Using mixed data in the Inverse Scattering Problem

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Abstract

Consider the fixed-\ell inverse scattering problem. We show that the zeros of the regular solution of the Schrödinger equation, $r_n(E)$, which are monotonic functions of the energy, determine a unique potential when the domain of the energy is such that the $r_n(E)$ range from zero to infinity. This suggests that the use of the mixed data of phase-shifts $\{\delta(\ell_0,k), k \geq k_0\} \cup \{\delta(\ell,k_0), \ell \geq \ell_0\}$, for which the zeros of the regular solution are monotonic in both domains, and range from zero to infinity, offers the possibility of determining the potential in a unique way.
1 Introduction

Approaches to the three-dimensional inverse scattering problem can be classified in two categories\[1, 2\]. In the first case, the fixed-\(E\) problem, Loeffel\[3\] obtained theorems predicting a unique potential from the knowledge of the phase-shifts \(\delta(\ell, k)\) at a specific energy \(E = k^2\), for all (non-discrete) non negative values of \(\lambda = \ell + 1/2\). When the set of data is reduced to discrete values of \(\lambda = \ell + 1/2\), \(\ell\) non-negative integers, Carlson’s theorem\[4\] predicts a unique potential \(V(E, r)\), provided that this latter belongs to a suitable class\[2, 3\]. The Newton series allow the construction of the potential \(V(E, r)\) from this latter set of discrete values\[3\].

In the second case, known also as fixed-\(\ell\) problem, (see Ref.[2] and references therein), an \(E\)-independent potential \(V_\ell(r)\) satisfying the integrability conditions

\[
\int_0^{+\infty} r|V_\ell(r)| \, dr < \infty, \quad \int_b^{+\infty} |V_\ell(r)| \, dr < \infty, \quad b > 0, \quad (1.1)
\]

can be constructed from the phase-shifts \(\delta(\ell, k)\), given for all momenta \(k \in (0, \infty)\) and from the discrete spectrum data (eigen-energies and the corresponding normalization constants).

Another class of Bargmann potentials can be constructed from input data which are partly \(E\)-dependent and partly \(\ell\)-dependent and where \(aE + b(\ell + 1)\) is a constant\[5\]. This latter construction is based on an extension of Newton’s method.

Historically, the idea of mixing the data was first explored by Grosse and Martin\[6\] for confining potentials. They conjectured that the knowledge of the ground state energies \(E^{(0)}_\ell\), for all non-negative integers \(\ell\), allows the recovery of the potential in an unique way. This problem has been studied numerically in Ref.[7].

In the present work, we are also concerned with a non-standard inverse problem, namely with the construction of the potential from a spectrum which involves both data coming from the \(E\)-fixed problem and the \(\ell\)-fixed problem, but where extensions of Newton’s method can no longer be applied. The set of mixed data considered here is \(\{\delta(\ell_0, k), k \geq k_0\} \cup \{\delta(\ell, k_0), \ell \geq \ell_0\}\), corresponding to the set of scattering parameters \(\{E \geq E_0, \ell = \ell_0\} \cup \{E = E_0, \ell \geq \ell_0\}\).

We will investigate in detail, for the fixed \(\ell\)-problem, to what extent the knowledge of the zeros \(r_n(E)\) of the regular solution allows the determination of the potential. We will show that the function \(r_n(E)\), being monotonic with respect to the energy, reflects a unique potential, provided that the domain of the energy is such that \(r_n(E)\) ranges from zero to infinity. This will lead to a uniqueness theorem. We also will depict a method which allows the recovery of the piecewise constant potentials from the knowledge of the function \(r_n(E)\). The advantage is that this latter method can be extended to our non-standard spectrum, given above, because the zeros of the regular solution are monotonic in both domains and still range from zero to infinity. The uniqueness theorem still applies and the piecewise constant potentials can be recovered from the zeros of the regular solution\[8\].

The next step consists in investigating to what extent the mixed data determines a unique potential. By analogy with the fixed \(\ell\)-problem we expect, as explained in detail below, that this should be the case, provided the potential satisfies the integrability conditions (1.1).
In Sec. 2 we discuss the formalism leading to the mixed inversion scheme presented in Sec. 3 and, as an example, we apply it to piecewise constant potentials. The uniqueness of the method, demonstrated in Ref. [8] by using a JWKB procedure will then be examined in the Born approximation. In Sec. 4 we will present our conclusions.

