1. Introduction

A Rota—Baxter algebra is a linear space $A$ over a field $k$ equipped with bilinear product $(a, b) \mapsto ab$, $a, b \in A$, and with a linear map $R : A \to A$ such that

$$R(a)R(b) = R(R(a)b + R(aR(b)) + \lambda R(ab),$$

where $\lambda$ is a constant from $k$. A linear operator $R$ satisfying (1) is called a Rota—Baxter operator of weight $\lambda$.

This notion initially appeared in analysis [1], and then in combinatorics [12] and quantum field theory [4]. We refer the reader to the book [9] and references therein for more details. There is a number of studies on associative and commutative Rota—Baxter algebras. Let us mention those that are close to the topic of this paper.

A linear basis of the free associative Rota—Baxter algebra was found in [6], where it was also shown that the universal enveloping Rota—Baxter algebra of a free dendriform (or tridendriform, for nonzero weight) algebra is free. A simpler proof of the same fact follows from [7]. Another method for finding this basis was applied in [2], in a more modern form this approach was exposed in [8].

The class of Rota—Baxter Lie algebras is of special interest since it is closely related with pre-Lie (left/right-symmetric) algebras. Namely, if $L$ is a Lie algebra with a product $[,]$ equipped with a Rota—Baxter operator $R$ then the same space $L$ with new operation $ab = [R(a), b]$, $a, b \in L$, is a pre-Lie algebra.

Moreover, there is a natural relation between Rota—Baxter operators and solutions of the classical Yang—Baxter equation (CYBE) [13]. Namely, if $L$ is a Lie algebra equipped with an symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ (not necessarily non-degenerate) then there is a natural map $L \to L^*$, $a \mapsto \langle a, \cdot \rangle$, and thus we have a map $\Phi : L \otimes L \to \text{End}(L)$. If $X \in L \otimes L$ is a skew-symmetric solution of CYBE

$$[X^{12}, X^{13}] + [X^{12}, X^{23}] + [X^{13}, X^{23}] = 0,$$

then $R = \Phi(X)$ is a Rota—Baxter operator on $L$.

This paper is devoted to combinatorial structure of Lie algebras with a Rota—Baxter operator. The main problem we solve is an analogue of the PBW Theorem for universal enveloping Rota—Baxter Lie algebra $U_{RB}(L)$ of an arbitrary Lie algebra $L$. We prove that $U_{RB}(L)$ carries natural filtration such that the corresponding associated graded algebra $\text{gr} U_{RB}(L)$ is isomorphic (as a Lie algebra) to the universal enveloping one in the class $\text{RALie}$ of Lie algebras with linear operator $R$.

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satisfying the identity $[R(x), R(y)] = 0$. We also note that the same statement is true in the varieties As and Com of associative and commutative algebras.

The main tool of the proof is a version of the Composition-Diamond Lemma (CD-Lemma) for Lie algebras with an additional operator. A more general approach to this Lemma (for Lie algebras with an arbitrary set of additional operators) was developed in [11]. In the proof of CD-Lemma we use terminology of [8] and some combinatorial results of [14]. For more detailed exposition of the latter results, see [3].

2. Algebras with additional operator

Suppose Var is a variety of linear algebras. Denote by RVar the variety of Var-algebras equipped with an additional linear operator $R$. Denote the natural forgetful functor from RVar to Var by $\Theta_R$, and let $U_R$ stands for its left adjoint functor from Var to RVar.

Obviously, for the free Var-algebra $\text{Var}(V)$ generated by a linear space $V$ the universal RVar-envelope $U_R(\text{Var}(V))$ is isomorphic to the free RVar-algebra $R\text{Var}(V)$.

Let us state the explicit construction of $U_R(A)$. Given $A \in \text{Var}$, denote by $\bar{A}$ the copy of the linear space $A$. Let $\rho: A \to \bar{A}$ stands for the isomorphism $a \to \bar{a}$.

Construct a series of algebras $\{A_n\}_{n \geq 0}$ by the following rule:

$$A_0 = A,$$

$$A_1 = A \ast \text{Var}(\bar{A}),$$

$$\cdots$$

$$A_n = A \ast \text{Var}(\bar{A}_{n-1}),$$

$$\cdots$$

where $\ast = \ast_{\text{Var}}$ denotes the free product in the variety Var.

As above, let $\bar{A}_n$ denotes a copy of the space $A_n$; denote the linear isomorphism $a \to \bar{a}$, $a \in A_n$, by $\rho_n$.

Construct a series of Var-homomorphisms $\tau_n: A_n \to A_{n+1}$, $n \geq 0$, as follows. Set $\tau_0$ be the canonical embedding of $A$ into the free product $A_1$ and proceed by induction:

$$\begin{array}{ccc}
A_{n-1} & \xrightarrow{\tau_{n-1}} & A_n \\
\downarrow & & \downarrow \\
\bar{A}_{n-1} & \xrightarrow{\bar{\tau}_{n-1}} & \bar{A}_n \\
\downarrow \subseteq & & \downarrow \subseteq \\
\text{Var}(\bar{A}_{n-1}) & \xrightarrow{\tau_0} & \text{Var}(\bar{A}_n) \\
\downarrow \subseteq & & \downarrow \subseteq \\
A \ast \text{Var}(\bar{A}_{n-1}) & \xrightarrow{\tau_n} & A \ast \text{Var}(\bar{A}_n),
\end{array}$$

where $\bar{\tau}_{n-1} = \rho_{n-1}^{-1} \circ \tau_{n-1} \circ \rho_n$ is a linear homomorphism, $\tau_0$ is the induced Var-homomorphism of free algebras, and $\tau_n$ is a Var-homomorphism that comes from the definition of free product.

Lemma 1. All homomorphisms $\tau_n$ are injective.
Proof. Assume \( \tau_{n-1} \) is injective. Consider

\[
\begin{array}{cccc}
\bar{A}_{n-1} & \xrightarrow{\bar{\tau}_{n-1}} & A_n & \xrightarrow{\varphi} & \bar{A}_{n-1} \\
\subseteq & & \subseteq & & \subseteq \\
\text{Var}(\bar{A}_{n-1}) & \longrightarrow & \text{Var}(A_n) & \longrightarrow & \text{Var}(\bar{A}_{n-1})
\end{array}
\]

where \( \varphi \) is a projection of the linear space \( A_n \) onto \( \bar{A}_{n-1} \): \( \varphi \bar{\tau}_{n-1} = id_{\bar{A}_{n-1}} \) (every such linear map extends to a homomorphism of free algebras). Then the universal property of free product (uniqueness) implies the existence of

\[
A_n = A * \text{Var}(\bar{A}_{n-1}) \xrightarrow{\tau_n} A * \text{Var}(A_n) \xrightarrow{\psi} A * \text{Var}(\bar{A}_{n-1}) = A_n,
\]

where \( \psi \tau_n = id_{A_n} \), so \( \tau_n \) is also injective. \( \square \)

**Lemma 2.** For every RVar-algebra \( B \) and for every homomorphism of Var-algebras \( \psi : A \to B \) there exists unique family \( \{ \psi_n \}_{n \geq 0} \) of Var-homomorphisms \( \psi_n : A_n \to B \) such that

\[
\rho_n \circ \psi_{n+1} = \psi_n \circ R
\]

and

\[
\psi_0 = \psi, \quad \psi_n = \tau_n \circ \psi_{n+1}.
\]

