Abstract

Purely magnetic spacetimes, in which the Riemann tensor satisfies $R_{abcd}u^b u^d = 0$ for some unit timelike vector $u^a$, are studied. The algebraic consequences for the Weyl and Ricci tensors are examined in detail and consideration given to the uniqueness of $u^a$. Some remarks concerning the nature of the congruence associated with $u^a$ are made.
1 Introduction

Let \((M, g)\) be a spacetime, that is \(M\) is a smooth, Hausdorff, paracompact, connected 4-dimensional manifold equipped with smooth metric \(g\) of signature +2. There has been some recent interest \([1, 2]\) in spacetimes whose Riemann tensor \(R^{a}_{bcd}\) satisfies (Latin indices take the values 1 to 4 throughout)

\[
R^{a}_{bcd}u^{b}u^{d} = 0
\]  

(1)

for some global smooth unit timelike vector field \(u^{a}\). Such spacetimes are known as purely magnetic. One can also define a spacetime to be purely Weyl magnetic if the Weyl tensor satisfies a condition analogous to (1) and clearly the two preceding definitions are equivalent in vacuum. In this paper some further remarks regarding the existence and uniqueness of solutions to (1), for \(u^{a}\), will be made and the relationship between purely magnetic and purely Weyl magnetic spacetimes clarified. In the vacuum case, some information regarding the rotation and shear of the timelike congruence associated with a solution of (1) will be obtained.

It should also be remarked that condition (1) expresses the fact that \(u^{a}\) is a non-generic vector field as studied recently by Beem and Harris \([3]\), who relate the existence of such vector fields to properties of the sectional curvature. The non-existence of non-generic vector fields is an assumption of some of the singularity theorems \([4, 5]\), in that it is required that if \(v^{b}\) is a tangent vector to a timelike geodesic then it satisfies \(R^{a}_{bcd}v^{b}v^{d} \neq 0\) at some point on the geodesic. If a solution \(u^{a}\) of (1) is tangent to a geodesic then this geodesic can have no conjugate points, as a simple application of the Jacobi equation will show.

A related concept is that of a purely electric spacetime in which the Riemann tensor satisfies

\[
^{*}R^{a}_{bcd}u^{b}u^{d} = 0
\]  

(2)

The asterisk denotes the usual dual operation, and is placed on the left to show that the dual is taken on the indices \(ab\) although it should be noted that the above equation is equivalent to the same equation with the dual placed on the right. It is clear that one can define purely Weyl electric spacetimes in an analogous fashion, by replacing the Riemann tensor in (2) by the Weyl tensor. Purely electric spacetimes were studied by Trümper \([6]\) who showed that they are necessarily purely Weyl electric. There is, however, no direct analogue of Trümper’s result for purely magnetic spacetimes, as illustrated
by an example discussed by Arianrhod et al. [2]. A theorem concerning the
relationship between purely magnetic and purely Weyl magnetic spacetimes
can, however, be formulated and will be given in section 3. A spacetime which
is purely Weyl magnetic and purely Weyl electric necessarily has vanishing
Weyl tensor [7], but it is possible for a non-flat, non-vacuum spacetime to
satisfy both conditions [1] and [2] simultaneously and this possibility will
be discussed in section 3.

2 Purely Magnetic Vacuum Spacetimes

Suppose that \((M, g)\) is a spacetime and at some \(p \in M\) the following equation
holds for the Weyl tensor \(C_{abcd}\) and a unit timelike vector \(u^a\).

\[
C_{abcd} u^b u^d = 0 \quad (3)
\]

The algebraic consequences of this condition on the Weyl tensor will now be
deduced.

