DETERMINATION OF NON-COMPACTLY SUPPORTED ELECTROMAGNETIC
POTENTIALS IN UNBOUNDED CLOSED WAVEGUIDE

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Abstract. We study the inverse problem of determining a magnetic Schrödinger operator in an unbounded closed waveguide from boundary measurements. We consider this problem with a general closed waveguide in the sense that we only require our unbounded domain to be contained into an infinite cylinder. In this context we prove the unique recovery of the magnetic field and the electric potential associated with general bounded and non-compactly supported electromagnetic potentials. By assuming that the electromagnetic potentials are known on the neighborhood of the boundary outside a compact set, we even prove the unique determination of the magnetic field and the electric potential from measurements restricted to a bounded subset of the infinite boundary. Finally, in the case of a waveguide taking the form of an infinite cylindrical domain, we prove the recovery of the magnetic field and the electric potential from partial data corresponding to restriction of Neumann boundary measurements to slightly more than half of the boundary. We establish all these results by mean of a new class of complex geometric optics solutions and of Carleman estimates suitably designed for our problem stated in an unbounded domain and with bounded electromagnetic potentials.

Keywords: Inverse problems, elliptic equations, electromagnetic potential, Carleman estimate, unbounded domain, closed waveguide, partial data.

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1. Introduction

1.1. Statement of the problem. Let \( \Omega \) be an unbounded open set of \( \mathbb{R}^3 \) corresponding to a closed waveguide. Here by closed waveguide we mean that there exists \( \omega \) a \( C^2 \) bounded open simply connected set of \( \mathbb{R}^2 \) such that the following condition is fulfilled

\[
\Omega \subset \omega \times \mathbb{R}.
\]

(1.1)

For \( A \in L^\infty(\Omega)^3 \), we define the magnetic Laplacian \( \Delta_A \) given by

\[
\Delta_A = \Delta + 2iA \cdot \nabla + id\text{iv}(A) - |A|^2.
\]

According to [19, Theorem 3.4 page 223], for any \( u \in H^1(\Omega) \) and \( \varphi \in C_0^\infty(\Omega) \), we have \( u\varphi \in W^{1,1}_0(\Omega) \), where \( W^{1,1}_0(\Omega) \) denotes the closure of \( C_0^\infty(\Omega) \) in \( W^{1,1}(\Omega) \). Therefore, using a density argument we can prove that, for any \( u \in H^1(\Omega) \) and \( A \in L^\infty(\Omega)^3 \), we have \( \text{div}(A)u \in D'(\Omega) \) and \( \Delta_A u \in D'(\Omega) \). Thus, for \( q \in L^\infty(\Omega; \mathbb{C}) \) and \( u \in H^1(\Omega) \), we can introduce the equation

\[
\Delta_A u + qu = 0, \quad \text{in } \Omega
\]

(1.2)

in the sense of distributions. Since we make no assumption on the boundary of \( \Omega \), in a similar way to [34], we define the trace map \( \tau \) on \( H^1(\Omega) \) by \( \tau u = [u] \) with \([u]\) the class of \( u \) in the quotient space \( H^1(\Omega)/H^1_0(\Omega) \), where \( H^1_0(\Omega) \) denotes the closure of \( C_0^\infty(\Omega) \) in \( H^1(\Omega) \). We associate to any solution \( u \in H^1(\Omega) \) of (1.2) the trace \( N_{A,q} u \in \left( \frac{H^1(\Omega)}{H^1_0(\Omega)} \right)' \), with \( \left( \frac{H^1(\Omega)}{H^1_0(\Omega)} \right)' \) the dual space of \( \frac{H^1(\Omega)}{H^1_0(\Omega)} \), defined by

\[
\langle N_{A,q} u, \tau g \rangle \left( \frac{H^1(\Omega)}{H^1_0(\Omega)} \right)' = -\int_\Omega (\nabla + iA)u \cdot (\nabla + iA)g dx + \int_\Omega qu \overline{g} dx, \quad g \in H^1(\Omega).
\]
Here, by using a density argument, one can prove that this map is well defined for \( u \) solving (1.2) since for \( g \in H^1_0(\Omega) \) the right hand side of this identity is equal to 0.

Recall that for \( \Omega = \omega \times \mathbb{R} \) one can identify \( H^1(\Omega) \) to \( H^1(\partial \omega \times \mathbb{R}) := L^2(\mathbb{R}; H^1(\partial \omega) \cap H^\frac{1}{2}(\mathbb{R}; L^2(\omega))). \) Then, for \( u \in H^1(\Omega) \) solving (1.2) and \( A \in W^{1,\infty}(\Omega)^3 \), we have \( \tau u = u_{|\partial \Omega} \) and

\[
N_{A,q}u = -\partial_\nu u = -\partial_\nu u - i(A \cdot \nu)u \in H^{1\frac{1}{2}}(\partial \omega \times \mathbb{R}) = (H^{1\frac{1}{2}}(\partial \omega \times \mathbb{R}))',
\]

with \( \nu \) the outward unit normal vector to \( \partial \omega \times \mathbb{R} \). This means that \( -N_{A,q} \) is the natural extension of the magnetic normal derivative in non smooth setting for general unbounded domains satisfying (1.1).

We introduce then the data

\[
D_{A,q} := \{ (\tau u, N_{A,q}u) : \ u \in H^1(\Omega), \ u \text{ solves (1.2)} \}.
\]  

Note that for \( \Omega = \omega \times \mathbb{R}, \ A \in W^{1,\infty}(\Omega)^3 \) and assuming that 0 is not in the spectrum of \( \Delta_A + q \) with Dirichlet boundary condition, \( D_{A,q} \) corresponds, up to the sign, to the graph of the so called Dirichlet-to-Neumann map associated with (1.2). In this paper we consider the simultaneous recovery of the magnetic field associated with \( A \) and \( q \) from the data \( D_{A,q} \). We consider both results with full and partial data.

1.2. Physical motivations. Let us first observe that the problem addressed in this paper is linked to the so called electrical impedance tomography (EIT in short) method and its applications in medical imaging and geophysical prospection (see [53] for more detail). The statement of the present inverse problem in an unbounded closed waveguide can be addressed in the context of problems of transmission to long distance or transmission through particular structures, with important ratio length-to-diameter, such as nanostructures. Here the goal of the inverse problem can be described as the unique recovery of an electromagnetic impurity transmission through particular structures, with important ratio length-to-diameter, such as nanostructures.

1.3. State of the art. We recall that the Calderón problem, addressed first in [5], has attracted many attention over the last decades (see for instance [11, 21] for an overview of several aspects of this problem). The first positive answer to this problem in dimension \( n \geq 3 \) has been addressed by Sylvester and Uhlmann in [38]. Here the authors introduced the so called complex geometric optics (CGO in short) solutions which remain one of the most important tools for the study of this problem. This last result has been extended in several way. For instance, we can mention the problem stated with partial data by [4] and improved by [27]. One of the first results about the recovery, modulo gauge invariance, of electromagnetic potentials has been addressed in [47] where the author proved the determination of magnetic field associated with magnetic potentials \( A \) lying in \( W^{2,\infty} \) by assuming that the magnetic field is sufficiently small. The smallness assumption of [47] was removed by [35] for smooth coefficients. Since then, [49] extends this result to magnetic potentials lying in \( C^1 \) and [31] extends it to magnetic potentials lying in a Dini class. To our best knowledge, the result with the weakest regularity assumption so far, for general bounded domain, is the one of [54] where the authors have considered bounded electromagnetic potentials. More recently, in the specific case of a ball in \( \mathbb{R}^3 \), [21] proved the recovery of unbounded magnetic potentials. Concerning results with partial data associated with this last problem, we mention the work of [17, 18] and concerning the stability issue, without being exhaustive, we refer to [3, 6, 7, 9, 39, 40, 50]. We mention also the work of [12, 22, 29] related to problems for hyperbolic and parabolic equations treated with an approach similar to the one considered for elliptic equations.

Note that all the above mentioned results have been stated in a bounded domain. Only a small number of articles studied such inverse boundary value problems in an unbounded domain. In [57], the authors combined unique continuation results with CGO solutions and a Carleman estimate borrowed from [41] in order to prove the unique recovery of compactly supported electric potentials of a Schrödinger operator in...
a slab from partial boundary measurements. This last result has been extended to magnetic Schrödinger operators by [33] and the stability issue has been addressed by [8]. We refer also to [24, 35, 36, 11, 22] for other related inverse problems stated in a slab. In [13, 15], the authors considered the stable recovery of coefficients periodic along the axis of an infinite cylindrical domain. More recently, [30] considered, for what seems to be the first time, the recovery of non-compactly supported and non-periodic electric potentials appearing in an infinite cylindrical domain. The results of [30] include also an extension of the work of [37] to the recovery of non-compactly supported coefficients in a slab. We mention also the work [1, 2, 16, 26, 28, 31, 32] treating the determination of coefficients appearing in different PDEs on an infinite cylindrical domain from boundary measurements.

1.4. Statement of the main results. Let us recall that there is an obstruction to the simultaneous recovery of $A, q$ from the data $D_{A,q}$ given by gauge invariance. More precisely according to [34, Lemma 3.1], which is stated for bounded domains but whose arguments can be extended without any difficulty to unbounded domains satisfying (1.1), the data $D_{A,q}$ satisfies the following gauge invariance.

$$ D_{A + \nabla \varphi, q} = D_{A, q}, \quad \varphi \in \{ h|_{\partial \Omega} : h \in W^{1,\infty}_{{\text{loc}}}(\mathbb{R}^3 : \mathbb{R}), \nabla_x h \in L^\infty(\mathbb{R}^3)^3, h|_{\mathbb{R}^3 \setminus \Omega} = 0 \}. $$

(1.4)

Taking into account this obstruction, for $A = (a_1, a_2, a_3)$, we consider the recovery of the magnetic field corresponding to the 2-form valued distribution $dA$ defined by

$$ dA := \sum_{1 \leq j < k \leq 3} (\partial_{x_j} a_k - \partial_{x_k} a_j) dx_j \wedge dx_k $$

and $q$. Assuming that $\Omega$ is simply connected and with some suitable regularity assumptions (see for instance Section 4.2), one can check that this result is equivalent to the recovery of the electromagnetic potential modulo gauge invariance.

This paper contains three main results. In the first main result, stated in Theorem 1.1, we consider the unique determination of electromagnetic potentials with low regularity from the full data $D_{A,q}$. In our second main result stated in Theorem 1.2, we prove, for electromagnetic potentials known on the neighborhood of the boundary outside a compact set, that measurements restricted to a bounded subset of $\partial \Omega$ can also recover uniquely the magnetic field and the electric potential. Finally, in our last result stated in Theorem 1.3, we give a partial data result by proving the unique recovery of a magnetic field and an electric potential associated with general class of electromagnetic potentials from restriction of the data $D_{A,q}$.

In our first main result we consider general class of bounded electromagnetic potentials and a general closed waveguide. This result can be stated as follows.

**Theorem 1.1.** Let $\Omega$ be an unbounded domain satisfying (1.1), let $A_1, A_2 \in L^\infty(\Omega)^3 \cap L^2(\Omega)^3$ be such that $A_1 - A_2 \in L^1(\Omega)^3$ and let $q_1, q_2 \in L^\infty(\Omega; \mathbb{C})$. Then the condition

$$ D_{A_1, q_1} = D_{A_2, q_2} $$

implies $dA_1 = dA_2$. Moreover, assuming that $q_1 - q_2 \in L^2(\Omega; \mathbb{C})$, (1.3) implies $q_1 = q_2$.

Let us remark that Theorem 1.1 is stated with boundary measurements in all parts of the unbounded boundary $\partial \Omega$. Despite the general setting of this problem, it may be difficult for several applications, like for transmission to long distance, to have access to such data. In order to make the measurements more relevant for some potential applications, we need to consider data restricted to a bounded portion of $\partial \Omega$. This will be the goal of our second result where we extend Theorem 1.1 to recovery of coefficients from measurements restricted to bounded portions of $\partial \Omega$. From now on, we assume that $\Omega$ is a domain with Lipschitz boundary. For all $s \in [0, \frac{1}{2}]$, we denote by $H^s_{{\text{loc}}}(\partial \Omega)$ the set of $f \in L^2_{{\text{loc}}}(\partial \Omega)$ such that for any $\chi \in C_0^\infty(\mathbb{R}^3)$, $\chi f \in H^s(\partial \Omega)$. For any $u \in H^1(\Omega)$, we can define $\tau_0 u = u|_{\partial \Omega}$ as an element of $H^\frac{1}{2}_{{\text{loc}}}(\partial \Omega)$. In the same way, for $U$ a closed (resp. open) subset of $\partial \Omega$ and for $u \in H^1(\Omega)$ solving $\Delta u + qu = 0$, with $A \in L^\infty(\Omega)^3$ and $q \in L^\infty(\Omega)$, we denote by $N_{A,q} u|_{U}$ the restriction of $N_{A,q} u$ to the subspace

$$ \{ g : g \in H^1(\Omega), \text{ supp}(\tau_0 g) \subset U \} $$
of $H^1_0(\Omega)$. Note that here $N_{A,q}u|_U$ is the natural extension of the restriction, up to the sign, of the magnetic normal derivative of $u$ to the set $U$. For $r > 0$ and $S_r = \partial \Omega \cap (\overbar{\omega} \times [-r, r])$, we can consider the restriction $D_{A,q,r}$ of the data $D_{A,q}$ given by

$$D_{A,q,r} := \{(\tau u, N_{A,q}u|_{S_r}) : u \in H^1(\Omega), \ u \text{ solves } (1.2), \ \text{supp}(\tau_0 u) \subset S_r\}. \quad (1.6)$$

In the spirit of [30, Corollary 1.3], fixing $\delta \in (0, r/2)$, we will apply Theorem 1.1 in order to prove the recovery of coefficients known on a neighborhood of the boundary outside $\Omega \cap (\omega \times (\delta - r, r - \delta))$ from the data $D_{A,q,r}$. For this purpose we need the following assumption on $\Omega$ and the admissible coefficients.

**Assumption 1:** For $j = 1, 2$, and for any $F \in L^2(\Omega)$ the equations $\Delta_{A_j} u_j + \nabla_{\overbar{\omega}} u_j = F$ and $\Delta_{A_j} u_j + q_j u_j = F$ admit respectively a solution $u_j \in H^1_0(\Omega)$.

We mention that Assumption 1 will be fulfilled if for instance $\Omega = \omega_1 \times \mathbb{R}$, with $\omega_1$ a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary, and if $0$ is not in the spectrum of the operators $\Delta_{A_j} + q_j$ and $\Delta_{A_j} + \nabla_{\overbar{\omega}}$, $j = 1, 2$, with Dirichlet boundary condition.

Let $n$ be the outward unit normal vector of $\partial \Omega$. Since $\Omega$ is only subjected to the condition $\Omega \subset \Omega_1$ this is why we use a different notation for the outward unit normal vector of $\Omega_1$ and $\Omega$. Before we state our result, let us also recall that for any $A \in L^\infty(\Omega)^3$ satisfying $\text{div}(A) \in L^1(\Omega)$, we can define the trace map $A \cdot n$ as the unique element of

$$\mathcal{B} \left( \frac{H^1(\Omega)}{H^0_0(\Omega)} ; \frac{H^1_0(\Omega)}{H^0_0(\Omega)} \right)$$

defined by

$$\langle (A \cdot n) g, \tau h \rangle_{\frac{H^1(\Omega)}{H^0_0(\Omega)}} := \int_\Omega \text{div}(A) h \overbar{\sigma} dx + \int_\Omega A \cdot \nabla h \overbar{\sigma} dx + \int \Omega h (A \cdot \overbar{\sigma} g) dx, \ g, h \in H^1(\Omega). \quad (1.7)$$

Again, by a density argument, one can easily check the validity of this definition by noticing that the right hand side of the identity vanishes as soon as $g \in H^1_0(\Omega)$ or $h \in H^0_0(\Omega)$. Here we use again the fact that, for $u \in H^1(\Omega)$ and $\varphi \in C^\infty_0(\Omega)$, we have $u \varphi \in W^{1,1}_0(\Omega)$.

Assuming that Assumption 1 is fulfilled, we state our second main result as follows.

