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The Topology of the Spectrum for Gelfand Pairs on Lie Groups.

Fabio Ferrari Ruffino

Summary. – Given a Gelfand pair of Lie groups, we identify the spectrum with a suitable subset of $\mathbb{C}^n$ and we prove the equivalence between Gelfand topology and euclidean topology.

1. – Introduction.

Let $(G, K)$ be a Gelfand pair with $G$ a connected Lie group and $K$ a compact subgroup. The Gelfand spectrum $\Sigma$ of $L^1(G)$, the commutative convolution algebra of bi-$K$-invariant integrable functions on $G$, is identified, as a set, with the set of bounded spherical functions. The Gelfand topology on $\Sigma$ is, by definition, the weak-$*$ topology, which coincides with the topology of uniform convergence on compact sets.

Since $G$ is a connected Lie group, the spherical functions on $G$ are characterized as the joint eigenfunctions of the algebra $U(G/K)$ of differential operators on $G/K$ invariant by left $G$-translation. Being this algebra finitely generated, we identify $\Sigma$ with a subset of $\mathbb{C}^s$ assigning to each function the $s$-tuple of its eigenvalues with respect to a finite set of generators. Hence one can define on $\Sigma$ also the Euclidean topology induced from $\mathbb{C}^s$. In this article we prove that the two topologies coincide.

2. – Gelfand pairs and spherical functions.

We briefly recall the general theory of Gelfand pairs, that can be found in [4]. Let $(G, \cdot)$ be a locally compact group, with a fixed left Haar measure $dx$. Let $K \leq G$ be a compact subgroup with normalized Haar measure $dk$. 

Sunto. – Data una coppia di Gelfand di gruppi di Lie, identifichiamo lo spettro con un opportuno sottoinsieme di $\mathbb{C}^n$ e dimostriamo l’equivalenza tra la topologia di Gelfand e la topologia euclidea.
Definition 1. – A function \(f : G \to \mathbb{C}\) is said to be bi-invariant under \(K\) if it is constant on double cosets of \(K\), i.e., if:
\[
f(k_1 x k_2) = f(x) \quad \forall k_1, k_2 \in K, \forall x \in G
\]

Let \(C_c(G)^\vee\) (resp. \(L^1(G)^\vee\)) be the set of continuous compactly-supported (resp. \(L^1\)) functions \(f : G \to \mathbb{C}\) that are bi-invariant under \(K\). It is easy to verify that it is a subalgebra of \(C_c(G)\) (resp. of \(L^1(G)\)) with respect to the convolution in \(G\).

Definition 2. – \((G, K)\) is said to be a Gelfand pair if \(C_c(G)^\vee\) is a commutative algebra.

One can easily prove that \(C_c(G)^\vee\) is dense in \(L^1(G)^\vee\), therefore \(C_c(G)^\vee\) is a commutative algebra if and only if \(L^1(G)^\vee\) is.

Given a function \(\varphi \in C(G)\) (not necessarily compactly-supported), we consider the linear functional:
\[
\chi_\varphi : C_c(G) \to \mathbb{C}
\]
\[
\chi_\varphi(f) = \int_G f(x) \varphi(x^{-1})dx
\]

Definition 3. – A function \(\varphi \in C(G), \varphi \neq 0\), is said to be spherical if it is bi-invariant under \(K\) and \(\chi_\varphi\) is a character of \(C_c(G)^\vee\), i.e.:
\[
\chi_\varphi(f * g) = \chi_\varphi(f) \cdot \chi_\varphi(g) \quad \forall f, g \in C_c(G)^\vee
\]

One proves that \(\varphi\) is spherical if and only if:
\[
\int_K \varphi(kxy)dk = \varphi(x)\varphi(y) \quad \forall x, y \in G
\]
(see [4] prop. I.3 p. 319). In particular, this implies that \(\varphi(1_G) = 1\).

Theorem 1. – The dual space of \(L^1(G)^\vee\) is \(L^\infty(G)^\vee\). In fact, every continuous functional on \(L^1(G)^\vee\) has the form:
\[
\chi_\varphi : f \to \int_G f(x) \varphi(x^{-1})dx
\]
with \(\varphi \in L^\infty(G)^\vee\) unique and such that \(\|\chi_\varphi\| = \|\varphi\|_{\infty}\).

