Generalized Hukuhara-Clarke Derivative of Interval-valued Functions and its Properties

Ram Surat Chauhan  
Indian Institute of Technology BHU

Debas Ghosh  (✉ debdas.mat@iitbhu.ac.in)  
Indian Institute of Technology BHU  https://orcid.org/0000-0003-2419-7082

Jaroslav Ramik  
Silesian University

Amit Kumar Debnath  
Indian Institute of Technology BHU

Research Article

Keywords: Interval-valued functions, Upper gH-Clarke derivative, Sublinear IVF, gH-Lipschitz function

Posted Date: June 30th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-277609/v1

License: ☕️ This work is licensed under a Creative Commons Attribution 4.0 International License.  
Read Full License
Generalized Hukuhara-Clarke Derivative of Interval-valued Functions and its Properties

Ram Surat Chauhan · Debdas Ghosh · Jaroslav Ramík · Amit Kumar Debnath

Received: date / Accepted: date

Abstract This paper is devoted to the study of \(gH\)-Clarke derivative for interval-valued functions. To develop the properties of \(gH\)-Clarke derivative, the concepts of limit superior, limit inferior, and sublinear interval-valued functions are studied in the sequel. It is proved that the upper \(gH\)-Clarke derivative of a \(gH\)-Lipschitz continuous interval-valued function (IVF) always exists. Further, it is found that for a convex and \(gH\)-Lipschitz IVF, the upper \(gH\)-Clarke derivative coincides with the \(gH\)-directional derivative. It is observed that the upper \(gH\)-Clarke derivative is a sublinear IVF. Several numerical examples are provided to support the study.

Keywords Interval-valued functions · Upper \(gH\)-Clarke derivative · Sublinear IVF · \(gH\)-Lipschitz function.

1 Introduction

Clarke derivative [9] is applied in nonsmooth analysis where a function do not have a unique linear approximation at a point. Advances of nonsmooth analysis [6, 29] show the essential need of this derivative to handle nondifferentiable functions, especially in the absence of convexity. The topics of optimization [21], control theory [21], variational...
method [1], generalized convexity [28], etc. are wide application areas of Clarke derivative.

The analysis of interval-valued functions (IVFs) enables one to effectively deal with the errors/uncertainties that appear while modeling practical problems. Thus, in this paper, we aim to develop Clarke derivative for IVFs.

As the topic of this study is Clarke derivatives for IVFs, we describe below the literature on calculus of Interval-Valued Functions (IVFs). It is to be noted that there is a relatively large joint intersection of the literature survey of this paper with the recent paper by Ghosh et al. [14]. However, the proposed work in this paper is completely different than that in [14]. In [14], the properties of IVFs that are differentiable in the sense of $gH$-Gâteaux and $gH$-Fréchet derivatives have been studied. On the other hand, we attempt to derive the properties upper $gH$-Clarke differentiable IVFs in this article.

1.1 Literature Survey

In the study of interval analysis, in addition to interval arithmetic, ordering of intervals and the calculus of IVFs play key roles. As a linear ordering of intervals is not found so far, the development of optimization theory with interval-valued function is not a trivial extension of the conventional optimization theory. Most often [4, 14, 32, 37], IOPs have been analyzed with respect to a partial ordering [20]. Some researchers [3, 11] used ordering relations of intervals based on the parametric comparison of intervals. In [7], an ordering relation of intervals is defined by a bijective correspondence between intervals and the points in $\mathbb{R}^2$. However, these ordering relations [3, 11, 7] of intervals can be derived from the relations described in [20]. Sengupta et al. [30] proposed an acceptability function for intervals, just like a fuzzy membership function. Recently, Ghosh et al. [15] investigated variable ordering relations for intervals and used them in IOPs.

To observe the properties of an IVF, calculus plays an essential role. Initially, in order to develop the calculus of IVFs, Hukuhara [19] introduced the concept of differentiability ($H$-differentiability) of IVFs with the help of the $H$-difference. However, $H$-differentiability is restrictive [4]. For instance, for a given interval $I$ and a differentiable function $h : \mathbb{R} \rightarrow \mathbb{R}$ with $h'(x) < 0$, the IVF $H(x) = I \odot h(x)$ is not $H$-differentiable [2]. To remove the deficiencies of $H$-differentiability, Bede and Gal [2] defined strongly generalized derivative ($G$-derivative) for IVFs and derived a Newton-Leibnitz-type formula. In order to formulate the mean-value theorem for IVFs, Markov [25] introduced a new concept of difference of intervals and defined differentiability of IVFs by this difference. In [32], Stefanini and Bede defined the generalized Hukuhara differentiability ($gH$-differentiability) of IVF by using the concept of generalized Hukuhara difference. Some fractional calculus for IVFs is reported in [24]. In defining the calculus of IVFs, the concepts of $gH$-partial derivative, $gH$-gradient, and $gH$-differentiability for IVFs have been developed in [4, 10, 25, 32, 34].

To derive a Karush-Kuhn-Tucker (KKT) condition for interval optimization problems (IOPs), Guo et al. [17] defined $gH$-symmetric derivative for IVFs. Ghosh [12]
analyzed the notion of $gH$-differentiability of multi-variable IVFs to propose a Newton-type method for IOPs. The concept of second order differentiability of IVFs is used in [35] to find existence-uniqueness results on solving interval differential equations. Lupulescu [23] defined delta generalized Hukuhara differentiability by using $gH$-difference and Riemann delta integrability for IVFs. Chalco et al. [5] introduced the concept of $\pi$-derivative for IVFs that generalizes Hukuhara derivative and $G$-derivative, and it is proved that this derivative is equivalent to $gH$-derivative. In [33], Stefanini and Bede defined level-wise generalized Hukuhara differentiability for IVFs and generalized fuzzy differentiability with LU-parametric representation for fuzzy-valued functions. Kalani et al. [22] analyzed the concept of interval-valued fuzzy derivative for perfect and semi-perfect interval-valued fuzzy mappings to derive a method for solving interval-valued fuzzy differential equations. Recently, Ghosh et al. [14] have provided the idea of $gH$-directional derivative, $gH$-Gâteaux derivative, and $gH$-Fréchet derivative of IVFs to derive the optimality conditions for IOPs.

1.2 Motivation and Contribution

From the literature on the analysis of IVFs, one can notice that the study of traditional generalized derivative (Clarke derivative) for IVFs have not been developed so far. However, the basic properties of generalized derivatives might be beneficial for characterizing and capturing the optimal solutions of IOPs with nonsmooth IVFs. To define and find properties of Clarke derivative of IVFs, we need to establish the notions of limit superior and sublinearity for IVFs. In this article, after illustrating the concept of limit superior, limit inferior, and sublinearity of IVFs, we define upper and lower $gH$-Clarke derivatives of IVFs. Although both of the upper and lower $gH$-Clarke derivatives of IVFs are defined in this article, only the properties of the upper $gH$-Clarke derivative are studied since the results for the lower derivative can be used analogously. It is shown that if an IVF is upper $gH$-Clarke differentiable at a point, then its derivative is a sublinear IVF. We further prove that the upper $gH$-Clarke derivative exists at any point if the IVF is convex $gH$-Lipschitz and the derivative is equal to $gH$-directional derivative.

1.3 Delineation

The rest of the article is demonstrated as this sequence. The next section covers some basic terminology and notions of convex analysis and interval analysis, followed by the convexity and calculus of IVFs that are required in this paper. Also, a few properties of intervals, the $gH$-directional of an IVF is discussed in Section 2. The concepts of limit superior, sublinear IVF and their properties are given in Section 3. In the same section, we define upper $gH$-Clarke derivative, lower $gH$-Clarke derivative of IVFs, and prove that a $gH$-Lipschitz IVF is always upper $gH$-Clarke differentiable. Also, it is found that upper $gH$-Clarke derivative coincides with $gH$-directional derivative for a convex $gH$-Lipschitz continuous IVF. Further, the sublinearity of the upper $gH$-Clarke derivative is shown in the same section.
2 Preliminaries and Terminology

This section is devoted to some basic notions on intervals and the convexity and calculus of IVFs. We use the following notations in the entire paper.

