Test vectors for some ramified representations

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Abstract

We give an explicit construction of test vectors for $T$-equivariant linear functionals on representations $\Pi$ of $GL_2$ of a $p$-adic field $F$, where $T$ is a non-split torus. Of particular interest is the case when both the representations are ramified; we completely solve this problem for principal series and Steinberg representations of $GL_2$, as well as for depth zero supercuspidals over $F = \mathbb{Q}_p$. A key ingredient is a theorem of Casselman and Silberger, which allows us to quickly reduce almost all cases to that of the principal series, which can be analyzed directly. Our method shows that the only genuinely difficult cases are the characters of $T$ which occur in the primitive part (or “type”) of $\Pi$ when $\Pi$ is supercuspidal. The method to handle the depth zero case is based on modular representation theory, motivated by considerations from Deligne-Lusztig theory and the de Rham cohomology of Deligne-Lusztig-Drinfeld curves. The proof also reveals some interesting features related to the Langlands correspondence in characteristic $p$. We show in particular that the test vector problem has an obstruction in characteristic $p$ beyond the root number criterion of Waldspurger and Tunnell, and that it exhibits an unexpected dichotomy related to the weights in Serre’s conjecture and the signs of standard Gauss sums.
1 Introduction

Let $F$ be a finite extension of $\mathbb{Q}_p$, for a rational prime $p$. Let $G$ denote the algebraic group $GL_2$ over $F$ and let $T \subset GL_2$ denote a maximal torus. Let $\chi$ denote a 1-dimensional character of $T(F)$. Let $\Pi$ denote an irreducible smooth admissible infinite dimensional representation of $G(F)$, with central character $\omega$. Suppose that $\chi$ agrees with $\omega$ on $F^* \subset T(F)$. The space $\text{Hom}_T(\Pi, \chi)$ has dimension 1 when it non-empty, which we assume to be the case. Let $\ell_{\chi}$ be a nonzero element of $\text{Hom}_T(\Pi, \chi)$. Then a test vector for $\ell_{\chi}$ is a vector $v \in \Pi$ such that $\ell_{\chi}(v)$ is nonzero. The study of test vectors was initiated by Gross and Prasad in [7]. The main results in [7] treat the cases where the ramification of $\Pi$ and $\chi$ is disjoint. It was already well-known, even before Gross-Prasad, that the case where $\Pi$ and $\chi$ are both ramified is significantly more complicated. Indeed, this problem comes up in producing explicit formulae for the local Rankin-Selberg integrals of a tensor product $\Pi_1 \otimes \Pi_2$ when $p$ is a bad prime for both $\Pi_1$ and $\Pi_2$. Many of these ramified cases were treated in [5], [2], and [9], but the general problem remains open.

From the point of view of number theory, having a recipe to produce a test vector at bad primes is highly desirable, in order to give completely explicit formulae of Gross-Zagier type. The consideration of bad places is unavoidable in Iwasawa theory and $p$-adic deformation theory, but many results of Gross-Zagier type impose unpleasant restriction on the level, or are proven only up to a nonzero constant, arising from unknown local integrals at the bad places. Furthermore, applications to number
theory demand that the test vector arising should be related in some elementary way
to the newform theory of Atkin-Lehner. As expressed in [5], one would like to exhibit
a test vector in the form of an explicit translate of the Atkin-Lehner new vector, and
one would like further to have a translate whose definition depends only on elementary
ramification data such as the conductor and central character of Π and the conductor
of the character χ. In the original setting of [7], the emphasis was placed on finding a
test vector that was fixed under the group of units of a suitably chosen Eichler order.

The original motivation for this paper was to produce test vectors in some cases
that were needed in the construction of [13], since these results seemed unavailable
in the literature. In the process of filling in the missing cases, we discovered that
it was possible to give substantially simpler proofs of some of the results of [5], and
also give a number of new results that were not covered in [5] at all. The results of
[5] are comprehensive in the case of a split torus T; our results in this paper pertain
to the case where the torus T corresponds to a quadratic field extension K/F. For
the remainder of this paper, we assume that T = K∗, for a quadratic field extension
K/F.

Our first result is quite general. We exhibit, in full generality, a test vector vχ such
that ℓχ(vχ) ≠ 0, provided that the ramification of χ is large compared to that of Π.
The test vector we supply is an obvious translate of the new vector, and the definition
depends only on elementary ramification data. This result is already contained in [5],
but our proof is simpler, and gives a pleasant explicit bound on the ramifications in
question. The key new ingredient is systematic use of a theorem of Silberger [12] and
Casselman [4]. Casselman and Silberger show that the restriction of Π to GL2(OF)
decomposes as the sum of a finite dimensional piece (the primitive piece, or “type”),
which we denote as V(Π), plus an infinite dimensional piece, denoted V(ω), that
depends only on the central character ω. The torus T(F) is compact mod centre,
under our assumptions, and one obtains a corresponding decomposition of Π into
characters of T. Each character appears with multiplicity one so we can classify the
characters of T which appear according to whether they appear in V(Π) or V(ω).

Then, our first result solves the test vector problem for those characters of T
which appear in V(ω). The constituents of V(ω) as a representation of GL2(OF) are
essentially induced, and producing test vectors is an elementary problem. Since V(ω)
depends only on $\omega$, our results here are valid for all $\Pi$. One way of interpreting this result is that the decomposition $V = V(\Pi) \oplus V(\omega)$ is a manifestation of the phenomenon that the epsilon factors of twists of $\Pi$ stabilize once the twist is sufficiently ramified; since the existence of a nonzero $\chi$-equivariant functional $\ell_\chi$ is governed by the value of a certain epsilon factor, one sees that the criterion which governs the existence of $\ell_\chi$ is automatic once $\chi$ is sufficiently ramified, and the existence of a good test vector is more or less automatic as well. Since the results depend only on $\omega$, one has only to analyze the principal series, and the results are given in Section 2 below. Another perspective was pointed out by Prasad – there is only one singular point for the group $PGL_2$ which is the identity, and if ones subtracts the characters of two representations, one gets a function which has no singularity and is locally constant everywhere, and whose restriction to any compact open subgroup such as $GL_2(\mathcal{O})$ is the character of a finite dimensional representation.

The more interesting case is that of characters $\chi$ which occur in the type $V(\Pi)$. Here the root number condition controlling the existence of $\ell_\chi$ is non-empty, and not easy to make explicit. Any construction of the test vector has to take into account the root numbers, and cannot simply depend on the levels of the characters and representations. This kind of analysis is not too hard for the case of special or principal series representations, where the type $V(\Pi)$ is rather simple, but it is highly nontrivial in the case of supercuspidals, where the finite dimensional representation $V(\Pi)$ is inflated from an irreducible cuspidal representation of the finite group $GL_2(\mathcal{O}_F/\varpi^n\mathcal{O}_F)$, for suitable $n$. Here $\varpi$ is a uniformizer of $\mathcal{O}_F$.

The most interesting results of this paper give a detailed analysis of the test vector problem for characters $\chi$ of an unramified quadratic extension $K/\mathbb{Q}_p$ and the case of depth zero supercuspidals $\Pi$. This means that $V(\Pi)$ has dimension $p - 1$, and is inflated from an irreducible cuspidal representation of the $GL_2(\mathbb{F}_p)$. To give an idea of how basic the test vector problem is, consider the case that the central character $\omega$ is trivial; then the trivial character of $T$ occurs in $V(\Pi)$, as shown by Gross [6], but the results of this paper are the first to give a natural test vector for the unique $T$-equivariant linear functional on $\Pi$. We remark here that this case is briefly treated in [7] and [6] but the test vector there is not related in any way to the invariants of an Eichler order, and is not connected in any way to the newforms of number theory.
Our results are quite surprising, and are intimately related to representation theory in characteristic $p$.

To state the results we recall some background material on representations of $GL_2$ over a finite field, as given in the book [10]. It is shown in that book that the cuspidal complex representations of $GL_2(F_q)$ are parametrized by characters $\nu : F_q^* \to \mathbb{C}^*$ such that $\nu \neq \nu^q$. Any such $\nu$ gives rise to a $q - 1$-dimensional representation $\rho(\nu) : GL_2(F_q) \to GL_{q-1}(\mathbb{C})$ with central character $F_q^* \to \mathbb{C}^*$ given by the restriction of $\nu$ to $F_q^*$. We have $\rho(\nu) \cong \rho(\nu^q)$, and this is the only possible isomorphism between the various representations $\rho(\nu)$, for varying $\nu$.

Now let $F/K$ denote the unramified quadratic extension. Since $p$ is odd, we may and do identify $K$ with the subgroup of matrices $\begin{pmatrix} a & b \\ b\xi & a \end{pmatrix}$ with $a, b \in F, a^2 - \xi b^2 \neq 0$ where $\xi \in F^*$ is a non-square unit. Reduction modulo $\varpi$ gives rise to an embedding $F_{q^2}^* \subset GL_2(F_q)$. Let $\nu$ denote a character of $F_{q^2}^*$ as above, and let $\omega$ be any character of $F^*$ which agrees with (the inflation of) $\nu$ on $O_F^*$. We remark that $\nu$ is determined on $O_K^*$ by inflation, but that we can extend $\nu$ to $K^*$ by requiring it to agree with $\omega$ on $F^*$. Then there exists an irreducible smooth admissible representation $\Pi$ of $GL_2(F)$ with central character $\omega$, such that the type $V(\Pi)$ is isomorphic to $\rho(\nu)$.

Consider a character $\chi$ of $T = K^*$ such that $\chi$ agrees with $\omega$ on $F^*$. Assume further that $\chi$ is trivial on the subgroup $1 + \varpi K O_K$ of principal units in $K$. Then we may identify $\chi$ with a character of $F_{q^2}^*$ which agrees with $\nu$ on $F_q^*$ (the central character being fixed). It is a basic fact about cuspidal representations of $GL_2(F_q)$ that each such character $\chi \neq \nu, \nu^q$ occurs with multiplicity one in the representation $\rho(\nu)$. Thus, for each choice of $\chi$, we get a distinguished line in the space $V(\Pi)$. As was observed by Gross and Prasad, a vector $v$ in $V$ is a test vector for $\chi$ if and only if it has nonzero projection on this distinguished line (here projection means orthogonal projection with respect to the action of the compact group $O_K^*$).

Now let $v^{\text{new}} \in \Pi$ denote the new vector provided by Casselman’s local version of Atkin-Lehner theory [3]. By definition, $v^{\text{new}}$ is a nonzero vector on the unique line in $V$ such that the subgroup $\begin{pmatrix} a & b \\ \varpi^2 c & d \end{pmatrix}$, with $a, d \in O_F^*, b, c \in O_F$, acts via the character $\omega(a)$. A basic observation is that $v^{\text{new}}$ lies in $V(\omega)$ and has projection zero
to $V(\Pi)$. Thus $v^{\text{new}}$ is never a test vector for any character $\chi$ appearing in $V(\Pi)$, and to produce a plausible candidate for test vector, one has to translate $v^{\text{new}}$ to put it inside $V(\Pi)$. Thus, consider the vector

$$v_1 = \left( \begin{array}{c} \omega^{-1} \\ 0 \\ 1 \end{array} \right) \cdot v^{\text{new}},$$

where the action of the matrix is given by the representation $\Pi$. An easy calculation show that $v_1$ is invariant under the group of matrices congruent to the identity modulo $\omega$. Thus $v_1 \in V(\Pi)$ and in fact $v_1$ is an eigenvector for the subgroup $D^0$ of matrices of the form

$$\left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right), \quad a \in O_F^*,$$

with eigenvalues given by $\omega(a)$. Thus, to determine whether $v_1$ is a test vector for $\chi$, one has to determine whether an eigenvector for the split diagonal torus has nonzero projection on an eigenspace of a non-split torus. It is this tension between the split and non-split tori that makes the problem hard.

