Dynamics of radiation dominated branes:
Vacuum dynamics from radiation

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Abstract

I point out a symmetry, between equations of state for polytropic fluids, in the equation of motion of a spherically symmetric singular shell embedded in 4-d and 5-d vacuum spacetimes. In particular the equation of motion of a shell consisting of radiation has the same form as for a vacuum shell or domain wall.

1 Introduction

In general relativity geometry and energy are non-linearly connected, and to study its dynamics is in general very complicated. Therefore we introduce symmetries. I shall consider spherically symmetric static $d+1$ dimensional spacetimes, $\mathcal{M}$, for $d = 3$ and $d = 4$; this could be Minkowski, Schwarzschild, de-Sitter and anti de-Sitter spacetimes. In these spacetimes we embed matter or energy confined to a $d$ dimensional surface or hypersurface (also denoted as a $d$-1 brane, referring to the number of spatial dimensions), i.e. the length scales are such that the thickness of the surface can be ignored. And the equation of motion is reduced to the 1-dimensional evolution of the radius of the surface.

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A very useful formalism to study such singular surfaces in general relativity was introduced by Israel [1] and has been extensively studied both for static and non-static spacetimes, see [2] and references therein.

The surface does not in general follow the geodesics of the background or bulk spacetime. This gives rise to some interesting results, e.g. radiation, material and vacuum shells that have no gravitational field. Radiation (photonic) shells have been studied in a cylindrical spacetime [3, 4]. In a 4-dimensional spherical spacetime of pure vacuum (Λ = 0) we show that for a radiation shell we get the potential of a domain wall [5] and discuss the solutions.

Recent observations [6, 7] indicate that the expansion of the universe is accelerating. Indications for this was also found in 1966 [8]. Motivated by string theory one considers 5-dimensional spacetimes where a singular shell or brane represents the universe. (See [9] for a review on brane cosmology.) The acceleration of the universe could be explained by a form of dark energy. This can be described in three ways: by a cosmological constant Λ, an ultralight scalar field or by a modification of Einstein’s equations, e.g. [10] where a radiation dominated brane is found to have accelerated expansion. The main new point of the present paper is that electro-magnetic radiation can contribute to Λ without any corrections to the field equations. Here I do not invoke the $Z_2$ symmetry.

2 Equation of motion

To study singular shells or branes in general relativity we use the metric junction method of Israel [1]. Hence, the spacetime manifold, $\mathcal{M}$, is split into two parts $\mathcal{M}^+$ and $\mathcal{M}^-$ separated by a common boundary $\Sigma$. The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu}, \quad (1)$$

$\kappa^2$ is the gravitational coupling constant in $d + 1$ dimensions. In the 4-dimensional spacetime of general relativity we have $\kappa^2 = \frac{8\pi G}{c^4}$. We use greek indices to denote the $d + 1$ coordinates of the bulk $\mathcal{M}^\pm$ and latin indices for the $d$ coordinates on the hypersurface $\Sigma$.

The equation of motion for the hypersurface is found from evaluating the Einstein field equations [1] across the boundary and is contained in the
Lanczos equation, which for a \( d + 1 \) dimensional spacetime is

\[
[K_{ij}] = \kappa^2 \left( S_{ij} - \frac{1}{d-1} S_{gij} \right),
\]

(2)

where \([K_{ij}]\) is the discontinuity of the extrinsic curvature tensor at the surface. \( K_{ij} \) is defined as the covariant derivative of the normal vector \( n \) to the hypersurface: \( K_{ij} = -e_i \cdot \nabla_j n \). \( S_{ij} \) is the energy-momentum tensor for the surface. For an ideal fluid \( S_{ij} \) is given by

\[
S_{ij} = (\rho + p) u_i u_j + p g_{ij},
\]

(3)

where \( \rho \) is the mass (energy) density of the surface and \( p \) is the tangential pressure of the surface. For a radiation dominated brane we have \( S_{ij} = 0 \). Thus, for radiation \( p = \frac{1}{d-1} \rho \).

