RELATIVISTIC IDEAL CLOCK

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Abstract. Two particularly simple ideal clocks exhibiting intrinsic circular motion with the speed of light and opposite spin alignment are described. The clocks are singled out by singularities of an inverse Legendre transformation for relativistic rotators of which mass and spin are fixed parameters. Such clocks work always the same way, no matter how they move. When subject to high accelerations or falling in strong gravitational fields of black holes, the clocks could be used to test the clock hypothesis.

An ideal clock is a mathematical abstraction of a nearly perfect material clocking mechanism. The clock hypothesis asserts that an ideal clock measures its proper time. This means that the number of consecutive cycles registered by the clock increases steadily with the affine parameter of the worldline of the clock’s center of mass (CM). On the dimensional grounds, we may expect that the hypothesis could be violated for extreme accelerations or falling in strong gravitational fields of black holes, the clocks could be used to test the clock hypothesis.

Now, it can be deduced what the invariant circle must be. From \( w \propto \langle p \wedge k \wedge n \rangle \) it follows that \( \langle kw \rangle = 0 \). This means that the image point of \( k \) moves about the image circle of \( w \). As so, \( k \) may be thought of as representing the clock’s pointer and \( w \) as representing the clock’s dial (see figure). In free motion in Minkowski spacetime, the vectors \( k_\alpha \) are parallel transported. Then a Lorenz invariant phase can be assigned to \( k \) between instants \( \tau_0 \) and \( \tau \) through:

\[
\phi (\tau, \tau_0) = i \ln \left( \frac{\kappa (\tau) - \kappa_+}{\kappa (\tau) - \kappa_-} \right)
\]

The phase \( \phi \) is a real number. In free motion of the clock, a rotation through \( \phi = 2\pi \) represents a single full clocking cycle.

2. Dynamical requirements and the Hamiltonian

The intuition derived from the theory of Eulerian rigid bodies suggests that the above clock will be insensitive to external influences if both its mass and size is fixed. This requirement can be fulfilled in an invariant way by imposing constraints on the Casimir invariants of the Poincaré group:

\[
\Psi_0 : \langle pp \rangle - m^2 = 0, \quad \Psi_2 : \langle ww \rangle = \frac{1}{4} m^4 \ell^2 \equiv 0.
\]

Here, constants \( m \) and \( \ell \) are fixed parameters with the dimension of mass and that of length. These constraints should be regarded as primary, i.e., implied by the form of the Lagrangian.

Motivated by devising an ideal clock, Staruszkiewicz observed \([3]\) that (unlike unitarity) the irreducibility of quantum systems has a classical counterpart realized in postulating Eq.\([3]\) as a means to singling out physically appealing Lagrangians. This postulate is in essence equivalent to the earlier, strong conservation idea due to Kuzenko, Lyakhovich and Segal \([4]\), introduced as a basic dynamical principle for devising Lagrangians suitable for geometric models of particles with spin.

As established in Sec.\([1]\) the phase space of the simplest clock can be parameterized using components of the position fourvector \( x \) and three tangent fourvectors \( p, k, \pi \) bound to satisfy constraints Eqs.\([13]\) where

\[
\langle pw \rangle \equiv -\text{Det} \begin{pmatrix} \langle pp \rangle & \langle pk \rangle & \langle pn \rangle \\ \langle kp \rangle & \langle kk \rangle & \langle kn \rangle \\ \langle np \rangle & \langle nk \rangle & \langle nn \rangle \end{pmatrix} \approx \langle pk \rangle^2 \langle nn \rangle.
\]

Between these dynamical variables we assume a Poisson bracket \([U, V] \equiv \{ \partial_U U_\pi V_\pi \} = \{ \partial_U U_\pi V_\pi \} + \{ \partial_U U_\pi V_\pi \} - \{ \partial_U U_\pi V_\pi \} \). Eq.\([1\)\]