More generally, one has to solve the following problem in finite group theory. Let $\nu : F_q^* \to \mathbb{C}^*$ be a character such that $\nu \neq \nu^q$, and let $\rho(\nu)$ denote corresponding cuspidal representation of $GL_2(F_q)$. Let $\mu$ denote a character of $F_q^*$, and let $v_\mu$ denote an eigenvector for the group $D^0 = \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right), \quad a \in F_q^*$, with eigenvalues under $\rho(\nu)$ given by

$$\left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \cdot v_\mu = \mu(a)v_\mu.$$

Such an eigenvector always exists, for any $\mu$. In fact, $\rho(\nu)$ decomposes as the sum of $q - 1$ invariant lines, each being an eigenspace for a unique character $\mu$. Let $\chi : F_q^* \to \mathbb{C}^*$ denote a character of the nonsplit torus $T$ such that $\chi$ agrees with $\nu$ on $F_q^*$, such that $\chi \neq \nu, \nu^q$. Here we identify the nonsplit torus with $2 \times 2$ matrices as explained above. Then, is the projection of $v_\mu$ to the $\chi$-eigenspace of $T$ nonzero? It is this problem that we solve in this paper, under the key hypothesis that $q = p$, so that we are dealing with the prime field. While our methods give some information for general $q$, the combinatorics are quite complicated, and we have not pursued a general theorem.

To give the reader a taste of our results, we give a sample in an illuminating special case, which arises from elliptic curves with conductor $p^2$. Suppose that $\Pi$ is
a depth zero supercuspidal representation of $GL_2(\mathbb{Q}_p)$ with trivial central character $\omega$. Then one can ask whether or not the vector $v_{\text{triv}}$ corresponding to the trivial character $\mu = 1$ is a test vector for some given $\chi$, for instance, for $\chi = 1$. Another obvious candidate for a test vector is the vector $v_{\text{quad}}$ corresponding to the choice of $\mu = \mu^{\text{quad}}$, the nontrivial quadratic character of $F_p^*$. The somewhat surprising answer is that both choices $v_{\text{triv}}$ and $v_{\text{quad}}$ work, but only half the time.

To state our result precisely, we need some way to label the characters of $T$ which arise. Thus fix any $\chi$ such that $\chi$ is trivial on $\mathbb{Q}_p^*$ and on the principal units $1 + p\mathcal{O}_K$. Any such character is defined on $\mathcal{O}_K^*/1 + p\mathcal{O}_K \cong F_p^*$ and takes values in the group of $\mu_{p^2-1}$ of complex $p^2 - 1$-th roots of unity. If we fix a prime $p$ of $\overline{\mathbb{Q}}$ above $p$, we may identify $\chi$ with a character $F_p^* \rightarrow F_p^*$. Then $\chi$ can be expressed in the form $x \mapsto x^{\nu^p}$, where $0 \leq a, b \leq p - 1$. This expression is unique except for the case $(a, b) = (0, 0)$ and $(a, b) = (p - 1, p - 1)$. We fix the former choice in that case. We write $\chi = \chi(a, b)$. We remind the reader that the integers $(a, b)$ depend on a choice of $p$. Given a representation $\rho(\nu)$ attached to a character $\nu$ such that $\nu \neq \nu^p$, the integers $a, b$ determined by $\nu$ must satisfy $a \neq b$, and so we may assume, by switching $\nu$ and $\nu^p$ if necessary, that $a > b$. If $\nu$ is trivial on $F_p^*$, then we must have $a + b \equiv 0 \pmod{p - 1}$, namely, $a = p - 1 - b$. Note that $a = b = 0$ is excluded since $\nu \neq \nu^p$.

Now let $\chi$ denote any character of $K^*/\mathbb{Q}_p^*(1 + p\mathcal{O}_K) \cong F_p^*/F_p^*$, and write $\chi = \chi(r, s)$ as above, with $0 \leq r, s \leq p - 1$ as above. Of course, for general $\chi$ we may have $r = s$ or $r < s$. However, we still have $r + s \equiv 0 \pmod{p - 1}$, which means $r + s = p - 1$ unless $r = s = 0$. In particular, $\chi$ is determined by the value of $r$ except for $(0, 0)$ and $(0, p - 1)$. Thus there are $p + 1$ choices for $\chi$, as expected, including $\chi = \nu, \nu^p$. Two of the $p + 1$ possible characters $\chi$ factor through the norm, namely, the trivial character and the unique nontrivial character of order 2. The remaining $p - 1$ characters are primitive in the sense that they do not factor through the norm.

To get a better label for $\chi(r, s)$ define $k_{\chi} = r + sp$, so that $\chi(x) = x^{k_{\chi}}$. The integer $k_{\chi}$ is closely related to the weight arising in Serre’s conjecture, but we will not need this connection here. Note that $k_{\nu} = a + bp$, and $k_{\nu^p} = b + ap$. Since $\nu \neq \nu^p$, we must have $k_{\nu} \neq k_{\nu^p}$; in fact, one has $k_{\nu^p} - k_{\nu} = (a - b)(p - 1) > 0$, so $k_{\nu^p} > k_{\nu}$. Furthermore, if $\rho(\nu)$ has trivial central character, then $\nu$ is trivial on $F_p^*$, so that $k_{\nu} = bp + a \equiv 0 \pmod{p - 1}$. Thus the integers $k_{\chi}$ run through the integers $0, p - 1, 2(p - 1), \ldots, p(p - 1)$.~

7
Since \( a + b = p - 1 \), one has \( k_\nu = a + bp = p - 1 - b + bp = (p - 1)(b + 1) \) and \( k_{\nu\rho} = b + ap = (p - 1)(a + 1) \). We label \( \chi \) with the integer \( k_\chi \), and we shall say that \( \chi \) has Type 1 if
\[
k_\nu < k_\chi < k_{\nu\rho}.
\]
Otherwise we say \( \chi \) is of Type 2. Observe that the trivial character \( \chi \) has \( k_\chi = 0 \) and has Type 2, while the nontrivial quadratic character has \( k_\chi = (p^2 - 1)/2 \) and has Type 1.

Write \( \psi \) to denote a nontrivial additive character of \( \mathbf{F}_p \). Then if \( \chi \) is as above, let \( G(\chi) \) denote the standard finite field Gauss sum with respect to \( \psi \) for \( \chi \) (viewed as a complex character of \( \mathbf{F}_p^* \)) so that \( G(\chi) = \sum_{u \in \mathbf{F}_p^2} \chi(u) \psi(\text{Tr}(u)) \). Then one has the following lemma, which will be proved in Section 3.

**Lemma 1.1** Suppose \( \chi \) does not factor through the norm, and that \( \chi \) is trivial on \( \mathbf{F}_p^* \). Then \( \epsilon_p(\chi) := G(\chi)_p = \chi(\sqrt{\xi}) = \sqrt{\xi}^{k_\chi} \), where \( \sqrt{\xi} \in \mathbf{F}_p^* \) is any element of trace zero whose square is a non-square element \( \xi \) in \( \mathbf{F}_p^* \).

With this lemma in hand, we extend the definition of \( \epsilon_p(\chi) \) to all \( \chi \) by setting \( \epsilon_p(\chi) = \chi(\sqrt{\xi}) \) when \( \chi \) is either trivial or quadratic. Then our result is as follows.

**Theorem 1.2** Let \( \Pi \) denote a depth zero supercuspidal representation of \( GL_2(\mathbf{Q}_p) \) associated to a character \( \nu : \mathbf{F}_p^* \to \mathbf{C}^* \). Assume that \( \Pi \) has trivial central character, so the type \( \rho(\nu) \) is such that \( \nu = \nu(a,b) \) with \( a > b \) and \( a + b = p - 1 \). Let \( \chi \) denote a character of \( \mathbf{K}^* / \mathbf{Q}_p^*(1 + p\mathbf{O}_K) \), identified with a character of \( \mathbf{F}_p^* / \mathbf{F}_p^* \). Then the following statements hold.

1. The vector \( v_{\text{triv}} \) is a test vector for the trivial character \( \chi = 1 \) if and only if \( \epsilon_p(\chi) \neq \epsilon_p(\nu) \). The character \( \chi = 1 \) is of Type 2.

2. The vector \( v_{\text{quad}} \) is a test vector for the nontrivial quadratic character \( \chi \) of \( \mathbf{K}^* / \mathbf{Q}_p^*(1 + p\mathbf{O}_K) \) if and only if \( \epsilon_p(\chi) \neq \epsilon_p(\nu) \). The nontrivial quadratic character \( \chi \) is of Type 1.

3. More generally, the vector \( v_{\text{triv}} \) is a test vector for all Type 2 characters \( \chi \) such that \( \epsilon_p(\chi) \neq \epsilon_p(\nu) \) and the vector \( v_{\text{quad}} \) is a test vector for all Type 1 characters.
χ such that εp(χ) ≠ εp(ν).

4. Neither ν-triv nor ν-quad is a test vector for any character χ with εp(χ) = εp(ν).

One can give a similar result for general depth zero supercuspidals when F = Qp, as follows. Let Π denote a depth zero supercuspidal representation of GL2(Qp) with central character ω, associated to the type ρ(ν). Write ν = υ(a, b) as above. Here we understand that ν is a character of F∗p2, extended to K∗ by virtue of the central character ω, and the integers a, b depend on a choice of prime above p in Q, and an identification of the complex p2 − 1 roots of unity with F∗p2. In particular, if x ∈ F∗p, then we have ν(x) = ω(x) = x(a+pb) = xa+b, by definition. Given a character χ of \(\mathcal{O}_K/(1 + p\mathcal{O}_K) = F^*_p\) we extend it to K∗, since compatibility with ω specifies the value χ(p) = ω(p). It will be notationally convenient to deal only with characters of the finite groups F∗p and F∗p2, and we generally will do this without comment in the sequel, since ω will be fixed.

Once again, we have p+1 characters χ, including ν, νp of Fp2 which agree with ν on F∗p. We extend χ to characters of K∗ as above. Write χ = χ(r, s), with 0 ≤ r, s ≤ p−1 as explained above, and set kχ = r + sp, then kχ ≡ kν (mod p−1), and we may write kχ = kν + tχ(p−1) for some tχ which is uniquely determined modulo p+1. Once again we can classify characters of Type 1 and Type 2, with the exactly the same definition as before, namely, by requiring that the characters of Type 1 are those whose weights lie between the weights of ν and νp.

Assume first that ω is even, in the sense that ω(−1) = 1, or − equivalently − that a + b is even. Let μ1 and μ2 denote the two possible characters of F∗p such that μ2 = ω, and let χi = N · μi be the composition with the norm. For concreteness, we take μ1(x) = x(a+b)/2 and μ2(x) = x(a+b+p−1)/2. Let μ3 denote the character of F∗p defined by x ↦ x(a+b)/2−1 and let μ4(x) = x(a+b+p−1)/2−1.

**Theorem 1.3** Let Π denote the depth zero supercuspidal representation associated to a character ν : Fp2* → C∗. Let ν = ν(a, b) with a+b even, a > b. Let χ be a character of the unramified quadratic extension K/Qp which agrees with ν on Fp2*. Then the following statements hold:
1. The vector \( v_{\mu_1} \) is a test vector for the character \( \chi_1 \) if and only if \( t_{\chi_1} \) is odd. The character \( \mu_1 \) is of Type 1.

2. The vector \( v_{\mu_2} \) is a test vector for the character \( \chi_2 \) if and only if \( t_{\chi_2} \) is odd. The character \( \mu_2 \) is of Type 2.

3. More generally, the vector \( v_{\mu_1} \) is a test vector for all characters \( \chi \) of Type 1 such that \( t_{\chi} \) is odd, and the vector \( v_{\mu_2} \) is a test vector for all characters \( \chi \) of Type 2 such that \( t_{\chi} \) is odd.

4. Neither \( v_{\mu_1} \) nor \( v_{\mu_2} \) is a test vector for any character \( \chi \) with \( t_{\chi} \) even.

5. The vector \( v_{\mu_3} \) is a test vector for all characters \( \chi \) of Type 1 such that \( t_{\chi} \) is even.

6. The vector \( v_{\mu_4} \) is a test vector for all characters \( \chi \) of Type 2 such that \( t_{\chi} \) is even.

We remark that the statements in the theorem above degenerate when \( p = 3 \), and \( \rho(\nu) \) has dimension 2, in the sense that some of the statements are empty. There are only two candidates for \( \chi \) in this case, namely, the trivial character and the nontrivial quadratic.

Finally, we give the theorem in the odd case. Let \( \nu : F_p^* \to C^* \) be a character such that \( \nu = \nu(a,b) \) with \( a + b \) odd and \( \nu \neq \nu^p \). Let \( \omega \) be a character of \( Q_p^* \) such that the composition of \( \omega \) with the Teichmüller lift of \( F_p^* \) agrees with \( \nu \). Let \( \Pi \) denote the depth zero supercuspidal associated to \( \nu \) and the choice of \( \omega \). Let \( \chi \) denote a character of the unramified quadratic extension \( K \) of \( Q_p \) which agrees with \( \omega \) on \( Q_p^* \). We can identify \( \chi \) with a character of \( F_{p^2}^* \) as above, and define the notion of Type 1 and Type 2, exactly as before.

