TWO WEIGHT ESTIMATES WITH MATRIX MEASURES FOR WELL
LOCALIZED OPERATORS

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Abstract. In this paper, we give necessary and sufficient conditions for weight ed
$L^2$ estimates with matrix-valued measures of well localized operators. Nam ely, we seek estimates
of the form:

$$\|T(Wf)\|_{L^2(V)} \leq C\|f\|_{L^2(W)}$$

where $T$ is formally an integral operator with additional structure, $W, V$ are matrix mea-
sures, and the underlying measure space possesses a filtration. The characterization we
obtain is of Sawyer-type; in particular we show that certain natural testing conditions ob-
tained by studying the operator and its adjoint on indicator functions suffice to determine
boundedness. Working in both the matrix weighted setting and the setting of measure spaces
with arbitrary filtrations requires novel modifications of a T1 proof strategy; a particular
benefit of this level of generality is that we obtain polynomial estimates on the complexity
of certain Haar shift operators.

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0. Introduction

In this paper, we give necessary and sufficient conditions for two weight $L^2$ estimates with
matrix-valued measures of the so-called well localized operators. We seek estimates of the
form:

$$\|T(Wf)\|_{L^2(V)} \leq C\|f\|_{L^2(W)},$$

† Research supported in part by National Science Foundation DMS grant #1448846.
* Research supported in part by National Science Foundation DMS grants #1301579, 1600139.
‡ Research supported in part by National Science Foundation DMS grant #1500509.
where $T$ is formally an integral operator with additional structure and $W, V$ are matrix measures. The main examples we have in mind are Haar shifts and their different generalizations, considered in the matrix weighted spaces. For details concerning matrix measures, generalized Haar shifts, and well localized operators, see Sections 1–3.

Our main results, Theorems 4.1 and 4.2, basically say that Sawyer-type testing conditions are necessary and sufficient for the boundedness of well localized operators. In other words, for the boundedness of such operators, it is sufficient to check the estimates of the operator and its adjoint on characteristic functions of cubes.

One of the main motivations for the paper is the matrix $A_2$ conjecture. This is an important open question in the matrix-weighted setting, which asks whether for a Calderón–Zygmund operator $T$

$$\|T\|_{L^2(W) \to L^2(W)} \leq C [W]_{A_2};$$

here $C$ depends on $T$ but not $W$, and

$$[W]_{A_2} := \sup_I \left\| \left( |I|^{-1} \int_I W(x) \, dx \right)^{1/2} \left( |I|^{-1} \int_I W(x)^{-1} \, dx \right)^{1/2} \right\|^2 < \infty$$

is the $A_2$ characteristic of $W$. For previous work on this problem, see [1, 3, 10, 13]. Currently, the best known dependence of the norm of $T$ on the $A_2$ characteristic of $W$ in the matrix setting is $[W]_{A_2}^{3/2}$, which was established, first by the third author and collaborators, [11] and then by the second author and collaborators using some techniques from the preprint [11] coupled with a new method, [4]. In the scalar case, the first solution for particular cases (martingale multipliers, Hilbert Transform, Riesz Transforms, etc.) and the first solution for all Calderón–Zygmund operators [6] used two weight techniques to get one weight estimates. In particular, in [6] a two weight estimate of Haar shifts, similar to a scalar version of our Theorem 4.5, was one of the ingredients of the proof.

The proof of the main results follows along the lines of [12] and is outlined in Section 6. The main part of the operator is estimated by a corresponding weighted paraproduct, see Section 5, and the estimate of the paraproduct is done using the Carleson Embedding Theorem. For the case of matrix-valued measures, one needs a matrix-weighted version of this theorem, with matrix-valued weights both in the domain and in the target space. Such a theorem appeared only recently in [5], making this paper possible.

In this paper, we adapt our main result Theorem 4.2 to Haar shifts with a finite number of terms, which allows us to simplify the testing requirements. Indeed, Theorem 4.5 reduces the matrix $A_2$ conjecture for this class of operators to establishing a single testing condition and its dual. Earlier, similar results were obtained (only in the scalar case) in [7, 14] with significant extra work from the results of [12] or by modifying the proofs from [12].

This paper not only establishes two-weight theorems in the dyadic matrix weighted setting, but also considers the problem in a much more general situation and establishes better testing bounds. Indeed, we treat the case of very general filtrations, not just the standard dyadic one and so, our cubes are allowed to have an arbitrary number of children. This requires us to slightly alter the definition of well localized and be more careful when estimating the “easier” part of our operator, see Section 6. As a benefit of working in this level of generality, we are able to get better estimates and stronger results than the ones in [12] or [2], even in the case of scalar measures. Indeed, our arguments give polynomial dependence on the complexity of the Haar shift (or band of the well localized operator), while the results from [12], [2] only give exponential dependence. While the theorems we obtain also require one to test on a
slightly more complicated class of functions, this additional condition can be removed if we consider generalized Haar shifts and assume that measures satisfy a joint $A_2$ condition, see Theorem 4.5. Thus, a dedicated reader will find the paper interesting even if they restrict themselves only to the scalar setting and to the setting when the underlying filtration is the standard dyadic lattice. Finally, we should mention that related matrix weighted results for different operators and with different estimates appear in [2, 8, 9].

1. Setup: matrix-valued measures and weighted estimates

1.1. Atomic filtered spaces. Let $({\mathcal X}, F, \sigma)$ be a sigma-finite measure space with a filtration $\{F_n\}$, that is, a sequence of increasing sigma-algebras $F_n \subset F$. Here, $F$ is the smallest sigma-algebra containing $\cup F_n$. We make the assumption that $F_n$ is atomic, meaning that there exists a countable collection $D_n$ of disjoint sets $Q$ of finite measure (which we call atoms or cubes) with the property that every set of $F_n$ is a union of atoms (cubes) $Q \in D_n$.

Denote by $D$ the collection of all atoms, $D = \bigcup_{n \in Z} D_n$. A set $Q$ could belong to multiple generations $D_n$, so atoms $Q \in D_n$ should formally be represented as pairs $(Q, n)$. However, to simplify notation, we will suppress the dependence on $n$ and write $Q$ instead of $(Q, n)$; if the generation (or rank) $n$ is needed, it will be represented by $rk Q$, i.e. if $Q$ stands for the atom $(Q, n)$ then $n = rk Q$. The inclusion $R \subset Q$ for atoms is understood as set inclusion together with the inequality $rk R \geq rk Q$. In particular, for any $r \in Z$, $Ch^r Q$ stands for the collection of atoms $R \subset Q$ with $rk R = r + rk Q$. For $r = 1$, we write $Ch Q$ and avoid the superscript.

For a measurable set $E$, we will often use the notation $|E| = \sigma(E)$.

1.2. Examples.

Example 1.1. One motivating example is the standard dyadic filtration in $\mathbb{R}^d$ with Lebesgue measure. For $n \in Z$, let $D_n := \{2^{-n}((0,1]^d + k) : k \in Z^d\}$ be the collection of dyadic cubes with side length $2^{-n}$. Then each $F_n$ is the $\sigma$-algebra generated by $D_n$, and $F$ is the Borel $\sigma$-algebra. In this example, we do not have atoms of different ranks coinciding as sets.

The standard dyadic filtration also leads to more interesting examples.

Example 1.2. Consider a measurable $\mathcal{X} \subset \mathbb{R}^d$, again endowed with Lebesgue measure. For each $n \in Z$, define the collection of atoms $D_n$ as the collection of all non-empty intersections $Q \cap \mathcal{X}$, where $Q$ runs over all dyadic cubes of side length $2^{-n}$ from the previous example.

If, for example, $\mathcal{X} = Q_0 = (0,1]^d$, then $Q_0 \in D_n$ for all $n \leq 0$, and we have cubes of different ranks coinciding as sets. Taking more complicated $\mathcal{X}$, we can have more complicated structures of atoms and their ranks. We can make this example even more complicated by letting the underlying measure $\sigma$ be an arbitrary Radon measure.

1.3. Matrix-valued measures. Let $F_0$ be the collection of sets $E \cap F$ where $E \in F$ and $F$ is a finite union of atoms. A $d \times d$ matrix-valued measure $W$ on $\mathcal{X}$ is a countably additive function on $F_0$ with values in the set of non-negative linear operators on $\mathbb{F}^d$, where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$. Equivalently, $W = (w_{j,k})_{j,k=1}^d$ is a $d \times d$ matrix whose entries $w_{j,k}$ are (possibly signed or even complex-valued) measures that are finite on atoms, such that for any $E \in F_0$, the matrix $(w_{j,k}(E))_{j,k=1}^d$ is positive semidefinite. Note that the measure $W$ is always finite.
on atoms. Given such a measure $\mathbf{W}$ and measurable functions $f = (f_1, f_2, \ldots, f_d)^T$ and $g = (g_1, g_2, \ldots, g_d)^T$ on $\mathcal{X}$ with values in $\mathbb{R}^d$, we can define the integrals
\[
\int_{\mathcal{X}} \langle d\mathbf{W} f, g \rangle_{\mathbb{R}^d} := \sum_{j,k=1}^d \int_{\mathcal{X}} f_k \hat{g}_j d\mathbf{w}_{j,k}, \quad \int_{\mathcal{X}} d\mathbf{W} f,
\]
where the second integral is the vector whose $j$th coordinate is given by $\sum_{k=1}^d \int_{\mathcal{X}} f_k d\mathbf{w}_{j,k}$.

The weighted space $L^2(\mathbf{W})$ is the space of measurable, $\mathbb{R}^d$-valued functions on $\mathcal{X}$ satisfying
\[
\|f\|_{L^2(\mathbf{W})}^2 := \int_{\mathcal{X}} \langle d\mathbf{W} f, f \rangle_{\mathbb{R}^d} < \infty.
\]

Readers not comfortable with matrix-valued measures can always, without loss of generality, restrict themselves to working with absolutely continuous measures and matrix-valued functions. Namely, it is an easy corollary of the non-negativity of the matrix measure $\mathbf{W}$ that all of the measures $\mathbf{w}_{j,k}$ are absolutely continuous with respect to the trace measure $\mathbf{w} := \text{tr} \mathbf{W} := \sum_{k=1}^d \mathbf{w}_{k,k}$. Therefore, we can write $d\mathbf{W} = W d\mathbf{w}$, where $W$ is a $\mathbf{w}$-a.e. positive semidefinite $d \times d$ matrix-valued function on $\mathcal{X}$ and
\[
\int_{\mathcal{X}} d\mathbf{W} f = \int_{\mathcal{X}} W f d\mathbf{w}, \quad \int_{\mathcal{X}} \langle d\mathbf{W} f, g \rangle_{\mathbb{R}^d} = \int_{\mathcal{X}} \langle W f, g \rangle_{\mathbb{R}^d} d\mathbf{w}.
\]
The matrix-valued function $W$ is called the density of $\mathbf{W}$ with respect to $\mathbf{w}$.

