A Combinatorial Family of Near Regular LDPC Codes

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Abstract—An elementary combinatorial Tanner graph construction for a family of near-regular low density parity check (LDPC) codes achieving high girth is presented. The construction allows flexibility in the choice of design parameters like rate, average degree, girth and block length of the code and yields an asymptotic family. The complexity of constructing codes in the family grows only quadratically with the block length.

I. INTRODUCTION

The fact that iterative decoding on LDPC codes performs well when the underlying Tanner graph [6] has large girth is well known [7]. The recent revival of interest in LDPC codes owing to their near capacity performance on various channel models has resulted in considerable research on the construction of LDPC code families of high rate and large girth. These constructions may be classified as random code constructions (for example see [2], [12]), construction of codes based on projective and combinatorial geometries (see [13], [14], [15] and references therein), heuristic search based constructions [3], [4], constructions based on circulant matrices [10], algebraic constructions (see [17], [9]), code constructions based on expander graphs [16], [19], and edge growth constructions [18].

In this note, we present an elementary graph theoretic construction for a family of binary LDPC codes. These codes achieve high girth and are almost regular in the sense that the degree of a vertex is allowed to differ by at most one from the average. We shall refer to these codes as ARG (Almost Regular high Girth) codes. The construction gives flexibility in the design parameters of the code like rate, block-length, and average degree, and yields an asymptotic family. We prove bounds on code parameters achieved by the construction. The complexity of the graph construction algorithm grows only quadratically with the block length of the code.

The construction here is similar in spirit to the very general graph construction scheme called the progressive edge-growth (PEG) algorithm proposed in [18] and may be considered as being specially tuned for obtaining near regular graphs of large girth. However, in [18] no technique for simultaneously bounding the maximum left and right degrees of the graphs constructed is provided, and hence the girth bounds depend on the values of the degrees obtained experimentally. The authors report that good values of girth can be achieved in practice.

The bounds on the node degrees in the Tanner graph construction proposed here are achieved by adapting a high girth graph construction technique known in the graph theory literature [8]. The following section presents the construction and establishes the bound on the girth of the Tanner graph constructed. Simulations indicate that rate 1/2 ARG codes perform better than regular codes of the same block length reported in [1].

II. THE CODE CONSTRUCTION

Given a bipartite graph \( G = (L, R, E \subseteq L \times R) \), \( |L| = n \), \( |R| = m \), the \( m \times n \) parity check matrix \( H(G) = [h_{i,j}] \) defined by \( h_{i,j} = 1 \) if and only if \( (j, i) \in E \), \( 1 \leq j \leq n \), \( 1 \leq i \leq m \) specifies a binary linear code \( C(G) \). We say \( G \) is the Tanner graph for \( C(G) \). The code \( C(G) \) is an LDPC code if the maximum degree of any vertex in \( G \) is bounded by a constant. The length of the shortest cycle in \( G \) is called the girth of \( G \) denoted by \( g(G) \). In the following, we describe the construction of a bipartite Tanner graph and give bounds on the parameters of the code defined by the graph.

Let \( m, n, p, q \) be positive integers with \( n > m > 1 \), \( p < q \), \( np = mq \) and let \( d \) be constant with \( d \leq (m + 3)/3(p + q) \). We construct a bipartite graph \( G = (L, R, E) \) with average left degree \( dp \) and average right degree \( dq \) as follows. Initially \( L = \{1..n\} \), \( R = \{1..m\} \) and \( E = \emptyset \). We denote by \( deg(x) \) the degree of a vertex \( x \in L \cup R \). Denote by \( \delta(x, y) \) the length of the shortest path from \( x \) to \( y \) in \( G \). Clearly \( \delta(x, y) = 0 \) and \( \delta(x, y) = \infty \) for all \( x, y \in L \cup R \) initially.

We will add \( npd(=mqd) \) edges to \( G \) one by one. When the \( e^{th} \) edge is added for some \( 1 \leq e \leq npd \) we shall say that the algorithm is in phase \((i, j)\) where \( i = \lceil e/n \rceil \) and \( j = \lceil e/m \rceil \). We say that the edge belongs to left phase \( i \) and right phase \( j \). Thus the first \( m \) edges will be added during phase \((1, 1)\), edges \( m + 1 \) to \( \min \{n, 2m\} \) will be added during phase \((1, 2)\) and so on. Note that after left phase \( i \), the average left degree of the graph will be \( i \). Similarly, the average right degree will be \( j \) at the end of right phase \( j \). The algorithm terminates at the end of phase \((dp, dq)\).

