Description of the vector $G$-bundles over $G$-spaces with quasi-free proper action of discrete group $G$

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Abstract
We give a description of the vector $G$-bundles over $G$-spaces with quasi-free proper action of discrete group $G$ in terms of the classifying space.

1 The setting of the problem
This problem naturally arises from the Conner-Floyd’s description (2) of the bordisms with the action of a group $G$ using the so-called fix-point construction. This construction reduces the problem of describing the bordisms to two simpler problems: a) description of the fixed-point set (or, more generally, the stationary point set), which happens to be a submanifold attached with the structure of its normal bundle and the action of the same group $G$, however, this action could have stationary points of lower rank; b) description of the bordisms of lower rank with an action of the group $G$. We assume that the group $G$ is discrete.

Let $\xi$ be an $G$-equivariant vector bundle with base $M$.

\[
\begin{array}{ccc}
\xi & \downarrow \\
M & \\
\end{array}
\]  

(1)

Let $H < G$ be a normal finite subgroup. Assume that the action of the group $G$ over the base $M$ reduces to the factor group $G_0 = G/H$:

\[
\begin{array}{ccc}
G \times M & \rightarrow & M \\
\downarrow & & \| \\
G_0 \times M & \rightarrow & M \\
\end{array}
\]  

(2)

suppose, additionally, that the action $G_0 \times M \rightarrow M$ is free and there is no more fixed points of the action of the group $H$ in the total space of the bundle $\xi$.

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So, we have the following commutative diagram
\[
\begin{array}{ccc}
G \times \xi & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
G_0 \times M & \longrightarrow & M
\end{array}
\] (3)

**Definition 1** As in [6, p. 210], we shall say that the described action of the group \(G\) is quasi-free over the base with normal stationary subgroup \(H\).

Reducing the action to the subgroup \(H\), we obtain the simpler diagram:
\[
\begin{array}{ccc}
H \times \xi & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
M & = & M
\end{array}
\] (4)

Following [4], let \(\rho_k : H \longrightarrow \mathbf{U}(V_k)\) be the series of all the irreducible (unitary) representation of the finite group \(H\). Then the \(H\)-bundle \(\xi\) can be presented as the finite direct sum:
\[
\xi \approx \bigoplus_k \left( \xi_k \otimes V_k \right),
\] (5)
where the action of the group \(H\) over the bundles \(\xi_k\) is trivial, \(V_k\) denotes the trivial bundle with fiber \(V_k\) and with fiberwise action of the group \(H\), defined using the linear representation \(\rho_k\).

**Lemma 1** The group \(G\) acts on every term of the sum (5) separately.

**Proof.** Consider now the action of the group \(G\) over the total space of the bundle \(\xi\). Fix a point \(x \in M\). The action of the element \(g \in G\) is fiberwise, and maps the fiber \(\xi_x\) to the fiber \(\xi_{gx}\):
\[
\Phi(x, g) : \xi_x \longrightarrow \xi_{gx}.
\]

Also, for a pair of elements \(g_1, g_2 \in G\) we have:
\[
\Phi(x, g_1 g_2) = \Phi(g_2 x, g_1) \circ \Phi(x, g_2),
\] (6)
\[
\Phi(x, g_1 g_2) : \xi_x \longrightarrow \xi_{g_2 x} \longrightarrow \xi_{g_1 g_2 x}
\]

In particular, if \(g_2 = h \in H < G\), then \(g_2 x = hx = x\). So,
\[
\Phi(x, gh) : \xi_x \longrightarrow \xi_x \longrightarrow \xi_{gx}
\]
Analogously, if \( g_1 = h \in H < G \), then \( g_1gx = hgx = gx \). So
\[
\Phi(x, hg) : \xi_x \xrightarrow{\Phi(x, g)} \xi_{gx} \xrightarrow{\Phi(gx, h)} \xi_{gx}
\]
According to [4] the operator \( \Phi(x, h) \) does not depends on the point \( x \in M \),
\[
\Phi(x, h) = \Psi(h) : \bigoplus_k \left( \xi_{k,x} \otimes V_k \right) \rightarrow \bigoplus_k \left( \xi_{k,x} \otimes V_k \right),
\]
here, since the action of the group \( H \) is given over every space \( V_k \) using pairwise different irreducible representations \( \rho_k \), we have
\[
\Psi(h) = \bigoplus_k \left( \text{Id} \otimes \rho_k(h) \right).
\]
In this way, we obtain the following relation:
\[
\Phi(x, gh) = \Phi(x, g) \circ \Psi(h) = \Phi(x, ghg^{-1}g) = \Psi(ghg^{-1}) \circ \Phi(x, g).
\] (7)

