Common Fixed Point and invariant Approximations for $C_q$-Commuting Mappings in P-Normed Spaces

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Abstract. The purpose of this paper is to prove a common fixed point (c.f.p) theorems by using condition $d(S(x), T(y)) \leq \ell \max\{d(h(x), G(y)), d(h(x), S(x)), d(G(y), T(y)), d(h(x), T(y)), d(G(y), S(x))\}$ For two pairs of mappings in p-normed space (p-n.s) and also obtain the best approximation (b.a) result. In the last part of this paper, it is proved that the fixed point (f.p) problem for these mappings is well-posed (w-p).

1. Introduction

C.f.p theorems for generalized affine mapping and a class of I-non expansive non commuting mappings were proved by Nashine and Dewangan [9]. Cho, Hussain, and Pathak [3] c.f.p theorems and b.a results in normed linear spaces are proved. In 2013, Bari and Vetro [2] proved some c.f.p and coincidence point results for three or four mappings. Further, Singh [11] proved the c.f.p theorem for multivalued mappings generalized. In 2017 AL-saidy, Abed, and Ajeel [1] proved three common random f.p theorems for commuting random operators defined on a non-starshaped subset of a p-n.s X. In this research, some c.f.p theorems for two pairs mappings defined on non-star-shaped domain subset of a p-n.s are proved.

2. Preliminaries

We need the following definitions and facts:

Definition (2.1): Let $X$ be a linear space and $\| \cdot \|_p$ be a real-valued function on $X$ with $0 < p \leq 1$. The pair $(X, \| \cdot \|_p)$ is called a p-n.s if for all $a, b$ in $X$ and scalars $\zeta$:

i. $\|a\|_p \geq 0$ and $\|a\|_p = 0$ iff $a = 0$
ii. $\|\zeta a\|_p = |\zeta|^p \|a\|_p$
iii. $\|a + b\|_p \leq \|a\|_p + \|b\|_p$

Every p-normed space $X$ induces a metric space with $(a, b) = \|a - b\|_p$, for all $a, b$ in $X$. If $p = 1$, we have the concept of a normed space [5]. Since a p-n.s is not necessarily locally convex space and the continuous dual $X'$ of p-normed space, $X$ need not separate the point of $X$ [7].
Example (2.1): Let \( X = R^3 \) with \( \| (a_1, a_2, a_3) \|_p = \sum_{i=1}^{3} |a_i|^p \) (\( |.| \) is the absolute value), for any pair \((a_1, a_2, a_3)\) in \( X \) and \( 0 < p \leq 1 \), then \( X \) is \( p \)-normed space since it is satisfying all conditions of definition (2.1).

Definition (2.2): [12]: Let \( X \) be a metric space, A subset \( A \) of \( X \) is called starshaped if there exist at least one point \( q \in A \) such that
\[
[a,q] = \lambda a + (1 - \lambda)q \in A, \quad \text{for all} \quad a \in A \quad \text{and} \quad 0 \leq \lambda \leq 1. \quad \text{In this case,} \; q \text{ \; is said the starcenter of} \; A.
\]

Definition (2.3): [15]: A self-mapping \( h \) of a linear space \( X \) is said to be affine if for all \( a, b \) in \( X \) and for any \( \zeta, \; 0 \leq \zeta \leq 1 \), \( h[\zeta a + (1 - \zeta) b] = \zeta h(a) + (1 - \zeta) h(b) \).

And \( h \) is called \( q \)-affine if there is \( q \in X \) such that \( h[\zeta a + (1 - \zeta) q] = \zeta h(a) + (1 - \zeta) q \), for all \( \zeta \in [0,1] \) and all \( a \in X \).

