THE BUBBLE TRANSFORM: A NEW TOOL FOR ANALYSIS OF
FINITE ELEMENT METHODS

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Abstract. The purpose of this paper is to discuss the construction of a linear operator, referred to as the bubble transform, which maps scalar functions defined on \( \Omega \subset \mathbb{R}^n \) into a collection of functions with local support. In fact, for a given simplicial triangulation \( \mathcal{T} \) of \( \Omega \), the associated bubble transform \( B_\mathcal{T} \) produces a decomposition of functions on \( \Omega \) into a sum of functions with support on the corresponding macroelements. The transform is bounded in both \( L^2 \) and the Sobolev space \( H^1 \), it is local, and it preserves the corresponding continuous piecewise polynomial spaces. As a consequence, this transform is a useful tool for constructing local projection operators into finite element spaces such that the appropriate operator norms are bounded independently of polynomial degree. The transform is basically constructed by two families of operators, local averaging operators and rational trace preserving cut-off operators.

1. Introduction

Let \( \Omega \) be a bounded polyhedral domain in \( \mathbb{R}^n \) and \( \mathcal{T} \) a simplicial triangulation of \( \Omega \). The purpose of this paper is to construct a decomposition of scalar functions on \( \Omega \) into a sum of functions with local support with respect to the triangulation \( \mathcal{T} \). The decomposition is defined by a linear map \( B = B_\mathcal{T} \), referred to as the bubble transform, which maps the Sobolev space \( H^1(\Omega) \) boundedly into a direct sum of local spaces of the form \( \hat{H}^1(\Omega_f) \), where \( f \) runs over all the subsimplices of \( \mathcal{T} \) and \( \Omega_f \) denotes appropriate macroelements associated to \( f \). More precisely,

\[
B = \sum_{f \in \Delta(\mathcal{T})} B_f : H^1(\Omega) \to \sum_{f \in \Delta(\mathcal{T})} \hat{H}^1(\Omega_f),
\]

where the maps \( B_f : H^1(\Omega) \to \hat{H}^1(\Omega_f) \) are local and bounded linear maps with the property that for all values of \( r \geq 1 \), if \( u \) is a continuous piecewise polynomial of degree at most \( r \) with respect to the triangulation \( \mathcal{T} \), then \( B_f u \) is a continuous piecewise polynomial of degree at most \( r \) with respect to the restriction of the triangulation to \( \Omega_f \). Thus the map \( B \) is independent of a particular polynomial degree \( r \) and so does not depend on a particular finite element space.

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To motivate the construction of the bubble transform, let us recall that the construction of projection operators is a key tool for deriving stability results and convergence estimates for various finite element methods. In particular, for the analysis of mixed finite element methods, projection operators which commute with differential operators have been a central feature since the beginning of such analysis, cf. [7, 8]. Another setting where such operators potentially would be very useful, but hard to construct, is the analysis of the so-called \( p \)-version of the finite element method, i.e., in the setting where we are interested in convergence properties as the polynomial degree of the finite element spaces increases. For such investigations, the construction of projection operators which admit uniform bounds with respect to polynomial degree represents a main challenge. In fact, so far such constructions have appeared to be substantially more difficult than the more standard analysis of the finite element method, where the focus is on convergence with respect to mesh refinement.

Pioneering results on the convergence of the \( p \)-method applied to second order elliptic problems in two space dimensions were derived by Babuška and Suri [4]. An important ingredient in their analysis was the construction of a polynomial preserving extension operator. A generalization of the construction to three space dimensions in the tetrahedral case can be found in [17], while the importance of such extension operators for the Maxwell equations was argued in [10]. Further developments of commuting extension operators for the de Rham complex in three space dimensions are for example presented in [11, 12, 13, 14]. These constructions have been used to establish a number of convergence results for the \( p \)-method, not only for boundary value problems, but also for eigenvalue problems [6]. A crucial step in this analysis is the use of so-called projection based interpolation operators, cf. [5, Chapter 3] and [10, 11, 16]. However, this development has not led to local projection operators which are uniformly bounded in the appropriate Sobolev norms. Some extra regularity seems to be necessary, cf. [6, Section 6] or [16, Section 4], and, as a consequence, the theory for the \( p \)-method is far more technical than the corresponding theory for the \( h \)-method. This complexity represents a main obstacle for generalizing the theory for the \( p \)-method in various directions. The bubble transform introduced in this paper represents a new tool which will be useful to overcome some of these difficulties. In particular, the construction of projection operators onto the spaces of continuous piecewise polynomials, which are uniformly bounded in \( H^1 \) with respect to the polynomial degree, is an immediate consequence.

In practical computations, improved accuracy is often achieved by combining increased polynomial degree and mesh refinement, an approach frequently referred to as the \( hp \)-finite element method. However, for simplicity, throughout this paper we consider the triangulation \( T \) to be fixed. Although the discussion in this paper is restricted to scalar valued functions, it will be convenient to use some of the notation defined for the more general situation of the de Rham complex and differential forms in [1, 3]. In particular, we let \( \Delta_j(T) \) denote the set of subsimplexes of dimension \( j \) of the triangulation \( T \), while

\[
\Delta(T) = \bigcup_{j=0}^{n} \Delta_j(T)
\]
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is the set of all subsimplexes. Furthermore, the space $\mathcal{P}_r \Lambda^0(\mathcal{T}) \subset H^1(\Omega)$ is the space of continuous piecewise polynomials of degree $r$ with respect to the triangulation $\mathcal{T}$. We recall that the spaces $\mathcal{P}_r \Lambda^0(\mathcal{T})$ admit degrees of freedom of the form

$$
(1.1) \quad \int_f u \eta, \quad \eta \in \mathcal{P}_{r-1-\dim f}(f), \quad f \in \Delta(\mathcal{T}),
$$

where $\mathcal{P}_j(f)$ denotes the set of polynomials of degree $j$ on $f$. These degrees of freedom uniquely determine an element in $\mathcal{P}_r \Lambda^0(\mathcal{T})$. In fact, the degrees of freedom associated to a given simplex $f \in \Delta(\mathcal{T})$ uniquely determine elements in $\mathcal{P}_r(f)$, the space of polynomials of degree $r$ on $f$ which vanish on the boundary $\partial f$.

For each $f \in \Delta(\mathcal{T})$, we let $\Omega_f$ be the macroelement consisting of the union of the elements of $\mathcal{T}$ containing $f$, i.e.,

$$
\Omega_f = \bigcup \{T | T \in \mathcal{T}, \quad f \in \Delta(\mathcal{T})\},
$$

while $\mathcal{T}_f$ is the restriction of the triangulation $\mathcal{T}$ to $\Omega_f$. It is a consequence of the properties of the degrees of freedom that for each $f \in \Delta(\mathcal{T})$, there exists an extension operator $E_f : \mathcal{P}_r(f) \to \mathcal{P}_r \Lambda^0(\mathcal{T}_f)$. Here, $\mathcal{P}_r \Lambda^0(\mathcal{T}_f)$ consists of all functions in $\mathcal{P}_r \Lambda^0(\mathcal{T}_f)$ which are identically zero on $\Omega \setminus \Omega_f$. Furthermore, the space $\mathcal{P}_r \Lambda^0(\mathcal{T})$ admits a direct sum decomposition of the form

$$
(1.2) \quad \mathcal{P}_r \Lambda^0(\mathcal{T}) = \bigoplus_{f \in \Delta(\mathcal{T})} E_f(\mathcal{P}_r(f)) \subset \bigoplus_{f \in \Delta(\mathcal{T})} \mathcal{P}_r \Lambda^0(\mathcal{T}_f).
$$

The extension operators $E_f$ introduced above, defined from the degrees of freedom, will depend on the space $\mathcal{P}_r \Lambda^0(\mathcal{T})$. In particular, they depend on the polynomial degree $r$. However, it is a key observation that the macroelements $\Omega_f$ only depend on the triangulation $\mathcal{T}$, and not on $r$. So for all $r$, there exists a decomposition of the space $\mathcal{P}_r \Lambda^0(\mathcal{T})$ of the form (1.2), i.e., into a direct sum of local spaces $\mathcal{P}_r \Lambda^0(\mathcal{T}_f)$. Furthermore, the geometric structure of these decompositions, represented by the simplexes $f \in \Delta(\mathcal{T})$ and the associated macroelements $\Omega_f$, is independent of $r$, and this indicates that a corresponding decomposition may also exist for the space $H^1(\Omega)$ itself. More precisely, the ansatz is a decomposition of $H^1(\Omega)$ of the form $H^1(\Omega) = \bigoplus_{f} H^1(\Omega_f)$. The bubble transform, $\mathcal{B} = \mathcal{B}_\mathcal{T}$, which we will introduce below, produces such a decomposition. As noted above, the transform is a bounded linear operator

$$
\mathcal{B} : H^1(\Omega) \to \bigoplus_{f \in \Delta(\mathcal{T})} H^1(\Omega_f)
$$

that preserves the piecewise polynomial spaces of (1.2) in the sense that if $u \in \mathcal{P}_r \Lambda^0(\mathcal{T})$, then each component of the transform, $\mathcal{B}_f u$, is in $\mathcal{P}_r \Lambda^0(\mathcal{T}_f) \subset H^1(\Omega_f)$. In fact, $\mathcal{B}$ is also bounded in $L^2$. The transform depends on the given triangulation $\mathcal{T}$, but there is no finite element space present in the construction.

We should note that once the transformation $\mathcal{B}$ is shown to exist, the construction of local and uniformly bounded projections onto the spaces $\mathcal{P}_r \Lambda^0(\mathcal{T})$, with a bound independent of $r$, is straightforward. We just project each component $\mathcal{B}_f u \in H^1(\Omega_f)$ by a local projection into the subspace $\mathcal{P}_r \Lambda^0(\mathcal{T}_f)$. Since each local projection can be chosen to have norm equal to one, the global operator mapping $u$ to the local projections of $\mathcal{B}_f u$ will be bounded independently of the degree $r$. 

...
Furthermore, this process will lead to a projection operator since the transform preserves continuous piecewise polynomials.

