DUALITY FOR ABELIAN ANDERSON
T-MOTIVES, AND RELATED PROBLEMS

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Abstract. Let $M$ be an abelian Anderson T-motive. We introduce the notion of
duality for $M$. Main results of the paper (we consider uniformizable $M$ over $\mathbb{F}_q[T]$)
of rank $r$, dimension $n$, whose nilpotent operator $N$ is 0):
1. Algebraic duality implies analytic duality (Theorem 5). Explicitly, this means
that the lattice of the dual of $M$ is the dual of the lattice of $M$.
2. Let $n = r - 1$. There is a $1 - 1$ correspondence between pure T-motives (all
they are uniformizable), and lattices of rank $r$ in $\mathbb{C}_\infty^n$ having dual (Corollary 8.1.4).
3. Let $r = 2n$. We consider the lattice map in a neighbourhood of a T-motive
$M_0$ which is the $n$-th power of the Carlitz module of rank 2. We prove that this
map is surjective, and its fibre is discrete infinite (Proposition 8.2.3). To get locally
a $1 - 1$ correspondence between T-motives and lattices, we introduce a notion of
C-lattice which is a pair: \{a lattice + a fixed class of its bases\} (Definition 8.2.1).
Open problem: what is an analog of these results for the case $(r, n) = 1$?
4. Pure T-motives have duals which are pure T-motives as well (Theorem 10.3).
5. For a self-dual uniformizable $M$ a polarization form on its lattice $L(M)$ is
defined. For some $M$ this form is skew symmetric like in the number field case, but
for some other $M$ it is symmetric. An example is given.
6. We define Hodge filtration of cohomology and for the above self-dual $M$ we
formulate Hodge conjecture in codimension 1 (Section 9).
7. Some explicit results are proved for $M$ having complete multiplication. The
CM-type of the dual of $M$ is the complement of the CM-type of $M$. Moreover, for
$M$ having multiplication by a division algebra there exists a simple formula for the
CM-type of the dual of $M$ (Section 12).
8. We construct a class of non-pure T-motives (T-motives having the completely
non-pure row echelon form) for which duals are explicitly calculated (Theorem 11.5).
This is the first step of the problem of description of all abelian T-motives having
duals.

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0. Introduction.

Main results of the paper are given in the abstract. A more detailed list is the following. For simplicity, most results are proved for the ring $A = \mathbb{F}_q[T]$. The main definition of duality of abelian $T$-motives (definition 1.8 — case $A = \mathbb{F}_q[T]$ and definition 1.13 — general case) is given in Section 1. Section 2 contains the definition of the dual lattice. For the reader’s convenience, we give in Section 3 explicit formulas for the dual lattice (this section can be skipped at the first reading). Section 5 contains the statement and the proof of the main theorem 5 — coincidence of algebraic and analytic duality for the case $A = \mathbb{F}_q[T]$ (section 4 contains the statement of the corresponding conjecture for the case of general $A$). Section 6 contains the theorem 6.2 describing the lattice of the tensor product of two abelian $T$-motives (case $N = 0$; the proof for the general case was obtained, but not published, by Anderson). Section 7 contains the notion of polarization form, including example 7.7 of an abelian $T$-motive having symmetric polarization form. We discuss in Section 8 the problem of correspondence between uniformizable abelian $T$-motives and lattices. We define in Section 9 the notion of Hodge filtration, and we state the Hodge conjecture in codimension 1.

Remaining parts of the paper are of secondary importance, they are independent of the main results and between themselves. Lemma 1.10 gives the explicit matrix form of the definition of duality of abelian $T$-motives. Since Taguchi in [T] gave a definition of dual to a Drinfeld module, we prove in Proposition 1.12.3 that the definition of the present paper is equivalent to the original definition of Taguchi. Section 1.14 contains a definition of duality for abelian $\tau$-sheaves ([BH], Definition 2.1), but we do not develop this subject. We prove in Section 10 that pure $T$-motives have duals which are pure $T$-motives as well, and some related results (a proof that the dual of an abelian $\tau$-sheaf is also an abelian $\tau$-sheaf can be obtained using ideas of Section 10). In Section 11 we consider $T$-motives having the completely non-pure row echelon form, and we give an explicit formula for their duals. In Section 12 we consider abelian $T$-motives with complete multiplication, and we give for them a very simple version of the proof of the first part of the main theorem. Section 13 contains some explicit formulas for abelian $T$-motives of complete multiplication. In 13.1 we describe the dual lattice, in 13.2 we show that the results of Section 12 are compatible with (the first form of) the main theorem of complete multiplication. Section 13.3 contains an explicit proof of the main theorem for abelian $T$-motives with complete multiplication by two types of simplest fields. Section 13.4 gives us an application of the notion of duality to the reduction of abelian $T$-motives (subject in development, see [L2]).

Notations.

$q$ is a power of $p$;
$C_\infty$ is the completion of the algebraic closure of $\mathbb{F}_q((1/\theta))$
(it is the function field analog of $C$);
For the case $A = \mathbb{F}_q[T]$ we denote $K = \mathbb{F}_q(T)$, $K_\infty = \mathbb{F}_q((T^{-1}))$;
For the general case $K$ is a finite separable field extension of $\mathbb{F}_q(T)$,
$\infty$ is a fixed valuation of $K$ over the infinity of $\mathbb{F}_q(T)$;
$A \subset K$ is the subring of elements which are regular outside $\infty$;
$K_\infty$ is the completion of $K$ at $\infty$;
$\iota : A \rightarrow C_\infty (\iota(T) = \theta)$ is the standard map of generic characteristic;
\[ K_C = K \otimes \mathbb{C}_\infty, \ A_C = A \otimes \mathbb{C}_\infty. \]

\( M_r \) is the set of \( r \times r \) matrices. If \( C \in M_r(\mathbb{C}_\infty[T]) \) or \( C \in M_r(\mathbb{C}_\infty(T)) \) is a matrix then \( C^t \) is its transposed, \( C^{t^{-1}} = (C^t)^{-1} \), \( C^{(i)} \) is obtained by raising of all coefficients of polynomials or rational functions, which are entries of \( C \), in \( q^i \)-th power, and \( C^{(i)^{-1}} = (C^{(i)})^{-1} \).

\[ \sigma : \mathbb{C}_\infty \to \mathbb{C}_\infty \] the Frobenius automorphism of \( \mathbb{C}_\infty \), i.e. \( \sigma(z) = z^q \).

We extend \( \sigma \) to \( K_C \), resp. \( A_C \) by the formula \( \sigma_C(k \otimes z) = k \otimes \sigma(z), \) \( z \in \mathbb{C}_\infty \), \( k \in K \), resp. \( k \in A \).

If \( M \) is a \( K_C \)-module, resp. \( A_C \)-module, we define \( M^{(1)} \) as the tensor product \( M \otimes_{K_C, \sigma} K_C \), resp. \( M \otimes_{A_C, \sigma} A_C \) (this notation is concordant with the above notation \( C^{(1)} \): we consider \( M \mapsto M^{(1)} \) as a functor; if \( M \) is free \( \mathbb{C}_\infty[T] \)-module and \( \alpha : M \to M \) is a map whose matrix in some basis \( \{m_i\} \) is \( C \), then the matrix of \( \alpha^{(1)} : M^{(1)} \to M^{(1)} \) in the base \( \{m_i \otimes 1\} \) is \( C^{(1)} \)).

\( \mathcal{C} \) is a Drinfeld module of rank 1 over \( A \) (particularly, if \( A = \mathbb{F}_q[T] \) then \( \mathcal{C} \) is the Carlitz module).

For an abelian T-motive \( M \) we denote by \( E = E(M) \) the corresponding T-module (see \cite{G}, Theorem 5.4.11; Goss uses the inverse functor \( E \mapsto M = M(E) \)), and Lie \( (M) \) is Lie \( (E(M)) \) (\cite{G}, 5.4).

\( E_r \) is the unit matrix of size \( r \).

1. Definitions.

**Case** \( A = \mathbb{F}_q[T] \). Let \( \mathbb{C}_\infty[T, \tau] \) be the standard ring of non-commutative polynomials used in the definition of T-motives for the case \( A = \mathbb{F}_q[T] \), i.e. for \( a \in \mathbb{C}_\infty \)

\[ Ta = aT, \ T\tau = \tau T, \ \tau a = a^q \tau \tag{1.1} \]

We need also an extension of \( \mathbb{C}_\infty[T, \tau] \) — the ring \( \mathbb{C}_\infty(T)[\tau] \) which is the ring of non-commutative polynomials in \( \tau \) over the field of rational functions \( \mathbb{C}_\infty(T) \) with the same relations (1.1). For a left \( \mathbb{C}_\infty[T, \tau] \)-module \( M \) we denote by \( M_{\mathbb{C}_\infty[T]} \) the same \( M \) treated as a \( \mathbb{C}_\infty[T] \)-module with respect to the natural inclusion \( \mathbb{C}_\infty[T] \to \mathbb{C}_\infty[T, \tau] \). Analogously, we define \( M_{\mathbb{C}_\infty[\tau]} \); we shall use similar notations also for the left \( \mathbb{C}_\infty(T)[\tau] \)-modules.

Obviously we have:

(1.2) For \( C \in M_r(\mathbb{C}_\infty(T)) \) operations \( C^t, C^{-1} \) and \( C^{(i)} \) commute.

**Definition 1.3.** ([G], 5.4.2, 5.4.12, 5.4.10). An abelian T-motive \( M \) is a left \( \mathbb{C}_\infty[T, \tau] \)-module which is free and finitely generated as both \( \mathbb{C}_\infty[T] \)-, \( \mathbb{C}_\infty[\tau] \)-module and such that

\[ \exists m = m(M) \text{ such that } (T - \theta)^m M/\tau M = 0 \tag{1.3.1} \]

Abelian T-motives are main objects of the present paper. If we affirm that an object exists this means that it exists as an abelian T-motive if otherwise is not stated. We denote dimension of \( M \) over \( \mathbb{C}_\infty[\tau] \) (resp. \( \mathbb{C}_\infty[T] \)) by \( n \) (resp. \( r \)), these numbers are called dimension and rank of \( M \). Morphisms of abelian T-motives are morphisms of left \( \mathbb{C}_\infty[T, \tau] \)-modules.

To define a left \( \mathbb{C}_\infty[T, \tau] \)-module \( M \) is the same as to define a left \( \mathbb{C}_\infty[T] \)-module \( M_{\mathbb{C}_\infty[T]} \) endowed by an action of \( \tau \) satisfying \( \tau(Pm) = P^{(1)}\tau(m), P \in \mathbb{C}_\infty[T] \).

In this situation we can also treat \( \tau \) as a \( \mathbb{C}_\infty[T] \)-linear map \( M^{(1)} \to M \). This interpretation is necessary if we consider the general case \( A \supset \mathbb{F}_q[T] \).
We need two categories which are larger than the category of abelian T-motives.

**Definition 1.4.** A pré-T-motive is a left $\mathbb{C}_\infty[T,\tau]$-module which is free and finitely generated as $\mathbb{C}_\infty[T]$-module, and satisfies (1.3.1).

**Definition 1.5.** A rational pré-T-motive is a left $\mathbb{C}_\infty(T)[\tau]$-module which is free and finitely generated as $\mathbb{C}_\infty(T)$-module.

**Remark 1.6.** An analog of (1.3.1) does not exist for them.

There is an obvious functor from the category of abelian T-motives to the category of pré-T-motives which is fully faithful, and an obvious functor from the category of pré-T-motives to the category of rational pré-T-motives. We denote these functors by $i_1, i_2$ respectively. It is easy to see (Remark 10.2.3) that if $M$ is a pré-T-motive then the action of $\tau$ on $i_2(M)$ is invertible.

Let $M, N$ be rational pré-T-motives such that the action of $\tau$ on $M_{\mathbb{C}_\infty(T)}$ is invertible.

**Definition 1.7.** $\text{Hom}(M, N)$ is a rational pré-T-motive such that

$$\text{Hom}(M, N)_{\mathbb{C}_\infty(T)} = \text{Hom}_{\mathbb{C}_\infty(T)}(M_{\mathbb{C}_\infty(T)}, N_{\mathbb{C}_\infty(T)})$$

and the action of $\tau$ is defined by the usual manner: for $\varphi : M \to N$, $m \in M$

$$(\tau \varphi)(m) = \tau(\varphi(\tau^{-1}(m)))$$

**Definition 1.8.** Let $M$ be an abelian T-motive and $\mu$ a positive number. An abelian T-motive $M' = M^{\mu}$ is called the $\mu$-dual of $M$ (dual if $\mu = 1$) if $M' = \text{Hom}(M, \mathcal{C}^{\otimes \mu})$ as a rational pré-T-motive, i.e.

$$i_2 \circ i_1(M') = \text{Hom}(i_2 \circ i_1(M), \mathcal{C}^{\otimes \mu}) \quad (1.8.1)$$

**Remark.** This definition generalizes the original one of Taguchi ([T], Section 5), see 1.12 below.

1.9. We shall need the explicit matrix description of the above objects. Let $e_* = (e_1, ... e_n)^t$ be the vector column of elements of a basis of $M$ over $\mathbb{C}_\infty[\tau]$. There exists a matrix $A \in M_n(\mathbb{C}_\infty[\tau])$ such that

$$Te_* = Ae_*, \quad A = \sum_{i=0}^{l} A_i \tau^i \text{ where } A_i \in M_n(\mathbb{C}_\infty) \quad (1.9.1)$$

Condition (1.3.1) is equivalent to the condition

$$A_0 = \theta E_n + N \quad (1.9.2)$$

where $N$ is a nilpotent matrix, and the condition $m(M) = 1$ is equivalent to the condition $N = 0$.

Let $f_* = (f_1, ... , f_r)^t$ be the vector column of elements of a basis of $M$ over $\mathbb{C}_\infty[T]$. There exists a matrix $Q = Q(f_*) \in M_r(\mathbb{C}_\infty[T])$ such that

$$\tau f_* = Q f_* \quad (1.9.3)$$
Lemma 1.10. Let $M$ be as above. An abelian T-motive $M'$ is the $\mu$-dual of $M$ iff there exists a basis $f'_* = (f'_1, \ldots, f'_r)$ of $M'$ over $\mathbb{C}_\infty[T]$ such that its matrix $Q' = Q(f'_*)$ satisfies

$$Q' = (T - \theta)^\mu Q^{t-1}$$

(1.10.1)

\[\square\]

1.10.2. For further applications we shall need the following lemma. The above $f_*, f'_*$ are the dual bases (i.e. if we consider $f'_i$ as elements of $\text{Hom}(M, \mathcal{C})$ then $f'_i(f_j) = \delta^i_j$ where $\delta$ is the only element of a $\mathbb{C}_\infty[T]$-basis of $\mathcal{C}$ (we have $\tau f = (T - \theta)f$). Let $\gamma$ be an endomorphism of $M$, $D$ its matrix in the basis $f_*$ (i.e. $\gamma(f_*) = Df_*$) and $\gamma'$ the dual endomorphism.

Lemma 1.10.3. The matrix of $\gamma'$ in the basis $f'_*$ is $D^t$. $\square$

Remark 1.11.1. For any $M$ having dual there exists a canonical homomorphism $\delta : \mathcal{C} \to M \otimes M'$. This is a well-known theorem of linear algebra. Really, in the above notations we have $f \mapsto \sum_i f_i \times f'_i$. It is obvious that $\delta$ is well-defined, canonical and compatible with the action of $\tau$.

Remark 1.11.2. The $\mu$-dual of $M$ — if it exists — is unique, i.e. does not depend on base change. This follows immediately from Definition 1.8, but can be deduced easily from 1.10.1. Really, let $g_* = (g_1, \ldots, g_r)^t$ be another basis of $M$ over $\mathbb{C}_\infty[T]$ and $C \in GL_r(\mathbb{C}_\infty[T])$ the matrix of base change (i.e. $g_* = Cf_*$. Then $Q(g_*) = C^{(1)}QC^{-1}$. Let $g'_* = (g'_1, \ldots, g'_r)^t$ be a basis of $M'$ over $\mathbb{C}_\infty[T]$ satisfying $g'_* = C^{t-1}f'_*$. Elementary calculation shows that matrices $Q(g_*)$, $Q(g'_*)$ satisfy (1.10.1).

Remark 1.11.3. The operation $M \mapsto M'^\mu$ is obviously contravariant functorial. I leave as an exercise to the reader to give an exact definition of the corresponding category such that the functor of duality is defined on it, and is involutive (recall that not all abelian T-motives have duals, and the dual of a map of abelian T-motives is a priori a map of rational pré-T-motives).

1.12. The original definition of duality ([T], Definition 4.1; Theorem 5.1) from the first sight seems to be more restrictive than the definition 1.8 of the present paper, but really they are equivalent. We recall some notations and definitions of [T] in a slightly less general setting (rough statements; see [T] for the exact statements). Let $G$ be a finite affine group scheme over $\mathbb{C}_\infty$, i.e. $G = \text{Spec } R$ where $R$ is a finite-dimensional $\mathbb{C}_\infty$-algebra. Let $\mu : R \to R \otimes R$ be the comultiplication of $R$. Such group $G$ is called a finite $v$-module ([T], Definition 3.1) if there is a homomorphism $\psi : A \to \text{End}_{gr. \text{ sch.}}(G)$ satisfying some natural conditions (for example, an analog of 1.3.1). Further, let $\mathcal{E}_G$ be a $\mathbb{C}_\infty$-subspace of $R$ defined as follows:

$$\mathcal{E}_G = \{ x \in R \mid \mu(x) = x \otimes 1 + 1 \otimes x \}$$

The map $x \mapsto x^q$ is a $\mathbb{C}_\infty$-linear map $fr : \mathcal{E}_G^{(1)} \to \mathcal{E}_G$. Further, the map $\psi(T) : G \to G$ can be defined on $\mathcal{E}_G$. Let $v : \mathcal{E}_G \to \mathcal{E}_G^{(1)}$ be a map satisfying $fr_G \circ v = \psi(T) - \theta$.

We consider two finite $v$-modules $G$, $H$, the above objects fr, $v$ etc. will carry the respective subscript. Let $*$ be the dual in the meaning of linear algebra.
**Definition 1.12.1** ([T], 4.1). Two finite \( v \)-modules \( G, H \) are called dual if there exists an isomorphism \( \alpha : \mathcal{E}_H^* \to \mathcal{E}_G \) such that if we denote by \( \mathfrak{v} : \mathcal{E}_G \to \mathcal{E}_G^{(1)} \) a map which enters in the commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}_H^* & \xrightarrow{\text{fr}_H} & \mathcal{E}_H^{* (1)} \\
\alpha \downarrow & & \alpha^{(1)} \downarrow \\
\mathcal{E}_G & \xrightarrow{\mathfrak{v}} & \mathcal{E}_G^{(1)}
\end{array}
\]

then we have:

\[
\text{fr}_G \circ \mathfrak{v} = \psi_G(T) - \theta
\quad (1.12.2)
\]
i.e. \( \mathfrak{v} = v_G \).

Let \( M \) be an abelian \( T \)-motive having \( m(M) = 1 \), \( E = E(M) \) the corresponding \( T \)-module and \( a \in A \). We denote \( E_a \) — the set of \( a \)-torsion elements of \( E \) — by \( M_a \). It is a finite \( v \)-module.

**Proposition 1.12.3.** Let \( M, M' \) be abelian \( T \)-motives which are dual in the meaning of Definition 1.8. Then \( \forall a \in A \) we have: \( M_a, M'_a \) are dual in the meaning of [T], Definition 4.1.

**Proof.** Condition \( a \in \mathbb{F}_q[T] \) implies that multiplication by \( \tau \) is well-defined on \( M/aM \).

**Lemma 1.12.3.1.** We have canonical isomorphisms \( i : M/aM \to \mathcal{E}_{M_a}, i^{(1)} : M/aM \to \mathcal{E}_{M_a}^{(1)} \) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
M/aM & \xrightarrow{\tau} & M/aM \\
i^{(1)} \downarrow & & i \downarrow \\
\mathcal{E}_{M_a} & \xrightarrow{fr} & \mathcal{E}_{M_a} \\
\psi^r & \xrightarrow{fr} & \mathcal{E}_{M_a}
\end{array}
\]

**Proof.** Let \( R \) be a ring such that \( \text{Spec } R = M_a \). The pairing between \( M \) and \( E \) shows that there exists a map \( M \to R \) which is obviously factorized via an inclusion \( M/aM \to R \). It is easy to see that the image of this inclusion is contained in \( \mathcal{E}_{M_a} \), i.e. we get \( i \). Since \( \dim_{\mathbb{C}_\infty}(M/aM) = \deg a \cdot r(M) \) and \( \dim_{\mathbb{C}_\infty}(R) = q^{\deg a \cdot r(M)} \) we get from [T], Definition 1.3 that \( i \) is an isomorphism. Other statements of the lemma are obvious. \( \square \)

This lemma means that we can rewrite Definition 1.12.1 for the case \( G = M_a, H = N_a \) by the following way:

**1.12.3.2.** Two finite \( v \)-modules \( M_a, N_a \) are dual if there exists an isomorphism \( \alpha : (N/aN)^* \to M/aM \) such that after identification via \( \alpha \) of \( \tau^* : (N/aN)^* \to (N/aN)^* \) with a map \( \mathfrak{v} : M/aM \to M/aM \) we have on \( M/aM \):

\[
\tau \circ \mathfrak{v} = T - \theta
\quad (1.12.3.3)
\]

We need a
Lemma 1.12.3.4. For \( i = 1, 2 \) let \( N_i \) be a free \( \mathbb{C}_\infty[T] \)-module of dimension \( r \) with a base \( f_{i*} = (f_{i1}, ..., f_{ir}) \), let \( \varphi_i : N_i \to N_i \) be \( \mathbb{C}_\infty[T] \)-linear maps having matrices \( \Omega_i \) in \( f_{i*} \) such that \( \Omega_2 = \Omega_1^{-1} \), and let \( a \) be as above. Let, further, \( \varphi_{i,a} : N_i/aN_i \to N_i/aN_i \) be the natural quotient of \( \varphi_i \). Then there exist \( \mathbb{C}_\infty \)-bases \( f_{i*} \) of \( N_i/aN_i \) such that the matrix of \( \varphi_{1,a} \) in the base \( f_{i*} \) is transposed to the matrix of \( \varphi_{2,a} \) in the base \( \bar{f}_{2*} \).

Proof. We can identify elements of \( N_2 \) with \( \mathbb{C}_\infty[T] \)-linear forms on \( N_1 \) (notation: for \( x \in N_2 \) the corresponding form is denoted by \( \chi_x \)) such that \( \chi_{\varphi_2(x)} = \chi_x \circ \varphi_1 \). Any \( \mathbb{C}_\infty[T] \)-linear form \( \chi \) on \( N_1 \) defines a \( \mathbb{C}_\infty[T]/a\mathbb{C}_\infty[T] \)-linear form on \( N_1/aN_1 \) which is denoted by \( \chi_a \). Let now \( x \in N_2/aN_2, \bar{x} \) its lift on \( N_2 \), then \( \chi_{\bar{x},a} = (\chi_{\bar{x}})_a \) is a well-defined \( \mathbb{C}_\infty[T]/a\mathbb{C}_\infty[T] \)-linear form on \( N_1/aN_1 \). For \( x \in N_2/aN_2 \) we have

\[ \chi_{\varphi_2(x),a} = \chi_{x,a} \circ \varphi_{1,a} \]

Further, let \( \lambda : \mathbb{C}_\infty[T] \to \mathbb{C}_\infty \) be a \( \mathbb{C}_\infty \)-linear map such that

1.12.3.5. Its kernel does not contain any non-zero ideal of \( \mathbb{C}_\infty[T]/a\mathbb{C}_\infty[T] \).

(such \( \lambda \) obviously exist.) For \( x \in N_2/aN_2 \) we denote \( \lambda \circ \chi_{x,a} \) by \( \psi_x \), it is a \( \mathbb{C}_\infty \)-linear form on \( \mathbb{C}_\infty \)-vector space \( N_1/aN_1 \). Obviously condition (1.12.3.5) implies that the map \( x \mapsto \psi_x \) is an isomorphism from \( N_2/aN_2 \) to the space of \( \mathbb{C}_\infty \)-linear forms on \( \mathbb{C}_\infty \)-vector space \( N_1/aN_1 \), and we have

\[ \psi_{\varphi_2(x)} = \psi_x \circ \varphi_{1,a} \]

which is equivalent to the statement of the lemma. \( \square \)

Finally, the proposition follows immediately from this lemma multiplied by \( T - \theta \), formula 1.10.1 and 1.12.3.2. \( \square \)

Remark. Let \( a = \sum_{i=0}^{k} g_i T^i \), \( g_i \in \mathbb{F}_q \), \( g_k = 1 \). Taguchi ([T], proof of 5.1 (iv)) uses the following \( \lambda : \lambda(T^j) = 0 \) for \( j < k - 1 \), \( \lambda(T^{k-1}) = 1 \). It is easy to check that for \( x = (T^i + T^{i-1}g_{k-1} + T^{i-2}g_{k-2} + ... + g_{k-i})f_{2j} \) this \( \lambda \) we have: \( \psi_x(T^i f_{1j}) = 1 \), \( \psi_x(T^{i'} f_{1j'}) = 0 \) for other \( i', j' \).

1.13. Case of arbitrary \( \mathbf{A} \). An Anderson T-motive over \( \mathbf{A} \) is defined for example in [BH], p.1. Let us reproduce this definition for the case of characteristic 0. Let \( J \) be an ideal of \( \mathbf{A}_C \) generated by the elements \( a \otimes 1 - 1 \otimes i(a) \) for all \( a \in \mathbf{A} \). The ring \( \mathbf{A}_C[\tau] \) is defined by the formula \( \tau \cdot (a \otimes z) = (a \otimes z^q) \cdot \tau \), \( a \in \mathbf{A} \), \( z \in \mathbb{C}_\infty \).

Definition 1.13.1. An Anderson T-motive \( M \) over \( \mathbf{A} \) is a pair \((M, \tau)\) where \( M \) is a locally free \( \mathbf{A}_C \)-module and \( \tau \) is a \( \mathbf{A}_C \)-linear map \( M^{(1)} \to M \) satisfying the following analog of 1.3.1, 1.9.2:

\[ \exists m \text{ such that } J^m(M/\tau(M^{(1)})) = 0 \] (1.13.2)

Remark 1.13.3. We can consider \( M \) as a \( \mathbf{A}_C[\tau] \)-module using the following formula for the product \( \tau \cdot m \):

\[ \tau \cdot m = \tau(m \otimes 1) \]

where \( m \in M \), \( m \otimes 1 \in M^{(1)} \).
The rank of $M$ as a locally free $A_C$-module is called the rank of the corresponding T-motive $(M, \tau)$. If $A = \mathbb{F}_q[T]$ then $M^{(1)}$ is isomorphic to $M$, we can consider $M$ as a $C_\infty[T, \tau]$-module, and it is possible to show that in this case 1.13.2 implies that $M_{C_\infty[\tau]}$ is a free $C_\infty[\tau]$-module. In the general case, the dimension $n$ of $(M, \tau)$ is defined as $\dim_{C_\infty}(M/\tau(M^{(1)}))$.

