Examples of nonsingular cosmological models are presented on the basis of exact solutions to multidimensional gravity equations. These examples involve pure imaginary scalar fields, or, in other terms, "phantom" fields with an unusual sign of the kinetic term in the Lagrangian. We show that, with such fields, hyperbolic nonsingular models with a cosmological bounce (unlike spherical and spatially flat models) emerge without special relations among the integration constants, i.e., without fine tuning. In such models, the extra-dimension scale factors as well as scalar fields evolve smoothly between different finite asymptotic values. Examples of theories which create phantom scalar fields are obtained from string-inspired multidimensional field models and from theories of gravity in integrable Weyl space-times.

**Keywords:** multidimensional gravity, cosmology, singularities, scalar fields, dilaton, M-theory, Weyl integrable geometry.

### 1. Introduction

The recent years have been marked with a lot of violent events in both observational and theoretical cosmology. The discovery of an accelerated expansion of the Universe [1] has been one of the most important empirical findings. There followed a flood of theoretical works trying to interpret and to explain this acceleration, see, e.g., [2]. The majority of such constructions have a common feature: they involve various kinds of scalar fields. In many cases these are so-called phantom scalars, having a “wrong” sign of the kinetic term in their Lagrangians.

It is of interest that such scalar fields are able to suggest a solution to one more long-standing problem of theoretical cosmology, namely, the initial singularity problem. Inclusion of such fields makes it possible to circumvent the well-known singularity theorems and to prevent the formation of a cosmological singularity, keeping the curvature at sub-Planckian scales.

In these notes we will discuss in some detail this mechanism of avoiding a singularity in isotropic cosmological models at the level of classical field theory. Its efficiency is clear from the simplest example of a time-dependent scalar field in general-relativistic isotropic cosmology, described in Sec. 2. We begin with a discussion of the late-time behaviour of various models with accelerated expansion and show that a deceleration parameter $q_0 < -1$ may be obtained with phantom scalar fields without creating a final singularity (“big rip”) related to an infinite growth of the scale factor at finite time. It is then shown that an initial singularity can also be avoided with the aid of such fields: it turns out that a regular minimum of the scale factor is only possible in hyperbolic models with a phantom scalar field.

A similar mechanism works as well in more complex cases to be discussed in the further sections. Sec. 3 gives examples of multidimensional models [3], emerging in the field limit of some topical string-based unification theories (see [4, 5] and reviews on string cosmology [6]). Phantom type fields take place in such models with dimensions $D \geq 11$. We will not touch upon the whole wealth of exact solutions obtained in such models and restrict ourselves to simple models with a single dilatonic field, a single axionic type antisymmetric form and two scale factors — the “external” one, $a(t)$, and the “internal” one, $b(t)$. We shall see that nonsingular solutions in which, as $t \to \infty$, the scale factor $a(t)$ grows while $b(t)$ and the dilaton $\phi(t)$ tend to finite limits, exist in closed and spatially flat cosmologies ($K_0 = 0, +1$) with some special values of the integration constants only, i.e., require “fine tuning”, whereas in hyperbolic models with Lobachevsky 3-geometry ($K_0 = -1$) they emerge generically, without any fine tuning.

In Sec. 4, similar inferences are obtained for cosmological models of a certain class of gravitation theories with Weyl non-metricity which are, in many observational predictions, equivalent to scalar-tensor theories (STT) of gravity.
and can also contain phantom scalar fields [4, 8]. Their main difference from STT is a geometric interpretation of the scalar fields which makes their possible phantom character more natural than for usual, material scalar fields.

Sec. 5 contains some concluding remarks.

2. 4D scalar field cosmologies at late and early times

To illustrate the general properties of scalar field driven cosmologies, let us consider the standard 4D Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)ds_0^2, \quad ds_0^2 = \frac{dr^2}{1 - K_0r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $a(t)$ is the scale factor and $ds_0^2$ is the metric of a 3-space of constant curvature $K_0 = 0, \pm 1$. The corresponding Einstein-Friedmann equations are

$$\frac{3}{a^2}(a'^2 + K_0) = \kappa \rho, \quad \frac{1}{a^2}(2aa'' + a'^2 + K_0) = -\kappa p,$$

where $\kappa = 8\pi G$ is the gravitational constant, the prime denotes $d/dt$, $\rho = -T_0^0$ is the total density of matter and $p = T_1^1 = T_2^2 = T_3^3$ its pressure. The matter can be of any origin, but the symmetry of its stress-energy tensor $T_\mu^\nu = \text{diag}(-\rho, p, p, p)$ is determined by the choice of the metric (2).

The most important kinematic observational parameters characterizing the Universe evolution are the Hubble parameter $H(t) := a'/a$ and the deceleration parameter $q(t) := -aa''/a'^2$. (The parameter $q$ was introduced when it was believed that the expansion of the Universe was decelerating; its negative values correspond to an accelerating Universe.) The current observational estimates of these two parameters considerably vary from one paper to another but can be more or less reliably taken as [3]

$$H_0 \approx 0.71 \pm 4 \frac{\text{km}}{s \cdot \text{Mpc}}, \quad q_0 \approx -1 \pm 0.4,$$

where the subscript “0” refers to the present epoch.

