Computing Theta Functions with Julia

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Abstract

We present a new package \texttt{Theta.jl} for computing with the Riemann theta function. It is implemented in Julia and offers accurate numerical evaluation of theta functions with characteristics and their derivatives of arbitrary order. Our package is optimized for multiple evaluations of theta functions for the same Riemann matrix, in small dimensions. As an application, we report on experimental approaches to the Schottky problem in genus five.

1 Introduction

The \textit{Riemann theta function} is the holomorphic function

\[
\theta: \mathbb{C}^g \times \mathbb{H}_g \rightarrow \mathbb{C}, \quad \theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2} n^t \tau n + n^t z\right)
\]

where \(e(x) = e^{2\pi i x}\) and \(\mathbb{H}_g\) is the \textit{Siegel upper-half space}, which consists of all complex symmetric \(g \times g\) matrices with positive definite imaginary part. Theta functions occupy a central role throughout mathematics, appearing in fields as diverse as algebraic geometry \cite{BL04, Igu72}, number theory \cite{Mum07a, EZ85}, integrable systems \cite{KS13, Seg08}, discrete mathematics \cite{RSD17}, cryptography \cite{Gau07} and statistics \cite{AA19}.

We present a new package \texttt{Theta.jl} for numerical computations of theta functions, programmed in Julia \cite{BEKS17}. Our package is specialized for multiple evaluations of theta functions for the same Riemann matrix \(\tau \in \mathbb{H}_g\) and different \(z\), for small values of the genus \(g\). Our implementation is based on the algorithm from \cite{DHB+04}, which we extend to support computations of theta functions with characteristics and derivatives of arbitrary order. Our package is designed as a free and open-source alternative to existing packages such as \texttt{algcurves} \cite{DHB+04} in Maple and the MATLAB implementation in \cite{FJK19}. \texttt{Theta.jl} is most similar to \texttt{abelfunctions} \cite{SD16} in Sage, but we implement additional functionalities for computing theta functions with characteristics, and optimizations such as Siegel transformations for faster computations in small genus.

The main application that we had in mind when designing our package was for numerical approaches to the Schottky problem in genus five. The Schottky problem asks to recognize Jacobians of curves amongst principally polarized abelian varieties, and is one of the central questions in algebraic geometry since the 19th century \cite{Gru12}. The first nontrivial case
of the Schottky problem is in genus four, which is completely solved [Sch88, Igu81]. For a recent approach linking computations and tropical geometry see [CKS19]. In this paper, we describe computational approaches for studying the Schottky problem in genus five, using our new package. In particular, we use \texttt{Theta.jl} to compute the equations in [FGS17, Acc83] which give a weak solution to the Schottky problem in genus five. We also use our package for computations on the genus five Schottky problem for Jacobians with a vanishing theta null, which is described in our companion paper [AC19].

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2 Theta functions

We define theta functions with characteristics as follows. A characteristic is an element \( m \in (\mathbb{Z}/2\mathbb{Z})^{2g} \), which we represent as a vector \( m = \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \) where \( \varepsilon, \delta \in \{0, 1\}^g \). The Riemann theta function with characteristic \( m \) is defined as

\[
\theta[m](z, \tau) = \sum_{n \in \mathbb{Z}^g} e^\left( \frac{1}{2} \left( n + \frac{\varepsilon}{2} \right)^t \tau \left( n + \frac{\varepsilon}{2} \right) + \left( n + \frac{\varepsilon}{2} \right)^t \left( z + \frac{\delta}{2} \right) \right)
\]

and it is a holomorphic function \( \theta[m] : \mathbb{C}^g \times \mathbb{H}_g \rightarrow \mathbb{C} \). The Riemann theta function in (1) is a special case of (2), where the characteristic is the all-zero vector. The sign of a characteristic \( m \) is defined as \( e(m) = (-1)^{\varepsilon \cdot \delta} \), and we call a characteristic even or odd if the sign is 1 or \( -1 \) respectively. As a function of \( z \), \( \theta[m](z, \tau) \) is even (respectively odd) if and only if the characteristic \( m \) is even (respectively odd). There are \( 2^{g-1}(2^g + 1) \) even theta characteristics and \( 2^{g-1}(2^g - 1) \) odd theta characteristics.

The theta constants are the functions on \( \mathbb{H}_g \) obtained by evaluating the theta functions with characteristics at \( z = 0 \),

\[
\theta[m](\tau) = \theta[m](0, \tau).
\]

Theta constants corresponding to odd characteristics vanish identically.

The theta function satisfies a heat equation [BL04, Proposition 8.5.5],

\[
\frac{\partial^2 \theta[m]}{\partial z_j \partial z_k} = (1 + \delta_{jk}) \cdot 2\pi i \cdot \frac{\partial \theta[m]}{\partial \tau_{jk}},
\]

where \( \delta_{jk} \) is 1 if \( j = k \) and 0 otherwise.

Moreover, theta functions have some remarkable symmetries. First of all, they are quasi-periodic with respect to the lattice \( \mathbb{Z}^g \oplus \tau \mathbb{Z}^g \) defined by \( \tau \) [BL04, Remark 8.5.3]. For all
\(a, b \in \mathbb{Z}^g\) and \(z \in \mathbb{C}^g\), the following functional equation holds.

\[
\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](z + \tau a + b, \tau) = e^\left(\varepsilon' b - \delta' \tau a - \frac{1}{2} a' \tau a - z' a \right) \theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](z, \tau).
\]  

(5)

Theta functions also transform naturally under symplectic transformations. The group \(\Gamma_g = \text{Sp}(2g, \mathbb{Z})\) of integral symplectic transformations acts on \(\mathbb{H}_g\) as follows. For \(\gamma \in \Gamma_g\) and \(\tau \in \mathbb{H}_g\),

\[
\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.
\]  

(6)

This extends to an action on \(\mathbb{C}^g \times \mathbb{H}_g\) by

\[
\gamma \cdot (z, \tau) = ((C\tau + D)^{-1}z, \gamma \cdot \tau),
\]  

(7)

and there is a corresponding action on the set of characteristics \((\mathbb{Z}/2\mathbb{Z})^g\) by

\[
\gamma \cdot \left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right] = \left[\begin{smallmatrix} D & -C \\ -B & A \end{smallmatrix}\right] \left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right] + \left[\begin{smallmatrix} \text{diag}(CD^t) \\ \text{diag}(AB^t) \end{smallmatrix}\right].
\]  

(8)

We now state the Theta Transformation Formula [BL04]:

\[
\theta[\gamma \cdot m](\gamma \cdot (z, \tau)) = \phi(\gamma, m, z, \tau) \cdot \sqrt{\det(C\tau + D)} \cdot \theta[m](z, \tau)
\]  

(9)

where \(\phi(\gamma, m, z, \tau) \in \mathbb{C}^*\) is an explicit function of the parameters with the same sign ambiguity as \(\sqrt{\det(C\tau + D)}\).

3  Numerically approximating theta functions

We describe in this section the algorithm that we use to compute theta functions in \texttt{Theta.jl}. In our implementation, we modify the algorithm from [DHB+04], generalizing it for theta functions with characteristics and derivatives of arbitrary order.

3.1 Notation

We standardize here the notation for the whole section. We separate \(z \in \mathbb{C}^g\) and \(\tau \in \mathbb{H}_g\) into real and imaginary parts, by writing \(z = x + iy, \tau = X + iY\), where \(x, y \in \mathbb{R}^g\) and \(X, Y\) are real symmetric \(g \times g\) matrices. Let \(Y = T^t T\) be the Cholesky decomposition of \(Y\), where \(T\) is upper-triangular. For any real vector \(V \in \mathbb{R}^g\), we use \([V]\) to denote the vector whose entries are the entries of \(V\) rounded to the closest integers, and we denote \([V] = V - [V]\).

