COHERENT STATES ON THE CIRCLE

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Abstract
A careful study of the physical properties of a family of coherent states on the circle, introduced some years ago by de Bièvre and González in [DG 92], is carried out. They were obtained from the Weyl–Heisenberg coherent states in $L^2(\mathbb{R})$ by means of the Weil–Brezin–Zak transformation, they are labeled by the points of the cylinder $S^1 \times \mathbb{R}$, and they provide a realization of $L^2(S^1)$ by entire functions (similar to the well-known Fock–Bargmann construction). In particular, we compute the expectation values of the position and momentum operators on the circle and we discuss the Heisenberg uncertainty relation.
1. Introduction

This paper is devoted to the study of the physical properties of a family of coherent states (CS) defined on the circle (i.e., belonging to $L^2(S^1)$) and labeled by the points of the cylinder. These CS were introduced by de Bièvre and González in Ref. [DG 92] and [DG 93], where they were simply sketched. Therefore, we make here a deep study about them. Our aim is to contribute to the development of the quantum theory on periodic phase spaces. Among these phase spaces we pay special attention to the cylinder because of their relation with physical systems with periodic motion and their nontrivial topology, moreover the quantum formalism on the cylinder is far to be completely understood.

On the other hand, it has been proved that families of CS are relevant in the study of many quantum systems [KS 85] [PE 86], however this formalism presents some troubles when one wants to apply it to the cylinder. For instance, the cylinder can be seen as a coadjoint orbit of the Euclidean group of the plane but, in strict sense, the Perelomov method [PE 86] for constructing CS with this group does not work (de Bièvre [DB 89] and Isham and Klauder [IK 91] have showed two different ways to avoid this problem). Nevertheless, the CS here introduced have not been obtained by any of these procedures, but by decomposition of the standard Weyl–Heisenberg CS on $\mathbb{R}$ (i.e., CS belonging to $L^2(\mathbb{R})$); in other words, the CS on the circle are the image of the Weyl–Heisenberg CS by the WBZ transform. The machinery to carry out such a decomposition is the Weil–Brezin–Zak (WBZ) transform [JA 82] [FO 89], which was originally used for the study of periodic potentials [ZA 68] [RS 78]. This WBZ transform relates the quantum formalisms on the plane and on the cylinder (or on the torus as phase space [BB 96]). Roughly speaking, this procedure maps functions of one variable on quasiperiodic functions of two variables by a generalization of the Bloch functions.

As an application, these CS can be used to study a quantum particle on the circle as it has been made recently by Kowalski et al [KR 96]. Although they assume to use CS different from ours and to have obtained a better approach to this problem, it is easy to prove that their CS are a particular case of the CS used here, which shows the wider generality of our approach.

The paper is organized as follows. In Section 2 we review the main properties of the WBZ transform that plays a central role in our work. Section 3 is devoted to the CS on the circle, which are obtained by decomposition of the standard Weyl–Heisenberg CS on $\mathbb{R}$ (i.e., CS belonging to $L^2(\mathbb{R})$); in other words, the CS on the circle are the image of the Weyl–Heisenberg CS by the WBZ transform. These CS on the circle provide a realization of the space $L^2(S^1)$ in terms of entire functions as it is shown in Section 4, in analogy with the Fock–Bargmann representation of $L^2(\mathbb{R})$ provided by the Weyl–Heisenberg CS. A part of the results of Sections 3 and 4 has been published in [DG 92]. Section 5 presents a generalization of the CS on the circle to a $n$–dimensional torus $\mathbb{T}^n$, thus we will obtain a family of CS in $L^2(\mathbb{T}^n)$. The physical properties are studied in Section 6, paying special attention to the expectation values of the position and the (angular) momentum operators, and to the Heisenberg uncertainty relation. The last section is devoted to prove that the CS of Ref. [KR 96] agree with our CS for a particular value of the parameters that characterize the latter ones, and to present some conclusions.

2. The Weil-Brezin-Zak transform

It is a well known fact that $L^2(\mathbb{R})$ is isomorphic to $L^2(S^1 \times S^{1*})$, where $S^{1*}$ is the dual space of $S^1$. This result has been used, for instance, in solid state physics to construct the so-called Bloch functions.
theory to construct the Bloch functions [ZA 68] [RS 78], as well as for quantum description of periodic variables [ZA 69]. In this context, we call Weil-Brezin-Zak (WBZ) transform \( T \) to the unitary map from \( L^2(\mathbb{R}) \) to \( L^2(S^1 \times S^1^*) \) [JA 82] [FO 89]. If we identify \( S^1 \) with the interval \([0, a)\) and \( S^1^* \) with \([0, 2\pi/a)\), then \( T \) is explicitly given by
\[
(T\psi)(q, k) = \sum_{n=-\infty}^{\infty} e^{inak} \psi(q - na),
\]
(2.1)
for \( \psi \in L^2(\mathbb{R}) \), \( q \in S^1 \) and \( k \in S^1^* \). Conversely,
\[
\psi(q - na) = \frac{a}{2\pi} \int_{0}^{2\pi/a} dk \, e^{-inak} (T\psi)(q, k), \quad q \in S^1, \ n \in \mathbb{Z}.
\]
(2.2)
In this way, the functions \( T\psi \) are periodic in \( k \) and quasiperiodic in \( q \),
\[
(T\psi)(q + na, k + m\frac{2\pi}{a}) = e^{inak} (T\psi)(q, k), \quad n, m \in \mathbb{Z}.
\]
(2.3)
Note that if we fix a value of \( k \) (as we are going to do from now on) the operator given by (2.1) is a projection onto \( L^2(S^1) \), which we will denote by \( T^{(k)} \), and we get the so-called constant fiber direct integral decomposition [RS 78],
\[
L^2(\mathbb{R}) \cong \int_{S^1^*} dk \, L^2(S^1).
\]
(2.4)
In this case, we will frequently use the notation \( T^{(k)}\psi = \psi^{(k)} \), and we will say that \( \psi^{(k)} \) is obtained by decomposition of \( \psi \).

