STABLE GROUPS AND EXPANSIONS OF \((\mathbb{Z}, +, 0)\)

GABRIEL CONANT AND ANAND PILLAY

Abstract. We show that if \(G\) is a sufficiently saturated stable group of finite weight with no infinite, infinite-index, chains of definable subgroups, then \(G\) is superstable of finite \(U\)-rank. Combined with recent work of Palacín and Sklinos, we conclude that \((\mathbb{Z}, +, 0)\) has no proper stable expansions of finite weight. A corollary of this result is that if \(P \subseteq \mathbb{Z}^n\) is definable in a finite dp-rank expansion of \((\mathbb{Z}, +, 0)\), and \((\mathbb{Z}, +, 0, P)\) is stable, then \(P\) is definable in \((\mathbb{Z}, +, 0)\). In particular, this answers a question of Marker on stable expansions of the group of integers by sets definable in Presburger arithmetic.

1. Introduction and Summary of Main Results

The work in this paper is motivated by questions surrounding first-order expansions of the group \((\mathbb{Z}, +, 0)\), which are well-behaved with respect to some notion of model theoretic tameness (e.g. stability or NIP). The group \((\mathbb{Z}, +, 0)\) is a well-known example of a stable group, and so this program is a natural analog of the very fruitful study of “tame” (e.g. o-minimal or NIP) expansions of the real ordered field \((\mathbb{R}, +, \cdot, <, 0)\). Expansions of \((\mathbb{Z}, +, 0)\) have emerged in the context of definable subgroups of finitely generated free groups, as well as the general growing industry of research on ordered abelian groups satisfying notions of tameness coming from dp-rank in NIP first-order theories (e.g. \([7, 9, 24]\)). We will provide more detail on these contexts toward the end of the introduction. For now, we state an explicit question, originally asked by Marker in 2011.

Question 1.1 (Marker). Is there a set \(P \subseteq \mathbb{Z}^n\), definable in Presburger arithmetic \((\mathbb{Z}, +, <, 0)\), such that \((\mathbb{Z}, +, 0, P)\) is a proper stable expansion of \((\mathbb{Z}, +, 0)\)?

The focus on Presburger arithmetic in the previous question is not unnatural. Indeed, \((\mathbb{Z}, +, <, 0)\) is an ordered structure, and thus unstable, but is still well understood and very well behaved model theoretically (to be precise, its theory is NIP of dp-rank 1 \([8]\)). Our first main result will show that, in fact, these model theoretic notions completely control the answer to Marker’s question.

Theorem 1.2. If \(P \subseteq \mathbb{Z}^n\) is definable in a finite dp-rank expansion of \((\mathbb{Z}, +, 0)\), and \((\mathbb{Z}, +, 0, P)\) is stable, then \(P\) is definable in \((\mathbb{Z}, +, 0)\).

The notion of dp-rank in NIP theories has been an important tool in extending the work of stability theory to the unstable setting (see, e.g., \([23]\)), and so Theorem 1.2 establishes a fundamental fact about the behavior of NIP expansions of \((\mathbb{Z}, +, 0)\). The proof of this theorem will be obtained from a more general result on stable...
groups (Theorem 1.4 below), combined with the following result of Palacín and Sklinos [15].

**Fact 1.3.** [15] \((\mathbb{Z}, +, 0)\) has no proper stable expansions of finite \(U\)-rank.

We emphasize that, *a priori*, Fact 1.3 alone is not sufficient to answer Marker’s question, or obtain Theorem 1.2. In particular, while the dp-rank of a complete theory is *bounded above* by its \(U\)-rank, there is no further general relationship between these two ranks. Indeed, there are stable groups of dp-rank 1 and infinite or undefined \(U\)-rank (see Example 2.10). Therefore, the work involved in proving Theorem 1.2 consists of showing that if a stable expansion of \((\mathbb{Z}, +, 0)\) has finite dp-rank, then it must have finite \(U\)-rank. In fact, we will obtain this conclusion from a general characterization of superstable groups of finite \(U\)-rank, which exploits the notion of *weight* in stable theories. Before stating this result, we clarify the following terminology (full definitions are given in Section 2).

Let \(G\) be a group definable in a complete theory \(T\). Unless otherwise stated, we assume \(G\) is evaluated in a sufficiently saturated monster model. The \(U\)-rank of \(G\), denoted by \(U(G)\), is the supremum of the \(U\)-ranks of types containing a formula defining \(G\). Replacing \(U\)-rank with weight, we similarly define the weight of \(G\), denoted by \(\text{wt}(G)\). We say \(G\) is *stable* if \(T\) is stable. We let \(<_{\infty}\) denote the partial order on groups given by: \(H <_{\infty} K\) if \(H \leq K\) and \([K : H] = \infty\). The length of a finite chain \(K_0 <_{\infty} \ldots <_{\infty} K_n\) is \(n\). If \(G\) is superstable of finite \(U\)-rank then, by well-known facts, \(G\) necessarily has finite weight and no infinite \(<_{\infty}\)-chains of definable subgroups (see [18, Theorem 19.9] and [3, Corollary III.8.2]). Our second main result is that these conditions are also sufficient.

