THE WEIGHTED COMPLEXITY AND THE DETERMINANT FUNCTIONS OF GRAPHS

DONGSEOK KIM, YOUNG SOO KWON, AND JAEUN LEE

Abstract. The complexity of a graph can be obtained as a derivative of variation of the zeta function [J. Combin. Theory Ser. B, 74 (1998), pp. 408-410] or a partial derivative of its generalized characteristic polynomial evaluated at a point [arXiv:0704.1431[math.CO]].

A similar result for the weighted complexity of weighted graphs was found using a determinant function [J. Combin. Theory Ser. B, 89 (2003), pp. 17-26]. In this paper, we consider the determinant function of two variables and discover a condition that the weighted complexity of a weighted graph is a partial derivative of the determinant function evaluated at a point. Consequently, we simply obtain the previous results and disclose a new formula for the Bartholdi zeta function. We also consider a new weighted complexity, for which the weights of spanning trees are taken as the sum of weights of edges in the tree, and find a similar formula for this new weighted complexity. As an application, we compute the weighted complexities of the product of the complete graphs.

Let $G$ be a finite simple graph with vertex set $V(G)$, edge set $E(G)$. Let $\nu_G$ and $\varepsilon_G$ denote the number of vertices and edges of $G$, respectively. Let $\mathcal{A}(G)$ and $\mathcal{D}(G)$ be the adjacency matrix and degree matrix of $G$, respectively. Then the admittance matrix or Laplacian matrix $\mathcal{L}(G)$ of $G$ is $\mathcal{D}(G) - \mathcal{A}(G)$. For other general terms, we refer to [6].

One of classical problems in graph theory is to find the complexity $\kappa(G)$, the number of spanning trees in a graph $G$ [5, 8]. The celebrated Kirchhoff’s matrix tree theorem finds that $\kappa(G)$ is any cofactor of the admittance matrix (or Laplacian matrix) of $G$ which is a generalization of Cayley’s formula which provides $\kappa(K_n)$ of the complete graph $K_n$ on $n$ vertices. On the other hand, the polynomial invariants of graphs have played a key role in the study of graphs. For instance, the chromatic polynomial $p_G(\lambda)$, introduced by Birkhoff, is a very important invariant of $G$ that counts the number of $\lambda$-colorings of $G$ [3]. A generalization of the chromatic polynomial is the Tutte polynomial $T_G(x, y)$ of a graph $G$ [14, 15], most easily defined as

$$T_G(x, y) = R_G(x - 1, y - 1),$$

where $R_G(x - 1, y - 1)$ is the Whitney’s rank generating function [16] and one can see that $\kappa(G) = T_G(1, 1)$. There are a few more bridges between the complexity and the polynomial invariants of graphs [9, 10, 12]. In [12], Northshield found that

$$f_G'(1) = 2(\varepsilon_G - \nu_G)\kappa(G),$$

where $f_G(u) = \det[I - u \mathcal{A}(G) + u^2 (\mathcal{D}(G) - I)]$. In [9], a similar results was shown for the generalized characteristic polynomials introduced by Cvetkovic and et al. [6].

2000 Mathematics Subject Classification. 05C50, 05C25, 15A15, 15A18.

Key words and phrases. determinant functions, complexity, weighted complexity.

This research was supported by the Yeungnam University research grants in 2007.
A weighted graph is a pair $G_{\omega} = (G, \omega)$, where $\omega : E(G) \rightarrow R$ is a function on the set $E(G)$ of edges in $G$ and $R$ is a commutative ring with identity. We call $G$ the underlying graph of $G_{\omega}$ and $\omega$ the weight function of $G_{\omega}$. Given any weighted graph $G_{\omega}$, the adjacency matrix $A(G_{\omega}) = (w_{ij})$ of $G_{\omega}$ is the square matrix of order $\nu_G$ defined by

$$w_{ij} = \begin{cases} \omega(e) & \text{if } e = \{v_i, v_j\} \in E(G), \\ 0 & \text{otherwise}. \end{cases}$$

Notice that the adjacent matrix $A(G_{\omega})$ of $G_{\omega}$ is symmetric. The incidence matrix $I(G_{\omega}) = (i_{hk})$ of $G_{\omega}$, with respect to a given orientation, is defined by

$$i_{hk} = \begin{cases} \omega(e_k) & \text{if } v_h \text{ is the positive end of } e_k, \\ -\omega(e_k) & \text{if } v_h \text{ is the negative end of } e_k, \\ 0 & \text{otherwise}. \end{cases}$$