We also denote \(v(n) = \sqrt{\pi}T(n + [Y^{-1}y])\) and we define the lattice \(\Lambda = \{v(n) | n \in \mathbb{Z}^g\}\), letting \(\rho\) be the length of the shortest nonzero vector in \(\Lambda\). We denote by \(\Gamma(z, x) = \int_x^\infty t^{x-1}e^{-t}dt\) the incomplete Gamma function.
3.2 Pointwise and uniform approximations

In [DHB+04], Deconinck et al. derive a pointwise approximation of the theta function, which approximates \( \theta \) by a finite sum with a specified error, given inputs \( z, \tau \).

**Theorem 3.1** ([DHB+04, Theorem 2]). Fix \( z \in \mathbb{C}^g, \tau \in \mathbb{H}_g \) and \( \epsilon > 0 \). Let \( R \) be the greater of \( (\sqrt{2g} + \rho)/2 \) and the real positive solution of \( R^g = g^{2g-1} \Gamma(g/2, (R - \rho/2)^2)/\rho^g \). The Riemann theta function \( \theta(z, \tau) \) is approximated by

\[
e^{-\pi y Y^{-1} y} \sum_{n \in S_R} e \left( \frac{1}{2} (n - [Y^{-1} y])^t X (n - [Y^{-1} y]) + (n - [Y^{-1} y])^t x \right) e^{-\|v(n)\|^2}, \tag{10}
\]

with an absolute error \( \epsilon \) on the sum, where

\[
S_R = \{ n \in \mathbb{Z}^g | \|v(n)\| < R \}. \tag{11}
\]

Note that the ellipsoid \( S_R \) in (11) depends on the input \( z \). If we are evaluating the theta function at multiple inputs \( z \) for the same matrix \( \tau \), it would be more efficient to compute a bigger ellipsoid such that the approximation works for every \( z \), instead of computing a different ellipsoid for each \( z \). Although this increases the number of terms in the sum in (10), we would only need to compute the ellipsoid once, which is often preferable as the computation of the ellipsoid is usually expensive. This is the idea behind the following uniform approximation of the theta function.

**Theorem 3.2** ([DHB+04, Theorem 3]). Fix \( \tau \in \mathbb{H}_g \) and \( \epsilon > 0 \). Let \( R \) be defined as in Theorem 3.1. For any \( z \in \mathbb{C}^g \), the Riemann theta function \( \theta(z, \tau) \) is approximated by (10) with an absolute error \( \epsilon \) on the sum, but with the set \( S_R \) replaced by \( U_R \), where

\[
U_R = \{ n \in \mathbb{Z}^g | \pi(n - c)^t Y(n - c) < R^2, |c_j| < 1/2, \forall j = 1, \ldots, g \}. \tag{12}
\]

The set \( U_R \) in (12) can be thought of as a deformed ellipsoid, which is the union of all ellipsoids \( S_R \) from (11), as \( z \in \mathbb{C}^g \) varies. By taking this union, we get a uniform approximation of the theta function for all inputs \( z \).

3.3 Theta functions with characteristics

We extend Theorem 3.2 for computing theta functions with characteristics.

**Theorem 3.3.** Fix \( \tau \in \mathbb{H}_g \) and \( \epsilon > 0 \). Let \( R \) be defined as in Theorem 3.1. For any input \( z \in \mathbb{C}^g \) and characteristic \( \left[ \begin{array}{c} \varepsilon \\ \delta \end{array} \right] \in \{0, 1\}^{2g} \), the Riemann theta function with characteristic \( \theta \left[ \begin{array}{c} \varepsilon \\ \delta \end{array} \right] (z, \tau) \) is approximated by

\[
e^{-\pi y Y^{-1} y} \sum_{n \in C_R} e \left( \frac{1}{2} (n - \eta)^t X (n - \eta) + \frac{1}{2} \left( x + \frac{\delta}{2} \right) \right) e^{-\|v(n+\frac{\varepsilon}{2})\|^2}, \tag{13}
\]

with an absolute error \( \epsilon \) on the sum, where \( \eta = [Y^{-1} y] - \frac{\varepsilon}{2} \) and

\[
C_R = \{ n \in \mathbb{Z}^g | \pi(n - c)^t Y(n - c) < R^2, |c_j| < 1, \forall j = 1, \ldots, g \}. \tag{14}
\]
Proof. From (2), we can compute theta functions with characteristics in a similar way as the usual theta function, by translating \( z \) to \( z + \frac{\delta}{2} \), and translating the lattice points in the sum from \( n \) to \( n + \varepsilon \). Note that this only changes the real part of \( z \), while the imaginary part stays the same. Hence the pointwise approximation in Theorem 3.1 holds for theta functions with characteristics, if we replace (10) by the formula in (13), where we take the sum over

\[
S_{R,\varepsilon} = \left\{ n \in \mathbb{Z}^g \left| \| v\left(n + \frac{\varepsilon}{2}\right) \| < R \right. \right\}. 
\] (15)

To obtain a uniform approximation for any \( z \in \mathbb{C}^g \) and any characteristic, we take the union of the ellipsoids \( S_{R,\varepsilon} \) from (15) as \( z \) and \( \varepsilon \) vary. Since \( v(n + \frac{\varepsilon}{2}) = \sqrt{\pi} T(n + [Y^{-1}y] + \frac{\varepsilon}{2}) \), and the entries of \( [Y^{-1}y] + \frac{\varepsilon}{2} \) have absolute value at most 1, it follows that the deformed ellipsoid \( C_R \) from (14) is the union of the ellipsoids \( S_{R,\varepsilon} \).

Note that in (14), we use a larger deformed ellipsoid whose center wanders about a cube with side length twice as large as in (12), in order to get a uniform approximation for arbitrary characteristics.

3.4 Derivatives of theta functions

We denote the \( N \)-th order derivative of the theta function along the vectors \( k^{(1)}, \ldots, k^{(N)} \) as

\[
D \left( k^{(1)}, \ldots, k^{(N)} \right) \theta(z, \tau) = \sum_{i_1, \ldots, i_N=1}^g k^{(1)}_{i_1} \cdots k^{(N)}_{i_N} \frac{\partial^N \theta(z, \tau)}{\partial z_{i_1} \cdots \partial z_{i_N}}. 
\] (16)

In [DHB+04], formulae are given for the pointwise and uniform approximations of the first and second derivatives of the theta function. We generalize these to arbitrary order derivatives here.

First we will need the following lemma.

