ALL FUNCTIONS ARE (LOCALLY) $s$-HARMONIC
(UP TO A SMALL ERROR) – AND APPLICATIONS

ENRICO VALDINOCI

Abstract. The classical and the fractional Laplacians exhibit a number of similarities, but also some rather striking, and sometimes surprising, structural differences.

A quite important example of these differences is that any function (regardless of its shape) can be locally approximated by functions with locally vanishing fractional Laplacian, as it was recently proved by Serena Dipierro, Ovidiu Savin and myself.

This informal note is an exposition of this result and of some of its consequences.

1. Introduction

Given $s \in (0,1)$, we take into account the so-called $s$-fractional Laplacian

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, dy. \tag{1.1}$$

In this definition, $u$ is supposed to be a sufficiently smooth function (to make the integral convergent for small $y$), and with some growth control at infinity (to make the integral convergent for large $y$). Also, for the sake of simplicity, a normalizing constant is dropped in (1.1). It is also interesting to observe that, by splitting two integrals and changing variables, equation (1.1) can be written as

$$(-\Delta)^s u(x) = \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus B_\rho} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus B_\rho} \frac{u(x) - u(x-y)}{|y|^{n+2s}} \, dy \tag{1.2}$$

$$= 2 \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus B_\rho} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, dy$$

$$= 2 \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus B_\rho(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy$$

$$= \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy,$$

where the notation “P.V.” stands for “in the Cauchy Principal Value Sense (and the factor 2 will not be relevant for our purposes).

The fractional Laplacian is one of the most widely studied operators in the recent literature, probably in view of its intrinsic beauty (in spite of the first impression that the definition in (1.1) can produce), of the large variety of different problems related to it, and of its great potentials in modeling real-world phenomena in applied sciences.
The setting in (1.1) is clearly related to an “incremental quotient” of \( u \) which gets averaged in all the space. Indeed, roughly speaking, equation (1.1) combines together several special features related to the classical Laplacian:

(1) The classical Laplacian arises from a second order incremental quotient, namely, for a smooth function \( u \) and a small increment \( h \), denoting by \( \{e_j\}_{j=1,...,n} \) the standard Euclidean basis of \( \mathbb{R}^n \), it holds that

\[
2u(x) - u(x + he_j) - u(x - he_j) - \frac{1}{2} D^2 u(x)(he_j) \cdot (he_j) + o(h^2)
\]

and so

\[
\lim_{h \to 0} \frac{2u(x) - u(x + he_j) - u(x - he_j)}{h^2} = -\Delta u(x).
\]

Comparing this with (1.1), we recognize a structure related to incremental quotients in the definition of fractional Laplacian;

(II) The classical Laplacian compares the value of a function with its average. Indeed, for a small \( \rho > 0 \),

\[
\int_{B_\rho(x)} u(y) dy = \int_{B_\rho} u(x + y) dy
\]

\[
= \int_{B_\rho} \left( u(x) + \nabla u(x) \cdot y + \frac{1}{2} D^2 u(x) y \cdot y + o(|y|^2) \right) dy.
\]

Also, by odd symmetry we see that

\[
\int_{B_\rho} y_j dy = 0 \quad \text{for all } j \in \{1,\ldots,n\}
\]

and

\[
\int_{B_\rho} y_j y_k dy = 0 \quad \text{for all } j \neq k \in \{1,\ldots,n\}.
\]

Consequently, we can write (1.3) as

\[
\int_{B_\rho(x)} u(y) dy = u(x) + \frac{1}{2} \sum_{j=1}^n \partial^2_{jj} u(x) \int_{B_\rho} y_j^2 dy + o(\rho^2)
\]

\[
= u(x) + \frac{1}{2n} \sum_{j=1}^n \partial^2_{jj} u(x) \int_{B_\rho} |y|^2 dy + o(\rho^2)
\]

\[
= u(x) + \frac{\rho^2}{2(n+2)} \Delta u(x) + o(\rho^2)
\]

and therefore

\[
-\Delta u(x) = 2(n+2) \lim_{\rho \to 0} \int_{B_\rho(x)} \frac{u(x) - u(y)}{\rho^2} dy.
\]
Similarly,
\[
\int_{\partial B_\rho(x)} u(y) \, d\mathcal{H}^{n-1}(y) = \int_{\partial B_\rho} \left( u(x) + \nabla u(x) \cdot y + \frac{1}{2} D^2 u(x) y \cdot y + o(|y|^2) \right) \, d\mathcal{H}^{n-1}(y)
\]
\[
= u(x) + \frac{1}{2} \sum_{j=1}^n \partial_{jj}^2 u(x) \int_{\partial B_\rho} y_j^2 \, d\mathcal{H}^{n-1}(y) + o(\rho^2)
\]
\[
= u(x) + \frac{1}{2n} \sum_{j=1}^n \partial_{jj}^2 u(x) \int_{\partial B_\rho} |y|^2 \, d\mathcal{H}^{n-1}(y) + o(\rho^2)
\]
\[
= u(x) + \frac{\rho^2}{2n} \Delta u(x) + o(\rho^2)
\]
and therefore
\[
-\Delta u(x) = 2n \lim_{\rho \searrow 0} \int_{\partial B_\rho(x)} \frac{u(x) - u(y)}{\rho^2} \, d\mathcal{H}^{n-1}(y)
\]
\[
= 2n \lim_{\rho \searrow 0} \int_{\partial B_\rho(x)} \frac{u(x) - u(y)}{|x-y|^2} \, d\mathcal{H}^{n-1}(y).
\]
(1.5)

