Finding Trajectories with High Asymptotic Growth Rate for Linear Constrained Switching Systems via a Lift Approach

Yuhao Zhang, Xiangru Xu

Abstract—This paper investigates how to generate a sequence of matrices with an asymptotic growth rate close to the constrained joint spectral radius (CJSR) of the constrained switching system whose switching sequences are constrained by a deterministic finite automaton. Based on a matrix-form expression, the dynamics of a constrained switching system are proved to be equivalent to the dynamics of a lifted arbitrary switching system. By using the dual solution of a sum-of-squares optimization program, an algorithm is designed to produce a sequence of matrices with an asymptotic growth rate that can be made arbitrarily close to the joint spectral radius (JSR) of the lifted arbitrary switching system, or equivalently the CJSR of the original constrained switching system. Several numerical examples are provided to illustrate the better performance of the proposed algorithm compared with existing ones.

I. INTRODUCTION

A switched system is a dynamical system that consists of a set of subsystems and a logical rule that orchestrates switching between these subsystems [1], [2]. The discrete-time linear switched system associated with a finite set of matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$ can be modeled as

$$x_{k+1} = A_{\sigma_k}x_k, \quad \sigma_k \in \{1, \ldots, m\}$$

where $\sigma_k$ is the switching mode of the system and $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{Z}_{\geq 0}$. The system (1) is called an arbitrary switching system and denoted by $S(\mathcal{A})$ as there is no constrain on the switching sequence. The joint spectral radius (JSR), which characterizes the maximal asymptotic growth rate of an infinite product of matrices of the set $\mathcal{A}$, was firstly introduced in [3] by Rota and Strang and has been studied for decades because of its wide application in number theory, computer network, wavelet functions, and signal processing [4]. In particular, the value of JSR is related to the stability of switched systems since [1] is stable if and only if JSR < 1 (see Corollary 1.1 in [4]). There have been numerous algorithms (e.g., Gripenberg, lower brute force, conitope, and sum-of-squares (SOS)) proposed in the past decades to approximate the value of JSR (see, e.g., [4]).

In this work, we consider the switched system (1) whose switching sequences are constrained by a deterministic finite automaton $\mathcal{M}$. We refer to such a switching system as the constrained switching system, denoted as $S(\mathcal{A}, \mathcal{M})$. Similar to the concept of JSR that applies to the arbitrary switching system, the constrained joint spectral radius (CJSR) characterizes the stability of the constrained switching system [5]. The approximation of CJSR, however, is much more difficult than that of JSR. In [6], Philippe et al. propose a semi-definite programming based method to approximate CJSR, where the T-product lift and the M-path dependent lift methods are used to improve the approximation accuracy; in [7], Xu and Behçet propose a unified matrix-based formulation for arbitrary and constrained switching systems and prove that the CJSR of a constrained switching system is equivalent to the JSR of a lifted arbitrary switching system, such that the approximation of CJSR can be reduced to the approximation of JSR for which many off-the-shelf algorithms exist; see also [8], [9] and references therein.

This work investigates how to generate a sequence of matrices with an asymptotic growth rate close to the CJSR. Finding such sequences is useful in various applications, such as providing a more accurate lower bound for the CJSR and testing the stability of linear switched systems [10]. There are a few existing algorithms that can be utilized to produce the optimal asymptotic growth rate sequences for arbitrary switching systems. One of the simplest ideas is to fix a number $k$, compute all possible products of length $k$ and take the product that produces the maximal spectral radius. By increasing the value of $k$, this brute force algorithm will converge to the JSR, but it suffers from explosion of the number of products of length $k$ since $m^k$ products would need to be computed. In [11], Gripenberg proposes a branch-and-bound algorithm that improves the search by a priori fixed absolute error. Other branch-and-bound algorithms include the complex polytope algorithm in [12] and the conitope method in [13]. All of the above algorithms can only deal with arbitrary switching systems. For constrained switching systems, the only existing result known to us is [10], where Legat et al. propose to use the dual solution of a SOS program for generating the desired switching sequences.

In this paper, we consider linear constrained switching systems and propose a novel lift-based approach to find a sequence of matrices with an asymptotic growth rate close to the CJSR. We employ a matrix-based formulation for linear constrained switching systems by using the semi-tensor product of matrices. Under this matrix framework, a constrained switching system $S(\mathcal{A}, \mathcal{M})$ is equivalent to a lifted arbitrary switching system $S(\mathcal{A}_M)$, and the JSR of $S(\mathcal{A}_M)$ is equal to the CJSR of $S(\mathcal{A}, \mathcal{M})$ [7]. Based on the lifted system $S(\mathcal{A}_M)$, we propose an algorithm to find sequences of matrices with the asymptotic growth rate close to the JSR of $S(\mathcal{A}_M)$, or equivalently, the CJSR of the original system $S(\mathcal{A}, \mathcal{M})$. We prove that the sequences generated by the algorithm are always accepted by the
constraining automaton, and the asymptotic growth rate can be made arbitrarily close to the CJSR. Through several numerical examples, we show that our algorithm is able to generate switching sequences with a success rate higher than the algorithm in [10]. Another salient feature of the proposed lift-based method is that it enables one to leverage existing algorithms (such as those in [12] and [13]) that are once only applicable to arbitrary switching systems to constrained switching systems. The remainder of the paper is organized as follows. Section III introduces the formal definitions of JSR and CJSR and the STP-based matrix formulation for arbitrary and constrained switching systems. Section IV presents the approximation algorithm, the theoretical bound of matrix products generated by the algorithm, and several numerical examples. Section V provides some concluding remarks.

Notation. Denote the sets of positive integers and non-negative integers as $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$, respectively. Denote $[m]$ as the set $\{1,2,\ldots,m\}$. Denote $I_n$ as the $n \times n$ identity matrix. Denote $\delta_k$ as the standard basis vector, i.e., a vector of dimension $n$ with 1 in the $k$th coordinate and 0’s elsewhere for $k \in [n]$; denote $\delta_0^n := 0_n$ as the zero vector of dimension $n$; denote $\Delta_n := \{\delta_1^n,\ldots,\delta_n^n\}$, $\Delta'_n := \Delta_n \cup \delta_0^n$, and $\delta_{[i_1,i_2,\ldots,i_m]} := [\delta_{i_1}^{i_2},\delta_{i_2}^{i_3},\ldots,\delta_{i_m}]$ in $\mathbb{R}^{n \times m}$ where $\delta_{i_j}^{j} \in \Delta'_n$, $j \in [m]$.

