NEW BLOW-UP RESULT FOR THE WEAKLY COUPLED WAVE EQUATIONS WITH A SCALE-INvariant DAMPING AND TIME DERIVATIVE NONLINEARITY

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Abstract. We consider in this article the weakly coupled system of wave equations in the scale-invariant case and with time-derivative nonlinearities. Under the usual assumption of small initial data, we obtain an improvement of the delimitation of the blow-up region by obtaining a new candidate for the critical curve. More precisely, we enhance the results obtained in [18] for the system under consideration in the present work. We believe that our result is optimal in the sense that beyond the blow-up region obtained here we may conjecture the global existence of the solution.

1. Introduction

We consider in this article the following weakly coupled system of semilinear wave equations with damping in the scale-invariant case and nonlinearities of derivative type, namely

\[
\begin{cases}
  u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t = |\partial_t v|^p, & x \in \mathbb{R}^N, \; t > 0, \\
  v_{tt} - \Delta v + \frac{\mu_2}{1+t} v_t = |\partial_t u|^q, & x \in \mathbb{R}^N, \; t > 0, \\
  u(x,0) = \varepsilon f_1(x), \; v(x,0) = \varepsilon f_2(x), & x \in \mathbb{R}^N, \\
  u_t(x,0) = \varepsilon g_1(x), \; v_t(x,0) = \varepsilon g_2(x), & x \in \mathbb{R}^N,
\end{cases}
\]

(1.1)

where $\mu_1, \mu_2$ are two positive constants. Moreover, the parameter $\varepsilon$ is a positive number describing the size of the initial data, and $f_1, f_2, g_1$ and $g_2$ are positive functions which are compactly supported on $B_{\mathbb{R}^N}(0,R), R > 0$.

Throughout this article, we suppose that $p, q > 1$.

Note that the situation, in the scale-invariant context, for coupled damped wave equations is not a simple generalization of the case of a one damped wave equation. However, we will point out here the recent improvements in this direction in the purpose to show the difference between the two settings (a one equation and a system). First, we recall the Glassey conjecture which asserts that the critical power $p_G$ should be given.

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The above critical value, $p_G$, gives rise to two regions for the power $p$ ensuring the global existence (for $p > p_G$) or the nonexistence (for $p \leq p_G$) of a global small data solution; see e.g. [7, 8, 10, 20, 21, 24, 27].

We recall here the case of one damped equation with only one time-derivative nonlinearity, namely

$$
\begin{align*}
&u_{tt} - \Delta u + \frac{\mu}{1 + t}u_t = |u_t|^p, \quad \text{in } \mathbb{R}^N \times [0, \infty), \\
u(x, 0) = \epsilon f(x), \quad u_t(x, 0) = \epsilon g(x), \quad x \in \mathbb{R}^N.
\end{align*}
$$

In the context of one damped equation, Lai and Takamura prove in [13] a blow-up result for the solution of (1.3) and they give an upper bound of the lifespan. We stress the fact that in this case there is no restriction for $\mu$ in the blow-up region for $p$, namely $p \in (1, p_G(N + 2\mu)]$. Recently, Palmieri and Tu proved in [18], among many other interesting results, a more accurate blow-up interval for $p$ in relationship with the solution of (1.3) with a mass term. More precisely, it is proven that the solution of this problem blows up in finite time for $p \in (1, p_G(N + \sigma(\mu))]$ where

$$
\begin{align*}
\sigma(\mu) &= \begin{cases} 
2\mu & \text{if } \mu \in [0, 1), \\
2 & \text{if } \mu \in [1, 2), \\
\mu & \text{if } \mu \geq 2.
\end{cases}
\end{align*}
$$

Of course the problems studied in [18] are more general.

Thanks to a better understanding of the corresponding linear problem to (1.3), the blow-up interval, $p \in (1, p_G(N + \sigma(\mu))]$ ($\sigma(\mu)$ is given by (1.4)), obtained in [18] and previously in [13], is improved in [6] to reach the interval $p \in (1, p_G(N + \mu)]$, for $\mu \in (0, 2)$. Our result for (1.3) coincides with the one in [18], for $\mu \geq 2$. We continue in this work to take advantage of the same technique developed in [6] to obtain an upgrade of the blow-up region for the solution of (1.1).

Now, going back to the system (1.1) and let $\mu_1 = \mu_2 = 0$. Then, the system (1.1) describes the coupling between two wave equations with time derivative nonlinearity.
More precisely, (1.1) yields

\[
\begin{aligned}
&\begin{cases}
  u_{tt} - \Delta u = |\partial_t v|^p, & x \in \mathbb{R}^N, \ t > 0, \\
  v_{tt} - \Delta v = |\partial_t u|^q, & x \in \mathbb{R}^N, \ t > 0, \\
  u(x, 0) = \varepsilon f_1(x), & x \in \mathbb{R}^N, \\
  v(x, 0) = \varepsilon f_2(x), & x \in \mathbb{R}^N, \\
  u_t(x, 0) = \varepsilon g_1(x), & x \in \mathbb{R}^N, \\
  v_t(x, 0) = \varepsilon g_2(x), & x \in \mathbb{R}^N.
\end{cases}
\end{aligned}
\]

(1.5)

