Josephson junctions in a local inhomogeneous magnetic field

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A Josephson junction can be subjected to a local, strongly inhomogeneous magnetic field in various experimental situations. Here this problem is analyzed analytically and numerically. A modified sine-Gordon type equation in the presence of time-dependent local field is derived and solved numerically in static and dynamic cases. Two specific examples of local fields are considered: induced either by an Abrikosov vortex, or by a tip of a magnetic force microscope (MFM). It is demonstrated that time-dependent local field can induce a dynamic flux-flow state in the junction with shuttling, or unidirectional ratchet-like Josephson vortex motion. This provides a mechanism of detection and manipulation of Josephson vortices by an oscillating MFM tip. In a static case local field leads to a distortion of the critical current versus magnetic field, \( I_c(H) \), modulation pattern. The distortion is sensitive to both the shape and the amplitude of the local field. Therefore, the \( I_c(H) \) pattern carries information about the local field distribution within the junction. This opens a possibility for employing a single Josephson junction as a scanning probe sensor with spatial resolution not limited by its geometrical size, thus obviating a known problem of a trade-off between field sensitivity and spatial resolution of a sensor.

I. INTRODUCTION

Properties of Josephson junctions (JJ’s) with spatially uniform parameters in a homogeneous magnetic field are well studied. Also, spatially nonuniform JJ’s in a homogeneous field were considered earlier. However, in many experimental situations JJ’s are subjected to a local, strongly inhomogeneous magnetic field. For example, it can originate from a self-induced flux in JJ’s with a sign-reversal order parameter; appear in JJ’s containing ferromagnetic interlayers with spatially inhomogeneous thicknesses, nanoparticles or domain walls; in JJ’s with a local current injection; can be induced by a nearby Abrikosov vortex, by a sharp tip of a Magnetic Force Microscope (MFM), etc. Although such a situation has been considered previously, often this has been done without proper substantiation. To my knowledge, there is no established formalism for a general treatment of such a problem, especially in the dynamic case.

Another motivations of this work is related to a recent proposal to use a single planar JJ as a scanning probe sensor. The leading superconducting scanning technique today is the scanning SQUID (superconducting quantum interference device) microscopy. Despite many advantages, SQUID’s suffer from a trade-off problem between field sensitivity and spatial resolution. SQUID’s, as well as most other superconducting magnetic sensors, are measuring flux with a resolution \( \Phi \) determined by the flux quantum \( \Phi_0 \). Therefore, field sensitivity is inversely proportional to the sensor (pickup loop) area \( S \), \( \delta H = \delta \Phi / S \). On the other hand, spatial resolution is determined by the sensor size \( \delta x \sim S^{1/2} \). Consequently, the better is spatial resolution, the worse is field sensitivity, \( \delta H \sim 1/\delta x^2 \). In Ref. it was argued that a sensor based on a single planar JJ would be able to obviate the trade-off problem at least in one spatial direction. Similar to a SQUID, the field sensitivity of a planar JJ is also inversely proportional to the area. Therefore, obviation of the trade-off problem would require independence of spatial resolution on the junction size. That is, the junction should be able to resolve spatial variation of magnetic field at a scale significantly smaller than the junction length. This brings us again to the problem of a JJ in a local spatially inhomogeneous magnetic field.

In this work I consider analytically and numerically a response of a single JJ to a local inhomogeneous and time-dependent magnetic field. First, a modified sine-Gordon equation for this case is derived. The equation is then solved numerically both for short and long junctions, and both in static and dynamic cases. Two specific examples (without loosing generality) are considered with a local field induced by an Abrikosov vortex, or by a tip of MFM. It is demonstrated that a time-dependent local field can induce a dynamic flux-flow phenomenon with either shuttling or unidirectional ratchet-like motion of Josephson vortices in the junction, which provides a mechanism for detection of Josephson vortices by MFM. Analysis of the static case shows how the critical current versus magnetic field, \( I_c(H) \), modulation patterns are distorted with introduction of the local inhomogeneous field. Importantly, the shape of distorted \( I_c(H) \) patterns depends both on the shape, amplitude and position of the local field. Therefore, the \( I_c(H) \) pattern carries detailed information about the local field and it should be possible to extract field distribution within the junction using proper mathematical treatment. This would open a possibility for making a scanning probe sensor based on a single planar Josephson junction with spatial resolution not limited by its geometrical size, thus obviating the trade-off problem between sensitivity and resolution.