The Petrov types of the Weyl tensor correspond to the possible Jordan
canonical forms of the \(3 \times 3\) complex symmetric matrix \(P_{\alpha\beta}\) defined by the
following equation [8] (Greek letters run from 1 to 3).

\[
P_{\alpha\beta} = C_{\alpha4\beta4} + iC_{\alpha4\beta4} \quad (4)
\]

In the above \(C_{abcd}\) are the components of the Weyl tensor at \(p\) with respect to
any orthonormal frame in which the metric is given by \(g = \text{diag}(1, 1, 1, -1)\)
and, since the left and right duals of the Weyl tensor are equal, one can
place the asterisk as above without ambiguity. If there exists a solution \(u^a\) of
(3) then, in an orthonormal frame whose timelike member is \(u^a\), \(C_{\alpha4\beta4} = 0\)
and so the matrix \(P_{\alpha\beta}\) is a purely imaginary symmetric matrix, and hence
diagonalisable. This implies that the Petrov type of the Weyl tensor at \(p\)
is either I, D or O with purely imaginary Petrov scalars. Conversely if \(P_{\alpha\beta}\)
assumes diagonal form with purely imaginary entries in some orthonormal
frame then from (3) one has that \(C_{\alpha4\beta4} = 0\) and hence the timelike member
of this frame satisfies (3). The vector \(u^a\) in (3) is the timelike member of the
canonical Petrov tetrad and hence its uniqueness properties can be obtained
from (3). The algebraic information available from equation (3) is contained
in the following theorem, which extends results in [1, 3].
Theorem 1 Suppose that \((M, g)\) is a spacetime and \(p \in M\). It follows that the Weyl tensor at \(p\) satisfies (3) for a unit timelike vector field \(u^a\) if and only if it is of Petrov type \(I\), \(D\) or \(O\) with purely imaginary Petrov scalars. In the case of Petrov type \(I\), \(u^a\) is unique up to sign and in the case of Petrov type \(D\), \(u^a\) is determined up to arbitrary boosts in the 2-space spanned by the principal null directions.

In the case of a purely magnetic vacuum spacetime it can be shown using the Bianchi identities that the Weyl tensor cannot be of Petrov type \(D\) over a non-empty open set [1] (this result was also given by Hall [9] in a slightly different context). If this fact is combined with Theorem 1 then one has

Theorem 2 Suppose that \((M, g)\) is a purely magnetic vacuum spacetime which is non-flat in the sense that its Riemann tensor does not vanish over an non-empty open set. The Weyl tensor is then of Petrov type \(I\) almost everywhere (i.e. on an open dense subset of \(M\)) and the unit timelike vector \(u^a\) is unique up to sign.

The final result in this section will concern the kinematic properties of the timelike congruence associated with a solution \(u^a\) of (3). The covariant derivative of a unit timelike vector field \(u_a\) can be decomposed in the following well known way (see e.g. [7, 8])

\[
\begin{align*}
    u_{a;b} &= -\dot{u}_a u_b + \omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta h_{ab} \\
    \theta &= \frac{1}{3} \theta h_{ab}
\end{align*}
\]

In equation (5) \(\dot{u}_a \equiv u_{a;b} u^b; \ \omega_{ab} = \omega_{[ab]}; \ \sigma_{ab} = \sigma_{(ab)}\) and \(h_{ab} \equiv g_{ab} + u_a u_b\) where square brackets denote skew-symmetrisation, round brackets denote symmetrisation and a semi-colon denotes a covariant derivative. The quantities \(\omega_{ab}, \sigma_{ab}\) and \(\theta\) are known respectively as the rotation, shear and expansion of the congruence. If the shear and rotation of a timelike congruence in some spacetime vanish then the spacetime must be purely Weyl electric [8]. For purely magnetic vacuum spacetimes the properties of the timelike congruence associated with \(u^a\) are controlled by the following theorem.

Theorem 3 Let \((M, g)\) be a purely magnetic non-flat vacuum spacetime with \(u^a\) the (necessarily unique up to sign) unit timelike vector field satisfying (3). It then follows that the shear of the timelike congruence associated with \(u^a\) cannot vanish over a non-empty open set and if the rotation is identically zero then the shear necessarily assumes a diagonal form with respect to the canonical Petrov tetrad.
From (3) it follows that $(C_{abcd}u^b u^d)^a = 0$ and if the vacuum Bianchi identities are used together with the cyclic symmetry $C_{a[bcdf]} = 0$ then one obtains

$$C_{abcd}u^d (2u^{bc} - u^{cd}) = 0$$

(6)

Now if (5) is used to substitute for $u_{a;b}$ then it can be seen that

$$C_{abcd}u^d (3\omega^{bc} + \sigma^{bc}) = 0$$

(7)