**Theorem 1.4.** If \(G\) is stable then the following are equivalent.

(i) \(G\) is superstable of finite \(U\)-rank.
(ii) \(G\) has finite weight and no infinite \(<_{\infty}\)-chains of definable subgroups.
(iii) \(G\) has finite weight and no infinite \(<_{\infty}\)-chains of definable normal subgroups.

Theorem 1.4 will be obtained as an immediate consequence of the following more detailed statement, which also gives an upper bound on the \(U\)-rank of \(G\).

**Theorem 1.5.** Let \(G\) be a stable group of finite weight. If \(G\) has no infinite \(<_{\infty}\)-chains of definable normal subgroups then:

(i) there is a uniform finite bound on the length of a \(<_{\infty}\)-chain of definable subgroups of \(G\), and
(ii) if \(n\) is the maximal length of a \(<_{\infty}\)-chain of definable normal subgroups of \(G\), then \(U(G) \leq n \text{wt}(G)\).

The proof of this theorem involves a new application of Zilber indecomposability in the setting of weight (see Lemma 3.1). In Section 2 we will also recall some classical examples showing that the upper bound in this result cannot be improved in general. All three theorems stated above are proved in Section 3.

We end this section with a discussion of related work and open questions. The motivation for Question 1.1 partly arose from interest in the induced structure on proper definable subgroups of finitely-generated free groups, which are examples of stable groups [21]. In particular, the maximal proper definable subgroups of such groups are exactly the centralizers of some nontrivial element, and thus isomorphic as groups to \((\mathbb{Z}, +, 0)\) (see [16]). Therefore, studying stable expansions of \((\mathbb{Z}, +, 0)\)
was seen as an alternate approach toward the unpublished result of Perin that the induced structure on centralizers in the free group is always a pure group. Another proof of this has been recently given by Byron and Sklinos [4].

Beyond this connection to the free group, there has been a recent flurry of interest in expansions of \((\mathbb{Z}, +, 0)\). On the stable side, we have the following ambitious question (which is similar to a question of Goodrick quoted in [15]).

**Question 1.6.** Characterize the sets \(P \subseteq \mathbb{Z}^n\) such that \((\mathbb{Z}, +, 0, P)\) is stable.

On the unstable side, Dolich and Goodrick [7] have shown that the ordered group \((\mathbb{Z}, +, <, 0)\) has no proper strong expansions (which includes expansions of finite dp-rank). Concerning reducts of Presburger arithmetic, a recent result of the first author [6] is that there are no structures strictly between \((\mathbb{Z}, +, 0)\) and \((\mathbb{Z}, +, <, 0)\). In a different direction, Kaplan and Shelah [11] show that if \(P = \{ z \in \mathbb{Z} : |z| \text{ is prime} \}\) then \((\mathbb{Z}, +, 0, P)\) is unstable and, assuming a fairly strong conjecture in number theory, \((\mathbb{Z}, +, 0, P)\) is supersimple of SU-rank 1 (see also Remark 1.8(3) below).

The investigation of stable expansions of \((\mathbb{Z}, +, 0)\) also fits naturally into the general question of when good properties of a structure are preserved after adding a new predicate. For example Pillay and Steinhorn [17] proved that there are no proper o-minimal expansions of \((\mathbb{N}, <)\), while Marker [12] exhibited proper strongly minimal expansions of \((\mathbb{N}, x \mapsto x + 1)\). Zilber [26] showed that there are proper \(\omega\)-stable expansions of the complex field \((\mathbb{C}, +, \cdot, 0, 1)\) (in particular, adding a predicate for the roots of unity), while Marker [13] proved that there are no proper stable expansions of \((\mathbb{C}, +, \cdot, 0, 1)\) by a semialgebraic set.

