COMMENT ON ‘COMMENT ON ‘AFFINE DENSITY, VON NEUMANN DIMENSION AND A PROBLEM OF PERELOMOV”

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ABSTRACT. In the Letter ‘Comment on ‘Affine density, von Neumann dimension and a problem of Perelomov’ https://doi.org/10.48550/arXiv.2211.04879, by Prof. J. L. Romero, it is claimed that the main theorem of Ref2 := [Adv. Math. 407, Article ID 108564, 22 p. (2022)] is included in the prior research survey Ref1 := [Expo. Math., 40(2), 265-301, 2022]. We provide some background, and demonstrate with hard facts that this claim is false, as well as other claims in the Letter which, in most cases, either result from wrong mathematical reasoning or objectively differ from what is written in the published material. The author might be interested in publicly retracting and apologising for his comments.

1. Introduction

This Letter is the exercise of a right of answer to the comments in [1]. In [1], it has been claimed that the main theorem of Ref2 := [Adv. Math. 407, Article ID 108564, 22 p. (2022)] is included in the prior research survey Ref1 := [Expo. Math., 40(2), 265-301, 2022]. We will demonstrate that this claim is false, by providing evidence that there is not enough information in [Expo. Math., 40(2), 265-301, 2022] to conclude any of the original results in [Adv. Math. 407, Article ID 108564, 22 p. (2022)]. We will also comment on other misquote claims of [1], which in most cases either follow from errors in the logical reasoning, or objectively do not match what is written in the published material and therefore seem to have been concluded from a superficial reading of our paper. We have cited Ref1 := [Expo. Math., 40(2), 265-301, 2022] and properly acknowledged all information relevant to our work.

The letter is organized as follows. In the next section we provide the required background on the wavelet transform and define the spaces that are fundamental for the discussion, as well as some literature review about these spaces. Readers familiar with the topic can directly go to the third section, where we specifically argue that the claim in the abstract of [1] is false. We isolate this from the other comments, since it represents a plagiarism claim that will be categorically refuted with hard facts. In the fourth section, the remaining contents of the Letter [1] are discussed. Evidence is provided that the author’s further claims either follow from wrong mathematical reasoning, or do not conform with what is written in the published material. The Letter is finished with further comments and with a suggestion for a friendly solution of this bizarre situation.

2. Background on the wavelet transform

Consider the unitary representation of the affine group $G_a = \mathbb{R} \times \mathbb{R}^+ \equiv \mathbb{C}^+$, acting on the Hardy space $H^2(\mathbb{C}^+)$
The continuous wavelet transform of a function \( f \) with respect to a wavelet \( \psi \) is defined, for every \( z = x + iy \in \mathbb{C}^+ \), as

\[
W_\psi f(z) = \langle f, \pi(z)\psi \rangle_{H^2(\mathbb{C}^+)}.
\]  

\( (2.1) \)

Consider the Daubechies-Paul [12] mother wavelets chosen from the family \( \{\psi_n^\alpha\}_{n \in \mathbb{N}_0}, \alpha > 0 \), defined in terms of the Laguerre polynomials as follows

\[
(\mathcal{F}\psi_n^\alpha)(\xi) := \xi^n e^{-\xi} L_n^\alpha(2\xi), \quad L_n^\alpha(t) = \frac{t^{-\alpha}e^t}{n!} \left(\frac{d}{dt}\right)^n (e^{-t}t^{\alpha+n}), \quad \xi > 0.
\]  

\( (2.2) \)

We will write \( d\mu^+(z) = y^{-2}d\mu_{\mathbb{C}^+}(z) \), where \( d\mu_{\mathbb{C}^+}(z) \) is the Lebesgue measure in \( \mathbb{C}^+ \). The image of \( H^2(\mathbb{C}^+) \) under the map

\[
W_{\psi_n^\alpha} : H^2(\mathbb{C}^+) \rightarrow L^2(\mathbb{C}^+, d\mu^+)
\]

is a space with reproducing kernel \( k_{\psi_n^\alpha}(z,w) = \frac{1}{C_{\psi_n^\alpha}} W_{\psi_n^\alpha} \psi_n^\alpha(w^{-1}.z) \), (where \( . \) stands for the product on \( G_a \) and \( k_{\psi_n^\alpha}(z,w) = \frac{\|\psi_n^\alpha\|^2}{C_{\psi_n^\alpha}} \), where \( C_{\psi_n^\alpha} = 2\pi \|\mathcal{F}\psi_n^\alpha\|^2_{L^2(\mathbb{R}^+, t^{-1}dt)} \). We will denote this space by \( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \).

