Climb on the Bandwagon: Consensus and Periodicity in a Lifetime Utility Model with Strategic Interactions

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Abstract
What is the emergent long-run equilibrium of a society, where many interacting agents bet on the optimal energy to put in place in order to climb on the Bandwagon? In this paper, we study the collective behavior of a large population of agents being either Left or Right: The core idea is that agents benefit from being with the winner party, but, on the other hand, they suffer a cost in changing their status quo. At the microscopic level, the model is formulated as a stochastic, symmetric dynamic game with \( N \) players. In the macroscopic limit as \( N \to +\infty \), the model can be rephrased as a mean field game, whose equilibria describe the “rational” collective behavior of the society. It is of particular interest to detect the emerging long time attractors, e.g., consensus or oscillating behavior. Significantly, we discover that bandwagoning can be persistent at the macrolevel: We provide evidence, also on the basis of numerical simulations, of endogenously generated periodicity.

Keywords Consensus · Mean field games · Multiple Nash equilibria · Opinion dynamics · Social interactions

1 Introduction
The emergence of collective behavior in complex societies has been one of the most studied paradigms of social sciences in the last decades. Pioneering works, among the others, have been devoted to the study of segregation (see [30]), social innovation (see [31]), riot’s formation (see [18]), social distance (see [1]) or the emergence of prices in financial markets (see [15]).
Following the celebrated *micromotives and macrobehavior* paradigm by Shelling (see [29]), large attention has been paid to the mechanisms under which social norms emerge as an aggregate output of a large population of interacting agents. In [5], we find one of the first attempts to formally analyze the large limits for economies characterized by social externalities, whereas in [24] a similar results is formalized in a game-theoretical setting, and in [14] in the field of artificial societies and agent based models. One of the key factors underpinning all these models is the presence of *positive externalities*, meaning that the single agent benefits from aligning with social norms, or, put differently, from being with the majority. The behavioral attitude behind this assumption is, basically, conformism, imitation or peer pressure. Of course, depending on the applications, different behavioral assumptions can be made (see, for instance, [28] or [10] for nonconforming individuals or minority games, respectively).

One aspect that, to our opinion, has not been sufficiently considered in the aforementioned literature, is the fact that changing opinion may be *effort-demanding* (for a recent contribution on this topic, see [2]). Gather information, modify habits or practices, revise operational strategies, join a new technology or, in one word, *climbing on the bandwagon*, may result to be a costly operation. This is the main goal of this paper: provide a stylized model to describe a large population of conformist agents, in charge to optimally determine the level of energy required to stick with the majority. Our idea is simple: The higher the effort is, the more likely it is to be with the winner party and the higher is the associated cost.

Indeed, to dynamically study the aggregate behavior of the society, we rely on a lifetime utility maximization problem where the agent is in charge to optimally set the effort to put in place to change status. Two remarks are needed. Firstly, the lifetime setup immediately refers to a parallel strand of literature, related to more classical optimization problems for consensus formation (see [4,21,32]). The second remark pertains to the modeling structure of the society: in order to study in detail (and possibly to obtain closed-form solutions) the relationship between social interaction and frictions in changing opinion, we stick with the simplest possible geometry: a mean field model. This means, in particular, that aggregate statistics such as the empirical distribution (or the empirical mean) are sufficient to fully describe the Markovian system. With this respect, we move in the framework of *mean field games*. The recent theory in this field has put forward a class of dynamic games for which the limit behavior, as the number of agents increases to infinity, can be described in analytic terms (see [19,22,25]).

In a nutshell, in our model the state of each agent (Left or Right) evolves as a controlled Markov process; the discounted lifetime utility is formed by two components: a reward received only when agreeing with the majority and a quadratic instantaneous cost related to the effort put in place. Since each agent looks for the control maximizing her own utility, under strategic interactions, it is natural to consider Nash equilibria for the resulting dynamic game. When the limit of infinitely many players is considered, we formally obtain the usual *mean field game equation*, given by a system of two coupled equations: One is the Hamilton–Jacobi–Bellman equation for the value function, and the second is the master equation for the optimal evolution of the representative agent. The rigorous foundation of this formal limit is to a large extent an open problem. Rigorous convergence results have been obtained recently for diffusion models (see [6,17]) and for models with discrete state space (see [9]); these results, that are limited to the case of finite time horizon, require assumptions that guarantee the *uniqueness* of the solution of the mean field game equation. In the model considered here,
this uniqueness fails. As pointed out in [8], all solutions of the mean field game equation have “physical” significance for the $N$-player game: If the feedback control corresponding to one solution of the mean field game equation is applied by each player in the $N$-player game, it is an approximate Nash equilibrium, with the approximation error going to zero as $N \to + \infty$. However, some solutions of the mean field game may not be obtained as limit of Nash equilibria of the $N$-player game. For examples of non uniqueness and the related convergence problems, we refer to [3,7].