In fact, unisolvent degrees of freedom, generalizing (1.1), exist for all the finite element spaces of differential forms, referred to as $P_r \Lambda^k(T)$ and $P_r^- \Lambda^k(T)$ and studied in [1, 6]. As long as the triangulation $T$ is fixed, all these spaces admit degrees of freedom with a common geometric structure, independent of the polynomial degree $r$. Therefore, for all these spaces there exist degrees of freedom generalizing (1.1), and local decompositions similar to (1.2). So far these decompositions have been utilized to derive basis functions in the general setting, cf. [2], and to construct canonical, but unbounded, local projections [1 Section 5.2]. By combining these canonical projections with appropriate smoothing operators, bounded, but nonlocal projections which commute with the exterior derivative were also constructed in [9, 18] and [1, Section 5.4]. Furthermore, in [15] local decompositions and a double complex structure were the main tools to obtain local and bounded cochain projections for the spaces $P_r \Lambda^k(T)$ and $P_r^- \Lambda^k(T)$. However, none of the projections just described will admit bounds which are independent of the polynomial degree $r$, while the construction of projections with such bounds is almost immediate from the properties of the bubble transform, cf. Section 4.3 below. Therefore, it is our ambition to generalize the construction of the bubble transform given below to differential forms in any dimension, such that the transform is bounded in the appropriate Sobolev norms, it commutes with the exterior derivative, and it preserves the finite element spaces $P_r \Lambda^k(T)$ and $P_r^- \Lambda^k(T)$. However, in the rest of this paper we restrict the discussion to 0-forms, i.e., to ordinary scalar valued functions defined on $\Omega \subset \mathbb{R}^n$.

The present paper is organized as follows. In Section 2 we present the main properties of the transform and introduce some useful notation. The key tools needed for the construction are introduced in Section 3. The main results of the paper are derived in Section 4. However, the verification of some of the more technical estimates are delayed until Section 5.

2. Preliminaries

We will use $H^1(\Omega)$ to denote the Sobolev space of all functions $L^2(\Omega)$ which also have the components of the gradient in $L^2$, and $\| \cdot \|_1$ is the corresponding norm. If $\Omega' \subset \Omega$, then $\| \cdot \|_{1,\Omega'}$ denotes the $H^1$ norm with respect to $\Omega'$. The corresponding notation for the $L^2$-norms are $\| \cdot \|_0$ and $\| \cdot \|_{0,\Omega'}$. Furthermore, if $\Omega_f$ is a macroelement associated to $f \in \Delta(T)$, then

$$\tilde{H}^1(\Omega_f) = \{ v \in H^1(\Omega_f) \mid \tilde{E}_f v \in H^1(\Omega) \},$$

where $\tilde{E}_f : L^2(\Omega_f) \to L^2(\Omega)$ denotes the the extension by zero outside $\Omega_f$. For any $f \in \Delta(T)$, $\Delta(f)$ is the set of subsimplexes of $f$. In addition to the macroelements $\Omega_f$, we also introduce the extended macroelements, $\Omega_f^e$, given by

$$\Omega_f^e = \cup \{ \Omega_g \mid g \in \Delta_0(T) \}.$$ 

It is a simple observation that if $g \in \Delta(f)$ then $\Omega_g \supset \Omega_f$, while $\Omega_g^e \subset \Omega_f^e$. 

2.1. An overview of the construction. The construction of the transformation $B$ will be done inductively with respect to the dimension of $f \in \Delta(T)$. We are seeking a decomposition of the space $H^1(\Omega)$ with properties similar to (1.2). More precisely, we will establish that any function $u \in H^1(\Omega)$ can be decomposed into a sum, $u = \sum f u_f$, where each component $u_f \in H^1(\Omega_f)$. The map $u \mapsto u_f$ will be denoted $B_f$, and the collection of all these maps can be seen as a linear transformation $B = B_\mathcal{T} : H^1(\Omega) \to \bigoplus_{f \in \Delta(T)} H^1(\Omega_f)$ with the following properties:

(i) $u = \sum f B_f u$, where the component map $B_f$ is a local operator mapping $H^1(\Omega^f) \to \hat{H}^1(\Omega_f)$.

(ii) $B$ is bounded, i.e., there is a constant $c$, depending on the triangulation $\mathcal{T}$, such that

$$\sum f \|B_f u\|^2_{1,\Omega_f} \leq c\|u\|^2_{1}, \quad u \in H^1(\Omega).$$

(iii) $B$ preserves the piecewise polynomial spaces in the sense that

$$u \in P_r \Lambda^0(\mathcal{T}) \implies B_f u \in \hat{P}_r \Lambda^0(\mathcal{T}_f).$$

In the special case when $n = 1$ and $\Omega$ is an interval, say $\Omega = (0,1)$, a transform with the properties above is easy to construct. In this case, $\mathcal{T}$ is simply a partition of the form

$$0 = x_0 < x_1 < \ldots < x_N = 1.$$ 

The set $\Delta_0(\mathcal{T})$ is the set of vertices $\{x_j\}$, while $\Delta_1(\mathcal{T})$ is the set of intervals of the form $(x_{j-1}, x_j)$. If $f = x_j \in \Delta_0(\mathcal{T})$, then $\Omega_f = (x_{j-1}, x_{j+1})$, with an obvious modification near the boundary, while $\Omega_f = f$ for $f \in \Delta_1(\mathcal{T})$. Let $\lambda_i \in P_1 \Lambda^0(\mathcal{T})$ be the standard piecewise linear “hat functions,” characterized by $\lambda_i(x_j) = \delta_{i,j}$. For all $f = x_j \in \Delta_0(\mathcal{T})$, we let $B_f u = u(x_j)\lambda_j$. By construction, $B_f u$ has support in $\Omega_f$. Furthermore, the function

$$u^1 = u - \sum_{f \in \Delta_0(\mathcal{T})} B_f u$$

vanishes at all the vertices $x_j$. Therefore, if we let $B_f u = u^1|_f$ for all $f \in \Delta_1(\mathcal{T})$, then $B_f u \in \hat{H}^1(\Omega_f)$, and $u = \sum_{f \in \Delta(T)} B_f u$. In fact, it is straightforward to check that all the properties (i)–(iii) hold for this construction.

In general, for $n > 1$, $\operatorname{tr}_f u$, for $f \in \Delta(T)$, will not be well defined for $u \in H^1(\Omega)$. Therefore, the simple construction above cannot be directly generalized to higher dimensions. For example, when $f$ is the vertex $x_0$, to define $B_f u$, we introduce the average of $u$

$$U(x) = \frac{1}{|\Omega_f|} \int_{\Omega_f} u(\lambda_0(x) + [1 - \lambda_0(x)]y) \, dy,$$

where $\lambda_0(x)$ is now the $n$-dimensional piecewise linear function equal to one at $x_0$ and zero at all other vertices. Note that if $u$ is well-defined at $x_0$, then $U(x_0) = u(x_0)$, while if $x \in \Omega \setminus \Omega_f$, then $U(x)$ is just the average of $u$ over $\Omega_f$. In general, for $x \neq x_0$, $U(x)$ has pointwise values. Note that $U(x)$ depends only on $\lambda_0(x)$, so is constant on level sets of $\lambda_0(x)$.
Figure 2.1. The level set $\lambda_0(x) = 1/2$ in the macroelement $\Omega_{x_0}$.

In fact, if we replace $\lambda_0(x)$ by a variable $\lambda$ taking values in $[0,1]$ in the definition of $U(x)$ above, then we may view $U$ as a function of $\lambda$, which we will call $(A_f u)(\lambda)$. Hence, $(A_f u)(\lambda_0(x)) = U(x)$. It is easy to check that if $u$ is a piecewise polynomial in $x$, then $A_f u$ is a polynomial in $\lambda$. Finally, if we define
\begin{equation}
(B_f u)(x) = (A_f u)(\lambda_0(x)) - [1 - \lambda_0(x)](A_f u)(0),
\end{equation}
then $B_f u$ will have support on $\Omega_f$.

For simplices $f$ of higher dimension, the operators $B_f$ will be constructed recursively by a process of the form
\[B_f u = C_f (u - \sum_{g \in \Delta(T), \dim g < \dim f} B_g u),\]
where $C_f$ is a local trace preserving cut–off operator, i.e., designed such that $C_f v$ is close to $v$ near $f$, but at the same time $C_f v$ vanishes outside $\Omega_f$. To also have $C_f v$ in $H^1$ will in general require compatibility conditions of $v$ on $\partial f \subset \partial \Omega_f$. We will return to the precise definition of the operators $B_f$ and $C_f$ in Section 4 below.

2.2. Barycentric coordinates. If $x_j \in \Delta_0(T)$ is a vertex, then $\lambda_j(x) \in P_1(T)$ is the corresponding barycentric coordinate, extended by zero outside the corresponding macroelement. If $f \in \Delta_m(T)$ has vertices $x_0, x_1, \ldots, x_m$, then we write $[x_0, x_1, \ldots, x_m]$ to denote convex combinations, i.e.,
\[f = [x_0, x_1, \ldots, x_m] = \{ x = \sum_{j=0}^{m} \alpha_j x_j \mid \sum \alpha_j = 1, \alpha_j \geq 0 \} \}

The corresponding vector field $(\lambda_0, \lambda_1, \ldots, \lambda_m)$ with values in $\mathbb{R}^{m+1}$ is denoted $\lambda_f$. Hence, the map $x \mapsto \lambda_f(x)$, restricted to $f$, is a one-one map of $f$ onto $S_m$, where
\[S_m = \{ \lambda = (\lambda_0, \ldots, \lambda_m) \in \mathbb{R}^{m+1} \mid \sum_{j=0}^{m} \lambda_j = 1, \lambda_j \geq 0 \}. \]
To the simplex \( S_m \) we associate the simplex \( S^c_m = [S_m, 0] \), given by
\[
S^c_m = \{ \lambda = (\lambda_0, \ldots, \lambda_m) \in \mathbb{R}^{m+1} \mid \sum_{j=0}^{m} \lambda_j \leq 1, \lambda_j \geq 0 \}.
\]
Hence, \( S_m \) is an \( m \) dimensional subsimplex of \( S^c_m \). For \( \lambda \in S^c_m \), we define
\[
b(\lambda) = b_m(\lambda) = 1 - \sum_{j=0}^{m} \lambda_j,
\]
i.e., corresponding to the barycentric coordinate of the origin.

If \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T) \), then the macroelements \( \Omega_f \) and \( \Omega_f^r \) are given by
\[
\Omega_f = \bigcap_{j=0}^{m} \Omega_{x_j} \quad \text{and} \quad \Omega_f^r = \bigcup_{j=0}^{m} \Omega_{x_j}.
\]
The map \( x \mapsto \lambda_f(x) \) maps \( \Omega \) to \( S^c_m \), \( f \) to \( S_m \), and the boundary \( \partial \Omega_f \) to \( \partial S^c_m \setminus S_m \), cf. Figure 2.2.

\[\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The map \( x \mapsto \lambda_f(x) \) for \( n = 2 \) and \( m = 1 \)}
\end{figure}\]

In particular, \( \Omega \setminus \Omega_f^r \) is mapped to the origin. For each \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T) \) we also introduce the piecewise linear function \( \rho_f \) on \( \Omega \) by
\[
\rho_f(x) = 1 - \sum_{j=0}^{m} \lambda_j(x) = b(\lambda_f(x)).
\]
As a consequence, the simplex \( f \) can be characterized as the null set of \( \rho_f \), while \( \rho_f \equiv 1 \) on \( \Omega \setminus \Omega_f^r \).

