RADIAL SYMMETRY OF POSITIVE SOLUTIONS TO EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract

The aim of this paper is to study radial symmetry and monotonicity properties for positive solution of elliptic equations involving the fractional Laplacian. We first consider the semi-linear Dirichlet problem

\[ (-\Delta)^\alpha u = f(u) + g, \text{ in } B_1, \quad u = 0 \text{ in } B_1^c, \quad (0.1) \]

where \((-\Delta)^\alpha\) denotes the fractional Laplacian, \(\alpha \in (0, 1)\), and \(B_1\) denotes the open unit ball centered at the origin in \(\mathbb{R}^N\) with \(N \geq 2\). The function \(f : [0, \infty) \to \mathbb{R}\) is assumed to be locally Lipschitz continuous and \(g : B_1 \to \mathbb{R}\) is radially symmetric and decreasing in \(|x|\).

In the second place we consider radial symmetry of positive solutions for the equation

\[ (-\Delta)^\alpha u = f(u), \text{ in } \mathbb{R}^N, \quad (0.2) \]

with \(u\) decaying at infinity and \(f\) satisfying some extra hypothesis, but possibly being non-increasing.

Our third goal is to consider radial symmetry of positive solutions for system of the form

\[
\begin{cases}
(-\Delta)^{\alpha_1} u = f_1(v) + g_1, & \text{in } B_1, \\
(-\Delta)^{\alpha_2} v = f_2(u) + g_2, & \text{in } B_1, \\
u = v = 0, & \text{in } B_1^c,
\end{cases}
\quad (0.3)
\]

where \(\alpha_1, \alpha_2 \in (0, 1)\), the functions \(f_1\) and \(f_2\) are locally Lipschitz continuous and increasing in \([0, \infty)\), and the functions \(g_1\) and \(g_2\) are radially symmetric and decreasing.

We prove our results through the method of moving planes, using the recently proved ABP estimates for the fractional Laplacian. We
use a truncation technique to overcome the difficulty introduced by the non-local character of the differential operator in the application of the moving planes.

Key words: Fractional Laplacian, Radial Symmetry, Moving Planes.

1 Introduction

The purpose of this paper is to study symmetry and monotonicity properties of positive solutions for equations involving the fractional Laplacian through the use of moving planes arguments. The first part of this article is devoted to the following semi-linear Dirichlet problem

\[
\begin{cases}
(-\Delta)^\alpha u = f(u) + g, & \text{in } B_1, \\
u = 0, & \text{in } B_1^c,
\end{cases}
\]  

where \(B_1\) denotes the open unit ball centered at the origin in \(\mathbb{R}^N\), \(N \geq 2\) and \((-\Delta)^\alpha\) with \(\alpha \in (0, 1)\) is the fractional Laplacian defined as

\[
(-\Delta)^\alpha u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dy,
\]

\(x \in B_1\). Here \(P.V.\) denotes the principal value of the integral, that for notational simplicity we omit in what follows.

During the last years, non-linear equations involving general integro-differential operators, especially, fractional Laplacian, have been studied by many authors. Caffarelli and Silvestre \[5\] gave a formulation of the fractional Laplacian through Dirichlet-Neumann maps. Various regularity issues for fractional elliptic equations has been studied by Cabré and Sire \[2\], Caffarelli and Silvestre \[6\], Capella, Dávila, Dupaigne and Sire \[7\], Ros-Oton and Serra \[22\] and Silvestre \[25\]. Existence and related results were studied by Cabré and Tan \[4\], Dipierro, Palatucci and Valdinoci \[12\], Felmer, Quaas and Tan \[13\], and Servadei and Valdinoci \[24\]. Great attention has also been devoted to symmetry results for equations involving the fractional Laplacian in \(\mathbb{R}^N\), such as in the work by Li \[19\] and Chen, Li and Ou \[8, 9\], where the method of moving planes in integral form has been developed to treat various equations and systems, see also Ma and Chen \[20\]. On the other hand, using the local formulation of Caffarelli and Silvestre, Cabré and Sire \[3\] applied the sliding method to obtain symmetry results for nonlinear equations with
fractional laplacian and Sire and Valdinoci [28] studied symmetry properties for a boundary reaction problem via a geometric inequality. Finally, in [13] the authors used the method of moving planes in integral form to prove symmetry results for

\[-\Delta^\alpha u + u = h(u) \text{ in } \mathbb{R}^N,
\]

(1.3)

taking advantage of the representation formula for \( u \) given by

\[ u(x) = K * h(u)(x), \quad x \in \mathbb{R}^N, \]

where the kernel \( K \), associated to the linear part of the equation, plays a key role in the arguments. This approach is not possible to be used for problem (1.1), since a similar representation formula is not available in general.

The study of radial symmetry and monotonicity of positive solutions for non-linear elliptic equations in bounded domains using the moving planes method based on the Maximum Principle was initiated with the work by Serrin [23] and Gidas, Ni and Nirenberg [14], with important subsequent advances by Berestycki and Nirenberg [1]. We refer to the review by Pacella and Ramaswamy [21] for a more complete discussion of the method and its various applications. In [1] the Maximum Principle for small domain, based on the Aleksandrov-Bakelman-Pucci (ABP) estimate, was used as a tool to obtain much general results, specially avoiding regularity hypothesis on the domain. In a recent article Guillen and Schwab, [16], provided an ABP estimate for a class of fully non-linear elliptic integro-differential equations. Motivated by this work, we obtain a version of the Maximum Principle for small domain and we apply it with the moving planes method as in [1] to prove symmetry and monotonicity properties for positive solutions to problem (1.1) in the ball and in more general domains.

We consider the following hypotheses on the functions \( f \) and \( g \):

(\( F1 \)) The function \( f : [0, \infty) \to \mathbb{R} \) is locally Lipschitz.

(\( G \)) The function \( g : B_1 \to \mathbb{R} \) is radially symmetric and decreasing in \( |x| \).

Before stating our first theorem we make precise the notion of solution that we use in this article. We say that a continuous function \( u : \mathbb{R}^N \to \mathbb{R} \) is a classical solution of equation (1.1) if the fractional Laplacian of \( u \) is defined at any point of \( B_1 \), according to the definition given in (1.2), and if \( u \) satisfies the equation and the external condition in a pointwise sense. This notion of solution is extended in a natural way to the other equations considered in this paper.
Now we are ready for our first theorem on radial symmetry and monotonicity properties for positive solutions of equation (1.1). It states as follows:

**Theorem 1.1** Assume that the functions $f$ and $g$ satisfy (F1) and (G), respectively. If $u$ is a positive classical solution of (1.1), then $u$ must be radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0, 1)$.

The proof of Theorem 1.1 is given in Section §3, where we prove a more general symmetry and monotonicity result for equation (1.1) on a general domain $\Omega$, which is convex and symmetric in one direction, see Theorem 3.1.

We devote the second part of this article to study symmetry results for a non-linear equation as (1.1), but in $\mathbb{R}^N$ and with $g \equiv 0$. For the problem in $\mathbb{R}^N$, the moving planes procedure has to start a different way because we cannot use the Maximum Principle for small domain. We refer to the work by Gidas, Ni and Nirenberg [15], Berestycki and Nirenberg [1], Li [17], and Li and Ni [18], where these results were studied assuming some additional hypothesis on $f$, allowing for decay properties of the solution $u$. A general result in this direction was obtained by Li [17] for the equation

$$-\Delta u = f(u), \quad \text{in } \mathbb{R}^N,$$

with $u$ decaying at infinity and $f$ satisfying the following hypothesis:

$$\text{(F2)} \quad \text{There exists } s_0 > 0, \gamma > 0 \text{ and } C > 0 \text{ such that }$$

$$\frac{f(v) - f(u)}{v - u} \leq C(u + v)^\gamma \quad \text{for all } 0 < u < v < s_0. \tag{1.4}$$

Motivated by these results, we are interested in similar properties of positive solutions for equations involving the fractional Laplacian under assumption (F2). Here is our second main theorem.

**Theorem 1.2** Assume that $\alpha \in (0, 1)$, $N \geq 2$, the function $f$ satisfies (F1)–(F2) and $u$ is a positive classical solution for the equation

$$\begin{cases}
(-\Delta)^\alpha u = f(u) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0.
\end{cases} \tag{1.5}$$

Assume further that there exists

$$m > \max\left\{\frac{2\alpha}{\gamma}, \frac{N}{\gamma + 2}\right\} \tag{1.6}$$
such that \( u \) satisfies

\[
    u(x) = O\left(\frac{1}{|x|^m}\right), \quad \text{as} \quad |x| \to \infty,
\]

then \( u \) is radially symmetric and strictly decreasing about some point in \( \mathbb{R}^N \).

In \([13]\), Felmer, Quaas and Tan studied symmetry of positive solutions using the integral form of the moving planes method, assuming that the function \( f \) is such that \( h(\xi) \equiv f(\xi) + \xi \) is super-linear, with sub-critical growth at infinity and

\[
    h \in C^1(\mathbb{R}), \quad \text{increasing and there exists } \tau > 0 \text{ such that}
    \lim_{v \to 0} \frac{h'(v)}{v^\tau} = 0.
\]

We see that Theorem 1.2 generalizes Theorem 1.3 in \([13]\), since here we do not assume \( f \) is differentiable and we do not require \( h \) to be increasing. In Section §4 we present an extension of Theorem 1.2 to \( f(\xi) = \xi^p - \xi^q \), with \( 0 < q < 1 < p \), that is not covered by the results in \([13]\) either, see Theorem 4.1. This non-linearity was studied by Valdebenito in \([27]\), where decay and symmetry results were obtained using local extension as in Caffarelli and Silvestre \([5]\) and regular moving planes.

