Simple variables for AdS$_5 \times S^5$ superspace

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Abstract

We introduce simple variables for describing the AdS$_5 \times S^5$ superspace, i.e. PSU$(2, 2|4)$ \times SO$(4,1)$ × SO$(5)$. The idea is to embed the coset superspace into a space described by variables which are in linear (ray) representations of the supergroup PSU$(2, 2|4)$ by imposing certain supersymmetric quadratic constraints (up to two overall U(1) factors). The construction can be considered as a supersymmetric generalisation of the elementary realisations of the AdS$_5$ and the $S^5$ spaces by the SO$(4,2)$ and SO$(6)$ invariant quadratic constraints on two six-dimensional flat spaces.

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String theory in $\text{AdS}_5 \times S^5$ has been studied extensively in recent years, because of the \text{AdS}/CFT correspondence \cite{1}. The theory is also a prime example of string theories with non-zero Ramond-Ramond fields in their backgrounds, and has a very high degree of symmetry, in particular, the maximal supersymmetry $\text{PSU}(2,2|4)$. The classical action of the theory \cite{2} in the Green-Schwarz formalism \cite{3,4} describes the propagation of the strings in the target superspace $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$, which is expressed as a coset space.

It is the purpose of the present note to point out that this superspace has a simple realisation in which the supersymmetry and the fermionic variables are represented in a particularly clear manner. Fermionic and bosonic variables are treated on an equal footing in this formalism.

The realisation can be considered as a supersymmetric generalisation of the standard definition of the $\text{AdS}_5$ and the $S^5$ spaces (with radii $R$) by embedding them into two six-dimensional flat spaces,

$$\eta_{\dot{i}\dot{j}} X^{\dot{i}} X^{\dot{j}} = -R^2, \quad \eta_{I'J'} Y^{I'} Y^{J'} = R^2.$$  

Here $X^{\dot{i}}$'s and $Y^{I'}$'s are six-dimensional real vectors. The indices $\dot{i}, \dot{j} = 0, 1, \ldots, 5$ refer to $\text{SO}(4,2)$ vector indices and the metric is given by $\eta_{\dot{i}\dot{j}} = \text{diag}(-1,1,1,1,1,-1)$; $I', J' = 1, 2, \ldots, 6$ are $\text{SO}(6)$ vector indices, $\eta_{I'J'} = \text{diag}(1,\ldots,1)$. The manifolds defined by these equations are equivalent to the coset spaces $\text{SO}(4,2)/\text{SO}(4,1)$ and $\text{SO}(6)/\text{SO}(5)$ respectively up to global issues which we shall ignore throughout this note. This type of embedding by quadratic constraints is often useful, in particular, to make the symmetry properties more transparent, as was originally pointed out by Dirac \cite{5}.

In our construction we will use linear representations (more precisely linear ray representations) of $\text{PSU}(2,2|4)$ and introduce certain quadratic constraints on the representation spaces. The supermanifold defined by these constraints will be shown to be equivalent to $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$.

**Representations**

As the supersymmetrisations of $X^{\dot{i}}$ and $Y^{I'}$, we use two sets of variables $X^{AB}$ and $Y^{AB}$. They belong to the (super-)anti-symmetric and symmetric products of the fundamental ray representations of $\text{PSU}(2,2|4)$,

$$X^{AB} = -(-1)^{AB} X^{BA},$$  

$$Y^{AB} = +(-1)^{AB} Y^{BA}.$$  

Our notation is as follows. Indices $A,B,\ldots$ are those for the $\text{PSU}(2,2|4)$ fundamental ray representation and take eight values. They consist of four $\text{SU}(2,2)$ “bosonic” components $\dot{a}, \dot{b}, \ldots$ and four $\text{SU}(4)$ “fermionic” components $\dot{a}', \dot{b}', \ldots$. The $A,B,\ldots$ indices on the exponent of $(-1)$ should be understood as either 0 for the bosonic components or 1 for the fermionic components. More explicitly we have

$$X^{\dot{a}} = -X^{\dot{b}}, \quad X^{\dot{a}} = -X^{\dot{b}'} = X^{\dot{b}'}, \quad X^{\dot{a}' \dot{b}'} = +X^{\dot{b}' \dot{a}'},$$  

$$Y^{\dot{a}} = +Y^{\dot{b}}, \quad Y^{\dot{a}} = +Y^{\dot{b}'} = Y^{\dot{b}'}, \quad Y^{\dot{a}' \dot{b}'} = -Y^{\dot{b}' \dot{a}'}. \quad (6)$$

\footnote{1 The assignment of the odd Grassmann parity to the $\text{SU}(4)$ part is purely conventional.}
The $\dot{a}\dot{b}$ and $a'b'$ components of $X$'s and $Y$'s are commuting and the $\dot{a}b'$ and $a'\dot{b}$ components are anti-commuting.