In this paper, we do not provide a rigorous convergence result in the number of players. We, rather, devote our attention to the long-run behavior of the asymptotic model and its properties. In particular, for the class of models introduced below, we find cases in which the long-run behavior of the mean field game leads to consensus, other in which the limit system admits periodic and non-constant solutions. This rhythmic behavior, a sort of macrobandwagoning of the society, emerges in the absence of external periodic signals, and it is endogenously produced by the micromotives behind the strategic behavior of agents. Investigation of periodic behavior of multi-agent systems has been recently studied, for instance, in [16,35]. In the context of mean field games, however, periodic behavior has been often predicted but, to our knowledge, proved only for the rather celebrated Mexican wave model (see [19]). It must be remarked that the Mexican wave model possesses a continuous symmetry, which allows the appearance of traveling wave solution. The model we propose below has a discrete (actually binary) space structure, so there is no continuous symmetry. Recent years have seen a formidable effort in the attempt of explaining rigorously the emergence of collective periodicity in noisy systems of interacting units. Given the impossibility of accounting for the huge related literature, we only mention the inspiring work [26], and few available rigorous results in [11–13,33]. In these works a key role in the emergence of periodicity is played by delay in the information transmission (see [13,33]) and dissipation (see [11,12]). One of the main purposes of this paper is to show that collective periodic behavior can also result from agents’ utility optimization.

2 The Microscopic Model

In this section, we introduce the $N$-player game. Later we introduce its formal limit as $N \to + \infty$. Although only the limit model will be analyzed, the $N$-player game motivates the limit model and provides a more direct interpretation of the parameters involved. Indeed, we will come back to the microscopic $N$-dimensional model at the end of Sect. 4 when discussing the interpretation of two possible specifications of the model.

Consider a network of $N$ interacting agents, each possessing a binary state $\sigma_i(t) \in \{-1, 1\}$ at time $t \geq 0$. Every agent can control her own state by means of the control $u_i = (u_i(t))_{t \geq 0}$. We assume here closed-loop controls under complete information:

$$u_i(t) = \varphi_i(t, \sigma(t))$$

for some function $\varphi_i$ which is right-continuous in $t$ and depending on the whole state $\sigma(t) = (\sigma_j(t))_{j=1}^N$ at time $t$. The controlled stochastic dynamics are given by

$$\mathbb{P}\left(\sigma_i(t + h) = -\sigma_i(t) \mid \sigma(s), s \leq t\right) = u_i(t)h + o(h).$$

(1)
In other words, $u_i(t)$ is the probability rate of flipping the state $\sigma_i$. Let

$$m_N(t) := \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t)$$

be the average state of the network at time $t$. The instantaneous reward of agent $i$ at time $t$ is given by

$$R_i(t) := \sigma_i(t)m_N(t) - \frac{1}{2\mu(\sigma_i(t), m_N(t))}u_i^2(t).$$

The two summands in the reward $R_i$ are easy to interpret. The term $\sigma_i(t)m_N(t)$ favors imitation: Agents are conformist; they gain when aligned with the majority. The term $-\frac{1}{2\mu}u_i^2(t)$ is an energy cost: A rapid change of the state would require high values for $u_i$, which are costly. The factor $\mu(\sigma_i(t), m_N(t))$, that we assume to be strictly positive (i.e., $\mu(\sigma, m) \geq c > 0$), modulates the relevance of this cost term: Large values of $\mu$ allow high mobility to the agents, who can rapidly adapt to a change in the majority. Conversely, small values of $\mu$ reduce the adaptive response of agents. We allow $\mu$ to depend on the state of agent $i$ and on the average state of the network. Note that the state dependence of $\mu$, which is equivalent to introduce an explicit symmetric state dependence of the form

$$\mathbb{P}\left(\sigma_i(t+h) = -\sigma_i(t) \mid \sigma(s), s \leq t\right) = A(\sigma_i(t), m_N(t))u_i(t)h + o(h)$$

in the single agent dynamics, allows a suitable level of generality and will be included in one of the example we present in next section.

