INTEGRAL REPRESENTATIONS OF APPELL’S $F_2$, $F_3$, HORN’S $H_2$ AND OLSSON’S $F_P$ FUNCTIONS

Katsuisha MIMACHI

(Received 11 January 2018 and revised 30 May 2019)

Abstract. We give integral representations of Euler type for Appell’s hypergeometric functions $F_2$, $F_3$, Horn’s hypergeometric function $H_2$ and Olsson’s hypergeometric function $F_P$. Their integrands are the same (up to a constant factor), and only the regions of integration vary.

0. Introduction

Appell’s hypergeometric function $F_2$ is the analytic continuation of

$$F_2(a, b_1, b_2, c_1, c_2; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{m!n!(c_1)_m(c_2)_n} x^m y^n, \quad |x| + |y| < 1,$$

where $(a)_n = \Gamma(a + n)/\Gamma(a)$, and satisfies the system $E_2$ of rank four:

$$E_2 \left\{ \begin{align*}
    & (1-x) \frac{\partial^2}{\partial x^2} - xy \frac{\partial^2}{\partial x \partial y} + (c_1 - (a + b_1 + 1)x) \frac{\partial}{\partial x} - b_1 y \frac{\partial}{\partial y} - ab_1 \\
    & (1-y) \frac{\partial^2}{\partial y^2} - xy \frac{\partial^2}{\partial x \partial y} + (c_2 - (a + b_2 + 1)y) \frac{\partial}{\partial y} - b_2 x \frac{\partial}{\partial x} - ab_2
\end{align*} \right\} F = 0,
$$

which is defined on the space

$$\mathbb{C}^2 \setminus \{\{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{x + y = 1\}\} \subset (\mathbb{P}^1)^2.$$

From the works by Appell [AKdF], Olsson [Ol], Sekiguchi [S], Takayama [T], and Diekema and Koornwinder [DK], it is known that Appell’s hypergeometric function $F_3$, Horn’s hypergeometric function $H_2$ and Olsson’s hypergeometric function $F_P$ also appear as solutions of $E_2$. Here Appell’s $F_3$, Horn’s $H_2$ and Olsson’s $F_P$ are analytic continuations of

$$F_3(a_1, a_2, b_1, b_2, c; x, y) = \sum_{m, n \geq 0} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{m!n!(c)_m+n} x^m y^n, \quad |x| < 1, \quad |y| < 1,$$

$$H_2(a, b, c, d, e; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{m!n!(e)_m} x^m y^n, \quad |x| < 1, \quad |y| < (|x| + 1)^{-1}$$

2010 Mathematics Subject Classification: Primary 33C60; Secondary 33C65, 33C70.

Keywords: Appell’s hypergeometric functions; Horn’s hypergeometric function; Olsson’s hypergeometric function; integral representations.
and,

\[
F_p(a, b_1, b_2, c_1, c_2; x, y) = \sum_{m,n \geq 0} \frac{(a)_{m+n}(a-c_2+1)_m(b_1)_m(b_2)_n}{m!n!(a+b_2-c_2+1)_m+c_1)_m} x^m (1-y)^n, \quad |x| < 1, \quad |y - 1| < 1.
\]

As a solution of $E_2$, each of the functions $F_2$, $F_3$, $H_2$ and $F_p$ is characterized as follows:

1. The holomorphic solution $f(x, y)$ around $(0, 0)$ such that $f(0, 0) = 1$ is unique when $c_1, c_2, c_1 - c_2 \notin \mathbb{Z}$ and it is given by $F_2(a, b_1, b_2, c_1, c_2; x, y)$.

2. The solution $f(x, y)$ around $(\infty, \infty)$ such that $f(x, y) = x^{b_1} y^{-b_2} (1 + \cdots)$ as $(x, y) \to (\infty, \infty)$ is unique and given by $x^{b_1} y^{-b_2} F_3(b_1, b_2, b_1 - c_2 + 1, 1 - a + b_1 + b_2; x^{-1}, y^{-1})$.

3. The solution $f(x, y)$ around $(0, 0)$ such that $f(x, y) = y^{-b_2} (1 + \cdots) as (x, y) \to (0, \infty)$ is unique when $c_1, a - b_2, a - b_2 - c_1 \notin \mathbb{Z}$ and it is given by $y^{-b_2} H_2(a - b_2, b_1, b_2 - c_2 + 1, b_2, c_1; x, -y^{-1})$.

4. The solution $f(x, y)$ around $(0, 1)$ such that $f(x, y) = x^{1-c_1} (1 + \cdots)$ as $(x, y) \to (0, 1)$ is unique when $c_1, a + b_2 - c_2, a + b_2 - c_1 - c_2 \notin \mathbb{Z}$ and it is given by $x^{1-c_1} F_p(a - c_1 + 1, b_1 - c_1 + 1, b_2, 2 - c_1, c_2; x, y)$.