Lemma 3.4 ([DHB+04 Lemma 2]). Let \( \Lambda \) be a \( g \)-dimensional affine lattice in \( \mathbb{R}^g \), and let \( p \in \mathbb{Z} \) be positive. Let \( \rho \) be the length of the shortest nonzero vector in \( \Lambda \), and let \( R > \frac{\rho}{2} + \frac{1}{2} \sqrt{g + 2p + \sqrt{g^2 + 8p}} \). Then

\[
\sum_{y \in \Lambda, \| y \| \geq R} \| y \|^p e^{-\| y \|^2} \leq \frac{g}{2} \left( \frac{2}{\rho} \right)^{\frac{g+j}{2}} \Gamma \left( \frac{g+j}{2}, \left( R - \frac{\rho}{2} \right)^2 \right). 
\] (17)

3.4.1 Pointwise approximation

We first give a formula for a pointwise approximation of the theta function, with derivatives of arbitrary order.
Theorem 3.5. Fix \( \tau \in \mathbb{H}_g \) and \( \epsilon > 0 \). Let \( R \) be the greater of \( \frac{1}{2} \sqrt{g + 2p + \sqrt{g^2 + 8p + \frac{\rho}{2}}} \) and the real positive solution of \( R \) in

\[
\epsilon = (2\pi)^Ng \left( \frac{2}{\rho} \right)^g \|k(1)\| \cdots \|k(N)\| \left[ \sum_{j=0}^{N} \binom{N}{j} \frac{1}{\pi^{j/2}} \|T^{-1}\|^j \|Y^{-1}y\|^N e^{-\frac{g+j}{2} \left( R - \frac{\rho}{2} \right)^2} \right].
\]

Let \( S_R \) be defined as in (11). The \( N \)-th derivative \( D(k^{(1)}, \ldots, k^{(N)})\theta(z, \tau) \) of the theta function is approximated by

\[
e^{\pi y^t Y^{-1}y} (2\pi i)^N \sum_{n \in S_R} (k^{(1)} \cdot (n - [Y^{-1}y])) \cdots (k^{(N)} \cdot (n - [Y^{-1}y]))
\]

\[
\times e \left( \frac{1}{2} (n - [Y^{-1}y])^t X(n - [Y^{-1}y]) + (n - [Y^{-1}y])^t x \right) e^{-\|v(n)\|^2},
\]

with an absolute error \( \epsilon \) on the product of \( (2\pi)^N \) with the sum. By \( v \cdot w \) we denote the usual scalar product of vectors.

Proof. Firstly, from [DHB+04] we can change the index of summation in

\[
D(k^{(1)}, \ldots, k^{(N)})\theta(z, \tau) = (2\pi i)^N \sum_{n \in \mathbb{Z}^g} (k^{(1)} \cdot n) \cdots (k^{(N)} \cdot n) e \left( \frac{1}{2} n^t \tau n + n^t z \right)
\]

to get the expression in (18), but with the set \( S_R \) replaced by \( \mathbb{Z}^g \). Thus the error in the approximation is

\[
\epsilon = \left| (2\pi i)^N \sum_{n \in \mathbb{Z}^g \setminus S_R} (k^{(1)} \cdot (n - [Y^{-1}y])) \cdots (k^{(N)} \cdot (n - [Y^{-1}y]))
\]

\[
\times e \left( \frac{1}{2} (n - [Y^{-1}y])^t X(n - [Y^{-1}y]) + (n - [Y^{-1}y])^t x \right) e^{-\|v(n)\|^2} \right|
\]

\[
\leq (2\pi)^N \|k^{(1)}\| \cdots \|k^{(N)}\| \sum_{n \in \mathbb{Z}^g \setminus S_R} \|n - [Y^{-1}y]\|^N e^{-\|v(n)\|^2}
\]

\[
= (2\pi)^N \|k^{(1)}\| \cdots \|k^{(N)}\| \sum_{n \in \mathbb{Z}^g \setminus S_R} \left( \frac{1}{\sqrt{\pi}} \|T^{-1}\| \cdot \|v(n)\| + \|Y^{-1}y\| \right)^N e^{-\|v(n)\|^2}
\]

\[
\leq (2\pi)^N \|k^{(1)}\| \cdots \|k^{(N)}\| \sum_{n \in \mathbb{Z}^g \setminus S_R} \sum_{j=0}^{N} \binom{N}{j} \frac{1}{\pi^{j/2}} \|T^{-1}\|^j \|v(n)\|^j \|Y^{-1}y\|^N e^{-\|v(n)\|^2}
\]

\[
= (2\pi)^N \|k^{(1)}\| \cdots \|k^{(N)}\| \sum_{j=0}^{N} \binom{N}{j} \frac{1}{\pi^{j/2}} \|T^{-1}\|^j \|Y^{-1}y\|^N \sum_{n \in \mathbb{Z}^g \setminus S_R} \|v(n)\|^j e^{-\|v(n)\|^2}
\]

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where we use the Cauchy-Schwarz and triangle inequalities. We then apply Lemma 3.4 to get the bound

\[
\varepsilon \leq (2\pi)^N \left| k^{(1)} \right| \cdots \left| k^{(N)} \right| \sum_{j=0}^{N} \left( \frac{N}{j} \right) \frac{1}{\pi^{j/2}} \| T^{-1} \| \left| y_{-1} \right|^j \left( \frac{g + j}{2} \right)^{\Gamma} \left( \frac{g + j}{2}, \left( R - \frac{\rho}{2} \right)^2 \right),
\]

\[
\leq (2\pi)^N \frac{g}{2} \left( \frac{2}{\rho} \right)^g \left| k^{(1)} \right| \cdots \left| k^{(N)} \right| \sum_{j=0}^{N} \left( \frac{N}{j} \right) \frac{1}{\pi^{j/2}} \| T^{-1} \| \left| y_{-1} \right|^j \left| Y_{-1} \right|^j \Gamma \left( \frac{g + j}{2}, \left( R - \frac{\rho}{2} \right)^2 \right).
\]

3.4.2 Uniform approximation

We now give a formula for a uniform approximation of derivatives of the theta function. First, we remark that by the quasi-periodicity of the theta function from (5), it suffices to consider inputs \( z \) of the form \( z = a + \tau b \), for \( a, b \in [0, 1)^g \), which is what we do here.

**Theorem 3.6.** Fix \( \tau \in \mathbb{H}_g \), \( \varepsilon > 0 \). Let \( k^{(1)}, \ldots, k^{(N)} \) be unit vectors, and let \( R \) be the greater of \( \frac{1}{2} \sqrt{g + 2p + \sqrt{g^2 + 8p + g}} \) and the real positive solution of \( R \) in

\[
(19)
\]

For inputs \( z \) of the form \( z = a + \tau b \), for \( a, b \in [0, 1)^g \), the \( N \)-th derivative \( D(k^{(1)}, \ldots, k^{(N)})\theta(z, \tau) \) of the theta function is approximated by (18) but with the set \( S_R \) replaced by \( U_R \) from (12), with an absolute error \( \varepsilon \) on the product of \( (2\pi i)^N \) with the sum.

**Proof.** For inputs \( z \) of the form \( z = a + \tau b \), we can write \( z \) as \( z = a + (X + iY)b = (a + Xb) + iYb = x + iy \). Then \( \| Y_{-1} \| = \| b \| \leq \sqrt{g} \). Substituting this and \( \| k^{(1)} \| = \cdots = \| k^{(N)} \| = 1 \) into the expression for \( \varepsilon \) in Theorem 3.5 the result follows.

3.4.3 Derivatives of theta functions with characteristics

We generalize Theorem 3.6 for derivatives of theta functions with characteristics. This follows from exactly the same argument as in Theorem 3.3 by computing the sum over a larger ellipsoid.

