Graphon Control of Large-scale Networks of Linear Systems

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Abstract—To achieve control objectives for extremely complex and very large scale networks using standard methods is essentially intractable. In this work, we propose and develop a theory of the approximate control of complex network systems by the use of graphon theory and the theory of infinite dimensional systems.

First, the graphon dynamical system models are formulated in an appropriate infinite dimensional space in order to represent arbitrary-size networks of linear dynamical systems, and to define the convergence of sequences of network systems with limits in the space. The exact controllability and the approximate controllability of graphon dynamical systems are then investigated.

Second, the minimum energy state-to-state control problem and the linear quadratic regulator problem for systems on complex networks are considered. The control problem for the graphon limit systems is solved in each case and an approximation is defined which yields control laws for the original control problems. Furthermore, the convergence properties of the approximation schemes are established. A systematic control design methodology is developed within this framework. Finally, numerical examples of complex networks with randomly sampled weightings are presented to illustrate the effectiveness of the graphon control methodology.

Index Terms—Graphon control, large networks, complex networks, graphons, infinite dimensional systems

I. INTRODUCTION

Complex network systems such as the Internet of Things (IoT), electric, neuronal, food web, epidemic, stock market and social networks, are ubiquitous, and the study of large scale networks has been the focus of much research over the past 20 years. In particular, researchers have been studying networks of interacting dynamical systems to learn which collective behaviours may emerge from system interactions over complex networks ([1]–[4]). Furthermore, in addition to the structural properties of networks, system theoretic notions such as controllability, observability, consensus dynamics and synchronization have been widely applied to systems on networks ([5]–[15]). However, to achieve general control objectives for extremely complex and very large scale networks (henceforth, complex networks) using these standard methods is essentially an intractable task.

Graphon theory, introduced and developed in recent years by L. Lovász, B. Szegedy, C. Borgs, J. T. Chayes, V. T. Sós, and K. Vesztergombi among others (see [16]–[20]), provides a theoretical tool to characterize complex graphs and graph limits. This work draws on graph theory, measure theory, probability, and functional analysis, and has been applied in different areas such as games [21], [22], signal processing [23] and crowd-sourcing [24].

We propose a graphon based control methodology for controlling complex network systems. The general graphon control strategy consists of the following steps:

1) Identify the graphon limit of the sequence $\hat{S}$ of networks as the number of nodes goes to infinity.
2) Solve the corresponding control problem for the limit graphon dynamical system.
3) Approximate the control law for the limit system so as to generate approximate control laws for finite network systems.
4) Apply the resulting control laws to the networks of systems along the sequence $\hat{S}$.

Specifically, in this paper, the minimum energy state-to-state control problem and the linear quadratic regulator problem are solved for complex network systems using this graphon control strategy.

The main contributions of this paper include:

• the formulation of graphon differential equations and graphon dynamical systems, which form fundamental components of this study of arbitrary-size networks of linear systems.
• the development of graphon state-to-state control methodology to solve state-to-state control problem on complex networks.
• the proposed graphon linear quadratic regulation methodology to solve linear quadratic regulator problems on complex networks.

Preliminary versions of this work appear in [25]–[27].

The paper is organized as follows: In Section II, the fundamentals of graphon theory are presented, followed by the development of the graphon unitary operator algebra and graphon differential equations. Section III introduces the network system model and and its equivalent representation by the graphon dynamical system. In Section IV, we study the properties of graphon dynamical systems, including existence and uniqueness of the solution and controllability. In Section V and Section VI, the graphon control strategies for state-to-state control problem and linear quadratic regulator problem are presented respectively. For each problem, the approximation method is developed and the corresponding convergence properties are established. Section VII contains numerical examples to illustrate the graphon control methodology.
Notation

Functions, graphons, and operators are represented by bold face letters to differentiate them from vectors, graphs, and matrices.

II. Preliminaries

A. Graphs, Adjacency Matrices and Pixel Pictures

The underlying structure of a network can be described by a graph $G = (V, E)$ specified by a vertex set $V$ and an edge set $E$ which represents the connections between vertices. An equivalent representation of a graph $G = (V, E)$ by a matrix called an adjacency matrix is defined to be the square $|V| \times |V|$ matrix $A$ such that an element $A_{ij}$ is one when there is an edge from vertex $i$ to vertex $j$, and zero otherwise. If the graph is a weighted graph where edges are associated with weights, then an equivalent representation of a graph $G$ is the space of graphons. Let $\mathcal{G}^\text{sp}$ denote the space of all symmetric measurable functions $W : [0, 1]^2 \to \mathcal{R}$.

The cut norm of a graphon $W \in \mathcal{G}_1^\text{sp}$ is then defined as

$$\|W\|_\square = \sup_{M,T \subseteq [0,1]} \left| \int_{M \times T} W(x,y) dx dy \right|$$

with the supremum taking over all measurable subsets $M$ and $T$ of $[0,1]$. Evidently, the following inequalities hold between norms on a graphon $W$:

$$\|W\|_\square \leq \|W\|_1 \leq \|W\|_2 \leq \|W\|_\infty \leq 1,$$

where the second to the forth norms are given by the corresponding $L^p$ norms on $\mathcal{G}^\text{sp}_1$. Denote the set of measure preserving bijections from $[0,1]$ to $[0,1]$ by $S_{[0,1]}$. The cut metric between two graphons $V$ and $W$ is then given by

$$\delta_\square(W, V) = \inf_{\phi \in S_{[0,1]}} \|W^\phi - V\|_\square,$$

where $W^\phi(x, y) = W(\phi(x), \phi(y))$. We see that the cut metric $\delta_\square(\cdot, \cdot)$ is given by measuring the maximum discrepancy between the integrals of two graphons over measurable subsets of $[0,1]$, then minimizing the maximum discrepancy over all possible measure preserving bijections.

Strictly speaking the cut metric is not a metric since the distance between two distinct graphons under the cut metric can be zero (see e.g. [18]). However, by identifying functions $V$ and $W$ for which $\delta_\square(V, W) = 0$, we can construct the metric space $\mathcal{G}_1^\text{sp}$ which denotes the image of $\mathcal{G}_1^\text{sp}$ under this identification. Similarly we construct $\mathcal{G}_0^\text{sp}$ from $\mathcal{G}_1^\text{sp}$ and $\mathcal{G}^\text{sp}$ from $\mathcal{G}_1^\text{sp}$.

We define the $L^2$ metric for any graphons $W$ and $V$ as

$$d_{L^2}(W, V) = \|W - V\|_2$$

and the $\delta_2$ metric as

$$\delta_2(W, V) = \inf_{\phi \in S_{[0,1]}} d_{L^2}(W^\phi, V) = \inf_{\phi \in S_{[0,1]}} \|W^\phi - V\|_2;$$

similarly, we define the $L^1$ metric as

$$d_{L^1}(W, V) = \|W - V\|_1,$$

and the $\delta_1$ metric as

$$\delta_1(W, V) = \inf_{\phi \in S_{[0,1]}} d_{L^1}(W^\phi, V) = \inf_{\phi \in S_{[0,1]}} \|W^\phi - V\|_1.$$

For any two graphons $W$ and $V$ the following inequalities hold immediately:

$$\delta_\square(W, V) \leq \delta_1(W, V) \leq \delta_2(W, V) \leq d_{L^2}(W, V).$$

The $\delta_2$ (or $\delta_1$) metric and $\delta_\square$ metric share the same equivalence classes under the measure preserving transformations [20 Corollary 8.14]. Clearly, the $\delta_2$ (or $\delta_1$) metric is also well defined on $\mathcal{G}_1^\text{sp}$. 

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Fig. 1. Dürer Graph, Adjacency Matrix, Pixel Diagram

Fig. 2. Graph Sequence Converging to Its Limit
C. Compactness of the Graphon Space

Theorem 1 ([20]). The space \((G_0^{sp}, \delta_{1})\) is compact.

This remains valid if \(G_0^{sp}\) is replaced by any uniformly bounded subset of \(G^{sp}\) closed in the cut metric [20].

Theorem 2 ([20]). The space \((G_0^{sp}, \delta_{1})\) is compact.

Sets in \(G_1^{sp}\) (or \(G_0^{sp}\)) compact with respect to the \(\delta_1\) metric are compact with respect to the cut metric. It follows immediately from [8] and Theorem 2 (or Theorem 1), if a graphon sequence is Cauchy in the \(\delta_2\) metric then it is also a Cauchy sequence in the cut metric and under both metrics, the limits are identical in \(G_1^{sp}\) (or \(G_0^{sp}\)).

Define the \(L^p\) closed ball in \(G^{sp}\) with radius \(C > 0\) as \(B_{L^p}(C) := \{W : \|W\|_p \leq C, W \in G^{sp}\}\).

Theorem 3 ([28]). The space \((B_{L^p}(C), \delta_{1})\) with \(1 < p \leq \infty\) is compact.

By compactness, infinite sequences of graphons will necessarily possess one or more sub-sequential limits under the cut metric.

Henceforth we only consider the \(L^2\) topology on the space of graphons. Consequently the convergence of a sequence of graphons will be interpreted as convergence in the complete space of graphons in the \(L^2\) metric. By the ordering of metrics given in [8] this further implies convergence in the compact space (of equivalence classes) of graphons under in the weaker cut metric topology.

Examples of sets of graphons which have common limits in the \(L^2\) metric and cut metric topologies are given by the so-called monotone families of graphons (see Appendix A). These correspond to graphs to which nodes or edges are recursively added at each of an infinite set of discrete time instants.

D. Step Functions in the Graphon Space

Graphons generalize weighted graphs in the following sense. A function \(W \in \tilde{G}_1^{sp}\) is called a step function if there is a partition \(Q = \{Q_1, \ldots, Q_k\}\) of \([0,1]\) into measurable sets such that \(W\) is constant on every product set \(Q_i \times Q_j\). The sets \(Q_i\) are the steps of \(W\). For every weighted graph \(G\) (on node set \(V(G)\)), a step function \(S_G \in \tilde{G}_1^{sp}\) is given as follows: partition \([0,1]\) into \(n\) measurable sets \(Q_1, \ldots, Q_n\) of measure \(\mu(Q_i) = \frac{1}{n}\), then for \(x \in Q_i\) and \(y \in Q_j\), we let \(S_G(x,y) = \beta_{ij}(G)\), where \(\alpha_i\) denotes the node weight of the \(i^{th}\) node, \(\alpha(G) = \sum \alpha_i\) and \(\beta_{ij}(G)\) denotes the weight of the edge from node \(i\) to node \(j\) (i.e., \(\beta_{ij}\) is the \(ij^{th}\) entry in the adjacency matrix of \(G\)). Evidently the function \(S_G\) depends on the labelling of the nodes of \(G\). We define the uniform partition \(P^N = \{P_1, P_2, \ldots, P_N\}\) of \([0,1]\) by setting \(P_k = \left[\frac{k-1}{N}, \frac{k}{N}\right), k \in \{1, N-1\}\) and \(P_N = \left[\frac{N-1}{N}, 1\right]\). Then \(\mu(P_i) = \frac{1}{N}, i \in \{1, 2, \ldots, N\}\). Under the uniform partition, the step functions can be represented by the pixel diagram on the unit square. See [20].