**Proof.** Let us show existence and uniqueness by induction. Given \( \psi_n : A_n \to B \), construct

\[
\begin{array}{cccc}
A_n & \xrightarrow{\rho_n} & \bar{A}_n & \xrightarrow{\subseteq} & \text{Var}(\bar{A}_n) & \xrightarrow{\subseteq} & A_{n+1} = A * \text{Var}(\bar{A}_n) \\
\psi_n & & \bar{\psi}_n & & \psi_n^0 & & \psi_n^1
\end{array}
\]

Here the rightmost vertical arrow is the canonical embedding of \( A \) into the free product which coincides with \( \tau_0 \circ \cdots \circ \tau_n \). \( \bar{\psi}_n = \rho_n^{-1} \circ \psi_n \circ R \) is a linear map, \( \psi_n^0 \) is a homomorphism of Var-algebras induced by \( \bar{\psi}_n \). The right-hand square in the diagram above induces Var-homomorphism \( \psi_{n+1} : A * A_n^0 \to B \).

Why \( \psi_n = \psi_{n+1} \tau_n \)? For \( n = 0 \), it follows from the definition of \( \psi_1 \). Assume \( n > 0 \) and \( \psi_{n-1} = \psi_n \tau_{n-1} \). Then for all \( y \in \bar{A}_{n-1} \)

\[
\bar{x}_{n-1}(y) = \rho_n \tau_{n-1} \rho_n^{-1}(y)
\]

Since \( \bar{\psi}_n \rho_n(z) = R\psi_n(z) \) for all \( z \in A_n \), we have

\[
\bar{\psi}_n \bar{x}_{n-1}(y) = \psi_n \rho_n \tau_{n-1} \rho_n^{-1}(y) = R\psi_n \tau_{n-1} \rho_n^{-1}(y) = R\psi_{n-1} \rho_n^{-1}(y) = \bar{\psi}_{n-1}(y).
\]

Therefore, the induced Var-homomorphisms are related in the same way:

\[
\psi_{n-1}^0 = \psi_n^0 \tau_{n-1}^0.
\]

Now, for all \( x \in \text{Var}(\bar{A}_{n-1}) \subseteq A_n \) we have

\[
\tau_n(x) = \tau_{n-1}^0(x).
\]

Hence,

\[
\psi_{n+1} \tau_n(x) = \psi_n \tau_{n-1}^0(x) = \psi_{n-1}^0(x) = \psi_n(x)
\]

by definition of \( \psi_{n+1} \). Since \( \psi_n \) is uniquely determined by its action on \( \bar{A}_{n-1} \) (uniqueness property of the universal map on free product), we have the required equality \( \psi_{n+1} \tau_n = \psi_n \) on the entire \( A_n \). \( \square \)
The chain

$$A \xrightarrow{\tau_0} A_1 \xrightarrow{\tau_1} A_2 \xrightarrow{} \ldots \xrightarrow{} A_n \xrightarrow{\tau_n} A_{n+1} \xrightarrow{} \ldots$$
naturally defines direct system of Var-algebras. Let

$$A_\infty = \lim_{\to} A_n,$$

$$\rho : A_\infty \to A_\infty, \quad \rho = \lim_{\to} \rho_n.$$

**Theorem 1.** The Var-algebra $A_\infty$ with linear map $\rho$ is isomorphic to the universal RVar enveloping $U_R(A)$.

**Proof.** The universal property of $(A_\infty, \rho)$ follows from Lemma 2. \hfill \square

Let us consider the particular case $\text{Var} = \text{Lie}$. Recall that if $Y$ is a well-ordered set of generators then the linear basis of $\text{Lie}(Y)$ may be constructed in the following way [4]. A word $u \in Y^*$ is called an (associative) Lyndon—Shirshov word (LS-word) if either $u \in Y$ or for every presentation $u = vw$, $v, w \in Y^*$, we have $u > vw$ lexicographically. Denote the set of all such words by $\text{LS}(Y)$. For every $u \in \text{LS}(Y)$ there exists standard bracketing $[u]$ such that $[u] = ([v][w])$, where $w$ is the longest proper LS-suffix of $u$ (then $v$ is also an LS-word, $[v]$ and $[w]$ are standard bracketings on these shorter words). The set $\{[u] : u \in \text{LS}(Y)\}$ is a linear basis of $\text{Lie}(Y)$.

It is not hard to construct the linear basis of free RLie-algebra $\text{RLie}(X)$ for a given well-ordered set $X$ of generators. Let $\text{RLS}_0(X) = \{[u] : u \in \text{LS}(X)\}$ be the basis of $\text{Lie}(X)$ equipped with leg-lex ordering:

$$[u] < [v] \iff u <_{\text{deglex}} v$$

Assume the set $\text{RLS}_n(X)$ is already constructed and equipped with a well order.

Consider the alphabet $U_n = X \cup \{R([u]) : [u] \in \text{RLS}_n(X)\}$ with the following order: $x < R([u])$ for all $x \in X$, $[u] \in \text{RLS}_n(X)$; $R([u]) < R([v]) \iff [u] < [v]$, $[u], [v] \in \text{RLS}_n(X)$. Then

$$\text{RLS}_{n+1}(X) := \{[w] : w \in \text{LS}(U_n)\}$$
equipped with deglex order.

Obviously, $\text{RLS}_n(X) \subset \text{RLS}_{n+1}(X)$ for all $n \geq 0$.

**Corollary 1.** The set

$$\text{RLS}(X) = \bigcup_{n \geq 0} \text{RLS}_n(X)$$
is a linear basis of $\text{RLie}(X)$.

**Proof.** Let $L = \text{Lie}(X)$, $U_R(L) \simeq \text{RLie}(X) \xrightarrow{\theta} L_\infty$. Consider the images of RLS-words as elements of $\text{RLie}(X)$ under the isomorphism $\theta$ induced by $x \mapsto x$, $x \in X$. By definition, $\theta(\text{RLS}_0(X))$ is the basis of $L_0 = L$. Assume $\theta(\text{RLS}_k(X))$ is a basis of $L_k$ for all $k \leq n$, and the embedding $\text{RLS}_{k-1}(X) \subset \text{RLS}_k(X)$ is compatible with $\tau_{k-1} : L_{k-1} \to L_k$, $k = 1, \ldots, n$. Then $\theta(\tau(R(\text{RLS}_n(X)))) = \rho_n(\theta(\text{RLS}_n(X)))$ is the set of free generators for $\text{Lie}(L_n)$. Moreover, $\theta$ is compatible with $\tau_{n+1}$.

Recall that a linear basis of a free product of two free Lie algebras is the free Lie algebra generated by disjoint union of the generating sets. In our case, one of these sets is $X$, other is $\theta(R(\text{RLS}_n(X)))$. Therefore, $\theta(\text{RLS}_{n+1}(X))$ is the linear basis of $L_{n+1}$. \hfill \square
In particular, $\text{RLie}(X)$ as a Lie algebra is isomorphic to $\text{Lie}(U)$, where $U = \bigcup_{n \geq 0} U_n$. Therefore, $\text{RLie}(X)$ has a natural ascending filtration

$$\text{RLie}^{(n)}(X) = \{ f \in \text{RLie}(X) \mid \deg f \leq n \},$$

where $\deg f$ is the degree of $f \in \text{Lie}(U)$ relative to the alphabet $U$.

