Error-disturbance relations for finite dimensional systems

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Abstract

We propose an error-disturbance relation for general observables on finite dimensional Hilbert spaces based on operational notions of error and disturbance. For two-dimensional systems we derive tight inequalities expressing the trade-off between accuracy and disturbance.

1 Introduction

The uncertainty principle can broadly be understood in two different ways: (a) as the impossibility of preparing a state such that two non-commuting observables $A$ and $\bar{A}$ are both sharply defined, or (b) as the fact that measuring $A$ will disturb any subsequent attempt to measure $\bar{A}$. The Robertson-Schödinger inequality,

$$\sigma_A^2 \sigma_{\bar{A}}^2 \geq \frac{1}{2} \left| \langle [A, \bar{A}] \rangle \right|,$$

is the usual textbook example of the uncertainty principle in the first sense, whereas there does not seem to be a broad consensus of how to express (b) in general. A third possible formulation is that (b') it is impossible to measure both $A$ and $\bar{A}$ simultaneously. There is a certain sense in which (b) and (b') can be regarded as equivalent\(^1\). Intuitively, given any apparatus capable of measuring $A$ without disturbing $\bar{A}$, it is clear that one could construct an apparatus for the joint measurement of $A$ and $\bar{A}$. Conversely, if one could measure both $A$ and $\bar{A}$, one could bring the system back to the appropriate eigenstate of $\bar{A}$ after the measurement, thus effectively measuring $A$ without disturbing $\bar{A}$. See Appendix B for a quantitative discussion.

We will use the term error-disturbance relation to denote an inequality expressing (b) or (b'). For simplicity, we will mainly focus on the (b') formulation.

\(^{1}\)Of course, quantitative expressions of (b) and (b') will not necessarily be identical.
Consider two non-degenerate observables on a $N < \infty$ dimensional Hilbert space $\mathcal{H}$. Let us write them as

$$A = \sum_{a=1}^{N} \lambda_a P_a, \quad \bar{A} = \sum_{a=1}^{N} \bar{\lambda}_a \bar{P}_a,$$

(2)

where

$$P_a = |a\rangle\langle a|, \quad \bar{P}_a = |\bar{a}\rangle\langle \bar{a}|$$

(3)

are projection operators. Now a measurement of $A$, say, will yield an eigenvalue $\lambda$, but that is clearly equivalent to specifying the integer $a$ such that $\lambda_a = \lambda$ (here we use that $A$ is non-degenerate). We will thus forget about the eigenvalues in the following, and focus on the projection operators (equivalently the eigenbases).

We describe the apparatus for (approximate) joint measurement of $A$ and $\bar{A}$ by a Positive Operator Valued Measure (POVM). In more detail, we consider $N^2$ positive (positive will always mean positive semidefinite for operators) operators $F_{ab}$ such that

$$\sum_{a,b} F_{ab} = 1,$$

(4)

and define the probability of getting output $(a,b)$ given a state $\rho$ to be

$$\mathcal{P}_\rho(a,b) := \text{tr}(F_{ab}\rho).$$

(5)

An output $(a,b)$ should intuitively be understood as indicating the joint measurement result $A = \lambda_a$ and $\bar{A} = \bar{\lambda}_b$.

We now want to quantify how good the apparatus is at measuring $A$ and $\bar{A}$. Let us define two POVMs corresponding to the first and second output of the apparatus:

$$M_a := \sum_b F_{ab}, \quad \bar{M}_a := \sum_b F_{ba}.$$ 

(6)

We consider the apparatus to be ‘good’ if these should are, respectively, close to the projective POVMs $P$ and $\bar{P}$. We thus need to introduce a metric, $d(M,P)$, to define what we mean by ‘close’. We will consider two different possibilities in the following.

The most basic way of testing the apparatus one could think of is to prepare the system in an eigenstate, $\rho = P_a$, say, and then checking how often the apparatus gives the right answer. This procedure naturally leads to the definition

$$d_c(M,P) := \sup_{a=1, \ldots, N} (1 - \text{tr}(M_a P_a)).$$

(7)

In words, $d_c(M,P)$ is the worst case error probability if the system is prepared in an eigenstate of $A$. It is clear that

$$d_c(M,P) \geq 0,$$

(8)

2This is similar to the approach in e.g. the case of entropic uncertainty relations [5, 13].
and it is easy to show that $d_c(M, P) = 0$ iff $M_a = P_a$ for all $a$. We will call $d_c(M, P)$ the calibration error, since it is similar in spirit to the one used in [4] for continuous variables.

For a fixed state $\rho$, $M$ and $P$ each define a probability distribution, so another approach would be to compare these distributions. A natural distance measure on probability distributions is the total variation distance:

$$\frac{1}{2} \sum_a |\text{tr}(M_a\rho) - \text{tr}(P_a\rho)|. \quad (9)$$

We then define the variation error by taking the supremum over all states:

$$d_v(M, P) := \sup \left\{ \frac{1}{2} \sum_a |\text{tr}(M_a\rho) - \text{tr}(P_a\rho)| : \rho \geq 0, \text{tr} \rho = 1 \right\}. \quad (10)$$

