Weak lensing generated by vector perturbations and detectability of cosmic strings

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Abstract. We study the observational signature of vector metric perturbations through the effect of weak gravitational lensing. In the presence of vector perturbations, the non-vanishing signals for B-mode cosmic shear and curl-mode deflection angle, which have never appeared in the case of scalar metric perturbations, naturally arise. Solving the geodesic and geodesic deviation equations, we drive the full-sky formulas for angular power spectra of weak lensing signals, and give the explicit expressions for E-/B-mode cosmic shear and gradient-/curl-mode deflection angle. As a possible source for seeding vector perturbations, we then consider a cosmic string network, and discuss its detectability from upcoming weak lensing and CMB measurements. Based on the formulas and a simple model for cosmic string network, we calculate the angular power spectra and expected signal-to-noise ratios for the B-mode cosmic shear and curl-mode deflection angle. We find that the weak lensing signals are enhanced for a smaller intercommuting probability of the string network, \( P \), and they are potentially detectable from the upcoming cosmic shear and CMB lensing observations. For \( P \sim 10^{-1} \), the minimum detectable tension of the cosmic string will be down to \( G\mu \sim 5 \times 10^{-8} \). With a theoretically inferred smallest value \( P \sim 10^{-3} \), we could even detect the string with \( G\mu \sim 5 \times 10^{-10} \).
1 Introduction

In standard cosmology, the vector mode of metric perturbations is thought to be a very minor component, and it does not serve as a seed of structure formation. One of the main theoretical reasons why we usually neglect vector perturbation is that in the absence of sources, vector perturbations decay away, and rapidly become negligible as the universe expands. It is,
however, known that vector perturbations are generated via a variety of mechanisms in the early universe. Possible sources to generate vector perturbations include topological defects such as cosmic strings [1–7], anisotropic stress of magnetic field [8–12], massive neutrinos [10, 12], second-order primordial density perturbations [13–18], and modification of vector sector of gravity such as Einstein-Aether theory [19–23]. In particular, there are active mechanisms that continuously generate vector perturbations even at late-time epoch. One such example is those produced by topological defects. Hence, even with a tiny fraction, active seeds can induce the non-vanishing signals of vector perturbations at present time, which might be potentially detectable through precision cosmological observations. A search for those tiny signals is thus very interesting and valuable, and the detection and/or measurement of vector perturbation offers an important clue to probe the physics and history of the very early universe beyond the last scattering surface.

In this paper, among various cosmological observations, we are particularly interested in the weak lensing observations, which can provide a direct evidence for the intervening vector perturbations along a line of sight by measuring the spatial patterns on the deformation of photon path. The weak lensing measurements of background sources such as galaxies and cosmic microwave background (CMB) have been widely studied and now been accepted as a standard cosmological technique [24–33] (for reviews, see [34–37]). There are a number of planned wide and deep weak lensing surveys, including Subaru Hyper Suprime-Cam (HSC) survey [38], Dark Energy Survey (DES) [39], and Large Synaptic Survey Telescope (LSST) [40]. They will provide a high-precision measurement of the deformation of the distant-galaxy images, namely cosmic shear fields. On the other hand, ongoing and upcoming CMB experiments such as PLANCK [41], POLARBEAR [42], ACTPol [43], SPTPol [44], CMBPol [45], and COrE [46], offer a unique opportunity to probe the gravitational lensing deflection of the CMB photons, called CMB-lensing signals, with unprecedented precision.

With the increasing interest in the precision weak lensing measurements, in this paper, we intend to clarify the observational signature of vector perturbations on the weak lensing experiment. The spatial pattern of cosmic shear fields is generally described by a two-dimensional symmetric trace-free field on the sky, and it can be decomposed into two parts; even-parity mode (E-mode) and the odd-parity mode (B-mode) (e.g., [32, 33]). Similarly, the deflection angle is decomposed into a gradient of scalar lensing potential (gradient-mode) and a rotation of pseudo-scalar lensing potential (curl-mode) (e.g., [32, 47]). The symmetric argument implies that the B-mode shear and the curl-mode deflection angle are produced by the vector and tensor perturbations, but not by the scalar perturbations. Hence, the non-vanishing B-mode or curl-mode signal on large angular scales would be a direct evidence for non-scalar metric perturbations. The weak lensing effect by the tensor perturbations has been previously studied in the cases of primordial gravitational wave (GW) [48–50] and secondary GW generated by the second-order primordial density perturbations [51], but the effect turns out to be very small and difficult to observe (but see Refs. [52, 53]). Here, we consider the weak lensing generated by vector perturbations, and derive the useful formulas for angular power spectra of E-/B-mode cosmic shear [eqs. (3.12), (3.14)–(3.18)], and the gradient-/curl-mode deflection angle [eqs. (3.24), (3.26), and (3.27)]. As a prospect for detecting non-zero B-mode cosmic shear or curl-mode deflection angle, we consider a cosmic string network as a possible source for seeding vector perturbations, and discuss its detectability.

It is known that the cosmic strings might have emerged in the early universe through spontaneous symmetry breakings [54–57]. Recently, another possibility to produce cosmic string has been pointed out in the context of superstring theory, and it is called cosmic
superstring. The properties of cosmic superstrings are quite similar to those of ordinary cosmic strings [58–62] (for reviews, see [63–70]), except for the fact that the intercommuting probability between strings is relatively low. Thus, not only the string tension, $\mu$, but also the intercommuting probability, $P$, are the important parameters to characterize the dynamics of cosmic string, as well as to distinguish between the conventional cosmic strings and the cosmic superstrings [71–74, 86]. Currently, the tightest observational constraint on $\mu$ and $P$ are obtained from CMB observations through Gott-Kaiser-Stebbins (GKS) effect [75–77], which is basically imprinted on small angular scales [1–7, 74, 78]. Theoretically, the parameter $P$ is expected to lie in $10^{-3} \lesssim P \lesssim 1$ [79–81] (though the range of parameters strongly depends on the type of strings and the detail of the model), and the current observation is not enough to constrain a wide parameter range of $P$. In this paper, based on the formula for weak lensing power spectra and a simple model of cosmic string network, we calculate the power spectra of B-mode cosmic shear and curl-mode deflection angle. The possibility to detect the weak lensing signals from the vector perturbations is discussed in detail for specific weak lensing and CMB measurements (see also [82–86]).

The paper is organized as follows. In section 2, we give basic equations for the weak lensing, and derive the expression for the deflection angle and the Jacobi map induced by the vector perturbations. In section 3, we investigate the properties of the shear fields and the deflection angle, and derive the formulas of the angular power spectra for the E-/B-mode cosmic shear and the gradient-/curl-mode deflection angle. Based on the formulas, in section 4, prospects for measuring the B-mode cosmic shear or curl-mode deflection angle are discussed, especially focusing on the cosmic string network. Finally, section 5 is devoted to summary and conclusion. In this paper, we assume a flat $\Lambda$CDM cosmological model with the cosmological parameters: $\Omega_b h^2 = 0.022$, $\Omega_m h^2 = 0.13$, $\Omega_{\Lambda} = 0.72$, $h = 0.7$, $n_s = 0.96$, $A_s = 2.4 \times 10^{-9}$, and $\tau = 0.086$ [78]. In Table 1, we summarize the definition of the quantities used to calculate the angular power spectrum.

## 2 Basic equations for weak lensing

In this section, we give the notation for the unperturbed and perturbed quantities, and derive the geodesic equation and geodesic deviation equations in the presence of vector metric perturbations. After the definitions of unperturbed and perturbed quantities in section 2.1, we give the basic equations which govern the gravitational lensing effect from vector perturbations and discuss the vector-induced gravitational lensing effects in section 2.2 and section 2.3.

### 2.1 Perturbed universe

Throughout the paper we consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe with the metric given by

$$ds^2 = a^2(\eta) \tilde{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) (g_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu,$$

where $a(\eta)$ corresponds to the conventional scale factor of a homogeneous and isotropic universe, $h_{\mu\nu}$ is a small metric perturbation, $\tilde{g}_{\mu\nu}$ is the conformal flat metric which include the spacetime inhomogeneity,

$$g_{\mu\nu} dx^\mu dx^\nu = -d\eta^2 + \tilde{g}_{ij} dx^i dx^j = -d\eta^2 + d\chi^2 + \chi^2 \omega_{ab} d\theta^a d\theta^b,$$
Table 1. Notations for quantities used in this paper.

| Symbol | eq. | Definition |
|--------|-----|------------|
| $g_{\mu\nu}$ | (2.1) | 4-dimensional metric on conformal transformed spacetime |
| $\gamma_{ij}$ | (2.2) | 3-dimensional spatial metric |
| $\omega_{ab}, e_{ab}$ | - | Metric/Levi-Civita pseudo-tensor on unit sphere |
| semi-colon ( ; ) | - | Covariant derivative associated with $g_{\mu\nu}$ |
| vertical bar ( | ) | - | Covariant derivative associated with $\gamma_{ij}$ |
| colon ( : ) | (2.13) | Covariant derivative associated with $\omega_{ab}$ |
| $\nabla^2$ | - | Laplace operator on the unit sphere |
| $\hat{n} = (\theta, \varphi)$ | - | Observed position on the sky |
| $k^\mu = E(1, -e_\chi)$ | (2.5) | Null vector on background spacetime |
| $w^\mu = (1, 0)$ | - | Observer’s 4-velocity |
| $e_\pm^i, e_\pm^a = e_i^a e_\pm^i$ | (3.1) | Basis of spin-weight $\pm 1$ |
| $\lambda, \chi = E(\lambda_0 - \lambda)$ | - | Affine parameter on background spacetime |
| $\sigma_{g,i}$ | (2.4) | Gauge-invariant vector perturbations |
| $0 \sigma_g = \sigma_{g,i} e_\chi^i$ | - | Spin-0 part of vector perturbations |
| $\sigma_{g,a} = \sigma_{g,i} e_\chi^i$ | - | Projected vector perturbations |
| $\pm 1 \sigma_g = \sigma_{g,i} e_\pm^i$ | - | Spin-$\pm 1$ part of vector perturbations |
| $P_{\sigma_g}$ | (3.11) | Auto-power spectrum for $\sigma_g$ |
| $\xi^a$ | - | Deviation vector field |
| $\Delta^a$ | (2.18) | Deflection angle on unit sphere |
| $x = \phi, \varpi$ | (2.19) | Scalar/pseudo-scalar lensing potentials |
| $\mathcal{D}^a_b$ | (2.23) | Jacobi map |
| $\mathcal{T}^a_b$ | (2.24) | Symmetric optical tidal matrix |
| $\gamma_{ab}$ | (2.27) | Symmetric trace-free part of Jacobi map |
| $g^X(X = E, B)$ | (3.7) | E-/B-mode reduced cosmic shear |
| $C_{XX}^X$ | (3.12), (3.24) | Angular auto-power spectrum for X |
| $S_{\text{vector}}^{X,\ell,\ell'}$ | (3.14), (3.15) | Transfer function for E- and B-modes |
| $\epsilon_\ell^X(m), \phi_\ell^X(m)$ | (3.16)-(3.18) | Radial E, B functions |
| $S_{\text{vector}}^{X,\ell,\ell'}$ | (3.26), (3.27) | Transfer function for scalar and pseudo-scalar lensing potential |

where $\omega_{ab} d\theta^a d\theta^b \equiv d\theta^2 + \sin^2 \theta d\varphi$ is the metric on the unit sphere.

With the metric (2.2), vector perturbations are generally given by

$$h_{00} = 0, \quad h_{0i} = B_i, \quad h_{ij} = H_{ij} + H_{ji}, \quad (2.3)$$

where both $B_i$ and $H_i$ are divergence-free three-vectors and the vertical bar ( | ) denotes the covariant derivative with respect to the three dimensional metric $\gamma_{ij}$. For convenience, we introduce the gauge-invariant vector perturbations:

$$\sigma_{g,i} \equiv \dot{H}_i - B_i, \quad (2.4)$$
where the dot denotes the derivative with respect to the conformal time \( \eta \). Using the gauge freedom for the vector perturbations, we adopt the gauge \( H_1 = 0 \), so-called conformal Newton gauge, hereafter. Appendix B summarizes the Christoffel symbols and Riemann tensors from the vector perturbations \( h_{0} = -\sigma_{E,i} \). We now set the metric perturbations at the observer position to zero because they can be absorbed into the homogeneous mapping.

We consider two geodesics \( x^\mu(v) \) and \( \tilde{x}^\mu(v) = x^\mu(v) + \xi^\mu(v) \), where \( v \) is the affine parameter and \( \xi^\mu(v) \) is the deviation vector field. Two geodesics lie in the past light cone of an observer \( O \). Since null geodesic is not affected by conformal transformations, it is sufficient to consider the spacetime without the Hubble expansion. Hence, we introduce a tangent vector \( k^\mu \) along the geodesic \( x^\mu(\lambda) \) on the conformally transformed spacetime with the affine parameter \( \lambda \), defined by [37, 87]

\[
k^\mu \equiv a^2 \frac{dx^\mu}{dv} = \frac{dx^\mu}{d\lambda}.
\] (2.5)

This null vector satisfies the equations:

\[
g_{\mu\nu}k^\mu k^\nu = 0, \quad k^\mu \partial_\mu k^\nu = \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0.
\] (2.6)

where the semi-colon ( ; ) and \( \Gamma^\mu_{\rho\sigma} \) are the covariant derivative and the Christoffel symbols associated with the unperturbed metric \( g_{\mu\nu} \), respectively. The geodesic equation at the zeroth-order in metric perturbations reads that \( x^\mu(\lambda) = E(\lambda, \lambda_{O} - \lambda) e^\mu_{\chi} \), where \( E \) and \( e^\mu_{\chi} \) represent the photon energy and the photon propagation direction measured from the observer in the background flat spacetime, \( \lambda_{O} \) denotes the affine parameter at \( O \). Note that the vector \( e^\mu_{\chi} \) is the unit vector tangent to a geodesic on the flat three-space, satisfying \( e^\mu_{\chi} e^\mu_{\chi} = 1 \) and \( (e^\mu_{\chi})_{\nu j} e^\mu_{\chi} = 0 \). For convenience, we now switch from \( \lambda \) to \( \chi \equiv E(\lambda_{O} - \lambda) \), hereafter.