Lets write the operator \( \Phi(x, g) \) using matrices to decompose the space \( \xi_x \) as the direct sum
\[
\xi_x = \bigoplus_k \left( \xi_{k,x} \otimes V_k \right):
\]
\[
\Phi(x, g) = \begin{pmatrix}
\Phi(x, g)_{1,1} & \cdots & \Phi(x, g)_{1,k} & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
\Phi(x, g)_{k,1} & \cdots & \Phi(x, g)_{k,k} & \cdots \\
\vdots & \ddots & \vdots & \ddots
\end{pmatrix}
\] (8)
If \( k \neq l \) then \( \Phi(x, g)_{k,l} = 0 \), i.e. the matrix \( \Phi(x, g) \) its diagonal,
\[
\Phi(x, g) = \bigoplus_k \Phi(x, g)_{k,k} : \bigoplus_k \left( \xi_{k,x} \otimes V_k \right) \rightarrow \bigoplus_k \left( \xi_{k,x} \otimes V_k \right),
\]
\[
\Phi(x, g)_{k,k} : \left( \xi_{k,x} \otimes V_k \right) \rightarrow \left( \xi_{k,x} \otimes V_k \right),
\]
as it was required to prove.

2 Description of the particular case \( \xi = \xi_0 \otimes V \)

Here we will consider the particular case of a \( G \)-vector bundle \( \xi = \xi_0 \otimes V \) with base \( M \).
\[
\xi \downarrow \quad M
\]
where the action of the group $G$ is quasi-free over the base with finite normal stationary subgroup $H < G$.

We will assume that the group $H$ acts trivially over the bundle $\xi_0$. By $V$ we denote the trivial bundle with fiber $V$ and with fiberwise action of the group $H$ given by an irreducible linear representation $\rho$.

**Definition 2** A canonical model for the fiber in a $G$-bundle $\xi = \xi_0 \otimes V$ with fiber $F \otimes V$ is the product $G_0 \times (F \otimes V)$ with an action of the group $G$

\[
G \times (G_0 \times (F \otimes V)) \xrightarrow{\phi} G_0 \times (F \otimes V)
\]

where $\mu$ denotes the natural left action of $G$ on its quotient $G_0$, and

\[
\phi([g], g_1) : [g] \times (F \otimes V) \rightarrow [g_1 g] \times (F \otimes V)
\]

is given by the formula

\[
\phi([g], g_1) = \text{Id} \otimes \rho(u(g_1 g)u^{-1}(g)). \tag{9}
\]

where

\[
u : G \rightarrow H
\]

is a homomorphism of right $H$-modules by multiplication, i.e.

\[
u(gh) = u(g)h, \quad \nu(1) = 1, \quad g \in G, \quad h \in H.
\]

**Lemma 2** The definition (9) of the action of $G$ is well-defined.

**Proof.** It is enough to prove that that a) the formula (9) defines an action, i.e.

\[
\phi([g], g_2 g_1) = \phi([g_1 g], g_2) \circ \phi([g], g_1),
\]

and b) that the formula (9) does not depends on the chosen representative $gh \in [g]$:

\[
\text{Id} \otimes \rho(u(g_1 g)u^{-1}(g)) = \text{Id} \otimes \rho(u(g_1 gh)u^{-1}(gh))
\]

for every $g \in G$ and $h \in H$.

In fact,

\[
\phi([g], g_2 g_1) = \text{Id} \otimes \rho(u(g_2 g_1 g)u^{-1}(g)) =
\]

\[
\text{Id} \otimes \rho(u(g_2 g_1 g)u(g_1 g)u^{-1}(g_1 g)u^{-1}(g)) =
\]

\[
= \text{Id} \otimes \rho(u(g_2 g_1 g)u(g_1 g)) \circ \text{Id} \otimes \rho(u^{-1}(g_1 g)u^{-1}(g)) =
\]

\[
= \phi([g_1 g], g_2) \circ \phi([g], g_1),
\]

what proves a), and, recalling the equation $u(gh) = u(g)h$ for every $g \in G$ and $h \in H$, it is clear that

\[
u(g_1 gh)u^{-1}(gh) = u(g_1 g)hh^{-1}u^{-1}(g) = u(g_1 g)u^{-1}(g),
\]

