Extensions of Hermite–Hadamard inequalities for harmonically convex functions via generalized fractional integrals

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Abstract
In the paper, the authors establish some new Hermite–Hadamard type inequalities for harmonically convex functions via generalized fractional integrals. Moreover, the authors prove extensions of the Hermite–Hadamard inequality for harmonically convex functions via generalized fractional integrals without using the harmonic convexity property for the functions. The results offered here are the refinements of the existing results for harmonically convex functions.

Keywords: Harmonically convex functions; Generalized fractional integrals; Hermite–Hadamard inequalities

1 Introduction
The Hermite–Hadamard inequality, which is the first basic result of convex mappings with a natural geometric interpretation and extensive use, has attracted attention with great interest in elementary mathematics. Many mathematicians have devoted their efforts to standardization, refining, imitation, and expansion into various categories of works such as convex mappings.

Inequalities found by C. Hermite and J. Hadamard for convex mappings are very important in literature (see [1]). These inequalities state that if \( F : I \to \mathbb{R} \) is a convex function on the interval \( I \) of real numbers and \( \kappa_1, \kappa_2 \in I \) with \( \kappa_1 < \kappa_2 \), then

\[
F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(x) \, dx \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}.
\]

(1.1)

Both inequalities hold in the reversed direction if \( F \) is concave. For further studies of this area, one can consult [2–22].

For brevity, in the upcoming results, we use the subsequent notations: Mappings \( \Lambda, \Lambda^*, \Psi, \Psi^* : [0, 1] \to \mathbb{R} \) are defined by

\[
\Lambda(x) = \int_0^x \frac{\psi((\kappa_2 - \kappa_1) \tau)}{\tau} \, d\tau < +\infty, \quad \Lambda^*(x) = \int_0^x \frac{\psi(\frac{\kappa_2 - \kappa_1}{\tau})}{\tau} \, d\tau < \infty,
\]

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\[
\Psi(x) = \int_{0}^{\infty} \frac{\varphi((x_2-x_1)\tau)}{x_2-x_1} \, d\tau < +\infty, \quad \Psi^*(x) = \int_{0}^{\infty} \frac{\varphi((x_2-x_1)\tau)}{\tau} \, d\tau < \infty,
\]

and
\[
\mathcal{G}(x) = \frac{1}{x}, \quad \phi(x) = F\left(\frac{\kappa_1\kappa_2}{x}\right),
\]

\[
m = \inf_{\tau \in [\kappa_1, \kappa_2]} \phi''(\tau), \quad M = \sup_{\tau \in [\kappa_1, \kappa_2]} \phi''(\tau).
\]

Now we give the definition of the generalized fractional integrals (GFIs) given by Sarikaya and Ertuğral in [23].

**Definition 1** The left-sided and right-sided GFIs are denoted by \( \kappa_1^+I_\varphi \) and \( \kappa_2^-I_\varphi \) and defined as follows:

\[
k_1^+I_\varphi F(x) = \int_{k_1}^{x} \frac{\varphi(x-\tau)}{x-\tau} F(\tau) \, d\tau, \quad x > k_1, \tag{1.2}
\]

\[
k_2^-I_\varphi F(x) = \int_{x}^{k_2} \frac{\varphi(\tau-x)}{\tau-x} F(\tau) \, d\tau, \quad x < k_2, \tag{1.3}
\]

where a function \( \varphi : [0, \infty) \to [0, \infty) \) satisfies the condition \( \int_{0}^{1} \frac{\varphi(\tau)}{\tau} \, d\tau < \infty \).

Recently, the authors gave some refinements of Hermite–Hadamard inequalities for GFIs under the condition of convexity, as follows.

**Theorem 1** ([23]) For a convex function \( F : [\kappa_1, \kappa_2] \to \mathbb{R} \) on \([\kappa_1, \kappa_2]\) with \( \kappa_1 < \kappa_2 \), the subsequent inequalities hold for GFIs:

\[
F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{2\Lambda(1)} \left[ \kappa_1^+I_\varphi F(\kappa_2) + \kappa_2^-I_\varphi F(\kappa_1) \right] \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}. \tag{1.4}
\]

**Theorem 2** ([24]) For a convex function \( F : [\kappa_1, \kappa_2] \to \mathbb{R} \) on \([\kappa_1, \kappa_2]\) with \( \kappa_1 < \kappa_2 \), the subsequent inequalities hold for GFIs:

\[
F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{2\Psi^*(1)} \left[ \kappa_1^+I_\varphi F(\kappa_2) + \kappa_2^-I_\varphi F(\kappa_1) \right] \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}. \tag{1.5}
\]

**Theorem 3** ([25]) For a convex function \( F : [\kappa_1, \kappa_2] \to \mathbb{R} \) on \([\kappa_1, \kappa_2]\) with \( \kappa_1 < \kappa_2 \), the subsequent inequalities hold for GFIs:

\[
F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{2\Psi^*(1)} \left[ \kappa_1^+I_\varphi F\left(\frac{\kappa_1 + \kappa_2}{2}\right) + \kappa_2^-I_\varphi F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \right] \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}. \tag{1.6}
\]

In [26], İşcan gave the following definition of harmonically convex functions and Hermite–Hadamard inequalities for harmonically convex functions.
**Definition 2** ([26]) A mapping \( F : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is called harmonically convex if
\[
F\left( \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \leq \tau F(\kappa_2) + (1 - \tau)F(\kappa_1) \tag{1.7}
\]
for all \( \kappa_1, \kappa_2 \in I \) and \( \tau \) in \([0,1]\). If inequality (1.7) holds in the reversed direction, then \( F \) is called a harmonically concave function.

**Theorem 4** ([26]) For a harmonically convex mapping \( F : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \), the following double inequality holds:
\[
F\left( \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \leq \kappa_1 \kappa_2 \int_{\kappa_1}^{\kappa_2} \frac{F(x)}{x^2} \, dx \leq \frac{F(\kappa_1) + F(\kappa_2)}{2} \tag{1.8}
\]
where \( \kappa_1, \kappa_2 \in I \) and \( \kappa_1 < \kappa_2 \).

In [27], İşcan and Wu gave the inequalities of Hermite–Hadamard type for harmonically convex functions via Riemann–Liouville fractional integrals.

**Theorem 5** ([27]) Let \( F : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a function such that \( F \in L([\kappa_1, \kappa_2]) \), where \( \kappa_1, \kappa_2 \in I \) with \( \kappa_1 < \kappa_2 \). If \( F \) is a harmonically convex function on \([\kappa_1, \kappa_2]\), the following double inequality holds for the Riemann–Liouville fractional integrals:
\[
F\left( \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \leq \frac{\Gamma(\alpha + 1)}{2} \left( \frac{\kappa_1 \kappa_2}{\kappa_2 - \kappa_1} \right)^{\alpha} \left\{ \int_{\kappa_1}^{\kappa_2} (F \circ G)\left( \frac{1}{k_2} \right) \, dx + \int_{\kappa_1}^{\kappa_2} (F \circ G)\left( \frac{1}{k_1} \right) \, dx \right\} \leq \frac{F(\kappa_1) + F(\kappa_2)}{2} \tag{1.9}
\]
where \( \alpha > 0 \).

In [28], Zhao et al. gave the following Hermite–Hadamard type inequalities for harmonically convex functions by utilizing GFIs.

**Theorem 6** Let \( F : I \subseteq (0, +\infty) \rightarrow \mathbb{R} \) be a mapping such that \( F \in L([\kappa_1, \kappa_2]) \). If \( F \) is a harmonically convex mapping on \([\kappa_1, \kappa_2]\), then the following inequalities hold for the GFIs:
\[
F\left( \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \leq \frac{1}{2\Lambda^*(1)} \left[ \int_{\kappa_1}^{\kappa_2} (F \circ G)\left( \frac{1}{k_2} \right) \, dx + \int_{\kappa_1}^{\kappa_2} (F \circ G)\left( \frac{1}{k_1} \right) \, dx \right] \leq \frac{F(\kappa_1) + F(\kappa_2)}{2} \tag{1.10}
\]

In [29], F. Chen gave the following useful lemma and the lower and upper bounds of the left- and right-hand sides of inequalities (1.9) as follows.

**Lemma 1** A mapping \( F : [\kappa_1, \kappa_2] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is called harmonically convex if and only if \( \phi(x) \) is convex on \([\kappa_1, \kappa_2]\).

