A series expansion for generalized harmonic functions

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Abstract
We consider a class of generalized harmonic functions in the open unit disc in the complex plane. Our main results concern a canonical series expansion for such functions. Of particular interest is a certain individual generalized harmonic function which suitably normalized plays the role of an associated Poisson kernel.

Keywords Harmonic function · Power series · Poisson kernel · Hypergeometric function

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0 Introduction

Let \( \mathbb{D} \) be the open unit disc in the complex plane \( \mathbb{C} \) and denote by \( \partial_z = \partial / \partial z \) and \( \bar{\partial}_z = \partial / \partial \bar{z} \) the usual complex partial derivatives. This work is concerned with second order partial differential operators of the form

\[
L_{p,q} = (1 - |z|^2)\partial_z \bar{\partial}_z + pz\partial_z + q\bar{z}\bar{\partial}_z - pq, \quad z \in \mathbb{D},
\]

(0.1)

where \( p, q \in \mathbb{C} \) are complex parameters. Of particular interest are solutions of the associated homogeneous equation

\[
L_{p,q}u = 0 \quad \text{in} \ \mathbb{D}.
\]

We say that a function \( u \) is \((p, q)\)-harmonic if \( u \) is twice continuously differentiable in \( \mathbb{D} \) (in symbols \( u \in C^2(\mathbb{D}) \)) and \( L_{p,q}u = 0 \) in \( \mathbb{D} \), where \( L_{p,q} \) is as in (0.1). Notice
that a function \( u \) is \((p, q)\)-harmonic if and only if its complex conjugate \( \bar{u} \) is \((\bar{q}, \bar{p})\)-harmonic. Observe also that a \((0, 0)\)-harmonic function is a harmonic function in \( \mathbb{D} \) in the usual sense.

An interesting example of a \((p, q)\)-harmonic function is the function

\[
u_{p, q}(z) = \frac{(1 - |z|^2)^{p+q+1}}{(1 - z)^{p+1}(1 - \bar{z})^{q+1}}, \quad z \in \mathbb{D}
\]

(see Theorem 1.4). Here powers are defined in the usual way using the principal branch of the logarithm, that is, we require that \( \log(1) = 0 \). Notice that the above functions \( u_{p, q} \) have the hermitian symmetry property that \( \bar{u}_{p, q} = u_{\bar{q}, \bar{p}} \) for \( p, q \in \mathbb{C} \).

Recall that elements of the unit circle \( T = \partial \mathbb{D} \) act on the unit disc \( \mathbb{D} \) as rotations.

On a function level we consider rotation operators

\[
R_{e^{i\theta}}u(z) = u(e^{i\theta}z), \quad z \in \mathbb{D},
\]

for \( e^{i\theta} \in T \) acting on functions \( u \) in \( \mathbb{D} \). A basic observation concerning the differential operator \( L_{p, q} \) is the commutativity relation

\[
L_{p, q}R_{e^{i\theta}}u = R_{e^{i\theta}}L_{p, q}u, \quad u \in C^2(\mathbb{D}),
\]

for \( e^{i\theta} \in T \). This latter commutativity relation suggests an analysis of \((p, q)\)-harmonic functions using concepts natural to classical Fourier analysis on the unit circle.

Let \( \mathbb{Z} \) be the set of integers. For a suitably smooth function \( u \) in \( \mathbb{D} \) we define its \( m \)-th homogeneous part by the formula

\[
u_m(z) = \frac{1}{2\pi} \int_T e^{-im\theta} u(e^{i\theta}z) d\theta, \quad z \in \mathbb{D},
\]

for \( m \in \mathbb{Z} \). Notice that the \( m \)-th homogeneous part \( u_m \) of \( u \) is the \( m \)-th Fourier coefficient of the vector-valued function

\[
T \ni e^{i\theta} \mapsto R_{e^{i\theta}}u,
\]

where \( R_{e^{i\theta}} \) is as in (0.3).

We set \( C^\infty(\mathbb{D}) = \bigcap_{n=0}^{\infty} C^n(\mathbb{D}) \), where \( C^n(\mathbb{D}) \) is the space of \( n \)-times continuously differentiable functions in \( \mathbb{D} \) for \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \). We topologize these spaces in the usual way using the semi-norms

\[
\|u\|_{j, k; K} = \max_{z \in K} |\partial^j \bar{\partial}^k u(z)|,
\]

where \( j, k \in \mathbb{N} \) are non-negative integers and \( K \subset \mathbb{D} \) is a compact subset of \( \mathbb{D} \).

Let us recall the classical hypergeometric function defined by

\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad z \in \mathbb{D},
\]

(0.5)
for $a, b, c \in \mathbb{C}$ such that $c \neq -1, -2, \ldots$. Here $(a)_0 = 1$ and

$$(a)_n = a(a + 1) \ldots (a + n - 1)$$

for $n = 1, 2, \ldots$ are Pochhammer symbols.

Let us return to a $(p, q)$-harmonic function $u$. We show that the $m$-th homogeneous part of $u$ has the form

$$u_m(z) = c_m F(-p, m - q; m + 1; |z|^2)z^m, \quad z \in \mathbb{D},$$

for some $c_m \in \mathbb{C}$ when $m \in \mathbb{N}$ is a non-negative integer (see Theorem 4.3). Here $F$ is the hypergeometric function (0.5). Hermitian symmetry leads to a similar formula for $u_m$ when $m \in \mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{N}$ is a negative integer (see Corollary 4.4). Notice that $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$.

A principal result of the present paper concerns the asymptotic behavior of the $m$-th homogeneous part $u_m$ of a $(p, q)$-harmonic function $u$. We show that

$$\limsup_{|m| \to \infty} \frac{\|u_m\|}{|m|^{1/jkK}} < 1$$

for all $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact (see Theorem 4.6). This result enables us to use the classical root test to establish absolute convergence of the function series $\sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$ (see Corollary 4.7).

A further analysis leads to a function series characterization of $(p, q)$-harmonic functions. A function $u$ in $\mathbb{D}$ is $(p, q)$-harmonic if and only if it has the form

$$u(z) = \sum_{m=0}^{\infty} c_m F(-p, m - q; m + 1; |z|^2)z^m$$

$$+ \sum_{m=1}^{\infty} c_{-m} F(-q, m - p; m + 1; |z|^2)\bar{z}^m, \quad z \in \mathbb{D}, \quad (0.6)$$

for some sequence $\{c_m\}_{m=-\infty}^{\infty}$ of complex numbers such that

$$\limsup_{|m| \to \infty} |c_m|^{1/|m|} \leq 1$$

(see Theorem 5.1). As mentioned above, the sums in (0.6) are absolutely convergent in the space $C^\infty(\mathbb{D})$. As a consequence we have that $u \in C^\infty(\mathbb{D})$ if $u$ is $(p, q)$-harmonic (see Corollary 4.9). This characterization of $(p, q)$-harmonic functions improves on a result by Ahern et al. [1, Theorem 2.1] when specialized to our setting.

We then turn to coefficient formulas for $(p, q)$-harmonic functions. We show that

$$c_m = \partial^m u(0)/m! \quad \text{and} \quad c_{-m} = \bar{\partial}^m u(0)/m!$$
for \( m \in \mathbb{N} \), where the \( c_m \)’s are as in (0.6) (see Theorem 5.3). As a consequence, we have that

\[
\begin{align*}
  u(z) &= \sum_{m=0}^{\infty} \frac{\partial^m u(0)}{m!} F(-p, m - q; m + 1; |z|^2) z^m \\
  &+ \sum_{m=1}^{\infty} \frac{\partial^m u(0)}{m!} F(-q, m - p; m + 1; |z|^2) z^m,
\end{align*}
\]

whenever \( u \) is a \((p, q)\)-harmonic function (see Theorem 5.4) as well as a corresponding uniqueness result for such functions (see Corollary 5.5).

Let us return to the function \( u_{p, q} \) in (0.2). A calculation of partial derivatives at the origin of the function \( u_{p, q} \) leads to the series expansion

\[
\begin{align*}
  u_{p, q}(z) &= \sum_{m=0}^{\infty} \frac{(p + 1)m}{m!} F(-p, m - q; m + 1; |z|^2) z^m \\
  &+ \sum_{m=1}^{\infty} \frac{(q + 1)m}{m!} F(-q, m - p; m + 1; |z|^2) z^m,
\end{align*}
\]

(see Theorem 6.3). This latter series expansion generalizes a well-known partial fraction decomposition formula for the classical Poisson kernel for \( \mathbb{D} \) which is obtained for \( p = q = 0 \).

A main contribution of this paper concerns series expansion of \((p, q)\)-harmonic functions. Of particular mention is a limit theorem for associated hypergeometric functions:

\[
\lim_{m \to \infty} F(-p, m - q; m + 1; z) = (1 - z)^p,
\]

(see Theorem 2.6). Apart from its intrinsic interest, this limit theorem provides an efficient tool for the study of limit properties of homogeneous parts of \((p, q)\)-harmonic functions.

The results of this paper have applications to Poisson integral representations of \((p, q)\)-harmonic functions which is possible when \( p, q \in \mathbb{C} \setminus \mathbb{Z} \) are such that \( \text{Re}(p) + \text{Re}(q) > -1 \). In the final section of the paper we comment briefly on the connection to such theory.

