On an extended Majda–Biello system

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Abstract

In this paper, we begin with extended Majda–Biello system (BSAB equations):
\[
\begin{align*}
0 &= A_t - DA_3 + \mu A_1 + \Gamma_S B_1^S + \Gamma_A B_1^A + (AB^S)_x \\
0 &= B_t^S - B_3^S + \Gamma_S A_1 + \lambda B_1^S + \sigma B_1^A + AA_1 \\
0 &= B_t^A - B_3^A + \Gamma_A A_1 + \sigma B_1^S - \lambda B_1^A
\end{align*}
\]

We conclude global well-posedness in $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by Brougain’s method and the stability of solitary wave solutions by putting it in a framework of generalised KdV type system with three components, where Hamiltonian structure plays an important role. Both of them are bases for numerical tests.

Last but not least, we explore the effect of interaction of two solitary waves in Majda–Biello system in a novel way:

While fixing initial data for one soliton $U$, we point out the effect on $U$ decays, to some extent and in certain range, in a polynomial way.

Since effect of interaction of two solitary waves are practically interesting, such kind of analysis, as we have explained, is likely be fundamental for generalised KdV type systems.

Keywords: Global well–posedness, Stability, Asymptotic behavior

1 Introduction

Nonlinearly coupled KdV type systems with Hamiltonian structure, sometimes derive from 2D Euler incompressible flows, say, Majda–Biello systems \(^{(1,1)}\) derived from \(^{3}\), Gear–Grimshaw systems \(^{15}\). By improving Rossby wave train solutions from
\[
\psi = -B(x - c_{BT} t) \sin(ly), \quad c_{BT} = \frac{1}{l^2}
\]

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\[ \psi = -B^S(x - c_{HT}t) \sin(l y) - B^A(x - c_{HT}t) \cos(l y), \quad c_{HT} = -\frac{1}{l^2}, \]

[7] presents BSAB system with three modes:

\[
\begin{cases}
0 = A_t - DA_3 + \mu A_1 + \Gamma_S B^S_1 + \Gamma_A B^A_1 + (AB^S)_x \\
0 = B^S_t - B^S_3 + \Gamma_S A_1 + \lambda B^A_1 + \sigma B^A_1 + A A_1 \\
0 = B^A_t - B^A_3 + \Gamma_A A_1 + \sigma B^S_1 - \lambda B^A_1
\end{cases}
\tag{1.1}
\]

an extension of Majda–Biello system with two modes:

\[
\begin{cases}
\triangle_1 = A_t - DA_3 + (AB)_x \equiv 0 \\
\triangle_2 = B_t - B_3 + AA_1 \equiv 0
\end{cases}
\]

On the other hand, putting physical backgrounds aside, there are several ways to study nonlinear nonintegrable PDE systems, especially one of KdV type:

- Sobolev space methods on well-posedness and ill-posedness of Cauchy problems with initial data in different functional spaces. Some methods, for instance, Bourgain’s method and its variant I–method [21, 20, 9, 18] originating in [12, 13]; and various asymptotic behaviors, say, [8] of variation of Gear–Grimshaw system has been studied;

- numerical stimulations [6, 19] on Majda–Biello system, [14, 15] for Gear–Grimshaw system, and some comparision of numerical methods [29];

- Lie symmetry analysis and conservation laws, or even integrability – more geometric point of view, for Majda–Biello system [27], for Gear–Grimshaw system [23], and others , say, [1];

- …

Different from systems with two modes, ones with more modes are less intensively studied. While it being rarely appeared, BSAB system does naturally occur and further physical description and analysis of this nonlinear Hamiltonian fluid dynamical system convince us that it virtually makes difference from one with two mode. In the future, models of KdV type with more modes are likely to emerge.

In this paper, we first propose global well-posedness of Cauchy problem of (1.1) with initial data in \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \), which is practically enough for situation of our later numerical tests. We also make sure that Bourgain’s
method works well for the system with three components as well as Hamiltonian, since, as a matter of fact, we put the routine for generalised system (2.2), which may include more systems of the kind in the future (just like the situation that ones with two modes share things in common).

Next, the theory of stability in [2] is successfully applied with sufficient conditions proposed for the generalised system (2.2), and can be regarded as a theoretical supplement for previous numerical tests, say, "their stability under collisions are also described" in [7]. The Hamiltonian structure also plays an important role.

Last but not least, we do numerical tests on interaction of solitary waves but of Majda–Biello system and only one collision is allowed. Such consideration is due to following reasons:

• practically speaking, as [7] has pointed out, $B^A$ with a linear dispersion equation in (1.1), shall assume zero initial data and only nonlinearly affects $B^A$ itself;

• BSAB system (1.1) with additional $B^A$ can be regarded as regular perturbation of Majda Biello system (4.1), where major phenomena can be observed by the latter;

• as [6] points out, Majda–Biello system (4.1) has a neat KdV form, with respect to $U,V$:

\[
\begin{align*}
U_t - \frac{(1+D)}{2} U_{xxx} + UU_x &= \frac{(1-D)}{2} V_{xxx} + \left(\frac{UV}{2}\right)_x \\
V_t - \frac{(1+D)}{2} V_{xxx} + VV_x &= \frac{(1-D)}{2} U_{xxx} + \left(\frac{UV}{2}\right)_x
\end{align*}
\]

which is superior to BSAB equation (1.1) and Gear–Grinshaw’s;

• in atmospheric background, radiation should not disturbed solitary structure after one interaction – were several interaction happens, this could not be realised (the author has consulted the author of [6], that periodic appearance and collision in [6] is more or less for the sake of numerical pseudo–spectral methods used).

After briefly recapitulate and explain previous numerical results in [6] involved, we explore the asymptotic behavior of the effect of interaction of two solitary waves on $U$, whose initial fixed – to author’s knowledge, such exploration is novel and later on, we show to be successful; the loss of energy for solitary structure due to interaction, is also an important topic for models themselves.
Parameters we endow \(U, V\) are, separately, \(\lambda_U, \lambda_V\) (here \(\lambda_U = 1\) is fixed), in order to follow notation of initial data in [6]:

\[
U(x, t) = S(x - \sigma t), V(x, t) = 0 \text{ or } U(x, t) = 0, V(x, T) = S(x - \sigma T)
\]

where

\[
S(x) \equiv -\frac{12}{\lambda} \cosh^{-2} \left( \frac{x - x_0}{\lambda \sqrt{(1 + D)/2}} \right), \sigma = -\frac{4}{\lambda^2}
\]

We propose that effect of interaction decays in the manner \(O(\lambda_U^I), I \simeq \text{text as } \lambda_U \to 0^+\) by numerically study of solitary structure of \(U\) after the collision, on three appropriate measurements, namely, \(T_1^U \equiv \int_R^L U \, dx, T_2^U \equiv \int_R^L U^2 \, dx, T_3^U \equiv \int_R^L U^2 + \frac{U^3}{3} \, dx\) i.e. mass, momentum, energy which has counterparts of distinguished KdV equation, which we are to elaborate.

