HOMOLOGY AND HOMOTOPY COMPLEXITY IN NEGATIVE CURVATURE

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Abstract. A classical theorem of Gromov states that the Betti numbers, i.e. the size of the free part of the homology groups, of negatively curved manifolds are bounded by the volume. We prove an analog of this theorem for the torsion part of the homology in all dimensions \( d \neq 3 \). Thus the total homology is controlled by the volume. This applies in particular to the classical case of hyperbolic manifolds. In dimension 3 the size of torsion homology cannot be bounded in terms of the volume.

As a byproduct, in dimension \( d \geq 4 \) we give a somewhat precise estimate for the number of negatively curved manifolds of finite volume, up to homotopy, and in dimension \( d \geq 5 \) up to homeomorphism.

These results are based on an effective simplicial thick-thin decomposition which is of independent interest.

1. Introduction

1.1. Bounding the homology. The topological complexity of Hadamard manifolds is controlled, to some extent, by the volume. This phenomenon is most nicely illustrated in the case of surfaces of constant negative curvature. Indeed, the Gauss–Bonnet theorem implies that the volume coincides (up to a normalization) with the Euler characteristic, which in turn determines the homeomorphism type of the manifold. In much greater generality, an important theorem of Gromov whose proof was worked out by Ballmann, Gromov and Schroeder in [3] says that the Betti numbers of a negatively curved manifold are bounded by the volume.

Notation: By normalized bounded negative curvature we mean that the sectional curvature is contained in a closed sub-interval of \([-1,0)\).

Theorem 1.1 ([3]). For every \( d \in \mathbb{N} \) there exists \( C = C_d > 0 \) such that for every complete \( d \)-dimensional Riemannian manifold of normalized bounded negative curvature and for every degree \( k \),

\[
\text{rank} H_k(M;\mathbb{Z}) \leq C \text{vol}(M).
\]

That is, the abelian group \( H_k(M;\mathbb{Z}) \) is isomorphic to \( \mathbb{Z}^{b_k} \oplus \text{tors}_k \) where \( b_k \leq C \text{vol}(M) \) and \( \text{tors}_k \) denotes the torsion part. In recent years there has been a growing interest in the size of the torsion part \( \text{tors}_k(M) \) motivated

\footnote{As noted in [14] below a general version of this theorem holds for analytic non-positively curved manifolds.}
by number theory and topology \[5, 28\]. However, torsion is much harder to control than \( b_k \). By a result of Gromov \[25, \S 1.8\] it is known that the group torsion \( \text{tors} H_k(M; \mathbb{Z}) \) is finite. In the same paper Gromov shows that in dimension 3 the size of torsion \( \text{tors} H_1(M) \) cannot be bounded in terms of the volume \( \text{vol}(M) \) (see \( \S 1.4 \) below for more results in this direction). Our main contribution is to show that in all other dimensions the logarithm of the torsion is linearly bounded by the volume.

**Theorem 1.2.** For every \( d \neq 3 \), there exists \( C = C_d > 0 \) such that for every complete \( d \)-dimensional Riemannian manifold \( M \) of normalized bounded negative curvature and for every degree \( k \),

\[
\log |\text{tors} H_k(M; \mathbb{Z})| \leq C \text{vol}(M).
\]

**Remark 1.3.** Note that the theorem is void in the case that \( M \) has infinite volume. Theorem 1.2 applies in particular to the classical case of hyperbolic manifolds of dimension \( d \neq 3 \). Prior to this work no effective bounds were known for the torsion part of the homology of hyperbolic manifolds.

1.2. **A simplicial thick-thin decomposition.** A crucial ingredient in our proof of Theorem 1.2 is an effective simplicial thick-thin decomposition, which is provided in Theorem 1.4 below. In \[19\] it is explained how to bound the complexity of the thick part associated with the classical thick-thin decomposition of a non-positively curved manifold. Theorem 1.4 below is of the same spirit and its proof relies on the method and techniques developed in \[19\]. A novel aspect of Theorem 1.4 is that it provides a control not only of the complexity of the thick part itself, but also on the inclusion of its boundary. This is crucial in the proof of Theorem 1.2 in the case of non-compact manifolds.

A \((D, V)\)-simplicial complex is a simplicial complex such that the number of its vertices is at most \( V \) and the degree of any vertex is at most \( D \). By a simplicial pair \((R, R_0)\) we mean a simplicial complex \( R \) and a subcomplex \( R_0 \). That is, every simplex in \( R_0 \) also appears in \( R \). A simplicial pair \((R, R_0)\) is called a \((D, V)\)-simplicial pair if \( R \) is a \((D, V)\)-simplicial complex.

The following result is a refinement of related results in \[19\] (cf. Theorem 7.4 and \( \S 8 \) there) in the normalized bounded negative curvature situation.

**Theorem 1.4** (see Theorem 4.1). Let \( d > 1 \) be an integer. There are constants \( D, c > 0 \) with the following properties. Every complete \( d \)-dimensional Riemannian manifold \( M \) of normalized bounded negative curvature has a compact \( d \)-dimensional submanifold \( M_+ \) with boundary \( \partial M_+ \) such that the pair \((M_+, \partial M_+)\) is homotopy equivalent to a \((D, c \cdot \text{vol}(M))\)-simplicial pair \((R, R_0)\) and the closure of \( M \setminus M_+ \) consists of at most \( c \cdot \text{vol}(M) \) many connected components, each of which is either homeomorphic to its boundary times \([0, \infty)\) or to a \(D^{d-1}\)-bundle over \( S^1\).

1.3. **Counting manifolds.** In \[25\], following \[13\] and \[30\], Gromov shows that for \( d \neq 3 \) the number of homeomorphism classes of closed \( d \)-dimensional
Riemannian manifolds of normalized bounded negative curvature with volume bounded by some $V > 0$ is finite. In [11] Burger, Gelander, Mozes and Lubotzky give precise bounds for the homotopy type of hyperbolic manifolds of bounded volume (see also [24] for estimates on the number of manifolds up to commensurability, and fundamental groups up to quasi-isometries). These were extended to general locally symmetric spaces in [19]. We extend these results to the setting of negatively curved manifolds of variable curvature. Let $\Gamma_d(v)$ denote the number of homotopy classes of complete $d$-dimensional Riemannian manifolds of normalized bounded negative curvature and volume $\leq v$. In §5.2 we prove:

**Theorem 1.5.** For every $d \geq 4$ there are $\alpha, \beta > 0$ such that

$$\alpha v \log v \leq \log \Gamma_d(v) \leq \beta v \log v$$

for sufficiently large $v > 0$.

Let $\mathcal{P}_d$ denote the family of complete Riemannian $d$-manifolds of finite volume with normalized bounded negative curvature. Let $\mathcal{P}_d(v)$ denote the number of homeomorphism classes of elements in $\mathcal{P}_d$ of volume at most $v$.

Provided $d \geq 5$ two manifolds in $\mathcal{P}_d$ are homotopy equivalent if and only if they are homeomorphic, by the celebrated work of Farrell–Jones [17, Corollary 7.5] on the Borel conjecture. Hence Theorem 1.5 above implies:

**Corollary 1.6.** For every $d \geq 5$ there are $\alpha, \beta > 0$ such that

$$\alpha v \log v \leq \log \mathcal{P}_d(v) \leq \beta v \log v$$

for sufficiently large $v > 0$.

1.4. **Some 3-dimensional examples.** It is well known that Theorem 1.2 fails in dimension 3.

Already in his 1978 paper [25], Gromov obtained a sequence $(M_i)$ of pairwise non-homeomorphic 3-dimensional closed manifolds of bounded negative curvature and bounded volume such that the size of torsion $\text{tors}_1(M_i)$ tends to $\infty$. Thurston’s theory of 3-dimensional hyperbolic geometry allows to construct similar examples which are all hyperbolic manifolds.

Theorems 1.7 and 1.8 below nicely illustrate the dramatic failure of Theorem 1.2 in dimension 3, which is caused by the existence of Dehn fillings. We record these results here because they belong to the complete picture of torsion homology in negative curvature and are not stated explicitly elsewhere. Theorem 1.7 is well known among experts, and we claim no credit. Finally, Theorem 1.8 is based on a modification of a construction by Brock and Dunfield [10].

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2. This result also applies to the number of diffeomorphism classes of such manifolds. For $d \neq 4$ this is due to the fact that every topological manifold has only finitely many smooth structures, for $d = 4$ a separated argument is given in [13].

3. In view of Mostow rigidity theorem, in dimension $> 2$, homotopic locally symmetric manifolds are isometric, thus the estimates of [11,19] apply also to the number of isometry types.
Theorem 1.7. There is a family $(M_{p,q})_{(p,q)\in \mathbb{N}^2}$ of pairwise non-homotopy equivalent closed hyperbolic 3-manifolds $M_{p,q}$ satisfying

- $\text{vol}(M_{p,q}) < 2.03$
- $H_1(M_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$.

Theorem 1.7 reveals two facts. Letting $p \to \infty$, one deduces that the torsion homology cannot be bounded in terms of the volume. Fixing $p$ and letting $q \to \infty$, demonstrates that bounding both the volume and the homology is not enough to impose a finiteness result.

For a fixed symmetric space $S$, it was shown in [1] that for a sequence of finite volume $S$-manifolds $M_n$ which converges to $S$ in the Benjamini–Schramm topology, the corresponding sequence of normalized Betti numbers $b_k(M_n)/\text{vol}(M_n)$ also converges, and its limit was identified with the $k$-th $L^2$-Betti number of $S$. In particular, for $S = \mathbb{H}^3$, we get

$$\lim_{n \to \infty} b_1(M_n)/\text{vol}(M_n) = 0.$$  

It was speculated that also the normalized torsion of the homology of such a sequence will exhibit a similar phenomenon.

However, the next theorem we discuss shows that the normalized torsion can be unbounded. In [10], Brock and Dunfield constructed a sequence of hyperbolic 3-manifolds which are homology spheres and converges to $\mathbb{H}^3$ in the Benjamini–Schramm topology. In particular, the torsion vanishes along that sequence. Building on their example one can construct a sequence of closed hyperbolic 3-manifolds which converges to $\mathbb{H}^3$ in the Benjamini–Schramm topology and for which the normalized torsion tends to an arbitrary value in $[0, \infty]$.

Theorem 1.8. For every $\alpha \in [0, \infty]$ there exists a sequence of closed hyperbolic 3-manifolds $M_n^\alpha$ which are all rational homology spheres, such that the sequence $M_n^\alpha$ converges in the Benjamini–Schramm topology to $\mathbb{H}^3$ and

$$\lim_{n \to \infty} \frac{\log |\text{tors} H_1(M_n^\alpha, \mathbb{Z})|}{\text{vol}(M_n^\alpha)} = \alpha.$$  

In Theorem 6.3, which is weaker than Theorem 1.8, we provide a simpler construction of a Benjamini–Schramm convergent sequence of hyperbolic 3-manifolds with explosive torsion.

The sequences given in the above theorems are non-arithmetic. Moreover, they are not uniformly discrete, i.e. the minimal injectivity radius in $M_n$ tends to 0. It is conjectured that the normalized analytic torsion in a residual tower of coverings of a closed hyperbolic 3-manifold $M$ converges to $\tau^{(2)}(M)/\text{vol}(M) = 1/6\pi^2$, where $\tau^{(2)}(M)$ is the $L^2$-torsion of $M$ [20, Question 13.73 on p. 487]. That the normalized size of the torsion in first homology in such a residual tower converges to the same value is related to the asymptotic vanishing of regulators. Bergeron and Venkatesh conjecture this to be the case provided $M$ is arithmetic [5, Conjecture 1.3]. We discuss these issues further in §6.7.
1.5. **From negative to non-positive curvature.** Recall that the analog of Theorem 1.1 was proved in [3] for non-positively curved manifolds without Euclidean factors, under the assumption that the metrics are analytic. It is natural to ask whether the results of this paper can be extended as well from the class of negatively curved to the class of non-positively curved manifolds.

By the rank-rigidity theorem [2, 12] an irreducible Hadamard manifold is either of Jacobi rank one or a locally symmetric space of non-compact type of real rank at least two. Jacobi rank one manifolds resemble in many ways negatively curved manifolds and it might be a challenge to generalize our results to that class, and the analysis of [3, Appendix II] by V. Schroeder is relevant to that case. A higher rank complete manifold of finite volume is arithmetic by the Margulis’ arithmeticity theorem. Recall Conjecture 1.3 from [19]:

**Conjecture 1.9.** Let $X$ be a symmetric space of non-compact type. Then there are $\alpha > 0$ and $\delta > 0$ such that every arithmetic $X$-manifold $M$ is homotopy equivalent to a $(\delta, \alpha \text{vol}(M))$-simplicial complex.

In view of Lemma 5.2, Conjecture 1.9 implies the conjectured bound on the homology. Recall that Conjecture 1.9 was proved in [19] for non-compact arithmetic manifolds and in particular the homology bounds follow in the non-compact arithmetic case. Moreover, a weak version of Conjecture 1.9 was obtained in [19] for all symmetric spaces with the three exceptions of $X = \mathbb{H}^3, \mathbb{H}^2 \times \mathbb{H}^2, \text{SL}_3(\mathbb{R})/\text{SO}(3)$. This weak version is enough to deduce the counting results (see [19] Theorem 1.5, Theorem 1.11] and [20] Corollary 1.4] for the exceptional cases above excluding $\mathbb{H}^3$).

However, to extend Theorem 1.2 to compact higher rank locally symmetric manifolds by purely geometric and topological means seems a challenging task. In particular, trying to adopt the lines of [3] one encounters many difficulties which do not appear in [3] since they do not have an impact on the rational homology. More specifically, recall that the basic idea of [3] is to define an appropriate function for which one can apply the Morse lemma, and argue by induction on the dimension of the singular set. In our situation, it may very well happen that the singular set has a 3-dimensional factor in which case we loose control on the torsion completely (cf. Theorem 1.7).

1.6. **Lattices in rank 1 simple groups.** Some of the results stated above are novel already in the setting of locally symmetric manifolds. In particular, the following is an immediate application of Theorem 1.2.

