Hydrodynamic interactions in polar liquid crystals on evolving surfaces

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We consider the derivation and numerical solution of the flow of polar liquid crystals whose molecular orientation is subjected to a tangential anchoring on an evolving curved surface. The underlying model is a simplified surface Ericksen-Leslie model, which is derived as a thin-film limit of the corresponding three-dimensional equations with appropriate boundary conditions. A finite element discretization is considered and the effect of hydrodynamics on the interplay of topology, geometric properties and defect dynamics is studied on various surfaces. Additionally we propose a surface formulation for an active polar viscous gel.

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I. INTRODUCTION

Liquid crystals (LC) are partially ordered materials that combine the fluidity of liquids with the orientational order of crystalline solids (Chaikin and Lubensky 1995; de Gennes and Prost 1993). Topological defects are a key feature of LC if considered under external constraints. In particular on curved surfaces these defects are important and have been intensively studied on a sphere (Bates et al. 2010; Dhakal et al. 2012; Dzubiella et al. 2000; Koning et al. 2013; Napoli and Vergori 2012; Shin et al. 2008) and under more complicated constraints (Martinez et al. 2014; Prinsen and van der Schoot 2003; Stark 2001). LC on curved surfaces can be realized on various levels. One possibility is to prepare a double emulsion of two concentric droplets (Fernandez-Nieves et al. 2007) for which the intervening shell is filled with a molecular or colloidal LC which shows a planar anchoring at the two curved interfaces (Liang et al. 2013, 2011; Lopez-Leon et al. 2012). Also air bubbles covered by microrods have been prepared and studied in real-space (Zhou et al. 2009). Moreover, topological defects for charged colloidal spheres confined on a sphere were experimentally studied (Guerra et al. 2018). Ellipsoidal colloids bound to curved fluid-fluid interfaces with negative Gaussian curvature (Liu et al. 2018) and spherical droplets covered with aspherical surfactants (Yang et al. 2018) were explored. Even living and motile “particles” like cells (Bade et al. 2017) and suspensions of microtubules and kinesin (Ellis et al. 2018; Keber et al. 2014) were recently studied on surfaces of different curvatures. In all these studies a tight coupling between topology, geometric properties and defect dynamics is observed. In equilibrium defects are positioned according to geometric properties of the surface (Kralj et al. 2011; Lubensky and Prost 1992; Nelson 2002). Creation and annihilation of defects can result from geometric interaction, leading to different realizations of the Poincaré-Hopf theorem on topologically equivalent but geometrically different surfaces (Nestler et al. 2018). Also changes in the phase diagram can be induced by the geometry, e.g. allowing for coexistence of isotropic and nematic phases in surface LC (Nitschke et al. 2018). In active systems the observed phenomena are even richer, including, e.g. oscillating defect patterns (Alaimo et al. 2017; Keber et al. 2014) and circulating band structures (Sklepnek and Henkes 2015).

Most theoretical studies of these phenomena use particle methods. Despite the interest in such methods a continuous description would be more essential for predicting and understanding the macroscopic relation between type and position of the defects and geometric properties of the surface. Also the influence of hydrodynamics and dynamic shape changes on these relations would be much more appropriate to study within a continuous approach. However, a coherent model, which accounts for the complex interplay between topology, geometry and dynamics, is still lacking. We here propose a minimal continuous surface hydrodynamic LC model, which contains the evolution of the surface, tangential polar ordering and surface hydrodynamics. The model is derived as a thin-film limit of the simplified Ericksen-Leslie model (Lin and Liu 2000). We describe a numerical approach to solve this model on general surfaces and demonstrate by simulations various expected and some unexpected...
phenomena on ellipsoidal and toroidal surfaces. These phenomena result from the tight coupling of the geometry with the velocity and the director field. However, a full exploration of the rich nonlinear phenomena resulting from this relation goes beyond the scope of the paper. We here only highlight the importance of the newly introduced geometric coupling terms in the equations. We also propose a more general model of surface active polar viscous gels, which can be derived and solved using the same concepts. The model follows as a thin-film limit of a three-dimensional active polar viscous gel model, which combines a more general Ericksen-Leslie model with active components (Marth et al. 2015; Tjhung et al. 2012).

II. THE ERICKSEN-LESLIE MODEL

The Ericksen-Leslie model (Ericksen 1976, 1961; Leslie 1968) is an established model for LC, whose relaxation dynamics are affected by hydrodynamics. In (Lin and Liu 2000), a simplified model was introduced and analyzed. This system already retains the main properties of the original Ericksen-Leslie model (Hu and Wu 2013; Huang et al. 2014; Lin et al. 2010; Wang et al. 2013) and will be considered as a starting model to derive a surface hydrodynamic LC model by means of a thin-film limit, see appendix B. Thus, the resulting simplified surface Ericksen-Leslie model (cf. eqs. (B30) - (B32)) reads

\[ \pi_S \partial_t \mathbf{v} + \nabla_S \mathbf{v} = \nu_n (B \mathbf{v} + \nabla_S \mathbf{v}_n) - \nabla V S S - \nu \Delta_{\text{dir}} \mathbf{v} + \nu (2 \kappa \mathbf{v} + \nabla_S (v_n \mathcal{H}) - 2 \text{div}_S (v_n B)) = \pi_S \partial_t \mathbf{p} + \nabla_S \mathbf{p} = \eta (\Delta_{\text{DG}} \mathbf{p} - B^2 \mathbf{p}) - \omega_n (\| \mathbf{p} \|_S^2 - 1) \mathbf{p} \]