- $\mathcal{X}$ denotes a real normed linear space with the norm $\| \cdot \|
- \mathcal{S}$ represents a nonempty subset of $\mathcal{X}$
- $B(\bar{x}, \delta)$ denotes the open ball centered at $\bar{x} \in \mathcal{X}$ with radius $\delta$
- $\overline{B}(\bar{x}, \delta)$ denotes $B(\bar{x}, \delta) \setminus \{\bar{x}\}$
- $\mathbb{R}$ represents the set of real numbers
- $\mathbb{R}_+$ is the set of nonnegative real numbers
- $I(\mathbb{R})$ represents the set of compact intervals

2.1 Arithmetic of Intervals and their Dominance Relation

Throughout the article, bold letters $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$, are used for denoting the elements of $I(\mathbb{R})$. An element $\mathbf{X}$ of $I(\mathbb{R})$ is presented by the corresponding small letter: $\mathbf{X} = [x, \bar{x}]$.

Let $\mathbf{X}, \mathbf{Y} \in I(\mathbb{R})$ and $\delta \in \mathbb{R}$. Moore’s [26,27] interval addition, subtraction, product, division and scalar multiplication are denoted by $\mathbf{X} \oplus \mathbf{Y}, \mathbf{X} \ominus \mathbf{Y}, \mathbf{X} \odot \mathbf{Y}, \mathbf{X} \otimes \mathbf{Y}$, and $\delta \odot \mathbf{X}$, respectively. In defining $\mathbf{X} \otimes \mathbf{Y}$, it is assumed that $0 \not\in \mathbf{Y}$. Since $\mathbf{X} \otimes \mathbf{X} \neq 0$ for any nondegenerate interval $\mathbf{X}$, we use the following concept of difference of intervals in this article.

**Definition 1** $(gH$-difference [31]). The $gH$-difference of $\mathbf{X}, \mathbf{Y} \in I(\mathbb{R})$ is an interval $\mathbf{Z}$ such that

$$\mathbf{X} = \mathbf{Y} \oplus \mathbf{Z} \text{ or } \mathbf{Y} = \mathbf{X} \ominus \mathbf{Z}. $$

We denote then $\mathbf{Z} = \mathbf{X} \ominus_{gH} \mathbf{Y}$. For $\mathbf{X} = [x, \bar{x}]$ and $\mathbf{Y} = [y, \bar{y}]$,

$$\mathbf{X} \ominus_{gH} \mathbf{Y} = \left[\min\{x - y, \bar{x} - \bar{y}\}, \max\{x - y, \bar{x} - \bar{y}\}\right] \text{ and } \mathbf{X} \ominus_{gH} \mathbf{X} = 0.$$

In the following, we provide a domination relation on intervals that is used throughout the paper. We remark that domination in the following definition is based on a minimization type optimization problems: the smaller value the better.

**Definition 2** (Dominance of intervals [36]). Consider two intervals $\mathbf{X} = [x, \bar{x}]$ and $\mathbf{Y} = [y, \bar{y}]$ in $I(\mathbb{R})$.

(i) $\mathbf{Y}$ is said to be dominated by $\mathbf{X}$ if $x \leq y$ and $\bar{x} \leq \bar{y}$, and then we write $\mathbf{X} \preceq \mathbf{Y}$;

(ii) $\mathbf{Y}$ is called strictly dominated by $\mathbf{X}$ if either $x < y$ and $\bar{x} < \bar{y}$ or $\bar{x} < y$ and $\bar{x} \leq y$,

(iii) if $\mathbf{Y}$ is not dominated by $\mathbf{X}$, then we write $\mathbf{X} \not\preceq \mathbf{Y}$;

(iv) if either $\mathbf{X}$ is dominated by $\mathbf{Y}$ or $\mathbf{Y}$ is dominated by $\mathbf{X}$, then it will be said that $\mathbf{X}$ and $\mathbf{Y}$ are comparable,

(v) if $\mathbf{X} \not\preceq \mathbf{Y}$ and $\mathbf{Y} \not\preceq \mathbf{X}$, then it will be said that none of $\mathbf{X}$ and $\mathbf{Y}$ dominates the other, or $\mathbf{X}$ and $\mathbf{Y}$ are not comparable.

Notice that if $\mathbf{Y}$ is strictly dominated by $\mathbf{X}$, then $\mathbf{Y}$ is dominated by $\mathbf{X}$. Moreover, if $\mathbf{Y}$ is not dominated by $\mathbf{X}$, then $\mathbf{Y}$ is not strictly dominated by $\mathbf{X}$. 
2.2 Few Results of Intervals

This subsection covered by some results of compact intervals that are used later in the paper. In the rest of the paper, by the norm of an interval, we refer to the following definition.

**Definition 3 (Norm on I(R) [26])**. Let \( Y = [\underline{y}, \overline{y}] \). A function \( \| \cdot \|_{I(R)} : I(R) \to \mathbb{R}^+ \), defined by

\[
\| Y \|_{I(R)} = \| [\underline{y}, \overline{y}] \|_{I(R)} = \max \{ |\underline{y}|, |\overline{y}| \},
\]

is a norm \( I(R) \).

**Lemma 1** Let \( Z \in I(R) \). Then, for all \( a, b \in \mathbb{R} \)

(i) \( Z \geq 0 \) \( \implies \) \( [a + b] \circ Z \leq |a| \circ Z \circ [b] \circ Z \) and

(ii) \( Z \leq 0 \) \( \implies \) \( [a + b] \circ Z \geq |a| \circ Z \circ [b] \circ Z \).

**Proof** See Appendix A.

**Lemma 2** For all \( A, B, C, D \in I(R) \),

(i) \( (A \oplus_{gH} C) \oplus (C \oplus_{gH} B) \neq A \oplus_{gH} B \).

(ii) \( B \preceq A \oplus [L, L] \), where \( L = \|B \oplus_{gH} A\|_{I(R)} \), and

(iii) \( \| (A \oplus_{gH} B) \oplus_{gH} (C \oplus_{gH} D) \|_{I(R)} \leq \| A \oplus_{gH} C \|_{I(R)} \oplus \| B \oplus_{gH} D \|_{I(R)} \).

**Proof** See Appendix B.

**Remark 1** The following two points are noticeable.

(i) For two elements \( A \) and \( B \) of \( I(R) \), if \( B = A \oplus (B \oplus_{gH} A) \), then (ii) of Lemma 2 is an obvious property since \( B \oplus_{gH} A \preceq [L, L] \). However, \( (A \oplus (B \oplus_{gH} A)) \) is not always equal to \( B \). For instance, for \( A = [4, 10] \) and \( B = [-3, 2] \),

\[
A \oplus (B \oplus_{gH} A) = [4, 10] \oplus [-8, -7] = [-4, 3] \neq B.
\]

Therefore, (ii) of Lemma 2 is not a trivial property.

(ii) For any \( A, B \) and \( C \) in \( I(R) \), if

\[
B \oplus_{gH} A \preceq C \implies B \preceq A \oplus C,
\]

then replacing \( C \) by \([L, L]\), we see that (ii) of Lemma 2 is an obvious property. However, (2.1) is not always true. For instance, if \( B = [-3, 2] \), \( A = [4, 10] \) and \( C = [-7.5, -6] \), then

\[
B \oplus_{gH} A = [-8, -7] \text{ and } A \oplus C = [-3.5, 4].
\]

Hence, \( B \oplus_{gH} A \preceq C \), but \( B \) and \( A \oplus C \) are not comparable. Thus, (ii) of Lemma 2 is not an obvious property.
2.3 Convexity and Calculus of IVFs

A function \( \mathbf{G} \) from \( S \) to \( I(\mathbb{R}) \) is known as an IVF. For each \( x \in S \), \( \mathbf{G} \) can be presented by intervals

\[
\mathbf{G}(x) = [\underline{g}(x), \overline{g}(x)],
\]

where \( \underline{g} \) and \( \overline{g} \) are real-valued functions on \( S \) such that \( \underline{g}(x) \leq \overline{g}(x) \) for all \( x \in S \).

If \( S \) is convex, then the IVF \( \mathbf{G} \) is said to be convex [36] on \( S \) if for any \( x, y \in S \),

\[
\mathbf{G}(\delta_1 x + \delta_2 y) \leq \delta_1 \mathbf{G}(x) \odot \delta_2 \mathbf{G}(y)
\]

for all \( \delta_1, \delta_2 \in [0, 1] \) with \( \delta_1 + \delta_2 = 1 \).

The IVF \( \mathbf{G} \) is \( gH \)-continuous \([10]\) at an interior point \( \bar{y} \in S \) if

\[
\lim_{\|v\| \to 0} (\mathbf{G}(\bar{y} + v) \circ_{gH} \mathbf{G}(\bar{y})) = 0.
\]

If \( \mathbf{G} \) is \( gH \)-continuous at each \( \bar{y} \in S \), then \( \mathbf{G} \) is said to be \( gH \)-continuous on \( S \).

