Index theory and dynamical symmetry enhancement of M-horizons

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ABSTRACT: We show that near-horizon geometries of 11-dimensional supergravity preserve an even number of supersymmetries. The proof follows from Lichnerowicz type theorems for two horizon Dirac operators, the field equations and Bianchi identities, and the vanishing of the index of a Dirac operator on the 9-dimensional horizon sections. As a consequence of this, we also prove that all M-horizons with non-vanishing fluxes admit a $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of symmetries.
1 Introduction

It is well known that the near horizon geometries of supersymmetric black holes and branes exhibit supersymmetry enhancement. There are many examples of this, which include the RN black hole as well as the D3-, M2- and M5-branes, see eg [1]. In all these cases, while the black hole and brane configuration preserve half the supersymmetry, the near horizon geometries are maximally supersymmetric. If supersymmetry enhancement near the horizons is universal\(^1\), all near horizon geometries must preserve at least two supersymmetries.

\(^1\)Throughout this paper we consider supergravity theories without higher curvature corrections.
The near horizon geometries as expressed in the Gaussian null coordinates of \([2, 3]\) exhibit two continuous symmetries. For example, the M-horizon fields in \((2.1)\) are invariant under the continuous symmetries generated by the vector fields \(\partial_u\) and \(u\partial_u - r\partial_r\). However in all known examples, the near horizon geometries exhibit additional symmetries, and the two above symmetries enhance at least to \(\mathfrak{sl}(2, \mathbb{R})\). The presence of these additional symmetries cannot be explained from the kinematics of the system. Therefore, it is a consequence of the dynamics and arises after implementing the field equations.

To our knowledge despite the overwhelming evidence there is for symmetry enhancement for near horizon geometries, there is no general proof which demonstrates this. For example, it is not known whether the near horizon geometries which preserve one supersymmetry automatically preserve two or more. The same applies for the additional bosonic symmetries, and in particular \(\mathfrak{sl}(2, \mathbb{R})\), of near horizon geometries.

Symmetry enhancement is instrumental in both understanding the properties of black holes as well as in AdS/CFT \([6]\). In the context of black holes, the presence of additional supersymmetries near the horizon would imply the existence of additional isometries as well as the presence of fundamental forms associated with a more refined \(G\)-structure than that associated with a single Killing spinor. This typically leads to more geometric restrictions on the horizon sections which in turn may lead to the classification of the horizons which preserve sufficiently large number of supersymmetries. For some historical references on uniqueness theorems of black holes and some new results see \([7]-[21]\). In the context of AdS/CFT, the enhancement of the bosonic symmetries to \(\mathfrak{sl}(2, \mathbb{R})\) is the minimal required to assert that the dual field theory is conformal.

In this paper, we shall demonstrate that M-horizons with smooth fields\(^2\) preserve an even number of supersymmetries. The geometry of M-horizons that preserve one supersymmetry have been investigated in \([4, 5]\). Here we shall show that they admit a second supersymmetry. Our proof\(^3\) is topological in nature, and utilizes in an essential way the field equations of 11-dimensional supergravity and the properties of two horizon Dirac operators \(D^{(\pm)}\). The horizon Dirac operators are defined on the horizon sections, which are compact 9-dimensional manifolds without boundary, and are constructed from the supercovariant connection of 11-dimensional supergravity after integrating out the lightcone directions. As a result, they exhibit couplings associated to the 4-form field strength of 11-dimensional supergravity, and arise naturally from the investigation of the Killing spinor equations (KSEs) for near horizon geometries. Our proof that the number of supersymmetries of near horizon geometries is even proceeds with establishing of Lichnerowicz type theorems for both horizon Dirac operators. These theorems relate the number of Killing spinors to the zero modes of the horizon Dirac operators and they are valid subject to the field equations and Bianchi identities of 11-dimensional supergravity. The number of supersymmetries, \(N > 0\), of a near horizon geometry is given by \(N = \dim \text{Ker } D^{(+)} + \dim \text{Ker } D^{(-)}\). The proof continues \(^2\)The near horizon geometry of NS5-branes preserves the same number of supersymmetries as the NS5-brane but the dilaton is not constant at the lightcone.

\(^3\)We do not assume the bilinear matching condition, ie the identification of the stationary Killing vector field of the black hole with the vector constructed as a Killing spinor bilinear, which has extensively been used in most of the literature on near horizon geometries. However see \([22]\).
by utilizing the vanishing of the index of Dirac operators on odd dimensional manifolds which can be used to demonstrate that \( \dim \text{Ker} \mathcal{D}^{(+)} = \dim \text{Ker} \mathcal{D}^{(-)} \). This in turn implies that the number of supersymmetries preserved by near horizon geometries is even.

Our theorem implies that the near horizon geometries of 11-dimensional supergravity preserve at least two supersymmetries. We shall demonstrate that a consequence of this is that all near horizon geometries of 11-dimensional supergravity with non-trivial fluxes admit a \( \mathfrak{sl}(2, \mathbb{R}) \) subgroup of isometries. The orbits of the \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry on the near horizon spacetime are either 2- or 3-dimensional. We find that if the orbits are only 2-dimensional, then the near horizon geometry is static and it is a warped product of \( AdS_2 \) with the near horizon section \( S \). Static M-horizons have been investigated before in [4] and are related \( AdS_2 \) backgrounds in M-theory initiated in [23].

To prove our results, we have used details of the supersymmetry transformations, field equations and Bianchi identities of 11-dimensional supergravity. However, the methodology used can be applied to all supergravity theories. Therefore, it is likely that all near horizon geometries of odd-dimensional supergravity theories preserve at least two supersymmetries and their symmetry algebra includes \( \mathfrak{sl}(2, \mathbb{R}) \). This assertion is supported by a similar result demonstrated for the horizons of 5-dimensional minimal gauged supergravity in [24].

This paper has been organized as follows. In section 2, we describe the field content of M-horizons and express the Bianchi identities and field equations of the theory on the horizon sections. In section 3, we integrate the KSEs along the lightcone directions and establish the independent KSEs on the horizon sections. In section 4, we prove the Lichnerowicz type theorems for two horizon Dirac operators and explore the index theorem for the Dirac operator to prove that the number of supersymmetries is even. In section 5, we investigate further the Killing spinors of M-horizons. In section 6, we explore the consequences for the geometry and topology of M-horizons to admit at least two supersymmetries and demonstrate that all M-horizons admit an \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry subalgebra. In section 7, we give our conclusions. In appendices A and B, we give more details about the proof of the Lichnerowicz type theorems we use, and present an alternative proof of one of the Lichnerowicz type theorems using the maximum principle, respectively.

2 Near Horizon Geometry

2.1 Near horizon fields

Adapting Gaussian null coordinates\([2, 3]\), the near horizon metric and 4-form field strength\(^4\) of 11-dimensional supergravity can be written \([4, 5]\) as

\[
\begin{align*}
\text{ds}^2 &= 2e^+e^- + \delta_{ij}e^i e^j = 2du(dr + rh - \frac{1}{2} r^2 \Delta du) + \text{ds}^2(S), \\
F &= e^+ \wedge e^- \wedge Y + re^+ \wedge d_h Y + X,
\end{align*}
\]

where we have introduced the frame

\[
\begin{align*}
e^+ &= du, \quad e^- = dr + rh - \frac{1}{2} r^2 \Delta du, \quad e^i = e^i J dy^J; \quad g_{IJ} = \delta_{ij} e^i e^j J,
\end{align*}
\]

\(^4\)Let \( \omega \) be a k-form, then \( d_h \omega = d \omega - h \wedge \omega. \)
and

\[ ds^2(S) = \delta_{ij} e^i e^j , \]  

(2.3)

is the metric of the horizon section \( S \) given by \( r = u = 0 \). \( S \) is taken to be compact, connected and without boundary. The dependence on the coordinates \( r, u \) is given explicitly. In particular, \( h = h_i e^i \), \( \Delta \), \( e^i \), \( Y \) and \( X \) depend only on the coordinates \( y \) of \( S \). We choose the frame indices \( i = 1, 2, 3, 4, 6, 7, 8, 9, \# \) and we follow the conventions of [25]. Observe that the Killing vector field \( \partial_u \) is non-space-like everywhere as \( \Delta \geq 0 \), and becomes null at \( r = 0 \). There is no loss of generality in taking \( \Delta \geq 0 \) as this is implied by the KSEs.