**Corollary 1.10.** Let $G$ be a connected simple Lie group of real rank 1. Assume $G$ is not locally isomorphic to $SO(3,1)$. Then there exists $C = C_G > 0$ such that for every torsion free lattice $\Gamma < G$ and for every degree $k$,

$$\log |\text{tors } H_k(\Gamma; \mathbb{Z})| \leq C \text{vol}(G/\Gamma).$$

By Theorem 1.7 the restriction that $G$ is not locally isomorphic to $SO(3,1)$ is indeed necessary. We note that similar analogues of Theorem 1.1 and
Theorem 1.5 were established for all lattices (i.e., with no torsion-free assumption) in [34] and [20], respectively. We do not know if one can omit the assumption that the lattices are torsion-free in Corollary 1.10.

1.7. Structure of the paper. In the next section we will collect some homotopy theoretical preliminaries. The technical heart of this paper is §3 in which we prove Theorem 3.1. Theorem 1.4 will be proven in §4 and our main results will be proven in §5. Finally, §6 will be devoted to the study of torsion in dimension 3.

1.8. Acknowledgment. We thank Ian Biringer, Yair Minsky and Juan Souto for their advice regarding 3-dimensional geometry. We thank Vikram Aithal and Hartwig Senska for helpful mathematical and stylistic comments. We thank the referee for a detailed and extremely helpful report.

We are grateful to the organizers of the conference Manifolds and Groups 2015 in Ventotene where a lot of progress on this project took place. U.B and T.G acknowledge the support of ISF-Moked grant 2095/15. U.B acknowledges the support of ERC grant 306706.

2. Homotopy-theoretic preliminaries

2.1. Some facts about cofibrations. On a technical level, it will be important for us to extend homotopies from subspaces and to detect homotopy equivalences locally. We collect some well known results about cofibrations that deal with these issues. Recall that a continuous map \( i: A \to X \) is a cofibration if it has the homotopy extension property meaning that any continuous map \((X \times \{0\} \cup_{i \times id} A \times [0, 1]) \to Y\) can be extended to a continuous map \(X \times [0, 1] \to Y\).

Remark 2.1. We assume that all spaces are Hausdorff. Then a cofibration is always an inclusion with closed image. If \(A \subset Y\) is a closed subspace and \((Y, A)\) is an NDR pair, i.e. there is a continuous function \(u: Y \to [0, 1]\) with \(u^{-1}(0) = A\) and a map \(H: Y \times [0, 1] \to Y\) such that

1. \(H(a, t) = a\) for \(a \in A\) and \(t \in [0, 1]\),
2. \(H(x, 0) = x\) for \(x \in Y\),
3. \(H(x, 1) \in A\) if \(u(x) < 1\),

then \(A \subset Y\) is a cofibration. For a reference see [31, Section 6.4]. From that one easily sees that the inclusion of a boundary of a manifold is a cofibration. Another important example of a cofibration is the inclusion of a simplicial subcomplex [15 Proposition 8.3.9 on p. 206].

The next theorem [15 Proposition 5.3.4 on p. 112] says that being a homotopy equivalence is a local property in the presence of cofibrations.
Theorem 2.2. A commutative diagram

\[
\begin{array}{ccc}
A & 
\xleftarrow{j} & B \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
A' & 
\xleftarrow{B} & B' \\
\end{array}
\xrightarrow{\simeq} \begin{array}{ccc}
\downarrow{\simeq} & & \\
C & 
\end{array}
\]

where the hooked arrows are cofibrations and the vertical arrows are homotopy equivalences induces a homotopy equivalence between the pushouts of the rows.

The next theorem [15, cf. Exercise 8 on p. 207] says that the homotopy type of a pushout only depends on the homotopy type of the attaching map in the presence of a cofibration.

Theorem 2.3. For \(i \in \{1, 2\}\) let

\[
\begin{array}{ccc}
A & \xrightarrow{g_i} & B \\
\downarrow{j} & & \downarrow{\simeq} \\
X & \xrightarrow{\simeq} & Y_i \\
\end{array}
\]

be a pushout diagram with a cofibration \(j\). If \(g_1\) is homotopic to \(g_2\) then \(Y_1\) is homotopy equivalent to \(Y_2\).

The following is a fundamental result in homotopy theory which is also used in the proof of the previous two theorems.

Theorem 2.4 ([31, Proposition on p. 47]). If in a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{\simeq} & B \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
X & \xrightarrow{\simeq} & Y \\
\end{array}
\]

the horizontal maps are homotopy equivalences and the vertical maps are cofibrations then \((X, A)\) and \((Y, B)\) are homotopy equivalent as pairs of spaces.

Lemma 2.5. Assume \(j : A \rightarrow X\) is a cofibration, and that the left square below commutes up to homotopy. Then there is a map \(F' : X \rightarrow Y\) homotopic to \(F\) such that the right square below (strictly) commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{j} & & \downarrow{g} \\
X & \xrightarrow{\simeq} & Y \\
\end{array}
\xrightarrow{\simeq} \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{j} & & \downarrow{g} \\
X & \xrightarrow{\simeq} & Y \\
\end{array}
\]

In particular, if \(F\) is a homotopy equivalence then \(F'\) is one as well.

Proof. Let \(h : A \times [0, 1] \cup X \times \{0\} \rightarrow Y\) be a homotopy from \(F \circ j\) to \(g \circ f\) on \(A \times [0, 1]\) and \(F\) on \(X \times \{0\}\). By the cofibration property \(h\) extends to a homotopy \(H : X \times [0, 1] \rightarrow Y\). Set \(F' := H_1\). \(\square\)
2.2. Revisiting the nerve construction. A family of subsets of a space $Y$ is called an open cover of $Y$ if the union of their interiors cover $Y$. We recall two constructions of spaces associated to an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of a space $Y$. This material is needed in the final step of the proof of Theorem 3.1 in §3.2.

The nerve of $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$ whose simplices correspond to tuples in $\mathcal{U}$ with nonempty intersection, i.e. every set of $\mathcal{U}$ corresponds to a vertex and two sets form an edge if they intersect, etc. More precisely, $N(\mathcal{U})$ is the quotient space of the disjoint union of copies of $\Delta^n$ ranging over $n \geq 0$ and the subsets $\{i_0, \ldots, i_n\} \subset I$ of cardinality $n + 1$ such that $U_{i_0} \cap \ldots \cap U_{i_n} \neq \emptyset$ with identifications over the faces of $\Delta^n$ corresponding to dropping an element in $\{i_0, \ldots, i_n\}$.

In a similar spirit, one defines the space $Y_\mathcal{U}$ as in Hatcher’s book [27, Section 4G] as the quotient space of the disjoint union of products 

$$(U_{i_0} \cap \ldots \cap U_{i_n}) \times \Delta^n$$

ranging over $n \geq 0$ and the subsets $\{i_0, \ldots, i_n\} \subset I$ of cardinality $n + 1$ such that $U_{i_0} \cap \ldots \cap U_{i_n} \neq \emptyset$. The identifications in the quotient are over the faces of $\Delta^n$ corresponding to dropping an element in $\{i_0, \ldots, i_n\}$.

Next we discuss some easy functorial properties. Let $A \subset Y$ be a subspace. Let $J \subset I$ be a subset, and let $\mathcal{V} = \{V_j\}_{j \in J}$ be an open cover of $A$ such that $V_j \subset U_j$ for every $j \in J$. The inclusion maps

$$V_{j_0} \cap \ldots V_{j_n} \times \Delta^n \to U_{j_0} \cap \ldots U_{j_n} \times \Delta^n$$

are compatible with the identification and induce a map

(1) \[ A_\mathcal{V} \to Y_\mathcal{U}. \]

Similarly we get an embedding of simplicial complexes

(2) \[ N(\mathcal{V}) \to N(\mathcal{U}). \]

Further the left factor projection

$$(U_{i_0} \cap \ldots \cap U_{i_n}) \times \Delta^n \to U_{i_0} \cap \ldots U_{i_n} \subset Y$$

and the right factor projection

$$(U_{i_0} \cap \ldots \cap U_{i_n}) \times \Delta^n \to \Delta^n$$

induce maps

(3) \[ N(\mathcal{U}) \leftarrow Y_\mathcal{U} \to Y. \]

Recall that an open cover $\mathcal{U}$ of a topological space $Y$ is called good if every nonempty intersection of sets of the cover is contractible. The following result is taken from Hatcher’s book [27, Section 4.G] in the case that the elements of $\mathcal{U}$ are open. Hatcher does not assume that $\mathcal{U}$ is locally finite. An open cover in our sense (i.e. the interiors of the elements of $\mathcal{U}$ cover the space but the elements are not required to be open) is numerable on paracompact spaces provided it is locally finite, and one can appeal to [15, Section 13.3] for the general case.
Remark 2.6. If the cover in question consists of open sets rather than sets whose interiors cover the space the assumption of local finiteness can be dropped in the following results.

Theorem 2.7. Let $\mathcal{U}$ be a locally finite open cover of a paracompact space $Y$. Then $Y_{\mathcal{U}} \to Y$ is a homotopy equivalence. If, in addition, $\mathcal{U}$ is good then also $Y_{\mathcal{U}} \to N(\mathcal{U})$ is a homotopy equivalence.

Clearly, we obtain the following well known result as a consequence [15, Theorem 13.3.1 on p. 324; 27, Corollary 4G.3].

Corollary 2.8 (Nerve lemma). The nerve of a locally finite good open cover of a paracompact space $Y$ is homotopy equivalent to $Y$.

Next we record a relative version of the previous result. The method of its proof will be used in a slightly more complicated context in Step 4 of the proof of Theorem 3.1.

Lemma 2.9 (Relative nerve lemma). Let $A \subset X$ be a subspace of a paracompact Hausdorff space $X$ such that $A \hookrightarrow X$ is a cofibration. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite open cover of $X$. Let $J \subset I$ be a subset, and let $\mathcal{V} = \{V_j\}_{j \in J}$ be a locally finite open cover of $A$ such that $V_j \subset U_j$ for each $j \in J$. If $\mathcal{U}$ and $\mathcal{V}$ are good covers of $X$ and $A$, respectively, then $(X, A)$ and $(N(\mathcal{U}), N(\mathcal{V}))$ are homotopy equivalent as pairs.

Proof. Consider the following commutative diagram involving the maps (1), (2) and (3).

\[
\begin{array}{ccc}
A & \xrightarrow{\sim} & A_{\mathcal{V}} \xrightarrow{\sim} N(\mathcal{V}) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sim} & X_{\mathcal{U}} \xrightarrow{\sim} N(\mathcal{U})
\end{array}
\]

By Remark 2.1 and Theorem 2.7 the hooked arrows are cofibrations and horizontal arrows are homotopy equivalences. Denote the middle vertical map by $j$. We do not know whether $j$ is a cofibration in general. But we can replace $j$ by a cofibration via the mapping cylinder $\text{cyl}(j)$, i.e. we obtain a factorization of $j$

\[
A_{\mathcal{V}} \hookrightarrow \text{cyl}(j) \xrightarrow{\sim} X_{\mathcal{U}}
\]

into a cofibration and a homotopy equivalence. So we obtain a commutative diagram with horizontal homotopy equivalences and vertical cofibrations:

\[
\begin{array}{ccc}
A & \xrightarrow{\sim} & A_{\mathcal{V}} \xrightarrow{\sim} N(\mathcal{V}) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sim} & \text{cyl}(j) \xrightarrow{\sim} N(\mathcal{U})
\end{array}
\]

By Theorem 2.4 the left and the middle vertical inclusions are homotopy equivalent as pairs of spaces. Similarly, the left and the right vertical inclusions are homotopy equivalent as pairs of spaces. So the same holds for the left and right vertical inclusion which finishes the proof. \qed
3. A RESULT FOR HADAMARD MANIFOLDS

The main result of this section, Theorem 3.1, is a refinement of a variant of [19, Theorem 7.4]. First we will introduce some notation and conventions that will be used throughout the paper.

For a Riemannian manifold \( M \) and a point \( x \in M \) we denote the injectivity radius at \( x \) of \( M \) by \( \text{InjRad}_M(x) \). For \( \epsilon > 0 \) we let

\[
M_{\geq \epsilon} := \{ x \in M : \text{InjRad}_M(x) \geq \epsilon/2 \} \quad \text{and} \quad M_{< \epsilon} := M \setminus M_{\geq \epsilon}.
\]

These are called the \( \epsilon \)-thick part and the \( \epsilon \)-thin part of \( M \), respectively. For a subset \( A \subset M \) and \( \tau > 0 \) we let

\[
(A)_\tau = \{ x \in M : d(x, A) \leq \tau \}
\]
denote the \( \tau \)-neighborhood of \( A \). Fixing a universal cover \( \hat{X} \) of \( M \), for a subset \( A \subset M \) we denote by \( \tilde{A} \) the pre-image in \( \hat{X} \) under the covering map.

With regard to the deck transformation action of \( \Gamma = \pi_1(M) \) on \( \hat{X} \) the set \( \tilde{A} \) is \( \Gamma \)-invariant. For \( \gamma \in \Gamma \) we denote by \( d_\gamma \) the displacement function \( x \mapsto d(x, \gamma \cdot x) \) and let \( \{ d_\gamma < \epsilon \} = \{ x : d_\gamma(x) < \epsilon \} \) denote its \( \epsilon \)-sub-level set. Recall that a Hadamard space is a simply connected complete Riemannian manifold of non-positive sectional curvature. If \( X \) is a Hadamard space then the metric \( d : X \times X \to [0, \infty) \) is convex and smooth outside the diagonal. Therefore the \( \epsilon \)-sub-level sets are convex and, by the implicit function theorem, have smooth boundary. Finally, observe the equation

\[
\tilde{M}_{< \epsilon} = \bigcup_{\gamma \in \Gamma \setminus \{1\}} \{ d_\gamma < \epsilon \}.
\]

**Theorem 3.1.** Let \( M \) be a \( d \)-dimensional complete Riemannian manifold of finite volume with sectional curvature varying in \([-1,0]\). Let \( X \) be its universal covering which is a Hadamard manifold. Let \( \Gamma = \pi_1(M) \), so \( M = \Gamma \setminus X \). Fix constants \( 0 < \epsilon < \epsilon_0 \). Let \( M_+ \subset M_{\geq \epsilon} \) be a compact \( d \)-dimensional submanifold with boundary. Let \( M_- = M \setminus M_+ \). Suppose that there is a conjugation invariant subset \( \Gamma' \subset \Gamma \) and a conjugation invariant assignment of numbers \( \{ \epsilon_\gamma \}_{\gamma \in \Gamma'} \subset [\epsilon, \epsilon_0] \) such that the pre-image \( \tilde{M}_- \subset X \) of \( M_- \) in \( X \) is given by \( \tilde{M}_- = \bigcup_{\gamma \in \Gamma'} \{ d_\gamma < \epsilon_\gamma \} \). Furthermore, suppose that for every point \( x \notin \tilde{M}_- \) the group

\[
\langle \gamma \in \Gamma' : d_\gamma(x) \leq 4\epsilon_\gamma \rangle
\]

is commutative. Then \( (M_+, \partial M_+) \) is homotopy equivalent to a \((D, c \cdot \text{vol } M_+)\) simplicial pair, where the constants \( D, c > 0 \) depend only on \( d, \epsilon_0 \) and \( \epsilon \).

