Anisotropic Percolation on Slabs

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Abstract

We consider anisotropic independent bond percolation models on the slab $\mathbb{Z}^2 \times \{0, \ldots, k\}$, where we suppose that the axial (vertical) bonds are open with probability $p$, while the radial (horizontal) bonds are open with probability $q$. We study the critical curves for these models and establish their continuity and strict monotonicity.

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1 Introduction and Main Result

Given $k \in \mathbb{N}$, let $S^k = (\mathbb{V}, \mathbb{E})$ be the slab of thickness $k$, the graph where the vertex set is $\mathbb{V} = \mathbb{Z}^2 \times \{0, 1, \ldots, k\}$ and the set of bonds is $\mathbb{E} = \{(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)) : x, y \in \mathbb{V}, \|x - y\|_1 = 1\}$, where $\|x - y\|_1 = \sum_{i=1}^3 |x_i - y_i|$ is the usual graph distance in $\mathbb{Z}^3$. The set $\mathbb{E}$ is naturally partitioned in two disjoint subsets $\mathbb{E}_h$ and $\mathbb{E}_v$. Namely, $\mathbb{E}_h = \{(x, y) \in \mathbb{E} : x_3 = y_3\}$ and $\mathbb{E}_v = \{(x, y) \in \mathbb{E} : x_1 = y_1, x_2 = y_2\}$. We say that $e$ is a radial or axial edge according to $e \in \mathbb{E}_h$ or $\mathbb{E}_v$, respectively.

Given two parameters $p, q \in [0, 1]$, we consider a bond anisotropic percolation model on $S^k$. We associate to each bond $e \in \mathbb{E}$, the state open or closed independently, where each bond is open with probability $p$ or $q$, if it belongs to $\mathbb{E}_h$ or $\mathbb{E}_v$, respectively. Thus, this model is described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{p,q})$ where $\Omega = \{0, 1\}^{\mathbb{E}}$, $\mathcal{F}$ is the $\sigma$-algebra generated by the cylinder sets in $\Omega$ and $\mathbb{P}_{p,q} = \prod_{e \in \mathbb{E}} \mu(e)$ is the product of Bernoulli measures, where $\mu(e)$ is the Bernoulli measure with parameter $p$ or $q$ acording to $e \in \mathbb{E}_h$ or $e \in \mathbb{E}_v$, respectively. We denote a typical element of $\Omega$ by $\omega$. When $\omega(e) = 1$ we say that $e$ is open, if $\omega(e) = 0$, $e$ is closed.

Given two vertices $x, y \in \mathbb{V}$ we say that $x$ and $y$ are connected in the configuration $\omega$ if there exists a finite path of open edges connecting $x$ to $y$. We will use the short notation $\{x \leftrightarrow y\}$ to denote the set of configurations where $x$ and $y$ are connected.

Given the vertex $x$, the cluster of $x$ in the configuration $\omega$ is the set $C_x(\omega) = \{y \in \mathbb{V} ; x \leftrightarrow y \ on \ \omega\}$. We say that the vertex $x$ percolates when the cardinality of $C_x(\omega)$ is infinite; we will use the following standard notation $\{x \leftrightarrow \infty\} := \{\omega \in \Omega ; \|C_x(\omega)\| = \infty\}$, where $\|C_x(\omega)\|$ is the number of vertices in $C_x(\omega)$. We define the percolation probability as the function $\theta(p, q) : [0, 1]^2 \rightarrow [0, 1]$ with $\theta(p, q) = \mathbb{P}_{p,q}(0 \leftrightarrow \infty)$. Consider the box $B(n) = [-n, n]^2 \times \{0, 1, \ldots, k\}$, denote by $\partial B(n) = \{y = (y_1, y_2, y_3) \in B(n) ; \max\{\|y_1\|, \|y_2\|\} = n\}$

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the boundary of $B(n)$ and $A_n = \{ 0 \leftrightarrow \partial B(n) \}$ the event where 0 is connected to $\partial B(n)$. Denoting by

$$\theta_n(p,q) = \mathbb{P}_{p,q}(A_n)$$

we observe that $\{ 0 \leftrightarrow \infty \} = \cap_{n \geq 1} A_n$ and $\theta_n(p,q) \downarrow \theta(p,q)$ as $n$ goes to infinity.

