PROJEKTIVE MODULI FOR HITCHIN PAIRS

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INTRODUCTION

In the paper [2], Hitchin studied pairs \((E, \varphi)\), where \(E\) is a vector bundle of rank two with a fixed determinant on a curve \(C\) and \(\varphi: E \to E \otimes K_C\) is a trace free homomorphism, and constructed a moduli space for them. This moduli space carries the structure of a non-complete, quasi-projective algebraic variety. Later, Nitsure [5] gave an algebraic construction of moduli spaces of pairs \((E, \varphi)\) over a curve \(C\) consisting of a vector bundle \(E\) of fixed degree and rank and a homomorphism \(\varphi: E \to E \otimes L\) where \(L\) is some previously chosen line bundle. He also obtained non-complete moduli spaces. The most general results were obtained by Yokogawa [7]. In his paper, \(C\) is replaced by a relative scheme \(f: X \to S\) where \(f\) is a smooth, projective, geometrically integral morphism and \(S\) is a scheme of finite type over a universally Japanese ring, and \(L\) by a locally free sheaf \(F\) on \(X\).

It is the aim of our paper to compactify some of the spaces obtained by Yokogawa, namely those where \(S = \text{Spec} \mathbb{C}\) and \(F\) is again a line bundle. In order to avoid confusion with the objects studied e.g. by Simpson, we will call our objects \((\text{oriented})\) Hitchin pairs.

We shall also mention that, only recently, T. Hausel compactified the space of oriented Hitchin pairs of rank two with fixed determinant over a curve \(C\), using methods from symplectic geometry. This result and a detailed investigation of the resulting spaces will appear in a forthcoming preprint of his.

The structure of this note is as follows: In the first section we treat the case where \(X\) is a point. This case shows how to define Hitchin pairs correctly and suggests the definition of (semi)stability. Then we prove a boundedness result following [5], construct a projective parameter space for semistable Hitchin pairs and a universal family on this parameter space, and finally define a linearized \(\text{SL}(V)\)-action on this parameter space such that the moduli space is given as parameter space \(// \text{SL}(V)\). After these constructions, we prove the (semi)stability criterion.
At some places, the techniques of our notes are similar to those in [3]. Hence, we often omit or sketch only briefly arguments which were carried out in detail in [3] in an analogous situation.

Acknowledgements

I want to thank Professor Okonek and Dr. A. Teleman for suggesting the problem and discussing various details of the proof with me. The author acknowledges support by AGE — Algebraic Geometry in Europe, contract No. ER-BCHRXT 940557 (BBW 93.0187), and by SNF, Nr. 2000 – 045209.95/1.

1. Compactifying the categorical quotient of a vector space

Let $G$ be a reductive algebraic group acting linearly on a vector space $V$. Consider the categorical quotient $W := V//G = \text{Spec } C[V]^G$. The torus $C^*$ acts canonically on $V$, and this action commutes with the given action of $G$. Now, let $G$ act trivially on $C$ and let $C^*$ act on $C$ by multiplication. We obtain a $(G \times C^*)$-action on $V \oplus C$. Observe that the equivalence relation induced by the given action is the following:

$$(v_1, \varepsilon_1) \sim (v_2, \varepsilon_2) \iff \exists z \in C^*, g \in G : v_2 = z \cdot (g \cdot v_1); \varepsilon_2 = z \cdot \varepsilon_1.$$ 

The point is $(v, \varepsilon) \in V \oplus C$ is semistable if and only if $[v, \varepsilon] \in \mathbb{P}(V \oplus C)$ is $G$-semistable. By the Hilbert criterion, the latter happens if and only if either $\varepsilon \neq 0$ or $v \in V$ is $G$-semistable. The space $(V \oplus C)//(G \times C^*) = \mathbb{P}(V \oplus C)//G$ obviously is a projective variety containing $W$ as an open affine subvariety. Let $W^{ss}$ be the image of the $G$-semistable points in $V$. The $C^*$-action on $V$ induces a $C^*$-action on $W^{ss}$. We observe that we have compactified $W$ with $W^{ss}//C^*$.

Applying the above discussion to the case $G = \text{SL}_n(C)$ and $V = M_n(C)$ (this is the case of Hitchin pairs over a point) shows that $(m, \varepsilon) \in M_n(C) \oplus C$ is semistable if and only if either $\varepsilon \neq 0$ or $m$ is not nilpotent.

Remark 1.1. Comparing this with [3], Thm.2.8, for $r = p$ and $N = 1$ shows that the semistability criterion stated there is false for points at infinity.

Our general construction is basically a relative version of the above over a projective scheme.

2. Hitchin pairs

Throughout this paper, we will work over the field of complex numbers. Let $X$ be a smooth projective variety of dimension $n$. If $n > 1$, we fix an ample divisor $H$ on $X$ whose associated line bundle will be denoted by $O_X(1)$. We will use $H$ to compute degrees and Hilbert polynomials. The Hilbert polynomial of a coherent sheaf $\mathcal{F}$ will be denoted by $P_{\mathcal{F}}$. We also fix a line bundle $L$ and a Hilbert polynomial $P$. The degree and the rank given by $P$ will be denoted by $d$ and $r$, respectively. Let Pic$(X)$ be the Picard scheme of $X$. We fix a Poincaré sheaf
\[ \mathcal{L} \text{ on } \text{Pic}(X) \times X. \] Furthermore, for a coherent sheaf \( \mathcal{E} \), set \( \mathcal{L}[\mathcal{E}] := \mathcal{L}[\{[\det \mathcal{E}]\} \times X]. \) This sheaf depends only on the isomorphy class of \( \mathcal{E} \). Unlike the situation in [6], the sheaf \( \mathcal{L} \) will play no essential rôle in our considerations.