For each integer \( m \geq 0 \), we let \( \mathcal{I}_m \) be the set of all subindexes of \( (0, 1, \ldots, m) \), i.e., \( \mathcal{I}_m \) corresponds to all subsets of \( \{0, 1, \ldots, m\} \). In particular, we count the empty set as an element of \( \mathcal{I}_m \), such that \( \mathcal{I}_m \) is a finite set with \( 2^{m+1} \) elements. We will use \( |I| \) to denote the cardinality of \( I \). If \( 0 \leq i \leq m \) is an integer, then there are
exactly $2^m$ elements of $\mathcal{I}_m$ which contain $i$, and $2^m$ elements which do not contain $i$. For any $I \in \mathcal{I}_m$, we define $P_I : S^c_m \to S^c_m$ by

$$(P_I \lambda)_i = \begin{cases} 0, & i \in I, \\ \lambda_i, & i \notin I. \end{cases}$$

Hence if $I$ is nonempty, then $P_I$ maps the simplex $S^c_m$ to a portion of its boundary. In particular, if $I = \{0, 1, \ldots, m\}$, then $P_I$ maps $S^c_m$ into the origin of $\mathbb{R}^{m+1}$, while $P_I$ is the identity if $I$ is the empty set. Finally, for any $f \in \Delta_m(T)$ and $I \in \mathcal{I}_m$ we let $f(I) \in \Delta(f)$ denote the corresponding subsimplex of $f$ given by $f(I) = \{x \in f \mid P_I \lambda f(x) = \lambda f(x)\}$. Hence, if $I$ is the empty set, then $f(I) = f$, while $f(I)$ is the the empty subsimplex of $f$ if $I = (0, 1, \ldots, m) \in \mathcal{I}_m$.

### 3. Tools for the construction

The key tools for the construction are two families of operators, referred to as trace preserving cut-off operators and local averaging operators.

#### 3.1. The trace preserving cut-off operator on $S^c_m$.

Let $w$ be a real valued function defined on $S^c_m$. For the discussion in this section, we will assume that $w$ is sufficiently regular to justify the operations below in a pointwise sense. We will introduce an operator $K = K_m$ which maps such functions $w$ into a new function on $S^c_m$, with the property that the trace on $S_m$ is preserved, but such that the trace of $K_m w$ vanishes on the rest of the boundary of $S^c_m$. In fact, the operator $K_m$ resembles the extension operators discussed in [12]. However, in the present setting, where we will be working with functions which may not have a trace on $S_m$, trace preserving operators seem to be a more useful concept. The operator $K_m$ can be viewed as a sum of pullbacks, weighted by rational coefficients. However, the operator $K_m$ preserves polynomials in an appropriate sense, cf. Lemma 3.1 below.

The operator $K_m$ is defined by

$$K_m \omega(\lambda) = \sum_{I \in \mathcal{I}_m} (-1)^{|I|} K_{m}^I w = \sum_{I \in \mathcal{I}_m} (-1)^{|I|} \frac{b(\lambda)}{b(P_I \lambda)} w(P_I \lambda), \quad \lambda \in S^c_m.$$ 

When $m = 0$, the set $\mathcal{I}_0$ has only two elements, the empty set and $\emptyset$. Therefore, the function $K_0$ maps functions $w = w(\lambda)$, defined on $S^c_0 = [0, 1]$, to

$$K_0 \omega(\lambda) = w(\lambda) - (1 - \lambda) w(0),$$

such that $\mathcal{E}_m$ can be rewritten as $B \omega = (K_0 \circ A_j)(\omega(\lambda_0(·)))$. We observe that $K_0 w(1) = w(1)$, $K_0 w(0) = 0$, and if $w \in P_r$ then $K_0 w \in P_r$. Formally, we can also argue that $\text{tr}_{S^c_m}(w - K_m w) = 0$ for $m$ greater than zero. This just follows since all the terms in the sum defining $K_m$, except for the one corresponding to $I = \emptyset$, i.e., $I$ is the empty set, have vanishing trace on $S_m$ due to the appearance of the term $b(\lambda)$ in the nominator. A corresponding argument also shows that the trace of $K_m w$ vanishes on the rest of the boundary of $S^c_m$. Recall that the boundary of $S^c_m$ consists of $S_m$ and the subsimplexes

$$S_{m,i} = \{ \lambda \in S^c_m \mid \lambda_i = 0 \} \quad i = 0, 1, \ldots, m.$$
Furthermore, for a fixed \( i \), let \( I \in \mathcal{I}_m \) be any index such that \( i \notin I \), and let \( I' \in \mathcal{I}_m \) be given as \( I' = I \cup \{i\} \). For \( \lambda \in S_{m,i} \) we have \( P_{I'} \lambda = P_I \lambda \), and therefore

\[
K_{m}^I w(\lambda) - K_{m}^{I'} w(\lambda) = \frac{b(\lambda)}{b(P_I \lambda)} w(P_I \lambda) - \frac{b(\lambda)}{b(P_{I'} \lambda)} w(P_{I'} \lambda) = 0.
\]

However, for a fixed \( i \) the set \( \mathcal{I}_m \) is exactly equal to the union of indexes of the form \( I \) and \( I' \). As a consequence, we conclude that \( K_m w \) is identically zero on \( S_{m,i} \), and hence on \( \partial S_m^c \setminus S_m \). In particular, \( K_m w \) is zero at the origin.

The operator \( K_m \) preserves polynomials in the following sense.

**Lemma 3.1.** Assume that \( w \in \mathcal{P}_r(S_m^c) \) with \( \text{tr}_{S_m} w \in \mathcal{P}_r(S_m^c) \). Then \( K_m w \in \mathcal{P}_r(S_m^c) \), \( \text{tr}_{S_m}(K_m w - w) = 0 \), and \( \text{tr}_{\partial S_m^c \setminus S_m} K_m w = 0 \).

**Proof.** Assume that \( w \in \mathcal{P}_r(S_m^c) \), such that \( \text{tr}_{S_m} w \) vanishes on the boundary of \( S_m \). To show that \( K_m w \in \mathcal{P}_r(S_m^c) \), we consider each term in the sum defining \( K_m w \) of the form

\[
K_m^I w(\lambda) := \frac{b(\lambda)}{b(P_I \lambda)} w(P_I \lambda).
\]

If \( I = \emptyset \), then \( K_m^I w = w \), while if \( I \) is the maximum set, \( I = (0,1,\ldots,m) \), then \( K_m^I w(\lambda) = b(\lambda) w(0,0,\ldots,0) \) which is linear. Therefore, it is enough to consider the other choices of \( I \), i.e., when \( K_m^I w \) has an essential rational coefficient \( b(\lambda)/b(P_I \lambda) \).

Note that since \( \text{tr}_{S_m} w \) vanishes on the boundary of \( S_m \), we can conclude that \( w(P_I \lambda) \) vanishes on \( \{\lambda \in S_m^c | b(P_I \lambda) = 0 \} \). This means that \( w(P_I \lambda) \) must be of the form \( w(P_I \lambda) = b(P_I \lambda) w'(P_I \lambda) \), where \( w' \in \mathcal{P}_{r-1}(S_{m,I}) \). Here

\[
S_{m,I} = \{\lambda \in S_m^c | P_I \lambda = \lambda \}.
\]

As a consequence, \( K_m^I w = b(\lambda) w'(P_I \lambda) \in \mathcal{P}_r(S_m^c) \). Furthermore, \( \text{tr}_{S_m} K_m w = \text{tr}_{S_m} w \) since all the terms \( K_m^I w \) have vanishing trace on \( S_m \), except for the one corresponding to \( I = \emptyset \). Finally, the property that the trace of \( K_m w \) vanishes on the rest of the boundary of \( S_m^c \) follows from the discussion given above. \( \square \)

### 3.2. The local averaging operator.

Throughout this section we will assume that \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(\mathcal{T}) \), where we assume that \( 0 \leq m < n \). For \( v \in L^2(\Omega_f) \) and \( \lambda \in S_m^c \), we let \( A_f v(\lambda) \) be given by

\[
A_f v(\lambda) = \int_{\Omega_f} v(y + \sum_{j=0}^m \lambda_j (x_j - y)) dy,
\]

where the slash through an integral means an average, i.e., \( \frac{1}{|\Omega_f|} \int_{\Omega_f} \). If \( \lambda \in S_m \), then the integrand is independent of \( y \), and therefore \( A_f v(\lambda) = v(x) \), where \( x = \sum_j \lambda_j x_j \in f \). Hence, at least formally, the operator \( \lambda^*_f \circ A_f \), which is given by \( v \mapsto A_f v(\lambda_f(\cdot)) \), is the identity operator on \( f \).

We will find it convenient to introduce the function \( G = G_m : S_m^c \times \Omega_f \to \Omega_f \) given by

\[
G_m(\lambda, y) = y + \sum_{j=0}^m \lambda_j (x_j - y) = \sum_{j=0}^m \lambda_j x_j + b(\lambda)y, \quad \lambda \in S_m^c, \ y \in \Omega_f,
\]

where the slash through an integral means an average, i.e., \( \frac{1}{|\Omega_f|} \int_{\Omega_f} \).
so that the operator \( A_f \) can be expressed as

\[
A_f v(\lambda) = \int_{\Omega_f} v(G_m(\lambda, y)) \, dy = |\Omega_f|^{-1} \sum_{T \in T_f} \int_T v(G_m(\lambda, y)) \, dy.
\]

In fact, we observe that for each \( y \in \Omega_f \), the map \( G_m(\cdot, y) \) maps \( S_m^c \) to \( \Omega_f \), and the operator \( A_f \) is simply the average with respect to \( y \) of the pullbacks with respect to these maps. It is a property of the map \( G_m \) that if \( y \in T \), where \( T \in T_f \), then \( G_m(\lambda, y) \in T \). In fact, \( G_m(\lambda, y) \) is a convex combination of \( y \) and \( b(\lambda)^{-1} \sum_i \lambda_i x_i \in f \).

A key property of the operator \( A_f \) is that it maps the piecewise polynomial spaces \( \mathcal{P}_r\Lambda^0(T_f) \) into the polynomial spaces \( \mathcal{P}_r(S_m^c) \).

**Lemma 3.2.** If \( v \in \mathcal{P}_r\Lambda^0(T) \), then \( A_f v \in \mathcal{P}_r(S_m^c) \). Furthermore, if \( \lambda \in S_m \), then \( A_f v(\lambda) = v(x) \), where \( x = \sum_{j=0}^m \lambda_j x_j \in f \).

