Entanglement and alpha entropies for a massive Dirac field in two dimensions

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Abstract

We present some exact results about universal quantities derived from the local density matrix $\rho$, for a free massive Dirac field in two dimensions. We first find $\text{tr}\rho^n$ in a novel fashion, which involves the correlators of suitable operators in the sine-Gordon model. These, in turn, can be written exactly in terms of the solutions of non-linear differential equations of the Painlevé $V$ type. Equipped with the previous results, we find the leading terms for the entanglement entropy, both for short and long distances, and showing that in the intermediate regime it can be expanded in a series of multiple integrals. The previous results have been checked by direct numerical calculations on the lattice, finding perfect agreement. Finally, we comment on a possible generalization of the entanglement entropy c-theorem to the alpha-entropies.

1 Introduction

The trace of the vacuum state projector over the degrees of freedom corresponding to a spatial region, results in a mixed density matrix with a non-vanishing 'geometric' entropy. This kind of construction was proposed by some authors [1, 2, 3, 4] in an attempt to explain the black hole entropy as some sort of entanglement entropy of the corresponding vacuum state. However, since the role of gravity cannot be ruled out, this identification is still a conjecture.

In recent years, there has been a renewed interest on the properties of local reduced density matrices, specially for low dimensional systems (see for example [5] [6] [7] [8] [9] [10], [11] [12], and references therein). This was partially motivated by developments in quantum information theory and also on the density matrix renormalization group method in two dimensions [13]. These investigations have made it manifest that the entanglement entropy, as well as other measures of information for the reduced density matrices of the vacuum state are interesting quantities by their own right. Besides, they can yield a different view on certain aspects of quantum field theory, since the objects one has to calculate are quite different to the standard ones. That means, for example, that there is a different structure of divergences, and a nice interplay between geometry and UV behaviour.

In this paper, we study measures of information for the local density matrix $\rho_A$ for a Dirac fermion in two dimensions. This is achieved by tracing the vacuum state over the degrees of freedom outside the set $A$. Specifically, we shall consider the $\alpha$-entropies

$$S_\alpha(A) = \frac{1}{1-\alpha} \text{log} \text{tr}(\rho_A^\alpha),$$

(1)
and the entanglement entropy
\[ S(A) = -\text{tr}(\rho_A \log \rho_A) = \lim_{\alpha \to 1} S_\alpha(A). \] (2)

These two functions of \( \rho_A \) are perfectly well defined on a lattice, but their continuum limit are plagued by UV divergences. In two dimensions, however, the form of those divergences is particularly simple, being just an additive constant proportional to the logarithm of the cutoff and to the number of boundary points in \( A \). Indeed, since the divergences have their origin in the UV fixed-point (where masses can be neglected), this follows from the result for the conformal case \[5, 14\]. Remarkably, for the entanglement entropy, it also follows from the strong subadditive property of the entropy, when the spatial symmetries are taken into account \[6\].

Several universal quantities can be easily obtained from \( S(A) \) and \( S_\alpha(A) \). In particular, when \( A \) is a single interval of length \( r \), we define the dimensionless functions
\[ c_\alpha(r) \equiv r \frac{dS_\alpha(r)}{dr}, \quad c(r) \equiv c_1(r). \] (3)
The function \( c(r) \) is always positive and decreasing, and we shall call it ‘entropic c-function’, since it plays the role of Zamolodchikov’s c-function in the entanglement entropy c-theorem \[6\]. All the universal information which can be obtained from \( S_\alpha(r) \) and \( S(r) \) is encoded in \( c_\alpha(r) \) and \( c(r) \).

For more general sets, formed by several disjoint intervals, universal quantities can be constructed through the mutual information function \( I \), which for two non-intersecting sets \( A, B \) is given by
\[ I(A, B) = S(A) + S(B) - S(A \cup B). \] (4)
Remarkably, \( I \) remains finite in higher dimensions\(^1\).

The traces of powers of \( \rho \) involved in the \( \alpha \)-entropies, with \( \alpha = n \in \mathbb{Z} \), can be represented by a functional integral with the fields defined on an \( n \)-sheeted surface with conical singularities located at the boundary points of the set \( A \). This integral is quite difficult to deal with, except for the conformal case, where that surface can be conveniently transformed.

We shall use here a novel approach (see section 2), whereby the problem in the \( n \)-covered plane is mapped to an equivalent one in which an external gauge field couples to an \( n \)-component fermion field (defined on the plane) and its role is to impose the correct boundary conditions through its vortex-like singularities. Then the resulting theory is bosonized, to express \( \text{tr} \rho^n \) as a sum of correlators of local operators.

In the massless case, discussed in section 3, the bosonized theory becomes a massless scalar field, and thus we could obtain the entropies explicitly, with results in agreement with the ones of \[5\]. As a by-product we derived a quite convenient expression for \( I(A, B) \).