*For \( n = 0^* \) the space \( \mathcal{W}_{\psi_0^\alpha}(\mathbb{C}^+) \) is related to Bergman spaces of analytic functions [16] (see, for instance, section 2.1 in [7]). As proven in [18], this is the only choice leading to spaces of analytic functions. It can be shown that the only subspace of \( L^2(\mathbb{C}^+, d\mu^+) \) invariant under the classical projective representation \( \tau^a \) of the group \( \text{PSL}(2,\mathbb{R}) \) is the Bergman space of analytic functions. The von Neumann dimension of the space \( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \) *for \( n = 0^* \) was computed for the first time in the projective case by Radulescu [28, Section 3], using an extension of the Schur-type orthogonality formulas to irreducible projective representations with square integrable coefficients. A recent approach without requiring Schur-type orthogonality formulas (and thus neither irreducibility nor square integrability) was presented by Jones [20].

*For \( n > 0^* \) the spaces \( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \) are related to spaces of non-analytic functions (real analytic and polyanalytic [4] of order \( n \)) and for special choices of \( \alpha \), to spaces of Maass forms [22] (see [13] for a modern reference), or hyperbolic Landau Levels [3, 11, 24]. There are very few papers dealing with the wavelet spaces \( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \). Motivated by the connections with Maass forms, physical models [3], DPP’s [5], the large sieve [6] and polyanalytic function theory [4], the author of this Letter has been participating on a program to study several aspects of these spaces *for \( n > 0^* \), started in [3] and recently developed in [2, 5, 6]. This is a new research direction in wavelet analysis, and fundamental parts of it, like the computation of the matrix representation coefficients, have only been completed very recently in [6].

In our paper under discussion [2], a new projective representation of \( \text{PSL}(2,\mathbb{R}) \) acting on \( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \) (*for all non-negative integer \( n^* \)) has been introduced. In Theorem 2.5 of [2], the wavelets leading to the required \( \text{PSL}(2,\mathbb{R}) \) invariance are shown, under mild conditions, to be the functions (2.2). The invariance is then used to compute the von Neumann dimension of the space *\( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \) for \( n > 0^* \), using the method of Jones [20]. As an application, in [2] we obtain two original results about frames in the spaces \( \mathcal{W}_{\psi_n^\alpha}(\mathbb{C}^+) \) for \( n > 0 \) (Theorem 2.1 and Theorem 2.8) and four Corollaries with variations and reflecting the connections between different topics. The case \( n = 0 \) of our Theorem 2.1 in [2] has been considered before in [29]. This is acknowledged in the comments before Corollary 2.4 in [2].
3. The abstract claim

The following results are discussed in [1]:

**A** The statement below, concerning a locally compact, second countable unimodular group $G$. A discrete series $\sigma$-representation of $G$ is defined by the authors in [29] as an irreducible projective representation with square integrable coefficients, the same setting of Definition 3.1 in [28].

**Theorem 1** ([29, Theorem 7.4]). Let $\Gamma \subseteq G$ be a lattice and let $(\pi, \mathcal{H}_\pi)$ be a discrete series $\sigma$-representation of $G$ of formal dimension $d_\pi > 0$.

(i) If $\pi|\Gamma$ admits a cyclic vector, then $\text{vol}(G/\Gamma)d_\pi \leq 1$.

In particular, if $\pi|\Gamma$ admits a frame vector, then $\text{vol}(G/\Gamma)d_\pi \leq 1$.

(ii) If $\pi|\Gamma$ admits a Riesz vector, then $\text{vol}(G/\Gamma)d_\pi \geq 1$.

**B** The following definition and result.

In [2], we define the following new projective representation $\tau_\alpha^m$ of the group $\text{PSL}(2, \mathbb{R})$ acting on the space $W_\psi_\alpha = W_\psi_\alpha(\mathbb{C}^+)$, which is the image of the Hardy space $H^2(\mathbb{C}^+)$ under the wavelet transform with the special window function $\psi_\alpha \in H^2(\mathbb{R})$ defined in (2.2).

$$\tau_\alpha^m(m^{-1})F(z) = \left(\frac{cz + d}{cz + d}\right)^{2n+\alpha+1} F(m \cdot z), \quad z \in \mathbb{C}^+, \quad m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R}).$$  (3.1)

Below is our Theorem 2.1 from [2].

