Hardy’s argument and successive spin-s measurements

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We consider a hidden-variable theoretic description of successive measurements of noncommuting spin observables on a input spin-s state. In this scenario, the hidden-variable theory leads to a Hardy-type argument that quantum predictions violate it. We show that the maximum probability of success of Hardy’s argument in quantum theory is \((\frac{1}{2})^{4s}\), which is more than in the spatial case.

I. INTRODUCTION

The well-known contradiction between quantum theory and local realism theory was first pointed out by Einstein, Podolosky and, Rosen [1]. In 1964 John Bell [2] derived an inequality using local hidden-variable theories (LHVTs) [3], which could be construed as a form of local realism) and showed that quantum mechanics and LHVTs predicted statistically different experimental results for certain combination of correlations in two distant systems. According to Bell’s theorem it follows that any classical imitation of quantum mechanics (QM) is necessary nonlocal. Recently, Gröblacher et al. [4], based on Leggett’s inequality [5], have shown that a broad class of nonlocal hidden-variable theories (HVTs) is still inconsistent with QM’s predictions in both theory and experiment (for more discussion, see [3]).

In contrast, some investigators were trying to show a direct contradiction (without using inequality) of QM versus LHVT [6–8]. All proofs required a minimum total of six dimensions in Hilbert space rather than the four required by Bell in his proof. Hardy [9, 10] gave a proof of nonlocality without using inequality for two qubits and all pure entangled states except maximally entangled states. Hardy’s argument has been generalized for many qubits and higher dimensional bipartite systems [11–14]. Cabello gave another argument of Bell’s theorem without inequalities for GHZ and W states [15]. Based on Cabello’s logic structure, Liang et al. [16] provided an example of a two-qubit mixed state which shows nonlocality, still without inequality. In this sense, Hardy’s logical structure is a special case of Cabello’s structure. In contrast, the authors of Ref. [17] have proved that the nonlocality argument proposed by Cabello is more general than Hardy’s nonlocality argument. They showed that the maximum probability of success of the Hardy and Cabello nonlocality (for the two-qubit system) in QM is 0.09 and 0.1078 respectively.

Entanglement in time is not introduced in QM because of the different roles that time and space play in quantum theory. The meaning of locality in time is that the results of measurement at time \(t_2\) are independent of any measurement performed at some earlier time \(t_1\) or later time \(t_3\). In Ref. [18], the authors proposed and analyzed a particular scenario to account for the deviations of QM from “realism,” which involves correlations in the outputs of successive measurements of noncommuting operations in a spin-s state. The correlations of successive measurements have been used by Popescu [19] in the context of nonlocal quantum correlations, to analyze a class of Werner states [20], which are entangled but do not break bipartite Bell-type inequality. Leggett and Garg [21] have used consecutive measurements to challenge the applicability of QM to macroscopic phenomena (see also [22]). Bell-type inequalities with successive measurements were first considered by Brunkner et al. [23]. They have derived CHSH-type inequality [24] for two successive measurements on an arbitrary state of a single qubit and have shown that every such state would violate that inequality for proper choice of the measurement setting.

In the present paper we study Hardy’s nonlocality arguments for the correlations between the outputs of \(n\) successive measurements for all \(s\)-spin measurements. We show that the maximum probability of success of Hardys argument in the successive measurement is much higher than the spatial ones in a certain sense.

The paper is organized as follows. In Sec. II, we consider the basic scenario in detail. Section III explains the logical structure of Hardy’s argument on time locality. In Sec. IV, we show that no time-local stochastic HVT (SHVT) can simultaneously satisfy Hardy’s argument. The maximum probability of success of Hardy’s argument for \(n\)-successive spin-s measurements is given in Sec. V. Section VI reports the conclusion. A proof is given in the Appendix.

II. THE BASIC SCENARIO

Consider the following sequence of measurements. A quantum particle with spin \(s\), prepared in the initial state \(\rho_0\), is sent through a cascade of Stern-Gerlach (SG) measurements for the spin components along the directions given by the unit vectors \(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots, \hat{a}_n\) (i.e., measurement of observables of the form \(\hat{S}\hat{a}\), where \(\hat{S} = (S_x, S_y, S_z)\) is the vector of spin angular momentum op-
operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ and $\hat{a}$ is a unit vector from $\mathbb{R}^3$. Each measurement has $2s + 1$ possible outcomes. For the $i$-th measurement, we denote these outcomes (eigenvalues) $\alpha_i \in \{s, s - 1, \ldots, -s\}$.