3. Coherent states on the circle

In this section we are going to show that a family of CS in \( L^2(S^1) \) can be constructed by decomposition of the standard Weyl–Heisenberg CS in \( L^2(\mathbb{R}) \). The latter are given as an orbit under the Weyl–Heisenberg group [KS 85] [PE 86]:
\[
\eta_{y,p}(x) := \exp \left( \frac{i}{\hbar} (pQ - yP) \right) \eta_0(x) = \exp \left( \frac{i}{\hbar} p(x - \frac{y}{2}) \right) \eta_0(x - y),
\]
(3.1)
where \( x, y, p \in \mathbb{R} \) and \( \eta_0 \in L^2(\mathbb{R}) \) is a fiducial state, which is usually chosen to be a normalized gaussian:
\[
\eta_0(x) = \left( \frac{\omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{\omega}{2\hbar} x^2 \right).
\]
(3.2)
Now, we can use (2.1) to construct the functions \( \eta_{y,p}^{(k)} \in L^2(S^1) \) (or \( |y, p; k\) in Dirac’s notation), and it is natural to ask if, for each value of \( k \), this set of functions will be also a set of CS, labeled by suitable values of \( y \) and \( p \). The answer is positive, according to the generalized definition of CS given in [KS 85]: simply, a family of states depending continuously on a set of labels and fulfilling a resolution of the unity. They are not constructed by Perelomov’s method [PE 86], as an orbit under a Lie group representation. Actually, we have here a non–trivial example of the “reproducing triples” introduced in [AA 91].
(3.1) **Theorem.** For each \( k \in S^1_* \), the family \( \{ \eta_{q,p}^{(k)} \equiv |q,p;k \rangle | (q,p) \in S^1 \times \mathbb{R} \} \), where \( \eta_{q,p} \) is given by (3.1) with \( |\eta_0| = 1 \), is a set of coherent states in \( L^2(S^1) \); i.e., they verify the following resolution of unity:

\[
\frac{1}{2\pi \hbar} \int_0^a dq \int_{-\infty}^{\infty} dp |q,p;k\rangle \langle q,p;k| = I.
\] (3.3)

The proof consists of a simple calculation, using the definitions (2.1) and (3.1) [GO 96]. If we choose \( \eta_0 \) according to (3.2), these CS take the form

\[
\eta_{q,p}^{(k)}(q') = \left( \frac{\omega}{2\hbar} \right)^{1/4} \exp \left( i \frac{p z^*}{\hbar} \right) \exp \left( -\frac{1}{2\hbar} (z^* - \omega q')^2 \right)
\]

\[
\times \theta \left( i \frac{a}{2\hbar} (z^* - \omega q' - ik) ; \rho_1 \right),
\] (3.4)

where \( z^* = \omega q + ip \), \( \rho_1 = \exp \left( -\frac{a^2 \omega}{2\hbar} \right) \), and \( \theta(z;\rho) = \sum_{n=-\infty}^{\infty} \rho^n e^{inz} \), \( |\rho| < 1 \), is the Theta function (sometimes denoted by \( \theta_3 \)) [WW 27] [AS 72] [ER 81] [MU 83].

As a corollary to the preceding theorem, we present a typical property of every set of CS [KS 85].

(3.2) **Corollary.** The mapping \( W^{(k)} : L^2(S^1) \rightarrow L^2(S^1 \times \mathbb{R}) \) given by

\[
(W^{(k)} \varphi)(q,p) = \langle q,p;k | \varphi \rangle
\] (3.5)

is an isometry, and \( W^{(k)}(L^2(S^1)) \) is a reproducing kernel space, with kernel

\[
\frac{1}{2\pi \hbar} \langle q',p';k | q,p;k \rangle.
\]

To compute this kernel, let us consider the orthonormal basis in \( L^2(S^1) \):

\[
\left\{|n;k\rangle \equiv \frac{1}{\sqrt{a}} \exp \left( \frac{2\pi}{a} n + k \right) q, \quad n \in \mathbb{Z}, \quad k \in S^1_* \text{ fixed} \right\}.
\] (3.6)

Then we can write

\[
|q,p;k\rangle = \sum_{n=-\infty}^{\infty} c_n^{(q,p;k)} |n;k\rangle,
\] (3.7)

where the coefficients are

\[
c_n^{(q,p;k)} = \frac{\sqrt{2\pi \hbar}}{a} e^{i[p/(2\hbar) - (2\pi n/a + k)]q} \tilde{\eta}_0 \left( \frac{2\pi}{a} n + k \right) \hbar - p,
\] (3.8)

\( \tilde{\eta}_0 \) being the Fourier transform of \( \eta_0 \). Now, using (3.2) and (3.8), we easily obtain

\[
\langle q',p';k | q,p;k \rangle = \sum_{n=-\infty}^{\infty} \langle q',p';k | n;k \rangle \langle n;k | q,p;k \rangle = \sum_{n=-\infty}^{\infty} c_n^{(q',p';k)*} c_n^{(q,p;k)}
\]

\[
= \frac{2}{a} \sqrt{\frac{\pi \hbar}{\omega}} e^{ik(q' - q)} e^{i(qp - q'p')/2\hbar} e^{-[(\hbar k - p)^2 + (\hbar k - p')^2]/2\omega h}
\]

\[
\times \theta \left( \pi \left[ |q' - q| + i(2\hbar k - p - p') \right] ; \rho_2 \right),
\] (3.9)
where $\rho_2 = \exp \left( -\frac{4\pi^2 \hbar}{\omega a^2} \right)$.