Again we have that

$$d_v(M, P) \geq 0, \quad (11)$$

and that $d_v(M, P) = 0$ iff $M_a = P_a$ for all $a$. We also find the relation

$$d_v(M, P) \geq d_c(M, P) \quad (12)$$

by writing

$$d_c(M, P) = \sup \left\{ \frac{1}{2} \sum_a |\text{tr}(M_a\rho) - \text{tr}(P_a\rho)| : \rho = |a\rangle\langle a|, a = 1, \ldots, N \right\}. \quad (13)$$

Let $\mathcal{F}$ be the powerset of $\{1, \ldots, N\}$, then it is easy to see that

$$\frac{1}{2} \sum_a |\text{tr}(M_a\rho) - \text{tr}(P_a\rho)| = \sup_{X \in \mathcal{F}} |\text{tr}(M(X)\rho) - \text{tr}(P(X)\rho)|, \quad (14)$$

where

$$M(X) = \sum_{a \in X} M_a, \quad P(X) = \sum_{a \in X} P_a. \quad (15)$$

It follows that we can alternatively write $d_v$ as

$$d_v(M, P) = \sup_{X \in \mathcal{F}} \|M(X) - P(X)\|, \quad (16)$$

where $\| \cdot \|$ is the operator norm.

The choice between $d_c$ and $d_v$ will be dictated by the application of the error-disturbance relation. The calibration error has a very simple operational interpretation, while the variation error treats the input states in a more uniform way (i.e. without singling out the eigenstates). We will see in Section 2 that

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3The total variation distance was also considered (in the context of approximate measurements) in [3].
already for the qubit, the two error metrics lead to inequivalent optimal joint measurement schemes.

We are now ready to write down the general form of our error-disturbance relation. Denote the errors by

\[
\epsilon_\beta = d_\beta(M, P), \quad \bar{\epsilon}_\beta = d_\beta(\bar{M}, \bar{P}), \quad \beta \in \{c, v\},
\]

we then want to consider inequalities of the form

\[
G_\beta(\epsilon_\beta, \bar{\epsilon}_\beta) \geq B_\beta(P, \bar{P}),
\]

valid for all POVMs \( F \). We remark that this is a state independent relation. The expression \( G(\epsilon, \bar{\epsilon}) \) is meant to quantify the total error of the apparatus, and there seems to be many valid choices for functional form of \( G \). The Roberson-Schödinger-like choice \( G(\epsilon, \bar{\epsilon}) = \epsilon \bar{\epsilon} \), however, is not a good one, since there is an apparatus such that \( \epsilon = 0 \) and \( \bar{\epsilon} \) is finite (the errors are bounded in our case) ruling out any non-trivial bound in \( (18) \).

We will focus on the case \( G(\epsilon, \bar{\epsilon}) := \epsilon + \bar{\epsilon} \), see Appendix A for a heuristic motivation of this choice. The task thus becomes to determine the strongest possible bound\(^4\) \( B(P, \bar{P}) \) making

\[
\epsilon_\beta + \bar{\epsilon}_\beta \geq B_\beta(P, \bar{P})
\]

valid. Note that we will sometimes omit the index \( \beta \) for brevity.

In Appendix B we show how \( (19) \) can also be interpreted as

\[
\epsilon + \bar{\eta} \geq B(P, \bar{P}),
\]

where \( \epsilon \) is the error of a quantum instrument measuring \( A \) and \( \bar{\eta} \) is the disturbance of \( \bar{A} \) incurred by the instrument.

### 1.1 Results

In Section 2 we will compute the optimal value of \( B_\beta(P, \bar{P}) \) for the qubit. Specifically, we find the tight relations

\[
\epsilon_c + \bar{\epsilon}_c \geq \sin \left( \frac{\pi}{4} + \theta \right) - \frac{\sqrt{2}}{2}
\]

and

\[
\epsilon_c + \bar{\epsilon}_c \geq 2 \sin^2 \frac{\theta}{2}.
\]

Here \( 0 \leq 2\theta \leq \frac{\pi}{2} \) is the angle between \( |1\rangle \) and \( |\bar{1}\rangle \) on the Bloch sphere.

In Section 3 we consider the case where the bases \( |a\rangle \) and \( |\bar{a}\rangle \) are mutually unbiased. Using some results from Appendix C we derive

\[
\epsilon_c + \bar{\epsilon}_c \geq 2 \left( \frac{1}{4} (4\sqrt{2} - 5) \right)^2
\]

\(^4\)Formally we want to set \( B(P, \bar{P}) = \inf_F \{ \epsilon + \bar{\epsilon} \} \), but that is clearly only useful if we know how to compute the infimum.
for \( N = 3 \), and
\[
\epsilon_c + \bar{\epsilon}_c \geq 2 \left( \frac{1}{31} \left( 4 \sqrt{7} - 9 \right) \right)^2
\]
for \( N = 5 \). By \cite{12} these also hold for the variation error.

After the completion of this manuscript earlier work by Bush and Heinosaari \cite{6} came to our attention. Their results overlap with our results for the qubit. In particular they also derive (21).

### 1.2 Related work

The formulation of our error-disturbance relation is closest in spirit to the one by Bush, Lahti, Pearson and Werner \cite{18, 5, 4} for canonical position and momentum operators, but a direct comparison is not possible, since we work with finite dimensional systems. Here we will briefly compare our proposed inequality with two other proposals.

