Abstract. A result of Gersten states that if $G$ is a hyperbolic group with integral cohomological dimension $\text{cd}_\mathbb{Z}(G) = 2$ then every finitely presented subgroup is hyperbolic. We generalize this result for the rational case $\text{cd}_\mathbb{Q}(G) = 2$. In particular, our result applies to the class of torsion-free hyperbolic groups $G$ with $\text{cd}_\mathbb{Z}(G) = 3$ and $\text{cd}_\mathbb{Q}(G) = 2$ discovered by Bestvina and Mess.

1. Introduction

The cohomological dimension $\text{cd}_R(G)$ of a group $G$ with respect to a ring $R$ is less than or equal to $n$ if the trivial $RG$-module $R$ has a projective resolution of length $n$. Let $\mathbb{Q}$ denote the field of rational numbers. The main result of this note:

Theorem 1.1. Let $G$ be a hyperbolic group such that $\text{cd}_\mathbb{Q}(G) \leq 2$. If $H$ is a finitely presented subgroup, then $H$ is hyperbolic.

The analogous statement for $\text{cd}_\mathbb{Z}(G)$ is a result of Steve Gersten that we recover as a consequence of the inequality

$$\text{cd}_\mathbb{Q}(G) \leq \text{cd}_\mathbb{Z}(G).$$

Corollary 1.2 (Gersten). Let $G$ be a hyperbolic group such that $\text{cd}_\mathbb{Z}(G) = 2$. If $H$ is a finitely presented subgroup, then $H$ is hyperbolic.

The first motivation to generalize Gersten’s result to the rational case is the existence of hyperbolic groups of integral cohomological dimension three and rational cohomological dimension two. The nature of finitely presented subgroups of groups in this class was not known. The first examples of such groups where discovered by Bestvina and Mess [BM91] based on methods by Davis and Januszkiewicz [DJ91].

The class also contains finite index subgroups of hyperbolic Coxeter groups, examples that were discovered by Dranishnikov [Dra99, Corollary 2.3]. We recall the nature of Bestvina-Mess examples in the following corollary.

Corollary 1.3. Let $X$ be a finite polyhedral 3-complex such that

- $X$ admits piecewise constant negative curvature cellular structure satisfying Gromov link condition, and
- $X$ is 3-manifold (without boundary) in the complement of a single vertex whose link is a non-orientable closed surface.

If $G = \pi_1X$ then $\text{cd}_\mathbb{Q}(G) = 2$, $\text{cd}_\mathbb{Z}(G) = 3$ and any finitely presented subgroup of $G$ is hyperbolic.

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The statement of Corollary 1.2 is sharp in the sense that there exist hyperbolic groups of integral cohomological dimension three containing finitely presented subgroups that are not hyperbolic, the first example was found by Noel Brady [Bra99]. More recently, infinite families of hyperbolic groups of integral cohomological dimension three containing non-hyperbolic finitely presented subgroups have been constructed, see for example [KG18].

Corollary 1.4. If \( G \) is a hyperbolic group such that \( \text{cd}_\mathbb{Z}(G) = 3 \) and it contains a non-hyperbolic finitely presented subgroup, then \( \text{cd}_\mathbb{Q}(G) = \text{cd}_\mathbb{Z}(G) \).

A second motivation of this project was to generalize Gersten’s result to groups admitting torsion, specifically, to the class of hyperbolic groups \( G \) admitting a 2-dimensional classifying space for proper actions \( E_G \). Recall that a model for \( E_G \) is a \( G \)-CW-complex \( X \) with the property that for each subgroup \( H \) the subcomplex of fixed points is contractible if \( H \) is finite, and empty if \( H \) is infinite. The minimal dimension of a model for \( E_G \) is denoted by \( \text{gd}(G) \). Considering the cellular chain complex with rational coefficients of a model for \( E_G \) with minimal dimension shows that

\[
\text{cd}_\mathbb{Q}(G) \leq \text{gd}(G).
\]

This inequality implies the following corollary.

Corollary 1.5. If \( G \) is a hyperbolic group such that \( \text{gd}(G) \leq 2 \), then any finitely presented subgroup is hyperbolic.

The statement of Corollary 1.5 was known in the following cases:

- If \( G \) admits a \( \text{CAT}(-1) \) 2-dimensional model for \( E_G \), see [HMP14, Corollary 1.5].
- If \( G \) admits a 2-dimensional model for \( E_G \), and \( H \) is finitely presented with finitely many conjugacy classes of finite groups, a consequence of [MP17, Theorem 1.3].
- If \( G \) is a hyperbolic small cancellation group of type \( C(7) \), \( C(5)-T(4) \), \( C(4)-T(5) \), \( C(3)-T(7) \) or \( C'(1/6) \), see [Ger96b, Theorem 7.6].

We remark that for a group \( G \) satisfying the hypothesis of Corollary 1.5, the conclusion follows from Gersten’s result 1.2 if, in addition, \( G \) is assumed to be virtually torsion free. It is an outstanding question whether hyperbolic groups are virtually torsion free [KW00].

Homological filling functions and the Proof of Theorem 1.1. Let \( R \) be a subring of \( \mathbb{Q} \). The \((n+1)\)-dimensional homological Filling Volume function over \( R \) of a cellular complex \( X \) is a function \( \text{FV}^{n+1}_{X,R} : \mathbb{N} \to \mathbb{R} \) describing the minimal volume required to fill integral cellular \( n \)-cycles with cellular \((n+1)\)-chains with coefficients in \( R \).