For the particular case \( f(u) = u^p \), for some \( p > 1 \), we see that (H) is not satisfied, but that (F2) does hold. Thus, if we knew the solution of (1.5) satisfies decay assumption (1.7) in this setting, we would have symmetry results in these cases. See \([15]\) and \([17]\) for the proof of decay properties in the case of the Laplacian.

The third part of this paper is devoted to investigate the radial symmetry of non-negative solutions for the following system of non-linear equations involving fractional Laplacians with different orders,

\[
\begin{cases}
    (-\Delta)^{\alpha_1} u = f_1(v) + g_1, \quad \text{in } B_1, \\
    (-\Delta)^{\alpha_2} v = f_2(u) + g_2, \quad \text{in } B_1, \\
    u = v = 0, \quad \text{in } B_1^c,
\end{cases}
\]

where \( \alpha_1, \alpha_2 \in (0, 1) \). We have following results for system (1.8):

**Theorem 1.3** Suppose \( f_1 \) and \( f_2 \) are locally Lipschitz continuous and increasing functions defined in \([0, \infty)\) and \( g_1 \) and \( g_2 \) satisfy (G). Assume that \((u,v)\) are positive, classical solutions of system (1.8), then \( u \) and \( v \) are radially symmetric and strictly decreasing in \( r = |x| \) for \( r \in (0, 1) \).
We prove our theorems using the moving planes method. The main
difficulty comes from the fact that the fractional Laplacian is a non-local
operator, and consequently Maximum Principle and Comparison Results require
information on the solutions in the whole complement of the domain, not
only at the boundary. To overcome this difficulty, we introduce a new trun-
cation technique which is well adapted to be used with the method of moving
planes.

The rest of the paper is organized as follows. In Section §2, we recall the
ABP estimate for equations involving fractional Laplacian, as proved in [16]
and we prove a form of Maximum Principle for domains with small measure.
In Section §3, we prove Theorem 1.1 by the moving planes method and
we extend our symmetry results to general domains with one dimensional
convexity and symmetry properties. In Section §4, the radial symmetry
of solutions for equation (1.5) in \( \mathbb{R}^N \) is obtained. An extension to a non-
lipschitzian non-linearity is given. In Section §5, we complete the proof of
Theorem 1.3. And finally, Section §6 is devoted to discuss (1.1) for a non-
local operator with non-homogeneous kernel.

## 2 Preliminaries

A key tool in the use of the moving planes method is the Maximum Principle
for small domain, which is a consequence of the ABP estimate. In [16],
Guillen and Schwab showed an ABP estimate (see Theorem 9.1) for general
integro-differential operators. In this section we recall this estimate in the
case of the fractional Laplacian in any open and bounded domain. Then we
obtain the Maximum Principle for small domains.

We start with the ABP estimate for the fractional Laplacian, which is
stated as follows:

**Proposition 2.1** Let \( \Omega \) be a bounded, connected open subset of \( \mathbb{R}^N \).
Suppose that \( h : \Omega \to \mathbb{R} \) is in \( L^\infty(\Omega) \) and \( w \in L^\infty(\mathbb{R}^N) \) is a classical solution of

\[
\begin{aligned}
\Delta^\alpha w(x) &\leq h(x), \quad x \in \Omega, \\
w(x) &\geq 0, \quad x \in \mathbb{R}^N \setminus \Omega.
\end{aligned}
\]

Then there exists a positive constant \( C \), depending on \( N \) and \( \alpha \), such that

\[
- \inf_{\Omega} w \leq C d^\alpha \| h^+ \|_{L^\infty(\Omega)} \| h^+ \|^{\alpha}_{L^N(\Omega)},
\]

where \( d = \text{diam}(\Omega) \) is the diameter of \( \Omega \) and \( h^+(x) = \max\{h(x), 0\} \).

Here and in what follows we write \( \Delta^\alpha w(x) = -(-\Delta)^\alpha w(x) \).
We have the following corollary

**Corollary 2.1** Under the assumptions of Proposition 2.1, with \( \Omega \) not necessarily connected, we have

\[
- \inf_{\Omega} w \leq Cd^\alpha \| h^+ \|_{L^\infty(\Omega)} |\Omega|^{\frac{\alpha}{N}}.
\]

**Proof.** We let \( w_0 \in L^\infty(\mathbb{R}^N) \) be a classical solution of

\[
\begin{cases}
\Delta^\alpha w_0(x) = \| h^+ \|_{L^\infty(\Omega)} \chi_\Omega(x), & x \in B_d(x_0), \\
w_0(x) = 0, & x \in \mathbb{R}^N \setminus B_d(x_0),
\end{cases}
\]

where \( x_0 \in \Omega \) and \( \Omega \subset B_d(x_0) \). We observe that \( B_d(x_0) \) is connected and that \( w_0 \leq 0 \) in all \( \mathbb{R}^N \). By Comparison Principle, we see that

\[
\inf_{\mathbb{R}^N} w_0 \leq \inf_{B_d(x_0)} w_0 \leq C(2d)^\alpha \| h^+ \|_{L^\infty(\Omega)} |\Omega|^{\frac{\alpha}{N}}
\]

and then we conclude

\[
- \inf_{\Omega} w = - \inf_{\mathbb{R}^N} w_0 \leq Cd^\alpha \| h^+ \|_{L^\infty(\Omega)} |\Omega|^{\frac{\alpha}{N}}. \quad \square
\]

**Remark 2.1** We notice that, under a possibly different constant \( C > 0 \), the ABP estimate for problem (2.1) with \( \alpha = 1 \)

\[
\begin{cases}
\Delta w(x) \leq h(x), & x \in \Omega, \\
w(x) \geq 0, & x \in \partial \Omega,
\end{cases}
\]

is precisely (2.2) with \( \alpha = 1 \).

As a consequence of the ABP estimate just recalled, we have the Maximum Principle for small domain, which is stated as follows:

**Proposition 2.2** Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^N \). Suppose that \( \varphi : \Omega \to \mathbb{R} \) is in \( L^\infty(\Omega) \) and \( w \in L^\infty(\mathbb{R}^N) \) is a classical solution of

\[
\begin{cases}
\Delta^\alpha w(x) \leq \varphi(x) w(x), & x \in \Omega, \\
w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Then there is \( \delta > 0 \) such that whenever \( |\Omega^-| \leq \delta \), \( w \) has to be non-negative in \( \Omega \). Here \( \Omega^- = \{ x \in \Omega \mid w(x) < 0 \} \).
Proof. By (2.5), we observe that

\[
\begin{align*}
\Delta^\alpha w(x) &\leq \varphi(x)w(x), \quad x \in \Omega^- \\
w(x) &\geq 0, \quad x \in \mathbb{R}^N \setminus \Omega^-.
\end{align*}
\]

Then, using Corollary 2.1 with \( h(x) = \varphi(x)w(x) \), we obtain that

\[
\|w\|_{L^\infty(\Omega^-)} = -\inf_{\Omega^-} w \leq Cd_0^\alpha \|\varphi w\|_{L^\infty(\Omega^-)} \|\varphi\|_{L^\infty(\Omega)} |\Omega^-|^{\alpha/N},
\]

where constant \( C > 0 \) depends on \( N \) and \( \alpha \). Here \( d_0 = \text{diam}(\Omega^-) \). Thus

\[
\|w\|_{L^\infty(\Omega^-)} \leq Cd_0^\alpha \|\varphi\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega^-)} |\Omega^-|^{\alpha/N}.
\]

We see that, if \( |\Omega^-| \) is such that \( Cd_0^\alpha \|\varphi\|_{L^\infty(\Omega)} |\Omega^-|^{\alpha/N} < 1 \), then we must have that

\[
\|w\|_{L^\infty(\Omega^-)} = 0.
\]

This implies that \( |\Omega^-| = 0 \) and since \( \Omega^- \) is open, we have \( \Omega^- = \emptyset \), completing the proof. \( \square \)

3 Proof of Theorem 1.1.

In this section we provide a proof of Theorem 1.1 on the radial symmetry and monotonicity of positive solutions to equation (1.1) in the unit ball. For this purpose we use the of moving planes method, for which we give some preliminary notation. We define

\[
\Sigma_\lambda = \{x = (x_1, x') \in B_1 \mid x_1 > \lambda\},
\]

(3.1)

\[
T_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \lambda\},
\]

(3.2)

\[
u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x),
\]

(3.3)

where \( \lambda \in (0, 1) \) and \( x_\lambda = (2\lambda - x_1, x') \) for \( x = (x_1, x') \in \mathbb{R}^N \). For any subset \( A \) of \( \mathbb{R}^N \), we write \( A_\lambda = \{x_\lambda : x \in A\} \), the reflection of \( A \) with regard to \( T_\lambda \).

Proof of Theorem 1.1. We divide the proof in three steps.

