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A novel mathematical model for COVID-19 with remedial strategies

Shumaila Javeed, Subtain Anjum, Khurram Saleem Alingeer, M. Atif, Mansoor Shaukat Khan, W. Aslam Farooq, Atif Hanif, Hijaz Ahmad, Shao-Wen Yao

Corresponding author.
E-mail address: yaoshaowen@hpu.edu.cn (S.-W. Yao).

Coronavirus (COVID-19) outbreak from Wuhan, Hubei province in China and spread out all over the World. In this work, a new mathematical model is proposed. The model consists the system of ODEs. The developed model describes the transmission pathways by employing non constant transmission rates with respect to the conditions of environment and epidemiology. There are many mathematical models purposed by many scientists. In this model, “α" and “α", transmission coefficients of the exposed cases to susceptible and infectious cases to susceptible respectively, are included. “θ" as a governmental action and restriction against the spread of coronavirus is also introduced. The RK method of order four (RK4) is employed to solve the model equations. The results are presented for four countries i.e., Pakistan, Italy, Japan, and Spain etc. The parametric study is also performed to validate the proposed model.

Introduction
Coronaviruses are a family of viruses responsible for respiratory diseases. At the end of 2019, a new type of coronavirus identified in Wuhan, China. The World Health Organization (WHO) gives a name to coronavirus as COVID-19. At the early stage 55% of first 425 confirmed cases were linked to the Huanan Seafood market [1], after this it spread from human to human. The disease novel coronavirus (COVID-19) started to spread in China and all over the World. It becomes a global concern in a few weeks. The WHO declared the novel coronavirus as COVID-19 [2]. By 1st March 2020, COVID-19 quickly spread in all the provinces of China and also it spread to 58 different countries [3,4]. According to WHO at the end of March, there were 754,933 confirmed cases and 36,522 deaths, on 31th July 2020 there were 17,114,712 confirmed cases and 1,459,920 deaths as well. Now on 30th November 2020 there are 62,516,515 confirmed cases and 1,459,920 deaths in 220 countries or areas [5]. Currently, the outbreak of COVID-19 is ongoing and infection cases have been growing and deaths as well.

Modelling and simulations are very important decision tools that can be helpful to simulate and finally control epidemiological animal and human diseases[6–8]. However, every situation of disease exhibits it’s own particular biological behaviours and characteristics, the proposed models of diseases will be able to tackle the real situations. Coronavirus is a totally new virus and completely a new situation. COVID-19 gets the attention of scientists such as mathematicians who worked in mathematical modeling. In February and March 2020, some papers about this coronavirus were published [9–13]. Some investigations on spreading of novel coronavirus (COVID-19) in mainland China was useful to understand the outbreak of COVID-19 all over the world [14–18]. The paper [19] is based on three ODEs which is called the SIR model. There are many variants of the SEIR and SIR models found in the literature [20–24]. Many other models and solutions based on differential equations are presented in recent literature [25–30]. Some of the authors also considered isolation and quarantine to reduce the spread of disease [31–33]. Similar studies were also reported in the literature for Turkey, Pakistan and South Africa [34–36]. Few recommendations are also provided by the researchers working in this area to reduce the effects of COVID-19 [37–39].

In this research, the effects of governmental action, weather condi-
d\(G(t)\)
\[
\frac{dG(t)}{dt} = \delta S(t) - \mu G(t).
\]

Here
\[
S(0) = \phi_1, E(0) = \phi_2, I(0) = \phi_3, Q(0) = \phi_4, R(0) = \phi_5, G(0) = \phi_6,
\]
and
\[
\Lambda = a_0 N.
\]

In this model “\(a_0\)” is birth rate, “\(a_e\)” is transmission rate from human to human, of the exposed cases to susceptible and “\(a_i\)” is also transmission rate from human to human, of the infectious cases to susceptible. Here we also introduce “\(\delta\)” as a governmental action and restriction against the spread of coronavirus. We conduct detailed analysis of model by using numerical methods as Runge Kutta method of order 4 (RK4) and Euler method, then manifest its applications by publicly reported data of different countries about coronavirus. In this research, proposed model is validated by considering four test problems. We use publicly available data about this virus to estimate the parameters. In these test problems, we provide complete overview about COVID-19 of Pakistan, Italy, Japan and Spain and also compare respectively the numerical result with the real reported data of virus of these mentioned countries. The results will be compared for above-mentioned countries on the base of governmental policies to control coronavirus, weather conditions, new cases, and deaths, etc.

The main motivation for this research is to explore the data of COVID-19 for four different countries with quantitative parameters. Safe and danger zones will also be highlighted after the detailed analysis of model and real data. Further more, quantitative analysis will help any health professional to predict the spread of pandemic as well as suggest how to control it by increasing safety measures to appropriate quantitative levels.

**Model formulation**

To model the coronavirus COVID-19, we proposed a new mathematical model, by introducing six different categories by dividing total human population in six compartments, i.e. \(S(t)\), \(E(t)\), \(I(t)\), \(Q(t)\), \(R(t)\), \(G(t)\) at time \(t\) the number of susceptible cases, exposed cases, infectious cases, quarantined cases, removed cases (recover and death both) and isolated cases (due to govt action).

The terms of developed model are defined as below:

**Susceptible (S):** Population of the region who is at risk of infection from disease.

**Exposed (E):** A person who may or may not infected from disease but not able to transmit.

**Infectious (I):** A person who is able to transmit the disease to other persons.

**Quarantined (Q):** A person who is under treatment and not able to meet other persons.

**Removed (R):** A person who got recovered or lost the life from disease.

**Isolated (G):** A person who is far away from other persons, objects, or society by government policy. Corresponding ODEs of model are given below:

\[
\frac{dS(t)}{dt} = \Lambda - a_e S(t)E(t) - a_i S(t)I(t) - \delta S(t) - \mu S(t),
\]

\[
\frac{dE(t)}{dt} = a_e S(t)E(t) + a_i S(t)I(t) - \beta E(t) - \mu E(t),
\]

\[
\frac{dI(t)}{dt} = \beta E(t) - \gamma I(t) - \mu I(t),
\]

\[
\frac{dQ(t)}{dt} = \gamma I(t) - (r + d)Q(t) - \mu Q(t),
\]

\[
\frac{dR(t)}{dt} = (r + d)Q(t) - \mu R(t),
\]

\[
\frac{dG(t)}{dt} = \delta S(t) - \mu G(t).
\]
Analysis of Model

In this section, the stability analysis is discussed by using linearizing method. Consider,
\(S(t), E(t), I(t), Q(t), R(t), G(t) \in \mathbb{R}^6\).

The feasible region \(\Omega\)

\[
\frac{dN}{dt} = \Delta - \mu N, \quad \frac{dx}{dt} = \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dQ(t)}{dt} + \frac{dR(t)}{dt} + \frac{dG(t)}{dt}.
\]

After putting the values, we have

\[
\frac{dN}{dt} = \Lambda - \mu N, \quad \frac{dx}{dt} = \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dQ(t)}{dt} + \frac{dR(t)}{dt} + \frac{dG(t)}{dt}.
\]

The system of Eqs. (1)–(6) admits two equilibrium points \(P_1 = (1, 0, 0, 0, 0, 0)\) and \(P_2 = (x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60})\), where \(x_{10} = \frac{(\delta + \mu)Qx_2}{\beta x_2 + \mu}\), \(x_{20} = \frac{(\delta + \mu)Qx_2}{\beta x_2 + \mu}\), \(x_{30} = \frac{(\delta + \mu)Qx_2}{\beta x_2 + \mu}\), \(x_{40} = \frac{(\delta + \mu)Qx_2}{\beta x_2 + \mu}\), \(x_{50} = \frac{(\delta + \mu)Qx_2}{\beta x_2 + \mu}\), \(x_{60} = \frac{(\delta + \mu)Qx_2}{\beta x_2 + \mu}\).

