Generalization of Doob decomposition Theorem.

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Abstract In the paper, we introduce the notion of a local regular supermartingale relative to a convex set of equivalent measures and prove for it an optional Doob decomposition in the discrete case. This Theorem is a generalization of the famous Doob decomposition onto the case of supermartingales relative to a convex set of equivalent measures.

Keywords random process, convex set of equivalent measures, optional Doob decomposition, regular supermartingale, martingale.

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1 Introduction.

In the paper, we generalize Doob decomposition for supermartingales relative to one measure onto the case of supermartingales relative to a convex set of equivalent measures. For supermartingales relative to one measure for continuous time Doob’s result was generalized in papers [12, 13].

At the beginning, we prove the auxiliary statements giving sufficient conditions of the existence of maximal element in a maximal chain, of the existence of nonzero non-decreasing process such that the sum of a supermartingale and this process is again a supermartingale relative to a convex set of equivalent measures needed for the main Theorems. In Theorem 2 we give sufficient conditions of the existence of the optional Doob decomposition for the special case
as the set of measures is generated by finite set of equivalent measures with bounded as below and above the Radon-Nicodym derivatives. After that, we introduce the notion of a regular supermartingale. Theorem 3 describes regular supermartingales. In Theorem 4 we give the necessary and sufficient conditions of regularity of supermartingales. Theorem 5 describes the structure of non-decreasing process for a regular supermartingale. Then we introduce the notion of a local regular supermartingale relative to a convex set of equivalent measures. At last, we prove Theorem 6 asserting that if the optional decomposition for a supermartingale is valid, then it is local regular one. Essentially, Theorem 6 and 7 give the necessary and sufficient conditions of local regularity of supermartingale.

After that, we prove auxiliary statements needed for the description of local regular supermartingales. Theorem 8 gives the necessary and sufficient conditions for a special class of nonnegative supermartingales to be local regular ones. In Theorems 9 and 10 we describe a wide class of local regular supermartingales. On the basis of these Theorems we introduce a certain class of local regular supermartingales and prove Theorem 11 giving the necessary and sufficient conditions for nonnegative uniformly integrable supermartingale to belong to this class. Using the results obtained we give examples of construction of local regular supermartingales. At last, we prove also Theorem 12 giving possibility to construct local regular supermartingales.

The optional decomposition for supermartingales plays fundamental role for risk assessment on incomplete markets [6], [7], [8], [9]. Considered in the paper problem is generalization of corresponding one that appeared in mathematical finance about optional decomposition for supermartingale and which is related with construction of superhedge strategy on incomplete financial markets. First, the optional decomposition for supermartingales was opened by El Karoui N. and Quenez M. C. [2] for diffusion processes. After that, Kramkov D. O. [11], [5] proved the optional decomposition for nonnegative bounded supermartingales. Folmer H. and Kabanov Yu. M. [3], [4] proved analogous result for an arbitrary supermartingale. Recently, Bouchard B. and Nutz M. [1] considered a class of discrete models and proved the necessary and sufficient conditions for validity of optional decomposition. Our statement of the problem unlike the above-mentioned one and it is more general: a supermartingale relative to a convex set of equivalent measures is given and it is necessary to find conditions on the supermartingale and the set of measures under that optional decomposition exists. Generality of our statement of the problem is that we do not require that the considered set of measures was generated by random process that is a local martingale as it is done in the papers [1, 2, 11, 4] and that is important for the proof of the optional decomposition in these papers.

2 Discrete case.

We assume that on a measurable space \( \{ \Omega, \mathcal{F} \} \) a filtration \( \mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}, \ m = 0, \infty \), and a family of measures \( M \) on \( \mathcal{F} \) are given. Further, we assume that \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \). A random process \( \psi = \{ \psi_m \}_{m=0}^{\infty} \) is said to be adapted one
relative to the filtration $\{F_m\}_{m=0}^{\infty}$ if $\psi_m$ is $F_m$ measurable random value for all $m = 0, \infty$.

**Definition 1.** An adapted random process $f = \{f_m\}_{m=0}^{\infty}$ is said to be a supermartingale relative to the filtration $F_m$, $m = 0, \infty$, and the family of measures $M$ if $E^P|f_m| < \infty$, $m = 0, \infty$, $P \in M$, and the inequalities

$$E^P\{f_m|F_k\} \leq f_k, \quad 0 \leq k \leq m, \quad m = 1, \infty, \quad P \in M,$$

are valid.

We consider that the filtration $F_m$, $m = 0, \infty$, is fixed. Further, for a supermartingale $f$ we use as denotation $\{f_m, F_m\}_{m=0}^{\infty}$ and denotation $\{f_m\}_{m=0}^{\infty}$.

Below in a few theorems, we consider a convex set of equivalent measures $M$ satisfying conditions: Radon – Nicodym derivative of any measure $Q_1 \in M$ with respect to any measure $Q_2 \in M$ satisfies inequalities

$$0 < l \leq \frac{dQ_1}{dQ_2} \leq L < \infty, \quad Q_1, Q_2 \in M,$$

where real numbers $l$, $L$ do not depend on $Q_1$, $Q_2 \in M$.

**Theorem 1.** Let $\{f_m, F_m\}_{m=0}^{\infty}$ be a supermartingale concerning a convex set of equivalent measures $M$ satisfying conditions (2). If for a certain measure $P_1 \in M$ there exist a natural number $1 \leq m_0 < \infty$, and $F_{m_0-1}$ measurable nonnegative random value $\varphi_{m_0}$, $P_1(\varphi_{m_0} > 0) > 0$, such that the inequality

$$f_{m_0-1} - E^{P_1}\{f_{m_0}|F_{m_0-1}\} \geq \varphi_{m_0},$$

is valid, then

$$f_{m_0-1} - E^{Q}\{f_{m_0}|F_{m_0-1}\} \geq \frac{l}{1+L} \varphi_{m_0}, \quad Q \in M_{\bar{\varepsilon}_0},$$

where

$$M_{\bar{\varepsilon}_0} = \{Q \in M, \quad Q = (1-\alpha)P_1 + \alpha P_2, \quad 0 \leq \alpha \leq \bar{\varepsilon}_0, \quad P_2 \in M\}, \quad P_1 \in M,$$

$$\bar{\varepsilon}_0 = \frac{L}{1+L}.$$

**Proof.** Let $B \in F_{m_0-1}$ and $Q = (1-\alpha)P_1 + \alpha P_2, \quad P_2 \in M, \quad 0 < \alpha < 1. \quad \text{Then}$

$$\int B [f_{m_0-1} - E^{Q}\{f_{m_0}|F_{m_0-1}\}] dQ =$$

$$\int B E^{Q}\{[f_{m_0-1} - f_{m_0}]|F_{m_0-1}\} dQ =$$

$$\int B [f_{m_0-1} - f_{m_0}] dQ =$$
\[(1 - \alpha) \int_B [f_{m_0 - 1} - f_{m_0}] dP_1 + \]
\[\alpha \int_B [f_{m_0 - 1} - f_{m_0}] dP_2 = \]
\[(1 - \alpha) \int_B [f_{m_0 - 1} - E^{P_1} \{f_{m_0} | \mathcal{F}_{m_0 - 1}\}] dP_1 + \]
\[\alpha \int_B [f_{m_0 - 1} - E^{P_2} \{f_{m_0} | \mathcal{F}_{m_0 - 1}\}] dP_2 \geq \]
\[(1 - \alpha) \int_B [f_{m_0 - 1} - E^{P_1} \{f_{m_0} | \mathcal{F}_{m_0 - 1}\}] dP_1 = \]
\[(1 - \alpha) \int_B [f_{m_0 - 1} - E^{P_1} \{f_{m_0} | \mathcal{F}_{m_0 - 1}\}] \frac{dP_1}{dQ} dQ \geq \]
\[(1 - \alpha) l \int_B \varphi_{m_0} dQ \geq (1 - \varepsilon_0) l \int_B \varphi_{m_0} dQ = \frac{l}{1 + L} \int_B \varphi_{m_0} dQ. \]

Arbitrariness of \( B \in \mathcal{F}_{m_0 - 1} \) proves the needed inequality. \( \square \)

**Lemma 1.** Any supermartingale \( \{f_{m}, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to a family of measures \( M \) for which there hold equalities \( E^{P} f_{m} = f_{0} \), \( m = 1, \infty \), \( P \in M \), is a martingale with respect to this family of measures and the filtration \( \mathcal{F}_m \), \( m = 1, \infty \).

**Proof.** The proof of Lemma 1 see [10]. \( \square \)

**Remark 1.** If the conditions of Lemma 1 are valid, then there hold equalities
\[ E^{P} \{f_m | \mathcal{F}_k\} = f_k , \quad 0 \leq k \leq m , \quad m = 1, \infty , \quad P \in M . \]  

(3)

Let \( f = \{f_{m}, \mathcal{F}_m\}_{m=0}^{\infty} \) be a supermartingale relative to a convex set of equivalent measures \( M \) and the filtration \( \mathcal{F}_m \), \( m = 0, \infty \). And let \( G \) be a set of adapted non-decreasing processes \( g = \{g_{m}\}_{m=0}^{\infty} \), \( g_0 = 0 \), such that \( f + g = \{f_{m} + g_{m}\}_{m=0}^{\infty} \) is a supermartingale concerning the family of measures \( M \) and the filtration \( \mathcal{F}_m \), \( m = 0, \infty \).

Introduce a partial ordering \( \preceq \) in the set of adapted non-decreasing processes \( G \).
Definition 2. We say that an adapted non-decreasing process \( g_1 = \{g^1_m\}_{m=0}^{\infty}, \)
\( g_0^1 = 0, g_1 \in G, \) does not exceed an adapted non-decreasing process \( g_2 = \{g^2_m\}_{m=0}^{\infty}, \)
\( g_0^2 = 0, g_2 \in G, \) if \( P(g^2_m - g^1_m \geq 0) = 1, \ m = 1, \infty. \) This partial ordering we denote by \( g_1 \preceq g_2. \)

For every nonnegative adapted non-decreasing process \( g = \{g_m\}_{m=0}^{\infty} \in G \)
there exists limit \( \lim_{m \to \infty} g_m \) which we denote by \( g_\infty. \)

Lemma 2. Let \( \tilde{G} \) be a maximal chain in \( G \) and for a certain \( Q \in M \)
\( \sup_{g \in \tilde{G}} E^Q g = \alpha^Q < \infty. \) Then there exists a sequence \( g^s = \{g^s_m\}_{m=0}^{\infty} \in \tilde{G}, \)
\( s = 1, 2, \ldots, \) such that
\[
\sup_{g \in \tilde{G}} E^Q g = \sup_{s \geq 1} E^Q g^s,
\]
where
\[
E^Q g = \sum_{m=0}^{\infty} \frac{E^Q g_m}{2^m}, \quad g \in G.
\]

Proof. Let \( 0 < \varepsilon_s < \alpha^Q, \ s = 1, \infty, \) be a sequence of real numbers satisfying conditions \( \varepsilon_s > \varepsilon_{s+1}, \varepsilon_s \to 0, \) as \( s \to \infty. \) Then there exists an element \( g^s \in \tilde{G} \)
such that \( \alpha^Q - \varepsilon_s < E^Q g^s \leq \alpha^Q, \ s = 1, \infty. \) The sequence \( g^s \in \tilde{G}, \)
\( s = 1, \infty, \) satisfies Lemma 2 conditions.

Lemma 3. If a supermartingale \( \{f_m, F_m\}_{m=0}^{\infty} \) relative to a convex set of equivalent measures \( M \) is such that
\[
|f_m| \leq \varphi, \quad m = 0, \infty, \quad E^Q \varphi < T < \infty, \quad Q \in M, \tag{4}
\]
where a real number \( T \) does not depend on \( Q \in M, \) then every maximal chain \( \tilde{G} \subseteq G \) contains a maximal element.