5. The holomorphic solution $f(x, y)$ around $(0, 1)$ such that $f(0, 0) = 1$ is unique when $c_1, a + b_2 - c_2, a + b_2 - c_1 - c_2 \notin \mathbb{Z}$ and it is given by $F_p(a, b_1, b_2, c_1, c_2; x, y)$.

\text{(Result 1) is given in [AKdF]. The others (2)–(5) are obtained in [Ol]. See also [Ka, M, S] and [T].}

On the other hand, Theorem 2.1 in [MN] says that the function

\[
\int_C \Phi(x, y; u, v) \, du \, dv
\]

satisfies the system $E_2$ for any cycle $C \in H_2(T_{(x, y)}, L')$. Here $\Phi(u, v) = \Phi(x, y; u, v)$ is a multivalued function

\[
\Phi(x, y; u, v) = u^{b_1-1} v^{b_2-1} (1 - u)^{c_1-b_1-1} (1 - v)^{c_2-b_2-1} (1 - xu - yv)^{-a}
\]
on

\[T_{(x, y)} = \{ (u, v) \in \mathbb{C}^2 \mid u v (1 - u) (1 - v) (1 - xu - yv) \neq 0 \} \subset (\mathbb{P}^1)^2\]

for

\[(x, y) \in \{ (x, y) \in \mathbb{C}^2 \mid xy (1 - x) (1 - y) (1 - x - y) \neq 0 \} \subset (\mathbb{P}^1)^2,\]

and $L'$ is the locally constant sheaf (the local system) defined by $\Phi(u, v)$: the sheaf consisting of the local solutions of $dL = L \omega$ for $\omega = d \Phi(u, v) / \Phi(u, v)$.

By using this theorem and the characterization above, we give integral representations of the functions $F_2$, $F_3$, $H_2$ and $F_p$. (The results for $F_2$ and $F_3$ are known.)

When complex variables $x$ and $y$ are real, for the function $\Phi(u, v) = \prod_j f_j(u, v)^{\lambda_j}$ and a simply connected domain $D$ of the real manifold $T_{\mathbb{R}}$ (the real locus of $T_{(x, y)}$), we load $D$ with a section

\[
\Phi_D(u, v) = \prod_j (\epsilon_j f_j(u, v))^{\lambda_j}
\]
of $L'$ on $D$, where $\epsilon_j = \pm$ is so determined that $\epsilon_j f_j(u, v)$ is positive on $D$, and the argument of each $\epsilon_j f_j(u, v)$ is assigned to be zero. This choice of a section is said to be \textit{standard}. 

THEOREM.

(i) If \( b_1, b_2, c_1 - b_1, b_2 - c_2 \notin \mathbb{Z}_{\leq 0} \), then we have

\[
F_2(a, b_1, b_2, c_1, c_2; x, y) = \frac{\Gamma(c_1) \Gamma(c_2)}{\Gamma(b_1) \Gamma(c_1 - b_1) \Gamma(b_2) \Gamma(c_2 - b_2)} \int_{u=0}^{1} \int_{v=0}^{1} u^{b_1-1} v^{b_2-1}(1-u)^{c_1-b_1-1} \times (1-v)^{c_2-b_2-1}(1-xu - yv)^{-a} \, dv \, du, 
\]

(0.1)

where the integrand is standardly loaded when \( x \) and \( y \) are real numbers satisfying \( 0 < x < 1 \) and \( 0 < y < 1 \). See Figure 1 for regions corresponding to parts (i) to (v).

(ii) If \( 1 - a, b_1, b_2 \notin \mathbb{Z}_{\leq 0} \), then we have

\[
x^{-b_1} y^{-b_2} F_3(b_1, b_2, b_1 - c_1 + 1, b_2 - c_2 + 1, 1 - a + b_1 + b_2; x^{-1}, y^{-1}) = \frac{\Gamma(1 - a + b_1 + b_2)}{\Gamma(1 - a) \Gamma(b_1) \Gamma(b_2)} \int_{u=0}^{1/y} \int_{v=0}^{1/y} u^{b_1-1} v^{b_2-1}(1-u)^{c_1-b_1-1} \times (1-v)^{c_2-b_2-1}(1-xu - yv)^{-a} \, du \, dv, 
\]

(0.2)

where the integrand is standardly loaded and the arguments of \( x \) and \( y \) in \( x^{-b_1} y^{-b_2} \) are assigned to be zero when \( x \) and \( y \) are real numbers satisfying \( 1 < x \) and \( 1 < y \).

(iii) If \( b_1, b_2, 1 - a, c_1 - b_1 \notin \mathbb{Z}_{\leq 0} \), then we have

\[
y^{-b_2} H_2(a - b_2, b_1, b_2 - c_2 + 1, b_2, c_1; x, -y^{-1}) = \frac{\Gamma(1 - a + b_2) \Gamma(c_1)}{\Gamma(1 - a) \Gamma(b_2) \Gamma(c_1 - b_1) \Gamma(b_1)} \int_{u=0}^{1} \int_{v=0}^{(1-xu)/y} u^{b_1-1} v^{b_2-1}(1-u)^{c_1-b_1-1} \times (1-v)^{c_2-b_2-1}(1-xu - yv)^{-a} \, dv \, du, 
\]

(0.3)

where the integrand is standardly loaded and the argument of \( y \) in \( y^{-b_2} \) is assigned to be zero when \( x \) and \( y \) are real numbers satisfying \( 0 < x < 1 \) and \( 1 < y \).