The degree matrix $D(G_{\omega})$ of $G_{\omega}$ is the diagonal matrix whose $(i, i)$-th entry is $\omega_i^G$, the sum of the weights of edges adjacent to $v_i$ in $G$ for each $1 \leq i \leq \nu_G$. The admittance matrix or Laplacian matrix $L(G_{\omega})$ of $G_{\omega}$ is $D(G_{\omega}) - A(G_{\omega})$. Notice that every unweighted graph $G$ can be considered as the weighted graph whose weight function assigns 1 to each edge of $G$ and that $L(G_{\omega}) = D(G_{\omega}) - A(G_{\omega}) = I(G_{\omega})I(G)^t$, where $A^t$ is the transpose of the matrix $A$.

Mizuno and Sato [11] considered the weighted complexity, and generalized Northshield’s result by showing

$$F_{G_{\omega}}'(1) = 2(\omega(G) - \nu_G)\kappa(G_{\omega}),$$

where $F_{G_{\omega}}(u) = \det[I - uA(G_{\omega}) + u^2(D(G_{\omega}) - I)]$ and $\omega(S) = \sum_{e \in E(S)} \omega(e)$ for any subgraph $S$ of $G$.

Instead of considering these determinant functions individually, we start from the following general determinant function,

$$\Phi_{G_{\omega}}(\lambda, \mu) = \det[f(\lambda, \mu)I + g(\lambda, \mu)D(G_{\omega}) + h(\lambda, \mu)A(G_{\omega})],$$

then find a condition that one can obtain a generalization of matrix tree theorem. Now, we are set to provide the main result as follows.

**Theorem 1.** Let $G_{\omega}$ be a finite weighted graph with $\nu_G$ vertices and $\epsilon_G$ edges whose weights on edges are complex numbers. Let $f(\lambda, \mu)$, $g(\lambda, \mu)$, and $h(\lambda, \mu)$ be partial differentiable functions such that $f(\alpha, \beta) = 0$ and $g(\alpha, \beta) + h(\alpha, \beta) = 0$ for some $\alpha$ and $\beta$. Then

$$\frac{\partial \Phi_{G_{\omega}}}{\partial \lambda}(\alpha, \beta) = g(\alpha, \beta)^{\nu_G-1}[f_\lambda(\alpha, \beta)\nu_G + (g_\lambda(\alpha, \beta) + h_\lambda(\alpha, \beta))2\omega(G)]\kappa(G_{\omega}),$$

and

$$\frac{\partial \Phi_{G_{\omega}}}{\partial \mu}(\alpha, \beta) = g(\alpha, \beta)^{\nu_G-1}[f_\mu(\alpha, \beta)\nu_G + (g_\mu(\alpha, \beta) + h_\mu(\alpha, \beta))2\omega(G)]\kappa(G_{\omega}).$$

For Theorem 1, the definition of the weighted complexity $\kappa(G_{\omega})$, found in Lemma 3, use the weight of a spanning tree as the product of weights on edges in the tree. Although it fits well with many occasions, it is much more natural to consider the weight of a spanning tree.
as the sum of weights on edges in the tree. We call this new weighted complexity, the sigma weighted complexity, \( \kappa_\sigma(G_\omega) \). Then, we find Theorem 4 as a counterpart of Theorem 1 for this complexity, \( \kappa_\sigma(G_\omega) \).

The outline of this paper is as follows. In section 1 we first prove a couple of lemmas which show the weighted complexity of weighted graphs is any cofactor of the Laplacian matrix \( \mathcal{L}(G_\omega) \) of \( G_\omega \). Then we provide a proof of Theorem 1. We also consider a new weighted complexity for which the weights of spanning trees are taken as the sum of weights of edges in the tree, and obtain a similar formula for this new weighted complexity. We also explain how previous results can be obtained from our consummation. We also provide a new sequel for the Bartholdi zeta function. Finally, we compute the weighted complexities of the product of the complete graphs in section 2.

