Correlations between the interacting fragments of decaying processes

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Abstract
We study the correlations (and alignment as a particular case) existent between the fragments originated in a decaying process when the daughter particles interact. The interaction between the particles is modeled using the potential of coupled oscillators, which can be treated analytically. This approach can be considered as a first step towards the characterization of realistic interacting decaying systems, an archetypal process in physics. The results presented here also suggest the possibility of manipulating correlations using external fields, a technique that could be useful to provide sources of entangled massive particles.

Keywords: Interacting decaying systems; Correlations and entanglement; Coupled oscillators

1 Introduction
In classical mechanics two particles originated in a decaying process maintain perfect correlations between their positions and momenta during the subsequent evolution of the system. In quantum mechanics the situation is not so favorable. The spread of the wavefunction and the uncertainties in the initial positions and momenta can cause in many cases a rapid lost of the initial correlations [1, 2, 3]. In these cases the correlations are not useful for practical purposes. References [2, 3] studied free non-relativistic decaying systems. Real systems are not in general free, but present various types of interactions. We pose in this paper the question if, when these interactions are taken into account, the picture of the process and the behavior of the correlations, are modified. The study of real interactions is in general very complicated, and many times we must resort to perturbative methods. We propose here an alternative approach. We consider
a simple model of interaction that can be solved analytically by assuming that
the two particles interact as coupled oscillators. In all the areas of physics, but
in particular in quantum mechanics, many problems that are not analytically
treatable are reduced to harmonic oscillators. As we are dealing with two par-
ticles, coupled oscillators could be a good candidate for our problem. On the
other hand, coupled oscillators have been used in several branches of physics to
simulate more complex systems [4, 5, 6]. We remark that we are modeling the
evolution taking place after the decaying process, not the transformation of the
mother into two daughter particles, a non-unitary process incompatible with
the unitary evolution of coupled oscillators. The model of interaction not only
holds for the direct interaction between the two particles (the electromagnetic
interaction in the case of charged particles) but also for external fields, which
could be introduced with the aim of driving the evolution of the system (see, for
instance, Ref. [7] where external fields are used to prepare non-spreading wave
packets).

In addition to the more complete characterization of decaying systems (one
prototype example in physics) provided by the inclusion of interactions, the
analysis below allows for a better understanding of how correlations could be
manipulated. In connection with the last point and from a more practical point
of view, it must be remarked that decaying systems with good correlations could
provide a source of entangled massive particles, well suited for two-particle in-
terference experiments. Up to now, most two-particle interference arrangements
use entangled photons because of the difficulty in preparing massive entangled
states.

The plan of the paper is as follows. First, in Sect. 2, we briefly review
the basics of decaying systems. The model of interaction, based on coupled
oscillators, is presented in Sect. 3. In Sect. 4 we derive the states of minimum
uncertainty, which reach the best compromise between momentum and position
initial uncertainties. Our analysis is restricted to these states. The evolution
of the interacting states is evaluated in Sect. 5. Section 6 deals with the
comparison between perfect correlations and alignment for interacting and free
systems. In Sect. 7 we evaluate the entanglement present in the system using
the Schmidt number. Finally, in the Conclusions we discuss the principal results
of the paper.

2 Basics of decaying systems

We present in this Section a brief review of the problem. The classical variables
describing a decaying process are \(x + y\) and \(p_x + p_y\), with \(x\) and \(y\) the positions
of the particles and \(p_x\) and \(p_y\) their momenta. In classical theory the law of
momentum conservation implies that if initially the total momentum is \(p_x + p_y = 0\), it will remain so during the subsequent free evolution of the complete
system; as \(p_x = -p_y\) the particles will always be found in opposite directions.
In quantum theory \( \mathbf{x} + \mathbf{y} \) and \( \hat{\mathbf{p}}_x + \hat{\mathbf{p}}_y \) are replaced by the operators \( \hat{x} + \hat{y} \) and \( \hat{p}_x + \hat{p}_y \). The uncertainty relations for these operators are

\[ \Delta(\hat{\mathbf{x}}_i + \hat{\mathbf{y}}_i) \Delta(\hat{\mathbf{p}}_x + \hat{\mathbf{p}}_y) \geq \hbar \]  