Proof:

The system of Eqs. (1)–(6) is given as by replacing \(S(t) \rightarrow x_1, E(t) \rightarrow x_2, I(t) \rightarrow x_3, Q(t) \rightarrow x_4, R(t) \rightarrow x_5, G(t) \rightarrow x_6\):

\[
dx1 = \Delta - \alpha x_1 x_2 - \alpha x_1 x_3 - \delta x_1 - \mu x_1,
\]

\[
dx2 = \alpha x_1 x_2 + \alpha x_1 x_3 - \beta x_2 - \mu x_2,
\]

\[
dx3 = \beta x_2 - \gamma x_3 - \mu x_3,
\]

\[
dx4 = \gamma x_3 - (r + d)x_4 - \mu x_4,
\]

\[
dx5 = (r + d)Qx_4 - \mu x_5,
\]

\[
dx6 = \delta x_1 - \mu x_6.
\]

The equilibrium points \((P_1, P_2)\) satisfy the following relation:

\[
\begin{align*}
\Delta - \alpha x_1 x_2 - \alpha x_1 x_3 - \delta x_1 - \mu x_1 &= 0, \\
\alpha x_1 x_2 + \alpha x_1 x_3 - \beta x_2 - \mu x_2 &= 0, \\
\beta x_2 - \gamma x_3 - \mu x_3 &= 0, \\
\gamma x_3 - (r + d)x_4 - \mu x_4 &= 0, \\
(r + d)Qx_4 - \mu x_5 &= 0, \\
\delta x_1 - \mu x_6 &= 0.
\end{align*}
\]

The first point \(P_1 = (1, 0, 0, 0, 0, 0)\) is trivial in the sense that all the persons are healthy and stay healthy for all time. Now we find the second equilibrium point, for this we consider Eqs. (14)–(19).

Adding Eqs. (14) and (15), we obtain

\[
\Lambda - (\delta + \mu)x_1 - (\beta + \mu)x_2 = 0,
\]

\[
x_2 = \frac{\Lambda - (\delta + \mu)x_1}{(\beta + \mu)}.
\]

Putting the value \(x_2\) into Eq. (21), we have

\[
x_3 = \frac{\beta}{(\beta + \mu)}x_1.
\]

Putting the value \(x_3\) into Eq. 20, we obtain

\[
x_4 = \frac{1}{(\beta + \mu)}\left[\Lambda - (\delta + \mu)(\beta + \mu)(\gamma + \mu)\right] - \beta x_2 - \mu x_2 = 0.
\]

\[
x_5 = \frac{1}{(\beta + \mu)(\gamma + \mu)}\left[\Lambda - (\delta + \mu)(\beta + \mu)(\gamma + \mu)\right] - \beta x_2 - \mu x_2 = 0.
\]

From Eq. (17), we have

\[
x_6 = \frac{\gamma}{r + d + \mu}x_1.
\]

Putting the value \(x_6\), we get

\[
x_1 = \frac{\beta}{(r + d + \mu)(\beta + \mu)(\gamma + \mu)}\left[\Lambda - (\delta + \mu)(\beta + \mu)(\gamma + \mu)\right] - \beta x_2 - \mu x_2 = 0.
\]

From Eq. (18), we have

\[
x_5 = \frac{\beta}{(\beta + \mu)(\gamma + \mu)}\left[\Lambda - (\delta + \mu)(\beta + \mu)(\gamma + \mu)\right] - \beta x_2 - \mu x_2 = 0.
\]

The \(P_2 = (x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60})\) is the point that corresponds to the endemic state i.e. the COVID-19 disease persists in two population.
The calculated values of $x_{10}, x_{20}, x_{30}, x_{40}, x_{50}$ and $x_{60}$ are given below:

$$x_{10} = \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}}, \quad x_{20} = \frac{1}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right], \quad x_{30} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right] x_{40} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right], \quad x_{50} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right] x_{60} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right].$$

Hence it is proved that the system of Eqs. (1)–(6) has two equilibrium points $P_{1} = (1, 0, 0, 0, 0, 0)$ and $P_{2} = (x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60})$.

**Theorem**

1. The equilibrium point at $P_{1} = (1, 0, 0, 0, 0, 0)$ is a saddle point.
2. The equilibrium point at $P_{2} = (x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60})$ is asymptotically stable.

Here,

$$x_{10} = \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}}, \quad x_{20} = \frac{1}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right], \quad x_{30} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right] x_{40} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right], \quad x_{50} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right] x_{60} = \frac{\beta}{(\gamma + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right].$$

**Proof:**

For simplicity, we let:

$$\frac{1}{(\beta + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right] = \xi,$$

and

$$\frac{\beta}{(\beta + \mu)} \left[ \Lambda - \frac{(\beta + \mu)(\gamma + \mu)}{\beta \alpha_{1} (\gamma + \mu) a_{1} \alpha_{1}} \right] = \psi.$$

Then we have,

$$x_{10} = \xi, \quad x_{20} = \xi, \quad x_{30} = \Psi, \quad x_{40} = \frac{\xi}{(\beta + \mu) a_{1} \alpha_{1}}, \quad x_{50} = \frac{\xi}{(\beta + \mu) a_{1} \alpha_{1}}, \quad x_{60} = \frac{\xi}{(\beta + \mu) a_{1} \alpha_{1}}.$$

The parameters values given in Table 1. The values given in Table 1 are collected from WHO and calculated using statistical models [40–46].

1. For finding the variational matrix, we use the system of model Eqs. (7)–(12) at the first equilibrium point $P_{1}$, we get the following matrix.

$$\begin{pmatrix}
-\beta \xi & -\alpha_{2} & 0 & 0 & 0 & 0 \\
0 & -\alpha_{2} & \beta & 0 & 0 & 0 \\
0 & 0 & -\beta & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 \\
0 & 0 & 0 & 0 & 0 & -\beta \\
\end{pmatrix}$$

(28)

Using MAPLE for Eq. 28, the following characteristic equation is obtained.

$$\lambda^{6} - (6\mu + r + d + \gamma - \alpha_{2} + \beta + \delta) \lambda^{5} + (6\mu + 2d + 3\gamma + 2\beta + 4r + \delta + 5\alpha_{2} + 4\beta + 3d + 7\gamma + 15\mu + 2\delta + 5\mu - 5\alpha_{2} - 2\beta - 4\delta) \lambda^{4} + (4\beta + 3d + 6\delta + 7\gamma + 15\mu + 4\beta + 2d + 7\gamma + 15\mu - 5\alpha_{2} - 2\beta - 4\delta) \lambda^{3} + (4\beta + 3d + 6\delta + 7\gamma + 15\mu + 4\beta + 2d + 7\gamma + 15\mu - 5\alpha_{2} - 2\beta - 4\delta) \lambda^{2} + (4\beta + 3d + 6\delta + 7\gamma + 15\mu + 4\beta + 2d + 7\gamma + 15\mu - 5\alpha_{2} - 2\beta - 4\delta) \lambda + (4\beta + 3d + 6\delta + 7\gamma + 15\mu + 4\beta + 2d + 7\gamma + 15\mu - 5\alpha_{2} - 2\beta - 4\delta) = 0$$