Observe that if $q = 1$ the model is equivalent to the bond percolation on $\mathbb{Z}^2$ with parameter $s$ satisfying

$$1 - s = (1 - p)^{k+1}.$$ In this case we denote the horizontal critical value by $p_k$ where $p_k = 1 - \frac{1}{k+1}$. Therefore, a simple domination argument shows that $\theta(p,q) = 0$ for all $p \leq p_k$ and $q \in [0,1]$.

If $q = 0$ we have $k + 1$ disjoint copies of $\mathbb{Z}^2$. Then the critical value in this case is $\frac{1}{2}$ and if $p > \frac{1}{2}$ there is percolation, regardless the value of $q$. Using a standard coupling argument we have that $\theta(p,q)$ is non-decreasing function in the parameters $p$ and $q$. Then we define the function $q^k_c : [p_k, \frac{1}{2}] \mapsto [0,1]$ by

$$q^k_c(p) = \sup \{ q \in [0,1]; \theta(p,q) = 0 \}.$$ We will show that $q^k_c(\frac{1}{2}) = 0$ and $q^k_c(p_k) = 1$. As $\theta(p,q)$ is non-decreasing in $p$ then $q^k_c(p)$ is non-increasing.

In this note we prove that the critical curve $q^k_c(p)$ (that divides the regions $\theta(p,q) > 0$ and $\theta(p,q) = 0$) is continuous and strictly decreasing. Analogously we define the function $p^k_c : [0,1] \mapsto [p_k, \frac{1}{2}]$ where

$$p^k_c(q) = \sup \{ p \in [0,1]; \theta(p,q) = 0 \}. \quad (1.1)$$

We will omit the index $k$ when it is not necessary and we will write $q_c(p)$ and $p_c(q)$.

Observe that given some vertex $x \in \mathbb{V}$ the percolation function $\mathbb{P}_{p,q}(x \leftrightarrow \infty)$ is, in general, different from $\theta(p,q)$ but the critical functions $q_c(p)$ and $p_c(q)$ are the same. Now we can state the main result of this paper:

**Theorem 1.** The functions $q_c : [p_k, \frac{1}{2}] \mapsto [0,1]$ and $p_c : [0,1] \mapsto [p_k, \frac{1}{2}]$ are decreasing, continuous and $q_c$ is the inverse of $p_c$. Moreover, for a compact $[a,b] \subset (p_k, \frac{1}{2})$, there exists positive constants $c(a,b)$ and $C(a,b)$ such that

$$c(a,b)|p' - p| \leq |q_c(p') - q_c(p)| \leq C(a,b)|p' - p| \quad (1.2)$$

for all $p', p \in [a,b]$.

**Remarks:** 1) All the results of this paper can be generalized, with some minor modifications, for anisotropic percolation in the whole graph $\mathbb{Z}^3$. 2) Given any $p,p' \in [0,1]$, $p \geq p'$, using the results of Grimmett and Marstrand (see [3], they are still valid for anisotropic percolation), it may be shown that for all sufficiently large $k$ we have $\theta(p,p') \geq \theta(p',p)$, which says that the slab $S^k$ percolates better when the greater parameter is on radial bonds. We expect that this behavior is true for any $k$. Simulations in [4] indicates that, in anisotropic $\mathbb{Z}^3$, $q_c(p)$ is convex. If such fact is indeed true for $S^k$, by Theorem 1 we have that, if $p > p'$ and $\theta(p,p') = 0$ then $\theta(p',p') = 0$.

In the next section, we state three Lemmas and prove Theorem 1. In Section 3, we will prove the Lemmas stated in Section 2.

## 2 Preliminary Lemmas and proof of Theorem 1

The first lemma proves that for $p < \frac{1}{2}$ and $q$ small enough, we have exponential decay of the radius of the open cluster. As a consequence we have that $q_c(p) > 0$ for all $p \in [p_k, \frac{1}{2}]$.

**Lemma 1.** Fixed $p < \frac{1}{2}$, there exists $\delta = \delta(p) > 0$ with the following property: For any $0 \leq q < \delta$, there exists a constant $c = c(p,q) > 0$ such that

$$\mathbb{P}_{p,q}(|C_0| \geq n) \leq e^{-cn}, \ \forall n \geq 1.$$ Where $C_0$ is the open cluster of the origin in $S^k$. 

