LIOUVILLE PROPERTY FOR $f$-HARMONIC FUNCTIONS WITH POLYNOMIAL GROWTH

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Abstract. We prove a Liouville property for any $f$-harmonic function with polynomial growth on a complete noncompact smooth metric measure space $(M, g, e^{-f}dv)$ when the Bakry-Émery Ricci curvature is nonnegative and its diameter of geodesic sphere has sublinear growth.

1. Introduction

In this paper, we will study a Liouville property for any $f$-harmonic function with polynomial growth on complete smooth metric measure spaces with the nonnegative Bakry-Émery Ricci curvature.

Recall that an $n$-dimensional complete smooth metric measure space, customarily denoted by $(M, g, e^{-f}dv)$, is an $n$-dimensional Riemannian manifold $(M, g)$ together with a weighted volume element $e^{-f}dv$ for some $f \in C^\infty(M)$, and the volume element $dv$ induced by the Riemannian metric $g$. The $f$-Laplacian $\Delta_f$ on smooth metric measure space $(M, g, e^{-f}dv)$, self-adjoint with respect to $e^{-f}dv$, is defined by

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$

On smooth metric measure space $(M, g, e^{-f}dv)$, a smooth function $u$ is called weighted harmonic (or $f$-harmonic) if $\Delta_f u = 0$, and $f$-subharmonic if $\Delta_f u \geq 0$. It is easy to see that the absolute value of an $f$-harmonic function is a nonnegative $f$-subharmonic function. If $f$ is constant, all notions mentioned above reduce to the classical Riemannian case.

A natural generalization of Ricci curvature associated to smooth metric measure space $(M, g, e^{-f}dv)$ is called $m$-Bakry-Émery Ricci curvature, which is defined by

$$\operatorname{Ric}^m_f = \operatorname{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df$$

for some number $m > 0$. Bakry and Émery [1] extensively studied this curvature tensor and its relationship to diffusion processes. When $m$ is
finite, the Bochner formula for the $m$-Bakry-Émery Ricci curvature can be read as

$$\frac{1}{2} \Delta f |\nabla u|^2 \geq \frac{(\Delta f u)^2}{m+n} + (\nabla \Delta f u, \nabla u) + \text{Ric}_f^m(\nabla u, \nabla u),$$

which could be viewed as the Bochner formula for the Ricci curvature of an $(n+m)$-dimensional manifold. Hence many classical results for manifolds with Ricci curvature bounded below can be extended to smooth metric measure spaces with $m$-Bakry-Émery Ricci curvature bounded below, see for example [2, 14, 18, 20, 22, 27] for details. When $m = \infty$, one denotes

$$\text{Ric}_f = \text{Ric}_\infty^f = \text{Ric} + \nabla^2 f.$$ 

In particular, if

$$\text{Ric}_f = \lambda g$$ 

for some $\lambda \in \mathbb{R}$, then $(M, g, e^{-f} dv_g)$ is a gradient Ricci soliton, which arises from the singularity analysis of the Ricci flow [9]. When $\text{Ric}_f$ is bounded below, some analytical and geometric properties on smooth metric measure spaces were also possibly explored as long as the weight function $f$ is assumed to be some condition. We refer to the work of Dung [6], Lott [15], Wei and Wylie [21], and [23, 25, 26, 27] for further details.

Recently, many Liouville theorems for $f$-harmonic functions on smooth metric measure spaces have been studied. For example, Wei and Wylie [21] proved that any positive $f$-harmonic function with some growth condition with $\text{Ric}_f \geq K > 0$ must be constant. Brighton [3] generalized Yau’s gradient result [29] and derived a gradient estimate for positive $f$-harmonic functions. He also obtained a Liouville theorem for positive bounded $f$-harmonic functions. Inspired by Brighton’s argument, Munteanu and Wang [16] applied the De Giorgi-Nash-Moser theory to get a sharp gradient estimate and a Liouville-type result for positive $f$-harmonic functions on smooth metric measure spaces. Dung and Dat [7] proved that any positive weighted $p$-eigenfunction is constant if it is of sublinear growth on smooth metric measure spaces with $\text{Ric}_f^p \geq 0$. Recently, Wang et al. [19] proved positive weighted $p$-eigenfunction is constant when $\text{Ric}_f^p$ is bounded below by using the Moser’s iterative technique.