Each agent $i$ aims at maximizing the discounted lifetime utility

$$U_i := \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} R_i(t)dt\right],$$

where $\lambda > 0$ is a constant discount factor. We remark $U_i = U_i(u_1, u_2, \ldots, u_n)$ depends on the dynamics of the whole system, so on the vector of controls of every agent. A control $u^* = (u^*_1, u^*_2, \ldots, u^*_N)$ is called a Nash equilibrium if for every $i = 1, \ldots, N$,

$$U_i(u^*_1, \ldots, u^*_{i-1}, u^*_i, u^*_{i+1}, \ldots, u^*_N) \geq U_i(u^*_1, \ldots, u^*_{i-1}, u_i, u^*_{i+1}, \ldots, u^*_N)$$

for every other control $u_i$: In equilibrium, no agent has interest in changing her strategy. Note that this dynamic game is invariant for permutation of agents, so it falls within the domain of mean field games (see [23,25]).

### 3 The Macroscopic Model

The limit as $N \to +\infty$ of the dynamic game described above is easy to obtain at a heuristic level. One expects that the average state $m_N(t)$ obeys a Law of Large Numbers, so it converges to a deterministic limit $m(t)$. The representative agent, whose state is denoted, with a slight abuse of notation, by $\sigma(t) \in \{-1, 1\}$, moves according to the feedback controlled dynamics

$$\mathbb{P}\left(\sigma(t+h) = -\sigma(t) \mid \sigma(s), s \leq t\right) = u(t)h + o(h),$$

with a possibly random initial condition $\sigma(0)$; she aims at maximizing

$$J(u) := \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} \left(\sigma(t)m(t) - \frac{1}{2\mu(\sigma(t), m(t))}u^2(t)\right)dt\right].$$

(2)
Note that \( J(u) \) also depends on the distribution of \( \sigma(0) \), which is however omitted. An equilibrium control \( u^* \) must satisfy the following consistency relation: If we denote by \( \sigma^*(t) \) the process produced by the control \( u^* \), then

\[
m(t) = \mathbb{E}[\sigma^*(t)].
\]

This problem is solved in two steps: Firstly, one writes the dynamic programming equation corresponding to the maximization problem for \( J(u) \) given \( m(t) \); then, one imposes that \( m(t) \) is consistent with the master equation for the optimal process \( \sigma^*(t) \). Denoting by \( V(\sigma, t) \) the value function of the control problem of maximizing \( J(u) \), the dynamic programming equation reads, defining \( \nabla V(\sigma, t) := V(-\sigma, t) - V(\sigma, t) \),

\[
- \lambda V(\sigma, t) + \frac{\mu(\sigma, m(t))}{2} \left[ (\nabla V(\sigma, t))^+ \right]^2 + \frac{\partial V}{\partial t}(\sigma, t) + \sigma m(t) = 0,
\]

and yields the optimal (feedback) control

\[
u^*(t) = \mu(\sigma, m(t)) [\nabla V(\sigma, t)]^+.
\]

A derivation of the Hamilton–Jacobi–Bellman equation (HJB), together with a classical verification argument, are postponed to “Appendix A”. Substituting \( u^* \) in (1), one derives a differential equation for \( m(t) \). Since \( \sigma \) is a binary variable, without loss of generality, we can write \( \mu(\sigma, m) \) more conveniently in the form \( \mu(\sigma, m) = \sigma a(m) + b(m) \), and set \( z(t) := \nabla V(1, t) \). By (3) and (1) we obtain the following system of coupled equations:

\[
\begin{cases}
\dot{z}(t) = \frac{b(m(t))}{2} z(t)|z(t)| + \frac{a(m(t))}{2} z^2(t) + \lambda z(t) + 2m(t) \\
\dot{m}(t) = -(m(t)b(m(t)) + a(m(t)))|z(t)| \\
&\quad - (m(t)a(m(t)) + b(m(t)))z(t)
\end{cases}
\]

Some remarks are needed concerning Eq. (4). It is relevant to note that Eq. (4) should not be meant as an initial value problem: Only the initial \( m(0) \), i.e., the initial information on agents’ proportion, is assigned. On the other hand, the value function in this problem is necessarily bounded, so only bounded solutions of (4) matter. Conversely, every bounded solution of (4) determines an equilibrium \( u^* \) for the control problem associated to the functional given in (2).

### 4 Baseline Cases: Constant Mobility and Crowding Effects

In this section, we consider two significant specifications of the model, for which we determine and characterize the bounded solutions of (4). For the first example the proofs of the facts that we outline below are given in “Appendix B”. For what concerns the second example, we provide heuristics for the existence of a homoclinic bifurcation leading to the appearance of a periodic orbit. Although strongly supported by numerical evidence, we could not prove rigorously the occurrence of this bifurcation.