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Lemma 2. For all \( p \in (p_k, \frac{1}{2}) \), we have that \( q_c(p) < 1 \). Moreover, it holds that \( \lim_{p \uparrow \frac{1}{2}} q_c(p) = q_c(\frac{1}{2}) = 0 \).

Combining this result with Lemma 1, we have that \( 0 < q_c(p) < 1 \) for all \( p \in (p_k, \frac{1}{2}) \).

Lemma 3. Given \( \delta > 0 \), there are \( \phi = \phi(\delta), \psi = \psi(\delta) \in (0, \frac{\pi}{2}) \) such that \( \forall (p, q) \in [\delta, 1-\delta]^2 \) the function \( \theta(p, q) \) is non-decreasing in the directions of \((\cos \phi, -\sin \phi)\) and \((-\cos \psi, \sin \psi)\).

As a consequence of this last lemma, we can see that \( \lim_{p \uparrow \frac{1}{2}} q_c(p) = q_c(p_k) = 1 \). In fact, suppose that \( q_c(p_k) < 1 \) there exists \( \bar{q} \) such that \( \theta(p_k, \bar{q}) > 0 \). Taking \( \delta = \min\left\{ \frac{1-\bar{q}}{\sqrt{2}}, \frac{\bar{q} - 2p_k}{\sqrt{2}} \right\} \), we obtain from Lemma 3 that \( \theta(p, q) \) is non-decreasing in the direction \((-\cos \psi, \sin \psi)\). Therefore, it must be a pair \((p, q) \in [\delta, 1-\delta]^2\) with \( p < p_k \) and \( \theta(p, q) > 0 \), a contradiction, because there is no percolation if \( p < p_k \). Then, \( q_c(p_k) = 1 \). In the same manner we show that \( \lim_{p \downarrow \frac{1}{2}} q_c(p) = 1 \).

In resume, we have that \( \lim_{p \uparrow \frac{1}{2}} q_c(p) = q_c(p_k) = 1 \), \( \lim_{p \downarrow \frac{1}{2}} q_c(p) = q_c(\frac{1}{2}) = 0 \) and \( 0 < q_c(p) < 1 \), \( \forall p \in (p_k, \frac{1}{2}) \). In analogous way, for the function \( p_c : [0, 1] \mapsto [p_k, 1/2] \) defined in Equation (1.1), we have by the facts above that \( p_c(0) = 1/2 \), \( p_c(1) = p_k \) and \( p_c(q) \) is non-increasing in \( q \), then \( p_c(q) \in [p_k, \frac{1}{2}] \), \( \forall q \in [0, 1] \).

Now, we are able to prove Theorem 1.

Proof of Theorem 1. First we will show (2.2). Fixed \([a, b] \subset (p_k, \frac{1}{2})\) we have \( 0 < q_c(b) \leq q_c(a) < 1 \), so take \( \delta = \delta(a, b) > 0 \) such that the square \([2\delta, 1-2\delta]^2\) contains the points \((a, q_c(a))\) and \((b, q_c(b))\). As \( q_c \) is non increasing we have \((p, q_c(p)) \in [2\delta, 1-2\delta]^2\), \( \forall p \in [a, b] \). Let \( \phi = \phi(\delta) \) and \( \psi = \psi(\delta) \) be given by Lemma 3 and take \( \epsilon = \frac{\delta}{2} \min\{\tan^{-1}\phi, \tan^{-1}\psi\} \). Consider then \( p < p' \in [a, b] \) with \( |p' - p| \leq \epsilon \).

We observe that \( \forall 0 < \eta < \frac{\delta}{2} \)

\( i) \) \( (p, q_c(p) + \eta), (p', q_c(p) + \eta - |p' - p| \tan \phi) \in [\delta, 1-\delta]^2 \) and

\[
(p', q_c(p) + \eta - |p' - p| \tan \phi) = (p, q_c(p) + \eta) + \frac{|p' - p|}{\cos \phi} (\cos \phi, -\sin \phi)
\]

(2.1)

\( ii) \) \( (p, q_c(p) - \eta), (p', q_c(p) - \eta - |p' - p| \tan \psi) \in [\delta, 1-\delta]^2 \) and

\[
(p', q_c(p) - \eta - |p' - p| \tan \psi) = (p, q_c(p) - \eta) + \frac{|p' - p|}{\cos \psi} (\cos \psi, -\sin \psi)
\]