**Theorem 1.2.** Let $\Omega$ be a connected open set with Lipschitz boundary satisfying (1.1). For $j = 1, 2$, let $A_j \in L^\infty(\Omega)^3 \cap L^2(\Omega)^3$, $\text{div}(A_j) \in L^\infty(\Omega)$, $q_j \in L^\infty(\Omega; \mathbb{C})$, $A_1 - A_2 \in L^1(\Omega)^3$. In addition, let Assumption 1 be fulfilled and, for $A_j \cdot n$, $j = 1, 2$, defined by (1.7) with $A = A_j$, let the condition

$$A_1 \cdot n = A_2 \cdot n \quad (1.8)$$

be fulfilled. Assume also that there exist $\delta \in (0, r/2)$ and two open connected set $\Omega_{\pm} \subset \Omega$ with Lipschitz boundary such that

$$\partial \Omega \cap (\overbar{\omega} \times (-\infty, -r + \delta)] \subset \partial \Omega_-, \quad \partial \Omega \cap (\overbar{\omega} \times [r - \delta, +\infty)) \subset \partial \Omega_+, \quad (1.9)$$

$$A_1(x) = A_2(x), \quad q_1(x) = q_2(x), \quad x \in \Omega_- \cup \Omega_+. \quad (1.10)$$

Then, the condition

$$D_{A_1, q_1, r} = D_{A_2, q_2, r} \quad (1.11)$$

implies $dA_1 = dA_2$. Moreover, assuming that $q_1 - q_2 \in L^2(\Omega; \mathbb{C})$, (1.11) implies $q_1 = q_2$.

For our last main result we will consider the specific case where $\Omega = \omega \times \mathbb{R}$. This time we want to consider the recovery of the coefficients not from full boundary measurements but from partial boundary measurements without assuming the knowledge of the coefficients close to the boundary. We remark that $\partial \Omega = \partial \omega \times \mathbb{R}$ and that the outward unit normal vector $\nu$ to $\partial \Omega$ takes the form

$$\nu(x', x_3) = (\nu(x'), 0)^T, \quad x = (x', x_3) \in \partial \Omega,$$

\footnote{Since $\Omega$ is only subjected to the condition $\Omega \subset \Omega_1$ we may have $\Omega \neq \Omega_1$ this is the reason why we use a different notation for the outward unit normal vector of $\Omega_1$ and $\Omega$.}
with $\nu'$ the outward unit normal vector of $\partial \omega$. In light of this identity, from now on, we denote by $\nu$ both the exterior unit normal vectors normal to $\partial \omega$ and to $\partial \omega \times \mathbb{R}$. We fix $\theta_0 \in S^1 := \{ y \in \mathbb{R}^2; \ |y| = 1 \}$ and we introduce the $\theta_0$-illuminated (resp., $\theta_0$-shadowed) face of $\partial \omega$, defined by $$\partial \omega^-_{\theta_0} := \{ x \in \partial \omega; \ \theta_0 \cdot \nu(x) \leq 0 \} \quad \text{(resp., } \partial \omega^+_{\theta_0} = \{ x \in \partial \omega; \ \theta_0 \cdot \nu(x) \geq 0 \}).$$ From now on, we denote by $x \cdot y := \sum_{j=1}^k x_j y_j$ the Euclidean scalar product of any two vectors $x := (x_1, \ldots, x_k)^T$ and $y := (y_1, \ldots, y_k)^T$ of $\mathbb{C}^k$. We fix $V$ a portion of $\partial \omega$ taking the form $V := V' \times \mathbb{R}$, where $V'$ is an arbitrary open neighborhood of $\partial \omega^-_{\theta_0}$ in $\partial \omega$. We introduce also the set of data $$D_{A,q,V} = \{(\tau u, N_{A,q} u_V) : \ u \in H^1(\Omega), \ u \ \text{solves} \ (1.2)\}.$$ Then we can state our last main result as follows.

**Theorem 1.3.** Let $\Omega = \omega \times \mathbb{R}$ and, for $j = 1, 2$, let $A_j \in L^\infty(\Omega)^3 \cap L^2(\Omega)^3$, $\text{div}(A_j) \in L^\infty(\Omega)$, $q_j \in L^\infty(\Omega; \mathbb{C})$, $A_1 - A_2 \in L^1(\Omega)^3$. Let also $A_1$ and $A_2$ satisfy (1.5). Then the condition

\[ dA_1 = dA_2. \]

implies $dA_1 = dA_2$. Moreover, assuming that $q_1 - q_2 \in L^1(\Omega; \mathbb{C})$, (1.5) implies also that $q_1 = q_2$.

**1.5. Comments about our results.** To the best of our knowledge Theorem 1.3 is the first result of recovery of a magnetic field and an electric potential in an unbounded domain with such a general setting. This point can be seen through four different aspects of the theorem. First, Theorem 1.1 is stated in a general unbounded domain subject only to condition (1.1). This makes an important difference with other related results which, to our best knowledge, have all been stated in specific unbounded domains like a slab, the half space or a cylindrical domain (see [33] [37] [14] [15]). In particular, Theorem 1.1 holds true with domains having different types of geometrical deformations like bends or twisting, which are frequently used in problems of transmission for improving the propagation. Second, to the best of our knowledge, in contrast to all other results stated for elliptic equations in an unbounded domain, Theorem 1.1 requires no assumptions about the spectrum of the magnetic Schrödinger operator associated with the electromagnetic potential under consideration. Usually such conditions make some restrictions on the class of coefficients under consideration, here we avoid such constraints. Third, we prove, for what seems to be the first time, the recovery of electromagnetic potentials that are neither compactly supported nor periodic. Actually we consider a class of electromagnetic potentials admitting various types of behavior outside a compact set (roughly speaking we consider magnetic potentials lying in $L^1(\Omega)^3$ and electric potentials lying in $L^2(\Omega)$). Fourth, Theorem 1.1 seems to be the first result stated for an unbounded domain with electromagnetic potentials having regularity comparable to [34], where the recovery of electromagnetic potentials has been stated with the weakest regularity condition so far for general bounded domains.

The main tools in our analysis are CGO solutions suitably designed for unbounded domains satisfying (1.1). Here in contrast to [14] [15] [33] [37] we do not restrict our analysis to compactly supported or periodic coefficients where, by mean of unique continuation or Floquet decomposition, one can transform the problem stated on an unbounded domain into a problem on a bounded domain. Like [33], we introduce a new class of CGO solutions designed for infinite cylindrical domains. The difficulties in the construction of such solutions are coming both from the fact that we consider magnetic potentials that are not compactly supported and the fact that we need to preserve the square integrability of the CGO solutions, which is not guaranteed by the usual CGO solutions in unbounded domains. In addition, like in [34], we build CGO solutions designed for bounded magnetic potentials. The construction of our CGO solutions requires Carleman estimates in negative order Sobolev space that we prove by extending some results, similar to those of [18] [43], to infinite cylindrical domains.

Let us observe that the construction of CGO solutions satisfying the square integrability property works only for domains contained into an infinite cylinder. For instance, we can not apply our construction to domains like slab or half space. However, in a similar way to [34] Corollary 1.4], applying Theorem 1.1 and...
one can prove that the result of \[33\] can be extended to electromagnetic potentials supported in infinite cylinder.

In this paper we consider electric potentials \( q \) that can be complex valued but we consider magnetic potentials \( A \) that take value in \( \mathbb{R}^3 \). Like in \[33, 34\], we could state our result with magnetic potentials taking value in \( \mathbb{C}^3 \), but for simplicity we restrict our analysis to real valued magnetic potentials.

1.6. Outline. This paper is organized as follows. In Section 2, we derive some Carleman estimates that will be useful at the same time for building the CGO solutions and restricting the data in Theorem \[1,2\]. In Section 3, we use the Carleman estimates in order to build our CGO solutions. Combining all these tools, in Section 4, 5, 6 we prove respectively Theorem \[1,1\], Theorem \[1,2\] and Theorem \[1,3\]. Finally, in Section 7 we explain how our result can be extended to higher dimension.

2. Carleman estimates

From now on, we fix \( \Omega_1 = \omega \times \mathbb{R} \). We associate to every point \( x \in \Omega_1 \) the coordinates \( x = (x', x_3) \), where \( x_3 \in \mathbb{R} \) and \( x' := (x_1, x_2) \in \omega \). In a similar way to the discussion before the statement of Theorem \[1,3\] we denote by \( v \) both the exterior unit vectors normal to \( \partial \omega \) and to \( \partial \Omega_1 \). The goal of this section is to establish two Carleman estimates for the magnetic Laplacian operator in the unbounded cylindrical domain \( \Omega_1 \). We start with a Carleman estimate which will be our first main tool. Then, using this Carleman estimate we will derive a Carleman estimate in negative Sobolev space.

2.1. General Carleman estimate. In order to prove our Carleman estimates we introduce first a weight function depending on two parameters \( s, \rho \in (1, +\infty) \) and we consider, for \( \rho > s > 1 \) and \( \theta \in \mathbb{S}^2 \), the perturbed weight

\[
\varphi_{\pm, s}(x', x_3) := \pm \rho \theta \cdot x' - s \frac{(x' \cdot \theta)^2}{2}, \quad x = (x', x_3) \in \omega \times \mathbb{R} = \Omega_1.
\]

We define

\[
P_{A, q, \pm, s} := e^{-\varphi_{\pm, s}}(\Delta + 2i A \cdot \nabla + q)e^{\varphi_{\pm, s}}.
\]

Like in \[18, 13\], we consider convexified weight, instead of the linear weight used in \[30, Proposition 31\], in order to be able to absorb first order perturbations of the Laplacian. Our first Carleman estimates can be seen as an extension of \[18, Proposition 2.3\], stated with linear weight, to unbounded cylindrical domains. These estimates take the following form.

Proposition 2.1. Let \( A \in L^\infty(\Omega_1)^3 \) and \( q \in L^\infty(\Omega_1; \mathbb{C}) \). Then there exist \( s_1 > 1 \) and, for \( s > s_1 \), \( \rho_1(s) \) such that for any \( v \in C_0^\infty(\mathbb{R}^3) \cap H^1_0(\Omega_1) \) the estimate

\[
\rho \int_{\partial \omega_{\pm, s} \times \mathbb{R}} |\partial_v v|^2 |\theta \cdot v| d\sigma(x) + s \rho^2 \int_{\Omega_1} |\Delta v|^2 dx + s \int_{\Omega_1} |\nabla v|^2 dx + s \rho^2 \int_{\Omega_1} |v|^2 dx
\]

\[
\leq C \left[ \|P_{A, q, \pm, s} v\|_{L^2(\Omega_1)}^2 + \rho \int_{\partial \omega_{\pm, s} \times \mathbb{R}} |\partial_v v|^2 |\theta \cdot v| d\sigma(x) \right]
\]

holds true for \( s > s_1 \), \( \rho \geq \rho_1(s) \) with \( C \) depending only on \( \Omega_1 \) and \( M \geq \|q\|_{L^\infty(\Omega_1)} + \|A\|_{L^\infty(\Omega_1)^3} \).

Proof. We start by proving that for all \( s > 1 \) there exists \( \rho_1(s) \) such that for \( \rho > \rho_1(s) \) we have

\[
\|e^{-\varphi_{\pm, s}} \Delta e^{\varphi_{\pm, s}} v\|_{L^2(\Omega_1)}^2 \geq \rho \int_{\partial \omega_{\pm, s} \times \mathbb{R}} |\partial_v v|^2 |\theta \cdot v| d\sigma(x) - 8 \rho \int_{\partial \omega_{\pm, s} \times \mathbb{R}} |\partial_v v|^2 |\theta \cdot v| d\sigma(x) + s \int_{\Omega_1} |\nabla v|^2 dx
\]

\[
+ \frac{s \rho^2}{2} \int_{\Omega_1} |v|^2 dx + c \rho^2 \int_{\Omega_1} |\Delta v|^2 dx,
\]
with \(c\) depending only on \(\Omega\). Using this estimate, we will derive (2.14). The proof of this result being similar for \(e^{-\tilde{\varphi} + \varphi} \Delta e^\varphi + e^\varphi\) and \(e^{-\tilde{\varphi} - \varphi} \Delta e^{-\varphi} + e^{-\varphi}\), we will only consider it for \(e^{-\tilde{\varphi} + \varphi} \Delta e^\varphi + e^\varphi\). We decompose \(e^{-\tilde{\varphi} + \varphi} \Delta e^\varphi + e^\varphi\) into three terms

\[
e^{-\tilde{\varphi} + \varphi} \Delta e^\varphi + e^\varphi = P_{1,+} + P_{2,+} + P_{3,+},
\]

with

\[
P_{1,+} = \Delta' + |\nabla \varphi_{+,s}|^2 - \Delta' \varphi_{+,s} = \Delta' + \rho^2 - 2s \rho(x' \cdot \theta) + s^2(x' \cdot \theta)^2 + s,
\]

\[
P_{2,+} = \partial_{\tilde{z}_3}^2, \quad P_{3,+} = 2 \nabla' \varphi_{+,s} \cdot \nabla' + 2 \Delta' \varphi_{+,s} = 2(\rho - s(x' \cdot \theta)) \theta \cdot \nabla' - 2s.
\]

Here \(\Delta' := \partial_{\tilde{z}_3}^2 + \partial_{\tilde{z}_2}^2, \nabla' := (\partial_{x_1}, \partial_{x_2})^T \) and \(\theta \cdot \nabla' = \theta_1 \partial_{x_1} + \theta_2 \partial_{x_2}\). Using some arguments similar to Proposition 2.3, one can check that for all \(s > 1\) there exists \(\rho_2(s) > 1\) such that for \(\rho > \rho_2(s)\) and \(y \in C^\infty(\overline{\omega}) \cap H^1_0(\omega)\) we have

\[
2\Re \int_\omega P_{1,+} + yP_{3,+} \overline{y} dx' \geq \rho \int_\partial_{\omega_{\omega,\theta}} |\partial_{\nu} y|^2 |\theta \cdot \nu| d\sigma(x') - 8p \int_\partial_{\omega_{\omega,\theta}} |\partial_{\nu} y|^2 |\theta \cdot \nu| d\sigma(x') + s \rho \int_\omega |y|^2 dx' + s \int_\omega |\nabla y|^2 dx.
\]

Applying this estimate to \(v(\cdot, x_3) := x' \mapsto v(x', x_3), x_3 \in \mathbb{R}\), we obtain

\[
2\Re \int_\omega P_{1,+} + v(\cdot, x_3) P_{3,+} \overline{v(\cdot, x_3)} dx' \geq \rho \int_\partial_{\omega_{\omega,\theta}} |\partial_{\nu} v(\cdot, x_3)|^2 |\theta \cdot \nu| d\sigma(x') + s \int_\omega |\nabla v(\cdot, x_3)|^2 dx - 8p \int_\partial_{\omega_{\omega,\theta}} |\partial_{\nu} v(\cdot, x_3)|^2 |\theta \cdot \nu| d\sigma(x') + s \rho \int_\omega |v(\cdot, x_3)|^2 dx', \quad x_3 \in \mathbb{R}.
\]

Integrating this estimate with respect to \(x_3 \in \mathbb{R}\), we get

\[
\|P_{1,+} + v + P_{2,+} + P_{3,+} + v\|^2_{L^2(\Omega_1)} \geq \rho \int_\omega |\partial_{\nu} v|^2 |\theta \cdot \nu| d\sigma(x') + 8p \int_\omega |\nabla v|^2 dx + 8s \rho \int_\omega |v|^2 dx + \frac{8s \rho^2}{2} \int_\omega |\nabla v|^2 dx.
\]

On the other hand, integrating by parts with respect to \(x_3 \in \mathbb{R}\) and then with respect to \(x' \in \omega\), we find

\[
\Re \int_{\Omega_1} P_{2,+} + v P_{3,+} \overline{v} dx = - \int_{\Omega_1} (\rho - s(x' \cdot \theta)) \theta \cdot \nabla' |\partial_{x_3} v|^2 dx + 2s \int_{\Omega_1} |\partial_{x_3} v|^2 dx
\]

\[
= s \int_{\Omega_1} |\partial_{x_3} v|^2 dx.
\]

Moreover, fixing

\[
\tilde{c} = 4 \left( 3 + \sup_{x' \in \overline{\omega}} |x'| \right)^2, \quad \rho_1(s) = \rho_2(s) + \tilde{c}^{-1} \sqrt{s},
\]

we deduce that, for \(\rho > \rho_1(s)\), we have

\[
\|P_{1,+} + v + P_{2,+} + v\|^2_{L^2(\Omega_1)} \geq \tilde{c}^{-1} \rho^{-2} \|P_{1,+} + v + P_{2,+} + v\|^2_{L^2(\Omega_1)} \geq s(2\tilde{c})^{-1} \rho^{-2} \|\Delta v\|^2_{L^2(\Omega_1)} - \frac{s \rho^2}{2} \|v\|^2_{L^2(\Omega_1)}.
\]

Combining this with (2.10) - (2.17), we deduce (2.14). Now let us complete the proof of (2.14). For this purpose, we introduce

\[
P_{4,\pm} = 2iA \cdot \nabla + 2iA \cdot \nabla \varphi_{\pm,s} + q = 2iA \cdot \nabla + 2(\pm \rho - s(x' \cdot \theta)) iA' \cdot \theta + q,
\]
with \( A = (a_1, a_2, a_3) \) and \( A' = (a_1, a_2) \), and we recall that \( P_{A,q,+,s} = e^{-\varphi_{+,s}} \Delta e^{\varphi_{+,s}} + P_{4,+,s} \). We find
\[
\| P_{A,q,\pm,s} v \|_{L^2(\Omega_1)}^2 \\
\geq \frac{\| e^{-\varphi_{+,s}} \Delta e^{\varphi_{+,s}} v \|_{L^2(\Omega_1)}^2}{2} - \| P_{4,+,s} v \|_{L^2(\Omega_1)}^2 \\
\geq \frac{\| e^{-\varphi_{+,s}} \Delta e^{\varphi_{+,s}} v \|_{L^2(\Omega_1)}^2}{2} - 3 \| A \|_{L^\infty(\Omega_1)}^2 \int_{\Omega_1} |\nabla v|^2 \, dx - 3 \left( 16 \| A \|_{L^\infty(\Omega_1)}^2 \rho + \| q \|_{L^\infty(\Omega_1)}^2 \right) \int_{\Omega_1} |v|^2 \, dx.
\]
Fixing \( s_1 = 48 \| A \|_{L^\infty(\Omega_1)}^2 + 6 \), we deduce (2.14) from (2.15). \( \Box \)

A direct consequence of these Carleman estimates is the following result which will be useful for Theorem 1.3.