For a group \( G \) with a \( K(G,1) \) model \( X \) with finite \((n+1)\)-skeleton, the \((n+1)\)-dimensional homological Filling Volume function over \( R \) of \( G \), denoted by \( \text{FV}^{n+1}_{G,R} \), is defined as \( \text{FV}^{n+1}_{\tilde{X},R} \) where \( \tilde{X} \) is the universal cover of \( X \). This function depends only on the group \( G \) up to the equivalence relation on the set of non-decreasing functions \( \mathbb{N} \to \mathbb{R} \) defined as \( f \sim g \) if and only if \( f \preceq g \) and \( g \preceq f \), where \( f \preceq g \) means there is \( C > 0 \) such that for all \( k \in \mathbb{N} \),

\[
f(k) \leq Cg(Ck + C) + Ck + C.
\]
Recall that a group $G$ is of type $R$-$FP_n$ if the trivial $RG$-module $R$ admits a partial projective resolution

$$P_n \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow R \rightarrow 0$$

where each $P_i$ is a finitely generated $RG$-module. In [HMP16], it is shown that to define $FV_{G,R}^{n+1}$ it is enough to assume that the group $G$ is of type $Z$-$FP_{n+1}$. We prove that the same statement holds for $FV_{G,R}^{n+1}$ in Section 3. The main technical result of this note is the following.

**Theorem 1.6.** Let $R$ be a subring of $\mathbb{Q}$. Let $G$ be a group of type $R$-$FP_{n+1}$ and suppose $\text{cd}_R(G) = n + 1$. Let $H \leq G$ be a subgroup of type $R$-$FP_{n+1}$. Then there is a constant $C > 0$ such that for all $k$

$$FV_{H,R}^{n+1} \leq FV_{G,R}^{n+1}.$$  

This theorem generalizes the main result of [HMP16], by considering an arbitrary subring of the rational numbers instead of only the ring of integers, and by replacing the topological assumptions $F_{n+1}$ on $G$ and $H$ with the weaker hypothesis $R$-$FP_{n+1}$.

The main result of this note, Theorem 1.1, is a consequence of Theorem 1.6 and the characterization of hyperbolic groups stated below, which is credited to Gersten [Ger96]. This characterization was revised by Mineyev [Min02, Theorem 7, statements (0) and (2)], and it was also revisited by Groves and Manning in [GM08, Theorem 2.30].

**Theorem 1.7.** [Min02, Theorem 7] [GM08, Theorem 2.30] A group $G$ is hyperbolic if and only if $G$ is finitely presented and the rational filling function $FV_{G,\mathbb{Q}}^2$ is bounded by a linear function, i.e., $FV_{G,\mathbb{Q}}^2(k) \leq k$.

**Proof of Theorem 1.7.** Let $G$ be a hyperbolic group such that $\text{cd}_{\mathbb{Q}}(G) = 2$, and let $H$ be a finitely presented subgroup. Theorem 1.7 implies that $FV_{G,\mathbb{Q}}^2$ is bounded by a linear function. By Theorem 1.6 $FV_{H,\mathbb{Q}}^2 \leq FV_{G,\mathbb{Q}}^2$. It follows $FV_{H,\mathbb{Q}}^2$ is bounded by a linear function. Then Theorem 1.7 implies that $H$ is a hyperbolic group. □

In view of Theorem 1.7 we raised the following question.

**Question 1.8.** Let $G$ be a $\mathbb{Q}$-$FP_2$ group and suppose $FV_{G,\mathbb{Q}}^2$ is bounded by a linear function. Is $G$ a hyperbolic group?

The analogous question obtained by replacing $\mathbb{Q}$ with $\mathbb{Z}$ is known to have a positive answer [Ger96b, Theorem 5.2]. One motivation behind this question is that a positive answer would imply that in our main result Theorem 1.1 $H$ can be assumed to be $\mathbb{Q}$-$FP_2$ instead of being finitely presented. Recall that $\mathbb{Q}$-$FP_2$ condition is weaker than being finitely presented, see the examples in [BR97].

The rest of the note is devoted to the definition of homological filling function and the proof of Theorem 1.6. The argument is relatively self-contained, and use and simplify ideas from [HMP16]. The main contributions of the article beside the results stated above are:

1. The definition of filling functions for arbitrary subdomains of the rationals, since the definition in [HMP16] does not generalize directly, and
(2) the replacement of topological arguments in [HMP16] by algebraic ones that allow us to prove certain statements under the weaker homological finiteness condition $R\text{-}FP_{n+1}$ instead of the topological assumption $F_{n+1}$; see Proposition 4.1 which is a construction based on the homological mapping cylinders, and Remark 4.2.

**Organization.** Preliminary definitions are included in Section 2 specifically the notions of filling norms and bounded morphisms on modules over arbitrary normed rings. Section 3 discuss the generalization of homological filling functions defined over arbitrary subdomains of the rational numbers. The last section contains the proof of Theorem 1.6.

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## 2. Filling Norms, Bounded morphisms

Let $R$ be a ring and let $\mathbb{R}$ denote the ordered field of real numbers. A norm on $R$ is a function $| \cdot | : R \to \mathbb{R}$ such that for any $r, r' \in R$
- $|r| \geq 0$ with equality if and only if $r = 0$,
- $|r + r'| \leq |r| + |r'|$, and
- $|rr_2| \leq |r_1||r_2|$ for $r_1, r_2 \in R$.

A normed ring is a ring equipped with a norm.

From here on, assume that $R$ is a normed ring. A norm on an $R$-module $M$ is a function $\| \cdot \| : M \to \mathbb{R}$ such that for any $m, m' \in M$ and $r \in R$
- $\|m\| \geq 0$ with equality if and only if $m = 0$,
- $\|m + m'\| \leq \|m\| + \|m'\|$, and
- $\|rm\| \leq |r|\|m\|$.

A function $M \to \mathbb{R}$ that satisfies the last two conditions and has only non-negative values is called a pseudo-norm.

The $\ell_1$-norm on a free $R$-module $F$ with fixed basis $\Lambda$ is defined as

$$\|\sum_{x \in \Lambda} r_xx\|_1 = \sum_{x \in \Lambda} |r_x|.$$ 

A free $R$-module with fixed basis is called a **based free module**.