**Theorem 2** ([2, Theorem 2.1]). Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a Fuchsian group with fundamental domain $\Omega \subset \mathbb{C}^+$ (of finite volume) and $F \in W_\psi_\alpha(\mathbb{C}^+)$. If $\{\tau_\alpha^\gamma(m)F\}_{\gamma \in \Gamma}$ is a frame for $W_\psi_\alpha(\mathbb{C}^+)$, then

$$|\Omega| \leq \frac{C_{\psi_\alpha}}{\|\psi_\alpha\|^2} = \frac{4\pi}{\alpha},$$  (3.2)

where $|\Omega|$ is calculated via the measure $\mu$.

If $\{\tau_\alpha^\gamma(m)F\}_{\gamma \in \Gamma}$ is a Riesz sequence for $W_\psi_\alpha(\mathbb{C}^+)$, then

$$|\Omega| \geq \frac{C_{\psi_\alpha}}{\|\psi_\alpha\|^2} = \frac{4\pi}{\alpha}.$$  (3.3)

• After stating these two results, the first comment of [1] is:

‘Let us start by mentioning that in [29, Example 9.2] we discuss at length how Theorem 1 applies to the so-called holomorphic discrete series of $\text{PSL}(2, \mathbb{R})$, which exhaust up to complex conjugation and projective unitary equivalence all square integrable irreducible representations of $\text{PSL}(2, \mathbb{R})$ (including (3.1)).’

• This comment is mathematically wrong. While the so-called holomorphic discrete series of $\text{PSL}(2, \mathbb{R})$ do exhaust up to complex conjugation and projective unitary equivalence all square integrable irreducible representations of $\text{PSL}(2, \mathbb{R})$, this obviously does
not include (3.1), which is only a projective representation for non-integer $\alpha$. See Bargmann [8, 5g, II].

- For integer $\alpha$, the unitary equivalence between the projections simplifies the application of Theorem 1, since the irreducibility and square integrability (that we didn’t need to use with our method—see Theorem A below) of the representation follows automatically. But for general $\alpha$ this cannot be assured without a proof. See the very last point of this Letter to see why this cannot be assured by general results for the projective representation restricted to a Fuchsian group, not even by lifting to the universal cover (the relevant cohomology obstruction only vanishes for some Fuchsian groups).

Then, spread along the paper, there are some comments aiming at supporting the following claim from the abstract:

‘We point out that the main theorem of Ref2 := [Adv. Math. 407, Article ID 108564, 22 p. (2022)] is included in the prior research survey Ref1 := [Expo. Math., 40(2), 265-301, 2022].’

- By stating that the result is contained in the mentioned prior research, the author is making a plagiarism claim. We will demonstrate that this claim is false, by showing that there is not enough information in [Expo. Math., 40(2), 265-301, 2022] (nor, as far our knowledge goes, in other prior research) to obtain Theorem 2. This requires the following half-page argumentation.

To obtain Theorem 2 from Theorem 1 it is required to:

1. Find a new projective representation $\tau^\alpha_n$ of $\text{PSL}(2, \mathbb{R})$ naturally acting on the spaces $W_{\psi}^\alpha_n(\mathbb{C}^+)$, because the classic projective representation of $\text{PSL}(2, \mathbb{R})$ on analytic Bergman spaces does not leave $W_{\psi}^\alpha_n(\mathbb{C}^+)$ invariant for $n > 0$.
2. Show that the reproducing kernel space $W_{\psi}^\alpha_n(\mathbb{C}^+)$ is invariant under $\tau^\alpha_n$.
3. Show that the representation $\tau^\alpha_n$ has square integrable coefficients.
4. Show that the representation $\tau^\alpha_n$ is irreducible.
5. Compute the von Neumann dimension $\dim_{\pi|\Gamma} W_{\psi}^\alpha_n(\mathbb{C}^+) = \text{vol}(G/\Gamma)d_\pi$ of the space $*W_{\psi}^\alpha_n(\mathbb{C}^+)$ for $n > 0^*$, with the representation $\tau^\alpha_n$ (the $d_\pi = d_\pi(n)$ was first computed for $n > 0$ by us in [2]).

