Nonlinear seismic imaging via reduced order model backprojection
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SUMMARY

We introduce a novel nonlinear seismic imaging method based on model order reduction. The reduced order model (ROM) is an orthogonal projection of the wave equation propagator operator on the subspace of the snapshots of the solutions of the wave equation. It can be computed entirely from the knowledge of the measured time domain seismic data. The image is a backprojection of the ROM using the subspace basis for the known smooth kinematic velocity model. The implicit orthogonalization of solution snapshots is a nonlinear procedure that differentiates our approach from the conventional linear methods (Kirchhoff, RTM). It allows for the removal of multiple reflection artifacts. It also enables us to estimate the magnitude of the reflectors similarly to the true amplitude migration algorithms.

INTRODUCTION

To simplify the exposition we make a number of assumptions on our model. First, we consider an acoustic wave equation

$$u_{tt} = Au, \quad t \in [0, T],$$

for the pressure $u$. We treat the spatial operator $A = C^2\Delta$ as a matrix $A \in \mathbb{R}^{N\times N}$, a discretization on some fine grid. The diagonal matrix $C = \text{diag}(c)$ contains the acoustic velocity values $c \in \mathbb{R}^N$ at the $N$ nodes of the fine grid, while $\Delta$ discretizes the Laplacian on that grid.

Second, we assume that the sources can be modeled as an initial condition

$$u(0) = S, \quad u_t(0) = 0,$$

which can be easily achieved by considering the even part of the time domain solution of equation (1) and thus the even part of the data. The matrix $S \in \mathbb{R}^{N\times p}$ contains all $p$ sources and is localized near the surface.

Under the initial conditions of equation (2) the solution to equation (1) takes the form

$$u(t) = \cos(t\sqrt{-A})S,$$

where the cosine and the square root are understood as matrix functions. Note that the matrix function of time $u(t) \in \mathbb{R}^{N\times p}$ consists of $p$ columns that contain the solutions corresponding to each source in $S$.

Third, we suppose that the source matrix $S$ admits a representation

$$S = q^2(A)CE,$$

where $E \in \mathbb{R}^{N\times p}$ are $p$ point sources supported on the surface and $q^2(\omega)$ is the Fourier transform of the source waveform. Here we consider a Gaussian source wavelet

$$q^2(A) = e^{-\|A\|},$$

For small $\sigma$ all quantities in equation (3) are localized near the surface. We assume that the velocity near the surface is known, thus we know $S$.

Fourth, we assume that the sources and receivers are collocated. This assumption makes the construction of the reduced order model in the next section a lot more straightforward. However, our approach can be generalized to the much more realistic case of noncollocated sources and receivers. The particular form of the receiver matrix $R \in \mathbb{R}^{N\times p}$ that we use is

$$R = C^{-1}E.$$

Finally, under all the assumptions above we can write a data model

$$F(t) = R^T \cos(t\sqrt{-A})S,$$

which is a $p \times p$ matrix function of time.

The problem of seismic imaging that we solve here is to find an estimate of the unknown velocity $c$ from the knowledge of the time domain data $F(t)$ for $t \in [0, T]$ and a smooth kinematic velocity model $c_0$.

TIME-DOMAIN INTERPOLATORY REDUCED ORDER MODEL

At the core of our approach is the construction of the ROM that interpolates the measured seismic data. The use of model order reduction techniques in inversion was proposed in Druskin et al. (2013), Borcea et al. (2014) for parabolic (controlled source electromagnetic method, CSEM) inverse problems. Unlike the diffusive parabolic case, where the authors employed frequency (Laplace) domain interpolation, the appropriate setting for the wave equation inversion is the time domain.

The particular form of the source and receiver matrices in equations (3,4) allows us to rewrite the data model from equation (7) in the completely symmetric form

$$F(t) = \hat{B}^T \cos \left( t\sqrt{-\hat{A}} \right) \hat{B},$$

where the symmetrized spatial operator is $\hat{A} = CAC$ and the source/receiver matrix is given by $\hat{B} = q(A)E$. Here we used the fact that analytic matrix functions commute with similarity transforms and also that the symmetric operator $\hat{A}$ is similar to the original $A$ with a similarity transform $C$.

In practice, the time domain data is measured at discrete time instants that we denote by $t_k = k\tau$ with $k = 0, 1, \ldots, 2n - 1$, where $\tau$ is the sampling interval and $t_{2n-1} = T$ is the terminal time.