II. PRELIMINARIES

A. JSR & CJSR

Given a finite set of matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$ and a switching sequence $\sigma = \sigma_1 \ldots \sigma_k$ with $\sigma_1, \ldots, \sigma_k \in [m]$, we define $A_\sigma$ as

$$A_\sigma := A_{\sigma_k} \ldots A_{\sigma_1}. \quad (2)$$

**Definition 1:** [4] The JSR of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) = \limsup_{k \to \infty} \rho_k(\mathcal{A})^{1/k}$ where $\rho_k(\mathcal{A}) = \max_{\sigma \in [m]^k} \|A_\sigma\|$, and $\|\cdot\|$ is any given sub-multiplicative matrix norm on $\mathbb{R}^{n \times n}$.

In this work, the deterministic finite automaton (DFA) is used to represent the constraints on the switching sequences.

**Definition 2:** The DFA $\mathcal{M}$ is a 3-tuple $(Q, U, f)$ consisting of a finite set of states $Q = \{q_1, q_2, \ldots, q_t\}$, a finite set of input symbols $U = \{1, 2, \ldots, m\}$ and a transition function $f : Q \times U \to Q$.

For system (1), we say a finite switching sequence $\sigma = \sigma_1 \ldots \sigma_k$ is accepted by $\mathcal{M}$ or $\mathcal{M}$-accepted if $\sigma_1, \ldots, \sigma_k \in U$ and there exists a finite state sequence $q_{j_1}q_{j_2} \ldots q_{j_{k-1}} \in Q$ and $q_{j_{k-1}} = f(q_{j_k}, \sigma_k)$ are defined for $i = 1, \ldots, k$; an infinite switching sequence accepted by $\mathcal{M}$ is defined similarly by taking $k = \infty$ [6], [14]. Denote the set of switching sequences accepted by $\mathcal{M}$ as $L(\mathcal{M})$.

Formally, the constrained switching system $S(\mathcal{A}, \mathcal{M})$ is the linear switching system as shown in (1) where $A_i \in \mathcal{A}$ for $i \in [m]$ and the switching sequence $\sigma \in L(\mathcal{M})$ [6].

**Definition 3:** [6] The CJSR of a constrained switching system $S(\mathcal{A}, \mathcal{M})$ is defined as

$$\rho(\mathcal{A}, \mathcal{M}) = \limsup_{k \to \infty} \rho_k(\mathcal{A}, \mathcal{M})^{1/k}$$

where $\rho_k(\mathcal{A}, \mathcal{M}) = \max_{\sigma \in [m]^k, \sigma \in L(\mathcal{M})} \|A_\sigma\|$, and $\|\cdot\|$ is any given sub-multiplicative matrix norm on $\mathbb{R}^{n \times n}$.

B. Semi-Tensor Product of Matrices

**Definition 4:** (Def. 1 in [15]) Given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, their semi-tensor product (STP) is defined as

$$A \bowtie B := (A \otimes I_{s/n}) (B \otimes I_{s/p}) \quad (3)$$

where $s$ is the least common multiple of $n$ and $p$, and $\otimes$ is the Kronecker product.

The following two properties of Kronerke product will be used in later sections [16], [17]:

- Given matrices $A \in \mathbb{R}^{nA \times n\Lambda}, B \in \mathbb{R}^{mB \times nB}, C \in \mathbb{R}^{nA \times nC}, D \in \mathbb{R}^{nB \times nD}$, it holds that
  $$\rho(A \bowtie B) = \rho(C \bowtie D) = (AC) \bowtie (BD) \quad (4)$$
- Given two matrices $A \in \mathbb{R}^{nA \times n\Lambda}, B \in \mathbb{R}^{mB \times nB}$ and any sub-multiplicative norm $\|\cdot\|$, it holds that
  $$\|A \bowtie B\| \leq \|A\| \|B\| \quad (5)$$

For any $i \in [m]$, we identify $i$ with $\delta_{m}^i$, denoted as

$$i \sim \delta_{m}^i. \quad (6)$$

For the switching sequence of system (1) at time step $k$, $\sigma_k$, we define its vector form as a column vector $\sigma(k) \in \Delta_m$ and let $\sigma(k) = \delta_m^i$ when $\sigma(k) = i$ where $i \in [m]$. That is, we identify $\sigma_k = i \in [m]$ with $\sigma(k) = \delta_m^i \in \Delta_m$, denoted as $\sigma_k \sim \sigma(k). \quad (7)$

Based on (7), we express a finite switching sequence $\sigma = \sigma_0 \ldots \sigma_{k-1} \in [m]^k$ into its vector form

$$\tilde{\sigma} = \nu^{k-1} \sigma(k-1-i) \in \Delta_{mk} \quad (8)$$

where $\sigma_i \sim \sigma(i)$, $i \in \{0,1,\ldots,k-1\}$.

**Lemma 1:** (Lemma 1 in [7]) If $\sigma_i = j_i \sim \delta_{m}^{j_i}$ where $j_i \in [m], i \in \{0,1,\ldots,k-1\}$, then the sequence $\sigma = \sigma_0 \ldots \sigma_{k-1}$ is identified with its vector form

$$\tilde{\sigma} := \delta_{m}^{j_0} \times \delta_{m}^{j_1} \times \cdots \times \delta_{m}^{j_k} \in \Delta_{mk} \quad (9)$$

where

$$\tau = 1 + \sum_{i=1}^k (j_{k-i} - 1) m^{k-i} \in [m^k]. \quad (10)$$

Conversely, given a vector $\tilde{\sigma} \sim \delta_{m}^{j_0} \times \cdots \times \delta_{m}^{j_k} \in \Delta_{mk}$ where $\tau \in [m^k]$, a set of numbers $j_0,\ldots,j_k \in [m]$ satisfying (10) can be uniquely determined, which corresponds to a switching sequence $\sigma = \sigma_0 \ldots \sigma_{k-1} \in [m^k]$.

C. STP-based Matrix Formulation

Define $x(k) \in \mathbb{R}^n$ as the vector form of state of system (1) by letting $x(k) = x_k$ for any $k \in \mathbb{Z}_{\geq 0}$. As will be shown below, the one-to-one correspondence between the scalar $\sigma_k$ (or, $x_k$) and the vector $\sigma(k)$ (or $x(k)$) is the key to converting the algebraic equation as shown in (1) into the STP-based matrix formulations of DFA, arbitrary switching systems and constrained switching systems.
1) DFAs: We revisit the STP-based matrix expression for the DFA $M$ introduced in [18], [19].

