Braided algebras and the $\kappa$-deformed oscillators

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Abstract

Recently there were presented several proposals how to formulate the binary relations describing $\kappa$-deformed oscillator algebras. In this paper we shall consider multilinear products of $\kappa$-deformed oscillators consistent with the axioms of braided algebras. In general case the braided triple products are quasi-associative and satisfy the hexagon condition depending on the coassociator $\Phi \in A \otimes A \otimes A$. We shall consider only the products of $\kappa$-oscillators consistent with co-associative braided algebra, with $\Phi = 1$. We shall consider three explicit examples of binary $\kappa$-deformed oscillator algebra relations and describe briefly their multilinear coassociative extensions satisfying the postulates of braided algebras. The third example, describing $\kappa$-deformed oscillators in group manifold approach to $\kappa$-deformed fourmomenta, is a new result.

1 Introduction.

The notion of classical (pseudo)-Riemannian manifold as the geometric framework for the description of gravitational effects in quantized theory appears to be not valid. The dynamics of general relativity (Einstein equations) combined with uncertainty principle of quantum mechanics imply that the exact localization of space-time events is not possible. One can argue [1] that the quantum gravity effects cause the new algebraic uncertainty relation between the relativistic space-time coordinates (usually we assume that $\kappa$ is related with Planck mass $M_p$)

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{1}{\kappa^2} \Theta_{\mu\nu}(\kappa \hat{x}) ,$$

where using Taylor expansion one gets

$$\Theta_{\mu\nu}(\kappa \hat{x}) = \Theta_{\mu\nu}^{(0)} + \kappa \Theta_{\mu\nu}^{(1)} \hat{x}_\rho + \cdots .$$

The first term in (2) describes the canonical deformation, with constant value of the commutator (1), and second one introduces so-called Lie-algebraic deformation of Minkowski space geometry. The general formula (1) has been also derived as the relation characterizing the $D$-brane coordinates, which describe the end points of $D = 10$ string [2].

An interesting task is to consider the construction of dynamic theories, in particular QFT on the deformed Minkowski space [1]. In order to formulate the deformed perturbation expansion of interacting quantized fields it is important to study the theory...
of deformed free quantum fields. In this paper we shall consider the case of so-called \( \kappa \)-deformation [3]-[5] which is the well-known example of Lie-algebraic deformation.

In order to deform consistently the relativistic free quantum fields one should (see e.g.[6])

1. replace the space-time field arguments \( x_\mu \) by their suitably deformed noncommutative counterparts \( \hat{x}_\mu \).

2. in place of the classical field oscillators algebra (\( p = (\vec{p}, p_0 = \omega(\vec{p})) \))

\[
\begin{align*}
[a(p), a(q)] &= [a^\dagger(p), a^\dagger(q)] = 0, \\
[a(p), a^\dagger(q)] &= \delta^{(3)}(\vec{p} - \vec{q}),
\end{align*}
\]

introduce the deformed set of oscillator algebra relations.

Dealing with noncommutative space-time arguments can be avoided if we can represent homomorphically the algebra of noncommutative fields \( \varphi(\hat{x}) \) by the suitable \( \star \)-product (see e.g.[7]-[12]) of standard commutative fields \( \varphi(x) \)

\[
\varphi(\hat{x}) \chi(\hat{x}) \leftrightarrow \varphi(x) \star \chi(x),
\]

with the noncommutative structure contained in the proper choice of \( \star \)-product. It can be added that in order to describe e.g. in noncommutative field theory the \( n \)-point Green functions we should extend the binary representation (5) to the \( n \)-ary products

\[
\varphi_1(\hat{x}_1) \ldots \varphi_n(\hat{x}_n) \leftrightarrow \varphi_1(x_1) \star \ldots \star \varphi_n(x_n),
\]

where the \( n \)-tuple \( (\hat{x}_1,\ldots,\hat{x}_n) \) provides the extension of basic algebra describing simple noncommutative space-time point to the algebraic variety of \( n \) noncommutative space-time points.

The aim of this paper is to consider the deformations of the oscillator algebras [3]-[11] obtained in \( \kappa \)-deformed relativistic framework [3]-[5] with the noncommutative \( \kappa \)-Minkowski space [14], [4], [5]

\[
\begin{align*}
[\hat{x}_0, \hat{x}_i] &= \frac{i}{\kappa} \hat{x}_i, \\
[\hat{x}_i, \hat{x}_j] &= 0.
\end{align*}
\]

Formula (7) is the special case of generalized \( \kappa \)-deformed relation [15]-[17]

\[
[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa}(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \quad [\hat{x}_i, \hat{x}_j] = 0,
\]

if we put \( a_\mu = (1, 0, 0, 0) \).

The \( \kappa \)-deformed Poincare symmetries are described by dual pair of Hopf algebras, the \( \kappa \)-deformed Poincare algebra and the \( \kappa \)-deformed Poincare group, with their explicit form depending on the choice of basic generators. In this paper we shall use the following modified Majid-Ruegg basis [18], [19], [6]

a) algebraic sector \( \hat{M}_{\mu\nu} = (\hat{M}_i = \frac{1}{2} \epsilon_{ijk} \hat{M}_{jk}, \hat{N}_i = \hat{M}_{i0}) \)
\[ [\hat{M}_{\mu\nu}, \hat{M}_{\lambda\sigma}] = i \left( \eta_{\mu\sigma} \hat{M}_{\nu\lambda} - \eta_{\nu\sigma} \hat{M}_{\mu\lambda} + \eta_{\nu\lambda} \hat{M}_{\mu\sigma} - \eta_{\mu\lambda} \hat{M}_{\nu\sigma} \right), \tag{9} \]