E. Graphons as Operators

A graphon \(W \in \tilde{G}_1^{sp}\) can be interpreted as an operator \(W : L^2[0,1] \rightarrow L^2[0,1]\). The operation on \(v \in L^2[0,1]\) is defined as follows:

\[
[Wv](x) = \int_0^1 W(x, \alpha)v(\alpha)d\alpha. \tag{9}
\]

The operator product is then defined by

\[
[UV](x, y) = \int_0^1 U(x, z)W(z, y)dz, \tag{10}
\]

where \(U, W \in \tilde{G}_1^{sp}\). See [20] for more details.

Note that if \(U \in \tilde{G}_1^{sp}\) and \(W \in \tilde{G}_1^{sp}\), then \(UW \in \tilde{G}_1^{sp}\), since for all \(x, y \in [0,1]\)

\[
\|UW\|(x,y) \leq \int_0^1 \|U(x,z)W(z,y)\|dz \leq 1. \tag{11}
\]

Consequently, the power \(W^n\) of an operator \(W \in \tilde{G}_1^{sp}\) is defined as

\[
W^n(x,y) = \int_{[0,1]^n} W(x,\alpha_1) \cdots W(\alpha_n-1, y)d\alpha_1 \cdots d\alpha_{n-1}
\]

with \(W^n \in \tilde{G}_1^{sp}\) (\(n \geq 1\)). \(W^0\) is formally defined as the identity operator on functions in \(L^2[0,1]\), but we note that \(W^0\) is not a graphon.

For simplicity of notation, \(UW\) is used to denote the graphon given by the convolution in (10); similarly, \(Wv\) denotes the function defined by (9).

F. The Graphon Unitary Operator Algebra

It is evident that the operator composition defined in (10) above yields an operator algebra with a multiplicative binary operation possessing the associativity, left distributivity, right distributivity properties and compatibility with the scalar field \(\mathbb{R}\), that is, for any \(V, W, H\) in the vector space \(L^2[0,1]^2\) and \(a, b \in \mathbb{R}\),

\[
(VW)H = V(WH),
\]

\[
V(W + H) = VW + VH,
\]

\[
(W + H)V = WV + HV,
\]

\[
(\alpha W)(\beta H) = (\alpha \beta)WH.
\]

Thus we have an operator algebra \(\mathcal{A}_G\) over the field \(\mathbb{R}\) acting on elements of \(L^2[0,1]\) with operator multiplication as given in [9]. By adjoining the identity element \(I\) to the algebra \(\mathcal{A}_G\) (see e.g. [29]) we obtain a unitary algebra \(\mathcal{A}_{UZ}\). The identity element \(I\) is defined as follows: for any \(W \in L^2[0,1]^2\)

\[
[IW](x,y) = \int_0^1 W(z,y)\delta(x,z)dz = W(x,y), \tag{12}
\]

where \(\delta(\cdot,z)dz\) is the measure satisfying \(\int_0^1 u(z)\delta(x,z)dz = u(x)\) for all \(u \in L^2[0,1]\), and in particular \(\int_0^1 \delta(x,z)dz = 1\).

The graphon unitary operator algebra \(\mathcal{A}_{UZ}\) will be used in the definition of the controllability Gramian and the input operator. More specifically, we use the subset \(\mathcal{A}_{UZ}^1 = \{aI + A : A \in \mathcal{G}_A, a \in \mathbb{R}\}\) where \(\mathcal{G}_A^1\) is the subset of \(\mathcal{G}_A\) that corresponds to \(\tilde{G}_1^{sp}\).
G. Graphon Differential Equations

Let $X$ be a Banach space. A linear operator $A : D(A) \subset X \to X$ is closed if $\{(x, Ax) : x \in D(A)\}$ is closed in the product space $X \times X$ (see [30]). $L(X)$ denotes the Banach algebra of all linear continuous mappings $T : X \to X$. $L^p([a, b]; X)$ denotes the Banach space of equivalence classes of strongly measurable (in the Borel sense) mappings $[a, b] \to X$ that are $p$-integrable, $1 \leq p < \infty$, with norm

$$\|f\|_{L^p([a, b]; X)} = \left( \int_a^b |f(s)|^p ds \right)^{1/p}.$$

A mapping $S : \mathcal{R} \to L(X)$ is said to be a strongly continuous semigroup on $X$ if the following properties hold:

1) $S(0) = I$, $S(t+s) = S(t)S(s)$, $\forall t, s \geq 0$
2) for all $x \in X$, $S(\cdot)x$ is continuous on $\mathcal{R}$.

A uniformly continuous semigroup is a strongly continuous semigroup $S$ such that $\lim_{t \to 0^+} \|S(t) - I\| = 0$, with $\| \cdot \|$ as the operator norm on a Banach space. The infinitesimal generator $A$ of a strongly continuous semigroup $S$ is the linear operator in $X$ defined by $Ax = \lim_{t \to 0^+} \frac{1}{t}[S(t)x - x]$, for all $x \in D(A)$, where $D(A) = \{x \in X : \text{ s.t. } \lim_{t \to 0^+} \frac{1}{t}[S(t)x - x] \text{ exists}\}$.

Let $A : [0, 1]^2 \to [-1, 1]$ be a graphon and hence a bounded and closed linear operator from $L^2([0, 1])$ to $L^2([0, 1])$. Following [51], $A$ is the infinitesimal generator of the (hence strongly) continuous semigroup

$$S_A(t) := e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

Therefore, the initial value problem of the graphon differential equation

$$\dot{y}_t = A y_t, \quad y_0 \in L^2[0, 1]$$ (13)

has a solution given by $y_t = e^{At}y_0$.

Theorem 4 (Appendix C). Let $\{A_N\}_{N=1}^{\infty}$ be a sequence of graphons such that $A_N \to A$, as $N \to \infty$ in the $L^2$ metric. Then for all $x \in L^2[0, 1]$, $e^{A_Nx} \to e^{Ax}$ as $N \to \infty$ in the $L^2$ metric where the convergence is pointwise in time and uniform on any time interval $[0, T]$.

III. NETWORK SYSTEMS AND THEIR LIMIT SYSTEMS

A. Network System Model

Consider an interlinked network $S^N$ of linear (symmetric) dynamical subsystems $\{S_i^N : 1 \leq i \leq N\}$, each with an $n$ dimensional state space. The subsystem $S_i^N$ at the node $V_i$ in the network $G_N(V, E)$ has interactions with $S_j^N$, $1 \leq j \leq N$, specified as below:

$$S_i^N : \dot{x}_i^N = \frac{1}{nN} \sum_{j=1}^{N} A_{ij} x_j^N + \frac{1}{nN} \sum_{j=1}^{N} B_{ij} u_i^N,$$

with $A_N = [A_{ij}], B_N = [B_{ij}] \in \mathbb{R}^{n \times n \times n}$, the (symmetric) block-wise adjacency matrices of $G_N(V, E)$ and of the input graph, where $A_{ij} = [0]$ if $S_j^N$ has no connection to $S_j^N$ and similarity for $B_{ij}$. Then the (symmetric) linear dynamics for the network system $S^N(A_N, B_N, G_N)$ can be represented by

$$S^N : \dot{x}^N = A_N \circ x^N + B_N \circ u^N,$$

where $\circ$ denotes the so called averaging operator given by $A_N \circ x = \frac{1}{(nN)} A_N x$. Let $S = \times_{N=1}^{\infty} S^N$ where $S^N = \cup_{A_N, B_N, G_N} S^N(A_N, B_N, G_N)$. For simplicity, we require the elements of $A_N$ and $B_N$ to be in $[-1, 1]$ for each $N$ (note that in general $A_N$ and $B_N$ have elements that are bounded real numbers for which case we would achieve similar results).

In addition, we note that if we take the supremum norm on vectors in $R^{nN}$, i.e. $\|x\|_\infty = \sup_{i} |x_i|$, and the corresponding operator norm of $A$, i.e. $\|A\|_{op, \infty} = \sup_{\|x\|_\infty \neq 0} \|Ax\|_\infty$, then $\|A\|_{op, \infty} \leq 1$.

B. Network Systems Described by Step Functions

Let $\{(A_N; B_N)\}_{N=1}^{\infty} \subseteq \mathcal{S}$ be a sequence of systems with the node averaging dynamics each of which is described according to (14). Let $|A_{Nij}| \leq 1$ and $|B_{Nij}| \leq 1$ for all $i, j \in \{1, ..., nN\}$. Let $A_s^{[N]}, B_s^{[N]} \in \mathbb{G}_1^{op}$ be the step functions corresponding one-to-one to $A_N$ and $B_N$; these are specified using the uniform partition $\mathcal{P}^{nn}$ of $[0, 1]$ by the following matrix to step function mapping $M_G$: for all $i, j \in \{1, 2, ..., nN\}$,

$$A_s^{[N]}(x, y) := A_{Nij}, \quad \forall (x, y) \in P_i \times P_j, \quad (15)$$

and similar for $B_s^{[N]}$.

Define a piece-wise constant (PWC) function on $\mathcal{R}$ to be any function of the form $\sum_{k=1}^{nN} \alpha_k \psi_{j_k}$ where $\alpha_1, ..., \alpha_l$ are complex numbers and each $I_k$ is a bounded interval (open, closed, or half-open). Let $L^2_{\text{pwc}}[0, 1]$ denote the space of piece-wise constant $L^2[0, 1]$ functions under the uniform partition $\mathcal{P}^{nn}$.

Let $u_s^i \in L^2_{\text{pwc}}[0, 1]$ correspond one-to-one to $u_t \in \mathcal{R}^{nN}$ via the following vector to PWC function mapping also denoted by $M_G$: for all $i \in \{1, ..., nN\}$,

$$u_s^i(\alpha) := u_s(i), \quad \forall \alpha \in P_i, \quad (16)$$

and $x_s^i \in L^2_{\text{pwc}}[0, 1]$ similarly correspond one-to-one to $x_t \in \mathcal{R}^{nN}$.

Lemma 1 (Appendix C). The trajectories of the system in (14) correspond one-to-one under the mapping $M_G$ to the trajectories of the system

$$\dot{x}_s^i = A_s^{[N]} x_s^i + B_s^{[N]} u_s^i,$$

$$x_s^i, u_s^i \in L^2_{\text{pwc}}[0, 1], A_s^{[N]}, B_s^{[N]} \in \mathbb{G}_1^{op} \subset \mathcal{G}_1^{AT}$$ (17)

with graphon operations defined according to (9).