The continuity equation for the brane is

\[
S_{ij} |_j + [T_{i\mu} n^\mu] = 0,
\]

(4)

where \( | \) denotes covariant derivative with respect to the metric connection on the brane. For a vacuum bulk we have \([T_{i\mu} n^\mu] = 0\), i.e. there are no normal forces exerted on the brane by the surroundings. We also have:

\[
S_{ij} [K^{ij}] = \kappa^2 \left( S_{ij} S^{ij} - \frac{1}{d-1} S^2 \right),
\]

(5)

\[
S_{ij} \{K^{ij}\} = -[T_{\mu\nu} n^\mu n^\nu].
\]

(6)

### 3 4-dimensional spacetimes

For spherically symmetric static 4-dimensional spacetimes the metric in \( \mathcal{M}^\pm \) can be written

\[
ds_\pm^2 = -f_\pm dt_\pm^2 + f_\pm^{-1} dR^2 + R^2 d\Omega^2.
\]

(7)

We shall look at a shell embedded in vacuum so that outside the shell we have Schwarzschild/de-Sitter spacetimes:

\[
f_\pm = 1 - \frac{2m_\pm}{R} - \frac{\Lambda_\pm}{3} R^2.
\]

(8)

The Schwarzschild mass \( m_- \) gives a mass at the center and \( m_+ = m_- + m \), with \( m \) the Schwarzschild mass of the shell.
For intrinsic coordinates we use proper time, $\tau$, on the shell. Thus, the metric on the shell is
\[ ds^2_\Sigma = -d\tau^2 + R^2 d\Omega^2. \] (9)

The normal vector to the shell is
\[ n_\alpha^\pm = (-\dot{R}, \dot{T}_\pm, 0, 0). \] (10)

From requiring that we induce the same metric on $\Sigma$ from $\mathcal{M}^-$ and $\mathcal{M}^+$ we have $\dot{T}_\pm = f_\pm^{-1} \sqrt{f_\pm + \dot{R}^2}$, where $\dot{\equiv} \frac{d}{d\tau}$. In this case the continuity equation (4) for an ideal fluid (3) gives
\[ \dot{\rho} = -2(\rho + p) \frac{\dot{R}}{R}. \] (11)

For an equation of state $p = \omega \rho$ with $\omega$ constant, this is readily solved. Thus, $\rho$ in terms of the trajectories $R(\tau)$ of the shell is
\[ \rho = b R^{-2(1+\omega)}, \quad b = constant. \] (12)

For a dust shell, $\omega = 0$, the constant $b$ gives a measure of the total rest mass of the shell.

The components of the extrinsic curvature tensor are: $K^{\pm}_\theta \theta = \zeta_\pm R \sqrt{f_\pm + \dot{R}^2}$ and $K^{\pm}_\tau \tau = \frac{-\zeta_\pm}{\sqrt{f_\pm + \dot{R}^2}}(\dot{R} + \frac{1}{2} \frac{\partial f_\pm}{\partial R})$, where $\zeta = \pm 1$. For static spacetimes the sign given by $\zeta$ gives the spatial topology. The sign $\zeta_\pm$ can be found from squaring the angular component of (2):
\[ K^{\pm}_\theta \theta = \frac{1}{8\pi \rho} (f_+ - f_-) \pm 2\pi R^2 \rho \] (13)

where the $\pm$ superscript corresponds to the $\pm$ in the equation respectively. An expression for the acceleration not containing discontinuities or averages is given by [2]
\[ K^{\pm}_{\tau \tau} = -\frac{\left[T_{\mu \nu} \dot{n}^\mu \dot{n}^\nu\right]}{\rho} - \frac{2p}{R^2 \rho} K^{\pm}_{\theta \theta} \pm \frac{1}{2} \kappa^2 \left(\frac{1}{2} \rho + 2p\right), \] (14)

