Grasping rules and semiclassical limit of the geometry in the Ponzano–Regge model

Jonathan Hackett and Simone Speziale

Perimeter Institute, 31 Caroline St. N, Waterloo, ON N2L 2Y5, Canada
E-mail: jhackett@perimeterinstitute.ca and sspeziale@perimeterinstitute.ca

Received 27 November 2006, in final form 30 January 2007
Published 6 March 2007
Online at stacks.iop.org/CQG/24/1525

Abstract
We show how the expectation values of geometrical quantities in 3D quantum gravity can be explicitly computed using grasping rules. We compute the volume of a labelled tetrahedron using the triple grasping. We show that the large spin expansion of this value is dominated by the classical expression, and we study the next to leading order quantum corrections.

PACS number: 04.60.Pp

1. Introduction

The spinfoam non-perturbative approach to quantum gravity [1] describes a microscopical quantum geometry, where the geometrical observables have discrete values, expressed in terms of half-integers (spins). These spins characterize ‘atoms’ of spacetime. A possible way to study the semiclassical limit consists of studying the expansion for large values of the spins. In this paper, we apply this procedure to the volume of a tetrahedron in the Ponzano–Regge (PR) model for 3D quantum gravity with Euclidean signature [2].

The spins, which are the fundamental variables of the model, label the irreducible representations (irreps) of $SU(2)$, the gauge group of 3D GR. Using the generating functional introduced in [3], the geometrical expectation values can be defined via the action of grasping operators. This action can be evaluated using the recoupling theory of $SU(2)$, and thus the geometrical values given in terms of purely algebraic quantities. Here we show that the relevant graspings can be identified starting from the discretization of the classical observables, and we focus on the first non-trivial observable, the quantum volume of a tetrahedron. Its value is obtained by triple graspings acting on the $\{6j\}$ symbol associated with the tetrahedron. We identify all the relevant graspings, and evaluate their action to write explicitly the value of the quantum volume in terms of algebraic quantities. This is our first result. The relevance of this result concerns the construction of spinfoam models of matter coupled to quantum gravity. The matter action typically contains a volume term, and thus knowing the value of the
quantum volume of a labelled tetrahedron is needed for constructing the coupled model. For instance, this result can be used in [5] and [6]. Furthermore, a power series of triple grasping acting on the $SU(2)\{6j\}$ symbol can be used to reconstruct the quantum $\{6j\}$ symbol used in the Turaev–Viro model [4].

Studying the large spin expansion of this value, we identify the dominant and subdominant graspings, and show that the expansion is remarkably dominated by the classical formula. However, the exact result for the volume has an extra factor multiplying the classical formula. This factor can be understood in terms of the well-known feature of the PR model to sum over both orientations of spacetime. The consequences of this fact for the volume were discussed in [4], and the factor here obtained agrees with their results. Motivated by the analysis of [4], we then consider the squared volume, and show that the correct classical formula is this time directly reproduced. This is our second result. The relevance of this result is to support the idea that the semiclassical limit of spinfoam models can be studied considering the large spin expansion. This idea is also supported by the recent calculations of the 2-point function [7–9]. This is the main open problem in the formalism, and other remarkable ideas to address it include defining coarse graining procedures [10], constructing effective field theories [11], or rewriting conventional quantum field theories in the language of spinfoams [12].

This paper is organized as follows. In section 2, we briefly recall how the Ponzano–Regge model is constructed, and we introduce the generating functional to compute expectation values of the geometry. In section 3, we show how to introduce the geometrical observables using the discrete variables of the model, and how to relate this observables to grasping operators. We compute the values of quantum lengths and angles, corresponding to quadratic graspings, and the value of the quantum volume, corresponding to the triple grasping. In section 4, we analyse the leading order of the large spin expansion for the volume and the squared volume. In section 5, we compute the next to leading order quantum corrections. In the final section 6 we summarize our results. All the calculations used in the paper are explicitly reported in the appendix, where we also fix our phase convention for the evaluation of spin networks, and show how this is related to the grasping rules.

Throughout we define the Planck length as $\ell_P := 16\pi\hbar G$.

2. Ponzano–Regge model and generating functional

We consider here the PR model on a fixed triangulation $\Delta$ of the spacetime manifold $M$. This can be obtained from a path integral quantization of the first order triad action for Riemannian GR. In the continuum, the fundamental fields are the $SU(2)$ connection $\omega^I_J$ and the triad $e^I$, and the action for GR reads

$$S_{GR}[e^I, \omega^I_J] = \frac{1}{16\pi G} \int_M \text{Tr} e \wedge F(\omega),$$

where the trace Tr is over the $SU(2)$ indices, and $F(\omega)$ is the curvature of $\omega$. This action can be discretized on $\Delta$ as follows. Consider an embedding $i : \Delta \rightarrow M$, which allows us to think of $\Delta$ as a cellular decomposition of $M$. Using this embedding, we can define the vectors $\ell_P^I \sim \int_s dx^I$, tangent to the segments $s$ of $\Delta$. Analogously, we can define the vectors $\ell_P^\mu \sim \int_e dx^\mu$ tangent to the edges $e$ of the dual triangulation $\Delta^*$ (which we recall are in one to one correspondence with triangles of $\Delta$).

We then discretize the triad field with an $\mathfrak{su}(2)$ algebra element, associated with the segments of $\Delta$, and the connection with an $SU(2)$ element, associated with the edges of $\Delta^*$:

$$e^I(x) \mapsto X^I_e := \frac{1}{\ell_P} e^I(x) \ell_P^I \sim \frac{1}{\ell_P} \int_s e^I(s) dx^I.$$

(2)
\[ \omega^{IJ}_\mu (x) \mapsto e^{\omega^{IJ}_\mu} \sim e^{\varepsilon \omega^{IJ}_\mu}. \]  

We also introduce the quantities \( g_f = \prod_{e \in \partial f} g_e \) associated with the faces \( f \) of \( \Delta^* \). We see from (3) that these quantities discretize the curvature, \( g_f \sim \exp \int f F(\omega) \). Recalling that faces of \( \Delta^* \) are in one to one correspondence with segments of \( \Delta \), we will use from now on the notation \( g_s \equiv g_f \).

Consequently, we write the discrete action as

\[ S[X_\ell, g_s] = \hbar \sum_s \text{Tr}[X_s g_s]. \]  

When the embedding is sufficiently refined, and the coordinate areas consequently small, we can expand \( g_s \sim 1 + \int f F(\omega) \). Recalling that \( \text{Tr} T = 0 \) for any \( T \in su(2) \), we see that (4) reduces to (1).

The quantum theory can be constructed from the partition function

\[ Z = \prod_e \int_{SU(2)} d g_e \prod_s \int_{su(2)} d X_s \exp \left( i \sum_s \text{Tr}[X_s g_s] \right). \]  

This quantity can be evaluated using the harmonic analysis of \( SU(2) \) (for details, see for instance [1]), and one obtains

\[ Z = \sum_{j_\ell} \prod_s d_{j_\ell} \prod_{\tau} \{6j\}, \]  

where the sum is over all possible assignments of half-integers \( j \) to the segments of \( \Delta \). The half-integers, or spins, label the irreducible representations of \( SU(2) \). The quantity \( d_j := 2j + 1 \) is the dimension of the representation. Finally, a \( \{6j\} \) symbol is associated with each tetrahedron \( \tau \) of \( \Delta \). For more discussion of the PR model, see [13]. The \( \{6j\} \) symbol is the key object of the recoupling theory of \( SU(2) \), and it depends only on the six \( j \)s attached to the segments of the tetrahedron. It can be written in terms of the Wigner 3m-symbols as

\[ \{j_1 j_2 j_3 j_4 j_5 j_6\} = \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_4 & j_5 & j_6 \\ m_4 & m_5 & m_6 \end{array} \right). \]  

The \( \{6j\} \) symbol has a well-known asymptotic behaviour, namely that if we rescale the half-integers entering the \( \{6j\} \) symbol as \( j_i = N k_i \), we have [2, 14, 15]

\[ \lim_{N \to \infty} \{6j\} = \frac{1}{\sqrt{12 \pi N^3 V(k_s)}} \cos \left( \sum_s \left( N k_s + \frac{1}{2} \right) \theta_s(N k_s) + \frac{\pi}{4} \right), \]  

where \( V(j_i) \) is the classical volume of the tetrahedron with segment lengths given by \( j + \frac{1}{2} \), and \( \theta_s \) are the corresponding dihedral angles, namely the angles between the normals to the triangles. As it will be useful in the following, we recall here that the volume of a tetrahedron can be expressed in terms of a dihedral angle using the formula

\[ V = \frac{2}{3} \frac{A_1 A_2}{\ell_s} \sin \theta_s, \]  

where \( A_1 \) and \( A_2 \) are the areas of the base triangles, and \( \ell_s \) is the length of the side opposite to the dihedral angle \( \theta_s \).
where $A_1$ and $A_2$ are the areas of the two triangles $t_1$ and $t_2$ sharing the segment $s$, as shown in the above figure. The areas can be expressed in terms of the segment lengths, using Heron’s formula (see the appendix).