**Theorem 7** Let \( F : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \rightarrow \mathbb{R} \) be a positive twice differentiable function with \( \kappa_1 < \kappa_2 \) and \( F \in L([\kappa_1, \kappa_2]) \). If \( \phi'' \) is bounded in \([\kappa_1, \kappa_2]\), then the following inequalities hold
for the Riemann–Liouville fractional integrals:

\[
\frac{m\alpha}{2(k_2 - k_1)^\alpha} \int_{x_1}^{x_1 + \alpha \Delta x} \left( \frac{k_1 + k_2}{2} - x \right)^{\alpha} \left[ (k_2 - x)^{\alpha - 1} + (x - k_1)^{\alpha - 1} \right] dx \\
\leq \frac{\Gamma(\alpha + 1)}{2} \left( \frac{k_1 + k_2}{k_2 - k_1} \right)^{\alpha} \left[ I_{\Delta x, \alpha} (\mathcal{F}(\mathcal{G})(1/k_1)) + I_{\Delta x, \alpha} (\mathcal{F}(\mathcal{G})(1/k_2)) \right] - \mathcal{F} \left( \frac{2k_1}{k_1 + k_2} \right) \\
\leq \frac{Ma}{2(k_2 - k_1)^\alpha} \int_{x_1}^{x_1 + \alpha \Delta x} \left( \frac{k_1 + k_2}{2} - x \right)^{\alpha} \left[ (k_2 - x)^{\alpha - 1} + (x - k_1)^{\alpha - 1} \right] dx. \quad (1.11)
\]

**Theorem 8** Let \( \mathcal{F} : [k_1, k_2] \subseteq (0, +\infty) \rightarrow \mathbb{R} \) be a positive twice differentiable function with \( k_1 < k_2 \) and \( \mathcal{F} \in L([k_1, k_2]) \). If \( \phi^\alpha \) is bounded in \([k_1, k_2]\), then the following inequalities hold for the Riemann–Liouville fractional integrals:

\[
\frac{-Ma}{2(k_2 - k_1)^\alpha} \int_{x_1}^{x_1 + \alpha \Delta x} (k_2 - x)(x - k_1) \left[ (k_2 - x)^{\alpha - 1} + (x - k_1)^{\alpha - 1} \right] dx \\
\leq \frac{\Gamma(\alpha + 1)}{2} \left( \frac{k_1 + k_2}{k_2 - k_1} \right)^{\alpha} \left[ I_{\Delta x, \alpha} (\mathcal{F}(\mathcal{G})(1/k_1)) + I_{\Delta x, \alpha} (\mathcal{F}(\mathcal{G})(1/k_2)) \right] - \mathcal{F}(k_1) + \frac{\mathcal{F}(k_2)}{2} \\
\leq \frac{-ma}{2(k_2 - k_1)^\alpha} \int_{x_1}^{x_1 + \alpha \Delta x} (k_2 - x)(x - k_1) \left[ (k_2 - x)^{\alpha - 1} + (x - k_1)^{\alpha - 1} \right] dx. \quad (1.12)
\]

In [30], Budak et al. gave the following inequalities for harmonically convex mappings.

**Theorem 9** Let \( \mathcal{F} : [k_1, k_2] \subseteq (0, +\infty) \rightarrow \mathbb{R} \) be a positive twice differentiable function with \( k_1 < k_2 \) and \( \mathcal{F} \in L([k_1, k_2]) \). If \( \phi^\alpha \) is bounded in \([k_1, k_2]\), then the following inequalities hold for GFIs:

\[
\frac{m}{2^\alpha (x)} \int_{x_1}^{x_2} \left( \frac{k_1 + k_2}{2} - x \right) \left[ \phi \left( \frac{x - k_1}{k_1 k_2} \right) + \frac{\phi \left( \frac{x - k_2}{k_1 k_2} \right)}{k_2 - x} \right] dx \\
\leq \frac{1}{2^\alpha (x)} \left[ \frac{1}{k_1} I_{\alpha} (\mathcal{F}(\mathcal{G})(1/k_1)) \left( \frac{1}{k_2} \right) + \frac{1}{k_2} I_{\alpha} (\mathcal{F}(\mathcal{G})(1/k_2)) \left( \frac{1}{k_1} \right) \right] - \mathcal{F} \left( \frac{2k_1 k_2}{k_1 + k_2} \right) \\
\leq \frac{M}{2^\alpha (x)} \int_{x_1}^{x_2} \left( \frac{k_1 + k_2}{2} - x \right) \left[ \phi \left( \frac{x - k_1}{k_1 k_2} \right) + \frac{\phi \left( \frac{x - k_2}{k_1 k_2} \right)}{k_2 - x} \right] dx
\]

and

\[
\frac{m}{2^\alpha (x)} \int_{x_1}^{x_2} \left[ \phi \left( \frac{x - k_1}{k_1 k_2} \right) (k_2 - x) + (x - k_1) \phi \left( \frac{k_2 - x}{k_1 k_2} \right) \right] dx \\
\leq \frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2} - \frac{1}{2^\alpha (x)} \left[ \frac{1}{k_1} I_{\alpha} (\mathcal{F}(\mathcal{G})(1/k_1)) \left( \frac{1}{k_2} \right) + \frac{1}{k_2} I_{\alpha} (\mathcal{F}(\mathcal{G})(1/k_2)) \left( \frac{1}{k_1} \right) \right] \\
\leq \frac{M}{2^\alpha (x)} \int_{x_1}^{x_2} \left[ \phi \left( \frac{x - k_1}{k_1 k_2} \right) (k_2 - x) + (x - k_1) \phi \left( \frac{k_2 - x}{k_1 k_2} \right) \right] dx.
\]

For recent findings and implications for integral inequalities via harmonically convex mappings and other classes of functions, see ([31–42]) and the references given therein.
2 Hermite–Hadamard type inequalities

In this portion, we deal with some new inequalities of Hermite–Hadamard type for harmonically convex mappings by applying GFIs.

Theorem 10 Let $F : I \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a function such that $F \in L([\kappa_1, \kappa_2])$. If $F$ is a harmonically convex function on $[\kappa_1, \kappa_2]$, then the following inequalities hold for the GFIs:

$$
F\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) \leq \frac{1}{2\psi(1)} \left[ (\frac{\kappa_2 - \kappa_1}{2\kappa_1\kappa_2}) \int_0^1 F(\frac{\kappa_2 - \kappa_1}{2\kappa_1\kappa_2}) d\tau \right]
\leq F(\kappa_1) + F(\kappa_2)
$$

(2.1)

Proof From harmonic convexity, we have

$$
F\left(\frac{2xy}{x + y}\right) \leq \frac{1}{2} [F(x) + F(y)].
$$

For $x = \frac{2\kappa_1\kappa_2}{\tau\kappa_1 + (2 - \tau)\kappa_2}$ and $y = \frac{2\kappa_1\kappa_2}{(2 - \tau)\kappa_1 + \tau\kappa_2}$, we get

$$
2F\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) \leq F\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1 + (2 - \tau)\kappa_2}\right) + F\left(\frac{2\kappa_1\kappa_2}{(2 - \tau)\kappa_1 + \tau\kappa_2}\right).
$$

(2.2)

Multiplying by $\frac{\psi\left(\frac{(\kappa_2 - \kappa_1)}{2\kappa_1\kappa_2}\right)}{\tau}$ both sides of inequality (2.2) and integrating the resultant one with respect to $\tau$ over $[0, 1]$, we obtain

$$
2F\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) \int_0^1 \frac{\psi\left(\frac{(\kappa_2 - \kappa_1)}{2\kappa_1\kappa_2}\right)}{\tau} d\tau
\leq \int_0^1 \frac{\psi\left(\frac{(\kappa_2 - \kappa_1)}{2\kappa_1\kappa_2}\right)}{\tau} F\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1 + (2 - \tau)\kappa_2}\right) d\tau
+ \int_0^1 \frac{\psi\left(\frac{(\kappa_2 - \kappa_1)}{2\kappa_1\kappa_2}\right)}{\tau} F\left(\frac{2\kappa_1\kappa_2}{(2 - \tau)\kappa_1 + \tau\kappa_2}\right) d\tau.
$$

(2.3)

For $\frac{1}{u} = \frac{2\kappa_1\kappa_2}{\tau\kappa_1 + (2 - \tau)\kappa_2}$ and $\frac{1}{v} = \frac{2\kappa_1\kappa_2}{(2 - \tau)\kappa_1 + \tau\kappa_2}$, we obtain

$$
2F\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) \psi(1)
\leq \int_{\frac{1}{u}}^{\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}} \psi\left(\frac{1}{u} - \frac{1}{\kappa_1}\right) F\left(\frac{1}{u}\right) du + \int_{\frac{1}{v}}^{\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}} \psi\left(\frac{1}{v} - \frac{1}{\kappa_2}\right) F\left(\frac{1}{v}\right) dv
= \left[ \frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2} \right] \int_0^1 F(\frac{\kappa_2 - \kappa_1}{2\kappa_1\kappa_2}) d\tau + \left[ \frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2} \right] \int_0^1 F(\frac{\kappa_2 - \kappa_1}{2\kappa_1\kappa_2}) d\tau.
$$

