Nearly Optimal Space Efficient Algorithm for Depth First Search

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Abstract

We design a space-efficient algorithm for performing depth-first search traversal (Dfs) of a graph in $O(m + n \log^* n)$ time using $O(n)$ bits of space. While a normal Dfs algorithm results in a Dfs-tree (in case the graph is connected), our space bounds do not permit us even to store such a tree. However, our algorithm correctly outputs all edges of the Dfs-tree.

The previous best algorithm (which used $O(n)$ working space) took $O(m \log n)$ time (Asano, Izumi, Kiyomi, Konagaya, Ono, Otachi, Schweitzer, Tarui, Uehara (ISAAC 2014) and Elmasry, Hagerup, Krammer (STACS 2015)). The main open question left behind in this area was to design faster algorithm for Dfs using $O(n)$ bits of space. Our algorithm answers this open question as it has a nearly optimal running time (as the Dfs takes $O(m + n)$ time even if there is no space restriction).

1 Introduction

In analyzing algorithms, mostly we concentrate on minimizing the running time, or the quality of the solution (if the problem is hard). After we have optimized the above parameters, we then look to reduce the space taken by the algorithm, if possible. An excellent theoretical question is: Given a problem $P$, design an algorithm that solves it in as low space as possible. These algorithms are called space-efficient algorithms as we want to optimize on the space taken by the algorithm while not increasing the running time by much (compared to the best algorithm for the problem with no space restriction).

Recently, designing space-efficient algorithms has gained importance because of the rapid growth in the use of mobile devices and other hand-held devices which come with limited memory (e.g., the devices like Raspberry Pi, which are widely used in IoT applications). Another crucial reason for the increasing importance of the space-efficient algorithms is the rate and the volume at which huge datasets are generated (“big data”). Areas like machine learning, scientific computing, network traffic monitoring, Internet search, signal processing, etc., need to process big data using as less memory as possible.

Algorithmic fields like Dynamic Graph Algorithm [10, 21, 25, 26, 28] and Streaming algorithm [2, 3, 12, 27, 3, 11, 2] mandate low space usage by the algorithm. In a streaming algorithm, the mandate is mentioned upfront. In a dynamic graph algorithm, this mandate is implied as we want the update time of the algorithm to be as low as possible. Low update time implies that we don’t have enough time to look at our data-structure. Thus, we want our data-structure to be as compact as possible. Motivated by the growing body of work in the field of space-efficient algorithms, this paper focuses on optimizing the space taken by the DFS algorithm, which is one of the fundamental graph algorithms.

However, one needs to be slightly cautious about the definition of space. For a graph problem, it would take $O(m + n)$ space just to represent the graph. So, it seems that any graph problem requires $\Omega(m + n)$ bits. To avoid such trivial answers, we first define our model of computation.
1.1 Model of Computation: Register Input Model [19]

Frederickson [19] introduced the register input model in which the input (graph – in this case) is given in a read-only memory (thus, it cannot be modified). Also the output of the algorithm is written on a write-only memory. Along with the input and the output memory, a random-access memory of limited size is also available. Similar to the standard RAM model, the data on the input memory and the workspace is divided into words of size $\Theta(\log n)$ bits. Any arithmetic, logical and bitwise operations on constant number of words take $O(1)$ time.

When we say that our algorithm uses $O(n)$ bits, this is the space on the random-access memory used by our algorithm. The above model takes care of the case when the input itself takes a lot of space — by designating a special read-only memory for the input.

We highlight some results that make use of the register input model. Pagter and Rauhe [23] described a comparison-based algorithm for sorting $n$ numbers: for every given $s$ with $\log n \leq s \leq n/\log n$, an algorithm that takes $O(n^2/s)$ time using $O(s)$ bits. A matching lower bound of $\Omega(n^2)$ for the time-space product was given by Beame [11] for the strong branching-program model. Please see references for other problems in this model [13, 14, 18, 22, 24, 4, 6, 9, 8, 15]. In this paper, our main focus is on the Depth First Search Problem.

1.2 DFS Problem

The problem of space efficient DFS has received a lot of attention recently. Asano et al. [5] designed an algorithm that can perform DFS in (unspecified) polynomial time using $n + o(n)$ bits. If the space is increased to $2n + o(n)$ bits then their running time decreases to $O(mn)$. They also showed how to perform DFS in $O(m \log n)$ time using $O(n)$ bits. Elmasry et al. [17] improved this result by designing an algorithm that can perform DFS in $O(m + n)$ time using $O(n \log \log n)$ bits. Banerjee et al. [7] proposed an efficient DFS algorithm that takes $O(m + n)$ time using $O(m + n)$ space. Note that this is a strict improvement (over the Elmasry et al. [17] result) only if the graph is sparse. The following open question was raised by Asano et al. [5] in their paper:

Using $O(n)$ space, can DFS be done in $o(m \log n)$ time?

Recently, Hagerup [20] claimed an algorithm that finds DFS in $O(m \log^* n + n)$ time using $O(n)$ bits of space. We improve upon this algorithm giving a near optimal running time for DFS — it is almost linear in $m + n$. Our result can be succinctly stated as follows:

Theorem 1. There exists a randomized algorithm that can perform DFS of a given graph in $O(m + n \log^* n)$ time with a high probability ($(1 - 1/n^c)$ (where $c \geq 1$)) using $O(n)$ bits of space. (Note that our algorithm is randomized because we use succinct dictionaries that use random bits)

The succinct dictionary (used by our algorithm) performs insertion/deletion in $O(1)$ time with a probability of $(1 - 1/n^c)$ (where $c \geq 3$). Our algorithm performs at most $O(n \log^* n + m)$ insertions/deletions across all dictionaries. Hence, the probability that our algorithm takes more than $O(1)$ time for any of these $O(n \log^* n + m)$ insertions/deletions is $O(1/n^{c-2})$ (by union bound).

2 Overview

We will assume that vertices of input graph $G$ are numbered from 1 to $n$. Let $\mathcal{N}(v)$ denote the neighborhood of the vertex $v$ and $\mathcal{N}(v)[k]$ denote the $k$-th neighbor of the vertex $v$, where $1 \leq k \leq |\mathcal{N}(v)|$. As in [17], we will assume that $\mathcal{N}(v)$ is an array. So, we have random access to any element in this array. Also, we implicitly know the degree of $v$, $\deg(v) = |\mathcal{N}(v)|$. 

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Normally, the DFS algorithm outputs the DFS tree. Given the space bounds, we cannot store the DFS tree, but, we output the edges of the DFS tree as soon as we encounter them. We view that the problem is solved if the output edges form a valid DFS tree.

We first give a quick overview of the non-recursive implementation of the DFS algorithm. Let $G(V,E)$ be the input graph having $n$ vertices and $m$ edges. For this implementation, we will use a stack $S$. Initially, all vertices are colored white and assume that we start the DFS from a vertex $u$. So, $u$ is added to the stack $S$. The algorithm then processes all elements of the stack till it becomes empty. Thus, the top vertex, say $u$, is popped from the stack and is processed as follows: each neighbor of $u$ is explored. If a white vertex $v$ is found, then $u$ is pushed on to the stack and processing of $v$ starts. If none of the neighbors of vertex $u$ are white, then $u$ is colored black. Whenever $u$ discovers a white vertex $v$, we push a tuple $(u, u.NEXT)$ on to $S$, where the second entry in the tuple tells us which neighbor of vertex $u$ to explore once processing of $u$ resumes.

Now, let us formally define the second entry in the tuple $(u, u.NEXT)$

**Definition 2.** For any vertex $u$, if $(u, u.NEXT)$ is an entry on the stack $S$, then $N(u)[u.NEXT]$ denotes the first neighbor of the vertex $u$ which is still not explored while processing $u$.

### Algorithm 1: INITIALIZE()

1 for $i \leftarrow 1$ to $n$ do
2    | COLOR($i$) $\leftarrow$ white;
3 end
4 foreach $u \in V$ do
5    if $u$ is white then
6        | Process($u$);
7    end
8 end

### Algorithm 2: Process($u$)

1 $S$.push($u, 1$);
2 while $S$ is not empty do
3    | ($v, k$) $\leftarrow$ top element of $S$;
4    | COLOR($v$) $\leftarrow$ gray;
/* scan neighbors of $v$ */
5    if $k \leq \text{deg}(v)$ then
6        | $S$.push($v, k + 1$)
7        | if COLOR($N(v)[k]$) is white then
8            | output edge ($v, N(v)[k]$);
9        | $S$.push($N(v)[k], 1$);
10    end
11 else
12    | COLOR($v$) $\leftarrow$ black ;
13 end
14 end
The space required to represent the first and second term of each tuple in the stack $S$ is $O(\log n)$ bits. As there are $n$ vertices in the graph, the size of the stack can reach $\Omega(n)$ in the worst case. So, the total space taken by the trivial algorithm is $O(n \log n)$ bits.

Our algorithm closely follows [17]. So, we first give a brief overview of their approach and later, we will explain our improvement over their approach.