where \( \mathbf{v}(t) \in T S(t) \) denotes the tangential surface velocity, \( \mathbf{p}(t) \in T S(t) \) the tangential director field, representing the averaged molecular orientation, \( p_S(x, t) < R \) the surface pressure and \( \sigma_{S}^E = (\nabla_S \mathbf{p})^T \nabla_S \mathbf{p} + (\mathbf{B}) \otimes (\mathbf{B}) \) the extrinsic surface Ericksen stress tensor. The model is defined on a compact smooth Riemannian surface \( S(t) \). We consider initial conditions \( \mathbf{v}(x, t = 0) = v_0(x) \in T_x S(0) \) and \( \mathbf{p}(x, t = 0) = p_0(x) \in T_x S(0) \). The positive constants \( \nu, \eta \) stand for the fluid viscosity, the competition between kinetic energy and elastic potential energy, and the elastic relaxation time for the molecular orientation field. \( \kappa \) is the Gaussian curvature, \( \mathcal{H} \) the mean curvature, \( \mathcal{B} \) the shape operator, \( v_n \) a prescribed normal velocity of the surface and \( \omega_n \) a penalization parameter to enforce \( \| \mathbf{p} \|_S = 1 \) weakly. \( T_x S(t) \) is the tangent space on \( x \in S(t), T S(t) = \bigcup_{x \in S(t)} T_x S(t) \) the tangent bundle, \( \pi_S \) the projection to the tangential space w.r.t. the surface \( S(t) \) and \( \nabla_S, \text{div}_S, \Delta_{\text{dir}} \) as well as \( \Delta_{\text{DG}} \) the covariant directional derivative, covariant gradient, surface divergence, Laplace-deRham operator and Bochner Laplacian, respectively. The system combines an incompressible surface Navier-Stokes equation (Jankuhn et al. 2017; Miura 2017; Reuther and Voigt 2015; Yavari et al. 2016) with a weak surface Frank-Oseen model (Nestler et al. 2018) on an evolving surface. For \( \lambda = 0 \) eqs. (1) and (2) are the surface Navier-Stokes equation for an incompressible surface fluid on an evolving surface. These equations can be obtained as a thin-film limit of the three-dimensional Navier-Stokes equation in an evolving domain (Miura 2017) or by a variational derivation (Yavari et al. 2016). If only a stationary surface is considered, \( v_n = 0 \), the equations reduce to the incompressible surface Navier-Stokes equation considered in (Arroyo and DeSimone 2009; Ebin and Marsden 1970; Mitrea and Taylor 2001; Nitschke et al. 2017, 2012; Reuther and Voigt 2015, 2018). Compared with its counterpart in flat space, not only the operators are replaced by the corresponding surface operators, also an additional contribution from the Gaussian curvature arises. This additional term results from the surface divergence of the surface strain tensor, see (Arroyo and DeSimone 2009; Jankuhn et al. 2017). The unusual sign results from the definition of the surface Laplace-deRham operator (Abraham et al. 1988). Eq. (1) with \( \mathbf{v} = 0 \), \( v_n = 0 \) and the Laplace-deRham operator \( \Delta_{\text{dir}} \) instead of the Bochner Laplacian \( \Delta_{\text{DG}} \) has been derived as a thin-film limit in (Nestler et al. 2018) and models the \( L^2 \)-gradient flow of a weak surface Frank-Oseen energy. The different operators result from different one-constant approximations in the Frank-Oseen energy, see appendix B for details. Again an additional geometric term enters in this equation if compared with the corresponding model in flat space. The term with the shape operator \( \mathcal{B} \) results from the influence of the embedding (Napoli and Vergori 2012; Nestler et al. 2018; Segatti et al. 2014). The coupled system eqs. (1) - (3) with \( v_n = 0 \) can be considered as the surface counterpart of the model in (Lin and Liu 2000). A related surface model has been proposed and analyzed in (Shkoller 2002). However, this model is derived from a variational principle on a stationary surface and thus only contains intrinsic terms. It differs from eqs. (1) - (3) with \( v_n = 0 \) by the extrinsic term \( \mathcal{B}^2 \mathbf{p} \) and the extrinsic contribution in the surface Ericksen stress tensor \( (\mathbf{B}) \otimes (\mathbf{B}) \). We will numerically demonstrate the influence of the additional geometric terms. If \( v_n \neq 0 \) further coupling terms occur. For a general discussion on transport of vector-valued quantities on evolving surfaces we refer to (Nitschke and Voigt 2018). The used formulation with the projection operator \( \pi_S \) requires the presence of an embedding space, which is in our case \( \mathbb{R}^3 \), see appendix B for details.
III. NUMERICAL METHOD

Eqs. (1) - (3) is a system of vector-valued surface PDEs. Numerical approaches have been developed for such equations on general surfaces only recently, see (Olshanski et al. 2018; Reuther and Voigt 2018) for the surface (Navier-)Stokes equation, (Nestler et al. 2018) for the surface Frank-Oseen model and for a surface vector-Laplace equation (Hansbo et al. 2016). Earlier approaches using vector spherical harmonics, see (Freeden and Schreiner 2008; Nestler et al. 2018; Pratoreius et al. 2018), are restricted to a sphere or radial manifold shapes (Gross and Atzberger 2018) and approaches which rewrite the surface Navier-Stokes equation in a surface vorticity-stream function formulation (Mickelin et al. 2018; Nitschke et al. 2012; Reusken 2018; Reuther and Voigt 2015) are limited to surfaces with genus $g(S) = 0$, see (Nestler et al. 2017; Reuther and Voigt 2018) for details. For the numerical solution of eqs. (1) - (3) we combine the methods in (Nestler et al. 2018; Reuther and Voigt 2018) in an operator splitting approach. The idea behind these methods is to extend the variational space from vectors in $T_S$ to vectors in $\mathbb{R}^3$, while penalizing the normal components. This allows to split the vector-valued surface PDE into a set of coupled scalar-valued surface PDEs for each component for which established numerical methods are available, see the review (Dziuk and Elliott 2013).

The corresponding extended problem to eqs. (1) - (3) reads

$$\pi_S \partial_t \tilde{v} + \nabla^S \tilde{v} = v_n (B \tilde{v} + \nabla_S v_n) - \nabla_S p_S - v \hat{\Delta}^r \tilde{v} + \nu (2 k \tilde{v} + \nabla_S (v_n H) - 2 \nabla_S (v_n B)) - \lambda \nabla S \hat{\sigma}^E_S - \alpha_\nu (\tilde{v} \cdot \nu) \nu$$

$$\text{div}_S \tilde{v} = v_n H$$

$$\pi_S \partial_t \tilde{p} + \nabla^S \tilde{p} = \eta \left( \Delta^D \tilde{p} - B^\mathcal{P} \right) - \omega \left( \| \tilde{p} \|^2 - 1 \right) \tilde{p} - \alpha_\rho (\nu \cdot \tilde{p}) \nu$$

with $\tilde{v} = v_x e^x + v_y e^y + v_z e^z$ and $\tilde{p} = p_x e^x + p_y e^y + p_z e^z \in \mathbb{R}^3$, not necessarily tangential to the surface and $\hat{\sigma}^E_S = (\nabla_S \tilde{p})^T \nabla_S \tilde{p} - (B \tilde{p}) \otimes (B \tilde{p})$. We further use $\text{div}_S \tilde{v} = \nabla \cdot \tilde{v} - \nu \cdot (\nabla \nu \cdot \nu)$, $\text{rot}_S \tilde{v} = -\text{div}_S (\nu \times \tilde{v})$ and $\Delta^D \tilde{p} = \text{div}_S \nabla_S \tilde{p}$ and $\Delta^r \tilde{v} = -\text{rot}_S \text{rot}_S \tilde{v} - \nabla_S (v_n H)$ since $\text{div}_S \tilde{v} = -v_n H$. The normal components $\tilde{v} \cdot \nu$ and $\tilde{p} \cdot \nu$ are penalized by the additional terms $\alpha_\nu (\nu \cdot \tilde{v}) \nu$ and $\alpha_\rho (\nu \cdot \tilde{p}) \nu$. For convergence results in $\alpha_\nu$ and $\alpha_\rho$ for the surface Navier-Stokes and the surface Frank-Oseen problem we refer to (Nestler et al. 2018; Reuther and Voigt 2018). Eqs. (1) - (6) can now be solved for each component $v_x, v_y, v_z, p_x, p_y, p_z$ and $p_S$ using standard approaches for scalar-valued problems on surfaces, such as the surface finite element method (Dziuk and Elliott 2007ab, 2013), level set approaches (Bertalmio et al. 2001; Dziuk and Elliott 2008; Stöcker and Voigt 2008) or diffuse interface approximations (Rätz and Voigt 2006). We consider a simple operator splitting approach and solve eq. (4)-(5) and eq. (6) iteratively in each time step, employing the same surface finite element discretizations as in (Nestler et al. 2018; Reuther and Voigt 2018). A semi-implicit Euler discretization in time is used. Thereby, the nonlinear transport term in eq. (4) and the normal-1 penalization term in eq. (6) are linearized in time by a Taylor-1 expansion and the transport term in eq. (6) as well as the term including the surface Ericksen stress tensor in eq. (4) are coupling terms in the operator splitting scheme. We additionally employ an adaptive time-stepping scheme which is based on the combination of changes in the surface Frank-Oseen energy and the Courant-Friedrichs-Lewy (CFL) condition. For more details we refer to appendix A. The resulting discrete equations are implemented in the FEM-toolbox AMDiS (Verf and Voigt 2017; Witkowski et al. 2015), where we additionally use a domain decomposition ansatz to efficiently distribute the workload on many cores systems.

