Curvature Formulas Related to a Family of Stable Higgs Bundles

Zhi Hu\textsuperscript{1,2,3}, Pengfei Huang\textsuperscript{4}

\textsuperscript{1} School of Science, Nanjing University of Science and Technology, Nanjing 210094, China. E-mail: halfask@mail.ustc.edu.cn; huz@uni-mainz.de
\textsuperscript{2} Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{3} Department of Mathematics, Mainz University, 55128 Mainz, Germany
\textsuperscript{4} Mathematisches Institut, Ruprecht-Karls Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany. E-mail: pfhwang@mathi.uni-heidelberg.de; pfhwangmath@gmail.com

Received: 13 December 2020 / Accepted: 31 May 2021
Published online: 8 July 2021 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract: In this paper, we investigate the geometry of the base complex manifold of an effectively parametrized holomorphic family of stable Higgs bundles over a fixed compact Kähler manifold. The starting point of our study is Schumacher–Toma/Biswas–Schumacher’s curvature formulas for Weil–Petersson-type metrics, in Sect. 2, we give some applications of their formulas on the geometric properties of the base manifold. In Sect. 3, we calculate the curvature on the higher direct image bundle, which recovers Biswas–Schumacher’s curvature formula. In Sect. 4, we construct a smooth and strongly pseudo-convex complex Finsler metric for the base manifold, the corresponding holomorphic sectional curvature is calculated explicitly.

Contents

1. Introduction ........................................ 1491
   1.1 Setup and some notations ........................ 1493
   1.2 Main results ..................................... 1496
2. Some applications of Biswas–Schumacher’s curvature formula ........ 1498
   2.1 Some identities ................................... 1498
   2.2 Some applications ................................. 1498
3. Curvature on direct image bundles .......................... 1500
4. Curvature of Finsler metric .............................. 1504

1. Introduction

Studying the moduli space of certain geometric objects from the viewpoint of differential geometry is an important approach to understand the geometry of the moduli space. A basic starting point is to endow the moduli space with a suitable Riemannian metric. If the parametrized geometric object is equipped with a metric, in general, the moduli
space could inherit a natural metric as a functional of the metric on the geometric object. Such metric on the moduli space is usually called the \textit{Weil–Petersson-type metric}. There are two typical examples of applying such ideas.

(i) The moduli space of certain compact polarized complex manifolds:

- For the moduli space $\mathcal{M}_g$ (and the Teichmüller space $\mathcal{T}_g$) of compact Riemann surfaces of genus $g \geq 2$, Ahlfors showed that the Weil–Petersson metric on $\mathcal{T}_g$ is a Kähler metric whose Ricci and holomorphic sectional curvatures are negative [1, 2] (see also [9] by Fischer and Tromba). Later Royden proved that the holomorphic sectional curvature of the Weil–Petersson metric is bounded away from zero [17]. Afterwards Wolpert showed that the holomorphic sectional curvature of the Weil–Petersson metric is bounded above by $-\frac{1}{2\pi(g-1)}$ [27], which confirms a conjecture of Royden [17]. This immediately implies the moduli space $\mathcal{M}_g$ is Kobayashi hyperbolic.

- Due to the famous results of Aubin [3] and Yau [29], every compact canonically polarized complex manifold admits a unique Kähler–Einstein metric of negative Ricci curvature unique up to a positive multiplicative constant. Then the moduli space of compact canonically polarized complex manifolds can be equipped with a Weil–Petersson-type metric, and Siu computed the corresponding curvature tensor [21].

- Also by Yau’s solution to the Calabi conjecture [29], there is a Weil–Petersson-type metric on the moduli space of compact polarized Calabi–Yau manifolds. Strominger gave the curvature formula for the case of Calabi–Yau threefolds using the Yukawa couplings [22].

(ii) The moduli space of stable bundles over a fixed compact Kähler manifold:

- Thanks to many author’s (Narasimhan-Seshadri [16], Donaldson [7], Uhlenbeck-Yau [25]) work on the existence of Hermitian–Einstein metrics on stable vector bundles, we also have the Weil–Petersson-type metric on the moduli space of stable vector bundles over a fixed compact Kähler manifold. In this case, Schumacher and Toma calculated the corresponding curvature tensor [19].

- The notion of Higgs field on a vector bundle was introduced by Hitchin [13] for Riemann surface case and by Simpson [20] in general, they also showed that stable Higgs bundles admit Hermitian–Einstein metrics. The work of Schumacher and Toma is then generalized to the moduli space of stable Higgs bundles over a fixed compact Kähler manifold by Biswas and Schumacher [4].

In order to capture more geometric information of moduli spaces, the method mentioned above is developed by constructing certain suitable Finsler metric on the moduli space. If the moduli space has an initial Riemannian metric, there is a way to construct the Finsler metric by recursive introducing the higher order parts from the terms in the curvature with fixed sign. For the moduli space of certain manifolds, this idea has been carried out. For example, Schumacher [18] and To-Yeung [24] constructed a Finsler metric on the moduli space of compact canonically polarized complex manifolds based on Siu’s curvature formula, and Deng [5] constructed a Finsler metric on the moduli space of compact polarized complex manifolds with semi-ample canonical bundles based on Griffiths’s curvature formula of Hodge bundles [11]. As remarkable applications, they can show the hyperbolicity in certain sense for these moduli spaces by calculating the holomorphic sectional curvature with respect to the Finsler metric.

This paper investigates the geometry of the base complex manifold of an effectively parametrized holomorphic family of stable Higgs bundles over a fixed compact Kähler
manifold by calculating the curvature with respect to suitable Weil–Petersson-type metric or Finsler metric.

1.1. Setup and some notations. Let \( X \) be an \( n \)-dimensional compact Kähler manifold equipped with a Kähler form \( \omega_X = \sqrt{-1} g_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta \) in terms of the local holomorphic coordinates \((z^1, \ldots, z^n)\), and let \( E \) be a vector bundle over \( X \) with a Hermitian metric \( h \). We introduce the following notations

- For any smooth \( \text{End}(E) \)-valued \((p, q)\)-form \( \theta \in \mathcal{A}_{X}^{p,q} (\text{End}(E)) \), its Hermitian conjugate \( \theta^{*h} \in \mathcal{A}_{X}^{q,p} (\text{End}(E)) \) is defined by
  \[
  h(\theta(u), v) = h(u, \theta^{*h}(v))
  \]
  for any \( u, v \in C^\infty(E) \).
- For any two smooth \( \text{End}(E) \)-valued \( r \)-forms \((r \leq n) \) \( \varphi, \psi \in \mathcal{A}_{X}^{r}(\text{End}(E)) \), writing
  \[
  \varphi = \sum_{p+q=r} \varphi_{\alpha_1 \cdots \alpha_p \, \bar{\beta}_1 \cdots \bar{\beta}_q} dz^{\alpha_1} \wedge \cdots \wedge d\bar{z}^{\alpha_p} \wedge d\bar{z}^{\bar{\beta}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\beta}_q},
  \]
  \[
  \psi = \sum_{p+q=r} \psi_{\gamma_1 \cdots \gamma_p \, \bar{\delta}_1 \cdots \bar{\delta}_q} d\bar{z}^{\gamma_1} \wedge \cdots \wedge d\bar{z}^{\gamma_p} \wedge d\bar{z}^{\bar{\delta}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\delta}_q},
  \]
  we define their inner product with respect to the induced metric \( \tilde{h} \) on \( \mathcal{A}_{X}^{r}(\text{End}(E)) \) as
  \[
  \tilde{h}(\varphi, \psi) = \sum_{p+q=r} g_{\bar{\gamma}_1 \alpha_1} \cdots g_{\bar{\gamma}_p \alpha_p} g_{\bar{\beta}_1 \bar{\delta}_1} \cdots g_{\bar{\beta}_q \bar{\delta}_q} \text{Tr}(\varphi_{\alpha_1 \cdots \alpha_p \, \bar{\beta}_1 \cdots \bar{\beta}_q} \psi_{\gamma_1 \cdots \gamma_p \, \bar{\delta}_1 \cdots \bar{\delta}_q}).
  \]
- For any operator \( \Xi : \mathcal{A}_{X}^{p,q} (\text{End}(E)) \rightarrow \mathcal{A}_{X}^{r,s}(\text{End}(E)) \), its formal adjoint \( \Xi^{\dagger h} : \mathcal{A}_{X}^{r,s}(\text{End}(E)) \rightarrow \mathcal{A}_{X}^{p,q} (\text{End}(E)) \) with respect to the induced metric \( \tilde{h} \) is defined by
  \[
  \int_X \tilde{h}(\Xi \varphi, \psi) \frac{\omega_X^n}{n!} = \int_X \tilde{h}(\varphi, \Xi^{\dagger h} \psi) \frac{\omega_X^n}{n!}
  \]
  for any \( \varphi \in \mathcal{A}_{X}^{p,q} (\text{End}(E)) \) and \( \psi \in \mathcal{A}_{X}^{r,s}(\text{End}(E)) \).

Let \( S \) be an \( m \)-dimensional complex manifold, and let \( \{(\mathcal{E}_s, \Phi_s)\}_{s \in S} \) be a holomorphic family of stable Higgs bundles of rank \( r \) on \( X \) parametrized by \( S \), namely a Higgs bundle \( (\mathcal{E}, \Phi) \) over \( \tilde{X} := X \times S \) such that \( (\mathcal{E}, \Phi)|_{X \times \{s\}} = (\mathcal{E}_s, \Phi_s) \) for each \( s \in S \). The relative Higgs complex associated to \( (\mathcal{E}, \Phi) \) is given by

\[
\mathcal{H}_r : 0 \rightarrow \mathcal{E} \xrightarrow{\Phi^\wedge} \mathcal{E} \otimes \Omega^{1}_{\tilde{X}/S} \xrightarrow{\Phi^\wedge} \cdots.
\]

It is known that the \( d \)-th direct image sheaf \( \mathbb{R}^d \pi_* \mathcal{H}_r \) is coherent over \( S \), and is thus locally free outside a proper analytic subvariety \( Z^{(d)} \) of \( S \), where \( \pi : \tilde{X} \rightarrow S \) is the projection onto the second factor. Moreover, \( \mathbb{R}^d \pi_* \mathcal{H}_r|_s = \mathbb{H}^d(X, \mathcal{H}_s) \) for any \( s \in X \setminus Z^{(d)} \), where

\[
\mathcal{H}_s : 0 \rightarrow \mathcal{E}_s \xrightarrow{\Phi^\wedge} \mathcal{E}_s \otimes \Omega^{1}_{X} \xrightarrow{\Phi^\wedge} \cdots
\]
is the Higgs complex associated to the Higgs bundle \((\mathcal{E}_s, \Phi_s)\). In particular, if both \(X\) and \(S\) are Riemann surfaces, then Donagi–Pantev–Simpson’s result \([6, \text{Theorem 3.6}]\) implies that \(\mathbb{R}^d \tau_s \mathbf{H}_r\) is locally free over \(S\).