Let \( \mu_1, \mu_2 \) be characters of \( F_p^* \) defined by \( \mu_1(x) = x^{(a+b-1)/2} \) and \( \mu_2(x) = x^{(a+b+p)/2} \), and let \( v_{\mu_i} \) denote corresponding eigenvectors in the type \( \rho(\nu) \) of \( \Pi \).

**Theorem 1.4** Let \( \Pi \) denote the depth zero supercuspidal representation associated to a character \( \nu : F_{p^2}^* \to C^* \). Let \((a, b)\) denote the pair associated to \( \nu \) and assume that
$a + b$ is odd, $a > b$. Let $\chi$ be a character of $K^*$ which agrees with $\nu$ on $\mathbb{Q}_p^*$ and which is distinct from $\nu, \nu^p$ on $\mathbb{F}_p^*$. Then the following statements hold if $a - b > 1$:

1. The vector $v_{\mu_1}$ is a test vector for $\chi$ if $\chi$ has Type 1.

2. The vector $v_{\mu_2}$ is a test vector for $\chi$ if $\chi$ has Type 2.

If $a - b = 1$ then the vector $v_{\mu_2}$ is a test vector for all $\chi$.

**Deligne-Lusztig theory, Serre weights, and the reduction to characteristic $p$**

We conclude the introduction with some comments on the proofs of the theorems in the supercuspidal case, and about the restriction to $F = \mathbb{Q}_p$ in the results. As we have remarked in the abstract, and as should be clear from the statements of the theorems, the results are based on reduction to characteristic $p$. This may seem somewhat artificial, but it may be motivated as follows. As we have explained, the fact that $v_{\mu}$ is a test vector for the character $\chi$ boils down to showing that $v_{\mu}$ has nonzero projection on the $\chi$-eigenspace of $T$. In other words, one has to show that the product of the two idempotents $e_{\chi}e_{\mu}$ is nonzero on $\rho(\nu)$, where $e_{\mu} = \frac{1}{q-1} \sum_{\sigma \in D_0} \mu^{-1}(\sigma)\sigma$, and $e_{\chi} = \frac{1}{q-1} \sum_{\sigma \in T} \chi^{-1}(\sigma)\sigma$. The construction of Deligne-Lusztig (which is due to Drinfeld in this case) shows how to realize the representation $\rho(\nu)$ (or rather, it’s restriction to $SL_2(\mathbb{F}_q)$) in the $\ell$-adic étale cohomology of the smooth plane curve $C: X^qY - Y^qX = 1$ over $\overline{\mathbb{F}}_q$. Here the action of $SL_2(\mathbb{F}_q)$ on $C$ is via the standard linear action on $(X, Y)$. Since $SL_2(\mathbb{F}_q)$ acts on $C$, it also acts on the Jacobian of $C$, and we find that there is a realization of $\rho(\nu)$ in the $\ell$-adic Tate module of $J$. Our problem is therefore to compute the image of $e_{\chi}e_{\mu}$ in the $\rho(\nu)$-isotypic part of $T_\ell(J) \otimes \overline{\mathbb{Q}}_\ell$, and show that it is nonzero.

So far this idea is mere tautology – computing $e_{\mu}e_{\chi}$ on the Jacobian is no different from trying to compute it directly, since the realization via Deligne-Lusztig doesn’t give any explicit model. However, the point is that one can think of $e_{\mu}e_{\chi}$ as a kind of endomorphism of $J$, and to prove that an endomorphism $e$ is nonzero, it is enough to compute $e$ on the tangent space of $J$, or what amounts to the same, on the space of holomorphic differentials on $C$. In other words, it is sufficient to check non-vanishing in the characteristic $p$ vector space given by de Rham cohomology rather
than the characteristic zero étale cohomology. The de Rham cohomology of $J$ is a representation of $SL_2(F_q)$ over $F_p$, and one can hope that it is easier to compute with than the original representation in characteristic zero. Happily, this turns out to be just the case.

Let us assume henceforth that $q = p$ is prime. Since $C$ is a smooth plane curve of degree $p + 1$, the classical method of adjoints shows that the sheaf of regular differentials is isomorphic to the twisting sheaf $O(p - 2)$, and $H^0(C, \Omega^1)$ is the span of the space of monomials $X^iY^j$ with degree $d = i + j \leq p - 2$. The action of $SL_2(F_p)$ is simply linear. Clearly the space of polynomials of a given degree is invariant under $SL_2(F_p)$, and we get a decomposition of $H^0(C, \Omega^1)$ into the sum of $p - 1$ representations, each of which is known to be irreducible. These representations are essentially the set of possible Serre weights for $SL_2(F_p)$. The eigenvectors for the split torus $D^0$ are just the monomials $X^iY^j$. To find the eigenvectors of the nonsplit torus $T$, one has to diagonalize matrices of the form \[
\begin{pmatrix}
 a & b \\
 b\xi & a
\end{pmatrix}
\] where $a, b \in F_p, a^2 - \xi b^2 \neq 0$ and $\xi$ is a nonsquare; this can be accomplished by considering monomials in $U = X - \sqrt{\xi}Y, V = X + \sqrt{\xi}Y$, and the composite $e_\chi e_\mu$ can be computed on any given polynomial simply by making the corresponding change of variables.

This geometric approach was the one taken in an earlier version of this work. However, the details are somewhat involved, even though the underlying idea is very simple, since one has to pass somehow from $SL_2$ to $GL_2$, and keep track of the various representations in characteristic zero and characteristic $p$. Furthermore, one has to deal with the fact that $e_\chi$ and $e_\mu$ will in general have non-rational coefficients, and are not genuine endomorphisms of the Jacobian. Nevertheless, it is quite natural to compute with Deligne-Lusztig curves, given the realization of the discrete series of $SL_2(F_p)$ in the Drinfeld curve, and it is also natural in view of the appearance in our results of the weights in Serre’s conjecture. This connection also arises in results of Herzig [8], who deals with the Serre weights which show up in the de Rham cohomology of Deligne-Lusztig varieties for $GL_n$. In general, it seems like an interesting idea to compute the de Rham cohomology of a Deligne-Lusztig variety as a representation space for the group in question, and compare the results with those in characteristic zero.
In this paper we use a simpler and more elementary method: we simply reduce the characteristic zero representations modulo $p$, and compute $e_\chi e_\mu$ on the reduced representation. This turns out to be enough for our purposes, and avoids any complications coming from geometry. The reduction of a discrete series representation of $GL_2(F_p)$ has generically 2 components, and this gives rise to our Type 1/Type 2 dichotomy. However, implementing this method reveals that it is significantly more complicated for residue field $F_q \neq F_p$ – the reduction of a discrete series representation of $GL_2(F_q)$, $q = p^d$ typically has $2^d$ components, parametrized by subsets of the $d$ embeddings of $F_q$ in $F_p$. Thus we get correspondingly more types of characters: one has a notion of Type 1/Type 2 at each embedding of $F_q$, and the statements of the results become correspondingly more complicated. We have not pursued this avenue, in this paper, principally to keep the length of the article reasonable. However, it seems clear that the same method would work for general $q$, at least in principle.

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**Notation:**

Let $\mathcal{O}$ denote the ring of integers in $F$, and let $P$ denote the maximal ideal, generated by the uniformizer $\varpi$. The ring of integers of $F$ is written as $\mathcal{O}$. The residue field of $F$ is $k_F = F_q$. If considering a finite extension $K/F$ we indicate the corresponding objects with a subscript $K$. For a ring $R$, let $G(R)$ denote the group $GL_2(R)$. For an integer $s \geq 0$, let $G(s)$ denote the principal congruence subgroup of $G(\mathcal{O})$, and let $B(s)$ denote the subgroup with of matrices with lower left corner divisible by $P^s$. We let $T$ denote a maximal torus of $G$, assumed non-split, and let $\Pi$ denote an irreducible infinite dimensional representation of $G(F)$ which is ramified, in the sense that the conductor of $\Pi$ is a nontrivial ideal of $\mathcal{O}$. Equivalently, the space of invariants of $G(\mathcal{O})$ in $\Pi$ is trivial. If $K$ is any finite extension of $F$ and $\chi$ is a character of $F^\times$, then the conductor of $\chi$ is the the ideal $(\varpi_K^r)$, where $r$ is the smallest non-negative integer such that $\chi$ is trivial on the set $1 + \varpi_K^r \mathcal{O}_K$. We will sometimes refer to $r$ itself as the conductor, and hope this will not ruffle too many feathers.
2 K-TYPES AND THE PRINCIPAL SERIES

In this section we use a key result due to Silberger Casselman, which states that any smooth irreducible admissible infinite dimensional representation of \( GL_2(F) \) is the sum of a primitive piece of finite dimension (the “type”) and another piece which depends only on the central character. Thus we will analyze test vectors for principal series representations, and deduce the consequences for an arbitrary representation sharing the same central character.

Consider an irreducible admissible smooth infinite dimensional representation of \( GL_2(F) \) with central character \( \omega \). Let \( P^r \) denote the conductor of \( \Pi \), so that \( r \) is the smallest non-negative integer such that \( \Pi \) contains a nonzero vector \( v_{\text{new}} \) such that \( g \cdot v_{\text{new}} = \omega(a)v_{\text{new}} \), for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(r) \). For each positive integer \( s \geq r \), we let \( u_{P^s}(\omega) \) denote the representation defined by Casselman \[3\], page 312. Thus \( u_{P^r}(\omega) = \text{Ind}_{B(r)}^{G(O)}(\omega) \), while for \( s > r \) it is is defined as the complement of \( \sum_{r \leq t \leq s} \text{Ind}_{B(t)}^{G(O)}(\omega) \) inside \( \text{Ind}_{B(s)}^{G(O)}(\omega) \). Again, we view the character \( \omega \) as a character of \( B(s) \) via \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \omega(a) \), which makes sense as long as \( s \geq r \). Each representation \( u_{P^s}(\omega) \) irreducible, by Proposition 1(b) of \[3\]. Each \( u_{P^s}(\omega) \) contains a distinguished vector \( u_s \) on which \( B(s) \) acts via the character \( \omega \). The vector \( u_s \) is unique up to scalars, and \( u_{P^s}(\omega) \) is characterized as the unique irreducible representation of \( G(O) \) containing a \( \omega \)-eigenvector for \( B(s) \) and which is trivial on \( G(s) \) but not on \( G(s-1) \). Then Casselman’s theorem from \[4\] is the following:

**Theorem 2.1** The restriction of \( \Pi \) to \( G(O) \) decomposes as \( \Pi = V(\pi) \oplus V(\omega) \), where \( V(\pi) \) is finite dimensional, and \( V(\omega) = \oplus_{s \geq r} u_{P^s}(\omega) \). The space \( V(\pi) \) is the space of invariants under the principal congruence subgroup \( G(r-1) \).

Consider a ramified principal series representation \( \Pi \) of \( GL_2(F) \) that has minimal conductor amongst its twists. Thus \( \Pi = \Pi(\nu,1) \) for some character \( \nu : F^* \rightarrow \mathbb{C}^* \). Let \( P^r \) denote the conductor of \( \nu \). We assume that \( r \geq 1 \). Then the conductor of \( \Pi \) is the ideal \( P^r \), which is the same as the conductor of \( \nu \).
Proposition 2.2 The restriction of $\Pi$ to $G(\mathcal{O})$ decomposes as $\bigoplus_{s \geq r} u_{P^s}(\nu)$.

Proof It is clear that $\bigoplus_{s \geq r} u_{P^s}(\nu)$ occurs inside $\Pi|_{G(\mathcal{O})}$. According to Theorem 1 in [3], the complement is the space of vectors fixed by $G(r - 1)$. However, this is zero, since the central character of $\nu$ has conductor $P^r$.

Thus the first irreducible piece of $\Pi$ is the $(q + 1)q^{r-1}$-dimensional representation $\text{Ind}_{G(\mathcal{O})}^{G(\mathcal{O})}(\nu)$, generated by the new vector.

Now consider a quadratic field extension $K/F$. We let $\mathcal{O}_K$ denote the ring of integers in $K$. We assume given an embedding of $\mathcal{O}_K^*$ into $G(\mathcal{O})$. It is well-known that such embeddings exist, since $G(\mathcal{O})$ is the group of units in a maximal order of $M_2(F)$. Thus, the restriction of $\Pi$ to $\mathcal{O}_K^*$ factors through the decomposition in the proposition above. Let $\mathcal{O}_{K,s}$ denote the order $\mathcal{O} + P^s\mathcal{O}_K$, for a positive integer $s$.