Motivated by the classical works of [12, 13, 28], many $L^p_f$-Liouville ($0 < p < \infty$) theorems on smooth metric measure spaces were extensively studied. Here $L^p_f$-Liouville theorem means that every $L^p_f$-integrable $f$-harmonic function on smooth metric measure spaces is constant. Recall that $u$ is called $L^p_f$-integrable, i.e. $u \in L^p_f$, if its $L^p_f$-norm, defined as

$$||u||_p := \left( \int_M |u|^p e^{-f} dv \right)^{1/p},$$

is finite. In [14] Li proved an $L^1_f$-Liouville theorem for $f$-subharmonic functions on smooth metric measure spaces with $\text{Ric}_f^m \geq -c(1 + r(x)^2)$, where $r(x) := d(o, x)$ is the distance function starting from a base point $o \in M$. 


0 < m < ∞. Pigola, Rimoldi, and Setti [17] proved a sharp \( L^p_f \)-Liouville theorem with \( p > 1 \) for \( f \)-subharmonic functions on smooth metric measure spaces without any curvature assumption. When \( f \) is bounded, the author [23] derived some \( L^p_f \)-Liouville theorems with \( 0 < p \leq 1 \) for \( f \)-subharmonic functions on smooth metric measure spaces with some Bakry-Emery curvature assumptions. When \( f \) is not bounded, P. Wu and the author [25, 26, 27] systematically studied various \( L^p_f \)-Liouville theorems with \( 0 < p \leq 1 \) on smooth metric measure spaces under different Bakry-Emery curvature conditions. In particular, we obtained a sharp \( L^1_f \)-Liouville theorem when \( \text{Ric}_f \geq 0 \) by using \( f \)-heat kernel Gaussian upper estimates on smooth metric measure spaces (see also [5]).

In this paper, we prove a new Liouville theorem for any \( f \)-harmonic function with polynomial growth when the Bakry-Emery Ricci curvature is non-negative and its diameter of geodesic sphere is sublinear. The proof uses a local \( f \)-Neumann Poincaré inequality, and an iterated argument combined with a cut off function technique, using a similar argument of [4] (see also [25, 26]).

**Theorem 1.1.** Let \((M, g, e^{-f} dv)\) be an \( n \)-dimensional complete noncompact smooth metric measure space with

\[
\text{Ric}_f \geq 0 \quad \text{and} \quad \sup_{x \in M} |f(x)| < +\infty.
\]

For a base point \( o \in M \) and \( R > 0 \), if the diameter of geodesic sphere \( B_o(R) \) has a sublinear growth:

\[
\text{diam} \partial B_o(R) := \sup_{x, y \in \partial B_o(R)} d(x, y) = o(R), \quad R \to \infty,
\]

then any \( f \)-harmonic function with polynomial growth is constant.

**Remark 1.2.** Munteanu and Wang [16] proved the Liouville property for all positive \( f \)-harmonic functions on smooth metric measure spaces. Our Liouville theorem hold for non-positive \( f \)-harmonic functions.

**Remark 1.3.** In the proof of Theorem 1.1, the finiteness of function \( f \) makes the iterated procedure to work efficiently. It is interesting to ask whether this condition can be removed.

**Remark 1.4.** For smooth Riemannian manifolds whose sectional curvature satisfies a quadratic decay lower bound and whose geodesic spheres have sublinear growth, A. Kasue [10, 11] proved that any harmonic function with polynomial growth is constant (see also G. Carron [4] for further generalizations). In some sense, Theorem 1.1 generalizes their results to the setting of smooth metric measure spaces.

This rest of the paper is organized as follows. In Section 2, we recall some known results, such as \( f \)-volume comparison theorems, local Poincaré...
inequalities for the Bakry-Émery Ricci curvature bounded below. In Section 3, we adapt an iterated argument to prove Theorem 1.1.

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2. Preliminaries

Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. For any point $x \in M$ and $R > 0$, we denote

$$A(R) := A(x, R) = \sup_{y \in B_x (3R)} |f(y)|.$$  

We often write $A$ for short. In [21] Wei and Wylie proved a relative $f$-volume comparison theorem.