Hence, we proved the first inequality. To prove the second inequality of (2.1), first we note that since $F$ is a harmonically convex function, we have

$$
F\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1 + (2 - \tau)\kappa_2}\right) \leq \left(\frac{2 - \tau}{\tau}\right) F(\kappa_1) + \left(\frac{\tau}{2}\right) F(\kappa_2)
$$

(2.4)
and
\[ F\left( \frac{2k_1k_2}{(2-\tau)k_1 + \tau k_2}\right) \leq \frac{\tau}{2} F(k_1) + \left( \frac{2-\tau}{2}\right) F(k_2). \]  
(2.5)

Adding (2.4) and (2.5), we get
\[ F\left( \frac{2k_1k_2}{(2-\tau)k_1 + \tau k_2}\right) + F\left( \frac{2k_1k_2}{(2-\tau)k_1 + \tau k_2}\right) \leq F(k_1) + F(k_2). \]  
(2.6)

Multiplying by \( \frac{\psi((\kappa_2-x)/\tau)}{\tau} \) both sides of inequality (2.6) and integrating the resultant one with respect to \( \tau \) over \([0,1]\), we obtain
\[
\int_0^1 \frac{\psi((\kappa_2-x)/\tau)}{\tau} F\left( \frac{2k_1k_2}{(2-\tau)k_1 + \tau k_2}\right) d\tau + \int_0^1 \frac{\psi((\kappa_2-x)/\tau)}{\tau} F\left( \frac{2k_1k_2}{(2-\tau)k_1 + \tau k_2}\right) d\tau 
\leq \left[ F(k_1) + F(k_2) \right] \int_0^1 \frac{\psi((\kappa_2-x)/\tau)}{\tau} d\tau.
\]

By changing the variables of integration, we have the second inequality of (2.1).

**Remark 1** Under the assumptions of Theorem 10, if we put \( \psi(\tau) = \tau \), then Theorem 10 reduces to Theorem 4.

**Corollary 1** Under the assumptions of Theorem 10, if we set \( \psi(\tau) = \Gamma(\tau) \), then we have the following inequality for Riemann–Liouville fractional integrals:

\[
F\left( \frac{2k_1k_2}{k_1 + k_2}\right) 
\leq 2^{\alpha-1} \Gamma(\alpha + 1) \left( \frac{k_1k_2}{k_2 - k_1}\right)^\alpha \left[ J_{(2k_1k_2)}^\alpha (\mathcal{F} \circ \mathcal{G})(1/k_1) + J_{(2k_1k_2)}^{\alpha+1} (\mathcal{F} \circ \mathcal{G})(1/k_2) \right] 
\leq \frac{F(k_1) + F(k_2)}{2}.
\]

**Corollary 2** Under the assumptions of Theorem 10, if we set \( \psi(\tau) = \frac{\tau}{\Gamma(\tau)} \), then we have the following inequality for the \( k \)-Riemann–Liouville fractional integrals:

\[
F\left( \frac{2k_1k_2}{k_1 + k_2}\right) 
\leq 2^{\alpha-1} \Gamma_k(\alpha + k) \left( \frac{k_1k_2}{k_2 - k_1}\right)^\alpha \left[ J_{(2k_1k_2)}^\alpha (\mathcal{F} \circ \mathcal{G})(1/k_1) + J_{(2k_1k_2)}^{\alpha+1} (\mathcal{F} \circ \mathcal{G})(1/k_2) \right] 
\leq \frac{F(k_1) + F(k_2)}{2}.
\]

**Theorem 11** Let \( F : I \subseteq (0, \infty) \to \mathbb{R} \) be a function such that \( F \in L([k_1, k_2]) \). If \( F \) is a harmonically convex function on \([k_1, k_2]\), then the following inequalities hold for the GFIs:

\[
F\left( \frac{2k_1k_2}{k_1 + k_2}\right) \leq \frac{1}{2} \psi(1) \left[ \frac{1}{k_1} I_{\psi}(\mathcal{F} \circ \mathcal{G})\left( \frac{k_1 + k_2}{2k_1k_2}\right) + \frac{1}{k_2} I_{\psi}(\mathcal{F} \circ \mathcal{G})\left( \frac{k_1 + k_2}{2k_1k_2}\right) \right].
\]
Adding (2.9) and (2.10), we have

\[
\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}.
\] (2.7)

**Proof.** Since \( \mathcal{F} \) is a harmonically convex function on \([\kappa_1, \kappa_2] \), we have

\[
\mathcal{F}\left(\frac{2xy}{x+y}\right) \leq \frac{1}{2}\left[\mathcal{F}(x) + \mathcal{F}(y)\right].
\]

For \( x = \frac{2\kappa_1 k_2}{(1-\tau)k_1 + (1+\tau)k_2} \) and \( y = \frac{2\kappa_1 k_2}{(1+\tau)k_1 + (1-\tau)k_2} \), we get

\[
2\mathcal{F}\left(\frac{2\kappa_1 k_2}{\kappa_1 + \kappa_2}\right) \leq \mathcal{F}\left(\frac{2\kappa_1 k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right) + \mathcal{F}\left(\frac{2\kappa_1 k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right). \quad (2.8)
\]

Multiplying by \( \frac{\psi((\kappa_2 - \kappa_1)/(1+\tau))}{\kappa_2} \) both sides of inequality (2.8) and integrating the resultant one with respect to \( \tau \) over \([0, 1]\), we obtain

\[
2\mathcal{F}\left(\mathcal{F}\left(\frac{2\kappa_1 k_2}{\kappa_1 + \kappa_2}\right) \Psi(1) \right) \leq \int_0^1 \frac{\psi((\kappa_2 - \kappa_1)/(1+\tau))}{\kappa_2} \mathcal{F}\left(\frac{2\kappa_1 k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right) d\tau + \int_0^1 \frac{\psi((\kappa_2 - \kappa_1)/(1+\tau))}{\kappa_2} \mathcal{F}\left(\frac{2\kappa_1 k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right) d\tau.
\]

By setting \( \frac{1}{\kappa_1} = \frac{2\kappa_1 k_2}{(1-\tau)k_1 + (1+\tau)k_2} \) and \( \frac{1}{\kappa_2} = \frac{2\kappa_1 k_2}{(1+\tau)k_1 + (1-\tau)k_2} \), we have

\[
2\mathcal{F}\left(\frac{2\kappa_1 k_2}{\kappa_1 + \kappa_2}\right) \Psi(1) \leq \int_{\frac{1}{\kappa_1}}^{\frac{1}{\kappa_2}} \frac{\psi(u - \frac{\kappa_1 + \kappa_2}{2})}{\kappa_1} \mathcal{F}\left(\frac{1}{u}\right) du + \int_{\frac{1}{\kappa_2}}^{\frac{1}{\kappa_1}} \frac{\psi(v - \frac{\kappa_1 + \kappa_2}{2})}{\kappa_2} \mathcal{F}\left(\frac{1}{v}\right) dv
\]

\[
= \left[ \frac{1}{\kappa_1} I_\psi (\mathcal{F} \circ \mathcal{G})(\frac{1}{\kappa_1 + \kappa_2}) \right] + \left[ \frac{1}{\kappa_2} I_\psi (\mathcal{F} \circ \mathcal{G})(\frac{1}{\kappa_1 + \kappa_2}) \right].
\]

Hence we have the first inequality in (2.7).

To prove the second inequality in (2.7), first we note that \( \mathcal{F} \) is a harmonically convex function, we get

\[
\mathcal{F}\left(\frac{2\kappa_1 k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right) \leq \frac{1 + \tau}{2} \mathcal{F}(\kappa_1) + \frac{1 - \tau}{2} \mathcal{F}(\kappa_2) \quad (2.9)
\]

and

\[
\mathcal{F}\left(\frac{2\kappa_1 k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right) \leq \frac{1 - \tau}{2} \mathcal{F}(\kappa_1) + \frac{1 + \tau}{2} \mathcal{F}(\kappa_2). \quad (2.10)
\]

Adding (2.9) and (2.10), we have

\[
\mathcal{F}\left(\frac{2\kappa_1 k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right) + \mathcal{F}\left(\frac{2\kappa_1 k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right) \leq \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2). \quad (2.11)
\]
Multiplying by $\frac{\psi((x_2-x_1)/t)}{t}$ both sides of inequality (2.11) and integrating the resultant one with respect to $\tau$ over $[0, 1]$, we obtain

$$
\int_0^1 \frac{\psi((x_2-x_1)/t)}{t} \left[ F\left(\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1 + (1+\tau)\kappa_2}\right) + F\left(\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1 + (1-\tau)\kappa_2}\right) \right] d\tau \leq \Psi(1)\left[F(\kappa_1) + F(\kappa_2)\right].
$$

By changing the variable of integration, we have the second inequality in (2.7).

**Remark 2** Under the assumptions of Theorem 11, if we put $\varphi(\tau) = \tau$, then Theorem 10 reduces to Theorem 4.

