EXTENSIONS BY $K_2$ AND FACTORIZATION LINE BUNDLES

JAMES TAO AND YIFEI ZHAO

Abstract. Let $X$ be a smooth, geometrically connected curve over a perfect field $k$. Given a connected, reductive group $G$, we prove that central extensions of $G$ by the sheaf $K_2$ on the big Zariski site of $X$, studied in Brylinski-Deligne [BD01], are equivalent to factorization line bundles on the Beilinson-Drinfeld affine Grassmannian $Gr_G$. Our result affirms a conjecture of Gaitsgory-Lysenko [GL16] and classifies factorization line bundles on $Gr_G$.

CONTENTS

Introduction .......................... 1
1. Factorization line bundles on $Gr_T$ .......................... 6
2. Compatibility with the Brylinski-Deligne classification .......................... 9
3. The main theorem .......................... 19
References .......................... 24

INTRODUCTION

0.1. Origin of the problem.

0.1.1. This paper compares two kinds of data relevant to the metaplectic Langlands theory: one is $K$-theoretic, and the other has to do with factorization structures on the affine Grassmannian $Gr_G$. Let us first explain how these structures arise in the Langlands theory, and why one should expect them to be related.

0.1.2. While the classical (global) Langlands program concerns automorphic functions defined on the adelic points of a reductive group $G(\mathbb{A})$, the metaplectic theory incorporates more general topological groups which arise as coverings of $G(\mathbb{A})$. The work of J.-L. Brylinski and P. Deligne [BD01] shows that a large class of such covering groups can be obtained from certain $K$-theoretic data, which we will refer to as *Brylinski-Deligne* data. By the work of M. Weissman [We15], their L-groups can be defined and used to parametrize irreducible representations in many contexts.

0.1.3. In the geometric Langlands theory, one replaces automorphic functions by sheaves on the moduli stack $\text{Bun}_G$ of $G$-bundles over an algebraic curve $X$. This theory also has a metaplectic extension, where one uses a gerbe $\mathcal{G}$ on $\text{Bun}_G$ to form a twisted category of sheaves. In the context of $\ell$-adic sheaves, $\mathcal{G}$ is a gerbe on the étale site of $\text{Bun}_G$, whereas if one works with $\mathcal{D}$-modules, then $\mathcal{G}$ is supposed to be a $G_m$-gerbe on its de Rham prestack. In this latter context, the gerbe $\mathcal{G}$ forms part of the data defining an algebra of twisted differential operators (TDOs) on $\text{Bun}_G$, whose study goes back to the classical text of A. Beilinson and V. Drinfeld [BD91].
0.1.4. The gerbes on $\text{Bun}_G$ that are relevant for the Langlands program are supposed to be compatible with an additional structure, that of factorization. To explain it informally, let us recall that the affine Grassmannian $\text{Gr}_G$ serves as a local avatar for $\text{Bun}_G$ and is naturally defined over the base curve $X$. The factorization structure of $\text{Gr}_G$ describes how its fibers over $X$ merge as points collide. The pullback of $\mathcal{G}$ along the projection $\text{Gr}_G \rightarrow \text{Bun}_G$ is supposed to be compatible with this merging behavior; we call such gerbes factorization gerbes on $\text{Gr}_G$. R. Reich [Re12] classified all factorization gerbes on $\text{Gr}_G$. As a consequence of his classification, we know that any such gerbe descends canonically to $\text{Bun}_G$. Therefore, we may regard factorization gerbes on $\text{Gr}_G$ as geometric metaplectic data.

0.1.5. The metaplectic extensions for arithmetic vis-à-vis geometric Langlands theories involve structures that appear quite different at first sight, so the natural question to ask is whether they are related. In other words, is there a direct path between the $K$-theoretic data of Brylinski-Deligne and factorization gerbes on $\text{Gr}_G$?

\[ \text{automorphic functions on covering groups} \quad \text{twisted automorphic sheaves} \]
\[ \text{Brylinski-Deligne data} \quad \text{factorization gerbes} \]

(0.1)

0.2. A conjecture of Gaitsgory-Lysenko.

0.2.1. We now turn to a context that exists in the overlap of the arithmetic and geometric Langlands theories, and explain the work of D. Gaitsgory and S. Lysenko [GL16], [Ga18] that provides a precise answer to the question above.

0.2.2. Suppose $X$ is a smooth, geometrically connected, projective curve over $\mathbb{F}_q$. Let $\mathbb{A}$ (resp. $\mathbb{O}$) denote the topological ring of adèles (resp. integral adèles) of $X$. We fix a connected, reductive group $G$. The covering groups one studies are central extensions of the topological group $G(\mathbb{A})$ by the units in a coefficient field $\mathbb{A}$:

\[ 1 \rightarrow A^\times \rightarrow \tilde{G} \rightarrow G(\mathbb{A}) \rightarrow 1, \]

(0.2)

which are equipped with a canonical splitting over $G(\mathbb{O})$.

0.2.3. The Brylinski-Deligne data in this context are central extensions of $G$ by (the sheafified) $K_2$, as sheaves on the big Zariski site of $X$:

\[ 1 \rightarrow K_2 \rightarrow E \rightarrow G \rightarrow 1. \]

(0.3)

Such extensions form a Picard groupoid $\text{CExt}(G, K_2)$. Fixing a character $\mathbb{F}_q^\times \rightarrow A^\times$, they give rise to extensions (0.2) by taking the residue map of algebraic $K$-theory (see [BD01, §10.4-10.7]). In the same paper, the authors classified extensions (0.3) by giving a hands-on description of the Picard groupoid $\text{CExt}(G, K_2)$ that does not involve $K$-theory.

0.2.4. It is observed in [GL16] that the association of covering groups to Brylinski-Deligne data passes through the Picard (2-)groupoid of factorization gerbes on the affine Grassmannian $\text{Gr}_G$, referred to in loc.cit. as the (2-)category of geometric metaplectic data. More precisely, there is a functor

\[ \Phi_G : \text{CExt}(G, K_2) \rightarrow \text{Pic}^{\text{fact}}(\text{Gr}_G) \]

(0.4)

from Brylinski-Deligne data to the category of factorization line bundles on $\text{Gr}_G$. From a factorization line bundle on $\text{Gr}_G$, one can extract a factorization gerbe using the Kummer exact sequence. The latter is shown to be equivalent to a multiplicative, factorization gerbe on
the loop group by the work of R. Reich [Re12]. Finally, one obtains (0.2) by taking the trace of Frobenius.

0.2.5. The association of covering groups to Brylinski-Deligne data is thus seen to factor as the following composition of functors:

\[
\begin{array}{cccc}
\text{Brylinski-Deligne data} & \xrightarrow{\Phi_G} & \text{factorization line bundles on } Gr_G & \xrightarrow{\text{Kummer}} \xrightarrow{\text{Reich}} \text{factorization gerbes on } Gr_G & \xrightarrow{\text{Tr(Frob)}} \text{multiplicative gerbes on the loop group } Gr_G & \xrightarrow{\text{covering groups}} \\
\end{array}
\]

The authors of [GL16] then conjectured that \( \Phi_G \) is an equivalence of Picard groupoids (Conjecture 3.4.2 of loc.cit.). In other words, one expects that no information is lost when we pass from \( K \)-theoretic metaplectic data to factorization structures on \( Gr_G \). This is the conjecture that we affirm in the present paper.

0.3. Our results.

0.3.1. In order to state our results in a broader context, let us first remark that the construction of \( \Phi_G \) is purely geometric. In [Ga18], D. Gaitsgory defined the functor \( \Phi_G \) (0.4) over an algebraically closed ground field \( k \) and any smooth, connected curve \( X \) (not necessarily projective) over \( k \). Since Galois descent holds for line bundles and for Brylinski-Deligne data (as follows from their classification), the functor \( \Phi_G \) exists over any perfect ground field \( k \).

0.3.2. There is only a small caveat that the characteristic of \( k \) cannot divide a certain integer \( N \) that depends on the group \( G \) (see §0.1.8 of loc.cit.). As is explained there, \( N = 1 \) for \( GL_n \) and \( Sp_{2n} \), but not in general.

0.3.3. We can now state our main theorem (Theorem 3.1), which essentially asserts that whenever the functor \( \Phi_G \) is defined, it is an equivalence of Picard groupoids:

**Theorem 0.1.** Suppose \( k \) is a perfect field, \( X \) is a smooth, geometrically connected curve and \( G \) is a connected reductive group over \( k \). If the characteristic of \( k \) does not divide \( N \), then there is a canonical equivalence of Picard groupoids

\[
\text{CExt}(G, K_2) \xrightarrow{\sim} \text{Pic}^{\text{fact}}(Gr_G).
\]

0.3.4. Since \( \text{CExt}(G, K_2) \) admits a hands-on description, our main result implies a classification of factorization line bundles on \( Gr_G \). More precisely, there is a commutative triangle of equivalences (appearing as (2.27) in the text):

\[
\begin{array}{ccc}
\text{CExt}(G, K_2) & \xrightarrow{\phi_G} & \text{Pic}^{\text{fact}}(Gr_G) \\
\phi_{\text{BD}} & & \theta_G(\Lambda_T) \\
\theta_G(\Lambda_T) & & \Psi
\end{array}
\]

mapping to the classification data of Brylinski-Deligne \( \theta_G(\Lambda_T) \), and we shall see that the functor \( \Psi \) can be defined quite explicitly.

For \( G = T \) a torus, \( \theta_G(\Lambda_T) \) identifies with the (even) \( \theta \)-data considered by A. Beilinson and V. Drinfeld [BD04]; for \( G \) semisimple and simply connected, it is the (discrete) abelian group of \( W \)-invariant, integral quadratic forms on \( \Lambda_T \); and the general case is a combination of both.
0.3.5. Another application of our theorem is the following:

**Corollary 0.2.** Suppose we are under the hypothesis of Theorem 0.1 and $X$ is furthermore projective. Then every factorization line bundle on $\text{Gr}_G$ canonically descends to $\text{Bun}_G$.

Indeed, this follows from the fact that the composition:

$$\text{CExt}(G, K_2) \xrightarrow{\Phi_G} \text{Pic}^{\text{fact}}(\text{Gr}_G) \to \text{Pic}(\text{Gr}_G)$$

factors through $\text{Pic}(\text{Bun}_G)$ (c.f. [Ga18, §2.4]).

0.4. **Our strategy.**

0.4.1. We should say first and foremost that our proof of the equivalence (0.4) depends heavily on the work of Brylinski-Deligne, and fairly lightly on the nature of the functor $\Phi_G$. This has several implications:

(a) One does not need to know how $\Phi_G$ is defined in order to understand our proof; in fact, as long as $\Phi_G$ gives the correct value on regular test schemes $S \to \text{Gr}_G$ (where it is easily specified using Gersten’s resolution of $K_2$) and satisfies some basic properties, then our proof will run through.

(b) After all functors in (0.5) are defined, checking that the triangle commutes is an essential step towards the proof, and takes up a large part of our work.

(c) If there was any “deep connection” between algebraic $K$-theory and factorization structures on the affine Grassmannian, it was not revealed in our proof.

For the reason mentioned in (c), a proof of the equivalence (0.4) without using the Brylinski-Deligne classification would certainly be desirable, but the authors could not find one. \footnote{As of now, even the definition of $\Phi_G$ appeals to the Brylinski-Deligne classification, see [Ga18, §5.1].}

0.4.2. Assuming the commutativity of (0.5) (which will be proved in §2), our proof of the main theorem proceeds by checking that $\Psi$ is an equivalence for various kinds of reductive groups $G$. We summarize the key insights and make attributions below (although the main text is organized somewhat differently):

**Step 1:** $G = T$ is a (split) torus. This case amounts to showing that $\text{Pic}^{\text{fact}}(\text{Gr}_T)$ is equivalent to $\theta$-data for the lattice $\Lambda_T$. This is the content of §1. In fact, we will show that the same is true for factorization line bundles on various versions of $\text{Gr}_T$. This part of the proof relies on A. Beilinson and V. Drinfeld’s classification of factorization line bundles on $\Lambda_T$-colored divisors of $X$ (see [BD04]) and the Pic-contractibility of $\text{Ran}(X)$.

**Step 2:** $G$ is semisimple and simply connected. This case is essentially reduced to classifying line bundles on $\text{Gr}_G$ at a point of the curve $X$, and the latter has been worked out by G. Faltings [Fa03]. Since this case is also needed in proving the commutativity of (0.5), it will appear along with it in §2.

**Step 3:** The derived subgroup $G_{\text{der}}$ is simply connected. This case essentially follows from the two previous ones. More precisely, let $T_1$ be the torus $G/G_{\text{der}}$. We observe that $\text{Gr}_G$ is an étale-locally trivial fiber bundle over $\text{Gr}_{T_1}$, with typically fiber $\text{Gr}_{G_{\text{der}}}$. We then use our knowledge from Step 2 to study when a factorization line bundle on $\text{Gr}_G$ descends to $\text{Gr}_{T_1}$, and we use Step 1 to classify the ones that do come from the base.

**Step 4:** An arbitrary reductive group $G$. This follows from the previous cases, by $h$-descent of line bundles on derived schemes. \footnote{Aside from this descent technique, which is suggested to us by D. Gaitsgory, our paper lives entirely within classical (i.e., non-derived) algebraic geometry.} Steps 3 and 4 form the content of §3.
0.5. Notations.

0.5.1. We do not need the theory of $\infty$-categories. Hence terms such as categories, groupoids, prestacks, etc., are all taken in the classical sense.

0.5.2. Throughout the paper, we let $k$ be an algebraically closed field; as noted before, the more general case of a perfect field is handled using Galois descent.

0.5.3. We let $X$ be a connected, smooth algebraic curve over $k$.

0.5.4. Let $G$ be a connected, reductive group over $k$. We write $G_{\text{der}}$ for the derived subgroup of $G$, and $\tilde{G}_{\text{der}}$ for its universal cover.

