PLASTIC DAMPING OF ALFVÉN WAVES IN MAGNETAR FLARES AND DELAYED AFTERGLOW EMISSION

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ABSTRACT

Magnetar flares generate Alfvén waves bouncing in the closed magnetosphere with energy up to \( \sim 10^{46} \, \text{erg} \). We show that on a timescale of 10 ms the waves are transmitted into the star and form a compressed packet of high energy density. This packet strongly shapes the stellar crust and initiates a plastic flow, heating the crust and melting it hundreds of meters below the surface. A fraction of the deposited plastic heat is eventually conducted to the stellar surface, contributing to the surface afterglow months to years after the flare. A large fraction of heat is lost to neutrino emission or conducted into the core of the neutron star.

Key words: dense matter – magnetic fields – stars: magnetars – stars: neutron – waves

1. INTRODUCTION

Magnetars are luminous, slowly rotating neutron stars powered by the decay of ultraradient magnetic fields \( B = 10^{14} - 10^{16} \, \text{G} \) (see, e.g., Woods & Thompson 2006; Mereghetti 2008 for reviews). They have hot surfaces, and produce nonthermal magnetospheric radiation and strong bursts of hard X-rays. Occasionally, magnetars produce giant flares with energies of \( 10^{44} - 10^{46} \, \text{erg} \). To date three giant flares have been observed from three magnetars. The main peak of the giant flare lasts \( \sim 0.3 \, \text{s} \) and can reach huge luminosities \( \mathcal{L} \sim 10^{47} \, \text{erg} \, \text{s}^{-1} \). Less powerful flares with \( \mathcal{L} < 10^{42} \, \text{erg} \, \text{s}^{-1} \) (often called “bursts”) occur much more frequently.

The flares are associated with a sudden change in the magnetospheric configuration, which could be triggered by an instability inside or outside the neutron star (Thompson & Duncan 1996). This cataclysmic event involves strong deviations from the magnetostatic equilibrium, launching waves of large amplitudes. Part of the released magnetic energy is promptly dissipated and converted to radiation, and part is stored in the excited waves.

In particular, Alfvén waves are generated with a total energy up to \( \sim 10^{46} \, \text{erg} \). They are trapped on the closed magnetic field lines, as the group velocity of Alfvén waves is parallel to the magnetic field. The fate of their energy is poorly known. It was proposed that the Alfvén waves can be damped through nonlinear processes (Thompson & Blaes 1998), which become efficient at very large amplitudes of the waves (see Section 5.2).

In this paper, we propose another mechanism of the Alfvén wave dissipation, which results from the wave interaction with the star. The waves are ducted along the magnetic field lines at nearly the speed of light and reach the stellar surface on a millisecond timescale. In Section 2 we examine the wave interaction with the star and find that a significant fraction of the wave energy is transmitted into the stellar crust. The reflected waves keep bouncing in the magnetosphere, but in a few tens of milliseconds most of their energy is drained and deposited into the crust, in the form of a compressed shear wave packet. In Section 3, we show that this packet causes strong plastic heating of the crust. In Section 4 we investigate the fate of heat deposited by the plastic damping of Alfvén waves. In particular, we evaluate the heat flux conducted back to the surface and the resulting surface luminosity, which should emerge long after the flare. Our results are discussed in Section 5.

2. WAVE TRANSMISSION INTO THE CRUST

The crust is nearly incompressible and supports shear waves that can be excited by the Alfvén waves impinging from the magnetosphere. Excitation of two-fluid crustal modes can be neglected, and the recent claim that magnetospheric Alfvén waves transform into crustal Hall waves (Lyutikov 2015) is incorrect. Hall waves propagating parallel to the magnetic field \( B \) with frequency \( \omega \) in the crust of density \( \rho \) have refractive index

\[
N = \frac{c k}{\omega} = \frac{\omega_{pe}}{\sqrt{\omega \omega_{ri}}} \approx 10^9 \frac{B_{15}^{1/2}}{\omega_{5}^{1/2}} B_{14}^{1/2} \quad \text{(we use the standard notation } X_{mn} = X/10^m \text{ for a quantity } X \text{ in cgs units). This implies a huge impedance mismatch with the magnetospheric Alfvén waves, which have } N \approx 1, \text{ and therefore their transformation to Hall waves is suppressed.} \]

No significant separation between electron and ion velocities can occur on millisecond timescales, and the response of the crust to the external disturbance is essentially single-fluid.

2.1. Transmission Coefficient

Consider a magnetospheric Alfvén wave of frequency \( \omega \) impinging on the crust of the neutron star. For simplicity let us assume that the initial (unperturbed) magnetic field \( B_{c} \) is uniform and vertical, so the wave is propagating vertically along the \( z \)-axis, and the horizontal displacement \( \xi(x, y) \) is along the \( y \)-axis. The plasma-filled magnetosphere and the crust are excellent conductors; therefore the magnetic field is frozen in the medium and the horizontal field \( B_{y} \) is related to the displacement by \( B_{y} / B_{c} = \partial \omega_{i} \omega_{y} \). The wave speed in the magnetosphere is close to the speed of light \( c \), and the wavelength is \( \lambda_{0} = 2\pi c / \omega \).

The wave propagation is described by the equation

\[
\rho(z) + \frac{B_{c}^{2}}{4\pi c^{2}} \frac{\partial^{2} \xi}{\partial t^{2}} = \frac{B_{y}^{2}}{4\pi} \frac{\partial^{2} \xi}{\partial z^{2}} - \frac{\partial \sigma}{\partial z},
\]

where \( \xi \) is the electric potential, \( \rho \) is the density, \( \sigma \) is the conductivity of the plasma, and \( B_{y} \) is the horizontal magnetic field. The wave equation is parabolic in the crust, and the wavelength is much greater than the thickness of the crust.