### 2.1 Previous Approach (Elmasry et. al. [17])

The trivial Dfs algorithm does not work for Elmasry et al.[17] because the stack $S$ itself takes $O(n \log n)$ bits of space. Hence, stack $S$ is not implemented — but, is referred to as an *imaginary* stack. Let the stack $S$ be divided into segments of size $\frac{n}{\log n}$ — the first segment is the bottommost $\frac{n}{\log n}$ vertices of $S$, the second segment is the next $\frac{n}{\log n}$ vertices of $S$ and so on. A new stack $S_1$ is implemented, which contains vertices from at most top two segments of the imaginary stack $S$. Each entry of the stack $S_1$ is a tuple: $(v, v.$Next$)$ where $v \in V$. The space required to represent these two terms is at most $2 \log n$. Thus, the total space required for $S_1$ is $O(\frac{n}{\log n} \times \log n) = O(n)$ bits. Since, the size of $S_1$ is very small as compared to the imaginary stack $S$, the main problem arises when an element is to be pushed on $S_1$ but it is full or when $S_1$ becomes empty (but $S$ contains vertices). Thus, there is a need to make space in $S_1$ or a way to restore vertices in $S_1$.

To handle the case when $S_1$ is full, Elmasry et al.[17] remove the bottom half elements of $S_1$. So, a new entry can now be pushed on to $S_1$, and the Dfs algorithm can proceed as usual.

Handling the second case (when $S_1$ is empty) requires to restore the top segment of $S$ in $S_1$. It turns out that the restoration process is the main bottleneck of this Dfs algorithm. To aid the restoration process, Elmasry et al.[17] propose an elegant solution by maintaining an additional stack $T$, called a trailer stack. The top-most element of each segment in $S$ is called as a *trailer* element. The stack $T$ stores the trailer element of each segment in $S$ — except trailers of those segments which are already present in $S_1$.

The stack $T$ is crucially used in the restoration process. Let $(u, u.$Next$)$ be the second top most entry in stack $T$. This implies that the first vertex of top segment of $S$ is $N(u)[u.$Next$-1]$. Now, a Dfs-like algorithm is run starting from the vertex $N(u)[u.$Next$-1]$ to restore the top segment of $S$ in $S_1$ as follows:

*Temporarily the meaning of gray and white vertex is changed.* Then, process $v \leftarrow N(u)[u.$Next$-1]$ to find $(v.$Next$-1)$ as follows: find the first gray neighbor $w$ of $v$, mark it white, push $(v, \ell+1)$ (where $N(v)[\ell] = w$), and then start processing of $w$. Elmasry et al. [17] show that this restoration process correctly restores the top segment of $S$.

Some explanation is in order about the above procedure. Once we have found $v$, we want to find $v.$Next. Analogously, we can say that we want to find $v.$Next$-1$. This vertex, $w \leftarrow N(v)[v.$Next$-1]$, was a white vertex encountered while processing $v$. Due to $w$, we stopped the processing of $v$, put $(v, v.$Next$)$ on $S$ and start the processing of $w$.

Even though the above algorithm is correct, it is still slow. Finding the first gray neighbor of a vertex $v$ takes $O(\deg(v))$ time. To overcome this difficulty, Elmasry et al.[17] suggest the use of two more data-structures. The first data-structure $D$ is an array of size $n$ that contains the following information for each vertex $v$: if $v$ is an element of $S$, then $D(v)$ contains

- The segment number in which $v$ lies.
- The approximate position of $v.$Next$-1$ in $N(v)$.

Since there are $\log n$ segments of $S$ (as each segment is of size $O(n/\log n)$), it requires $\log \log n$ bits to represent the first quantity. Similarly, storing the approximate position also takes $O(\log \log n)$ bits. Thus the space required for $D$ is $O(n \log \log n)$ bits.
We give a brief overview of our approach. In [17], the array \( D \) plays a critical role in the restoration process. While restoring the top segment, \( D(v) \) provides the required information for each vertex \( v \) which is a part of the top-most segment. However, \( D(v) \) takes \( O(n \log \log n) \) bits – a space we cannot afford. Our main observation is that we do not require information related to all vertices while restoring \( S_1 \). Indeed, storing information about vertices in the top-most segment suffices. Unfortunately, it is not easy to keep information related to vertices in top-most segment efficiently in \( O(n) \) space. To overcome this difficulty, along with the stack \( S \) we implement \( S_2 \) (a dynamic dictionary – as described in Lemma 3) which contains information about top vertices \( \frac{2n}{(\log \log \log n)^2} \) vertices of the imaginary stack \( S \). For each vertex in \( S_2 \), we store \( O(\log \log n) \) bits of information that will help us when we restore \( S_1 \) (remember that the size of \( S_1 \) is much less than the size of \( S_2 \)). We can show that the size of \( S_2 \) is \( \approx O\left(\frac{2n}{(\log \log \log n)^2} \times \log \log n\right) = O\left(\frac{n}{\log \log n}\right) \) bits. Thus, we have successfully reduced the size of \( S_2 \) (named \( D \) in [17]).

Since \( S_2 \) does not store the information of all the vertices in stack \( S \), it faces the restoration problem as well. If top \( \frac{2n}{(\log \log n)^2} \) vertices are popped out of \( S \), those are also deleted from \( S_2 \). Thus, we need to restore \( S_2 \). To aid in the restoration of \( S_2 \), we implement another data-structure \( S_3 \), which contains the information top \( \frac{2n}{(\log \log \log n)^2} \) vertices of \( S \). For each vertex in \( S_3 \), we will store \( O(\log \log \log n) \) bits of information. The size of \( S_3 \) can be shown to be \( O\left(\frac{n}{\log \log \log n}\right) \) bits. It is not hard to see that this process goes on recursively and we have many data-structures \( S_i \) where the last data-structure is \( S_{\log^* n} \). \( S_{\log^* n} \) stores information about \( \frac{2n}{n^\alpha} \) vertices, where \( \alpha \geq 1 \) is some constant. But, the restoration problem does not disappear yet. Now the question is how do we restore \( S_{\log^* n} \)? Beyond this, we do not create any more data-structure. We restore \( S_{\log^* n} \) using the most trivial strategy, that is by running DFS all over again. Our main claim is that throughout our algorithm \( S_{\log^* n} \) is restored at most \( \alpha^2 \) times. We will show that the time taken to restore \( S_{\log^* n} \) is \( O(m + n) \). Thus the total time taken to restore \( S_{\log^* n} \) is \( O(\alpha^2(m + n)) = O(m + n) \) (since

\[ \frac{2n}{(\log \log \log n)^2} \]

Figure 1: A pictorial description of the approach in [17]. The restoration of stack \( S_1 \) depends on \( D \). The size of \( D \) is \( O(n \log \log n) \) and our aim is to reduce this size. Note that all the data-structures [17] are not shown in the figure.

The second term in \( D(v) \) helps to fasten the search process for \( v \). Next – 1 only if the degree of \( v \) is sufficiently small. However, to take care of high degree vertices, the trailer stack \( T \) is extended to include not only trailers but also all the pair \((u, u.\text{NEXT})\), where \( u \) is a high degree vertex. Finally, Elmasry et al. [17] show that the extended trailer stack \( T \) takes \( O(n) \) bits. Moreover, using \( D \) and the extended \( T \) restores \( S_1 \) correctly and efficiently.

2.2 Our Approach

We give a brief overview of our approach. In [17], the array \( D \) plays a critical role in the restoration process. While restoring the top segment, \( D(v) \) provides the required information for each vertex \( v \) which is a part of the top-most segment. However, \( D(v) \) takes \( O(n \log \log n) \) bits – a space we cannot afford. Our main observation is that we do not require information related to all vertices while restoring \( S_1 \). Indeed, storing information about vertices in the top-most segment suffices. Unfortunately, it is not easy to keep information related to vertices in top-most segment efficiently in \( O(n) \) space. To overcome this difficulty, along with the stack \( S \) we implement \( S_2 \) (a dynamic dictionary – as described in Lemma 3) which contains information about top vertices \( \frac{2n}{(\log \log \log n)^2} \) vertices of the imaginary stack \( S \). For each vertex in \( S_2 \), we store \( O(\log \log n) \) bits of information that will help us when we restore \( S_1 \) (remember that the size of \( S_1 \) is much less than the size of \( S_2 \)). We can show that the size of \( S_2 \) is \( \approx O\left(\frac{2n}{(\log \log \log n)^2} \times \log \log n\right) = O\left(\frac{n}{\log \log n}\right) \) bits. Thus, we have successfully reduced the size of \( S_2 \) (named \( D \) in [17]).

