A PATH INTEGRAL APPROACH TO CURRENT

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ABSTRACT

Discontinuous initial wave functions or wave functions with discontinuous derivative and with bounded support arise in a natural way in various situations in physics, in particular in measurement theory. The propagation of such initial wave functions is not well described by the Schrödinger current which vanishes on the boundary of the support of the wave function. This propagation gives rise to a uni-directional current at the boundary of the support. We use path integrals to define current and uni-directional current and give a direct derivation of the expression for current from the path integral formulation for both diffusion and quantum mechanics. Furthermore, we give an explicit asymptotic expression for the short time propagation of initial wave function with compact support for both the cases of discontinuous derivative and discontinuous wave function. We show that in the former case the probability propagated across the boundary of the support in time $\Delta t$ is $O\left(\Delta t^{3/2}\right)$ and the initial uni-directional current is $O\left(\Delta t^{1/2}\right)$. This recovers the Zeno effect for continuous detection of a particle in a given domain. For the latter case the probability propagated across the boundary of the support in time $\Delta t$ is $O\left(\Delta t^{1/2}\right)$ and the initial uni-directional current is $O\left(\Delta t^{-1/2}\right)$. This is an anti-Zeno effect. However, the probability propagated across a point located at a finite distance from the boundary of the support is $O\left(\Delta t\right)$. This gives a decay law.
1. Introduction

Detection of the coordinate of a quantum particle can give the negative result that the particle is not in a given domain, for example, it is not in a half space bounded by a given plane. This means that at the moment of measurement the wave function of the undetected particle vanishes identically in the given domain, for example, beyond the separating plane. It follows that the wave function of the particle is supported outside the measured domain. The boundary of the support is the boundary of the domain, for example, it can be the plane separating the measured half space from the other half. The instantaneous killing of the wave function in a domain \( G \) for a given time interval \([t_0, t_0 + \Delta t]\) can be accomplished mathematically by introducing in the Schrödinger equation the (dimensionless) time dependent potential

\[
V(x, t) = \infty \cdot \chi_G(x) \chi_{[t_0, t_0+\Delta t]}(t)
\]

where

\[
\chi_G(x) = \begin{cases} 
1 & \text{if } x \in G \\
0 & \text{otherwise.}
\end{cases}
\]

After the instantaneous negative detection the particle in the time interval \([t_0, t_0 + \Delta t]\) it is free to propagate into the domain, that is, the potential \( V(x, t) \) is turned off instantaneously. A discontinuous wave function with compact support can be realized mathematically by introducing a similar potential that contains the spatial derivative of the delta function on the boundary of the domain and an infinite potential inside the domain.

The propagation of the wave function of the particle after the negative measurement is not well described by the Schrödinger current because the compactly supported initial wave function after the measurement gives rise to an initially vanishing Schrödinger current at the boundary of the support. Indeed, assuming the wave function is continuous, since

\[
\psi(x, t)|_{\partial G} = 0, \quad \text{for all } t_0 \leq t \leq t_0 + \Delta t,
\]

where \( t_0 \) is the instant of the measurement and \( \partial G \) is the boundary of the measured domain, we must have

\[
J_{\partial G}(x, t) = \frac{\hbar}{m} \Im \psi(x, t) \nabla \bar{\psi}(x, t) = 0, \quad \text{for all } t_0 \leq t \leq t_0 + \Delta t,
\]

It is obvious, however, that despite the vanishing current on \( \partial G \) there is propagation across \( \partial G \) after the instant \( t_0 + \Delta t \). This is in general the case in Schrödinger’s equation if the initial wave function has compact support. The case of a discontinuous initial wave function is problematic as well.
It is the purpose of this paper to calculate the short time propagation of the wave function across the boundary of the support of the initial wave function. This propagation gives rise to an instantaneous uni-directional current into the measured domain $G$. Our main result is an explicit asymptotic expression for the wave function at time $t_0 + \Delta t$ for small $\Delta t$. Our analysis is one-dimensional, however the generalization to higher dimensions is straightforward.

From this expression, we find that for a continuous wave function the probability that propagates across $\partial G$ in time $\Delta t$ is $O\left(\Delta t^{3/2}\right)$ and the rate of propagation is $O\left(\Delta t^{1/2}\right)$. The latter means that the initial uni-directional current is $O\left(\Delta t^{1/2}\right)$. We also determine the probability $P_c$ that propagates in time $\Delta t$ beyond a distance $c$ from $\partial G$,

$$P_c = O\left(\left(\frac{\Delta t}{c}\right)^3\right).$$

This expansion is valid for $c \geq \sqrt{\alpha}$. For example, if $c = O\left(\Delta t^{1/3}\right)$, we obtain $P_c = O\left(\Delta t^2\right)$. For $c = O\left(\sqrt{\Delta t}\right)$ the propagated probability is $O\left(\Delta t^{3/2}\right)$. This gives an estimate on the dependence of the probability of detection on the size of the detector. Detection at a point or in a domain is discussed in the context of absorption in separate papers [1]-[4].

Our result recovers the Zeno effect [5, 6] that a quantum particle cannot arrive at a point under continuous observation (the particle is “frozen” or “reflected back” by the continuous observation). This result is in agreement with the result of [7] (see Section 6 for more detailed discussion).

For a discontinuous initial wave function, we find that the probability that propagates across $\partial G$ in time $\Delta t$ is $O\left(\Delta t^{1/2}\right)$ and the rate of propagation is $O\left(\Delta t^{-1/2}\right)$. The latter means that the initial uni-directional current is $O\left(\Delta t^{-1/2}\right)$ and is initially infinite. This result is an anti-Zeno effect [8]. The probability $P_c$ for this case is given by

$$P_c = O\left(\frac{\Delta t}{c}\right).$$

This implies that the survival probability decays exponentially (see eq. (5.1)).