1 Only electrons move in a Hall wave (analogous to a whistler in plasma physics) while ions are static. The velocity of the electron fluid \( v_{e} = j / e n_{e} \) is related to electric current \( j = (c/4\pi) \nabla \times B \), which gives a tiny \( v_{e} \) because of the high electron density \( n_{e} \) in the crust. Therefore, the “two-fluid” (electron–ion) description is useful only for slow phenomena in the crust.
Here $\rho(z)$ is the mass density and $\rho + B_z^2/4\pi c^2$ can be thought of as the effective inertial mass density of the magnetized medium. The first term on the right-hand side describes the restoring force of magnetic tension, and the last term describes the force due to the shear stress in the medium.

In particular, if the medium is elastic with a shear modulus $\mu$ then $\sigma = -\mu s$, where $s = \partial \xi/\partial z$ is the strain of the elastic deformation. In this case, Equation (1) becomes a simple wave equation with the wave speed given by

$$v^2(z) = \frac{B_z^2/4\pi + \mu(z)}{B_z^2/4\pi c^2 + \rho(z)}. \quad (2)$$

In the magnetosphere, we will neglect the mass density $\rho$ and the shear modulus $\mu$, which gives $v = c$. In the crust, we will use the profiles $\rho(z)$ and $\mu(z)$ shown in Figure 1. The density profile is obtained from the relativistic hydrostatic equation using the SLy equation of state (Haensel & Potekhin 2004) for a neutron star with mass $M = 1.4M_\odot$. The radius of the star is $R = 11.7$ km, and its surface gravitational acceleration is $g = (GM/R^2)(1 - r_g/R)^{1/2} = 1.7 \times 10^{14}$ cm s$^{-2}$ where $r_g = 2GM/c^2$. For the shear modulus $\mu$ we use the fitting formula given by Piro (2005) and Sotani et al. (2007) for low and high densities.

As the wave propagates into the deeper crust, its speed is reduced and its wavelength is compressed,

$$\lambda(z) = \frac{\lambda_0 v(z)}{c}, \quad \lambda_0 = \frac{2\pi c}{\omega}. \quad (3)$$

The reflection of the wave occurs in the region where the characteristic scale-height for the change of $v(z)$,

$$h(z) = \frac{v}{|dv/dz|}, \quad (4)$$

is smaller than the wavelength $\lambda(z)$. Figure 2 shows $\lambda(z), h(z)$, and the depth $z_1$ where they are equal. The typical value of $z_1$ is around 200 m below the surface; its exact value depends on $B_z$.

The transmitted wave below $z_1$ has $\lambda \ll h$ and can be described in the WKB approximation. Then the wave displacement takes the form (e.g., Fitzpatrick 2013),

$$\xi(z) = \text{const} \frac{Z^{1/2}(z)}{Z z_0} \cos \left[ \omega \left( t - \int_0^z \frac{dz'}{v(z')} \right) \right], \quad (5)$$

where $Z(z)$ is the impedance,

$$Z(z) = \left[ \frac{B_z^2}{4\pi} + \mu(z) \right]^{1/2} \left[ \frac{B_z^2}{4\pi c^2} + \rho(z) \right]^{1/2}. \quad (6)$$

A simple estimate for the transmission coefficient is obtained using the impedance at $z_1$ (Blaes et al. 1989),

$$T \sim \frac{4\pi v(z_1)}{Z(z_1) + Z(0)} \approx \frac{4\pi v(z_1)}{c} \approx \frac{4\pi v(z_1)}{c}. \quad (7)$$

For instance, for $B_z = 3 \times 10^{14}$ G, Equation (7) gives $T \sim 5\%$. A more accurate transmission coefficient is obtained by solving the wave equation numerically, which gives a higher value of $T = 12\%$ (the smoothness of the crustal density variation as the wave approaches $z_1$ enhances the transmission). The numerically calculated $T(B_z)$ is shown in Figure 3. It is comparable to 0.1 for typical magnetar fields.
The reflection coefficient $\mathcal{R} = 1 - T \approx 0.9$ is large, and the reflected Alfvén waves will bounce many times in the magnetosphere. Their amplitudes decrease by $T \sim 10^{-6}$ every time they bounce from the surface. The repeated transmission events form a train of compressed waves in the crust. This train propagates into the crust with velocity $v \sim 10^{-2} c$.

One can show from Equation (5) that the strain $s = \partial \xi / \partial z$ in the transmitted wave evolves as $|s| \propto v^{-1} \zeta^{1/2} \propto \rho^{1/4}$. It increases as the wave propagates into the deeper and denser crust. This has a simple physical reason: the wave decelerates, and hence its energy density $U_w$ grows as $v^{-1}$ (so that the wave continues to carry its energy flux $F_w = U_w v = \text{const}$). The wave energy oscillates between the kinetic energy and the horizontal field plus elastic energy of the crust. Therefore, $U_w$ may be written in two ways: $U_w \sim (\rho + B^2/4\pi \epsilon^2) \xi^2 \omega^2$ (kinetic) or $U_w \sim s^2 (B^2/4\pi + \mu)$ (magnetic-elastic). In the region where $\rho \xi^2 > B^2/4\pi$ and $\mu < B^2/4\pi$ this requires $\xi^2 \propto (\rho v)^{-1}$ and $s^2 \propto v^{-1}$. Then the relation $s \sim \xi / \lambda \propto \xi / v$ gives

$$\xi \propto v^{1/2}, \quad v \propto \rho^{-1/2}, \quad s \propto \rho^{1/4}.$$  

In the lower crust where $\mu \gtrsim B^2/4\pi$ and $v \approx (\mu / \rho)^{1/2} \approx \text{const} \approx 10^8 \text{cm s}^{-1}$ one finds $s \propto \rho^{-1/2}$. In this region the wave strain significantly decreases. This evolution of $s$ with depth (increase and then decrease) may be observed in the numerical simulation presented below.

