NORMALIZER, DIVERGENCE TYPE, AND PATTERSON MEASURE FOR DISCRETE GROUPS OF THE GROMOV HYPERBOLIC SPACE

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Abstract. For a non-elementary discrete isometry group $G$ of divergence type acting on a proper geodesic $\delta$-hyperbolic space, we prove that its Patterson measure is quasi-invariant under the normalizer of $G$. As applications of this result, we have: (1) under a minor assumption, such a discrete group $G$ admits no proper conjugation, that is, if the conjugate of $G$ is contained in $G$, then it coincides with $G$; (2) the critical exponent of any non-elementary normal subgroup of $G$ is strictly greater than half of that for $G$.

1. Introduction

The Patterson–Sullivan theory of Kleinian groups studies the dynamics and geometry of discrete isometry groups of the hyperbolic space $\mathbb{H}^{n+1}$ or $(n+1)$-dimensional complete hyperbolic manifolds using invariant conformal measures on the boundary $\mathbb{S}^n$ at infinity (see [16, 17]). Recently, they have often been generalized to simply connected Riemannian manifolds with variable negative curvature bounded above or, more generally, to CAT($-1$)-spaces (see [13]). Following the great success of this theory, it was extended to discrete groups acting on other metric spaces of hyperbolic nature and their boundary at infinity. The Gromov hyperbolic space is a typical object to which the theory of the classical hyperbolic space is generalized; in fact, the Patterson–Sullivan theory was developed for discrete isometry groups on a proper geodesic $\delta$-hyperbolic space by Coornaert [2].

Among other important results on Kleinian groups in this field, the investigation of normal subgroups of a Kleinian group (equivalently, normal covers of a hyperbolic manifold) has considerably progressed. For instance, the characterization of the amenability of the covering of a convex compact manifold in terms of certain geometric invariants has been proved. This was originally due to Brooks, and a recent account oriented toward the Patterson–Sullivan theory can be found in [15]. The Patterson measure is the characteristic invariant conformal measure of a Kleinian group and its invariance under the normalizer was shown by the present authors [8] in the following form. The survey article

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also explained a consequence of this theorem and a relation to the problem of the critical exponents of normal subgroups.

**Theorem 1.1.** Let $\Gamma$ be a non-elementary Kleinian group acting on the hyperbolic space $\mathbb{H}^{n+1}$ whose Poincaré series diverges at the critical exponent. Then, the Patterson measure for $\Gamma$ is invariant under the normalizer $N(\Gamma)$ of $\Gamma$.

In this study, we generalize this theorem to a discrete isometry group $\Gamma$ of a proper geodesic $\delta$-hyperbolic space $(X, d)$. As a counterpart to the conformal invariant measure, the quasiconformal measure of quasi-invariance was introduced in [2]. Roughly speaking, a concept on the usual geometry is defined on the Gromov hyperbolic space with controllable ambiguity; thus, the notion of invariance of a conformal measure must be appropriately weakened. We say that an $s$-dimensional quasiconformal measure $\mu$ on the boundary $\partial X$ is $\Gamma$-quasi-invariant if there is a constant $D \geq 1$ independent of $\gamma \in \Gamma$ such that the Radon–Nikodym derivative of the pull-back $\gamma^* \mu$ to $\mu$ is comparable with the $s$-dimensional magnification rate of $\gamma$ with multiplicative error factor $D$.

On the contrary, the critical exponent of a discrete isometry group $\Gamma \subset \text{Isom}(X, d)$ is determined in exactly the same way as the exponential growth rate of the orbit, and in both classical and modern cases, a conformal measure or a quasiconformal measure of the dimension at the critical exponent reflects the geometry of $\Gamma$. This is what we defined as the Patterson measure. Moreover, the divergence of the Poincaré series at the critical exponent is a distinguished property for $\Gamma$ and ensures uniqueness of the Patterson measure in a certain sense. If $\Gamma$ satisfies this property, then $\Gamma$ is said to be of divergence type.

The main theorem of this study is the following. It will be proved in Section 6.

**Theorem 1.2.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group of divergence type acting on a proper geodesic $\delta$-hyperbolic space $(X, d)$. Then, the Patterson measure for $\Gamma$ is quasi-invariant under the normalizer $N(\Gamma)$ of $\Gamma$.

We present two applications of this theorem in Sections 7 and 8. For Kleinian groups, we used Theorem 1.1 for a problem of proper conjugation in [8] and for a new proof of the theorem by Roblin [14] concerning the lower bound of the critical exponents of normal subgroups in [7]. Our applications correspond to these results.

**Theorem 1.3.** Let $G \subset \text{Isom}(X, d)$ be a non-elementary discrete isometry group of divergence type that acts on $X$ uniformly properly discontinuously. If $\alpha G \alpha^{-1} \subset G$ for $\alpha \in \text{Isom}(X, d)$, then $\alpha G \alpha^{-1} = G$.

When $G$ is quasi-convex cocompact, the same conclusion was proved in [9]. Theorem 1.3 is an extension of this case. For technical reasons, we assume a certain uniformity property of the properly discontinuous action. As we mentioned in our previous paper, we can apply this theorem to the problem of proper conjugation for a subgroup $G$ of any hyperbolic group.
Theorem 1.4. Let $G \subset \text{Isom}(X, d)$ be a discrete group of divergence type and let $\Gamma \subset G$ be a non-elementary normal subgroup. Then, the critical exponent of $\Gamma$ is strictly greater than half of that for $G$.

The arguments in [7] for Kleinian groups require only basic geometry on the hyperbolic space $\mathbb{H}^{n+1}$ except those for showing the strict inequality. The basic geometric properties can be adjusted to discrete isometry groups of the Gromov hyperbolic space. To obtain the strict inequality, one should assume that $G$ is of divergence type and apply Theorem 1.2.

The fundamental fact for proving Theorem 1.2 is that the Patterson measure for a discrete group $\Gamma$ has a certain uniqueness property if $\Gamma$ is of divergence type. We call this quasi-uniqueness and it is formulated such that if $\mu$ and $\mu'$ are two Patterson measures for $\Gamma$, then they are mutually absolutely continuous, and the Radon–Nikodym derivative $d\mu'/d\mu$ is bounded from above and away from zero almost everywhere on $\partial X$. This follows from the ergodicity of the action of $\Gamma$ on $\partial X$ with respect to the Patterson measure, as was shown in [2].

However, we have to obtain more explicit bounds in terms of the quasi-invariance constants $D$ and $D'$ for $\mu$ and $\mu'$ as well as their total masses, which is presented in Section 5. This is due to the fact that it is not sufficient for Theorem 1.2 that a measure $\mu'$ given by the pull-back of $\mu$ under $g \in N(\Gamma)$ is also a Patterson measure for $\Gamma$.

For the quasi-invariance under $N(\Gamma)$, we must establish the uniformity of the bounds of $d\mu'/d\mu$ independent of $g \in N(\Gamma)$. Moreover, to estimate the total mass of $\mu'$, which is the total mass of the quasiconformal measure $\mu$ with the reference point changed by $g \in N(\Gamma)$, we take $\mu$ as the Patterson measure obtained by the canonical construction from the Poincaré series of $\Gamma$. The advantage of this construction is that the invariance of the Poincaré series under the normalizer $N(\Gamma)$ is reduced to $\mu$; hence, we see that the total mass of $\mu'$ is comparable with that of $\mu$. This is an idea for the proof of Theorem 1.2 that is presented in Section 6.

In the next three sections (2–4), we carry out preliminary work toward the main theorem. The fact that a discrete group $\Gamma$ of divergence type acts on $\partial X$ ergodically with respect to the Patterson measure $\mu$ is a consequence of the condition that $\mu$ has positive measure on the conical limit set $\Lambda_c(\Gamma) \subset \partial X$. These are well-known arguments for Kleinian groups. In fact, this fact originates in the Hopf–Tsuji problem for Fuchsian groups and the Lebesgue measure. Sullivan [16, 17] generalized this to Kleinian groups and their Patterson measures by considering the geodesic flow on the hyperbolic manifold $\mathbb{H}^{n+1}/\Gamma$. Later, Tukia [18] presented an elementary proof without the argument of the geodesic flow. One can expect that his proof is applicable to discrete isometry groups of the Gromov hyperbolic space if necessary changes are made. We do this in Section 3 where Tukia’s original arguments will also be clarified.

There are several methods for showing the ergodicity of a Kleinian group $\Gamma$ with respect to the Patterson measure $\mu$ when $\mu(\Lambda_c(\Gamma)) > 0$. An intuitive explanation is to rely on the density point theorem (see [10, Theorem 4.4.4]). Namely, if we replace the reference point
of $\mu$ with orbit points tending to the density point of $\Lambda_c(\Gamma)$ conically, then the measure of $\Lambda_c(\Gamma)$ increases and hence must be of full measure by the invariance under $\Gamma$. There are various versions of the density point theorem originating from Lebesgue’s theorem. In Section 4, we verify that the version from Federer [6] is suitable for finite Borel measures on the boundary $\partial X$ of the Gromov hyperbolic space and a family of shadows as covering subsets.

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2. Preliminaries

In this section, we summarize several properties of Gromov hyperbolic spaces, their discrete isometry groups, and quasi-invariant quasiconformal measures that are necessary in our arguments of this study. We present them here by dividing the entire section into subsections.

2.1. Gromov hyperbolic space and its boundary. A geodesic metric space $(X, d)$ is called $\delta$-hyperbolic for $\delta \geq 0$ if, for every geodesic triangle $(\alpha, \beta, \gamma)$ in $X$, any edge, say $\alpha$, is contained in the closed $\delta$-neighborhood of the union $\beta \cup \gamma$ of the other edges. We call a $\delta$-hyperbolic space $(X, d)$ for some $\delta \geq 0$ a Gromov hyperbolic space. Henceforth, we further assume that a $\delta$-hyperbolic space $(X, d)$ is proper and has a fixed base point $o \in X$. Concerning the fundamental properties of the Gromov hyperbolic space mentioned in this subsection, the reader may refer to the lecture note by Coornaert, Delzant, and Papadopoulos [3].

We consider geodesic rays $\sigma : [0, \infty) \to X$ with arc length parameter starting from the base point $o$. Two such geodesic rays $\sigma_1$ and $\sigma_2$ are regarded as asymptotically equivalent if there is some constant $K$ such that $d(\sigma_1(t), \sigma_2(t)) \leq K$ for all $t \geq 0$. Then, the space of all geodesic rays based at $o$ modulo the asymptotic equivalence defines a boundary $\partial X$ of $X$, which yields the compactification $\overline{X} = X \cup \partial X$ under the compact-open topology on the space of geodesic rays. We see that $\overline{X}$ is a compact Hausdorff space satisfying the second countability axiom. Every isometric automorphism of $X$ extends to a self-homeomorphism of $\overline{X}$.

The characterization of $\delta$-hyperbolicity by triangles also provides the following properties of geodesics possibly of infinite length (see Ohshika [11, Proposition 2.61]).

**Proposition 2.1.** For a $\delta$-hyperbolic space $(X, d)$, there is a constant $\kappa(\delta) \geq 0$ depending only on $\delta$ that satisfies the following properties:

1. for an ideal geodesic triangle $(\alpha, \beta, \gamma)$ in $X$, some of whose vertices are on $\partial X$, any edge $\alpha$ is contained in the closed $\kappa(\delta)$-neighborhood of the union $\beta \cup \gamma$ of the other edges;
(2) for any geodesics \( \alpha \) and \( \beta \) with the same endpoints possibly on \( \partial X \), one is contained in the closed \( \kappa(\delta) \)-neighborhood of the other.

Hereafter, the constant \( \kappa(\delta) \) will always refer to that in the above proposition.

Another metric can be defined on the compactification \( \overline{X} = X \cup \partial X \). We choose a so-called visual parameter \( a \in (1, a_0(\delta)) \) where \( a_0(\delta) \) is some constant depending only on \( \delta \). We fix this \( a \) and do not move it throughout this paper. Then, there is a visual metric \( d_a \) on \( \overline{X} \) with respect to the base point \( o \) that satisfies the following properties.

1. The topology on \( \overline{X} \) induced by the visual metric \( d_a \) coincides with the topology of the compactification of \( (X,d) \).
2. There exists a constant \( \lambda = \lambda(\delta,a) \geq 1 \) such that for any geodesic line \( (\xi,\eta) \) connecting any \( \xi,\eta \in \partial X \), we have
   \[
   \lambda^{-1} a^{-d(o,(\xi,\eta))} \leq d_a(\xi,\eta) \leq \lambda a^{-d(o,(\xi,\eta))}.
   \]

This is an analog of the Euclidean metric on \( \mathbb{B}^{n+1} \cup \mathbb{S}^n \) for the unit ball model \( \mathbb{B}^{n+1} = \{ x \in \mathbb{R}^{n+1} \mid |x| < 1 \} \) of the hyperbolic space \( \mathbb{H}^{n+1} \) and its boundary \( \mathbb{S}^n = \{ |x| = 1 \} \).

**Remark.** We often use the following notation. For \( A > 0 \), \( B > 0 \), and \( c \geq 1 \), the notation \( A \asymp_c B \) implies that \( c^{-1} A \leq B \leq c A \). Thus, the above (2) can be rewritten as
   \[
   d_a(\xi,\eta) \asymp_{\lambda} a^{-d(o,(\xi,\eta))}.
   \]

2.2. Horospherical structure. For a \( \delta \)-hyperbolic space \( (X,d) \) with base point \( o \in X \), we can define an analog of a horosphere of \( (\mathbb{B}^{n+1},d_H) \) as a level set of the Busemann function. For a given point \( \xi \in \partial X \), let \( \sigma : [0,\infty) \to X \) be a geodesic ray such that \( \sigma(0) = o \) and \( \sigma(\infty) = \lim_{t \to \infty} \sigma(t) = \xi \). Then, the **Busemann function** at \( \xi \) is defined as
   \[
   h_\xi(z) = \lim_{t \to \infty} (d(z,\sigma(t)) - d(o,\sigma(t))).
   \]

This depends on the choice of the geodesic ray \( \sigma \), but the difference is uniformly bounded by some constant depending only on \( \delta \).

We define the **Poisson kernel** by \( k(z,\xi) = a^{-h_\xi(z)} \) for \( z \in X \) and \( \xi \in \partial X \), because it plays a similar role to the Poisson kernel \( (1 - |z|^2)/|z - \xi|^2 \) of \( \mathbb{B}^{n+1} \). Then, the analog of the conformal derivative of an isometric automorphism \( \gamma \in \text{Isom}(X,d) \) at \( \xi \in \partial X \) is given by
   \[
   j_\gamma(\xi) = a^{-h_\xi(\gamma^{-1}(o))} = k(\gamma^{-1}(o),\xi).
   \]

We remark that \( k(z,\xi) \) is defined by the choice of a family of geodesic rays \( \sigma \) and not necessarily a measurable function of \( \xi \in \partial X \) in this definition; however, this is not a problem in our arguments.

**Proposition 2.2.** For any \( z \in X \), \( \xi \in \partial X \), and \( \gamma \in \text{Isom}(X,d) \), the Poisson kernel satisfies
   \[
   a^{-2\kappa(\delta)} \frac{k(z,\xi)}{j_\gamma(\xi)} \leq k(\gamma(z),\gamma(\xi)) \leq a^{2\kappa(\delta)} \frac{k(z,\xi)}{j_\gamma(\xi)}.
   \]
Proof. Let $\sigma$ be a geodesic ray from $o$ to $\xi$ and let $\sigma'$ be a geodesic ray from $o$ to $\gamma(\xi)$. Then
\[ h_{\gamma(\xi)}(\gamma(z)) = \lim_{t \to \infty} (d(\gamma(z), \sigma'(t)) - d(o, \sigma'(t))) = \lim_{t \to \infty} (d(z, \gamma^{-1} \circ \sigma'(t)) - d(\gamma^{-1}(o), \gamma^{-1} \circ \sigma'(t))). \]
Here, $\gamma^{-1} \circ \sigma'$ is a geodesic ray from $\gamma^{-1}(o)$ to $\xi$. Because $\sigma$ has the same endpoint $\xi$ as $\gamma^{-1} \circ \sigma'$, Proposition 2.1 implies that we can replace $\gamma^{-1} \circ \sigma'$ with $\sigma$ in the formula of $h_{\gamma(\xi)}(\gamma(z))$ with additive error $2\kappa(\delta)$. On the contrary, by
\[ h_{\xi}(z) = \lim_{t \to \infty} (d(z, \sigma(t)) - d(o, \sigma(t))) = \lim_{t \to \infty} (d(z, \sigma(t)) - d(\gamma^{-1}(o), \sigma(t))) + \lim_{t \to \infty} (d(\gamma^{-1}(o), \sigma(t)) - d(o, \sigma(t))), \]
we see that
\[ |h_{\gamma(\xi)}(\gamma(z)) - h_{\xi}(z) + h_{\xi}(\gamma^{-1}(o))| \leq 2\kappa(\delta), \]
which implies the required estimate. \qed

Moreover, $h_{\xi}(z)$ is approximated by the difference of the distances to $z$ and $o$ from a point $x$ sufficiently close to $\xi$. This can be found in Coornaert [2, Lemme 2.2].