Introducing the observer’s 4-velocity at \( O \), \( u^\mu \), we define orthonormal spacelike basis along the light ray, \( e^\mu_{a} \) with \( a = \theta, \varphi \), which satisfies

\[
g_{\mu\nu} e^\mu_{a} e^\nu_{b} = \omega_{ab}, \quad g_{\mu\nu} k^\mu e^\nu_{a} = g_{\mu\nu} u^\mu e^\nu_{a} = 0.
\] (2.7)

They are parallely transported along the geodesics as \( u^\mu ;_{\nu} k^\nu = 0, \) \( e^\mu_{a ;\nu} k^\nu = 0 \). For a static observer, we have \( u^\mu = (1, 0, 0, 0) \) and \( e^\mu_{a} = (0, e^\mu_{a}) \), and the spatial basis vectors on the background spacetime in the Cartesian coordinate can be written as

\[
e^\mu_{\chi}(\hat{n}) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \chi = \theta \sin \varphi, \cos \theta, \quad \varphi = \sin \theta \sin \varphi, \cos \theta.
\] (2.8)

\[
e^\mu_{\theta}(\hat{n}) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), \quad \theta = \theta, \varphi = \sin \theta \sin \varphi, \cos \theta.
\] (2.9)

\[
e^\mu_{\varphi}(\hat{n}) = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0),
\] (2.10)

where \( \hat{n} = (\theta, \varphi) \) is the observed position on the sky. With these notations, we have

\[
e^{i}_{\chi} \partial_i = \partial_\chi, \quad e^{i}_{\theta} \partial_i = \frac{1}{\chi} \partial_\theta, \quad e^{i}_{\varphi} \partial_i = \frac{1}{\chi} \partial_\varphi,
\] (2.11)

and we can evaluate

\[
\chi e^i_{\alpha} \partial_j e^i_{\beta} = e^i_{\alpha}, \quad \chi^2 e^i_{\alpha} \partial_j \partial_k e^i_{\chi} = -\omega_{ab} e^i_{\alpha}, \quad e^i_{\chi} \partial_j e^i_{\alpha} = 0,
\]

\[
\chi e^i_{\alpha} \partial_j e^i_{\beta} = \chi e^i_{\alpha} \partial_j e^i_{\beta} = \cot \theta e^i_{\varphi}, \quad \chi e^i_{\theta} \partial_j e^i_{\varphi} = -e^i_{\chi},
\]

\[
\chi e^i_{\varphi} \partial_j e^i_{\varphi} = -\sin \theta \sin \theta e^i_{\chi} + \cos \theta e^i_{\varphi}).
\] (2.12)
We then define the intrinsic covariant derivative of a two-vector on the unit sphere, \( X_a = X_i e^i_a \), in terms of the polarization basis as

\[
X_{a:b} = \chi e^i_b \partial_j X_a - (2) \Gamma^c_{ab} X_c ,
\]

where \((2) \Gamma^c_{ab}\) is the two dimensional Christoffel symbol defined on the unit sphere, and we have introduced the colon ( : ) as the covariant derivative with respect to the unit sphere metric \( \omega_{ab} \). Here the polarization indices \((a, b, \cdots)\) are raised or lowered with respect to \( \omega_{ab} \).

### 2.2 Geodesic equation

At the linear-order in metric perturbations, the gravitational lensing effect appears on the spatial components of the geodesic equation for the photon ray. The geodesic equation for the linear-order geodesic equation, we expand the Christoffel symbols as \( \tilde{\Gamma}^a_{\mu \nu} \). Here the polarization indices \((a, b, \cdots)\) are raised or lowered with respect to \( \omega_{ab} \).

The linear-order spatial geodesic equation for \( \tilde{\chi}^a(\chi) \) is given by

\[
\frac{d^2 \tilde{\chi}^a}{d\chi^2} + 2 \Gamma^i_{jk} e^j_a \frac{d \tilde{\chi}^k}{d\chi} + \delta \Gamma^i_{\mu \nu} \frac{dx^\mu}{d\chi} \frac{dx^\nu}{d\chi} = 0 ,
\]

where \( \Gamma^i_{\mu \nu} \) is the Christoffel symbol associated with the unperturbed metric \( g_{\mu \nu} \). To derive the linear-order geodesic equation, we expand the Christoffel symbols as \( \tilde{\Gamma}^i_{\mu \nu} = \Gamma^i_{\mu \nu} + \delta \Gamma^i_{\mu \nu} \), where \( \Gamma^i_{\mu \nu} \) is the Christoffel symbols associated with the unperturbed metric \( g_{\mu \nu} \) (see Appendix B). The linear-order spatial geodesic equation for \( \tilde{\chi}^a \) becomes

\[
\frac{d^2 \tilde{\chi}^a}{d\chi^2} + 2 \Gamma^i_{jk} e^j_a \frac{d \tilde{\chi}^k}{d\chi} + \delta \Gamma^i_{\mu \nu} \frac{dx^\mu}{d\chi} \frac{dx^\nu}{d\chi} = 0 ,
\]

where \( d/d\chi \equiv e^i_\chi \partial_i - \partial_\chi \). To extract the angular components of the deviation vector, \( \xi^a = \xi^i e^a_i \), we multiply \( e^a_i \) in both side of eq. (2.15). Since the unperturbed Christoffel symbols satisfies \( \Gamma^i_{jk} = 0 \) in the Cartesian coordinate system, with the condition for the parallel transportation, \( (d/d\chi) e^a_i = 0 \), we obtain the equation for \( \xi^a \):

\[
\frac{d^2 \xi^a}{d\chi^2} = 1 \omega^{ab} \left\{ (\sigma_{g,b})_0 - \frac{d}{d\chi} (\sigma_{g,b}) \right\} ,
\]

where we have used eqs. (2.12), (B.5), and defined \( \sigma_{g,i} \equiv g_{i\chi} e^i_\chi \), \( \sigma_{g,a} \equiv g_{a\chi} e^a_\chi \). Imposing the initial conditions, \( \xi^a|_0 = 0 \) and \( (d\xi^a/d\chi)|_0 = \delta \theta^a_0 \), where \( \delta \theta^a_0 \) denotes the angular coordinate at \( O \), the solution for eq. (2.16) becomes

\[
\frac{\xi^a(\chi S)}{\chi S} = \delta \theta^a_0 + \omega^{ab} \int_0^{\chi S} \frac{d\chi}{\chi} \left\{ \frac{\chi S - \chi}{\chi S} (\sigma_{g,b})_0 - \sigma_{g,b} \right\} \right|_{(\phi - \chi S, \chi S e^i_\chi)} .
\]

In the above, the integral at the right-hand-side is evaluated along the unperturbed light path according to the Born approximation. We can confirm that this result exactly matches the one of Ref. [88].

Provided the deviation vector at the both end points, the deflection angle, \( \Delta^a \), can be estimated through \[32\]

\[
\Delta^a = \frac{\xi^a(\chi S)}{\chi S} - \delta \theta^a_0 .
\]
(\varpi) lensing potentials. Then, the deflection angle is described by the sum of the two terms (e.g., [47]):

$$\Delta_a = \phi_a + \varpi_b \epsilon^b_a,$$

(2.19)

where \( \epsilon^b_a \) denotes the two dimensional Levi-Civita pseudo-tensor. Hereafter, we call the first and second terms in the right-hand-side of eq. (2.19) gradient- and curl-modes, respectively [71]. The scalar-/pseudo-scalar lensing potentials in the case of the vector perturbations can be written as

$$\nabla^2 \phi = \Delta_{,b}^a \delta^b_a = \int_0^{\chi_S} \frac{d\chi}{\chi} \left\{ \frac{\chi S - \chi}{\chi S} \nabla^2 (0\sigma_g) - \sigma_g^{a, a} \right\},$$

(2.20)

$$\nabla^2 \varpi = \Delta_{,b}^a \epsilon^b_a = -\int_0^{\chi_S} \frac{d\chi}{\chi} \sigma_g^{a, b} \epsilon^b_a,$$

(2.21)

where \( \nabla^2 \) is the Laplace operator on the sphere, namely \( \nabla^2 \phi = \phi_{,ab} \omega^{ab} = \phi_{,a,a} \). Note that eq. (2.21) coincides with eq. (1.4) of Ref. [71]. The curl component of the deflection becomes non-vanishing in the presence of the divergence-free component of \( \sigma_g^{a, a} \). In section 3.2, using eqs. (2.20) and (2.21), we will derive the explicit formulas for the gradient-/curl-mode deflection angle and their angular power spectrum.

### 2.3 Geodesic deviation equation

Here, we introduce the Jacobi map which characterizes the deformation of light bundle. In terms of the projected deviation vector \( \xi^a \), the geodesic deviation equation in the conformal transformed spacetime can be written as [26, 30]

$$\frac{d^2 \xi^a}{d\chi^2} = T^a_{\ b} \xi^b; \quad T^a_{\ b} = -\frac{1}{E^2} R_{\mu\rho\sigma\nu} k^\mu k^\nu \epsilon^{\rho a} e^\sigma_b,$$

(2.22)

where \( T^a_{\ b} \) is called the symmetric optical tidal matrix, and \( R_{\mu\rho\sigma\nu} \) is the Riemann tensor of the metric \( g_{\mu\nu} \). Given the initial conditions at the observer, \( \xi^a|_O = 0 \) and \( (d\xi^a/d\chi)|_O = \delta^a_\delta \), the solution of eq. (2.22) is generally rewritten in the following form:

$$\xi^a(\chi) \equiv D^a_{\ b}(\chi) \delta^b_\delta,$$

(2.23)

where \( D^a_{\ b} \) is the Jacobi map and it satisfies

$$\frac{d^2 D^a_{\ b}}{d\chi^2} = T^a_{\ c} D^c_{\ b},$$

(2.24)

with the initial condition at the observer \( O \) rewritten with \( D^a_{\ b}|_O = 0 \) and \( (d/d\chi)D^a_{\ b}|_O = \delta^a_\delta \).

We are particularly concerned with the perturbed Jacobi matrix induced by vector perturbations. To get the expressions relevant for the weak lensing measurements, we expand eq. (2.24) as \( D^a_{\ b} = D^a_{\ b} + \delta D^a_{\ b} \), and \( T^a_{\ b} = T^a_{\ b} + \delta T^a_{\ b} \). Since the tidal matrix vanishes in the unperturbed spacetime, \( T^a_{\ b} = 0 \), the zeroth-order solution of Jacobi map becomes \( D^a_{\ b} = \chi \delta^a_\delta \). Plugging this expression into eq. (2.24) and solving the linear-order equation, we obtain the expression valid up to the linear-order in metric perturbations:

$$D^a_{\ b}(\chi_S) = \chi_S \delta^a_\delta + \int_0^{\chi_S} d\chi \ (\chi_S - \chi) \chi \delta T^a_{\ b}(\chi) + \mathcal{O}(h^2).$$

(2.25)
In the above, an important observation is that the resultant Jacobi map is always symmetric. Hence, the anti-symmetric part of the Jacobi map, which is directly related to the rotation mode, does not appear at the linear order.

Note that Eq. (2.25) is still general in the sense that the Jacobi map includes the deformation arising from all of the metric perturbations. To specifically derive the expression relevant for the vector perturbations, we explicitly write down the tidal matrix. Using eqs. (B.7), (2.12), and (2.13), we have

$$\delta T_{ab}^{\text{vector}} = -e^i_v e^j_v \partial_j \partial_k h_{0i} + \frac{d}{d \chi} \left( e^i_v e^j_v \partial_j \partial_k h_{0i} \right)$$

$$\equiv \frac{1}{\chi^2} \left\{ (0 \sigma_g)_{;ab} - \frac{d}{d \chi} \left( \chi \sigma_g(0;ab) \right) + \chi \omega_{ab}(0;\sigma_g) \right\} ,$$

(2.26)

where we have introduced the symmetric operation defined by

$$A_{(a} B_{b)} \equiv \frac{1}{2} (A_{a} B_{b} + A_{b} B_{a}) .$$

Since we are interested in the shear fields, it is sufficient to consider the symmetric trace-free part of the Jacobi map, $D_{(ab)}$, where the angle bracket $\langle \cdots \rangle$ denotes the symmetric trace-free part taken in the two-dimensional space: $X_{(ab)} \equiv \langle X_{ab} + X_{ba} - X_{cc} \omega_{ab} \rangle / 2$. For the vector perturbations, the symmetric trace-free part of eq. (2.25) reduces to (see also [24, 25, 88])

$$\gamma_{ab} \equiv \frac{1}{\chi_S} D_{(ab)}(\chi_S) = \int_0^{\chi_S} \frac{d \chi}{\chi} \left\{ \chi - \chi_S \left( (0 \sigma_g)_{;ab} - \sigma_g(0;ab) \right) \right\} \bigg|_{(0;\sigma_g,\chi_S e_i)} .$$

(2.27)

Strictly speaking, the affine parameter $\lambda$ and/or the conformal distance $\chi$ are not direct observables, and we should express the Jacobi map as a function of the redshift of the source, $z_S$, taking the perturbation of the observed redshift into account. At first-order, however, the perturbation of the redshift affects only the trace part of the Jacobi map [25]. Thus, simply relating $\chi$ with redshift $z$ through $\chi = \int_0^z dz' / H(z')$, the expression (2.27) still remains relevant, and we will use it to derive the formulas for angular power spectra of the E-/B-mode cosmic shear.

Finally, we note that eqs. (2.17) or (2.18) and (2.27) lead to the following simple relation:

$$\gamma_{ab} = \Delta_{(a;b)} .$$

(2.28)

This relation is valid at linear order, and exactly coincides with the one empirically defined in [32]. On the other hand, no such expression is obtained for the relation between $D_{ab}$ and $\Delta_{a;b}$ because of the non-vanishing trace part.

3 Weak lensing observables induced by vector perturbations

In this section, we derive the full-sky formulas for the angular power spectra of the E-/B-mode cosmic shear and the gradient-/curl-mode deflection angle generated by vector metric perturbations. The formula for the angular power spectra are respectively given in section 3.1 and 3.2, for the E-/B-mode cosmic shear [eqs. (3.12), (3.14)–(3.18)], and the gradient-/curl-mode deflection angle [eqs. (3.24), (3.26), and (3.27)]. We then discuss an interesting relation for angular power spectra between the shear and the cosmic shear in section 3.3.
3.1 Cosmic shear

In discussing the spatial patterns of shear fields on celestial sphere, it is useful to introduce the spin-weighted quantities. A quantity \( sX \) that transforms as \( sX \rightarrow e^{iαs} sX \) under a rotation of \((e^i_θ, e^i_ϕ)\) by an angle \( α \) is called spin-weighted quantity with spin-\( s \). According to [89], we define the basis of spin-weight ±1 as

\[
\hat{n}^i = e^i_θ \sinθ e^i_ϕ \hat{n}.
\]  

(3.1)

With this basis, the Jacobi map can be decomposed into the spin-0 and spin-±2 components [25]:

\[
\begin{align*}
0D &= D^{ab} e^a_+ e^b_- , \\
\pm 2D &= D^{ab} e^a_± e^b_± ,
\end{align*}
\]  

(3.2)

where we have defined the projected basis \( e^a_± = e^a_+ i \sinθ e^a_- \) (3.1).