2 Formalism

Consider a potential $V(r)$ satisfying the integrability conditions\cite{2}

$$
\int_0^a r |V_\ell(r)| \, dr < \infty, \quad a < \infty
$$
$$
\int_b^{+\infty} |V_\ell(r)| \, dr < \infty, \quad b > 0
$$

and the Schrödinger equation

$$(\frac{d^2}{dr^2} + E - V(r) - \frac{(\ell + 1/2)^2 - 1/4}{r^2}) \psi_\ell(E, r) = 0. \quad (2.2)$$

The regular solutions $\psi_\ell(E, r)$ satisfy the Cauchy condition $\lim_{r \to 0} r^{\ell-1} \psi_\ell(E, r) = 1$ and take on the asymptotic form $\psi_\ell \propto \sin(\ell \pi/2 + \delta(\ell, k))$ with $\delta(\ell, k)$ being the phase shifts. (Recall that here $E = k^2$.)

Let us define $r(\ell, E)$ such that the regular solution $\psi_\ell(E, r)$ vanishes for $r = r(\ell, E) \neq 0$. For a fixed energy and a fixed $\ell$, there is then a countable number of zeros, each position being denoted by $r_n(\ell, E)$.

Furthermore, for any potential, the zeros satisfy the monotonicity properties:

- $E = k^2 \mapsto r_n(\ell, E)$ is a decreasing function, as has been shown by Sturm in 1830’s\cite{9} and,

- $\ell \mapsto r_n(\ell, E)$ is an increasing function. This has been shown in Ref.\cite{8}.

Recall that for potentials satisfying (2.1) the function $r_n(\ell, E)$ is such that $r_n(\ell, E) \to 0$ for $E \to \infty$\cite{10}. In the absence of bound states $r_n(\ell, E) \to \infty$ for $E \to 0$. However, in the presence of one bound state, $r_1(\ell, E) \to \infty$ for $E \to E_1$. Here $E_1 < 0$ denotes the ground state energy\cite{1}.

For $N$ bound states, $E_1 < E_2 \ldots < E_N$, if $n \leq N$, $r_n(\ell, E) \to \infty$ for $E \to E_n$\cite{10}, whereas for $n > N$, $r_n(\ell, E) \to \infty$ as $E \to 0$.

This is illustrated in figure 1 where we have drawn the four first zeros of the regular $s$-wave solution for the Bargmann transparent potential, Eq.(27) of Ref.\cite{11} with $\ell = 0$ and $\gamma^2 = 10$, which has a bound state at the energy $E = -10$ in $1/L^2$ units. The vertical axis $E = 0$ is explicitly depicted to show that all zeros but the first one go to infinity as $E$ goes to 0. The first zero goes to infinity when $E \to E_1 = -10$.

In the fixed-$\ell$ inversion, if all the zeros are assumed to be known, i.e. $E \mapsto r_n(\ell, E)$ is known and $r_n(\ell, E)$ describes the entire interval $[0, \infty[$, the potential $V_\ell(r)$ is then uniquely

\footnote{We adopt this notation to denote the ground state by $E_1$ instead of the traditionally used $E_0$ in order to be consistent with the meaning given to $n$, namely to denote zeros of the wave function}
Figure 1: Zeros $r_n(E), n = 1, 2, 3$ and 4, of the regular s-wave solution for a Bargmann potential (see text).

determined. This can be checked easily. Consider the following Sturm-Liouville problem on $[0, R]$, i.e., namely the equation

$$\psi''_\ell(r) + \left( E - V_\ell(R - r) - \frac{\ell(\ell + 1)}{(R - r)^2} \right) \psi_\ell(r) = 0,$$

(2.3)
coupled with the Dirichlet conditions

$$\psi_\ell(0) = \psi_\ell(R) = 0.$$

The spectral data are the eigenvalues, namely the $E^*_n$'s such that $R = r_n(\ell, E^*_n)$, together with the normalization constants$^{[12]}

$$\rho_n = \frac{\int_0^R \psi_\ell(r')^2 dr'}{\psi'_\ell(0)^2},$$

here given by the positive values

$$\rho_n = -(dr_n/dE)(\ell, E_n^*).$$

The potential, assumed to be square integrable on $[0, R]$ is uniquely determined on $[0, R]$ by the spectral data $\{E^*_n, \rho_n\}^{[13]}$. The technique of constructing $V$ from this set of data is well-known$^{[2, 12, 14, 15]}$.