Proposition 2.3. For every $\xi \in \partial X$ and every $z \in X$, there is a neighborhood $U_{\xi} \subset X$ of $\xi$ such that
\[ |h_{\xi}(z) - (d(z, x) - d(o, x))| \leq c(\delta) \]
for every $x \in U_{\xi} \cap X$, where $c(\delta) \geq 0$ is a constant depending only on $\delta$.

Hereafter, the constant $c(\delta)$ will always refer to that in the above proposition.

2.3. The Poincaré series and the limit set. For a Gromov hyperbolic space $(X, d)$, we denote the group of all isometric automorphisms of $(X, d)$ by $\text{Isom}(X, d)$. We say that a subgroup $\Gamma \subset \text{Isom}(X, d)$ is discrete if it acts on $X$ properly discontinuously.

Definition. For a discrete subgroup $\Gamma \subset \text{Isom}(X, d)$, the Poincaré series $P_{\Gamma}^s(z, x)$ of dimension (or exponent) $s \geq 0$ with respect to the visual parameter $a$ is given by
\[ P_{\Gamma}^s(z, x) = \sum_{\gamma \in \Gamma} a^{-sd(z, \gamma(x))}. \]
We call $z \in X$ the reference point and $x \in X$ the orbit point. The convergence or divergence of $P_{\Gamma}^s(z, x)$ is independent of the choice of $z$ and $x$. The critical exponent of $\Gamma$ is defined by
\[ e_a(\Gamma) = \inf \{ s \geq 0 \mid P_{\Gamma}^s(z, x) < \infty \}. \]

We remark that in contrast with the case of Kleinian groups, the critical exponent is possibly infinite. In this study, we are only interested in the case where it is finite. The divergence of the Poincaré series at the finite critical exponent is a remarkable property.
Definition. If the critical exponent $e_a(\Gamma)$ of a discrete group $\Gamma \subset \text{Isom}(X,d)$ is finite and the Poincaré series $P^\Gamma(z,x)$ of dimension $s = e_a(\Gamma)$ diverges, then $\Gamma$ is said to be of divergence type.

For example, every quasiconvex cocompact group $\Gamma$ with $e_a(\Gamma) < \infty$ is of divergence type (see Coornaert [2, Corollaire 7.3]).

Here, we present certain properties of the Poincaré series that are used later. See [8, Proposition 2.1] in the case of Kleinian groups. There is no difference in the present case.

Proposition 2.4. We have the following properties:

(1) $P^\Gamma_s(z,x) = P^\Gamma_s(x,z)$;
(2) $P^\Gamma_s(g(z), g(x)) = P^\Gamma_s(x,z)$ for every $g \in \text{Isom}(X,d)$ with $g\Gamma g^{-1} = \Gamma$.

Proof. Property (1) follows from the equality $d(z, \gamma(x)) = d(x, \gamma^{-1}(z))$. Property (2) follows from the equality $d(g(z), \gamma g(z)) = d(z, \tilde{\gamma}(x))$ for $\tilde{\gamma} \in \Gamma$ with $\gamma g = g\tilde{\gamma}$. □

Next, we define the limit set $\Lambda(\Gamma)$ of a discrete group $\Gamma$ as the set of all accumulation points $\xi$ of the orbit $\Gamma(x)$ of $x \in X$ in $\partial X$. This is independent of the choice of $x$; hence, we may take $x = o$. It is known that $\Lambda(\Gamma)$ is a $\Gamma$-invariant closed subset of $\partial X$. If $\#\Lambda(\Gamma) \geq 3$, then we say that $\Gamma$ is non-elementary.

Definition. For a discrete group $\Gamma \subset \text{Isom}(X,d)$, $\xi \in \Lambda(\Gamma)$ is called a conical limit point if there is some geodesic ray $\beta$ toward $\xi$ and a constant $\rho > 0$ such that the orbit $\Gamma(o)$ accumulates to $\xi$ within the closed $\rho$-neighborhood of some geodesic ray $\beta$ toward $\xi$. The set of all conical limit points of $\Gamma$ is called the conical limit set and denoted by $\Lambda^c(\Gamma)$.

We utilize the exhaustion of $\Lambda^c(\Gamma)$ by a sequence of $\Gamma$-invariant subsets defined by $\rho$. Namely, for a fixed $\rho > 0$, $\xi \in \Lambda^c(\Gamma)$ belongs to the conical limit subset $\Lambda^c(\rho)(\Gamma)$ if $\Gamma(o)$ accumulates to $\xi$ within the closed $\rho$-neighborhood of some geodesic ray $\beta$ toward $\xi$. Then $\Lambda^c(\Gamma) = \bigcup_{\rho > 0} \Lambda^c(\rho)(\Gamma)$.

2.4. Quasiconformal measure. In the Kleinian group case, conformal measures on the boundary at infinity play a central role in the Patterson–Sullivan theory. On the Gromov hyperbolic space, such measures are allowed to have some ambiguity, which are called quasiconformal measures. They were introduced by Coornaert [2].

We first define this concept for a family of measures labeled by all points in the Gromov hyperbolic space $X$, and then formulate its quasi-invariance under a subgroup $\Gamma \subset \text{Isom}(X,d)$. For Kleinian groups, this way of defining those measures can be found in Nicholls [10].

Definition. A family $\{\mu_z\}_{z \in X}$ of finite positive Borel measures on $\partial X$ is called a quasi-conformal measure family of dimension $s \geq 0$ if the following conditions are satisfied:

(1) $\mu_z$ and $\mu_{z'}$ are mutually absolutely continuous for any $z, z' \in X$;
(2) there is a constant $C \geq 1$ such that the Radon–Nikodym derivative satisfies
\[ C^{-s}k(z, \xi)^s \leq \frac{d\mu_z}{d\mu_o}(\xi) \leq C^sk(z, \xi)^s \quad \text{(a.e. } \xi \in \partial X) \]
for every $z \in X$.

We call $C$ the quasiconformal constant.

**Definition.** For a subgroup $\Gamma \subset \text{Isom}(X, d)$, an $s$-dimensional quasiconformal measure family $\{\mu_z\}_{z \in X}$ is called $\Gamma$-quasi-invariant if there is a constant $D \geq 1$ such that
\[ D^{-1} \leq \frac{d(\gamma^*\mu_z)}{d\mu_z}(\xi) \leq D \quad \text{(a.e. } \xi \in \partial X) \]
for every $z \in X$ and for every $\gamma \in \Gamma$. Here, $\gamma^*\mu$ denotes the pull-back of the measure $\mu$ by $\gamma$. To specify the quasi-invariance constant $D$, we call it $(\Gamma, D)$-quasi-invariant.

The condition that $\{\mu_z\}_{z \in X}$ is $(\Gamma, D)$-quasi-invariant implies a condition on a single positive finite Borel measure $\mu = \mu_o$ at the base point as follows.

**Proposition 2.5.** If an $s$-dimensional quasiconformal measure family $\{\mu_z\}_{z \in X}$ is $(\Gamma, D)$-quasi-invariant with quasiconformal constant $C \geq 1$, then $\mu = \mu_o$ satisfies
\[ \tilde{D}^{-1}j_\gamma(\xi)^s \leq \frac{d(\gamma^*\mu)}{d\mu}(\xi) \leq \tilde{D}j_\gamma(\xi)^s \quad \text{(a.e. } \xi \in \partial X) \]
for every $\gamma \in \Gamma$ with $\tilde{D} = C^sD$.

**Proof.** The $(\Gamma, D)$-quasi-invariance of $\{\mu_z\}_{z \in X}$ for $z = \gamma^{-1}(o)$ implies
\[ D^{-1} \leq \frac{d(\gamma^*\mu)}{d\mu_{\gamma^{-1}(o)}}(\xi) \leq D, \]
and the quasiconformality with constant $C$ implies
\[ C^{-s}k(\gamma^{-1}(o), \xi)^s \leq \frac{d\mu_{\gamma^{-1}(o)}}{d\mu_o}(\xi) \leq C^sk(\gamma^{-1}(o), \xi)^s. \]

These two formulae with $k(\gamma^{-1}(o), \xi) = j_\gamma(\xi)$ show the assertion. \qed

If a positive finite Borel measure $\mu$ on $\partial X$ satisfies the condition in Proposition 2.5, then we also call it a $\Gamma$-quasi-invariant or, more precisely, $(\Gamma, \tilde{D})$-quasi-invariant quasiconformal measure of dimension $s$.

**2.5. The Patterson measure.** Quasi-invariant quasiconformal measures of dimension at the critical exponent are the main tools in our study.

**Definition.** For a non-elementary discrete group $\Gamma \subset \text{Isom}(X, d)$, a $\Gamma$-quasi-invariant quasiconformal measure $\mu$ or measure family $\{\mu_z\}_{z \in X}$ of dimension at the critical exponent $e_a(\Gamma) < \infty$ with support in the limit set $\Lambda(\Gamma)$ is called a Patterson measure (family).
Remark. In this paper, by the support of a measure, we mean the smallest closed set that is of full measure. As the limit set $\Lambda(\Gamma)$ is the minimal non-empty closed $\Gamma$-invariant subset when $\Gamma$ is non-elementary (see [2, Théorème 5.1]), if a $\Gamma$-quasi-invariant quasiconformal measure has its support in $\Lambda(\Gamma)$, then it coincides with $\Lambda(\Gamma)$.

The existence of a Patterson measure can be verified by the construction due to Patterson. For a discrete group $\Gamma$ of divergence type, this is given by a weak-$*$ limit of a sequence of weighted Dirac masses $m^s_{z,x}$ on $\overline{X}$ defined by the Poincaré series $P^s_\Gamma(z,x)$ as $s$ tends to $e_a(\Gamma)$. This construction naturally produces a $\Gamma$-quasi-invariant quasiconformal measure family. We discuss the canonical Patterson measures obtained in this way in Section 6. In addition, the lower bound of the dimensions of quasi-invariant quasiconformal measures for $\Gamma$ is equal to the critical exponent $e_a(\Gamma)$, which is a consequence of the shadow lemma stated in the following subsection. These results were proved by Coornaert [2, Théorème 5.4, Corollaire 6.6] as follows.

**Theorem 2.6.** Assume that a non-elementary discrete group $\Gamma \subset \text{Isom}(X,d)$ has finite critical exponent $e_a(\Gamma)$. Then, a Patterson measure for $\Gamma$ exists. Moreover, the dimension $s$ of any $\Gamma$-quasi-invariant quasiconformal measure is at least $e_a(\Gamma)$.

We note that if $\Gamma$ is of divergence type, then every $\Gamma$-quasi-invariant quasiconformal measure $\mu$ of dimension $e_a(\Gamma)$ must have its support in the limit set $\Lambda(\Gamma)$, which was mentioned in [9, Lemma 3.7]. This means that $\mu$ is a Patterson measure.

**Proposition 2.7.** Assume that a discrete group $\Gamma \subset \text{Isom}(X,d)$ is of divergence type. If $\mu$ is a $\Gamma$-quasi-invariant quasiconformal measure of dimension $e_a(\Gamma)$, then the support of $\mu$ is on the limit set $\Lambda(\Gamma)$.

In Section [4], we show that any Patterson measure $\mu$ has full measure on the conical limit set $\Lambda_c(\Gamma)$ when $\Gamma$ is of divergence type.

The following is an immediate consequence of Proposition 2.7.

**Corollary 2.8.** Assume that a non-elementary discrete group $G \subset \text{Isom}(X,d)$ contains a discrete subgroup $\Gamma$ of divergence type. If $\Lambda(G) \supseteq \Lambda(\Gamma)$, then $e_a(G) > e_a(\Gamma)$.

**Proof.** Suppose that $e_a(G) = e_a(\Gamma)$. Then, the Patterson measure $\mu$ for $G$ whose support coincides with $\Lambda(G)$ is also a $\Gamma$-quasi-invariant quasiconformal measure of dimension $e_a(G) = e_a(\Gamma)$. By Proposition 2.7, the support of $\mu$ is in $\Lambda(\Gamma)$. This implies that $\Lambda(G) = \Lambda(\Gamma)$. □

### 2.6. The shadow lemma.

The shadow of a ball in the Gromov hyperbolic space $X$ connects the measure on the boundary $\partial X$ with the geometry of $X$. The shadow lemma is a fundamental tool in the Patterson–Sullivan theory.

**Definition.** Let $B(x,r)$ be the closed ball of center $x \in X$ and radius $r \geq 0$. For a light source $\omega \in \overline{X}$, the shadow is defined by

$$S_\omega(x,r) = \{\xi \in \partial X \mid \forall[\omega, \xi] \cap B(x,r) \neq \emptyset\},$$
where \( \forall [\omega, \xi] \) refers to every geodesic line or ray connecting \( \omega \) and \( \xi \). In addition, the extended shadow is given by

\[
\tilde{S}_\omega(x, r) = \{ y \in \overline{X} \mid \forall [\omega, y] \cap B(x, r) \neq \emptyset \}.
\]

The shadow lemma is based on the following estimate for the Poisson kernel. If we take \( z = \gamma^{-1}(o) \) for \( \gamma \in \text{Isom}(X, d) \), this turns out to be the estimate for \( j_\gamma(\xi) \). This was essentially given in [2, Lemme 6.1].

**Lemma 2.9.** There is a constant \( C = C(\delta, a) \geq 1 \) such that if \( o \notin \tilde{S}_\omega(z, r) \), then

\[
C^{-1}a^{d(o,z)-2r} \leq k(z, \xi) \leq Ca^{d(o,z)}
\]

for every \( \xi \in S_\omega(z, r) \).

**Proof.** For a tree \( X \), that is, a 0-hyperbolic space, the statement can be easily verified. Then, we apply approximation by trees as in [2, Théorème 1.1]. \( \square \)

The complement of a shadow can be arbitrarily small if we make the radius sufficiently large. This geometric observation [2, Lemme 6.3] is also used in the proof of the shadow lemma below. For later purposes, we extend it slightly and provide a proof. Here, \( \text{diam}_a \) denotes the diameter with respect to the visual metric \( d_a \).

**Proposition 2.10.** For every \( \varepsilon > 0 \), there is a constant \( r(\varepsilon) > 0 \) such that if \( r \geq r(\varepsilon) \), then

\[
\text{diam}_a(\partial X - S_\omega(o, r)) \leq \varepsilon
\]

for every \( \omega \in \overline{X} \).

**Proof.** We may assume that \( \omega \notin B(o, r) \). For any \( \xi, \eta \in \partial X - S_\omega(o, r) \), we take some geodesic lines or rays \([\omega, \xi]\) and \([\omega, \eta]\) that do not intersect \( B(o, r) \). For any geodesic line \((\xi, \eta)\), we consider the geodesic triangle \( \Delta(\omega, \xi, \eta) \) with edges \((\xi, \eta)\), \([\omega, \xi]\), and \([\omega, \eta]\). As \((\xi, \eta)\) is within a distance \( \kappa(\delta) \) from the union \([\omega, \xi] \cup [\omega, \eta]\) by Proposition 2.1, we have \( d(o, (\xi, \eta)) \geq r - \kappa(\delta) \). By the estimate for the visual metric,

\[
d_a(\xi, \eta) \leq \lambda a^{-d(o,(\xi,\eta))} \leq \lambda a^{\kappa(\delta)-r}
\]

for the constant \( \lambda = \lambda(\delta, a) \geq 1 \). Hence, we can choose \( r(\varepsilon) \) so that \( \lambda a^{\kappa(\delta)-r(\varepsilon)} \leq \varepsilon \). \( \square \)

The following theorem was proved in [2, Proposition 6.1].

**Theorem 2.11** (Shadow lemma). Let \( \Gamma \subset \text{Isom}(X, d) \) be a non-elementary discrete group and let \( \mu \) be a \( \Gamma \)-quasi-invariant quasiconformal measure of dimension \( s \). We fix a light source \( \omega \in \overline{X} \). Then, there are constants \( L \geq 1 \) and \( r_0 > 0 \) such that

\[
L^{-1}a^{-sd(o,\gamma^{-1}(o))} \leq \mu(S_\omega(\gamma^{-1}(o), r)) \leq La^{-2r}a^{-sd(o,\gamma^{-1}(o))}
\]

for every \( \gamma \in \Gamma \) with \( o \notin \tilde{S}_\omega(\gamma^{-1}(o), r) \) and for every \( r \geq r_0 \).
The conical limit set $\Lambda_c(\Gamma)$ can be described by the limit superior of the family of shadows $\{S_\rho(\gamma(o), r)\}_{\gamma \in \Gamma}$. More precisely, by setting $r = \rho + \kappa(\delta)$ for each $\rho > 0$, Proposition 2.1 implies that
\[
\limsup_{\gamma \in \Gamma} S_\rho(\gamma^{-1}(o), \rho) \subset \Lambda^c(\rho)(\Gamma) \subset \limsup_{\gamma \in \Gamma} S_\rho(\gamma^{-1}(o), r).
\]
Then, $\Lambda_c(\Gamma) = \bigcup_{\rho > 0} \Lambda^c(\rho)(\Gamma)$ coincides with the limit of the right-hand side (left-hand side) as $r \to \infty$ ($\rho \to \infty$).

By this description of $\Lambda_c(\Gamma)$ and Theorem 2.11, we have the following claim. In Section 3 we show that the converse of this statement is also true.

