Abstract. The paper deals with Sturm–Liouville-type operators with frozen argument of the form
\[ \ell y := -y''(x) + q(x)y(a), \quad y^{(\alpha)}(0) = y^{(\beta)}(1) = 0, \]
where \( \alpha, \beta \in \{0, 1\} \) and \( a \in [0, 1] \) is an arbitrary fixed rational number. Such nonlocal operators belong to the so-called loaded differential operators, which often appear in mathematical physics. We focus on the inverse problem of recovering the potential \( q(x) \) from the spectrum of the operator \( \ell \). Our goal is two-fold. Firstly, we establish a deep connection between the so-called main equation of this inverse problem and Chebyshev polynomials of the first and the second kinds. This connection gives a new perspective method for solving the inverse problem. In particular, it allows one to completely describe all non-degenerate and degenerate cases, i.e. when the solution of the inverse problem is unique or not, respectively. Secondly, we give a complete and convenient description of iso-spectral potentials in the space of complex-valued integrable functions.

Key words: Sturm–Liouville operator, functional-differential operator, frozen argument, Chebyshev polynomials, Jacobi matrices, inverse spectral problem, iso-spectral potentials.

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1. Introduction

Consider the boundary value problem \( \mathcal{L} := \mathcal{L}(q(x), a, \alpha, \beta) \) of the form
\[ \ell y := -y''(x) + q(x)y(a) = \lambda y(x), \quad 0 < x < 1, \]
\[ y^{(\alpha)}(0) = y^{(\beta)}(1) = 0, \]
where \( \lambda \) is the spectral parameter, \( q(x) \) is a complex-valued function in \( L(0, 1) \), to which we refer as potential, and \( \alpha, \beta \in \{0, 1\} \), while \( a \in [0, 1] \). The operator \( \ell \) is called the Sturm–Liouville-type operator with frozen argument.

Denote by \( \{\lambda_n\}_{n \geq 1} \) the spectrum of \( \mathcal{L} \) and consider the following inverse problem.

Inverse Problem 1. Given \( \{\lambda_n\}_{n \geq 1}, a, \alpha \) and \( \beta \); find \( q(x) \).

Nonlocal operators of the form \([1], [2]\) belong to the so-called loaded differential operators (see, e.g., \([3],[4]\)), which often appear in mathematical physics. For example, some models of
Among purely mathematical applications, we illustrate here the so-called method of reduction to loaded equations (see, \[5\]-\[6\]). For this purpose, let us aim to study the boundary value problem for the integro-differential equation
\[-y''(x) + \int_0^1 H(x, t)y(t) \, dt = \lambda y(x), \quad 0 < x < 1,\]
subject to boundary conditions \[2\] for some \(\alpha\) and \(\beta\). The method consists in replacing equation \[3\] with the loaded one
\[-y''(x) + \sum_{\nu=1}^N q_\nu(x) y(a_\nu) = \lambda y(x), \quad 0 < x < 1,\]
possessing frozen arguments \(a_1, \ldots, a_N\), where the sum is an appropriate quadrature formula for approximating the integral in \[3\]. For example, Simpson’s rule (see, e.g., \[10\])
\[\int_{x_1}^{x_2} f(x) \, dx \approx \frac{x_2 - x_1}{6} \left( f(x_1) + 4f\left(\frac{x_1 + x_2}{2}\right) + f(x_2) \right)\]
in the case \(\alpha = \beta = 0\) leads to equation \[4\] with \(q(x) = 2H(x, a)/3\) and \(a = 1/2\).

Various aspects of Inverse Problem 1 in the case \(q(x) \in L_2(0, 1)\) were studied in \[11\]-\[14\]. In \[11\]-\[13\], diverse cases of the triple \((a, \alpha, \beta)\) with rational \(a\)’s were considered. In particular, it was established that the inverse problem may be uniquely solvable or not depending on the parameters \(\alpha, \beta\) and also on the parity of the integers \(k, j\) or \(j+k\) taken from the representation \(a = j/k\) under the assumption that \(j\) and \(k\) are mutually prime. According to this, there were highlighted two cases: non-degenerate and degenerate ones, respectively. Moreover, a complete characterization of the spectrum \(\{\lambda_n\}_{n \geq 1}\) was given, which includes the asymptotics for large modulus eigenvalues along with a special additional condition in the degenerate case. Specifically, it was established that, in the degenerate case, asymptotically \(k\)-th part of the spectrum degenerates, i.e. each \(k\)-th eigenvalue carries no information on the potential.

For example, in the case when \(\alpha = \beta = 0\) and \(q(x) \in L_2(0, 1)\), the spectrum is completely characterized by the relations
\[\lambda_n = (\pi n)^2 + \varkappa_n, \quad \{\varkappa_n\} \in l_2, \quad \lambda_{kn} = (\pi kn)^2, \quad n \in \mathbb{N},\]
i.e. each \(k\)-th eigenvalue \(\lambda_{kn}\) degenerates. In this case, for the unique solvability of Inverse Problem 1, one should specify the potential on one of the subintervals \(((\nu-1)/k, \nu/k), \nu = 1, k\). Thus, the smaller part of the spectrum degenerates, the less additional information on the potential is required, while in the non-degenerate case no extra information is required at all (see \[17\] and \[18\] below for a complete description of degenerate and non-degenerate subcases).

This causes instable informativity of the spectrum with respect to \(a\), which was first revealed in \[12\] (see also \[13\]). For example, while a half of the spectrum degenerates as soon as \(a = 1/2\), for \(a = a_k := (k - 1)/(2k)\) with even \(k\) so does only its \(2k\)-th part, but \(a_k \to 1/2\) as \(k \to \infty\).

Thus, returning to the method of reduction to loaded equations mentioned above, one can note that this method is sensitive to choosing the nodes \(a_1, \ldots, a_N\) in \[4\] at least when approximating the spectrum of the initial integro-differential operator \[3\] subject to \[2\].
Concerning irrational values of \( a \), it was established in \([14]\) that all they correspond to the non-degenerate case for all pairs \((\alpha, \beta)\), i.e. the solution of Inverse Problem 1 is always unique as soon as \( a \notin \mathbb{Q} \). However, the question of the spectrum characterization still remains open.

In \([15, 16]\) and other works, in connection with the theory of diffusion processes, the case \( a = 1 \) was investigated but with the special nonlocal boundary conditions

\[
y(0) - \alpha y(1) = y'(1) - \alpha y'(0) + \int_{0}^{1} y(t)q(t) \, dt = 0, \quad \alpha \in \{0, 1\},
\]

guaranting the self-adjointness of the corresponding operator generated by \([1] \) and \([5]\). However, such settings never entail the uniqueness of recovering the function \( q(x) \) from the spectrum.

In \([9]\), the case of the quasi-periodic boundary conditions of the form

\[
y(0) - \gamma y(1) = y'(0) - \gamma y'(1) = 0
\]

for any possible \( \gamma \in \mathbb{C} \setminus \{0\} \) was studied, and a complete solution was obtained for the inverse problem of recovering the potential \( q(x) \) from the corresponding spectrum (spectra). Further aspects of recovering the operator \( \ell \) as well as its spectral properties were studied in \([17–20]\).