0.5.5. We let $\text{Ran}(X) := \lim_{I \in \text{fSet}_{\text{surj}}} X^I$ denote the Ran space associated to $X$, where the index category is that of finite (nonempty) sets with surjections. It has the following functor of points: for every affine scheme $S$ over $k$, the set $\text{Maps}(S, \text{Ran}(X))$ classifies finite subsets $x^I = \{x^{(1)}, \ldots, x^{(|I|)}\}$ of $\text{Maps}(S, X)$.

0.5.6. A prestack $Y$ over $\text{Ran}(X)$ is a factorization prestack if its pullback $\sqcup^* Y$ along the map of disjoint union:

$$\sqcup : \text{Ran}(X)^{\times I}_{\text{disj}} \to \text{Ran}(X)$$

is identified with the restriction $Y^{\times I}_{|\text{Ran}(X)^{\times I}_{\text{disj}}}$ for each $I$, together with natural compatibility data for compositions; we refer the reader to [GL16, §2.2] for the precise definitions.

0.5.7. Let $Y$ be a factorization prestack over $\text{Ran}(X)$. A factorization line bundle on $Y$ is a line bundle $L$ together with an isomorphism $\sqcup^* L \simto \sqcup^* Y^{\times I}_{|\text{Ran}(X)^{\times I}_{\text{disj}}}$ over the factorization isomorphism $\sqcup^* Y \simto \sqcup^* Y^{\times I}_{|\text{Ran}(X)^{\times I}_{\text{disj}}}$ and with natural compatibility data for compositions, see loc.cit..

0.5.8. We write $\text{Gr}_G$ for the Beilinson-Drinfeld affine Grassmannian associated to $G$. The set $\text{Maps}(S, \text{Gr}_G)$ classifies triples $(\{x^I\}, \mathcal{P}_G, \alpha)$, where:

(a) $x^I$ is a finite subset of $\text{Maps}(S, X)$;
(b) $\mathcal{P}_G$ is a(n étale-locally trivial) $G$-bundle over $S \times X$;
(c) $\alpha$ is a trivialization of $\mathcal{P}_G$ over $S \times X - \bigcup_{i \in I} \Gamma_{x(i)}$, where $\Gamma_{x(i)}$ denotes the graph of $x^{(i)}$.

The morphism $\text{Gr}_G \to \text{Ran}(X)$ is ind-schematic and of ind-finite type, and realizes $\text{Gr}_G$ as a factorization prestack over $\text{Ran}(X)$. The base change of $\text{Gr}_G$ along $X^I \to \text{Ran}(X)$ will be denoted by $\text{Gr}_{G, X^I}$.

0.5.9. We let $\mathcal{L}G$ (resp. $\mathcal{L}^+G$) be the loop (resp. arc) group. They are factorization group prestacks over $\text{Ran}(X)$. Furthermore, the projection $\mathcal{L}G \to \text{Ran}(X)$ is ind-schematic and $\mathcal{L}^+G \to \text{Ran}(X)$ is schematic. The stack $\text{Gr}_G$ can be realized as the quotient $\mathcal{L}G/\mathcal{L}^+G$ of fpqc sheaves.

0.5.10. For a closed point $x \in X$, we denote by $\mathcal{O}_x$ the completed local ring at $x$ and $\mathcal{K}_x$ its localization at a uniformizer. The fibers of the above prestacks at a closed point $x \in X$ will be denoted by $\text{Gr}_{G,x}, \mathcal{L}_x G$, and $\mathcal{L}^+_x G$. Thus $\mathcal{L}_x G(k) \cong G(\mathcal{K}_x)$ and $\mathcal{L}^+_x G(k) \cong G(\mathcal{O}_x)$. 

Acknowledgements. We thank D. Gaitsgory for suggesting this problem to us, and for many insights that played a substantial role in its solution. We also benefited from discussions with Justin Campbell, Elden Elmanto, Quoc P. Ho, and Xinwen Zhu.

1. Factorization line bundles on \( \text{Gr}_T \)

In this section, we prove that factorization line bundles on various versions of \( \text{Gr}_T \) (e.g., combinatorial, rational) are all classified by \( \theta \)-data.

1.1. The many faces of \( \text{Gr}_T \). Suppose \( T \) is a torus over \( k \). We let \( \Lambda_T \) denote its co-character lattice. The objects we will introduce are summarized in the following commutative diagram:

\[
\begin{array}{ccc}
\text{Gr}_{T, \text{comb}} & \longrightarrow & \text{Gr}_T \\
& & \downarrow \\
& & \text{Div}(X) \otimes \Lambda_T \\
\downarrow & & \\
\text{Gr}_{T, \text{rat}} & \longrightarrow & \text{Gr}_{T, \text{lax}}
\end{array}
\]  

(1.1)

1.1.1. \( \text{Gr}_{T, \text{comb}} \). Consider an index category whose objects are pairs \((I, \lambda(I))\), where \( I \) is a finite set, and \( \lambda(I) \) is an \( I \)-family of elements in \( \Lambda_T \) (its element corresponding to \( i \in I \) is denoted by \( \lambda(i) \)). A morphism \((I, \lambda(I)) \to (J, \lambda(J))\) in this category consists of a surjective map \( \varphi : I \to J \) such that \( \lambda(j) = \sum_{i \in \varphi^{-1}(j)} \lambda(i) \) for all \( j \in J \). We set:

\[
\text{Gr}_{T, \text{comb}} := \text{colim}_{(I, \lambda(I))} X^I.
\]

\( \text{Gr}_{T, \text{comb}} \) is a factorization prestack over \( \text{Ran}(X) \). Furthermore, we have a canonical map \( \text{Gr}_{T, \text{comb}} \to \text{Gr}_T \) sending an \( S \)-point \( x^I : S \to X^I \) corresponding to \((I, \lambda(I))\) to the triple \((\{x^I(i)\}, \otimes_{i \in I} O_{X^I}(\lambda(i) \Gamma_x), \alpha)\) where \( \alpha \) is the tautological trivialization.

1.1.2. \( \text{Gr}_{T, \text{lax}} \). We let \( \text{Gr}_{T, \text{lax}} \) denote the lax prestack\(^3\) whose value at \( S \) is the category whose objects are triples \((x^I, \mathcal{P}_T, \alpha)\) as in \( \text{Gr}_T(S) \), but there is a morphism:

\[
(x^I, \mathcal{P}_T, \alpha) \to (x^J, \mathcal{P}_T', \alpha'),
\]

whenever \( x^I \subset x^J \), \( \mathcal{P}_T \cong \mathcal{P}_T' \), and the trivialization \( \alpha \) restricts to \( \alpha' \) over the complement of \( \bigcup_{j \notin I} \Gamma_x(j) \). Such a morphism is non-invertible when \( x^I \subset x^J \) is a proper inclusion.

\( \text{Gr}_{T, \text{lax}} \) is a factorization lax prestack over the lax version of the Ran space \( \text{Ran}(X)_{\text{lax}} \). Furthermore, we have a canonical map \( \text{Gr}_T \to \text{Gr}_{T, \text{lax}} \) sending \((x^I, \mathcal{P}_T, \alpha)\) to the very same object.

1.1.3. \( \text{Gr}_{T, \text{rat}} \). We define \( \text{Gr}_{T, \text{rat}} \) as a prestack whose value at \( S \) is the groupoid of \( T \)-bundles \( \mathcal{P}_T \) over \( S \times X \) equipped with a rational trivialization, i.e., for some open \( U \subset S \times X \) which is schematically dense after arbitrary base change \( S' \to S \), the \( T \)-bundle \( \mathcal{P}_T \) admits a trivialization over \( U \); we regard two rational trivializations as equivalent if they agree on the overlaps.

Even though \( \text{Gr}_{T, \text{rat}} \) does not live over any version of the Ran space, one can still make sense of factorization line bundles (or any other gadget) over \( \text{Gr}_{T, \text{rat}} \). Namely, it is a line bundle \( \mathcal{L} \) over \( \text{Gr}_{T, \text{rat}} \) together with isomorphisms:

\[
\psi_{\mathcal{P}^{(1)}_T} : \mathcal{L} \big|_{\mathcal{P}^{(1)}_T} \cong \mathcal{L} \big|_{\mathcal{P}^{(2)}_T} \otimes \mathcal{L} \big|_{\mathcal{P}^{(1)}_T},
\]

whenever \( \mathcal{P}^{(1)}_T \) (resp. \( \mathcal{P}^{(2)}_T \)) admits a trivialization over \( U^{(1)} \) (resp. \( U^{(2)} \)) such that the complements of \( U^{(1)} \) and \( U^{(2)} \) are disjoint, and \( \mathcal{P}_T \) is the gluing of \( \mathcal{P}^{(1)}_T \big|_{U^{(2)}} \) and \( \mathcal{P}^{(2)}_T \big|_{U^{(1)}} \) along

\(^3\)See [Ga15, §2] for an introduction to lax prestacks.
\(U^{(1)} \cap U^{(2)}\), where they are both trivialized. The isomorphisms \(c_{\text{pr}(1), \text{pr}(2)}\) are required to satisfy the obvious compatibility conditions in the presence of three \(T\)-bundles.

**Remark 1.1.** The objects \(\text{Gr}_{T, \text{rat}}\) and \(\text{Gr}_{T, \text{rat}}\) have analogues for a general group \(G\), but we will not use them in this paper.

1.1.4. \(\text{Div}(X) \otimes \Lambda_T\). Recall the prestack \(\text{Div}(X)\) whose value at \(S\) is the abelian group of Cartier divisors of \(S \times X\) relative to \(S\). We take \(\text{Div}(X) \otimes \Lambda_T\) as its extension of scalars to \(\Lambda_T\). There is a morphism \(\text{Div}(X) \to \text{Gr}_{\mathbb{Z}, \text{rat}}\) defined by associating to a Cartier divisor \(D\) the line bundle \(\mathcal{O}_{S \times X}(D)\). It extends to a morphism \(\text{Div}(X) \otimes \Lambda_T \to \text{Gr}_{T, \text{rat}}\).

As in the previous case, we make sense of factorization line bundles over \(\text{Div}(X) \otimes \Lambda_T\) as follows. It is a line bundle \(\mathcal{L}\) together with isomorphisms:

\[
c_{D_1, D_2} : \mathcal{L}|_{D_1 + D_2} \xrightarrow{\sim} \mathcal{L}|_{D_1} \otimes \mathcal{L}|_{D_2},
\]

whenever the support of \(D_1\) and \(D_2\) are disjoint. The isomorphisms \(c_{D_1, D_2}\) are required to satisfy the obvious compatibility conditions for three divisors.

1.2. **Classification statements.**

1.2.1. \(\theta\)-data. We recall the notion of \(\theta\)-data for a lattice \(\Lambda\) due to Beilinson-Drinfeld [BD04, §3.10.3]. The Picard groupoid \(\theta(\Lambda)\) consists of triples \((q, \mathcal{L}(\lambda), c_{\lambda, \mu})\) where:

(a) \(q \in Q(\Lambda, \mathbb{Z})\) is an integral valued quadratic form on \(\Lambda\); we use \(\kappa\) to denote its symmetric bilinear form, defined by the formula: \(\kappa(\lambda, \mu) := q(\lambda + \mu) - q(\lambda) - q(\mu)\);

(b) \(\mathcal{L}(\lambda)\) is a system of line bundles on \(X\) parametrized by \(\lambda \in \Lambda\), and

(c) \(c_{\lambda, \mu}\) are isomorphisms:

\[
c_{\lambda, \mu} : \mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) \xrightarrow{\sim} \mathcal{L}(\lambda + \mu) \otimes \omega^\kappa(\lambda, \mu),\quad (1.2)
\]

which are associative, and satisfy a \(\kappa\text{-twisted}\) commutativity condition, i.e.

\[
c_{\lambda, \mu}(a \otimes b) = (-1)^{\kappa(\lambda, \mu)} \cdot c_{\mu, \lambda}(b \otimes a).
\]

**Remark 1.2.** The authors of [BD04] work in the setting of \(\mathbb{Z}/2\mathbb{Z}\)-graded line bundles, so what we call \(\theta\)-data corresponds to what they call even \(\theta\)-data.

1.2.2. Shifted \(\theta\)-data. For later purposes, we also introduce a Picard groupoid \(\theta^+(\Lambda)\) consisting of pairs \((q, \mathcal{L}(\lambda), c^\kappa_{\lambda, \mu})\), where we replace \((1.2)\) by isomorphisms \(c^\kappa_{\lambda, \mu} : \mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) \xrightarrow{\sim} \mathcal{L}(\lambda + \mu)\) and also demand that they are associative and satisfy the \(\kappa\)-twisted commutativity condition. Clearly, we have an equivalence:

\[
\theta(\Lambda) \xrightarrow{\sim} \theta^+(\Lambda), \quad (q, \mathcal{L}(\lambda)) \mapsto (q, \mathcal{L}(\lambda) \otimes \omega^q(\lambda)).
\]

**Lemma 1.3.** There is a canonical equivalence of Picard groupoids \(\text{Pic}^\text{fact}(\text{Gr}_{T, \text{comb}}) \xrightarrow{\sim} \theta(\Lambda_T)\).

