kL)^2} \leq C\|h_x\|_{L^2(\Omega)},
$$

(1.9)
and
\[ \sigma_{i2} = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{K\beta h_x(t, y)}{x-y} dy, \]
whose bound can be evaluated by employing Theorem A.1 suppose that \( h_x \in L^2(\Omega) \).

**Remark.** We would like to point out another way to interpret the singular integral \( \sigma_i \) with the following modification:
\[ \sigma_i(t, x) = \text{P.V.} \int_{-\infty}^{\infty} \frac{K\beta h_x(t, x)}{D(x-x_1)} dx_1 =: \sigma_{i1} + \sigma_{i2}, \] (1.10)
where \( D(x-x_1) \) is defined by
\[ D(x-x_1) = \begin{cases} \text{sgn}(x_1-x)|x-x_1|^{1+\zeta}, & \text{if } x \not\in [a,d]; \\ x-x_1, & \text{if } x \in [a,d] \end{cases} \] (1.11)
with an arbitrarily given positive constant \( \zeta \). We can then conclude that the series \( \sigma_{i1} \) converges and the singular integral \( \sigma_{i2} \) may be treated as in the previous method. We will not use this method in the proofs in this paper.

### 1.2 Main results

We first perform nondimensionalization of the equation. Using \( M_d \mu \) as the time unit, \( \mu \) the unit of \( \sigma_i \), \( \tau \) and \( \Psi \), \( L_0 \) the unit of the length scale of the continuum equation, and \( \mu L_0 \) the unit of \( \gamma \), we have the dimensionless form of the equation. Further introducing parameters
\[ \alpha_1 = \gamma H, \quad \alpha_2 = b, \quad \alpha_3 = \tau b + \Psi H, \]
where all the quantities are in dimensionless form, equation (1.1) can be written as
\[ h_t - \frac{\alpha_1}{2} (|h_x|h_x + 2Bh_x)_x + (\alpha_2\sigma_i + \alpha_3)(|h_x| + B) = 0. \] (1.12)
Here we have used the formula \( (|y|y)' = 2|y| \). From now on, we will use this nondimensionalized equation with the dimensionless parameters described above.

To define weak solutions to the initial-boundary value problem (1.1) – (1.3), we denote by \( \Omega = (a, d) \) a bounded open interval with constants \( a < d \), by \( T_e > 0 \) an arbitrary constant, and by \( Q_{T_e} \) the domain \((0, T_e) \times \Omega \). Define
\[ (v_1, v_2)_Z = \int_Z v_1(y)v_2(y)dy \]
for \( Z = \Omega \) or \( Z = Q_{T_e} \). Moreover, if \( v \) is a function defined on \( Q_{T_e} \), we use \( v(t) \) to represent the mapping \( x \mapsto v(t, x) \) and sometimes write \( v = v(t) \) for convenience.

**Statement of the main results.** Our main results are concerned with the existence and uniqueness of weak solution to an initial-boundary value problem.

**Definition 1.1** Let \( h_0 \in L^1(\Omega) \). A function \( h \) with
\[ h \in L^2(0, T_e; H^1_{per}(\Omega)) \] (1.13)
is called a weak solution to problem (1.1) – (1.3), if for all \( \varphi \in C^\infty_0((-\infty, T_e) \times \Omega) \), there holds

\[
(h, \varphi_t)_{Q_{T_e}} - \frac{\alpha_1}{2} \left( |h_x| h_x + 2Bh_x, \varphi_x \right)_{Q_{T_e}} - \left( (\alpha_2 \sigma_i + \alpha_3) (|h_x| + B), \varphi \right)_{Q_{T_e}} + (h_0, \varphi(0))_{\Omega} = 0.
\]

(1.14)

We then have

**Theorem 1.1** Assume that \( \gamma H \) is sufficiently greater than \( b \), and \( h_0 \in H^1_{\text{per}}(\Omega) \). Then there exists a unique weak solution \( h \) to problem (1.1) – (1.3) with \( B > 0 \), which in addition to (1.13), satisfies

\[
h \in L^\infty(0, T_e; H^1_{\text{per}}(\Omega)), \quad h_x \in L^2(0, T_e; H^1_{\text{per}}(\Omega)) \cap L^3(Q_{T_e}),
\]

(1.15)

\[
h_t \in L^\frac{5}{4}(Q_{T_e}), \quad |h_x|h_x \in L^\frac{4}{3}(0, T_e; W^{1,\frac{4}{3}}_{\text{per}}(\Omega)).
\]

(1.16)

We are also interested in the limit as \( B \to 0 \).

**Definition 1.2** Let \( h_0 \in L^1(\Omega) \). A function \( h \) with

\[
h \in L^2(0, T_e; H^1_{\text{per}}(\Omega))
\]

(1.17)

is called a weak solution to problem (1.1) – (1.3) with \( B = 0 \), if for all \( \varphi \in C^\infty_0((-\infty, T_e) \times \Omega) \), there holds

\[
(h, \varphi_t)_{Q_{T_e}} - \frac{\alpha_1}{2} \left( |h_x| h_x, \varphi_x \right)_{Q_{T_e}} - \left( (\alpha_2 \sigma_i + \alpha_3)|h_x|, \varphi \right)_{Q_{T_e}} + (h_0, \varphi(0))_{\Omega} = 0.
\]

(1.18)

We denote a solution to problem (1.1) – (1.3) by \( h_B \), then \( h_B \) converges \( h \) almost everywhere \((t, x)\) over \( Q_{T_e} \), and \( h \) satisfies (1.18).

**Theorem 1.2** Assume that \( \gamma H \) is sufficiently greater than \( b \), and \( h_0 \in H^1_{\text{per}}(\Omega) \). Then there exists a weak solution \( h \) to problem (1.1) – (1.3) with \( B = 0 \), which in addition to (1.17), satisfies

\[
h \in L^\infty(0, T_e; H^1_{\text{per}}(\Omega)), \quad h_x \in L^3(Q_{T_e}),
\]

(1.19)

\[
h_t \in L^\frac{5}{4}(Q_{T_e}), \quad |h_x|h_x \in L^\frac{4}{3}(0, T_e; W^{1,\frac{4}{3}}_{\text{per}}(\Omega)),
\]

(1.20)

\[
(|h_x|h_x)_t \in L^1(0, T_e; H^{-2}_{\text{per}}(\Omega)).
\]

(1.21)

**Remarks.** 1. In the original units, the assumption that \( \gamma H \) is sufficiently greater than \( b \) means that \( \gamma \gg (b/H)\mu L_0 \), where \( L_0 \) is the length scale of the continuum equation.

2. For the regularity of the solution \( h \), we have a more regular weak solution in the case of \( B > 0 \) than that in the case of \( B = 0 \). This result agrees with the numerical results obtained in Ref. [33] in which sharp corners were developed in \( h \) in the case of \( B = 0 \).

**Notations.** \( C, C(\cdot) \) denote, respectively, universal constants which may vary from line to line. and \( C(\cdot) \) depends on its argument(s). Greek letters \( \varepsilon, \zeta \) are small positive numbers which are normally assumed to be small. \( \kappa \) is taken in \((0, 1]\), which will be sent to zero. \( T_e \) (or \( t_e \)) denotes a positive constant related to time, the life of a solution.
Let $p$, $q$ be real numbers such that $p$, $q \geq 1$. Let $\mathbb{N}$ be the set of natural number and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^d$ be $d$-dimensional Euclidean space.

$\Omega$ denotes an open, bounded, simple-connected domain in $\mathbb{R}^d$ with natural number $d$, with smooth boundary $\partial \Omega$. It represents the material points of a solid body. $Q_t = (0, t) \times \Omega$, and its parabolic boundary $\mathcal{P}Q_t$ is defined by $\mathcal{P}Q_t := (\partial \Omega \times [0, t)) \cup (\Omega \times \{0\})$.

$L^p(\Omega)$ are the Sobolev spaces of $p$-integrable real functions over $\Omega$ endowed with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p \, dx\right)^{\frac{1}{p}}, \text{ if } p < \infty; \quad \|f\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |f(x)|.$$ 

Throughout this article, the norm of $L^2(\Omega)$ is denoted by $\| \cdot \|$, and the norm of $L^2(Q_T)$ is denoted by $\| \cdot \|_{Q_T}$.

Let $\Omega$ be an $n$-dimensional cuboid. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index and $|\alpha|$ be its length, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $D^\alpha f$ is the $|\alpha|$-th order weak derivatives. Define the space $W^{m,p}_{\text{per}}(\Omega) = \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \leq m, \text{ and } \gamma_j f|_{\text{on one face}} = (-1)^j \gamma_j f|_{\text{on the corresponding face}}, \text{ for } j = 0, 1, \cdots, m-1\}$ endowed with norm

$$\|f\|_{W^{m,p}_{\text{per}}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}},$$

where $\gamma_j$ are the trace operators. And $W^{m,p}_{0,\text{per}}(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the norm $\| \cdot \|_{W^{m,p}_{\text{per}}(\Omega)}$. For $p = 2$, $H^{m}_{\text{per}}(\Omega) := W^{m,2}_{\text{per}}(\Omega)$, $H^{-m}_{\text{per}}(\Omega)$ denotes the dual space of $H^{m}_{\text{per}}(\Omega)$. $H^m_0(\Omega) := W^{m,2}_0(\Omega)$.

