A homotopy aspect of representation theory

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Abstract

We bring a linkage from representation theory of Lie groups to homotopy theory for maps between flag manifolds. As applications we derive from representation theory abundant families of distinguishable homotopy classes of maps between flag manifolds.

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1. Introduction

A current topic in algebraic geometry is the relationship between representation theory of Lie groups and Schubert calculus in flag manifolds (cf. [LG, Chapter 4], [K, VIII], [T]). It seems not aware in the existing literature that there is a direct connection between representation theory and homotopy theory for maps between flag manifolds, implicitly contained in the classical works of Borel and Hirzebruch [BH]. This connection may be summarized as “character factors through homotopy” and is presented in the Theorem in §5. Applying this idea we obtain abundant families of maps between flag manifolds whose homotopy classes are distinguishable by the characters of representations (Corollary 1–3 in §5).

We begin with a problem concerning homotopy classification for maps between flag manifolds (§2), followed by a brief introduction to what we need from representation theory (§3,§4). The main results are stated and established respectively in §5 and §6. The final section §7 is devoted to further comments.

2. A realization problem in homotopy theory

For a compact connected semi–simple Lie group $G$ with a maximal torus $T \subset G$, write $G/T$ for the flag manifold of $G$ modulo $T$. If $G$ is the unitary group $U(n)$ of order $n$ and if $T^n \subset U(n)$ is the diagonal subgroup, we write $F(n)$ instead of $U(n)/T^n$. 

The primary objective of our study is the set \([G/T, F(n)]\) of all homotopy classes of maps from \(G/T\) to \(F(n)\) (cf. §7.1 for references on the topic). For this purpose one may examine the correspondence that sends a map \(f\) to the induced cohomology homomorphism \(f^*\).

\[
(A) \quad r : [G/T, F(n)] \to \text{Hom}[H^2(F(n)), H^2(G/T)]
\]

(resp. \(r_1 : [G/T, F(n)] \to \text{Hom}[H^*(F(n)), H^*(G/T)]\)),

where \(\text{Hom}[H^*(F(n)), H^*(G/T)]\) is the set of all ring maps from \(H^*(F(n))\) to \(H^*(G/T)\). It follows from rational homotopy theory \([GH]\) that \(r_1\) is finite to one;

We recall also from Bott and Samelson \([BS]\) that

the ring \(H^*(G/T)\) is torsion free and over rationals generated multiplicatively by elements from \(H^2(G/T)\).

For these reasons a solution to the following problem is an important step towards understanding the set \([G/T, F(n)]\).

**Problem.** Which homomorphisms \(h : H^2(F(n)) \to H^2(G/T)\) are induced by maps \(G/T \to F(n)\)?

3. **Notions**

For a compact connected Lie group \(G\) with a fixed maximal torus \(T\), consider the transgression

\[
(B) \quad \tau_G : H^1(T) = \text{Hom}(T, S^1) \to H^2(G/T)
\]

in the fibration \(T \hookrightarrow G \to G/T\) \([BH, \S 10.1]\), where \(S^1\) is the circle group, and where as indicated the 1–dimensional cohomology \(H^1(T)\) is canonically identified with \(\text{Hom}(T, S^1)\), the set of 1–dimensional unitary representation of \(T\) (cf. Lemma 6 in §6).

If \(G = U(n)\) is the unitary group, the diagonal subgroup \(T^n \subset U(n)\) has the factorization \(T^n = \text{diag}\{e^{i\theta_1}, \cdots, e^{i\theta_n}\}\). Let \(t'_k \in H^1(T^n) = \text{Hom}(T^n, S^1)\) be the projection onto the \(k\)th factor, \(1 \leq k \leq n\), and put \(t_k = \tau_{U(n)}(t'_k) \in H^2(F(n))\).

**Lemma 1.** In \(H^2(F(n))\) one has \(t_1 + \cdots + t_n = 0\). Further, the classes \(t_1, \cdots, t_{n-1}\) form a basis of \(H^2(F(n))\).