Definition 5: [18] Given a DFA $M = (Q, U, f)$ where $Q = \{q_1, \ldots, q_4\}$, $U = \{1, \ldots, m\}$, its transition structure matrix is defined as

$$F = [F_1, F_2, \ldots, F_m] \in \mathbb{R}^{\ell \times m \ell} \quad (11)$$

where $F_j \in \mathbb{R}^{\ell \times \ell}$ is defined as follows: for $j \in [m],\n$$F_{j(i, \ell)} = \begin{cases} 1, & \text{if } q_k = f(q_j, j); \\ 0, & \text{otherwise}. \end{cases} \quad (12)$$

The DFA $M$ can be considered as a discrete-time dynamical system as follows: given an initial state $q_{0j}$ and an input sequence $\sigma = \sigma_0 \sigma_1 \ldots \sigma_k \sigma_{k+1} \ldots \sigma_{k+1}$ evolves according to $q_{j+1} = f(q_j, \sigma_i)$ if the transition function $f(q_j, \sigma_i)$ is defined, where $q_{0j} = j_0, j_1, \ldots, j_\ell \in [\ell], j_0, j_1, \ldots, j_\ell \in [m]$. We identify each state $q_j \in Q$ with its vector form $\delta^q_j$ where $i \in [\ell]$, denoted as $q_i \sim \delta^q_i$, so that $Q$ is identified with $\Delta_\ell$. Similarly, we identify the input $u \in U$ with its vector form $\delta^\ell u$ where $j \in [m]$, denoted as $u_i \sim \delta^\ell u_i$, so that $U$ is identified with $\Delta_m$. Let $q(k) \in \Delta^q_\ell$ and $\sigma(k) \in \Delta_u_m$ be the vector forms of the state and the input of $M$ at time step $k$, respectively. We let $\sigma_k = \delta^\ell m_k$ for some $\kappa \in [m]$ if the input $\sigma_k = \delta^\ell m_k$; similarly, we let $q(k) = \delta^q_i$ for some $s \in [\ell]$ if the state $q_k = \delta^q_i$ and let $q(k) = \delta^q_i$ if the state $q_k$ is undefined. Note that if $f(q_j, \sigma_i)$ is undefined for some $i \in \mathbb{Z}_{>0}$, then $q_{j+1}, q_{j+1+i}$ are all undefined.

Lemma 2: (Theorem 1 in [18]) The matrix expression of the dynamics of $M$ is

$$q(k+1) = F \times \sigma(k) \times q(k) \quad (13)$$

where $F$ is defined in (11), $q(k) \in \Delta^q_\ell$ and $\sigma(k) \in \Delta_u_m$ are the vector forms of the state and input of $M$, respectively.

Lemma 3: (Corollary 1 in [7]) Given a switching sequence $\sigma = \sigma_0 \ldots \sigma_{k-1} \in [m]^k, \sigma \in L(M)$ if and only if $F_{\sigma_{k-1}} \ldots F_{\sigma_0} \neq 0$.  

2) Arbitrary Switching Systems: The following lemma presents a STP-based matrix formulation for arbitrary switching systems.

Lemma 4: (Proposition 1 in [7]) Given a finite set of matrices $A = \{A_1, \ldots, A_m\}$, with

$$A_1 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.73 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.46 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.73 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.46 \end{pmatrix},$$

and the DFA $M = (Q, U, f)$ is given by $Q = \{q_1, q_2, q_3, q_4\}, U = \{1, 2, 3, 4\}$ with its transition map shown in Fig 1. Using $12$, we can compute the transition structure matrix $F = [F_1, F_2, F_3, F_4]$ where $F_1 = \delta_{\ell} [3, 3, 3, 3], F_2 = \delta_{\ell} [1, 1, 0], F_3 = \delta_{\ell} [2, 0, 2, 0], F_4 = \delta_{\ell} [0, 0, 4, 0]$. We can calculate matrices $\Phi_i = F_i \otimes A_i$ for $i \in [4]$ and define the set of matrices $A_M = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$. Then by Lemma 4, $\rho(A, M) = \rho(AM)$ holds.

3) Constrained Switching Systems

Definition 6: [18] Given a switching system $S(AM)$ can be considered as a lifted system of the constrained switching system $S(A, M)$. The following result from [7] reveals that the CJSR of $S(A, M)$ and the JSR of $S(AM)$ are equivalent.

Lemma 6: (adopted from Theorem 2 in [7]) The following equality holds:

$$\rho(A, M) = \rho(AM) \quad (20)$$

The importance of Lemma 6 lies in that it enables one to convert the problem of approximating the CJSR of $S(A, M)$ into the problem of approximating the JSR of its lifted system $S(AM)$, for which many off-the-shelf algorithms exist. It has been shown in [7] that this lift method can generate the CJSR with significantly higher accuracy within a much shorter computational time.

Example 1: Consider a constrained switching system $S(A, M)$ given in [6] where the set $A = \{A_1, A_2, A_3, A_4\}$ with

$$A_1 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.73 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.46 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.73 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0.94 & 0.56 \\ -0.53 & 0.46 \end{pmatrix},$$

and the DFA $M = (Q, U, f)$ is given by $Q = \{q_1, q_2, q_3, q_4\}, U = \{1, 2, 3, 4\}$ with its transition map shown in Fig 1. Using $12$, we can compute the transition structure matrix $F = [F_1, F_2, F_3, F_4]$ where $F_1 = \delta_{\ell} [3, 3, 3, 3], F_2 = \delta_{\ell} [1, 1, 0], F_3 = \delta_{\ell} [2, 0, 2, 0], F_4 = \delta_{\ell} [0, 0, 4, 0]$. We can calculate matrices $\Phi_i = F_i \otimes A_i$ for $i \in [4]$ and define the set of matrices $A_M = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$. Then by Lemma 4, $\rho(A, M) = \rho(AM)$ holds.