The IVF \( \mathbf{G} \) is \( gH \)-Lipschitz continuous \([14]\) at \( \bar{y} \in S \) if there exist constants \( K' > 0 \) and \( \delta > 0 \) such that

\[
\| \mathbf{G}(\bar{y}) \circ_{gH} \mathbf{G}(y) \|_{I(\mathbb{R})} \leq K'\| \bar{y} - y \| \text{ for all } y \in S \cap B(\bar{y}, \delta).
\]

The constant \( K' \) is called a Lipschitz constant of \( \mathbf{F} \) at \( \bar{x} \). If there exists a \( K > 0 \) such that

\[
\| \mathbf{G}(x) \circ_{gH} \mathbf{G}(y) \|_{I(\mathbb{R})} \leq K\| x - y \| \text{ for all } x, y \in S,
\]

then the IVF \( \mathbf{G} \) is \( gH \)-Lipschitz continuous on \( S \) with Lipschitz constant \( K \).

Lemma 3 (See [36]). If \( \mathbf{G} \) is a convex IVF on a convex set \( S \subseteq X \), then \( \underline{g} \) and \( \overline{g} \) are convex on \( S \) and vise-versa.

Lemma 4 Let \( \mathbf{G} \) be an IVF defined on \( S \). Then,

(i) if \( \mathbf{G} \) is \( gH \)-continuous on \( S \), then \( \underline{g} \) and \( \overline{g} \) are continuous on \( S \) and vise-versa.

(ii) if \( \mathbf{G} \) is \( gH \)-Lipschitz continuous on \( S \), then \( \underline{g} \) and \( \overline{g} \) are Lipschitz continuous on \( S \) and vise-versa.

(iii) if \( \mathbf{G} \) is a \( gH \)-Lipschitz continuous on \( S \), then \( \mathbf{G} \) is \( gH \)-continuous on \( S \).

Proof (i) See [16],

(ii) Let \( \mathbf{G} \) be \( gH \)-Lipschitz continuous on \( X \). Thus, there exists \( K > 0 \) such that for any \( x, y \in X \) we have

\[
\| \mathbf{G}(x) \circ_{gH} \mathbf{G}(y) \|_{I(\mathbb{R})} \leq K\| x - y \|
\]

\[
\implies \max \{ \| \underline{g}(x) - \underline{g}(y) \|, \| \overline{g}(x) - \overline{g}(y) \| \} \leq K\| x - y \|
\]

\[
\implies \| \underline{g}(x) - \underline{g}(y) \| \leq K\| x - y \| \text{ and } \| \overline{g}(x) - \overline{g}(y) \| \leq K\| x - y \|.
\]

Hence, \( \underline{g} \) and \( \overline{g} \) are Lipschitz continuous on \( X \).

Conversely, let the functions \( \underline{g} \) and \( \overline{g} \) be Lipschitz continuous on \( X \). Thus, there exist \( K_1, K_2 > 0 \) such that for all \( x, y \in X \),

\[
\| \underline{g}(x) - \underline{g}(y) \| \leq K_1\| x - y \| \text{ and } \| \overline{g}(x) - \overline{g}(y) \| \leq K_2\| x - y \|
\]

\[
\implies \max \{ \| \underline{g}(x) - \underline{g}(y) \|, \| \overline{g}(x) - \overline{g}(y) \| \} \leq \bar{K}\| x - y \|, \text{ where } \bar{K} = \max\{K_1, K_2\}
\]

\[
\implies \| \mathbf{G}(x) \circ_{gH} \mathbf{G}(y) \|_{I(\mathbb{R})} \leq \bar{K}\| x - y \|.
\]

Hence, \( \mathbf{G} \) is \( gH \)-Lipschitz continuous IVF on \( X \).
(iii) Let $G$ be $gH$-Lipschitz continuous on $X$. Then, there exists an $K > 0$ such that for all $x, y \in X$, we have

$$\|G(y) \ominus_{gH} G(x)\|_{l(\mathbb{R})} \leq K\|y - x\|.$$ 

For $h = y - x \in X$,

$$\|G(x + h) \ominus_{gH} G(x)\|_{l(\mathbb{R})} \leq K\|h\|$$

$$\implies \lim_{\|h\| \to 0} \frac{1}{\|h\|} \|G(x + h) \ominus_{gH} G(x)\|_{l(\mathbb{R})} = 0$$

$$\implies \lim_{\|h\| \to 0} (G(x + h) \ominus_{gH} G(x)) = 0.$$ 

Hence, $G$ is $gH$-continuous at $x \in X$.

A consequence of Lemma 4 is that $gH$-continuity and $gH$-Lipschitz continuity of IVFs can be defined classically, i.e., without the concept of $gH$-difference. Then, the prefix $gH$- in continuity and Lipschitz continuity could be omitted.

Remark 2 Converse of (iii) of Lemma 4 is not true. For example, consider $X$ as the Euclidean space $\mathbb{R}$, $S = [0, 10]$, and the IVF $G : S \to l(\mathbb{R})$, which is defined by

$$G(y) = \sqrt{y} \ominus [2, 5].$$

Since $g(y) = 2\sqrt{y}$ and $\overline{g}(y) = 5\sqrt{y}$ are continuous on $S$, $G$ is $gH$-continuous on $S$ by (i) of Lemma 4. If $G$ is $gH$-Lipschitz continuous on $S$, then by (ii) of Lemma 4, $g$ and $\overline{g}$ are Lipschitz continuous on $S$, which is not true. Consequently, $G$ is not $gH$-Lipschitz continuous on $S$.

Definition 4 ($gH$-directional derivative [14,34]). Let $T$ be an IVF on $S$. Then, $T$ is $gH$-directional differentiable at $\bar{y} \in S$ in direction $v \in X$ if the limit

$$\lim_{\delta \to 0^+} \frac{1}{\delta} \ominus (T(\bar{y} + \delta v) \ominus_{gH} T(\bar{y}))$$

exists finitely. The limit is denoted by $T_{gH}(\bar{y})(h)$, is known as $gH$-directional derivative of $T$ at $\bar{y}$ in the direction $v$.

3 $gH$-Clarke Derivative of IVF

In this section, extended concepts of the $gH$-directional derivative, namely upper and lower $gH$-Clarke derivatives, for IVFs are given. A short discussion on the required notions of limit superior and sublinearity for IVFs is provided.

Definition 5 (Supremum and limit superior of an IVF). Let $G$ be an IVF defined on $S$. Then, the supremum of $G$ over $S$ is defined by

$$\sup_{S} G = \left[ \sup_{S} g, \sup_{S} \overline{g} \right],$$
where \( \sup_{\mathcal{S}} g = \sup \{ g(y) : y \in \mathcal{S} \} \) and \( \sup_{\mathcal{S}} \overline{g} = \sup \{ \overline{g}(y) : y \in \mathcal{S} \} \).

The limit superior of \( G \) at a limit point \( \bar{y} \) of \( \mathcal{S} \) is defined by

\[
\limsup_{y \to \bar{y}} G(y) = \left[ \limsup_{y \to \bar{y}} g(y), \limsup_{y \to \bar{y}} \overline{g}(y) \right],
\]

where \( \limsup_{y \to \bar{y}} g(y) = \lim_{\delta \to 0} \left( \sup_{y \in B(y, \delta) \cap \mathcal{S}} g(y) \right) \) and \( \limsup_{y \to \bar{y}} \overline{g}(y) = \lim_{\delta \to 0} \left( \sup_{y \in B(y, \delta) \cap \mathcal{S}} \overline{g}(y) \right) \).

**Definition 6** (Infimum and limit inferior of an IVF). Let \( G \) be an IVF defined on \( \mathcal{S} \). Then, the infimum of \( G \) over \( \mathcal{S} \) is defined by

\[
\inf_{\mathcal{S}} G = \left[ \inf_{\mathcal{S}} g, \inf_{\mathcal{S}} \overline{g} \right],
\]

where \( \inf_{\mathcal{S}} g = \inf \{ g(y) : y \in \mathcal{S} \} \) and \( \inf_{\mathcal{S}} \overline{g} = \inf \{ \overline{g}(y) : y \in \mathcal{S} \} \).