We now concentrate on the case of the zeros of the regular solution. For a fixed $\ell$, the nth zero of the regular solution can be considered as a function of the energy $r_n(\ell, E)$, monotonic and such that $\psi_\ell(E, r_n(\ell, E)) \equiv 0$. It defines a line of zeros. Moreover $E \mapsto r_n(\ell, E)$ admits an inverse function $r \mapsto E_{n,\ell}(r)$ which is also monotonic and is the inverse of this line of zeros. For example consider the potential $V \equiv 0$ in the s-wave. The regular solution is proportional to $\sin(\sqrt{E} r)$. The lines of zeros are given by $r_n(0, E) = n\pi/\sqrt{E}$ and the inverses of these lines are $E_{n,0} = n^2\pi^2/r^2$.

We have shown above that if we know all lines $r \mapsto E_{n,\ell}(r), n \geq 1$, for $r$ running from 0 to $\infty$, then we can construct the desired potential, assumed to be locally $L^2(\mathcal{R})$. If only a single line $r \mapsto E_{n_1,\ell}(r)$ is known, no method is available to recover the potential, except
in the special case of piecewise constant potentials. What we can show is a uniqueness property, if \( E_{n,\ell}(r) \) is known for all positive \( r \).

In the special case of piecewise constant potentials, having discontinuities at values of \( r = a_j, j = 1, \ldots, j_{\text{max}}, \) and being zero for \( r > a_{j_{\text{max}}} \), it is easy to show that there is a one to one correspondence between the discontinuities of the third derivative of \( E_{n,\ell}(r) \) with respect to \( r \) and the discontinuities of \( V \). This suggests the following lemma:

**Lemma 2.1** For a piecewise constant potential, the knowledge of a single line of zeros allows the reconstruction of the potential in an unique way provided that the line is monotonic with respect to the energy and runs over the entire positive axis.

The proof is based on the consideration of the third derivative of

\[
\psi(E(r), r) \equiv 0,
\]

where the function \( r \mapsto E(r) \) denotes the inverse function of the function \( E \mapsto r(E) \) which is the single line of zeros \( r(E) \) considered. To be specific, if \( E''' \) has a discontinuity at \( r = a \) then

\[
\frac{d^3E}{d r^3}(a^+) - \frac{d^3E}{d r^3}(a^-) = -2 \frac{dE}{dr}(a) \left[ V(a^+) - V(a^-) \right]. \tag{2.4}
\]

This is equivalent to the relation

\[
\frac{d^3r}{dE^3}(E_a^+) - \frac{d^3r}{dE^3}(E_a^-) = 2 \left( \frac{dr}{dE}(E_a) \right)^3 \left[ V(r(E_a^+)) - V(r(E_a^-)) \right]. \tag{2.5}
\]

Since the potential is zero for \( r > a_{j_{\text{max}}} \), the relation (2.4) allows us to reconstruct the potential between \( a_{j_{\text{max}}} \) and \( a_{j_{\text{max}}-1} \). The procedure can be repeated at each \( a_j \), and the potential is obtained at \( r \neq a_j \) by summing the successive values at each discontinuity appearing beyond \( r \).

As an illustration consider for instance the explicit potential defined by

\[
V(r) = \begin{cases} 
-2 & r < 2 \\
-1 & 2 < r < 3 \\
0 & r > 3 
\end{cases} \tag{2.6}
\]

In the figure 2, we have drawn the function \( r \mapsto -E'''(r)/(2E'(r)) \), related to the inverse \( E(r) \) of the first line of zeros for the \( s \)-wave regular solution of the Schrödinger equation involving the potential (2.7).

Clearly the discontinuities of \( r \mapsto -E'''(r)/(2E'(r)) \) happen at the points where \( V \) has discontinuities, namely \( r = 2 \) and \( r = 3 \), and equation (2.4) is satisfied. We know that the potential is zero beyond \( r = 3 \). So we have \( 0 = V(3^+) \). From the curve \( V(3^+) = V(3^-) + 1 \) then \( V(3^-) = -1 \). As \( V \) is piecewise constant we have \( V(3^-) = V(2^+) \) and from the curve \( V(2^+) = V(2^-) + 1 \) so that \( V(2^-) = -2 \). We then recover the potential of Eq. (2.7).