1. MAIN RESULTS

Even though the matrix tree theorem of an unweighted graph \( G \) finds that any cofactor of the Laplacian matrix of \( G \) are the same [7], it was previously known that all principal cofactors of the Laplacian matrix \( \mathcal{L}(G_\omega) \) of \( G_\omega \) are the same [4, 13]. These common values were defined as the weighted complexity \( \kappa(G_\omega) \) of a weighted graph \( G \) [11]. In the following Lemma 2 we extend this definition that any cofactor of the Laplacian matrix \( \mathcal{L}(G_\omega) \) of \( G_\omega \) is the weighted complexity \( \kappa(G_\omega) \), i.e., not only the principal cofactors but also any cofactor of the Laplacian matrix \( \mathcal{L}(G_\omega) \) of \( G_\omega \) are the same. In the proof of the main theorem, the ring \( R \) is the polynomial ring over real numbers, but the following lemmas can be proven for a commutative ring with identity.

**Lemma 2.** Let \( R \) be a commutative ring with identity and let \( \omega : E(G) \to R \) be a weight function of a graph \( G \). Let \( U \) be a subset of \( E(G) \) having \( \nu_G - 1 \) edges and let \( \langle U \rangle \) be the spanning subgraph of \( G \) induced by \( U \). Let \( \mathcal{I}(\langle U \rangle) \) be the matrix obtained by removing \( i \)-th row of \( \mathcal{I}(\langle U \rangle) \). Then, for each \( i = 1, 2, \ldots, \nu_G \),

\[
\det(\mathcal{I}(\langle U \rangle)_i) = (-1)^{i-1} \det(\mathcal{I}(\langle U \rangle)_1) = (-1)^{i-1} \prod_{e \in U} \omega(e) \det(\mathcal{I}(\langle U \rangle)_1),
\]

where \( \mathcal{I}(\langle U \rangle) \) is the incidence matrix of the underlying tree \( \langle U \rangle \) of \( \langle U \rangle \). In particular, for each \( i = 1, 2, \ldots, \nu_G \),

\[
\det(\mathcal{I}(\langle U \rangle)_i) = (-1)^{i-1} \det(\mathcal{I}(\langle U \rangle)_1).
\]

**Proof.** For convenience, let \( \mathcal{I}(\langle U \rangle) = (r_1, r_2, \ldots, r_\nu_G)^t \). Then \( r_1 + r_2 + \cdots + r_\nu_G = 0 \). From this fact and properties of the determinant function, we can have that

\[
det(\mathcal{I}(\langle U \rangle)_i) \\
= \det(r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_\nu)^t \\
= (-1) \det(-r_1 - r_2 - \cdots - r_{i-1} - r_{i+1} - \cdots - r_\nu, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_\nu)^t \\
= (-1) \det(r_2, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_\nu)^t \\
= (-1)^{i-1} \det(r_2, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_\nu)^t \\
= (-1)^{i-1} \det(\mathcal{I}(\langle U \rangle)_1).
\]
Since, for each edge \( e \) in \( U \), \( \omega(e) \) is a common factor of the column of \( \mathcal{I}(T_\omega)_1 \) corresponding to the edge \( e \), we have
\[
\det(\mathcal{I}([U]_1)) = \left( \prod_{e \in E(T)} \omega(e) \right) \det(\mathcal{I}([U])_1).
\]

It completes the proof. \( \square \)

**Lemma 3.** Let \( R \) be a commutative ring with identity and let \( \omega : E(G) \to R \) be a weight function of a graph \( G \). Then
\[
\mathcal{L}(G_\omega)_{ij} = \sum_{T \in T(G)} \left( \prod_{e \in E(T)} \omega(e) \right),
\]
for each \( 1 \leq i, j \leq \nu_G \), where \( T(G) \) is the set of all spanning trees in \( G \) and \( A_{ij} \) is the \( ij \)-cofactor of a matrix \( A \).