Attention will now be restricted to a generic point $p \in M$ where it may be assumed (by theorem 2) that the Weyl tensor is of Petrov type I and (by a theorem of Brans [10]) that all the Petrov scalars are non-zero. Let $x^a, y^a, z^a, u^a$ be a canonical Petrov tetrad (unique up to reflections) and fix an orientation such that this tetrad is positively oriented. A basis for the space of 2-forms at $p$ is given by $F_{ab} = u_{(a} x_{b)}$, $G_{ab} = u_{(a} y_{b)}$, $H_{ab} = u_{(a} z_{b)}$ and their duals. The canonical form for a Petrov type I Weyl tensor with purely imaginary Petrov scalars is, with respect to the above basis [11]:

$$C_{abcd} = \lambda_1 (F_{ab} F_{cd}^* + F_{cd} F_{ab}^*) + \lambda_2 (G_{ab} G_{cd}^* + G_{cd} G_{ab}^*) + \lambda_3 (H_{ab} H_{cd}^* + H_{cd} H_{ab}^*)$$

(8)

where $\lambda_1, \lambda_2$ and $\lambda_3$ are real and satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$. If equation (8) is substituted into (7) then one obtains the following three equations

$$3(\lambda_3 + \lambda_2) z_{(b} y_{c)} \omega^{bc} + (\lambda_2 - \lambda_3) z_{(b} y_{c)} \sigma^{bc} = 0$$

(9)

$$3(\lambda_1 + \lambda_3) x_{(b} z_{c)} \omega^{bc} + (\lambda_3 - \lambda_1) x_{(b} z_{c)} \sigma^{bc} = 0$$

(10)

$$3(\lambda_2 + \lambda_1) y_{(b} x_{c)} \omega^{bc} + (\lambda_1 - \lambda_2) y_{(b} x_{c)} \sigma^{bc} = 0$$

(11)

Since $p$ is a generic point then one cannot have $\lambda_i = \pm \lambda_j$ for $i \neq j$. Suppose the shear $\sigma_{ab}$ is identically zero on some non-empty open subset $U \subset M$. Equations (9) to (11) will then show that the rotation vanishes almost everywhere on $U$ and hence is identically zero on $U$ by continuity. It then follows [1] that the spacetime is purely electric on $U$ and hence $C_{abcd} = 0$ on $U$, contradicting the non-flat assumption. The shear is therefore non-zero on an open dense subset of $M$.

If the rotation is identically zero on $M$ then equations (9) to (11) show that $\sigma_{ab}$ takes the form $\alpha x_a x_b + \beta y_a y_b + \gamma z_a z_b$ for some real functions $\alpha, \beta$ and $\gamma$. $\square$
3 Purely Magnetic non-vacuum Spacetimes

In this section consideration will be given to the existence and uniqueness of solutions to equation (1) for non-vacuum spacetimes. Unfortunately it has not proved possible to give a complete solution to the problem as is provided by theorems 1 and 2 in the vacuum case but, nevertheless, some algebraic information is available about the Riemann tensor of purely magnetic non-vacuum spacetimes. The question of under what circumstances equation (1) implies equation (3) is answered by theorem 5 and the consequences of (1) for the type of matter field present are given by theorem 7.

The first point to be remarked upon is that the terminology ‘purely magnetic’ is a little misleading in the non-vacuum case. The terminology arose due to the fact that one can decompose the Weyl tensor in a formally similar fashion to the decomposition of the Maxwell bivector into electric and magnetic components. The electric and magnetic components of the Weyl tensor (with respect to a given observer \(u^a\)) are, respectively, the real and imaginary parts of the matrix \(P_{\alpha\beta}\) defined by equation (4), where \(u^a\) is the timelike member of the orthonormal frame. The Weyl tensor can then be described as ‘purely magnetic’ if its electric part vanishes, and vice-versa. If a similar construction were to be attempted for the (non-vacuum) Riemann tensor then it would be possible for the corresponding \(P_{\alpha\beta}\) to vanish for a non-zero Riemann tensor. The Weyl tensor decomposition relies on the property \(\star C_{abcd} = C_{abcd}\), which is not enjoyed by the Riemann tensor.