The goal of this section is a proof of Theorem 3.1. We start with a preliminary subsection and then conclude the proof of Theorem 3.1 in four steps. The proof relies heavily on results and constructions given in §3, §4 and §7 of [19]. We recall that in [19] it is globally assumed that \( X \) is a symmetric space of non-compact type. However, §3, §4, §7 only rely on the

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4 Caution: \( \tilde{A} \) does not denote a universal cover of \( A \).
assumption that $X$ is a Hadamard manifold. Accordingly, we are free to use the results of these sections in [19].

3.1. An estimate for essential volume. We define

$$N(d, r, R) := \frac{\text{vol}_{\mathbb{H}^d} B(o, R + r/2)}{\text{vol}_{\mathbb{E}^d} (B(0, r/2))}$$

as the ratio of volumes of the corresponding balls in the hyperbolic $d$-space $\mathbb{H}^d$ and the Euclidean $d$-space $\mathbb{E}^d$, respectively.

**Lemma 3.2.** Let $X$ be a $d$-dimensional Hadamard manifold with sectional curvature varying in $[-1, 0]$. For $0 < r < R$ and any $x \in X$, the cardinality of an $r$-discrete subset of $B_X(x, R)$ is bounded above by $N(d, r, R)$.

Indeed, the $r/2$ balls centered at the points of such an $r$-discrete set are disjoint and, on the other hand, contained in the $(R + r/2)$ ball around $x$. Therefore, the lemma is a straightforward consequence of the following volume estimate which follows by combining the Bishop–Gunt her (lower bound) and the Bishop–Gromov (upper bound) comparison theorems:

**Theorem 3.3** ([18, Theorem 3.101 on p. 169 and Theorem 4.19 on p. 214]). Let $X$ be a Hadamard manifold of dimension $d$ and sectional curvature varying in the segment $[-1, 0]$. For every $R > 0$ and every $x \in X$ we have

$$\text{vol}_{\mathbb{E}^d} (B(0, R)) \leq \text{vol}_X (B(x, R)) \leq \text{vol}_{\mathbb{H}^d} B(o, R).$$

**Corollary 3.4.** Let $M = \Gamma \backslash X$ be a $d$-dimensional complete Riemannian manifold of sectional curvature varying in the segment $[-1, 0]$. Let $\epsilon > 0$ and $x \in \tilde{M}_{\geq \epsilon}$. Then for every $R > 0$ we have:

$$\# \{ \gamma \in \Gamma : d_\gamma (x) \leq R \} \leq N(d, \epsilon, R).$$

**Proof.** Indeed, the set $\{ \gamma \cdot x : \gamma \in \Gamma, d_\gamma (x) \leq R \}$ is $\epsilon$-discrete and contained in the ball $B(x, R)$. \qed

3.2. Proof of Theorem [3.1]

**Step 1: Obtuse angles between sub-level sets.** The goal of the first step is to show the following variant of [19, Lemma 7.1].

**Lemma 3.5.** Let $x \in (\tilde{M}_-, \setminus \tilde{M}_-).$ Consider the commutative subgroup

$$A := \langle \gamma \in \Gamma' : d_\gamma (x) \leq 4\epsilon \rangle \subset \Gamma.$$

Then we have

$$\nabla d(\cdot, \{ d_\alpha < \epsilon \})_x \cdot \nabla d(\cdot, \{ d_\beta < \epsilon \})_x \geq 0$$

for every $\alpha, \beta \in A \cap \Gamma'$.

The special case where $x \in \partial \{ d_\alpha < \epsilon \} \cap \partial \{ d_\beta < \epsilon \}$ follows directly from [19, Lemma 7.1]. However, we will need the result for points which are not necessarily on the boundary. The proof relies on tools from [19, §7].
\[ \nabla d(\cdot, \{ d_\beta < \epsilon_\beta \})|_x \]

\[ \nabla d(\cdot, \{ d_\alpha < \epsilon_\alpha \})|_x \]

\[ d_\alpha < \epsilon_\alpha \]

\[ d_\beta < \epsilon_\beta \]

\textbf{Figure 1.} If \( \alpha \) and \( \beta \) commute, the angle formed by the sub-level sets is obtuse.

\textbf{Remark 3.6.} Note that \( A \) is not empty. Indeed, because of \( x \in (\tilde{M}_-)_e \setminus \tilde{M}_- \) we have \( d(x, \tilde{M}_-) \leq \epsilon \), so there exists at least one \( \gamma \in \Gamma' \) such that

\[ x \in (\{ d_\gamma < \epsilon_\gamma \})_e \subset \{ d_\gamma < \epsilon_\gamma + 2\epsilon \} \subset \{ d_\gamma < 4\epsilon_\gamma \}. \]

The basic fact that stands behind Lemma 3.5, is given in the following:

\textbf{Lemma 3.7.} Let \( X \) be a Hadamard manifold, \( \Omega \subset X \) a convex subset with smooth boundary, and \( \alpha \) an isometry of \( X \) which preserves \( \Omega \). Let \( x \in \partial \Omega \) be a point on the boundary of \( \Omega \), let \( \hat{f} \in T_x(X) \) be the external normal to \( \Omega \) at \( x \). Then \( \nabla d_\alpha|_x \cdot \hat{f} \geq 0 \).

Note that \( \hat{f} = \nabla d(\cdot, \Omega)|_x \) and that the vectors \( \nabla d_\alpha|_x \) and \( \nabla d(\cdot, \{ d_\alpha \leq \epsilon_\alpha(x) \})|_x \) only differ by a positive scalar.

\textbf{Proof of Lemma 3.7.} Let \( \pi_\Omega \) be the nearest point projection from \( X \) to \( \Omega \). Let \( c(t) \) be the geodesic ray starting at \( x \) with \( \dot{c}(0) = \hat{f} \), and let \( c'(t) = \alpha \cdot c(t) \) (see Figure 3.2). Note that \( \pi_\Omega(c(t)) = x \) and \( \pi_\Omega(c'(t)) = \alpha \cdot x \). Since the sectional curvature is non-positive \( \phi(t) := d(c(t), c'(t)) \) is a convex function. Since \( \pi_\Omega \) is also a contraction \( \phi(t) \) is monotonically non-decreasing. Thus,

\[
0 \leq \frac{d}{dt} \phi(t)|_{t=0} = \frac{d}{dt} d(c(t), \alpha \cdot c(t))|_{t=0} \\
= \frac{d}{dt} d_\alpha(c(t))|_{t=0} = \nabla d_\alpha|_x \cdot \dot{c}(0) = \nabla d_\alpha|_x \cdot \hat{f}.
\]

\[ \alpha \cdot c(t) \]

\[ \alpha \cdot x \]

\[ x \]
Lemma 3.8. Let $\Theta \subset X$ be a closed subset. Let $t \geq 0$. For a point $z$ outside the interior of $(\Theta)_t$ we have

\[(6) \quad d(z, \Theta) = d(z, (\Theta)_t) + t,\]

and

\[(7) \quad \nabla d(\cdot, (\Theta)_t)|_z = \nabla d(\cdot, \Theta)|_z.\]

Proof. The second assertion (7) follows from the first (6), which we show next. Let $z_\Theta$ be a point in $\Theta$ of minimal distance to $z$. Let $c$ be a minimizing geodesic from $z$ to $z_\Theta$. Let $z_t$ be the first point of intersection of $c$ with $(\Theta)_t$. Then $d(z_t, z_\Theta) = t$ and $d(z, z_\Theta) = d(z, z_t) + d(z_t, z_\Theta)$. If $d(z, (\Theta)_t) < d(z, z_t)$ then there would be a point $z'_t \in (\Theta)_t$ with $d(z, z'_t) < d(z, z_t)$ and thus $d(z, \Theta) \leq d(z, z'_t) + t < d(z, z_t) + t = d(z, \Theta)$, which is absurd. Hence $d(z, (\Theta)_t) = d(z, z_t)$ which implies (6).

Proof of Lemma 3.7. Denote $t_\alpha = d(x, \{d_\alpha \leq \epsilon_\alpha\})$ and $t_\beta = d(x, \{d_\beta \leq \epsilon_\beta\})$. Without loss of generality suppose $t_\alpha \leq t_\beta$. Set

$$\Omega := \underbrace{(\{d_\beta \leq \epsilon_\beta\})}_{(t_\beta - t_\alpha)},$$

unless $t_\alpha = t_\beta$ in which case we set $\Omega = \{d_\beta \leq \epsilon_\beta\}$. Since $\alpha$ and $\beta$ commute, $\Omega$ is $\alpha$-invariant. Lemma 3.7 implies that

$$\nabla d_\alpha|_y \cdot \nabla d(\cdot, \Omega)|_y \geq 0$$

for every $y \in \partial \{d_\alpha \leq \epsilon_\alpha\} \cap \partial \Omega$. Using the remark following Theorem 3.7 and Lemma 3.8 we obtain that

$$\nabla d(\cdot, \{d_\alpha \leq \epsilon_\alpha\})|_y \cdot \nabla d(\cdot, \{d_\beta \leq \epsilon_\beta\})|_y \geq 0$$

for every $y \in \partial \{d_\alpha \leq \epsilon_\alpha\} \cap \partial \Omega$. Denoting the maximal angle by

$$\varphi_0 = \sup \angle (\nabla d(\cdot, \{d_\alpha \leq \epsilon_\alpha\})|_y, \nabla d(\cdot, \Omega)|_y) : y \in \partial \{d_\alpha \leq \epsilon_\alpha\} \cap \partial \Omega,$$

we deduce that $\varphi_0 \leq \frac{\pi}{2}$. Now for $t > 0$ we let

$$\varphi_t = \sup \angle (\nabla d(\cdot, \{d_\alpha \leq \epsilon_\alpha\})|_y, \nabla d(\cdot, \Omega)|_y) : y \in \partial \{d_\alpha \leq \epsilon_\alpha\}_t \cap \partial (\Omega)_t.$$

Observe that since the curvature is non-positive both $\{d_\alpha \leq \epsilon_\alpha\}$ and $\Omega$ are convex, so we may apply [19, Lemma 7.2] and deduce that $\varphi$ is non-increasing. By assumption,

$$d(x, \{d_\alpha \leq \epsilon_\alpha\}) = d(x, \Omega) = t_\alpha.$$

It follows that

$$\angle (\nabla d(\cdot, \{d_\alpha \leq \epsilon_\alpha\})|_x, \nabla d(\cdot, \Omega)|_x) \leq \varphi_{t_\alpha} \leq \frac{\pi}{2}.$$

Therefore

$$\nabla d(\cdot, \{d_\alpha \leq \epsilon_\alpha\}) \cdot \nabla d(\cdot, \{d_\beta \leq \epsilon_\beta\}) = \nabla d(\cdot, \{d_\alpha \leq \epsilon_\alpha\})|_x \cdot \nabla d(\cdot, \Omega)|_x \geq 0. \square$$
Step 2: Setting the constants $b$ and $\delta$. In this step we define constants $b = b(d)$ and $\delta = \delta(d,b,\epsilon)$ which will be used throughout the proof and discuss their properties.

**Definition 3.9.** For every dimension $d$ let $b = b(d)$ be the maximal cardinality of a 1-discrete subset of unit vectors in $\mathbb{R}^d$. For every dimension $d$ and every $\epsilon > 0$ choose $\delta_0(\epsilon/4, b + 1, d) > 0$ that satisfies the statement of Lemma 3.10 below. Then we define $\delta = \min\{\delta_0(\epsilon/4, b + 1, d), \epsilon/(2(b + 1))\}$.

**Lemma 3.10** (cf. [19, Proposition 4.7]). For all $\alpha > 0$, $\beta > 0$ and for every dimension $d > 0$ there exists $\delta_0 = \delta_0(\alpha, \beta, d) > 0$ satisfying the following property: For $\delta \leq \delta_0$ and every $d$-dimensional Hadamard manifold $X$ with sectional curvature $\geq -1$, any point $x \in X$, any ball $C \subset X$ of radius $\alpha$ with $x \in \partial C$, and any point $y \in B(x, \beta \alpha) \setminus (C)_{\delta/2}$, the inner product of the external normal vector of $C$ at $x$ with the tangent at $x$ to the geodesic segment $[x,y]$ is positive.

**Proof.** The statement is a variant of [19, Proposition 4.7]. For convenience we sketch the proof. Assume the contrary. Then there is a sequence $\delta_n \to 0$, $d$-dimensional Hadamard manifolds $X_n$, points $x_n \in X_n$ and $y_n \in B(x_n, \delta_n \beta) \setminus (C_n)_{\delta_n/2}$, and $\alpha$-balls $C_n$ violating the above statement.

Let $d_n$ be the metric of $X_n$. Let $B_n := B(x_n, \delta_n \beta)$. The sequence of pointed metric spaces $(B_n, x_n)$ with scaled metric $\frac{1}{\delta_n \beta} d_n$ converges in the Gromov–Hausdorff topology to the (pointed) standard Euclidean unit ball $(B_0, 0)$. The sequence $(C_n \cap B_n, x_n)$ converges to a half ball $C_0$ in $B_0$ because the scaled radius of $C_n$ is at least $\alpha/\delta_n \beta$, thus tends to $\infty$. Upon passing to a subsequence, we may also assume that $(y_n)$ converges to a point $y_0 \in B_0 \setminus (C_0)_{\frac{1}{\delta_n \beta}}$. The inner product of $y_0$ and the external normal to $C_0$ is $\geq \frac{1}{\delta_n \beta} > 0$ which contradicts the assumption.

Our choice of $b$ was made so that the following statement holds true.