Regularity of the horizon requires that \( \Delta, h, Y \) and \( X \) are globally defined and smooth 0-, 1-, 2- and 4-forms on \( S \), respectively. This is our smoothness assumption and it is required to establish our result.

### 2.2 Bianchi identities and field equations

The Bianchi identities and field equations of 11-dimensional supergravity [26] can be decomposed along the lightcone directions and those of the horizon section \( S \). For the Bianchi identity of \( F \), \( dF = 0 \), such a decomposition yields

\[ dX = 0 , \]  

(2.4)

ie \( X \) is a closed form on \( S \).

Similarly, the field equation of the 3-form gauge potential is

\[ d \ast_{11} F - \frac{1}{2} F \wedge F = 0 , \]  

(2.5)

where \( \ast_{11} \) is the Hodge star operation of 11-dimensional spacetime, which can be decomposed as

\[ - \ast_9 d_h Y - h \wedge \ast_9 X + d \ast_9 X = Y \wedge X , \]  

(2.6)

and

\[ - d \ast_9 Y = \frac{1}{2} X \wedge X , \]  

(2.7)

where \( \ast_9 \) is the Hodge star operation on \( S \). The spacetime volume form is chosen as \( \epsilon_{11} = e^+ \wedge e^- \wedge \epsilon_S \), where \( \epsilon_S \) is the volume form of \( S \). Equivalently, in components, one has

\[ \nabla^i X_{i\ell_1 \ell_2 \ell_3} + 3 \nabla_{[\ell_1} Y_{\ell_2 \ell_3]} = 3h_{[\ell_1 Y_{\ell_2 \ell_3]} + h^i X_{i\ell_1 \ell_2 \ell_3} - \frac{1}{48} \epsilon_{\ell_1 \ell_2 \ell_3}^{q_1 q_2 q_3 q_4 q_5 q_6} Y_{q_1 q_2} X_{q_3 q_4 q_5 q_6} \]  

(2.8)

and

\[ \nabla^j Y_{j\ell} - \frac{1}{1152} \epsilon_{q_1 q_2 q_3 q_4 q_5 q_6 q_7 q_8} X_{q_1 q_2 q_3 q_4} X_{q_5 q_6 q_7 q_8} = 0 , \]  

(2.9)

where \( \nabla \) is the Levi-Civita connection of the metric \( ds^2(S) \) on the near horizon section \( S \).
The Einstein equation is
\[ R_{MN} = \frac{1}{12} F_{ML_1L_2L_3} F_{NL_1L_2L_3} - \frac{1}{144} g_{MN} F_{L_1L_2L_3L_4} F^{L_1L_2L_3L_4} \] (2.10)
This decomposes into a number of components. In particular along \( S \), one finds
\[ \tilde{R}_{ij} + \tilde{\nabla}_i h_j - \frac{1}{2} h_i h_j = -\frac{1}{2} Y_{\ell_1 \ell_2 \ell_3} X^\ell_1 \ell_2 \ell_3 + \delta_{ij} \left( \frac{1}{12} Y_{\ell_1 \ell_2 \ell_3} X^\ell_1 \ell_2 \ell_3 - \frac{1}{144} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} \right) , \] (2.11)
where \( \tilde{R}_{ij} \) is the Ricci tensor of \( S \). The \( + - \) component of the Einstein equation gives
\[ \tilde{\nabla}^i h_i = 2\Delta + h^2 - \frac{1}{3} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} - \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} , \] (2.12)
Similarly, the \( ++ \) and \( + i \) components of the Einstein equation can be expressed as
\[ \frac{1}{2} \tilde{\nabla}^i \tilde{\nabla}_i \Delta - \frac{3}{2} \Delta \tilde{\nabla}_i h_i + \Delta h^2 + \frac{1}{4} d h_{ij} d h_{ij} = \frac{1}{12} (d_h Y)_{\ell_1 \ell_2 \ell_3} (d_h Y)^{\ell_1 \ell_2 \ell_3} , \] (2.13)
and
\[ - \frac{1}{2} \tilde{\nabla}^i d h_{ij} + h^2 (d h)_{ij} - \tilde{\nabla}_i \Delta - \Delta h_i = \frac{1}{12} X_{\ell_1 \ell_2 \ell_3 \ell_4} (d_h Y)_{\ell_1 \ell_2 \ell_3} - \frac{1}{4} (d_h Y)_{\ell_1 \ell_2} Y_{\ell_1 \ell_2} , \] (2.14)
respectively. Although we have included the \( ++ \) and the \( + i \) components of the Einstein equations for completeness, it is straightforward to show that both (2.13) and (2.14) hold as a consequence of (2.4), the 3-form field equations (2.6) and (2.7) and the components of the Einstein equation in (2.11) and (2.12). This does not make use of supersymmetry, or any assumptions on the topology of \( S \). Hence, the conditions on \( ds^2(\mathcal{S}) \), \( \Delta \), \( h \), \( Y \) and \( X \) simplify to (2.4), (2.6), (2.7), (2.11) and (2.12).

3 Killing spinor equations
The KSE of 11-dimensional supergravity [26] is
\[ \nabla_M \epsilon + \left( -\frac{1}{288} \Gamma_M^{L_1L_2L_3L_4} F_{L_1L_2L_3L_4} + \frac{1}{36} F_{ML_1L_2L_3} \Gamma^{L_1L_2L_3} \right) \epsilon = 0 , \] (3.1)
where \( \nabla \) is the spacetime Levi-Civita connection. As for the field equations above, the KSE can be decomposed along the light-cone and \( S \) directions. This already has been done in [4] and [5]. Here we shall repeat some of the steps as our analysis is different from that in the above references. In particular, we shall not assume that there is bi-linear matching, ie that the Killing vector field \( \partial_a \) is identified with the vector constructed as bi-linear of the Killing spinor of backgrounds preserving one supersymmetry.
3.1 Integration of KSEs along the lightcone

To solve the KSEs along the light-cone directions, we decompose the Killing spinor as

\[ \epsilon = \epsilon_+ + \epsilon_-, \quad \Gamma_\pm \epsilon_\pm = 0. \]  

(3.2)

Then after some computation, see [4, 5], we find that

\[ \epsilon_+ = \eta_+, \quad \epsilon_- = \eta_- + r \Gamma_- \Theta_\pm \eta_+, \]  

(3.3)

and

\[ \eta_+ = \phi_+ + u \Gamma_+ \Theta_\phi_-, \quad \eta_- = \phi_-, \]  

(3.4)

where

\[ \Theta_\pm = \left( \frac{1}{4} h_i \Gamma^i + \frac{1}{288} X_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2} \ell_3 \pm \frac{1}{12} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right), \]  

(3.5)

and \( \phi_\pm = \phi_\pm (y) \) do not depend on \( r \) or \( u \).