Wave functions with compact support and discontinuous derivatives appear in other applications as well [2, 9]. They represent actual physical situations and are of both theoretical and practical interest.

The notion of uni-directional current can be used to define the notion of “time of arrival” and measurement in quantum mechanics. A review of different approaches to the definition of uni-directional current related to time of arrival in quantum mechanics is given in [10, 11]. Our approach to the definition of a uni-directional current is different from all the above mentioned approaches. It is defined only for wave functions that vanish identically beyond the detector, that is, its support is initially bounded by the detector.
and is defined only at the boundary of the support. At such points the uni-directional current and the current are the same.

Our approach to the calculation of the uni-directional current is based on a Feynman integral approach. This seems to be a new approach that is fundamentally different than the above mentioned approaches in that it is not approximate and gives the full asymptotic behavior of the wave function for short times. A direct derivation of the Schrödinger current from the Feynman integral is given and extended to the case of a uni-directional current.

In this approach uni-directional currents appear when the wave function satisfies the following conditions,

- The initial wave function has compact support.
- At the boundary of the support either the wave function or its derivative has a discontinuity.

Initial wave functions that satisfy these conditions are denoted ICSWF (initial compact support wave function). In general, the solutions of Schrödinger’s equation do not develop discontinuities or non-smoothness. However, there are situations where non-smoothness does occur [9]. If an infinite potential barrier is introduced, the gradient of the wave function has a discontinuity across the barrier. Indeed, in our formalism for the description of an absorbing wall such a discontinuity arises in a natural way.

In Sections 2, the case of compact support in diffusion theory is reviewed. The Wiener path integral formulation of diffusion is shown to lead to the classical definitions of diffusion current and to a uni-directional diffusion current at an absorbing boundary. This derivation seems to be new. In Section 3, a Feynman integral formulation is used to define quantum probability current and a new direct derivation of the Schrödinger current from Feynman’s integral is given. This definition leads to the definition of a uni-directional current at the boundary of the support of the initial wave function. In Section 4, the main results of the paper are presented. These are short time asymptotic expansions for the propagation of an ICSWF and the uni-directional current. Finally, Section 6 contains a discussion and summary of the results.

2. Current and uni-directional current in diffusion

We consider a diffusion process, \( x(t) \), with noise coefficient \( \sigma(x, t) \) and drift \( b(x, t) \). The transition probability density of the process is denoted \( p(y, t \mid x, s) \). This is the probability density

\[
p(y, t \mid x, s) = \frac{\partial}{\partial y} \Pr \{ x(t) < y \mid x(s) = x \}.
\]
It satisfies the Fokker-Planck equation
\[
\frac{\partial}{\partial t} p(y, t \mid x, s) = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[ \sigma^2(y, t) p(y, t \mid x, s) \right] - \frac{\partial}{\partial y} \left[ b(y, t) p(y, t \mid x, s) \right]
\] (2.1)
with the initial condition
\[
\lim_{t \downarrow s} p(y, t \mid x, s) = \delta(y - x).
\] (2.2)
At an absorbing boundary at \( y = 0 \), say, the function \( p(y, t \mid x, s) \) satisfies the boundary condition
\[
p(0, t \mid x, s) = 0.
\] (2.3)