**The Constant-Mobility Model**

When \( \mu(\sigma, m) = \mu = \text{const} \), Eq. (4) takes the form:

\[
\begin{cases}
\dot{z}(t) = \frac{\mu}{2} z(t)|z(t)| + \lambda z(t) + 2m(t) \\
\dot{m}(t) = -\mu m(t)|z(t)| - \mu z(t)
\end{cases}
\]

(5)
This specification of $\mu$ is the simplest one and resembles classical models, where the cost of effort is not state dependent. We are interested in finding bounded solutions to (5). Note that $(z^*, m^*) = (0, 0)$ is always an equilibrium. Moreover, two different regimes are detected under which the behavior of the system is completely different:

(a) **Low-mobility regime**: $\mu \leq \frac{\lambda^2}{8}$. For every $m(0) \in [-1, 1]$ Eq. (5) admits a unique bounded solution. For $m(0) \neq 0$ consensus occurs: $\lim_{t \to +\infty} m(t) = \text{sign}(m(0)) \in [-1, 1]$.

(b) **High-mobility regime**: $\mu > \frac{\lambda^2}{8}$. For $|m(0)| \neq 0$ sufficiently small, there is more than one bounded solution to (5). All such solutions reach consensus ($\lim_{t \to +\infty} m(t) \in \{-1, 1\}$), but exhibit a transient oscillatory regime, in which the orbits of the solutions spiral around $(0, 0)$ before reaching consensus.

In Fig. 1 (top panel), we plot the stable manifolds related to the two fixed points $P$ and $Q$ of (5), different from the origin, for $\lambda = 1$ and $\mu = 0.1$. These values of the parameters fall under the low-mobility regime. In Fig. 1 (bottom panel), the values of the parameters are $\lambda = 1$ and $\mu = 1$. In this latter case, being under the high-mobility regime, the manifolds are spiraling around the origin before reaching the consensus. Therefore, under this regime, the equilibrium control may be not unique: There are possibly multiple equilibrium controls leading to transient oscillating behavior.

### 4.1 Introducing Crowding Effects

Here we set $\mu(\sigma, m) := \mu(1 + \epsilon \sigma | m |)$, for some $\mu > 0$ and $\epsilon \in (0, 1)$. Mobility is, now, asymmetric: Changing state is more costly for an agent than belonging to the minority, the cost to reinforce the majority, in effect, increases the more the society is polarized. Put differently, the marginal cost to attract more people on the bandwagon is higher when the majority is more pronounced and, on the opposite, it is easier to loose some of them. As the title suggests, the interpretation of this cost structure resembles the idea of *crowding*. Suppose the agents aim at reaching a very popular region of some (abstract) space. The first movers are suffering a lower cost, in relative terms, compared to late movers; as $m$ increases, it becomes more and more costly to reach the desired region. On the opposite, it becomes quite cheap to move to the minority region (although, maybe, not desirable). The proposed definition of $\mu$ is the simplest one that matches this intuition. Under this new assumption, Eq. (4) becomes:

\[
\begin{align*}
\dot{z}(t) &= \frac{\mu}{2} z(t)|z(t)| + \frac{\mu \epsilon m}{2} z^2(t) + \lambda z(t) + 2m(t) \\
\dot{m}(t) &= -(1 + \epsilon) \mu m(t) |z(t)| - \mu (1 + \epsilon m^2(t)) z(t)
\end{align*}
\]

(Differently from the previous case, for certain values of the parameters, an equilibrium control leading to permanent oscillatory behavior is detected. Indeed, a new threshold level $\hat{\mu}$ for the mobility parameter, with $\frac{\lambda^2}{8} < \hat{\mu} < +\infty$ appears. There are, therefore, three possible regimes:

(a) **Low-mobility regime**: $\mu \leq \frac{\lambda^2}{8}$. For every $m(0) \in [-1, 1]$ Eq. (6) admits a unique bounded solution. For $m(0) \neq 0$ consensus occurs: $\lim_{t \to +\infty} m(t) = \text{sign}(m(0)) \in [-1, 1]$.