1.4. **Weighted estimates with matrix weights.** This paper deals with two weight estimates of discrete “integral” operators $T$ that are represented (at least formally) as $T f(x) = \int_{\mathcal{X}} K(x, y) f(y) d\sigma(y)$, where the kernel $K(x, y) = k(x, y) \otimes I_d$, for a scalar-valued kernel $k(x, y)$. We are interested in estimates of the form
\[
\|T(\mathbf{W} f)\|_{L^2(\mathbf{V})} \leq C \|f\|_{L^2(\mathbf{W})}
\]
with matrix-valued measures $\mathbf{V}$ and $\mathbf{W}$. Here $T(\mathbf{W} f)$ is defined for the integral operator $T$ by
\[
T(\mathbf{W} f)(x) = \int_{\mathcal{X}} d\mathbf{W}(y) K(x, y) f(y) = \int_{\mathcal{X}} K(x, y) W(y) f(y) d\mathbf{w}(y),
\]
where $W$ is the density of $\mathbf{W}$ with respect to the scalar trace measure $\mathbf{w}$. We will use the symbol $T_w$ for the operator $f \mapsto T(\mathbf{W} f)$ and the symbol $T_{\mathbf{w}}$ for the operator $f \mapsto T(f \mathbf{w})$, where
\[
T_{\mathbf{w}} f(x) := \int_{\mathcal{X}} K(x, y) f(y) d\mathbf{w}(y).
\]
This operator $T_{\mathbf{w}}$ is defined for both scalar-valued functions and functions with values in $\mathbb{R}^d$; we will use the same notation for both cases, although formally in the latter case, we should write $T_{\mathbf{w}} \otimes I_d$. If $d\mathbf{W} = W d\mathbf{w}$ and $d\mathbf{V} = V d\mathbf{v}$ for any scalar measures $\mathbf{w}, \mathbf{v}$ defined on $\mathcal{F}$ and positive semidefinite functions $W, V$, we can rewrite estimate (1.1) as
\[
\|V^{1/2} T_{\mathbf{w}} W^{1/2} f\|_{L^2(\mathbf{V})} \leq C \|f\|_{L^2(\mathbf{W})},
\]
A particularly interesting case is when the measures $\mathbf{V}$ and $\mathbf{W}$ are absolutely continuous with respect to the underlying measure $\sigma$. Then we can write $d\mathbf{W} = W d\sigma$ and $d\mathbf{V} = V d\sigma$ and in (1.4), we can just take $\mathbf{v} = \mathbf{w} = \sigma$. 
2. Setup: generalized band operators and Haar shifts

2.1. Expectations and martingale differences. Let us introduce some notation and terminology. We call a measurable function \( f \) locally integrable if it is integrable on every atom \( Q \in \mathcal{D} \). For an atom \( Q \) and a locally integrable function \( f \), we denote by \( \langle f \rangle_Q \) its average (with respect to the underlying measure \( \sigma \))

\[
\langle f \rangle_Q := \sigma(Q)^{-1} \int_Q f \, d\sigma,
\]

with the convention that \( \langle f \rangle_Q = 0 \) if \( \sigma(Q) = 0 \). Define the averaging operator (expectation) \( E_Q \) by

\[
E_Q f := \langle f \rangle_Q 1_Q
\]

and the martingale difference operator \( \Delta_Q \) by

\[
\Delta_Q := \sum_{R \in \text{Ch} Q} E_R - E_Q.
\]

Note that \( E_Q \) and \( \Delta_Q \) are orthogonal projections in \( L^2(\sigma) \) and that the subspaces generated by the \( \Delta_Q \) are orthogonal to each other. We think of \( E_Q \), \( \Delta_Q \) as operators in Lebesgue spaces (\( E_Q f \), \( \Delta_Q f \) are defined \( \sigma \)-a.e.), so if for atoms \( Q_1 \subset Q_2 \) we have \( \sigma(Q_2 \setminus Q_1) = 0 \), then \( \Delta_{Q_2} = 0 \).

2.1.1. Generalized band operators. To the collection \( \mathcal{D} \), associate a tree structure where each \( Q \) is connected to the elements of the collection \( \text{Ch} Q \). Given this tree, let \( d_{\text{tree}}(Q, R) \) denote the “tree distance” between atoms \( Q \) and \( R \), namely, the number of edges of the shortest path connecting \( Q \) and \( R \). If \( Q \) and \( R \) share no common ancestors, then \( d_{\text{tree}}(Q, R) = \infty \).

The operators of interest possess a band structure related to this tree distance, as defined below. These operators are called generalized band operators because they generalize the band operators studied in [12, 2].

Definition 2.1. A bounded operator \( T : L^2(\sigma) \to L^2(\sigma) \) is a generalized band operator of radius \( r \) if \( T \) can be written as

\[
T = \sum_{j,k=1 \atop j,k \in \mathcal{D}} \sum_{Q,R \in \mathcal{D}} P^j_R T^{j,k}_R P^k_Q \quad T^{j,k}_R : P^k_Q L^2(\sigma) \to P^j_R L^2(\sigma),
\]

where each \( T^{j,k}_R \) is a bounded operator on \( L^2(\sigma) \) satisfying \( T^{j,k}_R = 0 \) if \( d_{\text{tree}}(R, Q) > r \) and for any \( Q \in \mathcal{D} \), the projections are defined as \( P^1_Q := \Delta_Q \) and \( P^2_Q := E_Q \).

In general, convergence is in the weak operator topology with respect to some ordering of the pairs \( Q, R \). We typically assume that the sum in (2.3) has only finitely many nonzero terms once we collect the blocks in groups as in (2.12).

Then each block \( \widetilde{T}^{j,k}_R \) is a bounded operator in \( L^2(\sigma) \) and can be represented as an integral operator with kernel \( K^{j,k}_{R,Q} \). The kernel \( K^{j,k}_{R,Q} \) can be computed as follows: for
For a numerical sequence \(a = \{a_Q\}_{Q \in \mathcal{D}}\) define the “dyadic” operator \(T_a\) on \(L^2(\sigma)\) by
\[
T_a f = \sum_{Q \in \mathcal{D}} a_Q E_Q f.
\]
Trivially, \(T_a\) is a generalized band operator of radius \(r = 0\) as long as \(T_a\) is bounded on \(L^2(\sigma)\).

**Remark.** For a sequence \(|a| := \{|a_Q|\}_{Q \in \mathcal{D}}\) one can easily see that the pointwise estimate
\[
|(T_a f)(x)| \leq (|T_a| f)(x) \quad \forall x \in \mathcal{X}
\]
holds. So, in the scalar case, the two weight estimates for \(T_a\) follow from the two weight estimates for \(T_{|a|}\). Thus, in the scalar case, the operators with all \(a_Q \geq 0\) (the so-called positive dyadic operators) play a special role in weighted estimates.

In the case of matrix-valued measures, it is not clear that the weighted estimates of \(T_{|a|}\) imply the corresponding estimates for \(T_a\) (we suspect that this is not true), so we do not reserve any special place for the positive dyadic operators.

**Example 2.3.** For \(r \in \mathbb{Z}_+\) and \(b\) a locally integrable function, define the paraprodut \(\Pi = \Pi'_b\) of order \(r\) on \(L^2(\sigma)\) by
\[
\Pi f = \sum_{Q \in \mathcal{D}} E_Q f \sum_{R \in \text{Ch}' Q} \Delta_R b = \sum_{Q \in \mathcal{D}} \sum_{R \in \text{Ch}' Q} \left(\Delta_R M_b E_Q\right) f,
\]
where \(M_b\) is multiplication by \(b\). Clearly, as long as \(\Pi\) is bounded on \(L^2(\sigma)\), it is a generalized band operator of radius \(r\).

**Remark.** Since \(\Pi f\) is defined by the sum of an orthogonal series, the convergence of the sum defining \(\Pi\) in the weak operator topology implies its unconditional convergence in the strong operator topology.
Example 2.4. A Haar shift of complexity \((m, n)\) is an operator \(T : L^2(\sigma) \rightarrow L^2(\sigma)\) defined by

\[
T = \sum_{Q \in \mathcal{D}} \sum_{R \in \text{Ch}^n(Q), S \in \text{Ch}^m(Q)} \Delta_R T_{R,S} \Delta_S, \quad T_{R,S} : \Delta_S L^2(\sigma) \rightarrow \Delta_R L^2(\sigma), \tag{2.6}
\]

where for each bounded operator \(\tilde{T}_{R,S} := \Delta_R T_{R,S} \Delta_S\), its canonical kernel \(K_{R,S}\) (defined by (2.4) and supported on \(R \times S\)) satisfies the estimate

\[
\|K_{R,S}\|_{\infty} \leq |Q|^{-1}, \tag{2.7}
\]

which in this paper, means \(|K_{R,S}(x, y)| \leq |Q|^{-1}\) for all \(x, y \in \mathcal{X}\). If \(T\) is a Haar shift of complexity \((m, n)\), then trivially its adjoint \(T^*\) is also a Haar shift of complexity \((n, m)\). Any Haar shift of complexity \((m, n)\) (in fact, any bounded operator given by (2.6)) is a generalized band operator of radius \(r = m + n\).

Remark. An operator defined by (2.6) is bounded if and only if all blocks \(T_Q\)

\[
T_Q := \sum_{R \in \text{Ch}^n(Q), S \in \text{Ch}^m(Q)} \Delta_R T_{R,S} \Delta_S
\]

are uniformly bounded: in this case the series in (2.6) converges unconditionally (independently of the ordering) in the strong operator topology.

Note, that the normalization condition (2.7) implies that \(\|T_Q\| \leq 1\). Indeed, (2.7) implies that the block \(T_Q\) can be represented as an integral operator with kernel \(K_Q\) (supported on \(Q \times Q\)) satisfying \(\|K_Q\|_{\infty} \leq |Q|^{-1}\), so \(\|K_Q\|_{L^2(Q \times Q)} \leq 1\).

The concept of Haar shifts can be generalized.

Definition 2.5. A generalized Haar shift of complexity \((m, n)\) is an operator \(T : L^2(\sigma) \rightarrow L^2(\sigma)\) defined by

\[
T = \sum_{j,k=1}^{2} \sum_{Q \in \mathcal{D}} \sum_{R \in \text{Ch}^n(Q), S \in \text{Ch}^m(Q)} P^j R^k T^j k R, S \Delta_S, \quad T^j k R, S : P^k S L^2(\sigma) \rightarrow P^j R L^2(\sigma), \tag{2.8}
\]

where each \(T^j k R, S\) is bounded, the projections are \(P^1_Q := \Delta_Q\) and \(P^2_Q := \mathbb{E}_Q\), and the kernel \(K^j k R, S\) of \(\tilde{T}^j k R, S := P^j R^k T^j k R, S \Delta_S\) satisfies

\[
\|K^j k R, S\|_{\infty} \leq |Q|^{-1}. \tag{2.9}
\]

We typically assume that the sum in (2.8) has only finitely many nonzero \(Q\) terms. In general, convergence is in the weak operator topology with respect to some ordering of the pairs \(R, S\).
It is convenient to present an alternate representation of a (generalized) Haar shift by grouping the terms $\tilde{T}_{R,Q}^{j,k}$. Namely, denoting

$$T_Q = \sum_{j,k=1}^{2} \sum_{R \in \text{Ch}^n(Q), S \in \text{Ch}^m(Q)} P_{R} T_{R,S}^{j,k} P_{S}$$

(or taking the inner sum in (2.6) for regular Haar shifts), we can represent a generalized Haar shift as $\sum_{Q \in D} T_Q$. Note that the kernel $K_Q$ of the integral operator $T_Q$ is supported on $Q \times Q$, constant on $R \times S$, $R, S \in \text{Ch}^{r+1} Q$, $r = \max\{m, n\}$. Since the sets $R \times S$, $R \in \text{Ch}^n Q$, $S \in \text{Ch}^m Q$ are disjoint, the kernel $K_Q$ satisfies the estimate

$$\|K_Q\|_\infty \leq |Q|^{-1} \quad (2.10)$$

for Haar shifts, and the estimate $\|K_Q\|_\infty \leq 4|Q|^{-1}$ for generalized ones. We need the constant “4” here because for each pair $R \in \text{Ch}^n Q$, $S \in \text{Ch}^m Q$, there are four operators $T_{R,S}^{j,k}$. This discussion motivates the following general object of study:

**Definition 2.6.** A generalized big Haar shift of complexity $r$ is a bounded operator $T : L^2(\sigma) \rightarrow L^2(\sigma)$ defined by

$$T = \sum_{Q \in D} T_Q,$$

where each block $T_Q$ is an integral operator with kernel $K_Q$, where $K_Q$ is supported on $Q \times Q$, constant on $R \times S$ with $R, S \in \text{Ch}^{r+1} Q$, and satisfies the estimate (2.10). If in addition, each block $T_Q$ and its adjoint $T_Q^*$ annihilate constants $1_Q$, we will call the operator simply a big Haar shift of complexity $r$, without the word generalized. Finally, if an operator $T$ admits the above representation but without the estimate (2.10), we will say that the operator $T$ has the structure of a (generalized) big Haar shift.