The uni-directional probability current (flux) density at a point \( x_1 \) is the probability per unit time of diffusing trajectories that propagate from the ray \( x < x_1 \) into the ray \( x > x_1 \). It is therefore given by
\[
J_{LR}(x_1, t) = \lim_{\Delta t \to 0} J_{LR}(x_1, t, \Delta t),
\] (2.4)
where
\[
J_{LR}(x_1, t, \Delta t) = \\
\frac{1}{\Delta t} \int_{x_1}^{\infty} dx \int_{-\infty}^{x_1} dy \frac{dy}{\sqrt{2\pi \Delta t \sigma(y, t)}} \exp \left\{ -\frac{[x - y - b(y, t) \Delta t]^2}{2\sigma^2(y, t) \Delta t} \right\} p(y, t - \Delta t)
\] (2.5)
(note that the dependence of \( p \) on the backward variables \( x, s \) has been suppressed). The integral (2.5) can be calculated by the Laplace method [13] at the saddle point \( x = y = x_1 \). First, we change variables in (2.5) to
\[
x = x_1 + \xi \sqrt{\Delta t}, \quad y = x_1 - \eta \sqrt{\Delta t}
\]
to obtain
\[
J_{LR}(x_1, t, \Delta t) = \int_0^\infty d\xi \int_0^\infty \frac{d\eta}{\sqrt{2\pi \Delta t \sigma(x_1 - \eta \sqrt{\Delta t}, t)}} \times \\
\exp \left\{ -\frac{[\xi + \eta - b(x_1 - \eta \sqrt{\Delta t}, t) \sqrt{\Delta t}]^2}{2\sigma^2(x_1 - \eta \sqrt{\Delta t}, t)} \right\} p(x_1 - \eta \sqrt{\Delta t}, t - \Delta t)
\] and changing the variable in the inner integral to \( \eta = \zeta - \xi \), we get
\[
J_{LR}(x_1, t, \Delta t) = \int_0^\infty d\xi \int_\xi^\infty \frac{d\zeta}{\sqrt{2\pi \Delta t \sigma(x_1 - (\zeta - \xi) \sqrt{\Delta t}, t)}} \times \\
\exp \left\{ -\frac{[\zeta - b(x_1 - (\zeta - \xi) \sqrt{\Delta t}, t) \sqrt{\Delta t}]^2}{2\sigma^2(x_1 - (\zeta - \xi) \sqrt{\Delta t}, t)} \right\} p(x_1 - (\zeta - \xi) \sqrt{\Delta t}, t - \Delta t).
\] (2.6)
Next, we expand the exponent in powers of $\sqrt{\Delta t}$ to obtain
\[
\left[ \zeta - b(x_1 - (\zeta - \xi) \sqrt{\Delta t}, t) \sqrt{\Delta t} \right]^2 = \frac{\zeta^2}{2\sigma^2(x_1, t)} + \left[ \frac{\zeta^2 (\zeta - \xi) \sigma^2(x_1, t)}{2 \sigma^4(x_1, t)} - \frac{\zeta b(x_1, t)}{\sigma^2(x_1, t)} \right] \sqrt{\Delta t} + O(\Delta t),
\]
the pre-exponential factor,
\[
\frac{1}{\sigma(x_1 - (\zeta - \xi) \sqrt{\Delta t}, t)} = \frac{1}{\sigma(x_1, t)} \left[ 1 + \frac{\sigma'(x_1, t)}{\sigma(x_1, t)} (\zeta - \xi) \sqrt{\Delta t} + O(\Delta t) \right],
\]
and the pdf
\[
p \left( x_1 - (\zeta - \xi) \sqrt{\Delta t}, t - \Delta t \right) = p(x_1, t) - \frac{\partial p(x_1, t)}{\partial t} (\zeta - \xi) \sqrt{\Delta t} - \frac{\partial p(x_1, t)}{\partial t} \xi \Delta t + O(\Delta t^{3/2}).
\]
Using the expansions (2.7)-(2.9) in (2.6), we obtain
\[
J_{LR}(x_1, t, \Delta t) = \int_0^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \frac{\zeta^2}{2\pi \Delta t \sigma(x_1, t)} \exp \left\{ -\frac{\zeta^2}{2\sigma^2(x_1, t)} \right\} \left\{ p(x_1, t) - \frac{\partial p(x_1, t)}{\partial t} \right\} \Delta t
\]
\[
= -\sqrt{\Delta t} \left[ \left( \frac{\zeta^2 (\zeta - \xi) \sigma'(x_1, t)}{\sigma^3(x_1, t)} - \frac{\zeta b(x_1, t)}{\sigma^2(x_1, t)} - \frac{\sigma'(x_1, t)(\zeta - \xi)}{\sigma(x_1, t)} \right) p(x_1, t)
\]
\[
+ (\zeta - \xi) p'(x_1, t) \right] + O(\Delta t).
\]
Similarly,
\[
J_{RL}(x_1, t) = \lim_{\Delta t \to 0} J_{RL}(x_1, t, \Delta t),
\]
where
\[
J_{RL}(x_1, t, \Delta t) = \frac{1}{\Delta t} \int_{-\infty}^{x_1} dx \int_{x_1}^{\infty} dy \frac{y}{\sqrt{2\pi \Delta t \sigma(y, t)}} \exp \left\{ -\frac{[x - y - b(y, t) \Delta t]^2}{2\sigma^2(y, t) \Delta t} \right\} p(y, t - \Delta t).
\]
The change of variables in (2.11)
\[
x = x_1 - \xi \sqrt{\Delta t}, \quad y = x_1 + \eta \sqrt{\Delta t}
\]
Since \( p \) gives the order of integration, in deriving eq. (2.13) use has been made of the following identities, obtained by changing the order of integration

\[
J_{\text{Fokker-Planck}}(x, t) = \int_0^\infty \frac{d\zeta}{\sqrt{2\pi \Delta t}} \sigma(x, t) \exp \left\{ -\frac{\zeta^2}{2\sigma^2(x, t)} \right\} \left\{ p(x, t) - \frac{\partial p(x, t)}{\partial t} \Delta t \right\} 
\]

\[
+ \sqrt{\Delta t} \left[ \left( \frac{\zeta^2 (\zeta - \xi) \sigma'(x, t)}{\sigma^3(x, t)} \right) - \frac{\zeta b(x, t)}{\sigma^2(x, t)} - \frac{\sigma'(x, t) (\zeta - \xi)}{\sigma(x, t)} \right] p(x, t) 
\]

\[
- (\zeta - \xi) p'(x, t) \right] + O(\Delta t).
\]

Since \( p(x, t) > 0 \), both \( J_{LR}(x, t) \) and \( J_{RL}(x, t) \) are infinite, however, the net flux density is finite and is given by

\[
J_{\text{net}}(x, t) = \lim_{\Delta t \to 0} \{ J_{LR}(x, t, \Delta t) - J_{RL}(x, t, \Delta t) \} = 
\]

\[
-2 \int_0^\infty \frac{d\xi}{\sqrt{2\pi \Delta t}} \sigma(x, t) \exp \left\{ -\frac{\zeta^2}{2\sigma^2(x, t)} \right\} \times 
\]

\[
\left[ \left( \frac{\zeta^2 (\zeta - \xi) \sigma'(x, t)}{\sigma^3(x, t)} \right) - \frac{\zeta b(x, t)}{\sigma^2(x, t)} - \frac{\sigma'(x, t) (\zeta - \xi)}{\sigma(x, t)} \right] p(x, t) + (\zeta - \xi) p'(x, t) \right] 
\]

\[
= -\frac{\partial}{\partial x} \left. \frac{\sigma^2(x, t)}{2} p(x, t) + b(x, t) p(x, t) \right| _{x=x_1} \right. 
\]