By the short exact sequence

\[
0 \rightarrow \pi^* \Omega^1_S \otimes \Omega^{p-1}_{X/S} \rightarrow \Omega^p\mathbb{P}(\pi^* \Omega^2_S \otimes \Omega^{p-2}_{X/S}) \rightarrow \Omega^p_{\mathbb{P}(X/S)} \rightarrow 0,
\]

this is a connecting morphism in the corresponding long exact sequence

\[
\rho : \mathbb{R}^1 \tau_s \mathbf{H}_r \rightarrow \mathbb{R}^2 \tau_s (\pi^* \Omega^1_S \otimes \mathbf{H}_r[-1]) = \Omega^1_S \otimes \mathbb{R}^1 \tau_s \mathbf{H}_r,
\]
or equivalently,

\[
\rho : TS \rightarrow \text{End}(\mathbb{R}^1 \tau_s \mathbf{H}_r),
\]
hence when restricting at each point \(s \in S\setminus Z^{(1)}\), we have a map called the Kodaira-Spencer map

\[
\rho_s : T_s S \rightarrow \mathbb{H}^1(X, \mathbf{E}H_s),
\]
where \(\mathbb{H}^1(X, \mathbf{E}H_s)\) is the first hypercohomology of the complex

\[
\mathbf{E}H_s : 0 \rightarrow \text{End}(\mathcal{E}_s) \xrightarrow{\Phi_s \otimes \text{Id} + \text{Id} \otimes \Phi_s} \text{End}(\mathcal{E}_s) \otimes \Omega^1_X \rightarrow 
\]

Actually, the Kodaira-Spencer map \(\rho_s\) can be defined for all \(s \in S\) \([4, 19]\).

In this paper, we always assume our family \((\mathcal{E}_s, \Phi_s)\) is effectively parametrized, namely the Kodaira-Spencer map \(\rho_s\) is injective for any \(s \in S\).

By the Kobayashi-Hitchin correspondence for Higgs bundles \([13, 20]\), on each fiber \((\mathcal{E}_s, \Phi_s)\) for \(s \in S\), there is a Hermitian–Einstein metric \(h_s\), and the family \(\{h_s\}_{s \in S}\) of metrics induces a Hermitian metric \(h\) on \(\mathcal{E}\). We introduce the operators \(d, d^\dagger, \Box\), which are fiberwisely defined as follows

- \(d|_s = \overline{\partial}_{\mathcal{E}_s} + \Phi_s\), where \(\overline{\partial}_{\mathcal{E}_s}\) stands for the holomorphic structure on \(\mathcal{E}_s\);
- \(d^\dagger|_s = (\overline{\partial}_{\mathcal{E}_s})^\dagger h_s + (\Phi_s)^\dagger h_s\);
- \(\Box|_s = d|_s d^\dagger|_s + d^\dagger|_s d|_s\).

Let \(\mathbf{TH}_s\) be the total complex associated to the Dolbeault resolution of \(\mathbf{H}_s\), then the hypercohomology \(\mathbb{H}^* (X, \mathbf{H}_s)\) is computed by the usual cohomology \(H^* (X, \mathbf{TH}_s)\), and by applying harmonic theory via the metric \(h_s\) one finds the unique harmonic representative for the cohomology class in \(\mathbb{H}^* (X, \mathbf{H}_s)\) (more detailed can be found in the Section 3 of [4]). Let \(\mathbf{t}\) be a harmonic representative of \(\mathbb{H}^d (X, \mathbf{H}_s)\), then we have

\[
d|_s \mathbf{t} = d^\dagger|_s \mathbf{t} = 0,
\]
or equivalently, expressing

\[
\mathbf{t} = \sum_{p+q=d} t^{p,q} = \sum_{p+q=d} t^{a_1 \ldots a_p \bar{b}_1 \ldots \bar{b}_q} dz^{a_1} \wedge \ldots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{b}_1} \wedge \ldots \wedge d\bar{z}^{\bar{b}_q}
\]

we have

\[
\overline{\partial}_{\mathcal{E}_s} t^{p,q} + \Phi_s (t^{p-1,q+1}) = 0,
\]
Curvature Formulas Related to a Family

\[ [\Lambda_{\omega_X}, \partial_{\xi_s}^h] t^{p-1,q+1} + [\Lambda_{\omega_X}, \Phi_s^h] t^{p,q} = 0 \]

for any $1 \leq p, q \leq n$, where the second equation is due to the Kähler identity.

Let $U \subseteq S$ be an open neighborhood of $s \in S$ with local holomorphic coordinates given by $(s^1, \cdots, s^m)$, the curvature form with respect to the metric $h$ can be expressed locally as:

\[ R = R_{\alpha\bar{\beta}} d\bar{z}^\alpha \wedge dz^\beta + R_{i\bar{\alpha}} ds^i \wedge d\bar{z}^\alpha + R_{\alpha i} dz^\alpha \wedge d\bar{s}^i + R_{ij} ds^i \wedge d\bar{s}^j. \]

We write $\Phi = \Phi_a dz^a$, $\Phi^h = \Phi_{\bar{a}}^h d\bar{z}^\alpha$. In particular, the Hermitian–Einstein condition at each $s \in S$ is given by

\[ g^{\beta\bar{\alpha}} (R_{\alpha\bar{\beta}} + [\Phi_{\alpha}, \Phi_{\bar{\beta}}^h])|_s = \lambda_s \cdot \text{Id}_{E_s} \]

for some real constant $\lambda_s$ determined by the slope of $E_s$ and the volume of $X$.

Again by the Dolbeault resolution and harmonic theory, we have the unique harmonic representative for the cohomology class in $H^\bullet(X, EH_s)$. In [4], Biswas and Schumacher gave the harmonic representative for the class of the image of the Kodaira-Spencer map. More precisely, let $\nabla_i$ and $\bar{\nabla}_i$ be the covariant derivatives along the directions $\partial/\partial s^i$ and $\partial/\partial \bar{s}^i$ determined by the connection of the metric $h$, respectively, and define a $C^\infty(\text{End}(E))$-valued 1-form

\[ \eta_i = R_{i\bar{a}} d\bar{z}^\alpha + \nabla_{\bar{i}} \Phi_a dz^a, \]

then for any $s \in U$, $\eta_i|_s$ is the harmonic representative of $\rho_s(\partial/\partial s^i)|_s$ in $H^1(X, EH_s)$ due to the Hermitian–Einstein condition. In particular, we have

\[ d|_s(\eta_i|_s) = d^\dagger h|_s(\eta_i|_s) = 0 \]

for any $s \in U$.

**Definition 1.1.** Let $\xi = \xi^i \partial/\partial s^i$ be a local holomorphic vector field over $U$.

(1) The Kodaira-Spencer image of $\xi$ is defined as

\[ \mathbb{H}(\xi) = \xi^i \eta_i = \xi^i (R_{i\bar{a}} d\bar{z}^\alpha + \nabla_{\bar{i}} \Phi_a dz^a), \]

namely $\mathbb{H}(\xi)|_s$ is the harmonic representative of $\rho_s(\partial/\partial s^i)|_s$ in $H^1(X, EH_s)$ for each $s \in U$.

(2) $\xi$ is called relatively harmonic if

\[ d|_s[\mathbb{H}(\xi)|_s, G|_s(\mathbb{H}(\xi)|_s)\hat{h}, \mathbb{H}(\xi)|_s] = d^\dagger h|_s[\mathbb{H}(\xi)|_s, G|_s(\mathbb{H}(\xi)|_s)\hat{h}, \mathbb{H}(\xi)|_s] = 0 \]

for any $s \in U$, where $G = \Box^{-1}$ denotes the Green operator that corresponds to $\Box$.

---

1 Throughout this paper, we use the lowercase Greek letters $\alpha, \beta, \gamma, \cdots$ and Roman letters $i, j, k, \cdots$ for the coordinates on $X$ and $S$, respectively.
1.2. Main results. With the notations introduced above, we consider a holomorphic family of stable Higgs bundles of rank \( r \) over \( X \) effectively parametrized by \( S \). The Weil–Petersson-type metric \( G_{ij}^{\text{WP}} \) on \( S \) is defined as follows [4]

\[
G_{ij}^{\text{WP}} = G^{\text{WP}} \left( \frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j} \right) = \int_X \tilde{h}(\eta_i, \eta_j) \frac{\omega_X^n}{n!}
\]

\[
= \int_X g^{\bar{\alpha} \bar{\beta}} \text{Tr}(R_{i\bar{\alpha}} R_{j\bar{\beta}} + \nabla_i \Phi_{\bar{\beta}} \nabla_j \Phi^*_{\bar{\alpha}}) \frac{\omega_X^n}{n!}.
\]

The assumption of effective parametrization guarantees the positive-definiteness of \( G^{\text{WP}} \). It is easy to check that this metric is Kähler. Biswas and Schumacher calculated the curvature of \( G^{\text{WP}} \) as [4]

\[
R_{ijkl} = \int_X \text{Tr}(R_{ij} \Box R_{kl} + R_{il} \Box R_{kj}) \frac{\omega_X^n}{n!}
\]

\[
- \int_X \text{Tr}([\eta_i \wedge \eta_k] \wedge G([\eta_j^* \wedge \eta_l^*]) \wedge \frac{\omega_X^{n-2}}{(n-2)!}),
\]  

(1.2)

where \([\bullet \wedge \bullet]\) stands for the exterior product of forms with values in an endomorphism bundle combined with the Lie bracket.

In Sect. 3, we consider the locally free higher direct image sheaf \( \mathbb{R}^d \pi_* \mathbf{H}_r \) of rank \( r_d \) over \( S \setminus Z^{(d)} \) with respect to the relative Higgs complex \( \mathbf{H}_r \) introduced above. \( \mathbb{R}^d \pi_* \mathbf{H}_r \) is equipped with an \( L^2 \)-metric \( H \) as follows

\[
H(t, t') := \int_X \tilde{h}_s(t, t') \frac{\omega_X^n}{n!},
\]

where \( t, t' \) are harmonic representatives of \( \mathbb{H}^d(X, \mathbf{H}_s) \), and

\[
\tilde{h}_s(t, t') = \sum_{p+q=d} g^{\bar{\gamma}_1 a_1} \ldots g^{\bar{\gamma}_p a_p} \bar{g}^{\bar{\beta}_1 \delta_1} \ldots \bar{g}^{\bar{\beta}_q \delta_q} \tilde{h}_s(t_{a_1 \ldots a_p} \bar{b}_1 \ldots \bar{b}_q, t_{1' \ldots q'} \gamma_1 \ldots \gamma_p \delta_1 \ldots \delta_q).
\]

We calculate the corresponding curvature tensor, which generalizes the results of To–Weng in [23] and Geiger–Schumacher in [10].