Let us fix $C = (C, \tau_C)$ — an Anderson T-motive of rank 1 over $A$. For an Anderson T-motive $M = (M, \tau_M)$ an Anderson T-motive $M' = (C, \tau_C)$ — the $C$-dual of $M$ — is defined as follows. We put $M' = \Hom_{A_C}(M, C)$. Since for any locally free $A_C$-modules $M_1$, $M_2$ we have

$$\Hom_{A_C}(M_1, M_2)^{(1)} = \Hom_{A_C}(M_1^{(1)}, M_2^{(1)})$$

we can define $\tau(M')$ by the following formula:

For $\varphi \in \Hom_{A_C}(M, C)^{(1)}$ we have $\tau(M')(\varphi) = \tau_C \circ \varphi \circ \tau_M^{-1}$

1.14. Duality for abelian $\tau$-sheaves. We use notations of [BH], Definition 2.1 if they do not differ from the notations of the present paper; otherwise we continue to use notations of the present paper (for example, $\mathcal{F}$ (resp. $\mathcal{F}^{(1)}$) of [BH] is $n$ (resp. $\mathcal{F}^{(1)}$) of the present paper). For any abelian $\tau$-sheaf $\mathcal{F}$ we denote its $\Pi_i$, $\tau_i$ by $\Pi_i(\mathcal{F})$, $\tau_i(\mathcal{F})$ respectively. If $M$, $N$ are invertible sheaves on $X$ and $\rho : M \to N$ a rational map then we denote by $\rho^{inv} : N \to M$ the rational map which is inverse to $\rho$ with respect to the composition. We define $\tau_{i, i-1}(\mathcal{F})$ (the rational $\tau_i$) as the composition map $\tau_{i-1}(\mathcal{F}) \circ \Pi_{i-1}^{(1)}(\mathcal{F})$, it is a rational map from $\mathcal{F}_{i}^{(1)}$ to $\mathcal{F}_i$.

Let $\mathcal{O}$ be a fixed abelian $\tau$-sheaf having $r = n = 1$. The $\mathcal{O}$-dual abelian $\tau$-sheaf $\mathcal{F}' = \mathcal{F}'_{\mathcal{O}}$ is defined by the formulas

$$\mathcal{F}'_0 = \Hom_X(\mathcal{F}_0, \mathcal{O}_0)$$

where $\Hom$ is the sheaf’s one, and the map $\tau_{i, i-1}(\mathcal{F}') : \mathcal{F}'_0^{(1)} \rightarrow \mathcal{F}_0'$ is defined as follows. We have $\mathcal{F}'_0^{(1)} = \Hom_X(\mathcal{F}_0^{(1)}, \mathcal{O}_0^{(1)})$. Let $\gamma \in \Hom_X(\mathcal{F}_0^{(1)}, \mathcal{O}_0^{(1)})(U)$ where $U$ is a sufficiently small affine subset of $X_{C_\infty}$, such that $\gamma : \mathcal{F}_0^{(1)}(U) \rightarrow \mathcal{O}_0^{(1)}(U)$.

1.14.1. We define: $[[\tau_{i, i-1}(\mathcal{F}')](U)](\gamma)$ is the following composition map:

$$\mathcal{F}_0(U) \xrightarrow{[\tau_{i, i-1}(\mathcal{F}')](U)} \mathcal{F}_0^{(1)}(U) \xrightarrow{\gamma} \mathcal{O}_0^{(1)}(U) \xrightarrow{[\tau_{i, i-1}(\mathcal{O})](U)} \mathcal{O}_0(U) \in \Hom_X(\mathcal{F}_0, \mathcal{O}_0)(U)$$

Clearly that this definition and the definitions 1.8, 1.13 are compatible with the forgetting functor $M(\mathcal{F})$ from abelian $\tau$-sheaves to pure Anderson T-motives of [BH], Section 3, page 8.

2. Analytic and algebraic duality.

We shall consider until the end of the Section 9 only the case of $M$ having $N = 0$ (exception: statement of some results of Anderson). In the present section we consider the case of arbitrary $A \supset \mathbb{F}_q[T]$. We denote by $A$ the image $\iota(A) \subset C_\infty$. Condition $N = 0$ implies that an element $a \in A$ acts on $\text{Lie } (M)$ by multiplication.
by \( \iota(a) \). We have an inclusion \( \mathcal{A} \hookrightarrow K_\infty \), we prolonge \( \iota \) to the inclusion \( K_\infty \hookrightarrow \mathbb{C}_\infty \), and we denote the image \( \iota(K_\infty) \subset \mathbb{C}_\infty \) by \( K_\infty \) as well. Hence, we have a

**Definition 2.1.** Let \( V \) be the space \( \mathbb{C}^n \). A locally free \( r \)-dimensional \( \mathcal{A} \)-submodule \( L \) of \( V \) is called a lattice if

(a) \( L \) generates \( V \) as a \( \mathbb{C}_\infty \)-module and

(b) The \( K_\infty \)-linear envelope of \( L \) has dimension \( r \) over \( K_\infty \).

Numbers \( n, r \) are called the dimension and the rank of \( L \) respectively. Attached to \((L, V)\) is the tautological inclusion \( \varphi = \varphi(L, V) : L \rightarrow V \). We shall consider the category of pairs \( (L, V) \); a map \( \psi : (L, V) \rightarrow (L_1, V_1) \) is a \( \mathbb{C}_\infty \)-linear map \( \psi_V : V \rightarrow V_1 \) such that \( \psi_V(L) \subset L_1 \). Inclusion \( \varphi \) can be extended to a map \( L \otimes \mathbb{C}_\infty \rightarrow V \) (which is surjective by 2.1 a), we denote it by \( \varphi = \varphi(L, V) \) as well.

We can also attach to \((L, V)\) another exact sequence

\[ 0 \rightarrow \text{Ker } \varphi \rightarrow L \otimes \mathbb{C}_\infty \rightarrow V \rightarrow 0 \]  

(2.2)

Let \( \mathcal{I} \in \text{Cl}(\mathcal{A}) \) be a class of ideals; we shall use the same notation \( \mathcal{I} \) to denote a representative in the \( \iota \)-image of this class.

**Definition 2.3.** A structure of \( \mathcal{I} \)-duality on \((L, V)\) is a triple \( (D, L', V') \) where \( L' \) is a lattice in \( V' \), we denote the inclusion \( L' \hookrightarrow V' \) by \( \varphi' \); and \( D \) is a structure of a perfect \( \mathcal{I} \)-pairing \( <*,*>_D \) between \( L \) and \( L' \). These objects must satisfy the below property 2.4. To formulate it we fix an isomorphism \( \alpha : \mathcal{I} \otimes \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \). The \( \mathcal{I} \)-pairing \( <*,*>_D \) induces (thanks to a fixed \( \alpha \)) an isomorphism \( \gamma_D : (L \otimes \mathbb{C}_\infty)^* \rightarrow L' \otimes \mathbb{C}_\infty \) (here and below for any object \( W \) we denote \( W^* = \text{Hom}_{\mathbb{C}_\infty}(W, \mathbb{C}_\infty) \)).

**Property 2.4.** There exists an isomorphism from \((\text{Ker } \varphi)^*\) to \( V' \) making the following diagram commutative:

\[
\begin{array}{ccc}
0 & \rightarrow & V^* \\
\downarrow & & \downarrow \gamma_D \\
0 & \rightarrow & \text{Ker } \varphi' \\
& & \downarrow \varphi' \\
& & L' \otimes \mathbb{C}_\infty \\
& & \varphi' \\
& & V' \\
& & 0
\end{array}
\]

(2.5)

Clearly 2.4 does not depend on a choice of \( \alpha \), and it is equivalent to the following two conditions:

2.6. \( \dim V' = r - n \);

2.7. The composition map \( \varphi' \circ \gamma_D \circ \varphi^* : V^* \rightarrow V' \) is 0.

This is proved by elementary diagram search.

**Remark 2.8.** Forgetting \( D \) in the triple \((D, L', V')\) we get a notion of a dual pair. It is easy to see that the functor \((L, V) \mapsto (L', V')\) is well-defined on a subcategory (not all lattices have duals, see below) of the category of the pairs \((L, V)\), it is contravariant and involutive, and \( L' \) is of dimension \( r - n \) and rank \( r \). We need to define not only the dual pair \((L', V')\) but also a structure of duality because of Theorem 5 where a canonical \( D \) naturally appears.
3. Explicit formulas.

Here we consider the case \( A = \mathbb{F}_q[T] \). In this case \( \text{Cl}(A) = 0 \), and \( L \)-dual is called simply dual. The coordinate description of the dual lattice is the following. Let \( e_1, \ldots, e_r \) be a \( \mathbb{F}_q[\theta] \)-basis of \( L \) such that \( \varphi(e_1), \ldots, \varphi(e_n) \) form a \( C_\infty \)-basis of \( V \). Let \( e_1, \ldots, e_r \) be a \( \mathbb{F}_q[\theta] \)-basis of \( L \) such that \( \varphi(e_1), \ldots, \varphi(e_n) \) form a \( C_\infty \)-basis of \( V \). Like in the theory of abelian varieties, we denote by \( Z = (z_{ij}) \) the Siegel matrix whose lines are coordinates of \( \varphi(e_{n+1}), \ldots, \varphi(e_r) \) in the basis \( \varphi(e_1), \ldots, \varphi(e_n) \), more exactly, the size of \( Z \) is \( (r - n) \times n \) and

\[
\forall i = 1, \ldots, r - n \quad \varphi(e_{n+i}) = \sum_{j=1}^{n} z_{ij} \varphi(e_j)
\]

(3.1)

\( Z \) defines \( L \), we denote \( L \) by \( \mathcal{L}(Z) \).

**Proposition 3.2.** \([\mathcal{L}(Z)]' = \mathcal{L}(-Z^t)\), i.e. a Siegel matrix of the dual lattice is the minus transposed Siegel matrix.

**Proof.** Follows immediately from the definitions. Really, let \( f_1, \ldots, f_r \) be a basis of \( L' \), we define the pairing by the formula

\[
< e_i, f_j > = \delta^j_i
\]

(3.3)

and the map \( \varphi' \) by the formula

\[
\forall i = 1, \ldots, n \quad \varphi'(f_i) = \sum_{j=1}^{r-n} -z_{ji} \varphi'(f_{n+j})
\]

(3.4)

(minus transposed Siegel matrix). Ker \( \varphi \) is generated by elements

\[
v_i = e_{n+i} - \sum_{j=1}^{n} z_{ij} e_j, \quad i = 1, \ldots, r - n
\]

and Ker \( \varphi' \) is generated by elements

\[
w_i = f_i + \sum_{j=1}^{r-n} z_{ji} f_{n+j}, \quad i = 1, \ldots, n
\]

(3.4)

It is sufficient to check that \( \forall i, j \) we have \( < v_i, w_j > = 0 \); this follows immediately from 3.3. \( \square \)

**Remark 3.5.** \( L' \) exists not for all \( L \). Trivial counterexample: case \( n = r = 1 \). To get another counterexamples, we use that for \( n = 1 \) (lattices of Drinfeld modules) a Siegel matrix is a column matrix \( Z = (z_1 \ldots z_{r-1})^t \) and

\[
\mathcal{L}(Z) \text{ is not a lattice} \iff 1, z_1, \ldots, z_{r-1} \text{ are linearly dependent over } K_{\infty}
\]

(3.6)

while for \( n = r - 1 \) a Siegel matrix is a row matrix \( Z = (-z_1 \ldots -z_{r-1}) \) and

\[
\mathcal{L}(Z) \text{ is not a lattice} \iff \forall i \quad z_i \in K_{\infty}
\]

(3.7)
Since condition (3.7) is strictly stronger than (3.6) we see that all lattices having
\( n = 1, r > 1 \) have duals while not all lattices having \( n = r - 1, r > 2 \) have duals.

It is clear that almost all matrices have duals. Here ”almost all” has the same
meaning that as ”Almost all matrices \( Z \) are a Siegel matrice of a lattice”, i.e. if we
choose an (infinite) basis of \( \mathbb{C}_\infty /K_\infty \), then coordinates of the entries of \( Z \) in this
basis must satisfy some polynomial relations in order that \( Z \) is not a Siegel matrice of a lattice.

**Remark 3.8.** The coordinate proof of the theorem that the notion of the
dual lattice is well-defined, is the following. Two Siegel matrics \( Z, Z_1 \) are called
equivalent iff there exists an isomorphism of their pairs \( (\mathfrak{L}(Z), V), (\mathfrak{L}(Z_1), V_1) \).

Let us consider the matrix
\[
A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
de the \( \mathfrak{L}(\mathbb{Z}[\theta]) \) (\( A, B, C, D \) are the \( n \times n \), \( n \times r - n \),
\( r - n \times n \), \( r - n \times r - n \)-blocks of \( \gamma \) respectively; we shall call this block structure
by the \( (n, r - n) \)-block structure) such that
\[
C + DZ = Z_1(A + BZ)
\]  
(3.8.1)

Let \( A_1, B_1, C_1, D_1 \) be the \( (n, r - n) \)-block structure of the matrix \( \gamma^{-1} \). The
equality
\[
-C_1^t + A_1^t Z = Z_1^t(D_1^t - B_1^t Z)
\]  
(3.8.2)
shows that if \( Z, Z_1 \) are equivalent then \(-Z^t, -Z_1^t \) are equivalent. [Proof of (3.8.2):
(3.8.1) implies \( Z_1 = (C + DZ)(A + BZ)^{-1} \); substituting this value of \( Z_1 \) to the
transposed (3.8.2), we get \(-C_1 + ZA_1 = (D_1 - ZB_1)(C + DZ)(A + BZ)^{-1} \), or
\((-C_1 + ZA_1)(A + BZ) = (D_1 - ZB_1)(C + DZ) \). This formula follows immediately
from \( \left( \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right) \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} E_n & 0 \\ 0 & E_{r-n} \end{array} \right) \).

Further, let \( \alpha : (L_1 \subset \mathbb{C}_\infty^n) \rightarrow (L_2 \subset \mathbb{C}_\infty^n) \) be a map of lattices. If \( L_1', L_2' \)
exsit, then the map \( \alpha' : (L_2' \subset \mathbb{C}_{r-n}^n) \rightarrow (L_1' \subset \mathbb{C}_{r-n}^n) \) is defined by the following
formulas. Let \( Z_i \) be the Siegel matrices of \( L_i \) in the bases \( e_{i1},...e_{ir} \) of \( L_i \) \( (i = 1, 2) \).
Let us consider the matrix \( \mathfrak{M} = (m_{ij}) \in M_r(\mathbb{F}_q[\theta]) \) of \( \alpha \) in the bases \( e_{i1},...e_{ir} \) (i.e.
\( \alpha(e_{i1}) = \sum_j m_{ij}e_{2j} \)). Let \( f_{i1},...f_{ir} \) be the dual base of \( L_i' \) (see 3.3) and \( e_{i1}',...e_{ir}' \)
another base of \( L_i' \) defined by
\[
e_{ij}' = f_{i,j+n}, \quad j + n \mod r
\]  
(3.8.3)

Formulas (3.8.3), (3.4) show that an analog of 3.1 is satisfied for both bases \( e_{i1}',...e_{ir}' \),
their Siegel matrices are \(-Z_i'\).

Let
\[
\mathfrak{M} = \begin{pmatrix} \mathfrak{M}_{11} & \mathfrak{M}_{12} \\ \mathfrak{M}_{21} & \mathfrak{M}_{22} \end{pmatrix}
\]
be the \( (n, r - n) \)-block structure of \( \mathfrak{M} \). The matrix of \( \alpha' \) in the bases \( f_{i1},...f_{ir} \) is
\( \mathfrak{M}' \), and using the matrix 3.8.3 of change of base, we get that \( \mathfrak{M}' \) — the matrix of
\( \alpha' \) in the bases \( e_{i1}',...e_{ir}' \) — has the following \( (r - n, n) \)-block structure:
\[
\mathfrak{M}' = \begin{pmatrix} \mathfrak{M}_{22}' & \mathfrak{M}_{12}' \\ \mathfrak{M}_{21}' & \mathfrak{M}_{11}' \end{pmatrix}
\]  
(3.8.4)
The property that $\mathfrak{M}$ comes from a $C_\infty$-linear map $C_\infty^n \to C_\infty^n$ implies that $\mathfrak{M}'$ comes from a $C_\infty$-linear map $C_\infty^{r-n} \to C_\infty^{r-n}$. This follows immediately from the definition of dual lattice, or can be easily checked algebraically.

**Remark 3.9.** Taking $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we get that $Z$ is equivalent to $-Z$, hence $Z'$ is also a Siegel matrix of the dual lattice.

### 4. Main conjecture for arbitrary $A$

The main result of the paper is the following Theorem 5 on coincidence of algebraic and analytic pairing. We formulate it as a conjecture 4.1 for any $A$, but we prove it only for the case $A = F_q[T]$. Let $M$ be a uniformizable abelian $T$-motive. Its lattice $L(M)$ is really a lattice in the meaning of Definition 2.1, because $[A]$, Corollary 3.3.6 (resp. [G], Lemma 5.9.12) means that it satisfies 2.1a (resp. 2.1b); recall that we consider the case $N = 0$, i.e. the action of $T$ on $\text{Lie}(M)$ is simply multiplication by $\theta$. Let us fix (like in 1.13) $C = (C, \tau_C)$ — an Anderson $T$-motive of rank 1 over $A$, and let $L(C)$ be its lattice. It is an $A$-module (we identify $A$ and $A$ via $\iota$). $\Omega = \Omega(A)$ is also an $A$-module, hence the notion of the $L(C) \otimes \Omega$-duality exists.

**Conjecture 4.1.** Let $M$ be a uniformizable abelian $T$-motive such that its $C$-dual $M'$ exists. Then $M'$ is uniformizable, and there exists a canonical structure of perfect $L(C) \otimes \Omega$-duality between $(L(M), \text{Lie}(M))$ and $(L(M'), \text{Lie}(M'))$.

**Remark 4.2.** It is possible to generalize the above conjecture to the case of non-uniformizable $M$, $M'$. The pairing is defined between $\text{Hom}_{A_C[T]}(M, Z_1)$ and $\text{Hom}_{A_C[T]}(M', Z_1)$ (see (5.2.1a) for the definition of $Z_1$), or, the same, between $M_a$ and $M'_a$ for any $a \in A$ (see 5.1.2).

### 5. Main theorem for $A = F_q[T]$

**Theorem 5.** Let $A = F_q[T]$, and $M$ a uniformizable abelian $T$-motive such that its dual $M'$ exists. Then $M'$ is uniformizable, and there exists a canonical structure of perfect duality between $(L(M), \text{Lie}(M))$ and $(L(M'), \text{Lie}(M'))$.

**Corollary 5.1.1.** If $A = F_q[T]$ then a Siegel matrix of $M'$ is the minus transpose of a Siegel matrix of $M$.

In the section 8 below we give a corollary of this theorem and some conjectures related to the problem of $1 - 1$ correspondence between $T$-motives and lattices. The proof of Theorem 5 consists of two steps. We formulate and prove Step 1 for the case of arbitrary $A$.

**Step 1.** For the above $M, M'$ we have:

(A) Uniformizability of $M$ implies uniformizability of $M'$.

(B) There exists a canonical $A$-linear perfect $L(C) \otimes \Omega$-pairing $<*, *>_M$ between $L(M)$ and $L(M')$. It is functorial.

**Remark 5.1.2.** Practically, (B) comes from [T], Theorem 4.3 (case $A = F_q[T]$). Really, to define a pairing between $L(M)$ and $L(M')$ it is sufficient to define (concordant) pairings between $L(M)/aL(M)$ and $L(M')/aL(M')$ for any $a \in A$. Since...

---

1The proof of this theorem was inspired by a result of Anderson, see Section 6 for details.
\[ M_a := E(M)_a = L(M)/aL(M) \] and because of Proposition 1.12.3 which affirms that \( M_a \) and \( M_a' \) are Taguchi-dual, we see that [T], Theorem 4.3 gives exactly the desired pairing.

We give two versions of the proof of Step 1: the first one — for the general case of arbitrary \( A \) and the second one — for the case \( A = \mathbb{F}_q[T] \) — is based on explicit calculations, it is used for the proof of Step 2.

5.2. **Proof: Step 1, Version 1.** Here we consider the general case of arbitrary \( A \). Let \( \Omega = \Omega(\mathbb{A}/\mathbb{F}_q) \) be the module of differential forms; we can consider it as an element of \( \text{Cl}(\mathbb{A}) \). We use formulas and notations of [G], Section 5.9 modifying them to the case of arbitrary \( K \). For example, \( \mathbb{A} \) (resp. \( K \)) of [G], 5.9.16 is \( \mathbb{A} \) (resp. \( K_\infty \)) of the present paper (recall that \( \bar{K} \) (resp. \( K[T, \tau] \)) of [G] is \( \mathbb{C}_\infty \) (resp. \( \mathbb{A}_C[\tau] \), see 1.13) of the present paper). Hence, we denote \( K\{T\} \) of [G], Definition 5.9.10 by \( \mathbb{C}_\infty \{T\} \). For the general case it must be replaced by a ring \( Z_0 \) defined by the formula

\[ Z_0 := \mathbb{A} \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty \{T\} \]  \hspace{1cm} (5.2.1)

\( Z_0 \) is a \( \mathbb{A}_C[\tau] \)-module, i.e. \( \tau \) acts on \( Z_0 \), and \( Z_0^\tau = \mathbb{A} \).

\( Z_1 \) for the present case is defined by the same formula [G], 5.9.22. Explicitly,

\[ Z_1 := \text{Hom}^{cont}(K_\infty/\mathbb{A}, \mathbb{C}_\infty) \] \hspace{1cm} (5.2.1a)

It is a locally free \( Z_0 \)-module of dimension 1 (the module structure is compatible with the action of \( \tau \); see [G], p. 168, lines 3 - 4 for the case \( A = \mathbb{F}_q[T] \)). We have: \( Z_1^\tau \) is a \( Z_0^\tau \)-module (\( = \mathbb{A} \)-module) which is isomorphic to \( \Omega(\mathbb{A}) \) (see the last lines of the proof of [G], Corollary 5.9.35 for the case \( A = \mathbb{F}_q[T] \)), and \( Z_1 \) is isomorphic to \( Z_0 \otimes \mathbb{A} \Omega(\mathbb{A}) \).

We shall consider \( M \) as a \( \mathbb{A}_C[\tau] \)-module, like in 1.13.3. We denote \( M\{T\} := M \otimes Z_0 \) (\( = [G] \), Definition 5.9.11.1 for the case \( A = \mathbb{F}_q[T] \)) and \( H^1(M) := M\{T\}^\tau \) like in [G], Definition 5.9.11.2. Analogous to [G], Corollary 5.9.25 we get that for the present case

\[ \text{Hom}_{\mathbb{A}_C[\tau]}(M, Z_1) = H_1(M) = L(M) \]

\( (H_1(M) = H_1(E) \) of [G], 5.9). Particularly, for \( M = \mathcal{C} \) we have

\[ L(\mathcal{C}) = \text{Hom}_{\mathbb{A}_C[\tau]}(\mathcal{C}, Z_1) \]

**Lemma 5.2.2.** \( H_1(M') = H^1(M) \otimes L(\mathcal{C}) \).

**Proof.** By definition, \( \text{Hom}_{\mathbb{A}_C}(M', Z_1) = \text{Hom}_{\mathbb{A}_C}(\text{Hom}_{\mathbb{A}_C}(M, \mathcal{C}), Z_1) \). Further,

\[ \text{Hom}_{\mathbb{A}_C}(\text{Hom}_{\mathbb{A}_C}(M, \mathcal{C}), Z_1) = (M \otimes Z_0) \otimes (\text{Hom}_{\mathbb{A}_C}(\mathcal{C}, Z_1)) \] \hspace{1cm} (5.2.3)

(an equality of linear algebra). In order to show that we can consider \( \tau \)-invariant subspaces, we need the following objects. Let \( I \) be an ideal of \( \mathbb{A} \), \( \mathcal{M}_0 = IZ_0 \). It is clear that \( \mathcal{M}_0^\tau = I \). Further, let \( \mathcal{M}_1 \) be a locally free \( Z_0 \)-module. We have a formula:

\[ (\mathcal{M}_0 \otimes \mathcal{M}_1)^\tau = \mathcal{M}_0^\tau \otimes \mathcal{M}_1^\tau \] \hspace{1cm} (5.2.4)
Really, $\mathcal{M}_0 \otimes_{Z_0} \mathcal{M}_1 = I\mathcal{M}_1$, and

$$(I\mathcal{M}_1)^\tau = I\mathcal{M}_1^\tau$$

(5.2.5)

where this formula is true by the following reason. Obviously $(I\mathcal{M}_1)^\tau \supset I\mathcal{M}_1^\tau$. Let $J$ be an ideal of $A$ such that $IJ$ is a principal ideal. We have $(IJ(J^{-1}\mathcal{M}_1))^\tau = IJ(J^{-1}\mathcal{M}_1)^\tau$ and $(IJ(J^{-1}\mathcal{M}_1))^\tau \supset I(J(J^{-1}\mathcal{M}_1))^\tau \supset IJ(J^{-1}\mathcal{M}_1)^\tau$, hence all these objects are equal and we get 5.2.5 and hence 5.2.4.

The action of $\tau$ on both sides of 5.2.3 coincide. Considering $\tau$-invariant elements of both sides of 5.2.3 and taking into consideration 5.2.4 (be an ideal of $M$ and $M$ and $M$ pairing of $H$, we get the lemma. □

This lemma proves (A) of Step 1.

**Lemma 5.2.6.** Let $\mathcal{M}_i$ ($i = 0, 1$) be two locally free $Z_0$-modules with $\tau$-action satisfying $\tau(cm) = (c)(\tau(m)$ ($c \in Z_0$, $m \in \mathcal{M}_i$), and $\psi : \mathcal{M}_0 \otimes_{Z_0} \mathcal{M}_1 \rightarrow Z_1$ a perfect pairing of $Z_0$-modules with $\tau$-action. Let, further, both $\mathcal{M}_i$ satisfy $\mathcal{M}_i^\tau \otimes_{A} Z_0 = \mathcal{M}_i$. Then the restriction of $\psi$ to $\mathcal{M}_0^\tau \otimes_{A} \mathcal{M}_1^\tau \rightarrow \Omega$ is a perfect pairing as well.