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1 A massless system (\( \langle pp \rangle = 0 \)) would be structurally something different, because for a parabolic homography preserving a null direction \( p \) and an orthogonal to it spatial direction \( w \) the analogous phase is not a Lorentz scalar.
form a system of independent first class constraints with respect to this bracket. In line with Dirac method \[5\], the most general Hamiltonian is a linear combination of all first class constraints with arbitary functions \(u\)'s as coefficients. It is convenient that the combination be taken as \[2\]

\[
H = \frac{u_1}{2m} \left( \frac{\langle pp \rangle - \langle x^2 \rangle}{2} \right) + \frac{u_2}{2m} \left( \frac{\langle pp \rangle + \frac{4}{l^2} \langle \Pi^2 \rangle}{2} \right) + u_3 \langle k \Pi \rangle + u_4 \langle k k \rangle
\]

The equations \(\partial_t H = 0\) form a system of first class constraints equivalent to Eq. [3]. Next, we introduce velocities \(\dot{q} = \partial_t \dot{\phi}\) \[5\]:

\[
\dot{x} \equiv (u_1 + u_3) \frac{p}{m} + u_2 \frac{4 \langle k \Pi \rangle}{l^2} + u_4 \Pi \approx \frac{m}{\sqrt{\langle pp \rangle}} \frac{4 \langle k \Pi \rangle}{l^2} \approx \frac{m}{\sqrt{\langle pp \rangle}} k, \quad k \equiv u_2 \frac{4 \langle k \Pi \rangle}{l^2} + u_4 \Pi
\]

By taking projective derivatives, defined recursively by \(b_t^{\mu+\sigma} := b_t (b_t^{\nu+\rho}),\) where \(b_t^{\mu+\sigma} := \partial_t^{\nu+\rho} p\), and \(\nu+\rho\), it can be shown that a curvature \(\kappa\) (defined by analogy with Frenet-Serret formulas) is fixed: \(\kappa = -m (\vec{b}_t \times \vec{b}_u \hat{c})^T \langle \vec{b}_x, \vec{b}_y \hat{c} \rangle^T \approx \frac{4}{l^2} / l,\) and that torsion vanishes on account of \(\vec{b}_x, \vec{b}_y, \hat{c}\), and \(\vec{b}_z\) being coplanar as \(p\)-orthogonal linear combinations of \(\vec{p}, \vec{t}, \vec{n}\). Hence, the trajectory perceived in the CM frame is a circle of a fixed radius \(l/2\) (without constraints Eq. [3] the radius would vary with the actual state \[3\]). Correspondingly, the worldline's path is winding up around a fixed space-time cylinder, the main axis of which represents the CM inertial motion. To measure the rate of change of the unit spatial vector \(n\) in the CM frame (see Eq. [5] for the definition of \(n\), a frequency scalar \(\Omega\) can be introduced:

\[
\Omega^2 := - \frac{\langle \Pi^2 \rangle}{\langle pp \rangle^2}; \quad \Omega = -\frac{\langle k \Pi \rangle}{\langle \Pi^2 \rangle} \text{ if } p = 0.
\]

For solutions, it reduces to a ratio: \(\Omega \approx \frac{\sqrt{\langle \Pi^2 \rangle}}{\langle pp \rangle}\) \[5\] if \(u_1 \approx |u_4|\) \[5\] if \(u_1 \approx |u_4|\). The equations of motion \(\dot{\Pi} \approx \frac{\langle kp \Pi \rangle}{\langle pp \rangle} = 0\) if \(\Omega = 0\) \[5\]. Both \(\Omega\) and \(\Lambda\) are reparameterization-invariant scalars with obvious physical meaning. On the other hand, \(\Omega\) and \(\Lambda\) are functions of the arbitrary ratio \(\rho / \Pi\). Thus the motion is indeterminate. To solve this paradox, this ratio needs to be set based on a sound guiding principle, so as not to introduce arbitrary features into the dynamics.

3. **Singularities in the inverse Legendre transformation**

The form of a Lagrangian \(L = \frac{1}{2} \langle \dot{\phi}^2 + \dot{\kappa}^2 \rangle - H\) corresponding to the Hamiltonian Eq. [3] is subject to invertibility of the map Eq. [3] restricted to a submanifold determined by the constraints Eq. [3]. For the purpose of the invertibility analysis, it must suffice to focus upon Lorentz scalars only. On the submanifold of interest, we may consider a map between two sets of variables: \(u_1, u_2, u_3, \langle kp \rangle, \langle pm \rangle\) and \(\langle k \Pi \rangle, \langle k k \rangle, \langle k x \rangle, \langle k \rangle\), \(\langle k \Pi \rangle\):

\[
\langle k \Pi \rangle \equiv \frac{\langle kp \rangle}{m} \langle u_1 + u_2 \rangle, \quad \langle k \Pi \rangle \equiv \frac{4 \langle kp \rangle}{m^2} \langle u_1 + u_3 \rangle
\]