Proof. – If \(\varphi \in L^\infty(G)^\vee\), \(\chi_\varphi\) is a continuous functional on \(L^1(G)\), hence on its closed subspace \(L^1(G)^\vee\), and \(\|\chi_\varphi\| \leq \|\varphi\|_{\infty}\).
For the converse, let \( \chi \) be a continuous functional on \( L^1(G) \). By the Hahn-Banach theorem, we can extend \( \chi \) to all of \( L^1(G) \) without altering its norm. So, being \( L^\infty(G) \) the dual of \( L^1(G) \), we have:

\[
\chi(f) = \int_G f(x)\psi(x^{-1})\,dx
\]

for some \( \psi \in L^\infty(G) \), with \( \|\chi\| = \|\psi\|_\infty \). Let \( \varphi = \psi^k \) be the radialization of \( \psi \), i.e.:

\[
\psi^k(x) = \int_{K \times K} \psi(k_1xk_2)\,dk_1dk_2
\]

It is easy to see that \( \varphi \in L^\infty(G) \), and \( \|\varphi\|_\infty \leq \|\psi\|_\infty = \|\chi\| \). Moreover, \( \psi \) and \( \varphi \) induce the same functional on \( L^1(G) \). In fact, if \( f \in L^1(G) \), we have:

\[
\int_G f(x)\varphi(x^{-1})\,dx = \int_G f(x)\int_{K \times K} \psi(k_1x^{-1}k_2)\,dk_1dk_2\,dx
\]

\[
= \int_{K \times K} \int_G f(x)\psi(k_1x^{-1}k_2)\,dx\,dk_1dk_2
\]

\[
= \int_{K \times K} \int_G f(k_2\varphi k_1)\psi(x^{-1})\,dx\,dk_1dk_2
\]

\[
= \int_G f(x)\psi(x^{-1})\,dx
\]

Hence every continuous functional on \( L^1(G) \) has the form \( \chi_\varphi \) for \( \varphi \in L^\infty(G) \), with \( \|\varphi\|_\infty \leq \|\chi_\varphi\| \). Since we have also proved that \( \|\chi_\varphi\| \leq \|\varphi\|_\infty \), we can conclude that \( \|\chi_\varphi\| = \|\varphi\|_\infty \). We now prove that \( \varphi \) is unique: by linearity of \( \chi_\varphi \) in \( \varphi \), we have to prove that \( \chi_\varphi = 0 \Rightarrow \varphi = 0 \). But \( \chi_\varphi = 0 \Leftrightarrow \|\chi_\varphi\| = 0 \Leftrightarrow \|\varphi\|_\infty = 0 \Leftrightarrow \varphi = 0 \). \( \square \)

**Theorem 2.** (See [4] Th. I.5 p. 320 or [7] Lemma 3.2 p. 408). – An element \( \varphi \) of \( L^\infty(G) \) defines a character of \( L^1(G) \) if and only if \( \varphi \) is a bounded spherical function. \( \square \)

**Corollary 3.** – A bounded spherical function has \( \infty \)-norm equal to 1.

**Proof.** – If \( \varphi \in L^\infty(G) \) is spherical, it determines a character \( \chi_\varphi \) of \( L^1(G) \), which is a commutative Banach algebra, with \( \|\chi_\varphi\| = \|\varphi\|_\infty \). Hence, \( \|\varphi\|_\infty = 1 \). \( \square \)

Let \( \Sigma \) be the spectrum of \( L^1(G) \), i.e., for the previous theorem, the set of bounded spherical functions. We define the *Fourier spherical transform*...
(see [4] p. 333):

\[ \hat{f} : \Sigma \rightarrow \mathbb{C} \]

\[ \hat{f}(\varphi) = \chi_{\varphi}(f) = \int_{G} f(x)\varphi(x^{-1}) \, dx \]

We can introduce on \( \Sigma \) the \textit{Gelfand topology}, i.e., the weak-* topology.

**Theorem 4.** – The Gelfand topology on \( \Sigma \) is equal to the topology of uniform convergence on compact sets (or locally uniform convergence). \( \Box \)

(The proof is similar to the one given in [8] p. 10-11.)

3. – The case of Lie groups.