In the present paper, we return to Inverse Problem 1. In \([11–13]\), this problem was reduced to some linear functional equation with respect to the potential \( q(x) \) (see equation \((10)\) in the next section), which was referred to as main equation of the inverse problem. For rational values of \( a \), the main equation was reduced to linear system \((13)\) with a special \( k \times k \) matrix \( A_{j,k}^{(\alpha,\beta)} \), whose rank appeared to be ranging between \( k - 1 \) and \( k \). These two possibilities, in turn, correspond to the degenerate and non-degenerate cases, respectively. In the works \([11, 13]\), various approaches to studying this matrix and calculating its determinant were developed.

Here, we establish a deep connection between the matrix \( A_{j,k}^{(\alpha,\beta)} \) and Chebyshev polynomials of the first and the second kinds, which gives another approach for studying Inverse Problem 1. Using this new approach, we obtain a complete and convenient description of all iso-spectral complex-valued potentials in \( L(0,1) \) for the degenerate case.

The paper is organized as follows. In the next section, we provide some necessary information on the boundary value problem \( \mathcal{L} \). In Section 3, we represent characteristic determinants of the matrices \( A_{1,k}^{(\alpha,\beta)} \) via Chebyshev polynomials. In Section 4, we establish that the matrices \( A_{j,k}^{(\alpha,\beta)} \) can be obtained after substituting \( A_{j,k}^{(\alpha,\beta)} \) into appropriate Chebyshev polynomials, which allows one to study their spectra for \( j > 0 \). In Section 5, we construct iso-spectral potentials in the degenerate case. In Section 6, we provide some illustrative examples.

2. Preliminary information

Let \( C(x, \lambda) \) and \( S(x, \lambda) \) be solutions of equation \([1]\) under the initial conditions

\[
C(a, \lambda) = S'(a, \lambda) = 1, \quad S(a, \lambda) = C'(a, \lambda) = 0. \tag{6}
\]

By substitution, it can be easily checked that

\[
C(x, \lambda) = \cos \rho(x - a) + \int_{a}^{x} \frac{\sin \rho(x - t)}{\rho} q(t) \, dt, \quad S(x, \lambda) = \frac{\sin \rho(x - a)}{\rho}, \quad \rho^2 := \lambda. \tag{7}
\]

Since these solutions are uniquely determined by conditions \((6)\), eigenvalues of the problem \( \mathcal{L} \) coincide with zeros of the entire function

\[
\Delta_{\alpha,\beta}(\lambda) = \begin{vmatrix} C^{(\alpha)}(0, \lambda) & S^{(\alpha)}(0, \lambda) \\ C^{(\beta)}(1, \lambda) & S^{(\beta)}(1, \lambda) \end{vmatrix}, \tag{8}
\]
which is called characteristics function of \( \mathcal{L} \).

Without loss of generality, we always assume that \( 0 \leq a \leq 1/2 \) since the spectrum of the problem \( \mathcal{L}(q(x), a, \alpha, \beta) \), obviously, coincides with the one of \( \mathcal{L}(q(1-x), 1-a, \beta, \alpha) \).

Substituting (7) into (8) one can obtain the following representations (see, e.g., [13]):

\[
\Delta_{\alpha,\alpha}(\lambda) = \rho^{2\alpha}\left(\frac{\sin \rho}{\rho} + \int_0^1 W_{\alpha,\alpha}(x) \frac{\cos \rho x}{\rho^2} \, dx\right), \quad \Delta_{\alpha,\beta}(\lambda) = (-1)^\alpha \cos \rho + \int_0^1 W_{\alpha,\beta}(x) \frac{\sin \rho x}{\rho} \, dx
\]  

(9)

for \( \alpha \neq \beta \), where the functions \( W_{\alpha,\beta}(x) \) have the form

\[
W_{\alpha,\beta}(x) = \frac{(-1)^{\alpha\beta}}{2} \begin{cases} 
q(1-a+x) + dq(1-a-x), & x \in (0,a), \\
cq(1-a-x) + dq(1-a,x), & x \in (a,1-a), \\
c\left(q(1+a-x) + q(x-1+a)\right), & x \in (1-a,1),
\end{cases}
\]

(10)

while the numbers \( c \) and \( d \) are determined by the formulae

\[
c = (-1)^{\beta+1}, \quad d = (-1)^{\alpha+\beta}.
\]

(11)

Assuming the function \( W_{\alpha,\beta}(x) \) to be known, one can consider (10) as a linear functional equation with respect to \( q(x) \). Since each function \( \Delta_{\alpha,\beta}(\lambda) \) is uniquely determined by its zeros:

\[
\Delta_{\alpha,\beta}(\lambda) = (-1)^\alpha(\lambda_1 - \lambda)^{\alpha\beta} \prod_{n=1+\alpha\beta}^{\infty} \frac{\lambda - \lambda_n}{(n - \alpha - \beta)^2 \pi^2},
\]

(12)

Inverse Problem 1 is equivalent to this functional equation (10), which is called main equation of the inverse problem.

If \( a \) is rational, i.e. there exist mutually prime integers \( j \) and \( k \) such that \( a = j/k \), then the main equation can be represented in the following way:

\[
W_{\alpha,\beta}(x) = \frac{(-1)^{\alpha\beta}}{2} Q^{-1} A_{j,k}^{(\alpha,\beta)} R q(x), \quad 0 < x < 1,
\]

(13)

where \( A_{j,k}^{(\alpha,\beta)} \) is a square matrix of order \( k \), while \( Q \) and \( R \) are bijective operators mapping \( L(0,1) \) onto \( (L(0,b))^k \), \( b := 1/k \), and acting by the formulae

\[
Qf := (f, Q_2 f, \ldots, Q_k f)^T, \quad Rf := (R_1 f, R_2 f, \ldots, R_k f)^T.
\]

(14)

Here, \( T \) is the transposition sign, while \( Q_{\nu} \) and \( R_{\nu} \) are shift and involution operators mapping \( L(0,1) \) onto \( L(0,b) \), which are determined by the formulae

\[
Q_{\nu} f(x) = \begin{cases} 
(f((\nu - 1)b + x) & \text{for odd } \nu, \\
f(\nu b - x) & \text{for even } \nu,
\end{cases} \quad R_{\nu} f(x) = \begin{cases} 
f((k - \nu)b + x) & \text{for even } j + \nu, \\
f((k - \nu + 1)b - x) & \text{for odd } j + \nu.
\end{cases}
\]

(15)

where \( x \in (0,b) \) and \( \nu = \frac{1}{1-b} \).

