Magnetic monopole solutions with a massive dilaton

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September 3, 2018

Abstract

Static, spherically symmetric monopole solutions of a spontaneously broken SU(2) gauge theory coupled to a massive dilaton field are studied in detail in function of the dilaton coupling strength and of the dilaton mass.

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In this paper we present some results on finite energy solutions in an SU(2) Yang-Mills (YM) and YM-Higgs (YMH) theory (with the Higgs field in the adjoint representation) coupled to a massive dilaton field. The present work is an extension of previous investigations with a massless dilaton field \[1,2,3\]. In Refs. \[1,2,3\] it has been found that the SU(2) dilaton-YM (DYM) theory admits static, finite energy (‘particle-like’) solutions (absent in the pure YM case). They are in close analogy to the particle-like solutions found by Bartnik and McKinnon in an Einstein-YM (EYM) system \[3\]. In Ref. \[3\] it has been shown that in the SU(2) dilaton-YMH (DYMH) theory in addition to the analogue of the (nonabelian) ’t Hooft-Polyakov monopole \[3\] there is a discrete family of finite energy solutions, which can be interpreted as radial excitations of the monopole. The mass scale of the radial excitations is inversely proportional to the dilaton coupling.

In Ref. \[3\] it has also been found that there is a maximal dilaton coupling, \(\alpha_{\text{max}}\), above which only an abelian solution exists. Although the abelian solution (which has a simple analytical form) is singular at the origin, its total energy is finite. The numerical results of Ref. \[3\] indicate that the nonabelian monopole merges with the abelian one for a critical value of the dilaton coupling, \(\alpha\). The critical dilaton coupling, \(\alpha_{\text{crit}}\), depends on the value of the Higgs self-coupling strength, \(\beta\). As found in Ref. \[3\] when \(\beta\) is sufficiently small the largest possible value of the dilaton coupling, \(\alpha_{\text{max}}\), for which a nonabelian solution still exists is different from the critical value, \(\alpha_{\text{crit}}\), i.e. we have \(\alpha_{\text{max}} > \alpha_{\text{crit}}\). This implies that if \(\alpha \in [\alpha_{\text{crit}}, \alpha_{\text{max}}]\) a bifurcation takes place and there are two different monopole solutions for the same value of \(\alpha\). All these findings are again in close analogy with the results of Refs. \[3,4\] found in the EYM case.

A massless dilaton, which necessarily appears in string theories \[8,9,10\], violates the equivalence principle, and therefore its (dimensionful) coupling strength is expected to be extremely weak, of the order of \(1/M_{\text{Pl}}\) where \(M_{\text{Pl}}\) is the Planck mass. It is very natural to assume, however, that the dilaton gets a mass (possibly related to supersymmetry breaking) then, however, there is no strong experimental constraint on the dilaton coupling. Therefore it might be of some importance to study the effect of mass of the dilaton in DYMH theories. Regular and black hole solutions in EYM theories coupled to a massive dilaton and axion have investigated in Ref. \[11\].

We have carried out a rather thorough numerical investigation and we have found that even if the dilaton is massive, most phenomena associated with the presence of the dilaton found in the massless case persist (the existence of radial excitations, \(\alpha_{\text{max}} \neq \alpha_{\text{crit}}\)). Also we have good numerical evidence that \(\alpha_{\text{crit}}\) (where the solution become abelian) is independent of the dilaton mass. We have also investigated in detail the limit \(m \to \infty\) of the mass of the dilaton and we were able to show its expected decoupling. Our numerical results show that \(\alpha_{\text{max}}\) grows as \(m\) increases, consistently with the expectation that for \(m \to \infty\) \(\alpha_{\text{max}}(m) \to \infty\).

We have also found that not just a single maximal dilaton coupling, \(\alpha_{\text{max}}\) exists, but there are several local extrema of \(\alpha\) too, e.g. \(\alpha_{\text{max}1} > \alpha_{\text{max}2} > \alpha_{\text{crit}} > \alpha_{\text{min}1}\). These bifurcation points are more clearly distinguishable as the dilaton mass becomes larger. This implies that there are certain values of the dilaton coupling, \(\alpha\), where three or even more different monopole solutions exists for the same value of \(\alpha\) with different masses.