**Proof.** Let $A : \mathcal{M}_0^\tau \rightarrow \Omega$ be an $A$-linear map. We prolonge it to a map $\alpha : \mathcal{M}_0 \rightarrow Z_1$ by $Z_0$-lineararity. By perfectness of $\psi$, there exists $m_1 \in \mathcal{M}_1$ such that $\alpha(m_0) = \psi(m_0 \otimes m_1)$. It is easy to see that $m_1$ is $\tau$-invariant (we use the fact that $\tau : Z_0 \rightarrow Z_0$ is surjective). □

**Lemma 5.2.7.** There is a natural perfect $A$-linear $\Omega$-valued pairing between $H_1(M)$ and $H^1(M) : H_1(M) \otimes H^1(M) \rightarrow \Omega$.

**Proof.** For the case $A = \mathbb{F}_q[T]$ this is [G], Corollary 5.9.35. General case: we have a perfect $Z_0$-pairing

$$\text{Hom}_{A_c}(M, Z_1) \otimes_{Z_0} (M \otimes_{A_c} Z_0) \rightarrow Z_1$$

Now we take $\mathcal{M}_0 = \text{Hom}_{A_c}(M, Z_1)$, $\mathcal{M}_1 = M \otimes_{A_c} Z_0$ and we apply Lemma 5.2.6. □

Step 1 of the theorem follows from these lemmas.

**Remark 5.2.8.** The pairing can be defined also as the composition of

$$H_1(M) \otimes H_1(M') = \text{Hom}_{A_c[\tau]}(M, Z_1) \otimes \text{Hom}_{A_c[\tau]}(M', Z_1) \rightarrow \text{Hom}_{A_c[\tau]}(M \otimes_{A_c} M', Z_1 \otimes_{Z_0} Z_1) = L(\mathcal{C}) \otimes \Omega$$

(5.2.9)

where the second map comes from a canonical map $\delta : \mathcal{C} \rightarrow M \otimes M'$ of Remark 1.11.1 (more exactly, of its analog for arbitrary $A$).

**Remark 5.2.10.** Recall that the explicit formula for functoriality is the following. Let $\alpha : M_1 \rightarrow M_2$ be a map of abelian T-motives, $\alpha' : M'_2 \rightarrow M'_1$ the dual map
and $L(\alpha) : L(M_2) \to L(M_1)$, $L(\alpha') : L(M'_1) \to L(M'_2)$ the corresponding maps on lattices. For any $l'_1 \in L(M'_1)$, $l_2 \in L(M_2)$ we have:

$$< L(\alpha)(l_2), l'_1 >_{M_1} = < l_2, L(\alpha')(l'_1) >_{M_2} \quad (5.2.11)$$

5.3. Proof: Step 1, Version 2. Case $A = \mathbb{F}_q[T]$. We identify $Z_1$ with $\mathbb{C}_\infty \{T\}$ (see [G], p.168, lines 3 – 4) and $A$ with $\Omega$; we shall construct an $A$-linear, $A$-valued pairing which is canonical up to multiplication by elements of $\mathbb{F}_q^*$. We have (under identification of $T$ and $\theta$ via $\iota$)

$$L(M) = \text{Hom}_{\mathbb{C}_\infty[T, \tau]}(M, Z_1) \quad (5.3.1)$$

Let $\varphi : M \to Z_1$, $\varphi' : M' \to Z_1$ be elements of $L(M)$, $L(M')$ respectively, and let $f_*, f'_*$ be from 1.9.3, 1.10. We denote

$$\varphi(f_*) = v_* \quad (5.3.2)$$

where $v_* \in (Z_1)^r$ is a vector column (it is a column of the scattering matrix ([A], p. 486) of $M$, see 5.4.1 below). The same notation for the dual: $\varphi'(f'_*) = v'_*$. Condition that $\varphi, \varphi'$ are $\tau$-homomorphisms is equivalent to

$$Qv_* = v_*^{(1)}, \quad Q'v'_* = v'_*^{(1)} \quad (5.3.3)$$

(analog of the formula for scattering matrices [A], (3.2.2)). Let us consider $\Xi = \sum_{i=0}^{\infty} a_i T^i \in \mathbb{C}_\infty \{T\} \subset \mathbb{C}_\infty [[T]]$ of [G], p. 172, line 1; recall that it is the only element (up to multiplication by $\mathbb{F}_q^*$) satisfying

$$\Xi = (T - \theta)\Xi^{(1)}, \quad \lim_{i \to \infty} a_i = 0, \quad |a_0| > |a_i| \quad \forall i > 0 \quad (5.3.4)$$

(see [G], p. 171, (*); there is a formula $\Xi = a_0 \prod_{i \geq 0} (1 - T/\theta^i)$ where $a_0$ satisfies $a_0 q^{-1} = -1/\theta$). Finally, we define

$$< \varphi, \varphi' > = \Xi v_*^t v'_* \quad (5.3.5)$$

Obviously $< \varphi, \varphi' >$ does not depend on a choice of a basis $f_*$.

Lemma 5.3.6. $< \varphi, \varphi' > \in \mathbb{F}_q[T]$.

Proof. Firstly, this element belongs to $\mathbb{F}_q[[T]]$, because

$$\Xi v_*^t v'_* - (\Xi v_*^t v'_*)^{(1)} = \Xi (v_*^t v'_* - (T - \theta)^{-1} v_*^{(1)t} v'_*^{(1)}) = \Xi v_*^t (E_r - (T - \theta)^{-1} Q' Q') v'_*$$

because of (5.3.3). Further (see (1.10.1) — the definition of $Q'$)

$$E_r - (T - \theta)^{-1} Q' Q' = 0$$

Finally, let $< \varphi, \varphi' >= \sum_{i=0}^{\infty} c_i T^i$. Since coefficients of all factors of (5.3.5): $\Xi$, $v_*$ and $v'_*$ — tend to 0, we get that $c_i$ also tend to 0. But $c_i \in \mathbb{F}_q$, i.e. they are almost all 0. □

Lemma 5.3.7. The above pairing is perfect.
Proof. We have an isomorphism

$$\alpha : \text{Hom}_{C_\infty[T,\tau]}(M, Z_1) \to \text{Hom}_{F_u[T]}(M\{T\}^\tau, F_u[T])$$ (5.3.8)

defined as the composition of the maps

$$\text{Hom}_{C_\infty[T,\tau]}(M, Z_1) = \text{Hom}_{C_\infty[T]}(M, Z_1)^\tau \xrightarrow{\beta'} \text{Hom}_{C_\infty\{T\}}(M\{T\}, \mathbb{C}_\infty\{T\})^\tau$$

where $M\{T\} = M \otimes_{C_\infty[T]} \mathbb{C}_\infty\{T\}$ with the natural action of $\tau$ (see [G], Definition 5.9.11). $\beta : \text{Hom}_{C_\infty[T]}(M, Z_1) \to \text{Hom}_{C_\infty\{T\}}(M\{T\}, \mathbb{C}_\infty\{T\})$ is the natural map and $\beta'$ is the restriction of $\beta$ on $\tau$-invariant elements. Using the Anderson’s criterion of uniformizability of $M$ (see, for example, [G], 5.9.14.3 and 5.9.13) we get immediately that both $\gamma$, $\beta$, and hence $\beta'$, and hence $\alpha$ are isomorphisms. Further, let us consider a homomorphism

$$i : \text{Hom}_{C_\infty[T,\tau]}(M', Z_1) \to M\{T\}^\tau$$ (5.3.9)

defined as follows. Let $\varphi'$, $f'_i$, $v'_i$ be as above. We set

$$i(\varphi') = \Xi v'_i f'_i \in M \otimes_{C_\infty[T]} \mathbb{C}_\infty[[T]]$$

Since $\Xi \in \mathbb{C}_\infty\{T\}$, we get that $\Xi v'_i f'_i \in M\{T\}$. A simple calculation (like in the Lemma 5.3.6, but simpler) shows that $i(\varphi')$ is $\tau$-invariant, hence $i$ really defines a map from $\text{Hom}_{C_\infty[T,\tau]}(M', Z_1)$ to $M\{T\}^\tau$. Obviously it is an inclusion. Let us prove that $i$ is surjective. Really, let $c_* \in (Z_1)^\tau$ be a column vector such that $c_* f'_i \in M\{T\}^\tau$. An analog of the above calculation shows that if we define $\varphi'$ by the formula $\varphi'(f'_i) = \Xi^{-1} c_*$ then $\varphi' \in \text{Hom}_{C_\infty[T,\tau]}(M', Z_1)$, and $i(\varphi') = c_* f'_i \in M\{T\}^\tau$. Finally, the combination of isomorphisms (5.3.8) and (5.3.9) corresponds to the pairing (5.3.5). □

5.4. Step 2 – End of the proof of Theorem 5. It is easy to see that the converse of the Corollary 5.1.1 (taking into consideration Proposition 3.2) is also true, i.e. in order to prove Theorem 5 it is sufficient to prove that a Siegel matrix of $M'$ is $-Z'$ where $Z$ is a Siegel matrix of $M$. Let us consider a basis $l_1, ..., l_r$ of $L(M)$ and for each $l_i$ we consider the corresponding (under identification 5.3.1) $\varphi_i \in \text{Hom}_{C_\infty[T,\tau]}(M, Z_1)$. Let $\Psi$ be the scattering matrix of $M$ with respect the bases $l_1, ..., l_r, f_1, ..., f_r$, and we denote $\varphi_i(f_i)$ by $v_{i*}$ (notations of 5.3.2).

Lemma 5.4.1. $v_{i*}$ is the $i$-th column of $\Psi$ ($Z_1$ is identified with $\mathbb{C}_\infty\{T\}$, see the proof).

Proof. Follows from the definitions. The isomorphism 5.3.1 is the composition of 2 isomorphisms $i_1 : L(M) \to \text{Hom}_{A}(K/A, E)$ ([G], 5.9.19) and $i_2 : \text{Hom}_{A}(K/A, E) \to \text{Hom}_{C_\infty[T,\tau]}(M, \text{Hom}(K/A, \mathbb{C}_\infty))$ ([G], 5.9.24; recall that $Z_1 = \text{Hom}(K/A, \mathbb{C}_\infty)$). For $l_i \in L(M)$ we have $(i_1(l_i))(T^{-k}) = \exp(\theta^{-k} l_i)$ ([G], line above the lemma 5.9.18) and $((i_2 \circ i_1(l_i))(f_j))(T^{-k}) = \langle f_j, \exp(\theta^{-k} l_i) \rangle \in [G]$, two lines above the lemma 5.9.24). Using the identification of $Z_1$ and $\mathbb{C}_\infty\{T\}$ ([G],
Let $l'_1, ..., l'_r$ be a basis of $L(M')$ which is dual to a basis $l_1, ..., l_r$ of $L(M)$ with respect to the pairing $5.3.5.$

**Lemma 5.4.2.** The scattering matrix of $M'$ with respect to the bases $l'_1, ..., l'_r$, $f'_1, ..., f'_r$ is $\Xi^{-1}\Psi^{-1}$.

**Proof.** Follows immediately from 5.4.1 applied to both $M, M'$, and formula 5.3.5. □

**Remark 5.4.3.** An alternative proof for the case of pure $M$ (for some basis of $L(M')$) is the following. We denote $\Xi^{-1}\Psi^{-1}$ by $\Psi_1$. It satisfies $\Psi_1(1) = (T - \theta)Q^{-1}\Psi_1$ and other conditions of [A], 3.1. According [A], Theorem 5, p. 488, there exists a pure uniformizable $T$-motive $M_1$ with $\sigma$-structure such that its scattering matrix is $\Psi_1$. Since $\Psi_1$ satisfies

$$\Psi_1^{(1)} = Q'\Psi_1$$

we get that $Q(M_1) = Q'$, i.e. $M_1 = M'$. □

We return to the case $N = 0$. Let us consider the $(T - \theta)$-Laurent series for $\Psi'$ and $\Xi^{-1}$:

$$\Psi' = \sum_{i=k(M')}^{\infty} A'_i(T - \theta)^i, \quad \Xi^{-1} = \sum_{i=k(\xi)}^{\infty} a_i(T - \theta)^i$$
Since for both \( M, M' \) we have \( N = 0 \), we get \( k(M) = k(M') = -1 \). An elementary calculation shows that \( k(\xi) \) is also \(-1\). Hence, equality \( \Psi'\Psi^t = \Xi^{-1} \) implies that \( A'_1A'^{-1} = 0 \).

Further, there exist \( n \) columns of \( A_1 \) which are \( \mathbb{C}_\infty \)-linerly independent (they are \( \psi_E \)-images of elements of \( L(M) \) which form a \( \mathbb{C}_\infty \)-basis of \( \text{Lie} \ (M) \)) and all other columns of \( A_1 \) are their linear combinations. Interchanging columns of \( A_1 \) if necessary we can assume that these columns are the first \( n \) columns. We denote by \( A_{1,le} \) (resp. \( A_{1,ri} \)) the \( r \times n \) (resp. \( r \times (r - n) \)) matrix formed by the first \( n \) (resp. the last \( r - n \)) columns of \( A_1 \). There exists a matrix \( C \) such that \( A_{1,ri} = A_{1,le}C \). Again according Proposition 3.3.2, [A], we have:

\[
C = Z^t \tag{5.4.5}
\]

where \( Z \) is a Siegel matrix of \( L(M) \) (see 5.5, 5.6 below for a proof of this formula in the present case).

Analogous objects are defined for \( A'_1 \). We denote by \( A'_{1,le} \) (resp. \( A'_{1,ri} \)) the \( r \times n \) (resp. \( r \times (r - n) \)) matrix formed by the first \( n \) (resp. the last \( r - n \)) columns of \( A'_1 \). Since \( A'_{1,le}A'_{1,le}^t = A'_{1,le}A'_{1,le}^t + A'_{1,ri}A'_{1,ri} \) we get that \( A'_{1,le}A'_{1,le}^t + A'_{1,ri}C^tA'^{-1} = 0 \). Since \( A'_{1,le} \) is a \( n \times r \)-matrix of rank \( n \), it is not a zero-divisor from the right, so

\[
A'_{1,le} = -A'_{1,ri}C^t \tag{5.4.6}
\]

Since the rank of \( A'_{1} \) is \( r - n \) and \( A'_{1,ri} \) is a \( r \times (r - n) \) matrix, (5.4.6) implies that columns of \( A'_{1,ri} \) are linerly independent, and by (5.4.6) and Proposition 3.3.2, [A] we get that \(-C^t \) is \( Z' \) where \( Z' \) is a Siegel matrix of \( M' \). □

**Remark 5.5.** Since the notations of [A] differ from the ones of the present paper, for the reader's convenience we give here a sketch of the proof for the case \( N = 0 \) of two crucial facts: Corollary 5.4.4 and 5.4.5 ([A], Theorem 3.3.2).

Let \( \alpha : \text{Lie} \ (M) \to E(M) \) be a linear isomorphism which is the first term of the series for \( \exp : \text{Lie} \ (M) \to E(M) \), and let \( l \in \text{Lie} \ (M), f \in M \) be arbitrary. We consider the \((T - \theta)\)-Laurent series \( \sum_{i=k}^{\infty} b_i(T - \theta)^i \) of \( \sum_{j=0}^{\infty} <\exp(\frac{1}{g_{j+1}}l), f > T^j \).

**Lemma 5.6.** If \( N = 0 \) then \( k = -1 \), and \( b_{-1} = -<\alpha(l), f > \) (this is [A], 3.3.4).

**Sketch of the proof.** For \( z \in \text{Lie} \ (M) \) we denote \( \exp(z) - \alpha(z) \) by \( \varepsilon(z) \), hence \( \sum_{j=0}^{\infty} <\exp(\frac{1}{g_{j+1}}l), f > T^j = \mathfrak{A} + \mathfrak{E} \), where

\[
\mathfrak{A} = \sum_{j=0}^{\infty} <\alpha(\frac{1}{g_{j+1}}l), f > T^j; \quad \mathfrak{E} = \sum_{j=0}^{\infty} <\varepsilon(\frac{1}{g_{j+1}}l), f > T^j
\]

We consider their \((T - \theta)\)-Laurent series:

\[
\mathfrak{A} = \sum_{i=k(\mathfrak{A})}^{\infty} a_i(T - \theta)^i; \quad \mathfrak{E} = \sum_{i=k(\mathfrak{E})}^{\infty} c_i(T - \theta)^i
\]

Since we have \( \exp(z) = \sum_{i=0}^{\infty} C_i z^i \) where \( C_0 = E_n \) we get that \( \varepsilon(z) = \sum_{i=1}^{\infty} C_i z^i \).

This means that for large \( j \) the element \( \varepsilon(\frac{1}{g_{j+1}}l) \) is small, and hence \( k(\mathfrak{E}) = 0 \),
because finitely many terms having small $j$ do not contribute to the pole of the 
$(T-\theta)$-Laurent series of $\mathcal{E}$ (the reader can prove easily the exact estimations himself, 
or to look [A], p. 491). Since $\alpha$ is $\mathbb{C}_\infty$-linear, equality $\sum_{j=0}^{\infty} \frac{1}{j+1} T^j = -(T-\theta)^{-1}$ 
implies that $k(\mathfrak{M}) = -1$ and $a_{-1} = -<\alpha(l), f>$ (and other $a_i = 0$), hence the 
lemma. □

This lemma obviously implies Corollary 5.4.4. Further, elements $f_1, \ldots, f_r$ generate 
the $\mathbb{C}_\infty$-space $M/\tau M$, because multiplication by $T$ on $M/\tau M$ coincides with 
multiplication by $\theta$, hence the fact that $f_1, \ldots, f_r$ $\mathbb{C}_\infty[T]$-generate $M/\tau M$ implies 
that they $\mathbb{C}_\infty$-generate $M/\tau M$.

Let $l_1, \ldots, l_n$ form a $\mathbb{C}_\infty$-basis of $\text{Lie} (M)$. Since the pairing $< *, * >$ between 
$E(M)$ and $M/\tau M$ is non-degenerate and $\alpha$ is an isomorphism, we get that columns 
$< \alpha(l_1), f_*>, \ldots, < \alpha(l_n), f*>$ are linearly independent. Again since $\alpha$ is an 
isomorphism and the pairing with $f_*$ is linear, we get that 

\[(<\alpha(l_{n+1}), f_*> \ldots <\alpha(l_r), f*>) = (<\alpha(l_1), f*> \ldots <\alpha(l_n), f*>) Z^t\]

Applying the lemma 5.6 to this formula we get immediately 5.4.5.

6. Tensor products.

There exists an analog of the Theorem 5 for the case of tensor products of abelian 
T-motives. It describes the lattice $L(M_1 \otimes M_2)$ in terms of $L(M_1)$, $L(M_2)$. This 
is a theorem of Anderson; it is formulated in [P], end of page 3, but its proof was 
not published. We recall its statement for the case of arbitrary $N \neq 0$, and we give 
its proof for the case $N = 0$ (case of arbitrary $N$ can be obtained easily using the 
same ideas).

If $M$ is an uniformizable abelian T-motive such that $N \neq 0$ then $\text{Lie} (M)$ (resp. 
$L(M)$) is a $\mathbb{C}_\infty[T]$ (resp. $\mathbb{F}_q[T]$)-module, where $T$ acts as $\theta + \iota(N)$, hence $\text{Lie} (M)$ 
is a $\mathbb{C}_\infty[[T-\theta]]$-module, and there is an exact sequence of $\mathbb{C}_\infty[[T-\theta]]$-modules 

\[0 \to q(M) \to L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[T-\theta]] \to \text{Lie} (M) \to 0 \tag{6.1}\]

The kernel $q = q(M)$ carries information on the pair $(L(M), \text{Lie} (M))$.

**Theorem 6.2** (Anderson). Let $M$, $\bar{M}$ be any two uniformizable abelian T- 
motives. Then 

\[q(M \otimes \bar{M}) = q(M) \otimes_{\mathbb{C}_\infty[[T-\theta]]} q(\bar{M}) \tag{6.2.1}\]

**Remark.** $M \otimes \bar{M}$ is a uniformizable abelian T-motive ([G], Corollary 5.9.38).

**Proof (case $N = 0$).** We define notations for $M$, and all notations for $\bar{M}$ 
will carry bar. Let $e_i$ and $Z$ be from 3. We consider $e_i$ as a $\mathbb{C}_\infty[[T-\theta]]$-basis 
of $L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[T-\theta]]$. Elements $b_i := (T-\theta)e_i$, $i = 1, \ldots, n$ and $b_{n+i} := 
\sum_{j=1}^{n} z_{ij} e_j$, $i = 1, \ldots, n$ form a $\mathbb{C}_\infty[[T-\theta]]$-basis of $q$. We need a

**Lemma 6.2.2.** $\Psi(M \otimes \bar{M}) = \Psi(M) \otimes \Psi(\bar{M})$ where $\Psi(M)$ (resp. $\Psi(\bar{M})$; $\Psi(M \otimes \bar{M})$) is taken with respect to bases $e_\ast$ of $L(M)$, $f_\ast$ of $M_{\mathbb{C}_\infty[T]}$ (resp. $\bar{e}_\ast$, $\bar{f}_\ast$ of $\bar{M}_{\mathbb{C}_\infty[T]}$; $e_\ast \otimes \bar{e}_\ast$ of $L(M \otimes \bar{M})$, $f_\ast \otimes \bar{f}_\ast$ of $(M \otimes \bar{M})_{\mathbb{C}_\infty[T]}$) (see the proof for the notations).
Proof. We consider a map

$$\alpha : \text{Hom}_{C_{\infty}[T]}(M, Z_1)^\tau \otimes \text{Hom}_{C_{\infty}[T]}(\bar{M}, Z_1)^\tau \to \text{Hom}_{C_{\infty}[T]}(M \otimes \bar{M}, Z_1)^\tau$$

defined as follows: for $\varphi \in \text{Hom}_{C_{\infty}[T]}(M, Z_1)^\tau$, $\bar{\varphi} \in \text{Hom}_{C_{\infty}[T]}(\bar{M}, Z_1)^\tau$ we let $[\alpha(\varphi \otimes \bar{\varphi})](f \otimes \bar{f}) = \varphi(f) \cdot \bar{\varphi}(\bar{f})$ (it is obvious that $\alpha(\varphi \otimes \bar{\varphi})$ is $\tau$-stable). Since $e_1, \ldots, e_r$ (resp. $\bar{e}_1, \ldots, \bar{e}_r$) is a basis of $\text{Hom}_{C_{\infty}[T]}(M, Z_1)^\tau$ (resp. $\text{Hom}_{C_{\infty}[T]}(\bar{M}, Z_1)^\tau$; we identify $L(M)$, resp. $L(\bar{M})$ with $\text{Hom}_{C_{\infty}[T]}(M, Z_1)^\tau$ (resp. $\text{Hom}_{C_{\infty}[T]}(\bar{M}, Z_1)^\tau$) we get (using Lemma 5.4.1) that $\Psi(M)$, $\bar{\Psi}(\bar{M})$ are non-degenerate. Since their product is also non-degenerate, we get $\alpha(e_i \otimes \bar{e}_i)$ are linearly independent and hence a basis of $\text{Hom}_{C_{\infty}[T]}(M \otimes \bar{M}, Z_1)^\tau$. Applying once again Lemma 5.4.1 we get the lemma. $\square$

If $A, B$ are two matrices then columns of $A \otimes B$ are indexed by pairs $(k, l)$ where $k$ (resp. $l$) is the number of a column of $A$ (resp. $B$). We denote by $A_k, B_l$, $A \otimes B_{(k,l)}$ the respective columns. Obviously we have: $A \otimes B_{(k,l)} = A_k \otimes B_l$ (tensor product of column matrices).

Let us prove that for $i = 1, \ldots, r - n$, $\bar{i} = 1, \ldots, \bar{r} - \bar{n}$ the element $b_{n+i} \otimes \bar{b}_{\bar{n}+\bar{i}} \in q(M \otimes \bar{M})$. According [A], Proposition 3.3.2, it is sufficient to prove that the corresponding linear combination (see 6.2.3 below) of the columns of the matrix $\Psi_{M \otimes \bar{M}}^-$ is 0. Since

$$b_{n+i} \otimes \bar{b}_{\bar{n}+\bar{i}} = \sum_{j, \bar{j}} z_{ij} \bar{z}_{\bar{i}j} e_j \otimes \bar{e}_{\bar{j}} - \sum_{j} z_{ij} e_j \otimes \bar{e}_{\bar{n}+\bar{i}} - \sum_{j} \bar{z}_{\bar{i}j} e_{n+i} \otimes e_j + e_{n+i} \otimes \bar{e}_{\bar{n}+\bar{i}}$$

we get the explicit form of this linear combination: it is sufficient to prove that for all $i, \bar{i}$ we have

$$\sum_{j, \bar{j}} z_{ij} \bar{z}_{\bar{i}j} (\Psi_{M \otimes \bar{M}}^-(j, \bar{j})) - \sum_{j} z_{ij} (\Psi_{M \otimes \bar{M}}^-)(j, n+i)$$

$$- \sum_{\bar{j}} \bar{z}_{\bar{i}j} (\Psi_{M \otimes \bar{M}}^-)(n+i, \bar{j}) + (\Psi_{M \otimes \bar{M}}^-)(n+i, \bar{n}+\bar{i}) = 0 \quad (6.2.3)$$

Further, 6.2.2 implies that

$$(\Psi_{M \otimes \bar{M}}^-)(k, k) = \frac{A_{-1,k} \otimes \bar{A}_{-1,k}}{(T - \theta)^2} + \frac{A_{-1,k} \otimes \bar{A}_{0,k} + A_{0,k} \otimes \bar{A}_{-1,k}}{T - \theta}$$

hence 6.2.3 becomes

$$\sum_{j, \bar{j}} z_{ij} \bar{z}_{\bar{i}j} \left(\frac{A_{-1,j} \otimes \bar{A}_{-1,\bar{j}}}{(T - \theta)^2} + \frac{A_{-1,j} \otimes \bar{A}_{0,\bar{j}} + A_{0,j} \otimes \bar{A}_{-1,\bar{j}}}{T - \theta}\right)$$

$$- \sum_{j} z_{ij} \left(\frac{A_{-1,j} \otimes \bar{A}_{-1,n+i}}{(T - \theta)^2} + \frac{A_{-1,j} \otimes \bar{A}_{0,n+i} + A_{0,j} \otimes \bar{A}_{-1,n+i}}{T - \theta}\right)$$

$$- \sum_{\bar{j}} \bar{z}_{\bar{i}j} \left(\frac{A_{-1,n+i} \otimes \bar{A}_{-1,\bar{j}}}{(T - \theta)^2} + \frac{A_{-1,n+i} \otimes \bar{A}_{0,\bar{j}} + A_{0,n+i} \otimes \bar{A}_{-1,\bar{j}}}{T - \theta}\right)$$

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\[
\frac{A_{-1,n+i} \otimes \bar{A}_{-1,\bar{n}+\bar{i}}}{(T-\theta)^2} + \frac{A_{-1,n+i} \otimes \bar{A}_{0,\bar{n}+\bar{i}} + A_{0,n+i} \otimes \bar{A}_{-1,\bar{n}+\bar{i}}}{T-\theta} = 0 \quad (6.2.4)
\]

It is easy to see that 6.2.4 follows immediately from the equalities

\[A_{-1,n+i} = \sum_j z_{ij} A_{-1,j} \quad (6.2.5)\]

\[\bar{A}_{-1,\bar{n}+\bar{i}} = \sum_j \bar{z}_{ij} \bar{A}_{-1,j} \]

For example, the left hand side of (6.2.4) has 2 terms containing \(\bar{A}_{0,j}\) (in the middle of the first and the third lines of (6.2.4)). Multiplying (6.2.5) by \(\bar{z}_{ij} \bar{A}_{0,j}\) we get that the sum of these 2 terms of (6.2.4) is 0. For other pairs of terms of (6.2.4) the situation is the same.