Fig. 1. DFA $M$ in Example 1.
III. Main Results

In this section, we will present Algorithm [1] based on a dual SOS program of the lifted arbitrary switching system \( S(\mathcal{A}_M) \) where \( \mathcal{A}_M \) is defined in [19]. We will prove that the sequence of matrices \( \Phi_{\sigma_1}, \ldots, \Phi_{\sigma_k} \) generated by Algorithm [1] has an asymptotic growth rate \( \lim_{k \to \infty} \| \Phi_{\sigma_1} \cdots \Phi_{\sigma_k} \|_2^p \) that can be made arbitrarily close to \( \rho(\mathcal{A}_M) \), and the switching sequence \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \) is always accepted by the constraining automaton (i.e., \( \sigma \in L(\mathcal{M}) \)). Based on that, we will prove the corresponding sequence of matrices \( \mathcal{A}_{\sigma_1}, \ldots, \mathcal{A}_{\sigma_k} \) has an asymptotic growth rate \( \lim_{k \to \infty} \| \mathcal{A}_{\sigma_1} \cdots \mathcal{A}_{\sigma_k} \|_2^{\frac{p}{2}} \) that can be made arbitrarily close to \( \rho(\mathcal{A}_M, \mathcal{A}_S) \). We will also demonstrate the effectiveness of our algorithm using several numerical examples.

A. Dual SOS Program

Consider the lifted arbitrary switching system \( S(\mathcal{A}_M) \) associated with the original constrained switching system \( S(\mathcal{A}_S, \mathcal{A}_M) \). Based on Theorem 2.2 in [20], the following Program 1 provides a SOS-based approximation algorithm for \( \rho(\mathcal{A}_M) \).

**Program 1:** (Primal)

\[
\inf_{p(x) \in \mathbb{R}_{2d}[x], \gamma \in \mathbb{R}} \gamma \quad \text{s.t.} \quad \begin{align*}
\gamma &> 0, \\
\rho(p(x) - \rho(\Phi_i x)) &\leq \gamma^{2d} & \forall i \in [m],
\end{align*}
\]

where \( \Phi_i \) is defined in [19], \( d \) is a fixed positive integer and \( \mathbb{R}_{2d}[x] \) is the set of homogeneous polynomials of degree \( 2d \) [21].

Denote \( \rho_{SOS,2d}(\mathcal{A}_M) \) as the solution of Program 1. By Theorem 2.2 in [20], a feasible solution of Program 1 provides an upper bound for \( \rho(\mathcal{A}_M) \):

\[
\rho(\mathcal{A}_M) \leq \rho_{SOS,2d}(\mathcal{A}_M) \tag{22}
\]

The dual variables of Program 1 are linear functionals over homogeneous polynomials of degree \( 2d \). The dual of the feasibility version of Program 1 is given by the following Program 2.

**Program 2:** (Dual)

\[
\begin{align*}
&\sum_{i=1}^{m} \tilde{E}_i [p(\Phi_i x)] \geq \gamma^{2d} \sum_{i=1}^{m} \tilde{E}_i [p(x)], \quad \forall p(x) \in \Sigma_{2d}, \\
&\sum_{j=1}^{n} \tilde{E}_i \left[ \sum_{i=1}^{m} x_{ij}^{2d} \right] = 1, \\
&\tilde{E}_i \in \Sigma_{2d}, \quad i \in [m],
\end{align*}
\]

where \( \Sigma_{2d} \) is the cone of homogeneous SOS polynomials of degree \( 2d \), \( \Sigma_{2d} \) is the dual of \( \Sigma_{2d} \), and \( \tilde{E}_i \) is the pseudo-expectation. Here, the pseudo-expectation can be seen as the expectation of a pseudo-distribution \( \tilde{\mu} \) on \( \mathbb{R}^n \) and satisfies:

\[
\langle \tilde{E}_i, p(x) \rangle = \tilde{E}_i [p(x)] = \int_{\mathbb{S}^{n-1}} p(x) d\tilde{\mu}
\]

where \( \mathbb{S}^{n-1} \) is the \((n-1)\)-dimensional sphere [10], [22]. Since \( p(x) \) is a homogeneous SOS polynomials of degree \( 2d \) and \( \tilde{E}_i \) belongs to the dual of the SOS cone, \( \tilde{E}_i [p(x)] \geq 0 \) holds.

Note that for any matrix \( \Phi_i \in \mathbb{R}^{n \times n} \), polynomial \( p(x) \) depends linearly on \( p(\Phi_i x) \). So we can define \( \mathcal{A}_M^* \) as the adjoint linear map such that \( \langle \mathcal{A}_M^* \tilde{E}_i, p(x) \rangle = \langle \tilde{E}_i, p(\Phi_i x) \rangle \) for all \( \tilde{E}_i \in \Sigma_{2d} \) and \( p(x) \in \Sigma_{2d} \). Therefore, the dual constraint (23) is equivalent to:

\[
\sum_{i=1}^{m} \mathcal{A}_M^* \tilde{E}_i \geq \gamma^{2d} \sum_{i=1}^{m} \tilde{E}_i, \quad \tilde{E}_i \in \Sigma_{2d}^*. \tag{26}
\]

Thus, with any positive integer \( d \) and any \( \gamma < \rho_{SOS,2d}(\mathcal{A}_M) \), a set of dual variables \( \{ \tilde{E}_1, \ldots, \tilde{E}_m \} \) can be obtained by solving Program 2 without specifying \( p(x) \).

**Remark 1:** In this work, Program 1 is mainly used to introduce its dual problem, Program 2. We will use other algorithms, rather than Program 1, to approximate \( \rho(\mathcal{A}_M) \). For example, consider the system \( S(\mathcal{A}_M) \) and its lifted system \( S(\mathcal{A}_M) \) shown in Example 1. By choosing the Gripenbergs algorithm and the conitope algorithm in the jsr function of the JSR toolbox in [23], we obtain the following bounds in 8.9 seconds in a computer with 3.7 GHz CPU and 32GB memory:

\[
0.974817197937 \leq \rho(\mathcal{A}_M) \leq 0.974817295434. \tag{27}
\]

If we choose \( 2d = 2 \) and use the jsr.opti sos function that employs Program 1, then it takes about 2.0 seconds to obtain the following bounds on the same computer:

\[
0.591401347649 \leq \rho(\mathcal{A}_M) \leq 1.18398668198. \tag{28}
\]

If we choose \( 2d = 4 \), then it takes about 108 seconds to obtain the following bounds:

\[
0.696738367695 \leq \rho(\mathcal{A}_M) \leq 0.986323172193. \tag{29}
\]

Clearly, the bounds shown in (27) is more accurate than (29) with a much shorter computational time.