**Proof.** If \( v \in \mathcal{P}_r\Lambda^0(T) \), then the restriction of \( v \) to each triangle in \( T_f \) is a polynomial of degree \( r \). Furthermore, the map \( y \mapsto G_m(\lambda, y) \) maps each \( T \) to itself, and depends linearly on \( \lambda \). Therefore, \( v(G_m(\lambda, y)) \in \mathcal{P}_r(S_m^c) \) for each fixed \( y \). Taking the average over \( \Omega_f \) with respect to \( y \) preserves this property, so \( A_f v \in \mathcal{P}_r(S_m^c) \). The second result follows from the fact that the integrand is independent of \( y \), and equal to \( v(\sum_j \lambda_j x_j) \), for \( \lambda \in S_m \). \( \square \)

We will also need mapping properties of the operator \( \lambda_f^* \circ A_f \). Since \( \lambda_f \) maps all of \( \Omega \) into \( S_m^c \), the operator \( \lambda_f^* \circ A_f \) maps a function \( v \) defined on \( L^2(\Omega_f) \) to \( A_f(\lambda_f(\cdot)) \) defined on all of \( \Omega \). It is a key result that this operator is bounded in \( L^2 \) and \( H^1 \). In fact, we even have the following.

**Lemma 3.3.** Assume that \( f \in \Delta_m(T) \) and \( I \in \mathcal{I}_m \), with \( m < n \). The operator \( \lambda_f^* \circ P_f^* \circ A_f \) is bounded as an operator from \( L^2(\Omega_f) \) to \( L^2(\Omega) \), as well as from \( H^1(\Omega_f) \) to \( H^1(\Omega) \).

The arguments involved to establish these boundedness results are slightly more technical than the discussion above. Therefore, we will delay the proof of this lemma, and the proofs of the next three results below, to the final section of the paper.

As we have observed above, the operator \( \lambda_f^* \circ A_f \) formally preserves traces on \( f \). A weak formulation of this result is expressed in the next lemma.

**Lemma 3.4.** Assume that \( f \in \Delta_m(T) \) with \( m < n \). Then

\[
\int_{\Omega} \rho_f^2(x)|v(x) - A_f v(\lambda_f(x))|^2 \, dx \leq c\|v\|_{L^2}^2, \quad v \in H^1(\Omega),
\]

where the constant \( c = c(\Omega, T) \) is independent of \( v \).

Since the function \( \rho_f(x) \) is identically zero on \( f \), this result shows that for any \( v \in H^1(\Omega_f) \) “the error,” \( v - A_f v \), has a decay property near \( f \).

The next result shows that the operator \( \lambda_f^* \circ P_f^* \circ A_f \) preserves such decay properties.
Lemma 3.5. Assume that \( f \in \Delta_m(T) \) and \( I \in \mathcal{I}_m \), with \( m < n \), and let \( g = f(I) \in \Delta(f) \). There is a constant \( c = c(\Omega, T) \), independent of \( v \), such that

\[
\int_{\Omega} \rho_g^{-2}(x)|Afv(P_I\lambda_f(x))|^2 \, dx \leq c \left[ \int_{\Omega} \rho_g^{-2}(x)|v(x)|^2 \, dx + \|\nabla v\|_0^2 \right]
\]

for all \( v \in H^1(\Omega) \), such that \( \rho_g^{-1}v \in L^2(\Omega) \).

Finally, the following lemma will be a key ingredient in the proof of Lemma 3.5 to follow.

Lemma 3.6. Assume that \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T) \) and \( I \in \mathcal{I}_m \), with \( m < n \) and such that \( 0 \notin I \). Furthermore, let \( I' = (0, I) \). Then

\[
\int_{\Omega} \lambda_0^{-2}(x)(Af v(P_I\lambda_f(x)) - Af v(P_I\lambda_f(x)))^2 \, dx \leq c \|\nabla v\|_{0,\Omega_f}^2 \quad v \in H^1(\Omega_f),
\]

where the constant \( c = c(\Omega, T) \) is independent of \( v \).

We remark that \( Af v(P_I\lambda_f(x)) - Af v(P_I\lambda_f(x)) = 0 \) outside \( \Omega_{x_0} \). Therefore, the integrand in the integral above should be considered to be zero outside \( \Omega_{x_0} \).

4. Precise definitions and main results

The transform \( \mathcal{B} = B_T \) will be defined by an inductive process which we now present.

4.1. Definition of the transform. We will define the map \( \mathcal{B} \) by a recursion with respect to the dimension of subsimplices \( f \in \Delta(T) \). The map \( \mathcal{B} \) can be defined on the space \( L^2 \), but the more interesting properties appear when it is restricted to \( H^1 \). The main tool for constructing the operator \( \mathcal{B} \) are trace preserving cut-off operators \( C_f \) which map functions defined on \( \Omega_f \) into functions defined on all of \( \Omega \). The operators \( C_f \) are defined by utilizing the corresponding operators \( K_m \) defined on \( S_m \). If \( f \in \Delta_m(T) \), with \( m < n \), then

\[
C_f\nu = (\lambda^*_f \circ K_m \circ Af)\nu = (K_m \circ Af)\nu(\lambda_f(\cdot)).
\]

A more detailed representation of the operator \( C_f \) is given by

\[
C_f\nu(x) = \sum_{I \in \mathcal{I}_m} (-1)^{|I|} \frac{\rho_{f(I)}(x)}{\rho_{f(I)}(x)} Afv(P_1\lambda_f(x)),
\]

where we recall that \( f(I) = \{x \in f \mid P_1\lambda_f(x) = \lambda_f(x)\} \). Observe that \( \lambda_f = (0, \ldots, 0) \) outside \( \Omega^+_f \) and that all functions of the form \( K_mw \) are zero at the origin in \( \mathbb{R}^{m+1} \). As a consequence, supp(\( C_f\nu \)) is contained in the closure of \( \Omega^+_f \). For the final case when \( f \in \Delta_0(T) = T \), we simply define the operator \( C_f \) to be the restriction to \( f \), i.e., \( C_f\nu = \nu|_f \).

If \( f \in \Delta_0(T) \), i.e., \( f \) is a vertex, then \( B_f = C_f \). More generally, for each \( f \in \Delta_m(T) \) we define

\[
B_fu = C_fu^m, \quad \text{where} \quad u^m = (u - \sum_{g \in \Delta_f(T) \setminus f} B_gu).
\]
Alternatively, the functions $u^m$ satisfy $u^0 = u$ and the recursion

$$u^{m+1} = u^m - \sum_{f \in \Delta_m(T)} C_f u^m = u^m - \sum_{f \in \Delta_m(T)} B_f u.$$ 

As a consequence of the definition of the operator $C_f$ for $\dim f = n$, it follows by construction that $u = \sum_f B_f u$. Furthermore, from the corresponding property of the operator $C_f$, it also follows that $\text{supp}(B_f u)$ is in the closure of $\Omega_f$. Also, by Lemma 3.3 and from the fact that $\rho_f/\rho_f(1) \leq 1$, it follows directly that the operator $B_f$ is bounded in $L^2$. However, it is more challenging to establish that $B_f$ is bounded in $H^1$, and that $B_f u \in H^1(\Omega_f)$ for $u \in H^1(\Omega)$.

4.2. Main properties of the transform. The main arguments needed for verifying the properties (i)–(iii) of the transform $B$, stated in Section 2 above, will be given here. We will first establish that the piecewise polynomial space, $P_r \Lambda^0(T)$, is preserved by the transform, i.e., we will show property (iii).

**Theorem 4.1.** If $u \in P_r \Lambda^0(T)$, then $B_f u \in P_r \Lambda^0(T_f)$ for all $f \in \Delta(T)$.

**Proof.** Assume that $u \in P_r \Lambda^0(T)$. We will show that for all $m$, $0 \leq m \leq n$, the following properties hold:

$$u^m \in P_r \Lambda^0(T), \quad \text{with } \text{tr}_g u^m = 0, \quad g \in \Delta_j(T), \quad j < m,$$

and

$$B_g u \in P_r \Lambda^0(T_g), \quad g \in \Delta_j(T), \quad j < m.$$ 

Here the function $u^m$ is defined by (4.2). The proof of (4.3) and (4.4) goes by induction on $m$. Note that for $m = 0$, these properties hold with $u^0 = u$. Assume now that (4.3) and (4.4) hold for a given $m$, $m < n$. Let $v \equiv u^m \in P_r \Lambda^0(T)$. Then, for any $f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T)$, we have $\text{tr}_f v \in P_r(f)$. Therefore, it follows from Lemma 3.2 that

$$A_f v \in P_r(S_m^f) \quad \text{and } \text{tr}_{S_m} A_f v \in P_r(S_m).$$

In fact, if $\lambda \in S_m$, then $A_f v(\lambda) = v(x)$, where $x = \sum_{j=0}^m \lambda_j x_j \in f$. But from Lemma 3.1 we can then conclude that

$$(K_m \circ A_f)v \in P_r(S_m^f), \quad \text{with } \text{tr}_{S_m} (I - K_m)A_f v = 0, \quad \text{tr}_{S_m \setminus S_m} (K_m \circ A_f)v = 0.$$ 

However, this implies that

$$B_f u = C_f^m u^m = (K_m \circ A_f)v(\lambda_f(\cdot)) \in P_r \Lambda^0(T_f),$$

and with $\text{tr}_f B_f u = \text{tr}_f u^m$. This property holds for all $f \in \Delta_m(T)$. Therefore, since

$$u^{m+1} = u^m - \sum_{f \in \Delta_m(T)} B_f u,$$

we can conclude that (4.3) and (4.4) hold with $m$ replaced by $m + 1$. This completes the induction argument. In particular, we have shown that $B_f u \in P_r \Lambda^0(T_f)$ for all $f \in \Delta_m(T)$, $m < n$. Furthermore, $\text{tr}_f u^m = 0$ for all $f \in \Delta_{n-1}(T)$. This means that

$$u^n = \sum_{T \in \mathcal{T}} u^n_T, \quad u^n_T \in P_r \Lambda^0(T), \quad T \in \mathcal{T}.$$ 

Since $B_f u = u^n_T$ for any $T \in \Delta_n(T) = \mathcal{T}$, the proof is completed. \qed
The next result will be a key step for showing properties (i) and (ii) of the transform.

**Lemma 4.2.** Assume that \( f \in \Delta_m(T) \), with \( m < n \), and that \( v \in H^1(\Omega_f) \) with \( \rho_g^{-1}v \in L^2(\Omega_f) \), where \( g = f(I) \) for \( I \in \mathcal{I}_m \). Define \( w = \frac{\partial f}{\partial g} A_f v(P_I \lambda_f(\cdot)) \). Then \( w \in H^1(\Omega) \) and \( \rho_f^{-1}w \in L^2(\Omega) \).