Proof. Let \( g = \{g_m\}_{m=0}^{\infty} \) belong to \( G, \) then
\[
E^Q (f_m + \varphi + g_m) \leq f_0 + T, \quad m = 0, \infty, \quad Q \in M.
\]
Then inequalities \( f_m + \varphi \geq 0, \ m = 0, \infty, \) yield
\[
E^Q g_m \leq f_0 + T, \quad m = 0, \infty, \quad \{g_m\}_{m=0}^{\infty} \in G.
\]
Introduce for a certain \( Q \in M \) an expectation for \( g = \{g_m\}_{m=0}^{\infty} \in G \)
\[
E^Q g = \sum_{m=0}^{\infty} \frac{E^Q g_m}{2^m}, \quad g \in G.
\]
Let \( \tilde{G} \subseteq G \) be a certain maximal chain. Therefore, we have inequality
\[
\sup_{g \in \tilde{G}} E^Q g = \alpha^Q_0 \leq f_0 + T < \infty,
\]
where \( Q \in M \) and is fixed. Due to Lemma 2,
\[
\sup_{g \in \hat{G}} E_1^Q g = \sup_{s \geq 1} E_1^Q g^s.
\]

In consequence of the linear ordering of elements of \( \hat{G} \),
\[
\max_{1 \leq s \leq k} g^s = g^{s_0(k)}, \quad 1 \leq s_0(k) \leq k,
\]
where \( s_0(k) \) is one of elements of the set \( \{1, 2, \ldots, k\} \) on which the considered maximum is reached, that is, \( 1 \leq s_0(k) \leq k \), and, moreover,
\[
g^{s_0(k)} \leq g^{s_0(k+1)}.
\]

It is evident that
\[
\max_{1 \leq s \leq k} E_1^Q g^s = E_1^Q g^{s_0(k)}.
\]

So, we obtain
\[
\sup_{s \geq 1} E_1^Q g^s = \lim_{k \to \infty} \max_{1 \leq s \leq k} E_1^Q g^s = \lim_{k \to \infty} E_1^Q g^{s_0(k)} = E_1^Q g^0,
\]
where \( g^0 = \lim_{k \to \infty} g^{s_0(k)} \), and that there exists, due to monotony of \( g^{s_0(k)} \). Thus,
\[
\sup_{s \geq 1} E_1^Q g^s = E_1^Q g^0 = \alpha_0^Q.
\]

Show that \( g^0 = \{g^0_m\}_{m=0}^\infty \) is a maximal element in \( \hat{G} \). It is evident that \( g^0 \) belongs to \( G \). For every element \( g = \{g_m\}_{m=0}^\infty \in \hat{G} \) two cases are possible:
1) \( \exists k \) such that \( g \preceq g^{s_0(k)} \).
2) \( \forall k \quad g^{s_0(k)} \prec g \).
In the first case \( g \preceq g^0 \). In the second one from 2) we have \( g^0 \preceq g \). At the same time
\[
E_1^Q g^{s_0(k)} \leq E_1^Q g. \quad (5)
\]

By passing to the limit in (5), we obtain
\[
E_1^Q g^0 \leq E_1^Q g. \quad (6)
\]

The strict inequality in (6) is impossible, since \( E_1^Q g^0 = \sup_{g \in \hat{G}} E_1^Q g \). Therefore,
\[
E_1^Q g^0 = E_1^Q g. \quad (7)
\]

The inequality \( g^0 \preceq g \) and the equality (7) imply that \( g = g^0 \). \( \square \)
Let $M$ be a convex set of equivalent probability measures on $\{\Omega, \mathcal{F}\}$. Introduce into $M$ a metric $|Q_1 - Q_2| = \sup_{A \in \mathcal{F}} |Q_1(A) - Q_2(A)|$, $Q_1$, $Q_2 \in M$.

**Lemma 4.** Let $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ be a supermartingale relative to a compact convex set of equivalent measures $M$ satisfying conditions (2). If for every set of measures $\{P_1, P_2, \ldots, P_\alpha\}$, $s < \infty$, $P_i \in M$, $i = 1, \ldots, \alpha$, there exist a natural number $1 \leq m_0 < \infty$, and depending on this set of measures $\mathcal{F}_{m_0-1}$ measurable nonnegative random variable $\Delta_{m_0}$, $P_1(\Delta_{m_0} > 0) > 0$, satisfying conditions

$$f_{m_0-1} - E^{P_1}\{f_{m_0} | \mathcal{F}_{m_0-1}\} \geq \Delta_{m_0}, \quad i = 1, \ldots, \alpha,$$

then the set $G$ of adapted non-decreasing processes $g = \{g_m\}_{m=0}^{\infty}$, $g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^{\infty}$ is a supermartingale relative to the set of measures $M$ contains nonzero element.

**Proof.** For any point $P_0 \in M$ let us define a set of measures

$$M^{P_0, \bar{\varepsilon}_0} = \{Q \in M, \ Q = (1 - \alpha)P_0 + \alpha P, \ P \in M, \ 0 \leq \alpha \leq \bar{\varepsilon}_0\}, \quad (9)$$

$$\bar{\varepsilon}_0 = \frac{L}{1 + L}.$$  

Prove that the set of measures $M^{P_0, \bar{\varepsilon}_0}$ contains some ball of a positive radius, that is, there exists a real number $\rho_0 > 0$ such that $M^{P_0, \bar{\varepsilon}_0} \supseteq C(P_0, \rho_0)$, where $C(P_0, \rho_0) = \{P \in M, \ |P_0 - P| < \rho_0\}$.

Let $C(P_0, \bar{\rho}) = \{P \in M, \ |P_0 - P| < \bar{\rho}\}$ be an open ball in $M$ with the center at the point $P_0$ of a radius $0 < \bar{\rho} < 1$. Consider a map of the set $M$ into itself given by the law: $f(P) = (1 - \bar{\varepsilon}_0)P_0 + \bar{\varepsilon}_0 P$, $P \in M$.

The mapping $f(P)$ maps an open ball $C(P_2, \delta) = \{P \in M, \ |P_2 - P| < \delta\}$ with the center at the point $P_2$ of a radius $\delta > 0$ into an open ball with the center at the point $(1 - \bar{\varepsilon}_0)P_0 + \bar{\varepsilon}_0 P_2$ of the radius $\bar{\varepsilon}_0 \delta$, since $|(1 - \bar{\varepsilon}_0)P_0 + \bar{\varepsilon}_0 P_2 - (1 - \bar{\varepsilon}_0)P_0 - \bar{\varepsilon}_0 P| = \bar{\varepsilon}_0 |P_2 - P| < \bar{\varepsilon}_0 \delta$. Therefore, an image of an open set $M_0 \subseteq M$ is an open set $f(M_0) \subseteq M$, thus $f(P)$ is an open mapping. Since $f(P_0) = P_0$, then the image of the ball $C(P_0, \bar{\rho}) = \{P \in M, \ |P_0 - P| < \bar{\rho}\}$ is a ball $C(P_0, \bar{\varepsilon}_0 \bar{\rho}) = \{P \in M, \ |P_0 - P| < \bar{\varepsilon}_0 \bar{\rho}\}$ and it is contained in $f(M)$. Thus, inclusions $M^{P_0, \bar{\varepsilon}_0} \supseteq f(M) \supseteq C(P_0, \bar{\varepsilon}_0 \bar{\rho})$ are valid. Let us put $\bar{\varepsilon}_0 \bar{\rho} = \rho_0$. Then we have $M^{P_0, \bar{\varepsilon}_0} \supseteq C(P_0, \rho_0)$, where $C(P_0, \rho_0) = \{P \in M, \ |P_0 - P| < \rho_0\}$. Consider an open covering $\bigcup_{P_0 \in M} C(P_0, \rho_0)$ of the compact set $M$. Due to compactness of $M$, there exists a finite subcovering

$$M = \bigcup_{i=1}^{v} C(P_0^i, \rho_0) \quad (10)$$

with the center at the points $P_0^i \in M$, $i = 1, \ldots, v$, and a covering by sets $M^{P_0^i, \bar{\varepsilon}_0} \supseteq C(P_0^i, \rho_0)$, $i = 1, \ldots, v$,

$$M = \bigcup_{i=1}^{v} M^{P_0^i, \bar{\varepsilon}_0}. \quad (11)$$
Consider the set of measures $P_i^0 \in M$, $i = 1, v$. From Lemma 4 conditions, there exist a natural number $1 \leq m_0 < \infty$, and depending on the set of measures $P_i^0 \in M$, $i = 1, v$, measurable nonnegative random variable $\Delta_{m_0}^v$, $P_0^1(\Delta_{m_0}^v > 0) > 0$, such that

$$f_{m_0-1} - E_{P_0^i}\{f_{m_0} | F_{m_0-1}\} \geq \Delta_{m_0}^v, \quad i = 1, v.$$ (12)

Due to Theorem 1, we have

$$f_{m_0-1} - E_{Q}\{f_{m_0} | F_{m_0-1}\} \geq \frac{l}{1 + L} \Delta_{m_0}^v = \varphi_{m_0}^v, \quad Q \in M.$$ (13)

The last inequality imply

$$E_{Q}\{f_{m_0} | F_s\} - E_{Q}\{f_{m_0} | F_s\} \geq E_{Q}\{\varphi_{m_0}^v | F_s\}, \quad Q \in M, \quad s < m_0.$$ (14)

But $E_{Q}\{f_{m_0-1} | F_s\} \leq f_s$, $s < m_0$. Therefore,

$$f_s - E_{Q}\{f_{m_0} | F_s\} \geq E_{Q}\{\varphi_{m_0}^v | F_s\}, \quad Q \in M, \quad s < m_0.$$ (15)

Since

$$f_{m_0} - E_{Q}\{f_{m} | F_{m_0}\} \geq 0, \quad Q \in M, \quad m \geq m_0,$$ (16)

we have

$$E_{Q}\{f_{m_0} | F_s\} - E_{Q}\{f_{m} | F_{m_0}\} \geq 0, \quad Q \in M, \quad s < m_0, \quad m \geq m_0.$$ (17)

Adding (17) to (15), we obtain

$$f_s - E_{Q}\{f_{m} | F_s\} \geq E_{Q}\{\varphi_{m_0}^v | F_s\}, \quad Q \in M, \quad s < m_0, \quad m \geq m_0,$$ (18)

or

$$f_s - E_{Q}\{f_{m} | F_s\} \geq E_{Q}\{\varphi_{m_0}^v | F_s\} \chi_{[m_0, \infty)}(m) - \varphi_{m_0}^v \chi_{[m_0, \infty)}(s), \quad Q \in M, \quad s \leq m_0, \quad m \geq m_0.$$ (19)

Introduce an adapted non-decreasing process

$$g_{m_0}^m = \{g_{m_0}^m\}_{m=0}^\infty, \quad g_{m_0}^m = \varphi_{m_0}^v \chi_{[m_0, \infty)}(m),$$

where $\chi_{[m_0, \infty)}(m)$ is an indicator function of the set $[m_0, \infty)$. Then (19) implies that

$$E_{Q}\{f_{m} + g_{m_0}^m | F_k\} \leq f_k + g_{k_0}^m, \quad 0 \leq k \leq m, \quad Q \in M.$$
In the Theorem 2 a convex set of equivalent measures

\[ M = \{ Q, Q = \sum_{i=1}^{n} \alpha_i P_i, \ \alpha_i \geq 0, \ i = 1, n, \ \sum_{i=1}^{n} \alpha_i = 1 \} \]  \hspace{1cm} (20)

satisfies conditions

\[ 0 < l \leq \frac{dP_i}{dP_j} \leq L < \infty, \quad i, j = 1, n, \]  \hspace{1cm} (21)

where \( l, L \) are real numbers.

Denote by \( G \) the set of all adapted non-decreasing processes \( g = \{ g_m \}_{m=0}^{\infty} \), \( g_0 = 0 \), such that \( \{ f_m + g_m \}_{m=0}^{\infty} \) is a supermartingale relative to all measures from \( M \).

**Theorem 2.** Let a supermartingale \( \{ f_m, F_m \}_{m=0}^{\infty} \) relative to the set of measures (20) satisfy the conditions (4), and let there exist a natural number \( 1 \leq m_0 < \infty \), and \( F_{m_0-1} \) measurable nonnegative random value \( \varphi_{m_0}^n, P_1(\varphi_{m_0}^n > 0) > 0 \), such that

\[ f_{m_0-1} - E^P_i \{ f_{m_0} | F_{m_0-1} \} \geq \varphi_{m_0}^n, \quad i = 1, n. \]  \hspace{1cm} (22)

If for the maximal element \( g^0 = \{ g_m^0 \}_{m=0}^{\infty} \) in a certain maximal chain \( \tilde{G} \subseteq G \) the equalities

\[ E^P_i (f_{\infty} + g^0_{\infty}) = f_0, \quad P_1 \in M, \quad i = 1, n, \]  \hspace{1cm} (23)

are valid, where \( f_{\infty} = \lim_{m \to \infty} f_m, g^0_{\infty} = \lim_{m \to \infty} g^0_m \), then there hold equalities

\[ E^P \{ f_m + g^0_m | F_k \} = f_k + g^0_k, \quad 0 \leq k \leq m, \quad m = 1, \infty, \quad P \in M. \]  \hspace{1cm} (24)

**Proof.** The set \( M \) is compact one in the introduced metric topology. From the inequalities (22) and the formula

\[ E^Q \{ f_{m_0} | F_{m_0-1} \} = \frac{\sum_{i=1}^{n} \alpha_i E^P_i \{ \varphi_i | F_{m_0-1} \} E^P_i \{ f_{m_0} | F_{m_0-1} \}}{\sum_{i=1}^{n} \alpha_i E^P_i \{ \varphi_i | F_{m_0-1} \}}, \quad Q \in M, \]  \hspace{1cm} (25)

where \( \varphi_i = \frac{dP_i}{dP_1} \), we obtain

\[ f_{m_0-1} - E^Q \{ f_{m_0} | F_{m_0-1} \} \geq \varphi_{m_0}^n, \quad Q \in M. \]  \hspace{1cm} (26)

The inequalities (21) lead to inequalities

\[ \frac{1}{nL} \leq \frac{dQ}{dP} \leq nL, \quad P, Q \in M. \]  \hspace{1cm} (27)