**Corollary 2.2.** Let \( A \in L^\infty(\Omega_1)^3 \) and \( q \in L^\infty(\Omega_1; \mathbb{C}) \). There exists \( \rho_1' > 0 \) such that for any \( u \in C^2_0(\mathbb{R}^3) \cap H^1_0(\Omega_1) \) the estimate
\[
\rho \int_{\omega_{x_1} \times \mathbb{R}} e^{-2\rho \cdot x'} |\partial_{x_1} u|^2 |\partial \cdot \nu(x')| \, d\sigma(x) + \rho^2 \int_{\Omega_1} e^{-2\rho \cdot x'} |u|^2 \, dx + \int_{\Omega_1} e^{-2\rho \cdot x'} |\nabla u|^2 \, dx \\
\leq C \left( \int_{\Omega_1} e^{-2\rho \cdot x'} \left| (-\Delta + 2iA \cdot \nabla + q) u \right|^2 \, dx + \rho \int_{\omega_{x_1} \times \mathbb{R}} e^{-2\rho \cdot x'} |\partial_{x_1} u|^2 |\partial \cdot \nu(x')| \, d\sigma(x) \right)
\]
holds true for \( \rho > \rho_1' \) with \( C \) depending only on \( \Omega_1 \) and \( M \geq \| q \|_{L^\infty(\Omega_1)} + \| A \|_{L^\infty(\Omega_1)} \).

**Proof.** We fix \( u \in C^2_0(\mathbb{R}^3) \cap H^1_0(\Omega_1) \) and we set \( v = e^{-\varphi_{+,s}} u \) such that
\[
\int_{\Omega_1} e^{-2\rho \cdot x'} |(-\Delta + 2iA \cdot \nabla + q) u|^2 \, dx = \int_{\Omega_1} |P_{A,q,+,s} v|^2 \, dx.
\]
The fact that \( v \in H^1_0(\Omega_1) \) implies \( \partial_{x_1} v \big|_{\partial \Omega_1} = e^{-\rho \cdot x'} \frac{e^{\varphi_{+,s}}}{\rho} \partial_{x_1} u \big|_{\partial \Omega_1} \) and we deduce that
\[
\int_{\omega_{x_1} \times \mathbb{R}} |\partial_{x_1} v|^2 |\partial \cdot \nu(x')| \, d\sigma(x) \geq \int_{\omega_{x_1} \times \mathbb{R}} e^{-2\rho \cdot x'} |\partial_{x_1} u|^2 (\cdot \cdot \cdot) \, d\sigma(x)
\]
and
\[
\int_{\omega_{x_1} \times \mathbb{R}} |\partial_{x_1} u|^2 |\partial \cdot \nu(x')| \, d\sigma(x) \geq e^{sb^2} \int_{\omega_{x_1} \times \mathbb{R}} e^{-2\rho \cdot x'} |\partial_{x_1} u|^2 (\cdot \cdot \cdot) \, d\sigma(x),
\]
with \( b = (2 + 2 \sup_{x_1 \in \omega} |x'|) \). Moreover, since
\[
\nabla u(x) = \nabla (e^{\varphi_{+,s}} v) = (\rho - sx \cdot \theta) u + e^{\varphi_{+,s}} e^{-\frac{(x_1 - s)^2}{2}} \nabla v, \quad x = (x', x_3) \in \omega \times \mathbb{R},
\]
we obtain
\[
\int_{\Omega_1} e^{-2\rho \cdot x'} |\nabla u|^2 \, dx \leq 2 \rho e^{sb^2} \int_{\Omega_1} |v|^2 \, dx + 2 e^{sb^2} \int_{\Omega_1} |\nabla v|^2 \, dx.
\]
Combining this estimates with (2.11) and (2.19) - (2.20), for \( s \geq s_1 \) and \( \rho > \rho_1(s) \), we get
\[
\int_{\Omega_1} e^{-2\rho \cdot x'} |\nabla u|^2 \, dx + \rho^2 \int_{\Omega_1} e^{-2\rho \cdot x'} |u|^2 \, dx + \rho \int_{\omega_{x_1} \times \mathbb{R}} e^{-2\rho \cdot x'} |\partial_{x_1} u|^2 (\cdot \cdot \cdot) \, d\sigma(x) \\
\leq \rho e^{sb^2} \int_{\omega_{x_1} \times \mathbb{R}} e^{-2\rho \cdot x'} |\partial_{x_1} u|^2 (\cdot \cdot \cdot) \, d\sigma(x) + Ce^{sb^2} \int_{\Omega_1} e^{-2\rho \cdot x'} |(-\Delta + 2iA \cdot \nabla + q) u|^2 \, dx.
\]
From this last estimate we deduce (2.18) by fixing \( s = s_1 + 1 \) and \( \rho_1' = \rho_1(s_1 + 1) \). \( \Box \)

**Remark 2.3.** By density the result of Proposition 2.1 and Corollary 2.2 can be extended to any \( v \in H^1_0(\Omega_1) \) satisfying \( \Delta v \in L^2(\Omega_1) \) and \( \partial_{x_1} v \in L^2(\partial \Omega_1) \).
2.2. Carleman estimate in negative order Sobolev space. The goal of this subsection is to apply the result of Proposition 2.4 in order to derive Carleman estimates in negative order Sobolev space which will be one of the most important ingredient in the construction of the CGO solutions. We recall first some preliminary tools and we derive a Carleman estimate in Sobolev space of negative order. In a similar way to Proposition 2.1 we decompose
\[
R := (\Delta + \rho^2 + 2.s.(x' \cdot \theta) + s^{2}(x' \cdot \theta)^2 + s, \quad P := 2(\rho - s(x' \cdot \theta)) \theta \cdot \nabla - 2s.
\]
with the norm
\[
\|u\|_{H^m_{\rho}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} (|\xi|^2 + \rho^2)^m |\hat{u}(\xi)|^2 d\xi.
\]
Here for all tempered distributions \( u \in \mathcal{S}'(\mathbb{R}^3) \), we denote by \( \hat{u} \) the Fourier transform of \( u \), which, for \( u \in L^1(\mathbb{R}^3) \), is defined by
\[
\hat{u}(\xi) := \mathcal{F}u(\xi) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x) dx.
\]
From now on, for \( m \in \mathbb{R} \) and \( \xi \in \mathbb{R}^3 \), we set
\[
\langle \xi, \rho \rangle = (|\xi|^2 + \rho^2)^\frac{1}{2}
\]
and \( \langle D_x, \rho \rangle^m u \) defined by
\[
\langle D_x, \rho \rangle^m u = \mathcal{F}^{-1}((\xi, \rho)^m \mathcal{F} u).
\]
For \( m \in \mathbb{R} \) we define also the class of symbols
\[
S^m_{\rho} = \{ c_\rho \in \mathcal{C}_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3) : |\partial_x^\alpha \partial_\xi^\beta c_\rho(x, \xi)| \leq C_{\alpha, \beta} \langle \xi, \rho \rangle^{m-|\beta|}, \quad \alpha, \beta \in \mathbb{N}^3 \}.
\]
Following [43] Theorem 18.1.6, for any \( m \in \mathbb{R} \) and \( c_\rho \in S^m_{\rho} \), we define \( c_\rho(x, D_x) \), with \( D_x = -i \nabla \), by
\[
c_\rho(x, D_x) \hat{y}(x) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} c_\rho(x, \xi) \hat{y}(\xi) e^{ix \cdot \xi} d\xi, \quad y \in \mathcal{S}(\mathbb{R}^3).
\]
For all \( m \in \mathbb{R} \), we set also \( OpS^m_{\rho} := \{ c_\rho(x, D_x) : \ c_\rho \in S^m_{\rho} \} \). We fix
\[
P_{A,q,\pm} := e^{\mp \rho x' \cdot \theta} (\Delta_A + q)e^{\pm \rho x' \cdot \theta},
\]
and, in the spirit of [39] estimate (2.14) and [43] Lemma 2.1, we consider the following Carleman estimate.

**Proposition 2.4.** Let \( A \in L^\infty(\Omega_1)^3 \) and \( q \in L^\infty(\Omega_1; \mathbb{C}) \). Then, there exists \( \rho_2 > 1 \) such that for all \( v \in C_0^\infty(\Omega_1) \), we have
\[
\rho^{-1} \|v\|_{H^m_{\rho}(\mathbb{R}^3)} \leq C \|P_{A,q,v}\|_{H^{-m}_{\rho}(\mathbb{R}^3)}, \quad \rho > \rho_2,
\]
with \( C > 0 \) depending on \( \Omega_1 \) and \( \|q\|_{L^\infty(\Omega_1)} + \|A\|_{L^\infty(\Omega_1)^3} \).

**Proof.** Since this result is similar for \( P_{A,q,v} \) and \( P_{A,q,-v} \), we will only prove it for \( P_{A,q,+v} \). For \( \varphi_{\pm,s} \) given by
\[
(A_{\varphi_{\pm,s}} + q) v \leq C \|v\|_{H^m_{\rho}(\mathbb{R}^3)},
\]
and in a similar way to Proposition 2.1 we decompose \( A_{\varphi_{\pm,s}} \) into three terms
\[
A_{\varphi_{\pm,s}} = A_{\varphi_{\pm,s}} + A_{\varphi_{\pm,s}} + A_{\varphi_{\pm,s}} + A_{\varphi_{\pm,s}},
\]
where we recall that
\[
P_{\varphi_{\pm,s}} = \Delta + \rho^2 - 2s(x' \cdot \theta) + s^2(x' \cdot \theta)^2 + s, \quad P_{\varphi_{\pm,s}} = 2(\rho - s(x' \cdot \theta)) \theta \cdot \nabla - 2s.
\]
We pick \( \omega \) a bounded \( C^1 \) open set of \( \mathbb{R}^2 \) such that \( \partial \subset \omega \) and we extend the function \( A \) and \( q \) to \( \mathbb{R}^3 \) with \( A = 0, q = 0 \) on \( \mathbb{R}^3 \setminus \Omega_1 \). We consider also \( \Omega = \omega \times \mathbb{R} \). We start with the Carleman estimate
\[
\rho^{-1} \|v\|_{H^m_{\rho}(\mathbb{R}^3)} \leq C \|R_{A,q,+v}\|_{H^{-m}_{\rho}(\mathbb{R}^3)}, \quad v \in C_0^\infty(\Omega_1).
\]
For this purpose, we fix \( w \in H^3(\mathbb{R}^3) \) satisfying \( \text{supp}(w) \subset \tilde{\Omega} \) and we consider the quantity
\[
\langle D_x, \rho \rangle^{-1} (P_{1, +} + P_{2, +}) (D_x, \rho) w.
\]
In all the remaining parts of this proof \( C > 0 \) denotes a generic constant depending on \( \Omega_1 \) and \( \| A \|_{L^\infty(\Omega_1)^3} + \| q \|_{L^\infty(\Omega_1)} \). Applying the properties of composition of pseudodifferential operators (e.g. \[23\] Theorem 18.1.8), we find
\[
\langle D_x, \rho \rangle^{-1} (P_{1, +} + P_{2, +}) (D_x, \rho) = P_{1, +} + P_{2, +} + S_{\rho}(x, D_x),
\]
where \( S_{\rho} \) is defined by
\[
S_{\rho}(x, \xi) = \nabla_\xi \langle \xi, \rho \rangle^{-1} \cdot D_x (p_{1, +}(x, \xi) + p_{2, +}(x, \xi)) \langle \xi, \rho \rangle + o_{\langle \xi, \rho \rangle \to \infty}(1),
\]
with
\[
p_{1, +}(x, \xi) = -|\xi|^2 + \rho^2 - 2s(\theta \cdot \theta) + s^2(\theta \cdot \theta)^2 \geq s, \quad p_{2, +}(x, \xi) = 2i(\rho - s(\theta \cdot \theta))|\theta \cdot \xi' - 2s, \quad \xi = (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R}.
\]
Therefore, we have
\[
S_{\rho}(x, \xi) = \frac{-2i\rho s + 2is^2\theta \cdot \xi + 2s(\theta \cdot \xi')|\xi \cdot \xi'|}{|\xi|^2 + \rho^2} + o_{\langle \xi, \rho \rangle \to \infty}(1)
\]
and it follows
\[
\| S_{\rho}(x, D_x) w \|_{L^2(\mathbb{R}^3)} \leq C s^2 \| w \|_{L^2(\mathbb{R}^3)}.
\]
On the other hand, applying \[2.14\] to \( w \), which is permitted according to Remark \[2.3\] with \( \Omega_1 \) replaced by \( \Omega \) and \( A = 0, q = 0 \), we get
\[
\| P_{1, +} w + P_{2, +} w \|_{L^2(\mathbb{R}^3)} \geq C \left( s^{1/2} \rho^{-1} \| \Delta w \|_{L^2(\mathbb{R}^3)} + s^{1/2} \| \nabla w \|_{L^2(\mathbb{R}^3)} + s^{1/2} \| w \|_{L^2(\mathbb{R}^3)} \right).
\]
Combining this estimate with \[2.21\] - \[2.23\], for \( \frac{2s}{s} \) sufficiently large, we obtain
\[
\| (P_{1, +} + P_{2, +}) (D_x, \rho) w \|_{H^{-1}_0(\mathbb{R}^3)} = \left\| (D_x, \rho)^{-1} (P_{1, +} + P_{2, +}) (D_x, \rho) w \right\|_{L^2(\mathbb{R}^3)} \geq C s^{1/2} \rho^{-1} \| \Delta w \|_{L^2(\mathbb{R}^3)} + \| \nabla w \|_{L^2(\mathbb{R}^3)} + \rho \| w \|_{L^2(\mathbb{R}^3)}.
\]
On the other hand, using the fact that \( w \in H^2(\tilde{\Omega}) \cap H^1_0(\tilde{\Omega}) \), the elliptic regularity for cylindrical domain (e.g. \[13\] Lemma 2.2) implies
\[
\| w \|_{H^2(\tilde{\Omega})} = \| w \|_{H^2(\tilde{\Omega})} \leq C (\| \Delta w \|_{L^2(\tilde{\Omega})} + \| w \|_{L^2(\tilde{\Omega})}).
\]
Combining this with the previous estimate, for \( s \) sufficiently large, we find
\[
\| (P_{1, +} + P_{2, +}) (D_x, \rho) w \|_{H^{-1}_0(\mathbb{R}^3)} \geq C s^{1/2} \rho^{-1} \| w \|_{H^2(\mathbb{R}^3)}.
\]
Moreover, we have
\[
\| P_{3, +, A} (D_x, \rho) w \|_{H^{-1}_0(\mathbb{R}^3)} \leq \left\| 2i(\rho - s(\theta \cdot \theta)) A \cdot \theta + (q - |A|^2) (D_x, \rho) w \right\|_{H^{-1}_0(\mathbb{R}^3)} + 2 \| A \cdot \nabla (D_x, \rho) w \|_{H^{-1}_0(\mathbb{R}^3)} + \| \text{idiv}(A) (D_x, \rho) w \|_{H^{-1}_0(\mathbb{R}^3)}.
\]
For the first term on the right hand side of this inequality, we have
\[
\| 2i(\rho - s(\theta \cdot \theta)) A \cdot \theta + (q - |A|^2) (D_x, \rho) w \|_{H^{-1}_0(\mathbb{R}^3)} \leq \rho^{-1} \left\| 2i(\rho - s(\theta \cdot \theta)) A \cdot \theta + (q - |A|^2) (D_x, \rho) w \right\|_{L^2(\mathbb{R}^3)} \leq C \| (D_x, \rho) w \|_{L^2(\mathbb{R}^3)} \leq C \| (D_x, \rho) w \|_{L^2(\mathbb{R}^3)} = C \| w \|_{H^1(\mathbb{R}^3)},
\]
(2.28)
with $C$ depending only on $\|A\|_{L^\infty(\Omega)}^3 + \|q\|_{L^\infty(\Omega)}$. For the second term on the right hand side of (2.24), we get
\[
\|A \cdot \nabla (D, \rho) w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \leq \rho^{-1} \|A \cdot \nabla (D, \rho) w\|_{L^2(\mathbb{R}^3)} \\
\leq \rho^{-1} \|A\|_{L^\infty(\Omega)} \|\nabla (D, \rho) w\|_{L^2(\mathbb{R}^3)} \\
\leq \rho^{-1} \|A\|_{L^\infty(\Omega)} \|w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}.
\]

