On the 1-density of Unit Ball Covering

Binhai Zhu *

Abstract

Motivated by modern applications like image processing and wireless sensor networks, we consider a variation of the Kepler Conjecture. Given any infinite set of unit balls covering the whole space, we want to know the optimal (lim sup) density of the volume which is covered by exactly one ball (i.e., the maximum such density over all unit ball covers, called the optimal 1-density and denoted as $\delta_d$, where $d$ is the dimension of the Euclidean space). We prove that in 2D the optimal 1-density $\delta_2 = (3\sqrt{3} - \pi)/\pi \approx 0.6539$, which is achieved through a regular hexagonal covering. In 3D, the problem is widely open and we present a Dodecahedral Cover Conjecture which states that the optimal 1-density in 3D, $\delta_3$, is bounded from above by the 1-density of a unit ball whose Voronoi polyhedron is a regular dodecahedron of circum-radius one (determined by twelve extra unit balls). We show numerically that this 1-density $\delta_3(dc) \approx 0.315$.

Keywords: Unit ball covering, 1-density, unit ball packing, the Kepler conjecture, wireless sensor networks, digital halftoning

*Department of Computer Science, Montana State University, Bozeman, MT 59717-3880, USA. Email: bhz@cs.montana.edu.

December 20, 2007
1 Introduction

The Kepler conjecture asserts that the density of a packing of unit balls in three dimensions (3D) is no more than $\frac{\pi}{\sqrt{18}} \approx 0.74048$. This is known as the oldest (unsolved?) problem in discrete geometry and is listed as part of the Hilbert’s 18th problem. About ten years ago, Hales published a series of papers concerning the proof of the Kepler conjecture, using a vast amount of computer verification [7]. While many believe that the conjecture has been proven, some others believe it has not (as the referees for *Annals of Mathematics* claimed that “The news from the referees is bad... They have not been able to certify the correctness of the proof, and will not be able to certify it in the future” [12]). In fact, some of Hales’ computer programs (total 3GB!) need to re-checked, which will take another 20 years [http://en.wikipedia.org/wiki/Kepler_conjecture](http://en.wikipedia.org/wiki/Kepler_conjecture).

In this paper, we make no attempt in re-proving the Kepler conjecture. In fact, this research is initiated by looking at the Kepler conjecture from another perspective: What is the origin of the Kepler conjecture? Application! It is known that T. Hariot first investigated the Kepler conjecture in 1591 while working for Sir Walter Raleigh, who asked him to determine formulas for the number of cannonballs regularly packed. Later on Hariot communicated the problem to Kepler. This problem on unit ball packing is known as the Kepler conjecture since [7].

Looking at the story, it is not difficult for one to realize that applications do drive theoretical research. In the cannonball packing problem, obviously cannonballs cannot overlap. But in some modern applications, the situation could be different. Example 1: in wireless sensor networks, many homogeneous sensors, whose sensing regions are modeled as unit disks in 2D, are placed in a region $R$ (we can assume that the diameter and width of $R$ are much bigger than one). As sensors have limited battery time, to extend network coverage lifetime (which is the time that the region is completely covered) one needs to schedule the sensors, i.e., turn some on and some off. The requirement is that all the time when the network is alive one must make sure that every point in $R$ is covered by some sensor. The abstract version of this problem has been shown to be NP-complete [4], but the complexity of the geometric version of the problem is still open (though it seems possible to extend the technique in [3] to obtain a PTAS). In any case, in this wireless sensor network problem clearly the sensing regions can overlap.

Another related example is regarding the conversion of continuous-tone images into binary images for printers, formally known as the *digital-halftoning* technique. Given a continuous-tone image, one can compute the spatial frequency distribution by applying the Laplacian or Sobel differential operators. Then we have a grid of points of possibly slightly different radii (or disks centered at grid points), and the problem is to compute a subset of disks which maximize the area
of the regions covered by exactly one disk. The complexity of the problem is still open, a factor-5.83 approximation was given in [1]. When the ratio of the radius of the largest disk over the radius of the smallest disk is a constant, then the problem admits a PTAS [3].