**Definition 2.1** (Filling norm). A **filling norm** on a finitely generated $R$-module $M$ is defined as follows. Let $\rho: F \to M$ be a surjective morphism of $R$-modules where $F$ is a finitely generated free $R$-module with fixed basis $\Lambda$ and induced $\ell_1$-norm $\| \cdot \|_1$. The **filling norm on $M$ induced by $\rho$ and $\Lambda$** is defined as

$$\|m\|_M = \inf\{\|x\|_1 : x \in F, \rho(x) = m\}.$$

**Remark 2.2.** The following statements can be easily verified.

1. An $\ell_1$-norm $\| \cdot \|_1$ on a finitely generated free $R$-module $F$ is a filling norm.
2. A filling norm $\| \cdot \|$ on a finitely generated $R$-module $M$ is a pseudo-norm, and is regular in the sense that

$$\|rm\| = |r|\|m\|$$
for any any $m \in M$ and $r \in R$ such that $r$ is a unit and $|r|r^{-1}| = 1$.

**Definition 2.3** (Bounded Morphism). A morphism $f: M \to N$ between $R$-modules with norms $\| \cdot \|_M$ and $\| \cdot \|_N$ respectively is called bounded (with respect to these norms) if there exists a fixed constant $C > 0$ such that $\|f(a)\|_N \leq C\|a\|_M$ for all $a \in M$.

The following lemma appears in [MP17] for the case that $R$ is a group ring. The proof for an arbitrary ring is analogous, we have included the argument for the convenience of the reader.

**Lemma 2.4.** [MP17, Lemma 4.6] Morphisms between finitely generated $R$-modules are bounded with respect to filling norms.

**Proof.** First observe that if $\tilde{\varphi}: A \to B$ is a morphism between finitely generated based free $R$-modules, then for $a \in A$,

$$\|\tilde{\varphi}(a)\|_B \leq C\|a\|_A,$$

where $\| \cdot \|_A$ and $\| \cdot \|_B$ are the corresponding $\ell_1$-norms, the constant $C$ is defined as $\max\{\|\tilde{\varphi}(a)\|_B : a \in \Lambda\}$ where $\Lambda$ is the fixed basis of $A$.

Now we prove the statement of the lemma. Let $\varphi: P \to Q$ be a morphism between finitely generated $R$-modules, and let $\| \cdot \|_P$ and $\| \cdot \|_Q$ denote filling norms on $P$ and $Q$ respectively. Suppose $A$ is a finitely generated based free $R$-module and that $\rho: A \to P$ induces the filling norm $\| \cdot \|_P$, and analogously assume that $\rho': B \to Q$ induces the filling norm $\| \cdot \|_Q$. Then, since $A$ is free, there is a morphism $\tilde{\varphi}: A \to B$ such that $\varphi \circ \rho = \rho' \circ \tilde{\varphi}$. Let $C$ be the constant for $\tilde{\varphi}$ defined above. Let $p \in P$ and note that for any $a \in A$ such that $\rho(a) = p$,

$$\|\varphi(p)\|_Q \leq \|\tilde{\varphi}(a)\|_B \leq C\|a\|_A.$$

Hence $\|\varphi(p)\|_Q \leq C\|p\|_P$. □

Two norms $\| \cdot \|$ and $\| \cdot \|^\prime$ on an $R$-module $M$ are said to be equivalent if there exists a constant $C > 0$ such that for all $m \in M$,

$$C^{-1}\|m\| \leq \|m\|^\prime \leq C\|m\|.$$

By considering the identity function on a finitely generated module $M$, the previous lemma implies:

**Corollary 2.5.** Any two filling norms on a finitely generated $R$-module $M$ are equivalent.

**Remark 2.6.** Let $M$ be a free $R$-module with basis $\Lambda$, and let $N$ be a free $R$-submodule generated by a finite subset $\Lambda' \subseteq \Lambda$. Consider the induced $\ell_1$-norms $\| \cdot \|_M$ and $\| \cdot \|_{\Lambda'}$ on $M$ and $N$ respectively.

1. The projection map $\pi: M \to N$ is bounded with respect to the induced $\ell_1$-norms.
2. The inclusion map $\iota: N \to M$ preserves the induced $\ell_1$-norms, in particular, it is bounded.

**Lemma 2.7.** Let $N$ be a finitely generated module with filling norm $\| \cdot \|_N$. Suppose that $N$ is an internal direct summand of a free module $F$ with an $\ell_1$-norm $\| \cdot \|_1$. Then $\| \cdot \|_N \sim \| \cdot \|_1$ on $N$. 
Proof. Since $N$ is a finitely generated module contained in $F$, there exist a finitely generated free submodule $I$ of $F$ which is an internal summand, $F = I \oplus J$, such that $N \subseteq I$, and the restriction of $\| \cdot \|_1$ to $I$ is an $\ell_1$-norm on $I$. Let $\iota: N \to I$ denote the inclusion and $\phi: F \to N$ denote the projection. By Lemma 2.4 both $\phi|_I: I \to N$ and $\iota: N \to I$ are bounded morphisms with respect to the norms $\| \cdot \|_1$ and $\| \cdot \|_N$; let $C_1$ and $C_2$ be the corresponding constants. Then

$$\|n\|_N = \|\phi(\iota(n))\|_N \leq C_1\|\iota(n)\|_1 \leq C_2C_1\|n\|_N$$

for all $n \in N$, and hence $\| \cdot \|_N \sim \| \cdot \|_1$ on $N$. \qed

For the rest of this section, let $G$ be a group, let $H$ be a subgroup, and as above let $R$ be a ring with norm $| \cdot |$.

Remark 2.8. Let $M$ be a free $RG$-module with $\ell_1$-norm $\| \cdot \|_\Lambda$ induced by a free basis set $\Lambda$. Then $M$ is a free $RH$-module and there exist a free $RH$-basis $\Lambda_H$ of $M$ such that the induced $\ell_1$-norms $\| \cdot \|_\Lambda$ and $\| \cdot \|_{\Lambda_H}$ are equal.

Indeed, if $S$ is a right transversal of the subgroup $H$ in $G$, then $\Lambda_H = \{gx : x \in \Lambda, g \in S\}$ is a free $RH$-basis of $M$ as an $H$-module, and the statement about the $\ell_1$-norms holds.