IV. RESULTS

In the following simulations we use $\lambda = 0.5$, $\alpha_\nu = 10^2$, $\omega \eta = 10^2$ and $\alpha_p = 10^4$, where all parameters are treated as non-dimensional. We compare the solution of eqs. (4)-(6) (the so-called wet case) and the solution of eq. (6) with $\tilde{v} = 0$ (the so-called dry case). To highlight the differences we take the surface Frank-Oseen energy $F^P$ and the surface kinetic energy $F^{\text{kin}}$ into account, which read in the extended form incorporating the penalization term,

$$F^P := \int_S \frac{\eta}{2} \left( \| \nabla_S \tilde{p} \|^2 + (B \tilde{p})^2 \right) + \frac{\omega \eta}{4} (\| \tilde{p} \|^2 - 1)^2 dS + \frac{\alpha_\rho}{2} \int_S (\tilde{p} \cdot \nu)^2 dS$$

$$F^{\text{kin}} := \frac{1}{2} \int_S \tilde{v}^2 dS$$

First, we consider eqs. (4)-(6) on a stationary, $v_n = 0$, ellipsoidal shape with major axes parameter $(0.7, 0.7, 1.2)$. We use the trivial solution as initial condition for the velocity and for the director field $\tilde{p}^0 = \nabla_S \psi^0 / \| \nabla_S \psi^0 \|$ with $\psi^0 = x_0 / 10 + x_1 / 10$ and $x = (x_0, x_1, x_2)^T$ the Euclidean coordinate vector. The latter generates a vector field with two +1 defects – to be more precise a source and a sink defect – and an out-of-equilibrium solution. Furthermore, we use $\nu = 2$, $\eta = 0.6$ and $h_\text{m} = 1.32 \cdot 10^{-2}$, where $h_\text{m}$ denotes the maximum mesh size. Figure 3 shows the influence of the hydrodynamics on the dynamical evolution of the director field. The two defects, which fulfill the Poincaré-Hopf theorem, evolve towards the geometrically favorable positions of...
high Gaussian curvature, the director field aligns with the minimal curvature lines of the geometry and as in flat space the hydrodynamics enhances the evolution towards the equilibrium configuration, which coincides for the dry and the wet case.

In the next example we consider a stationary torus with major radius $R = 2$, minor radius $r = 0.5$ and the $x_2$-axis as symmetry axis. Again we use the trivial solution as initial condition for the velocity $\hat{v}$ and a random (normalized) vector field for the director field $\hat{p}$. We here use the simulation parameters $\nu = 1$ and $\eta = 0.4$. The maximum mesh size is fixed at $h_m = 2.74 \cdot 10^{-2}$. All other parameters are equal to that used in Figure 1. In Figure 2 we focus on the annihilation of defects in one realization, (top) shows the evolution of the director field $\hat{p}$ for the dry and below for the wet case. Again in the wet case the dynamic is enhanced, which is quantified by the stronger overall decay of the surface Frank-Oseen energy, cf. Figure 2 (middle). Additionally, in Figure 2 (bottom) the corresponding flow field $\hat{v}$ is shown for the considered annihilation of a source (+1) and a saddle (−1) defect in the director field $\hat{p}$. After all defects are annihilated, which again is in accordance with the Poincaré-Hopf theorem, the velocity field $\hat{v}$ decays to zero and the director field $\hat{p}$ aligns with the minimal curvature lines of the geometry. The reached equilibrium configuration coincides for both the dry and the wet case.

While in the two previous examples the expected minimal energy configuration was reached, we now consider an initial condition for which only a local min-
FIG. 3 Top: Schematic defect positions of the initial condition (left) and the final configuration (right) on a torus with the analytical initial condition for the director field $\hat{p}$ and zero initial condition for the velocity field $\hat{v}$. Red dots are indicating $+1$ defects (sources or sinks) and blue dots are indicating $-1$ defects (saddle). Bottom: Evolution of the director field $\hat{p}$ for $t = 1, 5, 25$ (left to right).

FIG. 4 Surface Frank-Oseen energy $F^P$ and surface kinetic energy $F^{\text{kin}}$ vs. time $t$ for the analytical initial condition for the director field $\hat{p}$ and the killing vector field as initial condition for the velocity $\hat{v}$.

ima can be reached. We use $\hat{p}^0 = \nabla_S \psi^0 / \| \nabla_S \psi^0 \|$ with $\psi^0 = \exp (-\langle \mathbf{x} - \mathbf{m} \rangle^2/2)$ and $\mathbf{m} = (R, 0, r)^T$ as initial condition for the director field. This produces two $\pm 1$ defect pairs which are located in opposite position to each other w.r.t. to the symmetry axis of the torus, again fulfilling the Poincaré-Hopf theorem. Thereby, one pair is rotated by an angle of $\pi/2$ compared to the other along the circle with the small radius, see Figure 3. The parameters are adapted to $\nu = 1$, $\eta = 0.4$ and $h_m = 2.74 \cdot 10^{-2}$. In a flat geometry with zero curvature these two pairs would annihilate. However, due to the geometric interaction in the present case resulting from the difference of the Gaussian curvature inside and outside of the torus, the reached nontrivial defect configuration is stable and the two $\pm 1$ defect pairs remain over time. The $-1$ defects are attracted to regions with positive Gaussian curvature, i.e. the outer of the torus, see Figure 3. The reached configuration, is a local minimum with a significantly larger surface Frank-Oseen energy $F^P$ as the defect-free configuration. In this example we did not find any significant difference between the dry and the wet case, when the zero initial condition for the velocity $\hat{v}$ is used. However, if we use a Killing vector field for the velocity as initial condition, i.e. $\hat{v}^0 = 1/2(0, 0, -x_1)$, cf. (Nitschke et al., 2017; Reuther and Voigt, 2018), the four defects start to rotate and cause a damping of the flow field, which converges to zero. In other words, the defects in the director field break the symmetry in the deformation tensor and therefore the kinetic energy dissipates to zero, see Figure 4. Thus, the final configuration is just a rotation of the configuration reached with $\hat{v}^0 = 0$, with the rotation angle depending on the strength of the initial velocity and the viscosity.

We now let the ellipsoid from Figure 1 evolve by prescribing the normal velocity $v_n$, such that the ellipsoid changes to a sphere and afterwards to an ellipsoid with a different axis orientation and vice versa to obtain a shape oscillation. The surface area remains constant during the evolution. Figure 5 shows schematically the evolution of the geometry and the axes parameters for one period of oscillation. We here use the same simulation parameters and initial conditions as considered in Figure 1. The evolution of the director field $\hat{p}$ is shown in Figure 6 again for the dry (top) and the wet (bottom) case. The defect positions again reallocate at their geometrically favorable position. However, due to the change in geometry the time scale for the reallocation competes with the time-scale for the shape changes. The enhanced evolution towards the minimal energy configuration with hydrodynamics becomes even more significant in these situations. Already slight modifications of the geometry are enough
to push the defect after crossing the sphere configurations, with no preferred defect position, to the energetically favorable state. In the dry case there is a strong delay and much stronger shape changes are needed to push the defect to the energetically favorable position. First an energy barrier for reallocating the defect position has to be overcome, which is shown by the further increase of the red line after the blue line has already dropped after crossing the sphere configuration in Figure 6 (middle). The parameters and the initial condition are further chosen in such a way that the defects in the dry case not quite reach the position at the poles if the shape evolution crosses the sphere and in the wet case they have moved beyond. This results in a constant orientation in the dry case and a flipping of the orientation of the director field in the wet case in each oscillation. The final configuration in Figure 6 after completing one oscillation cycle is energetically equivalent for the dry and the wet case even if the orientation of the director field $\hat{\mathbf{p}}$ differs, see also the video in the supplementary material. This behavior clearly depends on the used parameters. However, it also demonstrates the strong influence hydrodynamics might have in such highly nonlinear systems, where the topology, geometric properties and defect dynamics are strongly coupled.