(iv) If \( 1 - a, b_2, a - c_1 - c_2 + 2, a - c_1 + 1 \notin \mathbb{Z}_{\leq 0} \), then we have

\[
x^{1-c_1} F_p(a - c_1 + 1, b_1 - c_1 + 1, b_2, 2 - c_1, c_2; x, y) = \frac{\Gamma(2 - c_1) \Gamma(a + b_2 - c_1 - c_2 + 2)}{\Gamma(1 - a) \Gamma(a - c_1 + 1) \Gamma(b_2) \Gamma(a - c_1 - c_2 + 2)} \times \int_{u=1/x}^{\infty} \int_{v=(1-xu)/y}^{0} u^{b_1-1} (-v)^{b_2-1}(u - 1)^{c_1-b_1-1}(1-v)^{c_2-b_2-1} \times (xu + yv - 1)^{-a} \, dv \, du, 
\]

(0.4)

where the integrand is standardly loaded and the argument of \( x \) in \( x^{1-c_1} \) is assigned to be zero when \( x \) and \( y \) are real numbers satisfying \( 0 < x < 1 \) and \( 0 < y \).
(v) If \( b_1, c_1 - b_1, b_2, a - c_2 + 1 \notin \mathbb{Z}_{\leq 0} \), then we have

\[
F_P(a, b_1, b_2, c_1, c_2; x, y) = \frac{\Gamma(a + b_2 - c_2 + 1)\Gamma(c_1)}{\Gamma(b_2)\Gamma(a - c_2 + 1)\Gamma(b_1)\Gamma(c_1 - b_1)} \int_{v=0}^{\infty} \int_{u=0}^{1} u^{b_1-1}(1-u)^{c_1-b_1-1}(-v)^{b_2-1} \times (1-v)^{c_2-b_2-1}(1-xu-yv)^{-a} \, du \, dv,
\]

where the integrand is standardly loaded when \( x \) and \( y \) are real numbers satisfying \( 0 < x, 0 < y \) and \( 0 < x + y < 1 \).

Remark. The formulas (0.1)–(0.4) correspond to (6.1)–(6.4) of [DK]. Integral representations of \( H_2 \) different from (0.3) are given in [Ki] and [Y]. The relationship between (0.3) and the integral representations in [Ki] and [Y] is studied well in [DK].

1. Proof of Theorem

1.1. Proof of (i)

For simplicity, we assume temporarily that \( x \) and \( y \) are real numbers satisfying \( 0 < x < 1 \) and \( 0 < y < 1 \).

When \( b_1, b_2, b_1 - c_1, c_2 - b_2 \notin \mathbb{Z} \), as a function of \( x, y \),

\[
\int_{\text{reg}(0<u<1) \times (0<v<1)} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1}(1-xu-yv)^{-a} \, dv \, du
\]
satisfies \( E_2 \) and is holomorphic around \((0, 0)\). Moreover, its value at \((0, 0)\) is

\[
\int_{\text{reg}(0<u<1)\times(0<v<1)} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1} \, dv \, du
\]

\[
= \int_{u=0}^{1} \int_{v=0}^{1} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1} \, dv \, du
\]

\[
= B(b_1, c_1 - b_1)B(b_2, c_2 - b_2).
\]

Here, to obtain the first equality we impose the conditions \(\text{Re}(c_1) > \text{Re}(b_1) > 0\) and \(\text{Re}(c_2) > \text{Re}(b_2) > 0\) but, as a result, we get rid of them by considering the analytic continuation with respect to the parameters \(b_1, b_2, c_1\) and \(c_2\).

On the other hand, under the conditions \(c_1, c_2, c_1 - c_2 \notin \mathbb{Z}\), the holomorphic solution of \(E_2\) around \((0, 0)\) is unique. Thus we reach the equality (0.1) under the conditions \(b_1, b_2, b_1 - c_1, b_2 - c_2, c_1, c_2, c_1 - c_2 \notin \mathbb{Z}\).

Once (0.1) holds, the conditions \(c_1, c_2, c_1 - c_2 \notin \mathbb{Z}\) can be dropped and the conditions \(b_1, b_2, b_1 - c_1, b_2 - c_2 \notin \mathbb{Z}\) can be relaxed to \(b_1, b_2, c_1 - b_1, c_2 - b_2 \notin \mathbb{Z}\leq 0\) by considering the analytic continuation with respect to the parameters \(b_1, b_2, c_1\) and \(c_2\). Finally, the analytic continuation with respect to the variables \(x\) and \(y\) leads to the result. This completes the proof.

1.2. Proof of (ii)

For simplicity, we assume temporarily that \(x\) and \(y\) are real numbers satisfying \(1 < x\) and \(1 < y\).

