Theory of Networked Minority Games based on Strategy Pattern Dynamics

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Abstract

We formulate a theory of agent-based models in which agents compete to be in a winning group. The agents may be part of a network or not, and the winning group may be a minority group or not. The novel feature of the present formalism is its focus on the dynamical pattern of strategy rankings, and its careful treatment of the strategy ties which arise during the system’s temporal evolution. We apply it to the Minority Game (MG) with connected populations. Expressions for the mean success rate among the agents and for the mean success rate for agents with \( k \) neighbors are derived. We also use the theory to estimate the value of connectivity \( p \) above which the Binary-Agent-Resource system with high resource level goes into the high-connectivity state.

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I. INTRODUCTION

Agent-based models form an important part of research on complex adaptive systems [1]. For example, the self-organization of an evolving population consisting of agents competing for a limited resource has potential applications in areas such as economics, biology, engineering, and social sciences [1, 2]. The bar-attendance problem proposed by Arthur [3, 4] constitutes a typical setting of such a system in which a population of agents decide whether to go to a popular bar having limited seating capacity. The agents are informed of the attendance in past weeks, and hence the agents share common information, interact through their actions, and learn from past experience. The problem can be simplified by considering binary games, either in the form of the minority game (MG) [5, 6] or in a binary-agent-resource (B-A-R) game [7, 8]. For modest resource levels in which there are more losers than winners, the minority Game proposed by Challet and Zhang [5, 9] represents a simple, yet non-trival, model that captures many of the essential features of such a competing population.

The MG considers an odd number \( N \) of agents. At each time step, the agents independently decide between two options ‘0’ and ‘1’. The winners are those who choose the minority option. The agents learn from past experience by evaluating the performance of their strategies, where each strategy maps the available global information (i.e., the record of the most recent \( m \) winning options) to an action. One important quantity in MG is the standard deviation \( \sigma \) of the number of agents making a particular choice. This quantity reflects the performance of the population as a whole in that a small \( \sigma \) implies on average more winners per turn, and hence a higher success rate per turn per agent. In the MG, \( \sigma \) exhibits a non-monotonic dependence on the memory size \( m \) of the agents [10, 11, 12]. When \( m \) is small, there is significant overlap between the agents’ strategies. This crowd effect [7, 13, 14] leads to a large \( \sigma \), implying the number of losers is high. This is the crowded, or informationally efficient phase of MG. In the informationally inefficient phase where \( m \) is large, \( \sigma \) is moderately small and the agents perform better than if they were to decide their actions randomly.

Theoretical analysis of the MG has been the focus of many studies [4, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Mapping the MG into the language of disordered spin systems makes the machinery in statistical physics of disordered system, most noticeably the replica trick,
useful in the study of models of competing population. For the MG, calculations based on
the replica method work well for the case of large strategy pool, i.e., when the strategy pool
is much larger than the strategies actually being used in making decisions. This is referred
to as the informationally inefficient phase because information is left in the resulting bit-
string patterns for a single realization of the game. In the informationally efficient phase,
the whole pool of strategies tend to be in play during the game. The crowd-anticrowd theory
gives a physically transparent quantitative theory of the observed features in this regime,
as well as in the inefficient regime. The crowd-anticrowd theory is based on the fact that
it is the difference in the numbers of agents playing a strategy $R$ and the corresponding
anti-correlated strategy $R'$ that plays the most important role in the understanding of the
fluctuations and hence performance of the whole population.

The crowd-anticrowd theory is a microscopic approach in the sense that it follows the
strategy play of the agents in the population. While this leads to microscopically correct
equations, in practice these equations are evaluated by simply time-averaging over the path
taken by the strategy rankings. A naive time-averaging over all histories becomes more
difficult to implement as the number of ties in strategy scores increases, since such ties affect
the number of agents playing a given ranking of strategy and hence must be incorporated
explicitly in the time-average. The effect of strategy ties becomes increasingly important
as $m$ decreases, i.e., size of strategy space decreases, since the chance of ties arising will
then increase. In the efficient phase at low $m$, therefore, there are frequent ties in the
strategy performance \[4, 21\], and hence the time-averagings within the crowd-anticrowd
theory require special care. In the present paper, we present a complementary theoretical
treatment for this regime in order to explicitly account for such strategy ties. The resulting
theory amounts to a non-trivial reorganization of the time-averagings within the crowd-
anticrowd theory. Like the original crowd-anticrowd theory, it is applicable to both non-
networked and networked populations \[21, 22\]. The theory we present is based on the idea
of following the patterns of the ranking of strategies as the game evolves in time, without
knowing the details on the ranking of each strategy. The effects of tied strategies are taken
into account by considering the number of strategies belonging to each rank as the game
evolves. To illustrate the applications of the theory, we show that the theory explains the
features observed in numerical results of a networked version of MG. The theory also allows
the evaluation of the success rate of agents with a given number of connected neighbors.
The latter is important in the study of the functionality of an underlying network in a competing population.