Step 1: We prove that if \( \lambda \in (0, 1) \) is close to 1, then \( w_\lambda > 0 \) in \( \Sigma_\lambda \). For this purpose, we start proving that if \( \lambda \in (0, 1) \) is close to 1, then \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \). If we define \( \Sigma^-_\lambda = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\} \), then we just need to prove that if \( \lambda \in (0, 1) \) is close to 1 then

\[
\Sigma^-_\lambda = \emptyset.
\]

(3.4)
We look at each of these integrals separately. Since \( u \) and we observe that \( u \in \Sigma^- \), we have

\[
\begin{align*}
\w_+(x) &= \begin{cases} 
  w_\lambda(x), & x \in \Sigma^- \\
  0, & x \in \mathbb{R}^N \setminus \Sigma^-,
\end{cases} 
\quad (3.5) \\
\w_-(x) &= \begin{cases} 
  0, & x \in \Sigma^- \\
  w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma^- 
\end{cases} 
\quad (3.6)
\end{align*}
\]

and we observe that \( \w_+(x) = \w_\lambda(x) - \w_-(x) \) for all \( x \in \mathbb{R}^N \). Next we claim that for all \( 0 < \lambda < 1 \), we have

\[
(-\Delta)^\alpha \w_-(x) \leq 0, \quad \forall x \in \Sigma^-.
\quad (3.7)
\]

By contradiction, we assume (3.4) is not true, that is \( \Sigma^- \neq \emptyset \). We denote

\[
\w_+(x) = \begin{cases} 
  w_\lambda(x), & x \in \Sigma^- \\
  0, & x \in \mathbb{R}^N \setminus \Sigma^-,
\end{cases} 
\quad (3.5) \\
\w_-(x) = \begin{cases} 
  0, & x \in \Sigma^- \\
  w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma^- 
\end{cases} 
\quad (3.6)
\]

By direct computation, for \( x \in \Sigma^- \), we have

\[
(-\Delta)^\alpha \w_-(x) = \int_{\mathbb{R}^N} \frac{w_\lambda(x) - w_\lambda(z)}{|x - z|^{N+2\alpha}} dz = -\int_{\mathbb{R}^N \setminus \Sigma^-} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz
\]

\[
= -\int_{(B_1 \setminus (B_1)\lambda) \cup ((B_1)\lambda \setminus B_1)} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz
\]

\[
-\int_{(\Sigma^- \setminus \Sigma^-) \cup (\Sigma^- \setminus \Sigma^-)\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz - \int_{(\Sigma^-)\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz
\]

\[
= -I_1 - I_2 - I_3.
\]

We look at each of these integrals separately. Since \( u = 0 \) in \((B_1)\lambda \setminus B_1\) and \( u_\lambda = 0 \) in \( B_1 \setminus (B_1)\lambda\), we have

\[
I_1 = \int_{(B_1 \setminus (B_1)\lambda) \cup ((B_1)\lambda \setminus B_1)} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz
\]

\[
= \int_{(B_1)\lambda \setminus B_1} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz - \int_{B_1 \setminus (B_1)\lambda} \frac{u(z)}{|x - z|^{N+2\alpha}} dz
\]

\[
= \int_{(B_1)\lambda \setminus B_1} w_\lambda(z) \left( \frac{1}{|x - z|^{N+2\alpha}} - \frac{1}{|x - z\lambda|^{N+2\alpha}} \right) dz \geq 0,
\]

since \( u_\lambda \geq 0 \) and \( |x - z\lambda| > |x - z| \) for all \( x \in \Sigma^- \) and \( z \in (B_1)\lambda \setminus B_1 \). In order to study the sign of \( I_2 \) we first observe that \( w_\lambda(z) = -w_\lambda(z) \) for any \( z \in \mathbb{R}^N \). Then

\[
I_2 = \int_{(\Sigma^- \setminus \Sigma^-) \cup (\Sigma^- \setminus \Sigma^-)\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz
\]

\[
= \int_{\Sigma^- \setminus \Sigma^-} \frac{w_\lambda(z)}{|x - z|^{N+2\alpha}} dz + \int_{\Sigma^- \setminus \Sigma^-} \frac{w_\lambda(z)}{|x - z\lambda|^{N+2\alpha}} dz
\]

\[
= \int_{\Sigma^- \setminus \Sigma^-} w_\lambda(z) \left( \frac{1}{|x - z|^{N+2\alpha}} - \frac{1}{|x - z\lambda|^{N+2\alpha}} \right) dz \geq 0,
\]
Let us define $\phi_g$ that assumption $(F)$ since $w$ and since $|x-z| > |x-z|$ for all $x \in \Sigma_\lambda^-$ and $z \in \Sigma_\lambda \setminus \Sigma_\lambda^-$. Finally, since $w_\lambda(z) < 0$ for $z \in \Sigma_\lambda^-$, we have

$$I_3 = \int_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz = \int_{\Sigma^-_\lambda} \frac{w_\lambda(z)}{|x-z\lambda|^{N+2\alpha}} dz$$

$$= -\int_{\Sigma^-_\lambda} \frac{w_\lambda(z)}{|x-z\lambda|^{N+2\alpha}} dz \geq 0.$$

Hence, we obtain (3.7), proving the claim. Now we apply (3.7) and linearity of the fractional Laplacian to obtain that, for $x \in \Sigma_\lambda^-$,

$$(-\Delta)^\alpha w_\lambda^+(x) \geq (-\Delta)^\alpha w_\lambda(x) = (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x).$$

(3.8)

Combining equation (1.1) with (3.8) and (3.5), for $x \in \Sigma_\lambda^-$ we have

$$(-\Delta)^\alpha w_\lambda^+(x) \geq (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x) = f(u_\lambda(x)) + g(x_\lambda) - f(u(x)) - g(x) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)}w_\lambda^+(x) + g(x_\lambda) - g(x).$$

Let us define $\varphi(x) = -(f(u_\lambda(x)) - f(u(x)))/(u_\lambda(x) - u(x))$ for $x \in \Sigma_\lambda^-$. By assumption $(F1)$, we have that $\varphi \in L^\infty(\Sigma_\lambda^-)$. By assumption $(G)$, we have that $g(x_\lambda) \geq g(x)$, since for all $x \in \Sigma_\lambda^-$ and $0 < \lambda < 1$, we have $|x| > |x_\lambda|$. Hence, we have

$$\Delta^\alpha w_\lambda^+(x) \leq \varphi(x)w_\lambda^+(x), \quad x \in \Sigma_\lambda^-$$

(3.9)

and since $w_\lambda^+ = 0$ in $(\Sigma_\lambda^-)^c$ we may apply Proposition 2.2. Choosing $\lambda \in (0,1)$ close enough to 1 we find that $|\Sigma_\lambda^-|$ is small and then

$$w_\lambda = w_\lambda^+ \geq 0 \quad \text{in} \quad \Sigma_\lambda^-.$$

But this is a contradiction with our assumption so we have

$$w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda.$$
Since \( x_0 \in \Sigma_\lambda \), we have \(|x_0| > |(x_0)_\lambda|\), then by assumption (G) we have \( g((x_0)_\lambda) \geq g(x_0) \) and thus
\[
(-\Delta)^\alpha w_\lambda(x_0) \geq 0. \tag{3.10}
\]
On the other hand, defining \( A_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda \} \), since \( w_\lambda(z_\lambda) = -w_\lambda(z) \) for any \( z \in \mathbb{R}^N \) and \( w_\lambda(x_0) = 0 \), we find
\[
(-\Delta)^\alpha w_\lambda(x_0) = -\int_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} \, dz - \int_{\mathbb{R}^N \setminus A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} \, dz
\]
\[
= -\int_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} \, dz - \int_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} \, dz
\]
\[
= -\int_{A_\lambda} w_\lambda(z) \left( \frac{1}{|x_0 - z|^{N+2\alpha}} - \frac{1}{|x_0 - z|^{N+2\alpha}} \right) \, dz.
\]
Since \(|x_0 - z_\lambda| > |x_0 - z|\) for \( z \in A_\lambda \), \( w_\lambda(z) \geq 0 \) and \( w_\lambda(z) \not\equiv 0 \) in \( A_\lambda \), from here we get
\[
(-\Delta)^\alpha w_\lambda(x_0) < 0, \tag{3.11}
\]
which contradicts (3.10), completing the proof of the claim.

**Step 2:** We define \( \lambda_0 = \inf \{ \lambda \in (0, 1) \mid w_\lambda > 0 \text{ in } \Sigma_\lambda \} \) and we prove that \( \lambda_0 = 0 \). Proceeding by contradiction, we assume that \( \lambda_0 > 0 \), then \( w_{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0} \) and \( w_{\lambda_0} \not\equiv 0 \) in \( \Sigma_{\lambda_0} \). Thus, by the claim just proved above, we have \( w_{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0} \).

Next we claim that if \( w_\lambda > 0 \) in \( \Sigma_\lambda \) for \( \lambda \in (0, 1) \), then there exists \( \epsilon \in (0, \lambda) \) such that \( w_{\lambda_\epsilon} > 0 \) in \( \Sigma_{\lambda_\epsilon} \), where \( \lambda_\epsilon = \lambda - \epsilon \). This claim directly implies that \( \lambda_0 = 0 \), completing Step 2.

