ON FIBERING AND SPLITTING OF 5-MANIFOLDS OVER THE CIRCLE

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Abstract. Our main result is a generalization of Cappell’s 5-dimensional splitting theorem. As an application, we analyze, up to internal $s$-cobordism, the smoothable splitting and fibering problems for certain 5-manifolds mapping to the circle. For example, these maps may have homotopy fibers which are in the class of finite connected sums of certain geometric 4-manifolds. Most of these homotopy fibers have non-vanishing second mod 2 homology and have fundamental groups of exponential growth, which are not known to be tractable by Freedman-Quinn topological surgery. Indeed, our key technique is topological cobordism, which may not be the trace of surgeries.

1. Introduction

The problem of whether or not a continuous mapping $f : M \to S^1$ to the circle from a closed manifold $M$ of dimension $> 5$ is homotopic to a fiber bundle projection was solved originally in the thesis of F. Thomas Farrell (cf. [Far72]). The sole obstruction lies in the Whitehead group of the fundamental group $\pi_1 M$ and has been reformulated in several ways [Sie70, Far71, HR96]. Precious little is known about the 5-dimensional fibering problem. The purpose of this paper is to provide more information using recent advances in rigidity. Our approach here blends together the systematic viewpoint of high-dimensional surgery theory and the more ad-hoc vanishing results known for certain geometric 4-manifolds.

First, we extend some surgery theory. The central theorem of this paper is a generalization of the Cappell–Weinberger theorem [Cap76, Wei87] for splitting compact 5-manifolds along certain incompressible, two-sided 4-submanifolds (Theorem 4.1). Indeed, the development of additional tools for our main splitting theorem motivated the author’s initial investigation of 4-manifolds [Kha07].

Then, we attack the fibering problem. A first application is a version of the Farrell fibering theorem for smooth $s$-block bundles (Definition 5.1, Theorem 5.8) over the circle $S^1$ with homotopy fiber $\mathbb{RP}^4$ (1.1); compare [CS76, FS81, HKT94]. The more central geometric applications are to topological $s$-block bundles (Theorem 5.6). Namely, we allow the fibers to be compact, orientable 4-manifolds whose interiors admit a complete, finite volume metrics of euclidean, real hyperbolic, or complex hyperbolic type (1.2). Moreover, we allow the fiber to be a finite connected sum of orientable surface bundles over surfaces of positive genus, and of $H$-bundles over the circle $S^1$ such that the compact irreducible 3-manifold $H$ either is $S^3$ or $D^3$, or is orientable with non-zero first Betti number (hence Haken), or has complete, finite volume hyperbolic interior (1.4). The hypotheses require smoothness of the total space and the conclusions assert smoothability of the fiber.
1.1. **Examples of fibers.** Our examples are chosen so that Farrell’s fibering obstruction in $K$-theory and Cappell’s splitting obstruction in $L$-theory vanish.

*First family of examples.* These fibers are certain non-orientable, smooth 4-manifolds with fundamental group cyclic of order two [HKT94]. Assume:

**Hypothesis 1.1.** Suppose $Q$ is a non-orientable DIFF 4-manifold of the form

$$Q = Q_0 \# Q_1$$

where:

1. $Q_0 = \# r(S^2 \times S^2)$ for some $r \geq 0$, and
2. $Q_1 = S^2 \times \mathbb{RP}^2$ or $Q_1 = S^2 \times \mathbb{RP}^2$ or $Q_1 = \# S \times n(\mathbb{RP}^4)$ for some $1 \leq n \leq 4$.

*Second family of examples.* These irreducible, possibly non-orientable fibers have torsion-free fundamental groups of exponential growth and have non-vanishing second homotopy group. Assume:

**Hypothesis 1.2.** Suppose $S$ is a compact, connected DIFF 4-manifold such that:

1. $S$ is the total space of a DIFF fiber bundle $S^2 \to S \to \Sigma$, for some compact, connected, possibly non-orientable 2-manifold $\Sigma$ of positive genus, or
2. $S$ is the total space of a DIFF fiber bundle $H \to S \to S^1$, for some closed, connected, hyperbolic 3-manifold $H$, or
3. the interior $S - \partial S$ admits a complete, finite volume metric of euclidean, real hyperbolic, or complex hyperbolic type.

Moreover, assume $H_1(S; \mathbb{Z})$ is 2-torsionfree if $S$ is non-orientable. Furthermore, in the fiber bundle $S \times_h S^3$ (resp. $S \times_\alpha S^1$) considered for types (2) and (3) in this section, assume $h : S \to S$ (resp. $\alpha$) is homotopic rel $\partial S$ to an isometry of $S - \partial S$.

**Remark 1.3.** According to [Hil02, Lemma 5.9], the isomorphism classes of fiber bundles $S^2 \to S \to \Sigma$ in type (1) are in bijective correspondence with the product $H^1(\Sigma; \mathbb{Z}_2) \times H^2(\Sigma; \mathbb{Z}_2)$. The orientable $S^2$-bundles over $\Sigma$ are classified by the second factor. The isomorphism classes of fiber bundles $H \to S \to S^1$ in type (2) are in bijective correspondence with $\pi_0(\text{Isom } H)$.

*Third family of examples.* These reducible, orientable fibers have torsion-free fundamental groups of exponential growth and have vanishing second homotopy groups. A simple example of such a fiber is $F = \# n(S^3 \times S^1)$, whose fundamental group $\pi_1(F)$ is the free group of rank $n$. Assume:

**Hypothesis 1.4.** Suppose $F$ is an orientable DIFF 4-manifold of the form

$$F = F_1 \# \cdots \# F_n$$

for some $n > 0$, under the following conditions on the compact, connected, orientable 4-manifolds $F_i$. Assume:

1. $F_i$ is the total space of a DIFF fiber bundle $H_i \to F_i \to S^1$, for some compact, connected, orientable 3-manifold $H_i$ such that:
   a. $H_i$ is $S^3$ or $D^3$, or
   b. $H_i$ is irreducible with non-zero first Betti number, or
2. $F_i$ is the total space of a DIFF fiber bundle $\Sigma^f_i \to F_i \to \Sigma^b_i$, for some compact, connected, orientable 2-manifolds $\Sigma^f_i$ and $\Sigma^b_i$ of positive genus.

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1 This hypothesis is required since Mostow rigidity fails for product geometries: the $\mathbb{E}^4$-manifold $T^2 \times T^2$ has monodromies made from non-conformal elements of $\pi_0 \text{Homeo}(T^2) = \text{PSL}_2(\mathbb{Z})$. 
1.2. Main results. The first splitting theorem is a specialization of the general splitting theorem (Theorem 4.1) to the mapping torus $X \times_h S^1$ of a homotopy self-equivalence $h : X \to X$ for certain classes of smooth 4-manifolds $X$.

Theorem 5.3. Let $X$ be any of the 4-manifolds $Q, S, F$ defined in (1.1, 1.2, 1.4). Let $h : X \to X$ be a homotopy equivalence which restricts to a diffeomorphism $\partial h : \partial X \to \partial X$. Suppose $M$ is a compact DIFF 5-manifold and $g : M \to X \times_h S^1$ is a homotopy equivalence which restricts to a diffeomorphism $\partial g : \partial M \to \partial X \times_{\partial h} S^1$.

Then $g$ is homotopic to a map $g'$ which restricts to a simple homotopy equivalence $g' : X' \to X$ such that the TOP inverse image $X' := (g')^{-1}(X)$ is homeomorphic to $X$ and the exterior $M'$ of $X'$ in $M$ is a smoothable TOP self $s$-cobordism of $X$.

In other words, cutting $M$ along the bicollared, smoothable TOP 4-submanifold $X' := (g')^{-1}(X) \approx X$ yields a smoothable TOP $s$-cobordism $(M', X, X)$ and a simple homotopy equivalence $(g'_s : g_0 : g'_1) : (M' : X, X) \to X \times (\Delta^1; 0, 1)$ of manifold triads such that $g'_1 = \alpha \circ g'_0$. Be aware that the existence of a smooth structure on $X'$ does not imply that $X'$ is a DIFF submanifold of $M$.

The second splitting theorem connects homotopy structures on mapping tori to smoothable $s$-cobordisms, homotopy self-equivalences, and the smoothing invariant.

Theorem 5.5. Let $X$ be any of the 4-manifolds $Q, S, F$ defined in (1.1, 1.2, 1.4). Let $\alpha : X \to X$ be a diffeomorphism. Then there is an exact sequence of based sets:

$$\pi_1^s(TOP^+(X), \overline{G}(X)) \xrightarrow{\cup} S^1_{TOP}(X \times_\alpha S^1) \xrightarrow{\text{ks}} F_2 \oplus H_1(X; F_2).$$

Our fibering theorem is proven using a key strategy of Tom Farrell [Par71]. If the smooth 4-manifold $X$ is closed and simply-connected, the analogous theorem was proven by J. Shaneson [Sha71 Thm. 5.1]. We do not assume $\partial X$ is connected.

Theorem 5.6. Let $X$ be any of the 4-manifolds $Q, S, F$ defined in (1.1, 1.2, 1.4). Let $M$ be a DIFF 5-manifold, and let $f : M \to S^1$ be a continuous map. Suppose $\partial X \to \partial M \to S^1$ is a DIFF fiber bundle and the homotopy equivalence $\partial X \to \text{hofiber}(\partial f)$ extends to a homotopy equivalence $X \to \text{hofiber}(f)$. Then $f : M \to S^1$ is homotopic rel $\partial M$ to the projection of a smoothable TOP $s$-block bundle with fiber $X$.

Remark. Let $X$ be an aspherical, compact, orientable DIFF 4-manifold with fundamental group $\pi$. Suppose the non-connective $L$-theory assembly map $H_n(\pi; \mathbb{L}^h) \to L_n^h(\mathbb{Z}[\pi])$ is an isomorphism for $n = 4, 5$. Then the general splitting and fibering theorems (4.1, 4.6) hold for $X$, with the inclusion of the standard high-dimensional algebraic $K$- and $L$-theory obstructions.

1.3. Techniques. Our methods employ geometric topology: topological transversality in all dimensions (Freedman–Quinn [FQ99]) and the prototype of a nilpotent normal cobordism construction for smooth 5-manifolds (Cappell [CS71 Cap76]). Our hypotheses are algebraic-topological in nature and come from the surgery characteristic class formulas of Sullivan–Wall [Wa99] and from the assembly map components of Taylor–Williams [TW79]. For the main application, the difficulty is showing that vanishing of algebraic $K$- and $L$-theory obstructions is sufficient for a solution to the topological fibering problem as an $s$-block bundle over the circle.

The reader should be aware that the topological transversality used in Section 2 produces 5-dimensional TOP normal bordisms $W \to X \times \Delta^1$ which may not be
smoothable, although \( \partial W = \partial_- W \cup \partial_+ W \) is smoothable. In particular, \( W \) may not admit a TOP handlebody structure relative to \( \partial_- W \). Hence \( W \) may not be the trace of surgeries on topologically embedded 2-spheres in \( X \). Therefore, \( W \) may not be produced by Freedman–Quinn surgery theory, which is developed only for fundamental groups \( \pi_1(X) \) of class \( \text{SA} \), containing subexponential growth [KQ00].