2.1. Oriented Hitchin Pairs. An oriented Hitchin pair of type \((\mathcal{L}, P, L)\) is a triple \((\mathcal{E}, \sigma, \varphi)\) consisting of a torsion free coherent sheaf \( \mathcal{E} \) with \( P_\mathcal{E} = P \), a homomorphism \( \sigma : \det(\mathcal{E}) \to \mathcal{L}[\mathcal{E}] \), and a homomorphism \( \varphi : \mathcal{E} \to \mathcal{E} \otimes L \). Two oriented Hitchin pairs \((\mathcal{E}_1, \sigma_1, \varphi_1)\) and \((\mathcal{E}_2, \sigma_2, \varphi_2)\) of type \((P, \mathcal{L}, L)\) are called equivalent, if there is an isomorphism \( \psi : \mathcal{E}_1 \to \mathcal{E}_2 \) such that \( \varphi_2 \circ \psi = (\psi \otimes \text{id}_L) \circ \varphi_1 \) and \( \sigma_1 = \sigma_2 \circ \det \psi \).

Remark 2.1. Of course, we can fix a line bundle \( \mathcal{L}_0 \) on \( X \) and consider oriented Hitchin pairs \((\mathcal{E}, \sigma, \varphi)\) such that \( \det \mathcal{E} \cong \mathcal{L}_0 \). Our proofs carry over to this situation.

Now, consider pairs \((\mathcal{E}, \varphi)\) consisting of a torsion free coherent sheaf \( \mathcal{E} \) with \( P_\mathcal{E} = P \) and a homomorphism \( \varphi : \mathcal{E} \to \mathcal{E} \otimes L \). We say that \((\mathcal{E}_1, \varphi_1)\) is equivalent to \((\mathcal{E}_2, \varphi_2)\) if and only if there is an isomorphism \( \psi : \mathcal{E}_1 \to \mathcal{E}_2 \) fulfilling \( \varphi_2 \circ \psi = (\psi \otimes \text{id}_L) \circ \varphi_1 \). Given a pair \((\mathcal{E}, \varphi)\), we can choose a non-zero orientation \( \sigma : \det \mathcal{E} \to \mathcal{L}[\mathcal{E}] \) in order to obtain an oriented Hitchin pair. We observe that the equivalence class of \((\mathcal{E}, \sigma, \varphi)\) does not depend on the choice of the orientation \( \sigma \). Therefore, we call a pair \((\mathcal{E}, \varphi)\) as above an oriented Hitchin pair of type \((\mathcal{L}, P)\).

Let \( S \) be a noetherian scheme. A family of oriented Hitchin pairs of type \((\mathcal{L}, P)\) parametrized by \( S \) is a pair \((\mathcal{E}_S, \varphi_S)\) where \( \mathcal{E}_S \) is a coherent sheaf on \( S \times X \) and \( \varphi_S \) is an element of \( H^0(S \times X, \text{End}(\mathcal{E}_S) \otimes \pi_X^* L) \) such that \((\mathcal{E}_S|_s \times X, \varphi_S|_s \times X)\) is an oriented Hitchin pair of type \((\mathcal{L}, P)\) for any closed point \( s \in S \). Two families \((\mathcal{E}_S^i, \varphi_S^i), i = 1, 2\), are said to be equivalent, if there is an isomorphism \( \psi_S : \mathcal{E}_S^1 \to \mathcal{E}_S^2 \otimes \pi_X^* L \) with \( \varphi_S^2 \circ \psi_S = ((\psi_S \otimes \text{id}_{\pi_X^* L}) \circ \varphi_S^1 \).

2.2. Hitchin Pairs. A Hitchin pair of type \((\mathcal{L}, P)\) is a triple \((\mathcal{E}, \varepsilon, \varphi)\) consisting of a torsion free coherent sheaf \( \mathcal{E} \) with \( P_\mathcal{E} = P \), a complex number \( \varepsilon \in \mathbb{C} \), and a homomorphism \( \varphi : \mathcal{E} \to \mathcal{E} \otimes L \). Two Hitchin pairs \((\mathcal{E}_1, \varepsilon_1, \varphi_1)\) and \((\mathcal{E}_2, \varepsilon_2, \varphi_2)\) are called equivalent, if there are an isomorphism \( \psi : \mathcal{E}_1 \to \mathcal{E}_2 \) and a complex number \( z \in \mathbb{C}^* \) such that \( \varphi_2 \circ \psi = \left( \psi \otimes (z \cdot \text{id}_L) \right) \circ \varphi_1 \) and \( \varepsilon_2 = z \varepsilon_1 \). Let \( S \) be a noetherian scheme. A family of Hitchin pairs of type \((\mathcal{L}, P)\) parametrized by \( S \) is a quadruple \((\mathcal{E}_S, \varepsilon_S, \varphi_S, \mathcal{M}_S)\) consisting of a coherent sheaf \( \mathcal{E}_S \) over \( S \times X \), an invertible sheaf \( \mathcal{M}_S \) over \( S \), a section \( \varepsilon_S \in H^0(S, \mathcal{M}_S) \), and an element \( \varphi_S \in H^0(S \times X, \text{End}(\mathcal{E}_S) \otimes \pi_X^* \mathcal{M}_S \otimes \pi_X^* L) \) such that its restriction to \( \{s\} \times X \) is a Hitchin pair of type \((\mathcal{L}, P)\) for any closed point \( s \in S \). The family \((\mathcal{E}_S^1, \varepsilon_S^1, \varphi_S^1, \mathcal{M}_S^1)\) is said to be equivalent to the family \((\mathcal{E}_S^2, \varepsilon_S^2, \varphi_S^2, \mathcal{M}_S^2)\) if there are isomorphisms \( \psi_S : \mathcal{E}_S^1 \to \mathcal{E}_S^2 \) and \( z_S : \mathcal{M}_S^1 \to \mathcal{M}_S^2 \) with \( \varphi_S^2 \circ \psi_S = \left( \psi_S \otimes \pi_X^* z_S \otimes \text{id}_{\pi_X^* L} \right) \circ \varphi_S^1 \) and \( \varepsilon_S^2 = \varepsilon_S^1 \circ z_S \).
2.3. (Semi)Stability. We call a Hitchin pair \((E, \varepsilon, \varphi)\) of type \((L, P)\) (semi)stable, if the following two conditions are satisfied:

1. For any \(\varphi\)-invariant subsheaf \(0 \neq F \subset E\) we have: \((P_F / \text{rk } F) \leq (P/r)\).
2. Either \(\varepsilon \neq 0\), or \((\varphi \otimes \text{id}_L \otimes r - 1) \circ \cdots \circ \varphi \neq 0\).

Remark 2.2. As usual, there are the corresponding notions of slope-(semi)stability. Slope-stability implies stability and semistability implies slope-semistability.

We are now able to define the functors \(M^{(s)}_{(L,P)}\) of equivalence classes of families of (semi)stable Hitchin pairs of type \((L, P)\). The functors of families of (semi)stable oriented Hitchin pairs of type \((L, P)\) are the open subfunctors \(\varepsilon \neq 0\).

3. Boundedness

It is the aim of this section to show that the family of isomorphy classes of torsion free coherent sheaves occuring in slope-semistable Hitchin pairs of type \((L, P)\) is bounded. We recall that any torsion free coherent sheaf \(E\) possesses a Harder-Narasimhan filtration

\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}
\]

where \(\mathcal{E}_i / \mathcal{E}_{i-1}\) is the subsheaf of maximal rank of \(\mathcal{E} / \mathcal{E}_{i-1}\) for which \(P_{\mathcal{E}_i / \mathcal{E}_{i-1}}\) is maximal. We have

\[
\mu(\mathcal{E}_i / \mathcal{E}_{i-1}) \geq \mu(\mathcal{E}_{i+1} / \mathcal{E}_i), \quad i = 1, \ldots, l - 1. \tag{1}
\]

A simple inductive argument shows that \(\mu(\mathcal{E}_i) > \mu(\mathcal{E}), i = 1, \ldots, l,\) when \(\mu(\mathcal{E}_1) > \mu(\mathcal{E})\). By a theorem of Maruyama \cite{4}, it is enough to bound \(\mu(\mathcal{E}_1)\) for torsion free coherent sheaves occuring in slope-semistable Hitchin pairs of type \((L, P)\):

**Theorem 3.1.** For any torsion free coherent sheaf which is part of a slope-semistable Hitchin pair of type \((L, P)\), we have

\[
\mu(\mathcal{E}_1) \leq \max\left\{\mu(\mathcal{E}), \mu(\mathcal{E}) + \frac{(r-1)^2}{r} \deg L\right\}.
\]

**Proof.** We follow the proof of \cite{4}, Prop.3.2, in the case of curves. Let \((\mathcal{E}, \varepsilon, \varphi)\) be a slope-semistable Hitchin pair of type \((L, P)\). If \(\mu(\mathcal{E}_1) \leq \mu(\mathcal{E})\), there is nothing to show. Otherwise, as we have seen above, \(\mu(\mathcal{E}_i) > \mu(\mathcal{E}), i = 1, \ldots, l\). By definition of slope-semistability, this means that the \(\mathcal{E}_i\) are not \(\varphi\)-invariant. Hence, the homomorphism \(\varphi_i: \mathcal{E}_i \rightarrow \mathcal{E}_i \otimes L\) is not trivial for \(i = 1, \ldots, l\). Let \(i \in \{0, \ldots, i - 1\}\) be maximal with \(\varphi_i(\mathcal{E}_i) = 0\) and \(\kappa \in \{i + 1, \ldots, l\}\) minimal with \(\varphi_i(\mathcal{E}_i) \subset (\mathcal{E}_\kappa / \mathcal{E}_i) \otimes L\). With these choices, the induced homomorphism from \(\mathcal{E}_{i+1} / \mathcal{E}_i\) to \(\mathcal{E}_\kappa / \mathcal{E}_{i-1} \otimes L\) is non-trivial. Both of these sheaves are slope-semistable, so that \(\mu(\mathcal{E}_{i+1} / \mathcal{E}_i) \leq \mu(\mathcal{E}_\kappa / \mathcal{E}_{i-1}) + \deg L\). By \cite{4}, we get

\[
\mu(\mathcal{E}_i / \mathcal{E}_{i-1}) \leq \mu(\mathcal{E}_{i+1} / \mathcal{E}_i) + \deg L. \tag{2}
\]
Since the assertion of the theorem follows. This gives

\[ \mu(E_1) \leq \mu(E/E_{i-1}) + (l-1) \deg L \leq \mu(E/E_{i-1}) + (r-1) \deg L. \]

Since \( \mu(E_1) + (r-1) \mu(E/E_{i-1}) \leq r \mu(E) \), i.e.,

\[ \mu(E/E_{i-1}) \leq \frac{d - \mu(E_1)}{r-1}, \]

the assertion of the theorem follows. \( \square \)

**Remark 3.2.** Fix a number \( m \) such that \( L \subset O_X(m) \). For any coherent sheaf \( F \), we obviously have \( P_{F \otimes L} \leq P_{F(m)} \). It is easy to see that there is a constant \( C' \) depending only on \( H \) and \( m \) with \( P_{F(m)} \leq P_F + Cx^{n-1} \).