The limit inferior of \( G \) at a limit point \( \bar{y} \) of \( \mathcal{S} \) is defined by

\[
\liminf_{y \to \bar{y}} G(y) = \left[ \liminf_{y \to \bar{y}} g(y), \liminf_{y \to \bar{y}} \overline{g}(y) \right],
\]

where \( \liminf_{y \to \bar{y}} g(y) = \lim_{\delta \to 0} \left( \inf_{y \in B(y, \delta) \cap \mathcal{S}} g(y) \right) \) and \( \liminf_{y \to \bar{y}} \overline{g}(y) = \lim_{\delta \to 0} \left( \inf_{y \in B(y, \delta) \cap \mathcal{S}} \overline{g}(y) \right) \).

**Lemma 5** Let \( H \) and \( G \) be two IVFs defined on \( \mathcal{S} \). Then, at any \( \bar{y} \in \mathcal{S} \), the following properties are true:

(i) \( \limsup_{y \to \bar{y}} (H(y) \oplus G(y)) \leq \limsup_{y \to \bar{y}} H(y) \oplus \limsup_{y \to \bar{y}} G(y) \),

(ii) \( \limsup_{y \to \bar{y}} (\delta \odot G(y)) = \delta \odot \limsup_{y \to \bar{y}} G(y) \) for all \( \delta \geq 0 \), and

(iii) \( \left\| \limsup_{y \to \bar{y}} G(y) \right\|_{I(\mathbb{R})} \leq \limsup_{y \to \bar{y}} \| G(y) \|_{I(\mathbb{R})} \).

**Proof** (i) Since

\[
\limsup_{y \to \bar{y}} (h(y) + g(y)) \leq \limsup_{y \to \bar{y}} h(y) + \limsup_{y \to \bar{y}} g(y)
\]

and

\[
\limsup_{y \to \bar{y}} (\overline{h}(y) + \overline{g}(y)) \leq \limsup_{y \to \bar{y}} \overline{h}(y) + \limsup_{y \to \bar{y}} \overline{g}(y),
\]

therefore

\[
\left[ \limsup_{y \to \bar{y}} (h(y) + g(y)), \limsup_{y \to \bar{y}} (\overline{h}(y) + \overline{g}(y)) \right] \leq \left[ \limsup_{y \to \bar{y}} h(y), \limsup_{y \to \bar{y}} \overline{h}(y) \right] \oplus \left[ \limsup_{y \to \bar{y}} g(y), \limsup_{y \to \bar{y}} \overline{g}(y) \right],
\]

which implies \( \limsup_{y \to \bar{y}} (H(y) \oplus G(y)) \leq \limsup_{y \to \bar{y}} H(y) \oplus \limsup_{y \to \bar{y}} G(y) \).
(ii) Since $g$ and $\overline{g}$ are define on $\mathbb{R}$, for any $\delta \geq 0$, we have
\[
\limsup_{y \to \bar{y}} (\delta g(y)) = \delta \limsup_{y \to \bar{y}} g(y) \text{ and } \limsup_{y \to \bar{y}} (\delta \overline{g}(y)) = \delta \limsup_{y \to \bar{y}} \overline{g}(y). \tag{3.1}
\]

Hence, for any $\delta \geq 0$,
\[
\limsup_{y \to \bar{y}} (\delta \circ G(y)) = \left[ \limsup_{y \to \bar{y}} (\delta g(y)), \limsup_{y \to \bar{y}} (\delta \overline{g}(y)) \right] \nonumber
\]
\[
= \delta \circ \limsup_{y \to \bar{y}} G(y) \text{ by (3.1)}.
\]

(iii) Let $g$ be define on $\mathbb{R}$. Then, $\left| \limsup_{y \to \bar{y}} g(y) \right| \leq \limsup_{y \to \bar{y}} |g(y)|$. By the definition of norm on $I(\mathbb{R})$,
\[
\left| \limsup_{y \to \bar{y}} G(y) \right| = \max \left\{ \left| \limsup_{y \to \bar{y}} g(y) \right|, \left| \limsup_{y \to \bar{y}} \overline{g}(y) \right| \right\} \nonumber
\]
\[
\leq \limsup_{y \to \bar{y}} \| G(y) \|_{I(\mathbb{R})}.
\]

**Definition 7 (Upper $gH$-Clarke derivative).** Let $G$ be an IVF defined on $S$. For $\bar{y} \in S$ and $v \in X$, if the limit superior
\[
\limsup_{y \to \bar{y}} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) = \lim_{\delta \to 0} \sup_{y \in \overline{S}(\bar{y}, \delta) \cap S, t \in (0, \delta)} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y))
\]
exists finitely, then the limit superior value is called upper $gH$-Clarke derivative of $G$ at $\bar{y}$ in the direction $v$, and it is denoted by $G_{\bar{e}}(\bar{y})(v)$. If this limit superior exists for all $v \in X$, then $G$ is said to be upper $gH$-Clarke differentiable at $\bar{y}$.

**Definition 8 (Lower $gH$-Clarke derivative).** Let $G$ be an IVF defined on $S$. For $\bar{y} \in S$ and $v \in X$, if the limit inferior
\[
\liminf_{y \to \bar{y}} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) = \lim_{\delta \to 0} \inf_{y \in \overline{S}(\bar{y}, \delta) \cap S, t \in (0, \delta)} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y))
\]
exists finitely, then the limit inferior value is called lower $gH$-Clarke derivative of $G$ at $\bar{y}$ in the direction $v$. If this limit inferior exists for all $v \in X$, then $G$ is said to be lower $gH$-Clarke differentiable at $\bar{y}$.

If $G$ has both upper and lower $gH$-Clarke derivatives at $\bar{y}$ and they are equal, then $G$ is called $gH$-Clark differentiable at $\bar{y}$.

**Remark 3** Conventionaly, for real valued-functions, the terminologies Clarke derivative [6, 21] and upper Clarke derivative [8] are interchangebably used. In fact, the upper Clarke derivative is usually referred to Clarke derivative. However, in order to avoid any confusion, we prefix upper and lower with the Clarke derivative corresponding to the values given by limit superior and limit inferior, respectively. In addition, throughout the article, we use the notation $G_{\bar{e}}$ to refer the upper $gH$-Clarke derivative of an IVF $G$. 

Remark 4 It is to note that $G$ is lower $gH$-Clark differentiable at $\bar{y}$ if and only if $[-1, -1] \circ G$ is upper $gH$-Clark differentiable at $\bar{y}$. That is why we deal only with the upper $gH$-Clark differentiability in this study.

Example 1 In this example, we calculate the upper $gH$-Clarke derivative at $\bar{y} = 0$ for the IVF $G(y) = |y| \circ C$, where $0 \preceq C \in L(\mathbb{R})$, $X$ is the Euclidean space $\mathbb{R}$, and $S = X$. For any $v \in X$, we see that

$$\limsup_{y \to 0} \frac{1}{t} \circ (G(y + tv) \odot gH G(y))$$

$$\preceq \limsup_{y \to 0} \frac{1}{t} \circ (|y| \circ C \odot |v| \circ C \odot gH |y| \circ C) \text{ by Lemma 1}$$

$$= |v| \circ C. \quad (3.2)$$

Further,

$$\limsup_{y \to 0} \frac{1}{t} \circ (|y + tv| \circ C \odot gH |y| \circ C)$$

$$\gtrsim \limsup_{y \to 0} \frac{1}{t} \circ (2|v| \circ C \odot gH t|v| \circ C) \text{ (taking } y = tv, v > 0)$$

$$= |v| \circ C. \quad (3.3)$$

Then, from the inequalities (3.2) and (3.3), we get $G_{\bar{y}}(v) = |v| \circ C$.

Lemma 6 If $g$ and $\bar{y}$ are upper Clarke differentiable at $\bar{y} \in S \subseteq X$, then the IVF $G$ is upper $gH$-Clarke differentiable at $\bar{y} \in S$.