**Theorem 1.2** (= Theorem 3.4). The curvature tensor \( \mathfrak{R} \) of \( (\mathbb{R}^d \pi_* \mathbf{H}_r, H) \) over \( S \setminus Z^{(d)} \) is given by

\[
\mathfrak{R}_{abij} = \left( \frac{1}{r \text{Vol}(X, \omega_X)} \int_X \text{Tr}(R_{ij}) \frac{\omega_X^n}{n!} \right) \cdot H_{ab} + \int_X \tilde{h}(G(\eta_j^* (ta)), \eta_i^* (tb)) \frac{\omega_X^n}{n!}
\]

\[
- \int_X \tilde{h}(G(\eta_j^* \eta_i) (ta), t_b) \frac{\omega_X^n}{n!} - \int_X \tilde{h}(G(\eta_i \wedge t_a), \eta_j \wedge t_b) \frac{\omega_X^n}{n!},
\]  

(1.3)

over an open neighborhood \( U \subseteq S \setminus Z^{(d)} \) for \( a, b = 1, \ldots, s \), \( i, j = 1, \ldots, m \), where \( \{t_1|_s, \ldots, t_r|_s\} \) forms a basis of \( \mathbb{H}^d(X, \mathbf{H}_s) \) with each \( t_a|_s \) being \( \Box|_s \)-harmonic for any \( s \in U \).
It is obvious that $G^\text{WP}$ can be viewed as a metric on the sheaf $\mathbb{R}^1\pi_*\mathcal{EH}_r$ over $S$, where $\mathcal{EH}_r$ is the relative Higgs complex associated to the Higgs bundle $(\text{End}(\mathcal{E}), \Phi \otimes \text{Id} + \text{Id} \otimes \Phi)$. Then by applying above curvature formula for $(\mathbb{R}^1\pi_*\mathcal{EH}_r, G^\text{WP})$, in other words, replacing $t_s$ by $\eta_i$ in (1.3), we can recover the curvature formula (1.2) of Biswas and Schumacher.

In Sect. 4, we continue to consider a holomorphic family of stable Higgs bundles of rank $r$ over $X$ effectively parametrized by $S$ from the viewpoint of Finsler geometry. When considering the holomorphic sectional curvature of the Weil–Petersson-type metric $G^\text{WP}$, one immediately finds that the first term on the right hand side of (1.2) is semipositive, and the second term becomes seminegative, which provides the higher-order part of certain Finsler metric according to the method of recursion mentioned above. The Finsler metric is constructed by adding this part into the Weil–Petersson-type metric, namely our Finsler metric $F_\kappa$ with a parameter $\kappa \geq 0$ is given by

$$F_\kappa = \sqrt{\frac{4}{\kappa} F_{(1)}^4 + \kappa F_{(2)}^4},$$

where

$$F_{(1)}(v) = \sqrt{\int_X \tilde{h}(\mathbb{H}(\xi), \mathbb{H}(\xi)) \frac{\omega^n}{n!}},$$

$$F_{(2)}(v) = \sqrt{\int_X \tilde{h}([\mathbb{H}(\xi) \wedge \mathbb{H}(\xi)], G[\mathbb{H}(\xi) \wedge \mathbb{H}(\xi)]) \frac{\omega^n}{n!}}$$

for $v = (s, \xi) \in TS$.

We show that $F_\kappa$ is a smooth and strongly pseudo-convex complex Finsler metric on $S$ (Proposition 4.2), and the corresponding holomorphic sectional curvature is calculated explicitly.

**Theorem 1.3** (= Theorem 4.5). Let $(s^1, \ldots, s^m)$ be the local holomorphic coordinate on $S$. The holomorphic sectional curvature of the Finsler metric $F_\kappa$ is given by

$$K_{F_\kappa} \left( \frac{\partial}{\partial s^i} \right) = \left( \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i], d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)^{-\frac{3}{2}}$$

$$\cdot \left[ 2 \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right] \left( 2 \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)$$

$$- \int_X \tilde{h}(d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i], d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}$$

$$+ \kappa \left( -5 \int_X \tilde{h}([G^{\tilde{\kappa}} G[\eta_i \wedge \eta_i], d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)$$

$$+ 5 \int_X \tilde{h}(Gd^{\tilde{\kappa}} [\eta_i, d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i], d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}$$

$$- 12 \text{Re} \int_X \tilde{h}(Gd^{\tilde{\kappa}} G[\eta_i \wedge \eta_i], d^{\tilde{\kappa}} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}$$

$$- 4 \int_X \tilde{h}([\eta_i \wedge dG[\eta_i \wedge \eta_i], [\eta_i \wedge dG[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}$$
\[ -\int_X \tilde{h}(G^2[[\eta_i \wedge \eta_i] \wedge \eta_i], [[\eta_i \wedge \eta_i] \wedge \eta_i]) \frac{\omega^n}{n!} \] 
\[ + \kappa^2 \left( \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(d^\dagger \tilde{h} G[\eta_i \wedge \eta_i], d^\dagger \tilde{h} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)^{-\frac{s}{2}} \] 
\[ \cdot \left[ \int_X \tilde{h}((d^\dagger \tilde{h} G[\eta_i \wedge \eta_i]) \wedge \eta_i), G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right] \] 
\[ + 2 \int_X \tilde{h}([\eta_i \wedge dG \tilde{h}^\dagger \eta_i], G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right]^2. \]

Applying this formula to the case when \( X \) is a compact Kähler surface, we immediately have the following corollary.

**Corollary 1.4.** Suppose \( X \) is a compact Kähler surface. Let \( \xi \) be a local holomorphic vector field over a neighborhood \( U \subseteq S \). Assume \( \xi \) is relatively harmonic and \( G(\mathbb{H}(\xi))^\dagger \mathbb{H}(\xi) \) is a negative-definite operator on \( A^1_X(\text{End}(E)) \) at some point \( s \in U \). If the holomorphic sectional curvature of \( G^{WP} \) is nonzero at \( s \), then there exists a smooth and strongly pseudo-convex Finsler metric \( F \) such that \( K_F(\xi)|_s > 0 \).

2. **Some applications of Biswas–Schumacher’s curvature formula**

2.1. **Some identities.** The following lemma collects some useful identities that will be used frequently later. The proof can be found in [4], basically, they follow from the Ricci identity and the Kähler identity.

**Lemma 2.1.** [4] We have the following identities

1. \( d\nabla_i \eta_j + [\eta_i \wedge \eta_j] = 0, \)
2. \( d^\dagger \eta_i \nabla_j \eta_j + \eta_j \eta_i = 0, \)
3. \( d^\dagger \eta_i \nabla_j \eta_j = d\nabla_j \eta_j = 0, \)
4. \( \nabla_j \eta_i = dR_{ij}, \)
5. \( \nabla_i \eta_j = -d^\dagger R_{ij}. \)

2.2. **Some applications.** Biswas–Schumacher’s curvature formula (1.2) provides a differential-geometric tool to study the geometric properties of the base manifold.

The following theorem collects some nice results which exhibit the deep relations between the birational geometry and the positivity of holomorphic sectional curvature of manifolds.

**Theorem 2.2.** [12, 28]

1. A compact Hermitian manifold with semipositive but not identically zero holomorphic sectional curvature has Kodaira dimension \( -\infty \).
2. A projective manifold with positive holomorphic sectional curvature is uniruled.

**Theorem 2.3.** (1) If \( (\mathcal{E}, \Phi) \) is an effectively parametrized holomorphic family of stable Higgs bundles on a fixed Riemann surface \( X \) parametrized by a compact complex manifold \( S \) such that \( (\nabla_i \Phi)|_S \neq 0 \) for some \( s \in S \), then the Kodaira dimension of \( S \) is \( -\infty \).
(2) If \((\mathcal{E}, \Phi)\) is an effectively parametrized holomorphic family of stable Higgs bundles of non-zero degree on a fixed Riemann surface \(X\) parametrized by a projective manifold \(S\) such that \((\nabla_i \eta_i)|_s \neq 0\) for each \(s \in S\), then \(S\) is uniruled.

(3) If \((\mathcal{E}, \Phi)\) is an effectively parametrized holomorphic family of stable Higgs bundles on a fixed Riemann surface \(X\) parametrized by a compact Kähler manifold \(S\), then either \(S\) is a torus, or the Kodaira dimension of \(S\) is \(-\infty\).

(4) If \(\mathcal{E}\) is an effectively parametrized holomorphic family of stable vector bundles with vanishing Chern classes on a fixed compact Kähler manifold \(X\) parametrized by a compact complex manifold \(S\) such that \((\nabla_i \eta_i)|_s \neq 0\) for some \(s \in S\), then the Kodaira dimension of \(S\) is \(-\infty\).

**Proof.** (1) If \(X\) is a Riemann surface, the sign of holomorphic sectional curvature of \(G^{WP}\) is the same as

\[
\mathcal{R}_{i\bar{i}i\bar{i}} = 2 \int_X \text{Tr}(R_{i\bar{i}} \Box R_{i\bar{i}})\omega_X,
\]

which is semipositive. To show (1), we show \(\mathcal{R}_{i\bar{i}i\bar{i}}(1 \leq i \leq m)\) can not be identically zero by contradiction. If \(\mathcal{R}_{i\bar{i}i\bar{i}}|_s = 0\) for some \(i\) at a point \(s \in S\), then we have

\[
\tilde{\partial}_s \mathcal{R}_{i\bar{i}}|_s = [\Phi_s, R_{i\bar{i}}|_s] = 0.
\]

It follows from Lemma 1.2 (4) that at each point \(s \in S\), \((\nabla_i \eta_i)|_s\) vanishes, which contradicts to our assumption. Then the conclusion follows from Theorem 2.2 (1).

(2) From the proof of (1) and Theorem 2.2 (2), we can easily obtain the conclusion.

(3) This claim follows from the fact that a compact Kähler manifold with identically zero holomorphic sectional curvature is a torus [28].