The matrix \( A_{j,k}^{(\alpha,\beta)} = (a_{m,n})_{m,n=1}^{1,k} \), in turn, is constructed in the following way. For \( k = 1 \), it consists of a single element \( a_{1,1} = 2(-1)^{\beta+1} \alpha \) as soon as \( j = 0 \), while, for \( k \geq 2j \geq 2 \), its
elements are determined by the formulae

\[
\begin{align*}
(i) \quad & a_{m,j-m+1} = 1, \quad m = 1, j, \\
(ii) \quad & a_{m,m+j} = d, \quad m = 1, k - j, \\
(iii) \quad & a_{m,m-j} = c, \quad m = j + 1, k, \\
(iv) \quad & a_{m,2k-m-j+1} = c, \quad m = k - j + 1, k, \\
(v) \quad & a_{m,n} = 0 \text{ for the remaining pairs } (m,n).
\end{align*}
\]

(16)

The items (i)–(iv) in (16) correspond to subdiagonals, consisting of equal elements: 1, c and d. For example, the the matrix \( A_{3,7}^{(\alpha,\beta)} \) has the form

\[
\begin{pmatrix}
\cdot & \cdot & 1 & d & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & d & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & d & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\
\cdot & \cdot & c & c & \cdot & \cdot & \cdot \\
\cdot & \cdot & c & c & \cdot & \cdot & \cdot \\
\end{pmatrix}
\]

where dots indicate zero elements.

In papers [11–13], various approaches were used for calculating the determinant and the rank of \( A_{j,k}^{(\alpha,\beta)} \), depending on generality of the situation. In particular, in [12], a reduction-type algorithm was suggested for the case \( \alpha = \beta = 0 \). This algorithm appeared to be equivalent to the Euclidean algorithm for finding the greatest common divisor of the numbers \( j \) and \( k \). Since \( j \) and \( k \) are mutually prime, the algorithm gave consecutive relations leading to the result:

\[
\text{rank } A_{j,k}^{(0,0)} = \text{rank } A_{j,k-j}^{(0,0)} + j = \text{rank } A_{k-2j,k-j}^{(0,0)} + j = \ldots = \text{rank } A_{0,1}^{(0,0)} + k - 1 = k - 1.
\]

Later, in [13], it was established that the rank of \( A_{j,k}^{(\alpha,\beta)} \) cannot be less than \( k - 1 \). For this purpose, a combinatorial approach for calculating the determinant of \( A_{j,k}^{(\alpha,\beta)} \) was suggested, which was based on studying properties of an undirected graph \( G \) corresponding to a special traversal of nonzero elements of \( A_{j,k}^{(\alpha,\beta)} \). It was established that \( G \) was a bipartite Eulerian cycle, which has led to representing \( \text{det } A_{j,k}^{(\alpha,\beta)} \) as a sum of precisely two products consisting of nonzero elements of \( A_{j,k}^{(\alpha,\beta)} \). This gave a complete classification of degenerate and non-degenerate cases corresponding to non-unique and unique solvability of Inverse Problem 1, respectively.

Specifically, the degenerate case occurs when one of the following groups of conditions is fulfilled:

\[
\begin{align*}
(I) \quad & \alpha = \beta = 0; \\
(II) \quad & \alpha = 0, \ \beta = 1 \text{ and } j \text{ is even}; \\
(III) \quad & \alpha = 1, \ \beta = 0 \text{ and } k + j \text{ is even}; \\
(IV) \quad & \alpha = \beta = 1 \text{ and } k \text{ is even};
\end{align*}
\]

(17)

while the non-degenerate case includes the remaining groups of conditions:

\[
\begin{align*}
(V) \quad & \alpha = 0, \ \beta = 1 \text{ and } j \text{ is odd}; \\
(VI) \quad & \alpha = 1, \ \beta = 0 \text{ and } k + j \text{ is odd}; \\
(VII) \quad & \alpha = \beta = 1 \text{ and } k \text{ is odd}.
\end{align*}
\]

(18)
This classification remains valid also for $a > 1/2$, i.e. it holds for all relevant $j \in \{0, \ldots, k\}$.

In the subsequent sections, we establish a deep connection between the matrix $A_{j,k}^{(\alpha,\beta)}$ and Chebyshev polynomials of the first and the second kinds. This connection gives, in particular, another approach for studying the main equation \([13]\). Using this approach one can easily give a complete description of iso-spectral potentials in the degenerate case (see Section 5).

3. Chebyshev polynomials and the case $j = 1$

First, we give some necessary information about Chebyshev polynomials $T_n(z)$ and $U_n(z)$ of the first and the second kinds, respectively, which can be defined by the formulae

$$T_n(\cos \theta) = \cos n\theta, \quad U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad n + 1 \in \mathbb{N}. \tag{19}$$

Alternatively, one can use the following recurrent relation

$$Y_{n+1}(z) = 2zY_n(z) - Y_{n-1}(z). \tag{20}$$

Then the polynomials of the first kind $T_n(z) = Y_n(z)$ are determined by the initial conditions

$$T_0(z) = 1, \quad T_1(z) = z, \tag{21}$$

while the initial conditions

$$U_0(z) = 1, \quad U_1(z) = 2z \tag{22}$$

determine the polynomials of the second kind $Y_n(z) = U_n(z)$. For more details, see, e.g., \[21\].

It is well known and also can be easily seen that Chebyshev polynomials may possess only simple zeros, and they are always odd or even functions in accordance with the parity of $n$. In particular, we have $T_n(0) = U_n(0) = 0$ as soon as $n$ is odd, and $T_n(0)U_n(0) \neq 0$ for even $n$.

Let us proceed with studying the matrix $A_{j,k}^{(\alpha,\beta)}$ for $k \geq 2$. In this section, we focus on the case $j = 1$. Consider the characteristic polynomial

$$p_k(z) := \det(zI - A_{1,k}^{(\alpha,\beta)}), \tag{23}$$

where $I$ is the unit matrix. The following lemma holds.

**Lemma 1.** The characteristic polynomial of the matrix $A_{1,k}^{(\alpha,\beta)}$ has the form

$$p_k(z) = (z - c)q_{k-1}(z) - cdq_{k-2}(z), \tag{24}$$

where $c$ and $d$ are determined by \([11]\), while the polynomials $q_n(z)$ can be found from the recurrent relations

$$q_0(z) = 1, \quad q_1(z) = z - 1, \quad q_{n+1}(z) = zq_n(z) - cdq_{n-1}(z), \quad n = 1, k - 2. \tag{25}$$

**Proof.** First, we note that $q_0(z)$ is the characteristic polynomial of the three-diagonal matrix $B_\nu$ that is obtained from $A_{1,\nu+1}^{(\alpha,\beta)}$ by removing the last column along with the last row, i.e.

$$q_\nu(z) = \det(zI - B_\nu) = \begin{vmatrix} z - 1 & -d & & \\ -c & z & -d & \\ & -c & \ddots & \ddots \\ & & \ddots & z & -d \\ & & & -c & z \end{vmatrix}, \tag{26}$$

$$...$$
where each of both subdiagonals consists of equal elements, while all elements of the main
diagonal starting from the second position are equal too. Indeed, for \( \nu = 1 \) formula (26)
is obvious. Further, let it hold for any \( \nu \leq n \). Then expanding the determinant in (26) for
\( \nu = n + 1 \) with respect to the elements of the last row we obtain the last equality in (25).
Finally, expanding the determinant

\[
\det(zI - A_{1,k}^{(\alpha,\beta)}) = \begin{vmatrix}
  z - 1 & -d & & \\
  -c & z - d & & \\
  & \ddots & \ddots & \\
  & & \ddots & z - d \\
  & & & -c & z - c
\end{vmatrix}
\]

with respect to the last row, we obtain representation (24).

□

The following corollary gives the classifications (17) and (18) for \( j = 1 \).