Lemma 2.9. Let $M$ be a finitely generated and projective $RG$-module with filling norm $\| \cdot \|_M$ and let $N$ be a finitely generated $RH$-module with filling norm $\| \cdot \|_N$. Suppose that $N$ is a internal direct summand of $M$ as an $RH$-module. Then $\| \cdot \|_N \sim \| \cdot \|_M$ on $N$.

Proof. Let $F$ be a finitely generated free based module with $\ell_1$-norm $\| \cdot \|_1$, and let $\phi: F \to M$ be a surjective $RG$-morphism inducing filling norm $\| \cdot \|_M$. Since $M$ is projective, there exist an $RG$-morphism $j: M \to F$ such that $j \circ \phi = \text{id}_M$. Lemma 2.4 implies that $j$ and $\phi$ are bounded $RG$-morphisms. Therefore $\| \cdot \|_M \sim \| \cdot \|_1$. Now consider $F$ as an $RH$-module with same $\ell_1$-norm $\| \cdot \|_1$, see Remark 2.8. Since $N$ is a direct summand of $M$ as an $RH$-module, it is a direct summand of $F$ as an $RH$-module. Then Lemma 2.7 implies $\| \cdot \|_N \sim \| \cdot \|_1$ on $N$. \qed

3. Definition of Homological Filling Functions

In this section $R$ denotes a subring of the rational numbers with the absolute value as a norm. Let $G$ be a group. The group ring $RG$ is a free abelian module over $R$, observe that $RG$ is a normed ring with $\ell_1$-norm induced by the free $R$-basis $G$. From now on, we consider $RG$ as a normed ring with this norm.

Definition 3.1 (Integral part). Let $P$ be a finitely generated $RG$-module. An integral part of $P$ is a $\mathbb{Z}G$-submodule $A$ which is finitely generated as a $\mathbb{Z}G$-module, and $A$ generates $P$ as an $RG$-module.

From here on, $[0, \infty]$ denotes the set of non-negative real numbers and infinity. The order relation as well as the addition operations are extended in the natural way.

Definition 3.2. The $n^{th}$-filling function of a group $G$ of type $RFP_{n+1}$,

$$FV_{G,R}^{n+1}: N \to [0, \infty],$$

is defined as follows. Let

$$P_{n+1} \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\partial_0} R \to 0,$$

(1)
be a partial projective resolution of finitely generated $RG$-modules of the trivial $RG$-module $R$. Let $K_n$ be an integral part for $\ker(\partial_n)$, let $\| \cdot \|_{P_n}$ and $\| \cdot \|_{P_{n+1}}$ be filling norms for $P_n$ and $P_{n+1}$ respectively. Then
\[
FV_{G,R}^{n+1}(k) = \max \left\{ \| \gamma \|_{\partial_{n+1}} : \gamma \in K_n, \| \gamma \|_{P_n} \leq k \right\},
\]
where
\[
\| \gamma \|_{\partial_{n+1}} = \inf \left\{ \| \mu \|_{P_{n+1}} : \mu \in P_{n+1}, \partial_{n+1}(\mu) = \gamma \right\}.
\]
By convention, define the maximum of the empty set as zero.

See Remark 3.8 on finiteness of $FV_{G,R}^{n+1}$. The rest of this section discusses the proof of the following theorem, which generalizes [HMP16, Theorem 3.5]. Consider the equivalence relation on the set of non-decreasing functions $\mathbb{N} \to [0, \infty]$ defined as $f \sim g$ if and only if $f \leq g$ and $g \leq f$, where $f \leq g$ means there is $C > 0$ such that for all $k \in \mathbb{N}$,
\[
f(k) \leq Cg(Ck + C) + Ck + C.
\]

**Theorem 3.3.** Let $G$ be a group of type $FP_{n+1}$. Then the $n$th-filling function $FV_{G,R}^{n+1}$ of $G$ is well defined up to equivalence relation $\sim$.

The proof of Theorem 3.3 relies on the following basic structure theorem for subrings of $\mathbb{Q}$.

**Proposition 3.4.** Let $R$ be a subring of $\mathbb{Q}$. Then there is a set $S$ of prime numbers in $\mathbb{Z}_+$ such that $R$ consists of all fractions $a/b$ where $a \in \mathbb{Z}$ and $b$ is product of powers of elements of $S$.

In the following proposition, which is a consequence of Proposition 3.4 we use the convention that for an element $a$ of an $RG$-module $A$, and any $r \in R$, $ra$ denotes the element $(re)a \in A$ where $e$ is the identity element of $G$.

**Proposition 3.5.** Let $P$ and $Q$ be finitely generated $RG$-modules. Then

1. If $A$ is an integral part, then for all units $r \in R$, $rA = \{ra : a \in A\}$ is an integral part.
2. If $f : P \to Q$ is a morphism of $RG$-modules, and $A$ and $B$ are integral parts of $P$ and $Q$ respectively, then there exists a positive integer $m$, which is a unit in $RG$, such that $f(mA) \subseteq B$.

**Proof.** The first statement is immediate from the definition.

For the second statement. Let $S$ be a finite generating set of $A$ as a $ZG$-module, and observe that $S$ generates $P$ as an $RG$-module. Let $F(S)$ be the free $RG$-module on $S$, let $\phi : F(S) \to P$, and let $C$ be the $ZG$-submodule of $F(S)$ generated by $S$, and observe that $\phi(C) = A$. Analogously, let $T$ be a finite generating set of $B$ as a $ZG$-module, let $\psi : F(T) \to Q$, and let $C'$ be the $ZG$-submodule of $F(T)$ generated by $T$, and note that $\psi(C') = B$.