On the other hand, the spin-±2 parts give the shear fields, \( γ \) and \( γ^* \), defined by

\[
\begin{align*}
γ &= \frac{-2D}{2χS} = \frac{1}{2} γ^{a} e^a_+ e^b_+ , \\
γ^* &= \frac{-2D}{2χS} = \frac{1}{2} γ^{a} e^a_- e^b_- .
\end{align*}
\]  

(3.3)

In practice, what we observe is not directly the shear itself, but rather the ratio between the anisotropic and isotropic deformations, so-called reduced shear. The reduced shear is related to the spin-weighted Jacobi map as

\[
\begin{align*}
g &\equiv \frac{γ}{1 - κ} = \frac{-2D}{0D} , \\
g^* &\equiv \frac{-2D}{0D} .
\end{align*}
\]  

(3.4)

Note that at linear-order, the reduced shear fields is simply described by the shear fields, i.e., \( g \simeq γ \) and \( g^* \simeq γ^* \). Since the reduced shear fields transform as the spin-±2 quantities, they are decomposed on the basis of spin-±2 harmonics \( \pm 2Y_{ℓm}(\hat{n}) \) (see appendix C.1 for definition):

\[
\begin{align*}
g(\hat{n}) &= \sum_{ℓ=2}^{∞} \sum_{m=-ℓ}^{+ℓ} +2g_{ℓm} +2Y_{ℓm}(\hat{n}) , \\
g^*(\hat{n}) &= \sum_{ℓ=2}^{∞} \sum_{m=-ℓ}^{+ℓ} -2g_{ℓm} -2Y_{ℓm}(\hat{n}) .
\end{align*}
\]  

(3.5)

Then, E- and B-modes, as the two parity eigenstates, can be defined:

\[
\begin{align*}
g^E_{ℓm} &= -\frac{1}{2} \left( +2g_{ℓm} + -2g_{ℓm} \right) , \\
g^B_{ℓm} &= -\frac{1}{2i} \left( +2g_{ℓm} -2g_{ℓm} \right) .
\end{align*}
\]  

(3.6)

The auto- and cross-power spectra of these quantities are also defined as:

\[
C^{XX'}_ℓ \overset{XX'}{=} \frac{1}{2ℓ + 1} \sum_m \langle g^X_{ℓm} g^{X'}_{ℓm} \rangle ,
\]  

(3.7)

where \( X, X' = E, B \), and the angle bracket \( \langle \cdots \rangle \) denotes the ensemble average.
With the preliminary setup mentioned above, let us now derive the explicit expression for E-/B-mode power spectra. Multiplying \( e_+^a e_+^b \) in both side of eq. (2.27), the spin-+2 part of the reduced shear, eq. (3.5), is written as (see also [24, 25])

\[
g = -\frac{1}{2} \int_0^{\chi_S} \frac{d\chi}{\chi} \left\{ \chi_S - \chi \left( \sigma_{k, ab} e_+^a e_+^b - (\sigma_{g, ab}) e_+^a e_+^b \right) \right\}.
\]

(3.9)

In the above, the gauge-invariant vector perturbation contains statistical information for spatial randomness, which can be decomposed into the Fourier modes. For a given Fourier mode \( \mathbf{k} \), a convenient representation would be \( e_+^i(\hat{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \), where \( \hat{k} = \mathbf{k}/k \), and \( e_+^i(\hat{k}) \) is the basis vector perpendicular to \( \hat{k} \). We then write \( \sigma_g^i \) as

\[
\sigma_g^i(r, \eta) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}} \left\{ \sigma_g^{(+)}(\mathbf{k}, \eta) e_+^i(\hat{k}) + \sigma_g^{(-)}(\mathbf{k}, \eta) e_-^i(\hat{k}) \right\}.
\]

(3.10)

We note that the trace-free condition for the vector perturbations, namely \( \sigma_g^i |_{i=0} = 0 \), is automatically satisfied when we consider the Fourier expansion, eq. (3.10). The quantities \( \sigma_g^{(+)}(\mathbf{k}, \eta) \) and \( \sigma_g^{(-)}(\mathbf{k}, \eta) \), are responsible for the randomness arising from initial condition and/or late-time evolution. Assuming the un-polarized state of vector fluctuations, their statistical properties are characterized by

\[
\left\langle \sigma_g^{(s)}(\mathbf{k}, \eta) \sigma_g^{(s')} (\mathbf{k}', \eta') \right\rangle = \left\{ \begin{array}{ll} \frac{1}{2} P_{2\sigma_g}(k; \eta, \eta') (2\pi)^3 \delta^3(k - k') : s = s' = \pm 1, \\ \quad \quad \quad : s \neq s'. \end{array} \right.
\]

(3.11)

Substituting eq. (3.10) into the expression (3.9), we first explore the relation between the harmonic coefficients \( g_{\ell m}^{EB} \) and \( \sigma_g^{(\pm1)} \). Next plugging this into the definition (3.8), after lengthy calculation presented in appendix C.2, we obtain the formulas for the power spectrum of E-/B-mode cosmic shear. The resultant expressions become

\[
C_{XX}^X = \frac{2}{\pi} \int_0^{\infty} k^2 dk \int_0^{\chi_S} k d\chi \int_0^{\chi_S} k d\chi' S_{X, \ell}^{vector}(k, \chi) S_{X, \ell}^{vector}(k, \chi') P_{2\sigma_g}(k; \eta_0 - \chi, \eta_0 - \chi'),
\]

(3.12)

\[
C_{EB}^X = 0,
\]

(3.13)

where the quantities \( S_{X, \ell}^{vector}(k, \chi) \) are the transfer functions defined as

\[
S_{E, \ell}^{vector}(k, \chi) = \frac{\chi_S - \chi}{\chi_S} \epsilon_{\ell}^{(0)}(k\chi) - \epsilon_{\ell}^{(1)}(k\chi),
\]

(3.14)

\[
S_{B, \ell}^{vector}(k, \chi) = \beta_{\ell}^{(1)}(k\chi),
\]

(3.15)

with the coefficients \( \epsilon_{\ell}^{(0,1)}(k\chi) \) and \( \beta_{\ell}^{(1)}(k\chi) \) given by (see also [90])

\[
\epsilon_{\ell}^{(0)}(x) = \frac{1}{2} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!} \frac{(\ell + 2)!}{(\ell - 2)!}} x^{\ell + 1} j_\ell(x),
\]

(3.16)

\[
\epsilon_{\ell}^{(1)}(x) = \frac{1}{2} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!} \frac{(\ell + 2)!}{(\ell - 2)!}} \left( \frac{j_\ell(x)}{x^2} + \frac{j'_\ell(x)}{x} \right),
\]

(3.17)

\[
\beta_{\ell}^{(1)}(x) = \frac{1}{2} \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!} \frac{(\ell + 2)!}{(\ell - 2)!}} \frac{j_\ell(x)}{x}.
\]

(3.18)

Here, \( j_\ell(x) \) is the spherical Bessel function. These formulas [eqs. (3.12)-(3.18)] are one of the main results of this paper.


\subsection*{3.2 Deflection angle}

Based on the expressions (2.20) and (2.21), let us next consider the deflection angle. First notice that the metric on the sphere, $\omega^{ab}$, and the Levi-Civita pseudo-tensor, $\epsilon^{ab}$, can be rewritten in terms of the basis vectors $e^a_\pm$ with \[34\]

$$\omega^{ab} = e^a_+ e^b_-,$$
$$\epsilon^{ab} = i \epsilon^a_+ e^b_-,$$  

(3.19)

where we have introduced the anti-symmetric operation defined by $A[aB] = \frac{1}{2}(A_a B_b - A_b B_a)$. Then, the scalar and pseudo-scalar lensing potentials are recast as

$$\nabla^2 \phi = \int_0^{\chi_S} \frac{d\chi}{\chi} \left\{ \frac{\chi_S - \chi}{\chi_S} (\sigma_g)_{,ab} e^a_+ e^b_- - \sigma_g e^a_+ (0) e^b_+ \right\},$$
$$\nabla^2 \bar{\omega} = -i \int_0^{\chi_S} \frac{d\chi}{\chi} \sigma_g e^a_+ e^b_-.$$

(3.20)

(3.21)

Since the scalar/pseudo-scalar lensing potentials, $\phi$ and $\bar{\omega}$, transform as spin-0 quantities, they are decomposed with the spin-0 harmonics:

$$\phi(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell m} Y_{\ell m}(\hat{n}), \quad \bar{\omega}(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \bar{\omega}_{\ell m} Y_{\ell m}(\hat{n}).$$

(3.22)

The angular power spectra for the scalar/pseudo-scalar lensing potentials are defined as

$$C_{xx'}^{\ell} = \frac{1}{2\ell + 1} \sum_{m} \langle x_{\ell m} x'_{\ell m} \rangle,$$

(3.23)

with $x, x' = \phi, \bar{\omega}$.

Now, similar manner to the cosmic shear, we first derive the relation between $x_{\ell m}$ and $\sigma_g^{(\pm1)}$. This is done with a rather lengthy calculation in appendix C.3. The resultant expressions are summarized in equations (C.63) and (C.64). Then, the angular power spectra for the gradient- and curl-modes are obtained by plugging these expressions into (3.23). As a result, we have

$$C_{xx'}^{\ell} = \frac{2}{\pi} \int_0^\infty k^2 dk \int_0^{\chi_S} k d\chi \int_0^{\chi_S} k d\chi' S_{\phi,\ell}(k,\chi) S_{\phi,\ell}(k,\chi') P_{\sigma_g,\ell}(k; \eta_0 - \chi, \eta_0 - \chi'),$$
$$C_{\phi\bar{\omega}}^{\ell} = 0,$$

(3.24)

(3.25)

where $P_{\sigma_g,\ell}$ was defined in eq. (3.11). The transfer functions $S_{\phi,\ell}$ are defined by

$$S_{\phi,\ell}(k,\chi) = 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left( \frac{\chi_S - \chi}{\chi_S} \epsilon^{(0)}_\ell(k\chi) - \epsilon^{(1)}_\ell(k\chi) \right),$$
$$S_{\phi,\ell}(k,\chi) = 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \rho^{(1)}_\ell(k\chi),$$

(3.26)

(3.27)

with the quantities $\epsilon^{(0,1)}_\ell$ and $\rho^{(1)}_\ell$ given by eqs. (3.16)-(3.18). The formulas given above are also one of the main results of this paper.
3.3 Shear-deflection relation

Here, we briefly mention the relation between E-/B-mode cosmic shear and gradient-/curl-mode deflection angle. From eq. (2.28), we found that in the case of vector perturbations the symmetric trace-free part of the Jacobi map, namely the shear fields, are directly related to the deflection angle, which can be decomposed into the gradient and curl modes [see eq. (2.19)]:

$$\Delta_{(ab)} = \phi_{;(ab)} + \varpi_{;c}(a \varepsilon^c_b) = \gamma_{ab}.$$  \hspace{1cm} (3.28)

This implies that we can extract the vector-induced gradient-/curl-modes from the vector-induced shear fields. Taking the divergence and then taking the divergence or curl again on eq. (3.28), we find [32, 34]

$$\gamma_{;ab} = \frac{1}{2} \nabla^2 (\nabla^2 + 2) \phi, \quad \epsilon^{ca} \gamma_{;ab;c} = \frac{1}{2} \nabla^2 (\nabla^2 + 2) \varpi.$$  \hspace{1cm} (3.29)

The above relation can be further reduced to simplified forms if we move to the harmonic space. After lengthy calculation presented in appendix C.4, we obtain the explicit relation between $\phi_{\ell m}$, $\varpi_{\ell m}$ defined in eq. (3.22) and $g^E_{\ell m}$, $g^B_{\ell m}$ defined in eq. (3.7) as

$$\phi_{\ell m} = 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \varpi_{\ell m}, \quad \varpi_{\ell m} = 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} g^B_{\ell m}.$$  \hspace{1cm} (3.30)

Thus, we reach at the relation between angular power spectra for shear and deflection angle:

$$C_{\ell}^{\phi \phi} = 4 \frac{(\ell - 2)!}{(\ell + 2)!} C_{\ell}^{EE}, \quad C_{\ell}^{\varpi \varpi} = 4 \frac{(\ell - 2)!}{(\ell + 2)!} C_{\ell}^{BB}, \quad C_{\ell}^{\phi \varpi} = C_{\ell}^{EB} = 0.$$  \hspace{1cm} (3.31)

These relations are nontrivial, but are generally valid as long as the relation between the deflection angle and the Jacobi map, eq. (3.28), holds. Note that the relations given here does not hold for general distribution of background sources, if the source distribution for shear fields differs from that for the deflection angle.

4 Implications for cosmic string network

In this section, as an illustrative example for the application of the full-sky formulas, we consider a cosmic string network as a possible source for seeing vector perturbations. Based on a simple model of cosmic strings described in Sec. 4.1, signal-to-noise ratios for B-mode cosmic shear and curl-mode deflection angle are estimated, and the detectability of the string network from future observations is discussed in Sec. 4.2.

4.1 Vector perturbations generated by a cosmic string network

In most of the active generation mechanisms, the vector metric perturbations are sourced by the non-vanishing vector mode of stress-energy tensor. This is true for the case of cosmic string network. Thus, to compute the weak lensing signals, we must first evaluate the vector stress-energy tensor induced by the cosmic strings. Let us write the vector stress-energy as [90]:

$$\delta T^0_{\nu}(r, \eta) = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot r} \left\{ v^{(+1)}(k, \eta) e_{+}(\hat{k}) + v^{(-1)}(k, \eta) e_{-}(\hat{k}) \right\}.$$  \hspace{1cm} (4.1)
Through the linearized Einstein equation, this is related to the vector metric perturbation as
\[ \sigma_{g}^{(\pm 1)}(k, \eta) = \frac{16\pi G a^2}{k^2} v^{(\pm 1)}(k, \eta). \]  

Here, we have ignored the contribution of the anisotropic stress tensor from the cosmological fluids.