By using the minimal spherically symmetric and static ansatz

\[
A_0^a = 0, \quad A_i^a = \epsilon_{aik} \frac{1 - W(r)}{r} x^k, \quad \Phi^a = H(r) \frac{x^a}{r}, \quad \varphi = \varphi(r),
\]

where \(A_0^a\) is the gauge potential, \(\Phi^a\) is the Higgs triplet and \(\varphi\) is the dilaton field, the reduced energy functional reads:

\[
E(\alpha, \beta^2, m^2) = \int_0^\infty dr \left[ \frac{1}{2} r^2 \varphi'^2 + \frac{1}{2} m^2 r^2 \varphi^2 + e^{2\alpha \varphi} \left( W^2 + \frac{(W^2 - 1)^2}{2r^2} \right) \right. \\
\left. + \frac{H'^2 r^2}{2} + H^2 W^2 + \frac{\beta^2}{8} e^{-2\alpha \varphi} r^2 (H^2 - v^2)^2 \right].
\]
Varying the energy functional $\mathcal{L}$ we obtain the field equations:

$$
(r^2 \varphi')' = 2ae^{2\alpha\varphi} \left(W^2 + \frac{(W^2 - 1)^2}{2r^2}\right) - \frac{\alpha\beta^2}{4}e^{-2\alpha\varphi}r^2(H^2 - v^2)^2 + m^2r^2\varphi \tag{3a}
$$

$$
\left(e^{2\alpha\varphi} W'\right)' = W \left[e^{2\alpha\varphi} \frac{W^2 - 1}{r^2} + H^2\right] \tag{3b}
$$

$$
(r^2 H')' = 2H \left(W^2 + \frac{\beta^2}{4}e^{-2\alpha\varphi}r^2(H^2 - v^2)\right). \tag{3c}
$$

By introducing the following variables:

$$
R = re^{-\alpha\varphi} := r\phi, \quad \nu := \alpha\varphi, \quad \dot{\tau} = \frac{dr}{d\tau} = \frac{1}{\phi}, \tag{4}
$$

the field equations (3) can be written as:

$$
\dot{R} = 1 - R\dot{\nu}, \tag{5a}
$$

$$
\frac{1}{\alpha^2} \left((R^2\dot{\nu})' + (R^2\dot{\nu})\right) = 2 \left(W^2 + \frac{(W^2 - 1)^2}{2R^2}\right) - \frac{\beta^2}{4}R^2(H^2 - v^2)^2 + \frac{m^2}{\alpha^2}R^2\nu e^{2\nu}, \tag{5b}
$$

$$
\ddot{W} + W\dot{\nu} = W \left(\frac{W^2 - 1}{R^2} + H^2\right), \tag{5c}
$$

$$
(R^2\ddot{H})' + R^2\dot{H}\dot{\nu} = H \left(2W^2 + \frac{\beta^2}{2}R^2(H^2 - v^2)\right). \tag{5d}
$$

The field equations (5) written in this form are distinguished by the fact that for $m = 0$ they contain only variables which are invariant under the dilatational symmetry

$$
r \longrightarrow e^{\alpha\varphi}r, \quad \varphi \longrightarrow \varphi + c, \tag{6}
$$

which is only broken by the presence of the mass term.