Furthermore, the + and − components of the KSE impose the following algebraic conditions on the Killing spinors

\[ \left( \frac{1}{2} \Delta - \frac{1}{8} d_{h_{ij}} \Gamma^{ij} + \frac{1}{72} d_{h} X_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2} \ell_3 + \frac{1}{12} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right) \eta_+ = 0, \]  

(3.6)

\[ \left( \frac{1}{4} \Delta h_i \Gamma^i - \frac{1}{4} \partial_i \Delta \Gamma^i + \left( - \frac{1}{8} d_{h_{ij}} \Gamma^{ij} - \frac{1}{24} d_{h} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right) \Theta_+ \right) \eta_+ = 0, \]  

(3.7)

and

\[ \left( - \frac{1}{2} \Delta - \frac{1}{8} d_{h_{ij}} \Gamma^{ij} + \frac{1}{24} d_{h} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right) \phi_- = 0. \]  

(3.8)

Note that conditions (3.6) and (3.8) can be expanded out into terms independent of \( u \) and terms linear in \( u \), yielding four \( u \)-independent conditions. However, it will turn out to be most convenient to write the algebraic conditions in the form of (3.6) and (3.7).

Since we have separated the light-cone directions from the rest, the remaining KSEs have manifest \( Spin(9) \subset Spin(10,1) \) local gauge invariance. The remaining KSEs can be written as

\[ \hat{\nabla}_i \eta_+ + \left( \frac{1}{4} h_i - \frac{1}{8} \Gamma_i \ell_1 \ell_2 \ell_3 \ell_4 X_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{36} X_{\ell_1 \ell_2 \ell_3} \ell_4 \Gamma^{\ell_1 \ell_2} \ell_3 \right) \eta_+ = 0, \]  

(3.9)
\[ \nabla_i \zeta + \left( -\frac{1}{2} h_i + \frac{1}{4} \Gamma_i^\ell h_\ell - \frac{1}{24} X_{i\ell_1 \ell_2 \ell_3} \Gamma_i^\ell_1 \ell_2 \ell_3 + \frac{1}{8} \Gamma_i^\ell_1 \ell_2 Y_{\ell_1 \ell_2} \right) \zeta \\
+ \left( \frac{1}{4} \Delta_i \Gamma_i^\ell \ell_1 \ell_2 \ell_3 \ell_4 h_{\ell_1 \ell_2} - \frac{2}{16} d h_\ell \Gamma_\ell - \frac{1}{48} d h \Gamma_\ell Y_{\ell_1 \ell_2 \ell_3} \Gamma_i^\ell_1 \ell_2 \ell_3 \right) \eta_+ = 0 , \] (3.10)

and

\[ \nabla_i \phi_\pm + \left( \frac{1}{4} h_i - \frac{1}{288} \Gamma_i^\ell_1 \ell_2 \ell_3 \ell_4 X_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{36} X_{i\ell_1 \ell_2 \ell_3} \Gamma_i^\ell_1 \ell_2 \ell_3 \\
- \frac{1}{24} \Delta_i \ell_1 \ell_2 \ell_3 \ell_4 Y_{\ell_1 \ell_2} + \frac{1}{6} Y_{ij} \Gamma_j \right) \phi_\pm = 0 . \] (3.11)

where

\[ \zeta \equiv \Theta_+ \eta_+ . \] (3.12)

Again, we remark that equations (3.9) and (3.10) contain both terms independent of \( u \), and linear in \( u \), but it is most convenient to write these equations in the way stated above.

### 3.2 Independent KSEs on \( S \)

It is important to what follows to establish the independent KSEs on \( S \). In investigating supersymmetric backgrounds, it is customary to first solve all KSEs and then impose the field equations which are not implied as integrability conditions. Here, we shall adopt a different strategy. We shall use all the field equations and Bianchi identities of the theory to find the independent KSEs that one has to impose such that a near horizon geometry is supersymmetric.

First consider equations (3.6), (3.7), (3.9) and (3.10). Equations of this form have already been investigated in [5], in the special case for which \( \phi_\pm = 0 \). However, the form of these equations remains unchanged if one relaxes the condition that \( \phi_\pm \) vanishes. Hence using exactly the same reasoning set out in [5], it follows that (3.6), (3.7) and (3.10) are implied by (3.9) and the bosonic field equations and Bianchi identities.

Next we consider (3.9) in more detail. It will be particularly useful to establish the following result: if \( \phi_\pm \) is a \((u, r)\)-independent) spinor satisfying (3.11), then

\[ \phi_\pm' \equiv \Gamma_+ \Theta_\mp \phi_\pm \] (3.13)

satisfies (3.9) with \( \eta_+ \) replaced with \( \phi_\pm' \). To see this, first evaluate the LHS of (3.9) acting on \( \eta_\prime_+ \), and use (3.11) to eliminate the terms involving \( \nabla_i \phi_\pm \), and then remove the \( \Gamma_+ \) term by left-multiplication with \( \Gamma_- \). Then compare the resulting algebraic condition on \( \phi_- \) with the following expression

\[ \frac{1}{2} \Gamma^{ij} (\nabla_j \nabla_i - \nabla_i \nabla_j) \phi_- = \frac{1}{4} \tilde{R}_{ij} \Gamma^j \phi_- , \] (3.14)

where the LHS of the above is evaluated using (3.11), and the entire expression is then simplified using the bosonic field equations and Bianchi identities. After a rather involved
computation, one finds that the resulting algebraic condition on \( \phi_- \) obtained from (3.14) is identical to the algebraic condition on \( \phi_- \) obtained by substituting \( \phi'_+ \) into (3.9) as described above. We remark that the condition (3.8) was not used at any stage of the computation. Note that this result implies that the part of (3.9) which is linear in \( u \) is satisfied automatically as a consequence of (3.11) and the field equations and Bianchi identities.

Next consider (3.8). It will again be useful to return to (3.14), with the LHS evaluated using (3.11). On contracting the resulting condition with \( \Gamma^i \) and making extensive use of the field equations and Bianchi identities to simplify the expression, one obtains a condition equivalent to (3.8).

To summarize, we have demonstrated that on making use of the field equations and Bianchi identities the independent KSEs are

\[
\nabla_i^{(+)} \phi_+ \equiv \tilde{\nabla}_i^{(+)} \phi_+ + \Psi_i^{(+)} \phi_+ = 0 ,
\]

and

\[
\nabla_i^{(-)} \phi_- \equiv \tilde{\nabla}_i^{(-)} \phi_- + \Psi_i^{(-)} \phi_- = 0 ,
\]

where

\[
\Psi_i^{(\pm)} = \frac{1}{4} h_i - \frac{1}{288} \Gamma_i \ell_1 \ell_2 \ell_3 \ell_4 X_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{96} X_{i \ell_1 \ell_2 \ell_3} \Gamma_{\ell_1 \ell_2 \ell_3} \\
\pm \frac{1}{24} \Gamma_{i \ell_1 \ell_2} Y_{\ell_1 \ell_2} \mp \frac{1}{6} Y_{ij} \Gamma^j .
\]

Moreover on imposing the field equations and Bianchi identities, if \( \phi_- \) satisfies (3.16), then \( \phi'_+ \) given by (3.13) satisfies (3.15). This analysis has been entirely local, and has not made use of the compactness of \( S \).

