Generalized Floquet theory for open quantum systems

C. M. Dai, Hong Li, W. Wang, and X. X. Yi*  
Center for Quantum Science and School of Physics, Northeast Normal University, Changchun 130024, China  
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For a periodically driven open quantum system, the Floquet theorem states that the time evolution operator \( \Lambda(t,0) \) of the system can be factorized as \( \Lambda(t,0) = D(t)e^{\text{eff} \Lambda(t)} \) with micro-motion operator \( D(t) \) possessing the same period as the external driving, and time-independent operator \( \text{eff} \Lambda(t) \). In this work, we extend this theorem to open systems that follow a modulated periodic evolution, in which the fast part is periodic while the slow part breaks the periodicity. We derive a factorization for the time evolution operator that separates the long time dynamics and the micro-motion for the open quantum system. High-frequency expansions for the effective evolution operator control the long time dynamics, and the micro-motion operator is also given and discussed. It may find applications in quantum engineering with open systems following modulated periodic evolution.

I. INTRODUCTION

Floquet theory has very long history. It can be dated back to 1880s in mathematics due to Gaston Floquet who gave a canonical form of solution to periodic linear differential equations. The application of Floquet theory in physics ranges from classical [1] to quantum systems [2], covering a variety of time-dependent dynamics. In recent years, the concept of Floquet engineering has attracted much attention since in periodically driven systems exotic phenomena can emerge that are absent in their undriven counterparts. The Floquet engineering is based on the fact that when a quantum system subject to periodically driving fields, the time evolution is governed by a time-independent effective Hamilton apart from a micro-motion operator [3, 4]. This concept provides us with a versatile tool to manipulate quantum systems and has been employed successfully in experiments, such as the control of superfluid-to-Mott-insulator transition [5, 6], the realization of artificial magnetic fields and topological band structures [7, 8] as well as the modulation of spin-orbit couplings [9, 10], to mention a few of them.

The Floquet theory has found its application not only in closed quantum system where the dynamics is governed by unitary evolution, but also in open quantum systems undergoing non-unitary evolution. Formally, an open system can be described as follows, when a quantum system is coupled to an environment, the evolution of the whole system (system plus bath) governed by the total Hamiltonian \( H_{\text{total}}(t) \) is unitary. We can get the exact system state by tracing over the bath degrees of freedom [20],

\[
\rho(t) = \text{Tr}_{\text{bath}}[U_{\text{total}}(t)\rho_{sb}(0)U_{\text{total}}^\dagger(t)], \tag{1}
\]

where \( \rho(t) \) is the reduced density matrix for the system, \( \rho_{sb}(0) = \rho(0) \otimes \rho_b(0) \) is the uncorrelated initial system-bath state, and \( U_{\text{total}}(t) = \text{Tr}_{\text{bath}}[\text{exp}(-i \int_0^t H_{\text{total}}(t')dt')] \) (\( \text{T} \) denotes time-ordering here and hereafter). It is possible to cast Eq. 1 into the convolutionless form by certain approximations [20],

\[
\partial_t \rho = \mathcal{L}(t)\rho(t). \tag{2}
\]

An example is the time-dependent Markovian process governed by the generator \( \mathcal{L}(t) \) in the Lindblad form [21]

\[
\partial_t \rho = -i[H(t), \rho(t)] + \sum_{\alpha} \{V_\alpha(t)\rho(t)V_\alpha^\dagger(t) - \frac{1}{2} \{V_\alpha^\dagger(t)V_\alpha(t), \rho(t)\}\}, \tag{3}
\]

where \( V_\alpha(t) (\alpha = 1, 2, 3, ...) \) are time-dependent operators determined by the system-bath interaction, and \( H(t) \) is the time-dependent effective Hamiltonian of the open system. It has been demonstrated that the dynamics given by Eq. 3 can always be embedded in a time-dependent Markovian dynamics on an appropriate extended state space [22]. A large class of non-Markovian quantum processes in open systems can also be described by Eq. 3.

Eq. 2 yields a two-parameter map \( \Lambda(t,s) \) defined by chronological time-ordering operator \( T \)

\[
\Lambda(t,s) = T \text{exp} \int_s^t \mathcal{L}(\tau)d\tau, t \geq s \geq 0, \tag{4}
\]

and it satisfies

\[
\Lambda(t,s)\Lambda(s,t') = \Lambda(t,t'), t \geq s \geq t'. \tag{5}
\]

In terms of these maps, the solution to the master equation Eq. 2 can be written as \( \rho(t) = \Lambda(t,0)\rho(0) \). This means \( \Lambda(t,s) \) propagates the density matrix at time \( s \) to the density matrix at time \( t \).

When the generator \( \mathcal{L}(t) \) possesses discrete time translation symmetry, namely \( \mathcal{L}(t+T) = \mathcal{L}(t) \), here \( T \) is the period. According to the Floquet theorem [19, 24, 25], \( \Lambda(t,s) \) can be decomposed into two parts, one can be given by an effective time-independent generator that controls the long time evolution and another is periodic in time describing the periodic micro-motion of the driven system (called the micro-motion operator) [19, 25].
Physically, such periodic time dependence of generator \( \mathcal{L}(t) \) can be realized, for example, by coupling a static system to periodic driving field \([26, 27]\) or via periodic modulation of the coupling strength between different parts of the system \([28]\). In practical situations, however, the time periodic dynamics may be changed in different ways. For instance, a time periodic driving might be turned on at some instance of time and the amplitude of the driving needs a time to ramp up to a certain value. In this case, the dynamics of the system is not perfectly periodic. It has been demonstrated experimentally that different ramping protocols can influence the Floquet state population \([29]\). The other example is that, consider atoms driven by laser pulses, there may have chirp in the pulses, leading to frequency change in the pulses. This again breaks the periodicity of the dynamics.