In order to discuss Ozawa’s uncertainty relation \cite{15, 16, 11, 1}, we need to introduce an auxiliary Hilbert space \( \mathcal{H}_{aux} \) and a fixed state \( \rho_{aux} \) on \( \mathcal{H}_{aux} \). The measurement apparatus is then described by two commuting observables \( A, \bar{A} \) on \( \mathcal{H} \otimes \mathcal{H}_{aux} \). Define the errors (the \( O \) is part of the name, not an index)
\[
\epsilon^2_{O,\rho} = \langle (A - A \otimes I_{aux})^2 \rangle_{\rho \otimes \rho_{aux}}, \quad \bar{\epsilon}^2_{O,\rho} = \langle (\bar{A} - \bar{A} \otimes I_{aux})^2 \rangle_{\rho \otimes \rho_{aux}},
\]
and the standard deviations
\[
\sigma^2_{\rho} = \langle (A - \langle A \rangle_\rho)^2 \rangle_\rho, \quad \bar{\sigma}^2_{\rho} = \langle (\bar{A} - \langle \bar{A} \rangle_\rho)^2 \rangle_\rho.
\]
One can then derive the following error-disturbance relation\cite{16}
\[
\epsilon_{O,\rho} \epsilon_{\bar{O},\rho} + \epsilon_{O,\rho} \bar{\sigma}_{\rho} + \sigma_{\rho} \bar{\epsilon}_{O,\rho} \geq \frac{1}{2} |\langle [A, \bar{A}] \rangle_\rho|.
\]

Another relation is due to Hofmann\cite{12}. Consider a POVM (the \( H \) is not an index)
\[
\sum_m F_{H,m} = 1,
\]
where \( m \) ranges over some set of measurement outcomes (the number of outcomes does not have to be related to the dimension \( N \) of \( \mathcal{H} \)), and introduce the errors
\[
\epsilon^2_{H,m} = \langle (A - \langle A \rangle_{\rho_m})^2 \rangle_{\rho_m}, \quad \bar{\epsilon}^2_{H,m} = \langle (\bar{A} - \langle \bar{A} \rangle_{\rho_m})^2 \rangle_{\rho_m}.
\]

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5One usually writes \( A = U^\dagger (1 \otimes A_{aux}) U \) (and similarly for \( \bar{A} \)), where \( A_{aux} \) is an observable on \( \mathcal{H}_{aux} \) and \( U \) is a unitary describing some interaction between the system of interest and the auxiliary system.

6In Ref. \cite{11} Hall derives a very similar inequality, but with \( \sigma \) (\( \bar{\sigma} \)) defined in terms of \( A \) (\( \bar{A} \)). A tight variant of \cite{27} is given by Branciard \cite{1}. See also Weston et al. \cite{19} and Lu et al. \cite{13} for related inequalities.
Here $\rho_m$ is the “retrodictive” state corresponding to the measurement outcome $m$, explicitly

$$\rho_m = \frac{F_{H,m}}{\text{tr}[F_{H,m}]}.$$  

We then have the relation \[12\] (see also \[9\])

$$\epsilon_{H,m} \bar{\epsilon}_{H,m} \geq \frac{1}{2} |\langle [A, \bar{A}] \rangle_{\rho_m}|.$$  

An obvious difference between these error-disturbance relations and ours is that here the spectra of $A$ and $\bar{A}$ plays a role, while in \[19\] only the spectral bases appear. A more substantial difference is that the RHS of \[27\] and \[31\] depends on a state (the (pre-measurement) system state in the Ozama case and the retrodictive state in the Hofmann case). For the Hofmann relation the RHS further depends on the details of the measurement apparatus. In contrast, the RHS of \[19\] only depends on the operators we want to measure. This means that our relation is non-trivial as long as $A$ and $\bar{A}$ do not commute\footnote{It follows from the results in Appendix C that $B(P, \bar{P})$ can be taken to non-zero when $A$ and $\bar{A}$ do not commute.}, independent of the system state and apparatus, which is not in general the case for \[27\] and \[31\].

2 The qubit

We will now consider the simplest non-trivial case, that is $N = 2$. We will see that it is possible to determine the optimal bound $B$ in \[19\]. Even for qubit measurements, it takes 12 real parameters to specify the $F$s, so computing the infimum directly would be demanding. Fortunately, almost all the degrees of freedom can be eliminated using symmetry. To see how this works, let us choose the axis on the Bloch sphere such that

$$P_1 = \frac{1}{2} \left( \mathbb{1} + (\cos \theta) \sigma_z + (\sin \theta) \sigma_x \right),$$  

and

$$\bar{P}_1 = \frac{1}{2} \left( \mathbb{1} + (\cos \theta) \sigma_z - (\sin \theta) \sigma_x \right),$$  

where $0 \leq \theta \leq \frac{\pi}{4}$ (we might have to relabel, say, $|1\rangle \leftrightarrow |2\rangle$ to allow this). We see that a 180° rotation along the $z$-axis will exchange the barred and unbarred basis. If we combine the rotation with a relabeling $|1\rangle \leftrightarrow \bar{1}$ and $|2\rangle \leftrightarrow |\bar{2}\rangle$ we thus get a new measurement scheme. Taking the average of the original and the rotated measurement scheme, as in Appendix A we get a measurement scheme which is symmetric, and at least as good. We will now make this more precise.