Lemma 2.3 We have $\mathcal{O}_K \cap G(2) = \{ u \in \mathcal{O}_{K,r} | u \equiv 1 \pmod{P^r} \}$.

Proof Suppose $u \in \mathcal{O}_K \cap G(2)$. Then $\frac{u - 1}{\varpi} \in M_2(\mathcal{O}) \cap K$, where $\varpi$ is a uniformizer for $P \subset \mathcal{O} \subset \mathcal{O}_K$. But $M_2(\mathcal{O}) \cap K = \mathcal{O}_K$, so we are done.

Now we assume that $K/F$ is unramified. We want to restrict the decomposition given by Casselman above to the group $\mathcal{O}_K^* \subset G(\mathcal{O})$, and for this we need some way to label the characters of $\mathcal{O}_K^*$ whose restriction to $\mathcal{O}^*$ coincides with the central character $\nu$ of $\Pi$. Consider any such character $\chi$. Define the conductor of $\chi$ to be the smallest positive integer $s$ such that $\chi$ is trivial on the set $\{ u \in \mathcal{O}^*_K | u \equiv 1 \pmod{P^s} \}$.

Clearly we must have $s \geq r$. The order of $(\mathcal{O}_K/P^s\mathcal{O}_{K})^*$ is $(q^2 - 1)q^{2(s-1)}$, and the order of $(\mathcal{O}/P^s)^*$ is $(q - 1)q^{s-1}$, so there are $(q + 1)q^{s-1}$ characters of conductor up to $s$. Note also that since $K/F$ is unramified, a character of $K$ is determined uniquely by its restriction to $\mathcal{O}_K^*$ and $\mathcal{F}^*$.

Proposition 2.4 Let $s \geq r$. Then the restriction of $u_{P^s}(\nu)$ to $\mathcal{O}_K^*$ is the sum of characters of level precisely $s$ that agree with $\nu$ on $\mathcal{F}^*$.

Proof Start with the case of $s = r$. It follows from the lemma above that the
restriction of \( u_{P^r}(\nu) \) to \( O_K^* \) is trivial on the set of elements congruent to 1 modulo \( P^r \). Equivalently, this restriction is the sum of characters of conductor at most \( P^r \), and any character that shows up is equal to \( \nu \) on the centre \( F^* \). But the dimension of \( u_{P^r}(\nu) \) is \((q + 1)q^{r-1}\), which is the number of possible characters. Since we know from a theorem of Tunnell that the restriction is multiplicity free, it follows that the restriction of \( u_{P^r}(\nu) \) to \( O_K^* \) is precisely the set of characters of \( K^* \) of conductor \( r \) that agree with \( \nu \) on the centre.

It remains to deal with the case of \( s > r \). This is a simple induction, using the fact that the dimension of \( u_{P^s}(\nu) \) is equal to the number of possible characters, together with the facts that the restriction is known to be multiplicity free, and that \( u_{P^s} \) is trivial on \( G(s) \).

The following theorem produces the requisite test vectors for minimal principal series representations and unramified \( K/F \).

**Theorem 2.5** Let \( K/F \) be unramified. Let \( \chi \) denote a character of \( K^* \) such that \( \chi \) agrees with \( \nu \) on \( F^* \) and such that \( \chi \) has level \( s > r \). Let \( u_s \) denote a nonzero vector on the unique line in \( u_{P^s}(\nu) \) that is a \( \nu \)-eigenspace for \( B(s) \). Then \( u_s \) is a test vector for \( \chi \). If \( s \leq r \), then the new vector \( v_r \) is a test vector for \( \chi \).

**Proof** By virtue of the proposition above, we know where to look for test vectors for characters of level \( s \): we must look in the representation \( u_{P^s}(\nu) \). Now, it is known that each representation \( u_{P^s}(\nu) \) contains a vector \( u_s \) such that \( g \cdot u_s = \nu(g)u_s \) for \( g \in B(s) \). Furthermore, we have \( G(O/P^s) = T(O/P^s) \cdot B(O/P^s) \), where \( T(O/P^s) \) and \( B(O/P^s) \) denote the images of \( O_K^* \) and \( B(s) \) in \( G(O/P^s) \). (The latter claim can be checked by observing that \( O_K^* \cap B(O) = O^* \), and by counting elements.) Thus, the translates of \( v_s \) by elements of \( O_K^* \) span the entire space \( u_{P^s}(\nu) \). Thus, if \( \ell_\chi(v_s) = 0 \), we must have \( \ell_\chi(\Pi(t)v_s) = \chi(t)\ell_\chi(v_s) = 0 \) for all \( t \in O_K^* \), and consequently, \( \ell_\chi \) must vanish identically on the entire space \( u_{P^s}(\nu) \). But this is impossible, since \( \chi \) appears in \( u_{P^s}(\nu) \). The proof of the statement in the case \( s \leq r \) is similar.

**Remark 2.6** In general, the vector \( u_s \) is not a translate of the new vector, unless \( r = s \). However, it is easy to check that since \( u_s \) is a test vector for \( \chi \) of conductor
s, then the translate \( v_s = \begin{pmatrix} \omega^{s-r} \\ 0 \\ 1 \end{pmatrix} \) is a test vector as well. We leave the details to the reader; the point is that \( v_s \) has nonzero projection on to \( u_s \), since it is not contained in \( \sum t < su_{P^r}(\nu) \).

2.7 Now we want to look at the case where \( K/F \) is ramified. In this case, there are two distinct maximal orders of \( M_2(F) \) that contain \( O_K \), and these two are swapped by the action of a uniformizer of \( O_K \). Equivalently, \( O_K \) is contained in an Eichler order \( R \) of level \( P \). We set \( H = R^* \).

Once again, we want to work out what sort of characters of \( K \) show up in the irreducible piece \( u_{P^r}(\nu) \). Let us write \( \omega_K \) and \( \omega \) for the uniformizers of \( O_K \) and \( O \) respectively, and consider a character \( \chi \) of \( K^* \) such that \( \chi \) agrees with \( \nu \) on \( F^* \).

Define the conductor of \( \chi \) to be the smallest positive integer \( t \) such that \( \chi \) is trivial on the set \( \{ u \in O_{K^*}^* | u \equiv 1 \pmod{\omega_{K}^{2t+1}} \} \). Clearly, we must have \( t \geq 2r - 1 \), just by looking at the central character, which has conductor \( r \) as a character of \( F^* \).

**Lemma 2.8** Let \( \chi \) denote a character of \( O_{K^*}^* \) such that \( \chi \) agrees with \( \nu \) on \( O_{K^*}^* \). Then the conductor of \( \nu \) is either equal to \( 2r - 1 \), or an even integer \( 2s \), where \( s \geq r \).

**Proof** It is clear that the conductor of \( \chi \) is at least \( 2r - 1 \), where \( P^r \) is the conductor of \( \nu \). Suppose that \( \chi(u) = 1 \) for the set of elements \( u \in O_{K^*}^* \) such that \( u \equiv 1 \pmod{\omega_{K}^{2t+1}} \), with \( t \geq r \). Then since \( \chi \) agrees with \( \nu \) on \( F^* \), \( \chi \) is also trivial on the set of elements \( u' \in F^* \) such that \( u' \equiv 1 \pmod{\omega^t} \). Now if \( u_1 \in O_{K^*}^* \) is any element that is congruent to 1 modulo \( p^t \), we may write \( u_1 = 1 + a\omega_{K}^{2t} + b\omega_{K}^{2t+1} \), with \( a \in O \) and \( b \in O_K \), since the residue fields of \( O_K \) and \( O_F \) are equal. Now select \( c \in O \) such that \( c\omega^t \equiv a\omega_{K}^{2t} \pmod{\omega_{K}^{2t+1}} \). Such \( c \) exists because \( \omega_{K}^{2t} \) and \( \omega^t \) are associate in \( O_K \), and the residue fields of \( O_K \) and \( O_F \) are equal. Then \( \chi(1 - c\omega^t) = \nu(1 - c\omega^t) = 1 \), and \( \chi(u_1) = \chi((1 - c\omega^t)u_1) = 1 \), because \( (1 - c\omega^t)u_1 \equiv 1 \pmod{\omega_{K}^{2t+1}} \). Thus the conductor of \( \chi \) is at most \( 2t \), and we are done.

We can now enumerate the characters \( \chi \) of \( K^* \) which agree with \( \nu \) on \( F^* \).

**Lemma 2.9** There are exactly \( 2q^{r-1} \) characters of \( K^* \) of conductor \( 2r - 1 \) that agree
with ν on $F^*$, and these restrict to $q^{r-1}$ distinct characters of $O^*_K$ of conductor $2r - 1$ that agree with ν on $O^*$.

If $s \geq r$, there are exactly $2(q^s - q^{s-1})$ characters of $K^*$ of conductor $2s$ that agree with ν on $F^*$, and these restrict to $q^s - q^{s-1}$ distinct characters of $O^*_K$ of conductor $s$ that agree with ν on $O^*$.

**Proof** It suffices to count the number of distinct characters of $O^*_K$ of given conductor which agree with ν on $O^*$, since each such obviously extends in two different ways to $K^*$, according to the value on the uniformizer $\varpi_K$, and the latter is determined up to sign by the restriction to $F^*O_K$.

In the case of minimal conductor $2r - 1$, we see that $(O^*_K/P^{2r-1}O_K)^*$ has cardinality $(q - 1)q^{2r-2}$. If $r = 1$, then the value of the character is determined by the values on elements of $O^*$, so it is unique. If $r > 1$, then the value of the character is determined on the elements coming from $O^*$. Observe that elements of $O$ have $\varpi_K$-adic expansions involving only even powers of $\varpi_K$, and are of the form $a_0 + a_1\varpi_K^2 + \ldots a_{r-1}\varpi_K^{2r-2}$ modulo $\varpi_K^{2r-1}$, which gives $(q - 1)q^{r-1}$ elements. Thus there are $q^{r-1}$ possible values for the character, if its value is fixed on $O^*$.

The remaining cases over even exponent $2s$ are calculated by a simple induction, starting with $r = s$. If $s \geq r$, then to calculate the number of characters of $O^*_K$ of conductor up to $2s$ that agree with ν on $O^*$, we simply observe that $(O^*_K/P^sO_K)^*$ has cardinality $(q - 1)q^{2s-1}$, whereas $(O/P^sO)^*$ has cardinality $(q - 1)q^{s-1}$.

**2.10** We know abstractly that the representation $\Pi$ decomposes as the direct sum of all characters $\chi$ of $K^*$ which agree with ν on $F^*$, and that this restriction is multiplicity-free. Thus if we restrict further to a representation of $O^*_K$, we get each possible character with multiplicity two. If the maximal order $O_K$ is optimally embedded in the maximal order $M_2(\mathbb{Z}_p)$, then this decomposition is compatible with the decomposition in (2.2), and our tasks is to locate the various eigenspaces for the full torus $K^*$ in terms of (2.2). Each eigenspace for $O^*_K$ is a two dimensional subspace inside (2.2), although we shall see below that it is typically not contained in any single summand. The uniformizer $\varpi_K$ acts nontrivially on every such eigenspace, and splits it in to two eigenspaces for distinct characters of the full group $K^*$. However, $\varpi_K$ is
not contained in $G(\mathcal{O})$, so we have to be a bit careful in determining its action, with respect to the decomposition (2.2).