In this manuscript, we consider a generalized Floquet formalism to handle the aforementioned problems, where the periodicity of generator \( \mathcal{L}(t) \) is disturbed slightly via other (slow-varying) time-dependent parameters, and the frequency can be chirped and changes slowly. In the generalized formalism, we show that the propagator \( \Lambda(t,s) \) can also be factorized into two parts, one is the long time evolution part given by an effective generator with slow time dependence, and another is micro-motion part with additional slowly changing terms.

The remainder of this manuscript is organized as follows. In Sec. II, we present a generalized Floquet formalism that separates the long time evolution and micro-motion. In Sec. III, we calculate the effective generator and micro-motion operator by high frequency expansions. In Sec. IV, we demonstrate our results with two examples. Conclusions and discussions are presented in Sec. V.

II. FORMULISM

We start with the dynamic equation Eq. (2) for the density matrix. In our case, \( \mathcal{L}(t) = \mathcal{L}[\omega t, p(t)] \) and \( \mathcal{L} \) is periodical with period \( 2\pi \) with respect to the first argument, namely, \( \mathcal{L}[\omega t + \omega \tau, p(t)] = \mathcal{L}[\theta + 2n\pi, \omega t, p(t)] \) with integer \( n \). Where the periodic time dependence of generator \( \mathcal{L}(t) \) is introduced through \( \theta \to \theta + \omega \tau \), and \( p(t) \) represents a set of time-dependent parameters that disturb the periodicity of \( \mathcal{L}(t) \). In Sec. IV, we will present two examples to show how \( \omega \tau \) and \( p(t) \) enter the dynamics of open systems. In the situation of chirped frequency, \( \omega \) depends on time. Here we define \( \omega_{eff} = \partial_t (\omega t) \) (called effective instantaneous frequency) that we will use later.

The formal solution of Eq. (2) can be given by the propagator in Eq. (1)

\[
\tilde{\rho}(t, \theta) \equiv \mathcal{T} \text{exp} \left[ \int_0^t \mathcal{L}[\omega t + \theta, p(\tau)] d\tau \right] \rho(0),
\]

or in the differential form with initial condition,

\[
\partial_t \tilde{\rho}(t, \theta) = \mathcal{L}[\omega t + \theta, p(\tau)] \tilde{\rho}(t, \theta), \tilde{\rho}(0, \theta) = \rho(0).
\]

We can think of Eq. (1) as a family of equations parameterized by initial phase \( \theta \), the corresponding propagators \( \Lambda_\theta(t,s) = \mathcal{T} \text{exp} \left[ \int_0^t \mathcal{L}[\omega t + \theta, p(\tau)] d\tau \right] \) are also dependent on parameter \( \theta \).

We extend the physical Hilbert space \( \mathcal{H} \) to Floquet space \( \mathcal{H} \otimes \mathcal{F} \) \([4, 30, 37]\). Where \( \mathcal{F} \) is the space of square-integrable functions on the circle of length \( 2\pi \) with scalar product defined by \( \langle \xi_1 | \xi_2 \rangle = \int_0^{2\pi} \xi_1(\theta)\xi_2(\theta) d\theta / 2\pi \). The space \( \mathcal{F} \) can be spanned by the orthonormal basis \( \{ e^{in\theta} \} \) with \( n \in \mathbb{Z} \) (all integers).

On the Floquet space \( \mathcal{H} \otimes \mathcal{F} \), we can define a Floquet generator and that the evolution it generates is essentially equivalent with Eq. (6). The Floquet generator is defined as,

\[
\mathcal{L}_\mathcal{F}(t) = -\omega_{eff} \frac{\partial}{\partial \theta} + \mathcal{L}[\theta, p(t)].
\]

the corresponding propagator satisfies equation,

\[
\partial_t \Lambda_\mathcal{F}(t,s) = \mathcal{L}_\mathcal{F}(t) \Lambda_\mathcal{F}(t,s), \Lambda_\mathcal{F}(s,s) = 1.
\]

The relation between \( \Lambda_\theta(t,s) \) and \( \Lambda_\mathcal{F}(t,s) \) is given by

\[
\Lambda_\theta(t,s) = S(\omega t) \Lambda_\mathcal{F}(t,s) S(-\omega s),
\]

where

\[
S(\omega t) = e^{\omega t/\partial \theta}.
\]

is the shift operator respect to \( \theta \). This relation can be easily verified by the definition and notice that \( \partial_t S(\omega t) = S(\omega t) \omega_{eff} S(\omega t) \), \( S(-\omega t) \mathcal{L}[\omega t + \theta, p(\tau)] S(\omega t) = \mathcal{L}[\theta, p(t)] \).

We can see from the relation Eq. (10) that the density matrix with initial condition \( \tilde{\rho}(0, \theta) = \rho(0) \) propagates by generator \( \mathcal{L}_\mathcal{F}(t) \) is equivalent to by \( \mathcal{L}[\omega t + \theta, p(\tau)] \) up to a shift transformation. More specifically, we define \( \tilde{\rho}_\mathcal{F}(t, \theta) \) by equation,

\[
\partial_t \tilde{\rho}_\mathcal{F}(t, \theta) = \mathcal{L}_\mathcal{F}(t) \tilde{\rho}_\mathcal{F}(t, \theta), \tilde{\rho}_\mathcal{F}(0, \theta) = \rho(0),
\]

and we have

\[
\tilde{\rho}(t, \theta) = S(\omega t) \tilde{\rho}_\mathcal{F}(t, \theta).
\]

By this transformation, we can transfer the dynamics to a frame that is independent of \( \omega t \). Namely, the periodic time dependence introduced by \( \omega t \) can be eliminated by the transformation (this elimination holds even \( \omega \) is time dependent). Ideally, for fixed \( \omega_{eff} \) and \( p(t) \), \( \mathcal{L}_\mathcal{F} \) is time independent. The slow variation of \( \omega_{eff} \) and \( p(t) \)
will introduce a slow time dependence to the Floquet generator.