II. THEORETICAL ANALYSIS

Figure represents a sketch of the studied problem. Let us consider a Josephson junction in applied uniform magnetic
Here the last term represents the displacement current density in the z-direction, \( J_b \), as
\[
\frac{\partial B_{z\phi}}{\partial y} = -\frac{4\pi}{c} J_b. \tag{5}
\]

This is how the bias term, which plays the role of the driving force for junction dynamics, enters the sine-Gordon equation. From Eqs. (15) we obtain
\[
J_d + J_n + J_s = c \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_y^*}{\partial y} \right) + J_b. \tag{6}
\]

The y-component of magnetic field, going through the junction, induces a phase gradient in the junction,
\[
\frac{\partial \varphi}{\partial x} = \frac{2\pi d_{eff}}{\Phi_0} B_y. \tag{7}
\]

Here \( d_{eff} \) is the so-called magnetic thickness of the junction. \( B_y \) is the total (screened) induction in the junction subjected both to the applied uniform field \( H \) and the local nonuniform field \( B^\ast \). It is generally not known and should be determined. To do so we separate the phase shift \( \varphi' \) caused solely by \( B^\ast \).

\[
\varphi = \phi + \varphi', \tag{8}
\]

\[
\frac{\partial \varphi^*}{\partial x} = \frac{2\pi d_{eff}}{\Phi_0} (B_y - B_y^*), \tag{9}
\]

\[
\frac{\partial \varphi}{\partial x} = \frac{2\pi d_{eff}}{\Phi_0} (B_y - B_y^*). \tag{10}
\]

\( \varphi'(x) \) is a known function, determined (up to an integration constant) by integration of Eq. (9) along the junction length.

Using Eqs. (7,10) we can write
\[
J_d + J_n + J_s = \frac{c \Phi_0}{8\pi^2 d_{eff}} \frac{\partial^2 \phi}{\partial x^2} + \frac{c}{4\pi} \left( \frac{\partial B_y^*}{\partial x} - \frac{\partial B_y^*}{\partial y} \right) + J_b. \tag{11}
\]

Note that the second term in the right-hand-side represents \( z \)-component of \( rot B^\ast \). Since \( B^\ast \) does not induce vacuum currents, \( rot B^\ast = 0 \), this term vanishes. Substituting Eqs. (1,2,4) in Eq. (11) we obtain the desired modified sine-Gordon-type equation:
\[
\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} - \alpha \frac{\partial \phi}{\partial t} = \sin(\phi + \varphi') - \frac{\partial \varphi'}{\partial t} + \frac{\partial^2 \varphi'}{\partial t^2} + \alpha \frac{\partial \varphi'}{\partial t}. \tag{12}
\]

Here space, \( \tilde{x} = x/\lambda_j \), is normalized by the Josephson penetration depth, \( \lambda_j = \sqrt{\frac{\Phi_0 C}{2\pi e d_{eff}}} \), and time, \( \tilde{t} = \omega_p t \), by the inverse plasma frequency \( \omega_p^{-1} = \sqrt{\frac{\Phi_0 C}{2\pi e d_{eff}}} \), \( \alpha = (\omega_p R_a C)^{-1} \) is the damping parameter and \( j_b = J_b/\Phi_0 \). Eq. (12) should be solved with respect to \( \phi \) for the known \( \varphi'(x,t) \) with boundary conditions at the junction edges \( x = 0, L_x \):
\[
\frac{\partial \varphi}{\partial x}(0, L_x) = \frac{2\pi d_{eff}}{\Phi_0} \lambda_j H. \tag{13}
\]