Corollary 1. The determinant of the matrix \( A_{1,k}^{(\alpha,\beta)} \) can be calculated by the formula

\[
\det A_{1,k}^{(\alpha,\beta)} = \begin{cases}
  (-cd)^{(k-1)/2}(1 + c) & \text{if } k \text{ is odd}, \\
  c(-cd)^{k/2 - 1}(1 - d) & \text{if } k \text{ is even}.
\end{cases}
\]  

(27)

Proof. According to (23) and (24), we have \( \det A_{1,k}^{(\alpha,\beta)} = (-1)^{(k+1)}c(q_{k-1}(0) + dq_{k-2}(0)) \). The first two formulae in (25) give
\( q_0(0) = 1 \) and \( q_1(0) = -1 \). Assume that

\[
q_{2\nu}(0) = (-cd)^\nu, \quad q_{2\nu+1}(0) = -(-cd)^\nu, \quad 0 \leq \nu \leq l,
\]  

(28)

for some \( l \in \mathbb{N} \). Then the last relation in (25) implies

\[
q_{2(l+1)}(0) = -cdq_{2l}(0) = (-cd)^{l+1}, \quad q_{2(l+1)+1}(0) = -cdq_{2l+1}(0) = -(-cd)^{l+1}.
\]

Hence, (28) holds for all \( \nu \geq 0 \). Substituting (28) into the first formula of this proof, we arrive at

\[
\det A_{1,k}^{(\alpha,\beta)} = \begin{cases}
  c(q_{2\nu}(0) + dq_{2\nu-1}(0)) = (-cd)^\nu(1 + c) & \text{for } k = 2\nu + 1, \\
  -c(q_{2\nu+1}(0) + dq_{2\nu}(0)) = c(-cd)^\nu(1 - d) & \text{for } k = 2\nu + 2,
\end{cases}
\]

which finalizes the proof.

□

The main result of this section is contained in the following theorem.

Theorem 1. The following representations hold:

\[
\det(zI - A_{1,k}^{(0,0)}) = i^{k-1}zU_{k-1}\left(\frac{z}{2i}\right),
\]  

(29)

\[
\det(zI - A_{1,k}^{(0,1)}) = 2i^kT_k\left(\frac{z}{2i}\right) - 2i^{k-1}U_{k-1}\left(\frac{z}{2i}\right),
\]  

(30)

\[
\det(zI - A_{1,k}^{(1,0)}) = 2T_k\left(\frac{z}{2}\right),
\]  

(31)

\[
\det(zI - A_{1,k}^{(1,1)}) = (z - 2)U_{k-1}\left(\frac{z}{2}\right).
\]  

(32)
Proof. First, we note that, for $Y_n(z) = T_n(z)$ and $Y_n(z) = U_n(z)$, relations (20)–(22) give

\[
2zU_n(z) = \begin{cases} 
2z, & n = 0, \\
4z^2, & n = 1, 
\end{cases}
\]

and

\[
2T_{n+1}(z) + 2iU_n(z) = \begin{cases} 
2z + 2i, & n = 0, \\
4z^2 + 4iz - 2, & n = 1, 
\end{cases}
\]  

(33)

Comparing this with (33), we get

\[
2T_{n+1}(z) = \begin{cases} 
2z, & n = 0, \\
4z^2 - 2, & n = 1, 
\end{cases}
\]

and

\[
(2z - 2)U_n(z) = \begin{cases} 
2z - 2, & n = 0, \\
4z^2 - 4z, & n = 1. 
\end{cases}
\]  

(34)

Let $\alpha = 0$. Then formulae (11) give $cd = -1$. Then (23) and (24) imply

\[
\det(zI - A^{(0,\beta)}_{1,k}) = (z - c)q_{k-1}(z) + q_{k-2}(z). 
\]  

(35)

Put $Y_n(z) := i^{-n}q_n(2iz)$, $n = 0, k - 1$. Using (25), one can easily check that the polynomials $Y_n(z)$ satisfy the recurrent relations (20). Substituting $q_n(z) = i^nY_n(z/(2i))$ into (35), we get

\[
\det(zI - A^{(0,\beta)}_{1,k}) = i^k \left( \left( \frac{2z}{2i} + ic \right) Y_{k-1} \left( \frac{z}{2i} \right) - Y_{k-2} \left( \frac{z}{2i} \right) \right). 
\]  

(36)

Using (25), we calculate: $Y_{-1}(z) = -i$, $Y_0(z) = 1$ and $Y_1(z) = 2z + i$. Hence, we obtain

\[
(2z + ic)Y_n(z) - Y_{n-1}(z) = \begin{cases} 
2z, & n = 0, \\
4z^2, & n = 1, 
\end{cases} \quad \beta = 0, 
\]

and

\[
\begin{cases} 
2zU_n(z), & \beta = 0, \\
2T_{n+1}(z) + 2iU_n(z), & \beta = 1, 
\end{cases} 
\]  

(37)

which along with (36) gives (29) and (30).

Further, let $\alpha = 1$. Then $cd = 1$, and formulae (23) and (24) imply

\[
\det(zI - A^{(1,\beta)}_{1,k}) = (z - c)q_{k-1}(z) - q_{k-2}(z). 
\]  

(38)

Put $Y_n(z) := q_n(2z)$, $n = 0, k - 1$. By virtue of (25), these polynomials $Y_n(z)$ satisfy the recurrent relations (20). Substituting $q_n(z) = Y_n(z/2)$ into (37), we get

\[
\det(zI - A^{(1,\beta)}_{1,k}) = \left( 2\frac{z}{2} - c \right) Y_{k-1} \left( \frac{z}{2} \right) - Y_{k-2} \left( \frac{z}{2} \right), 
\]

where

\[
(2z - c)Y_n(z) - Y_{n-1}(z) = \begin{cases} 
2z, & n = 0, \\
4z^2 - 2, & n = 1, 
\end{cases} \quad \beta = 0, 
\]

and

\[
\begin{cases} 
2zU_n(z), & \beta = 0, \\
2z - 2, & \beta = 1, 
\end{cases} 
\]  

(39)

Comparing this with (34) and using (38), we arrive at (31) and (32). \qed
Corollary 2. Denote by $\sigma(A)$ the spectrum of the matrix $A$. Then

$$\sigma(A_{1,k}^{(0,0)}) = \{0\} \cup \left\{ 2i \cos \frac{\nu \pi}{k} \right\}_{\nu=1,k-1}, \quad (39)$$

$$0 \notin \sigma(A_{1,k}^{(0,1)}), \quad (40)$$

$$\sigma(A_{1,k}^{(1,0)}) = \left\{ 2 \cos \frac{(2\nu + 1)\pi}{2k} \right\}_{\nu=0,k-1}, \quad (41)$$

$$\sigma(A_{1,k}^{(1,1)}) = \left\{ 2 \cos \frac{\nu \pi}{k} \right\}_{\nu=0,k-1}. \quad (42)$$

Proof. It is well known and also can be obtained as a simple corollary from (19) that the sets of zeros of the polynomials $T_n(z)$ and $U_n(z)$ have the forms

$$T_n := \left\{ \cos \frac{(2\nu + 1)\pi}{2n} \right\}_{\nu=0,n-1}, \quad U_n := \left\{ \cos \frac{\nu \pi}{n + 1} \right\}_{\nu=1,n}, \quad (43)$$

respectively. Thus, (39), (41) and (42) follow directly from (29), (31) and (32). Concerning (40), it is sufficient to recall that, in (30), $T_k(0)U_{k-1}(0) = 0$, while $T_k(0) \neq U_{k-1}(0)$. \hfill \Box

4. The case $j > 1$

In this section, we establish connections between the matrices $A_{j,k}^{(\alpha,\beta)}$ and $A_{1,k}^{(\alpha,\beta)}$, which allow one to reduce studying the case $j > 1$ to the case $j = 1$. Namely, the following theorem holds.