Inequalities (26) and (27) imply that conditions of Lemma 4 are satisfied for any set of measures \( Q_1, \ldots, Q_s \in M \). Hence, it follows that the set \( G \) contains
nonzero element. Let \( \tilde{G} \subseteq G \) be a maximal chain in \( G \) satisfying condition of Theorem 2. Denote by \( g^0 = \{g^0_m\}_{m=0}^\infty \), a maximal element in \( \tilde{G} \subseteq G \). Theorem 2 and Lemma 3 yield that as \( \{f_m\}_{m=0}^\infty \) and \( \{g^0_m\}_{m=0}^\infty \) are uniformly integrable relative to each measure from \( M \). There exist therefore limits
\[
\lim_{m \to \infty} f_m = f_\infty, \quad \lim_{m \to \infty} g^0_m = g^0_\infty
\]
with probability 1. Due to Theorem 2 condition, in this maximal chain
\[
E^{P_i}(f_\infty + g^0_\infty) = f_0, \quad P_i \in M, \quad i = 1, n.
\]
Since \( \{f_m + g^0_m\}_{m=0}^\infty \) is a supermartingale concerning all measures from \( M \), we have
\[
E^{P_i}(f_m + g^0_m) \leq E^{P_i}(f_k + g^0_k) \leq f_0, \quad k < m, \quad m = 1, \infty, \quad P_i \in M. \tag{28}
\]
By passing to the limit in (28), as \( m \to \infty \), we obtain
\[
f_0 = E^{P_i}(f_\infty + g^0_\infty) \leq E^{P_i}(f_k + g^0_k) \leq f_0, \quad k = 1, \infty, \quad P_i \in M. \tag{29}
\]
So, \( E^{P_i}(f_k + g^0_k) = f_0 \), \( k = 1, \infty \), \( P_i \in M \), \( i = 1, n \). Taking into account Remark 1 we have
\[
E^{P_i}\{f_m + g^0_m|F_k\} = f_k + g^0_k, \quad 0 \leq k \leq m, \quad m = 1, \infty, \quad P_i \in M, \quad i = 1, n. \tag{30}
\]
Hence,
\[
E^P\{f_m + g^0_m|F_k\} =
\]
\[
\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i|F_k\} E^{P_i}\{f_m + g^0_m|F_k\} =
\]
\[
\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i|F_k\} = f_k + g^0_k, \quad 0 \leq k \leq m, \quad P \in M. \tag{31}
\]
where \( \varphi_i = \frac{dP_i}{dP_1} \), \( i = 1, n \). Theorem 2 is proven. \( \Box \)

Let \( M \) be a convex set of equivalent measures. Bellow, \( G_s \) is a set of adapted non-decreasing processes \( \{g^0_m\}_{m=0}^\infty \), \( g^0_0 = 0 \), for which \( \{f_m + g^0_m\}_{m=0}^\infty \) is a supermartingale relative to all measures from
\[
\hat{M}_s = \{Q, Q = \sum_{i=1}^s \gamma_i \hat{P}_i, \quad \gamma_i \geq 0, \quad i = 1, s, \quad \sum_{i=1}^s \gamma_i = 1\}, \tag{32}
\]
where \( \hat{P}_1, \ldots, \hat{P}_s \in M \) and satisfy conditions
\[
0 < l \leq \frac{d\hat{P}_i}{d\hat{P}_j} \leq L < \infty, \quad i, j = 1, s, \tag{33}
\]
l, \( L \) are real numbers depending on the set of measures \( \hat{P}_1, \ldots, \hat{P}_s \in M \).
Definition 3. Let a supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to a convex set of equivalent measures \( \mathcal{M} \) satisfy conditions (4). We call it regular one if for every set of measures (32) satisfying conditions (33) there exist a natural number \( 1 \leq m_0 < \infty \), and \( \mathcal{F}_{m_0-1} \) measurable nonnegative random value \( \varphi_{m_0}^s, \hat{P}_i(\varphi_{m_0}^s > 0) > 0 \), such that the inequalities

\[
 f_{m_0-1} - E^{\hat{P}_i}\{f_m | \mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^s, \quad i = 1, s,
\]

hold and for the maximal element \( g^s = \{g_m^s\}_{m=0}^{\infty} \) in a certain maximal chain \( \tilde{G}_s \subseteq G_s \) the equalities

\[
 E^{\hat{P}_i}\{f_m + g_m^s | \mathcal{F}_k\} = f_k + g_k^s, \quad 0 \leq k \leq m, \quad i = 1, s, \quad m = 1, \infty, \quad (34)
\]

are valid. Moreover, there exists an adapted nonnegative process \( \bar{g}^0 = \{\bar{g}_m^0\}_{m=0}^{\infty} \), \( \bar{g}_0^0 = 0 \), \( E^{P_0}\bar{g}_m^0 < \infty, \quad m = 1, \infty, \quad P_0 \in \mathcal{M} \), not depending on the set of measures \( \hat{P}_1, \ldots, \hat{P}_s \) such that

\[
 E^{\hat{P}_i}\{g_m^s - g_{m-1}^s | \mathcal{F}_{m-1}\} = E^{\hat{P}_i}\{\bar{g}_m^0 | \mathcal{F}_{m-1}\}, \quad m = 1, \infty, \quad i = 1, s. \quad (35)
\]

The next Theorem describes regular supermartingales.

Theorem 3. Let \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) be a regular supermartingale relative to a convex set of equivalent measures \( \mathcal{M} \). Then for the maximal element \( g^0 = \{g_m^0\}_{m=0}^{\infty} \) in a certain maximal chain \( \tilde{G} \subseteq G \) the equalities

\[
 E^{P_0}(f_m + g_m^0) = f_0, \quad m = 1, \infty, \quad P_0 \in \mathcal{M},
\]

are valid. There exists a martingale \( \{\tilde{M}_m, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to the family of measures \( \mathcal{M} \) such that

\[
 f_m = \tilde{M}_m - g_m^0, \quad m = 1, \infty.
\]

Moreover, for the martingale \( \{\tilde{M}_m, \mathcal{F}_m\}_{m=0}^{\infty} \) the representation

\[
 \tilde{M}_m = E^{P_0}(f_\infty + g_\infty | \mathcal{F}_m), \quad m = 1, \infty, \quad P_0 \in \mathcal{M},
\]

holds, where \( f_\infty + g_\infty = \lim_{m \to \infty} (f_m + g_m) \).

Proof. For any finite set of measures \( P_1, \ldots, P_n, \quad P_i \in \mathcal{M}, \quad i = 1, n \), let us introduce into consideration two sets of measures

\[
 M_n = \{P, \quad P = \sum_{i=1}^{n} \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 1, n, \quad \sum_{i=1}^{n} \alpha_i = 1\},
\]

\[
 \tilde{M}_n = \{P, \quad P = \sum_{i=1}^{n} \alpha_i P_i, \quad \alpha_i > 0, \quad i = 1, n, \quad \sum_{i=1}^{n} \alpha_i = 1\}.
\]
Let $\hat{P}_1, \ldots, \hat{P}_s$ be a certain subset of measures from $\hat{M}_n$. For every measure $\hat{P}_i \in \hat{M}_n$ the representation $\hat{P}_i = \sum_{k=1}^{n} \alpha^i_k P_k$ is valid, where $\alpha^i_k > 0$, $i = \overline{1,s}$, $k = \overline{1,n}$. The representation for $\hat{P}_i$, $i = \overline{1,s}$, imply the validity of inequalities

\[
0 < l = \min_{i,j} \frac{\min_k \alpha^i_k}{\max_k \alpha^j_k} \leq \frac{d\hat{P}_i}{dP_j} \leq \max_{i,j} \frac{\max_k \alpha^i_k}{\min_k \alpha^j_k} = L < \infty, \quad i, j = \overline{1,s}.
\]

Denote by $G_s$ a set of adapted non-decreasing processes $\{g_m\}_{m=0}^{\infty}$, $g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^{\infty} = \{\Phi_m\}^s_{m=0}$ is a supermartingale relative to all measures from 

\[
\hat{M}_s = \{Q, \ Q = \sum_{i=1}^{s} \gamma_i \hat{P}_i, \ \gamma_i \geq 0, \ i = \overline{1,s}, \ \sum_{i=1}^{s} \gamma_i = 1\}.
\]

In accordance with the definition of a regular supermartingale, there exist a natural number $1 \leq m_0 < \infty$, and $\mathcal{F}_{m_0-1}$ measurable nonnegative random value $\varphi^{s}_{m_0}$, $\hat{P}_i(\varphi^{s}_{m_0} > 0) > 0$, such that the inequalities there hold

\[
f_{m_0-1} - E^{\hat{P}_i} \{f_{m_0} | \mathcal{F}_{m_0-1}\} \geq \varphi^{s}_{m_0}, \quad i = \overline{1,s},
\]

and for a maximal element $g^s = \{g^s_{m}\}_{m=0}^{\infty}$ in a certain maximal chain $\bar{G}_s \subseteq G_s$ there hold equalities (34), (35). Equalities (35) yield the equalities

\[
E^Q \{g^s_{m} - g^s_{m-1} | \mathcal{F}_{m-1}\} = \sum_{i=1}^{s} \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | \mathcal{F}_{m-1}\} E^{\hat{P}_i} \{g^s_{m} - g^s_{m-1} | \mathcal{F}_{m-1}\} = \sum_{i=1}^{s} \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | \mathcal{F}_{m-1}\}
\]

\[
= \sum_{i=1}^{s} \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | \mathcal{F}_{m-1}\} E^{\hat{P}_i} \{g^0_{m} | \mathcal{F}_{m-1}\} = E^Q \{g^0_{m} | \mathcal{F}_{m-1}\},
\]

\[
m = \overline{1,\infty}, \quad Q \in \hat{M}_s.
\]

where $\hat{\varphi}_i = \frac{d\hat{P}_i}{dP_1}$, $i = \overline{1,n}$. Taking into account the equalities (34), we obtain

\[
E^Q \{f_m + g^s_{m} | \mathcal{F}_{m-1}\} = \sum_{i=1}^{s} \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | \mathcal{F}_{m-1}\} E^{\hat{P}_i} \{f_m + g^s_{m} | \mathcal{F}_{m-1}\} = \sum_{i=1}^{s} \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | \mathcal{F}_{m-1}\}
\]
\[ f_{m-1} + g^{s}_{m-1}, \quad m = 1, \infty, \quad Q \in \hat{M}_s. \] (37)

Thus, we have

\[ E^Q \{ g^{s}_{m} - g^{s}_{m-1} | \mathcal{F}_{m-1} \} = E^Q \{ \bar{g}^0_m | \mathcal{F}_{m-1} \}, \quad m = 1, \infty, \quad Q \in \hat{M}_s. \] (38)

\[ E^Q \{ f_m + g^{s}_{m} | \mathcal{F}_{m-1} \} = f_{m-1} + g^{s}_{m-1}, \quad m = 1, \infty, \quad Q \in \hat{M}_s. \] (39)

Let us introduce into consideration a random process \( \{ N_m, \mathcal{F}_m \}_{m=0}^{\infty} \), where

\[ N_0 = f_0, \quad N_m = f_m + \sum_{i=1}^{m} \bar{g}^0_i, \quad m = 1, \infty. \]

It is evident that \( E^Q |N_m| < \infty, \quad m = \frac{1}{\infty}, \quad Q \in \hat{M}_s \). The definition of \( \{ N_m, \mathcal{F}_m \}_{m=0}^{\infty} \) and the formulae (38), (39) yield

\[ E^Q \{ N_{m-1} - N_m | \mathcal{F}_{m-1} \} = E^Q \{ f_{m-1} - f_m - \bar{g}^0_m | \mathcal{F}_{m-1} \} = E^Q \{ \bar{g}^s_m - g^{s}_{m-1} - \bar{g}^0_m | \mathcal{F}_{m-1} \} = 0, \quad m = 1, \infty, \quad Q \in \hat{M}_s. \]

The last equalities imply

\[ E^Q \{ N_m | \mathcal{F}_{m-1} \} = N_{m-1}, \quad m = \frac{1}{\infty}, \quad Q \in \hat{M}_s. \]

Due to arbitrariness of the set of measures \( \hat{P}_1, \ldots, \hat{P}_s, \hat{P}_t \in \hat{M}_n \), we have

\[ E^P \{ N_m | \mathcal{F}_{m-1} \} = N_{m-1}, \quad P \in \hat{M}_n, \quad m = \frac{1}{\infty}. \] (40)

So, the set \( G_0 \) of adapted non-decreasing processes \( \{ g_m \}_{m=0}^{\infty}, \quad g_0 = 0, \) for which \( \{ f_m + g_m \}_{m=0}^{\infty} \) is a supermartingale relative to all measures from \( \hat{M}_n \) contains nonzero element \( \tilde{g}^0 = \{ \bar{g}^0_m \}_{m=0}^{\infty}, \quad \bar{g}^0_0 = 0, \quad \bar{g}^0_m = \sum_{i=1}^{m} \bar{g}^0_i, \quad m = \frac{1}{\infty}, \)

which is a maximal element in a maximal chain \( \tilde{G}_0 \) containing this element. Really, if \( g^0 = \{ \bar{g}^0_m \}_{m=0}^{\infty}, \quad \bar{g}^0_0 = 0, \) is a maximal element in the maximal chain \( \tilde{G}_0 \subseteq G_0 \), then there hold inequalities

\[ E^{P_0} \{ f_m + g^0_m | \mathcal{F}_k \} \leq f_k + g^0_k, \quad m = \frac{1}{\infty}, \quad 1 \leq k \leq m, \quad P_0 \in \hat{M}_n, \] (41)

\[ E^{P_0} (f_m + g^0_m) \leq f_0, \quad m = \frac{1}{\infty}, \quad P_0 \in \hat{M}_n. \] (42)

and inequality \( \tilde{g}^0 \leq g^0 \) meaning that \( \bar{g}^0_m \leq g^0_m, \quad m = \frac{0}{\infty} \). Equalities (40) yield

\[ E^{P_0} (f_m + \bar{g}^0_m) = f_0, \quad m = \frac{1}{\infty}, \quad P_0 \in \hat{M}_n. \] (43)