Proof Since $g$ and $\bar{y}$ are upper Clarke differentiable at $\bar{y}$. Therefore, both of the following limits exist,

$$\limsup_{y \to \bar{y}} \frac{1}{t} l_1(t) \text{ and } \limsup_{y \to \bar{y}} \frac{1}{t} l_2(t)$$

exist, where $l_1(t) = g(y + tv) - g(y)$ and $l_2(t) = \bar{g}(y + tv) - \bar{g}(y)$. Thus,

$$\limsup_{y \to \bar{y}} \frac{1}{t} \left( l_1(t) + l_2(t) \right) \text{ and } \limsup_{y \to \bar{y}} \frac{1}{t} |l_1(t) - l_2(t)| \text{ exist}$$

$$\implies \limsup_{y \to \bar{y}} \frac{1}{2t} \left( l_1(t) + l_2(t) - |l_1(t) - l_2(t)| \right) \text{ and }$$

$$\limsup_{y \to \bar{y}} \frac{1}{2t} \left( l_1(t) + l_2(t) + |l_1(t) - l_2(t)| \right) \text{ exist}$$

$$\implies \limsup_{y \to \bar{y}} \frac{1}{t} \left( \min \{l_1(t), l_2(t)\} \right) \text{ and } \limsup_{y \to \bar{y}} \frac{1}{t} \left( \max \{l_1(t), l_2(t)\} \right) \text{ exist}$$

$$\implies \limsup_{y \to \bar{y}} \frac{1}{t} \circ (G(y + tv) \odot gH G(y)) \text{ exists.}$$

Hence, $G$ is upper $gH$-Clarke differentiable IVF at $\bar{y} \in S$.  

10
Remark 5 Let \( \mathbf{G} \) be upper \( gH \)-Clarke differentiable IVF at a point \( \bar{z} \) in \( S \). Then, \( \mathbf{G} \) may not be \( gH \)-directional differentiable at \( \bar{z} \in S \). For example, take \( \mathcal{X} \) as the Euclidean space \( \mathbb{R} \), \( S = \mathcal{X} \) and the IVF \( \mathbf{G} : S \to I(\mathbb{R}) \), which is defined by

\[
\mathbf{G}(z) = \begin{cases} 
\sin^2 \frac{z}{z} \circ \mathbf{C} & \text{if } z \neq 0 \\
\frac{z}{5} \circ \mathbf{C} & \text{if } z = 0,
\end{cases}
\]

where \( \mathbf{C} \in I(\mathbb{R}) \) with \( \mathbf{C} \geq 0 \). For all nonzero \( v \) in \( \mathcal{X} \), we obtain

\[
\lim_{t \to 0^+} \frac{1}{t} \circ (\mathbf{G}(z + tv) \ominus gH \mathbf{G}(z)) = \mathbf{v} \circ \mathbf{C},
\]

which implies

\[
\limsup_{t \to 0^+} \frac{1}{t} \circ (\mathbf{G}(z + tv) \ominus gH \mathbf{G}(z)) = \mathbf{v} \circ \mathbf{C}.
\]

Thus,

\[
\lim_{z \to 0^+} \frac{1}{t} \circ (\mathbf{G}(z + tv) \ominus gH \mathbf{G}(z)) = \mathbf{v} \circ \mathbf{C},
\]

Hence, \( \mathbf{G} \) is upper \( gH \)-Clarke differentiable at \( \bar{z} = 0 \). However, the limit

\[
\lim_{t \to 0^+} \frac{1}{t} \circ (\mathbf{G}(\bar{z} + tv) \ominus gH \mathbf{G}(\bar{z})) = \mathbf{v} \circ \mathbf{C} = \mathbf{0},
\]

does not exist at \( \bar{z} = 0 \). Consequently, \( \mathbf{G} \) is not \( gH \)-directional differentiable at \( \bar{z} \).

Remark 6 Let \( \mathbf{G} \) be \( gH \)-directional differentiable IVF at a point \( \bar{y} \) in \( S \). Then, \( \mathbf{G} \) is not necessarily upper \( gH \)-Clarke differentiable at \( \bar{y} \in S \). For instance, take \( \mathcal{X} \) as the Euclidean space \( \mathbb{R}^2 \), \( S = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0, y_2 \geq 0\} \) and the IVF \( \mathbf{G} : S \to I(\mathbb{R}) \), which is defined by

\[
\mathbf{G}(y_1, y_2) = \begin{cases} 
\bar{y}^2 (1 + \frac{1}{y_2}) \circ [3, 8] & \text{if } y = (y_1, y_2) \neq (0, 0) \\
0 & \text{otherwise}.
\end{cases}
\]

Then, at \( \bar{y} = (0, 0) \) and \( v = (v_1, v_2) \in \mathcal{X} \) such that for sufficiently small \( t > 0 \) so that \( \bar{y} + tv \in S \), we have

\[
\lim_{t \to 0^+} \frac{1}{t} \circ (\mathbf{G}(\bar{y} + tv) \ominus gH \mathbf{G}(\bar{y})) = \begin{cases} 
\frac{v_2^2}{v_2} \circ [3, 8] & \text{if } v_2 \neq 0 \\
0 & \text{otherwise}.
\end{cases}
\]
Hence, $F$ is a $gH$-directional differentiable at $\bar{y}$ in every direction $v \in \mathcal{X}$.

Again, for $y = (y_1, y_2) \in S$ and $v = (v_1, v_2) \in \mathcal{X}$, we have

\[
\lim_{t \to 0^+} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) = \lim_{t \to 0^+} \frac{1}{t} \circ \left( (y_1 + tv_1)^2 \left( 1 + \frac{1}{y_2 + tv_2} \right) \ominus [3, 8] \ominus_{gH} y_1^2 \left( 1 + \frac{1}{y_2} \right) \ominus [3, 8] \right)
\]

\[
= \min \left\{ 3 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right), 8 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right) \right\}, \max \left\{ 3 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right), 8 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right) \right\}
\]

Along $y_2 = my_1$, where $m$ is any real number,

\[
\lim_{y \to y_0} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) = \left\{ \min \left\{ 3 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right), 8 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right) \right\}, \max \left\{ 3 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right), 8 \left( \frac{2v_1}{m} - \frac{v_2}{m^2} \right) \right\} \right\}
\]

Hence, for $v_2 > 0$, $(\frac{2v_1}{m} - \frac{v_2}{m^2}) \to -\infty$ as $m \to 0$. Consequently,

\[
\limsup_{y \to y_0} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) \text{ does not exist.}
\]

This implies that $G$ has no upper $gH$-Clarke derivative at $\bar{y} \in S$.

The following theorem extends the well-known result from [21] for Lipschitz continuous functions to $gH$-Lipschitz continuous IVFs with the help of Lemma 6.

**Theorem 1** Let $S \subset \mathcal{X}$ and $G : S \to I(\mathbb{R})$ be a $gH$-Lipschitz continuous IVF at an interior point $\bar{y}$ with a Lipschitz constant $K'$. Then, $G$ is upper $gH$-Clarke differentiable at $\bar{y}$ and

\[
\| G'_{\bar{y}}(v) \|_{I(\mathbb{R})} \leq K' \| v \| \text{ for all } v \in \mathcal{X}.
\]

**Proof** Since $G$ is $gH$-Lipschitz continuous at $\bar{y} \in S$, for any $v \in \mathcal{X}$, we get for $t > 0$ that

\[
\left\| \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) \right\|_{I(\mathbb{R})} \leq \frac{1}{t} K' \| y + tv - y \| = K' \| v \|,
\]

if $y$ and $t$ are sufficiently close to $\bar{y}$ and 0, respectively. From inequality (3.4) we have

\[
\left| \frac{1}{t} (g(y + tv) - g(y)) \right| \leq K' \| v \| \text{ and } \left| \frac{1}{t} (\overline{g}(y + tv) - \overline{g}(y)) \right| \leq K' \| v \|.
\]

Hence, the limit superior $g_{\bar{y}}(v)$ and $\overline{g}_{\bar{y}}(v)$ exist at $\bar{y}$ (cf. p. 69 of [21]). By Lemma 6, the limit superior $G'_{\bar{y}}(v)$ exists. Furthermore, by $gH$-Lipschitz continuity of $G$ on $S$, we have the following for all $v \in \mathcal{X}$:

\[
\| G'_{\bar{y}}(v) \|_{I(\mathbb{R})} = \left\| \limsup_{t \to 0^+} \frac{1}{t} \circ (G(y + tv) \ominus_{gH} G(y)) \right\|_{I(\mathbb{R})}
\]
Due to convexity of $G$, since $G$ is $gH$-directionally differentiable at $\bar{y}$ in every direction $v \in \mathcal{X}$. Also, as $G$ is $gH$-Lipschitz continuous at $\bar{y}$, from Theorem 1, we get that $G$ is upper $gH$-Clarke differentiable at any $\bar{y}$ in every direction $v \in \mathcal{X}$. Thus, by Definitions 4 and 7, we observe that

$$G_{\varphi}(\bar{y}) (v) \leq G_{\varphi}(\bar{y}) (v) \text{ for all } v.$$ 

For the reverse of inequality (3.5), we write

$$G_{\varphi}(\bar{y}) (v) = \lim_{\delta \to 0+} \sup_{\|v - \bar{y}\| < \delta} \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)).$$