We can now carry out the proof of [3.1] for semistable Hitchin pairs and Hilbert polynomials, where we replace (2) by

\[ \frac{P_{E_i/E_{i-1}}}{\text{rk} E_i - \text{rk} E_{i-1}} \leq \frac{P_{E_{i+1}/E_i} + Cx^{n-1}}{\text{rk} E_i - \text{rk} E_{i-1}} \leq \frac{P_{E_{i+1}/E_i}}{\text{rk} E_i - \text{rk} E_{i-1}} + Cx^{n-1}. \]

This gives

\[ \frac{P_{E_i}}{\text{rk} E_i} \leq \frac{P + (r-1)^2 Cx^{n-1}}{r}. \]

### 4. A parameter space for semistable Hitchin pairs

For \( \mu \in \mathbb{N} \), we define \( P_\mu \) by \( P_\mu(x) := P(x + \mu) \). Twisting by \( O_X(\mu) \) yields an isomorphism between the functors \( M^{(s)}_{(L,P)} \) and \( M^{(s)}_{(L,P,\mu)} \). By Theorem 3.1, we may assume that any torsion free coherent sheaf \( E \) appearing in a semistable Hitchin pair of type \( (L, P) \) fulfills the following conditions:

1. \( E \) is globally generated.
2. \( H^i(X, E) = 0 \) for every \( i > 0 \).

Let \( p := P(0) \), \( V \) be a complex vector space of dimension \( p \), and \( Q \) the projective \( \text{Quot} \) scheme of (all) quotients of \( V \otimes O_X \) with Hilbert polynomial \( P \). On the product \( Q \times X \), there is a universal quotient

\[ q_Q : V \otimes O_{Q \times X} \rightarrow E_Q. \]

We choose \( m \) large enough, so that \( L \subset O_X(m) \) and so that \( O_X(m) \) is globally generated. Furthermore, we choose \( \nu \) large enough, so that \( q_Q(\nu) \) induces a closed embedding \( Q \subset \mathcal{E} := \text{Gr}(V \otimes H^0(O_X(\nu)), P(\nu)) \) and so that the multiplication map \( H^0(O_X(\nu)) \otimes H^0(O_X(m)) \rightarrow H^0(O_X(\nu m)) \) is surjective. We set \( N := H^0(O_X(\nu)), M := H^0(O_X(m)) \), and \( W := V \otimes N \). By our choice of \( \nu \), for any Hitchin pair \( (E, \varepsilon, \varphi) \), the map \( V \otimes N \rightarrow H^0(E(\nu)) \) is surjective. It follows that \( \varphi \otimes \text{id}_{O_X(\nu)} \) is induced by an element \( f \in W^\vee \otimes W \otimes M \). Set \( P := \mathbb{P}(\mathbb{C} \oplus W^\vee \otimes W \otimes M') \), and let

\[ s : O_P \rightarrow [\mathbb{C} \oplus W^\vee \otimes W \otimes M] \otimes O_P(1) \]
be the tautological section. First, we can construct a subscheme \( \tilde{\mathfrak{P}} \subset Q \times P \) whose closed points are those \( s = ([q], \tilde{s}) \in Q \times P \) for which the second component of \( \pi_0^*s \) induces a homomorphism \( E_{\mathfrak{Q}([q]) \times X}(\nu) \rightarrow E_{\mathfrak{Q}([q]) \times X}(\nu) \otimes H^m \). Let \( E_{\tilde{\mathfrak{P}}} \) be the restriction of \( \pi_0^*E \) to \( \tilde{\mathfrak{P}} \times X \) and

\[
h_{\tilde{\mathfrak{P}}}: E_{\tilde{\mathfrak{P}}} \rightarrow E_{\tilde{\mathfrak{P}}} \otimes \pi_X^*(O_X(m)/L)
\]

be the induced homomorphism. We then define \( \mathfrak{P} \) as the closed subscheme of \( \tilde{\mathfrak{P}} \) whose closed points are those \( s \in \tilde{\mathfrak{P}} \) for which \( h_{\tilde{\mathfrak{P}}}|_{\mathfrak{Q}} \equiv 0 \). The scheme \( \mathfrak{P} \) is a parameter space for pairs \( ([q]: V \otimes O_X \rightarrow \mathcal{E}], [\varepsilon, \varphi]) \) with \( [q] \in \mathfrak{Q} \), \( [\varepsilon, \varphi] \in \mathbb{P}(\mathbb{C} \oplus H^0(End\mathcal{E} \otimes L)^\vee) \). On \( \mathfrak{P} \times X \), there exists a universal family \( (E_\mathfrak{P}, \varepsilon_\mathfrak{P}, \varphi_\mathfrak{P}, M_\mathfrak{P}) \).