which is a sufficient condition for b) to be true.
As it is well known, for the actions we are studying, we can always consider over the base $M$ an atlas of equivariant charts $\{O_\alpha\}$,

$$M = \bigcup \alpha O_\alpha,$$

$$[g]O_\alpha = O_\alpha, \quad \forall [g] \in G_0.$$

If the atlas is fine enough, then every chart can be presented as a disjoint union of its subcharts:

$$O_\alpha = \bigsqcup_{[g] \in G_0} [g]U_\alpha \approx U_\alpha \times G_0,$$

i.e. $[g]U_\alpha \cap [g']U_\alpha = \emptyset$ if $[g] \neq [g']$, and when $\alpha \neq \beta$, if $U_\alpha \cap [g_\alpha\beta]U_\beta \neq \emptyset$, then the element $g_\alpha\beta$ is the only one for which that intersection is non-empty, i.e. if $[g] \neq [g_\alpha\beta]$, then $U_\alpha \cap [g]U_\beta = \emptyset$, i.e.

$$O_\alpha \cap O_\beta \approx (U_\alpha \cap [g_\alpha\beta]U_\beta) \times G_0,$$

for every $\alpha, \beta$. We use these facts and notations to formulate the next theorem.

**Theorem 1** The bundle $\xi = \xi_0 \otimes V$ is locally homeomorphic to the cartesian product of some chart $U_\alpha$ by the canonical model. More precisely, for a fine enough atlas, there exist $G$-equivariant trivializations

$$\psi_\alpha : O_\alpha \times (F \otimes V) \to \xi|_{O_\alpha}$$

where

$$O_\alpha \times (F \otimes V) \approx U_\alpha \times (G_0 \times (F \otimes V))$$

and the diagram

$$
\begin{array}{ccc}
\xi|_{O_\alpha} & \xrightarrow{g} & \xi|_{O_\alpha} \\
\uparrow \psi_\alpha & & \uparrow \psi_\alpha \\
U_\alpha \times (G_0 \times (F \otimes V)) & \xrightarrow{\text{Id} \times \phi(g)} & U_\alpha \times (G_0 \times (F \otimes V))
\end{array}
$$

is commutative where $g \in G$, $\text{Id} : U_\alpha \to U_\alpha$, and $\phi(g)$ denotes the canonical action.

**Proof.** Using an atlas as in the remarks at the beginning of the theorem, we shall construct the trivialization (10) starting from an arbitrary trivialization

$$\psi_\alpha : U_\alpha \times (F \otimes V) \to \xi|_{U_\alpha}$$

in such a way, that the diagram

$$
\begin{array}{ccc}
\xi|_{U_\alpha} & \xrightarrow{g} & \xi|[g]U_\alpha \\
\uparrow \psi_\alpha & & \uparrow \psi_\alpha \\
U_\alpha \times (F \otimes V) & \xrightarrow{[g]} & [g]U_\alpha \times (F \otimes V)
\end{array}
$$

is commutative.
commutes for every $g \in [g]$, where the left and upper arrows are given and we have to construct the down and right arrows.

From such a construction, the equivariance will follow automatically and the proof of the theorem reduces to show that the constructed down arrow coincides with that on (11).

Evidently, for a given trivialization $\psi_\alpha : U_\alpha \times (F \otimes V) \to \xi_{|U_\alpha}$, there are several ways to define a trivialization $\psi_\alpha : [g]U_\alpha \times (F \otimes V) \to \xi_{|[g]U_\alpha}$, since there are several elements $g \in G$ sending $\xi_{|U_\alpha}$ to $\xi_{|[g]U_\alpha}$.

Thus, consider a set-theoretic cross-section $p' : G_0 \longrightarrow G$ to the projection $p$ in the exact sequence of groups

$$1 \longrightarrow H \longrightarrow G \overset{p}{\longrightarrow} G_0,$$

$$p \circ p' = \text{Id} : G_0 \overset{p'}{\longrightarrow} G \overset{p}{\longrightarrow} G_0.$$ 

Put

$$g' = p' \circ p : G \longrightarrow G.$$ 

Without loss of generality, we can take $g'(1) = 1$.

