Hidden Spacetime Symmetries and Generalized Holonomy in M-theory\footnote{Research supported in part by DOE Grant DE-FG02-95ER40899.}

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Abstract

In M-theory vacua with vanishing 4-form $F_{(4)}$, one can invoke the ordinary Riemannian holonomy $H \subset \text{SO}(10,1)$ to account for unbroken supersymmetries $n = 1, 2, 3, 4, 6, 8, 16, 32$. However, the generalized holonomy conjecture, valid for non-zero $F_{(4)}$, can account for more exotic fractions of supersymmetry, in particular $16 < n < 32$. The conjectured holonomies are given by $H \subset G$ where $G$ are the generalized structure groups $G = \text{SO}(d-1,1) \times G(\text{spacelike})$, $G = \text{ISO}(d-1) \times G(\text{null})$ and $G = \text{SO}(d) \times G(\text{timelike})$ with $1 \leq d < 11$. For example, $G(\text{spacelike}) = \text{SO}(16)$, $G(\text{null}) = [\text{SU}(8) \times \text{U}(1)] \rtimes \mathbb{R}^{56}$ and $G(\text{timelike}) = \text{SO}^*(16)$ when $d = 3$. Although extending spacetime symmetries, there is no conflict with the Coleman-Mandula theorem. The holonomy conjecture rules out certain vacua which are otherwise permitted by the supersymmetry algebra.
1 Introduction

M-theory not only provides a non-perturbative unification of the five consistent superstring theories, but also embraces earlier work on supermembranes and eleven-dimensional supergravity \[1\]. It is regarded by many as the dreamed-of final theory and has accordingly received an enormous amount of attention. It is curious, therefore, that two of the most basic questions of M-theory have until now remained unanswered:

i) \textit{What are the symmetries of M-theory?}

ii) \textit{How many supersymmetries can vacua of M-theory preserve?}

The first purpose of this paper is to argue that M-theory possesses previously unidentified hidden spacetime (timelike and null) symmetries in addition to the well-known hidden internal (spacelike) symmetries. These take the form of generalized structure groups \(G\) that replace the Lorentz group \(SO(10, 1)\).

The second purpose is to argue that the number of supersymmetries preserved by an M-theory vacuum is given by the number of singlets appearing in the decomposition of the 32-dimensional representation of \(G\) under \(G \supset H\) where \(H\) are generalized holonomy groups.

The equations of M-theory display the maximum number of supersymmetries \(N=32\), and so \(n\), the number of supersymmetries preserved by a particular vacuum, must be some integer between 0 and 32. But are some values of \(n\) forbidden and, if so, which ones? For quite some time it was widely believed that, aside from the maximal \(n = 32\), \(n\) is restricted to \(0 \leq n \leq 16\) with \(n = 16\) being realized by the fundamental BPS objects of M-theory: the M2-brane, the M5-brane, the M-wave and the M-monopole. The subsequent discovery of intersecting brane configurations with \(n = 0, 1, 2, 3, 4, 5, 6, 8, 16\) lent credence to this argument. In \[2\], on the other hand, it was shown that all values \(0 \leq n \leq 32\) are allowed by the M-theory algebra \[3\], and examples of vacua with \(16 < n < 32\) have indeed since been found. Following \[4\] and \[5\], we here put forward a \textit{generalized holonomy conjecture} according to which the answer lies somewhere in between. Evidence in favor of this conjecture includes the observations that there are no known counterexamples and that a previously undiscovered example predicted in \[5\], namely \(n=14\), has recently been found \[6\].

As we shall see, these conjectures are based on a group-theoretical argument which applies to the fully-fledged M-theory. To get the ball rolling, however, we begin with the low energy limit of M-theory, namely \(D = 11\) supergravity. The unique \(D = 11\) supermultiplet is
comprised of a graviton $g_{MN}$, a gravitino $\Psi_M$ and 3-form gauge field $A_{MNP}$, where $M = 0, 1, \ldots, 10$, with 44, 128 and 84 physical degrees of freedom, respectively. In section 2 we conjecture that the supergravity equations of motion for this set of fields admit hidden timelike and null symmetries (in addition to previously demonstrated hidden spacelike ones). Then in section 3 we propose that, so long as the $D = 11$ Killing spinor equation has such hidden symmetries, we may enlarge the tangent space group into a generalized structure group. This allows us to analyze the number of supersymmetries based on a generalized holonomy conjecture. Partial justification for this conjecture is presented in section 4 in the context of a dimensionally reduced theory. In section 5 we discuss some consequences of generalized holonomy for classifying supersymmetric vacua, and finally conclude in section 6.