**Proof.** This follows from the description of the ring \(H^*(F(n))\) due to Borel \([B]\): \(H^*(F(n)) = \mathbb{Z}[t_1, \cdots, t_n]/< e_i | 1 \leq i \leq n >\), where the \(e_i \in \mathbb{Z}[t_1, \cdots, t_n]\) is the \(i\)th elementary symmetric function in the \(t_1, \cdots, t_n\). \(\square\)
Consider next a compact connected semi–simple $G$ with a fixed maximal torus $T$. Let $p : \tilde{G} \to G$ be the universal cover of $G$ and let $\tilde{T} \subset \tilde{G}$ be the maximal torus that corresponds to $T$ under $p$. Then $p$ induces an isomorphism $\tilde{G}/\tilde{T} \cong G/T$ between flag varieties. It is known that ([BH,§10.1], [DZZ,Theorem 2])

**Lemma 2.** A set $\{\omega_i \in \text{Hom}(\tilde{T}, S^1) \mid 1 \leq i \leq m = \dim \tilde{T}\}$ of fundamental dominant weights of $\tilde{G}$ relative to $\tilde{T}$ constitutes a basis of $H^1(\tilde{T}) = \text{Hom}(\tilde{T}, S^1)$ (cf. Lemma 6 in §6), and the transgression $\tau_{\tilde{G}} : H^1(\tilde{T}) \to H^2(G/T)$ is an isomorphism.

Consequently, if we make no difference in notation between elements in $H^1(\tilde{T})$ and their images under $\tau_{\tilde{G}}$ in $H^2(G/T)$ (as in [BH]), we have

**Lemma 3.** The set $\{\omega_i \in H^2(G/T) \mid 1 \leq i \leq m\}$ of fundamental dominant weights of $\tilde{G}$ is a basis of $H^2(G/T)$.

**Remark 1.** The classes $\omega_i \in H^2(G/T), 1 \leq i \leq m,$ are of particular interests in the algebraic intersection theory on $G/T$. With respect to the classical Schubert cell decomposition of $G/T$ [BGG], they are precisely the special Schubert classes on $G/T$ (cf. [DZZ, §5.2]).

In view of Lemma 1 and 3, every homomorphism $h : H^2(F(n)) \to H^2(G/T)$ has a canonical numerical characterization as

(C) $h(t_k) = a_{k,1}\omega_1 + \cdots + a_{k,m}\omega_m, 1 \leq k \leq n - 1,$

where $a_{k,i} \in \mathbb{Z}$. As a result, if we let $\mathbb{Z}^+$ be the set of non–negative integers, and let $\mathbb{Z}^+ [\omega_1, \cdots, \omega_m, \rho]$ be the semiring of polynomials in the variables $\omega_1, \cdots, \omega_m, \rho$ with coefficients in $\mathbb{Z}^+$ that is subject to the relation

(D) $\omega_1 \cdots \omega_m \rho = 1,$

we may introduce a map

$s : \text{Hom}(H^2(F(n)), H^2(G/T)) \to \mathbb{Z}^+ [\omega_1, \cdots, \omega_m, \rho]$

by the simple algorithm: if $h : H^2(F(n)) \to H^2(G/T)$ is given as that in (C), we put

$$s(h) = \sum_{1 \leq k \leq n - 1} \omega_1^{a_{k,1}} \cdots \omega_m^{a_{k,m}} + \omega_1^{b_1} \cdots \omega_m^{b_m},$$

where $b_i = -(a_{i,1} + \cdots + a_{n-1,i})$. We note that although the $s(h)$ may appear as an element in the semiring $\mathbb{Z}^+ [\omega_1^\pm, \cdots, \omega_m^\pm]$ of Laurent polynomials, the obvious relations $\omega_k^{-1} = \rho \prod_{j \neq k} \omega_j, 1 \leq k \leq m,$ from (D) are sufficient to convert it to an element in $\mathbb{Z}^+ [\omega_1, \cdots, \omega_m, \rho].$
4. Preliminaries from representation theory

Denote by $R^+(G)$ the semiring of isomorphism classes $\{V\}$ of $G-$Modules $V$ over $\mathbb{C}$. As in §3 let $\tilde{G}$ be the universal cover of a semi–simple $G$ with $\tilde{T} \subset \tilde{G}$ the maximal torus that corresponds to $T$ in $G$. The character homomorphism of $\tilde{G}$ is the semiring map

$$\chi : R^+(\tilde{G}) \rightarrow R^+(\tilde{T}) = \mathbb{Z}^+[\omega_1, \cdots, \omega_m, \rho],$$

given by restriction $\{V\} \rightarrow \{V \mid \tilde{T}\}$, where the identification

$$R^+(\tilde{T}) = \mathbb{Z}^+[\omega_1, \cdots, \omega_m, \rho]$$

follows from Lemma 2.