2 Global well–posedness with initial data in \(L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\)

The global well–posedness and ill–posedness of Cauchy problem of nonintegrable KdV type systems with initial data in various functional spaces, have received intensive studies: about Gear–Grimshaw equations see examples [9, 4]; about Majda–Biello equations see examples [22, 21, 20].

In this section, we propose global well–posedness of Cauchy problem of (1.1) with initial data in \(L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\), which is practically enough for situation of our later numerical tests. We also make sure that Brougain’s method works well for the system with three components as well as Hamiltonian, since, as a matter of fact, we put the routine for generalised system (2.2), which may include more systems of the kind in the future (just like the situation that ones with two modes share things in common).

**Notation 1.** For the sake of simplicity, we follow notations:

1. \(\psi_j \equiv D^j_x \psi, \psi \in \{u, v, w, A, B, \ldots\}, j = 0, 1, 2 \ldots \text{ j times total derivative with respect to } x \) (generally \(\psi \equiv \psi_0\));

2. \(r \equiv \sum_{j=0}^{l_0} \left(1 + \frac{j}{2}\right) \alpha_j + \sum_{j=0}^{l_0} \left(1 + \frac{j}{2}\right) \beta_j, \text{ rank of } \psi^{\alpha_0} \psi_1^{\alpha_1} \ldots \psi^{\alpha_k} \xi_1^{\beta_1} \ldots \xi_\ell^{\beta_\ell};\)
3. $T^r$, conserved density with rank $r$, and $X^r$ (or $-X^r$), associated flux. Then the conservation law (CLs) in the canonical form is

$$T^r_t + X^r_x = 0 \quad (2.1)$$

By contrast, $a_j, b_j, c_j, j = 0, 1, \ldots$ are coefficients rather than derivative of functions.

System (1.1) can be generalized as KdV type system

$$\triangle_j = 0, j = 1, 2, 3 \quad (2.2)$$

where

$$
\begin{align*}
\begin{cases}
a_0 \Delta_1 &= a_0 u_t + a_1 u_3 + \beta w_3 + \gamma v_3 + [u(a_2 v + a_3 w)]_x + h(w)_x \\
&+ c_2 w w_1 + b_1 v v_3 + a_4 u u_1 + (a_5 u_1 + b_6 w_1 + c_6 v_1) \\
b_0 \Delta_2 &= b_0 v_t + b_1 v_3 + \gamma u_3 + \alpha w_3 + [v(b_2 u + b_3 v)]_x + h(w)_x \\
&+ c_2 u u_1 + c_3 w w_1 + b_4 v v_1 + (b_5 v_1 + c_6 u_1 + a_6 w_1) \\
c_0 \Delta_3 &= c_0 w_t + c_3 u_3 + \alpha v_3 + \beta u_3 + [w(c_2 u + c_3 v)]_x + h(w)_x \\
&+ b_2 v v_1 + a_3 u u_1 + c_4 w w_1 + (c_5 v_1 + a_6 u_1 + b_6 w_1)
\end{cases}
\end{align*}
$$

$(a_0 > 0, b_0 > 0, c_0 > 0$ and $\det \mathbb{A} \equiv \det \begin{pmatrix} a_1 & \gamma & \beta \\ \gamma & b_1 & \alpha \\ \beta & \alpha & c_1 \end{pmatrix} \neq 0)$.

That plays an important role in our proof of well–posedness, as we will point out, is the Hamiltonian structure of (2.2):

$$U_t = D \delta \mathcal{H} = D \delta \int T^3 dx$$

$$= D E_U(T^3) = D \begin{pmatrix} E_u(T^3) \\ E_v(T^3) \\ E_w(T^3) \end{pmatrix} \quad (2.4)$$

with the skew–adjoint operator $D = \mathbb{H}^{-1} \text{diag}\{\partial_x, \partial_x, \partial_x\}$ and

$$2T^3 = a_1 u_1^2 + b_1 v_1^2 + c_1 w_1^2 + 2\alpha u_1 w_1 + 2\beta u_1 v_1 + 2\gamma v_1 v_1$$

$$- u^2(a_2 v + a_3 w) - v^2(b_2 u + b_3 v) - w^2(c_2 u + c_3 v) - \frac{a_4 u^3 + b_4 v^3 + c_4 w^3}{3}$$

$$- h u v w - (a_5 u^2 + b_5 v^2 + c_5 w^2) - (a_6 u v + b_6 u w + c_6 v w) \quad (2.5)$$

with

$$U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \mathbb{H} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}$$
In addition, following conserved densities (writing in the form of CLs)

\[ T^{1u} = u \]  \hspace{1cm} (2.6a)

\[ a_0 X^{1u} = a_1 u_2 + \beta v_2 + \gamma v_3 + u(a_2 v + a_3 w) + h u w + \frac{a_2 u^2 + a_3 y^2 + a_4 y^2}{2} + (a_5 u + b_6 w + c_6 y) \]  \hspace{1cm} (2.6b)

\[ T^{1v} = v \]  \hspace{1cm} (2.7a)

\[ b_0 X^{1v} = b_1 v_2 + \gamma u_2 + \alpha v_2 + v(b_2 w + b_3 y) + h u w + \frac{a_2 u^2 + c_3 y^2 + b_4 y^2}{2} + (b_5 v + c_6 y + a_6 y) \]  \hspace{1cm} (2.7b)

\[ T^{1w} = w \]  \hspace{1cm} (2.8a)

\[ a_0 X^{1w} = c_1 w_2 + \alpha v_2 + \beta y_2 + w(c_2 u + c_3 v) + h w w + \frac{b_2 u^2 + a_3 y^2 + c_4 y^2}{2} + (c_5 w + a_6 y + b_6 y) \]  \hspace{1cm} (2.8b)

\[ T^2 = \frac{a_0 u^2 + b_0 y^2 + c_0 y^2}{2} \]  \hspace{1cm} (2.9a)

\[ X^2 = a_1 u_1 u_2 + b_1 v_1 v_2 + c_1 w_1 y_2 - \frac{a_1 u_1^2 + b_1 v_1^2 + c_1 w_1^2}{2} + \frac{a_4 y^3 + b_4 y^3 + c_4 y^3}{3} \]
\[ + u^2 (a_2 v + a_3 w) + v^2 (b_2 w + b_3 y) + w^2 (c_2 u + c_3 v) + 2 h u w w \]
\[ + \frac{a_5 u^2 + b_5 y^2 + c_5 y^2}{2} + a_6 y w + b_6 w y + c_6 w y \]
\[ + \alpha (v w_2 + v_2 w - v_1 y_1) + \beta (u w_2 + v_2 w - u_1 y_1) + \gamma (w w_2 + u_2 w - u_1 y_1) \]  \hspace{1cm} (2.9b)

\[ \text{[27]} \] has implied the counterparts of Majda–Biello system are unique of the kind; we may expect no more CL.