**Proof.** Since \( g \in \Delta(f) \), \( \rho_f/\rho_g \leq 1 \). Therefore, it follows directly from Lemma 3.3 that \( w \in L^2(\Omega) \). We also have from Lemma 3.3 that

\[
\int_{\Omega} |\rho_f^{-1}w|^2 \, dx = \int_{\Omega} |\rho_g^{-1} A_f v(P_I \lambda_f(x))|^2 \, dx \\
\leq c \left[ \int_{\Omega_f} |\rho_g^{-1} v(x)|^2 \, dx + \| \text{grad} v \|^2_{L^2(\Omega_f)} \right] < \infty,
\]

so the desired decay property of \( w \) follows. It remains to show that \( w \in H^1(\Omega) \). From the identity

\[
\text{grad}(\rho_f/\rho_g) = \rho_g^{-1}(\text{grad} \rho_f - \frac{\partial f}{\partial g} \text{grad} \rho_g),
\]

we obtain that \( |\text{grad}(\rho_f/\rho_g)| \leq c_0 \rho_g^{-1} \), where \( c_0 = c_0(\Omega, T) \). Therefore, we can conclude that

\[
\int_{\Omega_f} (|\text{grad}(\rho_f/\rho_g)|) A_f v(P_I \lambda_f(x)) |^2 \, dx \leq c_0^2 \int_{\Omega_f} |\rho_g^{-1} A_f v(P_I \lambda_f(x)) |^2 \, dx.
\]

Together with Leibnitz’ rule and the result of Lemma 3.3, this will imply that \( w \in H^1(\Omega) \). This completes the proof. \( \square \)

**Lemma 4.3.** Let \( f \in \Delta_m(T) \) with \( x_0 \in \Delta_g(f) \). Assume that \( v \in H^1(\Omega_f) \), with the property that \( \rho_g^{-1}v \in L^2(\Omega_f) \) for all \( g \in \Delta_j(f) \), \( j < m \). Then \( \lambda_0^{-1} C_f v \in L^2(\Omega) \).

**Proof.** Assume first that \( m < n \). Let \( I \in \mathcal{I}_m \) be any index set such that \( 0 \notin I \). Furthermore, let \( I' = (0, I) \in \mathcal{I}_m \). In other words, \( x_0 \in \Delta(g) \) while \( x_0 \notin \Delta(g') \), where \( g = f(I) \) and \( g' = f(I') \). The desired result will follow if we can show that

\[
\lambda_0^{-1} \left[ \frac{\partial f}{\partial g} A_f v(P_I \lambda_f(\cdot)) - \frac{\rho_f}{\rho_g} A_f v(P_I \lambda_f(\cdot)) \right]
\]

\[
= \lambda_0^{-1} \frac{\partial f}{\partial g'} [ A_f v(P_I \lambda_f(\cdot)) - A_f v(P_I \lambda_f(\cdot)) ] + \frac{\rho_f}{\rho_g \rho_g'} A_f v(P_I \lambda_f(\cdot)) \in L^2(\Omega).
\]

However, Lemma 3.0 and the fact that \( \rho_f/\rho_g \leq 1 \) implies that the first term on the right hand side is in \( L^2 \). Furthermore, it follows by assumption that \( \rho_g^{-1}v \in L^2 \), and therefore Lemma 3.0 implies that the second term is in \( L^2 \).

If \( m = n \), then we recall that \( C_f v \) is just \( v \) restricted to \( f \). If \( f = [x_0, x_1, \ldots, x_n] \) and \( g = [x_1, \ldots, x_n] \), then \( \rho_g^{-1}v = \lambda_0^{-1}v \in L^2 \) by assumption. This completes the proof. \( \square \)

**Lemma 4.4.** Let \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T) \) and assume that \( v \in H^1(\Omega_f) \), with the property that \( \rho_g^{-1}v \in L^2(\Omega_f) \) for \( g \in \Delta_j(f) \), \( j < m \). Define \( w = C_f v \). Then \( w|_{\Omega_f} \in H^1(\Omega_f) \) and \( w \equiv 0 \) on \( \Omega \setminus \Omega_f \).
Lemma 4.4. It remains to show the decay property, i.e., that $\rho u \in H^1(\Omega_f)$. The proof goes by induction on $m < n$. Therefore, it is in $\mathring{H}^1(\Omega_{x_0})$. Since the numbering of the vertices of $f$ is arbitrary, this will in fact imply that

$$w \in \cap_{j=0}^m \mathring{H}^1(\Omega_{x_j}) = \mathring{H}^1(\Omega_f).$$

However, the property that $w \in \mathring{H}^1(\Omega_{x_0})$ is a consequence of the decay results expressed in Lemmas \[4.3\] i.e., that $\lambda_0^{-1} w \in L^2$. For any $\epsilon > 0$, let $\phi_\epsilon$ be a smooth function on $\mathbb{R}$ such that $\phi_\epsilon \equiv 0$ on $(-\epsilon/2, \epsilon/2)$, $\phi_\epsilon \equiv 1$ on the complement of $(-\epsilon, \epsilon)$, and such that $\phi_\epsilon'(\lambda) \lambda$ is uniformly bounded, i.e.,

\[
|\phi_\epsilon'(\lambda)| \leq c/|\lambda|, \quad \frac{\epsilon}{2} \leq |\lambda| \leq \epsilon,
\]

for some constant $c$. By construction, the functions $v_\epsilon \equiv \phi_\epsilon(\lambda_0(\cdot))w$ are in $\mathring{H}^1(\Omega_{x_0})$, and to show that $w$ belongs to the same space, it is enough to show that the $v_\epsilon$ converge to $w$, as $\epsilon$ tends to zero, in $H^1(\Omega_{x_0})$. However,

$$\int_{\Omega_{x_0}} |v_\epsilon - w|^2 \, dx = \int_{\Omega_{x_0}} |(\phi_\epsilon(\lambda_0(\cdot)) - 1)w|^2 \, dx \leq \int_{\Omega_{x_0}} |w|^2 \, dx \to 0,$$

where $\Omega_{x_0,\epsilon} = \{x \in \Omega_{x_0} \mid \lambda_0(x) \leq \epsilon\}$. This shows the $L^2$ convergence. Furthermore,

$$\int_{\Omega_{x_0}} |\text{grad}(v_\epsilon - w)|^2 \, dx \leq 2 \int_{\Omega_{x_0,\epsilon}} |\text{grad} w|^2 \, dx + 2 \int_{\Omega_{x_0,\epsilon}} |(\phi_\epsilon(\lambda_0(\cdot)))w|^2 \, dx.
$$

The first term goes to zero by the $H^1$ boundedness of $w$, and, as a consequence of (4.5) and the $L^2$ property of $\lambda_0^{-1}w$ established in Lemma 4.3, the second term goes to zero with $\epsilon$. By completeness of $\mathring{H}^1(\Omega_{x_0})$, it follows that $w \in \mathring{H}^1(\Omega_{x_0})$ and therefore it is in $\mathring{H}^1(\Omega_f)$.

We recall from the definition of the operator $C_f$ that $w$ is identically zero on $\Omega \setminus \Omega_f$. Hence, it remains to show that $w$ is identically zero on $\Omega_f^m \setminus \Omega_f$ when $m < n$. However, at each point in $\Omega_f^m \setminus \Omega_f$, at least one of the extended barycentric coordinates associated to $f$ is zero. Therefore, $w$ in this region corresponds to a pullback of $w$ from $\partial_S^c \setminus S_m$, and this is zero since $\text{tr}_{\partial \Omega_f} w = 0$.

\[\square\]

Lemma 4.5. Let $u \in H^1(\Omega)$ and define the functions $u^m$, $0 \leq m \leq n$, by (4.2). Then $u^m \in H^1(\Omega)$ and $\rho_f^{-1}u^m \in L^2(\Omega)$ for all $f \in \Delta_j(\mathcal{T})$, $j < m$.

\[\text{Proof.}\] The proof goes by induction on $m$. For $m = 0$ the result holds with $u^0 = u$. Furthermore, if the result holds for a given $m < n$, then $u^{m+1} \in H^1(\Omega)$ by Lemma 4.4. It remains to show the decay property, i.e., that $\rho_f^{-1}u^{m+1} \in L^2(\Omega)$ for all $f \in \Delta_j(\mathcal{T})$ for $j < m$. For any $f \in \Delta_m(\mathcal{T})$ we have

$$\rho_f^{-1}(u^m - C_f u^m)
= \rho_f^{-1}[u^m - A_f u^m(\lambda_f(\cdot))] - \rho_f^{-1} \sum_{i \in \mathbb{Z}_m, I \neq \emptyset} (-1)^{|I|} \frac{\rho_f}{\rho_f(I)} A_f u^m(P_I \lambda(\cdot)).$$

However, the first term on the right side is in $L^2$ as a consequence of Lemma 3.4, while Lemma 4.2 and the induction hypothesis implies that all the terms in the
sum are in $L^2$. We can therefore conclude that for $f \in \Delta_m$, $\rho^{-1}_f (u^m - C_f u^m)$ is in $L^2(\Omega)$. To show that $\rho^{-1}_f u^{m+1}$ is in $L^2$, we express this as

\begin{equation}
\rho^{-1}_f u^{m+1} = \rho^{-1}_f (u^m - C_f u^m) + \sum_{g \in \Delta_m(T) \atop g \neq f} \rho^{-1}_g u^m.
\end{equation}

Recall that by definition, $C_g u^m$ is identically zero outside $\Omega^e_g$. On the other hand, if $g \in \Delta_m(T)$ and $g \neq f$, then on each $T \in T$, such that $f \cap T \neq \emptyset$ and $g \cap T \neq \emptyset$, there exists a vertex $x_0 \in g \cap T$ which is not in $f$. Then $\lambda_0 \leq \rho_f$ on $T$, which implies that

$$|\rho^{-1}_f C_g u^m| \leq |\lambda_0^{-1} C_g u^m| \text{ on } T.$$ 

By repeating this for all $T \subset \Omega_f^e$, and by applying Lemma 4.3, we obtain that all the terms in the sum (4.6) are in $L^2$. Since $f \in \Delta_m(T)$ is arbitrary, this shows the desired decay result for all $f \in \Delta_m(T)$. However, if $g \in \Delta(f)$, then $\rho^{-1}_g (x) \leq \rho^{-1}_f (x)$, and therefore $\rho^{-1}_f u^{m+1} \in L^2$ for all $f \in \Delta_j(T), j \leq m$. This completes the induction argument and therefore the proof of the lemma.

