We consider the power of an collisional Penrose process with an unbound energy extraction from an extreme Reissner-Nordström black hole. It takes infinite time in a time coordinate at a constant radial coordinate outside of the black hole in the process. For a practical use of black holes as a power plant, the power of the process for an observer far away from the black hole can be useful. We define the power as the energy gain from the extreme Reissner-Nordström black hole divided by time interval of the process in a coordinate time and we find the upper bound of the power in a near-horizon limit while the efficiency of the process can be arbitrary large in the limit. Thus, we conclude that there is no trade-off relation between the efficiency and power in the collisional Penrose process in the extreme Reissner-Nordström spacetime.

I. INTRODUCTION

In 1969, Penrose suggested a process, which is called Penrose process, to extract rotational energy from a Kerr black hole by dropping a particle with a negative energy in the black hole [1] and Denardo and Ruffini pointed out that the electromagnetic counterpart of the Penrose process occurs in Reissner-Nordström spacetime [2]. Blandford and Znajek considered the electromagnetic extraction of rotational energy from the Kerr black hole in an astrophysical situation [3] and it is considered that the electromagnetic extraction of energy from black holes can explain high energy jets near black holes.

Piran, Shaham, and Katz in 1975 [4] and T. Piran and J. Shaham in 1977 [5] investigated particle collision near the Kerr black hole. They pointed out that the center-of-mass energy of the two particle can be arbitrary high if the Kerr black hole has an extremal event horizon and one of two particles has a critical angular momentum. The particle collision with infinite center-of-mass energy called Bañados-Silk-West process since it is rediscovered by Bañados, Silk, and West in 2009 [6].

After the rediscovering the BSW process, several authors sent critical looks toward the process with the arbitrary high center-of-mass energy. There is an upper bound of a rotation of the Kerr black hole and it cannot be extreme rotation black hole in an astrophysical situation and The gravitational radiation and black-reaction constrain the center-of-mass energy for the particle collision [2]. A infinite proper time of a falling particle to reach the extreme event horizon is required to obtain unbound center-of-mass energy [8]. Self-gravity of falling objects bounds the center-of-mass energy of collisions [8, 10].

The BSW’s letter [6] stimulates several authors to investigate detail of the collisional Penrose process [6]. The upper limit of energy extraction by the collision Penrose process after the BSW collision near the extreme Kerr black hole is very modest [11, 12]. Schnittman found a collisional Penrose process with energy gain can be more than ten times the energy of incident particles [13, 17]. A collisional Penrose process after the head-on collision of two particles near an extreme Kerr black hole [18–21], collisional Penrose processes with spinning particles [22, 23], and collisional Penrose processes in wormhole spacetimes [24, 25] and in an overspinning Kerr black hole spacetime [26] were investigated.

Zaslavskii found the electromagnetic counterpart of BSW collision in an extreme charged Reissner-Nordström black hole [27]. Zaslavskii pointed out that energy extraction from an extreme Reissner-Nordström black hole in a collisional Penrose process after the BSW collision can be unbound [28, 29] while the energy extraction from the extreme Kerr black hole is very modest. The Reissner-Nordström spacetime is more tractable than the Kerr black hole due to spherical symmetry of the spacetime. A finite center-of-mass energy of BSW collisions of two shells includes their self-gravity was shown in Ref. [9] and a upper bound of energy extraction from the extreme Reissner-Nordström black hole by fully taking into account the self-gravity of the colliding shells was obtained in Ref. [30].

On this paper, inspired by a trade-off relation between efficiency and power of a heat engine by Shiraishi, Saito, and Tasaki [31], we consider the power of the collisional Penrose process which gives infinite efficiency in the extreme Reissner-Nordström spacetime after the BSW collision. Is there a trade-off relation between efficiency and power of the collisional Penrose process? To answer the

1 A multiple BSW collision and multiple Penrose process in Reissner-Nordström spacetime [31, 32] and a BSW collision in higher-dimensional Reissner-Nordström spacetime [33] were also investigated.
question, we define the power of the collisional Penrose process as the energy extraction divided by a coordinate time and we discuss the maximum of the power in the process.