**Example 2.1.** Indeed, when we set

\[
\begin{align*}
    a_0 &= 1, a_1 = -\sigma, a_2 = 1, a_3 = a_4 = \alpha = 0, a_5 = \mu, a_6 = \sigma \\
    b_0 &= 1, b_1 = -1, b_2 = b_3 = b_4 = \beta = 0, b_5 = \lambda, b_6 = \Gamma_A \\
    c_0 &= 1, c_1 = -1, c_2 = c_3 = c_4 = \gamma = 0, c_5 = -\lambda, c_6 = \Gamma_S \\
    h &= 0
\end{align*}

\]  \hspace{1cm} (2.10)

\[ \text{[2.2]} \] turns into \([1.1]\).

**Example 2.2.** Set

\[
\begin{align*}
    a_0 &= a_1 = 1, a_2 = a_3 = a_5 = a_6 = 0, a_4 = 3, \alpha = 0 \\
    b_0 &= b_1 = 1, b_2 = b_3 = b_5 = b_6 = 0, b_4 = 3, \beta = 0 \\
    c_0 &= c_1 = 2, c_2 = c_3 = 3, c_5 = c_6 = \gamma = 0 \\
    h &= 0
\end{align*}

\]  \hspace{1cm} (2.11)
we get a system has good integrability (mentioned in [17, 1]):

\[
\begin{cases}
0 = u_t + u_3 + 3uu_1 + 3ww_1 \\
0 = v_t + v_3 + 3vv_1 + 3ww_1 \\
0 = w_t + w_3 + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x
\end{cases}
\tag{2.12}
\]

**Example 2.3.** Set

\[
\begin{align*}
    a_0 &= a_1 = 1, a_2 = \tilde{a}_2, a_4 = 1, a_3 = a_5 = a_6 = \alpha = 0 \\
b_0 &= \tilde{b}_1, b_1 = b_4 = 1, b_3 = \tilde{a}_1, b_5 = \frac{r}{b_2}, b_2 = b_6 = \beta = 0 \\
c_2 &= c_3 = c_6 = 0, \gamma = \tilde{a}_3 \\
h &= 0, \text{ arbitrarily choose } c_0, c_1, c_4, c_5
\end{align*}
\tag{2.13}
\]

\(\triangle_1, \triangle_2\) of (2.2) turn into Gear–Grimshaw system in [15]:

\[
\begin{cases}
0 = u_t + u_3 + uu_1 + \tilde{a}_1 vv_1 + \tilde{a}_2 (uv)_1 + \tilde{a}_3 v_3 \\
0 = b_1 v_t + v_3 + vv_1 + rv_1 + \tilde{b}_1 [\tilde{a}_1 (uv)_1 + \tilde{a}_2 uu_1 + \tilde{a}_3 uv]
\end{cases}
\tag{2.14}
\]

**Definition 2.4.** Cauchy problem of a PDE system is locally (globally) well–posed in the function space \(X\) if it induces a dynamical system in \(X\) by generating a continuous local (global) flow.

**Notation 2.** By \(\mathcal{F}\), the Fourier transform while \(\mathcal{F}^{-1}\) its inverse; \(a + 0\) means \(a + \epsilon\) for arbitrarily small \(\epsilon > 0\); with \(H^s(\mathbb{R})\), the \(L^2(\mathbb{R})\) based Sobolev space, i.e. with the norm

\[
\| f \|_{H^s(\mathbb{R})} = \| (1 + |k|)^s \mathcal{F}(f) \|_{L^2(\mathbb{R})}
\]

\(X_s(T) = C(0, T; H^s(\mathbb{R}))) \cap C^1(0, T; H^{s-1}(\mathbb{R}))\)

Sobolev spaces \(X^p_{a_0, b_0, c_0}\) the completion of Schwartz space \(\mathcal{S}(\mathbb{R}^3)\), correspondingly define \(\| (u, v, w) \|_{k,p} = \left( \int (a_0 u_k^2 + b_0 v_k^2 + c_0 w_k^2)^\frac{p}{2} dx \right)^\frac{1}{p}\) where \(a_0, b_0, c_0 > 0, k, p \in \mathbb{R}\).

In order to present the role the Hamiltonian structure plays in, we first show an estimate similar to ones in [9] which holds for (2.2) (for the sake of simplicity, \(s\) an positive integer):

**Lemma 2.5.** If \((u, v, w) \in X_s(T) \times X_s(T) \times X_s(T)\) is a local solution of the initial value problem (2.3) with initial \((u_0, v_0, w_0)\), then for any \(k, 1 \leq k \leq s,\)

\[
\begin{align*}
sup_{[0, T]} \| (u(t), v(t), w(t)) \|_{k,2} \\
\leq \| (u_0, v_0, w_0) \|_{k,2} \exp \left( C \int_0^T (\| u_0 \|_{\infty} + \| v_0 \|_{\infty} + \| w_0 \|_{\infty}) \, dx \right)
\end{align*}
\tag{2.15}
\]
Proof. Considering

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int \left[ a_0 u_1^2 + b_0 v_1^2 + c_0 w_1^2 \right] \, dx \\
= a_0 u_k \frac{\partial^k \Delta_1}{\partial x^k} + b_0 v_k \frac{\partial^k \Delta_2}{\partial x^k} + c_0 w_k \frac{\partial^k \Delta_3}{\partial x^k} \\
= -a_2 \int u_k (uv)_{k+1} + v_k \left( \frac{u^2}{2} \right)_{k+1} \, dx - a_3 \int u_k (uw)_{k+1} + w_k \left( \frac{u^2}{2} \right)_{k+1} \, dx \\
- b_2 \int v_k (vw)_{k+1} + w_k \left( \frac{v^2}{2} \right)_{k+1} \, dx - b_3 \int v_k (vu)_{k+1} + u_k \left( \frac{v^2}{2} \right)_{k+1} \, dx \\
- c_2 \int w_k (uw)_{k+1} + u_k \left( \frac{w^2}{2} \right)_{k+1} \, dx - c_3 \int w_k (vw)_{k+1} + v_k \left( \frac{w^2}{2} \right)_{k+1} \, dx \\
- h \int [u_k (vw)_{k+1} + v_k (uv)_{k+1} + w_k (uw)_{k+1}] \, dx.
\end{align*}

(2.16)