2 Hidden spacetime symmetries of D=11 supergravity

Long ago, Cremmer and Julia [7] pointed out that, when dimensionally reduced to $d$ dimensions, $D = 11$ supergravity exhibits hidden symmetries. For example $E_7(\text{global}) \times SU(8)(\text{local})$ when $d = 4$ and $E_8(\text{global}) \times SO(16)(\text{local})$ when $d = 3$. The question was then posed [8]: do these symmetries appear magically only after dimensional reduction, or were they already present in the full uncompactified and untruncated $D = 11$ theory? The question was answered by de Wit and Nicolai [9, 10] who made a $d/(11 - d)$ split and fixed the gauge by setting to zero the off-diagonal components of the elfbein. They showed that in the resulting field equations the local symmetries are indeed already present, but the global symmetries are not. For example, after making the split $SO(10,1) \supset SO(3,1) \times SO(7)$, we find the enlarged symmetry $SO(3,1) \times SU(8)$. There is no global $E_7$ invariance (although the 70 internal components of the metric and 3-form may nevertheless be assigned to an $E_7/SU(8)$ coset). Similar results were found for other values of $d$: in each case the internal subgroup $SO(11 - d)$ gets enlarged to some compact group $G(\text{spacelike})$ while the spacetime subgroup $SO(d - 1, 1)$ remains intact\(^4\). In this paper we ask instead whether there are hidden spacetime symmetries. This is a question that could have been asked long ago, but we suspect that people may have been inhibited by the Coleman-Mandula theorem which forbids combining spacetime and internal symmetries [11]. However, this is a statement about

\(^4\)We keep the terminology “spacetime” and “internal” even though no compactification or dimensional reduction is implied.
\[ d/(11 - d) \mid \mathcal{G} = \text{SO}(d - 1, 1) \times G(\text{spacelike}) \mid \epsilon \text{ representation} \]

| \( d/(11 - d) \) | \( \mathcal{G} \) | \( \epsilon \text{ representation} \) |
|---------------------|-------------------|-----------------|
| 10/1                | \( \text{SO}(9, 1) \times \{1\} \) | \( 16 + \overline{16} \) |
| 9/2                 | \( \text{SO}(8, 1) \times \text{SO}(2) \) | \( 16_{\pm 1/2} \) |
| 8/3                 | \( \text{SO}(7, 1) \times \text{SO}(3) \times \text{SO}(2) \) | \( (8_s, 2)_{1/2} + (8_c, 2)_{-1/2} \) |
| 7/4                 | \( \text{SO}(6, 1) \times \text{SO}(5) \) | \( (8, 4) \) |
| 6/5                 | \( \text{SO}(5, 1) \times \text{SO}(5) \times \text{SO}(5) \) | \( (4, 4, 1) + (\overline{1}, 1, 4) \) |
| 5/6                 | \( \text{SO}(4, 1) \times \text{USp}(8) \) | \( (4, 8) \) |
| 4/7                 | \( \text{SO}(3, 1) \times \text{SU}(8) \) | \( (2, 1, 8) + (1, 2, \overline{8}) \) |
| 3/8                 | \( \text{SO}(2, 1) \times \text{SO}(16) \) | \( (2, 16) \) |
| 2/9                 | \( \text{SO}(1, 1) \times \text{SO}(16) \times \text{SO}(16) \) | \( (16, 1)_{1/2} + (1, 16)_{-1/2} \) |
| 1/10                | \( \{1\} \times \text{SO}(32) \) | \( 32 \) |

Table 1: Generalized structure groups: spacelike case. The last column denotes the representation of \( \epsilon \) under \( \mathcal{G} \).

Poincare symmetries of the S-matrix and here we are concerned with Lorentz symmetries of the equations of motion, so there will be no conflict.

The explicit demonstration of \( G(\text{spacelike}) \) invariance by de Wit and Nicolai is very involved, to say the least. However, the result is quite simple: one finds the same \( G(\text{spacelike}) \) in the full uncompactified \( D = 11 \) theory as was already found in the spacelike dimensional reduction of Cremmer and Julia. Here we content ourselves with the educated guess that the same logic applies to \( G(\text{timelike}) \) and \( G(\text{null}) \): they are the same as what one finds by timelike and null reduction, respectively. So we propose that, after making a \( d/(11 - d) \) split, the Lorentz subgroup \( G = \text{SO}(d - 1, 1) \times \text{SO}(11 - d) \) can be enlarged to the generalized structure groups \( \mathcal{G} = \text{SO}(d - 1, 1) \times G(\text{spacelike}) \), \( \mathcal{G} = \text{ISO}(d - 1) \times G(\text{null}) \) and \( \mathcal{G} = \text{SO}(d) \times G(\text{timelike}) \) as shown in Tables 1, 2, and 3.

Some of the noncompact groups appearing in the Tables may be unfamiliar, but a nice discussion of their properties may be found in [12]. For \( d > 2 \) the groups \( G(\text{spacelike}) \), \( G(\text{timelike}) \) and \( G(\text{null}) \) are the same as those obtained from the spacelike dimensional reductions of Cremmer and Julia [7], the timelike reductions of Hull and Julia [13]5, and the null reduction of section 8.2, respectively. For our purposes, however, their physical

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5Actually, for the 8/3 split, we have the factor \( \text{SO}(1, 1) \) instead of their \( \text{SO}(2) \).
Table 2: Generalized structure groups: null case. The last column denotes the representation of \( \epsilon \) under the maximum compact subgroup of \( \mathcal{G} \).