Even more generally, Theorem 1.4 fits into the investigation of when stronger forms of stability can be proved for stable groups satisfying various assumptions on definable subgroups. For example, in [2], Baldwin and Pillay prove that if \(G\) is superstable of finite \(U\)-rank, and \(G\) has no proper connected type-definable normal abelian subgroups, then \(G\) is \(\omega\)-stable. In [10], Gagelman proves that if \(G\) is superstable of finite \(U\)-rank and satisfies the descending chain condition on definable subgroups, then \(G\) is \(\omega\)-stable. It would be interesting to know if the finiteness conditions on weight and \(U\)-rank in Theorem 1.4 can be relaxed to obtain a characterization of superstable groups of a similar flavor. In particular, it is well known that if \(G\) is a superstable group, then every type in \(G\) has finite weight (i.e. \(G\) is strongly stable) and \(G\) has no infinite descending \(<_\infty\)-chains of definable subgroups (i.e. \(G\) satisfies the superstable descending chain condition). Therefore, we ask the following question, which is an analog of Theorem 1.4 for superstable groups.

**Question 1.7.** Suppose \(G\) is a strongly stable group satisfying the superstable descending chain condition. Is \(G\) is superstable?

We end with some important remarks.

**Remark 1.8.**

1. Many of the results above on \((\mathbb{Z}, +, 0)\) do not hold if one considers expansions of structures elementarily equivalent to \((\mathbb{Z}, +, 0)\). For example, there are models \((M, +, 0)\) of Th(\((\mathbb{Z}, +, 0)\)) with proper stable expansions of finite \(U\)-rank.
2. Theorem 1.2 also holds with inp-rank in place of dp-rank, since these ranks coincide in the stable case (see [11]). Therefore the theorem can be applied in the more general class of NTP\(_2\) theories.
(3) Fact [13] does not hold if stable is replaced by simple. For example, by work of Chatzidakis and Pillay [5], there are “generic” subsets \( P \subseteq \mathbb{Z} \) such that \((\mathbb{Z}, +, 0, P)\) is unstable, but supersimple of \(SU\)-rank 1.

2. Preliminaries

The purpose of this section is to collect the preliminary tools and facts that we will need in the proofs of our main results. Our intent is to include sufficient detail so as to make this paper accessible to a wider audience beyond those researchers well-versed in stability theory. For example, Lemma [2.4] and Proposition [2.12] are folkloric facts, which seem to be used primarily in the superstable context, and to not appear in the literature in more general settings. Therefore we have included proofs suitable for the general stable case.

Throughout this section, \( T \) is a stable first-order theory, and we assume \( T = T^{eq} \).

We work in a sufficiently saturated monster model \( \mathbb{M} \) of \( T \), and use letters \( A, B, \ldots \) for small parameter sets in \( \mathbb{M} \), where a parameter set \( A \) is small (written \( A \subset \mathbb{M} \)) if \( \mathbb{M} \) is \(|T(A)|^{+}\)-saturated. In general, a cardinal \( \kappa \) is small or bounded if \( \mathbb{M} \) is \( \kappa^{+}\)-saturated. We use letters \( X, Y, \ldots \) for definable or type-definable sets, and we always identify such a set \( X \) with its set of realizations \( X(\mathbb{M}) \) in the monster model. As usual, by a type-definable set we mean an intersection of a small collection of definable sets. Given a type \( p \), and a type-definable set \( X \), we write \( p \models X \) if \( p \) extends a type defining \( X \). We use \( \perp \) for the nonforking independence relation in \( T \). We assume familiarity with stability and \( U \)-rank.

**Definition 2.1.**

1. Given a sequence \( (\bar{b}_{i})_{i \in I} \) of tuples and \( C \subset \mathbb{M} \), we say \( (\bar{b}_{i})_{i \in I} \) is \( C \)-independent if \( \bar{b}_{i} \perp_{C} \{ \bar{b}_{j} : j \neq i \} \) for all \( i \in I \).

2. Given \( C \subset \mathbb{M} \) and \( p \in S(C) \), define the weight of \( p \), denoted \( \text{wt}(p) \), to be the supremum over cardinals \( \kappa \) for which there is some \( B \supseteq C \), a realization \( \bar{a} \models p \), and a \( B \)-independent sequence \( (\bar{b}_{i})_{i < \kappa} \) such that \( \bar{a} \perp_{B} \bar{b}_{i} \) for all \( i < \kappa \).

3. Let \( \text{rk} \) denote either \( U \)-rank or weight.
   (i) If \( \bar{a} \in \mathbb{M} \) and \( C \subset \mathbb{M} \) then \( \text{rk}(\bar{a}/C) \) denotes \( \text{rk}(tp(\bar{a}/C)) \).
   (ii) If \( X \) is type-definable, then \( \text{rk}(X) = \sup\{ \text{rk}(p) : p \models X \} \).

The final notion of rank discussed in the introduction is \( \text{dp-rank} \), which we calculate for type-definable sets in the same way. In particular, if \( X \) is type-definable then \( \text{dp}(X) = \sup\{ \text{dp}(p) : p \models X \} \) where we set \( \text{dp}(p) \) to be the supremum over cardinals \( \kappa \) such that the relation “\( \text{dp}(p) \geq \kappa \)” holds, as defined in [23, Chapter 4].