### 2.2. Numerical Model

To illustrate the transmission process we set up a simple one-dimensional simulation of waves bouncing in the magnetosphere between the footpoints of a closed magnetic flux tube. The Alfvén waves are ducted along the magnetic field lines and the problem can be made one-dimensional by pretending that the flux tube is straight, and by placing its two opposite footpoints on the $z$-axis, separated by distance $L$. Here $L$ represents the length of magnetospheric field lines. The stellar crust with the density profile $\rho(z)$ is placed symmetrically at the two ends of the computational box. The crust thickness $\sim 1 \text{ km}$ is much smaller than $L$.

For any initial shear distortion of the field lines, one can calculate the subsequent dynamics of the generated Alfvén waves by solving Equation (1) numerically. In our numerical models, we take the initial distortion of the form

$$\xi_0(z) = A \exp \left[ -\frac{(z - z_0)^2}{2z^2} \right],$$  

which is localized in the middle of the box $z_0$, far away from the crust, with $l < L$. The distortion immediately splits into two waves propagating toward the opposite ends of the box. The strain profile of each wave $s(z) = \partial \xi / \partial z$ is determined by the initial distortion. An important parameter of the wave is its initial maximum strain,

$$s_0 = \frac{1}{2} \max[\partial z \xi_0(z)] = \frac{A}{2l} v^{-1/2}.$$  

The total energy initially stored in the two waves is

$$E_0 = \frac{\sqrt{\pi} A S}{8\pi} \frac{B^2}{l},$$  

where $S$ is the cross-sectional area of the flux tube.

We follow the evolution of the waves and their interaction with the crust until almost all the wave energy $E_0$ has been drained from the magnetosphere; this typically takes tens of light-crossing times $L/c$. The wave Equation (1) is solved on a grid with 1000 points in the magnetosphere (uniformly spaced) and a much finer grid in the crust (one point per meter). Convergence tests have been done to ensure that the grid is sufficiently large to resolve the wave dynamics.

The snapshot of the simulation in Figure 4 shows the distortion of the crust at time $t = 4.6L/c$, when the magnetospheric waves have bounced four times. The parameters of this sample model are $L = 40 \text{ km}$, $l = 5/\sqrt{2} \text{ km}$, $A = 5 \text{ km}$, and $B_c = 3 \times 10^{14} \text{ G}$. The corresponding $s_0 = (2e)^{-1/2} \approx 0.43$ and $E_0/S \approx 4.5 \times 10^{32} \text{ erg cm}^{-2}$. In the snapshot shown in the figure, about 1/3 of the wave energy $E_0$ has already been transmitted into the crust. The transmitted wave has been decelerated to $v \approx 10^8 \text{ cm s}^{-1}$ and compressed by the factor of...
\(c/v \approx 3 \times 10^2\). The compression creates a high energy density of the horizontal magnetic field at \(z = 200-500\, \text{m}\), \((s B_z)^2/8\pi \approx 10(s_B/s_0)^2/8\pi\), and strain \(s \approx 3s_0\).

3. PLASTIC HEATING

The description of the wave dynamics in Section 2 is incomplete because it assumes the elastic response \(\sigma = -\mu s\) everywhere in the crust. The more realistic model must take into account two facts. (1) When the solid crust is deformed by the shear wave beyond a critical stress \(\sigma_{cr}\) its response becomes plastic rather than elastic. (2) The crustal temperature may be high enough to reduce \(\sigma_{cr}\) or even melt the crust, leading to \(\sigma_{cr} \approx 0\). Therefore, the model should keep track of the crustal temperature.

3.1. Pre-flare Temperature Profile

The typical persistent surface temperature of magnetars is \(T_s \sim (3-4) \times 10^6\, \text{K}\) (Woods & Thompson 2006, pp. 547–586). It corresponds to the radiation flux \(F = \sigma_{SB} T_s^4 \approx 10^{22}\, \text{erg}\, \text{s}^{-1}\), where \(\sigma_{SB} = 5.67 \times 10^{-5}\, \text{erg}\, \text{s}^{-1}\, \text{cm}^{-2}\, \text{K}^{-4}\) is the Stefan–Boltzmann constant. A usual way to estimate the subsurface temperature profile of neutron stars \(T(z)\) assumes that the surface flux \(F\) is supplied by quasi-steady diffusion of heat from the crust. Then \(T(z)\) is given by the equation

\[
\kappa(T, z) \frac{dT}{dz} = F = \sigma_{SB} T_s^4,
\]

where \(\kappa\) is the effective conductivity, which is dominated by degenerate electrons at densities \(\rho > 10^6\, \text{g}\, \text{cm}^{-3}\) and by radiation in the low-density layers near the surface. Note that \(\kappa\) depends on the local magnetic field. Combining Equation (12) with the hydrostatic equation \(dp/dz = \rho g\) gives

\[
\frac{d \log T}{d \log \rho} = \frac{3}{16} \frac{PK T_s^4}{g T_s^4}.
\]

Here \(P\) is the pressure, \(g = (GM/R^2)(1 - r_s/R)^{-1/2}\) is the surface gravitational acceleration, and \(K = 16\sigma_{SB} T_s^3/3K\rho\) are the effective opacity; all quantities are measured in the local rest frame of the crust. The surface luminosity and temperature measured by a distant observer are \(L^\infty = (1 - r_s/R)L^*\) and \(T^\infty = (1 - r_s/R)^{3/2} T_s\) (Thorne 1977).

Equation (13) assumes that the temperature profile had enough time to relax to the steady state at depths of interest, which typically takes \(\sim 1\) year. In a true steady state, the relatively high surface temperature of magnetars requires a steady source of heat in the crust (Kaminker et al. 2006) or in the core. Alternatively, one may view this temperature profile as quasi-steady, slowly cooling after a previous heating episode.

If one accepts this thermal model for the pre-flare state of the crust, one can find \(T(z)\) from Equation (13) and determine the melting depth \(z_{mel}\) above which the crust is melted. We use the code of Potekhin (1999) to calculate the thermal conductivity and the melting point \(T_{mel}(\rho)\) of the crustal material. An approximate result is sufficient for the purposes of this paper and we do not discuss here the poorly known chemical composition of the magnetar crust. For simplicity, we assume an iron crust with a small impurity parameter.