**Proof.** Let \( \mathcal{I}(G_\omega)_i \) be the matrix obtained by removing \( i \)-th row of \( \mathcal{I}(G_\omega) \). Then \( \mathcal{L}(G_\omega)_{ij} = (-1)^{i+j} \det(\mathcal{I}(G_\omega)_i, \mathcal{I}(G)_j) \). By applying Binet-Cauchy theorem and Lemma 2 we can see that
\[
\det(\mathcal{I}(G_\omega)_i, (\mathcal{I}(G)_j)^t) = \sum_{|U|=\nu_G-1} \det([\mathcal{I}(G_\omega)_i]_U) \det(([\mathcal{I}(G)_j]^t)_U)
\]
\[
= \sum_{|U|=\nu_G-1} (-1)^{i+j} \left( \prod_{e \in U} \omega(e) \right) \det([\mathcal{I}(G)_1]_U)^2,
\]
where \( [\mathcal{I}(G)_i]_U \) is the square submatrix of \( \mathcal{I}(G)_i \) whose \( \nu_G - 1 \) columns corresponding to the edges in a subset \( U \) of \( E(G) \). It is known that \( \det([\mathcal{I}(G)_i]_U) \neq 0 \) if and only if \( \det([\mathcal{I}(G)_1]_U) \neq 0 \). Moreover, if \( \langle U \rangle \) is a tree, then \( \det([\mathcal{I}(G)_i]_U) = \pm 1 \). (For example, see [2, Propositions 5.3 and 5.4]). Form this, it can be shown that
\[
\det(\mathcal{I}(G_\omega)_i, (\mathcal{I}(G)_j)^t) = \sum_{T \in T(G)} (-1)^{i+j} \left( \prod_{e \in E(T)} \omega(e) \right).
\]
It completes the proof. \( \square \)

Using Lemma 3 one can define the weighted complexity \( \kappa(G_\omega) \) of a weighted graph \( G_\omega \) by
\[
\kappa(G_\omega) \equiv \mathcal{L}(G_\omega)_{ij}.
\]

Now we are set to proceed the proof of Theorem 1. For convenience, let
\[
\Phi_{G_\omega}(\lambda, \mu) = \det[c_1(\lambda, \mu), c_2(\lambda, \mu), \ldots, c_{\nu_G}(\lambda, \mu)].
\]
Then
\[
\frac{\partial \Phi_{G_\omega}}{\partial \lambda}(\lambda, \mu) = \sum_{i=1}^{\nu_G} \det[c_1(\lambda, \mu), c_2(\lambda, \mu), \ldots, c_{i-1}(\lambda, \mu), (c_i)_1(\lambda, \mu), c_{i+1}(\lambda, \mu), \ldots, c_{\nu_G}(\lambda, \mu)],
\]
where

\[(c_i)_\lambda(\alpha, \beta) = [h_\lambda(\alpha, \beta)w_{i1}, \ldots, h_\lambda(\alpha, \beta)w_{i-1i}, f_\lambda(\alpha, \beta) + g_\lambda(\alpha, \beta)\omega^G_i, h_\lambda(\alpha, \beta)w_{ii}, \ldots, h_\lambda(\alpha, \beta)w_{i\nu_G}]'.\]

Since \(f(\alpha, \beta) = 0\) and \(g(\alpha, \beta) + h(\alpha, \beta) = 0\) for some \(\alpha\) and \(\beta\), we can see that the expansion of the determinant

\[\det[c_1(\alpha, \beta), c_2(\alpha, \beta), \ldots, c_{i-1}(\alpha, \beta), (c_i)_\lambda(\alpha, \beta), c_{i+1}(\alpha, \beta), \ldots, c_{\nu_G}(\alpha, \beta)]\]

with respect to the \(i\)-th column is

\[g(\alpha, \beta)^{\nu_G-1} \left\{ f_\lambda(\alpha, \beta) + g_\lambda(\alpha, \beta)\omega^G_i \right\} L(G_\omega)^{ii} + h_\lambda(\alpha, \beta) \sum_{k=1, k \neq i}^{\nu_G} \omega_{ki} L(G_\omega)^{ki} \].

Since \(L(G_\omega)^{ij} = \kappa(G_\omega)\) for each \(1 \leq i, j \leq \nu_G\), we have

\[
\frac{\partial \Phi_{G_\omega}(\alpha, \beta)}{\partial \lambda} = \sum_{i=1}^{\nu_G} \det[c_1(\alpha, \beta), c_2(\alpha, \beta), \ldots, (c_i)_\lambda(\alpha, \beta), \ldots, c_{\nu_G}(\alpha, \beta)] \\
= \sum_{i=1}^{\nu_G} g(\alpha, \beta)^{\nu_G-1} \left\{ f_\lambda(\alpha, \beta) + (g_\lambda(\alpha, \beta) + h_\lambda(\alpha, \beta))w^G_i \right\} \kappa(G_\omega) \\
= g(\alpha, \beta)^{\nu_G-1} \left\{ f_\lambda(\alpha, \beta)\nu_G + (g_\lambda(\alpha, \beta) + h_\lambda(\alpha, \beta))2\omega(G) \right\} \kappa(G_\omega).
\]