The items (1)-(5) are not contained in [Expo. Math., 40(2), 265-301, 2022] (nor, as far our knowledge goes, in other published research). It is actually enough to say that the projective representation $\tau^\alpha_n$ has been considered for the first time in our paper [Adv. Math. 407, Article ID 108564, 22 p. (2022)]. Because without the explicit form of the projective representation $\tau^\alpha_n$ it is obviously not possible to state (1)-(5) (not even to define the coefficients of the frame in Theorem 2), and therefore not possible to prove these required properties. It should be noticed that the spaces $W_{\psi}^\alpha_n(\mathbb{C}^+)$ for $n > 0$ are not even mentioned in [Expo. Math., 40(2), 265-301, 2022].

Our approach in [2] is simpler than the one using Theorem 1, because it does not require to prove irreducibility and square integrability. It uses two elementary results on von Neumann
algebras, which, according to Jones [20, Theorem 3.5, (viii),(ix)] are due to Murray and von Neumann [25] (1936). These results are stronger (and of course, older) than Theorem 1 i) and, combined with the relation between Riesz and separating vectors (a novel result from [29] that we use and properly quote), imply Theorem 2 requiring only conditions (1), (2) and (5). We invite the readers to our simple proof of Theorem 2.1 in section 4.3 of [2]. For a sample of the strength of the results we used, observe that, for a general von Neumann algebra $M$ acting on a Hilbert space $H$, the following holds:

**Theorem A** [Murray and von Neumann [25] (1936)]. $M$ admits a cyclic vector if and only if $\dim_M H \leq 1$.

Take $M = \pi|\Gamma$. Then $\dim_M H = vol(G/\Gamma)d_\pi$ and that’s it. We obtain, from a 86 years old result:

**Theorem A’** [Murray and von Neumann [25] (1936)]. $\pi|\Gamma$ admits a cyclic vector if and only if $vol(G/\Gamma)d_\pi \leq 1$.

We observe that **Theorem A’ is stronger than Theorem 1 i)**: it gives the same conclusions without the discrete series $\sigma$-representation assumptions on the projective representation. Therefore, in the final part of our argument, we use Theorem A instead of Theorem 1 (this seems to be the main concern of the author of [1]) because: it’s simpler, stronger, and older.

**Conclusion.** The plagiarism claim made in the abstract of [1] is false.

4. Comments on the Letter

4.1. **On section 1.1.** In this section the author presents a computation of $d_\pi$, alternative to ours in [2]. There are not many comments to be made, since this was not included in [Expo. Math., 40(2), 265-301, 2022]. But perhaps one should draw to the attention of the author that his proof is incomplete. The author is confused about the definition of discrete series $\sigma$-representation from his own work with van Velthoven: after sorting the Jargon and notation in the beginning of Section 1.1, this is defined simply as a projective representation and the square integrability and irreducibility, fundamental in the definition of discrete series $\sigma$-representation are forgotten. For this reason, the author didn’t prove the Schur-type orthogonality relations [1, (1.5)], corresponding to Definition 3.1 in [28], which only hold for irreducible projective representations with square integrable coefficients (in Romero and van Velthoven’s parlance, this is exactly a discrete series $\sigma$-representation). Neither the irreducibility nor the square integrability of the representation $\tau^\alpha_n$ is shown in [1]. This amounts to some extra work and is essential for the suggested approach. We have actually done this work, but since our approach in [2] does not require these two properties, the proof is omitted and left as an exercise to the interested reader.

4.2. **On section 2.** In this section the author reproduces some of the proofs of [29], perhaps to simplify the work of the reader. I have no comments to make about this section.

4.3. **On section 3.2, ‘Our work’.** The section starts with a variation on a previous theme:
‘Firstly, our research survey [29] discusses at length the application of Theorem 1 - and its counterpart concerning the existence of coherent systems - to the holomorphic series of $\text{PSL}(2, \mathbb{R})$, which exhaust up to complex conjugation and projective unitary equivalence all square integrable irreducible representations of $\text{PSL}(2, \mathbb{R})$. The statements in Theorem 2 are of course invariant under unitary equivalence.’

- For those who missed the previous related comment in Section 3, here it goes again. While the so-called holomorphic discrete series of $\text{PSL}(2, \mathbb{R})$ do exhaust up to complex conjugation and projective unitary equivalence all square integrable irreducible representations of $\text{PSL}(2, \mathbb{R})$, this obviously does not include (3.1), which, for non-integer $\alpha$, is only a projective representation. Thus, the above comment has no relevance for the current discussion. See also the last point of this Letter.