To process, we expand \(\hat{\rho}_F(t, \theta)\) by basis \(\{e^{i n \theta}\}\) in the Floquet space \(\mathcal{H} \otimes \mathcal{F}\),

\[
\hat{\rho}_F(t, \theta) = \sum_{n=-\infty}^{\infty} \hat{\rho}_F^{(n)}(t) e^{i n \theta}.
\]  

(14)

We obtain a set of equations for the expansion coefficients \(\hat{\rho}_F^{(n)}(t)\) in the physical space \(\mathcal{H}\).

\[
\partial_t \hat{\rho}_F^{(n)}(t) = \sum_{m=-\infty}^{\infty} \mathcal{L}_F^{(n,m)}(t) \hat{\rho}_F^{(m)}(t).
\]  

(15)

where \(\mathcal{L}_F^{(n,m)}(t) \equiv \int_0^{2\pi} e^{-i n \theta} \mathcal{L}_F(t) e^{i m \theta} d\theta/2\pi = \mathcal{L}_F^{(n-m)}[p(t)] - i n \omega_{\text{eff}} \delta_n^m\) with \(\mathcal{L}_F^{(n)}[p(t)] \equiv \int_0^{2\pi} \mathcal{L}_F[t, \theta, p(t)] e^{-i n \theta} d\theta/2\pi\).

Define a vector by the coefficients \(\hat{\rho}_F^{(n)}(t)\),

\[
\hat{\rho}_F(t) = [\cdots, \hat{\rho}_F^{(-1)}(t), \hat{\rho}_F^{(0)}(t), \hat{\rho}_F^{(1)}(t), \cdots],
\]  

we can write Eq. (15) as

\[
\partial_t \hat{\rho}_F(t) = \hat{\mathcal{L}}_F(t) \hat{\rho}_F(t),
\]  

(16)

here we use the same symbol \(\hat{\mathcal{L}}_F(t)\) to represent the matrix form of \(\hat{\mathcal{L}}_F(t)\) in Eq. (8) with the basis \(\{e^{i n \theta}\}\).

With the shift matrix \(R_l^{(n,m)} = \delta_{n+l}^m\) and the number matrix \(N^{(n,m)} = n \delta_n^m\), \(\mathcal{L}_F(t)\) can be written in a more compact form,

\[
\hat{\mathcal{L}}_F(t) = \sum_{n=-\infty}^{\infty} \mathcal{L}_F^{(n)}[p(t)] R_n - i \omega_{\text{eff}} N.
\]  

(17)

This means that \(\hat{\mathcal{L}}_F(t)\) takes the following form

\[
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]  

where

\[
\mathcal{L}_F^{(n)}[p(t)] = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]  

(18)

The matrix \(R_n\) and \(N\) in Eq. (17) satisfy commutation relations \([R_l, R_k] = 0, [R_l, N] = -i R_k\) and \(R_l R_k = R_{l+k}\) with \(l\) and \(k\) arbitrary integers. These relations are useful when we derive a high frequency expansion in the Sec. III.

We shall find a transformation \(\mathcal{D}(t)\) that block diagonalize \(\hat{\mathcal{L}}_F(t)\). This means with the transformation defined by \(\mathcal{D}(t)\),

\[
\hat{\rho}_F(t) = \mathcal{D}(t) \hat{\rho}_D(t),
\]  

(19)

the dynamic of \(\hat{\rho}_D(t)\) is governed by a generator \(\hat{\mathcal{L}}_D(t)\) of block diagonal form in the Floquet space \(\mathcal{H} \otimes \mathcal{F}\),

\[
\partial_t \hat{\rho}_D(t) = \hat{\mathcal{L}}_D(t) \hat{\rho}_D(t),
\]  

(20)

with

\[
\hat{\mathcal{L}}_D(t) = \mathcal{L}_{\text{eff}}(t) R_0 - i \omega_{\text{eff}} N,
\]  

(21)

where \(\mathcal{L}_{\text{eff}}(t)\) represents an effective generator in the physical space \(\mathcal{H}\). Combining Eq. (16), Eq. (19) and Eq. (20), \(\hat{\mathcal{L}}_F(t)\) and \(\hat{\rho}_D(t)\) satisfy the following equation,

\[
(\partial_t \mathcal{D}^{-1}) \mathcal{D} + \mathcal{D}^{-1} \hat{\mathcal{L}}_F(t) \mathcal{D} = \hat{\mathcal{L}}_D(t).
\]  

(22)

Thus \(\hat{\mathcal{L}}_F(t)\) and \(\mathcal{L}_{\text{eff}}(t)\) are connected through relation,

\[
(\partial_t \mathcal{D}^{-1}) \mathcal{D} + \mathcal{D}^{-1} \hat{\mathcal{L}}_F(t) \mathcal{D} = \mathcal{L}_{\text{eff}}(t) R_0 - i \omega_{\text{eff}} N.
\]  

(23)

Due to the special structure of Floquet generator \(\hat{\mathcal{L}}_F(t)\) Eq. (17), if we find a transformation \(\mathcal{D}(t)\) and the corresponding \(\mathcal{L}_{\text{eff}}(t)\) satisfy Eq. (23), \(\mathcal{D}(t) = \mathcal{R}_l^{-1} D(t) \mathcal{R}_l\) is also a solution of Eq. (23) with the same \(\mathcal{L}_{\text{eff}}(t)\). It is sufficient to consider that \(\mathcal{D}(t)\) is invariant under shift transformation \(\mathcal{R}_l\) [152], this is the case when \(\mathcal{D}(t)\) takes following form,

\[
\mathcal{D}(t) = \sum_{n=-\infty}^{\infty} \mathcal{D}^{(n)}(t) \mathcal{R}_n.
\]  

(24)

When \(\hat{\mathcal{L}}_F(t)\) is time independent, \(\mathcal{D}(t)\) can be chosen to be time independent [4131922] and by solving Eq. (23) we obtain the time independent effective generator \(\mathcal{L}_{\text{eff}}(t)\). This is exactly the situation of conventional Floquet theory with the generator \(\mathcal{L}(t)\) having perfect periodic time dependence. For the more general situation we consider here, the solution \(\mathcal{D}(t)\) and \(\mathcal{L}_{\text{eff}}(t)\) of Eq. (23) may have explicit time dependence.