Similarly, we can have the second equation. It completes the proof. \(\Box\)

Next, we will obtain another key theorem for which the weight of a spanning tree \(T\) is defined by the sum of weights of edges in \(T\), different from that of \(\kappa(G_\omega)\). For a weighted graph \(G_\omega\), the \(\sigma\)-weighted complexity, denoted by \(\kappa_\sigma(G_\omega)\), is the sum of all weights in the edges of spanning trees in \(G\), that is,

\[\kappa_\sigma(G_\omega) = \sum_{T \in T(G)} \left( \sum_{e \in T} \omega(e) \right) = \sum_{T \in T(G)} \omega(T).
\]

Then, for any constant weight function \(\omega = c\), it is clear that \(\kappa_\sigma(G_\omega) = c(\nu_G - 1)\kappa(G)\).

In particular, \(\kappa_\sigma(G) = (\nu_G - 1)\kappa(G)\) for any graph \(G\). For a weighted graph \(G_\omega\) with \(\omega : E(G) \to \mathbb{C}\), we define a new weight function \(\omega_\sigma : E(G) \to \mathbb{C}[x]\) by \(\omega_\sigma(e) = x^\omega(e)\). Then \(\kappa(G_\omega_\sigma)'(1) = \kappa_\sigma(G_\omega)\). Now, by using a method similar to the proof of Theorem 4, we have the following theorem.

**Theorem 4.** Let \(G_\omega\) be a finite weighted graph with \(\nu_G\) vertices and \(\epsilon_G\) edges whose weights on edges are complex numbers. Let \(f(\lambda, \mu), g(\lambda, \mu), \) and \(h(\lambda, \mu)\) be partial differentiable functions such that \(f(\alpha, \beta) = 0\) and \(g(\alpha, \beta) + h(\alpha, \beta) = 0\) for some \(\alpha\) and \(\beta\). Then

\[
\frac{\partial^2 \Phi_{G_\omega}(\alpha, \beta, 1)}{\partial x \partial \lambda} = g(\alpha, \beta)^{\nu_G-1} \left[ g_\lambda(\alpha, \beta) + h_\lambda(\alpha, \beta) \right] 2\omega(G)\kappa(G) \\
+ g(\alpha, \beta)^{\nu_G-1} \left[ f_\lambda(\alpha, \beta)\nu_G + (g_\lambda(\alpha, \beta) + h_\lambda(\alpha, \beta))2\epsilon_G \right] \kappa_\sigma(G_\omega),
\]
and
\[
\frac{\partial^2 \Phi_{G^\omega}(\alpha, \beta, 1)}{\partial x \partial \mu} = g(\alpha, \beta)^{\nu_G-1} [g_\mu(\alpha, \beta) + h_\mu(\alpha, \beta)] 2\omega(G) \kappa(G) \\
+ g(\alpha, \beta)^{\nu_G-1} [f_\mu(\alpha, \beta) \nu_G + (g_\mu(\alpha, \beta) + h_\mu(\alpha, \beta)] 2e_G] \kappa_\sigma(G^\omega).
\]

We will show how the previous results can be obtained from Theorem 1 and 4. For \(F_{G^\omega}(u)\), one can choose \(f(\lambda, \mu) = 1 - \lambda^2, g(\lambda, \mu) = \lambda^2, h(\lambda, \mu) = -\lambda\) and \((\alpha, \beta) = (1, 0)\). Consequently, we find the following corollaries from Theorem 1.

**Corollary 5** ([12]). Let \(f_G(u)\) be a variation of the zeta function defined as
\[
f_G(u) = \det [I - uA(G) + u^2(D(G) - I)].
\]
Then,
\[
f_G'(1) = 2(\epsilon_G - \nu_G) \kappa(G).
\]

**Corollary 6** ([11]). Let \(F_{G^\omega}(u)\) be a variation of the zeta function of weighted graph \(G^\omega\) defined as
\[
F_{G^\omega}(u) = \det [I - uA(G) + u^2(D(G) - I)].
\]
Then,
\[
F_{G^\omega}'(1) = 2(\omega(G) - \nu_G) \kappa(G^\omega).
\]

By setting \(f(\lambda, \mu) = \lambda, g(\lambda, \mu) = \mu, h(\lambda, \mu) = -1\) and \((\alpha, \beta) = (0, 1)\), we find the following corollary.

