Tighter monogamy and polygamy relations using Rényi-\(\alpha\) entropy

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We investigate monogamy relations related to the Rényi-\(\alpha\) entanglement and polygamy relations related to the Rényi-\(\alpha\) entanglement of assistance. We present new entanglement monogamy relations satisfied by the \(\mu\)-th power of Rényi-\(\alpha\) entanglement with \(\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2\) for \(\mu \geq 2\), and polygamy relations satisfied by the \(\mu\)-th power of Rényi-\(\alpha\) entanglement of assistance with \(\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2\) for \(0 \leq \mu \leq 1\). These relations are shown to be tighter than the existing ones.

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I. INTRODUCTION

One fundamental property of quantum entanglement is in its limited shareability in multi-party quantum systems \cite{1}. For example, if the two subsystems are more entangled with each other, then they will share a less amount of entanglement with the other subsystems with specific entanglement measures. This restricted shareability of entanglement is named as the monogamy of entanglement (MoE). The concept of monogamy is an essential feature allowing for security in quantum key distribution \cite{3}. It also plays an important role in many field of physics such as condensed matter physics \cite{4, 5}, statistical mechanics \cite{6}, and even black-hole physics \cite{7, 8}. Monogamy inequality was first built for three-qubit systems using tangle as the bipartite entanglement measure \cite{9}, and generalized into multi-qubit systems in terms of various entanglement measures \cite{10}.

On the other hand, the assisted entanglement, which is a dual concept to bipartite entanglement measures, is known allowing for security in quantum key distribution \cite{3}. It also plays an important role in many field of physics such as condensed matter physics \cite{4, 5}, statistical mechanics \cite{6}, and even black-hole physics \cite{7, 8}. One fundamental property of quantum entanglement is in its limited shareability in multi-party quantum systems \cite{11}. For example, if the two subsystems are more entangled with each other, then they will share a less amount of entanglement with the other subsystems with specific entanglement measures. This restricted shareability of entanglement is named as the monogamy of entanglement (MoE). The concept of monogamy is an essential feature allowing for security in quantum key distribution \cite{3}. It also plays an important role in many field of physics such as condensed matter physics \cite{4, 5}, statistical mechanics \cite{6}, and even black-hole physics \cite{7, 8}. Monogamy inequality was first built for three-qubit systems using tangle as the bipartite entanglement measure \cite{9}, and generalized into multi-qubit systems in terms of various entanglement measures \cite{10}.

II. TIGHTER MONOGAMY RELATIONS FOR RÉNYI-\(\alpha\) ENTANGLEMENT

Let \(H_X\) denote a discrete finite-dimensional complex vector space associated with a quantum subsystem \(X\). For a bipartite pure state \(|\psi\rangle_{AB}\) in vector space \(H_A \otimes H_B\), the \(\alpha\) Rényi entropy is defined as \cite{22}

\[
E_\alpha(|\psi\rangle_{AB}) := S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2(\text{tr} \rho_A^\alpha),
\]

where \(\rho_A := \text{tr}_B |\psi\rangle_{AB}\langle \psi|\) denotes the reduced density matrix of subsystem \(A\) of \(|\psi\rangle_{AB}\).
where the Rényi-$\alpha$ entropy is $S_\alpha(\rho_A) = [\log_2(\sum \lambda_i^\alpha)]/(1-\alpha)$ with $\alpha$ being a nonnegative real number and $\lambda_i$ being the eigenvalue of reduced density matrix $\rho_A$. The Rényi-$\alpha$ entropy $S_\alpha(\rho)$ converges to the von Neumann entropy when the order $\alpha$ tends to 1. For a bipartite mixed state $\rho_{AB}$, the RaE is defined via the convex-roof extension

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(\ket{\psi_i}_{AB}),$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i \ket{\psi_i}_{AB} \bra{\psi_i}$.