(29)

Using MAPLE for Eq. 29, the following characteristic equation is obtained.

$$\lambda^{6} - (6\mu + \delta + \beta - \gamma + \alpha_{2} + \alpha_{2} + \beta + d + \delta) \lambda^{5} + (6\mu + \delta + \beta - \gamma + \alpha_{2} + \beta + d + \delta) \lambda^{4} + (6\mu + \delta + \beta - \gamma + \alpha_{2} + \beta + d + \delta) \lambda^{3} + (6\mu + \delta + \beta - \gamma + \alpha_{2} + \beta + d + \delta) \lambda^{2} + (6\mu + \delta + \beta - \gamma + \alpha_{2} + \beta + d + \delta) \lambda + (6\mu + \delta + \beta - \gamma + \alpha_{2} + \beta + d + \delta) = 0$$

(30)
Fig. 1. Proposed Model of COVID-19 with inclusion of government policy.

Fig. 2. In Pakistan: Tests performed and confirmed cases.

Fig. 3. In Pakistan: Recovered cases and deaths Cases.
Table 2
Study of relative errors of infectious cases for Pakistan.

| Date   | Relative errors | Date   | Relative errors | Date   | Relative errors |
|--------|-----------------|--------|-----------------|--------|-----------------|
| 1st    | 0.00353714      | 11th   | 0.018611967     | 21st   | 0.022530718     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 2nd    | 0.05933762      | 12th   | 0.04961982      | 22nd   | 0.014370193     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 3rd    | 0.016483681     | 13th   | 0.03904853      | 23rd   | 0.017295136     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 4th    | 0.07782919      | 14th   | 0.043724446     | 24th   | 0.004993105     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 5th    | 0.066274092     | 15th   | 0.016915839     | 25th   | 0.026576735     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 6th    | 0.02744429      | 16th   | 0.006657531     | 26th   | 0.012788158     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 7th    | 0.004037946     | 17th   | 0.012635892     | 27th   | 0.015288762     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 8th    | 0.026160779     | 18th   | 0.008455499     | 28th   | 0.03361372      |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 9th    | 0.072072323     | 19th   | 0.005058541     | 29th   | 0.004347435     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 10th   | 0.044731812     | 20th   | 0.002280837     | 30th   | 0.006717684     |
| Nov    |                  | Nov    |                  | Nov    |                  |

Table 3
Study of relative errors of removed cases for Pakistan.

| Date   | Relative errors | Date   | Relative errors | Date   | Relative errors |
|--------|-----------------|--------|-----------------|--------|-----------------|
| 1st    | 0.022848439     | 11th   | 0.001014855     | 21st   | 0.014629661     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 2nd    | 0.03479109      | 12th   | 0.007550067     | 22nd   | 0.010417078     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 3rd    | 0.055548059     | 13th   | 0.032253553     | 23rd   | 0.015949148     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 4th    | 0.013282583     | 14th   | 0.006472368     | 24th   | 0.031151726     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 5th    | 0.005404622     | 15th   | 0.040744057     | 25th   | 0.02244561      |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 6th    | 0.030001821     | 16th   | 0.041790411     | 26th   | 0.017771562     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 7th    | 0.083562877     | 17th   | 0.050551446     | 27th   | 0.007295753     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 8th    | 0.018183534     | 18th   | 0.040348327     | 28th   | 0.011238798     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 9th    | 0.009819484     | 19th   | 0.018725169     | 29th   | 0.000512259     |
| Nov    |                  | Nov    |                  | Nov    |                  |
| 10th   | 0.003797866     | 20th   | 0.004873214     | 30th   | 0.000952095     |
| Nov    |                  | Nov    |                  | Nov    |                  |
Now we find the eigenvalues using MATLAB from the above equation for Pakistan and Spain.

The eigenvalues for Pakistan are as follows:
\[ \lambda_1 = -1.90508 \times 10^5, \lambda_2 = -0.0355144028, \lambda_3 = -1.90508 \times 10^5, \]
\[ \lambda_4 = -5.08220963929699 \times 10^5, \]
\[ \lambda_5 = -0.000394150800098247, \lambda_6 = -0.1410769508. \]

The eigenvalues for Spain are as follows:
\[ \lambda_1 = -2.54601 \times 10^5, \lambda_2 = -0.1186935411, \lambda_3 = -2.54601 \times 10^5, \]
\[ \lambda_4 = -2.1955.2092485067, \]
\[ \lambda_5 = -0.00138325809703833, \lambda_6 = -0.169372760014153. \]

All the eigenvalues are negative at \( P_2 \), thus the point \( P_2 \) is asymp-
totically stable for both Pakistan and Spain. The population will have COVID-19 disease.

**Positivity of Solution**

The modelling of COVID-19 will be meaningful, if the solution of the system are non-negative initial condition will remain positive for all $t > 0$.

**Theorem**

For all $t > 0$ and initial conditions $P(0) \geq 0$ where $P(t) = (S, E, I, Q, R, G)$ the solution of the model are positive for all $t > 0$.

**Proof:**

Consider Eq. 1 of the COVID-19 model,

$$\frac{dS(t)}{dt} = \Lambda - \alpha E(t)S(t) - \alpha I(t)S(t) - \beta S(t) - \mu S(t)$$
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\[ \frac{dS}{dt} = - (\alpha E(t) + \alpha I(t) + \mu) S(t) - \delta S(t) + \Lambda \]

Now we let \( \lambda_1(t) = (\alpha E(t) + \alpha I(t) + \mu) \) and \( \lambda_2(t) = \Lambda \), then above equation become,

\[ \frac{dS}{dt} = - (\lambda_1(t)) S(t) - \delta S(t) + \lambda_2(t) \]

\[ \frac{dS}{dt} = - (\lambda_1(t) + \delta) S(t) + \lambda_2(t) \]  

(30)

Taking integration on both sides of Eq. (30),

\[ \frac{d}{dt} \left[ S(t) \exp \left\{ dt + \int_0^t \lambda_2(S) dS \right\} \right] = \lambda_2(t) \exp \left\{ \delta t + \int_0^t \lambda_1(S) dS \right\} \]

Hence

\[ \left[ S(t) \exp \left\{ dt + \int_0^t \lambda_2(S) dS \right\} - S(0) \right] = \int_0^t \lambda_2(t) \exp \left\{ \delta S + \int_0^t \lambda_1(S) dS \right\} dS \]

Thus the solution of the above equation is

Fig. 8. Variations of the susceptible, isolated, exposed, infectious, quarantined and removed population for the different values of \( \alpha_E \). The values of other parameters are \( \alpha_I = 3.245065087, \beta = 0.0003351, \gamma = 0.1240579, \delta = 0.003505, r = 0.058306850 \) and \( d = 0.00414502 \).
Fig. 9. Variations of the susceptible, isolated, exposed, infectious, quarantined and removed population for the different values of $\alpha_I$. The values of other parameters are $\alpha_E = 0.004253392$, $\beta_1 = 0.0003351$, $\gamma_1 = 0.1240579$, $\delta = 0.003505$, $r = 0.058306850$ and $d = 0.00414502$. 

