Traces of a Quantum Anti Resonance in a Driven System

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It has been previously shown, that classically chaotic kicked systems, whose unperturbed spectrum possesses one energy scale, admits a quantum anti-resonance (QAR) behavior. In this study we extend the conditions under which this QAR occurs for the case of a two-sided kicked 1D infinite potential well. It is then shown by a perturbative argument that this QAR effects the behavior of the equivalent driven well, i.e., the number of periods needed to leave the initial state has a sharp peak around the QAR. We give a numerical evidence that the anti resonance persists even for large values of the perturbation parameter.

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The study of “quantum chaos”, i.e., understanding the fingerprints of classical chaos in quantum mechanics [1], has led to the discovery of a variety of new quantum-dynamical phenomena. Several such phenomena occur in time-periodic systems described by the general Hamiltonian

$$H = H_0 + H_1 f(t),$$

where $H_0$ is some time-independent Hamiltonian, $H_1$ represents a perturbation, and $f(t)$ is periodic with period $T$, $f(t + T) = f(t)$. In many cases, $f(t)$ is chosen, for simplicity, as a periodic delta function, $f(t) = \Delta_T(t) \equiv \sum_{s=-\infty}^{\infty} \delta(t - sT)$, giving the well-studied class of “kicked” systems. Representative models in this class are the kicked rotor (KR) [2–6], the kicked Harper model (KHM) [7–10], and the kicked harmonic oscillator (KHO) [9,11].

In recent years a different kind of phenomenon for time-periodic systems has been studied: exactly periodic recurrences. This phenomenon is defined, in general, by

$$U^p = e^{-i\beta},$$

where $U$ is the one-period evolution operator for (1), $e^{-i\beta}$ is some constant phase factor (a complex number), and $p$ is the smallest positive integer for which (2) is satisfied. Thus $pT$ is the recurrence period.

The class of systems considered are the two-sided kicked rotors (TKRs) [12,13], defined by the Hamiltonian

$$H = \frac{L^2}{2I} + \hat{k} V(\theta) \sum_{s=-\infty}^{\infty} (-1)^s \delta \left( t - \frac{sT}{2} \right),$$

where $I$ is the moment of inertia, $\hat{k}$ is the kicking parameter, $T$ is the time period, and $V(\theta)$ is a general periodic and analytic function of the angle $\theta$. Two-sided kicking perturbations such as in (3) were considered in several physical contexts [14] as approximations of sinusoidal driving potentials corresponding to ac electromagnetic fields. By increasing $\hat{k}$ in the classical TKR, one observes the typical transition from bounded to global chaos [12], as in the KR case. The quantum dynamics is governed, as usual, by the evolution operator $U$ in one period, e.g., from $t = 0$ to $t = T$.
where \( \hat{n} \equiv L/h = -id/d\theta, \tau \equiv hT/(4I), \) and \( k \equiv \hat{k}/h. \)

A most distinctive feature of the quantum TKR is that \( U \) becomes the identity operator for \( \tau = 2\pi m, \) an integer (since the operator \( \exp(-i\tau \hat{n}^2) \) in (1) is clearly the identity in this case). This implies \textit{exactly} periodic recurrences (with period 1) of an arbitrary wave-packet. This phenomenon is referred to as quantum antiresonance (QAR).

The previously described QAR was based on the fact that the unperturbed evolution between successive kicks was described by the identity operator such that the opposite sign kicks are cancelled. It is therefore clear that the same phenomenon will occur for a two sided kicked 1D infinite potential well. In the present study, we show the existence of a new anti-resonance phenomena for the case of the linearly kicked one dimensional potential well. The new QAR occurs when the period of the kicks is \textit{half} of the period needed for the more general QAR, as described in Eq. (4). Thus, this QAR is, in some sense, a “period halving” of the former, and specific to this special case, as we will discuss below. It is then shown that this new QAR, namely the one which occurs when a complete period of the perturbation is compatible with the level spacing frequencies, affects the behavior of the corresponding driven system. In particular, in the region of the QAR condition, the driven system becomes nearly periodic. This may be considered as a ‘trace’ of the exact QAR in the kicked system. Numerical evidence for this effect are presented.