**Remark 2:** Compared with the dual SOS program in [10], Program 2 above contains a smaller number of pseudo-expectations, but involves \( x \) with a larger dimension because the matrix \( \Phi_i \in \mathbb{R}^{n \times n} \) has a larger dimension than \( A_i \in \mathbb{R}^{n \times n} \). Therefore, the SOS constraint corresponding to (23) involves a higher order of SOS polynomials, which renders the corresponding SDP more difficult to solve, especially when the degree \( 2d \) is chosen to be large. A possible method to alleviate this issue is to replace the dual constraint (23) with the diagonally-dominant-sum-of-squares (DSOS) constraint or the scaled-diagonally-dominant-sum-of-squares (SDDSOS) constraint [24]. Nonetheless, the numerical examples in Section III-C will show that Program 2 can normally generate satisfying results with small values of \( d \). Other advantages of Program 2 will be discussed in later subsections.

B. Sequence Generating Algorithm

Given a set of matrices \( \mathcal{A}_M = \{ \Phi_1, \ldots, \Phi_m \} \) with \( \Phi_i \) defined in [19] and a set of dual variables \( \{ \tilde{E}_1, \ldots, \tilde{E}_m \} \) that are obtained by solving Program 2, Algorithm 1 below...
generates a switching sequence $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ such that the product of matrices $\Phi_1 \cdots \Phi_{\sigma_k}$ has an asymptotic growth rate close to $\rho(A_{\mathcal{M}})$. Algorithm 1 takes a positive integer $h$ as an input parameter, which can be considered as the “horizon” of the algorithm. Instead of generating each switching mode $\sigma_i$ one by one, Algorithm 1 generates $h$ switching modes simultaneously at each iteration step. Therefore, the total length of sequences generated by the algorithm, $k$, will be a multiple of $h$.

**Algorithm 1:** Generate a sequence of matrices $\Phi_{\sigma_1}, \cdots, \Phi_{\sigma_k}$ with an asymptotic growth rate close to $\rho(A_{\mathcal{M}})$

**Input:** a set of matrices $\{\Phi_1, \ldots, \Phi_m\}$, a set of dual variables $\{\bar{E}_1, \ldots, \bar{E}_m\}$, three positive integers $h, k, d$ where $k$ is a multiple of $h$

**Output:** a switching sequence $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$

Choose an arbitrary polynomial $p_0(x) \in \text{int}(\Sigma_{2d})$ Set $p_1(x) \leftarrow p_0(x)$

for $i = 1, h + 1, 2h + 1, \ldots, k - h + 1$

\[ \sigma_1, \ldots, \sigma_{i+h-1} \leftarrow \arg \max_{\sigma_1, \ldots, \sigma_{i+h-1}} \bar{E}_{\sigma_h} \left[ p_i(\Phi_{\sigma_1} \cdots \Phi_{\sigma_{i+h-1}}) \right] \]

\[ p_{i+h}(x) \leftarrow p_i(\Phi_{\sigma_1} \cdots \Phi_{\sigma_{i+h-1}}) \]

end

To start with, we choose an arbitrary polynomial $p_0(x)$ in the interior of the cone of SOS homogeneous polynomials of degree $2d$, i.e. $p_0(x) \in \text{int}(\Sigma_{2d})$. Given the set of matrices $\{\Phi_1, \ldots, \Phi_m\}$ and the set of dual variables $\{\bar{E}_1, \ldots, \bar{E}_m\}$ computed from Program 2 we define

\[ \theta_k := \bar{E}_{\sigma_k} \left[ p_0(\Phi_{\sigma_1} \cdots \Phi_{\sigma_k}) \right]. \] \hspace{1cm} (30)

Then, the “for loop” generates a switching sequence such that $\theta_k$ remains large for increasing $k$. Algorithm 1 terminates with $i = k - h + 1$, from which we obtain

\[ p_{k+1}(x) = p_k(\Phi_{\sigma_1} \cdots \Phi_{\sigma_k}). \] \hspace{1cm} (31)

Note that the order of mode subscripts in the matrix product of Algorithm 1 and (30), (31) as well, is reversed from that of (29).

The following lemma provides an inequality on $\theta_k$ using the dual constraint (23).

**Lemma 7:** Consider a finite set of matrices $\mathcal{A}_M = \{\Phi_1, \ldots, \Phi_m\}$. For any $d, h \in \mathbb{Z}_{>0}$ and any $\gamma < \rho_{SOS, 2d}(\mathcal{A}_M)$, Algorithm 1 produces a sequence that satisfies the following inequality for all $k \in \mathbb{Z}_{>0}$ that is a multiple of $h$:

\[ \theta_k \geq \frac{\gamma^{2dh}}{m^h} \theta_{k-h} \]

**Proof:** Since $p_0(x)$ is SOS, $p_{k-h+1}(x)$ is also SOS. Using (23), we have

\[ \sum_{\sigma_1, \ldots, \sigma_{k-h} \in [m]^h} \bar{E}_{\sigma_h} \left[ p_{k-h+1}(\Phi_{\sigma_1} \cdots \Phi_{\sigma_h}) \right] \]

\[ \geq \gamma^{2dh} \sum_{\sigma_1, \ldots, \sigma_{h-1} \in [m]^{h-1}} \bar{E}_{\sigma_{h-1}} \left[ p_{k-h+1}(\Phi_{\sigma_1} \cdots \Phi_{\sigma_{h-1}}) \right] \]

\[ \vdots \]

\[ \geq \gamma^{2dh} \theta_{k-h} \]

Since the left-hand side expression has $m^h$ positive terms and Algorithm 1 picks the term with the highest value as $\theta_k$, we have $m^h \theta_k \geq \gamma^{2dh} \theta_{k-h}$. The conclusion follows immediately.

The following two lemmas are from [10].

**Lemma 8:** [10] For any matrix $\Phi \in \mathbb{R}^{n \times n}$ and symmetric positive semidefinite matrix $P$, the following inequality holds:

\[ \rho(\Phi^T P \Phi) \leq \rho(P) \rho(\Phi^T \Phi) \] \hspace{1cm} (32)

**Lemma 9:** [10] For any polynomial $p(x) \in \Sigma_{2d}$ and any matrix $\Phi \in \mathbb{R}^{n \times n}$, there exists a positive constant $\beta$ that does not depend on $\Phi$ such that

\[ \beta \|p(\Phi)x - p(\Phi)x\| \leq \text{SOS}. \]

**Proposition 1:** Consider a finite set of matrices $\mathcal{A}_M = \{\Phi_1, \ldots, \Phi_m\}$. For any $d, h \in \mathbb{Z}_{>0}$ and any $\gamma < \rho_{SOS, 2d}(\mathcal{A}_M)$, Algorithm 1 produces a switching sequence $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ such that the following inequality holds:

\[ \lim_{k \to \infty} \|\Phi_{\sigma_1} \cdots \Phi_{\sigma_k}\| \geq \frac{\gamma}{m^h} \] \hspace{1cm} (34)

**Proof:** Since there are finite switching modes, there must be a mode $\bar{\sigma} \in [m]$ that appears infinitely many times in the sequence at multiples of $h$. Let $k_1$ be the smallest multiple of $h$ such that $\sigma_{k_1} = \bar{\sigma}$ and let $g(x) = p_{k_1}(x)$. Since $p_0(x) \in \text{int}(\Sigma_{2d})$, we know that $g(x) \in \text{int}(\Sigma_{2d})$.