When \(\text{Re}(1-a) > 0\) and \(\text{Re}(b_1) > 0\) and \(\text{Re}(b_2) > 0\), the change of integration variables such as \((u, v) \mapsto (u/x, v/y)\) implies that

\[
\int_{v=0}^{1/y} \int_{u=0}^{(1-uv)/x} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1}(1-xu-yv)^{-a} \, dv \, du
\]

\[
= x^{1-b_1}y^{1-b_2} \int_{v=0}^{1} \int_{u=0}^{1-v} u^{b_1-1}v^{b_2-1}(1-u-v)^{-a}
\]

\[
\times (1-u/x)^{c_1-b_1-1}(1-v/y)^{c_2-b_2-1} \, dv \, du,
\]

and the integral on the right-hand side is a holomorphic function in the variables \(1/x\) and \(1/y\) and tends to

\[
\int_{v=0}^{1} \int_{u=0}^{1-v} u^{b_1-1}v^{b_2-1}(1-u-v)^{-a} \, dv \, du = \frac{\Gamma(1-a)\Gamma(b_1)\Gamma(b_2)}{\Gamma(1-a+b_1+b_2)}
\]
as \((x, y) \rightarrow (\infty, \infty)\).

On the other hand, as a function of \(x\) and \(y\),

\[
\int_{\text{reg}((0<v<1/y)\times(0<u<1-uv)/x))} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1}
\]

\[
\times (1-xu-yv)^{-a} \, dv \, du
\]

satisfies \(E_2\) if \(a, b_1, b_2 \notin \mathbb{Z}\), and is equal to

\[
\int \int_{(0<v<1/y)\times(0<u<1-uv)/x)} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1}(1-xu-yv)^{-a} \, dv \, du
\]
if \( \text{Re}(1 - a) > 0, \text{Re}(b_1) > 0 \) and \( \text{Re}(b_2) > 0 \). By the same argument as in (i), we obtain the result. This completes the proof.

### 1.3. Proof of (iii)

We assume temporarily that \( x \) and \( y \) are real numbers satisfying \( 0 < x < 1 \) and \( 1 < y \).

When \( a, b_1, b_2, b_1 - c_1 \notin \mathbb{Z} \),

\[
\int_{\text{reg}((0 < u < 1) \times (0 < v < (1 - xu)/y))} u^{b_1 - 1} v^{b_2 - 1} (1 - u)^{c_1 - b_1 - 1} (1 - v)^{c_2 - b_2 - 1} \
\times (1 - xu - yv)^{-a} \, dv \, du
\]
satisfies \( E_2 \).

If \( a, b_1, b_2, b_1 - c_1 \notin \mathbb{Z} \) and \( \text{Re}(b_1) > 0, \text{Re}(b_2) > 0, \text{Re}(1 - a) > 0 \) and \( \text{Re}(c_1 - b_1) > 0 \), the change of integration variables \((u, v) \mapsto (u, v/y)\) implies that

\[
\int_{\text{reg}((0 < u < 1) \times (0 < v < (1 - xu)/y))} u^{b_1 - 1} v^{b_2 - 1} (1 - u)^{c_1 - b_1 - 1} (1 - v)^{c_2 - b_2 - 1} \
\times (1 - xu - yv)^{-a} \, dv \, du
\]

\[
= \int_{(0 < u < 1) \times (0 < v < (1 - xu)/y)} u^{b_1 - 1} v^{b_2 - 1} (1 - u)^{c_1 - b_1 - 1} (1 - v)^{c_2 - b_2 - 1} \
\times (1 - xu - yv)^{-a} \, dv \, du
\]

\[
= y^{-b_2} \int_{u=0}^{1} \int_{v=0}^{1-xu} u^{b_1 - 1} v^{b_2 - 1} (1 - u)^{c_1 - b_1 - 1} \left(1 - \frac{v}{y}\right)^{c_2 - b_2 - 1} (1 - xu - yv)^{-a} \, dv \, du.
\]

Here the integral on the right-hand side expresses a holomorphic function around \((0, \infty)\) in the variables \( x \) and \( 1/y \) and tends to

\[
\int_{u=0}^{1} \int_{v=0}^{1} u^{b_1 - 1} v^{b_2 - 1} (1 - v)^{-a} (1 - u)^{c_1 - b_1 - 1} \, dv \, du = B(b_1, c_1 - b_1)B(b_2, 1 - a)
\]
as \((x, y) \to (0, \infty)\). Therefore, analytic continuation with respect to the parameters \( a, b_1, b_2, c_1 \) and \( c_2 \) and the variables \( x \) and \( y \) leads to the result. This completes the proof.

### 1.4. Proof of (iv)

We assume temporarily that \( x \) and \( y \) are real numbers satisfying \( 0 < x < 1 \) and \( 0 < y \).