\footnote{With our sign convention the area of the shell is increasing in the normal direction for $\zeta = -1$ and decreasing for $\zeta = +1$. $\zeta$ can also be identified with the angle $\arctan\frac{v}{u}$ in Kruskal-Szekers type coordinates, see [3]: for $\zeta = -1$ the angle is increasing along the shells trajectory and for $\zeta = +1$ the angle is decreasing along the trajectory.}
where the normal component of the four acceleration is given by \( a^\pm \equiv a_{\mu}n^\mu|_\pm = K^\pm_{\tau\tau} \). The first term gives the difference in the normal force on the shell from the bulk, and we have \([T_{\mu\nu}n^\mu n^\nu]\) = 0. The second term is due to the pressure of the shell and the geometry of the embedding. The last term gives the self gravitational attraction. Thus, from (14) and (13) we have for \( \Lambda_-=\Lambda_+=\Lambda \):

\[
a^\pm = -\frac{2\omega}{R^2} \left( -\frac{m}{4\pi R\rho} \pm 2\pi R^2 \rho \right) \pm 4\pi \rho \left( \frac{1}{2} + 2\omega \right),
\]

independent of \( m_- \) and \( \Lambda \). From (13) we find that \( \zeta_- = -1 \) for all \( R \), while \( \zeta_+ \) will change sign during the history of the shell. (\( \zeta_- \) changes sign if \( \Lambda_- > \Lambda_+ \)).

From the angular component of (2) the equation of motion for a surface consisting of an ideal fluid is:

\[
\dot{R}^2 = \frac{1}{(\kappa^2 R\rho)^2} (f_+ - f_-)^2 - \frac{1}{2} (f_+ + f_-) + \frac{1}{2} (\kappa^2 R\rho)^2.
\]

(16)

Taking the derivative of this and using the continuity equation (11) gives the time component of the Lanczos equation (2), i.e. the dynamics is contained in (16).

From eq. (16) using eq. (12) we get\(^2\)

\[
\dot{R}^2 = \left( \frac{m}{4\pi b} \right)^2 R^{4\omega} + (2\pi b)^2 R^{-(4\omega+2)} + \frac{m + 2m_-}{R} + \frac{\Lambda}{3} R^2 - 1.
\]

(17)

The first term comes from the embedding, the second term is from the gravitational self interaction of the shell, and the last terms are due to the background.

We observe an interesting symmetry in eq. (17). For equation of states given by \( \omega_1 \) and \( \omega_2 \) where

\[
\omega_2 = -\left( \omega_1 + \frac{1}{2} \right)
\]

(18)

the equation of motion (17) is invariant under the transformation \( \omega_1 \to \omega_2 \) with the interchange

\[
\frac{m}{4\pi b} \leftrightarrow 2\pi b.
\]

\(^2\)We have set \( \kappa^2 = 8\pi \), i.e. in geometrical units \( c = 1 \) and \( G = 1 \).
That is, a radiation shell $\omega = \frac{1}{2}$ is symmetric to vacuum shell (or domain wall) $\omega = -1$, while a dust shell, $\omega = 0$, is symmetric to a shell with $\omega = -\frac{1}{2}$.

For $\omega_1 > \frac{1}{2}$ we have $\omega_2 < -1$, and for equation of state $-\frac{1}{4}$ we get $\omega_1 = \omega_2$.

Let us look at solutions of eq. (17) with $\Lambda = 0$ for small and large $R$. For a radiation shell at small $R$ the motion is dictated by the second term, i.e. $R \propto \tau^{\frac{1}{3}}$. For large $R$ the first term dominates giving exponential evolution, $\ln R \propto \tau$. A dust shell at small $R$ evolves with $R \propto \sqrt{\tau}$, while at large $R$ we find $R \propto \tau$.