We are justified in avoiding the full definition of \( \text{dp-rank} \) because of the following fact about stable theories.

**Fact 2.2** ([11], [14], [23]). If \( X \subseteq \mathbb{M} \) is type-definable then \( \text{wt}(X) = \text{dp}(X) \).

We will use the following basic properties of \( U \)-rank and weight.

**Fact 2.3.** Let \( \text{rk} \) denote either \( U \)-rank or weight.
(a) Given \( \bar{a} \in \mathbb{M} \) and \( C \subset \mathbb{M} \), \( \text{rk}(\bar{a}/C) = 0 \) if and only if \( \bar{a} \in \text{acl}(C) \).
(b) Fix \( \bar{a}, \bar{b} \in \mathbb{M} \) and \( C \subset \mathbb{M} \). If \( \bar{a} \in \text{acl}(\bar{b}/C) \) then \( \text{rk}(\bar{a}/C) \leq \text{rk}(\bar{b}/C) \).
(c) Suppose \( X \) is type-definable and \( f \) is a definable function with domain containing \( X \). Then \( \text{rk}(f(X)) \leq \text{rk}(X) \).
Proof. These are straightforward exercises. Parts (b) and (c) follow easily from
part (a) together with Lascar’s inequality for $U$-rank (see [13 Theorem 19.4]), and
a sufficiently similar inequality for weight (see [22 Lemma V.3.11(2)]).

In a superstable theory, the weight of a type $p$ is bounded by the sum of the
integer coefficients in the Cantor normal form of $U(p)$ (see [13 Theorem 19.9]). In
particular, one has $\text{wt}(p) \leq U(p)$, which still holds for stable theories in general.

Lemma 2.4. If $C \subseteq M$ and $p \in S(C)$, then $\text{wt}(p) \leq U(p)$.

Proof. Fix $p \in S(C)$. Suppose we have a set $B \supseteq C$, a realization $\bar{a} \models p$, and a $B$-

-independent sequence $(\bar{b}_i)_{i < \kappa}$, for some cardinal $\kappa$, such that $\bar{a} \downarrow_C B$ and $\bar{a} \not\subseteq B \bar{b}_i$ for all $i < \kappa$. We prove $U(\bar{a}/B) \geq \kappa$, which implies $U(p) \geq \kappa$. Given $i \leq \kappa$, define $B_i = B \cup \{b_j : i \leq j\}$ (so $B_\kappa = B$). We prove, by induction on $i \leq \kappa$, that

$U(\bar{a}/B_i) \geq i$. Given this, we will then have $U(\bar{a}/B) = U(\bar{a}/B_\kappa) \geq \kappa$.

The base case is trivial; so suppose $\lambda \leq \kappa$ is a limit ordinal and $U(\bar{a}/B_i) \geq i$ for all

$i < \lambda$. For any $i < \lambda$, we have $B_\lambda \subseteq B_i$, and so $U(\bar{a}/B_\lambda) \geq U(\bar{a}/B_i) \geq i$. Therefore

$U(\bar{a}/B_\lambda) \geq \lambda$. Finally, fix $i < \kappa$ and suppose $U(\bar{a}/B_i) \geq i$. Since $B_i+1 \subseteq B \bar{b}_i$ and \n
$\bar{a} \not\subseteq B_i \bar{b}_i$, we have $\bar{a} \not\subseteq B_i+1 \bar{b}_i$ by transitivity. Therefore $U(\bar{a}/B_i+1) \geq i + 1$. \qed

For general stable theories, Lemma 2.3 is the most one can say concerning the
relationship between weight and $U$-rank for arbitrary types (see Example 2.10).
However, when working “close” to types of $U$-rank 1, weight and $U$-rank coincide.
This will be a key tool in the proof of our main result.

Proposition 2.5. Fix $C \subseteq M$, and suppose $X \subseteq M$ is such that $U(\bar{a}/C) \leq 1$ for

all $a \in X$. If $\bar{b}$ is a finite tuple in $\text{acl}(XC)$ then $U(\bar{b}/C) = \text{wt}(\bar{b}/C)$.

Proof. First, we observe that by Lascar’s inequality and Fact 2.3(b), $U(\bar{b}/C)$ exists

(and is in fact finite) for any finite tuple $\bar{b}$ from $\text{acl}(XC)$. In particular, for any

such $\bar{b}$ and any $C \subseteq B \subseteq A$, we have $\bar{b} \not\subseteq B \bar{a}$ if and only if $U(\bar{b}/A) = U(\bar{b}/B)$.