The only substantial difference appears as the term on the last line:

\[ | \int [u_k (vw)_{k+1} + v_k (uv)_{k+1} + w_k (uw)_{k+1}] \, dx | \]  

(2.17)

which, when applying Leibniz’s rule and integrations by parts and Holder’s inequality, can also be controlled by

\[ C \left( \| u_1 \|_{\infty} + \| v_1 \|_{\infty} + \| w_1 \|_{\infty} \right) \left( a_0 \| u_k \|^2 + b_0 \| v_k \|^2 + c_0 \| w_k \|^2 \right) \]  

(2.18)

(an example is when \( k = 1 \):

\[ \int [u_1 (vw)_2 + v_1 (uw)_2 + w_1 (uv)_2] \, dx \]

\[ = - \int [u_2 (vw)_1 + v_2 (uw)_2 + w_2 (uv)_1] \, dx \]

\[ = - \int [u(v_1 w_1)_1 + v(u_1 w_1)_1 + w(u_1 v_1)_1] \, dx = 3 \int u_1 v_1 w_1 \, dx \]

)

Therefore we conclude,

\begin{align*}
&\frac{1}{2} \frac{d}{dt} \int \left[ a_0 u_1^2 + b_0 v_1^2 + c_0 w_1^2 \right] \, dx \\
&\leq C \left( \| u_1 \|_{\infty} + \| v_1 \|_{\infty} + \| w_1 \|_{\infty} \right) \left( a_0 \| u_k \|^2 + b_0 \| v_k \|^2 + c_0 \| w_k \|^2 \right)
\end{align*}

(2.19)

Gronwall’s inequality applied to (2.16) we obtain the results.

Remark 2.6. This similar result holds, more or less, due to existence of Hamiltonian structure as well as the similarity between \( X^2 \) and \( T^3 \) – more explicit.
for the latter, ratios of coefficients of \( vw, wu, uv \) are the same, both 1 : 1 : 1 in \( X^2 \) and \( T^3 \); existence of \( T^3 \) ensures combination of corresponding term into

\[
\int [u_k(vw)_{k+1} + v_k(uw)_{k+1} + w_k(uv)_{k+1}] \, dx \quad (2.20)
\]

About the priori estimate in \([9, 4, 25]\), linear theory does work: since the matrix \( H^{-1}A \) is obviously diagonalizable with \( \lambda_j, j = 1, 2, 3 \) as eigenvalues, as well as the transforming matrix \( T \)

\[
TH^{-1}A = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}
\]

Then (2.2) turns into

\[
0 = TUt + \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}T + H^{-1}B_2TU_1 + H^{-1}C(U)T
\]

where

\[
H_2 = THT^{-1}, T^{-1}B_2T = B = \begin{pmatrix}
  a_5 & c_6 & b_6 \\
  c_6 & b_5 & a_6 \\
  b_6 & a_6 & c_5
\end{pmatrix}
\]

and

\[
T^{-1}C(U)T = \begin{pmatrix}
  a_4 & a_2 & a_3 & u \\
  a_2 & b_3 & h & v \\
  a_3 & h & c_2 & w \\
  b_3 & b_4 & b_2 & u \\
  h & b_2 & c_3 & v \\
  a_3 & h & c_2 & u \\
  h & b_2 & c_3 & v \\
  c_2 & c_3 & c_4 & w
\end{pmatrix}
\]

**Remark 2.7.** Notice \( \lambda_1\lambda_2\lambda_3 \neq 0 \iff \text{det } A \neq 0 \); diagonalizability of \( H^{-1}A \) is due to existence of Hamiltonian structure. Furthermore, \( \lambda_j > 0(<0), j = 1, 2, 3 \iff A \) is positive (negative) definite—since we can switch \( U \leftrightarrow -U \), two cases are substantially the same.

Following exactly the same routine as in \([9,4,25]\), we know that the following holds:

**Theorem 2.8.** There is a unique solution of system (2.2) with \( (u_0, v_0, w_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) as initial data such that

\[
(u, v, w) \in C([0, \infty], L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))
\]
Proof. That \( \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \) makes (2.21) into

\[
0 = V_t + \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}V_3 + H^{-1}_2B_2V_1 + H^{-1}_2C(V)V_1
\]

where \( H^{-1}_2B \) is also diagonalizable in \( \mathbb{R} \) and where we keep \( C(V) \equiv C(U) \) regrading no confusion here and that details not important; furthermore, after same performance in \([25, 4]\) of

\[ u^2(x, t) = \tilde{u}(\frac{x}{\lambda_1}, t), v^2(x, t) = \tilde{v}(\frac{x}{\lambda_2}, t), w^2(x, t) = \tilde{w}(\frac{x}{\lambda_3}, t) \]

\[ \bullet \text{ scale change} \]

we can reduce study of our problem to

\[
0 = U_t + U_3 + \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} U_1 + f(U)_x \quad (2.22)
\]

where \( B_2 \) is diagonalizable in \( \mathbb{R} \) and where without any confusion, \( U = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \)

and that

\[
f(U) = \mathcal{G}_{3 \times 6}(u^1 u^1, u^1 u^2, u^1 u^3, u^2 u^2, u^2 u^3, u^3 u^3), \quad \mathcal{G}_{3 \times 6} \in \mathbb{R}^{3 \times 6}
\]

certain norms of whom the crucial bilinear estimates can control, the norms are corresponding weighted norms. So proved the result. \( \square \)

**Corollary 2.9.** There is a unique solution of system (2.2) with \((A_0, B_0^S, B_0^A) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})\) as initial data such that

\((A, B^S, B^A) \in C([0, \infty], L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}))\)

### 3 Stability of solitary waves

The symmetry group of (2.2) may be generated by the vector field of the form

\[
v = \xi^1(x, t, A, B) \frac{\partial}{\partial x} + \xi^2(x, t, A, B) \frac{\partial}{\partial t}
\]

\[
+\eta^1(x, t, A, B^S, B^A) \frac{\partial}{\partial A} + \eta^2(x, t, A, B^S, B^A) \frac{\partial}{\partial B^S} + \eta^3(x, t, A, B^S, B^A) \frac{\partial}{\partial B^A}
\]

10
Example 3.1. By direct calculation of GeM package, we can figure out
\[
v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial t}, v_3 = \frac{\partial}{\partial B^A}
\]
are infinitesimal generators with characteristics
\[
Q_1 = -\begin{pmatrix} A_1 \\ B_1^A \\ B_1^S \end{pmatrix}, Q_2 = -\begin{pmatrix} A_t \\ B_t^A \\ B_t^S \end{pmatrix}, Q_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.1}
\]