If \( G \) and \( K \) are Lie groups, we can characterize Gelfand pairs and spherical functions by a differential point of view. Given a differential operator \( D \) on a manifold \( M \) (see [7] p. 239) and a diffeomorphism \( \phi \) of \( M \), we say that \( D \) is \( \phi \)-invariant if \( D(f \circ \phi) = Df \circ \phi \) \( \forall f \in C_c^\infty(M) \).

On a Lie group \( G \) we have a special family of diffeomorphisms, the left translations by elements of \( G \): \( \phi_g(x) = gx \). Remembering that a Lie group always admits an \textit{analytic} structure compatible with the operations, we can construct a unique analytic structure also on the space of left cosets \( G/K \) (with the quotient topology) such that the \( G \)-action on \( G/K \):

\[ L : G \times G/K \rightarrow G/K \]

\[ L(x, gK) = xgK \]

is analytic (see [6] p. 113).

Let \( C^\infty_K(G) \) be the set of functions in \( C^\infty(G) \) such that \( f(xk) = f(x) \)

\( \forall k \in K, g \in G \). We have an isomorphism of algebras between \( C^\infty(G/K) \) and \( C^\infty_K(G) \) given by the projection \( \pi \).

We consider three algebras of differential operators (see [7] p. 274-287 and [6] p. 389-398):

\[ \mathbb{D}(G) = \{ \text{diff. op. on } G \text{ invariant by left } G\text{-translation} \} \]

\[ \mathbb{D}_K(G) = \{ \text{diff. op. in } \mathbb{D}(G) \text{ invariant also by right } K\text{-translation} \} \]

\[ \mathbb{D}(G/K) = \{ \text{diff. op. on } G/K \text{ invariant by left } G\text{-translation} \} \]
We also consider the algebra:

\[ D^K_K(G) = D_K(G)/A, \quad A = \{ D \in D_K(G) : Df = 0 \, \forall f \in C^\infty_K(G) \} \]

We can think of \( D^K_K(G) \) as the algebra of differential operators in \( D_K(G) \) acting only on \( C^\infty_K(G) \); in fact, if \( D \) and \( E \) coincide on \( C^\infty_K(G) \), we have \( D - E \in A \).

One can prove that \( D^K_K(G) \cong D(G/K) \), with the isomorphism given by the projection \( \pi \) (see [6] lemma 2.2 p. 390).

**Theorem 5.** (See [7] p. 485 ex. 13). – Let \( G \) be a connected Lie group and let \( K \) be a compact subgroup. Then, \( (G, K) \) is a Gelfand pair if and only if \( D^K_K(G) \) is a commutative algebra. \( \square \)

**Theorem 6.** (See [7] prop. 2.2 p. 400). – Let \( (G, K) \) be a Gelfand pair of Lie groups and \( f \in C(G) \). Then, \( f \) is spherical if and only if:

- \( f \in C^\infty(G) \);  
- \( f(1_G) = 1 \);  
- \( f \) is an eigenfunction of all the operators in \( D^K_K(G) \):  
  \[ Df = \lambda_D f \quad \forall D \in D^K_K(G) \] \( \square \)

**Remark.** – The proof of the theorem shows that a spherical function is necessarily analytic.

It can be proved that, being \( K \) compact, \( D^K_K(G) \) is a finitely-generated algebra (see [6] cor. 2.8 p. 395 and th. 5.6 p. 421). Let \( D_1, \ldots, D_s \) be generators. Of course, \( \phi \) is an eigenfunction of all the operators in \( D^K_K(G) \) if and only if it is an eigenfunction of the generators. In this way, we can associate to each spherical function the \( s \)-uple of eigenvalues \((\lambda_1, \ldots, \lambda_s)\) with respect to the generators. We can also prove that this association is injective, because the analyticity implies that two spherical functions having the same eigenvalues \((\lambda_1, \ldots, \lambda_s)\) must coincide (see [7] cor. 2.3 p. 402).

4. – The topology of the spectrum.

In this way, we identify a spherical function, and in particular a bounded one, with a point in \( C^\circ \). So we identify the spectrum \( \Sigma \) of \( L^1(G)^\circ \) with a subset \( A \subseteq C^\circ \). Now, on \( \Sigma \) we have the Gelfand topology, and on \( A \) we have the induced euclidean topology. It is natural to ask if these two topologies coincide.