We use two irreducible representations of $PSU(2,2|4)$, rather than one irreducible representation. At first sight, it might seem that the use of the two variables ($X$'s and $Y$'s) would make the superspace a direct product of two superspaces. Actually, the constraints we introduce below intertwine the two variables so that the final superspace cannot be written as a direct product of two spaces. This is consistent with the fact that while the bosonic part of the superspace $AdS_5 \times S^5$ is written as a direct product, the full superspace $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ is not.

In order to formulate the constraints we introduce further conventions and notations on the supersymmetric tensor calculus. We use the standard “left derivative” convention for supersymmetric tensor indices, such that

$$v^A w_A = (-1)^{A^2} w_A v^A$$  \hspace{1cm} (7)

is a scalar: the indices $A$ should be contracted in this manner. In this convention, Kronecker’s delta has the index structure

$$\delta_B^A.$$  \hspace{1cm} (8)

By applying complex conjugation to (7) it follows that

$$\overline{v^A w_A} = \overline{w_A v^A} = \overline{w_A} \overline{v^A}$$  \hspace{1cm} (9)

is a scalar. We have introduced indices $\bar{A}, \bar{B}, \ldots$ by defining $\overline{w_A} = \overline{w_{\bar{A}}}$, $\overline{v^A} = \overline{v^{\bar{A}}}$. The $\bar{A}, \bar{B}, \ldots$ indices should be contracted in the manner indicated in the above formula. The fundamental ray representation of $PSU(2,2|4)$ is equipped with a “hermitian metric”

$$\eta_{AB} = \text{diag}(-1,-1,1,1;1,1,1)$$  \hspace{1cm} (10)

and its inverse

$$\eta^{\bar{A}\bar{B}} = \text{diag}(-1,-1,1,1;1,1,1).$$  \hspace{1cm} (11)

They can be used to lower and raise the indices.

An element of the fundamental ray representation of the $PSU(2,2|4)$ supergroup transforms as

$$v^A \mapsto v^B U_B^A$$  \hspace{1cm} (12)

where $U_{\dot{a}\dot{b}}, U_{a'b'}$‘s are commuting and $U_{\dot{a}b'}, U_{a'\dot{b}}$ are anti-commuting. The variables $X, Y$’s transform under $PSU(2,2|4)$ transformations by the transformation rule,

$$X^{AB} \mapsto X^{CD} U_C^A U_D^B (-1)^D(A+C),$$  \hspace{1cm} (13)

$$Y^{AB} \mapsto Y^{CD} U_C^A U_D^B (-1)^D(A+C).$$  \hspace{1cm} (14)

The condition

$$U_B^A \eta_{AB} \overline{U_C^B} \eta^{CE} = \delta_B^E$$  \hspace{1cm} (15)

defines the $U(2,2|4)$ supergroup. A further constraint

$$\text{sdet } U = 1$$  \hspace{1cm} (16)
defines the $SU(2,2\mid 4)$ supergroup \[\boxed{6}\]. Finally, by identifying two $U$’s related by an overall $U(1)$ transformation
\[
U \sim e^{i\alpha}U,
\] (17)
we obtain the $PSU(2,2\mid 4)$ supergroup. This identification implies that the fundamental representation should be considered as a ray (or projective) representation, namely elements of the representation space should be identified as follows
\[
v^A \sim e^{i\alpha}v^A.
\] (18)
As a consequence, the spaces described by the variables $X$, $Y$ also have natural identifications
\[
X^{AB} \sim e^{i\alpha}X^{AB}, \quad Y^{AB} \sim e^{i\beta}Y^{AB}.
\] (19)
Alternatively, we may also speak about linear representations of $SU(2,2\mid 4)$ or $U(2,2\mid 4)$, without introducing the identification, though $PSU(2,2\mid 4)$ is the physically interesting case.