**Lemma 3.11** ([19, Proposition 4.2]). For every $s > 0$ and $t > 0$ with $s + t < \epsilon$, we have

$$M \setminus (M_{-})_s \subset (M \setminus (M_{-})_{s+t})_{bt}.$$

**Proof.** Up to different notation the lemma is identical with [19, Proposition 4.2]. Note that [19, Proposition 4.2] is a step in the proof of [19, Lemma 4.1] and some assumptions of the former are implicit in its formulation — they are stated explicitly in the formulation of the latter. However, not all the assumptions of the latter are actually used in the proof of the former. Hence our task is to compare the setting of [19, Lemma 4.1 and 4.2] with the present one. The correspondence\(^5\) is $S = X$, $M' = M_+$, $I = \Gamma'$, $X_\gamma = \{d_i < \epsilon_s\}$ for $\gamma \in \Gamma'$, $\tau = s$, and $\delta = t$. Then we have $M' = M_+ \subset M_{2s}$, and the cover of $M_+$ by the $X_i$ is indeed a locally finite

\(^5\)Recall that the results of §4 of [19] apply to a general Hadamard manifold.
cover by convex open sets with a smooth boundary. Finally, Lemma 3.12 below provides the (not necessarily continuous) vector field \( \hat{n} \) with the exact property assumed in [19, Lemma 4.1]. The properties above are the only ones used in the proof of [19, Proposition 4.2]. Thus we get the result. \( \Box \)

**Lemma 3.12.** For every \( x \in (\tilde{M}_-) \setminus \tilde{M} = \tilde{M}_+ \cap (\tilde{M}_-) \) there is a unit vector \( \hat{n}(x) \in T_x X \) such that

\[
d_{\gamma}(x) \leq 4e_\gamma \ \Rightarrow \ \hat{n}(x) \cdot \nabla d(\cdot, \{d_{\gamma} < e_\gamma\})|_x > 1/b.
\]

for every \( \gamma \in \Gamma' \).

**Proof.** Let \( I := \{ \gamma \in \Gamma' : d_{\gamma}(x) \leq 4e_\gamma \} \). Let \( \Delta(x) \) be a maximal 1-discrete subset of the vectors \( \nabla d(\cdot, \{d_{\gamma} < e_\gamma\})|_x, \gamma \in I \). The set \( \Delta(x) \) has at most \( b = b(d) \) elements. Let \( \hat{n}(x) \) be the sum of the vectors of \( \Delta(x) \) normalized to unit length. The statement about \( \hat{n}(x) \) is then easily verified (see [19, Lemma 7.3]). \( \Box \)

**Step 3: Constructing a deformation retract.** We define

\[
\tilde{M}_+ = X \setminus (\tilde{M}_-)_{\frac{\varepsilon}{2}} \text{ and } M'_+ = \Gamma \setminus \tilde{M}'_+.
\]

The main task of this step is to show that the shrinking \( M'_+ \) of \( M_+ \) has the same homotopy type as \( M_+ \). Indeed there is a strong deformation retract from \( M_+ \) to \( M'_+ \) with nice properties. Recall that a strong deformation retract from a space \( W \) to a subset \( A \subset W \) is a homotopy \( h : W \times [0, 1] \to W \) from \( \text{id}_W \) to a map into \( A \) such that \( h_t|_A = \text{id}_A \) for every \( t \in [0, 1] \).

**Lemma 3.13.** There exists a strong deformation retract \( f : M_+ \times [0, 1] \to M_+ \) from \( M_+ \) to \( M'_+ \) with the following properties:

(i) The time 1 map, \( f_1 : M_+ \to M'_+ \), restricts to a homeomorphism \( f_1 : \partial M_+ \to \partial M'_+ \).

(ii) For every \( y \in M'_+ \) with \( d(y, (M_-)_{\frac{\varepsilon}{2}}) > \delta \), the ball \( B(y, (b+1)\delta) \) is stable under \( f \) in the sense that if \( (x, t) \in M_+ \times [0, 1] \) and \( f(x, t) \in B(y, (b+1)\delta) \) then \( f(x, s) \in B(y, (b+1)\delta) \) for all \( s \in [t, 1] \).

Lemma 3.13 could be deduced from the analysis in [19, §3]. However, we cannot refer directly to statements from [19, §3] but require adjustments in the arguments. Therefore we shall give a self contained proof which is in a sense more straightforward.

**Proof.** We will construct a \( \Gamma \)-equivariant strong deformation retract from \( \tilde{M}_+ \) to \( \tilde{M}'_+ \); by passing to \( \Gamma \)-quotients we obtain the desired map \( f \). To this end, we will construct a smooth \( \Gamma \)-equivariant vector field \( \tilde{V} \) on \( X \) with the following properties.

a) The image of the support of \( \tilde{V} \) in \( M = \Gamma \setminus X \) is compact. Hence \( \tilde{V} \) possesses a global flow \( c : X \times \mathbb{R} \to X \).

b) On \( \{(x, t) \in X \times [0, \infty) \mid d(c(x, t), \tilde{M}_-) \in (0, \varepsilon/2)\} \) the function \( t \mapsto d(c(x, t), \tilde{M}_-) \) has lower Lipschitz constant 1.
c) Let $y \not\in (\tilde{M}_-)_{\epsilon/2+\delta}$. For $x \in B(y, (b+1)\delta) \cap (\tilde{M}_-)_{\epsilon/2}$ the angle between $\tilde{V}(x)$ and the geodesic from $x$ to $y$ is acute.

Before constructing $\tilde{V}$ let us explain how, given $\tilde{V}$, we obtain the desired deformation retract.

For $x \in X$ with $d(x, \tilde{M}_-) \leq \epsilon/2$ define $T(x)$ to be the minimal time $t \in [0, \infty)$ such that $d(c(x, T(x)), \tilde{M}_-) = \epsilon/2$, that is, $T(x)$ is the entrance time of the flow line through $x$ into $\tilde{M}_+$. We claim that the entrance time is continuous. Because of (b) we have $T(x) \leq \frac{\epsilon}{2}$. Let $\epsilon_1 > 0$. The function

$$g: \{x \in X : d(x, \tilde{M}_-) \leq \epsilon/2\} \times [0, \frac{\epsilon}{2}] \to \mathbb{R}, \ (x, t) \mapsto d(c(x, t), \tilde{M}_-)$$

is continuous. Thus every point whose distance to $\tilde{M}_-$ is at most $\epsilon/2$ has a neighborhood $U$ so that there is $\delta_1 > 0$ such that $d(x, x') < \delta_1$ with $x, x' \in U$ implies $|g(x, t) - g(x', t)| < \epsilon_1$ for every $t \in [0, 1]$. Consider $x, x' \in U$ with $d(x, x') < \delta_1$. Assume that $T(x) \leq T(x')$. Then $d(c(x', T(x)), \tilde{M}_-) \geq \epsilon/2 - \epsilon_1$ and thus $T(x) - T(x') \leq \epsilon_1$ according to (b). Thus indeed, $T$ is continuous.

The map $f: \tilde{M}_+ \times [0, 1] \to \tilde{M}_+$ defined by

$$f(x, t) = \begin{cases} c(x, tT(x)) & \text{if } d(x, \tilde{M}_-) \leq \epsilon/2, \\ x & \text{otherwise,} \end{cases}$$

is a deformation retract.
is the desired $\Gamma$-equivariant strong deformation retract from $\tilde{M}_+$ to $\tilde{M}_+$. It is clear that $f_1$ restricts to a map $\partial\tilde{M}_+ \to \partial\tilde{M}_+$. The inverse of $f_1$ is obtained by flowing along $-\tilde{V}$ to $\partial\tilde{M}_+$ in a similar way as above.

The stability of balls, i.e. the fact that flow lines do not leave a ball $B(y, (b + 1)\delta)$ once they enter it, follows directly from property (c).

Next we turn to the construction of $\tilde{V}$. For $\gamma \in \Gamma'$ we define the function

$$\phi_\gamma(x) = d(x, \{d_\gamma < \epsilon_\gamma\})$$

which is convex and smooth. It satisfies the equivariance property

$$\phi_\gamma(\lambda x) = \phi_{\lambda \gamma \lambda^{-1}}(x).$$

We choose smooth functions $\tau_\gamma : X \to [0, 1]$ such that $\tau_\gamma \equiv 1$ on $\{d_\gamma < \epsilon_\gamma\}_{\epsilon/2} \cap \tilde{M}_+$ and $\tau_\gamma$ is supported in $\{d_\gamma < \epsilon_\gamma\}_{\epsilon/2 + \delta/2} \cap (\tilde{M}_+)_{\delta/2}$, and the collection $\{\tau_\gamma\}$ is $\Gamma$-equivariant, i.e. $\tau_\gamma(\lambda x) = \tau_{\lambda \gamma \lambda^{-1}}(x)$, $\forall \gamma, \lambda \in \Gamma$. Now we set

$$\tilde{V}(x) = \sum_{\gamma \in \Gamma'} \tau_\gamma(x) \cdot \nabla \phi_\gamma|_x.$$

Since for any $r > 0$ and $x \in X$ the subset $\{g \in \text{Isom}(X) : d_g(x) < r\}$ is compact and $\Gamma < \text{Isom}(X)$ is discrete, the above sum only involves finitely many summands. Since $M$ has finite volume, Theorem 3.3 easily implies that $M$ is compact.

Property (b) follows since for every $x$ with $d(x, \tilde{M}_-) \in (0, \epsilon/2]$ and for every $\gamma \in \Gamma'$ with

$$d(x, \tilde{M}_-) = d(x, \{d_\gamma < \epsilon_\gamma\}),$$

we have

$$\langle \tilde{V}(x), \nabla \phi_\gamma|_x \rangle \geq \langle \nabla \phi_\gamma|_x, \nabla \phi_\gamma|_x \rangle = 1$$

by Lemma 3.5 and (5). Let $y \notin (\tilde{M}_-)_{\epsilon/2 + \delta}$. Let $\tilde{U}(x) \in T_xX$ be the tangent to the geodesic from $x$ to $y$ in $X$. The stability of $B(y, (b + 1)\delta)$ will follow once we show:

$$x \in B(y, (b + 1)\delta) \cap (\tilde{M}_-)_{\epsilon/2} \Rightarrow \tilde{V}(x) \cdot \tilde{U}(x) > 0.$$  

Let $x \in B(y, (b + 1)\delta) \cap (\tilde{M}_-)_{\epsilon/2}$. The implication (9) will follow from

$$\nabla \phi_\gamma|_x \cdot \tilde{U}(x) > 0 \quad \text{for every } \gamma \text{ with } \phi_\gamma(x) \leq \epsilon/2 + \delta/2.$$  

Let $z$ be a point on the shortest geodesic from $x$ to $\{d_\gamma \leq \epsilon_\gamma\}$ satisfying $d(z, x) = \epsilon/4$. Let $\pi_\gamma$ be the nearest point projection to $\{d_\gamma \leq \epsilon_\gamma\}$. We let $C = B(z, \epsilon/4)$. For $w \in C$ note that

$$d(w, \pi_\gamma(x)) \leq d(w, z) + d(z, \pi_\gamma(x)) \leq \epsilon/4 + d(x, \pi_\gamma(x)) - \epsilon/4 = \phi_\gamma(x).$$

Hence $C \subset \{\phi_\gamma \leq \phi_\gamma(x)\}$. Hence $\nabla \phi_\gamma|_x$ is the external normal vector of $C$ at $x$. Now (10) follows by applying Lemma 3.10 for $\alpha = \epsilon/4$ and $\beta = b + 1$ provided $d(y, C) > \delta/2$. Since $\phi_\gamma(x) \leq \epsilon/2 + \delta/2$ we have

$$d(z, \tilde{M}_-) \leq \phi_\gamma(z) = \phi_\gamma(x) - \epsilon/4 < \epsilon/4 + \delta/2.$$
So
\[ d(y, C) > d(y, \tilde{M}_-) - (\epsilon/4 + d(z, \tilde{M}_-)) > \delta/2 \]
since \( y \not\in (\tilde{M}_-)_\epsilon/2. \)
\[ \square \]

**Step 4: Completion of the proof of Theorem 3.1.** In the final step of the proof of Theorem 3.1 we construct suitable open covers of \( M_+ \) and \( \partial M_+ \) that yield the desired simplicial pair via the nerve construction. Let \( pr: X \to M \) be the universal cover. We define

- \( Z \) to be a maximal \( \delta \)-discrete subset in \( M \setminus (M_-)_{\frac{\epsilon}{2} + \delta} \) and for every \( z \in Z \) the point \( \bar{z} \in X \) to be a choice of lift of \( z \),
- \( N_+ \) to be the union of the balls \( B(z, (b + 1)\delta) \) with \( z \in Z \),
- \( N_0 := N_+ \cap (M_-)_{\epsilon/2} \),
- \( W \subset Z \times \Gamma' \) to be the subset of pairs \((z, \gamma)\) for which \( V(z, \gamma) := pr(B(\bar{z}, (b + 1)\delta) \cap \{d_\gamma < \epsilon_\gamma\})_{\epsilon/2} \neq \emptyset \),
- \( \mathcal{V} \) to be the family of sets \( V(z, \gamma) \) indexed over \( W \),
- \( \mathcal{U} \) to be the family of sets \( (B(z, (b + 1)\delta))_{z \in Z} \) and \( (V(z, \gamma))_{(z, \gamma) \in W} \)

indexed over \( Z \cup W \).

The family \( \mathcal{V} \) is an open cover of \( N_0 \) in the sense of §2.2, that is, the sets in \( \mathcal{V} \) are not necessarily open in \( N_0 \) but the union of their \( N_0 \)-interiors cover \( N_0 \). The family \( \mathcal{U} \) is an open cover of \( N_+ \). Both \( \mathcal{V} \) and \( \mathcal{U} \) are good since their elements are convex.

In the sequel we refer to the deformation retract \( f \) constructed in the previous step. The deformation retract at time \( t \) is denoted by \( f_t \).

Let \( z \in Z \). Since \( (b + 1)\delta < \epsilon/2 \) and \( d(z, M_-) \geq \epsilon/2 + \delta \) we get \( B(z, (b + 1)\delta) \cap M_- = \emptyset \), thus \( B(z, (b + 1)\delta) \subset M_+ \) for every \( z \in Z \). Hence
\[ N_+ \subset M_+. \]

By Lemma 3.11 we have
\[ M'_+ \subset N_+ \]
and hence also \( \partial M'_+ \subset N_0 \).