Let $S$ be an unitary or antiunitary operator on $\mathcal{H}$. Then

$$\rho \mapsto S\rho S^\dagger$$  

\[34\]
maps the space of states bijectively onto itself. It follows that (here \((S^\dagger MS)_a = S^\dagger M_a S\), etc.)

\[
d_B(S^\dagger MS, P) = d_B(M, SPS^\dagger). \tag{35}
\]

If we now take \(S = \sigma_z\), and set (it is clear that \(F'\) is a POVM)

\[
F'_{11} = \sigma_z F_{11} \sigma_z, \quad F'_{12} = \sigma_z F_{21} \sigma_z, \quad F'_{21} = \sigma_z F_{12} \sigma_z, \quad F'_{22} = \sigma_z F_{22} \sigma_z, \tag{36}
\]

then

\[
\epsilon' = d(M', P) = d(\bar{M}, \bar{P}), \quad \bar{\epsilon}' = d(M', \bar{P}) = d(M, P). \tag{38}
\]

By taking the average of \(F\) and \(F'\), we see that the infimum of \(G(\epsilon, \bar{\epsilon}) = \epsilon + \bar{\epsilon}\) is achieved by \(F_s\) satisfying

\[
F_{11} = \sigma_z F_{11} \sigma_z = \sigma_y F_{22} \sigma_y, \quad F_{12} = \sigma_z F_{21} \sigma_z = \sigma_y F_{21} \sigma_y, \quad F_{21} = \sigma_z F_{12} \sigma_z = \sigma_y F_{12} \sigma_y, \quad F_{22} = \sigma_z F_{22} \sigma_z = \sigma_y F_{11} \sigma_y. \tag{39a}
\]

The second equal sign in the equations follows from similar considerations with \(S = \sigma_y\). Finally, by letting \(S\) be the antilinear operator corresponding to reflection in the \(x-z\) plane, we find that we can further restrict to \(F_s\) with no \(\sigma_y\) components, i.e.

\[
\text{tr}(F_{ab} \sigma_y) = 0 \quad [\text{for all } a \text{ and } b]. \tag{39c}
\]

In this symmetric subspace \(\epsilon = \bar{\epsilon}\), so the task becomes to find the smallest possible value of \(\epsilon\). We will now do this for each of the error metrics.

### 2.1 Variation error

By (39c), we can parametrize \(M_1\) as

\[
M_1 = \frac{1}{2}([1 + (\cos \theta + 2\epsilon \cos \chi)\sigma_z + (\sin \theta + 2\epsilon \sin \chi)\sigma_x]). \tag{40}
\]

The use of \(\epsilon\) is consistent, since a simple calculation shows that

\[
d_v(M, P) = \epsilon. \tag{41}
\]

Note that the symmetries (39) fix the other three marginals, but the \(F_s\) are not completely fixed. We parametrize the freedom by \(a\), setting

\[
F_{11} = aP_+ + (a - \kappa)P_-, \tag{42}
\]

and consequently

\[
F_{12} = \frac{1}{2}([1 + \kappa - 2a]P_+ + [\sin \theta + 2\epsilon \sin \chi]P_x), \tag{43}
\]

\footnote{With the standard choice for the Pauli matrices this is simply complex conjugation.}
in terms of the projections
\[ P_\pm = \frac{1}{2}(\mathbb{1} \pm \sigma_z), \]  
and with
\[ \kappa = \cos \theta + 2\epsilon \cos \chi. \]

It is clear that $F_{11}$ is positive iff $a \geq 0$ and $a - \kappa \geq 0$.

We assume that we have found a point $(\epsilon, \chi, a)$ such that $\epsilon$ is minimal and the $F$s are positive. Note that $F_{11}$ ($F_{12}$) is positive iff $F_{22}$ ($F_{21}$) is positive. We will assume that $\kappa > 0$ for now. Then we can set $a = \kappa$, since making $a$ smaller makes $F_{12}$ more positive, thus
\[ F_{11} = \kappa P_+, \]  
and
\[ F_{12} = \frac{1}{2}([1 - \kappa] \mathbb{1} + [\sin \theta + 2\epsilon \sin \chi] \sigma_z). \]

If $F_{12}(\epsilon, \chi)$ was strictly positive, then, by continuity, one could find a $\epsilon' < \epsilon$ such that $F_{12}(\epsilon', \chi)$ would still be positive, thus this cannot be the case. We conclude that at least one of the eigenvalues of $F_{12}$ must be zero, or
\[ \det F_{12} = 0. \]  
If now
\[ \frac{\partial \det F_{12}}{\partial \chi} \neq 0 \]  
then we could find a $\chi'$ near $\chi$ such that both the eigenvalues would be strictly positive (with the same $\epsilon$), a contradiction by the above argument, thus
\[ \frac{\partial \det F_{12}}{\partial \chi} = 0. \]

It is now straightforward to find the solutions to (48) and (50). The one with the smallest $\epsilon$, amongst those satisfying $\text{tr} F_{12} \geq 0$, had
\[ \chi_{\text{optimal,} v} = \frac{5\pi}{4}, \]  
and
\[ \epsilon_{\text{optimal,} v} = \frac{1}{2} \left( \sin \left( \frac{\pi}{4} + \theta \right) - \frac{\sqrt{2}}{2} \right). \]

For $\kappa \leq 0$ it follows directly from (45) that
\[ \epsilon \geq \frac{\cos \theta}{2}; \]