**Remark 2.7.** It is easy to see that a (generalized) band operator of radius $r$ has the structure of a (generalized) big Haar shift of complexity $r$. To see that, we can just define

$$T_Q := \sum_{R \in \text{Ch}^r Q} \sum_{S \in \text{Ch}(Q) \cap \text{Ch}^m Q} \tilde{T}_{R,S}^{j,k} + \sum_{R \in \text{Ch}^r Q} \sum_{S \in \text{Ch}^m Q} \tilde{T}_{R,S}^{j,k}.$$

Moreover, if the kernels $K_{R,S}^{j,k}$ of the blocks $\tilde{T}_{R,S}^{j,k}$ admit the estimate $\|K_{R,S}^{j,k}\|_\infty \leq 1/4|Q|^{-1}$ (or the estimate $\|K_{R,S}\|_\infty \leq |Q|^{-1}$ for kernels of the blocks $\tilde{T}_{R,S}$ for the case of a band operator), then the operator $T$ is a (generalized) big Haar shift.

### 3. Setup: Weighted Martingale Differences and Well Localized Operators

#### 3.1. Weighted martingale differences

For the matrix measure $W$ (or $V$) discussed above, one can define the $W$-weighted expectation $E_Q^W$ and the martingale difference $\Delta_Q^W$ by

$$E_Q^W f = \langle f \rangle_Q^W 1_Q, \quad \langle f \rangle_Q^W := W(Q)^{-1} \left( \int_Q dW f \right)$$

(3.1)
\[
\Delta^W_Q = \sum_{R \in \text{Ch} Q} E^W_R - E^W_Q
\]

respectively, for all atoms \(Q \in \mathcal{D}\).

Initially, this definition only makes sense if \(W(Q)\) is invertible. However if \(W(Q)\) is not invertible, we can interpret \(W(Q)^{-1}\) as the Moore–Penrose pseudoinverse of \(W(Q)\). Here, the Moore–Penrose pseudoinverse of a matrix \(A\) is the unique matrix \(A^+\) defined as follows: on \(\ker A\), it is the zero operator and on \((\ker A)^\perp = \text{Ran} A\), it is the inverse of \(A|_{\text{Ran} A}\). For example, if \(W(Q) = 0\), then \(W(Q)^{-1} = 0\).

The standard computations for \(E^W_Q\) and \(\Delta^W_Q\) still work in this general case because \(\int_Q dWf \in \text{Ran} W(Q)\). To see this, write \(W = W(x)dw\), where \(w\) is the trace measure of \(W\), and observe that a vector \(e \in \ker W(Q)\) if and only if \(1_Q e\) equals the zero function in \(L^2(W)\). One can use this to show \(\int_Q dWf \perp \ker W(Q)\) and using \(W(Q)\) self-adjoint, conclude \(\int_Q dWf \in \text{Ran} W(Q)\).

Then, it is not too hard to see that \(E^W_Q\) is the orthogonal projection in \(L^2(W)\) onto the subspace of constants \(\{1_Q e : e \in \mathbb{F}^d\}\). To prove this fact, one needs to use properties of the Moore–Penrose pseudoinverse to show that \(1_Q e = E^W_Q(1_Q e)\) in \(L^2(W)\) for all \(Q \in \mathcal{D}, e \in \mathbb{F}^d\). It can also be shown that \(E^W_Q \Delta^W_Q = \Delta^W_Q E^W_Q = 0\), that \(\Delta^W_Q\) is an orthogonal projection, and that the subspaces generated by \(\Delta^W_Q\) and \(\Delta^W_R\) are orthogonal whenever \(Q \neq R\).

### 3.2. Well localized operators

To state and prove the main results, it is convenient to introduce the formalism of well localized operators between weighted spaces, rather than work directly with operators that have the structure of a (generalized) big Haar shift. Earlier, we defined the operator \(T_w, T_w f := T(Wf)\) as the integral (1.2), provided that the integral is well defined. But to verify boundedness, we only need to know the bilinear form of the operator on a dense set.

Let \(L\) denote the set of finite linear combinations of functions of the form \(1_Q e\), with \(Q \in \mathcal{D}\) and \(e \in \mathbb{F}^d\). By Lemma 8.1, this set is dense in \(L^2(W)\) and so, it suffices to know how to compute

\[
\langle T_w 1_Q e, 1_R v \rangle_{L^2(V)} = \iint_{X \times X} \left( dW(y)K(x,y)1_Q(y)e, dV(x)1_R(x)v \right)_{\mathbb{F}^d}^{\mathbb{F}d}
= \iint_{X \times X} \left( W(y)K(x,y)1_Q(y)e, V(x)1_R(x)v \right)_{\mathbb{F}^d}^{\mathbb{F}d} \, dw(y)dv(x)
\]

for all \(Q, R \in \mathcal{D}\) and \(e, v \in \mathbb{F}^d\). To be precise, we say:

**Definition 3.1.** An operator \(T_w\) acts formally from \(L^2(W)\) to \(L^2(V)\) if the bilinear form

\[
\langle T_w 1_Q e, 1_R v \rangle_{L^2(V)}
\]

is well defined for all \(Q, R \in \mathcal{D}\) and all \(e, v \in \mathbb{F}^d\). Then, the formal adjoint \(T_w^*\) is given by

\[
\langle T_w 1_Q e, 1_R v \rangle_{L^2(V)} = \langle 1_Q e, T_w^* 1_R v \rangle_{L^2(W)}.
\]
As part of the definition, we also require a very weak continuity property, namely that

\[
\langle T^*_W 1_Q^e, 1_R^v \rangle_{L^2(\mathcal{V})} = \sum_{S \in \text{Ch} Q} \langle T^*_W 1_{S}^e, 1_R^v \rangle_{L^2(\mathcal{V})} = \sum_{S \in \text{Ch} R} \langle T^*_W 1_Q^e, 1_S^v \rangle_{L^2(\mathcal{V})};
\]

this property is nontrivial only if \(Q, R\) have infinitely many children.

Consider the set \(\mathcal{L}\) of finite linear combinations of functions of the form \(1_Q^e\), with \(Q \in \mathcal{D}\) and \(e \in \mathbb{F}^d\). If the bilinear form (3.3) is defined, then

\[
\langle T^*_W f, g \rangle_{L^2(\mathcal{W})} = \langle f, T^*_W g \rangle_{L^2(\mathcal{W})}
\]

is well defined for all \(f, g \in \mathcal{L}\). Since \(\Delta^W f \in \mathcal{L}\) for \(f \in \mathcal{L}\), the expression \(\langle T^*_W \Delta^W f, \Delta^V g \rangle_{L^2(\mathcal{V})}\) is also well defined for all \(f, g \in \mathcal{L}\). Thus the expression \(\Delta^V T^*_W \Delta^W\) is well defined, in the sense that its bilinear form is well defined for \(f, g \in \mathcal{L}\).

**Definition 3.2.** An operator \(T^*_W\) acting formally from \(L^2(\mathcal{W})\) to \(L^2(\mathcal{V})\) is said to be **localized** if for all \(e, v \in \mathbb{F}^d\),

\[
\langle T^*_W 1_Q^e, 1_R^v \rangle_{L^2(\mathcal{V})} = 0
\]

whenever \(Q, R \in \mathcal{D}\) share no common ancestors.

**Definition 3.3.** An operator \(T^*_W\) acting formally from \(L^2(\mathcal{W})\) to \(L^2(\mathcal{V})\) is called **\(r\)-lower triangular** if for all \(R, Q \in \mathcal{D}\) and \(e \in \mathbb{F}^d\),

\[
\Delta^V T^*_W 1_Q^e = 0
\]

if either

(i) \(R \not\in Q\) and \(\text{rk} R \geq r + \text{rk} Q\), or

(ii) \(R \not\in Q^{(r+1)}\) and \(\text{rk} R \geq \text{rk} Q - 1\);

here \(Q^{(r+1)}\) is the order \(r + 1\) ancestor of \(Q\), i.e. \(Q^{(r+1)}\) is the unique atom with \(Q \subset Q^{(r+1)}\) and \(\text{rk} Q^{(r+1)} = \text{rk} Q - (r + 1)\).

As mentioned earlier, \(T^*_W\) is defined via a bilinear form. So, in Definition 3.3, the statement \(\Delta^V T^*_W 1_Q^e = 0\) should be interpreted as

\[
\langle \Delta^V T^*_W 1_Q^e, g \rangle_{L^2(\mathcal{V})} = \langle T^*_W 1_Q^e, \Delta^V g \rangle_{L^2(\mathcal{V})} = 0,
\]

for all \(g \in \mathcal{L}\). More generally, for any \(f \in \mathcal{L}\), the notation \(T^*_W f\) or \(\Delta^V T^*_W f\) should be interpreted in terms of the bilinear form.

**Definition 3.4.** An operator \(T^*_W\) acting formally from \(L^2(\mathcal{W})\) to \(L^2(\mathcal{V})\) is said to be **well localized with radius \(r\)** if it is localized and if both \(T^*_W\) and its formal adjoint \(T^*_V\) are \(r\)-lower triangular.

**Remark 3.5.** This definition is very similar to the definition of well localized operators from [12], with two exceptions. First, there was a typo in [12], and in the language of this paper, the definition in [12] only required that \(\text{rk} R \geq \text{rk} Q\) in condition (ii) of the above Definition 3.3. This was a typo; the inequality \(\text{rk} R \geq \text{rk} Q\) is not sufficient to get the results in [12]. For more details, see the discussion in [2].
The other difference, which is more essential, is that in [12], it was not required for the operator $T_W$ to be localized in the sense of the above Definition 3.2. In this paper, by requiring our operators to be localized, we are able to get better estimates than those in [12]. In particular, we do not require any bounds on the number of children of a cube $Q \in \mathcal{D}$. In [12], it was assumed that each cube had at most $N$ children for some fixed $N \in \mathbb{N}$, and the estimates depended on this bound.

Since all of the examples we have in mind (all of them were presented earlier) give rise to localized operators, we included this requirement in the definition of well localized operators. Thus, we are able to get better estimates than those in [12], even for the case of scalar measures.

3.3. From band operators to well localized operators. Now we will show that if $T$ has the structure of a generalized big Haar shift of complexity $r$, see Definition 2.6, and if $T$ also satisfies several additional assumptions, then the operator $T_W$, $T_Wf = T(Wf)$ is a well localized operator of radius $r$.