**Lemma 3.14.** The inclusions \( M'_+ \hookrightarrow N_+ \) and \( \partial M'_+ \hookrightarrow N_0 \) are homotopy equivalences.

**Proof.** We claim that
\[ r: N_+ \hookrightarrow M_+ \xrightarrow{f_1} M'_+ \]
is a homotopy inverse of the inclusion \( j: M'_+ \hookrightarrow N_+ \). It is clear that \( r \circ j = id \).

By the stability of balls under \( f \) (see Lemma 3.13) we have
\[ f_t(B((b + 1)\delta, z)) \subset B((b + 1)\delta, z) \]
for every \( t \in [0,1] \) since this is obviously true for \( t = 0 \). In particular, \( f_1 \) preserves \( N_+ \) for every \( t \in [0,1] \). Thus \( f \) restricts to a homotopy between \( id \) and \( j \circ r: N_+ \to N_+ \).

Since \( f_t \) also restricts to \( (M_-)_{\epsilon/2} \) we may argue similarly for \( \partial M'_+ \hookrightarrow N_0 \). \[ \square \]
Consider the following commutative diagram

$$
\begin{array}{c}
\partial M_+ \xrightarrow{\approx} \partial M'_+ \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
M_+ \xrightarrow{\approx} M'_+ \xrightarrow{\approx} N_+ \xrightarrow{\approx} (N_+)_{\mathcal{V}} \xrightarrow{\approx} N(\mathcal{V})
\end{array}
$$

where $\approx$ indicates a homeomorphism, $\cong$ indicates a homotopy equivalence and the hooked arrows are inclusions. The horizontal maps on the right hand side are homotopy equivalences since the open covers $\mathcal{V}$ and $\mathcal{U}$ are good (see Theorem 2.7). It was shown in Step 3 that its restriction $f_1: \partial M_+ \to \partial M'_+$ is a homeomorphism, in particular, a homotopy equivalence. The outer vertical maps in (11) are cofibrations, being the inclusion of the boundary of a manifold and the inclusion of a subcomplex, respectively. We avoid proving that the inner vertical maps are cofibrations, in which case we could appeal directly to the relative nerve lemma as a blackbox (see Lemma 2.9). Instead we will use below the method of the proof of the relative nerve lemma.

We factorize $j$ into a cofibration $j'$ and a homotopy equivalence $r$ via the mapping cylinder $\text{cyl}(j)$ of $j$:

$$
j: N_0 \xrightarrow{j'} \text{cyl}(j) \xrightarrow{r} N_+
$$

We choose a homotopy inverse $r': N_+ \to \text{cyl}(j)$ of $r$. Next consider the following three squares:

$$
\begin{array}{c}
\partial M_+ \approx N_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
M_+ \approx N_+
\end{array}
\quad
\begin{array}{c}
\partial M_+ \approx N_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
M_+ \approx \text{cyl}(j)
\end{array}
\quad
\begin{array}{c}
\partial M_+ \approx N_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
M_+ \approx \text{cyl}(j)
\end{array}
$$
The left square arises from composing the left two squares in (11) and thus commutes. The middle square, and in particular its lower horizontal arrow, arises from composition with $r'$. The middle square only commutes up to homotopy. From an application of Lemma 2.5 we obtain the commutative right square. By Theorem 2.4 applied to the right square we obtain a homotopy equivalence of pairs

\[(M_+, \partial M_+) \simeq (\text{cyl}(j), N_0).\]

Next we apply the mapping cylinder construction to the map $k$ in (11) and get a factorization of $k$ as

\[\begin{align*}
(N_0)_V \xrightarrow{k'} \text{cyl}(k) \xrightarrow{s} (N_+)_{[U].}
\end{align*}\]

By pre-composing the lower horizontal map in the third square of (11) with the homotopy equivalence $s$ and post-composing it with $r'$ we obtain a homotopy-commutative diagram with vertical cofibrations:

\[\begin{array}{ccc}
N_0 & \xleftarrow{\simeq} & (N_0)_V \\
\downarrow{j'} & & \downarrow{k'} \\
\text{cyl}(j) & \xleftarrow{\simeq} & \text{cyl}(k)
\end{array}\]

As above, we can use Lemma 2.5 to replace its lower horizontal arrow by another homotopy equivalence making the square strictly commute and then apply Theorem 2.4 to get a homotopy equivalence of pairs

\[(\text{cyl}(j), N_0) \simeq (\text{cyl}(k), (N_0)_V).\]

By composing the lower horizontal arrow in the right hand square of (11) with $s$ we obtain the commutative diagram

\[\begin{array}{ccc}
(N_0)_V & \xrightarrow{\simeq} & N(V) \\
\downarrow{k'} & & \downarrow{} \\
\text{cyl}(k) & \xrightarrow{\simeq} & N(U)
\end{array}\]

to which we apply Theorem 2.4. We thus obtain a homotopy equivalence of pairs

\[(\text{cyl}(k), (N_0)_V) \simeq (N(U), N(V)).\]

The desired homotopy equivalence of pairs

\[(M_+, \partial M_+) \simeq (N(U), N(V))\]

is obtained by combining (12) and (13) and (14).

Referring to (4) we define two constants:

\[c := \frac{(N(d, \epsilon, 4\epsilon_0) + 1)}{\text{vol}_{\mathbb{R}^4}(B(0, \delta/2))}\]

\[D := (N(d, \epsilon, 4\epsilon_0) + 1)N(d, \delta, 2(b + 1)\delta)\]

The proof of Theorem 3.1 will be completed by the following lemma.
Lemma 3.15. The nerve $N(U)$ is a $(D, c \text{vol}(M_+))$-simplicial complex.

Proof. First we show that the number of vertices in $N(U)$, that is $\#Z + \#W$, is at most $c \text{vol}(M_+)$. Since $Z$ is a $\delta$-discrete subset of $M_+$, Theorem 3.3 implies that

$$\#Z \leq \frac{\text{vol}(M_+)}{\text{vol}_e(B(0, \delta/2)).}$$

Let $z \in Z$. If $\gamma \in \Gamma'$ is such that $B(\bar{z}, (b+1)\delta)$ intersects $\overline{(\{d\gamma < \epsilon_\gamma\})_{\epsilon/2}}$ then

$$d_\gamma(\bar{z}) \leq \epsilon_\gamma + 2(\frac{\epsilon}{2} + (b+1)\delta) < 4\epsilon_0.$$ 

Hence Corollary 3.3 implies that

$$\#\{\gamma \in \Gamma' : B(\bar{z}, (b+1)\delta) \cap \overline{(\{d\gamma < \epsilon_\gamma\})_{\epsilon/2}} \neq \emptyset\} \leq N(d, \epsilon, 4\epsilon_0).$$

Thus $\#W \leq \#Z \cdot N(d, \epsilon, 4\epsilon_0)$ from which the desired bound $\#Z + \#W \leq c \text{vol}(M_+)$ follows.

Next we show that the degree of a vertex in $N(U)$ is at most $D$. Consider an element $U_0 \in U$ which represents a vertex in $N(U)$. Let $z_0 \in Z$ or $(z_0, \gamma_0) \in W$, respectively, be the index of the element $U_0 \in U$. The degree of $U_0$ is bounded using (16) by

$$\deg U_0 \leq \#\{z \in Z : B(z_0, (b+1)\delta) \cap B(z, (b+1)\delta) \neq \emptyset\}$$

$$+ \#\{(z, \gamma) \in Z \times \Gamma' : B(z_0, (b+1)\delta) \cap \overline{(\{d\gamma < \epsilon_\gamma\})_{\epsilon/2}} \neq \emptyset \text{ and } B(z_0, (b+1)\delta) \cap B(z, (b+1)\delta) \neq \emptyset\}$$

$$\leq (N(d, \epsilon, 4\epsilon_0) + 1) \cdot \#\{z \in Z : B(z_0, (b+1)\delta) \cap B(z, (b+1)\delta) \neq \emptyset\}.$$ 

The latter factor is bounded the cardinality of $Z \cap B(z_0, 2(b+1)\delta)$. With Lemma 3.2 the desired bound $\deg U_0 \leq D$ follows. □

4. A SIMPLICIAL DECOMPOSITION OF NEGATIVELY CURVED MANIFOLDS

The main goal of this section is to prove the following theorem.

Theorem 4.1 (see Theorem 1.4). Let $d > 1$ be an integer. There are constants $D, c > 0$ with the following properties. Every complete $d$-dimensional Riemannian manifold $M$ of normalized bounded negative curvature has a compact $d$-dimensional submanifold $M_+$ with boundary $\partial M_+$ such that the pair $(M_+, \partial M_+)$ is homotopy equivalent to a $(D, c \cdot \text{vol}M)$-simplicial pair $(\mathcal{R}, \mathcal{R}_0)$ and the closure of $M \setminus M_+$ consists of at most $c \cdot \text{vol}(M)$ many connected components, each of which is either homeomorphic to its boundary times $[0, \infty)$ or to a $D^{d-1}$-bundle over $S^1$.

A similar result, without the information on $\partial M^+$ is given in [19] §4.7.8. The preparation made in the previous section allows us to control the inclusion of the boundary $\partial M^+$ in $M^+$. An additional complication comes from the fact that we deal with general negatively curved manifolds whose universal cover is not necessarily homogeneous. This is the reason we will not work with the usual thick-thin decomposition but with a variant which gives ‘more weight’ to loops corresponding to elements with larger centralisers.
Remark 4.2. Note that the connected components of $\partial M_+$ correspond to the connected components of $R_0$. In particular, if $N \subset \partial M_+$ is a union of several connected components then $(M_+, N)$ is homotopy equivalent to a $(D, c \cdot \text{vol} M)$ simplicial pair $(R, R'_0)$.

4.1. The Margulis lemma. Fix $d > 1$. Let $X$ be a $d$-dimensional Hadamard space and $\Gamma$ a discrete group of isometries of $X$. For $\gamma \in \Gamma$ we denote by $d_\gamma$ the displacement function $x \mapsto d(x, \gamma \cdot x)$. For $\epsilon > 0$ and $x \in X$ we let

$$\Sigma_{x, \epsilon} = \{ \gamma \in \Gamma \setminus \{1\} : d_\gamma(x) < \epsilon \}, \quad \text{and } \Gamma_{x, \epsilon} = \langle \Sigma_{x, \epsilon} \rangle.$$ 

In negative curvature, the classical Margulis lemma has the following form:

**Theorem 4.3 (Margulis lemma [3, 10.3]).** Given $d > 1$ there are $\epsilon(d) > 0$ and $m = m(d) \in \mathbb{N}$ such that if $X$ is a $d$-dimensional Hadamard manifold with normalized bounded negative curvature then for every torsion-free discrete group $\Gamma$ of isometries of $X$, any $\epsilon \leq \epsilon(d)$ and every $x \in X$, the group $\Gamma_{x, \epsilon}$ is either

- trivial,
- a cyclic group generated by a loxodromic element, or
- consisting of parabolic elements sharing a common fixed point at $X(\infty)$ and admitting a normal nilpotent subgroup of index $\leq m$.

We shall refer to the second and third types as ‘loxodromic’ and ‘parabolic’ types respectively. We shall refer to the constant $\epsilon(d)$ as the Margulis constant.

4.2. Nilpotent subgroups of small elements. Let $X$ be a $d$-dimensional Hadamard space of normalized bounded negative curvature and let $\epsilon \leq \epsilon(d)$.

Let $\Gamma$ be a torsion free discrete group of isometries of $X$. For every $x \in X$ we denote by $N_x = N_{x, \epsilon}$ the (unique) maximal normal nilpotent subgroup of $\Gamma_x = \Gamma_{x, \epsilon}$. It satisfies

$$N_x = \prod \{ H : H \lhd \Gamma_x \text{ is normal nilpotent of index } \leq m \}.$$ 

The group $N_x$ is a nilpotent characteristic subgroup of $\Gamma_x$ of index $\leq m$ satisfying $N_{\gamma \cdot x} = \gamma N_x \gamma^{-1}$ for every $x \in X$ and $\gamma \in \Gamma$.

When $\Gamma_x$ is of loxodromic type we have that $N_x = \Gamma_x$. When $\Gamma_x$ is of parabolic type, it fixes a unique point $\zeta \in X(\infty)$ and preserves the horospheres around it. In that case, we set

$$\Gamma_\zeta := \langle \gamma \in \Gamma : \gamma \cdot \zeta = \zeta \rangle,$$ 

We shall call such $\zeta$ a $\Gamma$-parabolic point.

Suppose now that $M = \Gamma \setminus X$ has finite volume. Given a $\Gamma$-parabolic point $\zeta \in X(\infty)$ and fixing a horosphere $H$ centered at $\zeta$ we get that $\Gamma_\zeta$ acts cocompactly on $H$. It follows that $\Gamma_\zeta$ is finitely generated. In particular, we see that if $c(t)$ is a geodesic in $X$ with $c(\infty) = \zeta$ then for all $t$ sufficiently large we have $\Gamma_{c(t)} = \Gamma_\zeta$ and hence we may set $N_{c(t)} = N_\zeta$. Thus $N_\zeta$ is a
characteristic subgroup of $\Gamma_\zeta$ of index at most $m$. It follows in particular that $N_{\gamma}\zeta = \gamma N_\zeta\gamma^{-1}$ for every $\gamma \in \Gamma$ and $\zeta \in N(\infty) \Gamma$-parabolic. Moreover, since $N_\zeta$ acts cocompactly on $H$ and hence its cohomological dimension is $d-1$. Since the cohomological dimension of a torsion free nilpotent group coincides with the Hirsch rank \[6, \text{Theorem 7.10}\], and since the nilpotency degree is obviously bounded by the Hirsch rank, we obtain:

**Lemma 4.4.** Given a $\Gamma$-parabolic point $\zeta \in N(\infty)$, the nilpotency degree of $N_\zeta$ is at most $d-1$.

### 4.3. The thick-thin decomposition

We will require the following variant of the classical thick-thin decomposition due to Thurston.

