Superradiance from BEC vortices: a numerical study

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The scattering of sound wave perturbations from vortex excitations of Bose-Einstein condensates (BEC) is investigated by numerical integration of the associated Klein-Gordon equation. It is found that, at sufficiently high angular speeds, sound wave-packets can extract a sizeable fraction of the vortex energy through a mechanism of superradiant scattering. It is conjectured that this superradiant regime may be detectable in BEC experiments.

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Recent years have witnessed a growing interest in pursuing analogue models of gravitational physics in condensed matter systems. The rationale for such models traces back to a seminal observation by Unruh, who noted a close analogy between sound wave propagation in an inhomogeneous background flow and field propagation in curved space-time. The analogy goes on by observing that, much like superfluid hydrodynamics is a large-scale effective theory of microscopic superfluids, field theory on a curved space-time might also be regarded as a large-scale limit of a possible microscopic formulation of quantum gravity. The crucial point is that, whereas microscopic theories of quantum gravity are still largely a matter of speculation, the microscopic theory of superfluids is well developed. It can thus be hoped that the wide body of knowledge available for the latter can be brought to the benefit of the former. For instance, assessing the mechanisms of sound radiation from ‘terrestrial black holes’ beyond the hydrodynamic picture may in principle offer new insights into the microscopic origin of cosmic black hole radiance, the Hawking effect, and other cosmological phenomena.

A key step along this long-term program is the study of scattering and radiance phenomena from black holes whose background space-time can be associated with fluid excitations such as vortices. A model of fluid flow which seems particularly well suited to pursue the ’analogue gravity’ program is the so-called draining-bathtub geometry, namely a three-dimensional flow with a sink (vortex) at the origin. The flow field induced by the vortex is associated with an acoustic metric with two crucial ingredients of black hole physics: an event horizon and an ergosphere. The former is a spatial surface which allows only one-way propagation of physical signals (from the outside into the vortex), while the latter is a region from which part of the the vortex energy can be extracted via the mechanism of superradiance.

Such a phenomenon was first studied by Zel’ dovich with regard to the generation of waves by a rotating body and was then analysed as stimulated emission in black-hole radiance. Superresonance is an acoustic-wave version of the Penrose process, whereby a plane-wave solution of a scalar massless field in the black hole background is scattered from the ergosphere with an amplification at the expenses of the rotational energy of the black hole. Such process has been shown to occur in a certain class of analogue (2+1)-dimensional rotating black holes. Later studies have discussed the frequency dependence of the amplification factor in superresonant scattering of acoustic perturbations from a rotating acoustic black hole by deriving the reflection coefficient as a function of the frequency $\sigma_0$ of the incoming monochromatic wave. It is found that in the range $0 < \sigma_0 < m\Omega$ the reflection coefficient is greater than unity, with $m$ being the azimuthal wavenumber and $\Omega$ the angular frequency of the acoustic horizon.

The main purpose of this paper is to present a quantitative investigation of superradiant scattering from sonic holes associated with BEC-like vortex configurations. Although superradiant scattering from hydrodynamic vortices has been discussed in the recent literature, we believe that this is the first quantitative assessment of such a phenomenon under specific BEC-like conditions.

In the limit of zero temperature, gaseous Bose-Einstein condensates are well described by the Gross-Pitaevskii equation (GPE)

$$ih\partial_t \Phi = \left( -\frac{\hbar^2}{2M} \nabla^2 + V_{\text{ext}} + \frac{4\pi\hbar^2a_s}{M} |\Phi|^2 \right) \Phi, \quad (1)$$

where $\Phi(\vec{r},t)$ is the wavefunction of the condensate normalised to the total number of bosons $N$, $a_s$ being the s-wave scattering length and $M$ the mass of the atoms. If we now use the Madelung representation $\Phi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}e^{i\theta(\vec{r},t)/\hbar}$ in Eq. 1), where $\rho(\vec{r},t) = |\Phi(\vec{r},t)|^2$ is the condensate density, the GPE equation takes a hydrodynamic form: the imaginary part is a continuity equation for an irrotational fluid flow of velocity $\vec{v}(\vec{r},t) = \nabla\theta(\vec{r},t)$ and density $\rho(\vec{r},t)$, and the real part is a Hamilton-Jacobi equation whose gradient leads to the Euler equation. As is well known, the GPE is equivalent to irrotational inviscid hydrodynamics.