Let $q, p \in \mathbb{R}$ such that $q, p \geq 1$.

$L^q(0, T; L^p(\Omega)) := \left\{f \mid f \text{ is Lebesgue measurable such that } \|f\|_{L^q(0, T; L^p(\Omega))} := \left(\int_0^T \left(\int_{\Omega} |f|^p \, dx\right)^{\frac{q}{p}} \, d\tau\right)^{\frac{1}{q}} < \infty\right\},$

and $L^q(0, t; W^{m,p}_{\text{per}}(\Omega)) := \{f \in L^q(0, t; L^p(\Omega)) \mid \int_0^t \|f(\cdot, \tau)\|_{W^{m,p}_{\text{per}}}^q \, d\tau < \infty\}$. See, e.g., [18].

We also need some function spaces: For non-negative integers $m, n$, real number $\alpha \in (0, 1)$ we denote by $C^{m+\alpha}(\overline{\Omega})$ the space of $m$-times differentiable functions on $\overline{\Omega}$, whose $m$th derivative is Hölder continuous with exponent $\alpha$. The space $C^{\alpha, \frac{m}{2}}(\overline{Q}_T)$ consists of all functions on $\overline{Q}_T$, which are Hölder continuous in the parabolic distance

$$d((t, x), (s, y)) := \sqrt{|t - s| + |x - y|^2}$$

$C^{m,n}(\overline{Q}_T)$ and $C^{m+\alpha, n+\frac{\alpha}{2}}(\overline{Q}_T)$, respectively, are the spaces of functions, whose $x$-derivatives up to order $m$ and $t$-derivatives up to order $n$ belong to $C(\overline{Q}_T)$ or to $C^{\alpha, \frac{m}{2}}(\overline{Q}_T)$, respectively.

**Organization of rest of this article.** The main results of this article are Theorem 1.1 and Theorem 1.2. The remaining sections are devoted to the proofs of these theorems. In Section 2, we construct an approximate initial-boundary value problem, and prove by employing the Leray-Schauder fixed-point theorem, the existence of classical solutions to this problem. Then we derive in Section 3 a priori estimates which are uniform in a
small parameter, in which we used some results about singular integrals. In Section 4, by making use of the Aubin-Lions lemma and properties of strong convergence, we prove the existence and uniqueness of weak solution to the original initial-boundary value problem when \( B \) is a positive constant. Section 5 is devoted to the study of asymptotic behavior as \( B \) goes to zero, to this end we establish \textit{a priori} estimates that are independent of \( B \), consequently we prove Theorem 1.2.

2 Existence of solutions to the modified problem

To prove Theorem 1.1 we construct the following approximate initial-boundary value problem.

\[
\begin{align*}
\eta - \alpha_1 \left( \int_0^{h_x} |p|_\kappa dp + Bh_x \right)_x &+ (\alpha_2 \sigma^\kappa_i + \alpha_3)(|h_x|_\kappa + B) = 0, \text{ in } Q_T, \\
\eta|_{x=a} & = h|_{x=d}, \quad h|_{x=x} = h|_{x=x}, \text{ on } \partial \Omega \times [0, T_e], \\
\eta(0, x) & = \eta^0(x), \text{ in } \Omega.
\end{align*}
\]

(2.1)

Here \( \kappa > 0 \) is a constant, we used the notation

\[
|p|_\kappa := \sqrt{|p|^2 + \kappa^2}
\]

(2.4)

to replace the function \( |p| \) to smooth the coefficient of the principal term in (2.1) and to guarantee that equation is uniformly parabolic from below. And \( \sigma^\kappa_i = \sigma^\kappa_{i1} + \sigma^\kappa_{i2} \), where \( \sigma^\kappa_{i1} \) is obtained via replacing \( h \) in \( \sigma^\kappa_i \) by a function \( \eta^\kappa_i \in L^2(0, T_e; H^2_{per}(\Omega)) \), and \( \sigma^\kappa_{i2} \) is defined, for this \( \eta^\kappa_i \), by

\[
\sigma^\kappa_{i2} = K\beta \int_{\{y > \kappa\} \cap \Omega} \frac{h^\kappa_i(t, x - y)}{y} dy.
\]

(2.5)

The initial data \( \eta^0_i(x) \) is chosen such that \( \eta^\kappa_i \in C^\infty(\Omega) \) and

\[
\|\eta^\kappa_i - \eta_0\|_{H^1_{per}(\Omega)} \to 0.
\]

We now state the existence of classical solution to problem (2.1) – (2.3) as follows.

**Theorem 2.1** Suppose that \( \gamma H \) is sufficiently greater than \( b \) and the initial data \( \eta^0_i \) satisfies the compatibility conditions \( \eta^0_i(a) = h^0_i(d), \quad \eta^0_i(a) = h^0_i(d), \quad \eta^0_i(a) = h^0_i(d), \) and \( \eta^0_i(0, a) + \alpha_2 \sigma^\kappa_i(0, a)(|\eta^0_{0x}(a)| + B) = h^\kappa_i(0, d) + \alpha_2 \sigma^\kappa_i(0, d)(|\eta^0_{0x}(d)| + B) \). Then there exists a classical solution \( \eta^\kappa_i \) to problem (2.1) – (2.3) such that

\[
\eta^\kappa_i \in L^2(Q_T), \quad \|\eta^\kappa_i\|_{C^{\beta/2,1+\beta}(Q_T)} \leq C_\kappa.
\]

We now present the strategy of the proof of Theorem 2.1. Since there is a non-local term \( \sigma^\kappa_i \) in Eq. (2.1), we shall employ the Leray-Schauder fixed-point theorem to prove this theorem. To this end, we first modify that equation as

\[
\begin{align*}
\eta - \alpha_1 \eta_{xx}(|\eta_x|_\kappa + B) &+ (\alpha_2 \lambda \sigma^\kappa_i + \alpha_3)(|\eta_x|_\kappa + B) = 0, \text{ in } Q_T, \\
\eta|_{x=a} & = h|_{x=d}, \quad \eta|_{x=x} = h|_{x=x}, \text{ on } \partial \Omega \times [0, T_e], \\
\eta(0, x) & = \lambda \eta^0_i(x), \text{ in } \Omega.
\end{align*}
\]

(2.6)
where $\lambda \in [0, 1]$, $\hat{\sigma}^\kappa_i = \hat{\sigma}^\kappa_{i1} + \hat{\sigma}^\kappa_{i2}$, and $\hat{\sigma}^\kappa_{i2}$ is defined by

$$
\hat{\sigma}^\kappa_{i2} = K\beta \int_{\{\|y\|_\kappa\}^\cap \Omega} \frac{\hat{h}_x(t, x-y)}{y} dy.
$$

We take $0 < \alpha < 1$ and define for any $\hat{h} \in B := C^{\alpha/2, 1+\alpha}(\bar{Q}_{t_e})$ a mapping $P_\lambda : [0, 1] \times B \to B$; $\hat{h} \mapsto h$ where $h$ is the solution to problem (2.6) – (2.8), and the existence of solutions to this problem can be found, e.g., in Theorem 4.1, P. 558, Ladyshenskaya et al [18] with slight modifications.

Next we derive a priori estimates which may depend on the parameter $\kappa$. In the rest part of this section, we assume that the conditions in Theorem 2.1 are met, and there exists a unique solution to (2.6) – (2.8), which means that $\hat{\sigma}^\kappa_i$ in (2.6) is replaced by $\sigma^\kappa_i$.

**Lemma 2.1** There holds for any $t \in [0, T_e]$ that

$$
\|h^\kappa(t)\|_{H^1(\Omega)}^2 + \int_0^t \|h^\kappa_{xx}(\tau)\|_2^2 d\tau \leq C_\kappa.
$$

Here $C_\kappa$ is a constant which may depend on $\kappa$.

This estimate is easier to obtain than those in Section 3 that are independent of $\kappa$, so we omit the details of the derivation for most of them. We also need the following estimates.

**Lemma 2.2** There holds for any $t \in [0, T_e]$ that

$$
\|h^\kappa_{xx}(t)\|^2 + \|h^\kappa_x(t)\|^2 + \int_0^t \|h^\kappa_{xx}(\tau)\|^2 d\tau \leq C_\kappa.
$$

This estimate is not necessary for the proof of the existence of weak solutions, thus we give the main idea on deriving it. For the sake of reader’s convenience, we present some tools as follows. First we recall the Gronwall lemma.