Theorem 2. For $\beta = 0, 1$ and $j = \lfloor \frac{1}{\nu} \rfloor n_k$, where $n_k = \lfloor k/2 \rfloor$, the following relations hold:

$$A_{j,k}^{(0,\beta)} = U_{j-1} \left( -\frac{c}{2} A_{1,k}^{(1,1-\beta)} \right) A_{1,k}^{(0,\beta)}, \quad (44)$$

$$A_{j,k}^{(1,\beta)} = 2cT_j \left( \frac{c}{2} A_{1,k}^{(1,\beta)} \right). \quad (45)$$

Here, $\lfloor x \rfloor$ denotes the integer part of $x$ and, as before, $c = (-1)^{1+\beta}$.

Proof. For $j = 1$, the assertion is obvious. According to the formulae $U_1(z) = 2z$ and $T_2(z) = 2z^2 - 1$, each of relations (44) and (45) for $j = 2$ is equivalent to the common relation

$$A_{2,k}^{(\alpha,\beta)} = dA_{1,k}^{(1,\gamma)} A_{1,k}^{(\alpha,\beta)} - 2acI, \quad \gamma := \begin{cases} 1-\beta, & \alpha = 0, \\ \beta, & \alpha = 1, \end{cases} \quad (46)$$

Consider a column vector $X = (x_1, \ldots, x_k)^T$ and denote $[X]_n := x_n$ for $n = 1, \ldots, k$. Then, by virtue of (16), we have

$$[A_{j,k}^{(\alpha,\beta)} X]_m = \begin{cases} x_{j-m+1} + dx_{j+m}, & m = \lfloor j \rfloor, \\ cx_{m-j} + dx_{j+1}, & m = j + 1, k - j, \\ c(x_{m-j} + x_{2k-m-j+1}), & m = k - j + 1, k. \end{cases} \quad (47)$$

In particular, this gives the formulae

$$[A_{1,k}^{(\alpha,\beta)} X]_m = \begin{cases} x_1 + dx_2, & m = 1, \\ cx_{m-1} + dx_{m+1}, & m = 2, k - 1, \\ c(x_{k-1} + x_k), & m = k. \end{cases} \quad (48)$$
since \( \gamma = \alpha \beta + (1 - \alpha)(1 - \beta) = 2\alpha \beta - \alpha - \beta + 1 \) and, hence, \((-1)^{1+\gamma} = (-1)^{\alpha + \beta} = d\). Substituting \(A_{1,k}^{(\alpha,\beta)}X\) given by (48) into (49) instead of \(X\), we get the relation

\[
[A_{1,k}^{(\alpha,\beta)}X]_m = \begin{cases} 
\phantom{\times} x_1 + dx_2, & m = 1, \\
\phantom{\times} d(x_{m-1} + x_{m+1}), & m = 2, \nu - 1, \\
\phantom{\times} d(x_{k-1} + x_k), & m = k, 
\end{cases} \tag{49}
\]

Comparing this with (47) for \(j = 2\) and taking into account that \(c + d = 2\alpha c\), we arrive at (46).

Assume now that (44) and (45) are valid when \((\nu,\nu), (\nu,\nu), (\nu,\nu), \ldots, (\nu,\nu)\) for some \(\nu \in \{2, \ldots, n_k - 1\}\). Then, according to (20), relation (44) for \(j = \nu + 1\) is equivalent to

\[
A_{\nu+1,k}^{(0,\beta)} = -cA_{1,k}^{(1,\beta)}U_{\nu-1}\left(-\frac{c}{2} A_{1,k}^{(1,\beta)}\right) A_{1,k}^{(0,\beta)} - U_{\nu-2}\left(-\frac{c}{2} A_{1,k}^{(1,\beta)}\right) A_{1,k}^{(0,\beta)}
= -cA_{1,k}^{(1,\beta)} A_{\nu,k}^{(0,\beta)} - A_{\nu-1,k}^{(0,\beta)}, \tag{50}
\]

while (45) for \(j = \nu + 1\) takes the form

\[
A_{\nu+1,k}^{(1,\beta)} = 2A_{1,k}^{(1,\beta)} T_\nu\left(\frac{c}{2} A_{1,k}^{(1,\beta)}\right) - 2cT_{\nu-1}\left(\frac{c}{2} A_{1,k}^{(1,\beta)}\right) = cA_{1,k}^{(1,\beta)} A_{\nu,k}^{(1,\beta)} - A_{\nu-1,k}^{(1,\beta)}. \tag{51}
\]

Using the relation \((-1)^{\nu+1}c = d\) along with the definition of \(\gamma\) in (46), one can rewrite (50) and (51) in the following common form:

\[
A_{\nu+1,k}^{(\alpha,\beta)} = dA_{1,k}^{(1,\gamma)} A_{\nu,k}^{(\alpha,\beta)} - A_{\nu-1,k}^{(\alpha,\beta)}, \quad \alpha, \beta = 0, 1. \tag{52}
\]

Thus, it remains to prove relation (52). Using (47), we calculate

\[
[(A_{\nu+1,k}^{(\alpha,\beta)} + A_{\nu-1,k}^{(\alpha,\beta)})X]_m = \begin{cases} 
\phantom{\times} x_{\nu-m+2} + dx_{\nu-1+m} + x_{\nu-m} + dx_{\nu-1+m}, & m = 1, \nu-1, \\
\phantom{\times} x_{\nu-m+2} + dx_{\nu+m-1} + x_{\nu-m} + dx_{\nu+m-1}, & m = \nu, \nu+1, \\
\phantom{\times} cx_{\nu-m+1} + dx_{\nu+m} + cx_{\nu-m} + dx_{\nu-m}, & m = \nu+2, k - \nu - 1, \tag{53}
\end{cases}
\]

Further, substituting \(A_{\nu,k}^{(\alpha,\beta)}X\) given by (47) into (49) instead of \(X\), we get the relation

\[
[A_{1,k}^{(1,\gamma)} A_{\nu,k}^{(\alpha,\beta)}X]_m = \begin{cases} 
\phantom{\times} d(x_{\nu-m+2} + dx_{\nu-1+m} + x_{\nu-m} + dx_{\nu+1+m}), & m = 1, \nu-1, \\
\phantom{\times} d(x_{\nu-m+2} + dx_{\nu-1+m} + cx_{\nu-m+1} + dx_{\nu+1+m}), & m = \nu, \nu+1, \\
\phantom{\times} d(cx_{\nu-m+1} + dx_{\nu-1} + cx_{\nu-m} + dx_{\nu+1}), & m = \nu+2, k - \nu - 1, \\
\phantom{\times} d(cx_{\nu-m} + dx_{\nu-m+1} + cx_{\nu-m+1} + dx_{\nu-1}), & m = k - \nu, k - \nu + 1, \\
\phantom{\times} d(cx_{\nu-m} + dx_{\nu-m+1} + cx_{\nu-m+1} + x_{2k-m-\nu}), & m = k - \nu + 2, k. 
\end{cases} \tag{54}
\]

Comparing this with (53), we arrive at (52). \(\square\)
Corollary 3. For $1 \leq j \leq [k/2]$, the matrix $A^{(\alpha,\beta)}_{j,k}$ is degenerate, i.e. $\det A^{(\alpha,\beta)}_{j,k} = 0$, if and only if one of the four conditions in (17) is fulfilled.