The number of new constraints for velocities depends on the rank of the Jacobian matrix of this map. Non-zero minors of maximal rank 4 for this Jacobian are: \(j_1 = \frac{1}{2} \langle k^2 \rangle \langle u_1 + u_3 \rangle \langle u_2 \rangle - \langle u_1 \rangle \langle u_3 \rangle - \langle u_2 \rangle^2\)

\[2\] The original KLS Hamiltonian \[4\] involved a complex variable \(\zeta, (\zeta \equiv \tilde{2} \langle \bar{c}^4 \rangle)\), inherited from a primary Lagrangian. Starting with a related Lagrangian expressed in terms of \(k\), a Hamiltonian analogous in form to Eq. [4] was arrived at in \[2\] (upon earlier reducing an extended phase space). Our approach goes in the opposite direction. We start with a Hamiltonian deduced from first principles. In \[2\] we generalized this method onto systems described by a collection of fourvectors.

\[3\] This difference is already seen for a material point described by a general Lagrangian \(L = \frac{1}{2} \langle \dot{\phi}^2 + \dot{\kappa}^2 \rangle - m \langle \Pi^2 \rangle\). The equation \(\partial_t L = 0\) implies two qualitatively distinct regimes: 1) in which \(\Pi\) is a function of \(\dot{\phi}\), then \(\dot{\phi} = m^{-1} \langle \Pi \rangle\), and 2) in which \(\Pi\) is independent of \(\dot{\phi}\), requiring \(m = 0\) and a constraint \(\langle \Pi \rangle = 0\). The resulting Lagrangians are: 1) that of a massive particle \(L = m \langle \Pi^2 \rangle\) with \(p = m \dot{\Pi}\) and 2) that of a massless particle \(L = \frac{1}{2} \langle \Pi^2 \rangle\) by \(\Pi = p = \dot{\Pi}\) and an arbitrary \(\Pi\) transforming as \(\Pi = \Pi \omega + \Pi \omega\) under a reparameterization \(\dot{\Pi} = \omega \Pi\). The analytic form of \(L = m \langle \Pi^2 \rangle\) would not be suitable in a region containing the surface \(\langle \Pi \rangle = 0\), where the corresponding \(p\) would be divergent.
Dirac equation, that a measurement of the electron’s instantaneous motion is certain to give the speed of light, which Dirac mentions in his Principles and asserts this result to be generally true in a relativistic theory.

The Dirac observation in conjunction with previous findings tempts one to conjecture that worldlines of classical analogs of quantum elementary particles should be null.

4. Lagrangians of the first kind

In the sub-luminal sector \((u_1^2 > u_2^2)\) let \(u_1 = \rho \sqrt{\psi} \), \(u_2 = \rho \sqrt{\phi} \), \(\rho > 0 \), \(|\psi| < \infty\). Then from Eq. (2) \(\rho = \sqrt{\psi} \text{ with } \psi = \frac{\pm 1}{\sqrt{-1}} \). With the resulting \(u_1, u_2\) substituted in Eq. (2) two Lagrangians follow: \(L = m \sqrt{(\psi \phi)} \left(\frac{1}{k} \pm \frac{2 \lambda \langle(kk) \rangle}{(\langle k \rangle)^2} \right) + \lambda (kk)\), with their respective Lagrange multipliers \(\lambda\). The sub-luminal Lagrangian \(L_s\) is that of the Fundamental Relativistic Rotator [3]. With the Lagrangian \(L_\perp\) we could consider both sub- or super-luminal motions.