\[ [\hat{M}_i, \hat{P}_j] = i \varepsilon_{ijk} \hat{P}_k, \tag{10} \]

\[ [\hat{N}_i, \hat{P}_j] = i \delta_{ij} e^{\hat{P}_k} \left[ \frac{\kappa}{2} \left( 1 - e^{-\frac{2\hat{P}_k}{\kappa}} \right) + \frac{1}{2\kappa} e^{-\frac{\hat{P}_k}{\kappa}} \hat{P}_k \right] - \frac{i}{2\kappa} e^{-\frac{\hat{P}_k}{\kappa}} \hat{P}_i \hat{P}_j, \tag{11} \]

\[ [\hat{M}_i, \hat{P}_0] = 0, \quad [\hat{N}_i, \hat{P}_0] = i e^{-\frac{\hat{P}_0}{\kappa}} \hat{P}_i, \quad [\hat{P}_\mu, \hat{P}_\nu] = 0, \tag{12-13} \]

b) coalgebra sector

\[ \Delta(\hat{M}_i) = \hat{M}_i \otimes 1 + 1 \otimes \hat{M}_i, \tag{14} \]

\[ \Delta(\hat{N}_i) = \hat{N}_i \otimes 1 + e^{-\frac{\hat{P}_0}{\kappa}} \otimes \hat{N}_i + \frac{1}{\kappa} \varepsilon_{ijk} e^{-\frac{\hat{P}_j}{\kappa}} \hat{P}_j \otimes \hat{M}_k, \tag{15} \]

\[ \Delta(\hat{P}_0) = \hat{P}_0 \otimes 1 + 1 \otimes \hat{P}_0, \tag{16} \]

\[ \Delta(\hat{P}_i) = \hat{P}_i \otimes e^{\frac{\hat{P}_0}{\kappa}} + e^{-\frac{\hat{P}_0}{\kappa}} \otimes \hat{P}_i, \tag{17} \]

c) antipodes

\[ S(\hat{M}_i) = -\hat{M}_i, \quad S(\hat{N}_i) = -e^{-\frac{\hat{P}_0}{\kappa}} \hat{N}_i + \varepsilon_{ijk} e^{-\frac{\hat{P}_j}{\kappa}} \hat{P}_j \hat{M}_k - \frac{1}{\kappa} \hat{P}_i e^{\frac{\hat{P}_0}{\kappa}}, \tag{18} \]

\[ S(\hat{P}_i) = -\hat{P}_i, \quad S(\hat{P}_0) = -\hat{P}_0. \tag{19} \]

One can show that the \(\kappa\)-deformed mass Casimir takes the form

\[ C^\kappa_2(\hat{P}_\mu) = \left( 2\kappa \sinh \left( \frac{\hat{P}_0}{2\kappa} \right) \right)^2 \hat{P}_i \hat{P}_i = m^2 \quad \Rightarrow \quad p_0 = \omega_\kappa(p^2) = 2\kappa \arcsinh \left( \frac{\sqrt{p^2 + \hat{M}^2}}{2\kappa} \right). \tag{20} \]

We mention that the relations (9)-(20) can be obtained from the ones in standard \(\kappa\)-deformation of \(\kappa\)-deformation of Majid-Ruegg (MR) basic [4] if we redefine the three-momentum generators \(\hat{P}_i = e^{-\hat{P}_0/2\kappa} \hat{P}_i^{MR} \).

The basic problem which should be addressed in the procedure of \(\kappa\)-deformation of relations \(3-4\) is the non-Abelian structure of the \(\kappa\)-deformed addition law of momenta. It follows from \(17\) that the two-particle \(\kappa\)-deformed momenta compose in the following way:

\[ \vec{p}^{(1+2)} = \vec{p}^{(1)} e^{p_0^{(2)}/2\kappa} + e^{-p_0^{(1)}/2\kappa} \vec{p}^{(2)}, \quad p_0^{(1+2)} = p_0^{(1)} + p_0^{(2)}. \tag{21} \]

The formulae \(21\) are derived from the Hopf-algebraic action of the symmetry generators \(\hat{g}\) on the products of representation modules \(\hat{a}(\vec{p})\)

\[ \hat{g}_A \triangleright (\hat{a}(\vec{p}) a(\vec{q})) = \left( \hat{g}_A^{(1)} \triangleright a(\vec{p}) \right) \left( \hat{g}_A^{(2)} \triangleright a(\vec{q}) \right), \tag{22} \]

where \(\Delta(\hat{g}_A) = \hat{g}_A^{(1)} \otimes \hat{g}_A^{(2)}\) (we put \(\hat{g}_A \equiv \hat{P}_\mu\), and \(\hat{P}_\mu \triangleright a(\vec{p}) = p_\mu a(\vec{p})(p_\mu = (\vec{p}, p_0))\).
We point out here that by introduction of $\kappa$-deformed oscillators we shall obtain their $\kappa$-deformed algebra covariant under the noncommutative translations, generated by $\hat{P}_\mu$.

We shall consider here three ways of writing consistent $\kappa$-deformation of oscillator algebra (3)-(4).

1. By introducing new multiplication in the standard oscillator algebra (3)-(4) [20], [6], [21].

2. By $\kappa$-deforming the flip operator $\tau_0: \tau_0[a(\vec{p})a(\vec{q})] = a(\vec{q})a(\vec{p})$ [22]-[25], [21] in a way consistent with the addition law (21).