C. Limits of Sequences of Network Systems

Now the sequence of network systems with the node averaging dynamics can be described by the sequence of step function operators as $\{(A_s^{[N]}, B_s^{[N]})\}_{N=1}^{\infty}$. Let the graphon sequences $\{A_s^{[N]}\}$ and $\{B_s^{[N]}\}$ be Cauchy sequences of step functions in $L^2[0, 1]^2$ (under the same measure preserving bijection). Due to the completeness of $L^2[0, 1]^2$, the respective graphon limits $A$ and $B$ exist and these will then necessarily be the limits in the cut metric (see Section II-C and (20)).

In fact, we can generalized the control input operator $B$ to $\mathcal{G}_A$, i.e., $B$ can consists of the identity operator part and the graphon part as $B = B_I + B_B$.  

Consider a sequence of systems \( \{(A^N, B^N) \in \tilde{G}_1^{\text{gp}} \times \tilde{G}_A^{1} \}_{N = 1}^{\infty} \). Decompose the input operator into the identity part and the graphon part as \( B^N = \beta N I + B^N_s \).

Definition 1. A sequence of systems \( \{(A^N, B^N) \in \tilde{G}_1^{\text{gp}} \times \tilde{G}_A^{1} \}_{N = 1}^{\infty} \) is convergent if

1) there exist \( \beta \in \mathcal{R} \) such that \( \lim_{N \to \infty} \beta N = \beta \)
2) there exist \( A, B \in \tilde{G}_1^{\text{gp}} \) such that \( \{(A^N, B^N)\} \)
   converges to \( (A; B) \) in the \( L^2 \) metric, i.e. \( A^N \to A \)
   and \( B^N_s \to B \) under the same sequence of measure preserving bijections in the \( L^2 \) metric.

Then the limit system is represented by \( (A; B) \) where \( B = \beta I + B \). With an abuse of notation, in the following sections we use \( B \) and \( B^N_s \) to represent input operators in \( \tilde{G}_A^{1} \).

IV. The Limit Graphon System and Its Properties

A. Limit Graphon Systems

We follow \([30]\) and specialize both the Hilbert space of states \( H \) and the Hilbert space of controls \( U \) appearing there to the space \( L^2(\mathcal{R}; L^2[0,1]) \). We formulate an infinite dimensional linear system as follows:

\[
LS_\infty : \quad \dot{x}_t = Ax_t + Bu_t, \quad x_0 \in L^2[0,1],
\]

where \( A \in \tilde{G}_1^{\text{gp}} \), \( B \in \tilde{G}_A^1 \), and are hence bounded operators on \( L^2[0,1] \), \( x_t \in L^2[0,1] \) is the system state at time \( t \) and \( u_t \in L^2[0,1] \) is the control input at time \( t \).

B. Uniqueness of the Solution

A solution \( x_{t(\cdot)} \in L^2(\mathcal{R}; L^2[0,1]) \) is a (mild) solution of \( (18) \) if \( x_t = e^{(t-s)A}x_s + \int_s^t e^{(t-s)A}Bu_s ds \) for all \( a \) and \( t \in \mathcal{R} \), taken to be \( a \leq t \) (see \([30]\)). Following \([30]\), the assumptions on the operators \( A \) and \( B \) are

\[
(H1) \quad \begin{cases} 
(i) \quad A \text{ generates a strongly continuous semigroup } e^{tA} \text{ on } L^2[0,1], \\
(ii) \quad B \in L(L^2[0,1]),
\end{cases}
\]

where the Hilbert space \( U(\text{control space}) \) in the present case is \( L^2[0,1] \).

Theorem 5. The graphon system \( LS_\infty \) in \( (18) \) has a unique solution \( x \in C([0,T]; L^2[0,1]) \) for all \( x_0 \in L^2[0,1] \) and all \( u \in L^2([0,T]; L^2[0,1]) \).

Proof. Since \( A \) as a graphon operator generates a uniformly continuous semigroup, \( H1(i) \) is satisfied. Moreover \( B \in \tilde{G}_A^1 \) as a linear operator is bounded and hence is a continuous linear mapping from control space \( L^2[0,1] \) to the state space \( L^2[0,1] \) satisfying \( H1(ii) \). Therefore, \( (H1) \) is satisfied and following \([30]\), the system \( (18) \) has a unique solution \( x \in C([0,T]; L^2[0,1]) \) for all \( x_0 \in L^2[0,1] \) and all \( u \in L^2([0,T]; L^2[0,1]) \).

C. Controllability

A system \((A; B)\) is exactly controllable on \([0,T]\) if for any initial state \( x_0 \in L^2[0,1] \) and any target state \( x_f \in L^2[0,1] \), there exists a control \( u \in L^2([0,T]; U) \) driving the system from \( x_0 \) to \( x_f \), i.e. \( x_T = x_f \) with \( x_T = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu_t dt \).

A system \((A; B)\) is approximately controllable on \([0,T]\) if for any initial state \( x_0 \in L^2[0,1] \), any target state \( x_f \in L^2[0,1] \) and any \( \varepsilon > 0 \), there exists a control \( u \in L^2([0,T]; U) \) driving the system from \( x_0 \) to points in the state space within an \( \varepsilon \)-distance from \( x_f \), i.e. \( \|x_T - x_f\| \leq \varepsilon \).

The controllability Gramian operator \( W_t \) : \( L^2[0,1] \rightarrow L^2[0,1] \) is defined as

\[
W_t := \int_0^t e^{A(t-s)}BB^T e^{A^T(t-s)} ds, \quad t > 0.
\]

A necessary and sufficient condition for exact controllability on \([0,T]\) is the uniform positive definiteness of \( W_T \):

\[
(W_T h, h) \geq c_T \|h\|^2
\]

for all \( h \in L^2[0,1] \), where \( c_T > 0 \) and \( \| \cdot \| \) is the \( L^2[0,1] \) norm. The positive definiteness of the controllability Gramian operator \( W_T \) as a kernel is equivalent to the approximate controllability of the corresponding system (see \([30], [32]\)).

Define the kernel space (or null space) for a linear operator \( L \) on \( L^2[0,1] \) as \( \ker(L) := \{ x \in L^2[0,1] : Lx = 0 \} \). The spectrum \( \sigma(L) \) of a linear bounded operator \( L \) on \( L^2[0,1] \) is the set of all (complex or real) scalars \( \lambda \) such that \( L - \lambda I \) is not invertible. Thus \( \lambda \in \sigma(L) \) if and only if at least one of the following two statements is true:

(i) The range of \( L - \lambda I \) is not all of \( L^2[0,1] \).
(ii) \( L - \lambda I \) is not one-to-one.

If \( (\text{ii}) \) holds, \( \lambda \) is said to be an eigenvalue of \( L \); the corresponding eigenspace is \( \ker(L - \lambda I) \); each \( x \in \ker(L - \lambda I) \) (except \( x = 0 \)) is an eigenvector of \( L \); it satisfies the equation \( Lx = \lambda x \). See \([33]\).

Theorem 6 (Appendix D). Let \( A \) be a graphon in \( \tilde{G}_1^{\text{gp}} \) and let \( B \) be a bounded linear operator on \( L^2[0,1] \). The linear system \((A; B)\) is exactly controllable on a finite time horizon \([0,T]\) if all the values in the spectrum of \( BB^T \) are lower bounded by a strictly positive constant.

Proposition 1 (Appendix D). Let \( A \) and \( B \) be graphons in \( \tilde{G}_1^{\text{gp}} \). Then \((A; B)\) is not exactly controllable on any finite time horizon \([0,T]\).

The results in Theorem 6 and Proposition 1 generalize to the case where \( A \) lies in any uniformly bounded subset of \( \tilde{G}^{\text{gp}} \) (that is, any set of symmetric measurable functions \( W : [0,1]^2 \rightarrow I \), where \( I \) is a bounded interval in \( \mathcal{R} \)).

V. Graphon State-to-State Control of Large-Scale Networks

A. Approximation of \( L^2[0,1] \) Input Functions via Piece-wise Constant Functions

The following basic result will be employed in the analysis.
Theorem 7 (p.198). Let $\lambda$ be any measure on $\mathcal{R}$ and let $1 \leq p < \infty$. Then piece-wise constant functions on $\mathcal{R}$ form a dense subset of $L^p(\mathcal{R}, B_\lambda, \lambda)$.

Piece-wise constant functions can approximate $L^2$ functions arbitrarily. In this paper we wish to approximate the control input $u_i(\cdot) \in L^2([0,1], 0 \leq t \leq T)$, through a piece-wise constant function in $L^2([0,1])$ denoted by $u_i^N(\cdot)$. Specifically, the approximation of an input $u_i(\cdot)$ by the function $u_i^N(\cdot)$ with the partition $Q = \{Q_1, Q_2, \ldots, Q_{nN}\}$ of $[0,1]$ is given as follows: for all $Q_i, i \in \{1, 2, \ldots, nN\}$,

$$u^N_i(\alpha) = \frac{1}{\mu(Q_i)} \int_{Q_i} u_i(\beta) d\beta, \quad \forall \alpha \in Q_i, \tag{19}$$

where $\mu(Q_i)$ denotes the measure of $Q_i$.

B. Limit Control for Network Systems with General Graphon Input Mappings

Consider a finite dimensional system $(A_N; B_N)$ with node averaging dynamics as in $(14)$ and $(A^*_N; B^*_N)$ as its equivalent step function system according to $(4)$. Theorem 8 (Appendix E). Consider a sequence of network systems $\{(A^*_N; B^*_N)\}$ converging to a graphon system $(A; B)$ in the following sense: $A^*_N \to A$ in the $L^2$ metric and $B^*_N \to B$ in the $L^2$ metric as $N \to \infty$. Consider the problem of driving the systems from the origin to some target state. Then for any $T > 0$:

1) there exists a control $v^N$ for $(A^*_N; B^*_N)$ approximating the control $v$ for $(A; B)$ such that

$$\|x_T(v) - x^N_T(v^N)\|_2 \leq \|A^*_N\|_2 \|B_N\|_2 \int_0^T e^{T - \tau}(T - \tau) \|v^N\| d\tau + \|A^*_N\|_2 \|B_N\|_2 \int_0^T e^{T - \tau}\|A^*_N\|_2 \|v^N\| d\tau,

2) furthermore, for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that for each $N \geq N_\varepsilon$,

$$\|x_T(v) - x^N_T(v^N)\|_2 < \varepsilon,$$

where $x_T(v)$ represents the terminal state of $(A; B)$ under control $v$, $x^N_T(v^N)$ represents the terminal state of $(A^*_N; B^*_N)$ under control $v^N$, and the control approximation is given in the following: $v^N_i(\alpha) = nN \int_{P_i} v_i(\beta) d\beta$, for all $\alpha \in P_i, t \in [0, T]$, with the uniform partition $P^{nN} = \{P_1, \ldots, P_{nN}\}$.