We also assume that the pre-flare magnetic field is not far from vertical. This is a reasonable assumption for the melted layer \((z \lesssim 100\, \text{m}, \text{see below})\) where the crustal magnetic field should match the force-free magnetosphere. A strong toroidal field could be stored in the deeper crust or the core of the neutron star. Equation (13) can be solved numerically as described in detail in previous works, which calculated the relation between the surface effective temperature \(T_s\) and the internal temperature \(T_b\) measured at neutron-drip depth \(z_b \approx 400\, \text{m}\) (\(\rho_b = 4 \times 10^{11}\, \text{g}\, \text{cm}^{-3}\)). In particular, Potekhin & Yakovlev (2001) provide a fitting formula for \(T_b(T_s)\) for various magnetic fields and assuming an iron crust. We use this relation to impose the condition \(T = T_b\) at \(z = z_b\), and then reconstruct the profile \(T(z)\) in the region of interest \(\rho > 10^6\, \text{g}\, \text{cm}^{-3}\) (above or below \(z_b\)) by integrating Equation (13) from \(z_b\). Thus we avoid integration in the shallow surface layers where thermal conductivity is dominated by radiation, and so we only use the electron conductivity in our calculations.

For a given \(T_b\), this calculation gives the subsurface temperature profile \(T(z)\) and the melting depth \(z_{mel}\). The melting temperature is approximately given by

\[
T_{mel} \approx 2.4 \times 10^9 \rho_{12}^{1/3}\, \text{K}.
\]

The exact value of melting depth \(z_{mel}\) depends on the magnetic field and its orientation relative to the stellar surface. A strong field increases the thermal conductivity along \(B\) and decreases it perpendicular to \(B\). Therefore, a vertical field tends to reduce the internal temperature, thus decreasing \(z_{mel}\). A horizontal field would hamper the heat flow in the radial direction and increase \(z_{mel}\). A typical \(z_{mel}\) in magnetars with non-horizontal surface fields is \(\sim 100\, \text{m}\).

3.2. Plastic Flow

As the wave packet propagates into the crust below \(z_{mel}\), it starts to interact with the solid phase (lattice). The response of the lattice is elastic as long as its strain is below a critical value \(s_{cr}\). The maximum \(s_{cr} \approx 0.1\) is comparable to the yielding threshold for an ideal crystal (Horowitz & Kadau 2009). The actual strain in the wave \(s \sim s_b(T_c/v_s)^{1/2}\) is much higher than \(s_{cr}\), and so the wave initiates a strong plastic flow with the high frequency \(\omega\). In contrast to a fluid Alfvén wave or elastic shear wave, the plastic flow is dissipative, i.e., it converts wave energy to heat, reducing its amplitude. Below we include this process in our wave propagation model.

The plastic heating rate per unit volume is

\[
\frac{dU_{pl}}{dt} = -\sigma_{pl},
\]

where \(s_{pl} = s - s_{el}\) is the plastic part of the strain, \(s_{el}\) is the elastic part, and \(\sigma\) is the shear stress sustained by the plastic flow. In the plastic regime \(|\sigma| > \sigma_{cr}\), where \(\sigma_{cr} = \mu s_{cr}\). The simple model of “viscoplastic solid” (e.g., Irgens 2008) gives the stress of the plastic flow in the form

\[
|\sigma| = \sigma_{cr} + \eta |s_{pl}|,
\]

where \(\eta\) is a viscosity coefficient.

The crystal becomes “soft” (i.e., \(\sigma_{cr}\) drops) if it is heated to a temperature comparable to the melting point \(T_{mel}\). The softening effect is responsible for the thermoplastic instability that can release internal magnetic stresses in magnetars (Beloborodov & Levin 2014). This instability, however, develops on a timescale much longer than 10 ms and does
not affect the dynamics considered in this paper. Here the plastic flow is driven by the strong external magnetic stress from the flare (rather than developing spontaneously inside the crust) and immediately reaches huge strains $|s| \gg s_{\text{cr}}$ and high temperatures. Because $|s| \gg s_{\text{cr}}$, the detailed behavior of $s_{\text{cr}}(T)$ and $\sigma_{\text{cr}}(T)$ is not important; our calculation should merely take into account the fact that plastic heating switches off when $T$ approaches $T_{\text{melt}}$.

This effect is included as follows: the stress $\sigma$ of the plastic flow is multiplied by the factor $1 - U_{\text{lin}}/U_{\text{melt}}$, where $U_{\text{lin}}(\rho, T)$ is the thermal energy density and $U_{\text{melt}} = U_{\text{lin}}(\rho, T_{\text{melt}})$. This prescription enforces $\sigma = 0$ when $T = T_{\text{melt}}$.

The stress in the elastic regime $\sigma = -\mu s$ must match the plastic stress at $s = s_{\text{cr}}$. This condition is automatically satisfied for the cold crystal. For a hot crystal the reduction of $\sigma_{\text{cr}}(T)$ may be interpreted as the reduction of shear modulus $\mu$ or the reduction of $s_{\text{cr}}$ (or both). These details are not important for our model, because the plastic flow has $|s| \gg s_{\text{cr}}$. The numerical models presented below assume $s_{\text{cr}}(T) = 0.1 = \text{const}$ and use the following prescription for $\sigma$,

$$\sigma = \left(1 - \frac{U_{\text{lin}}}{U_{\text{melt}}}\right) \times \begin{cases} -\mu s, & \text{elastic} \\ (0.1\mu + \eta s_{\text{el}})\text{sign}(-s), & \text{plastic} \end{cases} (17)$$

where $\mu$ is the shear modulus at $T \ll T_{\text{melt}}$ shown in Figure 1, and one may think of $(1 - U_{\text{lin}}/U_{\text{melt}})\mu$ as the shear modulus reduced by heating. We verified that practically the same results are obtained if we choose a temperature-dependent $s_{\text{cr}} = 0.1(1 - U_{\text{lin}}/U_{\text{melt}})$ with shear modulus unchanged by heating.