By the way, we see that these CS are not normalized. It follows immediately from (3.9) that
\[ \langle q, p; k | q, p; k \rangle = \frac{2}{a} \sqrt{\frac{\pi \hbar}{\omega}} e^{-(hk-p)/\hbar} \theta \left( \frac{2\pi}{\omega a} (hk-p) ; \rho_2 \right). \] (3.10)

Taking into account the identity
\[ \theta(z; \rho_2) = \frac{a}{2} \sqrt{\frac{\omega}{\pi \hbar}} e^{-\omega a^2 z^2/(4\pi^2 \hbar)} \theta \left( -i \frac{\omega a^2}{4\pi^2 \hbar} z ; \rho_1^{1/2} \right), \] (3.11)
which is easily deduced from the so-called functional equation of $\theta$ [ER 81] [MU 83], we obtain as well the expression
\[ \langle q, p; k | q, p; k \rangle = \theta \left( \frac{a}{2\hbar} (hk-p) ; \rho_1^{1/2} \right). \] (3.12)

4. A REALIZATION OF $L^2(S^1)$ BY ANALYTIC FUNCTIONS

Let us consider again the isometry $W^{(k)}$ given by (3.5). If we define the new mapping
\[ (B^{(k)} \varphi)(z) = \exp \left( \frac{i}{2\omega \hbar} pz \right) (W^{(k)} \varphi)(q, p) \] (4.1a)
\[ = \exp \left( \frac{i}{2\omega \hbar} pz \right) \langle q, p; k | \varphi \rangle, \quad \varphi \in L^2(S^1), \] (4.1b)
with $z = \omega q - ip$, then $B^{(k)} \varphi$ is an analytic function on $S^1 + i\mathbb{R}$ (because $\theta(z; \rho)$ is an entire function of $z$). This suggests us to search for a representation of $L^2(S^1)$ by entire functions, similar to the standard Fock–Bargmann representation of $L^2(\mathbb{R})$ [BA 61] [PE 86] [FO 89]. In this context, it is quite natural to define a new set of CS, labeled by $z^* = \omega q + ip$, by
\[ |z^*; k \rangle = \exp \left( -\frac{i}{2\omega \hbar} pz^* \right) |q, p; k \rangle \] (4.2a)
\[ \eta_{z^*}^{(k)}(q') = \left( \frac{\omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{(z^* - \omega q')^2}{2\omega \hbar} \right) \theta \left( \frac{i}{2\hbar} \frac{a}{\omega} (z^* - \omega q' - ik\hbar) ; \rho_1 \right) \] (4.2b)
such that we simply have
\[ (B^{(k)} \varphi)(z) = \langle z; k | \varphi \rangle, \quad \varphi \in L^2(S^1). \] (4.3)

Note that we write $|z^*; k \rangle^\dagger = \langle z; k \rangle$. Since $|z^* + \omega a; k \rangle = e^{-iak} |z^*; k \rangle$, as it is easy to check, we can extend $(B^{(k)} \varphi)(z)$ to the whole of $\mathbb{C}$, getting so an entire function of $z$, $\forall \varphi \in L^2(S^1)$. Moreover, the CS $|z^*; k \rangle$ fulfill the resolution of the unity:
\[ \frac{1}{2\pi} \int_0^a dq \int_0^\infty dp \ e^{-p^2/(\omega \hbar)} |z^*; k \rangle \langle z; k \rangle = I. \] (4.4)
Hence \( B^{(k)} \) is an isometry from \( L^2(S^1) \) into the space

\[
F = \left\{ \psi(z) \text{ entire, } \psi(z + \omega a) = e^{ik\omega} \psi(z) \text{ and } \right. \\
\left\| \psi \right\|^2_F = \frac{1}{2\pi \hbar} \int_0^a dq \int_{-\infty}^{\infty} dp \, e^{-p^2/(\omega \hbar)} |\psi(z)|^2 < \infty, \ z = \omega q - ip \right\}. \tag{4.5}
\]

We see that the space \( F \) is similar to the usual Fock space. Since \( B^{(k)} \) also maps \( L^2(S^1) \) onto \( F \), as we will see, we have a complete analogy with the standard case. Obviously, we can define the following orthonormal set in \( F \):

\[
\{ \psi_n(z) := (B^{(k)}|n; k))(z) = (z|n; k) | n \in \mathbb{Z} \}, \tag{4.6}
\]

and it is not hard to compute the functions

\[
\psi_n(z) = \left( \frac{4\pi \hbar}{a^2 \omega} \right)^{1/4} \exp\left(-\frac{\hbar}{2\omega}(\frac{2\pi}{a} n + k)^2\right) \exp\left(i \frac{2\pi}{a} n + k \right) \psi(z), \quad (4.7)
\]

To prove that \( B^{(k)} \) is onto is equivalent to prove that these functions form a basis for \( F \). But, after (4.7), this amounts to the existence of a Fourier series for any \( \psi \in F \), as it is really the case, i.e.,

\[
\psi(z) = \sum_{n=-\infty}^{\infty} a_n e^{i(2\pi n/a + k)z/\omega}, \quad \forall \psi \in F, \tag{4.8}
\]

because of the quasiperiodicity of the functions in \( F \) (we have introduced, for convenience, a factor \( e^{ikz/\omega} \) in the usual Fourier series). Using (4.7) and the orthonormality of the set \{ \psi_n \}, the expression (4.8) becomes

\[
\psi(z) = \sum_{n=-\infty}^{\infty} (\psi_n|\psi)_F \psi_n(z), \quad \forall \psi \in F, \tag{4.9}
\]

where \((\cdot|\cdot)_F\) denotes the inner product of \( F \) and

\[
(\psi_n|\psi)_F = a_n \left( \frac{a^2 \omega}{4\pi \hbar} \right)^{1/4} e^{h(2\pi n/a + k)^2/(2\omega)}. \tag{4.10}
\]

Clearly, there is a one-to-one correspondence between the coefficients \((\psi_n|\psi)_F\) and \(a_n\), so the set \{ \psi_n \} is a basis and \( B^{(k)} \) is onto.