2.11 When $K/F$ is ramified the order $\mathcal{O}_K$ is contained in two distinct maximal orders of the matrix algebra. Thus $\mathcal{O}_K$ is contained in an Eichler order of level $p$, and we may assume that this order is the standard one consisting of matrices whose lower left entry is divisible by $\varpi$. Denote this order as $R(P)$. It will be handy in the sequel to have an explicit description of the embedding at hand, which may be given as follows. Since $K/F$ is ramified and quadratic, it is tamely ramified if the residue characteristic is odd. In this case, we may assume that $\varpi^2_K = \varpi$ is a uniformizer for $\mathcal{O}$, and the required embedding is given by

$$\varpi_K \mapsto \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$  

In the case that the residue characteristic is two, we may still write $= F(\sqrt{\xi})$ for some $\xi$, where $\xi$ is a non-square in $F$, and $\text{ord}_P(\xi) = 0, 1$. If $\text{ord}_P(\xi) = 1$, then it is a uniformizer for $\mathcal{O}$, and we may take the same embedding as above. If $\xi$ is a unit, then since we are assuming that $K/F$ is ramified, this implies that $\text{ord}_P(\xi - 1) = 1$ and $\varpi = \xi - 1$ is a uniformizer for $\mathcal{O}$. In this case, our embedding is given by

$$\varpi_K \mapsto \begin{pmatrix} 1 - \xi & -\xi \\ \xi - 1 & \xi - 1 \end{pmatrix}.$$  

With these embeddings in hand, we will attempt to decompose the representation of $\text{Ind}_{\mathcal{B}(r)}^{G(\mathcal{O})}(\nu)$ according to the characters of $\mathcal{O}_K^*$ that it contains. By definition, this representation is realized in the space of translates of a new vector $v_r$, on which $\mathcal{B}(r)$ acts via the character $\nu$, where we view $\nu$ as a character of $\mathcal{B}(r)$ by evaluating $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \nu(a)$, as always.

**Lemma 2.12** The subgroup $1 + \varpi^{2r - 1}_K \mathcal{O}_K \subset \mathcal{B}(r)$ acts trivially on $v_r$. The subspace of $\text{Ind}_{\mathcal{B}(r)}^{G(\mathcal{O})}(\nu)$ generated by the vectors $t \cdot v_r$, for $t \in \mathcal{O}_K^*$ has dimension $q^r - 1$, and is the direct sum of distinct characters of $\mathcal{O}_K^*$ of conductor $2r - 1$. 

19
Proof The first statement is a direct computation, using the embeddings given above, since elements of $1 + \varpi^{2r-1}O_K$ are represented by matrices which have have diagonal elements that are congruent to 1 modulo $\varpi^r$, and $\nu$ is trivial on such elements. The second statement follows from the fact that $O_K \cap R(P^r)$ is the order $O_{K,r-1}$, and thus the vectors $t \cdot v_r$ for $t \in O^*_K/O^*_{K,r-1}$ are linearly independent in the irreducible induced representation $\text{Ind}^{G(O)}_{B(r)}(\nu)$. The fact that the characters of $O^*_K$ that appear are distinct follows from the linear independence of the vectors $t \cdot v_r$, which implies that they realize a quotient of the regular representation of $O^*_K/O^*_{K,r-1}$.

Let $V_r$ denote the space spanned by the vectors $t \cdot v_r$, for $t \in O^*_K$. Let $w_r = \varpi_K \cdot v_r$, and let $W_r = \varpi_K \cdot V_r$.

Lemma 2.13 The following assertions hold:

1. The space $W_r$ is spanned by the vectors $t \cdot w_r$.

2. The space $W_r$ is contained in the representation $\text{Ind}^{G(O)}_{B(r)}(\nu)$.

3. The spaces $V_r$ and $W_r$ generate a $2q^{r-1}$-dimensional space inside $\text{Ind}^{G(O)}_{B(r)}(\nu)$ in which each character of $O^*_K$ of conductor $2r - 1$ which agrees with $\nu$ on $O^*_K$ appears with multiplicity 2.

Proof The first assertion is obvious, since $K^*$ is commutative. For the rest, we may argue as follows. According to Casselman, the space $\text{Ind}^{G(O)}_{B(r)}(\nu)$ is the space of invariants in $\Pi$ under the principal congruence subgroup $G(r)$. Thus it suffices to show that $w_r$ is fixed under $G(r)$. Let $g \in G(r)$. Then direct computation with the matrix representing $\varpi_K$ shows that $\varpi_K^{-1}g \varpi_K$ is a matrix in $B(r+1)$, with diagonal entries that are 1 modulo $\varpi^r$. Since $w_r = \varpi_K \cdot v_r$, and $B(r+1) \subset B(r)$ acts on $v_r$ via a character, the second statement of the lemma follows.

As for the final statement, it follows from the fact that $\text{Ind}^{G(O)}_{B(r)}(\nu)$ is irreducible of dimension $(q + 1)q^{r-1}$ that $\text{Ind}^{B(1)}_{B(r)}(\nu)$ is irreducible, of dimension $q^{r-1}$ as a representation of $B(1)$. Using the explicit embeddings given above, once checks that $\varpi_K B(1) \varpi_K^{-1} = B(1)$, so $W_r$ is also a representation space for $B(1)$, and is also irreducible. Thus either $V_r = W_r$ or $V_r \cap W_r = 0$. We claim that the latter must hold.
To verify this, observe that the vector $w_r \in W_r$ is stable under $B_r$, and realizes the representation \[
\begin{pmatrix} a & b \\ cp^r & d \end{pmatrix} \mapsto \nu(d) \] (as opposed to $\nu(a)$, which picks out $v_r$). This can be checked directly, by using the specified embeddings. Thus, it suffices to prove that $\text{Ind}_{B(r)}^{B(1)}(\nu)$ contains no such vector. To do this, we have to find the dimension of the space of functions $f$ on $B(1)$ such that $f \left( \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} g \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \right) = \nu(a')\nu(d)f(g)$.

It is well-known that $B(r) \backslash B(1)/B(r)$ has $r$ elements, represented by the elements \[
\begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix}, \text{ for } 1 \leq j \leq r,
\] and using these representatives, we may compute as in [3], page 305, to see that no such function exists. Thus $V_r \cap W_r = 0$, and we are done.

Now we can find test vectors for the characters of $K^*$ with minimal level $2r - 1$. We remind the reader that we are assuming that $O_K$ is contained in the Eichler order $R(P)$, and embedded as above.

**Proposition 2.14** Let $\chi$ denote any character of conductor $2r - 1$ on $O_K^*$ which agrees with $\nu$ on $F^*$. Let $v_r$ denote a nonzero vector such that $g \cdot v_r = \nu(g)v_r$, for $g \in B(r) = R(P^r)^*$ (i.e. $v_r$ is a new vector). Then $v_r$ is a test vector for the unique $\chi$-linear form $\ell_\chi$ on $\Pi$.

**Proof** Suppose that $\ell_\chi(v_r) = 0$. Then $\ell_\chi(\Pi(t)v_r) = 0$ as well, for all $t \in K^*$. In particular, $\ell_\chi$ vanishes on the entire $2q^{r-1}$-dimensional space spanned by $W_r$ and $V_r$, since they are generated by $K^*$-translates of $v_r$. But this is impossible, since the character $\chi$ occurs in this space.

**Remark 2.15** Note that we have not used any particular property of $\pi(\varpi_K)$, other than the fact that $\varpi_K$ normalizes $B(1)$. One could of course use the fact that $\Pi$ is induced as a representation of $G(F)$ to compute $\varpi_K \cdot v_r$ explicitly, but we prefer not to do this, as we want an argument that depends only on the components of the restriction of $\Pi$ to $G(O)$.

It is now easy to see how to get test vectors for the characters of next level up. The
representation $\text{Ind}_{B(r)}^{G(O)}(\nu)$ has dimension $(q + 1)q^{r-1}$, and there is a $2q^{r-1}$-dimensional subspace on which $\mathcal{O}_{K}^\ast$ acts via characters of conductor $2r - 1$. Let $V_{r+1}$ denote the complementary $\mathcal{O}_{K}^\ast$-invariant subspace, so that $V_{r+1}$ has dimension $q^r - q^{r-1}$.

**Lemma 2.16** The representation of $\mathcal{O}_{K}^\ast$ on $V_{r+1}$ is given as the direct sum of characters $\chi$, where $\chi$ runs over the characters of conductor $2r$ which agree with $\nu$ on $F^\ast$. In particular, each such character occurs with multiplicity 1.

**Proof** Consider the tree of $\text{PGL}_2$, whose vertices correspond to maximal compact subgroups of $G(F)$. The maximal subgroup $G(O)$ defines a vertex $x$ in the tree. This vertex has $p + 1$ neighbours, precisely one of which contains $O_{K}^\ast$, namely, the other endpoint of the edge corresponding to the standard Eichler order $R(P)$. Consider a vertex $y$ at distance $r$ from $x$, with corresponding maximal compact $K_y$. Then $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r}^\ast$ or $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r-1}^\ast$, depending on whether or not $K_y$ contains the units of the standard Eichler order $R(P)$. Equivalently, $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r-1}^\ast$ if and only if the path from $y$ to $x$ contains the edge corresponding to $R(P)$. Thus, there are $q^{r-1}$ vertices $y$ which satisfy $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r-1}^\ast$, and $q^r$ that satisfy $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r}^\ast$.

Let $R_y$ denote the Eichler order of discriminant $P^r$ whose unit group is given by the intersection of $K_x$ and $K_y$. The orders $R_y$ are permuted transitively by the conjugation action of $G(O)$, since $G(O)$ acts transitively on the set of maximal orders at distance $r$. The line spanned by the vector $v_r$ appearing previously is fixed by the units of the standard order $R(P^r)$ whose lower left entries are divisible by $P^r$, which corresponds to $R_{y_0}$ for some vertex $y_0$ at distance $r$. It follows that the unit group of each order $R_y$ fixes a line spanned by a vector $v_y \in \text{Ind}_{B(r)}^{G(O)}(\nu)$. Since $\text{Ind}_{B(r)}^{G(O)}(\nu)$ has dimension $(q + 1)q^{r-1}$, the span of the vectors $v_y$ is stable under $G(O)$, it follows that the vectors $v_y$ are linearly independent and span $\text{Ind}_{B(r)}^{G(O)}(\nu)$. The space $V_r$ mentioned previously is the span of the vectors $v_y$ for the $q^{r-1}$ orders $R_y$ such that $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r-1}^\ast$.

Let $y$ be such that $\mathcal{O}_{K} \cap K_y = \mathcal{O}_{K,r}^\ast$. Let $v_y$ be a nonzero vector spanning a line fixed by $R_y^\ast$. Then the $\mathcal{O}_{K}^\ast$-orbit of $v_y$ is spanned by the vectors $v_{y'}$, corresponding to the lines fixed by $R_{y'}^\ast$, where $R_{y'}$ runs over the $\mathcal{O}_{K}^\ast$-orbit of $R_y$. But it is clear that this orbit has cardinality given by $\mathcal{O}_{K}^\ast/\mathcal{O}_{K,r}^\ast = (q - 1)q^{2r-2}/(q - 1)q^{r-1} = q^r$. 

22
It follows therefore that the space of $O_K$-translates of $v_y$ generate the representation of $O^*_K$ induced from a character of $O^*_{K,r} = O_K \cap R^*_y$. Thus we get all characters of $O^*_K$ of conductor up to $2r$ which agree with $\nu$ on the centre, each with multiplicity 1. The space $W_r$ accounts for $q^r - 1$ characters of conductor $2r - 1$, and the remaining $q^r - q^{r-1}$ characters give the space $V_{r+1}$, containing the ones of conductor $2r$, each with multiplicity one.

We may now repeat the same argument as was used in the minimal case, finding a disjoint space $W_{r+1} \subset \text{Ind}^{G(O)}_{B(r+1)}$ consisting of vectors spanning lines fixed by orders at distance $r$ from the other maximal order which contains $O^*_K$, and at distance $r + 1$ from $K_x$. The two spaces are interchanged by the action of the uniformizer $\varpi_K$ on $V_{r+1}$ to obtain a disjoint space which realizes the same set of characters, and the same argument then shows that the vector $v_y$, where $y$ is any vertex at distance $r$ from $x$ such that $R_x$ optimally contains $O_{K,r}$ is a test vector for the characters of conductor $2r$. Repeating this process, we get the following result:

**Theorem 2.17** Let $s \geq r$. Let $\chi$ denote any character of $K^*$ of conductor $2s$ which agrees with $\nu$ on $F^*$. Then if $S$ is an Eichler order of discriminant $P^s$ which optimally contains the order of conductor $P^s$, then any nonzero vector on the line fixed by $S$ is a test vector for $\chi$.