![Figure 1. A regular hexagonal covering.](image)

We feel that the reason why we still could not completely solve these kinds of geometric problems is that some fundamental question regarding disk covering, very surprisingly, is still untouched. We cover one such problem in this paper; namely, the 1-density of any unit disk cover which covers all the points in the plane (2D) [1]. We prove a theorem which is corresponding to Thue’s theorem on the density of unit disk packing; loosely speaking, the regular hexagonal covering achieves the optimal 1-density at 0.6539 (Figure 1). (A regular hexagonal covering is the configuration where the Voronoi polygon of any center of the unit disks is a congruent regular hexagon. Readers are referred to [11] for concepts and properties regarding Voronoi diagrams.)

We then consider the problem in 3D space, i.e., given any unit ball cover covering all the points in space, decide the corresponding optimal 1-density. To some extent, this is in fact similar to the Kepler conjecture, which is on unit ball packing. But the status of this 3D problem is widely open. Our numerical calculation with computer programs indicates that the 1-density of a unit ball whose Voronoi polyhedron is a regular dodecahedron with circum-radius one (determined by twelve extra unit balls) is around 0.315. We conjecture that this is the upper bound of the optimal 1-density $\delta_3$, which is also in parallel with the Dodecahedral Conjecture by L. Fejes Tóth [13, 8] regarding the Kepler conjecture. We will call this the Dodecahedral Cover Conjecture henceforth.

This paper is organized as follows. In Section 2, we present necessary definitions and a proof

---

1. Note that the density for finite unit disk (ball) covering has long been studied [2]. But this traditional density is not related to the 1-density we will discuss in this paper. The traditional density for unit ball covering refers to the maximum ratio of the volume of some convex region covered by the balls over the volume of the whole region covered by the balls.
for the 2D case. In Section 3, we show how the 1-density in the Dodecahedral Cover Conjecture can be computed (estimated) empirically. In Section 4, we discuss our conjecture and leave some other open problems.

2 Result in 2D

We first make some definitions for any $d$-dimension.

**Definition 2.1** A unit ball cover in $d$-dimensional space is an infinite set of unit balls which cover every point in the $d$-space.

**Definition 2.2** Given any unit ball cover $C$, let the subset of the balls in $C$ which covers a larger ball $B$ with a fixed center and a radius $r > 1$ be $C(B, r)$. Let $A_1(C(B, r))$ be the volume of the parts in $B$ which is covered by exactly one ball in $C(B, r)$. Let $A(B)$ be the volume of $B$. Then, the 1-density of $C$, $\delta_d(C)$, is defined as

$$\delta_d(C) = \lim_{r \to +\infty} \sup_{B, r} \frac{A_1(C(B, r))}{A(B)}.$$

Among all possible unit ball covers, the one achieving the maximum 1-density is denoted as $C^*$ and the corresponding optimal 1-density is denoted as $\delta_d$.

For the 2D case, a ‘ball’ is usually called a disk and ‘space’ is typically called a plane. The main theorem for the 2D case is based on the following lemmas.

**Lemma 2.1** Let $P = \{p_1, p_2, p_3, \ldots\}$ be an infinite set of points in the 2D plane such that $P$ lie on an infinite (or unbounded) number of lines. Let $VD(P)$ be the Voronoi diagram of $P$ such that every point $q$ in the plane is covered by some unit disk centered at $p_j$. Then the Voronoi polygon of $p_i$, $VP(p_i)$, is bounded for all $i$.

**Proof.** If $VP(p_i)$ is unbounded, then $p_i$ is on the convex hull of $P = \{p_1, p_2, p_3, \ldots\}$. Then some points in $VP(p_i)$ is not covered by any unit disk centered at $p_j$ for any $j$. A contradiction. \qed

Let $D = \{D_1, D_2, D_3, \ldots\}$ be an infinite number of disks of a uniform radius, with each disk $D_i$ centered at $o_i$. Let $VD(D)$ be the Voronoi diagram of the center $o_i$’s for the disks in $D$. For any $o_i$, let $v_i$ be the Voronoi vertex in the Voronoi polygon $VP(o_i)$ which is the farthest from $o_i$. Define $r_{\max} = \sup_k d(o_k, v_k)$. We have the following lemma.

**Lemma 2.2** $D = \{D_i | i = 1, 2, 3, \ldots\}$ is a unit disk cover if and only if $v_i$ is covered by $D_i$, for all $i$. 

Figure 2. Illustration for the proof of Lemma 2.2.