Definition (2.4): [10]: Let \( A \neq \emptyset \) subset of a p-n.s \( X \). The set of b.a to \( a \in X \), denoted as \( p_A(a) \) is defined by
\[
p_A(a) = \{ y \in A: \| a - y \|_p : \text{dist}(a, A) \}, \quad \text{where} \quad \text{dist}(a, A) = \inf \{ \| a - x \|_p : a \in A \}.
\]

Definition (2.5): Let \( A \neq \emptyset \) subset of a metric space \( X \) and let \( S \) and \( T \) be self-mappings of \( A \). A point \( a \in A \) is a c.f (coincidence(c)) point of \( S \) and \( T \) if \( Sx = Tx = x \) (\( Sx = Tx \)) [14]. The set of c.f.p of \( S \) and \( T \) is denoted by \( F(S,T) \), the set of c.p of \( S \) and \( T \) is denoted by \( C(S,T) \) and the closure of the set \( A \) is denoted by \( Cl(A) \).

A mapping \( S: A \to A \) is called:

1. Hemi-compact [4] if any sequence \( \{x_n\} \) in \( A \) has a convergent subsequence whenever \( d(x_n, S(x_n)) \to 0 \) as \( n \to \infty \);
2. completely continuous [4] if \( \{x_n\} \) weakly converges to \( a \) which implies that \( \{S(x_n)\} \) converges strongly to \( S(a) \);
3. Demi-closed at \( x \), if for every sequence \( \{x_n\} \) in \( A \) such that \( \{x_n\} \) converges weakly to \( x \) and \( \{S(x_n)\} \) converges strongly to \( y \), we have \( S(x) = y \) [8].

The pair \( (S,T) \) is said to be

1. R-weakly commuting mappings [4] if \( \forall a \in A, \exists \Re > 0 \) such that \( d(STa, TSa) < \Re d(Sa, Ta) \), if \( \Re = 1 \), then the mappings is said weakly commuting.
2. R-weakly compatible [13] if they commute at their coincidence points, that is, \( STa = TSa \) whenever \( Sa = Ta \).
3. \( C_q \)-commuting [4] if \( STa = TSa \) for all \( a \in C_q(S,T) \), where \( C_q(S,T) = \bigcup \{C(S,T_k): 0 \leq k \leq 1 \} \) and \( T_k(a) = (1 - k)q + kT(a) \).

Definition (2.6): Let \( X \) a p-normed space, \( A \subseteq X \), \( S: X \to X \) be a mapping we say that \( A \) has property \((a_t)\) if

i. \( S: A \to A \)
ii. \( (1 - k_n)q + k_n S(x) \in A \), for some \( q \in A \) and a fixed real sequence \( < k_n > \) converging to \( 1 \) and for each \( x \in A \).

Remark(2.1): Any \( q \)-starshaped set has the property \((a_1)\) w.r.t any mapping \( S: A \to A \), but the converse is not true in general.

Definition (2.7): Let \( X \) be a p-normed space, \( A \subseteq X \) and \( A \) has property \((a_1)\) w.r.t a mapping \( G: X \to X \), \( q \in A \), and sequence \( < k_n > \). A mapping \( h: X \to X \) is called have the property \((a_2)\) on \( A \) with property \((a_1)\) if \( h((1 - k_n)q + k_n G(x)) = (1-k_n) h(q) + k_n h(G(x)) \) for all \( x \in A \) and \( n \in N \).
in this paper we need the following theorems:

Theorem (2.1)[6]: Let $\varnothing \neq A \subseteq X$, with $S, T, h, G : A \rightarrow A$ such that $\forall x, y \in A$ and $0 \leq \ell < \frac{1}{2}$, the pair $(S, T)$ satisfy the following condition

\[
\begin{align*}
    d(S(x), T(y)) &\leq \ell \max\{d(h(x), G(y)), d(h(x), S(x)), d(G(y), T(y)), d(h(x), T(y)), d(G(y), S(x))\} \\
    \text{if } C_l(S(A)) \subseteq G(A), C_l(T(A)) \subseteq h(A) \quad \text{and one of the subsets } C_l(S(A)), C_l(T(A)), C_l(h(A)) \quad \text{or } C_l(G(A)) \quad \text{is complete, then } C(S, h) \neq \varnothing \quad \text{and } C(T, G) \neq \varnothing.
\end{align*}
\]

Theorem (2.2)[6]: Let $X, A, S, T, h, C_l(S(A)), C_l(T(A)), C_l(h(A))$ and $C_l(G(A))$ as in theorem (2.1). If the pairs $\{S, h\}$ and $\{T, G\}$ are weakly compatible (or R-weakly commuting), then $F(S) \cap F(h) \cap F(T) \cap F(G) \neq \varnothing$.