For the massive case (section 4), the dual theory is instead a sine-Gordon model at the free fermion point. In this case, we still could evaluate \( \text{tr} \rho^n \) exactly for a single interval of size \( r \), by expressing it as a sum of \( n \) correlators for exponential operators. A method to find that type of correlators was (fortunately) already available in the literature \[15\], and it could be adapted to our case after some minor modifications. The outcome is an

\(^1\)Formulae \[3\] make sense only when a translation invariant cutoff is used. This apparent limitation could be overcome, in two dimensions, by defining \( c(r) \) through the function \( F(A, B) = S(A) + S(B) - S(A \cap B) - S(A \cup B) \) used in \[5\] for overlapping sets. However, \( F(A, B) \) is not finite, in general, for dimensions greater than 2.
Figure 1: The plane with cuts along the intervals \((u_i, v_i), i = 1, ..., p\). The circuits \(C_{u_i}\) and \(C_{v_i}\) are used in the text to discuss the boundary conditions.

exact expression for tr\(\rho^n\) in terms of the solutions of second order non linear differential equations. Also for the massive fermion case, we obtained expansions for small and large values of \(mr\) (\(m\) is the fermion mass), and by analytical continuation in \(n\) we calculated the corresponding expansions for the entropy.

This article concludes with a summary of our results (section 5) and a discussion on a possible generalization of the \(c\)-theorem to the \(\alpha\)-entropies. This conjecture naturally suggests itself by the results of the explicit calculations.

2 Reduced density matrix for a Dirac fermion

We consider the reduced density matrix for a free Dirac fermion, which is defined by tracing the vacuum state over the degrees of freedom lying outside a given set \(A\). We specify \(A\) as a collection of disjoint intervals \((u_i, v_i), i = 1, ..., p\) (see figure 1).

Following [14], the density matrix \(\rho(\Psi_{\text{in}}, \Psi_{\text{out}})\) can be written as a functional integral on the Euclidean plane with boundary conditions \(\Psi = \Psi_{\text{in}}\), and \(\Psi = \Psi_{\text{out}}\) along each side of the cuts \((u_i, v_i)\)

\[
\rho(\Psi_{\text{in}}, \Psi_{\text{out}}) = \frac{1}{Z[1]} \int D\Psi e^{-S[\Psi]},
\]

where \(Z[1]\) is a normalization factor, introduced in order to have tr\(\rho = 1\).

In order to obtain tr\(\rho^n\), we consider \(n\) copies of the cut plane, sewing together the cut \((u_i, v_i)_{\text{out}}^k\) with the cut \((u_i, v_i)_{\text{in}}^{k+1}\), for all \(i = 1, ..., p\), and the copies \(k = 1, ..., n\), where the copy \(n+1\) coincides with the first one. The trace of \(\rho^n\) is then given by the functional integral \(Z[n]\) for the field in this manifold,

\[
\text{tr} \rho^n = \frac{Z[n]}{Z[1]^n}.
\]

For a fermion field, we have to remember that the trace requires to introduce a minus sign in the path integral boundary condition between the fields along the first and the last cut [16], as it happens for the fermion thermal partition function in the Matsubara formalism [17]. Besides, in the present case, for each copy there is an additional factor \(-1\). This is due to the existence of a non trivial Lorentz rotation around the points \(u_i\) and \(v_i\) which is present in the Euclidean Hamiltonian when expressing tr\(\rho^n\) as a path integral [18]. From these considerations, we finally get a total factor \((-1)^{(n+1)}\) connecting the fields along the first cut \((u_i, v_i)_{\text{in}}^1\) and the last one \((u_i, v_i)_{\text{out}}^n\) in (6).

Rather than dealing with field defined on a non trivial manifold, we find it more convenient to work on a single plane, although with an \(n\)-component field

\[
\bar{\Psi} = \begin{pmatrix}
\Psi_1(x) \\
\vdots \\
\Psi_n(x)
\end{pmatrix},
\]
where $\Psi_l(x)$ is the field on the $l^{th}$ copy. Of course, the singularities at the boundaries are still there, and we shall show a simple way of taking them into account now.

Note that the space is simply connected but the vector $\vec{\Psi}$ is not single-valued. In fact, turning around any of the $C_u_i$ circuits (see figure 1) it is multiplied by a matrix $T$, and after turning around the $C_v_i$ circuit it gets multiplied by the inverse matrix $T^{-1}$. Here,

$$T = \begin{pmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & \ldots & \ldots & 0 \\ (-1)^{(n+1)} & 0 & \ldots & \ldots & 0 \end{pmatrix},$$

which has eigenvalues $e^{i \frac{k}{n} 2\pi}$, with $k = -\frac{(n-1)}{2}, -\frac{(n-1)}{2} + 1, \ldots, \frac{(n-1)}{2}$.

Then, changing basis by a unitary transformation in the replica space, we can diagonalize $T$, and the problem is reduced to $n$ decoupled fields $\Phi^k$ living on a single plane. These fields are multivalued, since when encircling $C_u_i$ or $C_v_i$ they are multiplied by $e^{i \frac{k}{n} 2\pi}$ or $e^{-i \frac{k}{n} 2\pi}$, respectively.