Note that $U$ may not be a well-ordered set, e.g., $xy > R(xy) > R^2(xy) > \ldots$ for $x > y$. However, for every $n \geq 0$ the subset $U_n$ is obviously well-ordered.

For an RLS-word $[u]$, denote by $\deg_R(u)$ ($R$-degree) the total number of operators $R$ appearing in $u$. For $f \in \text{RLie}(X)$, set $\deg_R(f)$ to be the maximal $R$-degree among all its monomials. Note that for every $n \geq 0$ there exists $N$ such that $\{ [u] \in \text{RLS}(X) \mid \deg_R(u) \leq n \} \subseteq \text{LS}(U_N)$.

**Remark 1.** Denote by $\text{RAVar}$ the subvariety of $\text{RVar}$ defined by identity $R(x)R(y) = 0$ (image of $R$ is abelian). The following construction is completely similar to the one stated above.

For $A \in \text{Var}$, let $A_0 = A$ and $A_{n+1} = A \ast A_n^0$, $n \geq 0$, where $A_n^0$ stands for the same space as $A_n$ considered as an algebra with trivial operations. Then the universal enveloping $\text{RAVar}$-algebra $\text{UR}_{RA}(A)$ is isomorphic to $\text{lim} A_n$.

**Corollary 2.** The free $\text{RALie}$-algebra $\text{RALie}(X)$ is isomorphic as a Lie algebra to the partially commutative Lie algebra $\text{Lie}(U \mid uv = 0, \ u, v \in U \setminus X)$.

Let us denote by $\text{RLS}_n(X)$ the set of all $[w] \in \text{RLS}_n(X)$ such that $w$ do not contain subwords of the form $R([u])R([v])$, $u, v \in \text{RLS}_{n-1}(X)$, $[u] > [v]$. It is easy to see [15] that $\text{RLS}_n(X)$ is the linear basis of the Lie algebra $L_n$ constructed from $L_0 = \text{Lie}(X)$ as above. Therefore,

$$\text{RLS}(X) = \bigcup_{n \geq 0} \text{RLS}_n(X)$$

is the linear basis of $\text{UR}_{RA}(\text{Lie}(X)) \simeq \text{RALie}(X)$.

3. **CD-lemma for $\text{RLie}$ algebras**

Let us call elements of $\text{RLie}(X)$ by $\text{RLie}$-polynomials, and let $\bar{f} \in \text{RLie}(X)$ stand for the leading word (principle monomial) of an $\text{RLie}$-polynomial $f$.

Let us recall an important statement which plays an important role in the combinatorial theory of Lie algebras.

**Lemma 3** (Shirshov bracketing, [14] Lemma 4). Let $U$ be an ordered set, and $w, u \in \text{LS}(U)$. Suppose $u$ is a subword of $w$, i.e., $w = aub$, where $a$ and $b$ are some words in $U$ (either of them may be empty). Denote by $w_{u\leftarrow a} = a \ast b$, a word in the alphabet $U \cup \{ * \}$ obtained from $w$ by replacing this occurrence of $u$ by a new symbol $\ast$. Then there exists unique bracketing on $w_{u\leftarrow a}$, denoted by $\{ w_{u\leftarrow a} \}$, such that

$$\{ a[u]b \} = [w] + \sum_i \alpha_i[w_i], \quad \alpha_i \in k, \ w_i \in \text{LS}(U), \ [w_i] < [w].$$

Uniqueness of the Shirshov bracketing implies the following property: let $w, u, z \in \text{LS}(U)$, $u$ is a subword of $z$, and $z$ is a subword of $w$. Consider the words $w_{z\leftarrow u} = a \ast b$, $w_{u\leftarrow z}$, and $z_{u\leftarrow z}$ with the corresponding Shirshov bracketings $\{ \ldots \}$. Then

$$\{ a[z_{u\leftarrow z}]b \} = \{ w_{u\leftarrow z} \}.$$
Suppose $S$ is a set of monic RLiE polynomials. Construct $\hat{S}$ as follows. For every $f \in S$, $\bar{f} = [u]$, consider the associative word $u \in \text{LS}(U)$ and consider all $[w] \in \text{RLS}(X)$ such that the corresponding $w \in \text{LS}(U)$ contain $u$ as a subword: $w = aub$. Let $\{w_{\bar{u} \rightarrow \bar{f}}\} = \{a*b\}$ be the Shirshov bracketing. Denote by $\hat{S}_0$ the collection of all RLiE polynomials $\{w_{\bar{u} \rightarrow \bar{f}}\} = \{afb\}$ corresponding to all possible occurrences of $u$, $[u] = \bar{f}$, $f \in S$, in all RLS-words $[w]$. Then $\overline{w_{\bar{u} \rightarrow \bar{f}}} = [w]$ and $w_{\bar{u} \rightarrow \bar{f}}$ belongs to the ideal of the Lie algebra $\text{Lie}(U)$ generated by $S$. All these polynomials are monic, and $S \subset \hat{S}_0$.

Proceed by induction: given $\hat{S}_n = \Sigma$, define $\hat{S}_{n+1} = \Sigma \cup \overline{R(\Sigma)}_0 \supset \hat{S}_n$, and

$$\hat{S} = \bigcup_{n \geq 0} \hat{S}_n$$

**Lemma 4.** An RLiE polynomial $f$ belongs to the ideal $I_R(S)$ generated by $S$ in RLiE$(X)$ if and only if $f = \sum \alpha_i h_i$, $h_i \in \hat{S}$, $\alpha_i \in k$.

**Proof.** The ideal $I_R(S)$ in RLiE$(X)$ is the minimal $R$-invariant ideal in the Lie algebra $\text{Lie}(U)$ which contains $S$. By the construction, $I_R(S) \subseteq \hat{S}$.

Conversely, it follows from [13] Lemma 3 that an ideal $I(\Sigma)$ generated by a set $\Sigma$ in Lie$(U)$ coincides with the linear span of $\Sigma_0$. Hence, the linear span of $\hat{S}$ is an ideal in Lie$(U)$. Obviously, this ideal is $R$-invariant, so $I_R(S) \subseteq k\hat{S}$.  

Recall that a rewriting system is an oriented graph $G = (V, E)$ which has no infinite oriented paths. A vertex $v \in V$ is called terminal if there are no edges of the form $v \rightarrow w$ in $E$.

Define an oriented graph $G_R(X, S)$ on the set of vertices RLiE$(X)$ based on a set of monic RLiE-polynomials $S$, assuming that two RLiE-polynomials $f$ and $g$ are connected by an edge $f \rightarrow g$ if and only if $f = f_0 + \alpha[u] + f_1$ (all monomials of $f_0$ are larger than $[u]$ and $[u] > \bar{f}_1$, $\alpha \in k$, $\alpha \neq 0$) such that $[u] = \bar{h}$ for some $h \in \hat{S}$, and $g = f - \alpha h$. Every edge obviously corresponds to unique $h \in \hat{S}$, and therefore has a well-defined level which is the minimal $n$ such that $h \in \hat{S}_n$.