The plan of the paper is as follows. In Sec.II, we define the MG in non-networked and networked populations. In Sec.III, we discuss the different ranking patterns based on performance of the strategies as the game evolves, and the fraction of strategies in each rank. The number of agents using a strategy belonging to a particular rank, is derived for both non-networked and networked MG in Section IV. In Sec.V, we apply the theory to derive an expression for the mean success rate in the efficient phase as a function of the connectivity in the population and compare results with those obtained by numerical simulations. An alternative way to study the mean success rate is to decompose the population into agents with different numbers of connected neighbors. An expression for the mean success rate of agents with $k$ connected neighbors is derived in Sec.VI. Results are in agreement with numerical simulations. Section VII gives a discussion on the limit of validity of the theory and shows that the theory can be extended to study B-A-R models at high connectivity.

II. THE MINORITY GAME

The basic MG comprises of $N$ agents competing to be in a minority group at each time step. The only information available to the agents is the history. The history is a bit-string of length $m$ recording the minority option for the most recent $m$ timesteps. There are a total of $2^m$ possible history bit-strings. For example, $m = 2$ has $2^2 = 4$ possible global outcome histories: 00, 01, 10 and 11. At the beginning of the game, each agent picks $s$ strategies, with repetition allowed. They make their decisions based on their strategies. A strategy is a look up table with $2^m$ entries giving the predictions for all possible history bit-strings. Since each entry can either be ‘0’ or ‘1’, the full strategy pool contains $2^{2^m}$ strategies. Adaptation is built in by allowing the agents to accumulate a merit (virtual) point for each of her strategies as the game proceeds, with the initial merit points set to zero for all strategies. Strategies that predicted the winning (losing) action at a given timestep, are assigned (deducted) one (virtual) point. At each turn, the agent follows the prediction of her best-scoring strategy. A random choice will be made for tied strategies.

The networked Minority Game explores the functionality of an underlying network in the context of a population competing for limited resource. At each timestep $t$, each agent...
(node) decides on one or two options, as in the basic MG. Each agent decides in light of (i) *global information* which takes the form of the history of the $m$ most recent global outcomes as in the basic MG, and (ii) *local information* obtained via network connections. The connections here need not be physical – it only matters that the connected neighbors are those with whom an agent can communicate. Adaptation is introduced by randomly assigning $s$ strategies to each agent. At each timestep, each agent compares the score of its own best-scoring strategy (or strategies) with the highest-scoring strategy (or strategies) among the agents to whom he is connected. The agent adopts the action of whichever strategy is highest-scoring overall, using a coin-toss to break any ties. The network can be a classical random network or take on the geometry of a growing scale-free network \cite{23}. For simplicity, we here assume a random network, where the connection between any two agents (i.e., nodes) exists with a probability $p$. Numerical results \cite{21} show that the presence of connections lowers the global performance of the population, while ensuring fairness by lowering the spread in the success rates among the agents. Here, we aim at formulating a theory that can be applied to explain the features observed in the numerical simulations. Since the effect of such connections is typically to increase the chances of strategy ties, particularly at low $m$, this motivates the present theory’s approach of tracking strategy patterns in time.

III. RANKING THE STRATEGIES

The two key ingredients for the present theory are (i) the patterns of strategy rankings according to performance, based on the strategies’ virtual points as the game proceeds; and (ii) the fraction of strategies in each rank for each ranking pattern. In the following two subsections, we discuss these two points. The discussion is valid for populations either with or without connections.

A. Ranking Pattern

As a particular run of a given game evolves, the pattern of strategy rankings also evolves. The instantaneous strategy ranking depends on the number of history bit-strings that have occurred an *odd* number of times and the next outcome will depend on whether the current
history bit-string has occurred an odd or even number of times. Both of these factors are important in the calculation of the mean success rate of the population.

Suppose we are at a given moment in the run of a game. Let \( \mu \) be the current history bit-string that the agents are using for decisions. Let \( \{t^\nu_{\text{odd}}\} \) be the set of turns (i.e., timesteps) so far in which a history \( \nu \) has occurred an odd number of times (including the initial history bit-string that starts the game) while \( \{t^\nu_{\text{even}}\} \) is the set of turns so far in which a history \( \nu \) has occurred an even number of times \[24\]. For small values of \( m \), i.e., in the efficient phase of the MG, the outcome time series exhibits the feature of anti-persistence or double periodicity \[10, 11, 12, 25, 26\]. This feature implies that all history bit-strings occur with equal probabilities. It means that for a current history \( \mu \) based on which that the agents decide, if the winning side is \( \eta \) (\( \eta \) can be 0 or 1) when \( t \in \{ t^\mu_{\text{even}} \} \), the outcome is \( 1 - \eta \) with probability unity in the next occurrence of \( \mu \). This follows that no strategies could perform better than the others in an average over time, and the virtual points (VP) of the strategies cannot show a runaway behavior, i.e., the VPs of strategies will not keep on increasing or decreasing. This property is intimately related to the fact that the Eulerian Trail is an underlying quasi-attractor of the game in this efficient regime \[26\]. By focusing on whether a history has occurred an odd or even number of times during a run, we are picking out what is essentially the most important aspect of the outcome series.