**Proposition 2.12.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group and let $\mu$ be an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure on $\partial X$. If the Poincaré series $P_t^\Gamma(z, x)$ converges, then the measure of the conical limit set $\mu(\Lambda_c(\Gamma))$ is zero.

**Proof.** We choose the constant $r_0 > 0$ in Theorem 2.11 for $\Gamma$ and $\mu$ and prove that $\mu(\Lambda^c(\rho)(\Gamma)) = 0$ for every $\rho \geq r_0 - \kappa(\delta)$. As $P_t^\Gamma(o, o) < \infty$, Theorem 2.11 implies that
\[
\mu\left(\bigcup_{\gamma \in \Gamma'} S_o(\gamma^{-1}(o), r)\right) \leq \sum_{\gamma \in \Gamma'} \mu(S_o(\gamma^{-1}(o), r)) \leq L a^{2rs} \sum_{\gamma \in \Gamma'} a^{-sd(o, \gamma^{-1}(o))} < \infty
\]
for $r = \rho + \kappa(\delta) \geq r_0$, where $\Gamma'$ is $\Gamma$ minus possibly finitely many elements. Then, we see that the measure of the limit superior of $\{S_o(\gamma^{-1}(o), r)\}$ is zero. \qed

**Corollary 2.13.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group and let $\mu$ be a $\Gamma$-quasi-invariant quasiconformal measure on $\partial X$. If $\mu(\Lambda_c(\Gamma)) > 0$, then $\Gamma$ is of divergence type and $\mu$ is a Patterson measure for $\Gamma$.

**Proof.** Let $s$ be the dimension of $\mu$. By Proposition 2.12, we have $P_t^\Gamma(z, x) = \infty$; hence, $s \leq e_\rho(\Gamma)$. Then Theorem 2.6 implies that $s = e_\rho(\Gamma)$; therefore, $\Gamma$ is of divergence type. Moreover, by Proposition 2.7, $\mu$ should be a Patterson measure. \qed

We can also claim that $\mu$ has no atoms on $\Lambda_c(\Gamma)$.

**Proposition 2.14.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group and let $\mu$ be an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure on $\partial X$. Then, $\mu$ has no point mass on a conical limit point $\xi \in \Lambda_c(\Gamma)$.

**Proof.** There is some $r > 0$ and a sequence $\{\gamma_n\}_{n=1}^\infty \subset \Gamma$ such that $\gamma_n^{-1}(o)$ converge to $\xi$ as $n \to \infty$ and $\xi \in S_o(\gamma_n^{-1}(o), r)$ for every $n \in \mathbb{N}$. Then, by Proposition 2.5 and Lemma 2.9 there are constants $D \geq 1$ and $C \geq 1$ such that
\[
\mu(\{\gamma_n(\xi)\}) = (\gamma_n^* \mu)(\{\xi\}) \geq D^{-1} (C^{-1} a^{d(o, \gamma_n^{-1}(o)) - 2r})^s \mu(\{\xi\}).
\]
This implies that if $\mu(\{\xi\}) > 0$ and $s > 0$ then $\mu(\{\gamma_n(\xi)\}) \to \infty$ as $n \to \infty$. If $\mu(\{\xi\}) > 0$ and $s = 0$ (even though in fact, we have $s > 0$ by Proposition 8.6 later), then $\mu(\Gamma(\xi)) = \infty$. Both cases are impossible; hence, $\mu(\{\xi\}) = 0$. \qed
3. Divergence type and measure on the conical limit set

We have seen in Proposition 2.12 that if the $s$-dimensional Poincaré series $P_t^s(z, x)$ converges for a discrete group $\Gamma \subset \text{Isom}(X, d)$, then the conical limit set $\Lambda_c(\Gamma)$ has null measure for any $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure $\mu$ on $\partial X$. In this section, we prove the converse of this statement.

**Theorem 3.1.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group. If $\mu(\Lambda_c(\Gamma)) = 0$ for an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure $\mu$ on $\partial X$, then $P_t^s(z, x)$ converges.

For Kleinian groups, this result was proved by Sullivan [16, 17] by considering ergodicity of the geodesic flow (see also Roblin [13, Théorème 1.7] for a complete argument). Later, Tukia [18] gave an elementary proof for it. His arguments are applicable to discrete isometry groups of Gromov hyperbolic spaces if certain modifications are made. In what follows, we will carry this out respecting Tukia’s arguments.

As in Proposition 2.1 for a $\delta$-hyperbolic space $(X, d)$, we choose the constant $\kappa(\delta) > 0$ such that for every geodesic triangle or bi-angle possibly with vertices on the boundary $\partial X$, each edge is contained in the closed $\kappa(\delta)$-neighborhood of the union of the others.

We utilize shadows to prove Theorem 3.1. In this section, we always put the light source $\omega$ on the boundary $\partial X$. The following Lemma 3.2 provides a fundamental relation between two shadows. As this will also be used later in another case where $\omega$ is in $X$, it is generally assumed that $\omega \in X$ only in this lemma.

**Lemma 3.2.** For any constants $r' \geq r \geq \rho \geq \rho' \geq 0$ with $r - \rho \geq \kappa(\delta)$, if

$$B(x, \rho') \cap \widehat{S}_\omega(z, \rho) \neq \emptyset$$

for any $z, x \in X$ with $d(z, x) > 4r' + \kappa(\delta)$ and for any $\omega \in X$, then

$$B(x, r') \subset \widehat{S}_\omega(z, r).$$

**Proof.** We choose a point $x' \in B(x, \rho') \cap \widehat{S}_\omega(z, \rho)$ and any geodesic ray $(\omega, x')$ (or geodesic segment $[\omega, x']$) from $x'$ to $\omega$. Then, $(\omega, x') \cap B(z, \rho) \neq \emptyset$, so we can take a point $p$ in this intersection. It should be noted that $d(x, x') \leq \rho'$ and $d(z, p) \leq \rho$. For any $y \in B(x, r')$, we have that $d(x', y) \leq \rho' + r'$. We consider any triangle $\Delta(\omega, x', y)$ with the vertex $y$ and the edge $(\omega, x')$ containing $p$. This edge is contained in the closed $\kappa(\delta)$-neighborhood of the union of the other edges $(\omega, y) \cup [x', y]$. It follows that for every geodesic ray $(\omega, y)$ and for every geodesic segment $[x', y]$, there is a point $p' \in (\omega, y) \cup [x', y]$ such that $d(p, p') \leq \kappa(\delta)$. We want to have $p' \in (\omega, y)$.

To see this, we show that $d(p, [x', y]) > \kappa(\delta)$ for every geodesic segment $[x', y]$. As $d(x, x') \leq \rho'$ and $d(x', y) \leq \rho' + r'$, the distance from $x$ to each point in $[x', y]$ is at most $2\rho' + r' \leq 3r'$. Using this together with $d(z, x) > 4r' + \kappa(\delta)$ and $d(z, p) \leq \rho \leq r'$, we obtain $d(p, [x', y]) > \kappa(\delta)$. Hence, $p' \in (\omega, y)$.
Furthermore, as $d(z, p) \leq \rho, d(p, p') \leq \kappa(\delta)$, and $r - \rho \geq \kappa(\delta)$, we have $d(z, p') \leq r$, that is, $p' \in B(z, r)$. Hence, $(\omega, y] \cap B(z, r) \neq \emptyset$. As $y$ is an arbitrary point of $B(x, r')$ and this conclusion is valid for every geodesic ray $(\omega, y]$, we conclude that $B(x, r') \subset \hat{S}_\omega(z, r)$. □

In the next two claims, we consider the influence of slightly moving the light source $\omega \in \partial X$.

**Lemma 3.3.** For $\omega_0 \in \partial X$, let $D \subset X$ be a domain with $\omega_0 \notin \mathcal{D}$. Let $r \geq 0$ be any constant. Then, there exists a neighborhood $V \subset \partial X$ of $\omega_0$ such that if $\omega \in V$ and $x \in D$, then $\hat{S}_\omega(x, r) \subset \hat{S}_{\omega_0}(x, r')$ for any $r' \geq r + \kappa(\delta)$.

**Proof.** We can choose a neighborhood $V$ of $\omega_0$ in $\partial X$ so that the distance from every point in the closed $r$-neighborhood $N_r(D)$ of $D$ to some geodesic line with the endpoints $\omega_0$ and any $\omega \in V$ is greater than $\kappa(\delta)$. Then, every point $y \in N_r(D)$ on a geodesic line $(\xi, \omega)$ with endpoints $\xi \in \partial X$ and $\omega \in V$ is within distance $\kappa(\delta)$ of any geodesic line $(\xi, \omega_0)$ with the endpoints $\xi$ and $\omega_0$ by Proposition 2.1.

For any $\omega \in V$ and $x \in D$, we prove that $S_\omega(x, r)$ is contained in $S_{\omega_0}(x, r')$. We take an arbitrary $\xi \in S_\omega(x, r)$ and choose some $y \in (\xi, \omega) \cap B(x, r)$. As $y \in N_r(D)$, every geodesic line $(\xi, \omega_0)$ contains a point $y'$ with $d(y, y') \leq \kappa(\delta) \leq r' - r$. Hence, $y' \in B(x, r')$. In particular, $(\xi, \omega_0) \cap B(x, r') \neq \emptyset$. This implies that $\xi \in S_{\omega_0}(x, r')$, and thus the inclusion $S_\omega(x, r) \subset S_{\omega_0}(x, r')$ is proved. The required inclusion $\hat{S}_\omega(x, r) \subset \hat{S}_{\omega_0}(x, r')$ then follows from this. □

The neighborhood $V$ of $\omega_0 \in \partial X$ given in Lemma 3.3 for $D = \hat{S}_{\omega_0}(o, r)$ and for a constant $r \geq 0$ is denoted by $V(\omega_0, r)$.

**Proposition 3.4.** Let $\omega_0 \in \partial X$ and $r \geq 0$. If $B(x, r) \cap \hat{S}_\omega(z, r) \neq \emptyset$ is satisfied for $\omega \in V(\omega_0, r + \kappa(\delta))$, $z \in \hat{S}_\omega(o, r + \kappa(\delta))$, and $x \in X$ with $d(x, z) > 4r + 9\kappa(\delta)$, then $B(x, r') \subset \hat{S}_{\omega_0}(z, r)$ for $r' = r + 2\kappa(\delta)$.

**Proof.** By Lemma 3.3 applied to $D = \hat{S}_{\omega_0}(o, r + \kappa(\delta))$, we see that $\hat{S}_\omega(z, r + \kappa(\delta)) \subset \hat{S}_{\omega_0}(z, r + 2\kappa(\delta))$ for any $\omega \in V(\omega_0, r + \kappa(\delta))$ and $z \in \hat{S}_{\omega_0}(o, r + \kappa(\delta))$. On the contrary, by Lemma 3.2 we see that the condition $B(x, r) \cap \hat{S}_\omega(z, r) \neq \emptyset$ with $d(x, z) > 4r + 9\kappa(\delta)$ implies $B(x, r + 2\kappa(\delta)) \subset \hat{S}_\omega(z, r + \kappa(\delta))$. Hence, $B(x, r') \subset \hat{S}_{\omega_0}(z, r')$ follows for $r' = r + 2\kappa(\delta)$. □

The following notations will be used in the proof of Theorem 3.1. For $\omega \in \partial X$, $z \in X$, and $r > 0$, we consider the subset

$$A_\omega(z, r) = \{x \in \Gamma(o) \mid B(x, r) \subset \hat{S}_\omega(z, r)\}$$
of the orbit $\Gamma(o) \subset X$. For each integer $i \in \mathbb{N}$, we define subsets $A^i_\omega(z, r)$ and $\hat{A}^i_\omega(z, r)$ of $A_\omega(z, r)$ inductively as follows:

$$\hat{A}^1_\omega(z, r) = A_\omega(z, r);$$

$$A^1_\omega(z, r) = \{x^1 \in \hat{A}^1_\omega(z, r) \mid B(x^1, r) \not\subset \hat{S}_\omega(x, r) \ (\forall x \in \hat{A}^1_\omega(z, r))\};$$

$$\cdots$$

$$\hat{A}^i_\omega(z, r) = A_\omega(z, r) - \bigcup_{j=1}^{i-1} A^j_\omega(z, r);$$

$$A^i_\omega(z, r) = \{x^i \in \hat{A}^i_\omega(z, r) \mid B(x^i, r) \not\subset \hat{S}_\omega(x, r) \ (\forall x \in \hat{A}^i_\omega(z, r))\}.$$

This gives a stratification of the orbit by using the inclusion relation of shadows.

As in the stratification by distance, orbit points in each stratum have disjoint shadows if they are sufficiently apart.

**Lemma 3.5.** Assume that constants $r, \rho \geq 0$ satisfy $r - \rho \geq \kappa(\delta)$. For any $\omega \in \partial X$ and $z \in X$ and for any $i \in \mathbb{N}$, if $x, x' \in A^i_\omega(z, r)$ satisfy $d(x, x') > 4r + \kappa(\delta)$, then

$$\hat{S}_\omega(x, \rho) \cap \hat{S}_\omega(x', \rho) = \emptyset.$$

**Proof.** It is assumed toward a contradiction that $\hat{S}_\omega(x, \rho) \cap \hat{S}_\omega(x', \rho) \neq \emptyset$. As $d(x, x') > 2\rho$ follows from assumption, $B(x, \rho) \cap B(x', \rho) = \emptyset$. Hence, either $B(x', \rho) \cap \hat{S}_\omega(x, \rho) \neq \emptyset$ or $B(x, \rho) \cap \hat{S}_\omega(x', \rho) \neq \emptyset$ is satisfied. We assume the former. The other case is treated similarly. We apply Lemma 3.2 for $r' = r > \rho = \rho'$ with $r - \rho \geq \kappa(\delta)$ to obtain $B(x', r) \subset \hat{S}_\omega(x, r)$. However, this violates the condition that $x$ and $x'$ belong to the same stratum $A^i_\omega(z, r)$.

This property can be interpreted in terms of the number of orbit points in each stratum having intersecting shadows. For $r > 0$, let $M(r)$ be the number of orbit points $\Gamma(o)$ in the closed ball $B(o, r)$.

**Corollary 3.6.** For constants $r, \rho \geq 0$ with $r - \rho \geq \kappa(\delta)$, the family of shadows $\{S_\omega(x, \rho)\}$ taken over all $x \in A^i_\omega(z, r)$ are $M(4r + \kappa(\delta))$-disjoint, that is, for each shadow $S_\omega(x, \rho)$, the number of shadows $S_\omega(x', \rho)$ in the family with $S_\omega(x, \rho) \cap S_\omega(x', \rho) \neq \emptyset$ is at most $M(4r + \kappa(\delta))$.

**Proof.** If $S_\omega(x, \rho) \cap S_\omega(x', \rho) \neq \emptyset$, then $d(x, x') \leq 4r + \kappa(\delta)$ by Lemma 3.5.

We note here that if we go through sufficiently many strata, we can gain a definite distance.

**Lemma 3.7.** For constants $r \geq 0$ and $\ell \geq 0$, let $m \in \mathbb{N}$ be an integer greater than $M(\ell + 2r)$. Then, every point $x \in A^m_\omega(z, r)$ for any $z \in \Gamma(o)$ and $\omega \in \partial X$ satisfies $d(z, x) > \ell$. 

Proof. Suppose to the contrary that \(d(z, x) \leq \ell\). We take a sequence \(z = x_0, x_1, \ldots, x_m = x\) such that \(x_i \in A_{\omega}^i(z, r)\) and \(B(x_i, r) \subset \hat{S}(x_{i-1}, r)\) for \(1 \leq i \leq m\). We show that these points are all in \(B(z, \ell + 2r)\). This contradicts the way of choosing \(m\). Clearly \(z = x_0\) and \(x = x_m\) belong to \(B(z, \ell + 2r)\). We only have to show that \(d(z, x_i) \leq d(z, x) + 2r\) for every \(i = 1, \ldots, m - 1\).

Suppose that there is some \(i\) such that \(d(z, x_i) > d(z, x) + 2r\). Take a point \(x' \in [z, x] \cap \partial B(x, r)\), which satisfies \(d(z, x') = d(z, x) - r\). Similarly, \(d(z, x_i) - r = d(z, B(x_i, r))\).

From these three conditions, we have
\[
d(z, x') + 2r < d(z, B(x_i, r)).
\]
However, by considering a geodesic ray \((\omega, x')\), which intersects both \(B(z, r)\) and \(B(x_i, r)\), we can derive a contradiction. Indeed, taking a point \(z' \in (\omega, x'] \cap B(z, r)\), we can apply the inequality \(d(z', x') \geq d(z', B(x_i, r))\) to show that
\[
d(z, x') + 2r \geq d(z', x') - d(z, z') + 2r \\
\geq d(z', B(x_i, r)) + r \geq d(z, B(x_i, r)).
\]
This completes the proof. \(\square\)

Proposition \ref{prop:stability} and Lemma \ref{lem:shadowcoverage} imply the stability in a certain sense of the structure of the strata under small changes of the light source.

**Proposition 3.8.** Let \(r, r' \geq 0\) be constants such that \(r' = r + 2\kappa(\delta)\) and let \(m = M(6r + 9\kappa(\delta))\). Then,
\[
\hat{A}_\omega^m(o, r) \subset \hat{A}_\omega^i(o, r')
\]
for any \(\omega_0 \in \partial X\), \(\omega \in V(\omega_0, r)\), and \(i \in \mathbb{N}\).