4 Horizon Dirac Equations and a Lichnerowicz Theorem

4.1 Horizon Dirac Equations

Given the gravitino KSE in a supergravity theory which is a parallel transport equation for the supercovariant connection, \( \mathcal{D}_A \epsilon = 0 \), one can construct a “supergravity Dirac equation” as \( \Gamma^A \mathcal{D}_A \epsilon = 0 \). This can be adapted to the near horizon geometries. In particular, for each of the “horizon gravitino KSEs” on \( S \)

\[
\nabla_i^{(\pm)} \phi_{\pm} \equiv \nabla_i^{(\pm)} \phi_{\pm} + \Psi_i^{(\pm)} \phi_{\pm} = 0 ,
\]

given in (3.15) and (3.16), respectively, one can associate a “horizon Dirac equation” as

\[
\mathcal{D}^{(\pm)} \phi_{\pm} = \Gamma^i \nabla_i^{(\pm)} \phi_{\pm} + \Psi^{(\pm)} \phi_{\pm} = 0 ,
\]

where

\[
\Psi^{(\pm)} = \Gamma^i \Psi_i^{(\pm)} = \mp \frac{1}{4} h_i \Gamma^\ell + \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} .
\]

These Dirac equations, in addition to the Levi-Civita connection, also depend on the fluxes of the supergravity theory restricted on the horizon section \( S \).
4.2 A Lichnerowicz theorem

The horizon Dirac equations (4.2) can be used to give a new characterization of the Killing spinors. Clearly, the gravitino KSEs are more restrictive. Any solution of the gravitino KSEs (3.15), (3.16) is also a solution of a corresponding Dirac equation. In what follows, we shall explore the converse. In particular, we shall show that the zero modes of the horizon Dirac equation are parallel with respect to the horizon supercovariant derivatives (4.1). Instrumental in the proof are the field equations and Bianchi identities of 11-dimensional supergravity as reduced on the horizon section $S$.

Before proceeding with the analysis of the supergravity case, it is useful to recall the Lichnerowicz theorem. On any spin compact manifold $N$, one can show the equality
\[
\int_N \langle \Gamma^i \nabla_i \epsilon, \Gamma^j \nabla_j \epsilon \rangle = \int_N \langle \nabla_i \epsilon, \nabla^i \epsilon \rangle + \int_N \frac{R}{4} \langle \epsilon, \epsilon \rangle ,
\]
where $\nabla$ is the Levi-Civita connection, $\langle \cdot, \cdot \rangle$ is the Dirac inner product and $R$ is the Ricci scalar. Clearly if $R > 0$, the Dirac operator has no zero modes. Moreover, if $R = 0$, then the zero modes of the Dirac operator are parallel with respect to the Levi-Civita connection.

This theorem can be generalized for M-horizons with the standard Dirac operator replaced with the horizon Dirac operators in (4.2) and the Levi-Civita covariant derivative replaced with the horizon supercovariant derivatives (4.1). A version of this theorem has already been proven in [5] but here we shall consider both horizon Dirac operators $D^{(\pm)}$.

For this, let $\phi$ be a Majorana $Spin(9)$ spinor and consider
\[
I^{(\pm)} = \int_S \langle \nabla_i \phi_\pm + \Psi^{(\pm)}_i \phi_\pm, \nabla^i \phi_\pm + \Psi^{(\pm)i} \phi_\pm \rangle - \int_S \| \nabla_i \nabla^i \phi_\pm + \Psi^{(\pm)} \phi_\pm \|^2 ,
\]
where $\langle \cdot, \cdot \rangle$ is the Dirac inner product of $Spin(9)$ which can be identified with the standard Hermitian inner product on $\Lambda^4(\mathbb{C}^4)$ restricted on the real subspace of Majorana spinors and $\| \cdot \|$ is the associated norm. Therefore, $\langle \cdot, \cdot \rangle$ is a real and positive definite. The $Spin(9)$ gamma matrices are Hermitian with respect to $\langle \cdot, \cdot \rangle$.

Clearly, if the integrals $I^{(\pm)}$ vanish, all zero modes of the horizon Dirac operators $D^{\pm}$ are parallel with respect to the horizon supercovariant derivatives $\nabla^{\pm}$ and so Killing. To show that $I^{(\pm)}$ vanish, assume that $S$ is compact and without boundary, and that $\phi$ is globally well-defined and smooth on $S$. Then, on integrating by parts, one can rewrite
\[
I^{(\pm)} = \int_S \langle \phi_\pm, (\Psi^{(\pm)}_i) \nabla^i \phi_\pm + \Psi^{(\pm)i} \phi_\pm + (\Psi^{(\pm)}_i) \nabla^i \phi_\pm + (\Psi^{(\pm)i}) \nabla^i \phi_\pm \rangle + \Gamma^{ij} \nabla_i \nabla_j \phi_\pm + (\nabla_i \Psi^{(\pm)}_j - (\nabla^i \Psi^{(\pm)}_j)) \phi_\pm + (\nabla_i \Psi^{(\pm)} - (\Psi^{(\pm)} \nabla^i \phi_\pm) .
\]
Next, evaluating the RHS of the above equation using the Bianchi identity of $X$ (2.4), the field equation of the 4-form field strength (2.6) and (2.7), the Einstein equations along $S$.

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\footnote{In fact, it can be identified as the restriction of the $Spin(10, 1)$ invariant inner product on the Majorana representation as restricted on $Spin(9)$ spinor representations under the decomposition $\epsilon = \epsilon_+ + \epsilon_-$ in (3.2).}
(2.11), one finds that
\[
\mathcal{I}^{(\pm)} = \int_S \langle \phi_{\pm}, \left( \mp \frac{1}{2} h_i \Gamma^i - \frac{1}{144} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{6} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} \right) \left( \nabla_{\ell_1} \phi_{\pm} + \Psi^{(\pm)} \phi_{\pm} \right) \rangle + \int \frac{1}{2} \nabla^i h_i (1 \pm 1) \langle \phi_{\pm}, \phi_{\pm} \rangle .
\]

(4.7)

Further details of this computation are given in Appendix A.

On comparing (4.7) with (4.5), one immediately finds that if \( \phi_- \) is a solution of the \( \mathcal{D}^{(-)} \) horizon Dirac equation, then \( \phi_- \) is a solution of the \( \nabla^{(-)} \) horizon gravitino KSE (3.16).

Also, if \( \phi_+ \) is a solution of the \( \mathcal{D}^{(+)} \) horizon Dirac equation, and \( \langle \phi_+, \phi_+ \rangle = \text{const} \) then \( \phi_+ \) is a solution of the \( \nabla^{(+)} \) horizon graviton KSE (3.15). Furthermore, it is straightforward to prove that if \( \phi_+ \) satisfies the \( \mathcal{D}^{(+)} \phi_+ = 0 \), then one obtains the condition
\[
\nabla^i \nabla_i \| \phi_+ \|^2 - h^i \nabla_i \| \phi_+ \|^2 = 2 \langle \nabla^{(+)} \phi_+, \nabla^{(+)} \phi_+ \rangle .
\]

(4.8)

Details of the derivation of (4.8) are given in Appendix B. Upon using the maximum principle, this condition implies that \( \| \phi_+ \| = \text{const} \), and \( \nabla^{(+)} \phi_+ = 0 \). Note that this provides an alternative proof of the Lichnerowicz type of theorem for the \( \mathcal{D}^{(+)} \) operator.

To conclude, the results of this section can be summarized as
\[
\nabla^{(\pm)} \phi_{\pm} = 0 \iff \mathcal{D}^{(\pm)} \phi_{\pm} = 0 .
\]

(4.9)

Hence, the Killing spinors of the horizon section \( S \) can be identified with the zero modes of horizon Dirac operators. In turn, the Killing spinors of the near horizon spacetime can be expressed in terms of the zero modes of the horizon Dirac operators \( \mathcal{D}^{(\pm)} \).