### 3.1 Radiation shells

For a radiation shell in a Schwarzschild/de-Sitter background we get:

$$\dot{R}^2 = \alpha R^2 + \frac{(2\pi b)^2}{R^4} + \frac{m + 2m_-}{R} - 1,$$

(20)

where

$$\alpha = \frac{m^2}{(4\pi b)^2} + \frac{\Lambda}{3}.$$  

(21)

The acceleration can be written from (15)

$$a^\pm = \frac{m}{4\pi b} \pm \frac{4\pi b}{R^3},$$

(22)

we see that $a^-$ changes sign at $R^3 = (4\pi b)^2$.

The total constant mass, $m$, of the shell can be written

$$m = \frac{4\pi b}{R} \sqrt{f_- + \dot{R}^2 - \left(\frac{4\pi b}{m}\right)^2}$$  

$$= \frac{4\pi b}{2R} \left(\zeta_+ \sqrt{f_+ + \dot{R}^2 + \zeta_- \sqrt{f_- + \dot{R}^2}}\right).$$

(23)

If $\zeta_+ = +1$ ($\zeta_- = -1$) throughout the history of the shell the equation of motion allows for a shell with zero gravitational mass, i.e. the shell is embedded in a flat spacetime.

Let us consider the case where $\Lambda = 0$ and $m_- = 0$. The equation of motion for the shell can also be viewed as describing a particle in a 1-dimensional potential, and introducing the variables:

$$Z^3 = \frac{2m}{(4\pi b)^2} R^3$$

(24)

$$t = \frac{m}{4\pi b} \tau,$$

(25)
Figure 1: Graph (a) shows the potential $V(Z)$ for a radiation dominated thin shell. The dashed line gives the event horizon associated with this potential. In (b) we show the 4 different solutions. The dotted line is where the potential and the horizon are equal and where the sign $\zeta_+$ changes sign.

we obtain the same potential as in [5] for a domain wall\(^3\):

$$\left( \frac{dZ}{dt} \right)^2 + V = E, \quad (26)$$

where

$$V = - \left( \frac{Z^3 - 1}{Z^2} \right)^2 - \frac{4}{Z} \quad (27)$$

$$E = \frac{-4(4\pi b)^{\frac{3}{2}}}{(2m)^{\frac{3}{2}}} \quad (28)$$

The Schwarzschild horizon is given by $R_H = 2m$ which from eq. (24) and eq. (28) leads to $E(Z_H) = -\frac{4}{Z_H}$. The potential is shown in fig. (a). For all the details on this potential see [5].

\(^3\)In their notation the potential is for $\gamma = 2$ which is for $\chi = 0$, i.e. $\Lambda_\pm = 0$. 

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We have seen that \( \zeta_- = -1 \) for all \( R \), while for \( \zeta_+ \) we find changes from +1 to -1 at \( Z_+ = 1 \) for increasing \( Z \). Also, the mass for this energy level \( V(Z_+) = E(m_+) \) is
\[
m_+ = (\pi b)^{\frac{1}{2}},
\]i.e. if we have \( m > m_+ \) then \( \zeta_+ \) changes sign during the motion of the shell.

The maximum of the potential is at \( Z_m^3 = 2 \), see (22) and (24), and is found to be \( V(Z_m) = -\frac{9}{2^3} \). The critical mass for unbound motion, \( V(Z_m) = E(m_{cr}) \), is
\[
m_{cr} = \left( \frac{32}{27} \pi b \right)^{\frac{1}{2}} \approx 1.09 (\pi b)^{\frac{1}{2}}.
\]
For \( m > m_{cr} \) we have unbound motion, solution IV in fig 11. In all we have four solutions:

Solution I: This is a bound solution for a mass \( m < m_+ \). So that \( \zeta_+ = +1 \) throughout the history of the shell. The shell expands from \( R = 0 \) to a maximum \( R \) outside the Schwarzschild horizon and then contracts, see fig. 8 (b) in [5]. Also, this solution admits a zero gravitational mass, \( m = 0 \).

