CENTRAL LIMIT THEOREMS FOR MULTIVARIATE BESSEL PROCESSES IN THE FREEZING REGIME II: THE COVARIANCE MATRICES

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Abstract. Bessel processes \((X_{t,k})_{t \geq 0}\) in \(N\) dimensions are classified via associated root systems and multiplicity constants \(k \geq 0\). They describe interacting Calogero-Moser-Sutherland particle systems with \(N\) particles and are related to \(\beta\)-Hermite and \(\beta\)-Laguerre ensembles. Recently, several central limit theorems were derived for fixed \(t > 0\), fixed starting points, and \(k \to \infty\). In this paper we extend the CLT in the A-case from start in 0 to arbitrary starting distributions by using a limit result for the corresponding Bessel functions. We also determine the eigenvalues and eigenvectors of the covariance matrices of the Gaussian limits and study applications to CLTs for the intermediate particles for \(k \to \infty\) and then \(N \to \infty\).

1. Introduction

Integrable interacting particle systems of Calogero-Moser-Sutherland type on \(\mathbb{R}\) with \(N\) particles are described by multivariate Bessel processes on closed Weyl chambers in \(\mathbb{R}^N\). These processes are classified via root systems and a finite number of multiplicity parameters \(k\) which govern the interactions; see [CGY], [R], [RV1], [RV2], [DF], [DV] and references therein. Recently, several limit theorems were derived for these processes when one or several multiplicity parameters \(k\) tend to infinity; see [AKM1], [AKM2], [AV], [V], and [VW]. In particular, [V] contains central limit theorems for the root systems \(A_{N-1}\), \(B_N\), and \(D_N\) when the particles start in the origin \(0 \in \mathbb{R}\) or, in some cases, with an arbitrary starting distribution independent from \(k\). In [V], the CLTs for \(k \to \infty\) were derived for the A-case only when the processes start in 0, while in all other cases arbitrary starting distributions were possible. This shortcoming in [V] in the A-case was caused by the lack of a suitable limit result for the Bessel functions of type A for \(k \to \infty\). We shall derive the corresponding limit result for the Bessel functions below which then will lead to a CLT for arbitrary starting distributions in Section 2.

In all CLTs in [V] and in Section 2 below, the limits in the CLTs are essentially independent from the starting distributions, and usually, the limits are \(N\)-dimensional centered Gaussian distributions where the inverses \(S_N := \Sigma_N^{-1}\) of the covariance matrices \(\Sigma_N\) can be determined explicitly in terms of the zeros of certain classical orthogonal polynomials. For instance, in the case \(A_{N-1}\), the zeros of
the Hermite polynomial $H_N$ appear, and in the case $B_N$, the zeros of appropriate Laguerre polynomials appear. We determine the eigenvalues and eigenvectors of the matrices $S_N$ and thus of $\Sigma_N$ in these cases. The results are surprisingly simple. These diagonalizations of $S_N$ and $\Sigma_N$ may be applied to the limit behavior of the middle particle in the cases $A_{N-1}$ when we first take $k \to \infty$ and then $N \to \infty$. We carry out the diagonalizations for the $A$-cases by using the empirical distributions $\mu_N$ of the zeros of the Hermite polynomials $H_N$ in combination with the finite systems of orthogonal polynomials associated with the measure $\mu_N$ introduced in Section 3. Corresponding results for the $B$-case are presented in Section 4 for the multiplicities $(k_1, k_2) = (\nu \cdot \beta, \beta)$ with $\nu > 0$ fixed and $\beta \to \infty$. Here, the zeros of classical Laguerre polynomials (with a parameter depending on $\nu$) instead of Hermite polynomials appear. In both cases, i.e., the Hermite as well as the Laguerre case, we get finite systems of orthogonal polynomials depending on $N$ which converge for $N \to \infty$ to the Tchebychev-polynomials of second kind which are orthogonal with respect to Wigner’s semicircle distribution.

The results of this paper on Bessel processes with start in $0 \in \mathbb{R}^N$ are closely related with central limit theorems for $\beta$-Hermite and $\beta$-Laguerre ensembles for the spectra of tridiagonal random matrix models due to Dumitriu and Edelman [DE1]. In particular, our freezing results correspond in some cases to the limits $\beta \to \infty$ in [DE2]. In particular, [DE2] contains explicit formulas for the covariance matrices $\Sigma_N$ of the Gaussian limits while we here use explicit formulas for their inverses $S_N$ as in [V]. In general, most of our results below for the starting point $x = 0$ admit interpretations in random matrix theory; for the background here we refer to [D], [Me], as well as to [RRV] for some specific results.

We also mention that the Bessel processes are diffusions on Weyl chambers which satisfy some stochastic differential equations; see [CGY] and references there. These SDEs are used in [AV] and [VW] to derive strong laws of large numbers and functional central limit theorems for $X_{t,k}$ for $k \to \infty$ with strong rates of convergence, whenever the processes start in points of the form $\sqrt{k} \cdot x$ where $x$ is some point in the interior of the Weyl chamber. These limit theorems are even locally uniform in $t$. It should be noticed that while the CLTs in [AV] and [VW] may have different forms, in some cases similar Gaussian limits appear with covariance matrices which are closely related to the matrices $S_N$ and $\Sigma_N$. Hence, the diagonalization results below admit applications for the CLTs in [VW].

2. A CENTRAL LIMIT THEOREM IN THE $A$-CASE FOR ARBITRARY STARTING DISTRIBUTIONS

Consider the root system $A_{N-1}$ first. The associated Bessel processes $(X_{t,k})_{t \geq 0}$ live on the closed Weyl chamber

$$C_N^A := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \},$$

the generator of the transition semigroup is

$$L_A f := \frac{1}{2} \Delta f + k \sum_{i=1}^N \left( \sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial}{\partial x_i} f,$$

(2.1)

where we regard the multiplicity $k \in [0, \infty]$ as a parameter and we assume reflecting boundaries in the usual sense (see, for example, [KS, p. 97]).
We are interested in limit theorems for \((X_{t,k})_{t \geq 0}\) for fixed \(t > 0\) and \(k \to \infty\). For this we recall that by \([R, RV1, RV2]\), the transition probabilities are given for \(t > 0\), \(x \in C^A_N\), \(S \subset C^A_N\) a Borel set, by

\[
K_t(x, S) = c_k^A \int_S \frac{1}{t^{\gamma_A + N/2}} e^{-\|x\|^2/2t} J_k^A \left( \frac{x}{\sqrt{t}} \right) \cdot w_k^A(y) \, dy \tag{2.2}
\]

with

\[
w_k^A(x) := \prod_{i<j} (x_i - x_j)^{2k}, \quad \gamma_A = kN(N - 1)/2, \tag{2.3}
\]

and the Macdonald-Mehta-Opdam constant

\[
c_k^A := \left( \int_{C_N^A} e^{-\|y\|^2/2} \cdot \prod_{i<j} (y_i - y_j)^{2k} \, dy \right)^{-1} = \frac{N!}{(2\pi)^{N/2}} \cdot \prod_{j=1}^N \frac{\Gamma(1 + k)}{\Gamma(1 + jk)} \tag{2.4}
\]

Here, \(J_k^A\) is a multivariate Bessel function of type \(A\) with multiplicity \(k\); see e.g. \([R, AKM1]\). We here only recapitulate that \(J_k^A\) is analytic on \(C^N \times C^N\) with \(J_k^A(x, y) > 0\) for \(x, y \in \mathbb{R}^N\), and with \(J_k^A(x, y) = J_k^A(y, x)\) and \(J_k^A(0, y) = 1\) for \(x, y \in \mathbb{C}^N\). Further properties will be discussed below.

If we start in \(x = 0 \in \mathbb{R}^N\), then \(X_{t,k}\) has the density

\[
\frac{c_k^A}{t^{\gamma_A + N/2}} e^{-\|y\|^2/(2t)} \cdot w_k^A(y) \, dy \tag{2.5}
\]

on \(C^A_N\) for \(t > 0\), which is in particular well-known for \(k = 1/2, 1, 2\) and \(t = 1\) as the distribution of the ordered eigenvalues of Gaussian orthogonal, unitary, and symplectic ensembles; see e.g. \([D]\). For general \(k > 0\) it is known from the tridiagonal \(\beta\)-Hermite ensembles of \([DE1]\).