For a concrete model of non-vanishing stress-energy tensor, we consider a string network described by the velocity-dependent one-scale (VOS) model \[72, 91–93\]. The string network consists of a collection of string segments, whose length and velocity are respectively given by \( \xi = 1/(H \gamma_s) \) and \( v_{\text{rms}} \), where \( \gamma_s \) represents the correlation length of the string network. The string segments are assumed to be randomly oriented and Poisson-distributed. The intercommuting process provides an essential mechanism for a string network to lose its energy due to loop formation. It is widely believed that the energy-loss mechanism allows the network to relax towards a cosmological attractor solution, in which \( \gamma_s \) and \( v_{\text{rms}} \) remain constant: this is so-called scaling solution. Several groups developed the numerical codes to evolve a string network and concluded that there exists a scaling regime for long strings \[2, 3, 94–98\]. In the VOS model, \( \gamma_s \) and \( v_{\text{rms}} \) are approximately described by \( \gamma_s = (\pi \sqrt{2}/3 \tilde{c} P)^{1/2} \) and \( v_{\text{rms}}^2 = (1 - \pi/3 \gamma_s)/2 \) \[72\], where \( \tilde{c} \approx 0.23 \) quantifies the efficiency of loop formation \[91\] and \( P \) is the intercommuting probability. For a tractable analytic estimate, we assume that the correlations between the string segments are characterized by the simple model developed in \[77, 99, 100\]. Then, from Appendix D, we obtain the equal-time auto power spectrum for the vector perturbations, \( P^{\sigma_g \sigma_g}(k; \eta, \eta) \):

\[ P^{\sigma_g \sigma_g}(k; \eta, \eta) = \left( \frac{16\pi G \mu}{3(1 - v_{\text{rms}}^2)} \right)^4 \left( \frac{a}{k \xi} \right)^5 \text{erf} \left( \frac{k \xi}{2 \sqrt{6}} \right), \]  

(4.3)

where \( \text{erf}(x) \) is the error function, \( \text{erf}(x) = (2/\sqrt{\pi}) \int_0^x dy e^{-y^2} \). To compute the weak lensing power spectra, we further need the unequal-time auto-power spectrum. Here, as a crude estimate, we adopt the following approximation \[101–103\]:

\[ P^{\sigma_g \sigma_g}(k; \eta_1, \eta_2) = \sqrt{P^{\sigma_g \sigma_g}(k; \eta_1, \eta_1) P^{\sigma_g \sigma_g}(k; \eta_2, \eta_2)}. \]  

(4.4)

The model and assumptions given above would be simplistic and might not be realistic for a precision study of lensing signals. Further, eqs. (4.3) and (4.4) may have additional modifications from the contributions of the loop strings or non-negligible correlations between different string segments \[74\]. Though these effects are expected to be small, they would certainly enhance the B-mode or curl-mode signals, and the expected signal-to-noise ratios will be increased. In this respect, the analysis based on the simple model may give a rather conservative estimate for the detectability of cosmic strings. Nevertheless, we should keep in mind the possibilities that the contributions of loop strings or non-vanishing correlations are accompanied with changes to the other components of string stress-energy, together with the generation of tensor perturbations, which might eventually lead to the reduction of the B-mode shear and the curl-mode.

4.2 Angular power spectra and signal-to-noise ratios

4.2.1 B-mode cosmic shear

Let us first compute the B-mode power spectrum of cosmic shear field. To discuss the weak lensing measurement from imaging surveys, we assume the redshift distribution of galaxies,
\( N(\chi_S) \, d\chi_S = N_g \frac{3z_S^2}{2(0.64z_m)^3} \exp \left[ -\frac{z_S}{0.64z_m} \right] \, dz_S , \) \hspace{1cm} (4.5)

where \( z_m \) and \( N_g \) denote the mean redshift and the total number of galaxies per square arcminute, respectively. Taking account of the redshift distribution of galaxies, we recast the formula for B-mode power spectra:

\[
C_{\ell}^{BB} = \frac{1}{2\pi} \frac{(\ell - 1)!(\ell + 2)!}{(\ell - 2)!(\ell + 1)!} \int_0^\infty k^2 \, dk \int_0^\infty d\chi \int_0^\infty d\chi' \times W_1(\chi) \frac{j_\ell(k\chi)}{\chi} W_1(\chi') \frac{j_\ell(k\chi')}{\chi'} P_{\sigma_g\sigma_g}(k; \eta_0 - \chi, \eta_0 - \chi') , \hspace{1cm} (4.6)
\]

where we have introduced the weight function \( W_1 \), which is the normalized distribution function for galaxies along a line-of-sight:

\[
W_1(\chi) = \int_\chi^\infty d\chi_S \frac{N(\chi_S)}{N_g} . \hspace{1cm} (4.7)
\]

In the cosmic shear measurement, apart from systematics, the main noise contribution would come from the intrinsic ellipticity of galaxies, which is described by

\[
N_{\ell}^{BB} = \left\langle \frac{\gamma_{int}^2}{\hat{N}} \right\rangle , \hspace{1cm} (4.8)
\]

where \( \langle \gamma_{int}^2 \rangle^{1/2} \) is the root-mean-square intrinsic ellipticity, and \( \hat{N} \) is the number density of galaxies per steradians. We adopt the empirically derived value, \( \langle \gamma_{int}^2 \rangle^{1/2} = 0.3 \) \([106]\), and parameterize the number density of galaxies as \( \hat{N} = 3600N_g(180/\pi)^2 \) [str\(^{-1}\)]. Then, the statistical error of the B-mode power spectrum is estimated as \([48, 51]\)

\[
\Delta C_{\ell}^{BB} = \sqrt{\frac{2}{(2\ell + 1) f_{sky} \Delta \ell}} \left( C_{\ell}^{BB} + N_{\ell}^{BB} \right) , \hspace{1cm} (4.9)
\]

where \( \Delta \ell \) is the size of multipole bin. For illustrative purpose to show the angular power spectra, we set \( \Delta \ell = (i + 1)^3 - i^3 \) for \( i \)-th multipole bin. To discuss the detectability of the weak lensing signals from cosmic strings, we quantify the signal-to-noise ratio (S/N) for the angular power spectrum. For the B-mode measurement, it is defined by

\[
\left( \frac{S}{N} \right)_<^{BB} = \left[ \sum_{\ell' = 2}^\ell \frac{C_{\ell'}^{BB}}{\Delta C_{\ell'}^{BB}} \right]^{1/2} . \hspace{1cm} (4.10)
\]

Note that the signal-to-noise ratio does not depend on the size of multipole bins, but depend sensitively on the string parameters, \( P \) and \( G\mu \). To see the detectability of cosmic strings through upcoming weak lensing measurements, we consider the three representative surveys; HSC, DES, and LSST. Table 2 summarize the basic parameters for the survey designs.

Top panels in Fig. 1 show the angular power spectra for B-mode cosmic shear induced by a cosmic string network. Here, we set the fiducial values of the string parameters to
$G\mu = 10^{-8}$ and $P = 10^{-3}$. These fiducial values are still consistent with the small-scale CMB measurements via the GKS effect [74]. Typically, the B-mode spectrum has a large power with a flat shape at large angular scales $\ell \lesssim 100$, and it rapidly falls off at small angular scales. These features are irrespective of the survey design, and are determined by the properties of cosmic string network and the lensing kernel [eqs. (3.12) and (3.15) or eq. (4.6)]. On the other hand, the expected amplitude of the power spectrum depends not only on string parameters but also on the survey depth ($z_m$), and the resultant signal-to-noise ratio for B-mode spectrum is rather sensitive to the survey specification. Each panel of Fig. 1 shows the expected errors (top) and signal-to-noise ratios (bottom) for three representative surveys. As a result, a wide and deep survey with dense sampling by LSST is capable of detecting the cosmic strings with high signal-to-noise ratio $S/N \sim 30$. On the other hand, comparison of the results between HSC and DES indicate that in the cases with a limited sky coverage, a deep imaging survey has an advantage to detect the B-mode signal with high statistical significance. To see this clearly, we allow to vary the survey parameters ($z_m$, $N_g$), and the signal-to-noise ratio normalized by the sky coverage, $f_{\text{sky}}^{-1/2} (S/N)_{\ell < \ell_f}^{\text{BB}}$, is estimated. The results are shown in Fig. 2, where we set the maximum multipole to $\ell = 200$. From this, we roughly estimate the dependence of survey design as $f_{\text{sky}}^{-1/2} (S/N)_{\ell < \ell_f}^{\text{BB}} \propto z_m^{-0.7} N_2^{0.55}$.

Fig. 3 shows the potential impact of the weak lensing surveys on the search for cosmic string network. Here, varying the cosmic string parameters while keeping the fiducial survey setup in Table 2, we plot the $(S/N)_{\ell < 200}^{\text{BB}}$ as function of string tension $G\mu$ and intercommuting probability $P$. The resultant signal-to-noise ratio is rather sensitive to these two parameters, and the detectability would be enhanced for smaller $P$ and larger $G\mu$. This is mainly because the power spectrum of vector perturbation $P_{\sigma_0\sigma_0}$ roughly scales as $P_{\sigma_0\sigma_0} \propto (G\mu)^2 \xi^{-5} \propto (G\mu)^2 P^{-5/2}$, which reflects the fact that a small intercommuting probability increases the number density of string segments, and thereby the correlation length for string network is reduced. Note that through the GKS effect, the small-scale power of CMB temperature anisotropies is induced by the cosmic strings, and it roughly scales as $\propto (G\mu)^2 P^{-3/2}$ [74]. The different parameter dependence on the CMB temperature anisotropies can be understood as follows: the CMB temperature discontinuity induced by strings basically has a weak dependence on the length of the string segment $\xi$, and the power spectrum amplitude is mostly determined by the number density of the string network [74], which implicitly depends on $\xi$. As a result, the angular power spectrum for the GKS effect scales as $\propto (G\mu)^2 n_s \propto (G\mu)^2 \xi^{-3} \propto (G\mu)^2 P^{-3/2}$, leading to the different dependence on the intercommuting probability. Though the results shown here come from the specific model of a string network, these scalings are expected to be generic and would remain the same. The region covered by shade in each panel of Fig. 3 represents the parameters disfavored by the small-scale CMB measurements, which are obtained from the condition that the string-induced temperature anisotropies cannot exceed the measured power spectrum. The lower boundary of the shaded region indeed comes from the scaling $\propto (G\mu)^2 P^{-3/2}$. These different behaviors suggest that the weak lensing measurement can be a complementary probe of the cosmic strings, and is advantageous for detecting a string network with small intercommuting probability. That is, the combined analysis of the weak lensing and small-scale CMB measurements would be quite essential not only to obtain a tight constraint on string parameters, but also to break the parameter degeneracies. The precision measurement of the large angle CMB temperature anisotropies would be also helpful to obtain a tighter constraint on the parameters. We hope to come back these issues in a future publication. Fig. 3 implies that for $P \lesssim 10^{-1}$, B-mode signal of cosmic strings is detectable for a small string tension $G\mu \sim 5 \times 10^{-8}$. For theoretically
Table 2. Survey design for HSC, DES, and LSST. The sky coverage $f_{\text{sky}}$, the mean redshift $z_{\text{m}}$, and the number of the galaxies per square arcminute $N_g$ are shown.

| Survey | $f_{\text{sky}}$ | $z_{\text{m}}$ | $N_g$ [arcmin$^{-2}$] |
|--------|------------------|----------------|-----------------------|
| HSC    | 0.05 (2000 deg$^2$) | 1.0 | 35 |
| DES    | 0.125 (5000 deg$^2$) | 0.5 | 12 |
| LSST   | 0.5 (20000 deg$^2$)  | 1.5 | 100 |

Figure 1. The angular power spectra of the B-mode cosmic shear from the vector perturbations generated by the cosmic string network with $G_{\mu} = 10^{-8}$, $P = 10^{-3}$ for LSST (left panel), HSC (center panel), and DES (right panel). The error boxes in each figure show the expected variance of angular power spectrum from each experiments. The bottom panels show the signal-to-noise ratio as a function of maximum multipole.

Inferred smallest value $P \sim 10^{-3}$, we could even detect the signal for $G_{\mu} \sim 5 \times 10^{-10}$.

4.2.2 Curl-mode deflection angle

Let us next consider the curl-mode signals from the CMB measurements. We calculate the expected curl-mode signal and the signal-to-noise ratio for the upcoming and idealistic CMB experiments. We consider the combination of small- and large-scale CMB measurements by ACTPol and PLANCK (ACTPol+PLANCK) for a representative upcoming/on-going experiment, and the high-resolution full-sky experiment limited by the cosmic variance, just for illustrative purpose. We assume that the curl-mode deflection angle is reconstructed from the lensed CMB map based on the quadratic reconstruction technique [107–110]. Similar to the B-mode cosmic shear, we define the signal-to-noise ratio for curl-mode deflection angle:

$$\left( \frac{S}{N} \right)_{\text{curl}} \propto \left[ \sum_{\ell'=2}^{\ell} \left( \frac{C_{\ell'\ell'}}{\Delta C_{\ell'\ell'}} \right)^2 \right]^{1/2},$$

(4.11)
Figure 2. The contour of the signal-to-noise ratio, $f_{\text{sky}}^{-1/2}(S/N)_{c}^{\text{BB}}$, as the function of $z_m$ and $N_g$. For the string components, we take $G\mu = 10^{-8}$, $P = 10^{-3}$.

Figure 3. The contours of the signal-to-noise ratio, $(S/N)_{c}^{\text{BB}}$, as the function of the tension $G\mu$ and the intercommuting probability $P$ for LSST (left panel), HSC (center panel), and DES (right panel). The shaded region is excluded from the GKS effect [74].

with the error $\Delta C_{\ell}^{\text{ps}}$ given by

$$\Delta C_{\ell}^{\text{ps}} = \sqrt{\frac{2}{(2\ell + 1)f_{\text{sky}}\Delta \ell}} \left( C_{\ell}^{\text{ps}} + N_{\ell}^{\text{ps},(c)} \right).$$

(4.12)

Here, $N_{\ell}^{\text{ps},(c)}$ is the reconstruction noise spectrum for the optimal combination of the quadratic estimator [71]. In Appendix E, the explicit expression for the noise spectrum of the curl mode is presented, and its dependence on the experimental specification is briefly summarized.
Figure 4. The angular power spectrum of the curl-mode deflection angle from the vector perturbations generated by the cosmic string network with $G \mu = 10^{-8}$, $P = 10^{-3}$. The error boxes represent the expected variance of angular power spectrum from ACTPol+PLANCK (empty red), CV-limit (shaded green), with the multipole used in the reconstruction procedure, $\ell_{\text{max}} = 7000$ (see [71] and Appendix E). Just for illustration, we set the size of multipole bins to $\Delta \ell = (i+1)^3 - i^3$ for $i$-th bin. The bottom panel shows the signal-to-noise ratio as a function of maximum multipole for each survey.

Fig. 4 shows the expected curl-mode signal for the cosmic strings. Top and bottom panels respectively plot the angular power spectrum and the signal-to-noise ratio for the pseudo-scalar lensing potential, $\varpi$, assuming the string parameters $G \mu = 10^{-8}$ and $P = 10^{-3}$. At large-angular scales, the curl-mode signal is prominent and has the largest amplitude, but the power spectrum rapidly falls off at small scales. Considering the fact that a measurement of string-induced CMB anisotropies via the GKS effect is only available at small scales, CMB-lensing experiment can be also a complementary probe, and would be more suited for the detection of a cosmic string network. Fig. 4 suggests that for a definite detection with $S/N \gtrsim 10$, a full-sky lensing experiment would be ideal and the best, but even with the upcoming experiment of ACTPol+PLANCK, we can detect the signature of cosmic strings with $S/N \sim 3$. Recalling that there would be additional contributions to the angular power spectrum, leading to an enhancement of the power spectrum amplitude, the results shown here should be regarded as a rather conservative estimate, and the actual detectability might be increased. Finally, Fig. 5 shows the dependence of signal-to-noise ratio on the string parameters $G \mu$ and $P$. Similar to the B-mode cosmic shear, the signal-to-noise ratio for curl-mode signal scales as $(S/N)_{\ell} \propto (G \mu)^2 P^{-5/2}$, and the CMB-lensing experiment is capable of detecting a string network with small $P$. Since the small-scale CMB experiment is usually dominated by the contributions from point sources and the Sunyaev-Zel’dovich (SZ) effect, the curl-mode measurement would provide not only a direct probe of cosmic strings, but also a diagnosis helpful to check the systematics in the derived constraints from the GKS effect.
5 Summary

In this paper, we have discussed the observational signature of the vector metric perturbations through the effect of weak gravitational lensing. In the presence of vector perturbations, the non-vanishing signals for B-mode cosmic shear and curl-mode deflection angle naturally appears, and these would be a unique signature of vector perturbations. Solving the geodesic and geodesic deviation equations, we have derived the full-sky formulas for angular power spectra of weak lensing signals, and give the explicit expressions for E-/B-mode cosmic shear [expression (3.12) with eqs. (3.14)-(3.18)], and gradient-/curl-mode deflection angle [expression (3.24) with eqs. (3.26) and (3.27)].