The key point of (8) is the argument of the cosine: up to the factor $\frac{2}{7}$ (which does not change the equations of motion), this is the Regge action $S_R = \sum \ell_1 \ell_2 (\ell_3)$, a discrete approximation to classical GR. Therefore, the amplitude of $Z$ is dominated by exponentials of the Regge action, which makes it promising to study the semiclassical limit in this way. If this is correct, then also the geometrical quantities that one can evaluate in the model should reduce to their classical expressions, in the large $j$ limit defined above.

However one would expect a single exponential to arise in the semiclassical limit, namely $Z_{GR} \sim \int Dg \ e^{i\lambda R}$. The meaning of this difference is well studied, see for instance the discussion in [1]. We might then conclude that on a single tetrahedron $\tau Z(\tau) = Z_{GR}(\tau) + Z_{BF}(\tau)$. As we will see below, this fact has consequences for the expectation value of the volume.

To introduce the geometrical observables in the PR model, below we write them as gauge-invariant functions of the variables $X'_j$. Then, to compute their expectation values, it is convenient to introduce a generating functional,

$$Z[J] = \prod_{x} \int \frac{d g_{x}}{S U(2)} \prod_{J} \int \frac{d X_{x}}{S U(2)} \exp \left(i \sum_{x} \text{Tr} X_{x} (g_{x} + J_{x}) \right).$$

This can be evaluated as described in [3], to give

$$Z[J] = \sum_{\{J_{x}\}} \prod_{s} d_{s} \prod_{J} \tau \left(\begin{array}{c}
\delta J_{x} \\
\delta J_{x}
\end{array}\right) \times D_{m_{1} m_{2}}^{(j_{1})} (e_{1}^{j_{1}}) D_{m_{3} m_{4}}^{(j_{2})} (e_{2}^{j_{2}}) D_{m_{5} m_{6}}^{(j_{3})} (e_{3}^{j_{3}}) D_{m_{7} m_{8}}^{(j_{4})} (e_{4}^{j_{4}}) D_{m_{9} m_{10}}^{(j_{5})} (e_{5}^{j_{5}}) D_{m_{11} m_{12}}^{(j_{6})} (e_{6}^{j_{6}}).$$

Here the $D$ are representation matrices. The quantity (12) represents the $[6j]$ symbols with source insertions. The $J_{s}$ are attached to the segments of $\Delta$; they are the sources of the quantum excitations $J_{s}$. For all $J_{s} = 0$ (12) reduces to the expression for the $[6j]$ symbol given above, thus $Z_{BF} = Z[J]_{|J_{s}=0}$.

Consider now a gauge-invariant observable constructed from the $X'_j$ variables, $\Phi[X'_j]$. Using the generating functional, its expectation value can be written as

$$\langle \Phi \rangle = \prod_{x} \int \frac{d g_{x}}{S U(2)} \prod_{J} \int \frac{d X_{x}}{S U(2)} \Phi[X'_j] \exp \left(i \sum_{x} X'_j s'_{x} \right) = \Phi \left[-i \frac{\delta}{\delta J_{x}} \right] Z[J]_{|J_{s}=0}. (13)$$

In the following section, we will use this procedure to compute expectation values. To do so, we need to know the action of the algebra derivatives. Acting on a group element in the representation $j$, we have

$$\frac{\delta}{\delta J} D^{(j)} (e^{l}) \bigg|_{j=0} = -i T^{(j)} (e^{l}),$$

where $T^{(j)}$ is the $l$th generator in the representation $j$. By inspecting (12), we see that a derivative acting on $J_s$ attaches to the segment $s$ an algebra generator in the irrep $j_s$ labelling the segment. This action is called ‘grasping’, and it is described in more details in the appendix. In particular, we are interested in quadratic and cubic gauge-invariant functions $\Phi$, such as squared lengths and volumes, which are related to the action of quadratic and cubic graspings.
3. Graspings and quantum geometry

3.1. Quadratic graspings: lengths and angles

The classical geometrical observables can be described using the discrete variables $X_i^I$. For instance, the length of a segment $\tilde{s}$ can be written as

$$\ell_{\tilde{s}} = e_{\mu \nu} \ell^\mu_1 \ell^\nu_1 = \ell^I_1 X^I_1 X^I_1.$$ 

In the same way, we can study the angles between the segments. To this aim, we consider the two segments $s_1$ and $s_2$, sharing a vertex, and the segment $s_3$ closing the triangle. The angle can be read from the scalar product

$$\ell_{s_1} \cdot \ell_{s_2} = \ell^{I_1}_1 X^I_1 X^I_2.$$

Calculating the expectation value of quadratic functions of the $X_i^I$s is particularly simple. The expectation value of a scalar product is given by

$$\langle \ell_{s_1} \cdot \ell_{s_2} \rangle = \frac{1}{Z} \int \ldots \int dX_i X_i^I X_i^I e^{\sum_j x_j^I x_j^I} = \frac{1}{Z} \frac{\delta^2}{\delta J_i^I \delta J_i^I} Z[J] \bigg|_{J=0}.$$ 

(15)

To evaluate this quantity, we need the action of the double grasping, which is computed in the appendix.

For the case $s_1 = s_2 \equiv \tilde{s}$, the relevant grasping gives

$$\frac{\delta}{\delta J_i^I} \frac{\delta}{\delta J_i^I} = -C^2(j) \{6j\}.$$ 

(16)

where $C^2(j) = j(j + 1)$ is the Casimir of $SU(2)$. Consequently, we have

$$\langle \ell_{\tilde{s}}^2 \rangle = \frac{1}{Z} \sum_{\{j_i\}} \ell_{\tilde{s}}^2 C^2(j) \prod_j d_j \prod_j \{6j\}.$$ 

(17)

We see that the value of the quantum squared length of a labelled segment is given by $\ell_{\tilde{s}}^2 C^2(j)$. This is the basic result of quantum geometry, and we see that the origin of its discreteness lies in the compactness of the group $SU(2)$, which gives a discrete series of representations. Therefore, $\ell_{\tilde{s}}$ acquires only discrete values, consistently with the canonical result [1].

For the case when $s_1 \neq s_2$ share a vertex, the relevant grasping gives

$$\frac{\delta}{\delta J_i^I} \frac{\delta}{\delta J_i^I} = -\frac{1}{2} [C^2(j_i) + C^2(j_2) - C^2(j_3)] \{6j\}.$$ 

(18)

where $s_3$ closes the triangle with $s_1$ and $s_2$. Therefore, we see that the value of $\cos \theta_{s_1 s_2}$ is given by the expression

$$\frac{1}{2C(j_1)C(j_2)} [C^2(j_1) + C^2(j_2) - C^2(j_3)].$$ 

(19)

Note that this expression coincides with the classical one, once we identify the Casimir with the segment lengths. However, this does not mean that the angle behaves classically: as the Casimir is discrete, the angle cannot range continuously between 0 and $\pi$, instead only specific values are allowed. Among these, the equilateral case: if we set all $j_i = j_0$ in (19), we obtain $\frac{1}{2} = \cos \left( \frac{\pi}{3} \right)$, which correctly reproduces an equilateral triangle.
3.2. Triple grasping: volume

We now come to the explicit computation of the volume, which is the first non-trivial geometrical object in the PR model. This is defined as

\[ V_t := \int e \, d^3 x \approx \frac{1}{3!} e \int e \, d^3 x, \quad e = \frac{1}{3!} \epsilon^{\mu \nu \rho} \epsilon_{IJK} \epsilon^I \epsilon^J \epsilon^K. \]  

(20)

The reason for the factor \(3!\) lies in the fact the determinant is the infinitesimal (metrical) volume of a cube, and there are six tetrahedra in a cube. To express \(V_t\) in terms of the variables \(X^i_t\), we proceed as follows. Consider three segments sharing a point \(p\) of \(\tau\), with coordinate vectors \(\ell^i_t(p)\); the coordinate volume is \(\int e \, d^3 x = \det \ell^i_t(p)\), by definition of \(\ell^i_t(p)\), independently of the point considered. We can think of \(\ell^i_t(p)\) as a \(3 \times 3\) matrix, with inverse \(n^i_t(p)\) being defined by

\[ \sum_{s \in p(\tau)} \ell^i_t(p) n^i_t(p) = \delta^i_t. \]  

(21)

This resolution of the identity can now be inserted in the definition (20) of \(e\), in order to give the variables \(X^i_t\):

\[ V_t = \frac{1}{3!} e \, \text{det} \ell^i_t = \frac{\text{det} \ell^i_t}{3!} e = \frac{\text{det} \ell^i_t}{3!} e^{\epsilon^{\mu \nu \rho} \epsilon_{IJK}} \epsilon^I \epsilon^J \epsilon^K. \]

Note that from the definition of \(n^i_t(p)\) in (21), we have \(e^{\epsilon^{\mu \nu \rho} n^i_t n^i_t} = e^{\epsilon^{\mu \nu \rho} (\det e^I)}^{-1}\), thus

\[ V_t = \frac{\ell^i_t}{3!} \sum_{p \in \tau} \epsilon_{i,j,k} X^j_s X^k_s. \]

(22)

With an eye at the construction of its quantum version, it is convenient to symmetrize this expression. One possible way to do so is to take the same point \(p\) in each insertion of (21), and then sum over the four contributions:

\[ V_t = \frac{\ell^i_t}{3!} \sum_{p \in \tau} \epsilon_{i,j,k} X^j_s X^k_s. \]

(22)

where in the latter expression \(s_1, s_2\) and \(s_3\) are a fixed right-handed triple belonging to the given \(p\).