For a $\{V\} \in R^+(\tilde{G})$, the polynomial $\chi(V) \in \mathbb{Z}^+[\omega_1, \cdots, \omega_m, \rho]$ is called the character of the $\tilde{G}$–module $V$. We will be in particular interested in the subset $\Omega(G) = \text{Im} \chi$ of $R^+(\tilde{T})$. It admits the partition

$$\Omega(G) = \bigoplus_{1 \leq n} \Omega_n(G)$$

with $\Omega_n(G) = \{g \in \Omega(G) \mid g(1, \cdots, 1) = n\}$. Alternatively, the $\Omega_n(G)$ consists of all characters of $n$–dimensional representations of $\tilde{G}$.

Indeed, elements in $\Omega(G)$ (resp. in $\Omega_n(G)$) are abundant. To explain this we consider the set $\Lambda(G) = \{a_1\omega_1 + \cdots + a_m\omega_m \in H^2(G/T) \mid a_m \in \mathbb{Z}^+\}$ of dominant weights of $G$. For a $\lambda \in \Lambda(G)$ let $L_\lambda$ be the complex line bundle over $G/T$ with first Chern class $c_1(L_\lambda) = \lambda$, and let $V_\lambda$ be the $\tilde{G}$–module of holomorphic sections of $L_\lambda$. Since the set $\{V_\lambda \mid \lambda \in \Lambda(G)\}$ presents all finite dimensional complex representations of $\tilde{G}$ (cf. [LG, p. 69]) and since the $\chi$ is injective, we have

**Lemma 4.** The set of infinitely many polynomials

$$\{\chi(V_\lambda) \in \mathbb{Z}^+[\omega_1, \cdots, \omega_m, \rho] \mid \lambda \in \Lambda(G)\}$$

is a basis for the semiabelian group $\Omega(G)$ (over $\mathbb{Z}^+$).

We emphasis at this stage that there have been several methods computing the character $\chi(V_\lambda) \in \mathbb{Z}^+[\omega_1, \cdots, \omega_m, \rho]$ in terms of the $\lambda \in \Lambda(G)$. These are the Weyl character formula [Hu], Demazure character formula [D] and the Lakshmibai-Seshadri character formula ([LS], [Li]). Consequently, the $\Omega(G)$ admits concrete presentation as a set of polynomials. We present such an example for further use.

If $G = SU(m)$ is the special unitary group of order $m$, then $(\tilde{G}, \tilde{T}) = (G, T)$, $G/T = F(m)$. Consider the semiring map
\[ \alpha : R^+(T) = Z^+[\omega_1, \ldots, \omega_{m-1}, \rho] \to Z^+[y_1, \ldots, y_m] \]

by \( \omega_k \rightarrow y_1 \cdots y_k, 1 \leq k \leq m-1; \rho \rightarrow y_2 y_3 \cdots y_{m-1} \), where \( Z^+[y_1, \ldots, y_m] \) is the semiring of polynomials in the \( y_1, \ldots, y_m \) with coefficients in \( Z^+ \) that is subject to the relation \( y_1 \cdots y_m = 1 \).

For a partition \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0) \) let \( s_{\mu}(y_1, \ldots, y_m) \in Z^+[y_1, \ldots, y_m] \) be Schur symmetric function associated to \( \mu \) [M, p.40], and let \( S[y_1, \ldots, y_m] \subset Z^+[y_1, \ldots, y_m] \) be the semiabelian group generated over \( Z^+ \) by all Schur functions \( s_{\mu}(y_1, \ldots, y_m) \) associated to the \( \mu \) with last part \( \mu_m = 0 \).

Then (cf. [FH, Chapter 15], [LG; Chapter 5])

**Lemma 5.** \( \alpha \) maps \( \Omega(SU(m)) \) isomorphically onto \( S[y_1, \ldots, y_m] \).

For the geometries underlying the transformations \( \alpha \), we refer to [DZZ; Example 3] or [DZ2; (4.1)].

### 5. Character factors through homotopy

Let \( \text{Hom}^0(\tilde{G}, U(n)) \) be the set of all homomorphisms \( g : \tilde{G} \rightarrow U(n) \) that satisfy \( g(\tilde{T}) \subset T^n \). The obvious inclusion \( \iota_n : \text{Hom}^0(\tilde{G}, U(n)) \rightarrow R^+(\tilde{G}) \) is surjective onto the subset \( R_n^+(\tilde{G}) \) of isomorphism classes of \( n \)-dimensional complex representations of \( \tilde{G} \). On the other hand, since every \( g \in \text{Hom}^0(\tilde{G}, U(n)) \) preserves the maximal torus, there is a ready–made map

\[ \rho : \text{Hom}^0(\tilde{G}, U(n)) \rightarrow [G/T, F(n)] \]

defined by the commutivity of the diagram

\[
\begin{array}{ccc}
\tilde{G} & \xrightarrow{g} & U(n) \\
\downarrow & & \downarrow \\
G/T = \tilde{G}/\tilde{T} & \xrightarrow{\rho(g)} & F(n)
\end{array}
\]

where the vertical maps are the obvious quotients. We may now organize the relevant correspondences in the diagram below.