Fix $\bar{b} \in \text{acl}(XC)$. By Fact 2.3(a), we may assume that some coordinate of $\bar{b}$ is not

in $\text{acl}(C)$. Fix $\bar{a} = (a_1, \ldots, a_n) \in X$, algebraically independent over $C$, with

$\bar{b} \in \text{acl}(\bar{a}, C)$. Let $k \leq n$ be maximal such that, for some $i_1 < \ldots < i_k \leq n$, we have $\bar{b} \downarrow_C a_{i_1} \ldots a_{i_k}$ (since $\bar{b} \in \text{acl}(\bar{a}, C) \setminus \text{acl}(C)$ we must have $k < n$, and it is possible that $k = 0$). Without loss of generality, assume $\bar{b} \downarrow_C a_1 \ldots a_k$. Let $\bar{a}_1 = (a_1, \ldots, a_k) \bar{a}_2 = (a_{k+1}, \ldots, a_n)$. Since $\bar{b} \downarrow_C \bar{a}_1$, we have $U(\bar{b}/C, \bar{a}_1) = U(\bar{b}/C)$ and \n
$\text{wt}(\bar{b}/C, \bar{a}_1) = \text{wt}(\bar{b}/C)$ (see, e.g., [22 Lemma V.3.11]). So to prove the result,

it suffices to show $U(\bar{b}/C, \bar{a}_1) = \text{wt}(\bar{b}/C, \bar{a}_1)$.

Since $\bar{a}_2$ is algebraically independent over $C\bar{a}_1$, we have $U(a_i/C, \bar{a}_1) = 1 = \text{wt}(a_i/C, \bar{a}_1)$ for all $k < i \leq n$ by Fact 2.3(a) and Lemma 2.4. It follows from Lascar’s inequality and\n
[22 Lemma V.3.11(1)] that $U(\bar{a}_2/C, \bar{a}_1) = \text{wt}(\bar{a}_2/C, \bar{a}_1)$. So to prove $U(\bar{b}/C, \bar{a}_1) = \text{wt}(\bar{b}/C, \bar{a}_1)$, it suffices to show $U(\bar{b}/C, \bar{a}_1) = U(\bar{a}_2/C, \bar{a}_1)$ and \n
$\text{wt}(\bar{b}/C, \bar{a}_1) = \text{wt}(\bar{a}_2/C, \bar{a}_1)$. Since $\bar{b} \in \text{acl}(\bar{a}_2, \bar{a}_1, C)$, it suffices by Fact 2.3(b) to show $\bar{a}_2 \in \text{acl}(\bar{b}, \bar{a}_1, C)$.

For a contradiction, suppose there is $k < i \leq n$ such that $a_i \not\in \text{acl}(\bar{b}, \bar{a}_1, C)$. Then $U(a_i/C, \bar{a}_1, \bar{b}) = 1 = U(a_i/C, \bar{a}_1)$ and so $a_i \downarrow_C \bar{a}_1 \bar{b}$. Since $\bar{b} \downarrow_C \bar{a}_1 \bar{a}_i$ by symmetry and transitivity. This contradicts the maximality of $k$. \qed

Remark 2.6. The notion of weight also behaves nicely in simple theories. For example, after replacing all occurrences of $U$-rank with $SU$-rank, the statements
of Fact 2.3, Lemma 2.4, and Proposition 2.5 hold when $T$ is simple (with identical proofs).

We now turn to stable groups. Recall the following classical results.

**Fact 2.7.** Let $G$ be a group definable in a model of a stable theory.

(a) (Baldwin-Saxl, see [19, Proposition 1.4]) Let $\{H_i : i \in I\}$ be a family of uniformly definable subgroups of $G$, and set $H = \bigcap_{i \in I} H_i$. Then $H = \bigcap_{i \in I_0} H_i$ for some finite $I_0 \subseteq I$. In particular, $H$ is definable.

(b) (Poizat, see [19, Theorem 5.17]) Any type-definable subgroup of $G$ is the intersection of at most $|T|$ many definable subgroups of $G$.

**Remark 2.8.** Unlike the previous preliminaries, these facts do not immediately go through if $T$ is only assumed to be simple. Indeed, there are simple unstable groups where part (a) fails [25, Example 1]. On the other hand, whether part (b) holds for groups definable in simple theories is a well known open question.

For the rest of this section, when we say $G$ is a stable group, we mean $G = G(M)$ is a group definable in the monster model $M$ of a stable theory $T = T_{eq}$. Given a stable group $G$, we let $G^0$ denote the connected component of $G$, which is the intersection of all definable subgroups of $G$ of finite index. By stability (e.g., Fact 2.7), $G^0$ is the intersection of at most $|T|$ many definable subgroups of $G$ of finite index, and hence is type-definable (over the same parameters used to define $G$). We say $G$ is connected if $G = G^0$.