3. By replacing the composition of fourmomenta by suitable group addition law [26], [27] in consistency with the ideas of group field theory [28], [29].

Our main aim here is to discuss the extension of these three sets of binary relations to multilinear product of oscillators and show its consistency with the axioms of braided algebras [30], [31]. Firstly in Sect. 2, we shall recall these axioms, in particular the hexagon relation which is satisfied by triple braided products. We shall use only the hexagon relation with coassociator $\Phi = 1$, what leads to simpler version of braided framework. In Sect. 3 we shall show explicitly how looks the corresponding triple products for three mentioned above choices of $\kappa$-deformed algebras.

It should be mentioned that we shall not investigate here the problem of covariance under full $\kappa$-deformed Poincare symmetries (see however Sect. 4). It has been argued [32], [33] that the requirement of covariance under the $\kappa$-boosts leads necessarily to nontrivial choice of $\Phi$.

\section{Braid factors and hexagon relations.}

In standard field theory in D=4 there are two statistics, bosonic and fermionic, corresponding to the following two choices of tensor products

$$\Psi(a \otimes b) \equiv b \otimes a = \pm a \otimes b.$$  \hfill (23)

For general braided algebra $A$ there exists an isomorphism $\Psi$ which in general case does not satisfy the involutive condition $\Psi \circ \Psi = 1$. In such a case one can introduce second different braiding functor $\Psi^{-1}$which leads in alternative way to the transposed product $b \otimes a$. The restrictive choice provided if $\Psi$ is an involution

$$\Psi^{-1} = \Psi \quad \implies \quad \Psi \circ \Psi = 1,$$  \hfill (24)

leads to simpler symmetric braided category. For tensor products $A^{\otimes n}$ one can define the braid functor $\Psi_{(n,m)}$ transposing the tensor products of algebra $A$

$$\Psi_{(n,m)}[(a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_m)] = (b_1 \otimes \ldots \otimes b_m) \otimes (a_1 \otimes \ldots \otimes a_n),$$  \hfill (25)

and $\Psi_{(1,1)} \equiv \Psi$.

The triple products of general braided algebra can be transposed in two equivalent ways. The transposition in general case require the coassociator mapping $\Phi \in A \otimes A \otimes A$

$$\Phi^{-1}(a \otimes (b \otimes c)) \equiv (a \otimes b) \otimes c,$$  \hfill (26)
which should satisfy the pentagon relation \[31\]. The binary braiding factor \(\Psi\) and coassociator \(\Phi\) permits to relate the cyclic transposition

\[
a \otimes (b \otimes c) \rightarrow ((c \otimes a) \otimes b),
\]

in accordance with the hexagon diagram \[31\].

\[
\begin{array}{c}
a \otimes (b \otimes c) \\
\downarrow id \otimes \Psi \\
(a \otimes (c \otimes b)) \\
\downarrow \Phi^{-1} \\
(b \otimes (a \otimes c)) \\
\downarrow \Psi \otimes id \\
((c \otimes a) \otimes b) \\
\downarrow \Phi^{-1} \\
(b \otimes (c \otimes a)) \\
\end{array}
\]

Fig. 1. Hexagon condition

If \(\Phi \equiv 1\) in the triple product we need not to use the subbrackets

\[
a \otimes (b \otimes c) \equiv (a \otimes b) \otimes c = a \otimes b \otimes c, \tag{28}\]

and two hexagon relations degenerate into one triangle diagram

\[
\begin{array}{c}
a \otimes b \otimes c \\
\downarrow 1 \otimes \Psi \\
(a \otimes c \otimes b) \\
\downarrow \Psi \otimes 1 \\
c \otimes a \otimes b \\
\end{array}
\]

Fig. 2. Triangle diagram

i.e. we see that

\[
\Psi_{(2,1)} \equiv (\Psi \otimes 1) \circ (1 \otimes \Psi). \tag{29}\]

In general case one can consider the braided structure of two different braided algebras \(A, B\) provided we known the braided factor \(\tilde{\Psi}\) exchanging the elements \(a \in A\) and \(b \in B\).

In our discussion of \(\kappa\)-deformed oscillators we shall consider the deformations of algebra \[31\] in the category of associative braided algebras. Following \[29\], the braided factors \(\Psi_{(1,2)}, \Psi_{(2,1)}\) are the following products of binary braided factors

\[
\Psi_{(A,B;C)} = \Psi_{(A,B)} \Psi_{(A,C)}, \quad \Psi_{(A;B,C)} = \Psi_{(A,C)} \Psi_{(B,C)}. \tag{30}\]

It should be stressed that the quantum groups described by quasi-triangular Hopf algebras define braided algebras with uniquely deformed braid factor \(\Psi_H\) \[30\], \[31\]. The quasi-triangular Hopf algebra \((H, \Delta, e, S, R)\) is a Hopf algebra with universal \(R\)-matrix determining quantum quasi-triangular structure relations \((\Delta \equiv \Delta^{(1)} \otimes \Delta^{(2)}; \Delta^{op} \equiv \Delta^{(2)} \otimes \Delta^{(1)})\)

\[
(1 \otimes \Delta)R = R_{12}R_{13}, \quad (\Delta^{op} \otimes 1)R = R_{23}R_{13}, \tag{31}\]
where $R$ is defined by the relation

$$\Delta^p = R \Delta R^{-1},$$

(32)

and $R_{12} = R \otimes 1$ etc. It appears that the category of $H$-representations $C = \text{Rep}(H)$ is a braided tensorial category with the braid factor

$$\Psi_H(x \otimes y) = \tau(R \triangleright (x \otimes y)) = (R^{(1)} \triangleright y)(R^{(2)} \triangleright x),$$

(33)

where $x, y \in \text{Rep}(H)$ and $R = R^{(1)} \otimes R^{(2)}$. The braiding is covariant under the action of the Hopf algebra generators $h \in H$, i.e. one can show that

$$h \triangleright \Psi_H(x \otimes y) = \Psi_H[h \triangleright (x \otimes y)],$$

(34)

in virtue of the relations (32). We see therefore that the covariant representations of Hopf algebra $H$ ($H$-modules) are described by the category of braided monoidal category of modules with braid factor $\Psi_H$; usually such a module can be extended to a braided algebra.