3- Main result

Now, by using Theorem (2.1) and Theorem (2.2) [6] we show the following:

Theorem (3.1): Let $S, T, h, G$ be self-maps on a subset $A$ of a $p$-normed space $X$ suppose that $c_l(S(A)) \subseteq G(A), c_l(T(A)) \subseteq h(A), q \in F(h) \cap F(G)$, and $h$ and $G$ have property $(\alpha_2)$. Suppose that the pairs $\{S, h\}$ and $\{T, G\}$ are $c_q$-commuting and satisfy

\[
\begin{align*}
    \|S(a) - T(b)\|_p &\leq k \max \left\{ \|h(a) - G(b)\|_p, \text{dist}(h(a), [q, S(a)]), \text{dist}(G(b), [q, T(b)]), \text{dist}(h(a), [q, T(b)]), \text{dist}(G(b), [q, S(a)]) \right\} \\
    \text{if } C(S, h) \neq \varnothing \quad \text{and } C(T, G) \neq \varnothing.
\end{align*}
\]

For all $a, b \in A$ and $0 \leq k < \frac{1}{2}$ If $S$ and $T$ are continuous and $A$ have property $(\alpha_2)$ with respect to $S$ and $T$, then $F(S) \cap F(T) \cap F(h) \cap F(G) \neq \varnothing$, if one of the following conditions is satisfied:

i. $C_l(S(A))$ and $C_l(T(A))$ are compact and $h$ and $G$ are continuous;

ii. $A$ is complete, $F(h)$ and $F(G)$ are bounded, and $S$ and $T$ are compact maps;

iii. $A$ is bounded and complete, $S$ and $T$ are hemicompact, and $h$ and $G$ are continuous;

iv. $A$ is weakly compact, $X$ is complete, $h - S$ and $G - T$ are demiclosed at 0, and $h$ and $G$ are weakly continuous.

v. $X$ is complete, $A$ is weakly compact, $S$ and $T$ are completely continuous, and $h$ and $G$ are continuous.

Prof: Define $S_n; A \rightarrow A$ and $T_n; A \rightarrow A$ by $S_n(a) = (1 - k_n)q + k_nS(a)$ and $T_n(a) = (1 - k_n)q + k_nT(a)$ and a fixed sequence of real numbers $k_n (0 \leq k_n \leq 1)$ Converging to 1, for some $q \in A$ and all $a \in A$. Since $A$ has property $(\alpha_2)$ with respect to $S$ and $T$, $c_l(S(A)) \subseteq G(A), c_l(T(A)) \subseteq h(A), q \in F(h) \cap F(G)$, and $h$ and $G$ have property $(\alpha_2)$. Then for each $n$, $C_l(S_n(A)) \subseteq G(A)$ and $C_l(T_n(A)) \subseteq h(A)$. As the pairs $\{S, h\}$ and $\{T, G\}$ are $c_q$-commuting and $h$ and $G$ have property $(\alpha_2)$, with $q \in F(h) \cap F(G)$, then for each $a \in C_q(S, h) \cap C_q(T, G)$,

\[
\begin{align*}
    h(S_n(a)) &= h((1 - k_n)q + k_nS(a)) = (1 - k_n)h(q) + k_nh(S(a)) = (1 - k_n)q + k_nh(S(a)) \quad \text{if } \forall a \in C(S, h) \subseteq C_q(S, h) \quad \text{and } \quad a \in C(T, G) \subseteq C_q(T, G).
\end{align*}
\]