This method cannot be applied to a potential defined by a continuous function. Nevertheless, for such potentials, the following uniqueness theorem holds:
Theorem 2.2 Consider two potentials $V_1$ and $V_2$ satisfying (2.1) and locally constant in the vicinity of zero. Within the fixed-$\ell$ problem, assume that two integers $n_1$ and $n_2$ exist such that the $n_1$-th line of zeros for the regular solution for $V_1$ coincides with the $n_2$-th line of zeros for the regular solution for $V_2$, both lines describing the whole positive axis as the energy $E$ varies. Then $V_1 \equiv V_2$.

The proof is not reproduced here and given in Ref. [8].

3 Mixed Problem

Consider the set \{\(E \geq E_0, \ell = \ell_0\) \cup \{\(E = E_0, \ell \geq \ell_0\)\}. The zeros of the regular solution form a line with two parts. In the first part, the zeros go from $r = 0$ to $r(\ell_0, E_0)$ as the energy $E$ varies from $\infty$ to $E_0$ ($\ell_0$ being fixed); in the second part, they go from $\ell_0$ to $\infty$ ($E_0$ being fixed). The monotonicity property required in the lemma and the theorem is preserved on both domains. As it has been done in the section 2 we can define a line of zeros. The $n$th line of zeros $r_n(\ell, E)$ of the regular solution describes a line formed of two parts. It is defined as follows. In the first part the zeros $r_n(\ell_0, E_0)$ go from $r = 0$, ($E = \infty$) to $r_0 = r(\ell_0, E_0)$, ($E = E_0$) as the energy $E$ varies from $\infty$ to $E_0$ ($\ell_0$ being fixed); in the second part, $r_n(\ell, E_0)$ goes from $r_0 = r(\ell_0, E_0)$, ($\ell = \ell_0$) The inverse function of the line of zeros are defined as follows. For $r \leq r_n(\ell_0, E_0) = r_0$ it is given by $E$ such that $r = r_n(\ell_0, E)$. For $r \geq r_n(\ell_0, E_0)$ it is given by $\ell$ such that $r = r_n(\ell, E_0)$. It is continuous at $r = r_0$ with a discontinuous derivative at $r = r_0$.

In Ref. [8] we have shown that the above lemma is still valid for piecewise constant potentials and that the uniqueness theorem still works.

To summarize the situation, we have shown in Ref. [8] that a single line of zeros, which, for the data considered, always goes from zero to infinity and moreover is monotonic, determines the potential uniquely. The remaining question is to examine whether the set of mixed data

\[
\{\delta(\ell = \ell_0, k) \quad k \in [k_0, +\infty]\} \cup \{\delta(\ell, k_0) \quad \ell \in [\ell_0, +\infty]\}
\]  

(3.1)
associated to the set \( \{ \ell = \ell_0, k \in [k_0, \infty] \} \cup \{ k = k_0, \ell \in [\ell_0, \infty] \} \), determines a line of zeros, and thus the potential, in a unique way - which is suggested by the analogy with the \( \ell \)-fixed problem. In the absence of bound states, all lines of zeros \( E \leftrightarrow r_n(\ell, E) \), monotonic with respect to \( E \), range from zero (\( E \) infinite) to infinity (\( E = 0 \)) when \( E \) goes from infinity to zero. In this case, we know that the potential, when it satisfies (1.1) is recovered in a unique way, given the phase-shifts \( \delta(\ell, k) \) for all \( k \geq 0 \). In contrast to the condition (2.1), the condition (1.1) excludes all pathologies i.e. potentials behaving asymptotically like \( 1/r^2 \), encountered in particular in the presence of a zero energy bound state, or ghost components in the Jost function[11]. With the mixed data we are in the same situation, namely all the lines of zeros are monotonic and range from zero (\( E \) infinite, \( \ell = \ell_0 \)) to infinity (\( E = E_0, \ell \) infinite). So we expect that (3.1) is associated with a unique potential decreasing faster than \( 1/r^2 \) at infinity. We cannot prove this in the general case, but we have investigated the problem in a JWKB approach in Ref. [8], where we have shown that the mixed data (3.1) allow us to recover the potential, provided that the turning point is unique.