From these general considerations follows immediately that for any twisted quantum group $H_F$ it is possible to introduce the covariant braided algebra of $H_F$-modules. The twist factor $F \in A \otimes A$ determines the universal $R$-matrix by the formula

$$R = F^T F^{-1},$$

(35)

and from the relations (31)-(35) we obtain the explicit formula for the twist factor.

Unfortunately, in the case of $\kappa$-deformation of relativistic symmetries the $\kappa$-deformed Poincare-Hopf algebra cannot be obtained by twisting procedure. At present there were proposed however several models of $\kappa$-deformed oscillators, without the knowledge of universal $R$-matrix for $\kappa$-Poincare algebra. Our aim here is to show how these various models are incorporated into the framework of braided algebras. In our last Section 4 following 

we shall discuss briefly how one can look for the $\kappa$-deformation of oscillators algebra (3)-(4), which is as well covariant under the $\kappa$-Poincare transformations.

3 Three examples of $\kappa$-deformed braided oscillators

algebras

In ref. \cite{20} it was firstly observed that the consistency of $\kappa$-deformed oscillator algebra with $\kappa$-deformed four-momentum conservation law leads after the exchange of deformed oscillators to the modification of the fourmomentum dependence. In such momentum-dependent statistics, describing e.g. a pair of $\kappa$-deformed particles, the behavior of one particle depends on the energy of the second one. One can say that $\kappa$-deformation introduces some geometric interparticle interaction modifying standard factorization properties of free particle states.

In this Section we shall provide three versions of $\kappa$-deformed oscillator algebra.
Let us write down the general momentum-dependent $\kappa$-deformation of the commuting creation oscillators as the following change of classical exchange relations (see (3))

$$a_\kappa(\vec{P}, P_0)a_\kappa(\vec{Q}, Q_0) = a_\kappa(\vec{R}, R_0)a_\kappa(\vec{S}, S_0),$$

(36)

where

$$\vec{P} = \vec{P}(\vec{p}, \vec{q}), \quad \vec{Q} = \vec{Q}(\vec{p}, \vec{q}), \quad \vec{R} = \vec{R}(\vec{p}, \vec{q}), \quad \vec{S} = \vec{S}(\vec{p}, \vec{q}),$$

(37)

$$P_0 = \omega_\kappa^1(\vec{p}, \vec{q}), \quad Q_0 = \omega_\kappa^2(\vec{p}, \vec{q}), \quad R_0 = \bar{\omega}_\kappa^1(\vec{p}, \vec{q}), \quad S_0 = \bar{\omega}_\kappa^2(\vec{p}, \vec{q}).$$

(38)

We explicitly introduced for every oscillator the $\kappa$-deformed energy-momentum dispersion relations satisfying the conditions

$$\lim_{\kappa \to \infty} \omega_\kappa^1(\vec{p}, \vec{q}) = \lim_{\kappa \to \infty} \bar{\omega}_\kappa^1(\vec{p}, \vec{q}) = \omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2},$$

(39)

$$\lim_{\kappa \to \infty} \omega_\kappa^2(\vec{p}, \vec{q}) = \lim_{\kappa \to \infty} \bar{\omega}_\kappa^2(\vec{p}, \vec{q}) = \omega(\vec{q}) = \sqrt{\vec{q}^2 + m^2}.$$ 

(40)

Besides we have

$$\lim_{\kappa \to \infty} \vec{P}(\vec{p}, \vec{q}) = \lim_{\kappa \to \infty} \vec{S}(\vec{p}, \vec{q}) = \vec{p},$$

(41)

$$\lim_{\kappa \to \infty} \vec{Q}(\vec{p}, \vec{q}) = \lim_{\kappa \to \infty} \vec{R}(\vec{p}, \vec{q}) = \vec{q}.$$ 

(42)

We assume that $\vec{P}_\mu > a(p) = p_\mu a(p)$; from the consistency of (36) with the $\kappa$-deformed addition law for the fourmomenta (see (21), (22)) one gets the following constraints for the functions (37)-(38)

$$\vec{P} e^{\omega_\kappa^1/2\kappa} + \vec{Q} e^{-\omega_\kappa^1/2\kappa} = \vec{R} e^{\bar{\omega}_\kappa^1/2\kappa} + \vec{S} e^{-\bar{\omega}_\kappa^1/2\kappa},$$

(43)

and

$$P_0 + Q_0 = R_0 + S_0, \quad \iff \quad \omega_\kappa^1 + \omega_\kappa^2 = \bar{\omega}_\kappa^1 + \bar{\omega}_\kappa^2.$$ 

(44)

We shall now consider three classes of solutions of the eq. (43)-(44) determining three types of $\kappa$-deformed oscillators.

1. **The $\kappa$-deformation of oscillator algebra (3)-(4)** obtained by the modification of multiplication law [20, 21].