Inequalities (42) and equalities (43) imply

\[ f_0 \geq E^{P_0} (f_m + \bar{g}^0_m) \geq E^{P_0} (f_m + g^0_m) = f_0, \quad m = \frac{1}{\infty}, \quad P_0 \in \hat{M}_n. \] (44)
The last inequalities lead to equalities

\[ E^{P_0}(g^0_m - \tilde{g}^0_m) = 0, \quad m = 1, \infty, \quad P_0 \in \tilde{M}_n. \]  

(45)

But

\[ g^0_m - \tilde{g}^0_m \geq 0, \quad m = 0, \infty. \]  

(46)

The equalities (45) and inequalities (46) yield \( g^0_m = \tilde{g}^0_m, \ m = 0, \infty, \) or \( \tilde{g}^0 = g^0. \)

Prove that \( G_n = G_0, \) where \( G_n \) is a set of non-decreasing processes \( g = \{g_m\}_{m=0}^{\infty} \) such that \( \{f_m + g_m\}_{m=0}^{\infty} \) is a supermartingale relative to all measures from \( M_n. \) Really, if \( g = \{g_m\}_{m=0}^{\infty} \) is a non-decreasing process from \( G_n, \) then it belongs to \( G_0, \) owing to that \( M_n \supset \tilde{M}_n \) and \( G_n \subseteq G_0. \) Suppose that \( g = \{g_m\}_{m=0}^{\infty}, \ g_0 = 0, \) is a non-decreasing process from \( G_0. \) It means that

\[ E^Q \{f_m + g_m|F_k\} \leq f_k + g_k, \quad m = 1, \infty, \ 0 \leq k \leq m, \quad Q \in \tilde{M}_n. \]  

(47)

The last inequalities can be written in the form

\[ \sum_{i=1}^{n} \alpha_i \int_A (f_m + g_m)dP_i \leq \sum_{i=1}^{n} \alpha_i \int_A (f_k + g_k)dP_i, \quad m = 1, \infty, \ 0 \leq k \leq m, \]

\[ A \in F_k, \quad \alpha_i > 0, \quad i = 1, n. \]

By passing to the limit, as \( \alpha_j \to 0, \quad \alpha_j > 0, \quad j \neq i, \quad \alpha_i \to 1, \) we obtain

\[ \int_A (f_m + g_m)dP_i \leq \int_A (f_k + g_k)dP_i, \quad i = 1, n, \quad A \in F_k, \quad m = 1, \infty, \ 0 \leq k \leq m. \]

The last inequalities yield inequalities

\[ \sum_{i=1}^{n} \alpha_i \int_A (f_m + g_m)dP_i \leq \sum_{i=1}^{n} \alpha_i \int_A (f_k + g_k)dP_i, \quad m = 1, \infty, \ 0 \leq k \leq m, \]

\[ A \in F_k, \quad \alpha_i \geq 0, \quad i = 1, n, \]

or

\[ E^Q \{f_m + g_m|F_k\} \leq f_k + g_k, \quad m = 1, \infty, \ 0 \leq k \leq m, \quad Q \in M_n. \]

It means that \( g = \{g_m\}_{m=0}^{\infty} \) belongs to \( G_n. \) On the basis of the above proved, for the maximal element \( \tilde{g}^0 = \{\tilde{g}_m^0\}_{m=0}^{\infty} \) in the maximal chain \( \tilde{G}_0 \subseteq G_0 \) the equalities

\[ E^Q \{f_m + \tilde{g}_m^0|F_k\} = f_k + \tilde{g}_k^0, \quad m = 1, \infty, \ 1 \leq k \leq m, \quad Q \in \tilde{M}_n, \]  

(48)

\[ E^Q (f_m + \tilde{g}_m^0) = f_0, \quad m = 1, \infty, \quad Q \in \tilde{M}_n, \]  

(49)
are valid. From proved equality $G_n = G_0$, it follows that $\tilde{G}_0$ is a maximal chain in $G_n$.

As far as $G_0$ coincides with $G_n$ we proved that the maximal element $\tilde{g}^0$ in a certain maximal chain in $G_n$ satisfies equalities

$$E^P\{f_m + \tilde{g}_m^0|F_k\} = f_k + \tilde{g}_k^0, \quad m = 1, \infty, \quad 1 \leq k \leq m, \quad P_0 \in M_n, \quad (50)$$

$$E^P_0(f_m + \tilde{g}_m^0) = f_0, \quad m = 1, \infty, \quad P_0 \in M_n. \quad (51)$$

Due to arbitrariness of the set of measure $P_1, \ldots, P_n$, $P_i \in M$, the set $G$ contains nonzero element $\tilde{g}^0$ and in the maximal chain $\tilde{G} \subseteq G$ containing element $\tilde{g}^0$ the maximal element $g^0 = \{g_m^0\}_{m=0}^{\infty}, g_0^0 = 0$, coincides with $\tilde{g}^0$.

The last statement can be proved as in the case of maximal chain $\tilde{G}_0$. So,

$$E^P_0\{f_m + g_m^0|F_k\} = f_k + g_k^0, \quad m = 1, \infty, \quad 1 \leq k \leq m, \quad P_0 \in M, \quad (52)$$

$$E^P_0(f_m + g_m^0) = f_0, \quad m = 1, \infty, \quad P_0 \in M. \quad (53)$$

Denote by $\{\tilde{M}_m, F_m\}_{m=0}^{\infty}$ a martingale relative to the set of measures $M$, where $\tilde{M}_m = f_m + \tilde{g}_m^0, \ m = 1, \infty$. Due to Theorem 3 conditions, the supermartingale $\{f_m, F_m\}_{m=0}^{\infty}$ and non-decreasing process $g^0 = \{g_m^0\}_{m=0}^{\infty}$ are uniformly integrable relative to any measure from $M$, since for the non-decreasing process $g^0 = \{g_m^0\}_{m=0}^{\infty}$ there hold bounds $E^P g_m^0 < T + f_0, m = 1, \infty, \ P \in M$.

Therefore, the martingale $\{\tilde{M}_m, F_m\}_{m=0}^{\infty}$ is uniformly integrable relative to any measure from $M$. So, with probability 1 relative to every measure from $M$ there exist limits

$$\lim_{m \to \infty} \tilde{M}_m = M_\infty = f_\infty + g_\infty^0, \quad \lim_{m \to \infty} f_m = f_\infty, \quad \lim_{m \to \infty} g_m^0 = g_\infty^0.$$

Moreover, the representation

$$\tilde{M}_m = E^P \{(f_\infty + g_\infty^0)|F_m\}, \quad m = 1, \infty, \quad P \in M, \quad (54)$$

holds, where $\tilde{M} = \{\tilde{M}_m\}_{m=0}^{\infty}$ does not depend on $P \in M$. \qed

In the next theorem we give the necessary and sufficient conditions of regularity of supermartingales.

**Theorem 4.** Let a supermartingale $\{f_m, F_m\}_{m=0}^{\infty}$ relative to a convex set of equivalent measures $M$ satisfy conditions (4). The necessary and sufficient conditions for it to be a regular one is the existence of adapted nonnegative random process $\tilde{g}^0 = \{\tilde{g}_m^0\}_{m=0}^{\infty}, \tilde{g}_0^0 = 0, E^P \tilde{g}_m^0 < \infty, m = 1, \infty, \ P \in M$, such that equalities

$$E^P{f_{m-1} - f_m|F_{m-1}} = E^P\{g_m^0|F_{m-1}\}, \quad m = 1, \infty, \quad P \in M, \quad (55)$$

are valid.
Proof. Necessity. If \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a regular supermartingale, then there exist a martingale \( \{\bar{M}_m, \mathcal{F}_m\}_{m=0}^{\infty} \) and a non-decreasing nonnegative random process \( \{g_m, \mathcal{F}_m\}_{m=0}^{\infty}, \ g_0 = 0 \), such that
\[
f_m = \bar{M}_m - g_m, \quad m = 1, \infty. \tag{56}
\]
As before, equalities (56) yield inequalities \( E^P g_m \leq f_0 + T, \ m = 1, \infty \), and equalities
\[
E^P \{f_{m-1} - f_m | \mathcal{F}_{m-1}\} =
= E^P \{g_{m-1} - g_m | \mathcal{F}_{m-1}\} = E^P \{g_0 | \mathcal{F}_{m-1}\}, \quad m = 1, \infty, \quad P \in M, \tag{57}
\]
where we introduced the denotation \( \bar{g}_m = g_m - g_{m-1} \geq 0 \). It is evident that \( E^P \bar{g}_m \leq 2(f_0 + T) \).

Sufficiency. If there exists an adapted nonnegative random process \( \bar{g}_0 = \{\bar{g}_m\}_{m=0}^{\infty}, \ g_0 = 0, \ E^P \bar{g}_m < \infty, \ m = 1, \infty \), such that the equalities (55) are valid, then let us consider a random process \( \{\bar{M}_m, \mathcal{F}_m\}_{m=0}^{\infty} \), where
\[
\bar{M}_0 = f_0, \quad \bar{M}_m = f_m + \sum_{i=1}^{m} \bar{g}_m, \quad m = 1, \infty. \tag{58}
\]
It is evident that \( E^P |\bar{M}_m| < \infty \) and
\[
E^P \{\bar{M}_{m-1} - \bar{M}_m | \mathcal{F}_{m-1}\} = E^P \{f_{m-1} - f_m - \bar{g}_m | \mathcal{F}_{m-1}\} = 0.
\]
Theorem 4 is proven.

In the next Theorem we describe the structure of non-decreasing process for a regular supermartingale.

Theorem 5. Let a supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to a convex set of equivalent measures \( M \) satisfy conditions (4). The necessary and sufficient conditions for it to be regular one is the existence of a non-decreasing adapted process \( g = \{g_m\}_{m=0}^{\infty}, \ g_0 = 0, \) and adapted processes \( \Psi^j = \{\Psi^j_m\}_{m=0}^{\infty}, \ \Psi^0 = 0, \ j = 1, n, \) such that between elements \( g_m, \ m = 1, \infty \), of non-decreasing process \( g = \{g_m\}_{m=0}^{\infty} \) the relations
\[
g_m - g_{m-1} = f_{m-1} - E^{P_j} \{f_m | \mathcal{F}_{m-1}\} + \Psi^j_m, \quad m = 1, \infty, \quad j = 1, n, \tag{59}
\]
are valid for each set of measures \( P_1, \ldots, P_n \in M \), where \( E^{P_j} |\Psi^j_m| < \infty, \ E^{P_j} \{\Psi^j_m | \mathcal{F}_{m-1}\} = 0, \ j = 1, n, \ m = 1, \infty \).

Proof. The necessity. Let \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) be a regular supermartingale. Then for it the representation
\[
f_m + g_m = M_m, \quad m = 1, \infty, \quad j = 1, n, \tag{60}
\]
is valid, where \( \{g_m\}_{m=0}^{\infty}, \ g_0 = 0, \) is a non-decreasing adapted process, \( \{M_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a martingale relative to the set of measures \( M \). For any finite set of measures \( P_1, \ldots, P_n \in M \), we have
\[
E^{P_j} \{f_m + g_m | \mathcal{F}_{m-1}\} = f_{m-1} + g_{m-1}, \quad m = 1, \infty, \quad j = 1, n. \tag{61}
\]
Hence, we have
\[ E^{P_j} \{g_m - g_{m-1} | \mathcal{F}_{m-1}\} = f_{m-1} - E^{P_j} \{f_m | \mathcal{F}_{m-1}\}, \quad m = 1, \infty, \quad j = 1, n. \quad (62) \]
Let us put
\[ \Psi^j_m = g_m - g_{m-1} - E^{P_j} \{g_m - g_{m-1} | \mathcal{F}_{m-1}\}. \quad (63) \]
The assumptions of Theorem 5 and Lemma 3, the representation (63) imply
\[ E^{P_j} |\Psi^j_m| < 4(f_0 + T), \quad E^{P_j} \{\Psi^j_m | \mathcal{F}_{m-1}\} = 0, \quad j = 1, n, \quad m = 1, \infty. \]
This proves the necessity.