We denote the complex conjugate of $X_{AB}$ by
\[
\overline{X^{AB}} = \overline{X^{BA}}.
\] (20)
We define $X$ with lower indices by
\[
X_{AB} = (-1)^{(B+C)A}\eta_{BC}\eta_{AD}\overline{X^{BD}} = (-1)^{(B+C)A}\eta_{BC}\eta_{AD}\overline{X^{CD}}.
\] (21)
(The sign factor $(-1)^{(B+C)A}$ above equals 1 because $\eta$ is diagonal.) Similarly, we define
\[
Y_{AB} = (-1)^{(B+C)A}\eta_{BC}\eta_{AD}\overline{Y^{BD}} = (-1)^{(B+C)A}\eta_{BC}\eta_{AD}\overline{Y^{CD}}.
\] (22)

**Constraints** On the space described by the variables $X^{AB}$ and $Y^{AB}$, we introduce the following quadratic constraints,
\[
X^{AC}Y_{CB} = 0, \quad X^{AC}X_{CB} - Y^{AC}Y_{CB} = (-1)^{AB}R^2\delta^A_B.
\] (23, 24)
The factor $(-1)^{AB}$ in (24) is necessary to make the index structures of the LHS and the RHS match.

By construction, the LHS and the RHS of the constraints have the same transformation properties under $PSU(2,2\mid 4)$ transformations, which can also be verified directly using (13)-(15). Hence these constraints have invariant meanings under $PSU(2,2\mid 4)$ transformations.

The constraints are invariant also under the two overall $U(1)$ transformations
\[
X^{AB} \mapsto e^{i\alpha}X^{AB}, \quad Y^{AB} \mapsto e^{i\beta}Y^{AB}.
\] (25)
Hence the constraints \[\boxed{23}, \boxed{24}\] are consistent with the identifications \[\boxed{19}\]: the constraints are correctly defined on the ray representations. \[\boxed{2}\]

\[\boxed{2}\] Alternatively one may consider the constraints to be invariant under $U(2,2\mid 4)$ transformations, without introducing the identification \[\boxed{19}\].
Equivalent to \( \frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)} \) We will now show that the supermanifold defined by the constraints \( [23], [24] \) is equivalent to the coset superspace \( \frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)} \).

Any two points on the supermanifold which are related by a \( \text{PSU}(2,2|4) \) transformation are equivalent. It is therefore natural to start by choosing a representative point on the manifold and study the manifold in the vicinity of the point.

We first choose a pair of vectors \( X^I_0, Y^I_0 \) satisfying the constraints \([11], [12]\). The representative point is constructed from \( X^I_0, Y^I_0 \) using the Clebsch-Gordan coefficients relating \( \text{SO}(4,2) \) and \( \text{SU}(2,2) \), \( \Gamma^{i\bar{a}b} \), and \( \text{SO}(6) \) and \( \text{SU}(4) \), \( \Gamma^{j'a'b'} \),

\[
X^{i\bar{a}}_0 = X_0^{i\bar{a}} = (X_0 \cdot \Gamma)^{i\bar{a}}, \quad X^{i\bar{a}}_0 = 0, \quad X^{a'b'}_0 = 0, \quad X^{a'b'}_0 = 0, \quad (26)
\]

\[
Y^{i\bar{a}}_0 = 0, \quad Y^{a'b'}_0 = 0, \quad Y^{a'b'}_0 = 0, \quad Y^{a'b'}_0 = Y_0^{i\bar{a}} \Gamma^{j'a'b'} = (Y_0 \cdot \Gamma)^{i\bar{a}} \cdot (27)
\]

This point in the superspace satisfies the constraints \([23], [24], [25]\) which can be checked using

\[
X^{i\bar{a}}_0 = -X_0^{i\bar{a}} = -(X_0 \cdot \Gamma)^{i\bar{a}}, \quad (28)
\]

\[
Y_0^{a'b'} = Y_0^{i\bar{a}} \Gamma^{j'a'b'} = (Y_0 \cdot \Gamma)^{i\bar{a}} \cdot (29)
\]

\[
(X_0 \cdot \Gamma)^{i\bar{a}} (X_0 \cdot \Gamma)^{k\bar{c}} = X_0^i X_0^k \delta_{\bar{c}\bar{a}} = -R^2 \delta_{\bar{c}\bar{a}}, \quad (30)
\]

\[
(Y_0 \cdot \Gamma)^{i\bar{a}} (Y_0 \cdot \Gamma)^{k\bar{c}} = Y_0^j Y_0^k \delta_{\bar{c}\bar{a}} = R^2 \delta_{\bar{c}\bar{a}}, \quad (31)
\]

\[
X^{i\bar{a}}_0 Y_0^{k\bar{c}} = R^2 \delta_{\bar{a}\bar{c}}, \quad Y_0^{a'b'} Y_0^{k\bar{c}} = R^2 \delta_{\bar{a}\bar{c}}. \quad (32)
\]