It is interesting to note that such a model system can in fact be realized experimentally. Modern semiconductor technology has enabled the fabrication of 1D quantum wells. Such a quantum well is fabricated by varying the alloy composition in a compound semiconductor like \( Al_xGa_{1-x}As \) along one dimension. Conduction electrons in such structures experience an arbitrarily shaped effective potential in the growth direction while remaining free in the perpendicular plane. Quantum wells are typically 200 – 300 meV deep with level spacing \( \Delta E \) between several meV and 150 meV. These systems are of special interest since they can be treated in means of pure quantum mechanical considerations while they are still

\[
U = e^{-i\tau \hat{n}^2} e^{ikV(\theta)} e^{-i\tau \hat{n}^2} e^{-ikV(\theta)},
\]
experimentally accessible. Recently, there has been interest in the behavior of such systems under the influence of an electromagnetic field \[16–19\]. The quantum well structure can be considered as an analogue of an 1D atom, and thus a study of the driven well may help us learn about the interaction of atoms and high-field electromagnetic radiation. In the region where the electric field is strong with respect to the level spacing of the well, one obtains a system in which to study non-perturbative effects in light-matter interaction. The QAR behavior is therefore experimentally accessible. One should expect a sharp anti-peak in the absorption spectrum of the quantum well.

Consider the Hamiltonian

\[
H = \frac{p^2}{2m} + \lambda \bar{X} \sum_{m=-\infty}^{\infty} \{ \delta(t/T - m) - \delta(t/T - m + 1/2) \},
\]

(5)
defined in the infinite well \(x \in [0, L]\). We use the dimensionless form, defined by the transformation \(\tau = \frac{2\pi t}{T}\), and \(X = \bar{X}/L\) and obtain the Schrodinger equation

\[
\frac{i}{\hbar} \frac{d\psi}{d\tau} = \mathcal{H}\psi = \left\{ -\frac{\hbar_{\text{eff}}^2}{2} \frac{\partial^2}{\partial x^2} + \frac{\beta}{\hbar_{\text{eff}}} X \sum_{m=-\infty}^{\infty} \{ \delta(\tau - 2\pi m) - \delta(\tau - 2\pi (m + 1/2)) \} \right\} \psi,
\]

(6)
where \(\hbar_{\text{eff}} = \hbar/(m\omega L^2)\) and \(\beta = \lambda/(m\omega^2 L)\). The evolution operator for takes the form

\[
U = \exp(iF) = e^{i\pi \bar{h}_{\text{eff}}^2 \frac{\partial^2}{\partial x^2}} e^{-i\beta \frac{\hbar_{\text{eff}}}{\hbar} X} e^{i\pi \bar{h}_{\text{eff}}^2 \frac{\partial^2}{\partial x^2}} e^{i\beta \frac{\hbar_{\text{eff}}}{\hbar} X}.
\]

(7)

Evidently, since the eigenvalues of the operator \(-\partial_x^2\) are of the form \(n^2\pi^2\) with \(n\) integer, if \(\hbar_{\text{eff}}\) takes the values \(\hbar_{\text{eff}} = 4k/\pi^2\) \((k\) integer), the free evolution of the system between two kicks turns out to be the identity operator, and the fact that the two kicks are with opposite sign implies that the whole evolution operator is also the identity. This is the usual quantum anti-resonance. However, we now show that for this special case, a QAR exist also when \(\hbar_{\text{eff}} = 2k/\pi^2\). In order to do that, we point out that

\[
e^{i\pi \bar{h}_{\text{eff}}^2 \frac{\partial^2}{\partial x^2}} e^{-i\beta \frac{\hbar_{\text{eff}}}{\hbar} X} e^{i\pi \bar{h}_{\text{eff}}^2 \frac{\partial^2}{\partial x^2}} e^{-i\beta \frac{\hbar_{\text{eff}}}{\hbar} X} = e^{-i\beta \frac{\hbar_{\text{eff}}}{\hbar} X} e^{i\pi \bar{h}_{\text{eff}}^2 \frac{\partial^2}{\partial x^2}},
\]

(8)
where \((X_I)_{mn} = X_{mn} e^{-i\beta \frac{\hbar_{\text{eff}}}{\hbar} \pi^2 (m^2 - n^2)}\), since \(X_{mn}\) takes the form
\[ X_{mn} = \begin{cases} -\frac{8mn}{\pi^2(m^2-n^2)}, & m + n \text{ odd} \\ 0, & m + n \text{ even, } m \neq n \\ 1/2, & m = n \end{cases} \]  

it is clear that \((X_I)_{mn} = -X_{mn} + 1\), where 1 is the unit matrix, the whole evolution operator \(U\) is then

\[ U = e^{-iF} = e^{-i\beta/\hbar_{\text{eff}}}, \]  

which is the identity operator up to a constant phase. Thus, one obtains a “period halving” of the usual quantum anti-resonance.