According to the $M_G$ mapping, the control law $v^N(\cdot)$ for the finite network system $(A_N; B_N)$ is given by

$$v^N_i(i) = v^N_i(\alpha), \quad \forall i \in \{1, \ldots, nN\}, \forall \alpha \in P_i, t \in [0, T].$$

C. Limit Control for Network Systems with the Identity Input Mapping

In general, the control input mapping $B$ is not limited to be a graphon mapping. As long as the control input map is a linear continuous mapping from $L^2[0,1]$ to $L^2[0,1]$, the existence and uniqueness of solutions are guaranteed. Hence the system $(A; I)$ has a unique solution. We note that while the identity operator $I$ may be represented by a positive measure on the diagonal in $[0,1]^2$ and hence may formally be treated as an element of $L^1[0,1]^2$, it is not an element of $L^2[0,1]^2$ and hence does not lie in $G_1^{sp}$. Consider a sequence of finite dimensional systems $\{(A_N; I_N)\}$ with node averaging dynamics and $\{(A^*_N; I)\}$ as its equivalent step function system sequence according to $(15)$.

Theorem 9 (Appendix E). Suppose $A^*_N \to A$ in the $L^2$ metric as $N \to \infty$. Consider the problem of driving the systems from the origin to some target state. Then for each $N, 1 \leq N < \infty$, there exists a control $u^N$ for $(A^*_N; I)$ approximating the control $u$ for $(A; I)$ such that

$$\|x_T(u) - x^N_T(u^N)\|_2 \leq \|A^*_N\|_2 \|B_N\|_2 \int_0^T e^{T - \tau}(T - \tau)\|u^N\| d\tau$$

$$+ \|A^*_N\|_2 \|B_N\|_2 \int_0^T |u^N - u^N(\cdot)| d\tau, \tag{20}$$

where $A^*_N = A - A^*_N$, $x_T(u)$ represents the terminal state of $(A; I)$ under control $u$, $x^N_T(u^N)$ represents the terminal state of $(A^*_N; I)$ under control $u^N$, and the control approximation is given by $u^N_i(\alpha) = nN \int_{P_i} u^N_i(\beta) d\beta$, for all $\alpha \in P_i$, with the uniform partition $P^{nN} = \{P_1, \ldots, P_{nN}\}$.

Furthermore, for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that for each $N \geq N_\varepsilon$,

$$\|x_T(u) - x^N_T(u^N)\|_2 < \varepsilon.$$

Based on the result in Theorem 9, the control law $u^N(\cdot)$ for the finite network system $(A_N; I_N)$ is given by $u^N(i) = u^N_i(\alpha), \quad \forall i \in \{1, \ldots, nN\}, \forall \alpha \in P_i, t \in [0, T]$.

Note that $u^N$ always exists by definition since the control approximation given by $(19)$ uses the same uniform partition as the step function approximation in the graphon space.

D. The Graphon State-to-state Control (GSSC) Strategy

Consider the control problem of steering the states of each member of $\{(A_N; B_N)\}_{N=1}^\infty \subseteq \mathcal{S}$ to each of a sequence of desired states $\{x^N_T \in \mathcal{R}^{nN}\}_{N=1}^\infty$. The Graphon State-to-state Control (GSSC) Strategy consists of four steps:

S.1 Let $\{(A^*_N; B^*_N)\}_{N=1}^\infty \subseteq \mathcal{F}_1^{sp} \times \mathcal{G}_1^{AT}$ be the sequence of graphon dynamical systems equivalent to $\{(A_N; B_N)\}_{N=1}^\infty \subseteq \mathcal{S}$ under the mapping $M_G$ and assume that it converges to the graphon system $(A; B) \in \mathcal{G}_1^{sp} \times \mathcal{G}_1^{AT}$. Let $\{x^N_T \in L^2([0,1])\}_{N=1}^\infty \subseteq \mathcal{S}$ be the image of $\{x^N_T \in \mathcal{R}^{nN}\}_{N=1}^\infty$ under $M_G$, which is assumed to converge to some $x^*_{\infty} \in L^2([0,1])$ in the $L^2([0,1])$ norm.

S.2 Specify the corresponding state to state control problem $CP^{\infty}$ for $(A; B) \in \mathcal{G}_1^{sp} \times \mathcal{G}_1^{AT}$ with $x^*_{\infty}$ as the target terminal state and choose a tolerance $\varepsilon > 0$.

S.3 Find a control law $u^\infty := \{u_T \in L^2([0,1]), \tau \in [0, T]\}$ solving $CP^{\infty}$.

S.4 Then generate the control law $\{u^N\}$ according to Theorems 8, 9 for which the convergence of $\{x^N_T(u^N)\}$ to $x^*_{\infty}$ is guaranteed. Together with the assumed convergence of $\{x^N_T \in L^2([0,1])\}_{N=1}^\infty$ to $x^*_{\infty}$, it yields $N_\varepsilon$ such that
E. Minimum Energy State-to-state Control for Graphon Systems

A specific control law which may be used in S.2 of the GSSC strategy is described in this section.

1) Minimum Energy Control of Infinite Dimensional Systems: Define the energy cost by the control over the time horizon $[0, T]$ as $J(u) = \int_0^T \|u(t)\|^2 dt$, $(0 < T < \infty)$. The objective is to drive the system from some state $x_0 \in L^2[0,1]$ to some state $x_T \in L^2[0,1]$ using minimum control energy. A function $u^* \in L^2(0,T;U)$ is called an optimal control if $J(u^*) \leq J(u)$, for all $u \in L^2(0,T;U)$ which drive the system from $x_0$ to $x_T$.

2) Minimum Energy Control Law:

Theorem 10 (Appendix C). If the graphon system $(A;B)$ with $W_T$ as its graphon controllability Gramian operator is exactly controllable, then the inverse operator $W_T^{-1}$ exists and is a bounded operator.

Assume the system $(A;B)$ is exactly controllable, then $W_T^{-1}$ exists and the optimal control law that achieves the minimum energy control is given by

$$u^* = B^T e^{A^T(T-\tau)} W_T^{-1} (x_T - e^{A(T)} x_0), \quad \tau \in [0,T]. \quad (21)$$

The minimum energy for controlling the system in time horizon $[0,T]$ is

$$\|u^*\|^2 = \|x_T - e^{A(T)} x_0\|^2 W_T^{-1} [x_T - e^{A(T)} x_0]. \quad (22)$$

3) Inverse of the Controllability Gramian Operator: Since graphon $A$ is a compact operator, it has a discrete spectrum.

Assume the spectral decomposition of $A$ is as follows $A(x,y) = \sum_{\lambda \in \text{Spec}} \lambda f_\lambda(x)f_\lambda(y)$, where $f_\lambda$ is the normalized eigenfunction corresponding to the non-zero eigenvalues $\lambda$ and assume $B = I$. Then

1) the controllability Gramian operator is given by

$$W_T = TI + \sum_{\lambda} \left( \frac{1}{2\lambda} e^{2\lambda T} - 1 - T \right) f_\lambda f_\lambda^T;$$

2) the inverse of the controllability Gramian operator for $(A;B)$ is given by

$$W_T^{-1} = \frac{1}{T}I - \frac{1}{T} \sum_{\lambda} \left( \frac{1}{2\lambda} e^{2\lambda T} - 1 - T \right) f_\lambda f_\lambda^T.$$

Note that $\lim_{\lambda \to 0} \left( \frac{1}{2\lambda} e^{2\lambda T} - 1 - T \right) = 0$. To achieve state-to-state control of linear system in infinite dimensional state space requires the system $(A;B)$ to be exactly controllable. Approximate controllability is not sufficient to achieve state-to-state control since the inverse operator of $W_T$ might not be bounded in certain subspaces in $L^2[0,1]$ and then the energy required is unbounded.

VI. Graphon Linear Quadratic Regulation (LQR) of Network Systems

A. LQR Problems for Graphon Dynamical Systems

Let $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and the inner product in $L^2[0,1]$. For finite $T > 0$, consider the problem of minimizing the cost given by

$$J(u) = \int_0^T \left[ \|C x_T\|^2 + \|u_T\|^2 \right] dt + \langle P_0 x_T, x_T \rangle \quad (23)$$

over all controls $u \in L^2((0,T]; L^2(0,1))$ subject to the system model constrained in (18). The assumptions for $C$ and $P_0$ are:

$$(H2) \begin{cases} (i) \quad P_0 \in L(L^2[0,1]) \text{ is Hermitian and non-negative,} \\ (ii) \quad C \in L(L^2[0,1]; Y) \end{cases}$$

where $Y$ is the Hilbert space of observations, which in the current case is $L^2[0,1]$.

Finding the feedback control via dynamic programming consists of the following standard steps:

Step 1. Solve the Riccati equation

$$\dot{P} = A^T P + PA - PBB^T P + C^T C, \quad \quad P(0) = P_0 \quad (24)$$

Step 2. Given the solution $P$ to the Riccati equation, the optimal control $u^*$ is given by

$$u^*_t = -B^T P(T-t)x_t^*, \quad t \in [0,T] \quad (25)$$

and the optimal trajectory $x^*$ is then the solution to the closed loop equation

$$\dot{x}_t = Ax_t - BB^T P(T-t)x_t, \quad t \in [0,T], x_0 \in L^2[0,1]. \quad (26)$$

B. Existence and Uniqueness of Solutions to LQR Problems

Applying the results in (20) to $L^2[0,1]$ space, one can show, under the assumptions $(H1)$ and $(H2)$, the existence and uniqueness of the solution to the Riccati equation (24) and the existence and uniqueness of optimal solution pair $(u^*, x^*)$ in (25) and (26).
C. The Graphon-Network LQR (GLQR) Strategy

Consider the control problem of regulating the states of each member of \( \{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S} \). The Graphon-Network LQR (GLQR) Strategy is as follows:

S.1 Let \( \{(A_s^{[N]}; B_s^{[N]}) \in \mathcal{G}_1^{sp} \times \mathcal{G}_{N-1}^{L_2[0,1]} \}_{N=1}^{\infty} \) be the sequence of step function systems equivalent to \( \{(A_N; B_N)\}_{N=1}^{\infty} \in \mathcal{S} \) under the mapping \( M_G \), and assume that it converges to the graphon system \((A; B) \in \mathcal{G}_1^{sp} \times \mathcal{G}_{N-1}^{L_2[0,1]} \).