(2.29)

Finally, for the last term on the right hand side of (2.27), by duality, we find
\[
\|i\text{div}(A) (D, \rho) w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \leq \rho^{-1} \|A \cdot \nabla (D, \rho) w\|_{L^2(\mathbb{R}^3)} + \|(D, \rho) w\|_{L^2(\mathbb{R}^3)} \\
\leq 2\rho^{-1} \|A\|_{L^\infty(\Omega)} \|w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}.
\]

(2.30)

Combining (2.27), (2.30), we obtain
\[
\|P_{3,+} (D, \rho) w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \leq C \rho^{-1} \|w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}
\]

and combining this with (2.26) for $s > 1$ sufficiently large, we get
\[
\|P_{A,s} (D, \rho) w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \geq C \rho^s \rho^{-1} \|w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}.
\]

(2.31)

Now let us set $\omega_j, j = 1, 2$ two open subsets of $\bar{\omega}$ such that $\bar{\omega} \subset \omega_1, \bar{\omega_2} \subset \omega_2$ and $\omega_1 \cap \omega_2 \subset \bar{\omega}$. We fix $\psi_0 \in C_0^\infty(\bar{\omega})$ satisfying $\psi_0 = 1$ on $\bar{\omega}_2, w(x', x_3) = \psi_0(x') (D, \rho)^{-1} v(x', x_3)$ and for $\psi_1 \in C_0^\infty(\omega_1)$ satisfying $\psi_1 = 1$ on $\omega$, we get
\[
(1 - \psi_0) (D, \rho)^{-1} v = (1 - \psi_0) (D, \rho)^{-1} \psi_1 v,
\]

where $\psi_1 v$ denotes the function $v(x', x_3) = x \mapsto \psi_1(x') v(x)$. According to [23] Theorem 18.1.8, since $1 - \psi_0$ is vanishing in a neighborhood of $\text{supp}(\psi_1)$, we have $(1 - \psi_0) (D, \rho)^{-1} v \in OpS_0^\infty$ and it follows
\[
\rho^{-1} \|v\|_{H^1(\mathbb{R}^3)} \leq \rho^{-1} \|(D, \rho)^{-1} v\|_{H^\frac{1}{2}(\mathbb{R}^3)} \\
\leq \rho^{-1} \|w\|_{H^2(\mathbb{R}^3)} + \rho^{-1} \|(1 - \psi_0) (D, \rho)^{-1} \psi_1 v\|_{H^\frac{1}{2}(\mathbb{R}^3)} \\
\leq \rho^{-1} \|w\|_{H^2(\mathbb{R}^3)} + C \|v\|_{L^2(\mathbb{R}^3)} \rho^2.
\]

In the same way, we find
\[
\|P_{A,s} v\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \geq \|P_{A,s} (D, \rho) w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} - \|P_{A,s} (D, \rho) (1 - \psi_0) (D, \rho)^{-1} \psi_1 v\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \\
\geq \|P_{A,s} (D, \rho) w\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} - C \|v\|_{L^2(\mathbb{R}^3)} \rho^2.
\]

Combining these estimates with (2.31), we deduce that (2.28) holds true for a sufficiently large value of $\rho$. Then, fixing $s$, we deduce (2.22).

\[\square\]

3. CGO solutions

In this section we introduce a class of CGO solutions suitable for our problem stated in an unbounded domain for magnetic Schrödinger equations. Like in the previous section, we fix $\Omega_1 = \omega \times \mathbb{R}$. Our goal is to build CGO solutions for the equations (1.2) extended to the cylindrical domain $\Omega_1$ in order to consider their restrictions on $\Omega$ for proving Theorem 1.1 since according to (1.1) we have $\Omega \subset \Omega_1$.

We consider CGO solutions on $\Omega_1$ corresponding to some specific solutions $u_j \in H^1(\Omega_1), j = 1, 2$, of $\Delta A_1 u_1 + q_1 u_1 = 0, \Delta A_2 u_2 + q_2 u_2 = 0$ in $\Omega_1$ for $A_j \in L^\infty(\Omega_1) \cap L^2(\Omega_1)$ and $q_j \in L^\infty(\Omega_1; \mathbb{C})$. More precisely,
like in [30], we start by considering \( \theta \in S^1 := \{ y \in \mathbb{R}^2 : |y| = 1 \} \), \( \xi' \in \theta^\perp \setminus \{0\} \) with \( \theta^\perp := \{ y \in \mathbb{R}^2 : y \cdot \theta = 0 \} \), \( \xi := (\xi', \xi_3) \in \mathbb{R}^3 \) with \( \xi_3 \neq 0 \). Then, we define \( \eta \in S^2 := \{ y \in \mathbb{R}^3 : |y| = 1 \} \) by

\[
\eta = \frac{(\xi', -|\xi'|^2 \xi_3)}{\sqrt{|\xi'|^2 + |\xi_3|^2}}.
\]

It is clear that

\[
\eta \cdot \xi = (\theta, 0) \cdot \xi = (\theta, 0) \cdot \eta = 0.
\]

We set also \( \psi \in C_0^\infty(\mathbb{R}; [0, 1]) \) such that \( \psi = 1 \) on a neighborhood of 0 in \( \mathbb{R} \) and, for \( \rho > 1 \), we consider solutions \( u_j \in H^1(\Omega_1) \) of \( \Delta_{A_j} u_1 + q_1 u_1 = 0 \), \( \Delta_{A_j} u_2 + \overline{\theta} u_2 = 0 \) in \( \Omega_1 \) taking the form

\[
\begin{align*}
  \hspace{1cm} u_1(x', x_3) = e^{i\theta \cdot x'} \left( \psi (\rho^{-\frac{4}{3}} x_3) b_{1, \rho} e^{i\rho x \cdot \eta - i\xi \cdot x} + w_1(x', x_3) \right), & \quad x' \in \omega, \ x_3 \in \mathbb{R}, \\
  u_2(x', x_3) = e^{-i\theta \cdot x'} \left( \psi (\rho^{-\frac{4}{3}} x_3) b_{2, \rho} e^{i\rho x \cdot \eta} + w_2(x', x_3) \right), & \quad x' \in \omega, \ x_3 \in \mathbb{R}.
\end{align*}
\]

(3.33)

(3.34)

Here \( b_{j, \rho} \in C^\infty(\overline{\Omega_1}) \) and the remainder term \( w_{j, \rho} \in H^1(\Omega_1) \) satisfies the decay property

\[
\lim_{\rho \to +\infty} (\rho^{-1} \|w_{j, \rho}\|_{H^1(\Omega_1)} + \|w_{j, \rho}\|_{L^2(\Omega_1)}) = 0.
\]

(3.35)

This construction can be summarized in the following way.

**Theorem 3.1.** For \( j = 1, 2 \) and for all \( \rho > \rho_2 \), with \( \rho_2 \) the constant of Proposition 2.4, the equations

\[
\Delta_{A_j} u_1 + q_1 u_1 = 0, \quad \Delta_{A_j} u_2 + \overline{\theta} u_2 = 0
\]

admit respectively a solution \( u_j \in H^1(\Omega_1) \) of the form (3.33) with \( w_{j, \rho} \) satisfying the decay property (3.35).

**Remark 3.2.** Like in [30], we cannot consider CGO solutions similar to those on bounded domains since they will not be square integrable in \( \xi \). More precisely, if in our construction we use the expression for \( \xi \), we cannot extend CGO solutions similar to those on bounded domains since we will not have square integrability in \( \xi \).

In a similar way to [30], we start by considering \( q \in H^1(\Omega_1) \) of the form (1.20)-(1.21). For this purpose, we assume that \( b_{j, \rho}, \ j = 1, 2 \), is an approximation of a solution of the equations

\[
2(\overline{\theta} + i\eta) \cdot \nabla b_1 + 2i(\overline{\theta} + i\eta) \cdot A_1(x) |b_1| = 0, \quad 2(-\overline{\theta} + i\eta) \cdot \nabla b_2 + 2i(-\overline{\theta} + i\eta) \cdot A_2(x) |b_2| = 0, \quad x \in \Omega_1.
\]

(3.36)

Here \( \overline{\theta} := (\theta, 0) \in S^2 \). This approach, also considered in [2, 30, 34, 11], makes it possible to reduce the regularity assumption on the first order coefficients \( A_j \). Indeed, by replacing the functions \( b_1, b_2 \), whose regularity depends on the one of the coefficients \( A_1 \) and \( A_2 \), with their approximation \( b_{1, \rho}, b_{2, \rho} \), we can weaken the regularity assumption imposed on the coefficients \( A_j, j = 1, 2 \), from \( W^{2, \infty}(\Omega)^3 \) to \( L^\infty(\Omega)^3 \). Moreover, this approach requires also no information about the domain \( \Omega \) and the coefficients \( A_j, j = 1, 2 \), on \( \partial \Omega \).

More precisely, if in our construction we use the expression \( b_j \) instead of \( b_{j, \rho}, j = 1, 2 \), then, following our strategy, we can prove Theorem 1.1 only for specific domains and for coefficients \( A_1, A_2 \in W^{2, \infty}(\Omega)^3 \cap L^1(\Omega) \) satisfying

\[
\partial^\alpha A_1(x) = \partial^\alpha A_2(x), \quad x \in \partial \Omega, \ \alpha \in \mathbb{N}^3, \ |\alpha| \leq 1,
\]

where in our case we make no assumption on the shape of \( \Omega \) (except the condition \( \Omega \subset \omega \times \mathbb{R} \)) and about \( A_j \) at \( \partial \Omega \).

Let us also mention that comparing to results stated on bounded domains (e.g. [18, 33]), the magnetic potentials \( A_1, A_2 \) cannot be extended to compactly supported functions of \( \mathbb{R}^3 \). However, we can extend...
them into functions of $\mathbb{R}^3$ supported in infinite cylinder. Combining this with the fact that $A_j \in L^2(\Omega_1)^3$, we will prove how we can build CGO solutions having properties similar to the one of [34].

In order to define $b_{j,\rho}$, $j = 1, 2$, we start by introducing a suitable approximation of the coefficients $A_j$, $j = 1, 2$. For all $r > 0$, we define $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$ and $B'_r := \{x' \in \mathbb{R}^2 : |x'| < r\}$. We fix $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi \geq 0$, $\int_{\mathbb{R}^3} \chi(x)dx = 1$, supp$(\chi) \subset B_1$, and we define $\chi_\rho$ by $\chi_\rho(x) = \rho^{\frac{3}{2}}\chi(\rho^{\frac{3}{2}}x)$. Then, for $j = 1, 2$, we fix

$$A_{j,\rho}(x) := \int_{\mathbb{R}^3} \chi_\rho(x - y)A_j(y)dy.$$  

Here, we assume that, for $j = 1, 2$, $A_j = 0$ on $\mathbb{R}^3 \setminus \Omega_1$. For $j = 1, 2$, since $A_j \in L^2(\mathbb{R}^3)^3$, by density one can check that

$$\lim_{\rho \to +\infty} \|A_{j,\rho} - A_j\|_{L^2(\mathbb{R}^3)} = 0,$$

and, using the fact that $A_j \in L^\infty(\mathbb{R}^3)^3$, we deduce the estimates

$$\|A_{j,\rho}\|_{H^k(\mathbb{R}^3)} + \|A_{j,\rho}\|_{W^{k,\infty}(\mathbb{R}^3)} \leq C_k \rho^k,$$

with $C_k$ independent of $\rho$. We remark that

$$A_{\rho}(x) := \int_{\mathbb{R}^3} \chi_\rho(x - y)A(y)dy = A_{1,\rho}(x) - A_{2,\rho}(x),$$

with $A = A_1 - A_2$. Recall that, for $j = 1, 2$, supp$(A_{j,\rho}) \subset \overline{\Omega_1} + B_1 := \{x + y : x \in \overline{\Omega_1}, y \in B_1\}$. Moreover, fixing $R := \sup_{x' \in \overline{\Omega}} |x'|$, $R_1 := 2\sqrt{2(R + 2 + \frac{4\pi}{\rho^2})}$ and assuming that $|(s_1, s_2)| \geq R_1$, we find $|s_1| \geq \frac{R}{\sqrt{2}}$ or $|s_2| \geq \frac{R}{\sqrt{2}}$. In addition, since $\theta \cdot \xi' = 0$, we get

$$|(s_1, s_2)| \geq R_1 \implies |s_1\theta + s_2\xi'| = |(s_1, s_2)| |\xi'| \geq \max(|s_1|, |s_2|) > 2R + 4$$

and, for all $x = (x', x_3) \in B'_R \times \mathbb{R}$, we get

$$|(s_1, s_2)| \geq R_1 \implies |x' - s_1\theta - s_2\xi'| \geq |s_1\theta + s_2\xi' - |x'| \geq R + 3.$$  

Thus, for all $x = (x', x_3) \in B'_R \times \mathbb{R}$, the function

$$(s_1, s_2) \mapsto A_{j,\rho}(s_1\tilde{\theta} + s_2\eta + x)$$

will be supported in $B'_R$. Thus, we can define

$$\Phi_{1,\rho}(x) := -\frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{(\tilde{\theta} + i\eta) \cdot A_{1,\rho}(x - s_1\tilde{\theta} - s_2\eta)}{s_1 + is_2} ds_1 ds_2,$$

$$\Phi_{2,\rho}(x) := -\frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{(-\tilde{\theta} + i\eta) \cdot A_{2,\rho}(x + s_1\tilde{\theta} - s_2\eta)}{s_1 + is_2} ds_1 ds_2.$$  

(3.39)

Fixing

$$b_{1,\rho}(x) = e^{\Phi_{1,\rho}(x)}, \quad b_{2,\rho}(x) = e^{\Phi_{2,\rho}(x)},$$

we obtain

$$\nabla b_{1,\rho} + i(\tilde{\theta} + i\eta) \cdot A_{1,\rho}(x)b_{1,\rho} = 0, \quad (\tilde{\theta} + i\eta) \cdot \nabla b_{2,\rho} + i[(-\tilde{\theta} + i\eta) \cdot A_{2,\rho}(x)]b_{2,\rho} = 0, \quad x \in \Omega_1.$$  

(3.41)

Here, even if $A_{j,\rho}$, $j = 1, 2$, is not compactly supported, one can use the fact that the functions

$$(s_1, s_2) \mapsto A_{j,\rho}(s_1\theta + s_2\eta + s_3\xi), \quad s_3 \in \mathbb{R},$$

are compactly supported to deduce [3.41]. Moreover, using the fact that

$$(x - s_1\tilde{\theta} - s_2\eta) \notin \text{supp}(A_{j,\rho}), \quad x \in B'_R \times \mathbb{R}, \quad |(s_1, s_2)| > R_1, \quad j = 1, 2,$$

...
for all $x \in B'_{R+1} \times \mathbb{R}$, $j = 1, 2$, we deduce that

$$|\Phi_{j, \rho}(x)| \leq \frac{1}{2\pi} \int_{|{s_1, s_2}| \leq R_1} \frac{|A_{j, \rho}(x - s_1 \bar{\theta} - s_2 \eta)|}{|s_1 + i s_2|} ds_1 ds_2 \leq \frac{\|A_{j, \rho}\|_{L^\infty(\mathbb{R}^3)}}{2\pi} \int_{|{s_1, s_2}| \leq R_1} \frac{1}{|s_1 + i s_2|} ds_1 ds_2 \leq C,$$

with $C$ independent of $\rho$. This proves that

$$\|\Phi_{j, \rho}\|_{L^\infty(B'_{R+1} \times \mathbb{R})} \leq C.$$

In the same way, we can prove that

$$\|\Phi_{j, \rho}\|_{W^{k, \infty}(B'_{R+1} \times \mathbb{R})} \leq C_k \rho^{\frac{k}{4}}, \quad k \geq 0,$$

with $C_k$ independent of $\rho$. According to this estimate, we have

$$\|b_{j, \rho}\|_{W^{k, \infty}(B'_{R+1} \times \mathbb{R})} \leq C_k \rho^{\frac{k}{4}}, \quad k \geq 0.$$

Moreover, conditions (3.41), (3.43) and the fact that $C$ introduced in Proposition 2.4. The proof for the existence of the remainder term $F_{j, \rho}$ appears in (3.33)-(3.34) and satisfying the decay property (3.35). For this purpose, we will construct the remainder term $F_{j, \rho}$, $j = 1, 2$, we will complete the construction of the solutions $u_{j, \rho}$ of the form (3.33)–(3.34).