Denote by \( \mathfrak{P}^{iso} \) the open set of pairs \( ([q]: V \otimes O_X \rightarrow \mathcal{E}], [\varepsilon, \varphi]) \) for which \( H^0(q) \) is an isomorphism. It is not hard to see that any family of semistable Hitchin pairs of type \((L, P)\) is locally induced by morphisms to \( \mathfrak{P}^{iso} \).

5. The SL(V)-action on \( \mathfrak{P} \)

On the Quot scheme \( \mathfrak{Q} \), there is a natural action \( \rho: \mathfrak{Q} \times SL(V) \rightarrow \mathfrak{Q} \). Furthermore, there is a natural action of \( SL(V) \) from the right on the vector space \( W^* \otimes W \otimes M \). If we let \( SL(V) \) act trivially on \( \mathbb{C} \), we get an action of \( SL(V) \) from the right on the scheme \( \mathfrak{Q} \times P \). Finally, we remark that the \( SL(V) \)-action leaves the parameter space \( \mathfrak{P} \) invariant. Hence, there is an action from the right of \( SL(V) \) on \( \mathfrak{P} \). We deduce

**Proposition 5.1.** Let \( S \) be a noetherian scheme and \( \beta_i: S \rightarrow \mathfrak{P}^{iso} \) two morphisms. Suppose that the pullbacks via the maps \( (\beta_i \times id_X) \) of the universal family \( (E_\mathfrak{P}, \varepsilon_\mathfrak{P}, \varphi_\mathfrak{P}, M_\mathfrak{P}) \) are equivalent. Then there exist an étale covering \( \tau: T \rightarrow S \) and a morphism \( g: T \rightarrow SL(V) \) such that \( \beta_1 \circ \tau = (\beta_2 \circ \tau) \cdot g \).

6. The (semi)stable points in \( \mathfrak{P} \)

Suppose we are given a projective scheme \( S \) and an action of an algebraic group \( G \), linearized in an invertible sheaf \( M \). For a point \( s \in S \) and a one parameter subgroup \( \lambda: \mathbb{C}^* \rightarrow G \), set \( s_\infty := \lim_{z \rightarrow \infty} \lambda(z) \cdot s \). Then \( s_\infty \) is a fixed point of the \( \mathbb{C}^* \)-action given by \( \lambda \), and \( \mathbb{C}^* \) acts on \( M \otimes \mathbb{C}(s_\infty) \) with weight, say, \( \gamma \). We set \( \mu(s, \lambda) := -\gamma \). If \( G \) is reductive and \( M \) is ample, then the Hilbert-Mumford criterion says that \( s \) is (semi)stable if and only if \( \mu(s, \lambda) \geq 0 \) for every one parameter subgroup \( \lambda \) of \( G \). We will apply this criterion in our situation.

A one parameter subgroup of \( SL(V) \) is determined by the following data:

1. A basis \( v_1, \ldots, v_p \) of \( V \).
2. Weights \( \gamma_1 \leq \cdots \leq \gamma_p \) with \( \sum_i \gamma_i = 0 \).
We recall that a weight vector \((\gamma_1, \ldots, \gamma_p)\), satisfying \(\gamma_1 \leq \cdots \leq \gamma_p\) and \(\sum_i \gamma_i = 0\) is a \(\mathbb{Q}\)-linear combination with non-negative coefficients of the weight vectors
\[
\gamma^{(i)} := (i-p, \ldots, i-p, i, \ldots, i).
\]
More precisely,
\[
(\gamma_1, \ldots, \gamma_p) = \sum_{i=1}^{p-1} \frac{\gamma_{i+1} - \gamma_i}{p} \gamma^{(i)}.
\]
(3)