We have traced the study of \((p, q)\)-harmonic functions back to Daryl Geller [13]. Other significant contributions are those of Ahern and collaborators [1,2]. Our interest in those topics [18–21] arose in connection to standard weights and earlier work of Garabedian [12]. Borichev and Hedenmalm [6] have established a connection to polyharmonic theory. Other recent related papers are [5,8–10,14,16,17,22].

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The function $u_{p,q}$ is $(p, q)$-harmonic

Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Observe that the functions $u_{p,q}$ have the hermitian symmetry property $\bar{u}_{p,q} = u_{\bar{q}, \bar{p}}$. Indeed,

$$\bar{u}_{p,q}(z) = \frac{(1 - |z|^2)\bar{p} + \bar{q} + 1}{(1 - \bar{z})\bar{p} + 1(1 - \bar{z})\bar{q} + 1} = u_{\bar{q}, \bar{p}}(z)$$

for $z \in \mathbb{D}$. We shall discuss in this section some differentiation formulas for the functions $u_{p,q}$. In particular, we shall show that the function $u_{p,q}$ is $(p, q)$-harmonic (see Theorem 1.4).

**Lemma 1.1** Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then

$$z\partial u_{p,q}(z) = \left( - (p + q + 1) \frac{|z|^2}{1 - |z|^2} + (p + 1) \frac{z}{1 - z} \right) u_{p,q}(z)$$

for $z \in \mathbb{D}$.

**Proof** Differentiating using the product rule for differentiation we have that

$$\partial u_{p,q}(z) = \frac{1}{(1 - \bar{z})q + 1} \left( (p + q + 1)(1 - |z|^2)^{p+q}(-\bar{z}) \frac{1}{(1 - z)^p + 1} ight.
+ (1 - |z|^2)^{p+q+1}(-(p + 1)) \frac{1}{(1 - z)^q + 1} \left. (-1) \right)
= \left( - (p + q + 1) \frac{\bar{z}}{1 - |z|^2} + (p + 1) \frac{1}{1 - z} \right) u_{p,q}(z)$$

for $z \in \mathbb{D}$, where the last equality is straightforward to check. This yields the conclusion of the lemma. \(\square\)

We next turn to the $\bar{\partial}$-derivative of $u_{p,q}$.

**Lemma 1.2** Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then

$$\bar{z}\bar{\partial} u_{p,q}(z) = \left( - (p + q + 1) \frac{|z|^2}{1 - |z|^2} + (q + 1) \frac{z}{1 - \bar{z}} \right) u_{p,q}(z)$$

for $z \in \mathbb{D}$.

**Proof** We shall use the hermitian symmetry property $\bar{u}_{p,q} = u_{\bar{q}, \bar{p}}$. From Lemma 1.1 we have that

$$z\partial u_{\bar{q}, \bar{p}}(z) = \left( - (\bar{p} + \bar{q} + 1) \frac{|z|^2}{1 - |z|^2} + (\bar{q} + 1) \frac{z}{1 - \bar{z}} \right) u_{\bar{q}, \bar{p}}(z)$$

for $z \in \mathbb{D}$. A complex conjugation now yields the conclusion of the lemma. \(\square\)
Following earlier practice from [20] we denote by \( A = z\partial - \bar{z}\bar{\partial} \) the angular derivative. We next calculate the angular derivative of \( u_{p,q} \).

**Corollary 1.3** Let \( u_{p,q} \) be as in (0.2) for some \( p, q \in \mathbb{C} \). Then

\[
(z\partial - \bar{z}\bar{\partial})u_{p,q}(z) = \left((p + 1)\frac{z}{1 - z} - (q + 1)\frac{\bar{z}}{1 - \bar{z}}\right)u_{p,q}(z)
\]

for \( z \in \mathbb{D} \).

**Proof** The result is evident from Lemmas 1.1 and 1.2. \( \square \)

We mention that Corollary 1.3 generalizes [20, Theorem 1.11].

Recall the partial fraction formula

\[
\frac{1 - |z|^2}{|1 - z|^2} = \frac{z}{1 - z} + \frac{\bar{z}}{1 - \bar{z}} + 1 \quad (1.1)
\]

for the classical Poisson kernel for the unit disc. Formula (1.1) is straightforward to check.

**Theorem 1.4** Let \( u_{p,q} \) be as in (0.2) for some \( p, q \in \mathbb{C} \). Then \( L_{p,q}u_{p,q} = 0 \) in \( \mathbb{D} \), where \( L_{p,q} \) is as in (0.1).

**Proof** Recall Lemma 1.2. Notice that the differential operator \( z\partial \) satisfies the product rule for differentiation. It is straightforward to check that

\[
z\frac{\partial}{\partial z} \left( \frac{|z|^2}{1 - |z|^2} \right) = \frac{|z|^2}{(1 - |z|^2)^2}
\]

whenever the formula makes sense. Differentiating using the product rule we have that

\[
|z|^2\partial\bar{\partial}u_{p,q}(z) = -(p + q + 1)\frac{|z|^2}{(1 - |z|^2)^2}u_{p,q}(z)
\]

\[
+ \left(- (p + q + 1)\frac{|z|^2}{1 - |z|^2} + (q + 1)\frac{\bar{z}}{1 - \bar{z}}\right)z\partial u_{p,q}(z)
\]

for \( z \in \mathbb{D} \). From Lemma 1.1 we now have that

\[
|z|^2\partial\bar{\partial}u_{p,q}(z) = -(p + q + 1)\frac{|z|^2}{(1 - |z|^2)^2}u_{p,q}(z)
\]

\[
+ \left(- (p + q + 1)\frac{|z|^2}{1 - |z|^2} + (q + 1)\frac{\bar{z}}{1 - \bar{z}}\right)
\]

\[
\times \left(- (p + q + 1)\frac{|z|^2}{1 - |z|^2} + (p + 1)\frac{z}{1 - z}\right)u_{p,q}(z)
\]
for $z \in \mathbb{D}$. Expanding the above product we see that

$$|z|^2 \partial \bar{z} u_{p,q}(z) = \left( - (p + q + 1) \frac{|z|^2}{1 - |z|^2} + (p + q + 1)^2 \frac{|z|^4}{(1 - |z|^2)^2} \right) u_{p,q}(z)$$

$$- (p + q + 1) \frac{|z|^2}{1 - |z|^2} \left( (p + 1) \frac{z}{1 - z} + (q + 1) \frac{\bar{z}}{1 - \bar{z}} \right) u_{p,q}(z)$$

$$+ (p + 1)(q + 1) \frac{|z|^2}{1 - |z|^2} u_{p,q}(z)$$

for $z \in \mathbb{D}$. Multiplying by a factor $(1 - |z|^2)/|z|^2$ we see that

$$(1 - |z|^2) \partial \bar{z} u_{p,q}(z) = \left( (p + q + 1)^2 \frac{|z|^2}{1 - |z|^2} - (p + q + 1) \frac{1}{1 - |z|^2} \right) u_{p,q}(z)$$

$$- (p + q + 1) \left( (p + 1) \frac{z}{1 - z} + (q + 1) \frac{\bar{z}}{1 - \bar{z}} \right) u_{p,q}(z)$$

$$+ (p + 1)(q + 1) \frac{1 - |z|^2}{|1 - z|^2} u_{p,q}(z)$$

for $z \in \mathbb{D}$. Notice an appearance of the classical Poisson kernel in the rightmost term above. Using the partial fraction formula (1.1) we have that

$$(1 - |z|^2) \partial \bar{z} u_{p,q}(z) = (p + q + 1) \left( (p + q) \frac{|z|^2}{1 - |z|^2} - 1 \right) u_{p,q}(z)$$

$$+ \left( (p + 1)(q + 1) - (p + q + 1)(p + 1) \right) \frac{z}{1 - z} u_{p,q}(z)$$

$$+ \left( (p + 1)(q + 1) - (p + q + 1)(q + 1) \right) \frac{\bar{z}}{1 - \bar{z}} u_{p,q}(z)$$

$$+ (p + 1)(q + 1) u_{p,q}(z)$$

for $z \in \mathbb{D}$. A simplification of terms now leads to the formula

$$(1 - |z|^2) \partial \bar{z} u_{p,q}(z) = (p + q + 1) \left( (p + q) \frac{|z|^2}{1 - |z|^2} \right) u_{p,q}(z)$$

$$- p(p + 1) \frac{z}{1 - z} u_{p,q}(z) - q(q + 1) \frac{\bar{z}}{1 - \bar{z}} u_{p,q}(z) + pq u_{p,q}(z) \quad (1.2)$$

for $z \in \mathbb{D}$. Recall Lemmas 1.1 and 1.2. In view of these two lemmas our latter formula (1.2) says that $L_{p,q} u_{p,q} = 0$ in $\mathbb{D}$. □
2 A sequence of hypergeometric functions

Let us first consider a second order partial differential operator of the form

\[ L_{p,q;r} = (1 - |z|^2) \partial^2_z + p \partial_z + q \overline{z} \partial_{\overline{z}} - r, \quad z \in \mathbb{D}, \]  

(2.1)

where \( p, q, r \in \mathbb{C} \) are complex parameters. A principal case is when \( r = pq \). Notice that \( L_{p,q;pq} = L_{p,q} \), where \( L_{p,q} \) is as in (0.1). The introduction of an additional parameter \( r \in \mathbb{C} \) allows for more general operators appearing in the study of conductivity problems, see for instance Calderón [7] or Astala and Päivärinta [4].