The study of the existence or nonexistence of solutions to (1.5) has been the subject of several works in the literature. First, let us point out the blow-up results obtained by Deng [2] and Xu [25]. For a family of coupled systems larger than the one studied in the present work, Ikeda et al. [9] have stated and proved several nice results related to different combinations of the nonlinearities in the coupled systems under examination. The context of the present article (for the damped case) is limited to the nonlinearity of derivative type (as in (1.1)), however, other nonlinearities will be considered elsewhere; see for instance [1, 9, 15, 16]. On the other hand, for the existence of solutions to (1.5), we refer the reader to [11]. We notice here that, thanks to the above works, the situation is understood regarding the derivation of the curve describing the threshold between the blow-up and global existence regions for the solutions to (1.5). More precisely, the critical (in the sense of interface between blow-up and global existence) curve for \( p, q \) is given by

\[
\Upsilon(N, p, q) := \max(\Lambda(N, p, q), \Lambda(N, q, p)) = 0,
\]

where

\[
\Lambda(N, p, q) := \frac{p + 1}{pq - 1} - \frac{N - 1}{2}.
\]

(1.6)

Under some assumptions, the solution \((u, v)\) of (1.5) blows up in finite time \( T(\varepsilon) \) for small initial data (of size \( \varepsilon \)), namely

\[
T(\varepsilon) \leq \begin{cases}
C\varepsilon^{-\Upsilon(N, p, q)} & \text{if } \Upsilon(N, p, q) > 0, \\
\exp(C\varepsilon^{-(pq - 1)}) & \text{if } \Upsilon(N, p, q) = 0, \ p \neq q, \\
\exp(C\varepsilon^{-(p - 1)}) & \text{if } \Upsilon(N, p, q) = 0, \ p = q.
\end{cases}
\]

(1.8)

Recently, Palmieri and Takamura [17] proved a nice result on the blow-up of the solution of a weakly coupled system of semilinear damped wave equations of derivative type. They consider nonnegative and summable coefficients in the damping terms (the \textit{scattering} case). More precisely, using the multiplier’s technique and taking advantage of the fact that the multipliers in this case are bounded, the authors in [17] prove that
for the solutions of the following system:

\[
\begin{aligned}
    u_{tt} - \Delta u + b_1(t)u_t &= |\partial_t v|^p, & x \in \mathbb{R}^N, & t > 0, \\
    v_{tt} - \Delta v + b_2(t)v_t &= |\partial_t u|^q, & x \in \mathbb{R}^N, & t > 0, \\
    u(x, 0) &= \varepsilon f_1(x), & v(x, 0) &= \varepsilon f_2(x), & x \in \mathbb{R}^N, \\
    u_t(x, 0) &= \varepsilon g_1(x), & v_t(x, 0) &= \varepsilon g_2(x), & x \in \mathbb{R}^N,
\end{aligned}
\]  

the \((p, q)\)-critical curve interestingly remains unchanged in the scattering case. Moreover, it is proven that the solutions of (1.9) blow up in finite time, and the blow-up time is accordingly given by (1.11). Although the result obtained in [17] is the same as for the system (1.5), but, the situation for the damped system studied in [17] is much more difficult in view of the coupling by means of two nonlinearities of derivative type.

In the context of the present work and to the best of our knowledge the only result on the blow-up of the weakly damped system in the scale-invariant case, (1.1), that we found in the literature is due to Palmieri and Tu who proved [18], among many other interesting results, a blow-up result for a system similar to (1.1). Indeed, the authors in [18] studied (1.1) in a more general setting, namely by adding two mass terms, which make the analysis somehow more delicate.

More precisely, the authors in [18] proved that there is blow-up for the system (1.1) for \(p, q\) satisfying

\[
\Omega(N, \sigma(\mu_1), \sigma(\mu_2), p, q) := \max(\Lambda(N + \sigma(\mu_1), p, q), \Lambda(N + \sigma(\mu_2), q, p)) \geq 0,
\]

where \(\Lambda\) is given by (1.7) and \(\sigma(\mu_i), i = 1, 2\) is given by (1.4). Moreover, the solution \((u, v)\) of (1.1) blows up in finite time \(T(\varepsilon)\) for small initial data (of size \(\varepsilon\)), namely

\[
T(\varepsilon) \leq \begin{cases} 
    C\varepsilon^{-\Omega(N, \sigma(\mu_1), \sigma(\mu_2), p, q)} & \text{if } \Omega(N, \sigma(\mu_1), \sigma(\mu_2), p, q) > 0, \\
    \exp(C\varepsilon^{-(pq-1)}) & \text{if } \Omega(N, \sigma(\mu_1), \sigma(\mu_2), p, q) = 0, \\
    \exp(C\varepsilon^{-\min\left(\frac{pq-1}{p+1}, \frac{pq-1}{q+1}\right)}) & \text{if } \Lambda(N + \sigma(\mu_1), p, q) = \Lambda(N + \sigma(\mu_2), q, p) = 0.
\end{cases}
\]

However, we will compare here our result to the one in [18] by simply omitting the mass terms.

The emphasis in our work is the study of the Cauchy problem (1.1) and the influence of the parameter \(\mu_1, \mu_2\) on the blow-up result and the lifespan estimate.

Therefore, thanks to a better comprehension of the role of the weak damping term in (1.1) in the dynamics together with a deeper understanding of the corresponding linear problem inherited from the techniques developed in [5, 6], we will improve the bound of the blow-up region. Indeed, the result on the blow-up region, defined by (1.10) and obtained in [18], is improved here, under some assumptions to be announced in our main
result, as follows:

\[ \Omega(N, \mu_1, \mu_2, p, q) = \max(\Lambda(N + \mu_1, p, q), \Lambda(N + \mu_2, q, p)) \geq 0, \]

where \( \Lambda \) is given by (1.7).

The article is organized as follows. We start in Section 2 by introducing the weak formulation of (1.1) in the energy space. Then, we state the main theorem of our work. In Section 3 we prove some technical lemmas that we will use, among other tools, to conclude the proof of the main result which is contained in Section 4.