Now we prove the claim. Let \( D_\mu = \{ x \in \Sigma_\lambda \mid \text{dist}(x, \partial \Sigma_\lambda) \geq \mu \} \) for \( \mu > 0 \) small. Since \( w_\lambda > 0 \) in \( \Sigma_\lambda \) and \( D_\mu \) is compact, then there exists \( \mu_0 > 0 \) such that \( w_\lambda \geq \mu_0 \) in \( D_\mu \). By continuity of \( w_\lambda(x) \), for \( \epsilon > 0 \) small enough and denoting \( \lambda_\epsilon = \lambda - \epsilon \), we have that
\[
w_{\lambda_\epsilon}(x) \geq 0 \text{ in } D_\mu.
\]
As a consequence,
\[
\Sigma_{\lambda_\epsilon} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu
\]
and \( |\Sigma_{\lambda_\epsilon}| \) is small if \( \epsilon \) and \( \mu \) are small. Using (3.7) and proceeding as in Step 1, we have for all \( x \in \Sigma_{\lambda_\epsilon} \) that
\[
(-\Delta)^\alpha w_{\lambda_\epsilon}^+(x) = (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x) - (-\Delta)^\alpha w_{\lambda_\epsilon}(x)
\]
\[
\geq (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x)
\]
\[
= \varphi(x) w_{\lambda_\epsilon}^+(x) + g(x_\lambda) - g(x) \geq \varphi(x) w_{\lambda_\epsilon}^+(x),
\]
11
2.2 implies that \( w \) before we have \( x \) for \( u \).

Step 3: By Step 2, we have \( x \) plane \( \phi \) where \( \lambda \) > 0 and \( w_{\lambda} \neq 0 \) in \( \Sigma_{\lambda} \), as before we have \( w_{\lambda} > 0 \) in \( \Sigma_{\lambda} \), completing the proof of the claim.

Finally, we prove \( u(r) \) is strictly decreasing in \( r \in (0, 1) \). Let us consider \( 0 < x_1 < \tilde{x}_1 < 1 \) and let \( \lambda = \frac{x_1 + \tilde{x}_1}{2} \). Then, as proved above we have

\[
\lambda_{\lambda}(x) > 0 \quad \text{for} \quad x \in \Sigma_{\lambda}.
\]

Then

\[
0 < \lambda_{\lambda}(\tilde{x}_1, 0, \ldots, 0) = \lambda_{\lambda}(\tilde{x}_1, 0, \ldots, 0) - \lambda_{\lambda}(0, 0, \ldots, 0)
\]

that is \( u(x_1, 0, \ldots, 0) > u(\tilde{x}_1, 0, \ldots, 0) \). Using the radial symmetry of \( u \), we conclude from here the monotonicity of \( u \). \( \square \)

The proof of Theorem 1.1 can be applied directly to prove symmetry results for problem (1.1) in more general domains. We have the following definition

**Definition 3.1** We say that domain \( \Omega \subset \mathbb{R}^N \) is convex in the \( x_1 \) direction:

\[
(x_1, x'), (x_1, y') \in \Omega \Rightarrow (x_1, tx' + (1 - t)y') \in \Omega, \quad \forall t \in (0, 1).
\]

Now we state the more general theorem:

**Theorem 3.1** Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is an open and bounded set. Assume further that \( \Omega \) is convex in the \( x_1 \) direction and symmetric with respect to the plane \( x_1 = 0 \). Assume that the function \( f \) satisfies (F1) and \( g \) satisfies

(\(G\)) The function \( g : \Omega \rightarrow \mathbb{R} \) is symmetric with respect to \( x_1 = 0 \) and decreasing in the \( x_1 \) direction, for \( x = (x_1, x') \in \Omega, \ x_1 > 0 \).

Let \( u \) be a positive classical solution of

\[
\begin{cases}
(-\Delta)^{\mu} u(x) = f(u(x)) + g(x), & x \in \Omega, \\
u(x) = 0, & x \in \Omega^c.
\end{cases}
\tag{3.12}
\]

Then \( u \) is symmetric with respect to \( x_1 \) and it is strictly decreasing in the \( x_1 \) direction for \( x = (x_1, x') \in \Omega, \ x_1 > 0 \).
4 Symmetry of solutions in $\mathbb{R}^N$

In this section we study radial symmetry results for positive solution of equation (1.5) in $\mathbb{R}^N$, in particular we will provide a proof of Theorem 1.2. In the case of the whole space, the moving planes procedure needs to be started in a different way, because we cannot use the Maximum Principle for small domains. We use the moving plane method as for the second order equation as in the work by Li [17] (see also [21]).

In this section we use the notation introduced in (3.1)-(3.3) and we let $u$ be a classical positive solution of (1.5). In order to prove Theorem 1.2 we need some preliminary lemmas.

**Lemma 4.1** Under the assumptions of Theorem 1.2, for any $\lambda \in \mathbb{R}$, we have

$$\int_{\Sigma_\lambda} (f(u_\lambda) - f(u))^+(u_\lambda - u)^+ \, dx < +\infty.$$  

**Proof.** By our hypothesis, for any given $\lambda \in \mathbb{R}$, we may choose $R > 1$ and some constant $c > 1$ such that

$$\frac{1}{c|x|^m} \leq u(x), \quad u_\lambda(x) \leq \frac{c}{|x|^m} < s_0 \quad \text{for all } x \in B_R^c,$$

where $s_0$ is the constant in condition (F2).

If $u_\lambda(x) > u(x)$ for some $x \in \Sigma_\lambda \cap B_R^c$, we have $0 < u(x) < u_\lambda(x) < s_0$. Using (1.4) with $v = u_\lambda(x)$, then

$$\frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} \leq C(u(x) + u_\lambda(x))^\gamma \leq 2^\gamma Cu_\lambda^\gamma(x),$$

then

$$(f(u_\lambda(x)) - f(u(x)))^+(u_\lambda(x) - u(x))^+ \leq 2^\gamma Cu_\lambda^\gamma(x)[(u_\lambda(x) - u(x))^+]^2 \leq \tilde{C}u_\lambda^{\gamma+2}(x),$$

for certain $\tilde{C} > 0$. We observe that, if $u_\lambda(x) \leq u(x)$ for some $x \in \Sigma_\lambda \cap B_R^c$, then inequality above is obvious. Therefore,

$$(f(u_\lambda) - f(u))^+(u_\lambda - u)^+ \leq \tilde{C}u_\lambda^{\gamma+2} \quad \text{in } \Sigma_\lambda \cap B_R^c.$$  

Now we integrate in $\Sigma_\lambda \cap B_R^c$ to obtain

$$\int_{\Sigma_\lambda \cap B_R^c} (f(u_\lambda) - f(u))^+(u_\lambda - u)^+ \, dx \leq \tilde{C} \int_{\Sigma_\lambda \cap B_R^c} u_\lambda^{\gamma+2}(x) \, dx \leq C \int_{B_R^c} |x|^{-m(\gamma+2)} \, dx < +\infty,$$  

where the last inequality holds by (1.6). Since $u$ and $u_\lambda$ are bounded and $f$ is locally Lipschitz, we have

$$\int_{\Sigma_\lambda \cap B_R} (f(u_\lambda) - f(u))^+(u_\lambda - u)^+dx < +\infty$$

and the proof is complete. \qed

It will be convenient for our analysis to define the following function

$$w(x) = \begin{cases} (u_\lambda - u)^+(x), & x \in \Sigma_\lambda, \\ (u_\lambda - u)^-(x), & x \in \Sigma_\lambda^c, \end{cases} \quad (4.1)$$

where $(u_\lambda - u)^+(x) = \max\{(u_\lambda - u)(x), 0\}$, $(u_\lambda - u)^-(x) = \min\{(u_\lambda - u)(x), 0\}$. We have

**Lemma 4.2** Under the assumptions of Theorem 1.2, there exists a constant $C > 0$ such that

$$\int_{\Sigma_\lambda} (-\Delta)^\alpha (u_\lambda - u)(u_\lambda - u)^+dx \geq C(\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}}. \quad (4.2)$$

**Proof.** We start observing that, given $x \in \Sigma_\lambda$, we have

$$w(x_\lambda) = (u_\lambda - u)^-(x_\lambda) = \min\{(u_\lambda - u)(x_\lambda), 0\} = \min\{(u - u_\lambda)(x), 0\} = -\max\{(u_\lambda - u)(x), 0\} = -(u_\lambda - u)^+(x) = -w(x)$$

and similarly $w(x) = -w(x_\lambda)$ for $x \in \Sigma_\lambda^c$ so that

$$w(x) = -w(x_\lambda) \quad \text{for} \quad x \in \mathbb{R}^N. \quad (4.3)$$

This implies

$$\int_{\mathbb{R}^N} |w|^{\frac{2N}{N-2\alpha}} dx = \int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx + \int_{\Sigma_\lambda^c} |w|^{\frac{2N}{N-2\alpha}} dx = 2 \int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx. \quad (4.4)$$

Next we see that for any $x \in \Sigma_\lambda \cap \text{supp}(w)$ we have that $w(x) = (u_\lambda - u)(x)$ and

$$(-\Delta)^\alpha (u_\lambda - u)(x) \geq (-\Delta)^\alpha w(x), \quad \forall x \in \Sigma_\lambda \cap \text{supp}(w),$$

$$(-\Delta)^\alpha w(x) - (-\Delta)^\alpha (u_\lambda - u)(x) = \int_{\mathbb{R}^N} \frac{(u_\lambda - u)(z) - w(z)}{|x - z|^{N+2\alpha}}dz$$

$$= \int_{\Sigma_\lambda \cap \text{supp}(w)^c} \frac{(u_\lambda - u)(z)}{|x - z|^{N+2\alpha}}dz + \int_{\Sigma_\lambda^c \cap \text{supp}(w)^c} \frac{(u_\lambda - u)(z)}{|x - z|^{N+2\alpha}}dz$$

$$= \int_{\Sigma_\lambda \cap \text{supp}(w)^c} (u_\lambda - u)(z)(\frac{1}{|x - z|^{N+2\alpha}} - \frac{1}{|x - z\lambda|^{N+2\alpha}})dz \leq 0. \quad (4.5)$$