(2.2)

By Lemma 3 and (2.1) we have that \( \theta(p', q_c(p) + \eta - |p' - p| \tan \phi) \geq \theta(p, q_c(p) + \eta) > 0 \), so \( q_c(p') \leq q_c(p) - \eta - |p' - p| \tan \phi, \forall 0 < \eta < \frac{\delta}{2} \), we get

\[
|q_c(p') - q_c(p)| \geq |p' - p| \tan \phi
\]

(2.3)

Again, by Lemma 3 (using the consequence that \( \theta(p, q) \) is non-decreasing in the direction \((\cos \psi, -\sin \psi)\)) and (2.2), we have that \( \theta(p', q_c(p) - \eta - |p' - p| \tan \psi) \leq \theta(p, q_c(p) - \eta) = 0 \), so \( q_c(p') \geq q_c(p) - \eta - |p' - p| \tan \psi, \forall 0 < \eta < \frac{\delta}{2} \), we get

\[
|q_c(p') - q_c(p)| \leq |p' - p| \tan \psi
\]

(2.4)

We proved above the inequalities in (1.2) with the restriction \(|p' - p| \leq \epsilon \). Using that \( q_c \) is non-increasing we have that the second inequality in (1.2) is true for all \( p', p \in [a, b] \). So, take \( C(a, b) = \tan \psi \). For the first inequality, observe that if \( p' - p > \epsilon \), by taking \( p \leq p_1 < p_2 \leq p' \) with \( p_2 - p_1 = \epsilon \) we have

\[
|q_c(p') - q_c(p)| \geq |q_c(p_2) - q_c(p_1)| \geq \frac{|p_2 - p_1| |p' - p| \tan \phi}{|p' - p|} \geq \frac{\epsilon |p' - p| \tan \phi}{|b - a|}
\]

(2.5)
so, take \( c(a, b) = \frac{\epsilon \tan \phi}{b - a} \).

The argument above shows that \( q_c \) is Lipschitz-continuous and strictly decreasing on any compact \([a, b] \subset (p_k, \frac{1}{2})\). Combining with \( \lim_{p \uparrow p_k} q_c(p) = q_c(p_k) = 1 \) and \( \lim_{p \downarrow p_k} q_c(p) = q_c(\frac{1}{2}) = 0 \), we have that \( q_c \) is strictly decreasing and continuous in the whole interval \([p_k, \frac{1}{2}]\).

Analogously we can prove that \( p_c(q) \) is strict decreasing and continuous for \( q \in [0, 1] \).

Now, we will show that the function \( q_c \) is the inverse function of \( p_c \). Given any \( p \in (p_k, \frac{1}{2}) \), as \( q_c(p) \) is strict decreasing, it holds that \( \forall \epsilon > 0, \theta(p - \epsilon, q_c(p)) = 0 \) (so \( p_c(q_c(p)) = p \)) and \( \theta(p + \epsilon, q_c(p)) > 0 \) (so \( p_c(q_c(p)) \leq p \)). Whence we conclude that \( p_c(q_c(p)) = p \). In the same manner we show that \( q_c(p_c(q)) = q \).

That is, \( q_c \) is the inverse function of \( p_c \).

\[ \square \]

### 3 Proofs of the Lemmas

#### Proof of Lemma 1

We adapt to \( S^k \) the ideas contained in Section 3.5 of [1]. The key idea for this proof is the Lemma 11 of Section 3.5 in [1], which is a consequence of a general result of [2] comparing \( l \)-independent measures with product measures.

Let \( G \) be a graph, and let \( \tilde{P} \) be a site percolation measure on \( G \), i.e., a probability measure on the set of assignments of states (open or closed) to the vertices of \( G \). The measure \( \tilde{P} \) is \( l \)-independent if, whenever \( U \) and \( V \) are sets of vertices of \( G \) whose graph distance is at least \( l \), the states of the vertices in \( U \) are independent of the states of the vertices in \( V \). Observe that if \( \tilde{P} \) is 1-independent, it means that \( \tilde{P} \) is a product measure.

