Bounds on the Poincaré constant of ultra log-concave random variables

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Abstract

We consider the discrete Poincaré constant, which relates the variance of a function to the expected square of its finite difference. We give an explicit bound on the Poincaré constant of ultra log-concave random variables in terms of their first two moments, and discuss how this bound relates to calculations performed by other authors.

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1 Introduction

In this article, we consider the finite difference operator $\Delta$ defined by $(\Delta g)(x) = g(x + 1) - g(x)$ for functions $g : \mathbb{N} = \{0, 1, \ldots\} \rightarrow \mathbb{R}$. We discuss probability mass functions $P_X$ supported on $\mathbb{N}$ for which $\Delta g$ being small in $L^2(P_X)$ implies that $g$ is close to constant (in the same sense). We make the following definition:

Definition 1.1 Define the Poincaré (spectral gap) constant of a random variable $X$ with mass function $P_X$ supported on $\mathbb{N}$ to be

$$R_X = \sup_{g \in H(X)} \frac{\sum_{x=0}^{\infty} P_X(x)g(x)^2}{\sum_{x=0}^{\infty} P_X(x)(\Delta g)(x)^2},$$
where the supremum is taken over \( g \) in the set

\[
H(X) = \left\{ \sum_{x=0}^{\infty} P_X(x)g(x) = 0 \right\} \cap \left\{ \sum_{x=0}^{\infty} P_X(x)g^2(x) < \infty \right\} \cap \left\{ \sum_{x=0}^{\infty} P_X(x)(\Delta g)(x)^2 > 0 \right\}.
\]

The motivation for this article is to show that the ultra log-concave (ULC) random variables (see Definition 2.3 below) defined by Pemantle [12] and Liggett [11] have finite Poincaré constant. In fact we can give a bound in terms of the first two moments.

Theorem 1.2

(i) If \( X \) is an ultra log-concave random variable of degree \( \infty \) then it has Poincaré constant \( R_X \) satisfying

\[
R_X \leq \mathbb{E}X + \frac{1}{2} \pm \sqrt{\frac{1}{4} + \mathbb{E}X - \text{Var} \ X}.
\]

(ii) If \( X \) is an ultra log-concave random variable of degree \( n \) then it has Poincaré constant \( R_X \) satisfying

\[
R_X \leq \mathbb{E}X + \frac{1}{2} - \frac{\mathbb{E}X}{2n} \pm \sqrt{\frac{1}{4} + \mathbb{E}X - \text{Var} \ X - \frac{\mathbb{E}X}{2n}}.
\]

Proof. See Section 2 below.

The plan of the paper is as follows. In Section 2 we give some techniques for calculating Poincaré constants. In particular, Lemmas 2.1 and 2.2 give conditions on the tails of the random variable \( X \) under which the Poincaré constant \( R_X \) is bounded, allowing us to prove Theorem 1.2.

The idea of proving such discrete Poincaré inequalities is not a new one, with previous authors to consider the problem including Bobkov and Götzte [1], Bobkov and Ledoux [2], Cacoullos and Papathanasiou [5], Chen and Lou [6], Klaassen [9], Prakasa Rao and Sreehari [13] and Srivastava and Sreehari [14]. In Section 3 we discuss how our results compare with theirs.

In Section 4 we correct a mistake in Part (ix) of Theorem 2 of [3].

2 Calculating Poincaré constants

We require a direct equivalent of Theorem 1 of Borovkov and Utev [3].
Lemma 2.1 Let $X$ have probability mass function supported on $\mathbb{N}$ of the form
\[ P_X(x) = \alpha P_1(x) + (1 - \alpha)P_2(x), \]
where $0 < \alpha \leq 1$. Suppose that for some $x_0$ and $c > 0$, the inequalities
\[ \sum_{x=y+1}^{\infty} (x - x_0)P_X(x) \leq cP_1(y) \text{ for } y \geq x_0, \tag{1} \]
\[ -\sum_{x=0}^{y} (x - x_0)P_X(x) \leq cP_1(y) \text{ for } y < x_0. \tag{2} \]
hold. Then
\[ R_X \leq c/\alpha. \]