We choose

$$\vec{P}(\vec{p}, \vec{q}) = \vec{p} e^{-\omega_\kappa^2/2\kappa}, \quad \vec{Q}(\vec{p}, \vec{q}) = \vec{q} e^{\omega_\kappa^1/2\kappa},$$

(45)

$$\vec{S}(\vec{p}, \vec{q}) = \vec{p} e^{\bar{\omega}_\kappa^2/2\kappa}, \quad \vec{R}(\vec{p}, \vec{q}) = \vec{q} e^{-\bar{\omega}_\kappa^1/2\kappa}.$$ 

(46)

The relation (43) leads to the identity

$$\vec{p} + \vec{q} = \vec{q} + \vec{p},$$

(47)

i.e. we obtain classical Abelian addition law for the three-momenta.
In order to satisfy the relation (44) we have to put the energies \( \omega^1, \omega^2, \tilde{\omega}^1, \tilde{\omega}^2 \) on the mass shells (20) which are deformed in the following way

\[
C^0_\kappa (\vec{P} e^{\omega^2/2\kappa}, \omega^1) = m^2, \quad C^0_\kappa (\vec{Q} e^{-\omega^1/2\kappa}, \omega^2) = m^2, \quad C^0_\kappa (\vec{R} e^{\tilde{\omega}^1/2\kappa}, \omega^1) = m^2, \quad C^0_\kappa (\vec{R} e^{\tilde{\omega}^2/2\kappa}, \omega^2) = m^2.
\] (48)

The set of equations (48)-(49) provides \( \kappa \)-deformed energy-momentum dispersion relations, satisfying respectively two coupled pairs of nonlinear equations

\[
\omega^1 = \epsilon \omega_\kappa (\vec{P} e^{\omega^2/2\kappa}), \quad \omega^2 = \tilde{\epsilon} \omega_\kappa (\vec{Q} e^{-\omega^1/2\kappa}),
\]

\[
\tilde{\omega}^1 = \eta \omega_\kappa (\vec{S} e^{\tilde{\omega}^2/2\kappa}), \quad \tilde{\omega}^2 = \tilde{\eta} \omega_\kappa (\vec{R} e^{\tilde{\omega}^1/2\kappa}),
\] (50)

where \( \epsilon, \tilde{\epsilon}, \eta, \tilde{\eta} \) = ±1. If we insert in (50)-(51) the values (45)-(46) we obtain in terms of variables \( (\vec{p}, \vec{q}) \) the following expressions (see (20))

\[
\omega^1 = \tilde{\omega}^1 = \omega_\kappa (\vec{p}), \quad \omega^2 = \tilde{\omega}^2 = \omega_\kappa (\vec{q}),
\] (52)

and it is evident that the energy conservation rule (44) is satisfied identically.

The \( \kappa \)-deformed oscillators algebra can be written for the choice (45)-(46) in the form identical to the classical relations (3)-(4)

\[
[a_\kappa(p), a_\kappa(q)] = \delta^{(3)}(\vec{p} - \vec{q}),
\]

\[
[a_\kappa(p), a_\kappa(q)] = 0,
\] (53)

\[
[a_\kappa^\dagger(p), a_\kappa^\dagger(q)] = \delta^{(3)}(\vec{p} - \vec{q}),
\] (54)

where \([A,B]_o = A \circ B - B \circ A\) and the new \( \circ \)-multiplication is defined as follows

\[
a_\kappa(p) \circ a_\kappa(q) = a_\kappa\left(e^{-\omega_\kappa(\vec{P})/2\kappa} p, \omega_\kappa(\vec{P})\right) a_\kappa\left(e^{\omega_\kappa(\vec{Q})/2\kappa} q, \omega_\kappa(\vec{Q})\right),
\]

\[
a_\kappa^\dagger(p) \circ a_\kappa^\dagger(q) = a_\kappa^\dagger\left(e^{\omega_\kappa(\vec{P})/2\kappa} p, \omega_\kappa(\vec{P})\right) a_\kappa^\dagger\left(e^{-\omega_\kappa(\vec{Q})/2\kappa} q, \omega_\kappa(\vec{Q})\right),
\]

\[
a_\kappa(p) \circ a_\kappa(q) = a_\kappa\left(e^{-\omega_\kappa(\vec{P})/2\kappa} p, \omega_\kappa(\vec{P})\right) a_\kappa\left(e^{\omega_\kappa(\vec{Q})/2\kappa} q, \omega_\kappa(\vec{Q})\right),
\]

\[
a_\kappa^\dagger(p) \circ a_\kappa^\dagger(q) = a_\kappa^\dagger\left(e^{\omega_\kappa(\vec{P})/2\kappa} p, \omega_\kappa(\vec{P})\right) a_\kappa^\dagger\left(e^{-\omega_\kappa(\vec{Q})/2\kappa} q, \omega_\kappa(\vec{Q})\right).
\] (55)

Let us consider now the algebra \((A(a); \circ)\) of the creation oscillators with modified \( \circ \)-multiplication. It can be shown that the relation (55) (for simplicity we consider here only creation operators) can be extended to \( n \)-ary products as follows

\[
a_\kappa(p^{(1)}) \circ \cdots \circ a_\kappa(p^{(n)}) = \prod_{k=1}^{n} a_\kappa(\chi^{(k)}_n(p_0^{(1)}, \ldots, p_0^{(n)}) \vec{p}^{(k)}, \vec{P}^{(k)}),
\] (59)

where

\[
\chi^{(k)}_n(p_0^{(1)}, \ldots, p_0^{(n)}) = \exp \frac{1}{2\kappa} \left( \sum_{j=1}^{k-1} p_0^{(j)} - \sum_{j=k+1}^{n} p_0^{(j)} \right).
\] (60)