The interpretation is very different. They are here proposed as symmetries of the full \( D = 11 \) equations of motion; there is no compactification involved, whether toroidal or otherwise. This conjecture that these symmetries are present in the full theory and not merely in its dimensional reductions may be put to the test, however, as we shall later describe. For \( d \leq 2 \) it is less clear whether these generalized structure groups are actually hidden symmetries. See the caveats of section 4. The \( \text{SO}(16) \times \text{SO}(16) \) for \( d = 2 \) is also discussed by Nicolai [14].

### 3 Hidden Symmetries and Generalized Holonomy

We begin by reviewing the connection between holonomy and the number of preserved super-symmetries, \( n \), of supergravity vacua. This also serves to define our notation. Subsequently, we introduce a generalized holonomy which involves the hidden symmetries conjectured in the previous section.
\( d/(11-d) \) & \( \mathcal{G} = \text{SO}(d) \times G(\text{timelike}) \) & \( \epsilon \) representation  \\ 10/1 & \text{SO}(10) \times \{1\} & 16 + \overline{16}  \\ 9/2 & \text{SO}(9) \times \text{SO}(1,1) & 16_{\pm 1/2}  \\ 8/3 & \text{SO}(8) \times \text{SO}(2,1) \times \text{SO}(1,1) & (8_{s}, 2)_{1/2} + (8_{c}, 2)_{-1/2}  \\ 7/4 & \text{SO}(7) \times \text{SO}(3,2) & (8, 4)  \\ 6/5 & \text{SO}(6) \times \text{SO}(5,\mathbb{C}) & (4, 4) + (\overline{4}, 4)  \\ 5/6 & \text{SO}(5) \times \text{USp}(4,4) & (4, 8)  \\ 4/7 & \text{SO}(4) \times \text{SU}^{*}(8) & (2, 1, 8) + (1, 2, \overline{8})  \\ 3/8 & \text{SO}(3) \times \text{SO}^{*}(16) & (2, 16)  \\ 2/9 & \text{SO}(2) \times \text{SO}(16,\mathbb{C}) & 16_{1/2} + \overline{16}_{-1/2}  \\ 1/10 & \{1\} \times \text{SO}(16,16) & 32  \\ 

Table 3: Generalized structure groups: timelike case. The last column denotes the representation of \( \epsilon \) under \( \mathcal{G} \).

### 3.1 Riemannian Holonomy

We are interested in solutions of the bosonic field equations

\[
R_{MN} = \frac{1}{12} \left( F_{MPQR} F^P_{\phantom{P}QR} - \frac{1}{12} g_{MN} F^{PQRS} F_{PQRS} \right) \tag{1}
\]

and

\[
d \ast F_{(4)} + \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0, \tag{2}
\]

where \( F_{(4)} = dA_{(3)} \). The supersymmetry transformation rule of the gravitino reduces in a purely bosonic background to

\[
\delta \Psi_M = \tilde{D}_M \epsilon, \tag{3}
\]

where the parameter \( \epsilon \) is a 32-component anticommuting spinor, and where

\[
\tilde{D}_M = D_M - \frac{1}{288} (\Gamma_{M}^{NPQR} - 8 \delta_{M}^{N} \Gamma^{PQR}) F_{NPQR}, \tag{4}
\]

where \( \Gamma^A \) are the \( D = 11 \) Dirac matrices. Here \( D_M \) is the usual Riemannian covariant derivative involving the connection \( \omega_M \) of the usual structure group \( \text{Spin}(10,1) \), the double cover of \( \text{SO}(10,1) \),

\[
D_M = \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB}. \tag{5}
\]
The number of supersymmetries preserved by an M-theory background depends on the number of covariantly constant spinors,

$$D_M \epsilon = 0,$$

(6)
called *Killing* spinors. It is the presence of the terms involving the 4-form $F_{(4)}$ in (4) that makes this counting difficult. So let us first examine the simpler vacua for which $F_{(4)}$ vanishes. Killing spinors then satisfy the integrability condition

$$[D_M, D_N] \epsilon = \frac{1}{4} R_{MN}^{AB} \Gamma_{AB} \epsilon = 0,$$

(7)

where $R_{MN}^{AB}$ is the Riemann tensor. The subgroup of Spin(10, 1) generated by this linear combination of Spin(10, 1) generators $\Gamma_{AB}$ corresponds to the *holonomy* group $H$ of the connection $\omega_M$. The number of supersymmetries, $n$, is then given by the number of singlets appearing in the decomposition of the 32 of Spin(10, 1) under $H$. In Euclidean signature, connections satisfying (7) are automatically Ricci-flat and hence solve field equations when $F_{(4)} = 0$. In Lorentzian signature, however, they need only be Ricci-null [15] so Ricci-flatness has to be imposed as an extra condition. In Euclidean signature, the holonomy groups have been classified [16]. In Lorentzian signature, much less is known but the question of which subgroups $H$ of Spin(10, 1) leave a spinor invariant has been answered in [17]. There are two sequences according as the vector $v_A = \epsilon \Gamma_A \epsilon$ is timelike or null, as shown in Tables 4 and 5. Since $v^2 \leq 0$, the spacelike $v_A$ case does not arise. The timelike $v_A$ case corresponds to static vacua, where $H \subset \text{Spin}(10) \subset \text{Spin}(10, 1)$ while the null case to non-static vacua where $H \subset \text{ISO}(9) \subset \text{Spin}(10, 1)$. It is then possible to determine the possible $n$-values [18, 19] and one finds $n = 2, 4, 6, 8, 16, 32$ for static vacua, as shown in Table 4 and $n = 1, 2, 3, 4, 8, 16, 32$ for non-static vacua, as shown in Table 5.