(b) **Moderate-mobility regime**: $\frac{\lambda^2}{8} < \mu \leq \hat{\mu}$. For $|m(0)| \neq 0$ sufficiently small there is more than one bounded solution to (6). All such solutions reach consensus ($\lim_{t \to +\infty} m(t) \in \{-1, 1\}$), but, for $|m(0)|$ small enough, they exhibit a transient oscillatory regime, in which the orbits of the solutions spiral around $(0, 0)$ before reaching consensus.
Fig. 1 Stable manifolds for the low-mobility regime (top panel) and high-mobility regime (bottom panel) in the case of constant mobility

(c) High-mobility regime: $\mu > \hat{\mu}$. For every $m(0) \in [-1, 1]$ Eq. (6) admits two bounded solutions leading to consensus: $\lim_{t \to +\infty} m(t) \in \{-1, 1\}$. Moreover (6) admits a unique non-constant periodic orbit: thus, for $|m(0)|$ sufficiently small, there are two periodic solutions which differ for a time shift.

In Fig. 2 we plot the stable manifolds associated to the fixed points $P$ and $Q$ of (6), for $\lambda = 0.5$, $\epsilon = 0.5$ and $\mu = \hat{\mu} = 4.558$. Being $\mu$ exactly at its critical level, the two manifolds join at $P$ and $Q$. On the same graph, we have also depicted the periodic orbit obtained for $\lambda = 0.5$, $\epsilon = 0.5$ and with $\mu = 4.6 > \hat{\mu}$ (in red in the figure).
What is the intuition behind the behavior of the population choosing an equilibrium corresponding to a periodic orbit as in the \textit{high-mobility regime}? To favor the intuition, we discuss the \textit{probabilistic/physical interpretation} of the oscillatory behavior of the model; to this aim we go back to the \textit{microscopic} model as described in Sect. 2. Suppose that we are in a situation where, at time $t$, the large part of the population has $\sigma_i(t) = 1$, so that $m_N(t)$ is close to 1. Suppose, moreover, that the discount factor $\lambda$ (measuring impatience) is small compared to $\mu$, so that the actual gain of being with the majority is relatively small compared to (the expectation of) the future utility. Because of the probabilistic nature of the microscopic model, one possible future scenario is a downturn, where the majority of the agents switches to $-1$. As said, for these values of the parameters, the expectation of future scenarios has a greater impact on the utility compared to the actual gain of being with the majority. What happens now is that, somewhat surprisingly, the agents in the minority perceive that the cost of effort to jump with the majority is too high compared to the (short run) benefit and the ride to the bandwagon stops. Meanwhile, some of the agents with the majority perceive that the run to the consensus is going to stop and decide to leave the majority. Note that, because of the definition of $\mu$, to be the first mover is an advantage and, as a matter of fact, we see an abrupt downturn in the level of $m_N$. The same rationale translates to the macroscopic model. Indeed, we are in a situation in which $m(t)$ is close to 1 and $\dot{m}(t) > 0$, so that the society is on the way toward consensus. Because of the form of $\mu(\sigma, m)$, $u^*(t)$ is very high and the same holds for the associated quadratic cost. Therefore, the probabilistic mechanism described above translates into a decrease of $\dot{m}(t)$ which, at some point, changes its sign. The consequence is a collapse to a negative value of $m$. Of course, when $m$ is close to $-1$, the symmetric situation happens and the society continues to oscillate forever.
Fig. 3 $\hat{\mu}$ as a function of $\lambda$, for $\epsilon = 0.5$, under the crowding effects model. This curve separates the moderate and the high-mobility regimes.

**Remark** For both examples presented above, we can describe agents’ behavior in the limits as $\lambda \to +\infty$ or $\lambda \to 0$ (shortsighted versus farsighted agents). In both cases, if we let $\lambda \to +\infty$ only the low-mobility regime survives: There exists a unique equilibrium leading to consensus. Conversely, as $\lambda \to 0$ we are left with the high-mobility regime: This is clear for the constant-mobility model; for the second model, one should show that $\hat{\mu} \to 0$ as $\lambda \to 0$, for which we only have numerical evidence. In particular, when crowding effects are introduced, in the limit $\lambda \to 0$ a periodic orbit is present. As an example, in Fig. 3 we plot $\hat{\mu}(\lambda)$ for $\epsilon = 0.5$.

### 5 Conclusions

We have studied the dynamics of the collective behavior in a society formed by a large population of conformist agents in charge to control for the effort to put in place in order to *climb on the bandwagon*. Agents are remunerated by sticking with the majority, but suffer a quadratic cost related to the control they put in place. Indeed, the binary state of the agents is controlled by progressively measurable controls determining the probability of switching the state. We determine the value function as the bounded solution to the Hamilton–Jacobi–Bellman equation and the relative optimal feedback control, which, in turn, also represents the Nash equilibrium of the associated mean field game.