Finally, we must choose $\eta$, which is unknown for the crustal material. The transition between the plastic and elastic regimes is smooth if $\eta$ vanishes when $|s| = s_{\text{cr}}$. Therefore, we assume $\eta$ in the form $\eta = \alpha \mu |s| - s_{\text{cr}}$, where $\alpha$ is a constant. We tried various values of $\alpha$ and found that plastic heating weakly depends on it as long as $\alpha$ is sufficiently large, $\alpha > 3 \times 10^{-5}$ s. Our sample numerical models use $\alpha = 3 \times 10^{-5}$ s.

The dynamic system described by Equation (1) with $\sigma$ given by Equation (17) satisfies the energy conservation law,

$$\frac{dQ}{dt} = S \int \frac{dU_{\text{lin}}}{dz} dz = -\frac{d}{dt}(E_{\text{kin}} + E_{\text{th}} + E_{\text{el}}), (18)$$

where

$$E_{\text{kin}} = S \int \left(\rho + \frac{B^2}{4\pi c^2}\right)\frac{x^2}{2} dz, (19)$$

$$E_{\text{th}} = S \int \frac{x^2 B^2}{8\pi} dz, (20)$$

$$E_{\text{el}} = S \int \frac{\mu s_{\text{el}}^2}{2} dz, (21)$$

where $|s_{\text{el}}| < s_{\text{cr}}$ in the elastic zone and $|s_{\text{el}}| = s_{\text{cr}}$ in the plastic zone.

The plastic flow occurs where $|\sigma|$ exceeds $\sigma_{\text{cr}}$ and continues as long as $d|s|/dt > 0$. Whenever the local absolute value of the strain stops growing, the plastic flow switches to the elastic regime; at this point $s_{\text{el}}$ and $\sigma$ are reset to zero.

3.3. Wave Damping and Post-flare Crustal Temperature

We re-run the model described in Section 2.2 with the new expression for $\sigma$ that takes into account the plastic damping in the crust (Equation (17)). The initial state is assumed to have the surface temperature $T_s = 3 \times 10^6$ K. All other parameters are the same as in Section 2.2, in particular $B_z = 3 \times 10^{14}$ G and $s_0 = (2\pi)^{-1/2} \approx 0.43$. The spacetime diagram of the wave evolution in the crust is presented in Figure 5. It shows the wave displacement and strain, and indicates the elastic, plastic, and melted regions.

Figure 6 shows the history of the wave energy transmission from the magnetosphere to the crust and the plastic damping effect. One can see that most of the transmitted wave energy is promptly converted to heat. We have verified that our numerical simulation satisfies the conservation law (Equation (18)) with accuracy better than 1%. Each time the wave hits the surface, the transmission coefficient is approximately 12%, and almost all the wave energy $E_0$ is damped after $\sim 10$ ms, so most of $E_0$ becomes stored as crustal heat. This heating results in deep melting of the crust, down to 500 m.

Since plastic dissipation switches off at the melting point, the crust naturally acquires the “ceiling” temperature $T \approx T_m$ in an extended region below the surface. The resulting temperature profile immediately after the flare is shown in Figure 7. To investigate how the results depend on $B_z$ and $s_0$, we have calculated the models with $B_z/10^{14}$ G = 0.3, 1, 3, 10 and $s_0 = 0.13, 0.25, 0.43$. Stronger waves in stronger magnetic fields melt deeper layers of the crust, up to 600 m in the calculated models.
4. COOLING

After the flare, the hot crust will cool on a much longer timescale. Two main processes cool the crust: neutrino emission and heat conduction. The temperature evolution with time is described by the following equation:

$$C_V \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left( \kappa \frac{\partial T}{\partial z} \right) - q_v,$$

(22)

where $\kappa$ is the thermal conductivity and $C_V$ is the heat capacity of the crust; both are functions of local $\rho(z)$, $T(z, t)$, and $B$. The sample numerical models presented below assume a uniform vertical magnetic field $B = B_0 = \text{const}$. We use $\kappa(\rho, T, B)$ and $C_V(\rho, T, B)$ calculated by the code of Potekhin (1999). The term $q_v(\rho, T, B)$ is the rate of local cooling by neutrino emission. This rate is described in detail by Potekhin & Yakovlev (2001). They provide useful analytical approximations for four relevant channels of neutrino emission: plasmon decay, bremsstrahlung, synchrotron, and electron–positron annihilation. We use their formulas in our calculations.

Similar to previous simulations of time-dependent heat diffusion in magnetars (Kaminker et al. 2006; Brown & Cumming 2009; Pons et al. 2009), we separate the crust into two regions: a blanketing envelope and an interior region. Here we choose the envelope boundary at $z_b \approx 60$ m where $\rho = \rho_b = 10^9$ g cm$^{-3}$. The typical timescale of heat diffusion from this depth is $t_b \ll 10^6$ s. It is sufficiently short to give a quasi-steady state in the envelope, and so the steady-state solution may be used to determine the relation between $T_b = T(z_b)$ and the effective surface temperature $T_s$. Note that $T_s$ defines the energy flux $F = \sigma_0 T_s^4$ through the envelope $z < z_b$, and thus in essence the $T_b-T_s$ relation is a relation between $T_b$ and the heat flux $F = \kappa \partial T/\partial z$ at $z_b$. It serves as a boundary condition for our time-dependent heat diffusion problem at $z > z_b$. Since this boundary condition relies on the steady-state solution at $z < z_b$ it can only be accurate when $T(t, z > z_b)$ evolves on timescales longer than $t_b$.