Due to convexity of $G$ on $\mathcal{X}$ and Lemma 3.1 of [14], we have the following equality

$$G_{\varphi}(\bar{y}) (v) = \lim_{\epsilon \to 0+} \sup_{\|v - \bar{y}\| < \epsilon } \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)).$$

For an arbitrary $\alpha > 0$ and from (3.6), we obtain

$$G_{\varphi}(\bar{y}) (v) \leq \lim_{\epsilon \to 0+} \sup_{\|v - \bar{y}\| < \epsilon \alpha } \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)).$$

Since $G$ is $gH$-Lipschitz continuous at $\bar{y}$ and $\|y - \bar{y}\| < \epsilon \alpha$ for sufficiently small $\epsilon > 0$,

$$\left\| \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)) \right\|_{L(\mathbb{R})} \leq \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)),$$

by (ii) of Lemma 2

$$\leq \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)) \leq K' \|y - \bar{y}\|,$$ 

where $K'$ is the Lipschitz constant of $G$ at $\bar{y} \in \mathcal{X}$

$$\leq 2K' \alpha.$$

Then, by (ii) of Lemma 2, we have

$$G_{\varphi}(\bar{y}) (v) \leq \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \circ (G(y + \epsilon v) \ominus_{gH} G(y)) \ominus [2K' \alpha, 2K' \alpha].$$

$$= G_{\varphi}(\bar{y}) (v) \ominus [2K' \alpha, 2K' \alpha].$$
Due to arbitrariness of $\alpha > 0$, we get
\[ G_{e}(y)(v) \leq G_{\delta y}(v) \text{ for all } v. \quad (3.7) \]
From (3.5) and (3.7), we obtain
\[ G_{e}(y)(v) = G_{\delta y}(v). \]

**Definition 9.** (Sublinear IVF). Let $G$ be an IVF on a linear subspace $S$ of $\mathcal{X}$. Then, $G$ is sublinear on $S$ if

(i) $G(\delta w) = \delta \circ G(w)$ for all $w \in S$ and for all $\delta \geq 0$, and

(ii) $G(v + w) \not\in G(v) \oplus G(w)$ for all $v, w \in S$.

**Example 2.** Let $\mathcal{X}$ be the Euclidean space $\mathbb{R}$ and $S = \mathcal{X}$. Then, the IVF $G : S \to I(\mathbb{R})$ that is defined by
\[ G(y) = |y| \circ A, \text{ where } A \in I(\mathbb{R}) \text{ such that } A \neq 0, \]
is sublinear on $S$. The reason is as follows.

For all $y, z \in S$ and $\delta \geq 0$, we have

(i) $G(\delta y) = |\delta y| \circ A = \delta |y| \circ A = \delta \circ G(y)$.

(ii) $|y + z| \circ A \not\in (|y| + |z|) \circ A$ by (i) Lemma 1

Then, $G$ is a sublinear IVF on $S$.

**Example 3.** Let $Q$ be a real positive definite matrix of order $n \times n$ and $S \subset \mathcal{X}$ be a linear space. Consider the IVF $G : S \to I(\mathbb{R})$, which is defined by
\[ G(y) = (\sqrt{y^T Q y}) \circ A, \text{ where } A \neq 0. \]

Then, $G$ is a sublinear IVF on $S$. The reason is as follows.

We write $G(y)$ as $g(y) \circ A$, where $g(y) = \sqrt{y^T Q y}$. By Example 1.2.3 of [18], $g$ satisfies the following conditions:

(a) for $\delta \geq 0$ and $y \in S$,
\[ g(\delta y) = \delta g(y), \quad (3.8) \]
and

(b) for all $x, y \in S$
\[ g(x + y) \leq g(x) + g(y). \quad (3.9) \]

From (3.8), we have
\[ g(\delta y) \circ A = \delta g(y) \circ A, \text{ or, } G(\delta y) = \delta \circ G(y). \]

Since $A \neq 0$, from (3.9) and Lemma 1, we obtain
\[ g(x + y) \circ A \not\in (g(x) + g(y)) \circ A \]
\[ \Rightarrow (g(x + y)) \circ A \not\in g(x) \circ A \oplus g(y) \circ A \text{ since } g(x) \text{ and } g(y) \text{ are nonnegative} \]
\[ \Rightarrow G(x + y) \not\in G(x) \oplus G(y). \]

Hence, $G$ is a sublinear IVF on $S$. 

Chauhan, R. S. et al.
Example 4 Let $G$ be a convex IVF on a linear subspace $\mathcal{S}$ of $\mathcal{X}$ such that for all $v \in \mathcal{S}$,

$$G(\alpha v) = \alpha \circ G(v) \quad \text{for every } \alpha \geq 0.$$  \hspace{1cm} (3.10)

Then, $G$ is a sublinear IVF on $\mathcal{S}$. The reason is as follows. For $v, w \in \mathcal{S}$ and $\delta_1, \delta_2 > 0$, we have

$$G(\delta_1 v + \delta_2 w) = G\left(\delta \left(\frac{\delta_1}{\delta} v + \frac{\delta_2}{\delta} w\right)\right),$$

where $\delta = \delta_1 + \delta_2$

$$= \delta \circ G\left(\frac{\delta_1}{\delta} v + \frac{\delta_2}{\delta} w\right) \quad \text{by (3.10)}$$

$$\leq \delta_1 \circ G(v) + \delta_2 \circ G(w) \quad \text{by the convexity of } G.$$  

Taking $\delta_1 = \delta_2 = 1$, we obtain

$$G(v + w) \leq G(v) \circ G(w) \quad \text{for all } v, w \in \mathcal{S}.$$  

Hence, $G$ is a sublinear IVF on $\mathcal{S}$.

Remark 7 A sublinear IVF may not be convex. For instance, take $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = \mathcal{X}$ and the IVF $G : \mathcal{S} \rightarrow I(\mathbb{R})$ that is given by

$$G(y) = |y| \circ [-3, 2].$$

Clearly, by Example 2, $G$ is a sublinear IVF on $\mathcal{S}$. However, $g(y) = -3|y|$ is not convex on $\mathcal{S}$. Therefore, by Lemma 3, $G$ is not a convex IVF on $\mathcal{S}$.

Theorem 3 Let $G$ be an upper $gH$-Clarke differentiable IVF at an interior point $\bar{y}$. Then, the upper $gH$-Clarke derivative $G_{\bar{y}}(\delta v)$ of $G$ is a sublinear IVF on $\mathcal{S}$.

Proof For an arbitrary $v \in \mathcal{S}$ and $\delta \geq 0$, we have

$$\limsup_{y \rightarrow \bar{y}} \frac{1}{t} \circ (G(y + t\delta v) \circ gH F(y)) = \delta \circ (\limsup_{y \rightarrow \bar{y}} \frac{1}{t} \circ (G(y + t\delta v) \circ gH F(y)))$$

$$= \delta \circ G_{\bar{y}}(\delta v).$$

Thus, $G_{\bar{y}}(\delta v) = \delta \circ G_{\bar{y}}(\delta v)$. For all $v_1, v_2 \in \mathcal{S}$, we get

$$G_{\bar{y}}(v_1 + v_2)$$

$$= \limsup_{y \rightarrow \bar{y}} \frac{1}{t} \circ (G(y + t(v_1 + v_2)) \circ gH G(y))$$

$$\leq \limsup_{y \rightarrow \bar{y}} \frac{1}{t} \circ \left[G(y + tv_1 + tv_2) \circ gH G(y + tv_2) \oplus G(y + tv_2) \circ gH G(y)\right],$$

by (i) of Lemma 2

$$= G_{\bar{y}}(v_1) \oplus G_{\bar{y}}(v_2).$$

Hence, $G_{\bar{y}}$ is a sublinear IVF on $\mathcal{S}$. 

4 Conclusion and Future Directions

In this article, mainly three concepts on IVFs have been studied—limit superior of IVF (Definition 5), upper \(gH\)-Clarke derivative (Definition 7), and sublinear IVF (Definition 9). One can trivially notice that in the degenerate case, each of the Definitions 5, 7, and 9 reduces to the respective conventional definition for the real-valued functions. It has been observed that for a \(gH\)-Lipschitz continuous IVF, the upper \(gH\)-Clarke derivative always exists (Theorem 1). Also, for a \(gH\)-Lipschitz continuous IVF, it has been found that the \(gH\)-directional derivative of a convex IVF coincides with the upper \(gH\)-Clarke derivative (Theorem 2). It has been noticed that the upper \(gH\)-Clarke derivative at an interior point of the domain of an IVF is a sublinear IVF (Theorem 3).