Note that thanks to separation of variables, Eq. (8), the local nonuniform field drops out from the boundary conditions. This occurs because at the junction edges \( B_y(0, L_x) = 0 \) and Eq. (15) becomes:
\[
\frac{\partial \varphi}{\partial x}(0, L_x) \neq 0. \tag{14}
\]
Hand side of Eq. (14) to zero. From Eq. (13), we explicitly obtain \( \phi(x) \approx (2\pi f_B H/\Phi_0) x + \phi_0 \), where \( \phi_0 \) is the integration constant. The total supercurrent is calculated directly by integration of \( \sin[\phi(x) + \phi^*(x)]dx \). The critical current is obtained by maximization with respect to the integration constant \( \phi_0 \).

To demonstrate how local inhomogeneous field distorts the \( I_c(H) \) pattern we consider the case when the local field is created by stray fields from an Abrikosov vortex. This case has been described in details in a recent work [25], in which it was shown that vortex-induced Josephson phase shift is well described by the equation:

\[
\phi^*(x) = -V \arctan \left( \frac{x - x_v}{|z_v|} \right),
\]

where \( V \) is the vorticity (+1 for a vortex, -1 for an antivortex), \( x_v \) is the coordinate of the vortex along the junction and \( z_v \) is the distance to the junction. Figure 2 shows (a) vortex stray fields, \(-B^*(x)\), in the junction and (b) corresponding Josephson phase shifts \( \phi^*(x) \) for an Abrikosov vortex, \( V = 1 \), at four different distances \( z_v \) to the JJ along the junction middle line \( x_v = 0.5L_x \). The closer is the vortex to the junction, the sharper and larger is the local stray field \( B^*(x) \). Note that the sign of the stray field is opposite to that in the vortex, leading to the minus sign in Eq. (15) [25].

Figure 3 illustrates an evolution of \( I_c(H) \) patterns with changing local fields: row (a) upon approaching the vortex to the junction along the middle line \( x_v = 0.5L_x \); row (b) upon moving the vortex from the left to the right sides of the junction at \( z_v = 0.05L_x \); and row (c) upon increasing the vorticity for a fixed position \( x_v = 0.5L_x \). The quantity \( \Theta_{j1} \) represents the absolute value of the total vortex-induced Josephson phase shift \( \Theta_{j1} = |\phi^*(L_x) - \phi^*(0)| \), which yields the total local flux in the junction, \( (\Phi)' = (\Theta_{j1}/2\pi)|\Phi_0| \).

The left panel in row (a) represents the case with far away vortex, \( z_v = 100L_x \), and negligible local field, \( B^* \approx 0 \). In this case \( I_c(H) \) follows the standard Fraunhofer pattern. Figs. 3(a) demonstrate that the \( I_c(H) \) patterns get progressively distorted upon approaching the vortex to the JJ, accompanied by a subsequent sharpening and increasing of \( B^* \), as shown in Fig. 2 (a). Figs. 3(b) correspond to the case when only the position of \( B^* \) maximum, \( x_v \), is changes, while the amplitude and the shape of \( B^* \) remains the same. In Figs. 3(c) the position and the shape remain the same and only the amplitude of \( B^* \) changes. Importantly, the distortion of \( I_c(H) \) is individual and each pattern in Fig. 3(a) is clearly distinctive. As seen from Figs. 3(a-c) the \( I_c(H) \) patterns are sensitive to the shape. Figs. 2(a) and 3(a), the position, Figs. 2(b) and the amplitude, Figs. 3(c), of local field. Thus, the \( I_c(H) \) contains an encrypted information about local field distribution \( B^*(x) \) and it should be possible to extract it by proper analysis. This supports the statement of Ref. [28] that spatial resolution of a scanning probe sensor based on a single planar junction is potentially not limited by its size. Such a device could obviate a trade-off problem between field sensitivity and spatial resolution inherent for scanning SQUID sensors [29, 30], as mentioned in the Introduction.