Alternatively, we can consider all possible 16 non-coplanar triples of segments, and write

\[ V_t = \frac{\ell^i_t}{3!} \sum s_1, s_2, s_3 \epsilon_{i,j,k} X^j_s X^k_s. \]  

(23)

This formula was proposed in [3]. Note that this latter case is more generic: (22) can be obtained from (23) restricting the triplets in the sum. The two expressions are clearly classically equivalent, but lead to different quantum values. As we show below, the corresponding values in the quantum theory have the same semiclassical leading order, but different corrections.

As it is more generic, we consider first the expression (23). Using the generating functional as above, we have

\[ \langle V_t \rangle = \frac{1}{Z} \frac{1}{3!} \frac{\ell^i_t}{16} \int_{SU(2)} d g e \sum_{s} \int_{SU(2)} d X s_1, s_2, s_3 \epsilon_{i,j,k} X^j_s X^k_s. \]

(24)
Table 1. The different triple graspings. The labels for the segments are as in (11). Types 2 and 3 are the classical configurations; the others are configurations absent in the classical case. The details of the evaluations, as well as the explicit values of the coefficients $c_\tau$, $c_0$ and $c_3$, are reported in the appendix. With abuse of notation, we used the same symbol $c$ for the coefficients of types 2, 3 and 5; however, a permutation of the segment labels is involved in going from one type to the other. For each grasping type, a single configuration is shown; the others can be obtained by permutations of the segments. Finally, note that there are symmetries in the evaluation: for instance in type 2, we chose to write the final result in terms of $\{6j\}$ shifted in the variable $j_1$, but we could have just as well shifted $j_2$ or $j_3$, and the coefficients $c_0(j_1)$, $c_0(j_3)$ would have accordingly changed.

| Grasping | Evaluation | Leading order |
|----------|------------|----------------|
| 1. | $0$ | $0$ |
| 2. | $c_{-\tau}(j_1) \left\{ j_1 - 1 \atop j_4 \atop j_5 \atop j_6 \right\} + c_0(j_1) \left\{ j_1 \atop j_4 \atop j_5 \atop j_6 \right\} + c_\tau(j_1) \left\{ j_1 + 1 \atop j_4 \atop j_5 \atop j_6 \right\}$ | $j^3$ |
| 3. | $c_{-\tau}(j_1) \left\{ j_1 - 1 \atop j_4 \atop j_5 \atop j_6 \right\} + c_0(j_1) \left\{ j_1 \atop j_4 \atop j_5 \atop j_6 \right\} + c_\tau(j_1) \left\{ j_1 + 1 \atop j_4 \atop j_5 \atop j_6 \right\}$ | $j^3$ |
| 4. | $-\frac{1}{2} (C^2(j_1) + C^2(j_2) - C^2(j_3)) \left\{ j_1 \atop j_4 \atop j_5 \atop j_6 \right\}$ | $j^2$ |
| 5. | $\frac{c_{-\tau}(j_1)}{j_1 + 1} \left\{ j_1 - 1 \atop j_4 \atop j_5 \atop j_6 \right\} + c_0(j_1) \left\{ j_1 \atop j_4 \atop j_5 \atop j_6 \right\} + c_\tau(j_1) \left\{ j_1 + 1 \atop j_4 \atop j_5 \atop j_6 \right\}$ | $j^2$ |
| 6. | $-C^2(j_1) \left\{ j_1 \atop j_4 \atop j_5 \atop j_6 \right\}$ | $j^2$ |

The derivatives act only on the sources present in the tetrahedron $\bar{\tau}$. Therefore, $\langle V_\tau \rangle$ depends only on the six $j_\alpha$ entering $\bar{\tau}$. The quantity in round brackets gives rise to the triple grasping on the tetrahedron, as explained in the appendix. Depending on the configuration of the triplet $s_1, s_2, s_3$ considered, different types of grasping are involved. These types, together with the result of their evaluation, are listed in table 1. We report the details of the evaluation in the appendix. With abuse of notation, we used the same symbol $c$ for the coefficients of types 2, 3 and 5; however, a permutation of the segment labels is involved in going from one type to the other. For each grasping type, a single configuration is shown; the others can be obtained by permutations of the segments. Finally, note that there are symmetries in the evaluation: for instance in type 2, we chose to write the final result in terms of $\{6j\}$ shifted in the variable $j_1$, but we could have just as well shifted $j_2$ or $j_3$, and the coefficients $c_0(j_1)$, $c_0(j_3)$ would have accordingly changed.

The different triple graspings. The labels for the segments are as in (11). Types 2 and 3 are the classical configurations; the others are configurations absent in the classical case. The details of the evaluations, as well as the explicit values of the coefficients $c_\tau$, $c_0$ and $c_3$, are reported in the appendix. With abuse of notation, we used the same symbol $c$ for the coefficients of types 2, 3 and 5; however, a permutation of the segment labels is involved in going from one type to the other. For each grasping type, a single configuration is shown; the others can be obtained by permutations of the segments. Finally, note that there are symmetries in the evaluation: for instance in type 2, we chose to write the final result in terms of $\{6j\}$ shifted in the variable $j_1$, but we could have just as well shifted $j_2$ or $j_3$, and the coefficients $c_0(j_1)$, $c_0(j_3)$ would have accordingly changed.

The deriv
the permutation 123 456 \mapsto 231 564. With this understanding, we can write the action of (22) as

$$\frac{i}{3!} \epsilon_i^3 \sum_{p \in \tau} \left[ c_-(j_i) \begin{pmatrix} j_1 - 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} + c_0(j_i) \begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} \right]
+ c_+(j_i) \begin{pmatrix} j_1 + 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix}^{-1},$$

(25)

where the explicit values of the coefficients $c_-(j_i), c_0(j_i)$ are given in the appendix. Note that they vary when different configurations (here different points of the tetrahedron) are considered. For simplicity of notation we did not write explicitly the dependences of the coefficients on the configuration.

Type 2 is the only grasping entering (22), whereas the more generic definition (23) involves all the different graspings, and the result would be ($i$ times) the sum of all the grasping evaluations listed in the table. Note that in both cases the value of the volume is purely imaginary. This was anticipated in [4], and can be understood as follows. Firstly, the volume is odd under change of orientation, namely $V(\tau) = -V(-\tau)$. Secondly, the unitarity of the PR amplitude implies $Z(\tau) = Z(-\tau)$. Therefore $\langle V(\tau) \rangle = -\langle V(\tau) \rangle$ which implies $\text{Re}(\langle V(\tau) \rangle) = 0$. As argued in [4], this volume can be related to a real volume $\langle V(\tau) \rangle_{GR}$, computed using the $Z_{GR}$ partition function defined in section 2, which we recall satisfies $Z = 2 \text{Re} Z_{GR}$. Using $Z_{GR}(\tau) = Z_{GR}(-\tau)$, we can formally write

$$\langle V(\tau) \rangle Z(\tau) = \langle V(\tau) \rangle_{GR} Z_{GR}(\tau) Z_{GR}(\tau) = 2i \langle V(\tau) \rangle_{GR} \text{Im} Z_{GR}(\tau),$$

(26)

from which

$$\langle V(\tau) \rangle = i \frac{\text{Im} Z_{GR}(\tau)}{\text{Re} Z_{GR}(\tau)} \langle V(\tau) \rangle_{GR}.\quad (27)$$

Let us now restrict this formula to a single tetrahedron, and assume that in the semiclassical limit $Z_{GR}(\tau) \sim \exp i \left( S_{GR}(\tau) + \frac{\pi}{4} \right)$. Under this assumption, the large spin limit of (27) formally reads

$$\langle V(\tau) \rangle = i \tan \left( S_{GR}(j_i) + \frac{\pi}{4} \right) \langle V(\tau) \rangle_{GR}.\quad (28)$$

We expect the value of the volume in the PR model to obey a relation like the one above. Also, note that the same reasoning applied to the squared volume gives $\langle V(\tau)^2 \rangle = \langle V(\tau)^2 \rangle_{GR}$, thus the PR model should indeed reproduce the correct semiclassical limit of the squared volume.