For $\rho \neq 0$, one can always write $p = w\rho$ where $w$ is in general time-dependent. The simplest models are obtained for $w = \text{const}$ (the so-called barotropic matter). In this case the conservation law $\nabla_\mu T_\mu^\alpha = 0$ leads to the relation

$$\rho = \text{const} \cdot a^{-3(w+1)},$$

giving the well-known laws $\rho \sim a^{-3}$ for dust ($w = 0$) and $\rho \sim a^{-4}$ for disordered radiation ($w = 1/3$). Accelerated expansion ($q < 0$, however, requires a negative pressure. It is easily seen that, for $w < -1/3$ (which is needed for obtaining $q < 0$) and a large scale factor $a$, the term with $K_0$ in (2) is negligible as compared with $\rho$ and $p$, so that the scale factor behaviour at late times does not depend on the spatial curvature. The $t$ dependence of the scale factor is then described as follows:

a) if $-1/3 > w > -1$, so that the dominant energy condition holds, the expansion may be called power-law inflation:

$$a \sim t^{2/[3(w+1)]}, \quad q = -1 + \frac{3}{2}(w + 1) > -1;$$

b) if $w = -1$, which corresponds to a positive cosmological constant, $\rho = \text{const} > 0$, we obtain exponential inflation:

$$a \sim e^{Ht}, \quad H = \text{const}, \quad q = -1;$$

c) if $w < -1$, the matter may be called exotic, and we obtain hyper-inflation ending with a singularity related to a blowing-up scale factor:

$$a \sim (t_\ast - t)^{-2/[3|w+1|]}, \quad q = -1 - \frac{3}{2}|w + 1| < -1,$$

where $t_\ast$ is the singular time instant.

In case c), matter behaves exotically indeed: its density grows as the volume grows, and all this ends with a “big rip”, a singularity at finite physical time, where both $a$ and $\rho$ grow infinitely.
Such a sad future of our Universe may be, however, avoided even if the present value of \( w \) is smaller than -1 but if \( w = p/\rho \) is time-dependent. Indeed, suppose that matter (or its dominating part) is represented by a scalar field \( \phi \) with the Lagrangian

\[
L_s = -\frac{1}{2} \varepsilon g^{\mu \nu} \phi,_{\mu} \phi,_{\nu} - V(\phi),
\]

where \( \varepsilon = \pm 1 \) and \( V(\phi) \) is a potential. For \( \phi = \phi(t) \) we have

\[
\rho = \frac{1}{2} \varepsilon \phi'^2 + V, \quad p = \frac{1}{2} \varepsilon \phi'^2 - V, \quad w = p/\rho = -1 + \frac{2\varepsilon \phi'^2}{2V + \varepsilon \phi'^2}.
\]

Thus a normal scalar field (\( \varepsilon = +1 \)) with a positive potential \( V \) gives \( w > -1 \) whereas a phantom scalar field (\( \varepsilon = -1 \)) with \( V > 0 \) leads to \( w < -1 \). However, if at large \( t \) the scalar field tends rapidly enough to a minimum of the potential, \( V_{\text{min}} > 0 \), then \( \phi' \to 0 \) and \( w \to -1 \) as \( t \to \infty \); \( V_{\text{min}} \) behaves as an effective cosmological constant, and accordingly we obtain the de Sitter asymptotic (3).

Let us now discuss a possible nonsingular behaviour of the early Universe, such that the scale factor \( a(t) \) undergoes a small but nonzero regular minimum \( a_{\text{min}} \) at some instant \( t = t_0 \), so that \( a'(t_0) = 0, \ a''(t_0) > 0 \). Eqs. (2) then give

\[
\text{sign} \rho = K_0, \quad \rho + 3p < 0.
\]

Suppose again that the model dynamics is dominated by a scalar field \( \phi \) and that near \( a_{\text{min}} \) the potential is negligible compared to the kinetic term, \( |V| \ll |\phi'^2| \). Then, for \( \phi = \phi(t) \), we obtain \( p \approx \rho \approx \varepsilon \phi'^2 \), and it is straightforward to find that the conditions (10) hold only in case \( K_0 = -1, \ \varepsilon = -1 \). Thus, among massless scalar fields, only the “phantom” one can lead to a bouncing isotropic cosmology, and only in hyperbolic models with Lobachevsky 3-geometry.

One can note that the kinetic term, whose effective equation of state is \( p = \rho \), evolves like \( a^{-6} \) [see (4)] whereas the potential term behaves qualitatively as a cosmological constant, therefore its neglect at small \( a(t) \) is justified.

### 3. A multidimensional \( p \)-brane model

#### 3.1. General features

Consider the action of \( D \)-dimensional gravity interacting with a dilatonic scalar field \( \Phi \) and antisymmetric forms \( F_s, F_p \), which account for contributions from both the Neveu-Schwarz — Neveu-Schwarz (NS-NS) and Ramond-Ramond (RR) sectors:

\[
S_3 = \int d^Dx \sqrt{g} \left\{ R - \omega \left( \frac{\partial \Phi}{\phi'} \right)^2 - \sum_s \frac{1}{n_s!} F_s^2 - \sum_r \frac{1}{n_r!} F_r^2 \right\}.
\]

where \( R \) is the scalar curvature, \( g = |\det g_{MN}| \), \( (\partial \Phi)^2 = g^{MN} \partial_M \Phi \partial_N \Phi \), \( M, N = 0, \ldots, D-1 \), \( \omega \) is a (Brans-Dicke type) coupling constant, \( n_s \) and \( n_r \) are the ranks of antisymmetric forms belonging, respectively, to the NS-NS and RR sectors of the effective action; for each \( n \)-form, \( F^2_n = F_{n,M_1 \ldots M_n} F^{M_1 \ldots M_n} \).