By similarly away we can show that $G(T_n(a)) = T_n(G(a))$ thus,

\[
\begin{align*}
    h(S_n(a)) &= h(S_n(h(a))) \quad \text{and } G(T_n(a)) = T_n(G(a)) \quad \text{for each } a \in C(S_n, h) \subseteq C_q(S, h) \quad \text{and } \quad a \in C(T_n, G) \subseteq C_q(T_n, G). \quad \text{Hence the pairs } \{S_n, h\} \quad \text{and } \{T_n, G\} \quad \text{are weakly compatible } \forall \text{a. Also by } (3.1),
\end{align*}
\]

\[
\begin{align*}
    \end{align*}
\]

\[
\begin{align*}
    \end{align*}
\]
\[ \|S_n(a) - T_n(b)\|_p \leq |k_n|^p \|S(a) - T(b)\|_p \leq \|S(a) - T(b)\|_p \]

\[ \leq k \max \left\{ \|h(a) - G(b)\|_p, \text{dist}(h(a), [q, S(a)]), \right\} \]

\[ \leq k \max \left\{ \|h(a) - G(b)\|_p, \|h(a) - S_n(a)\|_p, \text{dist}(G(b), [q, S(a)]) \right\} \]

For all \( a, b \in A \).

i. Since \( Cl(S(A)) \) and \( Cl(T(A)) \) are compact, then \( Cl(S(A)) \) and \( Cl(T(A)) \) are also compact, hence all conditions of theorem (2.2) are satisfied on the mappings \( S_n, T_n, h \) and \( G \), therefore \( \exists a_n \in A \) s.t \( a_n = S_n(a_n) = T_n(a_n) = h(a_n) = G(a_n) \). Since \( Cl(S(A)) \) and \( Cl(T(A)) \) are compact, \( \{S(a_n)\} \) and \( \{T(a_n)\} \) sequence in \( S(A) \) and \( T(A) \) (respectively) \( S(A) \subseteq Cl(S(A)) \) and \( T(A) \subseteq Cl(T(A)) \) implies that, there exists a subsequence \( \{S(a_m)\} \) of \( S(a_n) \) and \( \{T(a_m)\} \) of \( T(a_n) \) s.t \( m \to \infty \)

\[ \lim S(a_m) = \lim T(a_m) = b \]

Since \( a_m = S_n(a_m) = (1 - k_m)q + k_mS(a_m) \) and \( a_m = T_n(a_m) = (1 - k_m)q + k_mT(a_m) \). We have \( \lim a_m = b \), hence by the continuity of \( h \) and \( G \) having \( b \in F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \). Therefore \( b \in F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \).

ii. As in (i), there exists \( a_n \in A \) s.t \( a_n = S_n(a_n) = T_n(a_n) = h(a_n) = G(a_n) \). Since \( S \) and \( T \) are compact and \( \{a_n\} \) being in \( F(h) \) and \( F(G) \) are bounded, so there exists a subsequence \( \{S(a_m)\} \) of \( S(a_n) \) and \( \{T(a_m)\} \) of \( T(a_n) \) such that \( m \to \infty \)

\[ \lim S(a_m) = \lim T(a_m) = b \]

The definition \( S_m(a_m) \) and \( T_m(a_m) \) \( \Rightarrow \lim a_m = b \), the continuity of \( S, T, h \) and \( G \) having \( b \in F(S) \cap F(T) \cap F(h) \cap F(G) \). Thus \( F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \).

iii. As in (i), \( \exists a_n \in A \) s.t \( a_n = S_n(a_n) = T_n(a_n) = h(a_n) = G(a_n) \), and \( A \) is bounded, so \( a_n - S(a_n) = (1 - k_m)q + k_mS(a_m) \rightarrow 0 \) as \( n \to \infty \). The hemi-compact of \( S \) and \( T \) implies that has \( \{a_n\} \) a subsequence \( \{a_m\} \) that \( m \to \infty \)

\[ \lim a_m = b \]

the continuity of \( S,T,h \) and \( G \) implies that \( b \in F(S) \cap F(T) \cap F(h) \cap F(G) \). Hence \( F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \).

iv. As in (i), \( \exists a_n \in A \) s.t \( a_n = S_n(a_n) = T_n(a_n) = h(a_n) = G(a_n) \). Since \( A \) is weakly compact, then \( \exists \) a subsequence \( \{a_m\} \) of \( \{a_n\} \) in \( A \) converging weakly to \( b \) as \( m \to \infty \), from weakly continuous of \( h \) and \( G \) we have \( b = h(b) = G(b) \). By (iii) \( h(a_m) - S(a_m) \) and \( G(a_m) - T(a_m) \) converging to \( 0 \) as \( m \to \infty \). The demi-closedness of \( h - S \) and \( G - T \) at \( 0 \) \( \Rightarrow S(b) = T(b) = h(b) = G(b) \).