The proof that for \(i = 1, \ldots, r-n, \bar{i} = 1, \ldots, \bar{n}\) the element \(b_{n+i} \otimes \bar{b}_{\bar{i}} \in q(M \otimes \bar{M})\) is analogous but simpler. We have

\[b_{n+i} \otimes \bar{b}_{\bar{i}} = (T-\theta)(-\sum_j z_{ij} e_j \otimes \bar{e}_{\bar{i}} + e_{n+i} \otimes \bar{e}_{\bar{i}})\]

The analog of (6.2.3)) is

\[(T-\theta)(-\sum_j z_{ij} (\Psi^{-}_{M\otimes \bar{M}})_{(j,\bar{i})} + (\Psi^{-}_{M\otimes \bar{M}})_{(n+i,\bar{i})}) = 0\]

and the analog of (6.2.4)) is

\[-\sum_j z_{ij} \frac{A_{-1,j} \otimes \bar{A}_{-1,i}}{T-\theta} + \frac{A_{-1,n+i} \otimes \bar{A}_{-1,\bar{i}}}{T-\theta} = 0\]

This equality follows immediately from (6.2.5).

Finally, elements \(b_i \otimes \bar{b}_i\) \((i = 1, \ldots, n, \bar{i} = 1, \ldots, \bar{n})\) obviously belong to \(q(M \otimes \bar{M})\).

So, we proved that \(q(M)_{C_{\infty}[[T-\theta]]} \otimes q(\bar{M}) \subset q(M \otimes \bar{M})\). Since the \(C_{\infty}\)-codimension of both subspaces in \(L(M) \otimes_{F_q[T]} L(\bar{M}) \otimes_{F_q[T]} C_{\infty}[[T-\theta]]\) is \(n\bar{n}\), they are equal. \(\Box\)

7. Polarization form.

Case A = \(F_{q[T]}\). Let \(M\) be a self-dual uniformizable abelian \(T\)-motive, i.e. there exists an isogeny \(\alpha : M \to M'\). It defines a \(F_q[T]\)-valued, \(F_q[T]\)-bilinear form \(<*,*>_{\alpha}\) on \(L(M')\) as follows:

\[<\varphi_1, \varphi_2>_{\alpha} = \langle L(\alpha)(\varphi_1), \varphi_2 >_M\]

5.2.11 implies that if \(\alpha' = -\alpha\) (resp. \(\alpha' = \alpha\)) then \(<*,*>_{\alpha}\) is skew symmetric (resp. symmetric).
Examples.

Let $e^*$ be from 1.9, and let $M$ given by the equation (here $A \in M_n(\mathbb{C}_\infty)$)

$$Te^*_\star = \theta e^*_\star + A\tau e^*_\star + \tau^2 e^*_\star$$  \hspace{1cm} \text{(7.1)}

be a T-motive of dimension $n$ and rank $2n$. Elements $f_i = e_i, f_{n+i} = \tau e_i$ ($i = 1, ..., n$) form a $\mathbb{C}_\infty[T]$-basis of $M$. We have (see, for example, Section 3): $M'$ is given by the equation

$$Te'_\star = \theta e'_\star - A^t \tau e'_\star + \tau^2 e'_\star$$

and if we define

$$f'_i = \tau e'_i, \quad f'_{n+i} = e'_i$$  \hspace{1cm} \text{(7.2)}

then bases $f^*_\star, f'_\star$ are dual in the meaning of Lemma 1.10.

Let $\alpha : M \to M'$ be given by the formula $\alpha(e_\star) = De'_\star$ where $D \in M_n(\mathbb{C}_\infty)$ (we consider only the case of constant map: elements of $D$ do not contain $\tau$). Condition that $\alpha$ is a $\mathbb{C}_\infty[T, \tau]$-map is equivalent to

$$D^{(2)} = D, \quad AD^{(1)} = -DA^t$$  \hspace{1cm} \text{(7.3)}

Further, we have

$$\alpha(f_\star) = Df'_f \quad \text{where} \quad D_f = \begin{pmatrix} 0 & D^{(1)} \\ D^{(1)} & 0 \end{pmatrix},$$

hence

$$\alpha' = \pm \alpha \iff D^{(1)}_f = \pm D_f \iff D^{(1)} = \pm D^t$$  \hspace{1cm} \text{(7.5)}

Let us fix $\varepsilon_0$ satisfying $\varepsilon_0^{q-1} = -1$. It is easy to see that if we take $A$ symmetric (particularly, if $n = 1$) then $D = \varepsilon_0 E_n$ satisfies 7.3, the sign in 7.5 is minus and hence for the corresponding $\alpha$ the form $<\cdot, \cdot >_{\alpha}$ is skew symmetric. If $A$ is skew symmetric then we can take $D = E_n$, the sign in 7.5 is plus and hence for the corresponding $\alpha$ the form $<\cdot, \cdot >_{\alpha}$ is symmetric.

Remark 7.6. The below statements are conjectures based on arguments similar to the ones which justify the below Conjecture 11.8.3. Since they are of secondary importance, we do not give any details of justification here.

A. Conjecture. If $n \geq 3$ then for a generic skew symmetric $A$ we have: $\text{End}(M) = \mathbb{F}_q[T]$.

B. Corollary. Conjecture A implies that the ”minimal” $\alpha : M \to M'$ is defined uniquely up to an element of $\mathbb{F}_q^*$, and hence the symmetric pairing $<\cdot, \cdot >_{\alpha}$ is also defined uniquely up to an element of $\mathbb{F}_q^*$.

C. Conjecture. If $n = 2, \alpha' = \alpha$ then $\text{End}M$ is strictly larger than $\mathbb{F}_q[T]$.

Other examples of a self-dual T-motive are $M \oplus M'$ where $M$ is any T-motive, but they do not give interesting examples of pairings.

D. Conjecture. There exist other (distinct from the ones defined by 7.1) self-dual T-motives $M$ having $\text{End}(M) = \mathbb{F}_q[T]$ (we can use a version of standard T-motives of Section 3).
Example 7.7. Case $A = 0, D = E_n$.

In this case we can find explicitly the matrix of the symmetric form $<\cdot, \cdot>_{\alpha}$ in some basis of $L(M')$. Let $\mathfrak{C}_2$ be the Carlitz module over the field $\mathbb{F}_{q^2}$ considered as a rank 2 Drinfeld module over $\mathbb{F}_q$ given by the equation

$$Te = \theta e + \tau^2 e$$

We have $M = \mathfrak{C}_2^{\otimes n}$. Let $\mathfrak{T}_T(\mathfrak{C}_2)$ be the convergent $T$-Tate module of $\mathfrak{C}_2$, i.e. the set of elements $\{z_i\} \in E(\mathfrak{C}_2) = \mathbb{C}_\infty$ ($i \geq -1, \ z_{-1} = 0$) such that

$$Tz_i = z_{i-1} \text{ for } i \geq 0 \text{ (i.e. } z_i^q - \theta z_i = z_{i-1}) \text{ and } z_i \to 0$$

It is a free 1-dimensional module over $\mathbb{F}_{q^2}[T]$. We choose and fix its generator; its $\{z_i\}$ satisfy (like in 5.3.4) $|z_0| > |z_i| \ \forall i > 0$. We denote $\sum_{k=0}^{\infty} z_k T^k$ by $z$.

Let $c$ be a fixed element of $\mathbb{F}_{q^2} - \mathbb{F}_q$. Formulas (5.3.3) show that the following elements $\varphi_i, \varphi_i' (i = 1, ..., 2n)$ form bases of $L(M), L(M')$ respectively ($j = 1, ..., n$; clearly that thanks to 7.2 we have $\varphi_i'(j') = \varphi_i(f_{n+j}), n + j \mod 2n$):

$$i \leq n : \varphi_i(f_j) = 3\delta_i^j, \varphi_i(f_{n+j}) = 3(1)\delta_i^j$$

$$i > n : \varphi_i(f_j) = c3\delta_{i-n}^j, \varphi_i(f_{n+j}) = c^q3(1)\delta_{i-n}^j$$

$$i \leq n : \varphi_i'(j') = 3(1)\delta_i^j, \varphi_i'(f_{n+j}) = 3\delta_i^j$$

$$i > n : \varphi_i'(j') = c^q3(1)\delta_{i-n}^j, \varphi_i'(f_{n+j}) = c3\delta_{i-n}^j$$

(by the way, it is clear that the same relation between elements of $\mathfrak{T}_T(M)$ and $\text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$ holds for all $M$). Formula 7.4 shows that $\alpha'(\varphi_i') = \varphi_{i+n}$, where $i + n \mod 2n$. Let us denote $\Xi = 3 \cdot 3(1) \in \mathbb{F}_q^*$ by $\gamma$. The above definitions and formulas show that the matrix of $<\cdot, \cdot>_{\alpha}$ in the basis $\varphi_1, \varphi_{n+1}, ..., \varphi_n, \varphi_{2n}$ consists of $n \ (2 \times 2)$-blocks (trace and norm of $\mathbb{F}_{q^2}/\mathbb{F}_q$)

$$\gamma \begin{pmatrix} \text{tr}(1) & \text{tr}(c) \\ \text{tr}(c) & \text{tr}(N(c)) \end{pmatrix} = \gamma \begin{pmatrix} 2 & c + c^q \\ c + c^q & 2c^{q+1} \end{pmatrix}$$

The determinant of this block is $-(c - c^q)^2\gamma^2$; it belongs to $\mathbb{F}_q^{*2} \iff q \equiv 3 \mod 4$ or $q$ is even. Since we have $n$ blocks, we have:

$$\det <\cdot, \cdot>_{\alpha} \notin \mathbb{F}_q^{*2} \iff q \equiv 1 \mod 4 \text{ and } n \text{ is odd}.$$

Remark 7.8 (Jorge Morales). There is a theorem of Harder (see e.g. W. Scharlau, ”Quadratic and Hermitian forms”, Springer-Verlag, Berlin, 1985, Chapter 6, Theorem 3.3) that states that a unimodular form over $k[X] - k$ being any field of characteristic not 2 — is the extension of a form over $k$, i.e. there is a basis in which all the entries of the associated symmetric matrix are constant. This means that the classification of the above quadratic forms over $\mathbb{F}_q[T]$ ($q$ odd) is very simple.
8. Relations between lattices and T-motives.

8.1. Firstly, we have

**Theorem 8.1.1.** ([H], Theorem 3.2). The dimension of the moduli set of pure T-motives of dimension \( n \) and rank \( r \) is \( n(r - n) \).

**Remark.** A tuple \((e_1, ..., e_r)\) of integers entering in the statement of this theorem in [H] is \((0, ..., 0, 1, ..., 1)\) with 0 repeated \( r - n \) times and 1 repeated \( n \) times for the case under consideration.

Since this number \( n(r - n) \) is equal to the dimension of the set of lattices of rank \( r \) and dimension \( n \), we can state an

**Open question 8.1.2.** Let \( r, n \) be given. Is it true that the moduli space of the pure uniformizable abelian T-motives of rank \( r \) and dimension \( n \) (maybe its irreducible component) is in 1–1 correspondence with (an open part of) the set of lattices of rank \( r \) and dimension \( n \), and is this correspondence functorial?

**Remarks 8.1.3.** A. The below Proposition 8.2.3 shows that if \((r, n) \neq 1\) then the answer is no.

B. We must exclude the case \( r = n = 1 \) from the statement of functoriality: the pair \((L_0, V_0)\) of rank and dimension 1 has many inclusions to any \((L, V)\) in the category of pairs, while for almost all \( M \) there is no inclusion \( C \rightarrow M \).

Theorem 5 implies that for \( n = r - 1 \) the answer to 8.1.2 is yes (the below Proposition 11.8.5 shows that most likely the condition of purity is essential):

**Corollary 8.1.4.** All pure T-motives of dimension \( r - 1 \) and rank \( r \) having \( N = 0 \) are uniformizable. There is a 1–1 functorial correspondence between pure T-motives of dimension \( r - 1 \) and rank \( r \) having \( N = 0 \) (\( r \geq 2 \)), and lattices of rank \( r \) in \( \mathbb{C}^{r-1}_\infty \) having dual.

**Proof.** Let \( L \) be a lattice of rank \( r \) in \( \mathbb{C}^{r-1}_\infty \) having dual \( L' \). There exists the only Drinfeld module \( M' \) such that \( L(M') = L' \), and let \( M \) be its dual. Theorem 5 implies that \( L(M) = L \). If there exists another pure T-motive \( M_1 \) of dimension \( r - 1 \) and rank \( r \) having \( N = 0 \) such that \( L(M_1) = L \) then by Corollary 10.4 (its proof is logically independent: there is no vicious circle) the dual \( M'_1 \) is a Drinfeld module, according Theorem 5 it satisfies \( L(M'_1) = L' \), hence \( M'_1 = M' \) and hence \( M_1 = M \). \( \square \)

**Remark 8.1.5.** Recall that lattices of rank \( r \) in \( \mathbb{C}^{r-1}_\infty \) having dual are described in 3.5 (formulas 3.6, 3.7).

**Remark 8.1.6.** Clearly for any \( r, n \) we have: if a lattice \( L \) of rank \( r \) and dimension \( n \) has no dual then \( L \neq L(M) \) for any pure uniformizable \( M \). I do not know whether Theorem 6.2 (which is an analog of Theorem 5 for another tensor operation) imposes a more strong similar restriction on the property of \( L \) to be the \( L(M) \) of some pure uniformizable \( M \), or not.

Further, for any uniformizable T-motive \( M \) we have a

**Corollary 8.1.7.** If the dual of \((L(M), \text{Lie}(M))\) does not exist then the dual of \( M \) does not exist. Example: the Carlitz module.

\[^2\text{I am grateful to Urs Hartl who indicated me this reference.}\]
8.2. Infinitesimal lattice map and C-lattices. For \( r = 2n, \ n > 1 \) the degree of the lattice map is infinite, hence in order to get a 1 - 1 correspondence it is necessary to introduce a new object called a C-lattice. Since I am not sure that this description is unknown, only sketches of proofs will be given. There is a C-lattice map from a neighbourhood of a T-motive \( M_0 \) to a neighbourhood of a C-lattice \( L_{C,0} \) (see below for the definition of \( M_0, L_{C,0} \)) which is 1 - 1.

For the case \((r, n) = 1\) I do not see an initial T-motive \( M_0 \) such that in its neighbourhood the degree of the lattice map is not 1 (see Remark 8.2.14 for more details), hence we can ask whether in this case the answer to 8.1.2 is yes. I think that this is few likely.

**Definition 8.2.1.** A C-lattice is a pair \((L, e_*)\) where

\[ L \subset V = \mathbb{C}_\infty^n \] is a lattice and

\[ e_1, \ldots, e_r \text{ is a } \mathbb{F}_q[\theta]\text{-basis of } L. \]

Further, two such pairs \((L; e_1, \ldots, e_r)\) and \((L'; e'_1, \ldots, e'_r)\) are called equivalent if there exists a \( \mathbb{C}_\infty\text{-linear map } \psi : V \to V \) such that \( \psi(L; e_1, \ldots, e_r) = (L'; e'_1, \ldots, e'_r) \), or \( L = L' \) and the matrix of the change of basis from \( e_1, \ldots, e_r \) to \( e'_1, \ldots, e'_r \) (which a priori belongs to \( GL_r(\mathbb{F}_q[\theta]) \) ) belongs to \( GL_r(\mathbb{F}_q) \).

The functor of forgetting the basis from C-lattices to lattices is denoted by \( i \). The notion of a Siegel matrix for a C-lattice is the same as for a lattice.

We consider infinitesimal degree of the lattice map in a neighbourhood of some distinguished (having many endomorphisms) T-motive \( M_0 \). We consider the case \( M_0 = \mathbb{C}_\infty^{2n} \) (see 7.7; it was denoted by \( M \) in 7.7). It is easy to see that \( M_0 \) is the (only) T-motive with complete multiplication by \( \mathbb{F}_{q^2}[\theta] \) with CM-type \( Id, fr^2, fr^4, \ldots, fr^{2n-2} \). Let \( \omega \in \mathbb{F}_{q^2} - \mathbb{F}_q \) be a fixed element. A Siegel matrix of \( M_0 \) is \( \omega E_n \). We denote the lattice (resp. the C-lattice) corresponding to \( \omega E_n \) by \( L_0 \) (resp. \( L_{C,0} \)). \( L_0 \) is the lattice of \( M_0 \). We consider 5 sets \( S_1, \ldots, S_5 \):

- **S_1.** The set of \( n \times n \) matrices \( A \).
- **S_2.** The set of T-motives \( M \) given by the equation
  \[ Te_* = \theta e_* + A \tau e_* + \tau^2 e_* \] (8.2.2)
  where \( A \in S_1, \ e_* = (e_1, \ldots, e_n)^t. \)
- **S_3.** The set of Siegel matrices \( Z \).
- **S_4.** The set of C-lattices of rank \( r = 2n \) in \( \mathbb{C}_\infty^n \).
- **S_5.** The set of lattices of rank \( r = 2n \) in \( \mathbb{C}_\infty^n \).

We consider initial elements \( 0, M_0, \omega E_n, L_{C,0}, L_0 \) of \( S_1, \ldots, S_5 \) respectively and open neighbourhoods \( U_i \subset S_i \) of these initial elements.

**Proposition 8.2.3.** There exist neighbourhoods \( U_2, U_4, U_5 \) such that

(a) The restriction of \( i \) to \( U_4 \) gives us an epimorphism \( U_4 \to U_5 \).

(b) there exists a 1 - 1 map \( \mu_{24} \) from \( U_2 \) to \( U_4 \) such that \( \mu_{25} := i \circ \mu_{24} \) (see the below diagram 8.2.4) is the lattice map from uniformizable abelian Anderson T-motives to lattices. Particularly, for \( n > 1 \) the fibre of \( \mu_{25} \) is discrete infinite.
Proof. It is sufficient to prove that there exists a commutative diagram

\[
\begin{array}{ccc}
U_1 & \xrightarrow{\mu_{12}} & U_2 \\
\mu_{13} \downarrow & & \mu_{24} \downarrow \xrightarrow{\mu_{25}} \\
U_3 & \xrightarrow{\mu_{34}} & U_4 \xrightarrow{\iota} U_5
\end{array}
\]  

(8.2.4)

where \(\mu_{12}(A)\) is the T-motive defined by 8.2.2, \(\mu_{34}(Z)\) is the C-lattice corresponding to a Siegel matrix \(Z\) and \(\mu_{13}\) is defined as follows. We identify \(\text{Lie}(\mathfrak{C}_2)\) with \(\mathbb{C}_\infty\) and hence \(\text{Lie}(M_0)\) with \(\mathbb{C}_\infty^n\). We consider the following basis \(l_0,1,...,l_0,2n\) of \(L_0 \subset \text{Lie}(M_0) = \mathbb{C}_\infty^n\): \(l_{0,i} = (0,...,0,1,0,...,0)\) (1 at the \(i\)-th place), \(l_{0,n+i} = \omega l_{0,i}, i = 1,...,n\). The Siegel matrix of this basis is \(\omega E_n\). Let \(A \in U_1, M\) its \(\mu_{12}\)-image and \(L\) its lattice, i.e. its \(\mu_{25} \circ \mu_{12}\)-image. For any \(l_0 \in L_0\) there exists a well-defined \(l \in L\) which is close to \(l_0\) (because entries of \(A\) are near 0). So, we consider a basis \(l_1,...,l_{2n}\) of \(L\) where any \(l_i\) is near the corresponding \(l_0,i\). The Siegel matrix corresponding to \(l_1,...,l_{2n}\) is exactly the \(\mu_{13}\)-image of \(M\). By definitions, the outer quadrangle is commutative.

We denote by \(d_{\alpha\beta}\) the degree of \(\mu_{\alpha\beta}\) at a generic point near the initial element of \(U_\alpha\). We must prove that there exists a map \(\mu_{24}\) preserving commutativity, and that \(d_{24} = 1\).

Lemma 8.2.5. \(d_{13} = 1\).

Proof. We can associate 8.2.2 the standard commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}_\infty^n & \xrightarrow{\text{Exp}} & \mathbb{C}_\infty^n \\
\theta \downarrow & & m_T \downarrow \\
\mathbb{C}_\infty^n & \xrightarrow{\text{Exp}} & \mathbb{C}_\infty^n
\end{array}
\]

where for \(Z \in \mathbb{C}_\infty^n\) we have \(m_T(Z) = \theta Z + AZ^{(1)} + Z^{(2)}, \text{Exp}(Z) = \text{Exp}_A(Z) = \sum_{i=0}^\infty C_i Z^{(i)}\) where \(C_i = C_i(A), C_0 = 1\).

For the reader’s convenience, we consider only the case \(n = 1\) (the general case does not require any new ideas), hence \(Z, A\) will be denoted by \(z, a\) respectively. We denote \(\theta_{ij} = \theta q^i - \theta q^j\). Recall that the exponent for \(\mathfrak{C}_2\) has the form

\[
\text{Exp}_0(z) = z + \frac{1}{\theta_{20}} z^2 + \frac{1}{\theta_{42} \theta_{40}} z^4 + ... 
\]

(8.2.6)

\[
(C_{2l}(0) = \frac{1}{\prod_{j=0}^{l-1} \theta_{2j+2j}}). We denote by \(y_0 \in \mathbb{C}_\infty\) a nearest-to-zero root to \(\text{Exp}_0(z) = 0\) (this is \(\xi\) of \(\mathfrak{C}_2\) in notations of \([G]\)). It is defined up to multiplication by elements of \(\mathbb{F}_{q^2}\), and it generates over \(\mathbb{F}_{q^2}[\theta]\) the lattice of the Carlitz module \(\mathfrak{C}_2\). We fix one such \(y_0\). If \(a\) is sufficiently small then there is the only root to \(\text{Exp}_a(z) = 0\) near \(y_0\), and there is the only root to \(\text{Exp}_a(z) = 0\) near \(\omega y_0\). We denote these roots by \(z = z(a), z' = z'(a)\) respectively, and we denote \(z = y_0 + \delta, z' = \omega y_0 + \delta'\). \(\delta\) (resp. \(\delta'\)) is a root to the power series

\[
\sum_{i,j=0}^\infty d_{ij} a^i \delta^j = 0 \quad (8.2.7)
\]
Moreover, $\delta$ that $d$ respectively. Exactly, both $\delta$ where $M$ on $(8.2.2)$ becomes a more complicated equation: terms having higher powers of $g$ an element $g$ of this group acts on the basis $\{e_i\}$ of $A$ and $\{\theta_i\}$ of $\theta_j$. This means that the approximate value of $\delta$ is

$$-d_{10}a, \ -d'_{10}a \tag{8.2.8}$$

respectively. Exactly, both $\delta$, $\delta'$ are power series in $a$ whose first term is given by (8.2.8). This means that

$$z = y_0 - d_{10}a + \sum_{i=2}^{\infty} k_i a^i$$

$$z' = \omega y_0 - d'_{10}a + \sum_{i=2}^{\infty} k'_i a^i$$

It is easy to see that $d'_{10} \neq \omega d_{10}$, hence the Siegel matrix $3 = z^{-1}z'$ is given by the formula

$$3 = \omega + \sum_{i=1}^{\infty} l_i a^i \tag{8.2.9}$$

and $l_1 \neq 0$. Since (for $n = 1$) $d_{13}$ is the minimal $i$ such that $l_i \neq 0$ we get that $d_{13} = 1$.

For $n > 1$ the calculation is the same (this is the main part of the proof, because for the case $n = 1$ the result is known). Analog of (8.2.9) is

$$3 = \omega E_n + l_1 A + P_{\geq 2}(A) \tag{8.2.10}$$

where $P_{\geq 2}(A)$ is a power series of entries of $A$ such that all its terms have degree $\geq 2$. Condition $l_1 \neq 0$ implies $d_{13} = 1$. □

We denote the monodromy group of $\mu_{12}$, resp. $\mu_{25}$ by $M_{12}$, resp. $M_{25}$. We have $M_{12} = GL_n(F_q^2)/F_q^*$. Really, the automorphism group of $M_0$ is $GL_n(F_q[T])$: an element $g$ of this group acts on the basis $e_*$ of $(8.2.2)$ if $A = 0$. But if $A$ is a generic matrix, then for $g \in GL_n(F_q[T]) - GL_n(F_q)$ the result of the action of $g$ on $(8.2.2)$ becomes a more complicated equation: terms having higher powers of $\tau$ appear, so these $g$ do not belong to $M_{12}$. Factorization by $F_q^*$ is obvious. Further, obviously $M_{25} = \{\gamma \in PGL_{2n}(F_q)|\gamma(\omega E_n) = \omega E_n\}$. The outer quadrangle of 8.2.4 defines a map from $M_{12}$ to $M_{25}$ which we denote by $\alpha$.

**Lemma 8.2.11.** $\alpha$ is injective, and $\text{im } \alpha = \{\gamma \in PGL_{2n}(F_q)|\gamma(\omega E_n) = \omega E_n\} = M_{25} \cap PGL_{2n}(F_q) \subset PGL_{2n}(F_q[\theta])$.  