2. Main Result

This section is devoted to the statement of our main result. For that purpose, we first start by giving the definition of the solution of (1.1) in the corresponding energy space. More precisely, the weak formulation associated with (1.1) reads as follows:

**Definition 2.1.** We say that \((u, v)\) is an energy solution of (1.1) on \([0, T)\) if

\[
\begin{align*}
&u, v \in C([0, T), H^1(\mathbb{R}^N)) \cap C^1([0, T), L^2(\mathbb{R}^N)), \\
&u_t \in L^q_{loc}((0, T) \times \mathbb{R}^N), \ v_t \in L^p_{loc}((0, T) \times \mathbb{R}^N)
\end{align*}
\]

satisfies, for all \(\Phi, \tilde{\Phi} \in C_0^\infty(\mathbb{R}^N \times [0, T))\) and all \(t \in [0, T)\), the following equations:

\[
\begin{align*}
&\int_{\mathbb{R}^N} u_t(x, t)\Phi(x, t)dx - \int_{\mathbb{R}^N} u_t(x, 0)\Phi(x, 0)dx - \int_0^t \int_{\mathbb{R}^N} u_t(x, s)\Phi_t(x, s)dx ds + \int_0^t \int_{\mathbb{R}^N} \nabla u(x, s) \cdot \nabla \Phi(x, s)dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}^N} \frac{\mu_1}{1+s} u_t(x, s)\Phi(x, s)dx ds = \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p\Phi(x, s)dx ds,
\end{align*}
\]

and

\[
\begin{align*}
&\int_{\mathbb{R}^N} v_t(x, t)\tilde{\Phi}(x, t)dx - \int_{\mathbb{R}^N} v_t(x, 0)\tilde{\Phi}(x, 0)dx \\
&\quad - \int_0^t \int_{\mathbb{R}^N} v_t(x, s)\tilde{\Phi}_t(x, s)dx ds + \int_0^t \int_{\mathbb{R}^N} \nabla v(x, s) \cdot \nabla \tilde{\Phi}(x, s)dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}^N} \frac{\mu_2}{1+s} v_t(x, s)\tilde{\Phi}(x, s)dx ds = \int_0^t \int_{\mathbb{R}^N} |u_t(x, s)|^q\tilde{\Phi}(x, s)dx ds.
\end{align*}
\]

The following theorem states the main result of this article.

**Theorem 2.2.** Let \(p, q > 1\) and \(\mu_1, \mu_2 > 0\) such that

\[ \Omega(N, \mu_1, \mu_2, p, q) \geq 0, \]

where the expression of \(\Omega\) is given by (1.10).

Assume that \(f_1, f_2 \in H^1(\mathbb{R}^N)\) and \(g_1, g_2 \in L^2(\mathbb{R}^N)\) are non-negative functions which are
2.2 compactly supported on $B_{\mathbb{R}^N}(0, R)$, and do not vanish everywhere. Let $(u, v)$ be an energy solution of (2.1)-(2.2) on $[0, T_\varepsilon]$ such that $\text{supp}(u), \text{supp}(v) \subset \{(x, t) \in \mathbb{R}^N \times [0, \infty) : |x| \leq t + R\}$. Then, there exists a constant $\varepsilon_0 = \varepsilon_0(f_1, f_2, g_1, g_2, N, R, p, q, \mu_1, \mu_2) > 0$ such that $T_\varepsilon$ verifies

\[
T(\varepsilon) \leq \begin{cases} 
C\varepsilon^{-\Omega(N, \mu_1, \mu_2, p, q)} & \text{if } \Omega(N, \mu_1, \mu_2, p, q) > 0, \\
\exp(C\varepsilon^{-(pq-1)}) & \text{if } \Omega(N, \mu_1, \mu_2, p, q) = 0, \\
\exp(C\varepsilon^{-\min\left(\frac{pq-1}{pq+1}, \frac{p-1}{q+1}\right)}) & \text{if } \lambda(N + \mu_1, p, q) = \lambda(N + \mu_2, q, p) = 0.
\end{cases}
\]

where $C$ is a positive constant independent of $\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0$.

Remark 2.1. We notice here that Theorem 2.2 shows that the critical curve for $p, q$ is in fact a shift of the dimension by $\mu_1, \mu_2$, respectively, in (1.6). We believe that this new blow-up region delimitation coincides with the critical one. Of course one has to rigorously confirm this assertion by proving a global existence result in the complementary region. This will be the subject of a forthcoming work.

Remark 2.2. The result in Theorem 2.2 holds true if we replace the linear damping term in (1.1), $\frac{\mu_i}{1+t}u_t$ (resp. $\frac{\mu_i}{1+t}v_t$), by $b_1(t)u_t$ (resp. $b_2(t)v_t$) with $[b_i(t) - \mu_i(1 + t)^{-1}]$; $i = 1, 2$, belongs to $L^1(0, \infty)$. The proof of this generalized damping case can be obtained by following the same steps as in the proof of Theorem 2.2 with the necessary modifications.

Remark 2.3. Note that the blow-up result (2.4) in Theorem 2.2 remains true for the solutions of the system (1.9) with e.g. $b_1(t)u_t = \frac{\mu_1}{1+t}u_t$ and $b_2(t)v_t$ is such that $[b_2(t) - \mu_2(1 + t)^{-1}]$ belongs to $L^1(0, \infty)$. It suffices to set in this case $\mu_2 = 0$ in (2.4). The proof can be carried out by combining the computations in the present article and [17].