When \( a, b_2, a - c_1, a - c_1 - c_2 \notin \mathbb{Z} \), as a function of \( (x, y) \),

\[
\int_{\text{reg}((1/x < u < \infty) \times ((1-xu)/y < v < \infty))} u^{b_1 - 1} (-v)^{b_2 - 1} (u - 1)^{c_1 - b_1 - 1} (1 - v)^{c_2 - b_2 - 1} \
\times (xu + yv - 1)^{-a} \, dv \, du
\]
satisfies \( E_2 \).

If \( a, b_2, a - c_1, a - c_1 - c_2 \notin \mathbb{Z} \) and \( \text{Re}(1 - a) > 0, \text{Re}(a - c_1 + 1) > 0, \text{Re}(b_2) > 0 \) and \( \text{Re}(2 + a - c_1 - c_2) > 0 \), the change of integration variables \((u, v) \mapsto (u/x, v)\) implies
that
\[
\int_{\text{reg}(1/|x|<\infty)\times((1-xu)/y<0)} u^{b_1-1}(-v)^{b_2-1}(u-1)^{c_1-b_1-1}(1-v)^{c_2-b_1-1}
\times (xu+yv-1)^{-a} \, dv \, du
\]
\[
= \int_{(1/|x|<\infty)\times((1-xu)/y<0)} u^{b_1-1}(-v)^{b_2-1}(u-1)^{c_1-b_1-1}(1-v)^{c_2-b_1-1}
\times (xu+yv-1)^{-a} \, dv \, du
\]
\[
= x^{1-c_1} \int_{u=1}^{\infty} \int_{v=0}^{(1-u)/y} u^{b_1-1}(-v)^{b_2-1}(u-x)^{c_1-b_1-1}(1-v)^{c_2-b_1-1}
\times (u+yv-1)^{-a} \, dv \, du.
\]
Here the integral on the right-hand side is a holomorphic function around \((0, 1)\) in the variables \(x\) and \(y\) and its value at \((0, 1)\) is
\[
\int_{u=1}^{\infty} \int_{v=0}^{1} u^{c_1-2}(-v)^{b_2-1}(1-v)^{c_2-b_1-1}(u+v-1)^{-a} \, dv \, du.
\]
The change of integration variables \((u, v) \to (u^{-1}(1-v)^{-1}, v(1-v)^{-1})\) implies that
\[
\int_{u=1}^{\infty} \int_{v=0}^{1} u^{c_1-2}(-v)^{b_2-1}(1-v)^{c_2-b_1-1}(u+v-1)^{-a} \, dv \, du
\]
\[
= \int_{0}^{1} \int_{0}^{1} u^{c_1-2}(1-u)^{-a}(-v)^{b_2-1}(1-v)^{1+a-c_1-c_2} \, du \, dv
\]
\[
= B(1-a, a-c_1+1)B(b_2, a-c_1-c_2+2).
\]
Therefore, analytic continuation with respect to the parameters \(a, b_1, b_2, c_1\) and \(c_2\) and the variables \(x\) and \(y\) leads to the result. This completes the proof.

1.5. Proof of \((v)\)

We assume temporarily that \(x\) and \(y\) are real numbers satisfying \(0 < x, 0 < y\) and \(0 < x+y < 1\).

When \(a - c_2, b_1, b_2, b_1 - c_1 \notin \mathbb{Z}\), as a function of \((x, y)\),
\[
\int_{\text{reg}(0 < u < 1) \times (|v| < v < 0)} u^{b_1-1}(1-u)^{c_1-b_1-1}(-v)^{b_2-1}(1-v)^{c_2-b_1-1}(1-xu-yv)^{-a} \, du \, dv
\]
satisfies \(E_2\).

If \(a - c_2, b_1, b_2, b_1 - c_1 \notin \mathbb{Z}\) and \(\text{Re}(b_1) > 0, \text{Re}(c_1 - b_1) > 0, \text{Re}(b_2) > 0\) and \(\text{Re}(a - c_2 + 1) > 0\), as a function of \((x, y)\),
\[
\int_{\text{reg}(0 < u < 1) \times (|v| < v < 0)} u^{b_1-1}(1-u)^{c_1-b_1-1}(-v)^{b_2-1}(1-v)^{c_2-b_1-1}
\times (1-xu-yv)^{-a} \, du \, dv
\]
\[
= \int_{0 < u < 1) \times (|v| < v < 0)} u^{b_1-1}(1-u)^{c_1-b_1-1}(-v)^{b_2-1}(1-v)^{c_2-b_1-1}
\times (1-xu-yv)^{-a} \, du \, dv
\]
is a holomorphic function around \((0, 1)\) and its value at \((0, 1)\) is
\[
\int_{(0<v<1)\times(\infty<v<1)} u^{b_1-1}(1-u)^{c_1-b_1-1}(1-v)^{b_2-1}(1-v)^{c_2-b_2-a-1} \, du \, dv
\]
\[
= \int_{0}^{1} \int_{0}^{1} u^{b_1-1}(1-u)^{c_1-b_1-1}v^{a-c_2}(1-v)^{b_2-1} \, du \, dv
\]
\[
= B(b_1, c_1 - b_1)B(a - c_2 + 1, b_2).
\]
Here the first equality follows from the change of integration variable \(v \to 1 - 1/v\).