Since $F(S)$ is free, there is an $RG$-morphism $\eta : F(S) \to F(T)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
F(S) & \xrightarrow{\eta} & F(T) \\
\phi \downarrow & & \downarrow \psi \\
P & \xrightarrow{f} & Q 
\end{array}
\]
Note that $\eta: F(S) \to F(T)$ is described by a finite matrix with entries in $RG$. By Proposition 3.4, there is an integer $m$, which is a unit in $R$, such that the morphism $m \eta: F(S) \to F(T)$ given by $\alpha \mapsto ma$ has the property that $\eta(C) \subseteq C'$. By commutativity of diagram $f \circ (m \phi) = \psi \circ (m \eta)$ and therefore $f(m A) \subseteq B$. □

The following lemma is an strengthening of Proposition 3.5 that will be used in the last section.

\textbf{Lemma 3.6.} Let $H \leq G$ be a subgroup and let $P$ and $Q$ be finitely generated RH and RG modules respectively. If $f: P \to Q$ is an RH-morphism, and $A$ and $B$ are integral parts of $P$ and $Q$ respectively, then there exists a positive integer $m$, which is a unit in $R$, such that $f(m A) \subseteq B$.

\textit{Proof.} Considering $Q$ as an RH-module, the proof proceeds similar to 3.5 except that here $F(T)$ is infinitely generated and so the matrix is infinite. But observe that only finitely many entries are non-zero, so the same argument holds. □

\textit{Proof of Theorem 3.3.} The proof is divided into two steps. The second step is a small variation of the argument in [HMP16, Proof of Theorem 3.5] for which we only remark the changes.

\textbf{Step 1.} $FV^n_{G}$ (up to equivalence) does not depend on the choice of integral part $K_n$.

Let $A$ and $B$ be two integral parts of $K_n$. By Proposition 3.5, there exists a positive integer $m$, that is a unit in $RG$, such that $m A \subseteq B$. Let $\gamma \in A$ such that $\|\gamma\|_{\rho_n} \leq k$. Then, since $m$ is a unit and $|m|^{-1}|m^{-1}| = 1$, $\|\gamma\|_{\delta_{n+1}} = \frac{1}{m} \|m \gamma\|_{\delta_{n+1}}$ and $\|m \gamma\|_{\rho_n} = m \|\gamma\|_{\rho_n} \leq mk$; see Remark 2.2. Observe that $m \gamma \in B$ therefore $\|\gamma\|_{\delta_{n+1}} \leq \frac{1}{m} FV_B(mk)$. Since $\gamma$ was arbitrary, $FV_A(k) \leq \frac{1}{m} FV_B(mk)$. By symmetry we get the other inequality.

\textbf{Step 2.} $FV^n_{G}$ (up to equivalence) does not depend on the choice of resolution $T$.

Let $(P, \partial_i)$ and $(Q, \delta_i)$ be a pair of resolutions as in (4). Since any two projective resolutions of $Q$ are chain homotopy equivalent, there exist chain maps $f_i: P_i \to Q_i$, $g_i: Q_i \to P_i$, and a map $h_i: P_i \to P_{i+1}$ such that

$$\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i = g_i \circ f_i - Id.$$ 

By Proposition 3.5, there exist integral parts $K_n$ and $K'_n$ of $\ker(\partial_n)$ and $\ker(\delta_n)$ respectively, such that $f_n(K_n) \subseteq K'_n$. This ensures that the same argument in [HMP16, Proof of Theorem 3.5] works except for a minor change in the choice of $\beta$. Replace it by the following: For $\epsilon < C$, choose $\beta \in Q_{n+1}$ such that $\delta_{n+1}(\beta) = f_n(\alpha)$ and $\|\beta\|_{\delta_{n+1}} < \|f_n(\alpha)\|_{\delta_{n+1}} + \epsilon$. The rest of the proof proceeds in the same manner. □

\textbf{Remark 3.7} (Topological interpretation of filling functions). Assume $G$ admits a $K(G, 1)$ model $X$ with finite $(n+1)$-skeleton. The augmented cellular complex $C_*(X, R)$ of the universal cover $\tilde{X}$ of $X$ is a projective resolution of the trivial $RG$-module $R$ by free modules. By considering the $\ell_1$-norm of $C_*(X, R)$ induced by the basis consisting of $i$-dimensional cells of $\tilde{X}$, the definition of $FV^n_{G,R}$ using this
resolution provides the interpretation $FV_{G,R}^{n+1}$ as the minimal volume required to fill integral $n$-cycles with $(n+1)$-cellular chains with coefficients in $R$. Observe that

$$FV_{G,R}^{n+1} \leq FV_{G,Z}^{n+1}$$

**Remark 3.8.** [Finiteness of $FV_{G,R}^{n+1}$] Assume that $G$ admits a $K(G,1)$ model $X$ with finite $(n+1)$-skeleton. By the main result of [FMP18], for every positive integer $k$, $FV_{G,Z}^{n+1}(k) < \infty$. Then equation (3) implies that $FV_{G,R}^{n+1}(k) < \infty$ for any $k \geq 0$.

A positive answer to the following question in the case that $R = \mathbb{Z}$ is given in [FMP18].

**Question 3.9.** Suppose that $G$ is of type $R$-$FP_{n+1}$. Is $FV_{G,R}^{n+1}(k) < \infty$ for all $k \in \mathbb{N}$?

**Remark 3.10** (On the use integral part in Definition 3.2) We note that the filling function $FV_{G,Z}^{n+1}$ was defined in [HMP16] by considering $\ker(\partial_n)$ in lieu of its integral part. This approach does not work to define $FV_{G,Q}^{n+1}$ as the following example illustrates. Consider the group presentation $G = \langle x, y \mid [x,y] \rangle$ and let $X$ be the universal cover of the presentation complex, i.e., the Cayley complex. In $X$ consider the following cycles with rational coefficients $a_n = \frac{1}{4^n}[x^n y^n]$ for $n \in \mathbb{N}$. Then $\|a_n\|_1 = 1$ and by regularity $\|a_n\|_0 = \frac{1}{4^n}$, in particular

$$\max\{\|\gamma\|_{\partial_2} : \gamma \in Z_n(\tilde{X}, Q), \|\gamma\|_1 \leq 1\} = \infty,$$

and hence the approach in [HMP16] does not yield a well defined $FV_{G,Q}^2(k)$. In contrast, using Definition 3.2, $FV_{G,Q}^2 \leq FV_{G,Z}^2 \sim k^2$.