That multivaluedness can now be easily disposed of, at the expense of coupling single-valued fields $\Phi^k$ to an external gauge field which is a pure gauge everywhere, except at the points $u_i$ and $v_i$ where it is vortex-like. Thus we arrived to the Lagrangian density

$$L_k = \bar{\Phi}^k \gamma^\mu \left( \partial_\mu + i A^k_\mu \right) \Phi^k + m \bar{\Phi}^k \Phi^k.$$ (9)

Indeed, the reverse step would be to get rid of the gauge field $A_\mu$ by performing a singular gauge transformation

$$\Phi^k(x) \to e^{-i \int_{x_0}^x dx' A^k_\mu(x')} \Phi^k(x),$$ (10)

(where $x_0$ is an arbitrary fixed point). Since the transformation is singular, one goes back to a multivalued field. By the same token, we learn that in order to reproduce the boundary conditions on $\Phi^k$, we should have

$$\oint_{C_u_i} dx^\mu A^k_\mu(x) = -\frac{2\pi k}{n},$$ (11)

$$\oint_{C_v_i} dx^\mu A^k_\mu(x) = \frac{2\pi k}{n}.$$ (12)

As a side remark, we note that the addition of terms of the form $2\pi q$, with $q$ an integer, to the right hand side of the above formulae does not change the total phase factor along the circuits. However, the winding number of the phase, and thus the boundary conditions the gauge field imposes, would be different. In fact, the free energy does depend on these integers and choosing $q \neq 0$ would not select the vacuum state.

Equations (11) and (12) hold for any two circuits $C_u_i$ and $C_v_i$ containing $u_i$ and $v_i$ respectively. Thus:

$$e^{\mu \nu} \partial_\nu A^k_\mu(x) = 2\pi \frac{k}{n} \sum_{i=1}^{n} \left[ \delta(x - u_i) - \delta(x - v_i) \right],$$ (13)

where the presence of a vortex-antivortex pair for each $k$ and each interval is explicit. The functional integral is then factorized:

$$Z[n] = \prod_{k=-(n-1)/2}^{(n-1)/2} Z_k,$$ (14)
where $Z_k$ can be obtained as vacuum expectation values in the free Dirac theory

$$Z_k = \left< e^{i \int A_k^\mu \bar{j}_k^\mu d^2x} \right>, \quad (15)$$

where $j_k^\mu$ is the Dirac current, $A_k^\mu$ satisfies (13), and we adopted a normalization such that $\langle 1 \rangle = 1$.

### 3 Bosonization and the massless case

In order to evaluate (15), it is quite convenient to use the bosonization technique [19], to express the current $j_k^\mu$ as

$$j_k^\mu \rightarrow \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi,$$  \quad (16)

where $\phi$ is a real scalar field. For a free massless Dirac field, the theory for the dual field $\phi$ is simply

$$L_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi.$$  \quad (17)

Therefore we have to evaluate:

$$Z_k = \left< e^{i \int A_k^\mu \epsilon^{\mu\nu} \partial_\nu \phi d^2x} \right> = \left< e^{-i \sqrt{4\pi/k} \sum_{i=1}^{p} (\phi(u_i) - \phi(v_i))} \right>, \quad (18)$$

where the vacuum expectation values correspond to the theory (17). Since $L_\phi$ is quadratic

$$\left< e^{-i \int f(x)\phi(x)d^2x} \right> = e^{-\frac{1}{2} \int f(x)G(x-y)f(y)d^2xd^2y},$$  \quad (19)

with the correlator

$$G(x-y) = -\frac{1}{2\pi} \log |x-y|,$$  \quad (20)

it follows that (18) can be written as

$$\log Z_k = -\frac{2k^2}{n^2} \Xi(u_i, v_j), \quad (21)$$

$$\Xi(u_i, v_j) = \sum_{i,j} \log |u_i - v_j| - \sum_{i<j} \log |u_i - u_j| - \sum_{i<j} \log |v_i - v_j| - p \log \varepsilon. \quad (22)$$

Here $\varepsilon$ is a cutoff introduced to split the coincidence points, $|u_i - u_i|, |v_i - v_i| \to \varepsilon$. Summing over $k$ and using (1) and (2) we obtain

$$S_n = \frac{1}{1-n} \log (\text{tr} \rho^n) = \frac{1}{1-n} \sum_k \log Z_k = \frac{1}{6} n + \frac{1}{n} \Xi(u_i, v_j), \quad (23)$$

$$S = \frac{1}{3} \Xi(u_i, v_j). \quad (24)$$

This agrees exactly with the general formula for the entanglement entropy for conformal theories obtained in [5]. The general case differs from (24) on a global factor of the Virasoro central charge $C$, where $C = 1$ for the Dirac field.

Equation (24) has an interesting corollary: recalling the definition (4) for the mutual information, it follows that, for non-intersecting sets $A, B$ and $C$

$$I(A, B \cup C) = I(A, B) + I(A, C).$$  \quad (25)
That is, in contrast to the entropy, the mutual information is extensive (in each of the sets separately) in the conformal case. This curious property does not hold in the non conformal case or in more dimensions. It can be written as

\[ I(A, B) = \frac{1}{3} \int_A dx \int_B dy \frac{1}{(x - y)^2} \]

(26)

for any two non intersecting \( A \) and \( B \). This formula for \( I(A, B) \) shows explicitly its model independent properties, that is, it is cutoff independent, positive and monotonically increasing with \( A \) and \( B \).

