NONLINEAR DISPERSION EQUATIONS: SMOOTH DEFORMATIONS, COMPACTONS, AND EXTENSIONS TO HIGHER ORDERS

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Abstract. The third-order nonlinear dispersion PDE, as the key model,
(0.1) \[ u_t = (uu_x)_xx \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \]
is studied. Two Riemann’s problems for (0.1) with initial data \( S_{\pm}(x) = \mp \text{sign} x \), create the shock \( (u(x, t) \equiv S_{\pm}(x)) \) and smooth rarefaction (for data \( S_+ \)) waves, [18].

The concept of “\( \delta \)-entropy” solutions (a “\( \delta \)-entropy test”) and others are developed for distinguishing shock and rarefaction waves by using stable smooth \( \delta \)-deformations of discontinuous solutions. These are analogous to entropy solutions for scalar conservation laws such as \( u_t + uu_x = 0 \), developed by Oleinik and Kruzhkov (in \( \mathbb{R}^N \)) in the 1950-60s.

The Rosenau–Hyman \( K(2, 2) \) (compacton) equation
\[ u_t = (uu_x)_{xx} + 4uu_x, \]
which has a special importance for applications, is studied. Compactons as compactly supported travelling wave solutions are shown to pass the \( \delta \)-entropy test. Shock and rarefaction waves are discussed for other NDEs such as
\[ u_t = (u^2u_x)_{xx}, \quad u_{tt} = (uu_x)_{xx}, \quad u_{tt} = uu_x, \quad u_{tt} = (uu_x)_{xx}, \quad u_t = (uu_x)_{xxxxxx}, \text{ etc.} \]

Dedicated to the memory of Professors O.A. Oleinik and S.N. Kruzhkov

1. Introduction: nonlinear dispersion PDEs and main results

1.1. NDEs: nonlinear dispersion equations in application and general PDE theory. The present paper continues the study begun in [18] of odd-order nonlinear dispersion (or dispersive) PDEs (NDEs). The canonical model is the third-order quadratic NDE (the NDE–3)
\[ (1.1) \quad u_t = A(u) \equiv (uu_x)_{xx} = uu_{xxx} + 3u_xu_{xx} \quad \text{in} \quad \mathbb{R} \times (0, T), \quad T > 0. \]
Posing for (1.1) the Cauchy problem includes locally integrable initial data
\[ (1.2) \quad u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}. \]
Frequently, we assume that \( u_0 \) is bounded and compactly supported. We will also deal with the initial-boundary values problem in \( (-L, L) \times \mathbb{R}_+ \) with Dirichlet boundary conditions.
Main applications concerning NDEs can be found in [14, 18]; see also [19, Ch. 4], so that we do not discuss these issues in detail. However, we need to stress the attention of the Reader to the compacton phenomena, which were not properly treated in the mathematical literature.

**Compact patterns and NDEs.** These are known for the Rosenau–Hyman (RH) equation

\[ u_t = (u^2)_{xxx} + (u^2)_x, \]

which is the \( K(2,2) \) equation from the general \( K(m,n) \) family of the NDEs:

\[ u_t = (u^n)_{xxx} + (u^m)_x \quad (u \geq 0). \]

References on physical applications of such NDEs are available in [18, §1] and in [19, §4.2]. We will check entropy properties of compactons for various NDEs of this type.

Further applied compacton-like models are discussed in [18]. A standard definition of weak solutions for (1.1) is also presented there, so that we are in a position to explain our main targets concerning entropy-like theory of shocks.

1.2. **Plan of the paper: entropy theory (a test) via smooth deformations and compactons.** As in [18], we begin with discussion of some auxiliary properties of the NDE–3.

**Smoothing for the NDE–3.** Firstly, we recall that the smoothing phenomena and results for sufficiently regular solutions of linear and nonlinear third-order PDEs are well known from the 1980-90s. For instance, infinite \( C^\infty \)-smoothing results were proved in [2] for a general linear equation of the form

\[ u_t + a(x,t)u_{xxx} = 0 \quad (a(x,t) \geq c > 0), \]

and in [3] for the corresponding fully nonlinear PDE

\[ u_t + f(u_{xxx}, u_{xx}, u_x, u, x, t) = 0 \quad (f_{u_{xxx}} \geq c > 0). \]

Namely, for a class of such equations, it is shown that, for data with minimal regularity and sufficient decay at infinity, there exists a unique solution \( u(x,t) \in C^\infty \) for arbitrarily small \( t > 0 \). Similar smoothing local in time results for unique solutions are available for

\[ u_t + f(D^3 u, D^2 u, Du, u, x, y, t) = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+; \]

see [25] and further references therein. Concerning unique continuation and continuous dependence properties, see [4] and references therein, and [36] for various estimates.

**The NDE: a conservation law in \( H^{-1} \).** Writing (1.1) as (see details in [18, §1.4])

\[ (-D_x^2)^{-1}u_t + uu_x = 0, \]

yields the first *a priori* uniform bound for data \( u_0 \in H^{-1}(\mathbb{R}) \). Namely, multiplying (1.8) by \( u \) in \( L^2 \) gives the conservation law

\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-1}}^2 = 0 \implies \|u(t)\|_{H^{-1}}^2 = \|u_0\|_{H^{-1}}^2 \quad \text{for all} \quad t > 0. \]

**Main results.** In the present paper, we propose some concepts for developing adequate mathematics of NDEs with shocks, which will be concluded in Section 5 by revealing
connections with other classes of nonlinear degenerate PDEs. It turns out that some NDE concepts has definite reliable common roots and can be put into the framework of much better developed theory of quasilinear parabolic equations. We restrict our attention to a key demand, how to distinguish the shock and rarefaction waves, and this is done by developing the so-called "\( \delta \)-entropy test" on solutions via smooth deformations. General uniqueness-entropy theory for NDEs such as (1.1) and others is shown to be illusive [15].

Concerning the simple canonical model (1.1), we do the following:

- (i) Reviewing local existence and uniqueness theory for the NDE (1.1) and, on its basis, developing an \( \delta \)-entropy test for distinguishing shock and rarefaction waves.

For the RH equation such as (1.3), we prove that:

- (ii) Rosenau’s compacton solutions are both \( \delta \)-entropy and G-admissible.

Some of related questions and results were previously discussed in a more applied and formal fashion in [13, § 7] and [19, Ch. 4].

### 1.3. On extensions and other nonlinear dispersion models.

The developed concepts cover a wide range of various NDEs. First of all, we should mention that the fact that (1.1) is degenerate at \( u = 0 \) and hence admits compactly supported solutions (which is an interesting pleasant feature) makes the analysis of \( \delta \)-entropy solutions and shocks much harder. However, shock waves exist for other non-degenerated NDEs with analytic coefficients. For instance, we study entropy shocks for the NDE with infinite propagation,

\[
(1.10) \quad u_t = ((1 + u^2)u_x)_{xx}.
\]

All our further NDEs admit analogous non-degenerate versions admitting shock and rarefaction waves, but no finite propagation and interfaces in the Cauchy problem.

Another related to (1.1) model to be discussed is the cubic fully divergent NDE

\[
(1.11) \quad u_t = (u^2 u_x)_{xx} \equiv \frac{1}{3} (u^3)_{xxx} \quad \text{ (the conservation law analogy is } u_t + u^2 u_x = 0).\]

We study (1.11) instead of less physically motivated “quadratic” model \( u_t = (|u|u_x)_{xx} \) that exhibits similar properties of shocks and rarefaction waves.

The results on \( \delta \)-entropy solutions and similarity patterns can be extended (Section 5) to truly quadratic non-fully divergent NDEs such as

\[
(1.12) \quad u_t = (uu_{xx})_x \equiv uu_{xxx} + u_x u_{xx},
\]

which we call the NDE–(2,1), where 2 and 1 stand for the number of the internal and external derivatives in this differential form. Notice that a standard concept of weak solutions hardly applies to (1.12), so that the shock \( S_-(x) \) is not a weak solution. In order to underline once more the fact that being weak is not a necessary demand, we consider a formal fully nonlinear NDE

\[
(1.13) \quad |u_t|^\gamma u_t = (uu_x)_{xx}, \quad \text{where } \gamma > -1.
\]

For \( \gamma = 0 \), this gives the original equation (1.1). Obviously, for \( \gamma \neq 0 \), (1.13) does not admit any weak formulation. Nevertheless, we show that (1.13) admits blow-up formation of shocks of \( S_-\)-type.
In Section 6, we discuss the shock formation mechanism for higher-order in time NDEs, \( u_t = (uu_x)_{xx} \) and \( u_{tt} = (uu_x)_{xx} \).

Several principal features remain the same for higher-order NDEs such as the quadratic fifth-order NDE \( \text{(NDE–5)} \)
\[ u_t = -(uu_x)_{xxxx} \] or, in general, \( u_t = (-1)^{m+1}D_x^{2m}(uu_x), \ m \geq 1; \)
see Section 7. These are conservation laws in \( H^{-2} \), or \( H^{-m} \). The mathematics of particular similarity solutions with shocks is developed in similar lines but technically becomes more involved, so we have to catch the similarity profiles numerically.

We also claim that some concepts such as smooth \( \delta \)-deformation and others, developed for models in 1D can be adapted to the \( N \)-dimensional NDEs. In particular, the basic NDE \( \text{(1.1)} \) in \( \mathbb{R}^N \) takes the form
\[ u_t = \Delta(u \frac{\partial u}{\partial x_1}) \equiv \frac{1}{2} \frac{\partial}{\partial x_1} \Delta u^2 \] in \( \mathbb{R}^N \times (0, T) \).

2. Conservation laws: smooth \( \delta \)-deformations define entropy solutions

From now on, being sufficiently informed about formation of crucial shock and other singularities in the NDEs, we will start to investigate the general questions on existence and uniqueness of entropy weak solutions of \( \text{(1.1)} \). As usual, we begin our discussion by stressing attention to key analogies with classic theory of first-order conservation laws such as Euler’s equation from gas dynamics
\[ u_t + uu_x = 0 \] in \( \mathbb{R} \times \mathbb{R}_+ \).

Entropy theory for such first-order PDEs was created by Oleinik [31, 32] and Kruzhkov [23] \((x \in \mathbb{R}^N)\) in the 1950–60s; see details on the history, main results, and modern developments in the well-known monographs [1, 5, 34]. Thus, we now apply smooth \( \delta \)-deformation concepts to these simpler PDEs considered now in \( Q_1 = \mathbb{R} \times (0, 1) \).

2.1. Preliminaries: entropy inequalities and solutions for conservation laws. It is known from the 1950’s that the Cauchy problem for general scalar conservation laws admits a unique entropy solution. We refer to first complete results by Oleinik (obtained in 1954-56), who introduced entropy conditions in 1D and proved existence and uniqueness results (see survey [31]), and by Kruzhkov (1970) [23], who developed general non-local theory of entropy solutions in \( \mathbb{R}^N \). In the general case, one of Oleinik’s local entropy condition has the form [31, p. 106]
\[ \frac{u(x_1,t) - u(x_2,t)}{x_1 - x_2} \leq K(x_1, x_2, t) \] for all \( x_1, x_2 \in \mathbb{R}, t \in [0, 1] \),
where \( K \) is a continuous function for \( t \in [0, 1] \). Oleinik’s local condition E (Entropy) introduced in [32], for the model equation \( \text{(2.1)} \) corresponds to the well-known principle of non-increasing entropy from gas dynamics,
\[ u(x^+, t) \leq u(x^-, t) \] in \( Q_1 = \mathbb{R} \times (0, 1) \),
with strict inequality on lines of discontinuity, [31, p. 101].
Kruzhkov’s entropy condition \([23]\) on solutions \(u \in L^\infty(Q_1)\) of (2.1) takes the form of the non-local inequality
\[
|u - k|_t + \frac{1}{2} \left[ \text{sign}(u - k)(u^2 - k^2) \right] x \leq 0 \quad \text{in } D'(Q_1) \quad \text{for any } k \in \mathbb{R}.
\]
This inequality is understood in the sense of distributions meaning that the sign \(\leq\) is preserved after multiplying the inequality by any smooth compactly supported cut-off function \(\varphi \in C^\infty_0(Q_1), \varphi \geq 0\), and integrating by parts. See clear presentation of these ideas in Taylor [37, p. 401]. Oleinik’s and Kruzhkov’s approaches are known to coincide in the 1D geometry. Both entropy conditions generate a semigroup of contractions in \(L^1\), so that if \(u\) and \(v\) are two solutions of (2.1), then
\[
\frac{d}{dt} \|u(t) - v(t)\|_{L^1} \leq 0.
\]
It is key that the unique entropy solution is constructed by the parabolic \(\varepsilon\)-approximation
\[
u_\varepsilon : \quad u_t + uu_x = \varepsilon u_{xx} \quad (\varepsilon > 0).
\]
Multiplying (2.6) by any smooth monotone increasing function \(E(u)\) (an approximation of sign \((u - k)\) for any \(k \in \mathbb{R}\)) yields on integration by parts the correct sign:
\[
\int \varepsilon u_{xx} E(u) = -\varepsilon \int E'(u)(u_x)^2 \leq 0.
\]
Hence, as \(\varepsilon \to 0\), this gives the necessary sign as in (2.4).