Example 3.2. As [27] has pointed out (and we have made the double-check), infinitesimal generators of Majda–Biello system (4.1) are
\[
v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial t}, v_3 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2A \frac{\partial}{\partial A} - 2B \frac{\partial}{\partial B} \tag{3.2}
\]
with corresponding characteristics
\[
Q_1 = -\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, Q_2 = -\begin{pmatrix} A_t \\ B_t \end{pmatrix}, Q_3 = -\begin{pmatrix} 2A + xA_1 + 3tA_t \\ 2B + xB_1 + 3tB_t \end{pmatrix} \tag{3.3}
\]

Generally, (2.2) has \(v_1, v_2\) as infinitesimal generators; while \(U(x, t) \equiv V(x - \sigma t)\), a traveling wave solution, corresponds to the translation group \(\langle \sigma v_1 + v_2 \rangle\), we can propose the stability of these solutions by the method [5, 2] which relies on the Hamiltonian structure – in fact, [2] studies
\[
0 = U_t + H^{-1} [\nabla N(U) - L U]_x \tag{3.4}
\]
where \(N(U)\), applied to (2.2), is a homogeneous function of degree 3\((\equiv p + 2)\), some terms of our Hamiltonian \(T^3\) in (2.5), i.e.
\[
N(U) = -u^2(a_2 v + a_3 w) - v^2(b_2 w + b_3 u) - w^2(c_2 u + c_3 v) - \frac{a_2 a_3}{3} u^3 + \frac{b_2 b_3}{3} v^3 + \frac{c_2 c_3}{3} w^3 - h u v w
\]
and \(L\), dispersion self–adjoint operator, is a matrix \(x\)–Fourier multiplier operator with Fourier symbol \(\mathcal{A} k^2 - i \mathcal{B} k\), i.e.
\[
\hat{L} \hat{U}(k) = [\mathcal{A} k^2 - i \mathcal{B} k] \hat{U}(k)
\]
The neat form of
\[
- \sigma \mathcal{H} V = \mathcal{B} V - \nabla N(V) \tag{3.5}
\]
is again \(\delta T^1(V) + \sigma \delta T^2(V) = 0\) which inspires further constrained minimization problem
\[
I_{\sigma, q} = \inf \{ (\sigma T^3 + T^2) (V) : V \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}), T^3(V) = q \} \tag{3.6}
\]
with solution set
\[ G_{\sigma,q} = \{ g \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}), T^3(g) = q \} \neq \emptyset \] (3.7)

Therefore, due to Remark 2.7 we can conclude from Theorem 2.2 of [2] Theorem 3.3.

Suppose \( A \) of (2.2) is definite, whatever positive or negative, then for each \( q > 0 \), the problem of minimizing \( T^3 \) subject to constraint \( T^2 = q \) has a nonempty solution set \( G_q \), and for each \( V \in G_q \), there exists \( \sigma > 0 \) such that \( V(x - \sigma t) \) is a solution of (2.2).

Moreover, the set \( G_q \) is stable in the sense that for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that there exists a global solution \( h(x,t) \) of (2.2) with initial data \( h(x,0) = h_0 \) and a map \( \psi : [0,\infty) \to G_q \) such that \( \| h(\cdot,t) - \psi(t) \|_1 < \epsilon \) as soon as \( h_0 \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) satisfying \( \| h_0 - V \|_1 < \delta \) for some \( V \in G_q \).

On the other hand, elementary routine of simplest equation method cannot find the traveling wave solutions of, say, Majda–Biello system (see A).

4 Numerical tests

To the author’s knowledge, numerical results for interaction of three solitary waves, say, in special cases of (1.1), has not been explored before: this may due to the fact

- they comprise CASEs with two components, may still be basic elements of general forms;
- system with three components rarely appear.

Under this philosophy, we restrict our discussion on numerical tests of Majda–Biello system:

\[
\begin{align*}
\triangle_1 &= A_t - DA_3 + (AB)_x = 0 \\
\triangle_2 &= B_t - B_3 + AA_1 = 0
\end{align*}
\] (4.1)

As [3] has pointed out, this may virtually be basic behaviors of interaction of BSAB system -B^S be small, may be regarded as small regular perturbation of Majda–Biello system.

Different from Gear–Grimshaw system, (4.1) has a equivalent coupled KdV form being numerically demonstrated in [3]:

\[
\begin{align*}
U_t - \frac{(1+D)}{2} U_{xxx} + UU_x &\equiv \frac{(1-D)}{2} V_{xxx} + \left( \frac{UV}{2} \right)_x \\
V_t - \frac{(1+D)}{2} V_{xxx} + VV_x &\equiv \frac{(1-D)}{2} U_{xxx} + \left( \frac{UV}{2} \right)_x
\end{align*}
\] (4.2)
This better demonstrates interaction of solitary waves since it more or less preserves KdV form were \( U = 0 \) or \( V = 0 \) in the initial data. Meanwhile, we constantly regard \( x \in \mathbb{R} \) and that were any periodic presentation, it is due to numerical convenience, substantially NOT for requirement of atmosphere models.

In this section, we first recall and explain previous numerical results, which will more or less relate with our later numerical study on asymptotic behavior for radiation decay as \( \lambda_V \to 0^+ \), with \( \lambda_U = 1 \) fixed.

4.1 Explanation for previous numerical phenomena in [6]

Study on interaction of two solitary waves amongst certain equations with solitary solutions has a long history since John Scott Russell reported "wave of translation" in 1844 ([23]). These include integrable PDE systems, say, sine–Gordon equations in [11], Boussinesq equations both belonging to KP hierarchy; nonintegrable ones, like

The radiation (rarefaction wave) behind the interaction/ collision of two solitary waves, implying nonintegrability, is similar to numerical results of BBM equation in [10].

The numerical method we used here is again pseudo–spectral method described in [29], the same as [3, 6, 7].

4.1.1 Previous numerical results

Previous numerical results involved, deserve recalls; meanwhile, we replenish some explanations.

Demonstration on localized soliton solutions, if a nonintegrable PDE system has ones, is a commonplace. [6] provides numerical demonstrations of \( D = 1 \) but no detailed explanation; on the other hand, when \( D = \frac{8}{9} \) localized solitons appear

\[
U(x,t) = S(x - \sigma t), V(x,t) = 0 \text{ or } U(x,t) = 0, V(x,T) = S(x - \sigma T) \quad (4.3)
\]

where

\[
S(x) = -\frac{12}{\lambda} \cosh^{-2} \left( \frac{x - x_0}{\lambda \sqrt{(1 + D)/2}} \right), \sigma = -\frac{4}{\lambda^2} \quad (4.4)
\]

who provide similar phenomena (see two cases in figure 4.1.1). \( \lambda = 1 \) for \( U \), \( \lambda = 0.75 \) for \( V \). However, one may argue whether the phenomena are due to numerical instability or the equation. There are, during interaction of two localized solitons, following phenomena, which can be explained:
• Split and phase shift

as two solitons getting close, with \( U_x < 0, V_x < 0, U < 0, V < 0 \), term \( \frac{(UV)_x}{2} > 0 \) forces both (due to symmetry of (4.2)) \( U \) and \( V \) upward;
as two 'solitons' moving apart, supposing with \( U \) slightly deformed from a solitary shape within a short time, then, term \( \frac{(UV)_x}{2} < 0 \) forces both \( U, V \) downward – were soliton structure of \( V \) temporarily disappeared due to first 'force' mentioned above, the structure reappears;

Above explains the process of phase shift.