### 4.3 Index theorem and supersymmetries of M-horizons

To proceed note that we have decomposed the spin bundle \( S \) of 11-dimensional supergravity as \( S = S_+ \oplus S_- \) on \( S \) using the projections \( \Gamma_{\pm} \) as in (3.2). Next observe that \( \mathcal{D}^{(+)} : \Gamma(S_+) \to \Gamma(S_+) \) and its adjoint \( (\mathcal{D}^{(+)})^\dagger : \Gamma(S_+) \to \Gamma(S_+) \), where \( \Gamma(S_+) \) are the smooth sections of \( S_+ \). The operator \( \mathcal{D}^{(+)} \) has the same principal symbol as the Dirac operator. Moreover , \( \mathcal{D}^{(+)} \) is defined on \( S \) which is an odd-dimensional manifold. It follows from Proposition 1 of [27] that the index of \( \mathcal{D}^{(+)} \) vanishes. As a result, we have that
\[
\dim \ker \mathcal{D}^{(+)} = \dim \ker (\mathcal{D}^{(+)})^\dagger .
\]

(4.10)

Observe that
\[
(\mathcal{D}^{(+)})^\dagger = - \Gamma^i \nabla_i - \frac{1}{4} h_i \Gamma^i + \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} .
\]

(4.11)

Next define \( \phi'_+ = \Gamma_+ \phi_- \) and observe that
\[
(\mathcal{D}^{(+)})^\dagger \phi'_+ = \Gamma_+ \mathcal{D}^{(-)} \phi_- .
\]

(4.12)
So, we conclude that \( \dim \ker(D^{(+)})^\dagger = \dim \ker D^{(-)} \). This together with the result from the index theorem (4.10) gives

\[
\dim \ker D^{(+)\dagger} = \dim \ker D^{(-)}.
\]  

(4.13)

The number of supersymmetries of a near horizon geometry is the number of parallel spinors of \( \nabla^{(\pm)} \) and so from the Lichnerowicz theorems and the above formula, one has

\[
N = \dim \ker D^{(+)\dagger} + \dim \ker D^{(-)} = 2 \dim \ker D^{(-)}.
\]  

(4.14)

This proves that the number of supersymmetries preserved by near M-horizon geometries is even.

5 Construction of \( \phi_+ \) from \( \phi_- \) Killing spinors

In section (3.2), we have demonstrated that if \( \phi_- \) is \( \tilde{\nabla}^{(-)} \)-parallel, then

\[
\phi_+ = \Gamma_+ \Theta_- \phi_-
\]  

(5.1)

satisfies \( \tilde{\nabla}^{(+)} \phi_+ = 0 \). Clearly this can be used to construct the \( \phi_+ \) solutions to the KSEs from the \( \phi_- \) solutions.

Since \( \phi_+ \) given in (5.1) is \( \tilde{\nabla}^{(+)} \)-parallel either is everywhere nonzero or vanishes identically. Consider the latter case that \( \phi_+ \) in (5.1) vanishes for \( \phi_- \neq 0 \). In this case, one must have

\[
\Theta_- \phi_- = 0 .
\]  

(5.2)

To proceed, note that (3.8) together with (5.2) imply that

\[
\langle \phi_-, \left( -\frac{1}{2} \Delta - \frac{1}{8} dh_{ij} \Gamma^{ij} + \frac{1}{24} dh Y_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3} \right) \phi_- \rangle = 0 .
\]  

(5.3)

This condition implies

\[
\Delta \langle \phi_-, \phi_- \rangle = 0
\]  

(5.4)

and hence

\[
\Delta = 0 ,
\]  

(5.5)

as \( \phi_- \) is no-where vanishing. Next, using (3.16), one finds

\[
\tilde{\nabla}_i \langle \phi_-, \phi_- \rangle = \frac{1}{2} h_i \langle \phi_-, \phi_- \rangle + \langle \phi_-, \left( \frac{1}{144} \Gamma_{\ell_1 \ell_2 \ell_3} \xi^{\ell_1 \ell_2 \ell_3} X_{\ell_1 \ell_2 \ell_3} \xi_{\ell_4} - \frac{1}{3} Y_{ij} \Gamma^j \right) \phi_- \rangle .
\]  

(5.6)

This expression can be further simplified, using \( \Theta_- \phi_- = 0 \), to eliminate the \( Y \) and \( X \) terms and to give

\[
\tilde{\nabla}_i \langle \phi_-, \phi_- \rangle = -h_i \langle \phi_-, \phi_- \rangle .
\]  

(5.7)
As $\phi_-$ is no-where zero, this implies that

$$dh = 0$$ \hspace{1cm} (5.8)

and (2.13) then implies that

$$d_h Y = 0$$ \hspace{1cm} (5.9)

as well. Returning to (5.7), on taking the divergence, and using (2.12) to eliminate the $\tilde{\nabla}i h_i$ term, one obtains

$$\tilde{\nabla}^i \tilde{\nabla}_i \langle \phi_-, \phi_- \rangle = \left( \frac{1}{3} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} + \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} \right) \langle \phi_-, \phi_- \rangle.$$ \hspace{1cm} (5.10)

On integrating both sides of this expression over $S$, the contribution from the LHS vanishes, so

$$\int_S \left( \frac{1}{3} Y_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} + \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} \right) \langle \phi_-, \phi_- \rangle = 0.$$ \hspace{1cm} (5.11)

Again, as $\phi_-$ is no-where vanishing, this implies that

$$Y = 0, X = 0.$$ \hspace{1cm} (5.12)

Next, (2.12) implies that

$$\tilde{\nabla}^i h_i = h^2$$ \hspace{1cm} (5.13)

and again integrating this expression over $S$, the contribution from the LHS vanishes, leading to

$$h = 0.$$ \hspace{1cm} (5.14)

Hence, if $\phi_+ \neq 0$, then $\Theta_- \phi_- = 0$ implies that $\Delta = 0, h = 0, Y = 0, X = 0$. In such a case, the near horizon geometry is locally isometric to $\mathbb{R}^{1,1} \times S$, where $S$ is a compact 9-dimensional Ricci flat manifold. Such manifolds are classified and $S$ is locally a product $S^1 \times X^8$, where in turn $X^8$ is a product of holonomy $Spin(7)$, $Sp(2)$, $G_2$, $SU(k)$, $k \leq 4$, and $\{1\}$ manifolds.

Therefore there are two possibilities. One possibility is that the conditions $\Delta = 0, h = 0, Y = 0, X = 0$ do not hold in which case the $\phi_+$ and $\phi_-$ Killing spinors are related by (5.1). This is a consequence of the index theorem which requires the number of zero modes of $D^{(-)}$ to be equal to those of $D^{(+)}$. The other possibility is whenever $\Delta = 0, h = 0, Y = 0, X = 0$. In such a case for every $\phi_-$ spinor satisfying $\tilde{\nabla}^{(-)} \phi_- = 0$, there is a spinor $\phi_+$ satisfying $\tilde{\nabla}^{(+)} \phi_+ = 0$ given by $\phi_+ = \Gamma_+ \phi_-$. In either case, the number of Killing spinors is even.
The dynamical \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry of M-horizons

6.1 Killing vectors

A priori near horizon geometries admit two Killing vector field generated by \( \partial_u \) and \( u \partial_u - r \partial_r \). However all known examples exhibit a larger symmetry algebra which always includes an \( \mathfrak{sl}(2, \mathbb{R}) \) subalgebra. Here we shall prove that this is a generic property of M-horizons with non-trivial fluxes and a direct consequence of supersymmetry. However, it should be stressed that the \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry is dynamical because it emerges after using the field equations of the theory.