**Proof.** For every \(x \in \hat{A}_\omega^m(o, r)\), we can choose a sequence \(\{x_0, x_1, \ldots, x_i\} \subset A_\omega(o, r)\) such that \(x_i = x\), \(x_0 = o\), and \(x_j \in \hat{A}_\omega^m(x_{j-1}, r)\) for every \(j = 1, 2, \ldots, i\). For \(\ell = 4r + 9\kappa(\delta)\), Lemma \ref{lem:shadowcoverage} asserts that \(d(x_{j-1}, x_j) > \ell\) for each \(j = 1, 2, \ldots, i\). Then, by the condition \(x_{j-1} \in \hat{S}_\omega(o, r + \kappa(\delta))\) given in Lemma \ref{lem:shadowcoverage} Proposition \ref{prop:stability} shows that \(B(x_j, r') \subset \hat{S}_\omega(x_{j-1}, r')\) for all such \(j\). This implies that \(x = x_i\) belongs to \(\hat{A}_\omega^i(o, r')\). \(\square\)

As a final step in the preparation, we note that \(X\) is covered by finitely many extended shadows \(\hat{S}_{\omega_j}(o, \rho)\) with \(\omega_j \in \partial X\) for \(j = 1, \ldots, k\). By the compactness of \(\overline{X}\), this is obvious if we know that every point in \(\overline{X}\) is covered by the interior of an extended shadow \(\hat{S}_{\omega}(o, \rho)\) for some \(\omega \in \partial X\) with a fixed radius \(\rho > 0\). However, this property is slightly different from the property that every closed ball centered at an orbit point is entirely contained in one of such extended shadows. We fill this gap with the following claim.

**Proposition 3.9.** Assume that constants \(r, \rho \geq 0\) satisfy \(r - \rho \geq \kappa(\delta)\). If there are finitely many points \(\omega_1, \ldots, \omega_k \in \partial X\) such that \(\bigcup_{j=1}^{k} \hat{S}_{\omega_j}(o, \rho) = \overline{X}\), then for every \(x \in \Gamma(o) - B(o, 4r + \kappa(\delta))\) there is some \(j = 1, \ldots, k\) such that \(B(x, r) \subset \hat{S}_{\omega_j}(o, r)\).
Proof. By assumption, every \( x = B(x, 0) \in \Gamma(o) \) belongs to some \( \widehat{S}_{\omega_j}(o, \rho) \). We apply Lemma 3.2 for \( r' = r > \rho > \rho' = 0 \). This yields that if \( d(o, x) > 4r + \kappa(\delta) \), then \( B(x, r) \subset \widehat{S}_{\omega_j}(o, r) \). □

The proof of Theorem 3.1 can now be carried out.

Proof of Theorem 3.1. Let \( \mu \) be an \( s \)-dimensional \( \Gamma \)-quasi-invariant quasiconformal measure on \( \partial X \) with \( \mu(\Lambda_c(\Gamma)) = 0 \). We choose \( \rho > 0 \) and extended shadows \( \widehat{S}_{\omega_j}(o, \rho) \) with \( \omega_j \in \partial X \) for \( j = 1, \ldots, k \) such that \( \bigcup_{j=1}^{k} \widehat{S}_{\omega_j}(o, \rho) = X \). By Proposition 3.9 if we set \( r \geq \rho + \kappa(\delta) \), then \( B(x, r) \) for every \( x \in \Gamma(o) - B(o, 4r + \kappa(\delta)) \) is contained in some \( \widehat{S}_{\omega_j}(o, r) \).

We prove that \( \sum_{x \in A_{\omega}(o, r)} \mu(S_{\omega}(x, r)) < \infty \) for any \( \omega \in \partial X \). By the shadow lemma (Theorem 2.11), this implies that \( \sum_{x \in A_{\omega}(o, r)} a^{-sd(o, x)} < \infty \).

As \( \Gamma(o) \) is the union of \( A_{\omega_1}(o, r) \) and \( A_{\omega_2}(o, r) \) except for the finitely many points contained in \( B(o, 4r + \kappa(\delta)) \), we obtain that \( P^s_{\Gamma}(o, o) = \sum_{x \in \Gamma(o)} a^{-sd(o, x)} < \infty \).

For a given \( \omega_0 \in \partial X \), we divide \( A_{\omega_0}(o, r) \) into \( \bigcup_{i=0}^{\infty} A^i_{\omega_0}(o, r) \). We set \( S_i = \bigcup_{x \in A^i_{\omega_0}(o, r)} S_{\omega_0}(x, r) = \bigcup_{x \in A^i_{\omega_0}(o, r)} S_{\omega_0}(x, r) \), which decreases as \( i \to \infty \). Then, \( \bigcap_i S_i \) is contained in \( \Lambda_c(\Gamma) \). As \( \mu(\Lambda_c(\Gamma)) = 0 \) by assumption, we see that \( \mu(S_i) \to 0 \) as \( i \to \infty \).

Lemma 3.10. Let \( r > 0 \) be a sufficiently large constant. For each \( \omega_0 \in \partial X \) and for any \( \alpha_0 > 0 \), there exists an integer \( I = I(\omega_0, \alpha_0) \in \mathbb{N} \) such that

\[
\sum_{x \in A^I_{\omega_0}(o, r)} \mu(S_{\omega}(x, r)) \leq \alpha_0 \mu(S_{\omega}(o, r))
\]

for every \( \omega \in V(\omega_0, r) \) and for every \( i \geq I \).

Proof. For an arbitrary \( \varepsilon > 0 \), we consider

\[
\bar{\varepsilon} = \varepsilon \inf \{ \mu(S_{\omega}(o, r)) \mid \omega \in V(\omega_0, r) \},
\]

which is positive for a sufficiently large \( r > 0 \). The above arguments for \( r' = r + 2\kappa(\delta) \) show that there is some \( i_0 \in \mathbb{N} \) such that

\[
\mu(\bigcup_{x \in A^i_{\omega_0}(o, r')} S_{\omega_0}(x, r')) \leq \bar{\varepsilon}
\]
for all $i \geq i_0$. By Proposition 3.8 we have
\[
\tilde{A}_\omega^{im}(o, r) \subset \tilde{A}_\omega^i(o, r')
\]
for any $\omega \in V(\omega_0, r)$ and $i \in \mathbb{N}$, where $m = M(6r + 9\kappa(\delta))$. Moreover, Lemma 3.3 yields
that $S_\omega(x, r) \subset S_{\omega_0}(x, r')$. Hence, by setting $I = m\bar{i}_0$, we have
\[
\mu(\bigcup_{x \in A^i_\omega(o, r)} S_\omega(x, r)) \leq \varepsilon \mu(S_\omega(o, r))
\]
for every $i \geq I$.

Here, we apply Corollary 3.6 for $\rho = r - \kappa(\delta)$. Then
\[
\sum_{x \in A^i_\omega(o, r)} \mu(S_\omega(x, \rho)) \leq M(4r + \kappa(\delta)) \mu\left(\bigcup_{x \in A^i_\omega(o, r)} S_\omega(x, \rho)\right)
\leq M(4r + \kappa(\delta)) \mu\left(\bigcup_{x \in A^i_\omega(o, r)} S_\omega(x, r)\right) \leq M(4r + \kappa(\delta))\varepsilon \mu(S_\omega(o, r)).
\]

Finally, by the shadow lemma (Theorem 2.11), if $r$ is sufficiently large we can find some constant $\tilde{L} \geq 1$ depending on $r - \rho = \kappa(\delta)$ such that $\mu(S_\omega(x, r)) \leq \tilde{L} \mu(S_\omega(x, \rho))$. The conclusion is
\[
\sum_{x \in A^i_\omega(o, r)} \mu(S_\omega(x, r)) \leq \tilde{L}M(4r + \kappa(\delta))\varepsilon \mu(S_\omega(o, r)).
\]
By choosing $\varepsilon > 0$ so that $\tilde{L}M(4r + \kappa(\delta))\varepsilon \leq \alpha_0$, we obtain the assertion. \qed

Hereafter, we choose a sufficiently large $r > 0$ that is applicable to the above lemma and fix it.

**Proposition 3.11.** For any $\alpha_0 > 0$, there exists an integer $I_0 = I_0(\alpha_0) \in \mathbb{N}$ such that
\[
\sum_{x \in A^i_\omega(o, r)} \mu(S_\omega(x, r)) \leq \alpha_0 \mu(S_\omega(o, r))
\]
for every $\omega \in \partial X$ and every $i \geq I_0$.

**Proof.** For each $\omega \in \partial X$, we take the neighborhood $V(\omega, r) \subset \partial X$. As $\partial X$ is compact, we can find finitely many such neighborhoods $\{V(\omega_i, r)\}_{i=1}^k$ that cover $\partial X$. For each $\omega_i$, we take the integer $I_i = I(\omega_i, \alpha_0)$ as in Lemma 3.10 and set $I_0 = \max\{I_i \mid 1 \leq i \leq k\}$. Then, this satisfies the required property. \qed

We prove that the uniform estimate in Proposition 3.11 is also valid even if we replace the base point $o$ with an arbitrary orbit point $z \in \Gamma(o)$.

**Lemma 3.12.** For any $\alpha > 0$, there exists an integer $I_* = I_*(\alpha) \in \mathbb{N}$ such that
\[
\sum_{x \in A^i_\omega(z, r)} \mu(S_\omega(x, r)) \leq \alpha \mu(S_\omega(z, r))
\]
for any $z \in \Gamma(o)$ and $\omega \in \partial X$ with $o \notin \tilde{S}_\omega(z, r)$ and for every $i \geq I_*$. 
Proof. We take any \( z \in \Gamma(o) \) and represent it by \( z = \gamma^{-1}(o) \) for \( \gamma \in \Gamma \). To show the required estimate, we consider the pull-back \( \gamma^* \mu \) of the measure \( \mu \). We note that \( \gamma(S_\omega(z, r)) = S_{\gamma(o)}(o, r) \) and \( \gamma(A^i_\omega(z, r)) = A^i_{\gamma(o)}(o, r) \). Hence, from Proposition 3.11, it follows that

\[
\sum_{x \in A^i_\omega(z, r)} (\gamma^* \mu)(S_\omega(x, r)) = \sum_{\gamma(x) \in A^i_{\gamma(o)}(o, r)} \mu(S_{\gamma(o)}(\gamma(x), r)) \leq \alpha_0 \mu(S_{\gamma(o)}(o, r)) = \alpha_0 (\gamma^* \mu)(S_\omega(z, r))
\]

for every \( i \geq I_0(\alpha_0) \). Thus, it suffices to show that the derivative \( (d(\gamma^* \mu)/d\mu)(\xi) \) is in a uniform range on the shadow \( S_\omega(z, r) \), which contains \( S_\omega(x, r) \) for all \( x \in A^i_\omega(z, r) \).

By the \( \Gamma \)-quasi-invariance of \( \mu \), we have

\[
\frac{d(\gamma^* \mu)}{d\mu}(\xi) \approx_D j_\gamma(\xi)^s = k(\gamma^{-1}(o), \xi)^s \quad (\text{a.e. } \xi \in \partial X)
\]

for some constant \( D \geq 1 \), where \( k(z, \xi) \) is the Poisson kernel. On the contrary, by Lemma 2.9, if \( o \notin \hat{S}_\omega(z, r) \), then

\[
C^{-1}a^{d(o, z) - 2r} \leq k(z, \xi) \leq Ca^{d(o, z)} \quad (\xi \in S_\omega(z, r))
\]

for some constant \( C \geq 1 \) independent of \( z = \gamma^{-1}(o) \). Therefore,

\[
\sum_{x \in A^i_\omega(z, r)} \mu(S_\omega(x, r)) \leq \alpha_0 D^2 C^{2s} a^{2sr} \mu(S_\omega(z, r))
\]

for every \( i \geq I_0(\alpha_0) \). For \( \alpha = \alpha_0 D^2 C^{2s} a^{2sr} \), we just set \( I_*(\alpha) = I_0(\alpha_0) \) to complete the proof. \( \square \)

Proof of Theorem 3.1 continued. Our goal is to prove that

\[
\sum_{x \in A_\omega(o, r)} \mu(S_\omega(x, r)) < \infty
\]

for any \( \omega \in \partial X \). For each \( i \in \mathbb{N} \), we set

\[
Q_i = \sum_{x \in A^i_\omega(o, r)} \mu(S_\omega(x, r)).
\]

For \( \alpha = 1/2 \), we choose the constant \( I_*(1/2) \in \mathbb{N} \) as in Lemma 3.12 and define it as \( I \). We can verify that

\[
\sum_{j=0}^{\infty} Q_{i+jI} \leq 2Q_i
\]
for each $i = 1, 2, \ldots, I$. To see this, we note that the condition $o \notin \widehat{S}_\omega(z, r)$ is satisfied for each $z \in A'_\omega(o, r)$. Then, Lemma 3.12 implies that

$$Q_{i+I} = \sum_{x \in A'_{i+I}(o, r)} \mu(S_\omega(x, r)) \leq \sum_{z \in A'_{i}(o, r)} \sum_{x \in A'_i(z, r)} \mu(S_\omega(x, r)) \leq \sum_{z \in A'_{i}(o, r)} \mu(S_\omega(z, r))/2 = Q_i/2.$$  

Inductively applying this inequality yields the estimate. Hence,

$$\sum_{x \in A_\omega(o, r)} \mu(S_\omega(x, r)) = \sum_{i=1}^{I} \sum_{j=0}^{\infty} Q_{i+jI} \leq 2 \sum_{i=1}^{I} Q_i.$$

Finally, we show that each $Q_i$ ($i = 1, 2, \ldots, I$) is finite. Indeed, Corollary 3.6 asserts that for $\rho = r - \kappa(\delta)$, the family $\{S_\omega(x, \rho)\}$ taken over all $x \in A'_\omega(o, r)$ is $M(4r + \kappa(\delta))$-disjoint. This, in particular, implies that

$$\sum_{x \in A'_\omega(o, r)} \mu(S_\omega(x, \rho)) \leq M(4r + \kappa(\delta))\mu(S_\omega(o, r)).$$

As before, the shadow lemma (Theorem 2.11) yields that $\mu(S_\omega(x, r)) \leq \widetilde{L}\mu(S_\omega(x, \rho))$ for some constant $\widetilde{L} \geq 1$. Therefore, we have

$$Q_i \leq \widetilde{L}M(4r + \kappa(\delta))\mu(S_\omega(o, r)) < \infty.$$  

This completes the proof of Theorem 3.1. □

Theorem 3.1 implies that if a non-elementary discrete group $\Gamma \subset \text{Isom}(X, d)$ is of divergence type, then $\mu(\Lambda_c(\Gamma)) > 0$ for any $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure $\mu$ on $\partial X$. In this situation, $\mu$ has full measure on $\Lambda_c(\Gamma)$.

Corollary 3.13. Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group of divergence type and let $\mu$ be an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure on $\partial X$. Then, $\mu(\Lambda_c(\Gamma)) = \mu(\partial X)$.

Proof. It is assumed toward a contradiction that $\mu(\partial X - \Lambda_c(\Gamma)) > 0$. Then, the measure $\mu' = \mu|_{\partial X - \Lambda_c(\Gamma)}$ obtained by restricting $\mu$ to $\partial X - \Lambda_c(\Gamma)$ is also an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure. Theorem 3.1 implies that $\mu'(\Lambda_c(\Gamma)) > 0$, but this is a contradiction. □

4. Ergodicity on the conical limit set

In this section, we prove that the action of a discrete group $\Gamma \subset \text{Isom}(X, d)$ on the conical limit set $\Lambda_c(\Gamma)$ is ergodic with respect to any $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure $\mu$. We note that this problem is non-trivial only when $\mu(\Lambda_c(\Gamma)) > 0$. Hence, we can assume that $\Gamma$ is of divergence type and $\mu$ is a Patterson measure for $\Gamma$ $(s = e_a(\Gamma))$ by Corollary 2.13.
Theorem 4.1. Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group and let $\mu$ be an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure with full measure on $\mu(\Lambda_c(\Gamma))$. If a measurable subset $E \subset \Lambda_c(\Gamma)$ in the conical limit set is $\Gamma$-invariant (for almost every $\mu$) and $\mu(E) > 0$, then $\mu(E) = \mu(\Lambda_c(\Gamma))$.

For Kleinian groups, one way to prove the corresponding result is to utilize the density point theorem (see Nicholls [10, Theorem 4.4.4] for example). In the other way, Roblin [13, pp.22–23] proved the result more generally for discrete isometry groups on CAT($-1$) spaces. His arguments are almost acceptable even in the case of discrete isometry groups of Gromov hyperbolic spaces; only few modifications such as those in Section 3 are required. Nevertheless, our proof here is again based on the density point theorem; our purpose is to show that the family of shadows can be adapted to elements of the density point theorem for Borel measures on metric spaces in general. Concerning this theorem, certain necessary concepts are introduced in the following from Federer [6].