14
where we used that $u_{\lambda} - u \leq 0$ in $\Sigma_{\lambda} \cap (\text{supp}(w))^c$ and $|x - z| \leq |x - z_{\lambda}|$ for $x, z \in \Sigma_{\lambda}$. From (4.5), using the equation and Lemma 4.1 we find that

$$\int_{\Sigma_{\lambda}} (\Delta)^{\alpha} w \, w \, dx \leq \int_{\Sigma_{\lambda}} (\Delta)^{\alpha} u_{\lambda} - u \, (u_{\lambda} - u)^+ \, dx$$

(4.6)

$$\leq \int_{\Sigma_{\lambda}} (f(u_{\lambda}) - f(u))^+ (u_{\lambda} - u)^+ \, dx < \infty.$$  

(4.7)

From here the following integrals are finite and, taking into account (4.3), we obtain that

$$\int_{\mathbb{R}^N} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx = \int_{\Sigma_{\lambda}} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx + \int_{\Sigma_{\lambda}^c} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx$$

(4.8)

$$= 2 \int_{\Sigma_{\lambda}} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx.$$  

Now we can use the Sobolev embedding from $H^{\alpha}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-2\alpha}}(\mathbb{R}^N)$ to find a constant $C$ so that

$$\int_{\Sigma_{\lambda}} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^N} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx \geq C \left( \int_{\mathbb{R}^N} |w|^{\frac{2N}{N-2\alpha}} \, dx \right)^{\frac{N-2\alpha}{N}}$$

(4.9)

On the other hand, from (4.3) and (4.6) we find that

$$\int_{\mathbb{R}^N} |(\Delta)^{\frac{\alpha}{2}} w|^2 \, dx = \int_{\mathbb{R}^N} (\Delta)^{\alpha} w \cdot w \, dx = 2 \int_{\Sigma_{\lambda}} (\Delta)^{\alpha} w \cdot w \, dx$$

$$\leq 2 \int_{\Sigma_{\lambda}} (\Delta)^{\alpha} (u_{\lambda} - u) (u_{\lambda} - u)^+ \, dx.$$  

(4.10)

From (4.9) and (4.10) the proof of the lemma is completed.□

Now we are ready to complete the Proof of Theorem 1.2. We divide the proof into three steps.

Step 1: We show that $\lambda_0 := \sup\{\lambda \mid u_{\lambda} \leq u\ in \ \Sigma_{\lambda}\}$ is finite. Using $(u_{\lambda} - u)^+$ as a test function in the equation for $u$ and $u_{\lambda}$, using (1.4) and Hölder inequality, for $\lambda$ big (negative), we find that

$$\int_{\Sigma_{\lambda}} (\Delta)^{\alpha} (u_{\lambda} - u) (u_{\lambda} - u)^+ \, dx = \int_{\Sigma_{\lambda}} (f(u_{\lambda}) - f(u)) (u_{\lambda} - u)^+ \, dx$$


\[
\int_{\Sigma_{\lambda}} \left[ \frac{f(u_{\lambda}) - f(u)}{u_{\lambda} - u} \right]^+ [(u_{\lambda} - u)^+]^2 dx \\
\leq C \int_{\Sigma_{\lambda}} u_{\lambda}^2 w^2 dx \leq \bar{C} \int_{\Sigma_{\lambda}} |x_{\lambda}|^{-2m_\gamma} w^2 dx \\
\leq \bar{C} \left( \int_{\Sigma_{\lambda}} |x_{\lambda}|^{-\frac{Nm_{\gamma}}{2a}} dx \right) \left( \int_{\Sigma_{\lambda}} |w| \frac{2N}{N-2a} dx \right)^{\frac{N-2a}{N}}.
\]

By Lemma 4.2, there exists a constant \( C > 0 \) such that
\[
\left( \int_{\Sigma_{\lambda}} |w|^{\frac{2N}{N-2a}} dx \right)^{\frac{N-2a}{N}} \leq C \left( \int_{\Sigma_{\lambda}} |x_{\lambda}|^{-\frac{Nm_{\gamma}}{2a}} dx \right)^{\frac{N-2a}{N}} \left( \int_{\Sigma_{\lambda}} |w|^{\frac{2N}{N-2a}} dx \right)^{\frac{N-2a}{N}},
\]
but we have
\[
\int_{\Sigma_{\lambda}} |x_{\lambda}|^{-\frac{Nm_{\gamma}}{2a}} dx \leq \int_{\Sigma_{\lambda}} |x|^{-\frac{Nm_{\gamma}}{2a}} dx \leq \int_{B_{r_{\lambda}}} |x|^{-\frac{Nm_{\gamma}}{2a}} dx = c |\lambda| \|N^2(w_{\lambda})\|^{2a-m_{\gamma}},
\]
so that, using (1.6), we can choose \( R > 0 \) big enough such that \( CR^{2a-m_{\gamma}} \leq \frac{1}{2} \), then we obtain
\[
\int_{\Sigma_{\lambda}} |w|^{\frac{2N}{N-2a}} dx = 0, \quad \forall \lambda < -R.
\]
Thus \( w = 0 \) in \( \Sigma_{\lambda} \) and then \( u_{\lambda} \leq u \) in \( \Sigma_{\lambda} \), for all \( \lambda < -R \), concluding that \( \lambda_0 \geq -R \). On the other hand, since \( u \) decays at infinity, then there exists \( \lambda_1 \) such that \( u(x) < u_{\lambda_1}(x) \) for some \( x \in \Sigma_{\lambda_1} \). Hence \( \lambda_0 \) is finite.

**Step 2:** We prove that \( u \equiv u_{\lambda_0} \) in \( \Sigma_{\lambda_0} \). Assuming the contrary, we have \( u \neq u_{\lambda_0} \) and \( u \geq u_{\lambda_0} \) in \( \Sigma_{\lambda_0} \). Assume next that there exists \( x_0 \in \Sigma_{\lambda_0} \) such that \( u_{\lambda_0}(x_0) = u(x_0) \), then we have
\[
(-\Delta)^{\alpha} u_{\lambda_0}(x_0) - (-\Delta)^{\alpha} u(x_0) = f(u_{\lambda_0}(x_0)) - f(u(x_0)) = 0. \tag{4.11}
\]

On the other hand,
\[
(-\Delta)^{\alpha} u_{\lambda_0}(x_0) - (-\Delta)^{\alpha} u(x_0) = - \int_{\mathbb{R}^N} \frac{u_{\lambda_0}(y) - u(y)}{|x_0 - y|^{N+2\alpha}} dy \\
= - \int_{\Sigma_{\lambda_0}} (u_{\lambda_0}(y) - u(y)) \left( \frac{1}{|x_0 - y|^{N+2\alpha}} - \frac{1}{|x_0 - y_{\lambda_0}|^{N+2\alpha}} \right) dy > 0,
\]
which contradicts (4.11). As a sequence, \( u > u_{\lambda_0} \) in \( \Sigma_{\lambda_0} \).

To complete Step 2, we only need to prove that \( u \geq u_{\lambda} \) in \( \Sigma_{\lambda} \) continues to hold when \( \lambda_0 < \lambda < \lambda_0 + \varepsilon \), where \( \varepsilon > 0 \) small. Let us consider then \( \varepsilon > 0 \), to be chosen later, and take \( \lambda \in (\lambda_0, \lambda_0 + \varepsilon) \). Let \( P = (\lambda, 0) \) and \( B(P, R) \) be the ball centered at \( P \) and with radius \( R > 1 \) to be chosen later. Define
\[ \hat{B} = \Sigma \cap B(P, R) \] and let us consider \((u_\lambda - u)^+\) test function in the equation for \(u\) and \(u_\lambda\) in \(\Sigma\), then from Lemma 4.2 we find

\[ \left( \int_{\Sigma \lambda} |w|^{\frac{2N}{N - 2m}} dx \right)^{\frac{N-2n}{N}} \leq C \int_{\Sigma \lambda} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx. \quad (4.12) \]

We estimate the integral on the right. Since \(f\) is locally Lipschitz, using Hölder inequality, we have

\[ \int_{\hat{B}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx \leq C \int_{\hat{B}} |w|^{\frac{2N}{N - 2m}} dx \left( \int_{\hat{B}} |\chi_{\text{supp}(u_\lambda - u)^+}|^{\frac{2N}{N - 2m}} dx \right)^{\frac{N-2n}{N}}. \quad (4.13) \]

On the other hand, for the integral over \(\Sigma \lambda \setminus \hat{B}\), we assume \(R\) and \(R_0\) are such that \(\Sigma \lambda \setminus \hat{B} \subset B^c(P, R) \subset B^c_{R_0}(0)\), proceeding as in Step 1, we have

\[ \int_{\Sigma \lambda \setminus \hat{B}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx \leq C \int_{\Sigma \lambda \setminus \hat{B}} u_\lambda^2 w^2 dx \leq C \left( \int_{\Sigma \lambda \setminus \hat{B}} |x|^{\frac{Nm\gamma}{2N - 2m}} dx \right)^{\frac{N}{N - 2m}} \left( \int_{\Sigma \lambda} |w|^{\frac{2N}{N - 2m}} dx \right)^{\frac{N-2n}{N}} \]

\[ \leq CR_0^{2\alpha - m\gamma} \left( \int_{\Sigma \lambda} |w|^{\frac{2N}{N - 2m}} dx \right)^{\frac{N-2n}{N}}. \quad (4.14) \]

Now we choose \(R_0\) such that \(CR_0^{2\alpha - m\gamma} < 1/2\), then choose \(R\) so that \(\Sigma \lambda \setminus \hat{B} \subset B^c(P, R) \subset B^c_{R_0}(0)\) and then choose \(\varepsilon > 0\) so that \(C |\hat{B} \cap \text{supp}(u_\lambda - u)^+|^{\frac{2N}{N - 2m}} < 1/2\). With this choice of the parameters, from (4.12), (4.13) and (4.14) it follows that \(w = 0\) in \(\Sigma \lambda\), which is a contradiction, completing Step 2.