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Proof. \( \alpha \) is defined by the condition: for any \( A \in S_1, \gamma \in M_{12} \) we have
\[
\mu_{13}(\gamma(A)) = (\alpha(\gamma))(\mu_{13}(A))
\] (8.2.12)
The explicit formula for \( \alpha \) is the following. For odd \( q \) we fix \( \omega \) satisfying \( \omega^2 \in \mathbb{F}_q^\ast \) (it is an easy exercise to find analog of the below formula for even \( q \)). Let \( \gamma = U + \omega V, \ U, V \in GL_n(\mathbb{F}_q) \) (we consider clearly a representative of \( \gamma \in GL_n(\mathbb{F}_q^2)/\mathbb{F}_q^\ast \) in \( GL_n(\mathbb{F}_q^2) \)). Then
\[
\alpha(\gamma) = \left( \begin{array}{cc} U & -\omega^2 V \\ -V & U \end{array} \right)^{-1}
\] (8.2.13)
(\( \alpha \) is an antihomomorphism, because the functor of lattice is contravariant).
It is checked immediately that (8.2.12) holds. We see that \( \text{im} \alpha = \{ \gamma \in PGL_{2n}(\mathbb{F}_q) | \gamma(\omega E_n) = \omega E_n \} \). □

Proposition 8.2.3 follows immediately from these lemmas. □

Remark 8.2.14. We see that if \( r = 2n, n > 1 \) then the lattice map \( \mu_{25} \) is not a local isomorphism near \( M_0 \). The origin of this phenomenon is reducibility of \( M_0 \) which implies that the monodromy group of \( \mu_{25} \) is much bigger then the one of \( \mu_{12} \). For other values of \( r, n \) a natural analog of \( M_0 \) is a T-motive with complete multiplication. Apparently if \( (r, n) = 1 \) then all pure T-motives with complete multiplication are irreducible (example: CM-field is \( \mathbb{F}_q^{\ast}[\theta] \)), and analog of Proposition 8.2.3 for this case shows that \( \mu_{25} \) is a local isomorphism near this T-motive.

Remark 8.2.15. From the first sight, for \( n = 1 \) Proposition 8.2.3 contradicts to a result of Drinfeld about \( 1 \) \( 1 \) correspondence between Drinfeld modules and lattices. Really, there is no contradiction: if \( n = 1 \) then \( \text{im} \alpha = M_{25} \), and — although \( i : S_4 \rightarrow S_5 \) clearly is not an isomorphism — its restriction to \( U_4 \) is an isomorphism \( U_4 \rightarrow U_5 \).

8.3. Duality of C-lattices. We have no analog of 2.2 for C-lattices, so we use an analog of 3.2 as a definition of duality. Namely, if \( Z \) is a Siegel matrix of a C-lattice \( (L, e_+) \) then its dual \( (L, e_+)' \) is a C-lattice whose Siegel matrix is \( -Z^t \) (or \( Z^t \) which is the same, but more convenient for further calculations, because \( (-\omega E_n)^t \neq \omega E_n \)). Equality 3.8.2 shows that this notion is well-defined (entries of \( A, B, C, D \) belong to \( \mathbb{F}_q \)).

An analog of Theorem 5 holds for C-lattices:

Theorem 8.3.1. Let \( M \in U_2 \). Then \( \mu_{24}(M') = \mu_{24}(M)' \).

Proof is completely analogous to the proof of Theorem 5, so we omit it. Alternatively, we can show that the exact form of (8.2.10) is
\[
3 = \omega E_n + \sum_{k=1}^{\infty} \sum_{d_1, \ldots, d_k} l_{d_1, \ldots, d_k} A^{(d_1)} \cdot \ldots \cdot A^{(d_k)}
\]
where coefficients \( l_{d_1, \ldots, d_k} \) satisfy
\[
l_{d_1, \ldots, d_k} = l_{d_k, \ldots, d_1}
\]
This obviously implies the theorem. □
9. Hodge spaces and conjecture.

We fix an uniformizable abelian $T$-motive $M$, we consider its pair $(L,V) = (L(M), \text{Lie } (M))$ and the map $\varphi : L \otimes \mathbb{C}_\infty \to V$. By analogy with the number field case we denote

$$H^{0,1}(M) := V^*, H^{1,0}(M) := (\text{Ker } \varphi)^*$$

i.e. we have an exact sequence

$$0 \to H^{0,1}(M) \to H^1(M, \mathbb{C}_\infty) \to H^{1,0}(M) \to 0$$

Further, we denote $H^i(M, \mathbb{C}_\infty) := \wedge^i(H^1(M, \mathbb{C}_\infty))$, $H^i(M) := \wedge^i(L(M))$. Since — unlike in the number field case — there is no complex conjugation on $H^1(M, \mathbb{C}_\infty)$, hence there is no section $H^{1,0}(M) \to H^1(M, \mathbb{C}_\infty)$ of the above exact sequence, we define a modified (increasing) Hodge filtration space $\mathfrak{F}^i(H^i(M, \mathbb{C}_\infty)) \subset H^i(M, \mathbb{C}_\infty)$ as the linear envelope of elements $a_1 \wedge \ldots \wedge a_i$ where for $k = 1, \ldots, i - j$ we have $a_k \in H^{0,1}(M)$. Particularly, $\mathfrak{F}^{-1}(H^1(M, \mathbb{C}_\infty)) = 0$, $\mathfrak{F}^i(H^i(M, \mathbb{C}_\infty)) = H^i(M, \mathbb{C}_\infty)$, and

$$H^{i,j}(M) = \mathfrak{F}^i(H^{i+j}(M, \mathbb{C}_\infty))/\mathfrak{F}^{i-1}(H^{i+j}(M, \mathbb{C}_\infty))$$

is a subquotient of $H^{i+j}(M, \mathbb{C}_\infty)$ but not its direct summand like in the number field case. An $i$-th Hodge class is an element of $\mathfrak{F}^i(H^{2i}(M, \mathbb{C}_\infty)) \cap H^{2i}(M)$.

Hodge conjecture for codimension 1. We have a

**Question 9.1.** Is it true that for pure $M$ of dimension $n$ and rank $2n$ the Hodge classes for $i = 1$ are in $1 - 1$ correspondence with the skew maps $M \to M' = \text{maps } \mathfrak{C} \to \wedge^2(M)$?

From the first sight, results of 8.2 imply that the answer to this question is no, and it is necessary to modify the notion of Hodge class (like we modify the notion of lattice in order to get the notion of C-lattice) in order to get a $1 - 1$ correspondence between Hodge classes and skew maps $M \to M'$. Really, the following proposition shows that maybe the answer to 9.1 in its present form is yes:

**Proposition 9.2.** For a generic $M$ of the form (8.2.2), as well as for $M = M_0$, the answer to 9.1 is yes.

**Proof.** For a generic $M$ both sets of Question 9.1 are empty. Further, for any $M$ it is obvious that existence of an element in $\mathfrak{F}^1(H^2(M, \mathbb{C}_\infty)) \cap H^2(M)$ is equivalent to existence of a skew map of pairs $(L(M), \text{Lie } (M)) \to (L(M'), \text{Lie } (M'))$, hence it is sufficient to prove that for $M = M_0$ the natural maps

$$\beta : \text{Hom}(M, M') \to \text{Hom}((L(M), \text{Lie } (M)), (L(M'), \text{Lie } (M')))$$

$$\beta^{\text{skew}} : \text{Hom}^{\text{skew}}(M, M') \to \text{Hom}^{\text{skew}}((L(M), \text{Lie } (M)), (L(M'), \text{Lie } (M')))$$

are isomorphisms (here $\text{Hom}^{\text{skew}}$ is the set of maps $\varphi$ satisfying $\varphi' = -\varphi$). We have $M_0' = M_0$, $\text{Hom}(M_0, M_0) = GL_n(\mathbb{F}_q[T])/\mathbb{F}_q^*$.

$\text{Hom}((L(M_0), \text{Lie } (M_0)), (L(M_0'), \text{Lie } (M_0'))) = M_{12}$. There is an obvious inclusion $M_{12} \to \text{Hom}(M_0, M_0)$ denoted by $\chi$. We have $\beta \circ \chi = \alpha$ where $\alpha$ is from 8.2.11. It is easy to check that the formula for $\beta$ practically coincides with 8.2.13. More exactly, we consider an isomorphism $\iota : \mathbb{F}_q[T] \to \mathbb{F}_q^2[\theta]$ (recall that $\iota(T) = \theta$).
Then (notations of 8.2.11, \( U, V \in \mathbb{F}_q[T] \)) \( \beta(U + \omega V) = \left( \begin{array}{cc} \iota(U) & -\omega^2 \iota(V) \\ -\iota(V) & \iota(U) \end{array} \right)^{-1} \).

We get that \( \beta \) is an isomorphism; it is clear that \( \beta^{skew} \) is also an isomorphism. \( \Box \)

I do not know what is the relation between \( i \)-th Hodge classes \( (i > 1) \) and maps \( C \to \wedge^{2i}(M) \), or maps \( \wedge^i(M) \to \wedge^i(M') \).

10. Duals of pures, and other elementary results.

Here we consider the case \( A = \mathbb{F}_q[T] \).

The definition 1.8 extends to the case of pré-T-motives, and remarks 1.11 hold for this case.

**Lemma 10.2.** Let \( M \) be a pré-T-motive, \( m = m(M) \) from its (1.3.1), and \( \mu \geq m \). Then \( M' \) — the \( \mu \)-dual of \( M \) — exists as a pré-T-motive, and \( m(M') \leq \mu \). If \( M' \) is a T-motive then \( \dim M' = r\mu - \dim M \) (\( r \) is the rank of \( M \)).

**Proof.** We must check that \( Q' \) has no denominators, and the condition (1.3.1). The module \( \tau M \) is a \( C_\infty[T] \)-submodule of \( M \) (because \( \alpha x = \tau a^{\frac{1}{q}} x \) for \( x \in M \)), hence there are \( C_\infty[T] \)-bases \( f_\ast = (f_i, \ldots, f_r)^{\ast} \), \( g_\ast = (g_1, \ldots, g_r)^{\ast} \) of \( M, \tau M \) respectively such that \( g_i = P_i f_i \), where \( P_1|P_2|\ldots|P_r, P_i \in C_\infty[T] \). Condition (1.3.1) means that \( \forall i \ (T - \theta)^{m_i} f_i \in \tau M \), i.e. \( P_i (T - \theta)^{m_i} \) where \( 0 \leq m_i \leq m_{i+1} \leq m \). There exists a matrix \( \Omega = \{ q_{ij} \} \in M_r(C_\infty[T]) \) such that

\[
\tau f_i = \sum_{j=1}^{r} q_{ij} g_j = \sum_{j=1}^{r} q_{ij} P_j f_j
\]

(10.2.1)

Although \( \tau \) is not a linear operator, it is easy to see that \( \Omega \in GL_r(C_\infty[T]) \) (really, there exists \( C = \{ c_{ij} \} \in M_r(C_\infty[T]) \) such that \( g_i = P_i f_i = \tau(\sum_{j=1}^{r} c_{ij} f_j) \), we have \( C^{(1)} \Omega = E_r \).

We denote the matrix \( \text{diag} (P_1, P_2, \ldots, P_r) \) by \( \mathfrak{P} \), so (10.2.1) means that

\[
Q = \Omega \mathfrak{P}
\]

(10.2.2)

**Remark 10.2.3.** Since \( \Omega \mathfrak{P} \in GL_r(C_\infty(T)) \), we get that the action of \( \tau \) on \( i_2(M) \) is invertible.

It is clear that if \( M \) is a T-motive then

\[
\dim M = \sum_{j=1}^{r} m_j
\]

(10.2.4)

(because \( \dim M = \dim_{C_\infty}(M/\tau M) \)). Further, (10.2.2) implies that for \( Q' = Q(M') \) we have

\[
Q' = \Omega^{t-1} \text{diag} ((T - \theta)^{\mu-m_1}, \ldots, (T - \theta)^{\mu-m_r})
\]

(10.2.5)

This means that elements of \( Q' \) have no denominators. The condition (1.3.1) for \( M' \) follows easily from (10.2.5) (because \( \Omega^{t-1} \in GL_r(C_\infty[T]) \)), and the dimension formula (for the case \( M' \) is a T-motive) follows immediately from (10.2.4) applied to \( M' \). \( \Box \)

A definition of a pure abelian T-motive ( = pure T-motive) can be found in [G] ((5.5.2), (5.5.6) of [G] + formula (1.3.1) of the present paper).
Theorem 10.3. Let $M$ be a pure T-motive and $m = m(M)$ from (1.3.1). Then (if $rm - n > 0$) its $m$-dual $M'$ exists, and it is pure.

Proof. The definition of pure ([G], (5.5.2)) is valid for pré-T-motives. We use its following matrix form. We denote $T^{-1}$ by $S$ and for any $C$ we let

$$C[i] = C^{(i-1)} \cdot C^{(i-2)} \ldots \cdot C^{(1)} \cdot C$$

Lemma 10.3.1. Let $Q \in M_r(\mathbb{C}_\infty[T])$ be a matrix such that formula (1.9.3) defines an abelian T-motive $M$. Then it is pure iff there exists $C \in GL_r(\mathbb{C}_\infty([S]))$ such that for some $q, s > 0$

$$S^q C^{(s)} Q[s] C^{-1} \in GL_r(\mathbb{C}_\infty[[S]])$$

i.e. iff $S^q C^{(s)} Q[s] C^{-1}$ is $S$-integer and its initial coefficient is invertible.

Proof. Elementary matrix calculations. We take $C$ as a matrix of base change of $f^*$ to a $C_\infty[[S]]$-basis of $W$ of (5.5.2) of [G]. □

Lemma 10.3.2. Let $\mu = m$. We have: $M = M'\mu$ of Lemma 10.2 is a pure pré-T-motive.

Proof. Let $q$, $s$ and $C$ be from Lemma 10.3.1. We have

$$Q'[s] = ((T - \theta)[s]^{\mu}) Q[s] t^{-1}$$

(we use (1.2)). We take $C' = C^t$. We have

$$S^{s\mu - q} C'^{(s)} Q'[s] C'^{-1} =$$

$$= S^{s\mu - q} C^{(s)} t^{-1} Q[s] t^{-1} ((\frac{1}{S} - \theta)[s]^{\mu}) C^t$$

$$= ((1 - S\theta)[s])^{\mu} S^{-q} C^{(s)} t^{-1} Q[s] t^{-1} C^t$$

$$= ((1 - S\theta)[s])^{\mu} (S^q C^{(s)} Q[s] C^{-1}) t^{-1}$$

We have: $q/s = n/r$ ([G], (5.5.6)), hence $(s\mu - q)/s = (r\mu - n)/r$ and $s\mu - q > 0$. Further, $(1 - S\theta)^{[s]} \mu \in GL_r(\mathbb{C}_\infty[[S]])$, and the result follows from Lemma 10.3.1. □

Remark. This result holds also for $\mu > m$.

The theorem 10.3 follows from Lemma 10.2, the above lemmas and the proposition that a pure pré-T-motive satisfying (1.3.1) is a T-motive ([G], (5.5.6), (5.5.7)). □

Corollary 10.4. Let $M$ be an abelian T-motive such that $m = 1$, $n = r - 1$. Then $M$ has dual $\iff M$ is pure $\iff M$ is dual to a Drinfeld module.

Proof. Dimension formula shows that $M'$ (if it exists) is a Drinfeld module, and they are all pure. □

Example 10.5. Let $M$ be given by

$$A_0 = \theta E_2, \quad A_1 = \begin{pmatrix} a_{11} & 0 \\ a_{12} & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
This $M$ has $m = 1$, $n = 2$, $r = 3$, and it is easy to see that it has no dual. Really, for this $M$ we have (notations of 1.9) $f_1 = e_1$, $f_2 = \tau e_1$, $f_3 = e_2$,

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ T - \theta & -a_{111} & 0 \\ 0 & -a_{121} & T - \theta \end{pmatrix}, \quad Q' = \begin{pmatrix} a_{111} & T - \theta & a_{121} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The last line of $Q'$ means that $\tau f'_3 = f'_3$. This is a contradiction to the property that $M'_{C_\infty[\tau]}$ is free. It is possible also to show (Proposition 11.3.4) that $M$ is not pure, and to use 10.4 in order to prove that it has no dual.

Later (Section 11) we shall construct examples of non-pure abelian T-motives which have dual. Considerations of 11.8 predict that there is enough such T-motives.

**Theorem 10.6.** For any abelian T-motive $M$ there exists $\mu_0$ such that for all $\mu \geq \mu_0$ the object $M'^{\mu}$ exists as an abelian T-motive. For these $\mu$ we have

$$M'^{\mu+1} = M'^{\mu} \otimes \mathcal{C} \quad (10.6.1)$$

**Proof.** (10.6.1) holds at the level of pré-T-motives, because $Q(\mathcal{C}) = (T - \theta)E_1$. According [G], Lemma 5.4.10 it is sufficient to prove that $M'^{\mu}$ is finitely generated as a $C_\infty[\tau]$-module. We shall use notations of Lemma 10.2. We take

$$\mu_0 = 1 + \{\text{the maximum of the degrees of entries of } Q(M) \text{ as polynomials in } T\} + \max(m_k)$$

Let $f'_1, ... f'_r$ be the basis of $M'^{\mu}$ over $C_\infty[T]$ dual to $f_1, ... f_r$. It is sufficient to prove the

**Lemma 10.6.2.** Let $i_0 = \mu - \min(m_k)$. Then elements $T^i f'_j$, $i < i_0$, $j = 1, ..., r$, generate $M'^{\mu}$ as a $C_\infty[\tau]$-module.

**Proof of the lemma.** By induction, it is sufficient to show that for all $\alpha \geq i_0$ the equation

$$\tau x = (T - \theta)^\alpha f'_j \quad (10.6.3)$$

(equality in $M'^{\mu}$) has a solution

$$x = \sum_{k=1}^r C_k f'_k$$

where $C_k \in C_\infty[T]$, $\deg(C_k) < \alpha$. According (10.2.5), the solution to (10.6.3) is given by

$$(C_1^{(1)}, ..., C_r^{(1)}) = (0, ... 0, (T - \theta)^{\alpha - \mu + m_j}, 0, ... 0) \mathcal{Q}^\ell$$

(the non-0 element of the row matrix is at the $j$-th place). Unequalities satisfied by $\mu$ and $\alpha$ show that all $C_k^{(1)}$ are polynomials of degree $< \alpha$. Since $c \mapsto c^q$ is surjective on $C_\infty$, we get the desired. □
10.7. Virtual abelian T-motives.\(^3\) We need two elementary lemmas.

**Lemma 10.7.0.**\(^4\) If \(M\) is an abelian T-motive then \(M \otimes \mathfrak{C}\) is also an abelian T-motive.

**Proof.** Let \(f_j (j = 1, \ldots, r)\) be a \(\mathbb{C}_\infty[T]\)-basis of \(M_{\mathbb{C}_\infty[T]}\) and \(\mathfrak{f}\) from 1.10.2, so \(f_j \otimes \mathfrak{f}\) is a \(\mathbb{C}_\infty[T]\)-basis of \((M \otimes \mathfrak{C})_{\mathbb{C}_\infty[T]}\). It is sufficient to prove that \((M \otimes \mathfrak{C})_{\mathbb{C}_\infty[r]}\) is finitely generated. Since \(M_{\mathbb{C}_\infty[r]}\) is finitely generated, it is easy to see that there exists a \(\alpha\) such that elements

\[(T - \theta)^i f_j, \; i = 0, \ldots, a, \; j = 1, \ldots, r\]

generate \(M_{\mathbb{C}_\infty[r]}\). This means that \(\forall j = 1, \ldots, r\) there exist \(c_{ijkl} \in \mathbb{C}\) such that

\[(T - \theta)^{a+1} f_j = \sum_{i=0}^a \sum_{k=0}^\gamma \sum_{l=1}^r c_{ijkl}(T - \theta)^i \tau^k f_l \quad (10.7.0.1)\]

where \(\gamma\) is a number.

Let us multiply (10.7.0.1) by \((T - \theta)^\gamma\). Taking into consideration the formula of the action of \(\tau\) on \(M \otimes \mathfrak{C}\) we get that the result gives us the following formula in \(M \otimes \mathfrak{C}\):

\[(T - \theta)^{a+\gamma+1} f_j \otimes \mathfrak{f} = \sum_{i=0}^a \sum_{k=0}^\gamma \sum_{l=1}^r c_{ijkl}(T - \theta)^{i+\gamma-k} \tau^k \cdot (f_l \otimes \mathfrak{f}) \quad (10.7.0.2)\]

This proves that for all \(j\) the element \((T - \theta)^{a+\gamma+1} f_j \otimes \mathfrak{f}\) is a linear combination of

\[(T - \theta)^i f_l \otimes \mathfrak{f}, \; i = 0, \ldots, a + \gamma, \; l = 1, \ldots, r \quad (10.7.0.3)\]

in \((M \otimes \mathfrak{C})_{\mathbb{C}_\infty[r]}\). Multiplying (10.7.0.2) by consecutive powers of \(T - \theta\) we get by induction that elements of 10.7.0.3 generate \((M \otimes \mathfrak{C})_{\mathbb{C}_\infty[r]}\). \(\square\)

**Lemma 10.7.1.** If \(M_1 \otimes \mathfrak{C}\) is isomorphic to \(M_2 \otimes \mathfrak{C}\) then \(M_1\) is isomorphic to \(M_2\).

**Proof.** Let \(f_{i*} (i = 1, 2)\) be a \(\mathbb{C}_\infty[T]\)-basis of \((M_i)_{\mathbb{C}_\infty[T]}\), \(Q_i\) from 1.9.3, \(\alpha : M_1 \otimes \mathfrak{C} \to M_2 \otimes \mathfrak{C}\) an isomorphism and \(C \in GL_r(\mathbb{C}_\infty[T])\) the matrix of \(\alpha\) in \(f_{i*} \otimes \mathfrak{f}, f_{2*} \otimes \mathfrak{f}\). The matrix of the action of \(\tau\) on \(M_i \otimes \mathfrak{C}\) in the base \(f_{i*} \otimes \mathfrak{f}\) is \((T - \theta)Q_i\), and the condition that \(\alpha\) commutes with multiplication by \(\tau\) is

\[(T - \theta)Q_1 C = C^{(1)}(T - \theta)Q_2\]

Dividing this equality by \(T - \theta\) we get that the map \(\alpha_0\) from \(M_1\) to \(M_2\) having the same matrix \(C\) in the bases \(f_{i*}\), commutes with \(\tau\), i.e. defines an isomorphism from \(M_1\) to \(M_2\). \(\square\)

Using Lemma 10.7.1 we can state the following

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\(^3\)This notion was indicated me by Taguchi.

\(^4\)Anderson proved (not published) that the tensor product of any abelian T-motives is also an abelian T-motive.
**Definition.** A virtual abelian T-motive is an object $M \otimes \mathcal{C}^\otimes \mu$ where $M$ is an abelian T-motive and $\mu \in \mathbb{Z}$, with the standard equivalence relation (here $\mu_1 \geq \mu_2$):

$$M_1 \otimes \mathcal{C}^\otimes \mu_1 = M_2 \otimes \mathcal{C}^\otimes \mu_2 \iff M_2 = M_1 \otimes \mathcal{C}^\otimes (\mu_1 - \mu_2)$$

$$\iff \exists \mu \text{ such that } \mu + \mu_1 \geq 0, \mu + \mu_2 \geq 0 \text{ and } M_1 \otimes \mathcal{C}^\otimes (\mu + \mu_1) = M_2 \otimes \mathcal{C}^\otimes (\mu + \mu_2)$$

Lemma 10.7.1 shows that these conditions are really equivalent.

**Corollary 10.7.2.** The $\mu$-dual of a virtual abelian T-motive is well-defined and always exists as a virtual abelian T-motive. $\square$

**Proposition 10.8.** The following formula is valid at the level of pré-T-motives: for any $\mu_1, \mu_2$, if $M_1^{\mu_1}, M_2^{\mu_2}$ exist then $(M_1 \otimes M_2)^{\tau(\mu_1 + \mu_2)}$ exists and

$$(M_1 \otimes M_2)^{\tau(\mu_1 + \mu_2)} = M_1^{\mu_1} \otimes M_2^{\mu_2}$$

**Proof.** This is a functorial equality; also we can check it by means of elementary matrix calculations. $\square$

**Proposition 10.9.** Let $P \in \mathbb{A}$ be an irreducible element. The Tate module $T_P(M^{\mu})$ is equal to

$$T_P(\mathcal{C})^\otimes \mu \otimes T_P(M)$$

(equality of Galois modules) where $T_P(M)$ is the dual Galois module.

**Proof.** It is completely analogous to the proof of the corresponding theorem for tensor products ([G], Proposition 5.7.3, p. 157). All modules in the below proof will be the Galois modules, and equalities of modules will be equalities of Galois modules. Recall that $E = E(M)$. Since $T_P(M) = \text{invlim}_n E_{P^n}$, it is sufficient to prove that for any $a \in \mathbb{A}$ we have $E(M^{\mu})_a = E(\mathcal{C}^\otimes \mu)_a \otimes \hat{E}_a$, where $\hat{E}_a$ is the dual of $E_a$ in the meaning of $[T]$, Definition 4.1. We have the following sequence of equalities of modules:

$$M^{\mu}/aM^{\mu} = \text{Hom}_{\mathcal{C}_\infty[T]}(M/aM, \mathcal{C}^\otimes \mu/a\mathcal{C}^\otimes \mu) \quad (10.9.2)$$

such that the action of $\tau$ on both sides of this equality coincide (to define the action of $\tau$ on the right and side of (10.9.2) we need the action of $\tau^{-1}$ on $M/aM$; it is well-defined, because the determinant of the action of $\tau$ on $M$ is a power of $T - \theta$, hence its image in $\mathcal{C}_\infty[T]/a\mathcal{C}_\infty[T]$ is invertible). $10.9.2$ follows immediately from the definition of $M^{\mu}$:

$$(M^{\mu}/aM^{\mu})^\tau = \text{Hom}_{\mathcal{C}_\infty[T]}((M/aM)^\tau, (\mathcal{C}^\otimes \mu/a\mathcal{C}^\otimes \mu)^\tau) \quad (10.9.3)$$

This follows from $10.9.2$ and the Lang’s theorem

$$\mathfrak{M}/a\mathfrak{M} = (\mathfrak{M}/a\mathfrak{M})^\tau \otimes_{\mathbb{F}_q[T]/a\mathbb{F}_q[T]} \mathcal{C}_\infty[T]/a\mathcal{C}_\infty[T]$$

applied to both $\mathfrak{M} = M, \mathfrak{M} = M^{\mu}$ (we use that both $M, M^{\mu}$ are free $\mathcal{C}_\infty[T]$-modules). Finally, we have a formula

$$E(\mathfrak{M})_a = \text{Hom}_{\mathbb{F}_q}((\mathfrak{M}/a\mathfrak{M})^\tau, \mathbb{F}_q)$$
11. An explicit formula.