In the clock context, it is appropriate to recall an earlier result [10] published in [11] that the Lagrangians Eq. (2) can be alternatively arrived at by adopting a physically dubious condition that the Hessian matrix \(\delta^2 L\) for a general Lagrangian \(L = f(\x) \sqrt{\langle \psi \phi \rangle}\) expressed in terms of only the 5 degrees of freedom characteristic of a rotator – Cartesian \(x(t)\) and spherical \(\delta(t)\), \(\psi(t)\) (considered as functions of \(x^0 \equiv t\)) must be zero. As shown therein, this leads to a differential equation for \(f: \frac{d^2 f}{d t^2} + 2 \xi (\dot{f}^2 + f \xi f) = 0\). As a direct consequence of this, the clock frequency becomes indeterminate. This conforms with what has been concluded in Sec. [2]. For reasons described in Sec. [2], with the Lagrangian Eq. (2) there would be no privileged velocity constraint suitable to set this frequency so as to make the motion determinate, while conditions involving \(\langle \psi \phi \rangle = 0\) (such as \(\xi = 0\) or \(\xi = 1\)) would not be compatible with the analytic structure of these Lagrangians (the canonical momenta \(\partial_t L\) would involve indeterminate forms 0/0). For these reasons we must come to the conclusion that Eq. (2) does not describe a clock at all.

It seems that neither considering more complicated systems nor introducing interactions would help to remove this indeterminacy of motion. For example, in the electromagnetic field, the consistency requirements \(\{\psi_1, H\} = 0\) (with \(p - eA\) substituted for \(p\) by the minimal coupling principle) lead to a secondary constraint \(F_{\mu \nu} \psi \kappa^\mu = 0\), which for rotators reduces to a condition \(F_{\mu \nu} = 0\) strictly connected with the Hessian singularity alluded to above. Although this condition might lead to a unique motion in some situations (e.g., with appropriate initial data in a uniform magnetic field [13]) this may not to be so in general (see, a toy model [14]).

5. Ideal Clocks

5.1. Second kind Lagrangian

The new velocity constraints arranged to forms of the first degree in the velocities read:

\[
\left\langle \frac{\dot{x}}{x} \right\rangle \approx 0, \quad \left\langle \frac{\dot{k}}{k} \right\rangle + \langle k \rangle \approx 0.
\]

By eliminating these constraints from Eq. (7) one finds \(u_1 = \chi, u_2 = \nu, (kp) = \frac{\pm 1}{\sqrt{-1}}\) and \((\langle k \rangle)^2 = \langle \psi \rangle \langle \phi \rangle\), where \(\chi, \nu\) are arbitrary functions. After discarding a total derivative involving \(\langle kk \rangle\) and the higher order terms in the velocity constraints (irrelevant for the Dirac variational procedure [5]), the resulting Lagrangian can be arranged in a form with a new independent variable \(x = (\langle kp \rangle)\) and a Lagrange multiplier \(\lambda\):

\[
L = \frac{m}{2} \left\langle \frac{\dot{x}}{x} \right\rangle^2 + \frac{\langle k \rangle^2}{2} \left\langle \frac{\dot{k}}{k} \right\rangle + \langle k \rangle \lambda + \lambda \langle kk \rangle.
\]

As expected, this Lagrangian is linear in the velocity constraints, with function of momenta as coefficients. In view of the equation \(\partial_t L = 0\), the conditions Eq. (11) can be regarded as consequences of one another, and hence, only \(\langle xx \rangle > 0\) may be imposed as a subsidiary condition. Then \(x\) becomes arbitrary. Conversely, if \(\partial_t L = 0\) is to be satisfied for arbitrary \(x\), then both conditions in Eq. (11) follow. The Casimir invariants \((\langle pp \rangle = \frac{\pm 1}{\sqrt{-1}}(1+\xi)\) and \((\langle uu \rangle = \frac{\pm 1}{\sqrt{-1}}(1+\lambda)\) are bound to satisfy only a single constraint \((\langle pp \rangle = \frac{\pm 1}{\sqrt{-1}}(1+\xi)\) and \((\langle uu \rangle = \frac{\pm 1}{\sqrt{-1}}(1+\lambda)\) 0) they would be functions of the velocities. But for Eq. (11) the principal conditions are satisfied on the basis of Hamilton’s principle, either supplemented with the null worldlines principle or with the condition that \(k\) be independent of the velocities. The latter requirement is crucial, since otherwise, by solving \(\partial_t L = 0\) for \(k\), one would end up with a qualitatively different Lagrangian \(L = \frac{m}{2} \left\langle \frac{\dot{x}}{x} \right\rangle^2 + f_0 + \lambda \langle k \rangle\) whose analytic form is not admissible on the surface Eq. (10) (the momenta \(\partial_t L\) would involve indeterminate forms 0/0).