We shall evaluate the operator \( L_{p,q;r} \) on a complex-valued function \( u \) in the punctured disc \( \mathbb{D}\setminus\{0\} \) of the form

\[ u(z) = f(|z|^2) z^m, \quad z \in \mathbb{D}\setminus\{0\}, \]  

(2.2)

for some \( f \in C^2(0,1) \) and \( m \in \mathbb{Z} \).

We introduce also the ordinary differential operator

\[ H_{a,b;c} = (1 - x) \frac{d^2}{dx^2} + (c - [a + b + 1]x) \frac{d}{dx} - ab, \]  

(2.3)

where \( a, b, c \in \mathbb{C} \) are complex parameters. Notice that the famous hypergeometric ordinary differential equation

\[ (1 - x)xy''(x) + (c - [a + b + 1]x)y'(x) - aby(x) = 0 \]

takes the form \( H_{a,b;c} y = 0 \) using the operator \( H_{a,b;c} \).

**Theorem 2.1** Let \( L_{p,q;r} \) be as in (2.1) for some \( p, q, r \in \mathbb{C} \). Let \( u \) be a function of the form (2.2) for some \( f \in C^2(0,1) \) and \( m \in \mathbb{Z} \). Then

\[ L_{p,q;r} u(z) = z^m H_{a,b;c} f(|z|^2), \quad z \in \mathbb{D}\setminus\{0\}, \]

where \( c = m + 1 \) and

\[ \begin{cases} a + b = m - p - q, \\ ab = r - pm. \end{cases} \]  

(2.4)

**Proof** A differentiation shows that

\[ \partial u(z) = mz^{m-1} f(|z|^2) + z^m \overline{z} f'(|z|^2) \]

for \( z \in \mathbb{D}\setminus\{0\} \), and similarly that

\[ \overline{\partial} u(z) = z^{m+1} f'(|z|^2) \]
for \( z \in \mathbb{D} \setminus \{0\} \). Another differentiation gives that
\[
\partial \bar{\partial} u(z) = (m + 1)z^m f'(\|z\|^2) + z^{m+1} \bar{z} f''(\|z\|^2)
\]
for \( z \in \mathbb{D} \setminus \{0\} \). A calculation using these formulas gives that
\[
L_{p,q,r} u(z) = (1 - \|z\|^2)((m + 1)z^m f'(\|z\|^2) + z^m |z|^2 f''(\|z\|^2)) + q \bar{z} z^{m+1} f'(\|z\|^2)
\]
\[
+ pz(mz^{m-1} f(\|z\|^2) + z^m \bar{z} f'(\|z\|^2)) - rz^m f(\|z\|^2)
\]
\[
= z^m \left( |z|^2 (1 - |z|^2) f''(\|z\|^2) + (m + 1 - [m + 1 - p - q] |z|^2) f'(\|z\|^2)\right)
\]
\[
- (r - pm) f(\|z\|^2)
\]
\[
= z^m H_{a,b;c} f(\|z\|^2)
\]
for \( z \in \mathbb{D} \setminus \{0\} \), where \( c = m + 1 \), the numbers \( a \) and \( b \) are as in (2.4) and \( H_{a,b;c} \) is as in (2.3). This yields the conclusion of the theorem.

Equations (2.4) say that \( a \) and \( b \) are the zeros of the quadratic polynomial
\[
P_{p,q;r;m}(\lambda) = \lambda^2 - (m - p - q)\lambda + r - pm, \quad \lambda \in \mathbb{C}.
\]

Notice that
\[
P_{p,q;r;m}(\lambda) = (\lambda + p)(\lambda + q - m) + r - pq, \quad \lambda \in \mathbb{C}.
\]
In particular, the zeros of \( P_{p,q;r;m} \) are \(-p\) and \( m - q \) in the principal case when \( r = pq \).

Theorem 2.1 suggests a natural construction of \((p, q)\)-harmonic functions.

**Proposition 2.2** Let \( p, q \in \mathbb{C} \). Consider the function
\[
u_m(z) = F(-p, m - q; m + 1; |z|^2) z^m, \quad z \in \mathbb{D},
\]
where \( m \in \mathbb{N} \) and \( F \) is the hypergeometric function (0.5). Then \( \nu_m \) is a \((p, q)\)-harmonic function.

**Proof** Clearly \( \nu_m \in C^\infty(\mathbb{D}) \). It is well-known that the hypergeometric function \( y = F(a, b; c; \cdot) \) satisfies the hypergeometric equation \( H_{a,b;c} y = 0 \) (see [3, Section 2.3]). The result now follows by Theorem 2.1.

**Corollary 2.3** Let \( p, q \in \mathbb{C} \). Consider the function
\[
u_m(z) = F(-q, |m| - p; |m| + 1; |z|^2) z^{|m|}, \quad z \in \mathbb{D},
\]
where \( m \in \mathbb{Z}^- \) and \( F \) is the hypergeometric function (0.5). Then \( \nu_m \) is a \((p, q)\)-harmonic function.
\textbf{Proof} We consider the complex conjugate
\[ u_m(z) = F(-\bar{\bar{q}}, |m| - \bar{\bar{p}}; |m| + 1; |z|^2)z^{|m|}, \quad z \in \mathbb{D}. \]

By Proposition 2.2 we have that \( \bar{u}_m \) is a \((\bar{\bar{q}}, \bar{\bar{p}})\)-harmonic function. From hermitian symmetry we conclude that \( u_m \) is a \((p, q)\)-harmonic function. \( \square \)

Following earlier practice, a function \( u \) in \( \mathbb{D} \setminus \{0\} \) is said to be homogeneous of order \( m \in \mathbb{Z} \) with respect to rotations if it has the property that
\[ u(e^{i\theta}z) = e^{im\theta}u(z), \quad z \in \mathbb{D} \setminus \{0\}, \]
for \( e^{i\theta} \in \mathbb{T} \). Notice that every function \( u \) of the form (2.2) is homogeneous of order \( m \) with respect to rotations.

\textbf{Theorem 2.4} Let \( p, q \in \mathbb{C} \) and \( m \in \mathbb{N} \). Let \( u \in C^2(\mathbb{D}) \) be homogeneous of order \( m \) with respect to rotations. Then \( u \) is \((p, q)\)-harmonic if and only if it has the form
\[ u(z) = cF(-p, m - q; m + 1; |z|^2)z^m, \quad z \in \mathbb{D}, \quad (2.5) \]
for some \( c \in \mathbb{C} \), where \( F \) is the hypergeometric function (0.5).

\textbf{Proof} From Proposition 2.2 we know that every function \( u \) of the form (2.5) is \((p, q)\)-harmonic.

Assume next that \( u \) is \((p, q)\)-harmonic. Since \( u \) is homogeneous of order \( m \), we can put \( u \) on the form (2.2) for some \( f \in C^2(0, 1) \). By Theorem 2.1 we have that
\[ H_{-p, m - q, m + 1} f(x) = 0, \quad 0 < x < 1, \]
where \( H_{a, b; c} \) is as in (2.3). Below we shall check that
\[ (m + 1)f'(x) + p(m - q)f(x) = o(1/x^{m+1}) \quad (2.6) \]
as \( x \to 0 \). Condition (2.6) allows us to apply [18, Proposition 1.3] to conclude that \( f \) is a constant multiple of the hypergeometric function \( F(-p, m - q; m + 1; \cdot) \). This will then complete the proof of the theorem.

We proceed to check (2.6). Recall formula (2.2). Since \( u \) is bounded near the origin we have that \( f(x) = O(1/x^{m/2}) \) as \( x \to 0 \). A differentiation of (2.2) gives that
\[ \bar{\partial}u(z) = z^{m+1}f'(|z|^2), \quad z \in \mathbb{D} \setminus \{0\}. \]
Since \( \bar{\partial}u \) is bounded near the origin we have that \( f'(x) = O(1/x^{(m+1)/2}) \) as \( x \to 0 \). We have now checked (2.6). \( \square \)

Theorem 2.4 and its proof are modeled on [18, Theorem 2.1]. We have merely supplied some details.
We shall use the fact that the hypergeometric functions are analytic in $\mathbb{D}$. Let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. The space $H(\mathbb{D})$ is topologized in the usual manner using the semi-norms

$$
\| f \|_K = \max_{z \in K} |f(z)|
$$

for $K \subset \mathbb{D}$ compact. Convergence in the space $H(\mathbb{D})$ is usually referred to as normal convergence in $\mathbb{D}$.

Recall the terminology that a subset $\mathcal{F}$ of $H(\mathbb{D})$ is called a normal family if every sequence of functions of $\mathcal{F}$ has a subsequence which converges in $H(\mathbb{D})$. The limit function is not required to belong to $\mathcal{F}$. We refer to Conway [11, Chapter VII] for background.

Recall also the binomial series:

$$
F(a, b; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = \frac{1}{(1 - z)^a}
$$

for $z \in \mathbb{D}$.