Thus \( F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \).

v. As in (iv) there is a subsequence \( \{a_m\} \) of \( \{a_n\} \) in \( A \) converging weakly to \( b \) as \( m \to \infty \), by \( S \) and \( T \) are completely continuous, then \( S(a_m) \to S(b) \) and \( T(a_m) \to T(b) \) as \( m \to \infty \). Since \( k_m \to 1 \) and \( a_m = S_m(a_m) = T_m(a_m) = (1 - k_m)q + k_mS_m(a_m) = (1 - k_m)q + k_mT(a_m) \), therefore \( a_m \to S(b) = T(b) \) as \( m \to \infty \) \( \Rightarrow S(a_m) \to S(b) \) and \( T(a_m) \to T(b) \) as \( m \to \infty \), but \( S(a_m) \to S(b) \) and \( T(a_m) \to T(b) \) as \( m \to \infty \) therefore \( S(b) = S(S(b)) \) and \( T(b) = T(T(b)) \), since \( S(b) = T(b) \), then \( S(b) = S(S(b)) = T(b) = T(T(b)) \), implies that \( c = S(c) = T(c) \), where \( c = T(b) = S(b) \). Also, since \( a_n \to c \) as \( m \to \infty \) and \( h \) and \( G \) are continuous mappings then \( c = h(c) = G(c) \).

Therefore \( F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \).

Theorem(3.2): Let \( \emptyset \neq A \subseteq X \) and \( S, T, h, G : X \to X \) be mappings s.t \( a_n \in F(S) \cap F(T) \cap F(h) \cap F(G) \) for some \( a_n \in X, S(\partial A \cap A) \subseteq A \) and \( T(\partial A \cap A) \subseteq A \). Assume that \( h(P_A(a_n)) = G(P_A(a_n)) = P_A(a_n) \) and the pairs \( \{S, h\} \) and \( \{T, G\} \) are \( C_q \)-commuting and continuous on \( P_A(a_n) \) and satisfies for all \( x \in P_A(a_n) \cup \{a_n\} \),

\[ \|S(a) - T(b)\|_p \leq \max \left\{ \|h(a) - G(b)\|_p \text{ if } b = a_n, \right\} \]

\[ \|h(a) - G(b)\|_p, \text{dist}(h(a), [q, S(a)]), \]

\[ \{\|\text{dist}(G(b), [q, T(b)]), \text{dist}(h(a), [q, T(b)]), \text{dist}(G(b), [q, S(a)])\} \]
if \( b \in P_A(a) \)

Suppose that \( P_A(a) \) is closed, has property (a_1), with respect to \( S \) and \( T \) with \( q \in F(h) \cap F(G) \) and \( h \) and \( G \) has property (a_2). then \( P_A(a) \cap F(S) \cap F(T) \cap F(h) \cap F(G) \neq \emptyset \) if one of the following conditions is satisfied:

i. \( Cl(S(P_A(a))) \) and \( Cl(T(P_A(a))) \) are compact and \( h \) and \( G \) are continuous.

ii. \( P_A(a) \) is complete, \( F(h) \) and \( F(G) \) are bounded, and \( S \) and \( T \) are compact maps.

iii. \( P_A(a) \) is bounded and complete, \( S \) and \( T \) are hemi-compact, and \( h \) and \( G \) are continuous.

iv. \( P_A(a) \) is weakly compact, \( X \) is complete, \( h - S \) and \( G - T \) are demiclosed at \( o, h, \) and \( G \) are weakly continuous.

v. \( X \) is complete, \( P_A(a) \) is weakly compact, \( S \) and \( T \) are completely continuous, and \( h \) and \( G \) are continuous.