We, then, solve in details two specifications of the model, where the mobility parameter is either constant or crowded-dependent (in the sense that the cost to reach the majority increases the more polarized is the society). Interestingly enough, we find regimes of the parameters under which consensus is asymptotically reached and regimes under which bandwagoning
is persistent at the macrolevel: The system is trapped in periodic solutions, and hence it oscillates perpetually.

This paper sheds some light on a couple of open issues raised in the context of social interaction and collective behavior. Firstly, the possibility of detecting collective periodicity in complex social systems seemed to be unfeasible (see [28]). Moreover, differently from [28], our agents solve a lifetime optimization problem and can be, somewhat surprisingly, satisfied even with a permanently oscillating society. Secondly, our model provides an example of collective behavior on the short-time horizon which still accounts for diversity in the long-run: Consensus is not necessarily the unique outcome of the society. With this respect, our model is in line with previous literature discussing long-term cultural diversity and short-term collective behavior (see [34]).

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A Derivation of the Mean Field HJB Equation

Define

$$V(\sigma, t) := \sup_u \mathbb{E}_{\sigma, t} \left[ \int_t^{\infty} e^{-\lambda(r-t)} R_u(r) dr \right],$$

where

$$R_u(t) := \sigma(t)m(t) - \frac{1}{2\mu(\sigma(t), m(t))} u^2(t),$$

$u(t)$ is the rate at which $\sigma(t)$ flips to $-\sigma(t)$, and $u = (u(t))_{t \geq 0}$ ranges over right-continuous nonnegative closed-loop controls, i.e., $u(t) = \varphi_u(t, \sigma(t))$ with $\varphi_u : [0, +\infty) \times [-1, 1] \to [0, +\infty]$ right continuous in $t$. Let

$$J_{\sigma, t}(u) := \mathbb{E}_{\sigma, t} \left[ \int_t^{\infty} e^{-\lambda(r-t)} R_u(r) dr \right].$$

If $u^*$ is an optimal control for (2), then by the Bellman principle $V(\sigma, t) = J_{\sigma, t}(u^*)$ for all $\sigma, t$. For $t$ fixed and $h > 0$, denote by $u^{h, \alpha}$ the control defined on $[t, +\infty)$,

$$u^{h, \alpha}(s) = \begin{cases} \alpha & \text{for } t \leq s < t + h \\ u^*(s) & \text{for } s \geq t + h. \end{cases}$$

Observe that

$$J_{\sigma, t}(u^{h, \alpha}) = \mathbb{E}_{\sigma, t} \left[ \int_t^{t+h} R_{u^{h, \alpha}}(r) dr + e^{-\lambda h} V(\sigma(t+h), t+h) \right]$$

$$= h \left[ \sigma(t)m(t) - \frac{1}{2\mu(\sigma(t), m(t))} \alpha^2 \right]$$

$$+ \mathbb{E}_{\sigma, t} \left[ e^{-\lambda h} V(\sigma(t+h), t+h) \right] + o(h). \quad (7)$$
Moreover
\[ J_{\sigma,t}(u^{h,\alpha}) \geq V(\sigma, t) \] (8)
for every \( \alpha \), while
\[ J_{\sigma,t}(u^{h,u^*(t,\sigma)}) = V(\sigma, t) + o(h), \] (9)
where right continuity is used in this last estimate. It follows that
\[ -\lim_{h \downarrow 0} \frac{\mathbb{E}_{\sigma,t} [e^{-\lambda h} V(\sigma(t + h), t + h) - V(\sigma, t)]}{h} \geq \sigma(t)m(t) - \frac{1}{2\mu(\sigma(t), m(t))}\alpha^2, \] (10)
for every \( \alpha \geq 0 \), with the equality being attained at \( \alpha = u^*(t, \sigma) \). By standard results on continuous time Markov chains
\[ \lim_{h \downarrow 0} \frac{\mathbb{E}_{\sigma,t} [e^{-\lambda h} V(\sigma(t + h), t + h) - V(\sigma, t)]}{h} = \frac{\partial V}{\partial t}(\sigma, t) + \alpha [V(-\sigma, t) - V(\sigma, t)] - \lambda V(\sigma, t), \]
and (3) follows.