The discrete data samples $F_k = F(t_k)$ admit a representation

$$F_k = F(k\tau) = \hat{B}^T \cos \left( k\arccos \cos \tau \sqrt{-\hat{A}} \right) \hat{B} = \hat{B}^T T_k(\hat{P}) \hat{B},$$

where
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where $T_k$ are Chebyshev polynomials of the first kind and $\hat{P}$ is the propagator given by

$$\hat{P} = \cos \left( \tau \sqrt{\Delta A} \right).$$

(10)

We wish to construct a reduced order model of size $np \ll N$ that matches the $2n$ measured data samples exactly

$$F_k = \hat{B}^T T_k(\hat{P}) \hat{B} = \hat{B}^T T_k(\hat{P}) \hat{B}, \quad k = 0, 1, \ldots, 2n-1,$$

(11)

where $\hat{P} \in \mathbb{R}^{np \times np}$ and $\hat{B} \in \mathbb{R}^{np \times p}$. Since we are solving the inverse problem of seismic imaging the ROM $(\hat{P}, \hat{B})$ should be computable from the knowledge of the sampled data $F_k$ only.

It appears that the solution to the data interpolation problem of equation [11] can be found in the projection form

$$\hat{P} = V^T \hat{P} V, \quad \hat{B} = V^T \hat{B},$$

(12)

where the columns of $V \in \mathbb{R}^{N \times np}$ constitute an orthonormal basis for the subspace spanned by the discrete time snapshots of solutions

$$\hat{u}_k = \hat{u}(t_k) = \cos \left( k \tau \sqrt{\Delta A} \right) \hat{B} = T_k(\hat{P}) \hat{B} \in \mathbb{R}^{N \times p}.$$

(13)

If we introduce the matrix of solution snapshots

$$U = [\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{n-1}] \in \mathbb{R}^{N \times np},$$

(14)

then $V$ is defined simply by

$$\text{colspan } V = \text{colspan } U, \quad V^T V = I.$$ 

(15)

Note that the above definition is not unique as $V$ is defined up to an orthonormal change of variables in the projection subspace. For the purpose of imaging some choices of $V$ are better than others.

The orthogonalization of snapshots $U$ must respect the causality and the propagating nature of the time domain solutions of the wave equation. Thus, each snapshot should be orthogonalized only against the previous ones. In linear algebra this is known as Gram-Schmidt procedure or the QR decomposition. Since our snapshots are matrices with $p$ discrete columns corresponding to all sources/receivers, we need a block version of QR decomposition

$$U = VL^T,$$

(16)

where $L^T \in \mathbb{R}^{np \times np}$ is block upper triangular with blocks of size $p$.

Obviously, we cannot simply use equation [16] since the snapshots $U$ are unknown to us. However, from the data we can obtain the inner products between the snapshots. A basic multiplication property of Chebyshev polynomials

$$T_i(x)T_j(x) = \frac{1}{2} \left( T_{i+j}(x) + T_{|i-j|}(x) \right)$$

(17)

combined with equation [13] immediately implies that

$$(U^T U)_{ij} = \hat{u}_i^T \hat{u}_j = \frac{1}{2} (F_{i+j} + F_{|i-j|}).$$

(18)

Applying equation [17] twice we can also obtain

$$(U^T \hat{P} U)_{ij} = \hat{u}_i^T \hat{P} \hat{u}_j = \frac{1}{4} (F_{i+j+1} + F_{|i-j+1|} + F_{i+j-1} + F_{|i-j-1|}).$$

(19)

The knowledge of Gram matrix $U^T U$ from equation [18] allows us to compute the block lower triangular factor $L$ in equation [16] via a block Cholesky decomposition

$$U^T U = LL^T.$$ 

(20)

Once the Cholesky factor is known we use equation [19] to obtain the final expression for the ROM

$$\hat{P} = L^{-1}(U^T \hat{P} U)L^{-T},$$

(21)

entirely from the sampled data $F_k$.

BACKPROJECTION IMAGING

After the reduced order model of equation [21] is obtained from the measured data we need to extract from it the information about the velocity $c$. The first step is to go from the ROM for the propagator $\hat{P}$ to the reduced model for the symmetrized spatial operator $\hat{A} = \hat{C} \hat{A} \hat{C}$ by approximately inverting equation [10] using the first two terms in the Taylor’s expansion

$$\hat{A} = \frac{2}{\tau^2} (\hat{P} - I) \approx V^T \hat{A} V.$$ 

(22)

There are multiple ways to obtain the estimate of the velocity from the knowledge of $\hat{A}$. One may employ optimization to solve for $c$ by minimizing the ROM misfit. Such procedure is superior to the conventional full waveform inversion (FWI) which minimizes the data misfit. However, in this work we are interested in a non-iterative imaging algorithm that assumes the knowledge of a smooth kinematic background model denoted by $c_0$.

If the projection subspace colspan $V$ is sufficiently rich, then the backprojection must be a good approximation of the spatial operator

$$\hat{A} \approx V \hat{A} V^T.$$ 

(23)

However, we have no direct access to the orthonormal basis $V$. We approximate it with a known basis $V_0$ for the smooth kinematic velocity model $c_0$.