Given an integer \( m > 0 \), denote by \( C_m(A) \) the event where some vertex in \( A \) is connected by an open path to a vertex at distance \( m \) from \( A \), where the distance is given by the maximum norm. Given \( m > k \) an integer, consider the box \( S_m = [0, m - 1]^2 \times \{0, 1, ..., k\} \) in \( S^k \). We will show first that given \( \epsilon > 0 \) (to be chosen later) and \( p < \frac{1}{2} \) there are \( m \) and \( \delta > 0 \) such that \( \mathbb{P}_{p,q}(C_m(S_m)) < \epsilon, \forall q \in [0, \delta] \).

For all \( i \in \{0, 1, ..., k\} \), we define \( P^i \) as the plan \( \mathbb{Z}^2 \times \{i\} \), \( S_m^i = [0, m - 1]^2 \times \{i\} \subset P^i \). Let \( Q(S_m) \) be the event where all vertical bonds of the box \( \tilde{S}_m = [-m, 2m - 1]^2 \times \{0, ..., k\} \) are closed.

Observe that \( C_m(S_m) \cap Q(S_m) \subset \bigcup_{i=0}^k C_m(S_m^i) \). Since the events \( C_m(S_m^i) \) have the same probability, we have \( \mathbb{P}_{p,q}(C_m(S_m) \cap Q(S_m)) \leq (k + 1)\mathbb{P}_{p,q}(C_m(S_m^0)) \). As \( p < \frac{1}{2} \), we use the exponential decay in the subcritical phase of \( Z^2 \simeq Z^2 \times \{0\} \) (see Theorem 5.4 in [3]). Then, there is some constant \( \psi(p) > 0 \) such that \( \mathbb{P}_p(0 \leftrightarrow [-n, n]^2) \leq e^{-\psi(p)n} \), where \( \mathbb{P}_p \) is the probability measure for ordinary bond Bernoulli percolation with parameter \( p \) on \( Z^2 \). Then

\[
\mathbb{P}_{p,q}(C_m(S_m^0)) = \mathbb{P}_p(C_m(S_m^0)) \leq \sum_{v \in S_m^0} \mathbb{P}_p(v \leftrightarrow \partial B(v, m)) \leq m^2 e^{-\psi(p)m} \xrightarrow{m \to \infty} 0,
\]

where \( B(v, m) \) is the ball of center \( v \) and radius \( m \) in the maximum norm in \( \mathbb{Z}^2 \times \{0\} \). Then, we can take \( m \) large enough such that \( \mathbb{P}_{p,q}(C_m(S_m^0)) \leq \frac{1}{2k+1} \). Thereby \( \mathbb{P}_{p,q}(C_m(S_m) \cap Q(S_m)) \leq \frac{1}{2} \).

Observing that \( \mathbb{P}_{p,q}(Q(S_m)) = (1 - q)^N \) where \( N = k(3m)^2 \) is the number of vertical bonds in \( \tilde{S}_m \), we can choose \( \delta > 0 \) small enough such that \( \mathbb{P}_{p,q}(Q(S_m)) > 1 - \frac{\epsilon}{2} \) for all \( p \in [0, 1] \) and \( q \in [0, \delta] \).

Then fixed \( p < \frac{1}{2} \) and \( q \in [0, \delta] \), it holds that

\[
\mathbb{P}_{p,q}(C_m(S_m)) = \mathbb{P}_{p,q}(C_m(S_m) \cap Q(S_m)) + \mathbb{P}_{p,q}(C_m(S_m) \cap Q(S_m)^c) \leq \frac{\epsilon}{2} + \mathbb{P}_{p,q}(Q(S_m)^c) \leq \epsilon.
\]
Now, we will define a site percolation measure $\tilde{\mathbb{P}}$ on $\mathbb{Z}^2$. We declare each vertex $v = (x, y) \in \mathbb{Z}^2$ as open if and only if the event $C_m(S_{v,m})$ holds for the $m$ by $m$ square $S_{v,m} = [mx+1, mx+m] \times [my+1, my+m] \times \{0, ..., k\}$.