Low-frequency perturbations around the stationary
state are essentially sound waves (zero sound) and obey the Bogoliubov set of differential equations for the density perturbation $\rho^{(1)}$ and the phase perturbation $\theta^{(1)}$ in terms of the local speed of sound $c(\vec{r}) = \sqrt{4\pi\hbar^2 a_s (\vec{r}) / M^2}$. These equations, within the limit of validity of the hydrodynamic approximation can be reduced to a single second-order equation for the phase perturbation $\theta$. This differential equation for $\theta^{(1)} = \Psi$ has the form of a relativistic Klein-Gordon equation $\partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \Psi) = 0$, with $g = \det g_{\mu\nu}$ in a curved space-time whose metric $g_{\mu\nu}$ is determined by the local speed of sound $c$ and the background stationary velocity $\vec{v}$.

It should be noted that the linearization suppresses the quantum nature of the GPE so that, within the linear perturbation theory, the circulation of vortices is not quantised as in BEC systems. The calculations reported below are aimed at examining what fraction of the energy can be extracted through superradiant scattering of a sound wave from a vortex described by BEC-like parameters. As discussed further below, the full nonlinear GPE will have to be used for a quantitative assessment of the extraction of energy quanta from BEC vortices.

For a single vortex with a drain at $r = 0$ and angular velocity $\Omega$ in the draining-bathtub model, the velocity field of the flow is

$$\vec{v} = \nabla \theta(r, \phi) = \left(-ca\dot{r} + \Omega a^2 \dot{\phi}\right)/r.$$  

(2)

where $\dot{r}$ and $\dot{\phi}$ denote unit vectors in polar coordinates, $a$ is the radius of the event horizon, and the background density $\rho_0$ of the fluid and the speed of sound $c$ are taken as constant throughout the flow. The acoustic metric associated with this configuration is

$$ds^2 = -(c^2 - (a^2 c^2 + a^4 \Omega^2 / r^2)) dt^2 + (2ca/r) dt dr - 2\Omega a^2 dt d\phi + dr^2 + r^2 d\phi^2 + dz^2.$$  

(3)

It is readily checked that this metric has an ergosphere whose radius is $r_{\text{erg}} = a \sqrt{1 + \Omega^2 a^2 / c^2}$. The growth of the ergosphere with increasing $\Omega$ allows an increasing extraction of energy from the vortex in superradiance conditions.

Linear perturbations of the velocity potential $\Psi$ satisfy the massless Klein-Gordon scalar wave equation on this background, i.e.

$$\left[\left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{2a}{cr} \frac{\partial}{\partial t} \frac{\partial}{\partial r} - \frac{2a^2 \Omega^2}{c^2 r^2} \frac{\partial^2}{\partial \phi^2} \right) + \left(1 - \frac{a^2}{c^2} \right) \frac{\partial^2}{\partial r^2} + \frac{2a^3 \Omega^2}{cr^3} \frac{\partial}{\partial r} \frac{\partial}{\partial \phi} + \frac{c^2 r^2 - a^2 \Omega^2}{c^2 r^4} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + \frac{r^2 + a^2}{r^3} \frac{\partial}{\partial r} - \frac{2a^3 \Omega}{cr^4} \frac{\partial}{\partial \phi}\right] \Psi = 0.$$  

(4)

Equation (4) is solved by means of numerical methods developed in [13] for the integration of massless scalar-field perturbations on a rotating Kerr black-hole background. To this purpose, Eq. (4) is conveniently recast into a system of first-order (strongly) hyperbolic equations through the definition of two conjugate fields $\Xi_i = \partial \Psi / \partial x^i$ and $\Pi = -(1/\alpha) (\partial \Psi / \partial t - \beta^2 \Xi_i)$, where $\beta^2 = (ac\Omega^{-1}, -a^2 \Omega^{-2}, 0)$ and $\alpha = c$ are the space and time shifts of the acoustic metric. By setting $\Psi = \psi_1(r,t)e^{im\phi}e^{ikz}$, $\Pi = \pi_1(r,t)e^{im\phi}e^{ikz}$, $\Xi_1 = \zeta_1(r,t)e^{im\phi}e^{ikz}$, $\Xi_2 = im\Psi$, and $\zeta_2 = i\kappa \Psi$, where $(k, m)$ are the axial and azimuthal wavenumbers, the hyperbolic system reads as follows:

$$\partial_t \pi_1 + c \partial_r (\xi_1 - a\pi_1/r) = (ac - ima^2 \Omega) \pi_1/r^2 + +c (k^2 + m^2/\alpha^2) \psi_1 - c\zeta_1/r$$

$$\partial_t \zeta_1 - c \partial_r (a\psi_1/r) = (ac - ima^2 \Omega) \psi_1/r^2 - c\pi_1$$  

(5)

The set of Eqs. (5) is augmented with the constraint $|C| = |\partial_r \psi_1 - \zeta_1| = 0$, which is used to monitor the quality of the numerical results. One-way inward propagation from the horizon is accounted for by an ingoing-radiation boundary condition, imposed through an excision technique. Details of the numerical procedure will be given in a forthcoming publication [16].

Following the standard prescription for scattering processes in Kerr black holes [17], the initial condition is chosen as a Gaussian pulse centered at $r = r_0$ and modulated by a monochromatic wave,

$$\psi_1(r,0) = A \exp[-(r - r_0 + ct)^2 / b^2 - i\sigma(r - r_0 + ct)/c]\big|_{t=0}.$$  

(6)

The corresponding power spectrum is a Gaussian distribution $P(\omega) = P_{\text{max}} \exp[-(\omega - \sigma)^2 b^2 / 4c^2]$, centered at frequency $\sigma$ with spectral width $1/b$. The superradiant regime is $0 < \sigma / \Omega < m$ for $m \geq 1$.

As already remarked, the main purpose of our calculations is to assess the amount of energy that can be extracted from a BEC-like sonic hole as a function of its angular velocity $\Omega$. We use as reference a set of parameters relevant to a BEC of $^{87}$Rb atoms [18], with vortex core radius $a \sim 0.2$ µm and angular speed $\Omega \sim 18$ KHz. It is interesting to note that the inverse transversal time of the BEC vortex, $c/a \sim 15$ KHz, is very close to the corresponding value for a cosmic black hole of radius $a \sim 10$ km. Such a quantitative match stems from the very low speed of sound in BEC’s, of the order of a few mm/s.

In the following we take $c = \hbar / (\sqrt{2} M)$ and $a = \xi$, where $\xi = (8\pi\rho_{0a})^{-1/2}$ is the healing length. The integration of Eqs. (5) is performed in the space-time domain $r \in [0, 150]$ and $t \in [0, 150]$, in units of $a = 1$ and $a/c = 1$ in space and time. The angular frequency is analysed in the range $0.14 < \Omega / c < 14$, corresponding to a frequency range $1.8 < \Omega < 18$ KHz and a density range $5 \times 10^{13} < \rho_0 < 5 \times 10^{14}$ cm$^{-3}$. The initial Gaussian pulse is centered at $r_0 = 50a$ and $\sigma = 0.5\Omega$, with amplitude $A = 0.3c$ and variance $b = 10a$. We perform our study for $m = 1$ and $k = 0.02/a$, corresponding to a condensate axial extent $H = 0.9$ mm. Violations of the
constraint $C(t) = 0$ are monitored over the entire space domain and are found to be consistently below $10^{-6}$ for all sets of parameters under investigation.

Fig. 1 shows a density map of the real part of $\psi_1$ in the range $r/a \in [1, 20]$ and $tc/a \in [0, 10]$. The initial Gaussian pulse moves towards the vortex horizon placed at $r = a$ and its trajectory is bent by the potential outside the horizon. The bending of the trajectory, with lightcones heading towards the horizon, is consistent with similar findings in numerical relativity [7, 19] as shown in the diagram in Fig. 1.