S.2 Define the linear quadratic cost for systems that are sufficiently close to the limit graphon system and that the corresponding LQR problem for the (limit) of finite network systems converges to a limit graphon system state trajectory are close to those achieved by the optimal LQR (see [30]) by \( C \) for the finite dimensional network system:

\[
J(u) = \int_0^T \left[ \|Cx\|^2 + \|u_s\|^2 \right] dt + \langle P_0 x_T, x_T \rangle
\]

and the linear quadratic cost for \((A_s^{[N]}; B_s^{[N]}) as:

\[
J(u^{[N]}) = \int_0^T \left[ \|C_s x_s^{[N]}\|^2 + \|u_t^{[N]}\|^2 \right] dt + \langle P_{s0} x_T^{[N]}, x_T^{[N]} \rangle
\]

where it is assumed that \( C_s^{[N]} \to C \) and \( P_{s0}^{[N]} \to P \) in the strong operator sense. Solve the infinite dimensional Riccati equation for \((A; B)\) to generate the solution \( P \).

S.3 Approximate \( P \) to generate \( \tilde{P}_N \) and hence the control law \( u_t^{[N]} = -B_s^{[N]}(T-t)x_s^{[N]} \) for \((A_s^{[N]}; B_s^{[N]})\).

Parallel to the state-to-state control problem, we take the notion of the effectiveness of the GLQR strategy for a sequence of network systems to be that (1) the regulation cost and the state trajectory are close to those achieved by the optimal LQR control; (2) the computation for generating the control law is tractable.

Again in analogy with the state-to-state control problem, the basic assumptions for the GLQR strategy are that the sequence of finite network systems converges to a limit graphon system (as in Definition 1) or that a given instance of the network sequence can be closely approximated by a graphon system, and that the corresponding LQR problem for the (limit) graphon system is tractable.

These assumptions, together with Theorem 13, guarantee the effectiveness of the GLQR strategy for the finite network systems that are sufficiently close to the limit graphon system for sufficiently large node cardinality.

D. Control Law Approximations

By approximating the Riccati equation solution \( P \) for \((A; B)\) we can generate \( \tilde{P}_N \) that provides the control law for the finite dimensional network system:

\[
u_t^{[N]} = -B_s^{[N]}(T-t)x_s^{[N]}
\]

1) Basic Notation: Let

\[
\Sigma(L^2[0,1]) = \{ T \in L(L^2[0,1]) : T \text{ is Hermitian} \}
\]

and

\[
\Sigma^+(L^2[0,1]) = \{ T \in \Sigma(L^2[0,1]) : \langle T x, x \rangle \geq 0, \forall x \in L^2[0,1] \}.
\]

Denote the topological space of all strongly continuous mappings \( F : I \to \Sigma(L^2[0,1]) \) endowed with strong convergence (see [50]) by \( C_s(I; \Sigma(L^2[0,1])) \).

2) Approximation of the Solution to the Riccati Equation: We need to extend the step function approximation to step function approximation by local integration against measures.

First, we construct the equivalent representation of the linear operator \( P \) in \( C_s([0, T]; \Sigma^+(L^2[0,1])) \) by integration against measures, that is, we first represent \( P \) by

\[
P(x, y) = \int_{S_i \times S_j} P(x, y) d\sigma(x, y) / \mu(S_i) \mu(S_j) \], \forall (x, y) \in S_i \times S_j,
\]

where \( \sigma(x, y) \) represents the measure (which can be a singular measure, a Lebesgue measure or a mixed measure).

Second, we introduce a method to approximate the operator \( P \) by local integration with respect to measures over partitions. The local step function approximation against measures of \( P \) is performed by integration against measures as follows:

\[
\tilde{P}_N(x, y) = \frac{\int_{S_i \times S_j} P(x, y) d\sigma(x, y) / \mu(S_i) \mu(S_j)}{\nu(x, y) \in S_i \times S_j},
\]

where \( S_i, S_j \subset [0, 1], \mu(S_i) \) represents the length of the interval \( S_i \) and \( \sigma(x, y) \) represents the measure (which can be a singular measure, a Lebesgue measure or a mixed measure).

3) Approximation of the Riccati Solution and Its Convergence to the Optimal Riccati Solution: Based on the definition of the step function approximation against measures, \( \tilde{P}_N(x) \) is the step function approximation of \( P(x) \) in \( L^2([0,1]) \), and hence it is the case that for any \( x \in L^2([0,1]) \),

\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \| \tilde{P}_N(t)x - P(t)x \| = 0.
\]

Therefore we obtain the following lemma.

Lemma 2. Let \( \tilde{P}_N \) be generated by step function approximation against measures from \( P \) via an \( N \times N \) uniform partition of \([0, 1]^2\). Then

\[
\lim_{N \to \infty} \tilde{P}_N = P, \text{ in } C_s([0, T]; \Sigma(L^2[0,1])).
\]

Theorem 11 (Appendix F). Let \( \tilde{P}_N \) be generated by step function approximation against measures from \( P \) via \( N \times N \) uniform partition of \([0, 1]^2\). For any \( x \in L^2([0,1]) \), for any \( t \in [0,T] \),

\[
\lim_{N \to \infty} \| \tilde{P}_N(t)x - P^{[N]}(t)x \| = 0,
\]

where \( P^{[N]} \) is the solution to Riccati equation of \((A^{[N]}_s; B^{[N]}_s)\) that converges strongly to the solution \( P \).

4) Continuous Dependence of Riccati Equation Solution with Respect to the Data: Let \( Ricc(A, B, C, P_0) \) denote the following Riccati equation

\[
\begin{align*}
\dot{P} &= A^T P + PA - PPBB^T P + C^T C, \\
P(0) &= P_0.
\end{align*}
\]

Assumption 1. (A1)

1) For any \( N \geq 1 \), \((A^{[N]}_s; B^{[N]}_s, C^{[N]}_s, P^{[N]}_{s0})\) satisfies (H1) and (H2).

2) The system sequence \( \{(A^{[N]}_s; B^{[N]}_s) \} \) converges to \((A; B)\) as in Definition 1.

3) The sequences \( \{C^{[N]}_s\} \) and \( \{P^{[N]}_{s0}\} \) converges strongly to \( C \) and \( P_0 \), respectively, as \( N \to \infty \).
4) C and $C_s[N]$ are self-joint linear operators.

**Theorem 12.** Consider a sequence of network systems $(A_N; B_N)_{N=1}^{\infty}$ with $(A[N]; B[N]) \in \mathcal{G}^{(p)} \times \mathcal{G}^{(p)}_{1}$ as the step function representation. Let $P$ and $P_s[N]$ be the solution to $Riccati(A, B, C, P_a)$ and $Riccati(A_s[N], B_s[N], C_s[N], P_s[N])$ respectively. If $(A_s[N], B_s[N])$ hold, then for any finite horizon $[0, T]$,

$$\lim_{N \to \infty} P_s[N] = P, \quad \text{in } C_s([0, T]; \Sigma(L^2[0, 1])).$$

**Proof.** From Theorem 4 we know for all $T > 0$ and all $x \in L^2[0, 1]$, $\lim_{N \to \infty} e^{A_s[N]T}x = e^{A[N]T}x$ uniformly in $[0, T]$. Since the system sequence $(A_s[N], B_s[N])$ converges to $(A; B)$ as in Definition 1, $(B_s[N])$ converges to $B$ in the strong operator sense. We may now apply Theorem 2.2, Part IV. [3], specialized to the Hilbert space $L^2[0, 1]$. Its hypotheses are then satisfied in the present case, the desired result follows.

5) **Convergence of States and Convergence of Costs.** Let $P_s[N]$ denote the solution to the Riccati equation for $(A_s[N], B_s[N])$ that converges strongly to the solution $P$ of the Riccati equation for $(A; B)$. Let $\hat{P}_N$ be the step function approximation against measures for $P$ generated via the $N \times N$ uniform partition of $[0, 1]^2$.

**Theorem 13 (Appendix F).** Consider the time horizon $[0, T]$. Let the optimal linear quadratic control law for $(A_s[N], B_s[N])$ be generated by

$$u_t^{N,*} = -B_s[N]^T \hat{P}_s[N](T-t)x_t^{N,*},$$

where the optimal state trajectory is given by $x_t^{N,*}$, and let the graphon approximate control law for $(A_s[N], B_s[N])$ be

$$u_t^{[N]} = -B_s[N]^T \hat{P}_s[N](T-t)x_t^{[N]},$$

where the corresponding state trajectory is given by $x_t^{[N]}$. Then

$$\forall t \in [0, T], \quad \lim_{N \to \infty} \|x_t^{N,*} - x_t^{[N]}\|_2 = 0,$$

and

$$\lim_{N \to \infty} |J(u_t^{N,*}) - J(u_t^{[N]})| = 0.$$

**VII. Numerical Examples**

**A. Network Systems with Sampled Weightings**

The generation of a randomly sampled network of size $N$ from a graphon $U$ is specified as follows:

1) Sample $N$ points from a uniform distribution in $[0, 1]$. Sort the sample points in the decreasing order of their values and label them from node 1 to node $N$. Denote the node set by $V_N$ and the value of node $i \in V_N$ by $v_i$.

2) Connect the nodes $i, j \in V_N$ with edge weight $U(v_i, v_j)$ to generate the network $G_N$. Then $A_{N_{ij}} = U(v_i, v_j)$ is the $ij$th element of the adjacency matrix of $G_N$.

Consider a network system evolving according to node averaging dynamics with $G_N$ describing the dynamic interactions. Suppose each node has an independent input channel. Denote the system by $(A_N; I_N)$, where $A_N$ is the adjacency matrix of $G_N$ and $I_N$ is the identity input mapping. The network system $(A_N; I_N)$ with node averaging dynamics is therefore described by

$$\dot{x}_t^i = \frac{1}{N} \sum_{j=1}^{N} A_{N_{ij}}x_t^j + u_t^i, \quad x_t^i, u_t^i \in R, i \in \{1, ..., N\},$$

(30)

where $A_{N_{ij}}$ is sampled from the graphon.

If $u$ is almost everywhere continuous, then the step function $A_s[N]$ of $A_N = [A_{N_{ij}}]$ converges to $U$ in the $\delta_1$ metric as $N \to \infty$ (see e.g. [25]), that is,

$$\delta_1(A_s[N], U) \to 0 \quad \text{as } N \to \infty.$$  

(31)

Further if we assume that $U$ is uniformly bounded, then (31) implies $\delta_2(A_s[N], U) \to 0$, as $N \to \infty$. By the generation procedure, we obtain the labeling that achieves the minimum distance between the network and the limit, and hence a sequence of networks converge in the $L^2$ metric to the limit $U$. It follows that if $U$ is almost everywhere continuous and uniformly bounded, then we can apply the graphon control strategy to the sampled network systems.