For an arbitrarily large integer $K$, there exists a $k \geq K$ and $k$ is a multiple of $h$ such that $\sigma_{k+h} = \bar{\sigma}$. Let $s_k = (\sigma_{k_1+1}, \ldots, \sigma_{k_1+k})$. By Lemma 7 we have

\[ \bar{E}_{\sigma} \left[ g(\Phi_{s_k} x) \right] \geq \frac{\gamma^{2dk}}{m^h} \bar{E}_{\sigma} [\Phi_{s_k} x]. \]

where $\Phi_{s_k} := \Phi_{s_{k_1+1}} \cdots \Phi_{s_{k_1+k}}$. By Lemma 9 there exists a constant $\beta$ that does not depend on $\Phi_{s_k}$ such that

\[ \beta \|\Phi_{s_k}\|_2 g(x) - g(\Phi_{s_k} x) \leq \text{SOS}. \]

Therefore,

\[ \bar{E}_{\sigma} \left[ \beta \|\Phi_{s_k}\|_2 g(x) \right] \geq \bar{E}_{\sigma} [g(\Phi_{s_k} x)] \] \hspace{1cm} (37)

and by (35),

\[ \beta \|\Phi_{s_k}\|_2 \bar{E}_{\sigma} [g(x)] \geq \frac{\gamma^{2dk}}{m^h} \bar{E}_{\sigma} [g(x)] \] \hspace{1cm} (38)

Since $\bar{E}_{\sigma} [g(x)] > 0$, we divide both sides of (38) by $\bar{E}_{\sigma} [g(x)]$ and get

\[ \beta \|\Phi_{s_k}\|_2 \geq \frac{\gamma^{2dk}}{m^h} \] \hspace{1cm} (39)
or equivalently,
\[ \| \Phi_{s_k} \|_2^2 \geq \frac{\gamma}{\beta^2 m} \]  

(Taking the limit of \( K \) to \( \infty \), we have)
\[ \lim_{k \to \infty} \| \Phi_{s_{k+1}} \cdots \Phi_{s_{k+t}} \|_2^2 \geq \frac{\gamma}{m^{\frac{1}{2}}} . \]  

(41)

Using the fact that \( \lim_{k \to \infty} \| \Phi_{s_k} \|_2 \) is accepted by the DFA \( M \), we have
\[ \lim_{k \to \infty} \| A_{s_1} \cdots A_{s_k} \|_2^2 = \lim_{k \to \infty} \| \Phi_{s_1} \cdots \Phi_{s_k} \|_2^2 \geq \frac{\gamma}{m^{\frac{1}{2}}} \]  

Therefore, by (42), we have
\[ \lim_{k \to \infty} \| A_{s_1} \cdots A_{s_k} \|_2^2 = \lim_{k \to \infty} \| \Phi_{s_1} \cdots \Phi_{s_k} \|_2^2 \geq \frac{\gamma}{m^{\frac{1}{2}}} \]  

where the last inequality is from Proposition 1. This completes the proof.

In practice, switching sequences generated by Algorithm 1 are periodic after some step. Therefore, instead of generating sequences of the maximum length by Algorithm 1 we can compute the average spectral radius for all the cycles with length smaller than some maximum length and choose the cycle corresponding to the largest average spectral radius. Formally,
\[ \rho_T(A,M) \leq \rho(A,M) \]

where \( \rho_T(A,M) = \max\{\rho(A_{c_1}, \ldots, A_{c_{\ell}})^{1/\ell} : c = c_1 c_2 \cdots c_\ell \in L(M), \ell \) is a cycle\}. The spectral radius of the matrices corresponding to the cycle provides a lower bound for \( \rho(A,M) \).

**Remark 3:** The equivalence between the constrained switching system \( S(A,M) \) and the lifted arbitrary switching system \( S(A,M) \) also allows us to leverage some existing algorithms that can generate high-growth sequences for arbitrary switching systems. For example, the Gripenberg algorithm in [11] and the conitope algorithm in [13] both search for the spectrum maximizing product to approximate the lower bounds of the JSR.

### C. Numerical Examples

In [10], algorithms are proposed for generating a sequence of matrices with a high asymptotic growth rate based on a dual SOS program for \( S(A,M) \). In this subsection, we will use several numerical examples to illustrate the effectiveness of the lift-based method proposed in this work and its comparisons with the method in [10].

**Example 2:** Consider a constrained switching system \( S(A,M) \) where the set \( A \) and the DFA \( M \) are given in Example 1. We apply Algorithm 1 to the lifted system \( S(A,M^l) \) to generate a switching sequence of a given length and look through all the circles in the sequence. The optimal cycle found depends on the initial choice of parameters, but most of the time we conclude that the first cycle coincides with the second one if the resulting labels “1,” ”3” of the first cycle are moved to the end.

\[
1, 1, 2, 1, 2, 3, 1, 1, 1, 1, 1 \\
1, 2, 1, 2, 3, 1, 1, 1, 1, 1
\]  

whose 8th roots of the corresponding spectral radius are both 0.9748717937. The average computation time of Algorithm 1 is 0.152 seconds on the computer with 3.7 GHz CPU and 32GB memory. Note that the value of this spectral radius is equal to the lower bound of \( \rho(A,M) \) given in [27]. Also note that the two cycles shown in (44) are essentially the same since the first cycle coincides with the second one if the beginning labels “1,” ”1” of the first cycle are moved to the end.
Example 7 in [10] considers the same constrained switching system \( S(\mathcal{A}, \mathcal{M}) \) as above. By using a dual SOS program for \( S(\mathcal{A}, \mathcal{M}) \), the following \( \mathcal{M} \)-accepted cycle is produced:

\[(3, 1, 2), (1, 3, 1), (3, 1, 2), (2, 1, 3), (2, 3, 1), (3, 3, 1)^3 \quad (45)\]

where the triplet \((u, v, w)\) denotes the edge between node \(u\) and node \(v\) with label \(w\) in the automaton \(\mathcal{M}\), and “3” in the exponent means that the edges is taken 3 times. The average computation time of Algorithm 2 in [10] is 0.0848 seconds, which is comparable with that of Algorithm [1].