Once again, the factors \(2(n+2)\) and \(2n\) in (1.4) and (1.5) are not important for our purposes, but the similarities between (1.2), (1.4) and (1.5) are evident and suggest that the fractional Laplacian is a suitably weighted average distributed in the whole of the space.

(III) The classical Laplace operator is variational and stems from a Dirichlet energy of the form
\[
\int |\nabla u(x)|^2 \, dx.
\]
(1.6)

Similarly, the fractional Laplacian is variational and the corresponding energy is the Gagliardo-Slobodeckij-Sobolev seminorm
\[
\iint \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.
\]
(1.7)

The integral in (1.6) usually ranges in a “domain” \(\Omega \subseteq \mathbb{R}^n\) which should be considered as the region of space where “action takes place”, or, better to say the complement of the region in which no action takes place (that is, the domain \(\Omega\) is the complement of the region \(\mathbb{R}^n \setminus \Omega\), where the data of \(u\) are fixed). The fractional counterpart of this is to take as “natural domain” for (1.7) the complement (in \(\mathbb{R}^{2n}\)) of the set \((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)\) where the data of \(u(x) - u(y)\) are fixed, that is, it is common to integrate (1.7) over the “cross domain”

\[
Q_\Omega := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^n \setminus \Omega)) \cup ((\mathbb{R}^n \setminus \Omega) \times \Omega)
\]
\[
= \mathbb{R}^{2n} \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)).
\]

(IV) Most importantly, the fractional Laplacian enjoys “elliptic” features that are similar to the ones of the classical Laplacian, e.g. in terms of maximum principle. The regularizing effects of the fractional Laplacian can be somewhat “guessed” from the singularity of the integral kernel in (1.1): indeed, on the one hand, to make sense of the integral in (1.1), one needs the function \(u\) to be “smooth enough” near \(x\); on the
other hand, and somehow conversely, if the integral in (1.1) is finite, the function \( u \) needs to have some regularity property near \( x \), in order to compensate the singularity of the kernel.

Several classical and recent publications presented the fractional Laplacian from different perspectives. See in particular [Ste70, Lan72, Sil05, DNPV12, BV16]. In our postmodern world some excellent online expositions of this topic have also become available, see in particular the very useful webpage https://www.ma.utexas.edu/mediawiki/index.php/Fractional_Laplacian

We also recall that the fractional Laplacian can also be framed into the context of probability and harmonic analysis, thus leading to different possible approaches and several possible definitions, see [Kwa17], and it is also possible to provide a suitable setting in order to define the fractional Laplacian for functions with polynomial growth at infinity, see [DSVa].

In spite of the extremely important similarities between the classical and the fractional Laplacian, several structural differences between these operators arise. See e.g. [AV] for a collection of some of these basic differences. Some of these differences have also extremely deep consequences on some recent results in the theory of nonlocal equations, see https://www.ma.utexas.edu/mediawiki/index.php/List_of_results_that_are_fundamentally_different_to_thelocal_case

In this note, we recall one of the basic differences between the classical and the fractional Laplacian, which has been recently discovered in [DSV17] and which presents a source of interesting consequences. This difference deals with the so called “s-harmonic functions”, which are the (rather surprising) counterpart of classical harmonic functions.

The parallelism between classical harmonic functions and s-harmonic functions lies in their definition, since \( u \) is said to be harmonic (respectively, s-harmonic) at \( x \) if \((-\Delta) u(x) = 0 \) (respectively, if \((-\Delta)^s u(x) = 0 \).

Already from the definition, a basic difference between the classical and the fractional case arises, since the definition of harmonic function at \( x \) only requires the function to be defined in an arbitrarily small neighborhood of \( x \), while the definition of s-harmonic function requires the function to be globally defined in \( \mathbb{R}^n \). This difference, which is somehow the counterpart of the structural differences between (1.2) on one side and (1.4) and (1.5) on the other side, turns out to be perhaps deeper than what may look at a first glance. As a matter of fact, the classical Laplacian is a very “rigid” operator, and for a function to be harmonic some very restrictive geometric conditions must hold (in particular, harmonic functions cannot have local minima). In sharp contrast with this fact, the fractional Laplacian is very flexible and the “oscillations of a function that come from far” can locally produce very significant contributions.