with \( i = 1, 2, 3 \) the three components of each operator and \( \Delta \hat{x}_i = \langle (\hat{x}_i - \langle \hat{x}_i \rangle)^2 \rangle^{1/2} \). Because of these relations it is not possible to prepare the system in a state with perfect correlations in momentum, as it was the case in classical theory (\( \Delta(\hat{x}_i + \hat{y}_i) \) would have an unbounded value). The most one can expect in realistic conditions is to have a sharp distribution centered around \( \mathbf{p}_x + \mathbf{p}_y = 0 \). For instance, in Ref. [2] the wavefunction of the system is taken as

\[ \psi(\mathbf{x}, \mathbf{y}, t) = \int \int f(\mathbf{p}_x, \mathbf{p}_y) \exp \left( i(\mathbf{p}_x \cdot \mathbf{x} + \mathbf{p}_y \cdot \mathbf{y} - Et)/\hbar \right) d^3\mathbf{p}_x d^3\mathbf{p}_y \]  

with \( E \) the energy of the system and \( f \) the momenta distribution, sharply centered around \( \mathbf{p}_x + \mathbf{p}_y = 0 \).

As a consequence of this uncertainty in the initial momenta and the inherent spreading of the wavefunction the correlations present in the classical case can be lost in many situations in the quantum realm. We discuss this point at extent in Sect. 6.

3 Interaction: coupled oscillators

Up to our knowledge in all the studies of decaying systems presented so far in the literature the subsequent evolution of the system is assumed to be free. However, in realistic conditions, in most of the cases there is some type of interaction between the products of the process. For instance, if a neutral particle decays into two particles with opposite charges there is an electromagnetic interaction which, at least at the initial times, is not negligible. The exact description of the system taking into account the electromagnetic interaction is, however, too complex.

We propose here a different approach. Instead of studying particular forms of realistic interactions we shall analyze an interaction that can be treated analytically. This approach can be considered as a first approximation to the problem. As signaled in the introduction harmonic oscillators are extensively used in classical and quantum theory to study complex systems not analytically solvable. Then we model the inter-particle interaction via the coupled oscillator potential.

There are other more physical reasons for this choice. Imagine we try to introduce some external interaction, for instance through the presence of some external fields, aimed to preserve the initial correlations. Examples can be found in the literature of the use of external fields to manipulate the form of the wavefunction. For instance, an imaginary potential can compensate the spreading of the wavefunction resulting in a non-spreading wavepacket [7]. Good correlations are obtained when \( \mathbf{x}_{\text{mea}} \approx -\mathbf{y}_{\text{mea}} \), i.e., when the measured positions
of the particles are close to the perfect (classical) correlation \( x_{\text{mea}} = -y_{\text{mea}} \).

Then if we introduce a potential of the type

\[
V(x, y) = \kappa(x + y)^2
\]

we have an interaction that acts when the particles tend to deviate from the good correlations. The strength of the interaction is proportional to the magnitude of the deviation, representing the constant \( \kappa \) the value of this proportionality. This is the ideal form of an interaction aimed to maintain good correlations. It can be seen as the ideal limit towards which any realistic interaction with this aim must tend. Other acceptable choices should have the form \( \kappa(x + y)^{2n} \), with \( n \) any positive integer number. However, we shall restrict our considerations to the case \( n = 1 \).

The form of Eq. (2) is that of the interaction term of two coupled oscillators. Classically, the Hamiltonian of two coupled oscillators is:

\[
H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + C_1 x_1^2 + C_2 x_2^2 + C_{12} x_1 \cdot y
\]

The quantum Hamiltonian is obtained from the classical one by the usual quantization procedure. Our case corresponds to the choice \( C_1 = C_2 = \kappa \) and \( C_{12} = 2\kappa \). As signaled in the introduction coupled oscillators have been used to simulate other physical systems [4, 5, 6].

The choice of a coupled oscillator to simulate the interaction of the decaying particles could be criticized because oscillators are usually associated with states of systems that remain bound, whereas decaying systems become free. However, it must be noted that the oscillator-type force only acts on the center of mass coordinate \( x + y \), remaining the relative position coordinate \( x - y \) free. This way we can have a coupled oscillator interaction and, at the same time, free evolution. The particles can become well separated as can easily be tested from the solutions of the problem (see comment after Eq. (20)).