The action (11) is written in the so-called Jordan conformal frame where the field \( \Phi \) is nonminimally coupled to gravity. This form is actually obtained in the weak field limit of many underlying theories as the framework describing the motion of fundamental objects, therefore we will interpret the metric \( g_{MN} \) appearing in (11) as the physical metric. Thus, if the fundamental objects are strings, one has in any dimension \( \omega = -1 \), while in cases where such objects are \( p \)-branes, one finds [4]

\[
\omega = -\frac{(D-1)(p-1) - (p+1)^2}{(D-2)(p-1) - (p+1)^2},
\]

where \( p \) is the brane dimension and \( D \) is the space-time dimension. The NS-NS sector of string theory predicts a Kalb-Ramond type field with \( n_s = 3 \); the type IIA superstring effective action contains RR terms with \( n_r = 2, 4 \), while type IIB predicts \( n_r = 3, 5 \). The action (11) may also represent the bosonic sectors of theories like 11-dimensional supergravity (where the dilaton is absent, and there is a 4-form gauge field), or 10-dimensional supergravity (there is a dilaton and a 3-form gauge field), or 12-dimensional “field theory of F-theory” (10), admitting the bosonic sector of 11-dimensional supergravity as a truncation. The model (10) contains a dilaton and two \( F \)-forms of ranks 4 and 5; it admits electric 2- and 3-branes and magnetic 5- and 6-branes.

The standard transformation

\[
g_{MN} = \Phi^{-2/(D-2)} g_{MN}
\]
leads to a theory reformulated in the Einstein conformal frame, more convenient for solving the field equations:

\[
S_E = \int d^Dx \sqrt{g_E} \left\{ \mathcal{R} - \eta_\omega (\partial \varphi)^2 - \sum_s \frac{n_s}{n_s!} e^{2\lambda_s \varphi} F_s^2 - \sum_r \frac{n_r}{n_r!} e^{2\lambda_r \varphi} F_r^2 \right\}
\]  

(14)

where all quantities are written in terms of the Einstein-frame metric \( \overline{g}_{MN} \); \( g_E = |\det \overline{g}_{MN}| \); for the scalar field we have denoted

\[
\Phi = e^{\varphi/\omega_1}, \quad \omega_1 = \sqrt{\omega + D - 2}; \quad \eta_\omega = \text{sign} \left( \omega + \frac{D-2}{2} \right).
\]  

(15)

while the coupling constants \( \lambda_s \) and \( \lambda_r \) are

\[
\lambda_s = \frac{n_s - 1}{\omega_1(D-2)} \quad \text{(NS-NS sector)}; \quad \lambda_r = \frac{2n_r - D}{2\omega_1(D-2)} \quad \text{(RR sector)}.
\]  

(16)

The sign factor \( \eta_\omega \) distinguishes “normal” theories \((\eta_\omega = +1)\), such that the kinetic term of the \( \varphi \) field in \( (14) \) has the normal sign corresponding to positive energy, from anomalous theories where this sign is “wrong” \((\eta_\omega = -1)\).

The factor \( \eta_\omega \) is thus quite similar to \( \varepsilon \) used in Sec. 2. It should be noted that many theories with \( \eta_\omega = -1 \) involve \( \eta_\omega = -1 \). According to \( (15) \),

\[
\frac{\eta_\omega}{\omega_1^2} = (D-2) \left[ 1 - \frac{(D-2)(p-1)}{(p+1)^2} \right].
\]  

(17)

Evidently, under the condition \((D-2)(p-1) > (p+1)^2\) we have \( \eta_\omega = -1 \). For \( p = 2, 5 \) this happens when \( D > 11 \), and for \( p = 3, 4 \) when \( D > 10 \).

The following table gives the values of \( \omega \) and \( \eta_\omega/\omega_1^2 \) for some particular space-time and brane dimensions.

| \( D \) | \( p \) | \( \omega \) | \( \eta_\omega/\omega_1^2 \) | \( D \) | \( p \) | \( \omega \) | \( \eta_\omega/\omega_1^2 \) |
| --- | --- | --- | --- | --- | --- | --- | --- |
| any | 1 | -1 | \( D - 2 \) | 12 | 2 | -2 | -10/9 |
| 10 | 2 | 0 | 8/9 | 12 | 3 | -3/2 | -5/2 |
| 10 | 3 | \( \infty \) | 0 | 12 | 4 | -8/5 | -2 |
| 10 | 4 | 2 | 8/25 | 12 | 5 | -2 | -10/9 |
| 10 | 5 | 0 | 8/9 | 12 | 6 | -6 | -10/49 |
| 10 | 6 | -4/9 | 72/49 | 12 | 7 | 1/2 | 5/8 |
| 11 | 2 | \( \infty \) | 0 | 12 | 8 | -4/11 | 110/81 |
| 11 | 3 | -2 | -9/8 | 14 | 2 | -4/3 | -4 |
| 11 | 4 | -5/2 | -18/25 | 14 | 6 | -16/11 | -132/49 |
| 11 | 5 | \( \infty \) | 0 | 26 | 3 | -17/16 | -48 |
| 11 | 6 | 1/4 | 36/49 | 26 | 4 | -50/47 | -1128/25 |