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\footnote{Incidentally, the POVM also solves the equations, but with $\epsilon = \sin(\theta/2)$, which is worse than (52).}
which is worse than \([52]\). We thus have an optimal bound

\[
d_v(M, P) + d_v(\tilde{M}, \tilde{P}) \geq 2\epsilon_{\text{optimal}, v} = \sin \left( \frac{\pi}{4} + \theta \right) - \sqrt{2}\frac{1}{2}
\]  

(54)

Explicitly, the POVM takes the form

\[
F_{11} = \frac{1}{2}(1 + \cos \theta - \sin \theta)P_+ \quad F_{12} = \frac{1}{4}(1 - \cos \theta + \sin \theta)(1 + \sigma_x), \quad (55a)
\]

\[
F_{21} = \frac{1}{4}(1 - \cos \theta + \sin \theta)(1 - \sigma_x) \quad F_{22} = \frac{1}{2}(1 + \cos \theta - \sin \theta)P_-.
\]

(55b)

2.2 Calibration error

The calculations in the case of the calibration error are similar to those of the previous section, so we give fewer details. We set

\[
M_1 = \frac{1}{2}(1 + \xi \sigma_z + h \sigma_x),
\]

(56)

with

\[
\xi = \frac{1 - 2\epsilon - h \sin \theta}{\cos \theta},
\]

(57)

and, for \(\xi > 0\),

\[
F_{11} = \xi P_+.
\]

(58)

The analogs of \([48]\) and \([50]\) hold (with \(h\) instead of \(\chi\)), and we find \([10]\)

\[
h_{\text{optimal}, c} = 0,
\]

(59)

and

\[
\epsilon_{\text{optimal}, c} = \sin \frac{\theta}{2}.
\]

(60)

The case of \(\xi \leq 0\) is also easily treated, and is seen to lead to suboptimal solutions (this is also intuitive, since we would have \(F_{11} \propto P_\) \(\)). As before, we deduce the tight bound

\[
d_c(M, P) + d_c(\tilde{M}, \tilde{P}) \geq 2\epsilon_{\text{optimal}, c} = 2\sin^2 \frac{\theta}{2}.
\]

(61)

The optimal POVM is just a projective measurement along the \(z\)-axis, i.e.

\[
F_{11} = P_+ \quad F_{12} = 0, \quad (62a)
\]

\[
F_{21} = 0 \quad F_{22} = P_-.
\]

(62b)

\[\text{At the point } \theta = \pi/4 \text{ the symmetry of the problem is enhanced, and the solution is optimal for any } h \in [0, 1].\]
3 Mutually unbiased bases

Two bases are said to be mutually unbiased if

$$|\langle a|\bar{b}\rangle|^2 = \frac{1}{N}$$  \hspace{1cm} (63)

for all $a$ and $b$. Here we will show that the symmetry condition \((84)\), which basically says that given a POVM one can construct a new one with $\epsilon$ and $\bar{\epsilon}$ interchanged (see Appendix A), holds for unbiased bases in three and five dimensions. This allows us to deduce explicit error-disturbance relations using the results of Appendix C.

Given a basis $|a\rangle$ we can construct a new basis $|\bar{a}\rangle$ by Fourier transformations,

$$\langle a|\bar{b}\rangle = \frac{1}{\sqrt{N}} e^{i\frac{2\pi ab}{N}},$$  \hspace{1cm} (64)

and it is clear that \((63)\) is satisfied. We will first show that \((84)\) holds when the bases are related as in \((64)\). Define the unitary operator $U$ by

$$|\bar{a}\rangle = U|a\rangle,$$  \hspace{1cm} (65)

then it follows from \((64)\) that

$$U^2|a\rangle = U|\bar{a}\rangle = |-a\rangle,$$  \hspace{1cm} (66)

where $-a$ is understood modulo $N$. Given a POVM $F_{ab}$, define a new one by

$$F'_{ab} = U^\dagger F_{-b,a} U.$$  \hspace{1cm} (67)

Using the invariance of $d$, Eq. \((35)\), we have

$$\epsilon' = d(M', P) = d(\bar{M}, \bar{P}) = \bar{\epsilon},$$  \hspace{1cm} (68)

and

$$\bar{\epsilon}' = d(M', \bar{P}) = d(\{M_{-a}\}_a, \{P_{-a}\}_a) = \epsilon,$$  \hspace{1cm} (69)

which demonstrates \((84)\).

Given a pair of bases that satisfy \((63)\) they will not generally be related by \((64)\), but sometimes they can be brought into that form by symmetry operations. Specifically, it is clear that permuting the basis vectors and changing their phases will leave $\epsilon$ and $\bar{\epsilon}$ invariant when combined with the appropriate permutation of the $F$s. More formally, let the $N \times N$ matrix $H$ be given by (the reason for the normalization will become clear)

$$H_{ab} = \sqrt{N} \langle a|\bar{b}\rangle.$$  \hspace{1cm} (70)

We now want to know whether we can find diagonal unitary matrices $D_1, D_2$ and permutation matrices $T_1, T_2$ such that

$$(D_1 T_1 H T_2 D_2)_{ab} = e^{i\frac{2\pi ab}{N}}.$$  \hspace{1cm} (71)
This problem has been studied. In fact, it is clear that
\[ |H_{ab}| = 1, \quad HH^\dagger = N 1, \quad (72) \]
which makes \( H \) a complex Hadamard matrix (see e.g. \[17\]), and the equivalence modulo permutations and phase appearing in (71) is exactly the equivalence that Hadamard matrices are classified up to. For \( N = 3 \) and \( N = 5 \) it is known \[10, 17\] that one can always find \( Ds \) and \( Ts \) such that (71) is satisfied. In other words, unbiased bases in three and five dimensions are essentially Fourier pairs (in the sense of (64)).