For a particular turn \( t \), we define the ranking of the strategies according to their performance up to that point in time based on the VPs of the strategies. The Rank-1 strategy or strategies have the highest VPs. The Rank-2 strategies are the second best-performing (having the second highest VPs), and so on. For small \( m \) (efficient phase), the ranking pattern of the strategies depends on the number of histories that have occurred an odd number of times. It is illustrative to consider an example for the case of \( m = 2 \) where there are 4 possible histories (00), (01), (10), (11). At \( t = 0 \), all strategies are assigned the same VP. There is only one rank, called Rank-1, of the strategies, with all the strategies belonging to this rank. This is also the case when the system returns to a situation equivalent to \( t = 0 \) after visiting every possible path from one history to another an equal number of times. At \( t = 1 \), let 00 be the corresponding history (without loss of generality, the random seed history is taken to be 00). The agents decide in a random fashion as the history has not occurred before (or has occurred an even number of times before). The outcome would be 1 (or 0) with probability 1/2. Let the outcome be 1, for example. The history bit-string will
become 001. Prior to the current $m = 2$ bit-string of 01, one history bit-string (namely 00) occurred once. The strategies are now divided into 2 ranks with

- **Rank-1**: including strategies that predict 1 for history 00
- **Rank-2**: including strategies that predict 0 for history 00

The strategy VP pattern thus consists of two ranks corresponding to assigning +1 VP for those strategies in Rank-1 and −1 for those in Rank-2.

If the outcome is also 1 at $t = 2$, the strategies that predict 1 for the history 01 will have a higher VP. Note that the history bit-string is now 0011. Prior to the current bit-string of 11, two $m = 2$ bit-strings 00 and 01 occurred once. The strategies will then be divided into three ranks after this timestep with

- **Rank-1**: including strategies that predict 1 for both histories 00 and 01
- **Rank-2**: including strategies that predict 1 for one of the two histories 00 and 01
- **Rank-3**: including strategies that predict 0 for both histories 00 and 01

The strategy VP pattern thus consists of three ranks corresponding to a VP of +2 for those strategies in Rank-1, 0 for those in Rank-2, and −2 for those in Rank-3.

If at some time $t$, the history 01 happens again, i.e., the history occurred an odd number of times prior to the one under consideration, the outcome will be 0 due to the crowd effect as the outcome was 1 in the last occurrence of the history. The Rank-1 strategies will lose and the Rank-3 strategies will win. As a result, the ranking of the strategies are then reduced to two ranks with

- **Rank-1**: including strategies that predict 1 for history 00
- **Rank-2**: including strategies that predict 0 for history 00

It is important to note that for a given $m$ in the efficient phase, there are only a finite number of patterns for the ranking of the strategies performance. In general, we have the following result for the strategy performance ranking pattern.

*If a number of $\kappa$ histories occurred an odd number of times, the strategies will be divided into $\kappa + 1$ ranks.* The ranking is as follows:

- **Rank-1**: including strategies that predicted the correct outcome for all $\kappa$ histories concerned
- **Rank-2**: including strategies that predicted the correct outcome for $\kappa - 1$ histories concerned

...
Rank-$\ell$: including strategies that predicted the correct outcomes for $\kappa-\ell+1$ histories concerned

..

Rank-$\kappa+1$: including strategies that predicted the correct outcome for 0 histories concerned

For a given value of $m$, $0 \leq \kappa \leq 2^m$ as there are $2^m$ possible histories. In the efficient phase, while the numbers of occurrence for every history are the same when averaged over a long time, $\kappa$ ($\kappa = 0, 1, 2, \ldots, 2^m$) histories may occur an odd number of times in each timestep as the game evolves. Therefore, the current strategy ranking pattern can be characterized by the parameter $\kappa$. For a timestep corresponding to $\kappa = 0$, i.e., all the histories had occurred an even number of times, there is only one rank and all the strategies lie in the same rank since they have tied VPs (zero VPs). In other words, there is only one (i.e., $C_{2^m}^0 = 1$) way to achieve a ranking pattern that consists only of Rank-1.

Next we deduce the probability $P(\kappa)$ of having $\kappa$ histories occur an odd number of times, without invoking too much known details of the dynamics. Assuming that each history has probability $1/2$ to appear as one that has occurred an odd number of times, then out of a total of $2^m$ history bit-strings, the probability $P(\kappa)$ of having $\kappa$ histories occur an odd number of times is

$$ P(\kappa) = C_{2^m}^\kappa \left(\frac{1}{2}\right)^{2^m} = C_{2^m}^\kappa / 2^{2^m}. \tag{1} $$

As the game evolves, the system maps out a path in the history space [26]. As the game goes from one history to another, it also makes transitions from one strategy performance ranking pattern to another. There may be frequent ties in the strategies’ performance. A merit of the present approach is that we take explicit account of possible ties in performance among the strategies by grouping them into the same rank. This is important in the efficient phase where there are frequent tied VPs among the strategies.