2. Five-Dimensional Assembly on 4-Manifolds

Let \((X, \partial X)\) be a based, compact, connected, TOP 4-manifold with fundamental group \( \pi = \pi_1(X) \) and orientation character \( \omega = w_1(X) : \pi \to \mathbb{Z}^\times \). Recall, for any \( \alpha \in \pi \) and \( \beta \in \pi_2(X) \), that there is a Whitehead product \([\alpha, \beta] \in \pi_2(X)\) which vanishes if and only if the loop \( \alpha \) acts trivially on \( \beta \). The \( \pi \)-coinvariants are the abelian group quotient \( \pi_2(X)_\pi := \pi_2(X)/\langle [\alpha, \beta] | \alpha \in \pi, \beta \in \pi_2(X) \rangle \).

**Hypothesis 2.1.** Suppose that the following homomorphism is surjective:

\[
(\iota, \kappa_3) : H_1(\pi; \mathbb{Z}^\omega) \oplus H_3(\pi; \mathbb{Z}_2) \longrightarrow L^h_5(\mathbb{Z}[\pi]^{\omega})
\]

and that the following induced homomorphism is injective:

\[
\text{Hurewicz} : (\pi_2(X) \otimes \mathbb{Z}_2)_\pi \longrightarrow H_2(X; \mathbb{Z}_2).
\]

**Theorem 2.2.** Assume Hypothesis \([2.1]\). Then the following surgery obstruction map is surjective:

\[
(2.2.1) \quad \sigma_* : N^\text{TOP}_5(X \times \Delta^1) \longrightarrow L^h_5(\mathbb{Z}[\pi]^{\omega}).
\]

Following Sylvain Cappell’s work on the Novikov conjecture, Jonathan Hillman obtained the same conclusion under different, group-theoretic hypotheses for a square-root closed graph of certain class of groups [HI02, Lem. 6.9].

**Proof.** There is a commutative diagram

\[
\begin{array}{ccc}
N^\text{TOP}_5(X \times \Delta^1) & \xrightarrow{\sigma_*} & L^h_5(\mathbb{Z}[\pi]^{\omega}) \\
\cap [X]_\cdot & \\nH_5(X; G/\text{TOP}^{\omega}) & \xrightarrow{u_*} & H_5(\pi; G/\text{TOP}^{\omega})
\end{array}
\]

Since \( A_\pi(1) = I_1 + \kappa_3 \) is surjective and \( u_1 : H_1(X; \mathbb{Z}^\omega) \to H_1(\pi; \mathbb{Z}^\omega) \) is an isomorphism, it suffices to show that \( u_3 : H_3(X; \mathbb{Z}_2) \to H_3(\pi; \mathbb{Z}_2) \) is surjective.

Consider the Leray–Serre spectral sequence, with \( \pi \)-twisted coefficients, of the fibration \( \tilde{X} \to X \xrightarrow{u} B\pi \), where \( \tilde{X} \) is the universal cover of \( X \). Then the map \( u_3 \) is an edge homomorphism with image subgroup \( E^\infty_{3,0} \). Note

\[
E^\infty_{3,0} = \text{Ker} \left( d^3_{3,0} : H_3(B\pi; \mathbb{Z}_2) \longrightarrow (\pi_3(X) \otimes \mathbb{Z}_2)_\pi \right).
\]

There is an exact sequence involving the associated graded groups \( E^\infty_{0,2} \) and \( E^\infty_{2,0} \) and inducing the classical Hopf sequence:

\[
0 \to \text{Cok}(d^3_{3,0}) \xrightarrow{\text{Hurewicz}} H_2(X; \mathbb{Z}_2) \xrightarrow{u_*} H_2(B\pi; \mathbb{Z}_2) \to 0.
\]

It follows from the second part of the hypothesis that the transgression \( \partial = d^3_{3,0} \) is zero. Therefore \( \text{Im}(u_3) = E^\infty_{3,0} = H_3(\pi; \mathbb{Z}_2) \), hence \( \sigma_* \) is surjective. 

\( \square \)
Some families of reducible examples of the theorem are obtained as finite connected sums of certain compact, aspherical 4-manifolds $X_i$ which are constructed from non-positively curved manifolds. Recall that the interior of any compact surface $\Sigma$ of positive genus has the structure of a complete, finite volume, euclidean or hyperbolic 2-manifold; hence $\Sigma$ is aspherical. The following corollary gives a rich source of examples, including $X = \#n(S^3 \times S^1)$, whose fundamental group is free.

**Corollary 2.3.** Suppose $X$ is a $\text{TOP}$ 4-manifold of the form

$$X = X_1 \# \cdots \# X_n$$

for some $n > 0$ and some compact, connected 4-manifolds $X_i$ with (torsion-free) fundamental groups $\Lambda_i$ such that:

1. $X_i$ is the total space of a fiber bundle $\Sigma^f_i \to X_i \to \Sigma_b^i$, for some compact, connected 2-manifolds $\Sigma^f_i$ and $\Sigma_b^i$ of positive genus,

2. $X_i$ is the total space of a fiber bundle $H_i \to X_i \to S^1$, for some compact, connected, irreducible 3-manifold $H_i$ such that:
   (a) $H_i$ is $S^3$ or $D^3$,
   (b) $H_i$ is orientable with non-zero first Betti number (i.e. $H^1(H_i;\mathbb{Z}) \neq 0$, e.g. the boundary $\partial H_i$ is non-empty), or
   (c) the interior $H_i - \partial H_i$ admits a complete, finite volume metric of hyperbolic type.

Then the topological 5-dimensional surgery obstruction map (2.2.1) is surjective. Moreover, $X_i$ only needs to have type (1), (2), or (3) up to homotopy equivalences which respect orientation characters.

**Proof.** Let $\Lambda_i := \pi_1(X_i)$ be the fundamental group of $X_i$ with orientation character $\omega_i : \Lambda_i \to \mathbb{Z}^\times$. Consider the connective assembly map

$$A_{\Lambda_i}(1) = (\iota \ast \kappa) : H_5(B\Lambda_i;G/\text{TOP}^{\omega_i}) \to L^h_5(\mathbb{Z}[\Lambda_i]^{\omega_i}).$$

In order to verify Hypothesis 2.1 and apply Theorem 2.2 it suffices to show that:

(i) $\pi_2(X_i) = 0$,

(ii) $H_d(B\Lambda_i;\mathbb{Z}) = 0$ for all $d > 4$, and

(iii) the non-connective assembly map is an isomorphism:

$$A_{\Lambda_i} : H_5(B\Lambda_i;\mathbb{L}^{\omega_i}) \to L^h_5(\mathbb{Z}[\Lambda_i]^{\omega_i}).$$

Then $A_{\Lambda_i}(1)$ is an isomorphism. So, since the trivial group 1 is square-root closed in the torsion-free groups $\Lambda_i$, the UNil-groups associated to the free product $\pi = \ast_{i=1}^n \Lambda_i$ vanish, by [Cap74b, Corollary 4], which was proven in [Cap76, Lemmas II.7,8,9]. Therefore, by the Mayer–Vietoris sequence in $L$-theory [Cap74b, Thm. 5(ii)], Proposition 5.9 for $h$-decorations, the five-lemma, and induction on $n$, we obtain that $A_\pi(1)$ is an isomorphism. Moreover, by the Mayer–Vietoris sequence in singular homology and the Hurewicz theorem applied to the universal cover $\tilde{X}$, we obtain that $\pi_2(X) = 0$.

There are three types of connected summands $X_i$.

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2 Positive genus: implies torsion-free; each surface $\Sigma^f_i$ and $\Sigma^h_i$ is a finite connected sum of at least either one torus $T^2$ or two real projective planes $\mathbb{R}P^2$, with arbitrary punctures. The first non-orientable example is the Klein bottle $Kl = \mathbb{R}P^2 \# \mathbb{R}P^2$. 

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Type 1. Since $X_i - \partial X_i$ is covered by $\mathbb{H}^4$ or $\mathbb{C} \mathbb{H}^2$, we obtain $X_i$ is aspherical. That is, the compact 4-manifold $X_i$ is model for $BA_i$. Then (i) and (ii) are satisfied. Since $X - \partial X_i$ is complete, homogeneous, and has non-positive sectional curvatures, by [FJ98 Proposition 0.10], condition (iii) is satisfied.

Type 2. Since the surfaces $\Sigma^i_1$ and $\Sigma^i_2$ are aspherical, by the homotopy fibration sequence, we obtain that the compact 4-manifold $X_i$ is aspherical. Then (i) and (ii) are satisfied. By a result of J. Hillman [Hil91 Lemma 6] for closed, aspherical surface bundles over surfaces, condition (iii) is satisfied. Indeed, the Mayer–Vietoris argument extends to compact, aspherical surfaces with boundary: each circle $C_j$ in the connected-decomposition of the aspherical surface $\Sigma^b_k = F_1 \# \cdots \# F_r$ generates an indivisible element in the free fundamental group of the many-punctured torus or Klein bottle $F_k$, hence each inclusion $\pi_1(C_j) \to \pi_1(F_k)$ of fundamental groups is square-root closed (see [CHS06 Thm. 2.4] for detail).

Type 3. There are three types of fibers $H_i$.

Type 3a. Conditions (i)–(iii) are immediately satisfied.

Type 3b. Since $H_i$ is a compact, connected, irreducible, orientable 3-manifold and $\pi_1(H_i)$ is infinite, using the Sphere Theorem of Papakyriakopoulos and the Hurewicz theorem, it can be shown that $H_i$ is aspherical. Then $X_i$ is aspherical, so conditions (i) and (ii) are satisfied. Since $H_i$ is irreducible and $H_i \neq D^3$, no connected component of $\partial H_i$ is a 2-sphere. If $\partial H_i$ is non-empty, then it can be shown that $H_i$ is Haken [Hem76 Lem. 6.8]. So, by theorems of S. Roushon, the non-connective assembly map $A_{\pi_1(H_i)}$ is an isomorphism in dimensions 4 and 5: if $\partial H$ is non-empty, this follows from [Rou00b Theorem 1.1(1)], and if $\partial H$ is empty, this follows from [Rou02 Theorem 1.2]. Therefore, by the Ranicki–Shaneson sequence [Ran73 Thm. 5.2], Proposition 5.9 for $h$-decorations, and the five-lemma, we obtain that condition (iii) is satisfied.

Type 3c. Since $\mathbb{H}^3$ is the universal cover of $H_i - \partial H_i$, the interior $H_i - \partial H_i$ is aspherical. But, since $\partial H_i$ has a collar implies that $H_i - \partial H_i \to H_i$ is a homotopy equivalence, we obtain that $H_i$ is also aspherical. Then $X_i$ is aspherical, so conditions (i) and (ii) are satisfied. Since $\mathbb{H}^3$ isometrically covers $H_i - \partial H_i$, by a result of Farrell and Jones [FJ98 Prop. 0.10], the non-connective assembly map $A_{\pi_1(H_i)}$ is an isomorphism in dimensions 4 and 5. Therefore, by the Ranicki–Shaneson sequence [Ran73 Thm. 5.2], Proposition 5.9 for $h$-decorations, and the five-lemma, we obtain that condition (iii) is satisfied.

Here is a family of non-aspherical examples $X$ of the theorem.