**Proof.** Given a factorization line bundle over \(\text{Gr}_{T, \text{comb}}\), we denote its pullback along the inclusion \(X \to \text{Gr}_{T, \text{comb}}\) corresponding to \((\{1\}, \lambda)\) by \(\mathcal{L}(\lambda)\), and its pullback along \(X^2 \to \text{Gr}_{T, \text{comb}}\) corresponding to \((\{1, 2\}, (\lambda, \mu))\) by \(\mathcal{L}(\lambda, \mu)\). The factorization isomorphism shows that there is an isomorphism \(\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu)|_{x^2 - \Delta} \xrightarrow{\sim} \mathcal{L}(\lambda, \mu)\). It extends to an isomorphism

\[
\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu) \xrightarrow{\sim} \mathcal{L}(\lambda, \mu) \otimes \mathcal{O}_{X^2}(-\kappa(\lambda, \mu)\Delta),\quad (1.3)
\]

for some unique integer \(\kappa(\lambda, \mu)\); its dependency on \(\lambda, \mu\) is bilinear, by consideration of a triple of line bundles. Since the factorization isomorphisms are \(\Sigma_2\)-invariant, so are the isomorphisms \((1.3)\). One deduces from this fact that \(\kappa\) is also symmetric.
We now argue that $\kappa(\lambda, \lambda)$ is even. The $\Sigma_2$-invariance of the factorization isomorphism $L^{(\lambda)} \otimes L^{(\lambda)}|_{X^2 - \Delta} \xrightarrow{\sim} L^{(\lambda, \lambda)}$ allows us to descend it to an isomorphism of line bundles (with the same notation) over $\text{Sym}^2(X) - \Delta$, which then extends into an isomorphism:

$$L^{(\lambda)} \otimes L^{(\lambda)} \xrightarrow{\sim} L^{(\lambda, \lambda)} \otimes \mathcal{O}_{\text{Sym}^2(X)}(-q(\lambda)\Delta),$$

for some uniquely defined integer $q(\lambda)$. On the other hand, (1.4) pulls back to (1.3) along the map $X^2 \to \text{Sym}^2(X)$. Since the latter map pulls $\mathcal{O}_{\text{Sym}^2(X)}(\Delta)$ back to $\mathcal{O}_{X^2}(2\Delta)$, we find $q(\lambda) = \frac{1}{2}\kappa(\lambda, \lambda)$.

Since $L^{(\lambda, \mu)}$ restricts to $L^{(\lambda+\mu)}$ along $\Delta \hookrightarrow X^2$, the isomorphism (1.3) restricts to a system of isomorphisms $c_{\lambda, \mu}$ as in (1.2). However, the identification $\mathcal{O}_{X^2}(\Delta)|_{\Delta} \xrightarrow{\sim} \omega_X$ is only $\Sigma_2$-invariant up to a sign, which implies that the isomorphisms $c_{\lambda, \mu}$ satisfy the $\kappa$-twisted commutativity condition. Thus the triple $(q, L^{(\lambda)}, c_{\lambda, \mu})$ is a well-defined object of $\theta(\Lambda_T)$. Checking that the resulting functor $\text{Pic}_{\text{fact}}(\text{Gr}_{T, \text{comb}}) \to \theta(\Lambda_T)$ is an equivalence is straightforward. □

1.2.3. We can now state the main result of this section. By pulling back along the morphisms of (1.1), we obtain a diagram of Picard groupoids, where the leftmost equivalence comes from Lemma 1.3:

$$\begin{array}{ccc}
\theta(\Lambda_T) & \xrightarrow{\sim} & \text{Pic}_{\text{fact}}(\text{Gr}_{T, \text{comb}}) \\
& & \xrightarrow{(a)} \text{Pic}_{\text{fact}}(\text{Gr}_T) \\
& (c) & \downarrow \\
& & \text{Pic}_{\text{fact}}(\text{Gr}_{T, \text{lax}}) \\
& & \xrightarrow{(b)} \text{Pic}_{\text{fact}}(\text{Gr}_{T, \text{rat}})
\end{array}$$

**Proposition 1.4.** All morphisms in (1.5) are equivalences.

*Proof.* We shall deduce from existing literature how each of the labeled maps is an equivalence:

(a) By [BD04, §3.10.7, Proposition], the composition of the top row defines an equivalence: $\text{Pic}_{\text{fact}}(\text{Gr}_T) \xrightarrow{(a)} \text{Pic}_{\text{fact}}(\text{Gr}(X) \otimes \Lambda_T)$. This shows that the map $(a)$ has a left inverse.

(b) By [Ba12, Proposition 5.2.2], the map $\text{Gr}_{T, \text{lax}} \to \text{Gr}_{T, \text{rat}}$ induces an equivalence after fppf sheafification. Hence pulling back defines an equivalence $\text{Pic}(\text{Gr}_{T, \text{rat}}) \xrightarrow{\sim} \text{Pic}(\text{Gr}_{T, \text{lax}})$. One immediately checks that the additional data defining factorization structures on both are also equivalent. Hence $(b)$ is an equivalence.

(c) By [Zh16, Theorem 4.3.9(2)], pulling back along $\text{Gr}_T \to \text{Gr}_{T, \text{rat}}$ defines an equivalence on rigidified line bundles. On the other hand, every factorization line bundle on $\text{Gr}_T$ pulls back to one along the unit section $\text{Ran}(X) \to \text{Gr}_T$, which is canonically trivial by Lemma 1.3 (applied to the trivial group). Thus a factorization line bundle on $\text{Gr}_T$ descends to a line bundle on $\text{Gr}_{T, \text{rat}}$, and the result has a canonical factorization structure as well, so we have an equivalence $\text{Pic}_{\text{fact}}(\text{Gr}_{T, \text{rat}}) \xrightarrow{\sim} \text{Pic}_{\text{fact}}(\text{Gr}_T)$. This shows that $(c)$ is an equivalence.

The undecorated maps in (1.5) are now equivalences by the 2-out-of-3 property. □

**Remark 1.5.** When $X$ is proper, [Ca17, Theorem 2.3.3] shows that the map $\text{Div}(X) \otimes \Lambda_T \to \text{Gr}_{T, \text{rat}}$ is an isomorphism of prestacks, which immediately implies that factorization line bundles on them are equivalent.

**Remark 1.6.** We have the following equivalence for any smooth, fiberwise connected, affine group scheme $G$ over $X$:

$$\text{Pic}_{\text{fact}}(\text{Gr}_{G, \text{rat}}) \xrightarrow{\sim} \text{Pic}_{\text{fact}}(\text{Gr}_{G, \text{lax}}) \xrightarrow{\sim} \text{Pic}_{\text{fact}}(\text{Gr}_G).$$

This is because the results [Ba12, Proposition 5.2.2] and [Zh16, Theorem 4.3.9(2)] both hold in this general context.
2. Compatibility with the Brylinski-Deligne classification

In this section, we first summarize Brylinski-Deligne’s classification of central extensions of $G$ by $K_2$. Then we construct a functor from $\text{Pic}^{\text{act}}(\text{Gr}_G)$ to the same classification data and we prove that it is compatible with Gaitgory’s functor $\Phi_G$.

2.1. Extensions by $K_2$.

2.1.1. This subsection serves as a summary of the main result of [BD01]. Let $G$ be a connected, reductive group over $k$. Fix a maximal torus $T \subset G$. We recall the notations $\theta(\Lambda_T)$ and $\theta^+(\Lambda_T)$ for the $\theta$-data associated to $\Lambda_T$ (see §1.2.1-1.2.2).

2.1.2. We let $K_2$ denote the Zariski sheafification of the presheaf on $\text{Sch}^{\text{aff}}$ that sends any $S \to X$ to $K_2(S)$. For a connected, reductive group $G$, we let $\text{CExt}(G, K_2)$ denote the Picard groupoid of central extensions

$$1 \to K_2 \to E \to G \to 1,$$

in the category of Zariski sheaves of groups on $\text{Sch}^{\text{aff}}_X$. This is Picard groupoid of Brylinski-Deligne data.

2.1.3. We will first define a functor

$$\text{CExt}(T, K_2) \to \theta^+(\Lambda_T).$$

Indeed, given a central extension $E$ of $T$, we construct a triple $(q, \mathcal{L}^{(\lambda)}, c_{\lambda, \mu}^+)$ in $\theta^+(\Lambda_T)$ from the following procedure:

(a) The commutator in $E$ defines a map $\text{comm} : T \otimes T \to K_2$ of Zariski sheaves on $\text{Sch}^{\text{aff}}_X$.

For any $\lambda, \mu \in \Lambda_T$, the composition: $\mathbb{G}_m \otimes \mathbb{G}_m \xrightarrow{\lambda \otimes \mu} T \otimes T \to K_2$ is some integral multiple of the universal symbol $\{ - , - \}$ (c.f. §3.8 of loc.cit.). We call this integer $\kappa(\lambda, \mu)$. One then checks that $\kappa(-, -)$ is the bilinear form associated to some quadratic form $q$.

(b) Consider the projection $p : \mathbb{G}_m \times X \to X$. Using the vanishing result $R^1 p_* K_2 = 0$ of Sherman (c.f. §3.1 of loc.cit.), we find an exact sequence of Zariski sheaves on $X$:

$$1 \to p_* K_2 \to p_* E \to p_* T \to 1.$$ 

Pushing out along the symbol map $p_* K_2 \to K_1 \cong \mathcal{O}_X^\times$, we obtain a multiplicative $\mathcal{O}_X^\times$-torsor over $p_* T$. The line bundle $\mathcal{L}^{(\lambda)}$ then arises as the fiber of the section of $p_* T$ defined by $\lambda \in \Lambda_T$.

(c) Note that the aforementioned multiplicative $\mathcal{O}_X^\times$-torsor over $p_* T$ equips the system $\{ \mathcal{L}^{(\lambda)} \}$ with the multiplicative structure $c_{\lambda, \mu}^+$. Its failure of commutativity is measured by $\kappa$, as desired.

2.1.4. It is proved in loc.cit. that (2.2) is an equivalence of Picard groupoids. We record here the unshifted version of this equivalence:

$$\text{CExt}(T, K_2) \xrightarrow{\sim} \theta(\Lambda_T),$$

i.e., it is the composition of (2.2) with the equivalence of Picard groupoids $\theta^+(\Lambda_T) \xrightarrow{\sim} \theta(\Lambda_T)$ sending $\mathcal{L}^{(\lambda)}$ to $\mathcal{L}^{(\lambda)} \otimes \omega_X^{-q^{(\lambda)}}$.

2.1.5. We now turn to the general case. Note that there is always a functor

$$\text{CExt}(G, K_2) \xrightarrow{\text{res}} \text{CExt}(T, K_2) \xrightarrow{\sim} \theta(\Lambda_T) \to Q(\Lambda_T, \mathbb{Z}),$$

whose image lands in the $W$-invariant part of $Q(\Lambda_T, \mathbb{Z})$. Thus, we may speak of the quadratic form $q$ associated to an extension (2.1).
2.1.6. Suppose $G$ is semisimple and simply connected. Then Theorem 4.7 of \textit{loc.cit.} asserts that (2.4) defines an equivalence: \textbf{CExt}($G, K_2$) \iso $Q(\Lambda_T, Z)^W$. Thus for a semisimple, simply connected group $G$, there is a map which associates theta data to a $W$-invariant quadratic form:

$$Q(\Lambda_T, Z)^W \rightarrow \theta(\Lambda_T).$$

(2.5)

2.1.7. Let $\widetilde{G}_\text{der}$ be the simply connected cover of $G_\text{der}$. It contains a maximal torus $\widetilde{T}_\text{der}$ which is the preimage of $T_\text{der}$. We now let $\theta_G(\Lambda_T)$ denote the Picard groupoid classifying:

- a theta datum $(q, L^{(\lambda)}, c_{\lambda,\mu})$ for $\Lambda_T$, where $q$ is Weyl-invariant;
- an isomorphism $\varphi$ between the following theta data for $\Lambda_{T_{\text{der}}}$:
  - the restriction of $(q, L^{(\lambda)}, c_{\lambda,\mu})$ to $\Lambda_{T_{\text{der}}}$;
  - the theta data associated to $q|_{\Lambda_{T_{\text{der}}}}$ via (2.5).

In other words, $\varphi$ consists of isomorphisms between line bundles, preserving their ($\omega$-twisted) multiplicative structure. We shall call $\theta_G(\Lambda_T)$ the Picard groupoid of \textit{enhanced} theta data. By definition, we have a functor:

$$\Phi_{BD} : \textbf{CExt} (G, K_2) \rightarrow \theta_G(\Lambda_T),$$

(2.6)

obtained by restrictions to $T$ and $\widetilde{T}_{\text{der}}$. The main theorem of [BD01] is that (2.6) is an equivalence of Picard groupoids, i.e., central extensions of $G$ by $K_2$ are classified by enhanced theta data.

2.2. Gaitsgory’s functor $\Phi_G$.

2.2.1. Under the condition that the characteristic of $k$ does not divide an integer $N$ that depends on $G$, D. Gaitsgory [Ga18] constructed a functor:

$$\Phi_G : \textbf{CExt} (G, K_2) \rightarrow \text{Pic}^{\text{fact}}(\text{Gr}_G).$$

(2.7)

Only two features of $\Phi_G$ will be used in proving its compatibility with the Brylinski-Deligne classification. We first cast them in informal language:

- Given a central extension (2.1), its image under $\Phi_G$ is a line bundle $L$ over $\text{Gr}_G$ with additional factorization data; for a \textit{regular} affine scheme $S \rightarrow \text{Gr}_G$, we need the restriction $L|_S$ to be given by “taking the residue” along $S \times X \rightarrow S$.

- Suppose $G = T$ is a torus; we need the functor $\Phi_T$ to factor through the Picard groupoid of \textit{multiplicative} factorization line bundles on $LT$, and for a closed point $x \in X$, we need the multiplicative structure on $L_x T$ to be given by the “tautological” one.

We will make precise what features (a) and (b) mean in the rest of this subsection, and explain how they can be deduced from \textit{loc.cit.}

2.2.2. Let $S$ be a \textit{regular} affine scheme over $k$ and $\pi : X \rightarrow S$ be a smooth relative curve, whose fibers are geometrically connected. Furthermore, suppose we have a finite set $\{x^i\}$ of sections $x^i : S \rightarrow X$. Let $\Gamma_{x^i}$ denote the (scheme-theoretic) union of their images, and $U_{x^i} := X \setminus \Gamma_{x^i}$ be its complement.