Therefore, analytic continuation with respect to the parameters \(a, b_1, b_2, c_1\) and \(c_2\) and the variables \(x\) and \(y\) leads to the result. This completes the proof.

2. Another proof

We give another way to derive (i)–(iv) in our Theorem, which is by direct calculation.

2.1. Proof of (i)

We assume temporarily that \(x\) and \(y\) are real numbers satisfying \(0 < x < 1\) and \(0 < y < 1\) and that \(b_1, b_2, c_1\) and \(c_2\) satisfy \(\text{Re}(c_1) > \text{Re}(b_1) > 0\) and \(\text{Re}(c_2) > \text{Re}(b_2) > 0\).

The multinomial theorem
\[
(1 - X - Y)^{-a} = \sum_{m,n \geq 0} \frac{(a)_{m+n}}{m!n!} X^m Y^n
\]
implies that
\[
\int_{u=0}^{1} \int_{v=0}^{1} u^{b_1-1}v^{b_2-1}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1} (1-xu - yv)^{-a} \, dv \, du
\]
\[
= \sum_{m,n \geq 0} \frac{(a)_{m+n}}{m!n!} X^m Y^n \int_{u=0}^{1} \int_{v=0}^{1} u^{b_1-1+m}v^{b_2-1+n}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1} \, dv \, du
\]
\[
= \sum_{m,n \geq 0} \frac{(a)_{m+n}}{m!n!} X^m Y^n B(b_1 + m, c_1 - b_1)B(b_2 + n, c_2 - b_2)
\]
\[
= B(b_1, c_1 - b_1)B(b_2, c_2 - b_2) \sum_{m,n \geq 0} \frac{(a)_{m+n}(b_1)(b_2)_{m}}{m!n!(c_1)(c_2)_n} X^m Y^n
\]
\[
= B(b_1, c_1 - b_1)B(b_2, c_2 - b_2)F_2(a, b_1, b_2, c_1, c_2; x, y).
\]
Analytic continuation with respect to the parameters \(b_1, b_2, c_1\) and \(c_2\) and the variables \(x\) and \(y\) leads to the result. This completes the proof.

2.2. Proof of (ii)

We assume temporarily that \(x\) and \(y\) are real numbers satisfying \(1 < x\) and \(1 < y\) and that \(a, b_1\) and \(b_2\) satisfy \(\text{Re}(1 - a) > 0, \text{Re}(b_1) > 0\) and \(\text{Re}(b_2) > 0\).
The change of integration variables \((u, v) \mapsto (u/x, v/y)\) and the binomial theorem imply that
\[
\int_{v=0}^{1/y} \int_{u=0}^{(1-uv)/x} u^{b_1-1} v^{b_2-1} (1 - xu - yv)^{-a} (1 - u)^{c_1-b_1-1} (1 - v)^{c_2-b_2-1} \, du \, dv \\
= x^{-b_1} y^{-b_2} \int_{v=0}^{1} \int_{u=0}^{1-v} u^{b_1-1} v^{b_2-1} (1 - u - v)^{-a} \left( 1 - \frac{u}{x} \right)^{c_1-b_1-1} \left( 1 - \frac{v}{y} \right)^{c_2-b_2-1} \, du \, dv \\
= x^{-b_1} y^{-b_2} \sum_{m,n \geq 0} \frac{(b_1 - c_1 + 1)_m (b_2 - c_2 + 1)_n}{m!n!} \left( \frac{1}{x} \right)^m \left( \frac{1}{y} \right)^n \\
\times \int_{v=0}^{1} \int_{u=0}^{1-v} u^{b_1-1+m} v^{b_2-1+n} (1 - u - v)^{-a} \, du \, dv \\
= x^{-b_1} y^{-b_2} \sum_{m,n \geq 0} \frac{(b_1 - c_1 + 1)_m (b_2 - c_2 + 1)_n}{m!n!} \left( \frac{1}{x} \right)^m \left( \frac{1}{y} \right)^n \\
\times \frac{(b_1)_m (b_2)_n}{(1 - a + b_1 + b_2)_m+n} \frac{\Gamma(1-a) \Gamma(b_1) \Gamma(b_2)}{\Gamma(1-a + b_1 + b_2)} \\
= \frac{\Gamma(1-a) \Gamma(b_1) \Gamma(b_2)}{\Gamma(1-a + b_1 + b_2)} x^{-b_1} y^{-b_2} \\
\times \frac{\Gamma(1-a) \Gamma(b_1) \Gamma(b_2)}{\Gamma(1-a + b_1 + b_2)} x^{-b_1} y^{-b_2} \\
\times F_3(b_1, b_2, b_1 - c_1 + 1, b_2 - c_2 + 1, 1 - a + b_1 + b_2; x^{-1}, y^{-1}).
\]
Analytic continuation with respect to the parameters \(a, b_1\) and \(b_2\) and the variables \(x\) and \(y\) leads to the result. This completes the proof.