**Theorem 3.7.** Fix \( \tau \in \mathbb{H}_g \), \( \varepsilon > 0 \). Let \( k^{(1)}, \ldots, k^{(N)} \) be unit vectors, and let \( R \) be defined as in Theorem 3.6. For \( z \) of the form \( z = a + \tau b \), for \( a, b \in [0, 1)^g \), and \( \left\lceil \frac{\varepsilon}{\delta} \right\rceil \in \{0, 1\}^{2g} \), the \( N \)-th derivative \( D(k^{(1)}, \ldots, k^{(N)})\theta(\varepsilon) \) of the theta function with characteristic is
approximated by
\[
\sum_{n \in C_{R}} (k^{(1)} \cdot (n - \eta)) \cdots (k^{(N)} \cdot (n - \eta)) \times e^{\left(\frac{1}{2} (n - \eta)^t X (n - \eta) + (n - \eta)^t \left(\frac{1}{2} x + \delta \right)\right)} e^{-\|v(n+\tilde{x})\|^2},
\]
with an absolute error $\epsilon$ on the product of $(2\pi i)^N$ with the sum, where $\eta = |Y^{-1}y| - \frac{\varepsilon}{2}$ and $C_{R}$ is as defined in (14).

3.5 Siegel reduction

If we are interested in computing the theta function for a fixed $\tau$ at many values of $z$, it may be more efficient if we transform $\tau$ such that the ellipsoids in (12) or (14) are less eccentric, so that they contain fewer lattice points. This can be done via symplectic transformations, which modify the theta function according to the Theta Transformation Formula (9). Moreover, for many applications, such as those related to the Schottky problem in Section 5, it does not matter whether we perform computations with a Riemann matrix or its transform, so we can choose a transformation such that the computations are faster.

For this purpose, we use Siegel’s algorithm [Sie89], which iteratively finds a new matrix where the corresponding ellipsoid has a smaller eccentricity. Siegel’s original goal was to construct a fundamental region for Riemann matrices, and while the algorithm is not optimal, it is an approximation which is useful for speeding up numerical computations.

Siegel’s algorithm is implemented in [DHB+04]. A variant is described in [FJK19], and we implement the latter in our package. We describe Siegel’s result below, and give more details on our implementation in Section 4.2.2.

**Theorem 3.8** ([Sie89]). Every $\tau \in \mathbb{H}_g$ can be transformed to $\hat{\tau} = \hat{X} + i\hat{Y} \in \mathbb{H}_g$ via the action of the symplectic group, such that if $\hat{Y} = T'T$ with $T'$ upper triangular,

1. $|\hat{X}_{jk}| \leq \frac{1}{2}$ for $j, k = 1, \ldots, g$.

2. The length of the shortest lattice vector $\rho$ of the lattice generated by the columns of $T$ is bounded from below by $\sqrt{3}/2$.

**Proof.** The first condition can be achieved via the transformation $\tau \mapsto \tau - [X]$.

For the second condition, we first do a symplectic transformation on $T$, such that the shortest vector of the lattice generated by $T$ is in the first column of the resulting matrix, so $\rho = T_{11} = \sqrt{Y_{11}}$. We then apply the symplectic transformation $\gamma = (\begin{pmatrix} A & B \\ C & D \end{pmatrix})$ given by

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & I_{g-1} \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 0_{g-1} \end{pmatrix}, \\
C = \begin{pmatrix} 1 & 0 \\ 0 & 0_{g-1} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & I_{g-1} \end{pmatrix}.
\]
where $I_{g-1}$ denotes the $(g-1) \times (g-1)$ identity matrix and $0_{g-1}$ denotes the $(g-1) \times (g-1)$ matrix with all entries zero. Each such transformation changes the determinant of the imaginary part of the matrix as follows.

$$|\det(\hat{Y})| = \frac{|\det(Y)|}{|\det(C\tau + D)|^{2}} = \frac{|\det(Y)|}{|\tau_{11}|^{2}}.$$  \hspace{1cm} (22)

If $|\tau_{11}| < 1$, we iteratively apply the transformation $\tau \rightarrow \tau - [X]$, followed by the transformation (21) until $|\hat{\tau}_{11}| \geq 1$. Then $|\hat{\tau}_{11}|^{2} = \hat{X}_{11}^{2} + \hat{Y}_{11}^{2} \geq 1$, so $\rho = \sqrt{\hat{Y}_{11}} \geq \sqrt{3/2}$. The fact that this iteration terminates was shown by Siegel.

4 Computing theta functions in Julia

In this section, we describe the basic functionality of our Julia package Theta.jl, as well as the algorithms used in the implementation, and comparisons with existing packages for computing theta functions.

4.1 Interface

Our Julia package Theta.jl is available at the following website, which has instructions and a link to more detailed documentation.

https://github.com/chualynn/Theta.jl

We describe the basic interface of the package here. To use Theta.jl, we recommend installing Julia in version 1.1 or above. The package can be installed and used with the following commands in Julia.

```
julia> import Pkg
julia> Pkg.add("Theta")
julia> using Theta
```

Starting with a matrix $\tau \in \mathbb{H}_{g}$, we first construct a RiemannMatrix from it. This is a type in Theta.jl which contains information needed to compute the theta function with input $\tau$. This includes the ellipsoids used for computing the theta function and its derivatives, as well as the Siegel-transformed matrix. To construct a RiemannMatrix, we give as input $\tau$, a boolean flag siegel which specifies if we want to perform a Siegel transformation on $\tau$, a floating point number $\epsilon$ which specifies the error in computing the theta functions, and an integer nderivs which specifies the highest order of the derivative for which we want to compute the theta function.

As an example, we start with a genus 5 curve defined by the singular model

$$x^{6}y^{2} - 4x^{4}y^{2} - 2x^{3}y^{3} - 2x^{4}y + 2x^{3}y + 4x^{2}y^{2} + 3xy^{3} + y^{4} + 4x^{2}y + 2xy^{2} + x^{2} - 4xy - 2y^{2} - 2x + 1.$$ \hspace{1cm} (23)

We compute the Riemann matrix $\tau$ of the curve using the package [BSZ19] in Sage [The19], and we type it as an input in Julia.
We then construct a RiemannMatrix in Theta.jl, where we specify in the input the options to compute a Siegel transformation, an error of $10^{-12}$, and to compute derivatives up to the fourth order.

```julia
julia> R = RiemannMatrix(τ, siegel=true, ϵ=1.0e-12, nderivs=4);
```

We pick some input $z$ and compute the theta function $\theta(z, τ)$ as follows.

```julia
julia> z = [1.041+0.996im; 1.254+0.669im; 0.591+0.509im; -0.301+0.599im; 0.388+0.051im];
julia> theta(z, R)
-854877.6514446283 + 2.3935081163150463e6im
```

We can compute first derivatives of theta functions by specifying the direction using the optional argument `derivs`. For instance, to compute $\frac{∂θ}{∂z_1}(z, τ)$, we use

```julia
julia> theta(z, R, derivs=[[1,0,0,0,0]])
2.6212151525759254e7 + 9.28714502306052e6im
```

We specify higher order derivatives by adding more elements in the input to `derivs`, where each element specifies the direction of the derivative. For instance, to compute $\frac{∂^3θ}{∂z_1^2}(z, τ)$, we use

```julia
julia> theta(z, R, derivs=[[0,0,1,0,0],[1,0,1,0,0]])
1.0478325534969474e8 - 3.3699994411453768e8im
```

We can compute theta functions with characteristics using the optional argument `char`.

```julia
julia> theta(z, R, char=[[0,1,1,0,0],[1,0,1,1,0]])
1.8859811381826473e6 - 1.6046614411453768e6im
```

We can also compute derivatives of theta functions with characteristics.

```julia
julia> theta(z, R, derivs=[[1,0,0,0,0]], char=[[0,1,0,0,1],[1,1,0,0,1]])
-2.448093122926732e7 + 3.582557740667034e7im
```

### 4.2 Algorithms

We describe here some details of the algorithms and the design choices that we made in our implementation.
4.2.1 Choice of ellipsoid

The main application of our package is for computing theta functions at the same Riemann matrix \( \tau \), and for multiple choices of inputs \( z \), characteristics and derivatives. As such, in our implementation we use the algorithm for uniform approximations of theta functions described in Theorem 3.7, which allows us to compute derivatives of theta functions with characteristics, for inputs \( z \) of the form \( z = a + \tau b \), for \( a, b \in [0,1)^g \).