The algorithm repeatedly picks up a vertex of minimum degree (chosen alternately from \( L \) and \( R \)) and adds from it an edge to the farthest vertex on the opposite side in such a way that the vertex degrees are not allowed to become excessive. During phase \((i, j)\), the degree of a left vertex never exceeds \( i + 1 \) and the degree of a right vertex never exceeds \( j + 1 \). We will prove that at the end of left phase \( i \), every vertex in \( L \) has degree at least \( i - 1 \) and at the end of right phase
j every vertex in \( R \) has degree at least \( j - 1 \). Hence, when the algorithm terminates, the left and the right degrees are bounded above by \( pd + 1 \) and \( qd + 1 \) respectively, and bounded below by \( pd - 1 \) and \( qd - 1 \) respectively yielding a near-regular graph. The steps of the algorithm are formalized below:

- for \( e := 1 \) to \( npd \) do \{npd = mqd edges added\}
  - 1) \( i := \lceil e/n \rceil \quad j := \lceil e/m \rceil \) \{phase is \((i, j)\}\}
  - 2) if \( e \) is odd, choose a vertex \( x \) of minimum degree from \( L \). Let \( S = \{ z \in R : \delta(x, z) > 1 \) and \( \deg(z) < j + 1 \} \). Select a \( y \in S \) such that \( \delta(x, y) \geq \delta(x, z) \) for all \( z \in S \). Add \((x, y)\) to \( E \).
  - 3) else if \( e \) is even, choose a vertex \( x \) of minimum degree from \( R \). Let \( S = \{ z \in L : \delta(x, z) > 1 \) and \( \deg(z) < i + 1 \} \). Select a \( y \in S \) such that \( \delta(x, y) \geq \delta(x, z) \) for all \( z \in S \). Add \((x, y)\) to \( E \).

We shall call edges corresponding to odd and even values of \( e \) as odd edges and even edges respectively. Note that the algorithm may fail to progress if the set \( S \) becomes empty and no edge could be added during some intermediate phase. We shall rule out this possibility later.

**Theorem 1:** \( C(G) \) is an LDPC code with rate \( r \geq 1 - p/q \).

**Proof:** Since \( H(G) \) is an \( m \times n \) matrix, \( r \geq 1 - m/n \). Since \( m/n = p/q \) by assumption, the claim on rate follows.

By construction the left and right degrees of any node in \( G \) are bounded by \( pd + 1 \) and \( qd + 1 \). Since \( d \) is constant the graph is of low density.

The following lemma proved by induction establishes the key invariants maintained by the algorithm.

**Lemma 1:** For all \( 1 \leq i \leq pd \) and \( 1 \leq j \leq qd \) the following holds:

- If the algorithm completes left phase \( i \) then \( i - 1 \leq \deg(x) \leq i + 1 \) for all \( x \in L \) at the end of left phase \( i \).
- If the algorithm completes right phase \( j \) then \( j - 1 \leq \deg(y) \leq j + 1 \) for all \( y \in R \) at the end of right phase \( j \).

**Proof:** We shall prove the first statement using induction. Initially the hypothesis holds. Assume the statement true for some \( i \), \( 0 \leq i < pd \) and consider the the situation after completion of left phase \( i + 1 \). Let \( n^-, n^0 \) and \( n^+ \) be the number of vertices with degree \( i - 1 \), \( i \), and \( i + 1 \) respectively at the end of left phase \( i \). Since the average degree of a left node is \( i \) at the end of left phase \( i \), we have the following:

\[
(i - 1)n^- + in^0 + (i + 1)n^+ = in = i(n^- + n^0 + n^+) + (1).
\]

Canceling terms we have \( n^- = n^+ \leq \lceil n/2 \rceil \). Thus to satisfy the lower bound in the induction hypothesis at most \( \lceil n/2 \rceil \) edges need to be added to the \( n^- \) deficient vertices in \( L \) during left phase \( i + 1 \). Since out of the \( n \) edges added during left phase \( i + 1 \) at least \( \lceil n/2 \rceil \) must be from minimum degree vertices in \( L \) (because every odd edge will be added from a vertex of minimum degree in \( L \), all the \( n^- \) deficient vertices would have increased their degree by at least one and the lower bound on the left degree will be satisfied after phase \( i + 1 \).

Since the average degree of a left vertex at the end of left phase \( i + 1 \) is \( i + 1 \), there always will exist a vertex \( x \in L \) with \( \deg(x) < i + 1 \) before the completion of left phase \( i + 1 \). Hence the algorithm will never choose a left vertex of degree \( i + 1 \) for adding an edge when an odd edge is added during phase \( i + 1 \). Finally, the algorithm explicitly ensures that an even edge is added from a vertex \( y \in R \) to \( x \in L \) during phase \( i + 1 \) only if \( \deg(x) \leq i + 1 \) before the addition. Hence in all cases, the upper bound on vertex degree is also maintained during left phase \( i + 1 \). The second statement in the lemma is proved similarly.

It remains to be shown that the algorithm will indeed complete all the phases successfully. The algorithm may fail to complete phase \( i \) if at some stage the set \( S \) constructed by the algorithm is empty. The following lemma rules out this possibility.