Each of the $(2s + 1)^n$ possible outcomes which one gets after performing $n$ consecutive measurements corresponds to a particular combination of the results of the measurements at previous $n - 1$ steps and the result of the measurement at the $n$th step. The probability of each of these $(2s + 1)^n$ outcomes is the joint probability for such combinations. Note that even though the spin observables $\hat{S} \cdot \hat{a}_1, \hat{S} \cdot \hat{a}_2, \ldots, \hat{S} \cdot \hat{a}_n$, whose measurements are being performed at times $t_1, t_2, \ldots, t_n$, respectively (with $t_1 < t_2 < \ldots < t_n$), do not commute, the aforementioned joint probabilities for the outcomes are well defined because each of these spin observables acts on different states.

We emphasize that this is the joint probability for the results of $n$ actual measurements and not a joint probability distribution for hypothetical simultaneous values of $n$ noncommuting observables. Moreover, various sub-beams (i.e., wave functions) emerging from every SG apparatus [corresponding to $(2s + 1)$ outcomes] at each stage of measurement are separated, without any overlap or recombination between them. In other words, the eigenwave packet $\psi_{s-j, t, \hat{a}}(x)$, corresponding to the eigenvalue $s-j$ of the observable $\hat{S} \cdot \hat{a}_j$, measured at time $t_j$, will not have any part in regions where the SG setups, for measurement of the observables $\hat{S} \cdot \hat{a}_{i+1,s}, \hat{S} \cdot \hat{a}_{i+1,s-1}, \ldots, \hat{S} \cdot \hat{a}_{i+1,s-j-1}, \hat{S} \cdot \hat{a}_{i+1,s-j-1}, \ldots, \hat{S} \cdot \hat{a}_{i+1,s}$, are situated. We further assume that, between two successive measurements, the spin state does not change with time, that is, $\hat{S}$ commutes with the interaction Hamiltonian, if any. Also, throughout the string of measurements, no component (i.e., sub-beam) is blocked. It should be mentioned here that the time of each of the measurements is measured by a common clock.

Let us consider an SHVT consists of: (i) a set $\Lambda$ whose elements $\lambda$ are called hidden variables; (ii) a normalized and positive probability distribution $p(\lambda)$ defined on $\Lambda$; and (iii) a set of probability distributions $p_\lambda(s.\hat{a}_1 = \alpha_1, s.\hat{a}_2 = \alpha_2, \ldots, s.\hat{a}_n = \alpha_n)$ for the outcomes of $n$ successive measurements, such that:

$$p_{QM}(s.\hat{a}_1 = \alpha_1, s.\hat{a}_2 = \alpha_2, \ldots, s.\hat{a}_n = \alpha_n) = \int_{\lambda} d\lambda p(\lambda)p_\lambda(s.\hat{a}_1 = \alpha_1, s.\hat{a}_2 = \alpha_2, \ldots, s.\hat{a}_n = \alpha_n).$$

(1)

Here the quantities on the left-hand side are the probability distributions which QM attaches to the successive outcomes $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the considered measurements on spin $s$.

By using the generalized equation (31) (see the Appendix), we get:

$$p_{QM}(s.\hat{a}_1 = \alpha_1, \ldots, s.\hat{a}_n = \alpha_n) = [d_{\alpha_1\alpha_2}(\beta_1 - \beta_0)d_{\alpha_1\alpha_2}(\beta_2 - \beta_1) \cdots d_{\alpha_{n-1}\alpha_n}(\beta_n - \beta_{n-1})]^2$$

(2)

where $d_{\alpha\beta}(\beta) \equiv \langle s, \alpha| \exp(-i\hat{S} \cdot \hat{a}(\beta))|s, \beta \rangle$.