We calculated the $T_b-T_s$ relation at $\rho_b = 10^9$ g cm$^{-3}$ using the steady-state solutions obtained in Section 3.1. This gave a tabulated boundary condition $F(T_b)$ at the upper boundary of our computational box $z_b \approx 60$ m. The lower boundary is chosen at $z \approx 1$ km, near the bottom of the crust where $\rho \sim 10^{14}$ g cm$^{-3}$. The exact position of the lower boundary is not important as long as it is deep enough. The deep crust has a high thermal conductivity and the heat is absorbed by the (approximately isothermal) core of a huge heat capacity. We use the absorbing boundary condition at constant temperature $T \sim 3 \times 10^8$ K, neglecting the increase in the core temperature due to the absorbed heat.

The initial condition $T(z, 0)$ for Equation (22) is provided by the plastic heating model (Figure 7). The initial temperature increases along the melting curve $T_m(z)$, reaches maximum, and drops at greater depths. We evolve this initial temperature profile on a uniform grid of 1000 points and a time step of 10 s for $10^7$ steps. To speed up the simulation, the values of $\kappa$, $C_V$, and $q_v$ on the grid are updated every 500 time steps. Convergence tests verified that this resolution is sufficient to
obtain accurate results. We also verified that our simulation conserves energy with better than 1% accuracy. The thermal energy lost by the crust is partially carried away by neutrinos and partially conducted through the boundaries.

Figure 8 shows the gradual evolution of the temperature profile \(T(z)\) after the flare with the fiducial parameters (see Section 2.2 and Figures 5, 6). During the first month, the initial peak of temperature at \(z = 500\) m is reduced from \(\sim 5 \times 10^9\) K mainly due to neutrino losses. Then the peak continues to flatten and spread due to thermal conduction, forming a rather flat profile of \(T \lesssim 10^9\) K in a few years.

We find that plasmon decay and bremsstrahlung make the dominant contributions to neutrino cooling, and synchrotron neutrino emission becomes significant in stronger magnetic fields \(B_z \sim 10^{15}\) G. Electron–positron annihilation dominates neutrino cooling only in the shallow, low-density region of the crust, and its net contribution to the energy loss is negligible.

With increasing \(B_z\) (and at fixed amplitude \(s_0\) of the waves excited in the flare) the deposited plastic heat increases, which increases the role of neutrino cooling. As a result, the relative contributions of plasmon decay, bremsstrahlung, and synchrotron neutrino emission depend on \(B_z\). This dependence is shown in Figure 9, where we also show the energy fraction that is conducted to the stellar surface and radiated away. The remaining energy fraction (not shown in Figure 9) is conducted into the core of the neutron star.

One can see from Figure 9 that only a small fraction of the stored crustal heat is conducted to and radiated from the stellar surface. For example, at \(B_z = 3 \times 10^{14}\) G less than 1% is conducted to the surface; roughly half the heat is lost by neutrino emission and half is conducted to the core.

The heat radiated from the surface produces a delayed afterglow emission of the flare. The solution of Equation (22) gives the surface radiation flux \(F\) as a function of time. This flux is shown in Figure 10. One can see that the surface flux peaks with a significant delay after the flare—it takes the thermal conduction timescale (of months to years) to transport the crustal heat to the surface.

The core remains much colder than the plastically heated crust. The heat conducted to the core cannot significantly boost its temperature because (1) the core has a large heat capacity, in particular if there are non-superfluid baryons (Yakovlev & Pethick 2004), and (2) the core is cooled by neutrino emission, and the cooling rate grows quickly at high temperatures.

**Figure 8.** Evolution of the crustal temperature profiles in our fiducial flare model with \(B = B_z = 3 \times 10^{14}\) G and \(s_0 = 0.43\).

**Figure 9.** Fraction of the post-flare crustal heat lost through surface emission and various channels of neutrino emission, as a function of \(B = B_z\). The flare is assumed to excite a pair of Alfvén waves with \(s_0 = 0.43\) (Section 2.2) that are plastically damped in the crust.

**Figure 10.** Surface thermal flux caused by the plastic heating in the giant flare. Upper panel: fixed \(s_0 = 0.43\) and varying \(B = B_z\). Lower panel: fixed \(B_z = 3 \times 10^{14}\) G and varying \(s_0\).
5. DISCUSSION

5.1. Plastic Damping and Cooling

In this paper we have described the phenomenon of plastic damping of Alfvén waves generated in magnetar flares. Our results may be summarized as follows.

(1) Transmission. The flare generates magnetospheric Alfvén waves with energy density

\[ U_0 \sim \frac{\mu_B s_0^2}{2}, \quad (23) \]

where \( \mu_B = B^2/4\pi \) is the tension of magnetic field lines and \( s_0 \geq 0.1 \) is the shear strain of the field lines. The waves are quickly transmitted into the crust of the neutron star. The transmission coefficient is \( T \sim 0.1 \) (Figure 3), and most of the wave energy is transmitted after \( N \sim T^{-1} \sim 10 \) reflection events (Figure 6). The transmitted waves form a train of \( N \) oscillations propagating with velocity \( v \lesssim 10^{-2}c \) and compressed by the factor of \( c/v \) (Figure 4).

(2) Compression. The wave energy, which is initially spread in the magnetosphere, becomes compressed upon transmission. The energy density of the transmitted wave is

\[ U_w \approx T \frac{c}{v} U_0, \quad (24) \]

where \( v \) decreases to \( 10^8 \) cm s\(^{-1}\) as the wave train propagates toward the bottom of the crust. The transmission occurs at depths \( z \) of a few hundred meters where the crustal density \( \rho \sim 10^{10} - 10^{11} \) g cm\(^{-3}\). In this region, \( U_w \) exceeds the maximum energy that could be stored in the elastic deformation of the crust, \( U_{el} = \mu s_x^2/2 \), and the wave propagation is still sustained by the tension of magnetic field lines, \( \mu_B \). Therefore, the transmission also leads to the strain amplification: \( s^2/s_0^2 \approx U_w/U_0 \sim 10 \).

