A new method on deterministic construction of the measurement matrix in compressed sensing

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Abstract

Construction on the measurement matrix $A$ is a central problem in compressed sensing. Although using random matrices is proven optimal and successful in both theory and applications. A deterministic construction on the measurement matrix is still very important and interesting. In fact, it is still an open problem proposed by T. Tao. In this paper, we shall provide a new deterministic construction method and prove it is optimal with regard to the mutual incoherence.

Index Terms

Compressed sensing, measurement matrix, deterministic construction, mutual incoherence, sparse signal reconstruction.

I. INTRODUCTION

Sparsity and compressed sensing have attracted a great deal of attentions recently. The key idea in compressed sensing [3], [8] is that if a signal $x \in \mathbb{R}^N$ is sparse, then we can exactly recover it from much fewer measurements $b = Ax$, where $A \in \mathbb{R}^{m \times N}$ is the measurement matrix and usually $m \ll N$.

To be more precise, we say $x \in \mathbb{R}^N$ is $s$-sparse if $\|x\|_0 \leq s$, where $\|x\|_0$ is the number of nonzero entries of $x$. Also, we say $x$ is sparse if $x$ is $s$-sparse and $s \ll N$. In many applications like image processing, video processing etc, signals are often in a very high dimensional space, i.e., $x \in \mathbb{R}^N$ with a very large $N$. That is, a signal $x$ usually has a huge mount of entries.

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unknown. It would take lots of effort to measure these entries if we measure them one by one. Fortunately, due to their natural structure, many signals are sparse or can be well approximated by sparse signals, either under the canonical basis or other special basis/frames.

For simplicity, we assume that $x$ is sparse under the canonical basis in $\mathbb{R}^N$, that is, $x = \sum_{k=1}^{s} x_{j_k} e_{j_k}$ with $1 \leq j_1 < j_2 < \cdots < j_s \leq N$. An important remark is that usually we do not have any prior information or assumption about the exact location of these nonzero entries of $x$.

To retrieve such a sparse signal $x$, a natural method is to solve the following $l_0$ problem

$$\min_x \|x\|_0 \quad \text{subject to} \quad Ax = b$$

(1)

where $A$ and $b$ are known. To ensure the $s$–sparse solution is unique, we would like to use the restricted isometry property (RIP) which was introduced by Candès and Tao in [4]. A matrix $A$ satisfies the RIP of order $s$ with the restricted isometry constant (RIC) $\delta_s = \delta_s(A)$ if $\delta_s$ is the smallest constant such that

$$(1 - \delta_s) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s) \|x\|_2^2$$

(2)

holds for all $s$-sparse signal $x$.

If $\delta_{2s}(A) < 1$, the $l_0$ problem has a unique $s$-sparse solution [4]. The $l_0$ problem is equivalent to the $l_1$ minimization problem when $\delta_{2s}(A) < \sqrt{2}/2$, please see [2], [13], [1] and the references therein.

Now it is natural to ask how to construct a desired measurement matrix. Using random matrix is proven to be very successful. Candès and Tao have proven the following theorem:

**Theorem I.1.** If the elements of a matrix $A$ is independently drawn from the gaussian distribution $\mathcal{N}(0, m/N)$, then with very high probability, we have $\delta_{2s}(A) \leq m/(s \log N)$.

A further conclusion of the above theorem is: If the elements of $A$ is independently drawn from the gaussian distribution $\mathcal{N}(0, m/N)$ and $m \geq Cs \log N$ with some constant $C$, then with very high probability, we have $\delta_{2s}(A) < \sqrt{2}/2$.

Since reducing the number of measurements is essential in compressed sensing. It is highly desirable to construct a measurement matrix $A \in \mathbb{R}^{m\times N}$ with $m$ as small as possible while satisfying $\delta_{2s}(A) < \sqrt{2}/2$. From the other hand, it is also highly desirable to construct a measurement matrix $A$ with optimal instances. It is proven [?] that to satisfy optimal instance, we
must have \( m \geq Cs \log N \) with some constant \( C \). Therefore, using random matrix to construct the measurement matrix \( A \in \mathbb{R}^{m \times N} \) with \( m = Cs \log N \) is optimal with regard to optimal instances.

Although using random matrix is so successful, it is still very important and interesting to study deterministic constructions. In fact, it is still an open problem proposed by Tao.