This examples together with the demonstrated energy reduction by creation of additional defects in geometrically favored positions in (Nestler et al. 2018), which is expected to hold also for the wet case, leads to a very rich phase space, considering geometric and material properties, whose exploration is beyond the scope of this paper.

V. DISCUSSION

Eqs. (1) - (3) have been derived as a thin-film limit of a three-dimensional simplified Ericksen-Leslie model, see appendix B. In (Shkoller 2002) a similar model was proposed, which differs from eqs. (1) – (3) with $v_n = 0$ in the extrinsic contributions. Especially the surface Ericksen stress tensor is considered to be $\sigma_S^E = (\nabla S \mathbf{p})^T \nabla S \mathbf{p}$. To show the strong difference between this intrinsic and the extrinsic surface Ericksen stress tensor from above $\sigma_S^E = (\nabla S \mathbf{p})^T \nabla S \mathbf{p} + (\mathbf{B} \mathbf{p}) \otimes (\mathbf{B} \mathbf{p})$ we now come back to the stationary ellipsoid and with slightly different parameters, i.e. $\nu = 0.5$, $\eta = 0.3$ and $\lambda = 1$. These parameters lead to a damped oscillation of the defects around the energetically favorable positions before they reach the final state configuration as in Figure 1. In Figure 7 the differences in the time evolution of the surface Frank-Oseen energy as well as the surface kinetic energy for both cases the intrinsic and extrinsic surface Ericksen stress are shown. The influence of the hydrodynamics is much stronger for the extrinsic surface Ericksen stress. Together with the example in Figure 6 such differences in the dynamics might have a huge impact on the overall evolution if also shape changes are considered. We would also like to point out, that even if the thin-film limit of the individual equations of eqs. (1) - (3), namely the surface Navier-Stokes and the surface Frank-Oseen equation are known (Nestler et al. 2018; Reuther and Voigt 2015), a naive transformation of the missing cou-
point term from the three-dimensional formulation to its surface counterpart will not be sufficient to obtain the extrinsic surface Ericksen stress term. Only a thin-film limit of the complete model, with appropriate boundary conditions, will lead to the full model. Appendix B provides all necessary tools to do this also for more complicated systems. We here provide the formulation for a surface active polar viscous gel, see (Kruse et al. 2004; Simha and Ramaswamy, 2002) and (Marth et al. 2015; Tjhung et al. 2012) for the considered three-dimensional formulation, which correspond to eqs. (B1) - (B3) with boundary conditions (B4) - (B6), i.e.

\[
\partial_t V + \nabla_U V = -\nabla P_{th} + \nu \Delta V + \text{div} \sigma^A - \lambda \text{div} \sigma^E \\
\text{div} V = 0 \\
\partial_t P + \nabla_U P = H + \alpha D_P + \Omega_P
\]

with

\[
\sigma^A = \frac{1}{2} (PH^T - HP^T) - \frac{\alpha}{2} (PH + HP^T) + \beta PP^T \\
H = \eta \Delta P - \omega_n \left( \left\| P \right\|_{\Omega_h}^2 - 1 \right) P \\
D = \frac{1}{2} \left( \nabla V + (\nabla V)^T \right) \\
\Omega = \frac{1}{2} \left( \nabla V - (\nabla V)^T \right)
\]

and \( \alpha, \beta \in \mathbb{R} \). The Navier-Stokes equation now contains additional distortion and active stresses, combined in \( \sigma^A \), while in the director field equation additional contributions from the deformation tensor \( D \) and the vorticity tensor \( \Omega \) arise. The model is a more general Ericksen-Leslie model for polar LC with an active contribution \( \beta PP^T \). The corresponding thin-film limit reads

\[
\pi_S \partial_t \bar{V} + \nabla_S \bar{V} = \nu_n (B\bar{V} + \nabla_S \nu_n) + \nu (-\Delta^{\text{div}} \bar{V} + 2\kappa \bar{V}) \\
+ \nu (\nabla_S (\nu_n \mathcal{H}) - 2 \text{div}_S (\nu_n B)) \\
- \nabla_S \sigma^A_S + \text{div}_S \sigma^E_S - \lambda \text{div}_S \sigma^E_S \\
- \frac{1 - \alpha}{2} (p^T \mathcal{B} \mathcal{H} \bar{V} + \nabla_S \nu_n))p \\
+ \frac{1 + \alpha}{2} (p^T B\bar{V}p) \\
\text{div}_S \bar{V} = \nu_n \mathcal{H} \\
\pi_S \partial_t \bar{P} + \nabla_S \bar{P} = h + \alpha D_S \bar{P} + \Omega_S \bar{P} - \alpha v_n \mathcal{H} \bar{P}
\]

with

\[
\sigma^A_S = \frac{1}{2} (ph^T - hp^T) - \frac{\alpha}{2} (ph + hp^T) + \beta pp^T \\
h = \eta (\Delta^{\text{div}} \bar{P} - \mathcal{B}^2 \bar{P}) - \omega_n \left( \left\| \bar{P} \right\|_{\Omega_h}^2 - 1 \right) \bar{P} \\
D_S = \frac{1}{2} (\nabla S \bar{V} + (\nabla S \bar{V})^T) \\
\Omega_S = \frac{1}{2} (\nabla S \bar{V} - (\nabla S \bar{V})^T)
\]

Besides corresponding surface operators and the additional geometric coupling terms with the shape operator \( \mathcal{B} \) and the mean curvature \( \mathcal{H} \) we also obtain an explicit appearance of the normal velocity \( \nu_n \) in the director field equation. Overall the additional terms in the more general Ericksen-Leslie model lead to an even tighter coupling between geometric properties and dynamics. A detailed analysis of these additional coupling terms and the influence of the activity will be discussed elsewhere.

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Appendix A: Numerics

To efficiently solve the surface Navier-Stokes equation in (Reuther and Voigt, 2018) a heavy assembly workload was avoided by applying \( \nu \times \) to eq. (4) and considering the rotated velocity field \( \bar{w} := \nu \times \bar{V} \). Since only the Rot_\mathcal{S} rot_\mathcal{S} (\cdot) operator occurs as second order operator we here can also use the same idea to reduce the assembly costs. Thus, the rotated version of eqs. (4) and (5) with tangential penalization of the rotated velocity \( \bar{w} \) now reads

\[
\pi_S \partial_t \bar{w} + \nabla_S \bar{w} = \text{rot}_\mathcal{S} \bar{P} + \nu (\nabla_S \text{div}_S \bar{w} + 2\kappa \bar{w}) \\
+ f_\mathcal{S} + f_\mathcal{w} - \lambda \nu \times \text{div}_S \sigma^E_S \\
\text{rot}_\mathcal{S} \bar{w} = \nu_n \mathcal{H}
\]
where we used for the convenience the abbreviations
\[
\begin{align*}
    f_g &:= v_n \operatorname{rot}_S v_n + 2\nu (\mathcal{H} \operatorname{rot}_S v_n - \nu \times (B \nabla v_n)) \\
    f_{\Phi} &:= -v_n \nu \times (B (\nu \times \hat{w})) - \alpha_v (\hat{w} \cdot \nu) \nu
\end{align*}
\]
and the alternative form of the viscous terms proposed in [B23].