Solutions regular at $r = 0$ (or equivalently at $\tau = 0$) must satisfy the following boundary conditions:

$$
H = a\tau + O(\tau^3), \quad W = 1 - b\tau^2 + O(\tau^4),
$$

$$
R = \tau - \frac{\nu_1}{3}\tau^3 + O(\tau^5), \quad \nu = \nu_0 + \frac{\nu_1}{2}\tau^2 + O(\tau^4), \tag{7}
$$

with \( \nu_1 = \alpha^2 \left(\frac{4b^2 - \beta^2v^4}{12}\right) + \frac{m^2}{3}\nu_0e^{2\nu_0} \),

i.e. solutions with a regular origin depend on three free parameters \((a, b, \nu_0)\). The asymptotic behaviour \((r \to \infty)\) of regular solutions is almost identical to those of the \(m = 0\) case (Eqs. (10) in Ref. [4]), i.e. for \(r \to \infty\) the b.c. corresponding to regular solutions are

$$
\varphi(r) = -\frac{\alpha}{m^2r^4} + O\left(\frac{1}{r^6}\right), \quad m \neq 0, \tag{8a}
$$

$$
W(r) = Be^{-r} \left(1 + O\left(\frac{1}{r}\right)\right), \tag{8b}
$$

$$
H(r) = 1 - \frac{C}{r}e^{-\beta r} \left(1 + O\left(\frac{1}{r}\right)\right) \quad \text{for } \beta < 2 \tag{8c}
$$

$$
H(r) = 1 - \frac{2B^2}{(\beta^2 - 4)r^2}e^{-2\beta r} \left(1 + O\left(\frac{1}{r}\right)\right) \quad \text{for } \beta > 2, \tag{8d}
$$

where \((B, C)\) are free parameters and we have set \(v = 1\). We note that \(\varphi(r \to \infty)\) does not tend to zero as \(e^{-mr}\), which is the expected asymptotic behaviour of a massive field. This
‘pathological’ behaviour and the missing free parameter are due to the fact that in Eq. (3a) as \( W \to 0 \) the \( O(1/r^2) \) term dominates. (The asymptotic behaviour of the Higgs field for \( \beta > 2 \) in Eq. (8d) is determined by the nonlinear terms in a similar way.)

With the use of the field eqs. (3) and the boundary conditions (8) one can easily show that the energy of the solution, \( E(\alpha, \beta^2, m^2) \), is a monotonically increasing function of \( \beta^2 \) and \( m^2 \) and a monotonically decreasing function of \( \alpha \).

In the DYM limit of the theory, i.e. \( (v \to 0 \) and \( H \to 0) \) there are no monopoles, but our numerical results clearly indicate the existence of globally regular solutions with zero magnetic charge. The existence of these solutions is not surprising, as they are simply the generalizations of those found (also numerically) in the massless case \( (m = 0) \) in Refs. [1, 2]. Our results for the \( m = 0 \) case are in complete agreement with those of Refs. [1, 2].

We have found only a slight dependence on the dilaton mass. Although our numerical solutions exhibit clearly that the dilaton tends to zero more and more as \( m \) increases, \( W \) varies very little for the considered range of values of \( m \) \( (0 \leq m \leq 12) \) as it can be seen on Fig. 1.

**Figure 1:** \( m^2 \) dependence of the first two excited solution in the DYM limit.

eywhere forcing \( W \equiv 1 \). This limit is, however, singular and we expect that for finite values of \( m \) (no matter how large) an infinity family of regular solutions still exists.

Before considering nonabelian monopole solutions, we discuss first the abelian case, i.e. \( \varphi = \varphi_{ab}, W \equiv 0, H \equiv v \) (a Dirac monopole coupled to a massive dilaton field). There is a finite energy solution, with a (logarithmic) singularity at the origin. In the massless case \( \varphi_{ab} = \varphi^{(0)} \) and it is given by

\[
\varphi^{(0)} = -\frac{1}{\alpha} \ln \left| 1 + \frac{\alpha}{r} \right|,
\]

while for \( m \neq 0 \) its analytic form is not known. Note that for abelian solutions the only relevant parameter is \( m\alpha \) (for \( m \neq 0, \alpha \neq 0 \)). It seems very likely that the abelian solution is unique for any value of \( m \).

Near the origin the power series expansion of \( \varphi_{ab} \) can be written as

\[
\alpha \varphi_{ab} = (1 + \frac{m^2}{4} r^2) \ln \frac{r}{\alpha} + ar + (\frac{1}{2} a^2 - \frac{5}{16} m^2) r^2 + \ldots
\]

where \( a \) is a free parameter. So clearly if \( r \ll 1/m \) the mass term in (10) is negligible and the dilaton is well approximated by the regular solution with \( m = 0 \), see Fig. 2.