Definition. Let $(\Lambda, d)$ be a metric space and let $\mu$ be a Borel measure on $\Lambda$ for which every bounded measurable subset has finite measure. A covering relation $\mathcal{C}$ is a subset of the set of all such pairs $\{(\xi, S)\}$ that $S$ is a measurable subset of $\Lambda$ and $\xi$ is a point in $S$. We say that $\mathcal{C}$ is fine at $\xi \in \Lambda$ if
$$\inf \{\text{diam}(S) \mid (\xi, S) \in \mathcal{C}\} = 0.$$ For any measurable subset $E \subset \Lambda$, we define a family of subsets of $\Lambda$ by
$$\mathcal{C}(E) = \{S \subset \Lambda \mid (\xi, S) \in \mathcal{C} \ (\exists \xi \in E)\}.$$

Definition. A covering relation $\mathcal{V}$ is called a Vitali relation for a Borel measure $\mu$ on $\Lambda$ if $\mathcal{V}$ is fine at every $x \in \Lambda$ and if the following condition holds: if $\mathcal{C} \subset \mathcal{V}$ is fine at every point $\xi$ of a measurable subset $E \subset \Lambda$, then $\mathcal{C}(E)$ has a countable disjoint subfamily $\{S_n\}_{n=1}^\infty$ such that $\mu(E - \bigsqcup_{n=1}^\infty S_n) = 0$.

A general density point theorem can be stated as follows ([6, Theorem 2.9.11]).

Theorem 4.2. Let $\mathcal{V}$ be a Vitali relation for a measure $\mu$ on $\Lambda$ and let $E \subset \Lambda$ be a measurable subset. Then, for almost every point $\xi \in E$ with respect to $\mu$, one has
$$\lim_{n \to \infty} \frac{\mu(E \cap S_n)}{\mu(S_n)} = 1$$
for every sequence $\{S_n\}_{n=1}^\infty$ such that $(\xi, S_n) \in \mathcal{V}$ for all $n$ and $\text{diam} S_n \to 0$ as $n \to \infty$.

As a sufficient condition for a Vitali relation, we have the following ([6, Theorem 2.8.17]).

Lemma 4.3. Let $\mathcal{V} = \{(\xi, S)\}$ be a covering relation for a measure $\mu$ on $\Lambda$ such that every $S \in \mathcal{V}(\Lambda)$ is a bounded closed subset and $\mathcal{V}$ is fine at every $\xi \in \Lambda$. For a non-negative function $f$ on $\mathcal{V}(\Lambda)$ and a constant $\tau \in (1, \infty)$, let
$$\tilde{S} = \bigcup \{S' \in \mathcal{V}(\Lambda) \mid S' \cap S \neq \emptyset, \ f(S') \leq \tau f(S)\} \subset \Lambda$$
for each $S \in \mathcal{V}(\Lambda)$. Suppose that for almost every $\xi \in \Lambda$ with respect to $\mu$

$$
\limsup_{S \to \xi} \left\{ f(S) + \frac{\mu(S)}{\mu(S)} \right\}
$$

is finite, where the limit superior is taken over all sequences $\{S\}$ with $(\xi, S) \in \mathcal{V}$ and $\text{diam} S \to 0$. Then, $\mathcal{V}$ is a Vitali relation for $\mu$.

We apply these results to our case. For a discrete group $\Gamma \subset \text{Isom}(X, d)$ of divergence type, we adopt the conical limit subset $\Lambda^{(\rho)}_c(\Gamma)$ for a sufficiently large $\rho > 0$ with the restriction of the visual metric $d_a$ as the metric space $(\Lambda, d)$. Moreover, we define $\mu$ to be the restriction of an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure to $\Lambda^{(\rho)}_c(\Gamma)$, and $\mathcal{V}$ to be

$$
\mathcal{V}^{(\rho, r)} = \{ (\xi, S^{(\rho)}_o(x, r)) \mid x \in \Gamma(o), \xi \in S^{(\rho)}_o(x, r) \}
$$

for a fixed $r \geq \rho + \kappa(\delta)$, where $S^{(\rho)}_o(x, r) = S_o(x, r) \cap \Lambda^{(\rho)}_c(\Gamma)$. We note that $S^{(\rho)}_o(x, r) \neq \emptyset$ for each $x \in \Gamma(o)$. We also consider

$$
f(S^{(\rho)}_o(x, r)) = a^{-d(o,x)}
$$

as the non-negative function $f$.

We show that the covering relation $\mathcal{V}^{(\rho, r)}$ is a Vitali relation for $\mu$ when $r \geq \rho + \kappa(\delta)$ is sufficiently large. First, we see that $\mathcal{V}^{(\rho, r)}$ is fine at every $\xi \in \Lambda^{(\rho)}_c(\Gamma)$ for each $r \geq \rho + \kappa(\delta)$ from the following proposition.

**Proposition 4.4.** The diameter of the shadow satisfies

$$
\text{diam}_a S_o(x, r) \leq \lambda a^{2r} \cdot a^{-d(o,x)}
$$

for any $x \in X$ and $r > 0$, where $\lambda = \lambda(\delta, a) \geq 1$ is the constant given in Subsection 2.4.

**Proof.** Taking any two points $\xi, \eta$ in $S_o(x, r)$ and some geodesic line $(\xi, \eta)$, we choose sequences $\{\xi_n\}$ and $\{\eta_n\}$ on $(\xi, \eta)$ such that $\xi_n \to \xi$ and $\eta_n \to \eta$ as $n \to \infty$. Then, the geodesic segments $[\xi_n, \eta_n] \subset (\xi, \eta)$ clearly satisfy $d(o, (\xi, \eta)) = d(o, [\xi_n, \eta_n])$ for all sufficiently large $n$. We consider geodesic segments $[o, \xi_n]$ and $[o, \eta_n]$. Passing to subsequences if necessary, we may assume that $[o, \xi_n]$ converge to a geodesic ray $[o, \xi]$ and $[o, \eta_n]$ converge to a geodesic ray $[o, \eta]$ as $n \to \infty$. For an arbitrary $\varepsilon > 0$, we can find $\xi'_n \in [o, \xi)$ and $\eta'_n \in [o, \eta)$ such that $d(\xi_n, \xi'_n) \leq \varepsilon$ and $d(\eta_n, \eta'_n) \leq \varepsilon$ for some sufficiently large $n$. Hereafter, we fix this $n$.

The distance $d(o, (\xi, \eta)) = d(o, [\xi_n, \eta_n])$ is bounded from below by the Gromov product

$$
(\xi_o \mid \eta_o) := \frac{1}{2}(d(o, \xi) + d(o, \eta) - d(\xi_n, \eta_n)) \geq (\xi'_n \mid \eta'_n)_o - 2\varepsilon.
$$

As $[o, \xi)$ and $[o, \eta)$ intersect $B(x, r)$, the triangle inequality yields that

$$
d(o, \xi'_n) \geq d(o, x) + d(x, \xi'_n) - 2r; \quad d(o, \eta'_n) \geq d(o, x) + d(x, \eta'_n) - 2r.
$$
Hence,
\[ d(o, (\xi, \eta)) \geq (\xi_n | \eta_n)_o \geq d(o, x) + (\xi'_n | \eta'_n)_x - 2r - 2\varepsilon \geq d(o, x) - 2r - 2\varepsilon. \]
As \( \varepsilon \) is arbitrary, we conclude that \( d(o, (\xi, \eta)) \geq d(o, x) - 2r \). Then, the distance on \( \partial X \) is estimated as
\[ d_a(\xi, \eta) \leq \lambda a^{-d(o, (\xi, \eta))} \leq \lambda a^{2r} \cdot a^{-d(o, x)}. \]
Thus, \( \text{diam}_a S_o(x, r) \) is bounded by this value. \( \square \)

Now we are ready to accomplish our purpose.

**Lemma 4.5.** Let \( \Gamma \subset \text{Isom}(X, d) \) be a non-elementary discrete group and let \( \mu \) be an s-dimensional \( \Gamma \)-quasi-invariant quasiconformal measure. Then, \( \mathcal{V}^{(\rho, r)} \) is a Vitali relation for \( \mu \) if \( r \geq \max\{\rho + \kappa(\delta), r_0\} \), where \( r_0 \) is the constant that arises from the shadow lemma.

**Proof.** Let \( \tau = a^r > 1 \). Concerning the function \( f(S_o^{(\rho)}(x, r)) = a^{-d(o, x)} \) for each \( x \in \Gamma(o) \), we see that the condition \( f(S_o^{(\rho)}(x', r)) \leq \tau f(S_o^{(\rho)}(x, r)) \) is equivalent to that \( d(o, x') \geq d(o, x) - r \). Consider \( x' \in \Gamma(o) \) that holds this condition. Let \( \tilde{r} = r + \kappa(\delta) \). Lemma 3.2 implies that if \( d(x, x') > 5\tilde{r} \) and \( B(x', r) \cap \tilde{S}_o(x, r) \neq \emptyset \), then \( B(x', \tilde{r}) \subset \tilde{S}_o(x, \tilde{r}) \). Here, we see that the latter assumption can be replaced by the condition \( S_o(x', r) \cap S_o(x, r) \neq \emptyset \).

Indeed, \( B(x, r) \) and \( B(x', r) \) are disjoint in this case and the condition \( S_o(x', r) \cap S_o(x, r) \neq \emptyset \) is equivalent to that either \( B(x', r) \cap \tilde{S}_o(x, r) \neq \emptyset \) or \( B(x, r) \cap \tilde{S}_o(x', r) \neq \emptyset \). However, the assumption \( d(x', o) \geq d(x, o) - r \) rules out the latter case, and thus we have \( B(x', r) \cap \tilde{S}_o(x, r) \neq \emptyset \).

Now we show that \( S_o(x', r) \subset S_o(x, 6\tilde{r}) \) under the condition \( S_o(x', r) \cap S_o(x, r) \neq \emptyset \). If \( d(x, x') > 5\tilde{r} \), then the above argument concludes that \( B(x', r) \subset S_o(x, \tilde{r}) \). This implies, in particular, that \( S_o(x', r) \subset S_o(x, \tilde{r}) \subset S_o(x, 6\tilde{r}) \). Furthermore, if \( d(x, x') \leq 5\tilde{r} \), then \( B(x', r) \subset B(x, 6\tilde{r}) \), which also implies that \( S_o(x', r) \subset S_o(x, 6\tilde{r}) \).

To prove that \( \mathcal{V}^{(\rho, r)} \) is a Vitali relation for \( \mu \), we rely on Lemma 4.3. As \( \mathcal{V}^{(\rho, r)} \) is fine at every \( \xi \in \Lambda_c^{(\rho)}(\Gamma) \) by Proposition 4.4, it suffices to show that \( f(S) + \mu(\tilde{S})/\mu(S) \) is uniformly bounded for every \( S = S_o^{(\rho)}(x, r) \). Clearly \( f(S) \leq 1 \). On the contrary, \( \tilde{S} \) is contained in \( S_o^{(\rho)}(x, 6\tilde{r}) \) as we have seen above. Then, the shadow lemma for the s-dimensional \( \Gamma \)-quasi-invariant quasiconformal measure \( \mu \) restricted to \( \Lambda_c^{(\rho)}(\Gamma) \) gives
\[
\mu(\tilde{S}) \leq \mu(S_o^{(\rho)}(x, 6\tilde{r})) \leq La^{12s\tilde{r}} \cdot a^{-sd(o, x)};
\]
\[
\mu(S) = \mu(S_o^{(\rho)}(x, r)) \geq L^{-1} a^{-sd(o, x)},
\]
where \( L \geq 1 \) is a constant independent of \( x \in \Gamma(o) \). This implies that \( \mu(\tilde{S})/\mu(S) \leq L^2 a^{12s\tilde{r}} < \infty. \) \( \square \)

**Proof of Theorem 4.1.** We prove that \( \mu(E \cap \Lambda_c^{(\rho)}(\Gamma)) = \mu(\Lambda_c^{(\rho)}(\Gamma)) \) for all sufficiently large \( \rho > 0 \). Then, because \( \Lambda_c(\Gamma) = \bigcup_{\rho > 0} \Lambda_c^{(\rho)}(\Gamma) \), we have that \( \mu(E) = \mu(E \cap \Lambda_c(\Gamma)) = \mu(E \cap \bigcup_{\rho > 0} \Lambda_c^{(\rho)}(\Gamma)). \)
Theorem 4.2, in particular, asserts that there is a density point \( \xi \) of \( \Lambda_c(\Gamma) - E \) such that

\[
\lim_{n \to \infty} \frac{\mu(S_0^{(\rho)}(\gamma^{-1}_n(o), r) - E)}{\mu(S_0^{(\rho)}(\gamma^{-1}_n(o), r))} = 1, \quad \text{or} \quad \lim_{n \to \infty} \frac{\mu(E \cap S_0^{(\rho)}(\gamma^{-1}_n(o), r))}{\mu(S_0^{(\rho)}(\gamma^{-1}_n(o), r))} = 0,
\]

where \( \{\gamma_n\}_{n=1}^\infty \subset \Gamma \) is a sequence such that \( \gamma^{-1}_n(o) \) converge to \( \xi \) within distance \( \rho \) from some geodesic ray toward \( \xi \). We note that the above limit at \( \xi \) exists for a fixed \( r \). However, as there are such density points \( \xi \) in full measure for each \( r \), we can choose a common density point \( \xi \) where the limit exists for countably many integers \( r \geq \max\{\rho + \kappa(\delta), r_0\} \). By passing to a subsequence, we may assume that \( \gamma_n(o) \) converge to some \( \eta \in \partial X \). By Proposition 2.14, we see that \( \mu(\{\eta\}) = 0 \).

We take an arbitrary \( \overline{\varepsilon} > 0 \) such that \( \mu(\Lambda_c^{(\rho)}(\Gamma)) \geq 2\overline{\varepsilon} \). By the regularity of the finite Borel measure \( \mu \), there is an open ball \( D(\eta, \varepsilon) \subset \partial X \) centered at \( \eta \) with radius \( \varepsilon > 0 \) such that \( \mu(D(\eta, \varepsilon)) \leq \overline{\varepsilon} \). Then, by Proposition 2.10, there is \( r(\varepsilon) > 0 \) such that

\[
\mathrm{diam}_\mu(\partial X - S_{\gamma(o)}(o, r)) \leq \varepsilon
\]

for every \( \gamma \in \Gamma \) and every \( r \geq r(\varepsilon) \). Fixing such an \( r \geq r(\varepsilon) \), we see that \( \gamma_n(S_0(\gamma^{-1}_n(o), r)) = S_{\gamma_n(o)}(o, r) \) does not contain \( \eta \) for all sufficiently large \( n \); hence,

\[
\mu(\Lambda_c^{(\rho)}(\Gamma) - \gamma_n(S_0^{(\rho)}(\gamma^{-1}_n(o), r))) \leq \overline{\varepsilon}.
\]

By the choice of \( \overline{\varepsilon} \), this implies that \( \mu(\gamma_n(S_0^{(\rho)}(\gamma^{-1}_n(o), r))) \geq \overline{\varepsilon} \).

Now we fix some \( r \geq \max\{\rho + \kappa(\delta), r_0, r(\varepsilon)\} \) and apply the above result. By Lemma 2.9, there is a constant \( C \geq 1 \) independent of \( \gamma \in \Gamma \) such that

\[
C^{-1}a^{d(o, \gamma^{-1}(o)) - 2r} \leq j_\gamma(\xi) = k(\gamma^{-1}(o), \xi) \leq Ca^{d(o, \gamma^{-1}(o))}
\]

for every \( \xi \in S_0(\gamma^{-1}(o), r) \). Then

\[
\mu(E \cap \gamma_n(S_0^{(\rho)}(\gamma^{-1}_n(o), r))) = (\gamma_n^*\mu)(E \cap S_0^{(\rho)}(\gamma^{-1}_n(o), r)) \leq D(Ca^{d(o, \gamma^{-1}(o))} \ast \mu(E \cap S_0^{(\rho)}(\gamma^{-1}_n(o), r)));
\]

\[
\mu(\gamma_n(S_0^{(\rho)}(\gamma^{-1}_n(o), r))) = (\gamma_n^*\mu)(S_0^{(\rho)}(\gamma^{-1}_n(o), r)) \geq D^{-1}(C^{-1}a^{d(o, \gamma^{-1}(o)) - 2r}) \ast \mu(S_0^{(\rho)}(\gamma^{-1}_n(o), r)),
\]

where \( D \geq 1 \) is the constant for \( \Gamma \)-quasi-invariance of \( \mu \). From these estimates, we have

\[
\frac{\mu(E \cap \gamma_n(S_0^{(\rho)}(\gamma^{-1}_n(o), r)))}{\mu(\gamma_n(S_0^{(\rho)}(\gamma^{-1}_n(o), r)))} \leq D^2(Ca^r)^{2\rho} \frac{\mu(E \cap S_0^{(\rho)}(\gamma^{-1}_n(o), r))}{\mu(S_0^{(\rho)}(\gamma^{-1}_n(o), r))},
\]

which tend to 0 as \( n \to \infty \).
As $\mu(\gamma_n(S_o(\gamma_n^{-1}(o)), r))) \geq \tilde{\varepsilon}$ for all sufficiently large $n$, we see that

$$\mu(E \cap \gamma_n(S_o(\gamma_n^{-1}(o)), r))) \to 0 \quad (n \to \infty).$$

Combined with $\mu(\Lambda_c(\rho) - \gamma_n(S_o(\gamma_n^{-1}(o)), r))) \leq \tilde{\varepsilon}$, this implies that $\mu(E) \leq \tilde{\varepsilon}$. As we have this conclusion for any sufficiently small $\tilde{\varepsilon} > 0$, we obtain $\mu(E) = 0$. However, this contradicts the assumption $\mu(E) > 0$, and the proof is complete. □

**Corollary 4.6.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group of divergence type and let $\mu$ be a Patterson measure for $\Gamma$. Then, $\Gamma$ acts on $\partial X$ ergodically with respect to $\mu$.