Proof Equations (1) and (2) are based on those given by Borovkov and Utev [3], and are applied to the proof given by Klaasen [9]. For the sake of completeness, we give the whole argument here. The key is to consider Klaasen’s kernel function, given by Equation (2.17) of [9]:
\[ \chi(x, y) = I([x_0] \leq y < x) - I(x \leq y < [x_0]) - (x_0 - [x_0])I(y = [x_0]). \tag{3} \]
For any given integer $x$, by considering the cases \{ $x < x_0$ \}, \{ $x > x_0$ \} and \{ $x = x_0$ \} separately, we deduce that for any function $h$:
\[ \sum_{y=0}^{\infty} \chi(x, y)h(y) = \begin{cases} 
- \sum_{y=x}^{[x_0]-1} h(y) + ([x_0] - x_0)h([x_0]) & \text{for } x < x_0, \\
([x_0] - x_0)h([x_0]) & \text{for } x = [x_0], \\
\sum_{y=x_0}^{y-1} h(y) + ([x_0] - x_0)h([x_0]) & \text{for } x > x_0.
\end{cases} \tag{4} \]
Taking $h \equiv \Delta g$ we deduce that $\sum_{y=0}^{\infty} \chi(x, y)(\Delta g)(y) = g(x) - g^*$, where $g^* = g([x_0]) + (\Delta g)([x_0])(x_0 - [x_0])$. In particular, taking $h(y) \equiv 1$ we deduce that $\sum_{y=0}^{\infty} \chi(x, y) = (x - x_0)$. Observe that by Cauchy-Schwarz:
\[ (g(x) - g^*)^2 \leq \left( \sum_{y=0}^{\infty} \chi(x, y)(\Delta g)(y) \right)^2 \leq (x - x_0) \left( \sum_{y=0}^{\infty} \chi(x, y)(\Delta g)(y)^2 \right). \tag{5} \]
Further, by assumption, Equation (3) implies that for any $y$,
\[ \sum_{x=0}^{\infty} \chi(x, y)P_X(x)(x - x_0) \leq cP_1(y). \tag{6} \]
[Equation (6) follows by considering three cases separately, using Equations (1) and (2):]
(i) For $y > |x_0|$ the left-hand side becomes $\sum_{x=y+1}^{\infty} P_X(x)(x - x_0) \leq cP_1(y)$ by (1).

(ii) For $y = |x_0|$ the left-hand side becomes $(1 - x_0 + |x_0|) \sum_{x=y+1}^{\infty} P_X(x)(x - x_0) \leq (1 - x_0 + |x_0|)cP_1(y) \leq cP_1(y)$ by (1), since for any $x_0$, $0 < 1 - x_0 + |x_0| \leq 1$.

(iii) For $y < |x_0|$ the left-hand side becomes $\sum_{x=0}^{y} P_X(x)(x - x_0) \leq cP_1(y)$ by (2).

This means that (with the reversal of order of summation justified by Fubini):

$$\sum_{x=0}^{\infty} P_X(x)g(x)^2 \leq \sum_{x=0}^{\infty} P_X(x)(g(x) - c)^2$$

$$\leq \sum_{x=0}^{\infty} P_X(x)(x - x_0) \left( \sum_{y=0}^{\infty} \chi(x,y)(\Delta g)(y)^2 \right)$$

$$\leq \sum_{y=0}^{\infty} (\Delta g)(y)^2 \left( \sum_{x=0}^{\infty} P_X(x)(x - x_0)\chi(x,y) \right)$$

$$\leq c \sum_{y=0}^{\infty} (\Delta g)(y)^2 P_1(y)$$

$$\leq \frac{c}{\alpha} \sum_{y=0}^{\infty} (\Delta g)(y)^2 P_X(y),$$

and the result holds, since for all $y$, the $P_1(y) \leq P_1(y) + (1/\alpha - 1)P_2(y) = P_X(y)/\alpha$. Note that Equation (7) follows by Equation (5), and Equation (8) follows by Equation (5).

\[\square\]

Note that Cacoullos and Papathanasiou [4] prove a similar result, but under the more restrictive condition that $x_0 = \mathbb{E}X$.

**Lemma 2.2**

(i) If there exists $C$ such that

$$\rho_X(x) := \frac{xP_X(x)}{P_X(x-1)} \text{ is }\begin{cases} \geq C & \text{ for } x < C, \\ \leq C & \text{ for } x \geq C, \end{cases}$$

then the Poincaré constant $R_X \leq C$.