4. The semiclassical limit of the volume

All the non-zero graspings (plus all possible permutations) listed in table 1 contribute to the quantum volume. However, not all of them contribute to the leading order of the large spin limit. Indeed, the graspings have different overall scalings, which we reported for commodity in the column on the right. In particular we see that all the degenerate (classically absent) graspings do not contribute to the leading order.

Let us focus our attention to the first non-degenerate grasping, number 2 in the table. This, we recall, is the only grasping entering the definition (22) of the volume. As we see from the table, the evaluation of the grasping gives a superposition of $\{6j\}$s. To study the large spin limit, consider first the coefficients: as we show in the appendix, in the large spin limit we
have
\[ c_\pm(j_i) \simeq \mp \frac{2}{l_1} A_{123} A_{156}, \]  
(29)
\[ c_0(j_i) \sim j_i^2. \]  
(30)

Here \( l_1 = j_1 + \frac{1}{2} \) and \( A_{123} \) is the area of the triangle bound by the segments \( l_1, l_2 \) and \( l_3 \). The asymptotics (29) are crucial: if we compare them with (9), we see that we are on the right track, a factor \( \sin \theta \) is all that is missing to recover the classical formula of the volume. On the other hand, (30) shows that the ‘diagonal’ term, the one with \( j_1 \) unchanged in the \( \{6j\} \), can be neglected at the leading order. The leading order then emerges only from the superdiagonal and the subdiagonal, namely those with \( j_1 \pm 1 \) in the \( \{6j\} \). Interestingly, this is analogous to what happens for the volume operator in the canonical approach of loop quantum gravity (see for instance [16]).

Using (8) to expand the \( \{6j\} \)s entering the grasping 2, the result reads
\[
\epsilon_{IJK} \frac{\delta}{\delta J_{i_1}} \frac{\delta}{\delta J_{i_2}} \frac{\delta}{\delta J_{i_3}} \frac{\delta}{\delta J_{i_4}} \frac{\delta}{\delta J_{i_5}} \frac{\delta}{\delta J_{i_6}} \sin \left( \frac{S_R(j_i) + \pi}{4} \right) = \frac{2}{l_1} A_{123} A_{156} \sqrt{\frac{12}{\pi V(j_i)}} \cos \left( \frac{S_R(j_i - 1) + \pi}{4} \right) - \cos \left( \frac{S_R(j_i + 1) + \pi}{4} \right) + O(j_i^2). \]  
(31)

Here we have anticipated that the next to leading order is \( O(j_i^2) \), as shown in the next section. For large spins, the Regge actions can be expanded. Using the well-known property \( \partial S_R / \partial j_1 = \theta \), we have
\[ S_R(j_1 \pm 1) \simeq S_R(j_1) + \frac{\partial S_R}{\partial j_1} \delta j_1 = S_R(j_1) \pm \theta_1. \]  
(32)

Consequently,
\[ \cos \left( \frac{S_R(j_i - 1) + \pi}{4} \right) - \cos \left( \frac{S_R(j_i + 1) + \pi}{4} \right) \simeq 2 \sin \theta_1 \sin \left( \frac{S_R(j_i) + \pi}{4} \right). \]  
(33)

The expansion of the Regge action has produced the sine of the dihedral angle needed to recover the classical formula for the volume. Putting everything together, (31) reads
\[
\frac{i}{3!} \epsilon_{IJK} \frac{\delta}{\delta J_{i_1}} \frac{\delta}{\delta J_{i_2}} \frac{\delta}{\delta J_{i_3}} \frac{\delta}{\delta J_{i_4}} \frac{\delta}{\delta J_{i_5}} \frac{\delta}{\delta J_{i_6}} \sin \left( \frac{S_R(j_i) + \pi}{4} \right) = i \frac{V(j_i)}{\sqrt{12 \pi V(j_i)}} \sin \left( \frac{S_R(j_i) + \pi}{4} \right) + O(j_i^2). \]  
(34)

In the large spin limit, we then have
\[ \langle \mathcal{V} \rangle \sim i \frac{1}{Z} \sum_{j \geq 1} \prod_{x} d_{j_x} \prod_{x} \{6j\} V_t(j_i) \tan \left( \frac{S_R(j_i) + \pi}{4} \right). \]  
(35)

We see that the leading order of the quantum volume is indeed proportional to the classical formula (9), but there is the extra factor of the tangent of the Regge action. This was to be expected, and it is consistent with the formal manipulation (28).

As discussed above, we can consider the squared volume \( \mathcal{V}^2 \), given by (minus) the squared of the triple grasping in (24), to obtain a real expectation value and avoid the extra tangent factor. However, grasplings in general do not commute, thus the definition of their products requires an ordering prescription. For the objective of studying the semiclassical limit, the
appropriate ordering seems to take some sort of ‘temporal ordering’: we act with the two triple grasplings one after the other, without allowing them to self–intersect. With this ordering, it is easy to compute the value of $V^2$ starting from the results listed in table 1. In particular, acting twice in a row with the grasping type 2 and proceeding as above, we obtain the following leading order,

$$\left(\frac{2}{l_1}A_{123}A_{156}\right)^2 \left[\left\{j_1 - 2 j_2 j_3 j_6\right\} - 2 \left\{j_1 j_2 j_3 j_6\right\} + \left\{j_1 + 2 j_2 j_3 j_6\right\}\right] = -\hbar^2 (3)^2 \sum_{j \gg 1} \prod_s d_j \prod_{\tau} (6j) V_\tau^2(j_s).$$

This result can be generalized to arbitrary powers of the triple grasping: this ordering prescription allows us to immediately identify the (semiclassical limit of the) $n$th power of the triple grasping (which is giving raise to $\pm n$ shifts in the $(6j)$ symbol) with the $n$th power of the volume (times the factor $i$ tangent if $n$ is odd).

In conclusion, the asymptotics of (powers of) the triple grasping are dominated by (powers of) the classical formula (9) for the volume of a tetrahedron. It is interesting to note that the polynomial part of this formula arises from the coefficients of the grasping, see (29). The non-polynomial part, namely the sine of the dihedral angle, arises on the other hand from the expansion of the Regge action entering the modified $(6j)$s with $\pm 1$. Thus the fact that the triple grasping produces a superposition of $(6j)$s is crucial. Furthermore, let us remark again that the fact that the leading asymptotics come from the terms with $\pm 1$ is somewhat reminiscent of the fact that in the spin network basis the canonical 3D volume operator in loop quantum gravity has values only on the superdiagonal and the subdiagonal.

5. Quantum corrections

Having discussed the meaning of the factor $i$ tangent in (35), we now look at the next to leading order corrections to the quantum volume. Let us first consider the quantum corrections to the grasping of type 2. These are the only ones entering the definition (22) of the volume. There are three types of corrections:
Corrections from $O(l^2)$ terms in the coefficients $c_i(l_j)$. These can be read from the appendix, respectively from (B.22), (B.18) and (B.23).

$$c^{(1)}(j_i) \sqrt{12 \pi V(j_i)} \cos \left( S_R(j_i - 1) + \frac{\pi}{4} \right) + c_0(j_i) \sqrt{12 \pi V(j_i)} \cos \left( S_R(j_i) + \frac{\pi}{4} \right)
+ c^{(1)}_+(j_i) \sqrt{12 \pi V(j_i)} \cos \left( S_R(j_i + 1) + \frac{\pi}{4} \right).$$

Using the fact that $c^{(1)} := c^{(1)}_+ \equiv c^{(1)}_+$ (see the appendix) and expanding the Regge action as above, this correction can be simply written as

$$i \frac{2c^{(1)}(j_i) \cos \theta_1 + c_0(j_i)}{\sqrt{12 \pi V(j_i)}} \cos \left( S_R(j_i) + \frac{\pi}{4} \right). \quad (38)$$

Recall that $c^{(1)}$ and $c_0$ depend on the configuration chosen (i.e. the triplet grasped—here 126). The correction to the volume is obtained summing over all configurations (four different ones using (22), sixteen using (23), which restores a symmetric expression.