As a possible source for seeding vector perturbations, we then considered a cosmic string network, and discuss its detectability from upcoming weak lensing and CMB measurements. Based on the formulas and a simple model for cosmic string network, we calculated the angular power spectra, and the expected signal-to-noise ratios for the B-mode cosmic shear and curl-mode deflection angle were estimated. The string-induced signals typically have a large power at large-angular scales, and we found that the signals with small intercommuting probability $P$ are detectable from future lensing experiments. With the theoretically inferred smallest value $P \sim 10^{-3}$, we could even detect the cosmic strings with $G\mu \sim 5 \times 10^{-10}$. Therefore, the weak lensing measurement of the B-mode cosmic shear and curl-mode deflection angle would be an important probe for cosmic string network, and is complementary to the small-scale CMB experiment via the GKS effect.

Throughout the paper, we have assumed several idealizations; the weak lensing measurement are perfect without annoying masking effect, and free from the foreground contaminations. In practice, the B-mode cosmic shear would be contaminated by the E/B-mode mixing arising from the incomplete sky coverage [111], intrinsic alignment of galaxy images induced by the gravitational clustering [112], and the shear-intrinsic ellipticity correlations [113, 114]. Also, the curl-mode deflection angle from the CMB maps is affected not only by the point sources and SZ effect [115, 116], but also by the masking effect [117, 118] and the inhomog-
geneous noises \cite{119}. As for the detection of cosmic string network, we have discussed the expected weak lensing signals based on a very simple model, and the model prediction should be further improved for future application to the lensing measurements. There are several missing pieces, including the contribution of tensor perturbations \cite{120–124} and the correlation between different string segments \cite{74}, which may enhance the signal-to-noise ratio. We hope to come back these issues near future.

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A Useful formula

We summarize the formulas used in this paper.

A.1 Spherical Bessel function

The spherical Bessel functions, $j_\ell(x)$, are solutions to the differential equation:

\[ j''_\ell(x) + \frac{2}{x} j'_\ell(x) + \left( 1 - \frac{\ell(\ell+1)}{x^2} \right) j_\ell(x) = 0. \]  

(A.1)

The recursion relations of spherical Bessel functions are given by

\[ \frac{j_\ell(x)}{x} = \frac{1}{2\ell + 1} \left\{ j_{\ell-1}(x) + j_{\ell+1}(x) \right\}, \]  

(A.2)

\[ j'_\ell(x) = \frac{1}{2\ell + 1} \left\{ \ell j_{\ell-1}(x) + (\ell + 1)j_{\ell+1}(x) \right\} = \ell \frac{j_\ell(x)}{x} - j_{\ell+1}(x). \]  

(A.3)

A.2 Legendre polynomials

The associated Legendre functions are defined in terms of the Legendre polynomials by

\[ P_{\ell,m}(\mu) = (-1)^m (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_{\ell}(\mu), \]  

(A.4)

where the Legendre polynomials are solutions to the differential equation:

\[ \frac{d}{d\mu} \left( (1 - \mu^2) \frac{d}{d\mu} P_{\ell}(\mu) \right) + \ell(\ell + 1) P_{\ell}(\mu) = 0. \]  

(A.5)

We then take the integration of the associated Legendre functions:

\[ \int_{-1}^{1} d\mu \sqrt{1 - \mu^2} P_{\ell+1}(\mu) e^{-ix\mu} = - \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right) e^{-ix\mu} \frac{d}{d\mu} P_{\ell}(\mu) \]

\[ = -i \left( 2\partial_x + x \left( 1 + \partial_x^2 \right) \right) \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right) e^{-ix\mu} P_{\ell}(\mu) \]

\[ = 2(-i)^{\ell+1} \left( 2\partial_x + x \left( 1 + \partial_x^2 \right) \right) j_\ell(x) \]

\[ = 2(-i)^{\ell+1} \frac{(\ell + 1)!}{(\ell - 1)!} \frac{j_\ell(x)}{x}, \]  

(A.6)

where we have used the differential equation for the spherical Bessel function eq. (A.1).
A.3 Spherical harmonics

A scalar field on the sphere can be expanded in (spin-0) spherical harmonics, \( Y_{\ell m}(\hat{n}) \). Spherical harmonics can be written in terms of the Legendre polynomials as

\[
Y_{\ell m}(\hat{n}) = \sqrt{\frac{2\ell + 1}{4\pi}} \frac{(\ell - m)!}{(\ell + m)!} P_{\ell m}(\mu) e^{im\varphi},
\]

where \( \mu = \cos \theta \). Then one can verify the angular integration of the spherical harmonics:

\[
\int d^2 \hat{n} Y_{\ell m}^* (\hat{n}) e^{\pm i \varphi} e^{-ix\mu} = \int_{-1}^{1} d\mu \sqrt{1 - \mu^2} \int_{0}^{2\pi} d\varphi Y_{\ell m}^* (\hat{n}) e^{\pm i \varphi} e^{-ix\mu}
\]

\[
= \pm 2\pi \delta_{\ell m, \pm 1} \sqrt{\frac{2\ell + 1}{4\pi}} \frac{(\ell - 1)!}{(\ell + 1)!} \int_{-1}^{1} d\mu \sqrt{1 - \mu^2} P_{\ell, \pm 1}(\mu) e^{-ix\mu}
\]

\[
= \pm (-i)^{\ell + 1} \delta_{\ell m, \pm 1} \sqrt{4\pi (2\ell + 1)} \frac{(\ell + 1)!}{(\ell - 1)!} \frac{j_\ell(x)}{x}, \tag{A.8}
\]

where we have used the angular integration of the Legendre polynomials, eq. (A.6).

B Christoffel symbols and Riemann tensors

The Christoffel symbols on the unperturbed spacetime, namely Minkowski spacetime, in the Cartesian coordinate system are trivially \( \Gamma_{\mu \nu}^\rho = 0 \). Since the Christoffel symbols are not covariant quantities, the unperturbed Christoffel symbols in the spherical coordinate system can have the components:

\[
\Gamma_{ab}^\chi = -\chi \omega_{ab}, \quad \Gamma_{\chi b}^a = \frac{1}{\chi} \delta_{a b}, \quad \Gamma_{bc}^a = (2) \Gamma_{bc}^a, \quad \text{otherwise} = 0, \tag{B.1}
\]

where \( a, b, c = \theta, \varphi \) and the two-dimensional Christoffel symbols are

\[
^{(2)} \Gamma_{\varphi \varphi}^\theta = -\sin \theta \cos \theta, \quad ^{(2)} \Gamma_{\theta \varphi}^\varphi = \cot \theta, \quad \text{otherwise} = 0. \tag{B.2}
\]

We can calculate the linearized Christoffel symbols as \( \delta \Gamma_{\mu \nu}^\rho = \frac{1}{2} g^{\rho \sigma} (h_{\mu \nu} + h_{\sigma \mu} h_{\nu \sigma} - h_{\mu \nu}) \). Hence we have the components of the linearized Christoffel symbols induced by the vector perturbations, \( h_{0i} = -\sigma_{g,i} \), as

\[
\delta \Gamma_{00}^i = -\sigma_{g,i}, \quad \delta \Gamma_{0j}^i = \frac{1}{2} (\sigma_{g,j} |^i - \sigma_{g,i} |^j), \quad \delta \Gamma_{ij}^0 = \frac{1}{2} (\sigma_{g,i |j} + \sigma_{g,j |i}), \quad \text{otherwise} = 0. \tag{B.3}
\]

Recalling that the geodesic in the background spacetime can be solved as \( x^\mu(\chi) = (\eta_0 - \chi, \chi, e_\chi) \), we can calculate

\[
\delta \Gamma_{\mu \nu}^\rho \frac{dx^\mu}{d\chi} \frac{dx^\nu}{d\chi} = \delta \Gamma_{00}^i - 2\delta \Gamma_{0j}^i e_j^\chi + \delta \Gamma_{ij}^k e_k^\chi e_j^\chi = -\sigma_{g}^i + \sigma_{g,i} |^j e_j^\chi - \sigma_{g,j} |^i e_j^\chi. \tag{B.4}
\]

Multiplying \( e_i^a \) in both side of eq. (B.4) and using the definition of the covariant derivative associated with \( \omega_{ab} \), eqs. (2.12), (2.13), we obtain the angular component of eq. (B.4):

\[
e_i^a \delta \Gamma_{\mu \nu}^\rho \frac{dx^\mu}{d\chi} \frac{dx^\nu}{d\chi} = \frac{1}{\chi} \omega_{ab} \left( \frac{d}{d\chi} (\chi \sigma_{g,b}) -(0 \sigma_{g}, b) \right), \tag{B.5}
\]
where we have introduced $d/d\chi = e^{i}_X \partial_{\eta} - \partial_{\eta}$, $\sigma_{g,i} = \sigma_{g,i} e^{i}_X$, $\sigma_{g,a} = \sigma_{g,a} e^{a}_X$.

Since the background geometry is Minkowski spacetime, the Riemann tensor and the symmetric optical tidal matrix at the zeroth-order in metric perturbations are trivially given as $R_{\mu\rho\sigma}(g_{\alpha\beta}) = 0$ and $\mathcal{T}^{\alpha}_b(g_{\alpha\beta}) = 0$. Using the explicit expression for the linearized Riemann tensor as $\delta R_{\mu\rho\sigma} = \frac{1}{2} ( - h_{\mu\nu,\rho\sigma} - h_{\mu\sigma,\nu\rho} + h_{\nu\sigma,\mu\rho} + h_{\nu\rho,\mu\sigma} )$, we have

$$\delta R_{0i0j} = - \frac{1}{2} \left( \sigma_{g,i|j} + \sigma_{g,j|i} \right), \quad \delta R_{ik0j} = \frac{1}{2} \left( \sigma_{g,i|k} - \sigma_{g,k|i} \right), \quad \delta R_{ijk0} = 0.$$ (B.6)

With a help of eqs. (2.12), (2.13), we can calculate the linearized symmetric optical tidal matrix as

$$\delta \mathcal{T}_{ab} = - \frac{1}{E^2} \delta R_{\mu\rho\sigma} k^\mu k^\nu \epsilon^a_{\rho} \epsilon^b_{\sigma} = - \delta R_{0i0j} \epsilon^{i}_a \epsilon^{j}_b + 2 \delta R_{ki0j} \epsilon^{k}_a \epsilon^{i}_b$$

$$= e^k_X \epsilon^{i}_a \epsilon^{j}_b \sigma_{g,k|i} - \frac{d}{d\chi} \left( \epsilon^{i}_a \epsilon^{j}_b \sigma_{g,i|j} \right)$$

$$= \frac{1}{\chi^2} \left\{ (\sigma_{g})_{ab} - \frac{d}{d\chi} \left( \chi \sigma_{g,(a:b)} \right) + \chi \omega_{ab} (\sigma_{g}) \right\}. \quad \text{(B.7)}$$

C Derivation of angular power spectrum

In this section, we provide details of the derivation of the full-sky formulas for the angular power spectrum for the E-/B-mode cosmic shear and the gradient-/curl-mode deflection angle generated by the vector perturbations. We first define the spin-raising/lowering operators, spin-weighted spherical harmonics and present the explicit relation between the intrinsic covariant derivative on the unit sphere and the spin-raising/lowering operators in section C.1. In section C.2 and C.3 we present the formula for the angular power spectrum for the E-/B-mode cosmic shear and the gradient-/curl-mode deflection angle generated by the vector perturbations, following [125, 126]. In section C.4 we derive the explicit relation between the cosmic shear field and the deflection angle.

C.1 Spin operators and spin-weighted spherical harmonics

We define a pair of operator $\vartheta$ and $\vartheta$, known as spin-raising and lowering operators, respectively. These operators have the properties of increasing or decreasing the index of the spins by 1. For a spin-$s$ function $sX$, their explicit forms are [125, 126]

$$\vartheta(sX) \equiv - \sin^s \theta \left( \partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} \right) \sin^{-s} \theta (sX), \quad \text{(C.1)}$$

$$\vartheta(sX) \equiv \sin^{-s} \theta \left( \partial_{\theta} - \frac{i}{\sin \theta} \partial_{\phi} \right) \sin^{s} \theta (sX). \quad \text{(C.2)}$$

Since we are interested in the spin-$\pm 2$ quantities, acting twice with the spin-raising/lowering operators on the spin-$\pm 2$ fields, $\pm 2X$, gives

$$\vartheta^2 (\pm 2X) = \left( - \partial_{\mu} \mp \frac{i}{1 - \mu^2} \partial_{\phi} \right)^2 \left( 1 - \mu^2 \right) (\pm 2X), \quad \text{(C.3)}$$

$$\vartheta^2 (\pm 2X) = \left( - \partial_{\mu} \pm \frac{i}{1 - \mu^2} \partial_{\phi} \right)^2 \left( 1 - \mu^2 \right) (\pm 2X). \quad \text{(C.4)}$$
where we have used the directional cosine \( \mu = \cos \theta \) and \( \partial_\mu = \partial / \partial \mu \). To see the relation between the intrinsic covariant derivatives on the unit sphere and the spin-raising/lowering operators, one can verify

\[
\chi e_i^\pm \partial_\mu e_i^\pm = \cot \theta e_i^\pm, \quad \chi e_i^\pm \partial_\mu e_i^\pm = -2e_i^\pm - \cot \theta e_i^\pm.
\]  

(C.5)

where we have used eq. (2.12). In terms of the spin basis, these are reduced to

\[
e^a_\pm b e^b_\pm = \cot \theta e^a_\pm, \quad e^a_\pm b e^b_\pm = - \cot \theta e^a_\pm.
\]  

(C.6)

A spin-s function, \( sX \), can be written in terms of the spin basis and a symmetric trace-free rank-s tensor, \( X_{a_1 \ldots a_s} \), as

\[
sX = X_{a_1 \ldots a_s} e^a_+ \cdots e^a_s, \quad (s \geq 0), \quad sX = X_{a_1 \ldots a|s|} e^a_+ \cdots e^a_{|s|}, \quad (s < 0),
\]  