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\[ S(t_1) = S(0) \exp \left\{ - \left( d(t_1) + \int_0^{t_1} \lambda_2(S) dS \right) \right\} + \exp \left\{ - \left( d(t_1) \right) \right\} \]
\[ \times \int_0^{t_1} \lambda_1(S) \exp \left\{ \delta S + \int_0^{S} \lambda_1(S) dS \right\} dS > 0 \]

Similarly it can be shown that the quantities \( E, I, Q, R, G \) are positive for \( P > 0 \) and for all time \( t > 0 \).

Numerical test problems

The mathematical model is applied to study the first and the second wave of epidemic COVID-19 in different countries such as Pakistan, Italy, Japan and Spain. The second wave is stronger than first wave of COVID-19. We use the outbreak data daily published by World Health Organization (WHO) and other sources [40–46]. The collected data sets consist of susceptible population, exposed and infectious cases, new reported quarantined cases, recovered cases and deaths due to COVID-19 of above mentioned countries. The RK4 is employed to solve the
Fig. 11. Variations of the susceptible, isolated, exposed, infectious, quarantined and removed population for the different values of $\gamma^{-1}$. The values of other parameters are $\alpha = 0.004253392$, $\alpha_i = 3.245065087$, $\beta_1 = 0.0003351$, $\lambda = 0.003505$, $r = 0.058306850$ and $d = 0.00414502$. 
Suppose we have $m$ differential equations:

\[ y_1' = g_1(t, y_1, y_2, \ldots, y_m) \]
\[ y_2' = g_2(t, y_1, y_2, \ldots, y_m) \]
\[ \vdots \]
\[ y_m' = g_m(t, y_1, y_2, \ldots, y_m), \]

with the initial conditions,

\[ y_1(t_0) = y_{10}, \ y_2(t_0) = y_{20}, \ldots, \ y_m(t_0) = y_{m0}. \]
Fig. 13. In Italy: Tests performed and confirmed cases.

Fig. 14. In Italy: Recovered cases and deaths.

Fig. 15. In Italy: Simulated results and real data of April 2020.
Fig. 16. In Italy: Simulated results and real data of November 2020.

Table 4
Study of relative errors of infectious cases for Italy.

| Date     | Relative errors | Date     | Relative errors | Date     | Relative errors |
|----------|-----------------|----------|-----------------|----------|-----------------|
| 1st Nov  | 0.005837104     | 11th     | 0.010989666     | 21st     | 0.020836        |
| 2nd      | 0.025797482     | 12th     | 0.010825995     | 22nd     | 0.006932        |
| 3rd      | 0.011612837     | 13th     | 0.029570628     | 23rd     | 0.004111        |
| 4th      | 0.01128946      | 14th     | 0.006934419     | 24th     | 0.019193        |
| 5th      | 0.015502215     | 15th     | 0.007492452     | 25th     | 0.006465        |
| 6th      | 0.022972174     | 16th     | 0.01461214      | 26th     | 0.008803        |
| 7th      | 0.009043897     | 17th     | 0.01080272      | 27th     | 0.004014        |
| 8th      | 0.0017352       | 18th     | 0.001021074     | 28th     | 0.020968        |
| 9th      | 0.028329254     | 19th     | 0.020118045     | 29th     | 0.000759        |
| 10th     | 0.014030874     | 20th     | 0.003738704     | 30th     | 0.000106        |

Table 5
Study of relative errors of removed cases for Italy.

| Date     | Relative errors | Date     | Relative errors | Date     | Relative errors |
|----------|-----------------|----------|-----------------|----------|-----------------|
| 1st Nov  | 0.021289       | 11th     | 0.009953        | 21st     | 0.000747483     |
| 2nd      | 0.010234       | 12th     | 0.005955        | 22nd     | 0.008041242     |
| 3rd Nov  | 0.022164       | 13th     | 0.012284        | 23rd     | 0.004971367     |
| 4th Nov  | 0.018508       | 14th     | 0.002667        | 24th     | 0.003336089     |
| 5th Nov  | 0.001966       | 15th     | 0.009950        | 25th     | 0.012826909     |
| 6th Nov  | 0.000210       | 16th     | 0.001081        | 26th     | 0.003136554     |
| 7th Nov  | 0.010121       | 17th     | 0.000732        | 27th     | 0.004863319     |
| 8th Nov  | 0.005261       | 18th     | 0.012080        | 28th     | 0.00176126      |
| 9th Nov  | 0.012356       | 19th     | 0.010885        | 29th     | 0.000366077     |
| 10th     | 0.000081       | 20th     | 0.000091        | 30th     | 0.000256919     |

There is no derivative on the right hand side and all of these $m$ equations are of order one. RK4 formula is as follows:

$$y_{n+1} = y_n + \frac{h}{6}(L_{1} + 2L_{2} + 2L_{3} + L_{4}),$$

where,

$$L_{1} = g_{1}(t_{n}, y_{1n}, y_{2n}, ..., y_{mn}).$$

$$L_{2} = g_{2}(t_{n} + \frac{h}{2}, y_{1n} + \frac{h}{2}L_{1}, y_{2n} + \frac{h}{2}L_{1}, ..., y_{mn} + \frac{h}{2}L_{1}).$$

$$L_{3} = g_{3}(t_{n} + \frac{h}{2}, y_{1n} + \frac{h}{2}L_{2}, y_{2n} + \frac{h}{2}L_{2}, ..., y_{mn} + \frac{h}{2}L_{2}).$$

$$L_{4} = g_{4}(t_{n} + h, y_{1n} + hL_{3}, y_{2n} + hL_{3}, ..., y_{mn} + hL_{3}).$$

where $y_{n+1}$ is the RK4 approximation of $y(t_{n+1})$ and $h$ is step size.

For numerical simulation, we consider four test problems and their comparisons with each other.

1. Test problem 1: Pakistan
2. Test problem 2: Italy
3. Test problem 3: Japan
4. Test problem 4: Spain

Test Problem 1: Pakistan

In Pakistan, COVID-19 spread out mostly from China and Iran. In start four Pakistani students in China effected from coronavirus. After this, many students and other people came back from China, Iran and other countries with positive tests of coronavirus. They came back to home and met with other people, in this way, coronavirus started to spread in all over Pakistan. Due to high risk of COVID-19, government took actions to control international traffics and bound all the people in their homes. One time Government successfully controlled coronavirus and step wise took off lock-down but from end of October second wave of coronavirus started in Pakistan and all over the World. During first coronavirus wave from 15th March, 2020 to 31st August, 2020, Pakistan faces 2,95,849 positive cases, 97.8% people recovered and 2.2% people died. Up to 30th November, there were total 4,00,482 positive cases, at the same time 97.7% people recovered from coronavirus and 2.3% people died due to coronavirus. Fig. 1 shows the proposed model of COVID-19 with inclusion of government policy.
The current and complete overview of COVID-19 in all over Pakistan shown in Figs. 2 and 3. These Figures represents the real data of tests performed, confirmed cases, deaths cases and recovered cases. Data related to COVID-19 cases of Pakistan are taken from different sources [40,41,43]. Fig. 2 (first graph) depicts the number of tests performed from 1st April, 2020 to 30th November, 2020. Mostly tests are performed in September (highest 42,299 in one day and total 8,93,091 in a month). The total tests performed till 30th November, 2020 are 55,08,810. Fig. 2 (second graph) represents the confirmed cases from 1st April, 2020 to 30th November, 2020. Coronavirus started from 15th March in Pakistan, and continuously increased till mid of June, and then decreased till September. There were most confirmed cases in June (highest 6,825 in one day and total 1,41,010 in a month). The total confirmed cases in Pakistan are 4,00,482 till 30th November, 2020. The confirmed cases are still increasing. The government closed all the public points, implemented a strict lockdown and aware the masses to follows the SOPs. Due to government polices, the COVID-19 remained in control from end of June to mid of October. In October, the government took off the lockdown and coronavirus spread again. Fig. 3 (first graph) represents recovered cases from 1st April, 2020 to 30th November, 2020.
In total 3,43,282 recovered people till 30th November, 2020. Fig. 3 (second graph) represents the number of deaths from 1st April to 30th November, 2020. As most cases were observed in June and July thus most deaths occurred in June and July. In total 8,091 deaths till 30th November, 2020.