Next we investigate the nonabelian solutions. In the asymptotic region, \( r \to \infty \), the behaviour of the nonabelian solutions is determined by linearizing the field Eqs. (3) around the
critical point \( \{ W = 0, H = v, \varphi = 0 \} \) for \( \tau \to \infty \), yielding Eqs. (8). As in this case \( \varphi \to 0 \) \( R \approx r, r \approx \tau \), \( W \) and \( (H - 1)r \) are exponential functions of \( \tau \).

We solved the field equations (5) numerically by a combination of a Runge-Kutta (RK) method (4th order, adaptive stepsize) integrating out from \( \tau \) \( \tau \) \( \tau \) \( \tau \) \( \tau \) \( \tau \) \( \tau \) iteration the pertinent system of integral equations from \( \tau = \infty \) to \( \tau = \tau_0 \). In order to desingularize the field equations (5) at the origin resp. at \( \tau = \infty \), we have introduced the following set of variables: \( \{ (W - 1)/R, \dot{W}, H, (rH)' \}, \varphi, \varphi, R \} \) resp. \( \{ W = W \pm \dot{W}, h_\pm = \beta h \pm H_+, \varphi_\pm = mR\varphi \pm Y, R - \tau \} \) where \( h = (H - 1)R, H_+ = (rH)' - 1, Y = \varphi + R\dot{\varphi} \). (For \( \beta = 0 \) we used \( R^2 \dot{H} \) instead of \( h_\pm \), and for \( m = 0 \) we replaced \( \varphi_\pm \) by \( R^2 \dot{\varphi} \)). The corresponding eqs. for \( \beta \neq 0 \) and \( m \neq 0 \) are the following:

\[
W_\pm = \pm \int_{1/\tau_\pm}^{1/\tau} \left[ W \left( \frac{W^2 - 1}{R^2} + \frac{h}{R}(h/R + 2) \right) - \alpha \dot{W}\varphi \right] t^2 e^{\pm(\tau - 1/t)} dt \tag{11a}
\]

\[
h_\pm = \pm \int_{1/\tau_\pm}^{1/\tau} \left[ \frac{2W^2H}{R} + \frac{\beta^2h^2}{2R}(h/R + 3) \mp \alpha \beta h\dot{\varphi} \right] t^2 e^{\pm\beta(\tau - 1/t)} dt \tag{11b}
\]

\[
\varphi_\pm = \pm \int_{1/\tau_\pm}^{1/\tau} \left[ \frac{\alpha}{R} \left( \frac{2W^2}{R} + \frac{(W^2 - 1)^2}{R^2} - \frac{\beta^2h^2}{4}(h/R + 2)^2 \right) \right. \\
\left. + m^2K(e^{2\alpha\varphi} - 1) \mp mK\alpha\dot{\varphi} \right] t^2 e^{\pm m(\tau - 1/t)} dt + C_{\varphi_\pm} e^{\pm m(\tau - \tau_\pm)} \tag{11c}
\]

\[
R - \tau = \int_{1/\tau}^{1/\tau} \frac{\alpha R\dot{\varphi}}{t^2} dt + C_{R - \tau} \tag{11d}
\]

where \( \tau_+ = \infty, \tau_- = \tau_0 \). The \( C_{ks} \)s are arbitrary constants determined by the boundary conditions (8) ensuring regularity at \( \tau = 0 \) and at \( \tau = \infty \). In order to suppress the divergent modes in
\{W_+, h_+, \varphi_+\} at \tau = \infty one has to choose \(C_{W_+}, C_{h_+}, C_{\varphi_+} = 0\). The remaining four constants \(\{C_{W_-}, C_{h_-}, C_{\varphi_-}, C_{R_{-\tau}}\}\) are determined by \(C_{W_-} = W_-(\tau_0), C_{h_-} = h_-(\tau_0), C_{\varphi_-} = \varphi_-(\tau_0), C_{R_{-\tau}} = R(\tau_0) - \tau_0\), where \(\{W_-(\tau_0), h_-(\tau_0), \varphi_-(\tau_0), R(\tau_0) - \tau_0\}\) are given by the RK procedure, guaranteeing regularity of the functions at \(\tau = 0\). The integral eqs. (11) are then solved by iteration yielding a regular solution for \(\tau \in [\tau_0, \infty)\). Then the shooting parameters for the RK are adjusted so that the resulting values \(\{W_+(\tau_0), h_+(\tau_0), \varphi_+(\tau_0)\}\) match those calculated from the solution of the integral equations.