**Lemma 2.3 (Gronwall Lemma)** Let $y, A, B$ be functions satisfying that $A(t), B(t)$ are integrable over $[0, T_e]$ and $y(t) \geq 0$ is absolutely continuous function. Then

$$
y'(t) \leq A(t)y(t) + B(t), \text{ for a.e. } t,
$$

implies

$$
y(t) \leq y(0)e^{\int_0^t A(\tau) d\tau} + \int_0^t B(s)e^{\int_s^t A(\tau) d\tau} ds.
$$

**Lemma 2.4 (Aubin-Lions)** Let $B_0$ and $B_2$ be reflexive Banach spaces and let $B_1$ be a Banach space such that $B_0$ is compactly embedded in $B_1$ and that $B_1$ is embedded in $B_2$. For $1 \leq p_0, p_1 \leq +\infty$, define

$$
W = \left\{ f \mid f \in L^{p_0}(0, T_e; B_0), \frac{df}{dt} \in L^{p_1}(0, T_e; B_2) \right\}.
$$

(i) if $p_0 < +\infty$, then the embedding of $W$ into $L^{p_0}(0, T_e; B_1)$ is compact.

(ii) if $p_0 = +\infty$ and $p_1 > 1$, then the embedding of $W$ into $C([0, T_e]; B_1)$ is compact.
In the case that $1 \leq p_0 < \infty$ and $p_1 = 1$, the lemma is also called the **Generalized Aubin-Lions** which plays a crucial role in the investigation of the limit as $B \to 0$ in Section 5. The proof of Lemma 2.4, we refer to, e.g., [20, 24, 25]. We also will use the lemma on Hölder continuity.

**Lemma 2.5** Let $f(t, x)$ be a function, defined over $Q_{t_0}$, such that

(i) $f$ is uniformly (with respect to $x$) Hölder continuous in $t$, with exponent $0 < \alpha \leq 1$, that is $|f(t, x) - f(s, x)| \leq C|t - s|^{\alpha}$, and

(ii) $f_x$ is uniformly (with respect to $t$) Hölder continuous in $x$, with exponent $0 < \beta \leq 1$, that is $|f_x(t, x) - f_x(t, y)| \leq C|y - x|^{\beta}$.

Then $f_x$ is uniformly Hölder continuous in $t$ with exponent $\gamma = \alpha\beta/(1 + \beta)$, such that $|f_x(t, x) - f_x(s, x)| \leq C'|t - s|^{\gamma}$, $\forall x \in \bar{\Omega}$, $0 \leq s \leq t \leq t_0$, where $C''$ is a constant which may depend on $C$, $C'$ and $\alpha, \beta$.

We now turn back to the proof of Lemma 2.2.

**Proof of Lemma 2.2** Differentiating formally Eq. (2.6) (where $\hat{\sigma}^\kappa$ is replaced by $\sigma^\kappa_i$) with respect to $t$, multiplying by $h^\kappa_i$, using integration by parts, and invoking the boundary condition (2.7), we obtain

$$\frac{1}{2} \frac{d}{dt} ||h^\kappa_i(t)||^2 + \alpha_1 \int_{\Omega} (|h^\kappa_{i|x}| + B) |h^\kappa_{i|x}|^2 \, dx = - \int_{\Omega} (\alpha_2 \sigma^\kappa_i(t)|h^\kappa_{i|x}| + B) h^\kappa_i + (\alpha_2 \sigma^\kappa_i + \alpha_3)(|p|) |p = h^\kappa_{i|x}| h^\kappa_i \, dx =: I_1 + I_2. \tag{2.12}$$

Next we are going to estimate $I_1, I_2$. Noting the periodic boundary conditions, by using the Hölder and Young inequalities and the Sobolev embedding theorem, we get

$$|I_1| \leq \frac{K\beta}{\kappa} \left( \int_{\Omega} \left( \int_{\Omega} \alpha_2 |h^\kappa_{i|x}(t, x - y)| \, dy \right)^{\frac{\gamma}{2}} \right)^{\frac{2}{\gamma}} \|h^\kappa_{i|x} + B\|_{L^\infty(\Omega)} \|h^\kappa_i\|$$

$$\leq C\kappa \|h^\kappa_{i|x}\| \left( \|h^\kappa_{i|x} + 1\| \|h^\kappa_i\| \right)$$

$$\leq \varepsilon \|h^\kappa_i\|^2 + C(\|h^\kappa_{i|x}\|^2 + 1) \|h^\kappa_i\|^2. \tag{2.13}$$

To evaluate $I_2$, we invoke the Nirenberg inequality in the following form

$$\|f\|_{L^4} \leq C\|f\|^{\frac{1}{2}} \|f\|^\frac{1}{2} + C' \|f\|, \tag{2.14}$$

where $f$ will be replaced by $h^\kappa_i$ and $h^\kappa_{i|x}$. It is easy to see that $|(|p|)| \leq 1$, hence applying the Young inequality and recalling the estimates (2.10) we arrive at

$$|I_2| \leq C(\|\sigma^\kappa_i\|_{L^1} + 1) \|h^\kappa_{i|x}\| \|h^\kappa_i\|_{L^4}$$

$$\leq C(h^\kappa_{i|x} + 1) \|h^\kappa_i\| \|h^\kappa_i\|_{L^4}$$

$$\leq C(\|h^\kappa_{i|x}\|^\frac{1}{2} \|h^\kappa_i\|^\frac{1}{2} + \|h^\kappa_i\| + 1) \|h^\kappa_i\| \|h^\kappa_i\| \|h^\kappa_i\| \|h^\kappa_i\| \|h^\kappa_i\|$$

$$\leq C(\|h^\kappa_{i|x}\|^\frac{1}{2} + 1) \|h^\kappa_i\| \|h^\kappa_i\| + C\|h^\kappa_i\| \|h^\kappa_i\| \|h^\kappa_i\| + 1) \|h^\kappa_i\|$$

$$\leq \varepsilon \|h^\kappa_i\|^2 + C(\|h^\kappa_{i|x}\|^\frac{1}{2} + 1) \|h^\kappa_i\|^2 + \varepsilon \|h^\kappa_i\|^2 + C(\|h^\kappa_{i|x}\|^2 + 1) \|h^\kappa_i\|^2. \tag{2.15}$$
Combination of (2.12), (2.13) and (2.15) yields
\[
\frac{1}{2} \frac{d}{dt} \|h^e_t(t)\|^2 + \int_{\Omega} \left( \alpha_1 |h^e_{xx}|^\alpha + (\alpha_1 B - 3\varepsilon) \right) \|h^e_{xx}\|^2 dx \\
\leq C \left( \|h^e_{xx}\|^2 + \|h^e_{x}\|^4 + 1 \right) \|h^e_t\|^2. \tag{2.16}
\]

To apply Lemma 2.3, we define \(y(t) = \|h^e_t(t)\|^2\), \(A(t) = C(\|h^e_{xx}\|^2 + \|h^e_{x}\|^4 + 1)\) which is integrable over \([0, t_e]\) by (2.10), and \(B(t) = 0\), then we infer from (2.16) in which we choose \(\varepsilon = \frac{\alpha_1 B}{6}\), that (2.11), except \(\|h^e_{xx}\|^2 \leq C\) (which however can be obtained from equation (2.6) with the help of other estimates in (2.11)), is valid. Thus the proof of Lemma 2.2 is complete.

In what follows we will derive more regularities from (2.10) and (2.11). To this end, we recall Lemma 2.4 and let \(f = h^e_x\),

\[
p_0 = \infty, \quad p_1 = 2, \quad B_0 = H^1_{per}(\Omega) \subset B := C^\alpha_{per}(\Omega), \quad B_1 = L^2(\Omega),
\]

it then follows from (2.10) and (2.11) that

\[
h^e_x \in L^\infty(0, t_e; B_0), \quad \partial h^e_x / \partial t = h^e_{tx} \in L^2(0, t_e; B_1),
\]

and we arrive at

\[
h^e_x \in C([0, t_e]; B) = C([0, t_e]; C^\alpha_{per}(\Omega)). \tag{2.17}
\]

Invoking the Sobolev embedding theorem, we also have

\[
|h^e(t, x) - h^e(s, x)| = \left| \int_s^t h^e_x(\tau, x) d\tau \right| \leq \left| \int_s^t \|h^e_x(\tau)\|_{L^\infty(\Omega)} d\tau \right| \\
\leq \left| \int_s^t \|h^e_x(\tau)\|_{H^1_{per}(\Omega)} d\tau \right| \\
\leq \left( \int_s^t \|h^e_x(\tau)\|^2_{H^1_{per}(\Omega)} d\tau \right)^{\frac{1}{2}} \left( \int_s^t 1 d\tau \right)^{\frac{1}{2}} \\
\leq C|t - s|^{\frac{1}{2}}. \tag{2.18}
\]