Here we restrict ourselves by the case \( A_0 = \theta E_n \iff N = 0 \iff m = 1 \). Let \( e_*, A, A_i, l, n \) be from (1.9). We consider in the present section two simple types of abelian T-motives (called standard-1 and standard-2 abelian T-motives respectively) whose \( A_i \) have a row echelon form, and we give an explicit formula for the dual of some standard-1 abelian T-motives. Analogous formula can be easily obtained for more general types of abelian T-motives. These results are the first step of the problem of description of all abelian T-motives having duals.

11.1. For the reader’s convenience, we give here the definition of standard-1 abelian T-motives for the case \( n = 2 \) (here \( \lambda_1 \) and \( \lambda_2 \) satisfying \( \lambda_1 = l, \ l > \lambda_2 \geq 2 \) are parameters):

\[
A_0 = \theta E_2, \text{ for } 0 < i < \lambda_2 \ A_i \text{ is arbitrary,}
\]

\[
A_{\lambda_2} = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix}, \text{ for } \lambda_2 < i < l \ A_i = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}, \ A_l = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

11.2. To define standard-2 abelian T-motives of dimension \( n \), we need to fix

1. A permutation \( \varphi \in S_n \), i.e. a \( 1 \rightarrow 1 \) map \( \varphi : (1, ..., n) \rightarrow (1, ..., n) \);
2. A function \( k : (1, ..., n) \rightarrow \mathbb{Z}^+ \) where \( \mathbb{Z}^+ \) is the set of integers \( \geq 1 \).

**Definition.** A standard-2 abelian T-motive of the type \((\varphi, k)\) is an abelian T-motive of dimension \( n \) given by the formulas \((i = 1, ..., n)\):

\[
Te_{\varphi(i)} = \theta e_{\varphi(i)} + \sum_{\alpha=1}^{n} \sum_{j=1}^{k(\alpha)-1} a_{j,\varphi(i),\alpha} \tau^j e_\alpha + \tau^{k(i)} e_i \quad (11.2.1)
\]

where \( a_{j,\varphi(i),\alpha} \in \mathbb{C}_\infty \) is the \((\varphi(i), \alpha)\)-th entry of the matrix \( A_j \).

**Proposition 11.2.2.** Formula 11.2.1 really defines an abelian T-motive denoted by \( M = M(\varphi, k) = M(\varphi, k, a_{**}) \). Its rank is \( \sum_{\alpha=1}^{n} k(\alpha) \) and elements \( X_{\alpha j} := \tau^j e_\alpha \), \( \alpha = 1, ..., n \), \( j = 0, ..., k(\alpha) - 1 \), form its \( \mathbb{C}_\infty[T] \)-basis. \( \square \)

The group \( S_n \) acts on the set of types \((\varphi, k)\) and on the set of the above \( M \); clearly for any \( \psi \in S_n \) we have \( \psi(M) \) is isomorphic to \( M \). Particularly, we can consider only \( \varphi \) of the following form of the product of the \( i \) cycles \((\alpha_0 = 0, \ \alpha_i = n)\):

\[
\varphi = (\alpha_0 + 1, ..., \alpha_1)(\alpha_1 + 1, ..., \alpha_2)...(\alpha_{i-1} + 1, ..., \alpha_i)
\]

(standard notation of the theory of permutations, for \( \gamma \neq \alpha_j \) we have \( \varphi(\gamma) = \gamma + 1 \), for \( \gamma = \alpha_j \) we have \( \varphi(\alpha_j) = \alpha_{j-1} + 1 \)).

**Example 11.2.4.** Let \( \varphi \) be defined by 11.2.3, the quantity of cycles \( i \) is equal to 1 and all \( a_{**} = 0 \). Then the corresponding \( M \) is of complete multiplication by a CM-field \( \mathbb{F}_q \) \((T)\) and its CM-type \( \Phi \) is \( \{\text{Id}, fr^{k(1)}, fr^{k(1)+k(2)}, ..., fr^{k(1)+k(2)+...+k(n-1)}\} \)

where \( fr \) is the Frobenius homomorphism \( \mathbb{F}_q \rightarrow \bar{\mathbb{F}}_q \) (see 13.3, first case: formulas}
13.3.1, 13.3.2 coinside with 11.2.1 for the given $\varphi$ and $a_{***} = 0$; $i_j$ of 13.3.0 is $k(1) + k(2) + \ldots + k(j - 1)$ of the present notations).

**Definition 11.3.** A standard-1 abelian T-motive is a standard-2 abelian T-motive whose $\varphi$ is the identical permutation $Id$.

**11.3.0.** Let $M = M(Id, k)$ be a standard-1 abelian T-motive. Acting by $\psi \in S_n$ we can consider only the case of non-increasing $k(j)$. We introduce a number $m \geq 1$ — the quantity of jumps of $k(j)$, and two sequences $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_m = n$ (sequence of arguments of points of jumps of the function $k$) and $0 = \lambda_{m+1} < \lambda_m < \ldots < \lambda_2 < \lambda_1 = l$ (sequence of values of $k$ on segments $[\gamma_{i-1} + 1, \ldots, \gamma_i]$) by the formulas

$$
k(1) = \ldots = k(\gamma_1) = \lambda_1$$
$$k(\gamma_1 + 1) = \ldots = k(\gamma_2) = \lambda_2$$
$$\ldots$$
$$k(\gamma_{m-1} + 1) = \ldots = k(\gamma_m) = \lambda_m$$

(11.3.1)

**Example 11.3.2.** The T-motive $M$ of 11.1 is a standard-1 having $m = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $\lambda_1$, $\lambda_2$ as in 11.1. Its rank $r = \lambda_1 + \lambda_2$.

**Conjecture 11.3.3.** A standard-2 abelian T-motive of the type $(\varphi, k)$ (notations of 11.2.3) is pure iff $\forall j = 1, \ldots, i$ we have:

$$\frac{\alpha_j - \alpha_{j-1}}{\sum_{\gamma = \alpha_{j-1} + 1}^{\alpha_j} k(\gamma)} = \frac{n}{r}$$

This conjecture is obviously true if all $a_{***}$ are 0.

To simplify exposition, we prove here only the following particular case of this conjecture.

**Proposition 11.3.4.** Let $M$ be a standard-1 abelian T-motive having $m > 1$, defined over $\mathbb{F}_q(\theta)$, having a good reduction at a point of degree 1 of $\mathbb{F}_q(\theta)$ (i.e. a point $\theta + c$, $c \in \mathbb{F}_q$). Then $M$ is not pure.

**Proof.** Let $M$ be defined by 11.2.1, we use notations of 11.3.1. We consider the action of Frobenius on $\tilde{M}$ — the reduction of $M$ at $\theta + c$. According [G], Theorem 5.6.10, it is sufficient to prove that orders of the roots of the characteristic polynomial of Frobenius over $A$ are not equal. More exactly, we consider the valuation infinity on $A$ (defined by the condition $\text{ord}(T) = -1$); the order corresponds to a continuation of this valuation to $\text{End}(\tilde{M})$. The action of Frobenius on $\tilde{M}$ coincides with multiplication by $\tau$, because the degree of the reduction point is 1.

A basis $f_*$ of $M_{C_\infty}[T]$ is the set of $X_{\alpha} := \tau^j e_\alpha$ of 11.2.2. The matrix $Q(M)$ is defined by the following formulas for the action of $\tau$ on $X_{\alpha}$:

$$\tau(X_{\alpha}) = X_{\alpha,j+1} \text{ if } j < k(\alpha) - 1$$

(11.3.4.1)
\[
\tau(X_{\alpha,k(\alpha)-1}) = TX_{\alpha,0} - \sum_{\delta=1}^{m} \sum_{d=\lambda_{\delta}+1}^{\infty} \sum_{c=1}^{\gamma_{\delta}} a_{d\alpha c}X_{cd}
\]  

(11.3.4.2)

This means that if we arrange \(X_{\alpha j}\) in lexicographic order \((X_{\alpha_{1}j_{1}}\) precedes \(X_{\alpha_{2}j_{2}}\) if \(\alpha_{1} < \alpha_{2}\)) then the matrix \(Q(M)\) has the block form:

\[
Q(M) = (C_{ij}) \quad (i, j = 1, ..., n)
\]

where \(C_{ij}\) is a \(k(i) \times k(j)\)-matrix of the form

\[
C_{ii} = \begin{pmatrix}
0 & 1 & 0 & ... & 0 \\
0 & 0 & 1 & ... & 0 \\
... & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & 1 \\
T - \theta & * & * & ... & *
\end{pmatrix}, \quad C_{ij} = \begin{pmatrix}
0 & 0 & ... & 0 \\
... & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & 1 \\
0 & * & ... & *
\end{pmatrix} (i \neq j)
\]

where asterisks mean elements \(a_{***}\) (in some order). Following N.N.Luzin, we shall call a lightning the set of entries of a matrix such that in each row and column of the matrix there is exactly one element of the lightning, and the value of a lightning is the product of its elements (i.e. the determinant is the alternating sum of the values of all lightnings). We consider the characteristic polynomial \(P(X) \in (C_{\infty}[T])[X]\) of \(Q(M)\). We have

\[
C_{ii} -XE_{k(i)} = \begin{pmatrix}
-X & 1 & 0 & ... & 0 \\
0 & -X & 1 & ... & 0 \\
... & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & 1 \\
T - \theta & * & * & ... & * - X
\end{pmatrix}
\]

**Lemma 11.3.4.3.** If a non-zero lightning of \(C_{ii} - XE_{k(i)}\) contains the term \(T - \theta\), then it does not contain any term containing \(X\). □

Let \(J\) be a subset of the set \(1, ..., n\) and \(J'\) its complement.

**Corollary 11.3.4.4.** If a non-zero lightning of \(Q(M) - XE_{r}\) contains terms \(T - \theta\) of blocks \(C_{jj}, j \in J\), then its value is a polynomial in \(X\) of degree \(\leq \sum_{j' \in J'} k(j')\), and there exists exactly one such lightning (called the principal \(J\)-lightning) whose value is a polynomial in \(X\) of degree \(\sum_{j' \in J'} k(j')\). □

Since the characteristic polynomial of Frobenius of \(\tilde{M}\) is \(\tilde{P}\) (respectively the valuation infinity of \(C_{\infty}[T]\)), it is sufficient to prove that the Newton polygon of \(P(X)\) is not reduced to the segment \(((0, -n); (r, 0))\) defined by its extreme terms \((T - \theta)^n\) and \(X^r\). To do it, it is sufficient to find a point on its Newton polygon which is below this segment. We consider \(J_{min}\) = the set of all \(\gamma_{m} - \gamma_{m-1}\) diagonal blocks \(C_{ii} (i = \gamma_{m-1} + 1, ..., \gamma_{m})\) of \(Q(M)\) of minimal size \(\lambda_{m}\). The value of the principal \(J_{min}\)-lightning is \((T - \theta)^{\gamma_{m} - \gamma_{m-1}}\) times polynomial in \(X\) of degree \(d := r - (\gamma_{m} - \gamma_{m-1})\lambda_{m}\). Corollary 11.3.4.4 implies that if the value of any other lightning of \(Q(M) - XE_{r}\) contains a term whose \(X\)-degree is equal to \(d\), then the \(T\)-degree of this term is strictly less than \(\gamma_{m} - \gamma_{m-1}\). This means that if we write \(P(X) = \sum_{i=0}^{r} C_{i}X^{i}, C_{i} \in C_{\infty}[T]\), then \(ord_{\infty}(C_{d}) = - (\gamma_{m} - \gamma_{m-1})\), i.e. the point with coordinates \([- (\gamma_{m} - \gamma_{m-1}), d]\) belongs to the Newton diagram of \(P(X)\), i.e. it is
above (really, at) the Newton polygon of \( P(X) \). This point is below the segment \(((0, -n); (r, 0))\). □

**Remark 11.3.4.5.** It is easy to see that the Newton polygon of \( P(X) \) coincides with the Newton polygon of the direct sum of trivial Drinfeld modules of ranks \( \lambda_* \), i.e. with the Newton polygon of the polynomial

\[
\prod_{i=1}^{m} (X^{\lambda_i} - T)^{\gamma_i - \gamma_{i-1}}
\]

**11.4.** To formulate the below theorem 11.5 we need some notations. Let \( M \) be a standard-1 abelian T-motive defined by formulas 11.2.1, 11.3.1. We impose the condition \( \lambda_m \geq 3 \). Theorem 11.5 affirms that it has dual. To find explicitly the dual of \( M \), we need to choose an arbitrary function \( \nu : (i, j) \rightarrow \nu(i, j) \) which is a 1-1 map from the set of pairs \((i, j)\) such that

\[
1 \leq i \leq n; \quad 1 \leq j \leq k(i) - 2
\]  

(11.4.1)
to the set \([n+1, \ldots, r-n]\) (recall that \( r = \sum_{i=1}^{n} k(i) \)).

Let the \((r-n) \times (r-n)\)-matrices \( B_1, B_2 \) be defined by the following formulas (here and until the end of the proof of 11.5 we have \( i, \alpha = 1, \ldots, n \); \( b_{\beta \gamma \delta} \) is the \((\gamma \delta)\)-th entry of \( B_{\beta} \), all entries of \( B_1, B_2 \) that are not in the below list are 0):

**11.4.2.** \( b_{1i\alpha} = -a_{k(i)-1, \alpha, i} \);

\( b_{1, \nu(i, j), \alpha} = -a_{j, \alpha, i} \) for \( 1 \leq j \leq k(i) - 2 \);

\( b_{1, \nu(i, j+1), \nu(i, j)} = 1 \) for \( 1 \leq j \leq k(i) - 3 \);

\( b_{1, i, \nu(i, k(i)-2)} = 1 \);

\( b_{2, \nu(i, 1), i} = 1 \).

We let \( B = \theta E_{r-n} + B_1 \tau + B_2 \tau^2 \) and consider a T-motive \( M(B) \) (see 11.5.1 below). Formulas 11.4.2 mean that \( M(B) \) is standard-2, its \( \varphi = \varphi_B \) is a product of \( n \) cycles

\[
i \xrightarrow{\varphi_B} \nu(i, 1) \xrightarrow{\varphi_B} \nu(i, 2) \xrightarrow{\varphi_B} \ldots \xrightarrow{\varphi_B} \nu(i, k(i) - 2) \xrightarrow{\varphi_B} i
\]

and its \( k = k_B \) is defined by the formulas \( k_B(\gamma) = 2 \) for \( \gamma \in [1, \ldots, n] \), \( k_B(\gamma) = 1 \) for \( \gamma \in [n+1, \ldots, r-n] \).

**Theorem 11.5.** Let \( M \) be from 11.4 (i.e. a standard-1 abelian T-motive having \( \lambda_m \geq 3 \)). Then \( M' = M(B) \).

**Proof.** Let \( e'_* = (e'_1, \ldots, e'_{r-n})^t \) be the vector column of elements of a basis of \( M(B) \) over \( \mathbb{C}_\infty[\tau] \) satisfying

\[
Te'_* = Be'_*
\]  

(11.5.1)  

Let us consider the set of pairs \((j, \ell)\) such that either \( j = 1, \ldots, n, \ \ell = 0,1 \) or \( j = n+1, \ldots, r-n, \ \ell = 0 \). For each pair \((j, \ell)\) of this set we let (as in [T], p. 580) \( Y_{j\ell} = \tau^\ell e'_j \).

Formulas (11.4.2) show that these \( Y_{*\*} \) form a basis of \( M(B)_{\mathbb{C}_\infty[\tau]} \).

---

5 This proof is a generalization of the corresponding proof of Taguchi; we keep his notations.
and the action of $\tau$ on this basis is given by the following formulas (here $j = 1, \ldots, k(i) - 2$):

$$\tau(Y_{i,0}) = Y_{i,1}$$

(11.5.2.1)

$$\tau(Y_{i,1}) = (T - \theta)Y_{\nu(i,1),0} + \sum_{\gamma=1}^{n} a_{1\gamma i} Y_{\gamma,1}$$

(11.5.2.2)

$$\tau(Y_{\nu(i,j),0}) = (T - \theta)Y_{\nu(i,j+1),0} + \sum_{\gamma=1}^{n} a_{j+1,\gamma,i} Y_{\gamma,1} \text{ if } j < k(i) - 2$$

(11.5.2.3)

$$\tau(Y_{\nu(i,k(i)-2),0}) = (T - \theta)Y_{i,0} + \sum_{\gamma=1}^{n} a_{k(i)-1,\gamma,i} Y_{\gamma,1}$$

(11.5.2.4)

Let $X'_{**}$ be the dual basis to the basis $X_{**}$ of 11.2.2.

11.5.3. Let us consider the following correspondence between $X'_{**}$ and $Y_{**}$:

- $X'_{ij}$ corresponds to $Y_{\nu(i,j),0}$ for the pair $(i, j)$ like in (11.4.1),
- $X'_{i0}$ corresponds to $Y_{i1}$ for $1 \leq i \leq n$;
- $X'_{i,k(i)-1}$ corresponds to $Y_{i0}$ for $1 \leq i \leq n$.

Therefore, in order to prove the Theorem 11.5 we must check that matrices defined by the dual to (11.3.4.\*) and by (11.5.2.\*) satisfy (1.10.1) under identification (11.5.3). This is an elementary exercise. \(\Box\)

Remark 11.6. Clearly it is possible to generalize the Theorem 11.5 to a larger class of abelian T-motives — some subclass of standard-3 T-motives, see Definition 11.8.1. The below example of the proof of Proposition 11.8.7 shows that probably the condition $\lambda_m \geq 3$ of the Theorem 11.5 can be changed by $\lambda_m \geq 2$: it is necessary to modify slightly formulas 11.4.2. From another side, a standard-1 abelian T-motive of the Example 2.5 shows that this condition cannot be changed to $\lambda_m \geq 1$.

11.7. An elementary transformation. To formulate the proposition 11.7.3, we change slightly notations in 1.9.1, namely, instead of $A = \sum_{i=0}^{l} A_i \tau^i$ we consider polynomials $P_k(M)$ of $x_1, \ldots, x_n$ ($k = 1, \ldots, n$) defined by the formula

$$P_k(M) = \sum_{i=0}^{l} \sum_{j=1}^{n} a_{i k j} x_j^q$$

(11.7.1)

Particularly, if $E$ is the T-module associated to $M$ (see [G], 5.4.5), $x_* = (x_1, \ldots, x_n)^t$ an element of $E$ then 11.7.1 is equivalent to $Tx_* = P_*(x_*)$ where $P_* = (P_1(M), \ldots, P_n(M))^t$ is the vector column. For a standard-1 abelian T-motive $M$ (we use notations of 11.3.0) having $m \geq 2$ we denote vector columns $\Psi_1(M) = (P_1(M), \ldots, P_{\gamma_1}(M))^t$, $\Psi_2(M) = (P_{\gamma_1+1}(M), \ldots, P_{\gamma_2}(M))^t$. We use similar notations for $M'$.

11.7.2. Let $M$ be as above, we consider the case $\lambda_2 = \lambda_1 - 1$. Let $C$ be a fixed $\gamma_1 \times (\gamma_2 - \gamma_1)$-matrix. We define a transformed T-motive $M_1$ by the formulas

$$\Psi_1(M_1) = \Psi_1(M) + C\Psi_2(M)^q$$
$P_i(M_1) = P_i(M)$ for $i > \gamma_1$

**Proposition 11.7.3.** For $M$, $C$, $M_1$ of 11.7.2 the dual $M'_1$ of $M_1$ is described by the following formulas:

$$\Psi_2(M'_1) = \Psi_2(M') - C^t \Psi_1(M')^q$$

$$P_i(M'_1) = P_i(M')$$ for $i \not\in [\gamma_1 + 1, ..., \gamma_2]$

**Proof** is similar to the proof of the Theorem 11.5, it is omitted. □

**11.8. Non-pure T-motives.** Most results of this subsection are conditional. We shall show that under some natural conjecture the condition of purity in 8.1.2 and 8.1.4 is essential, and that for non-pure T-motives the notion of algebraic duality is richer than the notion of analytic duality.

We generalize slightly the definition 11.2.1 as follows. Let $\succ$ be a linear ordering on the set $[1, ..., n]$, $\varphi$, $k$ as in 11.2.

**Definition 11.8.1.** A standard-3 abelian T-motive of the type $(\varphi, k, \succ)$ is an abelian T-motive of dimension $n$ given by the formulas

$$Te_{\varphi(i)} = \theta e_{\varphi(i)} + \sum_{j=1}^{n} \sum_{l=1}^{k(j)-1} a_{i,\varphi(i),j} \tau^l e_j + \sum_{j \succ i} a_{k(j),\varphi(i),j} \tau^{k(j)} e_j + \tau^{k(i)} e_i$$

(11.8.2)

where $a_{***} \in \mathbb{C}_\infty$ are coefficients (the only difference with 11.2.1 is the term $\sum_{j \succ i} a_{k(j),\varphi(i),j} \tau^{k(j)} e_j$).

Let $M_1$, $M_2$ be two isomorphic standard-3 abelian T-motives of the same type $(\varphi, k, \succ)$ with $\mathbb{C}_\infty[\tau]$-bases $e_{1*}$, $e_{2*}$ respectively (we use notations of 11.8.2 for both $M_1$, $M_2$). There exists $C \in M_n(\mathbb{C}_\infty[\tau])$ such that the formula defining an isomorphism between $M_1$ and $M_2$ is the following: $e_{2*} = Ce_{1*}$.

**Conjecture 11.8.3.** For generic $M_1$, $M_2$ we have $C \in M_n(\mathbb{C}_\infty)$.

This conjecture is based on calculations in some explicit cases: if we fix $M_1$ and $C \in M_n(\mathbb{C}_\infty[\tau])$, $C \not\in M_n(\mathbb{C}_\infty)$ then formula for the action of $\tau$ on $e_{2*}$ is not a formula of the type 11.8.2 but something much more complicated.

We denote by $\mathcal{M}_u(r, n)$ the moduli space of uniformizable T-motives of the rank $r$ and dimension $n$, by $\mathcal{L}(r, n)$ the moduli space of lattices of the rank $r$ and dimension $n$ and by $\mathcal{L} : \mathcal{M}_u(r, n) \rightarrow \mathcal{L}(r, n)$ the functor of lattice associated to an uniformizable T-motive.

**Proposition 11.8.5.** Conjecture 11.8.3 implies that the dimension of the fibers of $\mathcal{L}$ is $> 0$ for $r = 3$, $n = 2$. Particularly, we cannot omit condition of purity in the statement of 8.1.2.

**Proof.** We consider standard-3 abelian T-motives of the type $n = 2$, $\varphi = Id$, $k(1) = 2$, $k(2) = 1, 2 \succ 1$. Such $M_1 = M_1(a_{111}, a_{112}, a_{121})$ is given by

$$A_0 = \theta E_2, \quad A_1 = \begin{pmatrix} a_{111} & a_{112} \\ a_{121} & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

40
(notations of Example 10.5). It has \( r = 3 \), it is not pure, hence it has no dual. A direct calculation shows immediately that for this \( M_1 \) there exists only finitely many \( M_2 \) of the same type such that \( e_{2*} = C e_{1*} \) where \( C \in M_n(C_\infty) \). Conjecture 11.8.3 implies that the dimension of the moduli space of these T-motives is 3 (because there are 3 coefficients \( a_{111}, a_{112}, a_{121} \)). Uniformizable T-motives form an open subset of this moduli space, while the moduli space of lattices of \( n = 2 \) and \( r = 3 \) has dimension 2. \( \square \)

**Remark.** Similar calculations are valid for any sufficiently large \( r, n \).

Standard-3 abelian T-motives of the above type have not dual. The following proposition shows that the same phenomenon holds for abelian T-motives having dual. We denote by \( M_{u,d}(r, n) \) the moduli space of uniformizable T-motives of the rank \( r \) and dimension \( n \) having dual, by \( L_d(r, n) \) the moduli space of lattices of the rank \( r \) and dimension \( n \) having dual, by \( D_d : M_{u,d}(r, n) \rightarrow \mathcal{L}_d(r, n) \) the functor of lattice and by \( D_M : M_{u,d}(r, n) \rightarrow M_{u,d}(r, r - n) \), \( D_L : \mathcal{L}_d(r, n) \rightarrow \mathcal{L}_d(r, r - n) \) the functors of duality on T-motives and lattices respectively. Practically, Theorem 5 implies that the following diagram is commutative:

\[
\begin{array}{ccc}
M_{u,d}(r, n) & \xrightarrow{D_M} & M_{u,d}(r, r - n) \\
\downarrow \mathcal{L}_d & & \downarrow \mathcal{L}_d \\
\mathcal{L}_d(r, n) & \xrightarrow{D_L} & \mathcal{L}_d(r, r - n)
\end{array}
\]  

(11.8.6)

**Proposition 11.8.7.** Conjecture 11.8.3 implies that the dimension of the fibers of \( \mathcal{L}_d \) in the diagram (11.8.6) is \( > 0 \) for \( r = 5, n = 2 \).

Practically, this means that the notion of algebraic duality is "richer" than the notion of analytic duality.

**Proof.** We consider standard-3 abelian T-motives of the type \( n = 2, \varphi = Id, k(1) = 3, k(2) = 2, 2 > 1, r = 5 \). Such \( M \) is given by

\[
A_0 = \theta E_2, \quad A_1 = \begin{pmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{211} & a_{212} \\ a_{221} & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

(notations of Example 10.5). It has dual. Really, we denote by \( A_{i*j} \) the \( j \)-th column of \( A_i \), and we denote by \( (C_1|C_2) \) the matrix formed by union of columns \( C_1, C_2 \). Then \( M' = M(B) \) is also a standard-3 T-motive, where

\[
B_1 = \begin{pmatrix}
-\det A_2 & -a_{221} & 1 \\
-\det(A_{1*2}A_{2*2}) & -a_{122} & 0 \\
-\det(A_{1*1}A_{2*2}) & -a_{121} & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 0 & 0 \\
-a_{212} & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

The same arguments as in the proof of Proposition 11.8.5 show that the conjecture 11.8.3 implies that the dimension of the moduli space of these T-motives is 7, while the moduli space of lattices of \( n = 2 \) and \( r = 5 \) has dimension 6. \( \square \)

As above, similar calculations are valid for any sufficiently large \( r, n \); clearly the dimension of fibers of \( \mathcal{L}_d \) becomes larger as \( r, n \) grow.

Let us mention two open questions related to the functor \( \mathcal{L} \). Firstly, let \( L \) be a self-dual lattice such that \( L \in \mathcal{L}(M_{u,d}(2n, n)) \). This means that \( D_M : \mathcal{L}_d^{-1}(L) \rightarrow \mathcal{L}_d^{-1}(L) \) is defined.
Open question 11.8.8. What can we tell on this functor, for example, what is the dimension of its stable elements?