Solution II: This is also a bound solution, with \( m \) in the range \( m_+ < m < m_{cr} \). So that now \( \zeta_+ = -1 \) when the shell crosses the horizon. See fig. 7 (b) in [5].

Solution III: This is a bounce solution for \( m < m_{cr} \). Here \( \zeta_+ = -1 \) throughout the history of the shell. The shell comes in from infinity stops at a minimum \( R \) outside the Schwarzschild horizon and heads out again. This corresponds to fig. 10 (b) in [5] with the shell in region I instead of region II.

Solution IV: This is an unbound solution, where \( m > m_{cr} \). Here \( \zeta_+ = -1 \) when the shell crosses the horizon. This corresponds to fig. 11 (b) in [5] with the shell moving into region I instead of region II.

Consider now a shell consisting of radiation and dust, \( \rho = \rho_r + \rho_d \). The radiation density goes as \( R^{-3} \) while the dust density goes with \( R^{-2} \). Hence,

\footnote{For a domain wall it is opposite, \( \zeta_+ \) goes from -1 to +1.}
for small $R$ the motion will be dictated by the radiation, and for large $R$ the motion is determined by the dust; i.e. there is no exponential expansion for large $R$ as for a domain wall containing dust.

4 5-dimensional spacetimes

We now consider a brane cosmological model without the $Z_2$ symmetry. We look at a brane that represents the boundary between Schwarzschild/(anti)de-Sitter spacetimes, see also III. The 5-dimensional spacetime metric can be written

$$ds^2 = -F dT^2 + F^{-1} dR^2 + R^2 d\Omega_4^2$$

(31)

with

$$d\Omega_4^2 = \frac{d\chi^2}{1-k\chi^2} + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)$$

(32)

and

$$F_\pm = k_\pm - C_\pm R^{-2} - \frac{\lambda_\pm}{6} R^2,$$

(33)

$C$ is the 5-dimensional equivalent of the Schwarzschild mass and $\lambda$ is the cosmological constant of the bulk. $k = 0, \pm 1$ gives the spatial curvature, planar or spherical/hyperboloidal.

The intrinsic metric on the brane is

$$ds^2_\Sigma = -d\tau^2 + R^2 d\Omega_4^2,$$

(34)

with $R = R(\tau)$ representing the expansion factor of the universe.

The continuity equation gives

$$\dot{\rho} = -3(\rho + p) \frac{\dot{R}}{R}$$

(35)

with solution for a polytropic equation of state, $p = \omega \rho$, given by

$$\rho = b R^{-3(\omega+1)}.$$  

(36)

The spatial components of the extrinsic curvature tensor are up to a sign

$$K_{ij}^\pm = \sqrt{f_\pm + \dot{R}^2} R g_{ij}.$$  

(37)
The resulting Friedmann equation has the same form as eq. \((16)\):

\[
\dot{R}^2 = \frac{1}{(\frac{7}{3}\kappa^2 R\rho)^2} (F_+ - F_-)^2 - \frac{1}{2} (F_+ + F_-) + \left(\frac{\kappa^2}{6} R\rho\right)^2.
\]  \(38\)

Here we have \(\rho^2\) and \(\rho^{-2}\) dependence. In \(Z_2\) symmetric brane cosmology the first term vanishes, i.e. we only have the \(\rho^2\) dependence.

Inserting for \(\rho\) from eq. \((36)\) we arrive at

\[
\dot{R}^2 = \left(\frac{3\mathcal{C}}{2\kappa^2 b}\right)^2 R^{6\omega} + \left(\frac{\kappa^2 b}{6}\right)^2 R^{-(6\omega+4)} + \frac{\Lambda}{6} R^2 + \frac{C}{2R^2} - k,
\]  \(39\)

with \(k = k_+ = k_-\) and \(\lambda = \lambda_+ = \lambda_-\), and where we have set \(\mathcal{C}_- = 0\). Thus, this gives the equation of motion for the boundary between (anti)de-Sitter or Minkowski and Schwarzschild/(anti)de-Sitter spacetimes. Equation \((39)\) is invariant under the transformation \(\omega_1 \to \omega_2\) where

\[
\omega_2 = -\left(\omega_1 + \frac{2}{3}\right)
\]  \(40\)

with the interchange

\[
\frac{3\mathcal{C}}{2\kappa^2 b} \leftrightarrow \frac{\kappa^2 b}{6}.
\]  \(41\)