Since \( S_2 \) does not store the information of all the vertices in stack \( S \), it faces the restoration problem as well. If top \( \frac{2n}{(\log \log n)^2} \) vertices are popped out of \( S \), those are also deleted from \( S_2 \). Thus, we need to restore \( S_2 \). To aid in the restoration of \( S_2 \), we implement another data-structure \( S_3 \), which contains the information top \( \frac{2n}{(\log \log \log n)^2} \) vertices of \( S \). For each vertex in \( S_3 \), we will store \( O(\log \log \log n) \) bits of information. The size of \( S_3 \) can be shown to be \( O\left(\frac{n}{\log \log \log n}\right) \) bits. It is not hard to see that this process goes on recursively and we have many data-structures \( S_i \) where the last data-structure is \( S_{\log^* n} \). \( S_{\log^* n} \) stores information about \( \frac{2n}{n^\alpha} \) vertices, where \( \alpha \geq 1 \) is some constant. But, the restoration problem does not disappear yet. Now the question is how do we restore \( S_{\log^* n} \)? Beyond this, we do not create any more data-structure. We restore \( S_{\log^* n} \) using the most trivial strategy, that is by running DFS all over again. Our main claim is that throughout our algorithm \( S_{\log^* n} \) is restored at most \( \alpha^2 \) times. We will show that the time taken to restore \( S_{\log^* n} \) is \( O(m + n) \). Thus the total time taken to restore \( S_{\log^* n} \) is \( O(\alpha^2(m + n)) = O(m + n) \) (since

\[ \frac{2n}{(\log \log \log n)^2} \]

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1 In our algorithm, size of \( S_1 \) is bit different than that in [17]. It is mentioned in Remark [1].
Figure 2: A pictorial description of our approach. We implement many "stacks" $S_1, S_2, \ldots, S_{\log^* n}$. The restoration of $S_i$ uses $S_{i+1}$ as $S_{i+1}$ contains the vertices to be restored in $S_i$. Note that all our data-structures are not shown in the figure.

$\alpha$ is a constant). For other $S_i$'s ($i \neq \log^* n$), our analysis is slightly different and it is the main technical contribution of this paper. We will show that the total time taken to restore $S_i$ over the entire course of the algorithm is $O\left(\frac{m}{\log^{(i)} n} + n\right)$ where $\log^{(i)} n := \log \log \log \ldots \log n$. Thus, the time taken to restore all $S_i$'s over the entire course of the algorithm is $O(m + n \log^* n)$.

Let us now briefly describe the space taken by our algorithm. Each $S_i$ stores information about at most top $\frac{2n}{(\log^{(i)} n)^2}$ vertices of $S$. Also, for each such vertex, we will only store $O(\log^{(i)} n)$ bits. Using succinct dictionary \[\ref{16}\], we will show that we can implement $S_i$ in $O\left(\frac{n}{\log^{(i)} n}\right)$ space. Thus, the total space taken by our algorithm is $O\left(\sum_{i=1}^{\log^* n} n \frac{n}{\log^{(i)} n}\right) = O(n)$ bits. Note that our algorithm will also use some other data-structures which we have not described till now. However, the major challenge in our work was to bound the size of $S_i$'s. All our other data-structures take $O(n)$ bits cumulatively. Thus, the total space taken by our algorithm is $O(n)$ bits. This completes the overview of our algorithm.

Remark 1. In the above description, each $S_i$ contains at most top $\frac{2n}{(\log^{(i)} n)^2}$ elements of $S$. Thus, the size of $S_1$ is $\frac{2n}{(\log n)^2}$. This is a crucial difference from the Elmasry et al. \[\ref{17}\] algorithm, where the size of $S_1$ was $\frac{2n}{\log n}$. The main reason for this change is to decreases the space taken by our algorithm. Indeed, the cumulative space taken by all $S_i$'s (in our algorithm) can be shown to be $\sum_{i=1}^{\log^* n} n \frac{n}{\log^{(i)} n} = O(n)$. In spite of this change, the running time of our algorithm does not suffer. To summarize, this is an important technical change from the previous work with the sole aim to decrease the space taken by the algorithm.

3 Preliminaries

In our algorithm, the following data-structure plays a crucial role.

**Lemma 3.** (Succinct Dynamic Dictionary \[\ref{16}\]) Given a universe $U$ of size $u$, there exists a dynamic dictionary that stores a subset $S \subset U$ of size at most $n$. Each element of $U$ has a satellite data of size $r$ where $r \in O(\log n)$. The time taken for membership, retrieval, insert, and delete any element (and its satellite data) is $O(1)$ with probability $(1 - 1/n^c)$ for some chosen constant $c$. The space taken by the data-structure is $n \log \frac{u}{n} + nr$ bits.
Note that a similar dictionary was also described in Lemma 2.1 of [17].

We define few basic notation/data-structures that will be used in the ensuing discussion.

- \(\log^{(i)} n := \log \log \log \ldots \log n\).
- \(\log^* n\) (iterated logarithm) is the number of times the logarithm function is iteratively applied till the result is \(\leq 2\). Define \(\alpha := \log \log \ldots \log n = \log^{(\log^* n)} n\). Note that \(1 < \alpha \leq 2\).

We divide the imaginary stack \(S\) into segments of size \(\left\lceil \frac{n}{(\log n)^2} \right\rceil\). An \(i\)-segment \((1 \leq i \leq \log^* n)\) contains vertices of \(\left(\frac{\log n}{\log^{(i)} n}\right)^2\) consecutive segments of \(S\). We divide the imaginary stack \(S\) into \(i\)-segments from bottom to top (only the topmost \(i\)-segment may contain less number of consecutive segments). The total number of vertices in an \(i\)-segment is at most \(\left\lceil \left(\frac{\log n}{\log^{(i)} n}\right)^2 \times \frac{n}{(\log n)^2} \right\rceil = \left\lceil \frac{n}{(\log^{(i)} n)^2} \right\rceil\) and the total number of \(i\)-segments is at most \((\log^{(i)} n)^2\).

For brevity, we will drop the ceil notation in the rest of the paper.

- Stack \(S_1\)
  A stack \(S_1\) will store the vertices present in at most top two segments of \(S\). Each cell of \(S_1\) contains the tuple of type \((v, v.\text{Next})\).

- Dynamic Dictionary for \(S_i\) \((2 \leq i \leq \log^* n)\)
  We will store information about vertices of at most top two \(i\)-segment in a dynamic dictionary \(S_i\) \((2 \leq i \leq \log^* n)\). This information will be crucial in restoring \(S_{i-1}\).

- Trailers
  In [17], the restoration algorithm uses the trailer stack to find a vertex from which the restoration of \(S_1\) should start. In our algorithm, as we have to restore \(S_1, S_2, \ldots, S_{\log^* n}\), we require many trailer stacks.

To this end, we implement a trailer stack for each \(S_i\). In the trailer stack \(T_i\) \((1 \leq i \leq (*)\)), we keep the bottommost element of the imaginary stack \(S\) and the top vertex of all \(i\)-segments of \(S\) that are not present in \(S_i\).

4 Our Algorithm

Our algorithm is nearly similar to the Elmasry et al.[17] algorithm. We initially color all the vertices white (the space taken by the \(\text{Color}\) array is \(O(n)\) bits as we color a vertex \text{WHITE}, \text{GRAY} or \text{BLACK} only). Then we take an arbitrary vertex, say \(u\), and do a \(\text{DFS}\) from \(u\). Like Elmasry et al.[17], initially \((u, 1)\) is pushed on to the stack. Additionally, we also insert \((u, 1)\) to all other \(S_i\)'s.

We then go over the stack \(S_1\) till it becomes empty. Analogously, we can say that we will process the stack \(S_1\) till the trailer \(T_1\) becomes empty — as \(T_1\) always contains the bottommost element of the imaginary stack \(S\). Our \texttt{WHILE} loop is similar to the standard \(\text{DFS}\) algorithm with the addition that we push and pop not only to \(S_1\) but insert to and delete from all \(S_i\)'s. Let \((v, v.\text{NEXT})\) be the top element of \(S_1\). We pop \(v\) from \(S_1\) and also delete it from all other \(S_i\)'s. Then we color \(v\) gray. We then check if the \((v.\text{NEXT})\)-th neighbor of \(v\), \(\mathcal{N}(v)[v.\text{NEXT}]\), is white or not. If it is white, then we first push \(v\) back on to the stack (and all other \(S_i\)'s). After that, \(\mathcal{N}(v)[v.\text{NEXT}]\) is pushed to
Algorithm 3: Dfs(u)

1 for i ← log* n to 1 do
2     INSERT(u, i, 1);
3 end
4 while trailer T1 is not empty do
5     (v, v.Next) ← top element of S1;
6     for i ← log* n to 1 do
7         DELETE(v, i);
8     end
9     COLOR(v) ← GRAY;
10    while v.Next ≤ deg(v) do
11        if COLOR(N(v)[v.Next]) is white then
12            for i ← log* n to 1 do
13                INSERT(v, i, v.Next + 1);
14                INSERT(N(v)[v.Next], i, 1);
15            end
16            break;
17        end
18        else
19            v.Next ← v.Next + 1;
20        end
21    end
22    else
23        COLOR(v) ← BLACK;
24    end
25 end

S1 and all the other relevant data-structure. When we have processed all the neighbors of v, it is colored BLACK.