Proof: Let \( a \in P_A(a) \), then \( \|a - a_0\|_p = dist(a_0, A) \).

Note that

\[
0 < k < 1, \|ka_0 - (1 - k)a - a_0\|_p = (1 - k)^p \|a - a_0\|_p < dist(a_0, A)
\]

Hence the line segment \( \{ka_0 - (1 - k)a: 0 < k < 1\} \) and the set \( A \) are disjoint. Therefore \( a \not\in int(A) \) and so \( a \in \partial A \).

Since \( S(\partial A \cap A) \subseteq A \) and \( T(\partial A \cap A) \subseteq A \), then \( S(a) \in A \) and \( T(a) \in A \).

Also, since \( h(P_A(a)) = G(P_A(a)) = P_A(a) \),

\[
a_0 = h(a_0) = G(a_0) = S(a_0) = T(a_0)
\]

and by using condition (3.2), we have

\[
\|S(a) - a\|_p = \|S(a) - T(a_0)\|_p \leq \|h(a) - G(a)\|_p = \|h(a) - a\|_p = dist(a, A)
\]

And \( \|T(a) - a\|_p = \|S(a) - T(a)\|_p \leq \|h(a) - G(a)\|_p = \|h(a) - a\|_p = dist(a, A) \)

Therefore

\[
S, T: P_A(a) \rightarrow P_A(a)
\]

Since \( P_A(a) \) is closed set then \( Cl(P_A(a)) = P_A(a) \), and since

\[
S(P_A(a)) \subseteq P_A(a) = Cl(P_A(a)),
\]

this implies \( cl(S(P_A(a))) \subseteq G(P_A(a)) \) and the result go ahead of theorem (3.1).

4. Well-posed Problem

Definition (4.1): Let \( (X, \| \cdot \|_p) \) be a p-normed space and \( T: X \rightarrow X \) a mapping, the f.p problem of \( T \) is said to be well-posed if:

i. \( T \) has a unique f.p \( \alpha \in X \);

ii. \( \forall \) sequence \( \{a_n\} \) in \( X \) such that \( \lim_{n \to \infty} \|T(a_n) - a_n\|_p = 0 \), we have \( \lim_{n \to \infty} \|a_n - \alpha\|_p = 0 \).

Definition (4.2): Let \( (X, \| \cdot \|_p) \) be a p-n.s and let \( T \) be a set of mappings in \( X \). The f.p of \( T \) is said to be w.p if:

\( T \) have a unique f.p \( \alpha \in X \);

for any sequence \( \{a_n\} \) of in \( X \) such that \( \lim_{n \to \infty} \|T(a_n) - a_n\|_p = 0 \), \( \forall T \in T \) we have \( \lim_{n \to \infty} \|a_n - \alpha\|_p = 0 \).

Theorem (4.1): If \( X, A, S, T, G, h, Cl(S(A)), Cl(T(A)), Cl(h(A)) \) and \( Cl(G(A)) \) as in theorem (2.2), then the c.f.p for the set mappings \( \{S, T, h, G\} \) is w-p.

Proof: By theorem (2.2), the mappings \( S, T, h, \) and \( G \) have a unique c.f.p \( b \) such that \( S(b) = T(b) = h(b) = G(b) = b \) (4-1)

Let \( \{a_n\} \) be a sequence in \( A \) such that

\[
\lim_{n \to \infty} \|S(a_n) - a_n\|_p = \lim_{n \to \infty} \|T(a_n) - a_n\|_p = \lim_{n \to \infty} \|h(a_n) - a_n\|_p = \lim_{n \to \infty} \|G(a_n) - a_n\|_p = 0
\]