**B. Minimum Energy Graphon State-to-state Control**

As an example, we consider the case where the graphon limit is given by

$$U(x, y) = \cos(2\pi(x - y)) + 0.5 \cos(4\pi(x - y)), \quad x, y \in [0, 1]$$

and solve the minimum energy control problem of driving the states of the network system $(A_N; I_N)$ to a Gaussian terminal state distribution $x_t^N$ from the origin over the time horizon $[0, T]$ with $T = 2$.

The system $(U; I)$ is controllable and the forward controllability Gramian operator is given by

$$W_T = \int_0^T e^{U(s)} e^{U(s)^T} ds = \int_0^T e^{2U(s)} ds.$$  

(32)

The minimum energy control for $(U; I)$ is given by

$$u_\tau^* = e^{U^T(\tau - \tau_1)} W_{\tau_1}^{-1} x_\tau, \quad \tau \in [0, T],$$  

(33)

Then the control law $u^N_\tau$ for the network system $(A_N; I_N)$ generated by $U$ comes from the following approximation:

$$u^N_\tau(i) = N \int_{P_i} u^*_\tau(d\beta), \quad \beta \in [0, T],$$

where $P_i$ is the $i$th element of the uniform partition $P_N$ of $[0, 1]$. The error $\|x_\tau(u) - x_\tau^N(u^{[N]}(u))\|_2$ is bounded as in [20] and converges to 0 as $N \to \infty$. The result of a simulation with a network system with 100 nodes using the proposed approximate control is shown in Figure 5.

**C. Graphon-Network LQR**

The control objective is to regulate randomly distributed network states around the origin with minimum LQR cost. As an example, we consider a sequence $\hat{S}$ of networks converging to the graphon limit $U(x, y) = 4 \cos(2\pi(x - y))$ for all $x, y \in [0, 1]$ as in figure (h) and solve the LQR problem over the time horizon $[0, T]$ with $T = 4$ for each network in the sequence.

In this simulation, as is shown in Figure 5 a network of size 320 along the sequence $\hat{S}$ is considered. The system is represented by $(A_{320}; I_{320})$ with $A_{320}$ as the adjacency matrix.
of the weighted network and $I_{320}$ as the identity input matrix of size 320. We set $B = I_{320}$, $C = \sqrt{2}I_{320}$, $P_0 = I_{320}$. The infinite dimensional limit Riccati equation can be solved with the solution given by $P_t = \alpha_t I + \beta_t U$, where $\alpha_t$ and $\beta_t$ satisfy

$$\dot{\alpha}_t = 2 - \alpha_t^2, \quad \dot{\beta}_t = 2\alpha_t + 16\beta_t - 2\alpha_t\beta_t - 8\beta_t^2, \quad t \in [0, 4]$$

with $\alpha_0 = 1, \beta_0 = 0$. The finite dimensional control law is generated by approximating the Riccati equation solution as in (28). As the networks increase in size and converge to the limit graphon, the strong convergence of the approximated graphon Riccati equation solution to the finite dimensional graphon Riccati equation solution is guaranteed by Theorem 11. Furthermore, the convergence of the state trajectory (and cost) to the optimal state trajectory (and the optimal cost) is guaranteed by Theorem 13.

Fig. 3. Minimum Energy Graphon State-to-state Control

Both the graphon-LQR control and the LQR optimal control regulate the system from the same random initial states to the origin as shown in figures (a) and (b). From figures (e) and (f), we see that the graphon-LQR control achieves remarkably similar performance to the LQR optimal control. The maximum trajectory difference from the optimal control is less than 4% of the maximum initial states. With the graph interpreted as an $L^2([0, 1]^2$ function, the distance between the graph and the graphon limit in $L^2([0, 1]^2$ is 0.000813. The Graphon-LQR control cost is only 0.133% higher than the optimal LQR control cost.

VIII. DISCUSSION

The basic assumptions justifying the application of graphon control strategies are, first, that a given sequence of finite

Fig. 4. Parameters for Riccati Equation Solution

Fig. 5. Simulation on a Network of 320 Nodes
network systems converges to a unique limit graphon system (as in Definition 1) or that a given instance can be closely approximated by a graphon system, along with the measure preserving bijections that achieve the best fit, and second, that the corresponding control problem for the (limit) graphon system is tractable.

Under these assumptions, Theorems 8, 9 and 13 guarantee the effectiveness graphon control strategies for the finite complex network systems.

A plausible empirical approach to model the required infinite limit graphon $G_\infty$ is to fit two dimensional Fourier series to the step function representation of the adjacency matrix. Such parametric modelling of empirical data could resemble parametric estimation in statistics and system identification. Moreover, due to the compactness of graphon operators, representations or approximations by simple spectral decomposition are possible and will be analysed in future work.

The generation of the graphon approximation models inevitably deals with relabelings. Although in the graphon control design methodology we do not restrict the labeling to be that of the best fit to the data, the control error still depends on the labeling of the nodes. Furthermore, the labeling of the nodes on the networks is necessary for control implementation. To find the best labelings for general graphs can be a complex combinatorial task. Consequently, we underline that it is assumed in this paper that the best labeling is known beforehand, either through a specific way of growing the networks with labels that ensure the best fit to the limit (as for monotonically increasing graphon sequences see Appendix A), or through graphon estimation methods.

IX. Conclusion

We propose a method to approximately control networks of linear systems using the inherent limit described by graphons. Important aspects requiring further investigations include: (1) the application of the proposed limit graphon control strategy to asymmetric network systems where the interactions of dynamics are described by directed networks; (2) an analysis of the important class of monotonically increasing graphon sequences; (3) the creation of an equivalent theory to the dense case developed here for sparse networks; (4) the generation of a methodology for systematically fitting bivariate analytic models to network data; (5) the application of graphon control to stochastic linear quadratic Gaussian problems; (6) the analysis of decentralized graphon control via Mean Field Game theory [22].

APPENDIX A
MONOTONICALLY INCREASING GRAPHONS

Consider a sequence of graphons associated with a partition $Q = \{Q_1, Q_2, \ldots\}$ of $[0, 1]$ satisfying $\sum_{i=1}^{\infty} |Q_i| = 1$. Define the $(n+1)^{th}$ strip graphon $U_{Q_{n+1}}$ as follows:

$$U_{Q_{n+1}}(x, y) = \begin{cases} c_{n+1}(x, y), & (x, y) \in Q_{n+1} \times \bigcup_{i=1}^{n} Q_i \\ c_{n+1}(y, x), & (x, y) \in \bigcup_{i=1}^{n} Q_i \times Q_{n+1} \\ l_{n+1}(x, y), & (x, y) \in Q_{n+1} \times Q_{n+1} \\ 0, & \text{otherwise} \end{cases}$$

where $c_{n+1} : Q_{n+1} \times \bigcup_{i=1}^{n} Q_i \to [-1, 1]$ is a measurable function and $l_{n+1} : Q_{n+1} \times Q_{n+1} \to [-1, 1]$ is a symmetric measurable function. Define a monotonically increasing graphon sequence $\{W_n\}$ recursively via

$$W_{n+1} = W_n + U_{Q_{n+1}}, n \geq 0, \quad W_0 = 0.$$

From the construction, the optimal measure preserving bijection $\phi$ which achieves the $\delta_2$ distance yields

$$\delta_2(W_{n+m}, W_n) = \|U_{Q_{n+1}} + \cdots + U_{Q_{n+m}}\|_2, m \geq 1, n \geq 0,$$

in other words $\phi$ is the identity mapping from $[0, 1]$ to $[0, 1]$.

If $\{c_n(\cdot, \cdot)\}$ and $\{l_n(\cdot, \cdot)\}$ are two sequences of constant functions, $\{W_n\}$ then corresponds to a sequence of graphs with non-homogenous node weights. An example of the size of the partitions is given by $|Q_n| = \frac{1}{4}(\frac{3}{4})^{n-1}$. Intuitively this means that nodes that come later have discounted node weights with respect to previous ones. However, if the partition of $W_n$ is interpreted as $\{Q_1, Q_2, \ldots Q_n, \cup_{i=1}^{\infty} Q_i\}$, a further refinement of this partition will yield a graph with uniform node weights that corresponds to $W_n$. Note that $W_n$ has only zero values on the last partition $\cup_{i=1}^{\infty} Q_i$.

Under the refinement of the partition, one optimal measure preserving bijection is still the identity mapping from $[0, 1]$ to $[0, 1]$. The measure preserving bijection can be considered as the permutation of the refined partitions which corresponds to the relabelings on nodes of the corresponding finite graph. With the refinement, one can apply the graphon dynamical system model directly without further modifications.

**Proposition 3.** Consider a monotonically increasing graphon sequence $\{W_n\}$ along with its partition $Q = \{Q_1, Q_2, \ldots\}$, $\sum_{i=1}^{\infty} |Q_i| = 1$. Then $\{W_n\}$ converges to a graphon limit $W$ in $G_{\infty}^P$ under the $L^2$ metric. Furthermore, $\{W_n\}$ converges to
where the $\delta_2$ metric and under the $\delta_\square$ metric.

Proof. $\sum_{i=1}^{\infty} |Q_i| = 1$ implies $\lim_{n \to \infty} |Q_n| = 0$. Then, by construction, $\{W_n\}$ forms a Cauchy sequence in $G_1^{sp}$ under the $L^2$ metric, and hence converges to a unique limit $W$ in $G_1^{sp}$. By the ordering of metrics given in [8], $\{W_n\}$ converges to (the equivalence class of) the graphon limit $W$ in $G_1^{sp}$ under the $\delta_2$ metric and under the $\delta_\square$ metric. □

APPENDIX B

LEMMA 3.6

Lemma 3. Consider a step function $A_s^{[N]} \in \mathcal{G}_1^{sp}$ defined via partition $\mathcal{P} = \{P_1, \ldots, P_{nN}\}$ and $u_r^{[N]} \in L^{3\cdot6}_{Pw}[0, 1]$ defined via the same partition $\mathcal{P}$ by

$$u_r^{[N]}(\alpha) = nN \int_{P_i} u_r(\beta) d\beta, \quad \forall \alpha \in P_i,$$

where $u_r \in L^2[0, 1]$. Then

$$A_s^{[N]} u_r^{[N]} = A_s^{[N]} u_r$$

and

$$(A_s^{[N]})^k u_r^{[N]} = (A_s^{[N]})^k u_r, \quad k \geq 1.$$  (35)