For a single interval we have that

\[ c_\alpha = \frac{\alpha + 1}{6\alpha} \quad , \quad c = \frac{1}{3} \]

(27)

which are constants.

4 The massive case

We consider here a massive fermion, focusing on the case of a single interval of length \( r \).

We can still use \[16\] to deal with \[15\] for the partition function. Now, however, the bosonization of the massive fermion theory leads to a sine-Gordon Lagrangian \[19\]

\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + \Lambda \cos(\sqrt{4\pi} \phi) \right), \]

(28)

where \( \Lambda \) is a mass parameter. Then, as in the previous section we have

\[ \log \left( \text{tr} \rho^n(r) \right) = \sum_{k=-(n-1)/2}^{(n-1)/2} \log \left( e^{-i\sqrt{4\pi} a \phi(r)} e^{i\sqrt{4\pi} a \phi(0)} \right), \]

(29)

where now the expectation value is evaluated in the sine-Gordon theory at the free fermion point given by the Lagrangian \[28\]. The correlators of exponential operators were studied in ref. \[15\], where it was shown that \( \langle e^{i\sqrt{4\pi} a \phi(r)} e^{i\sqrt{4\pi} a' \phi(0)} \rangle \) for \( \alpha, \alpha' \in [0, 1] \) can be parametrized by a function satisfying a nonlinear second order differential equation of the Painlevé type. This is done by mapping the sum over the form factors into the determinant of a Fredholm operator. The general relation of this type of determinants and differential equations is studied in \[20\]. The result of \[15, 20\] can not be directly applied here, since we need the correlator for \( \alpha' = -\alpha \). This case can be dealt with through a minor modification in those results (see Appendix A for more details).

To proceed, one introduces the function

\[ w_a(x) = r \frac{d}{dr} \log \left( e^{-i\sqrt{4\pi} a \phi(r)} e^{i\sqrt{4\pi} a \phi(0)} \right), \]

(30)

with

\[ w_a(x) = -\int_{x}^{\infty} y v_a^2(y) dy, \]

(31)

where \( v_a \) satisfies

\[ v_a' + \frac{1}{x} v_a' = -\frac{v_a}{1 - v_a^2} \left( v_a' \right)^2 + v_a - v_a^3 + \frac{4a^2}{x^2} \frac{v_a}{1 - v_a^2}. \]

(32)
Figure 2: Solid lines are plots of $c_n(x)$, obtained by solving differential equations (31-34) numerically. The values of $n$ are, from top to bottom, $n = 2, 3, 5$ and 50. These functions take the value $(n + 1)/(6n)$ at the origin, which cumulates at $1/6 = 0.166\ldots$ for large $n$. For large $x$, they decay exponentially fast. Dotted lines correspond to the $c_n(x)$ that results from putting the model on a lattice, for $n = 2$ and $n = 3$. This points are evaluated for set sizes ranging from 200 to 600 lattice points and inverse mass values ranging from 200 to 3200 lattice units. The fact that these points, computed for several different lattice mass values, tend to lie in a single continuous curve shows the universal character of $c_n(x)$.

Here we have defined $x \equiv r m$, and the boundary condition for (32) is

$$v_a(x) \sim \frac{2}{\pi} \sin(\pi a) K_{2a}(x) \quad \text{as} \quad x \to \infty,$$

where $K_{2a}(x)$ is the standard modified Bessel function.

Thus, (31), (32) and (33) give the exact value of $c_n(r)$ in terms of a finite number of Painlevé type functions

$$c_n(x) = \frac{1}{1 - n} \sum_{k=-(n-1)/2}^{(n-1)/2} w_{k/n}(x).$$

The functions $c_n(x)$ are shown in figure (2) for some values of $n$. They take the massless case value $\frac{n+1}{6n}$ at $x = 0$, and lie between $1/6$ for $n = \infty$ and $1/4$ for $n = 2$. At large $x = rm$ they decay exponentially fast. Note that the case $n = 1$ corresponds to the entropic $c$-function, which at the origin takes the value $1/3$ (see figure 3).

We have also made a direct numerical evaluations of $c_n(x)$ on the lattice, with a method which is described in Appendix B. Figure 2 shows the results for $n = 2$ and $n = 3$, which match the exact theoretical values given by (34). For higher values of $n$, the approach to the continuum limit of $c_n(r)$ is slower, and agreement within few percent requires lattice sets bigger than 1000 points, already for $n = 4$.