In particular, for a bipartite $2 \otimes d$ mixed state $\rho_{AB}$, the Rényi-$\alpha$ entanglement has an analytical expression [20]

$$E_\alpha(\rho_{AB}) = f_\alpha \left[ C^2(\rho_{AB}) \right],$$

where the order $\alpha$ ranges in the region $[(\sqrt{7}-1)/2,(\sqrt{13}-1)/2]$ and the function $f_\alpha(x)$ has the form

$$f_\alpha(x) = \frac{1}{1-\alpha} \log_2 \left[ \left( \frac{1-\sqrt{1-x}}{2} \right)^\alpha + \left( \frac{1+\sqrt{1-x}}{2} \right)^\alpha \right].$$

For any two-qubit state $\rho_{AB}$ with $\alpha \geq (\sqrt{7}-1)/2$, there also exist an analytic formula of RaE [22]

$$E_\alpha(\rho_{AB}) = f_\alpha (C(\rho_{AB})),$$

where the function $f_\alpha(x)$ has the form (6).

In Ref. [20], we have known that for an arbitrary three-qubit mixed state $\rho_{A_1A_2A_3}$, the $\mu$-th power Rényi-$\alpha$ entanglement obeys the monogamy relation

$$E_\alpha^\mu(\rho_{A_1|A_2A_3}) \geq E_\alpha^\mu(\rho_{A_1A_2}) + E_\alpha^\mu(\rho_{A_1A_3}),$$

where the order $\alpha \geq (\sqrt{7}-1)/2 \simeq 0.823$ and the power $\mu \geq 2$. Moreover, in $N$-qubit systems, the following monogamy relations is also satisfied

$$E_\alpha^\mu(\rho_{A_1B_2B_3\ldots B_{N-1}}) \geq \sum_{i=1}^{k-1} E_\alpha^\mu(\rho_{A_1B_i}) + E_\alpha^\mu(\rho_{A_1B_iB_{i+1}\ldots B_{N-1}}),$$

where the power $\mu \geq 2$ and the order $\alpha \in [\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$.

In fact, we can prove the following result for Rényi-$\alpha$ entanglement. Before this, we need to consider a Lemma for concurrence.

[Lemma 1] [23] For any $2 \otimes 2 \otimes 2^{n-2}$ mixed state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, if $C_{AB} \geq C_{AC}$, we have

$$C_\alpha^\alpha(\rho) \geq C_{AB}^\alpha + (2^{\frac{\alpha}{2}} - 1)C_{AC}^\alpha,$$

for all $\alpha \geq 2$.

[Proof]. Since it has been shown that $C_{AB}^\alpha \geq C_{AB}^2 + C_{AC}^2$ for arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state $\rho_{ABC}$ [24]. Then, if $C_{AB} \geq C_{AC}$, we have

$$C_\alpha C_{AB} \geq \left( C_{AB}^2 + C_{AC}^2 \right)^{\frac{\alpha}{2}} = C_{AB}^\alpha \left( 1 + \frac{C_{AC}^2}{C_{AB}^2} \right)^{\frac{\alpha}{2}} \geq C_{AB}^\alpha \left[ 1 + (2^{\frac{\alpha}{2}} - 1) \left( \frac{C_{AC}^2}{C_{AB}^2} \right)^{\frac{\alpha}{2}} \right] = C_{AB}^\alpha + (2^{\frac{\alpha}{2}} - 1)C_{AC}^\alpha$$

where the second inequality is due to $(1 + t)^{\alpha} \geq 1 + (2^\alpha - 1)t^\alpha$ for any real number $x$ and $t$, $0 \leq t \leq 1$, $x \in [1,\infty]$. As the subsystems $A$ and $B$ are equivalent in this case, we have assumed that $C_{AB} \geq C_{AC}$ without loss of generality. Moreover, if $C_{AB} = 0$ we have $C_{AB} = C_{AC} = 0$. That is to say the lower bound becomes trivially zero. ■
Theorem 1. For any $N$-qubit mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B_1 \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1} \cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1} \cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, the R\'enyi-$\alpha$ entanglement $E_\alpha(\rho)$ satisfies

\[
E_\alpha^m(\rho_{A|B_1B_2 \cdots B_{N-1}}) \\
\geq E_\alpha^m(\rho_{AB_i}) + (2^m - 1)E_\alpha^m(\rho_{AB_2}) + \cdots + (2^m - 1)^{m-1}E_\alpha^m(\rho_{AB_m}) \\
+ (2^m - 1)^m E_\alpha(\rho_{AB_{N-1}}),
\]

for $\mu \geq 2$ and $\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$. 