The obvious advantage of the conservation law (2.1) is that, for smooth initial data (1.2), the unique local continuous solution is obtained by method of characteristics and is given by the corresponding algebraic equation
\[
dt = \frac{dx}{u} \implies u(x,t) = u_0(x - u(x,t)t) \quad \text{for all } t \in [0, \Delta t),
\]
where \(\Delta t \leq 1\) is the first moment of time when a shock of the type \(S_-(x)\) (this type is guaranteed by (2.3)) occurs at some point or many points.

Thus, for \(t \geq \Delta t\), it is necessary to apply the entropy inequalities to select good (entropy) solutions. Using this, and bearing in mind that entropy solutions are continuous relative initial data (in \(L^1\), say), we propose the following construction which is fully based on algebraic relations (2.8):

2.2. Conservation laws: \(\delta\)-stable = entropy solutions. It is the obvious well-known and, nevertheless, crucial observation that, by the characteristic mechanism (2.8),
\[
\text{(2.9) non-entropy shocks of the shape } S_+ \text{ cannot appear evolutionary.}
\]
Indeed, differentiating (2.8) in \(x\) yields
\[
u_x(x,t) = \frac{u'_0(x - u(x,t)t)}{1 + u'_0(x - u(x,t)t)} \quad \text{so that}
\]
\[
u'_0 \geq 0 \implies \text{no blow-up of } u_x \text{ (“gradient catastrophe”) occurs.}
\]
Recalling the necessary evolution property in (2.10), given a small \(\delta > 0\) and a bounded (say, for simplicity, in \(L^1\) and in \(L^\infty\)) solution \(u(x,t)\) of the Cauchy problem (2.1), (1.2), we construct its \(\delta\)-deformation given explicitly by the characteristic method (2.8) as follows:
(i) we perform a smooth $\delta$-deformation of initial data $u_0 \in L^1 \cap L^\infty$ by introducing a suitable $C^1$ function $u_{0\delta}(x)$ such that

\begin{equation}
(2.11) \quad \int |u_0 - u_{0\delta}| < \delta.
\end{equation}

By $u_{1\delta}(t, x)$ we denote the unique local solution of the Cauchy problem with data $u_{0\delta}$, so that by (2.8), continuous function $u_{1\delta}(x, t)$ is defined algebraically on the maximal interval $t \in [t_0, t_1(\delta))$, where we denote $t_0 = 0$ and $t_1(\delta) = \Delta_{1\delta}$. It is important that, here and later on, smooth deformations are performed in a small neighborhood of possible discontinuities only leaving the rest of smooth profiles untouchable, so that these evolve along the characteristics, as usual.

Actually, this emphasizes the obvious fact that the shocks (on a set of zero measure) occur as a result of nonlinear interaction of the areas with continuous solutions, which hence cannot be connected without discontinuities.

(ii) Since at $t = \Delta_{1\delta}$ a shock of type $S_-$ (or possibly infinitely many shocks) is supposed to occur, since otherwise we continue the algebraic procedure, we perform another suitable $\delta$-deformation of the “data” $u_{1\delta}(x, \Delta_{1\delta})$ to get a unique continuous solution $u_{2\delta}(x, t)$ on the maximal interval $t \in [t_1(\delta), t_2(\delta))$, with $t_2(\delta) = \Delta_{1\delta} + \Delta_{2\delta}$, etc.

\[ \ldots \]

(k) With suitable choices of each $\delta$-deformations of “data” at the moments $t = t_j(\delta)$, when $u_{j\delta}(x, t)$ has a shock for $j = 1, 2, \ldots$, there exists a $t_k(\delta) > 1$ for some finite $k = k(\delta)$, where $k(\delta) \to +\infty$ as $\delta \to 0$. It is easy to see that, for bounded solutions, $k(\delta)$ is always finite. A contradiction is obtained while assuming that $t_j(\delta) \to \tilde{t} < 1$ as $j \to \infty$ for arbitrarily small $\delta > 0$ meaning a kind of “complete blow-up” that is impossible for conservation laws obeying the Maximum Principle.

This gives us a global $\delta$-deformation in $\mathbb{R} \times [0, 1]$ of the solution $u(x, t)$, which is a discontinuous orbit denoted by

\begin{equation}
(2.12) \quad u^\delta(x, t) = \{ u_{j\delta}(x, t) \} \text{ for } t \in [t_{j-1}(\delta), t_j(\delta)), \quad j = 1, 2, \ldots, k(\delta).
\end{equation}

Recall that the whole orbit (2.12) has been constructed by the algebraic characteristic calculus using (2.8) only. Finally, by an arbitrary smooth $\delta$-deformation, we will mean the function (2.12) constructed by any sufficiently refined finite partition $\{ t_j(\delta) \}$ of $[0, 1]$, without reaching a shock of $S_-$-type at some or all intermediate points $t = t_j(\delta)$.

We next say that, given a solution $u(x, t)$, it is stable relative smooth deformations, or simply $\delta$-stable ($\delta$deformation-stable), if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any finite $\delta$-deformation of $u$ given by (2.12),

\begin{equation}
(2.13) \quad \iint |u - u^\delta| < \varepsilon.
\end{equation}

Then we have the following simple observation:

**Proposition 2.1.** Let under given hypothesis, a weak solution $u(x, t)$ of the Cauchy problem (2.7), (1.2) be $\delta$-stable. Then it is entropy.

Indeed, if $u(x, t)$ is not entropy, then there exists $t_\ast \in (0, 1]$ such that $u(x, t_\ast)$ does not satisfy (2.3), i.e., this profile has a finite non-entropy shock of the type $S_+$ at some point...
\( x_\ast \in \mathbb{R} \). Since those shocks cannot be reproduced with arbitrary accuracy \( \varepsilon \) in \( L^1 \) by the characteristic system (2.3), any \( \delta \)-deformation \( u^\delta \) at \( t = t_\ast \) must stay \( \varepsilon_0 > 0 \) away from \( u(x, t_\ast) \) for arbitrarily small \( \delta > 0 \).

Of course, this construction does not play a role for conservation laws with well-developed entropy theory, which establishes existence of a semigroup of \( L^1 \)-contractions of entropy solutions. Obviously, this strong contractivity property guarantees also uniqueness of \( \delta \)-entropy solutions. The situation is different for the NDEs:

3. On \( \delta \)-entropy solutions (a test) of the NDE

Thus, we are going to develop and discuss some aspects of entropy solutions for (1.1), without using the idea of vanishing, \( \varepsilon \to 0 \), viscosity as in [13, § 7]

\[
(3.1) \quad u_t = (uu_x)_{xx} - \varepsilon u_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \quad (\varepsilon > 0).
\]

A direct verification that the \( \varepsilon \)-approximation (3.1) yields as \( \varepsilon \to 0 \) the correct Kruzhkov’s-type entropy solution leads to difficult open problems. We begin with:

3.1. (2m+1)th-order NDEs for any \( m \geq 1 \) DO NOT generate a semigroup of contractions in \( L^1 \). A first naive approach would be to try to create a standard entropy condition for the NDE of, say, the following form (cf. (2.4)):

\[
(3.2) \quad |u - k|_t - \frac{1}{2} \left[ \text{sign}(u - k)(u^2 - k^2) \right]_{xxx} \leq 0 \quad \text{in} \quad \mathcal{D}'(Q_1) \quad \text{for any} \quad k \in \mathbb{R}.
\]

Then Kruzhkov’s-type computations with (1.1) are supposed to be performed by using his fundamental idea of doubling the space dimension; see a clear presentation in [37, p. 402], with some obvious adaptations of test functions involved.

One should avoid doing this bearing in mind that this approach must end up with the contractivity property (2.5), which cannot be true for any PDE of order larger than two, since these are associated with manipulations based on the Maximum Principle for first-order or, at most, for second-order parabolic PDEs. This means that semigroups of contractions in \( L^1 \) are not available for such NDEs (1.15) with any \( m \geq 1 \).

3.2. On smooth solutions and odd-order operator theory. Thus, we return to the Cauchy problem for the NDE (1.1). As we have mentioned, unlike the first-order case (2.6), applying the \( \varepsilon \)-approximation as in (3.1) leads to a number of principal difficult problems and, in the maximal generality (excluding special cases), does not give neither existence of a solution via the family \( \{u_\varepsilon\} \) nor uniqueness of an “\( \varepsilon \)-entropy” solution, [13].

We will develop other concepts of solutions by different types of approximations, and then the concept of uniqueness will be attached to the nature of existence results.

On local semigroup of smooth solutions. Beforehand, it is of importance that, as the similarity solutions in [18, § 3] showed, the NDE (1.1) does not admit a global in time solution for any bounded \( L^1 \) data. This is in striking difference with the conservation laws (2.1), where such existence is guaranteed by the Maximum Principle. Therefore, we restrict our attention to weak solutions \( u(x, t) \) in \( Q_1 \), where

\[
(3.3) \quad u_0(x) \in C_0^\infty(\mathbb{R}) \quad \text{is sufficiently small}.
\]
Then, as the first step of a similar construction, we have to check that for such smooth initial data \( u_0 \), there exists a unique local classical \( C^{3,1} \) solution \( u(x,t) \) of (1.1). Recall that characteristic methods similar to that in (2.8) are not available for higher-order PDEs. This just means that (1.1) generates a standard local semigroup in the class smooth functions. These results are known for non-degenerate NDEs such as (1.6), and moreover the solutions are \( C^\infty \) locally in time, [2, 3, 25]. Actually, these smoothing results can be viewed in conjunction with classic methods of analytic semigroups in PDE theory; see [4] and references in a more recent paper [8]; see below.

**Uniqueness and continuous dependence: an illustration.** Actually, in our construction, we will need just a local semigroup of smooth solutions that is continuous in \( L^1 \). The fact that this is generated by third-order (or other odd-order NDEs) is illustrated by the following easy example. Consider, for definiteness, the NDE

\[
(3.4) \quad u_t = A(u) \equiv uu_{xxx}, \quad u(x,0) = u_0(x) \in H^7(\mathbb{R}),
\]

where, without loss of generality, we take into account the principal higher-order term only. According to the above results, we assume that \( u(x,t) \) satisfies

\[
(3.5) \quad \frac{1}{C} \leq u \leq C \quad (C > 1)
\]

and is sufficiently smooth, \( u \in L^\infty([0,T], H^7(\mathbb{R})) \) and \( u_t \in L^\infty([0,T], H^4(\mathbb{R})) \). See details on such uniqueness results in [3, § 3].