Let us now present our numerical results in some detail. First of all the fundamental non-abelian monopole solutions, which tend to the ‘t Hooft-Polyakov monopole as \(\alpha \to 0\) exists only up to a maximal value of the dilaton coupling strength \(\alpha, \alpha_{\max}(\beta, m)\). For \(\alpha > \alpha_{\max}\) only the abelian solution \(\{\varphi \equiv \varphi_{ab}, W \equiv 0, H \equiv 1\}\) seems to exist. The \(\alpha\) dependence of the solution is, however, quite complex as the nonabelian solution does not necessarily cease to exist for \(\alpha = \alpha_{\max}\). In fact that value of \(\alpha\) (which value we call critical, \(\alpha_{\text{crit}}\)) for which the nonabelian solution merges with the abelian one, does not necessarily coincide with \(\alpha_{\max}\). As \(\alpha\) approaches its critical value, \(\alpha_{\text{crit}}\), the dilaton field becomes divergent at the origin, whereas \(W(r) \to 0\) and \(H(r) \to 1\). This behaviour is shown on Fig. 3 where the fundamental monopole is plotted as \(\alpha\) varies from 0 to \(\alpha_{\text{crit}}\). Note that although the solution becomes singular its total energy remains finite.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{The \(\alpha\) dependence of the fundamental monopole solution for \(m = 1, \beta^2 = 0.1\). Above the exterior, below the interior solution is depicted.}
\end{figure}

The critical value of \(\alpha\) depends strongly on \(\beta\), but it does not seem to depend on the mass
of the dilaton, $m$. This latter phenomenon can be understood as follows. By performing the transformation \( \tilde{r} = r e^{-\alpha \varphi} \) we define $\varphi_{\text{sh}}(r) = \varphi(r) - \varphi_0$ where $\varphi_0 := \varphi(0)$ guaranteeing $\varphi_{\text{sh}}(0) = 0$ and then the field eq. (12) takes the form:

$$
(r^2 \varphi_{\text{sh}}')' = 2\alpha e^{2\alpha \varphi_{\text{sh}}} \left( W' + \frac{(W^2 - 1)^2}{2\tilde{r}^2} \right) - \frac{\alpha^2}{4} e^{-2\alpha \varphi_{\text{sh}}} \tilde{r}^2 (H^2 - 1)^2 + m^2 e^{2\alpha \varphi_{\text{sh}}} \tilde{r}^2 (\varphi_{\text{sh}} + \varphi_0),
$$

where $\tilde{r} = r e^{-\alpha \varphi_0}$ and the prime denotes now derivation with respect to $\tilde{r}$. Note that $\varphi_{\text{sh}}(\tilde{r}) \to -\varphi_0$ as $\tilde{r} \to \infty$. The remaining field eqs. (13b,13c) are form invariant.