From (59) one can show that the \( n \)-ary product is symmetric under the change of order of oscillators, i.e. it satisfies the classical bosonic relations

\[
a_\kappa(p_1) \circ \cdots \circ a_\kappa(p_i) \circ \cdots \circ a_\kappa(p_j) \circ \cdots \circ a_\kappa(p_n)
\]

\[
= a_\kappa(p_1) \circ \cdots \circ a_\kappa(p_j) \circ \cdots \circ a_\kappa(p_i) \circ \cdots \circ a_\kappa(p_n),
\] (61)
and is associative, i.e. $\Phi = 1$.

In particular for $n = 3$ the associative triple product is the following

$$a_\kappa(p) \circ a_\kappa(q) \circ a_\kappa(r) \equiv a_\kappa(p^\omega_{3k}(q) + \omega_3(r)) a_\kappa(q^\omega_{3k}(p) + \omega_3(r)) a_\kappa(r^\omega_{3k}(p) + \omega_3(r)), \omega_3 = 0$$

$$a_\kappa(p^\omega_{3k}(q) + \omega_3(r)) a_\kappa(q^\omega_{3k}(p) + \omega_3(r)) a_\kappa(r^\omega_{3k}(p) + \omega_3(r)), \omega_3 = 0$$

and leads to the set of classical trilinear symmetry relations

$$a_\kappa(p_1) \circ a_\kappa(p_2) \circ a_\kappa(p_3) = a_\kappa(p_1) \circ a_\kappa(p_2) \circ a_\kappa(p_3), \quad \text{(63)}$$

where $(i, j, k)$ is an arbitrary permutation of $(1, 2, 3)$.

The relation (62) can be extended to the algebra $(A(a, a^\dagger); \circ)$ of arbitrary products of creation and annihilation operators [20], [21] with the relations (61) valid as well for annihilation operators $a^\dagger(p)$. Due to the new $\kappa$-deformed multiplication (65-68) which leads to the symmetry relations (61), in the algebra $(A(a, a^\dagger); \circ)$ the braided structure is trivial. We add that using the relations (53-54) one can introduce also the normal products of $\kappa$-deformed field oscillators.

2 The $\kappa$-deformation introduced by nontrivial involutive braiding factor [22], [23].

Other choice of the functions occurring in (36) consistent with the relation (43), is the following

$$\vec{P}(\vec{p}, \vec{q}) = \vec{p}, \qquad \vec{Q}(\vec{p}, \vec{q}) = \vec{q}, \quad \text{(64)}$$

$$\vec{S}(\vec{p}, \vec{q}) = \vec{p} \epsilon^{\omega_2/\kappa}, \qquad \vec{R}(\vec{p}, \vec{q}) = \vec{q} \epsilon^{-\omega_2/\kappa}, \quad \text{(65)}$$

where we shall show below that $\omega_2 = \omega_2^2 = \omega_2^3 = \omega_2^3 = \omega_2^3$ (see (20) and (38)).

In order to obtain the conservation of energy (see (44)) we postulate that the relations (52) are valid. We obtain the following nonsymmetric addition law for the three-momenta (see also (31)), which as follows from (43) gives the same result after the exchange $1 \leftrightarrow 2$ of two particles

$$\vec{p}^{1+2} \equiv \vec{p} + \vec{q} = \vec{p} \epsilon^{\omega_1/\kappa} + \vec{q} \epsilon^{-\omega_1/\kappa} = \vec{p}^{2+1}. \quad \text{(66)}$$

The binary oscillator algebra (36) for the choice (54)-(55) of deformed momentum arguments takes the braided form (further we denote $a(\vec{p}) \equiv a(\vec{p}, p_0 = \omega_1(\vec{p})), a^\dagger(\vec{p}) \equiv a(\vec{p}, p_0 = -\omega_1(\vec{p}))$

$$a^\dagger(\vec{p}) a(\vec{q}) = \delta^3(\vec{p} \epsilon^{\omega_1/2\kappa} - \vec{q} \epsilon^{-\omega_1/2\kappa}) = \delta^3(\vec{p} \epsilon^{\omega_1/2\kappa}, \vec{q} \epsilon^{-\omega_1/2\kappa}) \equiv 0$$

where explicitly

$$\vec{r}[a^\dagger(\vec{p}) a(\vec{q})] = a(\vec{q} \epsilon^{-\omega_1(\vec{p})}) a(\vec{p} \epsilon^{\omega_1(\vec{p})}), \quad \text{(67)}$$

$$\vec{r}[a^\dagger(\vec{p}) a(\vec{q})] = a(\vec{q} \epsilon^{\omega_1(\vec{p})}) a(\vec{p} \epsilon^{-\omega_1(\vec{p})}), \quad \text{(68)}$$
The first relation (67) is the deformation of the relation (4), with the argument of Dirac delta deformed in agreement with (66). It is easy to see that from $\vec{p} - \vec{q} = 0$ does not follow that $\vec{p} - \vec{q} = 0$, as follows from nonsymmetric addition law (66) and replacement $\vec{q} \rightarrow -\vec{q}$. The second relation follows from (66), (64), (65) and the third relation is its complex conjugation. It can be shown that the braid (68) is involutive, i.e. $\tilde{\tau}^2 = 1$, as follows e.g. from the relation

$$\tilde{\tau}[a(\vec{q} e^{-\frac{\tau}{\kappa} \omega_\kappa(\vec{q})})a(\vec{p} e^{\frac{\tau}{\kappa} \omega_\kappa(\vec{p})})] = a(\vec{p})a(\vec{q}).$$