We now turn to study the driven version of this problem. This system is classically chaotic as shown in Figure 1. Its quantum behavior is described by the dimensionless Schrödinger equation

\[ i\frac{d\psi}{d\tau} = \mathcal{H}\psi = \left\{ -\frac{\hbar_{\text{eff}}}{2}\frac{\partial^2}{\partial x^2} + \frac{\beta}{\hbar_{\text{eff}}} X \cos(\tau) \right\} \psi. \]  

Clearly, due to the continuous character of the perturbation, the exact QAR described above is not possible for this Hamiltonian. However, as we are now to show, traces of the exact periodicity found in the kicked system in case of QAR are seen in the driven system as well. In fact, kicked systems are widely used as an approximation to cosinusoidally driven perturbations. This can be understood in terms of the relation

\[ \sum_{n=-\infty}^{\infty} \left[ \delta\left(\frac{t}{T} - n\right) - \delta\left(\frac{t}{T} - (n + \frac{1}{2})\right) \right] = 4 \sum_{n=1}^{\infty} \cos((2n - 1)\Omega t), \]  

through which, it is clear that in the limit where one may neglect the effect of the higher frequencies, the kicks are essentially the same as the cos.

It turns out that under the condition \(\hbar_{\text{eff}} = 2k/\pi^2\) the QAR manifests itself in the perturbative expansion of the time-dependent evolution operator. This can be shown as follows. The transition probability (to each order in \(\beta\)) for a complete period is given in terms of a time-ordered integral of the form
\[ A^{(k)}(m \to n) \sim (\beta/\hbar_{\text{eff}})^k T \int_0^{2\pi} dt_1 \cdots \int_0^{2\pi} dt_k X_I(t_1) \cdots X_I(t_k) \cos(t_1) \cdots \cos(t_k). \] (13)

The general form of \( X_I(t) \) is
\[ (X_I)_{mn} = X_{mn} e^{-i \frac{\hbar_{\text{eff}}}{2\pi} (m^2 - n^2)}, \] (14)
and thus, (13) takes the form
\[
A^{(k)}(m \to n) \sim (\beta/\hbar_{\text{eff}})^k \sum_{j_1, \ldots, j_{k-1}} X_{m j_1} X_{j_1 j_2} \cdots X_{j_{k-1} n}
\]
\[
T \int_0^{2\pi} dt_1 \cdots \int_0^{2\pi} dt_k \exp\left(-i t_1 \frac{\hbar_{\text{eff}}}{2} \pi^2 (m^2 - j_1^2)\right) \exp\left(-i t_2 \frac{\hbar_{\text{eff}}}{2} \pi^2 (j_1^2 - j_2^2)\right) \cdots \exp\left(-i t_k \frac{\hbar_{\text{eff}}}{2} \pi^2 (j_{k-1}^2 - n^2)\right) \cos(t_1) \cdots \cos(t_k).
\] (15)

In the anti resonance case, \( \hbar_{\text{eff}} = \frac{2k}{\pi^2} \), all the frequencies in the integrals are integers, and thus since the integral is over a complete period the only contribution comes from the zero mode terms. It is easy to see that even in the most 'soft' case, i.e. \( k = 1 \), the first contribution to the transition amplitudes is at least of order \( \beta^3 \) (if the ground states is populated). This can be seen due to the fact that to order \( \beta^k \) the integrand related to the transition \( m \to n \) contains the frequencies \( \omega_{mn} \pm 1 \pm 1 \ldots \pm 1 \) [\( k \) times]. Since the smallest frequency is \( \omega_{12} = 3\alpha \), the first and second order terms do not have any zero frequency components. Thus the integral over a whole period vanishes.

In order to confirm the above predictions, we have solved the time-dependent Schrödinger equation numerically, using a quality control Runge-Kutta method, for various values of \( \beta \) and \( \hbar_{\text{eff}} \) in the anti-resonance region. The results are shown in Figure 2. We plot the inverse of the probability to leave the initial state after one period. This quantity describes approximately the amount of time needed to leave the initial state. It is interesting to note that the QAR traces persist even for large values of \( \beta \), up to \( \beta = 2 \). However, as \( \beta \) increases, the position of the anti resonance is slightly shifted.

In conclusion, we have extended the concept of QAR for a case in which the entire period of the perturbation is in resonance with the level spacing frequencies. It has been shown by theoretical arguments and numerical results that this effect persist for driven systems.
as well. The conditions under which this effect occur can be realized in experiments on quantum wells in far infra-red radiation.
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FIGURES

FIG. 1. Stroboscopic map of phase space trajectories for the linearly driven well, for $\beta = 0.01$, and several initial conditions.

FIG. 2. The inverse of $\Delta$, the probability to escape from the initial state after one period of the driving force, as a function of $\hbar_{\text{eff}}$, for various values of $\beta$. (a) $\beta=0.1$; (b) $\beta=1.0$; (c) $\beta=2.0$. 