• Radiation

Although during major time of the collision \( \frac{(UV)_x}{2} \), the forcing term appears half positive, half negative, it \( \frac{(UV)_x}{2} \) rises and falls in a more complicated process (see figure (4.1.1)), many small 'solitons' appear – radiation appears majorly at the "beginning" and "ending" of the collision.

That radiation moves left (in opposite direction) concides the loss of mass of \( U \), see figure 5 and next discussion.

• Effect of the terms \( \frac{1+D}{2} U_{xxx} (\frac{1+D}{2} V_{xxx}) \)

Comparing case of \( D = 1, D = \frac{8}{9} \) in figure 4.1.1 these terms contributes more "ripples" near x-axis; \( D \rightarrow 1- \) can be regarded as a regular perturbation of CASE \( D = 1 \).

4.2 Behavior as \( \lambda_V \rightarrow 0^+ \) while \( \lambda_U = 1 \)

Improvement of numerical methods

Considering \( \lambda_V \rightarrow 0^+ \), we need to scale space interval \( dx = O(\lambda_V^2) \) while \( dt = O(\lambda_V) \). Consequently, we prefer adding semi–implicit operator into classical numerical methods mentioned above since numerical stability requires \( dt = O(\lambda_V^3) \) to ensure time integration \( | \frac{4}{2} | dt, | (AB)_x | dt \) under control; by comparision, the former methods may require \( dt \sim \lambda_V^6 \) due to the requirement of \( \frac{\Delta t}{(\Delta x)^3} < C \) (say, see [26, 29]); to some extent, the result proves to be stable (see B for CASE \( \lambda_V = 0.35, \lambda_U = 1 \)).
Figure 1: Demonstration of interacting term $\frac{(UV)_x}{2}$ ($\gamma = 0$): left is distribution of $\frac{1}{2} \max |(UV)_x|$ w.r.t. time $t$, right is snapshots during the collision: $\frac{1}{2} |(UV)_x|$ reaches maximum at $t = 4.40$ (red circle pointed on the left as well as the third snapshot on the right).

Figure 2: Demonstration of interacting term $\frac{(UV)_x}{2}$ ($D = 1$): left is the "beginning": evolution before $\frac{|(UV)_x|}{2}$ reaches around 2 at $t = 3.80$; right is the "ending": evolution after $\frac{|(UV)_x^2|}{2}$ decays below 2 at $t = 4.80$. 
Figure 3: Snapshots of interaction of solitary waves \(D = 1 - 2\gamma\). \(\gamma = 0\) first graph; \(\gamma = \frac{1}{9}\) the second): \(U, V\) with the \(\lambda = 1\) and \(\lambda = 0.75\)
Measurements of solitary structure

Due to stability mentioned in section 3, we can expect solitary structure after the collision. Numerically, we extract solitary structure of $U$ after the collision by

- obtain the position $P$ of the maximal absolute value of $U$;
- find position $L, R$ the nearest point with negative value of $U$, separately left and right with respect to $P$;
- regard part of $U$ between position $L, R$ is a "soliton" $\tilde{U}$;

Meanwhile, inspired by first three CLs of distinguished KdV equation, we use following measurements of the solitary structure in $\tilde{U}$

- $T^1_U \triangleq \int_L^R U \, dx$, the mass of $\tilde{U}$, with initial value $T^1_{U_0} = -24$;
- $T^2_U \triangleq \int_L^R U^2 \, dx$, the momentum of $\tilde{U}$, with initial value $T^2_{U_0} = 192$;
- $T^3_U \triangleq \int_L^R U^2 + \frac{U^3}{3} \, dx$, the energy of $\tilde{U}$, with initial value $T^3_{U_0} = -460.8$.

It is well-known that when $\lambda_V \sim \lambda_U = 1$, resonance appears with relatively large amount of radiation – indeed, the author has particularly tested on the case $\lambda_V = 0.99$ with the result that both $U, V$ reshape into two "solitons" with much larger difference in $\lambda$ (or a more apparent behavior, relatively larger difference in traveling speeds) after the collision; here we are going to demonstrate the behavior when $\lambda_V \to 0$.

After testing the CASE $\lambda_V = 0.75, 0.70, 0.65, \ldots, 0.25, 0.20$, behaviors of three measurements of $\tilde{U}$ after the collision, with respect to $\lambda_V$ has been shown as first graph in figure 5, even though $T^1_U$ stays on one side of the initial problem for $j = 1, 2, 3$, say, $T^1_U > T^1_{U_0}$ for every $\lambda_V$ tested, $T^1_U, T^2_U$ really show something different from $T^3_U$. This can be better presented when we turn to logarithmic coordinates, who are usually used to find polynomial asymptotic behaviors, i.e. second graph in figure 5, where we find

$$ R^1(\lambda_V) \triangleq T^1_U - T^1_{U_0} \simeq 0.4529\lambda_V^{2.5576} $$
$$ R^2(\lambda_V) \triangleq T^2_U - T^2_{U_0} \simeq 9.1188\lambda_V^{5.589} $$
$$ R^3(\lambda_V) \triangleq T^3_U - \frac{4}{3}T^3_{U_0} \simeq 35.5389\lambda_V^{5.5194} \tag{4.5} $$

To some extent, (4.5) manifests
Figure 4: maximal of $\text{abs}(U)$ – height of "soliton" $U$ after the collision, $\lambda_V = 0.55, 0.45, 0.25$
Figure 5: Three measurements $T_1, T_2, T_3$ with respect to $\lambda V \in [0.2, 0.75]$

**Proposition 4.1.** The effect of interaction over $U$ decays polynomially. More specifically, $R_j(\lambda V) = O(\lambda^I_j V)$, $j = 1, 2, 3$ as $\lambda V \in [0.2, 0.5]$ with powers $I_1 \simeq 2.5576, I_2 \simeq 5.5589, I_3 \simeq 5.5194$.