**Corollary 2.4.** Suppose $X$ is a compact TOP 4-manifold which is homotopy equivalent to the total space of a fiber bundle $S^2 \to E \to \Sigma$, for some compact, connected 2-manifold $\Sigma$ of positive genus. Then the topological 5-dimensional surgery obstruction map (2.2.1) is surjective.

**Proof.** By [Hil02 Theorem 6.16], $X$ is $s$-cobordant to $E$. Hence there is an induced (simple) homotopy equivalence $X \to E$ which respects orientation characters. The same methods of Corollary (2.3.2) show that $(i, \kappa_3)$ is surjective. Note that $\pi_1(E) = \pi_1(\Sigma)$ may not act trivially on $\pi_2(E) = \pi_2(S^2) = \mathbb{Z}$ but does acts trivially on $\pi_2(E) \otimes \mathbb{Z}_2 = \mathbb{Z}_2$. An elementary argument with the Leray–Serre spectral sequence shows that $H_2(E; \mathbb{Z}_2) = H_2(S^2; \mathbb{Z}_2) \oplus H_2(\Sigma; \mathbb{Z}_2)$. Therefore Hurewicz is injective, and we are done by Theorem 2.2. \qed
The difference between DIFF and TOP for $\sigma_*$ is displayed in [Kha07, Prop. 2.1]. Later, we shall refer to a hypothesis introduced by Cappell [Cap76, Thm. 5, Rmk.].

**Hypothesis 2.5.** Suppose that the following map is surjective:

$$\sigma_* : \mathcal{N}_{\text{DIFF}}(X \times \Delta^1) \to L^h_5(Z[\pi]^\omega).$$

**Remark 2.6.** Suppose $X$ is a DIFF 4-manifold and $\pi^\omega = (C_2)^-$. By [Wal99, Theorem 13A.1], the following surgery obstruction map is automatically surjective:

$$\sigma_* : \mathcal{N}_{\text{DIFF}}(X \times \Delta^1) \to L^h_5(Z[C_2]^-) = 0.$$  

Topological surjectivity fails for a connected sum $X \# X'$ of such manifolds: the Mayer–Vietoris sequence [Cap74b] shows that the cokernel is $\text{UNil}^h_5(Z; Z^-, Z^-) \cong \text{UNil}^h_3(Z; Z, Z)$, and this abelian group was shown to be infinitely generated [CR05].

3. **Exactness at 4-dimensional normal invariants**

For the convenience of the reader, we first recall the relevant hypotheses from the precursor [Kha07, §3]. Let $(X, \partial X)$ be a based, compact, connected TOP 4-manifold with fundamental group $\pi$ and orientation character $\omega : \pi \to \mathbb{Z}^\times$. Let $u : X \to B\pi$ be a based, continuous map that induces an isomorphism on fundamental groups. Denote the induced homomorphism

$$u_2 : H_2(X; \mathbb{Z}_2) \to H_2(B\pi; \mathbb{Z}_2).$$

Recall that $X$ satisfies Poincaré duality with a unique mod 2 orientation class $[X] \in H_4(X, \partial X; \mathbb{Z}_2)$. The second Wu class $v_2(X) \in H^2(X; \mathbb{Z}_2)$ is a homomorphism

$$v_2(X) : H_2(X; \mathbb{Z}_2) \to \mathbb{Z}_2$$

uniquely determined, for all cohomology classes $a \in H^2(X, \partial X; \mathbb{Z}_2)$, by the formula

$$(v_2(X), a \cap [X]) = \langle a \cup a, [X] \rangle.$$

Consider three cases for the orientation character $\omega$ below. The homomorphism

$$\kappa_2 : H_2(B\pi; \mathbb{Z}_2) \to L^h_4(Z[\pi]^\omega)$$

is the 2-dimensional component of the $L$-theory assembly map.

**Hypothesis 3.1.** Let $X$ be orientable. Suppose that $\kappa_2$ is injective on the subgroup $u_2(\text{Ker} v_2(X))$.

**Hypothesis 3.2.** Let $X$ be non-orientable such that $\pi$ contains an orientation-reversing element of finite order, and if $\text{CAT} = \text{DIFF}$, then suppose that orientation-reversing element has order two. Suppose that $\kappa_2$ is injective on all $H_2(B\pi; \mathbb{Z}_2)$, and suppose that $\text{Ker}(u_2) \subseteq \text{Ker}(v_2)$.

**Hypothesis 3.3.** Let $X$ be non-orientable such that there exists an epimorphism $\pi^\omega \to \mathbb{Z}^\times$. Suppose that $\kappa_2$ is injective on the subgroup $u_2(\text{Ker} v_2(X))$.

Next, we recall the relevant results from [Kha07, §4] used frequently in the later proofs in this paper. The subcategory $\text{TOP}_0 \subseteq \text{TOP}$ consists of those maps $f : M \to X$ with Kirby–Siebenmann stable smoothing invariant $\text{ks}(f) := \text{ks}(M) - \text{ks}(X) = 0 \in \mathbb{Z}_2$. All structure sets and normal invariants below are relative to a diffeomorphism on $\partial X$.

**Theorem 3.4.** Let $(X, \partial X)$ be a based, compact, connected, CAT 4–manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(X) : \pi \to \mathbb{Z}^\times$. 

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Suppose Hypothesis 3.1 or 3.2. Then the surgery sequence of based sets is exact at the smooth normal invariants:

\[ S_{\text{DIFF}}(X) \xrightarrow{\eta} N_{\text{DIFF}}(X) \xrightarrow{\sigma^*} L^h_4(\mathbb{Z}[\pi]^\omega). \]

(2) Suppose Hypothesis 3.1 or 3.2 or 3.3. Then the surgery sequence of based sets is exact at the stably smoothable normal invariants:

\[ S_{\text{TOP0}}(X) \xrightarrow{\eta} N_{\text{TOP0}}(X) \xrightarrow{\sigma^*} L^h_4(\mathbb{Z}[\pi]^\omega). \]

Corollary 3.5. Let \( \pi \) be a free product of groups of the form

\[ \pi = \star_{i=1}^n \Lambda_i \]

for some \( n > 0 \), where each \( \Lambda_i \) is a torsion-free lattice in either \( \text{Isom}(\mathbb{E}^{m_i}) \) or \( \text{Isom}(\mathbb{H}^{m_i}) \) or \( \text{Isom}(\mathbb{C}H^{m_i}) \) for some \( m_i > 0 \). Suppose the orientation character \( \omega \) is trivial. Then the surgery sequences (3.4.1) and (3.4.2) are exact.

Corollary 3.6. Suppose \( X \) is a DIFF 4–manifold of the form

\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]

for some \( n > 0 \) and \( r \geq 0 \), and each summand \( X_i \) is either \( S^2 \times \mathbb{R}P^2 \) or \( S^2 \times \mathbb{R}P^2 \) or \( \#_1 n(\mathbb{R}P^4) \) for some \( 1 \leq n \leq 4 \). Then the surgery sequences (3.4.1) and (3.4.2) are exact.

Corollary 3.7. Suppose \( X \) is a TOP 4–manifold of the form

\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]

for some \( n > 0 \) and \( r \geq 0 \), and each summand \( X_i \) is the total space of a fiber bundle

\[ H_i \longrightarrow X_i \longrightarrow S^1. \]

Here, we suppose \( H_i \) is a compact, connected 3–manifold such that:

1. \( H_i \) is \( S^3 \) or \( D^3 \), or
2. \( H_i \) is irreducible with non-zero first Betti number.

Moreover, if \( H_i \) is non-orientable, we assume that the quotient group \( H_1(H_i; \mathbb{Z})_{(\alpha_i)} \) of coinvariants is 2-torsionfree, where \( \alpha_i : H_i \rightarrow H_i \) is the monodromy homeomorphism. Then the surgery sequence (3.4.2) is exact.

Corollary 3.8. Suppose \( X \) is a TOP 4–manifold of the form

\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]

for some \( n > 0 \) and \( r \geq 0 \), and each summand \( X_i \) is the total space of a fiber bundle

\[ \Sigma^f_i \longrightarrow X_i \longrightarrow \Sigma^b_i. \]

Here, we suppose the fiber and base are compact, connected 2–manifolds, \( \Sigma^f_i \neq \mathbb{R}P^2 \), and \( \Sigma^b_i \) has positive genus. Moreover, if \( X_i \) is non-orientable, we assume that the fiber \( \Sigma^f_i \) is orientable and that the monodromy action of \( \pi_1(\Sigma^b_i) \) of the base preserves any orientation on the fiber. Then the surgery sequence (3.4.2) is exact.
4. Splitting of 5-manifolds

We generalize Cappell’s 5-dimensional splitting theorem [Cap76, Thm. 5, Remark], using the homological hypotheses developed in Sections 2—3. Our proof incorporates the possible non-vanishing of UNil. The DIFF and TOP cases are distinguished, and the results of this section are applied to the fibering problem in Section 5. The stable surgery version of the splitting theorem can be found in [CS71]. However, the stable splitting of 5-manifolds is not pursued here, since connecting sum a single fiber with \( S^2 \times S^2 \) destroys the fibering property over \( S^1 \).

Let \((Y, \partial Y)\) be a based, compact, connected CAT 5-manifold. Let \((Y_0, \partial Y_0)\) is a based, compact, connected CAT 4-manifold. Suppose \( Y_0 \) is an incompressible, two-sided submanifold of \( Y \). That is, the induced homomorphism \( \pi_1(Y_0) \to \pi_1(Y) \) is injective, and there is a separating decomposition

\[
Y = Y_– \cup_{Y_0} Y_+ \quad \text{with} \quad \partial Y = \partial Y_– \cup_{\partial Y_0} \partial Y_+
\]
or, respectively, a non-separating decomposition

\[
Y = \cup_{Y_0} Y_\infty \quad \text{with} \quad \partial Y = \cup_{\partial Y_0} \partial Y_\infty.
\]

The Seifert–van Kampen theorem identifies \( \pi_1(Y) = \Pi = \Pi_– \ast \Pi_0 \Pi_+ \) as the corresponding injective, amalgamated free product of fundamental groups, or, respectively,

\[
\pi_1(Y) = \Pi = \ast \Pi_0 \Pi_\infty
\]
as the corresponding injective, HNN-extension\(^3\) of fundamental groups.

A homotopy equivalence \( g \) to \( Y \) is CAT splittable along \( Y_0 \) if \( g \) is homotopic, relative to a CAT isomorphism \( \partial g \), to a union \( g_– \cup_{g_0} g_+ \) (resp. \( \cup_{g_0} g_\infty \)) of homotopy equivalences from compact CAT manifolds to \( Y_–, Y_0, Y_+ \) (resp. \( Y_0, Y_\infty \)) [Cap74a]. Under certain conditions, we show that the vanishing of high-dimensional obstructions in \( \text{Nil}_0 \) and \( \text{UNil}_0^* \) are sufficient for splitting. These two obstructions were formulated by Friedhelm Waldhausen (1960’s) and Sylvain Cappell (1970’s).