It is well-known (see \([AKM1]\) and also Section 6.7 of \([S]\)) that the density (2.5) is maximal on \(C^A_N\) precisely for \(y = \sqrt{2} \cdot z\) where \(z \in C^A_N\) is the vector with the ordered zeros of the classical Hermite polynomial \(H_N\) as entries where, as usual, the polynomials \((H_N)_{N \geq 0}\) are orthogonal w.r.t. the density \(e^{-z^2}\). More precisely, we have the following useful characterization of the vector \(z\); see \([AV]\):

**Lemma 2.1.** For \(z \in C^A_N\), the following statements are equivalent:

1. The function \(W_A(x) := \sum_{i,j<i<j} \ln(x_i - x_j) - \|x\|^2/2\) is maximal at \(z \in C^A_N\);
2. For \(i = 1, \ldots, N\): \(z_i = \sum_{j \neq i} 1/\sqrt{x_{i,j}}\);
3. \(z = (z_{1,N}, \ldots, z_{N,N})\) for the ordered zeros \(z_{1,N} > \ldots > z_{N,N}\) of \(H_N\).

This characterization was used in \([V]\) to prove the following central limit theorem (please notice that the limit \(N(0, t \cdot \Sigma_N)\) there must be replaced by \(N(0, \Sigma_N)\)):

**Theorem 2.2.** Consider the Bessel processes \((X_{t,k})_{t \geq 0}\) of type \(A_{N-1}\) on \(C^A_N\) for \(k \geq 0\) with start in \(0 \in C^A_N\). Then, for each \(t > 0\),

\[
\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k} \cdot (z_{1,N}, \ldots, z_{N,N})
\]

converges for \(k \to \infty\) to the centered \(N\)-dimensional distribution \(N(0, \Sigma_N)\) with the regular covariance matrix \(\Sigma_N\) with \(\Sigma_N^{-1} = S_N = (s_{i,j})_{i,j=1}^N\) and

\[
s_{i,j} := \begin{cases} 
1 + \sum_{l \neq i} (z_{i,N} - z_{l,N})^{-2} & \text{for } i = j \\
-(z_{i,N} - z_{j,N})^{-2} & \text{for } i \neq j
\end{cases} \tag{2.6}
\]

The matrix \(S_N\) satisfies \(\det S_N = N!\).
Moreover, the length of the partition, denoted \( l \), is more complicated than for the other root systems in [V], as the systems \( A_{N-1} \) are not reduced on \( \mathbb{R}^N \). This means that with the vector \( 1 := (1, \ldots, 1) \in \mathbb{R}^N \), the space \( \mathbb{R}^N \) can be decomposed into \( \mathbb{R} \cdot 1 \) and its orthogonal complement

\[
1^\perp = \{ x \in \mathbb{R}^N : \sum_i x_i = 0 \} \subset \mathbb{R}^N
\]

where the associated Weyl group (which is the symmetric group \( S_N \) here) acts on both spaces separately. It will turn out that the limit behavior of the CLT is slightly different on both components. To describe this, we denote the orthogonal projections from \( \mathbb{R}^N \) onto \( \mathbb{R} \cdot 1 \) and \( 1^\perp \) by \( \pi_1 \) and \( \pi_1^\perp \) respectively. In particular, for all \( x \in \mathbb{R}^N \), \( \pi_1(x) = \bar{x} 1 \) for the center of gravity \( \bar{x} = \frac{1}{N} \sum_{i=1}^N x_i \) of the particles.

**Theorem 2.5.** Consider the Bessel processes \((X_{t,k})_{t \geq 0}\) of type \( A_{N-1} \) on \( C^A_N \) for \( k \geq 0 \) with a fixed starting point \( x \in C^A_N \). Then, for each \( t > 0 \),

\[
\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k} \cdot (z_{1,N}, \ldots, z_{N,N})
\]

converges for \( k \to \infty \) to the \( N \)-dimensional normal distribution \( N(\pi_1(x/\sqrt{t}), \Sigma_N) \) with \( \Sigma_N \) as in Theorem 2.2.

For the proof of Theorem 2.5, we mainly follow the ideas of the proofs of Theorem 3.3 and Corollary 3.7 in [V] in the \( B \)-case. As main ingredient we need some facts on \( J^A_k \). We first recapitulate the following well-known decomposition; see e.g. [BF]:

**Lemma 2.4.** For all \( x, y \in \mathbb{R}^N \),

\[
J^A_k(x, y) = e^{\langle \pi_1(x), \pi_1(y) \rangle} \cdot J^A_k(\pi_1^\perp(x), \pi_1^\perp(y)) = e^{N\bar{x} \bar{y}} \cdot J^A_k(\pi_1^\perp(x), \pi_1^\perp(y)). \tag{2.7}
\]

We also need the following limit result for \( J^A_k \) for \( k \to \infty \) which is a consequence of Corollary 8 of [AM] on Dunkl kernels for arbitrary root systems. Here, we include a proof that is specific to the root system \( A_{N-1} \):

**Theorem 2.5.** For \( x, y \in 1^\perp \),

\[
\lim_{k \to \infty} J^A_k(\sqrt{2k} \cdot x, y) = \exp \left( \frac{||x||^2||y||^2}{N(N-1)} \right) \tag{2.8}
\]

locally uniformly.

**Proof.** From [BF] we have

\[
J^A_k(x, y) = \theta^{(1/k)}_0(x, y) = \sum_{n=0}^{\infty} \sum_{\tau : l(\tau) \leq N, |\tau| = n} c_\tau(1/k) \frac{P^{(1/k)}_\tau(x) P^{(1/k)}_\tau(y)}{(kN)^{1/k}} \tag{2.9}
\]

with \( P_\tau^{(\alpha)}(x) \) a Jack polynomial [Ma] and \( \tau \) an integer partition with dual partition \( \tau' \). In general, integer partitions are sequences of non-negative integers in non-strictly decreasing order, namely \( \tau = (\tau_1, \tau_2, \ldots) \) with \( \tau_i \geq \tau_j \) for every \( i < j \). Moreover, the length of the partition, denoted \( l(\tau) \), is the number of nonzero parts in the partition, and the sum of its parts is denoted \( |\tau| \). The dual partition \( \tau' \) is the partition with parts \( \tau'_i \) equal to the number of parts of \( \tau \) that are greater than
or equal to \( i \). Finally, the expression \((i, j) \in \tau\) means that both \( i \leq l(\tau) \) and \( j \leq \tau_i \) are satisfied. With this, we can give the definition of all remaining symbols,

\[
(a)_{\tau}^{(\alpha)} = \prod_{i=1}^{l(\tau)} \frac{\Gamma(a - (i - 1)/\alpha + \tau_i)}{\Gamma(a - (i - 1)/\alpha)},
\]

\[
c_{\tau}(\alpha) = \prod_{(i,j) \in \tau} (\alpha(\tau_i - j) + \tau'_j - i + 1),
\]

\[
c'_{\tau}(\alpha) = \prod_{(i,j) \in \tau} (\alpha(\tau_i - j + 1) + \tau'_j - i).
\]

We rewrite the generalized Pochhammer symbol as

\[
(a)_{\tau}^{(\alpha)} = \prod_{i=1}^{l(\tau)} \frac{\Gamma(a - (i - 1)/\alpha + \tau_i)}{\Gamma(a - (i - 1)/\alpha)} = \prod_{(i,j) \in \tau} (a - (i - 1)/\alpha + j - 1). \tag{2.10}
\]

Now we consider the large \( k \) limit for the coefficients of the sum,

\[
\frac{c_{\tau}(1/k)}{c'_{\tau}(1/k)(kN)^{1/k}} = \prod_{(i,j) \in \tau} \frac{\tau_i - j + k(\tau'_i - i + 1)}{(\tau_i - j + 1 + k(\tau'_j - i))(k(N - i + 1) + j - 1)}
\]

\[
= \prod_{(i,j) \in \tau; i < \tau'_j} \frac{\tau_i - j + k(\tau'_i - i)}{(\tau_i - j + 1 + k(\tau'_j - i))(k(N - i + 1) + j - 1)}
\]

\[
\times \prod_{(i,j) \in \tau; i = \tau'_j} \frac{\tau_i - j + k}{(\tau_i - j + 1)(k(N - i + 1) + j - 1)}
\]

\[
= \prod_{(i,j) \in \tau; i < \tau'_j} \left( \frac{1}{k}(\tau'_j - i)(N - i + 1) + O(k^{-2}) \right)
\]

\[
\times \prod_{(i,j) \in \tau; i = \tau'_j} \left( \frac{1}{(\tau_i - j + 1)(N - i + 1)} + O(k^{-1}) \right). \tag{2.11}
\]