B. Fraction of strategies in each rank

The fraction of strategies in a particular rank for given value of $\kappa$ can be calculated readily. It turns out that the ratio of the number of strategies in increasing ranks (recall Rank-1 corresponds to highest VP) follows the numbers in the Pascal triangle. When all the histories occurred an even number of times, there is only one rank with a fraction unity of
strategies, i.e., all strategies, belonging to the rank. If only one history \((\kappa = 1)\) occurred an odd number of times, there are two \((= \kappa + 1)\) ranks with half (fraction \(1/2\)) of the strategies in Rank-1 and the other half (fraction \(1/2\)) in Rank-2. The ratio of the fractions of strategies in the two ranks is \(1 : 1\). If two histories \((\kappa = 2)\) occurred an odd number of times, there are three \((= \kappa + 1)\) ranks, with a fraction \(1/4\) of the strategies in Rank-1, \(1/2\) in Rank-2, \(1/4\) in Rank-3. The ratio of the fractions is \(1 : 2 : 1\). For three histories occurring an odd number of times, there are four ranks with the ratio of fractions of strategies in the ranks given by \(1 : 3 : 3 : 1\), and so on. For \(\kappa\) histories occurring an odd number of times, the fraction of strategies in Rank-1 is \(C_{\kappa}^0/2^\kappa\), the fraction of strategies in Rank-2 is \(C_{\kappa}^1/2^\kappa\), and so on. In general, the fraction of strategies in Rank-\(l\) is \(C_{\kappa}^l/2^\kappa\), where the denominator comes from \(\sum_{i=1}^{\kappa+1} C_{\kappa}^i = 2^\kappa\). The ratio of the fractions of strategies in different ranks is thus given by \(C_{\kappa}^0 : C_{\kappa}^1 : \cdots : C_{\kappa}^{\kappa-1} : C_{\kappa}^{\kappa}\), which are the numbers in the Pascal triangle.

IV. NUMBER OF AGENTS USING A BEST STRATEGY BELONGING TO RANK-\(l\)

A. Non-connected population

Consider the case of a non-connected population, i.e. basic MG or \(p = 0\) in a networked MG. As an agent uses the best-scoring strategy up to the moment of making decision, he will use the strategy with the lowest rank among the \(s\) strategies that he was randomly assigned at the beginning of the game.

Let \(\kappa\) be the number of histories that occurred an odd number of times. It is convenient for later discussions to introduce the probability

\[
\alpha_j^\kappa \equiv \frac{1}{2^\kappa} \sum_{i=j+1}^{\kappa+1} C_{\kappa}^i = \frac{1}{2^\kappa} \sum_{i=j}^{\kappa} C_{\kappa}^i ;
\]

that an agent holds a strategy with performance worse than Rank-\(j\). For an agent using a Rank-1 strategy to decide, he must possess at least one rank-1 strategy. This happens with a probability \(1 - (\alpha_1^\kappa)^s\), where \((\alpha_1^\kappa)^s\) is the probability that the agent holds \(s\) strategies that are all worse than Rank-1.

Let \(N_l\) be the number of agents who hold a strategy in Rank-\(l\) as their best strategy, for a given \(\kappa\). In the basic MG, this is also the number of agents who will use a strategy in
Rank-$l$ to decide their action. For a population of $N$ agents, it follows that for given $\kappa$

$$N_1 = N [1 - (\alpha_1^\kappa)^s]. \quad (3)$$

Similarly, for an agent using a Rank-2 strategy, he must hold at least one Rank-2 strategy and must not hold any Rank-1 strategy. Therefore,

$$N_2 = N [(\alpha_1^\kappa)^s - (\alpha_2^\kappa)^s]. \quad (4)$$

In general the number of agents holding a Rank-$l$ strategy ($l = 1, 2, \ldots, \kappa$) as their best strategy, is given by

$$N_l = N \left[ (\alpha_{l-1}^\kappa)^s - (\alpha_l^\kappa)^s \right], \quad (5)$$

with $\alpha_0^s = 1$ as given by Eq. (2). For $l = \kappa + 1$,

$$N_{\kappa+1} = N (\alpha_0^\kappa)^s. \quad (6)$$

As an example, take $s = 2$, $N = 101$, and consider a moment in the game corresponding to $\kappa = 4$. Hence we have $\kappa + 1 = 5$ ranks. The ratio of strategies in these ranks is $1 : 4 : 6 : 4 : 1$. The average number of agents using strategies in each rank in these turns is given by $N_1 = 12.23$, $N_2 = 41.03$, $N_3 = 37.88$, $N_4 = 9.47$, and $N_5 = 0.39$. These numbers change with time as the game evolves to timesteps with different values of $\kappa$. Knowing the number of agents using each rank of strategies, it is then possible to evaluate analytically the average number of agents making a particular decision and the mean success rate of the agents, as we shall discuss in later sections.