Then,

‘the special case of Theorem 2 with $F = W_{\psi_n}^{\alpha} \psi_n^\alpha$ is attributed to us in [2], whereas the reader is led to understand that the generalization to arbitrary $F$ is the original contribution of [2]. While, indeed, in [[29], Section 9.1.3, “Perelomov’s problem with respect to other special vectors”] we discuss the application of Theorem 1 to the special vectors in question, the very same arguments apply to any other generator’

- This is a total mess. The original contribution of Theorem 2 is the extension to general $n$, not to general $F$. What we attribute to [29] is the special case of Theorem 2 with $F = W_{\psi_n}^{\alpha} f$ (case $n = 0$, general $F$! - this implies the result for $F = W_{\psi_n}^{\alpha} \psi_n^\alpha$ - the general $n$ case $F = W_{\psi_n}^{\alpha} f$ is not possible to obtain without the steps (1)-(5), in particular needs the von Neumann dimension of $W_{\psi_n}^{\alpha} (\mathbb{C}^+)$). Because what is proved in [29] is the case $n = 0$ of Theorem 2 (as recognized by us in the comments before Corollary 2.4 in [2], which is the result we attributed to [29]). More precisely, it is shown in [29, Section 9.1.2-9.1.3] that if $\{\pi_\alpha(\gamma) g \}_{\gamma \in \Gamma}$ is a frame (and therefore complete) in $A^2_\alpha (\mathbb{C}^+)$, where $A^2_\alpha (\mathbb{C}^+)$ stands for the Bergman space of analytic functions and $\pi_\alpha$ for the classical projective representation of $\text{PSL}(2, \mathbb{R})$ in $A^2_\alpha (\mathbb{C}^+)$, then the conclusion of Theorem 2 holds. Since the Bergman space is a weighted version of $W_{\psi_n}^{\alpha} (\mathbb{C}^+)$ (see for instance section 2.1 in [7]), this is precisely the statement of Theorem 2 for $n = 0$. The author of [1] should read more carefully the two papers under discussion.

Then:

Thirdly, with a certain physical motivation, the authors of [3] consider the above-mentioned affine coherent states and conclude that the density condition

$$|\Omega| \leq \frac{4\pi (n + 1)}{\alpha}$$
is necessary for completeness. In [[29], Section 9.1.3] we point out that coupling theory, as embodied in Theorem 1, offers the sharper bound
\[ |\Omega| \leq \frac{4\pi}{\alpha} \] (4.1)

- This is true! I have given before my congratulations to the authors of [29] for improving our 2015 results in [3], despite the general case \( \alpha \) not being completely executed (see the last remarks). And we have quoted this result before our formulation of Corollary 2.4 in [2]: ‘As recently observed by Romero and Velthoven in a different formulation...’

We are actually quoting more than what is proved in [29], since we also attribute the Riesz sequence condition. This was not proved in [29] but it follows from the material in [29] (in contrast with our Theorems 2.1, 2.5 and 2.8 in [2], which, as already explained, do not; the estimate (4.1) is the case \( n = 0 \) of our theorem 2.1; it was fully executed in [29] for the case of \( \alpha \) integer). In the paper with Balazs, de Gosson and Mouayn [3], where the problem was considered for the first time, we have used automorphic forms and this required introducing multiple zeros, whence the factor \((n + 1)\). The authors of [29] deserve all the credit for this amazing improvement of our early results (on our behalf, [3] was the first paper on the topic, and sometimes first papers don’t achieve best possible results).

Continuing,

The final remark in [[2], Section 2] on the “physical relevance of the results” may give the impression that this observation is original to [2]; we invite the interested reader to look into [[29], Section 9.1.3].’

- There is a scandalous mismatch between this claim and what is written in our paper. The author of [1] is once again advised to read more carefully the papers under discussion before posting public comments of this nature. Our final remark on the ‘physical relevance of the results’ [2, Page 9, line -6] concludes as follows: ‘Corollary 2.4 provides estimates for the number of particles distributed by the model on such a region’. It turns out that Corollary 2.4 is precisely the result we attributed to Romero and Velthoven [29] and this is the only result quoted in the whole paragraph! The reader may want to know how can this ‘give the impression that this observation is original to [2]’.

- We are actually using Corollary 2.7 from [2], which uses the connection to the Maass forms operator, but since this amounts to a specialization of the parameters in Romero and Velthoven’s result, we were extremely careful and, to respect their original contribution, we quoted instead Corollary 2.4.