**Lemma 2.5** Let $p, q \in \mathbb{C}$ and consider the functions

$$
f_m(z) = F(-p, m - q; m + 1; z), \quad z \in \mathbb{D},
$$

for $m \in \mathbb{N}$, where $F$ is the hypergeometric function (0.5). Then $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ is a normal family of analytic functions in $\mathbb{D}$.

**Proof** We shall prove that the functions in $\mathcal{F}$ are uniformly bounded on compact subsets of $\mathbb{D}$. The conclusion of the lemma then follows by a classical result of Montel (see Conway [11, Theorem VII.2.9]).

Let $K \subset \mathbb{D}$ be compact. Since $K \subset \mathbb{D}$ is compact there exists $0 < r < 1$ such that $\max_{z \in K} |z| < r$. Choose $N$ such that

$$
|1 - (q + 1)/(m + 1)| \leq 1/r
$$

for $m > N$. Observe that

$$
\frac{(m - q)_n}{(m + 1)_n} = \prod_{k=0}^{n-1} \frac{m - q + k}{m + 1 + k} = \prod_{k=0}^{n-1} \left( 1 - \frac{q + 1}{m + 1 + k} \right).
$$

Therefore $|(m - q)_n/(m + 1)_n| \leq 1/r^n$ for $n \in \mathbb{N}$ provided $m > N$.

We now estimate the $f_m$’s with $m > N$. From (0.5) we have that

$$
f_m(z) = \sum_{n=0}^{\infty} \frac{(-p)_n (m - q)_n}{(m + 1)_n} \frac{z^n}{n!}, \quad z \in \mathbb{D}.
$$
Notice in this sum an appearance of the quotient $(m - q)_n/(m + 1)_n$ considered in the previous paragraph. From the triangle inequality and the result of the previous paragraph we have that
\[
|f_m(z)| \leq \sum_{n=0}^{\infty} \frac{(|p|)_n}{n!} \frac{|z|^n}{r^n} = \frac{1}{(1 - |z|/r)^{|p|}}
\]
for $|z| < r$ and $m > N$, where the last equality follows by the binomial series (2.8). This proves that the functions in $\mathcal{F}$ are uniformly bounded on $K$. \qed

The following limit theorem will be much useful.

**Theorem 2.6** Let $p, q \in \mathbb{C}$. Then
\[
\lim_{m \to \infty} F(-p, m - q; m + 1; z) = (1 - z)^P, \quad z \in \mathbb{D},
\]
with normal convergence, where $F$ is the hypergeometric function (0.5).

**Proof** Let the $f_m$’s be as in Lemma 2.5. From Lemma 2.5 we have that the set $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ is a normal family. From (0.5) we have that
\[
f_m^{(n)}(0) = \frac{(-p)_n(m - q)_n}{(m + 1)_n}
\]
for $m, n \in \mathbb{N}$. Recall that the Pochhammer symbol $(\cdot)_n$ is a monic polynomial of degree $n$. Passing to the limit we have that $\lim_{m \to \infty} f_m^{(n)}(0) = (-p)_n$ for $n \in \mathbb{N}$.

From (2.8) we have that $f^{(n)}(0) = (-p)_n$ for $n \in \mathbb{N}$, where
\[
f(z) = (1 - z)^P, \quad z \in \mathbb{D}.
\]
A standard argument now yields that $f_m \to f$ in $H(\mathbb{D})$ as $m \to \infty$. Assume to reach a contradiction that there exists a compact set $K \subset \mathbb{D}$ such that $\{f_m\}_{m=0}^{\infty}$ does not converge uniformly to $f$ on $K$. Passing to a subsequence we can assume that
\[
\max_{z \in K} |f_{m_k}(z) - f(z)| \geq \delta > 0 \quad (2.9)
\]
for $k = 1, 2, \ldots$. Since the set $\mathcal{F}$ is a normal family, we can after passage to another subsequence if necessary, assume that $f_{m_k} \to g$ in $H(\mathbb{D})$ as $k \to \infty$ for some $g \in H(\mathbb{D})$. From the first paragraph of the proof we have $g^{(n)}(0) = (-p)_n$ for $n \in \mathbb{N}$, and a uniqueness argument gives that $g = f$ in $\mathbb{D}$. Thus $f_{m_k} \to f$ in $H(\mathbb{D})$ as $k \to \infty$, which contradicts (2.9). \qed

We emphasize that Theorem 2.6 appears much natural in view of the binomial series (2.8).
3 A generalized power series

From the product rule for differentiation we have that
\[(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}g^{(k)}\]
for, say, \(f, g \in C^n[0, 1]\).

**Lemma 3.1** Let \(j, k, m \in \mathbb{N}\) be such that \(m \geq j\). Let \(u\) be a function of the form (2.2) for some \(f \in C^\infty(0, 1)\). Then
\[\partial^j \bar{\partial}^k u(z) = z^{m+k-j} \sum_{l=0}^{j} \binom{j}{l} (m+k)! \frac{1}{(m+k+l-j)!} |z|^{2l} f^{(k+l)}(|z|^2)\]
for \(z \in \mathbb{D}\).

**Proof** Recall (2.2). Differentiating with respect to \(\bar{z}\) we have that
\[\bar{\partial}^k u(z) = z^{m+k} f^{(k)}(|z|^2)\]
for \(z \in \mathbb{D}\). Another differentiation using the product rule for differentiation gives that
\[\partial^j \bar{\partial}^k u(z) = \sum_{l=0}^{j} \binom{j}{l} (m+k)! \frac{1}{(m+k+l-j)!} z^{m+k-(j-l)} f^{(k+l)}(|z|^2)\bar{z}^l\]
\[= z^{m+k-j} \sum_{l=0}^{j} \binom{j}{l} (m+k)! \frac{1}{(m+k+l-j)!} |z|^{2l} f^{(k+l)}(|z|^2)\]
for \(z \in \mathbb{D}\). This completes the proof of the lemma. \(\square\)

Let \(C^\infty(\mathbb{D})\) be the set of smooth complex-valued functions in the unit disc \(\mathbb{D}\). The space \(C^\infty(\mathbb{D})\) is topologized by means of the family of semi-norms
\[\|u\|_{j,k;K} = \max_{z \in K} |\partial^j \bar{\partial}^k u(z)|, \quad (3.1)\]
where \(j, k \in \mathbb{N}\) and \(K \subset \mathbb{D}\) is compact. Recall that \(u_m \to u\) in \(C^\infty(\mathbb{D})\) as \(m \to \infty\) means that \(\lim_{m \to \infty} \|u_m - u\|_{j,k;K} = 0\) for all \(j, k \in \mathbb{N}\) and \(K \subset \mathbb{D}\) compact.

A (formal) series \(\sum_{m=0}^{\infty} u_m\) of functions \(u_m \in C^\infty(\mathbb{D})\) for \(m \in \mathbb{N}\) is said to be absolutely convergent in \(C^\infty(\mathbb{D})\) if
\[\sum_{m=0}^{\infty} \|u_m\|_{j,k;K} < +\infty\]
whenever \( j, k \in \mathbb{N} \) and \( K \subset \mathbb{D} \) is compact. By completeness of \( C^\infty(\mathbb{D}) \) we have that every series absolutely convergent in \( C^\infty(\mathbb{D}) \) is convergent in \( C^\infty(\mathbb{D}) \).

**Theorem 3.2** Let \( \{f_m\}_{m=0}^\infty \) be a sequence in \( C^\infty[0, 1) \) such that

\[
\limsup_{m \to \infty} \left( \max_{0 \leq \alpha \leq r} |f_m^{(n)}(\alpha)| \right)^{1/m} \leq 1
\]

for \( n \in \mathbb{N} \) and \( 0 < r < 1 \). Set

\[
u_m(z) = f_m(|z|^2)z^m, \quad z \in \mathbb{D},
\]

for \( m = 0, 1, 2, \ldots \). Then \( \limsup_{m \to \infty} \|u_m\|_{j,k;K}^{1/m} < 1 \) for all \( j, k \in \mathbb{N} \) and \( K \subset \mathbb{D} \) compact, where \( \|\cdot\|_{j,k;K} \) is as in (3.1).

**Proof** Fix \( j, k \in \mathbb{N} \) and \( K \subset \mathbb{D} \) compact. Set \( r = \max_{z \in K} |z| < 1 \). From Lemma 3.1 we have that

\[
\partial^j \partial^k \nu_m(z) = z^{m+k-j} \sum_{l=0}^{j} \binom{j}{l} \frac{(m+k)!}{(m+k+l-j)!} |z|^{2l} f_m^{(k+l)}(|z|^2)
\]

for \( z \in \mathbb{D} \) provided \( m \geq j \). We next apply the triangle inequality to see that

\[
\|\nu_m\|_{j,k;K} \leq r^{m+k-j} \left( \sum_{l=0}^{j} \binom{j}{l} \frac{(m+k)!}{(m+k+l-j)!} r^{2l} \right) \max_{k \leq n \leq j+k} \|f_m^{(n)}\|_{[0,r^2]}
\]

\[
\leq r^{m+k-j} (m+k)^j (1 + r^2)^j \max_{k \leq n \leq j+k} \|f_m^{(n)}\|_{[0,r^2]}
\]

(3.2)

for \( m \geq j \), where we have used the notation (2.7). We shall next apply the \( m \)-th root to (3.2) and pass to the limit as \( m \to \infty \). In view of the assumption on the sequence \( \{f_m\}_{m=0}^\infty \) we conclude from (3.2) that

\[
\limsup_{m \to \infty} \|u_m\|_{j,k;K}^{1/m} \leq r.
\]

Since \( 0 < r < 1 \), this yields the conclusion of the theorem. \( \Box \)

From the conclusion of Theorem 3.2 we have that \( \sum_{m=0}^\infty \|u_m\|_{j,k;K} < +\infty \) whenever \( j, k \in \mathbb{N} \) and \( K \subset \mathbb{D} \) is compact. Thus the series \( \sum_{m=0}^\infty u_m \) is absolutely convergent in \( C^\infty(\mathbb{D}) \).