We now show that, if \( (V(\sigma, t), m(t)) \) solve (3) coupled to the second equation in (4) and \( V(\sigma, t) \) is bounded, then
\[ u^*(t) = \mu(\sigma, m(t)) [\nabla V(\sigma, t)]^+ \]
maximizes (2). Note that the equation for \( m(t) \) guarantees that \( \mathbb{E}(\sigma^*(t)) = m(t) \), so \( u^* \) is an equilibrium control. To show that \( u^* \) maximizes (2) observe that
\[ 0 = -\lambda V(\sigma, t) + u^*(t)\nabla V(\sigma, t) - \frac{1}{2\mu(\sigma, m(t))}(u^*(t))^2 + \frac{\partial V}{\partial t}(\sigma, t) + \sigma m(t) \]
\[ = -\lambda V(\sigma, t) + \sup_a \left[ a\nabla V(\sigma, t) - \frac{1}{2\mu(\sigma, m(t))}a^2 \right] \]
\[ + \frac{\partial V}{\partial t}(\sigma, t) + \sigma m(t). \] (11)
Consider now an arbitrary feedback control \( u \), and denote by \( \sigma(t) \) the process with control \( u \). A standard application of Ito’s rule for Markov chains yields, for every \( t > 0 \),
\[ \mathbb{E} \left\{ e^{-\lambda t} V(\sigma(t), t) - V(\sigma(0), 0) - \int_0^t \left[ -\lambda e^{-\lambda s} V(\sigma(s), s) + e^{-\lambda s} \frac{\partial V}{\partial s}(\sigma(s), s) - e^{-\lambda s} u(s)\nabla V(\sigma(s), s) \right] ds \right\} = 0 \] (12)
Using (11):
\[ u(s)\nabla V(\sigma(s), s) \leq \lambda V(\sigma(s), s) + \frac{1}{2\mu(\sigma, m(s))}u^2(s) - \frac{\partial V}{\partial s}(\sigma(s), s) - \sigma(s)m(s), \]
which, inserted in (12) gives
\[
\mathbb{E}\left\{ e^{-\lambda t} V(\sigma(t), t) - V(\sigma(0), 0) \right\} \\
+ \int_0^t e^{-\lambda s} \left[ \sigma(s)m(s) - \frac{1}{2\mu(\sigma, m(s))} u^2(s) \right] ds \leq 0,
\]
where equality is attained for \( u = u^* \). Letting \( t \to +\infty \) and using the boundedness of \( V \), we obtain
\[
J(u) \leq \mathbb{E}[V(\sigma(0), 0)] = J(u^*),
\]
and the proof is complete.

B Derivation of Other Facts

Proof of Facts related to the constant-mobility model
We first observe that (5), besides the origin \( O \), admits two other equilibria \( P \) and \( Q \), symmetric with respect to the origin: \( \pm \left( -\left( \sqrt{\lambda^2 + 4\mu} - \lambda \right) / \mu, 1 \right) \). Linear analysis shows that \( P \) and \( Q \) are saddle points for all values of the parameters; the origin \( O \) is linearly unstable:

- for \( \mu \leq \frac{\lambda^2}{8} \) it is repellent, i.e., the eigenvalues of the linearized system are both negative reals;
- for \( \mu > \frac{\lambda^2}{8} \) is an unstable spiral, i.e., the eigenvalues of the linearized system have both negative real part, but nonzero imaginary part.

In order to perform a global analysis, we first consider the nullcline \( N \) given by the equation \( \frac{\mu}{2} z^2 + \lambda z + 2m = 0 \). Off the nullcline, solutions to (5) have trajectories that are locally graphs of a function \( m = m(z) \). By implicit differentiation, assuming \( (z, m) \in [0, +\infty) \times [-1, 1] \), it turns out that \( m''(z) > 0 \) if and only if \( \phi^-(z) < m < \phi^+(z) \), with
\[
\phi^\pm(z) = \frac{-z}{4} \left[ \lambda \mp \sqrt{\lambda^2 - 8\mu + 6\lambda \mu z + 4\mu^2 z^2} \right].
\]
For \( (z, m) \in (-\infty, 0) \times [-1, 1] \), similar convexity conditions are obtained by reflection w.r.t. the origin. Consider the fixed point \( Q \) and its stable manifold \( M_s \), i.e., the trajectory of a solution of (5) converging to \( Q \).