(2.13)

In deriving eq. (2.13) use has been made of the following identities, obtained by changing the order of integration,

\[
\int_0^\infty \frac{d\xi}{\xi} \frac{\zeta^2 (\zeta - \xi) \frac{d\zeta}{\sqrt{2\pi \sigma}} \exp \left\{ -\frac{\zeta^2}{2\sigma^2} \right\}}{\sqrt{2\pi \sigma}} = \int_0^\infty \frac{\zeta^4 d\zeta}{2\sqrt{2\pi \sigma}} \exp \left\{ -\frac{\zeta^2}{2\sigma^2} \right\} = \frac{3\sigma^4}{4} 
\]

\[
\int_0^\infty \frac{d\xi}{\xi} \frac{\zeta \frac{d\zeta}{\sqrt{2\pi \sigma}} \exp \left\{ -\frac{\zeta^2}{2\sigma^2} \right\}}{\sqrt{2\pi \sigma}} = \int_0^\infty \frac{\zeta^2 d\zeta}{\sqrt{2\pi \sigma}} \exp \left\{ -\frac{\zeta^2}{2\sigma^2} \right\} = \frac{\sigma^2}{2} 
\]

\[
\int_0^\infty \frac{d\xi}{\xi} \frac{(\zeta - \xi) \frac{d\zeta}{\sqrt{2\pi \sigma}} \exp \left\{ -\frac{\zeta^2}{2\sigma^2} \right\}}{\sqrt{2\pi \sigma}} = \frac{1}{2} \int_0^\infty \frac{\zeta^2 d\zeta}{\sqrt{2\pi \sigma}} \exp \left\{ -\frac{\zeta^2}{2\sigma^2} \right\} = \frac{\sigma^2}{4}. 
\]

Equation (2.13) is the classical expression for the probability (or heat) current in diffusion theory [4]. The Fokker-Planck equation (2.1) can be written in terms of the flux density function \( J(x, t) \) in the conservation law form

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} J(x, t). 
\]

(2.14)

Next, we calculate the uni-directional flux at the absorbing boundary \( x = 0 \). The absorbing boundary condition (2.3) implies that the pdf vanishes for all \( x \geq 0 \) so that its right derivatives at the origin vanish. It follows from eq. (2.11) that

\[
J_{RL}(0, t) = 0.
\]
On the other hand, eqs. (2.3) and (2.13) give
\[ J(0, t) = J_{LR}(0, t) = -\left. \frac{\partial}{\partial x} \frac{\sigma^2(x, t)}{2} p(x, t) \right|_{x=0}. \]
Since \( \sigma^2(x, t) > 0 \) and \( p(x, t) > 0 \) for \( x < 0 \), but \( p(0, t) = 0 \), it follows that \( J(0, t) > 0 \). This means that there is positive flux into the absorbing boundary so that the probability of trajectories that survive in the region to the left of the absorbing boundary, \( \int_{-\infty}^{0} p(x, t) \, dx \), must be a decreasing function of time. This can be seen directly from eq. (2.14) by integrating it with respect to \( x \) over the ray \( (-\infty, 0) \) and using the fact that \( \lim_{x \to -\infty} J(x, t) = 0 \),
\[ \frac{d}{dt} \int_{-\infty}^{0} p(x, t) \, dx = -J(0, t) < 0. \]

3. Current and uni-directional current in Feynman integrals

To keep the calculations simple, we consider a particle with an infinite potential for \( x > 0 \) and its free propagation after the infinite potential is turned off at time \( t = 0 \). There is no analog to uni-directional current in Feynman integrals due to the non-additivity of probability on sets of trajectories. First, we examine the notion of current in the usual Feynman integral. The current at a point is the net rate of change of probability on one side of the point. That is, the current at \( x = 0 \), say, is
\[ J(0, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{0} \left[ |\psi(x, t + \Delta t)|^2 - |\psi(x, t)|^2 \right] \, dx. \] (3.1)
According to the Feynman formalism, we write
\[ \psi(x, t + \Delta t) = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \int_{-\infty}^{\infty} \psi(y, t) \exp \left\{ \frac{im}{2\hbar \Delta t} (x - y)^2 \right\} \, dy \]
so that
\[ \int_{-\infty}^{0} |\psi(x, t + \Delta t)|^2 \, dx = \left. \int_{-\infty}^{\infty} \left| \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \int_{-\infty}^{\infty} \psi(y, t) \exp \left\{ \frac{im}{2\hbar \Delta t} (x - y)^2 \right\} \, dy \right|^2 \, dx. \]
Thus the current is the difference between the population of trajectories that propagates in the time interval \((t, t + \Delta t)\) from the entire line into the interval \((-\infty, 0)\) and the population there at time \( t \).

Expanding
\[ \psi(y, t) = \psi(x, t) + (y - x)\psi_x(x, t) + \frac{1}{2}(y - x)^2\psi_{xx}(x, t) + \cdots, \] (3.2)
we obtain

\[ \int_{-\infty}^{0} \left| \int_{-\infty}^{\infty} \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \psi(y, t) \exp \left\{ \frac{im}{2\hbar \Delta t} (x-y)^2 \right\} dy \right|^2 dx = (3.3) \]

\[ \int_{-\infty}^{0} \left( |\psi(x, t)|^2 + \Re \psi(x, t) \bar{\psi}_{xx}(x, t) \frac{i\hbar \Delta t}{m} \right) dx + o(\Delta t). \] (3.4)

It follows from integration by parts that

\[ J(0) = \frac{\hbar}{m} \Im \int_{-\infty}^{0} \psi(x, t) \bar{\psi}_{xx}(x, t) dx = \frac{\hbar}{m} \Im \psi(0, t) \bar{\psi}_x(0, t), \] (3.5)

because \( \psi_x(0, t) \bar{\psi}_x(0, t) \) is real valued. Equation (3.5) is identical to the Schrödinger current. This derivation seems to be new.