\[
\begin{array}{ccc}
\text{Hom}^0(\tilde{G}, U(n)) & \xrightarrow{\iota_n} & R^+(\tilde{G}) \\
\rho \downarrow & & \downarrow \chi \\
[G/T, F(n)] & \xrightarrow{s} & \text{Hom}[H^2(F(n)), H^2(G/T)] & \xrightarrow{\delta} & Z^+[\omega_1, \ldots, \omega_m, \rho]
\end{array}
\]

**Theorem** (Character factors through homotopy). \( \chi \circ \iota_n = s \circ r \circ \rho \).

The proof of the Theorem will be postponed until the next section and at this moment, we show how it leads to a partial solution to our problem. From the Theorem we obtain
Corollary 1. For any polynomial \( g \in \Omega_{n}(G) \), there exists a map \( f : G/T \rightarrow F(n) \) such that \( s(f^*) = g \). Furthermore, one may read the induced map \( f^* : H^2(F(n)) \rightarrow H^2(G/T) \) from the polynomial \( g \).

The first part of Corollary 1 may be rephrased as

Corollary 2. For a \( h \in \text{Hom}(H^2(F(n)), H^2(G/T)) \) with \( s(h) \in \Omega(G) \), there exists a map \( f : G/T \rightarrow F(n) \) such that \( f^* = h \).

In algebraic geometry, Schur functions appear as the polynomial representatives of Schubert classes in complex Grassmanian manifolds (cf. [L], [St]). In representation theory, Schur functions occur as the characters of irreducible representations of unitary groups [T], [FH]. Combining Lemma 5 with Corollary 1 we get the following result illustrating the fashion by which homotopy theory interacts with combinatorics of symmetric functions.

Corollary 3. For any symmetric function \( g \in S[y_1, \ldots, y_m] \) with \( g(1, \ldots, 1) = n \), there exists a map \( f : F(m) \rightarrow F(n) \) such that \( \alpha \circ s(f^*) = g \).

Remark 2. According to Corollary 3, every Schur function \( s_{\mu}(y_1, \ldots, y_m) \) with \( s_{\mu}(1, \ldots, 1) = n \) can arise as the \( \alpha \circ s \)-image of a linear map \( f_{\mu}^* : H^2(F(n)) \rightarrow H^2(F(m)) \).

We refer to [P] for three definitions of Schur functions in combinatorics. It would be natural to expect that useful properties of Schur functions can be derived from the linear maps \( f_{\mu}^* \).

6. Proof of the Theorem

The binary operation

\[
\text{Hom}(T, S^1) \times \text{Hom}(T, S^1) \rightarrow \text{Hom}(T, S^1)
\]

by \( \alpha \beta = \mu \circ (\alpha \times \beta) \circ \Delta \) furnishes the set \( \text{Hom}(T, S^1) \) with the structure of an abelian group, where \( \Delta : T \rightarrow T \times T \) is the diagonal embedding and where \( \mu : S^1 \times S^1 \rightarrow S^1 \) is the product in \( S^1 \). Consider the map

\[
\iota : \text{Hom}(T, S^1) \rightarrow H^1(T)
\]

by \( \iota(\alpha) = \alpha^*[S^1] \), where \( \alpha^* : H^1(S^1) \rightarrow H^1(T) \) is induced by \( \alpha \) and where \( [S^1] \in H^1(S^1) = \mathbb{Z} \) is the fundamental class of the oriented circle \( S^1 \). The following standard fact clarifies the identification \( H^1(T) = \text{Hom}(T, S^1) \) that we have adopted in (B) and in Lemma 2.

Lemma 6. The correspondence \( \iota \) is an isomorphism of abelian groups. In particular, one has
\[ \iota(\omega_1^a \cdots \omega_n^a) = a_1 \omega_1 + \cdots + a_m \omega_m \text{ (in } H^1(T)), \]

where we use \( \omega_i \in H^1(T) \) instead of \( \iota(\omega_i) \) (as in Lemma 2 for the sake of simplicity).