**Proof.** We first show the “if” part. Without loss of generality, let $r_{\text{max}} = 1 \geq r_i = d(o_i, v_i)$, for any $i$. By the definition of $v_i$, a disk $D'_i$ centered at $o_i$ with radius $r_i$ covers $VP(o_i)$. As $D_i$ is centered at $o_i$ and with a radius $r_{\text{max}} = 1 \geq r_i$, $D_i$ also covers $VP(o_i)$ (Figure 2). So every point in the plane is covered by some unit disk in $D$. $D$ is hence a unit disk cover.

We now show the “only if” part. If $D$ is a unit disk cover, then by definition, every point in the plane is covered by some disk $D_k$. Following Lemma 2.1, all the Voronoi polygons $VP(o_i)$ in $VD(D)$ are bounded, for all $i$. Without loss of generality, let $r_{\text{max}} = 1 = d(o_i, v_i)$, for some $i$. By definition, $v_i$ is the farthest Voronoi vertex in $VP(o_i)$ from $o_i$. Therefore the disk $D_i$ covers $v_i$ (and $VP(o_i)$).

**Theorem 2.1** $\delta_2 = (3\sqrt{3} - \pi)/\pi \approx 0.6539$.

**Proof.** Let $C = \{C_1, C_2, C_3, \ldots\}$ be any unit disk cover, with each disk $C_i$ centered at $o_i$. Without loss of generality, let $r_{\text{max}} = 1 = d(o_i, v_i)$, where $v_i$ is the farther Voronoi vertex on $VP(o_i)$ from $o_i$. By the definition of a Voronoi vertex, $v_i$ is shared by at least three Voronoi polygons. Assume that one of the other Voronoi polygons containing $v_i$ as a vertex is $VP(o_j)$, we have $d(o_i, v_i) = d(o_j, v_i) = 1$ (notice that $v_i$ is the farthest Voronoi vertex on $VP(o_j)$ from $o_j$, or we can imagine that $v_j = v_i$). Following Lemma 2.2 to cover the whole plane we must cover $v_i$ using a unit disk centered at $o_i$. Let $d(o_i, o_j) = 2x$. Denote the sector in $C_i$ bounded by rays $\overrightarrow{o_i v_i}$ and $\overrightarrow{o_i o_j}$ as $\langle v_i o_i o_j \rangle$ (Figure 3). Obviously, to maximize the 1-density of sector $\langle v_i o_i o_j \rangle$, it is reasonable to assume that no disk $C_k$ intersects the region in sector $\langle v_i o_i o_j \rangle$ which is out of the disk $C_j$. This part of the area in $\langle v_i o_i o_j \rangle$ which is only covered by $C_i$, is

$$x\sqrt{1-x^2} - \frac{\arccos x}{2}.$$  

By a simple calculation (taking the derivative over $x$ then solving a simple equation), this function is maximized at $\frac{3\sqrt{3} - \pi}{12}$ when we set $x = \frac{\sqrt{3}}{2}$, which implies that the angle between $\overrightarrow{o_i v_i}$ and $\overrightarrow{o_i o_j}$ is

---

2We do not assume that no three centers are collinear, as long as all the centers are on an infinite number of lines.
π/6 (Figure 3). Correspondingly, the optimal 1-density in the sector \( \langle v_i o_i o_j \rangle \) is

\[
\frac{3\sqrt{3} - \pi}{12}/\frac{\pi}{12} = \frac{3\sqrt{3} - \pi}{\pi},
\]

which is roughly 0.6539. By symmetry, the 1-density for \( C_i \) is optimal when \( C_j \) and the five other unit disks intersect \( C_i \) in a regular hexagonal setting (e.g., as in Figure 1). Clearly this configuration can be arranged at all unit disks.

![Figure 3. Illustration for the proof of Theorem 2.1.](image)

Therefore,

\[
\delta_2 = \frac{3\sqrt{3} - \pi}{\pi} \approx 0.6539.
\]

We comment that the above optimal regular hexagonal covering is similar to Thue’s theorem [14, 15], which states that the densest density of unit disk packing is achieved with a regular hexagonal packing, which admits a density at \( \frac{\pi}{\sqrt{12}} \approx 0.9068 \).

3 Result in 3D

For the 3D problem, the most relevant application seems to be placing sensors (or sonars) in the ocean to monitor the temperature change or the appearance of submarines, etc. It turns out that for the 3D problem, it might be difficult to apply the same technique as in Theorem 2.1 (as the 3D counterpart of the regular hexagons, i.e., regular dodecahedra, cannot even tile the whole space). Another reason is the difficulty in computing the volume of a non-convex body (i.e., it might not even be a polyhedron) analytically. This is very similar to the Kepler conjecture. However, different from the Kepler conjecture, the status on computing/estimating the optimal 1-density \( \delta_3 \) is widely open.