Thus, assuming that there exists the second smooth solution \( v(x,t) \), we subtract the equations and obtain for the difference \( w = u - v \) the following:

\[
(3.6) \quad w_t = uw_{xxx} + v_{xxx}w.
\]

We next divide (3.6) by \( u \geq \frac{1}{C} > 0 \), multiply by \( w \) in \( L^2 \), so, after integrating by parts,

\[
(3.7) \quad \int \frac{wu}{u} \, dt = \frac{1}{2} \int \frac{w^2}{u} \, du + \frac{1}{2} \int \frac{uw^2}{u} \, dw = \int \frac{v_{xxx}w^2}{u}.
\]

Therefore, using the assumed regularity yields

\[
(3.8) \quad \frac{1}{2} \frac{d}{dt} \int \frac{w^2}{u} \, dt = \int \left( \frac{u_{xxx}}{u} - \frac{1}{2} \frac{uw}{u^2} \right) w^2 \equiv \int \left( \frac{w_{xxx}}{u} - \frac{1}{2} \frac{uw}{u^2} \right) w^2 \leq C_1 \int \frac{w^2}{u},
\]

where we use the fact that \( u_{xxx}(\cdot,t), v_{xxx}(\cdot,t) \in L^\infty([0,T]) \). By Gronwall’s inequality, (3.8) implies that \( w(t) \equiv 0 \). As usual, this construction can be translated to the continuous dependence result in \( L^2 \) and hence in \( L^1_{\text{loc}} \).

**On degenerate NDEs.** For degenerate NDEs such as (1.1) and for solutions of changing sign, the unique local smooth solvability is a technical result, which we do not completely concentrate upon, and present below some rather formal comments justifying such a local continuation. One of the main difficulties of this local analysis, is that (1.1) admits solutions with finite interfaces and free boundaries, which represent “weak shocks” with quite tricky (smooth enough but not \( C^3 \)) behaviour.

Thus, in addition, except the shock waves, which we are mostly interested in, the NDE (1.1) is degenerate at \( \{ u = 0 \} \), so that the local existence of sufficiently smooth solution must include the demand of “transversality” of all the zeros (a finite number) of initial
data \( u_0(x) \) (or \( u(x, t_j(\delta)) \) later on). Here the transversality of the zero at, say, \( x = 0 \) has a standard meaning:

\[ u_0'(0) \neq 0. \]

For instance, for key applications, we may assume that \( u_0(x) \) is anti-symmetric, so \( u(-x, t) \equiv -u(x, t) \), and hence the only transversal zero is fixed at the origin \( x = 0 \) only, i.e.

\[ (3.9) \quad u(0, t) \equiv 0, \quad \text{and} \quad u(x, t) > 0 \text{ for } x < 0. \]

Then, according to regularity results for odd-order PDEs \( [2, 3, 21, 29] \) (cf. \( [4, 8, 27] \)), the linearization about sufficiently smooth \( u_0(x) \) yields that the possibility of local smooth extension of solution is governed by the good spectral properties of the third-order linear operator with the principal part

\[ (3.10) \quad P_3^1 = x \frac{d^3}{dx^3} \text{ for } x \approx 0^+. \]

This type of degeneracy is not sufficient to destroy good spectral properties of \( P_3^1 \) that still will admit a discrete spectrum and a compact resolvent in the corresponding weighted space \( \sim L^2_{1/x} \) for \( x > 0 \). Note that the singular point \( x = 0 \) starts to generate a continuous spectrum for the operator\( (3.11) \quad P_3^n = x^n \frac{d^3}{dx^3} \text{ (} x > 0 \text{) } \)

in the parameter range \( n \geq 3 \) only, i.e., for much stronger degeneracy than in \( (3.10) \). Indeed, then the change \( z = x^\alpha \) with \( \alpha = \frac{3-n}{3} > 0 \) transforms \( (3.11) \) into the regular operator with the constant principal part

\[ (3.12) \quad P_3 = D_z^3 \text{ for } z \approx 0^+, \]

for which all necessary spectral properties are obviously valid, \( [30] \). The finite interface behaviour will be shown to correspond to \( n = 2 \), so it is still in the good range. Our conclusions here are based on the well-known fact that the linear PDE

\[ (3.13) \quad u_t = u_{xxx} \]

generates a smooth (analytic in a properly weighted \( L^2 \)-space) semiflow given by

\[ (3.14) \quad u(x, t) = b(x - \cdot, t) * u_0(\cdot), \]

where \( b(x, t) \) is the fundamental solution

\[ (3.15) \quad b(x, t) = t^{-\frac{1}{2}} F(x/t^{\frac{3}{2}}), \text{ where } F = Ai(z), \quad F'' + \frac{1}{3} F z = 0, \quad \int F = 1. \]

Thus, for the degenerate NDE \( (1.1) \), the notion of “sufficiently smooth solutions” should also include the assumption of transversality, i.e., of local behaviour near zeros. Of course, this is not that essential hypothesis that has a local character, and, for instance, completely disappears for the related non-degenerate NDEs such as \( (1.10) \), which also admits shocks and needs proper entropy theory (to be treated also).
On odd-order ordinary differential operators. In the above analysis, we need a detailed spectral theory of third-order (or more generally, odd-) operators such as

\[ P_3 = a(z)D_z^3 + b(z)D_z^2 + c(z)D_z + d(z)I, \quad z \in (-L, L) \quad (a(z) \geq c > 0), \]

with bounded coefficients. This theory is available in Naimark’s classic book [30, Ch. 2]. It was shown that for regular boundary conditions (e.g., for periodic ones that are regular for any order and that suit us well), operators admit a discrete spectrum \( \{\lambda_k\} \), where the eigenvalues \( \lambda_k \) are all simple for \( k \gg 1 \), and a complete in \( L^2 \) subset of eigenfunctions \( \{\psi_k\} \) that create a Riesz basis\(^1\). This makes it possible to use standard eigenfunction expansion techniques; see necessary details and references at the end of Ch. 2 therein.

The eigenvalues of (3.16) have the asymptotics

\[ \lambda_k \sim (\pm 2\pi ki)^3 \quad \text{for all} \quad k \gg 1. \]

In particular, this means that \( P_3 - aI \) for any \( a \gg 1 \) is not a sectorial operator that makes suspicious referring to the analogies with analytic theory [4, 8, 27] that is natural for even-order parabolic flows.

Nevertheless, recall that (3.14) guarantees analyticity of solutions that is now associated with the Airy-type operator

\[ B_3 = D_z^3 + \frac{1}{3} zD_z + \frac{1}{3} I \quad \text{in} \quad L^2_\rho(\mathbb{R}), \quad \rho(z) = e^{a|z|^{3/2}}, \]

where \( a > 0 \) is sufficiently small; cf. a “parabolic” version of such a spectral theory in [7]. It turns out that (3.18) has the real spectrum (see [12, §9])

\[ \sigma(B) = \{ -\frac{i}{3}, l = 0, 1, 2, ... \}, \]

so that \( B - aI \) is sectorial for \( a \geq 0 \) (\( \lambda_0 = 0 \) is simple), and this justifies the fact that (3.14) is an analytic flow.

Note also that analytic smoothing effects are known for higher-order dispersive equations with operators of principal type, [35]. This suggests to treat (3.14) by classic approach as in Da Prato–Grisvard [4] by linearizing about a sufficiently smooth \( u_0 = u(t_0), \ t_0 \geq 0, \) by setting \( u(t) = u_0 + v(t) \) giving the linearized equation

\[ v_t = A'(u_0)v + A(u_0) + g(v), \quad t > t_0; \quad v(t_0) = 0, \]

where \( g(v) \) is a quadratic perturbation. Using good semigroup properties of \( e^{A'(u_0)t} \), this makes it possible to study local regularity properties of the integral equation

\[ v(t) = \int_{t_0}^{t} e^{A'(u_0)(t-s)}(A(u_0) + g(v(s))) \, ds. \]

It is key that the necessary smoothness of solutions demands the fast exponential decay of solutions \( v(x, t) \) as \( x \to \infty \), since one needs that \( v(\cdot, t) \in L^2_\rho \); cf. [25], where \( C^\infty \)-smoothing also needs an exponential-like decay. Equations such as (3.20) can be used to guarantee local existence of smooth solutions of a wide class of odd-order NDEs.

\(^1\)This is G.M. Kessel’man’s (1964) and V.P. Mikhailov’s (1962) result.
Thus, we state the following conclusion to be used later on:

\[(3.21)\] any sufficiently smooth solution \(u(x,t)\) of \((3.4), (3.5)\) at \(t = t_0\),

\[\text{can be uniquely extended to some interval } t \in (t_0, t_0 + \delta), \delta > 0.\]

3.3. **Global solutions by Galerkin method.** Here we demonstrate the application of another classic approach to nonlinear problems that, suddenly, in the present case of unclear entropy nature of solutions of NDEs and the open uniqueness problem, gives a partial answer to both. We mean the Galerkin method that was the most widely used approach for constructing weak solutions via finite-dimensional approximations; see Lions [26] with many applications therein.

Thus, by this classic theory of nonlinear problems, under the assumption \((3.3)\) and others, if necessary, let us perform a standard construction of a compactly supported (for simplicity) solution by Galerkin method using the basis \(\{\psi_k\}\) of eigenfunctions of the regular linear operator \(P_2 = D_x^2 < 0\) with the Dirichlet boundary conditions,

\[\psi'' = \lambda_k \psi, \quad \psi = 0 \text{ at } x = \pm L \implies \lambda_k \sim -k^2. \quad \text{and} \quad u = 0 \text{ at } x = L.\]

As an alternative, it is curious that, for our purposes, possible (and more convenient for some reasons) to use the eigenfunction set of the operator \(P_4 = -D_x^4 < 0\) again with the Dirichlet conditions

\[\psi = \psi_x = 0 \text{ at } x = \pm L.\]

Special Galerkin bases associated with higher-order operators \(P_6 = D_x^6 < 0\) are also may be convenient; see applications to third-order linear dispersion equations in [24].

In all these self-adjoint cases, the eigenfunctions form a complete and closed set in \(L^2\); see classic theory of ordinary differential operators in Naimark [30, p. 89].

On the other hand, looking more natural choice of the third-order operator \(P_3 = D_x^3\) for Galerkin approximation of (1.1) will cause a difficult problem, since for the third-order PDE with the principal operator as in (1.1),

\[\psi'' = \lambda_k \psi, \quad \psi = 0 \text{ at } x = \pm L \implies \lambda_k \sim -k^2. \quad \text{and} \quad u = 0 \text{ at } x = L.\]

as an alternative, it is curious that, for our purposes, possible (and more convenient for some reasons) to use the eigenfunction set of the operator \(P_4 = -D_x^4 < 0\) again with the Dirichlet conditions

\[\psi = \psi_x = 0 \text{ at } x = \pm L.\]

For \(a > 0\), proper setting for the IBV problem includes the Dirichlet conditions (see Faminskii [11] for details and a survey)

\[u = u_x = 0 \text{ at } x = -L \quad \text{and} \quad u = 0 \text{ at } x = L.\]

For \(a < 0\), the boundary conditions must be swapped, so that the proper setting of the problem depends on the unknown sign of solutions. Here, the fact that \(P_3 = D_x^3\) is not self-adjoint is not essential since, relative to adjoint basis \(\{\psi_k^*\}\), the closure and completeness of the bi-orthonormal generalized eigenfunction sets remain valid.