Lemma 2.1. Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n - 1) K$ for some constant $K \geq 0$, then

$$\frac{V_f (B_x (R_1, R_2))}{V_f (B_x (r_1, r_2))} \leq \frac{V_{K + 4A} (B_x (R_1, R_2))}{V_{K + 4A} (B_x (r_1, r_2))}$$

for any $0 < r_1 < r_2$, $0 < R_1 < R_2$, $r_1 \leq R_1$, $r_2 \leq R_2$, where $B_x (R_1, R_2) = B_x (R_2) \setminus B_x (R_1)$, and $A = A(x, \frac{1}{3} R_2)$. Here $V_{K + 4A} (B_x (r))$ denotes the volume of the ball in the model space $M_{K + 4A}$, i.e., the simply connected space form of dimension $n + 4A$ with constant sectional curvature $-K$.

For model space $V_{K + 4A} (B_x (r))$, if $K > 0$, this model space is the hyperbolic space; if $K = 0$, this model space is the Euclidean space. In any case, we have the following lower and upper estimates

$$\omega_{n + 4A} \cdot r^{n + 4A} \leq V_{K + 4A} (B_x (r)) \leq \omega_{n + 4A} \cdot r^{n + 4A} e^{(n-1+4A)\sqrt{K} r},$$

where $\omega_{n + 4A}$ is the volume of the unit ball in $(n + 4A)$-dimensional Euclidean space.

In [20], P. Wu and the author proved a local Neumann Poincaré inequality on complete smooth metric measure spaces.

Lemma 2.2. Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with

$$\text{Ric}_f \geq -(n - 1) K$$

for some constant $K \geq 0$. Fix a point $o \in M$ and $R > 0$. For any $x \in B(o, R)$, we have

$$\int_{B_x (r)} |c - c_{B_x (r)}|^2 e^{-f} dv \leq c_1 e^{c_2 A + c_3 (1 + A) \sqrt{K} r} \cdot r^2 \int_{B_x (r)} |\nabla c|^2 e^{-f} dv.$$
for all $0 < r < R$ and $\varphi \in C^\infty_0(B_x(r))$, where $\varphi_{B_x(r)} := \int_{B_x(r)} \varphi \, d\mu / \int_{B_x(r)} d\mu$; $A = \sup_{y \in B_o(3R)} |f(y)|$, and where the coefficient $c_i$ are constants, depending only on the dimension $n$.

We remark that the local $f$-Neumann Poincaré inequality and so called $f$-volume doubling property can imply a local Sobolev inequality and a mean value inequality for solutions to the $f$-heat equation on smooth metric measure spaces. Putting these inequalities together, we can furthermore prove Moser’s Harnack inequality by using the De Giorgi-Nash-Moser theory. For detailed discussions, the interested readers can refer to [26].

In this paper, we are concerned with some analytic and geometric results on $(M, g, e^{-f} \, dv)$ when $\text{Ric}_f \geq 0$. Under this curvature assumption, Lemma 2.1 implies that

**Proposition 2.3.** Let $(M, g, e^{-f} \, dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. Fix a point $o \in M$. If $\text{Ric}_f \geq 0$, then

$$V_f(B_o(R)) \leq C \, R^{n+4A}$$

for all $R > 1$, where $C$ is a constant, which only depends on $n$, $V_f(B_o(1))$ and $A := \sup_{y \in B_o(R)} |f(y)|$.

**Proof.** In Lemma 2.1, we let $r_1 = R_1 = 0$, $r_2 = 1$, $R_2 = R$ and $x = o$, and then

$$\frac{V_f(B_o(R))}{V_f(B_o(1))} \leq \frac{V_0^{n+4A}(B_o(R))}{V_0^{n+4A}(B_o(1))} \leq C(n, A) R^{n+4A},$$

where $A = \sup_{y \in B_o(R)} |f(y)|$. Note that we also used (2.2) in the above second inequality. Hence

$$V_f(B_o(R)) \leq C \, R^{n+4A}$$

for all $R > 1$, where $C$ is a positive constant depending on $n$, $A$ and $V_f(B_o(1))$. \hfill $\square$