### 3.2 Generalized holonomy

When we want to include vacua with $F_{(4)} \neq 0$ we face the problem that the connection in (4) is no longer the spin connection to which the bulk of the mathematical literature on holonomy groups is devoted. In addition to the Spin(10, 1) generators $\Gamma_{AB}$, it is apparent from (4) that there are terms involving $\Gamma_{ABC}$ and $\Gamma_{ABCDE}$. As a result, the connection takes its values in the full $D = 11$ Clifford algebra. Moreover, this connection can preserve exotic fractions of supersymmetry forbidden by the Riemannian connection. For example,
\[
\begin{array}{|c|c|c|}
\hline
\frac{d}{11-d} & H \subset SO(11-d) \subset Spin(10) & n \\
\hline
7/4 & SU(2) \cong Sp(2) & 16 \\
5/6 & SU(3) & 8 \\
4/7 & G_2 & 4 \\
3/8 & SU(2) \times SU(2) & 8 \\
& Sp(4) & 6 \\
& SU(4) & 4 \\
& Spin(7) & 2 \\
1/10 & SU(2) \times SU(3) & 4 \\
& SU(5) & 2 \\
\hline
\end{array}
\]

Table 4: Holonomy of static M-theory vacua with \( F_{(4)} = 0 \) and their supersymmetries.

the M-branes at angles in \cite{20} include \( n=5 \), the 11-dimensional pp-waves in \cite{21,22,23,24} include \( n = 18, 20, 22, 24, 26 \) (and \( n = 28 \) for Type IIB), the squashed \( N(1,1) \) spaces in \cite{25} and the M5-branes in a pp-wave background in \cite{26} include \( n=12 \) and the Gödel universes in \cite{27} include \( n = 18, 20, 22, 24 \).

However, we can attempt to quantify this in terms of generalized holonomy groups \( \mathcal{H} \subset \mathcal{G} \) where \( \mathcal{G} \) are the generalized structure groups discussed in section \cite{2}. The generalized holonomy conjecture \cite{4,5} states that one can assign a holonomy \( \mathcal{H} \subset \mathcal{G} \) to the generalized connection\(^6\) appearing in the supercovariant derivative \cite{1}. Here we propose that, after making a \( d/(11-d) \) split, the Lorentz subgroup \( G = SO(d-1,1) \times SO(11-d) \) can be enlarged to the generalized structure groups \( \mathcal{G} = SO(d-1,1) \times G(spacelike) \), \( \mathcal{G} = ISO(d-1) \times G(null) \) and \( \mathcal{G} = SO(d) \times G(timelike) \) as shown in Tables \cite{1,2} and \cite{3}. Note that in the right hand column of the tables we have listed the corresponding \( \mathcal{G} \) representations under which the 32 supersymmetry parameters \( \epsilon \) transform. The number of supersymmetries preserved by an M-theory vacuum is then given by the number of singlets appearing in the decomposition of these representations under \( \mathcal{G} \supset \mathcal{H} \).

\(^{6}\)A related conjecture was made in \cite{28}, where the generalized holonomy could be any subgroup of \( SO(16,16) \). This also appears in our conjectured hidden structure groups under the 1/10 split, though only in the timelike case \( \mathcal{G}(timelike) \).
\[
\begin{array}{|c|c|c|}
\hline
\d / (11 - d) & H \subset \text{ISO}(d - 1) \times \text{ISO}(10 - d) \subset \text{Spin}(10, 1) & n \\
\hline
10/1 & \mathbb{R}^9 & 16 \\
6/5 & \mathbb{R}^5 \times (\text{SU}(2) \times \mathbb{R}^4) & 8 \\
4/7 & \mathbb{R}^3 \times (\text{SU}(3) \times \mathbb{R}^6) & 4 \\
3/8 & \mathbb{R}^2 \times (G_2 \times \mathbb{R}^7) & 2 \\
2/9 & \mathbb{R} \times (\text{SU}(2) \times \mathbb{R}^4) \times (\text{SU}(2) \times \mathbb{R}^4) & 4 \\
& \mathbb{R} \times (\text{Sp}(4) \times \mathbb{R}^8) & 3 \\
& \mathbb{R} \times (\text{SU}(4) \times \mathbb{R}^8) & 2 \\
& \mathbb{R} \times (\text{Spin}(7) \times \mathbb{R}^8) & 1 \\
\hline
\end{array}
\]

Table 5: Holonomy of non-static M-theory vacua with \( F_{(4)} = 0 \) and their supersymmetries.