Now, recall that the Jack polynomials are homogeneous,

\[
\mathcal{P}^{(1/k)}_{\tau}(\sqrt{2kx}) = (2k)^{\tau_1/2} \mathcal{P}^{(1/k)}_{\tau}(x),
\]

and that they converge to the elementary symmetric polynomials,

\[
e_n(x) = \sum_{1 \leq l_1 < \ldots < l_n} \prod_{j=1}^{n} x_{l_j}, \quad e_{\tau}(x) = \prod_{j=1}^{l(\tau)} e_{\tau_j}(x),
\]

when \( k \to \infty, \)

\[
\lim_{k \to \infty} \mathcal{P}^{(1/k)}_{\tau}(x) = e_{\tau}(x).
\]
Then, we have
\[ J_k^A(\sqrt{2k} \cdot x, y) = \sum_{n=0}^{\infty} \sum_{\tau: |\tau| \leq n} \left( \frac{1}{k} \right)^{|\tau|-\tau_1} \times \prod_{(i,j) \in \tau : i < \tau_j} \left( \frac{\tau_j' - i + 1}{(\tau_j' - i)(N - i + 1)} + O(k^{-1}) \right) \times \prod_{(i,j) \in \tau : i = \tau_j} \left( \frac{1}{(\tau_i - j + 1)(N - i + 1)} + O(k^{-1}) \right) \times (2k)^{|\tau|/2} e_{\tau'}(x)e_{\tau}(y). \] (2.12)

However, since we have imposed \( \sum_{i=1}^{N} x_i = e_1(x) = 0 \), all terms for which any of the \( \tau_j' = 1 \) vanish automatically. Consequently, we must have \( \tau_1 = \tau_2 \) for all partitions, and the leading-order terms in \( k \) are those with partitions \( x \tau \) of length two with \( \tau_1 = \tau_2 \). Therefore,
\[ \lim_{k \to \infty} J_k^A(\sqrt{2k}x, y) = \sum_{n=0}^{\infty} \frac{2^{2n}}{n!N^n(N-1)^n} \left[ e_2(x)e_2(y) \right]^n = \exp \left[ \frac{4e_2(x)e_2(y)}{N(N-1)} \right]. \] (2.13)

Now, since
\[ 0 = \left( \sum_{i=1}^{N} x_i \right)^2 = \sum_{i=1}^{N} x_i^2 + 2 \sum_{1 \leq i < j \leq N} x_ix_j, \]
we have
\[ \|x\|^2 = -2e_2(x) \]
and a similar relation for \( y \). Finally, we obtain
\[ \lim_{k \to \infty} J_k^A(\sqrt{2k}x, y) = \exp \left[ \frac{\|x\|^2\|y\|^2}{N(N-1)} \right], \] (2.14)
as desired. \( \square \)

**Proof of Theorem 2.3.** By the definition of the transition kernels \( K_i \) in (2.2), the \( K_i \) admit the same space-time-scaling as Brownian motions. We thus may assume that \( t = 1 \) in the proof without loss of generality.

Moreover, (2.7) implies that the kernels \( K_i \) are partially translation invariant in the sense that
\[ K_i(x + c1, S + c1) = K_i(x, S) \quad \text{for } c \in \mathbb{R}, \ t > 0, \ x \in C_N^A, \ S \subset C_N^A. \] (2.15)
Thus, without loss of generality, we can add the assumption that the starting point \( x \in C_N^A \) satisfies \( x \in 1^+ \).

Then, \( X_{1,k} \) has the density
\[ c_k^A e^{-\|x\|^2/2-\|y\|^2/2} \cdot J_k^A(x, y) \cdot w_k^A(y) \]
on \( C_N^A \). Hence, \( X_{1,k} - \sqrt{2k} \cdot z \) has the density
\[ f_k^A(y) := c_k^A e^{-\|x\|^2/2} J_k^A(x, y + \sqrt{2k} \cdot z) \cdot \exp \left( -\|y + \sqrt{2k} \cdot z\|^2/2 \right) w_k^A(y + \sqrt{2k} \cdot z) \] (2.16)
on the shifted cone $C_N^A - \sqrt{2k} \cdot z$ with $f_k^A(y) = 0$ elsewhere on $\mathbb{R}^N$. Using the definition of $w_k^A$ we now write this density as

$$f_k^A(y) = \tilde{c}_k \cdot h_k(y),$$

with $\tilde{c}_k$ given by (see Appendix D in [AKM1])

$$\tilde{c}_k := e^{-\|x\|^2/2} \left( \frac{k}{e} \right)^{kN(N-1)/2} \frac{N!}{(2\pi)^{N/2}} \prod_{j=1}^{N} \frac{\Gamma(1+k)}{\Gamma(1+jk)} \prod_{m=1}^{N} m^k,$$

which is independent of $y$ (but dependent on $x, k$), and with

$$h_k(y) := J_k^A(x, y + \sqrt{2k} \cdot z),$$

$$\cdot \exp \left( -\|y\|^2/2 - \frac{1}{\sqrt{2k}} \sum_{i<j} \ln \left( 1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)} \right) \right)$$

$$= J_k^A(x, y + \sqrt{2k} \cdot z) \cdot \exp \left( -\|y\|^2/2 - \frac{1}{2} \sum_{i<j} \frac{(y_i - y_j)^2}{(z_i - z_j)^2} + O(k^{-1/2}) \right)$$

(2.17)

for $y \in C_N^A - \sqrt{2k} \cdot z$ and $h_k(y) = 0$ elsewhere. The last equality in (2.17) follows from the Taylor formula for $\ln(1+x)$ and from Lemma 2.1 precisely as in the proof of Eq. (2.8) of [V]. Next, we recall that $x \in \mathbb{R}^+$ (by our assumption) and $z \in \mathbb{R}^+$ (because $H_N$ has either even or odd symmetry). We thus conclude from Lemma 2.3 and Theorem 2.4 that for all $y \in \mathbb{R}^N$

$$\lim_{k \to \infty} J_k^A(x, y + \sqrt{2k} \cdot z) = \lim_{k \to \infty} J_k^A(x, \sqrt{2k}(z + y/\sqrt{2k}))$$

$$= \exp \left( \frac{\|x\|^2\|z\|^2}{N(N-1)} \right) = \exp(\|x\|^2/2) =: d(x)$$

(2.18)

where we have used

$$\sum_{k=1}^{N} z_{k,N}^2 = N(N-1)/2$$

(2.19)

(see (D.22) in [AKM1]). In summary,

$$\lim_{k \to \infty} h_k(y) = d(x) \cdot \exp \left( -\frac{\|y\|^2}{2} - \frac{1}{2} \sum_{i<j} \frac{(y_i - y_j)^2}{(z_i - z_j)^2} \right).$$

(2.20)

Now let $f \in C_b(\mathbb{R}^N)$ be a bounded continuous function. We shall show that (2.20) implies that

$$\lim_{k \to \infty} \int_{\mathbb{R}^N} f(y) \cdot h_k(y) dy = \int_{\mathbb{R}^N} f(y) \cdot \exp(\frac{||y||^2}{2} - \frac{1}{2} \sum_{i<j} \frac{(y_i - y_j)^2}{(z_i - z_j)^2}) dy.$$ (2.21)

For this we use dominated convergence. We consider the Taylor polynomial of $\ln(1+x)$ and notice that by the Lagrange remainder,

$$\ln(1 + \frac{y_i \pm y_j}{\sqrt{\beta}(z_i \pm z_j)}) = \frac{y_i \pm y_j}{\sqrt{\beta}(z_i \pm z_j)} - \frac{(y_i \pm y_j)^2}{2\beta(z_i \pm z_j)^2} \cdot w_{\pm}$$