5. Further comments

- Our work was strongly influenced by Jones last paper [20], which offers new methods, simpler than those in the previous literature. This was intended to be the first paper
on what he envisioned as a new field called Applied von Neumann algebra. I leave the YouTube talk link for those interested https://www.youtube.com/watch?v=s70vmKv4zfg. Before our paper, the relevant problems have only been considered in the analytic Bergman setting, the natural environment of automorphic forms. We offer in [2] a contribution for this field, by extending the Bergman space setting of [20] to an infinite sequence of wavelet spaces including the spaces of Maass forms and computing the von Neumann dimension for all of them. These spaces are central in mathematics, since the Maass Laplacian is the spinal cord of Selberg’s trace formula (see the clear presentation of Patterson [26]). Our projective representation (3.1) didn’t come from outer space, it was designed to match the automorphic coefficient of Maass forms [13, 22]. This opens a series of questions, like constructing the functions vanishing on the orbits of Fuchsian groups, whose existence is assured by Theorem 2.1 (see Corollary 2.7 in [2]). Such construction is expected to involve Maass forms in a nontrivial way. Deep problems arise if one attempts to extend other problems treated by Jones in [20], to our setting.

• On section 3.1. There are some comments about the status of Perelomov problem. We have clarified this in our Remark 2.6 of [2]. Abstract results can also be seen as a solution, this is really a personal perspective, and the in [1] quoted Bekka’s claim from [10] goes into this direction, but does not invalidate ours, which is concrete and even classify the concrete wavelets for which the problem can be considered (Theorem 2.5 in [2]). As we point out in Remark 2.6 of [2], the really difficult part is still open. We invite the interested reader to read the Remark, the fascinated reader to try to prove what is still open, and the interested and fascinated reader to give a go at Seip’s conjecture mentioned in the introduction. I would love to see a proof before leaving this existence.

• On section 3.1. Second paragraph. I had to read our whole paper to see where in ‘Sections 1.1 and 1.2 of [2] elaborate on the many technical challenges that the exponential growth of balls in the hyperbolic metric’. Besides the introductory sentence, and the mention in Section 1.1 that Landau-type [21, 23], comparison [17] and analytic [31] methods do not work here, nothing else is mentioned. Obviously, the interest of these methods is to circumvent such difficulties, which indeed do not show up in this approach. We have invested efforts in giving a broad idea of the state of the art of the motivating problems in the introduction.

• The author of [1] seems particularly fixed in our Theorem 2.1, suggesting it is not new (something already clarified in this Letter) and calling it ‘main result’. We don’t make such distinctions in our paper. We offer three theorems (2.1, 2.5, 2.8) and six corollaries. My favorite is Theorem 2.5, my coauthor prefers Theorem 2.8. All readers are invited to read [2] and make their own hit parade.

• The constructive criticism of this Letter towards [1] does not include Romero and van Velthoven’s work [29]. [29] is a beautiful piece that goes beyond its survey intents, with original contributions that were used and properly quoted in our work, and which fully achieves the authors’ stated goal in the introduction, since it really ‘motivates the non-specialist to delve deeper into operator-algebraic methods’. It also does a great job in providing a proof of Theorem 1 by elementary methods inspired in ‘Janssen’s classroom proof’ [19]. Therefore, we also invite all readers to read [29].
Despite the praise in the previous paragraph, I need to point out a small mistake in Example 9.3 of [29], which seems to be one of the many sources of the confusion that led the author of the Letter [1] to write strange things, since the same mistake appears twice in [1], as already pointed out above. For non-integer $\alpha$, $\pi_\alpha$ is just a projective representation, therefore not in the holomorphic discrete series (as claimed in Example 9.3 of [29]), since it has been shown by Bargmann [8, 5g, II] that this implies $\alpha$ integer (see also [15, 3.3]). Thus, the orthogonality relations cannot be obtained from the holomorphic discrete series. Thus, the argument of the authors’ only assures the bound (4.1) for $\alpha = n$ (thus, to be fully rigorous, only improve our results in [3] in these special cases). The general case of non-integer $\alpha$ is not completely executed. We have ignored this, and attributed the result with general $\alpha$ to [29] because, fortunately, this is not a serious mistake: it can be corrected by replacing the concept of holomorphic discrete series by irreducible square integrable projective representations with square integrable representations (Definition 3.1 in [28], this is what the authors call discrete series $\sigma$-representation), and then using the results of Section 3 of [28], the claimed Schur-type orthogonality holds, and the conclusion of the authors’ in Example 9.3 of [29] follows.