**Theorem 4.1.** Let \((Y, \partial Y)\) be a finite, simple Poincaré pair of formal dimension 5 [Wald99, §2]. Suppose \( \partial Y \) and \( Y_0 \) are compact DIFF 4-manifolds such that \((Y_0, \partial Y_0)\) is a connected, incompressible, two-sided Poincaré subpair of \((Y, \partial Y)\) with tubular neighborhood \( Y_0 \times [-1, 1] \). If CAT = DIFF, assume \( Y_0 \) satisfies Hypothesis \( \mathcal{H7} \) or \( \mathcal{H2} \) and satisfies Hypothesis \( \mathcal{H2} \). If CAT = TOP, assume \( Y_0 \) satisfies Hypothesis \( \mathcal{H1} \) or \( \mathcal{H2} \) or \( \mathcal{H3} \) and satisfies Hypothesis \( \mathcal{H1} \).

Suppose \( g : (W, \partial W) \to (Y, \partial Y) \) is a homotopy equivalence for some compact DIFF 5-manifold \( W \) such that the restriction \( \partial g : \partial W \to \partial Y \) is a diffeomorphism. Then \( g \) is CAT splittable along \( Y_0 \) if and only if

1. the cellular splitting obstruction, given by the image of the Whitehead torsion \( \tau(g) \in \text{Wh}_1(\Pi) \), vanishes:

\[
\text{split}_K(g; Y_0) \in \text{Wh}_0(\Pi_0) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\Pi_0]; \mathbb{Z}[\Pi_– - \Pi_0], \mathbb{Z}[\Pi_+ - \Pi_0])
\]

\(^3\)In the non-separating case, we write \( \Pi_–, \Pi_+ \) as the two monomorphic images of \( \Pi_0 \) in \( \Pi_\infty \).
or, respectively,
\[
\text{split}_K(g; Y_0) \in \text{Wh}_0(\Pi_0) \oplus \\tilde{\text{Nil}}_0(\mathbb{Z}[\Pi_0]; \mathbb{Z}[\Pi_\infty - \Pi_0^+], \mathbb{Z}[\Pi_\infty - \Pi_0^+], -\mathbb{Z}[\Pi_\infty]++)
\]
and subsequently
\[
(2) \text{ the manifold splitting obstruction, given by the algebraic position of discs in the fundamental subdomains of the } \Pi_0\text{-cover, vanishes:}
\]
\[
\text{split}_L(g; Y_0) \in \text{UNil}_0(\mathbb{Z}[\Pi_0]; \mathbb{Z}[\Pi_\infty - \Pi_0^+], \mathbb{Z}[\Pi_\infty - \Pi_0^+])
\]
or, respectively,
\[
\text{split}_L(g; Y_0) \in \text{UNil}_0^s(\mathbb{Z}[\Pi_0]; \mathbb{Z}[\Pi_\infty - \Pi_0^+], \mathbb{Z}[\Pi_\infty - \Pi_0^+]).
\]
Furthermore, if \(g\) is CAT splittable along \(Y_0\), then \(g\) is homotopic rel \(\partial W\) to a split homotopy equivalence \(g': W \to Y\) such that the CAT inverse image \((g')^{-1}(Y_0)\) is CAT isomorphic to \(Y_0\).

Our theorem mildly generalizes \cite[Theorem 5, Remark]{Cap76}, which included: if \(\Pi_0\) is a finite group of odd order, then \(H_2(\Pi_0; \mathbb{Z}_2) = 0\) and \(L_5^0(\mathbb{Z}[\Pi_0]) = 0\).

**Corollary 4.2** (Cappell). Suppose \(g: W \to Y\) is a homotopy equivalence of closed DIFF 5-manifolds. Assume:

1. \(Y_0\) is orientable,
2. \(H_2(\Pi_0; \mathbb{Z}_2) = 0\),
3. \(\Pi_0\) is square-root closed\(^4\) in \(\Pi\), and
4. the following surgery obstruction map is surjective (cf. Hyp. 2.5):
   \[
   \sigma_* : N_{\text{DIFF}}(X \times \Delta^1) \longrightarrow L_5^0(\mathbb{Z}[\Pi_0]).
   \]

Then \(g\) is DIFF splittable along \(Y_0\) if and only if the above image \(\text{split}_K(g; Y_0)\) of the Whitehead torsion \(\tau(g) \in \text{Wh}_1(\Pi)\) vanishes. \(\square\)

Define a decoration subgroup \(B \subseteq \text{Wh}_1(\Pi)\) as the image of \(\text{Wh}_1(\Pi_-) \oplus \text{Wh}_1(\Pi_+)\), respectively \(\text{Wh}_1(\Pi_\infty)\), under the homomorphism induced by inclusion. Recall that the **structure set** \(\mathcal{S}^B_{\text{CAT}}(Y)\) is defined as the set of equivalence classes of homotopy equivalences \(g : (W, \partial W) \to (Y, \partial Y)\) such that \(W\) is a compact CAT manifold and \(\partial g : \partial W \to \partial Y\) is a CAT isomorphism and \(g\) has Whitehead torsion \(\tau(g) \in B\), under the equivalence relation \(g \sim g'\) if there exists a CAT isomorphism \(h : W \to W'\) such that \(g' \circ h\) is homotopic to \(g\). The **split structure set** \(\mathcal{S}^\text{split}_{\text{CAT}}(Y; Y_0)\) is defined as the subset of \(\mathcal{S}^B_{\text{CAT}}(Y)\) whose elements are represented by homotopy equivalences CAT splittable along \(Y_0\). The abelian group \(\text{UNil}_0^s\) depends only on the fundamental groups \(\Pi_-, \Pi_0, \Pi_+\) (resp. \(\Pi_0, \Pi_\infty\)) with orientation character \(\omega\). \(\text{UNil}_0^s\) is algebraically defined and has zero decoration in \(\tilde{\text{Nil}}_0\) \cite{Cap74a, Cap76}.

**Definition 4.3.** Let \((Y, \partial Y)\) be a compact DIFF manifold. Define the **smoothable structure set** \(\mathcal{S}_{\text{TOP}^+}(Y)\) as the image of \(\mathcal{S}_{\text{DIFF}}(Y)\) under the forgetful map to \(\mathcal{S}_{\text{TOP}}(Y)\). That is, \(\mathcal{S}_{\text{TOP}^+}(Y)\) is the subset of \(\mathcal{S}_{\text{TOP}}(Y)\) consisting of the elements representable by homotopy equivalences \(g : (W, \partial W) \to (Y, \partial Y)\) such that \(W\) admits a DIFF structure extending the DIFF structure on \(\partial W\) induced by \(\partial g\).

A more succinct statement illuminates the method of proof in higher dimensions: Sylvain Cappell’s “nilpotent normal cobordism construction” \cite{Cap74a, Cap76}.

\(^4\)Square-root closed: if \(g \in \Pi\), then \(g^2 \in \Pi_0\) implies \(g \in \Pi_0\).
Theorem 4.4. Let \((Y, \partial Y)\) be a finite, simple Poincaré pair of formal dimension 5 [Wal99 §2]. Suppose \(\partial Y\) and \(Y_0\) are compact \(\text{DIFF}\) 4-manifolds such that \((Y_0, \partial Y_0)\) is a connected, incompressible, two-sided Poincaré subpair of \((Y, \partial Y)\) with tubular neighborhood \(Y_0 \times [-1, 1]\).

1. Assume \(Y_0\) satisfies Hypothesis 3.1 or 3.2 and satisfies Hypothesis 2.2. Then there is a bijection
   \[
   \text{ncc}^s : \text{S}^B_{\text{DIFF}}(Y) \longrightarrow \text{S}^\text{split}_{\text{DIFF}}(Y; Y_0) \times \text{UNil}_6^s
   \]
   such that composition with projection onto the first factor is a subset retraction, and composition with projection onto the second factor is the manifold splitting obstruction \(\text{split}_L\). Furthermore, \(g\) and \(\text{ncc}^s(g)\) have equal image in \(\mathcal{N}_{\text{DIFF}}(Y)\).

2. Assume \(Y_0\) satisfies Hypothesis 4.1 or 4.2 or 4.3 and satisfies Hypothesis 2.1. Then there is an injection
   \[
   \text{ncc}^s_+ : \text{S}^B_{\text{TOP}^+}(Y) \longrightarrow \text{S}^\text{split}_{\text{TOP}}(Y; Y_0) \times \text{UNil}_6^s
   \]
   such that composition with projection onto the first factor restricts to a subset inclusion \(\text{S}^\text{split}_{\text{TOP}^+}(Y; Y_0) \subseteq \text{S}^\text{split}_{\text{TOP}}(Y; Y_0)\), and composition with projection onto the second factor is the manifold splitting obstruction \(\text{split}_L\). Furthermore, \(g\) and \(\text{ncc}^s(g)\) have equal image in \(\mathcal{N}_{\text{TOP}}(Y)\).

4.1. Proof by cobordism. We simply extend Cappell’s modification [Cap76 Chapter V] of the Cappell–Shaneson proof [CSS71 Theorems 4.1, 5.1] of 5-dimensional splitting as to include the non-vanishing of \(\text{UNil}_6^s\). Our homological conditions eschew the performance of surgery on the 4-manifold \(Y_0\). Examples are given in §4.

Remark 4.5. Friedhelm Waldhausen had shown that \(\tilde{\text{Nil}}_0\) is a summand of \(\text{Wh}_1(\Pi)\) and that there is an exact sequence of abelian groups [Wal78]:
   \[
   \text{Wh}_1(\Pi_0) \to \text{Wh}_1(\Pi_-) \oplus \text{Wh}_1(\Pi_+) \to \text{Wh}_1(\Pi)/\tilde{\text{Nil}}_0 \to \text{Wh}_0(\Pi_+),
   \]
   or, respectively,
   \[
   \text{Wh}_1(\Pi_0) \to \text{Wh}_1(\Pi_+) \to \text{Wh}_1(\Pi)/\tilde{\text{Nil}}_0 \to \text{Wh}_0(\Pi_+),
   \]

Waldhausen showed that the cellular splitting obstruction is algebraically defined as the image \(\text{split}_K(g; Y_0) \in \text{Wh}_0(\Pi_0) \oplus \tilde{\text{Nil}}_0\) of the Whitehead torsion \(\tau(g) \in \text{Wh}_1(\Pi)\). It vanishes if and only if \(g\) is CW splittable along \(Y_0\) [Wal69 erratum].

Remark 4.6. Sylvain Cappell had shown that \(\text{UNil}_6^s\) is a summand of \(L_6^B(\Pi)\) and that there is an exact sequence of abelian groups [Cap74B]:
   \[
   L_6^B(\Pi_0) \to L_6^B(\Pi_-) \oplus L_6^B(\Pi_+) \to L_6^B(\Pi)/\text{UNil}_6^s \to L_5^B(\Pi)
   \]

or, respectively,
   \[
   L_6^B(\Pi_0) \to L_6^B(\Pi_+) \to L_6^B(\Pi)/\text{UNil}_6^s \to L_5^B(\Pi).
or, respectively,

\[ L^h_6(\Pi_0) \xrightarrow{i - i_+} L^h_6(\Pi_\infty) \xrightarrow{j_\infty} L^B_6(\Pi)/\text{UNil}_6^g \]

\[ \xrightarrow{\partial} L^h_6(\Pi_0) \xrightarrow{i - i_+} L^h_6(\Pi_\infty) \xrightarrow{j_\infty} L^B_6(\Pi). \]

If the cellular splitting obstruction vanishes, then Cappell showed that the manifold splitting obstruction is algebraically defined as \( \text{split}_L(g; Y_0) \in \text{UNil}_6^g \). It vanishes if \( g \) is CAT splittable along \( Y_0 \) [Cap74a]. We shall investigate the converse.