In order to evaluate $c_n(r)$ for non integer $n$, or to compute the entropic $c$ function, we would need an analytical continuation in $n$ of (34). In what follows we obtain these quantities for expansions at long and short distances.
4.1 The long distance expansion

A naive expansion for large $x$ in (33) is not sufficient to evaluate the entropy, which requires the $\alpha \to 1$ limit of $c_{\alpha}$. Indeed, expansions for large $x$ can be obtained from (33) by using the asymptotic form of the Bessel function for large values of its argument

$$K_{\alpha}(x) \sim e^{-x} \left( \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{x}} + \sqrt{\frac{\pi}{2}} \frac{1}{8} (4a^2 - 1) \frac{1}{x^2} + \ldots \right).$$

This gives for the sum of $v^2$

$$\sum_{k=-(n-1)/2}^{(n-1)/2} v_{k/n}^2(x) \sim \left( \frac{2}{\pi} \right)^2 \sum_{k=-(n-1)/2}^{(n-1)/2} \sin^2(\pi \frac{k}{n}) K_{2\alpha}^2(r)$$

$$\sim e^{-2x} \left[ n \frac{1}{\pi x} + \left( \frac{1}{12} \frac{n^2 - 4}{\pi n} + \frac{2 \cos(\frac{\pi}{n})}{\pi n \sin^2(\frac{\pi}{n})} \right) \frac{1}{x^2} + \ldots \right].$$

Now we can apply this formula for non integer values of $n = \alpha$. This provides a good expansion for fixed $\alpha$ and sufficiently large $x$, from which the expansion for $c_{\alpha}(x)$ follows by term by term integration

$$c_{\alpha}(r) \sim \frac{e^{-2x}}{\alpha - 1} \left[ \frac{\alpha}{2\pi} + \left( \frac{1}{24} \frac{\alpha^2 - 4}{\pi \alpha} + \frac{\cos(\frac{\pi}{\alpha})}{\pi \alpha \sin^2(\frac{\pi}{\alpha})} \right) \frac{1}{x} + \ldots \right].$$

However, we see from the way the coefficients in the expansion depend on $\alpha$ that the series does not converge unless $x \gg (\alpha - 1)^{-1}$ for $\alpha$ approaching 1.

Thus, (37) cannot be used to compute the entropy. This can be repaired using directly the first term in the form factor expansion for the correlators (see Appendix A) or, equivalently, the integral representation for the Bessel function

$$K_{\alpha}(x) = \int_1^\infty du e^{-xu} \frac{(u + \sqrt{u^2 - 1})^\alpha + (u + \sqrt{u^2 - 1})^{-\alpha}}{2\sqrt{u^2 - 1}}$$

in expression (36) and summing over $k$ inside the double integral. We have the following asymptotic expression

$$c_{\alpha}(x) \sim -\frac{2}{\pi^2} \int_x^\infty dy \int_1^\infty du \int_1^\infty dv \frac{e^{-y(u+v)}}{\sqrt{u^2 - 1}\sqrt{v^2 - 1}} \times$$

$$\times \left( F_{\alpha} \left( \left( u + \sqrt{u^2 - 1} \right) \left( v + \sqrt{v^2 - 1} \right) \right) + F_{\alpha} \left( \frac{u + \sqrt{u^2 - 1}}{v + \sqrt{v^2 - 1}} \right) \right),$$

where

$$F_{\alpha}(z) = \frac{1}{4(1-\alpha)} \left( z - \frac{1}{z} \right) \left( \frac{2 \cos(\frac{\pi}{\alpha})}{z^{\frac{1}{\alpha}} - z^{-\frac{1}{\alpha}}} + \frac{2}{z^{\frac{2}{\alpha}} - 2 \cos(\frac{2\pi}{\alpha})} \right).$$

Surprisingly enough, this function is proportional to a delta function in the limit $\alpha \to 1$. Indeed, for a fixed $z \neq 1$, $F_{\alpha}(z) \to 0$ as $\alpha \to 1$. However, for $z = 1$ this limit is singular. Around $z = 1$ and $\alpha = 1$ the singularity can be isolated by using polar coordinates in the $(\alpha - 1, z - 1)$ plane and expanding in the radial coordinate. The result is

$$F_{\alpha}(z) = -\pi^2 \frac{(\alpha - 1)}{2 \pi^2(\alpha - 1)^2 + (z - 1)^2} (1 + O((z - 1), (\alpha - 1))).$$
Figure 3: The dotted curve is the function \( c(r) \) evaluated on a lattice, with points obtained for different values of the mass and lattice distance (see caption of figure (2)). The continuity of the plot agrees with the universal character of \( c(r) \), which is in fact a function of \( x = mr \). The solid-line curves are the short and long distance leading terms we evaluated analytically.

Thus, we have

\[
\lim_{\alpha \to 1} F_\alpha(z) = -\frac{\pi^2}{2} \delta(z - 1),
\]

(41)

which yields

\[
c(x) \sim - \int_x^\infty dy \, y K_0(2y) = \frac{1}{2} x \, K_1(2x),
\]

(42)

in perfect agreement with the numerical results (see figure (3)).

This term corresponds to the first one in the form factor expansion, which is due to one soliton-antisoliton pair. In the fermion language it is the contribution from a single fermion-antifermion pair. It is interesting to note that (42) is \( 1/4 \) at the origin, and thus a single fermion-antifermion pair contributes more than 75% to the entanglement entropy function \( c(r) \) at all scales.