In analogy to the current study, future research can be carried out for other generalized directional derivatives for IVFs, e.g., Dini, Hadamard, Michel-Penot, etc., and their relationships \[8\]. In parallel to the proposed analysis of IVFs, another promising direction of future research can be the analysis of the fuzzy-valued functions (FVFs) as the alpha-cuts of fuzzy numbers are compact intervals \[13\]. Hence, in future, one can attempt to extend the proposed idea of \(gH\)-Clarke derivative for fuzzy-valued functions.

The proposed upper \(gH\)-Clarke derivative can be dealt with control systems and differential equations in a noisy or uncertain environment in the future. A noisy and uncertain environment in control system or a differential equation appears due to the incomplete information of demand for a product and changes in the climate. In future, we will try to solve a control problem in a noisy or uncertain environment with the help of \(gH\)-Clarke and \(gH\)-Fréchet derivatives.

Appendix A Proof of Lemma 1

Proof Let \(Z = [z, \tau]\).

(i) If \(Z \succeq 0\), then

\[
\begin{align*}
\bar{z} \geq 0 \text{ and } \bar{\tau} \geq 0 \\
\implies |a|z + |b|\bar{z} \geq |a + b|z \text{ and } |a|\bar{\tau} + |b|\bar{\tau} \geq |a + b|\bar{\tau} \\
\implies |a + b| \odot Z \preceq |a| \odot Z \oplus |b| \odot Z.
\end{align*}
\]

(ii) If \(Z \preceq 0\), then

\[
\begin{align*}
\bar{z} \leq 0 \text{ and } \bar{\tau} \leq 0 \\
\implies |a|z + |b|\bar{z} \leq |a + b|z \text{ and } |a|\bar{\tau} + |b|\bar{\tau} \leq |a + b|\bar{\tau} \\
\implies |a + b| \odot Z \succeq |a| \odot Z \oplus |b| \odot Z.
\end{align*}
\]

Appendix B Proof of Lemma 2

Proof Let \(A = [a, \bar{a}], B = [b, \bar{b}], C = [c, \bar{c}]\) and \(D = [d, \bar{d}]\).

(i) We have the following four possible cases.

- **Case 1.** Let \(\bar{a} - \bar{\tau} \geq a - c\) and \(\bar{b} - \bar{\tau} \geq c - b\). Then, \(\bar{a} - \bar{b} \geq a - b\) and

\[
(A \ominus_{gH} C) \ominus (C \ominus_{gH} B) = [a - c, \bar{a} - \bar{\tau}] \ominus [c - b, \bar{c} - \bar{\tau}] = [a - b, \bar{a} - \bar{b}] = A \ominus_{gH} B.
\]
• Case 2. Let $\pi - r \leq a - c$ and $r - b \leq c - b$. Therefore, $\pi - b \leq a - b$ and

$$\langle A \ominus_{gH} C \rangle \ominus \langle C \ominus_{gH} B \rangle = [\pi - \bar{c}, a - c] \ominus [\bar{c} - \bar{b}, c - \bar{b}] = [\pi - \bar{b}, a - \bar{b}] = A \ominus_{gH} B.$$  

• Case 3. Let $\pi - r < a - c$ and $r - b > c - b$. Therefore,

$$\langle A \ominus_{gH} C \rangle \ominus \langle C \ominus_{gH} B \rangle = [\pi - \bar{c}, a - c] \ominus [\bar{c} - \bar{b}, c - \bar{b}] = [\pi - \bar{c} + c - \bar{b}, a - \bar{b} + c - \bar{b}].$$

If possible, let

$$\langle A \ominus_{gH} C \rangle \ominus \langle C \ominus_{gH} B \rangle \prec A \ominus_{gH} B.$$  (Appendix B.1)

If $\pi - b \geq a - b$, then from (Appendix B.1) we get

$$[\pi - 2 + c - b, a - c + r - b] < [a - b, \pi - b]$$

$$\Rightarrow a - c + r - b \leq \pi - b$$

$$\Rightarrow a - c \leq \pi - r,$$  which is an impossibility.

Further, if $\pi - b \leq a - b$, then from (Appendix B.1), we have

$$[\pi - 2 + c - b, a - c + r - b] < [\pi - b, a - b]$$

$$\Rightarrow a - c + r - b \leq a - b$$

$$\Rightarrow \pi - b \leq c - b,$$  which is an impossibility.

Thus, (Appendix B.1) is not true.

• Case 4. Let $\pi - r > a - c$ and $r - b < c - b$. Proceeding as in Case 3 of (i) we can prove that

$$\langle A \ominus_{gH} C \rangle \ominus \langle C \ominus_{gH} B \rangle \neq A \ominus_{gH} B.$$  (Appendix B.2)

(ii) As $\|B \ominus_{gH} A\|_{I(\mathbb{R})} = \max\{|\tilde{b} - a|, |\bar{b} - \bar{a}|\}$, we break the proof in two cases.

• Case 1. If $(L =) \|B \ominus_{gH} A\|_{I(\mathbb{R})} = |\tilde{b} - a|$, then

$$|\tilde{b} - a| \geq |\bar{b} - \bar{a}| \Rightarrow |\tilde{b} - a| \geq |\bar{b} - \bar{a} \Rightarrow \bar{b} \leq \bar{a} + L.$$  (Appendix B.3)

Since $\tilde{b} - a \leq |\tilde{b} - a|$, then

$$\tilde{b} \leq a + L.$$  (Appendix B.4)

From (Appendix B.2) and (Appendix B.3), we have $B \leq A \oplus [L, L]$.

• Case 2. If $(L =) \|B \ominus_{gH} A\|_{I(\mathbb{R})} = |\bar{b} - \bar{a}|$, then

$$|\tilde{b} - a| \leq |\bar{b} - \bar{a}| \Rightarrow |\tilde{b} - a| \leq |\bar{b} - \bar{a} \Rightarrow \tilde{b} \leq \bar{a} + L.$$  (Appendix B.5)

Since $\bar{b} - \bar{a} \leq |\bar{b} - \bar{a}|$, then

$$\bar{b} \leq \bar{a} + L.$$  (Appendix B.6)

From (Appendix B.4) and (Appendix B.5), we obtain $B \leq A \oplus [L, L]$, where $L = \|B \ominus_{gH} A\|_{I(\mathbb{R})}$.

(iii) If possible, let there exist $A$, $B$, $C$ and $D$ in $I(\mathbb{R})$ such that

$$\|\langle A \ominus_{gH} B \rangle \ominus \langle C \ominus_{gH} D \rangle\|_{I(\mathbb{R})} > \|\langle A \ominus_{gH} C \rangle \ominus \langle B \ominus_{gH} D \rangle\|_{I(\mathbb{R})}.$$  (Appendix B.6)

According to the definition of $gH$-difference of two intervals,

either $A \ominus_{gH} B = [a - b, \pi - b]$ or $A \ominus_{gH} B = [\pi - \bar{b}, a - \bar{b}]$,

either $C \ominus_{gH} D = [c - d, \tau - \bar{d}]$ or $C \ominus_{gH} D = [\pi - \bar{d}, c - \bar{d}]$,

either $A \ominus_{gH} C = [a - c, \pi - c]$ or $A \ominus_{gH} C = [\pi - \bar{c}, a - \bar{c}]$,

and

either $D \ominus_{gH} B = [d - b, \bar{d} - \bar{b}]$ or $D \ominus_{gH} B = [\bar{d} - \bar{b}, d - \bar{b}]$.

Then, one of the following holds true:

(a) $(A \ominus_{gH} B) \ominus_{gH} (C \ominus_{gH} D) = [a - b - c + d, \pi - \bar{b} + \bar{d}]$
(b) \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [a - b - \pi + \theta, a - \pi - \xi + d]\)

(c) \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [\pi - \xi - \rho + \theta, a - \pi - \xi + d]\)

(d) \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [\pi - \xi - \rho + \theta, a - \pi - \sigma + \theta]\).

- **Case 1.** Let \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [a - b - \xi + d, a - \pi - \xi + d]\).

  (a) If \(\|(A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D)\|_{\ell_1(B)} = [a - b - \xi + d]\), then from equation (Appendix B.6), we have

  \[|a - b - \xi + d| > |a - d| + |b - d| > |a - b - \xi + d|,

  which is impossible.