Probably, the most striking example of this phenomenon is that such far-away oscillations can make the fractional Laplacian of any function to almost vanish at a point, and in fact any given function, without any restriction on its geometric properties, can be approximated arbitrarily well by an s-harmonic function. In this sense, we have:

**Theorem 1** (“All functions are locally s-harmonic up to a small error” [DSV17]). For any \( \varepsilon > 0 \) and any function \( \bar{v} \in C^2(B_1) \), there exists \( v_\varepsilon \) such that

\[
\begin{cases}
\| \bar{v} - v_\varepsilon \|_{C^2(B_1)} \leq \varepsilon, \\
(-\Delta)^s v_\varepsilon = 0 \text{ in } B_1.
\end{cases}
\]
A proof of this fact (in dimension 1 for the sake of simplicity) will be given in Section 3 (see the original paper [DSV17] for the full details of the argument in any dimension).

We stress that the phenomenon described in Theorem 1 is very general, and it arises also for other nonlocal operators, independently from their possibly “elliptic” structure (for instance all functions are locally \(s\)-caloric, or \(s\)-hyperbolic, etc.), see [DSV16].

It is interesting to remark that the proofs in [DSV17, DSV16] are not “quantitative”, in the sense that they are based on a contradiction argument, and the “shape” of the approximating \(s\)-harmonic (or \(s\)-caloric, or \(s\)-hyperbolic) function cannot be detected by our methods. On the other hand, for a very nice quantitative version of Theorem 1 see Theorem 1.4 in [RS17a]. See also [RS17b] for a quantitative approach to the parabolic case and [GSU16] for related results (and, of course, quantitative proofs are harder and technically more advanced than the one that we present here). In addition, results similar to Theorem 1 hold true for nonlocal operators with memory, see [Buc].

Theorem 1 possesses some simple, but quite interesting consequences. In the forthcoming Section 2 we present a few of them, related to

(i) The fractional Maximum Principle and Harnack Inequality;
(ii) The classification of stable solutions for fractional equations;
(iii) The diffusive strategy of biological populations.

2. Applications of Theorem 1

2.1. The fractional Maximum Principle and Harnack Inequality. One of the main features of the classical Laplace operator is that it enjoys the Maximum Principle. For instance, as well known, it holds that:

**Theorem 2.** Let \(u\) be a harmonic and nonnegative function in \(B_1\). If \(u(x_0) = 0\) for some \(x_0 \in B_1\), then \(u\) is necessarily constantly equal to 0 in \(B_1\).

A classical quantitative version of Theorem 2 was given by Axel von Harnack and can be stated as follows:

**Theorem 3.** If \(u\) is harmonic in \(B_1\) and nonnegative in \(B_1\), then, for every \(r \in (0, 1)\),

\[
\sup_{B_r} u \leq C_r \inf_{B_r} u,
\]

for some \(C_r > 0\) depending on \(n\) and \(r\).

The original manuscript by von Harnack is available at

https://ia902306.us.archive.org/9/items/vorlesunganwend00weierich/vorlesunganwend00weierich.pdf

The fractional counterpart of Theorem 3 goes as follows:

**Theorem 4.** If \(u\) is \(s\)-harmonic in \(B_1\) and nonnegative in the whole of \(\mathbb{R}^n\), then, for every \(r \in (0, 1)\),

\[
\sup_{B_r} u \leq C_r \inf_{B_r} u,
\]

for some \(C_r > 0\) depending on \(n\) and \(r\).
See [Kaß01, BL02, CS09, ST10, Kaß11, KRS14] and the references therein for a detailed study of the fractional Harnack Inequality. Of course, an important structural difference between Theorems 3 and 4 (besides the $s$-harmonicity versus the classical harmonicity) is the fact that in Theorem 4 one requires a global condition on the sign of the solution. Interestingly, if in Theorem 4 one replaces the assumption that $u$ is nonnegative in the full space with the assumption that $u$ is nonnegative just in the unit ball, then the result turns out to be false, as described by the following example:

**Theorem 5.** There exists a bounded function $u$ which is $s$-harmonic in $B_1$, nonnegative in $B_1$, not identically 0 in $B_1$, but such that

$$\inf_{B_{1/2}} u = 0.$$  

Theorem 5 suggests that some care has to be taken when dealing with Maximum Principles and oscillation results in the fractional case, and in fact the nonlocal character of the operator requires global conditions for this type of results to hold, in virtue of the contributions “coming from far away”.

A proof of Theorem 5 can be obtained directly from Theorem 1. Indeed, we take $n = 1$, $\bar{v}(x) := x^2$ and $\varepsilon := \frac{1}{16}$. Then, Theorem 1 provides a function $v$ which is $s$-harmonic in $(-1, 1)$ and such that

$$\|\bar{v} - v\|_{L^\infty((-1, 1))} \leq \|\bar{v} - v\|_{C^2((-1, 1))} \leq \frac{1}{16}.$$  

In particular, if $|x| \geq \frac{1}{2}$,

$$v(x) \geq \bar{v}(x) - \frac{1}{16} = |x|^2 - \frac{1}{16} \geq \left(\frac{1}{2}\right)^2 - \frac{1}{16} = \frac{3}{16},$$  

while

$$v(0) \leq \bar{v}(0) + \frac{1}{16} = \frac{1}{16}.$$  

Accordingly,

$$\inf_{(-1, 1)} v \leq \frac{1}{16} < \frac{3}{16} \leq \inf_{(-1, 1) \setminus (-1/2, 1/2)} v,$$

which gives that

$$\inf_{(-1, 1)} v = \min_{[-1/2, 1/2]} v =: \iota.$$  

Then the function $u := v - \iota$ satisfies the thesis of Theorem 5, as desired.