**Theorem 4.5.** Fix $0 < \epsilon \leq \epsilon(d)$. Assume $M = \Gamma \setminus X$ is a complete $d$-manifold of normalized bounded negative curvature. Suppose further that $M$ has finite volume. Suppose that we are given an assignment of numbers $\gamma \mapsto \epsilon_\gamma$, $\gamma \in \Gamma \setminus \{1\}$, which is conjugation invariant and valued in the interval $[\epsilon, \epsilon(d)]$. Let $M_- := \cup_{\gamma \in \Gamma \setminus \{1\}} \{d_\gamma < \epsilon_\gamma\}$, $M_+ := X \setminus M_-$, and set $M_+ = \Gamma \setminus M_+$ and $M_- = \Gamma \setminus M_-$. Then $M_+$ is a compact manifold with boundary and the connected components of $M_-$ are of two types (corresponding to the type of $\Gamma_\epsilon$):

- A tube: a tubular neighborhood of a short closed geodesic, homeomorphic to a ball bundle over the circle and homotopy equivalent to a circle.
- A cusp: homeomorphic to a compact $(d-1)$-manifold times a half line.

The number of components of $M_+$ is at most $C \cdot \text{vol}(M)$ where $C = C(d)$ is a constant depending on $d$. If $d > 2$ then $M_+$ is connected.

The difference between our formulation and the standard one is that in the standard one $\epsilon = \epsilon(d)$ and hence all the $\epsilon_\gamma$ are equal to $\epsilon$. However, the proof of the statement is the same as the original one. For instance the result follows mutatis mutandis from \[3, \S 8\].

**Remark 4.6.** In practice, it is useful to apply Theorem 4.5 when the assignment $\gamma \mapsto \epsilon_\gamma$ is defined on a proper (conjugation invariant) subset $\Gamma' \subset \Gamma \setminus \{1\}$. This is legitimate as long as one is able to show that the resulting $M_+$ is contained in the $\epsilon$-thick part $M_{\geq \epsilon}$. Indeed, in that situation one can extend the assignment to $\Gamma \setminus \{1\}$ by declaring $\epsilon_\gamma = \epsilon$ for every $\gamma \notin \Gamma'$, without changing $M_+$.

**Remark 4.7.** For a generic assignment $\epsilon_\gamma$, $\#\{\gamma : d_\gamma(x) = \epsilon_\gamma\} \leq d$ for every $x \in X$, in which case $M_+$ is a manifold with corners.
4.4. **Proof of Theorem 4.1** We assume $M$ is a manifold as in Theorem 3.1, $X$ is its universal cover and $\Gamma$ its fundamental group, which acts on $X$ by deck transformations.

We aim to apply Theorem 3.1. To this end, we have to define constants $0 < \epsilon < \epsilon_0$, a conjugation invariant subset $\Gamma' \subset \Gamma$ and a conjugation invariant assignment of constants $\epsilon_\gamma \in [\epsilon, \epsilon_0]$ for $\gamma \in \Gamma'$. We will then set

$$\tilde{M}_- = \cup_{\gamma \in \Gamma'} \{ d_\gamma < \epsilon_\gamma \} \subset X$$

and let $M_-$ be its image in $X$ and $M_+ = M \setminus M_-$. We will have to show that $M_+ \subset M_{\geq \epsilon}$ and that for every point $x \notin M_-$ the group

$$\langle \gamma \in \Gamma' : d_\gamma(x) \leq 4\epsilon_\gamma \rangle$$

is commutative.

We let $\epsilon(d)$ and $m(d)$ be as in Theorem 4.3. We set

$$\epsilon_0 = \epsilon(d) \text{ and } \epsilon = \frac{\epsilon(d)}{4m(d)17^d}.$$  

Having fixed $\epsilon$, we use the shorthand notation $\Gamma_x = \Gamma_{x,\epsilon}$ and for a $\Gamma$-parabolic point $\zeta \in X(\infty)$ we let $\Gamma_\zeta$ and $N_\zeta \triangleleft \Gamma_\zeta$ be the corresponding subgroup defined in 4.4. We let $\Gamma'$ be the union of all nontrivial loxodromic elements with minimal displacement $\leq \epsilon(d)$ and all nontrivial parabolic elements which belong to some $N_\zeta$ with $\zeta \in X(\infty)$ $\Gamma$-parabolic.

Aiming at defining $\epsilon_\gamma$ we start by defining $i(\gamma)$ for each $\gamma \in \Gamma'$. For a loxodromic $\gamma$ we set $i(\gamma) = 0$. For a parabolic $\gamma$ we let $i(\gamma)$ be its centrality rank at infinity. Let us make this precise. Assume $\gamma \in \Gamma'$ is parabolic and let $\zeta$ be its fixed point in $X(\infty)$. Let $C^i(N_\zeta)$ be the upper central series of $N_\zeta$. That is $C^0(N_\zeta)$ is the center of $N_\zeta$ and $C^i(N_\zeta)/C^{i-1}(N_\zeta)$ is the center of $N_\zeta/C^{i-1}(N_\zeta)$. We let $i(\gamma)$ be the minimal index such that $\gamma \in C^i(N_\zeta)$.

Finally, for every $\gamma \in \Gamma'$ we define

$$\epsilon_\gamma = \frac{\epsilon(d)}{4 \cdot 17^{i(\gamma)}}.$$  

Note that $\Gamma'$ and the assignment $\Gamma' \ni \gamma \mapsto \epsilon_\gamma$ are invariant under conjugation in $\Gamma$. We set

$$\tilde{M}_- = \cup_{\gamma \in \Gamma'} \{ d_\gamma < \epsilon_\gamma \}, \ M_- = \Gamma \setminus \tilde{M}_- \text{ and } M_+ = M \setminus M_-.$$  

We now argue to show that $M_+ \subset M_{\geq \epsilon}$. Fix $x$ in the preimage of $M_+$, that is $x \notin M_-$, and $\gamma \in \Gamma \setminus \{1\}$. We need to show that $d_\gamma(x) \geq \epsilon$. If $\gamma$ is loxodromic this follows immediately from the fact that $x \notin \{ d_\gamma < \epsilon_\gamma \}$ since $\epsilon < \epsilon_\gamma$. Thus we may suppose by negation that $d_\gamma(x) < \epsilon$ where $\gamma$ is parabolic, fixing a point $\zeta$ at infinity. Since $\gamma \in \Gamma_x \leq \Gamma_\zeta$, for some $k \leq m(d)$, $\gamma^k \in N_\zeta$. It follows that

$$d_{\gamma^k}(x) \leq m(d)d_\gamma(x) < m(d)\epsilon = \frac{\epsilon(d)}{4 \cdot 17^d} \leq \frac{\epsilon(d)}{4 \cdot 17^{i(\gamma^k)}} = \epsilon_{\gamma^k}.$$  

Thus $x \in \{ d_{\gamma^k} < \epsilon_{\gamma^k} \} \subset \tilde{M}_-$ and we get the desired contradiction.
Next we fix \( x \notin \tilde{M}_- \) and argue to show that the group \( \langle \gamma \in \Gamma' : d_\gamma(x) \leq 4\epsilon_\gamma \rangle \) is commutative. To this end it is enough to show that every \( \alpha, \beta \) in the generating set \( \{ \gamma \in \Gamma' : d_\gamma(x) \leq 4\epsilon_\gamma \} \) commute. We fix \( \alpha, \beta \) in this set and assume by negation that \( [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1} \neq 1 \). Since \( 4\epsilon_\alpha, 4\epsilon_\beta \leq \epsilon(d) \) we have in particular that \( \alpha, \beta \in \Gamma_{x, \epsilon(d)} \). By assumption \( \alpha \) and \( \beta \) do not commute hence \( \Gamma_{x, \epsilon(d)} \) is not abelian. According to Theorem 4.3 we therefore have that \( \Gamma_{x, \epsilon(d)} \) is parabolic. Thus \( \alpha, \beta \in \Gamma_\zeta \) for some \( \zeta \in X(\infty) \). Recalling that \( N_\zeta = \Gamma' \cap \Gamma_\zeta \), we deduce furthermore that \( \alpha, \beta \in N_\zeta \). Since \( N_\zeta \) is nilpotent,

\[
i([\alpha, \beta]) \leq \min\{i(\alpha), i(\beta)\} - 1.
\]

Therefore, assuming that \( i(\alpha) \leq i(\beta) \), we get

\[
d_{[\alpha, \beta]}(x) \leq 2d_\alpha(x) + 2d_\beta(x) \leq 8(\epsilon_\alpha + \epsilon_\beta) \leq \frac{16\epsilon(d)}{4 \cdot 17^i(\alpha)} < \frac{\epsilon(d)}{4 \cdot 17^i([\alpha, \beta])} = \epsilon([\alpha, \beta]),
\]

contradicting the assumption that \( x \) is not contained in \( \tilde{M}_- \).

We therefore meet the conditions of Theorem 5.1. We let \( D \) and \( c \) be the constants given there which depend only on \( d \), \( \epsilon_0 \) and \( \epsilon \). In fact, \( D \) and \( c \) depend only on \( d \), as \( \epsilon \) and \( \epsilon_0 \) were defined by means of the Margulis constants \( \epsilon(d) \) and \( m(d) \). We conclude that \( (M_+, \partial M_+) \) is homotopy equivalent to a \( (D, c \cdot \text{vol } M_+) \) simplicial pair \( (\mathcal{R}, \mathcal{R}_0) \).

The statement regarding the components of the complement \( M_- = M \setminus M_+ \) is given in Theorem 4.5. Thus, we have completed the proof of Theorem 4.1. \( \square \)

5. Homology and homotopy in dimension \( \neq 3 \)

5.1. Torsion homology bounds from simplicial approximations. A useful tool for bounding the torsion homology of simplicial complexes is the following lemma attributed to Gabber (see e.g. [16, 35, 36]). A proof can be found in a paper by Soulé [36, Lemma 1].

**Lemma 5.1.** Let \( A \) and \( B \) be finitely generated free \( \mathbb{Z} \)-modules. Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_m \) be \( \mathbb{Z} \)-bases of \( A \) and \( B \), respectively. We endow \( B^\mathbb{C} = \mathbb{C} \otimes \mathbb{Z} B \) with the Hilbert space structure for which \( b_1, \ldots, b_m \) is a Hilbert basis. Let \( f : A \to B \) be a homomorphism. Let \( I \subset \{1, \ldots, n\} \) be a subset such that \( \{f(a_i) \mid i \in I\} \) is a basis of \( \text{im}(f) \). Then

\[
|\text{tors coker}(f)| \leq \prod_{i \in I} |f(a_i)|.
\]

From the previous lemma we deduce an estimate of the torsion in relative homology groups analogous to the absolute case as in (cf. e.g. [16, 35, 36]):

**Lemma 5.2.** For \( p, D \in \mathbb{N} \) there is a constant \( C(D, p) > 0 \) with the following property. Let \( Y \) be a space, and let \( K \subset Y \) be a (possibly empty) subspace so that \( (Y, K) \) is homotopy equivalent to a \( (D, V) \)-simplicial pair. Then

\[
\log(|\text{tors } H_p(Y, K; \mathbb{Z})|) \leq C(D, p) \cdot V.
\]
Proof. Note that a \((D, V)\)-simplicial complex has at most \(D^p \cdot V\) \(p\)-simplices. The relative simplicial chain complex has hence a \(\mathbb{Z}\)-basis of size at most \(D^p \cdot V\) (corresponding to simplices whose boundary is not completely in the subcomplex). The \(\mathbb{Z}\)-basis induces a Hilbert basis on the complexification. The norm of the differential of a \(p\)-simplex is at most \((p + 1)\). The statement follows now from the previous lemma. \(\square\)

5.2. Conclusion of proofs of Theorems 1.2 and 1.5

Proof of Theorem 1.2. For \(d = 2\) the statement follows from the Gauss–Bonnet theorem. Fix \(d \geq 4\). Let \(M\) be a complete \(d\)-dimensional manifold of finite volume with normalized bounded negative curvature. According to Theorem 4.1 there are constants \(D = D(d) > 0\) and \(c = c(d) > 0\) and a \(d\)-dimensional compact submanifold \(M_+ \subset M\) with boundary \(M_0 = \partial M_+\) such that \((M_+, M_0)\) is homotopy equivalent to a \((D, c \cdot \text{vol}(M))\)-simplicial pair. The closure \(M_-\) of the complement of \(M_+\) in \(M\) decomposes into compact components which are ball bundles over circles and non-compact components. Let \(M_\infty\) and \(M_\infty^\infty\) be the union of the compact and the non-compact components, respectively. We consider the following subspaces:

\[
\begin{align*}
M_- &= M_0^- \cup M_\infty = \overline{M \setminus M_+} \\
M_0 &= \partial M_+ \\
M_0^\infty &= M_- \cap M_0, \\
M_0^c &= M_\infty^\infty \cap M_0, \\
M_c &= M_+ \cup M_\infty^c.
\end{align*}
\]

Note that \((M_+, M_0^\infty)\) is also homotopy equivalent to a \((D, c \cdot \text{vol}(M))\)-simplicial pair (see Remark 4.2). The space \(M_c\) is also a \(d\)-dimensional manifold with (possibly empty) boundary. The inclusion \(M_0^\infty \subset M_\infty^\infty\) is a strong deformation retract which induces a strong deformation retract \(M_c \subset M\).

By Lemma 5.2 there is a constant \(C = C(d) > 0\) such that

\[
\log |\text{tors} \, H_p(M_+; \mathbb{Z})| \leq C \cdot \text{vol}(M)
\]

in all degrees \(p\).

We claim that the inclusion \(M_0^\infty \subset M_\infty^c\) is \((d - 2)\)-connected: \(M_\infty^c\) is a topological sum of ball bundles over \(S^1\), thus the inclusion fits into commutative diagram with horizontal fiber sequences:

\[
\begin{array}{ccc}
\bigcup S^{d-2} & \longrightarrow & M_0^\infty \\
\downarrow & & \downarrow \\
\bigcup D^{d-1} & \longrightarrow & M_\infty^c \\
\end{array}
\]

\[
\bigcup S^1
\]

and

\[
\bigcup S^1
\]

\[
\bigcup S^1
\]
The long exact sequence in homotopy groups and the 5-lemma imply the connectivity claim. Next we apply the homotopy excision theorem [15, Proposition 6.10.1 on p. 152] to the pushout square

\[
\begin{array}{ccc}
M_0^c & \longrightarrow & M_+ \\
\downarrow & & \downarrow \\
M_c & \longrightarrow & M_c
\end{array}
\]

The left vertical map is a cofibration since it is the inclusion of a union of boundary components into a manifold with boundary. The left vertical map is \((d-2)\)-connected. By homotopy excision the right vertical map is also \((d-2)\)-connected. Since the inclusion \(M_c \subset M\) is a deformation retract we obtain that the inclusion \(M_+ \subset M\) is \((d-2)\)-connected. Hence

\[
(18) \quad \log |\text{tors} H_p(M; \mathbb{Z})| = \log |\text{tors} H_p(M_+; \mathbb{Z})| \leq C \cdot \text{vol}(M)
\]

for \(p \in \{0, \ldots, d-3\}\). Next we assume that \(p \geq d-2 \geq 2\).