This paper is organized as follows. In Sec. II, we review the motion of a charged particle in an extreme Reissner-Nordström spacetime. In Sec. III, we review the energy extraction from the extreme Reissner-Nordström black hole in a collision Penrose process and we investigate the power of the process. In Sec. IV, we conclude our result. In this paper, we use geometrical units in which the light speed and Newton’s constant are unity.

II. MOTION OF A CHARGED PARTICLE IN A REISSNER-NORDSTRÖM SPACETIME

A line element and a vector potential in Reissner-Nordström spacetime are expressed by

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{2.1} \]

\[ A_\mu dx^\mu = -\frac{Q}{r}dt, \tag{2.2} \]

where \( f(r) \) is given by

\[ f(r) \equiv 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.3} \]

and \( Q \) and \( M \) are an electrical charge and a mass, respectively. It is a black hole spacetime with an event horizon at \( r = r_H \equiv M + \sqrt{M^2 - Q^2} \) for \( |Q| \leq M \) and it is a spacetime with naked singularity for \( |Q| > M \). We assume an extreme charge \( Q = M > 0 \) since we are interested in a collisional Penrose process with an unbound energy extraction \([28, 29]\).

The four-momentum \( p^\mu \) of a particle with an electrical charge \( q \) is expressed as, from the Hamiltonian equation,

\[ p^\mu = \frac{\partial H}{\partial \pi_\mu} = \pi^\mu - qA^\mu, \tag{2.4} \]

where \( H \) is the Hamiltonian of the charged particle given by

\[ H \equiv \frac{1}{2} g^{\mu\nu}(\pi_\mu - qA_\mu)(\pi_\nu - qA_\nu), \tag{2.5} \]

and \( \pi^\mu \) is the canonical momentum of the charged particle conjugate to the coordinates \( x^\mu \). We assume that charged particles have vanishing angular momentum \( L \equiv \pi_\phi = 0 \) and they move only in a radial direction on an equatorial plane \( \theta = \pi/2 \).

From the \( t \) component of the four-momentum \( p^t = dx^t/d\lambda \), where \( \lambda \) is an affine parameter, and from Eq. (2.4), we get

\[ \frac{dt}{d\lambda} = \frac{1}{f} \left( E - \frac{qM}{r} \right), \tag{2.6} \]

where \( E \equiv -\pi_t \) is a conserved energy of the charged particle. The particle should satisfy a forward-in-time condition \( dt/d\lambda \geq 0 \). The condition is expressed by

\[ E - \frac{qM}{r} \geq 0 \tag{2.7} \]

and it yields, for \( r = r_H = M \),

\[ E = q. \tag{2.8} \]

We call a charged particle with \( E = q \) critical.

From \( g_{\mu\nu}p^\mu p^\nu = -m^2 \), where \( m \) is the mass of the charged particle, we obtain

\[ \left( \frac{dr}{d\lambda} \right)^2 + V(r) = 0, \tag{2.9} \]

where \( V(r) \) is the effective potential of the radial motion of the charged particle given by

\[ V(r) \equiv - \left( E - \frac{qM}{r} \right)^2 + m^2 f. \tag{2.10} \]

The charged particle can exist only in a region that the effective potential \( V(r) \) is non-positive. The radial component of the four momentum of a particle can be written in \( p^r = \sigma \sqrt{-V} \), where \( \sigma = -1 \) (\( \sigma = 1 \)) for a ingoing (outgoing) particle.

III. PARTICLE COLLISION AND ENERGY EXTRACTION FROM A BLACK HOLE

In this section, we review the energy extraction from the extreme Reissner-Nordström black hole in a collision Penrose process \([28, 29]\) and we investigate the power in the collision Penrose process. We consider that particles 1 and 2 collides near the black hole \( r = r_c \equiv M(1 + \epsilon) \), where \( 0 < \epsilon \ll 1 \), and particles 3 and 4 are produced. We set \( \sigma_1 = \sigma_2 = -1 \). Here and hereinafter, physical values with subscripts 1, 2, 3, and 4 denote physical values of particles 1, 2, 3, and 4, respectively. The center-of-mass energy \( E_{\text{CM}} \) of particles 1 and 2 at the collision is given by

\[ E_{\text{CM}}^2 \equiv -g_{\mu\nu}(p_1^\mu + p_2^\mu)(p_1^\nu + p_2^\nu) = m_1^2 + m_2^2 + \frac{2}{f(r_c)} \left( E_1 - \frac{q_1M}{r_c} \right) \left( E_2 - \frac{q_2M}{r_c} \right) + \sqrt{V_1(r_c)} \sqrt{V_2(r_c)} \tag{3.1} \]