**Proof.** This follows from Corollary 3.13 and Theorem 4.1. □

5. **Quasi-uniqueness of Patterson measures**

In this section, we prove that under the assumption of ergodicity of a discrete group $\Gamma \subset \text{Isom}(X, d)$ with respect to an $s$-dimensional $\Gamma$-quasi-invariant quasiconformal measure $\mu$, any such measure that is absolutely continuous with respect to $\mu$ is unique in a certain sense. We apply this “quasi-uniqueness” to Patterson measures for $\Gamma$ of divergence type.

For later purposes, the ambiguity of the uniqueness will be described in terms of the quasi-invariance constants and the total mass of the measures. For a measure $\mu$ in general, we denote its total mass by $\|\mu\|$.  

**Lemma 5.1.** If a discrete group $\Gamma \subset \text{Isom}(X, d)$ acts ergodically on $\partial X$ with respect to a $(\Gamma, D)$-quasiconformal measure $\mu$, then any $(\Gamma, D')$-quasiconformal measure $\nu$ that is absolutely continuous with respect to $\mu$ satisfies

$$(DD')^{-1}\frac{\|\nu\|}{\|\mu\|} \leq \frac{d\nu}{d\mu}(\xi) \leq DD'\frac{\|\nu\|}{\|\mu\|} \quad (\text{a.e. } \xi \in \partial X).$$

In particular, $\mu$ is absolutely continuous with respect to $\nu$.

**Proof.** For simplicity, we may assume that $\|\mu\| = \|\nu\| = 1$. Let

$$E = \bigcap_{\gamma \in \Gamma} \{\xi \in \partial X \mid (DD')^{-1} \leq \frac{d\nu}{d\mu}(\gamma(\xi)) \leq DD'\},$$

which is a $\Gamma$-invariant measurable subset of $\partial X$. By ergodicity, we have $\mu(E) = 0$ or $\mu(E) = 1$. We prove that $\mu(E) = 1$, which shows, in particular, that

$$(DD')^{-1} \leq \frac{d\nu}{d\mu}(\xi) \leq DD' \quad (\text{a.e. } \xi \in \partial X)$$

by taking $\gamma = \text{id.}$
It is assumed toward a contradiction that \( \mu(E) = 0 \), that is, \( \mu(E^c) = 1 \) for the complement \( E^c \) of \( E \). We divide \( E^c \) into two disjoint \( \Gamma \)-invariant measurable subsets:

\[
E^c_+ = \bigcup_{\gamma \in \Gamma} \{ \xi \in \partial X \mid \frac{d\nu}{d\mu}(\gamma(\xi)) > DD' \};
\]

\[
E^c_- = \bigcup_{\gamma \in \Gamma} \{ \xi \in \partial X \mid \frac{d\nu}{d\mu}(\gamma(\xi)) < (DD')^{-1} \}.
\]

Again by ergodicity, we have \( \mu(E^c_+) = 1 \) or otherwise \( \mu(E^c_-) = 1 \). For each \( n \in \mathbb{N} \), we define

\[
(E^c_+)_n = \bigcup_{\gamma \in \Gamma} \{ \xi \in \partial X \mid \frac{d\nu}{d\mu}(\gamma(\xi)) > DD' + \frac{1}{n} \};
\]

\[
(E^c_-)_n = \bigcup_{\gamma \in \Gamma} \{ \xi \in \partial X \mid \frac{d\nu}{d\mu}(\gamma(\xi)) < (DD')^{-1} - \frac{1}{n} \},
\]

which are also \( \Gamma \)-invariant. Then, each \( \{(E^c_\pm)_n\}_{n=1}^\infty \) is an increasing sequence converging to \( E^c_\pm = \bigcup_{n=1}^\infty (E^c_\pm)_n \). As \( \mu((E^c_\pm)_n) \) is either 0 or 1 for every \( n \), there is some \( n_0 \in \mathbb{N} \) such that either \( \mu((E^c_+)_n_0) = 1 \) or \( \mu((E^c_-)_n_0) = 1 \). Finally, we consider \( \Gamma \)-invariant measurable subsets

\[
F_+ = \bigcap_{\gamma \in \Gamma} \{ \xi \in \partial X \mid \frac{d\nu}{d\mu}(\gamma(\xi)) > 1 + \frac{1}{n_0 DD'} \};
\]

\[
F_- = \bigcap_{\gamma \in \Gamma} \{ \xi \in \partial X \mid \frac{d\nu}{d\mu}(\gamma(\xi)) < 1 - \frac{DD'}{n_0} \}.
\]

We see that \( F_\pm \) contains \( (E^c_\pm)_{n_0} \); hence, \( \mu(F_+) = 1 \) or otherwise \( \mu(F_-) = 0 \) is satisfied. Indeed, for almost every \( \xi \in (E^c_-)_{n_0} \) there is some \( \gamma_0 \in \Gamma \) such that \( (d\nu/d\mu)(\gamma_0(\xi)) < (DD')^{-1} - 1/n_0 \). Then, for every \( \gamma \in \Gamma \)

\[
\frac{d\nu}{d\mu}(\gamma(\xi)) = \frac{d(\gamma\gamma_0^{-1})^*\nu}{d(\gamma\gamma_0^{-1})^*\mu}(\gamma(\xi)) \leq \frac{D'}{D-1} \cdot \frac{d\nu}{d\mu}(\gamma_0(\xi)) < 1 - \frac{DD'}{n_0},
\]

which shows that \( \xi \in F_- \). The other case for \( F_+ \) is treated similarly. Then, \( \nu(F_-) = 1 \) because \( \nu(F_-) = 1 - \nu(F^c_-) \) and \( \nu \) is absolutely continuous with respect to \( \mu \). However, we have

\[
\nu(F_-) = \int_{F_-} \frac{d\nu}{d\mu}(\xi)d\mu(\xi) \leq \left( 1 - \frac{DD'}{n_0} \right) \mu(F_-) < 1,
\]

which is a contradiction. We also obtain \( \nu(F_+) > 1 \) in the other case where \( \mu(F_+) = 1 \), which leads to a contradiction.

The quasi-uniqueness is mainly applied to Patterson measures for \( \Gamma \) of divergence type.
Theorem 5.2. Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group of divergence type. Then, any two Patterson measures for $\Gamma$ are mutually absolutely continuous. If $\mu$ and $\mu'$ are Patterson measures for $\Gamma$ with quasi-invariance constants $D$ and $D'$ respectively, then

$$(DD')^{-1} \frac{d\mu'}{d\mu}(\xi) \leq DD' \frac{d\mu'}{d\mu}(\xi) \quad (\text{a.e. } \xi \in \partial X).$$

Proof. If $\mu$ and $\mu'$ are Patterson measures for $\Gamma$, then $\mu + \mu'$ is also a Patterson measure for $\Gamma$. By Corollary 4.6, $\Gamma$ acts ergodically on $\partial X$ with respect to $\mu + \mu'$. As $\mu$ and $\mu'$ are absolutely continuous with respect to $\mu + \mu'$, Lemma 5.1 implies that $\mu$ and $\mu'$ are mutually absolutely continuous via $\mu + \mu'$. Then, the required inequality also follows from Lemma 5.1. □

6. QUASI-INVARiance UNDER THE NORMALIZER

In the previous section, we have seen the "quasi-uniqueness" of Patterson measures for a divergence-type group $\Gamma \subset \text{Isom}(X, d)$. Using this property, we show that the Patterson measure is also quasi-invariant under the normalizer of $\Gamma$. The invariance under the normalizer is a property of the Poincaré series and we use the inheritance of this property to the quasi-unique Patterson measure. To this end, we have to fix the canonical construction of a Patterson measure family from the weighted Dirac masses defined by the Poincaré series, which is the so-called Patterson construction. See Nicholls [10] in the case of Kleinian groups.

We always assume that a discrete group $\Gamma \subset \text{Isom}(X, d)$ is non-elementary and of divergence type. For any reference point $z \in X$, orbit point $x \in X$, and exponent $s > e_a(\Gamma)$, we define a measure on $X$ by

$$m^s_{z, x} = \frac{1}{P^s_{\Gamma}(o, x)} \sum_{\gamma \in \Gamma} a^{-sd(z, \gamma(x))} D_{\gamma(x)},$$

where $D_x$ is the Dirac measure at $x \in X$. In fact, $m^s_{z, x} = m^s_{z, x'}$ if $x' \in \Gamma(x)$. We note that the total mass $\|m^s_{z, x}\|$ satisfies

$$a^{-sd(o, z)} \leq \|m^s_{z, x}\| = \frac{P^s_{\Gamma}(z, x)}{P^s_{\Gamma}(o, x)} \leq a^{sd(o, z)}; \quad \|m^s_{o, x}\| = 1.$$

The measure $m^s_{z, x}$ is precisely $\Gamma$-invariant in the sense that $g^*m^s_{g(z), x} = m^s_{z, x}$ for every $g \in \Gamma$. Indeed,

$$g^*m^s_{g(z), x} = \frac{1}{P^s_{\Gamma}(o, x)} \sum_{\gamma \in \Gamma} a^{-sd(g(z), \gamma(x))} D_{g^{-1}\gamma(x)} = \frac{1}{P^s_{\Gamma}(o, x)} \sum_{\gamma \in \Gamma} a^{-sd(g(z), \gamma(x))} D_{\gamma(x)} = m^s_{z, x}.$$

For any decreasing sequence of $s$ to $e_a(\Gamma)$, there is a subsequence $\{s_i\}_{i \in \mathbb{N}}$ such that $m^s_{z, x}$ converge to some measure on $X$ in the weak-* sense. We denote this limit measure by $m_{z, x}$, even though it also depends on the choice of the sequence $\{s_i\}$. However, $m^s_{z, x}$ is invariant when $x$ is replaced in the orbit $\Gamma(x)$; furthermore, it is $\Gamma$-invariant, as shown
above. Therefore, we can take the same sequence \( \{s_i\} \) for all \( \gamma(x) \) and for all \( \gamma(z) \) (\( \gamma \in \Gamma \)); we assume this choice hereafter. The total mass \( \|m_{z,x}\| \) satisfies the same inequalities for \( \|m_{z,x}\| \) after replacing \( s \) with \( e_\alpha(\Gamma) \) and, in particular, \( \|m_{0,x}\| = 1 \). Owing to the condition that \( \Gamma \) is of divergence type, it can be proved that the support of \( m_{z,x} \) is in the limit set \( \Lambda(\Gamma) \).

By fixing any orbit point \( x \), we have \( \{m_{z,x}\}_{z \in X} \), which we call the canonical measure family. When the orbit point \( x \) is not in question or is assumed to be the base point \( o \), we denote the canonical measure family by \( \{m_z\}_{z \in X} \) for brevity. As \( \{m_{z,x}\}_{z \in X} \) is \( \Gamma \)-invariant, so is the canonical measure family \( \{m_{z,x}\}_{z \in X} \).

We show that this is a Patterson measure family; we can call it the canonical Patterson measure family hereafter. The proof is a modification of that in Coornaert [2, Théorème 5.4].

**Lemma 6.1.** Let \( \Gamma \subset \text{Isom}(X,d) \) be a discrete group of divergence type. For any \( x \in X \), the canonical measure family \( \{m_{z,x}\}_{z \in X} \) is a quasiconformal measure family of dimension \( e_\alpha(\Gamma) \) with quasiconformal constant \( a^{c(\delta)} \geq 1 \). Hence, this is a Patterson measure family for \( \Gamma \) with quasi-invariance constant \( 1 \).

**Proof.** For every \( \xi \in \partial X \) and for every \( z \in X \), we choose a neighborhood \( U_\xi \subset X \) of \( \xi \) as in Proposition 2.3. Let \( f \) be any continuous function on \( U_\xi \) with compact support. Then

\[
m^s_{z,x}(f) = \int_{U_\xi} f(\zeta) dm^s_{z,x}(\zeta) = \frac{1}{P^s(\gamma(x))} \sum_{\gamma(x) \in U_\xi} a^{-sd(z,\gamma(x))} f(\gamma(x));
\]

\[
m^s_{0,x}(f) = \int_{U_\xi} f(\zeta) dm^s_{0,x}(\zeta) = \frac{1}{P^s(\gamma(x))} \sum_{\gamma(x) \in U_\xi} a^{-sd(o,\gamma(x))} f(\gamma(x)).
\]

It follows from Proposition 2.3 with the constant \( c(\delta) \) that

\[a^{-sc(\delta)} k(z,\xi)^s \leq m^s_{z,x}(f) / m^s_{0,x}(f) \leq a^{sc(\delta)} k(z,\xi)^s.\]

Taking the limit of some subsequence \( \{s_i\} \) as \( s \to e_\alpha(\Gamma) = e \), which may be different for \( m^s_{z,x} \) and for \( m^s_{0,x} \), we have

\[a^{-ec(\delta)} k(z,\xi)^e \leq m^e_{z,x}(f) / m^e_{0,x}(f) \leq a^{ec(\delta)} k(z,\xi)^e.\]

As \( f \) is arbitrary, this implies that \( (dm_{z,x}/dm_{0,x})(\xi) \approx a^{ec(\delta)} k(z,\xi)^e \). Thus, the quasiconformality with constant \( a^{e(\delta)} \) is proved.

**Remark.** Regarding the canonical Patterson measure family \( \{m_{z,x}\}_{z \in X} \), if we consider \( \mu = m_{0,x} \), then by Proposition 2.3, \( \mu \) is a \( (\Gamma, a^{e_\alpha(\Gamma)c(\delta)}) \)-quasi-invariant quasiconformal measure with total mass \( \|\mu\| = 1 \). We also call this the canonical Patterson measure. It is that given in [2, Théorème 5.4], where the convergence-type group case was also treated.
The canonical Patterson measure family is quasi-unique in the sense that it is independent of the choice of the orbit point $x \in X$ and the weak-* limit.

**Lemma 6.2.** The canonical Patterson measure families $\{m_{z,x}\}_{z \in X}$ and $\{m_{z,x'}\}_{z \in X}$ for $x, x' \in X$ satisfy $(dm_{z,x'}/dm_{z,x})(\xi) \asymp_K 1$ for $K = a^{4\kappa}(\Gamma)c(\delta)$. This includes the case where the weak-* limits $m_{z,x}$ and $m_{z,x'}$ are different even if $x = x'$.

**Proof.** We consider the canonical Patterson measures $\mu = m_{o,x}$ and $\mu' = m_{o,x'}$, which are $(\Gamma, a^{\kappa}(\Gamma)c(\delta))$-quasi-invariant as in the remark above. As $\|\mu\| = \|\mu'\| = 1$, Theorem 5.2 implies that $(d\mu'/d\mu)(\xi) \asymp a^{2\kappa}(\Gamma)c(\delta)$. Moreover, the quasiconformality by Lemma 6.1 yields that

$$\frac{dm_{z,x'}}{dm_{z,x}}(\xi) \asymp a^{2\kappa}(\Gamma)c(\delta) \frac{dm_{o,x'}}{dm_{o,x}}(\xi).$$

This proves the statement. \qed

Now we consider the quasi-invariance of the Patterson measure for $\Gamma$ under its normalizer. For a subgroup $\Gamma \subset \text{Isom}(X, d)$, this is denoted by

$$N(\Gamma) = \{g \in \text{Isom}(X, d) \mid g\Gamma g^{-1} = \Gamma\}.$$

First, we consider the pull-back of a $\Gamma$-quasi-invariant quasiconformal measure family by each element $g \in N(\Gamma)$.

**Proposition 6.3.** For a discrete group $\Gamma \subset \text{Isom}(X, d)$, let $\{\mu_z\}_{z \in X}$ be an $s$-dimensional $(\Gamma, D)$-quasi-invariant quasiconformal measure family with quasiconformal constant $C \geq 1$. Then, $\{g^*\mu_{g(z)}\}_{z \in X}$ is also an $s$-dimensional $(\Gamma, D)$-quasi-invariant quasiconformal measure family with quasiconformal constant $C' \geq 1$ for every $g \in N(\Gamma)$. Here $C' = a^{4\kappa}(\delta)C^2$; in particular, it is independent of $g \in N(\Gamma)$.