We will construct a functor, referred to hereafter as \textit{taking the residue} along $\pi$:

$$\left\{ \begin{array}{l} \text{K}_2\text{-gerbes } \mathcal{G} \text{ on } X \text{ with } \\
\text{neutralization } \gamma \text{ over } U_{x^i} \end{array} \right\} \rightarrow \text{Pic}(S).$$

(2.8)

Indeed, the datum $(\mathcal{G}, \gamma)$ is equivalent to a section of $\iota^*\text{K}_2[2]$ over $X$, where $\iota : \Gamma_{x^i} \hookrightarrow X$ is the closed immersion. On the other hand, the Gersten resolution of $\text{K}_2$ on $X$ shows that $\iota^*\text{K}_2[2]$ is
quasi-isomorphic to the complex concentrated in degrees $[-1, 0]$:

$$\bigoplus_{\nu \in \mathcal{I}} (t_{\eta(\nu)})_* K_1(\eta) \rightarrow \bigoplus_{\text{codim}(\nu) = 1} (t_{\nu})_* \mathbb{Z} \quad (2.9)$$

where $t_{\eta(\nu)}$ (resp. $t_{\nu}$) denotes the inclusion of the generic point of the $i$th section (resp. codimension-one point $\nu$ of $\Gamma_{x,i}$). On the other hand, $K_1[1]$ over $S$ is quasi-isomorphic to:

$$(t_{\eta})_* K_1(\eta) \rightarrow \bigoplus_{\text{codim}(\nu) = 1} (t_{\nu})_* \mathbb{Z}.$$

Thus the image of (2.9) under $\pi$ maps to $K_1[1]$ via summation. Hence a section of $\mathcal{K}_2[2]$ over $\mathfrak{X}$ gives rise to a section of $K_1[1] \cong \mathcal{O}_{X}^\times[1]$, i.e., a line bundle on $S$.

2.2.3. Given an extension $E (2.1)$ and a map $S \rightarrow \text{Gr}G$ specified by the triple $\{(x^I)\}, \mathcal{P}_G, \alpha$ where $\mathcal{P}_G$ is Zariski locally trivial, we obtain a (Zariski) $K_2$-gerbe $\mathcal{G}$ over $S \times X$, which classifies an $E$-torsor $\mathcal{P}_E$ equipped with an identification of its induced $G$-torsor $(\mathcal{P}_E)_G \sim \mathcal{P}_G$. The trivialization $\alpha$ gives rise to a neutralization $\gamma$ of $\mathcal{G}$ over $U_{x,i}$.

Suppose $S$ is regular, then $(\mathcal{G}, \gamma)$ produces a line bundle on $S$ by taking the residue (2.8) along $\pi : S \times X \rightarrow S$. This process also applies when $\mathcal{P}_G$ is only étale locally trivial, since étale locally on $S$ the bundle $\mathcal{P}_G$ becomes Zariski locally trivial (see [DS95]). The fact that $\Phi_G(E)|_S$ naturally agrees with this line bundle is the content of [Ga18, §2.3]; this is what we meant in part (a) of §2.2.1.

2.2.4. Recall that a multiplicative line bundle $\mathcal{L}$ on $\mathcal{L}G$ amounts to the additional isomorphism:

$$\text{mult}^* \mathcal{L} \sim \mathcal{L} \boxtimes \mathcal{L} \quad (2.10)$$

over $\mathcal{L}G \times \mathcal{L}G$ that satisfies the cocycle condition on the triple product. If $\mathcal{L}$ is a factorization line bundle, then being multiplicative amounts to an isomorphism (2.10) that is compatible with the factorization structures on both sides.

We let $\text{Pic}^{\text{fact, } \times}(\mathcal{L}G)$ (resp. $\text{Pic}^{\text{fact, } \times}_{\mathcal{L}G}(\mathcal{L}G)$) denote the Picard groupoid of multiplicative factorization line bundles on $\mathcal{L}G$ (resp. together with a trivialization as such over $\mathcal{L}^+G$). Clearly, there is a descent functor:

$$\text{Pic}^{\text{fact, } \times}_{/\mathcal{L}^+G}(\mathcal{L}G) \rightarrow \text{Pic}^{\text{fact}}(\text{Gr}G).$$

We now state part (b) of §2.2.1 as a lemma:

**Lemma 2.1.** (a) The functor $\Phi_T$ factors through $\text{Pic}^{\text{fact, } \times}_{/\mathcal{L}^+T}(\mathcal{L}T)$, i.e., $\Phi_T(E)$ has a canonical multiplicative structure over $\mathcal{L}T$, trivialized over $\mathcal{L}^+T$;

(b) Over a closed point $x \in X$, the restriction of the above multiplicative structure to the abstract group $T(\mathcal{K}_x)^4$ agrees with that on the $k^\times$-torsor coming from the push-out of

$$0 \rightarrow K_2(\mathcal{K}_x) \rightarrow E(\mathcal{K}_x) \rightarrow T(\mathcal{K}_x) \rightarrow 0 \quad (2.11)$$

along the residue map $K_2(\mathcal{K}_x) \rightarrow k^\times$. The same holds over any field extension $k \subset k'$.

**Remark 2.2.** Part (b) makes sense since $\Phi_T(E)|_T$ for $t \in T(\mathcal{K}_x)$ agrees with the $k^\times$-torsor induced from (2.11); this follows from the description of $\Phi_T(E)$ on regular test schemes (§2.2.3).

---

4I.e., the group of $k$-points of $\mathcal{L}_xT$. 
Proof of Lemma 2.1. Recall that $\mathcal{L} := \Phi_T(E)$ is constructed as follows. The datum $E$ can be interpreted as a pointed morphism $e : X \times BT \to B^2 K_2$. Let $K$ denote the full $K$-theory spectrum, regarded as a Zariski sheaf on $\text{Sch}^{aff}$. Then $e$ lifts (non-uniquely) to some $\tilde{e} : X \times BT \to K_{\geq 2}$ ([Ga18, §5.3.1]). Hence the data $\{(x^I), \mathcal{P}_T, \alpha\}$ of an $S$-point of $\text{Gr}_T$ (where we may again assume $\mathcal{P}_T$ to be Zariski-locally trivial) give us a section of $K_{\geq 2}$ over $S \times X$ with support on $\Gamma_x$. The line bundle $\mathcal{L}_{\tilde{e}|_S}$ is then constructed using the map:

\[ \tau^{\leq 0} \pi_* \mathcal{L}_x K_{\geq 2} \to O_S^*[1] \tag{2.12} \]

(c.f. (3.2.2) of loc.cit.). For two lifts $\tilde{e}$ and $\tilde{e}'$, we need to produce a canonical isomorphism $\mathcal{L}_{\tilde{e}} \sim \mathcal{L}_{\tilde{e}'}$. This is done as follows:

(a) for $S$ the spectrum of an Artinian $k$-algebra, (2.12) factors through $\tau^{\leq 0} \pi_* \mathcal{L}_x K_{\geq 2}$, so we obtain a canonical isomorphism $\mathcal{L}_{\tilde{e}}|_S \sim \mathcal{L}_{\tilde{e}'}|_S$.

(b) there exists an isomorphism $\mathcal{L}_{\tilde{e}} \sim \mathcal{L}_{\tilde{e}'}$ which restricts to the one in (a) for any $S$ the spectrum of an Artinian $k$-algebra ($§ 5.3.4-6$ of loc.cit.).

We now claim that $\mathcal{L}_{\tilde{e}}|_{\mathcal{L}_T}$ acquires a canonical multiplicative structure. Indeed, $\tilde{e}$ induces a morphism $X \times T \to K_{\geq 2}$ of group sheaves. Given $S$-points $t, t'$ of $\mathcal{L}T$ over the same point $x^I \in \text{Ran}(X)$, we may view them both as maps $D_{x^I} \to X \times T$. There is a canonical homotopy between $\tilde{e}(t) + \tilde{e}(t')$ and $\tilde{e}(tt')$ as maps $D_{x^I} \to K_{\geq 2}$. Under the map $K_{\geq 2} \to \pi_* \mathcal{L}_x K_{\geq 2}$, we obtain a canonical homotopy between the corresponding sections of $\mathcal{L}_{\tilde{e}'} \sim \mathcal{L}_{\tilde{e}'}$ under (2.12).

It remains to check that for two lifts $\tilde{e}$ and $\tilde{e}'$, the canonical isomorphism $\mathcal{L}_{\tilde{e}} \sim \mathcal{L}_{\tilde{e}'}$ is compatible with the multiplicative structures on both sides. This amounts to checking that the following diagram of line bundles over $\mathcal{L}T \times \mathcal{L}T$ commutes:

\[
\begin{array}{ccc}
\text{mult}^* \mathcal{L}_{\tilde{e}} & \longrightarrow & \mathcal{L}_{\tilde{e}} \boxtimes \mathcal{L}_{\tilde{e}} \\
\downarrow & & \downarrow \\
\text{mult}^* \mathcal{L}_{\tilde{e}'} & \longrightarrow & \mathcal{L}_{\tilde{e}'} \boxtimes \mathcal{L}_{\tilde{e}'}.
\end{array}
\]

It suffices to test the commutativity over $S$ the spectrum of an Artinian $k$-algebra. Note again that for such $S$, (2.12) factors through $\tau^{\leq 0} \pi_* \mathcal{L}_x K_{\geq 2}$, so the construction of the multiplicative structure does not require a lift of $e$. Therefore, we have equipped $\mathcal{L}$ with a canonical multiplicative structure over $\mathcal{L}T$.

Part (b) of the lemma is immediate from the above construction, applied to $S = \text{Spec}(k)$ (or $\text{Spec}(k')$ for a field extension $k \subset k'$). \qed

2.3. Compatibility: torus case.

2.3.1. Fix a torus $T$. Recall the equivalence of Proposition 1.4:

\[ \text{Pic}^{\text{fact}}(\text{Gr}_T) \sim \Theta(\Lambda_T). \tag{2.13} \]

The goal of this subsection is to prove:

**Lemma 2.3.** The following diagram of Picard groupoids commutes functorially in $T$:

\[ \text{CExt}(T, K_2) \xrightarrow{\Phi_T} \text{Pic}^{\text{fact}}(\text{Gr}_T) \xleftarrow{\Theta} \Theta(\Lambda_T). \tag{2.14} \]
2.3.2. Notations. Fix an object \( E \) of \( \mathbf{CExt}(T, K_2) \). We denote its image in \( \theta^+(\Lambda_T) \) under (2.2) by \( (q, L^{(\xi)}, c^{\pm}_{\mu, \nu}) \), and its image under \( \Phi_T \) by \( L \). The image of \( L \) in \( \theta(\Lambda_T) \) will be denoted by \( (q', L^{(\xi)}, c_{\mu, \nu}) \). We ought to show:

(a) \( q = q' \);

(b) there is a canonical system of isomorphisms:

\[
L^{(\xi)} \cong L^{(\xi)} \otimes \omega_X^{q(\xi)}
\]  

(2.15)

which respects \( c^+_{\mu, \nu} \) and \( c_{\mu, \nu} \).

2.3.3. Quadratic forms. We first show \( q = q' \) by checking that their bilinear forms \( \kappa \) and \( \kappa' \) agree. Fixing a closed point \( x \) in \( X \) and any co-character \( \mu \in \Lambda_T \), we will show that \( \kappa(-, \mu) \) and \( \kappa'(-, \mu) \) define the same character \( T(k') \to \mathbb{G}_m(k') \) for every field extension \( k \subset k' \); this will imply that \( \kappa = \kappa' \).

We now further fix a uniformizer of the completed local ring \( t \in O_x \). This provides an isomorphism \( k[t] \cong O_x \), so we regard \( t^\mu \) as an element of \( T(O_x) \). Consider the central extension (2.11) corresponding to \( x \in X \). Pushing-out along the residue map \( K_2(\mathcal{X}_x) \to k^\times \), we obtain central extension:

\[
0 \to k^\times \to E' \to T(\mathcal{X}_x) \to 0.
\]

So the conjugation action of \( T(O_x) \) on the fiber of \( E(\mathcal{X}_x) \to T(\mathcal{X}_x) \) at \( t^\mu \) induces a map:

\[
T(O_x) \to k^\times.
\]

(2.16)

We will calculate this map (and its variant for a field extension \( k \subset k' \)) in two ways.

**Step 1.** We first show that the map (2.16) is given by the composition:

\[
T(O_x) \xrightarrow{ev} T(k) \xrightarrow{\kappa(-, \mu)} k^\times.
\]

Indeed, recall from §2.1.3(a) that the composition \( \mathbb{G}_m \otimes \mathbb{G}_m \xrightarrow{\lambda \otimes \mu} \mathbb{G}_m \xrightarrow{ev} K_2 \) is the \( \kappa(\lambda, \mu) \)-multiple of the universal symbol. Thus the map:

\[
\mathbb{G}_m(\mathcal{X}_x) \otimes \mathbb{G}_m(\mathcal{X}_x) \xrightarrow{\lambda \otimes \mu} T(\mathcal{X}_x) \xrightarrow{ev} K_2(\mathcal{X}_x) \xrightarrow{res} k^\times
\]

is the \( \kappa(\lambda, \mu) \)-multiple of the Contou-Carrère symbol \( \{f, g\} := (\text{ord}(g)/\text{ord}(f))(0) \). Hence the conjugation action of \( f \in \mathbb{G}_m(O_x) \) (through \( \lambda \)) on \( E' \) is given by \( c' \sim \{f, t\}^{\kappa(\lambda, \mu)} c' \). Note that \( \{f, t\} = f(0), \) as required.