**Time discretization**

For the discretization in time we again use the same approach proposed in [Reuther and Voigt, 2018]. Let the time interval \([t_0, t_{end}]\) be divided into a sequence of discrete times \(0 < t_0 < t_1 < \ldots\) with time step width \(\tau^m = t^m - t^{m-1}\). Thereby, the superscript denotes the timestep number. The vector field \(\hat{w}^m(x)\) correspond to the respective rotated velocity field \(\hat{w}(x, t^m)\). All other quantities follow the same notation. The time derivative is approximated by a standard difference quotient and a Chorin projection method ([Chorin, 1968]) is applied to eqs. (A1) and (A2). Furthermore, we define the discrete time derivatives\(d^m_{\hat{w}} := \frac{1}{\tau^m} (\hat{w}^m - \pi_S \hat{w}^{m-1})\) and \(d^m\rho := \frac{1}{\tau^m} (\hat{p}^m - \pi_S \hat{p}^{m-1})\), with \(\pi_S\) the projection to the surface at time \(t^m\). Thus, we get a time-discrete version of eqs. (A1), (A2) and (0):

\[
\begin{align*}
    &d^m_{\hat{w}} + \nabla \hat{w} \cdot \hat{w} = \nu (\nabla_S \hat{w} - 2\kappa \hat{w}^* + f_g + f_{\Phi}), \\
    &-\lambda \nu \times \nabla_S \hat{w}^* = \tau^m \Delta_{S\rho} \hat{w}^* - \nu \mathcal{H} \\
    &\hat{w}^m = \hat{w}^* - \tau^m \nabla \hat{w} \cdot \hat{w} \rho \\
    &d^m\rho + \nabla \hat{w} \cdot \hat{w}^m = \eta \left( \Delta_{DG} \hat{w}^m - B^2 \hat{w}^m \right) \\
    &\quad - \omega_n (\|\hat{p}^{m-1}\|^2 - 1) \hat{p}^m \\
    &\quad - \alpha_p (\nu \cdot \hat{p}^m) \nu
\end{align*}
\]

where \(\hat{\sigma}^E\) is evaluated at the old timestep, i.e. \(\hat{\sigma}^E = (\nabla_S \hat{p}^{m-1})^T \nabla_S \hat{p}^{m-1} + (B \hat{p}^{m-1}) \otimes (B \hat{p}^{m-1})\). Note that for readability we used a Taylor-0 linearization of the transport term in (A3) and the norm-1 penalization term in (0). In the simulations from above we performed a Taylor-1 linearization, see [Nitschke et al., 2017] and [Nestler et al., 2018] for details.

**Spatial discretization**

The considered extension of the tangential vector fields to the Euclidean space allows us to apply the surface finite element method ([Dziuk and Elliott, 2013]) for each component of the respective vector field. Let \(S_h\) denote the interpolations of the respective surfaces \(S(t^m)\) such that \(S_h := \bigcup_{T \in S_h} T\) with a conforming triangulation \(S_h\). Furthermore, we introduce the finite element space

\[
V_h(S_h) = \{ v_h \in C^0(S_h) : v_h|_T \in \mathbb{P}_1, \forall T \in T \}
\]

which is used twice as trail and test space and the standard \(L_2\) scalar product on \(\mathbb{V}_h(S_h), (\alpha, \beta) := \int_{S_h} (\alpha, \beta) dS\). By using an operator splitting technique we decouple the hydrodynamic and the director field equation in the following way. First the surface finite element approximation of eqs. (A3), (A4) is solved, which reads:

\[
\begin{align*}
    &\left( d^m_{\hat{w}} + \nabla \hat{w} \cdot \hat{w}^* - 2\eta \nu \hat{w}^* - (f_g + f_{\Phi}), \xi \right) \\
    &= -\left( \nu \nabla_S \hat{w}^*, (\nabla_S \xi)_i \right) - \left( \nu \nabla_S \hat{w}^*, (\nabla_S \nu \times (\xi \epsilon_i)) \right) \\
    &\quad - \left( \lambda \hat{w}^*, (\nu \nabla \hat{w} \cdot \hat{w}) \nu, \eta \right)
\end{align*}
\]

for \(i = x, y, z\). The resulting vector field \(\hat{w}^m\) is then used to determine \(\hat{w}^m\) by the pressure correction step in eq. (A5). The transformation \(\tilde{v}^m = -\nu \times \hat{w}^m\) leads to the velocity field at the new timestep \(t^m\). Finally, the surface finite element approximation of eq. (A6) is solved, which reads:

\[
\begin{align*}
    &\left( d^m\rho_i + \nabla \hat{w} \cdot \hat{p}^m, \xi \right) \\
    &= -\left( \eta \nabla_S \hat{p}^m, (\nabla_S \xi)_i \right) - \left( \eta \nu \hat{p}^m, \xi \right) \\
    &\quad - \left( \omega_n (\|\hat{p}^{m-1}\|^2 - 1) \hat{p}^m + \alpha_p (\nu \cdot \hat{p}^m) \nu_i, \xi \right)
\end{align*}
\]

for \(i = x, y, z\).

**Pressure relaxation schemes**

In some situations it is useful to modify the Chorin projection scheme (A3), (A4) and (A5). To be more precise the resulting finite element matrix of the pressure equation (A4) is sometimes ill-conditioned, especially when the term \(\nabla \hat{w}^*\) is big compared to the others. The solution of eq. (A4) can be seen as the steady-state solution of a heat conduction equation where the heat source is determined by the right hand side of eq. (A4). Therefore, we add a relaxation scheme in form of a discrete time derivative on a different timescale to the left hand side of eq. (A4), i.e.

\[
\frac{1}{\tau^*} \left( p^{m+1,i+1} - p^{m+1,i} \right) = -\nabla \hat{w} \cdot \hat{w}^* + \nu \mathcal{H},
\]

where \(\tau^*\) denotes the timestep and \(l\) the timestep number on the different timescale. Instead of solving eq. (A4) the iterative process in eq. (A7) is performed until a steady-state is reached, which is then used in the correction step in eq. (A5).

**Appendix B: Thin film limit**

We assume a regular moving surface \(S(t) \subset \mathbb{R}^3\) without boundaries and a thin film \(\Omega_h(t) := S(t) \times [-h/2, h/2] \subset \mathbb{R}^3\).
\( \mathbb{R}^3 \) of sufficiently small thickness \( h \), such that the thin film parametrization \( X(t, y^1, y^2, \xi) = x(t, y^1, y^2) + \xi \nu(t, y^1, y^2) \) is injective for the surface parametrization \( x(t, \cdot, \cdot) \). Thereby, \( y^1 \) and \( y^2 \) denote the local surface coordinates, \( \nu(t, \cdot, \cdot) \) the surface normal field and \( \xi \in [-h/2, h/2] \) is the local normal coordinate. Since the thin film is moving according to the surface, the parametrization \( X \) is not unique, which arises from the choice of an observer within the thin film. For a pure Eulerian observer, i.e., for the observer velocity \( W = \partial_t X = 0 \), we are not able to formulate proper intrinsic physics at the surface \( S \) for \( h \rightarrow 0 \). To overcome this issue, we choose a transversal observer as the surface observer parametrization \( x \), i.e., Eulerian in the tangential space and Lagrangian in normal direction and hence \( \partial_t x = \partial_t X \big|_{S} = v_n \nu \), where \( v_n = W^k \big|_{S} \) is the normal surface velocity of \( S \). To ensure a constant thickness \( h \) of the thin film and that the surface \( S \) is located in the middle of the thin film over time, we stipulating the same transversal behavior for both boundary surfaces. Thus, we get \( \partial_t X \big|_{\partial\Omega_h} = v_n \nu + \frac{h}{2} \partial_t \nu = W^k \big|_{\partial\Omega_h} \nu \) and therefore \( W^k \big|_{\partial\Omega_h} = v_n \), since \( \partial_t \nu \) is always tangential on the boundaries \( \partial\Omega_h \).