We can write now some expressions for the inverse \( B^{-1} \) of \( B^{(k)} \):

\[
|B^{-1}\psi\rangle = \sum_{n=-\infty}^{\infty} (\psi_n|\psi)_F |n; k\rangle \tag{4.11a}
\]

\[
= \frac{1}{2\pi \hbar} \int_0^a dq \int_{-\infty}^{\infty} dp e^{-p^2/(\omega \hbar)} \psi(z) |z^*; k\rangle, \quad z = \omega q - ip. \tag{4.11b}
\]
5. **Coherent states on the torus**

All the results of the precedent sections can be easily generalized to a higher number of dimensions. To this purpose, let us take a unitary basis \( \{ \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \} \) in \( \mathbb{R}^n \) as well as a set of real numbers \( \{ a_1, a_2, \ldots, a_n \} \) and let us consider the associated lattice \( \mathcal{L} \) [RS 78], that is,

\[
\mathcal{L} = \{ \vec{a} \in \mathbb{R}^n \mid \vec{a} = \sum_{i=1}^{n} m_i a_i \vec{e}_i, \; m_i \in \mathbb{Z} \}. \tag{5.1}
\]

In the same way, we define the dual basis \( \{ \vec{e}_1, \ldots, \vec{e}_n \} \) by \( \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \) and the dual lattice by

\[
\mathcal{L}' = \{ \vec{b} \in \mathbb{R}^n \mid \vec{b} = \sum_{i=1}^{n} m_i \frac{2\pi}{a_i} \vec{e}_i, \; m_i \in \mathbb{Z} \}. \tag{5.2}
\]

The corresponding basic cells \( \mathbb{T}^n \) and \( (\mathbb{T}^n)' \) are \( n \)-dimensional torus,

\[
\mathbb{T}^n = \{ \vec{q} \in \mathbb{R}^n \mid \vec{q} = \sum_{i=1}^{n} q_i \vec{e}_i, \; 0 \leq q_i < a_i \}, \tag{5.3}
\]

\[
(\mathbb{T}^n)' = \{ \vec{k} \in \mathbb{R}^n \mid \vec{k} = \sum_{i=1}^{n} k_i \vec{e}_i, \; 0 \leq k_i < \frac{2\pi}{a_i} \}. \tag{5.4}
\]

We shall define the \( n \)-dimensional Weil-Brezin-Zak transform \( T \) as a unitary map from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{T}^n \times (\mathbb{T}^n)') \) [JA 82] [FO 89], given by

\[
(T\psi)(\vec{q}, \vec{k}) = \sum_{\vec{a} \in \mathcal{L}} e^{i\vec{a} \cdot \vec{k}} \psi(\vec{q} - \vec{a}), \quad \vec{z} \in \mathbb{T}^n, \; \vec{w} \in (\mathbb{T}^n)', \tag{5.5}
\]

with \( \psi \in L^2(\mathbb{R}^n) \). The functions \( T\psi \) verify

\[
(T\psi)(\vec{q} + \vec{a}, \vec{k} + \vec{b}) = e^{i\vec{a} \cdot \vec{k}} (T\psi)(\vec{q}, \vec{k}), \quad \vec{a} \in \mathcal{L}, \; \vec{b} \in \mathcal{L}'. \tag{5.6}
\]

From now on, we are going to fix a value of \( \vec{k} \), so that the expression (5.5) defines a projection \( T(\vec{k}) \) onto \( L^2(\mathbb{T}^n) \). We will use the notation \( T(\vec{k})\psi = \psi(\vec{k}) \).

Coherent states on the torus are obtained as the image under \( T(\vec{k}) \) of the \( n \)-dimensional Weyl–Heisenberg CS \( \eta_{\vec{q}, \vec{p}} \in L^2(\mathbb{R}^n) \), which we write as

\[
\eta_{\vec{q}, \vec{p}}(\vec{x}) = \exp\left( \frac{i}{\hbar} \vec{p} \cdot (\vec{x} - \frac{\vec{q}}{2}) \right) \eta_0(\vec{x} - \vec{q}), \tag{5.7}
\]

where \( \vec{x}, \vec{q}, \vec{p} \in \mathbb{R}^n \), and the fiducial state \( \eta_0 \in L^2(\mathbb{R}^n) \) is chosen to be a normalized gaussian:

\[
\eta_0(\vec{x}) = \left( \frac{\omega}{\pi \hbar} \right)^{n/4} \exp(-\frac{\omega}{2\hbar} \vec{x}^2). \tag{5.8}
\]

In this case, the functions \( \eta_{\vec{q}, \vec{p}}^{(\vec{k})} \) take the form [GO 96]

\[
\eta_{\vec{q}, \vec{p}}^{(\vec{k})}(\vec{q}') = \left( \frac{\omega}{\pi \hbar} \right)^{n/4} e^{-i\vec{p} \cdot \vec{q} / (2\hbar)} e^{i\vec{p} \cdot \vec{q}' / \hbar} e^{-\omega(\vec{q} - \vec{q}')^2 / (2\hbar)}
\]

\[
\times \Theta\left( \frac{1}{\hbar} \Delta[h\vec{k} - \vec{p} + i\omega(\vec{q} - \vec{q}')] \right), \tag{5.9}
\]

where \( \Delta \) is the characteristic function of the interval [GO 96].
where $G$ is the $n \times n$ symmetric matrix of the lattice, whose elements are $g_{ij} = \vec{e}_i \cdot \vec{e}_j$; $\Delta$ is a $n \times n$ diagonal matrix with elements $a_i$; $\Omega$ is another $n \times n$ matrix given by

$$\Omega = i \frac{\omega}{2\pi \hbar} \Delta G \Delta; \quad (5.10)$$

and $\Theta$ is the $n$–dimensional Theta function [MU 83]:

$$\Theta(\vec{z} | \Omega) = \sum_{\vec{m} \in \mathbb{Z}^n} \exp(i \pi \vec{m} \cdot \Omega \vec{m}) \exp(2i \pi \vec{m} \cdot \vec{z}). \quad (5.11)$$

We have thus the following $n$–dimensional version of Theorem (3.1):

(5.1) **Theorem.** For each $\vec{k} \in (\mathbb{T}^n)'$, the family of functions

$$\left\{ \eta_{\vec{q},\vec{p}}^{(\vec{k})} \equiv |\vec{q},\vec{p};\vec{k}\rangle \mid (\vec{q},\vec{p}) \in \mathbb{T}^n \times \mathbb{R}^n \right\}, \quad (5.12)$$

given by (5.9), is a set of coherent states in $L^2(\mathbb{T}^n)$; i.e., they verify the resolution of unity:

$$\frac{1}{(2\pi \hbar)^n} \int_{\mathbb{T}^n} d\vec{q} \int_{\mathbb{R}^n} d\vec{p} |\vec{q},\vec{p};\vec{k}\rangle \langle \vec{q},\vec{p};\vec{k}| = I. \quad (5.13)$$

Most generalizations of the one–dimensional results are straightforward [GO 96]. We simply are going to write here the expression for the product $\langle \vec{q}',\vec{p}';\vec{k}|\vec{q},\vec{p};\vec{k}\rangle$.