Equivalently, $\det A^{(\alpha,\beta)}_{j,k} \neq 0$ if and only if one of the three conditions in (18) is fulfilled.

Proof. Consider $\alpha = \beta = 0$ first. Then (39) and (44) imply $\det A_{0,0}^{(0,0)} = 0$ for any possible $j$ and $k$. Thus, the assertion of the corollary is proven for condition (I) in (17).

The rest part of the proof is based on the following well-known fact, which is valid both for Hermitian and non-Hermitian square matrices $A$, being a particular case of the corresponding abstract assertion (see, e.g., Theorem 3.3 on p. 16 in [22] or Theorem 10.28 on p. 263 in [23]).

Proposition 1. Let $P(z)$ be an algebraic polynomial and $A$ be a square matrix. Then

$$\sigma(P(A)) = \{P(z)\}_{z \in \sigma(A)}.$$ 

Moreover, if $X$ is an eigenvector corresponding to an eigenvalue $z_0$ of the matrix $A$, then $X$ is an eigenvector related to the eigenvalue $P(z_0)$ of $P(A)$.

Let us return to the proof of Corollary 3. For $\alpha = 0$ and $\beta = 1$, according to (11), (40) and (44), we have $\det A_{0,k}^{(0,1)} = 0$ if and only if $0 \in \sigma(U_{j-1}((-1/2)A_{1,k}^{(1,0)}))$. By virtue of Proposition 1, this inclusion is equivalent to the relation $U_{j-1} \cap \sigma((-1/2)A_{1,k}^{(1,0)}) \neq \emptyset$, where, as in the proof of Corollary 2, we use the designation $U_{n} = \{z : U_{n}(z) = 0\}$ similarly to $T_{n} = \{z : T_{n}(z) = 0\}$. Thus, according to (41) and (43), the latter intersection is not empty if and only if

$$\cos \frac{\nu \pi}{j} + \cos \frac{(2l+1)\pi}{2k} = 0$$

for a certain choice of $\nu \in \{1, \ldots, j-1\}$ and $l \in \{0, \ldots, k-1\}$. The latter, in turn, is equivalent to the relation

$$\nu \frac{\pi}{j} + (-1)^s \frac{2l+1}{2k} = 1 + 2m$$

for some integers $s$ and $m$. Obviously, (54) implies the evenness of $j$. Conversely, let $j$ be even. Then $k$ is odd, and (54) holds for $s = m = 0$ as soon as $\nu = j/2$ and $l = (k-1)/2$.

For $\alpha = 1$ and $\beta = 0$, relations (11) and (45) imply that $\det A_{j,k}^{(1,0)} = 0$ is equivalent to $T_{j} \cap \sigma((-1/2)A_{1,k}^{(1,0)}) \neq \emptyset$. By virtue of (41) and (43), this intersection is not empty if and only if

$$\cos \frac{(2\nu+1)\pi}{2j} + \cos \frac{(2l+1)\pi}{2k} = 0$$

for some $\nu \in \{0, \ldots, j-1\}$ and $l \in \{0, \ldots, k-1\}$, which is equivalent to the relation

$$\frac{2\nu+1}{2j} + (-1)^s \frac{2l+1}{2k} = 1 + 2m$$

with $s, m \in \mathbb{Z}$. In its turn, (55) implies the evenness of $j + k$. Conversely, let $j + k$ be even. Then $j$ and $k$ are odd, and (55) holds for $s = m = 0$ with $\nu = (j-1)/2$ and $l = (k-1)/2$.

Finally, let $\alpha = \beta = 1$. Then (11) and (45) imply that $\det A_{j,k}^{(1,1)} = 0$ is equivalent to $T_{j} \cap \sigma((1/2)A_{1,k}^{(1,1)}) \neq \emptyset$. By virtue of (42) and (43), the latter holds if and only if

$$\cos \frac{(2\nu+1)\pi}{2j} = \cos \frac{l\pi}{k}$$

for some $\nu \in \{0, \ldots, j-1\}$ and $l \in \{0, \ldots, k-1\}$, which, in turn, is equivalent to the relation

$$\frac{2\nu+1}{2j} + (-1)^s \frac{l}{k} = 2m$$

(56)
for some \( s, m \in \mathbb{Z} \). Obviously, (56) implies the evenness of \( k \). Conversely, let \( k \) be even. Then \( j \) is odd, and (56) holds for \( s = 1 \) and \( m = 0 \) with \( \nu = (j - 1)/2 \) and \( l = k/2 \).

5. Iso-spectral potentials

In this section, we return to Inverse Problem 1. The above results give an easy and convenient way for constructing iso-spectral potentials in the degenerate case.

Let the parameters \( \alpha, \beta \) and \( a = j/k \) with mutually prime \( j \) and \( k \) satisfy one of conditions (I)–(IV) in [17] and, for definiteness, also let \( a \in (0, 1/2] \). Fix a model complex-valued potential \( q_0(x) \in L(0, 1) \) and consider the corresponding eigenvalue problem \( L(q_0(x), \alpha, \beta, a) \) with the spectrum \( \Lambda := \{ \lambda_n \}_{n \geq 1} \). Consider the set \( \mathcal{M}_\Lambda \) of all corresponding iso-spectral potentials \( q(x) \), i.e. of such ones for which the spectrum of the problem \( L(q(x), \alpha, \beta, a) \) coincides with \( \Lambda \).

By virtue of (9), (12) and (13), we have the representation

\[
\mathcal{M}_\Lambda = \left\{ q_0(x) + g(x) : g(x) \in \mathcal{R}_{j,k}^{(\alpha, \beta)} \right\},
\]

(57)

where

\[
\mathcal{R}_{j,k}^{(\alpha, \beta)} = \left\{ R^{-1}F(x) : F(x) \in (L(0, b))^k \text{ and } A_{j,k}^{(\alpha, \beta)}F(x) = 0 \text{ a.e. on } (0, b) \right\}, \quad b = \frac{1}{k},
\]

i.e. the supplement \( g(x) \) in (57) is independent of \( q_0(x) \).

Thus, the question of describing all iso-spectral potentials is reduced to studying the kernel of the matrix \( A_{j,k}^{(\alpha, \beta)} \). The following lemma answers this question for \( j = 1 \).