These formulæ follow from properties of the Clebsch-Gordan coefficients summarised in the appendix. It is sometimes useful to specify further the point by choosing \( X^I_0 = (0, \ldots, 0, 1) \), \( Y^I_0 = (0, \ldots, 0, 1) \).

We next consider the orbit of this representative point under all possible \( \text{PSU}(2,2|4) \) transformations. The equivalence of the constrained superspace to the coset superspace will be shown in two steps. First we will show the equivalence of the coset space and the orbit space and then the equivalence of the orbit space and the constrained superspace.

The orbit and the coset are equivalent if the subgroup of \( \text{PSU}(2,2|4) \) which leaves the representative point fixed is precisely \( \text{SO}(4,1) \times \text{SO}(5) \).

It is sufficient to consider the infinitesimal transformations of the representative point specified by \( X^{AB}_0, Y^{AB}_0 \). An infinitesimal transformation \( U_A^B = \delta_A^B + \delta U_A^B \) satisfies, from \([13]\),

\[
0 = \delta U_A^C \eta_{CB} + \eta_{AC} \delta U_B^C. \quad (33)
\]

The infinitesimal transformation rules of \( X^I \) and \( Y^I \) are derived from \([13], [14]\),

\[
\delta X^{AB} = X^{AC} \delta U_C^B + X^{CB} \delta U_C^A (-1)^{(C+A)} B, \quad (34)
\]

\[
\delta Y^{AB} = Y^{AC} \delta U_C^B + Y^{CB} \delta U_C^A (-1)^{(C+A)} B. \quad (35)
\]

The bosonic transformations consist of \( \text{SU}(2, 2) \) transformations acting on \( \dot{a}, \dot{b} \) indices and \( \text{SU}(4) \) transformations acting on \( a', b' \) indices. \[3\] The only non-zero components on which a

\[3\] We fixed the relative sign factors in \([23]\) by the requirement that the point specified by \([20], [21], [11], [2]\) is a solution to \([24]\).

\[4\] The two \( U(1) \) transformations in \( U(2,2|4) \) which are eliminated by the \( P \) and \( S \) conditions \([19], [17]\) are absorbed precisely by the identifications \([19]\) or the transformations \([25]\).
SU(2,2) transformation can act are \(X_{(0)}^{ab}\). By the standard property of the Clebsch-Gordan coefficients, this action is equivalent to the action of the corresponding SO(4,2) transformation on \(X_{(0)}^{I}\). The SO(4,2) transformations which leave \(X_{(0)}^{I}\) (satisfying (11)) invariant are precisely those forming SO(4,1). Similarly, the subset of SU(4) transformations which leave \(X_{(0)}^{AB}\)'s and \(Y_{(0)}^{AB}\)'s invariant are equivalent to SO(5) transformations which leave \(Y_{(0)}^{I}\) invariant.

It therefore remains to be shown that under any fermionic transformations (with parameters \(\delta U_{a}^{b'}, \delta U_{a'}^{b}\), satisfying (33)), the representative point is not fixed. The representative point transforms under the fermionic transformations as,

\[
\delta X^{\hat{a}\hat{b}'} = X_{(0)}^{\hat{a}\hat{c}} \delta U_{\hat{c}'}^{b} = -\delta X^{b'\hat{a}},
\]

\[
\delta Y^{\hat{a}\hat{b}'} = -Y_{(0)}^{c'\hat{b}'} \delta U_{c'}^{\hat{a}} = +\delta Y^{b'\hat{a}}.
\]

Since \(X_{(0)}^{ab}\) and \(Y_{(0)}^{ab}\) are invertible (see (32)), it follows that the representative point is not fixed under any fermionic transformations. Thus the equivalence between the orbit and the coset is established.

From the covariance of the constraints (23), (24), it follows that all points on the orbit space will satisfy the constraints. Therefore the orbit space is contained in the space defined by the constraints.