For notational compactness of tensor algebra we use the thin film calculus presented in [Nitschke et al., 2018] and [Nestler et al., 2018] (Appendix) which is based on Ricci calculus, where lowercase indices \( i, j, k, \ldots \) denote components w.r.t. \( y^1 \) and \( y^2 \) in the surface coordinate system and uppercase indices \( I, J, K, \ldots \) denote components w.r.t. \( y^1, y^2 \) and \( \xi \) in the extended three dimensional thin film coordinate system. Metric quantities at the surface \( S \) are the metric tensor \( g_{ij} = \langle \partial_i x, \partial_j x \rangle_{\mathbb{R}^3} \) (first fundamental form), the shape operator \( B_{ij} = -\langle \partial_i x, \partial_j \nu \rangle_{\mathbb{R}^3} \) (second fundamental form), its square \( [B^2]_{ij} = B_{ik}B_{kj} = (\partial_i \nu, \partial_j \nu, \nu)_{\mathbb{R}^3} \) (third fundamental form), the mean curvature \( \mathcal{H} = tr B = B^i_i \), the Gaussian curvature \( \kappa = det B^2 = det \{B^i_j\} \) and the Christoffel symbols \( \Gamma^k_{ij} = \frac{1}{2} g^{kj}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \) for covariant differentiating (e.g. \( \nabla X \cdot \nabla \nu \big|_{S} = p^k\big|_{k} = \partial_k p^i + \Gamma^i_{kj}p^j \) for a contravariant vector field \( p \in T S = T^1 S \)). In the thin film \( \Omega_h \), the metric tensor \( G_{IJ} = (\partial_i X, \partial_j X)_{\mathbb{R}^3} \) and the Christoffel symbols \( \Gamma^K_{IJ} = \frac{1}{2} G^{KL}(\partial_K G_{IJ} + \partial_J G_{IK} - \partial_I G_{KL}) \) for covariant differentiating (e.g. \( \nabla X \cdot \nabla \nu \big|_{S} = p^k\big|_{K} = p^I_{K} + \Gamma^I_{KJ}p^J \) for a contravariant vector field \( p \in T \Omega_h = T^1 \Omega_h \)) can be developed at the surface by \( G_{ij} = g_{ij} - 2\xi B_{ij} + \xi^2 [B^2]_{ij} \), \( G_{\xi \xi} = 1 \), \( G_{\xi i} = G_{i \xi} = 0 \), \( G_\xi = g^\xi = 1 \), \( G_{\xi i} = G_{i \xi} = 0 \), \( \Gamma^i_{ij} = \Gamma^i_{ji} = \Gamma^i_{jk} + \partial_j \nu + \mathcal{O}(\nu) \), \( \Gamma^i_{\xi j} \equiv B_{ij} + \mathcal{O}(\nu) \), \( \Gamma^i_{\xi \xi} = \Gamma^i_{\xi i} = -B^k_{ij} + \partial_k \nu + \mathcal{O}(\nu) \) and \( \Gamma^k_{\xi i} = \Gamma^k_{i \xi} = \Gamma^k_{\xi \xi} = 0 \), see [Nitschke et al., 2018] and [Nestler et al., 2018] for details.

Our starting point is the simplified local three dimensional Ericksen-Leslie model ([Lin and Liu, 2000]), i.e.

\[
\begin{align*}
\partial_t V + \nabla_U V &= -\nabla P_{\Omega_h} + \nu \Delta V - \lambda \div \sigma^E \\
\div V &= 0
\end{align*}
\]  
\begin{align*}
\partial_t P + \nabla_U P &= \eta \Delta P - \omega_n (\|P\|_{\Omega_h}^2 - 1) P
\end{align*}

in \( \Omega_h \times \mathbb{R}^+ \) with fluid velocity \( V \in T \Omega_h \), relative fluid velocity \( U = V - W \in T \Omega_h \), with respect to the observer velocity \( W = \partial_t X \), director field \( P \in T \Omega_h \), pressure \( P_{\Omega_h} \), Ericksen stress tensor \( \sigma^E = (\nabla P)^T \nabla P \), fluid viscosity \( \nu \), competition between kinetic energy and elastic potential energy \( \lambda \) and elastic relaxation time for the molecular orientation field \( \eta \). Besides initial conditions, we consider homogeneous Dirichlet boundary conditions for the normal components and Neumann boundary conditions for the tangential components of the director and homogeneous Navier boundary conditions for the velocity field, i.e.

\[
\begin{align*}
(P, \nu)_{\partial \Omega_h} &= (\hat{P}, 0) \\
\nabla \nu \left( \pi^3_{\partial \Omega} P \right) &= \{P_{i \xi}, 0\} = 0 \\
\pi^3_{\partial \Omega}(\nu \cdot \mathbf{L} \mathbf{G}) &= \{V_{i \xi} + V_{\xi i}, 0\} = 0
\end{align*}
\]  

on \( \partial\Omega_h \), which follows from the special choice of the transversal observer from above.

As proposed in [Nitschke et al., 2018] the boundary quantities are continuous to the surface \( S \) by Taylor expansions at the boundaries, which e.g. results in

\[
\begin{align*}
P_{\xi} \big|_{S} &= \mathcal{O}(h^2) \\
P_{i \xi} \big|_{S} &= \mathcal{O}(h^2) \\
V_{i \xi} \big|_{S} &= \mathcal{O}(h^2) \\
V_{i \xi} + V_{\xi i} \big|_{S} &= \mathcal{O}(h^2)
\end{align*}
\]  

Note that the right identity of eq. \[B9\] is achieved by using \( V_{\xi} \big|_{X(\xi = -\frac{h}{2})} = V_{\xi} \big|_{X(\xi = \frac{h}{2})} = V_{\xi} \big|_{X(\xi = 0)} = v_n \), the related second order difference quotient and \( V_{i \xi} + V_{\xi i} = \partial^2_\xi V_{\xi} \), since vanishing Christoffel symbols \( \Gamma^K_{IJ} \).