(ii) If for some $n$ there exists $D$ such that

$$\rho_X^{(n)}(x) := \frac{xP_X(x)}{(n-x+1)P_X(x-1)} \text{ is }\begin{cases} \geq D/(1-D) & \text{ for } x < Dn, \\ \leq D/(1-D) & \text{ for } x \geq Dn, \end{cases}$$

then the Poincaré constant $R_X \leq Dn$. 

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Proof (i) If Equation (9) holds then taking $x_0 = C$, for $y \geq x_0$:
\[
\sum_{x=y+1}^{\infty} P_X(x)(x-x_0) \leq C \sum_{x=y+1}^{\infty} P_X(x-1) - C \sum_{x=y+1}^{\infty} P_X(x) = CP_X(y),
\]
and for $y < x_0$:
\[
\sum_{x=0}^{y} P_X(x)(x-x_0) \leq C \sum_{x=0}^{y} P_X(x) - C \sum_{x=0}^{y} P_X(x-1) = CP_X(y),
\]
so that Equations (1) and (2) hold with $c_1 = C$, so by Lemma 2.1, $R_X \leq C$.

(ii) If Equation (10) holds, then for $x \geq Dn$ then we can rearrange in a way suggested by Klaasen’s work [9], to obtain
\[
P_X(x)(x-Dn) \leq D(P_X(x-1)(n-x+1) - P_X(x)(n-x)),
\]
so taking $x_0 = Dn$, for $y \geq x_0$, since the sum collapses:
\[
\sum_{x=y+1}^{\infty} P_X(x)(x-Dn) = D \sum_{x=y+1}^{\infty} P_X(x-1)(n-x+1) - P_X(x)(n-x) = D(n-y)P_X(y),
\]
which is $\leq DnP_X(y)$. A similar argument works for $y < x_0$ with the signs reversed, so that Equations (1) and (2) hold with $c_1 = Dn$, so by Lemma 2.1, $R_X \leq Dn$. \qed

Lemmas 2.1 and 2.2 imply that the Poincaré constant is finite for the following classes:

**Definition 2.3** A random variable $X$ is ultra log-concave of degree $n$, or $\text{ULC}(n)$ for short, if $X$ is supported on $\{0, 1, \ldots, n\}$ and the function $P_X(x)/\binom{n}{x}$ is log-concave, or equivalently
\[
\rho_X^{(n)}(x) = \frac{xP_X(x)}{(n-x+1)P_X(x-1)} \text{ is non-increasing in } x.
\]
A random variable $X$ is ultra log-concave of degree $\infty$, or $\text{ULC}(\infty)$ for short, if the function $P_X(x)x!$ is log-concave, or equivalently
\[
\rho_X(x) = \frac{xP_X(x)}{P_X(x-1)} \text{ is non-increasing in } x.
\]

These classes were introduced by Pemantle [12] and Liggett [11] in order to capture properties of negative dependence. Liggett remarks that if a random variable is $\text{ULC}(n)$
then it is also ULC$(n + 1)$, and proves in Theorem 2 of [11] that if $X$ and $Y$ are independent and ULC$(n)$ and ULC$(m)$ respectively then their convolution is ULC$(n + m)$. Hence the sum of $n$ independent Bernoulli random variables has the ULC$(n)$ property. Clearly the Poisson random variables are ULC$(\infty)$.

Proof of Theorem 1.2 If $X$ is ULC$(\infty)$ then $\rho_X(x) = xP_X(x)/P_X(x - 1)$ is non-increasing in $x$, and $\tau(x) = x$ is strictly increasing in $x$, so Equation (9) must hold for some $C < \infty$, so by Lemma 2.2(i) $R_X \leq C$. We can analyse Equation (9) further to deduce bounds on $C$. Considering the cases $\{x < C\}$ and $\{x \geq C\}$ separately, we know that for each $x$ the summand is negative, so deduce that the sum

$$0 \geq \sum_{x=0}^{\infty} P_X(x - 1) \left( \frac{xP_X(x)}{P_X(x - 1)} - C \right) (x - C) = \mathbb{E}X^2 - C(2\mathbb{E}X + 1) + C^2.$$  