Completion of the Proof of Theorem 2.1. Now using (2.18) and (2.17) we may apply Lemma 2.5 to conclude that there exists a positive constant \(0 < \alpha < 1\) such that \(\|h^e_x\|_{C^{\alpha/2, \alpha}} \leq C_\kappa\). By the \textit{a priori} estimate of the Schauder type for parabolic equations, we thus obtain that

\[
\|h^e_x\|_{C^{1+\alpha/2, 2+\alpha}(Q_{t_e})} \leq C_\kappa.
\]

Invoking that \(C^{1+\alpha/2, 2+\alpha}(\tilde{Q}_{t_e}) \subset C^{\alpha/2, 1+\alpha}(\tilde{Q}_{t_e})\), we see the conditions for the Leray-Schauder fixed-point theorem are satisfied. By definition it is easy to see that \(P_0 h \equiv 0\). Thus we are in a position to apply the Leray-Schauder fixed-point theorem (see, e.g., [11]), and assert that \(P_1\) has a fixed point, i.e., \(P_1 h \equiv h\) and this implies that a classical solution exists globally. Hence the proof of Theorem 2.1 is complete.
3 A prior estimates

In this section we are going to derive a prior estimates for solutions to the modified problem (2.1) – (2.3), which are uniform with respect to $\kappa$. Since we shall take the limits of approximate solutions as $\kappa \to 0$, in what follows we may assume that

$$0 < \kappa \leq 1, \quad \gamma H \text{ is sufficiently greater than } b. \quad (3.1)$$

In this section, the letter $C$ stands for various positive constants independent of $\kappa$, but may depend on $B$.

**Lemma 3.1** There hold for any $t \in [0, T_e]$

$$\|h^\kappa(t)\|^2 + \int_0^t \int_\Omega \left( \int_0^{h^\kappa_x} |y|_\kappa dy + Bh^\kappa_x \right) h^\kappa_x dx d\tau \leq C, \quad (3.2)$$

$$\int_0^t \|h^\kappa_x(\tau)\|_{L^3(\Omega)}^3 d\tau \leq C. \quad (3.3)$$

**Proof.** Multiplying Eq. (2.1) by $h^\kappa$, making use of integration by parts, and invoking the boundary condition (2.2), we arrive at

$$\frac{1}{2} \frac{d}{dt} \|h^\kappa(t)\|^2 + \alpha_1 \int_\Omega \left( \int_0^{h^\kappa_x} |y|_\kappa dy + Bh^\kappa_x \right) h^\kappa_x dx$$

$$= - \int_\Omega (\alpha_2 \sigma^\kappa_i + \alpha_3) (|h^\kappa_x|_\kappa + B) h^\kappa_x dx$$

$$= - \int_\Omega \frac{\partial}{\partial x} \int_a^x (\alpha_2 \sigma^\kappa_i + \alpha_3) (|h^\kappa_x|_\kappa + B) dy h^\kappa_x dx$$

$$= \int_a^x (\alpha_2 \sigma^\kappa_i + \alpha_3) (|h^\kappa_x|_\kappa + B) dy h^\kappa_x dx$$

$$=: I. \quad (3.4)$$

For a term in the left-hand side of (3.4) we evaluate it as

$$\int_\Omega \int_0^{h^\kappa_x} |y|_\kappa dy h^\kappa_x dx \geq \int_\Omega \int_0^{h^\kappa_x} |y| dy h^\kappa_x dx$$

$$= \frac{1}{2} \int_\Omega |h^\kappa_x|^3 dx. \quad (3.5)$$

Note that the above inequality is obvious for $h^\kappa_x \geq 0$, otherwise one may replace $h^\kappa_x$ by $-|h^\kappa_x|$ and finds the same inequality. Applying the Young and Hölder inequalities and Theorem A.1, we obtain

$$|I| \leq \int_\Omega |(\alpha_2 \sigma^\kappa_i + \alpha_3) (|h^\kappa_x|_\kappa + B)| dx \int_\Omega |h^\kappa_x| dx$$

$$\leq \|\alpha_2 \sigma^\kappa_i + \alpha_3\|_{L^3(\Omega)} \|h^\kappa_x|_\kappa + B\|_{L^3(\Omega)} \|h^\kappa_x\|_{L^3(\Omega)} \|1\|_{L^3(\Omega)} \|h^\kappa_x\|_{L^3(\Omega)} \|1\|_{L^\frac{2}{3}(\Omega)}$$

$$\leq |\Omega| \left( \alpha_2 \|\sigma^\kappa_i\|_{L^3(\Omega)} + \alpha_3 |\Omega| \right) (\|h^\kappa_x\|_{L^3(\Omega)} + C) \|h^\kappa_x\|_{L^3(\Omega)}. \quad (3.6)$$

Here $|\Omega|$ denotes the measure of $\Omega$. We now estimate the term of $\sigma^\kappa_i$. From estimate (1.9) it follows that

$$\|\sigma_{i1}\|_{L^3(\Omega)} \leq C \|h_x\|_{L^2(\Omega)} \leq C \|h_x\|_{L^3(\Omega)}. \quad (3.7)$$
and from Theorem A.1 we infer that
\[ \| \sigma_i^\kappa \|_{L^3(\Omega)} \leq C \| h_x^\kappa \|_{L^3(\Omega)}. \] (3.8)

Thus we arrive at
\[ \| \sigma_i^\kappa \|_{L^3(\Omega)} \leq \| \sigma_0^\kappa \|_{L^3(\Omega)} + \| \sigma_1^\kappa \|_{L^3(\Omega)} \leq C \| h_x^\kappa \|_{L^3(\Omega)}, \] (3.9)

and hence
\[ |I| \leq |\Omega| \left( \alpha_2 C \| h_x^\kappa \|_{L^3(\Omega)} + \alpha_3 |\Omega| \right) \left( \| h_x^\kappa \|_{L^3(\Omega)} + C \right) \| h_x^\kappa \|_{L^3(\Omega)} \]
\[ \leq C |\Omega| \alpha_2 \| h_x^\kappa \|_{L^3(\Omega)} + \varepsilon \| h_x^\kappa \|_{L^3(\Omega)} + C \varepsilon. \] (3.10)

Combination of inequalities (3.4) and (3.10) yields
\[ \frac{1}{2} \frac{d}{dt} \| h^\kappa(t) \|^2 + \alpha_1 \int_\Omega h_x^\kappa \int_0^{h_x^\kappa} |y|_\kappa dy + \frac{\alpha_1}{4} \int_\Omega |h_x^\kappa|_\kappa^3 dx + B \alpha_1 \int_\Omega |h_x^\kappa|^2 dx \]
\[ \leq \hat{\alpha}_2 \| h_x^\kappa \|_{L^3(\Omega)} + \varepsilon \| h_x^\kappa \|_{L^3(\Omega)} + C \varepsilon, \] (3.11)

which implies
\[ \frac{1}{2} \frac{d}{dt} \| h^\kappa(t) \|^2 + \frac{\alpha_1}{2} \int_\Omega h_x^\kappa \int_0^{h_x^\kappa} |y|_\kappa dy + \left( \frac{\alpha_1}{4} - \hat{\alpha}_2 - \varepsilon \right) \int_\Omega |h_x^\kappa|_\kappa^3 dx + B \alpha_1 \int_\Omega |h_x^\kappa|^2 dx \]
\[ \leq C \varepsilon. \] (3.12)

Here \( \hat{\alpha}_2 := \text{meas}(\Omega)C_3 \alpha_2 \). Therefore choosing that \( \alpha_1 \) is sufficiently greater than \( \alpha_2 \), and \( \varepsilon \) is suitably small, integrating (3.12) with respect to \( t \), we arrive at
\[ \| h^\kappa(t) \|^2 + \int_0^t \int_\Omega \left( h_x^\kappa \int_0^{h_x^\kappa} |y|_\kappa dy + |h_x^\kappa|_\kappa^3 + |h_x^\kappa|^2 \right) dx d\tau \leq C + \| h_0^\kappa \|^2 \leq C. \] (3.13)

Thus the proof of this lemma is complete.