(2.22)

with $w_{\pm} \in [0, 1]$. As in the proof of Theorem 2.2 of [V] we obtain from Lemma 2.3 that for all $k > 0$

$$0 \leq h_k(y) \leq J_k^A(x, \sqrt{2k}(z + y/\sqrt{2k})) \cdot e^{-\|y\|^2/2}.$$ (2.23)
Next, we estimate $J_k^A$. For this we recapitulate from [RV2] that for all root systems and all multiplicities $k \geq 0$, the associated Bessel functions $J$ satisfy
\[ 0 < J(a, b) \leq \exp(\|a\| \cdot \|b\|) \quad \text{for all} \quad a, b \in \mathbb{R}^N. \]
In particular,
\[ 0 < J_k^A(x, y + \sqrt{2k} \cdot z) \leq \exp(\|x\| \cdot (\|y\| + \sqrt{2k} \cdot \|z\|)). \]
This shows that
\[ J_k^A(x, y + \sqrt{2k} \cdot z) \leq e^2 \|x\| \cdot \|y\| \quad \text{for } k > 0, \text{ and } y \in \mathbb{R}^N \text{ with } \|y\| \geq \sqrt{2k} \cdot \|z\|. \quad (2.24) \]
On the other hand, if $\|y\| \leq \sqrt{2k} \cdot \|z\|$, then $y/\sqrt{2k} + z$ is contained in a fixed compact set $C \subset \mathbb{R}^N$. Therefore we obtain from $x, z \in 1^\perp$, Lemma [2.4] and Theorem [2.5] that
\[ \sup_{y \in \mathbb{R}^N, k \geq 0: \|y\| \leq \sqrt{2k} \cdot \|z\|} J_k^A(x, y + \sqrt{2k} \cdot z) = \sup_{y \in \mathbb{R}^N, k \geq 0: \|y\| \leq \sqrt{2k} \cdot \|z\|} J_k^A(x, \pi_1(y) + \sqrt{2k} \cdot z) = \sup_{y \in 1^\perp, k \geq 0: \|y\| \leq \sqrt{2k} \cdot \|z\|} J_k^A(x, \sqrt{2k}(\frac{y}{\sqrt{2k}} + z)) < \infty. \]
This estimation, (2.24), and (2.23) readily imply that the dominated convergence theorem in (2.21) works as claimed.

If we take $f$ in Eq. (2.21) as the constant $1$, we obtain that the constants $c_k$ of the probability densities $f_k^A$ tend to
\[ \tilde{d}(x) := \left( d(x) \int_{\mathbb{R}^N} \exp\left( -\frac{\|y\|^2}{2} - \frac{1}{2} \sum_{i<j} \frac{(y_i - y_j)^2}{(z_i - z_j)^2} \right) dy \right)^{-1} \]
which can be expressed explicitly in terms of the determinant $S_N$. On the other hand, it follows from the proof of Theorem 2.2 in [V] (see in particular Eqs. (2.3)-(2.5) for the case $x = 0$) that in our generalized case
\[ \lim_{k \to \infty} \tilde{c}_k = e^{-\|x\|^2/2} \frac{\sqrt{N!}}{(2\pi)^{N/2}}. \]
A comparison of both limits shows that $\det S_N = N!$ as shown in Corollary 2.3 of [V], and that the constants depending on $x$ also fit.

If we take this convergence of the norming constants into account, we obtain from (2.21) that the probability measures $f_k^A(y) \, dy$ tend weakly to the normal distribution $N(0, \Sigma_N)$. This completes the proof.

We denote by $M^1(S)$ the set of probability distributions on a set $S$, and by $\mu_t$ the scaling of $\mu \in M^1(S)$ by a factor of $\sqrt{t}$, namely, $\mu_t(\{x\}) := t^{N/2} \mu(\{x \cdot \sqrt{t}\})$.

**Corollary 2.6.** Let $\mu \in M^1(C_N^A)$ be an arbitrary starting distribution on $C_N^A$. Consider the Bessel processes $(X_{t,k})_{t \geq 0}$ of type $A_{N-1}$ on $C_N^A$ for $k \geq 0$ with this starting distribution $\mu$. Then
\[ \frac{X_{t,k} - \sqrt{2k} \cdot (z_{1,N}, \ldots, z_{N,N})}{\sqrt{t}} \]
converges for $k \to \infty$ to the $N$-dimensional distribution $\pi_1(\mu_t) \ast N(0, \Sigma_N)$ with the normal distribution $N(0, \Sigma_N)$, the covariance matrix $\Sigma_N$ as in Theorem 2.2 and
the usual convolution $\ast$ of probability measures on $\mathbb{R}^N$, where $\pi_1(\mu_t)$ is the image measure of $\mu_t$ under the projection $\pi_1$.

Proof. If $\mu$ is a Dirac measure, say at $x \in C_N^A$, then the statement is precisely Theorem 2.2. This then leads easily to the general case; see the proof of Corollary 3.7 in [V]. \qed

3. The covariance matrices in the $A$-case

We now study the matrices $S_N = \Sigma_N^{-1}$ from Theorems 2.2 and 2.3 more closely. We first determine the eigenvalues and eigenvectors. The eigenvectors will be described in terms of a certain finite sequence of orthogonal polynomials. For this we introduce the empirical measures

$$\mu_N := \frac{1}{N}(\delta_{z_{1,N}} + \ldots + \delta_{z_{N,N}}) \in M_1^1(\mathbb{R})$$

(3.1)

of the zeros of $H_N$. We consider the associated finite sequence of orthogonal polynomials \( \{P^{(N)}_n\}_{n=0}^{N-1} \) with positive leading coefficients and with the normalizations

$$\sum_{i=1}^{N} P^{(N)}_n(z_{i,N})^2 = 1 \quad (n = 0, \ldots, N - 1).$$

(3.2)

These polynomials with $\deg[P^{(N)}_n] = n$ ($n = 0, \ldots, N - 1$) are determined uniquely by Gram-Schmidt orthogonalization and normalization from the monomials $x^n$ ($n = 0, \ldots, N - 1$) on the spaces $L^2(\mathbb{R}, \mu_N)$. For the background on finite sequences of orthogonal polynomials we refer to [C]. These orthogonal polynomials satisfy a three-term recurrence relation (see [C], Section I.4). The normalization (3.2) and the orthogonality of the polynomials $P^{(N)}_n$ ensure that for $N \in \mathbb{N}$ the matrices

$$T_N := (P^{(N)}_{j-1}(z_{i,N}))_{i,j=1,\ldots,N}$$

(3.3)

are orthogonal. In particular,

$$P^{(N)}_0 \equiv N^{-1/2}, \quad P^{(N)}_1(x) = \sqrt{\frac{2}{N(N-1)}} x,$$

and

$$P^{(N)}_2(x) = c_2(x^2 - (N - 1)/2), \quad c_2 = \frac{2}{\sqrt{N(N-1)(N-2)}}.$$

The expressions for $P^{(N)}_1(x)$ and $P^{(N)}_2(x)$ follow from orthogonality and from (2.19).

We have the following result about the eigenvalues and eigenvectors of $S_N$:

**Theorem 3.1.** For each $N \geq 2$, the matrix $S_N$ from Theorem 2.2 has the eigenvalues $1, 2, \ldots, N$. Moreover, for each $n = 1, \ldots, N$, the vector

$$\left(P^{(N)}_{n-1}(z_{1,N}), \ldots, P^{(N)}_{n-1}(z_{N,N})\right)^T$$

is an eigenvector of $S_N$ for the eigenvalue $n$, i.e., $S_N = T_N \cdot \text{diag}(1, 2, \ldots, N) \cdot T_N^T$.

Proof. In the first main step of the proof we show by induction on $n = 1, \ldots, N$ that $n$ is an eigenvalue of $S_N$, and that there exists some polynomial $q_n$ of degree $n - 1$ such that the vector

$$\left(q_n(z_{1,N}), \ldots, q_n(z_{N,N})\right)^T$$
is an associated eigenvector of $S_N$. In a short second step we then will identify the polynomials $q_n$.

We start our induction with $n = 1$. We observe that $(1, \ldots, 1)^T$ is clearly an eigenvector for the eigenvalue 1. Moreover, if we use Lemma 2.1(2), we also see easily that $(z_{1,N}, \ldots, z_{N,N})^T$ is an eigenvector for the eigenvalue 2. It can be also checked with this argument and an easy computation that

$$(P_2^{(N)}(z_{1,N}), \ldots, P_2^{(N)}(z_{N,N}))^T$$

as given above is an eigenvector for the eigenvalue 3.