**Corollary 3** Under the assumptions of Theorem 11, if we set $\varphi(\tau) = \frac{\alpha}{\Gamma(\alpha)}\tau$, then we have the following inequalities for the Riemann–Liouville fractional integrals:

$$
F\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)
\leq 2^{\alpha-1} \Gamma(\alpha + 1) \left[ j_{1/(\kappa_1)}^\alpha (F \circ G)\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) + j_{1/(\kappa_2)}^\alpha (F \circ G)\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) \right]
\leq \frac{F(\kappa_1) + F(\kappa_2)}{2}.
$$

**Corollary 4** Under the assumptions of Theorem 11, if we put $\varphi(\tau) = \frac{x^k}{\Gamma(x^k)}$, then we have the following inequalities for the $k$-Riemann–Liouville fractional integrals:

$$
F\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)
\leq 2^{\frac{k}{\alpha-1}} \Gamma_k(\alpha + 1) \left[ j_{1/(\kappa_1)}^{\alpha-k} (F \circ G)\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) + j_{1/(\kappa_2)}^{\alpha-k} (F \circ G)\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) \right]
\leq \frac{F(\kappa_1) + F(\kappa_2)}{2}.
$$

### 3 Extension of Hermite–Hadamard type inequalities

In this section, we give the following inequalities which give the above and below bounds for the left- and right-hand sides of inequalities (2.1) and (2.7). We prove inequalities (2.1) and (2.7) under the condition $\phi'(\kappa_1 + \kappa_2 - x) \geq \phi'(x)$ instead of the harmonic convexity of $F$.

**Theorem 12** Let $F : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \to \mathbb{R}$ be a positive twice differentiable function with $\kappa_1 < \kappa_2$ and $F \in L([\kappa_1, \kappa_2])$. If $\phi''$ is bounded in $[\kappa_1, \kappa_2]$, then the following inequalities hold for GFIs:

$$
m \frac{m}{2\Psi(1)} \int_{\kappa_1}^{\kappa_1+\kappa_2} \left(\frac{\kappa_1 + \kappa_2}{2} - x\right)^2 \frac{\phi'(x-k_1)}{x-k_1} dx
$$
\[ \leq \frac{1}{2\Psi(1)} \left[ \frac{\psi(\gamma x^2)}{\gamma x^2}, I_\nu(F \circ G)(1/\kappa_1) + \frac{\psi(\gamma x^2)}{\gamma x^2}, I_\nu(F \circ G)(1/\kappa_2) \right] - F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) \]

\[ \leq M \int_{x_1}^{x_2} \left( \frac{\kappa_1 + \kappa_2}{2} - x \right)^2 \frac{\psi(\frac{x-x_1}{\kappa_1\kappa_2})}{x - \kappa_1} \, dx. \]  

\text{(3.1)}

**Proof** By using the change of variables, we have

\[ \frac{1}{2\Psi(1)} \left[ \frac{\psi(\gamma x^2)}{\gamma x^2}, I_\nu(F \circ G)(1/\kappa_1) + \frac{\psi(\gamma x^2)}{\gamma x^2}, I_\nu(F \circ G)(1/\kappa_2) \right] \]

\[ \leq \frac{1}{2\Psi(1)} \left[ \int_{x_1}^{x_2} \frac{\psi(\gamma x^2)}{\gamma x^2}, F \left( \frac{1}{x} \right) \, dx + \int_{x_1}^{x_2} \frac{\psi(\gamma x^2)}{\gamma x^2}, F \left( \frac{1}{x} \right) \, dx \right] \]

\[ = \frac{1}{2\Psi(1)} \left[ \int_{x_1}^{x_2} \frac{\psi(\gamma x^2)}{\gamma x^2}, \frac{\kappa_1\kappa_2}{x} \, dx + \int_{x_1}^{x_2} \frac{\psi(\gamma x^2)}{\gamma x^2}, \frac{\kappa_1\kappa_2}{x - \kappa_1} \, dx \right] \]

\[ = \frac{1}{2\Psi(1)} \left[ \frac{\kappa_1\kappa_2}{x} + \frac{\kappa_1\kappa_2}{x - \kappa_1} \right] \psi \left( \frac{x-x_1}{\gamma x^2} \right) \psi \left( \frac{x-x_2}{\gamma x^2} \right) \, dx. \]  

\text{(3.2)}

By equality (3.2), we get

\[ \frac{1}{2\Psi(1)} \left[ \frac{\psi(\gamma x^2)}{\gamma x^2}, I_\nu(F \circ G)(1/\kappa_1) + \frac{\psi(\gamma x^2)}{\gamma x^2}, I_\nu(F \circ G)(1/\kappa_2) \right] - F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) \]

\[ = \frac{1}{2\Psi(1)} \int_{x_1}^{x_2} \frac{\kappa_1\kappa_2}{x} + \frac{\kappa_1\kappa_2}{x - \kappa_1} \psi \left( \frac{x-x_1}{\gamma x^2} \right) \psi \left( \frac{x-x_2}{\gamma x^2} \right) \, dx - F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) \]

\[ = \frac{1}{2\Psi(1)} \int_{x_1}^{x_2} \frac{\kappa_1\kappa_2}{x} + \frac{\kappa_1\kappa_2}{x - \kappa_1} - 2F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) \frac{\psi(\gamma x^2)}{\gamma x^2}, \frac{\kappa_1\kappa_2}{x - \kappa_1} \, dx. \]  

\text{(3.3)}

Using the fact that

\[ F \left( \frac{\kappa_1\kappa_2}{x} \right) - F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) = \phi(x) - \phi \left( \frac{\kappa_1 + \kappa_2}{2} \right) = - \int_{x}^{\gamma x^2 \phi} \phi' \, d\tau \]

and

\[ F \left( \frac{\kappa_1\kappa_2}{x - \kappa_1 + \kappa_2} \right) - F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) = \phi(x) - \phi \left( \frac{\kappa_1 + \kappa_2}{2} \right) = \int_{x}^{\gamma x^2 \phi} \phi' \, d\tau, \]

we have

\[ F \left( \frac{\kappa_1\kappa_2}{x} \right) + F \left( \frac{\kappa_1\kappa_2}{x - \kappa_1 + \kappa_2} \right) - 2F \left( \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \right) \]

\[ = \int_{x}^{\gamma x^2 \phi} \phi' \, d\tau - \int_{x}^{\gamma x^2 \phi} \phi' \, d\tau \]

\[ = \int_{x}^{\gamma x^2 \phi} \phi' \, d\tau - \int_{x}^{\gamma x^2 \phi} \phi' \, d\tau \]

\[ = \int_{x}^{\gamma x^2 \phi} \left[ \phi' \left( \kappa_1 + \kappa_2 - \tau \right) - \phi' \left( \tau \right) \right] \, d\tau. \]  

\text{(3.4)}
We also have
\[
\phi'(k_1 + k_2 - \tau) - \phi'(\tau) = \int^{k_1+k_2-\tau}_\tau \phi''(u) \, du.
\] (3.5)

By using equality (3.5) and the assumption which gives inequality (3.1), we obtain
\[
m \int^{k_1+k_2-\tau}_\tau du \leq \int^{k_1+k_2-\tau}_\tau \phi''(u) \, du \leq M \int^{k_1+k_2-\tau}_\tau du
\]
i.e.
\[
m(k_1 + k_2 - 2\tau) \leq \phi'(k_1 + k_2 - \tau) - \phi'(\tau) \leq M(k_1 + k_2 - 2\tau).
\] (3.6)

Integrating inequality (3.6) with respect to \(\tau\) on \([x, \frac{k_1+k_2}{2}]\), we get
\[
m\left(\frac{k_1 + k_2}{2} - x\right)^2 \leq \int^\frac{k_1+k_2}{2}_x \left[\phi'(k_1 + k_2 - \tau) - \phi'(\tau)\right] d\tau \leq M\left(\frac{k_1 + k_2}{2} - x\right)^2.
\]

By equality (3.4), we have
\[
m\left(\frac{k_1 + k_2}{2} - x\right)^2 \leq F\left(\frac{k_1k_2}{x}\right) + F\left(\frac{k_1k_2}{k_1 + k_2 - x}\right) - 2F\left(\frac{2k_1k_2}{k_1 + k_2}\right)
\]
\[
\leq M\left(\frac{k_1 + k_2}{2} - x\right)^2.
\] (3.7)

