Entanglement Entropy in the Two-Dimensional Random Transverse Field Ising Model

Rong Yu,1,2 Hubert Saleur,3,1 and Stephan Haas1

1Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089-0484
2Department of Physics and Astronomy, University of Tennessee, Knoxville, TN 37996-1200
3Service de Physique Théorique, CEN Saclay, Gif Sur Yvette, F-91191

The scaling behavior of the entanglement entropy in the two-dimensional random transverse field Ising model is studied numerically through the strong disordered renormalization group method. We find that the leading term of the entanglement entropy always scales linearly with the block size. However, besides this area law contribution, we find a subleading logarithmic correction at the quantum critical point. This correction is discussed from the point of view of an underlying percolation transition, both at finite and at zero temperature.

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The study of novel quantum phases and related quantum phase transitions (QPTs) is at the forefront of many recent developments in condensed matter physics. It relies heavily on the concept of entanglement entropy.

A state |Φ⟩ of a bipartite system A∪B is entangled if it cannot be described accurately in either subsystem A or B. A convenient measure of this entanglement is the entropy, $S_A = -\text{Tr} \rho_A \log \rho_A$, where $\rho_A = \text{Tr}_B |\Phi⟩⟨Φ|$. Denote the linear dimensions of A∪B and A as $M$ and $L$, respectively. An important question in quantum many-body systems is to study how $S_A(L)$ scales with $L$ in the limit of $M \to \infty$ in different quantum phases. This question has been extensively investigated in one dimension (1D) \cite{1,2,3,4,5,6,7}. There, it is now well understood that for non-critical systems, $S(L)$ saturates to a constant as $L \to \infty$; whereas in critical systems, a logarithmic modification stands out as the leading term: $S(L) \sim \ln L$, and its coefficient is associated with the central charge of the related (1 + 1) conformal field theory (CFT) \cite{12}. In higher dimensions, it is generally believed that an area law holds at least for non-critical systems: the entanglement entropy scales as the area of the boundary between subsystem A and B, $S(L) \sim L^{d-1}$. This has been confirmed by studies on bosonic harmonic lattice systems \cite{13,8}. For critical systems, the situation is more complicated. The area law is shown to be violated in free fermion systems with a finite Fermi surface \cite{10,9}. But it still holds for fermionic systems without a finite Fermi surface \cite{10}, and critical bosonic systems \cite{11}.

In d-dimensional where the area law holds, $S(L) \sim f_a L^{d-1}$. Here $f_a$ is a boundary free energy determined by the short-distance properties of the system, and is hence not universal. It is thus interesting to wonder about subleading \cite{12} terms in $S(L)$, where, maybe, universal coefficients depending only on the model and the topological properties of the system could appear. For instance, it was recently found that in two-dimensional (2D) gapped systems, a subleading constant contribution in $S(L)$ is related to the topological order \cite{13}. Also, for a class of $z = 2$ conformal quantum critical systems in 2D, a universal logarithmic correction to the area law term has been found \cite{14}. Clearly, the problem is not fully settled.

It is of course also possible to investigate entanglement in quantum disordered systems. In a series of studies in 1D based on strong disorder renormalization group (SDRG) techniques \cite{17}, it was found that for the class of 1D infinite randomness fixed points (IRFP), a $\ln L$ term in $S(L)$ is also present \cite{3,15}. Similar results were also discovered for 1D aperiodic systems \cite{16}.

In this paper, we report on our study of the 2D random transverse field Ising (RTFI) model, and the numerical calculation of the entanglement entropy using the SDRG technique. The model is defined on a 2D square lattice with linear dimension $M$ and open boundary condition. The subsystem $A$ is a $L \times L$ square region located in the center of the square lattice. The Hamiltonian reads

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i^x S_j^x - \sum_i h_i S_i^z.$$  \(1\)

The Ising coupling $J_{ij}$ and the transverse field $h_i$ take random values drawn from the following box shape distributions:

$$P(J) = \Theta(J) - \Theta(J - 1),$$

$$P(h) = \frac{1}{h_0} [\Theta(h) - \Theta(h - h_0)].$$  \(2\)