Hence, radiation, \(\omega = \frac{1}{3}\), corresponds to Lorentz invariant vacuum energy, \(\omega = -1\). For \(\omega_1 > \frac{1}{3}\) we get \(\omega_2 < -1\). A dust universe, \(\omega = 0\), is symmetric to \(\omega = -\frac{2}{3}\), which represent a topological defect in form of a domain wall. A cosmic string \(\omega = -\frac{1}{3}\) has no symmetry, \(\omega_1 = \omega_2\).

### 4.1 Radiation and dust branes

For a radiation dominated universe the Friedmann equation becomes

\[
\dot{R}^2 = \alpha R^2 + \beta R^{-6} + \frac{\mathcal{C}}{2R^2} - k
\]  \(42\)

with

\[
\alpha = \left(\frac{3\mathcal{C}}{2\kappa^2 b}\right)^2 + \frac{\lambda}{6}, \quad \beta = \left(\frac{\kappa^2 b}{6}\right)^2.
\]  \(43\)

For \(k = 0\), and defining \(X = R^4\) this can be written

\[
\frac{\dot{X}^2}{16} = \alpha X^2 + \frac{\mathcal{C}}{2} X + \beta.
\]  \(44\)
This has same form as a $Z_2$ symmetric brane with radiation and vacuum energy (with $C = 0$), see e.g. [9]. Integrating (42) with $\alpha = 0$ we have the expansion $R^4 = 2Ct^2 + 4\sqrt{\beta}t$. Thus, at late time we have standard evolution. But for a brane containing radiation and dust the equation of motion will at late times be determined by the dust component. To consider dust solutions we set $\lambda = 0$ and define $Y = R^2$ to find

$$\frac{Y^2}{4} = \alpha'Y + \beta Y^{-1} + \frac{C}{2}$$

(45)

with

$$\alpha' = \left(\frac{3C}{2k^2}\right)^2 - k.$$  

(46)

This is solved by elliptical integrals. We only look at the limits. For small $R$ we get the evolution $R \propto \tau^{\frac{1}{3}}$, for large $R$ we get $R \propto \tau$, i.e. we do not get the standard evolution of $\tau^{\frac{2}{3}}$.

5 Conclusions

We have studied spherically symmetric branes embedded in static spacetimes, and shown that the equation of motion in 4-dimensional spacetimes, eq. (17), and 5-dimensional spacetimes, eq. (39), have the same form for different polytropic fluids given by $\omega_1$ and $\omega_2$. Where $\omega_1$ and $\omega_2$ are related by eq. (18) and eq. (40) respectively. This symmetry comes from the embedding and is of second order in the Schwarzschild mass of the shell. An interesting case is that a radiation shell corresponds to a vacuum shell or domain wall.

In 4 dimensions we investigate the equation of motion for a radiation shell embedded in an empty spacetime, i.e. Minkowski and Schwarzschild spacetimes outside the shell. The potential describing the motion is identical to that of a domain wall studied exhaustively in [5]. We find different global properties. Also, in contrast to a domain wall, for a radiation shell with added dust the dust component will dominate for large radii.

In 5 dimensions we show that the equation of motion for a radiation brane has the same form as a mirror, $Z_2$, symmetric brane with vacuum energy and radiation. Where for late times we find standard cosmological evolution. For a dust brane we find that it evolves as Randall-Sundrum for early times will at late times the evolution is proportional to cosmic time $\tau$. 

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