**Lemma 3.2** There holds for any \( t \in [0, T_\varepsilon] \)
\[ \| h_x^\kappa(t) \|^2 + \int_0^t \int_\Omega (|h_x^\kappa|_\kappa + B) |h_x^\kappa|^2 dx d\tau \leq C. \] (3.14)

**Proof.** Multiplying Eq. (2.21) by \(-h_x^\kappa\), employing the technique of integration by parts with respect to \( x \), and invoking the boundary condition (2.2), we obtain formally for almost all \( t \) that
\[ \frac{1}{2} \frac{d}{dt} \| h_x^\kappa(t) \|^2 + \alpha_1 \int_\Omega (|h_x^\kappa|_\kappa + B) |h_x^\kappa|^2 dx = \int_\Omega (\alpha_2 \sigma_i^\kappa + \alpha_3 (|h_x^\kappa|_\kappa + B) h_x^\kappa dx \]
\[ = \int_\Omega \alpha_2 \sigma_i^\kappa (|h_x^\kappa|_\kappa + B) h_x^\kappa dx + \int_\Omega \alpha_3 (|h_x^\kappa|_\kappa + B) h_x^\kappa dx \]
\[ =: I_1 + I_2. \] (3.15)

We may employ the technique of finite difference to justify the formal computation in (3.15). It is quite standard so we omit the details.
Now we treat $I_1$ and $I_2$, and first estimate the easier term $I_2$. Applying the Young inequality with $\varepsilon$, we have

$$
|I_2| = \left| \int_{\Omega} \alpha_3(|h_x^\kappa| + B)^{\frac{1}{2}}(|h_t^\kappa| + B)^{\frac{3}{2}} h_{xx}^\kappa dx \right|
$$

$$
\leq C_\varepsilon \int_{\Omega} (|h_x^\kappa| + \varepsilon + \frac{\varepsilon}{2}) \int_{\Omega} (|h_t^\kappa| + B)|h_{xx}^\kappa|^2 dx
\leq C_\varepsilon \int_{\Omega} (|h_x^\kappa| + \varepsilon + \frac{\varepsilon}{2}) \int_{\Omega} (|h_t^\kappa| + B)|h_{xx}^\kappa|^2 dx
\leq C_\varepsilon \int_{\Omega} (|h_x^\kappa|^2 + C') dx + \frac{\varepsilon}{2} \int_{\Omega} (|h_t^\kappa| + B)|h_{xx}^\kappa|^2 dx. \tag{3.16}
$$

Here we used the simple inequality $|p| \leq |p| + \kappa$. To handle $I_1$, we recall Theorem A.1

$$
|I_1| = \alpha_2 \left| \int_{\Omega} \sigma_i^\kappa (|h_x^\kappa| + B)^{\frac{1}{2}} (|h_t^\kappa| + B)^{\frac{1}{2}} h_{xx}^\kappa dx \right|
$$

$$
\leq \alpha_2 \left( \int_{\Omega} \sigma_i^\kappa |^3 dx \right)^{\frac{1}{3}} \left( \int_{\Omega} (|h_x^\kappa| + B)^{\frac{1}{2}} dx \right)^{\frac{1}{3}} \left( \int_{\Omega} (|h_t^\kappa| + B)^{\frac{3}{2}} h_{xx}^\kappa dx \right)^{\frac{1}{3}}
\leq \alpha_2 \|h_x^\kappa\|_{L^3(\Omega)}^{1+\frac{1}{2}} \left( \int_{\Omega} (|h_x^\kappa| + B)^{\frac{1}{2}} dx \right)^{\frac{1}{3}} \left( \int_{\Omega} (|h_t^\kappa| + B)^{\frac{3}{2}} h_{xx}^\kappa dx \right)^{\frac{1}{3}}
\leq C_\varepsilon \|h_x^\kappa\|_{L^3(\Omega)}^3 + \frac{\varepsilon}{2} \int_{\Omega} (|h_t^\kappa| + B)|h_{xx}^\kappa|^2 dx. \tag{3.17}
$$

Combining (3.15) with (3.16) and (3.17), integrating it with respect to $t$, and making use of the Young inequality, we then arrive at

$$
\frac{1}{2} \|h_x^\kappa(t)\|^2 + \alpha_1 \int_0^t \int_{\Omega} (|h_x^\kappa| + B)|h_{xx}^\kappa|^2 dx \tau
\leq C_\varepsilon \int_0^t \|h_x^\kappa\|_{L^3(\Omega)}^3 d\tau + \varepsilon \int_0^t \int_{\Omega} (|h_t^\kappa| + B)|h_{xx}^\kappa|^2 dx \tau. \tag{3.18}
$$

Now we choose $\varepsilon$ small enough so that $\alpha_1 - \varepsilon > 0$, recall the estimates in Lemma 5.1, then

$$
\frac{1}{2} \|h_x^\kappa(t)\|^2 + (\alpha_1 - \varepsilon) \int_0^t \int_{\Omega} (|h_x^\kappa| + B)|h_{xx}^\kappa|^2 dx \tau
\leq C_\varepsilon \int_0^t \|h_x^\kappa\|_{L^3(\Omega)}^3 d\tau + C
\leq C. \tag{3.19}
$$

Therefore, the proof of Lemma 5.2 is complete.
Corollary 3.1 There hold for any \( t \in [0, T_e] \)
\[
\int_0^t \int_\Omega (|h_x^\kappa| |h_{xx}^\kappa|) \frac{1}{2} dx d\tau \leq C, \tag{3.20}
\]
\[
\int_0^t \int_\Omega (|h_x^\kappa h_{xx}^\kappa|) \frac{3}{2} dx d\tau \leq C, \tag{3.21}
\]
\[
\int_0^t \| (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq C, \tag{3.22}
\]
\[
\int_0^t \| h_x^\kappa \|_{L^\infty(\Omega)}^\frac{4}{3} d\tau \leq C. \tag{3.23}
\]

Proof. By Hölder’s inequality, we have for some \( 1 \leq p < 2, q = \frac{2}{p} \) and \( \frac{1}{q} + \frac{1}{q} = 1 \) that
\[
\int_0^t \int_\Omega |h_x^\kappa|^p dx d\tau = \int_0^t \int_\Omega |h_x^\kappa| \left( |h_x^\kappa| |h_{xx}^\kappa| \right)^p dx d\tau
\leq \left( \int_0^t \int_\Omega |h_x^\kappa|^{\frac{2p}{p-1}} dx d\tau \right)^{\frac{p-1}{p}} \left( \int_0^t \int_\Omega |h_x^\kappa|^{\frac{pq}{p-1}} |h_{xx}^\kappa|^{\frac{pq}{p-1}} dx d\tau \right)^{\frac{p-1}{q}}  \leq \left( \int_0^t \int_\Omega |h_x^\kappa|^{\frac{2p}{p-1}} dx d\tau \right)^{\frac{p-1}{p}} \left( \int_0^t \int_\Omega |h_x^\kappa|^{\frac{pq}{p-1}} dx d\tau \right)^{\frac{p-1}{q}}. \tag{3.24}
\]

Inequality (3.24) implies for \( \frac{p}{p-1} \leq 2 \), i.e., \( p \leq \frac{4}{3} \), that the right-hand side of (3.24) is bounded, hence (3.24) is true.

Invoking the basic fact that \(|p|_\kappa \geq |p|\), from (3.20) it follows that (3.21) holds. To prove (3.23), we apply the Poincaré inequality in the following form
\[
\| f - \bar{f} \|_{L^p(\Omega)} \leq C \| f_x \|_{L^q(\Omega)},
\]
where \( \bar{f} = \int_\Omega f(x) dx / |\Omega| \), choose \( p = \frac{4}{3} \), then from (3.21) we deduce that
\[
\| (h_x^\kappa)^2 - (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)} \leq \| (h_x^\kappa)^2 - (h_x^\kappa)^2 \|_{x} \| (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)} = 2 \| h_x^\kappa h_{xx}^\kappa \|_{W^{1,\frac{4}{3}}(\Omega)} \tag{3.25}
\]
hence
\[
\int_0^t \| (h_x^\kappa)^2 - (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq 2 \int_0^t \| h_x^\kappa h_{xx}^\kappa \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq C, \tag{3.26}
\]
which implies
\[
\int_0^t \| (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq \int_0^t \| (h_x^\kappa)^2 - (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau + \int_0^t \| (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq C + \int_0^t \| (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq C + \sup_{0 \leq \tau \leq t} \| (h_x^\kappa)^2 \|_{W^{1,\frac{4}{3}}(\Omega)} \int_0^t \| 1 \|_{L^{\frac{4}{3}}(\Omega)}^\frac{4}{3} d\tau \leq C. \tag{3.27}
\]

By the Sobolev embedding theorem we have \( W^{1,\frac{4}{3}}(\Omega) \subset L^\infty(\Omega) \), and (3.22) follows. Hence the proof of the corollary is complete.
Lemma 3.3 There holds for any \( t \in [0, T_e] \)

\[
\|h^\kappa_t\|_{L^4(Q_{T_e})} \leq C. \tag{3.28}
\]

Proof. Recalling the regularity about \( h^\kappa_t \) we use the integration by parts and obtain

\[
(h^\kappa_t, \varphi_t)_{Q_{T_e}} = \int_0^{T_e} \frac{d}{dt}(h^\kappa, \varphi)_{\Omega} dt - (h^\kappa_t, \varphi)_{Q_{T_e}} = (h^\kappa, \varphi)|_{t=0} - (h^\kappa_t, \varphi)_{Q_{T_e}} = -(h^\kappa(0), \varphi(0))_{\Omega} - (h^\kappa_t, \varphi)_{Q_{T_e}} \tag{3.29}
\]

thus one has

\[
(h^\kappa_t, \varphi)_{Q_{T_e}} \leq C \left( (\alpha_1 h^\kappa_{xx} - (\alpha_2 \sigma^\kappa_i + \alpha_3))(|h^\kappa_x| + B) \right)_{Q_{T_e}} \tag{3.30}
\]

Making use of Theorem [A.1] with \( p = \frac{4}{3} \), and estimates [3.20], [3.23], and the Hölder inequality, we have

\[
\| (h^\kappa_t, \varphi)_{Q_{T_e}} \| \leq C \left( \| h^\kappa_x \|_{L^4(Q_{T_e})} + \| h^\kappa_{xx} \|_{L^4(Q_{T_e})} \right) \| \varphi \|_{L^4(Q_{T_e})} + C \left( \| \sigma^\kappa_i \|_{L^4(Q_{T_e})} + 1 \right) \| \varphi \|_{L^4(Q_{T_e})} \leq C \left( 1 + \| h^\kappa_x \|_{L^4(Q_{T_e})} \right) \| \varphi \|_{L^4(Q_{T_e})} \]

for all \( \varphi \in L^4(Q_{T_e}) \). Thus [3.31] implies that \( h^\kappa_t \in L^4(Q_{T_e}) \) and [3.28] holds. The proof of the lemma is complete.