In Theorem 3.7 we approximate the theta function by taking the sum over a deformed ellipsoid \( C_R \) from (14). For a fixed \( \tau \in \mathbb{H}_g \) and error \( \epsilon > 0 \), \( C_R \) depends only on the order \( N \) of the derivative. Hence for each order of the derivative for which we are interested in computing the theta function, we compute an ellipsoid \( C_R \). Then we can compute theta functions with any input \( z = a + \tau b \), any characteristic, and \( N \)-th order derivatives along \( N \) unit vectors.

While \( C_R \) is larger than the ellipsoids for the other less general approximations in Section 3, we make this design choice as it is expensive to compute the ellipsoid relative to computing more terms in the sum in (20). Hence if we are computing multiple values of the theta function for a fixed matrix \( \tau \), it is faster to compute a bigger ellipsoid and use the same ellipsoid for every computation, rather than repeatedly computing a slightly smaller ellipsoid for each computation.

4.2.2 Lattice reductions

In \cite{DHB04}, the authors approximate the length \( \rho \) of the shortest vector of the lattice generated by \( T \) using the LLL algorithm by Lenstra, Lenstra and Lovász \cite{LLL82}. This is a reasonable choice if \( g \) is large, since computing the shortest vector is in general NP-hard under randomized reductions \cite{Ajt98} and is impractical for large dimensions. On the other hand, the LLL algorithm gives a polynomial time approximation, but with an error that grows exponentially with the dimension. In our implementation, since we focus on lattices with small dimensions \( g = 5 \), we compute the shortest vector exactly using the enumeration algorithm in \cite{SE94}, which is fast for small dimensions. Moreover, by computing \( \rho \) exactly, we obtain a smaller ellipsoid (14) than if we use an overestimation of \( \rho \) from the LLL algorithm.

We also compute the Siegel transformation from Section 3.5 once for each Riemann matrix, and work with the Siegel-transformed matrix for all computations. This helps us to achieve a lower amortized running time for computing many values of the theta function on a fixed Riemann matrix. We describe here the algorithm for Siegel reduction from \cite{DHB04, FJK19}.

**Algorithm 4.1** (Siegel reduction). **Input:** A Riemann matrix \( \tau \in \mathbb{H}_g \). **Output:** A symplectically transformed matrix \( \hat{\tau} \) satisfying the properties in Theorem 3.8. 

While \( |\tau_{11}| \geq 1 \) do

1. Let \( \tau = X + iY \). Compute the Cholesky decomposition \( Y = T^T T \), where \( T \) is upper-triangular.
2. Compute a unimodular transformation such that the shortest vector of the lattice generated by $T$ is in the first column of the transformed matrix, which should also be upper-triangular. Apply this transformation to $\tau$.

3. Apply the transformation $\tau \mapsto \tau - [X]$.

4. Apply the transformation in (21).

end

For step 2 of Algorithm 4.1 we use the algorithm for HKZ reduction in [ZQW12].

4.3 Comparisons with other packages

We compare Theta.jl with the other packages for computing theta functions that we are aware of, namely algcurves [DHB+04] in Maple, the MATLAB package in [FJK19] and abelfunctions [SD16] in Sage.

4.3.1 Functionality

The main advantage of Theta.jl in terms of functionality is that we support computations of theta functions with characteristics, as well as their derivatives, which to our knowledge is not implemented in other packages. Moreover, we make optimizations described in Section 4.2 for faster computations in applications where we do many computations with a fixed Riemann matrix of low genus.

4.3.2 Performance

We compare the performance of Theta.jl with the Sage package abelfunctions [SD16], by comparing the average time taken to compute the genus 5 FGSM relations of Section 5.2 as well as to compute the Hessian matrix of Section 5.4. We do not do these comparisons for the other packages as they were not available to us.

For our experiments, we sample matrices in the Siegel upper-half space as follows. First we sample $5 \times 5$ matrices $M_X, M_Y$ such that the entries are random floating point numbers between $-1$ and 1, using the random number generators in Julia and NumPy. Then we sample $\tau \in \mathbb{H}_5$ as $\tau = \frac{1}{2}(M_X + M_X^t) + M_Y^t M_Y i$. This is implemented in Theta.jl for general dimensions $g$, in the function random_siegel(g).

In each experiment, we sample 1000 random matrices using the routine described above, and do our computations on each matrix using Theta.jl and abelfunctions. We use a Lenovo Thinkpad T460p with a Intel Core i7-6820HQ processor (8MB Cache, up to 3.60GHz). We list in the table below the average time and standard deviation using Theta.jl and abelfunctions, for computing the FGSM relations in genus 5, and the genus 5 Hessian matrix.
| Experiment | Package    | Average time (s) | Standard deviation (s) |
|------------|------------|------------------|------------------------|
| FGSM       | Theta.jl   | 2.5              | 0.6                    |
|            | abelfunctions | 114.2           | 290.5                  |
| Hessian    | Theta.jl   | 0.7              | 0.2                    |
|            | abelfunctions | 20.3            | 58.0                   |

One major reason for the faster runtime on Theta.jl is the use of the Siegel transformation on the Riemann matrix, which is not implemented in abelfunctions. This also leads to the higher standard deviation in the computation for the latter.

5 Applications to the Schottky problem in genus five

Here we describe the main application that we had in mind when designing our package: experiments around the Schottky problem in genus five. We start with a brief account of the background of the problem, referring to [BL04, Mum07b, Igusa72] for more details.

5.1 Abelian varieties and Jacobians

An abelian variety is a projective variety that has the structure of an algebraic group, and it is a fundamental object in algebraic geometry. Especially important are principally polarized abelian varieties, which can all be described in terms of Riemann matrices. For every \( \tau \in \mathbb{H}_g \), we define a *principally polarized abelian variety* (ppav) as the quotient \( A_\tau = \mathbb{C}^g / \Lambda_\tau \), where \( \Lambda_\tau = \mathbb{Z}^g \oplus \tau \mathbb{Z}^g \) is a sublattice of \( \mathbb{C}^g \). The polarization on \( A_\tau \) is given by the *theta divisor*

\[
\Theta_\tau = \{ z \in A_\tau \mid \theta(z, \tau) = 0 \} .
\] (24)

This is well-defined on \( A_\tau \) because of the quasi-periodicity of the theta function in (5). Every such abelian variety is a group via the usual addition on \( \mathbb{C}^g \). At the same time, the theta functions with characteristics can be used to give an embedding of \( A_\tau \) inside \( \mathbb{P}^{3g-1} \), so that \( A_\tau \) is a projective variety as well.

It turns out that two ppavs \( A_\tau \) and \( A_{\tau'} \) are isomorphic if and only if the corresponding Riemann matrices are related via the action (6) of the symplectic group \( \Gamma_g = \text{Sp}(2g, \mathbb{Z}) \). Hence, the quotient \( A_g = \mathbb{H}_g / \text{Sp}(2g, \mathbb{Z}) \) is the *moduli space of principally polarized abelian varieties of dimension* \( g \). This is a quasi-projective variety of dimension \( \dim A_g = \dim \mathbb{H}_g = g(g+1)/2 \).