4. Proof of Theorem 1.6

The proof of Theorem 1.6 is discussed after the proof of the following proposition.

**Proposition 4.1.** Suppose that $cd_R(G) = n+1$, $G$ is of type $R$-$FP_{n+1}$, and $H$ is a subgroup of $G$ of type $R$-$FP_{n+1}$. Then for any partial projective resolution of the trivial $RH$-module $R$ of finite type

$$Q_{n+1} \rightarrow Q_n \rightarrow \cdots \rightarrow Q_0 \rightarrow R \rightarrow 0,$$

there is a projective resolution of the trivial $RG$-module $R$ of finite type

$$0 \rightarrow M_{n+1} \rightarrow M_n \rightarrow \cdots \rightarrow M_0 \rightarrow R \rightarrow 0,$$

an injective morphisms $i_i : Q_i \rightarrow M_i$ of $RH$-modules, $0 \leq i \leq n$, such that

$$Q_n \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow R$$

is a commutative diagram of $RH$-modules, and the short exact sequences of $RH$-modules

$$0 \rightarrow Q_i \rightarrow M_i \rightarrow S_i \rightarrow 0$$

split. In particular each $S_i$ is a projective $RH$-module.
Remark 4.2. Proposition 4.1 replaces topological arguments in [HMP16], based on work of Gersten [Ger96], that use topological mapping cylinders. The arguments there are relatively less involved. In the generality that we are working, it is not possible to rely on this type of constructions. We would need free cocompact actions on $(n+1)$-acyclic complexes for $G$ and $H$, they are not known to exist under our hypothesis. Specifically, recall that a group $G$ is of type $FH_2$ if $G$ admits a cocompact action on an $n$-acyclic space $X$; in this case the action of $G$ on the cellular chain complex of $X$ induces a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. Hence $FH_2$ implies $FP_n$. It is an open question whether groups of type $FP_n$ are of type $FH_n$ for $n \geq 3$, see [BBD97].

The proof of the Proposition 4.1 is an application of the mapping cylinder of chain complexes from basic homological algebra that we recall below. Let $B = \{B_i, d_i\}$ and $C = \{C_i, d'_i\}$ be two chain complexes of modules over some fixed ring, and let $f: B \to C$ be a chain map. Then the mapping cylinder $M_f = \{M_i, d''_i\}$ is a chain complex where $M_i = C_i \oplus B_i \oplus B_{i-1}$ with

$$d''_i = \begin{pmatrix} d'_i & 0 & -f_i \\ 0 & d_i & 1d_B \\ 0 & 0 & -d_i \end{pmatrix}$$

The natural inclusion $C \to M_f$ is a chain homotopy equivalence with homotopy inverse map $M_f \to C$ given by $(c, b, b') \mapsto c + f(b)$. Also note that, if both $B$ and $C$ consists of only finitely generated projective modules then the same holds for $M_f$. For background on mapping cylinders see [Wei94].

Proof of Proposition 4.1. We split the proof into four steps.

Step 1. Definition of the resolution $\{\}$ as a mapping cylinder

Since $cd_R(G) = n + 1$ and $G$ is of type $R$-$FP_{n+1}$, there is a projective resolution of $RG$-modules of finite type

$$(8) \quad 0 \to P_{n+1} \to P_n \to \ldots \to P_0 \to R \to 0,$$

see [Bro94] pg.199, Prop. 6.1].

The group ring $RG$ is a free right $RH$-module. It follows that the extension of scalars functor from left $RH$-modules to left $RG$-modules $M \mapsto RG \otimes_R M$ is exact. This functor also preserves finite generation and projectiveness. From the given resolution (4), we obtain a partial projective resolution of the $RG$-module $RG \otimes_R R$ of finite type

$$(9) \quad RG \otimes_R Q_n \to \cdots \to RG \otimes_R Q_0 \to RG \otimes_R R \to 0.$$

Consider the $RG$-morphism $\phi: RG \otimes_R R \to R$ induced by

$$(10) \quad \phi: RG \times R \to R, \quad (s, r) \mapsto \epsilon(s)r,$$

where $\epsilon: RG \to R$ is the augmentation map, $\epsilon(\sum r_i g_i) = \sum r_i$. Since each of the $RG$-modules $RG \otimes_R Q_i$ is projective, there are $RG$-morphisms $f_i: RG \otimes_R Q_i \to P_i$ such that

$$\begin{array}{cccccc}
RG \otimes_R Q_n & \longrightarrow & \cdots & \longrightarrow & RG \otimes_R Q_0 & \longrightarrow & RG \otimes_R R \\
\downarrow f_n & & & & \downarrow f_0 & & \downarrow \phi \\
P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & R.
\end{array}$$
is a commutative diagram, see [Bro94, pg.22, Lemma 7.4].

Let $M_i = (M_i)$ be the mapping cylinder the chain map $f = (f_i)$ where $f_i$ is the $RG$-morphism defined above for $0 \leq i \leq n$, $f_{n+1}$ is the morphism $0 \to P_{n+1}$, and $f_0$ is the morphism $0 \to 0$ for any other value of $i$.

Observe that

$$M_i = P_i \oplus (RG \otimes RH Q_1) \oplus (RG \otimes RH Q_{i-1})$$

for $1 \leq i \leq n$, $M_0 = P_0 \oplus (RG \otimes RH Q_0) \oplus 0$, $M_{n+1} = P_{n+1} \oplus 0 \oplus (RG \otimes RH Q_n)$, and $M_i = 0$ for any other value of $i$. Hence all $M_i$ are finitely generated and projective.

Let $P_\ast = (P_i)$ be the chain complex induced by $\mathcal{S}$, where $P_i = 0$ for $i > n+1$ and $i < 0$. Observe that $P_\ast$ is the target of the chain map $f$. Since $P_\ast$ and $M_\ast$ are chain homotopic,

$$0 \to M_{n+1} \to M_n \to \cdots \to M_0 \to R \to 0,$$

is a projective resolution of finite type of the trivial $RG$-module $R$.