Treating this as a quadratic in $C$, we know that this means that $C \in [C_-, C_+]$, for some roots $C_\pm$. Hence, we need simply calculate the value of $C_+$. If $X$ is ULC$(n)$ then $P_X^{(n)}(x) = xP_X(x)/(P_X(x - 1)(n - x + 1))$ is non-increasing in $x$, and $\tau(x) = x/(n - x)$ is strictly increasing for $x \leq n$, so Equation (10) must hold for some $D < \infty$, so by Lemma 2.2(ii) $R_X \leq Dn$. We can analyse Equation (10) in a similar way to deduce bounds on $D$. Considering the cases $\{x < Dn\}$ and $\{x \geq Dn\}$ separately, we deduce that the sum

$$0 \geq \sum_{x=0}^{\infty} (1 - D)P_X(x - 1)(n - x + 1) \left( \frac{xP_X(x)}{P_X(x - 1)(n - x + 1)} - \frac{D}{1 - D} \right) (x - Dn) = \mathbb{E}X^2 - Dn(2\mathbb{E}X + 1 - \mathbb{E}X/n) + D^2n^2.$$  

Treating this as a quadratic in $C = Dn$, we know that

$$C \leq \mathbb{E}X + \frac{1}{2} - \frac{\mathbb{E}X}{2n} + \sqrt{\frac{1}{4} + \mathbb{E}X - \text{Var}X - \frac{\mathbb{E}X}{2n} - \frac{(\mathbb{E}X)^2}{n} + \frac{(\mathbb{E}X)^2}{4n^2}}, \quad (11)$$

and the sum of the last two terms is negative, so they can be removed to give a neater bound. □

Remark 2.4 The upper bounds given in Theorem 1.2 are indeed real-valued, since the variances of ultra log-concave variables are constrained in terms of their means $\lambda$. Specifically, for $X$ in ULC$(n)$, since we can view it as the covariance of a decreasing and increasing function (with respect to probability mass function $(n - x)P_X(x)/(n - \lambda)$) the sum

$$\sum_{x=0}^{n} \frac{(n - x)P_X(x)}{n - \lambda} \left( \frac{P_X(x + 1)(x + 1)}{P_X(x)(n - x)} - \frac{\lambda}{n - \lambda} \right) x$$

is

$$\sum_{x=0}^{n} \frac{(n - x)P_X(x)}{n - \lambda} \left( \frac{P_X(x + 1)(x + 1)}{P_X(x)(n - x)} - \frac{\lambda}{n - \lambda} \right) x.$$
is negative, by Chebyshev’s rearrangement lemma. Rearranging, this tells us that $\mathbb{E}X(X - 1) \leq \lambda^2(n-1)/n$, or that $\mathbb{E}X - \text{Var}(X) \geq \lambda^2/n$. (The corresponding result is well known for $n = \infty$). Substituting in the bounds of Theorem 1.2, we deduce that the argument of the square roots are positive, and indeed the same is true for Equation (11).

3 Comparison with other results

We now discuss how our results compare with known bounds. First we mention a simple lower bound, the direct equivalent of Theorem 2(vi) of [3]:

**Lemma 3.1** For any random variable $X$ with finite variance: $R_X \geq \text{Var}(X)$.

**Proof** Consider the function $g(x) = (x - \mathbb{E}X)$, so that $\Delta g(x) = 1$. We know that $\mathbb{E}g(X)^2 = \text{Var} X$ and $\mathbb{E}(\Delta g)(X)^2 = 1$, so that $R_X = \sup_g(\mathbb{E}g(X)^2)/(\mathbb{E}(\Delta g)(X)^2) \geq \text{Var}(X)/1$.

We briefly discuss how our results differ from those of other authors to have considered discrete Poincaré constants.

Bobkov and Götze [1] give necessary and sufficient conditions for the discrete Poincaré constant to be finite. However, as we shall see, their methods can significantly overestimate the exact value of the constant.

**Theorem 3.2** ([1]) Given random variable $X$ with probability mass function $P_X$, define

$$C(P_X) := \sup_x \frac{F_X(x)(1 - F_X(x))}{P_X(x)},$$

(12)

where $F_X(x) = \sum_{y=0}^{x} P_X(y)$. Bobkov and Götze’s work implies that

$$C(P_X) \leq R_X \leq \frac{C(P_X)}{P_X(0)}.$$

**Proof** The left-hand side follows by considering $g_x(y) = I(y \leq x) - F_X(x)$ for each $x$. Then $g_x(y)^2 = I(y \leq x)(1 - 2F_X(x)) + F_X(x)^2$, so $\mathbb{E}g_x^2(Y) = F_X(x)(1 - F_X(x))$, and $(\Delta g_x)(y) = I(y = x)$ so $\mathbb{E}(\Delta g_x)(Y)^2 = P_X(x)$.