### 3.2. Remainder term of the CGO solutions.

In this subsection we will construct the remainder term $w_{j, \rho}$, $j = 1, 2$, appearing in (3.33)–(3.34) and satisfying the decay property (3.35). For this purpose, we will combine the Carleman estimate (2.22) with the properties of the expressions $b_{j, \rho}$, $j = 1, 2$, in order to complete the construction of these solutions. In this subsection, we assume that $\rho > \rho_2$ with $\rho_2$ the constant introduced in Proposition 2.4. The proof for the existence of the remainder term $w_{1, \rho}$ and $w_{2, \rho}$ being similar, we will only show the existence of $w_{1, \rho}$. Let us first remark that $w_{1, \rho}$ should be a solution of the equation

$$P_{A_{1, q_1} + w} = e^{-\rho \theta \cdot x'} (\Delta_{A_1} + q_1) e^{\rho \theta \cdot x'} w = e^{i \rho \eta \cdot x} F_{1, \rho}(x), \quad x \in \Omega_1,$$

with $F_{1, \rho}$ defined, for all $x = (x', x_3) \in B'_{R+1} \times \mathbb{R}$ (we recall that $B'_r := \{x' \in \mathbb{R}^2 : |x'| < r\}$ and $R = \sup |x'|$), by

$$F_{1, \rho}(x) = -e^{-\rho \theta \cdot x' - i \rho \eta \cdot x} (\Delta_{A_1} + q_1) \left[ e^{\rho \theta \cdot x' + i \rho \eta \cdot x} \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_{1, \rho} e^{-\xi \cdot x} \right]$$

$$= -\left( -|\xi|^2 + \text{div}(A_1 + q_1) \right) \psi \left( \rho^{-\frac{1}{2}} x_3 \right) + 2i \rho \psi \left( \rho^{-\frac{1}{2}} x_3 \right) - 2i \xi \rho \psi \left( \rho^{-\frac{1}{2}} x_3 \right) + b_{1, \rho} e^{-\xi \cdot x}$$

$$- \left[ \rho^{-\frac{1}{2}} \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_{1, \rho} + 2 \partial_{x_3} b_{1, \rho} \rho \psi \left( \rho^{-\frac{1}{2}} x_3 \right) - i 2 \xi \cdot \nabla b_{1, \rho} \psi \left( \rho^{-\frac{1}{2}} x_3 \right) \right] e^{-\xi \cdot x}$$

$$- 2 \rho (\bar{\theta} + i \eta) \cdot \nabla b_{1, \rho} + i [(\bar{\theta} + i \eta) - A_1] b_{1, \rho} \psi \left( \rho^{-\frac{1}{2}} x_3 \right) e^{-\xi \cdot x}.$$  

(3.47)
Here we consider $A_1$ as an element of $L^\infty(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$ satisfying $A_1 = 0$ on $\mathbb{R}^3 \setminus \Omega_1$. We fix $\varphi \in C_0^\infty(B_{R+1}^\prime; [0, 1])$ satisfying $\varphi = 1$ on $B_{R+\frac{1}{2}}^\prime$, and we define

$$G_\rho(x', x_3) := \varphi(x') F_{1, \rho}(x', x_3), \quad x' \in \mathbb{R}^2, \ x_3 \in \mathbb{R},$$

$$K_\rho(x) := G_\rho(x) - \varphi(x') \psi \left( \rho^{-\frac{4}{5}} x_3 \right) \text{div}(A_1) b_{1, \rho} e^{-i \xi \cdot x}, \quad x' \in \mathbb{R}^2, \ x_3 \in \mathbb{R}, \ x = (x', x_3).$$

From now on, combining (2.22) with (3.49), we will complete the construction of the remainder term $w_{1, \rho}$ by using a classical duality argument. More precisely, applying (2.22), we consider the linear form $\langle \cdot, \cdot \rangle_{H^{-1}_\rho(\mathbb{R}^3), L^2(\mathbb{R}^3)}$ and the fact that, using a change of variable, we find

$$\| \chi \left( \rho^{-\frac{4}{5}} x_3 \right) \|_{L^2(B_{R+1}^\prime \times \mathbb{R})} + \| \chi' \left( \rho^{-\frac{4}{5}} x_3 \right) \|_{L^2(B_{R+1}^\prime \times \mathbb{R})} + \| \chi'' \left( \rho^{-\frac{4}{5}} x_3 \right) \|_{L^2(B_{R+1}^\prime \times \mathbb{R})} \leq C \rho^{\frac{4}{5}},$$

we deduce that

$$\| K_\rho \|_{H^{-1}_\rho(\mathbb{R}^3)} \leq \rho^{-1} \| K_\rho \|_{L^2(\mathbb{R}^3)} = \rho^{-1} \| K_\rho \|_{L^2(B_{R+1}^\prime \times \mathbb{R})} \leq C(\| A_1 - A_{1, \rho} \|_{L^2(\mathbb{R}^3)^3} + \rho^{-\frac{4}{5}}). \quad (3.48)$$

In the same way, since supp(div$(A)) \subset \overline{\mathcal{M}} \times \mathbb{R} \subset B_{R+\frac{1}{2}}^\prime \times \mathbb{R}$, we have

$$\varphi(x') \psi \left( \rho^{-\frac{4}{5}} x_3 \right) \text{div}(A_1) b_{1, \rho} = \psi \left( \rho^{-\frac{4}{5}} x_3 \right) \text{div}(A_1) b_{1, \rho}.$$

Moreover, fixing

$$c_{1, \rho}(x) := \psi \left( \rho^{-\frac{4}{5}} x_3 \right) b_{1, \rho}(x), \quad x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R},$$

for any $h \in H^1_\rho(\mathbb{R}^3)$, we obtain

$$\left| \langle \text{div}(A_1) c_{1, \rho}, h \rangle_{H^{-1}_\rho(\mathbb{R}^3), H_\rho^1(\mathbb{R}^3)} \right| \leq \left| \langle A_1 \cdot \nabla c_{1, \rho}, h \rangle_{L^2(\mathbb{R}^3)} \right| + \left| \langle c_{1, \rho}, A_1 \cdot \nabla h \rangle_{L^2(\mathbb{R}^3)} \right| \leq \left| \langle A_1 \cdot \nabla c_{1, \rho}, h \rangle_{L^2(\mathbb{R}^3)} \right| + \left| \langle c_{1, \rho}, (A_1 - A_{1, \rho}) \cdot \nabla h \rangle_{L^2(\mathbb{R}^3)} \right| \leq \left( \| c_{1, \rho} \|_{W^{1, \infty}(\Omega_1)} \| A_1 \|_{L^2(\Omega_1)^3} \rho^{-1} + \| c_{1, \rho} \|_{L^\infty(B_{R+1}^\prime \times \mathbb{R})} \| A_1 - A_{1, \rho} \|_{L^2(\mathbb{R}^3)^3} \right) \| h \|_{H_\rho^1(\mathbb{R}^3)} + \left| \langle \text{div}(c_{1, \rho} A_1), h \rangle_{L^2(\mathbb{R}^3)} \right| \leq \left( \| c_{1, \rho} \|_{W^{1, \infty}(B_{R+1}^\prime \times \mathbb{R})} \| A_1 \|_{L^2(\Omega_1)^3} + \| A_{1, \rho} \|_{H^1(\mathbb{R}^3)^3} \right) \rho^{-1} + \| c_{1, \rho} \|_{L^\infty(B_{R+1}^\prime \times \mathbb{R})} \| A_1 - A_{1, \rho} \|_{L^2(\mathbb{R}^3)^3} \right) \| h \|_{H_\rho^1(\mathbb{R}^3)}.$$

Here we use the fact that supp$(A_{1, \rho}) \subset \Omega_1 + B_1 \subset B_{R+1}^\prime \times \mathbb{R}$. Combining this with (3.48) and (3.49), we find

$$\left| \langle \text{div}(A_1) c_{1, \rho}, h \rangle_{H^{-1}_\rho(\mathbb{R}^3), H_\rho^1(\mathbb{R}^3)} \right| \leq C(\rho^{-\frac{4}{5}} + \| A_1 - A_{1, \rho} \|_{L^2(\mathbb{R}^3)^3}) \| h \|_{H_\rho^1(\mathbb{R}^3)}$$

and it follows

$$\| \psi \left( \rho^{-\frac{4}{5}} x_3 \right) \text{div}(A_1) b_{1, \rho} \|_{H^{-1}_\rho(\mathbb{R}^3)} \leq C(\rho^{-\frac{4}{5}} + \| A_1 - A_{1, \rho} \|_{L^2(\mathbb{R}^3)^3})).$$

Then, (3.48) implies

$$\| G_\rho \|_{H^{-1}_\rho(\mathbb{R}^3)} \leq C(\| A_1 - A_{1, \rho} \|_{L^2(\mathbb{R}^3)^3} + \rho^{-\frac{4}{5}}). \quad (3.49)$$

From now on, combining (2.22) with (3.49), we will complete the construction of the remainder term $w_{1, \rho}$ by using a classical duality argument. More precisely, applying (2.22), we consider the linear form $T_\rho$ defined on $\mathcal{Q} := \{ P_{A_1, \overline{\mathcal{M}}}, w : w \in C_0^\infty(\Omega_1) \}$ by

$$T_\rho(P_{A_1, \overline{\mathcal{M}}}, w) := \langle G_\rho, e^{-i \eta \cdot \xi} v \rangle_{H^{-1}_\rho(\mathbb{R}^3), H_\rho^1(\mathbb{R}^3)}; \quad v \in C_0^\infty(\Omega_1).$$

Here and from now on we define the duality bracket $\langle \cdot, \cdot \rangle_{H^{-1}_\rho(\mathbb{R}^3), L^2(\mathbb{R}^3)}$ in the complex sense, which means that

$$\langle v, w \rangle_{H^{-1}_\rho(\mathbb{R}^3), L^2(\mathbb{R}^3)} = \langle v, w \rangle_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} v \overline{w} dx, \quad v \in L^2(\mathbb{R}^3), \ w \in H^1(\mathbb{R}^3).$$
Applying again (2.22), for all $v \in C_0^\infty(\Omega_1)$, we obtain
\[
|T_\rho(P_{A,\overline{\rho}}^{-1},v)| \leq \|G_\rho\|_{H^{-1}_\rho((R^3_3))} \|e^{-ip\rho \cdot x}v\|_{H^1_\rho((R^3_3))}
\leq 2\rho \|G_\rho\|_{H^{-1}_\rho((R^3_3))} \rho^{-1} \|v\|_{H^3_\rho((R^3_3))}
\leq C\rho \|G_\rho\|_{H^{-1}_\rho((R^3_3))} \|P_{A,\overline{\rho}}^{-1},v\|_{H^{-1}_\rho((R^3_3))},
\]
with $C > 0$ independent of $\rho$. Thus, applying the Hahn-Banach theorem, we deduce that $T_\rho$ admits an extension as a continuous linear form on $H^{-1}_\rho((R^3_3))$ whose norm will be upper bounded by $C\rho \|G_\rho\|_{H^{-1}_\rho((R^3_3))}$.

Therefore, there exists $w_{1,\rho} \in H^1_\rho((R^3_3))$ such that
\[
\langle P_{A,\overline{\rho}}^{-1},v, w_{1,\rho}\rangle_{H^{-1}_\rho((R^3_3)),H^1_\rho((R^3_3))} = T_\rho(P_{A,\overline{\rho}}^{-1},v) = \langle G_\rho, e^{-ip\rho \cdot x}v\rangle_{H^{-1}_\rho((R^3_3)),H^1_\rho((R^3_3))},
\]
$v \in C_0^\infty(\Omega_1)$, (3.50)
\[
\|w_{1,\rho}\|_{H^1_\rho((R^3_3))} \leq C\rho \|G_\rho\|_{H^{-1}_\rho((R^3_3))}.
\]
(3.51)

From (3.50) and the fact that, for all $x \in \Omega_1$, $G_\rho(x) = F_{1,\rho}(x)$, we obtain
\[
\langle P_{A,\overline{\rho}},w_{1,\rho}\rangle_{D'(\Omega_1),C_0^\infty(\Omega_1)} = \langle P_{A,\overline{\rho}}^{-1},v, w_{1,\rho}\rangle_{H^{-1}_\rho((R^3_3)),H^1_\rho((R^3_3))} = \langle G_\rho, e^{-ip\rho \cdot x}v\rangle_{H^{-1}_\rho((R^3_3)),H^1_\rho((R^3_3))} = \langle e^{ip\rho \cdot x}F_{1,\rho},v\rangle_{D'(\Omega_1),C_0^\infty(\Omega_1)}.
\]

It follows that $w_{1,\rho}$ solves $P_{A,\overline{\rho}}^{-1} + w_{1,\rho} = e^{ip\rho \cdot x}F_{1,\rho}$ in $\Omega_1$ and $u_1$ given by (3.49) is a solution of $\Delta_{A_1}u_1 + q_1u_1 = 0$ in $\Omega_1$ lying in $H^1(\Omega_1)$. In addition, from (3.49) and (3.51), we deduce that
\[
\rho^{-1}\|w_{1,\rho}\|_{H^1(\Omega_1)} + \|w_{1,\rho}\|_{L^2(\Omega_1)} \leq 2\rho^{-1}\|w_{1,\rho}\|_{H^1(\Omega_1)} \leq C(\|A_1 - A_{1,\rho}\|_{L^2(\Omega_3)}^3 + \rho^{-\frac{3}{2}})
\]
(3.52)
which implies the decay property (3.44). This completes the proof of Theorem 3.4.

4. Uniqueness Result

In this section we will use the result of the preceding section in order to complete the proof of Theorem 1.1.

Namely under the assumption of Theorem 1.1 we will show that (1.3) implies that $dA_1 = dA_2$. Then, assuming $A = A_1 - A_2 \in C(\Omega)$, we will prove that $q_1 = q_2$. For $j = 1, 2$, we assume that $A_j \in L^\infty(\Omega) \cap L^2(\Omega)^3$ and $q_j \in L^\infty(\Omega; C)$ with $A_1$ and $A_2$ extended by 0 on $\Omega \setminus \Omega$. We use here the notation of the previous sections and we assume that $A = A_1 - A_2 \in L^1(\Omega)$. We start with the recovery of the magnetic field.

4.1. Recovery of the magnetic field. In this subsection we will prove that (1.3) implies that $dA_1 = dA_2$.

Let us first remark that $A_\rho = A_{1,\rho} - A_{2,\rho} = \chi_\rho \ast A$ and, since $A \in L^1(\Omega)$, by density one can check that
\[
\lim_{\rho \to +\infty} \|A_\rho - A\|_{L^1(\Omega)} = 0.
\]
(4.53)

For $j = 1, 2$, we fix $u_j \in H^1(\Omega)$ a solution of $\Delta_{A_j}u_j + q_ju_j = 0$, $\Delta_{A_j}u_j + \overline{\mu_2}u_j = 0$ in $\Omega_1$ of the form (3.33) with $\rho > \rho_2$ and with $u_j \rho$ satisfying (3.30). In view of (1.1), we can see that the restriction of $u_1$ (resp. $u_2$) to $\Omega$ is lying in $H^1(\Omega)$ and it solves the equation $\Delta_{A_j}u_j + q_ju_j = 0$ (resp. $\Delta_{A_j}u_j + \overline{\mu_2}u_j = 0$) in $\Omega$. From now on, we consider the restriction to $\Omega$ of these CGO solutions initially defined on $\Omega$.