In order to obtain the relation (44) the energy-momentum dispersion relations expressing the energies $\omega_\kappa^1, \omega_\kappa^2, \tilde{\omega}_\kappa^1, \tilde{\omega}_\kappa^2$ as the functions of three-momenta $\vec{P}, \vec{Q}, \vec{R}, \vec{S}$ (see (36)) should be postulated as follows

$$C^\kappa_2(\vec{P}, \omega_\kappa^1) = m^2, \quad C^\kappa_2(\vec{Q}, \omega_\kappa^2) = m^2,$$
$$C^\kappa_2(\vec{S} e^{-\omega_\kappa^2/\kappa}, \tilde{\omega}_\kappa^1) = m^2, \quad C^\kappa_2(\vec{R} e^{\omega_\kappa^2/\kappa}, \tilde{\omega}_\kappa^1) = m^2. \quad \text{(70)} \quad \text{(71)}$$

After substitute of (64)-(65) we obtain the confirmation that $\omega_\kappa^1 = \tilde{\omega}_\kappa^1 = \omega_\kappa(\vec{p})$, $\omega_\kappa^2 = \tilde{\omega}_\kappa^2 = \omega_\kappa(\vec{q})$.

In order to extend the relations (67) to arbitrary products of the creation operators we should use the relations (29) for the braid $\Psi \equiv \tilde{\tau}$. For ternary product we obtain for example the following braiding relation

$$a(\vec{p})a(\vec{q})a(\vec{r}) \equiv \tilde{\tau}_{12}[\tilde{\tau}_{23}(a(\vec{p})a(\vec{q})a(\vec{r}))] = a(\vec{r} e^{-\frac{\kappa}{\tau}(\omega_\kappa(\vec{p})+\omega_\kappa(\vec{q}))})a(\vec{p} e^{\frac{\kappa}{\tau}(\omega_\kappa(\vec{p})-\omega_\kappa(\vec{q}))})a(\vec{q} e^{\frac{\kappa}{\tau}(\omega_\kappa(\vec{p})+\omega_\kappa(\vec{q}))}),$$

which is consistent with the hexagonal relation.

In general case, if we permutate the oscillators in $n$-ary product by following the permutation $(1, 2, \ldots, n) \rightarrow (j_1, j_2, \ldots, j_n)$, we should describe it as a product of simple braid factors corresponding to the flips $(1, 2, \ldots, k, k+1, \ldots, n) \rightarrow (1, 2, \ldots, k+1, k, \ldots, n)$. For every such simple flip we should introduce the binary braiding factors $\tilde{\tau}_{kk+1}$ given by the relations (68). In such a case we obtain $n$-ary braiding as expressed by suitable product of braiding factors.

3. Braided $\kappa$-deformed oscillators from $\kappa$-deformed fourmomentum group composition law.

Let us assume that in the relation (36) only one oscillator has modified three-momentum dependence. We postulate

$$\vec{P}(\vec{p}, \vec{q}) = \vec{p}, \quad \vec{Q}(\vec{p}, \vec{q}) = \vec{q} = \vec{R}(\vec{p}, \vec{q}).$$

It appears that the remaining three-momentum $\vec{S}$ can be calculated from the conservation law (43). If we assume further the relations (72) one obtains

$$\vec{S}(\vec{p}, \vec{q}) = [\vec{p} e^{\omega_\kappa(\vec{q})/2\kappa} - 2\vec{q} \sinh\frac{\omega_\kappa(\vec{p})}{2\kappa}] e^{\omega_\kappa(\vec{q})/2\kappa},$$

which can be also obtained by the group manifold approach to the composition of $\kappa$-deformed fourmomenta [26, 27]. We obtain the following relation

$$a(\vec{p})a(\vec{q}) \equiv \tilde{\rho}[a(\vec{p})a(\vec{q})] = a(\vec{q})a(\vec{S}(\vec{p}, \vec{q})).$$
The operator $\hat{\rho}$ describes a nonsymmetric braided factor, which introduces only the $\kappa$-deformed momentum dependence in one permuted oscillator. Further one can show that the braiding (75) is not involutive, i.e. $\hat{\rho}^2 \neq 1$.

The relation (75) is consistent with the nonAbelian three-momentum conservation law (66). If we introduce four functions ($\epsilon, \eta = \pm 1$)

$$\overline{S}^{(\epsilon, \eta)}(\overrightarrow{p}, \overrightarrow{q}) = [\hat{p}e^{\omega_{\kappa}(\overrightarrow{q})/2\kappa} - 2\epsilon \overrightarrow{q} \sinh \frac{\omega_{\kappa}(\overrightarrow{p})}{2\kappa}]e^{\eta\omega_{\kappa}(\overrightarrow{q})/2\kappa},$$

where $\overline{S} = \overline{S}^{(+,+)}$, one can supplement the relation (75)

$$\begin{align*}
a^\dagger(\overrightarrow{p})a^\dagger(\overrightarrow{q}) &\equiv \hat{\rho}[a^\dagger(\overrightarrow{p})a^\dagger(\overrightarrow{q})] = a^\dagger(\overrightarrow{q})a^\dagger(\overline{S}^{(-,-)}(\overrightarrow{p}, \overrightarrow{q})), \\
a^\dagger(\overrightarrow{p})a(\overrightarrow{q}) &\equiv \hat{\rho}[a^\dagger(\overrightarrow{p})a(\overrightarrow{q})] = a^\dagger(\overrightarrow{q})a^\dagger(\overline{S}^{(-,+)}(\overrightarrow{p}, \overrightarrow{q})).
\end{align*}$$