\section{4 States of minimum uncertainty}

As remarked in Refs. [2] and [3] and previous sections the initial uncertainty in position and momentum is one of the causes of alignment lost in decaying systems. This uncertainty can be minimized choosing the states of minimum uncertainty of the relevant operators of the problem, \( \hat{x} + \hat{y} \) and \( \hat{p}_x + \hat{p}_y \). These states can be easily calculated using well-known techniques [8]: given two operators \( Q_+ \) and \( Q_- \) they are determined by the equations,

\[
(\hat{Q}_+ - Q_+)|\psi> = -i\epsilon(\hat{Q}_- - Q_-)|\psi>
\]

with

\[
Q_\pm = <\hat{Q}_\pm> \psi \quad \epsilon = -<\psi|i[\hat{Q}_+, \hat{Q}_-]|\psi> \frac{1}{2(\Delta \psi Q_-)^2}
\]
where \([\hat{Q}_+^\dagger, \hat{Q}_-^\dagger]\) is the commutator of \(\hat{Q}_+\) and \(\hat{Q}_-\) and \(\Delta\psi\hat{Q}_- = \langle (\hat{Q}_- - \langle \hat{Q}_- \rangle^\dagger )^2 \rangle^{1/2}\) is evaluated in the state \(\psi\). In the position representation the operators can be written as \((\hat{Q}_+)_j = \hat{x}_j + \hat{y}_j = x_j + y_j\) and \((\hat{Q}_-)_j = (\hat{p}_x)_j + (\hat{p}_y)_j = -i\hbar\partial/\partial x_j - i\hbar\partial/\partial y_j\), with \(j = 1, 2, 3\). Then Eqs. (4) and (5) become:

\[
(x_j + y_j - (Q_+)_j - i(Q_-)_j\epsilon_j)\psi = -\epsilon_j\hbar \left( \frac{\partial\psi}{\partial x_j} + \frac{\partial\psi}{\partial y_j} \right) \tag{6}
\]

and

\[
\epsilon_j = -\frac{<\psi| - 2\hbar|\psi>}{2(\Delta\psi(\hat{p}_x_j + \hat{p}_y_j))^2} = \frac{\hbar}{2(\Delta\psi(\hat{p}_x_j + \hat{p}_y_j))^2} \tag{7}
\]

where, and from now on, we assume by simplicity \(\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon\).

In the particular case \(Q_+ = Q_- = 0\), it is immediate to verify by direct substitution that the solution of the above equation is

\[
\psi(x, y) \sim \exp\left(-\frac{(x + y)^2}{4\epsilon\hbar}\right) \tag{8}
\]

We note that this is an entangled wavefunction, which reflects a good initial correlation for position measurements \((x_{mea} \approx -y_{mea})\) with a dispersion of the order \(2\hbar^{1/2}\epsilon^{1/2}\).

5 Evolution of the system

The evolution of the state is ruled by Schrödinger’s equation (we assume by simplicity the mass of the two particles to be the same):

\[
i\partial\psi(x, y, t) = -\frac{\hbar^2}{2m}(\nabla_x^2 + \nabla_y^2)\psi(x, y, t) + \kappa(x + y)^2\psi(x, y, t) \tag{9}
\]

In order to solve this equation it is convenient to introduce the center of mass and relative position coordinates, defined by the relations:

\[
X = \frac{1}{2}(x + y) ; \ Y = x - y \tag{10}
\]

The Schrödinger equation becomes:

\[
i\hbar\partial\psi(X, Y, t) = -\left(\frac{\hbar^2}{2M}\nabla_X^2 + \frac{\hbar^2}{2\mu}\nabla_Y^2\right)\psi(X, Y, t) + \frac{M\omega^2}{2}X^2\psi(X, Y, t) \tag{11}
\]

where \(M = 2m\) is the total mass, \(\mu = m/2\) is the reduced mass and \(\omega^2 = 8\kappa/M\) is a habitual way of expressing the coefficient of the interaction potential.