Multiplying inequality (3.7) by \(\frac{\psi(k_{x+1/2})}{\Psi(1/2)}\) and integrating the resultant inequality with respect to \(x\) on \([k_1, \frac{k_1+k_2}{2}]\), we establish
\[
\frac{m}{2\Psi(1)} \int^{\frac{k_1+k_2}{2}}_{k_1} \left(\frac{k_1 + k_2}{2} - x\right)^2 \psi\left(\frac{x-k_1}{k_{x+1/2}}\right) \, dx
\]
\[
\leq \frac{1}{2\Psi(1)} \int^{\frac{k_1+k_2}{2}}_{k_1} \left[F\left(\frac{k_1k_2}{x}\right) + F\left(\frac{k_1k_2}{k_1 + k_2 - x}\right) - 2F\left(\frac{2k_1k_2}{k_1 + k_2}\right)\right] \psi\left(\frac{x-k_1}{k_{x+1/2}}\right) \, dx
\]
\[
\leq \frac{M}{2\Psi(1)} \int^{\frac{k_1+k_2}{2}}_{k_1} \left(\frac{k_1 + k_2}{2} - x\right)^2 \psi\left(\frac{x-k_1}{k_{x+1/2}}\right) \, dx.
\]

That is, we get
\[
\frac{m}{2\Psi(1)} \int^{\frac{k_1+k_2}{2}}_{k_1} \left(\frac{k_1 + k_2}{2} - x\right)^2 \psi\left(\frac{x-k_1}{k_{x+1/2}}\right) \, dx
\]
\[
\leq \frac{1}{2\Psi(1)} \left[I_\varphi(G)(1/k_1) + I_\varphi(G)(1/k_2)\right] - F\left(\frac{2k_1k_2}{k_1 + k_2}\right)
\]
\[
\leq \frac{M}{2\Psi(1)} \int^{\frac{k_1+k_2}{2}}_{k_1} \left(\frac{k_1 + k_2}{2} - x\right)^2 \psi\left(\frac{x-k_1}{k_{x+1/2}}\right) \, dx,
\]
which gives inequality (3.1). □
Remark 3 Under the assumptions of Theorem 12, if we put $\varphi(\tau) = \tau$, then we have the following inequalities:

$$
\frac{m(k_2 - k_1)^2}{24} \leq \frac{k_1k_2}{k_2 - k_1} \int_{x_1}^{x_2} \frac{\mathcal{F}(x)}{x^2} \, dx - \mathcal{F} \left( \frac{2k_1k_2}{k_1 + k_2} \right) \leq \frac{M(k_2 - k_1)^2}{24}.
$$

(3.8)

Corollary 5 Under the assumptions of Theorem 12, if we set $\varphi(\tau) = \frac{\tau}{\Gamma(\alpha + 1)}$, then we have the following inequalities for the Riemann–Liouville fractional integrals:

$$
\frac{m(k_2 - k_1)^2}{4(\alpha + 1)(\alpha + 2)} \leq 2^{\alpha + 1} \Gamma(\alpha + 1) \left( \frac{k_1k_2}{k_2 - k_1} \right) \alpha

\times \left[ \int_{x_1}^{x_2} \frac{\mathcal{F}(x) - \mathcal{F}(0)}{x^{1 + \alpha}} \, dx \right]^{\alpha}

\leq \frac{M(k_2 - k_1)^2}{4(\alpha + 1)(\alpha + 2)}

\leq \frac{M(k_2 - k_1)^2}{4(\alpha + 1)(\alpha + 2)}.
$$

Corollary 6 Under the assumptions of Theorem 12, if we put $\varphi(\tau) = \frac{\tau^2}{\Gamma(2)}$, then we have the following inequalities for the $k$-Riemann–Liouville fractional integrals:

$$
\frac{m(k_2 - k_1)^2}{4(\frac{\alpha}{2} + 1)(\frac{\alpha}{2} + 2)}

\leq 2^{\frac{\alpha}{2} + 1} \Gamma(\alpha + k) \left( \frac{k_1k_2}{k_2 - k_1} \right) \frac{\alpha}{2}

\times \left[ \int_{x_1}^{x_2} \frac{\mathcal{F}(x) - \mathcal{F}(0)}{x^{1 + \alpha}} \, dx \right]^{\alpha}

\leq \frac{M(k_2 - k_1)^2}{4(\frac{\alpha}{2} + 1)(\frac{\alpha}{2} + 2)}.
$$

Theorem 13 Let $\mathcal{F} : [k_1, k_2] \subseteq (0, +\infty) \to \mathbb{R}$ be a positive twice differentiable function with $k_1 < k_2$ and $\mathcal{F} \in L([k_1, k_2])$. If $\mathcal{F}''$ is bounded in $[k_1, k_2]$, then the following inequalities hold for the GPs:

$$
\frac{m}{2\Psi(1)} \int_{x_1}^{x_2} \frac{x - k_1}{k_2 - x} \varphi \left( \frac{x - k_1}{k_1k_2} \right) \, dx

\leq \frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2} - \frac{1}{2\Psi(1)} \left[ \int_{x_1}^{x_2} \frac{\mathcal{F}(x) - \mathcal{F}(0)}{x^{1 + \alpha}} \, dx \right]^{\alpha}

\leq \frac{M}{2\Psi(1)} \int_{x_1}^{x_2} \frac{x - k_1}{k_2 - x} \varphi \left( \frac{x - k_1}{k_1k_2} \right) \, dx.
$$

(3.9)

Proof By using the change of variables, we have

$$
\frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2} - \frac{1}{2\Psi(1)} \left[ \int_{x_1}^{x_2} \frac{\mathcal{F}(x) - \mathcal{F}(0)}{x^{1 + \alpha}} \, dx \right]^{\alpha}

= \frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2} - \frac{1}{2\Psi(1)}
$$
By using the equalities

\[
\mathcal{F}(\frac{k_1 k_2}{x}) - \mathcal{F}(k_1) = \phi(x) - \phi(k_1) = \int_{k_1}^{x} \phi'(\tau) \, d\tau
\]

and

\[
\mathcal{F}(k_2) - \mathcal{F}(\frac{k_1 k_2}{k_1 + k_2 - x}) = \phi(k_2) - \phi(k_1 + k_2 - x) = \int_{k_1 + k_2 - x}^{k_2} \phi'(\tau) \, d\tau,
\]

we have

\[
\phi(k_1) + \phi(k_2) - \phi(x) - \phi(k_1 + k_2 - x)
\]

\[
= \int_{k_1 + k_2 - x}^{k_2} \phi'(\tau) \, d\tau - \int_{k_1}^{x} \phi'(\tau) \, d\tau = \int_{k_1}^{x} [\phi'(k_1 + k_2 - \tau) - \phi'(\tau)] \, d\tau. 
\]  

(3.11)

Integrating (3.6) with respect to \(\tau\) over \([k_1, x]\), we get

\[
m \int_{k_1}^{x} (k_1 + k_2 - 2\tau) \, d\tau \leq \int_{k_1}^{x} [\phi'(k_1 + k_2 - \tau) - \phi'(\tau)] \, d\tau
\]

\[
\leq M \int_{k_1}^{x} (k_1 + k_2 - 2\tau) \, d\tau,
\]

which implies that

\[
m(x - k_1)(k_2 - x) \leq \mathcal{F}(k_1) + \mathcal{F}(k_2) - \mathcal{F}(\frac{k_1 k_2}{x}) - \mathcal{F}(\frac{k_1 k_2}{k_1 + k_2 - x})
\]

\[
\leq M(x - k_1)(k_2 - x). 
\]  

(3.12)

Multiplying inequality (3.12) by \(\frac{\phi(x-k_1)}{2\Psi(1/x-k_1)}\) and integrating the resultant inequality with respect to \(x\) on \([k_1, x]_{\frac{x_1 + x_2}{2}}\), we establish

\[
\frac{m}{2\Psi(1)} \int_{k_1}^{x} \left[ \mathcal{F}(k_1) + \mathcal{F}(k_2) - \mathcal{F}(\frac{k_1 k_2}{x}) - \mathcal{F}(\frac{k_1 k_2}{k_1 + k_2 - x}) \right] \frac{\phi(x-k_1)}{x-k_1} \, dx
\]

\[
\leq \int_{k_1}^{x} \left[ \mathcal{F}(k_1) + \mathcal{F}(k_2) - \mathcal{F}(\frac{k_1 k_2}{x}) - \mathcal{F}(\frac{k_1 k_2}{k_1 + k_2 - x}) \right] \frac{\phi(x-k_1)}{x-k_1} \, dx
\]
Remark 4 Under the assumptions of Theorem 13, if we put \( \varphi(x) = \frac{x}{\kappa_2 - x} \), then we have the following inequalities for the \( k \)-Riemann–Liouville fractional integrals which gives inequalities (3.9).