The consequence of the preceding remarks is that one can construct spacetimes which are purely electric and purely magnetic and such spacetimes are characterised by the existence of a timelike \(u^a\) satisfying

\[
R_{abcd}u^d = 0
\]  

(12)

The above equation has been studied in the context of the algebraic determination of the metric by the Riemann tensor \[12\] and the algebraic properties of a Riemann tensor satisfying \[12\] are derived in \[13\]. It is remarked that a spacetime which is 1+3 locally decomposable \[14\], where the three dimensional factor is spacelike, has timelike solutions to \[12\]. The following theorem summarises the results contained in the references given.

**Theorem 4** Suppose that \((M, g)\) is a spacetime and at some \(p \in M\) there is a unit timelike vector \(u^a\) satisfying equations (1) and (2). Then \(u^a\) satisfies
and the following two possibilities occur if the Riemann tensor is non-zero at $p$.

(i) Equation (12) has a unit timelike solution which is unique up to sign, the Weyl tensor is of Petrov type $I$, $D$ or $O$ and the Ricci tensor is diagonalisable.

(ii) Equation (12) admits 2 independent timelike solutions. The Weyl tensor is of Petrov type $D$ and the Ricci tensor is of Segre type $\{(11)(1,1)\}$. The solutions to (12) consist of all the timelike vectors in the 2-space spanned by the principal null directions of the Weyl tensor.

In addition, if condition (i) holds over a non-empty open set then it follows by an application of the Bianchi identities that the congruence associated with $u^a$ is geodesic.

The next result in this section will concern the relationship between purely magnetic and purely Weyl magnetic spacetimes. As has been observed by Arianrhod et al. [2], condition (1) does not in general imply that condition (3) holds. However if one imposes restrictions on the Ricci tensor then it is found that these two conditions are equivalent.

**Theorem 5** Let $(M, g)$ be a spacetime, $p \in M$ and $u^a$ a unit timelike vector at $p$. It follows that any two of the following conditions at $p$ implies the third.

(i) $C_{abcd}u^b u^d = 0$

(ii) $R_{abcd}u^b u^d = 0$

(iii) The Ricci tensor $(R_{ab} \equiv R^c_{\ abc})$ takes the form $u_{(a} q_{b)} - u^c q_c g_{ab}$ for some vector $q_a$.

**Proof** The Riemann tensor admits the standard decomposition (see e.g. [3])

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{1}{6} R g_{a[c} g_{d]b}$$

where in the above the following definitions have been used

$$E_{abcd} = S_{a[c} g_{d]b} - S_{b[c} g_{d]a}$$

$$S_{ab} = R_{ab} - \frac{R}{4} g_{ab}$$

$$R = R^a_a$$

(13)
If one contracts with \( u^b u^d \) and defines \( h_{ab} = g_{ab} + u_a u_b \) then after some calculation one obtains

\[
R_{abcd} u^b u^d = C_{abcd} u^b u^d - \frac{1}{2} h_c h^d_a [R_{bd} - (\frac{R}{3} + R_{ef} u^e u^f) g_{bd}] \tag{15}
\]

If conditions (i) and (ii) hold then (15) shows that the expression in the square brackets is of the form \( u(a q_b) \) for some \( q_b \) and, noting that (ii) gives \( R_{ab} u^a u^b = 0 \), a little rearrangement shows that \( R_{ab} \) takes the form given in (iii). If (iii) holds then equation (15) reduces to \( R_{abcd} u^b u^d = C_{abcd} u^b u^d \) and the rest of the theorem follows.\( \square \)