The model given in Eqs. (1)–(6) are solved using RK4. The simulated results of model equations (c.f Eqs. (1)–(6)) are presented. For the estimation of the values of parameters, the statistics terminologies are used. The parameters values are given in Table 1. The comparison of simulated results and real data of infectious and removed cases are provided in graphs.

**Problem 1:**
The comparison of simulated results and real data, during 1st coronavirus wave presented below. Fig. 4 represents comparison of simulated results and real data of infectious and removed cases from 1st May, 2020 to 30 May, 2020. The simulated results are close to the real data in infectious and removed cases as depicted in Fig. 4.

**Problem 2:**
In Fig. 5 represents the simulated results and real data of infectious and removed cases from 1st November to 30th November, 2020. The
Fig. 19. In Japan: Tests performed and confirmed cases.

Fig. 20. In Japan: Recovered cases and deaths.

Fig. 21. In Japan: Simulated results and real data of July 2020.
Table 6
Study of relative errors of infectious cases for Japan.

| Date   | Relative errors | Date   | Relative errors | Date   | Relative errors |
|--------|-----------------|--------|-----------------|--------|-----------------|
| 1st Nov| 0.021417        | 11th Nov| 0.013667       | 21st Nov| 0.003152       |
| 2nd Nov| 0.032025        | 12th Nov| 0.025980       | 22nd Nov| 0.011940       |
| 3rd Nov| 0.062096        | 13th Nov| 0.114169       | 23rd Nov| 0.015773       |
| 4th Nov| 0.046100        | 14th Nov| 0.007850       | 24th Nov| 0.011560       |
| 5th Nov| 0.019282        | 15th Nov| 0.066167       | 25th Nov| 0.015211       |
| 6th Nov| 0.029177        | 16th Nov| 0.010303       | 26th Nov| 0.049499       |
| 7th Nov| 0.116308        | 17th Nov| 0.024887       | 27th Nov| 0.049959       |
| 8th Nov| 0.038660        | 18th Nov| 0.052348       | 28th Nov| 0.081207       |
| 9th Nov| 0.031839        | 19th Nov| 0.046037       | 29th Nov| 0.043064       |
| 10th Nov| 0.034344       | 20th Nov| 0.002207       | 30th Nov| 0.006499       |

Table 7
Study of relative errors of removed cases for Japan.

| Date   | Relative errors | Date   | Relative errors | Date   | Relative errors |
|--------|-----------------|--------|-----------------|--------|-----------------|
| 1st Nov| 0.005460839     | 11th Nov| 0.002168027     | 21st Nov| 0.0022099304   |
| 2nd Nov| 0.00418358      | 12th Nov| 0.003456148     | 22nd Nov| 0.000166255    |
| 3rd Nov| 0.002964324     | 13th Nov| 0.005825422     | 23rd Nov| 0.004016489    |
| 4th Nov| 0.000982712     | 14th Nov| 0.008252693     | 24th Nov| 7.25122E-05    |
| 5th Nov| 0.008073763     | 15th Nov| 0.005905965     | 25th Nov| 0.003515107    |
| 6th Nov| 0.00130152      | 16th Nov| 0.002007694     | 26th Nov| 0.003280727    |
| 7th Nov| 0.00016718      | 17th Nov| 0.001519254     | 27th Nov| 0.003623707    |
| 8th Nov| 0.001257024     | 18th Nov| 0.004564607     | 28th Nov| 0.006613595    |
| 9th Nov| 0.002073573     | 19th Nov| 0.002818295     | 29th Nov| 0.000869262    |
| 10th Nov| 0.000156779     | 20th Nov| 0.002632602     | 30th Nov| 0.000338695    |

Fig. 22. In Japan: Simulated results and real data of November 2020.

Simulated results are close to the real data in infectious and removed cases, relative errors are given in the Table 2 and 3. Figs. 6 represents the simulated results of the model (susceptible, isolated, exposed, infectious, quarantined and removed cases) from 1st November to 30th November, 2020. The results shows the decrease in susceptible population and increase in infected and removed population. Similarly the exposed cases and isolated cases increases. The affected people are recovering from disease. When the people follow the SOPs, then decrease in infectious and quarantined cases, and increase in isolation. When the government strictly implements the SOP’s against the spread of coronavirus then isolated population increase rapidly.

Table 2 represents the relative errors from 1st November to 30th November, 2020 of infectious cases. We see that the relative errors are less than 1 for all days. Table 3 represents relative errors from 1st November to 30th November, 2020 of removed cases. We see that the relative errors are less than 1 for all days, which verify the correctness of model formulation. The developed mathematical model can be helpful to measure the coronavirus situations.

**Problem 3: Prediction for Next 6 Months**

The prediction of COVID-19 using the mathematical model is presented. Fig. 7 represents the prediction of COVID-19 for 180 days. The simulated results by developed model of COVID-19 (1st December, 2020 to 30th May, 2021) are presented. The results show that the number of infected cases are increasing almost 265%. As the infected increases, the suspected decreases 16%, which is clearly depicted in Fig. 7. Fig. 7 depicts that number of removed cases are increasing. According to the results, the infected population due to COVID-19 will increase and the educational system will remain on-line. The government has to implement strict strategies such as smart lock-down, reduction of timings in shops etc to control the disease. If people follow SOPs than coronavirus will be controlled otherwise its not possible.