For the sake of clarity, it is worth making a few remarks concerning weight and $U$-rank in stable groups. In particular, given a definable group $G$ and $A \subseteq M$, we let $S_G(A)$ denote the space of complete types, over parameters in $A$, which contain a formula defining $G$. Then, if $\text{rk}$ denotes either $U$-rank or weight, we can express $\text{rk}(G)$ as

$$\text{rk}(G) = \sup \{ \text{rk}(p) : p \in S_G(A) \text{ for some } A \subseteq M \}.$$  

We say $G$ has finite $U$-rank (respectively, finite weight) if $U(G) < \omega$ (respectively, $\text{wt}(G) < \omega$).

If $G$ is stable then $U(p) = U(G)$ for any generic type $p$ in $G$ (see [3, Lemma III.4.5(i)]). On the other hand, it is possible that all types in $G$ have finite weight, but $\text{wt}(G)$ is not finite (e.g. Example 2.10(1) below). Since our focus is on the case that $\text{wt}(G)$ is finite, we will not concern ourselves with this situation.

**Remark 2.9.** When considering examples of stable groups, it is often the case that the group $G$ is the whole structure (i.e. defined by the formula $x = x$). Therefore, given a group $G = (G, \cdot, 1, \ldots)$, when we speak of the $U$-rank or weight of $G$, we continue to mean as calculated in a monster model according the definitions and conventions above.

The following examples illustrate some of the possible variety concerning weight, $U$-rank, and $<\omega$-chains in stable groups.

**Example 2.10.**

(1) Let $G \models \text{Th}(\mathbb{Z}, +, 0, \{2^n : n \in \mathbb{N}\})$. Then $G$ is superstable of $U$-rank $\omega$ (see [15], [20]). Thus every type in $G$ has finite weight (i.e. $G$ is strongly stable), but $G$ does not have finite weight by Theorem 1.4.
Definition 2.11. if, for all type-definable subgroups $p$ of $\tilde{G}$, there exists a bounded set such that $H \subseteq \tilde{G}$, and so $H \subseteq \tilde{G}$ is type-definable over $A$. Then $\tilde{G}$ is strictly stable of weight 1 (this is again similar to part (2)).

Our final preliminary tools concern indecomposable sets in stable groups.

Proposition 2.12. Let $G$ be a stable group. Fix $A \subseteq M$ and a stationary type $p \in S_G(A)$. Let $X = p(M)$. Then $X \subseteq G$ is indecomposable.

Proof. Let $\mathcal{F}$ denote the family of type-definable subgroups $H \subseteq G$ such that $X/H$ is bounded. Let $H_0$ be the intersection of the elements of $\mathcal{F}$. Using Fact 2.7, it is a standard exercise to show that $H_0$ is a type-definable subgroup of $G$ and $X/H_0$ is bounded (i.e. $H_0 \in \mathcal{F}$). Note that $A$-invariance of $X$ implies $A$-invariance of $H_0$, and so $H_0$ is type-definable over $A$.

Let $\tilde{p} \in S_G(M)$ be the unique global nonforking extension of $p$. Let $C \subseteq X$ be a bounded set such that $X/H_0 = \{cH_0 : c \in C\}$, and fix a realization $u \in G$ of $\tilde{p}|_{AC}$. Then $u \in X$, and so $u \in cH_0$ for some $c \in C$, which means $\tilde{p} \models cH_0$. If $f \in \text{Aut}(M/A)$ then, by $A$-invariance of $H_0$ and $\tilde{p}$, we have $\tilde{p} \models f(c)H_0$, and so $f(cH_0) = f(c)H_0 = cH_0$. Consequently, $cH_0$ is type-definable over $A$, and so $p \models cH_0$. Therefore $X \subseteq cH_0$, which implies $X \subseteq cH$ for all $H \in \mathcal{F}$. □

A well-known result of Berline and Lascar is the Indecomposability Theorem for superstable groups [3, Theorem V.3.1]. In order to use this result without the assumption of superstability, we state the following corollary of its proof.

Fact 2.13. Suppose $G$ is a stable group and $\{X_i : i \in I\}$ is a family of indecomposable type-definable subgroups of $G$, each containing 1. Given $n > 0$ and $\sigma = (i_0, \ldots, i_n) \in I^{<\omega}$, let $X_\sigma = X_{i_0} \cdot X_{i_1} \cdot \ldots \cdot X_{i_n}$. Assume that there is a uniform finite bound on $U(\sigma)$, where $\sigma$ ranges over $I^{<\omega}$. Then $\bigcup_{\sigma \in I} X_\sigma$ generates a connected type-definable subgroup $H$ of $G$. In particular, there are $i_0, \ldots, i_n$ such that $H = X_{i_0} \cdot \ldots \cdot X_{i_n} \cdot X_{i_n}^{-1} \cdot \ldots \cdot X_{i_0}^{-1}$.