Secondly, let us consider $M_1, M_2$ of CM-type with CM-field $\mathbb{F}_{q^r}(T)$, see 13.3.

Open question 11.8.9. Let the CM-types $\Phi_1, \Phi_2$ of the above $M_1, M_2$ satisfy $\Phi_1 \neq \alpha\Phi_2$, where $\alpha \in \text{Gal}(\mathbb{F}_{q^r}(T)/\mathbb{F}_q(T))$. Are lattices $L(M_1), L(M_2)$ non-isomorphic?

Clearly the negative answer to this question implies the negative answer to the Question 8.1.2.

For any given $M_1, M_2$ the answer can be easily found by computer calculation. Really, let $M$ be one of $M_1, M_2, c_1, ..., c_r$ a basis of $\mathbb{F}_{q^r}/\mathbb{F}_q$ and $\alpha_1, ..., \alpha_n \subset \text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ the CM-type of $M$. We define matrices $\mathcal{M}, \mathcal{N}$ as follows: $(\mathcal{M})_{ij} = \alpha_j(c_i)$, $(i, j = 1, ..., n)$, $(\mathcal{N})_{ij} = \alpha_j(c_{n+i})$, $j = 1, ..., n, i = 1, ..., r - n$. The Siegel matrix $Z(M)$ is obviously $\mathcal{N}\mathcal{M}^{-1}$. So, we can find explicitly $Z(M_1)$, $Z(M_2)$ for both $M_1, M_2$. To check whether $Z(M_1), Z(M_2)$ are equivalent or not, it is sufficient to find a solution to 3.8.1 such that the entries of $A, B, C, D$ are in $M_{*, *}(\mathbb{F}_q)$ (this is obvious: the condition $\exists \gamma \in \text{GL}_r(\mathbb{F}_q[\theta])$ is equivalent to the condition $\exists \gamma \in \text{GL}_r(\mathbb{F}_q)$, because entries of $Z(M_1), Z(M_2)$ are in $\mathbb{F}_{q^r}$). The equation 3.8.1 is linear with respect to $A, B, C, D$, and we can check whether its solution satisfying $\det \gamma \neq 0$ exists or not.

For the case $q = 2, r = 4, n = 2$, CM-types of $M_1, M_2$ are $(\text{Id}, Fr), (\text{Id}, Fr^2)$ respectively, a calculation shows that the answer is positive: lattices $L(M_1), L(M_2)$ are not isomorphic.

12. Abelian T-motives having multiplications.

In this section we consider the case $A = \mathbb{F}_q[T]$. Let $\mathfrak{A}$ be a separable extension of $\mathbb{F}_q(T)$ such that $\mathfrak{A}_C := \mathfrak{A} \otimes \mathbb{C}_\infty$ is also a field, $\pi : X \to P^1(\mathbb{C}_\infty)$ the projection of curves over $\mathbb{C}_\infty$ corresponding to $\mathbb{C}_\infty(T) \subset \mathfrak{A}_C$. Let $\mathfrak{A}, X$ satisfy the condition: $\infty \in X$ is the only point on $X$ over $\infty \in P^1(\mathbb{C}_\infty)$. Let $A_{\mathfrak{A}}$ be the subring of $\mathfrak{A}$ consisting of elements regular outside of infinity. We denote $g = \dim A_{\mathfrak{A}}/\mathbb{F}_q(T)$ and $\alpha_1, ..., \alpha_g : \mathfrak{A} \to \mathbb{C}_\infty$ — inclusions over $i : \mathbb{F}_q(T) \to \mathbb{C}_\infty$ (recall that $i(T) = \theta$). Let $\mathcal{W}$ be a central simple algebra over $\mathfrak{A}$ of dimension $q^2$. Each $\alpha_i : \mathfrak{A} \to \mathbb{C}_\infty$ can be extended to a representation $\chi_i : \mathcal{W} \to M_q(\mathbb{C}_\infty)$.

12.1. Analytic CM-type. Let $(L, V)$ be as in 2 (recall that $A = \mathbb{F}_q[T]$) such that there exists an inclusion $i : \mathcal{W} \to \text{End}^0(L, V)$, where $\text{End}^0(L, V) = \text{End}(L, V) \otimes \mathbb{F}_q(T)$. It defines a representation of $\mathcal{W}$ on $V$ denoted by $\Psi$ which is isomorphic to $\sum_{i=1}^g \alpha_i \chi_i$ where $\{\alpha_i\}$ are some multiplicities (the CM-type of the action of $\mathcal{W}$ on $(L, V)$). [Proof: restriction of $\Psi$ on $\mathfrak{A}$ is a sum of one-dimensional representations, i.e. $V = \bigoplus_{i=1}^g V_i$ where $k \in \mathfrak{A}$ acts on $V_i$ by multiplication by $\alpha_i(k)$. Spaces $V_i$ are $\Psi$-invariant. We consider an isomorphism $\mathcal{W} \otimes \mathfrak{A} \mathbb{C}_\infty = M_q(\mathbb{C}_\infty)$ where the inclusion of $\mathfrak{A}$ in $\mathbb{C}_\infty$ is $\alpha_i$. We extend $\Psi|_{V_i}$ to $\mathcal{W} \otimes \mathfrak{A} \mathbb{C}_\infty$ by $\mathbb{C}_\infty$-linearity using the inclusion $\alpha_i$ of $\mathfrak{A}$ in $\mathbb{C}_\infty$. It remains to show that a representation of $M_q(\mathbb{C}_\infty)$ is a direct sum of its $q$-dimensional standard representations. We consider the corresponding representation of Lie algebra $\mathfrak{sl}_q(\mathbb{C}_\infty)$. It is a sum of irreducible representations. Let $\omega$ be the highest weight of any of these irreducible representations.]
ω is extended uniquely to the set of diagonal matrices of $M_q(\mathbb{C}_\infty)$, because ω is identical on scalars. Since our representation is not only of Lie algebra but of algebra $M_q(\mathbb{C}_\infty)$, we get that ω is a ring homomorphism \( \text{Diag} (M_q(\mathbb{C}_\infty)) \rightarrow \mathbb{C}_\infty \). There exists the only such ω corresponding to the $q$-dimensional standard representation\].

Further, we denote $m = \dim_W L \otimes \mathbb{F}_q(T)$ (g, q, Ψ, $r_i$, m are analogs of g, q, Φ, $r_i$, m of [Sh63] respectively). Clearly we have

$$n = q \sum_{i=1}^{g} r_i, \quad r = mgq^2 \quad (12.2)$$

By functoriality, we have the dual inclusion $i' : \mathcal{W}^{op} \rightarrow \text{End}^0 (L', V')$ where $\mathcal{W}^{op}$ is the opposite algebra.

**Remark.** A construction of Hilbert-Blumenthal modules ([A], 4.3, p. 498) practically is a particular case of the present construction: for Hilbert-Blumenthal modules we have $q = 1$, i.e. $\mathfrak{R} = \mathcal{W}$, and all $r_i = 1$. Anderson considers the case when $\infty$ splits completely; this difference with the present case is not essential.

**Proposition 12.3.** If the dual pair $(L', V')$ exists then the CM-type of the dual inclusion is \( \{mq-r_i\}, i = 1, \ldots, g \).

**Proof.** We have $L \otimes \mathbb{C}_\infty$ is isomorphic to $(\mathcal{W} \otimes \mathbb{C}_\infty)^m$ as a $\mathcal{W}$-module. Since the natural representation of $\mathcal{W}$ on $\mathcal{W} \otimes \mathbb{C}_\infty$ is isomorphic to $q \sum_{i=1}^{g} \chi_i$ we get that $L \otimes \mathbb{C}_\infty$ is isomorphic to $mq \sum_{i=1}^{g} \chi_i$ as a $\mathcal{W}$-module. Consideration of the exact sequence $0 \rightarrow V'^* \rightarrow L \otimes \mathbb{C}_\infty \rightarrow V \rightarrow 0$ gives us the desired. □

**Remark 12.4.1.** This result is an analog of the corresponding theorem in the number field case. We use notations of [Sh63], Section 2. Let $A$ be an abelian variety having endomorphism algebra of type IV, and $(r_\nu, s_\nu) = (r_\nu(A), s_\nu(A))$ are from [Sh63], Section 2, (8). Then

$$r_\nu(A') = mq - r_\nu(A) = s_\nu(A), \quad s_\nu(A') = mq - s_\nu(A) = r_\nu(A)$$

By the way, Shimura writes that the CM-types of $A$ and $A'$ coincide ([Sh98], 6.3, second line below (5), case $A$ of CM-type). We see that his affirmation is not natural: he considers the complex conjugate action of the endomorphism ring on $A'$. It is necessary to take into consideration this difference of notations comparing formulas of 12.3 and 13.2 with the corresponding formulas of Shimura.

**Remark 12.4.2.** According [L1], an abelian Anderson T-motive $M$ is an analog of an abelian variety $A$ with multiplication by an imaginary quadratic field $K$. The above consideration shows that this analogy holds for $M$ and $A$ having more multiplications. Really, if $A$ has more multiplications then (we use notations of [Sh63], Section 2) $F_0 = FK$, and numbers $(r_\nu(A), s_\nu(A))$ satisfy $n(A) = q \sum_{i=1}^{g} r_\nu(A)$, where $(n(A), \dim(A) - n(A))$ is the signature of $A$ treated as an abelian variety with multiplication by $K$. This is an analog of 12.2.

**12.5. Complete multiplication.** Here we consider the case $q = m = 1$, i.e. $\mathfrak{R} = \mathcal{W}$ and $g = r$. 

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Lemma 12.5.1. In this case the condition \( N = 0 \) implies that the CM-type
\[
\sum_{i=1}^{r} r_i \alpha_i \quad (12.5.2)
\]
of the action of \( \mathcal{R} \) on on \((L, V)\) has the property: all \( r_i \) are 0 or 1.

**Proof.** \( N = 0 \) means that the action of \( T \in \mathbb{A} \) on \( V \) is simply multiplication by \( \theta \). We write the CM-type \( \sum_{i=1}^{r} r_i \chi_i \) in the form \( \sum_{i=1}^{n} \chi_{\alpha_i} \) where \( \alpha_1, \ldots, \alpha_n \in [1, \ldots, r] \) are not necessarily distinct. Let \( l_1 \) be an (only) element of a basis of \( L \otimes \mathbb{A}_{\mathcal{R}} \) over \( \mathcal{R} \) and \( e_1, \ldots, e_n \) a basis of \( V \) over \( \mathbb{C}_\infty \) such that the action of \( \mathcal{R} \) on \( V \) is given by the formulas
\[
k(e_i) = \chi_{\alpha_i}(k)e_i, \quad k \in \mathcal{R}
\]
Multiplying \( e_i \) by scalars if necessary, we can assume that \( l_1 = \sum e_i \). Therefore, if \( \alpha_i = \alpha_j \) (i.e. not all \( r_i \) in (12.5.2) are 0, 1) then the \( e_{\alpha_i} \)-th coordinate of any element of \( L \) coincide with its \( e_{\alpha_j} \)-th coordinate, hence \( L \) does not \( \mathbb{C}_\infty \)-generate \( V \) — a contradiction. \( \square \)

12.5.3. Let \( M \) be an abelian \( T \)-motive of rank \( r \) and dimension \( n \) having multiplication by \( \mathbb{A}_{\mathcal{R}} \). Recall that we consider only the case \( N = 0 \). This means that the character of the action of \( \mathcal{R} \) on \( M/\tau M \) is isomorphic to \( \sum_{i=1}^{r} r_i \alpha_i \). Since \( E(M) = (M/\tau M)^* \) we get that the character of the action of \( \mathcal{R} \) on \( E(M) \) is the same. If
\[
\text{all } r_i \text{ are 0 or 1} \quad (12.5.4)
\]
we shall use the terminology that \( M \) has the CM-type \( \Phi \subset \{ \alpha_1, \ldots, \alpha_r \} \) where \( \Phi \) is defined by the condition \( \alpha_i \in \Phi \iff r_i = 1 \).

It is easy to see that this case occurs for uniformizable \( M \). Really, if \( M \) is uniformizable then the action of \( \mathcal{R} \) can be prolonged on \((L(M), V(M))\), and the character of the action of \( \mathcal{R} \) on \( V(M) \) coincides with the one on \( E(M) \). The result follows from Lemma 12.5.1.

**Lemma 12.5.5.** There exists a canonical isomorphism \( \gamma \) from the set of inclusions \( \alpha_1, \ldots, \alpha_r \) to the set of points \( \theta_{\alpha_1}, \ldots, \theta_{\alpha_r} \) of \( X \) over \( \theta \in P^1(\mathbb{C}_\infty) \).

**Proof.** A point \( t \in X \) over \( \theta \in P^1(\mathbb{C}_\infty) \) defines a function \( \varphi_t : \mathcal{R}_C \to P^1(\mathbb{C}_\infty) \) — the value of an element \( f \in \mathcal{R}_C \) treated as a function on \( X \) at the point \( t \). This function must satisfy the standard axioms of valuation and the condition \( \varphi_t(T) = \theta \). Let \( \alpha_i \) be an inclusion of \( \mathcal{R} \) to \( \mathbb{C}_\infty \) over \( t \). It defines a valuation \( \varphi_{\alpha_i} : \mathcal{R}_C \to P^1(\mathbb{C}_\infty) \) by the formula \( \varphi_{\alpha_i}(k \otimes f) = \alpha_i(k)f(\theta) \), where \( k \in \mathcal{R} \), \( f \in \mathbb{C}_\infty(T) \). We define \( \gamma(\alpha_i) \) by the condition \( \varphi_\gamma(\alpha_i) = \varphi_{\alpha_i} \); it is easy to see that \( \gamma \) is an isomorphism. \( \square \)

**Theorem 12.6.** For any above \( \{ \mathcal{R}, \Phi \} \) there exists an abelian \( T \)-motive \((M, \tau)\) with complete multiplication by \( \mathcal{R} \) having CM-type \( \Phi \).

**Proof (Drinfeld).** We denote the divisor \( \sum_{\alpha_i \in \Phi} \gamma(\alpha_i) \) by \( \theta_{\Phi} \). We construct a \( \mathcal{F} \)-sheaf \( F \) of dimension 1 over \( \mathcal{R} \) which will give us \( M \). Let \( fr \) be the Frobenius map on \( \text{Pic}_0(X) \). It is an algebraic map, and the \( fr - \text{Id} : \text{Pic}_0(X) \to \text{Pic}_0(X) \) is an algebraic map as well. Since the action of \( fr \) on the tangent space of \( \text{Pic}_0(X) \) at 0 is the zero map, the action of \( fr - \text{Id} \) on the tangent space of \( \text{Pic}_0(X) \) at 0 is the minus identical map and hence \( fr - \text{Id} \) is an isogeny of \( \text{Pic}_0(X) \). Particularly,
there exists a divisor $D$ of degree 0 on $X$ such that we have the following equality in $\text{Pic}_0(X)$:

$$fr(D) - D = -\theta_\Phi + n\infty \quad (12.6.0)$$

This means that if we let $F = F_\Phi = O(D)$ then there exists a rational map $\tau_X = \tau_{X,\Phi} : F^{(1)} \to F$ such that

$$\text{Div}(\tau_X) = \theta_\Phi - n\infty \quad (12.6.1)$$

The pair $(F_\Phi, \tau_{X,\Phi})$ is the desired $\mathcal{F}$-sheaf.

**Remark.** It is easy to see that if the genus of $X$ is $> 0$ then different CM-types $\Phi_1$, $\Phi_2$ give us different sheaves $F_{\Phi_1}$, $F_{\Phi_2}$, while if the genus of $X$ is 0 then $F_{\Phi_1} = F_{\Phi_2} = \mathcal{O}$, but the maps $\tau_{X,\Phi_1}$, $\tau_{X,\Phi_2}$ are clearly different.

Let $U_0 = X - \{\infty\}$ be an open part of $X$. We denote $F(U_0)$ by $\mathcal{M}$, hence $F^{(1)}(U_0) = \mathcal{M}^{(1)}$. Since the support of the negative part of the right hand side of 12.6.1 is $\{\infty\}$, we get that the (a priori rational) map $\tau_X(U_0) : \mathcal{M}^{(1)} \to \mathcal{M}$ is really a map of $\mathbb{A}_\mathbb{R}$-modules.

Let $M$ be a $\mathbb{C}_\infty[T]$-module obtained from $\mathcal{M}$ by restriction of scalars from $\mathbb{A}_\mathbb{R}$ to $\mathbb{C}_\infty[T]$. Construction $F \mapsto M$ is functorial, and we denote this functor by $\delta$.

Further, we denote by $\alpha$ the tautological isomorphism $\mathcal{M} \to M$. $M$ is a free $r$-dimensional $\mathbb{C}_\infty[T]$-module, and (because $\mathcal{M}^{(1)}$ is isomorphic to $M$) the same restriction of scalars of $\tau_X(U_0)$ defines us a $\mathbb{C}_\infty[T]$-skew map from $M$ to $M$ denoted by $\tau$ (skew means that $\tau( zm) = z^d \tau(m)$, $z \in \mathbb{C}_\infty$). $\tau$ is defined by the formula $\tau(m) = \alpha \circ \tau_X((\alpha^{-1}(m))^{(1)})$.

It is easy to check that $(M, \tau)$ is the required abelian T-motive. Really, $M$ is a $\mathbb{A}_\mathbb{R}$-module, and $\tau$ commutes with this multiplication. The fact that the positive part of the right hand side of 12.6.1 is $\theta_\Phi$ means that 1.13.2 holds for $M$ and that the CM-type of the action of $\mathbb{A}_\mathbb{R}$ is $\Phi$.

**Remark 12.6.2.** It is easy to prove for this case that $M$ is a free $\mathbb{C}_\infty[\tau]$-module. Really, it is sufficient to prove (see [G], Lemma 12.4.10) that $M$ is finitely generated as a $\mathbb{C}_\infty[\tau]$-module. We choose $D$ such that $\infty \not\in \text{Supp}(D)$. There exists $P \in \mathbb{A}_\mathbb{R}^n$, such that $\tau_X(U_0) : \mathcal{M}^{(1)} \to \mathcal{M}$ is multiplication by $P$ (recall that both $\mathcal{M}^{(1)}$, $M$ are $\mathbb{A}_\mathbb{R}$-submodules of $\mathbb{R}$). 12.6.0 implies that $-\text{ord}_\infty(P) = n$. There exists a number $n_1$ such that

(a) $h^0(X, \mathcal{O}(D + n_1\infty)) > 0$;  \quad (b) for any $k \geq 0$ we have

$$h^0(X, \mathcal{O}(D + (n_1 + k)\infty)) = h^0(X, \mathcal{O}(D + n_1\infty)) + k \quad (12.6.3)$$

$$h^0(X, \mathcal{O}(D^{(1)} + (n_1 + k)\infty)) = h^0(X, \mathcal{O}(D^{(1)} + n_1\infty)) + k \quad (12.6.4)$$

It is sufficient to prove that if $g_1, \ldots, g_k$ are elements of a basis of $H^0(X, \mathcal{O}(D + (n_1 + n)\infty))$, then for any $Q \in \mathcal{M}$ the element $\alpha(Q) \in M$ is generated by $\alpha(g_1), \ldots, \alpha(g_k)$ over $\mathbb{C}_\infty[\tau]$. We prove it by induction by $n_2 := -\text{ord}_\infty(Q)$. If $n_2 \leq n_1 + n$ the result is trivial. If not then 12.6.3, 12.6.4 imply that the multiplication by $P$ defines an isomorphism

$$H^0(X, \mathcal{O}(D^{(1)} + (n_2 - n)\infty)) / H^0(X, \mathcal{O}(D^{(1)} + (n_2 - n - 1)\infty)) \to$$

$$\to H^0(X, \mathcal{O}(D + n_2\infty)) / H^0(X, \mathcal{O}(D + (n_2 - 1)\infty))$$
This means that $\exists Q_1 \in H^0(X, \mathcal{O}(D^{(1)} + (n_2 - n)\infty))$, $-\text{ord}_\infty(Q_1) = n_2 - n$ such that $-\text{ord}_\infty(Q - PQ_1) \leq n_2 - 1$. An element $Q_1^{(-1)} \in \mathcal{M}$ exists; since $\alpha(Q) = \tau(\alpha(Q^{(-1)}_1)) + \alpha(Q - PQ_1)$, the result follows by induction. □

If $\mathcal{R}$ and $\Phi$ are given then the construction of the Theorem 12.6 defines $F$ uniquely up to tensoring by $\mathcal{O}(D)$ where $D \in \text{Div}(X(\mathcal{R}))$. We denote the set of these $F$ by $F(\{\mathcal{R}, \Phi\})$, and we denote by $M(\{\mathcal{R}, \Phi\})$ the set $\delta(F(\{\mathcal{R}, \Phi\}))$. Further, we denote by $\Phi' = \{\alpha_1, \ldots, \alpha_r\} - \Phi$ the complementary CM-type.

**Theorem 12.7.** Let $M \in M(\{\mathcal{R}, \Phi\})$. Then $M'$ exists, and $M' \in M(\{\mathcal{R}, \Phi'\})$. More exactly, if $F \in F(\{\mathcal{R}, \Phi\})$ then $F^{-1} \otimes \mathcal{D}^{-1} \in F(\{\mathcal{R}, \Phi'\})$ where $\mathcal{D}$ is the different sheaf on $X$, and if $M = \delta(F)$ then $M' = \delta(F^{-1} \otimes \mathcal{D}^{-1})$.

**Proof.** Let $G$ be any invertible sheaf on $X$. We have a

**Lemma 12.7.0.** There exists the canonical isomorphism $\varphi_G : \pi_*(G^{-1} \otimes \mathcal{D}^{-1}) \to \text{Hom}_{\mathcal{P}^1}(\pi_*(G), \mathcal{O})$.

**Proof.** At the level of affine open sets $\varphi_G$ comes from the trace bilinear form of field extension $\mathcal{R}/\mathbb{F}_q(T)$. Concordance with glueing is obvious. □

We need the relative version of this lemma. Let $G_1, G_2$ be invertible sheaves on $X$, $\rho : G_1 \to G_2$ any rational map. Obviously there exists a rational map $\rho^{-1} : G_2^{-1} \to G_1^{-1}$. Recall that we denote by $\rho^{\text{inv}} : G_2 \to G_1$ the rational map which is inverse to $\rho$ respectively the composition. The map $\pi_*(\rho^{-1} \otimes \mathcal{D}^{-1}) : \pi_*(G_2^{-1} \otimes \mathcal{D}^{-1}) \to \pi_*(G_1^{-1} \otimes \mathcal{D}^{-1})$ is obviously defined. The map (denoted by $\beta(\rho)$) from $\text{Hom}_{\mathcal{P}^1}(\pi_*(G_1), \mathcal{O})$ to $\text{Hom}_{\mathcal{P}^1}(\pi_*(G_2), \mathcal{O})$ is defined as follows at the level of affine open sets: let $\gamma \in \text{Hom}_{\mathcal{P}^1}(\pi_*(G_1), \mathcal{O})(U)$ where $U$ is a sufficiently small affine subset of $\mathcal{P}^1$, such that we have a map $\gamma(U) : \pi_*(G_1)(U) \to \mathcal{O}(U)$. Then $(\beta(\gamma))(U)$ is the composition map $\gamma(U) \circ \pi_*(\rho^{\text{inv}})(U)$:

$$\pi_*(G_2)(U) \xrightarrow{\pi_*(\rho^{-1}(U))} \pi_*(G_1)(U) \xrightarrow{\gamma(U)} \mathcal{O}(U)$$

**Lemma 12.7.1.** The above maps form a commutative diagram:

$$\begin{array}{ccc}
\pi_*(G_2^{-1} \otimes \mathcal{D}^{-1}) & \xrightarrow{\pi_*(\rho^{-1}(\otimes \mathcal{D}^{-1}))} & \pi_*(G_1^{-1} \otimes \mathcal{D}^{-1}) \\
\varphi_{G_2} & & \varphi_{G_1} \\
\xrightarrow{\beta(\rho)} & & \\
\text{Hom}_{\mathcal{P}^1}(\pi_*(G_1), \mathcal{O}) & \xrightarrow{\beta(\rho)} & \text{Hom}_{\mathcal{P}^1}(\pi_*(G_2), \mathcal{O})
\end{array}$$

We apply this lemma to the case $\{\rho : G_1 \to G_2\} = \{\tau_{X, \Phi} : F^{(1)} \to F\}$. We have:

$$\text{Div}(\tau_{X, \Phi}^{-1} \otimes \mathcal{D}^{-1}) = -\text{Div}(\tau_{X, \Phi}) = -\theta_\Phi + n\infty$$

Further, we multiply $\tau_{X, \Phi}^{-1} \otimes \mathcal{D}^{-1}$ by $T - \theta$. We have:

$$\text{Div}((T - \theta)\tau_{X, \Phi}^{-1} \otimes \mathcal{D}^{-1}) = \text{Div}(T - \theta) + \text{Div}(\tau_{X, \Phi}^{-1} \otimes \mathcal{D}^{-1}) = \theta_\Phi' - (r - n)\infty$$

i.e. $(T - \theta)\tau_{X, \Phi}^{-1} \otimes \mathcal{D}$ is one of $\tau_{X, \Phi}$, i.e. $F^{-1} \otimes \mathcal{D}^{-1} \in F(\{\mathcal{R}, \Phi'\})$. Further, $(T - \theta)\beta(\tau_{X, \Phi})$ is the map which is used in the definition of duality of $M$. This means that the lemma 12.7.1 implies the theorem. □
Remark 12.8. There exists a simple proof of the second part of the Theorem 5 for uniformizable abelian T-motives $M$ with complete multiplication by $A_{\mathbb{R}} \subset \mathbb{R}$. Recall that this second part is the proof of 2.7 for $M$. Really, let us consider the diagram 2.5. The CM-types of action of $\mathbb{R}$ on Lie $(M)$ and on $E(M)$ coincide, and the CM-types of action of $\mathbb{R}$ on a vector space and on its dual space coincide. This means that the CM-type of $V^*$ is $\Phi$ and the CM-type of $V'$ is $\Phi'$. Further, $\gamma_D$ of 2.5 commutes with complete multiplication: this follows immediately for example from a description of $\gamma_D$ given in Remark 5.2.8. Really, all homomorphisms of 5.2.9 commute with complete multiplication. For example, this condition for $\delta$ of 1.11.1 is written as follows: if $k \in \mathbb{R}$, $m_k(M)$, resp. $m_k(M')$ is the map of complete multiplication by $k$ of $M$, resp. $M'$, then $(m_k(M) \otimes Id) \circ \delta = (Id \otimes m_k(M')) \circ \delta$ — see any textbook on linear algebra.