(C.7)

with \( e^a_\pm \equiv e^a_1 e^a_\pm \). With these notations, we can easily prove the following useful relations:

\[
\langle 0X \rangle a_+ e^a_+ = -\bar{\phi} (0X), \quad \langle 0X \rangle a e^a = -\bar{\phi} (0X),
\]  

(C.8)

\[
X_{ab} e^a_+ e^b_+ = -\bar{\phi} (+1X), \quad X_{ab} e^a_- e^b_- = -\bar{\phi} (-1X),
\]  

(C.9)

\[
X_{ab} e^a_+ e^b_+ = -\bar{\phi} (0X), \quad X_{ab} e^a_- e^b_- = -\bar{\phi} (-1X),
\]  

(C.10)

\[
\langle 0X \rangle_{ab} e^a_+ e^b_+ = \bar{\phi}^2 (0X), \quad \langle 0X \rangle_{ab} e^a_- e^b_- = \bar{\phi}^2 (0X),
\]  

(C.11)

\[
\langle 0X \rangle_{ab} e^a_+ e^b_+ = \bar{\phi}\bar{\phi} (0X) = \bar{\phi}\bar{\phi} (0X).
\]  

(C.12)

With the spin operators, one can express the spin-weighted spherical harmonics, \( sY_{\ell m} \), in terms of the spin-0 spherical harmonics, \( Y_{\ell m} \),

\[
sY_{\ell m}(\hat{n}) = \sqrt{(\ell - s)! \ell + s)!} \bar{\phi}^s Y_{\ell m}(\hat{n}),
\]  

(C.13)

for \( 0 \leq s \leq \ell \), and

\[
sY_{\ell m}(\hat{n}) = \sqrt{(\ell + s)! \ell + s)!} \bar{\phi}^{-s} Y_{\ell m}(\hat{n}),
\]  

(C.14)

for \( -\ell \leq s \leq 0 \). One can also see the following useful properties of the spin-weighted spherical harmonics:

\[
\bar{\phi} \left(sY_{\ell m}(\hat{n})\right) = \sqrt{(\ell - s)(\ell + s + 1)} s+1Y_{\ell m}(\hat{n}),
\]  

(C.15)

\[
\bar{\phi} \left(sY_{\ell m}(\hat{n})\right) = -\sqrt{(\ell + s)(\ell - s + 1)} s-1Y_{\ell m}(\hat{n}).
\]  

(C.16)

### C.2 E-/B-mode cosmic shear

Since we are interested in the spin-\(\pm 2\) quantities, \( g \) and \( g^* \), it is convenient to introduce the spin-0 quantities, which are constructed from the spin-\(\pm 2\) quantities. For real space calculations it is useful to introduce the spin-0 quantities, \( \bar{\phi}^2 g(\hat{n}) \) and \( \bar{\phi}^2 g^*(\hat{n}) \) [125, 126]:

\[
\bar{\phi}^2 g(\hat{n}) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} 2g_{\ell m} \bar{\phi}^2 \left(+2Y_{\ell m}(\hat{n})\right) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{(\ell + 2)! (\ell - 2)!} \bar{\phi}^2 g_{\ell m} Y_{\ell m}(\hat{n}),
\]  

(C.17)

\[
\bar{\phi}^2 g^*(\hat{n}) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} -2g_{\ell m} \bar{\phi}^2 \left(-2Y_{\ell m}(\hat{n})\right) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{(\ell + 2)! (\ell - 2)!} -2g_{\ell m} Y_{\ell m}(\hat{n}),
\]  

(C.18)
where we have used eqs. (3.6), (C.15) and (C.16). Furthermore, we can introduce \( \tilde{g}^E(n) \) and \( \tilde{g}^B(n) \), constructed from the spin-0 quantities, \( \vartheta^2g \) and \( \vartheta^2g^* \):

\[
\tilde{g}^E(n) = -\frac{1}{2} \left( \vartheta^2 g(n) + \vartheta^2 g^*(n) \right) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{g}_{\ell m}^E Y_{\ell m}(n),
\]

\[
\tilde{g}^B(n) = -\frac{1}{2i} \left( \vartheta^2 g(n) - \vartheta^2 g^*(n) \right) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{g}_{\ell m}^B Y_{\ell m}(n).
\]

Combining with eqs. (C.17)-(C.20), one can see the explicit relation between the multipole coefficients of the spin-0 quantities, \( \tilde{g}^E \) and \( \tilde{g}^B \), and of the spin-\pm2 quantities, \( g^E \) and \( g^B \) (see eq. (3.7)), as

\[
\tilde{g}_{\ell m}^E = -\frac{1}{2} \sqrt{\frac{(\ell + 2)!!}{(\ell - 2)!!}} \left( \pm 2g_{\ell m} + 2g_{\ell m} \right) = \sqrt{\frac{(\ell + 2)!!}{(\ell - 2)!!}} \tilde{g}_{\ell m}^E,
\]

\[
\tilde{g}_{\ell m}^B = -\frac{1}{2i} \sqrt{\frac{(\ell + 2)!!}{(\ell - 2)!!}} \left( \pm 2g_{\ell m} - 2g_{\ell m} \right) = \sqrt{\frac{(\ell + 2)!!}{(\ell - 2)!!}} \tilde{g}_{\ell m}^B.
\]

Once we obtain the multipole coefficients of the spin-0 quantities, the E-/B-mode cosmic shear can be calculated by using eqs. (C.21) and (C.22).

Let us now derive the explicit expression for the reduced shear and E-/B-mode angular power spectra in terms of spin-weighted quantities. Using eqs. (C.9), (C.11), we can rewrite eq. (3.9) in terms of the spin operators:

\[
g = -\frac{1}{2} \int_0^{\pi} d\chi \left\{ \frac{\chi_8 - \chi_5}{\chi_8} \vartheta^2 (0\sigma_g) + \vartheta (1\sigma_g) \right\},
\]

where \( 0\sigma_g = \sigma_g, e^i_\chi \), and \( 1\sigma_g = \sigma_g, e^i_\pm \). To compute the angular power spectrum, we fix the coordinate system. Without loss of generality, we can always choose coordinates such that \( k^i \equiv k^i k = k(0, 0, 1) \), \( e^i_1(k) = (1, \pm i, 0) \). It is useful to introduce the directional cosine \( \mu = \hat{k}^i e^i_\chi(n) = \cos \theta \) rather than \( \theta \). We then evaluate

\[
e^i_\chi, (n) e^{i_\pm} (k) = \sqrt{1 - \mu^2} e^{\pm i\varphi},
\]

\[
e^i_{\pm, i}(n) e^{i_\pm}_\pm (k) = (\mu \mp 1) e^{\pm i\varphi},
\]

Decomposing the vector perturbations into the Fourier modes (see eq. (3.10)), we multiply \( \sigma_g, i \) by \( e^i_1(n) \) and \( e^i_\pm(n) \). With a help of eqs. (C.24), (C.25), we obtain

\[
0\sigma_g \equiv \sigma_g, i e^i_\chi = \int \frac{d^3k}{(2\pi)^3} e^{-ik\chi} \sqrt{1 - \mu^2} \left( \sigma_g^{(1)}(k, \eta) e^{i\varphi} + \sigma_g^{(-1)}(k, \eta) e^{-i\varphi} \right),
\]

\[
\pm 1\sigma_g \equiv \sigma_g, i e^i_\pm = \int \frac{d^3k}{(2\pi)^3} e^{-ik\chi} \left( \sigma_g^{(1)}(k, \eta) (\mu \mp 1) e^{i\varphi} + \sigma_g^{(-1)}(k, \eta) (\mu \pm 1) e^{-i\varphi} \right).
\]
Since the quantities $0\sigma_g$ and $\pm 1\sigma_g$ transform as spin-0 and $\pm 1$ quantities, acting with the spin-raising/lowering operators, eq. (C.1), on $0\sigma_g$ and $\pm 1\sigma_g$ leads to

$$
\phi^2 (0\sigma_g) = (1 - \mu^2) \left( -\partial_\mu + \frac{i}{1 - \mu^2} \partial_\varphi \right)^2 (0\sigma_g)
$$

$$
= \int \frac{d^3k}{(2\pi)^3} \left\{ \left( \partial_\mu e^{-ik\chi_\mu} \right) (1 - \mu^2)^{3/2} \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right) - 2 \left( \partial_\mu e^{-ik\chi_\mu} \right) \sqrt{1 - \mu^2} \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right) \right\}, \tag{C.28}
$$

$$
\phi^2 (\pm 1\sigma_g) = (1 - \mu^2) \left( -\partial_\mu + \frac{i}{1 - \mu^2} \partial_\varphi \right)^2 (\pm 1\sigma_g)
$$

$$
= \int \frac{d^3k}{(2\pi)^3} \left\{ \left( \partial_\mu e^{-ik\chi_\mu} \right) (1 - \mu^2)^{3/2} \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right) - 2 \left( \partial_\mu e^{-ik\chi_\mu} \right) \sqrt{1 - \mu^2} \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right) \right\}, \tag{C.29}
$$

and

$$
\phi (\pm 1\sigma_g) = - (1 - \mu^2) \left( -\partial_\mu + \frac{i}{1 - \mu^2} \partial_\varphi \right) \frac{1}{\sqrt{1 - \mu^2}} (\pm 1\sigma_g)
$$

$$
= \int \frac{d^3k}{(2\pi)^3} \left( \partial_\mu e^{-ik\chi_\mu} \right) \sqrt{1 - \mu^2} \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right), \tag{C.30}
$$

$$
\phi (0\sigma_g) = - (1 - \mu^2) \left( -\partial_\mu + \frac{i}{1 - \mu^2} \partial_\varphi \right) \frac{1}{\sqrt{1 - \mu^2}} (0\sigma_g)
$$

$$
= \int \frac{d^3k}{(2\pi)^3} \left( \partial_\mu e^{-ik\chi_\mu} \right) \sqrt{1 - \mu^2} \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right), \tag{C.31}
$$

where $\partial_\mu \equiv \partial/\partial \mu$. Plugging eqs. (C.28) and (C.30) into eq. (C.23), we obtain the reduced shear induced by vector perturbations:

$$
g = -\frac{1}{2} \int_0^{\chi_S} \frac{d\chi}{\chi} \int_0^{\chi_S} \frac{d\chi}{\chi} \left\{ \frac{\chi S - \chi}{\chi S} \left( \partial_\mu e^{-ik\chi_\mu} \right) \left( \sigma^{(1)}_g (\mu - 1) e^{i\varphi} + \sigma^{(-1)}_g (\mu + 1) e^{-i\varphi} \right) \right\}. \tag{C.32}
$$

Following the same step as derived in the case of the spin-+2 part, we multiply $e^a e^b$ in both side of eq. (2.27) to construct the spin-+2 part of the reduced shear eq. (3.5):

$$
g^* = -\frac{1}{2} \int_0^{\chi_S} \frac{d\chi}{\chi} \left\{ \frac{\chi S - \chi}{\chi S} \phi^2 (0\sigma_g) + \phi (0\sigma_g) \right\}, \tag{C.33}
$$

where we have used eqs. (C.9), (11). Plugging eqs. (29) and (31) into eq. (C.33), we have

$$
g^* = -\frac{1}{2} \int_0^{\chi_S} \frac{d\chi}{\chi} \left\{ \frac{\chi S - \chi}{\chi S} \phi^2 (0\sigma_g) + \phi (0\sigma_g) \right\}. \tag{C.34}
$$
We use the spin-0 quantities, $\hat{g}^E$ and $\hat{g}^B$ defined in eqs. (C.19) and (C.20), to compute the angular power spectrum. Since $g$ and $g^*$ transform as spin-$\pm 2$ quantities, we can apply the formula eqs. (C.3), (C.4) to eqs. (C.32) and (C.34) and calculate $\hat{g}^E$ and $\hat{g}^B$ as

\[
\hat{g}^E = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \sqrt{1 - \mu^2} \left[ \left( \sigma_\ell(x) + \left[ 1 - 2\frac{\chi_S - \chi}{\chi_S} \right] \hat{E}_\ell(x) \right) e^{-i x \mu} \right]_{x = k \chi},
\]

\[
\hat{g}^B = - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \sqrt{1 - \mu^2} \left( \sigma_\ell(x) - \left[ 1 - 2\frac{\chi_S - \chi}{\chi_S} \right] \hat{B}_\ell(x) \right) e^{-i x \mu} \left|_{x = k \chi} \right.,
\]

where we have introduced the operators $\hat{E}_{0,1}(x)$ and $\hat{B}(x)$ satisfying the following equations:

\[
\left( -\partial_\mu \pm \frac{1}{1 - \mu^2} \right)^2 \left[ \left( 1 - \mu^2 \right)^{3/2} \partial_\mu e^{-i x \mu} \right] = \sqrt{1 - \mu^2} \left( \sigma_\ell(x) \mp 2i \delta \hat{B}(x) \right) e^{-i x \mu},
\]

\[
\left( -\partial_\mu \pm \frac{1}{1 - \mu^2} \right)^2 \left[ \left( 1 - \mu^2 \right)^{3/2} \partial_\mu e^{-i x \mu} \right] = \sqrt{1 - \mu^2} \left( \sigma_\ell(x) \mp i \delta \hat{B}(x) \right) e^{-i x \mu}.
\]

It follows that

\[
\hat{E}_0(x) = x^2 \left[ 4 + 20\partial_\mu^2 + 10x (\partial_\mu + \partial_\mu^2) + x^2 (1 + \partial_\mu^2)^2 \right],
\]

\[
\hat{E}_1(x) = x \left[ 12\partial_\mu - x^2 (\partial_\mu^2 + \partial_\mu) - 4x (1 + 2\partial_\mu^2) \right],
\]

\[
\hat{B}(x) = x^2 \left[ 4\partial_\mu + x (1 + \partial_\mu^2) \right].
\]