**The sufficiency.** For any set of measures \( P_1, \ldots, P_n \in M \) the representation (59) for a non-decreasing adapted process \( g = \{g_m\}_{m=0}^\infty, \quad g_0 = 0 \), is valid. Hence, we obtain (62) and (61). The equalities (62), (61) and the formula
\[
E^P \{f_m + g_m | \mathcal{F}_{m-1}\} = \sum_{i=1}^n \alpha_i E^P \{\varphi_i | \mathcal{F}_{m-1}\} E^P \{f_m + g_m | \mathcal{F}_{m-1}\}
\]
implies
\[
E^P \{f_m + g_m | \mathcal{F}_{m-1}\} = f_{m-1} + g_{m-1}, \quad m = 1, \infty, \quad P \in M_n.
\]
Arbitrariness of the set of measures \( P_1, \ldots, P_n \in M \) and fulfllment of the condition (4) for the supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) imply its regularity. \( \square \)

Further, we consider a class of supermartingales \( F \) satisfying conditions
\[
\sup_{P \in M} E^P |f_m| < \infty, \quad m = 0, \infty.
\]

**Definition 4.** A supermartingale \( f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty \in F \) is said to be local regular one if there exists an increasing sequence of nonrandom stopping times \( \tau_{k_s} = k_s, \quad k_s < \infty, \quad s = 1, \infty, \quad \lim_{s \to \infty} k_s = \infty \), such that the stopped process \( f^{\tau_{k_s}} = \{f_{m \wedge \tau_{k_s}}, \mathcal{F}_m\}_{m=0}^\infty \) is a regular supermartingale for every \( \tau_{k_s} = k_s, \quad k_s < \infty, \quad s = 1, \infty \).

**Theorem 6.** Let \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) be a supermartingale relative to a convex set of equivalent measures \( M \), belonging to the class \( F \), for which the representation
\[ f_m = M_m - g_m^0, \quad m = 0, \infty, \quad (64) \]
is valid, where \( \{M_m\}_{m=0}^\infty \) is a martingale relative to a convex set of equivalent measures \( M \) such that
\[
E^P |M_m| < \infty, \quad m = 0, \infty, \quad P \in M,
\]
g\( g^0 = \{g_m^0\}_{m=0}^\infty, \quad g_0^0 = 0 \), is a non-decreasing adapted process. Then \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) is a local regular supermartingale.
Proof. The representation (64) and assumptions of Theorem 6 imply inequalities $E^P g^0_m < \infty$, $m = 1, \infty$, $P \in M$. For any measure $P \in M$, therefore we have

$$E^P \{ f_m + g^0_m | \mathcal{F}_{m-1} \} = M_{m-1} = f_{m-1} + g^0_{m-1}, \quad m = 1, \infty. \tag{65}$$

Consider a sequence of stopping times $\tau_s = s$, $s = 1, \infty$. Equalities (65) yield

$$E^P \{ f_{m \wedge \tau_s} + g^0_{m \wedge \tau_s} | \mathcal{F}_{m-1} \} = M_{(m-1) \wedge \tau_s} = f_{(m-1) \wedge \tau_s} + g^0_{(m-1) \wedge \tau_s}, \quad m = 1, \infty, \quad P \in M. \tag{66}$$

For the stopped supermartingale $\{ f_{m \wedge \tau_s}, \mathcal{F}_m \}_{m=0}^\infty$, the set $G$ of adapted non-decreasing processes $g = \{ g_m \}_{m=0}^\infty$, $g_0 = 0$, such that $\{ f_{m \wedge \tau_s} + g_m, \mathcal{F}_m \}_{m=0}^\infty$ is a supermartingale relative to a convex set of equivalent measures $M$ contains nonzero element $g^{0, \tau_s} = \{ g^0_{m \wedge \tau_s} \}_{m=0}^\infty$, $g_0^0 = 0$. Consider a maximal chain $\tilde{G} \subseteq G$ containing this element and let $g = \{ g_m \}_{m=0}^\infty$, $g_0 = 0$, be a maximal element in $\tilde{G}$ which exists, since the stopped supermartingale $\{ f_{m \wedge \tau_s}, \mathcal{F}_m \}_{m=0}^\infty$ is such that $| f_{m \wedge \tau_s} | \leq \sum_{i=0}^s | f_i | = \varphi$, $m = 0, \infty$, $E^P \varphi \leq \sum_{i=0}^s \sup_{P \in M} E^P | f_i | = T < \infty$. Then

$$E^P \{ f_{m \wedge \tau_s} + g_m | \mathcal{F}_{m-1} \} \leq f_{(m-1) \wedge \tau_s} + g_{m-1}, \quad m = 1, \infty. \tag{67}$$

Equalities (66) and inequality $g^{0, \tau_s} \leq g$ imply

$$f_0 = E^P \{ f_{m \wedge \tau_s} + g^0_{m \wedge \tau_s} \} \leq E^P \{ f_{m \wedge \tau_s} + g_m \} \leq f_0, \quad m = 1, \infty, \quad P \in M. \tag{68}$$

The last inequalities yield

$$E^P \{ f_{m \wedge \tau_s} + g_m \} = f_0, \quad m = 1, \infty, \quad P \in M. \tag{69}$$

The equalities (69), inequality $g^{0, \tau_s} \leq g$, and equalities

$$E^P \{ f_{m \wedge \tau_s} + g^0_{m \wedge \tau_s} \} = M_0 = f_0, \quad m = 1, \infty, \quad P \in M, \tag{70}$$

imply that $g^{0, \tau_s} = g$.

So, we proved that the stopped supermartingale $\{ f_{m \wedge \tau_s}, \mathcal{F}_m \}_{m=0}^\infty$ is regular one for every stopping time $\tau_s$, $s = 1, \infty$, converging to the infinity, as $s \to \infty$. This proves Theorem 6.

Theorem 7. Let a supermartingale $\{ f_m, \mathcal{F}_m \}_{m=0}^\infty$ relative to a convex set of equivalent measures $M$ on a measurable space $\{ \Omega, \mathcal{F} \}$ belongs to a class $F$ and there exists a nonnegative adapted random process $\{ g^0_m \}_{m=1}^\infty$, $E^P g^0_m < \infty$, $m = 1, \infty, \quad P \in M$, such that

$$f_{m-1} - E^P \{ f_m | \mathcal{F}_{m-1} \} = E^P \{ g^0_m | \mathcal{F}_{m-1} \}, \quad m = 1, \infty, \quad P \in M, \tag{71}$$

then $\{ f_m, \mathcal{F}_m \}_{m=0}^\infty$ is a local regular supermartingale.
**Proof.** To prove Theorem 7 let us consider a random process

\[
\bar{M}_m = f_m + \sum_{i=1}^{m} \tilde{g}_i^0, \quad m = 1, \infty, \quad P \in M, \quad f_0 = \bar{M}_0.
\]

It is evident that \( E^P|\bar{M}_m| < \infty, \quad m = 1, \infty, \quad P \in M, \) and \( E^P\{\bar{M}_m|F_{m-1}\} = \bar{M}_{m-1}, \quad m = 1, \infty, \quad P \in M. \) Therefore, for \( f_m \) the representation

\[
f_m = \bar{M}_m - g_m, \quad m = 0, \infty,
\]

is valid, where \( g_m = \sum_{i=1}^{m} \tilde{g}_i^0. \) Supermartingale (72) satisfies conditions of the

Theorem 6. The Theorem 7 is proved. \(\square\)

Below we describe local regular supermartingales. For this we need some auxiliary statements. Denote by \( N_0 = \{1, 2, \ldots, \infty\} \) the set of positive natural numbers.

On a measurable space \( \{\Omega, \mathcal{F}\} \) let us consider two sub \( \sigma \)-algebras \( G_n \subset G_N \) of \( \sigma \)-algebra \( \mathcal{F}. \) We suppose that for \( N > n \) \( \sigma \)-algebra \( G_N \) is generated by sets \( E_s, \quad s = 1, \infty, \) satisfying conditions \( E_j \cap E_m = \emptyset, \quad j \neq m, \quad \bigcup_{s=1}^{\infty} E_s = \Omega. \)

We assume that \( G_n \) is generated by sets \( F_j, \quad j = 1, \infty, \) satisfying conditions \( F_j \cap F_m = \emptyset, \quad j \neq m, \quad \bigcup_{j=1}^{\infty} F_j = \Omega, \) and such that \( F_j = \bigcup_{s \in I_j} E_s, \quad j = 1, \infty, \) where \( I_j \) are subsets of the set \( N_0, \quad I_r \cap I_l = \emptyset, \quad r \neq l, \quad \bigcup_{j=1}^{\infty} I_j = N_0. \)

**Lemma 5.** Let \( P_1, \ldots, P_k \) be a set of equivalent measures on a measurable space \( \{\Omega, \mathcal{F}\}. \) If \( P_1(E_s) > 0, \quad s = 1, \infty, \) then the formulas

\[
E^{P_l}\left\{ \frac{dP_i}{dP_l} | G_N \right\} = \sum_{j=1}^{\infty} \sum_{s \in I_j} \frac{P_i(E_s)}{P_l(F_j)} \frac{P_l(F_j)}{P_i(E_s)} \chi_{E_s}(\omega), \quad l = 1, k,
\]

are valid.

**Proof.** It is evident that

\[
E^{P_l}\left\{ \frac{dP_i}{dP_l} | G_N \right\} = \sum_{s=1}^{\infty} \frac{1}{P_l(E_s)} \int_{E_s} \frac{dP_i}{dP_l} dP_l \chi_{E_s}(\omega) = \sum_{s=1}^{\infty} \frac{P_i(E_s)}{P_l(E_s)} \chi_{E_s}(\omega),
\]

\[
E^{P_l}\left\{ \frac{dP_i}{dP_l} | G_n \right\} = \sum_{j=1}^{\infty} \frac{P_i(F_j)}{P_l(F_j)} \chi_{F_j}(\omega).
\]

Since \( \chi_{F_j}(\omega) = \sum_{s \in I_j} \chi_{E_s}(\omega) \) we have

\[
E^{P_l}\left\{ \frac{dP_i}{dP_l} | G_n \right\} = \sum_{j=1}^{\infty} \sum_{s \in I_j} \frac{P_i(F_j)}{P_l(F_j)} \chi_{E_s}(\omega).
\]
Therefore,

\[
E^{P_i} \left\{ \frac{dP_i}{dP_i|G_N} | G_N \right\} = \sum_{s=1}^\infty \frac{P_i(E_s)}{P_i(E_s)} \chi_{E_s}(\omega) = \sum_{j=1}^\infty \sum_{s \in I_j} \frac{P_i(E_s)}{P_i(F_j)} \chi_{E_s}(\omega) = \sum_{j=1}^\infty \sum_{s \in I_j} \frac{P_i(E_s)}{P_i(F_j)} \chi_{E_s}(\omega).
\]  

(77)

The Lemma 5 is proved.

**Lemma 6.** Let a set of equivalent measures \( P_1, \ldots, P_k \) on \( \{\Omega, \mathcal{F}\} \) are such that for a certain \( 1 \leq i_0 \leq k \) conditional measures \( \frac{P_i(A_s)}{P_i(F_j)} \), \( A_s \subseteq F_j, j = 1, \infty, i = \overline{1, k} \), satisfy conditions

\[
\frac{P_i(A_s)}{P_i(F_j)} \leq \frac{P_{i_0}(A_s)}{P_{i_0}(F_j)}, A_s \subseteq F_j, \bigcup_{s \in I_j} A_s = F_j, j = 1, \infty, i = \overline{1, k}.
\]

(78)

Then the inequalities

\[
E^{P_i} \left\{ \frac{dP_i}{dP_i|G_n} | G_n \right\} \leq \frac{E^{P_{i_0}} \left\{ dP_{i_0} \right\}}{E^{P_{i_0}} \left\{ dP_{i_0} \right\}|G_n}, \quad i = \overline{1, k}, \quad l = \overline{1, k},
\]

(79)

are valid.

**Proof.** The proof of the Lemma 6 follows from the formulas (77).

**Definition 5.** A filtration \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1}, n = \overline{1, \infty} \), on a measurable space \( \{\Omega, \mathcal{F}\} \) satisfies **condition A**, if

1) \( \sigma \)-algebra \( \mathcal{F} \) coincides with minimal \( \sigma \)-algebra generated by the sets belonging to the set \( \bigcup_{n=0}^\infty \mathcal{F}_n \);

2) \( \mathcal{F}_n \) is generated by sets \( A_s^n \subseteq \mathcal{F} \), \( s = \overline{1, \infty}, n = \overline{1, \infty} \), such that

\[
A^n_m \cap A^n_j = \emptyset, \quad m \neq j, \quad \bigcup_{s=1}^\infty A^n_s = \Omega, \quad A^n_s = \bigcup_{j \in I^n_s} A^n_{j+1}, \quad s = \overline{1, \infty}
\]

\[
I^n_s \cap I^n_m = \emptyset, \quad s \neq m, \quad \bigcup_{s=1}^\infty I^n_s = N_0, \quad n = \overline{1, \infty}.
\]

**Definition 6.** On a measurable space \( \{\Omega, \mathcal{F}\} \) with filtration \( \mathcal{F}_n \) satisfying **condition A** a set of equivalent measures \( P_1, \ldots, P_k \) satisfies **condition B** if

\[
P_i(A^n_s) > 0, \quad s = \overline{1, \infty}, \quad n = \overline{1, \infty},
\]
and for a certain $1 \leq i_0 \leq k$ the inequalities

$$\frac{P_i(A_{j}^{n+1})}{P_i(A_{\infty}^{n})} \leq \frac{P_{i_0}(A_{j}^{n}+1)}{P_{i_0}(A_{\infty}^{n})}, \quad j \in I^n, \quad n = 1, \infty,$$

are valid.