When (84) holds, we can restrict attention to POVMs with \( \epsilon = \bar{\epsilon} \) as in Section 2. Using the results of Appendix C we then derive the following bound for mutually unbiased bases with \( N = 3 \),
\[ \epsilon_c + \bar{\epsilon}_c \geq 2 \left( \frac{1}{7} \left( 4\sqrt{2} - 5 \right) \right)^2, \quad (73) \]
and for \( N = 5 \) we have
\[ \epsilon_c + \bar{\epsilon}_c \geq 2 \left( \frac{1}{31} \left( 4\sqrt{7} - 9 \right) \right)^2. \quad (74) \]

4 Further work

The estimates in Appendix C are not optimal and possibly they are quite far from optimal, so a reasonable next step would be to attempt improve them. This would immediately lead to sharper bound in (73) and (74).

I would also be interesting to understand under which conditions (if any) the symmetry condition (84) fails.

While this manuscript was being finished, a preprint appeared introducing a information-theoretic error-disturbance relation[2]. Their approach seems similar to ours in some ways, and it would be interesting to explore the exact relationship.

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A The choice of \( G \)

We will demand that \( G \) should be monotone,
\[ G(\epsilon', \bar{\epsilon}') \leq G(\epsilon, \bar{\epsilon}) \quad (75) \]
\[ ^{11} \text{That is } H \sim H' \text{ iff } H' = D_1T_1HT_2D_2 \text{ for some } Ds \text{ and } Ts. \]
when \( \epsilon' \leq \epsilon \) and \( \bar{\epsilon}' \leq \bar{\epsilon} \), and symmetric,
\[
G(\epsilon, \bar{\epsilon}) = G(\bar{\epsilon}, \epsilon).
\] (76)

We show that, given a symmetry assumption on the space of measurements (Eq. (84)), the choice
\[
G(\epsilon, \bar{\epsilon}) = \epsilon + \bar{\epsilon}
\] (77)
is the strongest among the convex \( G \).

First let us note that given two POVMs \( F^{(1)} \) and \( F^{(2)} \), we can form a new one by convex combination:
\[
F^{(3)}_{ab} = \alpha F^{(1)}_{ab} + (1 - \alpha) F^{(2)}_{ab}, \quad 0 \leq \alpha \leq 1.
\] (78)

From (13) and (10) it is easy to see that the process of combining the POVMs does not introduce additional error, in the sense that
\[
\epsilon^{(3)} \leq \alpha \epsilon^{(1)} + (1 - \alpha) \epsilon^{(2)},
\] (79)
with
\[
\epsilon^{(i)} = d(M^{(i)}, P).
\] (80)

An identical statement holds for \( \bar{\epsilon} \), of course. It would be natural to choose \( G \) such that this property also holds for the combined error \( G(\epsilon, \bar{\epsilon}) \). This is insured if \( G \) is convex,
\[
G(\alpha \epsilon + (1 - \alpha) \epsilon', \alpha \bar{\epsilon} + (1 - \alpha) \bar{\epsilon}') \leq \alpha G(\epsilon, \bar{\epsilon}) + (1 - \alpha) G(\epsilon', \bar{\epsilon}'),
\] (81)
as we then indeed have
\[
G(\epsilon^{(3)}, \bar{\epsilon}^{(3)}) \leq \alpha G(\epsilon^{(1)}, \bar{\epsilon}^{(1)}) + (1 - \alpha) G(\epsilon^{(2)}, \bar{\epsilon}^{(2)}).
\] (82)

For a given pair of bases, let \( E(\epsilon, \bar{\epsilon}) \) abbreviate “there exists a POVM with \( \epsilon = d(M, P) \) and \( \bar{\epsilon} = d(M, \bar{P}) \)”, and set
\[
2\epsilon_{\text{inf}} = \inf \{ \epsilon + \bar{\epsilon} | E(\epsilon, \bar{\epsilon}) \}.
\] (83)

Let us assume the following symmetry
\[
E(\epsilon, \bar{\epsilon}) \iff E(\bar{\epsilon}, \epsilon),
\] (84)
which is shown to hold for the qubit in Section 2 and for mutually unbiased bases (in 3 and 5 dimension) in Section 3. Using (79) we then have
\[
E(\epsilon, \bar{\epsilon}) \Rightarrow E\left(\frac{1}{2} (\epsilon + \bar{\epsilon}), \frac{1}{2} (\epsilon + \bar{\epsilon})\right).
\] (85)

We thus have
\[
\epsilon_{\text{inf}} = \inf \{ \epsilon | E(\epsilon, \epsilon) \}.
\] (86)

\[\text{Here we also use that one can always modify a given POVM to make } \epsilon \text{ and } \bar{\epsilon} \text{ bigger (within the relevant limits).}\]
Consider now a convex function $G$ as just discussed, and set

$$G_{\text{inf}} = \inf \{ G(\epsilon, \bar{\epsilon}) | E(\epsilon, \bar{\epsilon}) \}. \quad (87)$$

If

$$\epsilon + \bar{\epsilon} \geq 2 \epsilon_{\text{inf}} \quad (88)$$

we have (the first inequality is by \(75\), \(76\) and \(81\))

$$G(\epsilon, \bar{\epsilon}) \geq G \left( \frac{1}{2}(\epsilon + \bar{\epsilon}), \frac{1}{2}(\epsilon + \bar{\epsilon}) \right) \quad (89)$$

$$\geq G(\epsilon_{\text{inf}}, \epsilon_{\text{inf}}) \quad (90)$$

$$\geq G_{\text{inf}}. \quad (91)$$

In other words, the inequality \(88\) implies any inequality (of the form \(18\)) with a convex $G$, assuming \(84\).