For a crash overview of the holomorphic discrete series in a signal analysis context, I would recommend Feichtinger-Gröchenig coorbit zero [14, Example 7.3], where the case of discrete series ($\alpha = n$) is worked out and, after this, the possibility of extension to projective representations is mentioned.

I invite the author of the Letter [1], after reading more carefully the material involved, for a blackboard mathematical discussion about the topics involved in these papers and Letters, at NuHAG’s discussion room at the University of Vienna, together with two Senior scientists from our group with the capacity of following the contents and arbitrate between facts and viewpoints in case this is needed. I believe Prof. Feichtinger and Prof. Gröchenig are more than qualified to assist us in this task. We can later invite Michael Speckbacher and Jordi van Velthoven for the discussion (and also present the material in NuHAG’s weekly seminar) but, at this stage, it would be ideal to first clarify things between the two of us.

The author might be interested in retracting the comments of [1] and to voluntarily making a public apology to the authors of [2] in arXiv, under the title Retraction and apology for ‘Comment on ‘Affine density, von Neumann dimension and a problem of Perelomov”.

A further remark about the projective representations. For groups with trivial Lie algebra cohomology as the case of the whole $SL(2, \mathbb{R})$, one could think about using Bargmann’s theorem [9] to lift the projective representations to representations in the universal cover, where they were classified by Pukanszky [27] and Sally [30]. But this is not possible to do for the restriction of the projective representation to an arbitrary Fuchsian group. Indeed, in Theorem A1 in [20, Appendix A], Jones proved that, for a Fuchsian group associated with a Riemann surface of genus $g$, the cohomology obstruction only vanishes (a requirement to lift the representation) if $\alpha$ is an integer multiple of $\frac{1}{g-1}$. Thus, for a Fuchsian group of genus $g = 2$, this requires $\alpha$ to be an integer and we are back in the holomorphic discrete series. Thus, there is no possibility of assuring unitary equivalences between the results of Theorem 2.1 by general results. The action of the projective representation (3.1) must be considered.
in its explicit form, acting on each Fuchsian group, in its associated space $W_{\psi_n}(\mathbb{C}^+)$. Thus, from any mathematical perspective, (3.1) is a new projective representation of $PSL(2,R)$.

Finally, it is with a profound sadness that I am forced to use a scientific repository to write a Letter like this. But the unannounced posting of [1] left no other choice. A plagiarism allegation from someone with the credibility of Prof. Romero, could easily raise doubts among my colleagues and end up with my career and reputation, as well of my coauthor’s, if left unanswered and unclarified. The situation is so unusual in mathematics that I doubt of the existence of a system prepared to solve it. The only possible definitive solution is to appeal to a consensus building among experts in mathematics.

Acknowledgements. The Letter has been written at the author’s sole responsibility. We thank the author of [1] for the correction of a typo in the definition of the representation (3.1) and a small mistake in the statement of Theorem 2.1 (as the author correctly pointed out, the compactness condition is not necessary). We will include these corrections, and a few more in a corrigendum to be submitted (the statements in these Letter already include them, namely a correction of the Nyquist rate, $\frac{2}{\alpha}$ should be replaced by $\frac{4\pi}{\alpha}$ an unfortunate mistake inherited from not taking into account the normalization of the Fourier transform used in [2]). The light notes of irony in one or two points of this letter are intended to soften down the tone of [1], which conveys a certain loftiness. Furthermore, the subject matter at stake is too serious to be taken seriously.