The solution of this equation at time \(t\) can be obtained by integration of the initial wavefunction using the kernel or propagator of the system [9]:

\[
\psi(X, Y, t) = \int d^3X_0 \int d^3Y_0 K(X, Y, t; X_0, Y_0, t_0)\psi_0(X_0, Y_0, t_0) \tag{12}
\]
with \( \psi_0(X_o, Y_o, t_o) = N_o \exp \left( -\frac{X_o^2}{a^2} \right) \) the initial wavefunction, given by the minimum uncertainty state studied in Sect. 4. \( N_o \) represents the normalization of this initial wavefunction (see later). All the integrations are carried out between the limits \(-\infty\) and \(\infty\). As Eq. (11) can be separated for variables \( X \) and \( Y \), the kernel is the product of the kernels corresponding to the two spatial variables [9]:

\[
K(X, Y, t; X_o, Y_o, t_o) = K_X(X, t; X_o, t_o) K_Y(Y, t; Y_o, t_o)
\] (13)

Now, we have two well-known kernels. From now on we take \( t_o = 0 \) by simplicity. For variable \( X \) it corresponds to the kernel of the harmonic oscillator [9]:

\[
K_X(X, t; X_o, t_o) = \left( \frac{M\omega}{2\pi i\hbar \sin \omega t} \right)^{3/2} \exp \left( \frac{iM\omega}{2\hbar \sin \omega t} \left( (X^2 + X_o^2) \cos \omega t - 2X \cdot X_o \right) \right)
\] (14)

On the other hand, for variable \( Y \) we have a free-particle evolution equation with kernel [9]:

\[
K_Y(Y, t; Y_o, t_o) = \left( \frac{\mu}{2\pi i\hbar} \right)^{3/2} \exp \left( \frac{i\mu}{2\hbar t} (Y - Y_o)^2 \right)
\] (15)

After a simple integration using the well-known relation \( \int dz \exp(\alpha z^2 + \beta z) = (\pi/\alpha)^{1/2} \exp(-\beta^2/4\alpha) \) valid for \( \text{Re}(\alpha) \leq 0 \), we obtain

\[
\psi(X, t) = N \exp \left( \frac{i\alpha}{2\pi i\hbar} \right) \exp \left( -\frac{M\pi \omega}{2\pi i\hbar a^{-2} \sin \omega t + M \pi \omega \cos \omega t} \right) \exp \left( -\alpha(t)X^2 \right)
\] (16)

where

\[
f(X, t) = \left( -\frac{M^3 \omega^3}{8\hbar^4 \sin^4 \omega t} \right) \left( \frac{\cos \omega t}{(\frac{1}{a^2} + \frac{M^2 \omega^2 \exp^2 \omega t}{4\hbar^2 \sin^2 \omega t})} + \frac{M \omega \cos \omega t}{2\hbar \sin \omega t} \right) X^2
\] (17)

and

\[
\alpha(t) = \frac{M^2 \omega^2 a^2}{4\hbar^2 \sin^2 \omega t + M^2 a^4 \omega^2 \cos^2 \omega t}
\] (18)

\( N \) is the normalization factor. On the other hand, \( f \) includes all the real functions that appear in the form \( \exp(if) \) in the wavefunction. They are of no interest because when calculating probabilities (the magnitudes of interest in next section) they only contribute as a constant term. Note that the wavefunction does not depend on the \( Y \) variable. The initial wavefunction \( \psi_0 \) does not depend on \( Y_o \). Then as \( K_Y \) is a free propagator cannot generate a dependence on \( Y \). In physical terms, if the initial state does not depend on the relative coordinates, the interaction term that is only function of the center of mass variables cannot introduce that type of dependence during the subsequent evolution.
The normalization factor is usually evaluated from the condition for the probability \( \int d^3x \int d^3y |\psi(x, y, t)|^2 = 1 \). However, wavefunction (16) cannot be normalized in an absolute sense. Effectively, a simple calculation gives

\[
\int d^3x \int d^3y |\psi(x, y, t)|^2 = |N|^2 |\mu(t)|^2 \left( \frac{\pi}{2\alpha} \right)^{3/2} \int d^3x
\]

where \( \mu(t) \) is the term in Eq. (16) going as the 3/2 power of the expression between parentheses.