**Consequences: test vectors for sufficiently ramified characters $\chi$**

We now take up the case of a general smooth irreducible admissible infinite dimensional representation $\Pi$ with central character $\omega$. Let $P^r$ denote the conductor of $\omega$. Then we have $\Pi = V(\pi) \oplus_{s \geq r} u_{P^s}(\omega)$, by Casselman’s theorem again. Each $u_{P^s}(\omega)$ contains a nonzero vector $v_s$, where $V_{r_1 + r_2} = v_{\text{new}}$ is the new vector, and $v_s$ is the translate of $v_{\text{new}}$ by the matrix

$$
\begin{pmatrix}
p^t & 0 \\
0 & 1
\end{pmatrix}
$$

for $t = s - (r_1 + r_2)$. The analysis of $u_{P^s}$ made above shows immediately that $v_s$ is a test vector for any $\chi$ occurring in $u_{P^s}$. Thus we get the following key theorem, which was proven by a more complicated method in [5]:

23
**Theorem 2.18** Let $\Pi$ be as above, and let $\chi$ be any character of $T$ that occurs in $u_{P^s}(\omega)$ for $s \geq r$. Then $v_s$ is a test vector for $\chi$. If $T$ is unramified, $\chi$ occurs in $u_{P^s}(\omega)$ if and only if $\chi$ has conductor $P^s$. Thus if $\chi$ is any character of $T$ which agrees with $\omega$ on $F^\times$, and $\chi$ has conductor $P^s$ with $s \geq r$, where $P^r$ is the conductor of $\omega$, then $v_s$ is a test vector for $\chi$.

In view of this result, the test vector problem will be completely solved if we can find test vectors for the characters $\chi$ occurring in the primitive part $V(\pi)$. We start with the case of a general principal series representation $\Pi = \Pi(\nu_1,\nu_2)$, where both $\nu_i$ are nontrivial. The central character is $\omega = \nu_1 \nu_2$. Let $\chi$ denote a character of $K^\times$ where $K/\mathbb{Q}_p$ is a quadratic field extension, possibly ramified. Assume that $\chi$ and $\omega$ agree on $F^\times$. Let $r$ denote the smallest positive integer such that $\omega$ is trivial on $1 + \mathfrak{O}_F$. Let $r_1$ and $r_2$ denote the corresponding quantities for $\nu_1$ and $\nu_2$, and that $r_1 \geq r_2$. Consider $\Pi' = \Pi \otimes \nu_2^{-1}$. Then $\Pi' = \Pi(\nu_1/\nu_2,1)$ is also irreducible, then the results proved above apply to $\Pi'$. Let $P''$ denote the conductor of $\omega' = \nu_1/\nu_2$. Then for $s \geq r'$, we have a vector $v'_s \in u_{P''}(\omega') \subset \Pi'$, and it is clear that $w_s = v'_s \otimes \nu_2 \in u_{P''}(\omega') \otimes \nu_2 \subset \Pi$ is a test vector for any character $\chi$ such that $\chi \otimes \nu_2^{-1}$ occurs in $u_{P''}(\Omega')$. This covers all possible $\chi$, since $\Pi' = \oplus u_{P''}(\omega')$, according to the analysis carried out above. In particular, we obtain test vectors for characters $\chi$ occurring in the primitive part $V(\Pi)$, which leads to the following general theorem:

**Theorem 2.19** Suppose that $\Pi = \Pi(\nu_1,\nu_2)$ is as above. Let $\chi$ denote any character of $T$ which agrees with $\omega = \nu_1 \nu_2$ on $F^\times$. Then, a test vector for $\chi$ is given by the vector $w_s = v'_s \otimes \nu_2$, where $v'_s \in u_{P''}(\omega')$ is the test vector at level $s$ for $\Pi' = \Pi(\nu_1/\nu_2,1) = \oplus u_{P''}(\omega')$, and $\omega = \nu_1/\nu_2$.

**Remark 2.20** Observe that if $s \geq r$, then we get test vectors in the case covered by Theorem 2.18 which are distinct from the test vectors provided by that result, as can be seen by considering the action of the diagonal torus.

The analogue of Theorem 2.19 for special representations is holds as well. In the case of the ‘minimal’ special representation of trivial central character, the space $V(\Pi)$ is zero and the result of Theorem 2.18 is comprehensive.
Thus, we have almost completely solved the test vector problem for special representations and the irreducible principle series, and a non-split torus $T$. The only question remaining is the following: if $\Pi$ is a non-minimal principal series, and $\chi$ is a character of $T$ which occurs in the primitive part $V(\Pi)$, then can we find a test vector for $\chi$ of the form
\[
\begin{pmatrix}
    p^t & 0 \\
    0 & 1
\end{pmatrix} \cdot v_{\text{new}},
\]
for some $t$ (necessarily negative)?

As for the supercuspidal representations, Theorem 2.18 gives the result for characters of conductor divisible by $P^r$, where $r$ is the conductor of $\Pi$. But there is no immediate way to deal with the characters of conductor $P^s$ with $s \leq r$. In particular, even the trivial character poses problems. (We remind the reader that the Gross-Prasad test vector for the trivial character is not fixed by the units in any Eichler order.) One can guess that test vectors of the form
\[
\begin{pmatrix}
    p^t & 0 \\
    0 & 1
\end{pmatrix} \cdot v'_{\text{extnew}}
\]
exist, but, as we shall see in the next section, the answer is surprisingly delicate, even in the case of depth zero supercuspidals of conductor $p^2$ of $GL_2(Q_p)$.

3 The depth zero supercuspidal representations of $GL_2(F)$.

Our goal in this section is to completely solve the test vector problem for depth zero supercuspidal representations of $GL_2(F)$ when $F = Q_p$, and the torus $T$ corresponding to an unramified quadratic extension $K/Q_p$. In this entire section, we assume that $p$ is odd.

We start by defining the groups and representations of interest. Let $G = GL_2(F_q)$, where $F_q$ is the residue field of $O = O_F$. Let $\nu$ denote a character $\nu : F_q^* \to C^*$ which does not factor through the norm $F_q^* \to F^*$, and let $\rho = \rho(\nu)$ denote the cuspidal representation of $G = GL_2(F_q)$ associated to $\nu$ by Piatetski-Shapiro [10]. Then there exists a representation $\Pi = \Pi(\nu)$ of $GL_2(F)$ such that the type of $\Pi$ is the inflation of $\rho$ to $GL_2(O)$. The representation $\Pi$ is called the depth zero supercuspidal representation associated to $\nu$. Our goal is to find a test vector for characters $\chi$ of the unramified torus $K/F$ and the representation $\Pi = \Pi(\nu)$ when $\chi$ occurs in the type, in the following sense.

Fix $\Pi = \Pi(\nu)$, with central character $\omega$. As we have observed already, $T = \omega^Z \times U$ where $U \subset GL_2(O)$ is compact and (since $T$ is unramified) $\omega$ is in $F^*$. If $\chi$ is a
character of $T$ which agrees with $\omega$ on $F^{*}$, then $\chi(\varpi)$ is fixed, and $\chi$ is determined by its restriction to $U$. On the other hand, Casselman’s theorem tells us that the restriction of $\Pi(\nu)$ to $GL_2(\mathcal{O})$ is the sum of the inflated representation $\rho(\nu)$ (the type) and the imprimitive part, so we get a corresponding decomposition of $\Pi(\nu)$ as an infinite direct sum of characters of $U$. We say that $\chi$ occurs in the type if there is a line in $\rho(\nu)$ on which $T$ acts by the character $\chi$. Our goal is to find a test vector for such $\chi$, since the characters occurring in the imprimitive part may be dealt with by the arguments in the first section of this paper. The point is that any $\chi$ which appears in the type is trivial on the principal units $u_1$ in $T$, since the type is inflated from a representation of $GL_2(F_q)$, which is trivial on $I + \varpi M_2(\mathcal{O})$. In other words, such a character $\chi$ may be identified with a character of $T(F_q) = U/U_1 \equiv F_{q^2} \subset GL_2(F_q)$, and finding a test vector for such $\chi$ is reduced to a problem in finite group theory for the representation $\rho(\nu)$ of the group $GL_2(F_q)$.

As for the candidate test vectors, we are looking for vectors related to local newforms. By Atkin-Lehner theory, the local newform $v^{\text{new}} \in \Pi(\nu)$ is a nonzero vector on the unique line in $\Pi(\nu)$ where the group $B(2)$ acts via the character $\left( \begin{array}{cc} a & b \\ p^2c & d \end{array} \right) \mapsto \omega(a)$. As explained in Section 1, this means that $v^{\text{new}} \in \Pi(\nu)$ lies in the first imprimitive piece of $\Pi(\nu)$. On the other hand, consider $w^{\text{new}} = \left( \begin{array}{cc} \varpi & 0 \\ 0 & 1 \end{array} \right) \cdot v^{\text{new}}$, where the action of the matrix on $v^{\text{new}}$ is given by the representation $\Pi(\nu)$. Then $w^{\text{new}}$ is invariant under $I + \varpi M_2(\mathcal{O})$, and therefore lies in the type. Furthermore, $w^{\text{new}}$ is an eigenvector for the matrices $\left( \begin{array}{cc} a & b\varpi \\ c\varpi & d \end{array} \right)$, which is to say, they are eigenvectors for the diagonal split diagonal torus $D$ of $GL_2(F_q)$. We want to know whether or not the unique $\chi$-equivariant functional on $\Pi(\nu)$ is nonzero on $w^{\text{new}}$, where $\chi$ is the character of $T$ such that $\chi = \omega$ on $F^{*}$ and such that the restriction of $\chi$ to $U$ occurs in the representation $\rho(\nu)$ of the group $GL_2(F_q)$. As remarked above, the fixed central character allows us to identify characters of $T$ which agree with $\omega$ on $F^{*}$ with characters of $T(F_q) \subset GL_2(F_q)$.

Thus, we are led to the following problem in the representation theory of the finite group $GL_2(F_q)$: suppose $\rho(\nu)$ is a cuspidal representation of $GL_2(F_q)$. Suppose that
χ is a character of the nonsplit torus $T(F_q)$ which occurs in the restriction of $\rho(\nu)$ to $T(F_q)$, and $\mu$ a character of the split torus $D$ which occurs in the restriction of $\rho(\nu)$ to $D$. Let $v_\chi$ and $v_\mu$ denote corresponding eigenvectors. We will see below that each of these characters $\chi, \mu$ occurs with multiplicity one, and it follows that the unique $\chi$-equivariant functional on $\rho(\nu)$ factors through the orthogonal projection to the line spanned by $v_\chi$. Thus, to determine whether or not $v_\mu$ is a test vector for $\chi$ is equivalent to determining whether or not the orthogonal projection of $v_\mu$ on to $v_\chi$ is nonzero. Here orthogonality is taken with respect to a $GL_2(F_q)$-invariant inner product.

**Discrete series representations of $GL_2(F_q)$ and test vectors modulo $p$**

In this subsection (and in this subsection only) we will use slightly different notation from the rest of this paper, since we are working solely with representations of $GL_2(F_q)$. Our goal to solve the problem described in the paragraph above.

3.1 Thus Let $\xi \in F_q^*$ denote a non-square element, and write $T$ for the subgroup of $GL_2(F_q)$ given by matrices of the form

\[
\begin{pmatrix}
a & b \\
b\xi & a
\end{pmatrix},
\]

where $a, b \in F_p, a \neq 0$, so that $T \cong F_q^2$. Let $D \subset S$ denote the subgroup

\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}
\]

with $a, b \in F_q^*$. Let $\omega: F_q^* \to C^*$ denote the central character of $\rho$. Thus $\omega$ is the restriction of $\nu$ to $F_q^* \subset F_q^{\times}$. Next we consider the restriction of $\rho$ to the subgroups $D$ and $T$. The restriction of $\rho$ to $D$ consists of the direct sum of the $q - 1$ characters $\mu : D \to C^*$ such that $\mu$ agrees with $\nu$ on the centre, each with multiplicity 1. The characters $\mu$ can be distinguished by their restrictions to the group $D^0 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in F_p^*$. We write $v_{\text{triv}}$ and $v_{\text{quad}}$ for the corresponding eigenvectors for the action of $D^0$.

On the other hand, the restriction of $\rho$ to $T$ is the sum of $q - 1$ characters $\chi : T \to C^*$ which agree with $\nu$ on the centre; there are $q + 1$ such characters, and each character other than $\nu$ and $\nu^q$ shows up with multiplicity one. Note that $\nu^q \neq \nu$, since $\nu$ does not factor through the norm. In this situation, we shall say that a vector $v \in \rho(\nu)$ is a test vector for $\chi$ if the projection of $v$ to the $\chi$-eigenspace of $T$ is nonzero.

27
To state the general results, it is important to have a way to label the various characters of $\mathbf{F}_{q^2}^*$. This is quite complicated for a general finite field, so from now on, we assume that $q = p$ (and as always that $p$ is odd).