We are now ready to establish the Theorem. Each representation \( g \in \text{Hom}_0(\tilde{G}, U(n)) \) induces a bundle map

\[
\begin{array}{ccc}
\tilde{T} & \hookrightarrow & \tilde{G} \\
g' & \downarrow & g \\
T^n & \hookrightarrow & U(n) \\
\rho(g) & \downarrow & F(n)
\end{array}
\]

where the \( g' \) is the restriction of \( g \) to \( \tilde{T} \). From the naturality of the transgression (B) we have the commutative diagram

\[
\begin{array}{ccc}
H^1(T^n) & \xrightarrow{\tau_{U(n)}} & H^2(F(n)) \\
g'^* & \downarrow & \rho(g)^* \\
H^1(\tilde{T}) & \xrightarrow{\tau_{\tilde{G}}} & H^2(G/T)
\end{array}
\]

With the \( \tau_{U(n)} \) and \( \tau_{\tilde{G}} \) being specified respectively in Lemma 1 and 2, we find that if the \( \rho(g)^* \) is given by

\[ \rho(g)^*(t_k) = a_{k,1}\omega_1 + \cdots + a_{k,m}\omega_m, \quad 1 \leq k \leq n - 1 \text{ (cf. (C))}, \]

in \( H^2(G/T) \), we have

\[ g'^*(t'_k) = a_{k,1}\omega_1 + \cdots + a_{k,m}\omega_m, \quad 1 \leq k \leq n - 1, \text{ and} \]

\[ g'^*(t'_n) = b_1\omega_1 + \cdots + b_m\omega_m \]

in \( H^1(\tilde{T}) \), where \( b_i = -(a_{1,i} + \cdots + a_{n-1,i}) \). However, as is standard, the character \( \chi(g) \) can be computed in terms of (E) as

\[ \chi(g) = \iota^{-1}[g'^*(t'_1)] + \cdots + \iota^{-1}[g'^*(t'_n)] \]

This agrees with \( s(\rho(g)^*) \) by Lemma 6.

### 7. Endnotes

This paper is by no means a final exposition in the topic, but is part of a larger effort to understand maps between flag manifolds by means of cohomology.

#### 7.1. During the past two decades many works were devoted to the study of self–maps of flag manifolds, cf. [GH], [H1], [H2], [HH], [P1], [P2], [DZ1], [Du].
However, the corresponding investigation into maps between different flag manifolds has not yet received as much attention as it deserves. The aim of present work is to demonstrate concrete examples (from representation theory) indicating the richness of the latter topic.

7.2. Our method can be extended to the study of homotopy classes of maps $G/T \to G'/T'$ between two arbitrary flag manifolds. For instance in the Problem in §2, one may replace the flag manifold $F(n)$ by $D(n) = SO(n)/T$ (resp. $T(n) = Sp(n)/T$), where the $SO(n)$ (resp. $Sp(n)$) is the special orthogonal group (resp. symplectic group) of order $n$ and where $T \subset SO(n)$ (resp. $T' \subset Sp(n)$) is a maximal torus. In these cases results analogous to the Theorem hold, only the character homomorphisms for the real and quaternionic representations will be involved and the correspondence $s$ in §3 would be modified accordingly.

7.3. Let $Map(G/T, F(n))$ be the space of all continuous maps $G/T \to F(n)$. For a $f \in Map(G/T, F(n))$, let $M_f \subset Map(G/T, F(n))$ be the path–connected component containing $f$. In [Me] and [S], W. Meier and S. Smith initiated the project to determine the rational homotopy type of the space $M_f$.

**Conjecture.** The rational homotopy type of $M_f$ is determined by the polynomial $s \circ r(f) \in Z^+[\omega_1, \cdots, \omega_m, \rho]$.

More precisely, if $g \in Hom^0(U(m), U(n))$ is such that

$$\alpha \circ \chi(g) = l(t_1 + \cdots + t_m) + (n - lm) \ (cf. \ Corollary \ 3),$$

then, modulo the products of odd dimensional spheres, the rational homotopy type of the space $M_{\rho(g)}$ has been determined by S. Smith [S]. We observe that the sum $t_1 + \cdots + t_m$ is the Schur function associated to the partition $\mu = (1, 0, \cdots, 0)$ ([M, P.42]). We note also from Lemma 5 that corresponding to any Schur function $s_\mu(y_1, \cdots, y_m)$ with $s_\mu(1, \cdots, 1) = n$, there is a representation $g_\mu \in Hom^0(U(m), U(n))$ with $\alpha \circ \chi(g_\mu) = s_\mu(y_1, \cdots, y_m)$.

**Problem.** For a partition $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$, determine the rational homotopy type of the space $M_{\rho(g_\mu)}$.

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