In this case

$$g'(g) = gu^{-1}(g),$$

where

$$u : G \longrightarrow H$$

is a homomorphism of right $H$-modules by multiplication, i.e.

$$u(gh) = u(g)h, \quad g \in G, h \in H.$$ 

In particular, this means that

$$g'(gh) = g'(g), \quad h \in H.$$ 

Lets

$$\tilde{\psi}_\alpha : U_\alpha \times F \longrightarrow \xi_0|U_\alpha$$

be some trivialization. We define the trivialization $\psi_\alpha$ in (10) by the rule: if $[g]x_\alpha \in [g]U_\alpha$, i.e. $x_\alpha \in U_\alpha$, then, the map

$$\psi_\alpha([g]x_\alpha) : [g]x_\alpha \times (F \otimes V) \longrightarrow \xi_{|[g]x_\alpha} \otimes V$$

is given by the formula

$$\psi_\alpha([g]x_\alpha) = \Phi(x_\alpha, g'(g)) \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right) = \Phi(x_\alpha, gu^{-1}(g)) \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right).$$

(12)
where, from the first equality, it is clear that the definition does not depend on
the representative \( g \in [g] \).

In particular, for \([g] = 1\), we recover the initial trivialization
\[
\psi_\alpha(x_\alpha) = \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id}
\]
since \( \Phi(x, g'(1)) = \Phi(x, 1) = 1 \).

Using this trivialization the action of the group \( G \) can be carried to the
cartesian product \( O_\alpha \times (F \otimes V) \):
\[
\Phi_\alpha(g) : O_\alpha \times (F \otimes V) \longrightarrow O_\alpha \times (F \otimes V) .
\]

Let \( x_\alpha \in U_\alpha, \ g \in G \), then
\[
\Phi_\alpha([g]x_\alpha, g_1) : [g]x_\alpha \times (F \otimes V) \longrightarrow [g_1]x_\alpha \times (F \otimes V)
\]
is given by the formula
\[
\Phi_\alpha([g]x_\alpha, g_1) = (\psi_\alpha([g_1]x_\alpha))^{-1} \Phi([g]x_\alpha, g_1) \psi_\alpha([g]x_\alpha) .
\]

Applying (12), we obtain
\[
\Phi_\alpha([g]x_\alpha, g_1) = \left( \Phi(x_\alpha, g_1 u^{-1}(g_1 g)) \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right) \right)^{-1} \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right) = \]
\[
= \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right)^{-1} \circ \Phi([g]x_\alpha, g_1) \circ \Phi(x_\alpha, g_1 u^{-1}(g)) \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right) = \]
\[
= \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right)^{-1} \circ \Phi(x_\alpha, u^{-1}(g_1 g)) \circ \Phi([g]x_\alpha, g_1) \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right) = \]
\[
= \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right)^{-1} \circ \Phi(x_\alpha, u^{-1}(g)) \circ \phi \circ \left( \tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id} \right) ;
\]
\[
\Phi_\alpha([g|x_\alpha, g_1]) = \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id}\right)^{-1} \circ \\
(\text{Id} \otimes \rho(u(g_1g))) \circ (\text{Id} \otimes \rho(u^{-1}(g))) \circ \\
(\tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id}) = \\
= \left(\tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id}\right)^{-1} \circ \\
(\text{Id} \otimes (\rho(u(g_1g)u^{-1}(g)))) \circ \\
(\tilde{\psi}_\alpha(x_\alpha) \otimes \text{Id}) = \\
= \text{Id} \otimes \rho(u(g_1g)u^{-1}(g)).
\]

The operator
\[
\Phi_\alpha([g|x_\alpha, g_1]) = \text{Id} \otimes \rho(u(g_1g)u^{-1}(g)) = \phi(g_1, [g]).
\]
does not depend on the point \(x_\alpha \in U_\alpha\). So, the theorem is proved.

By \(\text{Aut}_G(G_0 \times (F \otimes V))\) we denote the group of equivariant automorphisms of the space \(G_0 \times (F \otimes V)\) as a vector \(G\)-bundle with base \(G_0\), fiber \(F \otimes V\) and canonical action of the group \(G\).