For different approaches to the counterexamples to the local Harnack Inequality in the fractional setting see [BS05, Kaß07] and also Chapter 2.3 in [BV16].

---

\(^1\)We take this opportunity to amend a typo in Theorem 2.3.1 of [BV16], where $\inf_{B_1} u$ has to be replaced by $\inf_{B_{1/2}} u$. 
2.2. The classification of stable solutions for fractional equations. In the Calculus of Variations\(^2\) literature, a solution \(u\) is called “stable” if it is the critical point of an energy functional whose second variation is nonnegative definite at \(u\). For instance, local minimizers of the energy are stable solutions, and it is in fact often convenient to study stable solutions since the stability class is often preserved under suitable limit procedures and it is sometimes technically easier (or at least less difficult) to prove that a solution is stable rather than deciding whether or not it is minimal.

We refer to the very nice monograph [Dup11] for a throughout discussion of the notion of stability and for many related results.

A classical result in the framework of stable solutions of elliptic equations was obtained independently by Richard Casten and Charles Holland, on the one side, and Hiroshi Matano, on the other side, and it deals with the classification of stable solutions with Neumann data. A paradigmatic result in this case can be stated as follows:

**Theorem 6 ([CH78, Mat79]).** Let \(\Omega \subset \mathbb{R}^n\) be a bounded and convex domain with smooth boundary.

Suppose that \(u\) is a smooth solution of

\[
\begin{cases}
-\Delta u(x) + f(u(x)) = 0 & \text{for any } x \in \Omega \\
\frac{\partial u}{\partial \nu}(x) = 0 & \text{for any } x \in \partial \Omega,
\end{cases}
\]

for some smooth function \(f\), where \(\nu\) denotes the (external) unit normal of \(\Omega\).

Assume also that \(u\) is stable, namely

\[
\int_{\Omega} |\nabla \varphi(x)|^2 + f'(u(x)) |\varphi(x)|^2 \, dx \geq 0,
\]

for any \(\varphi \in H^1(\Omega)\).

Then, \(u\) is necessarily constant.

We remark that

\[
(2.3)
\]

when \(f\) vanishes identically then \((2.2)\) is automatically satisfied.

It is interesting to observe that, with respect to Theorem 6, the fractional case behaves very differently, and nonconstant stable solutions with Neumann data in convex domains do exist, according to the following result:

**Theorem 7 ([DSVb]).** Let \(s \in (0, 1)\). There exist an open interval \(I \subset \mathbb{R}\) and a nonconstant function \(u\) such that

\[
\begin{align*}
(2.4) & \quad (-\Delta)^s u = 0 \quad \text{in } I, \\
(2.5) & \quad \lim_{x \to x_0 \in \partial I} \frac{u(x) - u(x_0)}{x - x_0} = 0 \\
(2.6) & \quad \text{ and } \quad u' = 0 \quad \text{on } \partial I.
\end{align*}
\]

We observe that \((2.4)\) is the natural fractional counterpart of \((2.1)\) (with \(f := 0\), and \((2.3)\) guarantees a stability condition). Of course, an interval is a (onedimensional) convex set, hence the geometric setting of Theorem 6 is respected in Theorem 7. Also, formula \((2.6)\) can be seen as a classical Neumann condition, while formula \((2.5)\) can be seen as a fractional

\(^2\)Notice that the notion of “stability” differs from one scientific community to another. In particular, the notion of stability that we treat here does not agree with that in Dynamical Systems or Algebraic Geometry.
Neumann condition (say, of order $s$). Condition (2.5) is indeed quite exploited as a natural boundary condition in fractional problems, and it is compatible with the boundary regularity theory and with the sliding methods, see [ROS14, FJ15] (for another notion of fractional Neumann condition see [DROV17]).

In this sense, Theorem 7 can be considered as a “counterexample” for the fractional analogue of Theorem 6 to hold. The construction of Theorem 7 is in fact very general. It is based on Theorem 1 and provides a series of rather “arbitrary” counterexamples, see Section 1.7 in [DSVb] for additional details.

It has to be pointed out, however, that results similar to the original ones in [CH78, Mat79] hold true for a different type of fractional operator (the so-called “spectral” fractional Laplacian, see [SV14]). In particular, classification results for stable solutions of nonlocal operators which can be seen as the fractional counterpart of those in [CH78, Mat79] have been given in Sections 1.4, 1.5 and 1.6 in [DSVb]. This fact shows the very intriguing phenomenon, according to which “little” modifications in the fractional settings do produce rather different results, which are sometimes in agreement with the classical case, and sometimes not.