We have shown (3.2) that the most general Killing spinor takes the form

\[
\epsilon = \phi_+ + u \Gamma_+ \Theta_- \phi_+ + \phi_- + r \Gamma_- \Theta_+ \phi_+ + ru \Gamma_- \Theta_+ \Theta_- \phi_- .
\]  

(6.1)

The two Killing spinors of the near horizon geometries can be constructed from the pairs \((\phi_-, 0)\) and \((\phi_-, \phi_+)\) with \(\phi_+ = \Gamma_+ \Theta_+ \phi_-\). Implementing these, we find that

\[
\epsilon_1 = \phi_- + u \phi_+ + ru \Gamma_- \Theta_+ \phi_+ , \quad \epsilon_2 = \phi_+ + r \Gamma_- \Theta_+ \phi_+ .
\]  

(6.2)

It can be easily shown that for any two Killing spinors \(\zeta_1\) and \(\zeta_2\), the 1-form bilinear

\[
K = \langle (\Gamma_+ - \Gamma_-) \epsilon_1, \Gamma_A \epsilon_2 \rangle e^A ,
\]  

(6.3)

is associated with a Killing vector which also preserves that 4-form field strength of 11-dimensional supergravity. In particular for the two Killing spinors (6.2), one can construct three 1-form bi-linears. A substitution of (6.2) into (6.3) reveals

\[
K_1 = \langle (\Gamma_+ - \Gamma_-) \epsilon_1, \Gamma_A \epsilon_2 \rangle e^A = (2r (\Gamma_+ \phi_-, \Theta_+ \phi_+) + r^2 u \Delta \| \phi_+ \|^2) e^+ - 2u \| \phi_+ \|^2 e^- + V_i e^i ,
\]

\[
K_2 = \langle (\Gamma_+ - \Gamma_-) \epsilon_2, \Gamma_A \epsilon_2 \rangle e^A = r^2 \Delta \| \phi_+ \|^2 e^+ - 2 \| \phi_+ \|^2 e^- ,
\]

\[
K_3 = \langle (\Gamma_+ - \Gamma_-) \epsilon_1, \Gamma_A \epsilon_1 \rangle e^A = (2 \| \phi_- \|^2 + 4ru (\Gamma_+ \phi_-, \Theta_+ \phi_+) + r^2 u^2 \Delta \| \phi_+ \|^2) e^+ - 2u^2 \| \phi_+ \|^2 e^- + 2uV_i e^i ,
\]  

(6.4)

where we have set

\[
V_i = \langle \Gamma_+ \phi_-, \Gamma_i \phi_+ \rangle .
\]  

(6.5)

Moreover, we have used the identities

\[
- \Delta \| \phi_+ \|^2 + 4 \| \Theta_+ \phi_+ \|^2 = 0
\]  

(6.6)

which follows from (3.6), and

\[
\langle \phi_+, \Gamma_i \Theta_+ \phi_+ \rangle = 0 ,
\]  

(6.7)

which follows from \(\langle \phi_+, \phi_+ \rangle = \text{const} \) proved in [5] as a consequence of (3.15) and the compactness of \(\mathcal{S}\). By construction \(K_1, K_2\) and \(K_3\) are associated with vector fields which leave both the near horizon metric and the 4-form flux (2.1) invariant.

---

\[\text{The inner product we use to define } K \text{ is the Dirac inner product of } Spin(10,1) \text{ restricted on its Majorana representation.}\]
6.2 The geometry of $S$

6.2.1 $V \neq 0$

The symmetries generated by $K_1, K_2$ and $K_3$ restrict the geometry of $S$. To find the restrictions on $S$, one decomposes the Killing condition of $\mathcal{L}_{K_a} g = 0$ and $\mathcal{L}_{K_a} F = 0$, $a = 1, 2, 3$ conditions along the lightcone and transverse directions. After a somewhat long but straightforward computation, one finds that

$$\tilde{\nabla}_i (V_j) = 0, \quad \tilde{\nabla} V = 0, \quad \tilde{\nabla} \Delta = 0, \quad \tilde{\nabla} Y = 0, \quad \tilde{\nabla} X = 0. \quad (6.8)$$

Therefore, $S$ admits an isometry generated by $V$ which leaves $h, \Delta, Y$ and $X$ invariant.

In addition, one also finds some useful identities which follow from the field equations and KSEs we have stated already. These are

$$-2 \| \phi_+ \|^2 - h_i V^i + 2 \langle \Gamma_+ \phi_-, \Theta_+ \phi_+ \rangle = 0, \quad i_V (dh) + 2d \langle \Gamma_+ \phi_-, \Theta_+ \phi_+ \rangle = 0,$$

$$2 \langle \Gamma_+ \phi_-, \Theta_+ \phi_+ \rangle - \Delta \| \phi_- \|^2 = 0, \quad V+ \| \phi_- \|^2 h + d \| \phi_- \|^2 = 0. \quad (6.9)$$

Notice that the last equality in (6.9) expresses $V$ in terms of $h$. This is significant as it generalizes a relation derived in the context of heterotic horizons in [28], [29] which equates $h$ with $V$. Furthermore observe that one can show that

$$\mathcal{L}_V \| \phi_- \|^2 = 0. \quad (6.10)$$

The geometry of $S$ is further restricted. The existence of a no-where vanishing spinor $\phi_-$ reduces the structure group\(^7\) of $S$ to $Spin(7)$. The existence of a second Killing spinor $\phi_+$ reduces the structure group further. There are various possibilities that can arise depending in which subspace $\phi_+$ lies giving isotropy groups $Spin(7)$, $SU(4)$, $G_2$ and $SU(3)$. There are geometric restrictions on these structures which will be explored elsewhere.

6.2.2 $V = 0$

A special case arises whenever $V = 0$. In this case, the group action generated by $K_1, K_2$ and $K_3$ has only 2-dimensional orbits. A direct substitution of this condition in (6.9) reveals that

$$\Delta \| \phi_- \|^2 = 2 \| \phi_+ \|^2, \quad h = \Delta^{-1} d \Delta. \quad (6.11)$$

Since $dh = 0$ and $h$ exact such horizons are static. After a coordinate transformation $r \rightarrow \Delta r$, the near horizon geometry becomes a warped product of $AdS_2$ with $S$, $AdS_2 \times_w S$. There are further consequences of this. For example the $++$ Einstein equation (2.13) implies that $d_0 Y = 0$. Static M-horizons have been extensively investigated in [4] and they are related to M-theory $AdS_2$ backgrounds which have been initially explored in [23].

\(^7\)The isotropy group of non-trivial orbits of $Spin(9)$ in the 16-dimensional Majorana representation is $Spin(7)$. Note that $Spin(9)/Spin(7) = S^3$. 

- 14 -
6.3 \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry of M-horizons

To show that all M-horizons with non-trivial fluxes admit and \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry, we use the various identities derived in the previous section to write the vector fields associated to the 1-forms \( K_1, K_2 \) and \( K_3 \) (6.4) as

\[
K_1 = -2u \| \phi_+ \|^2 \partial_u + 2r \| \phi_+ \|^2 \partial_r + V_i \partial_i, \\
K_2 = -2 \| \phi_+ \|^2 \partial_u, \\
K_3 = -2u^2 \| \phi_+ \|^2 \partial_u + (2 \| \phi_- \|^2 + 4ru \| \phi_+ \|^2) \partial_r + 2uV^i \partial_i, 
\]

where we have used the same symbol for the 1-forms and the associated vector fields. A direct computation then reveals using (6.10) that

\[
[K_1, K_2] = 2 \| \phi_+ \|^2 K_2, \quad [K_2, K_3] = -4 \| \phi_+ \|^2 K_1, \quad [K_3, K_1] = 2 \| \phi_+ \|^2 K_3. 
\]

Therefore all M-horizons with non-trivial fluxes admit an \( \mathfrak{sl}(2, \mathbb{R}) \) symmetry subalgebra.

Away from the fixed points of \( V \), the orbits of the \( \mathfrak{sl}(2, \mathbb{R}) \) action are 3-dimensional. Therefore if \( V \neq 0 \), the \( \mathfrak{sl}(2, \mathbb{R}) \) action action must have some 3-dimensional orbits. If \( V = 0 \), we have shown that \( \mathfrak{sl}(2, \mathbb{R}) \) has only 2-dimensional orbits isometric to \( \text{AdS}_2 \).