**Proof of Theorem 4.1** (Necessity) Suppose \( g \) is CAT splittable along \( Y_0 \). Then \( \text{split}_K(g; Y_0) = 0 \) and \( \text{split}_L(g; Y_0) = 0 \) vanish by Remarks 4.5 and 4.6.

(Sufficiency) Suppose \( \text{split}_K(g; Y_0) = 0 \) and \( \text{split}_L(g; Y_0) = 0 \). Then \( g \) is CW splittable along \( Y_0 \) and \( g \in S^B_{\text{DIFF}}(Y) \) (resp. \( g \in S^B_{\text{TOP}}(Y) \)) by Remark 4.3. Since \( Y_0 \) satisfies the hypotheses in Sections 2–3 for exactness of the CAT surgery sequence, by Theorem 4.4 it follows that \( \text{nncc}^s(g) = (g, 0) \). In other words, \( g \) is CAT splittable along \( Y_0 \).

Furthermore, the normal bordisms over \( Y_0 \) in the proof of Theorem 4.3 depend only on the homotopy self-equivalences and normal self-bordisms of [Kha07, Proposition 3.5] and Section 2. Therefore \( g : W \to Y \) is CAT normally bordant to a split homotopy equivalence \( g' = g^4 \) such that the CAT restriction \( g' : (g')^{-1}(Y_0) \to Y_0 \) is a homotopy self-equivalence.

**Proof of Theorem 4.4** (Definition, I) Let \( g : (W, \partial W) \to (Y, \partial Y) \) be a homotopy equivalence with Whitehead torsion \( \tau(g) \in B \) and \( \partial g \) a CAT isomorphism, representing an element of \( S^B_{\text{CAT}}(Y) \). Our principal goal is to define a CAT normal bordism \( G' \) over \( Y \times \Delta^1 \) from \( g \) to a homotopy equivalence \( g' : (W', \partial W') \to (Y', \partial Y') \) such that \( h \) is CAT split along \( Y_0 \) and that \( G' \) has surgery obstruction \( \sigma_*(G') \in \text{UNil}_6^g \subseteq L^B_6(\Pi) \).

Define \( \text{nncc}^s(g) := (g', \sigma_*(G')) \in S^B_{\text{CAT}}(Y; Y_0) \times \text{UNil}_6^g \). (Well-definition; Projection properties) Note that \( \sigma_*(G') \) depends only on the normal bordism class of \( G' \) relative to \( \partial G' = g \cup g' \), and that \( \sigma_*(G') \) lies in \( L^B_6 \) since \( \tau(g), \tau(g') \in B \). Let \( Z := \mathbb{C}P^2 \# 2(S^3 \times S^5) \) be the closed CAT 8-manifold with Euler characteristic \( \chi(Z) = 1 \) and signature \( \sigma^*(Z) = 1 \) used by Weinberger for decorated periodicity [Wei72]. Cappell has shown

\[ \sigma_*(G' \times 1_Z) = \text{split}_L(g \times 1_Z; Y_0 \times Z) \]

for 13-dimensional homotopy equivalences [Cap76]. Note \( \sigma_*(G') = \sigma_*(G' \times 1_Z) \), by Kwon–Szczarba’s torsion product formula and Ranicki’s surgery product formula [Ran80, Prop. 8.1(ii)]. Also note \( \text{split}_L(g \times 1_Z; Y_0 \times Z) = \text{split}_L(g; Y_0) \), since these splitting obstructions in \( \text{UNil}_6^g \) coincide [Ran81, Prop. 7.6.2A] with the codimension-one quadratic signatures [Ran81, Prop. 7.2.2] of \( g \) and \( g \times 1_Z \) in the codimension-one Poincaré embedding groups \( L_{S_1} \) and \( L_{S_{12}} \), and since \( \times 1_Z : L_{S_1} \to L_{S_{12}} \) is an isomorphism [Wal99, Cor. 11.6.1]. So \( \sigma_*(G') = \text{split}_L(g; Y_0) \). Suppose \( G'' \) is another such CAT normal bordism from \( g \) to some split \( g'' \). Then \( \sigma_*(G'') = \text{split}_L(g; Y_0) \). So \( G' \cup_{g} -G'' \) is a normal bordism from \( g' \) to \( g'' \) with surgery obstruction \( 0 \in L^B_6 \).

By the 6-dimensional CAT s-cobordism theorem, it follows that \( g' \) and \( g'' \) are CAT isomorphic. Therefore \( \text{nncc}^s(g) = (g', \sigma_*(G')) \) is well-defined and satisfies the asserted projection properties.
(Bijectivity) Consider Wall’s action \[ \text{[Wal99 Thm. 5.8]} \] of the abelian group \( L^6_5 \) on the 5-dimensional structure set \( S^6_{\text{CAT}}(Y) \). It follows from the properties defining \( ncc^* \) that the restriction of Wall’s action is the inverse function of \( ncc^* \):
\[
\text{act} : S^6_{\text{CAT}}(Y; Y_0) \times \text{UNil}_6^6 \longrightarrow S^6_{\text{CAT}}(Y).
\]

(Definition, II) First, we normally\footnote{The stable normal CAT microbundle on \( Y \) and on \( Y_0 \) is \( \langle \mathcal{G} \rangle^* (\nu_W) \) and \( \langle \mathcal{G} \rangle^* (\nu_W) \big| Y_0 \), lifting the Spivak normal spherical fibration on \( Y \), where \( \mathcal{G} : Y \to W \) is a homotopy inverse of \( g : W \to Y \).} cobord \( g \) to a degree one normal map \( g^1 \) so that the restriction to \( (g^1)^{-1} (Y_0) \) is a homotopy equivalence. By general position, we may assume that \( g : W \to Y \) is DIFF transversal to \( Y_0 \). Consider the degree one normal map
\[
g_0 := g|_{W_0} : W_0 \longrightarrow Y_0
\]
where the DIFF 4-manifold \( W_0 := g^{-1}(Y_0) \) is the transverse inverse image of \( Y_0 \).

Denote \( \hat{W} = \hat{W}_- \cup_{Y_0} \hat{W}_+ \) as the based cover of \( Y \) corresponding to the subgroup \( \Pi_0 \). The \( \mathbb{Z}[\Pi_0] \)-submodule \( P := K_0(\hat{W}_- \times Z) \) is a finitely generated, projective lagrangian of the \( \mathbb{Z}[\Pi_0] \)-equivariant intersection form on the surgery kernel
\[
K_0(\hat{W}_- \times Z) = \text{Ker} (g \times 1_2 : H_0(\hat{W}_- \times Z) \longrightarrow H_0(Y_0 \times Z)),
\]
where we can homotope \( g \times 1_2 : W \times Z \to Y \times Z \) so that \( g\big|_{(g^{-1}(Y_0 \times Z)} \) is 6-connected, degree one, normal map between 12-dimensional manifolds \[ \text{[Cap76, Lemma II.1]} \]. Furthermore, the projective class \([P] \in Wh_0(\Pi_0)\) satisfies \([P^*] = -[P] \) \[ \text{[Cap76, Lemma II.2]} \] and is the homomorphic image of the Whitehead torsion \( \tau(g) = \tau(g \times 1_2) \in Wh_1(\Pi) \) under Waldhausen’s connecting map \( \partial \) (see Remark \[ \text{[Wal99, \S9]} \]). But \( \tau(g) \in B \) implies that \([P] \in \partial(\tau(g)) = 0 \). Therefore the stably free surgery obstruction vanishes by decorated periodicity:
\[
\sigma_*(g_0) = \sigma_*(g_0 \times 1_2) = 0 \in L^8_5(\Pi_0).
\]

Then, by Theorem \[ \text{[Wal99 \S9]} \], there exists a CAT normal bordism \( G^1 \) from \( g_0 \) to a homotopy equivalence \( g^1 : W_0^1 \to Y_0 \). So the union
\[
G^1 := (g \times [0, 1]) \cup_{g_0 \times 1 \times [-1, 1]} (G^1_0 \times [-1, 1])
\]
is a CAT normal bordism from the homotopy equivalence \( g : W \to Y \) to a degree one normal map \( g^1 : W^1 \to Y \) with transversal restriction \( g_0^1 = (g^1)|_{(g^1)^{-1}(Y_0)} \) a homotopy equivalence.

Second, we normally cobord \( g^1 \) relative \( (g^1)^{-1}(Y - Y_0) \) to a degree one normal map \( g^2 \) so that the restriction to \( (g^2)^{-1}(Y - Y_0) \) has vanishing surgery obstruction. Since \( g_{1_2}^1 = (g^1)|_{(g^1)^{-1}(Y_- \cup Y_+)} \) (resp. \( g_\infty^1 = (g^1)|_{(g^1)^{-1}(Y_- \cup Y_+)} \)) restricts to a homotopy equivalence \( g_\infty^2 \) on \( \partial \) the boundary, and since \( G^1 \) is a normal bordism of source and target from \( g_1^1 \) (resp. \( g_\infty^1 \)) to the \( B \)-torsion homotopy equivalence \( g \) over the reference space \( K(\Pi, 1) \), the image of surgery obstruction vanishes \[ \text{[Wal99, \S9]} \]:
\[
(j_- + j_+)(\sigma_*(g_1^1)) = 0 \in L^8_5(\Pi)
\]
or, respectively,
\[
(j_\infty)(\sigma_*(g_\infty^1)) = 0 \in L^8_5(\Pi).
\]

Therefore, by Cappell’s \( L \)-theory Mayer–Vietoris exact sequence (Remark \[ \text{[Wal99 \S6]} \]), there exists \( a \in L^8_5(\Pi_0) \) such that
\[
(i_- - i_+)(a) = -\sigma_*(g_\infty^1) \in L^8_5(\Pi_-) \oplus L^8_5(\Pi_-)
\]
or, respectively,

\[ (i_+ - i_-)(a) = -\sigma_*(g^3_\infty) \in L^h_5(\Pi_\infty). \]

Then, by Theorem 2.2 if CAT = TOP, or by Hypothesis 2.5 if CAT = DIFF, there exists a CAT normal bordism \( G_0^5 \) from the homotopy equivalence \( g^3_0 \) to itself realizing this surgery obstruction: \( \sigma_*(G^3_0) = a \in L^h_5(\Pi_0) \). So the union

\[ G^2 := (g^1 \times [0, 1]) \cup_{\tilde{g}^5_\infty \times 1 \times [-1, 1]} (G^2_0 \times [-1, 1]) \]

is a CAT normal bordism from the degree one normal map \( g^1 : W^1 \to Y \) to another degree one normal map \( g^2 : W^2 \to Y \) such that the transversal restriction \( g^2_0 : W^1_0 \to Y_0 \) is a homotopy equivalence and the transversal restriction \( g^2_\pm : W^2_\pm \to Y_\pm \) (resp. \( g^2_\infty : W^2_\infty \to Y_\infty \)) has vanishing surgery obstruction.