We conjecture that \( \delta_3 \) is bounded from above by the 1-density of a unit ball whose Voronoi polyhedron is a regular dodecahedron with circum-radius one (determined by twelve extra unit balls).
We denote this 1-density as $\delta_3(dc)$. As we do not know how to calculate $\delta_3(dc)$ analytically (which is related to compute/estimate the volume of a non-convex body), we first use a random sampling technique to estimate $\delta_3(dc)$. Incidentally, while there have been a lot of theoretical research on estimating the volume of a convex body, estimating/approximating the volume of a non-convex body has only been practically investigated recently by the author of this paper and his colleagues (though without a theoretical guarantee) [10]. (The problem of estimating/approximating the volume of a convex body, with some theoretical guarantee, has been well researched [9]. However, for non-convex geometric body, the problem is still open.) We will sketch some details of the method (called Generate-and-Probe) below and apply it to obtain an approximate $\delta_3(dc)$. We comment that Generate-and-Probe is a very general method, empirically it works even when no description of the body is given — as long as there is an oracle which tells whether a point is inside the body or not.

Following Figure 4, the unit ball $B_i$ centered at $o_i$ is a circumscribing ball for its Voronoi polyhedron $VP(o_i)$ — a regular dodecahedron. Computed by a computer program, the edge length $a$ of $VP(o_i)$ is $a=0.763934$. The distance from $o_i$ to one of the faces of $VP(o_i)$ is $H=0.760071$. Define $R$ as the radius of the circumscribing circle of a face of $VP(o_i)$, which is a pentagon. Then $R=0.649841$. So the ratio $\alpha_1$ of the volume of $VP(o_i)$ to the volume of $B_i$ is

$$\alpha_1 = \frac{12 \times (0.5 \times a \times R \times \sin(3.14159 \times (54.0/180.0)) \times 5) \times H/3}{4 \times 3.14159/3} = 0.728762.$$ 

To compute the 1-density of $\delta_3(dc)$, all we need to do now is to compute the ratio $\alpha_2$ of the volume inside a dodecahedron which is covered by only one ball to the volume of the dodecahedron. Then $\delta_3(dc) = \alpha_1 \alpha_2$.

By symmetry, we consider two unit balls centered at $o=(0,0,0)$ and $p_0=(0,0,2H)=(0,0,1.52014)$ respectively. The intersection of the two balls, at height $z = H = 0.760071$, contains a regular pentagon (face) of the regular dodecahedron corresponding to the unit ball centered at $p_0$. Let $p_2=(0,0,0.760071)$ be the center of this pentagon. Again, by symmetry, we only need to consider
a tetrahedron formed by \(p_0, p_2\) and two other neighboring vertices of the pentagon at height \(z = H\). We choose them as \(p_1=(0,0.649841,0.760071)\), \(p_3=(0.618035,0.200812,0.760071)\). Then the volume of the tetrahedron \(T = \diamond(p_0p_1p_2p_3)\) formed by \(p_0, p_1, p_2, p_3\), \(\text{vol}(T)\), is analytically computed as

\[
\text{vol}(T) = 0.050877.
\]

The part of \(T\), which is out of the ball centered at \(o=(0,0,0)\), or \(x^2 + y^2 + z^2 = 1\), is what we need to compute. We denote this part as \(S\) and its volume as \(\text{vol}(S)\). In Figure 4, we show the bottom-up view of the regular dodecahedron centered at \(p_0\), in (I); the side view along the y-axis in (II); and the 3D view in (III).

The volume of \(S\), which is not even a polyhedron, is hard to compute. We first use the Generate-and-Probe method of Liu, Zhang and Zhu [10]. The method starts by generating \(n\) random points in \(S\), and then computing a small probing ball (the radius of the probing ball is selected so that the probability that the ball contains at least some points in \(S\) goes to 1 when \(n \to +\infty\)). Then we use the probing ball to probe as many parts of \(S\) as possible and count the average ratio \(R\) of the number of random points enclosed by the probing balls over the volume of all the probing balls (of course, over a decent number of tries). Eventually, the volume of \(S\) is simply \(n/R\). We summarize our empirical results over three cases with \(n=80,000\), \(n=140,000\) and \(n=200,000\) respectively. The average volume of \(\text{vol}(S)\) is obtained over 100 tries of the Generate-and-Probe method and the standard deviation is also given.

| \(n\)     | 80,000  | 140,000 | 200,000 |
|-----------|---------|---------|---------|
| \(\text{avg}(\text{vol}(S))\) | 0.0219066 | 0.0219515 | 0.0221303 |
| \(\sigma(\text{vol}(S))\) | 1.66214e-07 | 6.56446e-08 | 6.14723e-08 |

Table 1. The average of \(\text{vol}(S)\) and its standard deviation \(\sigma(\text{vol}(S))\), obtained by Generate-and-Probe, over 100 tries.