In [8], A. Grigor’yan and L. Saloff-Coste introduced a useful notion: remote ball. Recall that, for a fixed point $o \in M$, a ball $B(x, \rho)$ in $M$ is called remote ball if

$$d(o, x) \geq 3\rho.$$ 

When $\text{Ric}_f \geq -c r^{-2}(x)$, A. Grigor’yan and L. Saloff-Coste [8] showed that all remote balls in manifold $M$ satisfy the volume doubling property, the Poincaré inequality and the parabolic Harnack inequality. In our case, for all remote balls $B(x, \rho)$, Lemma 2.2 can be restated as

**Proposition 2.4.** Let $(M, g, e^{-f} \, dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with

$$\text{Ric}_f \geq 0.$$
Lemma 3.1. Let \( \varphi \in C^\infty(B_x(\rho)) \), where \( \varphi_{B_x(\rho)} := \int_{B_x(\rho)} \varphi e^{-f} dv / \int_{B_x(\rho)} e^{-f} dv \) and \( \bar{A} = \sup_{y \in B_o(3R)} |f(y)| \). Here \( c_4 \) and \( c_5 \) are constants depending only on \( n \).

3. Proof of Theorem 1.1

In this section, we first follow the argument of [4] to give a useful gradient estimate in the integral sense. The proof depends on the cut-off function technique and iterated procedure. Then we apply this estimate to prove Theorem 1.1.

Lemma 3.1. Let \((M, g, e^{-f}dv)\) be an \( n \)-dimensional complete noncompact smooth metric measure space. For any a point \( o \in M \) and \( R > 0 \), let \( u \) be a \( f \)-harmonic smooth function on \( B_o(2R) \), and

\[
\text{diam} \, \partial B_o(r) \leq \epsilon r \quad \text{with} \quad \epsilon \in (0, 1/12)
\]

for all \( r \in [R, 2R] \). Assume that all remote balls \( B_x(\rho), x \in B_o(2R), \) satisfies a local Poincaré inequality:

\[
\int_{B_x(\rho)} |\varphi - \varphi_{B_x(\rho)}|^2 e^{-f} dv \leq c_4 e^{c_5 \bar{A}} \cdot R^2 \int_{B_x(\rho)} |\nabla \varphi|^2 e^{-f} dv
\]

for any \( \varphi \in C^\infty(B_x(\rho)) \), where \( \varphi_{B_x(\rho)} := \int_{B_x(\rho)} \varphi e^{-f} dv / \int_{B_x(\rho)} e^{-f} dv \) and \( \bar{A} := \sup_{y \in B_o(6R)} |f(y)| \). Then we have

\[
\int_{B_o(R)} |\nabla u|^2 e^{-f} dv \leq \delta^{\frac{n}{2}} \int_{B_o(2R)} |\nabla u|^2 e^{-f} dv,
\]

where \( \delta := \left( \frac{9c_4 e^{c_5 \bar{A}}}{1 + 9c_4 e^{c_5 \bar{A}}} \right)^{\frac{2}{b}}, \) and \( \bar{A} := \sup_{y \in B_o(6R)} |f(y)| \).

Proof. For any a point \( o \in M \) and \( R > 0 \), suppose that \( u : B_o(2R) \to \mathbb{R} \) be a \( f \)-harmonic function and \( c \in \mathbb{R} \) is a real number. Choosing

\[
r \in [R + 3\epsilon R, 2R - 3\epsilon R],
\]

where \( \epsilon \in (0, 1/12) \), the diameter hypothesis in lemma implies that there exists some \( x \in \partial B_o(r) \) such that

\[
B_o(r + \epsilon R) \setminus B_o(r) \subset B_x(\epsilon R + \epsilon R).
\]