If \(\mathcal{D}(t)\) has the form of Eq. (24), its inverse should also have the form \(\mathcal{D}^{-1}(t) = \sum_{n=-\infty}^{\infty} \mathcal{D}^{(n)}(t) \mathcal{R}_n\) (By the uniqueness of inverse and the equation \(\mathcal{D}^{-1} \mathcal{D} = 1\) is invariant under shift transformation \(\mathcal{R}_l\)) where \(\mathcal{D}^{(n)}(t)\) is the expansion coefficients of \(\mathcal{D}^{-1}(t)\) in the basis of shift matrix \(\{\mathcal{R}_n\}\).

Consider the block diagonal form of \(\hat{\mathcal{L}}_D(t)\), the formal solution for Eq. (20) is that

\[
\hat{\rho}_D^{(n)}(t) = e^{-i \omega t} \Lambda(t, t) \mathcal{D}^{(n)}(0) \rho(0),
\]  

(25)

where \(\Lambda(t, t) = \mathcal{T} \exp \left[ \int_0^t \mathcal{L}_{\text{eff}}(\tau) d\tau \right]\). What follows is

\[
\hat{\rho}_F^{(n)}(t) = \mathcal{D}(t) \hat{\rho}_D^{(n)}(t)^{\mathcal{D}}(t).
\]  

(26)

Restoring the basis \(\{e^{i n \theta}\}\) using Eq. (14) and performing the shift \(S(\omega t)\) respect to \(\theta\) using Eq. (15), we obtain

\[
\rho(t, \theta) = \mathcal{D}(\theta + \omega t, t) \Lambda(t, 0) \mathcal{D}^{-1}(\theta, 0) \rho(0).
\]  

(27)
where $D(\theta + \omega t, t) \equiv \sum_{n=-\infty}^{\infty} D^{(n)}(t)e^{in(\theta + \omega t)}$ is the micro-motion operator depends on the initial phase $\theta$. Here the relation $D^{-1}(\theta + \omega t, t) = \sum_{n=-\infty}^{\infty} D^{-1(n)}(t)e^{-in(\theta + \omega t)}$ is used.

The expression Eq. (27) represents a generalized Floquet theory. For $D$ on the Floquet space $\mathcal{H} \otimes F$ without explicit time dependence, the coefficients $D^{(n)}$ are time independent, the micro-motion operator $D(\theta + \omega t, t)$ has periodic time dependence with period $T = 2\pi/\omega$. Generally, $D(\theta + \omega t, t)$ will acquire additional time dependence due to the explicit time dependence of $D^{(n)}(t)$.

Once we obtain the effective generator $\mathcal{L}_{eff}(t)$ and micro-motion operator $D(\theta + \omega t, t)$, we can use Eq. (27) to find the time evolution of density matrix. When $\mathcal{L}_{eff}(t)$ changes sufficiently slowly, the long time evolution $\mathcal{T}exp[\int_0^t \mathcal{L}_{eff}(\tau)d\tau]$ can be treated as a dynamics governed by the effective generator. Adiabatic approximation [34, 35, 36] can be then applied straightforwardly.

Usually, $\mathcal{L}_{eff}(t)$ and $D(\theta + \omega t, t)$ can not be determined analytically. But for sufficiently high instantaneous frequency $\omega_{eff}$, i.e., the operator norm of off-diagonal elements of $\mathcal{L}_{eff}(t)$ is much smaller than the instantaneous frequency $||\mathcal{L}^{(n)}[p(t)]|| \ll \omega_{eff}$ ($n \neq 0$) and it changes little over one period $||\mathcal{L}^{(n)}[p(t)]|| \ll \omega_{eff}||\mathcal{L}^{(n)}[p(t)]||$, both $\mathcal{L}_{eff}(t)$ and $D(\theta + \omega t, t)$ can be represent as power series of inverse instantaneous frequency $\omega_{eff}^{-1}$ [38, 39], see the next section.

### III. HIGH FREQUENCY EXPANSION

In this section we calculate $D(\theta + \omega t, t)$ and $\mathcal{L}_{eff}(t)$ in the high frequency limit. Recall that matrix $D(t)$ can be written as an exponential form $D(t) = e^{\Omega(t)}$. Because $D(t)$ has skew diagonal form, $\Omega(t)$ should have the same form $\Omega(t) = \sum_{n=-\infty}^{\infty} \Omega^{(n)}(t)R_n$. Denote $\Omega(t)$ and $\mathcal{L}_{eff}(t)$ as a sum of different orders of $\omega_{eff}^{-1}$,

$$\Omega(t) = \sum_{n=1}^{\infty} \Omega^{(n)}(t),$$

$$\mathcal{L}_{eff}(t) = \sum_{n=0}^{\infty} \mathcal{L}_{eff(n)}(t),$$

where $n$-th terms $\Omega^{(n)}(t) = \sum_{m=-\infty}^{m=\infty} \Omega^{(m,n)}(t)R_m$ and $\mathcal{L}_{eff(n)}(t)$ are of the order of $\omega_{eff}^{-n}$ (for simplicity we omit the argument $t$ hereafter). Take these into Eq. (27) and expand the left hand side of Eq. (27) by identity [34, 35]

$$(\partial_t D^{-1})D^{-1} \mathcal{F} = \sum_{n=0}^{\infty} \frac{1}{(k+1)!} Ad_{\Omega^{(k)}}[-\tilde{\Omega}] + \sum_{k=0}^{\infty} \frac{1}{k!} Ad_{\Omega}^{k}[\tilde{\mathcal{F}}],$$

where $Ad_{\Omega}[\mathcal{X}]$ means,

$$\sum_{k=0}^{\infty} \frac{1}{k!} [\mathcal{X}, \cdots [\mathcal{X}, \mathcal{Y}], \cdots].$$

we can derive expressions for the expansion in the following way.