For a field extension \( k \subset k' \), the above computation holds without modification.

**Step 2.** We now calculate the map (2.16) alternatively as follows. Recall the canonical multiplicative structure on \( \mathcal{L} \big|_{\mathcal{LT}} \) from Lemma 2.1. It induces a strong \( \mathcal{L}^+T \)-equivariance structure on \( \mathcal{L} \) (over \( \text{Gr}_T \), c.f. [GL16, §7.3.4]) with respect to the trivial left \( \mathcal{L}^+T \)-action; in other words, the twisted product \( \mathcal{L} \otimes \mathcal{L} \) on the convolution Grassmannian \( \text{Gr}_{T, X^2} \) is identified with the pullback of \( \mathcal{L}^{(2)} \) along the action map \( \text{Gr}_{T, X^2} \to \text{Gr}_{T, X^2} \), in a way that is compatible with the factorization structure of \( \mathcal{L} \).

\[\text{Step 2.} \] Indeed, for every \( \lambda \in \Lambda_T \), suppose \( z \sim z^{\kappa(\lambda, \mu)} \) and \( z \sim z^{\kappa'(\lambda, \mu)} \) define the same map \( \mathbb{G}_m(k') \to \mathbb{G}_m(k') \) for all field extension \( k \subset k' \). By suitably choosing \( k' \), we can ensure that \( (k')^\times \) contains an element of infinite order. Thus \( \kappa(\lambda, \mu) \) agrees with \( \kappa'(\lambda, \mu) \).
Furthermore, its value at $\text{Gr}^m_{T,x}$ is given by the conjugation action (2.16). We claim now that the map (2.16) is given by

$$T(0_2) \xrightarrow{\kappa} T(k) \xrightarrow{\kappa(-\mu)} k^\kappa.$$  

Indeed, this follows from the fact that for a factorization line bundle $\mathcal{L}$ on $\text{Gr}_T$ with associated bilinear form $\kappa'$, every strong $\mathcal{L}^+T$-equivariance structure acts on $t^\mu \in \text{Gr}_{T,x}$ through the composition $\mathcal{L}^+T \xrightarrow{\kappa} T \xrightarrow{\kappa(-\mu)} \mathbb{G}_m$ (c.f. [GL16, §7.4]).

Again for a field extension $k \subset k'$, the above computation holds without modification. This finishes the proof that $\kappa = \kappa'$.

2.3.4. Isomorphisms of line bundles. We now construct the isomorphisms (2.15). The strategy is to first identify $\mathcal{L}^{(\lambda)}$ with the twist of $\mathcal{L}^{(\lambda)}_+$ by some power of the tangent sheaf $\mathcal{T}_X$, and then determine this power.

**Step 1.** Consider the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Define $\mathcal{G}^{(\lambda)}$ as the $\mathbf{K}_2$-gerbe on $X \times X$ classifying a $\text{pr}_1^* E$-torsor $\mathcal{P}_E$, together with an isomorphism $(\mathcal{P}_E)_T \xrightarrow{\sim} \mathcal{O}(\lambda \Delta)$. Then $\mathcal{G}^{(\lambda)}$ comes equipped with a neutralization $\gamma$ over $\mathcal{O}(X \times X - \Delta)$. The line bundle $\mathcal{L}^{(\lambda)}$ arises from $(\mathcal{G}^{(\lambda)}, \gamma)$ by taking the residue along $\text{pr}_1$ (c.f. §2.2.2).

Let $X \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$ be the deformation of the diagonal embedding to the normal cone, constructed as the blow-up of $X \times X \times \mathbb{A}^1$ along the diagonally embedded subscheme $X \times \{0\}$, where we then remove the strict transform of $X \times X \times \{0\}$. It has the following features:

(a) $X \times \{t\} \hookrightarrow X \times \{t\}$ identifies with $X \hookrightarrow X \times X$ for $t \neq 0$;
(b) $X \times \{0\} \hookrightarrow X \times \{0\}$ identifies with the embedding of $X$ as the zero section inside the total space of the tangent sheaf $T_X$.
(c) there is a canonical map $\mathcal{X} \xrightarrow{\text{pr}_1, \text{pr}_2} X \times X$ which is identity for $t \neq 0$, and the canonical projection $T_X \xrightarrow{\gamma} X \times X$ at $t = 0$.

Consider $\mathcal{Z} := X \times \mathbb{A}^1$ as a divisor inside $\mathcal{X}$. We define $\widetilde{\mathcal{G}}^{(\lambda)}$ as the $\mathbf{K}_2$-gerbe classifying a $\text{pr}_1^* E$-torsor $\mathcal{P}_E$ over $\mathcal{X}$, together with an isomorphism $(\mathcal{P}_E)_T \xrightarrow{\sim} \mathcal{O}(\lambda \mathcal{Z})$. Note that $\widetilde{\mathcal{G}}^{(\lambda)}$ is equipped with a neutralization over $\mathcal{O}(X \times \mathcal{Z})$, so we may take the residue along $\text{pr}_1$ to obtain a line bundle $\widetilde{\mathcal{L}}^{(\lambda)}$ over $X \times \mathcal{Z}$.

Tautologically, $\widetilde{\mathcal{L}}^{(\lambda)}|_{X \times \{t\}}$ identifies with $\mathcal{L}^{(\lambda)}$ for $t \neq 0$. On the other hand, every line bundle on $X \times \mathbb{A}^1$ canonically identifies with the pullback of a line bundle from $X$. Thus, we obtain an isomorphism $\mathcal{L}^{(\lambda)}|_{X \times \{t\}} \xrightarrow{\sim} \widetilde{\mathcal{L}}^{(\lambda)}|_{X \times \{t\}}$. This shows that $\mathcal{L}^{(\lambda)}$ arises from the residue of $(\mathcal{G}^{(\lambda)}_T, \gamma_T)_X$ along $p : T_X \rightarrow X$, where:

(a) $\mathcal{G}^{(\lambda)}_T$ is the $\mathbf{K}_2$-gerbe on $T_X$ classifying a $\rho E$-torsor $\mathcal{P}_E$, together with an isomorphism $(\mathcal{P}_E)_T \xrightarrow{\sim} \mathcal{O}(\lambda\{0\})$, where $\{0\}$ denotes the zero section $X \rightarrow T_X$; and
(b) $\gamma_T$ is the tautological neutralization of $\mathcal{G}^{(\lambda)}_T$ over $T_X - \{0\}$.

**Step 2.** In the above description, suppose we replaced $p : T_X \rightarrow X$ by the trivial line bundle $A^1_{\mathcal{X}} \rightarrow X$; then the line bundle arising from taking the residue of the analogously defined pair $(\mathcal{G}^{(\lambda)}_{A^1_{\mathcal{X}}}, \gamma_{A^1_{\mathcal{X}}})$ would identify with $\mathcal{L}^{(\lambda)}_+$. Indeed, this follows from comparing the construction of §2.2.2 with that of §2.1.3(b).

We now explain an alternative way to arrive at $\mathcal{L}^{(\lambda)}$ via twisting the line bundle $A^1_{\mathcal{X}} \rightarrow X$ in the above construction. Consider the $\mathbb{G}_m$-action on $A^1_{\mathcal{X}}$ by scaling. The pair $(\mathcal{G}^{(\lambda)}_{A^1_{\mathcal{X}}}, \gamma_{A^1_{\mathcal{X}}})$ admits a $\mathbb{G}_m$-equivariance structure. Hence $L^{(\lambda)}_+$ (the total space of $\mathcal{L}^{(\lambda)}_+$) is equipped with a fiberwise $\mathbb{G}_m$-action. Since $\mathcal{G}^{(\lambda)}_T$ identifies with the twisted product $\mathcal{G}^{0} \boxtimes \mathcal{G}^{(\lambda)}_T$ on the total space
$T^\times_X \times_{\mathbb{G}_m} \mathbb{A}^1_X$ (where $\mathbb{S}^0$ denotes the trivial gerbe), we find $L^{(\lambda)} \simeq T^\times_X \times_{\mathbb{G}_m} L^\lambda_+$. In other words, suppose the fiberwise $\mathbb{G}_m$-action on $L^\lambda_+$ is given by some character $q_1(\lambda) \in \mathbb{Z}$, then there is a canonical isomorphism:

$$L^{(\lambda)} \simeq T^\times_X \otimes L^\lambda_+.$$  \hfill (2.17)

**Step 3.** We now calculate the character $q_1(\lambda)$.

It suffices to do so at a closed point $x \in X$. The line $L_+^{(\lambda)} \mid_{x \in X}$ admits a simple description as follows (c.f. §2.1.3). Evaluating $E$ at $\mathbb{G}_{m,x} := \text{Spec}(k[t, t^{-1}])$, we obtain an exact sequence:

$$0 \to K_2(k[t, t^{-1}]) \to E(k[t, t^{-1}]) \to T(k_x[t, t^{-1}]) \to 0,$$  \hfill (2.18)

and consequently a $K_2(k[t, t^{-1}])$-torsor $E(z)$ at every point $z \in T(k[t, t^{-1}])$. The line $L_+^{(\lambda)} \mid_{x \in X}$ is the $k^\times$-torsor induced from $E(t^\lambda)$ along the residue map $K_2(k[t, t^{-1}]) \to k^\times$.

To unburden the notation, we again use $L_+^{(\lambda)}$ to denote this line; the $\mathbb{G}_m(k)$-action on it also admits a simple description. Take $a \in \mathbb{G}_m(k)$, the action by $a^{\eta(\lambda)}$:

$$a^{\eta(\lambda)} : L_+^{(\lambda)} \mid_{x \in X} \simeq L_+^{(\lambda)} \mid_{x \in X}$$  \hfill (2.19)

is given as follows.

(a) Consider the scaling map $k[t, t^{-1}] \to k[t, t^{-1}]$, $t \mapsto t \cdot a$. It induces a group automorphism $E(k[t, t^{-1}]) \xrightarrow{\sim} E(k[t, t^{-1}])$, covering the analogously defined automorphism on $T(k[t, t^{-1}])$.

In particular, we obtain a map $a_* : E(t^\lambda) \to E(t^\lambda a^\lambda)$ (incompatible with the $K_2(k[t, t^{-1}])$-torsor structures.)

After inducing to $k^\times$-torsors, we obtain a map compatible with the $k^\times$-torsor structures:

$$a_* : L_+^{(\lambda)} \to L_+(t^\lambda a^\lambda) := E(t^\lambda a^\lambda)_{k^\times},$$

since $a_* : K_2(k[t, t^{-1}]) \to K_2(k[t, t^{-1}])$ induces the identity on $k^\times$.

(b) On the other hand, every element in $T(k[t])$ admits a lift to $E(k[t])$, up to an element from $K_2(k[t])$ (as follows from $R^1 p_* K_2 = 0$ for $p : \mathbb{A}^1_S \to S$, c.f. [BD01, §3.1]) Hence we have another map $E(t^\lambda) \to E(t^\lambda a^\lambda)$, defined as right-multiplying by any lift of $a^\lambda \in T(k[t])$.

Inducing along $K_2(k[t, t^{-1}]) \to k^\times$, we again obtain a map of $k^\times$-torsors:

$$R_{a^\lambda} : L_+^{(\lambda)} \to L_+(t^\lambda a^\lambda).$$

Note that this map is independent of the choice of the lift.

(c) The automorphism (2.19) identifies with the composition $R_{a^\lambda}^{-1} \circ a_*$. 

**Step 4.** We shall now deduce two identities:

$$q_1(2\lambda) - \kappa(\lambda, \lambda) = 2 \cdot q_1(\lambda)$$  \hfill (2.20)

$$4 \cdot q_1(\lambda) = q_1(2\lambda)$$  \hfill (2.21)

The combination of these identities will show that $q_1(\lambda) = \frac{1}{2} \kappa(\lambda, \lambda) = q(\lambda)$. Then the desired isomorphism follows from (2.17).

---

6Caution: we do not yet know that $q_1(\lambda)$ depends quadratically on $\lambda$. 
Proof of (2.20). This follows from the multiplicative structure on $E(k[t,t^{-1}])$. Indeed, consider the following commutative diagrams:

$$
\begin{array}{ccc}
L_+^{(2\lambda)} & \xrightarrow{a_*} & L_+((2\lambda)2\lambda) \\
\downarrow & \cong & \downarrow \\
L_+^{(\lambda)} \otimes L_+^{(\lambda)} & \rightarrow & L_+((\lambda)\lambda) \otimes L_+((\lambda)\lambda)
\end{array}
$$

where vertical arrows witness the multiplicativity of $L_+^{(\lambda)}$. The first diagram commutes because $a_*$ defines a group homomorphism on $E(k[t,t^{-1}])$. The second diagram commutes because it calculates the commutator $[a^\lambda, t^\lambda] = E_+(2\lambda) - E_+(\lambda,\lambda)$ on $k^\times$, whose residue identifies with $a^\lambda(\lambda,\lambda)$.

Now, tracing through the horizontal arrows gives rise to the identity $a_{\lambda(2\lambda)-\lambda(\lambda)} = a^2 q_{\lambda(\lambda)}$ in $k^\times$. Since the same calculation is valid for any field extension $k \subset k'$, we obtain (2.20).

Proof of (2.21). This follows from the functoriality of $E(k[t,t^{-1}])$ with respect to the double covering map $sq(t) = t^2$ on $k[t,t^{-1}]$. Note that $sq_* : E(k[t,t^{-1}]) \rightarrow E(k[t,t^{-1}])$ induces a quadratic map of $k^\times$-torsors: $sq_* : L_+^{(\lambda)} \rightarrow L_+^{(2\lambda)}$.