Expression (19) is unbounded because the integration is between \( -\infty \) and \( \infty \). Then we must work with relative probability densities or probability densities per unit volume, which are given by \( \int d^3x \int d^3y |\psi(x, y, t)|^2 / \int d^3x \). It is simple to see that this relative probability is normalized to unity. The normalized (in this relative sense) wavefunction is:

\[
\psi(X, t) = \left( \frac{2\alpha(t)}{\pi} \right)^{3/4} \exp(-\alpha(t)X^2)
\]

Note that there is an additional pure exponential factor \( \exp(i\varphi_\mu(t)) \), where \( \varphi_\mu \) is the phase of \( \mu(t) \) (\( \mu(t) = |\mu(t)| \exp(i\varphi_\mu(t)) \)), but it is irrelevant for probabilities and can be included in \( f(X, t) \).

The above equation shows that, although we have used an oscillator-type interaction, the particles can become well separated. The solution only depends on \( X \). The variable \( Y \) is not constrained by Eq. (20) and can reach arbitrarily large values.

Now, we consider the decaying system in free evolution. Its mathematical form can be derived from the equations in interaction using the relation

\[
\lim_{\omega \to 0} \frac{\sin \omega t}{\omega} = t
\]

First, we note that from the term \( \exp(if) \) it is simple to see that the wavefunction at times \( t > 0 \) has no longer the form of a minimum uncertainty wavepacket (as it is well-known [8], the same behavior occurs for one-particle minimum uncertainty wavepackets). The normalized wavefunction (in the relative sense) is given by Eq. (20) with \( \alpha(t) \) replaced by \( \alpha_F(t) \), given by

\[
\alpha_F(t) = \frac{1}{a^2 + \left( \frac{2\hbar t}{Ma} \right)^2}
\]

### 6 Correlations

We analyze in this section if the introduction of the interaction improves the behavior of the correlations. Two types of measures for the correlations will be used. One is the probability of detection with perfect (classical) correlation, the other is the alignment at large times. We study them separately.
6.1 Perfect correlations

Two particles have perfect (classical) correlations when the positions of the particles found in a measurement process are opposite: \( x_{\text{mea}} = -y_{\text{mea}} \) or \( X_{\text{mea}} = 0 \).

Let us first consider the free evolution case. From the expression for the relative probability density for \( X = 0 \) we see that the probability of perfect correlation goes as \( \alpha^{3/2}(t) \), a decreasing function of time. For very large times, \( \alpha_F(t) \to 0 \) and the probability of perfect correlation becomes negligible.

We consider now the interacting case, where \( \alpha(t) \) shows an oscillatory behavior. It varies between \( \alpha_- = 1/a^2 \) and \( \alpha_+ = (M\omega a/2\hbar)^2 \). Depending on the strength of the interaction coupling \( \omega \) we have three different scenarios: when \( \omega > 2\hbar/Ma^2 \) we have \( \alpha_+ > \alpha_- \), whereas for \( \omega < 2\hbar/Ma^2 \) the relation is \( \alpha_+ < \alpha_- \). In the limiting case \( \omega = 2\hbar/Ma^2 \) we obtain \( \alpha_+ = \alpha_- \). The probability of perfect correlation goes as \( \alpha^{3/2}(t) \) and oscillates between \( \alpha_-^{3/2} \) and \( \alpha_+^{3/2} \). For instance, for \( \omega > 2\hbar/Ma^2 \) the probability of perfect correlation increases in the interval \( t \in (0, \pi/2) \) reaching its maximum value at \( t = \pi/2 \), starting a stage of decreasing behavior at this point. The perfect correlations are not completely lost even in the limit of very large times, as it was the case for free evolution. The analysis for \( \omega < 2\hbar/Ma^2 \) follows similar lines. The limiting situation occurs when \( \omega = 2\hbar/Ma^2 \). Now \( \alpha \) is constant and the probability of perfect correlation detection does not change with time. We conclude that the presence of the interaction improves the behavior of the perfect correlations, which become an oscillating or constant function instead of the decreasing one associated with free evolution.