(3) Flow. The shear strain of the transmitted wave, \( s \sim (Tc/v^3)^{1/2}s_0 \), exceeds the maximum possible strain of elastic deformation \( s_{el} \sim 0.1 \). Therefore, the wave induces a strong plastic flow of the crust, which dissipates the wave energy. The plastic stress \( \sigma \) is comparable to \( \mu s_x \), which gives the dissipated energy \( dq \sim \mu s_x |ds| \). As the wave propagates into denser layers \( \rho \sim 10^{12} \) g cm\(^{-3}\), the shear modulus of the lattice increases to \( \mu \sim 10^{48} \) erg cm\(^{-3}\) (Figure 1). The plastic heat density deposited by the wave train is given by

\[ U_{th} \sim \sigma s N \sim \mu s_x s N. \quad (25) \]

The high \( U_{th} \) given by this estimate implies that the wave energy density \( U_w \) converts to \( U_{th} \), i.e., efficient damping occurs.

(4) Melting. Damping of the wave is buffered by melting—plastic damping is inefficient where the heated crust becomes nearly liquid, and the wave continues to propagate to denser layers that have a higher \( T_{melt}(\rho) \). As a result, a simple temperature profile \( T \sim T_{melt}(\rho) \) is created by the plastic flow in an extended region of the crust (Figure 7).

Most of the wave damping occurs at depth \( z_{damp} \) where \( T_{melt} \) is so high that the wave dissipation becomes marginally capable of melting the crust. Thus \( z_{damp} \) is also the depth of the melted region. At this depth the following condition is satisfied:

\[ C_V T_{melt} \sim U_w. \quad (26) \]

We found \( z_{damp} \approx 500 \) m for a typical wave energy in magnetar giant flares and \( B_r \sim 3 \times 10^{14} \) G (Figure 7). Deeper melting \( z_{damp} \approx 700 \) m is possible if the giant flare occurs in a flux rope of a particularly strong field \( B_r \gtrsim 10^{15} \) G.

(5) Cooling. On a timescale of months to years, the deposited heat is mostly lost to neutrino emission and conducted into the core of the star (Figures 8 and 9). A modest energy \( E_{\nu} \) is conducted to the stellar surface and emitted in a delayed afterglow radiation. A typical energy radiated per unit area is \( E_{\nu}/\delta \sim 10^{36} \) erg cm\(^{-2}\). The timescale for the rise of afterglow luminosity is the thermal conduction time \( t_{cond} \sim 10^8 \) s. Over a broad range of the flare parameters, the peak flux of surface afterglow is \( P_{max} \sim (2-4) \times 10^{22} \) erg cm\(^{-2}\) s\(^{-1}\) (Figure 10).

There are ways to refine our model of surface afterglow from plastic damping of magnetospheric waves. All sample models shown in this paper assumed an approximately vertical (radial) magnetic field in the upper crust. A strongly inclined field would significantly reduce thermal conductivity in the radial direction and delay the crustal cooling. It could also bolster a high crustal temperature before the flare, which would give a deeper melted zone where plastic damping would be impossible. In this case, the flare could only cause heating of the deep crust where practically all heat is wasted to neutrino emission and inward conduction. Thus, a strong non-radial field component tends to reduce the expected afterglow emission.

Our presented models assumed an iron composition of the blanketing envelope. A light-element composition of the envelope would increase its thermal conductivity (Potekhin et al. 2003), decreasing the internal temperature and reducing the depth of the melted layer in the pre-flare crust. Therefore, if magnetars have a light-element envelope, their post-flare cooling occurs faster. This effect somewhat increases the afterglow flux, especially at early times, and may offset the opposite effect of the non-radial magnetic field.

5.2. Other Mechanisms of Alfvén Wave Damping

Nonlinear interactions of Alfvén waves in the magnetosphere provide an additional damping mechanism. The existing estimates (Thompson & Blaes 1998) suggest that this mechanism will be dominant at very high amplitudes of the waves, \( s_0 \gtrsim 1 \). The nonlinear interactions occur as the Alfvén waves bounce from the stellar surface and collide in the magnetosphere. The nonlinear terms in the electrodynamic equations show two types of wave interactions:

1. \( A + A \rightarrow F \): two Alfvén waves \( A \) convert into a fast magnetosonic wave \( F \) (which may escape the magnetosphere). The damping of Alfvén waves by this “3-wave” interaction occurs on the timescale,

\[ t_{damp} \sim \frac{\tau/c}{k_\perp} \sim \left( \frac{k_\parallel}{k_\perp} \right)^2 \frac{\tau/c}{s^2}, \quad (27) \]

where \( \lambda = 2\pi/k_\parallel \) is the wavelength along \( B \) (comparable to the length of the magnetospheric field line \( L \)), \( \xi \) is the characteristic displacement in the waves, and \( k_\parallel \) is the wavevector component perpendicular to the magnetic field. The Alfvén waves ducted along the curved magnetic field lines may be expected to have \( k_\parallel \sim k_\parallel \).

2. \( A + A \rightarrow A + A \): two Alfvén waves generate two new Alfvén waves. This “4-wave” interaction initiates a
cascade to high $k_z$, which may lead to wave dissipation on small scales (Thompson & Blaes 1998). The damping time due to this higher-order process is

$$t_{\text{damp}} \sim \frac{\lambda/c}{(k_z \xi)^4} \sim \left( \frac{k_z}{k_c} \right)^4 \frac{\lambda/c}{s^4}. \quad (28)$$

The time $t_{\text{damp}}$ given by Equations (27) and (28) should be compared with $T^{-1} L_c/c \sim 10 L_c/c$, the lifetime of the Alfvén waves to transmission and plastic damping in the crust. The numerical coefficients in Equations (27) and (28) have not been calculated, but estimates suggest that if the flare generates $s_0 \gtrsim 1$, the nonlinear wave interactions can reduce $s_0$ to a value $\lesssim 1$ before the waves are damped plastically in the crust.

The wave cannot be completely damped by the plastic mechanism. In particular, at strains $|s| < s_{\text{cr}}$ it propagates with no significant damping. The residual wave train will reach the bottom of the crust and enter the liquid core. It will travel through the core along the magnetic field lines, and after time $t \sim 2r/v$ (typically shorter than 1 s) the train will again emerge somewhere at the bottom of the crust and continue to propagate upward.