**Corollary 7** ([9]). Let \(F_G(\lambda, \mu)\) be the generalized characteristic polynomial defined as
\[
F_G(\lambda, \mu) = \det [\lambda I - (A(G) - \mu D(G))].
\]
Then,
\[
\frac{\partial F_G}{\partial \mu}(0, 1) = 2\epsilon_G \kappa(G).
\]

**Theorem 8.** Let \(B_G(t, u)\) be a variation of the Bartholdi zeta function [1] of \(G\) defined as
\[
B_G(t, u) = \det [I - A(G)u + (1 - t)(D(G) - (1 - t)I)u^2].
\]
Then the complexity \(\kappa(G)\) of \(G\) can be obtained as follows,
\[
\frac{\partial B_G}{\partial t}(1, 0) = 2(\nu_G - \epsilon_G) \kappa(G) \quad \text{and} \quad \frac{\partial B_G}{\partial \mu}(1, 0) = 2(\epsilon_G - \nu_G) \kappa(G).
\]

**Proof.** By setting \(f(t, u) = 1 - (1 - t)^2u^2, g(t, u) = (1 - t)u^2, h(t, u) = -u\) and \((\alpha, \beta) = (1, 0)\). Note that \(B_G(0, u) = f_G(u)\). \(\square\)

**Corollary 9.** Let \(\sigma_{G^\omega}(\mu) = \det [\mu I - (D(G^\omega) - A(G^\omega))]\) be the characteristic function of the Laplacian matrix. Then
\[
\sigma_{G^\omega}'(0) = (-1)^{\nu_G-1} \nu_G \kappa(G^\omega).
\]
2. The weighted complexities of the product of the complete graphs

To demonstrate Theorem \([\text{I}]\) and Theorem \([\text{II}]\) we consider the product of the complete graphs \(K_{m_1} \times K_{m_2} \times \ldots \times K_{m_n} \equiv G(m_1, m_2, \ldots, m_n)\) whose vertices are all \(n\)-tuples of numbers \(a_i\) where \(a_i \in \{1, 2, \ldots, m_i\}\) and \(i = 1, 2, \ldots, n\) and two vertices \(a = (a_1, a_2, \ldots, a_n)\) and \(b = (b_1, b_2, \ldots, b_n)\) are adjacent if and only if \(a\) and \(b\) differ in exactly one coordinate. We define a weight function \(\omega : E(G(m_1, m_2, \ldots, m_n)) \to \{\omega_1, \omega_2, \ldots, \omega_n\}\) by \(\omega(\{a, b\}) = \omega_i\) if \(a\) and \(b\) differ in the \(i\)-th coordinate. Then

\[
det (\lambda I - \mathcal{L}(G(m_1, m_2, \ldots, m_n))) = \lambda \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \left( \lambda - \sum_{s \in S} m_s \omega_s \right)^{\prod_{s \in S} (m_s-1)}.
\]

By applying Theorem \([\text{I}]\) and the fact that \((-1)^{\sum_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \prod_{s \in S} (m_s-1)} = (-1)^{m_1 m_2 \ldots m_n - 1} = (-1)^{m_G(m_1, m_2, \ldots, m_n)}\), we have

\[
\kappa(G(m_1, m_2, \ldots, m_n)) \left( \prod_{i=1}^n m_i \right) = \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \left( \sum_{s \in S} m_s \right)^{\prod_{s \in S} (m_s-1)},
\]

and

\[
\kappa(G(m_1, m_2, \ldots, m_n)) \left( \prod_{i=1}^n m_i \right) = \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \left( \sum_{s \in S} m_s \right)^{\prod_{s \in S} (m_s-1)}.
\]

In particular, if \(m_s = m\) for all \(s = 1, 2, \ldots, n\),

\[
\kappa(G(m, m, \ldots, m)) m^n = \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} m^{(m-1)^{|S|}} \left( \sum_{s \in S} \omega_s \right)^{(m-1)^{|S|}}
\]

\[
= \left( \prod_{k=1}^n m (\binom{n}{k}) (m-1)^k \right) \left[ \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \left( \sum_{s \in S} \omega_s \right) \right]^{(m-1)^{|S|}}.
\]

Thus, we find

\[
\kappa(G(m, m, \ldots, m)) = m^{n-n-1} \left[ \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \left( \sum_{s \in S} \omega_s \right) \right]^{(m-1)^{|S|}},
\]

and

\[
\kappa(G(m, m, \ldots, m)) = m^{n-n-1} \left( \prod_{k=1}^n (\binom{n}{k}) (m-1)^k \right).
\]