6.2 Alignment

Alignment is a weaker measure of correlations than perfect correlations. We say that two particles are aligned when they are detected in, approximately, opposite directions. A quantitative criterion for alignment can be introduced following Ref. [3]. The angle characterizing angular deviation (perfect alignment is given by \( \pi \)) can be expressed as:

\[
\tan \theta = \frac{TD(t)}{R(t)} = \frac{\Delta X_i}{<x_i>}
\]

(23)

where \( TD(t) \) is the transversal deviation and \( R(t) \) is the distance both particles have traveled. In statistical terms \( TD \) can be taken of the order \( \Delta X_i \), i.e., the variance of any of the components of the center of mass position \( (i = 1, 2 \text{ or } 3, \text{ the problem is isotropic}) \). On the other hand, \( R \) is of the order \( <x_i> \), the expectation value of the position of any of the particles.

We evaluate now these two variables. We use the Heisenberg picture, where the evolution of any operator \( A \) (we omit the operator symbol to simplify the notation) is ruled by the equation \( i\hbar dA/dt = [A, H] + i\hbar \partial A/\partial t \), with \( H \) the
Hamiltonian of the system. For the position and momentum operators these relations become
\[ m \frac{dx}{dt} = p_x \; ; \; m \frac{dy}{dt} = p_y \; ; \; \frac{dp_x}{dt} = -2\kappa (x + y) = \frac{dp_y}{dt} \] (24)
Using the center of mass coordinate \( X \) and the total momentum \( P = p_x + p_y \), the equations can be rewritten as
\[ m \frac{dX}{dt} = \frac{1}{2} P \; ; \; \frac{dP}{dt} = -8\kappa X \] (25)
The solution of these equations is
\[ X = X_o \cos \Omega t + \frac{P_o}{2m} \sin \Omega t \] (26)
and
\[ P = -2m\Omega X_o \sin \Omega t + P_o \cos \Omega t \] (27)
with \( \Omega = 4\kappa/m \) (note \( \omega^2 = 2\Omega^2 \)) and \( X_o \) and \( P_o \) the initial values of both operators.
Assuming the expectation values of both initial conditions to be null, \( < X_o >= 0 \) and \( < P_o >= 0 \), the variance of \( X \), can be expressed in the simple form:
\[ (\Delta X_i(t))^2 = (\Delta X_{oi})^2 \cos^2 \Omega t + (\Delta P_{oi})^2 \frac{\sin^2 \Omega t}{4M^2\Omega^2} + < X_{oi}P_{oi} > \frac{\sin \Omega t \cos \Omega t}{M\Omega} \] (28)
Finally, we evaluate \( < x_i(t) >= \). First, we note that from \( dp_{x_i}/dt = -2\kappa X_i \) we have
\[ p_{x_i}(t) = p_{x_i}(0) - \frac{2\kappa X_{oi}}{\Omega} \sin \Omega t + \frac{\kappa P_{oi}}{m\Omega^2} (\cos \Omega t - 1) \] (29)
Taking mean values and remembering our assumption \( < X_o >= 0 \) and \( < P_o >= 0 \) we have \( < p_{x_i}(t) >=< p_{x_i}(0) > \). Now, we use the relation \( m dx_i/dt = p_{x_i} \) that gives (assuming \( < x_i(0) >= 0 \), \( m < x_i(t) >=< p_{x_i} >= t/m \), or
\[ < x_i(t) >=< p_{x_i}(0) > t/m \] (30)
Using again the relation (21), we have for the free evolution
\[ (\Delta X^F_i(t))^2 = (\Delta X_{oi})^2 + (\Delta P_{oi})^2 \frac{t^2}{4M^2} + \frac{t}{M} < X_{oi}P_{oi} > \] (31)
On the other hand, Eq. (30) is valid for both free and interacting evolutions.
Now, we can evaluate the angular deviation. We shall concentrate on the large times behavior. For the free evolution of the minimum uncertainty state we have (the \( \infty \) superscript refers to \( t \rightarrow \infty \)):
\[ \tan \theta^F = \frac{m\Delta P_{oi}}{2M < p_{x_i}(0) >} = \frac{< \lambda >}{16\pi \Delta X_{oi}} \] (32)
where we have used the relation (1) and the equation \( < p_x(0) > = \hbar \langle \lambda \rangle \), with \( \langle \lambda \rangle \) the mean wavelength associated with the initial mean momentum. Equation (32) shows that at large times there is only alignment if \( \langle \lambda \rangle \ll \Delta X_{oi} \), i.e., if the mean initial wavelength of the particles is much smaller than the dispersion of the center of mass of the source.