3.2 We recall the labelling of characters of $\mathbf{F}_{p^2}^*$ that was given in the introduction. If $x \in \mathbf{F}_{p^2}$, we identify $x$ with its image under the Teichmüller lift to the ring $W = W(\mathbf{F}_{p^2})$ of Witt vectors for $\mathbf{F}_{p^2}$. Let $\pi = x^p$. Then $x \mapsto \pi$ is also a $W$-valued character of $\mathbf{F}_{p^2}^*$. If $\chi$ is an arbitrary character of $\mathbf{F}_{p^2}^*$ with values in $W$, we may write $\chi(x) = x^{a+b}$, for positive integers $a, b$ satisfying $0 \leq a, b < p - 1$, since $\chi$ has order dividing $p^2 - 1$. We call this character $\chi(a, b)$. Note that we can exclude $a = b = p - 1$, since $a + pb$ may be assumed to lie in the range $0 \leq a + pb < p^2 - 1$. We set $k_\chi = a + pb$, so that $\chi(x) = x^{k_\chi}$. We say that $\chi$ is even or odd according to the sign of $\chi(-1)$; equivalently, $\chi$ is even if and only if $a, b$ have the same parity. The character $\chi$ factors through the norm if and only if $a = b$, or – equivalently – if $k_\chi$ is divisible by $p + 1$. Note that switching $\nu$ and $\nu^p$ interchanges $a$ and $b$, so we may assume that $a > b$, if $\nu$ does not factor through the norm.

3.3 An important point for us will be to determine the various characters $\nu$ that coincide on $\mathbf{F}_{p}^*$: characters $\chi, \nu$ which agree on $\mathbf{F}_{p}^*$ are those with weights $k_\chi \equiv k_\nu \pmod{p - 1}$. If $\nu$ is a fixed character with $\nu \neq \nu^p$, and $\chi$ is a character which agrees with $\nu$ on $\mathbf{F}_{p}^*$, then we say that $\chi$ is of Type 1 if $k_\chi$ is between $k_\nu$ and $k_{\nu^p}$; else we say $\chi$ is of Type 2. In any case, we define $t_\chi = \frac{k_\chi - k_\nu}{p - 1}$. Note that $\frac{k_{\nu^p} - k_\nu}{p - 1}$ is even, so that $t_\nu$ and $t_{\nu^p}$ have the same parity.

With this enumeration of characters in hand, the result in the even case may be stated as follows. Consider the representation $\rho(\nu)$, where $\nu = \nu(a, b)$ is even. Let $\omega$ denote the central character of $\rho(\nu)$, so $\omega(x) = x^{a+b}$. Define characters $\mu_1, \mu_2$ according to the following prescription: $\mu_1(x) = x^{(a+b)/2}$, and $\mu_2(x) = x^{(a+b+p-1)/2}$. Further define $\mu_3(x) = x^{(a+b)/2-1}$ and $\mu_4(x) = x^{(a+b+p-1)/2-1}$. Let $v_{\mu_i}$ denote an eigenvector for $D^0$ with eigenvalue $\mu_i$. Note that $\mu_1, \mu_2$ are the two square roots of the central character $\omega$.

**Theorem 3.4** Consider the representation $\rho(\nu)$ for $\nu(a, b)$, with notation as above.
Assume $a, b$ are labelled so that $a > b$, and that $a - b$ is even. Let $\chi : T \to W^*$ denote a character such that $\chi = \nu$ on $F_p^*$, and $\chi \neq \nu, \nu^p$. Then the following statements hold:

1. Suppose $\chi$ is of Type 1 and $t_{\chi}$ is odd. Then the vector $v_{\mu_1}$ is a test vector for $\chi$.
2. Suppose $\chi$ is of Type 2 and $t_{\chi}$ is odd. Then the vector $v_{\mu_2}$ is a test vector for $\chi$.
3. Suppose $\chi$ is of Type 1 and $t_{\chi}$ is even. Then the vector $v_{\mu_3}$ is a test vector for $\chi$.
4. Suppose $\chi$ is of Type 2 and $t_{\chi}$ is even. Then the vector $v_{\mu_4}$ is a test vector for $\chi$.

Now consider the odd case, where $\nu = \nu(a, b)$ with $a - b = 2t + 1$. Define characters $\mu_1, \mu_2$ of $F_p^*$ by $\mu_1(x) = x^{(a+b-1)/2}$ and $\mu_2(x) = x^{(a+b+p)/2}$, and let $v_{\mu_i}$ denote eigenvectors for $D^0$ with eigenvalue $\mu_i$. Let $\omega(x) = x^{a+b}$, for $x \in F_p^*$. If $\chi$ is a character of $F_p^*$ which agrees with $\omega$ on $F_p^*$, we define the notion of Type 1 and Type 2 exactly as before.

**Theorem 3.5** Consider the representation $\rho(\nu)$ for $\nu(a, b)$ odd, with notation as above. Assume $a, b$ are labelled so that $a > b$ and $a - b$ is odd. Let $\chi : T \to W^*$ denote a character such that $\chi = \omega$ on $F_p^*$, and $\chi \neq \nu, \nu^p$. Suppose $a - b > 1$. Then the following statements hold:

- Suppose $\chi$ is of Type 1. Then the vector $v_{\mu_1}$ is a test vector for $\chi$.
- Suppose $\chi$ is of Type 2. Then the vector $v_{\mu_2}$ is a test vector for $\chi$.

If $a - b = 1$, then the vector $v_{\mu_2}$ is a test vector for $\chi$, for all $\chi$.

Once correctly formulated, these theorems are surprisingly easy to prove. The key point is that a cuspidal representation of $GL_2(F_p)$ is typically reducible modulo a prime above $p$, with Jordan-Holder filtration of length 2. The characters in question arise in one component or the other, hence the Type 1 and Type 2 of the theorems. There is one exception case of odd central character, where the representation is irreducible modulo $p$, leading to the final case of Theorem 3.5.
3.6 We start with the proof in the even case as stated in Theorem 3.4. Let \( \nu \) be a character as in the theorem, such that \( \rho(\nu) \) has even central character. Let \( L/F \) denote a finite extension of \( F \) such that \( \rho(\nu) \) may be realized via matrices with entries in \( L \). There exists an \( \mathcal{O}_L \)-lattice \( B \) stable under \( \rho(\nu) \), where \( \mathcal{O}_L \) is the ring of integers in \( L \), and the reduction of \( B \) modulo the maximal ideal \( \mathfrak{m}_L \) of \( \mathcal{O}_L \) gives a \( q - 1 \)-dimensional representation \( \overline{\rho}(\nu) \) of \( GL_2(F_p) \) with entries in \( k_L \), where \( k_L \) is the residue field of \( \mathcal{O}_L \). It follows from [11], Lemma 4.2, that this representation has Jordan-Hölder length 2, with composition factors 

\[
V_1 = \det^{b+1} \text{Sym}^{a-b-2}(k_L^2) \quad \text{and} \quad V_2 = \det^a \text{Sym}^{p-1-(a-b)}(k_L^2).
\]

Consider the representations of the two tori \( T \) and \( D^0 \) on the \( F_p \)-representations \( \overline{\rho}(\nu) \) and \( V_i \). Each \( V_i \), as well as \( \overline{\rho}(\nu) \), is semisimple for the actions of \( D^0 \) and \( T \) separately, since these groups have order prime to \( p \). Since each character in the decomposition of \( \rho(\nu) \) with respect to \( D \) or \( T \) occurs with multiplicity one, and since the groups have order prime to \( p \), it follows that the same is true for the decomposition of the reduced representation \( \overline{\rho}(\nu) \). Here we are identifying \( F_p \)-valued characters with their Teichmüller lifts. It follows that each of the \( V_i \) may be decomposed as the sum of certain characters of either \( T \) or \( D \), each with multiplicity one, and that we get partitions of the characters of \( D \) and \( T \) occurring in \( \rho(\nu) \), according to whether they occur in \( V_1 \) or \( V_2 \).

If \( \chi \) is a character of \( T \) and \( \mu \) a character of \( D \), we say that \( \chi \) and \( \mu \) are compatible if they occur in the same component \( V_i \) or not. If \( v_\mu \) is a vector in the lattice \( B \) which is an eigenvector for \( D^0 \) with eigenvector \( \mu \), and \( \chi \) character of \( T \), then we shall say \( v_\mu \) is a test vector for \( \chi \) if \( e_\chi(v_\mu) \) is nonzero modulo modulo \( \mathfrak{m}_L B \), namely, nonzero in the characteristic \( p \) representation obtained by reduction of \( B \). Clearly, a neccessary condition for this to occur is that \( \chi \) and \( \mu \) be compatible in the above sense.

Our task is therefore to determine the decomposition of the representations \( V_i \) to determine compatibility, and then to produce further conditions that exhibit test vectors (modulo \( p \)) in each component. The characters listed in cases (1) and (3) of the Theorem 3.4 are the characters \( \chi \) which occur in \( V_1 \), while the characters \( \chi \) of cases (2) and (4) are the ones occurring in \( V_2 \).

Consider first \( V_1 = \det^{b+1} \text{Sym}^{a-b-2}(k_L^2) \), which we realize as the space of homogeneous polynomials in 2 variables \( X, Y \) of degree \( d = a - b - 2 \). Observe here that since
a and b are of the same parity, a − b is even and positive. The group \( GL_n(F_p) \) acts
linearly on \( X, Y \). Each monomial \( X^iY^{d-i} \) is an eigenvector for \( D^0 \), with eigenvalues
given by the character \( x \mapsto X^i \), for \( b + 1 \leq i \leq a - 1 \). Thus \( e_\mu \) simply picks out the
coefficient of \( X^iY^{d-i} \), for suitable \( i \) depending on \( \mu \). The character \( \mu_1 \) in the theorem
corresponds to the monomial \( X^{d/2}Y^{d/2} \), while \( \mu_3 \) corresponds to \( X^{d/2-1}Y^{d/2+1} \).

To start with we compute \( e_\chi \) on the monomial \( X^{d/2}Y^{d/2} \). Observe that \( T \) is a
non-split torus in \( GL_2(F_p) \), and to find the eigenvectors for \( T \), we must diagonalize
the corresponding matrices \( \begin{pmatrix} a & b \\ b\xi & a \end{pmatrix} \). Evidently, the eigenvectors are the monomials
in variables \( U = X - \sqrt[2]{\xi}Y \) and \( V = X + \sqrt[2]{\xi}Y \), where \( \sqrt[2]{\xi} \) is some element of trace
zero whose square is the nonsquare element \( \xi \). Obviously, we have \( X = \frac{1}{2}(U + V) \) and
\( Y = \frac{1}{2\sqrt[2]{\xi}}(U - V) \), so that \( X^{d/2}Y^{d/2} = \left( \frac{-1}{2\xi} \right)^{d/2}(U^2 - V^2)^{d/2} = \left( \frac{-1}{2\xi} \right)^{d/2}\sum_i (d/2)U^{2i}V^{d-2i} \).
Each monomial is distinct, and appears with coefficient which is nonzero modulo \( p \). It follows that \( e_\chi(X^{d/2}Y^{d/2}) \) is nonzero for each character \( \chi \) corresponding to a
monomial in \( U, V \) which appears in the expression for \( X^{d/2}Y^{d/2} \). The action of \( x \in T \)
on \( U^{2i}V^{d-2i} \) is given by \( x \mapsto x^{(p+1)(b+1)+2i+p(d-2i)} = b + ap - (2i + 1)(p - 1) \).
Now the value of this exponent is \( b + ap - (p - 1) = k_\nu - (p - 1) \) when \( i = 0 \), and
\( b + ap - (p - 1)(a - b - 1) = k_\nu + (p - 1) \) when \( i = d/2 \), so we find that the
characters \( \chi \) that show up are the ones with \( k_\nu \) between \( k_\nu \) and \( k_\nu + p \), and of the form
\( k_\nu + (2j + 1)(p - 1) \), as asserted.

Moving on to Case 3, which is the other instance of the component \( V_1 \), consider
this time the monomial \( X^{d/2-1}Y^{d/2+1} \). Making the change of variable \( V = X + \sqrt[2]{\xi}Y \),
\( U = X - \sqrt[2]{\xi}Y \) as before, we find that \( X^{d/2-1}Y^{d/2+1} = C \cdot (U^2 - \xi V^2)^{d/2-1}(U - V)^2 = \)
\( C \cdot (U^2 - \xi V^2)^{d/2-1}(U^2 - 2UV + V^2) \). Since \( (U^2 - \xi V^2)^{d/2-1} \) is a sum of \( d/2 \) distinct
monomials (the binomial coefficients being prime to \( p \) with \( U, V \) occurring to even
powers, the expression \( C \cdot (U^2 - \xi V^2)^{d/2-1}(U^2 - 2UV + V^2) \) contains \( d/2 \) distinct
monomials where the exponent of both \( U \) and \( V \) is odd. The corresponding characters \( \chi \) are the ones listed in case 4 of the theorem, by calculating the corresponding
exponents, as in the previous case.