After a rather lengthy calculation, we obtain

$$\langle \vec{q}',\vec{p}';\vec{k}|\vec{q},\vec{p};\vec{k}\rangle = 2^n \sqrt{gA} \left( \frac{\pi \hbar}{4} \right)^{n/2} e^{i\pi \vec{q} \cdot \vec{p}' - (h\vec{k} - \vec{p}')^2 / (2\hbar)} e^{-i\pi \vec{q}' \cdot \vec{p} / (4\hbar)} \Theta \left( \pi \Delta^{-1} \left[ \vec{q}' - \vec{q} + \frac{i}{\omega} G^{-1}(2h\vec{k} - \vec{p} - \vec{p}') \right] \right) |\Omega'\rangle, \quad (5.14)$$

where $g = \det G$, $A = a_1 a_2 \cdots a_n$ and $\Omega' = -2\Omega^{-1}$. Therefore, we get also

$$\langle \vec{q},\vec{p};\vec{k}|\vec{q},\vec{p};\vec{k}\rangle = 2^n \sqrt{gA} \left( \frac{\pi \hbar}{4} \right)^{n/2} e^{-(h\vec{k} - \vec{p})^2 / (2\hbar)} \Theta \left( \frac{2\pi}{\omega} \Delta^{-1} G^{-1}(h\vec{k} - \vec{p}) \right) |\Omega'\rangle. \quad (5.15)$$

Finally, it can be shown [GO 96] that when the lattice is orthogonal, the CS $\eta_{\vec{q},\vec{p}}^{(\vec{k})}$ factorize out like a product of one–dimensional CS given by (3.4), i.e.,

$$\eta_{\vec{q},\vec{p}}^{(\vec{k})}(\vec{q}') = \prod_{i=1}^{n} \eta_{q_i,p_i}^{(k_i)}(q_i'). \quad (5.16)$$

6. Physical properties of the CS on the circle

This section is devoted to discuss the physical properties of the CS on the circle introduced in Section 3 (a complete study has been realized in [GO 96]). As these states have been constructed by decomposition of the standard Weyl–Heisenberg CS in $L^2(\mathbb{R})$, it seems to us that comparison between both cases could be illustrative.
Moreover, it is known that the Weyl–Heisenberg CS have very nice quasiclassical properties, for instance, to minimalize the Heisenberg uncertainty relation, and it would be of great interest to reproduce such behaviour on the circle. As a matter of fact, we are going to see that the physical properties of the CS on the circle depend mainly on some dimensionless parameter, related to the spread of the initial standard CS. If this spread was smaller than the length $a$ of the circle, we get CS on the circle very similar to the standard CS. But if such spread was comparable or bigger than $a$, the CS on the circle are rather like plane waves.

We will discuss also the relation between the CS parameters $q, p$ and the expectation values in these states of the position and momentum operators on $L^2(S^1)$. Whereas for standard CS in $L^2(\mathbb{R})$ both things are the same, this is not the case on the circle. So, firstly we will recall the correct definitions for the position and momentum operators on $L^2(S^1)$ (which show some significant differences from their analogous on the real line). Then, we will compute the expectation values of these operators and, finally, we will devote some attention to the Heisenberg uncertainty relation on the circle, but in a different and more suitable form of the usual one on the real line.

In order to provide an easier understanding of the somewhat complicated expressions, we are going to illustrate our results with several figures. In any case, it has been possible to realize a complete analytic study [GO 96].

6.1 Quantum mechanics on the circle.

The topology of the circle has important consequences for the quantum formalism on this configuration space. Indeed, experience shows that a direct translation of the formalism on the real line leads to serious inconsistencies [CN 68] [ZA 69] [LE 76]. For instance, it is known that the (angular) momentum operator on $L^2(S^1)$ has discrete spectrum. Moreover, functions in its domain have to verify the constraint $\varphi(a) = e^{ia{k}}\varphi(0)$, where $a$ is the length of the circle and $k \in [0, 2\pi/a]$ is a parameter as in Section 2 [RS 75]. Thus, in fact, there is not one but a family of momentum operators on $L^2(S^1)$, labeled by $k$ and which we denote by $P^{(k)}$. As a consequence, a canonical commutation relation as in $L^2(\mathbb{R})$

\[
[Q, P^{(k)}] = i\hbar, \quad (6.1)
\]

with position operator $Q$ defined as usual, is inconsistent in $L^2(S^1)$. Heisenberg’s uncertainty relation is even more troublesome, because of the compact spectrum of $Q$ on $L^2(S^1)$. In effect, this relation allows the position dispersion to be bigger than $a$, which has no physical meaning.

All these problems can be solved choosing the unitary operator $E = \exp(i2\pi Q/a)$ as a better representation for the position on the circle [LE 76]. It has precisely the circle as spectrum and its commutator with the momentum operator is

\[
[P^{(k)}, E] = \frac{2\pi\hbar}{a}E, \quad (6.2)
\]

which poses no domain problems. From this fundamental relation (6.2) we can also deduce an uncertainty relation more suitable for the circle [LE 76]. Since $E$ is unitary but not selfadjoint, the dispersion $\Delta E$ should be defined in the form

\[
(\Delta E)^2 := \langle E^\dagger E \rangle - |\langle E \rangle|^2 = 1 - |\langle E \rangle|^2, \quad (6.3)
\]
so that relation (6.2) yields, by the usual method, the following Heisenberg uncertainty relation:

\[ (\Delta P^{(k)})^2 \frac{(\Delta E)^2}{1 - (\Delta E)^2} \geq \left( \frac{\pi \hbar}{a} \right)^2. \]  

(6.4)

Note that now, when \( \Delta P^{(k)} = 0 \) we must have \( \Delta E = 1 \), which is a more appropriate result because of the compactness of the position variable on the circle. Moreover, relation (6.4) reduces to the usual Heisenberg uncertainty relation when \( \Delta E \ll 1 \) [LE 76]. Thus, we will call \( E \) the “angle” operator and from now on we will use it as the quantum representation for the position on the circle.