The low-amplitude waves will continue to travel through the magnetosphere and the star for a while. Their lifetime at any given amplitude $s$ is limited by the nonlinear interactions in the magnetosphere $t_{\text{damp}} \propto s^{-2}$. The Alfvén waves are also subject to gradual ohmic dissipation, as their propagation involves excitation of electric currents demanded by $\nabla \times \mathbf{B} \neq 0$. After the flare, the effective resistivity of the magnetosphere is controlled by the threshold voltage of electron–positron discharge that is self-organized to conduct the electric currents (Beloborodov & Thompson 2007).

5.3. Observed Afterglow

Sudden crustal heating followed by gradual crustal cooling was proposed to power the afterglow of the giant flare in SGR 1900+14 (Lyubarsky et al. 2002). The afterglow was extremely bright in the first hours after the flare, $L \sim 10^{37} - 10^{38}$ erg s$^{-1}$, and during the next month it showed a power-law decay $L \propto t^{-0.7}$ (Woods et al. 2001). Lyubarsky et al. (2002) explored how heat should be deposited to give the observed afterglow light curve and found that heating should be approximately uniform throughout the 500 m deep layer below the surface. This implies, in particular, enormous heating in the shallow layers $z \ll 100$ m. The heating mechanism in the low-density layers is unclear and certainly cannot be provided by plastic dissipation. Therefore we do not attempt to explain the early afterglow of SGR 1900+14 by crustal heating. We also note that the afterglow spectrum was nonthermal (Woods et al. 2001), which suggests a magnetospheric source.

Plastic damping of magnetospheric Alfvén waves produces a well defined temperature profile of the crust: $T \approx T_{\text{mech}}(z)$ down to $z_{\text{damp}}$. This leads to specific predictions for the afterglow light curves (Figure 10), with the surface flux $F \sim (2-4) \times 10^{22}$ erg cm$^{-2}$ s$^{-1}$ on a timescale $\gtrsim 100$ d. This flux and timescale appear to be consistent with observations of some less energetic “transient” magnetars after their bursting activity.

In particular, the luminosity of SGR 1627–41 after its outbursts in 1998 and 2008 showed a decay on a timescale of a year (Mereghetti et al. 2006; Esposito et al. 2008; An et al. 2012). The luminosity at $t \sim 100$ d was $L \sim 7 \times 10^{34} (d/11$ kpc)$^2$ erg s$^{-1}$ after the 1998 outburst and $L \sim 2 \times 10^{34} (d/11$ kpc)$^2$ erg s$^{-1}$ after the 2008 outburst, where the distance $d \approx 11$ kpc was inferred from the apparent location of SGR 1627–41 in a star-forming region (Hurley et al. 1999). The decay on a timescale of a year is consistent with the crust melting down to $z_{\text{damp}} \sim 300$ m, and the observed luminosity $L$ is consistent with the area of the melted crust occupying $\sim 10\%$ of the stellar surface.

Swift J1822.3–1606 provides another example. It produced afterglow emission following the outburst in 2011 (Rea et al. 2012; Scholz et al. 2012, 2014). Similar to the afterglow of SGR 1627–41, its light curve may be described as a double exponential, with the second (longer) exponential component visible after $\sim 100$ d. Scholz et al. (2014) used a crustal cooling model to describe both the early and late afterglow components in Swift J1822.3–1606. In their model, heat deposition is a phenomenological parameter adjusted to reproduced observations. We find that plastic damping of magnetospheric waves is only capable of explaining the late afterglow component, and a different heat source must be invoked for the early component. The late component has luminosity and decay timescales similar to those observed in SGR 1627–41, consistent with the crust melting down to $z_{\text{damp}} \sim 300$ m.

A reliable identification of the crustal afterglow is complicated by the presence of another, nonthermal, emission component. The nonthermal source is likely present during the afterglow of SGR 1627–41 (An et al. 2012), and nonthermal hard X-rays are unambiguously detected in the transient magnetar 1E 1547.0–5408 during its afterglow following the 2009 outburst (Enoto et al. 2012; Kuiper et al. 2012). The nonthermal activity is usually associated with the twisted equilibrium magnetosphere, which carries persistent electric currents (Thompson et al. 2002; Beloborodov 2013). The twist is ohmically dissipated over a timescale of a year, which happens to be comparable to the timescale of crustal cooling.

Another complication is the expected external heating of the stellar surface bombarded by magnetospheric particles. This heating occurs at the footpoint of the current-carrying magnetic field lines (“j-bundle”). As the magnetosphere slowly untwists, the j-bundle shrinks and so does its hot footpoint (Beloborodov 2009). Such shrinking hot spots have been observed in several transient magnetars, including the canonical transient magnetar XTE J1810–197. Following an outburst in 2003 it showed an X-ray afterglow decaying on a timescale of a year, with luminosity $L \sim 2 \times 10^{34}$ erg s$^{-1}$ at $t \sim 1$ year (Gottlieb & Halpern 2007). The observed decay $\dot{A}(t)$ of luminosity $\dot{L}(t)$ of the hot spot evolved in agreement with the predictions of the untwisting magnetosphere model. Similar shrinking hot spots were observed in 1E 1547.0–5408, CXOU J164710.2–455216, SGR 0501+4516, SGR 0418+5729 (see the data collection in Beloborodov 2011 and references therein), and more recently in Swift J1822.3–1606 (Scholz et al. 2014) and the Galactic Center magnetar SGR J1745–2900 (Coti Zelati et al. 2015).

Strong Alfvén waves and deep plastic heating are certainly expected in energetic events, in particular in giant flares. All three giant flares observed to date were emitted by persistently active magnetars, which maintain a high level of both magnetospheric activity and surface luminosity. It is possible
that plastic damping of Alfvén waves is the main mechanism that keeps the crust hot in these objects.

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