References

[1] J. L. Romero, Comment on "Affine density, von Neumann dimension and a problem of Perelomov", arXiv:2211.04879, (2022). https://doi.org/10.48550/arXiv.2211.04879
[2] L. D. Abreu, M. Speckbacher, Affine density, von Neumann dimension and a problem of Perelomov, Adv. Math., 407, 108564 (2022). https://doi.org/10.1016/j.aim.2022.108564.
[3] L. D. Abreu, P. Balazs, M. de Gosson, Z. Mouayn, Discrete coherent states for higher Landau levels, Ann. of Phys. 363 (2015) 337-353.
[4] L. D. Abreu, H. G. Feichtinger, Function spaces of polyanalytic functions, Harmonic and Complex Analysis and Its Applications, Springer (2014), pp. 1-38.
[5] L. D. Abreu, P. Balazs, S. Jaksic, The affine ensemble: determinantal point processes associated with the $ax+b$ group, J. Math. Soc. Japan, 88018801, 1-15, (2022).
[6] L. D. Abreu, M. Speckbacher, Donoho-Logan large sieve principles for the continuous wavelet transform, arXiv preprint arXiv:2210.13056, (2022). https://arxiv.org/pdf/2210.13056.pdf
[7] L. D. Abreu, Z. Mouayn, F. Voigtlaender, A fractal uncertainty principle for Bergman spaces and analytic wavelets, J. Math. Anal. Appl. 519 (1), 126699, (2023). https://www.sciencedirect.com/science/article/pii/S0022247X22007132
[8] V. Bargmann, Irreducible Unitary representations of the Lorentz group, Ann. Math., 48, 568-640, (1947).
[9] V. Bargmann, On unitary ray representations of continuous groups, Ann. Math., 59, 1-46, (1954).
[10] M. B. Bekka, Square integrable representations, lattices and von Neumann algebras, In Lie theory and its applications in physics, III (Clausthal, 1999), pages 27–40. World Sci. Publ., River Edge, NJ, 2000.
[11] A. Comtet, On the Landau levels on the hyperbolic plane, Ann. Phys., 173, 185-209, (1987).
[12] I. Daubechies, T. Paul, Time-frequency localization operators - a geometric phase space approach: II. The use of dilations, Inverse Probl., (4), 661-680, (1988).
[13] W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, Invent. Math., 92, 73-90, (1988).
[14] H. G. Feichtinger, K. Gröchenig, *A unified approach to atomic decompositions via integrable group representations*, Function spaces and applications, 52-73, (1988).
[15] F. M. Goodman, P. de la Harpe, V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer, New York, (1988).
[16] H. Hedenmalm, B. Korenblum, K. Zhu, *The Theory of Bergman Spaces*, (2000) Springer, ISBN 978-0-387-98791-0.
[17] C. Heil, G. Kutyniok, *The homogeneous approximation property for wavelet frames*, J. Approx. Theory, 147, 28-46, (2007).
[18] N. Holighaus, G. Koliander, Z. Průša, L. D. Abreu, *Characterization of analytic wavelet transforms and a new phaseless reconstruction algorithm*, IEEE Trans. on Signal Process. 67, (15), 3894-3908 (2019).
[19] A. J. E. M. Janssen, Classroom proof of the density theorem for Gabor systems. *Vienna, Erwin Schrödinger Institute (ESI) Preprint 1649*, 2005.
[20] V. F. R. Jones, *Bergman space zero sets, modular forms, von Neumann algebras and ordered groups*, arXiv preprint arXiv:2006.16419, Jun 29, (2020). [https://arxiv.org/pdf/2006.16419.pdf](https://arxiv.org/pdf/2006.16419.pdf)
[21] H. J. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math., 117, pp. 37-52, (1967).
[22] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann., 121, 141–183, (1949).
[23] M. Mitkovski, A. Ramirez, *Density results for continuous frames*, J. Fourier Anal. Appl., 26, Article number: 56 (2020).
[24] Z. Mouayn, *Characterization of hyperbolic Landau states by coherent state transforms*, J. Phys. A: Math. Gen., 36, 8071–8076, (2003).
[25] F. Murray, J. von Neumann, *On rings of operators*, Ann. Math. 37, 116-229, (1936).
[26] S. J. Patterson. *The Laplacian operator on a Riemann surface*. Compos. Math., 31, 83–107, 1975.
[27] L. Pukanszky, *The Plancherel formula for the universal covering group of SL(2,$\mathbb{R}$)*, Math. Ann., 156, (1964), 96-143.
[28] F. Radulescu, *The $\Gamma$-equivariant form of the Berezin quantization of the upper half plane*, Mem. Am. Math. Soc., 133, (630) (1998), p. viii+70. [https://art.torvergata.it/bitstream/2108/100654/2/copie](https://art.torvergata.it/bitstream/2108/100654/2/copie)
[29] J. L. Romero, J. T. van Velthoven, *The density theorem for discrete series representations restricted to lattices*, Expo. Math., 40, 265-301, (2022). [https://doi.org/10.1016/j.exmath.2021.10.001](https://doi.org/10.1016/j.exmath.2021.10.001)
[30] P. J. Sally, *Intertwining operators and the representations of SL(2,$\mathbb{R}$)*, J. Funct. Anal., 6.3, 441-453, (1970).
[31] K Seip, *Beurling type density theorems in the unit disk*, Invent. Math., 113 (1), 21-39, (1993).

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