**III. STATIC CASE**

In the static case the only current component is the supercurrent \( J_s \), Eq. (1), and Eq. (12) is reduced to

\[
\frac{\partial^2 \phi}{\partial x^2} = \sin(\phi + \phi^*) - j_b.
\]

According to Eqs. (8-10), the local supercurrent density \( J_s(x) \) directly depends on the local field \( B^*(x) \). Experimentally measurable quantity, however, is the critical current \( I_c \), which represents the maximum value of the integral of \( J_s \) along the junction length. The value of \( I_c \) alone does not disclose the \( B^*(x) \) distribution. However, as we will show below, the \( I_c(H) \) modulation pattern does carry information about distribution of magnetic induction in the junction.

**III A. Short junctions**

First we consider the simplest case of a short junction \( L_x \ll \lambda \). In this case we may neglect magnetic field screening in the junction, i.e. set the second derivative term in the left-hand side of Eq. (14) to zero. From Eq. (13), we explicitly obtain

\[
\phi(x) = (2\pi f_B H/\Phi_0) x + \phi_0.
\]

In what follows we will normalize magnetic field by \( H_0 = \Phi_0/\pi \Lambda_1 \Lambda_2 \) and voltage by \( V_0 = \Phi_0 \omega_p/2\pi c \), where \( \Lambda_1 = t_I + \lambda_1 \coth(d_1/\Lambda_1) + \lambda_2 \coth(d_2/\Lambda_2) \), \( t_I \) is the junction interlayer width, \( d_{1,2} \) are the widths and \( \Lambda_{1,2} \) are the London penetration depths of the two electrodes (for details see Ref. [22]).

![Fig. 2.](image-url) (Color online). (a) Abrikosov vortex-induced stray fields, \(-B^*(x)\), at four different distances \( z_v \) of the vortex to the junction and at \( z_v = 0.5L_x \) (in the semi-logarithmic scale). The minus sign is due to the opposite sign of the vortex stray field with respect to the vortex. (b) Corresponding Josephson phase shifts \( \phi^*(x) \).

![Diagram](image-url)
III B. Long junctions

For long JJ’s, $L_x \gg \lambda_J$, screening of magnetic field by the junction becomes significant. Simultaneously, Josephson vortices (JV’s) appear and start to affect junction properties. To obtain $I_c$, with static $B^\ast$ either an ordinary differential equation (14), or a dynamic partial differential equation (12) with time-independent $\varphi^\ast$ should be solved with boundary conditions, Eq. (13). Eq. (14) is solved by a finite difference method with successive iterations and $I_c$ is determined as a maximum bias current at which a solution converges. Eq. (12) is integrated explicitly using a central difference approximation and $I_c$ is determined using a threshold criterion for voltage. In case of a significant nonlinearity of $\varphi^\ast(x)$, the iterative solution of Eq. (14) may be quite sensitive to the initial approximation. On the other hand, the damping term in Eq. (12) allows less strict requirements to the initial approximation and usually provides faster convergence because for the considered here static case one can use a large $\alpha \approx 1$ to speed up calculations. Therefore, all simulations shown below are obtained by solving partial differential Eq. (12).

Figure 4(a) shows a simulated $I_c(H)$ pattern for a long JJ with $L_x = 10 \lambda_J$, in absence of local field $B^\ast = 0$. In contrast to the Fraunhofer $I_c(H)$ pattern for a short JJ, Fig. 3(a), it has a broad triangular central lobe, corresponding to the Meissner state [1]. Beyond it JV’s penetrate into the junction. Edge pinning of JV’s, due to interaction with their own images [25], leads to metastability and multiply-valued $I_c$. Some of the metastable states are seen in Figure 4(a). To obtain those states simulations are done by sweeping magnetic field back-and-forth in different field intervals.

Next we consider the case with a local field. Here we keep in mind another relevant case, when $B^\ast$ is induced by the MFM tip [26, 27]. A standard MFM tip is covered by a thin
ferromagnetic layer. Therefore $B^r$ from the MFM sensor has a sharp dipole-type peak, originating from the end of the tip, and a broad background from the ferromagnetic layer at the cantilever. To mimic it we approximate the tip-induced and a broad background from the ferromagnetic layer at the cantilever. To mimic it we approximate the tip-induced local field and repulsion force are shown by the blue peak and arrow, respectively. Lorentz forces exerted by bias currents of different signs are depicted by red arrows. It is seen that the asymmetrically located local field creates a left-right asymmetry for JV motion and thus makes positive and negative critical currents dissimilar, as seen in panel (c).