(4) By the assumption on vanishing of Chern classes, we have \(R_{\alpha\bar{\beta}}|_s = 0\) for any \(1 \leq \alpha, \beta \leq n\) and any \(s \in S\). Then we have

\[
(\tilde{\partial}_s^{\beta\alpha}([\nabla_i R_{\bar{\beta}}^{\beta\alpha}d\bar{z} \wedge d\bar{z}']|_s = -2(g^{\beta\alpha} \nabla_i [\nabla_i R_{\bar{\beta}}^{\beta\alpha}d\bar{z} \wedge d\bar{z}']|_s = -2(g^{\beta\alpha} (\nabla_i R_{\bar{\beta}}^{\beta\alpha} + \nabla_i \nabla_i R_{\bar{\beta}}^{\beta\alpha})d\bar{z} \wedge d\bar{z}')|_s = -2(g^{\beta\alpha} (\nabla_i R_{\bar{\beta}}^{\beta\alpha} + \nabla_i \nabla_i R_{\bar{\beta}}^{\beta\alpha})d\bar{z} \wedge d\bar{z}')|_s = 0.
\]

The identity in Lemma 1.2 (1) is locally written as [4]

\[
\tilde{\partial}_s^{\beta\alpha}((\nabla_i R_{\bar{\beta}}^{\beta\alpha}d\bar{z})|_s + ([R_{\bar{\beta}}^{\beta\alpha}, R_{\bar{\beta}}^{\beta\alpha}]d\bar{z} \wedge d\bar{z}')|_s = 0,
\]

then the operator \((\tilde{\partial}_s^{\beta\alpha})^{\bar{\kappa}}\) acting on both sides gives rise to \((\tilde{\partial}_s^{\beta\alpha})^{\bar{\kappa}}\tilde{\partial}_s^{\beta\alpha}((\nabla_i R_{\bar{\beta}}^{\beta\alpha}d\bar{z})|_s = 0,\) which implies

\[
\tilde{\partial}_s^{\beta\alpha}((\nabla_i R_{\bar{\beta}}^{\beta\alpha}d\bar{z})|_s = 0,
\]

namely \([\eta_i \wedge \eta_i]|_s = ([R_{\bar{\beta}}^{\beta\alpha}, R_{\bar{\beta}}^{\beta\alpha}]d\bar{z} \wedge d\bar{z}')|_s = 0.\) Therefore, again by Biswas–Schumacher’s curvature formula, we have

\[
\mathcal{R}_{i\bar{i}i\bar{i}} = 2 \int_X \text{Tr}(R_{i\bar{i}} \Box R_{i\bar{i}})\omega_X \geq 0.
\]

Then the conclusions also follow from the proof of (1). \(\square\)
Applying the above theorem to the moduli space of vector bundles on a fixed Riemann surface $X$, we have the following corollary.

**Corollary 2.4.** Assume $X$ is a Riemann surface of genus $g \geq 1$, and let $r$ and $d$ be two positive integers such that they are coprime. Denote by $N_X(r, d)$ the coarse moduli space of semistable vector bundles of rank $r$ and degree $d$ on $X$. Then $N_X(r, d)$ is either an abelian variety, or of Kodaira dimension $-\infty$.

**Proof.** When $r$ and $d$ are coprime, it is known that $N_X(r, d)$ is a smooth projective variety and is a fine moduli space such that there exists a (global) universal vector bundle $E$ over $X \times N_X(r, d)$. Then the conclusion follows from Theorem 2.3 (3). □

**Remark 2.5.** Assume $g \geq 2, r \geq 2$, let $L$ be a fixed line bundle of degree $d$ on $X$, and let $N_X(r, L)$ the coarse moduli space of semistable vector bundles of rank $r$ with determinant $L$ on $X$. It is known that $N_X(r, L)$ is uniruled (since $N_X(r, L)$ is simply-connected, the above corollary only implies it has Kodaira dimension $-\infty$), hence $N_X(r, d)$ contains a free rational curve $C$ whose normal bundle $NC$ is semipositive. From the adjunction formula it follows that $-K_X$ is positive on the curve $C$, where $K_X$ denotes the canonical line bundle of $X$. On the other hand, by a result of Drezet and Narasimhan [8], the Picard group Pic($N_X(r, L)$) is isomorphic to $\mathbb{Z}$, which implies $-K_X$ is ample. Therefore, $N_X(r, L)$ is a Fano variety. By determinant map, $N_X(r, d)$ is a fibration over a Picard variety of deg $d$ with Fano fibers. However, this result does not immediately imply the Kodaira dimension of $N_X(r, d)$ is $-\infty$.

In general, we guess Theorem 2.3 (3) also holds for higher dimensional $X$, namely the following conjecture is proposed.

**Conjecture 2.6.** If $(E, \Phi)$ is an effectively parametrized holomorphic family of stable Higgs bundles on a fixed compact Kähler manifold $X$ parametrized by a simply-connected compact complex manifold $S$, then the Kodaira dimension of $S$ is $-\infty$.

### 3. Curvature on direct image bundles

In this section, we will calculate the curvature formula on the locally free higher direct image sheaf $\mathbb{R}^d\pi_!H_t$ over $S\setminus Z^{(d)}$, where the notations for these objects can be found in Sect. 1. Our main aim of this section is to prove Theorem 1.2.

As a warm-up, we firstly calculate the corresponding curvature tensor for $(\pi_*H_t, H)$. Let $s_0$ be a point lying in $S\setminus Z^{(0)}$, we choose a holomorphic trivialization $\{t_1, \cdots, t_{r_0}\}$ of $\pi_*H_t$ over an open neighborhood $U \subseteq S\setminus Z^{(0)}$ containing $s_0$, and choose a normal coordinate $\{s^1, \cdots, s^m\}$ for $U$ so that the $L^2$-metric $H$ satisfying

$$\frac{\partial H_{ab}}{\partial s^i} \bigg|_{s_0} = 0$$

for any $1 \leq i \leq m, 1 \leq a, b \leq r_0$, where $H_{ab} = H(t_a, t_b)$. Note that $\{t_1|_s, \cdots, t_{r_0}|_s\}$ forms a basis of $\mathbb{H}^0(X, H_t)$. Therefore, the curvature tensor $\mathcal{R}$ for $(\pi_*H_t, H)$ at $s_0$ is given by

$$\mathcal{R}_{abij}|_{s_0} = -\frac{\partial^2 H_{ab}}{\partial s^i \partial s^j} \bigg|_{s_0} = -\left( \int_X h(\nabla_j \nabla_i t_a, t_b) \frac{\omega^n}{n!} \right) \bigg|_{s_0} - \left( \int_X h(\nabla_i t_a, \nabla_j t_b) \frac{\omega^n}{n!} \right) \bigg|_{s_0}.$$
Lemma 3.1. We have the following equality at $s_0$

$$\left( \int_X h(\nabla_i t_a, \nabla_j t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0} = \left( \int_X \tilde{h}(G(\eta_i(t_a)), \eta_j(t_b)) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0}.$$

Proof. Due to $\{t_a\}$ being a holomorphic frame over $U$, i.e. $\nabla_j t_a = 0$ for any $1 \leq a \leq r_0$, $1 \leq j \leq m$, fixing the indices $i$, $a$, we have

$$\left( \int_X h((\nabla_i t_a), t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0} = 0$$

for any $1 \leq b \leq r_0$, which implies that the harmonic projection $P((\nabla_i t_a)|_{s_0})$ of $(\nabla_i t_a)|_{s_0}$ with respect to the Laplacian $\Box|_{s_0}$ on $(\mathcal{E}_s, \Phi_s, h_s)$ vanishes. It follows that

$$\left( \int_X h((\nabla_i t_a), t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0} = \left( \int_X h(\Box G \nabla_i t_a, \nabla_j t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0}$$

$$= \int_X \tilde{h}_{s_0}(\tilde{d}|_{s_0}(G\nabla_i t_a)|_{s_0}, \tilde{d}|_{s_0}(\nabla_j t_b)|_{s_0}) \frac{\omega^n_X}{n!}$$

$$+ \int_X \tilde{h}_{s_0}(\tilde{d}|_{s_0}(G\nabla_i t_a)|_{s_0}, \tilde{d}|_{s_0}(\nabla_j t_b)|_{s_0}) \frac{\omega^n_X}{n!}$$

By the Ricci identity, we calculate

$$\tilde{d}|_{s_0}(\nabla_i t_a)|_{s_0} = - (\nabla_i(\Phi(t_a)))|_{s_0} - (R_{i\tilde{a}} d\tilde{z}_{\tilde{a}}(t_a))|_{s_0} + \Phi_{s_0}((\nabla_i t_a)|_{s_0})$$

$$= - \eta_i(t_a)|_{s_0},$$

which yields that

$$\left( \int_X h((\nabla_i t_a), \nabla_j t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0} = \left( \int_X \tilde{h}(G(\eta_i(t_a)), \eta_j(t_b)) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0}.$$

The desired equality is obtained. \qed

Lemma 3.2. We have the following equality at $s_0$

$$\left( \int_X h(\nabla_j \nabla_i t_a, t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0} = - \left( \frac{1}{r \text{Vol}(X, \omega_X)} \int_X \text{Tr}(R_{ij} \big|_{s_0}) \frac{\omega^n_X}{n!} \right) (H_{ab}|_{s_0})$$

$$+ \left( \int_X h(G((\eta_j \tilde{t}_i)(t_a)), t_b) \frac{\omega^n_X}{n!} \right) \bigg|_{s_0}. $$

Proof. Again by the holomorphicity of $t_a$ and by the Ricci identity, we have

$$(\nabla_j \nabla_i t_a)|_{s_0} = - R_{ij}(t_a)|_{s_0}.$$

Since $(\mathcal{E}_s, \Phi_s)$ is a stable Higgs bundle, the harmonic projection $P(R_{ij}|_s)$ of $R_{ij}|_s$ with respect to the induced Laplacian $\Box|_s$ on End$(\mathcal{E}_s)$ must satisfy

$$P(R_{ij}|_s) = c_{ij}|_s \cdot \text{Id}_{\mathcal{E}_s}.$$
where \( c_{ij} ds^i \wedge d\tilde{s}^j \) is a \((1,1)\)-form over \( S \) given by
\[
c_{ij} = \frac{1}{r \operatorname{Vol}(X, \omega_X)} \int_X \operatorname{Tr}(R_{ij}) \frac{\omega^n_X}{n!}.
\]
Therefore, we arrive at
\[
\left( \int_X h(\nabla_{\tilde{t}a} \nabla_i \tilde{t}b, \omega^n_X/n!) \right)_{s_0} = -\left( \int_X h(R_{ij} \tilde{t}a, \tilde{t}b, \omega^n_X/n!) \right)_{s_0}
\]
\[
= -\int_X h_{s_0}(P(R_{ij})_{s_0} + (G \Box R_{ij})_{s_0})(\tilde{t}a, \tilde{t}b) \frac{\omega^n_X}{n!}
\]
\[
= -(c_{ij} H_{ab})_{s_0} + \left( \int_X h(G((\eta_j^\tau_i \eta_i)(\tilde{t}a)), \tilde{t}b) \frac{\omega^n_X}{n!} \right)_{s_0},
\]
where we apply the identity in Lemma 1.2 (4) for the third equality. \( \square \)

Combining the above two lemmas together leads to the following theorem as an analog of Theorem 1 in [23]

**Theorem 3.3.** The curvature tensor \( \mathfrak{R} \) of \((\pi_a H_r, H)\) over \( S \setminus Z(0) \) is given by
\[
\mathfrak{R}_{abi} = \left( \frac{1}{r \operatorname{Vol}(X, \omega_X)} \int_X \operatorname{Tr}(R_{ij}) \frac{\omega^n_X}{n!} \right)_{s_0} \cdot H_{ab} - \int_X h(G((\eta_j^\tau_i \eta_i)(\tilde{t}a)), \tilde{t}b) \frac{\omega^n_X}{n!}
\]
\[
- \int_X \tilde{h}(G(\eta_i(\tilde{t}a)), \eta_j(\tilde{t}b)) \frac{\omega^n_X}{n!}
\]

Next we calculate the curvature tensor for higher-order direct image sheaf \( \mathbb{R}^d \pi_a H_r \). As before, for a point \( s_0 \in S \setminus Z(d) \), we choose a holomorphic trivialization \( \{t_1, \ldots, t_r\} \) of \( \mathbb{R}^d \pi_a H_r \) over an open neighborhood \( U \subseteq S \setminus Z(d) \), where each \( t_a \) lies in \( \mathcal{A}_{X \times U}^d(\mathcal{E}) \) and satisfies \( d t_a = 0 \). A key observation similar to Lemma 2 in [18] and Lemma 2.1 in [10] is that for \( \chi \in \mathcal{A}_{X \times U}^d(\mathcal{E}) \) satisfying \( d \chi = 0 \) there exists a form \( \theta \in \mathcal{A}_{X \times V}^{d-1}(\mathcal{E}) \) on some open neighborhood \( V \subseteq U \) of \( s_0 \) such that \( (\chi + d \theta)|s = P(\chi)|s \) for every \( s \in V \).