Let us turn to the general induction step for $n$. We use the $N \times N$-identity matrix $I_N$ and consider the vector

$$v_n := (z_{1,N}^{n-1}, \ldots, z_{N,N}^{n-1})^T.$$ 

Then the $i$-th coordinate of $(S_N - nI_N)v_n$ satisfies

$$(S_N - nI_N)v_n)_i = (1 - n)z_{i,N}^{n-1} + \sum_{j : j \neq i} \frac{z_{i,N}^{n-2} - z_{j,N}^{n-2}}{z_{i,N} - z_{j,N}}$$

$$= (1 - n)z_{i,N}^{n-1} + \sum_{j : j \neq i} \frac{z_{i,N}^{n-3} - z_{j,N}^{n-3}}{z_{i,N} - z_{j,N}}$$

$$+ \sum_{j : j \neq i} \frac{z_{i,N}^{n-4} - z_{j,N}^{n-4}}{z_{i,N} - z_{j,N}}$$

$$= - \sum_{m=1}^{n-2} \sum_{l=0}^{m-1} z_{i,N}^{n-3-l} \left( \sum_{j=1}^{N} z_{j,N}^l - z_{i,N}^l \right),$$

where the last equation follows from item (2) of Lemma 2.1. If we put

$$s_l := \sum_{j=1}^{N} z_{j,N}^l \quad (l = 0, 1, \ldots),$$

we notice that $s_l = 0$ whenever $l$ is odd due to the symmetry of the zeroes of $H_N$, and we obtain

$$(S_N - nI_N)v_n)_i = \sum_{m=1}^{n-2} \sum_{l=0}^{m-1} z_{i,N}^{n-3} - \sum_{l=0}^{n-3} \sum_{m=l+1}^{n-2} s_l z_{i,N}^{n-3-l}$$

$$= \frac{(n-1)(n-2)}{2} z_{i,N}^{n-3} - \sum_{l=0}^{\lfloor (n-3)/2 \rfloor} (n - 2(l + 1)) s_{2l} z_{i,N}^{n-1-2(l+1)}$$

$$= -\left( N - \frac{n-1}{2} \right) (n-2) z_{i,N}^{n-3} - \sum_{l=1}^{\lfloor (n-3)/2 \rfloor} (n - 2(l + 1)) s_{2l} z_{i,N}^{n-1-2(l+1)},$$

(3.5)

which is a polynomial with all terms either even or odd in $z_{i,N}$. Note that it is easy to confirm that

$$s_{2l} = \frac{1}{2} \sum_{m=0}^{l-1} s_{2(l-1-m)} s_{2m} - \frac{2l-1}{2} s_{2(l-1)}$$

(3.6)
with $s_0 = N$, meaning that the coefficients $s_l$ are functions of $N$ alone. We thus find a polynomial $r_{n-3}$ of order $n - 3$ with

$$(S_N - nI_N)v_n = (r_{n-3}(z_{1,N}), \ldots, r_{n-3}(z_{N,N}))^T. \quad (3.7)$$

On the other hand, by our induction assumptions, we have polynomials $q_1, \ldots, q_{n-2}$ with $\text{deg}[q_l] = l - 1 \ (l = 1, \ldots, n - 2)$ and

$$(S_N - nI_N)(q_l(z_{1,N}), \ldots, q_l(z_{N,N}))^T = -(n - l) \cdot (q_l(z_{1,N}), \ldots, q_l(z_{N,N}))^T. \quad (3.8)$$

As the $q_1, \ldots, q_{n-2}$ form a basis of the vector space $\mathbb{R}_{n-3}[x]$ of all polynomials of degree at most $n - 3$, we can find a polynomial $p_{n-3} \in \mathbb{R}_{n-3}[x]$ that satisfies

$$(S_N - nI_N)(p_{n-3}(z_{1,N}), \ldots, p_{n-3}(z_{N,N}))^T = (r_{n-3}(z_{1,N}), \ldots, r_{n-3}(z_{N,N}))^T. \quad (3.9)$$

Therefore, the monic polynomial $q_n(x) := x^{n-1} - p_{n-3}(x)$ has the required properties. This completes the induction.

We finally identify the $q_n$ more explicitly. As $S_N$ is symmetric, the vectors

$$(q_n(z_{1,N}), \ldots, q_n(z_{N,N}))^T \quad (n = 1, \ldots, N)$$

are orthogonal, i.e.,

$$\sum_{i=1}^{N} q_n(z_{i,N}) \cdot q_l(z_{i,N}) = 0 \quad (n, l = 1, \ldots, N, \ n \neq l).$$

Hence, $(q_n)_{n=1,\ldots,N}$ is just a finite sequence of orthogonal polynomials associated with the empirical measure $\mu_N$. This implies that the $q_n$ are equal to $P_{n-1}^{(N)}$ for $n = 1, \ldots, N$ up to normalizations. This completes the proof of the theorem. $\square$

**Remark 3.2.** The CLT 2.2 was also derived by Dumitriu and Edelman [DE2] for $t = 1$. We point out that their statement contains explicit formulas for the covariance matrix $\Sigma_N = (\sigma_{i,j}^2)_{i,j=1,\ldots,N}$ of the limit and not its inverse $S_N = \Sigma_N^{-1}$ as in (2.2). In fact, in our notations, Theorem 3.1 of [DE2] yields that

$$\sigma_{i,j}^2 = \sum_{l=0}^{N-1} \frac{H_l^2(z_{i,N})H_l^2(z_{j,N}) + \sum_{r=0}^{N-2} H_{l+1}(z_{i,N})H_l(z_{i,N})H_{l+1}(z_{j,N})H_l(z_{j,N})}{\sum_{l=0}^{N-1} H_l^2(z_{i,N}) \cdot \sum_{l=0}^{N-1} H_l^2(z_{j,N})} \quad (3.10)$$

with the orthonormal Hermite polynomials $(\tilde{H}_n)_{n \geq 0}$. Theorem 3.1 and a comparison of Theorem 2.2 above with Theorem 3.1 of [DE2] show that the matrix $\Sigma_N$ as in (3.10) has the form

$$\Sigma_N = T_N \cdot \text{diag}(1/2, \ldots, 1/N) \cdot T_N^T. \quad (3.11)$$

Even knowing these facts, we are unable to check this statement for general dimensions $N$ directly via (3.10) even in the simplest cases like the eigenvalue $1$ with eigenvector $(1, \ldots, 1)^T$.

Next, we study the polynomials $P_k^{(N)}$ more closely for large dimensions $N$. We recapitulate the well-known fact (see e.g. [G], [KM], or [D] for different proofs) that for $\mathbb{R}$-valued random variables $X_N$ with distributions $\mu_N$, the r.v.'s $\sqrt{N}X_N$ tend in distribution to the r.v. $X$ which obeys the semicircle law $\mu_{sc}$, namely, the probability measure given by the density

$$\mu_{sc}(x) = \frac{2}{\pi} \sqrt{1 - x^2} \cdot 1_{[-1,1]}(x).$$
For this we recall that the odd moments of $\mu_{sc}$ are zero while for $n \in \mathbb{N}$, the $2n$-th moments are given by $2^{-2n}C_n$ with the Catalan numbers

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n} \quad (n \geq 0);$$

see e.g. [D] or [G]. The convergence of the $\mu_N$ to $\mu_{sc}$ above can now be derived via the moment convergence theorem [FS]. In fact, the following rate of convergence for the moments was given in Theorem 2 of [KM]; please notice that [KM] use a different normalization for the Hermite polynomials in their arguments. We have translated their results to our setting:

**Proposition 3.3.** For all $n \in \mathbb{N}_0$ the $n$-th moment

$$m_N(n) := E(X_N^n) = \frac{1}{N} \sum_{i=1}^{N} z_{i,N}^n$$

of a random variable $X_N$ with the distribution $\mu_N$ in (3.1), satisfies

$$m_N(n) = \begin{cases} 
    (N/2)^{n/2}C_{n/2} + \frac{1}{N} \cdot f_n(N) & \text{for } n \text{ even} \\
    0 & \text{for } n \text{ odd}
\end{cases}$$

with polynomials $f_n$ of degree at most $n/2$.