On the other hand, we have the following commutative diagrams:

$$
\begin{array}{ccc}
L_+^{(\lambda)} & \xrightarrow{(a^2)_*} & L_+((\lambda)2\lambda) \\
\downarrow & \cong & \downarrow \\
L_+^{(2\lambda)} & \rightarrow & L_+((2\lambda)2\lambda)
\end{array}
\begin{array}{ccc}
L_+^{(\lambda)} & \xrightarrow{R_{2\lambda}} & L_+((\lambda)2\lambda) \\
\downarrow & \cong & \downarrow \\
L_+^{(2\lambda)} & \rightarrow & L_+((2\lambda)2\lambda)
\end{array}
$$

The first diagram commutes tautologically. The second diagram commutes because $a^{2\lambda}$ belongs to the subgroup $T(k) \hookrightarrow T(k[t,t^{-1}])$, and we may first lift $a^{2\lambda}$ to $E(k)$ so that its image in $E(k[t,t^{-1}])$ is fixed by the automorphism $sq_*$. Tracing through the horizontal maps and using the quadraticity of vertical maps, we find $a^{4 q_1(\lambda)} = a^{q_1(2\lambda)}$ in $k^\times$. Again because the same calculation is valid for any field extension $k \subset k'$, we obtain (2.21).

\(\square\) (Lemma 2.3)

2.4. Compatibility: general case.

2.4.1. We now return to the general case of a reductive group $G$. Appealing to the equivalence (2.13), we obtain a functor:

$$\text{Pic}^{\text{fact}}(\text{Gr}_G) \xrightarrow{\text{res}} \text{Pic}^{\text{fact}}(\text{Gr}_T) \xrightarrow{\sim} \theta(\Lambda_T) \rightarrow Q(\Lambda_T, \mathbb{Z}). \tag{2.22}$$

Proposition 2.5. Suppose $G$ is semisimple and simply connected. Then (2.22) defines an equivalence: $\text{Pic}^{\text{fact}}(\text{Gr}_G) \xrightarrow{\sim} Q(\Lambda_T, \mathbb{Z})^W$.

In this subsection, we will first prove Proposition 2.5, and then use it to deduce the general compatibility result between Gaitgory functor $\Phi_G$ and the Brylinski-Deligne classification.

2.4.2. We use the notation $\text{Pic}^c(\text{Gr}_G)$ to denote the Picard groupoid of line bundles on $\text{Gr}_G$ together with a rigidification at the unit section $e : \text{Ran}(X) \hookrightarrow \text{Gr}_G$; the notation $\text{Pic}^c(\text{Gr}_G, X_1)$ carries an analogous meaning. Since factorization line bundles on $\text{Ran}(X)$ are canonically trivial (c.f. Lemma 1.3), we have a forgetful functor $\text{Pic}^{\text{fact}}(\text{Gr}_G) \rightarrow \text{Pic}^c(\text{Gr}_G)$.

\(\text{i.e.}, \) the $k^\times$-action on the two lines intertwines $k^\times \rightarrow k^\times$, $a \mapsto a^2$. 

2.4.3. We first prove Proposition 2.5 in the case where \( G \) is simple and simply connected. We note that in this case, the abelian group \( Q(\Lambda_T, \mathbb{Z})^W \) is isomorphic to \( \mathbb{Z} \), where a generator is given by the minimal \( W \)-invariant quadratic form \( q_{\text{min}} \), uniquely specified by the property that \( q(\alpha) = 1 \) for any short coroot \( \alpha \).

We fix a point \( x \in X \). The calculation of Picard schemes \( \text{Pic}^e(\text{Gr}_{G,x}) \) in [Zh16, §3.4] proves that there are isomorphisms:

\[
\text{Pic}^\text{fact}(\text{Gr}_G) \sim \text{Pic}^e(\text{Gr}_G) \sim \text{Pic}^e(\text{Gr}_{G,x}),
\]

(2.23)
given by pulling back along \( \text{Gr}_{G,x} \to \text{Gr}_G \). On the other hand, the result of G. Faltings [Fa03] shows that \( \text{Pic}^e(\text{Gr}_{G,x}) \) is also isomorphic to \( \mathbb{Z} \) (in particular, it is discrete), and the generator of \( \text{Pic}^e(\text{Gr}_{G,x}) \) is a certain line bundle \( L_{\text{min}} \) satisfying the following property:

(*) Let \( L_{\text{det}} \) be the determinant line bundle on \( \text{Gr}_{G,x} \), whose fiber at an \( S \)-point \( (\mathcal{P}_G, \mathcal{P}_G|_{\text{D}_x}) \sim \mathcal{P}_G^0 \) is the relative determinant of the lattices \( \mathfrak{g}_{\mathcal{P}_G}, \mathfrak{g}_{\mathcal{P}_G}^0 \subset \mathfrak{g}(K_x) \). Then there is an isomorphism \( (L_{\text{min}}) \otimes h \sim L_{\text{det}} \).

In order to show that (2.22) is an isomorphism onto \( Q(\Lambda_T, \mathbb{Z})^W \), it suffices to show that for some nonzero integer \( d \), the image of \( (L_{\text{min}}) \otimes h \) (regarded as an element in \( \text{Pic}^\text{fact}(\text{Gr}_G) \) via (2.23)) equals \( d \cdot q \). We will prove this statement for \( d = 2h \) by calculating the image of \( L_{\text{det}} \).

Note that \( L_{\text{det}} \) has a natural factorization structure (c.f. [GL16, §5.2.1]). By tracing through the functors in (2.22), we see that its image is the quadratic form \( q_{\text{det}} \) whose associated bilinear form \( \kappa_{\text{det}} \) equals:

\[
\kappa_{\text{det}}(\lambda, \mu) = \sum_{\alpha \in \Phi} \langle \lambda, \alpha \rangle \langle \mu, \alpha \rangle = \text{Kil}(\lambda, \mu),
\]

where Kil stands for the Killing form. On the other hand, \( h \) is defined so that \( \text{Kil} = 2h \cdot \kappa_{\text{min}} \). Thus \( q_{\text{det}} = 2h \cdot q_{\text{min}} \) as desired.

2.4.4. In order to handle the general case, we first note a cohomological vanishing result that will also be useful later. We continue to fix a \( k \)-point \( x \in X \). Recall that for a dominant cocharacter \( \lambda \in \Lambda_T^+ \), we have the affine Schubert cell \( \text{Gr}_{G,x}^\leq \lambda \to \text{Gr}_{G,x} \) such that \( \text{Gr}_{G,x} \) is isomorphic to the infinite union colim \( \text{Gr}_{G,x}^\leq \lambda \). When \( G \) is semisimple and simply connected, each \( \text{Gr}_{G,x}^\leq \lambda \) is integral.

**Lemma 2.6.** Suppose \( G \) is semisimple and simply connected. Then for any \( \lambda \in \Lambda_T^+ \), we have \( H^i(\text{Gr}_{G,x}^\leq \lambda, \mathcal{O}) = 0 \) for \( i \geq 1 \).

**Proof.** Let \( I \) denote the Iwahori subgroup of \( L_x^+ G \) and \( \text{Fl}_{G,x} := L_x^+ G/I \) be the affine flag variety. The \( I \)-orbits of \( \text{Fl}_{G,x} \) are parametrized by the affine Weyl group \( W^{\text{aff}} \). Let \( \text{Fl}_{G,x}^w \) denote the orbit corresponding to \( w \in W^{\text{aff}} \) and \( \text{Fl}^\text{w-clos}_{G,x} \) its closure. We note that the projection \( \text{Fl}_{G,x} \to \text{Gr}_{G,x} \) is a flat-locally trivial fiber bundle with typical fiber \( G/B \). Furthermore, for any \( \lambda \in \Lambda_T^+ \), there is a Cartesian square:

\[
\begin{array}{ccc}
\text{Fl}^w_{G,x} & \to & \text{Fl}_{G,x} \\
\downarrow & & \downarrow \\
\text{Gr}^\leq \lambda_{G,x} & \to & \text{Gr}_{G,x}
\end{array}
\]

where \( w \) is the longest element in the double coset of \( \lambda \), after we identify \( \Lambda_T^+ \) with \( W/W^{\text{aff}}/W \). Since \( k \to \text{R} \Gamma(G/B, \mathcal{O}) \), we reduce the proof to showing \( k \to \text{R} \Gamma(\text{Fl}^w_{G,x}, \mathcal{O}) \).
We now make an argument similar to that for finite dimensional Schubert cells. Namely, for each simple (affine) reflection \( s \in W^{\aff} \), we let \( P_s := I \cup (IsI) \) denote the corresponding minimal parahoric subgroup. Suppose \( w = s_1 \cdots s_l \) is an reduced expression. Then we have an affine Bott-Samelson resolution:

\[
\Fl_G^{\leq w} := P_{s_1} \times P_{s_2} \times \cdots \times P_{s_l} / I \to \Fl_G^{\leq w},
\]

where the \( I \)-superscripts indicate quotients by anti-diagonal actions. Since each \( P_s / I \) is isomorphic to \( \PP^1 \), the scheme \( \Fl_G^{\leq w} \) is an iterated \( \PP^1 \)-bundle. Thus, we reduce to showing that \( \O_{\Fl_G^{\leq w}} \) has vanishing higher direct image along (2.24), and this follows from the same proof as the usual Bott-Samelson resolution, c.f. [Br04, Theorem 2.2.3].

\textbf{Remark 2.7.} Lemma 2.6 can be seen as an affine version of the Borel-Weil-Bott theorem and is likely to be known, but the authors could not find a reference.

2.4.5. We now prove Proposition 2.5 in the general case. Suppose \( G \) has simple factors \( \{ G_j \}_{j \in J} \). It suffices to prove that pulling back along the factors \( \Gr_{G_j} \hookrightarrow \Gr_G \) defines an equivalence of Picard groupoids:

\[
\Pic_{\text{fact}}(\Gr_G) \xrightarrow{\sim} \prod_{j \in J} \Pic_{\text{fact}}(\Gr_{G_j}).
\]  

(2.25)

Note that this morphism fits into a commutative diagram of Picard groupoids:

\[
\begin{array}{c}
\Pic_{\text{fact}}(\Gr_G) \\
\downarrow \text{(2.25)}
\end{array}
\begin{array}{c}
\Pic^c(\Gr_G) \\
\downarrow \text{(b)}
\end{array}
\begin{array}{c}
\Pic^c(\Gr_{G,x}) \\
\downarrow \text{(a)}
\end{array}
\]

\[
\prod_{j \in J} \Pic_{\text{fact}}(\Gr_{G_j}) \xrightarrow{\sim} \prod_{j \in J} \Pic^c(\Gr_{G_j}) \xrightarrow{\sim} \prod_{j \in J} \Pic^c(\Gr_{G_j,x})
\]

where the lower row consists of equivalences, c.f. (2.23). We note that the cohomological vanishing Lemma 2.6 for \( i = 1 \) implies that (a) is an equivalence.\(^8\) That (b) is an equivalence follows from [Zh16, Lemma 3.4.2] and the proof of [Zh16, Lemma 3.4.3]. Together, these facts imply that (c) is an equivalence.

2.4.6. Finally, we argue that the left square is Cartesian, which would imply that (2.25) is an equivalence. Concretely, this means that given a rigidified line bundle \( \mathcal{L} \) over \( \Gr_G \) (which passes to \( \boxtimes_{j \in J} \mathcal{L}_j \) over \( \prod_{j \in J} \Gr_{G_j} \) via the equivalence (c)), the datum needed to upgrade it to a factorization structure on \( \mathcal{L} \):

\[
\varphi : \mathcal{L}^{(2)}|_{X^2 - \Delta} \xrightarrow{\sim} \mathcal{L}^{(1)} \boxtimes \mathcal{L}^{(1)}
\]

is equivalent to that of factorization structures \( \varphi_j \) on each \( \mathcal{L}_j \). We note that the collection \( \{ \varphi_j \}_{j \in J} \) defines a factorization structure \( \boxtimes_{j \in J} \varphi_j \) on \( \mathcal{L} \) and conversely a factorization structure \( \varphi \) on \( \mathcal{L} \) defines \( \varphi_j \) by restriction to the \( j \)th unit section \( X^2 \times X^2 \times \cdots \times X^2 \hookrightarrow \Gr_{G_j, X^2} \). Thus it remains to show:

\textbf{Claim 2.8.} Any \( \mathcal{L} \in \Pic^c(\Gr_G) \) has at most one factorization structure compatible with its rigidification.

\(^8\)Recall: suppose \( X, Y \in \text{Sch}_K \) are connected schemes of finite type with base points, and \( X \) is integral, projective with \( H^1(X, \O_X) = 0 \). Then \( \Pic^c(X) \times \Pic^c(Y) \xrightarrow{\sim} \Pic^c(X \times Y) \) (see [Ha13, Exercise III.12.6]).
Indeed, any two such factorization structures differ by an automorphism \( \beta \) of \( \mathcal{L}(2) \mid_{X^2 - \Delta} \) that restricts to identity along the unit section. Since \( \text{Gr}_{G, X^2} \mid_{X^2 - \Delta} \) is an ind-integral ind-scheme over \( X^2 - \Delta \), it suffices to show that \( \beta \) becomes the identity after restricting to the fibers at \( k \)-points of \( X^2 - \Delta \). The latter follows from the discreteness of \( \text{Pic}^e(\text{Gr}_{G, x} \times \text{Gr}_{G, y}) \), which in turn follows from that of \( \text{Pic}^e(\text{Gr}_{G, x}) \) and Lemma 2.6. □ (Proposition 2.5)

2.4.7. For a semisimple and simply connected group \( G \), we obtain a map:

\[
Q(\Lambda_T, \mathbb{Z})^W \to \theta(\Lambda_T)
\]

by first lifting an element of \( Q(\Lambda_T, \mathbb{Z})^W \) to \( \text{Pic}^\text{fact}(\text{Gr}_G) \) using the isomorphism of Proposition 2.5, and then mapping to \( \theta(\Lambda_T) \). By Lemma 2.3, the above functor identifies with (2.5).