2.3. The diffusive strategy of biological populations. A classical problem in biomathematics consists in studying the evolution of a biological species with density $u = u(x,t)$ in $B_1 \ni x$, with prescribed boundary or external conditions. In this framework, the so-called logistic equations is based on the ansatz that the state of the population is due to three well distinguishable features:

- The population diffuses according to a stochastic motion;
- For small density, the population grows more or less linearly, thanks to some resources $\rho = \rho(x) > 0$;
- When the density overcomes a critical threshold $\sigma/\mu$, for some $\mu = \mu(x) > 0$, the population unfortunately dies (roughly speaking, because “there is no food for everybody”).

When the diffusion term is lead by the standard Brownian motion, the logistic equation that we describe takes the form

$$
\partial_t u = \Delta u + (\sigma - \mu u) u \quad \text{in } B_1 \times (0,T),
$$

for some $T > 0$. In particular, the study of the steady states of (2.7) leads to the equation

$$
- \Delta u = (\sigma - \mu u) u \quad \text{in } B_1.
$$

On the other hand, recent experiments have shown that several predators do not follow standard diffusion processes, but rather discontinuous processes with jumps whose distribution may exhibit a long (e.g. with a polynomial tail), see e.g. [VAB+96]. This fact, that may seem surprising, has indeed a sound motivation: for a predator it makes little sense to move randomly looking for prey, since, after a first attack, the other possible targets will rapidly escape from the dangerous area – conversely, a strategy of “hit and run”, based on quick hunts after long excursions, is more reasonable to be efficient and ensure more food to the predator.

In this sense, a natural nonlocal variation of (2.8) to be taken into account is the fractional logistic equation

$$
(-\Delta)^s u = (\sigma - \mu u) u \quad \text{in } B_1,
$$

where $s$ is the order of the fractional Laplacian and $\Delta^s$ denotes the fractional Laplacian.
with \( s \in (0, 1) \), see e.g. [MPV13, MV17, CM17, CDV17] and the references therein. Interestingly, different species in nature seem to exhibit different values of the fractional parameter \( s \), probably due to different environmental conditions and different morphological structures and it is an intriguing problem to understand what “the optimal exponent \( s \)” should be in concrete circumstances, see [SV17].

Another interesting special feature offered by nonlocal diffusion is the possibility for nonlocal populations to efficiently plan their distribution in order to consume all (or almost all) the given resources in a certain “strategic region”. That is, if the region of interest for the population is, say, the ball \( B_1 \), the species can artificially and appropriately settle its distribution outside \( B_1 \), in order to satisfy in \( B_1 \) a logistic equation as that in (2.9), for a resource that is arbitrarily close to the original one. The precise statement of this result is the following:

**Theorem 8 ([MV17, CDV17]).** Assume that \( \sigma, \mu \in C^2(B_1) \), with
\[
\inf_{B_1} \sigma > 0 \quad \text{and} \quad \inf_{B_1} \mu > 0.
\]
Then, for any \( \varepsilon > 0 \) there exist \( u_\varepsilon \) and \( \sigma_\varepsilon \) such that
\[
\begin{align*}
\| \sigma - \sigma_\varepsilon \|_{C^2(B_1)} & \leq \varepsilon, \\
u_\varepsilon & \geq \sigma_\varepsilon/\mu \quad \text{in} \quad B_1, \\
(-\Delta)^s u_\varepsilon & = (\sigma_\varepsilon - \mu u_\varepsilon) u_\varepsilon \quad \text{in} \quad B_1.
\end{align*}
\]

Once again, a proof of Theorem 8 may be performed by exploiting Theorem 1, see Section 7 in [CDV17].

### 3. Proof of Theorem 1

For simplicity, we focus on the one-dimensional case: the general case follows by technical modifications and can be found in the original article [DSV17].

The core of the proof is to show that the derivatives of \( s \)-harmonic functions have “maximal span” as a linear space (and we stress that this is not true for harmonic functions, since the second derivatives of harmonic functions satisfy a linear prescription).

We consider the set
\[
\mathcal{V} := \{ h : \mathbb{R} \to \mathbb{R} \text{ s.t. } h \text{ is smooth and } s \text{-harmonic in some neighborhood of the origin} \}.
\]
Notice that \( \mathcal{V} \) has a linear space structure, namely if \( h_1 \) is \( s \)-harmonic in some open set \( V_1 \) containing the origin and \( h_2 \) is \( s \)-harmonic in some open set \( V_2 \) containing the origin, then, for any \( \lambda_1, \lambda_2 \in \mathbb{R} \), we have that \( h_3 := \lambda_1 h_1 + \lambda_2 h_2 \) is \( s \)-harmonic in the open set \( V_3 := V_1 \cap V_2 \ni 0 \).

Then, given \( J \in \mathbb{N} \), we define
\[
\mathcal{V}_J := \{ (h(0), h(0), \ldots, h^{(J)}(0)) \text{ with } h \in \mathcal{V} \}.
\]
As customary, here \( h^{(J)} \) denotes the \( J \)th derivative of the function \( h \). In this way, we have that \( \mathcal{V}_J \) is a linear subspace of \( \mathbb{R}^{J+1} \) (roughly speaking, each element of \( \mathcal{V}_J \) is a \((J + 1)\)-dimensional array containing the first \( J \) derivatives of a locally \( s \)-harmonic function).