3. SOME AUXILIARY RESULTS

We define the following positive test function

\[
\psi_i(x, t) := \rho_i(t)\phi(x), \quad i = 1, 2,
\]

where

\[
\phi(x) := \begin{cases} 
\int_{S^{N-1}} e^{x \cdot \omega} d\omega & \text{for } N \geq 2, \\
e^x + e^{-x} & \text{for } N = 1.
\end{cases}
\]

Note that the function $\phi(x)$ is introduced in [26] and $\rho_i(t), \ [15, 19, 22, 23]$, is solution of

\[
\frac{d^2 \rho_i(t)}{dt^2} - \rho_i(t) - \frac{d}{dt}\left(\frac{\mu_i}{1 + t} \rho_i(t)\right) = 0, \quad i = 1, 2.
\]

The expression of $\rho_i(t)$ reads as follows (see the Appendix for more details):

\[
\rho_i(t) = (t + 1)^{\frac{\mu_i+1}{2}} K_{\frac{\mu_i-1}{6}}(t + 1), \quad i = 1, 2,
\]
where
\[ K_{\nu}(t) = \int_{0}^{\infty} \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \quad \nu \in \mathbb{R}. \]

Moreover, the function \( \phi(x) \) verifies
\[ \Delta \phi = \phi. \]

Note that the function \( \psi_i(x, t) \) satisfies the corresponding conjugate equation, namely we have
\[ \partial_t^2 \psi_i(x, t) - \Delta \psi_i(x, t) - \partial_t \left( \frac{\mu_i}{1 + t} \psi_i(x, t) \right) = 0. \]

Throughout this article, we will denote by \( C \) a generic positive constant which may depend on the data \((p, q, \mu_i, N, R, f_i, g_i)_{i=1,2}\) but not on \( \varepsilon \) and whose the value may change from line to line. Nevertheless, we will precise the dependence of the constant \( C \) on the parameters of the problem when it is necessary.

The following lemma holds true for the function \( \psi_i(x, t) \).

**Lemma 3.1** ([26]). Let \( r > 1 \). Then, there exists a constant \( C = C(N, R, r) > 0 \) such that
\[ \int_{|x| \leq t + R} \left( \psi_i(x, t) \right)^r dx \leq C \rho_i^r(t) e^{rt(1 + t)} 2^{(N-1)}(1 + t), \quad i = 1, 2, \quad \forall \ t \geq 0. \]

Now, we introduce the following functionals.
\[ F_1(t) := e^{-t} \int_{\mathbb{R}^N} u(x, t) \phi(x) dx, \quad F_2(t) := e^{-t} \int_{\mathbb{R}^N} v(x, t) \phi(x) dx, \]
and
\[ \tilde{F}_1(t) := e^{-t} \int_{\mathbb{R}^N} \partial_t u(x, t) \phi(x) dx, \quad \tilde{F}_2(t) := e^{-t} \int_{\mathbb{R}^N} \partial_t v(x, t) \phi(x) dx. \]

The next two lemmas give the first lower bounds for \( F_i(t) \) and \( \tilde{F}_i(t) \), \( i=1,2 \), respectively.

**Lemma 3.2.** Assume that the assumptions in Theorem 2.2 hold. Then, for \( i = 1, 2 \), we have
\[ F_i(t) \geq \frac{\varepsilon}{2m_i(t)} \int_{\mathbb{R}^N} f_i(x) \phi(x) dx, \quad \text{for all} \ t \in [0, T), \]
where \( m_i(t) := (1 + t)^{\mu_i} \).

**Lemma 3.3.** Under the same assumptions of Theorem 2.2, it holds that
\[ \tilde{F}_i(t) \geq \frac{\varepsilon}{2m_i(t)} \int_{\mathbb{R}^N} g_i(x) \phi(x) dx, \quad i = 1, 2, \quad \text{for all} \ t \in [0, T). \]
The proofs of the two above lemmas are based on the multiplier’s technique introduced in [14] and used in several works; see e.g. [5, 6, 12, 13, 17]. Note that the multipliers \((m_i(t))_{i=1,2}\) are not bounded.

**Remark 3.1.** We notice that the lower bounds obtained in Lemmas 3.2 and 3.3 are not optimal, however, the results there are sufficient since we only need the positivity of \(F_i(t)\) and \(\tilde{F}_i(t)\), \(i = 1, 2\), for all \(t > 0\), in the proof of our main result. Nevertheless, we will instead introduce new functionals for which we aim to obtain better lower bounds. This will be the subject of the next two lemmas.

We define now the functionals that we will use to prove the blow-up criteria later on:

\[
G_1(t) := \int_{\mathbb{R}^N} u(x, t)\psi_1(x, t)dx, \quad G_2(t) := \int_{\mathbb{R}^N} v(x, t)\psi_2(x, t)dx,
\]

and

\[
\tilde{G}_1(t) := \int_{\mathbb{R}^N} \partial_t u(x, t)\psi_1(x, t)dx, \quad \tilde{G}_2(t) := \int_{\mathbb{R}^N} \partial_t v(x, t)\psi_2(x, t)dx.
\]

The next two lemmas give the first lower bounds for \(G_i(t)\) and \(\tilde{G}_i(t)\) \((i = 1, 2)\), respectively. More precisely, we will prove that the functions \(G_i(t)\) and \(\tilde{G}_i(t)\) are coercive. This is the first observation which will be used to improve the main result of this article. Indeed, in comparison with (3.9) and (3.10), we obtain in the subsequent better lower bounds for the functionals \(G_1(t)\) and \(\tilde{G}_1(t)\); see (3.13) and (3.25) below, passing thus from the size \(\varepsilon/(1 + t)^{\mu_i/2}\) to the size \(\varepsilon\).

Although the techniques used here are similar to the ones in our previous work [6] in the case of a one single equation, but, the situation is slightly different for the system (1.1). So, we will include all the details about the proofs of the next two lemmas. However, we will only show the proofs for the solution \(u\), and for \(v\) the computations follow similarly.