The spectral radius corresponding to the cycle (45) is the same as that of cycle (44). However, the algorithms in [10] only produce one unique path that is accepted by \(\mathcal{M}\) since all the starting nodes and ending nodes have been specified; in comparison, Algorithm [1] in this work generates a set of \(\mathcal{M}\)-accepted sequences that have the same order of edge labels as Algorithm [1] only specifies the labels of edges - this salient feature is because the constrained switching system is lifted into an associated arbitrary switching system. For example, the cycle (44) may correspond to other path of the automaton \(\mathcal{M}\) such as (2, 1, 2), (1, 3, 1), (3, 1, 2), (2, 1, 2), (2, 3, 1), (3, 3, 1)^3. □

In both Algorithm [1] above and the algorithms in [10], an arbitrary polynomial \(p_0(x) \in \text{int}(\Sigma_{2d})\) needs to be picked. Compared with the algorithms in [10], Algorithm [1] proposed in this work tends to have a higher success rate in generating \(\mathcal{M}\)-accepted cycles with a larger asymptotic growth rate. The higher success rate of our algorithm might come from the fact that it only specifies the labels of edges, which is less restrictive in describing the switching sequences than algorithms in [10], as explained in Example 2.

**Example 3:** Consider a constrained switching system \( S(\mathcal{A}, \mathcal{M}) \) where the set \(\mathcal{A} = \{A_1, A_2, A_3, A_4\} \) is given by

\[
\begin{align*}
A_1 &= \begin{pmatrix} 0.55 & -0.69 \\ 0.43 & 0.25 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0.77 & 0.31 \\ -0.28 & 0.41 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} -0.86 & -0.63 \\ -0.95 & -0.79 \end{pmatrix}, & A_4 &= \begin{pmatrix} 0.16 & 0.44 \\ -0.14 & 0.55 \end{pmatrix},
\end{align*}
\]

and the DFA \(\mathcal{M} = (Q, U, f)\) with \(Q = \{q_1, q_2, q_3, q_4\}\), \(U = \{1, 2, 3, 4\}\) and its transition map shown in Fig 2. Note that this DFA is not strongly connected since there is no path from states \(q_2\) and \(q_3\) to either \(q_1\) or \(q_4\). According to Definition [5] the transition structure matrix is given by \(F = [F_1 \ F_2 \ F_3 \ F_4]\), with \(F_1 = \delta_1[0, 3, 3, 3], F_2 = \delta_2[1, 0, 1, 1], F_3 = \delta_3[2, 0, 2, 0], F_4 = \delta_4[0, 0, 0, 0]\). Again we can calculate matrices \(\Phi_i = F_i \otimes A_i, i \in [4]\) and define the set of matrices \(\mathcal{A}_M = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}\). Then we have \(\rho(\mathcal{A}, \mathcal{M}) = \rho(\mathcal{A}_M)\) by Lemma [6]. By choosing the Gripenbergs algorithm and the conitope algorithm in the jsr function of the JSR toolbox in [23], we obtain the following bounds for \(\rho(\mathcal{A}, \mathcal{M})\):

\[0.841354205739 \leq \rho(\mathcal{A}, \mathcal{M}) \leq 0.841354286369. \quad (46)\]

By choosing \(h = 1\) and \(2d = 2\), we run Algorithm [1] 100 times by randomly picking a polynomial \(p_0(x) \in \text{int}(\Sigma_{2d})\).

\[\frac{1}{\rho(\mathcal{A}, \mathcal{M})} \approx \frac{1}{0.841354205739} = 0.841354205739. \quad (47)\]

Algorithm [1] always finds the following cycle for the lifted system \(S(\mathcal{A}_M)\):

\[3, 1, 1, 1 \quad (47)\]

whose 4th root of the corresponding spectral radius is equal to 0.841354205739. The average computation time for each run is 0.143 seconds. Note that the cycle (47) is accepted by \(L(\mathcal{M})\). Therefore, the asymptotic growth rate \(\lim_{k \to \infty} \|A^3\|^k\|_2\) is equal to 0.841354205739, which is exactly the lower bound of \(\rho(\mathcal{A}, \mathcal{M})\) found in (46).

As a comparison, we use Algorithm 2 in [10] to produce a sequence of matrices with a high asymptotic growth rate for the same \(S(\mathcal{A}, \mathcal{M})\). By choosing \(h = 1\) and \(2d = 2\) as above, we run Algorithm 2 in [10] 100 times by randomly picking a polynomial \(p_0(x) \in \text{int}(\Sigma_{2d})\). It turns out that 48 times that algorithm finds the same sequence as (47) while 52 times that algorithm finds other sequences with a smaller average spectral radius. And the average computation time for each run is 0.094 seconds. The simulation result remains unchanged even if we increase \(h\), the algorithm “horizon” in Algorithm 2, to 5.

**Example 4:** Consider another constrained switching system \(S(\mathcal{A}, \mathcal{M})\) where the set \(\mathcal{A} = \{A_1, A_2, A_3, A_4\}\) is

\[
\begin{align*}
A_1 &= \begin{pmatrix} 0.95 & -0.03 \\ 0.43 & 0.51 \end{pmatrix}, & A_2 &= \begin{pmatrix} -0.71 & 0.91 \\ 0.28 & 0.69 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} -0.66 & 0.67 \\ 0.13 & -0.76 \end{pmatrix}, & A_4 &= \begin{pmatrix} -0.64 & 0.57 \\ -0.81 & 0.12 \end{pmatrix},
\end{align*}
\]

and the DFA \(\mathcal{M} = (Q, U, f)\) with \(Q = \{q_1, q_2, q_3, q_4\}\), \(U = \{1, 2, 3, 4\}\) and its transition map as shown in Fig 3. Similarly to Example 3 we can compute the transition structure matrices \(F_1 = \delta_1[0, 0, 3, 3], F_2 = \delta_2[1, 1, 1, 1], F_3 = \delta_3[2, 0, 0, 0], F_4 = \delta_4[4, 4, 4, 4]\). The set of matrices \(\mathcal{A}_M = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}\) is calculated by \(\Phi_i = F_i \otimes A\). Again by Lemma [6] we have \(\rho(\mathcal{A}, \mathcal{M}) = \rho(\mathcal{A}_M)\). Using the Gripenbergs algorithm and the conitope algorithm in the jsr function of the JSR toolbox in [23], we obtain the following bounds for \(\rho(\mathcal{A}, \mathcal{M})\):