Hence, in order to show that the orbit space and the constrained manifold are equivalent, it is sufficient to check that the constrained manifold does not contain “extra directions”.

Hence establishing that the constrained manifold contains the correct number of bosonic and fermionic dimensions is enough to ensure the equivalence of the constrained superspace and the orbit space.

Since all points on the manifold will be equivalent, it suffices to check this property in the vicinity of the representative point,

\[
X^{AB} = X_{(0)}^{AB} + \delta X^{AB}, \quad Y^{AB} = Y_{(0)}^{AB} + \delta Y^{AB}.
\]

The constraint (23) can be linearised to yield,

\[
0 = X_{(0)}^{\hat{a}\hat{c}} \delta Y_{\hat{c}'}^{b},
\]

\[
0 = \delta X^{\hat{a}\hat{c}'} Y_{(0)}^{c'b} + X_{(0)}^{\hat{a}\hat{c}} \delta Y_{\hat{c}'}^{b},
\]

\[
0 = \delta X^{c'd'} Y_{(0)}^{c'b},
\]

and (24) gives

\[
0 = \delta X^{\hat{a}\hat{c}} X_{(0)}^{\hat{c}b} + X_{(0)}^{\hat{a}\hat{c}} \delta X_{\hat{c}'}^{b},
\]

\[
0 = X_{(0)}^{\hat{a}\hat{c}} \delta X_{\hat{c}'}^{b} - \delta Y^{\hat{a}\hat{c}'} Y_{(0)}^{c'b'},
\]

\[
0 = \delta X^{c'd'} X_{(0)}^{\hat{c}b} - Y_{(0)}^{c'd'} \delta Y_{\hat{c}'}^{b'},
\]

\[
0 = -\delta Y^{c'd'} Y_{(0)}^{c'b'} - Y_{(0)}^{c'd'} \delta Y_{\hat{c}'}^{b'}.\]

The formulae (39) and (41) mean that the unwanted components belonging to the ten-dimensional symmetric representations of SU(4) (in \(\delta X\)) and of SU(2,2) (in \(\delta Y\)) are actually eliminated by the constraints. In order to understand the meaning of (42), (45), we write

\[
\delta X^{\hat{a}\hat{b}} = \delta X^{I} \Gamma^{\hat{a}\hat{b}}, \quad \delta Y^{c'd'} = \delta Y^{I} \Gamma^{c'd'}.\]
using the fact that each of $\Gamma^{i\dot{a}b}, \Gamma^{i'a'b'}$ spans a basis of $4 \times 4$ anti-symmetric matrices. In terms of this notation \cite{42,45} imply

\begin{equation}
\delta X^i \Gamma_i^{\dot{a}c} X_j^{(0)} \Gamma_{\dot{b}c} + X_i^{(0)} \Gamma_i^{\dot{a}c} \delta X^j \Gamma_{\dot{b}c} = 0,
\end{equation}

\begin{equation}
\delta Y^{i'} \Gamma_{i'}^{a'c'} Y_j^{(0)} \Gamma_{b'c'} + Y_i^{(0)} \Gamma_{i'}^{a'c'} \delta Y^{j'} \Gamma_{b'c'} = 0.
\end{equation}

Here $\delta X^i$ and $\delta Y^{i'}$ are six-dimensional complex vectors. By decomposing them into real and imaginary parts we obtain,

\begin{equation}
X_{(0)i} \Re \delta X^i = 0, \quad Y_{(0)i'} \Re \delta Y^{i'} = 0,
\end{equation}

\begin{equation}
X_{(0)i} \Im \delta X^i - X_{(0)j} \Im \delta X^j = 0, \quad Y_{(0)i'} \Im \delta Y^{i'} - Y_{(0)j'} \Im \delta Y^{j'} = 0.
\end{equation}

Thus, the imaginary parts of the vectors $\delta X^i$ and $\delta Y^{i'}$ are proportional to $X_{(0)i}$ and $Y_{(0)i'}$ respectively; they are related to the original representative point by infinitesimal $U(1)$ transformations \cite{25} and therefore should be neglected in the ray representations. The real parts of the vectors $\delta X^i$ and $\delta Y^{i'}$ are orthogonal to $X_{(0)i}$ and $Y_{(0)i'}$ respectively; they are nothing but the tangent spaces of $AdS_5$ and $S^5$ at the representative point. Thus the bosonic tangent space of the constrained manifold is just as it should be.