**Step 3:** By translation, we may say that \(\lambda_0 = 0\). An repeating the argument from the other side, we find that \(u\) is symmetric about \(x_1\)-axis. Using the same argument in any arbitrary direction, we finally conclude that \(u\) is radially symmetric.

Finally, we prove that \(u(r)\) is strictly decreasing in \(r > 0\), by using the same arguments as in the case of a ball. This completes the proof. \(\square\)

At the end of this section we want to give a theorem on radial symmetry of solutions for equation (1.5) in a case where \(f\) is only locally Lipschitz in \((0, \infty)\), see [11] and [10] for the case of the Laplacian. In precise terms we have
Theorem 4.1 Let $u$ be a positive classical solution of
\[
\begin{cases}
(-\Delta)^\alpha u = u^p - u^q & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0,
\end{cases}
\]
satisfying
\[
u(x) = O\left(|x|^{-\frac{N+2\alpha}{q}}\right) \quad \text{as } |x| \to \infty,
\]
where $\alpha \in (0, 1)$, $N \geq 2$ and $0 < q < 1 < p$. Then $u$ is radially symmetric and strictly decreasing about some point.

Proof. We denote $f(u) = u^p - u^q$ for $u > 0$, and consider $\gamma > 0$ and $s_0$ small enough, then for all $u, v$ satisfying $0 < u < v < s_0$, we have
\[
\frac{f(v) - f(u)}{v - u} < 0 \leq C(u + v)^\gamma,
\]
for some constant $C > 0$, so that (F2) holds. We also observe that for a positive classical solution $u$ of (4.15), $u \geq c$ in any bounded domain $\Omega$, for a constant $c > 0$ depending on $\Omega$ and then, in (4.13) we may use Lipschitz continuity of $f$ in the bounded interval $[c, \sup u]$. We set $m = \frac{N+2\alpha}{q}$ and $\gamma$ may be chosen so that (1.6) holds. The proof of Theorem 4.1 goes in the same way as that of Theorem 1.2. □

Remark 4.1 In a work by Valdebenito [27], the estimate (4.16) is obtained by using super solutions and Theorem 4.1 is proved using the local extension of equation (4.15) as given by Caffarelli and Silvestre in [5] and then using a regular moving planes argument as developed for elliptic equations with non-linear boundary conditions by Terracini [26].

5 Symmetry results for system

The aim of this section is to prove Theorem 1.3 by the moving planes method applied to a system of equations in the unit ball $B_1$. Let $\Sigma_\lambda$ and $T_\lambda$ be defined as in Section 3. For $x = (x_1, x') \in \mathbb{R}^N$ and $\lambda \in (0, 1)$ we let $x_\lambda = (2\lambda - x_1, x')$,
\[
u_\lambda(x) = u(x_\lambda), \quad w_{\lambda,u}(x) = u_\lambda(x) - u(x),
\]
\[
u_\lambda(x) = v(x_\lambda), \quad \text{and} \quad w_{\lambda,v}(x) = v_\lambda(x) - v(x).
\]

Proof of Theorem 1.3. We will split this proof into three steps.
Step 1: We start the moving planes proving that if \( \lambda \) is close to 1, then \( w_{\lambda,u} \) and \( w_{\lambda,v} \) are positive in \( \Sigma_{\lambda} \). For that purpose we define

\[
\Sigma_{\lambda,u}^{-} = \{ x \in \Sigma_{\lambda} \mid w_{\lambda,u}(x) < 0 \} \quad \text{and} \quad \Sigma_{\lambda,v}^{-} = \{ x \in \Sigma_{\lambda} \mid w_{\lambda,u}(x) < 0 \}.
\]

We show next that \( \Sigma_{\lambda,u}^{-} \) is empty for \( \lambda \) close to 1. Assume, by contradiction, that \( \Sigma_{\lambda,u}^{-} \) is not empty and define

\[
w_{\lambda,u}^+(x) = \begin{cases} w_{\lambda,u}(x), & x \in \Sigma_{\lambda,u}^{-} \backslash \Sigma_{\lambda,u}^- \smallskip \text{and} \\ 0, & x \in \mathbb{R}^N \backslash \Sigma_{\lambda,u}^- \end{cases}
\]

and

\[
w_{\lambda,u}^-(x) = \begin{cases} 0, & x \in \Sigma_{\lambda,u}^{-} \backslash \Sigma_{\lambda,u}^- \smallskip \text{and} \\ w_{\lambda,u}(x), & x \in \mathbb{R}^N \backslash \Sigma_{\lambda,u}^- \end{cases}
\]

Using the arguments given in Step 1 of the proof of Theorem 1.1 we get

\[
(-\Delta)^{\alpha_1} w_{\lambda,u}^+(x) \geq (-\Delta)^{\alpha_1} w_{\lambda,u}(x) \quad \text{and} \quad (-\Delta)^{\alpha_1} w_{\lambda,u}^-(x) \leq 0,
\]

for all \( x \in \Sigma_{\lambda,u}^- \). From here, using equation (1.8), for \( x \in \Sigma_{\lambda,u}^- \) we have

\[
(-\Delta)^{\alpha_1} w_{\lambda,u}^+(x) \geq (-\Delta)^{\alpha_1} u(x) - (-\Delta)^{\alpha_1} u(x) \\
= f_1(v_{\lambda}(x)) + g_1(x_{\lambda}) - f_1(v(x)) - g_1(x) \\
= \varphi_v(x) w_{\lambda,v}(x) + g_1(x_{\lambda}) - g_1(x) \\
\geq \varphi_v(x) w_{\lambda,v}(x),
\]

where \( \varphi_v(x) = (f_1(v_{\lambda}(x)) - f_1(v(x)))/(v_{\lambda}(x) - v(x)) \) and where we used that \( g_1 \) is radially symmetric and decreasing, with \( |x| > |x_{\lambda}| \). We further observe that, since \( f_1 \) is locally Lipschitz continuous, we have that \( \varphi_v(\cdot) \in L^{\infty}(\Sigma_{\lambda,u}^-) \).

Now we consider (5.4) together with \( w_{\lambda,v}^+ = 0 \) in \( (\Sigma_{\lambda,u}^-)^c \) and \( w_{\lambda,v}^+ < 0 \) in \( \Sigma_{\lambda,v}^- \), to use Proposition 2.1 to find a constant \( C > 0 \), depending on \( N \) and \( \alpha_1 \) only, such that

\[
\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C\|(-\varphi_v w_{\lambda,v})^+\|^1_{L^\infty(\Sigma_{\lambda,u}^-)} \|(-\varphi_v w_{\lambda,v})^+\|^\alpha_1_{L^\infty(\Sigma_{\lambda,u}^-)}
\]

We observe that \( diam(\Sigma_{\lambda,u}^-) \leq 1 \). Since \( f_1 \) is increasing, we have

\[
-\varphi_v w_{\lambda,v} = f_1(v) - f_1(v_{\lambda}) \leq 0 \quad \text{in} \quad (\Sigma_{\lambda,u}^-)^c \quad \text{and} \quad \text{(5.6)}
\]

\[
-\varphi_v w_{\lambda,v} = f_1(v) - f_1(v_{\lambda}) > 0 \quad \text{in} \quad \Sigma_{\lambda,v}^- \quad \text{and} \quad \text{(5.7)}
\]

Denoting \( \Sigma_{\lambda}^- = \Sigma_{\lambda,u}^- \cap \Sigma_{\lambda,v}^- \), from (5.5), (5.6) and (5.7), we obtain

\[
\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C\|(-\varphi_v w_{\lambda,v})^+\|^1_{L^\infty(\Sigma_{\lambda}^-)} \|(-\varphi_v w_{\lambda,v})^+\|^\alpha_1_{L^\infty(\Sigma_{\lambda}^-)}
\]