Since \(M_c^c\) is homotopically a 1-dimensional complex, the long exact homology sequence of \((M_c, M_c^c)\) implies that \(H_p(M_c; \mathbb{Z}) \rightarrow H_p(M_c, M_c^c; \mathbb{Z})\) is injective. By excision the latter module is isomorphic to \(H_p(M_c^+, M_0^c; \mathbb{Z})\) for which we have the torsion bound (17). This concludes the proof. \(\square\)

**Remark 5.3.** The proof above is easier if \(M\) is closed since we do not have to rely on the relative statement in Theorem 4.1. In this case we conclude (18) just from an analysis of the thick part. Then we appeal to Poincaré duality and the universal coefficient theorem to finish the proof of Theorem 1.2.

**Proof of Theorem 1.5.** We retain the setting of the previous proof. Let \(M\) be a \(d\)-dimensional complete Riemannian manifold of normalized bounded negative curvature and volume \(\leq v\).

We showed in the previous proof that \(M_+ \hookrightarrow M\) is a \(\pi_1\)-isomorphism. Since \(M_+\) is homotopy equivalent to a \((D, CV)\)-simplicial complex and the logarithm of the number of possible 2-skeleta of such simplicial complexes is bounded by \(\text{const} V \log V\) (cf. [11, Proposition on p. 1164]), the upper bound in the statement of Theorem 1.5 follows. The lower bound comes from considering hyperbolic manifolds alone and was already established in [4, 11, 24]. \(\square\)

### 6. Torsion in dimension 3

This section is devoted to the study of torsion in the first homology groups of 3-dimensional hyperbolic manifolds. §6.1 is devoted to the proof of Theorem 1.7. In the proceeding sections we give basic facts concerning the space of Invariant Random Subgroups of \(\text{PSL}_2(\mathbb{C})\) and the Benjamini–Schramm of hyperbolic 3-manifolds, which are necessary ingredients of the proofs of Theorem 6.3 and Theorem 1.8. In §6.5 we prove Theorem 6.3. In §6.6 we review a construction due to Brock and Dunfield [10] and prove Theorem 1.8. Lastly, in §6.7 we discuss analytic torsion.
6.1. Bounded volume but unbounded torsion – Proof of Theorem 1.7. In this subsection we prove Theorem 1.7. The manifolds $M_{p,q}$ are all obtained by different Dehn fillings of a knot complement of a fixed knot in $S^3$.

Let us first recall Dehn fillings. Let $M$ be a non-compact complete hyperbolic $3$-manifold. We assume for notational simplicity that $M$ has only one cusp. A typical example of such $M$ is a hyperbolic knot complement. Let $M^c$ be a compact core of $M$ obtained by chopping off horoball cusp neighborhoods. The boundary $\partial M^c$ is a torus and its intrinsic metric is flat. Let $f: S^1 \times S^1 \to \partial M^c$ be an isometry with respect to a suitably scaled flat metric on $S^1 \times S^1$. The manifold $M$ is homeomorphic to $M^c \cup_f (T^2 \times [0, \infty))$.

Let $\alpha$ be a closed simple geodesic in $\partial M^c$. Let $f_\alpha: S^1 \times S^1 \to \partial M^c$ be a diffeomorphism that maps $S^1 \times \{\ast\}$ to $\alpha$. Then $M_\alpha = M_c \cup_{f_\alpha} D^2 \times S^1$ is a closed manifold; we say $M_\alpha$ is obtained from $M$ by a Dehn filling along $\alpha$. By the $2\pi$-theorem of Gromov–Thurston, $M_\alpha$ admits a complete hyperbolic structure if the length of $\alpha$ is greater than $2\pi$ (see [3] for a detailed proof).

A similar discussion applies if $M$ has more than one cusp.

The next lemma describes the effect of Dehn fillings on the first homology.

**Lemma 6.1.** Let $M$ be a complete hyperbolic $3$-manifold of finite volume that has exactly one cusp. Let $(\mu, \lambda)$ be a basis of $H_1(\partial M^c; \mathbb{Z})$ such that $\lambda$ is in the kernel of $H_1(\partial M^c; \mathbb{Z}) \to H_1(M^c; \mathbb{Z})$. Let $\alpha_{p,q}$ be a simple closed geodesic representing $p\mu + q\lambda \in H_1(\partial M^c; \mathbb{Z})$. The manifold $M_{(p,q)}$ obtained from Dehn filling along $\alpha_{p,q}$ satisfies

$$|\text{tors } H_1(M_{(p,q)}; \mathbb{Z})| \geq p,$$

with equality in the case that $H_1(M, \mathbb{Z}) \cong \mathbb{Z}$ and it is generated by the image of $H_1(\partial M^c; \mathbb{Z})$. In particular, an equality holds in case $M$ is a knot complement.

**Proof.** The image $\mu' \in H_1(M^c; \mathbb{Z})$ of $\mu$ generates an infinite cyclic subgroup since the rank of the image of $H_1(\partial M^c; \mathbb{Z})$ in $H_1(M^c; \mathbb{Z})$ is 1 by Poincaré duality (“half lives half dies”, see [26] Lemma 3.5). In the case of a knot complement $\mu'$ is the generator of $H_1(M^c; \mathbb{Z})$ by Alexander duality. The Mayer–Vietoris sequence implies that $H_1(M_{\alpha_{p,q}}; \mathbb{Z})$ is the cokernel of the homomorphism

$$\text{incl}_c \oplus (f_{\alpha_{p,q}})_* : H_1(S^1 \times S^1) \to H_1(D^2 \times S^1) \oplus H_1(M^c).$$

This cokernel maps injectively into the cokernel of the map $\text{incl}_c \oplus (f_{\alpha_{p,q}})_*$ restricted to its image $H_1(D^2 \times S^1) \oplus \langle \mu' \rangle$. The latter map is represented by a matrix $A = \begin{pmatrix} 0 & p \\ 1 & * \end{pmatrix}$. The size of the cokernel of $A$ is $|\text{det}(A)| = p$. □

**Proof of Theorem 1.7.** If $M$ is a knot complement $S^3 - K$ then

$$H_1(M; \mathbb{Z}) \cong H^1(S^1) \cong \mathbb{Z}.$$
by Alexander duality. The compact core $M^c$ is homeomorphic to $S^3$ with a tubular neighborhood of $K$ removed. The first homology $H_1(\partial M^c; \mathbb{C})$ has a basis $(\mu, \lambda)$ where $\mu$, the meridian, represents the boundary of a disk in the solid torus around $K$, and $\lambda$ is the longitude being nullhomologous in $M$. Let us consider the specific case of the figure eight knot $K_8$. The knot complement $M_8 = S^3 - K_8$ is an arithmetic manifold, associated with an index 12 subgroup of $\text{PSL}_2(\mathbb{Z}[\omega])$, where $\omega$ is a primitive cube root of unity, and the volume of $M_8$ is twice the volume of a regular ideal simplex which is
\[
\text{vol}(M_8) = 6 \int_0^{\pi/3} -\log(2 \sin \theta) d\theta < 2.03.
\]
According to Thurston [37, Theorem 4.7] every closed manifold $M_{(p,q)}$ with coprime $p, q$ obtained from Dehn filling $M_8$ along $p\mu + q\lambda$ is hyperbolic except possibly for the values
\[
(\pm p, \pm q) \in \{(1,0), (0,1), (1,1), (2,1), (3,1), (4,1)\}.
\]
If so, the volume of $M_{(p,q)}$ is less than the volume of $M_8$ by Thurston’s hyperbolic Dehn filling theorem (see [9, Appendix B] for a detailed proof). By Lemma 6.1 applied to $M_8$, one gets that $H_1(M_{p,q}; \mathbb{Z}) = \mathbb{Z}/p$. Taking $p \in \mathbb{Z}$ arbitrary and $q$ coprime to $p$ with the exception of the finite list above we obtain the manifolds $M_{p,q}$ in the statement of Theorem 1.7.

6.2. Invariant Random Subgroups and the Benjamini–Schramm space. Let us recall the Benjamini–Schramm space $\text{BS}(\mathbb{H}^3)$ associated to $\mathbb{H}^3$. We refer the reader to [1,21] for details. A point in $\text{BS}(\mathbb{H}^3)$ is a random complete hyperbolic 3-manifold with a special point and a choice of frame at its tangent space. A finite volume complete hyperbolic 3-manifold $M$ corresponds to a point in $\text{BS}(\mathbb{H}^3)$ by normalizing the Riemannian volume of $M$ and picking a point in the frame bundle over $M$ at random. One may define the topology on $\text{BS}(\mathbb{H}^3)$ directly by defining an appropriate Gromov–Hausdorff topology on the space of framed manifolds and consider the space of probability measures on that. A quicker way however is to associate it with the space of invariant random subgroups of $\text{PSL}_2(\mathbb{C}) = \text{Isom}(\mathbb{H}^3)^\circ$.

Consider the space $\text{Sub}(\text{PSL}_2(\mathbb{C}))$ of closed subgroups of $\text{PSL}_2(\mathbb{C})$ equipped with the Chabauty topology. An IRS on $\text{PSL}_2(\mathbb{C})$ is a conjugation invariant Borel regular probability measure on $\text{Sub}(\text{PSL}_2(\mathbb{C}))$. An IRS is said to be discrete if it is supported almost surely on the set of discrete subgroups. We let $\text{IRS}_d(\text{PSL}_2(\mathbb{C}))$ denote the space of discrete IRS on $G$ equipped with the weak-* topology. Then $\text{IRS}_d(G)$ is compact (see [22] or [21, Sec. 3.2]). By fixing an origin and a tangent frame in $\mathbb{H}^3$, we obtain a map from the set of discrete subgroups $\text{Sub}_d(\text{PSL}_2(\mathbb{C}))$ to the space of framed hyperbolic 3-manifolds, $\Gamma \mapsto \Gamma \backslash \mathbb{H}^3$. Thus every IRS on $\text{PSL}_2(\mathbb{C})$ defines a point in $\text{BS}(\mathbb{H}^3)$ via pushing forward the measure. This map is one to one and its image can be characterized as the set of points in $\text{BS}(\mathbb{H}^3)$ which are invariant under the geodesic flow, denoted $\text{BS}(\mathbb{H}^3)^{\text{inv}}$. The inverse of this map is
defined via taking the deck-transformations associated to the fundamental group of a random manifold. These maps identify the compact topological space \( \text{IRS}_d(\text{PSL}_2(\mathbb{C})) \) with \( \text{BS}(\mathbb{H}^3)^{\text{inv}} \), and the weak-\( \ast \) topology on the first is the \textit{Benjamini–Schramm topology} (or short: BS-topology) on the latter.

6.3. Benjamini-Schramm convergent sequences of manifolds. The trivial IRS — the Dirac mass \( \delta_{(1)} \) at the identity of \( \text{PSL}_2(\mathbb{C}) \), corresponds to the random manifold which is almost surely \( \mathbb{H}^3 \). A sequence of complete finite volume hyperbolic manifolds \( M_n \) BS-converges to \( \mathbb{H}^3 \) if and only if the corresponding sequence of invariant random subgroups \( \mu_n \) converges to \( \delta_{(1)} \).

Recall the following characterization from [1]:

**Lemma 6.2.** A sequence \( M_n \) of complete hyperbolic 3-manifolds of finite volume BS-converges to \( \mathbb{H}^3 \) if and only if for every \( r > 0 \)

\[
\frac{\text{vol}\{x \in M_n \mid \text{InjRad}_{M_n}(x) > r\}}{\text{vol}(M_n)} \rightarrow 1.
\]

6.4. Metric on BS(\( \mathbb{H}^3 \)). It is well known that Sub(\( \text{PSL}_2(\mathbb{C}) \)) is metrizable. The following elegant way to define a metric was suggested by Ian Biringer [7]. Let \( B_r \) denote the \( r \) ball around the identity in \( \text{PSL}_2(\mathbb{C}) \) with respect to the matrix norm. We let \( \text{Hd} \) denote the Hausdorff distance between bounded sets in \( \text{PSL}_2(\mathbb{C}) \). The distance \( \rho \) on Sub(\( \text{PSL}_2(\mathbb{C}) \)) is defined as follow:

\[
\rho(H_1, H_2) := \int_{r>0} \text{Hd}(H_1 \cap B_r, H_2 \cap B_r)e^{-r}dr.
\]

As a consequence, also the space \( \text{BS}(\mathbb{H}^3)^{\text{inv}} \simeq \text{IRS}_d(\text{PSL}_2(\mathbb{C})) \) is metrizable. A concrete metric on it is given by the Kantorovich–Wasserstein metric (associated with the bounded metric \( \min\{\rho, 1\} \)), given by the formula

\[
\text{IRS}_d(\text{PSL}_2(\mathbb{C})) \ni \mu, \nu \mapsto \inf_{\eta \in J(\mu, \nu)} \int \min\{\rho(H_1, H_2), 1\}d\eta(H_1, H_2),
\]

where \( J(\mu, \nu) \) is the space of joinings of \( \mu \) and \( \nu \), i.e. the space of probability measures on Sub(\( \text{PSL}_2(\mathbb{C}) \))^2 with marginal measures \( \mu \) and \( \nu \).