Proof. First, for all $x \in P_i$,

$$[A_s^{[N]} u_r](x) = \int_0^1 A_s^{[N]}(x, y) u_r(y) dy = \sum_j \int_{P_j} A^{[N]}_{ij} u_r(y) dy,$$

where $A^{[N]}_{ij} = A^{[N]}(x, y)$, for all $(x, y) \in (P_i, P_j)$; then

$$[A_s^{[N]} u_r](x) = \sum_j \int_{P_j} A^{[N]}_{ij} u_r(y) dy, \quad \forall x \in P_i,$$

$$= \sum_j A^{[N]}_{ij} \int_{P_j} u_r(y) dy = \sum_j A^{[N]}_{ij} \cdot \mu(P_j) \cdot u_r^{[N]}(x)$$  (37)

Finally, $37$ and $38$ give the equality in $34$. An immediate implication of $34$ is that for $k \geq 1$

$$(A_s^{[N]})^k u_r^{[N]} = (A_s^{[N]})^k u_r.$$  (39)

Proof. An application of the Cauchy-Schwarz Inequality gives

$$||A_s \Delta(x, \cdot), u_r(\cdot)||^2 \leq \int_0^1 A_s^2(x, y) dy \cdot \int_0^1 u_r^2(y) dy,$$

where $A_\Delta$ denotes $(A - A_s)$. Hence

$$||A_s u_r - A_s u_r||^2 = ||(A_s - A_s) u_r||^2 = \int_0^1 \int_0^1 [A(x, y) - A_s(x, y)] u_r(y) dy]^2 dx$$

$$\leq \int_0^1 \int_0^1 A_\Delta^2(x, y) dy \int_0^1 u_r^2(y) dy \cdot dx = ||A_\Delta||^2 ||u_r||^2.$$  (41)

Lemma 5. Consider $A, A_s \in \mathcal{G}_1^{sp}$ and $u_r \in L^2[0, 1]$. Then for $k \geq 1$

$$||A^k - A_s^k||_2 \leq k ||A_\Delta||_2 \cdot ||u_r||_2$$

where $A_\Delta$ denotes $(A - A_s)$.

Proof. Define $P_k(x, y) = \sum_{i=0}^{k} x^{k-i} y^{i}$. Then

$$x^k - y^k = (x - y) P_{k-1}(x, y),$$

and we see that for $k \geq 1$

$$A^k - A_s^k = P_{k-1}(A, A_s)(A - A_s).$$  (43)

Since $A^{(k-1)} A_s^i \in \mathcal{G}_1^{sp}$, for all $i \in \{0, 1, \ldots k - 1\}$, we know that $||A^{(k-1)} A_s^i||_2 \leq 1$. Therefore, by the Minkowski inequality,

$$||P_{k-1}(A, A_s)||_2 \leq \sum_{i=0}^{k-1} ||A^{(k-1)} A_s^i||_2 \leq k \cdot 1.$$  (44)

Lemma 6. Consider $A_s \in \mathcal{G}_1^{sp}$. Then for $k \geq 1$

$$||A_s^k||_2 \leq ||A_s||_2.$$  (40)

Proof. Let $V$ and $W$ be any two graphons in $\mathcal{G}_1^{sp}$. Then for $k \geq 1$,

$$||VW||_2 = \int_0^1 \int_0^1 |V(x, \beta) W(\beta, y)| dy dx$$

$$\leq \int_0^1 \int_0^1 (V(x, \beta)) dy \cdot \int_0^1 |W(\alpha, y)| dy \cdot dx = ||V||_2^2 \cdot ||W||_2^2$$  (45)

Applying (45) to $A_s^k$ multiple times we have $||A_s^k||_2 \leq ||A_s||_2^k$. □

Results in Lemma 3, Lemma 4, and Lemma 6 generalize to functions in any uniformly bounded subsets of $G_1^{sp}$. 

A. Proof of Lemma 7

Proof. Since
\[ [A_s^{[N]}x_s](\alpha) = \int_0^1 A_s^{[N]}(\alpha, \beta)x_s(\beta)d\beta, \quad x_s \in L^2[0, 1], \]
it follows that for all \( \alpha \in P_1 \),
\[ A_s^{[N]}x_s(t)(\alpha) = \int_0^1 A_s^{[N]}(\alpha, \beta)x_s(\beta)d\beta \]
(46)
\[ = \sum_{j=1}^N \int_{P_j} A_s^{[N]}(\alpha, \beta)x_s(\beta)d\beta \]
\[ = \sum_{j=1}^N A_{ij}x_jd\beta = \sum_{j=1}^N \frac{1}{N} A_{ij}x_j \]
\[ = \frac{1}{N}[A_Nx]_t = [A_N \circ x]_t. \]
This implies that the step function \( A_s^{[N]} \) in the graphon space space, considered as an operator, represents a mapping in \( L^2[0, 1] \); this operator is equivalent to the matrix transformation \( A_N \) with \( \circ \) operation in \( RN \) and the corresponding mapping \( MG \). A similar conclusion holds for \( B_s^{[N]} \) and \( BN \).

We conclude that the trajectory of the system \((A_N; B_N)\) corresponds one-to-one to that of \((A_s^{[N]}, B_s^{[N]})\) under the corresponding vector-to-PWC-function mapping \( MG \).

B. Proof of Theorem 6

Proof. Let us define \( P_k(x, y) = \sum_{i=0}^k x^{k-i}y^i \), then
\[ x^k - y^k = (x - y)P_{k-1}(x, y), \]
and we see that for \( k \geq 1 \)
\[ A_{ij}^k - A_{ij}^k = P_{k-1}(A_N, A_s)(A_N - A_s). \]
For an arbitrary \( x \in L^2[0, 1] \) and finite \( t, 0 \leq t < \infty, \)
\[ \|e^{A_Nt}x - e^{A_xt}x\|_2 \leq \sum_{k=1}^\infty \frac{t^k}{k!}(A_{ij}^k - A_{ij}^k)x \|_2 \]
( by the property in Lemma 4 )
\[ \leq \sum_{k=1}^\infty \frac{t^k}{k!}\|P_{k-1}(A_N, A_s)\|_2 \|A_{ij}^k\|_2 \|x\|_2 \]
\[ = te^t \cdot \|A_{ij}^k\|_2 \|x\|_2. \]
Therefore, the left hand side goes to zero as \( \|A_{ij}^k\|_2 \rightarrow 0 \) \( \|A_{ij}^k\|_2 \rightarrow 0 \) goes to zero. It follows that for \( t \in [0, T] \)
\[ \|e^{A_Nt}x - e^{A_xt}x\|_2 \leq Te^t \cdot \|A_{ij}^k\|_2 \|x\|_2. \]
(49)
Hence the convergence is point-wise in time and uniform in \( t \) when \( t \) is in some finite interval \([0, T]\).
B. Proof of Proposition 7

Proof. By Lemma 7, since A and B are graphons in $\tilde{G}^{sp}$, there exists $c_1 \geq 0$ and $c_2 \geq 0$, such that

$$c_1 \geq \|A\|_2 \geq \|A\|_{op} \quad \text{and} \quad c_2 \geq \|B\|_2 \geq \|B\|_{op}.$$  \hfill (54)

Hence

$$\|e^{At}\|_{op} = \sup_{x \in L^2([0,1],x \|x\|_2=1)} \|e^{At}x\|_2 
\leq \sup_{x \in L^2([0,1],x \|x\|_2=1)} \sum_{k=0}^{\infty} \frac{1}{k!} \|A^k\|_{op} \|x\|_2 
= e^{\|A\|_{op}t} \leq e^{c_1 t}, \quad t \in [0, T].$$

Therefore

$$\|W_T\|_2 \leq \int_0^T \|e^{At}BB^Te^{At}t\|_{op} \|B\|_2^2 \|e^{At}B\|_2^2 \|e^{At}B\|_2^2 \|e^{At}B\|_2^2 dt 
\leq T(e^{c_2}e^{c_2})^2 < \infty,$$

which implies $W_T \in L^2([0,1]^2)$ and hence $W_T$ is a compact (and self-joint) operator on $L^2([0,1])$ functions (see e.g. [36 Chapter 2, Proposition 4.7]). This means that $W_T$ has a countable number of nonzero (real) eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ such that $\lambda_n \to 0$, and each eigenvalue has finite multiplicity (see e.g. [30]). Therefore $W_T$ is not uniformly positive definite and hence the system $(A; B)$ is not exactly controllable.

\[\square\]

APPENDIX E

PROOFS FOR STATE-TO-STATE GRAPHON CONTROL

A. Proof of Theorem 9

Proof. Denote the terminal state of $(A; B)$ under a control $v \in L^2([0, T]; U)$ by $G(v)$ and similarly the terminal state of $(A^N; B^N)$ under control $v \in L^2([0, T]; U)$ by $G^N(v)$. Then the difference between the two terminal states is given by $x_T(u) - x_T^N(u^N) = G(u) - G^N(u^N)$. We note that $(A^N; B^N)$ and $u^N$ are defined via the uniform partition $P = \{P_1, \ldots, P_n\}$. By Lemma 5 $A_s^N u^N_T = A_s^N u^N$.

For simplicity of notation, without causing confusion, we now set $A_s = A_s^N$, $u^N = u^N$ and $u^N_T = u^N_T$. Since the systems under consideration start with zero initial condition, we obtain

$$G^N(u) = \int_0^T e^{A_s(T-\tau)}u_\tau d\tau 
= \int_0^T \left[ u_\tau + \sum_{k=1}^{\infty} \frac{1}{k!} (T-\tau)^k A_s^k u_\tau \right] d\tau 
= \int_0^T [x_\tau - u^N_T] d\tau + \int_0^T e^{A_s(T-\tau)}u^N d\tau 
= \int_0^T [x_\tau - u^N_T] d\tau + G^N(u^N).$$

Therefore

$$G(u) - G^N(u^N) = G(u) - G^N(u) + \int_0^T [x_\tau - u^N_T] d\tau.$$ 

\hfill (58)

By Lemma 5 we have

$$\|(A^k - A_s^k)u_\tau\| \leq k\|A\|_2 \|u_\tau\|_2,$$  \hfill (59)

where $A_\Delta$ denotes $(A - A_s)$. Hence we have the following

$$\|G(u) - G^N(u^N)\|_2 \leq \int_0^T \sum_{k=0}^{\infty} \frac{(T-\tau)^k}{k!} \|A^k - A_s^k\| \|u_\tau\|_2 d\tau 
\leq \int_0^T \sum_{k=1}^{\infty} \frac{(T-\tau)^k}{k!} \|A_\Delta\|_2 \|u_\tau\|_2 d\tau 
= \|A_\Delta\|_2 \int_0^T e^{T-\tau}(T-\tau)\|u_\tau\|_2 d\tau.$$  \hfill (60)

Finally, the $L^2$ difference in terminal states is bounded by

$$\|x_T(u) - x_T^N(u^N)\|_2 \leq \|G(u) - G^N(u)\|_2 + \int_0^T \|u_\tau - u^N_\tau\|_2 d\tau$$

\hfill (61)

\[\square\]