We assume that we only have finitely many terms $T_Q$ in the representation (2.11) and that each block $T_Q$ is represented by an integral operator with a bounded kernel. Note that the latter assumption is always true if all $Q \in \mathcal{D}$ have finitely many children; for the generalized big Haar shifts, it is just postulated.

The above two assumptions imply that the bilinear form (3.5) is well defined for $f, g \in \mathcal{L}$. Moreover, the facts that the kernel is bounded and $W, V$ are finite on atoms can be used to show that $T_W$ and its formal adjoint satisfy the weak continuity property (3.4). Thus $T_W$ is well defined as an integral operator with a bounded, compactly supported kernel, it can be shown that the bilinear form (3.5) is well defined for all $f \in L^2(W)$, $g \in L^2(V)$, so in fact $T_W$ is a bounded operator $L^2(W) \to L^2(V)$.

Lemma 3.6. Let $T$ have the structure of a generalized big Haar shift of complexity $r$, satisfying the assumptions above. Then for matrix-valued measures $W$ and $V$, the operator $T_W$, $T_Wf = T(Wf)$, acting formally from $L^2(W)$ to $L^2(V)$ is a well localized operator of radius $r$.

Proof. The fact that the operator $T_W$ is localized, see Definition 3.2, is obvious. Now, we will show that $T_W$ is $r$-lower triangular; then by the symmetry, we obtain the same result for $T^*_V$.

Observe that if $Q \in \mathcal{D}, e \in \mathbb{F}^d$, the assumptions on $T$ imply that the function $T(W1_Qe)$ is well-defined and in $L^2(V)$. Then to prove that $T_W$ is $r$-lower triangular, it suffices to show that outside of $Q^{(r+1)}$, $T(W1_Qe)$ is constant on cubes $R$ with $\text{rk} R \geq \text{rk} Q - 1$, and that outside of $Q$, it is constant on cubes $R$ with $\text{rk} R \geq \text{rk} Q + r$.

Let us analyze when $T_S(W1_Qe)$ can be non-zero and how it behaves in that case. First, observe that $T_S(W1_Qe)$ is non-zero outside of $Q$ only if $Q \subset S$. Since $\text{rk} S \leq \text{rk} Q - 1$ for $Q \subset S$, the condition $\text{rk} R \geq \text{rk} Q + r$ implies that

$$\text{rk} R \geq \text{rk} Q + r \geq \text{rk} S + 1 + r.$$ (3.6)
We know that the kernel of $T_S$ is constant on sets $S' \times S''$ with $S', S'' \in \text{Ch}^{r+1} S$, and therefore, $T_S(W^1_Q e)$ is constant on cubes $R$ such that $\text{rk} R \geq \text{rk} S + r + 1$.

So, if $R \cap Q = \emptyset$ and $\text{rk} R \geq \text{rk} Q + r$, we can conclude from (3.6) that $T_S(W^1_Q e)$ is constant on $R$. Similarly, if $T_S(W^1_Q e)$ does not vanish outside of $Q^{(r+1)}$, then $Q^{(r+1)} \subseteq S$, so

$$\text{rk} S \leq \text{rk} Q^{(r+1)} - 1 = \text{rk} Q - (r + 1) - 1 = \text{rk} Q - r - 2,$$

or equivalently

$$\text{rk} S + r + 1 \leq \text{rk} Q - 1.$$

The condition $\text{rk} R \geq \text{rk} Q - 1$ then implies that

$$\text{rk} R \geq \text{rk} Q - 1 \geq \text{rk} S + r + 1. \quad (3.7)$$

But, as we discussed above, $T_S(W^1_Q e)$ is constant on cubes $R$ such that $\text{rk} R \geq \text{rk} S + r + 1$, so (3.7) implies that outside of $Q^{(r+1)}$, the function $T_S(W^1_Q e)$ is constant on cubes $R$ with $\text{rk} R \geq \text{rk} Q - 1$. \hfill \Box

4. Main results

4.1. Estimates of well localized operators. For a cube $Q \in \mathcal{D}$, define $D^r_Q$ to be the collection of functions

$$D^r_Q := \left\{ f_Q = \sum_{R \in \text{Ch}^r Q} \Delta^r_R f : f \in L \right\}.$$

Given this definition, we can state our first main result.

**Theorem 4.1.** Let $T^r_w$ be a well localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$. Then $T^r_w$ extends to a bounded operator from $L^2(W)$ to $L^2(V)$ if and only if the following conditions

(i) $\|1_Q T^r_w 1_Q e\|_{L^2(V)} \leq \mathcal{C}_1 \|1_Q e\|_{L^2(W)}$ for all $e \in \mathcal{F}^d$;

(ii) $\|1_Q T^r_w f_Q\|_{L^2(V)} \leq \mathcal{C}_2 \|f_Q\|_{L^2(W)}$ for all $f_Q \in D^r_Q$;

and their dual counterparts (corresponding conditions for $T^*_v$ with $V$ and $W$ interchanged) hold for all $Q \in \mathcal{D}$. Furthermore,

$$\|T^r_w\|_{L^2(W) \rightarrow L^2(V)} \leq \left( C(d)^{1/2} + 1/2 \right) (\mathcal{C}_1 + \mathcal{C}_1^*) + (r + 1)^{1/2} (\mathcal{C}_2 + \mathcal{C}_2^*);$$

here $\mathcal{C}_1^*, \mathcal{C}_2^*$ are the constants from the duals to the testing conditions (i), (ii) respectively, and $C(d)$ is the constant from the Matrix Carleson Embedding Theorem (Theorem 5.5 below).

Moreover, for the best possible bounds $\mathcal{C}_k, \mathcal{C}_k^*$ we trivially have

$$\mathcal{C}_k, \mathcal{C}_k^* \leq \|T^r_w\|_{L^2(W) \rightarrow L^2(V)} \quad k = 1, 2. \quad (4.1)$$

**Remark.** In the case when each $Q \in \mathcal{D}$ has at most $N$ children ($N < \infty$), condition (ii) follows from the testing condition $\|T^r_w 1_Q e\|_{L^2(V)} \leq \mathcal{C}_3 \|1_Q e\|_{L^2(W)}$. In this case, one can estimate $\mathcal{C}_2 \leq C(r, N) \mathcal{C}_3$ and obtain similar estimates for the dual condition. That is very similar to the approach used in [12].
Condition (i) in Theorem 4.1 can be slightly relaxed. Given a well localized operator $T_W$ and cube $Q \in D$, define its truncation $T^Q_W$ by

$$T^Q_W f = \sum_{R \in D(Q)} \Delta^V_R T_W f,$$

and similarly for the dual $T^*_V$.

**Theorem 4.2.** Let $T_W$ be a well localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$. Then $T_W$ extends to a bounded operator from $L^2(W)$ to $L^2(V)$ if and only if the conditions

(i) $\|T^Q_W 1_Q e\|_{L^2(V)} \leq \xi_1 \|1_Q e\|_{L^2(W)}$ for all $e \in \mathbb{R}^d$;

(ii) $\|T^Q_W f\|_{L^2(V)} \leq \xi_2 \|f\|_{L^2(W)}$ for all $f \in D^W_Q$,

their dual counterparts (corresponding conditions for $T^*_V$ with $V$ and $W$ interchanged) and the following weak type estimate

(iii) $|\langle T_W 1_Q e, 1_Q v \rangle|_{L^2(V)} \leq \xi_3 \|1_Q e\|_{L^2(W)} \|1_Q v\|_{L^2(V)}$ for all $e, v \in \mathbb{R}^d$,

hold for all $Q \in D$. Furthermore,

$$\|T_W\|_{L^2(W) \to L^2(V)} \leq C(d)^{1/2}(\xi_1 + \xi_1^*) + (r + 1)^{1/2}(\xi_2 + \xi_2^*) + \xi_3;$$

here $\xi_1^*, \xi_2^*$ are the constants from the duals to the testing conditions (i), (ii) respectively, and $C(d)$ is the constant from the Matrix Carleson Embedding Theorem (Theorem 5.5 below). Moreover, for the best possible bounds $\xi_k, \xi_k^*$ we trivially have

$$\xi_k, \xi_k^* \leq \|T_W\|_{L^2(W) \to L^2(V)}, \quad k = 1, 2, 3. \quad (4.3)$$

There is no dual condition to (iii) in this theorem because this condition is self-dual. Note that Theorem 4.1 follows immediately from Theorem 4.2, because trivially for all $f \in \mathcal{L}$,

$$\|T^Q_W f\|_{L^2(V)} \leq \|1_Q T_W f\|_{L^2(V)}.$$ 

Note also that condition (i) of Theorem 4.1 implies condition (iii) of Theorem 4.2, with the trivial estimate for the corresponding bounds

$$\xi_3 \leq (\xi_1 + \xi_1^*)/2;$$

here $\xi_3$ is the bound from Theorem 4.2, and $\xi_1, \xi_1^*$ are the bounds from condition (i) and its dual in Theorem 4.1.

**Remark 4.3.** Condition (iii) of Theorem 4.2 can be further relaxed. First, we do not need this condition to hold for all cubes $Q \in D$: it is sufficient if this condition holds for arbitrarily large cubes $Q$, meaning that for any $Q_0 \in D$ one can find $Q \in D$, $Q_0 \subset Q$ for which (iii) holds.

Secondly, if for any $Q_0 \in D$ we have for the increasing sequence of cubes $Q_n$, where $Q_{n+1}$ is the parent of $Q_n$, that $W(Q_n) \geq \alpha_n 1$, $\alpha_n \to +\infty$ and similarly for $V$, then condition (iii) can be removed from Theorem 4.2.
4.2. Applications to estimates of Haar shifts. While conditions (i) from Theorems 4.1 and 4.2 are pretty standard testing conditions, and condition (iii) from Theorem 4.2 is the standard weak boundedness condition, condition (ii) seems unnecessarily complicated. However, if the $d \times d$ matrix measures $V$, $W$ satisfy the two weight matrix $A_2$ condition

$$\sup_{Q \in D} |Q|^{-2} \|V(Q)^{1/2}W(Q)^{1/2}\|_2 =: [V, W]_{A_2} < \infty,$$  \hspace{1cm} (4.4)$$

and the operator $T$ is a generalized big Haar shift with finitely many terms, see Definition 2.6, then condition (ii) follows from a testing condition similar to (i) and the $A_2$ condition (4.4).