**Lemma 7** (See [3] p. 218). – Let \( X \) be a topological space such that every point has a countable fundamental system of neighborhoods. Then a subset of \( X \) is closed if and only if it is sequentially closed.
In particular, two such topologies coincide if and only if they induce the same notion of convergence on sequences. □

**Lemma 8.** – $L^1(G)^\circ$ is separable.

**Proof.** – We have only to prove that $L^1(G)$ is separable, because a subset of a separable space is separable (see [1] prop. III.22 p. 47). To see this, we choose a denumerable base of $G$ (which exists by definition of differential manifold) and we consider the subspace generated by the characteristic functions of these base-sets. Then we can argue as for $L^1(\mathbb{R}^n)$ (see [1] Th. IV.13 p. 62). □

**Remark 9.** – Being $K$ compact, one can construct on $G/K$ a riemannian metric invariant by the left action of $G$: hence, Laplace-Beltrami operator $\Delta$ with respect to this metric is invariant by left $G$-translations (see [6] Prop. 2.1 p. 387), i.e., $\Delta \in \mathcal{D}(G/K)$. This implies that, if $\varphi$ is a spherical function, $\pi : G \to G/K$ is the projection and $\varphi^\pi = \varphi \circ \pi^{-1}$, then $\varphi^\pi$ is an eigenfunction of $\Delta$, which is an elliptic operator.

**Theorem 10.** – The induced euclidean topology on $A$ and the Gelfand topology on $\Sigma$ coincide under the bijection $\varphi \in \Sigma \mapsto (\lambda_1, ..., \lambda_s) \in A$.

**Proof.** – Of course $A$ is a metric space, so every point of $A$ has a denumerable fundamental system of neighborhoods. By corollary 3, $\Sigma \subseteq B\left(\left(L^1(G)^\circ\right)\right)$ (where $B$ is the unit ball). Being $L^1(G)^\circ$ separable for lemma 8, the weak-* topology is metrizable on the unit ball (see [1] Th. III.25 p. 48), in particular the Gelfand topology on $\Sigma$ is metrizable. So, applying lemma 7, we have to prove that the two topologies we are considering induce the same notion of convergence.

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of spherical functions, and let $\mathcal{D}_K^G = \langle D_1, \ldots, D_s \rangle$. Let, $\forall i \in \{1, \ldots, s\}$:

\[
D_i \varphi_n = \lambda_{i,n} \varphi_n
\]

\[
D_i \varphi = \lambda_i \varphi
\]

We have to prove that if $\varphi_n \to \varphi$ locally uniformly, then $\lambda_{i,n} \to \lambda_i \ \forall i \in \{1, \ldots, s\}$. But $D_i \varphi_n(1_G) = \lambda_{i,n} \varphi_n(1_G) = \lambda_{i,n}$ and similarly $D_i \varphi(1_G) = \lambda_i$. So, being $D_1, \ldots, D_s$ generators, we have to prove that:

$$\varphi_n \to \varphi \text{ loc. unif. } \Rightarrow D \varphi_n(1_G) \to D \varphi(1_G), \quad \forall D \in \mathcal{D}_K^G$$

If $f$ is a spherical function, it is continuous and non-zero by hypothesis, so it is easy to construct a function $\rho \in C^\infty_c(G)$ such that:

$$\int_G f(x) \rho(x) dx \neq 0$$
(We have to choose a point \(x_0 \in G\) such that \(f(x_0) \neq 0\), choose by continuity a neighborhood \(U(x_0)\) such that \(\mathbb{R}f\) or \(3f\) has constant sign on \(U\), and construct \(\rho \geq 0\) such that \(\text{supp}(\rho) \subseteq U\) and \(\rho(x_0) = 1\). So we have, by the formula (1):

\[
f(x) \int_G f(y) \rho(y) \, dy = \int_G \rho(y) \left( \int_K f(xk) \, dk \right) \, dy
= \int_G \int_K \rho(y) f(xk) \, dy \, dk = \int_K \rho(k^{-1}x^{-1}y) f(y) \, dy \, dk
= \int_K \left( \int_G \rho(k^{-1}x^{-1}y) \, dk \right) f(y) \, dy
\]

Concretely, the last integral in \(dy\) is not extended to all of \(G\): indeed, the domain of integration is the set of \(y\) such that \(\exists k \in K : k^{-1}x^{-1}y \in \text{supp}(\rho)\), i.e., \(x \cdot K \cdot \text{supp}(\rho)\), which is compact because the product in \(G\) is continuous.