### B The disturbance caused by measurement

To discuss the disturbance incurred by an approximate measurement of $A$, we need to keep track of the post-measurement state. The correct notion is that of a quantum instrument (see e.g. \(7\)). A quantum instrument can though of as a quantum channel that also has a classical output (i.e. the outcome of the measurement). In more detail, we associate a completely positive$^{13}$ map

$$\rho \mapsto E_a(\rho) \quad (92)$$

to each measurement outcome $a$ which sends positive operators on $\mathcal{H}$ to positive operators on $\mathcal{H}$. The probability of a outcome given a state $\rho$ is

$$P_\rho(a) = \text{tr}[E_a(\rho)], \quad (93)$$

and the (normalized) post-measurement state is

$$\frac{E_a(\rho)}{\text{tr}[E_a(\rho)]}. \quad (94)$$

From (93) it follows that we must have

$$\sum_a \text{tr}[E_a(\rho)] = \text{tr}(\rho) = 1, \quad (95)$$

for all states $\rho$ (note that the individual $E_a$ do not preserve trace).

---

$^{13}$A map is positive iff it sends positive operators to positive operators. It is completely positive iff the induced (linear) map on operators on $\mathcal{H} \otimes \mathcal{H}_{\text{aux}}$ (defined by $\rho \otimes \rho_{\text{aux}} \mapsto E_a(\rho) \otimes \rho_{\text{aux}}$) is positive for all (finite dimensional) $\mathcal{H}_{\text{aux}}$. 
If we perform a projective measurement of $\bar{A}$ after the measurement of $A$, the joint probability distribution of the two measurements is

$$P_{\rho}(a, b) = \langle \bar{b}|E_a(\rho)|\bar{b}\rangle.$$  

(96)

It is a well known result due to Choi and Kraus that we can find operators $V_{ak}$ such that

$$E_a(\rho) = \sum_k V_{ak}\rho V_{ak}^\dagger, \quad \sum_k V_{ak}^\dagger V_{ak} = 1,$$  

(97)

where $k$ ranges over some finite set. We can thus write

$$P_{\rho}(a, b) = \text{tr}(F_{ab}\rho),$$  

(98)

where

$$F_{ab} = \sum_k V_{ak}^\dagger|\bar{b}\rangle\langle \bar{b}| V_{ak}.$$  

(99)

It is easy to check that $F_{ab}$ is a POVM.

We conclude that the joint measurement relation (19) translates to the error-disturbance relation (with the same bound $B(P, \bar{P})$)

$$\epsilon + \bar{\eta} \geq B(P, \bar{P})$$  

(100)

for the instrument

$$E_a(\rho) = \sum_k V_{ak}\rho V_{ak}^\dagger,$$  

(101)

where

$$\epsilon = d(M, P), \quad \bar{\eta} = d(\bar{M}, \bar{P}),$$  

(102)

and

$$M_a = \sum_k V_{ak}^\dagger V_{ak}, \quad \bar{M}_b = \sum_{a,k} V_{ak}^\dagger|\bar{b}\rangle\langle \bar{b}| V_{ak}.$$  

(103)

The interpretation of $\epsilon$ is as in Section 1, i.e. it quantifies how much the measurement defined by the instrument $E_a$ deviates from the ideal (projective) measurement of $A$. On the other hand, $\bar{\eta}$ is the error of a projective measurement of $\bar{A}$ performed after the instrument, relative to a projective measurement of $\bar{A}$ before the instrument. It is thus natural to interpret $\bar{\eta}$ as the disturbance of $\bar{A}$ caused by the measurement of $A$. This justifies calling (100) an error-disturbance relation.

Let us note that the notion of a quantum instrument is sufficiently general to include any error correction one could perform on the system after learning result of the $A$ measurement, so (100) holds even with error correction.