We can think of a function series

\[
\sum_{m=0}^\infty f_m(|z|^2)z^m
\]

(3.3)
as a vector-valued power series. The sequence of coefficients \( \{ f_m \}_{m=0}^\infty \) in (3.3) is now a sequence of functions in \( C^\infty[0, 1) \). From this point of view, Theorem 3.2 generalizes a well-known root criteria for power series (see for instance Conway [11, Theorem III.1.3]).

Of particular concern to us are sequences \( \{ f_m \}_{m=0}^\infty \) of the form

\[
    f_m(x) = F(-p, m - q; m + 1; x), \quad 0 \leq x < 1,
\]

for \( m \in \mathbb{N} \), where \( p, q \in \mathbb{C} \) and \( F \) is the hypergeometric function (0.5). We next observe that Theorem 2.6 guarantees that every such sequence \( \{ f_m \}_{k=0}^\infty \) satisfies the assumption of Theorem 3.2.

**Proposition 3.3** Let \( p, q \in \mathbb{C} \). Then

\[
    \limsup_{m \to \infty} \left( \max_{0 \leq x \leq r} |F^{(n)}(-p, m - q; m + 1; x)| \right)^{1/m} \leq 1
\]

for \( n \in \mathbb{N} \) and \( 0 < r < 1 \), where \( F \) is the hypergeometric function (0.5).

**Proof** Recall the fact that the complex derivative \( f \mapsto f' \) is continuous in the topology of normal convergence of analytic functions (see Conway [11, Theorem VII.2.1]). Recall Montel’s theorem characterizing normal families of analytic functions (see Conway [11, Theorem VII.2.9]). In view of these two results, the proposition follows from Theorem 2.6. \( \square \)

### 4 Analysis of homogeneous parts

Notice that the rotation operators \( \mathbb{T} \ni e^{i\theta} \mapsto R_{e^{i\theta}} \) from (0.3) have the group properties that \( R_1 = I \) is the identity and

\[
    R_{e^{i(\theta + \tau)}} = R_{e^{i\theta}} R_{e^{i\tau}}
\]

for \( e^{i\theta}, e^{i\tau} \in \mathbb{T} \).

Recall the notion of homogeneity with respect to rotations made precise in the paragraph before Theorem 2.4. Observe that a function \( u \) in \( \mathbb{D} \) is homogeneous of order \( m \in \mathbb{Z} \) with respect to rotations if and only if \( R_{e^{i\theta}} u = e^{im\theta} u \) for \( e^{i\theta} \in \mathbb{T} \).

**Proposition 4.1** Let \( u_m \) be the \( m \)-th homogeneous part of \( u \in C^n(\mathbb{D}) \) for some \( m \in \mathbb{Z} \), where \( n \in \mathbb{N} \cup \{ \infty \} \). Then \( u_m \in C^n(\mathbb{D}) \) and \( u_m \) is homogeneous of order \( m \) with respect to rotations.

**Proof** Differentiations under the integral in (0.4) show that \( u_m \in C^n(\mathbb{D}) \). From formula (0.4) we have that

\[
    u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\tau}} u d\tau \quad (4.1)
\]
in a vector-valued sense. Let \( e^{i\theta} \in \mathbb{T} \). From the group property of rotation operators we have that
\[
R_{e^{i\theta}} u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\theta}} R_{e^{i\tau}} u \, d\tau = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} e^{i\tau} R_{e^{i(\theta+\tau)}} u \, d\tau
\]
in a vector-valued sense. A change of variables now gives that
\[
R_{e^{i\theta}} u_m = e^{im\theta} \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\tau}} u \, d\tau = e^{im\theta} u_m.
\]
Since \( e^{i\theta} \in \mathbb{T} \) is arbitrary, this yields that \( u_m \) is homogeneous of order \( m \) with respect to rotations. \( \square \)

Below we shall make use of the commutativity relation
\[
L_{p,q} R_{e^{i\theta}} u = R_{e^{i\theta}} L_{p,q} u, \quad u \in C^2(\mathbb{D}), \tag{4.2}
\]
for \( e^{i\theta} \in \mathbb{T} \). In order to prove (4.2) it suffices to check that the differential operators \( z\partial \) and \( \bar{z}\partial \) commute with the rotations \( R_{e^{i\theta}} \) which is evident.

We now return to \( (p,q) \)-harmonic functions.

**Proposition 4.2** Let \( p, q \in \mathbb{C} \). Let \( u \) be a \( (p,q) \)-harmonic function and denote by \( u_m \) its \( m \)-th homogeneous part for some \( m \in \mathbb{Z} \). Then \( u_m \) is \( (p,q) \)-harmonic.

**Proof** From Proposition 4.1 we have that \( u_m \in C^2(\mathbb{D}) \) since \( u \in C^2(\mathbb{D}) \). Recall formula (4.1). Applying the operator \( L_{p,q} \) we have that
\[
L_{p,q} u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} L_{p,q} R_{e^{i\tau}} u \, d\tau
\]
in a vector-valued sense. We now use the commutativity relation (4.2) to conclude that
\[
L_{p,q} u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\tau}} L_{p,q} u \, d\tau = 0,
\]
where the last equality is evident since \( u \) is \( (p,q) \)-harmonic. \( \square \)

Let us next calculate the homogeneous parts of a \( (p,q) \)-harmonic function.

**Theorem 4.3** Let \( p, q \in \mathbb{C} \). Let \( u \) be \( (p,q) \)-harmonic function and denote by \( u_m \) its \( m \)-th homogeneous part for some \( m \in \mathbb{N} \). Then
\[
u_m(z) = c_m F(-p,m-q;m+1;|z|^2)z^m, \quad z \in \mathbb{D}, \tag{4.3}
\]
for some \( c_m \in \mathbb{C} \), where \( F \) is the hypergeometric function (0.5).
Proof By Proposition 4.1, the function $u_m$ is homogeneous of order $m$ with respect to rotations. By Proposition 4.2, the function $u_m$ is $(p, q)$-harmonic. The result now follows from Theorem 2.4. 

Corollary 4.4 Let $p, q \in \mathbb{C}$. Let $u$ be $(p, q)$-harmonic function and denote by $u_m$ its $m$-th homogeneous part for some $m \in \mathbb{Z} \setminus \mathbb{Z}^+$. Then
\begin{equation}
 u_m(z) = c_m F(-q, |m| - p; |m| + 1; |z|^2) z^{|m|}, \quad z \in \mathbb{D},
\end{equation}
for some $c_m \in \mathbb{C}$, where $F$ is the hypergeometric function (0.5).

Proof The complex conjugate $\bar{u}_m$ is the $-m = |m|$-th homogeneous part of the function $\bar{u}$. By hermitian symmetry, the function $\bar{u}$ is $(\bar{q}, \bar{p})$-harmonic. From Theorem 4.3 we have that
\begin{equation}
 \overline{u_m(z)} = \bar{a}_m F(-\bar{q}, |m| - \bar{p}; |m| + 1; |z|^2) z^{|m|}, \quad z \in \mathbb{D},
\end{equation}
for some $a_m \in \mathbb{C}$. A complex conjugation now yields (4.4) with $c_m = \bar{a}_m$. 

As a by-product from Theorem 4.3 and Corollary 4.4 we have that $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$ if $u$ is $(p, q)$-harmonic.

Recall formula (0.4) for the homogeneous parts of a function $u$ in $\mathbb{D}$. From the triangle inequality we have that
\begin{equation}
 \sup_{|z| = r} |u_m(z)| \leq \sup_{|z| = r} |u(z)|
\end{equation}
for $0 < r < 1$ and $m \in \mathbb{Z}$. Notice that (4.5) generalizes a well-known bound for Fourier coefficients.

We next estimate the constants $c_m$ appearing in Theorem 4.3 or Corollary 4.4.

Lemma 4.5 Let $p, q \in \mathbb{C}$. Let $u$ be a $(p, q)$-harmonic function and consider its $m$-th homogeneous part $u_m$ for $m \in \mathbb{Z}$. Let $c_m$ be as in (4.4) or (4.4) depending on whether $m \in \mathbb{N}$ or $m \in \mathbb{Z}^-$. Then $\limsup_{|m| \to \infty} |c_m|^{1/|m|} \leq 1$.