To simplify the results, we will set $\Omega^{(0)}(t) = 0$ to find a special solution for the effective generator, because $\mathcal{F}_{eff}(t)$ is not uniquely identified by Eq. (27). [32, 33, 34]. Collecting the same order terms in both sides of the Eq. (27), we get a series of equations. The first three equations are

$$\mathcal{L}_{eff(0)} \mathcal{F}_0 = \tilde{\mathcal{F}} + [\Omega^{(1)}, i\omega_{eff} \mathcal{N}],$$

$$\mathcal{L}_{eff(1)} \mathcal{F}_0 = -\Omega^{(1)} - [\Omega^{(1)}, \tilde{\mathcal{F}}] + [\Omega^{(2)}, i\omega_{eff} \mathcal{N}] - \frac{1}{2!} [\Omega^{(1)}, [\Omega^{(1)}, i\omega_{eff} \mathcal{N}]],$$

$$\mathcal{L}_{eff(2)} \mathcal{F}_0 = -\Omega^{(2)} + \frac{1}{2!} [\Omega^{(1)}, [\Omega^{(1)}, \tilde{\mathcal{F}}]] - [\Omega^{(2)}, [\Omega^{(3)}, i\omega_{eff} \mathcal{N}]] - \frac{1}{2!} [\Omega^{(1)}, [\Omega^{(2)}, i\omega_{eff} \mathcal{N}]] - \frac{1}{2!} [\Omega^{(2)}, [\Omega^{(3)}, i\omega_{eff} \mathcal{N}]] + \frac{1}{3!} [\Omega^{(1)}, [\Omega^{(1)}, [\Omega^{(1)}, i\omega_{eff} \mathcal{N}]]],$$

where $\tilde{\mathcal{F}} = \sum_{n=-\infty}^{\infty} \tilde{\mathcal{F}}^{(n)}[p(t)]R_n$. Comparing the coefficients in both sides of Eq. (30) in terms of shift matrix, we can get $\Omega(t)$ and $\mathcal{L}_{eff}(t)$ up to the second order in $\omega_{eff}^{-1}$ and the higher order terms can be obtained in a similar way. The results are,

$$\mathcal{L}_{eff(0)} = \tilde{\mathcal{F}},$$

$$\mathcal{L}_{eff(1)} = \frac{1}{\omega_{eff}} \sum_{n=0}^{\infty} \frac{1}{2n!} [\tilde{\mathcal{F}}^{(-n)}, \tilde{\mathcal{F}}^{(n)}],$$

$$\mathcal{L}_{eff(2)} = \frac{1}{\omega_{eff}} \sum_{n=0}^{\infty} \frac{1}{2n!} [\tilde{\mathcal{F}}^{(-n)}, \tilde{\mathcal{F}}^{(-n)}] + \frac{1}{3n!} [\tilde{\mathcal{F}}^{(-n)}, [\tilde{\mathcal{F}}^{(-n)}, \tilde{\mathcal{F}}^{(-n)}]]$$

$$- \sum_{m,n \neq 0} \frac{1}{3mn} [\tilde{\mathcal{F}}^{(m)}, [\tilde{\mathcal{F}}^{(n)}, \tilde{\mathcal{F}}^{(-m-n)}]].$$

In contrast with the earlier studies that the generator $\mathcal{F}(t)$ is periodic in time with fixed frequency, the $\omega$ here is replaced by the instantaneous frequency $\omega_{eff}$, all components $\mathcal{F}^{(n)}[p(t)]$ of $\tilde{\mathcal{F}}$ can have slow time dependence and a non-trivial term $[\tilde{\mathcal{F}}^{(n)}, \partial_t \tilde{\mathcal{F}}^{(-n)}]$ appears in the second order term. Thus $\mathcal{F}_{eff}(t)$ will acquire slow time dependence and generally not commute at different time. The approximate effective generator depends not only on the instantaneous values of parameters but also on the rate of changes. As for the expansions of $\Omega^{(1)}$ the derivative
of $\omega_{\text{eff}}$ also appears in the second order term,

$$\Omega^{(n)}_{(1)} = -\frac{i}{\omega_{\text{eff}}} \tilde{L}^{(n)},$$

$$\Omega^{(n)}_{(2)} = -\frac{1}{\omega_{\text{eff}}^2} \left[ \frac{1}{2m^2} \mathcal{L}^{(0)}, \mathcal{L}^{(n)} \right] + \sum_{m \neq 0} \frac{1}{2m^2} \mathcal{L}^{(n-m), \mathcal{L}^{(m)}} - \frac{1}{i} \left[ \partial_t \mathcal{L}^{(n)} - (\omega_{\text{eff}}/\omega_{\text{eff}}) \tilde{L}^{(n)} \right].$$  \(\text{(32)}\)

These equations together give the effective generator with slow time dependence $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}(0)} + \mathcal{L}_{\text{eff}(1)} + \mathcal{L}_{\text{eff}(2)} + O(\omega_{\text{eff}}^{-3})$ and the corresponding micro-motion operator $D(\theta + \omega t, t) = \exp[\Omega_{(1)}(\theta + \omega t, t) + \Omega_{(2)}(\theta + \omega t, t) + O(\omega_{\text{eff}}^{-3})]$ where $\Omega_{(n)}(\theta + \omega t, t) = \sum_{m \neq 0} \Omega^{(n)}_{(m)}(t)e^{i m(\theta + \omega t)}$ (We use the relation $\sum_{n = -\infty}^{\infty} \Omega^{(n)}(t)e^{i m(\theta + \omega t)} = \exp[\sum_{n = -\infty}^{\infty} \Omega^{(n)}(t)e^{i m(\theta + \omega t)}]$ when we calculate the summation).

Discussions on the present expansions are in order. When $\omega_{\text{eff}} \to \infty$, $\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}(0)}$, i.e., the generator takes the zeroth Fourier component, while the micro-motion $D(\theta + \omega t, t) \to I$ approaches identity in this case. When the frequency $\omega$ and slow-varying parameters $p$ are independent of time, the expansions Eq. (31) and Eq. (32) reduce to the time-independent form that are the same as that in previous works \[6, 7, 8, 11\], where the dynamics is strictly periodic.