### B. Networked Population

Let $p$ be the probability that two randomly chosen agents are connected. For $p \neq 0$, the agents may decide based on a strategy that they do not hold. As a result, the number of agents who actually use a strategy for decision in a particular rank is, in general, not equal to the number of agents $N_j$ who hold a best-scoring strategy belonging to that rank $[21]$. The number of agents $\tilde{N}_l(p)$ who decide by using a Rank-$l$ strategy can formally be expressed as a sum of two terms

$$\tilde{N}_l(p) = N_l + \sum_{j=l+1}^{\kappa+1} \Delta N_{j\ell}, \quad (7)$$
where $\bar{N}_l$ is the number of agents who hold a Rank-$l$ strategy as their best-performing strategy and are not linked to agents with a better (hence lower ranking) performing strategy, and the second term represents all those using a Rank-$l$ strategy due to the presence of links. Writing $q = 1 - p$, $\bar{N}_l$ is then given by

$$\bar{N}_l = N_l q \sum_{i=1}^{l-1} N_i,$$

with $N_l$ given by Eqs. (5) and (6). In Eq. (7), $\Delta N_{jl}$ is the number of agents who hold a Rank-$j$ strategy as their best performing strategy, but they use a Rank-$l$ strategy for deciding because they are linked to agents carrying such a strategy. Note that $j > l$ because an agent will use the best performing strategy among his own strategies and his connected neighbors’ strategies in our networked MG model. Now consider an agent who does not hold a strategy in Rank-$l$ but uses a Rank-$l$ strategy held by one of his neighbors. This happens only when (i) he is not linked to any agent who holds a strategy better than Rank-$l$ (the probability is thus $q \sum_{i=1}^{l-1} N_i$) and (ii) he is linked to at least one agent who holds a Rank-$l$ strategy (the probability is $(1 - q^{N_l})$). Hence we have

$$\Delta N_{jl} = N_j (q \sum_{i=1}^{l-1} N_i) (1 - q^{N_l}).$$

Equation (7) for $\bar{N}(p)$ coupled with $\bar{N}$ is given by Eq. (8), $\Delta N_{jl}$ is given by Eq. (9), and $N_j$ is given by Eqs. (5) and (6) gives the number of agents who use a strategy in Rank-$l$ for deciding their action in a connected population.

V. APPLICATION: MEAN SUCCESS RATE

The mean success rate $\langle w \rangle$ (or mean wealth) of the agents is the average number of winners per agent per turn. This quantity reflects the global performance of the population as a whole. This quantity is also closely related to the fluctuations (or standard deviation) in the number of agents choosing a particular option as the game proceeds. A smaller fluctuation implies a higher mean success rate. Figure 1 shows $\langle w \rangle$ as a function of connectivity $p$ obtained by numerical simulations for $m = 1$ and $m = 2$ (symbols) in a population of $N = 101$ agents with $s = 2$ strategies per agent. As $p$ increases, $\langle w \rangle$ decreases, together with a drop in the spread of the success rates among the agents. Thus in the networked MG model, while higher connectivity ensures fairness, the efficiency also decreases. Qualitatively,
the drop in $\langle w \rangle$ comes about from the enhanced crowd effect as $p$ increases. Here, we derive an expression for the mean success rate as a function of connectivity $p$. Consider a timestep $t$ corresponding to $\kappa$ histories having occurred an odd number of times. Given this, $t$ may belong to $\{t^\mu_{\text{even}}\}$ or $\{t^\mu_{\text{odd}}\}$ for the particular history bit-string $\mu$ that the population is facing when making a decision, since there are $2^m - \kappa$ histories which have occurred an even number of times.

If $t \in \{t^\mu_{\text{odd}}\}$, the mean number of agents choosing the last winning option of the corresponding history is given by

$$A_{\text{odd}}(\kappa) = \sum_{l=1}^{\kappa+1} \tilde{N}_l(p) \left( \frac{\kappa - l + 1}{\kappa} \right). \quad (10)$$

This is because the Rank-$l$ strategies must have made the correct predictions for $\kappa - l + 1$ out of the $\kappa$ histories concerned. Thus, the agents using a Rank-$l$ strategy have a probability $(\kappa - l + 1)/\kappa$ of choosing the previous winning option for the history $\mu$ based on what every agent decides. Due to crowd effect, this is also the probability that the agents using a Rank-$l$ strategy lose. Therefore, they will win with a probability $1 - (\kappa - l + 1)/\kappa = (l - 1)/\kappa$. The mean success rate $w_{\text{odd}}(\kappa)$ for a given $\kappa$ and $t \in \{t^\mu_{\text{odd}}\}$ is

$$w_{\text{odd}}(\kappa) = \sum_{l=1}^{\kappa+1} \left( \frac{\tilde{N}_l(p)}{N} \right) \left( \frac{l - 1}{\kappa} \right). \quad (11)$$