19
Similar to (5.1) and (5.2), we define
\[ w_{\lambda,v}^+(x) = \begin{cases} w_{\lambda,v}(x), & x \in \Sigma_{\lambda,v}^- \\ 0, & x \in \mathbb{R}^N \setminus \Sigma_{\lambda,v}^- \end{cases} \]
and
\[ w_{\lambda,v}^-(x) = \begin{cases} 0, & x \in \Sigma_{\lambda,v}^- \\ w_{\lambda,v}(x), & x \in \mathbb{R}^N \setminus \Sigma_{\lambda,v}^- \end{cases} \]
With this definition (5.8) becomes
\[ \| w_{\lambda,u}^+ \|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C \| w_{\lambda,v}^+ \|_{L^\infty(\Sigma_{\lambda,v}^-)} |\Sigma_{\lambda,v}^-|^\frac{\alpha_1}{\alpha_2}, \] (5.9)
where we used that \( \varphi_v \) is bounded and we have changed the constant \( C \), if necessary. At this point we observe that if \( w_{\lambda,v}^+ = 0 \) then \( w_{\lambda,u}^+ = 0 \) providing a contradiction. Thus we have that \( \Sigma_{\lambda,v}^- \neq \emptyset \) and we may argue in a completely analogous way to obtain
\[ \| w_{\lambda,u}^+ \|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C \| w_{\lambda,u}^+ \|_{L^\infty(\Sigma_{\lambda,u}^-)} |\Sigma_{\lambda,u}^-|^\frac{\alpha_1}{\alpha_2}, \] (5.10)
that combined with (5.9) yields
\[ \| w_{\lambda,u}^+ \|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C^2 |\Sigma_{\lambda,v}^-|^\frac{\alpha_1+\alpha_2}{\alpha_2} \| w_{\lambda,u}^+ \|_{L^\infty(\Sigma_{\lambda,u}^-)}, \]
\[ \| w_{\lambda,v}^+ \|_{L^\infty(\Sigma_{\lambda,v}^-)} \leq C^2 |\Sigma_{\lambda,u}^-|^\frac{\alpha_1+\alpha_2}{\alpha_2} \| w_{\lambda,v}^+ \|_{L^\infty(\Sigma_{\lambda,v}^-)}. \]
Now we just take \( \lambda \) close enough to 1 so that \( C^2 |\Sigma_{\lambda,v}^-|^\frac{\alpha_1+\alpha_2}{\alpha_2} < 1 \) and we conclude that \( \| w_{\lambda,u}^+ \|_{L^\infty(\Sigma_{\lambda,u}^-)} = \| w_{\lambda,v}^+ \|_{L^\infty(\Sigma_{\lambda,v}^-)} = 0 \), so \( |\Sigma_{\lambda,u}^-| = |\Sigma_{\lambda,v}^-| = 0 \) and since \( \Sigma_{\lambda,u}^- \) and \( \Sigma_{\lambda,v}^- \) are open we have that \( \Sigma_{\lambda,u}^-, \Sigma_{\lambda,v}^- = \emptyset \), which is a contradiction.

Thus we have that \( w_{\lambda,u} \geq 0 \) in \( \Sigma_{\lambda} \) when \( \lambda \) is close enough to 1. Similarly, we obtain \( w_{\lambda,v} \geq 0 \) in \( \Sigma_{\lambda} \) for \( \lambda \) close to 1. In order to complete Step 1 we will prove a bit more general statement that will be useful later, that is, given \( 0 < \lambda < 1 \), if \( w_{\lambda,u} \geq 0, w_{\lambda,v} \geq 0, w_{\lambda,u} \neq 0 \) and \( w_{\lambda,v} \neq 0 \) in \( \Sigma_{\lambda} \), then \( w_{\lambda,u} > 0 \) and \( w_{\lambda,v} > 0 \) in \( \Sigma_{\lambda} \). For proving this property suppose there exists \( x_0 \in \Sigma_{\lambda} \) such that
\[ w_{\lambda,u}(x_0) = 0. \] (5.11)
On one hand, by using similar arguments yielding (3.11) we find that
\[ (-\Delta)^{\alpha_1} w_{\lambda,u}(x_0) < 0. \] (5.12)
On the other hand, by our assumption we have that \( w_{\lambda,v}(x_0) = v_{\lambda}(x_0) - v(x_0) \geq 0 \) and since \( |x_0| > |(x_0)_{\lambda}| \), from the monotonicity hypothesis on \( f_1 \) and \( g_1 \), we obtain
\[
f_1(v_\lambda(x_0)) \geq f_1(v(x_0)), \quad g_1((x_0)_{\lambda}) \geq g_1(x_0).
\]
Thus, using (1.8), we find
\[
(-\Delta)^{\alpha_1} w_{\lambda,u}(x_0) = f_1(v_\lambda(x_0)) + g_1((x_0)_{\lambda}) - f_1(v(x_0)) - g_1(x_0) \geq 0,
\]
which is impossible with (5.12). This completes Step 1.

**Step 2:** We prove that \( \lambda_0 = 0 \), where
\[
\lambda_0 = \inf\{\lambda \in (0,1) \mid w_{\lambda,u} \ , \ w_{\lambda,v} > 0 \ \text{in} \ \Sigma_\lambda\}.
\]
If not, that is, if \( \lambda_0 > 0 \) we have that \( w_{\lambda_0,u}, w_{\lambda_0,v} \geq 0 \) and \( w_{\lambda_0,u}, w_{\lambda_0,v} \not= 0 \in \Sigma_{\lambda_0} \). If we use the property we just proved above, we may assume that \( w_{\lambda_0,u} > 0 \) and \( w_{\lambda_0,v} > 0 \in \Sigma_{\lambda_0} \). In what follows we argue that the plane can be moved to left, that is, that there exists \( \epsilon \in (0,\lambda) \) such that \( w_{\lambda,u} > 0 \) and \( w_{\lambda,v} > 0 \) in \( \Sigma_{\lambda} \), where \( \lambda_\epsilon = \lambda_0 - \epsilon \), providing a contradiction with the definition of \( \lambda_0 \).

Let us consider the set \( D_\mu = \{x \in \Sigma_\lambda \mid \text{dist}(x, \partial \Sigma_\lambda) \geq \mu\} \) for \( \mu > 0 \) small. Since \( w_{\lambda,u}, w_{\lambda,v} > 0 \) in \( \Sigma_\lambda \) and \( D_\mu \) is compact, then there exists \( \mu_0 > 0 \) such that \( w_{\lambda,u}, w_{\lambda,v} \geq \mu_0 \) in \( D_\mu \). By continuity of \( w_{\lambda,u}(x) \) and \( w_{\lambda,v}(x) \), for \( \epsilon > 0 \) small enough, we have that
\[
w_{\lambda_\epsilon,u}, \ w_{\lambda_\epsilon,v} \geq 0 \ \text{in} \ D_\mu.
\]
and, as a consequence, \( \Sigma_{\lambda_\epsilon,u}, \Sigma_{\lambda_\epsilon,v} \subset \Sigma_{\lambda} \setminus D_\mu \), and \( |\Sigma_{\lambda_\epsilon,u}| \) and \( |\Sigma_{\lambda_\epsilon,v}| \) are small if \( \epsilon \) and \( \mu \) are small.

Since \( f_1 \) and \( f_2 \) are locally Lipschitz continuous and increasing, \( g_1 \) and \( g_2 \) are radially symmetric and decreasing, we may repeat the arguments given in Step 1 to obtain
\[
\|w_{\lambda_\epsilon,u}^+\|_{L^\infty(\Sigma_{\lambda_\epsilon,u})} \leq C^2|\Sigma_{\lambda_\epsilon}|^{-\frac{\alpha_1+\alpha_2}{8}} \|w_{\lambda_0,u}^+\|_{L^\infty(\Sigma_{\lambda_0,u})},
\]
and
\[
\|w_{\lambda_\epsilon,v}^+\|_{L^\infty(\Sigma_{\lambda_\epsilon,v})} \leq C^2|\Sigma_{\lambda_\epsilon}|^{-\frac{\alpha_1+\alpha_2}{8}} \|w_{\lambda_0,v}^+\|_{L^\infty(\Sigma_{\lambda_0,v})},
\]
where \( \Sigma_{\lambda_\epsilon} = \Sigma_{\lambda_\epsilon,u} \cap \Sigma_{\lambda_\epsilon,v} \). Now we may choose \( \epsilon \) and \( \mu \) small such that
\[
C^2|\Sigma_{\lambda_\epsilon}|^{-\frac{\alpha_1+\alpha_2}{8}} < 1,
\]
then we obtain \( \|w_{\lambda_\epsilon,u}^+\|_{L^\infty(\Sigma_{\lambda_\epsilon,u})} = \|w_{\lambda_\epsilon,v}^+\|_{L^\infty(\Sigma_{\lambda_\epsilon,v})} = 0. \)
From here we argue as in Step 1 to obtain that \( w_{\lambda_\epsilon,u} \) and \( w_{\lambda_\epsilon,v} \) are positive in \( \Sigma_\lambda \), completing Step 2.

Finally, we obtain that \( u \) and \( v \) are radially symmetric and strictly decreasing respect to \( r = |x| \) for \( r \in (0,1) \) in the same way in Step 3 in the proof of Theorem 1.1. \( \square \)
6 The case of a non-local operator with non-homogeneous kernel.

The main purpose of this section is to discuss radial symmetry for a problem with a non-local operator \( \mathcal{L} \) of fractional order, but with a non-homogeneous kernel. The operator is defined as follows:

\[
\mathcal{L} u(x) = P.V. \int_{\mathbb{R}^N} (u(x) - u(y))K_\mu(x - y)dy,
\]

where the kernel \( K_\mu \) satisfies that

\[
K_\mu(x) = \begin{cases} 
\frac{1}{|x|^{N+2\alpha_1}}, & |x| < 1 \\
\frac{\mu}{|x|^{N+2\alpha_2}}, & |x| \geq 1
\end{cases}
\]

with \( \mu \in [0, 1] \) and \( \alpha_1, \alpha_2 \in (0, 1) \). Being more precise, we consider the equation

\[
\begin{cases}
\mathcal{L} u(x) = f(u(x)) + g(x), & x \in B_1, \\
u(x) = 0, & x \in B_1^c,
\end{cases}
\]

and our theorem states

**Theorem 6.1** Assume that the function \( f \) satisfies (F1) and \( g \) satisfies (G). If \( u \) is a positive classical solution of (6.3), then \( u \) must be radially symmetric and strictly decreasing in \( r = |x| \) for \( r \in (0, 1) \).