That can be written as

\[
\frac{m}{2\Psi(1)} \int_{\kappa_1}^{\kappa_1 + \kappa_2} (x - \kappa_1)(\kappa_2 - x) \frac{\varphi(x)}{\kappa_2 - x} dx.
\]

which gives inequalities (3.9). \( \square \)

Corollary 7 Under the assumptions of Theorem 13, if we set \( \varphi(x) = \frac{x}{\kappa_1} \), then we have the following inequalities for the Riemann–Liouville fractional integrals:

\[
\frac{m(k_2 - \kappa_1)^2}{12} \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} - \frac{\kappa_1 \kappa_2}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \frac{\mathcal{F}(x)}{x^2} dx \leq \frac{M(k_2 - \kappa_1)^2}{24}.
\] (3.13)

Corollary 8 Under the assumptions of Theorem 13, if we put \( \varphi(x) = \frac{x}{\kappa_1} \), then we have the following inequalities for the \( k \)-Riemann–Liouville fractional integrals:

\[
\frac{m(k_2 - \kappa_1)^2}{8(\alpha + 1)(\alpha + 2)} \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} - 2^{\alpha - 1} \Gamma(\alpha + 1) \left( \frac{\kappa_1 \kappa_2}{\kappa_2 - \kappa_1} \right)^\alpha \\
\times \left[ \int_{\kappa_1}^{\kappa_1 + \kappa_2} \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) \right] \leq M(k_2 - \kappa_1)^2(\alpha + 3)
\]
**Theorem 14** Let $F : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \to \mathbb{R}$ be a positive twice differentiable function with $\kappa_1 < \kappa_2$ and $F \in L([\kappa_1, \kappa_2])$. If $\phi'(\kappa_1 + \kappa_2 - x) \geq \phi'(x), \forall x \in [\kappa_1, \frac{\kappa_1 + \kappa_2}{2}]$, then we have the following inequalities for GFI's:

$$
\frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \leq \frac{1}{2\Psi(1)} \left[ \left( \frac{\kappa_1 + \kappa_2}{2} \right), \mathcal{I}_p(F \circ G)(1/\kappa_1) + \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right), \mathcal{I}_p(F \circ G)(1/\kappa_2) \right] - \frac{F(\kappa_1) + F(\kappa_2)}{2}.
$$

(3.14)

**Proof** From (3.3) and (3.4), one has

$$
\frac{1}{2\Psi(1)} \left[ \left( \frac{\kappa_1 + \kappa_2}{2} \right), \mathcal{I}_p(F \circ G)(1/\kappa_1) + \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right), \mathcal{I}_p(F \circ G)(1/\kappa_2) \right] - \frac{F(\kappa_1) + F(\kappa_2)}{2} = \frac{1}{2\Psi(1)} \int_{\kappa_1}^{\kappa_2} \left[ F \left( \frac{\kappa_1 \kappa_2}{x} \right) + F \left( \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2 - x} \right) - 2F \left( \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \right] \frac{\phi(\kappa_1 + \kappa_2 - \tau)}{x - \kappa_1} d\tau.
$$

which gives the first inequality in (3.14). On the other hand, by equalities (3.10) and (3.11), we have

$$
\frac{F(\kappa_1) + F(\kappa_2)}{2} - \frac{1}{2\Psi(1)} \left[ \left( \frac{\kappa_1 + \kappa_2}{2} \right), \mathcal{I}_p(F \circ G)(1/\kappa_1) + \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right), \mathcal{I}_p(F \circ G)(1/\kappa_2) \right] = \frac{1}{2\Psi(1)} \int_{\kappa_1}^{\kappa_2} \left[ F(\kappa_1) + F(\kappa_2) - F \left( \frac{\kappa_1 \kappa_2}{x} \right) - \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2 - x} \right] \frac{\phi(\kappa_1 + \kappa_2 - \tau)}{x - \kappa_1} d\tau.
$$

This gives the second inequality in (3.14) and completes the proof.  

**Theorem 15** Let $F : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \to \mathbb{R}$ be a positive twice differentiable function with $\kappa_1 < \kappa_2$ and $F \in L([\kappa_1, \kappa_2])$. If $\phi''$ is bounded in $[\kappa_1, \kappa_2]$, then the following inequalities hold for the GFI's:

$$
\frac{m}{2\Psi(1)} \int_{\kappa_1}^{\kappa_2} \left( \frac{\kappa_1 + \kappa_2}{2} - x \right) \phi \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right) - \frac{x}{\kappa_1 \kappa_2} dx.
$$

$$
\leq \frac{1}{2\Psi(1)} \left[ \left( \frac{\kappa_1 + \kappa_2}{2} \right), \mathcal{I}_p(F \circ G)(\kappa_1 + \kappa_2) + \frac{1}{\kappa_1 \kappa_2}, \mathcal{I}_p(F \circ G)(\kappa_1 + \kappa_2) \right] - \frac{F(\kappa_1 \kappa_2)}{\kappa_1 + \kappa_2}.
$$

(3.15)

**Proof** By using the change of variables, we have

$$
\frac{1}{2\Psi(1)} \left[ \left( \frac{\kappa_1 + \kappa_2}{2} \right), \mathcal{I}_p(F \circ G)(\kappa_1 + \kappa_2) + \frac{1}{\kappa_2}, \mathcal{I}_p(F \circ G)(\kappa_1 + \kappa_2) \right] - \frac{F(\kappa_1 \kappa_2)}{\kappa_1 + \kappa_2}.
$$
Using the fact that by equality (3.16), we get

\[
\int_{1}^{\frac{1}{2}x_{1}+x_{2}} \frac{x}{x_{1}+x_{2}} \frac{\phi'(x)}{\phi(x)} \, dx = \int_{1}^{\frac{1}{2}x_{1}+x_{2}} \frac{1}{x} \frac{\phi'(x)}{\phi(x)} \, dx - \int_{x_{1}+x_{2}}^{x} \frac{1}{x} \frac{\phi'(x)}{\phi(x)} \, dx.
\]

By equality (3.16), we get

\[
\int_{1}^{\frac{1}{2}x_{1}+x_{2}} \frac{x}{x_{1}+x_{2}} \frac{\phi'(x)}{\phi(x)} \, dx = \int_{1}^{\frac{1}{2}x_{1}+x_{2}} \frac{1}{x} \frac{\phi'(x)}{\phi(x)} \, dx - \int_{x_{1}+x_{2}}^{x} \frac{1}{x} \frac{\phi'(x)}{\phi(x)} \, dx.
\]

Using the fact that

\[
\mathcal{F}\left(\frac{\kappa_1 \kappa_2}{x}\right) - \mathcal{F}\left(\frac{2\kappa_1 \kappa_2}{k_1 + k_2}\right) = \phi(x) - \phi\left(\frac{k_1 + k_2}{2}\right) = -\int_{x}^{\frac{1}{2}x_{1}+x_{2}} \phi'(\tau) \, d\tau
\]

and

\[
\mathcal{F}\left(\frac{\kappa_1 \kappa_2}{x_{1}+x_{2}} - x\right) - \mathcal{F}\left(\frac{2\kappa_1 \kappa_2}{k_1 + k_2}\right) = \phi(k_1 + k_2 - x) - \phi\left(\frac{k_1 + k_2}{2}\right) = \int_{\frac{1}{2}x_{1}+x_{2}}^{x} \phi'(\tau) \, d\tau,
\]

we have

\[
\mathcal{F}\left(\frac{\kappa_1 \kappa_2}{x}\right) + \mathcal{F}\left(\frac{\kappa_1 \kappa_2}{x_{1}+x_{2}} - x\right) - 2\mathcal{F}\left(\frac{2\kappa_1 \kappa_2}{k_1 + k_2}\right)
\]

\[
= \int_{\frac{1}{2}x_{1}+x_{2}}^{x} \phi'(\tau) \, d\tau - \int_{x}^{\frac{1}{2}x_{1}+x_{2}} \phi'(\tau) \, d\tau
\]

\[
= \int_{x}^{\frac{1}{2}x_{1}+x_{2}} \phi'(k_1 + k_2 - u) \, du - \int_{x}^{\frac{1}{2}x_{1}+x_{2}} \phi'(\tau) \, d\tau
\]

\[
= \int_{x}^{\frac{1}{2}x_{1}+x_{2}} \left[ \phi'(k_1 + k_2 - \tau) - \phi'(\tau) \right] \, d\tau. \tag{3.18}
\]

We also have

\[
\phi'(k_1 + k_2 - \tau) - \phi'(\tau) = \int_{\tau}^{k_1 + k_2 - \tau} \phi''(u) \, du. \tag{3.19}
\]
By using equality (3.19) and the assumption \( m < \phi''(u) < M, \) \( u \in [k_1, k_2] \), we obtain
\[
m \int_{\tau}^{x_{k_1 + k_2 - \tau}} du \leq \int_{\tau}^{x_{k_1 + k_2 - \tau}} \phi''(u) du \leq M \int_{\tau}^{x_{k_1 + k_2 - \tau}} du
\]
i.e.
\[
m(k_1 + k_2 - 2\tau) \leq \phi'(k_1 + k_2 - \tau) - \phi'(\tau) \leq M(k_1 + k_2 - 2\tau).
\]
(3.20)