More formally, let $f : \Omega \mapsto \{0, 1\}^{\mathbb{Z}^2}$ be the function defined as $f(\omega) = (f_v(\omega))_{v \in \mathbb{Z}^2}$ where

$$
f_v(\omega) = \begin{cases} 1, & \text{if } \omega \in C_m(S_{v,m}) \\ 0, & \text{if } \omega \notin C_m(S_{v,m}). \end{cases} \quad (3.1)
$$

The function $f$ and the measure $\mathbb{P}_{p,q}$ induce a probability measure $\tilde{\mathbb{P}}$ on $\{0, 1\}^{\mathbb{Z}^2}$ given by $\tilde{\mathbb{P}}(\mathcal{A}) = \mathbb{P}_{p,q}(f^{-1}(\mathcal{A}))$ for any $\mathcal{A} \in \mathcal{A}$, where $\mathcal{A}$ is the $\sigma$-algebra generated by the cylinder sets of $\{0, 1\}^{\mathbb{Z}^2}$. This measure $\tilde{\mathbb{P}}$ give us a site percolation model on $(0, 1)^{\mathbb{Z}^2}$.

Since the event $C_m(S_{v,m})$ depends only on the states of sites within distance (in the graph distance) $m$ of $S_{v,m}$, then the measure $\tilde{\mathbb{P}}$ is 5-independent. Furthermore, each vertex $v \in \mathbb{Z}^2$ is open with probability $\tilde{\mathbb{P}}(v \text{ is open}) = \mathbb{P}_{p,q}(C_m(S_{v,m})) \leq \epsilon$.

From Lemma 11 of Section 3.5 in [1], we can take $\epsilon > 0$ and $a > 0$ (the constants $k$ and $\Delta$ in Lemma 11 are 5 and 4, respectively) such that if $\tilde{\mathbb{P}}(v \text{ is open}) \leq \epsilon$, then

$$
\tilde{\mathbb{P}}(|\check{C}_v| \geq n) \leq e^{-an}, \quad \forall n \geq 1.
$$

Where $\check{C}_v$ is the open cluster of the vertex $v$ in this 5-independent model induced on $\mathbb{Z}^2$.

If $|C_0| \geq (k+1)(4m+1)^2$ this implies that every site $u \in C_0$ is connected by an open path to some site at distance at least $2m$ from $u$, then $C_m(S_{v,m})$ occurs for every vertex $v \in S_{v,m} \cap C_0$, that is, the site $v$ is open in the 5-independent model induced on $\mathbb{Z}^2$, in particular $v \in \check{C}_0$. Hence, as each $S_{v,m}$ contains $(k+1)m^2$ sites, if $n \geq (k+1)(4m+1)^2$ we have

$$
\mathbb{P}_{p,q}(|C_0| \geq n) \leq \tilde{\mathbb{P}}\left(|\check{C}_0| \geq \frac{n}{(k+1)m^2}\right) \leq e^{\frac{-an}{(k+1)m^2}}.
$$

We conclude the proof of Lemma 1 taking $\epsilon = \frac{q}{(k+1)m^2}$.

**Proof of Lemma 2** First we will prove that $q_c(p) < 1$ for $p \in (p_k, 1/2]$. Since $q_c$ is non-increasing in $p$, it is sufficient to show that $q_c(p) < 1$ for $p$ close to $p_k$. Given any $\epsilon > 0$, we will show that there exists $q < 1$ such that que $\theta(p_k + \epsilon, q) > 0$, therefore $q_c(p_k + \epsilon) \leq q < 1$.

Let $u_1 = (1, 0, 0), u_2 = (0, 1, 0)$ and $u_3 = (0, 0, 1)$ be the unitary vectors. Consider the graph $G$ obtained from $S^k$ replacing each vertical bond $f_v = \langle v, v + u_3 \rangle$, with $v \in \mathbb{Z}^2 \times \{0, 1, ..., k-1\}$, by 4 parallel bonds (denoted by $f^{V}_v, f^I_1, f^I_2$ and $f^d_1$) connecting the vertices $v$ and $v + u_3$ and declaring each of these new bonds open with probability $\tilde{q}$ where $1 - q = (1 - \tilde{q})^4$. It means that each vertical bond in $S^k$ is closed if and only if the respective 4 parallel bonds of $G$ are closed. Observe that $G$ and $S^k$ have the same vertex set and this replacement does not affect the connective functions involving the vertices of $S^k$. We have that $\mathbb{P}_{p,q}(0 \leftrightarrow \partial B(n) \text{ in } S^k) = \mathbb{P}_{p,q}(0 \leftrightarrow \partial B(n) \text{ in } G)$ and $\theta^n(p, q) = \theta^G(p, \tilde{q})$.