4 Existence of solutions to the original IBVP

In this section we shall use the \textit{a priori} estimates established in Section 2.2 to investigate the convergence of \( h^\kappa \) as \( \kappa \to 0 \), and show that there exists a subsequence, which converges to a weak solution to the initial-boundary value problem \( 1.1 - 1.3 \), thereby prove Theorem [1.1].

Lemma 4.1 There exists a subsequence of \( h^\kappa_x \) (we still denote it by \( h^\kappa_x \)) such that

\[
\begin{align*}
    h^\kappa_x & \to h_x \text{ strongly in } L^2(Q_{T_e}), \tag{4.1} \\
    |h^\kappa_x| & \to |h_x| \text{ strongly in } L^2(Q_{T_e}), \tag{4.2} \\
    |h^\kappa_x|_0, h^\kappa_x & \to |h_x| h_x \text{ strongly in } L^1(Q_{T_e}) \tag{4.3}
\end{align*}
\]

as \( \kappa \to 0 \).

Proof. Let \( p_0 = 2, p_1 = \frac{4}{3} \) and

\[
B_0 = H^1_{\text{per}}(\Omega), \quad B_1 = L^2(\Omega), \quad B_2 = W^{-1,\frac{4}{3}}_{\text{per}}(\Omega).
\]

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These spaces satisfy the assumptions of Lemma 2.4. Since estimate (3.14) implies that $h^e_{xx} \in L^2(0,T_e;L^2(\Omega))$, then

$$h^e_x \in L^2(0,T_e;H^1_{\operatorname{per}}(\Omega)).$$

From estimate (3.28), we have

$$h^e_{xt} \in L^4(0,T_e;W^{-1,4}_{\operatorname{per}}).$$

It follows from Theorem 2.4 that

$$h^e_x \rightarrow h_x \text{ strongly in } L^2(Q_{T_e}),$$

as $\kappa \rightarrow 0$. This proves (4.1).

It is easy to see that

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$$

for all $x, y \in \mathbb{R}^+$. From this we deduce that as $\kappa \rightarrow 0$

$$\left\|h^e_x|\kappa - h_x\right\|^2_{L^2(Q_{T_e})} \leq \int_{Q_{T_e}} \left(\left|h^e_x\right|^2 - (h_x)^2\right) \, dx \, dt \leq \int_{Q_{T_e}} \left(\left|h^e_x\right|^2 - (h_x)^2 + \kappa^2\right) \, dx \, dt$$

$$\leq \left\|h^e_x - h_x\right\|_{L^2(Q_{T_e})} \left\|h^e_x - h_x\right\|_{L^2(Q_{T_e})} + \left\|\kappa^2\right\|_{L^2(Q_{T_e})} \rightarrow 0. \tag{4.6}$$

From this we infer that $|h^e_x|\kappa$ converges to $|h_x|$ strongly in $L^2(Q_{T_e})$ as $\kappa \rightarrow 0$. This proves (4.2). Combining (4.1) and (4.2), we get (4.3) immediately.

**Proof of Theorem 1.1.** We have $\|h^e\|_{L^\infty(0,T_e;H^1_{\operatorname{per}}(\Omega))} \leq C$, and $\|h^e\|_{L^2(0,T_e;H^\alpha_{\operatorname{per}}(\Omega))} \leq C$ by (3.14). This implies $h \in L^\infty(0,T_e;H^1_{\operatorname{per}}(\Omega)) \cap L^2(0,T_e;H^2_{\operatorname{per}}(\Omega))$, since we can select a subsequence of $h^e$ which converges weakly to $h$ in this space. Thus, $h$ satisfies (1.13).

It therefore suffices to show that problem (1.1) – (1.3) is fulfilled in the weak sense which means we need to prove the relation (1.14) holds. To this end, we employ the following equality

$$(h^e, \varphi_t)_{Q_{T_e}} - \alpha_3 \left(\int_0^{h^e} |p_x| \, dp + Bh^e_x, \varphi_x\right)_{Q_{T_e}} - ((\alpha_2\sigma^e_t + \alpha_3)(|h^e_x| + B), \varphi)_{Q_{T_e}} + (h_0, \varphi(0))_{\Omega} = 0. \tag{4.7}$$

From which we see that equation (1.14) follows if we show that

$$(h^e, \varphi_t)_{Q_{T_e}} \rightarrow (h, \varphi_t)_{Q_{T_e}}, \tag{4.8}$$

$$\left(\int_0^{h^e} \left|\varphi\right|_{\kappa}, \varphi_x\right)_{Q_{T_e}} \rightarrow \left(\frac{1}{2}|h_x|_{\kappa}, \varphi_x\right)_{Q_{T_e}}, \tag{4.9}$$

$$\left(|h^e_x|_{\kappa}, \varphi\right)_{Q_{T_e}} \rightarrow \left(|h_x|, \varphi\right)_{Q_{T_e}}, \tag{4.10}$$

$$\left(h^e_x, \varphi_x\right)_{Q_{T_e}} \rightarrow \left(h_x, \varphi_x\right)_{Q_{T_e}}, \tag{4.11}$$

$$\left(\sigma^e_t|h^e_x|_{\kappa}, \varphi\right)_{Q_{T_e}} \rightarrow \left(\sigma_t|h_x|, \varphi\right)_{Q_{T_e}}. \tag{4.12}$$
as $\kappa \to 0$. Now, the conclusions (4.8) and (4.11) follow easily from (3.14), and the relation (4.10) follows from (1.2). It remains to prove (4.9) and (4.12). To prove (4.9) we write

$$\int_0^{h^-_x} |y|\kappa dy - \frac{1}{2}|h_x|h_x = \left( \int_0^{h^-_x} |y|\kappa dy - \frac{1}{2}|h^+_x|\kappa h^+_x \right) + \frac{1}{2}(|h^+_x|\kappa h^+_x - |h_x|h_x)$$

$$:= I_1 + I_2.$$  \hfill (4.13)

The conclusion (4.3) implies

$$\|I_2\|_{L^1(Q_{T\kappa})} \to 0,$$  \hfill (4.14)

as $\kappa \to 0$. Next we handle $I_1$ as follows.

$$|I_1| = \left| \int_0^{h^-_x} |y|\kappa dy - \frac{1}{2}|h^+_x|\kappa h^+_x \right| \leq \left| \int_0^{h^-_x} |y|\kappa dy - \int_0^{h^-_x} |y|dy \right|$$

$$\leq \int_0^{h^-_x} |y|\kappa - |y|dy$$

$$\leq \int_0^{h^-_x} \kappa dy = \kappa|h^+_x|,$$  \hfill (4.15)

whence (3.14) implies

$$\|I_1\|_{L^1(Q_{T\kappa})} \leq C\|I_1\|_{L^\infty(0, T\kappa; L^2(\Omega))}$$

$$\leq C\|I_1\|_{L^\infty(0, T\kappa; H^1(\Omega))} \leq C\kappa \to 0,$$  \hfill (4.16)

as $\kappa \to 0$. From this relation and (4.13), (4.14) we obtain

$$\left\| \int_0^{h^-_x} |y|\kappa dy - \frac{1}{2}|h_x|h_x \right\|_{L^1(Q_{T\kappa})} \to 0,$$  \hfill (4.17)

which implies (4.9). Finally we prove (4.12). Applying the compactness lemma and Theorem A.1 with $p = 2$ we get that

$$\sigma^\kappa \to \sigma_i \text{ strongly in } L^2(Q_{T\kappa})$$  \hfill (4.18)

where

$$\sigma_i(t, x) = P.V. \int_{-\infty}^{\infty} \frac{K\beta h_x(t, x_1)}{x - x_1} dx_1.$$  

Then recalling (4.2) one concludes that

$$\sigma^\kappa|\kappa h^+_x| \to \sigma_i|\kappa x| \text{ strongly in } L^1(Q_{T\kappa}),$$  \hfill (4.19)

which implies (4.12). Thus (1.14) holds.