**Step 2. Definition of the injective $RH$-morphisms $i_\ast: Q_i \to M_i$.**

We have the following commutative diagram of $RH$-modules

\[
\begin{array}{cccccc}
Q_n & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 \\
\downarrow \tau_n & & & \downarrow \tau_1 & & \downarrow \tau_0 \\
RG \otimes RH Q_n & \longrightarrow & \cdots & \longrightarrow & RG \otimes RH Q_1 & \longrightarrow & RG \otimes RH Q_0 \\
\downarrow \jmath_n & & & \downarrow \jmath_1 & & \downarrow \jmath_0 \\
M_n & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0.
\end{array}
\]

(12)

where $\tau_k: Q_k \to RG \otimes RH Q_k$ is the natural inclusion given by $q \mapsto e \otimes q$ (here $e$ denotes the identity element of $G$), and the vertical arrows $\jmath_i: RG \otimes RH Q_i \to M_i$ are the natural inclusions. Then define

$$i_\ast = j_\ast \circ \tau_\ast$$

for $0 \leq i \leq n$, and observe that they are injective $RH$-morphisms.

**Step 3. Verifying commutative diagram (12).**

In view of the commutative diagram (12), we only need to verify that if $H_0(Q)$ and $H_0(M)$ denote the cokernels of $Q_1 \to Q_0$ and $M_1 \to M_0$ respectively then the $RH$-morphism $H(t_0): H_0(Q) \to H_0(M)$ induced by $t_0$ is an isomorphism.

Before the argument, we remark that this is not immediate, it depends on the choice of the $RG$-morphism $f_0$; the available choices for $f_0$ depend on the choice of the $RG$-morphism $\phi: RG \otimes RH R \to R$; our choice is defined by (10).

Let $H_0(P)$ denote the cokernel of $P_1 \to P_0$. Let $\tau_{-1}: R \to RG \otimes RH R$ be defined by $r \mapsto e \otimes r$ where $e$ denotes the identity element of $G$. Then $\phi \circ \tau_{-1}$ is the identity map on $R$. It follows that the induced $RH$-morphism $H_0(f_0 \circ \tau_0): H_0(Q) \to H_0(P)$ is an isomorphism. Since $\kappa: M_\ast \to P_\ast$ given by $(p, q, q') \mapsto p + f(q)$ is a chain homotopy equivalence, $H(\kappa_\ast): H_0(M) \to H_0(P)$ is an isomorphism. Observe that $H(f_0 \circ \tau_0)$ equals $H(\kappa_\ast) \circ H(t_0)$ and hence $H(t_0)$ is an isomorphism.

**Step 4. The exact sequence (7) splits, and each $S_i$ is a projective $RH$-module.**
This is immediate since $i_t: Q_i \to M_i$ is the inclusion of a direct summand of $M_i$ as an $RH$-module. Since restriction of scalars preserves projectiveness, $M_i$ is projective as an $RH$-module and hence $S_i$ is projective as well. 

\textbf{Proof of Theorem 1.6.} Consider projective resolutions as (4) and (5) as well as $RH$-morphisms $i_t: Q_i \to M_i$ as described in Proposition 4.1.

Let $M^* = (M_i, \delta^M_i)$ denote the chain complex induced by (5), with the assumption that $M_i = 0$ for $i > n$ and $i < 0$. Analogously, let $Q^* = (Q_i, \delta^Q_i)$ be the chain complex induced by (4), with the assumption that $Q_i = 0$ for $i > n$ and $i < 0$. Observe that we are not using the modules $Q_{n+1}$ and $M_{n+1}$ in the definition of $Q^*$ and $M^*$. Let $S^*$ be the quotient chain complex $M^*/Q^*$. Consider the induced chain map $i = (i_t): Q^* \to M^*$.

We use the following notation. The kernel of $\delta^Q_n$ is denoted by $Z_n(Q)$. The $n$-homology group of the complex $Q^*$ is denoted by $H_n(Q)$. Analogous notation is used for the other chain complexes.

\textbf{Step 1. The induced sequence}

\begin{equation}
0 \longrightarrow Z_n(Q) \xrightarrow{i_n} Z_n(M) \longrightarrow Z_n(S) \longrightarrow 0
\end{equation}

is exact and satisfies

- $Z_n(Q)$ is a finitely generated $RH$-module.
- $Z_n(M)$ is a finitely generated and projective $RG$-module.
- $Z_n(Q)$ is a direct summand of $Z_n(M)$ as an $RH$-module.

Observe that $H_{n+1}(Q)$ and $H_{n-1}(Q)$ are both trivial. The short exact sequence of chain complexes of $RH$-modules

\begin{equation}
0 \longrightarrow Q^* \xrightarrow{i} M^* \longrightarrow S^* \longrightarrow 0
\end{equation}

induces a long exact sequence

\begin{equation}
0 \longrightarrow H_n(Q) \xrightarrow{i_n} H_n(M) \longrightarrow H_n(S) \longrightarrow 0
\end{equation}

which is precisely (13).

The $RH$-module $Z_n(Q)$ is finitely generated since $Q_{n+1}$ is a finitely generated $RH$-module and $\delta^Q_{n+1}$ maps $Q_{n+1}$ onto $Z_n(Q)$.

That $Z_n(M_n)$ is a finitely generated and projective $RG$-module follows from a direct application of Schanuel's lemma [Bro94, pg.193, Lemma 4.4] to the exact sequences (5) and

\begin{equation}
0 \to Z_n(M) \to M_n \to \cdots \to M_0 \to R \to 0.
\end{equation}

Finally, to show that $Z_n(Q)$ is a direct summand of $Z_n(M)$ as an $RH$-module, we argue that that $Z_n(S)$ is projective $RH$-module. Consider the sequence of $RH$-modules induced by $S^*$

\begin{equation}
0 \to Z_n(S) \to S_n \to \cdots \to S_0 \to 0.
\end{equation}

Note that this sequence is exact by observing the long exact sequence of homologies induced by (17). Indeed, $H_i(Q)$ and $H_i(M)$ are trivial for $0 < i < n$, and $H(i): H_0(Q) \to H_0(M)$ is an isomorphism by (6). Since each $S_i$ is projective, exactness of (17) implies that $Z_n(S)$ is projective.