**Corollary 1** The transition functions on the intersection
\[
O_\alpha \cap O_\beta \approx (U_\alpha \cap [g_{\alpha\beta}]U_\beta) \times G_0,
\]
i.e. the homomorphisms \(\Psi_{\alpha\beta}\) on the diagram
\[
(U_\alpha \cap [g_{\alpha\beta}]U_\beta) \times (G_0 \times (F \otimes V)) \xrightarrow{\Psi_{\alpha\beta}} (U_\alpha \cap [g_{\alpha\beta}]U_\beta) \times (G_0 \times (F \otimes V)) \\
(U_\alpha \cap [g_{\alpha\beta}]U_\beta) \times G_0 \xrightarrow{\text{Id}} (U_\alpha \cap [g_{\alpha\beta}]U_\beta) \times G_0
\]
are equivariant with respect to the canonical action of the group \(G\) over the product of the base by the canonical model, i.e.
\[
\Psi_{\alpha\beta}(x) \circ \phi(g_1, [g]) = \phi(g_1, [g]) \circ \Psi_{\alpha\beta}(x)
\]
for every \(x \in U_\alpha \cap [g_{\alpha\beta}]U_\beta, g_1 \in G, [g] \in G_0\). In other words,
\[
\Psi_{\alpha\beta}(x) \in \text{Aut}_G(G_0 \times (F \otimes V)).
\]

Now we give a more accurate description of the group \(\text{Aut}_G(G_0 \times (F \otimes V))\). By definition, an element of the group \(\text{Aut}_G(G_0 \times (F \otimes V))\) is an equivariant mapping \(A^a\), such that the pair \((A^a, a)\) defines a commutative diagram
\[
(G_0 \times (F \otimes V)) \xrightarrow{A_0^a} G_0 \times (F \otimes V) \\
\xrightarrow{G_0} G_0 \xrightarrow{\alpha} G_0,
\]
which commutes with the canonical action, i.e. the map \( a \in \text{Aut}_G(G_0) \) satisfies the condition
\[
a \in \text{Aut}_G(G_0) \approx G_0, \quad a[g] = [ga], \quad [g] \in G_0,
\]
and the mapping \( A^a = (A^a[g])_{|g| \in G_0} \),
\[
A^a[g] : [g] \times (F \otimes V) \rightarrow [ga] \times (F \otimes V)
\]
satisfies a commutation condition with respect to the action of the group \( G \):
\[
\begin{bmatrix}
[g] \times (F \otimes V) & A^a[g] & [ga] \times (F \otimes V) \\
\phi(g_1, [g]) & [g_1ga] \\
[g_1g] \times (F \otimes V) & A^a[g_1g] & [g_1ga] \times (F \otimes V)
\end{bmatrix}
\]
\[
\phi(g_1, [ga]) \circ A^a[g] = A^a[g_1g] \circ \phi(g_1, [g]) \tag{14}
\]
i.e.
\[
(\text{Id} \otimes \rho(u(g_1ga)u^{-1}(ga)))A^a[g] = A^a[g_1g](\text{Id} \otimes \rho(u(g_1g)u^{-1}(g))) \tag{15}
\]
where \([g] \in G_0, \quad g_1 \in G\).

Lemma 3 One has an exact sequence of groups
\[
1 \rightarrow GL(F) \rightarrow \text{Aut}_G(G_0 \times (F \otimes V)) \rightarrow G_0 \rightarrow 1. \tag{16}
\]

Proof. To define a projection
\[
pr : \text{Aut}_G(G_0 \times (F \otimes V)) \rightarrow G_0
\]
we send the fiberwise map
\[
A^a : G_0 \times (F \otimes V) \rightarrow G_0 \times (F \otimes V)
\]
to its restriction over the base \( a : G_0 \rightarrow G_0 \), i.e. \( a \in \text{Aut}_G(G_0) \approx G_0 \). So, this is a well-defined homomorphism.

We need to show that \( pr \) is an epimorphism and that its kernel is isomorphic to \( GL(F) \). Lets calculate the kernel.

For \([a] = [1]\) we have
\[
(\text{Id} \otimes \rho(u(g_1g)u^{-1}(g)))A^1[g] = A^1[g_1g](\text{Id} \otimes \rho(u(g_1g)u^{-1}(g))) \tag{17}
\]
In the case \( g_1 = h \in H \), we obtain
\[
(\text{Id} \otimes \rho(u(hg)u^{-1}(g)))A^1[g] = A^1[g](\text{Id} \otimes \rho(u(hg)u^{-1}(g)))
\]
Since the representation $\rho$ is irreducible, by Schur’s lemma, we have

$$A^1[g] = B^1[g] \otimes \text{Id}.$$  

On the other side, assuming in (17) that $g = 1$, we have

$$(\text{Id} \otimes \rho(u(g)))A^1[1] = A^1[g](\text{Id} \otimes \rho(u(g))),$$

i.e.

$$(\text{Id} \otimes \rho(u(g)))(B^1[1] \otimes \text{Id}) = (B^1[g] \otimes \text{Id})(\text{Id} \otimes \rho(u(g))),$$

or

$$(B^1[g] \otimes \text{Id}) = (B^1[1] \otimes \text{Id}).$$

So, the kernel $\ker p r$ is isomorphic to the group $GL(F)$.