Integrating inequality (3.20) with respect to \( \tau \) on \( [\tau, \frac{k_1 + k_2}{2}] \), we get
\[
m \left( \frac{k_1 + k_2}{2} - x \right)^2 \leq \int_{\tau}^{\frac{k_1 + k_2}{2}} \left[ \phi'(k_1 + k_2 - \tau) - \phi'(\tau) \right] d\tau \leq M \left( \frac{k_1 + k_2}{2} - x \right)^2.
\]
(3.21)

By equality (3.18), we have
\[
m \left( \frac{k_1 + k_2}{2} - x \right)^2 \leq F \left( \frac{k_1 k_2}{x} \right) + F \left( \frac{k_1 k_2}{k_1 + k_2 - x} \right) - 2F \left( \frac{2k_1 k_2}{k_1 + k_2} \right)
\]
\[
\leq M \left( \frac{k_1 + k_2}{2} - x \right)^2.
\]

Multiplying inequality (3.21) by \( \frac{\phi' \left( \frac{k_1 + k_2}{2} \right) - \phi' \left( \frac{k_1 + k_2}{2} \right)}{2 \tau \left( \frac{k_1 + k_2}{2} \right) - x} \) and integrating the resultant inequality with respect to \( x \) on \( [\tau, \frac{k_1 + k_2}{2}] \), we establish
\[
m \int_{\tau}^{\frac{k_1 + k_2}{2}} \left( \frac{k_1 + k_2}{2} - x \right)^2 \phi'(\frac{k_1 + k_2}{2} - x) \left( \frac{k_1 + k_2}{2} - x \right) \left( \frac{k_1 + k_2}{2} - x \right) dx
\]
\[
\leq \frac{\psi \left( \frac{k_1 + k_2}{2} \right) - \psi \left( \frac{k_1 + k_2}{2} \right)}{2 \tau \left( \frac{k_1 + k_2}{2} \right) - x} \left[ \frac{k_1 k_2}{x} \right] \left( \frac{k_1 + k_2}{2} - x \right) \left( \frac{k_1 + k_2}{2} - x \right) dx
\]
\[
\leq \frac{M}{2 \psi \left( \frac{k_1 + k_2}{2} \right)} \int_{\tau}^{\frac{k_1 + k_2}{2}} \left( \frac{k_1 + k_2}{2} - x \right)^2 \phi'(\frac{k_1 + k_2}{2} - x) \left( \frac{k_1 + k_2}{2} - x \right) \left( \frac{k_1 + k_2}{2} - x \right) dx.
\]

That is, we get
\[
m \int_{\tau}^{\frac{k_1 + k_2}{2}} \left( \frac{k_1 + k_2}{2} - x \right)^2 \phi'(\frac{k_1 + k_2}{2} - x) \left( \frac{k_1 + k_2}{2} - x \right) \left( \frac{k_1 + k_2}{2} - x \right) dx
\]
\[
\leq \frac{\psi \left( \frac{k_1 + k_2}{2} \right) - \psi \left( \frac{k_1 + k_2}{2} \right)}{2 \tau \left( \frac{k_1 + k_2}{2} \right) - x} \left[ \frac{k_1 k_2}{x} \right] \left( \frac{k_1 + k_2}{2} - x \right) \left( \frac{k_1 + k_2}{2} - x \right) dx
\]
\[
\leq \frac{M}{2 \psi \left( \frac{k_1 + k_2}{2} \right)} \int_{\tau}^{\frac{k_1 + k_2}{2}} \left( \frac{k_1 + k_2}{2} - x \right)^2 \phi'(\frac{k_1 + k_2}{2} - x) \left( \frac{k_1 + k_2}{2} - x \right) \left( \frac{k_1 + k_2}{2} - x \right) dx,
\]
which gives inequality (3.15).

Remark 5 Under the assumptions of Theorem 15, if we put \( \psi(x) = \tau \), then inequality (3.15) reduces to inequality (3.8).
Corollary 9  Under the assumptions of Theorem 15, if we set \( \varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)} \), then we have the following inequalities for the Riemann–Liouville fractional integrals:

\[
\frac{m\alpha(k_2 - \kappa_1)^2}{8(\alpha + 2)} 
\leq 2^{\alpha-1}\Gamma(\alpha + 1)\left(\frac{\kappa_1k_2}{k_2 - \kappa_1}\right)^\alpha
\times \left[ \int_{(1/k)}^G (\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) + \int_{(1/k)}^G (\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) \right] - \mathcal{F} \left( \frac{2k_1k_2}{k_1 + k_2} \right) 
\leq \frac{M\alpha(k_2 - \kappa_1)^2}{8(\alpha + 2)}.
\]

Corollary 10  Under the assumptions of Theorem 15, if we put \( \varphi(\tau) = \frac{\varphi}{\Gamma(\kappa_2)} \), then we have the following inequalities for the \( k \)-Riemann–Liouville fractional integrals:

\[
\frac{m^2\alpha(k_2 - \kappa_1)^2}{8(\frac{\alpha}{k} + 2)} 
\leq 2^{\frac{\alpha}{k} - 1}\Gamma(\alpha + k)\left(\frac{k_1k_2}{k_2 - \kappa_1}\right)^\frac{\alpha}{k}
\times \left[ \int_{(1/k)}^G (\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) + \int_{(1/k)}^G (\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) \right] - \mathcal{F} \left( \frac{2k_1k_2}{k_1 + k_2} \right) 
\leq \frac{M^2\alpha(k_2 - \kappa_1)^2}{8(\frac{\alpha}{k} + 2)}.
\]

Theorem 16  Let \( \mathcal{F} : [k_1, k_2] \subseteq (0, +\infty) \rightarrow \mathbb{R} \) be a positive twice differentiable function with \( k_1 < k_2 \) and \( \mathcal{F} \in L([k_1, k_2]) \). If \( \psi'' \) is bounded in \([k_1, k_2] \), then the following inequalities hold for the GFIs:

\[
\frac{m}{2\Psi(1)} \int_{k_1}^{k_2} (k_2 - x)(x - k_1) \frac{\psi \left( \frac{k_1 + k_2}{2k_1k_2} \right)}{\frac{k_1 + k_2}{2} - x} \ dx 
\leq \frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2} - \frac{1}{2\Psi(1)} \left[ \frac{1}{2} I_{\varphi}(\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) + \frac{1}{2} I_{\varphi}(\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) \right] 
\leq \frac{M}{2\Psi(1)} \int_{k_1}^{k_2} (k_2 - x)(x - k_1) \frac{\psi \left( \frac{k_1 + k_2}{2k_1k_2} \right)}{\frac{k_1 + k_2}{2} - x} \ dx. \quad (3.22)
\]

Proof  By using the change of variables, we have

\[
\frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2} - \frac{1}{2\Psi(1)} \left[ \frac{1}{2} I_{\varphi}(\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) + \frac{1}{2} I_{\varphi}(\mathcal{F} \circ \mathcal{G}) \left( \frac{k_1 + k_2}{2k_1k_2} \right) \right] 
= \frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2}
\]

\[
- \frac{1}{2\Psi(1)} \left[ \int_{\frac{k_1}{2}}^{\frac{k_2}{2}} \left( \frac{1}{\frac{k_1}{2} + \frac{k_2}{2}} - x \right) \mathcal{F} \left( \frac{1}{x} \right) \ dx + \int_{\frac{k_2}{2}}^{\frac{k_1}{2}} \left( \frac{1}{\frac{k_1}{2} + \frac{k_2}{2}} - x \right) \mathcal{F} \left( \frac{1}{x} \right) \ dx \right] 
= \frac{\mathcal{F}(k_1) + \mathcal{F}(k_2)}{2}
\]
\begin{align*}
&- \frac{1}{2\psi(1)} \int_{x_1}^{x_2} \psi \left( \frac{\kappa x - \kappa_1 x_2^2}{x_1 x_2} \right) \mathcal{F} \left( \frac{\kappa_1 x_2}{x} \right) \, dx \\
&\quad + \int_{x_1}^{x_2} \psi \left( \frac{\kappa x - \kappa_1 x_2^2}{x_1 x_2} \right) \mathcal{F} \left( \frac{\kappa_1 x_2}{x} \right) \, dx \bigg] \\
&= \frac{1}{2\psi(1)} \int_{x_1}^{x_2} \left[ \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) - \mathcal{F} \left( \frac{\kappa_1 x_2}{\kappa + x_2 - x} \right) \right] \phi \left( \frac{\kappa x - \kappa_1 x_2^2}{x_1 x_2} \right) \, dx.
\end{align*}