The possible Segre types of a Ricci tensor of the form given in condition (iii) of the above theorem are related to the nature of the vector \( q_a \). To calculate these Segre types it will prove more convenient to work with the tensor \( \tilde{R}_{ab} \equiv R_{ab} - (R/3) g_{ab} \) which clearly has the same Segre type as \( R_{ab} \). It should first be noted that since \( \tilde{R}_{ab} = u_{(a} q_{b)} \) it necessarily has two spacelike eigenvectors with eigenvalue zero spanning the spacelike 2-space which is the orthogonal complement to the 2-space spanned by \( q_a \) and \( u_a \). If \( q_a \neq 0 \) then this must be the only spacelike eigen-2-space, or else it is contained in a (necessarily spacelike) eigen-3-space in which case the Segre type of \( \tilde{R}_{ab} \) is \( (111) \) with \( u^a \) the unique timelike eigenvector and \( u^{[a} q^{b]} = 0 \). If \( u_a \) is not parallel to \( q_a \) then any further eigenvectors must lie in the timelike 2-space spanned by \( u_a \) and \( q_a \). A short calculation will show that the Segre type of \( \tilde{R}_{ab} = u_{(a} q_{b)} \) is \( (112) \), \( (11) z \bar{z} \) or \( (11) 1, 1 \) according to whether \( q_b \) is null, spacelike or timelike respectively (where \( z \bar{z} \) denotes a pair of conjugate complex eigenvectors). An examination of the appropriate canonical forms given in [15] associated with these Segre types will show that no further degeneracies are possible (apart from \( q_a \) parallel to \( u_a \) or \( q_a = 0 \)). In addition the expression \( \tilde{R}_{ab} = u_{(a} q_{b)} \) uniquely determines the vectors \( u_a \) and \( q_a \) up to a possible scaling and interchange of \( u_a \) and \( q_a \) (clearly the interchange is only possible where \( u_a \) and \( q_a \) have the same nature.) The following theorem summarises the foregoing discussion.

**Theorem 6** Suppose that at some point of a spacetime the Ricci tensor \( R_{ab} \) can be written in the form given in condition (iii) of the previous theorem. Assuming \( u_a \) and \( q_a \) non-zero and non-parallel then the Segre type of \( R_{ab} \) is either \( (11) 1, 1 \), \( (11) z \bar{z} \) or \( (11) 2 \) depending on whether \( q_a \) is timelike, spacelike or null respectively. No further degeneracies are possible. The vectors \( u_a \) and \( q_a \) are unique up to scaling and/or swapping. If \( u_a \) and \( q_a \) are parallel then their direction is unique and the Segre type of \( R_{ab} \) is \( (111), 1 \).
It should be emphasised that if the Ricci tensor is of one of the Segre types given above then it does not necessarily take the form given in condition (iii) of theorem 5, as additional restrictions are placed on the eigenvalues by condition (iii).

The possible types of energy-momentum tensor in a purely magnetic spacetime are restricted by the fact that equation (1) can be contracted to give the necessary condition

$$R_{ab}u^au^b = 0$$  \hspace{1cm} (16)

It has been pointed out by Beem and Harris [3] that this equation cannot hold in a proper Einstein space and it follows from a result of Penrose and Rindler [16, p328] that (16) cannot hold in a (null or non-null) electromagnetic spacetime, with vanishing cosmological constant. Suppose that the Ricci tensor takes the algebraic form of a perfect fluid type with flow vector $v^a$, pressure $p$ and energy density $\mu$. The Ricci tensor can then be written as

$$R_{ab} = (\mu + p)v_av_b + \frac{1}{2}(\mu - p)g_{ab}$$ \hspace{1cm} (17)

Combining equations (17) and (16) gives that the necessary and sufficient condition for $R_{ab}$ to satisfy (16) is $2(v^av_a)^2 = (\mu - p)/(\mu + p)$. In particular if one wishes to have $u^a = v^a$ then the appropriate condition is $\mu + 3p = 0$ and in this case it follows that $R_{ab}$ satisfies condition (iii) of theorem 5 and hence the Weyl tensor $C_{abcd}$ satisfies $C_{abcd}u^bu^d = 0$. Conversely if $C_{abcd}u^bu^d = 0 = R_{abcd}u^bu^d$ when the Ricci tensor is of the form (17) then theorems 5 and 6 show that $u^{[a}v^{b]} = 0$. The following theorem summarises the possible physical types of purely magnetic spacetimes.

**Theorem 7** Suppose that $(M, g)$ is a spacetime and that equation (17) holds at some $p \in M$. It then follows that the Ricci tensor cannot be a non-zero multiple of the metric or be of electromagnetic type at $p$. If the Ricci tensor is of perfect fluid type with flow vector $v^a$, energy density $\mu$ and pressure $p$ then one has

$$\mu = \left( \frac{1 + 2(v^av_a)^2}{1 - 2(v^av_a)^2} \right) p$$ \hspace{1cm} (18)

In this case if $u^{[a}v^{b]} = 0$ if and only if the Weyl tensor is of petrov type I, D or O with purely imaginary Petrov scalars.
In the non-vacuum case the uniqueness of a solution \( u^a \) to (1) is, in general, much harder to predict than in the vacuum case. Some results in this respect have been given by Beem and Harris [3], for example if \( u^a, v^a \) and some linear combination of \( u^a \) and \( v^a \) satisfy (1) then all linear combinations of \( u^a \) and \( v^a \) will satisfy (1). Various generalisations of this result are given in [3].

