Technological equipment parts rough surfaces elastic-plastic strain under compression mathematical modelling

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Abstract. This article discusses dimensions and similarity analysis methods, which are the theoretical basis mathematical modeling in the study of the contact interaction of technological equipment parts fixed interfaces. In the solid mechanics mathematical modeling is used mainly in experimental studies of stress fields, strains and displacements arising under load in the model, geometrically similar to the real element in nature (prototype).

1. Introduction
Fixed interfaces and friction pairs of modern machines and technological equipment study is primarily aimed at their reliable operation at a given design operation period.
Production processes technological capabilities and the automation level increasing leads to the technological equipment quality and economic performance improvement. Operating modes intensification makes it necessary the quality requirements of the main mates and parts included in the machine technological system to increase. Contact stresses and the interacting surfaces shape are the most significant factors determine the fixed machine mates serviceability [1–5].

The stress-strain state mathematical modeling of using physical modeling on optically sensitive materials models by photomechanics, on full-scale metal samples models using holographic interferometry and speckle photography and other experimental methods is the most effective fixed interfaces contact interaction processes studying method.

Therefore, the paper describes in detail the similarity and modeling theory foundations in relation to elastic-plastic boundary value problems solution.

The contact deformation technological equipment parts rough surfaces during compression is an example of material elastic-plastic behavior [6–14]. Problems of this class are solved using experimental (optical) research methods [15–20]. The reason for this is the complexity of setting and obtaining theoretical solutions to these problems.

Similarity issues for elastic-plastic contact problems physical modeling are still poorly investigated.
The main mixed boundary elastic-plastic contact problem criteria analysis is carried out to overcome these limitations and obtain the appropriate similarity criteria, modeling conditions and formulas conversion similar values from the model to nature.

2. Main part
In this article, we consider elastic-plastic bodies’ small deformations.
The symmetric tensor of small deformations is expressed through the displacement vector $\ddot{u}_i$ in a rectangular Cartesian coordinate system according to the relation: $2\varepsilon(x) = u_{ij} + u_{ij}$. It is not essential whether the coordinates of the body belong to the deformed or to the undeformed state, in the considered approximation.

Similarly, the strain rate tensor $\varepsilon_{ij}$ is expressed in terms of displacement rates $\dot{u}_i$ by means of relations.

$$\varepsilon_{ij} = \dot{u}_{ij} + \dot{u}_{ji} \quad (i,j = 1,2,3).$$

(1)

In (1) the point above the corresponding value means the partial derivative in time

$$\varepsilon_{ij} = \frac{\partial \varepsilon_{ij}}{\partial t}, \quad \dot{u}_{ij} = \frac{\partial \varepsilon_{ij}}{\partial t} \left( \frac{\partial u_j}{\partial x_i} \right), \quad \ddot{u}_{ij} = \frac{\partial \left( \frac{\partial u_j}{\partial x_i} \right)}{\partial t}.$$  

Changes in geometry caused by deformation are not taken into composing equilibrium equations

$$\sigma_{ij}(x) + \rho f_{ij}(x) = 0 \quad (\text{symmetric with respect to } \sigma_{ij} \text{ and } \sigma_{ji}),$$

where $\rho f_{ij}(x) = 0$. In this reason, it is not important to the deformed or undeformed state referred symmetric stress tensor. The body element actual total deformations $\dot{e}_{ij}$ can be represented as a sum of elastic deformations $\varepsilon_{ij}^e$ and plastic deformations $\varepsilon_{ij}^p$, as well as their velocities.

$$\dot{e}_{ij}^p = \dot{e}_{ij}^e + \varepsilon_{ij}^p, \quad \dot{e}_{ij}^e = \dot{e}_{ij} + \dot{e}_{ij}^p.$$  

(2)

The relation (2) is based on the deformation theory, according to which elastic and plastic deformations are determined by introducing an intermediate non-stressed state obtained from the elastic-plastic configuration by means of a purely elastic unloading process. This configuration is not physically continuous, but it gives a logical separation of elastic and plastic deformation components. The elastic total deformation component is determined by Hooke's law.