**Lemma 2.** Each eigenvalue \( z_0 \) of the matrix \( A_{1,k}^{(\alpha, \beta)} \) has the geometric multiplicity one, while the corresponding eigenvector has the form

\[
X_0 = \left(1, dq_1(z_0), d^2q_2(z_0), \ldots, d^{k-1}q_{k-1}(z_0)\right)^T.
\]

(58)

**Proof.** According to (48), relation \( A_{1,k}^{(\alpha, \beta)}X_0 = z_0X_0 \) is equivalent to the system

\[
\begin{cases}
[X_0]_1 + d[X_0]_2 = z_0[X_0]_1, \\
c[X_0]_{m-1} + d[X_0]_{m+1} = z_0[X_0]_m, \quad m = 2, k - 1, \\
c([X_0]_{k-1} + [X_0]_k) = z_0[X_0]_k.
\end{cases}
\]

(59)

Thus, we have \([X_0]_1 \neq 0\) as soon as \( X_0 \) is an eigenvector, otherwise (59) would imply \( X_0 = 0 \). Without loss of generality, we put \([X_0]_1 = 1\). Then the first two lines in (59) give the relations

\[
[X_0]_2 = d(z_0 - 1), \quad [X_0]_{m+1} = z_0[X_0]_m - cd[X_0]_{m-1}, \quad m = 2, k - 1.
\]

(60)

Substituting \([X_0]_m =: d^{m-1}Y_m, m = 1, k, \) into (60), we arrive at the relations

\[
Y_1 = 1, \quad Y_2 = z_0 - 1, \quad Y_{m+1} = z_0Y_m - cdY_{m-1}, \quad m = 2, k - 1.
\]

Comparing this with (25), we get \( Y_m = q_{m-1}(z_0) \) and, hence, \([X_0]_m = d^{m-1}q_{m-1}(z_0)\) for \( m = 1, k, \) which finalizes the proof. \( \square \)

It can be easily seen that the last relation in (59) is fulfilled automatically as soon as \( z_0 \) is an eigenvalue of the matrix \( A_{1,k}^{(\alpha, \beta)} \). Indeed, by virtue of (58), this relation is equivalent to the relation \( cdq_{k-2}(z_0) = (z_0 - c)q_{k-1}(z_0) \), which, according to (24), is equivalent to \( p_k(z_0) = 0 \).
We also note that, according to (29), the algebraic multiplicity of the zero eigenvalue of the matrix $A_{1,k}^{(0,0)}$ may be equal to 2, while, by virtue of Lemma 2, the geometric one cannot.

**Lemma 3.** Let the values $\alpha$, $\beta$, $j$ and $k$ obey one of conditions (I)-(IV) in (17). Then the kernel of the matrix $A_{j,k}^{(\alpha,\beta)}$ coincides with a linear hull of the vector $X = (x_1, \ldots, x_k)^T$ determined in the following way:

\[
\begin{align*}
\alpha &= \beta = 0 : \quad x_\nu = (-1)^{\nu-1}, \quad \nu = \frac{1}{2}k; \\
\alpha &= 0, \quad \beta = 1 : \quad x_\nu = (-1)^{\lceil \frac{\nu}{2} \rceil}, \quad \nu = \frac{1}{2}k; \\
\alpha &= 1, \quad \beta = 0 : \quad x_\nu = (-1)^{\lceil \frac{\nu-1}{2} \rceil}, \quad \nu = \frac{1}{2}k; \\
\alpha &= \beta = 1 : \quad x_\nu = (-1)^{\lceil \frac{\nu}{2} \rceil}, \quad \nu = \frac{1}{2}k.
\end{align*}
\]

**Proof.** According to Remark 2 in [13], in the degenerate case, we have rank $A_{j,k}^{(\alpha,\beta)} = k - 1$, i.e. ker $A_{j,k}^{(\alpha,\beta)}$ is always one-dimensional. By virtue of (11) and (27), the matrix $A_{1,k}^{(0,0)}$ is degenerate if and only if $\beta = 0$. Hence, relation (44) along with Lemma 2 implies that ker $A_{j,k}^{(0,0)}$ is a linear hull of the vector $X = X_0$ determined by (58) for $\alpha = \beta = 0 = 0$. Moreover, by virtue of (45) along with Proposition 1 and Lemma 2, the kernel of $A_{j,k}^{(1,\beta)}$ for $\beta \in \{0, 1\}$ is a linear hull of the vector $X = X_0$ determined by (58) and the corresponding $\beta$ as well as $z_0 = 0$ since $j$ is odd in both subcases (III), (IV) and, hence, $T_j(0) = 0$. Thus, formulae (61), (63) and (64) for components of $X$ can be easily obtained using (11), (28) and (58).

Now let $(\alpha, \beta) = (0, 1)$. Then representation (44) takes the form

\[ A_{j,k}^{(0,1)} = U_{j-1} \left( -\frac{1}{2} A_{1,k}^{(1,0)} \right) A_{1,k}^{(0,1)}. \]

According to (II) in (17) as well as (V) in (18), we have det $A_{j,k}^{(0,1)} = 0$ if and only if $j$ is even. In the degenerate case, since det $A_{j,k}^{(0,1)} \neq 0$, we have det $U_{j-1}((-1/2) A_{1,k}^{(1,0)}) = 0$. Moreover, since $j$ and $k$ are mutually prime, the value $k$ is odd. Thus, by virtue of Proposition 1 along with the relation $U_{j-1}(0) = 0$, a unique up to a multiplicative constant eigenvector of the matrix $A_{j,k}^{(0,1)}$ corresponding to the zero eigenvalue satisfies the linear equation

\[ A_{1,k}^{(0,1)} X = X_0, \]

where $X_0$ is an eigenvector of the matrix $A_{1,k}^{(1,0)}$ related to the zero eigenvalue. By virtue of (63), we have $[X_0]_{\nu} = (-1)^{\lceil \frac{\nu-1}{2} \rceil}, \nu = \frac{1}{2}k$. Thus, according to (48), equation (65) is equivalent to the system of scalar equations

\[ x_1 - x_2 = 1, \quad x_{\nu-1} - x_{\nu+1} = (-1)^{\lceil \frac{\nu-1}{2} \rceil}, \quad \nu = 2, k - 1, \quad x_{k-1} + x_k = (-1)^{\lceil \frac{k-1}{2} \rceil}. \]

Summing up all equations in (66), we get $2x_1 = s_k$, where

\[ s_n = \sum_{\nu=1}^{n} (-1)^{\lceil \frac{\nu-1}{2} \rceil}. \]

Obviously, $s_{4l+1} = s_{4l+3} = 1$, $s_{4l+2} = 2$ and $s_{4l+4} = 0$ for all $l \geq 0$. Thus, since $k$ is odd, we have $s_k = 1$ and, hence, $x_1 = 1/2$. Rewrite the first $k - 1$ equations in (66) in the following way:

\[
\begin{align*}
\begin{cases}
x_1 - x_2 = x_1 - x_3 = 1, \\
x_{2\nu} - x_{2\nu+2} = (-1)^{\nu}, \\
x_{2\nu+1} - x_{2\nu+3} = (-1)^{\nu},
\end{cases} \quad \nu = 1, \frac{k-3}{2},
\end{align*}
\]
where $\Lambda$ is the spectrum of the problem $L$ in (61)--(64), while the function $f$ is different from $q_j$ that is determined by the corresponding formula in [17]. Thus, we arrive at the following representations depending on the parities of $q_j$:

\[ q(x) = q_0(x) + R^{-1}F(x), \quad F(x) = Xf(x). \quad (67) \]

Obviously, the obtained function $q(x)$ ranges over $\mathcal{M}_\Lambda$ as soon as so does $f(x)$ over $L(0,b)$, where $\Lambda$ is the spectrum of the problem $L(q_0(x), \alpha, \beta, j/k)$.