Proof. For $\mu \geq 2$, we have

\[
f_\alpha^m(x^2 + y^2) \geq (f_\alpha(x^2) + f_\alpha(y^2))^\mu \\
\geq f_\alpha^m(x^2) + (2^m - 1)f_\alpha^m(y^2),
\]

where the first inequality is due to the convex property of $f_\alpha(x)$ for $\alpha \geq (\sqrt{7} - 1)/2$ and the second inequality is obtained from a similar consideration in the proof of the second inequality in Lemma 1.

Let $\rho = \sum p_i |\psi_i\rangle \langle \psi_i|$ be the optimal decomposition of $E_\alpha(\rho_{A|B_1B_2 \cdots B_{N-1}})$ for the $N$-qubit mixed state $\rho$; then we have

\[
E_\alpha^2(\rho_{A|B_1B_2 \cdots B_{N-1}}) = \left[ \sum_i p_i E_\alpha(|\psi_i\rangle \langle \psi_i|_{A|B_1B_2 \cdots B_{N-1}}) \right]^2 \\
= \left[ \sum_i p_i E_\alpha[C_{A|B_1B_2 \cdots B_{N-1}}(|\psi_i\rangle)] \right]^2 \\
\geq \left[ E_\alpha \sum_i p_i C_{A|B_1B_2 \cdots B_{N-1}}(|\psi_i\rangle) \right]^2 \\
= E_\alpha^2[C_{A|B_1B_2 \cdots B_{N-1}}(\rho)],
\]

here we have used in the second equality the pure state formula of the RoE and taken the $E_\alpha(\cdot)$ as a function of the concurrence $C$ for $\alpha \geq (\sqrt{7} - 1)/2$; in the third inequality we have used the monotonically increasing and convex properties of $E_\alpha(C)$ as a function of the concurrence; in the fourth inequality we have used the convex property of concurrence for mixed states. Then from (13) we have

\[
E_\alpha^m(\rho_{A|B_1B_2 \cdots B_{N-1}}) \geq f_\alpha^m(C_{AB_1}^2 + C_{AB_2}^2 + \cdots + C_{AB_{m-1}}^2) \\
\geq f_\alpha^m(C_{AB_1}^2 + (2^m - 1)f_\alpha^m(C_{AB_2}^2 + \cdots + C_{AB_{m-1}}^2) \\
\geq f_\alpha^m(C_{AB_1}^2 + (2^m - 1)^2 f_\alpha^m(C_{AB_2}^2 + \cdots + C_{AB_{m-1}}^2) \\
\geq \cdots \\
\geq f_\alpha^m(C_{AB_1}^2 + (2^m - 1)^{m-1} f_\alpha^m(C_{AB_m}^2) \\
+ (2^m - 1)^m f_\alpha^m(C_{A|B_{m+1} \cdots B_{N-1}}^2),
\]

where we have used the monogamy inequality $C_x(\rho_{A|B_1B_2 \cdots B_{N-1}}) \geq C_x(\rho_{AB_1}) + C_x(\rho_{AB_2}) + \cdots + C_x(\rho_{AB_{N-1}})$ with $x \geq 2$ for $N$-qubit states $\rho$ and the monotonically increasing property of $f_\alpha(C^2)$ to obtain the first inequality. By using (12) repeatedly, we get the other inequalities.

Since $C_{AB_i} \geq C_{A|B_{i+1} \cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1} \cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, by using (12) and the similar consideration in the proof of the second inequality in Lemma 1, then we have

\[
f_\alpha^m(C_{ABm+1}^2 \cdots B_{N-1}^2) \geq (2^m - 1)f_\alpha^m(C_{ABm+1}^2) + f_\alpha^m(C_{ABm+2}^2 \cdots B_{N-1}^2) \\
\geq (2^m - 1) \left( f_\alpha^m(C_{ABm+1}^2) + \cdots + f_\alpha^m(C_{AB_{N-1}}^2) \right) \\
+ f_\alpha^m(C_{AB_{N-1}}^2).
\]

Since for any $2 \otimes 2$ quantum state $\rho_{AB_i}$, $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, $E_\alpha(\rho_{AB_i}) = f_\alpha[C^2(\rho_{AB_i})]$, therefore combining (14) and (15), we have Theorem 1.
Fig. 1: Behavior of the Rényi-α entanglement of |ψ⟩ and its lower bound, which are functions of µ plotted. The solid line represents the Rényi-α entanglement of |ψ⟩ in Example 1, the dot-dashed line represents the lower bound from our result, and the dashed line represents the lower bound from the result in (9) of [20].