For \(m = 2\), \(G(2, 2, \ldots, 2)\) is the \(n\)-dimensional hypercube \(Q_n\), and its weighted complexity is

\[
\kappa((Q_n)_\omega) = 2^{n-n-1} \prod_{\emptyset \neq S \subset \{1, 2, \ldots, n\}} \left( \sum_{s \in S} \omega_s \right),
\]
and its complexity is

$$\kappa(Q_n) = 2^{2n-n-1} \left( \prod_{k=1}^{n} k^{(n)} \right).$$

Similarly, by using Theorem \[4\] we have

$$\kappa_\sigma(G(m_1, m_2, \ldots, m_n)) \left( \prod_{i=1}^{n} m_i \right) = \sum_{\emptyset \neq S, \ \emptyset \neq T \neq S, \ T \subset \{1,2,\ldots,n\}} \left( \prod_{\emptyset \neq T \neq S, \ T \subset \{1,2,\ldots,n\}} \left( \sum_{t \in T} \right)^{m_t} \right)^{\Pi_{T \neq T} (m_t-1)} \Omega(S),$$

where

$$\Omega(S) = \left[ \prod_{s \in S} (m_s - 1) \left( \sum_{s \in S} m_s \right)^{-1+\Pi_{s \in S} (m_s-1)} \left( \sum_{s \in S} m_s \omega_s \right) \right].$$

and the sigma weighted complexity of \(G(m, m, \ldots, m)\) is

$$\kappa_\sigma(G(m, m, \ldots, m))$$

$$= m^{m^n-n-1} \sum_{\emptyset \neq S, \ S \subset \{1,2,\ldots,n\}} \left( \prod_{\emptyset \neq T \neq S, \ T \subset \{1,2,\ldots,n\}} |T|^{(m-1)^{|T|}} \right) \left( \sum_{s \in S} \omega_s \right) (m-1)^{|S|} |S|^{(m-1)^{|S|}-1}$$

$$= m^{m^n-n-1} \prod_{k=1}^{n} k^{(n)}(m-1)^k \left( \sum_{\emptyset \neq S \subset \{1,2,\ldots,n\}} (m-1)^{|S|} \sum_{s \in S} \omega_s \right)$$

$$= m^{m^n-n-1} \prod_{k=1}^{n} k^{(n)}(m-1)^k \left( \sum_{k=1}^{n} (m-1)^k \left( \frac{n}{k} \omega_1 + \omega_2 + \ldots + \omega_n \right) \right)$$

$$= m^{m^n-n-1} \left( \prod_{k=1}^{n} k^{(n)}(m-1)^k \right) \frac{m^n-1}{n} (\omega_1 + \omega_2 + \ldots + \omega_n).$$

We observe that every spanning tree in \(Q_{n}\omega\) contains at least one edge of weight \(\omega_i\) for each \(i = 1,2,\ldots,n\). Let \(\omega_1 \leq \omega_2 \leq \ldots \leq \omega_n\). By applying Kruskal’s algorithm to \(Q_{n}\omega\), we can find a minimum spanning tree whose edge set is \(E_1 \cup E_2 \cup \cdots \cup E_n\), where \(E_i = \{(0, \ldots, 0, 0, \ast), (0, \ldots, 0, 1, \ast)\} | \ast \in \{0, 1\}^{n-i} \text{ for each } i = 1,2,\ldots, n\). Since \(\omega(\epsilon) = \omega_i\) for all \(\epsilon \in E_i\) and \(|E_i| = 2^{n-i}\text{ for each } i = 1,2,\ldots, n\), we have

$$\min\{\kappa_\sigma(T) : T \text{ is a spanning tree of the weighted graph } Q_{n}\omega\} = \sum_{i=1}^{n} 2^{n-i} w_i.$$

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**Department of Mathematics, Kyungpook National University, Taegu, 702-201 Korea**

*E-mail address: dongseok@knu.ac.kr*

**Department of Mathematics, Yeungnam University, Kyongsan, 712-749, Korea**

*E-mail address: ysookwon@yu.ac.kr*

**Department of Mathematics, Yeungnam University, Kyongsan, 712-749, Korea**

*E-mail address: julee@yu.ac.kr*