**Lemma 3.4.** Assume that the assumptions in Theorem 2.2 hold. Let \((u, v)\) be an energy solution of (2.1)-(2.2). Then, for \(i = 1, 2\), there exists \(T_0 = T_0(\mu_1, \mu_2) > 1\) such that

\[
G_i(t) \geq C_{G_i} \varepsilon, \quad \text{for all } t \geq T_0,
\]

where \(C_{G_i}\) is a positive constant which depends on \(f_i, g_i, N, R\) and \(\mu_i\).

**Proof.** As mentioned before, we will prove the lemma for \(u\). The proof for \(v\) is similar. Let \(t \in [0, T)\), then using Definition 2.1 and performing an integration by parts in space
Now, substituting in (3.14), we obtain
\[
\int_{\mathbb{R}^N} u_t(x, t) \Phi(x, t) dx - \varepsilon \int_{\mathbb{R}^N} g_1(x) \Phi(x, 0) dx
\]
\[
- \int_0^t \int_{\mathbb{R}^N} \{ u_t(x, s) \Phi_t(x, s) + u(x, s) \Delta \Phi(x, s) \} \, dx \, ds + \int_0^t \int_{\mathbb{R}^N} \frac{\mu_1}{1 + s} u_t(x, s) \Phi(x, s) \, dx \, ds
\]
\[
= \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \Phi(x, s) \, dx \, ds, \quad \forall \Phi \in C^\infty_0(\mathbb{R}^N \times [0, T]).
\]

Now, substituting in (3.14) \( \Phi(x, t) \) by \( \psi_1(x, t) \), we infer that
\[
\int_{\mathbb{R}^N} u_t(x, t) \psi_1(x, t) dx - \varepsilon \int_{\mathbb{R}^N} g_1(x) \psi_1(x, 0) dx
\]
\[
- \int_0^t \int_{\mathbb{R}^N} \{ u_t(x, s) \partial_t \psi_1(x, s) + u(x, s) \Delta \psi_1(x, s) \} \, dx \, ds + \int_0^t \int_{\mathbb{R}^N} \frac{\mu_1}{1 + s} u_t(x, s) \psi_1(x, s) \, dx \, ds
\]
\[
= \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) \, dx \, ds.
\]
Performing an integration by parts for the first and third terms in the second line of (3.15) and utilizing (3.1) and (3.5), we obtain
\[
\int_{\mathbb{R}^N} \left[ u_t(x, t) \psi_1(x, t) - u(x, t) \partial_t \psi_1(x, t) + \frac{\mu_1}{1 + t} u(x, t) \psi_1(x, t) \right] dx
\]
\[
= \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) dx \, ds + \varepsilon C(f_1, g_1),
\]
where
\[
C(f_1, g_1) := \rho_1(0) \int_{\mathbb{R}^N} \left[ (\mu_1 - \frac{\rho_1'(0)}{\rho_1(0)}) f_1(x) \phi(x) + g_1(x) \phi(x) \right] dx.
\]
We notice that the constant \( C(f_1, g_1) \) is positive thanks to (5.2) and the fact that the function \( K_\nu(t) \) is positive (see (5.3) in the Appendix).

Hence, using the definition of \( G_1 \), as in (3.11), and (3.1), the equation (3.16) yields
\[
G_1'(t) + \Gamma_1(t) G_1(t) = \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) dx \, ds + \varepsilon C(f_1, g_1),
\]
where
\[
\Gamma_1(t) := \frac{\mu_1}{1 + t} - 2 \frac{\rho_1'(t)}{\rho_1(t)}.
\]
Multiplying (3.18) by \( \frac{(1 + t)^{\mu_1}}{\rho_1^2(t)} \) and integrating over \((0, t)\), we deduce that
\[
G_1(t) \geq G_1(0) \frac{\rho_1^2(t)}{(1 + t)^{\mu_1}} + \varepsilon C(f_1, g_1) \frac{\rho_1^2(t)}{(1 + t)^{\mu_1}} \int_0^t (1 + s)^{\mu_1} ds.
\]
Using (3.4) and the fact that $G_1(0) > 0$, the estimate (3.20) yields
\begin{equation}
G_1(t) \geq \varepsilon C(f_1, g_1)(1 + t)K_{\frac{1}{\mu_1}}^2(t + 1) \int_{t/2}^{t} \frac{1}{(1 + s)K_{\frac{1}{\mu_1}}^2(s + 1)} ds.
\end{equation}

From (5.3), we have the existence of $T_0^1 = T_0^1(\mu_1) > 1$ such that
\begin{equation}
(1 + t)K_{\frac{1}{2}}^2(t + 1) > \frac{\pi}{4} e^{-2(t + 1)} \quad \text{and} \quad (1 + t)^{-1}K_{\frac{1}{2}}^{-2}(t + 1) > \frac{1}{\pi} e^{2(t + 1)}, \forall t \geq T_0^1/2.
\end{equation}

Hence, we have
\begin{equation}
G_1(t) \geq \frac{\varepsilon}{4} C(f_1, g_1)e^{-2t} \int_{t/2}^{t} e^{2s} ds \geq \frac{\varepsilon}{8} C(f_1, g_1)e^{-2t}(e^{2t} - e^t), \forall t \geq T_0^1.
\end{equation}

Finally, using $e^{2t} > 2e^t, \forall t \geq 1$, we deduce that
\begin{equation}
G_1(t) \geq \frac{\varepsilon}{16} C(f_1, g_1), \forall t \geq T_0^1.
\end{equation}

Hence, similarly for $v$ we have the existence of $T_0^2 = T_0^2(\mu_2) > 1$. Finally, to conclude we take $T_0 = \max(T_0^1, T_0^2)$. This ends the proof of Lemma 3.4. \Box

Now we are in position to prove the following lemma.