The idea for Theorem 6.1 is to take advantage of the fact that the non-local operator \( \mathcal{L} \) differs from the fractional Laplacian by a zero order operator. Using this idea, we obtain a Maximum Principle for domains with small volume through the ABP-estimate given Proposition 2.1 and we are able to use the moving planes method as in the case of the fractional Laplacian. We prove first

**Proposition 6.1** Let \( \Sigma_\lambda \) and \( \Sigma_\lambda^- \) be defined as in the Section §3. Suppose that \( \varphi \in L^\infty(\Sigma_\lambda) \) and that \( w_\lambda \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) is a solution of

\[
\begin{cases}
-\mathcal{L} w_\lambda(x) \leq \varphi(x)w_\lambda(x), & x \in \Sigma_\lambda, \\
w_\lambda(x) \geq 0, & x \in \mathbb{R}^N \setminus \Sigma_\lambda,
\end{cases}
\]

where \( \mathcal{L} \) was defined in (6.1). Then, if \( |\Sigma_\lambda^-| \) is small enough, \( w_\lambda \) is non-negative in \( \Sigma_\lambda \), that is,

\[
w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda.
\]
Proof. We define $w_\lambda^+(x)$ as in (3.5), then we have

$$
\mathcal{L}w_\lambda^+(x) = \int_{B_1(x)} \frac{w_\lambda^+(x) - w_\lambda^+(z)}{|x - z|^{N+2\alpha_1}} dz + \mu \int_{\mathbb{R}^N \setminus B_1(x)} \frac{w_\lambda^+(x) - w_\lambda^+(z)}{|x - z|^{N+2\alpha_2}} dz
$$

$$
= (-\Delta)^{\alpha_1}w_\lambda^+(x)
$$

$$
+ \int_{\mathbb{R}^N \setminus B_1(x)} (w_\lambda^+(x) - w_\lambda^+(z))((\frac{\mu}{|x - z|^{N+2\alpha_2}} - \frac{1}{|x - z|^{N+2\alpha_1}}) dz
$$

$$
\leq (-\Delta)^{\alpha_1}w_\lambda^+(x) + 2C_0\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}, \quad x \in \Sigma^-_\lambda,
$$

where $C_0 = \int_{\mathbb{R}^N \setminus B_1} |\mu|_{|y|^{N+2\alpha_2}} - |y|^{N+2\alpha_1}|dy$. Thus we have

$$
\Delta^{\alpha_1}w_\lambda^+(x) \leq -\mathcal{L}w_\lambda^+(x) + 2C_0\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}, \quad x \in \Sigma^-_\lambda.
$$

(6.5)

Since $K_\mu$ is radially symmetric and decreasing in $|x|$, we may repeat the arguments used to prove (3.7) to get

$$
\mathcal{L}w_\lambda^-(x) \leq 0, \quad \forall \ x \in \Sigma^-_\lambda,
$$

(6.6)

where $0 < \lambda < 1$ and $w_\lambda^-$ was defined in (3.6). Using (6.5), the linearity of $\mathcal{L}$, (6.6) and equation (6.4), for all $x \in \Sigma^-_\lambda$, we have

$$
\Delta^{\alpha_1}w_\lambda^+(x) \leq -\mathcal{L}w_\lambda^+(x) + 2C_0\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}
$$

$$
\leq -\mathcal{L}w_\lambda^+(x) + 2C_0\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}
$$

$$
\leq \varphi(x)w_\lambda^+(x) + 2C_0\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)} \leq C_1\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)},
$$

(6.7)

where $C_1 = \|\varphi\|_{L^\infty(\Sigma^-_\lambda)} + 2C_0$ and we notice that $w_\lambda = w_\lambda^+$ in $\Sigma^-_\lambda$. Hence, we have

$$
\left\{
\begin{array}{l}
\Delta^{\alpha_1}w_\lambda^+(x) \leq C_1\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}, \quad x \in \Sigma^-_\lambda, \\
w_\lambda^+(x) = 0, \quad x \in \mathbb{R}^N \setminus \Sigma^-_\lambda.
\end{array}
\right.
$$

(6.8)

Then, using Proposition 2.1 with $h(x) = C_1\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}$, we obtain a constant $C > 0$ such that

$$
\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)} = -\inf_{\Sigma^-_\lambda} w_\lambda^+ \leq Cd^{\alpha_1}\|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)}|\Sigma^-_\lambda|^{\frac{2\alpha_1}{N}}
$$

where $d = \text{diam}(\Sigma^-_\lambda)$. If $|\Sigma^-_\lambda|$ is small enough we conclude that $\|w_\lambda\|_{L^\infty(\Sigma^-_\lambda)} = \|w_\lambda^+\|_{L^\infty(\Sigma^-_\lambda)} = 0$, from where we complete the proof.

Now we provide a proof for Theorem 6.1.

23
Proof of Theorem 6.1. The proof of this theorem goes like the one for Theorem 1.1 where we use Proposition 6.1 instead of Proposition 2.1 and \( \mathcal{L} \) instead of \( (-\Delta)^\alpha \). The only place where there is a difference is in the following property: for \( 0 < \lambda < 1 \), if \( w_\lambda \geq 0 \) and \( w_\lambda \not\equiv 0 \) in \( \Sigma_\lambda \), then \( w_\lambda > 0 \) in \( \Sigma_\lambda \).

For \( \mu \in (0,1] \), since \( K_\mu \) is radially symmetric and strictly decreasing, the proof of the property is similar to that given in Theorem 1.1. So we only need to prove it in case \( \mu = 0 \) so the kernel \( K_0 \) vanishes outside the unit ball \( B_1 \). Let us assume that \( w_\lambda \geq 0 \) and \( w_\lambda \not\equiv 0 \) in \( \Sigma_\lambda \) and, by contradiction, let us assume \( \Sigma_0 = \{ x \in \Sigma_\lambda \mid w_\lambda(x) = 0 \} \neq \emptyset \). By our assumptions on \( \lambda \) we have that \( \Sigma_\lambda \setminus \Sigma_0 = \{ x \in \Sigma_\lambda \mid w_\lambda(x) > 0 \} \) is open and nonempty. Let us consider \( x_0 \in \Sigma_0 \) such that

\[
dist(x_0, \Sigma_\lambda \setminus \Sigma_0) \leq 1/2, \tag{6.9}
\]

and observe that \( (\Sigma_\lambda \setminus \Sigma_0) \cap B_1(x_0) \) is nonempty. Using (6.9) we have

\[
\mathcal{L}w_\lambda(x_0) = \mathcal{L}u_\lambda(x_0) - \mathcal{L}u(x_0) = f(u_\lambda(x_0)) - f(u(x_0)) + g((x_0)_\lambda) - g(x_0) = g((x_0)_\lambda) - g(x_0) \geq 0, \tag{6.10}
\]

where the last inequality holds by monotonicity assumption on \( g \) and since \( |x_0| > |(x_0)_\lambda| \). On the other hand, denoting by \( A_\lambda = \{ (x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda \} \), since \( w_\lambda(x_0) = 0 \) and \( w_\lambda(z_\lambda) = -w_\lambda(z) \) for any \( z \in \mathbb{R}^N \), we have

\[
\mathcal{L}w_\lambda(x_0) = -\int_{A_\lambda} w_\lambda(z)K_0(x_0 - z)dz - \int_{\mathbb{R}^N \setminus A_\lambda} w_\lambda(z)K_0(x_0 - z)dz = -\int_{A_\lambda} w_\lambda(z)_0(x_0 - z)dz - \int_{A_\lambda} w_\lambda(z_\lambda)K_0(x_0 - z_\lambda)dz = -\int_{A_\lambda} w_\lambda(z)(K_0(x_0 - z) - K_0(x_0 - z_\lambda))dz.
\]

Since \( |x_0 - z_\lambda| > |x_0 - z| \) for \( z \in A_\lambda \), by definition of \( K_0, \Sigma_\lambda \) and \( \Sigma_0 \), we have that

\[
K_0(x_0 - z) > K_0(x_0 - z_\lambda) \quad \text{and} \quad w_\lambda(z) > 0 \quad \text{for} \quad z \in (\Sigma_\lambda \setminus \Sigma_0) \cap B_1(x_0),
\]

and we also have that \( w_\lambda(z) \geq 0 \) and \( K_0(x_0 - z) \geq K_0(x_0 - z_\lambda) \) for all \( z \in A_\lambda \), so that

\[
\mathcal{L}w_\lambda(x_0) < 0,
\]

contradicting (6.10). Hence \( \Sigma_0 \) is empty and then \( w_\lambda > 0 \) in \( \Sigma_\lambda \), completing the proof of the theorem. \( \square \)
Remark 6.1 The theorem we just proved can be extended to more general non-homogeneous kernels in the following class

\[ K(x) = \begin{cases} |x|^{-N-2\alpha}, & x \in B_r, \\ \theta(x), & x \in B^c_r, \end{cases} \tag{6.11} \]

here \( \alpha \in (0,1) \), \( r > 0 \) and the function \( \theta : B^c_r \to \mathbb{R} \) satisfies that

(C) \( \theta \in L^1(B^c_r) \) is nonnegative, radially symmetric and such that the kernel \( K \) is decreasing.

Acknowledgements: P.F. was partially supported by Fondecyt Grant # 1110291, BASAL-CMM projects and CAPDE, Anillo ACT-125. Y.W. was partially supported by Becas CMM.

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