II. MUTUAL INCOHERENCE

As mentioned before, \( \delta_{2s} < 2^{-1/2} \) is a sharp sufficient condition. However, according to its definition, R.I.C. is very hard to calculate. On the other hand, another constant, mutual incoherence, is much easier to calculate. For a measurement matrix \( A \), we denote \( \mu_A \) the mutual incoherence by

\[
\mu_A := \max_{1 \leq i < j \leq N} \frac{\langle Ae_i, Ae_j \rangle}{\| Ae_i \|_2 \| Ae_j \|_2}.
\]

(3)

It is proven by Tai and Wang [***] that if

\[
\mu_A < 1/(2s - 1),
\]

(4)

then the measurement matrix \( A \) is suitable for recovering every \( s \)-sparse signal \( x \) from \( b = Ax \) and the above condition is sharp. Now we will focus on how to construct a sensing matrix \( A \in \mathbb{R}^{m \times N} \) such that (4) is satisfied.

First of all, let us review the possible range of \( m \) when \( N \) and \( s \) are given. Define

\[
\mu_{m,N} := \min_{A \in \mathbb{R}^{m \times N}} \mu_A.
\]

(5)

By the famous Welch bound [***], we have

\[
\mu_{m,N} \geq \sqrt{\frac{N - m}{(N - 1)m}}.
\]

(6)

However, it is not a sharp bound in some situations. For instance, if we fix \( m \), then all the column vectors of \( A \) are in \( \mathbb{R}^m \). Now when \( N \to +\infty \), which means the number of column vectors of \( A \) goes to positive infinity, then the mutual incoherence of \( A \) will go to 1, since those column vectors of \( A \) are getting crowder and crowder in \( \mathbb{R}^m \). That is,

\[
\lim_{N \to +\infty} \mu_{m,N} = 1
\]

when \( m \) is fixed. If (6) is sharp, we would have

\[
\lim_{N \to +\infty} \mu_{m,N} = \sqrt{\frac{1}{m}}
\]

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which contradicts the above equality! As pointed out by [****], another bound is

\[ m \geq C \ln N \frac{(\frac{1}{\mu})^2}{\ln(\frac{1}{\mu})} \]  

(7)

where \( C \) is a constant independent of \( m, N \) and \( \mu \). A remark of above bound is that it implies \( \mu \to 1 \) when \( m \) is fixed and \( N \to +\infty \). Also, it is still unknown whether the above bound is sharp. For all known constructions in the literature, we have

\[ \sqrt{m} \ln m \geq C \ln N \frac{1}{\mu} \]  

(8)

Now we propose the following new construction method. This algorithm is a random algorithm, the possibility of this algorithm to succeed is very high. Moreover, if this random algorithm succeeds, we know for sure that the output matrix \( A \) will satisfy the condition \( \mu_A \leq \frac{1}{2s} \).

Algorithm: 1. Input \( N \) and \( s \). Fix the seed of a random generator. Choose \( m \geq \lceil 8s^2 \ln(2sN/\pi) \rceil + 2 \) and define \( j = 0, x_0 = e_1 \in \mathbb{R}^m \).

2. Repeat the following:

2.1 Let \( k := 0 \) and replace \( j \) by \( j + 1 \).

2.2 Replace \( k \) by \( k + 1 \). Use the random generator to get a unit vector \( y \in \mathbb{R}^m \), then calculate

\[ \mu_y := \max_{1 \leq i \leq j} |\langle x_i, y \rangle| \]

2.3 repeat 2.2 if \( \mu_y > \frac{1}{2s} \) and \( k < 10 \).

3. Repeat 2 if \( k < 10 \) and \( j < N \).

Now we claim the following theorem:

**Theorem II.1.** *The possibility that the above algorithm find the desired sensing matrix is at least \( 1 - 10^{-4} \). If the algorithm succeed, the worst computational complexity is \( 10mN + N(N - 1)/2 \).*

Proof: For a given unit vector \( x \in \mathbb{R}^m \), consider these two caps

\[ C_{x,s} := \{ y \in \mathbb{R}^m \| y \|_2 = 1 \ \text{and} \ \| \langle y, x \rangle \| \geq 1/(2s) \} \]

By direct calculation, one can verify that the surface area \( A_{s,m} \) of these two caps \( C_{x,s} \) satisfies

\[ A_{s,m} \leq 2s(1 - 1/(2s)^2)^{(m-1)/2}V_{m-1} \]

Therefore, the possibility of finding a wanted \( y \) such that

\[ |\langle y, x_i \rangle| \leq 1/(2s) \ \forall i = \{1, 2, \ldots, j\} \]
is at least $1 - jA_{s,m}/A_m \geq 1 - 2js(1-1/(2s)^2)\left(m-1\right)^{1/2}V_{m-1}/A_m \geq 1 - 2js\left(1-1/(2s)^2\right)\left(m-1\right)^{1/2}/(2\pi)$, which by direct calculation, is at least $1 - (j/(2N))^{10}$. Then the all claim can be verified by direct calculation.

**Remark II.2.** It is only a very rough draft, a refined version with suitable citations will be updated soon.

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