We now investigate the regularity properties of the solution stated in (1.15) and (1.16). For (1.15), we apply the estimates in Lemmas 3.1, 3.2 and the relation (4.4). The assertion $h_t \in L^4(Q_{T\kappa})$ is implied by (3.23). To verify the second assertion in (1.16), we use estimates (3.21) and (3.23) in Corollary 3.1, and also the strong convergence result in Lemma 4.1. Consequently (1.16) holds.
It remains to prove the uniqueness. To this end, we recall the regularity of \( h_t \), and definition (1.14), using integration by parts, to get
\[
-(h_t, \varphi)_{Q_T} - \alpha_1 \left( \frac{1}{2} h_x |h_x| + Bh_x, \varphi_x \right)_{Q_T} - ((\alpha_2 \sigma_i + \alpha_3)(|h_x| + B), \varphi)_{Q_T} = -(h(0), \varphi(0))_\Omega + (h_0, \varphi(0))_\Omega = 0. \tag{4.20}
\]
Suppose that there exist two solutions \( h_1, h_2 \), let \( h = h_1 - h_2 \), then from (4.20) we infer that
\[
(h_t, \varphi)_{Q_T} + \alpha_1 \left( h_1 x |h_1 x| - h_2 x |h_2 x|, \varphi_x \right)_{Q_T} + \alpha_1 (Bh, \varphi)_{Q_T} + \left((\alpha_2 \sigma_i^1 + \alpha_3)(|h_1 x| + B) - (\alpha_2 \sigma_i^2 + \alpha_3)(|h_2 x| + B), \varphi\right)_{Q_T} = 0. \tag{4.21}
\]
Here \( \sigma_i^j \) with \( j = 1, 2 \) stand for the formulas of \( \sigma_i \) in which \( h \) is replaced by \( h^j \), respectively. Since \( C^\infty_0(Q_t) \) is dense in \( L^2(Q_t) \), we can choose \( \varphi = h \), using the monotonicity property
\[
(x|x| - y|y|)(x - y) \geq 0
\]
to infer from (4.21) that
\[
\frac{1}{2} ||h(t)||^2 + \alpha_1 B ||h_x||_{Q_T}^2 + \left((\alpha_2 \sigma_i^1 + \alpha_3)(|h_1 x| + B) - (\alpha_2 \sigma_i^2 + \alpha_3)(|h_2 x| + B), h\right)_{Q_T} \leq \frac{1}{2} ||h(0)||^2 = 0. \tag{4.22}
\]
We write
\[
I := \left((\alpha_2 \sigma_i^1 + \alpha_3)(|h_1 x| + B) - (\alpha_2 \sigma_i^2 + \alpha_3)(|h_2 x| + B), h\right)_{Q_T} = \left(\int_a^x (\alpha_2 \sigma_i^1 + \alpha_3)(|h_1 x| - |h_2 x|) + \alpha_2 (\sigma_i^1 - \sigma_i^2)(|h_2 x| + B)dy, h_x\right)_{Q_T}, \tag{4.23}
\]
whence applying again Theorem A.1 and the Hölder inequality, we obtain
\[
|I| \leq C \int_0^t \int_a^d ((\sigma_i^1 + 1)|h_x| + |\sigma_i^1 - \sigma_i^2||(|h_2 x| + B)) dy ||h_x||d\tau \\
\leq C \int_0^t ((\sigma_i^1 + 1)||h_x|| + ||\sigma_i^1 - \sigma_i^2||(|h_2 x| + B)) ||h_x||d\tau \\
\leq C' \int_0^t (((h_1 x|) + ||h_2 x|| + 1)) ||h_x||^2d\tau \\
\leq C' \int_0^t ||h_x||^2d\tau. \tag{4.24}
\]
Now choosing \( \alpha_1 \) sufficiently large, we infer from (4.24) and (4.22) that
\[
\frac{1}{2} ||h(t)||^2 + (\alpha_1 B - C')||h_x||_{Q_T}^2 \leq 0,
\]
hence \( ||h(t)|| = 0 \) which implies that \( h = 0 \) for almost all \( (t, x) \in Q_T \), the uniqueness follows, and thus the proof of Theorem 1.1 is complete.
5 The limit of $h_B$ as $B$ vanishes

This section is devoted to the investigation the limit of $h_B$ as $B \to 0$, and to the proof of Theorem 1.2. We will denote the solution $h$ to problem (1.1) – (1.3) by $h_B$. Thus we need a priori estimates which are independent of $B$ and $B$ may be taken to meet

$$0 < B \leq 1.$$

Those estimates in Lemmas 3.1 and 3.2 and Corollary 3.1 are of this type. In this section a universal constant $C$ is independent of $B$.

To prove Theorem 1.2 we shall obtain more estimates as follows.

Lemma 5.1 There hold for any $t \in [0, T_e]$ and for any $\phi \in L^\infty(0, T_e; H^2_{\text{per}}(\Omega))$ that

$$
\begin{align*}
| \langle (h^\kappa_B \cdot h^\kappa_B)_t, \phi \rangle | &\leq C \| \phi \|_{L^\infty(0, T_e; H^2_{\text{per}}(\Omega))}, \\
\| (h^\kappa_B \cdot h^\kappa_B)_t \|_{L^1(0, t; L^2_{\text{per}}(\Omega))} &\leq C.
\end{align*}
$$

(5.1) (5.2)

Proof. For a rigorous procedure, we derive estimate (5.1) from Eq. (2.6), where $\hat{\sigma}^\kappa_i$ is replaced by $\sigma^\kappa_i$, thus we see the solution $h$ depends on both $\kappa$ and $B$. We shall write $h = h^\kappa_B(t, x)$. However as in Section 4 one can pass $h^\kappa_B$ to its limit as $\kappa \to 0$, and get solutions $h_B$, and hence we get estimates for $h_B$. To this end, we take an arbitrary $\phi \in L^\infty(0, t; H^2_{\text{per}}(\Omega))$, multiply $h^\kappa_B$ by $(h^\kappa_B \phi)_x$, and integrate the production with respect to $x, t$, and arrive at

$$
I := 2 \int_0^t \int_\Omega (h^\kappa_B)(h^\kappa_B \phi)_x dx d\tau = 2 \int_0^t \int_\Omega h^\kappa_B \left( \frac{h^\kappa_B h^\kappa_B}{|h^\kappa_B|^2} \phi + |h^\kappa_B| \phi_x \right) dx d\tau
=: I_1 + I_2.
$$

(5.3)

Here $|h^\kappa_B|^2 = \sqrt{|h^\kappa_B|^2 + \delta^2}$ with a small positive parameter $\delta \leq 1$. For the sake of notations’ simplicity, we still denote $h = h^\kappa_B$ by $h = h^\kappa$.

We now treat $I_1$ and $I_2$. First for $I_1$ we invoke Eq. (2.6) (where $\hat{\sigma}^\kappa_i$ is replaced by $\sigma^\kappa_i$), to get

$$
|I_1| \leq 2 \int_0^t \int_\Omega h^\kappa_B h^\kappa_B |h^\kappa_B| \phi dx \leq C \int_0^t \int_\Omega |h^\kappa_B h^\kappa_B \phi| dx d\tau
\leq C \int_0^t \int_\Omega \left( (|h^\kappa_B|^B + B)|h^\kappa_B|^2 |\phi| + (|\sigma^\kappa_i| + 1)(|h^\kappa_B|^B + B)|h^\kappa_B \phi| \right) dx d\tau
=: I_{11} + I_{12}.
$$

(5.4)

By using the estimates in Lemmas 3.1 and 3.2 and Corollary 3.1 one gets easily

$$
|I_{11}| \leq C \| \phi \|_{L^\infty(Q_t)} \int_0^t \int_\Omega (|h^\kappa_B|^B + B)|h^\kappa_B|^2 dx d\tau
\leq C \| \phi \|_{L^\infty(Q_t)} \leq C \| \phi \|_{L^\infty(0, t; H^0_{\text{per}}(\Omega))},
$$

(5.5)

By using the estimates in Lemmas 3.1 and 3.2 and Corollary 3.1 one gets easily
Using the estimates in Lemmas 3.1 and 3.2, we obtain

\[ |I_{12}| \leq C \| \phi \|_{L^\infty(Q_t)} \int_0^t \left( (\| \sigma_x^\kappa \|_{L^3(\Omega)} + 1) \left( \| h_x^\kappa \|_{L^2(\Omega)} + B \right) \| h_x^\kappa \|_{L^2(\Omega)} \right) d\tau \]

\[ \leq C \| \phi \|_{L^\infty(0,t;H^2_{per}(\Omega))} \int_0^t \left( \left( \| h_x^\kappa \|_{L^3(\Omega)} \right)^2 + (\| h_x^\kappa \|_{L^2(\Omega)} + B) \| h_x^\kappa \|_{L^2(\Omega)} \right) d\tau \]

\[ \leq C \| \phi \|_{L^\infty(0,t;H^2_{per}(\Omega))}. \quad (5.6) \]