3. Proofs of the main results

As in the previous section, when we say $G$ is a stable group we mean $G = G(M)$ is a group definable in the monster model $M$ of a stable theory $T = T^{\text{st}}$. Toward the proofs of Theorems 1.2, 1.3, and 1.5, we start with the following technical lemma concerning definable subgroups of infinite stable groups of finite weight.
Lemma 3.1. Let $G$ be an infinite stable group of finite weight.

(a) There is an infinite connected type-definable normal subgroup $H \leq G$, with $U(H) = \text{wt}(H)$.

(b) Assume $G$ has no infinite descending $<\omega$-chains of definable normal subgroups.

If $K < \omega G$ is definable and normal in $G$, then there is a definable normal subgroup $L \leq G$ such that $K < L$ and $U(G/K) \leq \text{wt}(G/K) \oplus U(G/L)$.

Proof. Part (a). Fix a stationary type $p \in S_G(A)$, for some $A \subseteq \mathbb{M}$, such that $U(p) = 1$. For example, choose $p$ minimal in the fundamental order among non-algebraic types in $S_G(A)$ (with $A$ varying over small parameter sets in $\mathbb{M}$), and then replace $p$ by a nonforking extension to a model.

Let $Y = p(\mathbb{M})$. Then $Y \subseteq G$ is indecomposable by Proposition 2.12. Fix some $u \in Y$, and set $X = u^{-1}Y$. Given $g \in G$, let $X_g^0 = gXg^{-1}$ and $X_g^i = g^{-1}X^{-1}g = (X_g^0)^{-1}$. Then $\{X_g^i : g \in G, \ i \in \{0,1\}\}$ is a family of indecomposable type-definable subsets of $G$, each of which contains $1$. By Fact 2.3(c), $U(X_g^0) = 1$ for all $g \in G$ and $i \in \{0,1\}$. Fix a sequence $\sigma = (g_0, \ldots, g_n)$ of elements of $G$, and set $X_\sigma = X_{g_0}^0 \cdot \cdots \cdot X_{g_n}^0$. In particular, $X_\sigma \subseteq \text{acl}(\bigcup_{i=0}^n X_{g_i}^0 \cup X_{g_i}^0)$ and so, by Proposition 2.3, $U(q) = \text{wt}(q)$ for any $q \models X_\sigma$. Therefore $U(X_\sigma) = \text{wt}(X_\sigma)$, and so $U(X_\sigma) \leq \text{wt}(G)$ by Fact 2.3(c).

Now we may apply Fact 2.13 to conclude that $\bigcup\{X_g^0 : g \in G, \ i \in \{0,1\}\}$ generates an infinite connected type-definable subgroup $H$ of $G$, which is normal by construction. Moreover, $H = X_\sigma$ for some $\sigma \in G^{<\omega}$, and so $U(H) = \text{wt}(H)$.

Part (b). Let $K < \omega G$ be definable and normal. We use $\rho$ to denote the pullback function on subgroups of $G/K$, i.e., given $H \leq G/K$ define $\rho(H) = \{g \in G : gK \subseteq H\} \leq G$.

By assumption and Fact 2.3(c), $G/K$ is an infinite stable group of finite weight. By part (a) applied to $G/K$, there is an infinite connected type-definable normal subgroup $H \leq G/K$, with $U(H) = \text{wt}(H)$. Since $H$ is type-definable, it is the intersection of a bounded family $(H_i)_{i \in I}$ of definable subgroups by Fact 2.7(b). Since $H$ is normal we may use Fact 2.7(a) to replace each $H_i$ with $\bigcap_{g \in G/K} gH_ig^{-1}$, and thus assume $H$ is the intersection of a bounded family of definable normal subgroups of $G/K$. Now, $G/K$ has no infinite descending $<\omega$-chains of definable normal subgroups since such a chain would pull back via $\rho$ to a chain in $G$. It follows that there is a definable normal subgroup $J$ of $G/K$ such that $H \leq J$ and $|J : H|$ is bounded. Since $H$ is type-definable and connected we then have $H = J^0$, which implies $U(J) = U(H) = \text{wt}(H)$ (see, e.g., [3, Sections III.4, IV.3]). By Fact 2.3(c), $U(J) \leq \text{wt}(G/K)$.

Now let $L = \rho(J)$. Then $L$ is a definable normal subgroup of $G$ and, since $J$ is infinite, $K < \omega L$. By definition of $L$, the groups $G/L$ and $(G/K)/J$ are definably isomorphic and so, by Lascar’s inequality for cosets [3, Corollary III.8.2],

$$U(G/K) \leq U(J) \oplus U(G/L) \leq \text{wt}(G/K) \oplus U(G/L).$$

We now prove the main results stated in the introduction.