Using the shifted field, $\varphi_{\text{sh}}$, in order to obtain a globally regular solution with some fixed asymptotic value of $\varphi_{\text{sh}}, -\varphi_0$, one also has to vary $\alpha$ in addition to the shooting parameters $a$, $b$. (One still has to suppress three divergent modes). Then the critical value of $\alpha$ is defined by taking the limit $\varphi_0 \to -\infty$. Then in Eq. (12) for any, fixed value of $\tilde{r}$ the terms proportional to $m^2$ tend to zero as $\varphi_0 \to -\infty$. From this it follows that $\alpha_{\text{crit}}$ is determined by the massless equations. Note that it is essential to keep $\tilde{r}$ fixed when taking the $\varphi_0 \to -\infty$ limit. In this limit one obtains nontrivial $\{W(\tilde{r}), H(\tilde{r})\}$ which are globally regular and a logarithmically divergent $\tilde{\varphi}(\tilde{r})$ as $\tilde{r} \to \infty$. Since for any fixed value of $\tilde{r}$, $r \to 0$ as $\varphi_0 \to -\infty$ this solution will be referred to as the ‘interior’ solution. It corresponds to a singular solution of Eqs. (13b,13c), $W(r) \equiv 0$, $H(r) \equiv 1$ for $r \neq 0$ with a logarithmically diverging dilaton field at $r = 0$ which we call ‘exterior’ solution. Note that while the interior solution is nonabelian, the exterior one is, however, abelian.

| $\alpha$ | $a$ | $b$ | $-\varphi_0$ | $\alpha E$ |
|---|---|---|---|---|
| 0 | 0.4821127 | 0.2381420 | 0.0000000 | 0.0000000 |
| 1 | 0.4481335 | 0.2370648 | 0.2787202 | 1.0605540 |
| 1.35 | 0.4021605 | 0.2368443 | 0.6065505 | 1.3627414 |
| 1.4 | 0.3874162 | 0.2378367 | 0.7807487 | 1.3967137 |
| 1.419 | 0.3750471 | 0.2400726 | 1.0195001 | 1.4078993 |
| 1.419 | 0.3719369 | 0.2411243 | 1.1136470 | 1.4078993 |
| 1.4 | 0.3650065 | 0.2464302 | 1.4078993 | 1.3999542 |
| 1.35 | 0.3662630 | 0.2550531 | 3.8698184 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 9.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 8.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 7.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 6.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 5.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 4.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 3.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 2.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 1.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 0.3420476 | 1.3849301 |
| 1.3495957 | 0.3662688 | 0.2546880 | 0.0000000 | 1.3849301 |

With our numerical procedure we found that in fact there is a local minimum of $\alpha$, $\alpha_{\text{min}}$,
and that even a second local maximum of $\alpha$ exists. We remark that some values (for $m = 0$) differ from our previous results as given in Table 2 of ref. [4]. The present data is more precise due to the use of the integral equations around infinity.

The numerical data show that the local minimum of $\alpha$ exists for a wide range of $\beta$, especially in the case of large dilaton masses. This local minimum exists even in the $m = 0$ case, but for larger $m$ the difference between the different extrema of $\alpha$ and of $\alpha_{\text{crit}}$ is more visible. This is the reason why $\alpha_{\text{min}}$ has not been found in Ref. [4], where the present matching procedure has not been used. It is quite possible that even more local extrema of $\alpha$ exist but our numerical data is not precise enough to see them. To investigate this question a better method is needed as one has to approach $\alpha_{\text{crit}}$ very closely where our numerical errors become too large. As $\beta$ increases (with $m$ fixed) $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$ gets closer. The mass and the $\beta$ dependence of the different extrema of $\alpha$ together with those of $\alpha_{\text{crit}}$ are given in Table 3. We also notice that $\alpha_{\text{max}}$ grows quite rapidly as $m$ increases while the critical alpha remains unchanged. This behaviour is illustrated on Fig. 4 where the $\alpha$ dependence of the shooting parameters $a$, $b$, $1/\phi_0$ and the energy of the solution, $E$, for different values of $m$ are plotted. It is clearly visible that they all become more and more constant for $\alpha \ll \alpha_{\text{max}}$ as $m$ grows. So the solution becomes more and more independent of $\alpha$ for $\alpha \in [0, \alpha_{\text{max}}]$ as $m$ increases and gets closer and closer to the 't Hooft-Polyakov monopole. The numerical results indicate that in the limit $m \to \infty$ the functions $W$ and $H$ approach those corresponding to the 't Hooft-Polyakov monopole while $\varphi \to 0$. 