Let’s return to our construction. Let \(\mathcal{O}_\Omega(1)\) be the restriction of the very ample line bundle on \(\mathcal{G}\) giving the Plücker embedding. We denote by \(\mathcal{O}(a_1, a_2)\) the restriction of the bundle \(\pi_\mathcal{G}^* \mathcal{O}_\mathcal{G}(a_1) \otimes \pi_\mathcal{P}^* \mathcal{O}_\mathcal{P}(a_2)\) to the parameter space \(\mathcal{P}\). The \(\text{SL}(V)\)-action on \(\mathcal{P}\) can be linearized in any of these sheaves. We will choose \(a_1, a_2 > 0\) with \(a_1 < (p-1)a_2\). For \([q]: V \otimes \mathcal{O}_X \to \mathcal{E}] \in \Omega\) and a subspace \(U \subset V\), \(\mathcal{E}_U\) is defined to be the subsheaf of \(\mathcal{E}\) which is generically generated by \(q(U \otimes \mathcal{O}_X)\) and for which \(\mathcal{E}/\mathcal{E}_U\) is torsion free. Given a basis \(v_1, \ldots, v_p\) of \(V\), we set \(\mathcal{E}_i := \mathcal{E}_{[(v_1, \ldots, v_i)]}\), so that we obtain a filtration
\[
\text{Tors} \mathcal{E} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{p-1} \subset \mathcal{E}_p = \mathcal{E}.
\]
Now, either \(\mathcal{E}_i = \mathcal{E}_{i+1}\) or \(\text{rk} \mathcal{E}_{i+1} = \text{rk} \mathcal{E}_i + 1\). For \(p = 1, \ldots, p\), we set \(k_\rho := \min_{i=1, \ldots, p}\{\text{rk} \mathcal{E}_i = \rho\}\) and \(k := (k_1, \ldots, k_p)\). Suppose we are given a one parameter subgroup \(\lambda\) of \(\text{SL}(V)\). For a point \([q]: V \otimes \mathcal{O}_X \to \mathcal{E}] \in \Omega\), set \([q_\infty]: V \otimes \mathcal{O}_X \to \mathcal{E}] := \lim_{z \to \infty} \lambda(z) \cdot [q]\). We denote the fibre of \(\mathcal{O}_\Omega(a_1)\) over \([q_\infty]\) by \(\Lambda\). Let \(v_1, \ldots, v_p\) be a basis of \(V\). If \(\lambda\) is the one parameter subgroup which is described by the weight vector \((\gamma_1, \ldots, \gamma_p)\), then \(\lambda\) acts on \(\Lambda\) with weight \(a_1 \gamma_k\) ([2], p.309) where we set
\[
\gamma_k := \gamma_{k_1} + \cdots + \gamma_{k_p}.
\]
In particular
\[
\gamma^{(i)}_k = (p-i) \text{rk} \mathcal{E}_i - i(\text{rk} \mathcal{E} - \text{rk} \mathcal{E}_i) = p \text{rk} \mathcal{E}_i - i \text{rk} \mathcal{E}.
\]
Now, consider the \(\text{SL}(V)\)-action on \(\mathcal{P}\). For a point \(\vec{s} \in \mathcal{P}\) and a one parameter subgroup \(\lambda\) of \(\text{SL}(V)\), define \(\vec{s}_\infty\) as above and let \(E\) be the fibre of \(\mathcal{O}_\mathcal{P}(a_2)\) over \(\vec{s}_\infty\). For the statement of the next lemma, we need the notion of a superinvariant subspace which will. Suppose we are given a homomorphism \(f: V \otimes N \to V \otimes N \otimes M\). A subspace \(U \subset V\) is called \(f\)-superinvariant, if \(U \otimes N \subset \ker f\) and if the induced homomorphism \(\overline{f}: (V/U) \otimes N \to (V/U) \otimes N \otimes M\) is identically zero. From now on, given an element \(s := ([q]: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])\) in \(\mathcal{P}\), the associated homomorphism in \(W^V \otimes W \otimes M\) will be denoted by \(f\). We have the following obvious

**Lemma 6.1.** Set \(s := ([q]: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])\). The one parameter subgroup \(\lambda\) which is given w.r.t. to the basis \(v_1, \ldots, v_p\) by the weight vector \(\gamma^{(i)}\) acts on \(E\) with weight
1. \(-a_2p\) if \(\langle v_1, \ldots, v_i \rangle\) is not \(f\)-invariant.
2. \(a_2p\) if \(\langle v_1, \ldots, v_i \rangle\) is \(f\)-superinvariant.
3. 0 in all the other cases.

An immediate consequence is:

**Corollary 6.2.** A necessary condition for a point
\(s := ([q: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])\)

\(\dim U \rk \mathcal{E} \leq p \rk \mathcal{E}_U.\)

**Corollary 6.3.** Let
\(s := ([q: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])\) be a point in \(\mathcal{Y}\) and suppose that either \(H^0(q)\) is not an isomorphism or that \(\mathcal{E}\) is not torsion free. Then \(s\) is not semistable.

**Proof.** Set \(U := \ker H^0(q)\) in the first case and \(U := H^0(\text{Tors} \mathcal{E})\) in the second case. Then \(U\) clearly violates the condition in Corollary 6.2. \(\square\)

We now state the main result of this section:

**Theorem 6.4.** For \(d\) sufficiently large the following assertion holds true: A point
\(s := ([q: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])\) is (semi)stable if and only if \(H^0(q)\) is an isomorphism, \(\mathcal{E}\) is torsion free, and \((\mathcal{E}, \varepsilon, \varphi)\) is a (semi)stable Hitchin pair.

We will need

**Proposition 6.5.** There is an integer \(k_0\) such that for any semistable Hitchin pair, any subsheaf \(\mathcal{F} \subset \mathcal{E}\), and any \(k \geq k_0:\)

\[rh^0(\mathcal{F}(k)) < (\rk \mathcal{F} + 1)P(k).\]

**Proof.** As in the proof on page 305 in [3], we conclude that for any sufficiently large constant \(\kappa\) there is an integer \(k_0\) such that for any Hitchin pair \((\mathcal{E}, \varepsilon, \varphi)\) and any subsheaf \(\mathcal{F} \subset \mathcal{E}\)

either \(|\deg \mathcal{F} - \rk \mathcal{F}\mu(\mathcal{E})| \leq \kappa\) or \(h^0(\mathcal{F}(k))/\rk \mathcal{F} < P(k)/r \ \forall k \geq k_0.\)

Let \(\mathcal{G}\) be the family of all saturated submodules of torsion free sheaves \(\mathcal{E}\) occurring in the family \(\mathcal{E}_\Omega\) which satisfy \(|\deg \mathcal{F} - \rk \mathcal{F}\mu(\mathcal{E})| \leq \kappa.\) Then this family is bounded ([3], Lemma 2.7). Hence, we may assume that all \(\mathcal{F} \in \mathcal{G}\) are globally generated and without higher cohomology. By the discussions following Remark 4.2

\[rh^0(\mathcal{F}(k)) \leq \rk \mathcal{F}(P(k) + (r - 1)^2Ck^{n-1})\]

\[= (\rk \mathcal{F} + 1)P(k) + [\rk \mathcal{F}(r - 1)^2Ck^{n-1} - P(k)]\]

\[\leq (\rk \mathcal{F} + 1)P(k) + [r(r - 1)^2Ck^{n-1} - P(k)].\]