In view of (1.1), we can find $v_2 \in H^1(\Omega)$ satisfying $\Delta_{A_j}v_2 + \overline{\mu_2}v_2 = 0$ with $\tau v_2 = \tau u_1$ and $N_{A_j,q_j}u_1 = N_{A_j,q_j}v_2$. Therefore, we have
\[
0 = \langle N_{A_1,q_1}u_1, \tau u_2 \rangle - \langle N_{A_2,q_2}u_2, \tau u_2 \rangle = \langle N_{A_1,q_1}u_1, \tau u_2 \rangle - \langle N_{A_2,q_2}u_2, \tau u_2 \rangle
= \langle N_{A_1,q_1}u_1, \tau u_2 \rangle - \langle N_{A_2,q_2}u_2, \tau u_1 \rangle
= i \int_{\Omega} (A_j \cdot \nabla u_1) \overline{\theta} dx - i \int_{\Omega} u_1 (A_j \cdot \nabla u_2) dx + \int_{\Omega} \overline{\theta} u_1 \overline{u_2} dx,
\]
Lemma 6.2] for its proof. Since here we deal with non-compact ly supported magnetic potentials, the proof of Lemma 4.1.

\[ \Phi \]

and we deduce that

\[ \Phi \]

Proof. For

\[ \Phi \]

\[ \Phi \]

\[ \Phi \]

\[ \Phi \]

\[ \Phi \]

\[ \Phi \]

\[ \Phi \]

\[ \Phi \]

Here we use (3.42) and the fact that by Lebesgue dominate convergence theorem

\[ \Phi \]

Combining this with (3.42) and (4.53), we obtain

\[ \Phi \]

On the other hand, one can easily check that

\[ \Phi \]

and we deduce that

\[ \Phi \]

Now let us consider the following intermediate result.

Lemma 4.1. We have

\[ \Phi \]

Proof. For \( \Phi \) compactly supported this result is well known and one can refer to [34] Proposition 3.3 or [12] Lemma 6.2 for its proof. Since here we deal with non-compactely supported magnetic potentials, the proof of the result will be required. From now on, to every \( \Phi \) \( \in \mathbb{R}^3 \), we associate the coordinate \( (\Phi, x) \in \mathbb{R}^2 \times \mathbb{R} \), with \( \Phi = (\Phi^1, \Phi^2, \Phi^3) = (\Phi, x) \). Recall that \( \Phi(\Phi, x) \subset B_{R+1} \times \mathbb{R} \) and, fixing \( \Phi(\Phi, x) : (\Phi, x) \mapsto A_p(x) \), in a similar way to Subsection 3.1, we find

\[ \Phi \]

Thus, fixing \( \Phi(\Phi, x) \mapsto \Phi(\Phi, x) \), for \( |x| > R_1 \) we have

\[ \Phi \]

It follows that

\[ \Phi \]

and in particular, for every \( x \in \mathbb{R} \), we get

\[ \Phi \]

On the other hand, using the fact that

\[ \Phi \]

and the fact that \( A_p \in L^1(\mathbb{R}^3) \), by Fubini’s theorem we find

\[ \Phi \]
Moreover, for all \( r > 0 \) fixing \( n = (n_1, n_2) \) the outward unit normal vector to \( B'_r \), we have

\[
\int_{|x''| < r} (\partial_{x'_1} + i \partial_{x'_2})e^{\Phi_\rho(x'', x_*)} dx'' = \int_{|x''| = r} e^{\Phi_\rho(x'', x_*)}(n_1 + in_2)ds(x'').
\]

Applying (4.56), we find

\[
e^{\Phi_\rho(x'', x_*)} = 1 + \Phi_\rho(x'', x_*) + O(|x''|^{-2})
\]

and it follows

\[
\int_{|x''| < r} (\partial_{x'_1} + i \partial_{x'_2})e^{\Phi_\rho(x'', x_*)} dx'' = \int_{|x''| = r} (n_1 + in_2)ds(x'') + \int_{|x''| = r} \Phi_\rho(x'', x_*)(n_1 + in_2)ds(x'') + O(r^{-1}).
\]

In addition, we get

\[
\int_{|x''| = r} (n_1 + in_2)ds(x'') = \int_{|x''| < r} (\partial_{x'_1} + i \partial_{x'_2})1dx'' = 0,
\]

\[
\int_{|x''| = r} \Phi_\rho(x'', x_*)(n_1 + in_2)ds(x'') = \int_{|x''| < r} (\partial_{x'_1} + i \partial_{x'_2})\Phi_\rho(x'', x_*)dx''
\]

and sending \( r \to +\infty \) in (4.58), we obtain

\[
\int_{\mathbb{R}^3} (A_\rho \cdot (\tilde{\theta} + i \eta))e^{\Phi_\rho}e^{-ix \cdot \xi} dx = i \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\partial_{x'_1} + i \partial_{x'_2})\Phi_\rho(x'', x_*)dx'' e^{-ix \cdot |\xi|} dx_*
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} (\tilde{\theta} + i \eta) \cdot \tilde{A}_\rho(x'', x_*)dx'' \right) e^{-ix \cdot |\xi|} dx_*.
\]

From this identity, we deduce (4.55). \( \square \)

Combining (4.53) and (4.54)-(4.55), we obtain

\[
(\tilde{\theta} + i \eta) \cdot \mathcal{F}(A)(\xi) = \lim_{\rho \to +\infty} (\tilde{\theta} + i \eta) \cdot \mathcal{F}(A_\rho)(\xi) = 0.
\]

In the same way, replacing \( \eta \) by \(-\eta\) in our analysis, we find \((\tilde{\theta} - i \eta) \cdot \mathcal{F}(A)(\xi) = 0\) and it follows \(\tilde{\theta} \cdot \mathcal{F}(A)(\xi) = \eta \cdot \mathcal{F}(A)(\xi) = 0\). Combining this with the fact that \((\tilde{\theta}, \eta)\) is an orthonormal basis of \(\xi^\perp = \{y \in \mathbb{R}^3 : y \cdot \xi = 0\}\), we find

\[
\zeta \cdot \mathcal{F}(A)(\xi) = 0, \quad \zeta \in \xi^\perp.
\]

Moreover, for \(1 \leq j < k \leq 3\), fixing \( \zeta = \xi_k e_j - \xi_j e_k \), with

\[
e_j = (0, \ldots, 0, \underset{\text{position } j}{1}, 0, \ldots, 0), \quad e_k = (0, \ldots, 0, \underset{\text{position } k}{1}, 0, \ldots, 0),
\]

(4.59) implies

\[
\xi_k \mathcal{F}(a_j)(\xi) - \xi_j \mathcal{F}(a_k)(\xi) = 0, \quad 1 \leq j < k \leq 3,
\]

where \(A = (a_1, a_2, a_3)\). Recall that so far, we have proved (4.60) for any \( \xi = (\xi', \xi) \in \mathbb{R}^2 \times \mathbb{R} \) with \(\xi' \neq 0\) and \(\xi_3 \neq 0\). Since \(A \in L^1(\mathbb{R}^3)^3\), we can extend this identity to any \( \xi \in \mathbb{R}^3 \) by using the continuity of \(\mathcal{F}(A)\).

Then, we deduce from (4.60) that

\[-i \mathcal{F}(\partial_{x'_1} a_j - \partial_{x'_2} a_k)(\xi) = \xi_k \mathcal{F}(a_j)(\xi) - \xi_j \mathcal{F}(a_k)(\xi) = 0, \quad 1 \leq j < k \leq 3, \quad \xi \in \mathbb{R}^3.
\]

This proves that in the sense of distribution we have \(dA = 0\) and \(dA_1 = dA_2\).
4.2. Recovery of the electric potential. In this subsection we assume that \( A \in L^\infty(\mathbb{R}^3)^3, dA = 0 \) are fulfilled and we will prove that \( q_1 = q_2 \). We start, with the following.

**Lemma 4.2.** Let \( A = (a_1, \ldots, a_3) \in L^\infty(\mathbb{R}^3)^3 \). Assume that \( dA = 0 \), and fix

\[
\varphi(x) := \int_0^1 A(sx) \cdot xds, \quad x \in \mathbb{R}^3.
\]  

(4.61)

Then, we have \( \varphi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^3) \) and \( \nabla \varphi = A \).

**Proof.** Note first that since \( A \in L^\infty(\mathbb{R}^3)^3 \), we have \( \varphi \in L^\infty_{\text{loc}}(\mathbb{R}^3) \). Let \( \psi \in C_0^\infty(\mathbb{R}^3) \) and consider \( j \in \{1, 2, 3\} \). We have

\[
\langle \partial_x \varphi, \psi \rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} = -\langle \varphi, \partial_x \psi \rangle_{L^2(\mathbb{R}^3)}
\]

\[
= -\sum_{k=1}^3 \int_{\mathbb{R}^3} \int_0^1 x_k a_k(sx) \partial_x \psi(x) ds dx
\]

\[
= -\sum_{k=1}^3 \int_{\mathbb{R}^3} \int_0^1 x_k a_k(sx) \partial_x \psi(x) dx ds.
\]

Applying the change of variable \( y = sx \) and then \( t = s^{-1} \), we obtain

\[
\langle \partial_x \varphi, \psi \rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} = -\sum_{j=1}^3 \int_0^1 t^{-4} \left( \int_{\mathbb{R}^3} y_j a_j(y) \partial_x \psi(t^{-1}y) dy \right) ds
\]

\[
= -\sum_{k=1}^3 \int_1^{t^{-1}} t^2 \left( \int_{\mathbb{R}^3} y_k a_k(y) \partial_x \psi(ty) dy dt
\]

\[
= \int_1^{t^{-1}} t \left\langle \partial_x \left( \sum_{k=1}^3 x_k a_k \right), \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} dt,
\]

with, for \( \tau \in \mathbb{R}, \psi(\tau \cdot) := x \mapsto \psi(\tau x) \). On the other hand, we have

\[
\left\langle \partial_x \left( \sum_{k=1}^3 x_k a_k \right), \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} = \left\langle a_j, \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} + \left\langle \sum_{k=1}^3 x_k \partial_x a_k, \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)}
\]

and using the fact that \( dA = 0 \), we get

\[
\left\langle \partial_x \left( \sum_{k=1}^3 x_k a_k \right), \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} = \left\langle a_j, \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} + \left\langle \sum_{k=1}^3 x_k \partial_x a_k, \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)}
\]

\[
= -2 \left\langle a_j, \psi(t \cdot) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)} - t \left\langle a_j, \left( \sum_{k=1}^3 x_k \partial_x \psi(t \cdot) \right) \right\rangle_{D'(\mathbb{R}^3), C_0^\infty(\mathbb{R}^3)}.
\]
It follows
\[
\left\langle \partial x_j \varphi, \psi \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)} = -\int_1^{+\infty} 2t \left\langle a_j, \psi(t) \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)} dt - \int_1^{+\infty} t^2 \partial_t \left\langle a_j, \psi(t) \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)} dt
\]
\[
= -\int_1^{+\infty} \partial_t \left[ t^2 \left\langle a_j, \psi(t) \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)} \right] dt
\]
\[
= \left\langle a_j, \psi \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)} - \lim_{t \to +\infty} t^2 \left\langle a_j, \psi(t) \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)} = \left\langle a_j, \psi \right\rangle_{D'(\mathbb{R}^3), C_c^\infty(\mathbb{R}^3)}.
\]
This proves that \( \nabla_x \varphi = A \) and it completes the proof of the lemma. \( \square \)

According to Lemma 4.2, the function \( \varphi \in W^{1,\infty}(\mathbb{R}^3) \) given by \( \text{(4.01)} \) satisfies \( \nabla \varphi = A \). Since \( \omega \) is simply connected \( \Omega_1 = \omega \times \mathbb{R} \) is also simply connected and \( \mathbb{R}^3 \setminus \Omega_1 \) is connected. Therefore, according to the fact that \( A = 0 \) in \( \mathbb{R}^3 \setminus \Omega_1 \), by extracting a constant to \( \varphi \) we may assume that \( \varphi = 0 \) on \( \mathbb{R}^3 \setminus \Omega_1 \). Thus, we have \( \varphi \vert_{\partial \Omega_1} = 0 \). Note also that by eventually extending \( \omega \), we may assume that \( \Omega_1 \) contains a neighborhood of \( \overline{\Omega} \). Now, for \( A \in L^\infty(\Omega_1)^3 \) and \( q \in L^\infty(\Omega_1) \) let us consider the set of data
\[
\mathcal{D}_{1,A,q} := \{(\tau_1 u, N_{1,A,q} u) : u \in H^1(\Omega_1), \Delta_A u + qu = 0\},
\]
where \( \tau_1 \) is the extension of the map \( u \mapsto u \vert_{\partial \Omega_1} \) and, for any solution \( u \in H^1(\Omega_1) \) of \( \Delta_A u + qu = 0 \) on \( \Omega_1 \), \( N_{1,A,q} u \) denotes the unique elements of \( H^{-1}(\partial \Omega_1) \) satisfying
\[
\langle N_{1,A,q} u, \tau_1 g \rangle = \int_{\Omega_1} \left( \nabla + iA \right) u \cdot \left( \nabla + iA \right) g dx + \int_{\Omega_1} qu \overline{g} dx, \quad g \in H^1(\Omega_1).
\]
Repeating some arguments of [33] Proposition 3.4 (see also [41] Lemma 4.2), one can easily check the following.

**Proposition 4.3.** For \( j = 1, 2 \), let \( A_j \in L^\infty(\Omega_1)^3 \), \( q_j \in L^\infty(\Omega_1) \) and assume that
\[
A_1(x) = A_2(x), \quad q_1(x) = q_2(x), \quad x \in \Omega_1 \setminus \Omega.
\]
Then the condition \( \text{(4.3)} \) implies that \( \mathcal{D}_{1,A_1,q_1} = \mathcal{D}_{1,A_2,q_2} \).

In view of this result and the fact that \( A_1 = A_2 = 0 \) and \( q_1 = q_2 = 0 \) on \( \Omega_1 \setminus \Omega \), we deduce that \( \mathcal{D}_{1,A_1,q_1} = \mathcal{D}_{1,A_2,q_2} \). Moreover, using the fact that \( A_1 - A_2 = \nabla \varphi \) with \( \varphi \in W^{1,\infty}(\Omega_1) \) satisfying \( \varphi \vert_{\mathbb{R}^3 \setminus \Omega_1} = 0 \), we obtain
\[
\mathcal{D}_{1,A_1,q_2} = \mathcal{D}_{1,A_2,v + \varphi,q_2} = \mathcal{D}_{1,A_2,q_2} = \mathcal{D}_{1,A_1,q_1}.
\]
Therefore, repeating the argumentation of Section 4.1, with \( A_1 = A_2 \), we find
\[
\lim_{\rho \to +\infty} \int_{\mathbb{R}^3} q(x) \psi^2(\rho^{-\frac{1}{4}} x_3) e^{-ix \cdot \xi} dx = 0,
\]
for all \( \xi = (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R} \) with \( \xi' \neq 0 \) and \( \xi_3 \neq 0 \). Here we have used the fact that, following our definition, \( A_{1,\rho} = A_{2,\rho} = \Phi_{2,\rho} = -\Phi_{1,\rho} \) and \( b_{1,\rho} b_{2,\rho} = 1 \). In \( \text{Lemma 4.2} \), we can assume for instance that \( \psi = 1 \) on \([-1, 1] \). We fix \( q_0(x', x_3) = q(x', x_3) \psi^2(\rho^{-\frac{1}{4}} x_3) \), \( (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \) and we remark that
\[
\|\mathcal{F}(q_0) - \mathcal{F}(q)\|_{L^2(\mathbb{R}^3)}^2 = \|q_0 - q\|_{L^2(\mathbb{R}^3)}^2 \leq \int_{\mathbb{R}^3} \left( 1 - \psi^2(\rho^{-\frac{1}{2}} x_3) \right) |q(x)|^2 dx
\]
\[
\leq \int_{|x_3| > \rho^{\frac{1}{4}}} \left( \int_{\mathbb{R}^2} |q(x', x_3)|^2 dx' \right) dx_3.
\]
Combining this with the fact that, according to Fubini’s theorem,
\[
x_3 \mapsto \left( \int_{\mathbb{R}^2} |q(x', x_3)|^2 dx' \right) \in L^1(\mathbb{R}),
\]
we deduce that
\[ \lim_{\rho \to +\infty} \|F(q_{\rho}) - F(q)\|_{L^2(\mathbb{R}^3)} = 0. \]
Thus, there exists a sequence \((\rho_k)_{k \in \mathbb{N}}\) such that \(\rho_k \to +\infty\) and for a.e. \(\xi \in \mathbb{R}^3\) we have
\[ \lim_{k \to +\infty} F(q_{\rho_k})(\xi) = F(q)(\xi). \]
Combining this with (4.62), we obtain that \(F(q) = 0\) which implies that \(q = 0\) and \(q_1 = q_2\). This completes the proof of Theorem 1.1.