4 Structure groups from dimensional reduction

In this section we provide partial justification for the conjectured hidden symmetries by demonstrating their presence in the gravitino variation of the dimensionally reduced theory. In particular, we consider a spacelike dimensional reduction corresponding to a \( d/(11 - d) \) split. Turning on only \( d \)-dimensional scalars, the reduction ansatz is particularly simple

\[
\begin{align*}
g^{(11)}_{MN} &= \begin{pmatrix} \Delta^{-1/(d-2)} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}, \\
A^{(11)}_{ijk} &= \phi_{ijk},
\end{align*}
\]

where \( \Delta = \det g_{ij} \). For \( d \leq 5 \), we must also consider the possibility dualizing either \( F_{(4)} \) components or (for \( d = 3 \)) Kaluza-Klein vectors to scalars. We will return to such possibilities below. But for now we focus on \( d \geq 6 \). In this case, a standard dimensional reduction of the \( D = 11 \) gravitino transformation, \([3]\), yields the \( d \)-dimensional gravitino transformation

\[
\delta \psi_\mu = [D_\mu + \frac{1}{4} Q_\mu^{ab} \Gamma_{ab} + \frac{1}{24} \partial_\mu \phi_{ijk} \Gamma^{ijk}] \epsilon.
\]

For completeness, we also note that the \( d \)-dimensional dilatinos transform according to

\[
\delta \lambda_i = -\frac{1}{2} \gamma^\mu [P_{ij} \Gamma^\mu - \frac{1}{36} (\Gamma^j \Gamma^k - 6 \delta^j_k \Gamma^k) \partial_\mu \phi_{ijkl}] \epsilon.
\]

In the above, the lower dimensional quantities are related to their \( D = 11 \) counterparts through

\[
\psi_\mu = \Delta^{\frac{1}{d-2}} \left( \Psi_\mu^{(11)} + \frac{1}{d-2} \gamma^\mu \Gamma^i \Psi_i^{(11)} \right), \quad \lambda_i = \Delta^{\frac{1}{d-2}} \Psi_i^{(11)},
\]
\[ \epsilon = \Delta^{(\frac{1}{4(d-2)})} \epsilon^{(11)}, \]
\[ Q^{ab}_{\mu} = e^{i[a} \partial_{\mu} e^{b]}, \quad P_{\mu ij} = e_{(i}^{\mu} \partial_{\mu} e_{j)} a. \] (11)

We now see that the lower dimensional gravitino transformation, (9), may be written in terms of a covariant derivative under a generalized connection

\[ \delta \psi_\mu = \hat{D}_\mu \epsilon, \quad \hat{D}_\mu = \partial_\mu + \frac{1}{4} \Omega_\mu, \] (12)

where

\[ \Omega_\mu = \omega_\mu^{\alpha\beta} \gamma_{\alpha\beta} + Q_{\mu}^{ab} \Gamma_{ab} + \frac{1}{3!} e^{ia} e^{jb} e^{kc} \partial_\mu \phi_{ijk} \Gamma_{abc}. \] (13)

Here \( \gamma_\alpha \) are SO\((d-1,1)\) Dirac matrices, while \( \Gamma_a \) are SO\((11-d)\) Dirac matrices. This decomposition is suggestive of a generalized structure group with connection given by \( \Omega_\mu \). However one additional requirement is necessary before declaring this an enlargement of SO\((d-1,1) \times \text{SO}(11-d)\), and that is to ensure that the algebra generated by \( \Gamma_{ab} \) and \( \Gamma_{abc} \) closes within itself. Along this line, we note that the commutators of these internal Dirac matrices have the schematic structure

\[ [\Gamma^{(2)}, \Gamma^{(2)}] = \Gamma^{(2)}, \quad [\Gamma^{(2)}, \Gamma^{(3)}] = \Gamma^{(3)}, \quad [\Gamma^{(3)}, \Gamma^{(3)}] = \Gamma^{(6)} + \Gamma^{(2)}. \] (14)

Here the notation \( \Gamma^{(n)} \) indicates the antisymmetric product of \( n \) Dirac matrices, and the right hand sides of the commutators only indicate what possible terms may show up. The first commutator above merely indicates that the \( \Gamma_{ab} \) matrices provide a representation of the Riemannian SO\((11-d)\) structure group.