A deterministic hidden-variable model is a particular instance of a stochastic one where all probabilities $p_\lambda$ can take only the values 0 or 1. We now analyze the consequences of SHVT for our scenario. In general, the outputs of $k$th and $l$th experiments may be correlated so that

$$p(s.\hat{a}_l = \alpha_l \& s.\hat{a}_k = \alpha_k) \neq p(s.\hat{a}_l = \alpha_l)p(s.\hat{a}_k = \alpha_k).$$

(3)

However, in SHVT we suppose that these correlations have a common cause represented by a stochastic hidden variable $\lambda$ so that

$$p_\lambda(s.\hat{a}_l = \alpha_l \& s.\hat{a}_k = \alpha_k) = p_\lambda(s.\hat{a}_l = \alpha_l)p_\lambda(s.\hat{a}_k = \alpha_k).$$

(4)

As a consequence of Eq. (1), the probability $p_\lambda(s.\hat{a}_k = \alpha_k)$ obtained in a measurement (say, $\hat{S} \cdot \hat{a}_k$) performed at time $t_k$ is independent of any other measurement (say, $\hat{S} \cdot \hat{a}_l$) made at some earlier or later time $t_l$. This is called locality in time [21, 22].

One should note that for a two-dimensional QM system, one can always assign values (deterministically or probabilistically) to the observables with the help of a HVT. Once the measurement is done, the system will be prepared in an output state (namely, an eigenstate of the observable), and the earlier HVT may or may not work to reproduce the values of the observables to be measured in that output state (prepared after the first measurement). In the present paper, we have considered the possibility of the existence of an HVT for every input qubit state which can reproduce the measurement outcomes of $n$ successive measurements.

III. THE LOGICAL STRUCTURE OF HARDY’S ARGUMENT ON TIME LOCALITY

Consider four yes/no-type events $A, A', B$ and $B'$, where $A$ and $A'$ may happen at time $t_1$, and $B$ and $B'$ may happen at another time, $t_2$ ($t_2 > t_1$). The joint probability that, at the first time ($t_1$), $A$ and, at the second time ($t_2$), $B$ are “yes” is 0. The joint probability that, at the first time ($t_1$), $A$ is “no” and, at the second time ($t_2$), $B'$ is “yes” is 0. The joint probability that, at the first time ($t_1$), $A'$ is “yes” and, at the second time ($t_2$), $B$ is “no”, is 0. The joint probability that both $A'$ and $B'$ are “yes” is nonzer. We can write this as follows

$$p(A = +1, B = +1) = 0,$$

$$p(A = -1, B' = +1) = 0,$$

$$p(A' = +1, B = -1) = 0,$$

$$p(A' = +1, B' = +1) = p \neq 0.$$ 

(5)

In the next section, we show that these four statements are not compatible with time-local realism. The nonzero
probability appearing in the argument is the measure of violation of time-local realism. It is interesting that two successive s-spin measurements violate time-local realism.

IV. HARDY’S ARGUMENT FOR n SUCCESSIVE MEASUREMENTS FOR ALL SPIN-s MEASUREMENTS

We deal with the case where the input state is a pure state whose eigenstates coincide with those of \( s\hat{a}_0 \) for some \( a_0 \) whose eigenvalues we denote \( \alpha_0 = j \). Hardy’s argument for a system of \( n \) successive spin-s measurements, in it’s minimal form \[28\], is given by following conditions:

\[
p(s\hat{a}_1 = j, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j) = 0, \tag{6}
\]
\[
p(s\hat{a}_1 = j - 1, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j) = 0,
\]
\[
p(s\hat{a}_1 = j - 2, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j) = 0,
\]
\[
p(s\hat{a}_1 = -j, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j) = 0, \tag{7}
\]
\[
p(s\hat{a}_1 = j, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j) = 0, \tag{8}
\]
\[
p(s\hat{a}_1 = j, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j - 1) = 0,
\]
\[
p(s\hat{a}_1 = j, s\hat{a}_2 = j, \ldots, s\hat{a}_n = j - 2) = 0,
\]
\[
p(s\hat{a}_1 = j, s\hat{a}_2 = j, \ldots, s\hat{a}_n = -j) = 0, \tag{9}
\]
\[
p(s\hat{a}_1^\prime = j, s\hat{a}_2^\prime = j, \ldots, s\hat{a}_n^\prime = j) = p. \tag{10}
\]