According to [14] and [15], we have the following formulae for $R^{-1}F(x)$, $x \in (0,1)$, with $F(t) = (f_1(t), \ldots, f_k(t))^T$, $t \in (0,b)$:

\[ R^{-1}F(x) = \begin{cases} f_\nu(x-(k-\nu)b) & \text{for even } j+\nu, \\ f_\nu((k-\nu+1)b-x) & \text{for odd } j+\nu, \end{cases} \quad x \in ((k-\nu)b, (k-\nu+1)b), \quad \nu = 1/k. \]

Thus, we arrive at the following representations depending on the parities of $j$ and $k$:

\[ R^{-1}F(x) = \begin{cases} f_k(x), & x \in (0,b), \\ f_{k-1}(2b-x), & x \in (b,2b), \\ f_{k-2}(x-2b), & x \in (2b,3b), \\ f_{k-3}(4b-x), & x \in (3b,4b), \\ f_{k-4}(x-4b), & x \in (4b,5b), \\ \vdots \\ f_2(1-b-x), & x \in (1-2b,1-b), \\ f_1(x-1+b), & x \in (1-b,1), \end{cases} \quad (68) \]

for odd $j$ and odd $k$;

\[ R^{-1}F(x) = \begin{cases} f_k(b-x), & x \in (0,b), \\ f_{k-1}(x-b), & x \in (b,2b), \\ f_{k-2}(3b-x), & x \in (2b,3b), \\ f_{k-3}(x-3b), & x \in (3b,4b), \\ f_{k-4}(5b-x), & x \in (4b,5b), \\ \vdots \\ f_2(x-1+2b), & x \in (1-2b,1-b), \\ f_1(1-x), & x \in (1-b,1), \end{cases} \quad (69) \]
for even $j$ and odd $k$;

$$R^{-1}F(x) = \begin{cases} f_k(b - x), & x \in (0, b), \\ f_{k-1}(x - b), & x \in (b, 2b), \\ f_{k-2}(3b - x), & x \in (2b, 3b), \\ f_{k-3}(x - 3b), & x \in (3b, 4b), \\ f_{k-4}(5b - x), & x \in (4b, 5b), \\ \vdots \\ f_3(x - 1 + 3b), & x \in (1 - 3b, 1 - 2b), \\ f_2(1 - b - x), & x \in (1 - 2b, 1 - b), \\ f_1(x - 1 + b), & x \in (1 - b, 1), \end{cases} \quad (70)$$

for odd $j$ and even $k$.

6. Illustrative examples

Finally, we give some examples illustrating the term $R^{-1}F(x)$ in (67) for all degenerate subcases (I)–(IV) in (17). We also provide the corresponding graphs of $R^{-1}F(x)$ taking the model function $f(x)$ in (67) of the following form:

$$f(x) = \frac{10x}{3b} - \frac{25x^2}{9b^2}, \quad b = \frac{1}{k}.$$ 

Example I. Let $\alpha = \beta = 0$. Then, for $(j, k) = (3, 7)$, formulae (61) and (68) give

$$F(x) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} f(x), \quad R^{-1}F(x) = \begin{cases} f(x), & x \in (0, 1/7), \\ -f(2/7 - x), & x \in (1/7, 2/7), \\ f(x - 2/7), & x \in (2/7, 3/7), \\ -f(4/7 - x), & x \in (3/7, 4/7), \\ f(x - 4/7), & x \in (4/7, 5/7), \\ -f(6/7 - x), & x \in (5/7, 6/7), \\ f(x - 6/7), & x \in (6/7, 1), \end{cases}$$

while, for $(j, k) = (3, 8)$, formulae (61) and (70) give the representations

$$F(x) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} f(x), \quad R^{-1}F(x) = \begin{cases} -f(1/8 - x), & x \in (0, 1/8), \\ f(x - 1/8), & x \in (1/8, 1/4), \\ -f(3/8 - x), & x \in (1/4, 3/8), \\ f(x - 3/8), & x \in (3/8, 1/2), \\ -f(5/8 - x), & x \in (1/2, 5/8), \\ f(x - 5/8), & x \in (5/8, 3/4), \\ -f(7/8 - x), & x \in (3/4, 7/8), \\ f(x - 7/8), & x \in (7/8, 1). \end{cases}$$
Example I: \( \alpha = \beta = 0, j = 3, k = 7 \).

Example II. Let \( \alpha = 0 \) and \( \beta = 1 \). Then, for \((j,k) = (2,7)\), formulae (62) and (69) give

\[
F(x) = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} f(x), \quad R^{-1}F(x) = \begin{cases} -f(1/7 - x), & x \in (0,1/7), \\ -f(x - 1/7), & x \in (1/7,2/7), \\ f(3/7 - x), & x \in (2/7,3/7), \\ f(x - 3/7), & x \in (3/7,4/7), \\ -f(5/7 - x), & x \in (4/7,5/7), \\ -f(x - 5/7), & x \in (5/7,6/7), \\ f(1 - x), & x \in (6/7,1). \end{cases}
\]

Example III. Let \( \alpha = 1 \) and \( \beta = 0 \). Then, for \((j,k) = (3,7)\), formulae (63) and (68) give

\[
F(x) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} f(x), \quad R^{-1}F(x) = \begin{cases} -f(x), & x \in (0,1/7), \\ f(2/7 - x), & x \in (1/7,2/7), \\ f(x - 2/7), & x \in (2/7,3/7), \\ -f(4/7 - x), & x \in (3/7,4/7), \\ -f(x - 4/7), & x \in (4/7,5/7), \\ f(6/7 - x), & x \in (5/7,6/7), \\ f(x - 6/7), & x \in (6/7,1). \end{cases}
\]

Example II: \( \alpha = 0, \beta = 1, j = 2, k = 7 \).

Example III: \( \alpha = 1, \beta = 0, j = 3, k = 7 \).
Example IV. Let $\alpha = 1$ and $\beta = 1$. Then, for $(j,k) = (3,8)$, formulae (64) and (70) give

$$F(x) = \begin{bmatrix}
1 \\
-1 \\
-1 \\
1 \\
1 \\
-1 \\
-1 \\
1
\end{bmatrix} f(x), \quad R^{-1}F(x) = \begin{cases}
f(1/8 - x), & x \in (0, 1/8), \\
-f(x - 1/8), & x \in (1/8, 1/4), \\
-f(3/8 - x), & x \in (1/4, 3/8), \\
f(x - 3/8), & x \in (3/8, 1/2), \\
f(5/8 - x), & x \in (1/2, 5/8), \\
-f(x - 5/8), & x \in (5/8, 3/4), \\
-f(7/8 - x), & x \in (3/4, 7/8), \\
f(x - 7/8), & x \in (7/8, 1).
\end{cases}$$

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