Moreover, for the case that \( C_{ABi} \geq C_{A|B_{i+1} \cdots B_{N-1}} \) for all \( i = 1, 2, \cdots, N - 2 \), we have a simple tighter monogamy relation for the Rényi-α entanglement:

**[Theorem 2]**. If \( C_{ABi} \geq C_{A|B_{i+1} \cdots B_{N-1}} \) for all \( i = 1, 2, \cdots, N - 2 \), we have

\[
E_\alpha^\mu(\rho_{A|B_1B_2 \cdots B_{N-1}}) \\
\geq E_\alpha^\mu(\rho_{AB_1}) + (2^\mu - 1)E_\alpha^\mu(\rho_{AB_2}) \cdots + (2^\mu - 1)^{N-2}E_\alpha^\mu(\rho_{ABN-1}),
\]

(16)

for \( \mu \geq 2 \) and \( \alpha \in [\sqrt{7} - 1/2, (\sqrt{13} - 1)/2] \).

**Example 1**. Let us consider the three-qubit state |ψ⟩ in the generalized Schmidt decomposition form [25, 26],

\[
|\psi⟩ = \lambda_0|000⟩ + \lambda_1|1\psi⟩ + \lambda_2|101⟩ + \lambda_3|110⟩ + \lambda_4|111⟩,
\]

(17)

where \( \lambda_i \geq 0 \), \( i = 0, 1, 2, 3, 4 \) and \( \sum_{i=0}^{4} \lambda_i^2 = 1 \). Set \( \lambda_0 = \lambda_1 = 1/2 \), \( \lambda_2 = \lambda_3 = \lambda_4 = \sqrt{6}/6 \). Since \( \alpha \in [\sqrt{7} - 1/2, (\sqrt{13} - 1)/2] \), we choose \( \alpha = (\sqrt{7} - 1)/2 \approx 0.823 \), we have \( E_\alpha(|\psi⟩_{AB}) = E_\alpha(|\psi⟩_{AC}) = 0.318620 \), \( E_\alpha(|\psi⟩_{ABC}) = 0.654205 \), and then \( E_\alpha^\mu(|\psi⟩_{ABC}) = (0.654205)^\mu \), \( E_\alpha^\mu(|\psi⟩_{AB}) + E_\alpha^\mu(|\psi⟩_{AC}) = 2(0.318620)^\mu \), \( E_\alpha^\mu(|\psi⟩_{AB}) + (2^\mu - 1)E_\alpha^\mu(|\psi⟩_{AC}) = 2^\mu(0.318620)^\mu \). It is easily verified that our result is better than the result in (9) for \( \mu \geq 2 \); see Fig 1.

**III. TIGHTER POLYGAMY RELATIONS FOR RÉNYI-α ENTANGLEMENT OF ASSISTANCE**

As a dual concept to Rényi-α entanglement, we define the Rényi-α entanglement of assistance (REoA) as

\[
E_\alpha^a(\rho_{AB}) := \text{max} \sum_i p_i E_\alpha (|\psi_i⟩_{AB}),
\]

(18)

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} = \sum_i p_i |ψ_i⟩_{AB} ⟨ψ_i| \).

In Ref. [21], we know that for any two-qubit state \( \rho_{AB} \) and \( \alpha \geq (\sqrt{7} - 1)/2 \), we have

\[
E_\alpha^a(\rho_{AB}) \geq f_\alpha(C^a(\rho_{AB})),
\]

(19)

where \( E_\alpha^a(\rho_{AB}) \) and \( C^a(\rho_{AB}) \) are the REoA and CoA of \( \rho_{AB} \), respectively. And for any \((\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2\) and the function \( f_\alpha(x) \) defined on the domain \( D = \{(x, y) | 0 \leq x, y \leq 1, 0 \leq x^2 + y^2 \leq 1 \} \), we have

\[
f_\alpha(x^2 + y^2) \leq f_\alpha(x) + f_\alpha(y).
\]