If the yield surface is regular in the vicinity of a point on this surface, that is, if it has a continuously varying tangent, then the equation of this surface $f(\sigma_{ij}) = 0$, where the sign of the yield function $f$ (symmetric with respect to $\sigma_{ij}$ and $\sigma_{ij}$) is chosen so that in the elastic region $f \leq 0$. Now the plastic deformation rates are determined by the flow law

$$\dot{e}_{ij}^p = \lambda \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}},$$

(3)

where $\lambda = 0$ if $f < 0$; $f = 0$, $\lambda = 0$ if $f > 0$.

The expression (3) is an associated Mises flow law with a flow function $f(\sigma_{ij})$. The field function serves as a plastic deformation rates potential, that is the plastic potential, it follows from (3). This means that the plastic strain rate tensor will have an external normal direction to the yield surface.

Material plastic incompressibility condition it is necessary to attach to equations $\sigma_{ij}(x) + \rho f_{ij}(x) = 0$ and (1–3) for completeness of the of the considered problem statement:

$$\dot{e}_{ij}^p = \dot{e}_{11}^p + \dot{e}_{22}^p + \dot{e}_{33}^p = 0.$$  

(4)

Mixed boundary-value elastic-plastic contact problem except the boundary conditions: $\sigma_{ij} \big|_{x_0} = \sigma_{ij}$ and $u \big|_{x_0} = u_{i0}$ must also include a boundary condition:
\[ \dot{u}_i = u(i = 1, 2, 3). \]  

According to [17], when interacting, for example, spherical surfaces, depending on the contact conditions, four type’s boundary value problems can occur.

We have 36 unknowns \( (\sigma_{ij}, e_{ij}, e'^{p}_{ij}, \alpha_{ij}, u_i, \dot{u}_i, F_i) \), which are uniquely determined from the system 36 equations (2–5) and (6).

\[ \sigma_{ij}(x_i) = \lambda \theta \delta_{ij} + 2\nu e_{ij}, \ \sigma_{ij}(x_i) + \rho F_i(x_i) = 0. \]  

The elastic-plastic contact problem boundary conditions in the General case, can be represented as:

\[ \sigma_{ij} n_{ij} / s = p_i; u_{ij} / s = u_{ij0} (i = 1, 2, 3); \]
\[ \dot{u}_{ij} = \dot{u}_{ij0}; \ [n_j \sigma_{ij}] = 0 (i = 1, 2, 3); \]
\[ \frac{\vec{F}_{FR}}{\vec{F}_{FR}} = - \frac{\vec{I} - \vec{n} \vec{n}}{\vec{I} - \vec{n} \vec{n}} \left( \frac{\vec{\tau}^+ - \vec{\tau}^-}{\vec{\tau}^+ - \vec{\tau}^-} \right). \]  

Here, indices \( I \) and \( II \) refer the corresponding value to the first or second contacting bodies, \( n_{ij} \) – normal to the surface, square brackets indicate the corresponding value gap, \( F_{FR} \) – friction, plus sign indicates the friction directions and the contact surfaces slip coincidence, minus sign – the mismatch of their directions, the line above indicates the vector, \( \vec{I} \) – spherical tensor of the second rank, \( \vec{n} \) – dyad.

In the axisymmetric contact problems particular case, the last boundary condition in the system (7) turns into Amonton’s law \( \tau_{ij} + \rho \sigma_{ij} = 0 \) in the meridional plane.

The three-dimensional contact elastic-plastic problem formulation in a closed form is determined by a equations system (1–3, 6, 7).