The following result shows that the transform satisfies properties (i) and (ii) above.

**Theorem 4.6.** Assume that $u \in H^1(\Omega)$. Then $u = \sum_{f \in \Delta(T)} B_f u$, where $B_f u \in H^1(\Omega_f)$ for each $f \in \Delta(T)$. Furthermore, the transformation $B_T : H^1(\Omega) \to \bigoplus_{f \in \Delta(T)} H^1(\Omega_f)$, with components $B_f$, is bounded.

**Proof.** We have already seen that $u = \sum_{f \in \Delta(T)} B_f u$. Furthermore, it is a consequence of Lemmas 4.2 and 4.5 that each $B_f u \in H^1(\Omega_f)$. Finally, the boundedness of the transformation can be seen by tracing the bounds derived in Lemmas 4.2, 4.5 and by utilizing the finite overlap property of the covering $\{\Omega_f\}$ of $\Omega$.  

**Corollary 4.7.** The transform $B_T$ is $L^2$ bounded, with $\text{supp } B_f u$ contained in the closure of $\Omega_f$ for all $u \in L^2(\Omega)$.

**Proof.** We have already seen that $B_T$ is $L^2$ bounded, and with $\text{supp } B_f u$ contained in the closure of the extended macroelement $\Omega_f^e$. However, due to the result of Theorem 4.6 and the density of $H^1(\Omega)$ in $L^2(\Omega)$, this implies that $\text{supp } B_f u$ is contained in the closure of $\Omega_f$.

### 4.3. Construction of projections

The result of Theorem 4.6 leads immediately to the construction of locally defined projections into the finite element spaces $\mathcal{P}_r \Lambda^0(T)$ which are uniformly bounded with respect to the polynomial degree $r$.

We just project each component $B_f u$ into the space $\mathcal{P}_r \Lambda^0(T_f)$ by a local projection $Q_{f,r}$. More precisely, the locally defined global projections $\pi = \pi_{T,r}$ will be of the form

$$\pi u = \sum_{f \in \Delta_m(T)} Q_{f,r} B_f u,$$

where $Q_{f,r}$ is a local projection onto $\mathcal{P}_r \Lambda^0(T_f)$. The operator $\pi$ will be a projection as a result of Theorem 4.4. If $Q_{f,r}$ is taken to be the local $H^1$-projection, with
corresponding operator norm equal to one, then Theorem 4.6 implies that \( \pi \) will be uniformly bounded in \( H^1 \) with respect to \( r \). On the other hand, if \( Q_{f,r} \) is taken to be the local \( L^2 \)-projection, then Corollary 4.7 implies uniform \( L^2 \) boundedness of \( \pi \) with respect to \( r \).

5. Proofs of Lemmas 3.3–3.6

To complete the paper, it remains to establish Lemmas 3.3–3.6 all related to properties of the averaging operators \( A_f \). Let \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T) \) be as above. Throughout this section we assume that \( 0 \leq m < n \). If \( T \in \mathcal{T}_f \) and \( \lambda \in \mathcal{S}_m \), we also let

\[
A_{f,T}v(\lambda) = \int_T v(G_m(\lambda, y)) \, dy,
\]
such that

\[
A_f v = \sum_{T \in \mathcal{T}_f} \frac{|T|}{|\Omega_f|} A_{f,T}v.
\]

Before we derive more properties of the operator \( A_f \) we will make some observations which will be useful below. A simple calculation shows that for any \( r \in \mathbb{R} \) we have

\[
\int_{S_m} b(\lambda)^r \, d\lambda = \int_{S_{m-1}} \int_0^{b(\lambda')} (b(\lambda') - \lambda_m)^r \, d\lambda_m \, d\lambda'
\]

\[
= \int_{S_{m-1}} \int_0^{b(\lambda')} z^r \, d\lambda' \, dz = \int_0^1 \int_{z \leq b(\lambda')} z^r \, d\lambda' \, dz = |S_{m-1}| \int_0^1 z^r (1 - z)^m \, dz.
\]

Hence, we can conclude that

\[
\int_{S_m} b(\lambda)^r \, d\lambda < \infty, \quad \text{for } r > -1.
\]

If \( f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T) \) and \( T \) is an element of \( \mathcal{T}_f \), we let \( f^*(T) \in \Delta_{n-m-1}(T) \) be the face opposite \( f \). In other words, if \( T = [x_0, x_1, \ldots, x_n] \), then

\[
f^*(T) = [x_{m+1}, \ldots, x_n] = \{ x \in T \mid \lambda_j(x) = 0, j = 0, 1, \ldots, m \}.
\]

Any point \( x \in T \) can be written uniquely as a convex combination of \( x_0, \ldots, x_m \) and a point \( q = q_f \in f^*(T) \), since

\[
x = \sum_{j=0}^n \lambda_j(x) x_j = \sum_{j=0}^m \lambda_j(x) x_j + \rho_f(x) q_f(x), \quad q_f(x) = \sum_{j=m+1}^n \lambda_j(x) x_j / \rho_f(x).
\]

Define \( f^* = \cup_{T \in \mathcal{T}_f} f^*(T) \). Then \( f^* \subset \partial \Omega_f \), and any \( x \in \Omega_f \) can be written as

\[
x = \sum_{j=0}^m \lambda_j(x) x_j + \rho_f(x) q_f(x), \quad q_f(x) \in f^*.
\]

The set \( f^* \) can alternatively be characterized as \( f^* = \partial \Omega_f^r \cap \partial \Omega_f \). An illustration of the geometry of \( f, \Omega_f, \) and \( f^* \) is given in Figure 5.1 below. In fact, if \( m = n-1 \), then \( f^* \) consist of two vertices in \( \Delta_0(T) \), while if \( m < n-1 \), \( f^* \) is a connected and piecewise flat manifold of dimension \( n - m - 1 \).
Figure 5.1. The macroelement $\Omega_f \subset \mathbb{R}^3$, where $f$ is the line from $x_0$ to $x_1$ and $f^*$ is the closed curve connecting $x_2, x_3, x_4$.

The map $x \mapsto (\lambda_f(x), q_f(x))$ defines a map from $\Omega_f$ to $\mathcal{S}_m^c \times f^*$, with an inverse given by

$$ (\lambda, q) \mapsto x = q + \sum_{j=0}^{m} \lambda_j (x_j - q) = G_m(\lambda, q). $$

The derivative of the map (5.3) can be expressed as the $n \times n$ matrix

$$ [x_0 - q, x_1 - q, \ldots, x_m - q, b(\lambda)Q], $$

where $Q$ is the piecewise constant $n \times (n-m-1)$ matrix representing the embedding of the tangent space of $f^*$ into $\mathbb{R}^n$. In other words, for each $T \in \mathcal{T}_f$ the columns of $Q$ can be taken to be an orthonormal basis for the tangent space of $f^*$ with respect to the ordinary Euclidean inner product of $\mathbb{R}^n$. Hence, by the scaling rule for determinants, the determinant of this matrix is of the form

$$ b(\lambda)^{n-m-1} \det([x_0 - q, x_1 - q, \ldots, x_m - q, Q]) := b(\lambda)^{n-m-1} J(f, q). $$

Furthermore, for a fixed mesh, the function $J(f, q)$ will be bounded from above and below. In other words, there exist constants $c_0 = c_1(\Omega, T)$, such that

$$ c_0 \leq J(f, q) \leq c_1, \quad f \in \Delta(T), \; q \in f^*. $$

The coordinates $(\lambda, q) \in \mathcal{S}_m^c \times f^*$ can be seen as generalized polar coordinates for the domain $\Omega_f$. The change of variables

$$ x \mapsto (\lambda_f(x), q_f(x)) \in \mathcal{S}_m^c \times f^* $$

leads to the identity

$$ \int_T \phi(\lambda_f(x), q_f(x)) \, dx = \int_{\mathcal{S}_m^c} \int_{f^*(T)} \phi(\lambda, q) J(f, q) \, dq \, b(\lambda)^{n-m-1} \, d\lambda, $$

with $\lambda \in (0, \lambda_{\max})$. Therefore, $J(f, q)$ define a change of variables $x \mapsto (\lambda_f(x), q_f(x))$ for the integral

$$ \int_T \phi(x) \, dx, $$

and $J$ is the Jacobian determinant of this transformation.
for any $T \in \mathcal{T}_f$, and any real valued function $\phi$ on $\mathcal{S}_m^c \times f^*(T)$. Furthermore, by summing over all $T \in \mathcal{T}_f$, we obtain

$$
\int_{\Omega_f} \phi(\lambda_f(x), q_f(x)) \, dx = \int_{\mathcal{S}_m^c} \int_{f^*(T)} \phi(\lambda, q) J(\lambda, q) \, dq \, b(\lambda)^{n-m-1} \, d\lambda.
$$

Here the integral over $f^*$ should be interpreted as a sum in the case $m = n - 1$, when $f^*$ consists of two points.

The function $G_m$ has the property that $G_m(\lambda_f(x), q_f(x)) = x$ and it satisfies the composition rule

$$
G_m(\lambda, G_m(\mu, y)) = G_m(\lambda', y) \quad \text{where } \lambda' = \lambda + b(\lambda)\mu.
$$

In particular, the matrix associated to the linear transformation $\lambda \mapsto \lambda'$ is $(m + 1) \times (m + 1)$ given by $I - \mu e^T$, where $e$ denotes the vector with all elements equal 1, and this matrix has determinant $b(\mu)$. Furthermore, $b(\lambda') = b(\lambda)b(\mu)$. Letting $y = G_m(\mu, q)$ and applying the identity (5.3) in the variable $y$, we can rewrite $A_{f,T}v(\lambda)$ as

$$
A_{f,T}v(\lambda) = |T|^{-1} \int_{\mathcal{S}_m} \int_{f^*(T)} v(G_m(\lambda, G_m(\mu, q))) J(\lambda, q) \, dq \, b(\mu)^{n-m-1} \, d\mu.
$$