We will define a bond percolation process on $\mathbb{Z}^2$ which is stochastically dominated by the bond percolation process on $G$, such that percolation on $\mathbb{Z}^2$ imply percolation on $G$. To simplify the notation we identify $\mathbb{Z}^2$ with $\mathbb{Z}^2 \times \{0\} \subset G$. To each bond $\langle v, v + u_1 \rangle$ of $\mathbb{Z}^2$ we define the paths $c^0_{v,u_1}$ on $G$, with $i \in \{0, 1, ..., k\}$ where $c^0_{v,u_1}$ is the bond $\langle v, v + u_1 \rangle$ and $c^i_{v,u_1}$, for $i = 1, \ldots, k$ is the path on $G$ that starts at $v$, takes the vertical path using the bonds $f^v_{v+ju_1}$, with $0 \leq j \leq i-1$, until the vertex $v + iu_3$, takes the bond $\langle v + iu_3, v + iu_3 + u_1 \rangle$, and get down vertically until the vertex $v + u_1$ using the bonds $f^d_{v+ju_3}$, with $0 \leq j \leq i-1$. Analogously we define the paths $c^i_{v,u_2}$ to the bond $\langle v, v + u_2 \rangle$, using the vertical bonds $f^v_{v+ju_3}$ and $f^d_{v+ju_3 + u_1}$. Declare each bond $e = \langle v, v + u_1 \rangle$ of $\mathbb{Z}^2$ as open if at least one of the respective paths $c^i_{v,u_1}$ is open. Analogously, we
do the same for the bond $e = (v, v + u_2)$ and the paths $c_i^{v, u_2}$. Observe that these paths were chosen in such way that we have an independent bond percolation process on $\mathbb{Z}^2$, with parameter $\mathcal{p} = \mathcal{p}(p, q)$ defined as:

$$
\mathcal{p} = \mathcal{p}(p, q) = p \sum_{j=0}^{k} \frac{[(1-p)q]^j}{1 - (1-p)q^k}
$$

reminding that $\hat{q} = 1 - (1-q)^{\frac{1}{k}}$. Taking $p = p_0 + \epsilon$ and $q = 1$ (so $\hat{q} = 1$) we get $\mathcal{p}(p_0 + \epsilon, 1) = 1 - (1 - \sqrt[1-k]{\hat{q}}\epsilon)^{k+1} > \frac{1}{2}$. As $\mathcal{p}(p, q)$ is a continuous function, there is $\delta > 0$ such that $\mathcal{p}(p_0 + \epsilon, q) > \frac{1}{2}$, for all $q \in (1 - \delta, 1]$. Then,  
we chose $q < 1$ such that $\mathcal{p}(p_0 + \epsilon, q) > \frac{1}{2}$. Therefore $\theta^{\mathcal{p}}(p, q) = \theta^{\mathcal{p}}(p, q) \geq \theta^{\mathcal{p}}(\mathcal{p}(p_0 + \epsilon, q)) > 0$.

To show that $\lim_{p \uparrow +} q_c(p) = 0$, we can suppose that $k = 1$ since $q_c^k$ is non-increasing in $k$. As $\hat{q} > 0$ is equivalent to $q > 0$, we have that for $q > 0$ and $p = \frac{1}{2}$, $\mathcal{p} = \frac{1}{2} + \frac{\epsilon^2}{4} > \frac{1}{2}$. Fixed $q = \epsilon > 0$, as $\mathcal{p}$ is a continuous function of $p$, we have that $\mathcal{p}(p, \epsilon) > \frac{1}{2} \forall p \in (\frac{1}{2} - \frac{\epsilon^2}{4}, \frac{1}{2}]$. Therefore, $\theta^{\mathcal{p}}(p, \epsilon) = \theta^{\mathcal{p}}(p, \epsilon) \geq \theta^{\mathcal{p}}(\mathcal{p}(p, \epsilon)) > 0$ and $q_c(p) \leq \epsilon \forall p \in (\frac{1}{2} - \frac{\epsilon^2}{4}, \frac{1}{2}]$. We have then $\lim_{p \uparrow +} q_c(p) = 0$ and $q_c(\frac{1}{2}) = 0$.

**Proof of Lemma 3** To prove this lemma we enunciated two classical results (without proof) which are adaptations to $\mathbb{R}^k$ of the Russo’s formula (see Theorem 2.25 in [2]) and, as consequence, an analogue of Lemma 3.5 of [2].