By the triangle inequality, \( b = S(b) \), (2.1) and (4.1) having
\[ \|b - a_n\|_p \leq \|b - T(a_n)\|_p + \|T(a_n) - a_n\|_p = \|S(b) - T(a_n)\|_p + \|T(a_n) - a_n\|_p \]

\[ \leq k \max \left\{ \|h(b) - G(a_n)\|_p, \|h(b) - T(a_n)\|_p, \|G(a_n), S(b)\|_p \right\} \]

\[ + \|T(a_n) - a_n\|_p \]

\[ \leq k \max \left\{ \|h(b) - G(a_n)\|_p, \|h(b) - S(b)\|_p \right\} \]

\[ + k \max \left\{ \|G(a_n) - T(a_n)\|_p, \|G(a_n) - S(b)\|_p \right\} \]

\[ + \|T(a_n) - a_n\|_p \]

\[ = k \max \{b - G(a_n)\|_p, \|G(a_n) - T(a_n)\|_p, \|b - T(a_n)\|_p \} + \|T(a_n) - a_n\|_p \]

\[ \leq k \max \{b - G(a_n)\|_p, \|G(a_n) - b\|_p + \|b - T(a_n)\|_p \} \]

\[ + \|T(a_n) - a_n\|_p = k \max \{\|G(a_n) - b\|_p + k\|b - T(a_n)\|_p + \|T(a_n) - a_n\|_p \} \]

\[ \leq k\|G(a_n) - a_n\|_p + k\|a_n - b\|_p + k\|b - a_n\|_p + k\|a_n - T(a_n)\|_p \]

\[ + \|T(a_n) - a_n\|_p \]

\[ \leq k\|G(a_n) - a_n\|_p + k\|a_n - b\|_p + \|a_n - T(a_n)\|_p (1 - 2k)\|a_n - b\|_p \]

\[ \leq k\|G(a_n) - a_n\|_p + (1 + k)\|T(a_n) - a_n\|_p \]

Thus we have, \( \lim_{n \to \infty} \|a_n - b\|_p = 0 \), hence c.f.p for the mappings \{S, T, h, G\} is w-p.

**Theorem (4.2):** If \( A, X, S, T, h \) and \( G \) as in theorem (3.1), then the c.f.p for the mappings \{S, T, h, G\} is well posed.

**Proof:** By theorem (3.1), the mappings \( S, T, h \) and \( G \) has a unique common fixed point \( b \).

Let \{ \( a_n \) \} sequence in \( A \) s.t \( \lim_{n \to \infty} \|S(a_n) - a_n\|_p = \lim_{n \to \infty} \|T(a_n) - a_n\|_p = \lim_{n \to \infty} \|h(a_n) - a_n\|_p = \lim_{n \to \infty} \|G(a_n) - a_n\|_p = 0 \)

By the triangle inequality, \( S(b) = b \). (2.2) and (4.1) we have:

\[ \|b - a_n\|_p \leq \|b - T(a_n)\|_p + \|T(a_n) - a_n\|_p \]

\[ \leq k \max \left\{ \|h(b) - G(a_n)\|_p, \|G(a_n) - T(a_n)\|_p \right\} \]

\[ + \|T(a_n) - a_n\|_p \]

\[ \leq k \max \left\{ \|G(a_n) - T(a_n)\|_p, \|G(a_n) - S(b)\|_p \right\} \]

\[ + \|T(a_n) - a_n\|_p \]

\[ = k \max \{b - G(a_n)\|_p, \|G(a_n) - T(a_n)\|_p \} + \|T(a_n) - a_n\|_p \]

\[ \leq k \max \{b - G(a_n)\|_p, \|G(a_n) - b\|_p + \|b - T(a_n)\|_p \} \]

\[ + \|T(a_n) - a_n\|_p = k \max \{\|G(a_n) - b\|_p + k\|b - T(a_n)\|_p + \|T(a_n) - a_n\|_p \} \]

\[ \leq k\|G(a_n) - a_n\|_p + k\|a_n - b\|_p + k\|b - a_n\|_p + k\|a_n - T(a_n)\|_p \]

\[ + \|T(a_n) - a_n\|_p \]

\[ \leq k\|G(a_n) - a_n\|_p + 2k\|a_n - b\|_p + \|a_n - T(a_n)\|_p \]

Thus we have, \( \lim_{n \to \infty} \|a_n - b\|_p = 0 \), having the c.f.p for the mappings \{S, T, h, G\} is w-p.

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