From now on, assume the following additional condition on $S$: $\deg_R \bar{s} \geq \deg_R s$ for every $s \in S$, i.e., the number of operators $R$ in the leading word $\bar{s}$ is greater or equal to $R$-degrees of all other monomials in $s$. Obviously, the same relation holds for $h \in \hat{S}$. In this case, $G_R(X, S)$ is a rewriting system since for every vertex $f \in \text{RLiE}(X)$ its cone (set of all vertices $g$ such that there exists an oriented path $f \rightarrow \cdots \rightarrow g$) belongs to $k\text{RLS}(X)$ for some $n$, and $U_n^*$ is well ordered. Terminal vertices of this rewriting system are also called $S$-reduced RLiE polynomials.

Let us denote by $f \sim_d g$ the fact that $f, g \in \text{RLiE}(X)$ are connected by a non-oriented path of length $d \geq 1$. Notation $f \sim g$ means that there exists $d \geq 1$ such that $f \sim_d g$.

The following two lemmas are almost obvious but we still state their proofs for readers’ convenience.

**Lemma 5.** Let $V$ be a subspace of $\text{RLiE}(X)$, and let $G(V)$ stand for the subgraph of $G_R(X, S)$ with vertices $V$. Then for every $f, g, h \in V$

$$f \sim g \text{ in } G(V) \iff f + h \sim g + h \text{ in } G(V).$$

**Proof.** It is enough to show ($\Rightarrow$). Suppose $f \sim_d g$ and proceed by induction in $d$.

In fact, we only need $d = 1$ since the induction step is obvious. Assume $f \rightarrow g$,
$f = f_0 + \alpha[u] + f_1$, $[u] = \hat{s}$, $s \in \hat{S}$, $g = f - \alpha s$ as in the definition of $\mathcal{G}_R(X, S)$. In particular, $s \in V$ Apply the same principle to write down decompositions of $h = h_0 + \beta[u] + h_1$ (for some $\beta \in \mathbb{k}$) and $g = f - \alpha s = g_0 + g_1$. Then $f + h = f_0 + h_0 + (\alpha + \beta)[u] + f_1 + h_1$, $g + h = g_0 + h_0 + \beta[u] + h_1 + g_1$. If $\alpha + \beta \neq 0$ and $\beta = 0$ then $f + h \to g + h$. If $\alpha + \beta \neq 0$ and $\beta \neq 0$ then $f + h \to f + h - (\alpha + \beta)s = g + h - \beta s \leftarrow g + h$.

so $f + h \leadsto g + h$. Finally, if $\alpha + \beta = 0$ then $f + h = f_0 + h_0 + f_1 + h_1 = g + \alpha s + h = g + h - \beta s \leftarrow g + h$.

\[\blacksquare\]

**Lemma 6.** In the notations of Lemma 5, the following statement holds: for every $f, g \in V$

$$f - g = \sum \alpha_i s_i, \quad \alpha_i \in \mathbb{k}, \ s_i \in \hat{S} \cap V,$$

if and only if $f \sim g$ in $\mathcal{G}(V)$.

**Proof.** ($\Rightarrow$) It follows from the definition of edges in $\mathcal{G}_R(X, S)$.

($\Leftarrow$) Assume $f - g = \alpha_1 s_1 + \cdots + \alpha_n s_n, \ s_i \in \hat{S} \cap V$. If $n = 1$ then we simply have $f - g \to 0$, so $f \sim g$ by Lemma 5. If $n > 1$ then $f - (g + \alpha_1 s_1) \sim 0$ by induction, so $f - g \sim \alpha_1 s_1 \to 0$ by Lemma 5.

By Lemma 4, the ideal $I_R(S)$ coincides with the linear span of $\hat{S}$. Therefore, connected components (in the non-oriented sense) of $\mathcal{G}_R(X, S)$ are exactly the elements of the quotient algebra $\text{RLie}(X)/I_R(S)$.

We will mainly use the following subspaces of $\text{RLie}(X)$:

$$V_n = \mathbb{k}\{[u] \in \text{RLS}(X) \mid \deg u \leq n\}, \quad n \geq 0,$$

$$V^{[w]} = \mathbb{k}\{[u] \in \text{RLS}(X) \mid [u] \leq [w]\}, \quad [w] \in \text{RLS}(X),$$

$$V^{[w]}_n = V_n \cap V^{[w]}.$$

Note that

$$V_n = \bigcup_{[w] \in \text{RLS}(X) \cap V_n} V^{[w]}_n,$$

and $\text{RLS}(X) \cap V_n$ is a well-ordered subset of $\text{RLS}(X)$.

Recall that a rewriting system is called **confluent** if for every vertex $v$ there exists unique terminal vertex $t$ such that $v$ is connected with $t$ by an oriented path (i.e., $v \to \cdots \to t$, or $v \sim t$). In particular, every non-oriented connected component of a confluent rewriting system contains unique terminal vertex.

Therefore, if the rewriting system $\mathcal{G}_R(X, S)$ is confluent then there exists unique normal form of an element of $\text{RLie}(X)/I_R(S)$ which may be found by straight-forward walk on the graph. The following statement is a well-known criterion of confluence.

**Theorem 2** (Diamond Lemma, [17]). A rewriting system $\mathcal{G} = (V, E)$ is confluent if and only if for every $v \in V$ and for every two edges $v \to w_1$, $v \to w_2$ there exists a vertex $u \in V$ such that $w_1 \sim u$ and $w_2 \sim u$.

$$\blacksquare$$

It is easy to see that rewriting system $\mathcal{G}_R(X, S)$ is confluent if and only if so is each subsystem $\mathcal{G}(V_n), \ n \geq 0$. The latter is confluent if and only if so is $\mathcal{G}(V^{[w]}_n), \ [w] \in \text{RLS}(X) \cap V_n$. 

ROTA—BAXTER LIE ALGEBRAS 7
Proposition 1. Let $S \subset \text{RLie}(X)$ be a set of monic RLie-polynomials, $n \geq 0$. Suppose the rewriting system $G(V_u) \subset G_R(X, S)$ has the following property: for every RLS-word $[w] \in V_n$ and for every pair of edges $[w] \rightarrow g_1, [w] \rightarrow g_2$ in $G(V_n)$ we have

$$g_1 - g_2 = \sum_i \alpha_i h_i, \quad h_i \in \hat{S} \cap V_n, \quad \bar{h}_i < [w].$$

(3) Then the system $G(V_n)$ is confluent.

Proof. Let us check the Diamond Condition from Theorem 2 for rewriting system $G_n^{[v]} = G(V_n^{[v]}) \subset G_R(X, S)$, $[v] \in \text{RLS}(X) \cap V_n$.

Proceed by induction on $[v]$. Assume the rewriting system $G_n^{[u]}$ is confluent for all $[u] \in V_n$, $[u] < [v]$, and consider an ambiguity in the graph $G_n^{[v]}$, i.e., a pair of edges $f \to g_1, f \to g_2$. Here $g_1 = f - \alpha h_1, g_2 = f - \beta h_2$, $h_i \in \hat{S} \cap V_n^{[v]}$. There are three possible cases:

Case 1: $h_1 \neq h_2$.