The theta constants \( \theta[m](0, \tau) \) give homogeneous coordinates on a finite cover of \( A_g \). First we consider the following subgroups of \( \Gamma_g \):

\[
\Gamma_g(4) = \{ \gamma \in \Gamma_g \mid \gamma \equiv \text{Id} \mod 4 \} ,
\] (25)

\[
\Gamma_g(4, 8) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(4) \mid \text{diag}(A'B) \equiv \text{diag}(C'D) \equiv 0 \mod 8 \right\} .
\] (26)

The group \( \Gamma_g(4, 8) \) is normal of finite index in \( \Gamma_g \), so the corresponding quotient \( A_g(4, 8) = \mathbb{H}_g / \Gamma_g(4, 8) \) is a finite Galois cover of \( A_g \). Moreover, the Theta Transformation Formula
shows that for every \( \gamma \in \Gamma _g(4, 8) \) and characteristic \( m \in (\mathbb{Z}/2\mathbb{Z})^g \) we have
\[
\theta [m](0, \gamma \cdot \tau) = \sqrt{\det(C\tau + D)} \cdot \theta [m](0, \tau).
\] (27)

Thus the even theta constants define a map to projective space,
\[
\mathcal{A}_g(4, 8) \longrightarrow \mathbb{P}^{2g-1(2g+1)-1}, \quad [\tau] \mapsto [\theta [m](0, \tau)]_{m \text{ even}}
\] (28)
which is actually an embedding, and realizes \( \mathcal{A}_g(4, 8) \) as an irreducible quasi-projective variety. By definition, polynomials in the homogeneous coordinates of \( \mathbb{P}^{2g-1(2g+1)-1} \) correspond to polynomials in the theta constants.

### 5.1.1 Jacobians of curves

Historically, abelian varieties arose from Jacobians of Riemann surfaces. For a Riemann surface \( C \) of genus \( g \), we define its Jacobian as the quotient
\[
J(C) = H^0(C, \omega_C)\vee / H_1(C, \mathbb{Z}),
\] (29)
where the lattice \( H^1(C, \mathbb{Z}) \) is embedded in \( H^0(C, \omega_C)\vee \) via the integration pairing
\[
H^0(C, \omega_C) \times H^1(C, \mathbb{Z}) \longrightarrow \mathbb{C}, \quad (\omega, \alpha) \mapsto \int_\alpha \omega.
\] (30)

The Jacobian is a principally polarized abelian variety, and the corresponding Riemann matrix \( \tau \in \mathcal{A}_g \) can be obtained by computing bases of \( H^0(C, \omega_C) \) and \( H^1(C, \mathbb{Z}) \), as well as the integration pairing. This is implemented numerically in the packages \texttt{abelfunctions} \cite{SD16} and \texttt{algcurves} \cite{DHB+04} in Sage, and \texttt{algcurves} \cite{DHB+04} in Maple. In this setting, the action of \( \Gamma _g \) on the Riemann matrices corresponds to a change of basis for \( H^0(C, \omega_C) \) or \( H^1(C, \mathbb{Z}) \).

The Jacobian construction defines the Torelli map from the moduli space \( \mathcal{M}_g \) of genus \( g \) Riemann surfaces to the moduli space \( \mathcal{A}_g \) of dimension \( g \) ppavs:
\[
\mathcal{J} : \mathcal{M}_g \longrightarrow \mathcal{A}_g, \quad [C] \mapsto [J(C)].
\] (31)

The image of this map is precisely the set of Jacobian varieties and its closure \( \mathcal{J}_g \) in \( \mathcal{A}_g \) is the Schottky locus. The Schottky problem asks for a characterization of \( \mathcal{J}_g \) inside \( \mathcal{A}_g \). It is one of the most celebrated questions in algebraic geometry, dating from the 19th century; we refer to \cite{Gru12} for a recent overview.

There are many possible interpretations and solutions to the Schottky problem. Here we focus on the most classical one, which asks for equations in the theta constants \( \theta [m](0, \tau) \) that vanish exactly on the Schottky locus. In terms of the projective embedding of \( \mathcal{A}_g(4, 8) \), this means determining the ideal generated by \( \mathcal{J}_g(4, 8) \) inside \( \mathbb{P}^{2g-1(2g+1)-1} \), where we denote by \( \mathcal{J}_g(4, 8) \) the pullback of the Schottky locus along the finite cover \( \mathcal{A}_g(4, 8) \rightarrow \mathcal{A}_g \).

In this form, the Schottky problem is completely solved only in genus 4, with an explicit equation given by Schottky \cite{Sch88} and Igusa \cite{Igu81}. A computational implementation and analysis of this solution was presented in \cite{CKS19}.
The weak Schottky problem asks for explicit equations that characterize Jacobians up to extra irreducible components. A solution to this problem was given in genus 5 by Accola \cite{Acc83}, and in a recent breakthrough, by Farkas, Grushevsky and Salvati Manni in all genera \cite{FGS17}. In the rest of this section, we present these two solutions, together with related algorithms that we implemented in \texttt{Theta.jl}. We also present a computational solution of a weak Schottky problem for genus five Jacobians with a theta null, from our companion paper \cite{ACT19}.

### 5.2 Farkas, Grushevsky and Salvati Manni’s solution

In a recent preprint \cite{FGS17}, H. Farkas, Grushevsky and Salvati Manni give a solution to the weak Schottky problem in arbitrary genus. To state their result, consider for every $g \geq 4$ and every $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{g-4}$ the following three monomials of degree 8 in theta constants (here we denote characteristics as row vectors for notational simplicity):

$$
RR_{34,\varepsilon}^1 = \theta \left[ \begin{array}{cccc}
E & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
0 & 0 & E & 0 \\
0 & 0 & 0 & E
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 0 & 1 & \varepsilon \\
0 & E & 0 & 1 \\
0 & 1 & \varepsilon & E \\
1 & 0 & E & \varepsilon
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 1 & 1 & 1 \\
1 & E & 1 & 1 \\
1 & 1 & E & 1 \\
1 & 1 & 1 & E
\end{array} \right]$

$$
RR_{34,\varepsilon}^2 = \theta \left[ \begin{array}{cccc}
E & 1 & 0 & \varepsilon \\
0 & E & 1 & 0 \\
0 & 1 & 0 & \varepsilon \\
0 & 0 & \varepsilon & 1
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 1 & 0 & \varepsilon \\
0 & E & 1 & 0 \\
0 & 1 & 0 & \varepsilon \\
0 & 0 & \varepsilon & 1
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 0 & 1 & \varepsilon \\
0 & E & 0 & 1 \\
0 & 1 & 0 & \varepsilon \\
1 & 0 & \varepsilon & 0
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 0 & 1 & \varepsilon \\
0 & E & 0 & 1 \\
0 & 1 & 0 & \varepsilon \\
1 & 0 & \varepsilon & 0
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 1 & 1 & 1 \\
1 & E & 1 & 1 \\
1 & 1 & E & 1 \\
1 & 1 & 1 & E
\end{array} \right]$

$$
RR_{34,\varepsilon}^3 = \theta \left[ \begin{array}{cccc}
E & 0 & 1 & \varepsilon \\
0 & E & 1 & 0 \\
0 & 1 & 0 & \varepsilon \\
0 & 0 & \varepsilon & 1
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 0 & 1 & \varepsilon \\
0 & E & 1 & 0 \\
0 & 1 & 0 & \varepsilon \\
0 & 0 & \varepsilon & 1
\end{array} \right] \theta \left[ \begin{array}{cccc}
E & 1 & 1 & 1 \\
1 & E & 1 & 1 \\
1 & 1 & E & 1 \\
1 & 1 & 1 & E
\end{array} \right]
$$

where for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{g-4})$, we denote $E = \varepsilon_1 + \cdots + \varepsilon_{g-4} \in \mathbb{Z}/2\mathbb{Z}$. Given three maps $a, b, c: (\mathbb{Z}/2\mathbb{Z})^{g-4} \to \{\pm 1\}$, define

$$
s_{34}^{a,b,c} = \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{g-4}} a_\varepsilon \sqrt{RR_{34,\varepsilon}^1} + b_\varepsilon \sqrt{RR_{34,\varepsilon}^2} + c_\varepsilon \sqrt{RR_{34,\varepsilon}^3}.\tag{33}
$$