If $t \in \{t^\mu_{\text{even}}\}$, the agents decide randomly and the mean number of agents choosing a particular option is $N/2$. In this case, the probability of having $n$ agents choose a particular option is

$$P_n = \binom{N}{n}/2^N, \quad (12)$$

as every agent has two options. For MG, the winners are those in the minority group. There are $n$ winners for $n < (N-1)/2$ and $(N-n)$ winners for $n \geq (N+1)/2$. The mean success rate $w_{\text{even}}$ for $t \in \{t^\mu_{\text{even}}\}$ is then given by

$$w_{\text{even}} = \sum_{n=0}^{(N-1)/2} P_n \frac{n}{N} + \sum_{n=(N+1)/2}^{N} P_n \frac{N - n}{N}. \quad (13)$$

We note that one may also make the crude approximation that $w_{\text{even}} = 1/2$, without taking into account of the fluctuations in the number of agents making identical decisions.

Given a value of $\kappa$, i.e., there are $\kappa$ histories which have occurred an odd number of times and $2^m - \kappa$ histories which occurred an even number of times, the probability of having a
timestep $t \in \{t^\mu_{\text{odd}}\}$ is $\kappa/2^m$. The probability of having a timestep $t \in \{t^\mu_{\text{even}}\}$ is $(1 - \kappa/2^m)$. The mean success rate $\langle w \rangle$ is obtained by averaging over the probabilities of having $t \in \{t_{\text{odd}}\}$ and $t \in \{t_{\text{even}}\}$ for given $\kappa$ and then averaging over the probability of having $\kappa$ odd-occurring strategies. The mean success rate is then formally given by

$$\langle w \rangle = \sum_{\kappa=0}^{2^m} P(\kappa) \left( \frac{\kappa}{2^m} w_{\text{odd}}(\kappa) + (1 - \frac{\kappa}{2^m}) w_{\text{even}} \right),$$

(14)

with $P(\kappa)$ given by Eq.(1). Equation (14) is a general expression for the mean success rate. It is valid for both non-networked and networked populations. Figure compares the analytic results (lines) for $\langle w \rangle$ from Eq.(14) as a function of $p$ for different values of $m = 1$ and $m = 3$. The results are in very good agreement with results obtained by numerical simulations. The present formalism also provides a physically transparent picture for the drop in $\langle w \rangle$ with $p$. Since there is no strategy that outperform others, those instantaneously better performing strategy or strategies have a higher chance of losing in immediate timesteps. Therefore, by forcing the agents to follow the better performing strategy of their connected neighbors actually lower their mean success rate. We emphasize that the present formalism is related to the crowd-anticrowd theory in that the agents using a strategy and those using the corresponding anticorrelated partner, have different success rates given by the term in the last parentheses in Eq.(11) since the pair of strategies must belong to different rankings.

VI. MEAN SUCCESS RATE OF AGENTS WITH DEGREE $k$

A useful way to describe the topological properties of a network is the degree distribution, which is the distribution of the number of connected neighbors among the nodes in a network. Statistical analysis has revealed that real world networks exhibit degree distributions of various kinds. For classical random graphs discussed in previous sections, the degree distribution is a Poisson distribution; while for growing networks with preferential attachment in its growth mechanism, the degree distribution exhibits power law behavior. While the analysis in the last section suffice for evaluating $\langle w \rangle$ in a random network, it will be useful to develop our formalism by focusing on agents with a given number of neighbors, i.e. a given degree. Here, we aim at studying the mean success rate of agents with degree $k$ in a networked MG.

Consider a particular agent having $k$ links to other agents. Recall that (see Eq.5) the
probability that an agent holding a Rank-\(l\) strategy as her best performing strategy is given by \((\alpha^\kappa_{l-1})^s - (\alpha^\kappa_l)^s\). Note that this is also the probability that her neighbor holds a Rank-\(l\) strategy as his best performing strategy. Combining these probabilities for an agent and his \(k\) neighbors, the probability \(\gamma(\kappa, k; l)\) of an agent with \(k\) neighbors using a Rank-\(l\) strategy is

\[
\gamma(\kappa, k; l) = (\alpha^\kappa_{l-1})^{(k+1)s} - (\alpha^\kappa_l)^{(k+1)s}.
\]  \hspace{1cm} (15)

This follows from the fact that an agent who has \(k\) links is equivalent to an agent who effectively has \((k + 1)s\) strategies in hand, with repetition allowed. Recall that the success rate or winning probability of a Rank-\(l\) strategy is \((l - 1)/\kappa\) for \(t \in \{\mu\}_{\text{odd}}\). The success rate of an agent with \(k\) links for timesteps \(t \in \{\mu\}_{\text{odd}}\) is given by

\[
w_{\text{odd}}(k, \kappa) = \sum_{l=1}^{\kappa+1} \gamma(\kappa, k; l) \frac{l - 1}{\kappa}.
\]  \hspace{1cm} (16)