First, we prove here that all time-local SHVTs predict \( p = 0 \). Suppose that a time-local SHVT reproducing, in accordance with Eq.\[4\], the quantum predictions exist. Accordingly, if we consider, for example, Eq.\[6\], we must have

\[
p(s\hat{a}_1^\prime = j, \ldots, s\hat{a}_l = j - 1, \ldots, s\hat{a}_n^\prime = j) = \int_\Lambda d\lambda \rho(\lambda)p_\lambda(s\hat{a}_1^\prime = j, \ldots, s\hat{a}_l = j - 1, \ldots, s\hat{a}_n^\prime = j)
\]
\[
= \int_\Lambda d\lambda \rho(\lambda)p_\lambda(s\hat{a}_1 = j) \ldots p_\lambda(s\hat{a}_l = j - 1) \ldots p_\lambda(s\hat{a}_n^\prime = j)
\]
\[
= 0, \tag{11}
\]

where the second equality is implied by the time-locality condition of Eq.\[4\]. The last equality in Eq.\[11\] can be fulfilled if and only if the product \( p_\lambda(s\hat{a}_1^\prime = j) \ldots p_\lambda(s\hat{a}_n^\prime = j) \) vanishes every time within \( \Lambda \). An equivalent result holds for Eqs.\[6, 7, 8, 9\], leading to:

\[
p_\lambda(s\hat{a}_1 = j)p_\lambda(s\hat{a}_2 = j) \ldots p_\lambda(s\hat{a}_n = j) = 0, \tag{12}
\]
\[
p_\lambda(s\hat{a}_1 = j - 1)p_\lambda(s\hat{a}_2 = j) \ldots p_\lambda(s\hat{a}_n = j) = 0,
\]
\[
p_\lambda(s\hat{a}_1 = j - 2)p_\lambda(s\hat{a}_2 = j) \ldots p_\lambda(s\hat{a}_n = j) = 0,
\]
\[
p_\lambda(s\hat{a}_1 = -j)p_\lambda(s\hat{a}_2 = j) \ldots p_\lambda(s\hat{a}_n = j) = 0, \tag{13}
\]
\[
p_\lambda(s\hat{a}_1 = j) \ldots p_\lambda(s\hat{a}_l = j - 1) \ldots p_\lambda(s\hat{a}_n = j) = 0,
\]
\[
p_\lambda(s\hat{a}_1 = j) \ldots p_\lambda(s\hat{a}_l = j - 2) \ldots p_\lambda(s\hat{a}_n = j) = 0,
\]
\[
p_\lambda(s\hat{a}_1 = -j) \ldots p_\lambda(s\hat{a}_n = j) = 0, \tag{14}
\]

\[\]
where the first $2jn + 1$ equations are supposed to hold almost every time within $\Lambda$, while the last equation has to be satisfied in a subset of $\Lambda$ whose measure according to the distribution $p(\lambda)$ is nonzero. To prove the more general result that no conceivable time-local SHVT can simultaneously satisfy Eqs. (12)–(16), a manipulation of those equations is required. To this end, let us sum all equations in each set. We obtain
\[
(1 - p(\lambda(\hat{a}_1 = j))) \left[ p(\lambda(s.\hat{a}_1' = j)) \ldots p(\lambda(s.\hat{a}_n' = j)) \right] = 0,
\]
where $p(\lambda(\hat{a}_1 = j)) = 1$ and $p(\lambda(s.\hat{a}_1' = j)) = 0$, for all $\lambda \in \Lambda$. Hence, we obtain a result leading to a contradiction.

Now let us partition the set of hidden variables $\Lambda$ and define the following subsets $A_1, A_2, \ldots, A_n$, and $B$ as:
\[
A_1 = \{ \lambda \in \Lambda | p(\lambda(s.\hat{a}_1 = j)) = 0 \},
\]
\[
A_i = \{ \lambda \in \Lambda | p(\lambda(s.\hat{a}_i = j)) = 0 \}, \quad i = 2, \ldots, n - 1,
\]
\[
A_n = \{ \lambda \in \Lambda | p(\lambda(s.\hat{a}_n = j)) = 0 \},
\]
\[
B = \Lambda - \{ A_1 \cup A_2 \cup \ldots \cup A_n \}.
\]