We take the product

$$
S_{34} = \prod_{a,b,c} s_{34}^{a,b,c}\tag{34}
$$

to get a polynomial in the theta constants of degree $4 \cdot 2^{3g-4-1} = 2^{3g-4+1}$. For any $3 \leq j < k \leq g$, let $RR_{jk,\varepsilon}^1, RR_{jk,\varepsilon}^2, RR_{jk,\varepsilon}^3, s_{jk}^{a,b,c}, S_{jk}$ be obtained from $RR_{34,\varepsilon}^1, RR_{34,\varepsilon}^2, RR_{34,\varepsilon}^3, s_{34}^{a,b,c}, S_{34}$ by swapping the columns 3, j and k in all the characteristics.

**Theorem 5.1** \cite{FGS17} [Main Theorem]. The equations \{$S_{jk}$\}_{3 \leq j < k \leq g} cut out a locus in $\mathcal{A}_g(4,8)$ that contains the Schottky locus as an irreducible component.

Observe that there are $\binom{g-2}{2} = \frac{(g-2)(g-3)}{2}$ equations $S_{jk}$, which is exactly the same number as the codimension of $\mathcal{J}_g$ inside $\mathcal{A}_g$.

**Remark 5.2.** In the genus 5 case, we have three equations $S_{34}$, $S_{35}$ and $S_{45}$. We know from a result by Donagi \cite{Don87} that these equations define extra components in addition to the Schottky locus, namely the intermediate Jacobian locus coming from cubic threefolds.
5.2.1 Numerical computations

For numerical computations, instead of checking that the product in (34) vanishes, we directly evaluate the expressions in (33) to reduce the numerical error. To determine if a matrix \( \tau \in H_g \) is in the vanishing locus of \( S_{jk} \), we compute the smallest absolute value of the expressions in (33) and check if it is smaller than some numerical tolerance. This procedure gives a real number for each \( S_{jk} \), and to determine if \( \tau \) is in the locus defined by all the \( S_{jk} \)'s, we take the maximum of these numbers and check if it is smaller than a numerical tolerance.

We implement this for genus 5 in the function \( \text{fgsm()} \) in \texttt{Theta.jl}. Using the same example matrix \( \tau \) from Section 4.1, the function \( \text{fgsm}(\tau) \) gives us the output \( 7.85046293418876e-16 \). This is expected since \( \tau \) is the Jacobian of a genus 5 curve.

5.3 Accola’s equations in genus 5

A solution to the weak Schottky problem in genus 5 was given already by Accola \[\text{Acc83}\] in 1983, in the form of eight equations in the theta constants whose zero locus contains the Schottky locus as an irreducible component. To describe these equations, we first introduce some definitions.

Definition 5.3 (Azygetic basis). An azygetic basis of \((\mathbb{Z}/2\mathbb{Z})^{2g}\) is an ordered set of distinct elements \((v_1, \ldots, v_{2g+1})\) such that

1. The \( v_i \) generate \((\mathbb{Z}/2\mathbb{Z})^{2g}\).
2. \( \sum_{i=1}^{2g+1} v_i = 0 \).
3. \( e(v_i, v_j) = -1 \) for all \( i \neq j \), where \( e(v, v') = (-1)^{e^*\delta - e''\delta} \).

Example 5.4. Mumford \[\text{Mum07b, Section 9}\] gives the following example of an azygetic basis.

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, & v_2 &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}, & \ldots, & v_g &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}, \\
v_{g+1} &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, & v_{g+2} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, & \ldots, & v_{2g} &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}, \\
M &= \sum_{i=1}^{2g} v_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.
\end{align*}
\]

The \( v_1, \ldots, v_g \) are odd whereas \( v_{g+1}, \ldots, v_{2g} \) and \( M \) are even. Any other azygetic basis can be obtained from this one via the action of the symplectic group \( \text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) \) on \((\mathbb{Z}/2\mathbb{Z})^{2g}\), acting as the subgroup of \( \text{GL}(2g, \mathbb{Z}/2\mathbb{Z}) \) with the symplectic form \( e(\cdot, \cdot) \).

Definition 5.5 (Hyperelliptic fundamental system). A hyperelliptic fundamental system in genus 5 is a set of eleven characteristics \( \{m_1, \ldots, m_{11}\} \subseteq (\mathbb{Z}/2\mathbb{Z})^{10} \) such that

1. The \( m_i \) are all even.
2. The set is azygetic, i.e. \( e(m_i + m_j + m_k) = -1 \) for all pairwise distinct \( i, j, k \).

3. The sum of an even number of the \( m_i \) is not zero.

**Example 5.6.** Let \( (v_1, \ldots, v_{10}, M) \) be the azygetic basis from Example 5.4. Then

\[
v_6, \ldots, v_{10}, v_1 + M, \ldots, v_4 + M, \alpha, \alpha + M,
\]

where \( \alpha = \sum_{i=1}^{4} v_i \), is a hyperelliptic fundamental system. We can write this explicitly as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

Given a hyperelliptic fundamental system \( m_1, \ldots, m_{11} \), we denote by \( n = \sum_{i=1}^{11} m_i \) the sum and we also denote \( a_i = n + m_i \). Now consider the subgroup

\[
G_{1234}^{1234} = \langle a_5 + a_6 + a_7 + a_8, a_5 + a_6 + a_9 + a_{10}, a_5 + a_7 + a_9 + a_{11} \rangle \subseteq (\mathbb{Z}/2\mathbb{Z})^g.
\]

This has 8 elements and \( e(m_i + s) = 1 \) for all \( s \in G_{1234}^{1234}, i = 1, 2, 3, 4 \). Define

\[
r_i(\tau) = \prod_{s \in G_{1234}^{1234}} \theta[m_i + s](0, \tau), \quad \text{for } i = 1, 2, 3, 4.
\]

These are monomials of degree 8 in the theta constants, with product

\[
SR_{1234} = \prod_{a,b,c = \pm 1} (\sqrt{r_1} + a \sqrt{r_2} + b \sqrt{r_3} + c \sqrt{r_4}) = \left( \sum_i r_i^2 - 2 \sum_{i < j} r_i r_j \right)^2 - 64 r_1 r_2 r_3 r_4.
\]

This is a polynomial of degree 4 in the \( r_i \)'s, hence of degree 32 in the theta constants. Moreover, for any \( k = 5, \ldots, 11 \), we can define the group \( G_{1234}^{1234} \) by swapping \( k \) with 4 in the definition of \( G_{1234}^{1234} \). This changes the monomials \( r_i \) in (39), to give us new polynomials \( SR_{123k} \).