With all these tools from above, we are able to realize a thin film limit of eqs. \[B1\] - \[B3\] for \( h \rightarrow 0 \), consistently, by Taylor expansion of the equations at the surface. The thin film \( \Omega_h \) is a flat Riemannian manifold and therefore the Riemannian curvature tensor vanish, i.e. covariant...
derivatives commute, and with the continuity equation \[ (\mathcal{L}_V G)_{|S} = v_{ij} + v_{li} - 2v_n B_{ij} \]
we obtain

\[
[\Delta V]_I = [\text{div} \nabla V + \text{div} \nabla V]_I = V_{I_{,K}} + V_{K_{,I}} = V_{I_{,K}} + V_{K_{,I}} = [\text{div} \mathcal{L}_V G]_I .
\]

(B12)

Hence, we have to develop three divergence terms of 2-tensors at the surface in eqs. (B1) - (B3), namely \( T_{\xi|S} \) for \( T \) being either \( \mathcal{L}_V G \), \( \sigma_{\xi}^E \) or \( \nabla P \). By using eqs. (B7) - (B11) it holds

\[
T_{\xi|S} = O(h^2) , \quad \partial_{\xi} T_{\xi|S} = O(h^2) , \nonumber \\
T_{\xi|\xi|S} = O(h^2) .
\]

(B13)

The covariant tangential components of the divergence at the surface are

\[
[\text{div} T]_I = T_{I_{,K}} = \left( \partial_{K} T_{I_{,K}} + \tau T_{I_{,K}} + \pi T_{I_{,K}} \right)\bigg|_S ,
\]

where it holds by using eq. (B13)

\[
\partial_{K} T_{I_{,K}} = \partial_K T_{I_{,K}} + O(h^2) , \quad \tau T_{I_{,K}} = \left( \tau T_{I_{,K}} + B_{ik} T_{\xi}^{k} \right)\bigg|_S + O(h^2) , \quad \pi T_{I_{,K}} = \left( \pi T_{I_{,K}} \right)\bigg|_S + O(h^2) .
\]

(B14)

(B15)

(B16)

Adding this up and taking the metric compatibility of \( \nabla S \) into account, we obtain

\[
[\text{div} T]_I = \left( T_{I_{,K}} \right)_{|S} - B_{ik} T_{\xi}^{k} + O(h^2) = \left( G_{ij} V_{ij} + \nabla S v_v \right) + O(h^2) .
\]

(B17)

Note that all normal derivatives vanished here, i.e. there is no need for a higher order expansion in \( \xi \) of the thin film Christoffel symbols as a consequence of the used boundary conditions. To substantiate the tensor \( T \in \Gamma(2) \Omega_h \), we first observe that

\[
V_{ij} = \left( \partial_{ij} V_{ij} - \Gamma_{ji}^{k} V_{k} - \Gamma_{ji}^{\xi} V_{\xi} \right)\bigg|_S = v_{ij} + v_{\xi} B_{ij} , \quad (B18)
\]

\[
P_{ij} = \left( \partial_{ij} P_{ij} - \Gamma_{ji}^{k} P_{k} - \Gamma_{ji}^{\xi} P_{\xi} \right)\bigg|_S = p_{ij} + O(h^2) , \quad (B19)
\]

\[
P_{\xi i} = \left( \partial_{ij} P_{\xi i} - \Gamma_{ji}^{k} P_{\xi k} \right)\bigg|_S = B_{kj} p_{k} + O(h^2) , \quad (B20)
\]

where \( v_i := v_{\xi i} \) and \( p_i := P_{ij} \), i.e. \( v := \pi S V_{ij} \in \Gamma(1) \Omega_h \) and in contravariant form \( p := \pi S P_{ij} \) \( \in \Gamma(1) \Omega_h \). We further obtain

\[
[\mathcal{L}_V G]_{|S} = v_{ij} + v_{li} - 2v_n B_{ij} = \left[ \mathcal{L}_V g - 2v_n B \right]_{|S} ,
\]

\[
[\mathcal{L}_V g - 2v_n B]_{|S} = \left[ \mathcal{L}_V g - 2v_n B \right]_{|S} ,
\]

\[
\left( \mathcal{E}^E_{|S} \right) = \left( \mathcal{E}^E_{|S} \right) + O(h^2) ,
\]

\[
\pi S \left( \left[ \mathcal{E}^E_{|S} \right] + O(h^2) \right) = \left( \mathcal{E}^E_{|S} \right) + O(h^2) ,
\]

\[
\pi S \left( \left[ \mathcal{E}^E_{|S} \right] + O(h^2) \right) = \left( \mathcal{E}^E_{|S} \right) + O(h^2) ,
\]

\[
\pi S \left( \left[ \mathcal{E}^E_{|S} \right] + O(h^2) \right) = \left( \mathcal{E}^E_{|S} \right) + O(h^2) .
\]

(B21)

which we define as the extrinsic surface Ericksen stress tensor \( \mathcal{E}_{|S} := \left( \mathcal{E}^E_{|S} \right) + O(h^2) \), and finally get

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

(B22)

Furthermore, we introduce the two curl operators \( \mathcal{E}_{|S} := \Gamma(1) \Omega_h \) for vector fields and \( \mathcal{L}_V g - 2v_n B \): \( \Gamma(1) \Omega_h \) for scalar fields, see (Nestler et al. 2018) and (Nitschke et al. 2017) for definitions. Therefore, rewriting eq. (B21) yields

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

\[
\pi S \left( \mathcal{E}^E_{|S} \right) = \mathcal{L}_V g - 2v_n B + O(h^2) ,
\]

(B23)

as a consequence of eq. (B22) and the Weizenböck machinery, i.e. interchanging covariant derivatives w.r.t. the Riemannian curvature tensor of \( S \), cf. (Arroyo and DeSimone 2009). With

\[
\| P \|_{\Omega_h}^2 = \left( G_{ij} P_{ij} + (P_{\xi})^2 \right)\bigg|_S = \left( P \right)_{S} = \left( P \right)_{S} - \left( P \right)_{S} + O(h^2) .
\]

we get for the remaining terms on the right hand sides of eqs. (B3) and (B11)

\[
\pi S \left( \left( P \right)_{S} - \left( P \right)_{S} + O(h^2) \right) = \left( P \right)_{S} - \left( P \right)_{S} + O(h^2) \nonumber \\
\pi S \left( \left( P \right)_{S} - \left( P \right)_{S} + O(h^2) \right) = \left( P \right)_{S} - \left( P \right)_{S} + O(h^2) .
\]

with the surface pressure \( p_S := P_{\Omega_h} \). Note that it holds for the relative velocity at the surface \( U := \pi S \in \Gamma(1) \), i.e.
The same term was proposed in (Yavari 2016) for the tangential part of the fluid acceleration in eq. \( \text{B1} \), Hence, eqs. \( \text{B18} \) and \( \text{B19} \) now read

\[
[\nabla_U V]_i|_S = v^k V_{i;k}|_S = [\nabla^S_v v - v_n \delta v]_i
\]

\[
[\nabla_U P]_i|_S = v^k P_{i;k}|_S = [\nabla^S_v p]_i + \mathcal{O}(h^2)
\]

By using \( \partial_i X = W = W_i \nu \), \( W_i|_S = v_n \) and \( V = V^k \partial_k X + V_i \nu \) (analogously for \( P \)) we obtain for the partial time derivatives

\[
[\partial_t V]_i|_S = \left( \partial_t, \partial_i X \right)_{\Omega_h}|_S
\]

\[
\quad = \left( \partial_t V^k \right) \partial_i X + V^k \partial_i W
\]

\[
\quad + \left( \partial_t V_\xi \right) \nu + V_\xi \partial_i \nu, \partial_i X \right)_{\Omega_h}|_S
\]

\[
\quad = g_{i k} \partial_t u^k + v^k \left( \partial_t W^k \right) \nu + v_n \partial_k \nu, \partial_i X \right)_{\Omega_h}|_S
\]

\[
\quad - v_n \left( \nu, \partial_i W \right)_{\Omega_h}|_S
\]

\[
\quad = g_{i k} \partial_t v^k - v_n \left( B_{i k} v^k + \partial_i v_n \right) \quad \text{(B24)}
\]

and

\[
[\partial_t P]_i|_S = \left( \partial_t, \partial_i X \right)_{\Omega_h}|_S
\]

\[
\quad = \left( \partial_t P^k \right) \partial_i X + P^k \partial_i W, \partial_i X \right)_{\Omega_h}|_S + \mathcal{O}(h^2)
\]

\[
\quad = g_{i k} \partial_t p^k - v_n B_{i k} p^k + \mathcal{O}(h^2) \quad \text{(B25)}
\]