As a consequence, we can assume the restriction \( t_a |s \) is harmonic for any \( s \in U \) and \( 1 \leq a \leq r_d \). The proof of Lemmas 3.1 and 3.2 continue to work for the case of higher-order direct image sheaf by minor modifications, therefore, we have the following theorem as an analog of Theorem 1 in [10].

**Theorem 3.4.** The curvature tensor \( \mathfrak{R} \) of \((\mathbb{R}^d \pi_a H_r, H)\) over \( S \setminus Z(d) \) is given by
\[
\mathfrak{R}_{abi} = \left( \frac{1}{r \operatorname{Vol}(X, \omega_X)} \int_X \operatorname{Tr}(R_{ij}) \frac{\omega^n_X}{n!} \right)_{s_0} \cdot H_{ab} + \int_X \tilde{h}(G((\eta_j^\tau_i \eta_i)(\tilde{t}a)), \eta_i^\tau_i (\tilde{t}b)) \frac{\omega^n_X}{n!}
\]
\[
- \int_X \tilde{h}(G(\eta_i(\tilde{t}a)), \eta_j(\tilde{t}b)) \frac{\omega^n_X}{n!} - \int_X \tilde{h}(G(\eta_i \wedge \tilde{t}a), \eta_j \wedge \tilde{t}b) \frac{\omega^n_X}{n!}.
\]

**Proof.** Since \( dt_a = 0 \), the same arguments as in Lemma 3.1 of [10] show that
\[
(\nabla_{\tilde{t}a} |s_0 = d |s_0 \delta_i^j(t_a)) |s_0,
\]
where \( \delta_i^j(t_a) \in \mathcal{A}_{X \times U}^{d-1}(\mathcal{E}) \). Hence, fixing the indices \( i, a \), we also have
\[
\left( \int_X \tilde{h}(\nabla_{\tilde{t}a} t_b, \frac{\omega^n_X}{n!}) \right)_{s_0} = 0
\]
for any \( 1 \leq b \leq r_d \). In addition, we have
\[
d^{\dagger}_{\tilde{h}}|_{s_0} (\nabla_j t_b)|_{s_0} = (\nabla_j d^{\dagger}_{\tilde{h}} t_b)|_{s_0} = 0.
\]
Therefore, the following identity as in Lemma 3.1 holds
\[
\left( \int_X \tilde{h}(\nabla_j t_a, \nabla_j t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} = \left( \int_X \tilde{h}(G(\eta_i \wedge t_a), \eta_j \wedge t_b) \frac{\omega^n_X}{n!} \right)|_{s_0}.
\]
To obtain the analog of Lemma 3.2, we need to calculate \((\nabla_j \nabla_j t_a)|_{s_0} = (\nabla_j \nabla_j t_a)|_{s_0} - (R_{i j}(t_a))|_{s_0}\). The first term of the right hand side can be computed as
\[
(\nabla_i \nabla_j t_a)|_{s_0} = (\nabla_i \delta_j(t_a))|_{s_0} = \frac{d}{d x_0}(\nabla_i \delta_j(t_a))|_{s_0} + \Phi_{s_0} \wedge \delta_j(t_a)|_{s_0} = \frac{d}{d x_0}(\nabla_i \delta_j(t_a))|_{s_0} + \eta_i|_{s_0} \wedge \delta_j(t_a)|_{s_0}.
\]
Then we have
\[
\left( \int_X \tilde{h}(\nabla_j \nabla_j t_a, t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} = \left( \int_X \tilde{h}(\eta_i \wedge \delta_j(t_a), t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} - \left( \int_X \tilde{h}(R_{i j}(t_a), t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} = \left( \int_X \tilde{h}(\delta_j(t_a), \eta_i \frac{\omega^n_X}{n!}) \right)|_{s_0} - \left( \int_X \tilde{h}(R_{i j}(t_a), t_b) \frac{\omega^n_X}{n!} \right)|_{s_0}.
\]
It follows from the harmonicity of \( t_a|_{s_0} \), the Ricci identity and the Kähler identity that
\[
d^{\dagger}_{\tilde{h}}|_{s_0} (\nabla_j t_b)|_{s_0} = -\sqrt{-1}i [\Lambda_{\omega_X}, \partial \epsilon_{s_0} + \Phi^{s_0}|_{s_0}] (\nabla_j t_b)|_{s_0} = -\sqrt{-1}i [\Lambda_{\omega_X}, -R_{i \bar{j}}|_{s_0} dj^\beta - (\nabla_j \Phi^{s_0})]t_b|_{s_0} = -\eta_i^{\dagger}_{\tilde{h}}(t_b)|_{s_0},
\]
which implies that
\[
\left( \int_X \tilde{h}(\nabla_j \nabla_j t_a, \nabla_j t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} = \left( \int_X \tilde{h}(\nabla_j \nabla_j t_a, \nabla_j t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} - \left( \int_X \tilde{h}(R_{i j}(t_a), t_b) \frac{\omega^n_X}{n!} \right)|_{s_0}.
\]
Moreover, since \((\nabla_j t_a)|_{s_0}\) is \(d|_{s_0}\)-exact, we have
\[
\left( \int_X \tilde{h}(\nabla_j t_a, \nabla_j t_b) \frac{\omega^n_X}{n!} \right)|_{s_0} = \int_X \tilde{h}(d|_{s_0} G|_{s_0}(d^{\dagger}_{\tilde{h}} t_a)|_{s_0} (\nabla_j t_a)|_{s_0} (\nabla_j t_b)|_{s_0} \frac{\omega^n_X}{n!} = \left( \int_X \tilde{h}(\eta_j^{\dagger}_{\tilde{h}}(t_a), \eta_i^{\dagger}_{\tilde{h}}(t_b)) \frac{\omega^n_X}{n!} \right)|_{s_0},
\]
which is a term automatic vanishing in the case of zero-order. According to above calculations, we finally get the theorem. \(\square\)
4. Curvature of Finsler metric

In this section we introduce a Finsler metric on $S$ and calculate the corresponding curvature. Our main aim of this section is to prove Theorem 1.3. Firstly, we recall some basic definitions of Finsler metric.

**Definition 4.1** [14, 24]. Let $S$ be an $m$-dimensional complex manifold, and $\pi : TS \to S$ be the holomorphic tangent bundle of $S$. The zero section of $TS$ is denoted by $o(S)$. A point $v$ lying in $TS$ is written as $v = (s, \xi)$ in terms of a local trivialization of $TS$ with $s = (s^1, \cdots, s^m)$, $\xi = \sum_{i=1}^m \xi^i \frac{\partial}{\partial s^i}$. The vertical bundle $V$ on $TS \setminus o(S)$ is defined by $V = \text{Ker}(d\pi: TTS \to TS)|_{TS \setminus o(S)}$, whose local frame at $v \in TS \setminus o(S)$ is given by $\left\{ \frac{\partial}{\partial \xi}, \cdots, \frac{\partial}{\partial \xi^m} \right\}$.

(1) A continuous real valued function $F: TS \to \mathbb{R}$ is called a complex Finsler metric on $S$ if the following conditions are satisfied:
   - $F(v) \geq 0$ for any $v \in TS$, and $F(v) = 0$ if and only if $v \in o(S)$,
   - $F(cv) = |c| F(v)$ for any $c \in \mathbb{C}^*$, where $c \cdot v = (s, c\xi)$.

(2) For a Finsler metric $F$, the Hermitian matrix $G_{ij} = \frac{\partial^2 F^2}{\partial \xi^i \partial \overline{\xi}^j}$ is called the Levi matrix associated to $F$.

(3) A Finsler metric $F$ is called smooth if for any open subset $U \subset S$ and any nontrivial $C^\infty$-section $u$ of $TS$, $F(u)$ is a smooth function on $U$.

(4) A Finsler metric $F$ is called strongly pseudo-convex if the Levi matrix defines a Hermitian metric on the vertical bundle $V$, namely for any nonzero $W = W^i \frac{\partial}{\partial \xi^i} \in V_v$, we have
   \[ G_{ij}(v)W^i \overline{W}^j > 0. \]

(5) $S$ is called a complex Finsler manifold if it is equipped with a complex Finsler metric that is smooth and strongly pseudo-convex.

(6) Given a smooth Finsler metric $F$ on $S$, for any holomorphic map $f: \mathbb{D} \to S$ from the unit disk $\mathbb{D}$ to $S$, there is an induced Hermitian metric $f^* F^2$ on $\mathbb{D}$, and the corresponding Gauss curvature $K(f, F)$ is given by
   \[ K(f, F) = -\frac{2}{f^* F^2} \frac{\partial^2 \log f^* F^2}{\partial t \partial \overline{t}}, \]
   where $\{t\}$ is the local holomorphic coordinate on $\mathbb{D}$. For a point $v \in TS$, let $D_v$ be the set consisting of holomorphic maps $f: \mathbb{D} \to S$ such that $f(0) = s$ and $f'(0) = cv$ for some nonzero constant $c$, then we introduce the holomorphic sectional curvature of the Finsler metric $F_k$ in the direction $v$ as
   \[ K_F(v) = \sup_{f \in D_v} K(f, F)|_0. \]

Return to our effectively parametrized family $(E, \Phi)$ over $S$. $S$ is endowed with a Finsler metric $F_k$ introduced in Sect. 1.