Since \(C\) does not depend on \(d\), we can achieve \([r(r - 1)^2Ck^{n-1} - P(k)] < 0\) for all \(k \geq k_0.\) \(\square\)
We choose $d$ large enough so that $k_0 = 0$, and so that all modules $F$ in the family $\mathcal{S}$ are globally generated and without higher cohomology. Since there are only finitely many possible Hilbert polynomials for sheaves in $\mathcal{S}$, the proof of 6.5 shows that we can assume that for any $F \subset E$, $E$ being a torsion free member of the family $\mathcal{E}_Q$, the inequality $P_F / \text{rk} F \leq P/r$ is equivalent to the inequality $h^0(F)/\text{rk} F \leq p/r$.

Proof of Theorem 6.4. First, let $(q: V \otimes \mathcal{O}_X \to \mathcal{E} \otimes \mathcal{O}_X, [\varepsilon, \varphi])$ be a (semi)stable point. Then, by $1.3$, $H^0(q)$ is an isomorphism and $\mathcal{E}$ is torsion free. Furthermore, $6.2$ shows that $(\mathcal{E}, \varepsilon, \varphi)$ is a (semi)stable Hitchin pair, provided $\varepsilon \neq 0$. We still have to show that $(\varphi \otimes \text{id}_{L^{\otimes 1}}) \circ \cdots \circ \varphi$ is not zero if $\varepsilon = 0$. For this, set $\mathcal{F}_i := \ker((\varphi \otimes \text{id}_{L^{\otimes 1}}) \circ \cdots \circ \varphi)$, $i = 1, \ldots, r$. We get a filtration

$$0 =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = \mathcal{E}$$

of $\mathcal{E}$. Choose a basis $v_1, \ldots, v_p$ of $V$ such that there are $\epsilon_i$ with $\langle v_1, \ldots, v_{\epsilon_i} \rangle = H^0(\mathcal{F}_i)$, $i = 1, \ldots, r$. Let $\lambda$ be the one parameter subgroup which is given by the weight vector $\sum \gamma^{(i)}$. The assumption $\mu(s, \lambda) \geq 0$ implies that there is an index $i$ with $a_1(rh^0(\mathcal{F}) - \text{rk} \mathcal{F}p) + a_2p \leq 0$, in particular

$$rh^0(\mathcal{F}_i) < (\text{rk} \mathcal{F}_i - 1)p.$$

This implies

$$rh^0(\mathcal{E}/\mathcal{F}_i) \geq (r - \text{rk} \mathcal{F}_i + 1)p.$$ 

Now, $(\varphi \otimes \text{id}_{L^{\otimes 1}}) \circ \cdots \circ \varphi$ maps $\mathcal{E}/\mathcal{F}_i$ isomorphically onto a $(\varphi \otimes \text{id}_{L^{\otimes 1}})$-invariant subsheaf of $\mathcal{E} \otimes L^{\otimes i}$. This sheaf can be identified with a $(\varphi \otimes \text{id}_{H^{\otimes im}})$-invariant subsheaf of $\mathcal{E} \otimes H^{\otimes im}$. But $\mathcal{E} \otimes H^{\otimes im}$ is also semistable, and the assumptions made before the beginning of the proof hold for this sheaf as well, so that

$$rh^0(\mathcal{E}/\mathcal{F}_i) \leq (r - \text{rk} \mathcal{F}_i)p_{E \otimes H^{\otimes im}} = (r - \text{rk} \mathcal{F}_i)p(\text{im}),$$

and, consequently,

$$(r - \text{rk} \mathcal{F}_i)(p(\text{im}) - p) \leq p.$$

But when $d$ is large, this is not possible.

Now, we prove the opposite direction: Let $s := ([q: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])$ be a point such that $H^0(q)$ is an isomorphism and $(\mathcal{E}, \varepsilon, \varphi)$ is a (semi)stable Hitchin pair. First, suppose $\varepsilon \neq 0$. Let $v_1, \ldots, v_p$ a basis of $V$. Let $\lambda$ be given by the weight vector $\gamma = \sum \alpha_i \gamma^{(i)}$. If all the spaces $\langle v_1, \ldots, v_i \rangle$ for which $\alpha_i \neq 0$ are $f$-invariant, then the (semi)stability condition implies $\gamma_{\mathcal{F}} \leq 0$. Together with $6.4$, this implies $\mu(s, \lambda) \geq 0$. In the other case, let $\alpha$ be the largest coefficient of a $\gamma^{(i)}$ for which $\langle v_1, \ldots, v_i \rangle$ is not $f$-invariant. By $6.3$, $\gamma_{\mathcal{F}} \leq p$ for $i = 1, \ldots, p - 1$ and, thus,

$$\mu(s, \lambda) \geq -a_1\alpha(p - 1)p + a_2\alpha p.$$

Now, the right hand expression is $> 0$, by our choice of $a_1$ and $a_2$. 