Proof For simplicity we consider the case $m \in \mathbb{N}$. The case $m \in \mathbb{Z}^-$ is analogous or follows by hermitian symmetry. Let $0 < r < 1$. From (4.4) and (4.5) we have that
\begin{equation}
 |c_m| F(-p, m - q; m + 1; r^2) |r^m| \leq \max_{|z| = r} |u(z)|
\end{equation}
for $m \in \mathbb{N}$. From Theorem 2.6 we have that
\begin{equation}
 \lim_{m \to \infty} F(-p, m - q; m + 1; r^2) = (1 - r^2)^p \neq 0.
\end{equation}

Therefore, a passage to the limit in (4.6) gives that
\begin{equation}
 \limsup_{m \to \infty} |c_m|^{1/m} \leq 1/r.
\end{equation}
Since $0 < r < 1$ is arbitrary we conclude that $\limsup_{m \to \infty} |c_m|^{1/m} \leq 1$. This yields the conclusion of the lemma. □

The following theorem is our main result about the asymptotic behavior of the homogeneous parts of a $(p, q)$-harmonic function.

**Theorem 4.6** Let $p, q \in \mathbb{C}$. Let $u$ be a $(p, q)$-harmonic function and denote by $u_m$ its $m$-th homogeneous part for $m \in \mathbb{Z}$. Then $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$ and

$$
\limsup_{m \to \infty} \|u_m\|_{1/m, j, k; K} < 1
$$

for all $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact.

**Proof** From Theorem 4.3 and Corollary 4.4 we have that $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$.

We consider first the case $m \in \mathbb{N}$. From Theorem 4.3 we have (4.4) for $m \in \mathbb{N}$, where $c_m \in \mathbb{C}$. Notice that the function $u_m$ in (4.4) is constructed from the function

$$
f_m(x) = c_m F(-p, m - q; m + 1; x), \quad 0 \leq x < 1,
$$

as in Theorem 3.2. From Proposition 3.3 and Lemma 4.5 we have that the assumption of Theorem 3.2 is satisfied. An application of Theorem 3.2 yields the conclusion that

$$
\limsup_{m \to \infty} \|u_m\|_{1/m, j, k; K} < 1
$$

for $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact. This completes the proof of the theorem. □

We point out that Theorem 2.6 forms an integral part in the proof of Theorem 4.6. Recall the notion of absolute convergence in $C^\infty(\mathbb{D})$, see the paragraph just before Theorem 3.2.

**Corollary 4.7** Let $p, q \in \mathbb{C}$. Let $u$ be a $(p, q)$-harmonic function with $m$-th homogeneous part $u_m$ for $m \in \mathbb{Z}$. Then the function series $\sum_{m=-\infty}^\infty u_m$ is absolutely convergent in $C^\infty(\mathbb{D})$.

**Proof** Theorem 4.6 allows us to apply the root test to conclude that

$$
\sum_{m=-\infty}^\infty \|u_m\|_{j, k; K} < +\infty
$$

for $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact. □

The following lemma is well-known but included for the sake of completeness.
Lemma 4.8 Let $u \in C^n(\mathbb{D})$ for some $n \in \mathbb{N}$ and denote by $u_m$ its $m$-th homogeneous part for $m \in \mathbb{Z}$. Then

$$u = \lim_{N \to +\infty} \sum_{m=-N}^{N} \left( 1 - \frac{|m|}{N+1} \right) u_m$$

in $C^n(\mathbb{D})$.

**Proof** It is straightforward to check that

$$\sum_{m=-N}^{N} \left( 1 - \frac{|m|}{N+1} \right) u_m = \frac{1}{2\pi} \int_{\mathbb{T}} K_N(e^{i\theta}) R_{e^{i\theta}} u \, d\theta$$

in a vector-valued sense, where

$$K_N(e^{i\theta}) = \sum_{m=-N}^{N} \left( 1 - \frac{|m|}{N+1} \right) e^{im\theta}, \quad e^{i\theta} \in \mathbb{T},$$

is the Fejér kernel. It is well-known that the $K_N$’s are non-negative and form an approximate identity as $N \to \infty$. Since $u \in C^n(\mathbb{D})$, it is well-known that the function

$$\mathbb{T} \ni e^{i\theta} \mapsto R_{e^{i\theta}} u \in C^n(\mathbb{D})$$

is continuous from $\mathbb{T}$ into $C^n(\mathbb{D})$. The proof is now completed by a standard argument. We refer to Katznelson [15, Section I.2] for details. \hfill \Box

We think of Lemma 4.8 as a version of Fejér’s theorem adapted to our context.

We next deduce that $(p, q)$-harmonic functions belong to the space $C^\infty(\mathbb{D})$.

**Corollary 4.9** Let $p, q \in \mathbb{C}$. Let $u$ be a $(p, q)$-harmonic function. Then $u \in C^\infty(\mathbb{D})$ and $u = \sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$, where $u_m$ is the $m$-th homogeneous part of $u$ for $m \in \mathbb{Z}$.

**Proof** From Lemma 4.8 we have that

$$u = \lim_{N \to +\infty} \sum_{m=-N}^{N} \left( 1 - \frac{|m|}{N+1} \right) u_m$$  \hfill (4.7)

in $C^2(\mathbb{D})$ since $u$ has such regularity. From Corollary 4.7 we know that the function series $\sum_{m=-\infty}^{\infty} u_m$ is absolutely convergent in $C^\infty(\mathbb{D})$. We can thus omit the convergence factors in (4.7) and deduce that $u = \sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$. By completeness we have that $u \in C^\infty(\mathbb{D})$. \hfill \Box
5 A series expansion of harmonic functions

The analysis from Sect. 4 leads to a natural function series description of \((p, q)\)-harmonic functions.

**Theorem 5.1** Let \(p, q \in \mathbb{C}\). Then \(u\) is a \((p, q)\)-harmonic function if and only if it has the form

\[
\begin{align*}
    u(z) &= \sum_{m=0}^{\infty} c_m F(-p, m-q; m+1; |z|^2) z^m \\
    &+ \sum_{m=1}^{\infty} c_{-m} F(-q, m-p; m+1; |z|^2) \overline{z}^m, \quad z \in \mathbb{D},
\end{align*}
\]

(5.1)

for some sequence \(\{c_m\}_{m=-\infty}^{\infty}\) of complex numbers such that

\[
    \limsup_{|m| \to \infty} |c_m|^{1/|m|} \leq 1,
\]

(5.2)

where \(F\) is the hypergeometric function \((0.5)\). Moreover, the sums in (5.1) are absolutely convergent in the space \(C^\infty(\mathbb{D})\) when (5.2) holds.

**Proof** Consider first a (formal) function series of the form (5.1) with coefficient sequence \(\{c_m\}_{m=-\infty}^{\infty}\) satisfying (5.2). From Theorem 3.2 and Proposition 3.3 we have that

\[
    \limsup_{|m| \to \infty} \frac{\|u_m\|_{1/j,k;K}}{1/|m|} < 1
\]

for \(j, k \in \mathbb{N}\) and \(K \subset \mathbb{D}\) compact, where

\[
    u_m(z) = c_m F(-p, m-q; m+1; |z|^2) z^m, \quad z \in \mathbb{D},
\]

for \(m \in \mathbb{N}\) and

\[
    u_m(z) = c_m F(-q, |m|-p; |m|+1; |z|^2) \overline{z}^{|m|}, \quad z \in \mathbb{D},
\]

for \(m \in \mathbb{Z}^-\). The root test now applies to show that the series \(\sum_{m=-\infty}^{\infty} u_m\) in (5.1) is absolutely convergent in \(C^\infty(\mathbb{D})\). This defines a function \(u = \sum_{m=-\infty}^{\infty} u_m\) by (5.1). From Proposition 2.2 and Corollary 2.3 we have that each term \(u_m\) in (5.1) is \((p, q)\)-harmonic. Since the series expansion (5.1) converges in the space \(C^\infty(\mathbb{D})\), we have that \(u \in C^\infty(\mathbb{D})\). Applying the operator \(L_{p,q}\) we have \(L_{p,q} u = \sum_{m=-\infty}^{\infty} L_{p,q} u_m = 0\) in \(C^\infty(\mathbb{D})\), which shows that \(u\) is \((p, q)\)-harmonic.

Consider next a \((p, q)\)-harmonic function \(u\). We proceed to derive (5.1). Denote by \(u_m\) the \(m\)-th homogeneous part of \(u\) for \(m \in \mathbb{Z}\). From Theorem 4.3 we have (4.4) for \(m \in \mathbb{N}\) and from Corollary 4.4 we have (4.4) for \(m \in \mathbb{Z}^-\), where \(c_m \in \mathbb{C}\) for \(m \in \mathbb{Z}\). From Corollary 4.9 we have (5.1) in the form that \(u = \sum_{m=-\infty}^{\infty} u_m\) in \(C^\infty(\mathbb{D})\).
By Corollary 4.7 we have that the function series \( \sum_{m=-\infty}^{\infty} u_m \) in (5.1) is absolutely convergent in \( C^\infty(D) \). From Lemma 4.5 we have that (5.2) holds. \( \square \)

We emphasize that the function series (5.1) has an interpretation of homogeneous expansion of the \((p, q)\)-harmonic function \( u \).

We mention that Theorem 5.1 improves on a result by Ahern et al. [1, Theorem 2.1] when specialized to the present setting. Theorem 5.1 contains also some earlier results of Olofsson and Wittsten. Theorem 5.1 with \((p, q) = (0, \alpha)\), \( \alpha \in \mathbb{R} \), yields [21, Theorem 1.2]. Theorem 5.1 with \((p, q) = (\alpha/2, \alpha/2)\), \( \alpha \in \mathbb{R} \), yields [18, Theorem 2.2].