**Lemma 7.** Let a filtration $\mathcal{F}_n$ and a set of equivalent measures $P_1, \ldots, P_k$ on a measurable space $\{\Omega, \mathcal{F}\}$ satisfy conditions $A$ and $B$, correspondingly. Then for every $1 \leq l \leq k$ and $1 \leq n \leq \infty$ the inequalities

$$\frac{dP_l}{dP_i}(F_{n}) \leq \frac{dP_l}{dP_{i_0}}(F_{n}), \quad i = 1, k, \quad l = 1, k,$$

are valid.

**Proof.** Taking into account Lemma 6 for every $1 \leq l \leq k$ and $N \geq n \geq 1$ we obtain the inequalities

$$\frac{E(P_{l}|F_{n})}{E(P_{l}|F_{N})} \leq \frac{E(P_{l}|F_{n})}{E(P_{l}|F_{N})}, \quad i = 1, k, \quad l = 1, k. \quad (80)$$

Since a random value $\frac{dP_{l}}{dP_{i}}$ is measurable one relative to the $\sigma$-algebra $\mathcal{F}$ and integrable with respect to the measure $P_{l}$, then the conditions of Levy Theorem are valid. It implies that with probability 1 $\lim_{N \to \infty} E(P_{l}|F_{n}) = \frac{dP_{l}}{dP_{i}}$. Passing to the limit in the inequalities (81), as $N \to \infty$, we obtain the inequalities (80) and the proof of Lemma 7.

Let $P_1, \ldots, P_k$ be a family of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$ and let us introduce denotation

$$M = \left\{ Q, \quad Q = \sum_{i=1}^{k} \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 1, k, \quad \sum_{i=1}^{k} \alpha_i = 1 \right\}. \quad (82)$$

**Lemma 8.** If $\xi$ is an integrable random value relative to the set of equivalent measures $P_1, \ldots, P_k$, then the formula

$$\sup_{Q \in M} E(Q|x|F_{n}) = \max_{1 \leq i \leq k} E(P_{i}|x|F_{n}) \quad (82)$$

is valid almost everywhere relative to the measure $P_1$.

**Proof.** Using the formula

$$E(Q|x|F_{n}) = \frac{\sum_{i=1}^{k} \alpha_i E(P_{i}|x|F_{n}) E(P_{i}|x|F_{n})}{\sum_{i=1}^{k} \alpha_i E(P_{i}|x|F_{n})}, \quad Q \in M, \quad (83)$$

we obtain the inequalities (80) and the proof of Lemma 7.
where $\varphi_i = \frac{dP_i}{dP_1}$, we obtain the inequality
\[
E^Q\{\xi | F_n\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi | F_n\},
\]
or
\[
\sup_{Q \in M} E^Q\{\xi | F_n\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi | F_n\}.
\]
On the other side
\[
E^{P_i}\{\xi | F_n\} \leq \sup_{Q \in M} E^Q\{\xi | F_n\}.
\]
Therefore,
\[
\max_{1 \leq i \leq k} E^{P_i}\{\xi | F_n\} \leq \sup_{Q \in M} E^Q\{\xi | F_n\}.
\]
The Lemma 8 is proved.

**Lemma 9.** Let $G$ be a sub $\sigma$-algebra of $F$ and $f_1, \ldots, f_n$ be nonnegative integrable random values relative to every measure from $M$. Then
\[
E^P\{\max\{f_1, \ldots, f_n\} | G\} \geq \max\{E^P\{f_1 | G\}, \ldots, E^P\{f_n | G\}\}, \quad P \in M. \quad (84)
\]
**Proof.** From inequalities
\[
\max_{1 \leq i \leq n} f_i \geq f_j, \quad j = 1, n,
\]
we have
\[
E^P\{\max_{1 \leq i \leq n} f_i | G\} \geq E^P\{f_j | G\}, \quad j = 1, n. \quad (86)
\]
The last imply
\[
E^P\{\max_{1 \leq i \leq n} f_i | G\} \geq \max_{1 \leq i \leq n} E^P\{f_i | G\}. \quad (87)
\]

In the next Lemma we present formula for calculation of conditional expectation relative to another measure from $M$.

**Lemma 10.** Let $M$ be a convex set of equivalent measures and let $\eta$ be an integrable random value relative to every measure from $M$ on a measurable space $\{\Omega, F\}$. Then the following formula
\[
E^{P_1}\{\eta | F_n\} = E^{P_2}\{\eta \varphi^{P_1}_{P_2} | F_n\}, \quad n = 1, \infty, \quad (88)
\]
is valid, where
\[
\varphi^{P_1}_{P_2} = \frac{dP_1}{dP_2} \left[ E^{P_2}\left\{\frac{dP_1}{dP_2} | F_n\right\}\right]^{-1}, \quad P_1, P_2 \in M.
\]
**Proof.** The proof of Lemma 10 is evident.
Lemma 11. Suppose that a filtration $\mathcal{F}_n$ and a set of equivalent measures $\{P_1, \ldots, P_k\}$ on $\{\Omega, \mathcal{F}\}$ satisfy conditions A and B, correspondingly. Let $\xi$ be a nonnegative bounded random value on a measurable space $\{\Omega, \mathcal{F}\}$. Then the formulae

$$E^{P_l}\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} = \max_{1 \leq i \leq k} E^{P_l}\{\xi\varphi_m^P|\mathcal{F}_m\}, \quad n > m, \quad l = 1, k, \quad (89)$$

are valid, where

$$\varphi_m^P = \frac{dP_i}{dP_l}\left[ E^{P_l}\left\{\frac{dP_i}{dP_l}\right\}\right]^{-1}.$$

Proof. From Lemma 10 we obtain

$$\max_{1 \leq i \leq k} E^{P_l}\{\xi|\mathcal{F}_n\} = \max_{1 \leq i \leq k} E^{P_l}\{\xi\varphi_m^P|\mathcal{F}_n\}, \quad l = 1, k.$$

Let us introduce denotation $T_i = \xi\varphi_m^P$. Then $T_i$ is an integrable random value and

$$\max_{1 \leq i \leq k} E^{P_l}\{\xi|\mathcal{F}_n\} = \max_{1 \leq i \leq k} E^{P_l}\{\xi\varphi_m^P|\mathcal{F}_n\} = \max_{1 \leq i \leq k} E^{P_l}\{T_i|\mathcal{F}_n\}, \quad l = 1, k.$$

Due to Lemma 9 we obtain the inequality

$$E^{P_l}\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} = E^{P_l}\{\max_{1 \leq i \leq k} E^{P_i}\{T_i|\mathcal{F}_n\}|\mathcal{F}_m\} \geq \max_{1 \leq i \leq k} E^{P_l}\{E^{P_i}\{T_i|\mathcal{F}_n\}|\mathcal{F}_m\} = \max_{1 \leq i \leq k} E^{P_l}\{T_i|\mathcal{F}_m\}.$$

Let us prove reciprocal inequality

$$E^{P_l}\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} \leq \max_{1 \leq i \leq k} E^{P_l}\{T_i|\mathcal{F}_m\}.$$

The last inequality follows from the fact that $\max_{1 \leq i \leq k} E^{P_l}\{T_i|\mathcal{F}_n\} = E^{P_l}\{T_{i_0}|\mathcal{F}_n\}$. Really,

$$E^{P_l}\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} = E^{P_l}\{E^{P_i}\{T_{i_0}|\mathcal{F}_n\}|\mathcal{F}_m\} = E^{P_l}\{T_{i_0}|\mathcal{F}_m\} \leq \max_{1 \leq i \leq k} E^{P_l}\{T_i|\mathcal{F}_m\}.$$

Lemma 11 is proved.

The next Lemma is a consequence of Lemma 11.

Lemma 12. Let a filtration $\mathcal{F}_n$ and the set of equivalent measures $\{P_1, \ldots, P_k\}$ on a measurable space $\{\Omega, \mathcal{F}\}$ satisfy conditions A and B, correspondingly and let $\xi$ be a nonnegative bounded random value on $\{\Omega, \mathcal{F}\}$. Then the equalities

$$E^{P_l}\{\xi \varphi_m^P|\mathcal{F}_n\} = \max_{1 \leq i \leq k} E^{P_l}\{\xi\varphi_m^P|\mathcal{F}_n\}, \quad l = 1, k, \quad n = 0, \infty, \quad (90)$$

are valid, where

$$\varphi_m^P = \frac{dP_i}{dP_l}\left[ E^{P_l}\left\{\frac{dP_i}{dP_l}\right\}\right]^{-1}.$$
Proof.

\[
\max_{1 \leq i \leq k} E^{P_i} \{\xi \varphi^P_n|\mathcal{F}_n\} \leq E^{P_i} \{\xi \max_{1 \leq i \leq k} \varphi^P_n|\mathcal{F}_n\} \leq E^{P_i} \{\xi \varphi^P_{n_0}|\mathcal{F}_n\} \leq \max_{1 \leq i \leq k} E^{P_i} \{\xi \varphi^P_n|\mathcal{F}_n\}, \quad l = 1, k, \; n = 0, \infty. \tag{91}
\]

The last inequalities prove Lemma 12. \qed

Lemma 13. Let a filtration \(\mathcal{F}_n\) and a set of equivalent measures \(\{P_1, \ldots, P_k\}\) on a measurable space \((\Omega, \mathcal{F})\) satisfy conditions \(A\) and \(B\), correspondingly. Then for every nonnegative integrable random value \(\xi\) relative to the set of measures \(P_1, \ldots, P_k\) the inequalities

\[
E^{P_l} \{\max_{1 \leq i \leq k} E^{P_i} \{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} \leq \max_{1 \leq i \leq k} E^{P_l} \{\xi|\mathcal{F}_m\}, \quad n > m, \; l = 1, k, \tag{92}
\]

are valid.

Proof. First, consider the case of bounded nonnegative random value \(\xi\). It is evident that the following equalities

\[
\bigcup_{i=1}^{k} \{\omega, \; E^{P_i} \left\{\frac{dP_i}{dP_l}\big|\mathcal{F}_n\right\} \geq E^{P_i} \left\{\frac{dP_i}{dP_l}\big|\mathcal{F}_m\right\}\} = \Omega, \; n > m, \tag{93}
\]

are valid. Due to (93) for every \(\omega \in \Omega\) there exist \(1 \leq i \leq k\) such that

\[
\frac{\xi \frac{dP_i}{dP_l}}{E^{P_i} \{\frac{dP_i}{dP_l}\big|\mathcal{F}_n\}} \leq \frac{\xi \frac{dP_i}{dP_l}}{E^{P_i} \{\frac{dP_i}{dP_l}\big|\mathcal{F}_m\}}. \tag{94}
\]

Therefore,

\[
\max_{1 \leq i \leq k} \frac{\xi \frac{dP_i}{dP_l}}{E^{P_i} \{\frac{dP_i}{dP_l}\big|\mathcal{F}_n\}} \leq \max_{1 \leq i \leq k} \frac{\xi \frac{dP_i}{dP_l}}{E^{P_i} \{\frac{dP_i}{dP_l}\big|\mathcal{F}_m\}}. \tag{95}
\]

From (95) we obtain inequality

\[
E^{P_l} \left\{\max_{1 \leq i \leq k} \frac{\xi \frac{dP_i}{dP_l}}{E^{P_i} \{\frac{dP_i}{dP_l}\big|\mathcal{F}_n\}}|\mathcal{F}_m\right\} \leq E^{P_l} \left\{\max_{1 \leq i \leq k} \frac{\xi \frac{dP_i}{dP_l}}{E^{P_i} \{\frac{dP_i}{dP_l}\big|\mathcal{F}_m\}}|\mathcal{F}_m\right\}. \tag{96}
\]

The Lemmas 11, 12 and inequality (96) prove Lemma 13, as \(\xi\) is bounded random value. Let us consider the case as \(\max_{1 \leq i \leq k} E^{P_i} \xi < \infty\). Let \(\xi_s, s = 1, \infty, \) be a sequence of bounded random values converging to \(\xi\) monotonously. Then

\[
E^{P_l} \{\max_{1 \leq i \leq k} E^{P_i} \{\xi_s|\mathcal{F}_n\}|\mathcal{F}_m\} \leq \max_{1 \leq i \leq k} E^{P_l} \{\xi_s|\mathcal{F}_m\}, \; l = 1, k. \tag{97}
\]

Due to monotony convergence of \(\xi_s\) to \(\xi\), as \(s \to \infty\), we can pass to the limit under conditional expectations on the left and on the right in inequalities (97) that proves Lemma 13. \qed
Lemma 14. Let a filtration $F_n$ and a set of equivalent measures $\{P_1, \ldots, P_k\}$ on a measurable space $\{\Omega, F\}$ satisfy conditions $A$ and $B$, correspondingly, and let $\xi$ be an integrable random value relative to the set of equivalent measures $P_1, \ldots, P_k$. Then the inequalities

$$E^Q\left\{ \sup_{P \in M} E^P\{\xi|F_n\}|F_m\right\} \leq \sup_{P \in M} E^P\{\xi|F_m\}, \quad n > m, \quad Q \in M,$$

are valid.