III C. Asymmetry of $I\_c(H)$ patterns

From comparison of Figs. 3 and 4(c-e) it can be seen that local fields lead to qualitatively different symmetries of $I\_c(H)$ patterns for short and long JJ’s. In the absence of local field, $B^r = 0$, the $I\_c(H)$ patterns for both short and long JJ’s (with uniform parameters) are symmetric both with respect to field and current directions. That is, positive, $I^r$, and negative, $I^l$, critical currents are the same for positive and negative fields: $I^r(H) = -I^l(-H)$, $I^l(H) = I^r(-H)$, see Figs. 3(a) and 4(a).

Local field $B^r \neq 0$ removes the space symmetry of the problem. In all cases this removes the symmetry with respect to field (space) inversion, $I\_c(H) \neq I\_c(-H)$. However, for short JJ’s the symmetry with respect to current (time) inversion is preserved, $I^r(H) = -I^l(-H)$. This occurs because for short JJ’s $I^r(H)$ and $I^l(-H)$ correspond to maxima and minima of the same integral of $\sin[(2\pi d_{JJ}H/\Phi_0)x+\varphi(x)+\phi_0]dx$. Those are achieved at some constant $\phi_0$ and $\phi_0 + \pi$, respectively, thus leading to $I^r(H) = -I^l(-H)$.

In long JJ’s local field removes the current reversal symmetry as well, $I^r(H) \neq -I^l(-H)$, see Figs. 4(c) and (d). This occurs because $I\_c$ in long JJ’s has a different nature: it can be considered as a depinning current for JV’s. Bias current exerts a Lorentz force on JV’s and leads to appearance of a flux-flow metastability of JJ’s inside the JJ. This leads to a more profound metastability of $I\_c(H)$ patterns as can be seen from comparison of Figs. 4(a) and (c-e).
FIG. 5. (Color online). Dynamics of a long junction $L_x = 10L_y$ at $H = -0.55$ induced by an oscillating MFM tip placed at the left edge of the junction $x_i = 0.01L_x$. Six time frames are shown (time increasing from left to right) within approximately one period of tip oscillation. For each frame the top panel shows a spatial distribution of the Josephson current, $\sin(\phi + \varphi)$, the middle panel - the voltage, $\partial(\phi + \varphi^*)/\partial t$, and the bottom panel - the inhomogeneous part of magnetic induction $B = H$. It is seen that the oscillating tip induces a shuttling motion of a single Josephson vortex, which enters and exits at the cite of the tip.

So far we considered perfectly uniform JJ’s. However, real JJ’s often contain some nonuniformities, e.g., they may have spatial variation of intrinsic junction parameters, such as the critical current density, electrode thickness, bias current density, self-field etc. In this case the $I_c(H)$ pattern in the absence of local field must only be centrosymmetric, $I_c^+(H) = -I_c^-(H)$ [5]. The latter is the consequence of space-time symmetry: simultaneous reversal of field (space) and current (time) is equivalent to flipping the junction upside-down, which should not affect the output of the experiment. For nonuniform junctions introduction of the local field $B^*$ removes all sorts of symmetry $I_c^+(H) \neq -I_c^-(H)$ and $I_c^+(H) \neq I_c^-(H)$ even for short junctions (not shown).

IV. DYNAMIC CASE

The local field can be time-dependent, as for example in case of MFM in the tapping mode [27]. The time-dependent local field $B^*(t)$ provides an additional driving force for junction dynamics, given by the last two terms in the right-hand side of Eq. (12). This can cause a flux-flow phenomenon induced by the oscillating MFM tip, as recently reported [27].