We have that, for all $\lambda$ belonging to $B$, $p(\lambda(s.\hat{a}_1 = j)p(\lambda(s.\hat{a}_2 = j) \ldots p(\lambda(s.\hat{a}_n = j)) \neq 0$. If set $B$ had a nonzero measure according to the distribution $\rho$, that is, if $\int_B d\lambda \rho(\lambda) \neq 0$, there would be violation of Eq. (16) and, consequently, of Eq. (15). Therefore, to fulfill Eq. (12), the set $A_1 \cup A_2 \cup \ldots \cup A_n$ must coincide with $\Lambda$ apart from a set of zero measure, and we are left only with hidden variables belonging to either $A_1$ or $A_2$ or ... or $A_n$. If $\lambda$ belongs to $A_1$, then, by definition, $p(\lambda(s.\hat{a}_1 = j)) = 0$, so that Eq. (17) can be satisfied only if $p(\lambda(s.\hat{a}'_1 = j) \ldots p(\lambda(s.\hat{a}'_n = j) = 0$. Hence, for any $\lambda \in \{ A_1 \cup A_2 \cup \ldots \cup A_n \}$, we obtain a result leading to a contradiction of Eq. (15), which requires that there is a set of nonzero $\rho$ measure within $\Lambda$ where both probabilities do not vanish.

To summarize, we have shown that it is not possible to exhibit any time-local hidden-variable model, satisfying Hardy’s logic for $n$ successive measurements.

In the next section, we show that in quantum theory for the $n$ successive spin measurements, sometimes $p > 0$.

V. HARDY’S ARGUMENT RUNS FOR $n$ SUCCESSIVE SPIN MEASUREMENTS BY QUANTUM MECHANICS

Hardy’s nonlocality argument is considered weaker than Bell inequalities in the bipartite case, as every maximally entangled state of two spin-$\frac{1}{2}$ particles violates Bell’s inequality maximally but none of them satisfies Hardy-type nonlocality conditions. The scenario in successive spin measurements is quite different, however. We showed in (15) (also see (13)) that all $n$ successive spin measurements break Bell-type inequalities, in contrast to the bipartite case, where only the entangled states break it. In this section, we prove that all $n$ successive spin measurements satisfy Hardy-type argument conditions. We now consider $n$ successive measurements in directions $\hat{a}_i$ ($i = 1, 2, \ldots, n$) on spin-$s$ particles. For a spin-$s$ system, we have (see the Appendix):
\[
|\langle \alpha_{k-1} | \alpha_k \rangle| = |\langle s.\hat{a}_{k-1} | s.\hat{a}_k \rangle| = d_{\alpha_{k-1}, \alpha_k}^s (\beta_k - \beta_{k-1}),
\]
where $\beta_k$ is the angle between the $\hat{a}_k$ and the $z$ axes. So, given the input state $|\alpha_0\rangle$, the (joint) probability that the measurement outcomes will be $|\alpha_1\rangle \in \{ +j, \ldots, -j \}$ in the first measurement, $|\alpha_2\rangle \in \{ +j, \ldots, -j \}$ in the second measurement, ... , $|\alpha_n\rangle \in \{ +j, \ldots, -j \}$ in the $n$-th measurement, is given by
\[
p(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Pi_{k=1}^n |\langle \alpha_{k-1} | \alpha_k \rangle| = \Pi_{k=1}^n d_{\alpha_{k-1}, \alpha_k}^s (\beta_k - \beta_{k-1}).
\]

We deal with the case where the input state is a pure state whose eigenstates coincide with those of $\hat{S}_z$, for some $\alpha_0$ whose eigenvalues we denote $\alpha_0 = j$. Now, by substituting Eq. (21) in the minimal form of Hardy’s argument [Eqs. (6)–(10)], we have
\[
d_{jj}(\beta_1) d_{jj}^2 (\beta_2 - \beta_1) \ldots d_{jj}^n (\beta_n - \beta_{n-1}) = 0,
\]
\[
d_{jj-1}(\beta_1) d_{jj-1}^2 (\beta_2 - \beta_1) \ldots d_{jj}^n (\beta_n - \beta_{n-1}) = 0,
\]
\[
d_{jj-2}(\beta_1) d_{jj-2}^2 (\beta_2 - \beta_1) \ldots d_{jj}^n (\beta_n - \beta_{n-1}) = 0,
\]
\[
\ldots
\]
\[
d_{jj-n}(\beta_1) d_{jj-n}^2 (\beta_2 - \beta_1) \ldots d_{jj}^n (\beta_n - \beta_{n-1}) = 0.
\]
From Eq (22), at least one of the factors must be 0. So
\[ d_{jj}(\beta_1) \ldots d_{j,j-1}(\beta_i - \beta_{i-1})d_{j,j-1}(\beta_{i+1} - \beta_i) \ldots d_{jj}(\beta_n - \beta_{n-1}) = 0, \]
\[ d_{jj}(\beta_1) \ldots d_{j,j-2}(\beta_i - \beta_{i-1})d_{j,j-2}(\beta_{i+1} - \beta_i) \ldots d_{jj}(\beta_n - \beta_{n-1}) = 0, \]
\[ d_{jj}(\beta_1') \ldots d_{j,j-1}(\beta_i' - \beta_{i-1})d_{j,j-1}(\beta_{i+1}' - \beta_i) \ldots d_{jj}(\beta_n' - \beta_{n-1}) = 0, \]