C A general bound

Here we will derive a bound for general dimension $N \geq 2$. We will first need some simple properties of positive operators.
Let $K \geq 0$ be an operator. If
\begin{equation}
\langle 1|K|1 \rangle, \langle 2|K|2 \rangle \leq \epsilon \tag{104}
\end{equation}
then
\begin{equation}
|\langle 1|K|2 \rangle| \leq \epsilon, \tag{105}
\end{equation}
while if
\begin{equation}
\langle 1|K|1 \rangle \leq 1, \text{ and } \langle 2|K|2 \rangle \leq \epsilon \tag{106}
\end{equation}
then
\begin{equation}
|\langle 1|K|2 \rangle| \leq \sqrt{\epsilon}. \tag{107}
\end{equation}
These relations can be derived by noting that
\begin{equation}
(\langle 1| + \alpha^* \langle 2|)K(\langle 1| + \alpha |2 \rangle) \geq 0 \tag{108}
\end{equation}
for all $\alpha \in \mathbb{C}$. It follows immediately that (with $\epsilon = \epsilon_c = d_c(M, P)$)
\begin{equation}
|\langle 1|M_1|2 \rangle| \leq \sum_a |\langle 1|F_a|2 \rangle| \leq 2\sqrt{\epsilon} + (N - 2)\epsilon. \tag{109}
\end{equation}
We now wish to find a lower bound for the same matrix element.
It is clear that
\begin{equation}
|\langle 1|\bar{M}_1|2 \rangle| \geq |\langle 1|\bar{1}\rangle\langle \bar{1}|2 \rangle\langle \bar{1}|\bar{M}_1|\bar{1} \rangle| - |A| - |B| - |C| - |D|, \tag{110}
\end{equation}
with
\begin{align}
A &= \sum_{a \geq 2} \langle 1|\bar{1}\rangle\langle \bar{a}|2 \rangle\langle \bar{1}|\bar{M}_1|\bar{a} \rangle, & B &= \sum_{a \geq 2} \langle 1|\bar{a}\rangle\langle \bar{1}|2 \rangle\langle \bar{a}|\bar{M}_1|\bar{1} \rangle, \tag{111a}
\end{align}
\begin{align}
C &= \sum_{a \geq 2} \langle 1|\bar{a}\rangle\langle \bar{a}|2 \rangle\langle \bar{a}|\bar{M}_1|\bar{a} \rangle, & D &= \sum_{a, b \geq 2 \atop a \neq b} \langle 1|\bar{a}\rangle\langle \bar{b}|2 \rangle\langle \bar{a}|\bar{M}_1|\bar{b} \rangle. \tag{111b}
\end{align}
To estimate $A$, let us consider the vector
\begin{equation}
|\psi \rangle = \delta e^{i\theta_1}|\bar{1} \rangle + \sum_{a \geq 2} e^{i\theta_a}|\bar{a} \rangle, \tag{112}
\end{equation}
with $\delta, \theta_a \in \mathbb{R}$, and the $\theta$s chosen such that
\begin{equation}
\langle \psi|1 \rangle = \delta|\langle \bar{1}|1 \rangle| + \sum_{a \geq 2} |\langle \bar{a}|1 \rangle|. \tag{113}
\end{equation}
By Cauchy-Schwarz
\begin{equation}
\langle \psi|1 \rangle \leq \|\psi\| = \delta^2 + N - 1. \tag{114}
\end{equation}
Combining (113) and (114), we find
\begin{equation}
\sum_{a \geq 2} |\langle \bar{a}|1 \rangle| \leq \sqrt{N - 1} \sqrt{1 - |\langle \bar{1}|1 \rangle|^2}. \tag{115}
\end{equation}
With the abbreviations
\[ t_a = |\langle \bar{1}|a \rangle|, \quad \tilde{t}_a = \sqrt{1-t_a^2}, \] (116)
we thus have
\[ |A| \leq \sqrt{N - 1} t_1 \tilde{t}_1 \sqrt{\epsilon}, \quad |B| \leq \sqrt{N - 1} t_2 \tilde{t}_2 \sqrt{\epsilon}, \] (117a)
and by similar considerations
\[ |C| \leq \tilde{t}_1 \tilde{t}_2 \sqrt{\epsilon}, \quad |D| \leq \sqrt{(N - 1)(N - 2)} \tilde{t}_1 \tilde{t}_2 \sqrt{\epsilon}. \] (117b)
For mutual unbiased bases, it is simple to see that we can improve the estimate of \(|D|\) to
\[ |D| \leq \frac{(N - 1)(N - 2)}{N} \epsilon. \] (118)
From (109), (110) and (117) it then follows that
\[ I(\epsilon, \bar{\epsilon}; |\langle \bar{1}|1 \rangle|, |\langle \bar{1}|2 \rangle|) \geq 0, \] (119)
with
\[ I(\epsilon, \epsilon'; t, t') = (N - 2)\epsilon + 2\sqrt{\epsilon} + \left( tt' + \tilde{t}\tilde{t}' \sqrt{(N - 1)(N - 2) + 1} \right) \epsilon' \]
\[ + \sqrt{N - 1}(tt' + \tilde{t}\tilde{t}') \sqrt{\epsilon'} - tt'. \] (120)
We thus have a non-trivial bound on \(\epsilon, \bar{\epsilon}\) as long as \(|\langle \bar{1}|1 \rangle| |\langle \bar{1}|2 \rangle| > 0\).
It is clear that we can combine the bounds (119) for different combinations of basis vectors. Formally, let us define the subsets
\[ H = \{ (\epsilon, \bar{\epsilon}) \in [0, \infty)^2 | I(\epsilon, \bar{\epsilon}; |\langle a|b \rangle|, |\langle a|c \rangle|) \geq 0 \text{ for all } a \text{ and } b \neq c \}, \] (121)
and
\[ H' = \{ (\epsilon, \bar{\epsilon}) \in [0, \infty)^2 | I(\epsilon, \bar{\epsilon}; |\langle a|\bar{b} \rangle|, |\langle a|\bar{c} \rangle|) \geq 0 \text{ for all } a \text{ and } b \neq c \}. \] (122)
then we have
\[ d_c(M, P) + d_c(\bar{M}, \bar{P}) \geq \inf \{ \epsilon + \bar{\epsilon} | (\epsilon, \bar{\epsilon}) \in H \cap H' \}. \] (123)

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