**Proof.** Let $\nu_z = g^*\mu_{g(z)}$ for brevity. We first prove that $\{\nu_z\}_{z \in X}$ is a quasiconformal measure family. This is done by

$$\frac{d\nu_z}{d\nu_o}(\xi) = \frac{d\mu_{g(z)}}{d\mu_{g(o)}}(g(\xi)) \asymp_{C^2} \frac{k(g(z), g(\xi))^s}{k(g(o), g(\xi))^s} \asymp_{a^{4\kappa}(\delta)} k(z, \xi)^s \quad (\text{a.e. } \xi \in \partial X),$$

where the last estimate follows from Proposition 2.2. The quasiconformal constant $C'$ can be chosen as $C' = a^{4\kappa}(\delta)C^2$. To see that $\{\nu_z\}_{z \in X}$ is $(\Gamma, D)$-quasi-invariant, we take an arbitrary $\gamma \in \Gamma$ and its conjugate $\tilde{\gamma} \in \Gamma$ satisfying $g\gamma = \tilde{\gamma}g$. Then,

$$\gamma^*\nu_{g(z)} = \gamma^* g^* \mu_{g\gamma(z)} = g^* \tilde{\gamma}^* \mu_{\gamma g(z)} \asymp_D g^* \mu_{g(z)} = \nu_z$$

yields the desired condition. \qed

Lemma 6.1 and Proposition 6.3 show the following consequence. Hereafter, if a Patterson measure family is $(\Gamma, D)$-quasi-invariant, then we call it a $(\Gamma, D)$-Patterson measure family for brevity.
Corollary 6.4. Let \( \{m_z\}_{z \in X} \) be the canonical Patterson measure family for a non-elementary discrete group \( \Gamma \) of divergence type. Then, for every \( g \in N(\Gamma) \), \( \{g^*m_{g(z)}\}_{z \in X} \) is a \((\Gamma, 1)\)-Patterson measure family with quasiconformal constant \( a^{4k(\delta) + 2c(\delta)} \).

We are now ready to explain our main result in this section. For a discrete group \( \Gamma \subset \text{Isom}(X,d) \) of divergence type, we take a Patterson measure family \( \{\mu_z\}_{z \in X} \) for \( \Gamma \). Then, by Proposition 6.3, every \( g \in N(\Gamma) \) yields a Patterson measure family \( \{g^*\mu_{g(z)}\}_{z \in X} \) for \( \Gamma \). Owing to the quasi-uniqueness by Theorem 5.2, this is comparable with the original \( \{\mu_z\}_{z \in X} \). If this comparison is uniform independently of \( g \in N(\Gamma) \), then we can conclude that \( \{\mu_z\}_{z \in X} \) is quasi-invariant under \( N(\Gamma) \). The problem is to show this uniformity; more precisely, to estimate the total mass of \( \{g^*\mu_{g(z)}\}_{z \in X} \). To this end, we have to utilize the canonical Patterson measure family \( \{m_z\}_{z \in X} \) rather than \( \{\mu_z\}_{z \in X} \).

Lemma 6.5. Let \( \{m_{z,x}\}_{z \in X} \) be any canonical Patterson measure family for a non-elementary discrete group \( \Gamma \) of divergence type. Then, for every \( g \in N(\Gamma) \) and for every \( x \in X \), the total mass of \( m_{g(o),x} \) satisfies \( \|m_{g(o),x}\| \preceq K^{3/2} 1 \), where \( K = a^{4e_a(\Gamma)c(\delta)} \) is the constant given in Lemma 6.2.

Proof. Suppose that \( m_{g(o),o} \) is the weak-* limit of \( m_{g(o),o}^{s_i} \) with \( s_i \searrow e = e_a(\Gamma) \) as \( i \to \infty \). Then,

\[
\|m_{g(o),o}\| = \lim_{i \to \infty} \|m_{g(o),o}^{s_i}\| = \lim_{i \to \infty} \frac{P_{\Gamma}^{s_i}(g(o),o)}{P_{\Gamma}^{s_i}(o,o)}.
\]

By Proposition 2.4, this ratio of the Poincaré series can be represented as

\[
\frac{P_{\Gamma}^{s_i}(g(o),o)}{P_{\Gamma}^{s_i}(o,o)} = \left( \frac{P_{\Gamma}^{s_i}(g(o),g(o))}{P_{\Gamma}^{s_i}(o,g(o))} \right)^{-1}.
\]

We choose a subsequence of \( \{s_i\} \) (denoted by the same \( s_i \)) so that \( m_{g(o),g(o)}^{s_i} \) converge to some \( m_{g(o),g(o)} \) in the weak-* sense. Then, the above ratio of the Poincaré series converges to \( \|m_{g(o),g(o)}\|^{-1} \) as \( i \to \infty \). This shows that \( \|m_{g(o),o}\| = \|m_{g(o),g(o)}\|^{-1} \). On the contrary, Lemma 6.2 implies that \( \|m_{g(o),o}\| \preceq K \|m_{g(o),g(o)}\| \). Hence, we have \( \|m_{g(o),o}\| \preceq K^{3/2} 1 \). By Lemma 6.2 again, \( \|m_{g(o),x}\| \preceq K^{3/2} 1 \). \( \square \)

Our main result, the quasi-invariance of the Patterson measure under the normalizer, is formulated as follows.

Theorem 6.6. Let \( \Gamma \subset \text{Isom}(X,d) \) be a non-elementary discrete group of divergence type and let \( \{\mu_z\}_{z \in X} \) be a \((\Gamma, D_0)\)-Patterson measure family with quasiconformal constant \( C_0 \). Then, there exists a constant \( D \geq 1 \) depending only on \( C_0, D_0, \delta, a, \) and \( e_a(\Gamma) \) such that

\[
D^{-1} \leq \frac{dg^{*}\mu_{g(z)}}{d\mu_z} (\xi) \leq D \quad (\text{a.e.} \, \xi \in \partial X)
\]

for every \( g \in N(\Gamma) \).
Proof. We first prove the result for the canonical Patterson measure family \( \{m_z\}_{z \in X} \). By Corollary 6.4 \( \{g^*m_{g(z)}\}_{z \in X} \) is a \((\Gamma, 1)\)-Patterson measure family with the quasiconformal constant \( C = a^{4e(\delta)+2c(\delta)} \). Let \( \nu_0 = g^*m_{g(o)} \), which is a \((\Gamma, C^e_0(\Gamma))\)-Patterson measure by the remark after Lemma 6.1. Its total mass is \( \|\nu_0\| = \|m_{g(o)}\| \asymp K^{-1/2} 1 \) for \( K = a^{4e(\Gamma)c(\delta)} \) by the proof of Lemma 6.5. We already know that \( m_o \) is also a \((\Gamma, C^e_0(\Gamma))\)-Patterson measure with \( \|m_o\| = 1 \) by the same remark mentioned above, where we set \( C' = a^e(\delta) \). Then, Theorem 5.2 asserts that

\[
(CC')^{-e_0(\Gamma)}K^{-1/2} \leq \frac{d\nu_0}{dm_o}(\xi) = \frac{dg^*m_{g(o)}}{dm_o}(\xi) \leq (CC')^{e_0(\Gamma)}K^{1/2} \quad (\text{a.e.} \, \xi \in \partial X).
\]

Finally, by using the quasiconformality of \( \{m_z\}_{z \in X} \) with the constant \( C' = a^e(\delta) \) and \( \{g^*m_{g(z)}\}_{z \in X} \) with the constant \( C = a^{4e(\delta)+2c(\delta)} \), we have

\[
(CC')^{-2e_0(\Gamma)}K^{-1/2} \leq \frac{dg^*m_{g(z)}}{dm_z}(\xi) \leq (CC')^{2e_0(\Gamma)}K^{1/2}.
\]

We now consider a \((\Gamma, D_0)\)-Patterson measure family \( \{\mu_z\}_{z \in X} \) in general with quasiconformal constant \( C_0 \). For the sake of simplicity, we may assume that \( \|\mu_o\| = 1 \). By Proposition 2.5 \( \mu_o \) is \((\Gamma, C^e_0(\Gamma) D_0)\)-quasi-invariant, and then Theorem 5.2 gives

\[
\frac{d\mu_o}{dm_o}(\xi) \asymp (C_oC^e(\Gamma) D_0)^{-1} 1.
\]

Similar to the process above, the quasiconformality then yields

\[
(C_oC')^{-2e_0(\Gamma)}D_0^{-1} \leq \frac{d\mu_z}{dm_z}(\xi) \leq (C_oC')^{2e_0(\Gamma)}D_0.
\]

By replacing \( z \) with \( g(z) \) and \( \xi \) with \( g(\xi) \) here, we also see that \( (dg^*\mu_{g(z)}/dg^*m_{g(z)})(\xi) \) is bounded from above and below by the same constants. Hence, the above three inequalities conclude that

\[
D^{-1} \leq \frac{dg^*\mu_{g(z)}}{dm_z}(\xi) \leq D \quad (\text{a.e.} \, \xi \in \partial X)
\]

for \( D = (CC')^{2e_0(\Gamma)}K^{1/2}(C_oC')^{4e_0(\Gamma)}D_0^2 \). \( \square \)

7. No proper conjugation for divergence-type groups

In this section, we consider the proper conjugation problem for discrete isometry groups of the Gromov hyperbolic space \((X, d)\). This is a continuation of our previous work [8, 9], where we proved the corresponding results for Kleinian groups of divergence type and convex cocompact subgroups of Isom\((X, d)\). A history of this problem and preceding results can be found in [8, 9] and the references therein.

First, we mention an assumption in our new theorem, which was not necessary in the previous theorems. For Kleinian groups, the Jørgensen theorem ensures that the geometric limit of a sequence of discrete groups is also discrete. To avoid these problems in the present arguments for discrete subgroups of Isom\((X, d)\) that are not necessarily
convex cocompact, we introduce the following additional assumption. This was already mentioned in [9].

**Definition.** We say that a discrete group $\Gamma \subset \text{Isom}(X, d)$ is uniformly properly discontinuous if there are a constant $r > 0$ and a positive integer $N \in \mathbb{N}$ such that the number of elements $\gamma \in \Gamma$ satisfying $\gamma(B(x, r)) \cap B(x, r) \neq \emptyset$ is bounded by $N$ for every $x \in X$.

We prepare some claims that are used in the arguments below. For a sequence of discrete subgroups $\{\Gamma_n\}$ of $\text{Isom}(X, d)$, we define the envelope denoted by $\text{Env}\{\Gamma_n\}$ to be the subgroup of $\text{Isom}(X, d)$ consisting of all elements $\gamma = \lim_{n \to \infty} \gamma_n$ given for some sequence $\gamma_n \in \Gamma_n$. We recall the following fact as in [9, Proposition 2.4].

**Proposition 7.1.** Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of subgroups of $\text{Isom}(X, d)$ that act uniformly properly discontinuously on $X$ where the uniformity is also independent of $n$. Then, $\text{Env}\{\Gamma_n\}$ also acts uniformly properly discontinuously on $X$.

In addition, lower semi-continuity of the critical exponents, which is known to be true for geometric convergence of Kleinian groups, is valid in the following form.

**Proposition 7.2.** Let $\{\Gamma_n\}_{n=1}^{\infty} \subset \text{Isom}(X, d)$ be a sequence of discrete groups of divergence type and let $\Gamma_\infty$ be a discrete subgroup of $\text{Env}\{\Gamma_n\}$. Then

$$\liminf_{n \to \infty} e_a(\Gamma_n) \geq e_a(\Gamma_\infty).$$

**Proof.** Let $e = \liminf_{n \to \infty} e_a(\Gamma_n)$. For each $\Gamma_n$, we take the canonical Patterson measure $\mu_n = (m_\alpha)_n$. Passing to a subsequence, we may assume that both $e_a(\Gamma_n)$ converge to $e < \infty$ and $\mu_n$ converge to some Borel measure $\mu$ on $\partial X$ with $\|\mu\| = 1$ in the weak*-sense as $n \to \infty$. Here, we see that $\mu$ is a $(\Gamma_\infty, a^{ec(\delta)})$-quasi-invariant quasiconformal measure of dimension $e$. Indeed, each canonical Patterson measure $\mu_n$ is $(\Gamma_n, a^{ec(\delta)})$-quasi-invariant by the remark after Lemma 6.1 and the weak*-limit $\mu$ preserves this quasi-invariance for the group $\text{Env}\{\Gamma_n\}$ and for the dimension $e$. By Theorem 2.6, the existence of such a measure $\mu$ for $\Gamma_\infty$ yields $e \geq e_a(\Gamma_\infty)$. \qed

The quasi-invariance of the Patterson measure under the normalizer (Theorem 6.6) will be used in the following situation. Although there is no essential difference, this slightly generalized formulation is more convenient.

**Proposition 7.3.** Let $\Gamma$ and $\tilde{\Gamma}$ be non-elementary discrete groups of divergence type in $\text{Isom}(X, d)$ such that $\Gamma \subset \tilde{\Gamma}$ and $e_a(\Gamma) = e_a(\tilde{\Gamma})$. Then, a Patterson measure (family) for $\Gamma$ is quasi-invariant under the normalizer $N(\tilde{\Gamma})$ of $\tilde{\Gamma}$. More precisely, if $\{\mu_z\}_{z \in X}$ is a $(\Gamma, D_0)$-Patterson measure family with quasiconformal constant $C_0$, then there exists a constant $D \geq 1$ depending only on $C_0, D_0, \delta, a$, and $e_a(\Gamma)$ such that

$$D^{-1} \leq \frac{dg^* \mu_0}{d\mu_z}(\xi) \leq D \quad (\text{a.e. } \xi \in \partial X)$$

for every $g \in N(\tilde{\Gamma})$.  

Proof. We take the canonical Patterson measure family \( \{ \tilde{\mu}_z \}_{z \in X} \) for \( \tilde{\Gamma} \), which is \( (\tilde{\Gamma}, 1) \)-quasi-invariant with quasiconformal constant \( a^e(\delta) \) by Lemma 6.1. Then, this is quasi-invariant under \( N(\tilde{\Gamma}) \) as in Theorem 6.6. Furthermore, because \( \Gamma \subset \tilde{\Gamma} \), \( \{ \tilde{\mu}_z \}_{z \in X} \) is a \((\Gamma, 1)\)-Patterson measure family with the quasiconformal constant \( a^e(\delta) \).

Let \( \{ \mu_z \}_{z \in X} \) be a \((\Gamma, D_0)\)-Patterson measure family with quasiconformal constant \( C_0 \). We may assume that \( \| \mu_o \| = 1 \). By Theorem 5.2 with the remark in the previous section, we have \( (d\mu_o/d\tilde{\mu}_o)(\xi) \asymp_{D_0(a^e(\delta)C_0)^{\gamma_o(\ell)}} 1 \) for every \( z \in X \). By the quasi-invariance of \( \{ \tilde{\mu}_z \}_{z \in X} \) under \( N(\tilde{\Gamma}) \), \( \{ \mu_z \}_{z \in X} \) is also \( N(\tilde{\Gamma}) \)-quasi-invariant. Moreover, the dependence of the constant \( D \) is as stated.

We state and prove the main theorem in this section. We say that \( G \subset \text{Isom}(X, d) \) admits proper conjugation if there is some element \( \alpha \in \text{Isom}(X, d) \) such that the conjugate \( \alpha G \alpha^{-1} \) is a proper subgroup of \( G \). Our result says that divergence-type groups do not permit such an unusual conjugation.

**Theorem 7.4.** Let \( G \subset \text{Isom}(X, d) \) be a non-elementary discrete group of divergence type that is uniformly properly discontinuous. If \( \alpha G \alpha^{-1} \subset G \) for \( \alpha \in \text{Isom}(X, d) \), then \( \alpha G \alpha^{-1} = G \).

**Proof.** Let \( \Gamma = \alpha G \alpha^{-1} \) and \( \Gamma_n = \alpha^{-n}\Gamma\alpha^n \) for each integer \( n \geq 0 \). Then \( \Gamma_0 = \Gamma \), \( \Gamma_1 = G \), and \( \{ \Gamma_n \}_{n \geq 0} \) is an increasing sequence of discrete subgroups of \( \text{Isom}(X, d) \) that are conjugate to \( G \). In particular, they are all uniformly properly discontinuous and they are of divergence type with the same critical exponent \( e = e_\alpha(G) \). We define \( \Gamma_\infty = \bigcup_{n \geq 0} \Gamma_n \), which coincides with the envelope \( \text{Env}(\Gamma_n) \) in this case. By Proposition 7.1, \( \Gamma_\infty \) is a discrete subgroup. As \( e_\alpha(\Gamma_n) = e \), Proposition 7.2 implies that \( e_\alpha(\Gamma_\infty) \leq e \). However, as the converse inequality is trivial by the inclusion relation of groups, we have \( e_\alpha(\Gamma_\infty) = e \). Moreover, \( \Gamma_\infty \) is clearly of divergence type because it includes \( \Gamma_n \). Furthermore, as the limit of \( \Gamma_{n-1} = \alpha \Gamma_n \alpha^{-1} \subset \Gamma_n \), we have \( \alpha \Gamma_\infty \alpha^{-1} = \Gamma_\infty \); thus, \( \alpha \in N(\Gamma_\infty) \).