6.2 Physical properties of the CS on the circle.

6.2.1 Probability density. Let us begin the study of the basic physical properties of the CS \( |q,p;k\rangle \) by computing its probability density \( P_{q,p;k}(q') \). The wave function \( \eta_{q,p}^{(k)}(q') \) of these states is given by expression (3.4). As they are not normalized, the probability density will be

\[ P_{q,p;k}(q') = \frac{|\eta_{q,p}^{(k)}(q')|^2}{\langle q,p;k|q,p;k \rangle}. \]  

(6.5)

that, making use of the expression (3.12), yields

\[ P_{q,p;k}(q') = \left( \frac{\omega}{\pi \hbar} \right)^{1/2} e^{-\omega(q-q')^2/\hbar} \frac{|\theta(a[(kh-p) + i\omega(q-q')]/(2\hbar); \rho_1)|^2}{\theta(a(kh-p)/(2\hbar); \rho_1/2)}. \]  

(6.6)

In order to clarify the notation, we are going to introduce two new variables,

\[ u := \frac{1}{a}(q' - q), \quad v := \frac{a}{2\pi \hbar}(p - kh), \]  

(6.7)

as well as the dimensionless parameter

\[ \alpha := \frac{a^2}{2\hbar} \omega. \]  

(6.8)

In this way, the probability density, from now on denoted by \( P_{\alpha}(u;v) \), looks like

\[ P_{\alpha}(u;v) = \frac{1}{a} \sqrt{\frac{2\alpha}{\pi}} e^{-2\alpha u^2} \frac{|\theta(\pi v + i\alpha u; e^{-\alpha})|^2}{\theta(\pi v; e^{-\alpha/2})}. \]  

(6.9)

This is a periodic function of \( u \) and \( v \), in both cases with period 1. This corresponds to a period \( a \) for \( q' - q \) and a period \( 2\pi \hbar/a \) for \( p \). To give a general idea of its main properties, we show in Figure 1 some significant cases. We observe that for high values of \( \alpha \) (approximately \( \alpha > 15 \)) the probability density is, with a good accuracy, a Gaussian regardless of the value of \( v \). That is, we have the same result as for the standard Weyl–Heisenberg CS. Note that, for these values of \( \alpha \), the width of the Gaussian is always smaller than \( a \). On the other hand, for small values of \( \alpha \) the probability density is no longer a Gaussian and its shape depends crucially on the value of \( v \). Only when \( v = 1/2 \) (i.e., \( p = ((2n + 1)\pi/a + k)\hbar \), with \( n \in \mathbb{Z} \)), it looks always as a “wave packet” for all the values of \( \alpha \) (right side of the Figure 1). In
Figure 1. The functions $a \mathcal{P}_\alpha(u - \frac{1}{2}, 0)$ (left) and $a \mathcal{P}_\alpha(u - \frac{1}{2}, \frac{1}{2})$ (right), for several values of $\alpha$. all the other cases it tends to be a plane wave when $\alpha \to 0$ (in the left side of the
6.2.2 **Expectation value of the angle operator.** To compute the expectation value of $E$ in the CS $|q, p; k\rangle$, we will make use of the following relation

$$E|q, p; k\rangle = e^{i\pi q/a}|q, p + \frac{2\pi}{a}h; k\rangle,$$

which is easily deduced from the obvious action of $E$ on the orthonormal basis $|n; k\rangle$ in $L^2(S^1)$ (see expression (3.6)),

$$E|n; k\rangle = |n + 1; k\rangle, \quad \forall n \in \mathbb{Z},$$

as well as from the expression (3.8) for the coefficients of the CS $|q, p; k\rangle$ in this basis. We will denote the expectation value of $E$ by $\langle E \rangle(u, v)$, with $v$ as in (6.7) but

$$u := \frac{q}{a},$$

from now on. We also continue using the parameter $\alpha$ defined in (6.8). Taking together the equations (6.10), (3.9), (3.11) and (3.12) we finally arrive at

$$\langle E \rangle(u, v) = \frac{\langle q, p; k|E|q, p; k\rangle}{\langle q, p; k|q, p; k\rangle} = e^{i2\pi u}e^{-\pi^2/(2\alpha)} \frac{\theta(\pi(v - \frac{1}{2}); e^{-\alpha/2})}{\theta(\pi v; e^{-\alpha/2})}.$$  

Of course, this is a periodic function of $u$ but also an even periodic function of $v$ with period 1. As all the factors excepting $e^{i2\pi u}$ are real positive [WW 27] [ER 81] [MU 83], we can write

$$\langle E \rangle(u, v) = e^{i2\pi u} |\langle E \rangle(v)|.$$  

We show the function $|\langle E \rangle(v)|$ in Figure 2, for some values of $\alpha$. Note that, in general, it is not possible to interpret the expectation value of $E$ as a measure of the average position of the CS on the circle because of the dependence on $v$. However, observe in Figure 2 that for high values of $\alpha$ the function $|\langle E \rangle(v)|$ is almost constant, so in these cases we would get the usual interpretation of the CS parameter $q$ as the average position of the quantum state. On the other hand, we have seen in Figure 1 that for small values of $\alpha$ most of the CS are nearly plane waves, hence it is not so important that the average position in these states cannot be well defined.

6.2.3 **Expectation value of the momentum operator.** We begin the calculation observing that the vectors of the basis $|n; k\rangle$ in $L^2(S^1)$ (see expression (3.6)) are eigenvectors of the momentum operator $P^{(k)}$,

$$P^{(k)}|n; k\rangle = \hbar \frac{2\pi}{a}n + k |n; k\rangle.$$  

Thus, we can write

$$\langle q, p; k|P^{(k)}|q, p; k\rangle = \hbar \sum_{n} \left|\frac{2\pi}{a}n + k\right|e_n^{(q,p; k)}|^2,$$
where the coefficients $c_n^{(q,p;k)}$ are given by the expressions (3.8) and (3.2). Using also the formulae (3.11) and (3.12) we finally find

$$\langle P^{(k)}(p) \rangle = \langle q, p; k | P^{(k)} | q, p; k \rangle = p + \frac{\hbar \alpha}{2a} \theta'(\pi v; e^{-\alpha/2}) \theta(\pi v; e^{-\alpha/2}),$$

where, for the sake of clarity, we use the two variables $p, v$ at the same time, and

$$\theta'(z; \rho) = \frac{d}{dz} \theta(z; \rho) = 2i \sum_{n=-\infty}^{\infty} n \rho^{n^2} e^{2in\rho}, \quad |\rho| < 1.$$