In the generic case, i.e. $[a] \neq 1$, we can compute the operator $A^a[g]$ in terms of its value at the identity $A^a[1]$ from the formula (15): assuming $g = 1$, we obtain (changing $g_1$ by $g$):

$$(\text{Id} \otimes \rho(u(ga)u^{-1}(a)))A^a[1] = A^a[g](\text{Id} \otimes \rho(u(g))),$$

i.e.

$$A^a[g] = (\text{Id} \otimes \rho(u(ga)u^{-1}(a)))A^a[1](\text{Id} \otimes \rho(u^{-1}(g))),$$

Therefore, the operator is completely defined by its value

$$A^a[1] : [1] \times (F \otimes V) \rightarrow [a] \times (F \otimes V)$$

at the identity $g = 1$.

Now we describe the operator $A^a[1]$ in terms of the representation $\rho$ and its properties.

We have a commutation rule with respect to the action of the subgroup $H$:

$$[1] \times (F \otimes V) \xrightarrow{A^a[1]} [a] \times (F \otimes V)$$

Equivalently

$$A^a[1] \circ \phi(h, [1]) = \phi(h, [a]) \circ A^a[1],$$

i.e.
\[ A^a[1] \circ (\text{Id} \otimes \rho(h)) = (\text{Id} \otimes \rho(g^{-1}(a)hg'(a))) \circ A^a[1] , \]
i.e.
\[ A^a[1] \circ (\text{Id} \otimes \rho(h)) = (\text{Id} \otimes \rho_{g'(a)}(h)) \circ A^a[1] . \]

The last equation means that the operator should \( A^a[1] \) permute these representations, or equivalently, such an operator exists only when the representations \( \rho \) and \( \rho_{g'(a)} \) are equivalent. Recalling the commutation rule (7), we see that this is the case we are being considering.

Thus, if the representations \( \rho \) and \( \rho_g \) are equivalent, we have an (inverse) splitting operator \( C(g) \), satisfying the equation
\[ \rho_g(h) = \rho(\bar{g}^{-1}h) = C(g)\rho(h)C^{-1}(g) . \]
for every \( g \in G \). The operator \( C(g) \) is defined up to multiplication by a scalar \( \mu_g \in SS^1 \subset C^1 \).

So
\[ A^a[1] \circ (\text{Id} \otimes \rho(h)) = (\text{Id} \otimes C(g'(a))) \circ \rho(h) \circ C^{-1}(g'(a)) \circ A^a[1] , \]
or
\[ (\text{Id} \otimes C^{-1}(g'(a))) \circ A^a[1] \circ (\text{Id} \otimes \rho(h)) = (\text{Id} \otimes \rho(h)) \circ (\text{Id} \otimes C^{-1}(g'(a))) \circ A^a[1] , \]

Then, by the Schur’s lemma,
\[ (\text{Id} \otimes C^{-1}(g'(a))) \circ A^a[1] = B^a[1] \otimes \text{Id} , \]
i.e.
\[ A^a[1] = B^a[1] \otimes C(g'(a)) , \]

Using the formula (19), we obtain
\[ A^a[g] = (\text{Id} \otimes \rho(u(ga)u^{-1}(a)))(B^a[1] \otimes C(g'(a)))(\text{Id} \otimes \rho(u^{-1}(g))), \]
i.e.
\[ A^a[g] = B^a[1] \otimes (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))) . \]

(21)
This means, that by defining the matrix $B^a[1]$, it is possible to obtain all the operators $A^a[g]$ satisfying the equation (19).

It remains to verify the commutation rule (15), i.e. in the formula

$$(\text{Id} \otimes \rho(u(g_1ga)u^{-1}(ga)))A^a[g] = A^a[g_1g](\text{Id} \otimes \rho(u(g_1g)u^{-1}(g)))$$

we substitute the expression (21):

$$(\text{Id} \otimes \rho(u(g_1ga)u^{-1}(ga)))\circ (B^a[1] \otimes (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g)))) =$$

$$= (B^a[1] \otimes (\rho(u(g_1ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g_1g)))) \circ (\text{Id} \otimes \rho(u(g_1g)u^{-1}(g)))$$

that is

$$B^a[1] \otimes \rho(u(g_1ga)u^{-1}(ga))) \circ (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))) =$$

$$= B^a[1] \otimes (\rho(u(g_1ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g_1g)))) \circ (\rho(u(g_1g)u^{-1}(g)))$$