Some comments are in order. First, the well-known result \( \omega = -1 \) for strings \((p = 1)\) in any dimension is recovered. Second, one obtains \( \omega = \infty \) for 2- and 5-branes in 11 dimensions, which conforms to the absence of a dilaton in 11D supergravity that predicts such branes. Third, in 12 dimensions one has \( \eta_\omega = -1 \) for \( p < 7 \), and such a theory \[10\] does contain a pure imaginary dilaton: the \( F \)-forms of ranks 4 and 5 are coupled to a dilaton field \( \varphi \) with the coupling constants \( \lambda_4 = -1/10 \) and \( \lambda_5 = -\lambda_1 \), respectively, while the product \( \lambda \varphi \) is real. As is concluded in Ref. \[10\], for \( D > 11 \) “imaginary couplings are exactly what is needed in order to make a consistent truncation to the fields of type IIB supergravity possible”. In our (equivalent) formulation, \( \varphi \) and \( \lambda \) are real and the unusual nature of the coupling is reflected in the sign factor \( \eta_\omega \). Supersymmetric models with \( D = 14 \) are also discussed \[11, 12\], while \( D = 26 \) is the well-known dimension for bosonic strings.

3.2. Solutions

There is a diversity of exact solutions (discussed, in particular, in Refs. \[13, 14, 15\], see also references therein) for the action \[14\] without \( L_m \), in space-times with the metric \( \overline{g}_{AB} \) of the form

\[
ds_E^2 = -e^{2\alpha(u)} du^2 + \sum_{i=0}^{\infty} e^{2\beta(u)} ds_i^2
\]  

(18)

where \( u \) is a time coordinate and \( ds_i^2 \) are \( u \)-independent metrics of internal \( d_i \)-dimensional factor spaces assumed to be Ricci-flat for \( i = 1, \ldots, n \) whereas \( ds_0^2 \) describes the “external” (observed) space of constant curvature \( K_0 = 0, \pm 1 \), corresponding to the three types of isotropic spaces.
We will be only interested here in cosmological solutions for a very simple special case: a single antisymmetric form $F_{[d_0]}$ from the NS-NS or RR sector, having a single (up to permutations) nontrivial component $F_{1 \ldots d_0}$ where the indices refer to the external space $M_0$ with the metric $ds_0^2$, a single internal space $M_1$ with the metric $ds_1^2$, so that in [15], $i = 0, 1,$ and $\varphi = \varphi(u)$. Then the field equations are easily integrated.

Let $u$ be a harmonic time coordinate for the metric [15], so that the coordinate condition is $\alpha = d_0 \beta^0 + d_1 \beta^1$.

The $F$-form is magnetic-type; the Maxwell-like equations due to [14] are satisfied trivially while the Bianchi identity $dF = 0$ implies

$$F_{1 \ldots d_0} = Q \sqrt{g_0}, \quad Q = \text{const},$$

where $g_0$ is the metric determinant corresponding to $ds_0^2$ and $Q$ is a charge, to be called the axionic charge since the only nonzero component of $F$ can be represented in terms of a pseudoscalar axion field in $d_0 + 1$ dimensions. The remaining unknowns are $\beta^0$, $\beta^1$ and $\varphi$.

In the Einstein equations $R^N_M - \frac{1}{2} \delta^N_M R = T^N_M$, written for the Einstein-frame metric [15], the stress-energy tensor $T^N_M$ has the property $T^u_u + T^z_z = 0$ (where $z$ belongs to $M_0$), and the corresponding Einstein equation has the Liouville form $\ddot{\alpha} - \beta^0 + K_0 (d_0 - 1)^2 e^{2\alpha - 2\beta^0}$, whence

$$\frac{1}{d_0 - 1} e^{\beta_0 - \alpha} = S(-K_0, k, u) \equiv \begin{cases} e^{ku}, & K_0 = 0, \ k \in \mathbb{R}; \\ k^{-1} \cosh ku, & K_0 = 1, \ k > 0; \\ k^{-1} \sinh ku, & K_0 = -1, \ k > 0; \\ u, & K_0 = 0, \ k = 0; \\ k^{-1} \sin ku, & K_0 = -1, \ k < 0, \\ k^{-1} \sinh ku, & K_0 = 0, \ k = 0; \\ k^{-1} \sin ku, & K_0 = 1, \ k < 0, \\ \end{cases}$$

(20)

where $k$ is an integration constant and one more constant is suppressed by a proper choice of the origin of $u$. Eq. (20) can be used to express $\beta^0$ in terms of $\beta \equiv \beta^1$.

It is helpful to consider the remaining unknowns as a vector $x^A = (\beta^1, \varphi)$ in the 2-dimensional target space $\mathbb{V}$ with the metric

$$(G_{AB}) = \begin{pmatrix} d_1 & 0 \\ 0 & \eta_\omega \end{pmatrix}, \quad (G^{AB}) = \begin{pmatrix} 1/(d_1) & 0 \\ 0 & \eta_\omega \end{pmatrix}, \quad d \overset{\text{def}}{=} \frac{D - 2}{d_0 - 1}. \quad (21)$$

The equations of motion then take the form

$$\ddot{x}^A = -\eta_\omega Q^2 Y^A c^{2y} \quad (22)$$

$$G_{AB} \ddot{x}^A \dot{x}^B + \eta_\omega Q^2 c^{2y} = \frac{d_0}{d_0 - 1} K, \quad K = \begin{cases} k^2 \text{sign} k, & K_0 = -1, \\ k^3, & K_0 = 0, +1. \\ \end{cases} \quad (23)$$

with the function $y(u) = d_1 \beta^1 + \lambda \varphi$, representable as a scalar product of $x^A$ and the constant vector $\vec{Y}$ in $\mathbb{V}$:

$$y(u) = Y_A x^A, \quad Y_A = (d_1, \ \lambda), \quad Y^A = (1/d_1, \ \eta_\omega \lambda). \quad (24)$$

Eq. (22) is a first integral of (22) that follows from the $y(u)$ component of the Einstein equations.