It remains to verify that the above result by Generate-and-Probe is accurate, as there is no theoretical guarantee for Generate-and-Probe. We try to verify this, again empirically. Luckily, the boundary of \(S\) can be specified explicitly. Let the tetrahedron which is obtained by cutting \(T\) with a halfspace \(z \geq 1\) be \(T'\) (the plane \(z = 1\) intersects the segment \(op_0\) at \(p_4=(0,0,1)\)). Then analytically, we can compute the volume of \(T'\), \(\text{vol}(T')\). It turns out that

\[
\text{vol}(T') = 0.0163051 < \text{vol}(S).
\]

The volume of the tetrahedron \(\diamond(p_0p_1p_3p_4)\) is \(0.0348169\), which is an upper bound of \(\text{vol}(S)\). To obtain a closer upper bound for \(\text{vol}(S)\), on the surface of \(x^2 + y^2 + z^2 = 1\) inside the tetrahedron
\(\diamond(p_0p_1p_3p_4)\) (note that \(p_1, p_3, p_4\) are the three corners of this spherical patch), we generate a set \(M\) of \(m\) points. We then compute the 3D convex hull of \(M \cup \{p_1, p_3, p_4\}\). We use the upper hull of \(CH(M \cup \{p_1, p_3, p_4\})\) to approximate the patch. The approximate volume of \(\text{vol}(S)\), which we still denote as \(\text{vol}(S)\), is computed by summarizing the volume of all the tetrahedra \(\diamond(p_0uvw)\) where \(\triangle(uvw)\) is a face on the upper hull of \(CH(M \cup \{p_1, p_3, p_4\})\). The results are summarized in Table 2.

| \(m\) | \(\text{avg}(\text{vol}(S))\) | \(\sigma(\text{vol}(S))\) | \(\max(\text{vol}(S))\) |
|------|------------------|-----------------|------------------|
| 2,000 | 0.0219467 | 6.52514e-11 | 0.0219879 |
| 4,000 | 0.0219645 | 1.91297e-11 | 0.0219956 |
| 6,000 | 0.0219810 | 8.92527e-12 | 0.0219994 |
| 8,000 | 0.0219843 | 5.62743e-12 | 0.0220002 |
| 10,000 | 0.0219873 | 4.46376e-12 | 0.0220016 |
| 12,000 | 0.0219914 | 3.10292e-12 | 0.0220046 |
| 14,000 | 0.0219931 | 2.59317e-12 | 0.0220046 |
| 16,000 | 0.0219947 | 2.63297e-12 | 0.0220096 |
| 18,000 | 0.0219966 | 1.48441e-12 | 0.0220060 |
| 20,000 | 0.0219970 | 2.07155e-12 | 0.0220060 |

Table 2. The average of \(\text{vol}(S)\), its standard deviation \(\sigma(\text{vol}(S))\) and its maximum \(\max(\text{vol}(S))\), obtained by the 3D Convex Hull method, over 100 tries.

From Table 2, it can also be seen that the maximum of \(\text{vol}(S)\), over all of our 1000 tries, has never been greater than 0.0220096. Therefore, with these rigorous computation and verification, it is safe to say that \(\text{vol}(S) \approx 0.022\) hence \(\alpha_2 \approx 0.432414\) and \(\delta_3/dc = \alpha_1\alpha_2 \approx 0.315\). The Convex Hull method has much smaller deviations, and is probably more accurate in this case, which is understandable. As we mentioned earlier, the Generate-and-Probe is a more general (interior-based) approach. Even when no exact description of the body is given, it still works (at least empirically). We hence have the following claim.