5.1.1. Connection with a family of Relativistic Rotators. To extend the construction in [3] so as to include also the case of Sec. [5], let a class of projection invariant Lagrangians of the first degree in the velocities be considered, whose form would be admissible on the surface \(\langle xx \rangle = 0\) and compatible with the condition \(\langle kk \rangle = 0\):

\[
L = \frac{m}{2} \left\langle \frac{\dot{x}}{x} \right\rangle^2 + \frac{\langle k \rangle^2}{2} \left\langle \frac{\dot{k}}{k} \right\rangle + f_0 + \lambda \langle k \rangle.
\]

The \(x\) must have appeared in this precise way for the dimensional grounds and it must transform as \(x \rightarrow x + \lambda k\) when \(k \rightarrow ak\) on account of the assumed projection invariance. Here, \(f_0\) is any function. If \(\partial_t L = 0\), the principal constraints reduce to \(f_0 = 0\) and \(\langle k \rangle = 0\) which are consistent with the analytic structure of these Lagrangians (the canonical momenta \(\partial_t L\) would involve indeterminate forms 0/0). For these reasons we must come to the conclusion that Eq. (12) does not describe a clock at all.

5.2. Third kind Lagrangian

Putting \(u_1 = -u_2\) consider for a while a restricted Legendre transformation with \(p \) left unaltered. Taking \(\eta = \mp \frac{1}{2} \sqrt{\langle k \rangle} \langle k \rangle \langle k \rangle \rangle + \lambda \langle k \rangle\) \(\approx 0\) and substituting in Eq. (7) into account (where \(\sqrt{\langle k \rangle} \langle k \rangle = \langle k \rangle\)) and integrating off the term linear in \(\langle kk \rangle\), one arrives at a Lagrangian:

\[
L = \langle kp \rangle \mp \frac{m}{2} \left\langle \frac{\dot{k}}{k} \right\rangle + \lambda \langle k \rangle.
\]
By making arbitrary variations w.r.t. \( p \) (\( \delta p \) must be independent of \( \delta \mathcal{L} \)), the result \( s = \frac{\mathcal{L}}{\langle \dot{\mathcal{L}} \rangle} = 0 \) following from Eqs. 5-7 can be re-obtained. It implies \( \langle \dot{\mathcal{L}} \rangle = 0 \) for any vector \( e \), and this fact can be used to eliminate \( p \) from \( \mathcal{L} \). Hence, the alternative Lagrangian takes on a form involving arbitrary \( e \) such that \( \langle \dot{e} \rangle \neq 0 \): \[
abla \mathcal{L} = m \nabla^2 \langle \dot{\mathcal{L}} \rangle + \lambda \langle \mathcal{L} \rangle. \tag{14}
\]

For \( \langle \dot{e} \rangle \) to be nonzero, it would suffice that \( e \) be obtained from any timelike vector by a two-parameter transformation group \( e \rightarrow \alpha(e + \beta k) \), with \( \alpha, \beta \) being arbitrary functions. This freedom in choosing \( e \) must be physically irrelevant, and this will be so if \( \partial \mathcal{L} = 0 \). This implies \( \nabla = \langle \dot{e} x \rangle \). Furthermore, \( p := \partial \mathcal{L} = 0 \) is collinear with \( e \) and is independent of the scale of \( e \). As so, \( p \) may be substituted in place of \( e \) in the expression for \( \partial \mathcal{L} \), hence \[
\frac{2}{l^2} \nabla^2 \langle \dot{\mathcal{L}} \rangle \Rightarrow \Omega = \frac{2}{l^2} \mathcal{L} \quad \text{(from Eq. 6)}.
\]
The constraint \( \langle pp \rangle - m^2 \equiv 0 \) does not follow from the Lagrangians Eqs. 5-7. Nevertheless, it is essential for consistency with the map Eq. 8. It must be regarded as a secondary first class constraint (whose purpose is to set \( \Omega \) to \( \Omega \) and the orbital radius to \( \rho \), consistently with the equations of motion).