Note that this identity does not depend on the particular matrix $B^a[1]$, thus, this means that we only need to verify the identity for arbitrary $a$, $g$ and $g_1$:

$$\rho(u(g_1ga)u^{-1}(ga))) \circ (\rho(u(ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))) =$$

$$= (\rho(u(g_1ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g_1g)))) \circ (\rho(u(g_1g)u^{-1}(g))),$$

which is obvious, after the natural simplifications

$$\rho(u(g_1ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))) =$$

$$= (\rho(u(g_1ga)u^{-1}(a)) \circ C(g'(a)) \circ \rho(u^{-1}(g))),$$

So, it follows, that for every element $[a] \in G_0$ there exist an element $(A^a, a) \in \text{Aut}_G (G_0 \times (F \otimes V))$. This means that the homomorphism

$$\text{Aut}_G (G_0 \times (F \otimes V)) \overset{pr}{\longrightarrow} G_0$$

is in fact an epimorphism, and the lemma is proved.

It is clear that there is an equivalence between $G$-vector bundles with fiber $G_0 \times (F \otimes V)$ over a (compact) base $X$, where $G$ acts trivially over the base and canonically over the fiber, and homotopy classes of mappings from $X$ to the space $B\text{Aut}_G (G_0 \times (F \otimes V))$.

Let denote by $\text{Vect}_G(M, \rho)$ the category of $G$-equivariant vector bundles $\xi = \xi_0 \otimes V$ with base $M$, where the action of the group $G$ is quasi-free over the base with finite normal stationary subgroup $H < G$, the group $H$ acts trivially over the bundle $\xi_0$ and $V$ denotes the trivial bundle with fiber $V$ and with
fiberwise action of the group $H$ given by an irreducible linear representation \( \rho \). Here we need to require for the representations \( \rho_g(h) = \rho(g^{-1}hg) \) to be equivalent for every \( g \in G \), in the other case, in view of the commutation rule, this category may be void.

This is a category because, in fact, we are just taking vector bundles over the space \( M \), then applying tensor product by the fixed bundle \( V \) and defining some action of the group \( G \) over the resulting spaces. The inclusion \( GL(F) \hookrightarrow Aut_G(G_0 \times (F \otimes V)) \) from lemma 2 ensures that the identities are included.

Denote by \( \text{Bundle}(X,L) \) the category of principal \( L \)-bundles over the base \( X \).

**Theorem 2** There is a monomorphism

\[
\text{Vect}_G(M,\rho) \longrightarrow \text{Bundle}(M/G_0, \text{Aut}_G(G_0 \times (F \otimes V))). \quad (22)
\]

**Proof.** By corollary 3, every element \( \xi \in \text{Vect}_G(M,\rho) \) is defined by transition functions

\[
\Psi_{\alpha\beta} : (U_{\alpha} \cap [g_{\alpha\beta}]U_{\beta}) \rightarrow \text{Aut}_G(G_0 \times (F \otimes V))
\]

where by construction, when \( [g] \neq [g_{\alpha\beta}] \), we have \( U_{\alpha} \cap [g]U_{\beta} = \emptyset \) and if \( [g] \neq 1 \), then \( U_{\alpha} \cap [g]U_{\alpha} = \emptyset \) and \( U_{\beta} \cap [g]U_{\beta} = \emptyset \). This means that the sets \( U_{\alpha} \) and \( U_{\beta} \) project homeomorphically to open sets under the natural projection \( M \rightarrow M/G_0 \). So, these transition functions are well-defined over an atlas of the quotient space \( M/G_0 \) and they form a \( G \)-bundle with fiber \( G_0 \times (F \otimes V) \) over this quotient space.

By the same arguments, it is obvious that every \( G \)-equivariant map

\[
h_{\alpha} : O_{\alpha} \times (F \otimes V) \rightarrow O_{\alpha} \times (F \otimes V) \quad (23)
\]

can be interpreted as a map

\[
h_{\alpha} : U_{\alpha} \times (G_0 \times (F \otimes V)) \rightarrow U_{\alpha} \times (G_0 \times (F \otimes V)) \quad (24)
\]

by means of the homeomorphism \( O_{\alpha} \approx U_{\alpha} \times G_0 \), where the set \( U_{\alpha} \) can be thought as an open set of the space \( M/G_0 \). Equivalently,

\[
h_{\alpha} : U_{\alpha} \rightarrow \text{Aut}_G(G_0 \times (F \otimes V)) \quad (25)
\]

where \( U_{\alpha} \) is homeomorphic to an open set of the space \( M/G_0 \). Therefore, the map \text{(22)} is well defined.