As for cases 2 and 4 of the theorem, and the component \( V_2 \), set \( d' = p - 1 - (a - b) \). Then one makes a similar computation with the monomials \( X^{d'/2}Y^{d'/2} \), which

31
corresponds to $\mu_2$, and $X^{d/2-1}Y^{d/2+1}$, which corresponds to $\mu_4$.

**Remark 3.7** It is natural to ask whether or not the conditions in Theorem 3.4 are necessary as well as sufficient. This is not entirely clear, since it may well be that the $\chi$-component of some $v_\mu$ is divisible by a prime above $p$, which means $v_\mu$ is a test vector in characteristic zero but not in characteristic $p$. However, it is clear from the proof of the theorem that the conditions stated are both necessary and sufficient to determine test vectors in characteristic $p$. In fact, one can say a bit more that Theorem 3.4 in characteristic zero as well; we refer the reader to Theorem 1.2 in the introduction for the statement, and the proof of that theorem below. We note also that there is some flexibility in the choice of test vector; one can just as well use the monomial $X^{d/2-1}Y^{d/2+1}$ instead of $X^{d/2-1}Y^{d/2+1}$, which leads to a different test vector.

**3.8** We now give the proof in the odd case. Let the notation be as in Theorem 3.5 and suppose that $a - b > 1$. We start with the component $V_1 = \det^{b+1} \text{Sym}^{a-b-2} (k_L^2)$. Set $t = (a - b - 1)/2$, and consider the monomial $X^tY^t$, corresponding to the character $\mu_1(x) = x^{a+b-1}/2$. In terms of $U,V$, we get

$$X^tY^t = \sum_{j=0}^{t-1} \binom{t-1}{j} C_j U^{2j} V^{2(t-1-j)} (U + V),$$

where $C_j$ is some nonzero constant. This expression contains $2t$ distinct monomials with nonzero coefficients, corresponding to distinct characters of $T$, as in the theorem. The argument for $V_2$ is similar, using the monomial $X^{t'-1}Y^{t'}$, where $2t' - 1 = p - 1 - (a - b)$, as is the case of $a - b = 1$, where there is only one component.

**Remark 3.9** Again, there is some flexibility in the choice of test vectors: for instance one could use the monomials $X^tY^{t-1}$ instead of $X^{t-1}Y^t$.

**Remark 3.10** A similar analysis to that carried out for the discrete series of $GL_2(F_p)$ can be made in the case of a nontrivial extension $F_{p'}$. It is evident that it would be more complicated to carry out such an analysis in full generality, since the discrete series representations of $F_q$ typically have many more components, and the binomial
coefficients which show up may in principle be divisible by $p$. However, it should be fairly straightforward to deal with any specific character, for instance the trivial character of $\mathbf{F}_{q^2}^*$. We have not pursued this question.

**Test vectors in characteristic zero: proof of the main theorems**

3.11 We now return to the situation and notation described in the introduction. Thus let $\Pi$ denote a depth zero supercuspidal representation of $GL_2(\mathbb{Q}_p)$, with central character $\omega$. Let $\rho : GL_2(\mathbb{Z}_p) \to GL_{q-1}(\mathbb{C})$ denote the type. Thus $\rho$ is the inflation from $GL_2(\mathbb{F}_p)$ of the $q-1$-dimensional representation $\rho(\nu)$ where $\nu : \mathbb{F}_{p^2}^* \to \mathbb{C}^*$ is a character which satisfies $\nu \neq \nu^p$. We write $\Pi = \Pi(\nu)$. Let $K/\mathbb{Q}_p$ denote the unramified quadratic extension of $\mathbb{Q}_p$, and let $\chi$ denote a character of $K^*$ such that $\chi$ agrees with $\omega$ on $\mathbb{Q}_p^*$, and assume that $\chi$ is trivial on the principal units $U = 1+p\mathcal{O}_K$.

As we have seen above, $\Pi(\nu)$ contains a $p-1$-dimensional subspace $V$ stable under $GL_2(\mathbb{F}_p)$ such that $GL_2(\mathbb{F}_p)$ acts via $\rho(\nu)$, inflated from $GL_2(\mathbb{F}_p)$. We may identify $\chi$ with a character of $\mathbb{F}_{p^2}^*$ as explained above, and $V$ contains a line where the action of $\mathbb{F}_{q^2}^*$ viewed as a subgroup of $GL_2(\mathbb{F}_p)$ is given by the character $\chi$.

We want to find a test vector $v \in V$, of the form $v = v_\mu$ as described in the previous paragraph. The notation is taken to mean that $v$ is an eigenvector for matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \subset GL_2(\mathbb{F}_p)$, with eigenvalue given by the character $a \mapsto \mu(a)$ of $\mathbb{F}_{p^2}^*$. This means simply that $v_\mu$ has nonzero projection on the $\chi$-eigenspace of the nonsplit torus $\mathbb{F}_{p^2}^*$. The theorems of the previous section give a recipe for $\mu$ in terms of $\chi$, provided that one has a way of labelling the characters correctly. Choose a generator $\sigma$ of the cyclic group $\mathbb{F}_{p^2}$, and identify $\sigma$ with its Teichmuller lift to the Witt ring $W = \mathcal{O}_K$. Let $\tau$ denote an arbitrary generator of $\mu_{p^{2-1}} \subset \mathbb{C}^*$. Let $\zeta = \nu(\sigma) \in \mu_{p^{2-1}}$. Then we can write $\zeta = \tau^{a+pb}$, with $0 \leq a, b \leq p-1$, and at the cost of interchanging $\nu$ and $\nu^p$, we may assume $a > b$. Further, we can write $\chi(\sigma) \in \mu_{p^{2-1}}$ in the form $\zeta = \tau^{r+ps}$ with $0 \leq r, s \leq p-1$. Thus we are in the situation of the previous paragraph, and the results given there lead immediately to the Theorems 1.3 and 1.4 of the Introduction.

3.12 We now want to give the proof of Theorem 1.2. Thus assume that $\Pi$ is such
that the central character $\omega$ is trivial. We want to determine when the vectors $v_{\text{triv}}$ and $v_{\text{quad}}$ are test vectors for a given $\chi$ of conductor $\leq 1$, and prove the necessary condition in the last statement of the theorem. Let $\xi$ denote a non-square element of $F_p^*$, and let $x_{\xi} = \begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix}$. Then we have

$$
\begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\xi & 0 \\ 0 & 1 \end{pmatrix}.
$$

(1)

We contend that the vectors $v_{\text{triv}}$ and $v_{\text{quad}}$ are eigenvectors for $x_{\xi}$, with eigenvalues given by $\nu(\sqrt{\xi}) = \sqrt{\xi}^{\nu}$ in each case. To see this, note first of all that $v_{\text{triv}}$ and $v_{\text{quad}}$ are eigenvectors for $\begin{pmatrix} -\xi & 0 \\ 0 & 1 \end{pmatrix}$, with eigenvalues 1 and $-\mu_{\text{quad}}(-1)$, respectively. We will show that each of $v_{\text{triv}}$ and $v_{\text{quad}}$ is an eigenvector for $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and to calculate the corresponding eigenvalue. This can be done by using the formula (12) on page 38 of [10], which gives a formula for the action of $w' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under $\rho(\nu)$ as follows:

$$(\rho(\nu)(w') \cdot \mu)(y) = \sum_{x \in F_p^*} j(yx) \mu(x)$$

where $j$ is the Bessel function whose definition we shall recall below. Note here that we are assuming that $\nu$ is trivial on $F_p^*$. Note also that Piatetski-Shapiro is writing $\mu$ for the vector we have denoted by $v_{\mu}$; the character $\mu$ itself is an eigenvector for $D^0$ in his model of $\rho(\nu)$ in the space of functions on $F_p^*$. To spell this out, let $\psi$ denote a nontrivial additive character of $F_p$. Then we have

$$j(u) = \frac{-1}{p} \sum_{N_x = u} \psi(Tr(x))\nu(x)$$

where $u \in F_p^*$ and the sum is taken over elements $x$ of $F_p$ with norm $N_x = u$. The reader should note that the minus sign in the formula for $j$ given above is missing from the definition in equation (4) in [10]; the sign makes a reappearance in the proof of the subsequent identity (6) – see the left side of the final formula in the proof on page 38, two lines above equation (12). The reader could also consult [1], page 427,
and notice the minus sign ($\epsilon = -1$) in the formula for the action of the Weyl element given there.

Thus we get

$$\left( \rho(\nu)(w') \cdot \mu \right)(y) = \frac{-1}{p} \sum_{x \in \mathbb{F}_p^*} \sum_{\text{Nu}=xy} \psi(\text{Tr}(u))\nu(u)\mu(x)$$

$$= \frac{-1}{p} \mu^{-1}(y) \sum_{x \in \mathbb{F}_p^*} \sum_{\text{Nu}=xy} \psi(\text{Tr}(u))\nu(u)\mu(xy)$$

$$= \frac{-1}{p} \mu^{-1}(y)G(\nu \cdot \mu \circ N, \psi).$$

Here $G(\nu \cdot (\mu \circ N), \psi)$ is the standard Gauss sum of the character $\nu \cdot \mu \circ N$ with respect to the additive character $\psi \circ \text{Tr}$. Now, if $\eta$ is any character of $\mathbb{F}_p^{*}$ which satisfies $\eta = 1$ on $\mathbb{F}_p^{*}$, then we calculate that

$$G(\eta, \psi) = \sum_{x \in \mathbb{F}_p^*} \psi(\text{Tr}(u))\eta(u)$$

$$= \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \psi(\text{Tr}(xy))\eta(xy)$$

$$= \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \psi(y\text{Tr}(x))\eta(x)$$

Suppose $\text{Tr}(x) \neq 0$. Then the sum on $y$ of $\psi(y\text{Tr}(x))$ has the value $-1$. On the other hand, if $\text{Tr}(x) = 0$, then the sum on $y$ has value $p - 1$. All such $x$ with trace zero are equal to $\sqrt{\xi}$, up to multiples of $\mathbb{F}_p^*$. Thus we find that

$$\sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \psi(y\text{Tr}(x))\eta(x) = (p - 1)\eta(\sqrt{\xi}) - \sum_{x \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}} \eta(x)\sum_{\text{Tr}(x) \neq 0}$$

where the final sum is taken over the $p - 1$ classes elements of nonzero trace, modulo scaling by $\mathbb{F}_p^*$. But now if we sum over all classes, then we get $\sum_{x \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}} \eta(x) = 0$ since $\eta$ is a nontrivial character which is trivial on $\mathbb{F}_p^{*}$, and so we conclude that $G(\eta, \psi) = p\eta(\sqrt{\xi})$.

Thus we obtain

$$\left( \rho(\nu)(w') \cdot \mu \right)(y) = -(\nu \cdot (\mu \circ N))(\sqrt{\xi})\mu^{-1}.$$
When $\mu^2 = 1$, we see that $v_\mu$ (or $\mu$, in Piatetski-Shapiro’s notation) is an eigenvector for $w'$ with eigenvalue $-(\nu \cdot \mu \circ N)(\sqrt{\xi}) = \pm 1$. When $\mu$ is trivial, this eigenvalue is $-\nu(\sqrt{\xi})$, and when $\mu$ is nontrivial quadratic, this eigenvalue is $-\nu(\sqrt{\xi}) \cdot \mu \circ N(\sqrt{\xi}) = \nu(\sqrt{\xi}) \mu(-\xi) = -\nu(\sqrt{\xi}) \mu(-1)$. Going back to equation (1), we find that the eigenvalue of $x_\xi$ on $v_{\text{triv}}$ and $v_{\text{quad}}$ is $\nu(\sqrt{\xi}) = -\epsilon_p(\nu)$ in each case.

In view of this computation, it is clear that the projection of $v_{\text{triv}}$ and $v_{\text{quad}}$ to the eigenspace of any character $\chi$ with $\epsilon_p(\chi) = \epsilon_p(\nu)$. This completes the proof of Theorem 1.2.

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