We introduce transformations scales into the specified equations system according to the linear-ambiguously according in the similar quantities of nature model, in accordance with the similarity theory and the dimensions analysis \( x_i = k_x x_i; \ldots; \dot{u}_i = k_\dot{u}_i; \dot{u}_i = k_{\dot{u}_i} ; \).

\[
\begin{align*}
\frac{k_x \sigma_{ij}}{k_i} + k_p \rho_{ij} &= 0; \ 2 k_x e_{ij} = k_x \left(u_{ij} + u_{ij} \right); \ 2 k_x \dot{e}_{ij} = k_x \left(u_{ij} + u_{ij} \right); \ k_p \sigma_{ij} n_{ij} = k_p p_{ij}; \\
k_x \dot{e}_{ij} = k_x e_{ij} + k_x e^p_{ij}; \ k_x \
\sigma_{ij} = k_x e_{ij} + k_x e^p_{ij}; \ k_x \
g_{ij} = k_x e_{ij} + k_x e^p_{ij}; \ k_x \dot{u}_{ij} = k_x \left(u_{ij} + u_{ij} \right); \ k \tau_{ij} + k_p \rho_{ij} &= 0.
\end{align*}
\]  

For the similarity in nature and model the elastic-plastic contacting bodies’ stress-strain state implementation must be equal to the unit of the ratio of the magnitude of the linear-to-one analogues of the system (8):

For the implementation on-similarity in nature and in model of the stress-strain state elastic-plastic contacting bodies must be equal to the unit of the ratio of the magnitude of the linear-to-one analogues of the system (8):
According to the \( \pi \)-theorem, no more than \( N - K \) independent dimensionless complexes and simplices can be composed of \( N \) dimensional quantities connected by a physical equation. In this case \( N = 16 \) (\( u, u, x, y, z, \dot{\varepsilon}, \dot{\varepsilon}^p, \sigma, p, \lambda, f, t, \rho, \tau \)).

According to (8), \( K = 3(l, p, t) \), therefore \( N - K = 16 - 3 = 13 \), which corresponds to the system (9).

The number of complexes \( \pi \) is equal to the difference between the number of quantities with unequal dimensions \( (u, u, \dot{\varepsilon}, \sigma, p, \lambda, f, t, \rho) \) \( n = 9 \) and the number \( K \) of quantities with independent dimensions: \( \pi = n - k = 9 - 3 = 6 \). This corresponds to the system of similarity indicators (9). The number of simplices \( N - n = 16 - 9 = 7 \), which is also in accordance with (9). Since the dimensionless quantities scales are equal to one, as the values of zero dimensions, then in (8) \( k_\varepsilon = 1 \) and \( k_\mu = 1 \). This in mind, the similarity indicator system (9) can be simplified:

\[
\frac{k_\varepsilon}{k_\mu} - \frac{k_\varepsilon}{k_\mu} - \frac{k_\varepsilon}{k_\mu} - \frac{k_\varepsilon}{k_\mu} - \frac{k_\varepsilon}{k_\mu} = 1
\]

(10)

The last similarity indicator in (10) shows that the values of the friction coefficients in nature (\( N \)) and the model (\( M \)) must coincide:

\[
k_\mu = \frac{\rho_N}{\rho_M} = 1,
\]

that is \( \rho_N = \rho_M \).

From the relations system (10) by replacing the similarity scale with the relations of the values included in them for nature and model, we obtain the following similarity criteria:

1. \( \left( \frac{u}{l} \right)_N = \left( \frac{u}{l} \right)_M = \text{idem} \)
2. \( \left( \frac{\dot{u}}{\dot{\varepsilon}} \right)_N = \left( \frac{\dot{u}}{\dot{\varepsilon}} \right)_M = \text{idem} \)
3. \( \left( \frac{\sigma}{j} \right)_N = \left( \frac{\sigma}{j} \right)_M = \text{idem} \)
4. \( \left( \frac{\sigma}{p} \right)_N = \left( \frac{\sigma}{p} \right)_M = \text{idem} \)
5. \( \left( \frac{t\sigma\dot{\varepsilon}^i}{f} \right)_N = \left( \frac{t\sigma\dot{\varepsilon}^i}{f} \right)_M = \text{idem} \)
6. \( \left( \frac{u}{l\dot{\varepsilon}^p} \right)_N = \left( \frac{u}{l\dot{\varepsilon}^p} \right)_M = \text{idem} \)
7. \( \left( \frac{u}{u_0} \right)_N = \left( \frac{u}{u_0} \right)_M = \text{idem} \)
8. \( \left( \frac{\dot{u}^i}{u^i} \right)_N = \left( \frac{\dot{u}^i}{u^i} \right)_M = \text{idem} \)
9. \( \left( \frac{\tau}{\sigma} \right)_N = \left( \frac{\tau}{\sigma} \right)_M = \text{idem} \)