We claim that
\[
\mathcal{V}_J = \mathbb{R}^{J+1}
\]
For this, we argue by contradiction and we suppose that \( \mathcal{V}_J \) is a linear subspace strictly smaller than \( \mathbb{R}^{J+1} \). That is, \( \mathcal{V}_J \) lies inside a \( J \)-dimensional hyperplane, say with normal \( \nu \). Namely, there exists
\[
(3.4) \quad \nu = (\nu_0, \ldots, \nu_J) \in \mathbb{R}^{J+1} \quad \text{with} \quad |\nu| = 1
\]
such that
\[
(3.5) \quad \mathcal{V}_J \subseteq \{ X = (X_0, \ldots, X_J) \in \mathbb{R}^{J+1} \text{ s.t. } \nu \cdot X = 0 \}
\]
Now, for any \( t > 0 \), we define
\[
h_t(x) := (x + t)^s_+.
\]
It is known that \( h_t \) is \( s \)-harmonic in \( (-t, +\infty) \) (see e.g. Chapter 2.4 in [BV16] for an elementary proof). Consequently, \( h_t \in \mathcal{V}' \) and then
\[
X_t := (h_t(0), \ldots, h_t^{(J)}(0)) \in \mathcal{V}_J.
\]
As a result, by (3.5),
\[
(3.6) \quad 0 = \nu \cdot X_t = \sum_{j=0}^J \nu_j h_t^{(j)}(0) = \sum_{j=0}^J \mu_{s,j} t^{s-j},
\]
where
\[
(3.7) \quad \mu_{s,j} := \nu_j \prod_{i=0}^{j-1} (s - i).
\]
Hence, multiplying the identity in (3.6) by \( t^{J-s} \), for any \( t > 0 \), it holds that
\[
\sum_{k=0}^J \mu_{s,J-k} t^k = 0,
\]
which, by the Identity Principle for Polynomials, implies that \( \mu_{s,0} = \cdots = \mu_{s,J} = 0 \) and accordingly\(^3 \) from (3.7) we get that \( \nu_0 = \cdots = \nu_J = 0 \). This is in contradiction with (3.4) and so the proof of (3.3) is complete.

Now, the proof of Theorem 1 follows by approximation and scaling. Given \( \bar{v} \in C^2(\overline{B_1}) \) and \( \varepsilon \in (0, 1) \), in view of the Stone-Weierstrass Theorem we take a polynomial \( P_\varepsilon \) such that
\[
(3.8) \quad \| \bar{v} - P_\varepsilon \|_{C^2(B_1)} \leq \frac{\varepsilon}{2}.
\]
We write
\[
P_\varepsilon(x) = \sum_{j=0}^{N_\varepsilon} c_{j,\varepsilon} x^j = \sum_{j=0}^{N_\varepsilon} m_{j,\varepsilon}(x),
\]
for some \( N_\varepsilon \in \mathbb{N} \) and some \( c_{1,\varepsilon}, \ldots, c_{N_\varepsilon,\varepsilon} \in \mathbb{R} \), where
\[
(3.9) \quad m_{j,\varepsilon}(x) := c_{j,\varepsilon} x^j.
\]
Without loss of generality, by possibly adding zero coefficients in the representation above, we can suppose that
\[
(3.10) \quad N_\varepsilon \geq 3.
\]
We set
\[ C_\varepsilon := \max_{j \in \{0, \ldots, N_\varepsilon\}} |c_{j,\varepsilon}|. \]

For any \( j \in \{0, \ldots, N_\varepsilon\} \), we let \( H_{j,\varepsilon} : \mathbb{R} \to \mathbb{R} \) be a function which is \( s \)-harmonic in a neighborhood of the origin and such that, for any \( i \in \{0, \ldots, N_\varepsilon\} \) it holds that
\[(3.11)\]
\[ H_{j,\varepsilon}^{(i)}(0) = \begin{cases} c_{j,\varepsilon} j! & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases} \]

Once again, \( H_{j,\varepsilon}^{(i)} \) denotes here the \( i \)th derivative of \( H_{j,\varepsilon} \). We stress that the existence of \( H_{j,\varepsilon} \) is a consequence of (3.3). We also set
\[(3.12)\]
\[ r_{j,\varepsilon} := \varepsilon 10 N_\varepsilon^2 \left( 1 + \sup_{x \in (-1,1)} |H_{j,\varepsilon}^{(N_\varepsilon+1)}(x)| \right) \in (0,1) \]
and \( \mathcal{H}_{j,\varepsilon}(x) := r_{j,\varepsilon}^{-j} H_{j,\varepsilon}(r_{j,\varepsilon}x) \).