Let us introduce some notation. For a generalized big Haar shift $T = \sum_{Q \in D} T_Q$ and cube $Q \in D$, define the operator $T^Q$ by

$$T^Q := \sum_{R \in D(Q)} T^Q_R.$$ \hspace{1cm} (4.5)$$

For a matrix measure $W$, define the weighted version $(T^Q)_W$ of $T^Q$ by

$$(T^Q)_W f = T^Q(Wf).$$ \hspace{1cm} \text{Note that $(T^Q)_W$ is different from $T^Q_W$ defined above in (4.2).}$$

Lemma 4.4. Let $T$ be a generalized big Haar shift of complexity $r$ with finitely many terms, and let the $d \times d$ matrix measures $V$ and $W$ satisfy the matrix $A_2$ condition (4.4). Assume that for all $Q \in D$

$$\|T^Q_1 f^e\|_{L^2(V)} \leq \|f^e\|_{L^2(W)}, \text{ for all } e \in \mathbb{F}^d. \hspace{1cm} (4.6)$$

Then for all $Q \in D$

$$\|T^Q_1 f^e\|_{L^2(V)} \leq \left(d^{1/2}; |V, W|_{A_2}^{1/2} + \|x\|\right) \|f^e\|_{L^2(W)}, \text{ for all } e \in \mathbb{F}^d; \hspace{1cm} (4.7)$$

$$\|T^Q_1 f_Q\|_{L^2(V)} \leq \left(d^{1/2}(2r + 1)|V, W|_{A_2}^{1/2} + \|x\|\right) \|f_Q\|_{L^2(W)}, \text{ for all } f_Q \in D^W_Q. \hspace{1cm} (4.8)$$

Moreover, for all sufficiently large $Q \in D$

$$\|T^Q_1 f^e\|_{L^2(V)} \leq \|f^e\|_{L^2(W)}. \hspace{1cm} (4.9)$$

Lemma 4.4 implies that for a generalized big Haar shift $T$ of complexity $r$ with finitely many terms, the bounds in the testing conditions in Theorem 4.2 can be estimated as

$$\mathcal{F}_1 \leq d^{1/2} |V, W|_{A_2}^{1/2} + \|x\|,$$

$$\mathcal{F}_2 \leq d^{1/2}(2r + 1)|V, W|_{A_2}^{1/2} + \|x\|,$$

$$\mathcal{F}_3 \leq \|x\|,$$

with the similar estimates for the dual bounds $\mathcal{F}_1^*, \mathcal{F}_2^*$. Note also that $\mathcal{F}_3 \leq \mathcal{F}_3^*$, so $\mathcal{F}_3 \leq (\mathcal{F} + \mathcal{F}_3^*)/2$. Using these estimates and applying Theorem 4.2, we get the following result.

Theorem 4.5. Let $T$ be a generalized big Haar shift of complexity $r$ with finitely many terms, and let the $d \times d$ matrix measures $V$ and $W$ satisfy the $A_2$ condition (4.4). Let

(i) $\|T^Q_1 f^e\|_{L^2(V)} \leq \|f^e\|_{L^2(W)}$, for all $Q \in D$ and all vectors $e \in \mathbb{F}^d$,
and let also the corresponding condition for \( T^* \) (with \( V \) and \( W \) interchanged) hold with constant \( \Xi^* \). Then,

\[
\| T_W \|_{L^2(W) \rightarrow L^2(V)} \leq \left( C(d)^{1/2} + (r + 1)^{1/2} + 1/2 \right) (\Xi + \Xi^*) + 2d^{1/2} \left( C(d)^{1/2}r + (2r + 1)(r + 1)^{1/2} \right) [V, W]^{1/2} ;
\]

here again \( C(d) \) is the constant from the Matrix Carleson Embedding Theorem (Theorem 5.5).

**Remark 4.6.** Under some additional assumptions on the filtration \( \{ \mathcal{F}_n \}_{n \in \mathbb{Z}} \), (i) of Theorem 4.5 (and its adjoint) is also necessary for the boundedness of the operator \( T_W : L^2(W) \rightarrow L^2(V) \). For example, it is necessary if there exists \( \kappa \in (0, 1) \) such that for any \( Q \in \mathcal{D} \) and its parent \( \hat{Q} \)

\[
|Q| \leq \kappa |\hat{Q}|.
\]

For the proof, see Lemma 7.3 below. This condition holds for the standard dyadic lattice, and for any homogeneous lattice, but it is in fact much weaker than homogeneity of the lattice.

Lemma 4.4 will also be proved later in Section 7.

**4.3. Remarks about norm dependence on the \( A_2 \) characteristic and complexity.** Let \( T \) be a Haar shift of complexity \( r \) with finitely many terms and let \( W \) be an \( A_2 \) weight. Then, Theorem 4.5 could potentially be used to estimate the dependence of the norm of \( T \) in the weighted space \( L^2(W) \) on the \( A_2 \) characteristic \([W]_{A_2}\) of the weight \( W \). Here, \([W]_{A_2}\) is the exactly the \( A_2 \) characteristic \([V, W]_{A_2}\) from (4.4) with \( dW = W d\sigma, dV = V^{-1} d\sigma \).

In the case of scalar weights \( w \), there is an estimate showing the norm of interest \( \| T \|_{L^2(w) \rightarrow L^2(w)} \) depends linearly on \([w]_{A_2}\), and this estimate is optimal, see [6]. In the matrix case, the best known estimate is \([W]_{A_2}^{3/2}\), which has been recently established independently by the third author and collaborators and the second author and collaborators. In any case, for Haar shifts with finitely many terms, Theorem 4.5 reduces the problem to finding the optimal estimate in the testing condition (i) and its dual.

Theorem 4.5 can also be used to study the dependence of \( \| T \|_{L^2(W) \rightarrow L^2(W)} \) on the complexity of \( T \). For scalar \( A_2 \) weights \( w \), the best known estimate for \( \| T \|_{L^2(w) \rightarrow L^2(w)} \) grows linearly in the complexity \( r \) of the Haar shift \( T \), see [15]. It appears that in the scalar case, Theorem 4.5 gives the growth rate \( r^{3/2} \) in terms of complexity, because testing constants similar to those in (i) and its dual are usually estimated by \( C \cdot (r + 1)[w]_{A_2} \).

However, then the standard splitting trick would allow us to get linear in complexity growth. Namely, one can split the operator \( T \) as \( T = \sum_{k=0}^r T_k \)

\[
T_k = \sum_{j \in \mathbb{Z}} \sum_{rkQ = k(r+1)j} T_{Q} ;
\]

then each \( T_k \) is a generalized big Haar shift of complexity zero, with respect to the rarefied filtration given by \( \sigma \)-algebras \( \mathcal{F}_{k+(r+1)n}, n \in \mathbb{Z} \).

In the scalar case, with the dyadic filtration, estimates of testing bounds like \( \Xi \) and \( \Xi^* \) in terms of \([w]_{A_2}\) do not appear to change if we restrict to rarefied filtrations, see [15]. As
5. Weighted paraproducts and their estimates

The essential part of the proof of the main results is the estimate of the associated weighted paraproducts, which is presented in this section.

5.1. Weighted paraproducts. Let \( f = 1_{S,e} \) be a characteristic function with \( S \in \mathcal{D} \) and \( e \in \mathbb{F}^d \). Then, for each fixed \( n \in \mathbb{Z} \), \( f \) has the orthogonal decomposition

\[
f = \sum_{Q \in \mathcal{D}, \text{rk} \, Q \geq -n} \Delta^W_Q f + \sum_{Q \in \mathcal{D}, \text{rk} \, Q = -n} E^W_Q f. \tag{5.1}
\]

To prove this equality, just observe that if \( \text{rk} \, Q \geq \text{rk} \, S \), then \( f 1_Q = E^W_Q f \) in \( L^2(W) \). So, if \( m \geq \text{rk} \, S \), we can use the definition of \( \Delta^W_Q \) to conclude

\[
f - \sum_{Q \in \mathcal{D}, \text{rk} \, Q \geq -n} \left( \Delta^W_Q f \right) - \sum_{Q \in \mathcal{D}, \text{rk} \, Q = -n} \left( E^W_Q f \right) = f - \sum_{Q \in \mathcal{D}, \text{rk} \, Q = m+1} \left( E^W_Q f \right) = 0
\]

in \( L^2(W) \). Letting \( m \to \infty \) gives the desired result. By orthogonality, it follows that

\[
\|f\|_{L^2(W)}^2 = \sum_{Q \in \mathcal{D}, \text{rk} \, Q \geq -n} \| \Delta^W_Q f \|_{L^2(W)}^2 + \sum_{Q \in \mathcal{D}, \text{rk} \, Q = -n} \| E^W_Q f \|_{L^2(W)}^2.
\]

For an operator \( T_w \) acting formally from \( L^2(W) \) to \( L^2(V) \), define the paraproduct \( \Pi^W = \Pi^W_T \) of complexity \( r \) as

\[
\Pi^W f = \sum_{Q \in \mathcal{D}, \text{rk} \, Q \in \text{Ch}^r Q} \Delta^V_Q \left( T_w E^W_Q f \right) = \sum_{Q \in \mathcal{D}, \text{rk} \, Q \in \text{Ch}^r Q} \Delta^V_Q \left( T_w \langle f \rangle_Q^W 1_Q \right); \tag{5.2}
\]

in the situations we consider, one can show that the bilinear form of the paraproduct \( \Pi^W \) is well defined for \( f, g \in \mathcal{L} \). Similarly for the adjoint \( T_{v^*} \) of \( T^W \), define the paraproduct \( \Pi^V = \Pi^V_{T^*} \) by

\[
\Pi^V g = \sum_{Q \in \mathcal{D}, \text{rk} \, Q \in \text{Ch}^r Q} \Delta^W_Q \left( T_{v^*} E^V_Q f \right).
\]

In later studies of these paraproducts, we will require the following simple lemma.

**Lemma 5.1.** Let \( T = T_w \) be a well localized operator of radius \( r \), acting formally from \( L^2(W) \) to \( L^2(V) \). Then for any cubes \( Q, S \in \mathcal{D} \) with \( Q \subset S \) and for any \( R \in \text{Ch}^r Q \) and for all \( e \in \mathbb{F}^d \),

\[
\Delta^r_R T_w 1_Q^e = \Delta^r_R T_w 1_s^e.
\]

**Remark 5.2.** The above Lemma 5.1 means that in the formula (5.2) for paraproducts, one can replace \( \langle f \rangle_Q^W 1_Q \) by \( \langle f \rangle_Q^W 1_S \) with an arbitrary cube \( S \supset Q \). So formally we can write in the right hand side of (5.2) the expression \( \langle f \rangle_Q^W 1 \) instead of \( \langle f \rangle_Q^W 1_Q \), which looks more in line with the definition of the paraproduct in the scalar case.
To make it even more similar to the scalar representation, we could use $T_w(1 \otimes I_{gd})$ instead of $T_w 1$ (to apply the operator $T_w$ to a matrix-valued function, one just needs to apply it to each column), and write the paraproduct $\Pi^w$ as

$$\Pi^w f = \sum_{Q \in \mathcal{D}} \sum_{R \in \text{Ch}_r Q} \left( T_w(1 \otimes I_{gd}) \right) \langle f \rangle^w_Q,$$

which is an alternate way of writing (5.2). The expression $T_w(1 \otimes I_{gd})$ should be understood as $T_w(1 \otimes I_{gd})$, where $S$ is an arbitrary cube with $Q \subset S$.

**Proof of Lemma 5.1.** Take a cube $P \neq Q$, $\text{rk } P = \text{rk } Q$. Since $T_w$ is $r$-lower triangular,

$$\Delta^V_R T_w 1_p e = 0,$$

for any cube $R \not\subset P$, $\text{rk } R \geq \text{rk } P + r$. In particular, that holds for $R \in \text{Ch}_r Q$.

Since for a cube $S \supset Q$ the set $S \setminus Q$ is a (countable) union of cubes $P$, $\text{rk } P = \text{rk } Q$, we conclude using the weak continuity property (3.4) that for any $R \in \text{Ch}_r Q$

$$\Delta^V_R T_w 1_{S \setminus Q} e = 0,$$

which proves the lemma. $\square$

**Remark 5.3.** As one can see, in the above proof we only used the fact that $T$ is $r$-lower triangular; more precisely, only a part of the definition was used.

The following lemma states that the paraproducts $\Pi^w$ and $\Pi^v$ exhibit the same behavior as $T_w$ and $T^*_v$ respectively.