We now calculate the running time of our Dfs algorithm in Algorithm 3. In the classical Dfs algorithm, a GRAY vertex is pushed onto the stack again after it finds a new WHITE vertex. This implies that vertices can be pushed on to the stack at most O(n) times. Our Dfs algorithm is nearly similar to the classical Dfs algorithm with the only difference that we insert/delete into log* n “stacks” instead of one. Thus we claim the following running time:

**Lemma 4.** Not accounting for the time taken by INSERT and DELETE procedures, the time taken by our Dfs algorithm in Algorithm 3 is O(m + n log* n).

In INSERT(v, i, v.Next) procedure, we add the information about vertex v to S_i. Remember that S_i is used to restore S_{i-1}. We will now describe S_i in detail.

5 Information in S_i

In [17], where we just have to restore S1, the following two pieces of information about each vertex is stored in D: (1) The segment number in which v lies. (2) The approximate position in N(v) where v.Next − 1 lies.
We try to generalize this idea. Unlike $D$, the dictionary $S_i$ in our algorithm contains information about vertices present in at most two top $i$-segments. For each such $v \in S_i$, let $S_i(v)$ denote the cell in which information related to $v$ is stored. We will store the following information related to $v$.

1. The $(i - 1)$-segment number in which $v$ lies.
   
   Remember that $S_i$’s main function is to restore $S_{i-1}$. Thus, for each vertex $v$, we will store the $(i - 1)$-segment to which $v$ belongs, let us denote it by $SEG_{i-1}(v)$. $SEG_{i-1}(v)$ will help the restore algorithm of $S_{i-1}$ to check whether $v$ indeed lies in the top $(i - 1)$-segment. Since the total number of $(i - 1)$-segment is $(\log^{(i-1)} n)^2$, $2\log^{(i)} n$ bits are required to represent $SEG_{i-1}(v)$.

2. The approximate position in $\mathcal{N}(v)$ where $v.NEXT - 1$ lies.
   
   The above information is used to find $v.NEXT - 1$ efficiently. It would have been nice if we could explicitly store $v.NEXT - 1$. However, this will require $O(\log n)$ bits for each vertex in $S_i$ — a space which we cannot afford. To overcome the space limitation, we divide $\mathcal{N}(v)$ into groups of appropriate size and store the group number in which $v$.NEXT $= 1$ lies.

The exact definition of the second term requires some more work. Note that $SEG_{i-1}(v)$ takes just $O(\log^{(i)} n)$ bits. We want the second term also to take $O(\log^{(i)} n)$ bits. Thus, the number of groups into which we divide $\mathcal{N}(v)$ should not be huge (it should be $\leq \log^{(i-1)} n$). However, if the number of groups is small, it implies that the group size, i.e., the number of vertices in each group, may be large. Thus, given the group number, finding $v.NEXT - 1$ in the group will take more time. Thus, we are faced with a dilemma where reducing the space increases the running time of our algorithm. To overcome this dilemma, we extend a strategy used in [17]. Elmasry et al. [17] divided the vertices into two sets — heavy and light. A light vertex has low degree — thus, its group size is small. For heavy vertices, they show that the total number of heavy vertices is small and for each heavy vertex $v$, $v.NEXT - 1$ can be stored explicitly without using too much space. We plan to extend this strategy. But unlike [17], we have a hierarchy of heavy and light vertices (since we have a hierarchy of $S_i$’s).

### 5.1 Light Vertices

**Definition 5.** A vertex $v$ is $i$-light if $\deg(v) \leq \frac{m(\log^{(i-1)} n)^2}{n}$ where $2 \leq i \leq \log^* n$. We define all the vertices in $V$ to be $1$-light.

We are now ready to define the second information related to $v$ stored in $S_i$. If $v$ is $i$-light, then we divide $\mathcal{N}(v)$ into groups of size $\frac{\deg(v)}{(\log^{(i-1)} n)^3}$.

**Definition 6.** If $v$ is $i$-light, then the second information of $v$ (approximate position of $v.NEXT - 1$ in $\mathcal{N}(v)$) stored in $S_i$ is $GROUP_{i-1}(v.NEXT - 1)$ defined as follows: $GROUP_{i-1}(v.NEXT - 1) := \ell$ if $\mathcal{L}_{\frac{\deg(v)}{(\log^{(i-1)} n)^3}} < v.NEXT - 1 \leq (\ell + 1)\frac{\deg(v)}{(\log^{(i-1)} n)^3}$.

The total number of groups of $\mathcal{N}(v)$ is $(\log^{(i-1)} n)^3$. Thus the total number of bits required to represent $GROUP(v.NEXT - 1)$ is $3\log^{(i)} n$ bits.

Remember that we partitioned the set of vertices into light and heavy only to make the group size small. We now bound the number of vertices in a group of a $i$-light vertex.

**Observation 7.** If $v$ is $i$-light, then the total number of vertices in each group of $\mathcal{N}(v)$ is $\leq \frac{\deg(v)}{(\log^{(i-1)} n)^3} \leq \frac{m(\log^{(i-1)} n)^2}{n(\log^{(i-1)} n)^3} = \frac{m}{n(\log^{(i-1)} n)}$.
We are now ready to formally define the information about vertex \( v \) stored in \( S_i \).

- If an \( i \)-light vertex \( v \) becomes a part of top \( i \)-segment of imaginary stack \( S \), then we store the following information about \( v \). \( S_i(v) = (\text{SEG}_{i-1}(v), \text{GROUP}_{i-1}(v, \text{NEXT} - 1)) \)

- If vertex \( v \) is not \( i \)-light, then \( S_i(v) = (\text{SEG}_{i-1}(v), 0) \), that is we just store the \((i - 1)\)-segment in which \( v \) resides.

Some explanation is in order. If \( v \) is an \( i \)-light vertex, then we can store the information \((\text{SEG}_{i-1}(v), \text{GROUP}_{i-1}(v, \text{NEXT} - 1))\) corresponding to \( v \). We have already shown that both these terms take \( O(\log(i) \cdot n) \) bits. Moreover, given the group number \( \text{GROUP}_{i-1}(v, \text{NEXT} - 1) \), we can find \( v, \text{NEXT} - 1 \) in \( O\left(\frac{m}{n \log(i-1) \cdot n}\right) \) time, as the number of vertices in each group of an \( i \)-light vertex is \( \leq \frac{m}{n \log(i-1) \cdot n} \) (using Observation 7).

However, if \( v \) is not \( i \)-light, then its group size may be \( > \frac{m}{n \log(i-1) \cdot n} \), which is not desirable (as this might increase the search time for \( v, \text{NEXT} - 1 \)). So, for such a vertex, we store \( \text{SEG}_{i-1}(v) \) only as there is no point in storing the second term (the second term 0 is just a dummy term). But for efficiency, we need to store some information regarding \( v, \text{NEXT} - 1 \) even for the vertex which is not \( i \)-light. In the next section, we describe a data-structure which will efficiently store information about all non \( i \)-light vertices.

### 5.2 Heavy Vertices

**Definition 8.** A vertex \( v \) is \( i \)-heavy if \( \frac{m(\log(i-1) \cdot n)^2}{n} < \deg(v) \leq \frac{m(\log(i-2) \cdot n)^2}{n} \) where \( 3 \leq i \leq \log^* n \). We define a 2-heavy vertex separately. A vertex \( v \) is said to be 2-heavy if \( \frac{m(\log(n))^2}{n} < \deg(v) \leq n \).

Note that our definition partitions the vertex set nicely. We prove this nice property in the following lemma:

**Lemma 9.** If \( v \) is not \( i \)-light (\( i \geq 2 \)), then it is \( j \)-heavy for some \( j \) where \( 2 \leq j \leq i \).

**Proof.** Since \( v \) is not \( i \)-light, \( \frac{m(\log(i-1) \cdot n)^2}{n} < \deg(v) \leq n \). Thus, there exists a \( j \) (\( 3 \leq j \leq i \)) such that \( \frac{m(\log(j-1) \cdot n)^2}{n} \leq \deg(v) < \frac{m(\log(j-2) \cdot n)^2}{n} \) or \( \frac{m(\log(j) \cdot n)^2}{n} \leq \deg(v) < n \) (the case when \( j = 2 \)). \( \square \)

We store the information related to an \( i \)-heavy vertex in a dynamic dictionary \( H_i \) where \( i \geq 2 \). Since degree of a \( i \)-heavy vertex \( v \) is \( \geq \frac{m(\log(i-1) \cdot n)^2}{n} \), total number of \( i \)-heavy vertices is \( O\left(\frac{n}{(\log(i-1) \cdot n)^2}\right) \). Similar to \( i \)-light vertices, we divide \( N(v) \) into groups of size \( \frac{\deg(v)}{(\log(i-2) \cdot n)^3} \). The only problem with this group size is that it is not defined for \( i = 2 \). If \( i = 2 \), then we divide \( N(v) \) into groups of size 1.