Third, we normally cobord \( g^2 \) relative \((g^2)^{-1}(Y_0)\) to a degree one normal map \( g^3 \) so that the restriction to \((g^3)^{-1}(Y - Y_0)\) is also a homotopy equivalence. Since \( \sigma_*(g^3_\infty) = 0 \in L^h_5(\Pi_\infty) \) (resp. \( \sigma_*(g^2_\infty) = 0 \in L^h_5(\Pi_\infty) \)), by exactness of the 5-dimensional surgery exact sequence at the CAT normal invariants \([Wal99, Thm. 10.3]\), there exists a CAT normal bordism \( G^4_\pm \) (resp. \( G^3_\infty \)) relative \( g_2 \times \{ -1, 1 \} \) from the degree one, CAT normal map \( g^2_\pm \) (resp. \( g^2_\infty \)) to a homotopy equivalence \( g^3_\pm \) (resp. \( g^3_\infty \)). So the union

\[ G^3 := (g^2 \times [0, 1] \times [-1, 1]) \cup_{\tilde{g}^3_\infty \times [0, 1] \times \{ -1, 1 \}} G^3_\pm \]

or, respectively,

\[ G^3 := (g^2 \times [0, 1] \times [-1, 1]) \cup_{\tilde{g}^3_\infty \times [0, 1] \times \{ -1, 1 \}} G^3_\infty \]

is a CAT normal bordism from the degree one normal map \( g^2 : W^2 \to Y \) to homotopy equivalence \( g^3 = g^2 \cup g^3_\infty : W^3 \to Y \) (resp. \( g^3 = \cup g^3_\infty : W_\infty \to Y \)) which is CAT split along \( Y_0 \).

Finally, we normally cobord the split homotopy equivalence \( g^3 = g' \) so that the normal bordism \( G' = G^3 \cup G^2 \cup G^5 \cup G^4 \) from \( g \) to \( g' \) has surgery obstruction in the subgroup \( \text{UNil}_6^* \) of the abelian group \( L^h_6 \).

Consider the surgery obstruction of the CAT normal bordism from \( g \) to \( g^3 \):

\[ b := -\sigma_*(G^1 \cup G^2 \cup G^3) \in L^h_6(\Pi). \]

Let \( c := \partial(b) \in L^h_5(\Pi_0) \) be the image in Cappell’s \( L \)-theory Mayer–Vietoris exact sequence (Remark 4.6). By Theorem 2.2 if CAT = TOP, or by Hypothesis 2.5 if CAT = DIFF, there exists a CAT normal bordism \( G^5_\pm \) from the homotopy equivalence \( g^3_0 \) to itself realizing this surgery obstruction: \( \sigma_*(G^5_0) = c \in L^h_5(\Pi_0) \). Then

\[ 0 = (i_+ - i_-)(c) = \sigma_*(g^3_\infty \cup G^5_0 \times \{ -1, 1 \}) \in L^h_5(\Pi_-) \oplus L^h_5(\Pi_+) \]

or, respectively,

\[ 0 = (i_+ - i_-)(c) = \sigma_*(g^3_\infty \cup G^5_0 \times \{ -1, 1 \}) \in L^h_5(\Pi_\infty). \]

Therefore, by exactness of the 5-dimensional surgery exact sequence at the CAT normal invariants \([Wal99, Thm. 10.3]\), there exists a CAT normal bordism \( G^5_\pm \) (resp. \( G^3_\infty \)) relative \( g^3_0 \times \{ -1, 1 \} \) from the degree one normal map \( g^3_\pm \cup G^3_0 \times \{ -1, 1 \} \) (resp. \( g^3_\infty \cup G^3_0 \times \{ -1, 1 \} \)) to a homotopy equivalence \( g^3_\pm : W^3 \pm \to Y_\pm \) (resp. \( g^3_\infty : W^\infty \to Y_\infty \)). So the union

\[ G^{3,5} := (G^{3,5}_0 \times \{ -1, 1 \}) \cup_{G^{3,5}_0 \times \{ -1, 1 \}} G^{3,5}_0 \]
or, respectively,
\[ G^{3.5} := (G^{3.5}_0 \times [-1, 1]) \cup_{G_{0}^{3.5} \times \{-1,1\}} G^{3.5}_{\infty} \]
is a CAT normal bordism from the split homotopy equivalence \( g^3 : W^3 \to Y \) to another split homotopy equivalence \( g^{3.5} : W^{3.5} \to Y \) such that
\[ \sigma_s(G^1 \cup G^2 \cup G^3 \cup G^{3.5}) = j(d) \oplus e \in L_0^B(\Pi) \]
for some \( d \in L_0^k(\Pi_-) \oplus L_0^h(\Pi_+) \) (resp. \( d \in L_0^h(\Pi_-) \oplus L_0^k(\Pi_+) \)) and \( e \in \text{UNil}_0^d \).

By Wall realization on 5-dimensional CAT structure sets [Va99 Thm. 10.5], there exists a CAT normal bordism \( G^4_\pm \) (resp. \( G^4_{\infty} \)) relative \( g_0^{3.5} \times \{-1,1\} \) from the homotopy equivalence \( g^{3.5}_\pm \) (resp. \( g^{3.5}_{\infty} \)) to another homotopy equivalence \( g^4_\pm \) (resp. \( g^4_{\infty} \)) such that \( \sigma_s(G^4_\pm) = -d \) (resp. \( \sigma_s(G^4_{\infty}) = -d \)). So the union
\[ G^4 := G^{3.5} \cup_{g^{3.5} \times 0} (g_0^{3.5} \times [0,1] \times [-1,1] \cup G^4_\pm) \]
or, respectively,
\[ G^4 := G^{3.5} \cup_{g^{3.5} \times 0} (g_0^{3.5} \times [0,1] \times [-1,1] \cup G^4_{\infty}) \]
is a CAT normal bordism from the split homotopy equivalence \( g^{3.5} : W^{3.5} \to Y \) to another split homotopy equivalence \( g^4 : W^4 \to Y \) such that
\[ \sigma_s(G^1 \cup G^2 \cup G^3 \cup G^4) = e \in \text{UNil}_0^d. \]
Thus the definition of \( \text{nncc}^s \) is complete.

5. Fibering and splitting over the circle

We approach the problem of fibered a 5-manifold \( W \) over the circle \( S^1 \) from Farrell’s point of view [Far71], which involves a finite domination of the infinite cyclic cover \( \tilde{W} \), the covering translation \( t : \tilde{W} \to \tilde{W} \), and a certain mapping torus.

Gluing the ends of a self \( h \)-cobordism \( (Y; X, X) \) by a self homeomorphism \( \alpha : X \to X \) yields an \( h \)-block bundle \( \cup_s Y \) over \( S^1 \) [Cha80 p. 306]. This is classically known as a pseudo-fibered over \( S^1 \) [Far72 Defn. 3.1] [Sie70 Defn. 4.2]. Consider the zero-torsion version of \( h \)-block bundles over \( S^1 \).

**Definition 5.1.** We call \( E \) the total space of a CAT \( s \)-block bundle over \( S^1 \) with homotopy fiber \( X \) if \( E \) is the compact CAT manifold obtained by gluing the ends of a self CAT \( s \)-cobordism \( (Y; X, X) \) rel \( \partial X \) by a CAT automorphism \( \alpha : X \to X \). We write \( E = \cup_s Y := Y/(x,0) \sim (\alpha(x),1) \), and a special case is the mapping torus \( X \times_\alpha S^1 := \cup_s X \times \Delta^1 \). The induced continuous map \( E \to S^1 \) is called a CAT \( s \)-block bundle projection, which is unique up to homotopy.

Fiber bundles are special cases of \( s \)-block bundles, and the converse is true if the \( s \)-cobordism theorem holds for the fiber in the given manifold category.

**Definition 5.2** (Quinn, compare [Nic82 §2.3]). Let \( X \) be a compact CAT manifold. The block homotopy automorphism space \( \tilde{G}^s(X) \) is the geometric realization of the Kan \( \Delta \)-set whose \( k \)-simplices are simple homotopy self-equivalences \( e : X \times \Delta^k \to X \times \Delta^k \) of \( (k+2) \)-ads which restrict to the identity over \( \partial X \). Note that \( \text{hAut}^s(X) = \pi_0 \tilde{G}^s(X) \). The basepoint is the identity \( 1_X : X \to X \).

Similarly, the block structure space \( \tilde{S}^s_{\text{CAT}}(X) \) is the geometric realization of the Kan \( \Delta \)-set whose \( k \)-simplices are simple homotopy equivalences \( Y \to X \times \Delta^k \) of CAT manifold \( (k+2) \)-ads in \( \mathbb{R}^\infty \) which restrict to CAT isomorphisms over \( \partial X \). Note the CAT \( s \)-bordism structure set is \( \tilde{S}^s_{\text{CAT}}(X) = \pi_0 \tilde{S}^s_{\text{CAT}}(X) \). We define the
A homotopy equivalence which restricts to a diffeomorphism except we do not require 4-dimensional Siebenmann ends on the infinite cyclic cover $\alpha$. Let $h$ be any homotopy self-equivalence $h : (X, \partial X) \rightarrow (X \times (\Delta^1; 0, 1))$ of CAT manifold triads such that the simple homotopy self-equivalences $h_i : X \rightarrow X$ satisfy $h_1 = \alpha \circ h_0$ and that $h$ restricts to a CAT isomorphism over $\partial X$. We define the $\alpha$-twisted fundamental set $\pi_1^\alpha(\mathcal{S}_\text{CAT}(X), \mathcal{G}^\alpha(X))$ as the set of homotopy classes of these $\alpha$-twisted loops. Note, if $\alpha$ is the identity automorphism, then this set is the first homotopy set of the pair. Define the union function

$$\cup : \pi_1^\alpha(\mathcal{S}_\text{CAT}(X), \mathcal{G}^\alpha(X)) \rightarrow \mathcal{S}_\text{CAT}(X \times_\alpha S^1)$$

as follows. Let $(h; h_0, h_1) : (Y; X, X) \rightarrow X \times (\Delta^1; 0, 1)$ be an $\alpha$-twisted simplicial loop. Then there is an induced simple homotopy equivalence, well-defined on homotopy classes of loops:

$$(\cup; [y]) \mapsto [h(y)]$$

Now we are ready to study three families of examples of homotopy fibers.

5.1. Statement of results. Our theorems below are crafted as to eliminate any algebraic $K$- or $L$-theory obstructions to splitting and fibering over the circle.

The first splitting theorem [5.4] is a special case of the general splitting theorem [4.1]. Here, for any homotopy self-equivalence $h : (X, \partial X) \rightarrow (X, \partial X)$, the mapping torus $Y = X \times_\alpha S^1$ is a Poincaré pair with $X$ a two-sided Poincaré subpair [Ran98, Prop. 24.4]. This level of abstraction is required to prove the fibering theorem [5.6].

**Theorem 5.4.** Let $X$ be any of the 4-manifolds $Q, S, F$ defined in [4.1, 4.2, 4.4]. Let $h : X \rightarrow X$ be a homotopy equivalence which restricts to a diffeomorphism $\partial h : \partial X \rightarrow \partial X$. Suppose $M$ is a compact DIFF 5-manifold and $g : M \rightarrow X \times_\alpha S^1$ is a homotopy equivalence which restricts to a diffeomorphism $\partial g : \partial M \rightarrow \partial X \times_\partial S^1$.

Then $g$ is homotopic to a map $g'$ which restricts to a simple homotopy equivalence $g' : X' \rightarrow X$ such that the TOP inverse image $X' : = (g')^{-1}(X)$ is homeomorphic to $X$ and the exterior $M'$ of $X'$ in $M$ is a smoothable TOP self $s$-cobordism of $X$.