Therefore, we have

\[
\pi_S \left[ \partial_t V + \nabla_U V \right]|_S = (\partial_t v^i) \partial_i x + \nabla^S_v v
\]

\[
\quad - v_n \left( 2Bv + \nabla_S v_n \right) \quad \text{(B26)}
\]

for the tangential part of the fluid acceleration in eq. \( \text{B1} \). The same term was proposed in (Yavari et al. 2016) by variation of the kinetic energy of a moving manifold in the context of Lagrangian field theory. Moreover, we also find this acceleration term in (Nitschke and Voigt 2018), where a covariant material derivative is derived in terms of covariant tensor transport through three dimensional moving spacetime embedded in a four dimensional absolute space. In this context eq. \( \text{B26} \) can be obtained by taking the spatial part of the covariant material derivative for the special case of velocity fields and a transversal observer. Evaluating the transport term for the director field \( P \) at the surface yields

\[
\pi_S \left[ \partial_t P + \nabla_U P \right]|_S = (\partial_t p^i) \partial_i x + \nabla^S_v p - v_n Bp
\]

which can also be found in (Nitschke and Voigt 2018), but as the spatial part of the covariant material derivative of a so-called instantaneous vector field from a transversal observers point of view. Finally, under boundary conditions \( \text{B14} \) – \( \text{B20} \) and \( h \to 0 \), eq. \( \text{B27} \) and the tangential parts of eqs. \( \text{B1} \) and \( \text{B3} \) reduces to

\[
(\partial_t v^i) \partial_i x + \nabla^S_v v - v_n \left( 2Bv + \nabla_S v_n \right)
\]

\[
= \nu \left( -\Delta_{\text{dir}} v^i + 2v_i v^j + \nabla_S v_n \right) - 2 \text{div}_S \left( v_n B \right)
\]

\[
- \nabla_{S\pi S} - \lambda \text{div}_S \delta_S^\nu
\]

\[\equiv \nabla_S v = v_n \pi S \]

\[
(\partial_t p^i) \partial_i x + \nabla^S_v p - v_n Bp
\]

\[
= \eta \left( \Delta_{\text{dir}} p - B^2 \right) - \omega_n \left( ||p||^2_2 - 1 \right) p
\]

in \( S \times \mathbb{R}_+ \). This system of PDEs has full rank, i.e. it contains five independent coupled equations with five degree of freedoms \( v^i \), \( v^2 \), \( p^1 \), \( p^2 \), depending on an arbitrary choice of local coordinates, and \( p_S \). A full discussion about the normal parts of eqs. \( \text{B1} \) and \( \text{B3} \) does not belong to this paper. Nevertheless, the normal parts would give us two additional equations, consistently w.r.t. \( h \), and two new free scalar valued quantities \( \gamma_1, \gamma_2 \in T^0S \). With our boundary conditions and assumption, w.r.t. the moving thin film geometry, this would be \( \gamma_1 := \partial_t \Omega_{\Omega_h}|_S \) and \( \gamma_2 := F_{\xi} \nu \xi \xi|_S = \partial^2 \Omega_{\Omega_h}|_S \). Both degree of freedoms would occur as zero order differential terms, i.e. the upcoming normal equations would not have any influence to eqs. \( \text{B27} \) – \( \text{B29} \). Therefore, the thin film limit of the normal part equations can be omitted as long as we are not interested in the quantities \( \gamma_1 \) and \( \gamma_2 \).

The partial time derivatives in eqs. \( \text{B27} \) and \( \text{B29} \) are realized only at the contravariant vector proxies of \( v \) and \( p \) w.r.t. local defined charts at the surface. The reason for the absence of an intrinsic covariant vector operator notation (similar to \( \nabla_S \)) for the time derivative is that the time \( t \) is not a coordinate of a moving space in a pure spatial perspective, especially for a moving surface with \( v \neq 0 \). Unfortunately, most of the numerical tools for solving surface PDEs do not work with a locally defined vector basis. They mimic vector-valued problems as a system of scalar-valued problems under the assumption of Euclidean coordinates. This means for the surface problem \( \text{B27} \) – \( \text{B29} \) that the tangential velocity field and the director field are considered to be vector fields in \( \mathbb{R}^3 \), which results in an under-determined problem. The two additional degrees of freedom can be handled in different ways, e.g. by penalty methods (Nestler et al. 2018; Reuther and Voigt 2018) or by using Lagrange multipliers (Jankuhn et al. 2017). The terms \( \partial_t v \) and \( \partial_t p \) makes certainly sense, if we consider \( v, p \in T^3 \), but note that in general \( \partial_t v \) as well as \( \partial_t p \) are no longer part of the tangential space of the surface \( S \). Nevertheless, we can use only the tangential part of \( \partial_t v = (\partial_j v^j) \partial_j x + v_j \partial_j \partial_i x \) for a transversal observer, i.e.

\[
[\partial_t v^i]_i = (\partial_t v, \partial_i x)|_S = g_{ij} \partial_j v^j - v_n B_{ij} v^j
\]

Analogously, the same holds for \( \partial_t p \). Finally, we obtain
by rewriting eqs. \[ B27 \] - \[ B29 \]

\[
\pi_S \partial_i \mathbf{v} + \nabla_S \mathbf{v} - v_n (B \mathbf{v} + \nabla_S v_n) = \nu \left( -\Delta^\mathbf{R} \mathbf{v} + 2k\mathbf{v} + \nabla_S (v_n \mathcal{H}) - 2 \text{div}_S (v_n B) \right) - \nabla_S p_S - \lambda \text{div}_S \mathbf{p}_S \quad \text{(B30)}
\]

\[
\text{div}_S \mathbf{v} = v_n \mathcal{H} \quad \text{(B31)}
\]

\[
\pi_S \partial_i \mathbf{p} + \nabla_S \mathbf{p} = \eta \left( \Delta^\mathbf{DG} \mathbf{p} - B^2 \mathbf{p} \right) - \omega_n \left( \| \mathbf{p} \|_S^2 - 1 \right) \mathbf{p} \quad \text{(B32)}
\]

in \( S \times \mathbb{R}_+ \), if \( \mathbf{v}, \mathbf{p} \in T_S \subset T \mathbb{R}^3 \).

Eq. \[ B33 \] together with the boundary condition \[ B5 \] is the \( L_2 \)-gradient flow along the material motion to minimize the Frank-Oseen energy functional with material constants \( \eta = K_{11} = K_{22} = K_{33} \) and \( K_{24} = 0 \), see \cite{Mori}. In the thin film limit, this leads to the minimization of the surface Frank-Oseen energy \( \frac{\nu}{2} \int_S \| \nabla_S p_S \|^2 \, dS \). This situation differs from \cite{Nestler}, where the one-constant approximation \( \eta = K_{11} = K_{22} = K_{33} = -K_{24} \) was assumed, which leads to minimizing the distortion energy \( \frac{\nu}{2} \int_S (\text{rot}_S \mathbf{p})^2 + (\text{div}_S \mathbf{p})^2 \, dS \). However, for the case \( \omega_n \to \infty \), where \( \| \mathbf{p} \|^2_S = 1 \) a.e., both energies only differ by a constant value \( \frac{2}{\nu} \int_S \kappa dS = \pi K \chi (S) \), where \( \chi (S) \) denotes the Euler characteristic. Thus, the minimizers of both energies are equal.

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