5. Recovery from Measurements on a Bounded Portion of \(\partial \Omega\)

In this section we will prove Theorem 1.2 and we assume that the conditions of this theorem are fulfilled. Recall that \(\tau_0\) denotes the extension of the map \(u \mapsto u|_{\partial \Omega}\) to \(u \in H^1(\Omega)\) which takes values in \(H^+_{loc}(\partial \Omega)\).

Consider the sets of functions
\[ Q_{A,q} := \{ u \in H^1(\Omega) : \Delta_A u + qu = 0 \}, \]
\[ Q_{A,q,r} := \{ u \in Q_{A,q} : \text{supp}(\tau_0 u) \subset S_r \}, \quad j = 1, 2. \]
Here we recall that \(S_r = \partial \Omega \cap (\mathcal{F} \times [r, +\infty))\). We have the following density result.

**Proposition 5.1.** The space \(Q_{A_1,q_1,r}\) (resp. \(Q_{A_2,q_2,r}\)) is dense in \(Q_{A_1,q_1}\) (resp. \(Q_{A_2,q_2}\)) for the topology induced by \(L^2(\Omega \setminus (\Omega_- \cup \Omega_+))\).

**Proof.** The proof of these two results being similar, we will only show the density of \(Q_{A_1,q_1,r}\) in \(Q_{A_1,q_1}\). We will prove the proposition by contradiction. Assume that \(Q_{A_1,q_1,r}\) is not dense in \(Q_{A_1,q_1}\). Then, there exist \(h \in L^2(\Omega \setminus (\Omega_- \cup \Omega_+))\) and \(v_0 \in Q_{A_1,q_1}\) such that
\[ \int_{\Omega \setminus (\Omega_- \cup \Omega_+)} h \varphi dx = 0, \quad v \in Q_{A_1,q_1,r}. \]  (5.63)
\[ \int_{\Omega \setminus (\Omega_- \cup \Omega_+)} h \varphi_0 dx \neq 0. \]  (5.64)
Let us mention that in contrast to several other related density result (e.g. [13] Proposition 3.1 and [30] Lemma 6.1) we consider a general unbounded domain \((\Omega)\) and we can not apply the Green formula as the usual sense. To avoid such difficulties, here we proceed differently than other related results.

From now on, we extend \(h\) by 0 to \(\Omega\). In view of Assumption 1, there exists \(u \in H_0^1(\Omega)\) such that \(\Delta_{A_1} u + q_1 u = h\). Then, condition (5.63) implies
\[ \int_{\Omega} (\Delta_{A_1} + q_1) uv dx = 0, \quad v \in Q_{A_1,q_1,r}. \]  (5.65)

Moreover, for any \(\varphi \in C_c^\infty(\Omega)\) and any \(w \in H^1(\Omega)\), we have
\[ 2i \int_{\Omega} (A_1 \cdot \nabla \varphi) \overline{w} dx = 2i \langle w A_1, \nabla \varphi \rangle_{L^2(\Omega)} + 2i \langle \overline{w} A_1, \varphi \rangle_{L^2(\Omega)} \]
\[ = -2i \langle \text{div}(w A_1), \varphi \rangle_{L^2(\Omega)} + 2i \int_{\Omega} \varphi (2iA_1 \cdot \nabla w) dx. \]  (5.66)
By density we can extend this identity to \(\varphi \in H_0^1(\Omega)\). Combining this with the fact that \(u \in H_0^1(\Omega)\), for any \(v \in Q_{A_1,q_1,r}\), we obtain
\[ \int_{\Omega} \Delta u \overline{v} dx - \int_{\Omega} u \overline{\Delta v} dx = \int_{\Omega} (\Delta_{A_1} + q_1) uv dx - \int_{\Omega} \overline{u} (\Delta_{A_1} + q_1) v dx \]
\[ = \int_{\Omega \setminus (\Omega_- \cup \Omega_+)} h \varphi_0 dx = 0. \]  (5.67)
On the other hand, in view of Assumption 1, for any \( F \in C_0^\infty(\mathbb{R}^3) \), satisfying \( \text{supp}(F|_{\partial \Omega}) \subset S_r \), we can define \( w_F \in H_0^1(\Omega) \) solving \( \Delta A_1 w_F + q_1 w_F = -\Delta A_1 F + q_1 F \) and \( v = w_F + F \in Q_{A_1,q_1,r} \). Using this choice for the element \( v \in Q_{A_1,q_1,r} \) in (5.67), we deduce that

\[
\int_{\Omega} \Delta u(w_F + F) dx - \int_{\Omega} u(\Delta w_F + \Delta F) dx = 0. \tag{5.68}
\]

In addition, since \( u \in H_0^1(\Omega) \) and \( w_F \in H_0^1(\Omega) \), one can check by density that

\[
\int_{\Omega} \Delta u w_F dx - \int_{\Omega} u \Delta w_F dx = -\int_{\Omega} \nabla u \cdot \nabla w_F dx + \int_{\Omega} \nabla u \cdot \nabla w_F dx = 0.
\]

Combining this with (5.68), we get

\[
\int_{\Omega} \Delta u F dx - \int_{\Omega} u \Delta F dx = 0, \quad F \in \{ G \in C_0^\infty(\mathbb{R}^3) : \text{supp}(G|_{\partial \Omega}) \subset S_r \}. \tag{5.69}
\]

We fix \( \gamma_1 \) an open set of \( \partial \Omega \) such that \( \gamma_1 \subset (S_r \setminus [\partial \Omega \cap (\mathcal{S} \times [\delta - r, r - \delta])] \). Then, we consider \( \Omega_* \) a bounded subset of \( \mathbb{R}^3 \setminus \Omega \) with no empty interior such that \( \Omega_* \cap \partial \Omega \subset \gamma_1 \) and such that \( \Omega_{-,*} := \Omega_- \cup \Omega_* \) is an open connected set of \( \mathbb{R}^3 \). Applying (5.69) and (5.68), we deduce that the extension of \( u \) by zero to \( \Omega_{-,*} \) satisfies

\[
\left\{ \begin{array}{ll}
(\Delta A_1 + q_i)u = 0 & \text{in } \Omega_{-,*}, \\
u \in H^1(\Omega_*) &= 0.
\end{array} \right.
\]

Then, applying the unique continuation property for elliptic equations (e.g. [20, Theorem 1.1] and [45, Theorem 1]), we deduce that \( u|_{\Omega_-} = 0 \). In the same way, we can prove that \( u|_{\Omega_+} = 0 \). Using these properties, we would like to prove the following identity

\[
\int_{\Omega} \Delta A_1 u v_0 dx = \int_{\Omega} u \Delta A_1 v_0 dx, \tag{5.70}
\]

where we recall that \( v_0 \) satisfies (6.64). For this purpose, we first recall that in a similar way to (5.67), we can show that

\[
\int_{\Omega} \Delta u \overline{v_0} dx - \int_{\Omega} u \Delta \overline{v_0} dx = \int_{\Omega} \Delta A_1 u \overline{v_0} dx - \int_{\Omega} u \Delta A_1 \overline{v_0} dx.
\]

Thus, we only need to prove that

\[
\int_{\Omega} \Delta u \overline{v_0} dx = \int_{\Omega} u \Delta \overline{v_0} dx, \tag{5.71}
\]

for showing (5.70). Let \( \varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^3) \) be such that \( \varphi_1 = 1 \) on \( \mathcal{S} \times [\frac{\delta}{2} - r, r - \frac{\delta}{2}] \), \( \varphi_2 = 1 \) on a neighborhood of \( \text{supp}(\varphi_1) \) and \( \text{supp}(\varphi_2) \cap \partial \Omega \subset (\mathcal{S} \times [\frac{\delta}{2} - r, r - \frac{\delta}{2}]) \). Since \( \text{supp}(\varphi_2 v_0) \cap \partial \Omega \subset S_r \) and

\[
\Delta A_1 (\varphi_2 v_0) = -q_1 \varphi_2 v_0 + 2 \nabla \varphi_2 \cdot \nabla v_0 + (\Delta A_1 \varphi_2) v_0 \in L^2(\Omega),
\]

in a similar way to (5.69), we can apply Assumption 1 and (5.63) in order to get

\[
\int_{\Omega} \Delta u \varphi_2 v_0 dx - \int_{\Omega} u \Delta (\varphi_2 v_0) dx = 0. \tag{5.72}
\]

In addition, using the fact that \( \varphi_2 = 1 \) on a neighborhood of \( \text{supp}(\varphi_1) \), we get

\[
\int_{\Omega} \Delta u((1 - \varphi_2) v_0) dx = \int_{\Omega} \Delta[(1 - \varphi_1) u (1 - \varphi_2) v_0] dx. \tag{5.73}
\]

On the other hand, using the fact that

\[
\Omega_- \cup \left( \omega \times [\frac{\delta}{2} - r, r - \frac{\delta}{2}] \cap \Omega \right) \cup \Omega_+
\]
corresponds to the intersection between a neighborhood of $\partial \Omega$ and $\Omega$, with the fact that

$$(1 - \varphi_1)u(x) = 0, \quad x \in \Omega - U \left( \omega \times \left[ \frac{\delta}{2} - r, \frac{\delta}{2} \right] \cap \Omega \right) \cup \Omega_+,$$

we deduce that the function $(1 - \varphi_1)u$ extended by zero to $\mathbb{R}^3$, satisfies $\nabla[(1 - \varphi_1)u] \in L^2(\mathbb{R}^3)$ and $\text{div}(\nabla[(1 - \varphi_1)u]) = \Delta[(1 - \varphi_1)u] \in L^2(\mathbb{R}^3)$. Moreover, combining (5.74) with the arguments used in the proof of Theorem 3.4 page 223, we can find a sequence of functions $G_k \in C_0^\infty(\Omega)^3$ such that

$$\lim_{k \to +\infty} \|G_k - \nabla[(1 - \varphi_1)u]\|_{L^2(\Omega)} = \lim_{k \to +\infty} \|\text{div}(G_k) - \Delta[(1 - \varphi_1)u]\|_{L^2(\Omega)} = 0.$$

Then, we have

$$\int_\Omega \text{div}(G_k)((1 - \varphi_2)v_0)dx = \langle (1 - \varphi_2)v_0, \text{div}(G_k) \rangle_{D'(\Omega), C_0^\infty(\Omega)}$$

$$= -\langle \nabla[(1 - \varphi_2)v_0], G_k \rangle_{(C_0^\infty(\Omega)^3)', C_0^\infty(\Omega)^3}$$

$$= -\int_\Omega G_k \cdot \nabla[(1 - \varphi_2)v_0]dx$$

and sending $k \to +\infty$, we obtain

$$\int_\Omega \Delta[(1 - \varphi_1)u](1 - \varphi_2)v_0 dx = -\int_\Omega \nabla[(1 - \varphi_1)u] \cdot (\nabla[(1 - \varphi_2)v_0])dx.$$

Then, using the fact that $(1 - \varphi_1)u \in H^1_0(\Omega)$, we find

$$\int_\Omega \Delta[(1 - \varphi_1)u](1 - \varphi_2)v_0 dx = -\int_\Omega \nabla[(1 - \varphi_1)u] \cdot (\nabla[(1 - \varphi_2)v_0])dx = \int_\Omega [(1 - \varphi_1)u](\Delta[(1 - \varphi_2)v_0])dx.$$

Combining this with (5.73) and applying again the fact that $\varphi_2 = 1$ on a neighborhood of $\text{supp}(\varphi_1)$, we find

$$\int_\Omega \Delta u(1 - \varphi_2)v_0 dx = \int_\Omega [(1 - \varphi_1)u](\Delta[(1 - \varphi_2)v_0])dx = \int_\Omega u(\Delta[(1 - \varphi_2)v_0])dx.$$

From this identity and (5.71), we deduce (5.74) and by the same way (5.70), we find

$$\int_\Omega u(v_0 dx) = \int_\Omega (\Delta_{A_1} + q_1)v_0 dx = \int_\Omega u(\Delta_{A_1} + q_1)v_0 dx = 0.$$

This contradicts (5.63). We have completed the proof of the proposition. \[\square\]

**Proof of the Theorem 1.2** Let $u_1 \in Q_{A_1, q_1, r}$ and $u_2 \in Q_{A_2, q_2, r}$. In a similar way to Section 4, we can prove that (1.11) implies

$$i \int_\Omega (A \cdot \nabla u_1) \overline{u_2} dx - i \int_\Omega u_1(A \cdot \nabla \overline{u_2}) dx + \int_\Omega \bar{q}u_1 \overline{u_2} dx = 0,$$

(5.75)

with $A = A_1 - A_2$ and $\bar{q} = |A_2|^2 - |A_1|^2 + q_1 - q_2$. On the other hand, according to (1.8), we have

$$\int_\Omega u_1(A \cdot \nabla \overline{u_2}) dx = -\int_\Omega (A \cdot \nabla u_1) \overline{u_2} dx - \int_\Omega \text{div}(A) u_1 \overline{u_2} dx.$$

Combining this with (5.75), we obtain

$$2i \int_\Omega (A \cdot \nabla u_1) \overline{u_2} dx + \int_\Omega [\bar{q} + i\text{div}(A)] u_1 \overline{u_2} dx = 0.$$

Then, (1.10) implies

$$2i \int_{\Omega \setminus (\Omega_+ U \Omega_-)} (A \cdot \nabla u_1) \overline{u_2} dx + \int_{\Omega \setminus (\Omega_+ U \Omega_-)} [\bar{q} + i\text{div}(A)] u_1 \overline{u_2} dx = 0.$$
Applying Lemma 5.1, we deduce by density that this last identity holds true for any $u_1 \in Q_{A_1,q_1, r}$ and any $u_2 \in Q_{A_2,q_2, r}$. Then applying again (1.8) and (1.10), we deduce that (5.75) holds true for any $u_1 \in Q_{A_1,q_1, r}$ and any $u_2 \in Q_{A_2,q_2, r}$. In the same way, applying (1.8) and (1.10), we can prove that (5.75) holds true for any $u_1 \in Q_{A_1,q_1, r}$ and any $u_2 \in Q_{A_2,q_2, r}$. Finally, choosing $u_1, u_2$ in a similar way to Section 4, we can deduce that $dA_1 = dA_2$. Then by repeating the arguments at the end of Section 4, we deduce that, for $q_1 - q_2 \in L^2(\Omega)$, we have $q_1 = q_2$. 