This proposition ensures that for all $n$,

$$E\left(\left(\frac{1}{\sqrt{2N}}X_N\right)^n\right) = E(X^n) = O(1/N) \quad (N \to \infty).$$

(3.12)

We now equip the vector space $\mathbb{R}[x]$ of all polynomials with the positive semi-definite products

$$\langle p, q \rangle_N := \frac{1}{N} \sum_{i=1}^{N} p\left(\frac{1}{\sqrt{2N}}z_{i,N}\right) \cdot q\left(\frac{1}{\sqrt{2N}}z_{i,N}\right) \quad (N \in \mathbb{N})$$

and

$$\langle p, q \rangle := \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2} \cdot p(x)q(x) \, dx = \int_{-1}^{1} pq \, d\mu_{sc}$$

and study the associated orthonormal polynomials. In the first case, the normalization (3.2) shows that these orthonormal polynomials $(\tilde{P}_n^{(N)})_{n=0,\ldots,N-1}$ satisfy

$$\tilde{P}_n^{(N)}(x) = \sqrt{N} \cdot P_n^{(N)}(\sqrt{2N} \cdot x) \quad (n = 0, \ldots, N-1).$$

(3.13)

Moreover, by Section 4.7 of [S], in the second case the orthonormal polynomials are the Tchebychev polynomials $(U_n)_{n \geq 0}$ of the second kind with

$$U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta} \quad (n \in \mathbb{N}_0).$$

(3.14)

Proposition 3.3 yields:

**Lemma 3.4.** For all $n \in \mathbb{N}_0$, and locally uniformly in $x \in \mathbb{R}$,

$$\tilde{P}_n^{(N)}(x) - U_n(x) = O(1/N) \quad (N \to \infty).$$
Proof. We first observe that \( \tilde{P}_0^{(N)} = 1 = U_0, U_1(x) = 2x \) and, by \((2.11)\), \( \tilde{P}_1^{(N)}(x) = 2\sqrt{\frac{N}{N-1}} \cdot x \). This proves the result for \( k = 0, 1 \).

The general case follows e.g. by induction on \( n \), Proposition \( 3.3 \) and the three-term-recurrence relation of the monic orthogonal polynomials associated with the orthonormal polynomials \( \tilde{P}_n^{(N)} \) and \( U_n \); see Section I.4 of \( \text{[C]} \). In both cases, the final orthonormalizations clearly preserve the order of convergence. \( \square \)

In the end of this section we briefly discuss some possible applications of Lemma \( 3.4 \) to the variances of particles of Calogero-Moser-Sutherland models, when we first take the limit \( k \to \infty \) and then the limit \( N \to \infty \). For this we choose an index \( i(N) \in \{1, \ldots, N\} \) for every \( N \) and consider the variances \( \sigma^2_{i(N),i(N)}(N) = \sigma^2_{i(N),i(N)}(N) \) of the \( i(N) \)-th particles. Using \( (3.11) \) and \( (3.13) \), we have

\[
\sigma^2_{i(N),i(N)}(N) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{n+1} \tilde{P}_n^{(N)}(z_{i(N)},N/\sqrt{2N})^2. \tag{3.15}
\]

By Lemma \( 3.4 \) \( \sigma^2_{i(N),i(N)}(N) \) should be approximately equal to

\[
\tilde{\sigma}^2_{i(N),i(N)}(N) := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{n+1} U_n(z_{i(N)},N/\sqrt{2N})^2. \tag{3.16}
\]

We discuss this heuristic idea for the particles in the middle of the models. To be more precise, we consider an odd number \( N = 2L - 1 \) \((L \in \mathbb{N})\) of particles and investigate the particle with number \( L \). In this case we use the representation \( (3.10) \) of \text{[DE2]} and get an exact asymptotic result for \( L \to \infty \). In fact, we use \( z_{2L-1} = 0, (3.10) \), as well as the formulas \((5.5.1)\) and \((5.5.4)\) of \text{[S]} on Hermite polynomials, as well as \( H_{2n+1}(0) = 0 \) for \( n \in \mathbb{N}_0 \). This and Stirling’s formula imply that

\[
\tilde{H}_{2l}(0)^2 = \frac{(2l)!}{(l!)^2 \sqrt{\pi} \cdot 2^{2l}} \sim \frac{1}{\pi \sqrt{l}}
\]

and thus

\[
\begin{align*}
\sigma^2_{2L-1,L}(2L-1) & = \sum_{l=0}^{L-1} \tilde{H}_{2l}(0)^4 \left( \sum_{l=0}^{L-1} \tilde{H}_{2l}(0)^2 \right)^2 \\
& \sim \sum_{l=1}^{L-1} \frac{1}{l} \left( \frac{1}{\sqrt{l}} \right)^2 \sim \frac{\ln L}{(2\sqrt{L})^2} = \frac{\ln L}{4L}
\end{align*}
\tag{3.17}
\]

for \( L \to \infty \). This and Theorem \( 2.2 \) lead to the following result:

**Corollary 3.5.** For \( L \in \mathbb{N} \) let \( X^{(L)}_{k} \) be the position of the \( L \)-th particle in the middle of a system with \( N = 2L - 1 \) particles with multiplicity \( k \). Then

\[
\frac{2\sqrt{L}}{\sqrt{\ln L}} \cdot X^{(L)}_{k,k}
\tag{3.18}
\]

tends in distribution to the standard normal distribution when first the limit \( k \to \infty \) and then the limit \( L \to \infty \) is taken.

On the other hand, we now study the approximation \( \tilde{\sigma}^2_{2L-1,L}(2L-1) \) of \( \sigma^2_{2L-1,L}(2L-1) \) above. In this case we use the polynomials \( U_1 \) as in \( (3.14) \) and consider the fixed
angle $\theta = \pi/2$ with $z_{L,2L-1} = 0 = \cos \theta$. Hence,

$$
\tilde{\sigma}_{L,L}^2(2L - 1) = \frac{1}{2L - 1} \sum_{k=0}^{2L-2} \frac{1}{k+1} \frac{\sin^2((k+1)\pi/2)}{\sin^2(\pi/2)} = \frac{1}{2L - 1} \sum_{k=0}^{L-1} \frac{1}{2k+1} \\
\sim \frac{\ln L}{4L}
$$

(3.19)

for $L \to \infty$ which fits perfectly with (3.17).

We finally mention that performing similar operations for the rightmost particle with number 1 does not yield the correct asymptotics for the corresponding variance. Here $z_{1,N}$ is the largest zero of $H_N$, and the Theorem of Plancherel-Rotach (see e.g. (6.3.9) of [S]) shows that

$$
z_{1,N}/\sqrt{2N} = 1 - \frac{i_1}{6^{1/3}(2N)^{2/3}} + o(N^{-2/3})
$$

(3.20)

with the first positive zero $i_1$ of the Airy function $Ai(-3^{1/3}x)$, where $Ai(x)$ is the solution of the differential equation

$$
d^2dx^2Ai(x) - xAi(x) = 0
$$

with the condition that $Ai(x) \to 0$ as $x \to \infty$. In particular, $z_{1,N}/\sqrt{2N} \in [0,1]$ for $N$ sufficiently large. For these $N$ we now choose $\theta_N \in [0,\pi]$ with $\cos \theta_N = z_{1,N}/\sqrt{2N}$. Then, by (3.20),

$$
1 - \frac{\theta^2_N}{2} + O(\theta^4_N) = \cos \theta_N = z_{1,N}/\sqrt{2N} = 1 - \frac{i_1}{6^{1/3}(2N)^{2/3}} + o(N^{-2/3})
$$

and thus

$$
\theta_N = \sqrt{\frac{21^{1/3}i_1}{6^{1/3}}} \cdot N^{-1/3} + o(N^{-1/3}).
$$

It can be now shown that

$$
\tilde{\sigma}_{1,1}^2(N) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{n+1} \frac{\sin^2((n+1)\theta_N)}{\sin^2 \theta_N} \\
\sim \frac{1}{N\theta_N} \sum_{n=0}^{N-1} \frac{\sin^2((n+1)\theta_N)}{(n+1)\theta_N} \\
\sim \frac{1}{2N\theta_N} \sum_{n=0}^{N-1} \frac{1}{(n+1)\theta_N} \\
 \sim \frac{\ln N}{2N\theta_N^2} \sim \frac{6^{1/3}}{i_1 2^{4/3}} \cdot \frac{\ln N}{N^{1/3}}
$$

(3.21)

As stated above, numerical experiments show that this rate does not seem to be the correct one for $\sigma_{1,1}^2(N)$ for $N \to \infty$. It also differs from the rate given in [DE2].