To obtain the multipole coefficients of $\hat{g}^E$ and $\hat{g}^B$, we perform the angular integration using eqs. (A.8):

\[
\hat{g}^E_{\ell m} = \int d^2 \hat{n} Y^*_{\ell m}(\hat{n}) \hat{g}^E(\hat{n})
\]

\[
= (-i)^{\ell+1} \sqrt{4\pi(2\ell + 1)(\ell + 1)!} \int \frac{d^3k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \left( \sigma_\ell(x) \delta_{\ell m, 1} - \sigma_\ell^{-1}(x) \delta_{\ell m, -1} \right)
\]

\[
\times \frac{1}{2} \left\{ \frac{\chi_S - \chi}{\chi_S} \hat{E}_0(x) + \left( 1 - 2\frac{\chi_S - \chi}{\chi_S} \right) \hat{E}_1(x) \right\} \left. \frac{j_\ell(x)}{x} \right|_{x = k \chi},
\]

\[
\hat{g}^B_{\ell m} = \int d^2 \hat{n} Y^*_{\ell m}(\hat{n}) \hat{g}^B(\hat{n})
\]

\[
= - (-i)^{\ell+1} \sqrt{4\pi(2\ell + 1)(\ell + 1)!} \int \frac{d^3k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \left( \sigma_\ell(x) \delta_{\ell m, 1} + \sigma_\ell^{-1}(x) \delta_{\ell m, -1} \right)
\]

\[
\times \left. \frac{1}{2} \hat{B}(x) \frac{j_\ell(x)}{x} \right|_{x = k \chi}.
\]
To proceed, we act with $\hat{\chi}_{0,1}(x)$ and $\hat{B}(x)$ on $j_\ell(x)/x$:

\[
\hat{\chi}_{0,1}(x) \frac{j_\ell(x)}{x} = \left\{ \frac{(\ell - 1)!}{(\ell + 1)!} \left\{ \frac{\ell + 2}{x} - 2j_\ell'(x) \right\} \right\}, \quad (C.44)
\]

\[
\hat{\chi}_{1}(x) \frac{j_\ell(x)}{x} = -\left\{ \frac{(\ell - 2)!}{(\ell + 2)!} \left\{ \frac{j_\ell(x)}{x} + j_\ell'(x) \right\} \right\}, \quad (C.45)
\]

\[
\hat{B}(x) \frac{j_\ell(x)}{x} = \frac{(\ell - 1)!}{(\ell + 2)!} j_\ell(x), \quad (C.46)
\]

where we have used the differential equation for the spherical Bessel function (see Appendix A.1). Since the multipole coefficients of the E-/B-mode cosmic shear are directly related to those of $g^E$ and $g^B$ through eqs. (C.21) and (C.22), we obtain

\[
E \frac{g_{\ell m}}{g_{\ell m}} = \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \frac{E}{g_{\ell m}}
\]

\[
= (-i)^{\ell+1} \sqrt{\pi(2\ell + 1)} \left\{ \frac{\ell - 1)!}{(\ell + 1)!} \right\} \int d^3k \int_0^{\chi_S} \frac{d\chi}{\chi} \left( \sigma_g^{(+)} \delta_{m,1} - \sigma_g^{(-)} \delta_{m,-1} \right)
\]

\[
\times \left\{ \chi_S - \chi \ell(\ell + 1) \frac{j_\ell(x)}{x} - \left\{ \frac{j_\ell(x)}{x} + j_\ell'(x) \right\} \right\} \bigg|_{x = k} \chi_S
\]

\[
= (-i)^{\ell+1} \sqrt{4\pi(2\ell + 1)} \int \frac{d^3k}{(2\pi)^3} \int_0^{\chi_S} k \chi \left( \sigma_g^{(+)} \delta_{m,1} - \sigma_g^{(-)} \delta_{m,-1} \right) S^\text{vector}_{E,\ell}(k, \chi),
\]

and

\[
B \frac{g_{\ell m}}{g_{\ell m}} = \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \frac{B}{g_{\ell m}}
\]

\[
= (-i)^{\ell+1} \sqrt{\pi(2\ell + 1)} \left\{ \frac{\ell - 1)!}{(\ell + 1)!} \right\} \int d^3k \int_0^{\chi_S} \frac{d\chi}{\chi} \left( \sigma_g^{(+)} \delta_{m,1} + \sigma_g^{(-)} \delta_{m,-1} \right) j_\ell(k \chi)
\]

\[
= (-i)^{\ell+1} \sqrt{4\pi(2\ell + 1)} \int \frac{d^3k}{(2\pi)^3} \int_0^{\chi_S} k \chi \left( \sigma_g^{(+)} \delta_{m,1} + \sigma_g^{(-)} \delta_{m,-1} \right) S^\text{vector}_{B,\ell}(k, \chi).
\]

where $S^\text{vector}_{E,\ell}$, $S^\text{vector}_{B,\ell}$ are defined as

\[
S^\text{vector}_{E,\ell}(k, \chi) = \frac{\chi_S - \chi}{\chi_S} \epsilon^{(0)}_\ell(k \chi) - \epsilon^{(1)}_\ell(k \chi),
\]

\[
S^\text{vector}_{B,\ell}(k, \chi) = \beta^{(1)}_\ell(k \chi),
\]
with the coefficients $\epsilon^{(0,1)}_\ell$ and $\beta^{(1)}_\ell$ given by
\[\epsilon^{(0)}_\ell (x) = \frac{1}{2} \sqrt{\frac{(\ell - 1)! (\ell + 2)!}{(\ell + 1)! (\ell - 2)!}} \frac{\ell!}{x^2} j_\ell(x), \quad (C.51)\]
\[\epsilon^{(1)}_\ell (x) = \frac{1}{2} \sqrt{\frac{(\ell - 1)! (\ell + 2)!}{(\ell + 1)! (\ell - 2)!}} \left( \frac{j_\ell(x)}{x^2} + \frac{j'_\ell(x)}{x} \right), \quad (C.52)\]
\[\beta^{(1)}_\ell (x) = \frac{1}{2} \sqrt{\frac{(\ell - 1)! (\ell + 2)!}{(\ell + 1)! (\ell - 2)!}} \frac{j_\ell(x)}{x}. \quad (C.53)\]

Substituting eqs. (C.47), (C.48) into eq. (3.8), we obtain the final expression for the angular power spectrum for the E-/B-mode cosmic shear:
\[C^{XX}_\ell = \frac{2}{\pi} \int_0^\infty k^2 \, dk \int_0^{\chi_S} k \, d\chi \int_0^{\chi_S} k \, d\chi' S^\text{vector}_{X,\ell}(k, \chi) S^\text{vector}_{X,\ell}(k, \chi') P_{\sigma\sigma}(k; \eta_0 - \chi, \eta_0 - \chi'), \quad (C.54)\]
\[C^{EB}_\ell = 0, \quad (C.55)\]
where we have used the condition for the un-polarized state of vector perturbations, eq. (3.11).

If we consider the polarized state of vector perturbations, the nonzero cross correlation between E- and B-mode would appear.

### C.3 Scalar-/pseudo-scalar lensing potential

In this subsection, we derive the explicit expression for the scalar-/pseudo-scalar lensing potentials and the gradient-/curl-mode angular power spectra induced by vector perturbations. The calculation of the angular power spectra are basically the same way as in the case of cosmic shear in previous subsection. In terms of the spin operators, eqs. (3.20) and (3.21) are
\[\nabla^2 \phi = \int_0^{\chi_S} \frac{d\chi}{\chi} \left[ \frac{\chi_S - \chi}{\chi_S} \tilde{\phi} (0\sigma_g) + \frac{1}{2} \left\{ \tilde{\phi} (+1\sigma_g) + \phi (-1\sigma_g) \right\} \right], \quad (C.56)\]
\[\nabla^2 \omega = \frac{1}{2k} \int_0^{\chi_S} \frac{d\chi}{\chi} \left[ \tilde{\phi} (+1\sigma_g) - \phi (-1\sigma_g) \right], \quad (C.57)\]

where $0\sigma_g = \sigma_{g,0} e^\chi_x$, $\pm 1\sigma_g = \sigma_{g, \pm 1} e^\chi_{\pm}$, and we have used eqs. (C.10), (C.12). In the coordinate $k^i \equiv k^\mu = (0, 0, 1)$ and $e^\chi_{\pm}(\hat{k}) = (1, \pm i, 0)$, we act with the spin-raising/lowering operators on $0\sigma_g$ and $\pm 1\sigma_g$ (eqs. (C.26), and (C.27)):
\[\tilde{\phi} (0\sigma_g) = \int \frac{d^3k}{(2\pi)^3} \sqrt{1 - \mu^2} \left( \sigma_{g}^{(1)} e^{i\varphi} + \sigma_{g}^{(-1)} e^{-i\varphi} \right) \left(1 - \mu^2\right) \partial^2 - 4\mu \partial_\mu - 2) e^{-ik\chi_\mu}, \quad (C.58)\]
\[\tilde{\phi} (+1\sigma_g) = \int \frac{d^3k}{(2\pi)^3} \sqrt{1 - \mu^2} \left( \partial_\mu e^{-ik\chi_\mu} \right) \left( \sigma_{g}^{(1)} (\mu - 1) e^{i\varphi} + \sigma_{g}^{(-1)} (\mu + 1) e^{-i\varphi} \right) + 2e^{-ik\chi_\mu} \left( \sigma_{g}^{(1)} e^{i\varphi} + \sigma_{g}^{(-1)} e^{-i\varphi} \right), \quad (C.59)\]
\[\tilde{\phi} (-1\sigma_g) = \int \frac{d^3k}{(2\pi)^3} \sqrt{1 - \mu^2} \left( \partial_\mu e^{-ik\chi_\mu} \right) \left( \sigma_{g}^{(1)} (\mu + 1) e^{i\varphi} + \sigma_{g}^{(-1)} (\mu - 1) e^{-i\varphi} \right) + 2e^{-ik\chi_\mu} \left( \sigma_{g}^{(1)} e^{i\varphi} + \sigma_{g}^{(-1)} e^{-i\varphi} \right). \quad (C.60)\]
Plugging eqs. (C.58)-(C.60) into eqs. (C.56), (C.57), we rewrite the scalar/pseudo-scalar lensing potentials with
\[
\nabla^2 \phi = \int \frac{d^3 k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \sqrt{1 - \mu^2} \left( \sigma^{(+1)}_g e^{i\varphi} + \sigma^{(-1)}_g e^{-i\varphi} \right) \\
\times \left\{ \frac{\chi_S - \chi}{\chi_S} \left( (1 - \mu^2) \partial_\mu^2 - 4 \mu \partial_\mu - 2 \right) + \left( \mu \partial_\mu + 2 \right) \right\} e^{-i\varphi} \bigg|_{x = k\chi}, \tag{C.61}
\]
\[
\nabla^2 \varpi = i \int \frac{d^3 k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \sqrt{1 - \mu^2} \left( \sigma^{(+1)}_g e^{i\varphi} - \sigma^{(-1)}_g e^{-i\varphi} \right) \\
\times \partial_\chi \left( \frac{\chi_S - \chi}{\chi_S} \left( (1 - \mu^2) \partial_\mu^2 - 4 \mu \partial_\mu - 2 \right) + \left( \mu \partial_\mu + 2 \right) \right) \bigg|_{x = k\chi}. \tag{C.62}
\]
Performing the angular integration and using the differential equation for the spherical Bessel function (see Appendix A.1), the multipole coefficients of \( \phi \) and \( \varpi \) are expressed as
\[
\phi_{\ell m} = \int d^2 \hat{n} \ Y^*_m(\hat{n}) \ \phi(\hat{n}) \\
= (-i)^{\ell + 1} \sqrt{\frac{4\pi (2\ell + 1)}{(\ell + 1)!}} \int \frac{d^3 k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \left( \sigma^{(+1)}_g \delta_{m, +1} - \sigma^{(-1)}_g \delta_{m, -1} \right) \\
\times \left\{ \frac{\chi_S - \chi}{\chi_S} \left( x^2 (1 + \partial_x^2) + 4x \partial_x + 2 \right) - \left( x \partial_x + 2 \right) \right\} \frac{j_\ell(x)}{x} \bigg|_{x = k\chi}, \tag{C.63}
\]
and
\[
\varpi_{\ell m} = \int d^2 \hat{n} \ Y^*_m(\hat{n}) \ \varpi(\hat{n}) \\
= - (-i)^{\ell + 1} \sqrt{\frac{4\pi (2\ell + 1)}{(\ell + 1)!}} \int \frac{d^3 k}{(2\pi)^3} \int_0^{\chi_S} \frac{d\chi}{\chi} \left( \sigma^{(+1)}_g \delta_{m, +1} - \sigma^{(-1)}_g \delta_{m, -1} \right) \frac{j_\ell(x)}{x} \\
\times \left\{ \frac{\chi_S - \chi}{\chi_S} \left( (\ell + 1) \partial_x^2 - \partial_x \right) + \left( \partial_x + \ell \right) \right\} \bigg|_{x = k\chi}, \tag{C.64}
\]
where we have used eq. (A.8) and the fact \( \nabla^2 Y_{\ell m} = -\ell (\ell + 1) Y_{\ell m} \). \( S_{\phi, \ell} \) and \( S_{\varpi, \ell} \) are defined by
\[
S_{\phi, \ell}(k, \chi) \equiv 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left( \frac{\chi_S - \chi}{\chi_S} \epsilon^{(0)}_\ell(k\chi) - \epsilon^{(1)}_\ell(k\chi) \right), \tag{C.65}
\]
\[
S_{\varpi, \ell}(k, \chi) \equiv 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \beta^{(1)}_\ell(k\chi), \tag{C.66}
\]

– 29 –
We then rewrite eqs. (C.4, C.61), and using the condition for the un-polarized state of vector perturbations eq. (C.11), we obtain the explicit expression for the angular power spectra for the gradient-/curl-mode deflection angle:

\[
C_{\ell}^{\ell m} = \frac{2}{\pi} \int_0^\infty k^2 \, dk \int_0^{\pi} k \, d\chi \int_0^{\pi} \sin \chi \, S^\text{vector}_{\chi,\ell} (k, \chi) S^\text{vector}_{\chi,\ell} (k, \chi') P_{\sigma \sigma} (k; \eta_0 - \chi, \eta_0 - \chi'),
\]

\[
C_{\ell}^{\phi \omega} = 0.
\]

**C.4 Derivation of shear-deflection relation**

In this subsection, we derive the explicit relation between cosmic shear and deflection angle in terms of the spin operators. The relation eq. (3.29) can be further reduced to simplified forms if we move to the harmonic space. With a help of \( \nabla^2 Y_{\ell m}(\hat{n}) = -\ell(\ell + 1)Y_{\ell m}(\hat{n}) \), we have

\[
\phi_{\ell m} = 2 \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \, Y_{\ell m}^{*} Y_{\ell m+2} \gamma_{ab} \varepsilon^{abc} \varepsilon_{bd},
\]

\[
\omega_{\ell m} = 2 \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \, Y_{\ell m}^{*} Y_{\ell m+2} \gamma_{ab} \varepsilon^{abc} \varepsilon_{bd}.
\]

Using eqs. (3.4), (3.5), and (C.11), we then rewrite the metric on the sphere, \( \omega_{ab} \), the Levi-Civita pseudo-tensor, \( e_{ab} \), in terms of the basis vector \( e_{\pm} \) (see eq. (3.19)) and spin operators:

\[
\phi_{\ell m} = 2 \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \, Y_{\ell m}^{*} Y_{\ell m+2} \gamma_{ab} e^{(a e}(b c)} e_{d)}
\]

\[
= - \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \left[ \left( \hat{\omega}^2 Y_{\ell m}(\hat{n}) \right)^{*} g(\hat{n}) + \left( \hat{\omega}^2 Y_{\ell m}(\hat{n}) \right)^{*} g^*(\hat{n}) \right],
\]

\[
\omega_{\ell m} = 2 \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \, Y_{\ell m}^{*} Y_{\ell m+2} \gamma_{ab} e^{(a e}(b c)} e_{d)}
\]

\[
= \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \left[ \left( \hat{\omega}^2 Y_{\ell m}(\hat{n}) \right)^{*} g(\hat{n}) - \left( \hat{\omega}^2 Y_{\ell m}(\hat{n}) \right)^{*} g^*(\hat{n}) \right],
\]

where we have used the traceless condition for the shear, namely \( \gamma_{ab} \omega^{ab} = \gamma_{ab} e^{a}_+ e^{b}_- = 0 \).