The simplest solution corresponds to $Q = 0$ (scalar vacuum):

$$\beta^1 = c^1 u + \xi^1, \quad \varphi = \varphi^u + \xi^u, \quad (25)$$

where $c^1$, $\xi^1$, $\varphi^u$ and $\xi^u$ are integration constants. Due to (23), the constants $c^A = (c^1, \ \varphi)$ are related by

$$c_A c^A = d_1 (c^1)^2 + \eta_\omega c^2 = \frac{d_0}{d_0 - 1} K. \quad (26)$$

If $Q \neq 0$, Eqs. (22) combine to yield an easily solvable (Liouville) equation for $y(u)$:

$$\ddot{y} + \eta_\omega Q^2 Y^2 c^{2y} = 0, \quad Y^2 = Y_A Y^A = \frac{d_1}{d} + \eta_\omega \lambda^2. \quad (27)$$

This is a special integrable case of the equations considered, e.g., in Refs. [13] [14] [15]. Assuming $Q^2 Y^2 > 0$, Eq. (27) gives

$$e^{-y(u)} = h^{-1} |Q| Y \cosh[h(u + u_1)] \quad (28)$$

Even for $\eta_\omega = -1$ one has $Y^2 > 0$ for fields from the NS-NS sector in any dimension and for fields from the RR sector if $D < 17$. 

where \( Y = |Y^2|^{1/2} \), \( h > 0 \) and \( u_1 \) are integration constants. The unknowns \( x^A \) are expressed in terms of \( y \) as follows:

\[
x^A = \frac{Y^A}{Y^2}y(u) + c^Au + c^A
\]

where the constants \( c^A = (c_1, c_\phi) \) and \( c^A = (c_1', c_\phi') \) satisfy the orthogonality relations

\[
c^AY_A = 0, \quad c^A Y_A = 0.
\]

Finally, the constraint \( \Delta_c \) leads to one more relation among the constants:

\[
\frac{h^2}{Y^2} + c_Ac^A = \frac{d_0}{d_0 - 1} K.
\]

3.3. Analysis of cosmological models

In what follows, we put \( d_0 = 3 \), so that \( d_1 = D - 4 \), and identify, term by term, the Jordan-frame metric \( ds_3^2 \) obtained in the above notations \( \Delta_{15}, \Delta_{18} \),

\[
ds_3^2 = \exp \left[ -\frac{2\varphi}{\omega_1(D - 2)} \right] \left\{ \frac{e^{-d_1b^1}}{2S(-K_0, k, u)} \left[ -\frac{d_1b^1}{4S^2(-K_0, k, u)} + ds_0^2 \right] + e^{2\beta_1^1}ds_1^2 \right\},
\]

where the function \( S(., ., .) \) is defined in \( \Delta_{20} \), with the familiar form of the metric

\[
ds_3^2 = -dt^2 + a^2(t)ds_0^2 + b^2(t)ds_1^2,
\]

so that \( a(t) \) and \( b(t) \) are the external and internal scale factors and \( t \) is the cosmic time.

To select nonsingular models, let us use the Kretschmann scalar \( \mathcal{K} = R_{MNPQ}R^{MNPQ} \), which is in our case a sum (with positive coefficients) of squares of all Riemann tensor components \( R_{MNPQ} \). Thus as long as \( \mathcal{K} \) is finite, all algebraic curvature invariants of this metric are finite as well. For the metric \( \Delta_{33} \) with \( d_0 = 3 \) one has (the primes denote \( d/dt \)):

\[
\mathcal{K} = 4 \left[ 3 \left( \frac{u''}{a} \right)^2 + d_1 \left( \frac{b''}{b} \right)^2 + 3d_1 \left( \frac{a'b'}{ab} \right)^2 \right] + 2 \left( \frac{K_0 + a^2}{a^2} \right)^2 + d_1(d_1 - 1) \left( \frac{b^4}{b^4} \right).
\]

By \( \Delta_{35} \), \( \mathcal{K} \to \infty \) and hence the space-time is singular when \( a \to 0 \), \( a \to \infty \), \( b \to 0 \) or \( b \to \infty \) at finite proper time \( t \). Accordingly, our interest will be in the asymptotic behaviour of the solutions at both ends of the range \( \mathbb{R}_u = (u_{\min}, u_{\max}) \) of the time coordinate \( u \), defined as the range where both \( a^2 \) and \( b^2 \) in \( \Delta_{33} \) are regular and positive. (Note that, as \( t \to \pm \infty \), a singularity does not occur when \( b(t) \to 0 \), or \( a \to 0 \) in case \( K_0 = 0 \).) At any \( u \in \mathbb{R}_u \) all the relevant functions are manifestly finite and analytical. The boundary values \( u_{\max} \) and \( u_{\min} \) may be finite or infinite; a finite value of \( u_{\max} \) or \( u_{\min} \) coincides with a zero of the function \( \Delta_{20} \).

Among regular solutions, of utmost interest are those in which \( a(t) \) grows while \( b(t) \) tends to a finite constant value as \( t \to \infty \). Any asymptotic may on equal grounds refer to the evolution beginning or end due to the time-reversal invariance of the field equations. We will for certainty speak of expansion or inflation, bearing in mind that the same asymptotic may mean contraction (deflation).