- Corrections from higher orders in the expansion (32) of the Regge action. These can be obtained as follows. First of all, we keep up to the second order term in the expansion of the Regge action,

$$S_R(j_i \pm 1) \approx S_R(j_i) \pm \theta_1 + G_{11}.$$

The exact form of the second derivative $G_{11} := \frac{1}{2} \frac{\partial^2 S_R}{\partial j^2}$ can be computed from elementary geometry (for examples, see [9]). From dimensional analysis, it is clear that $G_{11} \sim 1/j$, thus we expect this term to contribute to the corrections. The contribution can be easily computed. We have $\sin(\theta_1 + G_{11}) \approx \sin \theta_1 + G_{11} \cos \theta_1$ instead of (33), thus the extra piece gives a correction to (34) of

$$i G_{11} \frac{V(j_i)}{\sqrt{12 \pi V(j_i)}} \cos \left( S_R(j_i) + \frac{\pi}{4} \right). \quad (39)$$

Here again, one has to sum over all the relevant configurations to obtain the correction to the volume.

- Corrections from higher orders in the expansion (8) of the $\{6j\}$ symbol. Unfortunately, the next term in (8) is not known in the literature, thus we cannot pursue this analysis to the end. However, numerical investigations performed in [9] hint for a next term of the type $\frac{N}{\sqrt{6} V(j_i^{1/3})} \left[ \sin \left( \phi - \frac{\pi}{4} \right) \cos \left( S_R(j_i) + \frac{\pi}{4} \right) 
+ \cos \left( \phi - \frac{\pi}{4} \right) \sin \left( S_R(j_i) + \frac{\pi}{4} \right) \right]$. \quad (40)

The relative sign of this correction depends on the sign of the numerical constants $N$ and $\phi$. The fit obtained in [9] for the equilateral case when all $j_i$ are the same, gave $N = -0.36$, $\phi = 0.68$.

Summing (38)–(40) among themselves and over the four configurations of grasping type 2 (corresponding to grasping the four different vertices), we obtain the overall next to leading order correction to the quantum volume. This goes as $j^2$, as anticipated in (31).
The analysis of the corrections of the grasping of type 3 leads exactly to the same results above. However the two definitions (22) and (23) do differ at the level of quantum corrections, as the latter also receives contributions from the degenerate grasplings of type 4, 5 and 6.

Consider first the grasplings of type 5. Its leading order can be immediately read from type 2—provided we take into account the extra denominators reducing by one power of \( j \) the coefficients of the grasplings. We can thus write

\[
i \frac{c_0(j_s) - 4A_{121}A_{156} \cos \theta_1 j_1^2}{\sqrt{12\pi V(j_s)}} \cos \left( S_{R}(j_s) + \frac{\pi}{4} \right).
\] (41)

Finally, 4 and 6 can be recognized, using the results of section 3.1, to be the values of the various scalar products between segment vectors, \( \ell_s \cdot \ell_s' \). Including all the permutations, we simply have the overall correction

\[
- \frac{i}{6} \sum_{s,s'} \ell_s \cdot \ell_s' \cos \left( S_{R}(j_s) + \frac{\pi}{4} \right).
\] (42)

These extra corrections (41) and (42) allow us to distinguish and differentiate between the two definitions of the quantum volume.

6. Conclusions

By relating the classical observable to a grasping operator and using the recoupling theory of \( SU(2) \), we explicitly computed the value of the quantum volume of a labelled tetrahedron in the PR model. The various terms contributing to the quantum value are reported in table 1, and the details of the evaluation in the appendix.

Then, we studied the semiclassical limit by considering the large spin expansion. The leading order, given in (35), is indeed dominated by the classical formula, but an additional factor is present. This factor, \( i \) times the tangent of the Regge action, can be understood as a consequence of the fact that the PR model sums over both orientations of spacetime, and was indeed anticipated in [4]. Consistently, the value of the squared volume should not have this problem, and we confirmed this expectation: the value of the squared volume, given in (37), is dominated precisely by the classical formula. In doing so, we introduced an ordering prescription for the products of graspings. Using this prescription, it is easy to see that the \( n \)th power of the triple grasping has the asymptotics dominated by the \( n \)th power of the classical volume (times a factor \( i \) tangent if \( n \) is odd). The existence of such an ordering prescription could turn out to be important in studying the large spin expansion of spinfoam models of gravity coupled to matter field.

It is interesting to note that a key factor of the classical formula, namely the sine of the dihedral angle, does not come directly from the coefficients of the grasping, but from the fact that the triple grasping changes the initial \( \{6j\} \) into a superposition of \( \{6j\} \). This fact mimics the structure of the canonical volume operator.

We considered two different definitions of the quantum volume, related to classically equivalent different discretizations, and the results summarized above hold for both definitions. However, we also showed how one can compute the next to leading order quantum correction, and this is where the two versions of the quantum volume differ. We expect this to be a generic feature of constructing quantum observables starting from discrete classical quantities: different quantum observables corresponding to classically equivalent quantities, even if they have the same semiclassical leading order, could still differ at the next to leading order. The latter can be then used to distinguish them.

In particular, the next to leading order also depends on the second term in the expansion of the \( \{6j\} \) symbol, a term which is not analytically known at the moment. At this stage, we
Grasping rules and semiclassical limit of the geometry in the Ponzano–Regge model

are thus not yet able to say what is the relative sign of the correction, and so we leave the issue
open. However, let us stress that knowing the next term of the \(6j\) is also useful to study how
quantum gravity can give rise to a different perturbative expansion for the graviton propagator,
as suggested in [9], so computing this term analytically could be useful for several reasons.

The results reported here show that the large spin limit reproduces semiclassical geometry
in 3D quantum gravity. It is important to extend a similar analysis to the physically interesting
4D case. However, the situation is more subtle in 4D. In fact, the classical geometrical
quantities do not commute in 4D spinfoam models of quantum gravity. Therefore an eigenstate
will not in general have the semiclassical limit, just as well as one cannot study the semiclassical
limit of the harmonic oscillator on an eigenstate of the energy. It is necessary to first construct
appropriate semiclassical states, or coherent states. A class of semiclassical states for the
tetrahedron of loop quantum gravity were proposed in [17]. It would be interesting to apply
the logic described here to study the value of the quantum volume of these states.

Acknowledgments

We thank Carlo Rovelli for useful discussions on the construction of the volume grasping,
and Laurent Freidel for suggestions on the use of the recoupling theory. Research at the
Perimeter Institute is supported in part by the Government of Canada through NSERC and by
the Province of Ontario through MEDT.

Appendix A. Evaluation of \(SU(2)\) spin networks

A spin network \(s\) is a triple \(\{\gamma, j_l, i_n\}\) consisting of a graph \(\gamma\); a set of irreps \(j_l\) associated with
the links; a set of intertwiners \(i_n\) associated with the nodes. A spin network state \(\psi_s(g_l)\) is
constructed assigning a group element \(g_l\) to each link in the corresponding irrep \(j_l\), namely
assigning a representation matrix \(D(j_l)(g_l)\), and then contracting the indices of all the matrices
with the intertwiners on the nodes:

\[
\psi_s(g_l) = \prod_l D(j_l)(g_l) \prod_n I_n.
\]  

(A.1)

By construction, this is a gauge-invariant quantity, belonging to the space \(\text{Inv} \left[ \bigotimes \mathcal{H}_j \right]\). Upon
the interpretation of the group elements as parallel transports (3), (A.1) becomes a cylindrical
function of the connection [18, 19]. The spin networks span the kinematical Hilbert space of
LQG, which is given by

\[
L^2[A/G] \cong \bigotimes_j \text{Inv} \left[ \bigotimes_l \mathcal{H}_j \right].
\]  

(A.2)

Evaluating a spin network refers to taking all group elements to the identity, \(g_l \mapsto 1\) and defines a map \(L^2[A/G] \mapsto \mathbb{C}\).

To fix the normalization of the spin networks, consider the \(\theta\) graph \(\theta\). Using the
normalized Wigner \(3j\)-coefficients as intertwiners, (A.1) reads

\[
\psi(g_1, g_2, g_3) = D_{m_1, m_2}^{(j_1)}(g_1) D_{m_2, m_3}^{(j_2)}(g_2) D_{m_3, m_1}^{(j_3)}(g_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix},
\]  

(A.3)

and its evaluation gives

\[
\psi(1, 1, 1) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{cases} 1 & \text{if } |j_1 - j_2| \leq j_3 \leq j_1 + j_2, \\ 0 & \text{otherwise}. \end{cases}
\]  

(A.4)
The inequality that has to be satisfied is the usual Clebsch–Gordan condition. Given an arbitrary spin network, its evaluation is identically zero unless the Clebsch–Gordan condition hold at all nodes.