We remark that, for any \( i, j \in \{0, \ldots, N_\varepsilon\} \),
\[(3.13)\]
\[ \mathcal{D}_{j,\varepsilon}^{(i)}(x) := \mathcal{H}_{j,\varepsilon}(x) - c_{j,\varepsilon} x^i = \mathcal{H}_{j,\varepsilon}(x) - m_{j,\varepsilon}(x) \]
satisfies
\[(3.14)\]
\[ \mathcal{D}_{j,\varepsilon}^{(i)}(0) = 0 \text{ for all } i \in \{0, \ldots, N_\varepsilon\}. \]

In addition, for any \( x \in (-1,1) \) and any \( j \in \{0, \ldots, N_\varepsilon\} \),
\[ |\mathcal{D}_{j,\varepsilon}^{(N_\varepsilon+1)}(x)| = |\mathcal{H}_{j,\varepsilon}^{(N_\varepsilon+1)}(x)| \]
\[ \leq r_{j,\varepsilon}^{N_\varepsilon+1-j} |H_{j,\varepsilon}^{(N_\varepsilon+1)}(r_{j,\varepsilon}x)| \]
\[ \leq r_{j,\varepsilon} \sup_{x \in (-1,1)} |H_{j,\varepsilon}^{(N_\varepsilon+1)}(x)| \]
\[ \leq \frac{\varepsilon}{2N_\varepsilon^2}, \]
thanks to (3.12). This, (3.14) and a Taylor expansion give that, for any \( x \in (-1,1) \) and any \( i, j \in \{0, \ldots, N_\varepsilon\} \),
\[ |\mathcal{D}_{j,\varepsilon}^{(i)}(x)| \leq \sup_{x \in (-1,1)} |\mathcal{D}_{j,\varepsilon}^{(N_\varepsilon+1)}(x)| \leq \frac{\varepsilon}{10 N_\varepsilon^2}. \]

Hence, recalling (3.10)
\[ \sum_{j=0}^{N_\varepsilon} \|\mathcal{D}_{j,\varepsilon}\|_{C^2(-1,1)} \leq \frac{\varepsilon}{2}. \]

So, we define
\[ v_\varepsilon := \sum_{j=0}^{N_\varepsilon} \mathcal{H}_{j,\varepsilon}. \]
We have that $v_{\varepsilon}$ is $s$-harmonic in $(-1,1)$ and, recalling (3.8) and (3.13),

$$\|\bar{v} - v_{\varepsilon}\|_{C^2(-1,1)} \leq \|\bar{v} - P_{\varepsilon}\|_{C^2(-1,1)} + \|P_{\varepsilon} - v_{\varepsilon}\|_{C^2(-1,1)}$$

$$\leq \varepsilon + \left\| \sum_{j=0}^{N_{\varepsilon}} (m_{j,\varepsilon} - H_{j,\varepsilon}) \right\|_{C^2(-1,1)}$$

$$\leq \varepsilon + \sum_{j=0}^{N_{\varepsilon}} \|D_{j,\varepsilon}\|_{C^2(-1,1)}$$

$$\leq \varepsilon.$$ 

This establishes Theorem 1 in this setting.

**REFERENCES**

[AV] Nicola Abatangelo and Enrico Valdinoci, *Getting acquainted with the fractional Laplacian*, Preprint. ↑4

[BL02] Richard F. Bass and David A. Levin, *Harnack inequalities for jump processes*, Potential Anal. 17 (2002), no. 4, 375–388, DOI 10.1023/A:1016378210944. MR1918242 ↑6

[BOS05] Krzysztof Bogdan and Pawel Sttonyk, *Harnack’s inequality for stable Lévy processes*, Potential Anal. 22 (2005), no. 2, 133–150, DOI 10.1007/s11118-004-0590-x. MR2137058 ↑6

[Buc] Claudia Bucur, *Local density of Caputo-stationary functions in the space of smooth functions*, to appear on ESAIM Control Optim. Calc. Var. ↑5

[BV16] Claudia Bucur and Enrico Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016. MR3469920 ↑4, 6, 10

[CS15] Xavier Cabré and Yannick Sire, *Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. 367 (2015), no. 2, 911–941, DOI 10.1090/S0002-9947-2014-05906-0. MR3280032 ↑6

[CDV17] Luis Caffarelli, Serena Dipierro, and Enrico Valdinoci, *A logistic equation with nonlocal interactions*, Kinet. Relat. Models 10 (2017), no. 1, 141–170, DOI 10.3934/krm.2017006. MR3579567 ↑9

[CS09] Luis Caffarelli and Luis Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. 62 (2009), no. 5, 597–638, DOI 10.1002/cpa.20274. MR2494809 ↑6

[CM17] Giulia Carboni and Dimitri Mugnai, *On some fractional equations with convex-concave and logistic-type nonlinearities*, J. Differential Equations 262 (2017), no. 3, 2393–2413, DOI 10.1016/j.jde.2016.10.045. MR3582231 ↑9

[CH78] Richard G. Casten and Charles J. Holland, *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. Differential Equations 27 (1978), no. 2, 266–273, DOI 10.1016/0022-0396(78)90033-5. MR480282 ↑7, 8

[DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521–573, DOI 10.1016/j.bulsci.2011.12.004. MR2944369 ↑4