IV. EXAMPLES

In this section we illustrate our theory with two examples. The first example consists of a spin $\frac{1}{2}$ particle coupled to a time-dependent magnetic field, and the second one is a harmonic oscillator driven by a time-dependent field. Both of them are subject to decoherence.

The generators that describe the two examples have a similar form

$$\mathcal{L}(t)(\phi) = \hat{L}[\theta + \omega t, p(t)](\phi) = -i[H(t), \phi] + \gamma(2X_- \circ X_+ - \{X_+ X_-, \phi\}),$$  \(\text{(33)}\)

where $X_+ = X_1^\dagger$ represents system-bath interaction and $\gamma$ is decay rate (system-bath coupling strength), the Hamiltonian $H(t) = H^{(1)}[p(t)] + H^{(-1)}[p(t)]e^{-i(\theta + \omega t)} + H^{(1)}[p(t)]e^{i(\theta + \omega t)}$ with $H^{(n)}[p(t)]$, $n = -1, 0, 1$ depends on the slowly-varying parameters $p(t)$. Then expansion Eq. (31) reduces to (leave out trivial term $\mathcal{L}_{\text{eff}(0)}$)

$$\mathcal{L}_{\text{eff}(1)}(\phi) = \frac{1}{i\omega_{\text{eff}}} \{[H^{(1)}(H^{(-1)}), \phi],$$

$$\mathcal{L}_{\text{eff}(2)}(\phi) = \frac{1}{2\omega_{\text{eff}}^2} \sum_{n = -1} \{[[\partial_t H^{(-1)}, H^{(n)}], \phi]$$

$$+ i[H^{(n)}, [H^{(-1)}, H^{(0)}]], \phi]$$

$$+ \gamma \{[H^{(n)}, [H^{(n)}, X_+ X_-]], \phi]\}$$

$$+ 2[H^{(n)}, X_+ \circ X_+ H^{(-1)}]$$

$$+ 2[H^{(n)}, X_+ \circ X_- H^{(n)}]$$

$$+ 2[H^{(n)}, [X_-, H^{(-1)}]] \circ X_+$$

$$+ 2X_- \circ [H^{(n)}, [X_+, H^{(-1)}]].$$  \(\text{(34)}\)

As shown, at a long time scale, both the effective Hamiltonian and system-bath interaction are modified by driving field characterized by $H^{(-1)}[p(t)] = H^{(1)}[p(t)^\dagger]$ and frequency $\omega$.

A. Spin $\frac{1}{2}$ particle

For a spin $\frac{1}{2}$ particle in a fast oscillating magnetic field with additional slow modulation, the system can be described by the generator in Eq. (33) with $X_- = \sigma_-$, $X_+ = \sigma_+$ and $H(t) = \alpha B_{\text{total}}(t) \cdot \sigma$, where $\alpha$ is the coupling constant and magnetic field $B_{\text{total}}(t) = B(t) + B(t)^\dagger$. Here the slowly-varying parameter $p(t) = \{B(t), B(t)^\dagger\}$. $\omega$ is the angular frequency of the fast oscillating. Using Eq. (33), we get

$$\mathcal{L}_{\text{eff}(1)}(\phi) = \frac{2\alpha^2}{\omega_{\text{eff}}} [B \times B^\dagger] \cdot \sigma, \phi],$$

$$\mathcal{L}_{\text{eff}(2)}(\phi) = \frac{\alpha^2}{2\omega_{\text{eff}}} \{[-4i\alpha[B \times B^\dagger \times B_0] \cdot \sigma, \phi]$$

$$+ 2[\partial [B \times B^\dagger] \cdot \sigma, \phi]$$

$$- \gamma [i[B \times B^\dagger \times [e_+ \times e_-] \cdot \sigma, \phi]$$

$$+ 4[B \times e_-] \cdot \sigma \circ [e_+ \times B^\dagger] \cdot \sigma]$$

$$+ 2[B \times e_- \cdot [B + e_+ B^\dagger] \cdot \sigma \circ \sigma_+]$$

$$+ 2\sigma_+ \circ [B \times [e_+ \times B^\dagger] \cdot \sigma]$$

$$+ (B \leftrightarrow B^\dagger)\},$$  \(\text{(35)}\)

where $e_{\pm} = (e_{x} \pm i e_{y})/2$, $e_{x}$ and $e_{y}$ are the unit vectors along the $x$- and $y$-direction, respectively.

Consider the case that $B_{\text{total}}(t)$ rotates rapidly around $e_{x}$ in the $y$-$z$ plane with slowly-varying angular velocity, and denote the included angle between $B_{\text{total}}$ and $e_{y}$ as $\theta + \omega t$. We have $B_{\text{total}}(t) = [0, \cos(\theta + \omega t), \sin(\theta + \omega t)]$ (we will set $\theta = 0$ hereafter for concreteness), i.e., $B_0 = 0$, $B = \pm [0, 1, -i]$ and the angular velocity $\omega_{\text{eff}} = \partial_t (\theta + \omega t)$ changes slowly. Substituting these into Eq. (33), we get

$$\mathcal{L}_{\text{eff}(1)}(\phi) = i(\alpha^2/\omega_{\text{eff}})[\sigma_z, \phi].$$
Because the first two terms of $L_{eff}(2)$ vanish, the third term of $L_{eff}(2)$ proportional to $\omega_{eff}^{-2}$ is also negligible when $\omega_{eff}$ is relatively large, then $L_{eff} \approx L_{eff}(0) + L_{eff}(1)$ is a good approximation.