Proof of Theorem 1.3. Let $G$ be a stable group of finite weight, with no infinite $<\omega$-chains of definable normal subgroups. We will use Lemma 3.1(b) to construct an ascending $<\omega$-chain of definable normal subgroups of $G$. By assumption, this construction must terminate at some finite step, at which point we will make the desired conclusions (claims (i) and (ii) in the statement of the theorem).
To start the construction, let $K_0 = \{1\}$. Now fix $m < \omega$ and suppose we have constructed definable normal subgroups $K_0 <_{\infty} \ldots <_{\infty} K_m \leq G$ such that $U(G) \leq U(G/K_m) \oplus \sum_{i < m} \text{wt}(G/K_i)$. If $G/K_m$ is finite then we terminate the construction. Otherwise, if $K_m <_{\infty} G$ then we use Lemma 3.1(b) to find a definable normal subgroup $K_{m+1} \leq G$ such that $K_m <_{\infty} K_{m+1}$ and $U(G/K_m) \leq U(G/K_{m+1}) \oplus \text{wt}(G/K_m)$. By induction, $U(G) \leq U(G/K_{m+1}) \oplus \sum_{i < m} \text{wt}(G/K_i)$.

Since $G$ has no infinite ascending $<_{\infty}$-chains of normal subgroups, the above construction must terminate at some $\hat{m} < \omega$, meaning that $G/K_{\hat{m}}$ is finite. By construction and Fact 2.3 $U(G) \leq \sum_{i < \hat{m}} \text{wt}(G/K_i) \leq \hat{m} \text{wt}(G)$. Thus $U(G)$ is finite which, by Lascar’s inequality for cosets, immediately yields claim (i). For claim (ii), let $n$ be the maximal length of a $<_{\infty}$-chain of definable normal subgroups of $G$ (note that $n$ exists by (i)). We must have $\hat{m} \leq n$ and so $U(G) \leq n \text{wt}(G)$.

Remark 3.2. Note that in the proof of Theorem 1.5, the assumption of no infinite descending $<_{\infty}$-chains of definable normal subgroups is used when applying Lemma 3.1(b). The stable groups described in parts (3) and (4) of Example 2.10 illustrate that all assumptions in Theorem 1.5 are necessary.

As outlined in the introduction, Theorem 1.4 follows immediately from Theorem 1.5 and standard facts.

Proof of Theorem 1.4. (ii) ⇒ (iii) is trivial, and (iii) ⇒ (i) is by Theorem 1.5. For (i) ⇒ (ii), first recall that finite $U$-rank implies finite weight by Lemma 2.4. Moreover, if $G$ is superstable of finite $U$-rank then it follows from Lascar’s inequality for cosets that $G$ has no infinite $<_{\infty}$-chains of definable subgroups.

Finally, we apply Theorem 1.4 to prove our main result concerning $(\mathbb{Z},+,0)$, namely that there are no proper stable expansions of $(\mathbb{Z},+,0)$ of finite $dp$-rank.

Proof of Theorem 1.3. Suppose $P \subseteq \mathbb{Z}^n$ is definable in a finite $dp$-rank expansion of $(\mathbb{Z},+,0)$ and $(\mathbb{Z},+,0,P)$ is stable. We want to show $P$ is definable in $(\mathbb{Z},+,0)$. We work in $T = \text{Th}(\mathbb{Z},+,0,P)$, and let $G$ be a sufficiently saturated model of $T$. Since $dp$-rank cannot increase after taking a reduct, $G$ has finite $dp$-rank, and thus finite weight by Fact 2.2. We claim that $G$ has no nontrivial definable subgroup of infinite index. Indeed, otherwise in $\mathbb{Z}$ we obtain a family $(H_n)_{n < \omega}$ uniformly definable nontrivial subgroups of $Z$ such that $H_n$ has index at least $n$. In particular, the intersection $H = \bigcap_{n < \omega} H_n$ has infinite index in $\mathbb{Z}$, and thus $H = \{0\}$. But by Fact 2.7(a), $H$ is equal to a finite subintersection, which is a contradiction since the intersection of finitely many nontrivial subgroups of $\mathbb{Z}$ is infinite. Now we may apply Theorem 1.4 to $G$ and conclude $U(G)$ is finite. Then $T$ is superstable of finite $U$-rank and so $P$ is definable in $(\mathbb{Z},+,0)$ by Fact 1.3.

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Department of Mathematics, University of Notre Dame, Notre Dame, IN, 46656, USA