\textbf{Step 2.} $FV_{H,R}^{n+1} \leq FV_{G,R}^{n+1}$. 

Let $\| \cdot \|_{M_n}$ and $\| \cdot \|_{Z_n(M)}$ denote filling norms on the $RG$-modules $M_n$ and $Z_n(M)$ respectively. Similarly, let $\| \cdot \|_{Q_n}$ and $\| \cdot \|_{Z_n(Q)}$ denote filling norms on $RH$-modules $Q_n$ and $Z_n(Q)$. For the map $Z_n(Q) \to Z_n(M)$, by Lemma 2.9, there exist integral parts $K$ and $K'$ of $Z_n(Q)$ and $Z_n(M)$ respectively, such that $K$ maps into $K'$ by the morphism $t$.

Since $t: Q_n \to M_n$ is the inclusion of a direct summand of $M_n$ as an $RH$-module, and $M_n$ is a projective $RH$-module, Lemma 2.9 implies that $\| \cdot \|_{M_n} \sim \| \cdot \|_{Q_n}$ on $Q_n$. In particular, there is a constant $C_0$ such that

$$\| t_n(\gamma) \|_{M_n} \leq C_0 \| \gamma \|_{Q_n}$$

for every $\gamma \in Q_n$.

By Step 1, $t_n: Z_n(Q) \to Z_n(M)$ is the inclusion of a direct summand of $Z_n(M)$ as an $RH$-module, and $Z_n(M)$ is a projective $RH$-module. Lemma 2.9 implies $\| \cdot \|_{Z_n(M)} \sim \| \cdot \|_{Z_n(Q)}$ on $Z_n(Q)$. Hence there is $C_1 > 0$ such that

$$\| \gamma \|_{Z_n(Q)} \leq C_1 \| t_n(\gamma) \|_{Z_n(M)}$$

for every $\gamma \in Z_n(M)$, and $\rho \circ t$ is identity on $Z_n(Q)$.

Let $k \in \mathbb{N}$ and $\gamma \in K \subseteq Z(Q_n)$ such that $\| \gamma \|_{Q_n} \leq k$. Then

$$\| \gamma \|_{Z_n(Q)} \leq C_1 \| t_n(\gamma) \|_{Z_n(M)} \leq C_1 FV_{G,R}^{n+1}(\| t_n(\gamma) \|_{M_n}) \leq C_1 FV_{G,R}^{n+1}(C_0 \| \gamma \|_{Q_n})$$

Therefore $FV_{H,R}^{n+1}(k) \leq C_1 FV_{G,R}^{n+1}(C_0 k)$ for every $k$.

\section*{References}

[BB97] Mladen Bestvina and Noel Brady, \textit{Morse theory and finiteness properties of groups}, Invent. Math. 129 (1997), no. 3, 445–470. MR1465330

[BM91] Mladen Bestvina and Geoffrey Mess, \textit{The boundary of negatively curved groups}, J. Amer. Math. Soc. 4 (1991), no. 3, 469–481. MR1096169

[Bra99] Noel Brady, \textit{Branched coverings of cubical complexes and subgroups of hyperbolic groups}, J. London Math. Soc. (2) 60 (1999), no. 2, 461–480. MR1724853

[Bro94] Kenneth S. Brown, \textit{Cohomology of groups}, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz, \textit{Hyperbolization of polyhedra}, J. Differential Geom. 34 (1991), no. 2, 347–388. MR1131435

[Dra99] A. N. Dranishnikov, \textit{Boundaries of Coxeter groups and simplicial complexes with given links}, J. Pure Appl. Algebra 137 (1999), no. 2, 139–151. MR1684267

[FMP18] Joshua W. Fleming and Eduardo Martínez-Pedroza, \textit{Finiteness of homological filling functions}, Involve 11 (2018), no. 4, 569–583. MR3778913

[Ger96a] S. M. Gersten, \textit{A cohomological characterization of hyperbolic groups}, 1996.

[Ger96b] \textit{Subgroups of word hyperbolic groups in dimension 2}, J. London Math. Soc. (2) 54 (1996), no. 2, 261–283. MR1405055

[GM08] Daniel Groves and Jason Fox Manning, \textit{Dehn filling in relatively hyperbolic groups}, Israel J. Math. 168 (2008), 317–429. MR2448064

[HMP14] Richard Gaelan Hanlon and Eduardo Martínez-Pedroza, \textit{Lifting group actions, equivariant towers and subgroups of non-positively curved groups}, Algebr. Geom. Topol. 14 (2014), no. 5, 2783–2808. MR3276848

[HMP16] Richard Gaelan Hanlon and Eduardo Martínez-Pedroza, \textit{A subgroup theorem for homological filling functions}, Groups Geom. Dyn. 10 (2016), no. 3, 867–883. MR3551182

[KG18] Robert Kropholler and Giles Gardam, \textit{Hyperbolic groups with finitely presented subgroups not of type F3}, 2018. arXiv:1808.09505.

[KW00] Ilya Kapovich and Daniel T. Wise, \textit{The equivalence of some residual properties of word-hyperbolic groups}, J. Algebra 223 (2000), no. 2, 562–583. MR1735163

[Min02] Igor Mineyev, \textit{Bounded cohomology characterizes hyperbolic groups}, Q. J. Math. 53 (2002), no. 1, 59–73. MR1886760
[MP17] Eduardo Martínez-Pedroza, *Subgroups of relatively hyperbolic groups of Bredon cohomological dimension 2*, J. Group Theory 20 (2017), no. 6, 1031–1060. MR3719315

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

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