(11)

From the last system similarity criterion (11), it follows that normal and tangential stresses should be modeled on the same scale.

When the geometric similarity scale \( k_i = l_i / l_M \) of the system (11) first similarity criterion it follows that \( k_u = k_i \), that is, in the simulation must respect the equality of geometric similarity scale and displacement, which implies the dependence: \( (u_i)_N = k_i (u_i)_M \), \( (i = 1, 2, 3) \) to recalculate the displacement values from the model to nature.
Scale \( \dot{u} : k_u = k_w^0 \), that is, \( \dot{u}_w = k_w^0 \dot{u}_M \), determine on the basis of the seventh similarity criterion included in the system (11). You must set the initial offset rate \( \dot{u}_o \) for this.

The strain rate scale \( k_\varepsilon = k_u / k_i \) is determined from the second system similarity criterion (11). From this follows the strain rates conversion formula from model to nature \( \dot{\varepsilon}_M = (k_u / k_i) \dot{\varepsilon}_w \).

The third system similarity criterion (11) uniquely determines the identity of the values of stresses in nature and the model, which are caused by the action of the which are caused by the action of the own weight simulated bodies \( \sigma_N = [(j)_u / (j)_M] \sigma_M \).

The stress scale is uniquely determined by a set scale power similarity, the fourth similarity criterion of the system (11)

\[
\sigma_N = \frac{k_p}{k_i} \sigma_M = \frac{p_N}{p_M} \frac{l_M}{l_N} \sigma_0^{1-0} \frac{d}{n}
\]

(12)

for volumetric problems and

\[
\sigma_N = \frac{k_p}{k_i} \sigma_M = \frac{p_N}{p_M} \frac{l_M}{l_N} \frac{d_N}{d_M} \sigma_0^{1-0} \frac{d}{n}
\]

(13)

for the plane problems.

In formulas (12) and (13) are indicated: \( d \) – the model thickness, \( n \) – the interference fringes order, \( \sigma_0^{1-0} \) – the strip patterns on stresses price value, the magnitude of which is determined from calibration tests. From the sixth similarity criterion in (11), knowing the scale \( k_u, k_i \) and given the time scale \( k_r \), you can uniquely determine the velocities scale, plastic deformation \( \varepsilon^p = (k_u / k_i) \varepsilon^p_M \), and knowing the similarity scale \( k_\varepsilon, k_r, k_\sigma \) from the fifth criterion in (11), it follows an unambiguous definition of the fluidity functions scale \( f(\sigma_j) = k_a k_\varepsilon k_i \), which implies that \( f_N(\sigma_j) = k_a k_\varepsilon k_i f_M(\sigma_j) \).

The fluidity functions scale with the similarity power scale can be related dependency (12): \( k_j = k_\varepsilon k_\sigma, k_i, k_r \) or to obtain the stress \( \sigma_{ij} \) across the similarity extent: \( (\sigma_{ij})_N = [k_f / k_j, k_i, (\sigma_{ij})_M] \).