5 Afterward

As for evolutionary equations themselves, theory of stability provides neat form (3.4):

$$0 = U_t + \mathbb{H}^{-1} [\nabla N(U) - LU]_x$$

Whether they can be utilized for theory on other well-posedness/ill-posedness of Cauchy problem deserves further study; as for decay of the effect of collision, it may still be hasty to conjecture it, for certain polynomial degrees $I_j, j = 1, 2, 3$ an asymptotic behavior for $\lambda V \in (0, 0.5]$ as $\lambda V \to 0^+$-- errors and time costs make we halt at $\lambda V = 0.2$; it may be more convincing for we describe asymptotic behaviors more than numerical results.

Thanks Prof. Joseph A. Biello for encouraging me on numerical tests; I am very grateful for Trevor Halsted’s company for numerical study.
Appendix

A Exact solutions

Since equation in the form (4.2) better illustrates solitary structure, we use $U, V$ instead of $A, B$ in this section.

We show simplest equation method described in [16] indeed find new exact solutions (rather than in the case $U = V = 0$) for (4.2). After transformations $U(x, t) = U(\xi), V(x, t) = V(\xi), \xi = x - \sigma t$, and integration, (4.2) becomes

$$
-\sigma U - (1 - \gamma)U_{xx} - \gamma V_{xx} + \frac{U(U - V)}{2} \equiv c_1
$$

$$
-\sigma V - (1 - \gamma)V_{xx} - \gamma U_{xx} + \frac{V(V - U)}{2} \equiv c_2
$$

(A.1)

For the sake of (4.4), we let $\sigma < 0$.

A simple version of this method is to find solutions in the form

$$
U = \sum_{j=0}^{m} a_j w^j, V = \sum_{s=0}^{n} b_s w^s
$$

(A.2)

where $w$ obeys the Riccati equation, namely, $\frac{dw}{d\xi} = c_0(k) + c_1(k)w - c_2(k)w^2$ ($c_2(k)$ positive and differentiable). $m, n$ are predetermined by homogeneous balance:

- putting $c_0(k) \equiv k, c_1(k) \equiv 0, c_2(k) \equiv k$, we get traditional homogeneous balance method (HBM) in [28], which is narrated below;

- putting $c_0(k) \equiv 0, c_1(k) \equiv a, c_2(k) \equiv b$, Bernoulli differential equation with $n = 2$.

**Definition A.1** (from [16]). the Riccati equation, $\frac{dw}{d\xi} = c_0(k) + c_1(k)w - c_2(k)w^2$ is called the simplest equation.

The second special case has been applied in [1].

**Homogeneous balance method (HBM)**

HBM was initially applied to integrable systems, say, variant Boussinesq equations [28]. It is uncertain whether it could provide new information to any nonintegrable systems. Following we show that it CAN be applied here.

Balancing $U''$ with $U^2$, $V''$ with $V^2$, we get the following ansatz.

$$
U = a_0 + a_1 w + a_2 w^2
$$

$$
V = b_0 + b_1 w + b_2 w^2
$$

(A.3a)
which leads to

$$U_{xx} = 2k^2 \left( a_2 - a_1 w - 4a_2 w^2 + a_1 w^3 + 3a_2 w^4 \right)$$
$$V_{xx} = 2k^2 \left( b_2 - b_1 w - 4b_2 w^2 + b_1 w^3 + 3b_2 w^4 \right)$$  \hspace{1cm} (A.3b)

where

$$w' = k(1 - w^2)$$  \hspace{1cm} (A.3c)

(A.3c) has general solution

$$w = \tanh(k\xi - c_0)$$  \hspace{1cm} (A.4)

**Theorem A.2.** The system (4.2) has exact solutions

$$U(x,t) \equiv -2k^2(y + 1) \gamma \left[ 1 - \frac{\sigma}{4k^2} \frac{3 - y}{y + 1} - 3 \cosh^{-2}(k(x - \sigma t) - c_0) \right]$$
$$V(x,t) \equiv -2k^2(y + 1) \gamma \left[ 1 - \frac{\sigma}{4k^2} \frac{3y - 1}{y + 1} - 3 \cosh^{-2}(k(x - \sigma t) - c_0) \right]$$  \hspace{1cm} (A.5)

where

$$y = \pm \sqrt{\frac{\gamma}{2} - \frac{1}{y} + \frac{1}{\gamma}}.$$

**Proof.** Take (A.3b) and (A.3a) into (A.1):

1. Set coefficients of $w^4$ zero, we obtain

$$\frac{a_2(a_2 - b_2)}{2} = 6k^2(1 - \gamma)a_2 + 6k^2\gamma b_2$$
$$\frac{b_2(b_2 - a_2)}{2} = 6k^2(1 - \gamma)b_2 + 6k^2\gamma a_2$$  \hspace{1cm} (A.6)

which leads to

$$a_2 = \frac{12k^2((1 - \gamma)y + \gamma)}{y - 1} = \frac{12k^2y(y + 1)}{(y - 1)^2}$$
$$b_2 = \frac{12k^2((1 - \gamma)y + \gamma)}{y - 1} = \frac{12k^2(y + 1)}{(y - 1)^2}$$  \hspace{1cm} (A.7)

where $y^2 + \frac{2(1 - \gamma)}{\gamma}y + 1 \equiv 0$;

2. Set coefficients of $w^3$ zero

$$\frac{a_2(a_1 - b_1) + a_1(a_2 - b_2)}{2} = 2k^2 \left[ \gamma b_1 + (1 - \gamma)a_1 \right]$$
$$\frac{b_2(b_1 - a_1) + b_1(b_2 - a_2)}{2} = 2k^2 \left[ \gamma a_1 + (1 - \gamma)b_1 \right]$$  \hspace{1cm} (A.8)

take (A.7) inside we get

$$((5 + \gamma)y^2 + (5 - 2\gamma)y + \gamma - 4) a_1 = ((3 + \gamma)y^2 + (3 - 2\gamma)y + \gamma) b_1$$
$$(\gamma y^2 + (3 - 2\gamma)y + 3 + \gamma) a_1 = ((\gamma - 4)y^2 + (5 - 2\gamma)y + \gamma + 5) b_1$$
By denoting $\psi(z) \doteq (5 + \gamma)z^2 + (5 - 2\gamma)z + \gamma - 4, \eta(z) \doteq (3 + \gamma)z^2 + (3 - 2\gamma)z + \gamma$, the above turns into

$$\begin{pmatrix} \psi(y) & \eta(y) \\ \eta(y^{-1}) & \psi(y^{-1}) \end{pmatrix} \begin{pmatrix} a_1 \\ -b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (A.9)$$

Since $\det \begin{pmatrix} \psi(y) & \eta(y) \\ \eta(y^{-1}) & \psi(y^{-1}) \end{pmatrix} = \frac{\eta_0}{\gamma} \left( 2 - \frac{1}{\gamma} \right) \neq 0$ (considering the practical case $\gamma \neq \frac{1}{2}$), we have $a_1 = b_1 = 0$;

