Nonlinear Flux Diffusion and ac Susceptibility of Superconductors - Exact Numerical Results

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Abstract

The ac response of a slab of material with electrodynamic characteristics $E \sim j^{\kappa+1}$, $\kappa \geq 0$, is studied numerically. From the solutions of the nonlinear diffusion equation, the fundamental and higher-order components of the harmonic susceptibility are obtained. A large portion of the data for every $\kappa$ can be scaled by a single parameter, $\xi = t_{1/2}/D$, where $t$ is the period of the ac field at the surface, $H_0$ is its amplitude and $D$ is the slab thickness. This is, however, only an approximate scaling property: The field penetration into a nonlinear medium is a more complex phenomenon than in the linear case. In particular, the susceptibility values are not uniquely defined by a set of only two parameters, such as $\kappa$ and $\xi$, while one parameter, i.e. $t^{1/2}/D$, is sufficient to describe the electrodynamic response of a linear medium.

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1. Introduction: Nonlinear diffusion and ac susceptibility of superconductors.

The problem of nonlinear diffusion has recently attracted considerable attention in diverse fields of science. One example, which is considered in the present work, deals with the magnetization process of superconductors. In the case when the response of a superconductor to an applied ac field, \( H = H_0 \cdot \exp(i \omega t) \), is linear, the electrodynamic properties in the superconducting state can be described in terms of the complex conductivity, \( \sigma = \sigma_r + i \sigma_i \).

In the flux–flow regime of high–\( T_c \) superconductors, which occurs in a broad \( H - T \) range, it is usually justified to neglect the imaginary component of the complex conductivity. In that case, Maxwell’s equations, \( \nabla \times B = 4\pi/c \cdot j \) and \( c \nabla \times E = -\partial B/\partial t \), provide the following result for the field penetration into a material filling half-space, \( x > 0 \):

\[
B = H_0 \cdot \eta(x) \cdot \exp(i \omega t),
\]

where \( \eta(x) = \exp(-\lambda x) \), and \( \lambda^2 = 2i/\delta^2 \), where \( \delta^2 = c^2/(2\pi \omega \sigma_r) \). For a thin plate of thickness \( 2D \), with the ac field parallel to the large surface of the plate, the following equations hold [1]:

\[
4\pi \chi' = -1 + \frac{1}{a} \cdot \frac{\sinh(a) + \sin(a)}{\cosh(a) + \cos(a)},
\]

\[
4\pi \chi'' = \frac{1}{a} \cdot \frac{\sin(a) - \sinh(a)}{\cosh(a) + \cos(a)},
\]

where \( a = 2D/\delta \). A maximum in \( \chi'' \) results when \( \delta(\sigma, \omega) \) is comparable to the sample size. Experimental results rarely show susceptibility curves which correspond to the flux–flow result of ohmic-like behaviour. Rather, the effects are most often nonlinear. The dependence of the measured susceptibility curves on the excitation current and higher-order components in the harmonic susceptibilities are observed. In the limiting case of very strongly nonlinear response, the critical-state model may be used for the calculation of the ac susceptibility. Then, only one parameter is needed to construct the hysteresis curve for the magnetization, the field of the first full penetration to the sample center, \( H^* \). It is assumed that the harmonic susceptibility components, \( \chi'_m \) and \( \chi''_m \), are defined as Fourier components of the time-dependant magnetic hysteresis curve; \( \chi'_m = M(t)/D = \sum_m (\chi'_m \cdot \cos(m \cdot \omega t) + \chi''_m \cdot \sin(m \cdot \omega t)) \), where \( m \) is an integer. When, for instance, \( H_0 < H^* \), \( 4\pi \chi'_1 = -(1 - H_0/2H^*) \) is obtained then, \( 4\pi \chi'_m = 0 \) for every odd \( m > 1 \), and \( 4\pi \chi''_m = 2H_0/3\pi mH^* \) for all odd \( m > 0 \). To deal with situations which are more relevant to the description of real experimental results, it is necessary to investigate a nonlinear theory of the magnetic response.
which would bridge the two limiting cases observed: the linear-response and the critical-state one. A fruitful approach to this problem is based on studies of the electrodynamic response of a medium characterized by a power-law current-voltage dependence, \( j = \sigma(E) \cdot E = \sigma_0 \cdot E_0 \cdot (E/E_0)^{1/(\kappa+1)} \), where \( \kappa \geq 0 \). Using Maxwell’s equations, the nonlinear diffusion equation describing the penetration of fields into a slab of thickness \( 2D \) lying in the \( yz \) plane [1-3] can be derived as

\[
\frac{\partial \beta}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left( \frac{\partial \beta}{\partial \bar{x}} \cdot \left| \frac{\partial \beta}{\partial \bar{x}} \right|^\kappa \right), \quad \bar{t} = \frac{t}{\tau_0}, \quad \bar{x} = \frac{x}{x_0}, \tag{2}
\]