**Lemma 2:** If \( d \leq (m + 3)/3(p + q) \) the algorithm will complete all the phases.

**Proof:** Suppose that the algorithm fails at some phase \((i, j)\) because the set \( S \) becomes empty while trying to add an odd edge from a vertex \( x \in L \). By Lemma 1, \( x \) must have at least \( i - 2 \) neighbours, each of degree at least \( j - 2 \). Since \( x \) has at most \( i + 1 \) neighbours (by Lemma 1) and \( S = 0 \), there must be at least \( m - i - 1 \) non-neighbours of \( x \) in \( R \) with degree \( j + 1 \). Thus the total degree of all vertices in \( R \) must be at least \((m - i - 1)(j + 1) + (i - 2)(j - 2)\). However, before phase \((i, j)\) ends the average right degree is less than \( j \). Hence we have:

\[
(m - i - 1)(j + 1) + (i - 2)(j - 2) < mj
\]

(2)

After simplification this yields \((m + 3)/3 < (i + j)\). Considering the case when an even edge is added and applying similar arguments we get the condition \((n + 3)/3 < (i + j)\). Since \( i \leq pd, j \leq qd \) and \( m < n \), if \( d < (m + 3)/3(p + q) \) the failure condition will never occur and the algorithm will successfully complete phase \((pd, qd)\).

We are now ready to prove the bound on the girth.

**Theorem 2:** \( g(G) \geq 2\log_{pd}(1 + m(pqd^2 - 1)/2(pd + 1)) \)

**Proof:** Assume that a smallest length cycle in \( G \) of length \( g(G) = 2r \) was formed during phase \((i, j)\) of the algorithm. Assume \( x \in L \) had the least degree and was connected to \( y \in R \) during the addition of an odd edge causing the cycle. Let \( T = \{ z \in R : \delta(x, z) \geq g \} \). The node \( x \) had to be connected to \( y \) and not to any node in \( T \) because \( \deg(z) = j + 1 \) for all \( z \in T \). But there are at most \( m/2 \) nodes of degree \( j + 1 \) during right phase \( j \). Thus \(|T| \leq m/2 \). Hence \(|T - T| \geq m/2 \). But all nodes in \( R - T \) must be at a distance at most \( g - 1 = 2r - 1 \) from \( x \). Since the maximum left and right degrees of a node in \( G \) are bounded by \( pd + 1 \) and \( qd + 1 \) respectively, the number of such nodes is bounded above by \((pd + 1)(pd + 1)(pqd^2) + \ldots + (pd + 1)(pqd^2)^{r - 1} = (pd + 1)((pqd^2)^r - 1)/(pqd^2 - 1) \). Combining the lower and upper bounds we get:

\[
m/2 \leq (pd + 1)((pqd^2)^r - 1)/(pqd^2 - 1).
\]

(3)
Fig. 1. Performance of ARG (504,8,3) code

Fig. 2. Performance of ARG (1008,8,3) code
A similar argument for the case $x \in R$ and $y \in L$ for the addition of an even numbered edge yields the inequality:
\[ n/2 \leq (qd + 1)/((pqd^2) - 1)/(pqd^2 - 1). \] (4)

The statement of the theorem follows by noting that $m < n$ and taking the lower of the two bounds.

The following table summarizes the minimum values of block length required for achieving various values of girth and average left degree for codes of designed rate 1/2, obtained by setting $p = 1$ and $q = 2$. These values were obtained experimentally by varying the values of $d$ and $g$ given as input to the algorithm. The minimum value of block length required for achieving a given girth in actual experiments turns out to be lower than the bound proved in Theorem 2 indicating that the bound is not tight.

| Code Parameters for rate 1/2 ARG codes |
| Left-degree (Average) | Girth | Block length |
|-----------------------|-------|--------------|
| 3                     | 6     | 40           |
| 4                     | 6     | 80           |
| 5                     | 6     | 172          |
| 3                     | 8     | 252          |
| 4                     | 8     | 940          |
| 3                     | 10    | 1490         |

III. COMPLEXITY

Assuming an adjacency list representation for the graph, the selection of a farthest non-neighbour satisfying the degree bound necessary during each edge addition may be performed by a simple breadth first search in $O(n)$ time. Since the total number of edge additions is linear when $d$ is fixed constant, the overall construction complexity is $O(n^2)$.

IV. PERFORMANCE SIMULATIONS

We shall refer to the code of block length $n$ defined by a Tanner graph of girth $g$ and average left degree $d$ as an $(n, g, d)$ code. Performance simulations for $(504,8,3)$, $(1008,8,3)$ and $(4000,10,3)$ ARG codes of designed rate 1/2 (corresponding to $p = 1$, $q = 2$ in the algorithm) are reported below. The ARG codes perform slightly better than the MacKay and regular PEG codes of the same length [1]. The bit and word error rate curves for the regular MacKay and PEG codes of the same length are plotted together with those of the ARG code for easy reference.

V. CONCLUSION

We have presented an algorithm for constructing near-regular LDPC codes of large girth. From a theoretical point of view, the algorithm yields an asymptotic family with a provable $\Omega(\log n)$ girth bound and quadratic complexity. The algorithm also gives good performance in practice in
comparison with regular codes of the same length. The problem of improving the girth bound remains open for further investigation.

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