Low-mobility regime: \( \mu \leq \frac{\lambda^2}{8} \). In this case the graphs of \( \phi^+ \) and \( \phi^- \) meet at the origin (see Fig. 4, top panel). Moreover, the graph of \( \phi^- \) meets the nullcline \( N \) at the equilibrium point \( Q \). A linear analysis at \( Q \) and the study of the direction of the vector field of (5) at the points of the graph of \( \phi^- \) show that \( M_s \) is at the left of the graph of \( \phi^- \). In particular \( M_s \) is concave, so it cannot intersect the nullcline \( N \), that can be intersected only vertically by a solution of (5). It follows that \( M_s \) is within the area between \( N \) and the graph of \( \phi^- \). Since the origin is stable for the time-reversal of (5), necessarily \( M_s \) joins the origin with \( Q \). Moreover, in the area between \( N \) and the graph of \( \phi^- \), it easily checked that \( \frac{dm}{dz} = \frac{m}{z} < 0 \), so it is the graph of a strictly decreasing function. Thus, for every \( m_0 \in (-1, 0) \), there is a unique point of \( M_s \), with \( m = m_0 \), which is the starting point of a solution of (5) converging to \( Q \); in particular \( m(t) \to -1 \) as \( t \to +\infty \). It is actually the only bounded solution starting from a point of the form \((m_0, z)\). This can be seen as follows. The point \((m_0, z)\), with \( m_0 < 0 \), cannot belong to the stable manifold of \( P \), which is the image of \( M_s \) under reflection w.r.t the origin. Thus the solution starting from \((m_0, z)\) cannot converge to any fixed point. Moreover, since the
Fig. 4 Low-mobility regime (top panel) and high-mobility regime (bottom panel)

divergence of the vector field driving (5) is constantly equal to $\lambda > 0$, then periodic orbits are not allowed. Thus, by the Poincaré-Bendixon Theorem, the solution starting from $(m_0, z)$ must be unbounded.

**High-mobility regime:** $\mu > \frac{\lambda^2}{8}$. In this case the graphs of $\phi^+$ and $\phi^-$ do not reach the origin (see Fig. 4, bottom panel). As in the low-mobility regime, the stable manifold $M_s$, as departing from $Q$, forms a concave curve between $N$ and the graph of $\phi^-$. If we show that
MSC gets arbitrarily close to the origin, then the previous linear analysis implies that it must spiral around the origin, in particular it is not the graph of an injective function.

Thus we are left to show that MSC gets arbitrarily close to the origin. This amounts to show that the solution \((\hat{z}(t), \hat{m}(t))\) of the time-reversed system starting from a point in MSC close to \(Q\), converges to the origin as \(t \to +\infty\). Due to the spiraling around the origin, \((\hat{z}(t), \hat{m}(t))\) cannot converge to the origin following the graph of a monotone function. Thus it must intersect first the positive \(z\)-axis and then the positive \(m\) axis at some \(m^* > 0\). Suppose \(m^* < 1\). Note that MSC intersects the \(m\)-axis horizontally, so, again by convexity, after having touched \((0, m^*)\) it continues downward. Since MSC, in the half-plane \(z < 0\) cannot touch the stable manifold of \(P\), it follows it is trapped in a bounded region. Due to the absence of periodic orbits, necessarily \((\hat{z}(t), \hat{m}(t)) \to (0, 0)\) as \(t \to +\infty\).

Finally, we need to show that \(m^* < 1\). By continuity from the low-mobility regime, this is certainly true for \(\mu - \frac{2}{8}\) sufficiently small. If our claim is false, then there must be a value of \(\mu\) for which \(m^* = m^*(\mu) = 1\). In this situation, MSC continuous horizontally up to \(P\). It follows that the union of MSC with the stable manifold of \(P\) forms a closed curve, tangent to the vector field driving (5); this is impossible by the Divergence Theorem.

**Sketch of the proof of Facts related to the crowding effects model**

We first observe that (6) has three equilibria: the origin \(O\), whose linear properties are identical to those of the constant-mobility model treated in the previous section, and the points \(P\) and \(Q\) with coordinates \(\pm \left( -\frac{2}{\lambda}, 1 \right)\). Both \(P\) and \(Q\) are easily seen to be saddle points, for all values of the parameters. Similarly to the constant-mobility case, the manifolds of \(P\) and \(Q\) can be proved to be monotone functions in the low-mobility regime, while they spiral around the origin in the moderate-mobility regime. What fails here is that the divergence of the driving vector field is not of constant sign, so that limit cycles cannot be ruled out. Although we do not have a full proof about the existence of a limit cycle, we provide clear evidence based on arguments derived by numerical inspection. Our analysis suggests that the \(m\) coordinate of the first intersection of the stable manifold of \(Q\) with the \(m\)-axis is increasing in \(\mu\), and it equals 1 at some \(\mu = \hat{\mu}\). Then, the manifold continues horizontally to reach \(P\) (as depicted in Fig. 2). Thus, by symmetry, the two stable manifolds join to form a separatrix. By increasing \(\mu\) further, a periodic orbit bifurcates from the separatrix through a homoclinic bifurcation.

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