(20)

From Ref. [21], it has been shown that for \((\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2 \), \( 0 \leq \mu \leq 1 \), and any \( N \)-qubit state \( \rho_{AB_1B_2 \cdots B_{N-1}} \), we have

\[
[E_\alpha^a(\rho_{A|B_1B_2 \cdots B_{N-1}})]^\mu \leq [E_\alpha^a(\rho_{A|B_1})]^\mu + \cdots + [E_\alpha^a(\rho_{A|B_{N-1}})]^\mu.
\]

(21)
In the following, we study the polygamy relations of REoA for $N$-qubit generalized $W$-class state. For $N$-qubit generalized $W$-class state, $|\psi\rangle_{AB_1\cdots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$ defined by

$$|\psi\rangle_{AB_1\cdots B_{N-1}} = a|10\cdots0\rangle + b_1|01\cdots0\rangle + \cdots + b_{N-1}|00\cdots1\rangle,$$

with $|a|^2 + \sum_{i=1}^{N-1} |b_i|^2 = 1$, one has \[27\]

$$C(\rho_{AB_i}) = C_\alpha(\rho_{AB_i}), \quad i = 1, 2, \ldots, N-1,$$

(23)

where $\rho_{AB_i} = Tr_{B_{N-1}}(\rho_{AB_1\cdots B_{N-1}})$.

**[Theorem 3]** Let $\rho_{AB_1\cdots B_{N-1}}$ denote the $N$-qubit reduced density matrix of the $N$-qubit generalized $W$-class state $|\psi\rangle_{AB_1\cdots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$ if $C_{\alpha}(\rho_{AB_i}) > C_{\alpha}(\rho_{AB_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \ldots, m$, and $C_{\alpha}(\rho_{AB_j}) \leq C_{\alpha}(\rho_{AB_{j+1}\cdots B_{N-1}})$ for $j = m + 1, \ldots, N - 2, \forall 1 \leq m \leq N - 3, N > 4$, the Rényi-$\alpha$ entanglement of assistance $E^\alpha_{\alpha}(\rho)$ satisfies

$$[E^\alpha_{\alpha}(\rho_{AB_1B_2\cdots B_{N-1}})]^\mu \
\leq [E^\alpha_{\alpha}(\rho_{AB_1B_2})]^\mu + (2^\mu - 1) [E^\alpha_{\alpha}(\rho_{AB_3})]^\mu + \cdots + (2^\mu - 1)^{m-1} [E^\alpha_{\alpha}(\rho_{AB_m})]^\mu \
+ (2^\mu - 1)^m [E^\alpha_{\alpha}(\rho_{AB_{N-1}})]^\mu,$$

(24)

for $0 \leq \mu \leq 1$ and $\alpha \in [\sqrt{\mu} - 1/2, (\sqrt{\mu} - 1)/2]$.

**[Proof].** For $0 \leq \mu \leq 1$, we have

$$[f_\alpha(\sqrt{x^2 + y^2})]^\mu \
\leq [f_\alpha(x) + f_\alpha(y)]^\mu \
\leq f_\alpha^\mu(x) + (2^\mu - 1)f_\alpha^\mu(y),$$

(25)

where the first inequality is due to inequality (17) and the monotonically increasing property of $x^\mu$ for $0 \leq \mu \leq 1$, and the second equality is obtained from a similar consideration in the proof of the second inequality in Lemma 1. Here we note that $(1 + t)^\mu \leq 1 + (2^\mu - 1)t^\mu$ with $0 \leq t \leq 1$, $x \in [0, 1]$.