**Lemma 3.5.** Assume that the assumptions in Theorem 2.2 hold. Let $(u, v)$ be an energy solution of (2.1)-(2.2). Then, for $i = 1, 2$, there exists $T_i = T_i(\mu_1, \mu_2) > 1$ such that
\begin{equation}
\tilde{G}_i(t) \geq C_{\tilde{G}_i} \varepsilon, \text{ for all } t \geq T_i,
\end{equation}
where $C_{\tilde{G}_i}$ is a positive constant which depends on $f_i, g_i, N, R$ and $\mu_i$.

**Proof.** As before, we will prove the lemma for $u$. The proof for $v$ is similar.
Let $t \in [0, T)$, then using the definition of $G_1$ and $\tilde{G}_1$, given respectively by (3.11) and (3.12), (3.1) and the fact that
\begin{equation}
G_1'(t) - \frac{\rho_1'(t)}{\rho_1(t)}G_1(t) = \tilde{G}_1(t),
\end{equation}
the equation (3.18) yields
\begin{equation}
\tilde{G}_1(t) + \left(\frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)}\right)G_1(t)
= \int_0^t \int_{\mathbb{R}^N} |v_1(x, s)|^p \psi_1(x, s) dx ds + \varepsilon C(f_1, g_1).
\end{equation}

Differentiating the equation (3.27) in time, we get
\begin{equation}
\tilde{G}_1''(t) + \left(\frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)}\right)G_1'(t) - \left(\frac{\mu_1}{(1 + t)^2} + \frac{\rho_1''(t)\rho_1(t) - (\rho_1'(t))^2}{\rho_1^2(t)}\right)G_1(t)
= \int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx.
\end{equation}
Using (3.3) and (3.26), the identity (3.28) becomes

\[ \tilde{G}'_1(t) + \left( \frac{\mu_1}{1+t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) \tilde{G}_1(t) - G_1(t) = \int_{\mathbb{R}^N} |v_t(x,t)|^p \psi_1(x,t) dx. \]  

Remember the definition of \( \Gamma_1(t) \), given by (3.19), we obtain

\[ \tilde{G}'_1(t) + \frac{3 \Gamma_1(t)}{4} \tilde{G}_1(t) \geq \Sigma^1_1(t) + \Sigma^2_1(t) + \Sigma^3_1(t), \]

where

\[ \Sigma^1_1(t) := \left( -\frac{\rho_1'(t)}{2\rho_1(t)} - \frac{\mu_1}{4(1+t)} \right) \left( \tilde{G}_1(t) + \left( \frac{\mu_1}{1+t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) G_1(t) \right), \]

\[ \Sigma^2_1(t) := \left( 1 + \left( \frac{\rho_1'(t)}{2\rho_1(t)} + \frac{\mu_1}{4(1+t)} \right) \left( \frac{\mu_1}{1+t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) \right) G_1(t), \]

and

\[ \Sigma^3_1(t) := \int_{\mathbb{R}^N} |v_t(x,t)|^p \psi_1(x,t) dx. \]

Now, from (3.27) and (5.4), we deduce that there exists \( T^1_1 = T^1_1(\mu_1) \geq T_0 \) such that

\[ \Sigma^1_1(t) \geq \frac{\varepsilon}{8} C(f_1, g_1) + \frac{1}{4} \int_0^t \int_{\mathbb{R}^N} |v_t(x,s)|^p \psi_1(x,s) dx ds, \quad \forall t \geq T^1_1. \]

Moreover, form Lemma 3.4 and (5.4), we conclude the existence of \( \tilde{T}^1_1 = \tilde{T}^1_1(\mu_1) \geq T^1_1 \) such that

\[ \Sigma^2_1(t) \geq 0, \quad \forall t \geq \tilde{T}^1_1. \]

Combining (3.30), (3.33), (3.34) and (3.35), we obtain

\[ \tilde{G}'_1(t) + \frac{3 \Gamma_1(t)}{4} \tilde{G}_1(t) \geq \frac{\varepsilon}{8} C(f_1, g_1) + \frac{1}{4} \int_0^t \int_{\mathbb{R}^N} |v_t(x,s)|^p \psi_1(x,s) dx ds \]

\[ + \int_{\mathbb{R}^N} |v_t(x,t)|^p \psi_1(x,t) dx, \quad \forall t \geq \tilde{T}^1_1. \]

Ignoring the nonlinear terms yields

\[ \tilde{G}'_1(t) + \frac{3 \Gamma_1(t)}{4} \tilde{G}_1(t) \geq \frac{\varepsilon}{8} C(f_1, g_1), \quad \forall t \geq \tilde{T}^1_1. \]

Multiplying (3.37) by \( \frac{(1+t)^{3\mu_1/4}}{\rho_1^{3/2}(t)} \) and integrating over \( (\tilde{T}^1_1, t) \), we deduce that

\[ \tilde{G}_1(t) \geq \tilde{G}_1(\tilde{T}^1_1) \frac{(1+\tilde{T}^1_1)^{3\mu_1/4}}{\rho_1^{3/2}(\tilde{T}^1_1)} \frac{\rho_1^{3/2}(t)}{(1+t)^{3\mu_1/4}} \]

\[ \varepsilon C(f_1, g_1) \frac{\rho_1^{3/2}(t)}{(1+t)^{3\mu_1/4}} \int_{\tilde{T}^1_1}^t \frac{(1+s)^{3\mu_1/4}}{\rho_1^{3/2}(s)} ds, \quad \forall t \geq \tilde{T}^1_1. \]
Now, observe that \( \tilde{G}_1(t) = \rho_1(t)e^{\tilde{F}_1(t)} \) where \( \tilde{F}_1(t) \) is given by (3.8). Hence, using Lemma 3.3 we infer that \( \tilde{G}_1(t) \geq 0 \) for all \( t \geq 0 \).