We should also take into account cases corresponding to \(t \in \{\mu\}_{\text{even}}\) for which the mean success rate of an agent is given by \(w_{\text{even}}\) in Eq.(13). As a result, the success rate of an agent with degree \(k\) is given by

\[
\langle w(k) \rangle = \sum_{\kappa=0}^{2m} P(\kappa) \left( \frac{\kappa}{2m} w_{\text{odd}}(k, \kappa) + (1 - \frac{\kappa}{2m}) w_{\text{even}} \right).
\]  \hspace{1cm} (17)

For the particular case of classical random graphs, the probability of having \(k\) links in a system with \(N\) nodes (agents) for a given value of connectivity \(p\) is given by

\[
Y(k) = C_k^{N-1} p^k (1 - p)^{N-1-k}.
\]  \hspace{1cm} (18)

Combining with Eq.(17), the mean success rate in the population with connectivity \(p\) is formally given by

\[
\langle w \rangle = \sum_{k=0}^{N-1} Y(k) \langle w(k) \rangle.
\]  \hspace{1cm} (19)

Figure 2 shows the numerical and analytic results of \(\langle w(k) \rangle\) as a function of \(k\) for \(m = 1\) and \(m = 2\). The analytic results are, again, in good agreement numerical results. The numerical results are obtained from data in many runs with different values of \(p\) ranging from \(0 \leq p \leq 0.5\). For a given \(p\), data are obtained for values of \(k\) around the mean degree \(\langle k(p) \rangle\). We note that, for given degree \(k\) and fixed \(m\), \(\langle w(k) \rangle\) does not depend on \(p\), i.e., the success rate of isolated agents in a population with \(p = 0.01\) is the same as that for \(p = 0.02\) (if isolated agents exists). For the present version of networked MG, the isolated agents, i.e.,
those without any links, have the highest mean success rate. This drops in $\langle w(k) \rangle$ comes about from the fact agents with connected neighbors effectively hold a substantial portion of the strategies and hence they will join the crowd. By being isolated, one can avoid the crowd and hence achieve a higher success rate. We also checked that $\langle w \rangle$ obtained by Eq. (19) are nearly identical to those obtained by Eq. (18), for small values of $m$.

VII. DISCUSSION AND EXTENSION TO NETWORKED B-A-R MODEL

We formulated a theory applicable to agent-based models in which a population is competing to be in the minority group. The population may be networked or non-networked. The theory is based on the tendency that the system restores itself and avoids the existence of strategies that outperform others. This is the case for the efficient phase in the MG. By invoking the idea that the strategy performance ranking patterns changes as the game evolves and that only a finite number of patterns exist, it is possible to study the ranking patterns based on the number of history bit-strings that occurred an odd number of times. The fraction of strategies in each rank can be found, together with the number of agents using a strategy of Rank-$l$ in order to decide. For the case of networked populations, care must be taken to evaluate the number of agents using a strategy of Rank-$l$ through the connections. An expression for the mean success rate as a function of connectivity $p$ and $m$ can be derived. Results are found to be in good agreement with those obtained by extensive numerical simulations on networked MG. A geometrical property of networks is the degree distribution. We derived an expression for the mean success rate of agents for a given degree $k$ in a networked MG with the underlying network being a classical random graph. The results are found to be, again, in good agreement with numerical results. The present theory has the merit of taking into account possible ties in the strategies’ performance.

The validity of the formalism depends on the assumption that the system passes through quasi-Eulerian paths in the history space in the efficient phases of both non-networked and networked MG. The details of the dynamics are not important, only that we assume the equal probabilities of the occurrence of the possible outcomes. The formalism can also be applied or extended to other situations that exhibit similar features. To illustrate the idea, we consider the interesting situation in a Binary Agent Resource (B-A-R) game with high resource level in a highly connected population, i.e., for high values of $p$. The B-A-R
model in a networked population represents a networked binary version of Arthur’s El Farol Problem concerning bar-attendance $[2, 3, 4, 7]$. In the B-A-R model, the winning option is no longer decided by the minority side. Instead, there is a general global resource level $L$ ($L < N$) which is not announced to the agents. At each timestep $t$, each agent decides upon two possible options: whether to access resource $L$ (action ‘+1’) or not. The two global outcomes at each timestep, ‘resource over-used’ and ‘resource not over-used’, are denoted as ‘0’ and ‘1’. If the number of agents $n_{+1}[t]$ choosing action +1 exceeds $L$ (i.e. resource over-used and hence global outcome ‘0’) then the $N - n_{+1}[t]$ abstaining agents win. By contrast if $n_{+1}[t] \leq L$ (i.e. resource not over-used and hence global outcome ‘1’) then these $n_{+1}[t]$ agents win.