Actually, the choice of linear operators \(P_2 = D_x^2, P_4 = -D_x^4,\) or others, is not of principal importance if we are looking for compactly supported solutions

\[u \in C^\infty_0((-L, L) \times [0, 1]).\]

It should be noted that the control of finite propagation property in (1.1) is difficult and is an essential part of our further analysis. For instance, we also can fix periodic boundary conditions that are always regular, [30] Ch. 2 (it is curious that (3.24) are not).
Thus, we construct a sequence \( \{u_m\} \) of approximating Galerkin solutions of (1.1), (1.2) in the form of finite sums

\[
(3.26) \quad u_m(x, t) = \sum_{k=1}^{m} C_k(t) \psi_k(x),
\]

where \( \{C_j\} \) solve the quadratic dynamical systems

\[
(3.27) \quad C'_j = \sum_{(k,l)} C_k C_l J_{klj}, \quad \text{where} \quad J_{klj} = \langle \psi_k \psi'_l, \psi''_j \rangle = \lambda_j \langle \psi_k \psi'_l, \psi_j \rangle.
\]

For the conservation law (2.1), the DS takes the same form as in (3.27), with the only difference that

\[
(3.28) \quad J_{klj} = -\langle \psi_k \psi'_l, \psi_j \rangle.
\]

The identity (1.9) for \( u_m \) takes the form

\[
(3.29) \quad \sum_{(k)} \frac{1}{\nu_k^2} C^2_k(t) = c_{0m} = \sum_{(k)} \frac{1}{\nu_k^2} C^2_k(0), \quad t > 0.
\]

This guarantees global existence of the solutions \( u_m(x, t) \) showing that

\[
(3.30) \quad C_k(t) \quad \text{do not blow-up and exist for all} \quad t > 0.
\]

Since \( \psi_k \) are given by \( \sin(\lambda_k x) \) or \( \cos(\lambda_k x) \), a lot of coefficients \( J_{klj} \) vanish. For instance, if \( u_0(x) \) is odd, we take all the sin-functions,

\[
\psi_k(x) = \frac{1}{\sqrt{L}} \sin \left( \frac{k\pi x}{L} \right), \quad \lambda_k = \frac{-k^2 \pi^2}{L^2}, \quad k = 1, 2, \ldots.
\]

The non-zero coefficients \( J_{klj} \) occur iff \( k = j, \ l = 2j \), where (3.27) becomes simpler,

\[
(3.31) \quad C'_j = \frac{2\pi^4 j^4}{L^7} C_j C_{2j}, \quad j \geq 1.
\]

It is curious that (3.31) yields the following feature of a “maximum principle”:

\[
(3.32) \quad \text{sign} C_j(t) = \text{sign} C_j(0), \quad j \geq 1.
\]

Other \emph{a priori} estimates are obtained by multiplying (1.1) in \( L^2 \) by \( u \) and \( u_{xx} \) yielding the identities

\[
(3.33) \quad \frac{1}{2} \frac{d}{dt} \int u^2 = -\frac{1}{2} \int (u_x)^3, \quad \frac{1}{2} \frac{d}{dt} \int (u_x)^2 = -\frac{5}{2} \int u_x (u_{xx})^2.
\]

Then some interpolations of various terms in the identities (3.33) are necessary.

Thus, the sequence of “regularized” solutions (Galerkin approximations) \( \{u_m(x, t)\} \) is globally defined, and

\[
(3.34) \quad \{u_m\} \quad \text{is uniformly bounded in} \quad L^\infty([0, 1]; H^{-1}).
\]

Therefore, along a subsequence, \( \{u_m\} \) converges to \( u \) weakly-* in \( L^\infty([0, 1]; H^{-1}) \), and, in addition, strongly in \( H^{-1}([0, 1]; H^{-2}) \), in view of compact embedding. This gives a weak solution. As usual, the better regularity comes from the special choice of Galerkin’s basis employed. We do not stress attention to this (bearing in mind local \( C^\infty \)-smoothing for non-degenerate NDEs). See [24] for rather exotic Galerkin bases applied to KdV type equations. Recall that, globally, smoothing is not available, since this construction is specially oriented to include shocks of \( S_\cdot \)-type.
Remark 1. Obviously, the estimate (3.29) does not and cannot prevent gradient catastrophe, which means that
\[ \|u_x(t)\|_2^2 = \sum |\lambda_k| C_k^2(t) \to +\infty \quad \text{as} \quad t \to T^- \leq 1. \]
Notice that for (1.1) there is an opportunity to create blow-up of the solutions \( u(\cdot, t) \) itself (possibly together with (3.35)), where
\[ \|u(t)\|_2^2 = \sum C_k^2(t) \to +\infty \quad \text{as} \quad t \to T^- \leq 1. \]
This does not happen if a finite shock appears via the self-similar patterns such as [18]
\[ u_-(x, t) = g(z), \quad z = x/(-t)^{\frac{1}{3}}, \quad (gg')'' = \frac{1}{3} g'z, \quad f(\mp \infty) = \pm 1. \]
Indeed, by the first identity in (3.33), there appears an integrable singularity,
\[ \int_{-1}^{1} u^2 \sim -\frac{1}{2} \int_{-1}^{1} (u_x)^3 \sim (-t)^{-\frac{2}{3}} \int_{-(-t)^{-1/3}}^{0} (g')^3 \, dz \in L^1((-1, 0)), \]
so that \( \|u(0^-)\|_2^2 \) remains finite. Here in (3.38) one needs to use the asymptotics of the Airy function [18 § 3], so that the integral therein diverges but its rate,
\[ \left| \int_{-(-t)^{-1/3}}^{0} (g')^3 \, dz \right| \sim O((-t)^{-\frac{1}{10}}), \]
is sufficient for the integrability.

Remark 2. Using the dynamical system (3.27) instead of the NDE (1.1) suggests to develop a formal calculus of the corresponding sequences, where, on identification,
\[ u = \sum C_k \psi_k \implies u = \{C_k\} \]
belongs to the little Hilbert space \( h^{-1}_P \) with the metric
\[ \|u\|_P^{-1} = \sum \frac{1}{|\lambda_k|} C_k^2. \]
Then (3.29) guarantees that
\[ u(t) \in h^{-1}_P \quad \text{for all} \quad t \geq 0, \]
meaning global solvability. Moreover, the embedding \( h^{-2}_P \subset h^{-1}_P \) is compact since \( \frac{1}{|\lambda_k|} \sim \frac{1}{\pi} \) [28] (for \( h^{-2}_P \), the metric contains \( \frac{1}{|\lambda_k|^2} \) in (3.40)), so that we can use the same Galerkin approximation method to construct suitable solutions. In this space, the blow-up formation of shocks means (3.55).

Remark 3. Writing the \( N \)-dimensional NDE (1.16) for compactly supported \( u_0 \) as
\[ (-\Delta)^{-1} u_t = -\frac{1}{2} \frac{\partial}{\partial x_1} u^2 \]
with the standard definition of the linear operator \( (-\Delta)^{-1} \) in \( L^2(\Omega) \), \( \Omega \) is sufficiently large, and multiplying (3.42) by \( u \) yields the same conservation identity (1.9). Some concepts developed above can be also adapted to the equations in \( \mathbb{R}^N \), though shock wave formation phenomena become more involved and are in general unknown.
3.4. δ-entropy solutions (a test) for the NDE. Assuming that the local smooth solvability problem above is well-posed, we now present the corresponding definition that will be applied to particular weak solutions. Recall that the topology of convergence, $L^1_{\text{loc}}$ at present, for (1.1) was justified by a similarity analysis presented in [18, Prop. 3.2]. For other NDEs, the topology may be different that can be a difficult problem.

**Definition 3.1.** A weak solution $u(x,t)$ of the Cauchy problem (1.1), (1.2) is called δ-entropy if there exists a sequence of its smooth δ-deformations $\{u^{\delta_k}, k = 1, 2, \ldots\}$, where $\delta_k \to 0$, which converges in $L^1_{\text{loc}}$ to $u$ as $k \to \infty$.

Note that this is slightly weaker (but equivalent) to the condition of δ-stability. The construction of global δ-deformation of $u$ is performed along the lines of (i)–(k) in Section 2.2. The only difference is that local δ-deformations can lead to complete blow-up for the NDE (1.1), as explained in [18, § 4.2]. To avoid this, one needs either to impose the condition (3.3) or specially assume that complete blow-up cannot occur under slight deformation of the data, or while performing its δ-deformation with any sufficiently small δ > 0. We call such solutions δ-extensible (the definition assumes that $u$ is δ-extensible).

**On δ-entropy test and uniqueness.** First of all, we again note that any uniqueness (and entropy) results for such NDEs are not achievable in principle, [15]. Therefore, we use the above results as a basis of the so-called "δ-Entropy Test" for testing shock and rarefaction waves; see first applications below.

δ-entropy solutions: motivation of the term. Let us explain why solutions are called δ-entropy, while we do not use any evolution integro-differential inequality such as (2.4). It turns out that the NDE (1.1) itself contains the right evolution choice of the admitted type shocks in the class of smooth solutions (precisely this makes sense of Definition 3.1).

For instance, as a rough explanation, assume that at $x=0$ the shock $S_+$ is going to appear at $t=1^-$ from a smooth solution $u(x,t)$ such that $u(x,1^-)$ remains smooth everywhere except $x=0$; e.g., for simplicity, we assume that

$$u(x,1) \approx S_+(x) \quad \text{in a neighbourhood } x \in (-\delta, \delta),$$

(3.43)

together with necessary derivatives $u_x$ and $u_{xx}$ that are assumed to be small at $x = \pm \delta$. Here $\delta > 0$ is also a small constant, so our illustration is of local nature. Multiplying (1.1) by $u$ and integrating over $(-\delta, \delta)$ for $t \approx 1^-$ yields the following main terms:

$$\frac{1}{2} \frac{d}{dt} \int (u^2 - 1) = \frac{1}{2} \int (u_x)^2 + \ldots > 0 \quad \text{(or < 0 for } S_-(x) \text{ at } t = 1),$$

(3.44)

since $u_x$ must be essentially positive on profiles $u(x,t)$ that smoothly approximate $S_+(x)$. One can see that (3.44) evolutionary prohibits stabilization to $S_+(x)$ as $t \to 1^-$, when $u^2 \to 1$ in $L^1_{\text{loc}}$. More rigorously [13, § 7.2], the same negative result is established using the weaker topology of $H^{-1}$, where multiplication applies to the non-local equation (1.8).

Similarly, we arrive at no contradiction while using (3.44) to describe stabilization to $S_-(x)$, since then $u_x$ is essentially negative. In fact, (3.44) reflects a finite-time formation of the singular shock $S_-$ (the gradient catastrophe) for the NDE (1.1) that was described in [18, § 3] in greater detail.
Thus, using smooth deformations guarantees (via smoothness, that is important) the preservation of the natural local entropies such as inequalities like (3.44) and the opposite one for $S_-$, so we call the constructed solutions $\delta$-entropy.

**First easy application of $\delta$-entropy test.** As a first application, we have:

**Proposition 3.1.** Shocks $S_- (x)$ and $H (-x)$ are $\delta$-entropy.

The result follows from the properties of similarity solutions (3.37), which, by shifting the blow-up time $T \mapsto T + \delta$, can be used as their local smooth $\delta$-deformations at any point $t \in [0, 1)$. For $H (-x)$, we will need an extra approximation of similarity profile $g(z)$ with finite interface at some $z = z_0$, at which it is not $C^3$, by sufficiently smooth profiles.

Let us use the negation in the following form:

**Definition 3.2.** A weak solution $u(x, t)$ of the Cauchy problem (1.1), (1.2) is not $\delta$-entropy if it is not $\delta$-stable.

**Proposition 3.2.** Shocks $S_+ (x)$ and $H (x)$ are not $\delta$-entropy.

Indeed, taking initial data $S_+ (x)$ and constructing its smooth $\delta$-deformation via the self-similar solution [18, § 3.4]

$$u_+ (x, t) = g(z), \quad z = x/t^{\frac{1}{3}}, \quad (gg''')' = -\frac{1}{3} g' z, \quad f(\mp \infty) = \mp 1.$$  

Performing time-shifting $t \mapsto t + \delta$, we obtain the global $\delta$-deformation $\{u^\delta = u_+ (x, t + \delta)\}$ which goes away from $S_+$.

Thus, we have shown that, at least, the idea of $\delta$-deformations allows us to distinguish basic $\delta$-entropy and non-entropy shocks without any use of mathematical manipulations associated with standard entropy inequalities, which are illusive for higher-order NDEs (and nonexistent in principle [15]).