Figure 5 shows a time sequence of solutions of Eq. (12) with an oscillating MFM tip. Simulation parameters correspond to Fig. 4 (c): $L_x = 10L_y$, $x_i/L_x = 0.01$ at $H = -0.55$. The damping parameter is $\alpha = 0.5$. The top panels show spatial distributions of the Josephson current density, the middle panels - of voltage, and the bottom panels - magnetization $B = H$. We assume that the tip is oscillating harmonically, but the tip field is anharmonic due to the non-linear distance dependence of the dipole-like tip field: $B^*(t) = B^*(0)(1+a(1-cos(\omega t)^a))$ with $a = 0.5$ and $\omega = 0.05\omega_{Jc}$. Simulations are done for zero bias current $j_b = 0$. Thus, all the dynamics is induced solely by the oscillating local field. Junction dynamics is periodic in time with the period of tip oscillations $T = 2\pi/\omega \approx 125.7\omega_{Jc}^{-1}$. Time sequences in Fig. 5 are shown for approximately one period of tip oscillations.

The considered field $H = -0.55$ corresponds to a large $I_c^+$ and a small $I_c^-$, see Fig. 4 (c). As discussed above, see Fig. 4 (f), for a long junction the asymmetry $I_c^+(H) \neq -I_c^-(H)$ makes left and right directions for motion of JV’s inequivalent. This in turn can lead to a variety of unusual effects: the junction may act as a vortex-diode and rectify an external periodic or aperiodic signal [5]. For the case of a large $I_c^+$ and a small $I_c^-$, the easy direction of JV motion is from left to right. From Fig. 5 it can be seen that with increasing of the tip-induced field at $t \approx 60 \approx T/2$ a Josephson vortex enters the junction from the left side, where the tip is placed. After that it rapidly moves to the right, inducing a significant negative flux-flow voltage, see the middle panel at $t = 61$. The JV penetrates to $x \approx 3L_y$, see the frame at $t = 73$. As the tip retracts the JV exits through the left edge inducing a positive flux-flow voltage, see the frame at $t = 134$. A slight delay between tip and vortex oscillations is caused by the viscosity of flux-flow motion due to a significant damping $\alpha = 0.5$.

Fig. 5 illustrates that the oscillating local field can induce a flux-flow phenomenon in the junction. Depending on parameters it can be the shuttling in/out vortex motion, as in Fig. 5 or a more complex ratchet-like unidirectional motion with entrance of the JV from one side and exit from the other side of the junction. An example of such ratchet-like motion can be found in the Supplementary Material to Ref. [27]. In that case every cycle four JV’s enter a junction from the left edge but only three leave from that side while one exits through the right edge, thus creating a ratchet effect, i.e. a net rectified unidirectional flux-flow motion induced by a periodic (or aperiodic) perturbation [5,6]. The back action of the tip-induced flux-flow motion leads to an additional damping of MFM tip oscillations, which can be detected in experiment. As discussed in Ref. [27], this provides a mechanism for detection of Josephson vortices by the MFM technique.
CONCLUSIONS

To conclude, we derived and analyzed numerically equations describing behavior of a Josephson junction in local inhomogeneous magnetic field. As discussed in the Introduction, such situation may have many different reasons and experimental realizations [17, 22, 27]. It was demonstrated that time-dependent local field provides an additional driving force, which may induce flux-flow type dynamics in long junctions. This provides a mechanism for detection and manipulation of Josephson vortices by tapping-mode magnetic force microscope [27]. Local inhomogeneous field removes the space-time symmetry of the junction and leads to a distortion of $I_c(H)$ modulation patterns. Importantly, the distortion uniquely depends on the spatial distribution of local field $B^*(x)$ within the junction. Therefore, the information about local field profile is encoded into the shape of the $I_c(H)$ pattern and may in principle be reconstructed using an appropriate mathematical analysis. This strengthens an earlier argument that a single planar junction can be advantageously used as a scanning probe sensor [28]. The field sensitivity of such sensor would depend on the area, similar to SQUID, but the spatial resolution would not be limited by the junction size. Therefore, a planar junction sensor can obviate the trade-off problem between field sensitivity and spatial resolution inherent in scanning SQUID microscopy.

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