2.3. Proof of (iii)

We assume temporarily that \(x\) and \(y\) are real numbers satisfying \(0 < x < 1\) and \(1 < y\) and that \(a, b_1, b_2\) and \(c_1\) satisfy \(\text{Re}(b_1) > 0, \text{Re}(b_2) > 0, \text{Re}(1-a) > 0\) and \(\text{Re}(c_1 - b_1) > 0\).

For \(u\) such that \(0 < u < 1\), the change of integration variable \(v \mapsto (1/y)(1 - xu)v\) and the binomial theorem imply that
\[
\int_{v=0}^{(1-xu)/y} v^{b_2-1} (1 - xu - yv)^{-a} (1 - v)^{c_2-b_2-1} \, dv \\
= y^{-b_2} (1 - xu)^{b_2-a} \int_{v=0}^{1} v^{b_2-1} (1 - v)^{-a} \left( \frac{1}{y} \right)^{1-xu} (1 - v)^{c_2-b_2-1} \, dv \\
= y^{-b_2} (1 - xu)^{b_2-a} \sum_{n \geq 0} \frac{(b_2 - c_2 + 1)_n}{n!} \left( \frac{1}{y} \right)^n (1 - xu)^{1+n} \, dv \\
= y^{-b_2} (1 - xu)^{b_2-a} B(b_2, 1-a) \sum_{n \geq 0} \frac{(b_2 - c_2 + 1)_n (b_2)_n}{n!(1-a + b_2)_n} \left( \frac{1}{y} \right)^n (1 - xu)^{n},
\]
On the other hand, we have

\[
\int_0^1 u^{b_1-1}(1-u)^{c_1-b_1-1}(1-xu)^{b_2-a+n} du = \sum_{m \geq 0} \frac{(a-b_2-n)m}{m!} \int_0^1 u^{b_1-1+m}(1-u)^{c_1-b_1-1} du
\]

\[
= B(b_1, c_1 - b_1) \sum_{m \geq 0} \frac{(a-b_2-n)m(b_1)m}{m!(c_1)_m} x^m.
\]

Therefore, we have

\[
\int_0^1 \int_{v=0}^{(1-xu)/y} u^{b_1-1} v^{b_2-1}(1-xu - yv)^{-a}(1-u)^{c_1-b_1-1}(1-v)^{c_2-b_2-1} dv du
\]

\[
= B(b_1, c_1 - b_1)B(b_2, 1-a)y^{-b_2} \sum_{m,n \geq 0} \frac{(a-b_2-n)m(b_1)m(b_2-c_2+1)n(b_2)n}{m!(c_1)_m} x^m y^{-n}
\]

\[
= B(b_1, c_1 - b_1)B(b_2, 1-a)y^{-b_2} \sum_{m,n \geq 0} \frac{(a-b_2-m-n)m(b_1)m(b_2-c_2+1)n(b_2)n}{m!n!(c_1)_m} x^m (-y^{-1})^n
\]

\[
= B(b_1, c_1 - b_1)B(b_2, 1-a)y^{-b_2} H_2(a-b_2, b_1, b_2 - c_2 + 1, b_2, c_1; x, -y^{-1}).
\]

Here we have used the equality

\[
(a-b_2-n)_m = \frac{\Gamma(a-b_2-n+m)}{\Gamma(a-b_2-n)} = \frac{\Gamma(a-b_2-n+m)(a-b_2-n)(a-b_2-n+1) \cdots (a-b_2-1)}{\Gamma(a-b_2)} = \frac{\Gamma(a-b_2-n+m)(b_2-a+1)n(-1)^n}{\Gamma(a-b_2)} = (-1)^n(a-b_2)_m(n(b_2-a+1)_n).
\]

Analytic continuation with respect to the parameters \(a, b_1, b_2\) and \(c_1\) and the variables \(x\) and \(y\) leads to the result. This completes the proof.

2.4. Proof of (iv)

We assume temporarily that \(x\) and \(y\) are real numbers satisfying \(0 < x < 1, 0 < y \) and that \(a, b_2, c_1\) and \(c_2\) satisfy \(\text{Re}(1-a) > 0, \text{Re}(a-c_1+1) > 0, \text{Re}(b_2) > 0\) and \(\text{Re}(2+a-c_1-c_2) > 0\).

For \(v\) such that \(\infty < v < 0\), the change of integration variable

\[
u \mapsto \frac{1-yv}{x} \frac{1}{u}
\]

and the binomial theorem imply that

\[
\int_0^\infty \int_{(1-yv)/x} u^{b_1-1}(u-1)^{c_1-b_1-1}(xu + yv - 1)^{-a} du
\]

\[
= (1-yv)^{c_1-a-1}x^{1-c_1} \int_0^1 u^{a-c_1}(1-u)^{-a} \left(1 - \frac{x}{1-yv}u\right)^{c_1-b_1-1} du
\]
Therefore, we have

\[(1 - yv)^{c_1-a-1} x^{1-c_1} \sum_{m \geq 0} \frac{(b_1 - c_1 + 1)_m}{m!} \left( \frac{x}{1 - yv} \right)^m \int_0^1 u^{a-c_1+m} (1 - u)^{-a} \, du \]

\[= (1 - yv)^{c_1-a-1} x^{1-c_1} B(a - c_1 + 1, 1 - a) \times \sum_{m \geq 0} \frac{(b_1 - c_1 + 1)_m}{m! (2 - c_1)_m} \left( \frac{x}{1 - yv} \right)^m . \]