Next, let $\varepsilon = 0$. Since the definition of (semi)stability implies in that case $(\varphi_{id_{\mathbb{P}r}}) \circ \cdots \circ \varphi \neq 0$, every one parameter subgroup acts with weight $\leq 0$ on the “$\mathbb{P}$-component” of $s$. This allows us to argue in the same way as before. \qed

7. The moduli space of semistable Hitchin pairs

7.1. S-equivalence and the main result. We define $M^{(s)s}_{(L,P)} := \mathcal{P}^{(s)s}\!/\!\text{SL}(V)$. Then $M^{ss}_{(L,P)}$ is a projective scheme. In order to describe its closed points, we have to introduce the notion of S-equivalence: For any semistable Hitchin pair, we can construct a Jordan-Hölder filtration of $E_0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$ by $\varphi$-invariant subsheaves. We obtain stable Hitchin pairs $(E_i/E_{i-1}, \varepsilon, \varphi_i), i = 1, ..., l$. The associated graded object

$$\text{gr}(E, \varepsilon, \varphi) := \bigoplus (E_i/E_{i-1}, \varepsilon, \varphi_i)$$

is well-defined up to isomorphism. We say that two semistable Hitchin pairs $(E_1, \varepsilon_1, \varphi_1)$ and $(E_2, \varepsilon_2, \varphi_2)$ of type $(L, P)$ are S-equivalent, if the associated graded objects are equivalent Hitchin pairs. One can show that any semistable Hitchin pair degenerates into its associated graded object and that the associated graded object is polystable. We summarize the results of our discussions in:

Theorem 7.1. i) There is a natural transformation of functors

$$\tau: M^{ss}_{(L,P)} \rightarrow h_{M^{ss}_{(L,P)}}$$

such that for any other scheme $\tilde{M}$ and any natural transformation $\tau': M^{ss}_{(L,P)} \rightarrow h_{\tilde{M}}$ there is a uniquely determined morphism $\vartheta: M^{ss}_{(L,P)} \rightarrow \tilde{M}$ with $\tau' = h(\vartheta) \circ \tau$.

ii) $M^{s}_{(L,P)}$ is a coarse moduli space for the functor $M^{s}_{(L,P)}$.

iii) The closed points of $M^{ss}_{(L,P)}$ naturally correspond to the S-equivalence classes of semistable Hitchin pairs of type $(L, P)$.

7.2. The $\mathbb{C}^*$-action on $M^{ss}_{(L,P)}$. On the space $M := M^{ss}_{(L,P)}$ there is a natural $\mathbb{C}^*$-action given by multiplication of $\varphi$ by a constant. The fixed point set is the union of the part which corresponds to the Hitchin pairs $(E, \varepsilon, 0)$, i.e., the Gieseker moduli space, and the part $M_\infty$ which corresponds to pairs $(E, 0, \varphi)$. The closed subset $M_\infty$ is the part which compactifies the moduli space of semistable oriented Hitchin pairs. Let $M_{\neq 0}$ be the $\mathbb{C}^*$-invariant open subscheme of semistable oriented Hitchin pairs, i.e., the set described by $\varepsilon \neq 0$. We observe that $M_\infty = M_{\neq 0}/\mathbb{C}^*$. Here, we use that the GIT-quotient comes with a natural ample line bundle and that the $\mathbb{C}^*$ action is canonically linearized in this line bundle.
7.3. **The Hitchin map.** Suppose that $X$ is a curve. Let $\mathcal{P}^*$ be the open subset of the parameter space $\mathcal{P}$ parametrizing elements $([q: V \otimes \mathcal{O}_X \to \mathcal{E}], [\varepsilon, \varphi])$ for which $\mathcal{E}$ is torsion free and $H^0(q)$ is an isomorphism, and $\mathcal{P}^*_{\neq 0}$ the part of $\mathcal{P}^*$ lying in $\mathcal{Q} \times (V \otimes N)^\vee \otimes (V \otimes N \otimes M)$, i.e., the part parametrizing pairs with $\varepsilon \neq 0$. Since the Quot scheme is reduced in this case, the restriction of $\mathcal{E}_\mathcal{P}$ to $\mathcal{P}^*_{\neq 0} \times X$ is locally free. This allows us to define the characteristic polynomial map associated to $\varphi_{\mathcal{P}^*_{\neq 0} \times X}$:

$$\chi_{\mathcal{P}^*_{\neq 0}}: \mathcal{P}^*_{\neq 0} \to H^0(X, L^\otimes r) \oplus \cdots \oplus H^0(X, L).$$

The $\mathbb{C}^*$-action on $(V \otimes N)^\vee \otimes (V \otimes N \otimes L)$ induces a $\mathbb{C}^*$-action on the right hand vector space which is given on $H^0(X, L^\otimes i)$ by multiplication with $z^i$, $i = 1, ..., r$. Let $\mathbb{C}^*$ act on $\mathbb{C}$ by multiplication and form the weighted projective space

$$\widehat{\mathbb{P}} := [H^0(X, L^\otimes r) \oplus \cdots \oplus H^0(X, L) \oplus \mathbb{C}]//\mathbb{C}^*.$$

Then the map $\chi_{\mathcal{P}^*_{\neq 0}}$ can be extended to a map $\chi: \mathcal{P}^* \to \widehat{\mathbb{P}}$ which is invariant under the $\text{SL}(V)$-action. Thus we get a map

$$\chi_{\mathcal{M}}: \mathcal{M} \to \widehat{\mathbb{P}},$$

which we call *the Hitchin map*. We observe that $\chi_{\mathcal{M}}$ is proper by [1], II.4.8.(c), applied to $f = \chi_{\mathcal{M}}$ and $g: \widehat{\mathbb{P}} \to \{\text{pt}\}$.

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