So, if we restrict \(x\) to an open neighborhood \(V\) of \(1_G\) with \(V\) compact, we can assume that, for all such \(x\), the domain of integration is \(V \cdot K \cdot \text{supp}(\rho)\). We put:

\[
C = V \cdot K \cdot \text{supp}(\rho)
\]

\[
A = \frac{1}{\int_G f(y) \rho(y) \, dy}
\]

\[
\psi(x, y) = \int_K \rho(k^{-1}x^{-1}y) \, dk
\]

\(C\) is compact, \(A \neq 0\) and, being \(K\) compact, \(\psi(x, y) \in C^\infty(G \times G)\). So, in particular, \(\psi(\cdot, y) \in C^\infty(V) \forall y \in C\). We have, for \(D \in \mathcal{D}(G)\):

\[
f|_V(x) = A \int_C \psi(x, y) f(y) \, dy
\]

\[
Df|_V(x) = A \int_C [D^{(x)} \psi(x, y)] f(y) \, dy
\]

\[
Df|_V(1_G) = A \int_C \eta_D(y) f(y) \, dy
\]

with \(\eta_D(y) = (D^{(x)} \psi(x, y)|_{x=1_G})\). But \(\eta_D(y)\) is a continuous function, indeed \(\psi \in C^\infty(G \times G)\), so \(D^{(x)} \psi(x, y) \in C^\infty(G \times G)\), and, composing with the immersion \(y \to (1_G, y)\) we still obtain a \(C^\infty(G)\) function. So, the restriction of \(\eta_D\) to \(C\) is still continuous.
So, applying the previous formula to \( \varphi_n \) and \( \varphi \), we obtain:

\[
D\varphi_n(1_G) = A_n \int_{\mathcal{C}} \eta_{n,D}(y) \varphi_n(y) \, dy
\]

\[
D\varphi(1_G) = A \int_{\mathcal{C}} \eta_D(y) \varphi(y) \, dy
\]

But, by construction, we can suppose \( \eta_{n,D} = \eta_D \): in fact, we can begin the construction with \( \rho_n = \rho \). For this, being \( \varphi_n(1_G) = \varphi(1_G) = 1 \), we choose a neighborhood \( U(1_G) \) with compact closure such that \( \Re \varphi_U \geq \delta > 0 \). Then, being by hypothesis \( \varphi_n|_U \to \varphi|_U \) uniformly, we can suppose that \( \Re \varphi_n|_U > 0 \) \( \forall n \in \mathbb{N} \). So we take \( \rho_n = \rho \) such that \( \rho(1_G) = 1 \) and \( \rho = 0 \) outside \( U \). From this we deduce that \( \eta_{n,D} = \eta_D \) and \( C_n = C \).

If \( \varphi_n \to \varphi \) uniformly on compact sets, in particular uniformly on \( C \), being \( \eta_D \) continuous and hence bounded on \( C \), we have that \( \eta_D \cdot \varphi_n \to \eta_D \cdot \varphi \) uniformly on \( C \). So \( \int \eta_D(y) \varphi_n(y) \, dy \to \int \eta_D(y) \varphi(y) \, dy \). Moreover, \( A_n \to A \), in fact:

\[
\left| \int_G \varphi_n(x) \rho(x) \, dx - \int_G \varphi(x) \rho(x) \, dx \right| \leq \int \left| \varphi_n(x) - \varphi(x) \right| \rho(x) \, dx
\]

\[
= \int_{\text{supp}(\rho)} \left| \varphi_n(x) - \varphi(x) \right| \rho(x) \, dx \leq K \int_{\text{supp}(\rho)} \left| \varphi_n(x) - \varphi(x) \right| \, dx \to 0
\]

So \( D\varphi_n(1_G) \to D\varphi(1_G) \).