We plan to investigate the orthogonal polynomials $\tilde{P}_n^{(N)}(x)$ and the relations between $\sigma_{i(N),i(N)}^2(N)$ and $\tilde{\sigma}_{i(N),i(N)}^2(N)$ more closely in a forthcoming paper.
4. The $B$-case and Laguerre polynomials

We now study the covariance matrices of the Gaussian limit of Bessel processes $(X_{t,k})_{t \geq 0}$ of type $B$. The processes live in the closed Weyl chamber

$$C_N^B := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \cdots \geq x_N \geq 0 \},$$

and their transition semigroup generator is

$$L_B f := \frac{1}{2} \Delta f + k_1 \sum_{i=1}^N \frac{1}{x_i} \frac{\partial}{\partial x_i} f + k_2 \sum_{i,j=1}^N \left( \sum_{l \neq i} \frac{1}{x_i - x_l} + \frac{1}{x_i + x_l} \right) \frac{\partial}{\partial x_i} f. \quad (4.1)$$

As in Section 2, the multiplicities are non-negative real parameters which we take here as $(k_1, k_2) = (\beta \cdot \nu, \beta)$ with $\nu > 0$ fixed and $\beta \to \infty$; henceforth, $k$ will be regarded as an integer variable unrelated to the multiplicities. For all other related quantities, such as the transition probabilities, we refer the reader to [V]. In this case, the limit is related to the ordered zeroes $z_{1,N}^{(\nu-1)} \geq \cdots \geq z_{N,N}^{(\nu-1)}$ of the Laguerre polynomial $L_N^{(\nu-1)}$. These polynomials are orthogonal w.r.t. the density $e^{-x}x^{\nu-1}$ by [S]. We start with the following known analogue of Lemma 2.1 above from [S, AKM2].

**Lemma 4.1.** For $r \in C_N^B$, the following statements are equivalent:

1. The function
   $$W_B(y) := 2 \sum_{i,j : i < j} \ln(y_i^2 - y_j^2) + 2\nu \sum_i \ln y_i - \|y\|^2/2$$
   is maximal at $r \in C_N^B$.
2. For $i = 1, \ldots, N$, $r = (r_1, \ldots, r_N)$ satisfies
   $$\frac{r_i}{2} = \sum_{j,j \neq i} \frac{2r_i}{r_i^2 - r_j^2} + \frac{\nu}{r_i};$$
3. If $z_{1,N}^{(\nu-1)} > \cdots > z_{N,N}^{(\nu-1)} > 0$ are the ordered zeroes of $L_N^{(\nu-1)}$, then
   $$2(z_{1,N}^{(\nu-1)}, \ldots, z_{N,N}^{(\nu-1)}) = (r_1^2, \ldots, r_N^2).$$

Using this lemma and the vector $r$ there, we have the following central limit theorem by [V].

**Theorem 4.2.** Consider the Bessel processes $(X_{t,k})_{t \geq 0}$ of type $B$ on $C_N^B$ for $k = (k_1, k_2) = (\beta \cdot \nu, \beta)$ and $\beta, \nu > 0$ with start in $x \in C_N^B$. Then, for each $t > 0$,

$$\frac{X_{t,(\beta \cdot \nu, \beta)}}{\sqrt{t}} \to \text{the centered } N\text{-dimensional distribution } N(0, \Sigma_N)$$

with the regular covariance matrix $\Sigma_N$ with $\Sigma_N^{-1} = S_N = (s_{i,j})_{i,j=1,\ldots,N}$ given by

$$s_{i,j} := \begin{cases} 1 + \frac{2\nu}{r_i} + 2 \sum_{l \neq i} (r_i - r_l)^2 + 2 \sum_{l \neq i} (r_i + r_l)^2 & \text{for } i = j, \\ 2(r_i + r_j)^2 - 2(r_i - r_j)^2 & \text{for } i \neq j. \end{cases} \quad (4.2)$$

The matrix $S_N$ satisfies $\det S_N = N!2^N$. 


We now proceed as in the previous section and determine the eigenvectors and eigenvalues of $S_N$. It will be convenient for this to introduce the empirical probability measures

$$\mu_{N,\nu} := \frac{1}{2N(N+\nu-1)}(2z_{1,N}^{(\nu-1)} \delta_{2z_{1,N}} + \ldots + 2z_{N,N}^{(\nu-1)} \delta_{2z_{N,N}}). \quad (4.3)$$

As

$$\sum_{k=1}^{N} z_{k,N}^{(\nu-1)} = N(N+\nu-1) \quad (4.4)$$

by Appendix C of [AKM2], these measures are probability measures. Next, we study the family of orthogonal polynomials $(P_{k}^{(N,\nu)})_{k=0,\ldots,N-1}$ with \(\deg[P_{k}^{(N,\nu)}] = k\) and positive leading coefficients under the normalization

$$\sum_{i=1}^{N} 2z_{i,N}^{(\nu-1)} P_{k}^{(N,\nu)}(2z_{i,N}^{(\nu-1)})^2 = 1 \quad (k = 0, \ldots, N - 1). \quad (4.5)$$

This normalization, the notations of Lemma 4.1(3), and the orthogonality of the $P_{k}^{(N,\nu)}$ ensure that the matrices

$$T_{N} := (r_{i} \cdot P_{k}^{(N,\nu)}(r_{i}^2))_{i=1,\ldots,N,k=0,\ldots,N-1} \quad (4.6)$$

are orthogonal.

The polynomials $P_{k}^{(N,\nu)}$ can be computed explicitly for small degrees. We have in particular,

$$P_{0}^{(N,\nu)}(x) = c_{0}, \quad P_{1}^{(N,\nu)}(x) = c_{1}(x - 2(2N + \nu - 2)), \quad (4.7)$$

$$P_{2}^{(N,\nu)}(x) = c_{2}(x^2 - 4(2N + \nu - 3)x + 4[(2N + \nu - 3)(2N + \nu - 2) - N(N + \nu - 1)])$$

with the constants $c_{0}, c_{1},$ and $c_{2}$ given by

$$c_{0}^{-2} = 2N(N+\nu-1),$$

$$c_{1}^{-2} = 8N(N+\nu-1)[N(N+\nu-1) - (2N+\nu-2)],$$

$$c_{2}^{-2} = 32N(N+\nu-1)[N^2(N+\nu-1)^2 - N(N+\nu-1)(6N+3\nu-8) + 2(2N+\nu-2)(2N+\nu-3)]. \quad (4.8)$$

These formulae follow from direct calculations, and in particular the formula for $P_{1}^{(N,\nu)}$ stems from item (2) in Lemma 4.1.

We characterize the matrix $S_N$ of type B in the following theorem.

**Theorem 4.3.** For $N \geq 2$, the matrix $S_N$ in Theorem 4.2 has the eigenvalues $2, 4, \ldots, 2N$. Moreover, for $k = 0, 1, \ldots, N - 1$ and the eigenvalue $2(k+1)$, an eigenvector is given by

$$(r_{1} P_{k}^{(N,\nu)}(r_{1}^2), \ldots, r_{N} P_{k}^{(N,\nu)}(r_{N}^2))^{T}.$$  