The system is expected to follow the instantaneous steady state of $L_{eff}$ except the micro-motion. The instantaneous steady state of $L_{eff}$ up to the first order in $\omega_{eff}$ is

$$\rho_s = 1/2 - [\gamma \omega_{eff} \alpha^2 \sigma_y + (\gamma \omega_{eff})^2 \sigma_z)/(2\alpha^4 + (\gamma \omega_{eff})^2)].$$

We can see that the instantaneous steady state $\rho_s$ depends on the coupling constant $\alpha$ and the product of bath coupling strength $\gamma$ and effective frequency $\omega_{eff}$. When $\gamma \omega_{eff} \gg \alpha^2$ or $\gamma \omega_{eff} \ll \alpha^2$, $\rho_s \rightarrow \frac{1}{2} \sigma_x$ or $\frac{1}{2} \sigma_z \sigma_x \sigma_z$ respectively. The steady behavior can be manipulated by the driving frequency or the system-bath coupling $\gamma$.

Fig. 1 shows the difference of the average of $\sigma_z$ in states given by $L_{eff}(0) + L_{eff}(1)$, $L$ and $\rho_s$, respectively. Here we consider $\omega_{eff}$ changing in two different ways—

In the first case, $\omega_{eff}$ increases from $\omega_i$ to $\omega_f$, once the maximum has been reached, $\omega_{eff}$ remains constant. In another case, $\omega_{eff}$ changes periodically.

In Fig. 1 (a) and (c), $\omega_{eff}$ changes slowly enough compared with $1/\gamma$, the results obtained by instantaneous steady state $\rho_s$ are almost the same as that by $L_{eff}$. In Fig. 1 (b) and (d), $\omega_{eff}$ changes a little faster that introduce a small departure between the results obtained by $\rho_s$ and $L_{eff}$.

The results by $L$ and $L_{eff}$ are fairly consistent except for the last oscillation due to the micro-motion. To the first order in $\omega_{eff}$, the operator $\Omega$ can be calculated by Eq. (32).

$$\Omega(1) = -i(\alpha/\omega_{eff})[\sigma_y \sin \omega t - \sigma_z \cos \omega t, \sigma_z].$$

and the micro-motion operator by $D(\sigma) = \epsilon \Omega(1) + D(\omega_{eff}^{-2})$. Up to the first order of $\omega_{eff}^{-1}$, $D(\sigma)$ goes back to its initial value after $\omega t$ changing $2\pi$ with possible correction caused by slowly-varying of $\omega_{eff}$. As show in Fig. 1 (a) red thin line, when $\omega_{eff}$ increase, the amplitude of oscillation decrease gradually and the period of oscillation approximately equals to $2\pi/\omega_{eff}$. Fig. 2 shows the results obtained by the combination of $D(1) = \epsilon \Omega(1) + D_{eff}$ and $D_{eff}$. A comparison with the exact result is also carried out. It is clear that the first order micro-motion together with the effective Lindblad $L_{eff}$ can give a fairly accurate time evolution for the system. It is worth addressing that the higher order terms of $\Omega$ contain dissipative effect due to the system-bath interactions, though in the first order approximation the micro-motion governed by $\Omega$ is well approximate by a unitary evolution $D(1)$.

The situation would be quite different when we consider periodically modulated system-bath interaction, the major contribution of the micro-motion is dissipative [19, 27] that can also be calculated by Eq. (32).

To demonstrate the effect of the non-trivial term $\partial_t B \times B$ that depends on the change rate of slowly-varying parameters $p(t) = \{B_0(t), B(t)\}$. We may set $B_0 = 0$, $B = \frac{1}{2}[0, \cos \omega t, \sin \omega t]$ as an example. Such a specific choice corresponds a magnetic field $B_{total}(t) = \cos(\theta + \omega t)/0, \cos \omega t, \sin \omega t)$ that rotate slowly around $e_z$ in the $y-z$ plane with constant angular velocity $\omega$ and the strength of magnetic field changes quickly with fixed angular frequency $\omega$. By Eq. (35), the first order term of $L_{eff}$ vanishes and the second order term is

$$L_{eff}(2) = \frac{\alpha^2}{\omega^2} \{ -i [\frac{\omega}{2} \sigma_x, \sigma_z] - \gamma \sum_{i,j=0}^3 C_{ij} \sigma_i \sigma_j \}.$$
where the coefficient matrix takes
\[ C = 32 \cos^2 \omega t \begin{pmatrix} 0 & 0 & \tan \omega t & -1 \\ 0 & 2 & i & i \tan \omega t \\ \tan \omega t & -i & 0 & \tan \omega t \\ -1 & -i \tan \omega t & \tan \omega t & -2 \end{pmatrix} \]

When the frequency \( \omega \) is large, the explicit time dependence of \( \mathcal{L}_{\text{eff}} \) caused by matrix \( C \) can be neglected, because it represents a small correction to the decay in the zeroth order term of \( \mathcal{L}_{\text{eff}} \). The major contribution is the Hamiltonian part in \( \mathcal{L}_{\text{eff}}(2) \). In this case the steady state reads,

\[ \rho_{\infty} = 1/2 + (a^2 \omega \gamma \sigma_y - \gamma^2 \omega^4 \sigma_z)/(2 \gamma^2 \omega^4 + a^4 \omega^2) \]

As show in Fig.3, the average of \( \sigma_z \) obtained by \( \rho_{\infty} \) and \( \mathcal{L}_{\text{eff}} \) is very close, and \( \rho_{\infty} \) depends on \( a^2 \omega \gamma \) and \( \gamma^2 \omega^4 \). The change rate of slowly-varying parameters characterized by \( \omega \) can also affect the long time dynamics.

The operator \( \Omega \) up to the first order in \( \omega_{\text{eff}}^{-1} \) can be written as

\[ \Omega(\omega) = -2i(\alpha \sin \omega t/\omega)[B \cdot \sigma, \sigma] \]

and the micro-motion operator takes \( \mathcal{D}(\omega) = e^{\Omega(\omega)} + O(\omega^{-2}) \). For fixed \( B \), \( \mathcal{D}(\omega) = e^{\Omega(\omega)} \) is periodic in time with period \( 2\pi/\omega \). The slow variation of \( B \) introduces additional time dependence to the dynamics. The approximate micro-motion operator becomes identity map when \( t = n\pi/\omega \) with \( n \) positive integers. As shown in Fig.3(c), the red thin line touches the black thick line when \( t = n\pi/\omega \).