**Problem 4: Parametric study of COVID-19 in Pakistan**

The focus of this section is to study the effects of parameters by changing one parameter while keeping all other parameters fixed. The value of $\alpha_E$ transmission rate of the exposed cases to susceptible is varied (0.00163998, 0.00425339, 0.00695208) and all the others parameter values are kept fixed. The variation of susceptible, isolated, exposed, infectious, quarantined and removed population are shown in Fig. 8. It is found that as the quantitative values of $\alpha_E$ increase, the number of infected population also increases along with the increase in number of exposed, quarantined and removed population. The susceptible population decreases more rapidly. The decrease in the value of $\alpha_E$ shows a decline in the number of infected, due to less number of infected population. There also occurs a decrease in number of exposed, quarantined and removed population. The susceptible population decline slowly and
isolated remain constant. Similarly with the increase and decrease in the value of $\alpha_I$, transmission rate of the infectious cases to susceptible $(2.245065087, 3.245065087, 4.245065087)$, presented in Fig. 9. In Fig. 10, the value of $\beta_1$ (latent period) is considered as $(0.0004071, 0.0005071, 0.0006071)$ and all the other values of parameters are unchanged. The increase in $\beta_1$ bring the rise in number of infected population. Than the number of susceptible and exposed cases decreases, and increase in quarantined and removed cases, isolated population remain constant and vice versa. In Fig. 11, the value of $\gamma_1$ (quarantine delay) is considered as $(0.00710579, 0.10710579, 0.20710579)$ and all the other values of parameters are unchanged. The increase in $\gamma_1$ bring the fall in number of infected population. Similarly the number of susceptible, exposed, quarantined and removed cases are decreasing and vice versa. In Fig. 12, the value of $\delta$ (protection rate) is varied as $(0.002505, 0.003505, 0.004505)$ and all the other values of parameters are unchanged. The increase in $\delta$ bring the fall in number of infected population. Similarly the number of susceptible, exposed, quarantined and removed cases are decreasing and vice versa.
In this proposed model, we introduce $\alpha_E$ (transmission rate of the exposed to susceptible), $\alpha_I$ (transmission rate of the infectious to susceptible), and $\delta$ (protection rate). The value of these parameters in the current situation is 0.004253392, 3.245065087, and 0.003505 respectively. If the values of $\alpha_E$, $\alpha_I$, and $\delta$ are less than 0.001, 0.1, and greater than 0.01 respectively then the minimum number of new cases will report (Safe Zone). If the values of $\alpha_E$, $\alpha_I$, and $\delta$ are greater than 0.01, 5.0, and less than 0.0001 respectively then the maximum number of new cases will report (Danger Zone).

**Test Problem 2: Italy**

In the early stages, Italy was the 1st European nation that was affected from COVID-19. The northern Italy was mainly affected. On 30th January, 2020, two Chinese tourists were reported with positive test. After this, on 21st February, 2020 two more cases were reported. The number of deaths in the Lombardy region alone, is greater than the number of deaths in China. Italy became most affected region of the world. We see that one time Italian government controlled the spreading of coronavirus. Now, Italy is facing second wave of coronavirus that is
Fig. 25. In Spain: Tests performed and confirmed cases.

Fig. 26. In Spain: Recovered cases and deaths Cases.

Fig. 27. In Spain: Simulated results and real data of August 2020.
Table 8
Study of relative errors of infectious cases for Spain.

| Date    | Relative errors | Date    | Relative errors | Date    | Relative errors |
|---------|-----------------|---------|-----------------|---------|-----------------|
| 1st Nov | 0.000014        | 11th    | 0.018504        | 21st    | 0.009561        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 2nd     | 0.001751        | 12th    | 0.015874        | 22nd    | 0.012454        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 3rd     | 0.014795        | 13th    | 0.005287        | 23rd    | 0.002167        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 4th     | 0.017023        | 14th    | 0.031464        | 24th    | 0.011541        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 5th     | 0.003007        | 15th    | 0.011762        | 25th    | 0.012931        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 6th     | 0.006194        | 16th    | 0.007177        | 26th    | 0.004235        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 7th     | 0.010685        | 17th    | 0.000956        | 27th    | 0.015973        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 8th     | 0.015374        | 18th    | 0.008230        | 28th    | 0.006864        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 9th     | 0.008847        | 19th    | 0.008324        | 29th    | 0.010380        |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 10th    | 0.006876        | 20th    | 0.011750        | 30th    | 0.000408        |
| Nov     |                 | Nov     |                 | Nov     |                 |

Fig. 28. In Spain: Simulated results and real data of November 2020.

Table 9
Study of relative errors of removed cases for Spain.

| Date    | Relative errors | Date    | Relative errors | Date    | Relative errors |
|---------|-----------------|---------|-----------------|---------|-----------------|
| 1st     | 0.00000213      | 11th    | 0.000217077     | 21st    | 0.016671331     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 2nd     | 0.012492074     | 12th    | 0.00461542      | 22nd    | 0.012332486     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 3rd     | 0.02101917      | 13th    | 0.00209847      | 23rd    | 0.011836044     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 4th     | 0.026333605     | 14th    | 5.868196-06     | 24th    | 0.015308041     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 5th     | 0.015092313     | 15th    | 0.002903759     | 25th    | 0.015711163     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 6th     | 0.016035556     | 16th    | 0.005694897     | 26th    | 0.012592594     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 7th     | 0.010022301     | 17th    | 0.002728282     | 27th    | 0.008109654     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 8th     | 0.013570866     | 18th    | 0.001777407     | 28th    | 0.003540895     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 9th     | 0.001216548     | 19th    | 0.022646522     | 29th    | 0.001132337     |
| Nov     |                 | Nov     |                 | Nov     |                 |
| 10th    | 0.004912528     | 20th    | 0.019578519     | 30th    | 0.0000021565    |
| Nov     |                 | Nov     |                 | Nov     |                 |

Problem 1:

Fig. 15 represents comparison of simulated results and real data of infectious and removed cases from 1st April, 2020 to 30 April, 2020 (during 1st coronavirus wave). The simulated results are close to the real data in infectious and removed cases as depicted in Fig. 15.

Problem 2:

Mostly in Italy maximum coronavirus cases w.r.t all over the World were reported during first wave and than decreasing day by day. Most patients were recovering from disease but second wave of coronavirus started.
Fig. 16 represents the simulated results and real data of infectious and removed cases from 1st November to 30th November, 2020. The simulated results are close to the real data in infectious and removed cases, relative errors are given in Tables 4 and 5.

Fig. 17 represents the simulated results of the model (susceptible, isolated, exposed, infectious, quarantined and removed cases) from 1st to 30th November, 2020. The results shows the decrease in susceptible population and increase in infected and removed population. Similarly the exposed cases and isolated cases increases. The affected people are recovering from disease. When the people follow the SOPs, then decrease in infectious and quarantined cases, and increase in isolation. When the government strictly implements the SOPs against the spread of coronavirus then isolated population increases fastly.

Table 4 represents the relative errors from 1st November to 30th November, 2020 of infectious cases. We see that the relative errors are less than 1 for all days. Table 5 represents relative errors from 1st November to 30th November, 2020 of removed cases. We see that the relative errors are less than 1 for all days, which verify the correctness of model formulation. The developed mathematical model can be helpful to measure the coronavirus situations.
Problem 3: Prediction for Next 6 Months

The prediction of COVID-19 using the mathematical model is presented. Fig. 18 represents the prediction of COVID-19 for 180 days. The simulated results by developed model of COVID-19 (1st December, 2020 to 30th May, 2021) are presented. The results show that the number of infected cases are increasing almost 43%. As the infected increases, the suspected decreases 79%, which is clearly depicted in Fig. 18. Furthermore, Fig. 18 shows that number of removed cases are increasing. According to the results, the infected population due to COVID-19 will increase then decrease from April 2021. The government has to implement strict strategies such as smart lock-down, reduction of timings in shops etc to control the disease. If people follow SOPs then coronavirus will be controlled otherwise its not possible.

In this proposed model, we introduce $\alpha_E$ (transmission rate of the exposed to susceptible), $\alpha_I$ (transmission rate of the infectious to susceptible), and $\delta$ (protection rate). The value of these parameters in the current situation is 0.0273207, 0.1211833, and 0.0050605 respectively. If the values of $\alpha_E, \alpha_I$, and $\delta$ are less than 0.0001, 0.010, and greater than 0.020 respectively then the minimum number of new cases will report (Safe Zone). If the values of $\alpha_E, \alpha_I$, and $\delta$ are greater than 0.100, 1.00, and

Fig. 30. In Spain: Simulated results of susceptible, isolated, exposed, infectious, quarantined and removed cases from 1st December, 2020 to 30th May, 2021.
less than 0.0010 respectively then the maximum number of new cases will report (Danger Zone).