B. Proof of Theorem 8

Proof. Denote the terminal state of $(A; B)$ under a control $v \in L^2([0, T]; U)$ by $G(v)$ and similarly the terminal state of $(A^N; B^N)$ under control $v \in L^2([0, T]; U)$ by $G^N(v)$. Then $x_T(v) - x_T^N(v^N) = G(v) - G^N(v^N)$. We note that $(A^N; B^N)$ and $v^N$ are defined via the uniform partition $P = \{P_1, \ldots, P_n\}$. Then following Lemma 3 one can show that for $\tau \in [0, T]$

$$B^N_v^N v^N_T = B^N_v^N v^N\tau,$$
$$A^N_s B^N_s v^N_T = A^N_s B^N_s v^N\tau.$$  \hfill (62)

For simplicity of notation, without causing confusion, we now set $A_s = A^N_s$ and $B_s = B^N_s$, then

$$G^N(v) = \int_0^T \left[ 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (T-\tau)^k A^k_s B_s v^N_T \right] d\tau$$

\hfill (63)

$$= \int_0^T B_s [v_\tau - v^N_\tau] d\tau$$

\[\square\]
Therefore \(G(v) - G^N(v) = G(v) - G^N(v)\) and hence
\[
\|G(v) - G^N(v)\|_2 = \int_0^T e^{A(T-t)}Bv_\tau d\tau - \int_0^T e^{A_s(T-t)}B_s v_\tau d\tau\]
\[
\leq \int_0^T e^{A(T-t)} - e^{A_s(T-t)}Bv_\tau d\tau \|
\leq \int_0^T e^{A_s(T-t)}[B - B_s]v_\tau d\tau \|
\leq \|A_{\Delta}\|_2 \cdot \|B\|_2 \int_0^T e^{-\varepsilon(T-t)}\|\|v_\tau\|_2 d\tau,
\]
where \(A_{\Delta} = A - A_s^N\). The second part of the terminal state difference in (64) can be estimated as follows:
\[
\|\int_0^T e^{A_s(T-t)}[B - B^s]v_\tau d\tau\|
\leq \int_0^T \sum_{k=1}^\infty \frac{(T-t)^k}{k!} \|A^k_s\|[B - B^s]v_\tau \|
\leq \|B\|_2 \int_0^T e^{-\varepsilon(T-t)}\|v_\tau\|_2 d\tau,
\]
with \(B_{\Delta} = B - B_s\). Finally, we obtain
\[
\|x_T - x_N^T\|_2 \leq \|A_{\Delta}\|_2 \|B\|_2 \int_0^T e^{-(\varepsilon(T-t))}\|v_\tau\|_2 d\tau
\leq \|A_{\Delta}\|_2 \|B\|_2 \int_0^T e^{-(\varepsilon(T-t))}\|A^N\|_2 \|v_\tau\|_2 d\tau.
\]

**APPENDIX F**

**Proofs for Graphon-LQR**

**A. Proof of Theorem [7]**

**Proof.** Since \(\lim_{N \to \infty} P_N = P\), in \(C_s([0,T]; \Sigma(L^2[0,1]))\)
and \(\lim_{N \to \infty} P_s^N = P_s\), in \(C_s([0,T]; \Sigma(L^2[0,1]))\), we obtain the following: for any \(x \in L^2[0,1]\) and for any \(t \in [0,T]\),
\[
\lim_{N \to \infty} \|P_N(t)x - P(t)x\|_2 = 0
\]
\[
\lim_{N \to \infty} \|P(t)x - P_s^N(t)x\|_2 = 0.
\]
Since
\[
\|P_N(t)x - P_s^N(t)x\|_2 \leq \|P_N(t)x - P(t)x\|_2 + \|P(t)x - P_s^N(t)x\|_2,
\]
we obtain \(\lim_{N \to \infty} \|P_N(t)x - P_s^N(t)x\|_2 = 0\).

**B. Proof of Theorem [13]**

**Proof.** The closed loop system with the optimal control law is given by
\[
x^*_t = \left(A_s^N - B_s^N B_s^N T P_s^N(T-t)\right)x^*_t,
\]
\(t \in [0,T], x_0 \in L^2[0,1]\).

The closed loop system under the graphon approximate control law is given by
\[
x^*_t = \left(A_s^N - B_s^N B_s^N T \tilde{P}_N(T-t)\right)x^*_t,
\]
\(t \in [0,T], x_0 \in L^2[0,1]\).

Let \(x^*_t := x^*_t - x^*_N\). By (68) and (69), we obtain
\[
x^*_t = F_N(t)x^*_t + V_N(t),
\]
where
\[
F_N(t) = \left(A_s^N - B_s^N B_s^N T P_s^N(T-t)\right)x^*_t.
\]

The integral representation of (70) is given by
\[
x^*_t = \int_0^t V_N(s)ds + \int_0^t F_N(t)x^*_s ds.
\]

Hence we obtain
\[
\|x^*_t\|_2 \leq \psi_N(t) + \int_0^t \chi_N(t)\|x^*_s\|_2 ds.
\]

First, we note that for any \(t \in [0,T]\), \(F_N(t)\) is a bounded linear operator, i.e., \(\chi_N(t)\) is bounded and hence there exist \(C > 0\),
\[
\chi_N(t)e^{\int_0^t \chi_N(s)ds} du \leq C, \quad \forall t, s \in [0,T], t \geq s.
\]

Second, based on the fact that \(B_s^N\) is a bounded operator, \(x^*_s \in L^2[0,1]\) and the result in Theorem [11] we obtain, for any \(t \in [0,T]\),
\[
\lim_{N \to \infty} \|B_s^N B_s^N T (P_s^N(T-t) - \tilde{P}_N(T-t))x^*_t\|_2 = 0,
\]
that is, for all \(t \in [0,T]\), \(\lim_{N \to \infty} \psi_N(t) = 0\). Therefore (72) and (74) yield
\[
\|x^*_t\|_2 \leq \psi_N(t) + C \int_0^t \psi_N(s)ds,
\]
which by (75) gives \(\lim_{N \to \infty} \|x^*_s - x^*_N\|_2 = 0, t \in [0,T]\).
Considering the fact that with the result in Theorem 11, we obtain
\[ \lim_{N \to \infty} J(u^{N*}) - J(u^{[N]}) = 0. \]

Hence
\[ J(u^{N*}) = \int_0^T \| C\dot{x}_t^* \|_2^2 + \| u_t^{N*} \|_2^2 \, dt. \]

Let \( I \) and \( W \) bounded operators on \( X \).

Following this we prove the result on the existence of the inverse mapping of graphon controllability Gramian operator

\[ W_T^{-1} = \frac{1}{T^2} I - \frac{1}{T} \sum_{l=1}^{\infty} \left( \frac{1}{\lambda_l} \left[ e^{2\lambda_l T} - 1 \right] \right) f_l f_l^T. \]

\section*{APPENDIX G}

\section*{INVERSE OF THE CONTROLLABILITY GRAMIAN}

Let \( X \) represent a Hilbert space. Let \( T \) and \( S \) be linear bounded operators on \( X \).

\begin{proposition}
Assume that \( T \) and \( S \) are symmetric and nonnegative; then \( I + TS \) is one-to-one and onto. Moreover \( \| (I + TS)^{-1} \| \geq \| S^{-1} \| \), and \( \| (I + TS)^{-1} \| \leq 1 + \| T \| \| S \| \).
\end{proposition}

Following this we prove the result on the existence of the inverse mapping of graphon controllability Gramian operator when the system is exactly controllable.

\subsection*{A. Proof of Theorem 70}

\begin{proof}
If the graphon system \((A; B)\) is exactly controllable, then
\[ \forall h \in L^2[0, 1], \exists c_T > 0, \quad (W_T h, h) \geq c_T \| h \|_2^2. \]

Let \( I \) denote the identity operator from \( L^2[0, 1] \) to \( L^2[0, 1] \). Let \( M = W_M - \frac{1}{c_T^2} I \), then \( W_M = \frac{1}{c_T^2} (I + \frac{2}{c_T^2} M) \). By definition, the operator \( M \) is nonnegative and symmetric and hence \( \frac{1}{c_T^2} M \) is nonnegative and symmetric. By Proposition 4, (I + \( \frac{2}{c_T^2} M \)) is one-to-one and onto and the inverse operator is bounded. By a scaling factor \( \frac{1}{c_T^2} \), \( W_T = \frac{1}{c_T^2} (I + \frac{2}{c_T^2} M) \) is one-to-one and onto and hence the inverse operator \( W_T^{-1} \) exists. Since the scaling factor \( \frac{1}{c_T^2} \) is strictly positive and finite, the inverse operator \( W_T^{-1} \) is also bounded.
\end{proof}

\section*{APPENDIX H}

\section*{CALCULATIONS FOR NUMERICAL EXAMPLES}

\subsection*{A. Minimum Energy State-to-state Control Law}

Consider the target terminal state
\[ x_T = \frac{1}{\sqrt{2\pi}} e^{-50(\alpha - 0.5)^2}, \quad \alpha \in [0, 1]. \]

The normalized eigenfunctions of the graphon operator
\[ U(x, y) = \cos(2\pi(x - y)) + 0.5 \cos(4\pi(x - y)), x, y \in [0, 1] \]

are given by
\[ f_1 = \sqrt{2} \cos 2\pi(-), \quad f_2 = \sqrt{2} \cos 4\pi(-) \]

with corresponding eigenvalues \( \lambda_1 = \frac{1}{2} \) and \( \lambda_2 = \frac{1}{4} \). The inner products can be calculated by
\[ \langle x_T, f_1 \rangle = -0.116088, \quad \langle x_T, f_2 \rangle = 0.064211. \]

The minimum energy control law is given by
\[ u_t^i = e^{U(T-t)W_T^{-1}x_T}, \quad t \in [0, T]. \]

Since
\[ e^{U(T-t)} = I + \sum_{l=1}^{2} e^{\lambda_l(T-t)} f_l f_l^T \]
the control law can be obtained as (81).

\begin{align*}
    \mathbf{u}_t &= \frac{1}{T} \mathbf{x}_T \\
    &= \frac{1}{T} \sum_{l=1}^{2} \lambda_l \left( e^{\lambda_l(T-t)} - e^{\lambda_l(T-t)} \right) \mathbf{f}_t^T \\
    &= \frac{1}{T} \sum_{l=1}^{2} \lambda_l \left( e^{\lambda_l(T-t)} - e^{\lambda_l(T-t)} \right) \mathbf{f}_t^T \\
    &= \left( e^{\lambda_l(T-t)} - e^{\lambda_l(T-t)} \right) \mathbf{f}_t^T ,
\end{align*}

Further by replacing the parameters in (81) and approximating the control law for any finite network system along the sequence converging to the graphon limit system.

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