A.1. The \{6j\} symbol

On the tetrahedral graph \(\text{\includegraphics[width=0.1\textwidth]{tetrahedron}}\), the evaluation of the spin network gives the \{6j\} symbol defined in (7),

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{pmatrix} = \begin{pmatrix}
  m_1 & m_2 & m_3 \\
  m_4 & m_5 & m_6
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{pmatrix} = \begin{pmatrix}
  m_1 & m_2 & m_3 \\
  m_4 & m_5 & m_6
\end{pmatrix}
\]

This object has the symmetries associated with the geometrical symmetries of the tetrahedron pictured above and, more fundamentally, it satisfies the Biedenharn–Elliott identity,

\[
\begin{pmatrix}
  j_1 & k_2 & j_3 \\
  k_1 & j_2 & k_3
\end{pmatrix} \begin{pmatrix}
  l_1 & k_2 & l_3 \\
  k_1 & l_2 & k_3
\end{pmatrix} = \sum_j d_j \begin{pmatrix}
  j_1 & j_2 & k_3 \\
  j_2 & j_3 & k_4
\end{pmatrix} \begin{pmatrix}
  l_1 & j_2 & l_3 \\
  l_2 & j_3 & l_4
\end{pmatrix} \begin{pmatrix}
  j_3 & j_1 & k_2 \\
  j_1 & j_3 & k_4
\end{pmatrix} = (A.5)
\]

An analytic expression for the \{6j\} is provided by the Racah formula [21],

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{pmatrix} = (-1)^{j_1+j_2+j_3+j_4} \Delta(j_1, j_2, j_3)\Delta(j_1, j_5, j_6)\Delta(j_4, j_5, j_6)\Delta(j_4, j_2, j_6) \times \sum_k \frac{(-1)^k (k+1)!}{f(k)}, \quad (A.6)
\]

where

\[
\Delta(j_1, j_2, j_3) := \frac{(j_1 + j_2 - j_3)!(j_1 - j_2 + j_3)!(j_1 + j_2 + j_3)!}{(j_1 + j_2 + j_3)!} \quad (A.7)
\]

and

\[
f(k) := (k - j_1 - j_2 - j_3)! (k - j_1 - j_5 - j_6)! (k - j_4 - j_2 - j_6)! \times (k - j_4 - j_5 - j_6)! (j_1 + j_2 + j_4 + j_5 - k)! \times (j_2 + j_3 + j_5 + j_6 - k)! (j_1 + j_3 + j_4 + j_6 - k)! \quad (A.8)
\]

\(k\) is an integer, and the sum in (A.6) is over all admissible \(k\). From the graphical representation of the \{6j\} symbol, it is clear that the Clebsch–Gordan conditions must hold on all four nodes, so that the functions \(\Delta\) appearing in (A.6) are all well defined.

Some useful explicit values of (A.6) are the followings:

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  1 & j_3 & j_2
\end{pmatrix} = \frac{1}{2} \frac{C^2(j_2) + C^2(j_3) - C^2(j_1)}{\sqrt{d_jd_2d_3}C^2(j_3)}, \quad (A.9)
\]

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  1 & j_3 & j_2 + 1
\end{pmatrix} = \frac{1}{2} \frac{\sqrt{(1 + j_2 + j_3 - j_1)(1 + j_2 + j_3 + 1)(-j_2 - j_3 + j_1)(j_2 + j_3 + 1)}}{\sqrt{(j_2 + 1)d_jd_2d_{j_2+1}C^2(j_3)}}, \quad (A.10)
\]

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  1 & j_3 & j_2 - 1
\end{pmatrix} = \frac{1}{2} \frac{\sqrt{(j_2 + j_3 - j_1)(j_2 + j_3 + 1)(1 + j_2 + j_3 + 1)(1 + j_2 + j_3 + 1)}}{\sqrt{j_3d_jd_2d_{j_2-1}C^2(j_3)}}, \quad (A.11)
\]
from which we immediately read
\[
\begin{align*}
\{ j_1 j_2 1 \} = \frac{1}{\sqrt{6d_jC^2(j)}}, & \quad \{ j_1 j_2 1 j \} = \frac{j}{\sqrt{6d_jC^2(j)}}, \\
\{ j_1 j_2 1 j - 1 \} = -\frac{j+1}{\sqrt{6d_jC^2(j)}}.
\end{align*}
\] (A.12)

As it will be useful in the following, let us recall here the definition of the \( \{9j\} \) symbol,
\[
\{ j_1 j_2 j_3 j_4 j_5 j_6 j_7 j_8 j_9 \} = \sum_j d_j \{ j_1 j_2 j_3 \} \{ j_4 j_5 j_6 \} \{ j_7 j_8 j_9 \}.
\] (A.13)

Note that it is antisymmetric under exchange of rows, thus
\[
\{ j_1 j_2 j_3 j_1 j_2 j_3 j_7 j_8 j_9 \} \equiv 0.
\] (A.14)

The \( \{9j\} \) symbol is the evaluation of the spin network on the hexagonal graph \( \begin{array}{c}
\text{hexagon} \end{array} \).

Appendix B. Grasping rules and recoupling theory

The action of an algebra derivative on an element group in a representation \( j \) is given by
\[
\frac{\delta}{\delta J^I} D^{(j)}(e^I) \bigg|_{J=0} = -iT^I(j),
\] (B.1)

where \( T^I(j) \) is the \( I \)th generator, in the irrep \( j \). We picture this action as \( \begin{array}{c}
\text{insertion of algebra generator} \end{array} \). The dashed line represents the insertion of the algebra generator. A dashed 3-valent vertex \( \begin{array}{c}
\text{3-valent vertex} \end{array} \) represents the algebra structure constants \( i\epsilon_{ijk} \). The insertion of an algebra generator is equivalent to adding a link in the adjoint irrep, up to normalization. We fix the normalization\(^1\) as to match the grasping rules of Bar-Natan [22], namely
\[
\begin{array}{c}
\text{dashed line} \end{array} = -\frac{\sqrt{6}}{2} = C^2(j) \begin{array}{c}
\text{insertion} \end{array}.
\] (B.2)

To this end, we choose
\[
\begin{array}{c}
\text{dashed line} \end{array} = \sqrt{d_jC^2(j)} \frac{1}{j}, \quad \frac{\sqrt{6}}{2} \frac{1}{j} = -\sqrt{6} \frac{1}{j}.
\] (B.3)

With this normalization, the action of the double grasping is the insertion in the spin network of an additional link in the adjoint irrep, weighted by the labels of the two links grasped:
\[
-\frac{\delta}{\delta J^1} \frac{\delta}{\delta J^2} \left( \begin{array}{c}
\text{double grasping} \end{array} \right) \bigg|_{j=0} \equiv \left( \begin{array}{c}
\text{insertion of additional link} \end{array} \right) = \sqrt{d_jC^2(j_1)d_jC^2(j_2)} \left( \begin{array}{c}
\text{additional link} \end{array} \right).
\] (B.4)

Here and in the following, a line with no labels means a link coloured with the adjoint \( j = 1 \) irrep. In the same way, the triple grasping is the insertion of a 3-valent node in the adjoint

\(^1\) For a different normalization, see for instance [23].
irrep, weighted by the labels of the three links grasped:

\[
\varepsilon_{IJK} \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \frac{\delta}{\delta J_3} \left( \begin{array}{c} j_1 \\ j_2 \\ j_3 \end{array} \right) \bigg|_{J=0} = \left( \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right) = -\sqrt{6d_{j_1}C^2(j_1)d_{j_2}C^2(j_2)d_{j_3}C^2(j_3)} \left( \begin{array}{c} j_1 \\ j_2 \\ j_3 \end{array} \right).
\]

Therefore, evaluating the action of a grasping operator on a given spin network amounts to evaluating a new spin network, obtained by adding to the former the graph of the grasp in the adjoint irrep. To compute the action of the grasping operators, it is then convenient the use of the recoupling theory.

The key equation to be used is the recoupling theorem:

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\sum_k \dim k \begin{array}{c}
\begin{array}{c}
\{ \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
k \\
j_4 \\
j_5 \\
j_6 \\
\end{array} \end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\{ \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array} \end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\{ \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
k \\
j_4 \\
j_5 \\
j_6 \\
\end{array} \end{array}
\end{array}
\end{array}
\] (B.6)

From the definition of the \( \theta \) spin network and the Clebsch–Gordan conditions, it follows that

\[
\begin{array}{c}
\begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
= \frac{1}{\dim j_1} \delta_{j_1,j_4} \quad .
\] (B.7)

Using (B.6) repeatedly, it is easy to demonstrate the following 3-vertex contraction:

\[
\begin{array}{c}
\begin{array}{c}
j_2 \\
j_1 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j_2 \\
j_1 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
j_1 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j_2 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_4 \\
j_5 \\
j_6 \\
\end{array}
\end{array}
\] (B.8)

Note that if we add an extra node to close the links \( j_1, j_2 \) and \( j_3 \) on both sides of (B.8), and using the normalization of the \( \theta \) spin network, we get back the definition of the \{6j\} symbol as the evaluation of the tetrahedral spin network.