Next, we consider the case that the wave function at time \( t \) vanishes outside the interval \([-a, 0]\). As mentioned above, this situation arises, for example, if up to time \( t \) there is an infinite potential for \( x > 0 \) and \( x < -a \) and the potential for \( x > 0 \) is turned off instantaneously at time \( t \). This is a mathematical idealization of various physical situations (see, e.g., Chapter 6). Since \( \psi(0, t) = 0 \), the Schrödinger current at the point \( x = 0 \) vanishes at time \( t \), according to eq.(3.3). Yet, there is probability flux across \( x = 0 \). To see this, we evaluate the rate of population change, (3.1). In the case at hand the second term in the integrand of (3.1) vanishes at time \( t \). We assume first that \( x = -a \) is a reflecting wall, that is, \( \psi(x, t) = 0 \) for \( x \leq -a \) for some positive \( a \). Thus there is no propagation across \( x = -a \). The case \( a = \infty \) is considered in Section 5 below.

The probability of propagating from a given interval \([-a, 0]\) into the ray \([0, \infty)\) in the time interval \((t, t + \Delta t)\), starting with the wave function \( \psi(x, t) \) in the interval \([-a, 0]\) and 0 outside, is given by

\[ P_{\text{out}} = \int_{0}^{\infty} |\psi(y, t + \Delta t)|^2 dy. \] (3.6)

The wave function at time \( t + \Delta t \) is given by the free propagator

\[ \psi(y, t + \Delta t) = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \int_{-a}^{0} \psi(x, t) \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} \right\} dx. \] (3.7)

The uni-directional current from \([-a, 0]\) into the ray \([0, \infty)\) is the current

\[ J_{LR}(0, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{0}^{\infty} |\psi(x, t + \Delta t)|^2 dx. \] (3.8)

Note that if trajectories are not truncated for \( x > 0 \), the wave function at time \( t + \Delta t \) no longer vanishes at \( x = 0 \) and so does the Schrödinger current.
4. Short time propagation

To estimate the uni-directional current $J_{LR}(0, t)$ for short times, we first estimate the integral in eq. (3.8) for small $\Delta t$. We begin with an initial wave function $\psi(x, 0)$ that is a polynomial

$$Q(x) = \sum_{j=1}^{N} q_j x^j$$

in the interval $[-a, 0]$, such that $Q(-a) = Q(0) = 0$ and $\psi(x, 0) = 0$ otherwise, the free propagation from the interval $[-a, 0]$ is given by

$$\psi(y, \Delta t) = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \int_{-a}^{0} Q(x) \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} \right\} dx.$$ 

The boundary condition at the left end of the support of $\psi(x, 0)$ is written explicitly as

$$\sum_{j=1}^{N} q_j (-a)^j = 0.$$ 

Setting

$$\alpha = \frac{\hbar \Delta t}{m},$$

the probability mass propagated freely into the positive axis in time $\Delta t$ is given by

$$\int_{0}^{\infty} |\psi(y, t + \Delta t)|^2 dy = \frac{1}{2\pi} \int_{0}^{\infty} \left| \int_{-a}^{0} Q(x) e^{i(x-y)^2/2\alpha} dx \right|^2 dy.$$ 

We change variables by setting $x = \sqrt{\alpha} \xi$, $y = \sqrt{\alpha} \eta$, $\xi = \zeta + \eta$ to get

$$\int_{0}^{\infty} |\psi(y, t + \Delta t)|^2 dy = \frac{\alpha^{1/2}}{2\pi} \int_{0}^{\infty} \left| \int_{-a/\sqrt{\alpha} - \eta}^{-\eta} Q \left( \sqrt{\alpha} (\zeta + \eta) \right) e^{i\zeta^2/2} d\zeta \right|^2 d\eta.$$ 

First, we evaluate the inner integral,

$$I_N(\eta) = \sum_{j=1}^{N} q_j \sqrt{\alpha^j} \int_{-a/\sqrt{\alpha} - \eta}^{-\eta} (\zeta + \eta)^j e^{i\zeta^2/2} d\zeta.$$ 

Integration by parts gives

$$I_N(\eta) = -i \sum_{j=1}^{N} q_j \sqrt{\alpha^j} \int_{-a/\sqrt{\alpha} - \eta}^{-\eta} (\zeta + \eta)^j \frac{d}{d\zeta} e^{i\zeta^2/2} d\zeta = -i \sum_{j=1}^{N} q_j \sqrt{\alpha^j} \left[ \frac{(-a/\sqrt{\alpha} + \eta)^{j+1}}{a/\sqrt{\alpha} + \eta} \right] + i \sum_{j=1}^{N} q_j \sqrt{\alpha^j} \int_{-a/\sqrt{\alpha} - \eta}^{-\eta} e^{i\zeta^2/2} \frac{d}{d\zeta} \left( \frac{(\zeta + \eta)^j}{\zeta} - \frac{(\zeta + \eta)^j}{\zeta^2} \right) d\zeta.$$ 