**Proposition 4.2.** The Finsler metric $F_k$ is smooth and strongly pseudo-convex, thus $S$ is a complex Finsler manifold.
Proof. The Levi matrix associated to $F_k$ is given by

\[
G_{ij}(v) = \frac{(\int_X \tilde{h}(\mathbb{H}(\xi), \mathbb{H}(\xi)) \frac{\omega^n}{n!}) (\int_X \tilde{h}(\eta_i, \eta_j) \frac{\omega^n}{n!}) + (\int_X \tilde{h}(\eta_i, \mathbb{H}(\xi)) \frac{\omega^n}{n!}) (\int_X \tilde{h}(\mathbb{H}(\xi), \eta_j) \frac{\omega^n}{n!})}{F_k(v)}
\]

\[
+ 2 \kappa \frac{\int_X \tilde{h}((\eta_i \land \mathbb{H}(\xi)), G[\eta_j \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!}}{F_k(v)}
\]

\[
- \kappa \frac{(\int_X \tilde{h}(\mathbb{H}(\xi), \mathbb{H}(\xi)) \frac{\omega^n}{n!}) (\int_X \tilde{h}(\mathbb{H}(\xi) \land \eta_j)) \frac{\omega^n}{n!})}{F_k(v)}
\]

\[
- \kappa \frac{(\int_X \tilde{h}(\mathbb{H}(\xi), \mathbb{H}(\xi)) \frac{\omega^n}{n!}) (\int_X \tilde{h}(\mathbb{H}(\xi) \land \eta_j)) \frac{\omega^n}{n!})}{F_k(v)}
\]

\[
- \kappa \frac{(\int_X \tilde{h}((\eta_i \land \mathbb{H}(\xi)), G[\mathbb{H}(\xi) \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!}) (\int_X \tilde{h}(\mathbb{H}(\xi) \land \eta_j)) \frac{\omega^n}{n!})}{F_k(v)}
\].

We only need to show $G_{ii}(v) > 0$ for any $v \in T S \setminus o(S)$. 

Cauchy-Schwarz inequality provides us

\[
\left( \int_X \tilde{h}((\eta_i \land \mathbb{H}(\xi)), G[\mathbb{H}(\xi) \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!} \right) \left( \int_X \tilde{h}(\mathbb{H}(\xi) \land \eta_j) \frac{\omega^n}{n!} \right) \leq \left( \int_X \tilde{h}((\mathbb{H}(\xi) \land \mathbb{H}(\xi)), G[\mathbb{H}(\xi) \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!} \right) \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)
\]

and similarly

\[
\left| \left( \int_X \tilde{h}((\mathbb{H}(\xi) \land \eta_i), G[\mathbb{H}(\xi) \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!} \right) \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right) \right| \leq \left( \int_X \tilde{h}((\mathbb{H}(\xi) \land \mathbb{H}(\xi)), G[\mathbb{H}(\xi) \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!} \right)^{\frac{1}{2}} \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^{\frac{1}{2}}
\]

Then we have

\[
G_{ii}(v) \geq \frac{1}{F_k(v)} \left[ \left( \int_X \tilde{h}((\mathbb{H}(\xi) \land \mathbb{H}(\xi)), G[\mathbb{H}(\xi) \land \mathbb{H}(\xi)]) \frac{\omega^n}{n!} \right)^{3} \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^{3} \right.
\]
Corollary 4.3 [26, Proposition 2.5]. The holomorphic sectional curvature of the Finsler metric $F_\kappa$ is given by

$$K_{F_\kappa}(\nu) = 2\Re \frac{\xi^i \xi^j \xi^k \xi^l}{F^2_{\kappa}},$$

where

$$\Re_{ijkl} = -\frac{\partial^2 G_{ij}}{\partial s^k \partial s^l} + G_{ip} G_{jq} G_{jq} G_{jk}.$$
E = \left( \int_X \tilde{h}(\nabla_i^2 \eta_i, \nabla_i \eta_i) \frac{\omega^n}{n!} + \int_X \tilde{h}(\nabla_i \eta_i, \nabla_i \nabla_i \eta_i) \frac{\omega^n}{n!} \right)_{s_0} \left( \int_X \tilde{h}(\nabla_i \nabla_i \eta_i, \nabla_i \eta_i) \frac{\omega^n}{n!} + \int_X \tilde{h}(\nabla_i \eta_i, \nabla_i^2 \eta_i) \frac{\omega^n}{n!} \right)_{s_0}.

The key point is to express them in terms of Kodaira-Spencer image.

Lemma 4.4. The following identities hold:

(1) \( A = \left( -6 \int_X h(Gd^{h}_i \eta_i \nabla_i G[\eta_i \wedge \eta_i], d^{h}_i \eta_i \nabla_i G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \\
+ 12 \text{Re} \int_X h(Gd^{h}_i [\eta_i, G\eta_i \nabla_i \eta_i], d^{h}_i \eta_i \nabla_i G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \\
+ 5 \int_X \tilde{h}((G \eta_i^{\sharp \h}, d^{h}_i G[\eta_i \wedge \eta_i]), d^{h}_i G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)_{s_0}, \)

(2) \( B = \left( \int_X \tilde{h}(G^2 [[\eta_i \wedge \eta_i] \wedge \eta_i],[[\eta_i \wedge \eta_i] \wedge \eta_i]) \frac{\omega^n}{n!} \right)_{s_0}, \)

(3) \( C = \left( 4 \int_X h(P[\eta_i, G\eta_i \nabla_i \eta_i], P[\eta_i, G\eta_i \nabla_i \eta_i]) \frac{\omega^n}{n!} \\
+ 4 \int_X h(G[\eta_i \wedge dG \eta_i \nabla_i \eta_i], [\eta_i \wedge dG \eta_i \nabla_i \eta_i]) \frac{\omega^n}{n!} \\
+ \int_X h(Gd^{h}_i \eta_i \nabla_i G[\eta_i \wedge \eta_i], d^{h}_i \eta_i \nabla_i G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)_{s_0}, \)

(4) \( D = \left( 4 \int_X h(P[\eta_i, G\eta_i \nabla_i \eta_i], P[\eta_i, G\eta_i \nabla_i \eta_i]) \frac{\omega^n}{n!} \right)_{s_0}, \)

(5) \( E = \left( \int_X \tilde{h}([[d^{h}_i G[\eta_i \wedge \eta_i]] \wedge \eta_i], [G[\eta_i \wedge \eta_i]]) \frac{\omega^n}{n!} \\
+ 2 \int_X \tilde{h}([[d^{h}_i G\eta_i \nabla_i \eta_i]], [G[\eta_i \wedge \eta_i]]) \frac{\omega^n}{n!} \right)_{s_0}. \)

Proof. (1) Firstly, we introduce the following notations. Let \( A = \alpha + \beta \) with \( \alpha \in \mathcal{A}_{X}^{1,0}(\text{End}(E)) \) and \( \beta \in \mathcal{A}_{X}^{0,1}(\text{End}(E)) \), then we define

\[
\Lambda \cdot [A^{\sharp h} \wedge A] = -\sqrt{-1} \Lambda_\omega [\alpha^{\sharp h} \wedge \alpha] + \sqrt{-1} \Lambda_\omega [\beta^{\sharp h} \wedge \beta].
\]

It is clear that \( \Lambda \cdot [A^{\sharp h} \wedge A] = A^{\sharp h} A \) by the Kähler identities.

For our purpose, we calculate

\[
\eta_i^{\sharp h} \nabla_i \eta_i = \Lambda \cdot [\eta_i^{\sharp h} \wedge \nabla_i \eta_i] \\
= \Lambda \cdot \nabla_i [\eta_i^{\sharp h} \wedge \eta_i] - \Lambda \cdot [\nabla_i \eta_i^{\sharp h} \wedge \eta_i] \\
= -\Box \nabla_i R_{ij} - d^{h}_i [\eta_i, R_{ij}] + \Lambda \cdot [d^{h}_i R_{ij} \wedge \eta_i] \\
= -\Box \nabla_i R_{ij} - 2d^{h}_i [\eta_i, R_{ij}].
\]
Then, together with the Ricci identity, we have
\[
\int_X \tilde{h}(\nabla_i \nabla^2_i \eta_i, \nabla_i \eta_i) \frac{\omega^n}{n!} \\
= \int_X \tilde{h}(3 \nabla_i [\eta_i, R_{ij}] + d \nabla^2_i R_{ij}, \nabla_i \eta_i) \frac{\omega^n}{n!} \\
= -3 \int_X \tilde{h}([R_{ij}, \nabla_i \eta_i], \nabla_i \eta_i) \frac{\omega^n}{n!} + 3 \int_X \tilde{h}(\nabla_i R_{ij}, \eta_i \nabla_i \eta_i) \frac{\omega^n}{n!} \\
= 3 \int_X h(\nabla_i R_{ij}, D \nabla_i R_{ij}) \frac{\omega^n}{n!} \\
- 3 \int_X \tilde{h}([R_{ij}, \nabla_i \eta_i], \nabla_i \eta_i) \frac{\omega^n}{n!} - 6 \int_X \tilde{h}(d \nabla_i R_{ij}, [\eta_i, R_{ij}]) \frac{\omega^n}{n!}.
\]

Consequently, we arrive at
\[
A = \left( -6 \int_X \tilde{h}(d \nabla_i R_{ij}, d \nabla_i R_{ij}) \frac{\omega^n}{n!} - 12 \Re \int_X \tilde{h}(d \nabla_i R_{ij}, [\eta_i, R_{ij}]) \frac{\omega^n}{n!} \\
- 5 \int_X \tilde{h}([R_{ij}, d \nabla R_{ij}], [\eta_i, \nabla \eta_i]) \frac{\omega^n}{n!} \right) |_{s_0} \\
= \left( -6 \int_X h(\nabla_i d \nabla_i R_{ij}, d \nabla_i R_{ij}) \frac{\omega^n}{n!} + 6 \int_X \tilde{h}([\eta_i, R_{ij}], [\eta_i, R_{ij}]) \frac{\omega^n}{n!} \\
- 5 \int_X \tilde{h}([R_{ij}, d \nabla R_{ij}], [\eta_i, \nabla \eta_i]) \frac{\omega^n}{n!} \right) |_{s_0}.
\]