To evaluate the graspings, the strategy is the following: we use the definitions (B.4) and (B.5) to obtain new graphs; then we use (B.6) enough times to reduce the new graphs to the original ones. In particular in the next sections, we will apply this strategy to the tetrahedral graph, in order to study the spectra of geometrical observables in the Ponzano–Regge model.

**B.1. The double grasping on the \{6j\}**

Here we show how to evaluate the double grasping on the tetrahedral spin network. These results were used in section 3.1.
Computing the quadratic grasping on a single link is trivial, all we need is the bubble move (B.7):
\[
-\frac{\delta}{\delta J_{1}} \frac{\delta}{\delta J_{2}} = \begin{pmatrix} j_{1} & j_{2} & j_{3} \\ j_{3} & j_{2} & j_{1} \end{pmatrix} = d_{h} C^{2}(j_{1}) C^{2}(j_{2}) = C^{2}(j_{1})[6j]. \tag{B.9}
\]
This proves (16) used to compute the spectrum of lengths.

Computing the grasping on two different segments sharing a point is also very simple. We have
\[
-\frac{\delta}{\delta J_{1}} \frac{\delta}{\delta J_{2}} \frac{\delta}{\delta J_{3}} = \begin{pmatrix} j_{3} & j_{1} & j_{2} \\ j_{2} & j_{1} & j_{3} \end{pmatrix} = \sqrt{d_{h} C^{2}(j_{1}) d_{h} C^{2}(j_{2})} = \sqrt{d_{h} C^{2}(j_{1}) d_{h} C^{2}(j_{2})}. \tag{B.10}
\]
The graph obtained with the grasping can be straightforwardly reduced to the initial tetrahedral graph, simply using (B.8) once:
\[
\begin{pmatrix} j_{3} & j_{1} & j_{2} \\ j_{2} & j_{1} & j_{3} \end{pmatrix} = \begin{pmatrix} j_{3} & j_{1} & j_{2} \\ j_{2} & j_{1} & j_{3} \end{pmatrix} = \begin{pmatrix} j_{3} & j_{1} & j_{2} \\ j_{2} & j_{1} & j_{3} \end{pmatrix}. \tag{B.11}
\]
Therefore, using the explicit expression (A.9), we obtain (18),
\[
-\frac{\delta}{\delta J_{1}} \frac{\delta}{\delta J_{2}} \frac{\delta}{\delta J_{3}} \Bigg|_{J=0} = \frac{1}{2} [C^{2}(j_{1}) C^{2}(j_{2}) - C^{2}(j_{3})][6j]. \tag{B.12}
\]
which we used to compute the spectrum of angles.

B.2. The triple grasping on the \{6j\}

Here we show how to evaluate the triple grasping on the \{6j\} symbol, a result used in section 3.2. As shown in table 1, there are several types of grasplings that one has to consider. We proceed in the order given there.

\textit{Grasping 1.}

\[
\begin{pmatrix} \delta_{j_{1}j_{2}} & \delta_{j_{1}j_{3}} & \delta_{j_{2}j_{3}} \\ \delta_{j_{2}j_{1}} & \delta_{j_{2}j_{3}} & \delta_{j_{3}j_{1}} \end{pmatrix} \begin{pmatrix} j_{3} & j_{1} & j_{2} \\ j_{2} & j_{1} & j_{3} \end{pmatrix} = \begin{pmatrix} j_{3} & j_{1} & j_{2} \\ j_{2} & j_{1} & j_{3} \end{pmatrix} = -\sqrt{6d_{h} C^{2}(j_{1}) d_{h} C^{2}(j_{2}) d_{h} C^{2}(j_{3})}. \tag{B.13}
\]

If we use the symmetries of the \{6j\} symbol, we see that the coefficient obtained in front of the original \{6j\} matches the definition of the \{9j\} symbol introduced in (A.13),
\[
\sum_{k} d_{k} \begin{pmatrix} j_{1} & j_{2} & j_{3} \\ j_{3} & j_{1} & j_{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ k & j_{1} & j_{3} \end{pmatrix} = \begin{pmatrix} j_{1} & j_{2} & j_{3} \\ j_{3} & j_{1} & j_{2} \end{pmatrix} = 0. \tag{B.14}
\]
By symmetry, this result applies to all coplanar triples, and we recover the classical result: there is no contribution to the volume from coplanar segments.
As a useful shorthand, we will define the quantity

\[
\left( \epsilon^{ijk} \frac{\delta}{\delta J_{i}} \frac{\delta}{\delta J_{j}} \frac{\delta}{\delta J_{k}} \right)_{j=0} = \sum_{k} \frac{\epsilon_{ijk}}{J_{i} J_{j} J_{k}} \left( j_{s}, j_{t}, j_{a} \right) \frac{1}{J_{s} J_{t} J_{a}} = -\sqrt{6} d_{j} C^{2}(j_{1}) d_{j} C^{2}(j_{2}) d_{j} C^{2}(j_{6})
\]

\[
\times \sum_{k} d_{k} \left\{ j_{s} j_{t} j_{a} \right\} \left\{ j_{s} j_{t} j_{a} \right\} \left\{ j_{s} j_{t} j_{a} \right\} \left\{ j_{s} j_{t} j_{a} \right\} \left\{ j_{s} j_{t} j_{a} \right\} .
\]

(B.15)

The calculation yields substantial differences from the previous one. The main one is the new spin \( k \) replacing the original \( j_{i} \) in the \( \{6j\} \) (by symmetry, we could have replaced \( j_{2} \) or \( j_{6} \), without changing the final result). From the third \( \{6j\} \) in the coefficient above, we read that the sum ranges over \( k = j_{i} - 1, j_{i}, j_{i} + 1 \). We can thus write the result above as

\[
c_{-}(j_{i}) \left\{ j_{i} - 1 \right\} j_{s} j_{t} j_{a} + c_{0}(j_{i}) \left\{ j_{i} j_{s} j_{t} j_{a} \right\} + c_{+}(j_{i}) \left\{ j_{i} + 1 \right\} j_{s} j_{t} j_{a} \right\},
\]

(B.16)

where, using the formula reported in the appendix A,

\[
c_{-}(j_{i}) = \frac{(j_{i} + 1)}{4(j_{i} + 2)(j_{i} + 1)} \sqrt{(j_{i} + j_{s} - j_{a})(j_{i} - j_{s} + j_{a})(1 + j_{s} + j_{a})(1 + j_{i} + j_{s} + j_{a})}
\]

\[
\times \sqrt{(j_{i} + j_{s} - j_{a})(j_{i} - j_{s} + j_{a})(1 + j_{i} + j_{s} + j_{a})},
\]

\[
c_{0}(j_{i}) = -\frac{[C^{2}(j_{i}) + C^{2}(j_{s}) - C^{2}(j_{a})][C^{2}(j_{i}) + C^{2}(j_{s}) - C^{2}(j_{a})]}{4C^{2}(j_{i})},
\]

\[
c_{+}(j_{i}) = -\frac{j_{i}}{4(j_{i} + 1)(2j_{i} + 1)}
\]

\[
\times \sqrt{(1 + j_{i} + j_{s} - j_{a})(1 + j_{i} - j_{s} + j_{a})(1 + j_{i} + j_{s} + j_{a})(2 + j_{i} + j_{s} + j_{a})}
\]

\[
\times \sqrt{(1 + j_{i} + j_{s} - j_{a})(1 + j_{i} - j_{s} + j_{a})(1 + j_{i} + j_{s} + j_{a})(2 + j_{i} + j_{s} + j_{a})}.\]

(B.19)

Let us now discuss the large spin expansion of this result. First of all, recall that the spins are related to the (here adimensional) lengths through \( l_{i} = j_{i} + \frac{1}{2} \). Second, let us recall Heron’s formula [20] for the area of a triangle with side lengths \( a, b, c \):

\[
A_{abc} = \frac{1}{2} \sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}.
\]

(B.20)

As a useful shorthand, we will define the quantity

\[
V_{l_{1}} = \frac{2}{l_{1}} A_{l_{1}l_{2}l_{3}} A_{l_{1}l_{4}l_{5}},
\]

(B.21)

and generalize to arbitrary \( V_{l_{i}} \) by the symmetry of the tetrahedron. Note that from (9) we immediately have \( V = \frac{1}{2} V_{l_{1}} \sin \theta_{i} \).