\[1.013964084304 \leq \rho(\mathcal{A}, \mathcal{M}) \leq 1.013964185777. \quad (48)\]
To evaluate the performance of Algorithm 1 proposed in this work and Algorithm 2 in [10], we run both algorithms with $2d = 2$ and $h = 1$, $h = 2$, $h = 5$ for 100 times separately. The $M$-accepted cycle with the largest average spectral radius found among all the tests is

$$1, 4$$

whose 2nd root of the corresponding spectral radius is $1.013964084304$. The success rate that our Algorithm 1 generates cycle (49) is 40% when $h = 1$ and 100% when $h = 2$ or $h = 5$. However, Algorithm 2 in [10] can not find cycle (49) when $h = 1$ or $h = 2$ and finds the cycle with 6% success rate when $h = 5$. The average computation time for each run of the proposed Algorithm 1 and Algorithm 2 in [10] are comparable; however, it is evident that our algorithm has a much higher success rate in generating $M$-accepted cycles.

IV. CONCLUSION

In this paper, we proposed a novel algorithm, based on the matrix-form expression of the constrained switching system $S(A, M)$ and the dual solution of the SOS approximation, to generate a sequence of matrices with an asymptotic growth rate close to $\rho(A, M)$. We proved that the high asymptotic growth rate sequence generated by our algorithm for the lifted arbitrary switching system $S(A, M)$ is an equivalently satisfying sequence for the original constrained switching system $S(A, M)$. We also showed that compared with other existing algorithms, the lift-based algorithm has a higher success rate of generating a sequence of matrices with a large asymptotic growth rate in several cases. In future work, we plan to develop more efficient algorithms on the closeness of the asymptotic growth rate to $\rho(A, M)$ by incorporating the proposed lifting method and other existing JSR approximation algorithms.

REFERENCES

[1] H. Lin and P. J. Antsaklis, “Stability and stabilizability of switched linear systems: a survey of recent results,” *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, 2009.

[2] D. Liberzon, *Switching in systems and control*. Boston: Birkhauser, 2003.

[3] G.-C. Rota and W. Strang, “A note on the joint spectral radius,” *Proceedings of the Netherlands Academy*, vol. 22, pp. 379–381, 1960.

[4] R. Jungers, *The joint spectral radius: theory and applications*, ser. Lecture Notes in Control and Information Sciences. Berlin: Springer-Verlag, 2009, vol. 385.

[5] X. Dai, “A Gelfand-type spectral radius formula and stability of linear constrained switching systems,” *Linear Algebra and Its Applications*, vol. 436, no. 5, pp. 1099–1113, 2012.

[6] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers, “Stability of discrete-time switching systems with constrained switching sequences,” *Automatica*, vol. 72, pp. 242–250, 2016.

[7] X. Xu and B. Acikmese, “Approximation of the constrained joint spectral radius via algebraic lifting,” *IEEE Transactions on Automatic Control*, 2020, in press. [Online]. Available: [https://ieeexplore.ieee.org/document/9184211]

[8] Y. Wang, N. Roohi, G. E. Dullerud, and M. Viswanathan, “Stability analysis of switched linear systems defined by regular languages,” *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2568–2575, 2017.

[9] V. Kozjakin, “The Berger-Wang formula for the Markovian joint spectral radius,” *Linear Algebra and Its Applications*, vol. 448, pp. 315–328, 2014.

[10] B. Legat, R. M. Jungers, and P. A. Parrilo, “Generating unstable trajectories for switched systems via dual sum-of-squares techniques,” in *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, 2016, pp. 51–60.

[11] G. Gripenberg, “Computing the joint spectral radius,” *Linear Algebra and Its Applications*, vol. 234, pp. 43–60, 1996.

[12] N. Guglielmi and M. Zennaro, “An algorithm for finding extremal polytope norms of matrix families,” *Linear Algebra and Its Applications*, vol. 428, no. 10, pp. 2265–2282, 2008.

[13] R. M. Jungers, A. Cicone, and N. Guglielmi, “Lifted polytope methods for computing the joint spectral radius,” *SIAM Journal on Matrix Analysis and Applications*, vol. 35, no. 2, pp. 391–410, 2014.

[14] S. Eilenberg, *Automata, languages, and machines*. Academic press, New York, 1974.

[15] D. Laschos and M. Margaliot, “Controllability of boolean control networks via the perron–frobenius theory,” *Automatica*, vol. 48, no. 6, pp. 1218–1223, 2012.

[16] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, 1990.

[17] P. Lancaster and H. Falahat, “Norms on direct sums and tensor products,” *Mathematics of Computation*, vol. 26, no. 118, pp. 401–414, 1972.

[18] X. Xu and Y. Hong, “Matrix expression and reachability analysis of finite automata,” *Journal of Control Theory and Applications*, vol. 10, no. 2, pp. 210–215, 2012.

[19] ———, “Matrix approach to model matching of asynchronous sequential machines,” *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2974–2979, 2013.

[20] P. A. Parrilo and A. Jadbabaie, “Approximation of the joint spectral radius using sum of squares,” *Linear Algebra and Its Applications*, vol. 428, no. 10, pp. 2385–2402, 2008.

[21] P. A. Parrilo, “Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization,” Ph.D. dissertation, California Institute of Technology, 2000.

[22] B. Barak, F. G. Brandao, A. W. Harrow, J. Kelner, D. Steurer, and Y. Zhou, “Hypercontractivity, sum-of-squares proofs, and their applications,” in *Proceedings of the 44th annual ACM Symposium on Theory of Computing*, 2012, pp. 307–326.

[23] G. Vankeerberghen, J. Hendrickx, and R. M. Jungers, “JSR: A toolbox to compute the joint spectral radius,” in *Proceedings of the 17th International Conference on Hybrid Systems: Computation and Control*, 2014, pp. 151–156.

[24] A. A. Ahmadi and A. Majumdar, “DSOS and SDSOS optimization: Lp and SOCP-based alternatives to sum of squares optimization,” in *48th Annual Conference on Information Sciences and Systems (CISS)*. IEEE, 2014, pp. 1–5.