Finally, since $\Phi \cap \Phi' = \emptyset$ and the map $\varphi' \circ \gamma_D \circ \varphi^*$ commutes with complete multiplication, we get that it must be 0.

13. Miscellaneous.

Let now $(L, V)$ be from 12.1, case $q = m = 1$, i.e. $\mathbb{R} = \mathbb{W}$ and $r = g$, and let the ring of complete multiplication be the maximal order $A_{\mathbb{R}}$. We identify $A$ and $\mathbb{F}_q[\theta]$ via $\iota$, i.e. we consider $\mathbb{R}$ as an extension of $\mathbb{F}_q[\theta]$. Let $\Phi$ be the CM-type of the action of $\mathbb{R}$ on $V$. This means that — as an $A_{\mathbb{R}}$-module — $L$ is isomorphic to $I$ where $I$ is an ideal of $A_{\mathbb{R}}$. The class of $I$ in $\text{Cl}(A_{\mathbb{R}})$ is defined by $L$ and $\Phi$ uniquely; we denote it by $\text{Cl}(L, \Phi)$.

Remark. $\text{Cl}(L, \Phi)$ depends on $\Phi$, because the action of $A_{\mathbb{R}}$ on $V$ depends on $\Phi$. Really, let $a \in L \subset V$, $a = (a_1, ..., a_n)$ its coordinates, $\Phi = \{\alpha_{i1}, ..., \alpha_{in}\} \subset \{\alpha_1, ..., \alpha_r\}$ and $k \in A_{\mathbb{R}}$. Then $ka$ has coordinates $(\alpha_{i1}(k)a_1, ..., \alpha_{in}(k)a_n)$, i.e. depends de $\Phi$. Particularly, the $A_{\mathbb{R}}$-module structure on $L$ depends on $\Phi$, and hence $\text{Cl}(L, \Phi)$ depends on $\Phi$. For example, if $n = 1$, $r = 2$, $\Phi_1 = \{\alpha_1\}$, $\Phi_2 = \{\alpha_2\}$, then $\text{Cl}(L, \Phi_2)$ is the conjugate of $\text{Cl}(L, \Phi_1)$.

Theorem 13.1. $\text{Cl}(L', \Phi') = (\text{Cl}(\Phi))^{-1}(\text{Cl}(L, \Phi))^{-1}$ where $\Phi$ is the different ideal of the ring extension $A_{\mathbb{R}}/\mathbb{F}_q[\theta]$.

Proof. This theorem follows from the above results; nevertheless, I give here an explicit elementary proof. Let $a^*_t = (a_1, ..., a_r)^t$ be a basis (considered as a vector column) of $\mathbb{R}$ over $\mathbb{F}_q[\theta]$ and $b^*_t = (b_1, ..., b_r)^t$ the dual basis. Recall that it satisfies 2 properties:

\begin{align*}
(1) \quad \forall i \neq j \quad \alpha_i(a^*_t)\alpha_j(b^*_t) = 0 \quad (\text{i.e.} \quad \sum_{k=1}^{r} \alpha_i(a_k)\alpha_j(b_k) = 0) \quad (13.1.1)
\end{align*}

(2) For $x \in \mathbb{R}$ let $m_{x,a^*}$ (resp $m_{x,b^*}$) be the matrix of multiplication by $x$ in the basis $a^*$ (resp. $b^*$). Then for all $x \in \mathbb{R}$ we have

\begin{align*}
m_{x,a^*} = m_{x,b^*}^t \quad (13.1.2)
\end{align*}

We define $e_{n,r-n}$ as an $r \times r$ block matrix $\begin{pmatrix} 0 & E_{r-n} \\ -E_n & 0 \end{pmatrix}$, and we define a new basis $b^*_t = (\bar{b}_1, ..., \bar{b}_r)^t$ by

\begin{align*}
\bar{b}^*_t = e_{n,r-n}b^*_t \quad (13.1.3)
\end{align*}
(explicit formula: \((\tilde{b}_1, \ldots, \tilde{b}_r) = (b_{n+1}, \ldots, b_r, -b_1, \ldots, -b_n)\)).

We can assume that \(\Phi = \{\alpha_1, \ldots, \alpha_n\}\). Since \(L\) has multiplication by \(A_{\mathbb{R}}\) and the CM-type of this multiplication is \(\Phi\), it is possible to choose \(a_*\) such that \(L \subset \mathbb{C}_\infty\) is generated over \(\mathbb{F}_q[\theta]\) by \(e_1, \ldots, e_r\) where
\[
e_i = (\alpha_1(a_i), \ldots, \alpha_n(a_i)) \quad (13.1.4)
\]

Let \(\hat{L} \subset \mathbb{C}_\infty^{-r}\) be generated over \(\mathbb{F}_q[\theta]\) by \(\hat{e}_1, \ldots, \hat{e}_r\) where
\[
\hat{e}_i = (\alpha_{n+1}(\tilde{b}_i), \ldots, \alpha_r(\tilde{b}_i)) \quad (13.1.5)
\]

**Lemma 13.1.6.** \(L' = \hat{L}\).

**Proof.** Let \(A\) (resp. \(B\)) be a matrix whose lines are the lines of coordinates of \(e_1, \ldots, e_n\) (resp. \(e_{n+1}, \ldots, e_r\)) in 13.1.4, and \(C\) (resp. \(D\)) a matrix whose lines are the lines of coordinates of \(\hat{e}_1, \ldots, \hat{e}_{r-n}\) (resp. \(\hat{e}_{r-n+1}, \ldots, \hat{e}_r\)) in 13.1.5. By definition of Siegel matrix, we have \(L = \mathcal{L}(BA^{-1}), \hat{L} = \mathcal{L}(DC^{-1})\). So, it is sufficient to prove that \((BA^{-1})^t = DC^{-1}\), i.e. \(A^tD = B^tC\). This follows immediately from the definition of \(A, B, C, D\) and (13.1.1). \(\square\)

For \(x \in A_{\mathbb{R}}\) we denote by \(\mathcal{M}_x(L)\) the matrix of multiplication by \(x\) in the basis \(e_*\) (see the notations of Remark 3.8). Obviously \(\mathcal{M}_x(L) = m_{x,a_*}\).

Let now \(A_{\mathbb{R}}\) acts on \(C_\infty^{-n}\) (the ambient space of \(L')\) by CM-type \(\Phi\). According (13.1.2) and (13.1.3), the matrix of the action of \(x \in A_{\mathbb{R}}\) in the basis \(\tilde{b}_*\) is
\[
\mathcal{E}_{n,r-n}m^t_{x,a_*} \mathcal{E}^{-1}_{n,r-n} \quad (13.1.7)
\]

Let \(\mathcal{M}, \mathcal{M'}\) be from Remark 3.8. Formula 3.8.4 shows that
\[
\mathcal{M}' = \mathcal{E}_{n,r-n} \mathcal{M'} \mathcal{E}^{-1}_{n,r-n} \quad (13.1.8)
\]

Formulas (13.1.7) and (13.1.8) — because of Lemma 1.10.3 — prove the theorem. \(\square\)

**13.2. Compatibility with the weak form of the main theorem of complete multiplication.**

The reader can think that Theorem 13.1 is incompatible with the main theorem of complex multiplication, because of the \(-1\)-th power in its statement. The reason is a bad choice of notations of Shimura, he affirms that the CM-type of an abelian variety \(A\) over a number field coincides with the CM-type of \(A'\), while we see that it is really the complement. Since even an analog of the weak form of the main theorem of complex multiplication — Theorem 13.2.6 — for the functional field case is not proved yet, the main result of the present section — Theorem 13.2.8 — is conditional: it affirms that if this weak form of the main theorem — Conjecture 13.2.7 — is true for an abelian T-motive with complete multiplication \(M\), then it is true for \(M'\) as well. By the way, even if it will turn out that the statement of the Conjecture 13.2.7 is not correct, the proof of 13.2.8 will not be affected, because the main ingredient of the proof is the formula 13.2.10 "neutralizing" the \(-1\)-th power of the Theorem 13.1.

Let us recall some definitions of [Sh71], Section 5.5. We consider an abelian variety \(A = \mathbb{C}^n/L\) with complex multiplication by \(K\). The set \(\text{Hom}(K, \overline{\mathbb{Q}})\) consists
of \(n\) pairs of mutually conjugate inclusions \(\{\varphi_1, \bar{\varphi}_1, \ldots, \varphi_n, \bar{\varphi}_n\}\). \(\Phi\) is a subset of the set \(\text{Hom}(K, \overline{\mathbb{Q}})\) such that \(\forall i = 1, \ldots, n\) we have:
\[
\Phi \cap \{\varphi_i, \bar{\varphi}_i\} \text{ consists of one element.} \tag{13.2.1}
\]

It is defined by the condition that the action of complex multiplication of \(K\) on \(\mathbb{C}^n\) is isomorphic to the direct sum of the elements of \(\Phi\). Let \(F\) be the Galois envelope of \(K/\mathbb{Q}\),
\[
G := \text{Gal } (F/\mathbb{Q}), \quad H := \text{Gal } (F/K), \quad S := \bigcup_{\alpha \in \Phi} H\alpha \tag{13.2.2}
\]

(the elements of Galois group act on \(x \in F\) from the right, i.e. by the formula \(x^{\alpha \beta} = (x^\alpha)\beta\); for \(\alpha \in \Phi\) we denote by \(\alpha\) also a representative in \(G\) of the coset \(\alpha\)).

We denote
\[
H^{\text{ref}} := \{\gamma \in G | S\gamma = S\} \tag{13.2.3}
\]
and let \(K^{\text{ref}}\) be the subfield of \(F\) corresponding to \(H^{\text{ref}}\). We have:
\[
H^{\text{ref}} S^{-1} = S^{-1} \tag{13.2.4}
\]
i.e. \(S^{-1}\) is an union of cosets of \(H^{\text{ref}}\) in \(G\). We can identify these cosets with elements of \(\text{Hom}(K^{\text{ref}}, \mathbb{Q})\). \(\Phi^{\text{ref}} \subset \text{Hom}(K^{\text{ref}}, \mathbb{Q})\) is, by definition, the set of these cosets. There is a map \(\det \Phi^{\text{ref}} : K^{\text{ref}} \times \rightarrow K^{\times}\) defined as follows:
\[
\det \Phi^{\text{ref}} (x) := \prod_{\alpha \in \Phi} \alpha(x) \tag{13.2.5}
\]

(it follows easily from the above formulas and definitions that \(\det \Phi^{\text{ref}} (x)\) really belongs to \(K^{\times}\)). It can be extended to the group of ideles and factorized to the group of classes of ideals, we denote this map by \(\det_{\text{Cl}} \Phi^{\text{ref}} : \text{Cl}(K^{\text{ref}}) \rightarrow \text{Cl}(K)\).

Finally, let \(\theta^{\text{ref}} : \text{Gal } (K^{\text{ref}} \text{ Hilb}/K^{\text{ref}}) \rightarrow \text{Cl}(K^{\text{ref}})\) be an isomorphism defined by the Artin reciprocity law.

We consider the case \(\text{End}(A) = O_K\). In this case \(L\) is isomorphic to an ideal of \(O_K\), its class is well-defined by the class of isomorphism of \(A\), we denote it by \(\text{Cl}(A)\).

**Theorem 13.2.6.** \(A\) is defined over \(K^{\text{ref}} \text{ Hilb}\); For any \(\gamma \in \text{Gal } (K^{\text{ref}} \text{ Hilb}/K^{\text{ref}})\) we have
\[
\text{Cl}(\gamma(A)) = \det_{\text{Cl}} \Phi^{\text{ref}} \circ \theta^{\text{ref}} (\gamma)^{-1}(\text{Cl}(A)). \quad \square
\]

This is a weak form of [SH71], Theorem 5.15 — the main theorem of complex multiplication.

Now we define analogous objects for the functional field case in order to formulate a conjectural analog of Theorem 13.2.6. Let \(\mathcal{R}, \Phi\) be from 12.5.3. \(\mathcal{R}^{\text{ref}}, \Phi^{\text{ref}}, \det \Phi^{\text{ref}}\) are defined by the same formulas 13.2.2 – 13.2.5 like in the number field case (\(\mathbb{Q}\) must be replaced by \(\mathbb{F}_q(T)\)). The facts that 13.2.1 has no meaning in the functional field case and that the order of \(S\) is not necessarily the half of the order of \(G\) do not affect the definitions.

The \(\infty\)-Hilbert class field of \(\mathcal{R}\) (denoted by \(\mathcal{R}^{\text{Hilb }\infty}\)) is an abelian extension of \(\mathcal{R}\) corresponding to the subgroup
\[
\mathcal{R}^{\infty} \cdot \prod_{v \neq \infty} O_v^* \cdot \mathcal{R}^*
\]
of the idele group of \( \mathfrak{K} \). We have an isomorphism \( \theta : \text{Gal} (\mathfrak{K}_{\text{Hilb}} \rightarrow \mathbb{K}) \rightarrow \text{Cl}(\mathcal{A}_{\mathfrak{K}}) \).

We formulate the functional field analog of Theorem 13.2.6 only for the case when

\[ (*) \text{ There exists only one point over } \infty \in P^1(\mathbb{F}_q) \text{ in the extension } \mathfrak{R}_{\text{ef}}/\mathbb{F}_q(T). \]

In this case the field \( \mathfrak{R}^\text{ref} \mathbb{H}_{\text{Hilb}} \rightarrow \mathbb{K} \) and the ring \( \mathcal{A}_{\mathfrak{R}_{\text{ef}}} \) are naturally defined, and we have an isomorphism \( \theta^\text{ref} : \text{Gal} (\mathfrak{R}^\text{ref} \mathbb{H}_{\text{Hilb}} \rightarrow \mathbb{K}_{\text{ef}}) \rightarrow \text{Cl}(\mathcal{A}_{\mathfrak{R}_{\text{ef}}}) \).

Let \( M \) be an uniformizable abelian \( T \)-motive of rank \( r \) and dimension \( n \) having complete multiplication by \( \mathcal{A}_{\mathfrak{R}} \), and \( \Phi \) its CM-type. \( \text{Cl}(M) \) is defined like \( \text{Cl}(A) \) in the number field case, it is \( \text{Cl}(L, \Phi) \) of 13.1.

**Conjecture 13.2.7.** If \( (*) \) holds, then \( M \) is defined over \( \mathfrak{R}^\text{ref} \mathbb{H}_{\text{Hilb}} \rightarrow \mathbb{K} \), and for any \( \gamma \in \text{Gal} (\mathfrak{R}^\text{ref} \mathbb{H}_{\text{Hilb}} \rightarrow \mathbb{K}_{\text{ef}}) \) we have \( \text{Cl}(\gamma(M)) = \text{det}_{\mathfrak{R}^\text{ref}} \Phi \circ \theta^\text{ref}(\gamma)^{-1} \text{Cl}(M) \).

Now we can formulate the main theorem of this section.

**Theorem 13.2.8.** If conjecture 13.2.7 is true for \( M \) then it is true for \( M' \).

**Proof.** It follows immediately from the functional analogs of 13.2.2 – 13.2.4 that

\[ (\mathfrak{R}, \Phi')^\text{ref} = (\mathfrak{R}^\text{ref}, (\Phi^\text{ref})') \]  

(13.2.9)

Further,

\[ \text{det}_{\mathfrak{R}^\text{ref}} \Phi^\text{ref} = (\text{det}_{\mathfrak{R}^\text{ref}} \Phi^\text{ref})^{-1} \]

(13.2.10)

Really, \( \text{det} \Phi^\text{ref}(x) \cdot \text{det}(\Phi^\text{ref})'(x) = N_{\mathfrak{R}_{\text{ef}}/\mathbb{F}_q(T)}(x) \in \mathbb{F}_q(T)^X \), hence gives the trivial class of ideals (we use here (13.2.9). Finally, for \( \gamma \in \text{Gal} (\mathfrak{R}_{\text{ef}}) \) we have

\[ (\gamma(M))' = \gamma(M') \]

(13.2.11)

The theorem follows immediately from 13.1, 13.2.10, 13.2.11 (recall that \( \text{Cl}(M) \) is \( \text{Cl}(L, \Phi) \) of 13.1).

13.3. Some explicit formulas. We give here an elementary explicit proof of the theorem 12.7 in two simple cases: \( \mathfrak{R} = \mathbb{F}_{q^r}(T) \) and \( \mathbb{F}_q(T^{1/r}) \). By the way, since the extension \( \mathbb{F}_{q^r}(T)/\mathbb{F}_q(T) \) is not absolutely irreducible, formally this case is not covered by the theorem 12.7.

Case \( \mathcal{A}_{\mathfrak{R}} = \mathbb{F}_{q^r}[T] \). Let \( \alpha_i \), where \( i = 0, ..., r - 1 \), be inclusions \( \mathfrak{R} \rightarrow \mathbb{C}_\infty \). For \( \omega \in \mathbb{F}_{q^r} \) we have \( \alpha_i(\omega) = \omega^{q^i} \). Let

\[ 0 \leq i_1 < i_2 < ... < i_n \leq r - 1 \]  

(13.3.0)

be numbers such that \( \Phi = \{ \alpha_{ij} \}, \ j = 1, ..., n \). We consider the following \( T \)-motive \( M = M(\mathfrak{R}, \Phi) \). Let \( e_1, ..., e_n \) be a basis of \( M_{\mathbb{C}_\infty[r]} \) such that \( m_\omega(e_j) = \omega^{q^i} e_j \) and the multiplication by \( T \) is defined by formulas

\[ Te_1 = \theta e_1 + \tau^{i_1-i_n+r} e_1 \]

(13.3.1)

\[ Te_j = \theta e_j + \tau^{i_j-i_{j-1}} e_{j-1}, \ \ j = 2, ..., n \]

(13.3.2)

It is easy to check that \( M \) has complete multiplication by \( \mathcal{A}_{\mathfrak{R}} \), and its CM-type is \( \Phi \).
Remark. It is possible to prove that $M(\mathfrak{A}, \Phi)$ is the only T-motive having these properties; we omit the proof.

**Proposition 13.3.3.** For $A_{\mathfrak{A}} = \mathbb{F}_q[T]$ we have: $M(\mathfrak{A}, \Phi)' = M(\mathfrak{A}, \Phi')$.

**Proof.** Elements $\tau^j e_k$ for $k = 1, \ldots, n$, $j = 0, \ldots, i_{k+1} - i_k - 1$ for $k < n$ and $j = 0, \ldots, i_1 - i_n + r - 1$ for $k = n$ form a basis of $M_{C_\infty}[T]$. Let us arrange these elements in the lexicographic order $(\tau^j e_k)$ precedes to $(\tau^{j_2} e_{k_2})$ if $k_1 < k_2$ and make a cyclic shift of them by $i_1$ denoting $e_1$ by $f_{i_1+1}$, $\tau^{i_1} e_1$ by $f_{i_2}$ etc. until $\tau^{i_1} e_1 + r - 1 e_n = f_{i_1}$. Formulas 13.3.1, 13.3.2 become

$$
\tau(f_i) = f_{i+1} \text{ if } i \notin \{i_1, \ldots, i_n\}
$$

$$
\tau(f_i) = (T - \theta)f_{i+1} \text{ if } i \in \{i_1, \ldots, i_n\}
$$

$(i \mod r$, i.e. $f_{r+1} = f_1$). Formula 1.10.1 shows that in the dual basis $f'_s$ we have

$$
\tau(f'_i) = f'_{i+1} \text{ if } i \notin \{i_1, \ldots, i_n\}
$$

$$
\tau(f'_i) = (T - \theta)f'_{i+1} \text{ if } i \notin \{i_1, \ldots, i_n\}
$$

which proves the proposition. □

**Case** $A_{\mathfrak{A}} = \mathbb{F}_q[T^{1/r}]$, $(r, q) = 1$. In order to define $M(\mathfrak{A}, \Phi)$ we need more notations. We denote $\theta^{1/r}$ and $T^{1/r}$ by $s$ and $S$ respectively, and let $\zeta_r$ be a primitive $r$-th root of 1. Let $\alpha_i, i_1 < i_2 < \ldots < i_n$ and $\Phi$ be the same as in the case $A_{\mathfrak{A}} = \mathbb{F}_q[T]$. We have $\alpha_i(S) = \zeta^i r S$. Further, we consider an overring $C_\infty[S, \tau]$ of $C_\infty[T, \tau]$ ($S$ is in the center of this ring), and we consider the category of modules over $C_\infty[S, \tau]$ such that the condition 1.9.2 is changed by a weakened condition 13.3.4 (here $A_{S,0} \in M_n(C_\infty)$ is defined by the formula $Se_\ast = A_{S}e_\ast$, where $A_S \in M_n(C_\infty)[T]$, $A_S = \sum_{i=0}^n A_{S,i} \tau^i$):

$$
A_{S,0}^r = \theta E_n + N \quad (13.3.4)
$$

Let $\tilde{M}$ be a $C_\infty[S, \tau]$-module such that $\dim \tilde{M}_{C_\infty}[S, \tau] = 1$, $f_1$ the only element of a basis of $M_{C_\infty}[S]$ and

$$
\tau f_1 = (S - \zeta^{i_1} r s) \cdots (S - \zeta^{i_n} r s) f_1
$$

By definition, $M = M(\mathfrak{A}, \Phi)$ is the restriction of scalars from $C_\infty[S, \tau]$ to $C_\infty[T, \tau]$ of $\tilde{M}$. Like in the case $A_{\mathfrak{A}} = \mathbb{F}_q[T]$, it is easy to check that $M$ has complete multiplication by $A_{\mathfrak{A}}$ with CM-type $\Phi$, and it is possible to prove that it is the only T-motive having these properties.

**Proposition 13.3.5.** For $A_{\mathfrak{A}} = \mathbb{F}_q[T^{1/r}]$, $(r, q) = 1$ we have: $M(\mathfrak{A}, \Phi)' = M(\mathfrak{A}, \Phi')$.

**Proof.** For $i = 1, \ldots, r$ we denote $f_i = S^{i-1} f_1$. These $f_* = f_*(\Phi)$ form a basis of $M_{C_\infty}[T]$, and the matrix $Q = Q(f_*, \Phi)$ of multiplication of $\tau$ in this basis has the following description. We denote by $\sigma_k(\Phi)$ the elementary symmetric polynomial $\sigma_k(\zeta^{i_1}, \ldots, \zeta^{i_n})$.

The first line of $Q$ is

$$
\sigma_n(\Phi) s^n \quad \sigma_{n-1}(\Phi) s^{n-1} \quad \ldots \quad \sigma_1(\Phi) s \quad 1 \quad 0 \quad \ldots \quad 0
$$
and its \(i\)-th line is obtained from the first line by 2 operations:

1. Cyclic shift of elements of the first line by \(i - 1\) positions to the right;
2. Multiplication of the first \(i + n - r\) elements of the obtained line by \(T\).

We consider another basis \(g_\ast = g_\ast(\Phi)\) of \(M_{\mathcal{C}_\infty[T]}\) obtained by inversion of order of \(f_1\), i.e. \(g_i = f_{r+1-i}\). The elements of \(Q(g_\ast)\) are obtained by reflection of positions of elements of \(Q(f_\ast)\) respectively the center of the matrix.

The theorem for the present case follows from the formula

\[
Q(f_\ast, \Phi)Q(g_\ast, \Phi')^t = (T - \theta)E_r
\]

whose proof is an elementary exercise: let \(\Phi' = \{j_1, ..., j_{r-n}\}\); we apply equality

\[
\sigma_k(x_1, ..., x_r) = \sum_l \sigma_l(x_{i_1}, ..., x_{i_l})\sigma_{k-l}(x_{j_1}, ..., x_{j_{r-n}})
\]

to \(1, \zeta_r, ..., \zeta_r^{r-1}\).

**13.4. Reduction.** Let \(K\) be a finite extension of \(\mathbb{F}_q(\theta)\), \(\mathfrak{p}\) a valuation of \(K\) over a valuation \(P\) of \(\mathbb{F}_q(\theta)\). We consider the case \(P \neq \infty\), i.e. \(P\) is an irreducible polynomial in \(\mathbb{F}_q[\theta]\). Let \(M\) be an abelian \(T\)-motive defined over \(K\) having dual such that both \(M, M'\) have good ordinary reduction at \(\mathfrak{p}\). We denote reductions by tilde. We shall identify \(A\) and \(\mathbb{F}_q[\theta]\) via \(\iota\). Let \(M_{P,0}\) be the kernel of the reduction map \(M_P \rightarrow \tilde{M}_P\).

**Conjecture 13.4.1.** For the above \(M, M'\) we have:

1. \(M_{P,0} = (A/P)^n\);
2. \(M_{P,0}\) and \(M'_{P,0}\) are mutually dual with respect to the pairing of Remarks 4.2, 5.1.2.

**Proof for a particular case:** \(M\) is a Drinfeld module, \(P = \theta\).

(1.9.1) for \(M\) has a form

\[
Te = \theta e + a_1 \tau e + ...a_{r-1} \tau^{r-1} e + \tau^r e
\]

Condition of good ordinary reduction means \(a_i \in \mathbb{F}_q(\theta)\), \(\text{ord}_\mathfrak{p}(a_1) \geq 0\), \(\text{ord}_\mathfrak{p} a_1 = 0\).

Let \(x \in M_T\), \(y \in M'_T\); we can consider \(x\) (resp. \(y\)) as an element of \(\mathbb{C}_\infty\) (resp. \(\mathbb{C}^{r-1}_\infty\)) satisfying some polynomial equation(s). Considering Newton polygon of these polynomials we get immediately (1) for both \(M, M'\). Let \(y = (y_1, ..., y_{r-1})\) be the coordinates of \(y\); explicit formula (5.3.5) for the present case has the form

\[
<y, y>_{M} = \Xi(xy^q y_{r-1} + x^q y_1 + x^q y_2 + ... + x^q y_{r-1})
\]

The same consideration of the Newton polygon of the above polynomials shows that for \(x \in M_{P,0}\), \(y \in M'_{P,0}\) we have \(\text{ord}_\mathfrak{p} x, \text{ord}_\mathfrak{p} y_i \geq 1/(q-1)\). Since \(\text{ord}_\mathfrak{p} \Xi = -1/(q-1)\) we get that \(\text{ord}_\mathfrak{p}(<x, y>_M) > 0\) and hence (because \(<x, y>_M \in \mathbb{F}_q\) we have \(<x, y>_M = 0\). Dimensions of \(M_{P,0}\), \(M'_{P,0}\) are complementary, hence they are mutually dual.

**Remark.** Analogous explicit proof exists for any standard-2 \(M\) of Section 11.1.
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