For the $N$-qubit generalized $W$-class state $|\psi\rangle_{A|B_1B_2\cdots B_{N-1}}$ from Eq.(2), we have

$$C^2(\psi)_{A|B_1B_2\cdots B_{N-1}} \leq [C^\alpha(\rho_{AB_1})]^2 + \cdots + [C^\alpha(\rho_{AB_{N-1}})]^2.$$

(26)

Assuming that $C^2(\rho_{AB_1B_2\cdots B_{N-1}}) \leq [C^\alpha(\rho_{AB_1})]^2 + \cdots + [C^\alpha(\rho_{AB_{N-1}})]^2 \leq 1$, then

$$[E^\alpha_{\alpha}(\psi)_{A|B_1B_2\cdots B_{N-1}}]^\mu = f_\alpha^\mu(\sqrt{C(\rho_{AB_1B_2\cdots B_{N-1}})}) \
\leq f_\alpha^\mu\left(\sqrt{[C^\alpha(\rho_{AB_1})]^2 + \cdots + [C^\alpha(\rho_{AB_{N-1}})]^2}\right) \
= f_\alpha^\mu\left(\sqrt{C(\rho_{AB_1})} + \cdots + \sqrt{C(\rho_{AB_{N-1}})}\right)^2.\]n

\[(27)\]

where in the second inequality we have used the monotonically increasing property of $f_\alpha(x)$ for $\alpha \geq (\sqrt{\mu} - 1)/2$, and the third equality is due to (23). By using (25) repeatedly and the similar consideration in the proof of Theorem 1, we get the forth inequality. The last inequality is due to (19) and (23).

Then we consider the case $C^2(\rho_{AB_1B_2\cdots B_{N-1}}) \leq [C^\alpha(\rho_{AB_1})]^2 + \cdots + [C^\alpha(\rho_{AB_{N-1}})]^2$. There must exist $k \in \{1, \ldots, N-2\}$ such that $[C^\alpha(\rho_{AB_1})]^2 + \cdots + [C^\alpha(\rho_{AB_k})]^2 \leq 1, [C^\alpha(\rho_{AB_1})]^2 + \cdots + [C^\alpha(\rho_{AB_{k+1}})]^2 > 1$. By
defining \( T := [C^\alpha (\rho_{A|B_1})]^2 + \cdots + [C^\alpha (\rho_{A|B_{k+1}})]^2 - 1 > 0 \), we can derive
\[
E^\alpha_\alpha (\psi_{A|B_1B_2\ldots B_{N-1}}) = f^\alpha_\alpha (C (\rho_{A|B_1B_2\ldots B_{N-1}})) \leq f^\alpha_\alpha (1)
\]
\[
= f^\alpha_\alpha \left( \sqrt{[C^\alpha (\rho_{A|B_1})]^2 + \cdots + [C^\alpha (\rho_{A|B_{k+1}})]^2} - T \right)
\]
\[
\leq f^\alpha_\alpha \left( \sqrt{C (\rho_{A|B_1})]^2 + \cdots + [C (\rho_{A|B_{k+1}})]^2} \right),
\]
where we have used the monotonically increasing property of \( f_\alpha(x) \) in the second inequality, in the forth inequality we have used (23) and the monotonically increasing property of \( f_\alpha(x) \).

When \( k + 1 \leq m \), we have
\[
f^\alpha_\alpha \left( \sqrt{[C (\rho_{A|B_1})]^2 + \cdots + [C (\rho_{A|B_{k+1}})]^2} \right)
\]
\[
\leq f^\alpha_\alpha (C (\rho_{A|B_1})) + (2\mu - 1)f^\alpha_\alpha (C (\rho_{A|B_2})) + \cdots + (2\mu - 1)^k f^\alpha_\alpha (C (\rho_{A|B_{k+1}}))
\]
\[
\leq [E^\alpha_\alpha (\rho_{A|B_1})]^\mu + (2\mu - 1) [E^\alpha_\alpha (\rho_{A|B_2})]^\mu + \cdots + (2\mu - 1)^{m-1} [E^\alpha_\alpha (\rho_{A|B_{m}})]^\mu
\]
\[
+ (2\mu - 1)^m [E^\alpha_\alpha (\rho_{A|B_{N-1}})]^\mu,
\]
where we have used (25) repeatedly and the similar consideration in the proof of Theorem 1 in the first inequality, and the second inequality is due to (19) and (23).