Therefore, using the above observation and (3.4), we deduce that

\[
\tilde{G}_1(T_1) (1 + \tilde{T}_1^{3/4}) \geq 0, \quad \forall \ t \geq 0.
\]

Employing (3.22) and (3.39), the estimate (3.38) yields

\[
\tilde{G}_1(t) \geq C \varepsilon e^{-3t/2} \int_{t/2}^t e^{3s/2} ds, \quad \text{for all } t \geq T_1^1 := 2\tilde{T}_1^1.
\]

Hence, we have

\[
\tilde{G}_1(t) \geq C \varepsilon, \quad \forall \ t \geq T_1^1.
\]

Hence, similarly for \( v \) we have the existence of \( T_2^1 = T_2^1(\mu_2) > 1 \) and we take \( T_1 = \max(T_1^1, T_2^1) \). This concludes the proof of Lemma 3.5. \(\square\)

4. Proof of Theorem 2.2.

This section is devoted to the proof of Theorem 2.2 which is somehow related to the obtaining of the critical curve associated with the nonlinear problem (1.1). First, we perform a better understanding of the linear problem associated with (1.1) and use the results in Section 3. In fact, we proved in Lemma 3.5 that \( \tilde{G}_i(t) \) are coercive functions for \( i = 1, 2 \). This is a crucial observation that we will use to improve the blow-up result of (1.1). Thanks to the observation described above and by introducing some new functionals \( L_1(t) \) and \( L_2(t) \) (see (4.1) and (4.2) below), which verify two integral inequalities similar to the ones in [18, (25) and (26)] with \( \mu_1, \mu_2 \) in the present work instead of \( \sigma_1, \sigma_2 \) in [18], we improve the blow-up result in [18] for the solution of (1.1). The result of this work makes the blow-up region for (1.1) more precise. Our result for (1.1) enhances the corresponding one in [18] except if \( \mu_1 \geq 2 \) and \( \mu_2 \geq 2 \) where the two results coincide.

Now, setting

\[
L_1(t) := \frac{1}{8} \int_{T_2}^t \int_{\mathbb{R}^N} |v_t(x,s)|^p \psi_1(x,s) dx ds + \frac{C_6 \varepsilon}{8},
\]

and

\[
L_2(t) := \frac{1}{8} \int_{T_2}^t \int_{\mathbb{R}^N} |u_t(x,s)|^q \psi_2(x,s) dx ds + \frac{C_6 \varepsilon}{8},
\]

where \( C_6 = \min(C(f_1, g_1), C(f_2, g_2), 8C_{G_1}, 8C_{G_2}) \) (\( C_{G_1} \) and \( C_{G_2} \) are defined in Lemmas 3.4 and 3.5, respectively) and \( T_2 := T_2(\mu_1, \mu_2) > T_1 \) is chosen such that \( \frac{1}{4} - \frac{3 \Gamma_i(t)}{2} > 0 \) and \( \Gamma_i(t) > 0 \), for \( i=1,2 \), for all \( t \geq T_2 \) (this is possible thanks to (3.19) and (5.4)), and
let
\[ \mathcal{F}_i(t) := \tilde{G}_i(t) - L_i(t), \quad \forall \, i = 1, 2. \]

Hence, we have for \( \mathcal{F}_1 \),
\[
\mathcal{F}_1'(t) + \frac{3\Gamma_1(t)}{4} \mathcal{F}_1(t) \geq \left( \frac{1}{4} - \frac{3\Gamma_1(t)}{32} \right) \int_{T_2}^{t} \int_{\mathbb{R}^N} |v_1(x, s)|^p \psi_1(x, s) dx ds
\]
\[
+ \frac{7}{8} \int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx + C_6 \left( 1 - \frac{3\Gamma_1(t)}{32} \right) \varepsilon.
\]
(4.3)

Multiplying (4.3) by \( \frac{(1+t)^{3\mu_1/4}}{\rho_1^{3/2}(t)} \) and integrating over \((T_2, t)\), we deduce that
\[
\mathcal{F}_1(t) \geq \mathcal{F}_1(T_2) \frac{(1+T_2)^{3\mu/4}}{\rho_1^{3/2}(T_2)} \frac{\rho_1^{3/2}(t)}{(1+t)^{3\mu_1/4}}, \quad \forall \, t \geq T_2,
\]
where \( \rho_1(t) \) is defined by (3.4).

Therefore we have \( \mathcal{F}_1(T_2) = \tilde{G}_1(T_2) - \frac{C_6 \varepsilon}{8} \geq \tilde{G}_1(T_2) - C_6 \varepsilon \geq 0 \) thanks to Lemma 3.5 and the fact that \( C_6 = \min(C(f_1, g_1), C(f_2, g_2), 8C_{\tilde{G}_1}, 8C_{\tilde{G}_2}) \leq 8C_{\tilde{G}_1} \).