4. **Compactons are $\delta$-entropy solutions**

Without loss of generality, we treat this question for a particular NDE. Namely, consider the following $K(2, 2)$ equation:

$$u_t = (uu_x)_{xx} + 4uu_x \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+.$$  

Its compacton solution has the explicit form [33],

$$u_c(x, t) = f_c(x + 3t), \quad \text{where} \quad f_c(y) = \begin{cases} 2 \cos^2 \left( \frac{y}{2} \right) & \text{for} \ |y| \leq \pi, \\ 0 & \text{otherwise.} \end{cases}$$

This is an example of a compactly supported weak solution of equation (4.1). One can see that at the interface points $y = \pm \pi$, the profile $f_c(y)$ is just $C^{1,1}_{y,1}$, i.e., the first derivative $f'_c(y)$ is Lipschitz. Therefore, it is not a classical $C^{3,1}_{x,t}$ solution of the PDE and has weak singularities at $y = \pm \pi$, so one needs to check whether it is an entropy solution. In addition, the “flux” $(ff')'$ is continuous at those points, though this does prove nothing.

We now use the concept of $\delta$-entropy solutions from Sections 2 and 3.

**Proposition 4.1.** The compacton (4.2) is a $\delta$-entropy solution of the NDE (4.1).
Proof. We are going to show that there exists smooth \( \delta \)-deformations of \( u_c \) for arbitrarily small \( \delta > 0 \). The general TW solutions as in (4.2) with \( \lambda = -3 \) yields the ODE

\[
3f' = (ff')'' + 2(f^2)' \quad \Rightarrow \quad 3f = (ff')' + 2f^2 + C_\delta,
\]

where we chose the constant of integration to be

\[
C_\delta = 3\delta - 2\delta^2 > 0.
\]

One can see on the phase plane in the variables \( \{f^2, (f^2)\}' \) that the ODE (4.3), (4.4) has a strictly positive and hence analytic solution \( f_\delta \) satisfying

\[
f_\delta(y) \to \delta^+ \quad \text{as} \quad y \to \pm \infty \text{ exponentially fast}, \quad \text{and} \quad f_\delta \to f_c \quad \text{as} \quad \delta \to 0^+
\]

uniformly in \( \mathbb{R} \). According to Definition 3.1, (4.5) implies that \( u_c \) is an entropy solution of (4.1), as well as \( f_c \) is G-admissible for the third-order ODE in (4.3).

It is crucial that Proposition 4.1 justifies that the \( K(2,2) \) equation (4.1), the NDE (1.1), and many others with similar principle degenerate third-order operators possess finite propagation of interfaces for entropy solutions.

It is worth recalling again that, regardless the existence of such nice smooth compactons (4.2), the generic behaviour for the RH equation (4.1), for other data, includes formation of shocks in finite time, with the local similarity mechanism as in [18 § 3.1].

5. On extensions to other related NDEs

5.1. Shocks for the non-degenerate NDE. We begin with the simpler model (1.10) that appeared in Section 3.2 while we discussed the possibility of extensions of sufficiently smooth solutions for defining \( \delta \)-deformations. Indeed, for (1.10), this is much easier. On the other hand, obviously, as an NDE, this admits shocks via standard similarity solutions

\[
u_-(x,t) = g(z), \quad z = x/(-t)^{\frac{1}{4}} \quad \Rightarrow \quad ((1 + g^2)y')'' = \frac{1}{3}g'z.
\]

This ODE is studied as usual. Figure 1(a) shows a few similarity profiles satisfying

\[
g(z) \sim z^{-\frac{1}{4}}e^{-a_0z^{3/2}} \to 0 \quad \text{as} \quad z \to +\infty, \quad \text{where} \quad a_0 = \frac{2}{3\sqrt{3}}.
\]

that create as \( t \to 0^- \) the shocks \( \sim H(-x) \). By dotted lines, we indicate there other profiles \( g(z) \), for which \( g(+\infty) \neq 0 \). For the sake of comparison with compactons, in Figure 1(b), we present the soliton of the related NDE

\[
u_t = ((1 + u^2)u_x)_{xx} + (1 + u^2)u_x, \quad \text{where}
\]

\[
u_s(x,t) = f_s(y), \quad y = x - \lambda t \quad \Rightarrow \quad -\lambda f' = ((1 + f^2)f')'' + (1 + f^2)f'.
\]

The soliton profiles have now exponential decay for \( \lambda < -1 \),

\[
f_s(y) \sim e^{-a_0|y|} \to 0 \quad \text{as} \quad |y| \to +\infty, \quad a_0 = \sqrt{|1 + \lambda|}.
\]
Figure 1. Similarity profiles for the NDE (1.10): shock profiles satisfying the ODE in (5.1) (a), and TW soliton satisfying (5.4) with \( \lambda = -1 \).

5.2. \( \delta \)-entropy approach to the NDE–(2,1). For the non-fully divergent PDE (1.12) we also apply the \( \delta \)-entropy to prove existence and uniqueness via suitable approximations.

On Galerkin method. Constructing Galerkin approximations, we face a new technical difficulty in passing to the limit since a uniform estimate such as (3.29) is not available for solutions [18, § 4]

\[
(5.5) \quad u_\alpha(x, t) = (-t)^\alpha g(z), \quad z = x / (-t)^{\beta}, \quad \beta = \frac{1+\alpha}{3} \quad (\alpha \in \mathbb{R}).
\]

Nevertheless, we can establish some extra estimates by using the corresponding DS (3.27), where \( J_{klj} = -\lambda l \langle \psi_k \psi_l, \psi_j' \rangle \). E.g., for odd data, the simpler system similar to (3.31),

\[
C_j' = \gamma_0 j^3 C_j C_{2j}, \quad 1 \leq j \leq m,
\]

implies that, for \( m \) even,

\[
(5.6) \quad C_m' = \gamma_0 (\frac{m}{2})^3 C_m^2 C_m \text{ and } C_j' = 0, \quad j > \frac{m}{2} \implies C_m'(t) = C_m(0) e^{\gamma_0 (\frac{m}{2})^3 C_m(0) t}.
\]

Therefore, assuming that

\[
(5.7) \quad u_0 \in C_0^3 \implies |C_m(0)| \leq \frac{c_*}{m^3}, \quad m \geq 1 \quad (c_* > 0),
\]

we obtain from (5.6) a uniform bound on the Galerkin coefficients \( \{C_j\} \), and hence a local weak solution.

Shocks and compactons exist. On the other hand, regardless its non-full divergence and nonexistence of any obvious conservation laws, the NDE (1.12) allows a similar treatment of shocks and rarefaction wave as for (1.1). For instance, formation of finite shocks for (1.12) is described by the same self-similarity as (3.37), with the ODE,

\[
(5.8) \quad u_-(x, t) = g(z), \quad z = x / (-t)^{\frac{1}{2}} \implies (g g'')' = \frac{1}{4} g' z, \quad g(\mp \infty) = \pm 1.
\]

Existence and uniqueness for (5.8) is proved similar to [18 Prop. 3.1]. In Figure 2(a), we show a few similarity profiles that create as \( t \to 0^- \) the shocks. The profile for \( S_-(x) \) has
Figure 2. The ODE (5.8): the shock similarity profiles including the unique solution (boldface line) for data $S_-(x)$ (a), and for $H(-x)$ with finite right-hand interface (b).

Figure 3. The saw-type similarity solution of the ODE in (5.9) for $\alpha_c = -0.2384...$

the derivative at the origin

$$g'(0) = -0.702... \quad \text{instead of } g'(0) = -0.51... \text{ for the NDE–3 (1.1).}$$

In (b) explaining formation of $H(-x)$, the right-hand interface is situated at $z_0 = 1.297...$. As another known key feature, Figure 3 shows the saw-type profile for the ODE

$$u_-(x,t) = (-t)^\alpha g(z), \quad z = x / (-t)^{1/\alpha} \quad \implies \quad (gg'')' = \frac{1+\alpha}{3} \, g'z - \alpha g,$$

where $\alpha_c \approx -0.2384.$
The compacton equation associated with (1.12) takes the form
\[ u_t = (uu_{xx})_x + 2uu_x \]
and admits the TW solution with the same \( f_c \) as in (4.2), but now for \( \lambda = -1 \),
\begin{align*}
(5.10) \quad u_c(x, t) = f_c(x + t), \quad f_c(y) = \begin{cases} 
2 \cos^2\left(\frac{y}{2}\right) & \text{for } |y| \leq \pi, \\
0 & \text{for } |y| \geq \pi.
\end{cases}
\end{align*}
As for (1.1), it is \( \delta \)-entropy and G-admissible; Proposition 4.1 is proved similarly.

5.3. Shock similarity profiles for Harry Dym-type equations. Consider the NDE
\begin{align*}
(5.11) \quad u_t = |u|^{n-1} uu_{xxx} \quad (n > 0),
\end{align*}
which for \( n = 3 \) becomes the quasilinear *Harry Dym equation*
\begin{align*}
(5.12) \quad u_t = u^3 u_{xxx},
\end{align*}
which also belongs to the NDE family and is an exotic integrable soliton equation; see [19, § 4.7] for survey and references therein. It admits the same formation of shocks \( S_-(x) \) by the similarity solutions given in (5.8) with the ODE
\begin{align*}
(5.13) \quad |g|^{n-1} g g''' = \frac{1}{3} g'z.
\end{align*}
Figure 4 shows that such similarity profiles exist for \( n \in (0, 2) \) and vanish as \( n \to 2^- \) (proof is easy), so that for \( n = 3 \) (the Harry Dym case) such shocks are not available.
5.4. **Shocks for fully nonlinear NDE.** For the NDE (1.13), the basic blow-up similarity solutions are slightly different,

\[ u_-(x,t) = g(z), \quad z = x/(-t)^{\beta}, \quad \beta = \frac{1+\gamma}{3} \quad \implies \quad (gg')'' = \beta^{1+\gamma}|g' z|^{\gamma} g' z. \]

Mathematics of such ODEs is not much different than that for (3.37). In Figure 5, we show how the shock similarity profiles \( g(z) \) depend on \( \gamma > -1 \). All these profiles satisfy the anti-symmetry conditions at the origin,

\[ g(0) = g''(0) = 0, \]

and the following expansion holds:

\[ g(z) = C z + \beta^{1+\gamma} \frac{C^\gamma}{(2+\gamma)(3+\gamma)(4+\gamma)} |z|^{\gamma} z^3 + \ldots \]

Note that the linearization about the constant equilibrium \( C_- = 1 \) as \( z \to -\infty \), again yields a nonlinear ODE,

\[ g''' = \beta^{1+\gamma} |g'|^{\gamma} g'|z|^{\gamma} z + \ldots \quad (z \ll -1), \]

which deserves further study. Figure 5 shows that the solutions remain equally oscillatory for all \( \gamma > -1 \), i.e., this is not a manifestation of the oscillatory character of the linear Airy function that occurs at a single simplest value \( \gamma = 0 \) only. Thus, all ODEs (5.14) with \( \gamma > -1 \) contain a strong nonlinear mechanism of oscillations about constant equilibria.

5.5. **Shock similarity profiles for cubic NDEs.** Analogously, in a similarity fashion, the shock formation is studies for the cubic fully divergent NDE (1.11). The formation of
shocks $H(-x)$ is described by the similarity solutions (5.8), where
\begin{equation}
(g^2 g')'' = \frac{1}{3} g' z,
\end{equation}
which admits a similar rigorous study. Figure 6 shows similarity profiles with the finite interface at $z = z_0 > 0$ with the expansion as $z \to z_0$
\begin{equation}
g(z) = \frac{20}{6} (z_0 - z)_+ - \frac{1}{10} \frac{1}{\sqrt{6z_0}} (z_0 - z)^2_+ + \ldots,
\end{equation}
for which the flux $(g^2 g')' \equiv \frac{1}{3} (g^3)''$ is continuous at $z = z_0$, so these are weak solutions. The flux is not zero for a more singular expansion such as
\begin{equation}
g(z) = C(z_0 - z)^{\frac{3}{2}} + \frac{2}{3C} (z_0 - z)^{\frac{5}{2}} + \ldots (z < z_0, \ C > 0).
\end{equation}
Similar to [18, § 3.1], such blow-up similarity solutions describe the generic formation of shock waves of the type $\sim H(-x)$ for (1.11). These solutions are entropy, which is proved by regular analytic approximations of the ODE as in Section 4.