When \( k + 1 > m \), we have
\[
f^\alpha_\alpha \left( \sqrt{[C (\rho_{A|B_1})]^2 + \cdots + [C (\rho_{A|B_{k+1}})]^2} \right)
\]
\[
\leq f^\alpha_\alpha (C (\rho_{A|B_1})) + (2\mu - 1)f^\alpha_\alpha (C (\rho_{A|B_2})) + \cdots + (2\mu - 1)^{m-1} f^\alpha_\alpha (C (\rho_{A|B_{m}}))
\]
\[
+ (2\mu - 1)^{m+1} \left( f^\alpha_\alpha (C (\rho_{A|B_{m+1}})) + \cdots + f^\alpha_\alpha (C (\rho_{A|B_{k+1}})) \right)
\]
\[
+ (2\mu - 1)^m f^\alpha_\alpha (C (\rho_{A|B_{k+1}}))
\]
\[
\leq [E^\alpha_\alpha (\rho_{A|B_1})]^\mu + (2\mu - 1) [E^\alpha_\alpha (\rho_{A|B_2})]^\mu + \cdots + (2\mu - 1)^{m-1} [E^\alpha_\alpha (\rho_{A|B_{m}})]^\mu
\]
\[
+ (2\mu - 1)^m \left( [E^\alpha_\alpha (\rho_{A|B_{m+1}})]^\mu + \cdots + [E^\alpha_\alpha (\rho_{A|B_{N-1}})]^\mu \right),
\]
where we have used (25) repeatedly and the similar consideration of the proof of Theorem 1 in the first inequality, and the second inequality is due to (19) and (23).
Combing (28), (29) and (30), we have completed the proof of Theorem.

Moreover, for the case that $C_{AB_i} \geq C_{A|B_1,\cdots, B_{N-1}}$ for all $i = 1, 2, \cdots, N-2$, we have a simple tighter monogamy relation for the Rényi-$\alpha$ entanglement of assistance:

**Theorem 4.** If $C_{AB_i} \geq C_{A|B_1,\cdots, B_{N-1}}$ for all $i = 1, 2, \cdots, N-2$, we have

$$
\left[ E_\alpha^a \left( \rho_{A|B_1B_2,\cdots,B_{N-1}} \right) \right]^{\mu} 
\leq \left[ E_\alpha^a \left( \rho_{A|B_1} \right) \right]^{\mu} + (2^\mu - 1) \left[ E_\alpha^a \left( \rho_{A|B_2} \right) \right]^{\mu} + \cdots + (2^\mu - 1)^{N-2} \left[ E_\alpha^a \left( \rho_{A|B_{N-1}} \right) \right]^{\mu},
$$

for $0 \leq \mu \leq 1$ and $\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$. 

**Example 2.** Let us consider the $W$ state, $|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle)$. Set $\alpha = (\sqrt{7} - 1)/2 \approx 0.823$, then we have $E_\alpha^a(|W\rangle_{AB}) = E_\alpha^a(|W\rangle_{A(C}) = 0.607218, E_\alpha^a(|W\rangle_{A(B(C)}) = 0.932108,$ and then $[E_\alpha^a(|W\rangle_{A(B(C)})]^{\mu} = (0.932108)^{\mu}$, $[E_\alpha^a(|W\rangle_{A(B)})]^{\mu} + [E_\alpha^a(|W\rangle_{A(C)})]^{\mu} = 2(0.607218)^{\mu}$, $[E_\alpha^a(|W\rangle_{A(B)})]^{\mu} + [E_\alpha^a(|W\rangle_{A(C)})]^{\mu} = 2^{\mu}(0.607218)^{\mu}$ for $0 \leq \mu \leq 1$. It is easily verified that our results are better than the results in (21) for $0 \leq \mu \leq 1$; see Fig 2.

**IV. CONCLUSION**

Entanglement monogamy and polygamy relations are not only fundamental property of entanglement in multiparty systems but also provide us an efficient way of characterizing multipartite entanglement. We have presented monogamy relations satisfied by the $\mu$-th power of Rényi-$\alpha$ entanglement for $\mu \geq 2$ and $\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, and polygamy relations satisfied by the $\mu$-th power of Rényi-$\alpha$ entanglement of assistance for $0 \leq \mu \leq 1$ and $\alpha \in [\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$. They are tighter, at least for some classes of quantum states, than the existing entanglement monogamy and polygamy relations. Tighter monogamy and polygamy relations imply finer characterizations of the entanglement distribution. Our approach may also be used to further study the monogamy and polygamy properties related to other quantum correlations.

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