**Test Problem 3: Japan**

The 1st outbreak of the COVID-19 in the Japan was reported on 16th January, 2020 in a resident of Kanagawa Prefecture. He came back from Wuhan, China. The 2nd outbreak was reported between 11th to 23rd March, 2020 from returners and travelling passengers, who came from the United States and Europe. The major cause of spreading coronavirus in country is due to Europe and China. Japan government took action on early stages, the Prime Minister Shinzo Abe requested to the closure off all the Japanese schools on 27th Feb, 2020. In Japan, the death rate is lowest per capita with respect to the developed world. Now again Japan faces increasing in coronavirus in all over the country. During first coronavirus wave from 23rd March, 2020 to 30th June, 2020, Italy faces 18,612 positive cases, 94.5% people recovered and 5.5% people died. In second coronavirus wave till 30th September, 2020, Italy faces 83,591 positive cases, 97.9% people recovered and 2.1% people died. Up to 30th November, there were total 1,49,962 positive cases, at the same time 98.4% people recovered from coronavirus and 1.6% people died due to COVID-19.

The current and complete overview of COVID-19 in all over Japan shown in Figs. 19 and 20. These figures represent the real data of tests performed, confirmed cases, deaths cases and recovered cases. Data related to COVID-19 cases of Japan are taken from different sources [40,41,45], Fig. 19 (first graph) represents, the test performed from 1st March, 2020 to 30th November, 2020. Mostly tests are performed in August. The total tests performed till 30th November, 2020 are 41,06,169. Fig. 19 (second graph) represents confirmed cases from 1st March, 2020 to 30th November, 2020. Coronavirus start from 23rd March in Japan, and continuously increased till mid of April, then decreasing till July. In July, again coronavirus started to spread in Japan. There were most confirmed cases in August (highest 1762 in one day and total 32,162 in a month). The total confirmed cases in Pakistan are 1,49,962 till 30th November, 2020. The confirmed cases are still increasing. The government closed all the public points, implemented a strict lockdown and aware the masses to follows the SOPs. Due to government polices, the COVID-19 remained in control from end of May to July and in September and October. Government of Japan took off the lockdown and coronavirus spread again. When government took the lockdown than coronavirus controlled otherwise started to spread. Fig. 20 (first graph) represents recovered cases from 1st March, 2020 to 30th November, 2020. There were most recovered cases in August, during the 2nd wave of coronavirus. In total 1,27,289 recovered people till 30th November, 2020. Fig. 20 (second graph) represents the number of deaths from 1st March, 2020 to 30th November, 2020. There were most deaths in April and May (highest 29 in one day and total 832 in two month). In total 2,076 deaths till 30th November, 2020.

The model given in Eqs. (1)–(6) are solved using RK4 for Japan. The simulated results of model equations (c.f Eqs. (1)–(6)) are presented. For the estimation of the values of parameters the statistics terminologies are used. The comparison of simulated results and real data of infectious and removed cases are provided in graphs.

**Problem 1:**

Fig. 21 represents the comparison of simulated results and real data of infectious and removed cases from 1st July, 2020 to 30th July, 2020 of Japan. The simulated results are close to the real data in infectious and removed cases as depicted in Fig. 21. **Problem 2:**

Fig. 22 represent the simulated results and real data of infectious and removed cases from 1st November to 30th November, 2020. The simulated results are close to the real data in infectious and removed cases, relative errors are given in Table 6 and 7.

Fig. 23 represents the simulated results of the model (susceptible, isolated, exposed, infected, quarantined and removed cases) from 1st November to 30th November, 2020. The results shows the decrease in susceptible population and increase in infected and removed population. Similarly the exposed cases and isolated cases increases. The affected people are recovering from disease. When the people follow the SOPs, then decrease in infectious and quarantined cases, and increase in isolation. When the government strictly implements the SOPs against the spread of coronavirus then isolated population increases fastly.

**Table 6** represents the relative errors from 1st November to 30th November, 2020 of infectious cases. We see that the relative errors are less than 1 for all days. **Table 7** represents relative errors from 1st November to 30th November, 2020 of removed cases. We see that the relative errors are less than 1 for all days, which verify the correctness of model formulation. The developed mathematical model can be helpful to measure the coronavirus situations.

**Problem 3: Prediction for Next 6 Months**

The prediction of COVID-19 using the mathematical model is presented. Fig. 24 represents the prediction of COVID-19 for 180 days. The simulated results by developed model of COVID-19 (1st December, 2020 to 30th May, 2021) are presented. The results show that the number of infected cases are increasing almost 491%. As the infected increases, the suspected decreases 50%, which is clearly depicted in Fig. 24, Fig. 24 depicts that number of removed cases are increasing. According to the results, the infected population due to COVID-19 will increase. The government has to implement strict strategies such as smart lock-down, reduction of timings in shops etc to control the disease. If people follow SOPs than coronavirus will be controlled otherwise its not possible. In this proposed model, we introduce \( e_s \) (transmission rate of the exposed to susceptible), \( t_s \) (transmission rate of the infectious to susceptible), and \( \delta \) (protection rate). The value of these parameters in the current situation is 0.2416577, 0.1091595, and 0.0017900 respectively. If the values of \( e_s, t_s, \delta \) are less than 0.1, 0.01, and greater than 0.0030 respectively then the minimum number of new cases will report (Safe Zone). If the values of \( e_s, t_s, \delta \) and \( \delta \) are greater than 0.5, 1, 5, and less than 0.0005 respectively then the maximum number of new cases will report (Danger Zone).

**Test Problem 4: Spain**

The virus was firstly confirmed in Spain on 31st January, 2020 from a German tourist. On 13th March, 2020 coronavirus spread in all the 50 provinces of the country. The government of Spain imposed lockdown on 14th March, 2020. The Spain also had become world highest reported rate of coronavirus infection for doctors and nurses. Now Spain has lifted strictest lockdowns and has given permission to people for movement in country and has opened the borders to some countries. During first coronavirus wave from 31st January, 2020 to 30th June, 2020, Italy faces 2,54,117 positive cases, 85.2% people recovered and 14.8% people died. Up to 30th November, there were total 17,02,328 positive cases, at the same time 96.9% people recovered from coronavirus and 3.1% people died due to coronavirus.

The current complete overview of COVID-19 in all over the Spain shown in Figs. 25 and 26. These Figures represents the real data of tests performed, confirmed cases, deaths cases and recovered cases. Data related to COVID-19 cases of Pakistan are taken from different sources [40,41,46], Fig. 25 (first graph) depicts the number of test performed from 16th March, 2020 to 30th November, 2020. Mostly tests are performed from September. The total test performed till 30th November, 2020 are 1,67,72,604. Fig. 25 (second graph) represents the confirmed cases from 16th March, 2020 to 30th November, 2020. Coronavirus start from 31st January in Spain, and continuously increasing till April, and decreasing till mid of May. There were most confirmed cases in Spain during first coronavirus wave from 20th March to 20th April (highest 9,222 in one day and total 1,79,660 in 30 days). There were normal coronavirus cases in May and June. In Spain second coronavirus wave started from August and rapidly increases in confirmed cases from mid of October. The total confirmed cases in Spain are 17,02,328 till 30th November, 2020. The confirmed cases are still increasing. This figure
also depicts the effects of governmental actions, when Spain faces 1st wave of COVID-19 then government close all the public points, implements a strict lockdown and aware the masses to follows the SOPs. Due to government polices, the COVID-19 remained in control from end of May to August. In August, the government took off the lockdown and coronavirus spread again. Fig. 26 (first graph) represents recovered cases from 16th March, 2020 to 30th November, 2020. There were most recovered cases in September and October. In total 14,16,001 recovered cases till 30th November, 2020. Fig. 26 (second graph) represents increasing the number of deaths from 16th March to 30th November, 2020. As most cases were observed in April and November thus most deaths occurred in April and November. In total 46,038 deaths till 30th November, 2020.