Denote $G_i = \frac{\partial G}{\partial s^i}$, $P_i = \frac{\partial P}{\partial s^i}$, $\square_i = \frac{\partial \square}{\partial s^i}$, then the identity $G \square + P = \text{Id}$ leads to
\[
d G_i \square R_{ij} = -d G_i \square R_{ij} - d P_i R_{ij} \\
= -dd \nabla G_i R_{ij} - [\eta_i, P R_{ij}]
\]
\[
d P_i A = -[\eta_i, PA] 
\]
Therefore, we have
\[
(\nabla_i d R_{ij}) |_{s_0} \\
= (\nabla_i d G \square R_{ij}) |_{s_0} \\
= ([\eta_i, G \square R_{ij}] + d G_i \square R_{ij} + d G \nabla_i \square R_{ij}) |_{s_0} \\
= ([\eta_i, R_{ij}] - dd \nabla G_i R_{ij} - d G \Lambda \cdot \nabla_i [\eta_i, \nabla_i \eta_i]) |_{s_0} \\
= (P \eta_i - \eta_i d \nabla G_i R_{ij} + d G \Lambda \cdot [d \nabla R_{ij}] + d G \Lambda \cdot [d \nabla R_{ij}] + d G \Lambda \cdot [\nabla_i \eta_i R_{ij}] + \eta_i) \\
+ d G \Lambda \cdot [\nabla_i \eta_i R_{ij} + d \nabla G_i R_{ij}] |_{s_0} \\
= (P \eta_i - \eta_i d \nabla G_i R_{ij} - dd \nabla G_i R_{ij} - d G \Lambda \cdot [\nabla_i \eta_i R_{ij}] + \eta_i) |_{s_0},
\]
which yields

\[
\left( \int_X \tilde{h}(\nabla_i dR_{ij}, \nabla_i dR_{ij}) \frac{\omega^n}{n!} \right)_{s_0} = \left( \int_X \tilde{h}(P[\eta_i, R_{ij}], P[\eta_i, R_{ij}]) \frac{\omega^n}{n!} + \int_X \tilde{h}(G[\eta_i \wedge dR_{ij}], [\eta_i \wedge dR_{ij}] \frac{\omega^n}{n!} \\
+ \int_X h(Gd^\dagger h[\eta_i, R_{ij}], d^\dagger h[\eta_i, R_{ij}]) \frac{\omega^n}{n!} + \int_X h(Gd^\dagger h[\eta_i, R_{ij}], d^\dagger h[\eta_i, R_{ij}]) \frac{\omega^n}{n!} + 2 \text{Re} \int_X h(Gd^\dagger h[\eta_i, R_{ij}], d^\dagger h[\eta_i, R_{ij}]) \frac{\omega^n}{n!} \right)_{s_0}.
\]

In addition, we have

\[
\int_X \tilde{h}([\eta_i, R_{ij}], [\eta_i, R_{ij}]) \frac{\omega^n}{n!} - \int_X \tilde{h}(P[\eta_i, R_{ij}], P[\eta_i, R_{ij}]) \frac{\omega^n}{n!} = \int_X \tilde{h}(\square G[\eta_i, R_{ij}], \square G[\eta_i, R_{ij}]) \frac{\omega^n}{n!} = \int_X \tilde{h}(G[\eta_i \wedge dR_{ij}], [\eta_i \wedge dR_{ij}] \frac{\omega^n}{n!} + \int_X h(Gd^\dagger h[\eta_i, R_{ij}], d^\dagger h[\eta_i, R_{ij}]) \frac{\omega^n}{n!}.
\]

Combining the above calculations, the desired equality follows.

(2) By means of Lemma 2.1, one can easily check that

\[
d\nabla_i^2 \eta_i + [\nabla_i \eta_i \wedge \eta_i] = 0,
\]

\[
d^\dagger \nabla_i^2 \eta_i = 0.
\]

Hence, we have

\[
\int_X \tilde{h}(\nabla_i^2 \eta_i, \nabla_i^2 \eta_i) \frac{\omega^n}{n!} = \int_X \tilde{h}(P + Gd^\dagger h d) \nabla_i^2 \eta_i, \nabla_i^2 \eta_i) \frac{\omega^n}{n!} = \int_X \tilde{h}(P(\nabla_i^2 \eta_i), P(\nabla_i^2 \eta_i)) \frac{\omega^n}{n!} + \int_X \tilde{h}(Gd\nabla_i^2 \eta_i, d\nabla_i^2 \eta_i) \frac{\omega^n}{n!} = \int_X \tilde{h}(P(\nabla_i^2 \eta_i), P(\nabla_i^2 \eta_i)) \frac{\omega^n}{n!} + \int_X \tilde{h}(G([\nabla_i \eta_i \wedge \eta_i]), [\nabla_i \eta_i \wedge \eta_i]) \frac{\omega^n}{n!}.
\]

On the other hand, since \(G^{WP}\) is a Kähler metric, we always have \(\frac{\partial^2 G^{WP}}{\partial s^i \partial s^j} \big|_{s_0} = 0\) for any \(1 \leq i, j, k, l \leq m\). Hence

\[
\left( \int_X \tilde{h}(\eta_k, \eta_l) \frac{\omega^n}{n!} \right)_{s_0} = \left( \int_X \tilde{h}(\nabla_i^2 \eta_k, \eta_l) \frac{\omega^n}{n!} + 2 \int_X \tilde{h}(\nabla_i \eta_k, \nabla_i \eta_l) \frac{\omega^n}{n!} + \int_X \tilde{h}(\eta_k, \nabla_i^2 \eta_l) \frac{\omega^n}{n!} \right)_{s_0}.
\]
Then it follows from the calculations in (1) that

\[ \left( \int_X \tilde{h}(\nabla_i^2 \eta_k, \eta_i) \frac{\omega^n}{n!} + 2 \int_X \tilde{h}(d^\uparrow_i \nabla_i \eta_k, R_{ij}) \frac{\omega^n}{n!} + \int_X \tilde{h}(d^\uparrow_i \eta_k, \nabla_i R_{ij}) \frac{\omega^n}{n!} \right) |_{s_0} = 0, \]

which means that \( P ((\nabla_i^2 \eta_i)|_{s_0}) = 0. \) Therefore

\[
B = \left( \int_X \tilde{h}(G[\nabla_i \eta_i \wedge \eta_i], [\nabla_i \eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right) |_{s_0}
\]

\[= \left( \int_X \tilde{h}(G^2([d \nabla_i \eta_i] \wedge \eta_i), [d \nabla_i \eta_i] \wedge \eta_i) \frac{\omega^n}{n!} + \int_X \tilde{h}(G^2 d^\dagger_i \eta_i \wedge \eta_i, d^\dagger_i \eta_i \wedge \eta_i) \frac{\omega^n}{n!} \right) |_{s_0}
\]

\[= \left( \int_X \tilde{h}(G^2([\eta_i \wedge \eta_i] \wedge \eta_i), [\eta_i \wedge \eta_i] \wedge \eta_i) \frac{\omega^n}{n!} + \int_X \tilde{h}(G^2 \nabla_i \Box \eta_i \wedge \nabla_i \Box \eta_i) \frac{\omega^n}{n!} \right) |_{s_0}
\]

\[= \left( \int_X \tilde{h}(G^2([\eta_i \wedge \eta_i] \wedge \eta_i), [\eta_i \wedge \eta_i] \wedge \eta_i) \frac{\omega^n}{n!} + \int_X \tilde{h}((\Box \nabla_i^2 \eta_i + \Box_i \nabla_i \eta_i), \Box \nabla_i^2 \eta_i + \Box_i \nabla_i \eta_i) \frac{\omega^n}{n!} \right) |_{s_0}
\]

\[= \left( \int_X \tilde{h}(G^2([\eta_i \wedge \eta_i] \wedge \eta_i), [\eta_i \wedge \eta_i] \wedge \eta_i) \frac{\omega^n}{n!} \right) |_{s_0},
\]

where the last equality follows from

\[ \Box \nabla_i^2 \eta_i + \Box_i \nabla_i \eta_i = d^\dagger_i d \nabla_i^2 \eta_i + d^\dagger_i [\eta_i \wedge \nabla_i \eta_i] = 0. \]

(3) Again by the Ricci identity, we have

\[ \int_X \tilde{h}(\nabla_i \nabla_i \eta_i, \nabla_i \nabla_i \eta_i) \frac{\omega^n}{n!} = \left( \int_X \tilde{h}(\nabla_i d R_{ij}, \nabla_i d R_{ij}) \frac{\omega^n}{n!} + \int_X \tilde{h}((\nabla_i, R_{ij}), [\eta_i, R_{ij}] \frac{\omega^n}{n!}) + 2 \text{Re} \int_X \tilde{h}(\nabla_i d R_{ij}, [\eta_i, R_{ij}] \frac{\omega^n}{n!}) \right.
\]

Then it follows from the calculations in (1) that

\[
C = \left( \int_X \tilde{h}(P[\eta_i, R_{ij}], P[\eta_i, R_{ij}]) \frac{\omega^n}{n!} + \int_X \tilde{h}(G[\eta_i \wedge d R_{ij}], [\eta_i \wedge d R_{ij}] \frac{\omega^n}{n!}) + \int_X \tilde{h}(G d^\dagger_i ([\eta_i, R_{ij}] + d^\dagger_i G[\eta_i \wedge \eta_i]), d^\dagger_i ([\eta_i, R_{ij}] + d^\dagger_i G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)
\]
\[ + \int_X \tilde{h}(P[\eta_i, R_{i\bar{i}}], P[\eta_i, R_{i\bar{i}}]) \frac{\omega^n}{n!} + \int_X \tilde{h}(G[\eta_i \wedge dR_{i\bar{i}}], [\eta_i \wedge dR_{i\bar{i}}]) \frac{\omega^n}{n!} \]
\[ + \int_X \tilde{h}(Gd^{\bar{k}}[\eta_i, R_{i\bar{i}}], d^{\bar{k}}[\eta_i, R_{i\bar{i}}]) \frac{\omega^n}{n!} \]
\[ + 2 \int_X \tilde{h}(P[\eta_i, R_{i\bar{i}}], P[\eta_i, R_{i\bar{i}}]) \frac{\omega^n}{n!} + 2 \int_X \tilde{h}(G[\eta_i \wedge dR_{i\bar{i}}], [\eta_i \wedge dR_{i\bar{i}}]) \frac{\omega^n}{n!} \]
\[ - 2 \int_X \tilde{h}(Gd^{\bar{k}}[\eta_i, R_{i\bar{i}}], d^{\bar{k}}[\eta_i, R_{i\bar{i}}]) \frac{\omega^n}{n!} \]
\[ - 2\text{Re} \int_X \tilde{h}(Gd^{\bar{k}}[\eta_i, R_{i\bar{i}}], d^{\bar{k}}[\eta_i, G[\eta_i \wedge \eta_i]] \frac{\omega^n}{n!}) \right|_{\mathfrak{s}_0} \]
\[ = \left( 4 \int_X \tilde{h}(P[\eta_i, R_{i\bar{i}}], P[\eta_i, R_{i\bar{i}}]) \frac{\omega^n}{n!} + 4 \int_X \tilde{h}(G[\eta_i \wedge dR_{i\bar{i}}], [\eta_i \wedge dR_{i\bar{i}}]) \frac{\omega^n}{n!} \right) \]
\[ + \int_X \tilde{h}(Gd^{\bar{k}}[\eta_i, R_{i\bar{i}}], d^{\bar{k}}[\eta_i, G[\eta_i \wedge \eta_i]] \frac{\omega^n}{n!}) \right|_{\mathfrak{s}_0} \]

The desired equality follows.