Then $f$ may be written in the form with ordered monomials $f = f_0 + \alpha[u_1] + f_1 + \beta[u_2] + f_2$, $[u] = \bar{h}_i$. Suppose $[u_1] - h_1 = \gamma [u_2] h + h$, where $h$ does not contain monomial $[u_2]$, $\gamma \in \mathfrak{k}$. It is now easy to see that if $\gamma \alpha + \beta \neq 0$ then there exists an ambiguity $g_1 \to g', g' \to g$ in $G_n^{[v]}$, where $g = f_0 + f_1 + f_2 + \alpha h + (\beta - \gamma) [u_2] - h_2$. If $\gamma \alpha + \beta = 0$ then there exist edges $g_1 \to g, g_2 \to g'$ for the same $g'$. Therefore, in this case the Diamond Condition holds.

Case 2: $h_1 = h_2 < f$.

Then $f = f_0 + \alpha[u_1] + f_1$, $[u] = \bar{h}_i < \bar{f} \leq [v]$. Hence, there is an ambiguity $f' \to g'_1, f' \to g'_2$ in $G_n^{[v]}$, where $f' = \alpha[u_1] + f_1, g_i = f_0 + g_i$. By the inductive assumption, there exist two paths in $G_n^{[v]}$: $g'_i \to \cdots \to g'$, $i = 1, 2$. Therefore, $g_i \to \cdots \to f_0 + g'$ in $G_n^{[v]}$ since all monomials in $f_0$ are greater than $[u]$.

Case 3: $h_1 = h_2 = f$.

Without loss of generality, assume $\bar{f} = [v]$. Then $g_i = \alpha([v] - h_i) + f_1$ and the difference $g_1 - g_2 = \alpha(h_2 - h_1)$ coincides (up to scalar) with one that appears in the pair of edges $[v] \to [v] - h_i$, $i = 1, 2$. Therefore, the condition of the statement implies $g_1$ and $g_2$ are connected by a non-oriented path in $G_n^{[u]}(X, S)$ for some $[u] < [v]$. The last rewriting system is assumed to be confluent by induction, so there exist oriented paths $g_1 \to \cdots \to g, g_2 \to \cdots \to g$ in $G_n^{[u]}(X, S)$, and the Diamond Condition holds for $G_n^{[v]}$.

Recall the Shirshov’s definition of a composition [16] in the free Lie algebra Lie$(U)$.

Let $f, g \in \text{Lie}(U)$ be monic Lie-polynomials, $\bar{f} = [u], \bar{g} = [v]$. We say that $f$ and $g$ form a composition relative to a word $w$ if $u = u_1 u_2, v = v_1 v_2, w' = v_1$ ($u_i, v_i \in U^*$). Here $w = u_1 u_2 v_2 = u_1 v_1 v_2$ is a LS-word, and there are two Shirshov brackets:

$$\{w_{u-e-v} \} = \{ * v_2 \} 1, \quad \{ w_{v-e-v} \} = \{ v_1 * \} 2.$$  

The Lie polynomial

$$(f, g)_w = \{ f v_2 \} 1 - \{ u_1 g \} 2$$

is called a composition of $f$ and $g$ relative to $w$. It is important that

$$(f, g)_w < [w].$$

(4)
It follows from the definition that if \( f, g \in S \) then
\[
[w] \rightarrow g_1 = [w] - \{fv_2\}_1
\]
is an edge in \( G_R(X,S) \), and so is
\[
[w] \rightarrow g_2 = [w] - \{u_1g\}_2.
\]
Therefore, \( g_1 - g_2 = (f,g)_w \).

Suppose \( S \) is a set of monic RLie polynomials such that the rewriting system \( G_R(X,S) \) is reduced, i.e., it has the following property: for every vertex \( s \in S \) there is only one edge \( s \rightarrow 0 \) in \( G_R(X,S) \). We will say \( S \) is reduced if so is \( G_R(X,S) \).

**Proposition 2.** Let \( S \) be a reduced set of monic RLie-polynomials in RLie\((X)\). Suppose that all compositions of type \((s_1,s_2)_w\), \( s_1, s_2 \in S, [w] \in V_n \cap RLS(X) \), have the following presentation:
\[
(s_1,s_2)_w = \sum_i \alpha_i h_i, \quad h_i \in \hat{S} \cap V_n, \quad \bar{h}_i < [w].
\]
Then the rewriting system \( G(V_n) \subset G_R(X,S) \) is confluent.

**Proof.** Check the conditions of Proposition [1] for a word \([w] \in RLS(X) \cap V_n\). Assume there is a pair of edges in \( G(V_n) \): \([w] \rightarrow g_1, [w] \rightarrow g_2\).

There are several possible cases.

1) Both edges are of level 0. (This case is actually covered by the classical Composition-Diamond Lemma [10], but we prefer to consider it in our terminology to make the exposition complete.) Then
\[
g_1 = [w] - h_1, \quad g_2 = [w] - h_2,
\]
\( h_i \in \{s_i\}_0 \cap V_n, s_i \in S \). Let \( u = \bar{s}_1 \) and \( v = \bar{s}_2 \).

Recall the following

**Lemma 7.** Suppose \( u, v, w \in LS(U), w = auvbc, \) where \( a, b, \) and \( c \) are some words in \( U \) (either of them may be empty). Then there exists a bracketing \( \{a*b*c\} \) such that \( \{a[u]vbc\} = [w] + \sum \alpha_i [u_i], [u_i] < [w] \).

This statement also implicitly appears in [10].

1.1) Let the corresponding occurrences of subwords \( u \) and \( v \) in \( w \) do not intersect. Then \( w = auvbc, h_1 = \{as_1bvc\}_1, h_2 = \{aubs_2c\}_2, \) where \( \{\ldots\}_1 \) and \( \{\ldots\}_2 \) are the Shirshov bracketings on \( w_{u+} \) and \( w_{v+} \), respectively. Therefore,
\[
g_1 - g_2 = \{aubs_2c\}_2 - \{as_1bvc\}_1
= \{aubs_2c\}_2 - \{a[u]bs_2c\}_12 + \{as_1b[v]c\}_12 - \{as_1bvc\}_12
+ \{a[u]bs_2c\}_12 - \{as_1bs_2c\}_12 + \{as_1bs_2c\}_12 - \{as_1b[v]c\}_12
= (\{aubs_2c\}_2 - \{a[u]bs_2c\}_12) + (\{as_1b[v]c\}_12 - \{as_1bvc\}_12)
+ \{a[u] - s_1)bs_2c\}_12 + \{as_1b(s_2 - [v])c\}_12.
\]
where \( \{a*b*c\}_12 \) is the bracketing from Lemma 7. In the last expression, all summands belong to linear span of \( h_i \in \hat{S} \cap V_n \) with \( \bar{h}_i < [w] \), so \( g_1 - g_2 \) has the required presentation.