6.5. A simple construction of a sequence with explosive torsion.

**Theorem 6.3.** There exists a sequence of closed hyperbolic 3-manifolds \( M_n \) that converges in the Benjamini–Schramm topology to \( \mathbb{H}^3 \) such that

\[
\lim_{n \to \infty} \frac{\log |\text{tors } H_1(M_n, \mathbb{Z})|}{\text{vol}(M_n)} = \infty.
\]

Furthermore, for any function \( f: (0, \infty) \to (0, \infty) \) there is such a sequence \( M_n \) with

\[
\log |\text{tors } H_1(M_n, \mathbb{Z})| > f(\text{vol}(M_n)).
\]

\footnote{Since the frame bundle over \( \mathbb{H}^3 \) is homogeneous, we may allow ourselves to omit the special point and the frame from the random manifold associated to \( \delta_{(1)} \) and simply denote it by \( \mathbb{H}^3 \).}
Denote by $\mathcal{F}$ the set consisting of complete finite volume hyperbolic 3-manifolds viewed as a subset of $BS(\mathbb{H}^3)^{inv}$. Denote by $\mathcal{K}$ the subset of $\mathcal{F}$ consisting of compact manifolds and set $\mathcal{U} = \mathcal{F} - \mathcal{K}$. Denote by $\mathcal{U}_1$ the subset of $\mathcal{U}$ consisting of manifolds with exactly one cusp and for $C > 0$ denote

$$\mathcal{K}(C) = \{M \in \mathcal{K} : \log |\text{tors } H_1(M; \mathbb{Z})| \geq C \text{ vol}(M)\}.$$ 

The proof of Theorem 6.3 will follow from the following proposition.

**Proposition 6.4.** Let $\text{clos}(\_)$ denote the closure of a set in the BS-topology. The BS-topology satisfies the following properties:

1. $\mathbb{H}^3 \in \text{clos}(\mathcal{U})$.
2. $\text{clos}(\mathcal{U}_1) = \text{clos}(\mathcal{U})$.
3. For every $C > 0$, $\mathcal{U} \subset \text{clos}(\mathcal{K}(C))$.

**Proof of Theorem 6.3 from Proposition 6.4.** Let $d$ a metric that induces the BS-topology on $BS(\mathbb{H}^3)^{inv}$. For every $n \in \mathbb{N}$, pick $M'_n \in \mathcal{U}$ with $d(M'_n, \mathbb{H}^3) < 1/n$ according to (1), and pick $M_n \in \mathcal{K}(f(n))$ with $d(M_n, M'_n) < 1/n$ according to (3). The sequence $(M_n)$ will have the desired properties. □

**Lemma 6.5.** Let $\Gamma_0$ be a lattice in $\text{PSL}_2(\mathbb{C})$ and let $f_n : \Gamma_0 \to \text{PSL}_2(\mathbb{C})$, $n \in \mathbb{N}$ be homomorphisms such that $\Gamma_n := f_n(\Gamma_0)$ are lattices, and $f_n$ converges to the inclusion in the topology of $\text{Hom}(\Gamma_0, \text{PSL}_2(\mathbb{C}))$. Let $M_n = \Gamma_n \backslash \mathbb{H}^3$ for $n \in \mathbb{N}$. Then $M_n$ converges to $M_0$ in the BS-topology.

**Proof.** This is a consequence of [23 Proposition 11.2]. Note that [23 Proposition 11.2] is formulated for uniform lattices, but the proof given in [23] applies with no changes to the non-uniform case. Indeed, the proof relies on the fact that $\text{vol}(M_n) \to \text{vol}(M_0)$ which is proved in [23] for uniform lattices obtained by arbitrary deformation, and is valid in our situation due to Thurston’s hyperbolic Dehn filling theorem. □

**Proof of Proposition 6.4.**

1. (1) Any residual tower of non-compact hyperbolic 3-manifolds of finite volume BS-converges to $\mathbb{H}^3$. For concreteness, we could take $M_n = \mathbb{H}^3/\Gamma_n$ where $\Gamma_n$ is the kernel of $\text{SL}_2(\mathbb{Z}[i]) \to \text{SL}_2(\mathbb{Z}/n[i])$. Obviously $M_n \to \mathbb{H}^3$ (see [1] Section 5) for concrete estimates on the rate of convergence of congruence towers.

2. (2) Suppose that $M = \Gamma \backslash \mathbb{H}^3$ is of finite volume and non-compact. The boundary of the compact core $M^c$ of $M$ consists of $n$ tori $T_1, \ldots, T_n$. Fix bases $(\lambda_i, \mu_i)$ of $H_1(T_i; \mathbb{Z})$. For $k > 1$ we perform Dehn fillings in the first $(n-1)$ cusps along curves representing $\lambda_i + k\mu_i$ to obtain a manifold $M_k$. For $k$ large enough, $M_k$ admits a complete hyperbolic structure of finite volume by Thurston’s hyperbolic Dehn filling theorem. Hence $M_k \in \mathcal{U}_1$. Further, $M_k$ tends to $M$ as $k \to \infty$ in the topology of the representation variety by the same theorem. By Lemma 6.5 $M_k$ BS-converges to $M$.

3. (3) By part (2) it is enough to show that a complete hyperbolic 3-manifold $M$ with one cusp is in the closure of $\mathcal{K}(C)$. By [20 Lemma 3.5] there is a basis $(\lambda, \mu)$ of $H_1(\partial M^c; \mathbb{Z})$ where $M^c$ is the compact core of $M$ such that
the hypothesis of Lemma 6.1 is satisfied. Performing Dehn fillings along \( \lambda + k\mu \) we obtain closed manifolds \( M_k \). By Thurston’s hyperbolic Dehn filling theorem \( M_k \) is hyperbolic for \( k \) large enough and \( \text{vol}(M_k) \leq \text{vol}(M) \). As in (2) we also conclude that \( M_k \to M \) in the BS-topology. By Lemma 6.1

\[
|\text{tors } H_1(M_k;\mathbb{Z})|/\text{vol}(M_k) \geq k/\text{vol}(M) \to \infty \text{ as } k \to \infty.
\]

\[\square\]

6.6. Asymptotic density of the normalized torsion. In this subsection we use a result of Brock and Dunfield [10] in order to modify the construction given in Subsection 6.5. The following theorem is extracted from [10].

**Theorem 6.6** ([10, §2]). There exists a sequence \( M_n \) consisting of finite volume hyperbolic 3-manifolds which BS-converges to \( \mathbb{H}^3 \) such that for each \( n \), \( M_n \) has one cusp and \( H_1(M_n,\mathbb{Z}) \cong \mathbb{Z} \). Furthermore the first homology of the cusp surjects on \( H_1(M_n,\mathbb{Z}) \).

Unfortunately, neither the theorem, nor the sequence \( M_n \) appear explicitly in [10]. We explain below how to modify the construction that does appear there, in order to prove Theorem 6.6. We follow closely the construction given in [10, §2.4]. We advise the reader to keep this paper close. Further details and justifications could be found there.

**Proof of Theorem 6.6.** We start by fixing once and for all a Heegaard splitting of \( S^3 \), \( S^3 = H^+ \cup S \cup H^- \), where \( H^\pm \) are open and \( S = \partial H^+ = \partial H^- \) is of genus 2. We will identify below the inclusion of \( S \) in a neighborhood of it in \( S^3 \) with the inclusion of \( S \times \{0\} \) inside \( S \times [0,6] \). We also fix a pants decomposition \( P \) of \( S \) so that the pared manifolds \( (H^\pm, P) \) are acylindrical and a separating essential simple closed curve \( \gamma \) on \( S \) so that the pared manifold

\[ U = ((S \times [0,2] - (\gamma \times \{1\}), P \times \{0,2\}) \]

is acylindrical.

The reminder of the construction will be dependent on a parameter \( R > 0 \) which we now fix. In our notation below we will stress the dependence on \( R \), which is implicit in [10] §2.4. We pick a pseudo-Anosov diffeomorphism \( f(R) : S \to S \) such that the corresponding mapping torus \( M_{f(R)} \) has injectivity radius larger than \( R + 1 \) and define a family of links \( L_n(R) \) which lie in \( S \times [0,6] \) by

\[ L_n(R) = P \times \{1\} \cup f(R)^n(P) \times \{2\} \cup f(R)^n(\gamma) \times \{3\} \cup f(R)^n(P) \times \{4\} \cup P \times \{5\}. \]

By [10, Lemma 2.6], for any given \( R \), for any large enough \( n \) (depending on \( R \)), the manifold \( S^3 - L_n(R) \) has a finite volume hyperbolic manifold structure. Assuming \( n \) is indeed large enough, we denote this hyperbolic manifold by \( N_n(R) \). By [10, Lemma 2.6] we also get that

\[
\lim_{n \to \infty} \frac{\text{vol}(N_n(R) \prec R)}{\text{vol}(N_n(R))} = 0.
\]

\[\text{For comparison sake: in the notation of [10] §2.4, we specialize here to the case } N_0 = S^3, g = 2 \text{ and } A = \{0\}.\]
We now perform a diagonalizing argument: for every \( m \in \mathbb{N} \) we fix \( n_m \in \mathbb{N} \)
so that
\[
\frac{\text{vol}(N_{n_m}(m))}{\text{vol}(N_{n_m}(m))} < \frac{1}{m}
\]
and set \( N'_m = N_{n_m}(m) \). We conclude that the sequence \( N'_m \) BS-converges to \( \mathbb{H}^3 \) as \( m \) tends to \( \infty \).

For fixed \( m \) and \( k \) define the manifold \( M_{m,k} \) obtained from \( S^3 \) by performing \( 1/k \) Dehn filling along the links in \( L_{n_m}(m) \) of height 1 and 2 and \( -1/k \) Dehn filling along the links in \( L_{n_m}(m) \) of height 4 and 5. If we further make a \( 1/k \) Dehn filling along the link at height 3 we would get a manifold (denoted \( N_{n_m,k} \) in [10, §2.4] for the implicit fixed parameter \( R = m \)), which is an integral homology sphere by [10, Lemma 2.5]. We do not perform this last Dehn filling! Thus \( H_1(M_{m,k}, \mathbb{Z}) \cong \mathbb{Z} \) by Alexander duality, and the homology of the cusp surjects on it.

Fixing \( m \), by Thurston’s Theorem we get that for \( k \) large enough \( M_{m,k} \) has the structure of a finite volume hyperbolic manifold. Thus, for every \( m \) and for a large enough \( k \) (depending on \( m \)), the manifold \( M_{m,k} \) is a finite volume hyperbolic 3-manifolds which has one cusp and \( H_1(M_{m,k}, \mathbb{Z}) \cong \mathbb{Z} \) is generated by this cusp. Thurston’s Theorem also tells us that for a fixed \( m \), when \( k \) tends to \( \infty \), \( M_{m,k} \) tends to \( N'_m \) in the representation variety topology. As explained in Lemma [6.5] by [23, Proposition 11.2] we get that \( M_{m,k} \) also BS-converges to \( N'_m \) as \( k \) tends to \( \infty \). By the fact that \( N'_m \) itself BS-converges to \( \mathbb{H}^3 \) as \( m \) tends to \( \infty \), using once more a diagonal argument, this time on \( m \) and \( k \), we obtain the required sequence of finite volume hyperbolic 3-manifolds which BS-converges to \( \mathbb{H}^3 \), each having one cusp which generates its first homology group.

**Proof of Theorem 1.8.** We fix a sequence \( M_n \) as given in Theorem 6.6. In particular \( v_n = \text{vol}(M_n) \) tends to infinity. We let \( p_n = [\alpha v_n] \) and for every \( q \in \mathbb{Z} \) we consider the manifold \( N_{n,q} = (M_n)_{(p_n,q)} \) constructed in Lemma 6.1 by performing a \((p_n,q)\) Dehn filling along the cusp. By Lemma 6.1 for every \( q \), \( \| \text{tors} H_1(N_{n,q}; \mathbb{Z}) \| = p_n \) and by Lemma [6.5] and Thurston’s theorem, \( N_{n,q} \) BS-converges to \( M_n \) as \( q \to \infty \). In particular,
\[
\lim_{q \to \infty} \frac{\log \| \text{tors} H_1(N_{n,q}; \mathbb{Z}) \|}{\text{vol}(N_{n,q})} = \frac{p_n}{v_n} = \frac{[\alpha v_n]}{v_n}.
\]
Since \( \lim_{\alpha \to \infty} \frac{[\alpha v_n]}{v_n} = \alpha \), using a diagonalizing argument we may pick a sequence \( M_n^{\alpha} = N_{n,q_n}^{\alpha} \) with the required properties.

**6.7. On analytic torsion.** The Ray-Singer torsion \( \tau(M) \) of a Riemannian manifold is defined as
\[
\tau(M) = \frac{1}{2} \sum_{k=0}^{\dim M} (-1)^k k \log \det \left( \Delta_k \right)
\]
where \( \det' \) is the zeta-regularized product of eigenvalues of the Laplacian \( \Delta_k \) on smooth \( k \)-forms. The \( p \)-th regulator \( R_p(M) \) is the covolume of the free part of \( H^p(M; \mathbb{Z}) \) as a lattice in \( H^p(M; \mathbb{R}) \) with respect to the harmonic metric, i.e. the metric induced from the usual scalar product of harmonic \( p \)-forms. By the Ray-Singer conjecture, proved by Cheeger and Müller, the analytic torsion coincides with the Reidemeister torsion [14, 32]. For a 3-dimensional manifold \( M \) this implies the relation

\[
\tau(M) = -\log |\text{tors} H_1(M, \mathbb{Z})| + \log(\text{vol}(M)) + 2\log(R_1(M)).
\]

In particular, if \( M \) is a rational homology sphere, \( R_1(M) = 0 \) and we get

\[
\tau(M) = -\log |\text{tors} H_1(M, \mathbb{Z})| + \log(\text{vol}(M)).
\]

The following corollary is thus an immediate application of Theorem 1.8.

**Corollary 6.7.** For every \( \alpha \in [0, \infty] \) there exists a sequence of closed hyperbolic 3-manifolds \( M_n^\alpha \) which are all rational homology spheres, such that the sequence \( M_n^\alpha \) converges in the Benjamini–Schramm topology to \( \mathbb{H}^3 \) and for the Ray-Singer torsion we have

\[
\frac{-\tau(M_n^\alpha)}{\text{vol}(M_n^\alpha)} \rightarrow \alpha.
\]

Note that in contrast to the above corollary, if \( (M_n) \) is a residual tower of coverings of a fixed hyperbolic 3-manifold \( M \) then the proof of [5, Theorem 4.5] implies that

\[
\liminf_{i \to \infty} \frac{-\tau(M_n)}{\text{vol}(M_n)} \leq \frac{1}{6\pi}.
\]

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