The constraints for the fermionic components \cite{40,43,44} are actually all equivalent to

\begin{equation}
\delta Y_{\dot{a}c'} = -\frac{1}{R^2} X_{(0)a} \delta X^{\dot{a}c'} Y_{(0)c'}. \quad (51)
\end{equation}

Before imposing the constraints, the independent fermionic fluctuations $\delta X^{\dot{a}b'}$ and $\delta Y^{\dot{a}b'}$ have 32 complex components. The above constraint imposes a certain reality condition on them. Because of this we have 32 real components, which is the correct number for the superspace under consideration. Hence the constrained supermanifold captures correctly fermionic directions of the orbit, or equivalently the coset space.\footnote{One can also check this more directly. The supersymmetry variation of the representative point \cite{30}, \cite{37} satisfies \cite{61} under the condition \cite{53}. Conversely, for any variation $\delta X$ and $\delta Y$ satisfying \cite{54}, one can find the fermionic infinitesimal parameters satisfying \cite{53} which produce the variation by \cite{30}, \cite{37}.} Thus finally the equivalence of $PSU(2,2|4)_{SO(4,1)\times SO(5)}$ and the supermanifold defined by the constraints \cite{29}, \cite{21} is established.\footnote{Alternatively, if one does not introduce the identification \cite{14}, the same argument presented here establishes the equivalence of the constrained space to $U(2|2,2)_{SO(4,1)\times SO(5)}$.}

**Discussion** It should be possible to write down the superstring Green-Schwarz action using the variables $X^{AB}$, $Y^{AB}$ as fields defined on the string worldsheet. The constraints should be imposed by introducing $\delta$-functionals associated with the constraints, in the path integral in terms of the fields $X^{AB}$ and $Y^{AB}$. It may also be possible to take a linear sigma model type approach, in which one first studies unconstrained $X$, $Y$ fields with various coupling constants, and take an appropriate limit of these coupling constants to realise the constraints. One should take into account of the $U(1) \times U(1)$ identifications \cite{14} in order to ensure that no extra degrees of freedom enter. It may also be possible to (partially) eliminate the $U(1)$ degrees of freedom by introducing non-quadratic constraints constructed using the superdeterminant.
It is interesting to study variables similar to the ones presented in this note for other AdS
superspaces, in particular those associated with the supermembrane theory on the $AdS_4 \times S^7$
and $AdS_7 \times S^4$ spaces \[7\].

We hope that the present formulation may provide a point of view which simplifies and
clarifies the structure of supersymmetric theories on AdS spacetimes. The formalism may also
be useful for study of quantities controlled by the $PSU(2,2|4)$ symmetry such as observables
in $\mathcal{N} = 4$ Super-Yang Mills theory in four-dimension.

This note presents results of work done several years ago. I was stimulated to write up the
present results by two very recent papers \[8,9\] which develop a new formulation of superstring
theory on $AdS_5 \times S^5$ using a parametrisation of the superspace built along similar directions
to the approach proposed in this note. The bosonic degrees of freedom in \[8,9\] are represented
in a similar way as done in \[26, 27\], where we specify a part of the bosonic coordinates of the
representative point. The fermionic degrees of freedom are however introduced differently in
our formalism compared to that of \[8,9\]. The supersymmetry is realised on the (constrained)
coordinates of our superspace in a linear fashion, whereas in \[8,9\] a non-linear realisation of
the supersymmetry is used. The formalism presented here may be advantageous for some
applications, as in particular in quantum field theories linearly realised symmetries can often
be more straightforwardly dealt with compared to non-linearly realised symmetries.

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Appendix We use $\Gamma^{iab}$, $\Gamma_{iab}$ for Clebsch-Gordan coefficients relating $SO(4,2)$ and $SU(2,2)$,
and $\Gamma^{a'b'}$, $\Gamma_{a'b'}$ for those relating $SO(6)$ and $SU(4)$. They can be considered as $4 \times 4$
submatrices of the $8 \times 8$ $SO(4,2)$ and $SO(6)$ Gamma matrices. They are anti-symmetric,

\[
\Gamma^{iab} = -\Gamma^{jba}, \quad \Gamma_{iab} = -\Gamma_{jba},
\]

\[
\Gamma^{a'b'} = -\Gamma^{b'a'}, \quad \Gamma_{a'b'} = -\Gamma_{b'a'},
\]