In some circumstances the Riemann tensor may be invariant under certain Lorentz transformations of the tangent space (at some point \( p \)) and these transformations form the isotropy group of the Riemann tensor at \( p \). If, under the action of a member \( A \) of the isotropy group, \( u^a \) is mapped to some other vector \( v^a \) then \( v^a \) must also be a solution of (1). In the case where \( R_{abcd} = C_{abcd} \) and a unit timelike non-trivial solution \( u^a \) of (1) is admitted then all other solutions to this equation can be found by considering the orbit of \( u^a \) under the isotropy group of the Riemann tensor (cf theorem 2). Unfortunately such a method cannot be used in the general case as the following example will show. If \( x^a, y^a, z^a, v^a \) is an orthonormal frame for the cotangent space at \( p \) then define

\[
\tilde{R}_{abcd} = x[a]y[b]y[c]z[d] + y[a]z[b]x[c]y[d]
\]

(19)

It is easily seen that \( \tilde{R}_{abcd} \) has all the algebraic symmetries of a Riemann tensor. It may also be verified that if \( u^a \) is orthogonal to the 2-plane spanned by \( x^a \) and \( y^a \) or if \( u^a \) is orthogonal to the 2-plane spanned by \( y^a \) and \( z^a \) then \( \tilde{R}_{abcd}u^b u^d = 0 \). However if one calculates the Weyl tensor \( \tilde{C}_{abcd} \) of \( \tilde{R}_{abcd} \), by noting that \( \tilde{R}_{ab}^{ab} = 0 \) and hence \( 2\tilde{C}_{abcd} = \tilde{R}_{abcd} - *\tilde{R}^{*}_{abcd} \) then \( \tilde{C}_{abcd} \) is seen to be of Petrov type I with \( v^a \) the timelike member of the canonical Petrov tetrad. Now the space of type (0,4) tensors with Riemann symmetries decomposes under the action of the Lorentz group into three irreducible subspaces [16] which correspond to the decomposition given by equation (13). Consequently the isotropy of the Riemann tensor is contained in that of the Weyl tensor, which only consists of finite number of reflections in the case of Petrov type I [17]. The isotropy group of \( \tilde{R}_{abcd} \) given by (19) therefore cannot generate the infinite number of solutions of (1).

4 Conclusions

After having studied the algebraic properties of curvature in purely magnetic spacetimes, one question that may come to mind is ‘are there any physically
interesting examples? In fact there are very few known examples of purely magnetic spacetimes of any description. One example due to Misra et al. [18] is discussed by McIntosh et al. [1] and the latter suggest that there may be no purely magnetic non-flat vacuum solutions. The reasons why there are few purely magnetic spacetimes in the literature are essentially given by theorems 2 and 7. The former states that all vacuum purely magnetic solutions are of type I, of which there are few solutions known (the static type I solutions cannot be purely magnetic [3]), and the latter implies that most physically interesting matter solutions cannot be purely magnetic. Of course it is no easy matter to determine whether or not a given metric has a purely magnetic curvature tensor. There is, however, a method given by McIntosh et al. for deciding whether the Weyl tensor of a given metric admits solutions to (8), which is suitable for implementation within a computer algebra system. It would be a fairly straightforward matter to use such a computer system to check the literature for purely Weyl magnetic spacetimes.

Hawking and Ellis [5] have excluded purely magnetic spacetimes in the conditions of their singularity theorems and argue that such spacetimes are unphysical. However it should be remembered that their argument only applies when one wishes to model the entire universe. There is no obvious physical reason why a purely magnetic solution should not act as model for some portion of spacetime. It is hoped that theorem 3 may be of some help in the vacuum case as the restrictions it places on the derivative of $u^a$ could conceivably be used to simplify the field equations.

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