where \( \beta = B/E_0 \), \( x_0 = c/(4\pi\sigma_0) \) and \( \tau_0 = 1/(4\pi\sigma_0) \). Recently, studies of solutions of eq. (2) have been carried out by many authors [2-4]. The exact analytical description of the response of a superconductor to an abrupt change of external field has been given in references [1] and [2] and is compared with the results of non-logarithmic magnetization relaxation measurements on high-\( T_c \) materials [5]. Various aspects of the ac response of superconductors has been studied as well by Dorogovtsev [6] and van der Beek et al. [7]. Recent results of Gilchrist and Dombre [8] can be compared with numerical results described in the present work. The distinctive feature of solutions of eq. (2) is that the flux-profile penetration resembles that in the models of the critical-state; When a field change is applied, the profile of perturbation spreads out from the surface towards the sample center but a region exists where the field distribution is unchanged inside. If the response to a field change of \( H_0 \) is considered, the time after the front of the field change arrives to the center, \( t^* \), is given by,

\[
\frac{t^*}{\tau_0} = \frac{\kappa}{2(\kappa+1)(\kappa+2)} \left( \frac{\Gamma(1/\kappa+1)\Gamma(3/2)}{\Gamma(1/\kappa+3/2)} \right)^\kappa \left( \frac{D}{x_0} \right)^{\kappa+2}. \tag{3}
\]

The initial magnetization, at \( t < t^* \), is given by:

\[
4\pi M = -H_0 \cdot \left( 1 - \left( \frac{t}{t^*} \right)^{1/(\kappa+2)} \cdot \frac{\Gamma(1/\kappa+3/2)}{\Gamma(1/\kappa+2) \cdot \Gamma(1/2)} \right). \tag{4}
\]

Equations (3) and (4) imply that a single parameter, \( \xi \equiv t^{1/(\kappa+2)} \cdot H_0^{\kappa/(\kappa+2)} / D \), can parametrize the short-time magnetization relaxation. It is informative to determine the extent to which this scaling relation is valid.
with respect to the ac susceptibility (with the replacement of $t$ and $H_0$ by
the ac field period and the field amplitude, respectively).

2. Numerical modelling of nonlinear diffusion.

Most of the calculations of the nonlinear diffusion process presented here
have been performed on an array of dimension $50 \times 200$, containing mag-
netic induction values, $B$, at 50 time intervals and 200 space-intervals [9].
The magnetic field at the surfaces of the sample, $H_0 \cdot \sin(\omega \cdot t)$, determines
the boundary conditions. An average magnetic field $< B(t) >$ in the sam-
ple has been computed from the magnetic field distribution, every 50 time
steps. Next, a Fourier time-analysis of $< B(t) >$ has been performed and
the coefficients of the fundamental and higher-order terms of the harmonic
content have been found, $4\pi \chi'_m = < B'_m >$ and $4\pi \chi''_m = < B''_m >$, except
for the real component of the fundamental susceptibility, which is given by
$4\pi \chi'_1 = -1 + < B'_1 >$. The method of computation and its results have been
carefully tested. First, the magnetization relaxation process after an abrupt
change of the external field has been simulated and numerical results were
compared with the known exact analytical expressions derived by Koziol and
de Chatel [2]. Then, the validity of the modelling of the ac-response in the
limit of linear diffusion on the ac susceptibility, as given by eq. (1), was
checked. It was confirmed also that the ac susceptibility converges towards
the critical-state results for large $\kappa$. The time range for which the response
to the ac field becomes periodic (the initial response at short time does not
satisfies this condition) was also investigated. In most cases it is safe to
analyze the data taken after the initial 50000 steps in time evolution (this
time depends on $\kappa$ and $H_0$). An additional, more reliable, criterion of stable
periodicity is based on the criterion that the dc or second harmonic com-
ponents are not found. At large values of $H_0$, the calculations become unstable
abruptly. It is possible to overcome this difficulty but at significant expense
in computation time (the computation of one susceptibility point requires an
average of about 3 hours on an IBM-PC computer 486DX2-33MHz). There-
fore, we have concentrated on performing calculations for a larger number of
points at lower fields.

3. Results and discussion

The penetration of an alternating field resembles, in some ways, the
response to an abrupt change of external field; the amplitude of field changes
diminishes gradually in the material and, if the field amplitude at the surface
is not too large, there is no penetration to a volume separated by a certain distance from the sample surface. Whether the front of the flux profile in ac penetration propagates towards the center or not, is not an easy question to answer, since the initial very slow propagation which is observed might only be due to unstable initial conditions. Within the accuracy of calculations, the flux profile has a self-replicating shape of diminishing amplitude, with perfect periodicity in time at every point in space but with a phase shift which changes with the distance from the surface. The profiles obtained for one value of an ac field amplitude coincide with the profiles computed for another ac field amplitude, if the phase lag and spatial coordinates are shifted properly. Plots of $\chi''$ versus $\chi'$ shown in Figures 1 and 2 for different values of the nonlinearity parameter $\kappa$ converge to the limit of linear diffusion for $\kappa \to 0$ and to the limit given by the critical-state model for large $\kappa >> 1$.