In Fig. 2 we show a typical time evolution of the energy of wavepacket $E_p(t) = (\rho_0 M/2) \int_0^{2\pi} d\phi \int_0^H dz \int_a^{143a} v_1^2 r \, dr$ with $v_1 = \nabla g(t)$ (top curves), normalized to its initial value $E_p(0)$, as well as the (independently calculated) rate $F(t)$ of change of the energy (bottom curves), normalized to its initial value $F(0)$, for $\sigma = 0.7c/a$ and $\Omega = 1.4c/a$, within the superradiant regime ($m = 1$, solid lines) and outside it ($m = 0$, dashed lines). $F(t)$ includes the net flux across the surfaces at $r = a$ and $r = 143a$ as well as a term due to the bulk compressibility,

$$F(t) = \frac{dE_p(t)}{dt} = \int \vec{v}_1 \cdot \vec{\Omega}_1 \, dV = \frac{dE_p(t)}{dt} = \frac{1}{2} \left[ \int v_1^2 \vec{v} \cdot \hat{n} \, dS - \int v_1^2 \nabla \cdot \vec{v}_1 \, dV \right].$$

In the non-superradiant case, the energy of the scattered wavepacket goes asymptotically to zero, indicating that all the energy of the impinging wavepacket is lost to the vortex sink. In the superradiant case instead, the energy of the back-scattered wavepacket exceeds its initial value, indicating extraction of energy from the ergosphere at the expense of the rotational energy of the vortex. Consistently with this picture, the energy flux for the superradiant (non-superradiant) case lies above (below) its background value during the scattering event, approximately in the range $35 < tc/a < 55$. The energy gained via superradiance is by no means small, as it is seen to exceed in this case twenty percent of the initial value $E_p(0)$.

It is now of great interest to examine the dependence of the superradiant energy gain on the angular speed of the vortex, so as to possibly identify an optimal value at which such energy gain can be maximised. In Fig. 3 we show the time evolution of the energy gain $E_p(t)/E_p(0)$ for a series of values of $\Omega$ in the superradiant range $\Omega a/c \in [0.6, 14]$, as can be experimentally achieved by varying the density of the condensate. A sharp increase of the energy gain is observed in the region $\Omega > c/a$. This is plausible, since $\Omega > c/a$ marks a transition from the regime where the radius of the ergosphere remains within a factor two of the sonic horizon, to the regime where it grows linearly with $\Omega$, thereby creating a sizeable ergospheric shell where energy can be extracted from.

Since BEC vortices are quantized, one is naturally led...
to ask whether the wavepacket may annihilate the vortex by extracting all of its energy in a much more demanding 'break-even' condition, that is $E_p(\infty) - E_p(0) = E_b$ where $E_b$ denotes the energy of the background vortex. In Fig. 4 we show the background energy $E_b$ (dashed line) and the total energy gain $\Delta E_p = (E_p(\infty) - E_p(0))$ for three values of $\sigma/\Omega$ in the superradiant range (solid lines), as functions of $\Omega$. In the perturbative regime (for $\Omega \lesssim 3c/a$, say) the efficiency of energy extraction from the vortex grows much faster with $\Omega$ than the quadratic increase of the background energy. This is especially true at large values of the ratio $\sigma/\Omega$. Although substantial values of $\Delta E_p/E_b$ are - by definition - beyond the scope of the perturbative Klein-Gordon description used throughout this work, it appears that nonlinear effects may primarily determine the way in which the energy extraction behaves as it becomes comparable to the background energy. The possibility that substantial superradiance efficiencies, as they emerge from the Klein-Gordon analysis, may persist even in the non-perturbative quantum regime described by the GPE cannot be ruled out. Even though the condition $\Delta E_p = E_b$ may remain out of reach for a single wavepacket, one may still conjecture that a train of wavepackets could reach the goal. It would be interesting to test this conjecture by numerical and experimental means.

As a further development of the present model, it will also be interesting to consider a vortex with no drain and to apply our analysis to superradiant scattering from a giant vortex. Such vortices have been found to have up to 60 quanta of circulation and can therefore be well approximated within the classical limit. In summary, numerical simulations of the Klein-Gordon equation for sonic perturbations impinging on a BEC-like vortex suggest the possibility that, under typical conditions of BEC experiments, a significant fraction of the vortex energy may be extracted via the mechanism of superradiance. It would be very interesting to test the feasibility of such an exciting scenario both via the numerical solution of the Gross-Pitaevskii equation and by actual experiments on rotating Bose-Einstein condensates.

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