Next \( I_2 \) is evaluated as follows.

\[ |I_2| \leq C \int_0^t \int_\Omega |h_x^\kappa| |h_x^\kappa| |h_x^\kappa| |\phi_x| dx d\tau \]

\[ \leq C \int_0^t \int_\Omega \left( (|h_x^\kappa| + B)|h_x^\kappa| + (|\sigma_x^\kappa| + 1)(|h_x^\kappa| + B) \right) |h_x^\kappa| |\phi_x| dx d\tau \]

\[ \leq C \| \phi_x \|_{L^\infty(\Omega)} \int_0^t \int_\Omega \left( (|h_x^\kappa| + B)|h_x^\kappa| + (|\sigma_x^\kappa| + 1)(|h_x^\kappa| + B) \right) (|h_x^\kappa| + 1) dx d\tau \]

\[ := I_{21} + I_{22}. \quad (5.7) \]

Using the estimates in Lemmas 3.1 and 3.2, we obtain

\[ |I_{21}| \leq C \| \phi_x \|_{L^\infty(\Omega)} \int_0^t \int_\Omega \left( (|h_x^\kappa| + B)\frac{3}{2} |h_x^\kappa| + (|h_x^\kappa| + 1) \right)^2 dx d\tau \]

\[ \leq C \| \phi_x \|_{L^\infty(\Omega)} \int_0^t \int_\Omega \left( (|h_x^\kappa| + B)|h_x^\kappa| + (|\sigma_x^\kappa| + 1) \right)^3 \right) dx d\tau \]

\[ \leq C \| \phi \|_{L^\infty(0,t;H^2_{per}(\Omega))}. \quad (5.8) \]

and from the estimates in Corollary 3.1 it follows that

\[ |I_{22}| \leq C \| \phi_x \|_{L^\infty(\Omega)} \int_0^t \int_\Omega \left( (|\sigma_x^\kappa| + 1)(|h_x^\kappa|)^2 + 1 \right) dx d\tau \]

\[ \leq C \| \phi \|_{L^\infty(0,t;H^2_{per}(\Omega))} \int_0^t \int_\Omega \left( (|\sigma_x^\kappa| + 1)^3 + (|h_x^\kappa|)^2 + 1 \right)^2 dx d\tau \]

\[ \leq C \| \phi \|_{L^\infty(0,t;H^2_{per}(\Omega))}. \quad (5.9) \]

Therefore combining (5.3) - (5.9) together we arrive at

\[ |I| \leq C \| \phi \|_{L^\infty(0,t;H^2_{per}(\Omega))}. \quad (5.10) \]

We now rewrite \( I \) as

\[ I := \int_0^t \int_\Omega h_x^\kappa (|h_x^\kappa| \phi)_x dx d\tau \]

\[ \rightarrow -2 \int_0^t \int_\Omega h_x^\kappa |h_x^\kappa| \phi dx d\tau \] as \( \delta \to 0 \)

\[ = - \int_0^t \int_\Omega (|h_x^\kappa| h_x^\kappa)_t \phi dx d\tau \]

\[ = -(\langle (|h_x^\kappa| h_x^\kappa)_t, \phi \rangle_{Q_t}). \quad (5.11) \]
It then follows from (5.11) and (5.10) that
\[
\left| \left( (h^n_x | h^n_x)_t, \phi \right)_{Q_t} \right| = |I| \leq C \| \phi \|_{L^\infty(0, t; H^2_{\text{per}}(\Omega))}.
\] (5.12)

Since \( L^1(0, t; H^2_{\text{per}}(\Omega)) \) is isometrically imbedded into the dual space of \( L^\infty(0, t; H^2_{\text{per}}(\Omega)) \), we complete the proof of Lemma 5.1.

We are going to study the asymptotic behavior of solution \( h_B \) as \( B \) goes to zero. For this purpose we also need the following lemma.

**Lemma 5.2** Let \( (0, T_e) \times \Omega \) be an open set in \( \mathbb{R}^+ \times \mathbb{R}^n \). Suppose functions \( g_n, g \) are in \( L^q((0, T_e) \times \Omega) \) for any given \( 1 < q < \infty \), which satisfy
\[
\| g_n \|_{L^q((0, T_e) \times \Omega)} \leq C, \quad g_n \to g \quad \text{a.e. in} \quad (0, T_e) \times \Omega.
\]

Then \( g_n \) converges to \( g \) weakly in \( L^q((0, T_e) \times \Omega) \).

**Proof of Theorem 1.2.** Applying now the generalized case \((p_1 = 1)\), of Aubin-Lions Lemma, i.e., Lemma 2.3 to the series \( (h_B)_x \), we assert from Lemma 5.1 and Corollary 3.1 that
\[
|h_B|_x = L^{\frac{4}{3}}((0, T_e); W^{1, \frac{4}{3}}_{\text{per}}(\Omega)); \quad (h_B)_x \in L^1((0, T_e); H^{-2}_{\text{per}}(\Omega)).
\]

This suggests us to choose
\[
p_0 = \frac{4}{3}, \quad p_1 = 1; \quad B_0 = W^{1, \frac{4}{3}}_{\text{per}}(\Omega), \quad B = L^2(\Omega), \quad B_1 = H^{-2}_{\text{per}}(\Omega).
\]

We thus have \( B_0 \subset B \),
\[
|h_B|_x \in L^{p_0}(0, T_e; B_0); \quad (h_B)_x \in L^1((0, T_e); B_1),
\]
and conclude that \( (h_B)_x \) is compact in \( L^{\frac{4}{3}}((0, T_e); B) \). Hence we can select a subsequence, denoted by \( h_{B_n} \), of \( h_B \), such that
\[
|h_{B_n}|_x \to \chi, \quad a.e., \quad (t, x) \in Q_{T_e},
\]
\( B_n \to 0, \) as \( n \to \infty \). It is easy to see that the function \( F : y \mapsto |y|y \) is reversible, we obtain \( (h_{B_n})_x \to F^{-1}(\chi) \) as \( n \to 0 \). By uniqueness of weak limit, we assert that \( F^{-1}(\chi) = h_x \).

Recalling that \( h_B \) satisfies
\[
(h_B, \varphi)_Q - \frac{\alpha_1}{2} \left( (h_B)_x (h_B)_x + 2B(h_B)_x, \varphi_x \right)_{Q_t} = \left( (\alpha_2 \sigma_i + \alpha_3)(|h_B|_x + B), \varphi \right)_{Q_{T_e}} - (h_0, \varphi(0))_\Omega,
\] (5.13)

we need only study the limits of the most difficult terms, i.e., the nonlinear terms like \( (h_B)_x (h_B)_x, \phi)_Q \).

Employing Lemma 5.2, we can easily pass the nonlinear terms to their limits. Thus \( h \) is a solution, in the sense of Definition 1.2 to problem (1.1) – (1.3) with \( B = 0 \). And the proof of Theorem 1.2 is thus complete.
\section*{A Singular integrals}

For the sake of the readers’ convenience, we include the following theorem on the boundedness of singular integrals.

\textbf{Theorem A.1 (p. 48, Ref. \[26\])} Let \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^n \). Suppose that the kernel \( K(x) \) satisfies the conditions

\begin{equation}
|K(x)| \leq C|x|^{-n}, \quad \text{for all } |x| > 0,
\end{equation}

and

\begin{equation}
\int_{\{|x| \geq 2|y|\}} |K(x - y) - K(x)| \, dx \leq C, \quad \text{for all } |y| > 0,
\end{equation}

where \( C \) is a positive constant. Let \( 1 < p < \infty \), for \( f \in L^p(\mathbb{R}^n) \) we define

\begin{equation}
T_\varepsilon(f)(x) = \int_{\{|y| \geq \varepsilon\}} f(x - y)K(y) \, dy, \quad \varepsilon > 0.
\end{equation}

Then there holds

\begin{equation}
\|T_\varepsilon(f)\|_p \leq C_p \|f\|_p
\end{equation}

where \( C_p \) is a constant that is independent of \( f \) and \( \varepsilon \). Also for each \( f \in L^p(\mathbb{R}^n) \), \( \lim_{\varepsilon \to 0} T_\varepsilon(f) = T(f) \) exists in \( L^p \) norm. The operator \( T \) so defined also satisfies the inequality \( \text{(A.5)} \).

The cancellation property alluded to is contained in condition \( \text{(A.3)} \). This hypothesis, together with \( \text{(A.1), (A.2)} \), allows us to prove the \( L^2 \) boundedness and from this the \( L^p \) convergence of the truncated integrals \( \text{(A.4)} \).

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