It is interesting to note that when $v = n/2$, with $n \in \mathbb{Z}$, i.e. $p = (n\pi/a + k)\hbar$, the expression (6.17) reduces to $\langle P^{(k)}(p) \rangle = p$ as in the standard CS case. For another
values of $v$, the difference between $\langle P^{(k)} \rangle$ and $p$ depends on the parameter $\alpha$. To show that, let us first rewrite the equation (6.17) using only the variable $v$:

$$\langle P^{(k)} \rangle (v) = \frac{2\pi \hbar}{a} \left( v + \frac{\alpha}{4\pi} \frac{\theta' (\pi v; e^{-\alpha/2})}{\theta (\pi v; e^{-\alpha/2})} \right) + k\hbar. \quad (6.19)$$

Now, we represent the function $(a/2\pi \hbar) (\langle P^{(k)} \rangle (v) - k\hbar)$ in Figure 3, for some values of $\alpha$ (remember that $2\pi \hbar / a$ is the “natural unit” for $p$). We see that for high values of $\alpha$, the CS parameter $p$ is a good approximation for the expectation value of the momentum operator. But for small values of $\alpha$, this expectation value tends to take some discrete values for almost all the values of $v$ [GO 96]. These are the “plane wave” states of Figure 1.

6.2.4 Heisenberg uncertainty relation. We conclude the study of the basic physical properties of the CS $|q, p; k\rangle$ with some comments about the Heisenberg uncertainty relation for these states. In the following we will try to verify if some of the CS $|q, p; k\rangle$ minimize relation (6.4), which has to be used on the circle, as we remarked above.

Let us denote by $\Delta_{(q,p)}^{(k)} A$ the dispersion of an operator $A$ in the CS $|q, p; k\rangle$. We begin computing this dispersion for the angle operator $E$. According to expression (6.3) we get

$$(\Delta_{(q,p)}^{(k)} E)^2 = 1 - |\langle E \rangle (u, v)|^2 = 1 - |\langle E \rangle (v)|^2, \quad (6.20)$$

where $|\langle E \rangle (v)|$ can be obtained from expression (6.13).

On the other hand, the dispersion $\Delta_{(q,p)}^{(k)} P^{(k)}$ of the momentum operator requires a few more calculations. Firstly, we have to compute the expectation value $\langle (P^{(k)})^2 \rangle$, which after (6.15) can be written as

$$\langle (P^{(k)})^2 \rangle (p) = \frac{\hbar^2}{\langle q, p; k | q, p; k \rangle} \sum_{n=-\infty}^{\infty} \left( \frac{2\pi}{a} m + k \right)^2 |c_{n}^{(q,p;k)}|^2. \quad (6.21)$$

Hence, making use again of the formulae (3.8), (3.2), (3.11) and (3.12) we get, after a rather lengthy but straightforward calculation,

$$\langle (P^{(k)})^2 \rangle (p) = \left( \frac{\hbar \alpha}{2a} \right)^2 \frac{\theta'' (\pi v; e^{-\alpha/2})}{\theta (\pi v; e^{-\alpha/2})} + p \left( \frac{\hbar \alpha}{a} \frac{\theta' (\pi v; e^{-\alpha/2})}{\theta (\pi v; e^{-\alpha/2})} + p \right) + \frac{\hbar^2 \alpha}{a^2}, \quad (6.22)$$

with $\theta'' (z; \rho) = d^2 \theta (z; \rho) / dz^2$. Finally, equations (6.22) and (6.17) taken together yield

$$\left( \Delta_{(q,p)}^{(k)} P^{(k)} \right)^2 = \left( \frac{\hbar}{a} \right)^2 \left[ \frac{\alpha^2}{4} \left( \frac{\theta'' (\pi v; e^{-\alpha/2})}{\theta (\pi v; e^{-\alpha/2})} - \frac{\theta' (\pi v; e^{-\alpha/2})^2}{\theta (\pi v; e^{-\alpha/2})^2} \right) + \alpha \right]. \quad (6.23)$$

We are now able to discuss the uncertainty relation (6.4) for the CS $|q, p; k\rangle$. Firstly, we define the uncertainty function

$$\Delta (v) := \frac{a}{2\pi} \frac{\Delta_{(q,p)}^{(k)} E}{\sqrt{1 - (\Delta_{(q,p)}^{(k)} E)^2}} \Delta_{(q,p)}^{(k)} P^{(k)}$$

$$= \frac{a}{2} \left( \frac{1}{|\langle E \rangle (v)|^2} - 1 \right)^{1/2} \Delta_{(q,p)}^{(k)} P^{(k)}. \quad (6.24)$$
Figure 3. The function $\frac{a}{2\pi\hbar} (\langle P^{(k)}(v) \rangle - k\hbar)$, for several values of $\alpha$. 
Figure 4. The function $\frac{2}{\hbar} \Delta(v)$ for several values of $\alpha$.

In this way, relation (6.4) reduces to

$$\Delta(v) \geq \frac{\hbar}{2},$$

which looks more similar to standard Heisenberg uncertainty relation, making thus the present discussion more intuitive. In view of expressions (6.13) and (6.23) we arrive at the following formula:

$$\Delta(v)^2 = \left(\frac{\hbar}{2\pi}\right)^2 \alpha \left[ e^{\pi^2/\alpha} \frac{\theta(\pi v; e^{-\alpha/2})^2}{\theta(\pi(v - \frac{1}{2}); e^{-\alpha/2})^2} - 1 \right]$$

$$\times \left[ \alpha \left( \frac{\theta''(\pi v; e^{-\alpha/2})}{\theta'(\pi v; e^{-\alpha/2})} - \frac{\theta'(\pi v; e^{-\alpha/2})^2}{\theta''(\pi v; e^{-\alpha/2})} \right) + 1 \right].$$

(6.26)
We have represented the function $(2/h) \Delta(v)$ in Figure 4, for some values of $\alpha$. Observe its somewhat curious appearance. We remark that variable $v$ is related to the CS parameter $p$, and that parameter $\alpha$ measures if the CS $|q,p;k\rangle$ is or not similar to a standard CS. In Figure 4, the value 1 on the vertical scale would correspond to a minimum uncertainty state, and in fact we see that for high values of $\alpha$ the function $\Delta(v)$ tends to this minimum, regardless of the value of $v$ [GO 96]. Nevertheless, none of the CS $|q,p;k\rangle$ is a real minimum uncertainty state, although we can obtain states so close to this limit as we wish, taking $\alpha$ sufficiently high.