We shall next turn to formulas for the coefficients \( c_m \) appearing in (5.1).

**Proposition 5.2** Let \( p, q \in \mathbb{C} \). Let \( u \) be a \((p, q)\)-harmonic function of the form (5.1) for some sequence \( \{c_m\}_{m=-\infty}^{\infty} \) of complex numbers satisfying (5.2). Then

\[
c_m = \lim_{r \to 0} \frac{1}{2\pi r^{|m|}} \int_T u(re^{i\theta})e^{-im\theta} \, d\theta
\]

for \( m \in \mathbb{Z} \).

**Proof** We consider the case \( m \in \mathbb{N} \). Using the expansion (5.1) it is straightforward to check that

\[
\frac{1}{2\pi} \int_T u(re^{i\theta})e^{-im\theta} \, d\theta = c_m F(-p, m - q; m + 1; r^2)r^m
\]

for \( 0 < r < 1 \). We next divide by \( r^m \) and pass to the limit to see that

\[
\lim_{r \to 0} \frac{1}{2\pi r^m} \int_T u(re^{i\theta})e^{-im\theta} \, d\theta = c_m F(-p, m - q; m + 1; 0) = c_m.
\]

This yields the conclusion of the proposition when \( m \in \mathbb{N} \).

The remaining case \( m \in \mathbb{Z} \setminus \mathbb{N} \) is proved similarly or follows by hermitian symmetry. We omit the details. \( \square \)

The integral quantity

\[
I_m(r) = \frac{1}{2\pi r^{|m|}} \int_T u(re^{i\theta})e^{-im\theta} \, d\theta
\]

in Proposition 5.2 is naturally thought of as a line integral. In fact,

\[
I_m(r) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{u(z)}{z^{m+1}} \, dz \quad \text{and} \quad I_{-m}(r) = -\frac{1}{2\pi i} \oint_{|z|=r} \frac{u(z)}{z^{m+1}} \, d\bar{z}
\]

for \( m \in \mathbb{N} \) and \( 0 < r < 1 \), where the circle of integration is traversed once in positive direction. These formulas are straightforward to check.
Theorem 5.3 Let \( p, q \in \mathbb{C} \). Let \( u \) be a \((p, q)\)-harmonic function of the form (5.1) for some sequence \( \{c_m\}_{m=-\infty}^{\infty} \) of complex numbers satisfying (5.2). Then

\[
c_m = \partial^m u(0)/m! \quad \text{and} \quad c_{-m} = \bar{\partial}^m u(0)/m!
\]

for \( m \in \mathbb{N} \).

Proof We shall calculate the integral limit in Proposition 5.2. Let \( I_m(r) \) be as in (5.3). Recall from Corollary 4.9 that \( u \in C^\infty(\mathbb{D}) \). Consider the Taylor expansion of \( u \) at the origin of degree \( m \geq 0 \):

\[
u(z) = \sum_{j,k \geq 0 \atop j+k \leq m} \frac{1}{j!k!} \partial^j \bar{\partial}^k u(0) z^j \bar{z}^k + \mathcal{O}(|z|^{m+1})
\]
as \( z \to 0 \). We have that

\[
I_m(r) = \frac{1}{2\pi r^m} \int_{\mathbb{T}} \left( \sum_{j,k \geq 0 \atop j+k \leq m} \frac{1}{j!k!} \partial^j \bar{\partial}^k u(0) r^{j+k} e^{i(j-k)\theta} \right) e^{-im\theta} d\theta + \mathcal{O}(r)
\]
as \( r \to 0 \), where the last equality follows by cancellation. A passage to the limit as \( r \to 0 \) now yields that \( c_m = \partial^m u(0)/m! \).

We now turn to the formula for \( c_{-m} \). A similar analysis as in the previous paragraph shows that

\[
I_{-m}(r) = \bar{\partial}^m u(0)/m! + \mathcal{O}(r)
\]
as \( r \to 0 \). A passage to the limit as \( r \to 0 \) yields that \( c_{-m} = \bar{\partial}^m u(0)/m! \). This completes the proof of the theorem. \( \square \)

The coefficient formulas in Theorem 5.3 leads to an addendum to Theorem 5.1.

Theorem 5.4 Let \( p, q \in \mathbb{C} \). Let \( u \) be a \((p, q)\)-harmonic function. Then

\[
u(z) = \sum_{m=0}^{\infty} \frac{\partial^m u(0)}{m!} F(-p, m - q; m + 1; |z|^2) z^m
\]

\[
+ \sum_{m=1}^{\infty} \bar{\partial}^m u(0)/m! + \mathcal{O}(r)
\]

for \( z \in \mathbb{D} \), where \( F \) is the hypergeometric function (0.5).

Proof Recall the series expansion (5.1) and (5.2) established in Theorem 5.1. By Theorem 5.3 we have that \( c_m = \partial^m u(0)/m! \) and \( c_{-m} = \bar{\partial}^m u(0)/m! \) for \( m \in \mathbb{N} \). This yields the conclusion of the theorem. \( \square \)
As a by-product of Theorem 5.4 we have a uniqueness result of classical type.

**Corollary 5.5** Let \( p, q \in \mathbb{C} \). Let \( u \) be a \((p, q)\)-harmonic function. Assume that \( u(0) = 0 \) and

\[
\partial^m u(0) = \bar{\partial}^m u(0) = 0
\]

for \( m = 1, 2, \ldots \). Then \( u(z) = 0 \) for all \( z \in \mathbb{D} \).

**Proof** The result is evident by Theorem 5.4. \( \square \)

### 6 Further properties of the function \( u_{p,q} \)

Let \( u_{p,q} \) be as in (0.2) for some \( p, q \in \mathbb{C} \). In this section we derive the series expansion of \( u_{p,q} \) (see Theorem 6.3). Of interest are also properties of integral means of \( u_{p,q} \) (see Theorems 6.4 and 6.6).

**Lemma 6.1** Let \( u_{p,q} \) be as in (0.2) for some \( p, q \in \mathbb{C} \). Let \( m \in \mathbb{N} \). Then

\[
\partial^m u_{p,q}(z) = \frac{(p + 1)^m}{(1 - z)^m} u_{p,q}(z) + \bar{z} g_m(z), \quad z \in \mathbb{D},
\]

where \( g_m \in C^\infty(\mathbb{D}) \).

**Proof** We prove the lemma by induction on \( m \in \mathbb{N} \). Clearly (6.1) holds for \( m = 0 \) with \( g_0 = 0 \). By Lemma 1.1 we have that

\[
\partial u_{p,q}(z) = \frac{p + 1}{1 - z} u_{p,q}(z) + \bar{z} g_1(z), \quad z \in \mathbb{D},
\]

with

\[
g_1(z) = -\frac{p + q + 1}{1 - |z|^2} u_{p,q}(z), \quad z \in \mathbb{D}.
\]

This proves (6.1) for \( m = 1 \).

Assume next that (6.1) holds for some \( m = n \geq 1 \), that is,

\[
\partial^n u_{p,q}(z) = \frac{(p + 1)^n}{(1 - z)^n} u_{p,q}(z) + \bar{z} g_n(z), \quad z \in \mathbb{D},
\]

where \( g_n \in C^\infty(\mathbb{D}) \). Differentiating we have that

\[
\partial^{n+1} u_{p,q}(z) = (p + 1)n \frac{n}{(1 - z)^{n+1}} u_{p,q}(z) + \frac{(p + 1)n}{(1 - z)^n} \partial u_{p,q}(z) + \bar{z} \partial g_n(z)
\]
for \( z \in \mathbb{D} \). We next use (6.2) to see that

\[
\tilde{\partial}^{n+1} u_{p,q}(z) = (p + 1) \frac{n}{n+1} u_{p,q}(z) + \frac{p + 1}{1 - z} u_{p,q}(z) + \tilde{z} g_1(z)
\]

\[
+ \tilde{z} \tilde{\partial} g_n(z) = \frac{p + 1}{1 - z} u_{p,q}(z) + \tilde{z} g_{n+1}(z)
\]

for \( z \in \mathbb{D} \), where

\[
g_{n+1}(z) = \frac{(p + 1)n}{(1 - z)^n} g_1(z) + \tilde{g}_n(z), \quad z \in \mathbb{D}.
\]

This proves (6.1) for \( m = n + 1 \). The conclusion of the lemma now follows by the principle of induction. \( \square \)

Notice that formula (6.1) interprets naturally as a congruence in \( C_\infty(\mathbb{D}) \) modulo \( \tilde{z} C_\infty(\mathbb{D}) \).

**Theorem 6.2** Let \( u_{p,q} \) be as in (0.2) for some \( p, q \in \mathbb{C} \). Then

\[
\tilde{\partial}^m u_{p,q}(0) = (p + 1)_m \quad \text{and} \quad \tilde{\partial}^m u_{p,q}(0) = (q + 1)_m
\]

for \( m \in \mathbb{N} \).