By dashed lines in Figure 6 we denote other profiles, for which $C_+ = g(+\infty) > 0$, so that the corresponding blow-up similarity solutions (3.37) lead to more general shocks with different values $C_\pm$ as $z \to \pm \infty$ (with $C_+ > 0$). Then, as $z \to +\infty$, $g(z)$ approaches $C_+$ exponentially fast,
\begin{equation}
g(z) = C_+ + O(e^{-a_0 z^{3/2}}), \quad \text{where} \quad a_0 = \frac{2}{3\sqrt{3} C_+}.
\end{equation}
Thus, the above solutions with the behaviour (5.19) close to interfaces show finite propagation for the NDE (1.11). There are also TWs with finite interfaces given by
\begin{equation}
u(x, t) = f(x + t) \Rightarrow f = (f^2 f')'
\end{equation}
that are entropy and are approximated by the analytic family $\{f_\delta \geq \delta > 0\}$ satisfying
\begin{equation}
f - \delta = (f^2 f')' \quad (\delta > 0).
\end{equation}
For instance, the following TW with the interface at $y = 0$ is $\delta$-entropy:
\begin{equation}
f(y) = \begin{cases} \sqrt{2} (-y) & \text{for } y < 0, \\ 0 & \text{for } y \geq 0. \end{cases}
\end{equation}
Other discontinuous TWs may not admit smooth approximations via similar TWs.

The boldface line in Figure 6 indicates the profile that leads to $H(-x)$ as $t \to 0^-$. Here the shock $H(-x)$ is not a weak solution of the NDE (5.18). Recall that it is a $\delta$-entropy solution, i.e., there exists a converging sequence of its smooth $\delta$-deformations.

More advanced shock patterns are created by similarity solutions (5.5), with
\begin{equation}\beta = \frac{1+2a}{3} \quad \text{and} \quad (g^2 g')'' = \frac{1+2a}{3} g' z - \alpha g.
\end{equation}
The interface expansion (5.19) changes into
\begin{equation}
g(z) = \sqrt{\frac{(1+2a)z_0}{6}} (z_0 - z)_+ - \frac{1}{10} \frac{1}{\sqrt{6(1+2a)z_0}} (z_0 - z)^2_+ + \ldots.
\end{equation}
Figure 6. Shock similarity profiles satisfying (5.18), (5.19) for various \( z_0 > 0 \); the boldface profile leading to \( H(-x) \) has \( z_0 = 1.20... \); dotted lines denote shock profiles with \( g(+\infty) > 0 \).

Figure 7. Shock similarity profiles of the ODE in (5.20) satisfying expansion (5.21) for \( z_0 = 1.2 \) for various positive and negative \( \alpha \).

Figure 7 shows typical solutions of the ODE in (5.20) for \( \alpha > 0 \) and \( \alpha < 0 \). The most interesting “saw-type” profiles occurs at

\[
\alpha_c \approx -0.0715.
\]
On non-divergent cubic equation. Consider briefly the cubic NDE–(2,1),
\( u_t = (u^2 u_{xx})_x \),
which is similar, though it does not admit finite propagation at the degeneracy level \( \{u = 0\} \). This is seen by using TWs
\[
\begin{align*}
  u(x,t) &= f(x+t) \implies f' = (f^2 f'')', \\
  f'' &= \frac{1}{f} + \frac{C_1}{f^2} \implies \frac{1}{2} (f')^2 = \ln |f| - \frac{C_1}{f} + C_2.
\end{align*}
\]
Setting \( C_1 = 0 \) by assuming continuity of flux: \( f^2 f'' = 0 \) at \( f = 0 \), yields the ODE
\[
\frac{1}{2} (f')^2 = \ln |f| + C_2
\]
that does not allow any connection with the singular level \( \{f = 0\} \).

The shock similarity profiles for (5.22) exhibit the same form (5.8) and the ODE is
\[
(g^2 g'')' = \frac{1}{3} g' z.
\]
Typical strictly positive profiles with \( g(-\infty) = C_- > C_+ = g(+\infty) > 0 \) are shown in Figure 8 so these describe blow-up formation of more general entropy shocks.

More general blow-up similarity patterns (5.5) for (5.22) yields the ODE
\[
(g^2 g'')' = \frac{1+2\alpha}{3} g' z - \alpha g,
\]
which exhibits properties that are similar to (5.20). In Figure 9(a), we show typical solutions of (5.25) for \( \alpha = -\frac{1}{10} \). These profiles are strictly positive with
\[
g(z) \sim |z|^\frac{3\alpha}{1+2\alpha} \text{ as } z \to \pm\infty \quad (\alpha \in (-\frac{1}{2}, 0)).
\]
Figure 9. The ODE (5.25): the shock similarity profiles for \( \alpha = -\frac{1}{10} \) (a), and a “saw” profile for \( \alpha = \alpha_c = -0.12559... \) (b). In the bottom right-hand corner of (a), we present a number of “steep” solutions that quickly vanish (according to (5.23) with \( C_1 < 0 \)). These show that the asymptotics (5.26) is unstable in the direction of shooting from \( z = +\infty \).

In (b), we present a special profile that plays a role of the “saw-type” solution for \( \alpha = \alpha_c = -0.12559... \).

This is the best “saw” we can get numerically, though it is seen that there exists the first vanishing point while other “teeth” still stay away from zero. Anyway, we have checked that positive shock profiles cannot be extended to \( \alpha < \alpha_c \), so this is definitely a critical value of parameter.

Related compactons. Consider the following compacton equation (q.v. (1.11)):

\[ u_t = (u^2 u_x)_{xx} + 9u^2 u_x. \]

The explicit compacton solution is now easier,

\[ u_c(x,t) = f_c(x + 2t), \quad f_c(y) = \begin{cases} \cos y \text{ for } |y| \leq \frac{\pi}{2}, \\ 0 \text{ for } |y| \geq \frac{\pi}{2}. \end{cases} \]

Regardless the fact that it is not \( C^1 \) at the interface, this solution is \( \delta \)-entropy (note that (5.19) exhibits the same regularity). The proof uses regular approximations as in (4.3).

5.6. An analogy with parabolic problems. In a natural sense, an analogy of the difference between the NDE–3 (NDE–(0,3)) (1.1) and (1.12) can be observed in nonlinear parabolic theory. Namely, the fully divergent fourth-order diffusion equation (the DE–4, or DE–(0,4)),

\[ u_t = -(|u|u)_{xxxx} \quad \text{in} \quad Q_1 = (-L, L) \times (0, 1) \]

(recall that the nonlinearity \( |u|u \) keeps the parabolicity on solutions of changing sign), by classic parabolic theory [26, Ch. 2], admits a unique weak solution of the Cauchy–Dirichlet
problem with data $u_0$ such that $(u_0)^2 \in H^2$. Multiplying (5.27) by $(|u|u)_t$ in $L^2(Q_1)$ and integrating by parts yields the following \textit{a priori} estimates of such weak solutions:

$$|u|u \in L^\infty(0,T;H^2) \quad \text{and} \quad (\sqrt{|u|u})_t \in L^2(Q_1).$$

Uniqueness follows from the \textit{monotonicity} of the operator $-|u|u_{xxxx}$ in $H^{-2}$: for two weak solutions $u$ and $v$,

$$\frac{1}{2} \frac{d}{dt} \|u - v\|^2_{H^{-2}} = -\int (|u|u - |v|v)_{xxxx} (D^4_x)^{-1}(u - v)$$

$$= -\int (|u|u - |v|v)(u - v) \leq 0,$$

so that (5.28) guarantees continuous dependence of solutions on initial data.

On the other hand, the fourth-order thin film equation (TFE–4)

$$(5.29) \quad u_t = -(|u|u_{xxx})_x,$$

which has the distribution of the derivatives (3.1), does not admit such a simple treatment of continuous dependence and uniqueness as via (5.28). The Cauchy problem for the non-fully divergent TFE–4 (5.29) needs special approximation approaches. [9].

For non-fully divergent operators such as in (1.12) or fifth-order ones of the types (2,3), (3,2), (4,1), in the NDEs (see [14])

$$u_t = -(uu_{xx})_{xx}, \quad u_t = -(uu_{xxx})_x, \quad u_t = -(uu_{xxxx})_x,$$

we face a difficulty that is similar to that for the TFE (5.29). In both cases, the \(\delta\)-approximation concepts will play a role, quite similarly to the higher-order parabolic TFEs–6 such as (see [10] and references therein)

$$u_t = (|u|u_{xxxx})_x, \quad u_t = (|u|u_{xxxx})_x, \quad u_t = (|u|u_{xxx})_xxx, \quad \text{etc.}$$

6. On related higher-order in time NDEs

It is principal for PDE theory to justify that the ideas of similarity shock wave formation remain valid for other NDEs that are higher-order in time. We claim that the concept of smooth \(\delta\)-deformations can be developed for such quasilinear degenerate PDEs. Let us present a few comments in these directions.

6.1. Second-order in time NDE. As in [18 § 1.2], we begin with the simple observation: $S_{\pm}(x)$ are stationary weak solutions of the \textit{second-order in time NDE}

$$(6.1) \quad u_{tt} = (uu_x).$$

To distinguish the entropy one, as usual, we introduce the similarity solutions

$$(6.2) \quad u_-(x,t) = g(z), \quad z = x/(-t)^{\frac{3}{2}}, \quad \text{where}$$

$$(6.3) \quad (gg')'' = \frac{10}{9} g'z + \frac{1}{3} g''z^2 \quad \text{in} \quad \mathbb{R}, \quad f(\mp\infty) = \pm 1.$$

The study of this ODE is similar to that in [18 § 3], so we present the existence result for the shock $S_-(x)$ in Figure [11]. The dotted lines show nonexistence of similarity profiles for $S_+(x)$ (cf. a proof below). The boldface profile is unique and satisfies the anti-symmetry
Figure 10. The shock similarity profile as the unique solution of the problem (6.3).
so that the stationary shock $S_-(x)$ that appears as $t \to 0^-$ according to the similarity law (6.2) (according to “centre/stable manifold” behaviour in [18, §6]) will next disappear in the same smooth similarity manner (6.2), where $(-t)$ is replaced by $t$. As a next step, the concept of smooth $\delta$-deformations should be applied to (6.1) to produce a unique solution of the Cauchy problem, but this demands extra more technical study.

6.2. Third-order in time NDE. Consider the third-order in time NDE

(6.7) $u_{ttt} = (uu_x)_{xx}$, or

(6.8) \[
\begin{cases}
u_t = v_x, \\
v_t = w_x, \\
w_t = uu_x,
\end{cases}
\]

being a first-order system with the characteristic equation $\lambda^3 = u$, with one real and two complex eigenvalues for $u \neq 0$, so it not hyperbolic.

Quite analogously, $S_{\pm}(x)$ are stationary weak solutions of (6.7) for which the basic (with $\alpha = 0$) similarity solutions are

(6.9) \[u_{\pm}(x,t) = g(z), \quad z = x/(-t), \quad \text{where}\]

(6.10) $(gg')'' = (z^3 g')'' \equiv 6g'z + 6g''z^2 + g'''z^3$ in $\mathbb{R}$, $f(\mp\infty) = \pm 1$.

Integrating (6.10) twice yields

$$gg' = z^3g' + Az + B, \quad \text{with constants} \quad A, B \in \mathbb{R},$$

so that the necessary similarity profile $g(z)$ solves the first-order ODE

(6.11) \[\frac{dg}{dz} = \frac{Az}{g^{3/2}}, \quad \text{where} \quad A = (g'(0))^2 > 0.
\]

By the phase-plane analysis of (6.11), we easily get the following:
Proposition 6.1. The problem (6.10) admits a unique solution \( g(z) \) satisfying the anti-symmetry conditions (5.13) that is positive for \( z < 0 \), monotone decreasing, and is real analytic.