We then rewrite \( \hat{\omega}^2 Y_{\ell m} \) and \( \hat{\omega}^2 Y_{\ell m} \) in terms of the spin-\( \pm 2 \) spherical harmonics \( \pm 2 Y_{\ell m} \) (see eqs. (C.13)):

\[
\phi_{\ell m} = - \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \left\{ Y_{\ell m}^{*}(\hat{n}) \, g(\hat{n}) + Y_{\ell m}^{*}(\hat{n}) \, g^*(\hat{n}) \right\}
\]

\[
= - \frac{(-\ell - 2)!}{(\ell + 2)!} \left( 2g_{\ell m} + 2g_{\ell m}^* \right),
\]

\[
\omega_{\ell m} = i \frac{(-\ell - 2)!}{(\ell + 2)!} \int d^2 \hat{n} \left\{ Y_{\ell m}^{*}(\hat{n}) \, g(\hat{n}) - Y_{\ell m}^{*}(\hat{n}) \, g^*(\hat{n}) \right\}
\]

\[
= i \frac{(-\ell - 2)!}{(\ell + 2)!} \left( 2g_{\ell m} - 2g_{\ell m}^* \right).
\]
Recalling that the combination $( + 2 g_{\ell m} \pm - 2 g_{\ell m} )$ can be rewritten in terms of the $E_\gamma / B$-mode cosmic shear field (see eq. (3.7)), we obtain the explicit relations between $\phi_{\ell m}$, $\varpi_{\ell m}$, $g^E_{\ell m}$, and $g^B_{\ell m}$ (see also [32]):

$$\phi_{\ell m} = 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!} g^E_{\ell m}}, \quad \varpi_{\ell m} = 2 \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!} g^B_{\ell m}}.$$  \quad (C.75)

### D  Derivation of correlations of a cosmic string network

Let us consider a Nambu-Goto string segment at the position $r = r(\sigma, \eta)$ where $\eta$ and $\sigma$ are the time and position on the string worldsheet. In the transverse gauge, the stress-energy tensor for a string segment can be described as [127]

$$\delta T^{\mu\nu}(r, \eta) = \mu \int d\sigma \left( \frac{1}{r^j r^j} - r^i r^j \right) \delta^3(r - r(\sigma, \eta)), \quad (D.1)$$

where the dot (') and the prime (') denote the derivative with respect to $\eta$ and $\sigma$. Comparing to eqs. (4.1) and (D.1), the velocity perturbations, $v^{(\pm 1)}$, due to a segment are given by

$$v^{(\pm 1)}(k, \eta) = \mu \int d\sigma \, r^i(\sigma, \eta) e^{\pm i}(\hat{k}) e^{i k \cdot r(\sigma, \eta)}. \quad (D.2)$$

Since the correlations can be described by a summation of the contribution of each segment, we can estimate the equal-time auto-power spectrum for the vector perturbations as

$$P_{\sigma_\eta \sigma_\eta}(k; \eta, \eta) = \frac{2 (16 \pi G)^2 a^4}{k^4} \frac{1}{V} \left\langle v^{(\pm 1) +}(k, \eta) v^{(\pm 1) -}(k, \eta) \right\rangle$$

$$= \frac{2 (16 \pi G \mu)^2 a^4}{k^4} n_s dV \frac{1}{V} e^{+i}(\hat{k}) e^{i -j}(\hat{k})$$

$$\times \left\langle \int d\sigma_1 d\sigma_2 r^i(\sigma_1, \eta) r^j(\sigma_2, \eta) e^{i k \cdot (r(\sigma_1, \eta) - r(\sigma_2, \eta))} \right\rangle. \quad (D.3)$$

where $dV = 4 \pi \chi^2 / H$ is the differential comoving volume element, $n_s = a^3 \xi^{-3}$ is the comoving number density of string segments, and $V = (2 \pi)^3 \delta^3(0)$ is the comoving box size. For the string averaging, we can use a very simple model developed in [77, 99, 100]. The assumption in this model is that all correlators can be expressed in terms of two-point correlations for $r^i(\sigma, \eta)$ and $r^j(\sigma, \eta)$. Assuming that $r^i(\sigma, \eta)$ and $r^j(\sigma, \eta)$ are exactly Gaussian and isotropic distributed with mean zero and variances $\langle r^i(\sigma, \eta) r^j(0, \eta) \rangle \equiv \frac{1}{2} \delta^{ij} V_s(\sigma)$, and $\langle r^i(\sigma, \eta) r^j(0, \eta) \rangle \equiv \frac{1}{2} \delta^{ij} T_s(\sigma)$, we can compute the equal-time auto-power spectrum for the vector perturbations as

$$P_{\sigma_\eta \sigma_\eta}(k, \eta, \eta) = \frac{(16 \pi G \mu)^2 a^4}{k^4} n_s dV \frac{1}{3V} \int d\sigma_+ d\sigma_- V_s(\sigma_-) \exp \left[ - \frac{1}{6} k^2 \Gamma_s(\sigma_-) \right], \quad (D.4)$$

where $\sigma_+ = \sigma_1 \pm \sigma_2$, we have introduced $\Gamma_s(\sigma) = \int_0^\sigma d\sigma_3 d\sigma_4 T_s(\sigma_3 - \sigma_4)$. On scale larger than the correlation length, the correlators are expected to be damped, and the correlators on scale $\sigma < \xi / a$ can be approximated as $V_s \approx \dot{v}_m^2$, $\Gamma_s \approx (1 - \dot{v}_m^2) \sigma^2$. Once we determine the region of the integration, we can calculate the auto-power spectrum of the velocity perturbations by integrating eq. (D.4). Since the term $\int d\sigma_+ / V$ corresponds to the length of the string,
segment within the unit volume and the correlators, $V_s$ and $\Gamma_s$, are damped at $\sigma_\perp \gg \xi/a$, we take the region of the integration as $\int d\sigma_\perp/V = a^2/\xi^2 \sqrt{1 - v_{\text{rms}}^2}$ and $|\sigma_\perp| \leq \xi/a \sqrt{1 - v_{\text{rms}}^2}$ hereafter. Then, we have

$$P_{\sigma_x \sigma_y}(k; \eta, \eta) \approx (16\pi G\mu)^2 \frac{2\sqrt{6\pi} v_{\text{rms}}^2}{3(1 - v_{\text{rms}}^2)} \frac{4\pi^2 a^4}{H} \left( \frac{a}{k\xi} \right) \text{erf} \left( \frac{k\xi}{2\sqrt{6}} \right). \quad (D.5)$$

### E Reconstruction noise

We provide the brief summary of the reconstruction noise spectrum, following [71]. The reconstruction noise spectrum for the optimal combination of the minimum variance estimator is given by

$$N^{x,(\alpha)}_\ell = \left[ \sum_{\alpha,\beta} \left\{ (N^{x}_\ell)^{-1} \right\}^{-1} \right]^{-1}, \quad (E.1)$$

where $x = \phi, \varpi$, the subscripts $\alpha, \beta$ mean a pair of two CMB maps, and the component of the matrix $\{N^{x}_\ell\}_{\alpha\beta}$ is the covariance of the reconstruction noise, which is given by

$$N^{x,(\alpha,\beta)}_\ell = \frac{1}{2\ell + 1} \sum_{L,L'} \left( F^{x,(\alpha)}_{\ell,L,L'} \right)^* \left( \tilde{C}^{XX'}_{L'} \tilde{C}^{YY'}_{L'} + F^{x,(\beta)}_{\ell,L,L'} (-1)^{L+L'} \tilde{C}^{XY}_{L'} \tilde{C}^{XY'}_{L} \right), \quad (E.2)$$

**Table 3.** The functional forms of $f^{x,(\alpha)}_{\ell,L,L'}$. 

| Segment | $f^{\phi,(\alpha)}_{\ell,L,L'}$ | $f^{\varpi,(\alpha)}_{\ell,L,L'}$ |
|---------|-------------------------------|-------------------------------|
| $\Theta$ | $0S^{\phi}_{\ell,L,L'} C^{\Theta}_{L'} + 0S^{\varpi}_{\ell,L,L'} C^{\Theta}_{L'}$ | $0S^{\phi}_{L',L,L'} C^{\Theta}_{L'} - 0S^{\varpi}_{L',L,L'} C^{\Theta}_{L'}$ |
| $\Theta E$ | $0S^{\phi}_{\ell,L,L'} C^{\Theta E}_{L'} + 0S^{\varpi}_{\ell,L,L'} C^{\Theta E}_{L'}$ | $0S^{\phi}_{L',L,L'} C^{\Theta E}_{L'} - 0S^{\varpi}_{L',L,L'} C^{\Theta E}_{L'}$ |
| $\Theta B$ | $-0S^{\phi}_{\ell,L,L'} C^{\Theta E}_{L'}$ | $0S^{\varpi}_{L',L,L'} C^{\Theta E}_{L'}$ |
| $E E$ | $0S^{\phi}_{\ell,L,L'} C^{EE}_{L'} + 0S^{\varpi}_{\ell,L,L'} C^{EE}_{L'}$ | $0S^{\phi}_{L',L,L'} C^{EE}_{L'} - 0S^{\varpi}_{L',L,L'} C^{EE}_{L'}$ |
| $E B$ | $-0S^{\phi}_{\ell,L,L'} C^{BB}_{L'} - 0S^{\varpi}_{\ell,L,L'} C^{EE}_{L'}$ | $-0S^{\phi}_{L',L,L'} C^{BB}_{L'} + 0S^{\varpi}_{L',L,L'} C^{EE}_{L'}$ |
| $B B$ | $0S^{\phi}_{\ell,L,L'} C^{BB}_{L'} + 0S^{\varpi}_{\ell,L,L'} C^{BB}_{L'}$ | $0S^{\phi}_{L',L,L'} C^{BB}_{L'} - 0S^{\varpi}_{L',L,L'} C^{BB}_{L'}$ |
The instrumental noise is given by
\[ \sigma_{\ell,L} = \sum_{\nu} (N_{\ell,L}^{XX})^{-1} \frac{\sigma_{\nu,T}}{T_{CMB}} \exp \left[ \frac{(\ell+1)\theta^2}{8 \ln 2} \right], \] (E.9)

where \( \ell_{\text{max}} \) denotes the maximum multipole used in the reconstruction procedure. The quantity \( C_L^{XY} \) is the lensed CMB angular power spectrum including the contributions from instrumental noise and we have introduced the weight function \( F_{\ell,L,L'}^{x,(\alpha)} \) defined by
\[ F_{\ell,L,L'}^{x,(\alpha)} = N_{\ell}^{x,(\alpha)} g_{\ell,L,L'}, \] (E.3)

where
\[ N_{\ell}^{x,(\alpha)} = \left[ \sum_{L,L'} f_{\ell,L,L'}^{x,(\alpha)} g_{\ell,L,L'}^{x,(\alpha)} \right]^{-1}, \] (E.4)

\[ g_{\ell,L,L'}^{x,(\alpha)} = \frac{f_{\ell,L,L'}^{x,(\alpha)}}{\sum_{L,L'} f_{\ell,L,L'}^{x,(\alpha)} g_{\ell,L,L'}^{x,(\alpha)}}. \] (E.5)

The coefficients, \( f_{\ell,L,L'}^{x,(\alpha)} \), are expressed by the combination of the unlensed CMB angular power spectrum, \( C_L^{XX} \), and the quantities, \( \phi S_{\ell,L,L'}^x \), \( \odot S_{\ell,L,L'}^x \), and \( \odot S_{\ell,L,L'}^x \). We summarize \( f_{\ell,L,L'}^{x,(\alpha)} \) in Table 3. The quantities, \( 0, \odot, \odot S_{\ell,L,L'}^x \), are written in terms of the Wigner-3j symbols as
\[ \phi S_{\ell,L,L'} = \sqrt{\frac{(2L+1)(2L'+1)(2\ell+1)}{16\pi}} \left[ L(L+1) + L'(L'+1) - \ell(\ell+1) \right] \left( \begin{array}{c} \ell & L & L' \\ s & 0 & -s \end{array} \right), \] (E.6)

\[ \odot S_{\ell,L,L'} = \sqrt{\frac{(2L+1)(2L'+1)(2\ell+1)}{16\pi}} \left[ \sqrt{\frac{L'+1-s}{L'+1+s}} \left( \begin{array}{c} \ell & L & L' \\ s & 1 & -s \end{array} \right) - \sqrt{\frac{L'-s}{L'+s}} \left( \begin{array}{c} \ell & L & L' \\ s & 1 & -s \end{array} \right) \right], \] (E.7)

\[ \odot S_{\ell,L,L'} = \frac{1}{2} \left( 2S_{\ell,L,L'} + S_{\ell,L,L}' \right), \quad \odot S_{\ell,L,L'} = \frac{1}{2i} \left( 2S_{\ell,L,L'} - S_{\ell,L,L}' \right), \] (E.8)

The instrumental noise is given by
\[ \mathcal{N}_{\ell}^{XX} = \left[ \sum_{\nu} (\mathcal{N}_{\ell,L}^{XX})^{-1} \right]^{-1} \left( \sigma_{\nu,T} \right) \exp \left[ \frac{(\ell+1)\theta^2}{8 \ln 2} \right], \] (E.9)
where $T_{\text{CMB}} = 2.7K$, $\theta_\nu$, and $\sigma_\nu$ represent the mean temperature of CMB, the beam size, and the sensitivity of each channel. We summarize the basic parameters for PLANCK and ACTPol in Table 4. For the cosmic variance limit, we take $N_{XX} = 0$. The reconstruction noise spectrum for ACTPol+PLANCK is assumed to have the form:

$$N_{\ell:ACTPol+PLANCK} = \left( f_{\text{sky}}^{\text{ACTPol}} \left( N_{\ell:ACTPol}^{\nu,\text{(c)}} \right)^2 + f_{\text{sky}}^{\text{PLANCK}} - f_{\text{sky}}^{\text{ACTPol}} \left( N_{\ell:PLANCK}^{\nu,\text{(c)}} \right)^2 \right)^{-1/2},$$ (E.10)

where $f_{\text{sky}}^{\text{ACTPol}}$ and $f_{\text{sky}}^{\text{PLANCK}}$ are fractal sky coverage of ACTPol and PLANCK.

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