[DROV17] Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci, *Nonlocal problems with Neumann boundary conditions*, Rev. Mat. Iberoam. 33 (2017), no. 2, 377–416, DOI 10.4171/RMI/942. MR3651008 ↑8

[DSV17] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci, *All functions are locally $s$-harmonic up to a small error*, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 4, 957–966, DOI 10.4171/JEMS/684. MR3626547 ↑4, 5, 9

[DSV16], *Local approximation of arbitrary functions by solutions of nonlocal equations*, submitted, preprint at arXiv:1609.04438 (2016). ↑5
ALL FUNCTIONS ARE $s$-HARMONIC

[DSVa] ______, Definition of fractional Laplacian for functions with polynomial growth, to appear on Rev. Mat. Iberoam. ↑4

[DSVb] Serena Dipierro, Nicola Soave, and Enrico Valdinoci, On stable solutions of boundary reaction-diffusion equations and applications to nonlocal problems with Neumann data, to appear on Indiana Univ. Math. J. ↑7, 8

[Dup11] Louis Dupaigne, Stable solutions of elliptic partial differential equations, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 143, Chapman & Hall/CRC, Boca Raton, FL, 2011. MR2779463 ↑7

[FJ15] Mouhamed Moustapha Fall and Sven Jarohs, Overdetermined problems with fractional Laplacian, ESAIM Control Optim. Calc. Var. 21 (2015), no. 4, 924–938, DOI 10.1051/cocv/2014048. MR3395749 ↑8

[GSU16] T. Ghosh, M. Salo, and G. Uhlmann, The Calderón problem for the fractional Schrödinger equation, ArXiv e-prints (2016), available at 1609.09248. ↑5

[Kaß01] Moritz Kaßmann, Harnack-Ungleichungen für nichtlokale Differentialoperatoren und Dirichlet-Formen, Bonner Mathematische Schriften [Bonn Mathematical Publications], vol. 336, Universität Bonn, Mathematisches Institut, Bonn, 2001 (German). Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2000. MR1941020 ↑6

[Kaß11] ______, A new formulation of Harnack’s inequality for nonlocal operators, C. R. Math. Acad. Sci. Paris 349 (2011), no. 11-12, 637–640, DOI 10.1016/j.crma.2011.04.014 (English, with English and French summaries). MR2817382 ↑6

[Kaß07] ______, The classical Harnack Inequality fails for non-local operators, Preprint SFB 611, Sonderforschungsbereich Singuläre Phänomene und Skalierung in Mathematischen Modellen 360 (2007). ↑6

[KRS14] Moritz Kaßmann, Marcus Rang, and Russell W. Schwab, Integro-differential equations with nonlinear directional dependence, Indiana Univ. Math. J. 63 (2014), no. 5, 1467–1498, DOI 10.1512/iumj.2014.63.5394. MR3283558 ↑6

[Kwa17] Mateusz Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator, Fract. Calc. Appl. Anal. 20 (2017), no. 1, 7–51, DOI 10.1515/fca-2017-0002. MR3613319 ↑4

[Mat79] Hiroshi Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci. 15 (1979), no. 2, 401–454, DOI 10.2977/prims/1195188180. MR555661 ↑7, 8

[MPV13] Eugenio Montefusco, Benedetta Pellacci, and Gianmaria Verzini, Fractional diffusion with Neumann boundary conditions: the logistic equation, Discrete Contin. Dyn. Syst. Ser. B 18 (2013), no. 8, 2175–2202, DOI 10.3934/dcdsb.2013.18.2175. MR3082317 ↑9

[ROS14] Xavier Ros-Oton and Joaquim Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. (9) 101 (2014), no. 3, 275–302, DOI 10.1016/j.matpur.2013.06.003 (English, with English and French summaries). MR3168912 ↑8

[RS17a] A. Rüland and M. Salo, The fractional Calderón problem: low regularity and stability, ArXiv e-prints (2017), available at 1708.06294. ↑5

[RS17b] ______, Quantitative Approximation Properties for the Fractional Heat Equation, ArXiv e-prints (2017), available at 1708.06300. ↑5

[SV14] Raffaella Servadei and Enrico Valdinoci, On the spectrum of two different fractional operators, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014), no. 4, 831–855, DOI 10.1017/S0308210512001783. MR3233760 ↑8

[Sil05] Luis Enrique Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–The University of Texas at Austin. MR2707618 ↑4
[SV17] Jürgen Sprekels and Enrico Valdinoci, *A new type of identification problems: optimizing the fractional order in a nonlocal evolution equation*, SIAM J. Control Optim. 55 (2017), no. 1, 70–93, DOI 10.1137/16M105575X. MR3590646

[Ste70] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095

[ST10] Pablo Raúl Stinga and José Luis Torrea, *Extension problem and Harnack’s inequality for some fractional operators*, Comm. Partial Differential Equations 35 (2010), no. 11, 2092–2122, DOI 10.1080/03605301003735680. MR2754080

[VAB+96] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, *Lévy flight search patterns of wandering albatrosses*, Nature 381 (1996), 413–415, DOI 10.1038/381413a0.