First, we evaluate the inner integral,
because
\[ \sum_{j=1}^{N} q_j \sqrt{\alpha^j \left( -\frac{a}{\sqrt{\alpha}} \right)^j} = \sum_{j=1}^{N} q_j \left( -\frac{a}{\sqrt{\alpha}} \right)^j = 0. \]

For \( j = 1 \), we obtain
\[-i q_1 \sqrt{\alpha} \eta \int_{-\eta}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta. \]

The function
\[ \eta \int_{-\eta}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \]
is square integrable and its integral is independent of \( \alpha \) to leading order. Indeed, the integral
\[ \int_{-\infty}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \]
exists and near \( \eta = 0 \) it is bounded by
\[ \left| \int_{-\infty}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \right| \leq \int_{-\infty}^{-\eta} \frac{d\zeta}{\zeta^2} = \frac{1}{\eta} \]
so that
\[ \left| \eta \int_{-\eta}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \right| \leq \eta \int_{-\infty}^{-\eta} \frac{d\zeta}{\zeta^2} = 1. \]

For large \( \eta \), we have the asymptotic limit
\[ \int_{-\infty}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \sim i \frac{e^{i\eta^2/2}}{\eta^3}, \]
as is easily seen from l’Hôpital’s rule, so that
\[ \left| \eta \int_{-\eta}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \right|^2 \leq \frac{1}{\eta^4}. \]

This means that the function \( \left| \eta \int_{-\eta}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \right|^2 \) is integrable. It follows that its contribution to the integral (4.1) is to leading order
\[ \frac{\alpha^{3/2}}{2\pi} |q_1|^2 \int_{0}^{\infty} \left| \eta \int_{-\infty}^{-\eta} e^{i\zeta^2/2} \frac{e^{i\zeta^2/2}}{\zeta^2} d\zeta \right|^2 \eta \, d\eta = O \left( \alpha^{3/2} \right). \quad (4.3) \]

Now, we consider the term \( j = 2 \):
\[ iq_2 \alpha \int_{-\eta}^{-\eta} e^{i\zeta^2/2} \left( 1 - \frac{\eta^2}{\zeta^2} \right) d\zeta. \]
Proceeding as above, we find that both terms in the integral are uniformly square integrable functions of $\eta$ for all $\alpha > 0$ sufficiently small. It follows that the contribution of the term $j = 2$ to the integral (4.1) is $O\left(\alpha^{5/2}\right)$. The mixed term involving $q_1q_2$ contributes $O(\alpha^3/2)$.

Next, we consider the third order term:

$$\int_{-\eta}^{\eta} (\zeta + \eta)^2 \frac{2\zeta - \eta}{\zeta^3} d\zeta e^{i\zeta^2/2} =$$

$$= \left( -\frac{a}{\sqrt{\alpha}} \right)^2 \frac{2(a/\sqrt{\alpha} + \eta) - \eta}{(-a/\sqrt{\alpha} - \eta)^3} e^{i(a/\sqrt{\alpha} + \eta)^2/2} - \int_{-a/\sqrt{\alpha} - \eta}^{-\eta} e^{i\zeta^2/2} \frac{d}{d\zeta} \left[ (\zeta + \eta)^2 \frac{2\zeta - \eta}{\zeta^3} \right] d\zeta$$

$$= O\left(\frac{1}{\alpha}\right).$$

(4.4)

This term has a pre-factor of $q_3\alpha^{3/2}$ so that its contribution to (4.1) is $O\left(\alpha^{3/2}\right)$. Proceeding by induction, we find that all terms contribute $O\left(\alpha^{3/2}\right)$ to (4.1). It follows that

$$\int_0^\infty |\psi(y, t + \Delta t)|^2 dy = O\left(\left(\frac{\hbar \Delta t}{m}\right)^{3/2}\right).$$

(4.5)

If $a = \infty$ and all Fresnel-type integrals are interpreted as the limits

$$\int_0^y f(x)e^{ix^2/2} dx = \lim_{\epsilon \downarrow 0} \int_{-\infty}^y f(x)e^{(-\epsilon + i)x^2/2} dx,$$

the term $O(1/\alpha)$ in eq.(4.4) is replaced by $O(1)$. It follows that the higher order terms of the polynomial contribute higher order terms in the expansion of $I_N$ in powers of $\alpha$.

Obviously, if the polynomial is replaced by an analytic function that vanishes at the ends of the interval, the result remains unchanged. Furthermore, if $Q(x)$ is a square integrable analytic function on the negative axis, eq.(4.5) holds.

The asymptotic estimate (4.7) is valid when higher order terms can be neglected relative to lower order terms. To get an explicit bound on $\Delta t$ from this condition, we write $\psi(x, t)$ as a series of eigenfunctions

$$\psi(x, t) = \sum_{j=1}^{\infty} \psi_j(x) \exp \left\{-\frac{iE_n t}{\hbar}\right\},$$

(4.6)

where $\psi_n(x)$ are eigenfunctions that satisfy the boundary condition $\psi_n(0) = 0$ and $E_n$ are the corresponding eigen energies, to find that

$$\frac{\partial^j \psi(0,t)}{\partial x^j} = j!q_j(t)$$
and
\[
\left| \frac{\partial^{2j+1} \psi_{n}(0)}{\partial x^{2j+1}} \right| = \left| \frac{2mE_n}{\hbar^2} \right| \left| \frac{\partial \psi_{n}(0)}{\partial x} \right| \tag{4.7}
\]
\[
\left| \frac{\partial^{2j} \psi_{n}(0)}{\partial x^{2j}} \right| = \left| \frac{2mE_n}{\hbar^2} \right| \left| \psi_{n}(0) \right| = 0.
\]