Let us now enumerate the possible kinds of asymptotics.

**Type I**: \( u \to \pm \infty \),

\[
dt^2 \sim e^{(A - 2k)|u|}du^2, \quad a^2 \sim e^{A|u|}, \quad b^2 \sim e^{B|u|},
\]

with \( k > 0 \) and the constants \( A \) an \( B \), depending on the parameters of the solution. An asymptotic of interest for \( a(t) \) takes place if \( A \geq 2k > 0 \):

(i) \( A > 2k \): \( t \to \infty \), \( a \sim t^{A/(A - 2k)} \) (power-law inflation);
(ii) \( A = 2k \): \( t \sim |u| \to \infty \), \( a \sim e^{kt} \) (exponential inflation).

A reformulation for \( k < 0 \) is evident. The scale factor \( b(t) \) tends to a finite limit if \( B = 0 \), i.e., under a special condition on the model parameters (fine tuning).

**Type Ia**: a modification of type I when \( k = 0 \), so that at \( u \to \infty \)

\[
dt^2 \sim u^{-3}e^{Au}du^2, \quad a^2 \sim u^{-1}e^{Au}du^2, \quad b^2 \sim e^{Bu}
\]
If $A > 0$, we have, as desired, $t \to \infty$ and $a \to \infty$; the expansion may be called “slow inflation” since it is only slightly quicker than linear: the derivative $da/dt \sim u$, which behaves somewhat like $\ln t$. If $A \leq 0$, then $a \to 0$ at finite $t$ (singularity). As for $b(t)$, one may repeat what was said in case I.

**Type II:** $u \to 0$, where the function $\eta$ tends to zero, so that $S(-K_0,k,u) \sim u$, while other quantities involved are finite. In this case

$$
 dt^2 \sim 1/u^3, \quad a^2 \sim 1/u, \quad b^2 \to \text{const} > 0.
$$

According to (34), $t \to \pm \infty$, $a(t) \sim |t|$ (linear expansion or contraction), whereas both $b(t)$ and $\varphi(t)$ tend to finite limits since they do not depend on $S(-K_0,k,u)$.

The dilaton $\varphi$ in all cases behaves like $\ln b(t)$, but, in general, with another constant $B$ in each particular solution.

This exhausts the possible kinds of asymptotics for $Y^2 > 0$. Solutions with $Y^2 \leq 0$, which can emerge when $\eta_\omega = -1$, may have other asymptotics, but they are of lesser interest.

**Scalar-vacuum cosmologies**

The scalar-vacuum models (32), (25) depend on two input constants, $\eta$, $\omega$ limits since they do not depend on integration constants.

**Type II:** $K = K_0$, $S = k/\cosh k\nu$, $k > 0$, hence the solution has two type I asymptotics at $u \to \pm \infty$, with $k > 0$ and the following constants $A = A_\pm$:

$$
 A_\pm = -k \mp \left[ d_1 e^1 + c_\varphi / d\omega_1 \right],
$$

so that at least at one of the asymptotics $A < 0$ whence $a \to 0$ at finite $t$, a singularity. The behaviour of $b(t)$ is also singular.

**Spatially flat models, $K_0 = 0.$** One has simply

$$
 a^2(t) = e^{Au}, \quad dt \sim e^{(A-2k)u/2} du,
$$

where $A = -c_\varphi/(d\omega_1) - d_1 e^1 - k$, $k \in \mathbb{R}$, and again $b^2(t) = e^{Bu}$, $B = \text{const}$. Thus each of the scale factors is either constant, or evolves between zero and infinity, and $a = 0$ occurs at finite $t$.

**Hyperbolic models, $K_0 = -1.$** If $k > 0$ [note that, when $\eta_\omega = 1$, there is necessarily $k > 0$ due to (26)], one has in (92) $S = k^{-1}\sinh k\nu$. Hence the model evolves between a type I asymptotic at $u \to \infty$, with $A$ coinciding with $A_+$ in Eq. (48), and type II at $u = 0$. Since type II is regular, a necessary condition for having a nonsingular model is $A \geq 2k$.

To find out if and when it happens for $\eta_\omega = +1$, it is convenient to introduce, instead of the two constants $c^1$ and $c_\varphi$ connected by (26), an “angle” $\theta$ such that

$$
 c^1 = \sqrt{\frac{3}{2d_1}} k \cos \theta, \quad c_\varphi = \sqrt{\frac{3}{2}} k \sin \theta.
$$

The condition $A \geq 2k$ will be realized for a certain choice of the integration constants if $A_+$ given by (48) has, as a function of $\theta$, a maximum no smaller than $2k$. An inspection shows that it happens if

$$
 \omega_1^2 \leq 1/[d(6d - d_1)] = 1/[(D-1)(D-2)].
$$

This is the only example of a nonsingular (bouncing) vacuum model with $\eta_\omega = +1$.

In case $k > 0$, $\eta_\omega = -1$, a choice of $c_\varphi$ and $c^1$ subject to (26) such that $A > 2k$ is easily made for any $\omega_1$.

For $\eta_\omega = -1$, $k = 0$, the model evolves between type Ia and II asymptotics, where at the Ia end ($u \to \infty$)

$$
 A = -d_1 c^1 - c_\varphi / (d\omega_1), \quad B = 2c^1 - c_\varphi / (d\omega_1).
$$

The necessary condition for regularity, $A > 0$, is satisfied for proper $c_1$ and $c_\varphi$ which can be chosen without problems.