Note also that if the fluxes are trivial, the spacetime is isometric to \( \mathbb{R}^2 \times S \) and \( S \) admits at least one isometry. The symmetry group in this case has a \( \mathfrak{so}(1, 1) \oplus \mathfrak{u}(1) \) subalgebra.

7 Concluding remarks

We have demonstrated that all M-horizons preserve an even number of supersymmetries and as a consequence of this those with non-trivial fluxes admit an \( \mathfrak{sl}(2, \mathbb{R}) \) subalgebra of symmetries. To establish these results, we have shown that the KSEs of the near horizon geometries are implied from two parallel transport equations on the horizon sections which depend on the 4-form fluxes. Then the associated Dirac equations were considered and two Lichnerowicz type theorems were proven which related the parallel spinors on the horizon sections with the zero modes of the associated Dirac operators. Then the vanishing of the Dirac index on the 9-dimensional horizon section led to the conclusion that M-horizons preserve an even number of supersymmetries. The invariance of M-horizons under a \( \mathfrak{sl}(2, \mathbb{R}) \) subalgebra then followed as a consequence of the supersymmetry enhancement.

Instrumental in the proof of the above results were the field equations and Bianchi identities of 11-dimensional supergravity. Both the supersymmetry enhancement from one to at least two supersymmetries as well as the presence of a \( \mathfrak{sl}(2, \mathbb{R}) \) invariance subalgebra of M-horizons are dynamical, and cannot be proven without the use of field equations.

Although our calculation is based on the details of 11-dimensional supergravity, like its field content and field equations, our methodology is general and applies to all supergravities. Therefore, it is likely that our results generalize to all odd-dimensional supergravities leading to the conclusion that all odd-dimensional near horizon geometries preserve at least two supersymmetries and admit a \( \mathfrak{sl}(2, \mathbb{R}) \) invariance subalgebra. This assertion is further strengthened by the results in [24] where similar results were established for the horizons of 5-dimensional minimal gauged supergravity. Our methodology can also be adapted to
investigate the symmetries of brane horizons and AdS backgrounds of odd-dimensional supergravity theories.

Our results can also be applied to even-dimensional supergravity theories. However, there are some differences. Assuming that the required Lichnerowicz type theorems can be established relating the number of Killing spinors to zero modes of Dirac operators, one does not expect that the index of the Dirac operator vanishes on the even-dimensional horizon sections. However, the investigation of heterotic horizons and those of 6-dimensional supergravity in [28], [29] and [30] both find that those with non-trivial fluxes preserve an even number of supersymmetries and have an sl(2, R) invariance subalgebra, but also see [31], [32]. So there may be a generalization of our results to even-dimensional horizons. Alternatively, one may expect that there is an expression relating the number of supersymmetries preserved by the even-dimensional horizons to the index of a Dirac operator.

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A A Lichnerowicz Identity

To prove the Lichnerowicz identity of section 4, we use the spinor conventions of [25], appendix A.2. In this conventions, the Dirac spinors of Spin(9, 1) are identified with Λ∗(C^5) and the Majorana spinors span a real 32-dimensional subspace after an appropriate reality condition is imposed. The Dirac spinors of Spin(9) are identified with the subspace Λ∗(C^4) ⊂ Λ∗(C^5). In particular, if C^5 = C < e_1, ..., e_5 >, then C^4 = C < e_1, ..., e_4 >. The Majorana spinors of Spin(9) are those of Spin(10, 1) restricted on Λ∗(C^4). From this, it is straightforward to identify the gamma matrices of Spin(9) from those of Spin(10, 1) which have been given in [25].

As has been explained in section 4, the Spin(9) invariant inner product ⟨·, ·⟩ restricted on the Majorana representation is positive definite and real, and so symmetric. With respect to this, the skew-symmetric products Γ^[k] of k Spin(9) gamma matrices are Hermitian for k = 0 mod 4 and k = 1 mod 4 while they are anti-Hermitian for k = 2 mod 4 and k = 3 mod 4. Using this, we have that

Ψ(±)† = ±1/2 h_i - 1/288 Γ^[i]t_1t_2t_3t_4 X_{t_1t_2t_3t_4} + 1/36 X_{|i}^{|i}t_1t_2t_3t_4 Γ^[i]t_1t_2t_3t_4 + 1/24 Γ^[i]t_1t_2 Y_{t_1t_2} + 1/6 Y_{i} Γ^[i]j ,
Ψ(±)† = ±1/2 h_i Γ^[i] + 1/96 X_{t_1t_2t_3t_4} Γ^[i]t_1t_2t_3t_4 + 1/8 Y_{t_1t_2} Γ^[i]t_1t_2 ,

where † is the adjoint with respect to the Spin(9)-invariant inner product ⟨·, ·⟩. Next let us turn to the computation of the RHS of (4.6). The term involving Ψ(±)†Ψ(±) can be expanded out directly in terms quadratic in the fluxes h, Y, X. In par-
ticular
\[
\Psi^{(\pm)\dagger}\Psi^{(\pm)} = \frac{1}{16} h^2 + \frac{1}{576} h_{\ell_1} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{12} (i h Y)_i \Gamma^i \\
- \frac{5}{27648} X_{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{192} Y_{\ell_1 \ell_2} Y_{\ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{5}{96} Y^m n Y_{mn} \ .
\]  
(A.2)

The term in (4.6) involving \(\Gamma_{ij}\tilde{\nabla}_i \tilde{\nabla}_j\) can be rewritten using
\[
\Gamma_{ij}\tilde{\nabla}_i \tilde{\nabla}_j \phi_{\pm} = \frac{1}{4} \tilde{R} \phi_{\pm} ,
\]  
where \(\tilde{R}\) is the Ricci scalar of \(\mathcal{S}\). From the Einstein field equation (2.11), one has
\[
\tilde{R} = -\tilde{\nabla}^i h_i + \frac{1}{2} h^2 + \frac{1}{4} Y_m n Y^{mn} + \frac{1}{48} X_{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} \ .
\]  
(A.4)

It follows that
\[
\int_S \langle \phi_{\pm}, \Gamma_{ij}\tilde{\nabla}_i \tilde{\nabla}_j \phi_{\pm} \rangle = \int_S \langle \phi_{\pm}, \left( -\frac{1}{8} h^2 - \frac{1}{16} Y_m n Y^{mn} - \frac{1}{192} X_{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} \right) \phi_{\pm} \rangle \\
+ \int_S \frac{1}{4} \tilde{\nabla}^i h_i (\phi_{\pm}, \phi_{\pm}) .
\]  
(A.5)

To proceed with the evaluation of (4.6), observe that
\[
(\Psi^{(\pm)\dagger} - \Psi^{(\pm)} \tilde{\nabla}_i \phi_{\pm} - (\Psi^{(\pm)\dagger} - \Psi^{(\pm)}) \Gamma_{ij}\tilde{\nabla}_i \phi_{\pm} = \left( -\frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{6} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right) \tilde{\nabla}_i \phi_{\pm} \\
+ \left( \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{6} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} \right) \Gamma_{ij} \tilde{\nabla}_i \phi_{\pm} .
\]  
(A.6)

Using the fact that the Clifford algebra element of first term in the RHS of the above equation is hermitian, the Bianchi identity \(dX = 0\) and upon integrating by parts, one finds that
\[
\int_S \langle \phi_{\pm}, (\Psi^{(\pm)\dagger} - \Psi^{(\pm)} \tilde{\nabla}_i \phi_{\pm} - (\Psi^{(\pm)\dagger} - \Psi^{(\pm)}) \Gamma_{ij}\tilde{\nabla}_i \phi_{\pm} \rangle = \pm \int_S \langle \phi, \frac{1}{12} (\tilde{\nabla}_i Y_{i\ell}) \Gamma^\ell \phi \rangle \\
+ \int_S \langle \phi, \left( \frac{1}{72} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{6} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} \right) \Gamma_{ij} \tilde{\nabla}_i \phi \rangle .
\]  
(A.7)

The term involving \(\tilde{\nabla}_i Y_{i\ell}\) is then further rewritten as a term quadratic in \(X\) using the field equation (2.7). Next, we rewrite the second line in terms of the Dirac operator \(\Gamma_{ij}\tilde{\nabla}_i \phi_{\pm} + \Psi^{(\pm)} \phi_{\pm}\), with a compensating term \(-\Psi^{(\pm)} \phi_{\pm}\) which gives a term quadratic in the fluxes \(h, X, Y\), and which can be expanded out straightforwardly.