The coefficients of the functions in the lengths \( c_{\pm}(l_{i}) \) are polynomials of order 3. To second order, we have

\[
c_{-}(l_{i}) \approx V_{l_{1}} \left[ \left( \frac{1}{l_{1}} \right) \left( 1 - \frac{1}{4(l_{1} + l_{2} - l_{3})} - \frac{1}{4(l_{1} - l_{2} + l_{3})} + \frac{1}{4(-l_{1} + l_{2} + l_{3})} \right)
\]

\[
- \frac{1}{4(l_{1} + l_{2} + l_{3})} \left( 1 - \frac{1}{4(l_{1} + l_{5} - l_{6})} - \frac{1}{4(l_{1} - l_{5} + l_{6})} \right)
\]

\[
+ \frac{1}{4(-l_{1} + l_{5} + l_{6})} - \frac{1}{4(l_{1} + l_{5} + l_{6})} \right] + o \left( \frac{1}{l_{1}^{2}} \right).
\]

(B.22)
Grasping rules and semiclassical limit of the geometry in the Ponzano–Regge model

\[ c_\pm(l_a) \simeq -V_1 \left[ \frac{1}{2} \left( 1 + \frac{1}{4(l_1 + l_2 - l_3)} + \frac{1}{4(l_1 - l_2 + l_3)} - \frac{1}{4(-l_1 + l_2 + l_3)} \right) \right. \\
\left. + \frac{1}{4(l_1 + l_2 + l_3)} \left( 1 + \frac{1}{4(l_1 + l_5 - l_6)} + \frac{1}{4(l_1 - l_5 + l_6)} - \frac{1}{4(-l_1 + l_5 + l_6)} \right) \right]. \]  

(B.23)

Let us define the coefficients of the expansion as in \( c_\pm(j_a) = c^{(0)}_\pm(j_a) + c^{(1)}_\pm(j_a) + \cdots \). From the equations above, we see that \( c^{(0)}_\pm(j_a) = \mp V_1, \) and \( c^{(1)}_\pm(j_a) = c_\pm(j_a). \)

All terms in \( c_0(j_a) \), on the other hand, are of order \( j^2 \). In particular, note that \( c_0(j_a) = -(\ell_1 \cdot \ell_2)(\ell_1 \cdot \ell_6) / \ell_2^2 \).

The values from the other three vertex grasplings can be obtained by symmetry. The other three contributions can be obtained as above, considering the other three cases \((j_1, j_3, j_s), (j_2, j_3, j_a), \) and \((j_4, j_5, j_6)\).

**Grasping 3.**

\[ \left( \epsilon_{1JK} \delta \frac{\delta}{J_1 \delta J_2 \delta J_3} \right)_{J=0} = \begin{pmatrix} j_3 & j_1 \\ j_6 & j_4 \end{pmatrix} = -\sqrt{6d_j c^{2}(j_1) c^{2}(j_2) d_j c^{2}(j_4)} \]

\[ \times \sum_k d_k \begin{pmatrix} j_1 & 1 \\ 1 & j_1 \end{pmatrix} \begin{pmatrix} j_6 & j_2 \\ 1 & j_4 \end{pmatrix} \begin{pmatrix} j_2 & 1 \\ 1 & k \end{pmatrix} \begin{pmatrix} j_3 & j_1 \\ k & j_5 \end{pmatrix} \begin{pmatrix} j_5 & j_6 \\ k & j_6 \end{pmatrix}. \]  

(B.24)

Under the permutation \( 123456 \mapsto 213645 \), (B.24) is equivalent to (B.15). Therefore, this grasping reproduces the same results as the previous one. Note that this is in agreement with the classical result,

\[ V = \frac{1}{3!} e_1 \wedge e_2 \wedge e_6 = \frac{1}{3!} e_1 \wedge e_2 \wedge e_4. \]  

(B.25)

**Grasping 4.**

\[ \left( \epsilon_{1JK} \delta \frac{\delta}{J_1 \delta J_2 \delta J_3} \right)_{J=0} = \begin{pmatrix} j_5 & j_1 \\ j_3 & j_4 \end{pmatrix} = -\sqrt{6d_j c^{2}(j_2) d_j c^{2}(j_1)} \]  

\[ = -\frac{1}{2} [c^{2}(j_1) + c^{2}(j_2) - c^{2}(j_3)] \begin{pmatrix} j_1 & j_2 & j_3 \\ j_5 & j_4 & j_6 \end{pmatrix}. \]  

(B.26)

**Grasping 5.**

\[ \left( \epsilon_{1JK} \delta \frac{\delta}{J_1 \delta J_2 \delta J_3} \right)_{J=0} = \begin{pmatrix} j_5 & j_1 \\ j_3 & j_4 \end{pmatrix} = -\sqrt{6d_j c^{2}(j_1) d_j c^{2}(j_4) c^{2}(j_2)} \]

\[ \times \sum_k d_k \begin{pmatrix} j_2 & 1 \\ 1 & j_4 \end{pmatrix} \begin{pmatrix} j_6 & j_4 \\ 1 & j_1 \end{pmatrix} \begin{pmatrix} j_5 & 1 \\ 1 & j_1 \end{pmatrix} \begin{pmatrix} j_1 & 1 \\ 1 & j_4 \end{pmatrix} \begin{pmatrix} j_2 & j_3 \\ j_5 & j_6 \end{pmatrix} \begin{pmatrix} j_3 & j_6 \\ j_5 & k \end{pmatrix} \begin{pmatrix} j_5 & j_6 \\ k & j_6 \end{pmatrix}. \]
\[
\begin{align*}
\sum_j c_+ (j_2) \left\{ j_1 - 1 \atop j_4 \right\} \left\{ j_2 \atop j_5 \right\} + c_0 (j_3) \left\{ j_1 \atop j_4 \right\} \left\{ j_2 \atop j_5 \right\} + c_+ (j_2) \left\{ j_1 + 1 \atop j_4 \right\} \left\{ j_2 \atop j_5 \right\} \left\{ j_3 \atop j_6 \right\} \\
= (B.27)
\end{align*}
\]

Grasping 6.

\[
\begin{align*}
\left( \epsilon_{JJK} \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \frac{\delta}{\delta J_5} \right) \left| _{J=0} \right. \\
= -\sqrt{6} \left[ d_{j_1} C^2 (j_1) \right] \frac{1}{d_{j_1}} \left\{ j_1 \atop 1 \right\} \left\{ j_1 \atop 1 \right\} \left\{ j_1 \atop j_4 \right\} \left\{ j_2 \atop j_5 \right\} \left\{ j_3 \atop j_6 \right\} \\
= -C^2 (j_1) \left\{ j_1 \atop j_4 \right\} \left\{ j_2 \atop j_5 \right\} \left\{ j_3 \atop j_6 \right\}. \\
= (B.28)
\end{align*}
\]

References

[1] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[2] Ponzano G and Regge T 1968 Semiclassical limit of Racah coefficients Spectroscopy and Group Theoretical Methods in Physics ed F Bloch (Amsterdam: North-Holland)
[3] Freidel L and Krasnov K 1999 Spin foam models and the classical action principle Adv. Theor. Math. Phys. 2 1183 (Preprint hep-th/9807092)
[4] Freidel L and Krasnov K 1999 Discrete space-time volume for 3-dimensional BF theory and quantum gravity Class. Quantum Grav. 16 351 (Preprint hep-th/9804185)
[5] Fairbairn W 2006 Fermions in three-dimensional spinfoam quantum gravity Preprint gr-qc/0609040
[6] Freidel L, Rovelli C and Speziale S Perturbative expansion for 3d Yang–Mills theory coupled to quantum gravity in the spinfoam formalism, in preparation
[7] Livine E, Speziale S and Willis J 2006 Towards the graviton from spinfoams: higher order corrections in the 3d toy model Phys. Rev. D to appear Preprint gr-qc/0605123
[8] Livine E R and Speziale S 2006 Group integral techniques for the spinfoam graviton propagator J. High Energy Phys. JHEP11(2006)092 (Preprint gr-qc/0608131)
[9] Livine E, Speziale S and Willis J 2006 Towards the graviton from spinfoams: higher order corrections in the 3d toy model Phys. Rev. D to appear Preprint gr-qc/0605123
[10] Livine E R and Terno D R 2006 Reconstructing quantum geometry from quantum information: area renormalisation, coarse-graining and entanglement on spin networks Preprint gr-qc/0603008
Grasping rules and semiclassical limit of the geometry in the Ponzano–Regge model

[18] Ashtekar A and Lewandowski J 1995 Projective techniques and functional integration for gauge theories *J. Math. Phys.* **36** 2170 (Preprint gr-qc/9411046)

[19] Baez J C 1996 Spin network states in gauge theory *Adv. Math.* **117** 253 (Preprint gr-qc/9411007)

[20] Heron of Alexandria, Metrica, (I cen. AD). See Sir Thomas Heath 1981 *A History of Greek Mathematics* vol II (New York: Dover) p 320

[21] Varshalovich D A, Moskalev A N and Khersonsky V K 1988 *Quantum Theory of Angular Momentum* (Singapore: World Scientific)

[22] Bar-Natan D 1995 On the Vassiliev Knot Invariants *Topology* **34** 423

[23] De Pietri R 1997 On the relation between the connection and the loop representation of quantum gravity *Class. Quantum Grav.* **14** 53 (Preprint gr-qc/9605064)