If either of the particles is critical and the other is not critical, the center-of-mass energy diverges in a near-horizon limit \( \epsilon \to 0 \) \([27]\). For simplicity, we assume that particle 1 is critical \( E_1 = q_1 \) and particle 2 has no charge \( q_2 = 0 \). The center-of-mass energy is obtained as

\[ E_{\text{CM}}^2 = m_1^2 + m_2^2 + \frac{2(1 + \epsilon)}{\epsilon} |E_1 E_2| \]
The radial component of Eq. (3.4), we obtain immediately after the creation of the particles \([28, 29]\):

\[
-\sqrt{E_1^2 - m_1^2} \sqrt{E_2^2 - m_2^2} \left( \frac{\epsilon}{1 + \epsilon} \right)^2 \sim \frac{2A_1E_2}{\epsilon},
\]

where \(A_1\) is defined by \(A_1 \equiv E_1 - \sqrt{E_1^2 - m_1^2}\).

The conservation law of the charges before and after the particle collision is expressed by

\[
q_1 + q_2 = q_3 + q_4. \quad (3.3)
\]

The conservation law of the four-momentum of the particles at the moment of the collision is expressed by

\[
p_1^i + p_2^i = p_3^i + p_4^i. \quad (3.4)
\]

The \(t\) component of the conservation laws of the four-momentum (3.4) and the charges (3.3) gives the conservation laws of the conserved energy

\[
E_1 + E_2 = E_3 + E_4. \quad (3.5)
\]

We assume that particle 3 is a near-critical particle with \(q_3 = E_3(1 + c_3\epsilon)\), where \(0 < c_3 < 1\), and that particles 3 and 4 are ingoing particles with \(\sigma_3 = \sigma_4 = -1\) immediately after the creation of the particles. The radial component of Eq. (3.4), we obtain

\[
A_1 + E_3(\delta_3 - 1) = -\sqrt{E_3^2(1 - \delta_3)^2 - m_3^2}, \quad (3.6)
\]

From the square of Eq. (3.6), we obtain

\[
\delta_3 = 1 - \frac{m_3^2 + A_1^2}{2A_1E_3}. \quad (3.7)
\]

We assume that particle 3 is reflected at a turning point \(r = r_-\) and we assume that \(E_3 > m_3\) so that particle 3 goes to spacial infinity. For simplicity, we also assume that particle 4 does not interact with particle 3 after their particle production and particle 4 falls into the event horizon of the black hole. From \(V_3(r_-) = 0\), we obtain the turning point \(r = r_-\) as

\[
r_- = M \left( 1 + \frac{E_3\delta_3}{E_3 - m_3} \right). \quad (3.8)
\]

From \(r_- \leq r_c\), a condition

\[
E_3(1 - \delta_3) \geq m_3 \quad (3.9)
\]

must be satisfied. If the inequality (3.9) holds, the inside of the square root in (3.8) is positive and the lower bound of the mass of particle 3

\[
m_3 \geq A_1 \quad (3.10)
\]

must hold so that the left-hand side of the Eq. (3.6) is negative. For simplicity, we assume \(m_0 \equiv m_1 = m_2\) and \(E_1 = E_2 \geq m_0\). As discussed in Ref. [29], the mass and conserved energy of particle 3 can be \(m_3 \sim E_{\text{CM}} \sim m_0/\sqrt{\epsilon}\) and \(E_3 \sim m_0/\epsilon\), respectively.