6. The Partial Data Result

This section is devoted to the proof of Theorem 1.3. For all $y \in S^1$, $r > 0$, we set

$$
\partial \omega_{+,r,y} = \{ x \in \partial \omega : \nu(x) \cdot y > r \}, \quad \partial \omega_{-,r,y} = \{ x \in \partial \omega : \nu(x) \cdot y \leq r \}.
$$

We assume that $\Omega = \omega \times \mathbb{R}$ and, without lost of generality, we assume that there exists $\varepsilon > 0$ such that for any $\theta \in \{ y \in S^1 : |y - \theta| \leq \varepsilon \}$ we have $\partial \omega_{-,r,\theta} \subset V'$. We consider $\rho > \max(\rho_2, \rho'_1)$, with $\rho'_1$ given in Corollary 2.2 and $\rho_2$ defined in Proposition 2.2, and we fix $\bar{\theta} \in \{ y \in S^1 : |y - \theta| \leq \varepsilon \}$, $\xi := (\xi', \xi_3) \in \mathbb{R}^3$ satisfying $\xi_3 \neq 0$ and $\xi' \in \theta^\perp \setminus \{ 0 \}$. Then, we fix $u_1 \in H^1(\Omega)$ a solution of $\Delta_{A_1} u_1 + q_1 u_1 = 0$ in $\Omega$ and $u_2 \in H^1(\Omega)$ a solution of $\Delta_{A_2} u_2 + q_2 u_2 = 0$ in $\Omega$ of the form (3.33)-(3.34) with $\rho > \rho_2$ and with $w_{j,\rho}$ satisfying (3.35). Following the argumentation of Section 3, used for proving the decay property of $w_{j,\rho}$ which is given for $j = 1$ by (3.32), we can show that

$$
\rho^{-1} \| w_{j,\rho} \|_{H^1(\Omega)} + \| w_{j,\rho} \|_{L^2(\Omega)} \leq C(\| A_j - A_{j,\rho} \|_{L^2(\mathbb{R}^3)^3} + \rho^{-\frac{3}{2}})
$$

and assuming that $\rho^{-\frac{3}{2}}$ admits a faster decay than $\| A_j - A_{j,\rho} \|_{L^2(\mathbb{R}^3)^3}$ we get

$$
\rho^{-1} \| w_{j,\rho} \|_{H^1(\Omega)} + \| w_{j,\rho} \|_{L^2(\Omega)} \leq C \| A_j - A_{j,\rho} \|_{L^2(\mathbb{R}^3)^3}.
$$

(6.76)

In view of (1.12), there exists $v_2 \in H^1(\Omega)$ satisfying $\Delta_{A_2} v_2 + q_2 v_2 = 0$ and $\tau v_2 = \tau u_1$, $N_{A_2,q_2} v_2_{|V} = N_{A_1,q_1} u_1_{|V}$. Combining this with (1.8) we deduce that $u = v_2 - u_1$ solves the boundary value problem

$$
\begin{cases}
\Delta_{A_2} u + q_2 u = 2iA \cdot \nabla u_1 + (q + i\text{div}(A) + |A_2|^2 - |A_1|^2) u_1 & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(6.77)

In particular, we have

$$
\Delta u = -2iA \cdot \nabla u - (q_2 + i\text{div}(A_2) - |A_2|^2) u + 2iA \cdot \nabla u_1 + (q + i\text{div}(A) + |A_2|^2 - |A_1|^2) u_1 \in L^2(\Omega)
$$

and, in view of (1.3), Lemma 2.2, we deduce that $u \in H^2(\Omega)$.

Now let us show that $\partial_{\nu} u_{|V} = 0$. We fix $w \in H^2(\Omega)$ satisfying $\text{supp}(w_{|\partial \Omega}) \subset V$ and using the fact that $N_{A_2,q_2} v_2_{|V} = N_{A_1,q_1} u_1_{|V}$, we get

$$
0 = \langle N_{A_2,q_2} v_2, \tau w \rangle - (N_{A_1,q_1} u_1, \tau w) = \int_{\Omega} (\nabla + iA_2) u_1 \cdot (\nabla + iA_1) w \, dx - \int_{\Omega} q_1 u_1 \overline{w} \, dx - \int_{\Omega} (\nabla + iA_2) v_2 \cdot (\nabla + iA_1) \overline{w} \, dx + \int_{\Omega} q_2 v_2 \overline{w} \, dx
$$

$$
= - \int_{\Omega} (\nabla + iA_2) u \cdot (\nabla + iA_1) \overline{w} \, dx + \int_{\Omega} q_2 w \overline{w} \, dx + \int_{\Omega} [iu_1 A \cdot \nabla w - i(A \cdot \nabla u_1) \overline{w} - (|A_2|^2 - |A_1|^2) u_1 \overline{w}] \, dx.
$$
Applying (1.8) and the fact that $u \in H^1_0(\Omega)$, we get
\[
\int_{\partial\Omega} [i u_1 A \cdot \nabla w - i (A \cdot \nabla u_1) \overline{w}] - \int_\Omega \omega_j - \int_\Omega \omega_i \frac{\partial u}{\partial x} dx
\]
\[
\begin{align*}
&= -2i \int_{\Omega} (A \cdot \nabla u_1) \overline{w} dx - i \int_{\Omega} \text{div}(A) u_1 \overline{w} dx - \int_{\Omega} \frac{\partial u}{\partial x} [2|A_2|^2 - |A_1|^2] u_1 \overline{w} dx \\
&= - \int_{\Omega} (\Delta u - q) u \overline{w} dx
\end{align*}
\]
and it follows
\[
\int_{\partial\Omega} \partial_\nu u \overline{w} d\sigma(x) = \int_{\Omega} \Delta u \overline{w} dx + \int_{\Omega} \nabla u \cdot \nabla \overline{w} dx = 0.
\]
Allowing $w \in H^2(\Omega)$, satisfying $\text{supp}(w_{|\partial\Omega}) \subset V$, to be arbitrary, we deduce $\partial_\nu u_{|V} = 0$. In the same way, multiplying (6.77) by $\overline{w}$ and then applying (1.8) and the Green formula, we get
\[
\int_{\Omega} [2i A \cdot \nabla u_1 \overline{w} + (q + i \text{div}(A) + |A_2|^2 - |A_1|^2) u_1 \overline{w}] dx = \int_{\partial\Omega} \partial_\nu \overline{w} d\sigma(x).
\]
Moreover, we have $\partial_\nu u_{|V} = 0$ and we get
\[
\int_{\Omega} [2i A \cdot \nabla u_1 \overline{w} + (q + i \text{div}(A) + |A_2|^2 - |A_1|^2) u_1 \overline{w}] dx = \int_{\partial\Omega} \partial_\nu \overline{w} d\sigma(x). \quad (6.78)
\]
In view of (6.78), we have
\[
\|w_{2,\rho}\|_{L^2(\partial\Omega)} \leq C \|w_{2,\rho}\|_{H^1_0(\Omega)} \|w_{2,\rho}\|_{L^2(\Omega)} \leq C \rho \|A_2 - A_{2,\rho}\|_{L^2(\mathbb{R}^3)^3}. \quad (6.79)
\]
Here we use the estimate
\[
\|f\|_{L^2(\partial\Omega)} \leq C \|f\|_{H^1_0(\Omega)} \|f\|_{L^2(\Omega)}^{1/2}, \quad f \in H^1(\Omega),
\]
which can be proved, in a similar way to bounded domains, by using local coordinates associated with $\partial \omega$ in order to transform, locally with respect to $x' \in \overline{w}$ for $x = (x', x_3) \in \overline{w} \times \mathbb{R} = \overline{\Omega}$, $\overline{\Omega}$ into the half space. Applying (6.79) and the Cauchy-Schwarz inequality, we obtain
\[
\left| \int_{\partial\Omega \setminus V} \partial_\nu w d\sigma(x) \right| \leq \int_{\partial\Omega \setminus V} \left| \partial_\nu w \right| dx \leq \int_{\partial\Omega \setminus V} \left| \partial_\nu w \right| dx \leq C \int_{\partial\Omega \setminus V} \left| \partial_\nu w \right| dx
\]
for some $C$ independent of $\rho$. This estimate and the Carleman estimate (2.18) implies
\[
\int_{\Omega} [2i A \cdot \nabla u_1 \overline{w} + (q + i \text{div}(A) + |A_2|^2 - |A_1|^2) u_1 \overline{w}] dx \leq C \rho \|A_2 - A_{2,\rho}\|_{L^2(\mathbb{R}^3)^3} \int_{\partial\Omega \setminus V} \left| \partial_\nu w \right|^2 d\sigma(x).
\]
It is clear that

\[ F(\xi_1) = F(\xi_2) \]

because \( F(A_1) = F(A_2) \) and \( A_1, A_2 \) are supported in \( \Omega \) and \( \Omega \). Since \( F \) is an isometry, we have

\[ \| F(A_1) - F(A_2) \| = \| A_1 - A_2 \| \]

Therefore, we have

\[ \| A_1 - A_2 \| \leq C \rho \]

and multiplying this inequality by \( \rho^{-1} \) and sending \( \rho \to +\infty \) we obtain from (3.37) that

\[ \lim_{\rho \to +\infty} \rho^{-1} \int_\Omega \left| \nabla u_1 \cdot \nabla u_2 + (q + i\text{div}(A) + |A_2|^2 - |A_1|^2)u_1 u_2 \right| \leq 0. \]

Combining this identity with the arguments of Section 4, we deduce that

\[ \eta_k F(a_j)(\xi) = 0, \quad 1 \leq j < k \leq 3 \] (6.81)

for all \( (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R} \) such that \( \xi' \in \theta^{-1} \{0\}, \theta \in \{y \in S^1 : |y - \theta_0| \leq \varepsilon\}, \xi_3 \neq 0 \). Since \( A \in L^1(\mathbb{R}^3) \), we can extend by continuity the identity (6.81) to all \( (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R} \) such that \( \xi' \in \theta^{-1} \{0\}, \theta \in \{y \in S^1 : |y - \theta_0| \leq \varepsilon\}, \xi_3 \in \mathbb{R} \). Consider the Fourier transform in \( x' \) and \( x_3 \) given, for \( f \in L^1(\mathbb{R}^3) \), by

\[ \mathcal{F}(\xi, x_3) = (2\pi)^{-1} \int_{\mathbb{R}^2} f(x', x_3)e^{-ix\cdot\xi}dx', \quad \mathcal{F}_{x_3}(f)(x', \xi_3) = (2\pi)^{-1} \int_{\mathbb{R}} f(x', x_3)e^{-ix_3\xi_3}dx_3. \]

It is clear that \( F A = F[\mathcal{F}_{x_3} A] \) and using the fact that, for all \( \xi_3 \in \mathbb{R} \), \( x' \mapsto \mathcal{F}_{x_3} A(x', \xi_3) \) is supported in \( \mathbb{R} \) which is compact, we deduce that, for all \( j = 1, 2, 3 \), \( \xi' \mapsto \mathcal{F}_{x_3} a_j(\xi', \xi_3) \) is complex valued and analytic. Therefore, for all \( \xi_3 \in \mathbb{R} \), the function \( \xi' \mapsto \eta_k \mathcal{F}(a_j)(\xi) - \xi_3 \mathcal{F}(a_k)(\xi) \) is real analytic and it follows that the identity (6.81) holds true for all \( \xi \in \mathbb{R}^3 \). Thus, we have \( dA_1 = dA_2 \). Then in a similar way to Section 4, we can prove that we can apply the gauge invariance to get

\[ D_{A_1, q_1, \nu} = D_{A_2, q_2, \nu}. \]

Repeating the above argumentation (see also \[30\] Section 5) we deduce that

\[ \lim_{\rho \to +\infty} \int_{\mathbb{R}^3} \chi^2(\rho^{-1} x_3)q(x)e^{-iq\cdot x}dx = 0, \]

for all \( (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R} \) such that \( \xi' \in \theta^{-1} \{0\}, \theta \in \{y \in S^1 : |y - \theta_0| \leq \varepsilon\}, \xi_3 \neq 0 \). Then, using the fact that \( q \in L^1(\mathbb{R}^3) \), an application of the Lebesgue dominated convergence theorem implies that \( \mathcal{F}(q)(\xi) = 0 \), for all \( (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R} \) such that \( \xi' \in \theta^{-1} \{0\}, \theta \in \{y \in S^1 : |y - \theta_0| \leq \varepsilon\}, \xi_3 \in \mathbb{R} \). Then, using the fact that \( q \in L^1(\mathbb{R}^3) \) and supp\( q \subset \mathbb{R} \times \mathbb{R}, \) we can repeat the above arguments in order to deduce that \( q = 0 \) and \( q_1 = q_2 \). This completes the proof of Theorem 1.3

7. Extension to higher dimension

In this section we discuss about some possible extensions of our results to some class of domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 4 \). For this purpose, let \( n \geq 4 \) and consider \( n_1, n_2 \in \mathbb{N} \) such that \( n_1 + n_2 = n \) and \( n_1 \geq 3 \). We fix also \( \omega \) a bounded and \( C^2 \) open set of \( \mathbb{R}^{n_1} \). Then our claim can be stated as follows: all the results of the present paper can be extended to any open and unbounded set \( \Omega \) of \( \mathbb{R}^n \) satisfying

\[ \Omega \subset \Omega_2 := \omega \times \mathbb{R}^{n_2}. \] (7.82)

Let us explain why our results can also be extended to unbounded domains \( \Omega \) satisfying (7.82). The main ingredient are suitable CGO solutions for our problem. Once this is proved one can easily complete the proof of the uniqueness result by repeating our argumentation. Since here we know that \( \omega \) is a bounded
open set of $\mathbb{R}^{n_1}$ with $n_1 \geq 3$, instead of the construction of the present paper we will consider CGO solutions constructed by mean of a projection argument inspired by the analysis of [21, 25]. More precisely, we fix $\xi = (\xi', \xi'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and we consider $\eta, \theta \in \mathbb{S}^{n_1-1}$ such that $\eta \cdot \theta = \eta \cdot \xi' = \theta \cdot \xi' = 0$. For all $r > 0$, we denote by $B'_r$ the ball of center zero and of radius $r$ of $\mathbb{R}^{n_1}$, we fix also $R := \sup_{x' \in \mathbb{R}^{n_1}} |x'|$, $R_1 := 2\sqrt{3}(R+2)$, $\tilde{\theta} = (\theta, 0) \in \mathbb{R}^n$ and $\tilde{\eta} = (\eta, 0) \in \mathbb{R}^n$. We set $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi \geq 0$, $\int_{\mathbb{R}^n} \chi(x)dx = 1$, supp$(\chi) \subset \{x \in \mathbb{R}^n : |x| < 1\}$, and we define $\chi_{\rho}$ by $\chi_{\rho}(x) = \rho^\frac{\chi}{\rho^\frac{\pi}{\rho^\frac{\pi}{\rho^\frac{\pi}{\rho}}}}, \rho > 0$. We set $\varphi(\rho)^2 = 1$.

Consider a fixed constant $C > 0$, we will obtain functions satisfying properties similar to those described in Section 3.1. Now let us fix $\chi \in C_0^\infty(\mathbb{R}^{n_2})$ such that $\chi \geq 0$, $\int_{\mathbb{R}_+} \chi(x)dx = 1$, supp$(\chi) \subset \{x \in \mathbb{R}^{n_2} : |x| < 1\}$, and we define $\chi_{\rho}$ by $\chi_{\rho}(x) = \rho^\frac{\chi}{\rho^\frac{\pi}{\rho^\frac{\pi}{\rho^\frac{\pi}{\rho}}}}, \rho > 0$. We set $\varphi(\rho)^2 = 1$.

In a similar way to Section 3.1, one can check that for all $x = (x', x'') \in B'_{R+1} \times \mathbb{R}^{n_2}$ the function

$$(s_1, s_2) \mapsto \chi_{\rho}(s_1 \tilde{\theta} - s_2 \tilde{\eta} + x)$$

will be supported in $\{z \in \mathbb{R}^2 : |z| < R_1\}$. Thus, we can define

$$\Phi_1,\rho(x) := \frac{-i}{2\pi} \int_{\mathbb{R}^2} \frac{\eta + i\tilde{\eta}}{s_1 + is_2} \cdot A_1,\rho(x - \tilde{\theta} - \tilde{\eta}) ds_1 ds_2,$$

$$\Phi_2,\rho(x) := \frac{-i}{2\pi} \int_{\mathbb{R}^2} \frac{\eta - i\tilde{\eta}}{s_1 + is_2} \cdot A_2,\rho(x - \tilde{\eta}) ds_1 ds_2.$$

Fixing $b_1,\rho(x) = e^{\Phi_1,\rho(x)}$, $b_2,\rho(x) = e^{\Phi_2,\rho(x)}$, we will obtain functions satisfying properties similar to those described in Section 3.1. Now let us fix $\psi \in C_0^\infty(\mathbb{R}^{n_2})$ a real valued function. Applying the results of Section 3.2, which can be extended without any difficulty to this setting, one can construct solutions $u_j \in H^1(\Omega_2)$, $j = 1, 2$, of $A_j u_j + q_j u_j = 0$ on $\Omega_2$ of the form

$$u_1(x', x'') = e^{\rho \phi \cdot x'} \left( \psi(x'')b_1,\rho(x', x'')e^{i\rho \phi \cdot x''} + w_1,\rho(x', x'') \right), \quad x' \in \omega, \ x'' \in \mathbb{R}^{n_2},$$

$$u_2(x', x'') = e^{-\rho \phi \cdot x'} \left( \psi(x')b_2,\rho(x', x'')e^{i\rho \phi \cdot x''} + w_2,\rho(x', x'') \right), \quad x' \in \omega, \ x'' \in \mathbb{R}^{n_2},$$

with $w_j$ satisfying the decay property

$$\lim_{\rho \to +\infty} \rho^{-1} \|w_j,\rho\|_{H^1(\Omega_2)} + \|w_j,\rho\|_{L^2(\Omega_2)} = 0.$$

After that, allowing the cut-off function $\psi \in C_0^\infty(\mathbb{R}^{n_2})$ to be arbitrary and repeating the arguments of Section 4 we can prove that all the results of this paper remain true when $\Omega \subset \mathbb{R}^n$ satisfies [21-25].

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