We store the group number of \( v \) in the dynamic dictionary \( H_i \), that is \( \text{GROUP}_{i-2}(v, \text{NEXT} - 1) \) defined as follows: \( \text{GROUP}_{i-2}(v, \text{NEXT} - 1) := \ell \) if \( \ell \cdot \frac{\deg(v)}{(\log(i-2) \cdot n)^3} < v, \text{NEXT} - 1 \leq (\ell + 1) \cdot \frac{\deg(v)}{(\log(i-2) \cdot n)^3} \).

Since we divide \( \deg(v) \) into groups of size \( \frac{\deg(v)}{(\log(i-2) \cdot n)^3} \), the total number of groups is \((\log(i-2) \cdot n)^3\). This implies that total space required to represent the group number per vertex in \( H_i \) is \( 3 \log(i-1) \cdot n \) bits.

Using Observation 7 if a vertex \( v \) is \( i \)-light, then the associated group size (stored in \( S_i \)) is \( \frac{m}{n \log(i-1) \cdot n} \). The next lemma present a very crucial feature of our algorithm:

**Lemma 10.** Let \( v \) be a vertex in \( S_i \), then the group size associated with \( v \) is of size \( \leq 1 + \frac{m}{n \log(i-1) \cdot n} \).
Proof. If \( v \) is \( i \)-light, then we have already seen that the group size associated with \( v \) (and stored in \( S_i \)) is \( \frac{m}{n \log^{(i-1)} n} \). Using Lemma 9 if \( v \) is not \( i \)-light, then it is \( j \)-heavy for \( 2 \leq j \leq i \). Thus, the information about the group of \( v \) is stored in \( H_j \), that is \( \text{GROUP}_{j-2}(v.\text{NEXT} - 1) \). To this end, we divide \( \mathcal{N}(v) \) into group of size \( \frac{\deg(v)}{(\log^{(j-2)/n})^2} \). There are two cases:

1. \( j > 2 \)

   Since \( v \) is \( j \)-heavy, \( \deg(v) \leq \frac{m(\log^{(j-2)/n})^2}{n} \). This implies that the size of each group is \( \leq \frac{m}{n \log^{(j-2)/n}} \leq \frac{m}{n \log^{(i-1)/n}} \).

2. \( j = 2 \)

   By definition, the group size is exactly 1.

Thus, the group size associated with \( v \) is \( \leq 1 + \frac{m}{n \log^{(i-1)/n}} \).

The above lemma shows a crucial property of all vertices in \( S_i \). The associated group size of all these vertices is \( \leq 1 + \frac{m}{n \log^{(i-1)/n}} \) irrespective of their degree. Thus, whenever we are searching for \( v.\text{NEXT} - 1 \) for a vertex \( v \), we have to search at most \( 1 + \frac{m}{n \log^{(i-1)/n}} \). We will crucially exploit this property in the restoration algorithm. However, before that let us take a look at the insert and delete procedures.

6 Insert and Delete Procedures

| Algorithm 4: \text{INSERT}(v, i, v.\text{NEXT}) |
|---|
| 1 if \( |S_i| = \frac{2n}{(\log^{(i-1)/n})^2} \) then |
| 2 \quad \text{RESTORE-FULL}(i); |
| 3 end |
| 4 if \( v \) is \( i \)-light then |
| 5 \quad S_i.\text{INSERT}(v, (\text{SEG}_{i-1}(v), \text{GROUP}_{i-1}(v.\text{NEXT} - 1))) \) or \( S_1.\text{PUSH}(v, v.\text{NEXT}) \) (if \( i = 1 \)); |
| 6 end |
| 7 else |
| 8 \quad S_i.\text{INSERT}(v, (\text{SEG}_{i-1}(v), 0)) |
| 9 end |
| 10 if \( \mathcal{T}_i \) is empty or recently pushed element becomes the top element of an \( i \)-segment then |
| 11 \quad \mathcal{T}_i.\text{PUSH}(v, v.\text{NEXT}); |
| 12 end |
| 13 if \( v \) is \( i \)-heavy then |
| 14 \quad H_i.\text{INSERT}(v, \text{GROUP}_{i-2}(v.\text{NEXT} - 1)); |
| 15 end |

In the \text{INSERT} procedure, \( v \) is to be inserted in \( S_i \). But \( S_i \) may be full, that is, it has \( \frac{2n}{(\log^{(i-1)/n})^2} \) vertices. So, we call \text{RESTORE-FULL}(i) \) which basically aims at removing half of the elements of \( S_i \). After the restoration, \( S_i \) has the top \( \frac{n}{(\log^{(i)/n})^2} \) vertices of the imaginary stack \( S \). We then insert \( (\text{SEG}_{i-1}(v), \text{GROUP}_{i-1}(v.\text{NEXT} - 1)) \) in \( S_i \). If this newly added element becomes the top element of
an \(i\)-segment or the trailer itself is empty then we add \((v, v.\text{Next})\) to the trailer \(T_i\). Lastly, if \(v\) is \(i\)-heavy, then it is added to \(H_i\). Three details are missing from the pseudo code of \text{INSERT}. We list them now:

1. \textit{Calculating SEG}_{i-1}(v)
   
   Let \(k_1\) be the total number of vertices in trailer \(T_i\) and \(k_2\) be the total number of vertices in \(S_i\). We first calculate the total number of vertices below \(v\) in the imaginary stack \(S\). This is \(k = (k_1 - 1) \times \text{size of } i\text{-segment} + k_2 = (k_1 - 1) \times \frac{n}{(\log^{0.8} n)^2} + k_2\). Once we have calculated \(k\), finding \(\text{SEG}_{i-1}(v)\) is just a mathematical calculation.

2. \textit{Calculating group}_{i-1}(v.\text{Next} - 1) or group\_i-2(v.\text{Next} - 1)
   
   This is just a mathematical calculation once we know \(v.\text{Next}\) and \(\deg(v)\).

3. \textit{Finding if } v \text{ is a top element of an } i\text{-segment}
   
   This can be done by maintaining the number of elements currently present in the imaginary stack \(S\). Before inserting \(v\), if \(|S| = 0\) or \(|S| = \frac{cn}{(\log^{0.8} n)^2} - 1\) \((c \geq 1)\), then we insert \((v, v.\text{Next})\) on to the trailer \(T_i\).

\begin{algorithm}
\textbf{Algorithm 5:} \text{DELETE}(v, i)
\begin{align*}
1 & \text{if } |S_i| < \frac{n}{2(\log^{0.8} n)^2} \text{ and } T_i \text{ has at least two elements then} \\
2 & \quad \text{RESTORE-EMPTY}(i); \\
3 & \text{end} \\
4 & \text{if } v \text{ is } i\text{-heavy then} \\
5 & \quad H_i.\text{DELETE}(v); \\
6 & \text{end} \\
7 & \text{if } v \text{ is on the top of the trailer } T_i \text{ then} \\
8 & \quad T_i.\text{POP}(); \\
9 & \text{end} \\
10 & \text{return } S_i.\text{DELETE}(v) \text{ or } S_1.\text{POP}() \text{ (if } i = 1) \\
\end{align*}
\end{algorithm}

The \text{DELETE}(v, i)\ is nearly similar to the \text{INSERT} procedure. We first check if the number of elements in \(S_i\) is less. If yes, then we also have to check if the trailer itself has enough elements. If yes, then we call \text{RESTORE-EMPTY}(i)\. After its execution, \(S_i\) contains topmost \(\frac{n}{(\log^{0.8} n)^2}\) vertices of the imaginary stack \(S\). If \(v\) is \(i\)-heavy, then it is removed from \(H_i\). After this, the top element of \(S_i\) (and \(T_i\) if necessary) is removed.

The following lemma about the running time of \text{INSERT} and \text{DELETE} is immediate (due to our data-structure in Lemma 3).

\textbf{Lemma 11.} Apart from the time taken by \text{RESTORE-EMPTY} and \text{RESTORE-FULL}, the running time taken by \text{INSERT} and \text{DELETE} procedure is \(O(1)\) with high probability\footnote{Since we use the data-structure described in Lemma 3 at most \(\text{poly}(n)\) times, all insert and deletes are successful with probability \(\geq 1 - \frac{1}{n^c}\) where \(c\) is some constant.}.

\section{Restore Procedure}

We now move on to the most important part of our algorithm, that is the restoration of \(S_i\)'s. First, we describe our approach for restoring the last dictionary, that is, \text{RESTORE-EMPTY}(\log^4 n)\.
Remember that to restore the last dictionary, we do the most trivial thing, that is run the DFS algorithm again. So, we run the DFS algorithm again from the starting vertex $u$ ignoring all the black vertices (this process is similar to the one described in [17]). We mark all the gray vertices white and perform a DFS from $u$ till we hit the topmost trailer of $T_{\log^* n}$. Whenever we encounter a vertex of the top $\log^* n$-segment, we add it to $S_{\log^* n}$ after calculating relevant parameters (as similar to that in INSERT algorithm). Note that we can easily find if $v$ is a part of top $\log^* n$-segment by comparing the number of vertices processed by the restore algorithm to the number of elements in the imaginary stack $S$ (which we can easily maintain). We now show that our RESTORE-EMPTY($\log^* n$) procedure is correct. To this end, we will compare our algorithm with the DFS algorithm that works with the imaginary stack $S$. We will call this DFS algorithm as an imaginary DFS algorithm. We first observe the following:

**Observation 12.** Let $(v, v.\text{Next})$ be an entry on the imaginary stack $S$ when we call RESTORE-EMPTY($\log^* n$). Then, all vertices in $N(v)[1 \ldots v.\text{Next} - 2]$ are black or gray when the imaginary DFS algorithm pushes this entry on to $S$.