The collisional Penrose process is expressed in Figure 1:

Particle 1 at an initial position \(r = r_i\) falls toward the extreme charged black hole and it collides with particle 2 at \(r = r_c\) after the coordinate time of \(\Delta t_1\). Due to the collision, particles 3 and 4 are produced. Particle 3 is reflected at \(r = r_-\) and it escapes to \(r = r_i\) after the coordinate time of \(\Delta t_3\). On the other hand, particle 4 reaches to the extreme event horizon at \(r = r_H\). The time intervals \(\Delta t_1\) and \(\Delta t_3\) in the coordinate time are given by

\[
\Delta t_1 = \frac{E_1}{\sqrt{E_1^2 - m_0^2}} \int_{r_1}^{r_c} \frac{dr}{(1 - M/r)^2} \sim \frac{E_1}{\sqrt{E_1^2 - m_0^2}} \left( \frac{M}{\epsilon} + r_i \right), \quad (3.11)
\]

and

\[
\Delta t_3 = -\int_{r_c}^{r_-} \frac{E_3 - \frac{q_3M/r}{(1 - M/r)^2} \sqrt{-V_3}}{\sqrt{E_3^2 - m_3^2}} \, dr + \int_{r_-}^{r_1} \frac{E_3 - \frac{q_3M/r}{(1 - M/r)^2} \sqrt{-V_3}}{\sqrt{E_3^2 - m_3^2}} \, dr \sim \left( \frac{2}{\delta_3 - 2} \right) \frac{M}{\epsilon} + r_i, \quad (3.12)
\]

respectively.

We define the power of the collisional Penrose process as \(\Delta E/\Delta t\), where \(\Delta E\) and \(\Delta t\) are defined by \(\Delta t \equiv r_c - r_i\).
\[ \Delta t + \Delta t_3 \text{ and } \Delta E \equiv E_3 - E_1 - E_2 \sim m_0/\epsilon, \] respectively. From Eqs. (3.11) and (3.12), the power of the collisional Penrose process is given by

\[ \frac{\Delta E}{\Delta t} \sim \left( \frac{E_1}{\sqrt{E_1^2 - m_0^2}} + 2 \delta_3 - 2 \right) M + \left( \frac{E_1}{\sqrt{E_1^2 - m_0^2}} + 1 \right) r_1 \epsilon \]

(Eq. 3.13)

The power is plotted in Fig. 2. If \( \epsilon \ll M/r_1 < 1 \) is satisfied, by using \( E_3 \sim m_0/\epsilon \), \( m_3 \sim m_0/\sqrt{\epsilon} \) and Eq. (3.14), we get \( \delta_3 \sim 1 - m_0/(2A_1) \sim 1 - (e_1 + \sqrt{e_1^2 - 1}/2 \) and the power is given by

\[ \frac{\Delta E}{\Delta t} \sim \frac{1 - e_1^2 + (2 - e_1)\sqrt{e_1^2 - 1} m_0}{e_1^2 + 2e_1 - 2 + e_1\sqrt{e_1^2 - 1}} M. \] (3.14)

We notice that the collisional Penrose process has the upper bound of the power, the value of which is estimated to be Eq. (3.14) in the near-horizon limit \( \epsilon \to 0 \).

IV. CONCLUSION

We have defined the power of the collisional Penrose process as the energy gain from the extreme Reissner-Nordström black hole divided by time interval of the process in a coordinate time. We have found the upper bound of the power which is shown as Eq. (3.14) and Fig. 2 in the near-horizon limit \( \epsilon \to 0 \). On the other hand, the efficiency of the collisional Penrose process defined by \( \eta \equiv E_3/(E_1 + E_2) \) has an arbitrary high value \( \eta \sim 1/(2\epsilon) \) in the near horizon limit \( \epsilon \to 0 \) under our assumptions as discussed in Ref. [29]. Therefore, we conclude that there is no trade-off relation between efficiency and power in the collisional Penrose process in the extreme Reissner-Nordström spacetime under our treatment. One may, however, find a trade-off relation between efficiency and power if the effect of the self-gravity of falling particles is taken into account. The power of the collisional Penrose process including self-gravity of the particles is left as a future work.

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