The proof of the right-hand side is more complicated, but the result is stated as the third displayed equation on P.274 of [1].
Our Lemma 2.1 was designed to generalise the calculation of the Poincaré constant of a Poisson random variable, given in Table 2.2 of Klaasen’s paper [9] and Proposition 3.5 of Cacoullos and Papathanasiou [5] in the same year. Lemma 2.1 reduces to give the values they calculated in this case.

Example 3.3 Writing \( \Pi_\lambda(x) = e^{-\lambda x}/x! \) for the Poisson mass function, we know that \( x\Pi_\lambda(x)/\Pi_\lambda(x-1) = \lambda \) for all \( x \). Hence Equation (9) holds with \( C = \lambda \). Lemma 2.2(i) implies that \( R_{\Pi_\lambda} \leq \lambda \). Combining this with Lemma 3.1 we deduce that

\[
R_{\Pi_\lambda} = \lambda. \tag{13}
\]

Taking \( x = 0 \) in Equation (12) we obtain that \( C(\Pi_\lambda) \geq 1 - \Pi_\lambda(0) = 1 - \exp(-\lambda) \), so Bobkov and Götze’s Theorem 3.2 implies an upper bound that is at least \( \exp(\lambda) - 1 \). This is certainly greater than the \( \lambda \) given in Equation (13), and can be significantly larger for large \( \lambda \).

Prakasa Rao and Sreehari [13] show that the property \( R_X = \text{Var}(X) \) characterizes the Poisson distribution (up to integer shifts) and Srivastava and Sreehari [14] extend this to a characterization via weighted Poincaré constants.

Remark 3.4 Equation (13) in Example 3.3 can also be proved using Poisson–Charlier polynomials \( c_n^{(\lambda)}(x) \), under the normalisation that \( \sum_{x=0}^{\infty} \Pi_\lambda(x)c_n^{(\lambda)}(x)c_n^{(\lambda)}(x) = n!\lambda^n\delta_{mn} \). It is well known (see for example [15]) that these polynomials behave well with respect to \( \Delta \), with \( \Delta c_n^{(\lambda)}(x) = nc_n^{(\lambda)}(x) \). This approach exactly parallels the calculation using Hermite polynomials performed by Chernoff [7] to determine the Poincaré constant of the normal distribution.

Next, we show that our method reproduces the bound implied by Table 2.2 of Klaasen [9] for binomial random variables. (Our method was specifically intended to generalise the result Klaasen gave in this case).

Example 3.5 For \( P_X(x) \) the mass function of a binomial \( B(n, p) \) random variable,

\[
\rho_X^{(n)}(x) = \frac{xP_X(x)}{P_X(x-1)(n-x+1)} = \frac{p}{1 - p}. \tag{14}
\]

In other words, Equation (14) holds with equality for \( D = p \), so Lemma 2.2(ii) implies that the binomial \( B(n, p) \) has Poincaré constant \( \leq np \).

Comparing this with the lower bound \( np(1 - p) \) from Lemma 3.1, we can see this bound is reasonably accurate for a range of parameter values, certainly compared with the \( 1/(1 - p)^n - 1 \) that follows from Bobkov and Götze’s Theorem 3.2.
These calculations can presumably be performed in the spirit of Remark 3.4, using the polynomials orthogonal with respect to binomial weights derived by Young in [16]).

Klaasen’s method, and the similar results of Cacoullos and Papathanasiou [5], in general give a weighted Poincaré constant, while we prefer to consider the standard unweighted Poincaré constant.

Example 3.6 If $X$ is the sum of $n$ independent Bernoulli Bern ($p_i$) random variables, then $\mathbb{E}X = \sum_{i=1}^{n} p_i$ and $\text{Var}(X) = \sum_{i=1}^{n} p_i(1 - p_i)$, so Theorem 1.2(i) gives $R_X \leq \sum_{i=1}^{n} p_i + 1/2 + \sqrt{1/4 + \sum_{i=1}^{n} p_i^2}$ (tighter bounds can be found using Theorem 1.2(ii), but they are less transparent). In this case, Klaasen does not give a bound on $R_X$, and the upper bounds implied by Bobkov and Götze’s Theorem 3.2 are again not tight.