Given a configuration $\omega$, we consider the configurations $\omega_e$ and $\omega^c$ that coincide with $\omega$ if $f \neq e$, but $\omega_e(e) = 0$ and $\omega^c(e) = 1$. We say that $e$ is pivotal for an increasing event $A$ in the configuration $\omega$ if $\mathcal{I}_A(\omega_e) = 0$ but $\mathcal{I}_A(\omega_e) = 1$. We denote by $(e$ is pivotal for $A)$ the set of such configurations. Thus, as $A_0$ is an increasing event, $e$ is pivotal for $A_0$ if and only if $A_0$ does not occur when $e$ is closed but $A_0$ does occur when $e$ is open.

**Proposition 1. (Russo’s formula)** Consider the function $\theta_n(p, q)$, then

$$
\frac{\partial}{\partial p} \theta_n(p, q) = \sum_{f \in \mathcal{E}_n \cap B(n)} \mathbb{P}_{p, q}(e \text{ is pivotal for } A_n)
$$

and

$$
\frac{\partial}{\partial q} \theta_n(p, q) = \sum_{f \in \mathcal{E}_n \cap B(n)} \mathbb{P}_{p, q}(f \text{ is pivotal for } A_n)
$$

**Proposition 2.** There exists a positive integer $N$ and a continuous function $\beta : (0, 1)^2 \mapsto (0, \infty)$ such that $\forall p, q \in (0, 1)$ and $n \geq N$, it holds that

$$
\beta^{-1}(p, q) \frac{\partial}{\partial p} \theta_n(p, q) \geq \beta(p, q) \frac{\partial}{\partial q} \theta_n(p, q) \geq \beta(p, q) \frac{\partial}{\partial q} \theta_n(p, q).
$$

We show the existence of $\phi$, the proof for $\psi$ is analogue. By the first inequality in Proposition 2, $\frac{\partial}{\partial p} \theta_n(p, q) \geq \beta(p, q) \frac{\partial}{\partial q} \theta_n(p, q)$ for all large $n$, where $\beta(p, q)$ is continuous in $(0, 1)^2$. Therefore, given $\delta > 0$ take $m > 0$ such that $\beta(p, q) \geq m$ in $[\delta, 1 - \delta]^2$, and in this case $\frac{\partial}{\partial p} \theta_n(p, q) \geq m \frac{\partial}{\partial q} \theta_n(p, q)$. Take then $\phi \in (0, \pi/2)$ such that $\tan \phi = m$. Therefore

$$
\nabla \theta_n(p, q) \cdot (\cos \phi, -\sin \phi) = \frac{\partial}{\partial p} \theta_n(p, q) \cos \phi - \frac{\partial}{\partial q} \theta_n(p, q) \sin \phi =
$$

$$
= \cos \phi \left( \frac{\partial}{\partial p} \theta_n(p, q) - \frac{\partial}{\partial q} \theta_n(p, q) \tan \phi \right) \geq \cos \phi \left( m \frac{\partial}{\partial q} \theta_n(p, q) - \frac{\partial}{\partial q} \theta_n(p, q) \tan \phi \right) = 0
$$

(3.3)
since that \( m = \tan \phi \).

Let \((p', q') = (p, q) + a(\cos \phi, -\sin \phi)\) such that \((p, q)\) and \((p', q')\) are in \([\delta, 1-\delta]^2\). Let \(\alpha : [0, a] \rightarrow [\delta, 1-\delta]^2\), 
\(\alpha(t) = (p, q) + t(\cos \phi, -\sin \phi)\) the linear path joining \((p, q)\) to \((p', q')\). Integrating along the path \(\alpha\) we get
\[
\theta_n(p', q') - \theta_n(p, q) = \int_0^a \frac{d}{dt}\theta_n(\alpha(t))dt = \int_0^a \nabla \theta_n(p, q) \cdot (\cos \phi, -\sin \phi)dt \geq 0.
\]

Taking the limit as \(n \rightarrow \infty\) we have
\[
\theta(p', q') = \lim_{n \rightarrow \infty} \theta_n(p', q') \geq \lim_{n \rightarrow \infty} \theta_n(p, q) = \theta(p, q)
\]

So \(\theta(p, q)\) is non-decreasing in the direction \((\cos \phi, -\sin \phi)\).

\(\square\)

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