**Lemma 5.4.** Let $T_w$ be a well localized operator of radius $r$ acting formally from $L^2(W)$ to $L^2(V)$, and let $\Pi^w = \Pi^w_T$ be the paraproduct of complexity $r$ defined as above. Then for $Q, R \in \mathcal{D}$

(i) If $\text{rk } R \leq r + \text{rk } Q$, then

$$\Delta^V_R \Pi^w \Delta^w_Q = 0.$$

(ii) If $R \not\subset Q$, then

$$\Delta^V_R \Pi^w \Delta^w_Q = 0.$$

(iii) If $\text{rk } R > r + \text{rk } Q$, then

$$\Delta^V_R \Pi^w \Delta^w_Q = \Delta^V_R T_w \Delta^w_Q,$$

and in particular if $R \not\subset Q$, both sides of the equality are zero.

**Proof.** Using summation indices $Q'$ and $R'$, we have

$$\Pi^w \Delta^w_Q = \sum_{Q' \in \mathcal{D}} \sum_{R' \in \text{Ch}_r(Q')} \Delta^V_{R'} \left( T_w E_{Q'} \Delta^w_Q \right),$$

and since $\Delta^V_R$ is orthogonal to $\Delta^V_{R'}$ for all choices of $R'$ except for $R$, we have

$$\Delta^V_R \Pi^w \Delta^w_Q = \Delta^V_R T_w E_{Q'} \Delta^w_Q$$
where \( Q' = R^{(r)} \) is the \( r \)-th order ancestor of \( R \). Notice that \( \mathbb{E}^W_{Q'} \Delta^W_Q \neq 0 \) only if \( Q' \subset Q \). So, if \( \text{rk } R \leq r + \text{rk } Q \) then \( \text{rk } R^{(r)} \leq \text{rk } Q \), which implies \( \mathbb{E}^W_{Q'} \Delta^W_Q = 0 \), and consequently
\[
\Delta^V_R \Pi^W_Q \Delta^W_Q = 0,
\]
proving the first statement. Also, if \( R \not\subset Q \), then \( Q' = R^{(r)} \not\subset Q \). As above, this implies \( \mathbb{E}^W_{Q'} \Delta^W_Q = 0 \), and consequently
\[
\Delta^V_R \Pi^W_Q \Delta^W_Q = 0,
\]
which proves the second statement.

To prove the third statement, assume \( \text{rk } R > r + \text{rk } Q \). If \( R \not\subset Q \), we can use our previous result and the fact that \( T^W \) is well localized to conclude:
\[
\Delta^V_R \Pi^W_Q \Delta^W_Q = 0 = \Delta^V_R T^W \Delta^W_Q.
\]
It now suffices to consider the case \( R \subset Q \). Recall that \( Q' = R^{(r)} \). Since \( Q \cap Q' \neq \emptyset \), we can look at ranks to conclude that \( Q' \subset Q \). Choose \( \tilde{Q} \in \text{Ch } Q \) with \( Q' \subseteq \tilde{Q} \). Then, using the fact that \( T^W \) is \( r \)-lower triangular, we have
\[
\Delta^V_R T^W \Delta^W_Q = \Delta^V_R T^W \left( \sum_{S \in \text{Ch } Q} E^W_S - E^W_Q \right) = \Delta^V_R T^W \left( E^W_Q - E^W_{\tilde{Q}} \cdot 1_{\tilde{Q}} \right).
\]
Using earlier arguments and \( Q' \subset Q \), we can write \( \Delta^V_R \Pi^W_Q \Delta^W_Q \) as
\[
\Delta^V_R T^W E^W_{Q'} \Delta^W_Q = \Delta^V_R T^W \left( E^W_{\tilde{Q}} \cdot 1_{Q'} - E^W_Q \cdot 1_{Q'} \right) = \Delta^V_R T^W \left( E^W_{\tilde{Q}} - E^W_Q \cdot 1_{\tilde{Q}} \right),
\]
where the last equality follows by Lemma 5.1, completing the proof. \( \square \)

5.2. Estimates of the paraproducts. The following theorem by the second and third authors from [5] will be used to control the norms of the paraproducts:

**Theorem 5.5** (The matrix weighted Carleson Embedding Theorem). Let \( W \) be a \( d \times d \) matrix-valued measure and let \( \{A_Q\}_{Q \in \mathcal{D}} \) be a sequence of positive semidefinite \( d \times d \) matrices indexed by \( \mathcal{D} \). Then the following statements are equivalent:

1. \( \sum_{Q \in \mathcal{D}} \left\| A^{1/2}_Q \int_Q dW f \right\|^2 \leq A \|f\|_{L^2(W)}^2 \)
2. \( \sum_{Q \in \mathcal{D}(Q_0)} W(Q) A_Q W(Q) \leq B W(Q_0) \) for all \( Q_0 \in \mathcal{D} \).

Moreover, for the best constants \( A \) and \( B \) we have \( B \leq A \leq C(d)B \), where \( C(d) \) is a constant depending only on the dimension \( d \).

**Remark.** In [5], the authors obtained the value \( C(d) = e \cdot d^3(d + 1)^2 \), where \( e \) is the base of the natural logarithm. This might give the optimal asymptotic in terms of the dimension \( d \), but it seems unlikely.

Now, we bound the paraproducts as follows:
Lemma 5.6. Let $\Pi^W$ be the paraproduct defined earlier and assume that the well localized operator $T_w$ satisfies the testing condition
\[\sum_{R \in D(Q)} \left\| \Delta^R T_w 1_{Q} e \right\|_{L^2(V)}^2 \leq \bar{\Omega}^2 \left\| 1_{Q} e \right\|_{L^2(W)}^2 \tag{5.4}\]
for all $Q \in D$ and $e \in \mathbb{F}^d$. Then $\Pi^W$ is bounded from $L^2(W)$ to $L^2(V)$ and
\[\left\| \Pi^W \right\|_{L^2(W) \to L^2(V)} \leq C(d)^{1/2} \bar{\Omega},\]
where $C(d)$ is the constant in Theorem 5.5.

Remark 5.7. The testing condition (5.4) is clearly weaker than the testing condition (i) from Theorem 4.2; the constant $\bar{\Omega}$ from (5.4) is majorized by the corresponding constant from (i).

Proof of Lemma 5.6. Fix $f \in L^2(W)$ and in the dense set $\mathcal{L}$. Then by orthogonality,
\[\left\| \Pi^W f \right\|_{L^2(V)}^2 = \sum_{Q \in D} \sum_{R \in \mathcal{C}'(Q)} \left\| \Delta^R \left( T_w B_Q f \right) \right\|_{L^2(V)}^2.\]

To control this, we use Theorem 5.5. First, for each $Q \in D$, define the linear map $B_Q : \mathbb{F}^d \to L^2(V)$ by
\[B_Q e = \sum_{R \in \mathcal{C}'(Q)} \Delta^R T_w (W(Q)^{-1} 1_{Q} e), \quad \forall e \in \mathbb{F}^d,\]
where $W(Q)^{-1}$ is the Moore-Penrose psuedoinverse of $W(Q)$. Then defining $A_Q := B_Q^* B_Q : \mathbb{F}^d \to \mathbb{F}^d$ we can write
\[\left\| \Pi^W f \right\|_{L^2(V)}^2 = \sum_{Q \in D} \left\| A_Q^{1/2} \int_Q df \right\|_{L^2(V)}^2,\]
so we are in position to apply Theorem 5.5.

To prove condition (ii) in Theorem 5.5, fix $Q_0 \in D$, $e \in \mathbb{F}^d$ and use the definitions of $A_Q$, $B_Q$ to obtain
\[\sum_{Q \in D(Q_0)} \left\| A_Q^{1/2} W(Q)e \right\|_{L^2(V)}^2 = \sum_{Q \in D(Q_0)} \left\| B_Q W(Q)e \right\|_{L^2(V)}^2\]
\[= \sum_{Q \in D(Q_0)} \sum_{R \in \mathcal{C}'(Q)} \left\| \Delta^R \left( T_w 1_{Q} e \right) \right\|_{L^2(V)}^2.\]

Then using Lemma 5.1 and the testing condition (5.4) we get
\[\sum_{Q \in D(Q_0)} \left\| A_Q^{1/2} W(Q)e \right\|_{L^2(V)}^2 \leq \bar{\Omega}^2 \left\| 1_{Q_0} e \right\|_{L^2(W)}^2\]
by Lemma 5.1,
\[\leq \bar{\Omega}^2 \left\| 1_{Q_0} e \right\|_{L^2(W)}^2\]
by (5.4),
so condition (ii) of Theorem 5.5 is verified. Thus
\[\left\| \Pi^W f \right\|_{L^2(V)}^2 \leq C(d) \bar{\Omega}^2 \left\| f \right\|_{L^2(W)}^2,\]
which completes the proof. \qed
6. Estimates of well localized operators

In this section we will prove Theorem 4.2. Theorem 4.1 will follow automatically, since the bounds $\Xi_1, \Xi_2$ and their duals $\Xi_1^*, \Xi_2^*$ from Theorem 4.2 are trivially majorized by the corresponding bounds from Theorem 4.1, and the bound $\Xi_3$ from Theorem 4.2 is dominated by the minimum of $\Xi_1$ and $\Xi_2^*$ from Theorem 4.1. We will also explain Remark 4.3, claiming that the weak estimate (iii) of Theorem 4.2 can be relaxed and sometimes ignored.

To prove Theorem 4.2, we estimate the bilinear form of the operator $T_W$. Let $f \in L^2(W)$ and $g \in L^2(V)$, with $\|f\|_{L^2(W)} = \|g\|_{L^2(V)} = 1$, be from the dense set $\mathcal{L}$ of finite linear combinations of characteristic functions of atoms times vectors, i.e.

$$f = \sum_{j=1}^{N} 1_{Q_j} e_j \quad \text{and} \quad g = \sum_{k=1}^{M} 1_{R_k} v_k,$$

(6.1)

where each $Q_j, R_k \in \mathcal{D}$ and $e_j, v_k \in \mathbb{F}^d$. By Lemma 8.1, such functions are dense in $L^2(W)$ and $L^2(V)$ and so to obtain the result, we just need to show that

$$|\langle T_W f, g \rangle_{L^2(V)}| \leq C\|f\|_{L^2(W)}\|g\|_{L^2(V)},$$

(6.2)

Let us first do some simplifications. Define an equivalence relation $\sim$ on $\mathcal{D}$, by saying that $Q \sim R$ if $Q$ and $R$ have a common ancestor (i.e. if $Q, R \subset S$ for some $S \in \mathcal{D}$).

Since $T_W$ is a localized operator, $\langle T_W 1_Q e, 1_R \nu \rangle_{L^2(V)} = 0$ if $Q$ and $R$ are in different equivalence classes, for all $e, \nu \in \mathbb{F}^d$. Therefore, it is sufficient to prove (6.2) under the assumption that all $Q_j, R_k$ in the representation (6.1) are in the same equivalence class; then taking the direct sum over equivalence classes, we get the general case.

Let $Q_0 \in \mathcal{D}$ be a common ancestor of all $Q_j, R_k$ appearing in the representation (6.1). Then, by (5.1), we can write $f, g$ using the orthogonal decompositions:

$$f = \sum_{Q \in \mathcal{D}(Q_0)} \Delta_Q^W f + \mathbb{E}_{Q_0}^W f =: f_1 + f_2; \quad \text{(6.3)}$$

$$g = \sum_{R \in \mathcal{D}(Q_0)} \Delta_R^V g + \mathbb{E}_{Q_0}^V g =: g_1 + g_2. \quad \text{(6.4)}$$

We will estimate the four terms $\langle T_W f_j, g_k \rangle_{L^2(V)}$ for $1 \leq j, k \leq 2$ separately.