**Proof.** From the equality

$$\sup_{Q \in M} E^Q\{\xi|F_n\} = \max_{1 \leq i \leq k} E^{P_i}\{\xi|F_n\}$$

we obtain inequality

$$E^Q\left\{ \max_{1 \leq i \leq k} E^{P_i}\{\xi|F_n\}|F_m\right\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi|F_m\} = \sup_{P \in M} E^P\{\xi|F_m\}.$$

Lemma 14 is proved.

Lemma 15. Let a filtration $F_n$ and a set of equivalent measures $\{P_1, \ldots, P_k\}$ on a measurable space $\{\Omega, F\}$ satisfy conditions $A$ and $B$, correspondingly, and let $\xi$ be a nonnegative integrable random value with respect to this set of measures and such that

$$E^{P_i}\xi = M_0, \quad i = 1, k, \quad (98)$$

then the random process $\{M_m = \sup_{P \in M} E^P\{\xi|F_m\}, F_m\}_{m=0}^\infty$ is a martingale relative to a convex set of equivalent measures $M$.

**Proof.** Due to Lemma 14 a random process $\{M_m = \sup_{P \in M} E^P\{\xi|F_m\}, F_m\}_{m=0}^\infty$ is a supermartingale, that is,

$$E^P\{M_m|F_{m-1}\} \leq M_{m-1}, \quad m = 1, \infty, \quad P \in M.$$

Or, $E^P M_m \leq M_0$. From the other side

$$E^{P_s}\left\{ \max_{1 \leq i \leq k} E^{P_i}\{\xi|F_m\}\right\} \geq \max_{1 \leq i \leq k} E^{P_s} E^{P_i}\{\xi|F_m\} \geq M_0, \quad s = 1, k.$$

The above inequalities imply $E^{P_s} M_m = M_0, \quad m = 1, \infty, \quad s = 1, k$. The last equalities lead to equalities $E^P M_m = M_0, \quad m = 1, \infty, \quad P \in M$. The fact that $M_m$ is a supermartingale relative to the set of measures $M$ and the above equalities prove Lemma 15.
Theorem 8. Let a filtration $\mathcal{F}_n$ and a set of equivalent measures $\{P_1, \ldots, P_k\}$ on a measurable space $\{\Omega, \mathcal{F}\}$ satisfy conditions $A$ and $B$, correspondingly. Suppose that $\xi$ is a nonnegative integrable random value relative to this set of measures. If $\xi$ is $\mathcal{F}_N$-measurable one for a certain $N < \infty$, then a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$, where

$$f_m = \sup_{P \in M} E^P \{\xi | \mathcal{F}_m\}, \quad m = 1, \infty, \quad \max_{1 \leq i \leq k} E^{P_i} \xi < \infty,$$

is local regular one if and only if

$$E^{P_i} \xi = f_0, \quad i = 1, k.$$  \hspace{0.5cm} (99)

Proof. The necessity. Let $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ be a local regular supermartingale. Then there exists a sequence of nonrandom stopping times $\tau_s = n_s, s = 1, \infty$, such that for every $n_s$ there exists $\varphi = \sum_{m=1}^{n_s} \sum_{i=1}^{k} E^{P_i} \{\xi | \mathcal{F}_m\}$ satisfying inequalities

$$\max_{1 \leq j \leq k} E^{P_j} \varphi \leq \sum_{m=1}^{n_s} \sum_{i=1}^{k} \max_{1 \leq j \leq k} E^{P_j} E^{P_i} \{\xi | \mathcal{F}_m\} \leq \sum_{m=1}^{n_s} \sum_{i=1}^{k} \max_{1 \leq j \leq k} E^{P_j} \max_{1 \leq i \leq k} E^{P_i} \xi = n_s k \max_{1 \leq i \leq k} E^{P_i} \xi,$$

$$\sup_{P \in M} E^P \varphi \leq \max_{1 \leq j \leq k} E^{P_j} \varphi \leq n_s k \max_{1 \leq i \leq k} E^{P_i} \xi;$$

and nonnegative adapted random process $\{\bar{g}^0_m\}_{m=0}^{\infty}, \bar{g}^0_0 = 0, E^{P_i} \bar{g}^0_m < \infty, 0 \leq m \leq n_s$ such that

$$f_m + \sum_{i=1}^{m} \bar{g}^0_i = \bar{M}_m, \quad E^P \bar{M}_m = f_0, \quad 0 \leq m \leq n_s, \quad P \in M.$$  \hspace{0.5cm} (100)

If $n_s > N$, then

$$E^{P_i} (\xi + \sum_{i=1}^{N} \bar{g}^0_i) = E^{P_i} \xi + E^{P_i} \sum_{i=1}^{N} \bar{g}^0_i = f_0.$$

But there exists $1 \leq i_1 \leq k$ such that $E^{P_{i_1}} \xi = f_0$. Therefore, $E^{P_{i_1}} \sum_{i=1}^{N} \bar{g}^0_i = 0.$

Due to equivalence of measures $P_i, \ i = 1, k$, we obtain

$$E^{P_i} \xi = f_0, \quad i = 1, k,$$  \hspace{0.5cm} (100)
where $f_0 = \sup_{P \in {\mathcal M}} E^P \xi$.

**Sufficiency.** If conditions (100) are satisfied, then $\bar{M}_m = \sup_{P \in {\mathcal M}} E^P \{ \xi | {\mathcal F}_m \}$ is a martingale. The last implies local regularity of $\{f_m, {\mathcal F}_m \}_{m=0}^\infty$. The Theorem 8 is proved.

**Theorem 9.** Let a filtration $\mathcal F_n$ on a measurable space $\{ \Omega, \mathcal F \}$ satisfies condition A and let $M$ be a convex set of equivalent measures on this measurable space. Suppose that

$$P(A^n_s) > 0, \quad P \in M, \quad s = 1, \infty, \quad n = 1, \infty,$$

and for a certain measure $P_{i_0} \in M$ the inequalities

$$\frac{P(A^{n+1}_j)}{P(A^n_s)} \leq \frac{P_{i_0}(A^{n+1}_j)}{P_{i_0}(A^n_s)}, \quad i = 1, k, \quad A^{n+1}_j \subseteq A^n_s, \quad A^n_s = \bigcup_{j \in I^n} A^{n+1}_j, \quad P \in M,$$

are valid. If $G_0$ is a set of all integrable nonnegative random values $\xi$ satisfying conditions

$$E^P \xi = 1, \quad P \in M, \quad (101)$$

then the random process $\{E^P \{ \xi | {\mathcal F}_m \}, {\mathcal F}_m \}_{m=0}^\infty, \quad \xi \in G_0$, is a local regular supermartingale.

**Proof.** Let $P_1, \ldots, P_n$ be a certain subset of measures from $M$ containing the measure $P_{i_0}$. Denote by $M_n$ a convex set of equivalent measures

$$M_n = \{ P \in M, \quad P = \sum_{i=1}^n \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 1, n, \quad \sum_{i=1}^n \alpha_i = 1 \}. \quad (102)$$

Due to Lemma 15 $\{\bar{M}_m\}_{m=0}^\infty$ is a martingale relative to the set of measures $M_n$, where $\bar{M}_m = \sup_{P \in M_n} E^P \{ \xi | {\mathcal F}_m \}, \quad \xi \in G_0$. Let us consider an arbitrary measure $P_0 \in M$ and let

$$\tilde{M}_m^{P_0} = \{ P \in M, \quad P = \sum_{i=0}^n \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 0, n, \quad \sum_{i=0}^n \alpha_i = 1 \}. \quad (103)$$

Then $\{\tilde{M}_m^{P_0}\}_{m=0}^\infty$, where $\tilde{M}_m^{P_0} = \sup_{P \in M_n^{P_0}} E^P \{ \xi | {\mathcal F}_m \}$, is a martingale relative to the set of measures $M_n^{P_0}$. It is evident that

$$\bar{M}_m \leq \tilde{M}_m^{P_0}, \quad m = 0, \infty. \quad (104)$$

Since $E^P \bar{M}_m = E^P \tilde{M}_m^{P_0} = 1, \quad m = 0, \infty, \quad P \in M_n$, the inequalities (104) give $\bar{M}_m = \tilde{M}_m^{P_0}$. Analogously, $E^{P_0} \{ \xi | {\mathcal F}_m \} \leq \tilde{M}_m^{P_0}$. From equalities $E^{P_0} E^P \{ \xi | {\mathcal F}_m \} = E^{P_0} \tilde{M}_m^{P_0} = 1$ we obtain $E^{P_0} \{ \xi | {\mathcal F}_m \} = M_m^{P_0} = \bar{M}_m$. Since the measure $P_0$ is arbitrary it implies that $E^P \{ \xi | {\mathcal F}_m \}, \quad m = 0, \infty$, is a martingale relative to all measures from $M$. Due to Theorem 7, it is a local regular supermartingale with random process $g_m^0 = 0, m = 0, \infty$. The Theorem 9 is proved. \qed
Theorem 10. Let a filtration $\mathcal{F}_n$ on a measurable space $\{\Omega, \mathcal{F}\}$ satisfies condition $A$ and let $M$ be a convex set of equivalent measures on this measurable space. Suppose that

$$P(A_s^n) > 0, \quad P \in M, \quad s = \overline{1, \infty}, \quad n = \overline{1, \infty},$$

and for a certain measure $P_{i_0} \in M$ the inequalities

$$\frac{P(A_{j+1}^n)}{P(A_s^n)} \leq \frac{P_{i_0}(A_{j+1}^n)}{P_{i_0}(A_s^n)}, \quad i = 1, k, \quad A_{j+1}^n \subseteq A_s^n, \quad A_s^n = \bigcup_{j \in I^n} A_{j+1}^n, \quad P \in M,$$

are valid. If $\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}$ is an adapted random process satisfying conditions

$$f_m \leq f_{m-1}, \quad E^P|\xi| < \infty, \quad P \in M, \quad m = \overline{1, \infty}, \quad \xi \in G_0, \tag{105}$$

where $G_0 = \{\xi \geq 0, E^P\xi = 1, P \in M\}$, then the random process

$$\{f_mE^P(\xi|\mathcal{F}_m), \mathcal{F}_m\}_{m=0}^{\infty}, \quad P \in M, \tag{106}$$

is a local regular supermartingale relative to all measures from $M$.

**Proof.** Due to Theorem 9, the random process $\{E^P(\xi|\mathcal{F}_m), \mathcal{F}_m\}_{m=0}^{\infty}$ is a martingale relative to all measures from $M$. Therefore,

$$f_{m-1}E^P(\xi|\mathcal{F}_{m-1}) - E^P\{f_mE^P(\xi|\mathcal{F}_m)|\mathcal{F}_{m-1}\} =$$

$$E^P((f_{m-1} - f_m)E^P(\xi|\mathcal{F}_m)|\mathcal{F}_{m-1}), \quad m = \overline{1, \infty}. \tag{107}$$

So, if to put $\tilde{g}_m^0 = (f_{m-1} - f_m)E^P(\xi|\mathcal{F}_m), \quad m = \overline{1, \infty}$, then $\tilde{g}_m^0 \geq 0$ and it is $\mathcal{F}_m$-measurable and $E^P\tilde{g}_m^0 \leq E^P\xi(|f_{m-1}| + |f_m|) < \infty$. It proves the needed statement. $\square$

**Corollary 1.** If $f_m = \alpha, \quad m = \overline{1, \infty}, \quad \alpha \in R^1$, then $\{\alpha E^P(\xi|\mathcal{F}_m)\}_{m=0}^{\infty}$ is a local regular supermartingale. If $\xi = 1$, then $\{f_m\}_{m=0}^{\infty}$ is also a local regular supermartingale.

Denote by $F_0$ the set of adapted processes

$$F_0 = \{f = \{f_m\}_{m=0}^{\infty}, \quad P(|f_m| < \infty) = 1, \quad P \in M, \quad f_m \leq f_{m-1}, \quad m = \overline{1, \infty}\}.$$

For every $\xi \in G_0$ let us introduce the set of adapted processes

$$L_\xi =$$

$$\{\tilde{f} = \{f_mE^P(\xi|\mathcal{F}_m)\}_{m=0}^{\infty}, \quad \{f_m\}_{m=0}^{\infty} \in F_0, \quad E^P\xi |f_m| < \infty, \quad P \in M, \quad m = \overline{1, \infty}\},$$

and

$$V = \bigcup_{\xi \in G_0} L_\xi.$$
Corollary 2. Every random adapted process from the set $K$, where

$$K = \left\{ \sum_{i=1}^{m} C_i \bar{f}_i, \bar{f}_i \in V, \ C_i \geq 0, \ i = \overline{1,m}, \ m = \overline{1,\infty} \right\},$$

is a local regular supermartingale.