Let \( \eta(x) \) be a \( C^2 \) cut-off function with support in \( B_o(r + \epsilon R) \). That is,

\[
\eta(x) = \begin{cases} 
1 & \text{if } x \in B_o(r), \\
\frac{r + \epsilon R - d(o, x)}{\epsilon R} & \text{if } x \in B_o(r + \epsilon R) \setminus B_o(r), \\
0 & \text{if } x \in M \setminus B_o(r + \epsilon R).
\end{cases}
\]
We easily observe that
\[ |\nabla(\eta(u - c))|^2 = \eta^2 |\nabla(u - c)|^2 + 2\eta(u - c)\langle \nabla\eta, \nabla(u - c) \rangle + (u - c)^2 |\nabla\eta|^2. \]
Integrating this identity with respect to the weighted measure \( e^{-f} dv \), we have
\[
\int_M |\nabla(\eta(u - c))|^2 = \int_M \eta^2 |\nabla(u - c)|^2 + 2\eta(u - c)\langle \nabla\eta, \nabla(u - c) \rangle + (u - c)^2 |\nabla\eta|^2.
\]
Notice that
\[
(3.3)
\]
\[
\int_M \eta^2 |\nabla(u - c)|^2 + 2\eta(u - c)\langle \nabla\eta, \nabla(u - c) \rangle = \int_M \langle \nabla((u - c)\eta^2), \nabla(u - c) \rangle
\]
\[
= -\int_M (u - c)^2 \Delta_f (u - c)
\]
\[
= 0,
\]
where we have used integration by part with respect to \( e^{-f} dv \) and the fact that \( u \) is \( f \)-harmonic. Therefore,
\[
\int_M |\nabla(\eta(u - c))|^2 e^{-f} dv = \int_{B(o, \epsilon R)} (u - c)^2 |\nabla\eta|^2 e^{-f} dv.
\]
According to the definition of \( \eta(x) \), by the above equality, we get that
\[
\int_{B_o(r)} |\nabla u|^2 e^{-f} dv \leq \int_{B_o(r+\epsilon R)} |\nabla(\eta(u - c))|^2 e^{-f} dv
\]
\[
= \int_{B_o(r+\epsilon R)} (u - c)^2 |\nabla\eta|^2 e^{-f} dv
\]
\[
\leq \frac{1}{\epsilon^2 R^2} \int_{B_o(r+\epsilon R) \setminus B_o(r)} (u - c)^2 e^{-f} dv
\]
\[
\leq \frac{1}{\epsilon^2 R^2} \int_{B_o(\epsilon R + \epsilon r)} (u - c)^2 e^{-f} dv.
\]
For the right hand side of the above inequality, since \( \epsilon \leq 1/12 \), this implies that the ball \( B(x, \epsilon R + \epsilon r) \) is remote. Hence if choosing
\[
c = u_{B_o(\epsilon(R + r))} = \frac{1}{\int_{B_o(\epsilon(R + r))} e^{-f} dv \cdot \int_{B_o(\epsilon(R + r))} u e^{-f} dv},
\]
then using the local \( f \)-Poincaré inequality (3.1) and the fact that \( r + R \leq 3R \), we have that
\[
\int_{B_o(r)} |\nabla u|^2 e^{-f} dv \leq 9c_4 e^{c_3 \bar{A}} \int_{B_o(3\epsilon R)} |\nabla u|^2 e^{-f} dv,
\]
where \( \bar{A} = \sup_{y \in B_o(6R)} |f(y)| \). Also noticing that
\[
B_x(3\epsilon R) \subset B_o(r + 3\epsilon R) \setminus B_o(r - 3\epsilon R),
\]
hence we get
\[ \int_{B_o(r-3\epsilon R)} |\nabla u|^2 e^{-f} dv \leq \int_{B_o(r)} |\nabla u|^2 e^{-f} dv \]
\[ \leq 9c_4 e^{c_5 A} \int_{B_o(r+3\epsilon R) \setminus B_o(r-3\epsilon R)} |\nabla u|^2 e^{-f} dv. \]

That is, for all \( r \in [R, R - 6\epsilon R] \), we have
\[ \int_{B_o(r)} |\nabla u|^2 e^{-f} dv \leq \left( \frac{9c_4 e^{c_5 A}}{1 + 9c_4 e^{c_5 A}} \right)^N \int_{B_o(2R)} |\nabla u|^2 e^{-f} dv. \]

Iterating this inequality, we finally get
\[ \int_{B_o(R)} |\nabla u|^2 e^{-f} dv \leq \left( \frac{9c_4 e^{c_5 A}}{1 + 9c_4 e^{c_5 A}} \right)^N \int_{B_o(2R)} |\nabla u|^2 e^{-f} dv \]
provided that
\[ N6\epsilon R \leq R. \]