6.1. Estimate of the main part. To estimate $\langle T_W f_1, g_1 \rangle_{L^2(V)}$ let us first notice that by Lemma 5.6, the testing condition (i) of Theorem 4.2 and its dual counterpart imply that the paraproducts $\Pi^W = \Pi_T^W$ and $\Pi^V = \Pi_T^V$ are bounded and that

$$\|\Pi^W\|_{L^2(W) \to L^2(V)} + \|\Pi^V\|_{L^2(V) \to L^2(W)} \leq C \langle d \rangle^{1/2} (\Xi_1 + \Xi_1^*).$$

Thus, it is sufficient to estimate the operator $\tilde{T}_W := T_W - \Pi^W - (\Pi^V)^*$. Lemma 5.4 implies that

$$\Delta_R^V \tilde{T}_W \Delta_Q^W = \begin{cases} \Delta_R^V T_W \Delta_Q^W, & |\text{rk} Q - \text{rk} R| \leq r; \\ 0, & |\text{rk} Q - \text{rk} R| > r, \end{cases}$$
\[ \langle \tilde{T}_{W}^+, g_{1} \rangle_{L^2(V)} = \sum_{Q, R \in \mathcal{D}(Q_0) \mid r \leq Q - r \leq R} \langle T_{W} \Delta_{Q} f, \Delta_{R} g \rangle_{L^2(V)} \]

\[ = \sum_{Q, R \in \mathcal{D}(Q_0) \mid r \leq Q - r \leq R} \langle T_{W} \Delta_{Q} f, \Delta_{R} g \rangle_{L^2(V)} + \sum_{Q, R \in \mathcal{D}(Q_0) \mid r \leq Q - r \leq R + r} \langle T_{W} \Delta_{Q} f, \Delta_{R} g \rangle_{L^2(V)}. \]

Let us estimate the first sum. The second one is treated similarly, by considering the dual operator \( T_{W}^{\ast} \). To estimate the first sum, we need to estimate the operator

\[ \tilde{T}_{W}^{+} := \sum_{Q, R \in \mathcal{D}(Q_0) \mid r \leq Q - r \leq R} \Delta_{R} T_{W} \Delta_{Q}^{W}. \]

Since \( T_{W} \) is \( r \)-lower triangular, we can see that \( \Delta_{R} T_{W} \Delta_{Q}^{W} = 0 \) if \( r \leq Q \) and \( R \not\in Q^{(r)} \).

So, we can rewrite \( \tilde{T}_{W}^{+} \) as

\[ \tilde{T}_{W}^{+} = \sum_{s \in \mathcal{D}(Q_0^{(r)})} \sum_{r \leq Q \leq R \leq r + 2r} \sum_{Q \in \mathcal{D}(Q_0^{(r)}) \cap \mathcal{D}(Q_0^{(r)})} \Delta_{R} T_{W} \Delta_{Q}^{W} =: \sum_{s \in \mathcal{D}(Q_0^{(r)})} \tilde{T}_{W}^{+}, \]

where

\[ \tilde{T}_{W}^{+} := \sum_{r \leq Q \leq R \leq r + 2r} \sum_{Q \in \mathcal{D}(Q_0^{(r)}) \cap \mathcal{D}(Q_0^{(r)})} \Delta_{R} T_{W} \Delta_{Q}^{W}. \]

The testing condition (ii) of Theorem 4.2 implies that

\[ \| \tilde{T}_{W}^{+} \|_{L^2(W) \rightarrow L^2(V)} \leq \mathfrak{S}_2. \]  \hspace{1cm} (6.5)

Note that if \( S \cap S' = \emptyset \) or \( |r \leq Q \) and \( R \not\in Q^{(r)} | r \leq Q - r \leq R \), then

\[ \text{Ran} \tilde{T}_{W}^{+} \perp \text{Ran} \tilde{T}_{W}^{+}, \quad \big( \ker \tilde{T}_{W}^{+} \big)_{\perp} \perp \big( \ker \tilde{T}_{W}^{+} \big)_{\perp}. \]

Therefore for fixed \( k \in \mathbb{Z} \), the operator \( \tilde{T}_{W}^{+,k} \) defined by

\[ \tilde{T}_{W}^{+,k} := \sum_{j \in \mathbb{Z}} \sum_{s \in \mathcal{D}(Q_0^{(r)}) \cap \mathcal{D}(Q_0^{(r)})} \tilde{T}_{W}^{+} \]

is the direct sum of the corresponding operators \( \tilde{T}_{W}^{+,k} \), and the estimate (6.5) implies

\[ \| \tilde{T}_{W}^{+,k} \|_{L^2(W) \rightarrow L^2(V)} \leq \mathfrak{S}_2. \]  \hspace{1cm} (6.6)

Since \( \tilde{T}_{W}^{+} = \sum_{k=0}^{r} \tilde{T}_{W}^{+,k} \), we can easily conclude from (6.6) that

\[ \| \tilde{T}_{W}^{+} \|_{L^2(W) \rightarrow L^2(V)} \leq (r + 1) \mathfrak{S}_2. \]

However, by being more careful, we can obtain the following better dependence on \( r \):

\[ \| \tilde{T}_{W}^{+} \|_{L^2(W) \rightarrow L^2(V)} \leq (r + 1)^{1/2} \mathfrak{S}_2. \]  \hspace{1cm} (6.7)
To get this, observe that for \(0 \leq j < k \leq r\)

\[
\left( \ker \tilde{T}^{+,j}_{\mathbf{w}} \right)^\perp \perp \left( \ker \tilde{T}^{+,k}_{\mathbf{w}} \right)^\perp.
\]

Then, decomposing \(f_1 = \sum_{k=0}^{r} f^k\), where

\[
f^k := \sum_{n \in \mathbb{Z}} \sum_{S \in \mathcal{D}(Q_0^{(r)})} \sum_{Q \in \text{Ch}^r S} \Delta^W_Q f,
\]

we get that

\[
\|\tilde{T}^+_{\mathbf{w}} f_1\|_{L^2(\mathbf{V})} = \left\| \tilde{T}^+_{\mathbf{w}} \sum_{k=0}^{r} f^k \right\|_{L^2(\mathbf{V})} = \left\| \sum_{k=0}^{r} \tilde{T}^+_{\mathbf{w}} f^k \right\|_{L^2(\mathbf{V})} 
\leq \mathfrak{X}_2 \sum_{k=0}^{r} \|f^k\|_{L^2(\mathbf{w})} \leq \mathfrak{X}_2 (r+1)^{1/2} \left( \sum_{k=0}^{r} \|f^k\|_{L^2(\mathbf{w})}^2 \right)^{1/2};
\]

here the last inequality is by Cauchy–Schwarz.

### 6.2. Estimates of parts involving constant functions.

Estimates

\[
|\langle T_{\mathbf{w}} f_2, g_1 \rangle_{L^2(\mathbf{V})}| \leq \mathfrak{X}_1, \quad |\langle T_{\mathbf{w}} f_1, g_2 \rangle_{L^2(\mathbf{V})}| \leq \mathfrak{X}_1^*
\]

follow immediately from the testing condition (i) and its dual. Estimate

\[
|\langle T_{\mathbf{w}} f_2, g_2 \rangle_{L^2(\mathbf{V})}| \leq \mathfrak{X}_3
\]

is a direct corollary of the assumption (iii).

Note that in decompositions (6.3) and (6.4), we can replace \(Q_0\) by any of its ancestors, so, as we said in Remark 4.3 it is sufficient that the estimate (iii) holds only for sufficiently large cubes \(Q\) (meaning that for any \(Q_0 \in \mathcal{D}\) we can find \(Q \in \mathcal{D}, Q_0 \subset Q\) such that (iii) holds for \(Q\)).

Moreover, if for the increasing sequence of cubes \(Q_n, n \geq 0\), where \(Q_{n+1}\) is the parent of \(Q_n\), we have that \(\mathbf{W}(Q_n) \geq \alpha_n \mathbf{I}, \alpha_n \nearrow \infty\) then writing decomposition (6.3) with \(Q_n\) instead of \(Q_0\) and letting \(n \to +\infty\), we obtain

\[
f = \sum_{Q \in \mathcal{D}} \Delta^W_Q f =: f_1.
\]

The analogous condition for \(\mathbf{V}\) implies the similar representation for \(g\), so the theorem is reduced to estimating \(\langle T_{\mathbf{w}} f_1, g_1 \rangle_{L^2(\mathbf{V})}\), which was done using only testing conditions (i), (ii) and their duals.

### 7. Estimates of the Haar shifts

In this section we will prove Lemmas 4.4 and 7.2. Theorem 4.5 is then a simple corollary of Theorem 4.2. We will need the following lemma, which is well known to specialists; for the convenience of the reader we present its proof here.
Lemma 7.1. Let \( T \) be an integral operator with kernel \( K \), \( Tf(x) = \int K(x,y)f(y)d\sigma(y) \), where \( K \) is supported on \( Q \times Q \) \((Q \in \mathcal{D})\) and \( \|K\|_\infty \leq |Q|^{-1} \). If the \( d \times d \) matrix measures \( V, W \) satisfy the matrix \( A_2 \) condition (4.4), then the operator \( T_W, T_Wf := T(Wf) \) satisfies

\[
\|T_W\|_{L^2(W) \to L^2(V)} \leq d^{1/2}[V,W]^{1/2}_{A_2}.
\]

Proof. Take \( f \in L^2(W) \), \( g \in L^2(V) \), with \( \|f\|_{L^2(W)} = \|g\|_{L^2(V)} = 1 \). As we discussed above in Section 1.3, we can assume without loss of generality that the measures \( V \) and \( W \) are absolutely continuous with respect to the scalar trace measures \( v \) and \( w \) respectively, \( dV = Vdv \), \( dW = Wdw \). We then can write

\[
\left|\langle T_Wf, g \rangle_{L^2(V)}\right| \leq \iint_{Q \times Q} |\langle V(x)K(x,y)W(y)f(y), g(x) \rangle_{p_d}| \, dv(x) \, dw(y).
\]

The integral then can be estimated by

\[
\left|\langle T_Wf, g \rangle_{L^2(V)}\right| \leq \left( \iint_{Q \times Q} \|V^{1/2}(x)W^{1/2}(y)\| \|V^{1/2}(x)g(x)\|_{p_d} \|W^{1/2}(y)f(y)\|_{p_d} \, dv(x) \, dw(y) \right)^{1/2} \times \left( \left|\iint_{Q \times Q} \|V^{1/2}(x)W^{1/2}(y)\|^2 \, dv(x) \, dw(y) \right| \right)^{1/2}.
\]

In the last integral, we can replace the operator norm by the Frobenius (Hilbert–Schmidt) norm \( \| \cdot \|_{\mathcal{S}_2} \) (recall that \( \|A\|_{\mathcal{S}_2} = \text{tr}(A^*A) \)):

\[
\iint_{Q \times Q} \|V^{1/2}(x)W^{1/2}(y)\|^2 \, dv(x) \, dw(y) \leq \left( \iint_{Q \times Q} \|V^{1/2}(x)W^{1/2}(y)\|^2_{\mathcal{S}_2} \, dv(x) \, dw(y) \right)^2 \leq \text{tr} \left( \left( \langle V(x)W(y) \rangle \right) \right) \leq \text{tr} \left( V(Q)W(Q) \right) \leq \|V(Q)^{1/2}W(Q)^{1/2}\|_{\mathcal{S}_2}^2 \leq d \|V(Q)^{1/2}W(Q)^{1/2}\|_{A_2}^2.
\]

Combining this with the previous estimate, we get the conclusion of the lemma. \( \square \)

7.1. Comparison of different truncations. Let \( T \) be a generalized big Haar shift of complexity \( r \). As before, we will assume that the we only have finitely many terms \( T_Q \) in the representation (2.11) and that each block \( T_Q \) is represented by an integral operator with a bounded kernel.