It is of interest to investigate what happens in the Born approximation. In the seventies Reignier[16] used a Born approximation of the scattering amplitude to show that the knowledge of the phase-shift at a fixed energy say \( E_0 = k_0^2 \) for each integer \( \ell \) is equivalent to the knowledge of the Fourier sine transform of the potential \( V(r) \),

\[
g(q) = \int_0^\infty \sin(qr) \, r V(r) \, dr ,
\]

for \( q \leq 2k_0 \). The scattering amplitude is determined from the phase-shifts at fixed energy \( \delta(\ell, k_0) \) for \( \ell = 0, 1, 2, 3, ..., E = E_0 = k_0^2 \).

Generally, the integral is assumed to be zero for \( q > 2k_0 \) leading to potentials

\[
r V(r) = \frac{2}{\pi} \int_0^{2k_0} \sin(qr) \, g(q) \, dq ,
\]

such that \( r V(r) \) is an entire function of \( r \) of order 1. Other extensions of \( g(q) \) are studied in Ref. [18].

A possible way to extend our knowledge of \( g(q) \) beyond \( 2k_0 \) is to take the Born approximation for the missing phase shifts \( \delta(\ell = 0, k) \) for \( k \geq k_0 \). This is given by

\[
\int_0^\infty \sin(kr)^2 \, V(r) \, dr = -k \delta(\ell = 0, k) .
\]

The derivative with respect to \( k \) yields

\[
g(q) = \int_0^\infty \sin(qr) \, r V(r) \, dr = -\frac{d(k \delta(\ell = 0, k))}{dk} , \quad \forall q = 2k \geq 2k_0 .
\]

This implies that \( g(q) \), known for \( q \leq 2k_0 \) is now known for every positive \( q \), including \( q \geq 2k_0 \), and that \( V(r) \) is uniquely given by

\[
r V(r) = \frac{2}{\pi} \int_0^\infty \sin(qr) \, g(q) \, dq .
\]

Consequently, the knowledge of \( \{ \delta(\ell, k_0), \ell \in \mathcal{N} \} \cup \{ \delta(\ell = 0, k), k \geq k_0 \} \) allows us to recover the potential in Born approximation if \( k_0 \) is sufficiently high.
4 Conclusion

In the present work, we have been concerned with a non-standard inverse problem, namely with the construction of the potential from a spectrum which involves data coming from both the $E$-fixed problem and the $\ell$-fixed problem, but where extensions of Newton’s method can no longer be applied. For the $\ell$-fixed problem, we have investigated to what extent the knowledge of the zeros of the regular solution allows the determination of the potential, and have also given a uniqueness theorem. Furthermore, we have shown that the zeros of the regular solution of the Schrödinger equation, which are monotonic functions of the energy, $r_n(E)$, determine a unique potential when the domain of energy is such that the $r_n(E)$ range from zero to infinity. The knowledge of a single line of zeros does not allow us to recover the underlying potential, except in the special case of piecewise constant functions.

As an application of the method, we have considered the mixed data $\{\delta(\ell_0, k) \quad k \geq k_0\} \cup \{\delta(\ell, k_0) \quad \ell \geq \ell_0\}$ for which the zeros of the regular solution are monotonic in both domains, and range from zero to infinity. These mixed data offer the possibility of determining the potential in a unique way. Indeed we have shown that a single line of zeros underlies a unique potential, which can be extracted when the potential is a piecewise constant function. We know from Ref. $[8]$ that the mixed data yield a unique $\ell$- and $E$-independent potential, in the JWKB approximation and in the case of a single turning point.

We have shown that, in Born approximation the following mixed data

$$\{\delta(\ell = 0, k), k \in [k_0, +\infty] \} \cup \{\delta(\ell, k_0), \ell \in \mathcal{N}\}$$

lead to an unique potential, still assumed to be $\ell$- and $E$-independent, which is the inverse Fourier sine transform of a function deduced from the data.

To conclude, our method, which does not use any extension of the Newton’s method, applies to mixed set of data and/or to the ”generalized eigenvalue problem” namely the scattering problem where the function which involves the scattering parameter is no longer separable in the scattering parameter and the space coordinates as it happens for the fixed-$E$ (function $(\ell, r) \mapsto \ell(\ell + 1)/r^2$) and fixed-$\ell$ (function $(E, r) \mapsto E$) problems.

Given a set of phase-shifts corresponding to a domain where the scattering parameter(s) varies (vary), we conjecture that, if all lines of zeros of the regular solution are monotonic, continuous and range from zero to infinity when the scattering parameter(s) describes the domain considered, both limits 0 and $\infty$ being reached at the boundary of the domain, then there exists a unique potential satisfying (1.1) and corresponding to the set of phase-shifts.

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