**Claim.** \(\delta_3/dc \approx 0.315\).

## 4 Closing Remarks

The Kepler conjecture on the 3D unit ball packing has been bothering mathematicians for about 400 years and a long story has been developed on it. In this paper, based on modern applications in image processing and wireless sensor networks, we consider the problem of computing the optimal 1-density of a unit ball cover which is a variation of the Kepler conjecture. We prove a theorem for the
2D case that this optimal 1-density in 2D is \( \delta_2 = \frac{3\sqrt{3} - \pi}{\pi} \approx 0.6539 \). The 3D problem is widely open, though we think that the optimal 1-density is bounded from above by the 1-density of a unit ball whose Voronoi polyhedron is a regular dodecahedral of circum-radius one (i.e., this Voronoi polyhedron is determined by twelve extra unit balls). We denote this 1-density as \( \delta_3(dc) \) and we show numerically that \( \delta_3(dc) \approx 0.315 \). We conclude this paper with the following formal conjecture.

**The Dodecahedral Cover Conjecture** (for unit ball covering): The optimal 1-density in 3D, \( \delta_3 \), is bounded from above by \( \delta_3(dc) \approx 0.315 \).

It is also natural to say that when the dimension \( d \) is large, the optimal 1-density in \( d \)-dimension goes to zero, i.e., \( \lim_{d \to +\infty} \delta_d \to 0 \).

**Acknowledgments**

I thank Sheng Liu for helping running the Generate-and-Probe program to obtain some of the empirical results. Thanks also to Bin Fu and Marshall Bern for discussions regarding the Kepler conjecture and the Dodecahedral Conjecture. Special thanks to Joe O’Rourke for his 3D Convex Hull code in C (on which my Convex Hull method program is based).

**References**

[1] T. Asano, P. Brass, and S. Sasahara. Disc covering problem with application to digital halftoning. *Proc. International Conference on Computational Science and Applications (ICCSA’04)*, LNCS 3045, Assisi, Italy, May 14-17, 2004.

[2] K. Böröczky Jr., *Finite Ball Packing and Covering*. Cambridge University Press, 2004.

[3] Z. Chen, B. Fu, Y. Tang and B. Zhu. A PTAS for a disc covering problem using width-bounded separators. *J. Combinatorial Optimization*, 11:203-217, 2006.

[4] M. Cardei, M. Thai, Y. Li and W. Wu. Energy-efficient target coverage in wireless sensor networks. *Proc. IEEE INFOCOM’05*, IEEE Press, pp. 1976-1984, 2005.

[5] S. Funke, A. Kesselman, F. Kuhn, Z. Lotker and M. Segal. Improved approximation algorithms for connected sensor cover. *Wireless Networks*, 13:153-164, 2007.

[6] H. Gupta, S. Das and Q. Gu. Connected sensor cover: self-organization of sensor networks for efficient query execution. *Proc. ACM MobiHoc’03*, ACM, pp. 189-200, 2003.

[7] T. Hales, [http://www.math.lsa.umich.edu/~hales/kepler98](http://www.math.lsa.umich.edu/~hales/kepler98) 1998.
[8] T. Hales and S. McLaughlin. A proof of the Dodecahedral Conjecture. arXiv:math/9811079, 1998.

[9] L. Lovász and S. Vempala. Simulated annealing in convex bodies and an $O^*(n^4)$ volume algorithm. Proc. 44th IEEE Annual Symp. on Found. of Comp. Sci (FOCS), pages 650–659, 2003.

[10] S. Liu, J. Zhang and B. Zhu. Volume computation using a direct Monte Carlo method. Proc. 13th Intl. Conf. on Computing and Combinatorics (COCOON’07), LNCS 4598, pp. 198-209, 2007.

[11] F.P. Preparata and M.I. Shamos. Computational Geometry: An Introduction. Springer-Verlag, 1985.

[12] G. Szpiro, Kepler’s Conjecture. John Wiley & Sons, Inc., 2003.

[13] F. Tóth. Über die dichteste Kugellagerung. Math Z., 48:676-684, 1943.

[14] A. Thue. Om nogle geometrisk taltheoretiske Theoremer. Forandlingerneved de Skandinaviske Naturforskeres, 14:352-353, 1892.

[15] A. Thue. Über die dichteste Zusammenstellung von kongruenten Kreisen in der Ebene. Christina Vid. Selsk. Skr., 1:1-9, 1910.

[16] H. Zhang and J.C. Hou. Maintaining sensing coverage and connectivity in large sensor networks. Technical Report UIUCDCS-R-2003-2351, 2003.