In the testing conditions in Theorems 4.2 and 4.5, we used different truncations of the operator \( T_W \), namely \( T_W^Q \) and \( (T^Q)_W \) respectively. These operators are generally different, but their difference can be estimated.
Now to state the estimate, we will need some new notation. Let $P^V_Q$ be the orthogonal projection in $L^2(V)$ onto the subspace of functions supported on $Q$ and orthogonal to $\{1_Q e : e \in \mathbb{F}^d\}$. Then, by (5.1),

$$P^V_Q f = \sum_{R \in \mathcal{D}(Q)} \Delta^V_R f = 1_Q f - \mathbb{E}^V_Q f, \quad \forall f \in \mathcal{L},$$

and we can extend this to all $f \in L^2(W)$. Then in this notation, the operator $T^Q_W$ defined above can be written as $T^Q_W = P^V_Q T_W$.

**Lemma 7.2.** For operators $T^Q_W$ and $(T^Q)_W$ introduced above and $f \in L^2(W)$ supported on $Q$,

$$\left\| \left( T^Q_W - P^V_Q (T^Q)_W \right) f \right\|_{L^2(V)} \leq d^{1/2} [V, W]^{1/2} \|f\|_{L^2(W)}.$$  

**Proof.** For $f \in \mathcal{L}$ and supported on $Q$ we have

$$\left( T^Q_W - P^V_Q (T^Q)_W \right) f = \sum_{k=1}^{r} P^V_Q T^{(k)}_W (W f),$$

where $Q^{(k)}$ is the ancestor of $Q$ order $k$. Note that the terms $T^{(k)}_W (W f)$ with $k > r$ are annihilated by $P^V_Q$.

Each operator $T^{(k)}_W$ is an integral operator with kernel $K^{(k)}_Q$ supported on $Q^{(k)} \times Q^{(k)}$ and satisfying $\|K^{(k)}_Q\|_\infty \leq |Q^{(k)}|^{-1}$. Therefore applying Lemma 7.1 and using the fact that $P^V_Q$ is an orthogonal projection (and so a contraction) in $L^2(V)$ we get

$$\|P^V_Q T^{(k)}_W (W f)\|_{L^2(V)} \leq d^{1/2} [V, W]^{1/2} \|f\|_{L^2(W)}.$$  

Summation over $k$ completes the proof. \hfill \Box

**Lemma 7.3.** Assume there exists $\kappa \in (0, 1)$ such that for any $Q \in \mathcal{D}$ and its parent $\hat{Q}$,

$$|Q| \leq \kappa |\hat{Q}|.$$  

Further, assume the weights $V$, $W$ satisfy the matrix $A_2$ condition (4.4). Then for any generalized big Haar shift $T$ of order $r$ with finitely many non-zero terms and for any $f \in L^2(W)$ supported on $Q \in \mathcal{D}$

$$\|1_Q (T_W - (T^Q)_W) f\|_{L^2(V)} \leq (1 - \kappa)^{-1} d^{1/2} [V, W]^{1/2} \|f\|_{L^2(W)}.$$  

**Corollary 7.4.** Under the assumptions of Lemma 7.3, the condition (i) from Theorem 4.5 is necessary for the boundedness of the operator $T_W : L^2(W) \to L^2(V)$, and the constants $\underline{\xi}$, $\overline{\xi}$ from Theorem 4.5 satisfy

$$\underline{\xi}, \overline{\xi} \leq \|T\|_{L^2(W) \to L^2(V)} + (1 - \kappa)^{-1} d^{1/2} [V, W]^{1/2}.$$  

**Proof of Lemma 7.3.** Fix $Q \in \mathcal{D}$ and let $Q^{(k)}$ be the ancestor of order $k$ of $Q$. Then for $f$ supported on $Q$

$$(T_W - (T^Q)_W) f = \sum_{k \geq 1} T^{(k)}_W (W f).$$
This sum can be written as the integral operator

\[ \int Q K(x, y) W(y) f(y) dy, \]

where

\[ K(x, y) = \sum_{k \geq 1} K_{Q(k)}(x, y), \]

and \( K_{Q(k)} \) is the kernel of the integral operator \( T_{Q(k)} \). We can now apply Lemma 7.1 to this sum of integral operators to obtain the desired bound. \( \square \)

### 7.2. Proof of Lemma 4.4.

Let the testing condition (4.6) hold. Applying Lemma 7.2 with \( f = 1_Q e \) and noticing that

\[ \| P^V Q^W f \|_{L^2(V)} \leq \| (T^Q) W f \|_{L^2(V)} \leq \| f \|_{L^2(W)}, \]

we immediately get (4.7).

To get (4.8), we need a bit more work. We can write

\[ T^Q = T^{r+1} + \sum_{k=0}^r T_k, \]

where

\[ T^{r+1} = \sum_{R \in Ch^{r+1} Q} T_R, \quad T_k = \sum_{R \in Ch^k Q} T_R, \]

with the obvious agreement that \( Ch^0 Q = \{ Q \} \). Following the agreed notation, for a scalar integral operator \( T \) we denote by \( T_W \) the operator defined by

\[ T_W f := T(W f), \]

whenever this expression is defined.

The operators \( T_R \) are \( R \)-localized, meaning that \( T_R f = T_R(1_R f) \), and \( T_R f \) is supported on \( R \), and the same holds for \( T^R \).

The functions \( f_Q \in D^W,r_Q \) are constant on cubes \( R \in Ch^{r+1} Q \), so using the testing condition (4.6) and the fact that the operators \( T^R \) are \( R \)-localized, we get for \( f_Q \in D^W,r_Q \)

\[ \| (T^{r+1})_W f_Q \|_{L^2(V)}^2 = \sum_{R \in Ch^{r+1} Q} \| T^R (W 1_R f_Q) \|_{L^2(V)}^2 \]
\[ \leq \sum_{R \in Ch^{r+1} Q} \| 1_R f_Q \|_{L^2(W)}^2 \]
\[ = \| f_Q \|_{L^2(W)}^2. \]  

(7.1)

To estimate the operators \( T_k \), we estimate each block \( T_k \) by Lemma 7.1, and using the fact that \( T_R \) is \( R \)-localized we get for \( f_Q \in D^W,r_Q \)

\[ \| (T_k)_W f_Q \|_{L^2(V)}^2 = \sum_{R \in Ch^k Q} \| T_R (W 1_R f_Q) \|_{L^2(V)}^2 \]
\[ \leq \sum_{R \in Ch^k Q} d [V, W]_{A_2} \| 1_R f_Q \|_{L^2(W)}^2 \]
\[ = d [V, W]_{A_2} \| f_Q \|_{L^2(W)}^2. \]
Adding these estimates for \( k = 0, 1, \ldots, r \) and combining them with (7.1), we see that for any \( f_Q \in D^W_\sigma \)
\[
\| (T^Q)_W f_Q \|_{L^2(V)} \leq \left( d^{1/2}(r + 1)[V, W]^{1/2} + \Xi \right) \| f_Q \|_{L^2(W)}.
\]
Since the projection \( P^V_Q \) is a contraction in \( L^2(V) \), the same estimate holds for the norm \( \| P^V_Q(T^Q)_W f \|_{L^2(V)} \), so combining it with Lemma 7.2, we obtain (4.8).

Finally, to show that (4.9) holds, let us recall that \( T = \sum_{R \in R} T_R \), where \( R \subset D \) is some finite collection. Then for each \( Q_0 \in D \) we can find a cube \( Q \supset Q_0 \) which is not contained in any \( R \in R \). Then \( T_W 1_Q e = (T^Q)_W 1_Q e \), and (4.9) follows from (4.6).

8. Appendix: Density of simple functions

**Lemma 8.1.** Let \( \mathcal{F} \) be the smallest \( \sigma \)-algebra containing an increasing sequence of atomic \( \sigma \)-algebras \( \mathcal{F}_n \), with sets of atoms \( \mathcal{D}_n \). Let \( \mathcal{L} \) denote the space of linear combinations of functions \( 1_Q e \) with \( Q \in D = \bigcup_n D_n \) and \( e \in \mathbb{F}^d \). If \( W \) is a \( d \times d \) matrix valued measure defined on \( \mathcal{F} \), then \( \mathcal{L} \) is dense in \( L^2(W) \).

**Proof.** First, observe that the result is true for scalar measures. Indeed, if \( \sigma \) is a scalar measure defined on \( \mathcal{F} \), then linear combinations of sets \( 1_Q, Q \in D \) are dense in \( L^2(\sigma) \). To see this, observe that we can obtain \( \sigma \) by first starting with \( \sigma \) defined on \( D \) and then extending \( \sigma \) to \( \mathcal{P}(\mathcal{X}) \) via the outer measure
\[
\sigma^*(F) := \inf \left\{ \sum_{j=1}^\infty \sigma(Q_j) : Q_j \in D, F \subset \bigcup_{j=1}^\infty Q_j \right\}.
\]
Then the Carathéodory’s Theorem implies that \( \sigma^* \) restricts to a measure on a \( \sigma \)-algebra \( \mathcal{M} \), which contains \( \mathcal{F} \). We only consider this measure restricted to \( \mathcal{F} \) and uniqueness implies that this measure is \( \sigma \). Then (8.1) shows linear combinations of \( 1_Q, Q \in D \) are dense in the set of linear combinations of \( 1_F, F \in \mathcal{F} \), which are dense in \( L^2(\sigma) \).

Now consider the matrix setting. Let \( W \) be a \( d \times d \) matrix valued measure defined on \( \mathcal{F} \) with trace measure \( W := \sum w_{i,i} \). Then
\[
|w_{i,j}(F)| \leq dw(F) \quad \forall F \in \mathcal{F}.
\]
Thus, the Radon-Nikodym Theorem allows us to write \( W = W(x) dw \), where the entries of \( W \) are in \( L^\infty(w) \).

We claim that if \( f \in L^2(W) \) satisfies \( \langle f, e 1_Q \rangle_{L^2(W)} = 0 \) for all \( e \in \mathbb{F}^d, Q \in D \), then \( f \equiv 0 \) in \( L^2(W) \). To see this, fix \( e \in \mathbb{F}^d \) and suppose that for any \( Q \in D \),
\[
\int_X \langle W(x)f, e \rangle_{\mathbb{F}^d} 1_Q \, dw = \int_X \langle dWf, 1_Q e \rangle_{\mathbb{F}^d} = 0.
\]
The scalar result implies the function \( \langle W(x)f(x), e \rangle_{\mathbb{F}^d} = 0 \) \( w \)-a.e. Let \( W(x)^{-1} \) denote the Moore-Penrose pseudoinverse of \( W(x) \). Then
\[
W(x)^{-1}W(x)f(x) = 0 \quad w \text{-a.e.}
\]
as well. This immediately implies that
\[
\| f \|_{L^2(W)}^2 = \int_X \langle dWf, f \rangle = \int_X \langle W^{-1}Wf, Wf \rangle \, dw = 0,
\]
so $f$ is the zero element in the space $L^2(W)$.

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