For \( d \geq 6 \), the internal space is restricted to five or fewer dimensions. In this case, the antisymmetric product \( \Gamma^{(6)} \) cannot show up, and the algebra clearly closes on \( \Gamma^{(2)} \) and \( \Gamma^{(3)} \). Working out the extended structure groups for these cases results in the expected Cremmer and Julia groups listed in the first four lines of Table I. A similar analysis follows for \( d \leq 5 \). However, in this case, we must also dualize an additional set of fields to see the hidden symmetries. For \( d = 5 \), an additional scalar arises from the dual of \( F_{\mu\nu\rho\sigma} \); this yields an addition to (13) of the form \( \Omega_\mu^{\text{additional}} = \frac{1}{4!} e^{\mu\nu\rho\sigma\lambda} F_{\mu\nu\rho\sigma} \Gamma_{123456} \). This \( \Gamma^{(6)} \) term is precisely what is necessary for the closure of the algebra of (13). Of course, in this case, we must also make note of the additional commutators

\[ [\Gamma^{(2)}, \Gamma^{(6)}] = \Gamma^{(6)}, \quad [\Gamma^{(3)}, \Gamma^{(6)}] = \Gamma^{(7)} + \Gamma^{(3)}, \quad [\Gamma^{(6)}, \Gamma^{(6)}] = \Gamma^{(10)} + \Gamma^{(6)} + \Gamma^{(2)}. \] (15)
However neither $\Gamma^{(7)}$ nor $\Gamma^{(10)}$ may show up in $d = 5$ for dimensional reasons.

The analysis for $d = 4$ is similar; however here $\Omega^{\text{additional}}_{\mu} = \frac{1}{3!} \epsilon_{\mu}^{\nu \rho \sigma} e^{ia} F_{\nu \rho \sigma} \Gamma_a \Gamma_{1234567}$. Closure of the algebra on $\Gamma^{(2)}$, $\Gamma^{(3)}$ and $\Gamma^{(6)}$ then follows because, while $\Gamma^{(7)}$ may in principle arise in the middle commutator of [15], it turns out to be kinematically forbidden. For $d = 3$, on the other hand, in additional to a contribution $\Omega^{\text{additional}}_{\mu} = \frac{1}{2! 2!} \epsilon_{\mu}^{\nu \rho} e^{ia} e^{ib} F_{\nu \rho ij} \Gamma_{ab} \Gamma_{12345678}$, one must also dualize the Kaluza-Klein vectors $g_\mu$. Doing so gives rise to a $\Gamma^{(7)}$ in the generalized connection which, in addition to the previously identified terms, completes the internal structure group to SO(16).

The remaining two cases, namely $d = 2$ and $d = 1$, fall somewhat outside the framework presented above. This is because in these low dimensions the generalized connections $\Omega_{\mu}$ derived via reduction are partially incomplete. For $d = 2$, we find

$$\Omega_{\mu}^{(d=2)} = \omega_{\mu}^{\alpha \beta} \gamma_{\alpha \beta} + Q_{\mu}^{ab} \Gamma_{ab} + \frac{1}{9} (\delta_{\mu}^{\nu} - \frac{1}{2} \gamma_{\mu}^{\nu}) e^{ia} e^{jb} e^{kc} \partial_{\nu} \phi_{ijk} \Gamma_{abc},$$

(16)

where $\gamma_{\mu \nu} = -\frac{1}{2} \epsilon_{\mu \nu} (\epsilon^{\alpha \beta} \gamma_{\alpha \beta})$ is necessarily proportional to the two-dimensional chirality matrix. Hence from a two-dimensional point of view, the scalars from the metric enter non-chirally, while the scalars from $F_{(4)}$ enter chirally. Taken together, the generalized connection [16] takes values in $\text{SO}(16)_+ \times \text{SO}(16)_-$, which we regard as the enlarged structure group. However not all generators are present because of lack of chirality in the term proportional to $Q_{\mu}^{ab}$. Thus at this point the generalized structure group deviates from the hidden symmetry group, which would be an infinite dimensional subgroup of affine $E_8$. Similarly, for $d = 1$, closure of the connection $\Omega_{\mu}^{(d=1)}$ results in an enlarged SO(32) structure group. However this is not obviously related to any actual hidden symmetry of the 1/10 split.

Until now, we have considered the spacelike reductions leading to the generalized structure groups of Table 1. For a timelike reduction, we simply interchange a time and a space direction in the above analysis. This results in an internal Clifford algebra with signature $(10 - d, 1)$, and yields the extended symmetry groups indicated in Table 3. Turning finally to the null case, we may replace one of the internal Dirac matrices with $\Gamma_+$ (where $+, -$ denote light-cone directions). Since $(\Gamma_+)^2 = 0$, this indicates that the extended structure groups for the null case are contractions the corresponding spacelike (or timelike) groups. In

[7] By postulating that the generalized structure groups survive as hidden symmetries of the full uncompactified theory, we avoid the undesirable features associated with compactifications including a timelike direction such as closed timelike curves.
addition, by removing $\Gamma_+$ from the set of Dirac matrices, we essentially end up in the case of one fewer compactified dimensions. As a result, the $G(null)$ group in $d$-dimensions must have a semi-direct product structure involving the $G(spacelike)$ group in $(d+1)$-dimensions. Of course, these groups also contain the original $ISO(10-d)$ structure group as a subgroup. The resulting generalized structure groups are given in Table 2.