Figure 4: The $\alpha$ dependence of the shooting parameters $a$, $b$, $1/\phi_0$ and the energy of the fundamental monopole with $\beta^2 = 0.1$ for some $m^2$. 

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To understand the $m \to \infty$ limit somewhat better let us define $\tilde{\phi}(r)$ by
\[
\phi = \frac{\alpha \tilde{\phi}}{m^2}.
\]
Assuming now that $\phi(r) \ll 1$ and keeping only the leading terms in $m^2$ it follows from the field equations (3) that $W \approx W_{\text{thp}}, H \approx H_{\text{thp}}$ where $W_{\text{thp}}, H_{\text{thp}}$ denote the 't Hooft-Polyakov solution and $\tilde{\phi} \approx \tilde{\phi}_{\text{inf}}$ where
\[
\tilde{\phi}_{\text{inf}} = -\frac{1}{r^2} \left( 2W_{\text{thp}}^2 + (W_{\text{thp}}^2 - 1)^2/r^2 - \frac{\beta^2}{4}r^2(H_{\text{thp}}^2 - 1)^2 \right).
\]
The numerical results show that already for $m^2 = 10 \tilde{\phi}_{\text{inf}}$ approximates $\tilde{\phi}$ very well, see Fig. 5. The shooting parameters also tend to the appropriate ('t Hooft-Polyakov) values. Fig. 6 also exhibits the tendency that the shooting parameters and the energy of the solution depends less and less on $\alpha$ as $m$ increases.

So the above result supports the expectation that in the limit $m \to \infty$ the dilaton decouples while tending to zero and $W, H$ converging to the 't Hooft-Polyakov monopole solution. Note that the growing of the maximal $\alpha, \alpha_{\text{max}}$, as $m$ becomes larger is also consistent with the expected decoupling.

Figure 5: The $m^2$ dependence of the fundamental monopole solution for $\alpha = 1, \beta^2 = 0.1$. $\tilde{\phi} = m^2 \phi/\alpha$ and 'tHP denotes the 't Hooft-Polyakov monopole solution. The corresponding parameters are given in Table 4.

Table 4: Parameters of the fundamental monopole solution for $\alpha = 1, \beta^2 = 0.1$, together with the corresponding values for the 't Hooft-Polyakov monopole. The starred value stands for the limiting value of $-\tilde{\phi}(0)$ for $m^2 \to \infty$ i.e. for $-\tilde{\phi}_{\text{inf}}(0) = 12b^2 - \beta^2/4$.

| $m^2$ | $a$     | $b$     | $-\tilde{\phi}_0$ | $E$     |
|-------|---------|---------|--------------------|---------|
| 0     | 0.4292657 | 0.2386866 | 0.8816368          | 0.9001767 |
| 0.1   | 0.4321814 | 0.2384289 | 0.6263968          | 0.9845152 |
| 1     | 0.4481335 | 0.2370648 | 0.2787202          | 1.0605536 |
| 10    | 0.4726173 | 0.2373279 | 0.0539949          | 1.0988876 |
| 100   | 0.4808578 | 0.2380868 | 0.0063855          | 1.1054006 |
| 'tHP  | 0.4821127 | 0.2381420 | 0.6555393          | 1.1061955 |

In addition to the fundamental monopole there is a discrete family of nonabelian solutions characterized by the number of zeros of $W, n$. They can be interpreted as radial excitations of the fundamental ($n = 0$) monopole. See Fig. 6. One can understand these oscillating solutions by linearizing the field eqs. (3) around $\{W = 0, H = v, \phi = 1/\alpha \ln(r/\alpha)\}$ i.e. ($R \approx \alpha$) in the
region $r \ll 1/m$ and $r \ll \alpha$ corresponding to $\tau \approx \alpha \ln r + \text{const.}$ (For $m = 0 \{W = 0, H = v, R = \alpha, \nu = 1/\alpha\}$ is a critical point of Eqs. (3).) The linearization yields the following behaviour: \{\begin{align*}
W(\tau) &\propto e^{\lambda_W \tau}, \quad H(\tau) - 1 \propto e^{\lambda_H \tau}, \quad \varphi(\tau) \propto e^{\lambda_\varphi \tau}\end{align*}\}; where the exponents, $\lambda_i$, are given as
\begin{align*}
\lambda_W &= 1/\alpha(-1/2 + \sqrt{\alpha^2 - 3/4}), \\
\lambda_H &= 1/\alpha(1/2 + \sqrt{\alpha^2 \beta^2 + 1/4}), \\
\lambda_\varphi &= 1/\alpha(1, -2).\end{align*}
(13)