1.2) Let the corresponding occurrences of \( u \) and \( v \) in \( w \) intersect: \( u = u_1u_2, v = v_1v_2, u_2 = v_1 \). Then \( z = uv_2 = u_1v \) is a LS-word, \( w = azb = auv_2b, h_1 = \{as_1v_2b\}_1, h_2 = \{au_1s_2b\}_2, \) where \( \{a*v_2b\}_1 \) and \( \{au_1*b\} \) are the
Shirshov bracketings on \(w_{u \rightarrow x}\) and \(w_{v \rightarrow x}\), respectively. Consider also the Shirshov bracketings \(\{a \ast b\}_0\) on \(w_{z \rightarrow x}\) and \(\{*v_2\}_{01}\), \(\{u_1 \ast v_2\}_{02}\) on \(z_{u \rightarrow x}\) and \(z_{v \rightarrow x}\), respectively. Then

\[
g_1 - g_2 = \{au_1s_2b\}_2 - \{as_1v_2b\}_1
= \{a\{u_1s_2\}_{02}b\}_0 - \{a\{s_1v_2\}_{01}b\}_0
= -\{afb\}_0,
\]

where \(f = (s_1, s_2)_z\). Since \(f = \sum_i \alpha_i h_i, \tilde{h}_i < [z], h_i \in \hat{S} \cap V_n\), RLie polynomial \(g_1 - g_2\) may be presented as (3).

2) The edge \([w] \rightarrow g_1\) is of positive level \(d\), \([w] \rightarrow g_2\) is of level 0. In this case,

\[
w = a_1 \ldots a_m, \quad a_i \in U,
\]

\(a_k = R([v])\) for some \(k\), where \([v] = \tilde{h}, h \in \{s_1\}_{d-1}\), \(s_1 \in S\). Therefore, \(h_1 = [a_1 \ldots a_{k-1} R(h) a_{k+1} \ldots a_m]\). As above,

\[
w = aub, \quad [u] = \tilde{s}_2,
\]

and \(h_2 = \{as_2b\}\). Since \(S\) is reduced, the occurrence of letter \(a_k = R([v]) \in U\) considered above may appear in either of the subwords \(a\) or \(b\). Suppose \(a = ca_kc'\) (the second case in analogous). Then

\[
w = cR([v])c'ub, \quad c, c', b \in U^* \cup \{\epsilon\},
\]

Therefore,

\[
g_1 - g_2 = \{cR([v])c's_2b\} - \{cR(h)c'ub\}
= \{cR([v])c's_2b\} - \{cR(h)c's_2b\} + \{cR(h)c'ub\},
\]

and the same reasonings as in Case 1.1 show the required relation (3) holds.

3) Both edges \([w] \rightarrow g_1\), \([w] \rightarrow g_2\) have positive level. In this case, \(w = a_1 \ldots a_k \ldots a_l \ldots a_m, a_i \in U\), where \(a_k = R([u]), a_l = R([v]), [u] \rightarrow g'_1\)

\[h_1 = [a_1 \ldots R(g'_1) \ldots a_l \ldots a_m], \quad h_2 = [a_1 \ldots a_k \ldots R([v]) \ldots a_m].\]

3.1) If \(k \neq l\) then one may proceed as in Case 2.

3.2) If \(k = l\), proceed by induction on the level of edges. Consider \(a_k = a_l = R([u])\) with edges \([u] \rightarrow g'_1, [u] \rightarrow g'_2\) in \(\mathcal{G}(V_{n-1}) \subset \mathcal{G}(V_n)\). Inductive assumption claims \(g'_1 - g'_2 = \sum \alpha_i h_i'\), \(h_i' < [u]\). Therefore,

\[
g_1 - g_2 = [a_1 \ldots R(g'_1 - g'_2) \ldots a_m]
\]

also has a required presentation (3).

The entire system \(S\) is closed with respect to composition if for every \(s_1, s_2 \in S\) every their composition \((s_1, s_2)_w\) may be presented as

\[
(s_1, s_2)_w = \sum \alpha_i h_i, \quad h_i \in \hat{S}, \quad \tilde{h}_i < [w], \quad \deg_R h_i \leq \deg_R w.
\]

A reduced set of monic RLie polynomials in RLie\((X)\) which is closed with respect to composition is called a Gröbner—Shirshov basis (GSB) in RLie\((X)\).

**Theorem 3.** If \(S\) is a GSB in RLie\((X)\). Then the rewriting system \(\mathcal{G}_R(X, S)\) is confluent.
Proof. The statement follows from Propositions\textsuperscript{1} and\textsuperscript{2}. □

Corollary 3. If $S$ is a GSB in $\text{RLie}(X)$ then the set of $S$-reduced words forms a linear basis of the algebra $\text{RLie}(X \mid S) = \text{RLie}(X)/I_R(S)$.

Proof. Terminal vertices of $\mathcal{G}(X, S)$ are exactly linear combinations of $S$-reduced words. □

Example 1. Let $S$ consists of all $R([u])R([v])$, $[u], [v] \in \text{RLie}(X)$, $[u] > [v]$. Then $S$ is a reduced system closed with respect to compositions, and the set of $S$-reduced words coincides with $\text{RLie}(X)$.

Obviously, $\text{RLie}(X \mid S) \simeq \text{RALie}(X)$, so $\text{RLie}(X)$ is indeed the linear basis of $\text{RLie}(X)$.

More general, let $L$ be a Lie algebra, and let $X$ be a linear basis of $L$ which is linearly ordered in some way.

Example 2. The set $W$ of all words $[w] \in \text{RLie}(X)$ such that $w$ do not contain subwords of type $xy$, $x, y \in X$, $x > y$, form a linear basis of $U_{RA}(L)$.

It is easy to see that $S = \{R([u])R([w]) \mid [u], [w] \in W, u > w\} \cup \{xy - [x, y] \mid x, y \in X, x > y\}$ is a GSB, and $\text{RLie}(X \mid S) \simeq U_{RA}(L)$.

4. Rota—Baxter Lie algebras

Let $\text{RBLie}$ denotes the variety of Lie algebras equipped with a Rota—Baxter operator $R$ of weight $\lambda \in \mathbb{A}$, i.e., a linear map satisfying the following identity:

$$[R(x), R(y)] = R([R(x), y]) + R([x, R(y)]) + \lambda R([x, y]).$$

Consider the forgetful functor $\text{RBLie} \to \text{Lie}$. For every $L \in \text{Lie}$ there exists universal enveloping $U_{RB}(L) \in \text{RBLie}$: $L \subset U_{RB}(L)$ is a Lie subalgebra, and for every $B \in \text{RBLie}$ and homomorphism $\varphi : L \to B$ of Lie algebras there exists unique homomorphism of $\text{RBLie}$ algebras $\tilde{\varphi} : U_{RB}(L) \to B$ such that $\tilde{\varphi}|_L = \varphi$. In this Section, we clarify the structure of $U_{RB}(L)$ and prove an analogue of the Poincaré—Birkhoff—Witt Theorem.

Suppose $L$ is a Lie algebra with a linear basis $X$. Assume $X$ to be well ordered in some way. Consider

$$S^{(0)} = \{xy - [x, y] \mid x, y \in X, x > y\} \subset \text{Lie}(X) \subset \text{RLie}(X).$$

Here $[x, y]$ is a linear form in $X$ equal to the product of $x$ and $y$ in $L$. Then $S^{(0)}$ is a GSB in $\text{Lie}(X)$ and, therefore, in $\text{RLie}(X)$. Moreover, $L \simeq \text{Lie}(X \mid S^{(0)})$.