**Theorem 5.7.** [Acc83] The zero locus of the polynomials \( SR_{123k} \), for \( k = 4, \ldots, 11 \), contains the Schottky locus \( J_5 \) as an irreducible component.

**Remark 5.8.** It is not known whether Accola’s equations contain components apart from the Schottky locus. It would be interesting to study whether the equations vanish on the Intermediate Jacobian locus.
5.3.1 Numerical computations

Similarly to the equations in (34), instead of checking if the product in (40) vanishes, we directly evaluate the factors to reduce the numerical error. To determine if a matrix \( \tau \in H^5 \) is in the vanishing locus of (40), we compute the smallest absolute value of the eight factors and check if it is smaller than some numerical tolerance. This procedure gives a real number for each \( k \), and to determine if \( \tau \) is in the locus defined by all the \( SR_{123k} \), we take the maximum of these numbers and check if it is smaller than a numerical tolerance.

We implement this in the function `accola()` in Theta.jl. Again using the example \( \tau \) from Section 4.1, the function `accola(\( \tau \))` gives us the output 3.062334813867916e-9, which is expected since \( \tau \) is in the Schottky locus.

5.4 Schottky problem for Jacobians with a vanishing theta null

A variant of the Schottky problem focuses on two-torsion points on Jacobians. A two-torsion point on an abelian variety \( A_\tau \) is a point \( z \in A_\tau \) such that \( 2z = 0 \). These can be written as

\[
z = \frac{\varepsilon}{2} + \frac{\delta}{2}, \quad \text{for} \quad m = \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \in (\mathbb{Z}/2\mathbb{Z})^{2g}.
\]

Hence two-torsion points correspond to characteristics, and we say that such a point is even or odd if the corresponding characteristic is. Observe that

\[
\theta \left( \frac{\varepsilon}{2} + \frac{\delta}{2}, \tau \right) = 0 \quad \text{if and only if} \quad \theta \left[ \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \right] (0, \tau) = 0.
\]

Thus the two-torsion points in \( \Theta_\tau \) correspond to the characteristics \( m \) such that the corresponding theta constants \( \theta[m](0, \tau) \) vanish. For this reason, we say that \( A_\tau \) has a vanishing theta null if it has an even two-torsion point in the theta divisor. The abelian varieties with this property have been intensely studied [Mum83, Bea77, Deb92, GM07, GM08] and they form a divisor \( \theta_{null} \) in \( A_g \). The Jacobians with a vanishing theta null lie in the locus \( J_g \cap \theta_{null} \) and they correspond to Riemann surfaces with an effective even theta characteristic [Cor89, TiB88]. The Schottky problem in this case becomes that of recognizing \( J_g \cap \theta_{null} \) inside \( \theta_{null} \).

The first observation is that a vanishing theta null is automatically a singular point of the theta divisor, because the partial derivatives \( \frac{\partial \theta[m]}{\partial z} \) are odd. Hence one is led to study the local structure of \( \Theta_\tau \) around the singular point, and the first natural invariant is the rank of the quadric tangent cone. The quadric tangent cone is defined by the Hessian evaluated at the two-torsion point \( z \),

\[
Q_z \Theta_\tau \sim \begin{pmatrix}
\frac{\partial^2 \theta}{\partial z_1^2} & \frac{\partial^2 \theta}{\partial z_1 \partial z_2} & \cdots & \frac{\partial^2 \theta}{\partial z_1 \partial z_g} \\
\frac{\partial^2 \theta}{\partial z_1 \partial z_2} & \frac{\partial^2 \theta}{\partial z_2^2} & \cdots & \frac{\partial^2 \theta}{\partial z_2 \partial z_g} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 \theta}{\partial z_1 \partial z_g} & \frac{\partial^2 \theta}{\partial z_2 \partial z_g} & \cdots & \frac{\partial^2 \theta}{\partial z_g^2}
\end{pmatrix}.
\]
The rank of $Q_z \Theta_{\tau}$ is the rank of the Hessian. This leads to a stratification of $\theta_{\text{null}}$, first introduced by Grushevsky and Salvati Manni [GM08],

$$\theta_{\text{null}}^0 \subseteq \theta_{\text{null}}^1 \subseteq \cdots \subseteq \theta_{\text{null}}^{g-1} \subseteq \theta_{\text{null}}^g = \theta_{\text{null}}$$

where $\theta_{\text{null}}^h$ is the locus of abelian varieties with a vanishing theta null, with a quadric tangent cone of rank at most $h$. In particular, if a Jacobian has a vanishing theta null, then a result of Kempf [Kem73] shows that the quadric tangent cone has rank at most three, hence

$$\mathcal{J}_g \cap \theta_{\text{null}} \subseteq \theta_{\text{null}}^3.$$ (45)

Grushevsky and Salvati Manni proved in [GM08] that this inclusion is actually an equality in genus 4, confirming a conjecture of H. Farkas. In the same paper, they ask whether $\mathcal{J}_g \cap \theta_{\text{null}}$ is an irreducible component of $\theta_{\text{null}}^3$ in higher genera, which would imply a solution to the weak Schottky problem for Jacobians with a vanishing theta null. The main result of our companion paper [AC19] is an affirmative answer to this question in genus 5.

**Theorem 5.9.** [AC19] In genus five, the locus $\mathcal{J}_5 \cap \theta_{\text{null}}$ is an irreducible component of $\theta_{\text{null}}^3$.

We observe that the containment $\tau \in \theta_{\text{null}}^3$ can be checked explicitly. Indeed, the condition of having an even two-torsion point in the theta divisor can be checked by evaluating the finitely many theta constants $\theta[m](0, \tau)$, and then the rank of the Hessian matrix can be computed numerically. We present such a computation here, which is also in our companion paper [AC19].

**5.4.1 Numerical computations**

From the example in Section 4.1, we use the function `schottky_null(\tau)` from `Theta.jl`. The output gives the even characteristic

$$m = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$ (46)

where the theta constant vanishes. The output also gives the corresponding Hessian matrix

$$\begin{pmatrix}
-2.79665 + 5.29764i & -9.57825 - 9.04671i & 7.36305 + 2.28697i & 7.58338 + 5.34729i & 6.15667 - 1.90199i \\
-9.57825 - 9.04671i & 18.9738 + 8.34582i & -23.1027 - 3.10545i & -9.31944 - 0.82282i & 0.524289 - 3.64991i \\
7.36305 + 2.28697i & -23.1027 - 3.10545i & 16.8441 - 1.15986i & 13.9363 - 4.56541i & -3.32248 + 4.10698i \\
7.58338 + 5.34729i & -9.31944 - 0.82282i & 13.9363 - 4.56541i & 2.89309 + 1.21773i & 3.86617 - 0.54620i \\
6.15667 - 1.90199i & 0.524289 - 3.64991i & -3.32248 + 4.10698i & 3.86617 - 0.54620i & -12.9726 - 1.928i
\end{pmatrix}$$

The Hessian has the eigenvalues

$$\begin{align*}
47.946229109152995 + 9.491932144035298i \\
-15.491689246713147 + 3.340125597497958i \\
-9.512858919129267 - 1.0587349322052013i \\
-2.7271385943272036 \times 10^{-15} - 1.1117459994936022i \times 10^{-14} \\
-5.698014266322794 \times 10^{-15} + 6.342925068807627i \times 10^{-15}
\end{align*}$$

so it has rank 3 as expected.
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