The asymptotic evaluation of the integrals for \(y\) near or at 0 gives that the coefficient of \(q_{j}\) in the expansion is \(O\left(\left(\frac{\hbar \Delta t}{m}\right)^{j+1/2}\right)\). It follows that the condition for the validity of the expansion is that
\[
\frac{\hbar \Delta t}{m} \ll \left| \frac{q_{2j+1}(t)}{q_{2j+3}(t)} \right| \tag{4.8}
\]
for all \(j \geq 0\). Using eqs. (4.6)-(4.7) in (4.8), we obtain that the condition for the validity of the expansion is
\[
\frac{\hbar \Delta t}{m} \ll \left| \frac{(2j+1)(2j+3)}{\sum_{n=1}^{\infty} \left( \frac{-2mE_n}{\hbar^2} \right)^j \frac{\partial \psi_{n}(0)}{\partial x} e^{-iE_n t/\hbar} \right| \tag{4.9}
\]
for all \(j \geq 1\).

If, for example, the initial wave function is a single eigenfunction, the condition (4.9) reduces to
\[
E_n \Delta t \ll \hbar. \tag{4.10}
\]
The analysis of the continuous spectrum case is identical. The summation with respect to \(n\) in the condition (4.9) is replaced by integration with respect to \(n\).

It follows from eq. (4.5) that for short times
\[
J_{LR}(0, t) = O(\sqrt{t})
\]
so that the population in \(y > 0\) increases as \(O(t^{3/2})\) for short times. Obviously, once \(\psi(0, t) \neq 0\), the current becomes the usual Schrödinger current.

The probability mass that propagates in time \(\Delta t\) beyond a fixed point \(c > 0\) can be found from the above expansions. This probability is defined as
\[
P_c = \int_c^{\infty} |\psi(y, t + \Delta t)|^2 dy = \frac{\alpha^{1/2}}{2\pi} \int_{c/\sqrt{\alpha}}^{\infty} \int_{-\eta}^{\eta} Q\left(\sqrt{\alpha} (\zeta + \eta)\right) e^{i\zeta^2/2} d\zeta \ d\eta, \tag{4.11}
\]
rather than (4.1). The individual terms in the expansion (4.2) are estimated as above with the obvious changes. Inequality (4.3) becomes
\[
\frac{\alpha^{3/2}}{2\pi} |q_1|^2 \int_{c/\sqrt{\alpha}}^{\infty} \int_{-\eta}^{\eta} e^{i\zeta^2/2} d\zeta \ d\eta = O\left(\alpha^3\right). \tag{4.12}
\]
The same estimate applies to all terms in the expansion. It follows that

\[ P_c = O \left( \left( \frac{\alpha}{c} \right)^3 \right). \] (4.13)

This expansion is valid for \( c \geq \sqrt{\alpha} \). For example, if \( c = O \left( \alpha^{1/3} \right) \), we obtain \( P_c = O (\alpha^2) \). For \( c = O \left( \sqrt{\alpha} \right) \) the result (4.5) is recovered.

Now, we consider an initially discontinuous wave function

\[ \psi(x, 0) = \begin{cases} Q(x) & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}, \]

where \( Q(x) = \sum_{j=0}^{N} q_j x^j \) vanishes at \( x = -a \) and \( Q(0) \neq 0 \). First, we consider the propagation of the term \( q_0 \). Its contribution to the propagated wave function is

\[ \frac{\alpha^{1/2}}{2\pi} \int_0^\infty \left| q_0 \int_{-a/\sqrt{\alpha}}^{-\eta} e^{i\zeta^2/2} d\zeta \right|^2 d\eta = O \left( \alpha^{1/2} \right). \]

This gives rise to an infinite uni-directional current at the point of discontinuity. Next, we calculate the probability propagated beyond a distance \( c > 0 \) from the discontinuity. Again, the term \( q_0 \) contributes

\[ P_c = \frac{\alpha^{1/2}}{2\pi} \int_{c/\sqrt{\alpha}}^\infty \left| q_0 \int_{-a/\sqrt{\alpha}}^{-\eta} e^{i\zeta^2/2} d\zeta \right|^2 d\eta = \frac{|q_0|^2 \alpha}{2\pi c} + O \left( \frac{\alpha^{3/2}}{c} \right). \] (4.14)

It follows that for \( c = O \left( \alpha^{1/2-\varepsilon} \right) \) the propagated probability is \( O \left( \alpha^{1/2+\varepsilon} \right) \), so that the resulting uni-directional current is infinite for \( 0 < \varepsilon < 1/2 \). If \( c = O (1) \), the propagated probability is \( O (\Delta t) \) and gives rise to a finite current.

The analysis of the propagation of a polynomial shows that the other terms in the polynomial contribute higher order terms to the propagated probability.

5. Discussion and summary

The expression (4.5) can be applied to the following experiment. A particle with energy \( E_n \) is released between two perfectly reflecting walls placed at \( x = 0 \) and \( x = -a \) (\( a > 0 \)). A perfectly absorbing detector is placed at \( x_0 > 0 \). The reflecting wall at \( x = 0 \) is removed instantaneously for a time interval of length \( \Delta t \) and is then instantaneously reinstated. If the particle crosses 0 in this time interval, it gets registered by the detector. According to eq. (4.3), if this experiment is repeated \( N \) times, the number of particles registered by the detector will be proportional to \( N (\Delta t)^{3/2} \), if \( \Delta t \) satisfies the condition (4.10).
Note that the condition \((4.10)\) in this context has no relation to the energy-time uncertainty principle because \(\Delta t\) is not related to a measurement of the particle’s energy. The removal and reinstatement of the reflecting wall changes the energy of the particle so that it is not known what it is after the wall is reinstated.