Such basic anti-symmetric similarity profiles are shown in Figure 12. These satisfy the expansion near the origin, as \( z \to 0 \),

\[
g(z) = \sum_{k \geq 0} c_k z^{2k+1} = Cz + \frac{1}{4} z^3 + \frac{3}{32} C z^5 + \ldots \quad (C = g'(0) < 0).
\]

Substituting the expansion in (6.12) into (6.11) yields \( g'(g - z^3) = Az \Rightarrow \sum_{(k,j \geq 0)} (2k+1)c_k(c_j - \delta_{j1})z^{2(k+j)+1} = Az. \)

The corresponding algebraic system for the expansion coefficients \( \{c_k\} \) is uniquely solved giving the unique analytic solution. The boldface profile \( g(z) \) in Figure 12 (by (6.9) it gives \( S_-(x) \) as \( t \to 0^- \)) is non-oscillatory about \( \pm 1 \) with the algebraic convergence

\[
g(z) = \pm 1 + \frac{4}{z} + \ldots \quad \text{as} \quad z \to \mp\infty.
\]

Again, the fundamental solutions of the corresponding linear PDE

\[
u_{ttt} = u_{xxx}
\]
is not oscillatory as \( x \to \pm \infty \). The linear PDE (6.13) exhibits some finite propagation features with the corresponding test consisting of checking the TWs,

\[
u(x,t) = f(x - \lambda t) \quad \implies \quad -\lambda^3 f''' = f''', \quad \text{i.e.,} \quad \lambda = -1,
\]

where the profile \( f(y) \) disappears from. This is similar to a few other well-known canonical equations of mathematical physics such as

\[
u_t = u_x \quad \text{(dispersion, } \lambda = -1) \quad \text{and} \quad u_{tt} = u_{xx} \quad \text{(wave equation, } \lambda = \pm 1).
\]

Any finite propagation is not true for (6.4). The blow-up solution (6.9) gives in the limit \( t \to 0^- \) the shock \( S_-(x) \), and (6.5) holds. In Figure 12 we also show the results of shooting with \( g'(0) > 0 \) giving unbounded profiles \( g(z) \sim z^3 \) as \( z \to \pm \infty \). As usual, this means nonexistence of similarity blow-up profiles corresponding to \( S_+ \)-type shocks.

A key difference with the previous problems is that the original ODE (6.10) written as

\[
(g - z^3)g''' = 6g'z + 6g''z^2 - 3g'g''
\]

has, instead of \( \{g = 0\} \), another singular line (a kind of nonlinear “light cone”)

\[
L_0: \quad g(z) = z^3.
\]

Then, formally, the existence of global solutions of (6.14) depends on the possibility of a continuous transition through it. The simpler integrated form (6.11) shows that typical solutions do not cross \( L_0 \) (except at the analytic point \( z = 0 \)), so that “weak discontinuities” do not occur.

Since (6.7) has the same symmetry

\[
\begin{cases}
u \mapsto -u, \\ t \mapsto -t,
\end{cases}
\]

(6.16)
as \((1.1)\), similarity solutions \((6.9)\) with \(-t \mapsto t\) and \(g(z) \mapsto g(-z)\) also give the rarefaction waves for \(S_+(x)\), as well as other types of collapse of initial non-entropy discontinuities.

Using the known asymptotic properties of blow-up similarity solutions \((6.9)\) and those global with \(-t \mapsto t\), for convenience, we formulate the following

**Proposition 6.2.** The Cauchy problem for the equation \((6.7)\) admits:

(i) an analytic solution \(u_-(x, t)\) in \(Q_T = \mathbb{R} \times (0, T)\) that converges as \(t \to T^-\) to the shock \(S_-(x)\) in \(L^1_{\text{loc}}\) and a.e., and

(ii) for non-analytic singular initial data as \(t \to 0^+\) given by

\[
(6.17) \quad u(x, t) \to S_+(x), \quad u_t(x, t) \to \frac{A}{x}, \quad \frac{1}{t^2} u_{tt}(x, t) \to 3A \text{sign} x
\]

with uniform convergence as \(t \to 0\) on any compact subset from \(\mathbb{R} \setminus \{0\}\) (and in \(L^1_{\text{loc}}\) for \(u(x, 0)\)), there exists an analytic solution in \(\mathbb{R} \times \mathbb{R}_+\).

**Analytic \(\delta\)-deformations by Cauchy-Kovalevskaya theorem.** Eventually, we start to deal with the third-order in time NDE \((6.7)\) that turns out to be in the normal form, so it obeys the Cauchy–Kovalevskaya (C-K) theorem \([37, p. 387]\). Hence, for any analytic initial data \(u(x, 0), u_t(x, 0),\) and \(u_{tt}(x, 0)\), there exists a unique local in time analytic solution \(u(x, t)\). Thus, \((6.7)\) generates a local semigroup of analytic solutions, and this makes it easier to deal with smooth \(\delta\)-deformations that always can be chosen to be analytic. On the other hand, such nonlinear PDEs can admit other (say, weak) solutions that are not analytic. Actually, Proposition \(6.1\) shows that the shock \(S_-(x)\) is a \(\delta\)-entropy solution of \((6.7)\), which is obtained by finite-time blow-up as \(t \to 0^-\) from the analytic similarity solution \((6.9)\).

\[2\] In this connection, the result (ii) in Proposition \(6.2\) sounds unusual: for non-analytic and very singular data, there exists a global analytic solution.
**Shocks for non-degenerate NDE.** For the corresponding non-degenerate NDE

\[(6.18)\]

\[u_{ttt} = ((1 + u^2)u_x)_x,\]

the similarity solutions (6.9) lead, on double integration, to the ODE (cf. (6.11))

\[(6.19)\]

\[g' = \frac{Ax + B}{1 + g^2} \quad (A, B \in \mathbb{R}).\]

It is easy to show using the phase-plane, that for \(z_0 = -\frac{B}{A} > 1\) (this gives a necessary extra singular point \((z_0, g_0)\) of the flow, where \(g_0^2 = z_0^3 - 1\)), (6.19) admits analytic solutions \(g(z)\) satisfying \(g(\pm \infty) = C_\pm > 0\) with \(C_- > C_+\), so as \(t \to 0^-\), we obtain the shock.

### 6.3. Stationary entropy shocks for other higher-order in time NDEs.

We now very briefly check entropy properties of the shocks \(S_\pm(x)\) for the following NDEs of arbitrary order:

\[(6.20)\]

\[D_t^{2m+2}u = D_x^{2m}(uu_x) \quad (m \geq 1).\]

For \(m = 0\), this gives the following simple NDE:

\[(6.21)\]

\[u_{tt} = uu_x,\]

for which both shocks \(S_\pm\) are obviously weak solutions, so one needs to identify which ones are entropy. Note that, as (6.7), the PDEs (6.20) for any \(m \geq 1\) obey the Cauchy–Kovalevskaya theorem, so a unique local semigroup of analytic solutions does exist.

#### \(\delta\)-entropy \(S_-\) via analytic TWs.

For a change, we present \(\delta\)-deformations by TWs

\[(6.22)\]

\[u(x,t) = f_\lambda(x - \lambda t) \implies \lambda^{2m+2}f^{(2m+2)} = \frac{1}{2} (f^2)^{(2m+1)}, \quad \text{or} \quad \lambda^{2m+2}f' = -\frac{1}{2} (1 - f^2) \implies f_\lambda(y) = \frac{e^{-y/\lambda^{2m+2}} - 1}{e^{-y/\lambda^{2m+2}} + 1}.\]

We then observe that

\[(6.23)\]

\[f_\lambda(y) \to S_-(y) \quad \text{as} \quad \lambda \to 0 \quad \text{uniformly in } \mathbb{R},\]

so that the stationary shock wave \(S_-(x)\) is G-admissible and is \(\delta\)-entropy, where the necessary \(\delta\)-deformation is given by the TW (6.22) with \(\lambda = \delta\).

A similar (but not explicit) construction of \(\delta\)-entropy solutions with convergence (6.23) is performed for other normal NDEs such as

\[(6.24)\]

\[D_t^{2m+4}u = -D_x^{2m}(uu_x), \quad \text{or} \quad D_t^{2m+2k}u = (-1)^k D_x^{2m}(uu_x), \quad k \geq 1.\]

The corresponding analytic TW profiles \(f_\lambda(y)\) satisfying the convergence (6.23) in \(L^1_{\text{loc}}\) are described in [13, § 4].

**Remark:** \(S_+\) can be formally created by a classical but non-analytic blow-up self-similar solution. There exists a self-similar blow-up to \(S_+\) for the NDE (6.21) via

\[(6.25)\]

\[u_-(x,t) = g(z), \quad z = \frac{x}{(-t)^{4z}}, \implies g'z^2 = \frac{1}{4} g^2, \quad \text{so} \quad g(z) = \begin{cases} \frac{4z}{4z+1}, & z \geq 0, \\ \frac{4z}{1-4z}, & z \leq 0. \end{cases}\]
This ODE obeys the symmetry

\begin{equation}
\begin{cases}
g \mapsto -g, \\
z \mapsto -z
\end{cases}
\end{equation}

Note that \( g(z) \) is just \( C^1 \) (not \( C^2 \)) at \( z = 0 \), which is enough to represent a weak solution of the degenerate PDE (6.21) (though, as we know, being weak often means almost nothing). Moreover, (6.25) is a classical \( C^1_{x,t} \) solution of (6.21). Observe that the non-analyticity of \( g(z) \) is associated with the too strong degeneracy at \( z = 0 \) of the corresponding ordinary differential operator \( z^2 \frac{d}{dz} \). We suspect that (6.25) is not entropy at all. Moreover, one can see that \( g(z) \) is not an odd function, so it looks more like a solution of an IBVP for \( x > 0 \) with some boundary condition at \( x = 0 \). Nevertheless, we recall that it is a classical solution of the Cauchy problem. The NDEs (6.20) deserve deeper study.

7. On shocks for spatially higher-order NDEs

7.1. Fifth-order NDEs. The similarity mechanism of shock formation remains valid for higher-order NDEs, among which, as an illustration, we comment on the following three (including the NDE-5 (1.15)):

\begin{align}
&u_t = -(uu_x)_{xxxx}, \\
&u_{tt} = -(uu_x)_{xxxx}, \\
&u_{ttt} = -(uu_x)_{xxxx}.
\end{align}

Concerning application of such fifth and higher-order NDEs, see [14], [19, p. 166], and references therein. The blow-up similarity solutions of \( S_-\)-type are the same,

\begin{equation}
\begin{cases}
u_-(x,t) = g(z), & z = x/(-t)^\beta, \quad \text{where} \quad \beta = \frac{1}{5}, \quad \beta = \frac{2}{5}, \quad \beta = \frac{3}{5},
\end{cases}
\end{equation}

respectively. The ODEs are, respectively,

\begin{align}
&\left( gg' \right)^{(4)} = -\frac{1}{5} g'z, \\
&\left( gg' \right)^{(4)} = -\frac{2}{25} (7g'z + 2g''z^2), \\
&\left( gg' \right)^{(4)} = -\frac{3}{125} (64g'z + 57g''z^2 + 9g'''z^3).
\end{align}

These are much more complicated equations than all those studied before. We do not have a proof of existence of the \( S_-\)-type profiles \( g(z) \) to say nothing about uniqueness, though we can justify that the shooting procedure to get a solution is well-posed according to dimensions of stable and unstable manifolds of orbits at the singular points \( z = 0 \) (where \( g = 0 \)) and \( z = -\infty \) (where \( g = +1 \)). On the other hand, the same numerical methods give us a strong evidence of existence-uniqueness. In Figure 13 using \texttt{bvp4c} solver of \texttt{MatLab}, we present the unique solutions of the ODEs (7.3) satisfying the standard conditions

\begin{equation}
g(\pm \infty) = \mp 1 \quad \text{and} \quad g(0) = g''(0) = g^{(4)}(0) = 0 \quad \text{(anti-symmetry)}.
\end{equation}

Note that first two ODEs admit solutions that are oscillatory about the equilibrium \( g = 1 \) as \( z \to -\infty \), while the last one has monotone non-oscillatory solutions according to
the following asymptotics, respectively: for \( z \ll -1 \), neglecting lower-order algebraic multipliers in the second and third formulae,

\[
g(z) - 1 \sim |z|^{-\frac{3}{8}} \cos \left( \frac{4}{5\sqrt{2}} 5^{-\frac{1}{4}} |z|^\frac{5}{4} + c_0 \right),
\]

\[
g(z) - 1 \sim e^{-\frac{2}{5} \sqrt{2} (\frac{3}{5})^\frac{5}{2} |z|^\frac{5}{4}} \cos \left( \frac{3}{5\sqrt{2}} (\frac{3}{5})^\frac{5}{2} |z|^\frac{5}{4} + c_0 \right),
\]

\[
g(z) - 1 \sim e^{-\frac{2}{5} (\frac{3}{5})^\frac{5}{2} |z|^\frac{5}{4}}.
\]

The exponentially small oscillations in the second line are hardly seen in the figure and requires another, logarithmic scale for revealing those.