5.3. Comparison of the clocks. It is convenient to write down the Hamiltonian equations in the CM gauge: \( \langle \rho \rangle = 0, \langle k \rangle = m \) and to consider a space-like direction \( n \) defined in Eq. 3 which is collinear with the projection of \( k \) onto a subspace orthogonal to \( p \). Together with the consistency requirements \( 0 = \langle \langle k \rangle \rangle, H \rangle = 0 = \langle \langle p \rangle \rangle, H \rangle \), this implies for \( u_1 = \pm u_2 \) that \( u_1 = 1, u_2 = \pm 1, u_3 = 0 \) and \( u_4 = \pm \sqrt{2} \). This way the Hamiltonian equations reduce to \[
x = \frac{p \pm n}{m} \quad \rho = 0 = \frac{4}{m \rho^2} \quad \nabla = \pm n, \quad \nabla = \pm m n, \quad \nabla = \pm n, \]
with \( \langle \rho \rangle = -1, \langle n \rangle = 0 \) (then \( k = \frac{\langle k \rangle}{\langle k \rangle} + n \)). The equations for \( n \) and \( \pi \) imply a uniform motion with frequency \( \Omega = \frac{1}{2} \) about a great circle on the unit sphere: \( \dot{n} + \frac{1}{2} \dot{n} = 0 \). The null directions of the clocks’ velocity vectors \( \dot{x} \) are conjugate to one another by the reflection \( \frac{\langle k \rangle}{\langle p \rangle} \rightarrow \frac{2}{n} \frac{\langle n \rangle}{\langle p \rangle} - \frac{\langle k \rangle}{\langle p \rangle} \). Interestingly, the two clocks have opposite spin alignment:
\[
p \wedge k \wedge \pi = \pm \frac{m^2}{4} p \wedge n \wedge n = \pm m p \wedge x \wedge \dot{x},
\]
where \( x = \frac{\dot{t}}{2} \dot{\rho} + \frac{\rho}{2} \dot{t}, \quad \langle \dot{\rho} \rangle = -1, \quad \langle \dot{t} \rangle = 0, \quad \langle \dot{\rho} \rangle = \frac{1}{2} t \) (then \( n = \frac{\dot{t}}{2} \)). In a sense, the two clocks can be regarded as a limiting case of the Lagrangian Eq. 1 with \( \langle \dot{\mathcal{L}} \rangle \rightarrow 0 \) with \( a = \frac{\langle \dot{\mathcal{L}} \rangle}{2 m^2} \rightarrow -1 \) for the clock Eq. 11 or \( b \) \( \langle \mathcal{L} \rangle \rightarrow 0 \) for the clock Eq. 14.

6. Summary and future applications

In this paper were described two mathematical clocks which are relativistic rotators exhibiting intrinsic circular motion with the speed of light and opposite spin alignment. The Lagrangians of the clocks were distinguished by a singularity of an inverse Legendre map for rotators of which Casimir scalars are fixed parameters. Such clocks are perfect, they work always the same way, no matter how they move. In future works, the two ideal clocks can be used to test the clock hypothesis. In free motion, the phase associated with the intrinsic circular motion of these clocks increases steadily with the affine parameter of the center of mass (CM). But it is not a priori obvious (even in the limit \( t \rightarrow 0 \)) if this property will survive for accelerated motions of the CM, e.g. for a constrained motion along a strongly curved worldline. For such motions, the chronometric curve – that is, properly parameterized helical null worldline of an ideal clock – would undergo additional distortions and this could affect the steady clocking rate.

Testing the clock hypothesis requires introducing interactions. However, usual coupling with external fields may lead to inconsistencies, see [15]. In this context, it would be instructive to see the implications of the secondary constraint \( f_{p p} p^k = 0 \) appearing when ideal clocks are minimally coupled with the electromagnetic field. Furthermore, it would be interesting to study the motion in strong gravitational fields of black holes. In curved spacetimes, there might arise problems even with defining the rotation phase: when global teleparallelism is lost, the local reference frames, used to measure the infinitesimal phase increments at various instants, cannot be unambiguously connected; in addition, some disturbances in the phase could appear due to rotation of local inertial frames.

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