At last the desired result follows by choosing \( \delta = \left( \frac{9c_4 e^{c_5 A}}{1 + 9c_4 e^{c_5 A}} \right)^{\frac{1}{6}}. \)

Now we are ready to apply a similar argument of [4] to prove Theorem 1.1 by using Lemma 3.1

Proof of Theorem 1.1. Let \( u: M \rightarrow \mathbb{R} \) be a \( f \)-harmonic function with polynomial growth of order \( \nu \), namely,
\[ |u(x)| \leq C(1 + d(o, x))^\nu. \]

For \( R >> 1 \), we define
\[ I_R := \int_{B_o(R)} |\nabla u|^2 e^{-f} dv \]
and
\[ \epsilon(r) := \sup_{t \geq r} \frac{\rho(t)}{t}, \]
where \( \rho(t) := \sup_{x,y \in \partial B_o(t)} d(x, y) \).

To estimate \( I_R \), we shall introduce a \( C_0^2(M) \) cut off function \( \xi(x) \), which satisfies
\[ \xi(x) = \begin{cases} 
1 & \text{in } B_o(R), \\
\frac{2R - d(o, x)}{R} & \text{in } B_o(2R) \setminus B_o(R), \\
0 & \text{in } M \setminus B_o(2R).
\end{cases} \]
Hence we have

\[ I_R \leq \int_{B_o(2R)} |\nabla (\xi u)|^2 e^{-f} \, dv \]

\[ = \int_{B_o(2R)} [u^2 |\nabla \xi|^2 + \xi^2 |\nabla \xi|^2 + 2\xi \nabla \xi, \nabla u] \, e^{-f} \, dv \]

\[ = \int_{B_o(2R)} [u^2 |\nabla \xi|^2 e^{-f} \, dv \]

\[ \leq CR^{2\nu + n + 4\tilde{A} - 2}, \tag{3.4} \]

where \(C\) depends on \(n, \tilde{A} := \sup_{x \in M} |f(x)|\) and \(V_f(B_o(1))\). Here in the third line of (3.4), we used the same proposition as (3.3); in the last line of (3.4), we used Proposition 2.3.

On the other hand, if we iterate the inequality (3.2) proved in Lemma 3.1, we can show that for all \(R\) such that \(\epsilon(R) \leq 1/12\):

\[ I_R \leq \delta^{-\ell + \sum_{j=0}^{\ell-1} \frac{1}{\epsilon(2^j R)}} I_{2^\ell R}, \]

where \(\delta = \left(\frac{9c_4 e^{c_5 A}}{1 + 9c_4 e^{c_5 A}}\right)^{-\frac{1}{\nu}}\). Applying (3.4) to the right hand side of the above inequality yields

\[ I_R \leq C \cdot \exp \left\{ \ell \left[ \left( 1 - \frac{1}{\epsilon(2^\ell R)} \right) \cdot \ln \delta + (2\nu + n + 4\tilde{A} - 2) \cdot \ln 2 \right] \right\}, \tag{3.5} \]

where in the above inequality, \(\tilde{A} := \sup_{x \in M} |f(x)|\), \(\delta := \left(\frac{9c_4 e^{c_5 A}}{1 + 9c_4 e^{c_5 A}}\right)^{-\frac{1}{\nu}}\) and \(C := C(n, \nu, R, \tilde{A}, V_f(B_o(1)))\).

For all sufficiently large \(R\), the Cesaro convergence theorem shows that

\[ \lim_{\ell \to +\infty} \frac{1}{\ell} \sum_{j=0}^{\ell-1} \frac{1}{\epsilon(2^j R)} = +\infty, \]

since \(\epsilon(2^j R) \to 0\) when \(j \to +\infty\). Meanwhile, by our assumption:

\[ \sup_{x \in M} |f(x)| < +\infty, \]

we know that \(\tilde{A} < +\infty,\)

\[ 0 < \delta < 1 \quad \text{and} \quad \ln \delta < 0 \]

for any \(R\) (even though \(\ell \to +\infty\)). Therefore, if \(\ell \to +\infty\) in (3.5), then we conclude that

\[ I_R = 0 \]

for all sufficiently large \(R\). Therefore \(u\) is constant. \(\Box\)
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