To prove the statement, we suppose to the contrary that \( \Gamma \subsetneq G \) and set \( \ell = [G : \Gamma] \in [2, \infty] \). Let

\[
G = g_1\Gamma \sqcup g_2\Gamma \sqcup \cdots \sqcup g_{\ell}\Gamma \sqcup \cdots
\]

be a coset decomposition of \( G \) by \( \Gamma \). Accordingly, we decompose the weighted Dirac measures \( (m_G)_{o,o}^s \) given by the Poincaré series \( P_G^s(o, o) \) for \( s > e \) to be \( (m_G)_{o,o}^s = \sum_{k=1}^{\ell} \nu_k \), where

\[
\nu_k^s = \frac{1}{P_G^s(o, o)} \sum_{\gamma \in \Gamma} a^{-s d(o, g_k\gamma(o))} D_{g_k\gamma(o)}.
\]
Using the weighted Dirac measures \((m_{\Gamma})_{g_k^{-1}(o),o}^s\) given by the Poincaré series \(P_{\Gamma}^s(g_k^{-1}(o),o)\), we represent \(\nu_k^s\) by
\[
\nu_k^s = \frac{P_{\Gamma}^s(o,o)}{P_G^s(o,o) \cdot P_{\Gamma}^s(o,o)} \sum_{\gamma \in \Gamma} a_{\cdot \cdot \cdot 1}^s(g_k^{-1}(o),\gamma(o))(g_k^{-1})^*D_{\gamma}(o) = \frac{P_{\Gamma}^s(o,o)}{P_G^s(o,o) \cdot (g_k^{-1})^*(m_{\Gamma})_{g_k^{-1}(o),o}^s}.
\]
Moreover, by \(\Gamma = \alpha G\alpha^{-1}\) and Proposition \(2.4\) we have
\[
\frac{P_{\Gamma}^s(o,o)}{P_G^s(o,o)} = \frac{P_{\Gamma}^s(o,o)}{P_G^s(o,o)} = \frac{P_G^s(o,o)}{P_G^s(o,o)} \cdot \frac{P_G^s(o,o)}{P_G^s(o,o)}.
\]
The corresponding substitutions yield
\[
(m_{\Gamma})_{o,o}^s = \sum_{k=1}^{\ell} \frac{P_G^s(o,o)}{P_G^s(o,o)} \cdot \frac{P_G^s(o,o)}{P_G^s(o,o)} \cdot \frac{P_G^s(o,o)}{P_G^s(o,o)} \cdot (g_k^{-1})^*(m_{\Gamma})_{g_k^{-1}(o),o}^s.
\]
We take the limit of the above equality. We can choose a sequence \(s_i \searrow e\) such that all the involved terms are convergent because they are at most countably many. As a result, we obtain
\[
(m_{\Gamma})_{o,o} \geq \sum_{k=1}^{\ell} \cdot (g_k^{-1})^*(m_{\Gamma})_{g_k^{-1}(o),o}^s.
\]
where \(\{(m_{\Gamma})_{z,x}\}\) and \(\{(m_{\Gamma})_{z,x}\}\) stand for the canonical Patterson measure families for \(G\) and \(\Gamma\), respectively. Here, we use Proposition \(7.3\) for \(\Gamma = \Gamma_\infty\). Then, there is a constant \(D \geq 1\) independent of the elements of \(N(\Gamma_\infty)\) such that
\[
D^{-1} \leq \frac{d(g_k^{-1})^*(m_{\Gamma})_{g_k^{-1}(o),o}^s}{d(m_{\Gamma})_{o,o}}(\xi) \leq D \quad (\text{a.e. } \xi \in \partial X).
\]
In particular, the total mass satisfies \(\|(g_k^{-1})^*(m_{\Gamma})_{g_k^{-1}(o),o}^s\| \asymp_D 1\). Similarly, we have
\[
\|(m_{\Gamma})_{\alpha^{-1}(o),o}\| = \|(\alpha^{-1})^*(m_{\Gamma})_{\alpha^{-1}(o),o}\| \asymp_D \|(m_{\Gamma})_{o,o}\| = 1;
\]
\[
\|(m_{\Gamma})_{\alpha^{-1}(o),\alpha^{-1}(o)}\| = \|(\alpha^{-1})^*(m_{\Gamma})_{\alpha^{-1}(o),\alpha^{-1}(o)}\| \asymp_D \|(m_{\Gamma})_{o,o}\| = 1.
\]
Then, taking the total mass in the above inequality, we can make the assertion \(\ell = [G : \Gamma] \leq D^3\). If \(\ell = \infty\), this is a contradiction; we may assume that \(\ell < \infty\).

Finally, we choose \(j \in \mathbb{N}\) such that \(\ell^j > D^3\). We consider \(\alpha^j\) instead of \(\alpha\) and set \(\Gamma' = \alpha^j G\alpha^{-j}\), which is a proper subgroup of \(G\) with index \([G : \Gamma'] = \ell^j\). Then, we repeat the same arguments as above for \(G\) and \(\Gamma'\). The conclusion is that \([G : \Gamma'] \leq D^3\). We note that the constant \(D\) is unaffected by this replacement because the dependence of \(D\) as in Proposition \(7.3\) is irrelevant to the canonical Patterson measures. In this way, we derive the contradiction, and thus prove the result. \(\Box\)
8. THE LOWER BOUND OF THE CRITICAL EXPONENTS OF NORMAL SUBGROUPS

For Kleinian groups, there are numerous important studies on the critical exponents of non-elementary normal subgroups $\Gamma$. Among them, concerning the lower bound of such exponents, Falk and Stratmann [5] proved that they are bounded from below by half the exponent of the original group $G$. Later, Roblin [14] extended this result in a different manner and it was proved, in particular, that if $G$ is of divergence type, then the strict inequality holds. More recently, a simple proof for these results appeared in [7]. We generalize this argument to discrete isometry groups of the Gromov hyperbolic space $\text{Isom}(X, d)$ and prove the following theorem.

**Theorem 8.1.** Let $G \subset \text{Isom}(X, d)$ be a discrete group and let $\Gamma$ be a non-elementary normal subgroup of $G$. Then, $e_a(\Gamma) \geq e_a(G)/2$. Moreover, if $G$ is of divergence type, then the strict inequality $e_a(\Gamma) > e_a(G)/2$ holds.

This theorem was already expected in [7] when [8, Theorem 4.3], which was used for the proof of the strict inequality, was going to be generalized to the case of the Gromov hyperbolic space. This generalization is here carried out as a consequence of Theorem 6.6 in the following form.

**Theorem 8.2.** Let $G \subset \text{Isom}(X, d)$ be a discrete group and let $\Gamma$ be a non-elementary normal subgroup of $G$. If $\Gamma$ is of divergence type, then $e_a(G) = e_a(\Gamma)$; moreover, $G$ is also of divergence type.

**Proof.** Let $\mu$ be a Patterson measure for $\Gamma$. By Theorem 6.6, $\mu$ is quasi-invariant under $N(\Gamma)$. In particular, $\mu$ is a $G$-quasi-invariant quasiconformal measure of dimension $e_a(\Gamma)$. On the contrary, by Theorem 2.6, the lower bound of the dimensions of $G$-quasi-invariant quasiconformal measures is $e_a(G)$. Hence, we have $e_a(\Gamma) \geq e_a(G)$. As the converse inequality is trivial by $\Gamma \subset G$, we see that $e_a(G) = e_a(\Gamma)$. Moreover, the divergence of $\Gamma$ at $e_a(\Gamma)$ implies that of $G$ at the same dimension. \qed

**Remark.** Corollary 2.8 asserts that when $\Gamma$ is a subgroup of $G$ not necessarily normal and $\Gamma$ is of divergence type, $e_a(G) = e_a(\Gamma)$ implies $\Lambda(G) = \Lambda(\Gamma)$. The converse is not true in general, but Theorem 8.2 states that if $\Gamma$ is non-elementary and normal in $G$, which implies $\Lambda(G) = \Lambda(\Gamma)$, then $e_a(G) = e_a(\Gamma)$.

We add necessary modification to the claims in [7] to apply them to discrete isometry groups of the Gromov hyperbolic space $(X, d)$.

All non-trivial elements of $\text{Isom}(X, d)$ are classified into three types: hyperbolic, parabolic, and elliptic. We say that $\gamma \in \text{Isom}(X, d)$ is *hyperbolic* if it has exactly two fixed points on $\partial X$. The following are well-known properties of hyperbolic elements of discrete groups: see, for example, Tukia [19, Section 2]. We note that $\text{Isom}(X, d)$ acts on the boundary $\partial X$ as a convergence group.

**Proposition 8.3.** Let $\Gamma \subset \text{Isom}(X, d)$ be a non-elementary discrete group. Then, $\Gamma$ contains a hyperbolic element $h$. Moreover, the stabilizer $\text{Stab}_\Gamma(\text{Fix}(h))$ of the fixed point
set $\text{Fix}(h) \subset \partial X$ of $h$ is a finite index extension of the cyclic group $\langle h \rangle$. If $\gamma \in \Gamma$ commutes with $h$, then $\gamma$ belongs to $\text{Stab}_\Gamma(\text{Fix}(h))$.

The novelty of the proof in [7] is the use of the following fact. Once the above properties are verified, the proof of the lemma can be carried out without any change even in the case of the Gromov hyperbolic space.

**Lemma 8.4.** Let $G \subset \text{Isom}(X, d)$ be a discrete group and let $\Gamma$ be a non-elementary normal subgroup of $G$. For any hyperbolic element $h \in \Gamma$, the map

$$
\iota_h : \langle h \rangle \backslash G \to \Gamma
$$

defined by $[g] \mapsto g^{-1}h_g$ is well-defined and at most $k$ to $1$, that is, there is $k = k_h \in \mathbb{N}$ such that $\# \iota_h^{-1}(\gamma) \leq k_h$ for every $\gamma \in \Gamma$.

An essential step in the adaption of the arguments for Kleinian groups to discrete isometry groups on the Gromov hyperbolic space lies in the following claim.

**Lemma 8.5.** Let $G \subset \text{Isom}(X, d)$ be a discrete group and let $h \in G$ be a hyperbolic element. Then, for every $s > 0$, there is a constant $A_h(s) > 0$ depending on $s$ and $h$ such that

$$
\sum_{o \in G} a^{-sd(o,g(o))} \leq A_h(s) \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,[g](o))},
$$

where $d(o,[g](o))$ is the distance from $o$ to the set $[g](o) = \{h^ng(o) \mid n \in \mathbb{Z}\}$.

**Proof.** We take a geodesic segment $[o,h(o)]$ connecting $o$ and $h(o)$ and make a piecewise geodesic curve $\beta = \bigcup_{n \in \mathbb{Z}} h^n([o,h(o)])$ with arc length parameter. In fact, $\beta : (-\infty, \infty) \to X$ is a quasi-geodesic line, that is, there are constants $\lambda \geq 1$ and $c \geq 0$ such that

$$
|s-t| \leq \lambda d(\beta(s), \beta(t)) + c
$$

for all $s, t \in \mathbb{R}$ (see [3] Lemme 6.5). Let $\ell_h = d(o,h(o))$.

For any coset $[g] \in \langle h \rangle \backslash G$, we consider the set $[g](o)$ and choose a point in it, which we may assume to be $g(o)$ without loss of generality, so that $[o,h(o)]$ contains the nearest point $x$ (not necessarily unique) from $g(o)$ to $\beta$. Let $L_{[g]} = d(x,g(o)) = d(\beta, [g](o))$. Then, we have

$$
d(o,[g](o)) \leq d(o,g(o)) \leq d(o,x) + d(x,g(o)) \leq \ell_h + L_{[g]}.
$$

We consider $h^ng(o)$ for every $n \in \mathbb{Z}$. By the invariance under $\langle h \rangle$, $h^n(x)$ is the nearest point from $h^ng(o)$ to $\beta$. The above inequality implies that

$$
d(h^n(x),h^ng(o)) = L_{[g]} \geq d(o,[g](o)) - \ell_h.
$$

We choose a geodesic segment $\tilde{\beta}_n = [o,h^n(x)]$. As $\tilde{\beta}_n$ is within distance $r = r(\delta, \lambda, c) \geq 0$ from the quasi-geodesic segment in $\beta$ between $o$ and $h^n(x)$ (see [3] Théorème 1.3), we have

$$
d(\tilde{\beta}_n, h^n(g(o)) \geq d(h^n(x), h^ng(o)) - r = L_{[g]} - r.
$$
Concerning the left-hand side, Lemma 8.5 implies that
\[ \langle o | h^n(x) \rangle_{h^n g(o)} = \frac{1}{2} \{ d(o, h^n g(o)) + d(h^n(x), h^n g(o)) - d(o, h^n(x)) \} \geq d(\tilde{\beta}_n, h^n g(o)) - 4\delta. \]

These inequalities imply that
\begin{align*}
d(o, h^n g(o)) &\geq d(o, h^n(x)) + d(h^n(x), h^n g(o)) - 2r - 8\delta \\
&\geq d(o, h^n(o)) - \ell_h + L_{|g|} - 2r - 8\delta \\
&\geq d(o, h^n(o)) + d(o, [g](o)) - 2(\ell_h + r + 4\delta).
\end{align*}

Using this estimate, we compute the Poincaré series as follows:
\[
\sum_{g \in G} a^{-sd(o,g(o))} = \sum_{n \in \mathbb{Z}} \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,h^n g(o))} \\
\leq a^{2s(\ell_h + r + 4\delta)} \sum_{n \in \mathbb{Z}} a^{-sd(o,h^n(o))} \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,[g](o))}.
\]

Here, \( \sum_{n \in \mathbb{Z}} a^{-sd(o,h^n(o))} \) converges because \( \beta \) is a quasi-geodesic that satisfies
\[ d(o, h^n(o)) \geq \lambda^{-1} \ell_h n - c. \]

Hence, by setting \( A_h(s) = a^{2s(\ell_h + r + 4\delta)} \sum_{n \in \mathbb{Z}} a^{-sd(o,h^n(o))} \), we obtain the assertion. \( \square \)

**Proof of Theorem 8.7.** Choose a hyperbolic element \( h \in \Gamma \) and fix it. Concerning the map \( \iota_h \) in Lemma 8.4, we note that
\begin{align*}
d(o, \iota_h([g])(0)) &= d(o, g^{-1} h g(o)) \\
&\leq d(o, g^{-1}(o)) + d(g^{-1}(o), g^{-1} h(o)) + d(g^{-1} h(o), g^{-1} h(g(o))) \\
&= 2d(o, g(o)) + d(o, h(o)).
\end{align*}

As this is still valid after replacing \( g \) with \( h^n g \) \( (n \in \mathbb{Z}) \), we see that
\[ d(o, \iota_h([g])(0)) \leq 2d(o, [g](o)) + d(o, h(o)). \]

It follows that
\[ a^{-sd(o,[g](o))} \leq a^{sd(o,h(o))/2} \cdot a^{-sd(o,\iota_h([g])(o))/2}. \]

Taking the sum over \( [g] \in \langle h \rangle \backslash G \), we obtain
\[ \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,[g](o))} \leq a^{s\ell_h/2} \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,\iota_h([g])(o))/2}. \]

Concerning the right-hand side in the above inequality, Lemma 8.4 implies that
\[ \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,\iota_h([g])(o))/2} \leq k_h \sum_{\gamma \in \Gamma} a^{-sd(o,\gamma)(o)/2}. \]

Concerning the left-hand side, Lemma 8.5 implies that
\[ \sum_{g \in G} a^{-sd(o,g(o))} \leq A_h(s) \sum_{[g] \in \langle h \rangle \backslash G} a^{-sd(o,[g](o))}. \]
Combining these inequalities, we finally obtain the following estimate:

\[
\sum_{g \in G} a^{-sd(o,g(o))} \leq A_h(s)a^{s\ell_h/2}k_h \sum_{\gamma \in \Gamma} a^{-sd(o,\gamma(o))/2}.
\]

Now we put \( s = 2e_a(\Gamma) + \varepsilon \) for an arbitrary \( \varepsilon > 0 \) and consider the final estimate just above. The right-hand side converges; hence, so does the left-hand side. This shows that \( e_a(G) \leq 2e_a(\Gamma) + \varepsilon \). As \( \varepsilon > 0 \) is arbitrary, we have \( e_a(G) \leq 2e_a(\Gamma) \), which yields the first assertion of the theorem.

Next, we assume that \( G \) is of divergence type. Then, we put \( s = e_a(G) \) and consider the same inequality. In this case, the left-hand side diverges; hence, so does the right-hand side. To prove the strict inequality, it is assumed toward a contradiction that \( e_a(\Gamma) = e_a(G)/2 \). As the exponent of the series on the right-hand side is \( s/2 = e_a(G)/2 = e_a(\Gamma) \), \( \Gamma \) must be of divergence type under this assumption. Theorem \( \ref{divergence-type} \) then implies that \( e_a(G) = e_a(\Gamma) \). This is possible only when \( e_a(G) = e_a(\Gamma) = 0 \). However, this contradicts the next claim, which we can also find in [2, Corollaire 5.5].

**Proposition 8.6.** The critical exponent \( e_a(G) \) of a non-elementary discrete group \( G \subset \text{Isom}(X,d) \) is strictly positive.

**Proof.** Let \( h \in G \) be a hyperbolic element. Then, by the last part of the proof of Lemma \( \ref{divergence-type} \), we see that \( e_a(\langle h \rangle) = 0 \) and \( \langle h \rangle \) is of divergence type. As \( \Lambda(G) \supseteq \Lambda(\langle h \rangle) \), Corollary \( \ref{divergence-type} \) shows that \( e_a(G) > 0 \).

**Note added in the revision.** Recently, we have found the following references that are closely related to the results in this study. Das, Simmons, and Urbański [4, Theorem 1.4.1] proved Theorem \( \ref{divergence-type} \) in a more general setting. Arzhantseva and Cashen [1] proved a special case of Theorem \( \ref{divergence-type} \) for isometric actions not necessarily on the Gromov hyperbolic spaces.

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