  (b) If \(\|(A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D)\|_{\ell_1(B)} = [\pi - \xi - \rho + \theta]\), then from equation (Appendix B.6), we have

  \[|\pi - \xi - \rho + \theta| > |\pi - \rho - \sigma + \theta| > |\pi - \xi - \rho + \theta|,

  which is again impossible.

- **Case 2.** Let \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [\pi - \xi - \rho + \theta, a - b - \xi + d]\).

  For this case, two subcases are similar to the Case 1 of (iii) will lead to impossibilities.

- **Case 3.** Let \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [a - b - \pi + \theta, a - \pi - \xi + d]\). Then,

  \[
  \begin{align*}
  a - b &\leq \pi - \xi \\
  \pi - \xi &\leq \pi - \theta
  \end{align*}
  \] (Appendix B.7)

  (a) If \(\|(A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D)\|_{\ell_1(B)} = [\pi - \xi - \rho + \theta]\), then \(\pi - \pi - \pi + d \geq 0\).

  From equation (Appendix B.6), we have

  \[|\pi - \xi - \rho + \theta| > |\pi - \xi + \theta| \implies \pi - \theta > \xi - d,

  which is contradictory to (Appendix B.7).

  (b) If \(\|(A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D)\|_{\ell_1(B)} = [a - b - \xi + d]\), then \(a - b - \pi + \theta < 0\).

  From equation (Appendix B.6), we have

  \[-(a - b + \pi) = |a - b - \pi + \theta| > |a - d| + |b - d| \implies \pi - \theta > \xi - d,

  which is again contradictory to (Appendix B.7).

- **Case 4.** Let \((A \circ_{gH} B) \circ_{gH} (C \circ_{gH} D) = [\pi - \xi - \rho + \theta, a - b - \pi + \theta]\).

  All the two subcases for this case are similar to Case 3 of (iii).

Hence, (Appendix B.6) is wrong, and thus the result follows.

Acknowledgement

The first author is thankful for a research scholarship awarded by the University Grants Commission, Government of India.

Funding

Not applicable

Author Contributions

All authors contributed to the study conception and analysis. Material preparation and analysis were performed by Ram Surat Chauhan, Debidas Ghosh, Jaroslav Ramik and Amit Kumar Deb Nath. The first draft of the manuscript was written by Ram Surat Chauhan and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.
Conflicts of interest/Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and material

Not applicable

Code availability

Not applicable

References

1. Ansari, Q. H., Lalitha, C. S., and Mehta, M. Generalized Convexity, Nonsmooth Variational Inequalities, and Nonsmooth Optimization, CRC Press (2013).
2. Bede, B. and Gal, S. G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems, 151, 581–599 (2005).
3. Bhurjee, A. K. and Padhan, S. K. Optimality conditions and duality results for non-differentiable interval optimization problems, Journal of Applied Mathematics and Computing, 50(1–2), 59–71 (2016).
4. Chalco-Cano, Y., Rufian-Lizana, A., Román-Flores H., and Jiménez-Gamero, M. D. Calculus for interval-valued functions using generalized Hukuhara derivative and applications, Fuzzy Sets and Systems, 219, 49–67 (2013).
5. Chalco-Cano, Y., Román-Flores, H., & Jiménez-Gamero, M. D. Generalized derivative and π-derivative for set-valued functions, Information Sciences, 181(11), 2177–2188 (2011).
6. Clarke, F. H. Optimization and Nonsmooth Analysis, Vol. 5, Siam (1990).
7. Costa, T. M., Chalco-Cano, Y., Lodwick, W. A., and Silva, G. N. Generalized interval vector spaces and interval optimization, Information Sciences, 311, 74–85 (2015).
8. Demyanyev, V. F. The rise of nonsmooth analysis: its main tools, Cybernetics and Systems Analysis, 38(4), 527–547 (2002).
9. Dutta, J. Generalized derivatives and nonsmooth optimization, a finite dimensional tour, Top, 13(2), 185–279 (2005).
10. Ghosh, D. Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions, Journal of Applied Mathematics and Computing, 53, 709–731 (2017).
11. Ghosh, D., Ghosh, D., Bhuiya, S. K., and Patra, L. K. A saddle point characterization of efficient solutions for interval optimization problems, Journal of Applied Mathematics and Computing, 58(1–2), 193–217 (2018).
12. Ghosh, D. A Newton method for capturing efficient solutions of interval optimization problems, Opsearch, 53(3), 648–665 (2016).
13. Ghosh, D., and Chakraborty, D. An Introduction to Analytical Fuzzy Plane Geometry, Studies in Fuzziness and Soft Computing, Volume No. 381, Springer (2019).
14. Ghosh, D., Chauhan, R. S., Mesiar, R., and Deb Nath, A. K. Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions, Information Sciences, 510, 317–340 (2020).
15. Ghosh, D., Deb Nath, A. K., and Pedrycz, W. A variable and a fixed ordering of intervals and their application in optimization with interval-valued functions, International Journal of Approximate Reasoning, 121, 187–205 (2020).
16. Ghosh, D., Deb Nath, A. K., Chauhan, R. S., & Castillo, O. Generalized-Hukuhara-Gradient efficient-direction method to solve optimization problems with interval-valued functions and its application in least squares problems. arXiv preprint arXiv:2011.10462 (2020).
17. Guo, Y., Ye, G., Zhao, D., & Liu, W. gH-Symmetrically derivative of interval-Valued functions and applications in interval-valued optimization. Symmetry, 11(10), 1203 (2019).
18. Hiriart-Urruty, J. B. and Lemaréchal, C. Fundamentals of Convex Analysis, Springer Science & Business Media (2012).
19. Hukuhara, M. Intégration des applications mesurables dont la valeur est un compact convexe, Funkcialaj Ekvacioj, 10, 205–223 (1967).
20. Ishibuchi, H. and Tanaka, H. Multiobjective programming in optimization of the interval objective function, European Journal of Operational Research, 48(2), 219–225 (1990).
21. Jahn, J. Introduction to the Theory of Nonlinear Optimization, Springer Science and Business Media, Third edition (2007).
22. Kalani, H., Akbarzadeh-T, M. R., Akbarzadeh, A., & Kardan, I. Interval-valued fuzzy derivatives and solution to interval-valued fuzzy differential equations, Journal of Intelligent & Fuzzy Systems, 30(6), 3373–3384 (2016).
23. Lupulescu, V. Hukuhara differentiability of interval-valued functions and interval differential equations on time scales, Information Sciences, 248, 50–67 (2013).
24. Lupulescu, V. Fractional calculus for interval-valued functions, Fuzzy Sets and Systems, 265, 63–85 (2015).
25. Markov, S. Calculus for interval functions of a real variable, Computing, 22(4), 325–337 (1979).
26. Moore, R. E. Interval Analysis, Prentice-Hall, Englewood Cliffs, New Jersey (1966).
27. Moore, R. E. Method and Applications of Interval Analysis, Society for Industrial and Applied Mathematics (1987).
28. Ramík, J., Vlach, M. Generalized Concavity in Optimization and Decision Making, Vol. 305, Kluwer Academic Publishers, Boston-Dordrecht-London (2002).
29. Schirotzek, W. Nonsmooth Analysis, Universitext, Springer Science & Business Media (2007).
30. Sengupta, A., Pal, T. K., and Chakraborty, D. Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming, Fuzzy Sets and Systems, 119(1), 129–138 (2001).
31. Stefanini, L. A generalization of Hukuhara difference, In Soft Methods for Handling Variability and Imprecision, Advances in Soft Computing, pp. 203–210 (2008).
32. Stefanini, L. and Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis, 71, 1311–1328 (2009).
33. Stefanini, L., & Bede, B. Generalized fuzzy differentiability with LU-parametric representation, Fuzzy Sets and Systems, 257, 184–203 (2014).
34. Stefanini, L. and Arana-Jiménez, M. Karush–Kuhn–Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability, Fuzzy Sets and Systems, 302, 1–34 (2016).
35. Van Heuvel, N. The initial value problem for interval-valued second-order differential equations under generalized $H$-differentiability, Information Sciences, 311, 119–148 (2015).
36. Wu, H. C. The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function, European Journal of Operational Research, 176, 46–59 (2007).
37. Wu, H. C. On interval-valued non-linear programming problems, Journal of Mathematical Analysis and Applications, 358(1), 299–316 (2008).