2.4.8. Recall the Picard groupoid \( \theta_G(\Lambda_T) \) of §2.1. We will define a functor:

\[
\text{Pic}^\text{fact}(\text{Gr}_G) \to \theta_G(\Lambda_T) \quad (2.26)
\]

Given \( \mathcal{L} \in \text{Pic}^\text{fact}(\text{Gr}_G) \), we will construct a theta datum \((q, \mathcal{L}(\lambda), c_{\lambda, \mu})\) for \( \Lambda_T \) as well as an isomorphism \( \varphi \) of two corresponding theta data for \( \Lambda_{\tilde{T} \text{der}} \).

Indeed, \((q, \mathcal{L}(\lambda), c_{\lambda, \mu})\) is the image of \( \mathcal{L} \) under the first two maps of (2.22). On the other hand, \( \mathcal{L} \) restricts to a factorization line bundle on \( \text{Gr}_{\tilde{T} \text{der}} \); under the same two maps, we obtain a theta datum \((q|_{\Lambda_{\tilde{T} \text{der}}}, \tilde{\mathcal{L}}(\lambda), \tilde{c}_{\lambda, \mu})\). By §2.4.1, this is the theta datum associated to \( q|_{\Lambda_{\tilde{T} \text{der}}} \) under (2.5). Therefore, we obtain \( \varphi \) from the commutativity datum of the diagram:

\[
\begin{array}{ccc}
\text{Pic}^\text{fact}(\text{Gr}_G) & \xrightarrow{\text{res}} & \text{Pic}^\text{fact}(\text{Gr}_T) \\
\downarrow & & \downarrow \\
\text{Pic}^\text{fact}(\text{Gr}_{G \text{der}}) & \xrightarrow{\text{res}} & \text{Pic}^\text{fact}(\text{Gr}_{T \text{der}}) \\
\end{array} \sim \theta(\Lambda_T) \quad (2.27)
\]

2.4.9. We now state the main compatibility result, generalizing Lemma 2.3:

**Proposition 2.9.** The following diagram of Picard groupoids commutes functorially in \( G \):

\[
\begin{array}{ccc}
\text{CExt}(G, K_2) & \xrightarrow{\Phi_G} & \text{Pic}^\text{fact}(\text{Gr}_G) \\
\downarrow & & \downarrow \\
\theta_G(\Lambda_T) & & \theta_G(\Lambda_T) \\
\end{array} \quad (2.27)
\]

**Proof.** Given a central extension of \( G \) by \( K_2 \), we have to construct an isomorphism between two elements of \( \theta(\Lambda_T) \) and check that it respects the isomorphism denoted by \( \varphi \). The isomorphism comes from the commutativity datum of Lemma 2.3, and the required compatibility follows from its functoriality with respect to the map of tori \( \tilde{T} \text{der} \to T \). □

3. The main theorem

3.1. Statement and reduction.
3.1.1. In this section, we prove the main theorem of the paper:

**Theorem 3.1.** The functor $\Phi_G$ (2.7) is an equivalence of Picard groupoids.

Using the commutativity of (2.27) and the fact that $\Phi_{\text{BD}}$ is an equivalence, we have already obtained some special cases of Theorem 3.1:

(a) the case $G = T$ is a torus follows from Proposition 1.4, as $\theta_G(\Lambda_T)$ becomes $\theta(\Lambda_T)$;
(b) the case $G$ semisimple, simply connected follows from Proposition 2.5, as $\theta_G(\Lambda_T)$ becomes the (discrete) abelian group $Q(\Lambda_T, Z)^{W}$.

3.1.2. We now perform a reduction of Theorem 3.1 to the case where $G_{\text{der}}$ is simply connected. Choose an exact sequence of groups:

$$1 \to T_2 \to \tilde{G} \to G \to 1,$$

where $T_2$ is a torus, and $\tilde{G}$ is a reductive group whose derived subgroup is simply connected. The sequence (3.1) is called a $z$-extension, c.f. [MS82, Proposition 3.1]. Consider the simplicial system $\tilde{G} \times T^*_2$, where the $n$th simplex is given by $\tilde{G} \times T^{*n}_2$ and the boundary maps are multiplications. Since $T_2$ is central in $\tilde{G}$, these multiplication maps define morphisms of algebraic groups. As a consequence, we obtain a simplicial system of prestacks $\text{Gr}_{\tilde{G} \times T^*_2}$ over Ran$(X)$. Appealing to [Ga18, Corollary 5.2.7], the Picard groupoid $\text{Pic}^{\text{fact}}(\text{Gr}_G)$ identifies with the limit of the co-simplicial system $\text{Pic}^{\text{fact}}(\text{Gr}_{\tilde{G} \times T^*_2})$.

**Remark 3.2.** The cited result follows from $h$-descent of line bundles in the context of derived schemes. A proof is given there using the theory of ind-cogerent sheaves, but one can avoid it by using [HLP14, §4].

**Lemma 3.3.** The canonical map of Picard groupoids is an equivalence:

$$\text{CExt}(G, K_2) \xrightarrow{\sim} \lim \text{CExt}(\tilde{G} \times T^*_2, K_2).$$

**Proof.** We argue that the Picard groupoid of (not necessarily central) extensions $\text{Ext}(G, K_2)$ maps isomorphically to $\lim \text{Ext}(\tilde{G} \times T^*_2, K_2)$; the result would follow since a $K_2$-extension of $G$ is central if and only if its pullback to each $\tilde{G} \times T^*_2$ is central.

Since $\text{Ext}(G, K_2)$ identifies with homomorphisms from $G$ to $B K_2$, it suffices to show that $G$ identifies with $\text{colim}(\tilde{G} \times T^*_2)$ in the category of Zariski sheaves of groups (in spaces). This in turn follows from:

(a) the forgetful functor from Zariski sheaves of groups to plain Zariski sheaves is conservative and commutes with geometric realizations;
(b) $G$ identifies with $\text{colim}(\tilde{G} \times T^*_2)$ in the category of plain Zariski sheaves, since every $T_2$-torsor is Zariski-locally trivial (Hilbert 90).

In other words, Theorem 3.1 for $G$ follows from the same result for each $\tilde{G} \times T^*_2$. In proving Theorem 3.1, we may thus assume that $G_{\text{der}}$ is simply connected.

3.2. **Proof of Theorem 3.1 for $G_{\text{der}}$ simply connected.**

3.2.1. We now prove Theorem 3.1 in the case that $G_{\text{der}}$ is simply connected. Let $T_1 := G/G_{\text{der}}$. Then the fiber of $\theta_G(\Lambda_T) \to Q(\Lambda_{T_{\text{der}}}, Z)^{W}$ identifies with $\theta(\Lambda_{T_1})$. Let $\text{Pic}^{\text{fact}}_{\text{qder}=0}(\text{Gr}_G)$ be the full subgroupoid of $\text{Pic}^{\text{fact}}(\text{Gr}_G)$, consisting of objects whose images vanish under the following composition:

$$\text{Pic}^{\text{fact}}(\text{Gr}_G) \to \text{Pic}^{\text{fact}}(\text{Gr}_{G_{\text{der}}}) \xrightarrow{(2.22)} Q(\Lambda_{T_{\text{der}}}, Z).$$
We then have a commutative diagram of Picard groupoids:

\[
\begin{array}{ccc}
\text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G) & \xrightarrow{\Phi} & \text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G) \\
\downarrow & & \downarrow \\
\theta(\Lambda_{T_1}) & \cong & \theta_G(\Lambda_T) \\
\end{array}
\]

Inspecting this diagram, we see that it suffices to show that the first vertical map:

\[
\text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G) \to \theta(\Lambda_{T_1})
\]

is an equivalence.

3.2.2. Consider the projection \( p : \text{Gr}_G \to \text{Gr}_{T_1} \). It defines a pullback functor

\[
p^* : \text{Pic}_{q_{\text{der}}=0}(\text{Gr}_{T_1}) \to \text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G)
\]

such that the composition:

\[
\text{Pic}_{q_{\text{der}}=0}(\text{Gr}_{T_1}) \xrightarrow{p^*} \text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G) \xrightarrow{(3.2)} \theta(\Lambda_{T_1})
\]

canonical identifies with the equivalence (2.13). It therefore suffices to show that (3.3) is an equivalence.

3.2.3. We note that (3.3) factors through the full subcategory

\[
\text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G) \to \text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G)
\]

of factorization line bundles on \( \text{Gr}_G \) which are trivial along fibers of \( p \) over \( k \)-points. In the rest of this subsection, we shall show that

(a) the containment (3.4) is an equivalence.

(b) pullback along \( p \) defines an equivalence

\[
\text{Pic}_{q_{\text{der}}=0}(\text{Gr}_{T_1}) \to \text{Pic}_{q_{\text{der}}=0}(\text{Gr}_G).
\]

The combination of these two statements will imply Theorem 3.1.

3.2.4. In order to prove the above statements, we first study the geometric properties of the projection \( p \).

**Lemma 3.4.** The map \( p \) realizes \( \text{Gr}_G \) as an étale locally trivial \( \text{Gr}_{G,\text{der}} \)-bundle over \( \text{Gr}_{T_1} \).

In other words, for every affine scheme \( S \to \text{Gr}_{T_1} \), there is an étale cover \( \tilde{S} \to S \) and an isomorphism \( \text{Gr}_G \times \tilde{S} \cong \text{Gr}_{G,\text{der}} \times \text{Ran}(X) \).

**Proof of Lemma 3.4.** We first show that \( G \to T_1 \) splits. Indeed, the maximal (split) torus \( T \subset G \) surjects onto \( T_1 \), so it suffices to show that the kernel \( T \cap G_{\text{der}} \) is connected. The latter follows since \( T \cap G_{\text{der}} \) is a maximal torus of \( G_{\text{der}} \).

Given an \( S \)-point \( \tilde{S} \to \text{Gr}_{T_1} \), we denote by \( S \xrightarrow{\gamma} \text{Gr}_{T_1} \) the “neutral point” corresponding to \( \gamma \), i.e., the composition \( S \xrightarrow{\gamma} \text{Gr}_{T_1} \xrightarrow{\text{Ran}(X)} \text{Gr}_{T_1} \). Since \( \text{Gr}_G \times \text{Ran}(X) \) identifies with \( \text{Gr}_{G,\text{der}} \times \tilde{S} \), it suffices to produce an isomorphism:

\[
\text{Gr}_G \times \tilde{S} \cong \text{Gr}_G \times \text{Ran}(X)
\]
after passing to some étale cover \( \tilde{S} \to S \).

We choose \( \tilde{S} \to S \) such that the elements \( \gamma, \gamma_0 \in \text{Maps}_{\text{Ran}(X)}(\tilde{S}, \text{Gr}_T) \) differ by the action of some \( \alpha \in \text{Maps}_{\text{Ran}(X)}(\tilde{S}, \mathcal{L} T) \) (this is possible, for example, by lifting \( S \to \text{Gr}_T \) to \( \tilde{S} \to \mathcal{L} T \)). The above discussion shows that we have a splitting of the canonical projection \( \mathcal{L} G \to \mathcal{L} T \). Hence \( \alpha \) can be lifted to an element \( \tilde{\alpha} \in \text{Maps}_{\text{Ran}(X)}(\tilde{S}, \mathcal{L} G) \). The equivariance property of \( \mathfrak{p} \) shows that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Gr}_G \times \tilde{S} & \xrightarrow{\text{act}_\alpha} & \text{Gr}_G \times \tilde{S} \\
\downarrow & & \downarrow \\
\text{Gr}_T \times \tilde{S} & \xrightarrow{\text{act}_\tilde{\alpha}} & \text{Gr}_T \times \tilde{S}
\end{array}
\]

Since \( \text{act}_\alpha \) transforms the section \( \gamma : \tilde{S} \to \text{Gr}_T \times \tilde{S} \) to \( \gamma_0 \), we obtain the required isomorphism (3.6) as \( \text{act}_\tilde{\alpha} \times \text{id}_{\tilde{S}} \).

3.2.5. Proof of (a). We now show that every \( \mathcal{L} \in \text{Pic}^{\text{fact}}_{\text{der}=0}(\text{Gr}_G) \) is fiberwise trivial along the projection \( p : \text{Gr}_G \to \text{Gr}_T \). Since the question concerns only points on \( \text{Gr}_T \), it suffices to show that the base change of \( \mathcal{L} \) to the subscheme \( X^{(\lambda_1, \ldots, \lambda_{|I|})} \to \text{Gr}_T, X^I \) is fiberwise trivial.

We write \( \mathcal{P}^{(\lambda^I)} \) for the étale sheaf of relative Picard group of \( \text{Gr}_{G, X^I} \to \text{Gr}_T, X^I \) over \( X^{(\lambda_1, \ldots, \lambda_{|I|})} \), i.e., it associates to every étale map \( V \to X^{(\lambda_1, \ldots, \lambda_{|I|})} \) the abelian group \( \text{Pic}(\text{Gr}_{G, X^n} \times V)/\text{Pic}(V) \). Thus \( \mathcal{L} \) defines a global section \( l^{(\lambda^I)} \) of \( \mathcal{P}^{(\lambda^I)} \) for every \( n \)-tuple \( \lambda^I \). The goal is to show that all \( l^{(\lambda^I)} \) vanish.