In case $\eta_\omega = -1$, $k < 0$, the function (26) is simply $|k|^{-1}\sin |k|u$, and the model has two type II asymptotics at adjacent zeros of $S$, say, $u = 0$ and $u = \pi/|k|$. This model is automatically nonsingular for any further choice of integration constants.
We conclude that among vacuum models only some hyperbolic ones can be nonsingular. For $\eta_\omega = +1$ in such a case $a(t)$ evolves from linear decrease to inflation, or from deflation to linear growth. Only in the latter case both $b(t)$ and $\varphi$ tend to finite limits as $t \to \infty$ without any fine tuning.

For $\eta_\omega = -1$ there is a model interpolating between two asymptotics of the latter kind. Thus, as $t$ changes from $-\infty$ to $+\infty$, $a(t)$ bounces from linear decrease to linear increase (generically with a different slope) whereas $b(t)$ and $\varphi(t)$ smoothly change from one finite value to another. The latter model exists for generic values of the integration constants.

**Cosmologies with an axionic charge**

The solution contains, in addition to the input parameters $D$, $\omega$ and $\lambda$, three independent essential integration constants: the “scale parameter” $k$, the charge $Q$ and also $h$ and $c_\varphi$ connected by (31); the constant $c^1$ is excluded by the first relation (30)

$$d_1 c^1 + \lambda c_\varphi = 0$$

so that the quantity $c^1 c_A$, appearing in (31), is expressed as $c^1 c_A = \eta_\omega(d/dt)c_\varphi^2 Y^2$. The fourth constant, the “shift parameter” $u_1$, as well as $c^1$ and $c_\varphi$, connected by (31), are qualitatively inessential.

Let us begin with “normal” models, $\eta_\omega = +1$. The solution (29) has the form

$$\beta^1(u) = \frac{1}{dY^2} y(u) + c^1 u + \mathcal{A}, \quad \varphi(u) = \frac{\lambda}{Y^2} y(u) + c_\varphi u + \mathcal{B},$$

where the function $y(u)$ is fiven by (28). The condition (31) leads to $k > 0$ and strongly restricts the possible model behaviour. Thus, it can be shown [3] that for all $K_0$ one of the asymptotics belongs to type I with the constants

$$A = -k + \frac{h}{dY^2} \left( d_1 + \frac{\lambda \eta_\omega}{\omega_1} \right) + c_\varphi \left( \lambda - \frac{1}{d \omega_1} \right),$$

$$B = \frac{h}{dY^2} \left( -2 + \frac{\lambda \eta_\omega}{\omega_1} \right) - c_\varphi \left( \frac{2 \lambda}{d_1} + \frac{1}{d \omega_1} \right).$$

The necessary condition for regularity $A > 2k$ may be fulfilled for small $\omega_1$, satisfying the condition (11), just as in the vacuum case. The constant $B$ may then have any sign, and only a special choice of the ratio $c_\varphi/h$ (fine tuning) can lead to $B = 0$, providing a finite limit of $b(t)$.

The second asymptotic depends on $K_0$. For closed and flat models it is again type I, and a nonsingular behaviour is again acheived by fine tuning. For $K_0 = -1$, the second asymptotic belongs to type II and is always regular. Thus particular models with an axionic charge may be even regular for $\eta_\omega = 1$) and, in addition, may be inflationary as $t \to \infty$.

The “anomalous” models with $\eta_\omega = -1$, just as in the vacuum case, are more diverse due to arbitrariness in the sign of $k$. For models $K_0 = 0$, $-1$, as well as for hyperbolic ones with $k \geq 0$, the restriction (11) is no more valid, but the type I and Ia asymptotics are again only nonsingular for special values of the parameters.

Lastly, the models $K_0 = -1$, $\eta_\omega = -1$, $k < 0$, as their vacuum counterparts, interpolate between two type II asymptotics and have the same qualitative features.

### 4. Vacuum multidimensional models with integrable Weyl geometry

Let us discuss multidimensional cosmological models assuming that space-time possesses $D$-dimensional Weyl geometry characterized by the metric $g_{MN}$ and the connection

$$\Gamma^A_{BC} = \tilde{\Gamma}^A_{BC} - \frac{1}{2} \left( \sigma_B \delta^A_C + \sigma_C \delta^A_B - g_{BC} \sigma^A \right)$$

where $\tilde{\Gamma}^A_{BC}$ are the Christoffel symbols for the metric $g_{AB}$, $\sigma$ is a scalar field and $\sigma_A = \partial_A \sigma$. The gravitational field is described by the tensor $g_{AB}$ and the scalar $\sigma$, as in scalar-tensor theories (STT). Just as in STT, the gravitational Lagrangian may contain different invariant combinations of $g_{AB}$ and $\sigma$. Restricting ourselves to Lagrangians which are linear in the curvature and quadratic in $\sigma_A$, we can write:

$$L = f(\sigma) \mathcal{R} - h(\sigma) \sigma^A \sigma_A - 2\Lambda(\sigma) + L_m$$

where $\mathcal{R}$ is the Weyl scalar curvature, obtained from the connection (46), $f$, $h$ and $\Lambda$ are arbitrary functions, and $L_m$ is the non-gravitational matter Lagrangian.
The field equation are simplified if one expresses the Weyl curvature $\mathcal{R}$ in terms of the Riemannian curvature $R$, corresponding to the metric $g_{AB}$:

$$\mathcal{R} = R - (D-1)\Box - \frac{1}{4}(D-1)(D-2)\sigma^A\sigma_A$$

($R$ and $\Box$ are formed from the Riemannian connection $\tilde{\Gamma}^A_{BC}$) and passes to the Einstein conformal picture with the aid of the transformation $g_{MN} = f^{-2/(D-2)}\mathcal{R}_{MN}$. Omitting a total divergence, we arrive at the following form of the Lagrangian:

$$\mathcal{L} = \mathcal{R} - F(\sigma)g^{AB}\sigma_A\sigma_B + f^{-D/(D-2)}[-2\Lambda(\sigma) + L_m]$$

where $\mathcal{R}$ is the Riemannian scalar curvature for the metric $\mathcal{R}_{AB}$, $A_\sigma = dA/d\sigma$ and

$$F(\sigma) = \frac{1}{f^2}\left[ fh - (D-1)f\left( f_\sigma + \frac{D-2}{4}\right) + \frac{D-1}{D-2}f^2 \right].$$

Consider vacuum ($L_m = 0$) cosmological models with the metric $\mathcal{L}^2$, assuming $\Lambda \equiv 0$. One can easily find that the substitution $\sigma \to \varphi$, such that $d\varphi/d\sigma = \sqrt{|F(\sigma)|}$, leads the action with the Lagrangian $\mathcal{L}^2$ to a form coinciding with $\mathcal{L}$ without an $F$-form and with the sign factor $\eta_{\varphi}$ replaced with sign $F(\sigma)$. Therefore, in the Einstein picture, the vacuum cosmologies of the gravitation theory with Weyl integrable space-time are entirely identical to the scalar-vacuum models from Sec. 3 (both types of models are equivalent to those of multidimensional general relativity with a massless, minimally coupled scalar field).

A difference can appear after a transition to Jordan’s picture since the conformal factors $\Phi^{-2/(D-2)}$ in Sec. 3 and $f^{-2/(D-2)}$ in the present section are, in general, different. However, if one supposes that the function $f(\sigma)$ is finite and smooth in the whole essential range of $\sigma$, then the qualitative properties of the models (as regards their regular or singular behaviour) coincide in the Einstein and Jordan pictures. Accordingly, in both pictures, the conclusion that hyperbolic models in the presence of a phantom scalar field contain a class of nonsingular bouncing models, preserves its generality. Such models do not require any fine tuning, and in all of them both the scalar field and the extra-dimension scale factors change in finite limits.

Stability of the qualitative features of these models with respect to addition of other kinds of matter has been confirmed by a numerical study of Weyl cosmologies with one more scalar field representing ordinary matter $\mathcal{L}$.

5. Concluding remarks

Phantom scalar fields are rather widely discussed as one of the dark energy candidates, able to explain the present accelerated expansion of the Universe, see Sec. 2 and, in more detail, e.g., $\mathcal{L}$. Such fields, if any, may also dominate in the early Universe, at small values of the scale factor $a(t)$, above all, due to maximum stiffness ($\rho = p$) of the equation of state of a massless scalar field, so that their energy density $\rho$ grows with falling $a(t)$ more rapidly than for other kinds of matter: e.g., in 4D FRW models one has $\rho \sim a^{-6}$. For the same reason, massive scalar fields, or fields with potentials, should actually behave as massless ones at small $a(t)$. Therefore the above conclusion that an initial cosmological singularity may be avoided without fine tuning in hyperbolic models ($K_0 = -1$) due to phantom scalar fields seems to be rather general.

This conclusion proves to be even more important in multidimensional cosmologies. If extra dimensions, being an inevitable ingredient in modern unification theories, are considered dynamically, the singularity problem becomes even more involved since, in addition to the usual cosmological scale factor, the extra dimensions can collapse or blow up, leading to a curvature singularity. However, in open models of the type discussed here, the external scale factor $a(t)$ dynamically differs from other variables. Formally, this circumstance was described here by the last line in Eq. (20): $a^2(t) \sim 1/S(-K_0, k, u)$ tends to infinity at two finite values of the harmonic coordinate $u$, which correspond to infinite physical time, whereas all other variables, such as scalar fields and internal scale factors, remain finite at these values of $u$.

The above simple models certainly do not pretend to describe the whole evolution, but only try to guess the qualitative features of the bouncing process near the maximum density state. The well-known features of standard cosmology: inflation, nucleosynthesis, particle creation etc. may follow at later times, but the mechanism described here seems to automatically provide stable compactification of the extra-dimension scale factors (if any) and constant values of scalars which may be related to coupling constants in unification theories, such as the dilaton in string theory. Cosmological and astrophysical problems related to stable compactification are discussed in Ref. $\mathcal{L}^2$, see also references therein.

I would also like to mention that such an exotic matter as a phantom scalar field, violating all standard energy conditions, in case it is concentrated in comparatively small regions of space, is precisely what is needed to create...
wormholes — and hence maybe also time machines [18]. There are many exact wormhole solutions involving phantom scalars (those in Refs. [19, 20] are probably the earliest). If such fields do exist in nature, whatever be their origin, they are, in principle, a ready construction material for wormholes.

Observations are known to yield the total cosmological density factor $\Omega_0$ smaller or close to unity; meanwhile, the presently popular spatially flat cosmologies, most convenient for various calculations, require the precise equality $\Omega_0 = 1$, actually a sort of fine tuning. It is much more probable that the real Universe at least slightly violates this special requirement, leading to $K_0 = -1$ if $\Omega_0 < 1$.

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