On the other hand, when the interaction is present the angular deviation at large times for states with \( < p_x(0) > \neq 0 \) is:

\[
\tan \theta^\infty = \lim_{t \to \infty} \frac{m \Delta X_i}{< p_x(0) > t} = 0
\]

because \( \Delta X_i \) is a finite function of time due to its periodic behavior. With interaction there is alignment at large times independently of the relation between the mean initial wavelength and the dispersion of the center of mass of the source.

We remark that result (32) is only valid for minimum uncertainty states (or those obeying the approximate relation \( 2\Delta P_{oi}\Delta X_{oi} \approx \bar{\hbar} \)), whereas Eq. (33) is valid for a much larger class of states, those with \( < P_{oi} > = < X_{oi} > = 0 \) and \( < p_x(0) > \neq 0 \).

7 Entanglement

In the previous section we have analyzed the correlations existent in the system. Another way to study the problem is to consider the entanglement present in the wavefunction. The correlations are the manifestation of the entanglement in measurement processes. A measure of the entanglement degree is given by the Schmidt number \([10, 11, 12]\). It has been used in a series of studies with important resemblances with our work \([11, 12]\). In particular, in \([11]\) it was studied the entanglement existent between a photon and the atom that has emitted it.

The Schmidt number \([10, 11]\) is given by

\[
S = \frac{1}{Tr_x(\hat{\rho}_x^2)} = \frac{1}{Tr_y(\hat{\rho}_y^2)}
\]

(34)

with the reduced density matrices

\[
\hat{\rho}_x = Tr_z(|\psi > < \psi|) ; \quad \hat{\rho}_y = Tr_z(|\psi > < \psi|)
\]

(35)

where \( Tr_z \) denotes trace with respect to the variables associated with particle \( z \) and \( |\psi > \) is given by Eq. (20). Note that we have used the notation \( S \) for the Schmidt number instead the usual \( K \) one in order to avoid any confusion with the propagator.
In order to evaluate the above traces we must introduce the Fourier decomposition of the wavefunction
\[
\psi(x, y, t) = \frac{1}{(2\pi\hbar)^3} \int \int d^3p d^3q \phi(p, q, t) e^{ip \cdot x / \hbar} e^{iq \cdot y / \hbar} = 
\int \int d^3p d^3q \phi(p, q, t) < x|p > < y|q >
\] (36)
where we have used the usual expression \( < x|p > = (2\pi\hbar)^{-3/2} \exp\left(\frac{-p^2}{4\hbar^2}\right) \delta^3(p - q) \) (37)

A simple calculation gives
\[
\phi(p, q, t) = \sqrt{8} \left(\frac{\pi}{2\alpha(t)}\right)^{3/4} \exp\left(\frac{-q^2}{4\hbar^2\alpha(t)}\right) \delta^3(p - q)
\] (37)

On the other hand, we have
\[
\hat{\rho}_x = \text{Tr}_q(|\psi >= < \psi |) = \int d^3q < q|\psi >= < \psi |q > = 
\int \int \int d^3q d^3p d^3p^* \phi(p, q, t) \phi^*(p^*, q, t) < x|p > < p^*|x >
\] (38)
where the trace is over the momentum \( q \) associated with the variable \( y \) and we have used the relations \( < q| < y|Q > < Q, y > |q >= < y|y > \delta^3(q - Q) \delta^3(Q - q) \).

The second trace is
\[
\text{Tr}_p(\hat{\rho}_{x}^2) = \int d^3p < p|\hat{\rho}_{x}^2|p >=
\int \int \int d^3pd^3qd^3p^* \phi(q, p, t) \phi^*(q^*, p^*, t) < p^*|p > < q^*|q >
\] (39)
where the trace now is over the momentum \( p \) associated with the variable \( x \).