(24)

To satisfy all equations (23) - (25), we have the following conditions:

\[ (\beta_1 = 0) \text{ or } (\beta_2 = \beta_1) \]
and
\[ (\beta_2 = \beta_1') \text{ or } (\beta_3 = \beta_1) \]

(27)

\[ (\beta_i = \beta_{i-1}) \text{ or } (\beta_i = \beta_{i+1}) \]
and
\[ (\beta_n = \beta_{n-1}). \]

Now, we can calculate the maximum value \( p \) by using these conditions. For example, if we select \( \beta_1 = \pi \), so we must have \( \beta_2 = \beta_1 = \pi \). In this case,

\[ p = d_{jj}(\beta_1')d_{jj}(\pi - \beta_1) \ldots d_{jj}(\beta_n' - \beta_{n-1}). \]

(28)

By substituting \( d_{jj}(\beta) = \cos^{2j}(\beta / 2) \), we have

\[ p = \cos^{4j}(\beta_1') \cos^{4j}(\pi - \beta_1) \ldots \cos^{4j}(\beta_n' - \beta_{n-1}). \]

(29)

By selecting \( \beta_n' = \beta_{n-1}' = \ldots = \beta_4' = \pi \) and \( \beta_1' = \pi \),

\[ p \leq \left( \frac{1}{2} \right)^{4j}. \]

(30)

We can obtain this result in the general case. We choose \( |\beta_l - \beta_{l-1}| = \pi \), where \( 2 \leq l \leq n \). Without loss of generality, we select \( \beta_l = \pi \) and \( \beta_{l-1} = 0 \). From the results obtained with Eq (27), \( \beta_{l+1} = \beta_l = \pi \) or \( \beta_{l+1} = \beta_{l-1} = \beta_l = \pi \). Exactly for (1-1)th in Eq (27), we have \( \beta_1 = \beta_{l-1} = 0 \) or \( \beta_{l-2} = \beta_l = \beta_{l-1} = \pi \). So we have four cases:

(i) \( \beta_{l-1} = \pi \) and \( \beta_{l-2} = 0 \)
(ii) \( \beta_{l-1}' = \pi \) and \( \beta_{l-2}' = 0 \)
(iii) \( \beta_{l+1} = \pi \) and \( \beta_{l}' = 0 \)
(iv) \( \beta_{l+1}' = \pi \) and \( \beta_{l}' = 0 \)

It is easy to see that for the first three cases, the maximum value of \( p \) is 0, but in the fourth case, by selecting \( \beta_1' = \beta_2 = \ldots = \beta_{l-1}' = 0 \) and \( \beta_{l+2}' = \beta_{l+3}' = \ldots = \beta_n' = \pi \), we get

\[ p = \cos^{4j}(\frac{\beta_1'}{2}) \sin^{4j}(\frac{\beta_1'}{2}) \leq \left( \frac{1}{2} \right)^{4j}. \]

(31)

We see that \( p > 0 \) for all spins, and also, the maximum probability of success of Hardy’s non-time locality is independent of the number of successive measurements and decreases with \( s \).
We considered an HVT description of successive measurements of noncommuting spin observables on an input spin- \( s \) state. Although these spin observables are noncommuting, they act on different states and so the joint probabilities for the outputs of successive measurements are well defined. In Ref. 18, we account for the maximum violation of these inequalities by quantum correlations by varying the spin value and the number of successive measurements. Our approach can be used to obtain a measure of the deviation of QM from theory obeying realism and time locality and may lead to a sharper understanding of QM.