Conversely, if we start from mappings of the form \text{(24)} where the sets \( U_{\alpha} \) are open in \( M/G_0 \), by refining the atlas, if it is necessary, we can always think that the inverse image of the open sets \( U_{\alpha} \) under the quotient map \( M \rightarrow M/G_0 \) are homeomorphic to the product \( U_{\alpha} \times G_0 \) and then obtain mappings of the form \text{(23)}. Therefore, the map \text{(22)} is a monomorphism.

Of course, the map \text{(22)} its not in general an epimorphism, since, when we define the category \( \text{Vect}_G(M,\rho) \), we are automatically fixing a bundle \( M \rightarrow M/G_0 \), or equivalently, a homotopy class in \([M/G_0, BG_0]\).
Theorem 3 If the space $X$ is compact, then
\[
\text{Bundle}(X, \text{Aut}_G (G_0 \times (F \otimes V))) \approx \bigsqcup_{M \in \text{Bundle}(X,G_0)} \text{Vect}_G(M, \rho). \quad (26)
\]

Proof. By theorem 5, there is an inclusion
\[
\bigsqcup_{M \in \text{Bundle}(X,G_0)} \text{Vect}_G(M, \rho) \hookrightarrow \text{Bundle}(X, \text{Aut}_G (G_0 \times (F \otimes V))). \quad (27)
\]

Now we will construct an inverse to the map (27), so the fact that the last union is disjoint will follow. Let $\Psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \rightarrow \text{Aut}_G (G_0 \times (F \otimes V))$ be the transition functions of a bundle $\xi \in \text{Bundle}(X, \text{Aut}_G (G_0 \times (F \otimes V)))$. By lemma 2, there is a continuous projection of groups $pr : \text{Aut}_G (G_0 \times (F \otimes V)) \rightarrow G_0$. So, by composition with $pr$ we obtain a bundle with the discrete fiber $G_0$, and it is well known that $G_0$ acts fiberwise and freely over the total space $M$ of this bundle and that $M/G_0 = X$.

Also, we can assume that we have chosen an atlas such that there is a homeomorphism
\[
M \approx \bigcup_{\alpha} (U_\alpha \times G_0) \approx \bigcup_{\alpha} \left( \bigcup_{[g] \in G_0} [g]U_\alpha \right)
\]
where the intersections are defined by the rule
\[
[1]U_\alpha \cap [g_{\alpha\beta}]U_\beta \approx U_\alpha \cap U_\beta
\]
where $[g_{\alpha\beta}] = pr \circ \Psi_{\alpha\beta}$.

On the other hand, we have
\[
\xi \approx \bigcup_{\alpha} (U_\alpha \times (G_0 \times (F \otimes V)))
\]
where $U_\alpha \times (G_0 \times (F \otimes V))$ intersects $U_\beta \times (G_0 \times (F \otimes V))$ on the points $(x, g, f \otimes v) = (x, \Psi_{\alpha\beta}([g], f \otimes v)) = (x, [g_{\alpha\beta}g], A_{\alpha\beta}[g](f \otimes v))$ where $x \in U_\alpha \cap U_\beta$ and, once again, we are using lemma 2 for the description of the operators $\Psi_{\alpha\beta}$.

Taking into account the homeomorphism
\[
U_\alpha \times G_0 \approx \bigsqcup_{[g] \in G_0} [g]U_\alpha
\]
we can rewrite
\[
([g]x, f \otimes v) = ([gg_{\alpha\beta}]x, A_{\alpha\beta}[g](f \otimes v))
\]
Therefore, the projection
\[(U_\alpha \times G_0) \times (F \otimes V) \to U_\alpha \times G_0\]
extends to a well-defined and continuous projection
\[\xi \to M.\]

It is clear by the preceding formulas, that this projection will be $G$-equivariant, if $G$ acts canonically over the fibers and in by left translations on $G_0$ under the quotient map $G \to G/H = G_0$. So, we have $\xi \in \text{Vect}_G(M, \rho)$.

To end the proof, we make the remark that, by the theory of principal $G_0$-bundles, the construction of the space $M$ is up to equivariant homeomorphism. This means that the inverse to (27) is well defined.

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