By using the equalities
\[
\mathcal{F} \left( \frac{\kappa_1 x_2}{x} \right) - \mathcal{F}(\kappa_1) = \phi(x) - \phi(\kappa_1) = \int_{\kappa_1}^{\kappa} \phi'(\tau) \, d\tau
\]
and
\[
\mathcal{F}(\kappa_2) - \mathcal{F} \left( \frac{\kappa_1 x_2}{\kappa + x_2 - x} \right) = \phi(\kappa_2) - \phi(\kappa_1 + x_2 - x) = \int_{\kappa_1 + x_2 - x}^{\kappa_2} \phi'(\tau) \, d\tau,
\]
we have
\[
\phi(\kappa_1) + \phi(\kappa_2) - \phi(x) - \phi(\kappa_1 + x_2 - x)
\]
\[
= \int_{\kappa_1 + x_2 - x}^{\kappa_2} \phi'(\tau) \, d\tau - \int_{\kappa_1}^{\kappa} \phi'(\tau) \, d\tau = \int_{\kappa_1}^{\kappa_2} \phi'(\kappa_1 + x_2 - \tau) - \phi'(\tau) \, d\tau.
\]

Integrating (3.6) with respect to \( \tau \) over \([\kappa_1, x]\), we get
\[
m \int_{\kappa_1}^{\kappa} (\kappa_1 + x_2 - 2\tau) \, d\tau \leq \int_{\kappa_1}^{\kappa} \left[ \phi(\kappa_1 + x_2 - \tau) - \phi'(\tau) \right] \, d\tau \leq M \int_{\kappa_1}^{\kappa} (\kappa_1 + x_2 - 2\tau) \, d\tau,
\]
which implies that
\[
m(x - \kappa_1)(x_2 - x) \leq \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) - \mathcal{F} \left( \frac{\kappa_1 x_2}{\kappa + x_2 - x} \right) - \mathcal{F} \left( \frac{\kappa_1 x_2}{\kappa + x_2 - x} \right)
\]
\[
\leq M(x - \kappa_1)(x_2 - x).
\]

Multiplying inequality (3.25) by \( \frac{\psi x_1 x_2}{2\psi(1)(x_1 x_2^2 - x)} \) and integrating the resultant inequality with respect to \( x \) on \([\kappa_1, \kappa_1 x_2^2]\), we establish
\[
\frac{m}{2\psi(1)} \int_{x_1}^{x_1 x_2^2} (x - \kappa_1)(x_2 - x) \frac{\psi x_1 x_2}{2\psi(1)(x_1 x_2^2 - x)} \, dx
\]
\[
\leq \frac{1}{2\psi(1)} \int_{x_1}^{x_1 x_2^2} \left[ \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) - \mathcal{F} \left( \frac{\kappa_1 x_2}{\kappa + x_2 - x} \right) - \mathcal{F} \left( \frac{\kappa_1 x_2}{\kappa + x_2 - x} \right) \right] \frac{\psi x_1 x_2}{2\psi(1)(x_1 x_2^2 - x)} \, dx
\]
\[
\leq \frac{M}{2\psi(1)} \int_{x_1}^{x_1 x_2^2} (x - \kappa_1)(x_2 - x) \frac{\psi x_1 x_2}{2\psi(1)(x_1 x_2^2 - x)} \, dx.
\]
That can be written as
\[
\frac{m}{2\Psi(1)} \int_{x_1}^{x_2} (x - \kappa_1)(\kappa_2 - x) \frac{\psi\left(\frac{x_1 + x_2}{2\kappa_1\kappa_2} - x\right)}{\frac{x_1 + x_2}{2} - x} \, dx
\]
\[
\leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} - \frac{1}{2\Psi(1)} \left[ \frac{1}{\kappa_1} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) + \frac{1}{\kappa_2} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) \right]
\]
\[
\leq \frac{M}{2\Psi(1)} \int_{x_1}^{x_2} (x - \kappa_1)(\kappa_2 - x) \frac{\psi\left(\frac{x_1 + x_2}{2\kappa_1\kappa_2} - x\right)}{\frac{x_1 + x_2}{2} - x} \, dx,
\]
which gives inequalities (3.22).

**Remark 6** Under the assumptions of Theorem 16, if we put \(\psi(\tau) = \tau\), then inequality (3.22) reduces to inequality (3.13).

**Corollary 11** Under the assumptions of Theorem 16, if we set \(\psi(\tau) = \frac{\tau^n}{\Gamma(\alpha)}\), then we have the following inequalities for the Riemann–Liouville fractional integrals:
\[
m(\kappa_2 - \kappa_1)^2
\]
\[
= \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} - \frac{1}{2\Psi(1)} \left[ \frac{1}{\kappa_1} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) + \frac{1}{\kappa_2} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) \right]
\]
\[
= \frac{M(\kappa_2 - \kappa_1)^2}{4(\alpha + 2)}.
\]

**Corollary 12** Under the assumptions of Theorem 16, if we put \(\psi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}\), then we have the following inequalities for the k-Riemann–Liouville fractional integrals:
\[
m(\kappa_2 - \kappa_1)^2
\]
\[
= \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} - \frac{1}{2\Psi(1)} \left[ \frac{1}{\kappa_1} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) + \frac{1}{\kappa_2} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) \right]
\]
\[
= \frac{M(\kappa_2 - \kappa_1)^2}{4(\frac{\alpha}{2} + 2)}.
\]

Now, by using Theorems 15 and 16, we prove inequality (2.7) under the condition \(\phi'(\kappa_1 + \kappa_2 - x) \geq \phi'(x)\) instead of the harmonic convexity of \(\mathcal{F}\).

**Theorem 17** Let \(\mathcal{F} : [\kappa_1, \kappa_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}\) be a positive twice differentiable function with \(\kappa_1 < \kappa_2\) and \(\mathcal{F} \in L([\kappa_1, \kappa_2])\). If \(\phi'(\kappa_1 + \kappa_2 - x) \geq \phi'(x), \forall x \in [\kappa_1, \frac{\kappa_1 + \kappa_2}{2}]\), then we have the following inequalities for GFIs:
\[
\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right) \leq \frac{1}{2\Psi(1)} \left[ \frac{1}{\kappa_1} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) + \frac{1}{\kappa_2} I_{\phi}(\mathcal{F} \circ \mathcal{G})\left(\frac{\kappa_1 + \kappa_2}{2\kappa_1\kappa_2}\right) \right]
\]
\[ \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \] (3.26)

**Proof** From (3.17) and (3.18), we get

\[ \frac{1}{2\Psi(1)} \left[ \frac{1}{\tau^2} I_\rho(\mathcal{F} \circ \mathcal{G}) \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right) + \frac{1}{\tau} I_\rho(\mathcal{F} \circ \mathcal{G}) \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right) \right] - \mathcal{F} \left( \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \]

\[ = \frac{1}{2\Psi(1)} \int_x^{1+\epsilon} \left[ \mathcal{F} \left( \frac{\kappa_1 \kappa_2}{x} \right) + \mathcal{F} \left( \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2 - x} \right) - 2\mathcal{F} \left( \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2} \right) \right] \frac{x(\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon})}{2} dx \]

\[ = \frac{1}{2\Psi(1)} \int_x^{1+\epsilon} \left[ \int_0^{\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon}} \frac{\phi'(x \kappa_1 + \kappa_2 - \tau) - \phi'(\tau)}{\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon}} d\tau \right] \frac{x(\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon})}{2} dx \]

\[ \geq 0, \]

which proves the first inequality in (3.26). On the other hand, by equalities (3.23) and (3.24), we have

\[ \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} - \frac{1}{2\Psi(1)} \left[ \frac{1}{\tau^2} I_\rho(\mathcal{F} \circ \mathcal{G}) \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right) + \frac{1}{\tau} I_\rho(\mathcal{F} \circ \mathcal{G}) \left( \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \right) \right] \]

\[ = \frac{1}{2\Psi(1)} \int_x^{1+\epsilon} \left[ \mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2) - \mathcal{F} \left( \frac{\kappa_1 \kappa_2}{x} \right) - \mathcal{F} \left( \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2 - x} \right) \right] \frac{x(\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon})}{2} dx \]

\[ = \frac{1}{2\Psi(1)} \int_x^{1+\epsilon} \left( \int_{\kappa_1}^{\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon}} \phi'(x \kappa_1 + \kappa_2 - \tau) - \phi'(\tau) d\tau \right) \frac{x(\frac{1+\epsilon}{x} - \frac{1}{1+\epsilon})}{2} dx \]

\[ \geq 0. \]

This proves the second inequality in (3.26) and completes the proof. \[ \square \]

### 4 Conclusion

In this work, the authors established Hermite–Hadamard type inequalities for harmonically convex functions by using generalized fractional integrals. Furthermore, the authors proved some extensions of newly proved inequalities without using the condition of harmonic convexity for the functions. It is an interesting and new problem, and the upcoming researchers can offer similar inequalities for harmonically convex functions on the co-ordinates via generalized fractional integrals in their future research.

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### Availability of data and materials

Not applicable.

### Competing interests

The authors declare that they have no competing interests.
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