The amplitude of the oscillations of $W$ tends to zero as $1/\sqrt{r}$ and their frequency is independent from the mass of the dilaton. From Eqs. (13) one expects that solutions with an oscillating $W$ and a logarithmically growing dilaton (in the region where $r \ll \min\{1, 1/m\}$) exist only for $\alpha^2 < 3/4$. For larger values of $r$ the behaviour of $\varphi$ changes from $\varphi \propto \ln(r)$ to $\varphi \propto 1/r^4$ and $W$ stops to oscillate. We expect that globally regular solutions with an arbitrary number of oscillations exist for $\alpha^2 < 3/4$.

In the massless ($m = 0$) case we have found excited solutions (with $n \neq 0$) only up to $\alpha = \sqrt{3}/2$ independently of $\beta$ and $n$. But with an increasing mass we have also observed solutions with $n = 1$ for $\alpha > \sqrt{3}/2$ (E.g. $\alpha = 0.98$, $m^2 = 10$ for $\beta^2 = 0.1$). In fact there is also a maximal and a critical value of $\alpha$ for the excitations similarly to the $n = 0$ case. Our numerical data is, however, not precise enough to allow for a good determination of $\alpha_{\text{crit}}$ (for $n \neq 0$) but they are consistent with $\alpha_{\text{crit}} = \sqrt{3}/2$ independently of $m$ and $n$. This is in agreement with the argument given for the mass independence of $\alpha_{\text{crit}}$. We remark that the maximal value of $\alpha$ for solutions with $n = 1$ has always been less than one. The energy of the excitations increases with $n$ and it is always larger than that of the fundamental monopole. The first zero of an oscillating solution moves closer and closer to $r = 0$ as the number of oscillations increases. The
excited solutions depend only slightly on the dilaton mass, similarly to what has been seen in
the massive DYM theory ($v \to 0$ limit).

We expect that the excited solutions are unstable against spherical perturbations, because
of the established instability of their counterparts in the massless DYM theory [1, 2]. We also
expect the fundamental monopole solutions to become unstable after the bifurcation point at
$\alpha = \alpha_{\text{max}}$, analogously to the case of gravitating monopoles [3].

Acknowledgements

We would like to thank D. Maison and P. Breitenlohner for helpful discussions.

References

[1] P. Bizon, Phys. Rev. D 47 (1993) 1656
[2] G. Lavrelashvili, and D. Maison, Phys. Lett. B 295 (1992) 67
[3] R. Bartnik, and J. McKinnon, Phys. Rev. Lett. 61 (1988) 141
[4] P. Forgács, J. Gyürüsi Phys. Lett. B 366 (1996) 205
[5] G. ’t Hooft, Nucl. Phys. B 279 (1974) 276
    A.M. Polyakov, JETP Lett. 20 (1974) 194
[6] P. Breitenlohner, P. Forgács, and D. Maison, Nucl. Phys. B 383 (1992) 357
[7] P. Breitenlohner, P. Forgács, and D. Maison, Nucl. Phys. B 442 (1995) 126
[8] E. Witten, Phys. Lett. B 155 (1985) 151
[9] C.P. Burgess, A. Font and F. Quevedo, Nucl. Phys. B 272 (1986) 661
[10] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, Vol. 2 (C.U.P, Cambridge, 1988)
[11] C.M. O’Neil, Phys. Rev. D 50 (1994) 865
[12] H. Hollmann, Phys. Lett. B 338 (1994) 181