Note that in the original approach [10], \( S \) counted the number of effective modes in the Schmidt decomposition of \( |\psi > \). Here, we do not consider the Schmidt decomposition of the wavefunction but the decomposition on the plane wave basis. Now the inverse of \( \text{Tr}_p(\hat{\rho}_{x}^2) \) counts the number of effective modes in this basis. This number is still denoted as the Schmidt number (see Ref. [12] where this terminology is also used although there is not a Schmidt decomposition of the analyzed system).

A simple calculation gives
\[
\text{Tr}_p(\hat{\rho}_{x}^2) = \sqrt{8}(2\pi\hbar)^3 \left(\frac{\pi}{2\alpha(t)}\right)^{3/2}
\] (40)
This equation shows that \( \text{Tr}_p(\hat{\rho}_{x}^2) \) has dimensions of \( \text{momentum}^3\text{longitude}^6 \). In order to obtain a dimensionless Schmidt’s number the usual expression above
must be replaced. One dimensional parameter provided by the problem is \(\hbar\), which suggests to use \((2\pi\hbar)^3/Tr_p(\hat{\rho}_x^2)\). This expression still has dimensions of \(\text{longitude}^{-3}\), reflecting the fact that the wavefunction is normalized as a relative probability density per unit volume. Thus, associated with this wavefunction we cannot have entanglement but entanglement per unit volume. Then the entanglement in a region of volume \(\mathcal{V}\) is measured by the following generalization of Schmidt’s number

\[
S_{\mathcal{V}} = \frac{(2\pi\hbar)^3\mathcal{V}}{Tr_p(\hat{\rho}_x^2)} = \frac{\mathcal{V}}{\sqrt{8}} \left(\frac{2\alpha(t)}{\pi}\right)^{3/2}
\]

(41)

Using this equation we obtain

\[
S_{\mathcal{V}}(t) \sim \alpha(t)^{3/2}
\]

(42)

In particular, in the free case for large times, \(t \gg Ma/2\hbar\), we have

\[
(S_{\mathcal{V}})_{\mathcal{F}}(t) \sim t^{-3}
\]

(43)

Equation (42) shows that in the interacting case entanglement is an oscillatory function of time. It can be analyzed following the lines of subsection 6.1. This behaviour of entanglement explains the persistence of correlations. As a matter of fact, it shows the same dependence on time \((\alpha(t)^{3/2})\) that perfect correlations. On the other hand, in the free case, the entanglement decreases following a law of type \(t^{-3}\). After some time almost all the initial entanglement is lost. We conclude that the pictures of the system obtained using correlations and entanglement are similar.

Finally, we want to remark that in Ref. [11] the relation between entanglement and uncertainty relations was studied. A similar analysis could be carried along the same lines in our case. However, it would enlarge too much the paper. We plan to do it in future work. We only notice that in that reference it was also signaled how the uncertainty relations adopt different forms depending on the type of measurement done on the system, single-particle or coincidence one. In this context it must be remarked that in this work (in Sect. 2) we have only considered single-particle measurements, which are usual to the Heisenberg uncertainty relations. In a scheme with conditional measurements these relations would be different [11].

8 Discussion

We have analyzed in this paper how the behavior of the fragments originated by the decaying of a mother particle is modified when interactions between the daughter particles or external fields driving the system are taken into account. We have shown that the initial correlations present in the system (such as measured by perfect correlations and alignment) are much better conserved when
the interaction is taken into account. This property, if corroborated for more realistic interactions could provide us with sources of particles with good correlations in those cases where the free evolution degrades the initial correlations. In this context note that our analysis also provides a criterion (the relation in Eq. (32) between $\langle \lambda \rangle$ and $\Delta X_{\text{oi}}$) to determine when alignment is preserved by the free evolution.

Good correlations could be useful for various purposes, for instance, in interferometric experiments with two-particle massive systems. One example in foundational issues is Popper’s argument [13, 3], where the actual experiments [14] have been carried out with photons instead of massive particles. When the experiments are restricted to the framework of quantum optics we lose the possibility of studying the dependence of the system on the form of the wavefunction, which is specially important for interferometric experiments.

As signaled before decaying systems are archetypal in physics. In the quantum realm they are particularly well suited to analyze the influence of uncertainty and spreading on the evolution of the system and, in particular, on its correlations. Then the example presented here has also a pedagogical interest.

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