More terms in the expansion for the entropy can be written in terms of multiple integrals, by using the form factor series for the correlators (55). First we expand the logarithm of the correlator order by order, and then sum over \( k \) inside the integrals. This is easily done since we have a formula analogous to (41)

\[
\lim_{n \to 1} \frac{1}{1 - n} \sum_{k = -(n-1)/2}^{k=(n-1)/2} \sin^2 q \left( \frac{k}{n} \pi \right) z^{\frac{2k}{n}} = -\frac{\pi^2}{2} \Gamma \left( q - \frac{1}{2} \right) \delta(z - 1).
\]

(43)

We do not write explicitly more terms of the series here, since the expression for the multiple integrals, though straightforward to obtain, do not seem to be particularly illuminating. Besides they are difficult to evaluate numerically.
4.2 The short distance expansion

Close to the conformal limit, the best way to expand $v_a(x)$ is by a direct use of the differential equations. We have the series solution of (32) around the origin

$$v_a(x) = -2a \log(x) + b + x^2 \left( \frac{1}{4} (2a - 8a^3 + b - 8a^2b - 4ab^2 - b^3) + \frac{1}{2} (-a + 8a^3 + 8a^2b + 3ab^2) \log(x) + (-4a^3 - 3a^2b) \log^2(x) + 2a^3 \log^3(x) \right) + O(x^4) \quad (44)$$

which is of the general form

$$v_a(x) = \sum_{s=0}^{\infty} x^{2s} \sum_{t=0}^{2s+1} f_{s,t} \log^t(x). \quad (45)$$

The full expansion requires only the knowledge of the constant term $f_{0,0} = b$. It might be absorbed into the logarithm, replacing $\log(x)$ by $\log(xe^{-\frac{b}{2a}})$ and setting $b = 0$ everywhere in the above expression. Unfortunately we do not know the general expression for $b$ as a function of $a$. An exception is the case $a = 1/2$, where $b = -1 - \gamma_E + 3 \log(2)$, and $\gamma_E$ is the Euler constant. This follows from the results for the Ising model correlators [21] (see Appendix A). Here we consider only the leading term which is independent of $b$. The integration constant for $w_a$ is given by the conformal limit of section 2, $w_a(0) = -2a^2$. Thus we have

$$w_a = -2a^2 + 2a^2 x^2 \log^2(x) + O(x^2 \log(x)), \quad (46)$$

$$c_\alpha(x) = \frac{\alpha + 1}{6\alpha} (1 - x^2 \log^2(x)) + O(x^2 \log(x)), \quad (47)$$

$$c(x) = \frac{1}{3} - \frac{1}{3} x^2 \log^2(x) + O(x^2 \log(x)) \quad (48)$$

5 Final remarks

To summarize the results, we have found the exact function $c_\alpha(r)$ for a massive fermion field in two dimensions for integer values of $\alpha = n$. It is given in terms of the solutions of non linear differential equations. This completely determines the $\alpha$-entropies for integer $\alpha$, except for a non-universal, ultraviolet divergent, (additive) constant. We have also found the leading terms of the entanglement entropy $c$-function $c(r)$ for short and large distances showing how it can be expressed as a series of multiple integrals. In the massless case, we have calculated the entanglement and the $\alpha$-entropies exactly for an arbitrary set, with results that coincide with the ones of [5]. Surprisingly, the mutual information turns out to be extensive in the massless case.

There are also other interesting universal quantities which can be derived from the ones above. For example, the $\alpha$-entropies increase logarithmically in the conformal limit,

$$S_\alpha(r) = \frac{\alpha + 1}{6\alpha} \log(r) + k_0, \quad (49)$$

and saturate for $rm \gg 1$,

$$S_\alpha(r) \to k_\infty. \quad (50)$$
Although the saturation constant $k_\infty$ is cutoff dependent, its dependence on the mass is universal and can be expressed in terms of $c_\alpha$. Defining an interpolating function

$$k(r) = s_\alpha(r) - c_\alpha(r) \log(r),$$

we have $k(0) = k_0$, $k(\infty) = k_\infty$. Using $k'(r) = -c'_\alpha(r) \log(r)$ it follows that

$$\Delta k = \int_0^\infty k'(r)dr = -\int_0^\infty \log(r)c'_\alpha(mr)d(mr) = -\Delta c_\alpha \log(m) + \text{const}.$$  \hspace{1cm} (52)

This is the result obtained in [5] from the analysis of the properties of the energy momentum tensor on a conical space.

There is a very interesting point arising from the equation (31) which gives $w_\alpha$ as an integral of an explicitly negative quantity. This implies through equation (34) that the tum tensor on a conical space. This is the result obtained in [5] from the analysis of the properties of the energy momentum tensor on a conical space.

Remarkably, in that case each $c_\alpha$ would lead to an alternative form of the c-theorem in two dimensions. Note that the $c_\alpha(r)$ are universal quantities which have fixed point values proportional to the Virasoro central charge.

Recently, a different conjecture which also involves the renormalization group flow and the reduced density matrices has been presented in a series of papers [7, 8, 9, 10]. Those authors proposed several interesting majorization relations for the local density matrices and connect them to the renormalization group irreversibility. One of the implications of this proposal is

$$S_\alpha(r_1) > S_\alpha(r_2) \text{ for } r_1 > r_2.$$  \hspace{1cm} (54)

This means $c_\alpha(r) > 0$ in our notation. We note that, at least for the entanglement entropy, this relation follows from translation invariance only [22], while (53) with $\alpha = 1$ requires the full Poincaré group symmetry, which is an essential ingredient for the c-theorem [23].