3. Set coefficients of $w^2$ zero,

$$-\sigma a_2 + 8k^2 [(1 - \gamma)a_2 + \gamma b_2] + \frac{a_1(a_1 - b_1)}{2} + \frac{a_2(a_2 - b_0)}{2} + \frac{a_0(a_2 - b_2)}{2} = 0$$

$$-\sigma b_2 + 8k^2 [(1 - \gamma)b_2 + \gamma a_2] + \frac{b_1(b_1 - a_1)}{2} + \frac{b_2(b_2 - a_0)}{2} + \frac{b_0(b_2 - a_2)}{2} = 0$$

combined with $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ as well as $a_1 = b_1 = 0$, we have

$$a_0 = -\frac{k^2 y}{(y-1)^2} \left[ \frac{\sigma}{\gamma} (3 - y) + 8(y + 1) \right]$$

$$b_0 = -\frac{k^2 y}{(y-1)^2} \left[ \frac{\sigma}{\gamma} (3y - 1) + 8(y + 1) \right] \quad (A.11)$$

4. Set coefficients of $w$ zero,

$$\frac{a_0(a_1 - b_1)}{2} + \frac{a_1(a_0 - b_0)}{2} = \sigma a_1 - 2k^2 [(1 - \gamma)a_1 + \gamma b_1]$$

$$\frac{b_0(b_1 - a_1)}{2} + \frac{b_1(b_0 - a_0)}{2} = \sigma b_1 - 2k^2 [(1 - \gamma)b_1 + \gamma a_1]$$

which holds naturally due to $a_1 = b_1 = 0$;

5. Considering constant terms

$$\frac{a_0(a_0 - b_0)}{2} = \sigma a_0 + 2k^2 [(1 - \gamma)a_2 + \gamma b_2] + c_1$$

$$\frac{b_0(b_0 - a_0)}{2} = \sigma b_0 + 2k^2 [(1 - \gamma)b_2 + \gamma a_2] + c_2$$

Therefore, one exact solution is

$$U(x, t) \equiv \frac{4k^2 y}{(y-1)^2} \left[ y + 1 - \frac{\sigma}{4k^2} (3 - y) - 3(y + 1) \cosh^{-2}(k(x - \sigma t) - c_0) \right]$$

$$\equiv -2k^2 (y + 1) \gamma \left[ 1 - \frac{\sigma}{4k^2} \frac{3 - y}{y + 1} - 3 \cosh^{-2}(k(x - \sigma t) - c_0) \right]$$

$$V(x, t) \equiv \frac{4k^2}{(y-1)^2} \left[ y + 1 - \frac{\sigma}{4k^2} (3y - 1) - 3(y + 1) \cosh^{-2}(k(x - \sigma t) - c_0) \right]$$

$$\equiv -2k^2 (y + 1) \gamma \left[ 1 - \frac{\sigma}{4k^2} \frac{3y - 1}{y + 1} - 3 \cosh^{-2}(k(x - \sigma t) - c_0) \right]$$

(A.14)
where \( y = \pm \sqrt{\frac{1 - \gamma - 1 + \gamma}{\gamma}} = \pm \sqrt{16(m+1)^2m^2 - 4(m+1)m}, \gamma = \frac{1}{(2m+1)^2}, m = 1, 2, \ldots \). So the result reaches.

**Example A.3.** Provided \( c_1 = 0, y = -\sqrt{16(m+1)^2m^2 - 4(m+1)m} \), it happens that

\[
\frac{\sigma}{k^2} = \frac{4(y+1)}{3y - 1} \tag{A.15}
\]

**Example A.4.** In order to verify the exact solution [A.5], we further provide (an impractical case in original model) \( m = \frac{1}{4}, c_0 = 0 \), which leads to \( \gamma = \frac{4}{9}, y = -2, \frac{\sigma}{k^2} = \frac{4}{7} \) as well as \( a_0 = -\frac{8k^2}{7}, b_0 = \frac{4k^2}{3}, a_2 = \frac{8k^2}{3}, b_2 = -\frac{4k^2}{3} \)

\[
\begin{align*}
U(x,t) &\equiv \frac{8k^2}{21} \left[ 4 - 7 \cosh^{-2} \left( k \left( x - \frac{4k^2}{7} t \right) \right) \right] \\
V(x,t) &\equiv \frac{4k^2}{3} \cosh^{-2} \left( k \left( x - \frac{4k^2}{7} t \right) \right) \tag{A.16}
\end{align*}
\]

**Remark A.5.** Due to instability of this solution (this solution is not even in \( L^1(\mathbb{R}) \)), we cannot provide the evolutionary process numerically.

**Remark A.6.** We show that \( c_1 = c_2 = 0 \) cannot happen: otherwise, from [A.13] we have

\[
\left( (3y - 1) \frac{\sigma}{k^2} \right)^2 = 16(y + 1)^2, \left( (3 - y) \frac{\sigma}{k^2} \right)^2 = 16(y + 1)^2 \tag{A.17}
\]

which implies

\[
y = 1(\gamma = -\infty), \frac{\sigma}{k^2} = 4; y = -1(\gamma = \frac{1}{2}), \frac{\sigma}{k^2} = 0 \tag{A.18}
\]

both of them are not supposed in the atmospheric model [B].

**Remark A.7.** In the case \( \gamma = 0, c_1 = c_2 = 0 \), by following above routine, we may find \( a_0 = a_1 = a_2 = 0 \) or \( b_0 = b_1 = b_2 = 0 \) – homogenous balance method provide no more information about exact solutions.

**Remark A.8.** Exact solutions [A.5] correspond to ones of \( A, B \)

\[
\begin{align*}
A &= \frac{\sqrt{2k^2(1-y^2)} \gamma}{y} \left[ 1 + \frac{\sigma}{4k^2} - 3\cosh^{-2}(k(x - \sigma t) - c_0) \right] \\
B &= -\frac{k^2(y+1)^2 \gamma}{y} \left[ 1 + \frac{\sigma}{4k^2} \frac{y^2 - 6y + 1}{(y+1)^2} - 3\cosh^{-2}(k(x - \sigma t) - c_0) \right] \tag{A.19}
\end{align*}
\]
B Numerical stability of interaction of two waves 
while $\lambda_U = 1, \lambda_V = 0.35$

Here are two figures (at different range of y–axes) showing numerical stability while testing the case $\lambda_U = 1, \lambda_V = 0.35$ (see figure 6).

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Figure 6: $\lambda_V = 0.35 (\lambda_U = 1)$: first figure with larger $y$–range, while second smaller: separately three snapshots at $t = 0.00, 2.00, 4.00$
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