5 Counting supersymmetries

Having defined a generalized holonomy for vacua with $F_{(4)} \neq 0$, we now turn to some elementary examples. For the basic objects of M-theory, the M2-brane configuration may be placed under the 3/8 (spacelike) classification, as it has three longitudinal and eight transverse directions. Focusing on the transverse directions (which is the analog of looking at $\hat{D}_\mu$), the M2-brane has generalized holonomy $SO(8)$ contained in $SO(2,1) \times SO(16)$ \[^4\]. In this case, the spinor decomposes as $(2,16) = 2(8) + 16(1)$, indicating the expected presence of 16 singlets. For the M5-brane with 6/5 (spacelike) split, the generalized $\hat{D}_\mu$ holonomy is given by $SO(5)_+ \subset SO(5,1) \times SO(5)_+ \times SO(5)_-$, with the spinor decomposition $(4,4,1) + (\bar{4},1,4) = 4(4) + 16(1)$. Since the wave solution depends on nine space-like coordinates, we may regard it as a 1/10 (null) split. In this case, it has generalized $\hat{D}_M$ holonomy $\mathbb{R}^9 \subset [SO(16) \times SO(16)] \times \mathbb{R}^{256}_{(16,16)}$. The spinor again decomposes into 16 singlets. Note, however, that since the wave is pure geometry, it could equally well be categorized under a 10/1 split as $\mathbb{R}^9 \subset ISO(9)$. Finally, the KK monopole is described by a 7/4 (spacelike) split, and has $\hat{D}_\mu$ holonomy $SU(2)_+ \subset SO(6,1) \times SO(5)$, where the spinor decomposes as $(8,4) = 8(2) + 16(1)$. In all four cases, the individual objects preserve exactly half of the 32 supersymmetries. However each object is associated with its own unique generalized holonomy, namely $SO(8), SO(5), \mathbb{R}^9$ and $SU(2)$ for the M2, M5, MW and MK, respectively.

The supersymmetry of intersecting brane configurations may be understood in a similar manner based on generalized holonomy. For example, for a M5 and MK configuration sharing six longitudinal directions, we may choose a 6/5 split. In this case, the structure group is $SO(5,1) \times SO(5)_+ \times SO(5)_-$, and the $\hat{D}_\mu$ holonomies of the individual objects are $SO(5)_+$ and $SU(2) \subset SO(5)_{diag}$, respectively. The holonomy for the combined configuration \[^8\]The reduction of $D$-dimensional pure gravity along a single null direction was analyzed by Julia and Nicolai \[^29\].
turns out to be $\text{SO}(5)_+ \times \text{SU}(2)_-$, with the spinor decomposing as $(4, 4, 1) + (\overline{4}, 1, 4) = 4(4, 1) + 4(1, 2) + 8(1, 1)$. The resulting eight singlets then signify the presence of a $1/4$ supersymmetric configuration. In principle, this analysis may be applied to more general brane configurations. However one goal of understanding enlarged holonomy is to obtain a classification of allowed holonomy groups and, as a result, to obtain a unified treatment of counting supersymmetries. We now provide some observations along this direction.

We first note the elementary fact that a $p$-dimensional representation can decompose into any number of singlets between 0 and $p$, except $(p-1)$, since if we have $(p-1)$ singlets, we must have $p$. It follows that in theories with $N$ supersymmetries, $n = N - 1$ is ruled out, even though it is permitted by the supersymmetry algebra.

In some cases, additional restrictions on $n$ may be obtained. For example, if the supersymmetry charge transforms as the $(2, 16)$ representation of $G$ when $d = 3$, then $n$ is restricted to $0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 32$ as first noted in \[5\]. No new values of $n$ are generated by $d > 3$ reps. For example, the 4 of SO(5) can decompose only into 0, 2 or 4 singlets but not 1.

We note that all the even values of $n$ discussed so far appear in the list and that $n = 30$ is absent. This is consistent with the presence of pp-waves with $n = 16, 18, 20, 22, 24, 26$ (and $n = 28$ for Type IIB) but the absence of $n=30$ noted in \[24, 21, 22, 23\]. Of course a good conjecture should not only account for the existing data but should go on to predict something new. For example, Gell-Mann’s flavor SU(3) not only accounted for the nine known members of the baryon decuplet but went on to predict the existence of the $\Omega^-$, which was subsequently discovered experimentally. For M-theory supersymmetries, the role of the $\Omega^-$ is played by $n = 14$ which at the time of its prediction had not been discovered “experimentally”. We note with satisfaction, therefore, that this missing member has recently been found in the form of a G"odel universe \[6\].

The $d = 2$ and $d = 1$ cases are more problematic since SO(16) $\times$ SO(16) and SO(32) in principle allow any $n$ except $n = 31$. So more work is required to explain the presence of M-branes at angles with $n = 0, 1, 2, 3, 4, 5, 6, 8, 16$ but the absence of $n = 7$ noted in \[20\]. Presumably, a more detailed analysis will show that only those subgroups compatible with these allowed values of $n$ actually appear as generalized holonomy groups. The beginnings of a classification of all supersymmetric $D = 11$ solutions may be found in \[30\].