The basic field oscillator algebra relation (44) is $\kappa$-deformed as follows

$$a^\dagger(\overrightarrow{p})a(\overrightarrow{q}) - \hat{\rho}[a^\dagger(\overrightarrow{p})a(\overrightarrow{q})] = \delta^3(\overrightarrow{p}e^{\omega_{\kappa}(\overrightarrow{q})/2\kappa} - \overrightarrow{q}e^{-\omega_{\kappa}(\overrightarrow{p})/2\kappa}) \equiv \delta^3(\overrightarrow{p} - \overrightarrow{q}).$$

The triple product of $\kappa$-deformed oscillators which satisfies the hexagon condition is characterized by the braided relation

$$\begin{align*}
a(\overrightarrow{p})a(\overrightarrow{q})a(\overrightarrow{r}) &\equiv \hat{\rho}_{12}[\hat{\rho}_{23}(a(\overrightarrow{p})a(\overrightarrow{q})a(\overrightarrow{r})] \\
&= a(\overrightarrow{p})a(\overline{S}^{(+,+)}(\overrightarrow{p}, \overrightarrow{r}))a(\overline{S}^{(+,+)}(\overrightarrow{q}, \overrightarrow{r})).
\end{align*}$$

Similarly as in previous example (see (67)-(68), (72)) one gets the braiding relation for $n$-ary product of the $\kappa$-oscillators as product of binary braiding relations (75), (77), (78). In addition contrary to our first two examples of $\kappa$-deformed oscillators, the noncommutative braid $\hat{\rho}$ does not satisfy the relation characterizing the permutation group $(\hat{\rho}_{i+1}\hat{\rho}_{i}\hat{\rho}_{i+1} = \hat{\rho}_{i}\hat{\rho}_{i+1}\hat{\rho}_{i})$. This result is supplemented with the relation $\hat{\rho}^2 \neq 1$. In order to get the consistency with the relation (13), which describes the conservation of energy under braiding, one should again keep the energy values given by formula (72).

## 4 Outlook

The proper choice of the $\kappa$-deformation of field oscillators algebra should be covariant under the action of $\kappa$-Poincare algebra (see 34). In particular if $\kappa$-deformed algebra is described by the following braid relation

$$a(\overrightarrow{p})a(\overrightarrow{q}) = \overline{\tau}_\kappa \triangleright [a(\overrightarrow{p})a(\overrightarrow{q})],$$

it should have the same form in all $\kappa$-Poincare frames, i.e. $(\hat{g}_A \equiv \hat{P}_\mu, \hat{M}_{\mu\nu})$

$$\hat{g}_A \triangleright \{\overline{\tau}_\kappa \triangleright [a(\overrightarrow{p})a(\overrightarrow{q})]\} = \overline{\tau}_\kappa \triangleright \{\hat{g}_A \triangleright [a(\overrightarrow{p})a(\overrightarrow{q})]\}.$$

The relation (13) provides the equality (82) for $\hat{g}_A = \hat{P}_\mu$, i.e. it represents the $\kappa$-deformed translation invariance for the $\kappa$-deformed oscillators algebra. The relation (82) for $\hat{g}_A = \hat{M}_i = \frac{1}{2}\epsilon_{ijk}\hat{M}_{jk}$ is obvious. The explicit formula for $\overline{\tau}_\kappa$ which satisfies (82) for the boost generators $\hat{N}_i = \hat{M}_{i0}$, has been only calculated in lowest orders of $\frac{1}{\kappa}$ [24].
The straightforward way which would provide the $\kappa$-covariant deformation (81) is to construct the universal $R$-matrix for $\kappa$-deformed Poincare algebra. In present literature such construction has been achieved only in the category of triangular quasi-Hopf algebras $H(A,\cdot,\Delta,S,R,\Phi)$ considered in detail by Drinfeld [34], where the three-linear map $\Phi$ describes the coassociator modifying the relations (31)

$$(1 \otimes \Delta)R = \Phi^{-1}_{231}R_{13}\Phi_{213}R^{-1}_{123}, \quad (\Delta \otimes 1)R = \Phi^{-1}_{321}R_{13}\Phi^{-1}_{132}R_{23}\Phi_{123}. \quad (83)$$

In [33] there was calculated the coassociator $\Phi$ up to $\frac{1}{\kappa}$ terms, and further shown that the universal $R$-matrix $R$ is triangular, i.e. satisfies the condition $R_{21} = R^{-1}$. Because all triangular algebras can be described by twist, one can conclude that the $\kappa$-Poincare algebra is a triangular quasi-Hopf algebra which can be obtained by twist from undeformed Poincare algebra, but such a twist does not satisfy the two-cocycle condition [33], [34], [35].

In order to introduce the complete covariant $\kappa$-deformed oscillator algebra we should define the binary algebra relation by introducing braided factor expressed by the universal $R$-matrix, and incorporate the coassociator $\Phi$ in the construction of algebraic multi-linear relations. Binary twist factor and coassociator characterize the complete $\kappa$-deformed oscillator algebra [33]. For achieving such a goal in explicit form it is necessary the knowledge of universal $R$-matrix (see (33)) and coassociator to arbitrary order in powers of $\frac{1}{\kappa}$.

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