(4) We have
\[ \left( \int_X \tilde{h}(\nabla_i(P\nabla_i \eta_i), \nabla_i(P\nabla_i \eta_i)) \frac{\omega^n}{n!} \right) \bigg|_{\mathfrak{s}_0} \]
\[ = \left( \int_X \tilde{h}(P_i Gd^{\bar{k}}([\eta_i \wedge \eta_i]), P_i Gd^{\bar{k}}([\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}) \right) \bigg|_{\mathfrak{s}_0} \]
\[ = \left( \int_X \tilde{h}(Pd^{\bar{k}} \nabla_i(G[\eta_i \wedge \eta_i]), Pd^{\bar{k}} \nabla_i(G[\eta_i \wedge \eta_i])) \frac{\omega^n}{n!}) \right) \bigg|_{\mathfrak{s}_0} \]
\[ = 0, \]
where the second equality is due to the identity
\[ P_i d^{\bar{k}} + P \nabla_i d^{\bar{k}} = 0. \]

And similarly, we have
\[ \left( \int_X \tilde{h}(\nabla_i(P \nabla_i \eta_i), \nabla_i(P \nabla_i \eta_i)) \frac{\omega^n}{n!} \right) \bigg|_{\mathfrak{s}_0} \]
\[ = \left( \int_X \tilde{h}(P \nabla_i \nabla_i \eta_i, P \nabla_i \nabla_i \eta_i) \frac{\omega^n}{n!} + \int_X \tilde{h}(P \nabla_i d^{\bar{k}} G[\eta_i \wedge \eta_i], P \nabla_i d^{\bar{k}} G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right) \bigg|_{\mathfrak{s}_0} \]
\[ = \left( 4 \int_X \tilde{h}(P[\eta_i, R_{i\bar{i}}], P[\eta_i, R_{i\bar{i}}]) \frac{\omega^n}{n!} + \int_X \tilde{h}(P_i Gd^{\bar{k}}([\eta_i \wedge \eta_i]), P_i Gd^{\bar{k}}([\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}) \right) \bigg|_{\mathfrak{s}_0} \]
\[ = \left( 4 \int_X \tilde{h}(P[\eta_i, G^{\bar{k}} \eta_i \eta_i, P[\eta_i, G^{\bar{k}} \eta_i \eta_i]) \frac{\omega^n}{n!} \right) \bigg|_{\mathfrak{s}_0} \]
\[ + \int_X \tilde{h}(Pd^{\bar{k}}[\eta_i \wedge G^2 d^{\bar{k}}[\eta_i \wedge \eta_i]], Pd^{\bar{k}}[\eta_i \wedge G^2 d^{\bar{k}}[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}) \bigg|_{\mathfrak{s}_0} \]
\[ = \left( 4 \int_X \tilde{h}(P[\eta_i, G^{\bar{k}} \eta_i \eta_i, P[\eta_i, G^{\bar{k}} \eta_i \eta_i]) \frac{\omega^n}{n!}) \bigg|_{\mathfrak{s}_0} \right). \]

Then the desired equality is obtained.
(5) Since we have
\[
\left( \int_X \tilde{h}(\nabla_i^2 \eta_i, \nabla_i \eta_i) \frac{\omega^n}{n!} \right)_{s_0} = \left( \int_X \tilde{h}(Gd^h \eta_i, \nabla_i \eta_i) \frac{\omega^n}{n!} \right)_{s_0}
\]
\[
= \left( \int_X \tilde{h}(\nabla_i^2 \eta_i \wedge \eta_i, [\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)_{s_0}
\]
\[
= -\left( \int_X \tilde{h}([d^h \eta_i \wedge \eta_i] \wedge \eta_i, G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right)_{s_0},
\]
the desired equality follows.

\[\square\]

**Theorem 4.5.** Let \(\{s^1, \cdots, s^m\}\) be the local holomorphic coordinate on \(S\). The holomorphic sectional curvature of the Finsler metric \(F_k\) is given by

\[
K_{F_k} \left( \frac{\partial}{\partial s^i} \right) = \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(d^h G[\eta_i \wedge \eta_i], d^h G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}
\]
\[
\cdot \left[ \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(d^h G[\eta_i \wedge \eta_i], d^h G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right]
\]
\[
- \int_X \tilde{h}(d^h G[\eta_i \wedge \eta_i], d^h G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}
\]
\[
+ \kappa \left( -5 \int_X \tilde{h}([G^h_i \eta_i] \wedge \eta_i), G^h_i \eta_i) \frac{\omega^n}{n!} \right)
\]
\[
+ \kappa \left( -5 \int_X \tilde{h}([G^h_i \eta_i] \wedge \eta_i), G^h_i \eta_i) \frac{\omega^n}{n!} \right)
\]
\[
- 12\text{Re} \int_X \tilde{h}(d^h G[\eta_i \wedge \eta_i], d^h G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}
\]
\[
- 4 \int_X \tilde{h}(G[\eta_i \wedge dG[\eta_i \wedge \eta_i], [\eta_i \wedge dG[\eta_i \wedge \eta_i]] \frac{\omega^n}{n!}
\]
\[
- \int_X \tilde{h}(G^2 [\eta_i \wedge \eta_i] \wedge \eta_i), [\eta_i \wedge \eta_i] \wedge \eta_i) \frac{\omega^n}{n!}
\]
\[
+ \kappa^2 \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(d^h G[\eta_i \wedge \eta_i], d^h G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}
\]
\[
\cdot \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(d^h G[\eta_i \wedge \eta_i], d^h G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!}
\]
\[
+ 2 \int_X \tilde{h}(\eta_i \wedge G[\eta_i \wedge \eta_i], G[\eta_i \wedge \eta_i]) \frac{\omega^n}{n!} \right].
\]

**Proof:** Substitute the formulas in Lemma 4.4 into the following expression

\[
- \left( \frac{\partial^2}{\partial s^i \partial s^j} \log \left( \int_X \tilde{h}(\eta_i, \eta_i) \frac{\omega^n}{n!} \right)^2 + \kappa \int_X \tilde{h}(\eta_i \wedge \eta_i) \frac{\omega^n}{n!} \right) \right|_{s_0}
\]
\[
\frac{(2 \int_X \hat{h}(\eta_i, \eta_j) \frac{\omega^n}{n!})(2 \int_X \hat{h}(\eta_i, \eta_j, G \eta_i, \eta_j) \frac{\omega^n}{n!} - \int_X \hat{h}([\eta_i, \eta_j], G[\eta_i, \eta_j]) \frac{\omega^n}{n!})}{\kappa(A + B + C - D)} - \kappa(A + B + C - D)
\]

and then by the covariance we get the theorem.  

Acknowledgements  The author P. Huang acknowledges funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2181/1 - 390900948 (the Heidelberg STRUCTURES Excellence Cluster), and by Collaborative Research Center/Transregio (CRC/TRR 191; 281071066-TRR 191). The authors would like to express their deep gratitude to the anonymous referee for many valuable suggestions.

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Ahlfors, L.: Some remarks on Teichmüller’s space of Riemann surfaces. Ann. Math. 74, 171–191 (1961)
2. Ahlfors, L.: Curvature properties of Teichmüller’s space. J. Anal. Math. 9, 161–176 (1961)
3. Aubin, T.: Équations du type Monge-Ampère sur les variétés kählériennes compactes. C. R. Acad. Sci. Paris Sér. A-B 283 (1976), A119–A121
4. Biswas, I., Schumacher, G.: Geometry of moduli spaces of Higgs bundles. Comm. Anal. Geom. 14, 765–793 (2006)
5. Deng, Y.: On the hyperbolicity of base spaces for maximally variational families of smooth projective varieties. arXiv:1806.01666 to appear in Jour. Eur. Math. Soc
6. Donagi, R., Pantev, T., Simpson, C.: Direct image in non abelian Hodge theory. arXiv:1612.06388
7. Donaldson, S.: Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proc. London Math. Soc. 50, 1–26 (1985)
8. Drezet, J., Narasimhan, M.: Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. Invent. Math. 97, 53–94 (1989)
9. Fischer, A., Tromba, A.: On the Weil-Petersson metric on Teichmüller space. Trans. Am. Math. Soc. 284, 319–335 (1984)
10. Geiger, T., Schumacher, G.: Curvature of higher direct image sheaves. Adv. Stud. Pure Math. 74, 171–184 (2017)
11. Griffiths, P. (ed.): Topics in transcendental algebraic geometry. Ann. of Math. Studies 106, Princeton, NJ, Princeton Univ. Press (1984)
12. Heier, G., Wong, B.: Scalar curvature and uniruledness on projective manifolds. Comm. Anal. Geom. 20, 751–764 (2012)
13. Hitchin, N.: The self-duality equations on a Riemann surface. Proc. London Math. Soc. 55, 59–126 (1987)
14. Kobayashi, S.: Hyperbolic complex spaces, Grundlehren der Mathematischen Wissenschaften 318. Springer, Berlin (1998)
15. Mochizuki, T.: Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces. J. Topol. 9, 1021–1073 (2016)
16. Narasimhan, M., Sechadri, C.: Stable and unitary vector bundles on a compact Riemann surafce. Ann. Math. 82, 540–564 (1965)
17. Royden, H.: Intrinsic metrics on Teichmüller space, in Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974, Vol. 2) Canad. Math. Congress, Montreal, QC, pp. 217–221 (1975)
18. Schumacher, G.: Positivity of relative canonical bundles and applications. Invent. Math. 190, 1–56 (2012)
19. Schumacher, G., Toma, M.: On the Petersson-Weil metric for the moduli space of Hermitian-Einstein bundles and its curvature. Math. Ann. 293, 101–107 (1992)
20. Simpson, C.: Constructing of variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Am. Math. Soc. 1, 867–918 (1988)
21. Siu, Y.-T.: Curvature of the Weil-Petersson metric in the moduli spaces of compact Kähler-Einstein manifolds of negative first Chern class, In: Contributions to several complex variables, 261–298, ed. P.-M. Wong and A. Howard, Vieweg (1986)
22. Strominger, A.: Special geometry. Comm. Math. Phys. 133, 163–180 (1990)
23. To, W.-K., Weng, L.: Curvature of the $L^2$-metric on the direct image of a family of Hermitian-Einstein vector bundles. Am. J. Math. 120, 649–661 (1998)

24. To, W.-K., Yeung, S.-K.: Finsler metrics and Kobayashi hyperbolicity of the moduli spaces of canonically polarized manifolds. Ann. Math. 181, 547–586 (2015)

25. Uhlenbeck, K., Yau, S.-T.: On the existence of Hermitian-Yang-Mills connections on stable vector bundles. Comm. Pure Appl. Math. 39, 5257–5293 (1986)

26. Wan, X.: Holomorphic sectional curvature of complex Finsler manifolds. J. Geom. Anal. 29, 194–216 (2019)

27. Wolpert, S.: Chern forms and the Riemann tensor for the moduli space of curves. Invent. Math. 85, 119–145 (1986)

28. Yang, X.: Hermitian manifolds with semi-positive holomorphic sectional curvature. Math. Res. Lett. 23, 939–952 (2016)

29. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. Comm. Pure Appl. Math. 31, 339–411 (1978)

Communicated by H-T. Yau