B. Harmonic oscillator

For a driven harmonic oscillator coupled to a damped environment, \( \mathcal{L}(t) = -i[H(t), \sigma] + \gamma(2a a^\dagger - \{a^\dagger a, \sigma\}) \) with \( H(t) = \omega_0 \omega^2 a^\dagger a + f(t) \cos \omega_0 t (a + a^\dagger) \). Here \( f(t) \) and \( \omega_0 \) are the amplitude and frequency of driven field, respectively. We consider the situation where \( \omega_0 \approx \omega \). Transforming the master equation to the interaction picture, i.e., \( \rho(t) = e^{i\omega_0 a^\dagger a \rho(t)} e^{-i\omega_0 a^\dagger a \rho(t)} \), we have

\[ \partial_t \rho(t) = -i[H(t), \rho(t)] + \gamma(2a^\dagger \rho(t) a - \rho(t) a^\dagger a) \]

where \( H(t) = f(t) \cos \omega_0 t(\rho \omega^2 a^\dagger a + a^\dagger a \rho(t) a^\dagger a) \). Assume \( \omega_0 \approx \omega \), then \( H(t) \approx H(0) + H^{(1)} e^{-i\omega_0 t} + H^{(2)} e^{-i\omega_0 t} \) depends on slowly-varying parameter \( p(t) = \{f(t), e^{(i\omega_0 - \omega) t}\} \), and \( \omega \) in this case \( \omega = \omega_0 + \omega_0 \). Here, \( H^{(0)} = \frac{i}{2} f(t) (a^\dagger e^{i\omega_0 t} - h.c.), H^{(1)} = \frac{i}{2} f(t) a^\dagger, H^{(2)} = \frac{i}{2} f(t) a \). Substituting these equations into Eq.30 and setting \( X = a, X = a^\dagger, \) one can find that the first order and second order terms of \( \mathcal{L}_{\text{eff}} \) vanish. So up to the second order in \( \omega_{\text{eff}}^{-1} \), we have

\[ \mathcal{L}_{\text{eff}}(\sigma) = -i \frac{f(t)}{2} [e^{i(\omega_0 - \omega) t} + h.c.], \sigma + i \gamma(2a a^\dagger - \{a^\dagger a, \sigma\}] \]

and up to the first order in \( \omega_{\text{eff}}^{-1} \), we have

\[ \Omega(\omega) = -(f(t)/2\omega_{\text{eff}}) [a^\dagger e^{i\omega_0 t} - e^{-i\omega_0 t}, \sigma] \]

So \( \mathcal{D}(\omega) = e^{\Omega(\omega)} = e^{(a^\dagger - a) \sigma} e^{\sigma - a^\dagger a} \) just the displacement operator with parameter \( \alpha = -f(t) e^{i\omega_0 t}/2\omega_{\text{eff}} \).

We plot Fig.4(a) and (b) for fixed \( \omega_0 \) and \( f(t) \), and from the figures we find that though \( \mathcal{L}_{\text{eff}} \) is time-dependent with period \( 2\pi/|\omega_0 - \omega| \), the average number \( \langle a^\dagger a \rangle \) obtained by \( \mathcal{L}_{\text{eff}} \) reaches a steady value due to the asymptotic behavior of \( \rho_{\text{eff}}(t), \rho_{\text{eff}}(t) \rightarrow e^{i(\omega_0 - \omega) t a^\dagger a} e^{(i\omega_0 - \omega) t a^\dagger a} \) with \( \rho_0 \) governed by,

\[ \mathcal{L}_0(\sigma) = i[(\omega_0 - \omega_0) a^\dagger a - f/2(a^\dagger a, a^\dagger + 1) \sigma + i(2a^\dagger a - \{a^\dagger a, \sigma\}] \]

Because \( \rho_1(t) = \mathcal{D}(\rho_{\text{eff}}(t)) \approx \mathcal{D}_1(\rho_{\text{eff}}(t)) \), we have \( \mathcal{T}_r(\rho_{\text{eff}}(t)) = \mathcal{T}_r(\rho_0(\infty) a^\dagger a - 2a^\dagger a \cos(2\omega_0 t + \delta) + |a|^2) \) with \( \rho_0 = \mathcal{T}_r(\rho_0(\infty) a^\dagger a) \). The result is shown in Fig.4(b), see the red thin line that oscillates with period \( \pi/\omega_0 \).

For varying effective frequency, e.g., \( \omega_{\text{eff}} = \omega_0 + \partial_0 \omega_0 \) in Fig.4(c) and (d), the frequency \( \omega_0 \) slowly changes from the resonant point \( \omega_0 = \omega_0 \). The results obtained by effective generator \( \rho_{\text{eff}} \) is also consistent with the exact dynamics except the small oscillation given by the micro-motion operator.

V. CONCLUSION AND DISCUSSIONS

In this paper, we have extended the open-system Floquet theorem to a more general situation. The extended
that break down the periodicity considered in the earlier open-system Floquet theorem. This extension has been done by removing the fast periodic term from the time-evolution operator(or generator) to obtain an effective generator that depends on time slightly. The slow-varying generator leads to an asymptotic solution combining with the micro-motion operator. We also give a high-frequency expansion to the effective generator and micro-motion operator, showing that the first two orders of the expansions agree well with the exact dynamics.

Compared with the conventional Floquet formalism, the slow-varying parameter can play an important role to control the long time dynamics. A natural extension of our result is to consider a system with two periodically drivings. One is small while another is very large. In terms of frequencies, this case is, ω₁ ≫ ω₂, our results can be applied easily to this situation.

Finally, we would like to point out that the formulism presented in this paper is limited to weak system-bath couplings. As we use the master equation as the starting points of discussion.

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