On the other hand, the change of integration variable \( v \mapsto v/(v - 1) \) and the binomial theorem imply that

\[\int_0^1 v^{b_2-1} (1 - v)^{c_2-b_2-1} (1 - yv)^{c_1-a-1-m} \, dv \]

\[= \int_0^1 v^{b_2-1} (1 - v)^{a-c_1-c_2+1+m} (1 - (1 - y)v)^{c_1-a-1-m} \, dv \]

\[= \sum_{m \geq 0} \frac{(a - c_1 + 1 + m)_n}{n!} (1 - y)^n \int_0^1 v^{b_2-1+n} (1 - v)^{a-c_1-c_2+1+m} \, dv \]

\[= B(a - c_1 - c_2 + 2, b_2) \sum_{m \geq 0} \frac{(a - c_1 + 1 + m)_n (b_2)_n}{m! (a + b_2 - c_1 - c_2 + 2)_{m+n}} (1 - y)^m . \]

Therefore, we have

\[\int_0^1 u^{b_1-1} v^{b_2-1} (xu + yv - 1)^{-a} (u - 1)^{c_1-b_1-1} (1 - v)^{c_2-b_2-1} \, dv \, du \]

\[= B(a - c_1 + 1, 1 - a) B(a - c_1 - c_2 + 2, b_2) x^{1-c_1} \times \sum_{m, n \geq 0} \frac{(a - c_1 + 1)_m (a - c_1 - c_2 + 2)_m (b_1 - c_1 + 1)_m (b_2)_n}{m! n! (a + b_2 - c_1 - c_2 + 2)_{m+n} (2 - c_1)_m} x^m (1 - y)^n \]

\[= B(a - c_1 + 1, 1 - a) B(a - c_1 - c_2 + 2, b_2) x^{1-c_1} \times F_P(a - c_1 + 1, b_1 - c_1 + 1, b_2, 2 - c_1, c_2; x, y) . \]

Analytic continuation with respect to the parameters \( a, b_2, c_1 \) and \( c_2 \) and the variables \( x \) and \( y \) leads to the result. This completes the proof.

2.5. Proof of (v)

We assume temporarily that \( x \) and \( y \) are real numbers satisfying \( 0 < x, 0 < y \) and \( 0 < x + y < 1 \) and that \( a, b_1, b_2, c_1 \) and \( c_2 \) satisfy \( \Re(b_1) > 0, \Re(c_1 - b_1) > 0, \Re(b_2) > 0 \) and \( \Re(a - c_2 + 1) > 0 \).

For \( v \) such that \( \infty < v < 0 \), the binomial theorem implies that

\[\int_0^1 u^{b_1-1} (1 - u)^{c_1-b_1-1} (1 - xu - yv)^{-a} \, du \]

\[= (1 - yv)^{-a} \int_0^1 u^{b_1-1} (1 - u)^{c_1-b_1-1} \left( 1 - \frac{x}{1 - yv} u \right)^{-a} \, du . \]
K. Mimachi and M. Noumi. Solutions in terms of integrals of multivalued functions for the classical problem.

M. Kita. On hypergeometric functions in several variables. I. New integral representations of Euler type.

E. Diekema and T. H. Koornwinder. Integral representations for Horn’s $H_2$ function and Olsson’s $F_P$ function. Kyushu J. Math. 73 (2019), 1–24.

M. Kato. Connection formulas and irreducibility conditions for Appell’s $F_2$. Kyushu J. Math. 66 (2012), 325–363.

M. Kita. On hypergeometric functions in several variables. I. New integral representations of Euler type. Japan. J. Math. 18 (1992), 25–74.

K. Mimachi. Nakagiri’s monodromy representations associated with Appell’s hypergeometric functions $F_2$ and $F_3$. Funkcial. Ekvac. 60 (2017), 77–132.

K. Mimachi and M. Noumi. Solutions in terms of integrals of multivalued functions for the classical hypergeometric equations and the hypergeometric system on the configuration space. Kyushu J. Math. 70 (2016), 315–342.
P. O. Olsson. On the integration of the differential equations of five-parametric double-hypergeometric functions of second order. J. Math. Phys. 18 (1977), 1285–1294.

J. Sekiguchi. A global representation of the Appell function $F_2$. RIMS Kokyuroku (Kyoto Univ.) 773 (1991), 66–77.

N. Takayama. Propagation of singularities of solutions of the Euler–Darboux equation and a global structure of the space of holonomic solutions II. Funkcial. Ekvac. 36 (1993), 187–234.

M. Yoshida. Euler integral transformations of hypergeometric functions of two variables. Hiroshima Math. J. 10 (1980), 329–335.