We can apply similar logic to theories with fewer than 32 supersymmetries. Of course, if
M-theory really underlies all supersymmetric theories then the corresponding vacua will all be special cases of the above. However, it is sometimes useful to focus on such a sub-theory, for example the Type I and heterotic strings with \( N = 16 \). Here \( G(\text{spacelike}) = \text{SO}(d) \times \text{SO}(d) \), \( G(\text{null}) = \text{ISO}(d-1) \times \text{ISO}(d-1) \) and \( G(\text{timelike}) = \text{SO}(d-1, 1) \times \text{SO}(d-1, 1) \). If the supersymmetry charge transforms as a \((2,8)\) representation of the generalized structure group when \( d = 3 \), then \( n \) is restricted to 0, 2, 4, 6, 8, 10, 12, 16. No new values of \( n \) are generated from other \( d > 4 \) reps. Once again, the \( d = 2 \) and \( d = 1 \) cases require a more detailed analysis.

## 6 The full M-theory

We have focused on the low energy limit of M-theory, but since the reasoning that led to the conjecture is based just on group theory, it seems reasonable to promote it to the full M-theory\(^9\). When counting the \( n \) value of a particular vacuum, however, we should be careful to note the phenomenon of *supersymmetry without supersymmetry*, where the supergravity approximation may fail to capture the full supersymmetry of an M-theory vacuum. For example, vacua related by T-duality and S-duality must, by definition, have the same \( n \) values. Yet they can appear to be different in supergravity \cite{33,34}, if one fails to take into account winding modes and non-perturbative solitons. So more work is needed to verify that the \( n \) values found so far in \( D = 11 \) supergravity exhaust those of M-theory, and to prove or disprove the conjecture.

### Notes added

After this paper was posted on the archive, a very interesting paper by Hull appeared \cite{35} which generalizes and extends the present theme. Hull conjectures that the hidden symmetry of M-theory is as large as \( \text{SL}(32, \mathbb{R}) \) and that this is necessary in order to accommodate all possible generalized holonomy groups. We here make some remarks in the light of Hull’s paper:

\(^9\)Similar conjectures can be applied to M-theory in signatures \((9,2)\) and \((6,5)\) \cite{31}, the so-called M’ and M* theories \cite{32}, but the groups will be different.
Hidden symmetries:

Hull stresses that, as a candidate hidden symmetry, $SL(32, \mathbb{R})$ is background independent. However, the hidden symmetries displayed in Tables 1, 2 and 3 are also background independent. They depend only on the choice of non-covariant split and gauge in which to write the field equations. Hull’s proposal is nevertheless very attractive since $SL(32, \mathbb{R})$ contains all the groups in Tables 1, 2 and 3 as subgroups and would thus answer the question of whether all these symmetries are present at the same time.

One can accommodate $SL(32, \mathbb{R})$ by extending the $d/(11-d)$ split to include the $d = 0$ case. Then the same $SL(32, \mathbb{R})$ would appear in all three tables. At the other end, one could also include the $d = 11$ case. Then the same $SO(10, 1)$ would appear in all three tables. Our reason for not including the $d = 0$ case stems from the apparent need to make a non-covariant split and to make the corresponding gauge choice before the hidden symmetries become apparent. Moreover, from the point of view of guessing the hidden symmetries from the dimensional reduction, the $d = 0$ case would be subject to the same caveats as the $d = 1$ and $d = 2$ cases: not all group generators are present in the covariant derivative. $SL(32, \mathbb{R})$ requires $\{\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}, \Gamma^{(5)}\}$ whereas only $\{\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(5)}\}$ appear in the covariant derivative. This is an important issue deserving of further study. That $M$-theory could involve a $GL(32, \mathbb{R})$ has also been conjectured by Barwald and West.

Generalized holonomy:

Hull goes on to stress the importance of $SL(32, \mathbb{R})$ by finding solutions whose holonomy is contained in $SL(32, \mathbb{R})$ but not in Tables 1, 2 and 3. Although not all generators are present in the covariant derivative, they are all present in the commutator. So we agree with Hull that $SL(32, \mathbb{R})$ is necessary if one wants to embrace all possible generalized holonomies.

Indeed, since the basic objects of $M$-theory discussed in section 5 involve warping by a harmonic function, the $\hat{D}$ holonomy is smaller than the $\tilde{D}$ holonomy, which requires extra $\mathbb{R}^n$ factors. Interestingly enough, the $\hat{D}$ holonomy nevertheless yields the correct counting of supersymmetries.

Hull points out that, in contrast to the groups appearing in Tables 1, 2 and 3, $SL(32, \mathbb{R})$ does not obey the $n \neq N - 1$ rule of section 5, and hence $M$-theory vacua with $n = 31$ are in principle possible. Of course we do not yet know whether the required $\mathbb{R}^{31}$ holonomy

10The case for $n = 31$ has also been made by Bandos et al. in the different context of hypothetical preons of $M$-theory preserving 31 out of 32 supersymmetries.
actually appears. To settle the issue of which $n$ values are allowed, it would be valuable to do for supergravity what Berger \[16\] did for gravity and have a complete classification of all possible generalized holonomy groups. But this may prove quite difficult.

So we remain open-minded about a formulation of M-theory with $\text{SL}(32, \mathbb{R})$ symmetry, but acknowledge the need for $\text{SL}(32, \mathbb{R})$ from the point of view of generalized holonomy.

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