Lemma 3.1 gives a lower bound $R_X \geq \sum_{i=1}^{n} (p_i - p_i^2)$, so that upper and lower bounds are asymptotically within 1 of each other in the regime where $\sum_{i} p_i^2$ tends to zero. It is perhaps not surprising that Le Cam [10] proved Poisson approximation bounds in this regime.

In fact, tighter upper bounds can be found, using the following result:

Lemma 3.7 If $X$ and $Y$ are independent then $R_{X+Y} \leq R_X + R_Y$.

Proof We consider $g$ such that $\mathbb{E}g(X + Y) = 0$, and define $h(u) = \mathbb{E}_Y g(u + Y)$, which has the property that $\mathbb{E}h(X) = 0$. Now, as in the proof of Theorem 2(vii) of [3]:

$$\mathbb{E}g^2(X + Y) = \mathbb{E}_X [\mathbb{E}g^2(X + Y)|X]$$

$$= \mathbb{E}_X \text{Var}(g(X + Y)|X) + \mathbb{E}_X [\mathbb{E}g(X + Y)|X]^2$$

$$\leq R_Y \mathbb{E}_X \left[\mathbb{E}_Y(\Delta g)(X + Y)^2|X\right] + \mathbb{E}h(X)^2$$

$$\leq R_Y \mathbb{E}(\Delta g)(X + Y)^2 + R_X \mathbb{E}(\Delta h)(X)^2.$$

For each $x$,

$$\Delta h(x) = h(x + 1) - h(x) = \sum_{y} P_Y(y) \left(g(x + y + 1) - g(x + y)\right),$$

so using Cauchy-Schwarz:

$$(\Delta h)(x)^2 \leq \sum_{y} P_Y(y) \left(g(x + y + 1) - g(x + y)\right)^2,$$

and taking expectations using weights $P_X(x)$, we deduce that $\mathbb{E}(\Delta h)(X)^2 \leq \mathbb{E}(\Delta g)(X + Y)^2$, and the result follows. \qed
Example 3.8 As in Example 3.5, take $X$ to be the sum of independent Bernoulli Bern $(p_i)$ random variables. Since Example 3.5 shows that each Bern $(p_i)$ has $R_{\text{Bern}(p_i)} = p_i$, using Lemma 3.7 we deduce that $R_X \leq \sum_{i=1}^{n} p_i$.

The improvement to the resulting bounds is not surprising, since for example [2] shows that tensorization provides good bounds. However, we believe that the fact that Theorem 1.2 holds in general, for cases where such tensorization is not possible, makes it worthwhile despite the weaker bounds it produces in these special cases. [Note the different value of the Poincaré constant of Bern $(p_i)$ given in [2] is due to their restricting attention to functions $g$ with $g(2) = g(0)$].

In fact, in the spirit of [8], it should be possible to give stronger results than Lemma 3.7 in the case where $X$ and $Y$ are not Poisson, and hence improve the bounds given in Example 3.8. We do not pursue this route here.

4 Remark

We briefly mention a mistake in a previous paper:

Remark 4.1 Part (ix) of Theorem 2 of [3] considers mixtures of continuous random variables, defining $Z_\alpha$ by $P(Z_\alpha \leq z) = \alpha P(X \leq z) + (1-\alpha) P(Y \leq z)$. Although [3] claims that the (continuous) Poincaré constant satisfies $R_{Z_\alpha} \leq \max(R_X, R_Y)$, this result is not true, in fact there is a sign error in the proof.

The key observation is that if $R_X < \infty$ then the support of $X$ must be an interval. Considering $X$ and $Y$ to be uniform on $[0,1]$ and $[3,4]$ respectively, [3] shows that $R_X$ and $R_Y$ are finite. However $R_Z = \infty$, since we can take functions arbitrarily close to

$$g(x) = \begin{cases} 
-(1-\alpha) & \text{for } x \leq 1 \\
\alpha & \text{for } x \geq 3
\end{cases},$$

which is non-constant on the support of $Z_\alpha$, but has zero derivative on the support of the mixture.

A similar argument works in the $\mathbb{N}$-valued case by taking $X$ and $Y$ uniform on $\{0,1\}$ and $\{3,4\}$ respectively.

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