6 Appendix A: Correlator of exponential operators in the sine-Gordon model

The two-point correlation function of [30] can be expanded as a sum over form factors as follows [15]

$$\langle : e^{i\sqrt{4\pi}a\phi(r)} : e^{i\sqrt{4\pi}a'\phi(0)} : \rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_0^\infty du_1...du_{2n} \left( \prod_{i=1}^{2n} e^{-mr(u_i + 1/u_i)} \right) \times f_\alpha(u_1, ..., u_{2n}) f_{\alpha'}(u_{2n}, ..., u_1),$$  \hspace{1cm} (55)

where $f_\alpha(u_1, ..., u_{2n})$ is the form factor

$$f_\alpha(u_1, ..., u_{2n}) = (-1)^n(n-1)/2 \left( \frac{\sin(\pi a)}{i\pi} \right)^n \left( \prod_{i=1}^{n} \left( \frac{u_{i+n}}{u_i} \right)^a \right) \times \Delta(u_1, ..., u_{2n}),$$  \hspace{1cm} (56)
and
\[ \Delta(u_1, ..., u_{2n}) = \frac{\prod_{i<j\leq n}(u_i - u_j) \prod_{n+1\leq i<j}(u_i - u_j)}{\prod_{r=1}^{n} \prod_{s=n+1}^{2n}(u_r + u_s)}. \] (57)

This series can be expressed as a Fredholm determinant [15]
\[ \langle : e^{i\sqrt{4\pi a_0}(r)} :: e^{i\sqrt{4\pi a_0'}(0)} : \rangle = \det(1 - \lambda^2 R_{a-a'} R^{T}_{a-a'}), \] (58)

where \( R_\theta \) is an integral operator on the half-line \((0, \infty)\) with kernel
\[ \left( \frac{u}{v} \right)^{\theta/2} e^{\left( -\frac{\pi}{4}(u+u^{-1}+v+v^{-1}) \right)} \frac{u+v}{u+v}, \] (59)

\( R^{T}_{\theta} \) is the transpose of \( R_\theta \), \( x = mr \), and
\[ \lambda = \frac{1}{\pi} \left( \sin(\pi a) \sin(\pi a') \right)^{1/2}. \] (60)

Defining
\[ \tau = \log \det(1 - \lambda^2 R_{\theta} R^{T}_{\theta}) \] (61)
we have the following differential equations [20]
\[ \frac{d^2 \tau}{dx^2} + \frac{1}{x} \frac{d\tau}{dx} = -d^2, \] (62)
\[ d'' + \frac{1}{x} d' = \frac{d}{1 + d^2} \left( d' \right)^2 + \frac{\theta^2}{x^2} \frac{d}{1 + d^2}, \] (63)

with boundary condition
\[ d(x, \lambda) \sim 2\lambda K_{\theta}(x) \quad \text{as} \quad x \to \infty. \] (64)

In the case relevant for this work \( a' = -a \), \( \theta = 2a \) and
\[ \lambda = \frac{i}{\pi} \sin(\pi a) \] (65)

is purely imaginary. This translates into a purely imaginary \( d \). Thus, defining \( v_a = -i d \) equations (31), (32) and (33) follow.

Equation (63) has appeared in the literature in different forms. It adopts the standard Painlevé V form [24] by the transformation
\[ d = i \frac{1 + h}{1 - h}, \] (66)

which leads to
\[ h'' + \frac{1}{h} h' = \left( \frac{1}{2h} + \frac{1}{h-1} \right) \left( h' \right)^2 + \frac{\theta^2}{8} \frac{(h-1)^2}{x^2} \left( h - \frac{1}{h} \right) + \frac{2h(h+1)}{(h-1)}. \] (67)

Also, by the transformation \( d = \sinh(f) \) we get the equation
\[ f'' + \frac{1}{x} f' = \frac{1}{2} \sinh(2f) + \frac{\theta^2}{x^2} \frac{\tanh(f)}{\cosh^2(f)}, \] (68)
which differs however from the equation given in [15] by a factor 4 in the last term. It can be checked that the well-known differential equations for the Ising model spin and disorder correlators [21, 25], \( \langle \sigma(r)\sigma(0) \rangle \) and \( \langle \mu(r)\mu(0) \rangle \), can be obtained from the differential equations (62) and (63) for the sine-Gordon correlators through the identification [26]

\[
\langle \sigma(r)\sigma(0) \rangle^2 = \left\langle \sin\left(\frac{1}{2}\sqrt{4\pi}\phi(r)\right)\sin\left(\frac{1}{2}\sqrt{4\pi}\phi(0)\right) \right\rangle,
\]

\[
\langle \mu(r)\mu(0) \rangle^2 = \left\langle \cos\left(\frac{1}{2}\sqrt{4\pi}\phi(r)\right)\cos\left(\frac{1}{2}\sqrt{4\pi}\phi(0)\right) \right\rangle.
\]

7 Appendix B: Numerical evaluation in the lattice

The vacuum expectation value \( \langle O_A \rangle \) for any operator \( O_A \) localized inside a region \( A \) must coincide with \( \text{tr}(\rho_A O_A) \), where \( \rho_A \) is the local density matrix. This fact was used in [27] to give an expression for \( \rho_A \) in terms of correlators for free Boson and Fermion discrete systems. We use this method here to compute the entanglement entropy for a free Dirac field on a lattice.