Then, we have
\[
\tilde{G}_1(t) \geq L_1(t), \quad \forall \, t \geq T_2.
\]
(4.5)

Similarly, we have an analogous estimate for \( \tilde{G}_2(t) \), namely
\[
\tilde{G}_2(t) \geq L_2(t), \quad \forall \, t \geq T_2.
\]
(4.6)

By Hölder’s inequality and the estimates (3.6) and (3.25), we can bound the nonlinear term as follows:
\[
\int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx \geq \tilde{G}_2^p(t) \left( \int_{|x| \leq t+R} \psi_2^{\frac{p}{p-1}}(x, t) \psi_1^{\frac{-1}{p-1}}(x, t) dx \right)^{-(p-1)}
\]
\[
\geq C \tilde{G}_2^p(t) \rho_1(t) \rho_2^{-p}(t) e^{-(p-1)t} (1+t)^{-\frac{(N-1)(p-1)}{4}}.
\]
(4.7)

Using (3.4) and (3.22), we get
\[
\rho_1(t) e^t \leq C(1+t)^{\frac{\mu_1}{2}}, \quad \forall \, t \geq T_0/2,
\]
(4.8)

and a similar estimate holds for \( \rho_2(t) \).

Hence, plugging (4.8) in (4.7) yields
\[
\int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx \geq C(1+t)^{-\frac{(N-1)(p-1)}{2} + \frac{\mu_1}{2} - \frac{\mu_2}{p}} \tilde{G}_2^p(t), \quad \forall \, t \geq T_2.
\]
(4.9)

From the above estimate and (4.6), we infer that
\[
L_1(t) \geq C(1+t)^{-\frac{(N-1)(p-1)}{2} + \frac{\mu_1}{2} - \frac{\mu_2}{p}} L_2^p(t), \quad \forall \, t \geq T_2.
\]
(4.10)
Similarly, we obtain an analogous estimate for $L'_2(t)$,
\begin{equation}
L'_2(t) \geq C(1 + t)^{-\frac{(N-1)}{2}(q-1)+\frac{p}{2} \Delta - \frac{p}{2} q} L'_q(t), \quad \forall \ t \geq T_2.
\end{equation}
Integrating (4.10) and (4.11) on $(T_2, t)$, we obtain, respectively,
\begin{equation}
L_1(t) \geq \frac{C_6 \varepsilon}{8} + C_0 \int_{T_2}^{t} (1 + s)^{-\frac{(N-1)}{2}(p-1)+\frac{p}{2} \Delta - \frac{p}{2} q} L'_q(s) ds, \quad \forall \ t \geq T_2,
\end{equation}
and
\begin{equation}
L_2(t) \geq \frac{C_6 \varepsilon}{8} + C_0 \int_{T_2}^{t} (1 + s)^{-\frac{(N-1)}{2}(p-1)+\frac{p}{2} \Delta - \frac{p}{2} q} L'_q(s) ds, \quad \forall \ t \geq T_2.
\end{equation}
Using the fact that $\frac{1}{T_2}(T_2 + s) \leq 1 + s \leq T_2 + s$ for all $s \in (T_2, t)$, because $T_2 > 1$, we deduce that
\begin{equation}
L_1(t) \geq \frac{C_6 \varepsilon}{8} + C_1 \int_{T_2}^{t} (T_2 + s)^{-\frac{(N-1)}{2}(p-1)+\frac{p}{2} \Delta - \frac{p}{2} q} L'_q(s) ds, \quad \forall \ t \geq T_2,
\end{equation}
and
\begin{equation}
L_2(t) \geq \frac{C_6 \varepsilon}{8} + C_1 \int_{T_2}^{t} (T_2 + s)^{-\frac{(N-1)}{2}(p-1)+\frac{p}{2} \Delta - \frac{p}{2} q} L'_q(s) ds, \quad \forall \ t \geq T_2.
\end{equation}
Therefore, we end up with the same integral inequalities as in [18]; here (4.14) (resp. (4.15) corresponds to (25) (resp. (26)) in [18]. From these two integral inequalities we mimic the same steps, line by line, in [18, Sections 4.2 and 4.3] to prove the blow-up result with the iteration process applied to (4.14)-(4.15). However, we should take in consideration the fact that in the present work the shift of the dimension $N$ is with $\mu_i$ instead of $\sigma(\mu_i)$ in [18], where $\sigma(\mu_i)$ is defined by (1.4).

5. Appendix

In this appendix, we will recall some properties of the function $\rho_i(t)$, for $i = 1, 2$, the solution of (3.3). Hence, following the computations in [23] (with $\eta = 1$), we can write the expression of $\rho_i(t)$ as follows:
\begin{equation}
\rho_i(t) = (t + 1)^{\frac{\mu_i+1}{2}} K_{\frac{\mu_i-1}{2}}(t + 1), \quad i = 1, 2,
\end{equation}
where
\[ K_{\nu}(t) = \int_0^\infty \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \quad \nu \in \mathbb{R}. \]
Using the property of $\rho_i(t)$ in the proof of Lemma 2.1 in [23] (with $\eta = 1$), we infer that
\begin{equation}
\frac{\rho'_i(t)}{\rho_i(t)} = \frac{\mu_i}{1 + t} - \frac{K_{\frac{\mu_i+1}{2}}(t + 1)}{K_{\frac{\mu_i-1}{2}}(t + 1)}, \quad i = 1, 2.
\end{equation}
From [4], we have the following property for the function $K_{\mu_i}(t), \ i=1,2$,

\begin{equation}
K_{\mu_i}(t) = \sqrt{\frac{\pi}{2t}} e^{-t}(1 + O(t^{-1})), \quad \text{as } t \to \infty.
\end{equation}

Combining (5.2) and (5.3), we infer that, for $i = 1, 2$,

\begin{equation}
\frac{\rho_i'(t)}{\rho_i(t)} = -1 + O(t^{-1}), \quad \text{as } t \to \infty.
\end{equation}

Finally, we refer the reader to [3] for more details about the properties of the function $K_{\mu_i}(t)$.

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