From the invariant transformation of Hooke's law it follows that \( k_\varepsilon = k_a k_{(\varepsilon)} \) or \( k_{(\varepsilon)} / k_\varepsilon = k_a k_{(\varepsilon)} / k_\varepsilon \), from the linear relation between the tangent stress and the relative shift \( \tau_{xy} = \sigma_{xy} \), we have

\[
k_r = \frac{k_L k_\mu}{k_{(\varepsilon)}} = k_\sigma
\]

(14)

according to the last similarity indicator in the system (14). According to the equation \( k_\varepsilon = k_a / k_\varepsilon \); from the relation (6) we have

\[
k_\varepsilon = \frac{k_a}{k_{(\varepsilon)}}
\]

(15)

Substitute the values from (15) into in equation (13)

\[
\frac{k_{(\varepsilon)}}{k_\varepsilon} = \frac{k_p k_i}{k_\sigma}
\]

(16)

The ninth similarity indicator of the system (11) can be represented as \( k_\varepsilon k_a k_\varepsilon, k_a k_f = 1 \), because of \( k_i = k_a / k_\sigma \).
The displacement scale is equal to:

\[ k_s = \frac{k_1 k_f}{k_2 k_f}. \]

The relation (16) is transformed into equation between similarity scales for modeling the considered class of elastic-plastic contact problems

\[ \frac{k_1 k_f}{k_2 k_f} = \frac{k_{[u, \mu']}}{k_E}. \]  

(17)

If we denote the right part of the equation (17) as \( k_N / k_M \) then for the contact case of two (1 and 2) spherical surfaces it can be represented as the sum \( k_N = (k_1 + k_2)_N \) and \( k_M = (k_1 + k_2)_M \)

\[ \frac{k_{[u, \mu']}}{k_E} = \frac{(k_1 + k_2)_N}{(k_1 + k_2)_M}, \]

(18)

where \( k_1 = \frac{1 - \mu_1^2}{\pi E_1} \), \( k_2 = \frac{1 - \mu_2^2}{\pi E_2} \) because in view of the deformation of both spheres, the integral equation of the deformation of their contact surface

\[ R \int \sigma_r d\varphi ds = u_0 - \frac{r^2}{2 R_1}. \]  

(19)

invariant under replacement \( R_1 \) by \( k = k_1 + k_2 \).

In (19) is indicated: \( R_1 = (1 - \mu_l^2)/\pi E \); \( r = l/2 \) – half of the characteristic size of the contact area, \( R_1 \) – the radius of the upper pressing sphere, \( \varphi \) and \( s \) – angular and radial coordinates, respectively, \( \sigma_r \) – contact pressure, \( u_0 \) – the vertical movement of the sphere in the touch center. Taking into account the ratio (18), the equation (19) is presented in the final form

\[ \left[ \begin{array}{c} k_1 k_f \\frac{k_f}{k_2 k_f} \\ k_1 k_f \\frac{k_f}{k_2 k_f} \end{array} \right] = \left[ \begin{array}{c} (k_1 + k_2)_N \\ (k_1 + k_2)_M \end{array} \right]. \]

(20)

or in expanded form, by replacing the scale of similarity relations of the corresponding quantities in nature and model:

\[ \left[ \begin{array}{c} l_n f_1(\sigma_0) \\ p_1 \sigma_0 \hat{\varepsilon}_0^p \\ l_n f_2(\sigma_0) \\ p_2 \sigma_0 \hat{\varepsilon}_0^p \end{array} \right] = \left[ \begin{array}{c} (k_1 + k_2)_N \\ (k_1 + k_2)_M \end{array} \right]. \]

The equation (20) in the presented form shows the relationship between the characteristics of physical and mechanical properties of contacting bodies materials with the stress-strain state parameters.

3. Conclusion

The results of the analysis allow us to draw some conclusions. In the considered contact problem, a plastic region is formed on the contact surface in a more plastic sphere, that is, with a lower yield strength \( \sigma_y \), near the contact stresses nominal concentration.

Solutions of non-dimensional contact elastic-plastic problems are associated with significant mathematical difficulties. The use of physical modeling methods and polarization-optical methods of stress measurement solves these problems. Their application to the practical modeling of elastic-plastic
contact problems was constrained by the absence of similarity formulas and conditions for modeling this nonlinear class of deformable solid mechanics’s problems. The developments of these issues outlined here open up the principal possibilities of experimental studies of elastic-plastic contact problems.

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