Consider a lattice Hamiltonian of the form

\[
\mathcal{H} = \sum_{i,j} M_{ij} c_i^\dagger c_j,
\]

where the creation and annihilation operators \( c_i^\dagger, c_j \) satisfy the anticommutation relations \( \{c_i^\dagger, c_j\} = \delta_{ij} \). Let the correlator \( C_{ij} = \left\langle c_i^\dagger c_j \right\rangle \) be taken on any given fixed eigenvector of the Hamiltonian and call \( C_A^{ij} \) the correlator matrix restricted to \( A \). Then the reduced density matrix \( \rho_A \) has the form

\[
\rho_A = \prod_{l} \rho_l = \prod_{l} \frac{e^{-\epsilon_l} d_l^\dagger d_l}{1 + e^{-\epsilon_l}},
\]

where the \( d_l \) are independent fermion annihilation operators which can be expressed as linear combination of the \( c_i \) and \( c_i^\dagger \). The \( \epsilon_l \) are related to the eigenvalues \( \nu_l \) of \( C^A \) by

\[
e^{-\epsilon_l} = \frac{\nu_l}{1 - \nu_l}.
\]

Hence both the entropy and \( \log \text{tr}\rho^n \) can be evaluated as \( S_A = \Sigma S_l \) and \( \log \text{tr}\rho^n = \Sigma \log \text{tr}\rho_l^n \), where

\[
S_l = \log(1 + e^{-\epsilon_l}) + \epsilon_l e^{-\epsilon_l} = -(1 - \nu_l) \log(1 - \nu_l) - \log(\nu_l) \nu_l,
\]

\[
\log \text{tr}\rho_l^n = \log(1 + e^{-n\epsilon_l}) - n \log(1 + e^{-\epsilon_l}) = \log((1 - \nu_l)^n + \nu_l^n).
\]

We are interested in the vacuum (half filled) state of a Hamiltonian with symmetric spectrum around the origin. In this case

\[
C = \theta(-M),
\]

where \( \theta(x) = (1 + \text{sign}(x))/2 \).
The lattice Hamiltonian for a Dirac fermion follows from the discretization of the Hamiltonian for a Majorana field, to cope with fermion doubling. This is given by

\[ H = -\frac{i}{2} \sum_{n=0}^{N-1} (\Psi_{n+1}^1 \Psi_n^1 + \Psi_n^2 \Psi_{n+1}^2) + i m \sum_{n=0}^{N-1} \Psi_n^1 \Psi_n^2, \tag{77} \]

with \( \Psi_n^i = \Psi_{i,n}^i \), \( \Psi_N^i = \Psi_{i,0}^i \) and \( \{\Psi_{i,n}^i, \Psi_j^j\} = \delta_{mn} \delta_{ij} \). We have taken \( N \) lattice sites and set the lattice spacing to one. Redefining the fermionic operators as

\[ c_{2k} = \frac{1}{2} (\Psi_{2k}^1 + i \Psi_{2k}^2), \tag{78} \]
\[ c_{2k+1} = \frac{1}{2} (\Psi_{2k+1}^1 - i \Psi_{2k+1}^2); \tag{79} \]

with \( \{c_n, c_m\} = \delta_{mn} \) we have a Hamiltonian in the form

\[ H = -\frac{i}{2} \sum_{n=0}^{N-1} (c_{n+1}^1 c_n^1 - c_n^1 c_{n+1}^1) + m \sum_{n=0}^{N-1} (-1)^n c_n^1 c_n \tag{80} \]

with

\[ M_{ij} = -\frac{i}{2} (\delta(i, j - 1) - \delta(i, j + 1)) + m \delta(i, j) (-1)^n. \tag{81} \]

The correlators for the infinite lattice limit \( N \to \infty \) are then

\[ \langle c_i^1 c_j^1 \rangle = \frac{1}{2} \delta(i-j), 0 + (-1)^i \int_0^{1/2} dx \frac{m \cos(2\pi x(i-j))}{\sqrt{m^2 + \sin(2\pi x)^2}} \quad \text{for } i-j \text{ even}, \tag{82} \]
\[ \langle c_i^1 c_j^1 \rangle = i \int_0^{1/2} dx \frac{\sin(2\pi x)}{\sqrt{m^2 + \sin(2\pi x)^2}} \sin(2\pi x(i-j)) \quad \text{for } i-j \text{ odd}. \tag{83} \]

Similar expressions where used in \[8\].

From (74) and (75) we can obtain \( c_n(r) \) and \( c(r) \) by taking numerical derivatives. We have used

\[ f' \left( q + \frac{1}{2} \right) = \frac{1}{4} (f(q+2) + f(q+1) - f(q) - f(q-1)). \tag{84} \]

This form for the derivative smoothes small oscillations which appear for quantities depending on a lattice size \( q \) when evaluated between even an odd adjacent values of \( q \).

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