A possible physical approximate realization of this experiment consists in turning off and on again a detecting device, e.g., by illuminating the detection region (and using e.g., the Compton effect), a time interval of length \(\Delta t\). According to eq.\((4.5)\), the survival probability on the left of the barrier is

\[
S(\Delta t) = 1 - c (\Delta t)^{3/2},
\]

where \(c\) is a constant. If this experiment is run on \(N\) identical systems with \(N\Delta t = T\) and \(\Delta t\) satisfies the condition \((4.10)\), the probability that all survive by time \(\Delta t\), denoted \(S_T(N)\), is

\[
S_T(N) = \left(1 - c (\Delta t)^{3/2}\right)^N \approx \exp \left\{ - \frac{c}{\sqrt{N}} \right\} = 1 - \frac{c}{\sqrt{N}}, \quad N \gg 1.
\]

The expected number of systems that decay is

\[
\langle N \rangle = O\left(N\Delta t^{3/2}\right) = O\left(T\sqrt{\Delta t}\right) \to 0 \quad \text{for} \quad N \gg 1.
\]

The result \((4.5)\) is similar to that obtained in time dependent perturbation theory, known as the Zeno effect [5, 7], where the probability that an irreversible decay will occur before a short time \(t\) is \(O(t^2)\). As in the Zeno effect, the law \((4.5)\) indicates that propagation into a detection region under continuous observation makes it impossible for a particle to be observed. This phenomenon is referred to as the freezing of a particle in its initial state. This apparent paradox disappears in quantum theory with a measuring device, as shown in [2, 4].

We return now to the ideal detection experiment by illuminating the detection region \(x > x_0\) for a short time \(\Delta t\) at time intervals \(\Delta t\) apart. The detection probability is \(P_c\), given in eq.\((4.11)\). If \(x_0 = 0\), the detection probability is \(O\left(\alpha^{3/2}\right)\) per illumination pulse. The result remains the same if the illuminated region is the interval \((0, c)\), where \(c = O\left(\sqrt{\alpha}\right)\). If, however, one illumination pulse covers the region \(x > 0\) and the following one covers \(x > c\), where \(c = O(1)\) for \(\alpha\) satisfying eq.\((4.8)\) or, equivalently, \((4.10)\), the conditional probability of detecting the particle in the second pulse, given that it was not detected in the first one, is \(O(\alpha^3)\). The former result means that the width of the illuminated region has to be at least \(O\left(\sqrt{\alpha}\right)\) to achieve the maximal order of magnitude of the probability of detection per pulse.

The result eq.\((4.14)\) can be applied to the following ideal measurement experiment. A discontinuous wave function is created by introducing a potential \(V(x) = \delta'(x)\) [9]. If
the illumination region is \( x > 0 \), the probability of detection per pulse is \( O(\sqrt{\alpha}) \). If the regions \( x > 0 \) and \( x > c \) are illuminated alternatively, the conditional probability of detecting the particle in the second pulse, given that it was not observed during the first one, is \( O(\alpha) \), according to eq. (4.14). This means, that if in the second pulse the illumination of the region \( x > c \) is kept forever, the conditional survival probability \( S(t) \) (the probability of not observing the particle by time \( t \) after the beginning of the second pulse) is

\[
S(t) = O\left(\exp\left\{-\gamma \int_0^t |q_0(t')|^2 \, dt' \right\}\right)
\]  

(5.1)

for some \( \gamma > 0 \). The probability propagated across the boundary of the support in time \( \Delta t \) is \( O(\Delta t^{1/2}) \) and the initial uni-directional current is \( O(\Delta t^{-1/2}) \). This is an anti-Zeno effect [9]. It means that in \( N \) identical systems observed for time \( \Delta t = T/N \) the the probability that all survive by time \( \Delta t \) is

\[
S_T(N) = \left(1 - c(\Delta t)^{1/2}\right)^N \approx \exp\left\{-c\sqrt{N}\right\} \rightarrow 0 \quad \text{for } N \gg 1.
\]

The expected number of systems that decay is

\[
\langle N \rangle = O\left(N\Delta t^{1/2}\right) = O\left(\sqrt{T N}\right) \rightarrow \infty \quad \text{for } N \gg 1.
\]

It should be remarked that the leading order short time asymptotics is unaffected by the presence of a finite potential beyond the boundary of the support of the initial wave function. This suggests the possibility that the detection region beyond the support of the initial wave function can be characterized by a potential without essentially changing the above result.

In summary, we compared the notions of net and uni-directional fluxes in the Wiener and Feynman integrals. At points where the density does not vanish the uni-directional fluxes are infinite, though the net flux is finite and is given by the traditional expressions for flux in the diffusion and Schrödinger equations. At points where the density vanishes, for example at certain types of boundaries (absorbing for Wiener trajectories and at the boundary of the support of the wave function) the uni-directional fluxes are finite. In the Wiener integral the uni-directional flux at an absorbing boundary does not vanish, resulting in a decay of the total population at an exponential rate. In contrast, if a reflecting boundary for the Feynman integral is removed at time \( t = 0 \), the flux across the boundary increases as \( O(\sqrt{t}) \). If Feynman trajectories that propagate into a boundary are instantaneously absorbed there, the flux at such a boundary is proportional to the square of the local gradient of the wave function [3].

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