3.2.6. Recall the computation of the étale sheaf of relative Picard groups \( \text{Pic}(\text{Gr}_{G_{\text{der}}, X^I}/X^I) \) in [Zh16, §3.4]. It fits into an exact sequence of sheaves of abelian groups over \( X^I \):

\[
0 \to \text{Pic}(\text{Gr}_{G_{\text{der}}, X^I}/X^I) \to \bigoplus_{|I|=|I|-1} (\Delta_{I \to J}), \quad \Delta_{J \to I} \to \Delta_{J \to I}.
\]

Here, \( A \) denotes the abelian group \( \mathbb{Z} \times \text{rk}(G_{\text{der}}) \), and \( \Delta_X \) is its associated constant sheaf of groups over \( X \). Lemma 3.4 shows that the sheaf \( \mathcal{P}^{(\lambda^I)} \) is étale locally isomorphic to \( \text{Pic}(\text{Gr}_{G_{\text{der}}, X^I}/X^I) \) under the identification \( X^{(\lambda_1, \ldots, \lambda_{|I|})} \to X^I \). We note a simple Lemma:

**Lemma 3.5.** Let \( Y \) be a connected, Noetherian scheme and \( \mathcal{F} \) be an étale sheaf on \( Y \). Suppose furthermore that \( \mathcal{F} \) is étale locally isomorphic to a subsheaf of a constant sheaf. Then a section \( s \in \Gamma(Y, \mathcal{F}) \) vanishes if and only if it does so over some étale open \( V \to Y \).

**Proof.** One can pick finitely many étale maps \( V_i \to Y \) (\( i \in I \)) so that:

(a) each \( V_i \) is connected;
(b) \( \mathcal{F}|_{V_i} \) is isomorphic to a subsheaf of a constant sheaf;
(c) the images \( U_i \) of \( V_i \) collectively cover \( Y \).

We induct on the cardinality of \( I \) over all connected, Noetherian schemes admitting such a cover; the base case \( I = \emptyset \) is trivial. The image \( U \) of \( V \to Y \) must intersect some \( U_i \). The

\footnote{Recall that for an \( I \)-family of co-characters \( \lambda^I = (\lambda_1, \ldots, \lambda_{|I|}) \), there is a closed immersion \( X^I \to \text{Gr}_T, X^I \) whose image we call \( X^{(\lambda_1, \ldots, \lambda_{|I|})} \).}
condition (b) implies that the restriction $s_i \in \Gamma(U_i, \mathcal{F})$ vanishes. Now, let $\tilde{Y} := \bigcup_{i \neq i} U_i$. It is \textit{not} necessarily connected. However, the fact that $Y$ is connected shows that $U_i$ intersects every connected component of $\tilde{Y}$. We apply the induction hypothesis to each connected component of $\tilde{Y}$ to conclude that $s$ vanishes. \hfill \Box

3.2.7. Our proof that each $t^{(\lambda)}$ vanishes now proceeds as follows:

\textbf{Step 1:} $t^{(0)} = 0$. Indeed, since line bundles on $\text{Gr}_{G_{\text{der}}} X$ are classified by the quadratic form $q_{\text{der}}$, we see that $\mathcal{L}$ is trivialized when pulled back along $\text{Gr}_{G_{\text{der}}} X \to \text{Gr}_G X$. On the other hand, $\text{Gr}_{G_{\text{der}}} X$ appears as the fiber of $p$ along the unit map $X \to \text{Gr}_{T_1}$. Hence $t^{(0)} = 0$.

\textbf{Step 2:} $t^{(\lambda)} = 0$ for all $\lambda \in \Lambda_{\mathbb{T}}$. Consider the section $t^{(\lambda,-\lambda)}$ of $\mathbb{P}(\lambda,-\lambda)$. It is represented by some line bundle $\mathcal{L}^{(\lambda,-\lambda)}$ over $\text{Gr}_G X^2 \times_{\text{Gr}_{T_1} X^2} X^{(\lambda,-\lambda)}$. We know from Step 1 that the restriction of $\mathcal{L}^{(\lambda,-\lambda)}$ to the diagonal comes from the base $X^{(0)} \to X^{(\lambda,-\lambda)}$. Hence, over an étale neighborhood of $X^{(0)}$, the section $t^{(\lambda,-\lambda)}$ has to vanish by the identification of $\mathbb{P}(\lambda,-\lambda)$ with $\text{Pic}(\text{Gr}_{G_{\text{der}}} X^2/X^2)$. We then apply Lemma 3.5 to conclude that $t^{(\lambda,-\lambda)}$ vanishes.

Now, under the identification of $\mathbb{P}(\lambda,-\lambda)$ with $\mathbb{P}(\lambda) \oplus \mathbb{P}(-\lambda)$ away from the diagonal, the section $t^{(\lambda,-\lambda)}$ passes to $t^{(\lambda)} \otimes t^{(-\lambda)}$. The fact that $t^{(\lambda,-\lambda)} = 0$ now implies that $t^{(\lambda)}$ (and $t^{(-\lambda)}$) vanishes.

\textbf{Step 3:} $t^{(\lambda^I)} = 0$ for all $I$-tuple $\lambda^I$. When the cardinality of $I$ is at least 2, we may use the factorization property of $t^{(\lambda^I)}$ to see that $t^{(\lambda^I)}$ vanishes away from the union of the diagonals in $X^{(\lambda_1,\ldots,\lambda_n)}$. Hence by Lemma 3.5 again we have $t^{(\lambda^I)} = 0$.

This finishes the proof that (3.4) is an equivalence.

3.2.8. \textbf{Proof of (b).} We first recall some standard results.

\textbf{Lemma 3.6.} Suppose $\tilde{G}$ is semisimple and simply connected. Then the morphism $\text{Gr}_{\tilde{G}} \to \text{Ran}(X)$ has the property that for every affine scheme $S \to \text{Ran}(X)$, we have a presentation

$$\text{Gr}_{\tilde{G}} \times_{\text{Ran}(X)} S \to \colim_i Y_i$$

where each $Y_i$ is a scheme of finite type over $S$, satisfying:

(a) $Y_i$ is proper and faithfully flat over $S$;
(b) The fiber $(Y_i)_s$ at every $k$-point $s \in S$ is connected and $H^1((Y_i)_s, \mathcal{O}) \cong 0$.

\textbf{Proof.} Since each $S \to \text{Ran}(X)$ factors through some $X^I$, it suffices to produce such a presentation for $\text{Gr}_{G,X^I}$. For each $I$-tuple $\lambda$, elements of $\Lambda_{\mathbb{T}}^I$, we may consider the Schubert variety $\text{Gr}_{G,X^I}^{\leq \lambda}$, which is proper, surjective over $X^I$. The flatness is proved in [Zhi09, §1.2] for $I = \{1,2\}$ and the general case is similar. The property (b) of its fibers is a special case of Lemma 2.6. \hfill \Box

\textbf{Remark 3.7.} Lemma 3.6(b) fails for non-semisimple groups, since $\text{Gr}_{G}$ may not be ind-reduced. We do not know whether the flatness in part (a) holds more generally.

3.2.9. Suppose $p : X \to Y$ is a morphism of finite type schemes over $k$\textsuperscript{10} such that

(a) $p$ is proper and faithfully flat;
(b) its fiber $X_y$ at every $k$-point $y \in Y$ is connected and $H^1(X_y, \mathcal{O}) = 0$.

\textbf{Lemma 3.8.} Let $\mathcal{L}$ be a line bundle on $X$. Under the above hypotheses on $p : X \to Y$, the following are equivalent:

\textsuperscript{10}Recall that $k$ is assumed to be algebraically closed.
(a) $\mathcal{L}$ is trivial along the fibers of $p$;
(b) $p_*\mathcal{L}$ is a line bundle over $Y$, and the canonical map $p^*p_*\mathcal{L} \to L$ is an isomorphism.

Proof. We use the formulation of the “cohomology and base change” theorem in [Va, 28.1.6]. The fiberwise triviality of $\mathcal{L}$, together with the vanishing of $H^1(X_y, \mathcal{O}_{X_y})$, shows that the canonical map:
\[
R^1 p_*\mathcal{L}|_y \to H^1(X_y, \mathcal{L}|_{X_y})
\]
is surjective, for any $k$-point $y \in Y$. Hence part (i) of loc.cit. applies and we see that that (3.7) is an isomorphism. Since $R^1 p_*\mathcal{L}$ is coherent, it must vanish. In particular, part (ii) of loc.cit. applies and shows that the canonical map $p_*\mathcal{L}|_y \to H^0(X_y, \mathcal{L}|_{X_y})$ is surjective. Another application of part (i) then shows that $p_*\mathcal{L}$ is locally free near $y$ of rank $h^0(X_y, \mathcal{L}|_{X_y}) = h^0(X_y, \mathcal{O}) = 1$, i.e., it is a line bundle. The isomorphism $p^*p_*\mathcal{L} \to \mathcal{L}$ is then obvious.

3.2.10. Suppose $p : X \to Y$ is ind-schematic morphism, represented by morphisms $p_i : X_i \to Y$ of schemes satisfying the hypothesis of §3.2.9. Then $p^* : \text{Pic}(Y) \to \text{Pic}(X)$ has a partially defined right adjoint:
\[
p_*\mathcal{L} := \lim_i (p_i)_*\mathcal{L}_i,
\]
while representing $\mathcal{L}$ by the inverse system $\mathcal{L}_i \in \text{Pic}(X_i)$
which is well defined on the full subcategory of $\text{Pic}(X)$ consisting of line bundles which are trivial along the fibers of $p$, and we furthermore have an isomorphism $p^*p_*\mathcal{L} \simeq \mathcal{L}$. For any line bundle $M$ from the base $Y$, it is also clear that $M \to p_*p^*M$. Hence $p^*$ defines an equivalence from $\text{Pic}(Y)$ to the full subcategory of $\text{Pic}(X)$ consisting of fiberwise trivial line bundles.

3.2.11. The above discussion, together with Lemma 3.4 and 3.6, shows that $p^*$ defines an equivalence $\text{Pic}(\text{Gr}_{T}) \simeq \text{Pic}(\text{Gr}_G)$. To see that this upgrades to an equivalence of factorization line bundles, we simply note that the map $\text{Gr}_G \times_{\text{Bun}(X)} \text{Gr}_G \to \text{Gr}_{T_i} \times_{\text{Bun}(X)} \text{Gr}_{T_i}$ again satisfies the hypothesis of §3.2.10 after base change to a scheme. This finishes the proof that (3.5) is an equivalence.

\[\Box\text{(Theorem 3.1)}\]

\begin{thebibliography}{99}
\bibitem[Ba12]{Barlev} Barlev, Jonathan. “D-modules on spaces of rational maps and on other generic data.” arXiv preprint arXiv:1204.3469 (2012).
\bibitem[BD91]{Beilinson-Drinfeld} Beilinson, Alexander, and Vladimir Drinfeld. Quantization of Hitchin’s integrable system and Hecke eigensheaves. (1991): 1297-1301.
\bibitem[BD04]{Beilinson-Drinfeld} Beilinson, Alexander, and Vladimir G. Drinfeld. Chiral algebras. Vol. 51. American Mathematical Soc., 2004.
\bibitem[Br04]{Brion} Brion, Michel. “Lectures on the geometry of flag varieties.”Topics in cohomological studies of algebraic varieties. Birkhuser Basel, 2005. 33-85.
\bibitem[BD01]{Brylinski} Brylinski, Jean-Luc, and Pierre Deligne. “Central extensions of reductive groups by $K_2$.” Publications mathématiques de l’IHÉS 94.1 (2001): 5-85.
\bibitem[Ca17]{Campbell} Campbell, Justin. “Unramified geometric class field theory and Cartier duality.” arXiv preprint arXiv:1710.02892 (2017).
\bibitem[DS95]{Drinfeld-Simpson} Drinfeld, V. G., and Carlos Simpson. “B-structures on $G$-bundles and local triviality.” Mathematical Research Letters 2.6 (1995): 823-829.
\bibitem[Fal03]{Faltings} Faltings, Gerhard. “Algebraic loop groups and moduli spaces of bundles.” Journal of the European Mathematical Society 5.1 (2003): 41-68.
\bibitem[Ga15]{Gaitsgory} Gaitsgory, Dennis. “The Atiyah-Bott formula for the cohomology of the moduli space of bundles on a curve.” arXiv preprint arXiv:1505.02331 (2015).
\bibitem[Ga18]{Gaitsgory} Gaitsgory, D. “Parameterization of factorizable line bundles by $K$-theory and motivic cohomology” arXiv preprint arXiv:1804.02567 (2018).
\end{thebibliography}
[GL16] Gaitsgory, D., and S. Lysenko. “Parameters and duality for the metaplectic geometric Langlands theory.” arXiv preprint arXiv:1608.00284 (2016).

[HLP14] Halpern-Leistner, Daniel, and Anatoly Preygel. “Mapping stacks and categorical notions of properness.” arXiv preprint arXiv:1402.3204 (2014).

[Ha13] Hartshorne, Robin. Algebraic geometry. Vol. 52. Springer Science & Business Media, 2013.

[MS82] Milne, James S., and K-Y. Shih. “Conjugates of Shimura varieties.” Hodge cycles, motives, and Shimura varieties. Springer, Berlin, Heidelberg, 1982. 280-356.

[Re12] Reich, Ryan. “Twisted geometric Satake equivalence via gerbes on the factorizable Grassmannian.” Representation Theory of the American Mathematical Society 16.11 (2012): 345-449.

[We15] Weissman, Martin H. “L-groups and parameters for covering groups.” arXiv preprint arXiv:1507.01042 (2015).

[Va] Vakil, Ravi. “The Rising Sea: Foundations of Algebraic Geometry.” URL: http://math.stanford.edu/~vakil/216blog/FOAGdec3014public.pdf.

[Zh09] Zhu, Xinwen. “Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian.” Advances in Mathematics 221.2 (2009): 570-600.

[Zh16] Zhu, Xinwen. “An introduction to affine Grassmannians and the geometric Satake equivalence.” arXiv preprint arXiv:1603.05593 (2016).

E-mail address: jamestao@mit.edu
E-mail address: yifei@math.harvard.edu