In particular,

$$S_N = T_N \cdot \text{diag}(2,4,\ldots,2N) \cdot T_N^{T}.$$
Proof. The strategy of the proof is identical to that of Theorem 3.1, so we only specify the differences. In order to simplify the calculations that follow, we write down the action of the matrix $S_N$ on a generic vector $v$:

\[
(S_Nv)_i = \sum_{j=1}^{N} s_{i,j} v_j = \left(1 + \frac{2\nu}{r_i^2}\right)v_i + 2 \sum_{l : l \neq i} v_l \left(\frac{1}{(r_i + r_l)^2} + \frac{1}{(r_i - r_l)^2}\right) + 2 \sum_{j : j \neq i} v_j \left(\frac{1}{(r_i + r_j)^2} - \frac{1}{(r_i - r_j)^2}\right)
\]

\[
= \left(1 + \frac{2\nu}{r_i^2}\right)v_i + 4 \sum_{l : l \neq i} \frac{v_l (r_i^2 + r_l^2)}{(r_i^2 - r_l^2)^2} - 8 \sum_{j : j \neq i} v_j \frac{r_i r_j}{(r_i^2 - r_j^2)^2}
\]

\[
= \left(1 + \frac{2\nu}{r_i^2}\right)v_i + 4 \sum_{l : l \neq i} \frac{v_l (r_i^2 + r_l^2) - 2v_l r_i r_l}{(r_i^2 - r_l^2)^2}
\]

\[
= 2v_i - 4 \sum_{l : l \neq i} \frac{v_i}{r_i^2 - r_l^2} + 4 \sum_{l : l \neq i} \frac{v_l (r_i^2 + r_l^2) - 2v_l r_i r_l}{(r_i^2 - r_l^2)^2}
\]

\[
= 2 \left[v_i + 4 \sum_{l : l \neq i} \frac{v_l r_i - v_l r_l}{(r_i^2 - r_l^2)^2}\right]
\]

\[
= 2 \left[v_i + 4 \sum_{l : l \neq i} \frac{v_l r_i - v_l r_l}{(r_i^2 - r_l^2)^2}\right].
\]

We used item (2) in Lemma 4.1 in the fifth line of the calculation. The induction here starts with $k = 0$ and its corresponding eigenvector $(r_1, \ldots, r_N)^T$, giving 2 as the eigenvalue. For the eigenvalue 4, it can be easily verified that the corresponding eigenvector is given by

\[
(r_1 P_1^{(N,\nu)}(r_1^2), \ldots, r_N P_1^{(N,\nu)}(r_N^2))^T.
\]

In the induction step, we consider the vector

\[
v_{2k+1} := (r_1^{2k+1}, \ldots, r_N^{2k+1})^T,
\]
and for \( k > 1 \) we obtain the following using (4.9):

\[
\left( (S_N - 2(k + 1)I_N)v_{2k+1} \right)_i = 2 \left[ -kr_i^{2k+1} + 4r_i \sum_{l: l \neq i} r_i^{(k-1-m)} \sum_{l: l \neq i} \frac{r_i^{2(m+1)} - r_i^{2(1-n)}}{r_i^2 - r_i^2} \right]
\]

\[
= 2 \left[ -kr_i^{2k+1} - 4r_i \sum_{m=0}^{k-1} r_i^{2(k-1-m)} \sum_{l: l \neq i} \frac{r_i^{2(m+1)} - r_i^{2(1-n)}}{r_i^2 - r_i^2} \right]
+ 4r_i \sum_{m=0}^{k-1} r_i^{2k} \sum_{l: l \neq i} \frac{1}{r_i^2 - r_i^2}
\]

\[
= 2 \left[ -kr_i^{2k+1} - 4r_i \sum_{m=0}^{k-1} r_i^{2(k-1-m)} \sum_{l: l \neq i} r_i^{2(m-n)} + kr_i^{2k+1} (1 - 2\nu/r_i^2) \right]
\]

\[
= -4 \left[ 2r_i \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} r_i^{2(k-1-n)} \sum_{l: l \neq i} r_i^{2m} + kr_i^{2k-1} \right].
\]

For the fourth equality we have made use of item (2) in Lemma 4.1 again. Now, we introduce the sums \( s_n = \sum_{j=1}^{N} r_j^n \) (note that \( s_0 = N \)), and we write

\[
\left((S_N - 2(k + 1)I_N)v_{2k+1}\right)_i = -4 \left[ 2r_i \sum_{n=0}^{k-1} (k-n) r_i^{2(k-1-n)} (s_{2n} - r_i^{2n}) + kr_i^{2k-1} \right]
\]

\[
= -4 \left[ 2 \sum_{n=0}^{k-1} (k-n) s_{2n} r_i^{2(k-n)-1} - k(k+1)r_i^{2k-1} + kr_i^{2k-1} \right]
\]

\[
= -4 \left[ 2 \sum_{n=1}^{k-1} (k-n) s_{2n} r_i^{2(k-n)-1} + k(2N + \nu - k - 1)r_i^{2k-1} \right].
\]

We have used the requirement that \( k > 1 \) for the last equality. Clearly, each term in this polynomial is of odd degree. As before, it can be confirmed directly that

\[
s_{2l} = 2 \left[ \sum_{m=0}^{l-1} s_{2(l-1-m)} s_{2m} - (l - \nu) s_{2(l-1)} \right]
\]

for \( l > 0 \) with \( s_0 = N \), so all coefficients \( s_{2l} \) are functions of \( N \) and \( \nu \). Therefore, we have a polynomial \( p_{k-1} \) of degree \( k - 1 \) such that

\[
(S_N - 2(k + 1)I_N)v_{2k+1} = (r_1 p_{k-1}(r_1^2), \ldots, r_N p_{k-1}(r_N^2))^T
\]

The rest of the proof is virtually identical with that of Theorem 3.1, one only needs to keep track of the degrees of the polynomials in the induction step to obtain the (mutually orthogonal) eigenvectors of \( S_N \). The associated polynomials are then orthogonal with respect to the measure \( \mu_{N, \nu} \). □
Now, we study the polynomials \((P_k^{(N,\nu)})_k\) more closely for fixed \(k\) and \(\nu\) and large dimensions \(N\) as in the preceding section. For this we first conclude from Theorem 1 of Gawronski [G] that the discrete probability measures
\[
\frac{1}{N}(\delta_{z_{1,N}^{(\nu-1)}/4N} + \ldots + \delta_{z_{N,N}^{(\nu-1)}/4N})^k
\]
(4.14)
tend weakly to the beta distribution \(\beta(1/2, 3/2) \in \mathcal{M}^1([0,1])\), which has the density
\[
f(t) = \frac{2}{\pi} t^{-1/2}(1-t)^{1/2}1_{[0,1]}(t).
\]
As the zeroes \(z_{i,N}^{(\nu-1)}/4N\) are contained in some compact interval for all \(i, N\) (see e.g. Section 6.32 of [S]), we conclude readily from the definition of weak convergence that the measures
\[
\frac{1}{4N^2}(\delta_{z_{1,N}^{(\nu-1)}/4N} + \ldots + \delta_{z_{N,N}^{(\nu-1)}/4N})^k
\]
tend weakly to the measure on \([0, 1]\) with density
\[
\frac{2}{\pi} t^{1/2}(1-t)^{1/2}1_{[0,1]}(t)
\]
where this measure has the mass 1/4. Hence, after normalization, the probability measures
\[
\frac{1}{N(N+\nu-1)}(\delta_{z_{1,N}^{(\nu-1)}/4N} + \ldots + \delta_{z_{N,N}^{(\nu-1)}/4N})^k
\]
tend weakly to the probability measure on \([0, 1]\) with density
\[
\frac{8}{\pi} t^{1/2}(1-t)^{1/2}1_{[0,1]}(t).
\]
After the transformation \([0, 1] \to [-1, 1], t \to 2t-1\), the image of this measure is just the semicircle law \(\mu_{sc} \in \mathcal{M}^1([-1,1])\) of the preceding section. In summary we see that for random variables \(Z_N\) with the distributions \(\mu_{N,\nu} (\nu \text{ fixed})\) from (4.3), the transformed random variables \(2\frac{Z_N}{\sqrt{2N^2 - 1}} - 1 = 2\frac{Z_N}{\sqrt{2N^2 - \frac{1}{\nu}}} - 1\) tend to \(\mu_{sc}\) in distribution. This observation in combination with the normalizations of the \(P_k^{(N,\nu)}\) in (4.5) proves readily and in a way similar to Lemma 3.3 the following convergence result for the \(P_k^{(N,\nu)}\) when \(N \to \infty\) with the Tchebychev polynomials \((U_k)_{k \geq 0}\) from Section 3 as limit:

**Lemma 4.4.** For each \(\nu > 0\), and each integer \(k \geq 0\),
\[
\lim_{N \to \infty} \sqrt{2N(N+\nu-1)} \cdot P_k^{(N,\nu)}(4N(x+1)) = U_k(x)
\]
locally uniformly in \(x\).

We expect that this limit can be used to derive additional limit results when we first take \(\beta \to \infty\) and then \(N \to \infty\), much like in the end of Section 3.

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