Numerical results for a high resource level B-A-R model show interesting features as a function of the connectivity $p$. Figure 3 shows the dependence on the mean success rate for $L = 90$ in a $N = 101$ population as a function of $p$. For $L > 3N/4$ in a non-connected population ($p = 0$), the system is in a frozen state in the sense that the outcome is persistently 1, i.e., the resource is persistently not over-used with $3N/4$ winners per turn. It is observed that as $p$ increases, the system moves away from the frozen state $[21]$. A high-$p$ limit is eventually reached corresponding to a state of anti-persistence or double periodicity, characterized by an outcome time series with equal probability for the two possible outcomes and a mean success rate slightly lower than $1/4$. In particular, it is observed that the value of $p$ (denoted by $p_c(m)$) above which the system reaches the high-$p$ state, depends sensitively on $m$ and increases with $m$.

The present theory can be extended to estimate $p_c(m)$. To proceed, we propose a criteria that the system is anti-persistent only if $A_{\text{odd}}(\kappa) > L$ for all $\kappa$. This can be understood easily since anti-persistence implies that for $t \in \{ \mu_{\text{odd}} \}$ for the history $\mu$ concerned, the outcome will be opposite to that in the last occurrence of the history. However, in B-A-R model, for $A_{\text{odd}}(\kappa) < L$, the winning option in the last occurrence of the history wins again, and the system ceases to be anti-persistence. We further note that $A_{\text{odd}}(\kappa)$ is a monotonically decreasing function of $\kappa$, with a minimum at $\kappa = 2^m$ when all the possible histories occurred an odd number of times. This behavior follows from Eq. (10), and we have also checked it against numerical results.

For a given high resource level $L$, as $p$ decreases from the high-$p$ state, the difference between $A_{\text{odd}}(\kappa)$ and $L$ drops. Eventually when $A_{\text{odd}}(\kappa) < L$, the system is no longer anti-
persistent for some $\kappa$. As $A_{\text{odd}}(\kappa)$ takes on its minimum value at $\kappa = 2^m$, an estimate on the breakdown of anti-persistent behavior is then given by the condition

$$A_{\text{odd}}(2^m) = L.$$  \hspace{1cm} (20)

Equation (20) can be used to estimate the critical value $p_c(m)$ for fixed resource level. To test the validity, we take a system of $N = 101$, $L = 90$, and $s = 2$ (see Fig. 3). The values of $p_c$ turns out to be $p_c = 0.0220$ for $m = 1$, $p_c = 0.0592$ for $m = 2$, $p_c = 0.2738$ for $m = 3$, and $p_c = 0.9988$ for $m = 4$, as marked by the arrows in Fig. 3. The results capture the trend that $p_c(m)$ increases with $m$. For $m = 5$, our estimate shows that the system cannot achieve an anti-persistent high-$p$ state even if $p = 1$, a result again consistent with numerical results [21]. Similarly, one may vary the resource level $L$ at given $p$ and Eq. (20) can be used to estimate the critical resource level $L_c(m)$ above which the system starts to deviate from an anti-persistent state.

The formalism can also be applied to a non-networked B-A-R game with resource level $L \gtrsim N/2$, for which the outcome series and thus the history series also exhibit anti-persistence or doubly periodic features [8]. The formalism can also be extended to cases in which the outcome (hence history) time-series shows known features other than anti-persistence. For example, it can be modified to study each of the many states that a high resource level B-A-R game passes through from the frozen state at $p = 0$ to the high-$p$ state as the connectivity varies. The starting point is to give a known outcome series, e.g., 11101110... for $m = 1$ and 1111111011110... for $m = 2$ just off the frozen state. Once the pattern of history time series is known, the part of the full history space that matters is also known and thus the ranking pattern of the strategies can be worked out [31]. Similar situations also happen in the Networked B-A-R model [21] with high resources [32]. The formalism can also be extended to study different variations on the basic MG, such as the thermal MG [33, 34] and MG with biased strategy pools [35]; and to different versions of networked MG in which neighboring agents compare their wealth instead of strategy performance [36, 37].
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Figure Captions

Figure 1: The mean success rate $\langle w \rangle$ of the agents as a function of connectivity $p$ for $m = 2$ and $m = 1$. Other parameters are $N = 101$ and $s = 2$. The symbols are results obtained by numerical simulations and the lines are analytic results obtained by using Eq.(14).

Figure 2: The mean success rate $\langle w(k) \rangle$ of agents of degree $k$ as a function of $k$ for $m = 2$ and $m = 1$. Other parameters are $N = 101$ and $s = 2$. The symbols represent numerical results obtained by carrying simulations with the range $0 \leq p \leq 0.5$. The lines give the analytic results obtained by Eq.(17).

Figure 3: The mean success rate $\langle w \rangle$ as a function of connectivity $p$ in the B-A-R model at high resource level ($L = 90$) for $m = 1, 2, 3, 4$ obtained by numerical simulations. The lines are guides to eye. Other parameters are $N = 101$ and $s = 2$. The arrows indicate the estimate of $p_c(m)$ using Eq.(20) above which the system goes into a high-$p$ state.
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