On the contrary, when the value of $\alpha$ is small we can see that the behaviour of the uncertainty relation for the CS $|q,p;k\rangle$ depends on the particular value of $p$, that is, $v$. As $\Delta(v)$ is an even periodic function of $v$, we just need to consider the values $0 \leq v \leq \frac{1}{2}$. Thus, it can be proved [GO 96] that

$$\lim_{\alpha \to 0} \Delta(v) = \begin{cases} \frac{\sqrt{2}}{2} h, & \text{if } v = 0, \text{ i.e., } p = (2n \frac{\pi}{a} + k) h, \quad n \in \mathbb{Z}; \\ \frac{\sqrt{3}}{2} h, & \text{if } v = \frac{1}{2}, \text{ i.e., } p = ((2n + 1) \frac{\pi}{a} + k) h, \quad n \in \mathbb{Z}; \\ h, & \text{in any other case.} \end{cases}$$

(6.27)

In other words, the uncertainty function $\Delta$ is upperly bounded, at worst, by $h!$ Hence, we conclude that for all the family of CS $|q,p;k\rangle$ we have

$$h > \Delta(v) > \frac{h}{2},$$

(6.28)

that, although strictly speaking does not correspond to minimum uncertainty states, as a matter of fact shows a quite good behaviour of the CS $|q,p;k\rangle$ in this subject. The best behaviour is obtained for those states associated to the value $v = 0$.

7. Conclusions

As we mention in the Introduction, a family of CS on the circle has been introduced in Ref. [KR 96]. These new CS are a particular case of the CS studied here. The authors of [KR 96] have not realized this fact and, moreover, they write in the Introduction: “... The coherent states thus obtained are different from those defined in this paper (Ref. [DG 93]). Nevertheless, it seems to us that the approach presented herein is a better one”. These CS are defined as

$$|\xi\rangle = \sum_j \xi^{-j} e^{-j^2/2} |j\rangle,$$

(7.1)

where $\xi = e^{-l+i\phi}$, $l \in \mathbb{R}$, $\phi \in S^1$ and $|j\rangle$ are the eigenvectors of the angular momentum operator. Two cases are considered in [KR 96]: boson case when $j$ takes integer values, and fermion case when $j$ takes half-integer values.

In the following, we are going to prove that the CS (7.1) are particular cases of our CS $|z^*;k\rangle$. We have (see Section 4)

$$|z^*;k\rangle = \sum_{n=0}^{\infty} \psi_n(z)^* |n; k\rangle,$$

(7.2)
where \( z = \omega q - ip \), and after (4.7)

\[
\psi_n(z)^* = \left( \frac{4\pi \hbar}{a^2 \omega} \right)^{1/4} \exp\left( -\frac{\hbar}{2\omega} \left( \frac{2\pi}{a} n + k \right)^2 \right) \exp\left( -\frac{i}{\omega} \left( \frac{2\pi}{a} n + k \right) z^* \right).
\] (7.3)

By analogy with Ref. [KR 96], we set from now on \( \hbar = 1 \) and \( a = 2\pi \), so that \( k \in [0,1) \). If we also put \( \xi = \exp(i z^* / \omega) = \exp(-p/\omega + iq) \), then (7.2) finally becomes

\[
|z^*; k\rangle = \left( \frac{1}{\pi \omega} \right)^{1/4} \sum_{n=-\infty}^{\infty} e^{-\frac{(n+k)^2}{2\omega}} \xi^{-(n+k)} |n; k\rangle.
\] (7.4)

Now, simply comparing expressions (7.1) and (7.4) we see that both coincide (up to a constant factor) if we set \( \omega = 1 \) and \( k = 0 \) for the boson case, or \( k = \pi/a = 1/2 \) for the fermion case. Indeed, for \( k = 0 \) we get

\[
|z^*; 0\rangle = \left( \frac{1}{\pi} \right)^{1/4} \sum_{n=-\infty}^{\infty} e^{-n^2/2} \xi^{-n} |n; 0\rangle,
\] (7.5)

which obviously coincides with (7.1) when \( j \) takes integer values, because expression (6.15) shows that \( |n; 0\rangle \) are the boson eigenvectors of the angular momentum operator. In the same way, for \( k = 1/2 \) we get

\[
|z^*; \frac{1}{2}\rangle = \left( \frac{1}{\pi} \right)^{1/4} \sum_{n=-\infty}^{\infty} e^{-\frac{(n+1/2)^2}{2}} \xi^{-(n+1/2)} |n; \frac{1}{2}\rangle,
\] (7.6)

that also equals (7.1) when \( j \) takes half-integer values, since \( |n; 1/2\rangle \) are now the fermion eigenvectors of the angular momentum operator, as we can see in expression (6.15). This ends the proof of our statement.

From the study of the physical properties of these CS we can state that they are very similar to the Heisenberg CS on \( \mathbb{R} \), provided that the wideness of the wave function is small in comparison with the length of the configuration space \( S^1 \). Otherwise, the properties of these CS drastically depend on the values of \( p \). Moreover, all the physical properties have a periodic behaviour in terms of \( p \).

It is worthy to note that our CS are “quasi-minimal”, i.e., although they do not minimize the Heisenberg uncertainty relation, the product of the dispersions of the angle and momentum operators is upperly bounded by \( \hbar \).

Finally, we mention that these CS may be used to quantize the cylinder by means of the Weyl correspondence [DG 92] [DG 93] [GO 96]. Work in this direction is in progress and the results will be published elsewhere.

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$v = 0$

$\alpha = 0.2$

$\alpha = 5$

$\alpha = 15$

$\chi = 100$