**Proof.** Consider the step when the imaginary DFS algorithm pushes the entry $(v, v.\text{Next})$ on to the stack. This means that it has found a white vertex $N(v)[v.\text{Next} - 1]$. Thus, $v$ has already processed all vertices in $N(v)[1 \ldots v.\text{Next} - 2]$ and color of each processed vertex is either gray or black. □

We now use the above observation to prove that RESTORE-EMPTY($\log^* n$) is correct.

**Lemma 13.** Let $(v, v.\text{Next})$ be an entry on the imaginary stack $S$ when we call RESTORE-EMPTY($\log^* n$). Then, (1) RESTORE-EMPTY($\log^* n$) also processes the tuple $(v, v.\text{Next})$ and (2) color of all the non-black vertices is exactly same in the imaginary DFS algorithm and our RESTORE-EMPTY algorithm (after both algorithms process $v$).

**Proof.** First, note that we start our restoration process without touching the color of a black vertex. Thus, if a vertex is black in the imaginary DFS algorithm (at the time we call RESTORE-EMPTY($\log^* n$)), it is also black in our algorithm.
We now prove the statement of the lemma using induction. Consider the moment when the imaginary DFS algorithm put the entry \((u, u\text{.Next})\) on to imaginary stack \(S\) where \(u\) is the vertex with which we started our DFS. We now claim that there is no gray vertex in the graph at this point in the imaginary DFS algorithm. This is because all the gray vertices are always on the imaginary stack \(S\) and when \(u\) is processed, there are no vertices on the imaginary stack. Thus, all non-black vertices have white color before the first push. Now, we claim that (1) is true. This is because the color of all the vertices in \(N(u)[1 \ldots u\text{.Next} - 2]\) is black, thus same for both algorithms. Due to Observation [12] we correctly find \((u, u\text{.Next})\). Before pushing \((u, u\text{.Next})\) on to the stack, both the algorithms make \(u\) gray. After the processing of \(u\), both the algorithms have same colors for all the non-black vertices, thus (2) is also true.

We now show that the statement is true in general when we are inserting an element \((v, v\text{.Next})\) at the \(k^{th}\) iteration. Using the induction hypothesis, all the non-black vertices have same color at the end of the \((k - 1)\)-th iteration. Also, if a vertex is black in the imaginary DFS algorithm, it is also black at the start of our restore algorithm (since we donot touch black vertices). Since the imaginary DFS algorithm puts \((v, v\text{.Next})\) on to the stack, vertices in \(N(v)[1 \ldots v\text{.Next} - 2]\) are black or gray. Using the above arguments, the color of these vertices is same even in our algorithm. Thus, we also push \((v, v\text{.Next})\) in our algorithm. Thus, (1) is true. Before pushing \((v, v\text{.Next})\), both our algorithm and the imaginary DFS algorithm mark \(v\) gray – the only change in the color of a vertex. Thus even (2) is true. This completes the induction step.

The above lemma implies that at the end of the restoration, \(S_{\log^* n}\) contains vertices from the top \((\log^* n)\)-segment of \(S\) and the color of each vertex is also correctly restored. In the restoration process, we use the data-structure described in Lemma [3] at most \(\text{poly}(n)\) times, thus all insert and deletes are successful with probability \(\geq 1 - \frac{1}{c}\) where \(c\) is some constant. Thus, the algorithm succeeds with very high probability.

Since, we are basically running the imaginary DFS again to restore \(S_{\log^* n}\), the following lemma is immediate.

**Lemma 14.** The time taken to restore \(S_{\log^* n}\) is \(O(m + n)\) with high probability.

Let us now look at Restore-Empty\((i)\) where \(1 \leq i < \log^* n\). Before Restore-Empty\((i)\) is called, we will assume that \(S_{i+1}\) has enough elements. This assumption is required as the vertices to be restored in \(S_i\) need to be present in \(S_{i+1}\).

- \(S_{i+1}\) contains at least \(\frac{n}{2(\log^{(i+1)} n)^2}\) vertices (we will prove this crucial assumption in the analysis)

For restoring \(S_i\), we start from the second element from top in trailer \(T_i\) and basically try to run the DFS-like algorithm from it. Let \((w, w\text{.Next})\) be second element from top in trailer \(T_i\). It implies that the first vertex (to be restored) in \(S_i\) is \(N(w)[w\text{.Next} - 1]\). So, we start a DFS from \(N(w)[w\text{.Next} - 1]\) with one simple change (similar to Elmasry et al. [17]) – we change the meaning of white and gray vertices. This is because all the vertices to be restored in \(S_i\) are gray and should not be processed once they are added in \(S_i\).

Let \(v \leftarrow N(w)[w\text{.Next} - 1]\). Since the size of \(S_{i+1}\) is sufficiently larger than \(S_i\), \(v\) is present in \(S_{i+1}\). Using \(S_{i+1}\), we find the \(i\)-segment number to which \(v\) belongs. In addition, we also want to find \(v\text{.Next} - 1\). To this end, we check if \(v\) is \((i + 1)\)-light. If yes, then we can find \(l_v = \text{GROUP}_i(v\text{.Next} - 1)\), that is the approximate group in which \(v\text{.Next} - 1\) resides. However, if \(v\) is not \((i + 1)\)-light, then we use Lemma [9] to conclude that \(v\) is \(j\)-heavy for some \(j \leq i + 1\), and we find \(l_v = \text{GROUP}_{j-2}(v\text{.Next} - 1)\) where \(j \leq i + 1\). By Lemma [10], irrespective of the fact whether \(v\)
Algorithm 7: Restore-Empty($i$)

1. $(w, w.\text{Next}) \leftarrow \text{second top element in } T_i$;
2. $v \leftarrow \mathcal{N}(w)[w.\text{Next} - 1]$;
3. do
   4. $(\text{Seg}_i(v), l_v) \leftarrow S_{i+1}.\text{Search}(v)$;
   5. $k \leftarrow l_v \frac{\deg(v)}{\log^{(i)} n}$;
   6. if $v$ is $j$-heavy where $j \leq i$ then
      7. $l_v \leftarrow H_j.\text{Search}(v)$;
   8. $k \leftarrow l_v \frac{\deg(v)}{\log^{(i-2)} n}$ or $l_v$ (if $j = 2$)
   9. end
10. for $k' = k$ to $k + 1 + \frac{m}{n \log^{(i)} n}$ do
   11. $x \leftarrow \mathcal{N}(v)[k']$;
   12. if $x$ is gray and $x$ in present in $S_{i+1}$ then
      13. $(\text{Seg}_i(x), l_x) \leftarrow S_{i+1}.\text{Search}(x)$;
      14. if $\text{Seg}_i(v) = \text{Seg}_i(x)$ then
         15. break;
      16. end
   17. end
   18. if $v$ is $i$-light then
      19. $S_i.\text{Insert}(v, (\text{Seg}_{i-1}(v), \text{GROUP}_{i-1}(k')))$;
   20. end
   21. else
      22. $S_i.\text{Insert}(v, (\text{Seg}_{i-1}(v), 0))$;
   23. end
24. $v \leftarrow x$;
25. $\text{Color}(v) \leftarrow \text{white}$;
26. end
27. while $v$ is not equal to the top of trailer $T_i$;
28. recolor all white colored vertex during the above while loop gray again;

is $(i+1)$-light or $j$-heavy, the group in which $v.\text{Next} - 1$ lies contains at most $1 + \frac{m}{n \log^{(i-1)} n}$ vertices.

Now comes the most important part of our algorithm. We want to identify $v.\text{Next} - 1$ correctly once we have found the group in which $v.\text{Next} - 1$ resides. We will now use the following lemma which will help us in identifying $v.\text{Next} - 1$.

Lemma 15. Let $l_v$ be the group number that was found out in Restore-Empty($i$) procedure while processing $v$. Then $v.\text{Next} - 1$ is the index of the first gray vertex, say $x$, in this group such that $\text{Seg}_{i-1}(x)$ is equal to $\text{Seg}_{i-1}(v)$.