For the converse, we know that \( \Sigma \subseteq B \left( (L^1(G))^\prime \right) \), which is compact for the weak*-topology by the Alaoglu-Banach theorem (see [1] Th. III.15 p. 42). Being the Gelfand topology metrizable on \( B \left( (L^1(G))^\prime \right) \), compactness is equivalent to compactness by sequences (see [2] prop. 4.4 p. 188). We indicate with \( \xrightarrow{\ast} \) the convergence with respect to the euclidean topology on \( A \). So let us suppose that \( \{\varphi_n\}_{n \in \mathbb{N}} \subseteq \Sigma \) is such that \( \varphi_n \xrightarrow{\ast} \varphi \). By compactness, we can extract a convergent subsequence (with respect to the Gelfand topology) \( \varphi_{n_k} \to \tilde{\varphi} \), with \( \tilde{\varphi} \in B((L^1(G))\prime) \). But necessarily \( \tilde{\varphi} \in \Sigma \cup \{0\} \): indeed, \( \chi_{\varphi_{n_k}}(f \ast g) \to \chi_{\tilde{\varphi}}(f \ast g) \) by definition on Gelfand topology, but \( \chi_{\varphi_{n_k}}(f \ast g) = \chi_{\varphi_{n_k}}(f) \cdot \chi_{\varphi_{n_k}}(g) \to \chi_{\tilde{\varphi}}(f) \cdot \chi_{\tilde{\varphi}}(g) \).

By remark 9, the functions \( \varphi^\prime_n \) are solutions of the equation:

\[
(A - \lambda_{A,n})\varphi^\prime_n = 0
\]

with \( A \) elliptic. Morover, \( \lambda_{A,n} \to \lambda_A \) with \( \lambda_A \) defined by \( A\varphi^\prime = \lambda_A\varphi^\prime \). Choosing a local chart \((U, \zeta)\) in the origin of \( G/K \), we have, by [5] Th. 8.32 p. 210, with \( \Omega = U, \Omega' \subseteq \Omega \subseteq \Omega, f = g = 0, a = 0 \) and denoting by \( \| \cdot \|_s \) the Sobolev norm of order \( s \):

\[
\| (\varphi^\prime_n \circ \zeta^{-1}) |_{\zeta(\Omega')} \|_1 \leq C(\| (\varphi^\prime_n \circ \zeta^{-1}) |_{\zeta(\Omega)} \|_1) = C
\]

and one can easily verify that \( C \) is independent by \( n \) because \( \lambda_{A,n} \to \lambda_A \), hence the
sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ is bounded. This implies that the functions $\varphi_{n,1}^{i,\pi} \mid_{\Omega'}$, and in particular the functions $\varphi_{n,1}^{i,\pi} \mid_{\Omega'}$, are equicontinuous and, by Arzela-Ascoli theorem (see [9] Th. 11.28 p. 245), there is a subsequence $\varphi_{n,k}^{i,\pi} \mid_{\Omega'} \rightarrow \psi$ locally uniformly on $G'/K$. It is easy to deduce from this that $\varphi_{n,k}^{i,\pi} \mid_{\pi^{-1}(\Omega')} \rightarrow \psi$ locally uniformly on $G$. In particular, $\psi(1_G) = 1$ because $\varphi_{n,k}^{i,\pi}(1_G) = 1 \forall k \in \mathbb{N}$. We can choose $\Omega'' \subseteq \Omega' \subseteq \pi^{-1}(\Omega')$ neighborhood of $1_G$ such that $\Re \psi \mid_{\Omega''} \geq \delta > 0$: in particular, $\varphi_{n,k}^{i,\pi} \mid_{\Omega''} \rightarrow \psi \mid_{\Omega''}$ uniformly. Then, if $\chi_{\Omega''}$ is the characteristic function of $\Omega''$, we consider the function:

$$\zeta(x) = (\chi_{\Omega''})^{i}(x^{-1})$$

with $(\chi_{\Omega''})^{i}$ defined according to formula (2) pag. 3. We have:

$$\zeta(\varphi_{n,k}^{i,\pi}) = \int_{G} \zeta(x) \varphi_{n,k}^{i,\pi}(x^{-1})dx = \int_{G} (\chi_{\Omega''})^{i}(x^{-1}) \varphi_{n,k}^{i,\pi}(x^{-1})dx
$$

$$= \int_{G} (\chi_{\Omega''})^{i}(x) \varphi_{n,k}^{i,\pi}(x)dx = \int_{G} \chi_{\Omega''}(x) \varphi_{n,k}^{i,\pi}(x)dx
$$

$$= \int_{G} \chi_{\Omega''}(x) \varphi_{n,k}^{i,\pi}(x)dx = \int_{\Omega''} \varphi_{n,k}^{i,\pi}(x)dx
$$

$$\Re \left[ \zeta(\varphi_{n,k}^{i,\pi}) \right] \rightarrow \int_{\Omega''} \Re \psi(x)dx \geq \delta |\Omega''| > 0$$

Hence, by definition of Gelfand topology, it is not possible that $\varphi_{n,k} \rightarrow 0$, so that $\varphi_{n,k} \rightarrow \tilde{\varphi} \in \Sigma$.