The model given in Eqs. (1)–(6) are solved using RK4 for Spain. The simulated results of model equations (c.f. Eqs. (1)–(6)) are presented. For the estimation of the values of parameters the statistics terminologies are used. The parameters values are given in Table 1. The comparison of simulated results and real data of infectious and removed cases are provided in graphs.

**Problem 1:**
Fig. 27 represents the comparison of simulated results and real data of infectious and removed cases from 1st August, 2020 to 30th August, 2020. Of Spain. The simulated results are close to the real data in infectious and removed cases as depicted in Fig. 27.

**Problem 2:**
Fig. 28 represents the simulated results and real data of infectious and removed cases from 1st November to 30th November, 2020. The simulated results are close to the real data in infectious and removed cases, relative errors are given in Tables 8 and 9. Figs. 29 represents the simulated results of the model (susceptible, isolated, exposed, infectious, quarantined and removed cases) from 1st November to 30th November, 2020. The results shows the decrease in susceptible population and increase in infected population, after decrease. The exposed cases, removed cases and isolated cases increases. The affected people are recovering from disease. The quarantined population decreases. When the people follow the SOPs, then decrease in infectious and quarantined cases, and increase in isolation. When the government strictly implements the SOPs against the spread of coronavirus then isolated population increases fastly.

Table 8 represents the relative errors from 1st November to 30th November, 2020 of infectious cases. We see that the relative errors are less than 1 for all days. Table 9 represents relative errors from 1st November to 30th November, 2020 of removed cases. We see that the relative errors are less than 1 for all days, which verify the correctness of model formulation. The developed mathematical model can be helpful to measure the coronavirus situations.

**Problem 3: Prediction for Next 6 Months**
The prediction of COVID-19 using the mathematical model is presented. Fig. 30 represents the prediction of COVID-19 for 180 days. The simulated results by developed model of COVID-19 (1st December, 2020 to 30th May, 2021) are presented. The results show that the number of infected cases are increasing almost 34%, after a small decrease. As the infected increases, the suspected decreases 58%, which is clearly depicted in Fig. 30. Furthermore, Fig. 30 shows that the number of removed cases are increasing. According to the results, the infected population due to COVID-19 will increase. The government has to implement strict strategies such as smart lock-down, reduction of timings in shops etc to control the disease. If people follow SOPs then coronavirus will be controlled otherwise its not possible.

In this proposed model, we introduce $a_e$ (transmission rate of the exposed to susceptible), $a_i$ (transmission rate of the infectious to susceptible), and $\delta$ (protection rate). The value of these parameters in the current situation is 0.0261309, 0.2619047, and 0.0018373 respectively. If the values of $a_e$, $a_i$, and $\delta$ are less than 0.007, 0.150, and greater than 0.009 respectively then the minimum number of new cases will report (Safe Zone). If the values of $a_e$, $a_i$, and $\delta$ are greater than 0.15, 0.90, and less than 0.0005 respectively then the maximum number of new cases will report (Danger Zone).

**Conclusion**

In this study, developed model was solved using numerical methods. The model contains system of ODEs which incorporate the human population (susceptible, exposed, infectious, quarantined, removed (recover and death) and also, we include isolated cases (due to govt action)). Runge Kutta of order four (RK4) was applied to acquire the numerical solution. Four test problems were considered with publicly available data, simulated data is then compared with the real data in order to validate proposed model. It is found that numerically obtained results using proposed model are very close to the real data. It is also observed that for some conditions, irregularities in the real data was observed which was hard to cover by numerical methods. However, these variations are accommodated in the proposed model with the help of parametric values. Proposed model is validated on the basis of test problems and predict the future condition of Pakistan, Italy, Japan and Spain. Simulation results proposed that, for Pakistan, if the values of $a_e$, $a_i$, and $\delta$ are less than 0.001, 0.1, and greater than 0.010 respectively then the minimum number of new cases will report (Safe Zone). If the values of $a_e$, $a_i$, and $\delta$ are greater than 0.01, 5.0, and less than 0.0001 respectively then the maximum number of new cases will report (Danger Zone). The studies also proposes statistics for Italy that, if the values of $a_e$, $a_i$, and $\delta$ are less than 0.0001, 0.010, and greater than 0.020 respectively then the minimum number of new cases will report (Safe Zone). If the values of $a_e$, $a_i$, and $\delta$ are greater than 0.100, 1.00, and less than 0.0010 respectively then the maximum number of new cases will report (Danger Zone). The studies also presents statistics for Japan that, if the values of $a_e$, $a_i$, and $\delta$ are less than 0.1, 0.01, and greater than 0.0030 respectively then the minimum number of new cases will report (Safe Zone). If the values of $a_e$, $a_i$, and $\delta$ are greater than 0.5, 1.5, and less than 0.0005 respectively then the maximum number of new cases will report (Danger Zone). Finally for Spain it is proposed that, if the values of $a_e$, $a_i$, and $\delta$ are less than 0.007, 0.150, and greater than 0.009 respectively then the minimum number of new cases will report (Safe Zone). If the values of $a_e$, $a_i$, and $\delta$ are greater than 0.15, 0.90, and less than 0.0005 respectively then the maximum number of new cases will report (Danger Zone). Furthermore, parametric study was performed. The study recommends to take this epidemic decease as a serious decease. Effects of this decease can be minimized by following all the SOP’s governed by WHO and local governments of the region. Also to minimize all the unnecessary activities along with proper hygiene. It is also concluded that the developed model can be used to analyze the transmission of COVID-19 in other regions.

**Future recommendations**

This comprehensive study for four different countries is analysed for key parameters such as transmission coefficient of exposed cases to susceptible cases, transmission coefficient of infectious cases to susceptible, and govt actions to restrict the spread of pandemic. It is obvious that the transmission coefficients should have as less values as possible and govt actions should have large values to control the disease. The range of safe and danger zones are also highlighted in this paper.

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**CRediT authorship contribution statement**

Shumaila Javeed: Conceptualization, Formal analysis, Methodology,
Investigation, Supervision, Project administration, Writing - original draft. Subtain Anjum: Visualization, Data curation, Formal analysis, Investigation, Writing - original draft. Khurram Saleem Aliingeer: Writing - original draft, Writing-review & editing, Validation, Software, Investigation, Writing - review & editing. M. Atif: Supervision, Project administration, Resources, Funding acquisition. Mansoor Shaukat Khan: Visualization, Data curation, Writing - original draft. W. Aslam Farooq: Visualization, Data curation, Software. Atif Hanif: Visualization, Data curation, Software, Investigation. Hijaz Ahmad: Validation, Supervision, Project administration, Funding acquisition. Shao-Wen Yao: Supervision, Project administration, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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