**Proof** We shall use Lemma 6.1. A point evaluation at the origin in (6.1) shows that \( \tilde{\partial}^m u_{p,q}(0) = (p + 1)_m \) for \( m \in \mathbb{N} \). Moreover, by hermitian symmetry \( \tilde{u}_{p,q} = \tilde{u}_{q,p} \) we have that

\[
\tilde{\partial}^m u_{p,q}(0) = \tilde{\partial}^m \tilde{u}_{q,p}(0) = \tilde{\partial}^m \tilde{u}_{q,p}(0) = (q + 1)_m = (q + 1)_m
\]

for \( m \in \mathbb{N} \). \( \square \)

We can now derive the series expansion of the function \( u_{p,q} \).

**Theorem 6.3** Let \( p, q \in \mathbb{C} \). Then

\[
\frac{(1 - |z|^2)^{p+q+1}}{(1 - z)^{p+1}(1 - \bar{z})^{q+1}} = \sum_{m=0}^{\infty} \frac{(p + 1)_m}{m!} F(-p, m - q; m + 1; |z|^2) z^m
\]

\[
+ \sum_{m=1}^{\infty} \frac{(q + 1)_m}{m!} F(-q, m - p; m + 1; |z|^2) \bar{z}^m
\]

for \( z \in \mathbb{D} \), where \( F \) is the hypergeometric function (0.5).
Proof Let \( u_{p,q} \) be as in (0.2). From Theorem 1.4 we know that the function \( u_{p,q} \) is \((p,q)\)-harmonic. By Theorem 5.4 it has the function series expansion

\[
\begin{align*}
  u_{p,q}(z) &= \sum_{m=0}^{\infty} \frac{a^m u_{p,q}(0)}{m!} F(-p, m - q; m + 1; |z|^2) z^m \\
  &\quad + \sum_{m=1}^{\infty} \frac{\bar{a}^m u_{p,q}(0)}{m!} F(-q, m - p; m + 1; |z|^2) \bar{z}^m
\end{align*}
\]

for \( z \in \mathbb{D} \). The result now follows from Theorem 6.2.

We point out that Theorem 6.3 generalizes a well-known partial fraction decomposition formula for the classical Poisson kernel for \( \mathbb{D} \), see formula (1.1). Suitably specialized Theorem 6.3 yields [21, Theorem 2.5] and [18, Theorem 3.2].

Notice that the series expansion in Theorem 6.3 is absolutely convergent in \( C^\infty(\mathbb{D}) \).

We denote by \( \text{Re}(z) \) and \( \text{Im}(z) \) the real and imaginary parts of a complex number \( z \), respectively. The standard Gamma function is defined by

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt
\]

for \( \text{Re}(z) > 0 \). It is well-known that \( \Gamma \) continues to a meromorphic function in \( \mathbb{C} \) with simple poles at the points \( z = 0, -1, -2, \ldots \).

We next turn to \( L^1 \) means of \( u_{p,q} \).

Theorem 6.4 Let \( u_{p,q} \) be as in (0.2) for some \( p, q \in \mathbb{C} \) such that \( \text{Re}(p) + \text{Re}(q) > -1 \). Then

\[
\frac{1}{2\pi} \int_T |u_{p,q}(re^{i\theta})| \, d\theta \leq e^{\frac{\pi}{2} |\text{Im}(p) - \text{Im}(q)|} \frac{\Gamma(\text{Re}(p) + \text{Re}(q) + 1)}{\Gamma(\text{Re}(p) + \text{Re}(q) + \frac{1}{2})^2}
\]

for \( 0 \leq r < 1 \), where \( \Gamma \) is the Gamma function.

Proof From definition of powers we have that

\[
u_{p,q}(z) = e^{(p+q+1) \log(1-|z|^2)} e^{- (p+q+2) \log|1-z|} e^{-i(p-q) \arg(1-z)}
\]

for \( z \in \mathbb{D} \). Notice that the complex number \( 1-z \) lies in the open right half-plane when \( z \in \mathbb{D} \). Passing to absolute values we have that

\[
|u_{p,q}(z)| \leq e^{\frac{\pi}{2} |\text{Im}(p) - \text{Im}(q)|} |1-|z|^2|^{-\text{Re}(p) + \text{Re}(q) + \frac{1}{2}}
\]

for \( z \in \mathbb{D} \). We next apply [18, Theorem 3.1] with \( \alpha = \text{Re}(p) + \text{Re}(q) \) to conclude that

\[
\frac{1}{2\pi} \int_T |u_{p,q}(re^{i\theta})| \, d\theta \leq e^{\frac{\pi}{2} |\text{Im}(p) - \text{Im}(q)|} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha/2 + 1)^2}
\]

for \( \alpha > 0 \).
for $0 \leq r < 1$. This yields the conclusion of the theorem. \hfill \Box

**Remark 6.5** A similar analysis as in the proof of Theorem 6.4 shows that
\[
\lim_{r \to 1} \frac{1}{2\pi} \int_{\mathbb{T}} |u_{p,q}(re^{i\theta})| \, d\theta = +\infty
\]
if $\text{Re}(p) + \text{Re}(q) \leq -1$. We omit the details.

A classical result known as Gauss summation formula says that
\[
\lim_{x \to 1} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]  \hspace{1cm} (6.3)
when $a, b \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$ are such that $\text{Re}(c) > \text{Re}(a) + \text{Re}(b)$ (see [3, Theorem 2.2.2]).

**Theorem 6.6** Let $p, q \in \mathbb{C}$ be such that $\text{Re}(p) + \text{Re}(q) > -1$. Then
\[
\lim_{r \to 1} \frac{1}{2\pi} \int_{\mathbb{T}} u_{p,q}(re^{i\theta}) \, d\theta = \frac{\Gamma(p + q + 1)}{\Gamma(p + 1)\Gamma(q + 1)},
\]
where $\Gamma$ is the Gamma function.

**Proof** Recall the series expansion of the function $u_{p,q}$ established in Theorem 6.3. By cancellation of terms we have that
\[
\frac{1}{2\pi} \int_{\mathbb{T}} u_{p,q}(re^{i\theta}) \, d\theta = F(-p, -q; 1; r^2)
\]
for $0 \leq r < 1$. Since $\text{Re}(p) + \text{Re}(q) > -1$, we can apply Gauss summation formula (6.3) to calculate the limit of the above quantity as $r \to 1$. This yields the conclusion of the theorem. \hfill \Box

### 7 The Dirichlet problem: concluding remarks

In a restricted range of parameters, the class of $(p, q)$-harmonic functions can be analyzed in terms of their boundary values. We shall discuss in this section some rudiments of such theory.

Let $p, q \in \mathbb{C}\backslash\mathbb{Z}^-$ be such that $\text{Re}(p) + \text{Re}(q) > -1$. The $(p, q)$-harmonic Poisson kernel is defined by
\[
K_{p,q}(z) = c_{p,q} \frac{(1 - |z|^2)^{p+q+1}}{(1 - z)^{p+1}(1 - \bar{z})^{q+1}}, \quad z \in \mathbb{D},
\]
where
\[
c_{p,q} = \frac{\Gamma(p + 1)\Gamma(q + 1)}{\Gamma(p + q + 1)}.
\]
Notice that the constant $c_{p,q}$ is a non-zero complex number in this range of parameters. By Theorem 1.4 the function $K_{p,q}$ is $(p,q)$-harmonic. Theorems 6.4 and 6.6 ensure that the function $K_{p,q}$ satisfies some standard properties for approximate identities.

The $(p,q)$-harmonic Poisson integral is defined by

$$K_{p,q}[f](z) = \frac{1}{2\pi} \int_{\mathbb{T}} K_{p,q}(ze^{-i\theta}) f(e^{i\theta}) \, d\theta, \quad z \in \mathbb{D},$$

for integrable functions $f \in L^1(\mathbb{T})$ on $\mathbb{T}$.

Let $C(\mathbb{T})$ be the space of continuous functions on $\mathbb{T}$ and fix $\varphi \in C(\mathbb{T})$. By the $(p,q)$-harmonic Dirichlet problem for $\varphi$ we understand the problem of finding a $(p,q)$-harmonic function $u$ such that $\lim_{r \to 1} u_r = \varphi$ in $C(\mathbb{T})$, where

$$u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \mathbb{T},$$

for $0 < r < 1$. Following usual practice, we formulate this latter Dirichlet problem as

$$\begin{align*}
L_{p,q} u &= 0 \quad \text{in } \mathbb{D}, \\
u &= \varphi \quad \text{on } \mathbb{T},
\end{align*}$$

(7.1)

where $L_{p,q}$ is as in (0.1).

**Theorem 7.1** Let $p, q \in \mathbb{C}\setminus\mathbb{Z}^-$ be such that $\text{Re}(p) + \text{Re}(q) > -1$. Let $\varphi \in C(\mathbb{T})$. Then a function $u$ in $\mathbb{D}$ satisfies (7.1) if and only if it has the form

$$u(z) = K_{p,q}[\varphi](z), \quad z \in \mathbb{D}.$$

The proof of Theorem 7.1 follows a standard scheme for such results and is therefore omitted, see [18,21] for details.

The present paper suggests a finer study of $(p,q)$-harmonic functions. Earlier results of such type concern Poisson integral representations, pointwise boundary limits, Green functions and Lipschitz continuity of generalized harmonic functions, see [18–21]. We mention here also work of Ahern and collaborators [1,2].

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