and satisfy

\[
\eta_{i\bar{a}} \eta_{j\bar{b}} \Gamma^{i\bar{a}b} = \Gamma^{i\bar{j}b}, \quad \eta_{i\bar{a}} \eta_{j\bar{b}} \Gamma_{i\bar{a}b} = \Gamma_{i\bar{j}b},
\]

\[
\eta_{i\bar{a}} \eta_{j\bar{b}} \eta_{k\bar{c}} \eta_{l\bar{d}} \Gamma^{i\bar{a}l\bar{b}k\bar{c}} = -\Gamma^{i\bar{j}l\bar{b}k\bar{c}}, \quad \eta_{i\bar{a}} \eta_{j\bar{b}} \eta_{k\bar{c}} \eta_{l\bar{d}} \Gamma_{i\bar{a}l\bar{b}k\bar{c}} = -\Gamma_{i\bar{j}l\bar{b}k\bar{c}}.
\]

The matrices

\[
\Gamma^{i\bar{j}a\bar{b}} = \frac{1}{2} \left( \Gamma^{i\bar{a}c\bar{b}} \Gamma^{j\bar{c}} - \Gamma^{j\bar{a}c\bar{b}} \Gamma^{i\bar{c}} \right)
\]

are linearly independent, and so are

\[
\Gamma^{a'b'} = \frac{1}{2} \left( \Gamma^{a'c'd'} \Gamma^{b'c} - \Gamma^{b'a'c} \Gamma^{a'c'} \right).
\]
An explicit representation is,

\[
\Gamma^{\dot{i}\dot{a}b} = (i1 \otimes \sigma^2, -i\sigma^1 \otimes \sigma^2, \sigma^2 \otimes \sigma^1, \sigma^2 \otimes \sigma^3, i\sigma^2 \otimes 1, -\sigma^3 \otimes \sigma^2), \tag{58}
\]

\[
\Gamma^i_{\dot{a}\dot{b}} = (i1 \otimes \sigma^2, i\sigma^1 \otimes \sigma^2, \sigma^2 \otimes \sigma^1, \sigma^2 \otimes \sigma^3, -i\sigma^2 \otimes 1, \sigma^3 \otimes \sigma^2), \tag{59}
\]

\[
\Gamma^{i'd'b'} = (i\sigma^2 \otimes 1, \sigma^2 \otimes \sigma^3, \sigma^2 \otimes \sigma^1, -i\sigma^1 \otimes \sigma^2, i\sigma^3 \otimes \sigma^2, 1 \otimes \sigma^2), \tag{60}
\]

\[
\Gamma^{i'}_{a'b'} = (-i\sigma^2 \otimes 1, \sigma^2 \otimes \sigma^3, \sigma^2 \otimes \sigma^1, i\sigma^1 \otimes \sigma^2, -i\sigma^3 \otimes \sigma^2, 1 \otimes \sigma^2). \tag{61}
\]

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2 (1998) 231 [hep-th/9711200].

[2] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in AdS(5) x S**5 background,” Nucl. Phys. B 533 (1998) 109 [hep-th/9805028].

[3] M. B. Green and J. H. Schwarz, “Covariant Description of Superstrings,” Phys. Lett. B 136 (1984) 367.

M. B. Green and J. H. Schwarz, “Properties of the Covariant Formulation of Superstring Theories,” Nucl. Phys. B 243 (1984) 285.

[4] M. T. Grisaru, P. S. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, “N=2 Superstrings in a Supergravity Background,” Phys. Lett. B 162 (1985) 116.

[5] P. A. M. Dirac, “The Electron Wave Equation in De-Sitter Space,” Annals Math. 36 (1935) 657.

P. A. M. Dirac, “Wave equations in conformal space,” Annals Math. 37 (1936) 429.

[6] See, for example, L. Frappat, P. Sorba and A. Sciarrino, “Dictionary on Lie superalgebras,” hep-th/9607161.

[7] B. de Wit, K. Peeters, J. Plefka and A. Sevrin, “The M theory two-brane in AdS(4) x S**7 and AdS(7) x S**4,” Phys. Lett. B 443 (1998) 153 [hep-th/9808052].

[8] J. H. Schwarz, “New Formulation of the Type IIB Superstring Action in AdS_5 x S^5,” arXiv:1506.07706 [hep-th].

[9] W. Siegel, “Parametrization of cosets for AdS5xS5 superstring action,” arXiv:1506.08172 [hep-th].