A key property, which is a special case of Lemma 3.3, is that the operator $\lambda_f^T \circ A_{f,T}$ is bounded in $L^2$. To see this, observe that we obtain from (5.4), (5.6), (5.7), and Minkowski’s inequality in the form $\|f g(\mu)\| \leq \int \|g(\mu)\| \, d\mu$, that

$$
\|A_{f,T}v(\lambda_f(\cdot))\|_{0,\Omega_f} 
\leq c \int_{\mathcal{S}_m^c} \left( \int_{\Omega_f} \int_{f^*(T)} |v(G(\lambda_f(x), G(\mu, q)))|^2 \, dq \, dx \right)^{1/2} b(\mu)^{n-m-1} \, d\mu 
\leq c \int_{\mathcal{S}_m^c} \left( \int_{\Omega_f} \int_{f^*(T)} |v(G(\lambda, G(\mu, q)))|^2 \, dq \, dx \right)^{1/2} b(\mu)^{n-m-1} \, d\mu 
\leq c \int_{\mathcal{S}_m^c} \left( \int_{\Omega_f} \int_{f^*(T)} |v(G(\lambda', q))|^{2} dq \, dx \right)^{1/2} b(\mu)^{1+ (n-m)/2} \, d\mu,
$$

where we have substituted $\lambda' = \lambda + b(\lambda)\mu$. However, by letting $(\lambda', q) \mapsto x = G(\lambda', q)$, we obtain from (5.3) that

$$
\|A_{f,T}v(\lambda_f(\cdot))\|_{0,\Omega_f} \leq c \int_{\mathcal{S}_m^c} \left( \int_T |v(x)|^2 \, dx \right)^{1/2} b(\mu)^{-1+ (n-m)/2} \, d\mu 
= c \|v\|_{0,T} \int_{\mathcal{S}_m^c} b(\mu)^{-1+ (n-m)/2} \, d\mu \leq c_1 \|v\|_{0,T},
$$

where we have used (5.1) and the fact that the exponent satisfies $-1 + (n-m)/2 \geq -1/2$. This shows that the operator $\lambda_f^T \circ A_{f,T}$ is bounded as an operator from $L^2(T)$ to $L^2(\Omega_f)$. Furthermore, if $T' \in \Delta(T)$ such that $T' \subset \Omega_f$, but $T' \notin \mathcal{T}_f$, we let $g = f \cap T'$. Then $g \in \Delta(f)$ and $A_{f,T}v|_{T'} = A_{g,T}v|_{T'}$.

By utilizing the argument just given with respect to $g$ instead of $f$ we can conclude that $\lambda_f^T \circ A_{f,T}$ is bounded from $L^2(T)$ to $L^2(\Omega_f^c)$. In particular, on the boundary of $\Omega_f^c$, $(\lambda_f^T \circ A_{f,T})v$ is constant with value

$$
A_{f,T}v(0) = \int_T v(y) \, dy.
$$
THE BUBBLE TRANSFORM

Figure 5.2. The case when $T' \subset \Omega_f^r$, but $T' \notin T_f$ (enclosed in the thick lines). Here $g = f \cap T'$.

In fact, this is also the value of $(\lambda_f^* \circ A_{f,T})v$ in $\Omega \setminus \Omega_f^r$, and we can therefore conclude that $\lambda_f^* \circ A_{f,T}$ is bounded from $L^2(T)$ to $L^2(\Omega)$. Since the operator $A_f$ is a weighted sum of the operators $A_{f,T}$, we can therefore conclude that $\lambda_f^* \circ A_f$ is bounded from $L^2(\Omega_f^r)$ to $L^2(\Omega)$.

A completely analogous argument, essentially using that differentiation commutes with averaging, also shows that $\lambda_f^* \circ A_f$ is bounded from $H^1(\Omega_f^r)$ to $H^1(\Omega)$. We just observe that

$$\text{grad } A_{f,T}v(\lambda_f(\cdot)) = \int_T (DG_m)^T \text{grad } G_m(\lambda_f(\cdot), y) \, dy.$$  

Here $DG_m = DG_m(y)$ is the derivative of $G_m(\lambda_f(x), y)$ with respect to $x$, given as the $n \times n$ matrix

$$DG_m = \sum_{j=0}^m (x_j - y)(\text{grad } \lambda_j)^T,$$

and this matrix is uniformly bounded with respect to $y$. We have therefore established Lemma 3.3 in the special case when $I$ is the empty set.

Proof of Lemma 3.3. We need to show that the operators $\lambda_f^* \circ P_f^* \circ A_f$ are bounded from $L^2(\Omega_f^r)$ to $L^2(\Omega)$ and from $H^1(\Omega_f^r)$ to $H^1(\Omega)$ for all $I \in \mathcal{I}_m$. As in the discussion above, it is sufficient to consider each of the operators $\lambda_f^* \circ P_f^* \circ A_{f,T}$ for all $T \in T_f$. However, the operator $\lambda_f^* \circ P_f^* \circ A_{f,T}$ is equal to $\lambda_g^* \circ A_{g,T}$, where $g = f(I) = \{x \in f \mid P_f \lambda_f(x) = \lambda_f(x)\}$, and as a consequence, the desired result follows from the discussion above.

Proof of Lemma 3.4. Since the function $\rho_f$ is identically to one outside $\Omega_f^r$ and the operator $\lambda_f^* \circ A_f$ is bounded in $L^2$, it is enough to show that

$$\int_{\Omega_f^r} \rho_f^{-2}(x)|v(x) - A_f v(\lambda_f(x))|^2 \, dx \leq c\|\text{grad } v\|_{0,\Omega_f^r}^2, \quad v \in H^1(\Omega).$$
Furthermore, it is enough to show the corresponding result for each of the operators \( A_{f,T} \), i.e., to show that

\[
\int_{\Omega_f} \rho_f^{-2}(x) |v(x) - A_{f,T} v(\lambda_f(x))|^2 \, dx \leq c \| \text{grad} v \|^2_{0,\Omega_f}, \quad v \in H^1(\Omega),
\]

for all \( T \in \mathcal{T}_f \). In fact, it will actually be enough to show that

\[
\int_{\Omega_f} \rho_f^{-2}(x) |v(x) - A_{f,T} v(\lambda_f(x))|^2 \, dx \leq c \| \text{grad} v \|^2_{0,\Omega_f}, \quad v \in H^1(\Omega).
\]

For assume that (5.10) has been established. If \( \lambda \) and \( (\gamma, T) \) with \( \gamma \neq 0 \) other than if \( m < n \)
we start by introducing a new averaging operator \( \tilde{A}_{f,T} \) such that

\[
\tilde{A}_{f,T} v(\lambda) = \int_{f^*(T)} v(G_m(\lambda, q)) \, dq = \int_T v(G_m(\lambda, q(y))) \, dq.
\]

In fact, if \( n = m - 1 \) such that \( f^*(T) \) is just a single vertex, then \( \tilde{A}_{f,T} v = v \). On the other hand, if \( m < n - 1 \), then \( f^* \) is connected, and this is utilized below. We will estimate the two terms

\[
\int_{\Omega_f} \rho_f^{-2}(x) |v(x) - \tilde{A}_{f,T} v(\lambda_f(x))|^2 \, dx, \quad \int_{\Omega_f} \rho_f^{-2}(x) |\tilde{A}_{f,T} v(\lambda_f(x)) - A_{f,T} v(\lambda_f(x))|^2 \, dx.
\]

Note that

\[
\tilde{A}_{f,T} v(0) = \int_{f^*(T)} v(G_m(0, q)) \, dq.
\]

Since this operator reproduces constants on \( f^* \), it follows by Poincaré’s inequality that

\[
\int_f |v(q) - \tilde{A}_{f,T} v(0)|^2 \, dq \leq c \| \text{grad} v \|^2_{0,f^*},
\]

for all functions \( v \in H^1(f^*) \). A scaling argument now shows that for any \( \lambda \in S^*_m \) we have

\[
\int_f |v(G_m(\lambda, q)) - \tilde{A}_{f,T} v(\lambda)|^2 \, dq \leq cb(\lambda)^2 \| \text{grad} v(G_m(\lambda, \cdot)) \|^2_{0,f^*}.
\]

To see this, just introduce the function \( \hat{v} \) defined on \( f^* \) by

\[
\hat{v}(q) = v(G_m(\lambda, q)) \quad \text{with} \quad \text{grad} \hat{v}(q) = b(\lambda) \text{grad} v(G_m(\lambda, q)).
\]
Furthermore, \( \tilde{A}_{f,T}\hat{v}(0) = \tilde{A}_{f,T}v(\lambda) \). Therefore, the estimate (5.12) follows directly from (5.11). Furthermore, by using (5.6) and (5.12) we obtain
\[
\int_{\Omega_f} \rho_f(x)^{-2} |v(x) - \tilde{A}_{f,T}v(\lambda_f(x))|^2 \, dx
\]
for all \( v \) such that \( v(G_m(\lambda, \cdot)) \) is in \( H^1(f^*) \) for all \( \lambda \in \mathcal{S}_m^c \). In particular, this estimate holds if \( v \in H^1(\Omega_f) \) is smooth, and this is the desired estimate for \( v - \tilde{A}_{f,T}v \).

To complete the proof, we need a corresponding estimate for \( \tilde{A}_{f,T}v(\lambda_f(\cdot)) - A_{f,T}v(\lambda_f(\cdot)) \). For any \( \lambda \in \mathcal{S}_m^c \) we have
\[
\tilde{A}_{f,T}v(\lambda) - A_{f,T}v(\lambda) = -\int_T [v(G_m(\lambda, q_f(y)) - v(G_m(\lambda, y))] \, dy
\]
\[
= b(\lambda) \int_T \int_0^1 \nabla v(G_m(\lambda, (1-t)q_f(y) + ty)) \cdot (y - q(y)) \, dt \, dy.
\]
However, writing
\[
y = \sum_{j=0}^m \lambda_j(y) x_j + \rho_f(y) q_f(y),
\]
it is easy to check that
\[
G_m(\lambda, (1-t)q_f(y) + ty) = G_m(\lambda', q_f(y)),
\]
where \( \lambda' = \lambda'(\lambda, t, \lambda_f(y)) \) and
\[
\lambda'(\lambda, t, \mu) = \lambda + \lambda(t) \mu, \quad \lambda, \mu \in \mathcal{S}_m^c, \ t \in \mathbb{R}.
\]
Therefore, since \( y = G_m(\lambda_f(y), q_f(y)) \), we can use (5.3) to rewrite the representation of \( \tilde{A}_{f,T}v(\lambda) - A_{f,T}v(\lambda) \) in the form
\[
\tilde{A}_{f,T}v(\lambda) - A_{f,T}v(\lambda) = \frac{b(\lambda)}{|T|} \int_0^1 \int_{S_m^c} b(\mu)^{n-m-1} \int_{f^*(T)} \nabla v(G_m(\lambda', \lambda(t, \mu), q)) \cdot (y - q) \, dq \, d\mu \, dt,
\]
where \( \mu = \lambda_f(y) \) and \( q = q_f(y) \). Hence, it follows by Minkowski’s inequality and (5.3) that
\[
\left( \int_{\Omega_f} \rho_f^{-2}(x)(\tilde{A}_{f,T}v(\lambda(x)) - A_{f,T}v(\lambda(x)))^2 \, dx \right)^{1/2}
\]
\[
\leq c \int_0^1 \int_{S_m^c} b(\mu)^{n-m-1} \int_{f^*(T)} |\nabla v(G_m(\lambda'(\lambda_f(x), t, \mu), q))|^2 \, dq \, d\mu \, dt
\]
\[
\leq c \int_0^1 \int_{S_m^c} b(\mu)^{n-m-1} \int_{f^*} (\int_{S_m^c} b(\mu)^{n-m-1} \int_{f^*} |\nabla v(G_m(\lambda', q))|^2 \, dq \, d\lambda)^{1/2} \, d\mu \, dt,
\]
where $\lambda' = \lambda'(\lambda, t, \mu)$. To proceed, we make the substitution $\lambda \mapsto \lambda'$. The matrix associated to this transformation is $I - t\mu e^T$, with determinant $b(t\mu)$. Here, as above, $e$ is the vector with all components equal to one. Furthermore, $b(\lambda') = b(\lambda)b(t\mu)$. Since $b(t\mu) \geq b(\mu)$, it follows, again using (5.6), that