In order to get an imaging formula we also notice that the diagonal of $\hat{A}$ is proportional to the square of the velocity

$$c^2 \propto \text{diag}(\hat{A}) = \text{diag}(\hat{C} \hat{A} \hat{C}),$$

(24)

where the square of $c^2$ is understood componentwise. Similarly, for the difference between the unknown velocity and the kinematic model we can write

$$\delta c^2 = c^2 - c_0^2 \propto \text{diag}(\hat{A} - \hat{A}_0).$$

(25)

Replacing the symmetrized operators $\hat{A}$ and $\hat{A}_0$ in equation [25] with their backprojection approximations from equation [23]...
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and also using the approximation $\mathbf{V} \approx \mathbf{V}_0$ we arrive at the formula
\[
\delta \mathbf{c}^2 \propto \text{diag} \left( (\mathbf{V}_0 (\mathbf{A} - \mathbf{A}_0) \mathbf{V}_0^T) \right),
\]
where $\mathbf{A}$ is computed from the measured data and $\mathbf{A}_0$. $\mathbf{V}_0$ are easily found since $\mathbf{c}_0^2$ is known.

Observe that unlike the conventional imaging approaches (reverse time migration, Kirchhoff migration) the formula in equation 26 is nonlinear in the measured data. This is due to the nonlinearity of the block Cholesky decomposition in equation 20 and inversion of the Cholesky factor $\mathbf{L}$ in equation 21. The nonlinearity that amounts to the implicit orthogonalization of solution snapshots $\mathbf{U}$ allows for a better quality image. In particular, the orthogonalization removes the multiple reflection artifacts which are otherwise very difficult to deal with using conventional linear migration algorithms. This is illustrated in Figure 1 where we show a simple synthetic model with two layers. The backprojection images the layers correctly while suppressing the multiple reflection artifacts that are present in the RTM image as ghost layers below the actual ones.

Algorithm 1 (Nonlinear ROM backprojection imaging)

1. Choose the sampling time interval $\tau$ and measure the discrete time samples of the seismic data with equation 7 $\mathbf{F}_k = \mathbf{F}(\tau k)$ for $k = 1, 2, \ldots, 2n - 1$.
2. Using equation 18 compute the snapshot Gram matrix $\mathbf{U}^T \mathbf{U}$ from the data and perform its block Cholesky decomposition as in equation 20.
3. Using equation 19 compute the matrix $\mathbf{U}^T \mathbf{P} \mathbf{U}$ from the data and use Cholesky factor from step 2 to form the ROM $\mathbf{P}$ with equation 21.
4. Choose a smooth kinematic velocity model $\mathbf{c}_0$ and use the block QR decomposition of equation 12 to compute the orthonormal basis $\mathbf{V}_0$; project $\mathbf{P}_0$ on $\mathbf{V}_0$ using equation 22 to obtain the kinematic model ROM $\mathbf{P}_0$.
5. From the propagator ROMs $\mathbf{P}$ and $\mathbf{P}_0$ obtain the spatial operator ROMs $\mathbf{A}_0$ and $\mathbf{A}_0$ using equation 22.
6. With the operator ROMs from step 5 and the orthonormal basis from step 4 compute $\delta \mathbf{c}^2$ from equation 26 and use the imaging formula of equation 27 to form the final image $\mathbf{c}^\star$.

NUMERICAL EXAMPLE: MARMOUSI MODEL

We evaluate the performance of our method on the synthetic data computed for the Marmousi model by Bourgeois et al. (1991).

The model is on a $15 \text{ m}$ grid with $N = 900 \times 180 = 162,000$ nodes. The choice of the time interval $\tau$ is very important for our method. To make the orthogonalization procedure well conditioned, it should be chosen at a Nyquist rate for the given source wavelet. Here we use $\tau = 33.5 \text{ ms}$ ($n = 35$) which corresponds to the frequency of about $\omega = 15 \text{ Hz}$. The data is measured for $p = 90$ sources/receivers spaced uniformly every 150 m. The kinematic model $\mathbf{c}_0$ is obtained by convolving the true velocity $\mathbf{c}$ with a Gaussian kernel of width 465 m and height of 315 m.
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In Figure 3 we show the image $c^*$ and the difference $c^* - c_0$ along with the smooth kinematic and true Marmousi models. We observe very good recovery of all the model’s features down to 2.4 km. The very bottom is not imaged because we had to truncate the data sampling at $2n = 70$ to avoid the reflections from the bottom, which at the moment employs reflective boundary conditions instead of the PML.

We also show in Figure 2 a vertical well log. It demonstrates that our method performs well not only recovering the locations of the reflectors but also their strengths. We observe that the magnitude of the imaged velocity $c^*$ is in good agreement with the true model $c$. In this particular aspect our algorithm performs similarly to the true amplitude migration methods.

CONCLUSIONS AND DISCUSSION

We introduced a novel nonlinear seismic imaging method based on the backprojection of reduced order models computed directly from the measured time domain data. The results of the early numerical experiments with synthetic data for Marmousi model show great promise.

The main issue to be solved to make the method viable for the real field data is to remove the assumption that the sources and receivers are collocated. This is certainly possible if one uses different left (source) and right (receiver) subspaces in the projection equation [12]. Other possible improvements include the implementation of absorbing boundary conditions (PML) and more accurate source models.
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