Next, we find that
\[
(\Gamma^i \Psi^{(\pm)} - \Psi^{(\pm)} \Gamma^i) \tilde{\nabla}_i \phi_{\pm} = \left( \mp \frac{1}{2} h^i + \frac{1}{48} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} \pm \frac{1}{2} Y_{\ell_1 \ell_2} \Gamma^\ell \right) \tilde{\nabla}_i \phi_{\pm} \\
+ \left( \pm \frac{1}{2} h^i \Gamma^\ell - \frac{1}{48} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} \right) \Gamma^i \tilde{\nabla}_i \phi_{\pm} .
\]  
(A.8)
Similarly, using the hermiticity of the Clifford element in the first term in the RHS of the above equation, $dX = 0$ and upon integrating by parts, one finds

\[
\int_S \langle \phi_\pm, (\Gamma^i \Psi^{(\pm)} - \Psi^{(\pm)} \Gamma^i) \tilde{\nabla}_i \phi_\pm \rangle = \int_S \langle \phi_\pm, +\frac{1}{4} (\tilde{\nabla}^i Y_{\ell \ell}) \Gamma^\ell \phi \rangle \\
+ \int_S \langle \phi, (\pm \frac{1}{2} h_{i\ell} \Gamma^\ell - \frac{1}{48} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^i \tilde{\nabla}_i \phi) \rangle \\
\pm \frac{1}{4} \int S \tilde{\nabla}^i h_i \langle \phi_\pm, \phi_\pm \rangle .
\]

(A.9)

The term involving $\tilde{\nabla}^i Y_{\ell \ell}$ is then further rewritten as a term quadratic in $X$ using (2.7). The second line is also further rewritten in terms of the Dirac operator $\Gamma^i \tilde{\nabla}_i \phi_\pm + \Psi^{(\pm)} \phi_\pm$, with a compensating term involving $-\Psi^{(\pm)} \phi_\pm$ which gives a term quadratic in the fluxes $h, X, Y$.

Next note that

\[
(\Gamma^i \tilde{\nabla}_i \Psi^{(\pm)} - \tilde{\nabla}^i \Psi_i^{(\pm)}) \phi_\pm = \left(\mp \frac{1}{8} d_{ij} \Gamma^{ij} + \frac{1}{72} \tilde{\nabla}^i X_{i\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3} \\
\pm \frac{1}{12} \tilde{\nabla}_i Y_{\ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3} \pm \frac{5}{12} \tilde{\nabla}^i Y_{i\ell} \Gamma^\ell \right) \phi_\pm,
\]

(A.10)

and so

\[
\int_S \langle \phi_\pm, (\Gamma^i \tilde{\nabla}_i \Psi^{(\pm)} - \tilde{\nabla}^i \Psi_i^{(\pm)}) \phi_\pm \rangle = \int_S \langle \phi_\pm, \pm \frac{5}{12} \tilde{\nabla}^i Y_{i\ell} \Gamma^\ell \phi_\pm \rangle,
\]

(A.11)

as the rest of the terms are anti-hermitian, and hence the associated form bilinears vanish. The term involving $\tilde{\nabla}^i Y_{\ell \ell}$ is then again rewritten as a term quadratic in $X$ using (2.7).

### B Derivation of (4.8)

To derive (4.8), let us assume that $\phi_+$ satisfies the horizon Dirac equation $\mathcal{D}^{(+)} \phi_+ = 0$. Then

\[
\tilde{\nabla}^i \tilde{\nabla}_i \| \phi_+ \|^2 = 2 \langle \phi_+, \tilde{\nabla}^i \tilde{\nabla}_i \phi_+ \rangle + 2 \langle \tilde{\nabla}_i \phi_+ \tilde{\nabla}^i \phi_+ \rangle .
\]

(B.1)

It will be useful to note the following identity

\[
\tilde{\nabla}^i \tilde{\nabla}_i \phi_+ = \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \phi_+) - \Gamma^j \tilde{\nabla}_j \tilde{\nabla}_i \phi_+ \\
= \Gamma^i \tilde{\nabla}_i \left( \frac{1}{4} h_{ij} \Gamma^j - \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right) \phi_+ + \frac{1}{4} \tilde{\mathcal{R}} \phi_+.
\]

(B.2)

This then implies that

\[
\langle \phi_+, \tilde{\nabla}^i \tilde{\nabla}_i \phi_+ \rangle = \langle \phi_+, \left( \frac{1}{8} h_{ij} \Gamma^j - \frac{1}{16} X_{\ell_1 \ell_2} Y^{\ell_1 \ell_2} + \frac{1}{192} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{4} \tilde{\nabla}^i Y_{i\ell} \Gamma^\ell \right) \phi_+ \rangle \\
+ \langle \phi_+, \Gamma^i \left( \frac{1}{4} h_{ij} \Gamma^j - \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \right) \phi_+ \rangle,
\]

(B.3)
where the trace of (2.11) has been used. Also, one has
\[
\langle \tilde{\nabla}^i \phi_+, \tilde{\nabla}_i \phi_+ \rangle = \langle \tilde{\nabla}^{(+)i} \phi_+, \tilde{\nabla}^{(+)i} \phi_+ \rangle - 2\langle \phi_+, (\Psi^{(+)i})^\dagger \tilde{\nabla}_i \phi_+ \rangle - \langle \phi_+, \Psi^{(+)i} \tilde{\nabla}^i \phi_+ \rangle. 
\] (B.4)

Substituting (B.3) and (B.4) into (B.1), one obtains
\[
\tilde{\nabla}^i \tilde{\nabla}_i \langle \phi^+, \phi^+ \rangle = \langle \tilde{\nabla}^{(+)i} \phi_+, \tilde{\nabla}^{(+)i} \phi_+ \rangle + 2 \langle \phi_+, \left( \frac{1}{4} \hbar^2 + \frac{1}{8} Y_{\ell_1 \ell_2} Y_{\ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{2} \tilde{\nabla}^i Y_{ij} \Gamma^j - 2\Psi^{(+)i} \Psi^{(+)j} \right) \phi_+ \rangle
+ 2 \langle \phi_+, \left( \Gamma^i \left( \frac{1}{4} \hbar \Gamma^j - \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} \right) \right) \tilde{\nabla}^i \phi_+ \rangle. 
\] (B.5)

To proceed, note that
\[
\Gamma^i \left( \frac{1}{4} \hbar \Gamma^j - \frac{1}{96} X_{\ell_1 \ell_2 \ell_3 \ell_4} \Gamma_{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{8} Y_{\ell_1 \ell_2} \Gamma_{\ell_1 \ell_2} \right) - 2\Psi^{(+)i} \Psi^{(+)j} \right) \Gamma^i.
\] (B.6)

Substitute this expression into (B.5), and use the Dirac equation \( D^{(+)} \phi_+ = 0 \) to eliminate the \( \Gamma^i \tilde{\nabla}_i \phi_+ \) term in favour of terms algebraic in the fluxes. On expanding out the resulting expression and making use of (A.2) and (2.9), one then obtains (4.8).

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