![Graph showing $\chi''$ versus $\chi'$ for different values of $\kappa$ and $t$.](image)

**Figure 1.** The $\chi''$ versus $\chi'$ plots of the ac susceptibility for different values of the nonlinearity parameter $\kappa$ and different periods of the ac field $t$, $(\kappa, t)$: (0.667, 25000, ◦), (0.667, 5000, ∆), (2, 12500, •), (2, 5000, □), (3, 10000, ◦), (12, 6250, □). Solid lines represent the critical-state and the linear-response limits for a thin plate. The difference between the data for $\kappa = 2$ obtained for two different frequencies of the ac field should be noted.
Figure 2. The third harmonic susceptibility $\chi''_{3}$ versus $\chi'_{3}$ compared to the critical-state result represented by the solid line, for the following nonlinearity parameter $\kappa$ and periods of the ac field, $t$, ($\kappa, t$): (0.667, 25000, •), (0.667, 5000, ◊), (1, 5000, □) and (2, 12500, ◦).

An important feature of the present results is seen in the Figure 1; Susceptibility points computed for different frequencies of the ac field but the same value of $\kappa$, do not fall on the same curve. This is different from what it might be expected and seems to have been unnoticed in previous work [8]. In Figure 3, we show that a simple scaling of the susceptibility with the amplitude of the ac field holds for the data obtained in the range of incomplete flux penetration, $\chi \sim H_0^{\kappa/(\kappa+2)}$. In Figure 4, the real component of the first harmonic susceptibility is drawn as a function of $H_0^{\kappa/(\kappa+2)} \cdot t^{1/(\kappa+2)}/D$, for different values of the field amplitude $H_0$, period of the field and for a few sample sizes. This latter scaling method is not perfect; small differences in the slopes of the data computed for various frequencies is found. This effect may be explained by the fact that flux profiles have a shape which depends on the time of field penetration. One should expect that the parameter $\xi = t^{1/(\kappa+2)} \cdot H_0^{\kappa/(\kappa+2)}/D$ will become an exact scaling variable only for the cases when all the parameters, $t$, $H_0$ and $D$, are simultaneously scaled by a constant $\lambda$ in the following way: $D \rightarrow \lambda \cdot D$, $H_0 \rightarrow (\lambda \cdot H_0)^{\kappa/(\kappa+2)}$ and $t \rightarrow (\lambda \cdot t)^{1/(\kappa+2)}$. 
Figure 3. The real component of the third harmonic susceptibility as a function of the amplitude of the ac field, computed for an ac field of period equal to 25000. The slope of solid lines decreases for increasing values of $\kappa$, which are the following: 0.667, 0.8, 1.2, 1.333, 1.667, 1.8, 2.333, 2.7, 3.35, 3.7. Similar scaling property is observed for the imaginary component of the third harmonic susceptibility and for higher-order components as well.

4. Conclusions

When the ac magnetic field does not penetrate to the sample center, the magnetic susceptibility is well described by a simple scaling relation: $\chi \sim \xi$, with $\xi = t^{1/(\kappa + 2)} \cdot H_0^{\kappa/((\kappa + 2))}/D$. For $\kappa$ values close to 0, the overall dependence of $\chi''(\chi')$ closely resembles the dependence observed in the linear case. The conductivity, however, computed from $\chi''(\chi')$ data by using the assumption that the linear theory holds, will lead to false information and yield strongly overestimated values. An experimental criterion for detecting nonlinearity would be the observation of the amplitude dependence or the existence of higher order harmonics in the ac response. The susceptibility values of a nonlinear medium are not uniquely defined by a set of two parameters only, such as $\kappa$ and $\xi$. For experimental purposes, however, treating $\xi$ as a scaling variable offers a sufficiently accurate method of testing for nonlinear properties of materials.
Figure 4. Scaling of the flux penetration \((4\pi\chi'_1 + 1)\) by a function \(H_0^{\kappa/(\kappa+2)} \cdot t_1^{1/(\kappa+2)}/D\), where \(H_0\) gives the field amplitude, \(t\) gives the period and \(D\) gives the sample thickness. Each of the solid lines passes through the data corresponding to the following values of the nonlinearity parameter \(\kappa\): 0.667, 1, 2, 3 and 5, for lines with the smallest to largest slope. \(D\) is equal to 20 or 100 (there is no distinction between the symbols of the data corresponding to different values of \(D\)), while \(t\) is 2500 (○), 5000 (□), 6250 (◇), 10000 (△), 12500 (▽), 25000 (●) and 100000 (□).

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