This model is known to have a quantum phase transition which is governed by an IRFP \cite{18,19}. Here the critical point is tuned by $h_0$. Starting from the original Hamiltonian Eq. \(1\), the SDRG finds the ground state by successively eliminating the highest energy degrees of freedom \cite{17,20}. At each RG step, we look for the largest term in the Hamiltonian; its coupling (or field) is defined as the energy scale $\Omega$ at this step. If $\Omega = h_i$, the local spin is frozen in the eigenstate of $S_i^z$ by the local field. It is then eliminated from the system, and an effective coupling $J_{jk}' \approx \max(J_{jk}, J_{ji}, J_{ik}/h_i)$ is introduced between its two neighboring spins at sites $j$ and $k$. If $\Omega = J_{ij}$, the two spins involved respond to the field uniformly, so that they are combined into a new
effective spin (or a cluster). Then we effectively eliminate one spin degree of freedom, and the local field at the new effective spin is $h_i' = h_i h_j / J_{ij}$. These two decimation procedures are illustrated in Fig. 1. Numerically the RG is processed until only one cluster is left in the system. The ground state then consists of independent clusters each of which is frozen into a GHZ state: $|C(n)\rangle = \frac{1}{\sqrt{2}} (|1\rangle^{\otimes n} + |-\rangle^{\otimes n})$. Since each GHZ state will contribute either 1 (we take logarithms in base two) to the entropy if it consists of degrees of freedom in both subsystems $A$ and $B$, or 0 otherwise, calculating the entanglement entropy between two subsystems is reduced to a pure cluster counting problem: $S(L)$ is proportional to the number of clusters $N(L)$ that cross the boundary between the two subsystems. Finally $S(L)$ is averaged over different disorder configurations. In practice, $10^5 - 10^6$ configurations are used.

![FIG. 1: (color online) Basic RG transformations (see text for detail). Upper: energy scale is a field; lower: energy scale is an Ising coupling.](image)

The foregoing technique has been applied with success in the 1D case. The technical difference in 2D is first that the quantum critical point is not exactly known. To locate it accurately we study the scaling behavior of the average magnetization $m(M)$ [21]. At the critical field $h_0'$,

$$\frac{m(2M)}{m(M)} \bigg|_{h_0 = h_0'} = 2^{-x_m}, \quad (3)$$

is independent of $M$, where $x_m$ is the anomalous dimension of the bulk magnetization. Our result is given in Fig. 2(a). The critical field is estimated to be $h_0' = 5.37 \pm 0.03$, with $x_m = 1.01 \pm 0.05$, which is consistent with a previous RG study [22]. The entropy $S(L)$ is calculated for various values of $h_0$. For both critical and noncritical $h_0$, we find that the area law holds: $S(L) \sim L$ in the leading term. The result of $S(L)/L$ for different system sizes at critical $h_0' = 5.35$ is shown in Fig. 2(b).

This conclusion is quite different from the one in a recent study [23], where a double-logarithmic modification of the area law in the same model was reported at the critical point. We find that for small systems the double-logarithmic fit is reasonable, but that for system size $M \geq 128$, $S(L)/L$ increases definitely slower than $\ln \ln L$ for $L > 48$, strongly suggesting $S(L) \sim L$ in the limit of $L \to \infty$, without modification, and that the observation of Ref. [23] is biased by finite size effects. We also note that our results are largely independent of the distribution of couplings, confirming the idea of universal behavior for $S(L)$.

![FIG. 2: (color online) (a): Finite-size scaling of magnetization ratio given in Eq. 3. (b): entropy per surface $S(L)/L$ v.s. $\ln \ln L$ at critical field $h_0' = 5.35$, The dashed line is a linear fit in $\ln \ln L$ scale.](image)

Having established the validity of the area law in this system, it is natural to investigate subleading terms. We thus consider $\delta S(L) = 2S(L) - S(2L)$, in which the terms linear in $L$ cancel exactly. We find in both disordered and ordered phases that $\delta S(L)$ saturates to a constant term, indicating $S(L) = aL + c$. Meanwhile, at the critical point, we find that $\delta S(L)$ scales linearly as $\ln L$, suggesting

$$S(L) = aL + b \ln L + c, \quad (4)$$

i.e., a logarithmic correction to the area law. The coefficient of this logarithmic correction is determined to be $b = -0.019\pm0.005$ through finite-size scaling in Fig. 2(a).