Proof. The proof is evident. \qed

Theorem 11. Let a filtration $\mathcal{F}_n$ on a measurable space $\{\Omega, \mathcal{F}\}$ satisfies condition $A$ and let $M$ be a convex set of equivalent measures on this measurable space. Suppose that

$$P(A^n_s) > 0, \ P \in M, \ s = \overline{1,\infty}, \ n = \overline{1,\infty},$$

and for a certain measure $P_0 \in M$ the inequalities

$$\frac{P(A^{n+1}_j)}{P(A^n_s)} \leq \frac{P_0(A^{n+1}_j)}{P_0(A^n_s)}, \quad i = \overline{1,k}, \ A^{n+1}_j \subseteq A^n_s, \ A^n_s = \bigcup_{j \in I^n_s} A^{n+1}_j, \ P \in M,$$

are valid. If $\{f_m\}_{m=0}^{\infty}$ is a nonnegative uniformly integrable supermartingale relative to the set of measures from $M$, then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set $K$.

Proof. Necessity. It is evident if $\{f_m\}_{m=0}^{\infty}$ belongs to $K$ then it is a local regular supermartingale.

Sufficiency. Suppose that $\{f_m\}_{m=0}^{\infty}$ is a local regular supermartingale. Then there exists nonnegative adapted process $\{g_m^0\}_{m=1}^{\infty}$, $E^P g_m^0 < \infty, \ m = \overline{1,\infty}$, and a martingale $\{M_m\}_{m=0}^{\infty}$, such that

$$f_m = M_m - \sum_{i=1}^{m} g_i^0, \ m = 0, \infty.$$

Then $M_m \geq 0, \ m = 0, \infty, \ E^P M_m < \infty, \ P \in M$. Since $0 < E^P M_m = f_0 < \infty$ we have $E^P \sum_{i=1}^{m} g_i^0 < f_0$. Let us put $g_\infty = \lim_{m \to \infty} \sum_{i=1}^{m} g_i^0$. Using uniform integrability of $f_m$ we can pass to the limit in the equality

$$E^P(f_m + \sum_{i=1}^{m} g_i^0) = f_0, \ P \in M,$$

as $m \to \infty$. Passing to the limit in the last equality, as $m \to \infty$, we obtain

$$E^P(f_\infty + g_\infty) = f_0.$$

Introduce into consideration a random value $\xi = \frac{f_\infty + g_\infty}{f_0}$. Then $E^P \xi = 1, \ P \in M$. From here we obtain that $\xi \in G_0$ and

$$M_m = f_0 E^P(\xi|\mathcal{F}_m), \ m = 0, \infty.$$
Let us put \( \bar{f}_2^2 = -\sum_{i=1}^{m} \bar{g}_i^0 \). It is easy to see that an adapted random process \( \bar{f}_2 = \{ \bar{f}_2^m \}_{m=0}^{\infty} \) belongs to \( F_0 \). Therefore, for the supermartingale \( f = \{ f_m \}_{m=0}^{\infty} \) the representation

\[
f = \bar{f}_1 + \bar{f}_2,
\]

is valid, where \( \bar{f}_1 = \{ f_0 E^P \{ \xi | \mathcal{F}_m \} \}_{m=0}^{\infty} \) belongs to \( L_\xi \) with \( \xi = \frac{f_{\infty} + g_{\infty}}{f_0} \) and \( f_m = f_0, m = 0, \infty \). The same is valid for \( \bar{f}_2 \) with \( \xi = 1 \). This implies that \( f \) belongs to \( K \). The Theorem 11 is proved.

Below we present some results needed for the description of the set \( G_0 \). We consider the case, as conditions of the Theorems 9, 10, 11 are valid. Let us consider the set of equations for a certain fixed \( n \geq 1 \)

\[
\sum_{j=1}^{\infty} P_i(A_j^n) \xi_j = 1, \quad i = 1, k. \tag{108}
\]

If there exists nonnegative solution \( \{ \xi_j \}_{j=1}^{\infty} \) of the set of equations (108), then the random value \( \xi = \sum_{j=1}^{\infty} \xi_j \chi_{A_j^n} \) is \( \mathcal{F}_n \)-measurable and belongs to the set \( G_0 \).

If to put \( a_j = \{ P_i(A_j^n) \}_{j=1}^{k}, \quad j = 1, \infty \), then the set of equations (108) can be written in the form

\[
\sum_{j=1}^{\infty} a_j \xi_j = a_0 \tag{109}
\]

with the vector \( a_0 = \{ e_i \}_{i=1}^{k}, \quad e_i = 1, \quad i = 1, k \). It is evident that homogeneous set of equations

\[
\sum_{j=1}^{\infty} a_j \xi_j = 0 \tag{110}
\]

has always a bounded nonzero solution. Then if to denote it by \( u = \{ u_j \}_{j=1}^{\infty} \), then due to boundedness of this solution, that is, \( |u_j| \leq C < \infty, \quad j = 1, \infty \), there exists a real number \( t > 0 \) such that \( \xi_j = 1 - tu_j \geq 0, \quad j = 1, \infty \). Such a vector \( \{ \xi_j \}_{j=1}^{\infty} \) is a nonhomogeneous nonnegative solution to the set of equations (109).

Below we prove Theorem 12 helping us to describe strictly positive solutions of the set equations (109).

**Definition 7.** A vector \( a_0 \in R^k_+ \) belongs to the interior of the nonnegative cone generated by vectors \( a_j \in R^k_+, \quad j = 1, \infty \), if there exist positive numbers \( \alpha_j > 0, \quad j = 1, \infty \), such that

\[
\sum_{j=1}^{\infty} \alpha_j a_j = a_0. \tag{111}
\]
The next Theorem generalizes a Theorem from [6] and describes all strictly positive solutions to the set of equations (109).

**Theorem 12.** Let a vector $a_0$ belongs to the interior of the cone generated by vectors $a_j \in \mathbb{R}^k$, $j = 1, \infty$, were dimension of the cone is $1 \leq r \leq k$, and let $r$ linear independent vectors $a_1, \ldots, a_r$ be such that the vector $a_0$ belongs to the interior of the cone generated by these vectors. Then there exists infinite number of linear independent nonnegative solutions $z_i$, $i = r, \infty$, of the set of equations (109), where

$$z_r = \{\langle a_0, f_1 \rangle, \ldots, \langle a_0, f_r \rangle, 0, 0, \ldots, \},$$

$$z_i = \{\langle a_0, f_1 \rangle - \langle a_i, f_1 \rangle y_i^*, \ldots, \langle a_0, f_r \rangle - \langle a_i, f_r \rangle y_i^*, 0, \ldots, 0, y_i^*, 0, \ldots, \},$$

$$y_i^* = \begin{cases}
\min_{l \in K_i} \frac{\langle a_0, f_l \rangle}{\langle a_i, f_l \rangle}, & K_i = \{l, \langle a_i, f_l \rangle > 0\}, \\
1, & \langle a_i, f_l \rangle \leq 0, \forall l = 1, r,
\end{cases}$$

\{f_1, \ldots, f_k\} is a set of linear independent vectors satisfying conditions

$$\langle f_i, a_j \rangle = \delta_{ij}, \quad i, j = 1, r, \quad \langle f_i, a_j \rangle = 0, \quad j = 1, r, \quad i = r + 1, k. \quad (112)$$

The set of strictly positive solutions of the set of equations (109) is given by the formula

$$z = \sum_{i=r}^{\infty} \gamma_i z_i, \quad (113)$$

where the vector $\gamma = \{\gamma_r, \ldots, \gamma_i, \ldots, \}$ satisfies conditions

$$\sum_{i=r}^{\infty} \gamma_i = 1, \quad \gamma_i > 0, \quad i = r + 1, \infty, \quad \sum_{i=r+1}^{\infty} a_i \gamma_i y_i^* < \infty,$$

$$\left\langle a_0 - \sum_{i=r+1}^{\infty} a_i \gamma_i^* y_i^*, f_k \right\rangle > 0, \quad k = 1, \infty. \quad (114)$$

**Proof.** In the Theorem 12 without loss of generality we assume that $r$ linear independent vectors $a_1, \ldots, a_r$ are such that the vector $a_0$ belongs to the interior of the cone generated by these vectors. If it is not the case and such vectors are $a_{i1}, \ldots, a_{ir}$, then by the renumbering the set of the vectors $a_j$, $j = 1, \infty$, we come to the case of the Theorem 12.

Let us indicate the necessary conditions of the existence of strictly positive solution to the set of equations (109). Due to existence of nonnegative solution of (109), the series $\sum_{i=1}^{\infty} \xi_i a_i$ is convergent one. Since $a_i \in \mathbb{R}_+^d$ we have that the series $\sum_{i=r+1}^{\infty} \xi_i a_i$ is also convergent one. Denote by \{f_1, \ldots, f_d\} a set of
vectors that satisfy conditions (112). We obtain that a set of equations (109) is equivalent to the set of equations

$$\left\langle \sum_{i=r+1}^{\infty} \xi_i a_i, f_j \right\rangle + \xi_j = \langle a_0, f_j \rangle > 0, \quad j = 1, r.$$  

(115)

where $\langle a, b \rangle$ denotes a scalar product of vectors $a$ and $b$. From here we have

$$\left\langle a_0 - \sum_{i=r+1}^{\infty} \xi_i a_i, f_j \right\rangle = \xi_j, \quad j = 1, r.$$  

(116)

It implies that inequalities

$$\left\langle a_0 - \sum_{i=r+1}^{\infty} \xi_i a_i, f_j \right\rangle > 0, \quad j = 1, r,$$  

(117)

are valid. If strictly positive vector $\{\xi_{r+1}, \ldots, \xi_m, \ldots\}$ is such that the series $\sum_{i=r}^{\infty} \xi_i a_i$ is convergent one and inequalities (117) are valid, then the vector

$$z = \left\{ \left\langle a_0 - \sum_{i=r+1}^{\infty} \xi_i a_i, f_1 \right\rangle, \ldots, \left\langle a_0 - \sum_{i=r+1}^{\infty} \xi_i a_i, f_r \right\rangle, \xi_{r+1}, \ldots, \xi_l, \ldots \right\}$$

is a general strictly positive solution of the set of equations (109). It is evident that nonnegative solution $z_l$ we obtain from the general strictly positive solution of (109), if to put $\{\xi_{r+1}, \ldots, \xi_m, \ldots\}$ such that $\xi_i = 0, \ i \neq l, \xi_l = y_l^*$. These solutions are nonnegative and linear independent. It is evident if to choose the vector $\gamma = \{\gamma_r, \ldots, \gamma_i, \ldots\}$ such that

$$\sum_{i=r+1}^{\infty} a_i y_i^* \gamma_i < \infty, \quad \sum_{i=r}^{\infty} \gamma_i = 1, \quad \gamma_i > 0, \quad i = r, \infty,$$  

(118)

then we obtain that inequalities

$$\left\langle a_0 - \sum_{i=r+1}^{\infty} a_i y_i^* \gamma_i, f_j \right\rangle > 0, \quad j = 1, r.$$  

(119)

are valid. From here a vector $\sum_{i=r}^{\infty} \gamma_i z_i$ is strictly positive solution of the set of equations (109).

It is evident that these conditions are also sufficient. Theorem 12 is proved. \qed

It is easy to see that the vector $a_0$ belongs to the interior of the cone generated by vectors $a_j = \{P_i(A_l^n)\}_{i=1}^{k}, \ j = 1, \infty$. The existence of $r$ linear independent subset of vectors $\{a_{i_1}, \ldots, a_{i_r}\}$ from the set of vectors $a_j = \ldots$
\{P_i(A^n_j)\}_{i=1}^k, \ j = 1, \infty, \text{ such that the vector } a_0 \text{ belongs to the interior of the cone generated by this subset of vectors is the conditions on the set of measures } \{P_1, \ldots, P_k\}. \text{ A simple criterion of verifying of belonging to the interior of the cone a certain vector } a_0 \text{ is contained in } [6].

At last, let us give an example of measurable space \(\{\Omega, \mathcal{F}\}\) and filtration on it and also a set of measures \(P_1, \ldots, P_k\) satisfying conditions \(A\) and \(B\). Let us put \(\Omega = [0, 1)\). Choose any monotonously increasing sequence \(\{x_k\}_{k=0}^\infty\), such that \(x_0 = 0, x_k < x_{k+1}, \lim_{k \to \infty} x_k = 1\). Denote by \(A^1_s = [a^1_s, b^1_s) = [x_{s-1}, x_s), s = 1, \infty\). The sets \(A^2_s, s = 1, \infty\), we construct by dividing in half intervals \(A^1_s\) and so on. Let us give measures \(P_1, \ldots, P_k\) on \(\mathcal{F}_n\) generated by sets \(A^n_s, s = 1, \infty\). On Borel \(\sigma\)-algebra \(\mathcal{B}([0, 1))\) of the set \([0, 1)\) let us give a set of measures \(P_1, \ldots, P_k\) by their Radon-Nicodym derivatives \(\frac{dP_i}{dP_1} = ix^{i-1}, x \in [0, 1), i = 1, k\), where \(P_1\) is Lebesgue measure on \([0, 1)\). Consider restrictions of this measures on the \(\sigma\)-algebra \(\mathcal{F}_n\). It easy to see that so given measures on \(\mathcal{F}_n\) satisfy condition \(B\) with index \(i_0 = 1\).

Applications of the results obtained to Mathematical Finance will be given in separated paper.

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