The second splitting theorem [5.5] connects homotopy TOP structures on mapping tori to smoothable $s$-cobordisms, homotopy self-equivalences, and the stable smoothing invariant of Kirby and Siebenmann [KS77].

**Theorem 5.5.** Let $X$ be any of the 4-manifolds $Q, S, F$ defined in [4.1, 4.2, 4.4]. Let $\alpha : X \rightarrow X$ be a diffeomorphism. Then there is an exact sequence of based sets:

$$\pi_1^\alpha(\mathcal{G}^\alpha(X)) \cup \rightarrow \mathcal{S}_\text{TOP}(X \times_\alpha S^1) \xrightarrow{\text{ks}} \mathbb{F}_2 \oplus H_1(X; \mathbb{F}_2) \alpha.$$
**Theorem 5.6.** Let $X$ be any of the 4-manifolds $Q, S, F$ defined in (1.1, 1.3, 1.4). Let $(M, f)$ be a connected DIFF 5-manifold band such that $\partial X \to \partial M \to S^1$ is a DIFF fiber bundle and the homotopy equivalence $\partial X \to \text{hofiber}(f)$ extends to a homotopy equivalence $X \to \text{hofiber}(f)$. Then $f : M \to S^1$ is homotopic rel $\partial M$ to the projection of a smoothable TOP $s$-block bundle with fiber $X$.

**Remark 5.7.** Topological splitting and fibering of 5-manifolds $W$ over the circle $S^1$ with fibers like $T^4$ or $Kl \times S^2$ can be established by Weinberger’s splitting theorem [Wei87], since the fundamental groups fibering problems for such $W$ remain unsolved, even as DIFF $s$-block bundle maps.

Now let us state the smooth result promised in the Introduction.

**Theorem 5.8.** Consider the closed, non-orientable, smooth 4-manifolds $Q$ (1.1). Let $(M, f)$ be a connected DIFF 5-manifold band such that $Q$ is homotopy equivalent to hofiber($f$). Then $f : M \to S^1$ is homotopic to the projection of a DIFF $s$-block bundle with fiber $Q$.

**5.2. Vanishing of lower Whitehead groups.** We start by showing that every homotopy equivalence under consideration has zero Whitehead torsion. Comparable results are found by J. Hillman [Hi102 §6.1] and intensely use [Wal78 §19].

**Proposition 5.9.** Let $X$ be any of the 4-manifolds $Q, S, F$ defined in (1.1, 1.3, 1.4). Let $h : X \to X$ be a homotopy self-equivalence which restricts to a diffeomorphism $\partial h : \partial X \to \partial X$. Suppose $Y = X \times_h S^1$ is the total space of a mapping torus fibration $X \to Y \to S^1$ with monodromy $h$. Then the Whitehead groups vanish for all $* \leq 1$:

$$\text{Wh}_*(\pi_1(X)) = \text{Wh}_*(\pi_1(Y)) = 0.$$  

Note if $h$ is a diffeomorphism, then $Y$ is the total space of a DIFF fiber bundle.

**Proof.** Each case of fiber shall be handled separately.

**Case of fiber $Q$ (1.1).** Let $Q$ be any of the non-orientable 4-manifolds listed. There are no non-identity automorphisms of the cyclic group $\pi_1(Q) = C_2$ of order two, so $\pi_1(Y) = C_2 \times C_\infty$. S. Kwasik has shown that $\text{Wh}_1(\pi_1(Q)) = \text{Wh}_1(\pi_1(Y)) = 0$, using the Rim square for the ring $\mathbb{Z}[C_\infty][C_2]$, the Bass–Heller–Swan decomposition, and assorted facts [Kwa86 pp. 422–423].

**Case of fiber $S$ (1.2).** There are three types of $S$.

**Type 1.** Suppose $S$ is the total space of a fiber bundle $S^2 \to S \to \Sigma$ such that $\Sigma$ is a compact, connected, possibly non-orientable 2-manifold of positive genus. Then, by [Wal78 Theorem 19.5(5)], the fundamental group $\pi_1(S) = \pi_1(\Sigma)$ is a member of Waldhausen’s class Cl of torsion-free groups. So, by [Wal78 Proposition 19.3], the HNN-extension $\pi_1(Y) = \pi_1(S) \times C_\infty$ is a member of Cl. Therefore, by [Wal78 Theorem 19.4], we obtain that $\text{Wh}_*(\pi_1(S)) = \text{Wh}_*(\pi_1(Y)) = 0$.

**Type 2.** Suppose $S$ is the total space of a fiber bundle $H \to S \to S^1$ such that $H$ is a closed, connected, hyperbolic 3-manifold. By Mostow rigidity, we may select the monodromy diffeomorphism $H \to H$ to be an isometry [Mos68] up to smooth isotopy [GMT03]. Since $H \to S \to S^1$ has isometric monodromy implies that $S$
is isometrically covered by $\mathbb{H}^3 \times \mathbb{E}^1$, the curvature matrix is constant, hence $S$ is $A$-regular. By hypothesis, $h$ is homotopically rel $\partial S$ to an isometry of $S - \partial S$. Then $Y - \partial Y$ is isometrically covered by $\mathbb{H}^3 \times \mathbb{E}^2$, hence $Y - \partial Y$ is $A$-regular. Therefore, by [FJ98, Lemma 0.12], we obtain that $Wh_* = Wh_*$ of euclidean type (resp. real hyperbolic or complex hyperbolic). Since $S$ is isometrically covered by $\mathbb{E}^4$ (resp. $\mathbb{H}^4$ or $\mathbb{CH}^2$), the curvature matrix is constant, hence $S - \partial S$ is $A$-regular. By hypothesis, $h$ is homotopically rel $\partial S$ to an isometry of $S - \partial S$. Then $Y - \partial Y$ is isometrically covered by $\mathbb{E}^4 \times \mathbb{E}^1$ (resp. $\mathbb{H}^4 \times \mathbb{E}^1$ or $\mathbb{CH}^2 \times \mathbb{E}^1$), the curvature matrix is constant, hence $Y - \partial Y$ is $A$-regular. Therefore, by [FJ98, Lemma 0.12], we obtain that $Wh_* (\pi_1 S) = Wh_* (\pi_1 Y) = 0$ for all $* \leq 1$.

**Type 3.** Suppose the interior $S - \partial S$ admits a complete, finite volume metric of euclidean type (resp. real hyperbolic or complex hyperbolic). Since $S - \partial S$ is isometrically covered by $\mathbb{E}^4$ (resp. $\mathbb{H}^4$ or $\mathbb{CH}^2$), the curvature matrix is constant, hence $S - \partial S$ is $A$-regular. By hypothesis, $h$ is homotopically rel $\partial S$ to an isometry of $S - \partial S$. Then $Y - \partial Y$ is isometrically covered by $\mathbb{E}^4 \times \mathbb{E}^1$ (resp. $\mathbb{H}^4 \times \mathbb{E}^1$ or $\mathbb{CH}^2 \times \mathbb{E}^1$), the curvature matrix is constant, hence $Y - \partial Y$ is $A$-regular. Therefore, by [FJ98, Lemma 0.12], we obtain that $Wh_* (\pi_1 S) = Wh_* (\pi_1 Y) = 0$ for all $* \leq 1$.

**Case of fiber $F$.**

**Type 1.** Suppose $F_i$ is the total space of a fiber bundle $H_i \to F_i \to S^1$ such that the compact, connected, irreducible, orientable 3-manifold $H_i$ either is $S^3$ or $D^4$ or has non-zero first Betti number. Then, by [Wal78] Proposition 19.5(6,8)], we obtain that $\pi_1 (H_i)$ is a member of Cl. So, by [Wal78] Proposition 19.3, the HNN-extension $\pi_1 (F_i) = \pi_1 (H_i) \rtimes C_\infty$ is a member of Cl.

**Type 2.** Suppose $F_i$ is the total space of a fiber bundle $\Sigma_i^f \to F_i \to \Sigma_i^b$ such that the fiber and base are compact, connected, orientable 2-manifolds of positive genus. Then, by a theorem of J. Hillman [Hill91, Thm. 1], we obtain that $\pi_1 (F_i)$ is a member of Cl. Indeed, the direct algebraic proof of Cavicchioli–Hegenbarth–Spaggiari [CHS06, Thm. 3.12] uses a Mayer–Vietoris argument for a connected-sum decomposition of the base $\Sigma_i^b$, which extends to aspherical, compact, possibly non-orientable surfaces with possibly non-empty boundary.

**Conclusion.** Now, since $\pi_1 (F_i)$ is a member of Cl, by [Wal78] Proposition 19.3], the fundamental group $\pi_1 (F) = \bigstar_{i=1}^n \pi_1 (F_i)$ of the connected sum $F = F_1 \# \cdots \# F_n$ is a member of Cl. So the HNN-extension $\pi_1 (Y) = \pi_1 (F) \rtimes C_\infty$ is a member of Cl. Therefore, by [Wal78, Theorem 19.4], we obtain $Wh_* (\pi_1 F) = Wh_* (\pi_1 Y) = 0$. □

### 5.3. Proof of main theorems over the circle.

**Proof of Theorem 5.4.** By the general splitting theorem (Thm. 4.1), it suffices to show that the following conditions hold, in order:

1. $X$ satisfies Hypothesis 3.1 or 3.2 or 3.3 and Hypothesis 2.1
2. the obstructions split $\omega (g; X)$ and $\omega (g; X)$ vanish, and
3. the $h$-cobordism $(M'; X', X')$ and homotopy equivalence $g' : X' \to X$ have zero Whitehead torsion.

**Condition (1). Case of $X = Q$.** By Corollary 3.6 $Q$ satisfies Hypothesis 3.2. Since [Wal99, Theorem 13A.1] implies $L^2_b (\mathbb{Z} / C_2) = 0$, $Q$ fulfills Theorem 2.2.

**Case of $X = S$.** There are three types of $S$. If $S$ is non-orientable, then, since the abelianization $H_1 (S; \mathbb{Z})$ is 2-torsionfree, there exists a lift $\hat{\omega} : \pi_1 (S) \to \mathbb{Z}^\times$ of the orientation character $\omega : \pi_1 (S) \to \mathbb{Z}^\times$.

**Type 1.** By Corollary 2.3 $S$ satisfies Hypothesis 2.1. If $S$ is orientable, then, by Corollary 3.5 $S$ satisfies Hypothesis 3.1. Otherwise, if $S$ is non-orientable, then, by Corollary 3.8 $S$ satisfies Hypothesis 3.2.

**Type 2.** Since $H$ is irreducible, by Corollary 2.3 $S$ satisfies Hypothesis 2.1. Recall that $S$ can be isometrically covered by $\mathbb{H}^3 \times \mathbb{E}^1$, by Mostow rigidity [Mos68, GMT03]. Then, by [FJ98, Proposition 0.10] and the argument of Corollary 3.5, it follows that
\( \kappa_2 \) is injective. If \( S \) is orientable, then \( S \) satisfies Hypothesis \( \ref{hyp:orientable} \). Otherwise, if \( S \) is non-orientable, then, since the lift \( \tilde{\omega} \) exists, \( S \) satisfies Hypothesis \( \ref{hyp:nonorientable} \).