Proof. We know that $v.\text{Next} - 1$ lies in the group $l_v$. Let $x \leftarrow \mathcal{N}(v)[v.\text{Next} - 1]$. We first discuss the properties of vertex $x$. Since we are restoring the top $i$-segment, $x$ should lie in the same segment as $v$, that is $\text{Seg}_i(v) = \text{Seg}_i(x)$. In the imaginary DFS algorithm, consider the step at which $v$ discovers $x$. Using Observation 12 we claim that at that point $x$ is the first white vertex of the group. Indeed, if there is another white vertex lying before $x$ in $\mathcal{N}(v)$, then that vertex will be processed first by the imaginary DFS algorithm.
Since the meaning of white and gray vertices are changed during the restoration, this means that \( x \) is the first gray vertex of the group during the restoration. This completes our proof. ■

The above lemma greatly simplifies our work, we just find the first gray vertex \( x \) such that \( \text{SEG}_{i}(x) = \text{SEG}_{i}(v) \). Once we have found \( x \), then we insert \( v \) in \( S_{i} \) by calculating all the relevant parameter and then move on to process \( x \). We now find the running time of Restore-Empty. We list the steps in this algorithm that dominates its running time.

1. Finding the \( j \) for which \( v \) is \( j \)-heavy (Step 6).

   An easy (but sub-optimal space) solution for this problem will be to store this information for each vertex in an array, say \( A \), of size \( n \). However, the space required by \( A \) will be \( O(n \log^{*} n) \) (as \( 2 \leq j \leq \log^{*} n \)). Since we do not have this much space, we use another strategy.

   If \( v \) is \( 2 \)-heavy, then we can find it in \( O(1) \) time. So, assume that \( 3 \leq j \leq \log^{*} n \). If \( v \) is \( j \)-heavy, then \( \frac{m(\log^{(j-1)} n)^2}{n} < \deg(v) \leq \frac{m(\log^{(2)} n)^2}{n} \) or \( \frac{m(\log^{(2)} n)^2}{n} \leq \frac{n \deg(v)}{m} \leq (\log^{(2)} n)^2 \).

   We make an array \( A \) of size \( O(\log^{2} n) \), such that each cell \( A[k] \) has \( A[k] = j \). Given any \( v \), if the content of the cell \( \frac{n \deg(v)}{m} \) of \( A \) is \( j \), then \( v \) is \( j \)-heavy. Since we probe \( A \) once, the time taken for this step is \( O(1) \) time.

   Note that the space taken by the array \( A \) is \( O(\log^{2} n \log(\log^{*} n)) \) which is subsumed in the \( O(n) \) notation.

2. Searching for \( v \). Next \(-1 \) (the for loop inside the while loop (step 10-18))

   Once we have found the starting vertex of the group (that is \( k \)) in the while loop, the time taken in the for loop is \( O\left(1 + \frac{m}{n \log^{(i)} n}\right) \). This is due to Lemma 10 which states that the group size associated with \( v \) has \( 1 + \frac{m}{n \log^{(i)} n} \) vertices.

3. Recoloring the vertices (Step 28).

   To this end, we should maintain all the vertices that are colored white by our restore algorithm and then enumerate them. Fortunately, there already exists a space-efficient data-structure that does this job.

   **Lemma 16.** (Succinct Enumerate Dictionary [7]) A set of elements from a universe of size \( n \) can be maintained using \( n + o(n) \) bits to support insert, delete, search and findany operations in constant time. We can enumerate all elements of the set (in no particular order) in \( O(k + 1) \) time where \( k \) is the number of elements in the set.

   We implement a enumerate dictionary in which we add all the vertices that are colored white by our restore algorithm. At the end of the while loop of the restore algorithm, we use the enumerate dictionary to enumerate all such vertices. We recolor each such vertex gray again and delete it from the succinct dictionary. Using the above lemma, the extra space taken by the enumerate dictionary is \( O(n) \).

   We now put everything together to calculate the total running time of Restore-Empty(\( i \)). Since, we restore vertices in topmost \( i \)-segment only, we process only \( \frac{n}{(\log^{(i)} n)^2} \) vertices in the while loop of Restore-Empty(\( i \)). Thus the while loop of Restore-Empty(\( i \)) take \( O\left((1 + \frac{m}{n \log^{(i)} n}) \times \right) \).
time. Also, time taken by the recoloring step is proportional to the number of vertices processed by \( \text{RESTORE-EMPTY}(i) \), that is \( O\left(\frac{n}{(\log^\ast n)^2}\right) \).

In the restoration process, we use the data-structure described in Lemma 3 at most \(\text{poly}(n)\) times, thus all insert and deletes are successful with probability \( \geq 1 - \frac{1}{n^c} \) where \( c \) is some constant. Thus, the algorithm succeeds with very high probability.

**Lemma 17.** The time taken to restore \( S_i \) in \( \text{RESTORE-EMPTY}(i) \) is \( O\left(\left(1 + \frac{m}{n \log^\ast n}\right) \times \frac{n}{(\log^\ast n)^2}\right) \) with high probability.

**Algorithm 8: Restore-Full**

1. \((w, w.\text{Next}) \leftarrow \text{top element in } T_i;\)
2. \(v \leftarrow \mathcal{N}(w)[w.\text{Next} - 1];\)
3. \(\text{counter} \leftarrow 0;\)
4. do
5. \((\text{SEG}_i(v), l_v) \leftarrow \text{SEG}_{i+1}.\text{SEARCH}(v);\)
6. \(k \leftarrow l_v\frac{\text{deg}(v)}{\log^\ast n};\)
7. if \( v \text{ is } j\text{-heavy where } j \leq i + 1 \) then
8. \(l_v \leftarrow \mathcal{H}_j.\text{SEARCH}(v);\)
9. \(k \leftarrow l_v\frac{\text{deg}(v)}{\log^\ast n} \text{ or } l_v \text{ (if } j = 2);\)
10. end
11. for \( k' = k \text{ to } k + 1 + m\frac{n}{n \log^\ast n} \) do
12. \(x \leftarrow \mathcal{N}(u)[k'];\)
13. if \( x \text{ is GRAY and } x \text{ in present in } S_{i+1} \) then
14. \((\text{SEG}_i(x), l_x) \leftarrow \text{SEG}_{i+1}.\text{SEARCH}(x);\)
15. if \( \text{SEG}_i(v) = \text{SEG}_i(x) \) then
16. break;
17. end
18. end
19. \(S_i.\text{DELETE}(v);\)
20. \(\text{COLOR}(v) \leftarrow \text{WHITE};\)
21. if \( \text{counter} = \frac{n}{(\log^\ast n)^2} \) then
22. \(\text{Add } (v, k') \text{ on the top of stack } T_i;\)
23. break;
24. end
25. \(v \leftarrow x;\)
26 while true;
27. recolor all WHITE colored vertex during the above while loop GRAY again;

Our last procedure \( \text{RESTORE-FULL}(i) \) is called when \( S_i \) is full, that is, it contains vertices from the top two \( i \)-segments of \( S \). The aim of \( \text{RESTORE-FULL}(i) \) is to remove the vertices from the second top most \( i \)-segment of \( S \). Thus, at the end of \( \text{RESTORE-FULL}(i) \), \( S_i \) contains vertices of top \( i \)-segment of \( S \). The procedure \( \text{RESTORE-FULL}(\log^\ast n) \) is same as \( \text{RESTORE-EMPTY}(\log^\ast n) \). For \( i < \log^\ast n \), the procedure \( \text{RESTORE-FULL}(i) \) is similar to \( \text{RESTORE-EMPTY}(i) \), we describe it next.
We start with the top-most element of the trailer, say \( w \). Thus, the first vertex from the second topmost segment of \( S_i \) is \( v \leftarrow N(w)[w,\text{Next} - 1] \). Thus, we know that we have to delete \( v \) from \( S_i \). However, before we delete \( v \), we first find \( N(v)[v,\text{Next} - 1] \). The process to find this is same as done in \text{RESTORE-EMPTY}(i) \). Then, we delete \( v \) from \( S_i \) and set \( v \leftarrow N(v)[v,\text{Next} - 1] \). This process is carried out till we process all the vertices in the second topmost segment of \( S \). Thus, after our counter hits \( \frac{n}{(\log^i(n))^2} \), we have deleted all the vertices from the second topmost segment of \( S \). Before we finish, we push the last processed vertex — which is the trailer vertex of the second topmost segment of \( S \) — on top of trailer \( T_i \).

The time taken by \text{RESTORE-FULL}(i) \) is same as the time taken by \text{RESTORE-EMPTY}(i) \). This is because the process to find \( v,\text{Next} - 1 \) (given \( v \)) is same for both the procedures. Also, the total number of vertices processed in both the procedures is same, that is \( \frac{n}{(\log^i(n))^2} \). Thus, the time taken to restore \( S_i \) in \text{RESTORE-FULL}(i) \) is also \( O\left(\left(1 + \frac{m}{n\log(n)}\right) \times \frac{n}{(\log^i(n))^2}\right) \) with high probability.

**Lemma 18.** The time taken to restore \( S_i \) in \text{RESTORE-FULL}(i) \) is \( O\left(\left(1 + \frac{m}{n\log(n)}\right) \times \frac{n}{(\log^i(n))^2}\right) \) with high probability.