Now, consider

$$\rho(x, y) = R(x)R(y) - R(R(x)y) + R(R(y)x) - \lambda R([x, y]), \quad x, y \in X, x > y,$$

and set $S^{(2)} \subset \text{RLie}(X)$ to be the union of $S^{(0)}$ set of all $\rho(x, y)$. Denote by $\mathcal{G}^{(2)}$ the subgraph $\mathcal{G}(V_2)$ of $\mathcal{G}_R(X, S^{(2)})$. Obviously, $S^{(2)}$ is a GSB: it is reduced, and the graph $\mathcal{G}_R(X, S^{(2)})$ has no ambiguities.

Proceed by induction on $R$-degree. Assume a reduced system $S^{(n)}$, $n \geq 2$, is already constructed in such a way that the subgraph $\mathcal{G}^{(n)} = \mathcal{G}(V_n) \subset \mathcal{G}_R(X, S^{(n)})$ is a confluent rewriting system. Denote by $T_n$ the set of terminal vertices of $\mathcal{G}^{(n)}$, and let $t_n : V_n \to T_n$ be the linear map that turns an $\text{RLie}$ polynomial $f$, $\deg_R f \leq n$, into the terminal vertex $t_n(f)$ connected with $f$. 


For every two terminal words \(a, b \in T_n \cap \text{RLS}(X)\), \(\deg_R a + \deg_R b = n - 1\), \(a > b\), consider
\[
\rho(a, b) = R(a)R(b) - R(t_n([R(a), b])) + R(t_n([R(b), a])) - \lambda R(t_n([a, b]))\]
where \([, , \cdot]\) stands for the product in \(\text{RLie}(X)\). Construct
\[
S^{(n+1)} = S^{(n)} \cup \{\rho(a, b) \mid a, b \in T_n \cap \text{RLS}(X), \ \deg_R a + \deg_R b = n - 1, \ a > b\}.
\]
It is easy to see from the construction that the subgraph \(G(V_n) \subset G_R(X, S^{(n+1)})\) coincides with \(G^{(n)}\).

We have to resolve the following questions:

- Prove that \(S^{(n+1)}\) is confluent (assuming so is \(S^{(n)}\));
- Show \(\text{RLie}(X | S) \simeq U_{RB}(L)\), where \(S\) is the union of all \(S^{(n)}\);
- Describe the set of \(S\)-reduced words in \(\text{RLS}(X)\).

**Lemma 8.** Let \(f, g \in \text{RLie}(X)\), \(\deg_R f + \deg_R g = n\). Then \(t_n([f, t_n(g)]) = t_n([f, g])\).

**Proof.** It follows from Lemma 8 that \([f, g]\) and \([f, t_n(g)]\) belong to the same connected component of \(G^{(n)}\). Since the latter is confluent, \(t_n([f, g]) = t_n([f, t_n(g)])\).

**Lemma 9.** The rewriting system \(G^{(n+1)} = G(V_n+1) \subset G_R(X, S^{(n+1)})\) is confluent.

**Proof.** Here we assume by induction that \(G^{(n)} = G(V_n) \subset G_R(X, S^{(n+1)})\) is confluent. It is enough to check the conditions of Proposition 3 for compositions \((s_1, s_2)_w, s_1, s_2 \in S^{(n+1)}, [w] \in \text{RLS}(X), \ \deg_R w = n + 1\).

Suppose \(s_1 = \rho(a, b), s_2 = \rho(b, c), w = R(a)R(b)R(c), a, b, c \in T_n \cap \text{RLS}(X), a > b > c\). Denote
\[
\rho(a, b) = R(a)R(b) - \sum_i \gamma_i R(c_i), \quad \rho(b, c) = R(b)R(c) - \sum_j \alpha_j R(a_j), \quad \rho(a, c) = R(a)R(c) - \sum_l \beta_l R(b_l),
\]
where \(\deg_R c_i, \ \deg_R a_j, \ \deg_R b_l < n - 1\). Then
\[
(s_1, s_2)_w = [\rho(a, b), R(c)] - [R(a), \rho(b, c)] = [R(a)R(b), R(c)] - [R(a), R(b)R(c)]
- \sum_i \gamma_i [R(c_i), R(c)] + \sum_j \alpha_j [R(a), R(a_j)]
= -[R(b), \rho(a, c)] - \sum_l \beta_l [R(b), R(b_l)]
- \sum_i \gamma_i [R(c_i), R(c)] + \sum_j \alpha_j [R(a), R(a_j)]
= -[R(b), \rho(a, c)] + K(a, b, c).
\]
Here \(h = [R(b), \rho(a, c)] \in \hat{S}^{(n)}, \ \hat{h} = [R(b)R(a)R(c)] < [w], \) and all monomials in
\[
K(a, b, c) = \sum_j \alpha_j [R(a), R(a_j)] - \sum_l \beta_l [R(b), R(b_l)] - \sum_i \gamma_i [R(c_i), R(c)]
\]
are smaller than \([w]\) since they are of degree two in \(U\). Straightforward computations show

\[ K(a, b, c) = \sum_k \xi_k h_k + R(J(a, b, c)), \]

where

\[
J(a, b, c) = t_n([R(a), t_n([R(b), c] + [b, R(c)] + \lambda[b, c])] + \lambda[a, t_n([R(b), R(c)])]
+ \lambda[a, t_n([R(b), c] + [b, R(c)] + \lambda[b, c]])
- [R(b), t_n([R(a), c] + [a, R(c)] + \lambda[a, c])] - [b, t_n([R(a), R(c)])]
- [\lambda[b, t_n([R(a), c] + [a, R(c)] + \lambda[a, c])]
- [t_n([R(a), R(b)]), c] - [t_n([R(a), b] + [a, R(b)] + \lambda[a, b]), R(c)]
- \lambda[t_n([R(a), b] + [a, R(b)] + \lambda[a, b]), c]).
\]

Indeed, \([R(b), R(c)] \rightarrow R(t_n([R(b), c] + [b, R(c)] + \lambda[b, c)])\) is an edge in \(G^{(n)}\), so \(t_n([R(b), R(c)]) = \sum_j \alpha_j R(a_j)\). Moreover, \(t_n(R(x)) = R(t_n(x))\) for all \(x \in V_n - 1\).

It remains to apply Lemma 8 to conclude

\[
J(a, b, c) = t_n\left( \text{Jac}(R(a), R(b), c) + \text{Jac}(R(a), b, R(c)) + \text{Jac}(a, R(b), R(c)) + \lambda \text{Jac}(R(a), b, c) + \lambda \text{Jac}(a, R(b), c) + \lambda \text{Jac}(a, b, R(c)) \right.
+ \lambda^2 \text{Jac}(a, b, c) \bigg) = 0,
\]

where \(\text{Jac}(x, y, z) = [x, [y, z]] - [y, [x, z]] - [[x, y], z]\) is the Jacobian. Hence, \((s_1, s_2)_w\) has a required presentation \((5)\).

Denote \(S = \bigcup_{n \geq 1} S^{(n)}\). Obviously, \(S\) is a GSB. Denote by \(T\) the set of terminal vertices in \(G_R(X, S), T = \bigcup_{n \geq 1} T_n\).