But, for the first part of the theorem, it must be $\varphi_{n,k} \rightarrow \tilde{\varphi}$, so $\tilde{\varphi} = \varphi$. Hence, we have proved that for every sequence $\varphi_{n} \rightarrow \varphi$, we can find a subsequence $\varphi_{n,k} \rightarrow \varphi$ uniformly on compact sets. Let us suppose that $\varphi_{n} \rightarrow \varphi$: then, we can find a compact set $C \subseteq G$, $\varepsilon > 0$ and a subsequence $\varphi_{n,k}$ such that $\sup_{x \in C} |\varphi_{n,k}(x) - \varphi(x)| > \varepsilon$ $\forall k \in \mathbb{N}$. But, of course, $\varphi_{n,k} \rightarrow \varphi$, so, applying the previous argument, we can find a sub-subsequence $\varphi_{n,j} \rightarrow \varphi$ uniformly on compact sets, in particular uniformly on $C$: a contradiction. \hfill $\Box$

From the proof of the previous theorem one can also conclude that:

**Corollary 11.** $- A$ is closed in $C^s$.

**Proof.** Let $\{z_n\}_{n \in \mathbb{N}} = \{\lambda_{1,n}, \ldots, \lambda_{s,n}\}$ be a sequence in $A$, with $z_n \rightarrow z = (\lambda_1, \ldots, \lambda_s) \in C^s$. Let $\varphi_n \in \Sigma$ be the spherical function associated to $z_n$. We have that $\lambda_{A,n} = P(\lambda_{1,n}, \ldots, \lambda_{s,n})$ with $P$ polynomial, hence $\lambda_{A,n} \rightarrow \lambda_A = P(\lambda_1, \ldots, \lambda_s)$: the sequence $\{\lambda_{A,n}\}$ is then bounded, hence, arguing as in the proof
of the theorem, we can extract a subsequence \( \varphi_{n_k} \to \tilde{\varphi} \in \Sigma \). But necessarily \( \varphi_{n_k} \rightharpoonup \tilde{\varphi} \), hence \((\lambda_1, \ldots, \lambda_n)\) is the point of \( A \) associated to \( \tilde{\varphi} \), so \( z \in A \). \( \square \)

**Corollary 12.** – If \( \varphi_n \to \varphi \) in \( \Sigma \) then \( D\varphi_n \to D\varphi \) uniformly on compact sets for every differential operator \( D \).

**Proof.** – For every \( x_0 \in G \), we have, for \( V \) neighborhood of \( x_0 \) with \( \overline{V} \) compact, \( C \) compact and \( A_n \neq 0 \):

\[
\varphi_n|_V(x) = A_n \int_C \psi(x,y)\varphi_n(y) \, dy
\]

\[
D\varphi_n|_V(x) = A_n \int_C \left[D(x)\psi(x,y)\right]\varphi_n(y) \, dy
\]

\[
D\varphi_n|_V(x) = A_n \int_C \eta_D(x,y)\varphi_n(y) \, dy
\]

with \( \eta \) continuous. Similarly:

\[
D\varphi|_V(x) = A \int_C \eta_D(x,y)\varphi(y) \, dy
\]

By continuity, \( \eta_D \) is bounded on \( \overline{V} \times C \), so in particular on \( V \times C \), so, being \( \varphi_n|_C \to \varphi|_C \) uniformly, we have that \( \int_C \eta_D(x,y)\varphi_n(y) \, dy \to \int_C \eta_D(x,y)\varphi(y) \, dy \) uniformly on \( x \). Moreover, \( A_n \to A \), so \( D\varphi_n|_V \to D\varphi|_V \) uniformly. \( \square \)

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