8 Analysis

8.1 Correctness of our Algorithm

To prove the correctness, we just need to show our assumption during the restoration procedure is true, that is \( S_{i+1} \) contains sufficient elements when \( S_i \) is restored.

**Lemma 19.** When \( S_i \) is restored, \( S_{i+1} \) contains at least top \( \frac{n}{2(\log^{i+1}(n))^2} \) vertices of imaginary stack \( S \), where \( 1 \leq i \leq \log^* n \).

**Proof.** First, we note a crucial aspect of our algorithm. In Algorithm \( 8 \), \text{INSERT} or \text{DELETE} occurs in \( S_{i+1} \) before \( S_i \).

We will now prove the lemma by induction on \( i \) where \( i \) decreases from \( \log^* n \) to 1. Let us first show the base case, that is \( S_{\log^* n} \) always contains \( \frac{n}{2^x} \) elements. We have already seen that \( S_{\log^* n} \) is correctly restored if it either becomes full or empty. So, \( S_{\log^* n} \) always contains top \( \frac{n}{2^x} \) of the imaginary stack \( S \).

Now, using induction hypothesis, we assume that all stacks \( S_j \) \( i + 1 \leq j \leq \log^* n \) contains at least top \( \frac{n}{2(\log^{j+1}(n))^2} \) vertices of the imaginary stack \( S \). Now we will prove the statement of the lemma for stack \( S_i \).

We will use the fact that we \text{INSERT} or \text{DELETE} in \( S_{i+1} \) before \( S_i \). Thus, whenever we are restoring \( S_i \), (using induction hypothesis) \( S_{i+1} \) contains top \( \frac{n}{2(\log^{i+1}(n))^2} \) vertices of the imaginary stack \( S \).

In order to restore \( S_i \) correctly, the only non-trivial requirement was that \( S_{i+1} \) contains enough vertices during the restoration of \( S_i \). Thus, we claim \( S_i \) is always restored correctly. This completes the correctness of the restore algorithm.

8.2 Space taken by our Algorithm

We now calculate the space taken by our algorithm. We list all our major data-structures and calculate their space.
1. **Color array**

The **Color** array is of size $n$ and each cell contains only three colors **black**, **gray**, or **white**. Thus each cell takes 2 bits. Thus, the space taken by the **Color** array is $O(n)$.

2. **Stack $S_1$**

$S_1$ contains vertices of at most 2 segments of the imaginary segment $S$. Thus, it contains at most $\frac{2n}{(\log(i) n)^2}$ vertices. Each entry of the stack is of size $O(\log(i) n)$. Thus, the total space taken by $S_1$ is $O\left(\frac{n}{\log n}\right)$.

3. **Dynamic Dictionary $S_i$ ($2 \leq i \leq \log^* n$)**

Since $S_i$ stores vertices from at most top two $i$-segment of the imaginary stack $S$, the number of vertices in $S_i$ is at most $\frac{2n}{(\log(i) n)^2}$. In Section 5, we saw that the information associated with each vertex of $S_i$ is $O(\log(i) n)$. Using Lemma 3, the space taken by $S_i$ is $\frac{2n}{(\log(i) n)^2} \times \log\left(\frac{n}{(2/\log(i) n)^{\log(i) n}}\right) + \frac{2n}{(\log(i) n)^2} \times \log(i) n = O\left(\frac{n}{\log(i) n}\right)$. Thus, the cumulative size all $S_i$’s is of $\sum_{i=2}^{\log^* n} O\left(\frac{n}{\log(i) n}\right) = O(n)$ bits.

4. **Trailers**

Since each $i$-segment contains $\frac{n}{(\log(i) n)^2}$ vertices, the total number of $i$-segment is $O((\log(i) n)^2)$. Thus, the number of elements in $T_i$ is $\leq O((\log(i) n)^2)$. In each cell $T_i$, we explicitly store the entry $(v, v.\text{Next})$. The total size of $T_i$ is thus $O(\log n(\log(i) n)^2)$ bits. The cumulative size of all $T_i$ is thus $O(\log n \sum_{i=1}^{\log^* n} (\log(i) n)^2)$ bits which are very small compared to our claimed space of $O(n)$ bits.

5. **Dictionary for Heavy vertices, $H_i$ ($2 \leq i \leq \log^* n$)**

We store the group number of $v$ in a dynamic dictionary $H_i$, that is $\text{GROUP}_{i-2}(v.\text{NEXT} - 1)$. Since we divide $\deg(v)$ into groups of size $\frac{\deg(v)}{(\log(i-2) n)^3}$, the total number of groups is $(\log(i-2) n)^3$. This implies that total space required to represent the group number per cell in $H_i$ is $3 \log(i-1)$ bits. Also, by definition, each vertex in $H_i$ has degree $\geq \frac{m(\log(i-1) n)^2}{n}$. Thus, the total number of vertices in $H_i$ can at most be $O\left(\frac{2n}{(\log(i-1) n)^2}\right)$. Using Lemma 3, the space taken for $H_i$ is $\frac{2n}{(\log(i-1) n)^2} \times \log\left(\frac{n}{(2/\log(i-1) n)^{\log(i-1) n}}\right) + \frac{2n}{(\log(i-1) n)^2} \times 3 \log(i-1) n = O\left(\frac{n}{\log(i) n}\right)$. Thus the cumulative size of all $H_i$’s is $\sum_{i=2}^{\log^* n} O\left(\frac{n}{\log(i) n}\right) = O(n)$ bits.

The reader can check that the total size of our algorithm is $O(n)$. We now find the total running time of our algorithm.

### 8.3 Running Time

Using Lemma 4, we know that our main DFS algorithm (Algorithm B) takes $O(m + n \log^* n)$ time. The $n \log^* n$ term is due to the fact that we call **INSERT** and **DELETE** procedure at most $n \log^* n$ times in our algorithm. Except the restoration part, the **INSERT** and **DELETE** procedure takes $O(1)$ time (Lemma 11). Thus the total running time of our algorithm (except the restoration procedure)
is $O(m + n \log^* n)$. To complete the analysis, we need to find the total running time of our restore algorithm.

Using Lemma 17 and 18 the time taken to restore $S_i$ ($2 \leq i \leq \log^* n$) is $O \left( \left(1 + \frac{m}{n \log^{(i-1)} n} \right) \times \frac{n}{(\log^{(i)} n)^2} \right)$. We will count the number of times $S_i$ is restored after it restored for the first time (this is just to simplify the analysis). Whenever $S_i$ is restored via Restore-Full, take a look at last time $S_i$ was restored. At that point there were exactly $\frac{n}{(\log^{(i)} n)^2}$ elements in $S_i$. Thus, at least $\frac{n}{(\log^{(i)} n)^2}$ vertices must be freshly added to $S_i$. All these freshly added vertices must have changed their color from white to gray. Since a vertex can change its color from black to gray only once in our DFS algorithm (when not processed in the restore procedure), $S_i$ can be restored via Restore-Full at most $(\log^{(i)} n)^2$ times. Similarly, if $S_i$ is restored via Restore-Empty, take a look at the step at which it was restored previously. At that time, $S_i$ had exactly $\frac{n}{(\log^{(i)} n)^2}$. This implies that at least $\frac{n}{2(\log^{(i)} n)^2}$ have been deleted from $S_i$. The only reason for deleting a vertex (when not processing it in a restore procedure) is that it has turned black. Since a vertex can change its color from gray to black only once in our DFS algorithm (when not processed in the restore procedure), the total number of times $S_i$ is restored via Restore-Empty is $O((\log^{(i)} n)^2)$. Thus, the total time taken in restoring $S_i$'s is as follows:

1. $i = \log^* n$

Remember that $\log^{(\log^* n)} n = \alpha$ where $\alpha$ is some constant. Using Lemma 14 the time taken to restore $S_{\log^* n}$ is $O(m + n)$. Thus the time taken for all restorations of $S_{\log^* n}$ is $O(\alpha^2(m + n)) = O(m + n)$.

2. $1 \leq i < \log^* n$

Using Lemma 17 and 18 the time taken to restore $S_i$ ($2 \leq i \leq \log^* n$) is $O \left( \left(1 + \frac{m}{n \log^{(i-1)} n} \right) \times \frac{n}{(\log^{(i)} n)^2} \right)$. Thus the time taken for all the restoration of $S_i$ is $O \left( \left(1 + \frac{m}{n \log^{(i)} n} \right) \times \frac{n}{(\log^{(i)} n)^2} \right) = O \left(n + \frac{m}{\log^{(i)} n}\right)$. Hence, the total time taken to restore all $S_i$'s ($1 \leq i \leq \log^* n$) is $O \sum_{i=1}^{\log^* n - 1} \left(n + \frac{m}{\log^{(i)} n}\right) = O(n \log^* n + m)$.

Thus, the total time taken by our algorithm is $O(m + n \log^* n)$. This proves our main result, that is Theorem 1.

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This is the reason we left out the first restoration, as given any restoration we want to look back at the step when the previous restoration happened.
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