7.2. On a seventh-order NDE. For completeness and convenience of comparison, Figure 13 also gives the shock similarity profiles (the dashed line) for the NDE–7,

\[
u_t = (uu_x)_{xxxxxx}, \quad \text{where}
\]

\[
u_-(x,t) = g(z), \quad z = x/(-t)^\frac{1}{7} \quad \Rightarrow \quad (gg')(6) = \frac{1}{7} g'z, \quad g(\pm \infty) = \mp 1.
\]

The shock profile is very similar to that for the NDE–5 in (1.15), so that a general geometry of these shock profiles does not essentially depend on the order, \((2m + 1)\), of the PDEs (1.15) for \( m \geq 1 \); the oscillatory behaviour also changes slightly with \( m \) and always has the type given in the first line in (7.5).

These results show that, for all the above higher-order NDEs, canonical shocks of \( S_- \)-type are obtained by blow-up in finite time from smooth classical solutions. According to our \( \delta \)-entropy approach, this confirms a correct entropy nature of such shock waves.
Let us describe other types of shocks and rarefaction waves for \((7.6)\) driven by blow-up similarity patterns
\[
(7.8) \quad u_-(x,t) = (-t)^\alpha g(z), \quad z = x/(-t)^\beta, \quad \beta = \frac{1}{7} + \frac{\alpha}{7} \quad \Rightarrow \quad (gg')^{(6)} = \frac{1}{7} (1+\alpha) g' z - \alpha g.
\]
These similarity profiles are presented in Figure 14. This shows that the profiles get more oscillatory for \(\alpha < 0\), but we failed to detect a “saw”-type profile as in [18, § 4.3] for such a seventh-order ODE by using any numerical method.

Finally, the analysis of the ODE in \((7.8)\) on the invariant subspace (cf. the invariant subspace in [18, § 4.3]) \(W_4 = \text{Span}\{z, z^3, z^5, z^7\}\) shows that a nontrivial dynamics exists for the critical exponent
\[
\alpha_c = \frac{415}{2574} = 0.161228..., \quad \text{and that the explicit solutions are given by}
\]
\[
g(z) = Cz + \frac{6}{13} z^7, \quad \text{where} \quad C \in \mathbb{R} \text{ is arbitrary.}
\]

8. On changing sign compactons for higher-order NDEs

Finally, we return to the compacton solutions of the NDEs. First time, we discussed the entropy properties of compactons in Section 4 for the NDE–3, where the entropy nature of such solutions was successfully justified. It turns out that the fact that these compactons are \(\delta\)-entropy, i.e., are constructed by smooth \(\delta\)-deformations, can be proved by a purely ODE approach, by smooth positive approximations of compactons via analytic solutions.
We must admit that this ODE approach cannot be extended in principle to higher-order NDEs, so we need either to return to the original PDE δ-entropy method as in Section 3.2 or to adapt the ODE approach to non-positive but less singular approximations (that we actually intend to do).

8.1. Compacton for a cubic fifth-order NDE. For introducing a new model, unlike most of previous cases (excluding (1.11) in Section 5.5), without any hesitation, we consider the cubic NDE–5

\[ u_t = -(u^2 u_x)_{xxxx} + u^2 u_x \text{ in } \mathbb{R} \times \mathbb{R}_+. \tag{8.1} \]

We take the following TW compacton with the specially chosen wave speed \( \lambda = -\frac{1}{3} \):

\[ u_c(x, t) = f(y), \quad y = x + \frac{1}{3} t \quad \implies \quad -(f^3)'' + f^3 = f \text{ in } \mathbb{R}. \tag{8.2} \]

We next perform the natural change leading to a simpler semilinear ODE,

\[ F = f^3 \quad \implies \quad F^{(4)} = F - F^{\frac{1}{3}} \text{ in } \mathbb{R}. \tag{8.3} \]

This easy looking equation admits a nontrivial countable set of various compactly supported solutions that are analyzed by variational methods based on Lusternik–Schirel’man category and Pohozaev’s fibering theory, [16]. Here we stress our attention to the primary facts that are connected with the proposed concepts of entropy solutions.

The first and simplest compacton solution of the ODE (8.2) is shown in Figure 15 that was obtained numerically with the tolerances and regularization parameters

\[ \text{Tols} = 10^{-10} \quad \text{and} \quad F^{\frac{1}{3}} \mapsto (\nu^2 + F^2)^{-\frac{1}{2}} F \text{ with also } \nu = 10^{-10}. \]

Figure 15. The first compacton profile of the ODE (8.2).
8.2. Oscillatory structure near interfaces: periodic orbits. In general, it looks that this compacton profile does not differ from those considered before as the explicit solutions in (4.2) or (5.10). However, there is a fundamental difference that changes the mathematics of such solutions: for the fifth-order NDE (8.1), the profiles \( f(y) = F^ \frac{1}{3} (y) \) are oscillatory and are of changing sign near finite interfaces. In Figure 16, we show first three zeros near the interface at \( y = y_0 > 0 \) of the compacton profile from Figure 15.

In order to describe key features of such oscillatory behaviour at the right-hand interface, as \( y \to y_0 > 0 \), when \( F(y) \to 0 \), we perform an extra scaling by setting in the two leading terms of the ODE

\[
F^{(4)} = -F^ \frac{1}{3} \quad \implies \quad F(y) = (y_0 - y)^6 \varphi(s), \quad s = \ln(y_0 - y),
\]

where the oscillatory component \( \varphi(s) \) solves the following ODE:

\[
P_4(\varphi) \equiv e^{-2s}[e^{-s}(e^{-s}(e^{-s}(e^{-s}(e^{-s}(e^{-s}(e^{-s}(\varphi'))'))')))'] \\
\equiv \varphi^{(4)} + 16\varphi''' + 119\varphi'' + 342\varphi' + 360\varphi = -\varphi^ \frac{1}{3}.
\]

It turns out that the oscillatory behaviour near the interface at \( y = y_0^- \) (i.e., at \( s = -\infty \)) is given by a periodic solution \( \varphi_*(s) \) of the ODE (8.5). Namely, we list the following properties that lead to existence of a periodic orbit of changing sign:

**Proposition 8.1.** The fourth-order dynamical system (8.5) satisfies:

(i) no orbits are attracted to infinity as \( s \to +\infty \);

(ii) it is a dissipative system with a bounded absorbing set; and

(iii) a nontrivial periodic orbit \( \varphi_*(s) \) exists.

**Proof.** (i) The operator in (8.5) is asymptotically linear [22, p. 77] with the derivative at the point at infinity \( P_4 \) that has the characteristic equation

\[
p_4(\lambda) = \lambda^4 + 16\lambda^3 + 119\lambda^2 + 342\lambda + 360 \equiv (\lambda + 6)(\lambda + 5)(\lambda + 4)(\lambda + 3) = 0.
\]
Therefore, all eigenvalues are real negative, $-6, -5, -4, \text{ and } -3$, so infinity cannot attract orbits as $s \to +\infty$. (ii) is a corollary of (i) after an extra scaling.

(iii) Existence of a periodic orbit for dissipative systems is a standard result of degree theory; see [22, p. 235]. We complete the proof of existence by using a shooting argument as in [9, § 7.1].

It turns out that the periodic solution $\varphi_*(s)$ is exponentially stable as $s \to +\infty$ (this is not easy to see from the ODE (8.5) by linearization and interpolation of the third term with the coefficient 119). The asymptotic stability of this periodic orbits is illustrated in figures in [19, p. 187].

Thus, at the singular end point $y = y_0^-$, the dynamical system (8.3) generates a two-dimensional bundle of orbits with the behaviour

\begin{equation}
F(y) = (y - y_0)^6[\varphi_*(s + s_0) + \ldots], \quad y_0 \in \mathbb{R}_+, \quad s_0 \in \mathbb{R},
\end{equation}

where $s_0$ is an arbitrary phase shift of the periodic motion. Thus, the interface point \{\(y = y_0\), \(F = F' = F'' = F''' = 0\)\} is a complicated singular point (a zero) of the dynamical system (8.3), so one needs to check whether it corresponds to an entropy solution-compacton. It is worth mentioning that the 2D bundle (8.6) matches with precisely two symmetry conditions at the origin, $F'(0) = F'''(0) = 0$, and the existence of the compacton is confirmed by variational methods, [16, § 5].

8.3. **Compactons are $\delta$-entropy: a formal illustration.** The oscillatory behaviour (8.6) of the compacton near finite interfaces makes impossible to use the positive analytic $\delta$-approximation as for the NDE–3 in (4.3). Indeed, the same procedure for $F = f^3$ now leads to the “regularized” ODE

\begin{equation}
F_\delta: \quad F^{(4)} = F - F' + C_\delta, \quad \text{where} \quad C_\delta = \frac{\delta^4}{4} - \delta, \quad \text{so} \quad F_\delta(y) \to \delta > 0, \quad y \to \infty.
\end{equation}

This gives the family \(\{F_\delta\}\) consisting of functions $F_\delta(y)$ that change sign finitely many times for all sufficiently small $\delta > 0$. These approximations $F_\delta$ are less singular than the limit compacton profile $F(y)$, which according to (8.6) is infinitely oscillatory as $y \to y_0^-$. The solvability of the approximating problem (8.7) can be traced out by the same variational method. Then the convergence

\begin{equation}
F_\delta(y) \to F(y) \quad \text{as} \quad \delta \to 0^+ \quad \text{uniformly}
\end{equation}

is associated with the stability of critical values of functionals; see [22, p. 387]. This $\delta$-approximation is shown in Figure 17(a), where the convergence (8.8) is rather slow and is observed starting from $\delta = 10^{-3}$ only, with the accuracy about 0.2. For $\delta = 10^{-2}$, the approximating profile $F_\delta(y)$ is still almost four times less than $F(y)$ at the origin. The accuracy 0.1 is achieved for $\delta = 10^{-5}$. In (b), up to $\delta = 10^{-8}$, we show the zero structure of $F_\delta(y)$ close to $y_0$, which, since $F(y) \approx \delta > 0$ for $y \gg 1$, is finite and each zero is transversal. These confirm that the approximating sequence $\{F_\delta\}$, though is of changing sign, is less singular than the compacton profile $F(y)$ itself.

An alternative approximating approach of such compactons is developed in [17], where $F$ is approximated as $\varepsilon \to 0^+$ by the analytic family $\{F_\varepsilon\}$ of solutions of the regularized
Incidentally, this approach makes it possible to trace out the Sturmian index of some solutions by a homotopic connection to variational problems with known ordered set of critical points and known number of zeros for each of them, [16].

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