To our knowledge, this is the first instance of such behavior in disordered 2D systems, and the second instance in all 2D systems after the examples in Ref. [14] for a class of conformal quantum critical models with dynamical exponent $z = 2$. There is no reason to expect that the $\ln L$ term we find in the critical RTFI model has much to do with the latter. This can be substantiated by calculating the amplitudes of the logarithmic term for different geometries, which obey some precise relations in the case of Ref. [14]. As an example, we considered a cross shape geometry as shown in Fig. 2(b). In this case as well, we can resolve a $\ln L$ term in $S(L)$ in addition to the area law contribution, with the coefficient $b_{cross} = -0.08 \pm 0.01$. We can then calculate...
and compare the ratios in our model, where we obtain \( b_{\text{cross}}/b_{\text{square}} \approx 4 \), and in the conformal quantum critical models where \( b_{\text{cross}}/b_{\text{square}} \approx 3 \) exactly. This implies that the \( \ln L \) term in \( \mathcal{S}(L) \) in our model most probably has a different origin, a non surprising conclusion since, for the IRFP, \( z \to \infty \).

The number of clusters crossing the boundary between two subsystems can be investigated at these percolation transitions as well (even though it does not correspond to an entanglement entropy except when \( h_0 = h_0^\infty \)). It is easy to see then that a \( \ln L \) correction to the area law is expected and related to conformal invariance, even though it depends on more complicated parameters that the central charge and the topology (in contrast with the example in Ref. [14]).

To better understand the \( \ln L \) term in \( \mathcal{S}(L) \) at the IRFP in 2D RTFI model, we notice that there is a striking difference between the model in 2D and in 1D. In 2D for any \( h_0 < h_0^\infty \) there is a finite-temperature phase transition at \( T_c(h_0) \) [18]: the IRFP in 2D can then be considered as an extension of this finite-temperature transition right down to \( T = 0 \). Through the SDRG, the transition to a ferromagnetically ordered phase can be mapped to a percolation transition in 2D [19]: the magnetic transition corresponds to the development of an infinite percolating spin cluster during RG. It is widely expected that this percolation process at the IRFP (which occurs at energy scale \( \Omega_\infty = 0 \)) is different from the one at finite temperature, since at \( h_0 = h_0^\infty \) the critical behavior is controlled by quantum fluctuations. This leads one to think of the IRFP as a type of “quantum percolation”, with fractal dimension \( d_f = 2 - x_m \approx 1.0 \). For \( h_0 < h_0^\infty \) meanwhile, the percolation takes place at finite energy scale \( \Omega_\infty \sim T_c \) in the RG, and is expected to be in the universality class of conventional classical percolation [19]. We have confirmed the classical percolation picture at finite energy scale by studying the scaling of largest active cluster size during RG. Some numerical results at \( h_0 = 3.2 \) are presented in Fig. 3(a). The percolation threshold is at \( \Gamma_{\infty} = \ln(\Omega_0/\Omega_\infty) = 5.27 \pm 0.02 \), where the extracted exponents \( \beta \) and \( \nu \) take the values of classical percolation indeed.

To see this, consider the “baby” case where the subsystem \( A \) takes the geometry of a line interval of length \( L \) on the boundary of the lattice, so the boundary between the two subsystems is \( A \) itself. Now \( N(L) \) is simply equal to the number of clusters touching \( A \). Its scaling can be studied using CFT techniques. For this, consider first the problem on the upper half complex plane, with \( A \) on the real axis. Use the well known expansion of the partition function of the critical \( Q \)-state Potts model in terms of clusters, or equivalently, dense loops [24]:

\[
\mathcal{Z} = \sum_P \sqrt{Q}^{N_P} \tag{5}
\]

where every loop gets the same weight \( \sqrt{Q} \). Now introduce a boundary conditions changing operator (BCCO) [25] \( \phi_y \) such that the two point function of \( \phi \) is defined through a sum similar to the one for \( Z \), but loops touching the boundary between the two insertions \( \zeta_1 \) and \( \zeta_2 \) get a different weight \( y \) instead of \( \sqrt{Q} \):

\[
< \phi_y(\zeta_1)\phi_y(\zeta_2) > = \frac{1}{Z} \sum \sqrt{Q}^{N_P} \left( \frac{y}{\sqrt{Q}} \right)^{N_P(\zeta_1,\zeta_2)} \tag{6}
\]
where \( N_P^b(\zeta_1, \zeta_2) \) gives the number of loops touching the boundary between \( \zeta_1 \) and \( \zeta_2 \) located on the real axis. We expect the two point function to have the following scaling form:

\[
\langle \phi_\beta(\zeta_1)\phi_\beta(\zeta_2) \rangle \sim e^{-f(y)|\zeta_1 - \zeta_2|^{-2b(y)}},
\]

where \( f(y) \) is the boundary free energy induced by the modified weight on the boundary, and the exponent \( h(y) \) is the anomalous dimension of BCCO. Now differentiate the two point function of BCCOs with respect to the weight \( y \), then take the limit \( y = \sqrt{Q} \). This leads to

\[
N_P^b(L) = aL + b \ln L,
\]

where

\[
b = -2\sqrt{Q} \frac{\partial h(y)}{\partial y} \bigg|_{y=\sqrt{Q}},
\]

and \( L = |\zeta_1 - \zeta_2| \).

With the exact expression of \( h(y) \) we obtain

\[
b = \frac{1}{2\pi p} \sqrt{Q(4-Q)},
\]

For percolation \((Q = 1)\), \( b_{\text{perc}} = \frac{\sqrt{2}}{4\pi} \approx 0.1378 \), reproduces an early result by Cardy. But in Eq. 10 we generalize Cardy’s result to general \( Q \), and it is interesting to see that the \( \ln L \) term vanishes at \( Q = 0 \) and \( Q = 4 \). It is also remarkable to see that \( b \) is related to the derivative of the anomalous dimension of BCCO, but not the central charge. This \( \ln L \) term is also observed in our RG calculation at finite \( \Gamma_\infty \). In Fig. 3(c) we show the scaling of \( \delta L \equiv 2N(L) - N(2L) \) at \( \Gamma_\infty = 5.27 \) for \( h_0 = 3.2 \). \( b \) is estimated to be \( 0.15 \pm 0.02 \), in agreement with the analytical result. This further confirms that the universality class at finite \( \Gamma_\infty \) is classical percolation. Interestingly, we find numerically for this case that \( |b| < 0.01 \) at the quantum critical point, consistent again with the idea of a different universality class when \( \Gamma_\infty \rightarrow \infty \).

Going back to the original problem where the subsystem \( A \) takes the geometry of a \( L \times L \) square, we have not derived a similar analytical result for ordinary percolation. \( N(L) \) of course cannot be calculated numerically. To get better scaling for large systems, we turn to a direct study of percolation. In Fig. 3(d), \( \delta N \) data at percolation threshold are shown. As well expected, \( \delta N \sim \ln L \) is resolved for both bond and site percolation, and the coefficient of the \( \ln L \) term takes the same value \( b = -0.06 \pm 0.01 \), in agreement with the idea that this term is universal. Note that we observe a negative value of \( b \) for subsystem \( A \) a square, just as in the case of the RTFI model. This is opposite to the sign of \( b \) in classical percolation when \( A \) is an interval.

The observation of a \( \ln L \) term in \( N(L) \) in percolation makes the presence of a similar term at the IRFP most likely: there will in fact always be such a term at energy scale \( \Omega_\infty \). When this scale is finite, the coefficient \( b \) takes the value of classical percolation, \( b = -0.06 \pm 0.01 \). But when \( \Omega_\infty \rightarrow 0 \), i.e., at the quantum critical point, quantum fluctuations become dominant, leading to a quantum percolation belonging to different universality class. A different \( b \) value, \( b = -0.019 \pm 0.005 \), reflecting this difference is then observed.

In summary, we have calculated the entanglement entropy of a 2D RTFI model by using a numerical SDRG method. In contrast to what is claimed in a recent preprint, we find that the leading term of the entropy follows the area law and depends linearly on the block size \( L \) in both critical and non-critical phases. However, a \( \ln L \) correction to the area law is discovered at criticality. While the presence of this correction may not have been expected from the entanglement point of view, it is very natural once the problem is reformulated geometrically. Indeed, the problem of counting clusters touching a boundary in 2D classical percolation is easily argued to give rise to sub logarithmic corrections, while the entanglement entropy in the RTFI model at criticality can be reformulated as a similar problem but in a different, “quantum percolation” universality class.

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