In the present paper, we have studied Hardy’s argument for the correlations between the outputs of \( n \) successive measurements for all \( s \)-spin measurements. We have shown that the maximum probability of success of Hardy’s argument for \( n \) successive measurements is \( \left( \frac{1}{2} \right)^{4s} \), which is independent of the number of successive measurements of spin \( (n) \) and decreases with increase of \( s \). This can be compared with the correlations corresponding to measurement of spin observables in a space-like separated two-particles scenario where only the non-maximally entangled states of any spin- \( s \)-bipartite system respond to Hardy’s nonlocality test. The authors of Ref. 28 showed that the maximum nonlocalities for a given pair of noncommuting spin- \( s \) observables per site turned out to be the same \( (0.0901099) \) for the \( s = \frac{1}{2}, 1, \text{and} \frac{3}{2} \).

So the maximum amount of nonlocality that can be obtained by Hardy’s nonlocality test on bipartite systems of higher spin values also remains the same.

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APPENDIX

Let us consider a situation where an ensemble of systems prepared in state \( \rho = | s, \alpha \rangle \langle s, \alpha | > | s, \beta \rangle \langle s, \beta | \) at time \( t = 0 \), is subjected to a measurement of the observable \( A(t_1) = s, \alpha \) at time \( t_1 \) followed by a measurement of the observable \( B(t_2) = s, \beta \) at time \( t_2 \) \( (t_2 > t_1 > 0) \), where we have adopted the Heisenberg picture of time evolution. Further, let us assume that both \( A(t_1) \) and \( B(t_2) \) have purely discrete spectra. Let \( \{ \alpha_1 \} = \{ -s, -s+1, \ldots, s \} \) and \( \{ \alpha_2 \} = \{ -s, -s+1, \ldots, s \} \) denote the eigenvalues and \( P^A(t_1)(\alpha_1) = | \langle s, \alpha \rangle |^2 \) and \( P^B(t_2)(\alpha_2) = | \langle s, \beta \rangle |^2 \) the corresponding eigenvectors of \( A(t_1) \) and \( B(t_2) \) respectively. Then the joint probability that a measurement of \( A(t_1) \) yields the outcome \( \alpha_1 \) and a measurement of \( B(t_2) \) yields the outcome \( \alpha_2 \) is given by

\[
P_R^\rho_{A(t_1),B(t_2)}(\alpha_1, \alpha_2) = \text{Tr} \left[ P^B(t_2)(\alpha_2) P^A(t_1)(\alpha_1) \rho P^A(t_1)(\alpha_1) P^B(t_2)(\alpha_2) \right] = | \langle s, \alpha \rangle |^2 | \langle s, \beta \rangle |^2.
\]

We know that \( | s, \alpha \rangle = \alpha \rangle = \sum_{m=-s}^{s} d_{\alpha,m}^{(s)}(\beta) | m \rangle \) where \( d_{\alpha,m}^{(s)}(\beta) \equiv \langle s, \alpha | e^{-i \mathcal{A}(\beta)} | m \rangle \) and it obtains Wigner’s formula [29] and \( \alpha_1 \in -s, \ldots, s \) and \( \beta_1 \) is the angle between the \( \alpha_1 \) and the \( z \) axes. In contrast,

\[
\langle s, \alpha_1 | s, \beta_2 \rangle = \sum_{m} m | d_{\alpha_1,m}^{(s)}(\beta_1) \sum_{m'} d_{\alpha_2,m'}^{(s)}(\beta_2) | m' > = \sum_{m} d_{\alpha_1,m}^{(s)}(\beta_1) d_{\alpha_2,m'}^{(s)}(\beta_2) = d_{\alpha_1,\alpha_2}^{(s)}(\beta_2 - \beta_1).
\]

So, we obtain

\[
\rho_{\text{QM}}(\alpha_1, \alpha_2) = | d_{\alpha_1,\alpha_2}^{(s)}(\beta_2 - \beta_1) |^2 | d_{\alpha_1,\alpha_2}^{(s)}(\beta_2 - \beta_1) |^2.
\]
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