**Lemma 10.** RLie\((X | S)\) is a Rota—Baxter Lie algebra.

**Proof.** We have to prove

\[(6) \quad [R(f), R(g)] - R([R(f), g]) - R([f, R(g)]) - \lambda R([f, g]) \in I_R(S)\]

for all \(f, g \in \text{RLie}(X)\). Since for every \(f \in \text{RLie}(X)\) there exists \(t \in T\) such that \(f - t \in I_R(S)\), it is enough to check \((6)\) for \(f = a, g = b\), where \(a, b \in T \cap \text{RLS}(X)\).

Assume \(a \in T_n, b \in T_m\). Then \([R(a), R(b)]\) and \(R([R(a), b] + [a, R(b)] + \lambda[a, b])\) have the same terminal form in \(G^{(n+m+2)}\), so they are connected by a non-oriented path in \(G_R(X, S)\). Hence, \((6)\) holds. \(\square\)

**Corollary 4.** RLie\((X | S) \simeq U_{RB}(L)\).

**Proof.** Let \(B \in \text{RBLie}\), and let \(\varphi : L \rightarrow B\) be a homomorphism of Lie algebras. Identify \(L\) with the Lie subalgebra in \(\text{RLie}(X)\) spanned by \(X\). Then there exists unique homomorphism of \(\text{RLie}\) algebras \(\psi : \text{RLie}(X) \rightarrow B\) such that \(\psi(x) = \varphi(x)\) for \(x \in X\). Denote by \(\tau\) the natural homomorphism \(\text{RLie}(X) \rightarrow \text{RLie}(X | S)\), \(\text{Ker} \tau = I_R(S)\). Since for every \(f \in V_n\) we have \(f - t_n(f) \in \text{Ker} \tau\), Lemma 10 implies \(S^{(n)} \subset \text{Ker} \tau\). Therefore, there exists a homomorphism of \(\text{RLie}\) algebras \(\tilde{\varphi} : \text{RLie}(X | S) \rightarrow B, \tilde{\varphi}(x) = \psi(x) = \varphi(x)\) for \(x \in X\). \(\square\)
Remark 2. For associative algebras the statement of Theorem 4 is easy to show by means of Gröbner—Shirshov bases technique for associative Rota—Baxter algebras is ideologically similar to the classical Poincaré—Birkhoff—Witt Theorem.

Theorem 4. \( \text{gr } U_{RB}(L) \simeq U_{RA}(L) \) as Lie algebras.

Proof. It is enough to compare Gröbner—Shirshov bases of \( U_{RA}(L) \) and \( U_{RB}(L) \). The principal parts of these relations coincide, they are of degree 2. For the latter algebra, the right-hand sides of relations are of degree 1.

Remark 2. For associative algebras the statement of Theorem 4 is easy to show by means of Gröbner—Shirshov bases technique for associative Rota—Baxter algebras: multiplication table of an associative algebra \( A \) is closed under all compositions in the free associative Rota—Baxter algebra.

Remark 3. For commutative algebras, an analogue of Theorem 4 also holds. Moreover, there is an explicit construction of the universal enveloping commutative Rota—Baxter algebra \( U_{RB}(A) \) for a given commutative algebra \( A \) (mixed shuffle algebra).

Let us briefly state the construction from [9] (in the case of zero weight) in more natural terms. Consider

\[
\text{III}(A) = A^\# \otimes B^\#, \quad B = \text{preCom}(A^\#)^{(+)} ,
\]

where \( \text{preCom}(A^\#) \) is the free pre-commutative (Zinbiel) algebra generated by the space \( A^\# \), \( B \) is the anti-commutator algebra of \( Z \) (it is an associative and commutative algebra), and \( A^\# = A \oplus k1_A \), \( B^\# = B \oplus k1_B \) are obtained by joining external units.

Define the linear operator on \( \text{III}(A) \):

\[
R(a \otimes 1_B) = 1_A \otimes a, \quad a \in A^\#, \\
R(a \otimes b) = 1_A \otimes ab, \quad a \in A^\#, \ b \in B.
\]

The Zinbiel identity \( (xy)z = x(yz) + x(zy) \) on \( \text{preCom}(A^\#) \) implies \( R \) to be a Rota—Baxter operator on \( \text{III}(A) \). For example,

\[
R(a_1 \otimes 1_B)R(a_2 \otimes b) = (1_A \otimes a_1)(1_A \otimes a_2b) \\
= 1_A \otimes (a_1a_2b) + (a_2b)a_1 = 1_A \otimes a_1(a_2b) + 1_A \otimes a_2(ba_1) + 1_A \otimes a_2(ab_1).
\]

On the other hand,

\[
R((a_1 \otimes 1_B)R(a_2 \otimes b) + R(a_1 \otimes 1_B)(a_2 \otimes b)) = R((a_1 \otimes 1_B)(1_A \otimes a_2b) + (1_A \otimes a_1)(a_2 \otimes b)) \\
= R(a_1 \otimes a_2b + a_2 \otimes (a_1b + ba_1)) = 1_A \otimes a_1(a_2b) + 1_A \otimes a_2(a_1b) + 1_A \otimes a_2(ba_1).
\]

It is easy to check (see [9]) that the embedding

\[
A \to \text{III}(A), \quad a \mapsto a \otimes 1_B, \ a \in A,
\]

may be extended to a homomorphism of Rota—Baxter algebras \( U_{RB}(A) \to \text{III}(A) \). Suppose \( X \) is a linear basis of \( A \) and consider the following elements of \( U_{RB}(A) \):

\[
u = R^{s_1}((x_1R^{s_2}((x_2R^{s_3}(\ldots R^{s_{n-1}}(x_{n-1}R^{s_n}(x_n))\ldots))), \quad x_i \in X, \ s_1 \geq 0, \ s_2, \ldots s_n > 0, n \geq 1.
\]
Images of these elements $\mathcal{I}(A)$ are linearly independent since the linear base of $\text{preCom}(\langle A \rangle^\#)$ is given by $x_1(x_2(x_3(\ldots(x_{n-1}x_n)\ldots)))$, $x_1 \in X \cup \{1_A\}$. On the other hand, the set of (7) obviously span $\mathcal{U}_{RB}(A)$. Therefore, (7) is a linear basis of $\mathcal{U}_{RB}(A)$ as well as of $\mathcal{U}_{RA}(A)$ from Remark 1.

For nonzero weight, it is enough to replace $\text{preCom}(\langle A \rangle^\#)$ with $\text{postCom}(\langle A \rangle^\#)$ (commutative tridendriform algebra, or CTD-algebra), and set $B$ to be the associated commutative algebra [15, p. 26].

**Remark 4.** The same statement holds for algebras with a Nijenhuis operator, i.e., a linear map $N$ such that

$$[N(x), N(y)] = N([N(x), y]) + N([x, N(y)]) - N^2([x, y]).$$

The route of the proof is completely similar to stated above. The key computation of a composition is based on the following relation which is easy to check by straightforward computation:

$$\text{Jac}(N(a), N(b), N(c)) = N\left(\text{Jac}(a, N(b), N(c)) + \text{Jac}(N(a), b, N(c))\right)$$

$$+ \text{Jac}(N(a), N(b), c) - N^2\left(\text{Jac}(a, b, N(c)) + \text{Jac}(a, N(b), c)\right)$$

$$+ \text{Jac}(N(a), b, c) + N^3\left(\text{Jac}(a, b, c)\right).$$

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