**Type 3.** By Corollary \( \ref{cor:orientability} \) \( S \) satisfies Hypothesis \( \ref{hyp:orientable} \). If \( S \) is orientable, then, by Corollary \( \ref{cor:orientability} \) \( S \) satisfies Hypothesis \( \ref{hyp:orientable} \). Otherwise, suppose \( S \) is non-orientable. The argument in Corollary \( \ref{cor:nonorientability} \) for the injectivity of \( \kappa_2 \) is the same as if \( S \) were orientable. Then, since \( \tilde{\omega} \) exists, \( S \) satisfies Hypothesis \( \ref{hyp:nonorientable} \).

**Case of \( X = F \).** There are two types of connected summands \( F_i \). By Corollary \( \ref{cor:orientability} \) \( F_i \) satisfies Hypothesis \( \ref{hyp:orientable} \). Note that \( F_i \) satisfies Hypothesis \( \ref{hyp:orientable} \) by Corollary \( \ref{cor:orientability} \) for Type 1 and by Corollary \( \ref{cor:orientability} \) for Type 2.

**Condition (2).** Recall the notation \( \Pi_0 := \pi_1(X) \) and \( \Pi := \pi_1(X \times_h S^1) \cong \pi_1(M) \). The \( K \)-theory obstruction \( \text{split}_K(g; X) \) lies in

\[
\text{Wh}_0(\Pi_0) \oplus \text{Nil}_0(\mathbb{Z}[\Pi_0]; 0, 0, -\mathbb{Z}[\Pi_0]_+, +\mathbb{Z}[\Pi_0]_-).
\]

By [Wal78, Theorem 2], the second factor is a summand of \( \text{Wh}_1(\Pi) \). Both \( \text{Wh}_0(\Pi_0) \) and \( \text{Wh}_1(\Pi) \) vanish by Proposition \( \ref{prop:vanishing} \). The \( L \)-theory obstruction \( \text{split}_L(g; X) \) lies in \( \text{UNil}_0(\mathbb{Z}[\Pi_0]; 0, 0) \), which vanishes by definition [Cap74b, §1].

**Condition (3).** The torsions of the \( h \)-cobordism \( (M'; X', X') \) and the homotopy equivalence \( g' \colon X' \to X \) lie in \( \text{Wh}_1(\Pi_0) \), which vanishes by Proposition \( \ref{prop:vanishing} \). \( \Box \)

**Proof of Theorem \( \ref{thm:main} \).** Let \( [g] \in S^h_{\text{TOP}}(X \times_\alpha S^1) \). Then \( [g] \) is an \( h \)-bordism class of some homotopy equivalence \( g \colon M \to X \times_\alpha S^1 \) such that \( M \) is a compact TOP 5-manifold and the restriction \( \partial g : \partial M \to \partial X \times_\partial_\alpha S^1 \) is a homeomorphism. Since \( \alpha : X \to X \) is a diffeomorphism, it implies that the mapping torus \( X \times_\alpha S^1 \) is a DIFF 5-manifold, we have

\[
\text{ks}[g] = g \circ \text{ks}(M) \in F_2 \oplus H_1(X; F_2)_{\alpha} \cong H^4(X \times_\alpha S^1, \partial X \times_\partial_\alpha S^1; \mathbb{F}_2).
\]

Here, the isomorphism is obtained from Poincaré duality and the Wang sequence. Note, by definition, that the basepoints are respected by \( \cup \) and \( \text{ks} \), and that the composite \( \text{ks} \circ \cup \) vanishes.

Suppose \( \text{ks}[g] = 0 \). Since \( \partial g \) induces a DIFF structure on \( \partial M \) and \( \text{ks}(M) = 0 \) and \( \dim(M) > 4 \), by a consequence of Milnor’s Lifting Criterion and the Product Structure Theorem, the DIFF structure on \( \partial M \) extends to a DIFF structure on \( M \). So, by Theorem \( \ref{thm:extension} \) we obtain that \( g \) is homotopic to a TOP split homotopy equivalence \( g' : M \to X \times_\alpha S^1 \) such that the restriction \( g' : (g')^{-1}(X) \to X \) is a homotopy self-equivalence. Moreover, the restriction of \( g' \) to the exterior of \( X \) yields a smoothable TOP \( s \)-cobordism \( (M'; X, X) \) and an \( \alpha \)-twisted simplicial loop \( (g'_{\alpha}; g_0', g_1') : (M'; X, X) \to X \times (\Delta^1; 0, 1) \) in \( \text{S}_{\text{TOP}^+}(X, G^s(X)) \). Therefore \( [g] = [g'] = \cup [g'_{\alpha}; g_0', g_1'] \). Thus exactness is proven at \( S^h_{\text{TOP}}(X \times_\alpha S^1) \). \( \Box \)

**Proof of Theorem \( \ref{thm:second} \).** The hypothesis gives a homotopy equivalence \( d : (X, \partial X) \to (\overline{M}, \partial \overline{M}) \) of pairs with homotopy inverse \( u : (\overline{M}, \partial \overline{M}) \to (X, \partial X) \) such that \( \partial u \circ \partial d = 1_{\partial X} \). In particular, \( d \) is a domination of \( (\overline{M}, \partial \overline{M}) \) by \( (X, \partial X) \). That is, there is a homotopy \( H : [0, 1] \times \overline{M} \to \overline{M} \) such that \( H_0 = d \circ u \) and \( H_1 = 1_\overline{M} \).

Recall \( \overline{M} = f^*(\mathbb{R}) \). Let \( \mathcal{F} : \overline{M} \to \mathbb{R} \) be the sub-projection covering \( f : M \to S^1 \).

Let \( t : \overline{M} \to \overline{M} \) be the unique covering transformation such that \( \mathcal{F}t(x) = \mathcal{F}(x) + 1 \).

Then the following composite is a homotopy self-equivalence of pairs:

\[
h := u \circ t \circ d : (X, \partial X) \to (X, \partial X).
\]

So, by cellular approximation of \( h \) and [Ran98, Proposition 24.4], the mapping torus \( X \times_\alpha S^1 = X \times [0, 1]/(x, 0) \sim (h(x), 1) \) is a finite Poincaré pair of formal
dimension 5, such that $X \cong X \times [1/2]$ is a two-sided Poincaré subpair with tubular neighborhood $X \times [-1, 1] \cong X \times [1/3, 2/3]$. Furthermore, since $\partial d : \partial X \to \partial \overline{M}$ and $\partial u : \partial \overline{M} \to \partial X$ are the 0-section and projection from $\partial \overline{M} = \partial X \times \mathbb{R}$, we obtain that the homotopy self-equivalence $\partial h : \partial X \to \partial X$ is in fact a self-diffeomorphism on each connected component. In particular, $\partial M = \partial X \times_\partial h S^1$.

Observe that the Borel construction fits into a fiber bundle $\mathbb{R} \to \overline{M} \times_Z \mathbb{R} \to M$ and similarly for $\partial M$. Then the projection $g_1 : \overline{M} \times_Z \mathbb{R} = \overline{M} \times_t S^1 \to M$ is a homotopy equivalence of manifold pairs. Note $\{H_s \circ t \circ d\}_{s \in [0,1]}$ is a homotopy from $d \circ h$ to $t \circ d : X \to \overline{M}$. Define a continuous map

$$g_2 : X \times_h S^1 \to \overline{M} \times_t S^1; \quad [x,s] \mapsto [H(s, td(x)), s].$$

By cyclic permutation of the composition factors of $h$, and by the adjunction lemma (see [Sie70]), the map $g_2$ is a homotopy equivalence of manifold pairs. Let $\overline{g}_1$ be a homotopy inverse of $g_1$, for $i = 1, 2$. Then we obtain a homotopy equivalence

$$g := \overline{g}_2 \circ \overline{g}_1 : (M, \partial M) \longrightarrow (X \times_h S^1, \partial X \times_\partial h S^1).$$

Furthermore, since $\partial X \to \partial M \to S^1$ is already a fiber bundle, the homotopy inverse $\overline{\eta}$ is homotopic to the above diffeomorphism $\partial X \times_\partial h S^1 \to \partial M$. By Theorem 5.4, the homotopy equivalence $g$ is homotopic rel $\partial M$ to a map $g'$ such that the TOP transverse restriction $g' : X' := (g')^{-1}(X) \to X$ is a simple homotopy equivalence and there is a homeomorphism $X' \approx X$. Moreover, $M = \cup_1 Q \cdot M'$ is obtained by gluing the ends of the smoothable TOP self $s$-cobordism $M' := M - X' \times (-1, 1)$ by the identity map.

Define quotient maps

$$q : X \times_h S^1 \to S^1; \quad [x,s] \mapsto [s]$$

$$q' : M \to S^1; \quad q' := q \circ g'.$$

Note $\partial q' = \partial f : \partial M \to S^1$ is the fiber bundle projection. Therefore, by obstruction theory, the continuous map $f : M \to S^1$ and the TOP $s$-block bundle projection $q' : M \to S^1$ are homotopic rel $\partial M$ if and only if they determine the same kernel subgroup of $\pi_1(M)$. Then, by covering space theory, it suffices to show that the isomorphism $g_* : \pi_1(M) \to \pi_1(X \times_h S^1)$ maps the subgroup $\text{Ker}(f_*) = p_* \pi_1(\overline{M})$ onto the subgroup $\text{Ker}(q_*) = p'_* \pi_1(X \times \mathbb{R})$. Here, $p : \overline{M} \to M$ and $p' : X \times \mathbb{R} \to X \times_h S^1$ are the infinite cyclic covers. Observe that the $\pi_1$-isomorphism induced by the split homotopy equivalence $g_2 : X \times_h S^1 \to \overline{M} \times_t S^1$ maps the subgroup $\text{Ker}(q_*) = \pi_1(X)$ onto $\pi_1(\overline{M})$, and that the $\pi_1$-isomorphism induced by the homotopy equivalence $g_1 : \overline{M} \times_t S^1 \to M$ maps the subgroup $\pi_1(\overline{M})$ onto $\text{Ker}(f_*)$. So, since $g_1 \circ g_2 = \overline{g}$ is the homotopy inverse of $g$, we are done. 

**Proof of Theorem 5.3** The proof of Theorem 5.6 constructs homotopy equivalences $h : Q \to Q$ and $g : M \to Q \times_h S^1$. Observe Corollary 3.6 implies that $Q$ satisfies Hypothesis 3.2, and Remark 2.6 implies that $Q$ satisfies Hypothesis 2.5. Recall that Conditions (2) and (3) of Proof 5.4 hold. Then, by Theorem 5.4, the homotopy equivalence $g$ is homotopic to a map $g'$ such that the DIFF transverse restriction $g' : Q' := (g')^{-1}(Q) \to Q$ is a simple homotopy equivalence and there is a diffeomorphism $Q' \approx Q$. Moreover, the DIFF 5-manifold $M = \cup_1 Q \cdot M'$ is obtained by gluing the ends of the DIFF self $s$-cobordism $M' := M - Q' \times (-1, 1)$ by the identity map. The remainder of Proof 5.6 shows that $f : M \to S^1$ is homotopic to the DIFF $s$-block bundle projection $q' : \cup_1 Q(M'; Q, Q) \to S^1$ obtained from $g'$. 


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