\[
\begin{align*}
\left( \int_{\Omega} \rho^{-2}_g(x)(A_f T v(\lambda_f(x) - A_f T v(\lambda_f(x))^2 dx \right)^{1/2} \\
\leq c \int_0^1 \int_{S_m} \frac{b(\mu)^{n-m-1}}{b(t\mu)^{(n-m)/2}} \left( \int_{S_m} b(\lambda')^{n-m-1} \int_{f^*} |\text{grad} v(G_m(\lambda', q))|^2 dq d\lambda' \right)^{1/2} d\mu dt \\
\leq c \int_{S_m} b(\mu)^{-1+(n-m)/2} \left( \int_{S_m} b(\lambda')^{n-m-1} \int_{f^*} |\text{grad} v(G_m(\lambda', q))|^2 dq d\lambda' \right)^{1/2} d\mu \\
\leq c \| \text{grad} v \|_{0, \Omega} \int_{S_m} b(\mu)^{-1+(n-m)/2} d\mu \leq c \| \text{grad} v \|_{0, \Omega_f}.
\end{align*}
\]

Together with (5.12), this completes the proof of (5.10) and hence the lemma is established. \hfill \square

**Proof of Lemma** 5.6. For $f \in \Delta_m(T)$ and $I \in \mathcal{I}_m$, with $m < n$, we have to show

\[
\int_{\Omega} \rho^{-2}_g(x)|A_f v(P_t \lambda_f(x))|^2 dx \leq c \int_{\Omega} \rho^{-2}_g(x)|v(x)|^2 dx + \| \text{grad} v \|_{0, \Omega}^2,
\]

where $g = f(I) \in \Delta(f)$. We observe that

\[
A_f v(P_t \lambda_f) = \sum_{T \in T_f} \frac{|T|}{|\Omega_f|} A_{g,T}(\lambda_g).
\]

However, by (5.9) we have

\[
\int_{\Omega} \rho^{-2}_g(x)|v(x) - A_{g,T} v(\lambda_g(x))|^2 dx \leq c \| v \|_{1, \Omega}^2,
\]

and by the triangle inequality this implies that

\[
\int_{\Omega} \rho^{-2}_g(x)|A_{g,T} v(\lambda_g(x))|^2 dx \leq c \int_{\Omega} \rho^{-2}_g(x)|v(x)|^2 dx + \| \text{grad} v \|_{0, \Omega}^2.
\]

The desired result follows by summing over $T \in T_f$. \hfill \square

**Proof of Lemma** 5.7. Let $m < n$, $f = [x_0, x_1, \ldots, x_m] \in \Delta_m(T)$, $I \in \mathcal{I}_m$ with $0 \notin I$ and $I' = (0, I)$. We must show that

\[
\int_{\Omega_{x_0}} \lambda^{-2}_0(x)(A_f v(P_t \lambda_f(x)) - A_f v(P_t \lambda_f(x)))^2 dx \leq c \| \text{grad} v \|_{0, \Omega}^2, \quad v \in H^1(\Omega_f).
\]

We recall that for any $T \in T_f$ we have $A_{f,T} v(P_t \lambda_f(\cdot)) = A_{g,T} v(\lambda_g(\cdot))$, where $g = f(I) \in \Delta(f)$. Similarly, $A_{f,T} v(P_t \lambda_f(\cdot)) = A_{g,T} v(P\lambda_g(\cdot))$, where $(P\lambda_g)_0 = 0$, and $(P\lambda_g)_i = (\lambda_g)_i$ for $i \neq 0$. The desired estimate will follow if we can show

\[
\int_{\Omega_{x_0}} \lambda^{-2}_0(x)(A_{g,T} v(\lambda_g(x)) - A_{g,T} v(P\lambda_g(x)))^2 dx \leq c \| \text{grad} v \|_{0, \Omega}^2.
\]
for all $v \in H^1(T)$, $T \in T_f$. In fact, it is enough to show that
\begin{equation}
\int_{\Omega_g} \lambda_0^{-2}(x)(A_{g,T}v(\lambda_g(x)) - A_{g,T}v(P\lambda_g(x)))^2 \, dx \leq c \| \nabla v \|^2_{0,T}.
\end{equation}

To see this, assume that $\hat{T} \in T_{\Omega_0}$ such that $\hat{T} \notin T_g$. Let $\hat{g} = g \cap \hat{T}$. Then $\hat{T} \in T_g$, and $(\lambda_{\hat{g}})_i = (\lambda_g)_i$ for all the components of $\lambda_g$ which are not identically zero on $\hat{T}$. Therefore (5.14), applied to $\hat{g}$ and $(\lambda_{\hat{g}})$, we obtain (5.13).

The rest of the proof is devoted to establishing (5.14). Without loss of generality we can assume that $g = [x_0, x_1, \ldots, x_j]$ such that
\[ A_{g,T}v(P\lambda_g) = \int_T v(G_j(\lambda_g, y) + \lambda_0(y - x_0)) \, dy. \]

We have
\[ A_{g,T}v(P\lambda_g) - A_{g,T}v(\lambda_g) = \int_T [v(G_j(\lambda, y) + \lambda_0(y - x_0)) - v(G_j(\lambda, y))] \, dy \]
\[ = \lambda_0 \int_T \int_0^1 \nabla v(G_j(\lambda, y) + t\lambda_0(y - x_0)) \cdot (y - x_0) \, dt \, dy, \]

where $\lambda = \lambda_g \in S^j$. If we express $y$ as $y = G_j(\mu, q)$, where $\mu = \lambda_g(y)$ and $q = q_g(y)$, we further obtain that
\[ G_j(\lambda, y) + t\lambda_0(y - x_0) = \sum_{i=0}^j \lambda_i x_i + (t\lambda_0 + b(\lambda))y - t\lambda_0x_0 \]
\[ = \sum_{i=0}^j \lambda_i x_i + (t\lambda_0 + b(\lambda))(\sum_{i=0}^j \mu_i x_i + b(\mu)q) - t\lambda_0x_0 \]
\[ = \sum_{i=0}^j \lambda'_i x_i + b(\lambda')q = G_j(\lambda', q), \]

where $\lambda' = \lambda(\lambda, t, \mu)$ is given by
\[ \lambda'_0 = (1 - t)\lambda_0 + (t\lambda_0 + b(\lambda))\mu_0 \]
and where
\[ \lambda'_i = \lambda_i + (t\lambda_0 + b(\lambda))\mu_i, \quad i > 0. \]

Using the identity (5.15), we therefore have
\[ A_{g,T}v(P\lambda_g) - A_{g,T}v(\lambda_g) \]
\[ = \frac{\lambda_0}{|T|} \int_{S^j} b(\mu)^{n-j-1} \int_0^1 \int_T \nabla v(G_j(\lambda', q)) \cdot (G_j(\mu, q) - x_0) \, dq \, dt \, d\mu, \]
where $\lambda' = \lambda'(\lambda, t, \mu)$ and $\lambda = \lambda_g$. The matrix associated to the linear transformation $\lambda \mapsto \lambda'$ is given by

$$I - \mu e^T + t(\mu - e_0)e_0^T = (I - \mu e^T)(I - t e_0 e_0^T),$$

with determinant $(1 - t)b(\mu)$.

From Minkowski’s inequality and (5.5) we now have

$$\left( \int_{\Omega_g} \lambda_g^{-2}(x)|A_{g,T}v(P\lambda_g(x)) - A_{g,T}v(\lambda_g(x))|^2 \, dx \right)^{1/2}$$

$$\leq c \int_{S_j} b(\mu)^{n-j-1} \int_0^1 \left( \int_{\Omega_f} \int_{S^*(T)} |\text{grad} \, v(G_j(\lambda'(x), q))|^2 \, dq \, dx \right)^{1/2} \, dt \, d\mu$$

$$\leq c \int_{S_j} b(\mu)^{n-j-1} \int_0^1 \left( \int_{S^*(T)} |\text{grad} \, v(G_j(\lambda'(x), q))|^2 \, dq \, d\lambda \right)^{1/2} \, dt \, d\mu,$$

where $\lambda' = \lambda'(\lambda, t, \mu)$ is given above, and $\lambda'(x) = \lambda'(\lambda_g(x), t, \mu)$. To proceed we make the substitution $\lambda \mapsto \lambda'$. We note

$$b(\lambda') = b(\lambda)b(\mu) + t\lambda_0 b(\mu) \geq b(\lambda)b(\mu),$$

and that $\lambda$ can be regarded as function of $\lambda', t$ and $\mu$. Therefore, we obtain

$$\left( \int_{\Omega_g} \lambda_g^{-2}(x)|A_{g,T}v(P\lambda_g(x)) - A_{g,T}v(\lambda_g(x))|^2 \, dx \right)^{1/2}$$

$$\leq c \int_{S_j} \int_0^1 \frac{b(\mu)^{n-j-3/2}}{(1 - t)^{1/2}} \left( \int_{S^*(T)} |\text{grad} \, v(G_j(\lambda'(x), q))|^2 \, dq \, d\lambda' \right)^{1/2} \, dt \, d\mu$$

$$\leq c \int_{S_j} \int_0^1 \frac{b(\mu)^{-1+(n-j)/2}}{(1 - t)^{1/2}} \left( \int_{S^*(T)} |\text{grad} \, v(G_j(\lambda'(x), q))|^2 \, dq \, d\lambda' \right)^{1/2} \, dt \, d\mu$$

$$\leq c \left( \int_T |\text{grad} \, v(\lambda')|^2 \, dx \right)^{1/2},$$

where (5.5) has been used for the final inequality, and where the integrals in $\mu$ and $t$ are easily seen to be bounded. This completes the proof of (5.14), and hence of the lemma.

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