Thirty-three deformation classes of compact hyperkähler orbifolds

Grégoire Menet

November 29, 2022

In honor of Professor Dimitri Markushevich for his 60 (±2) birthday

Abstract

As their smooth analogue the irreducible symplectic varieties appear as elementary bricks in the generalizations of the Bogomolov decomposition theorem ([2], [11]). Let $S$ be a K3 surface; generalizing the Fujiki construction [14], we investigate the irreducible symplectic varieties with simply connected smooth locus that can be obtained as terminalizations of quotients of the product $S^n$. In dimension 4, we compute the singularities for 29 orbifolds examples which appear to be independent under deformation. We also provide 4 additional orbifolds examples in dimension 6.

1 Introduction

1.1 Motivation and main results

Irreducible symplectic orbifolds (or compact hyperkähler orbifolds) can be seen as the simplest generalization of compact hyperkähler manifolds; they are orbifolds with a simply connected smooth locus carrying a unique (up to scalar) non-degenerate holomorphic 2-form and with a singular locus in codimension 4. In the last few years, the theory of irreducible symplectic (IHS) orbifolds have been developed especially with the generalization of the global Torelli theorem [25] and of the Kähler cone knowledge [27]. In this context, it appears natural to search for examples of such orbifolds. In a more general perspective, the irreducible symplectic orbifolds are a particular case of the irreducible symplectic (IHS) varieties which are getting attention in the last years especially because of recent important generalizations as the Bogomolov decomposition theorem [2] and the global Torelli theorem [3].

In a perspective of producing examples of irreducible symplectic orbifolds a huge amount of combinations seems possible to explore. Let $X$ be a projective symplectic manifold endowed with a finite automorphism group $G$ such that $X/G$ has a unique (up to scalar) non-degenerate holomorphic 2-form on its smooth locus. Let $Y \to X/G$ be a terminalization (see Section 2.2 for definition and existence). Let $Y_{\text{reg}}$ be the smooth locus of $Y$. If we assume that $\pi_1(Y_{\text{reg}}) = 0$, then $Y$ is an irreducible symplectic variety with singular locus of codimension 4 (see Definition 2.4 and Proposition 2.11). Therefore, it is natural to ask which conditions we need on $X$ and $G$ in order to have $\pi_1(Y_{\text{reg}}) = 0$.

Proposition 1.1. Assume that $X$ is simply connected then $\pi_1(Y_{\text{reg}}) = 0$ if and only if $G$ is generated by automorphisms with fixed locus of codimension 2.

This proposition corresponds to Proposition 2.13 in Section 2.3.

Already in dimension 4, several cases of the previous simple construction can be studied. The variety $X$ can be:

(1) the product of two K3 surfaces;

(2) a manifold of $K3^{[2]}$-type (equivalent by deformation to a Hilbert scheme of 2 points on a K3 surface);
(3) a fourfold of Kummer type (equivalent by deformation to a generalized Kummer fourfold);

(4) a complex torus of dimension 4 (however Proposition 1.1 do not apply in this case).

Forty years ago, Fujiki partially investigated the case (1) in [13, Section 13]. The main objective of this paper is to generalize and complete the Fujiki investigations.

Fujiki has provided his construction in dimension 4. It can be generalized for every dimension as follows. Let $S$ be a projective K3 surface and $G$ a finite non-trivial symplectic automorphism group. Let $\theta : G \to G$ be an involution (which can be id). Let $n \geq 2$ be an integer. We set $j_\theta : G \to \text{Aut}(S^n)$ defined by

$$j_\theta(g)(x_1, x_2, x_3, ..., x_n) = (g(x_1), \theta(g)(x_2), x_3, ..., x_n).$$ (1)

I.e. $j_\theta(g)$ is given by the diagonal action with $g$ on the first factor, $\theta(g)$ on the second and id on the other factors. The permutation group $S_n$ also acts naturally on $S^n$.

**Definition 1.2.** Let

$$Y \to S^n/\langle j_\theta(G), S_n \rangle$$

be a terminalization of the quotient $S^n/\langle j_\theta(G), S_n \rangle$. Such a variety $Y$ is called a Fujiki variety, moreover we denote $S(G)_\theta^{[n]} := Y$. This construction of varieties is called the Fujiki construction.

**Remark 1.3.** Note that all the Fujiki varieties are primitively symplectic varieties with singularities in codimension 4 (see Definition 2.4 and Proposition 3.12).

**Remark 1.4.** Since $S(G)_\theta^{[n]}$ necessarily contains at least one exceptional divisor (see Proposition 2.29), the Fujiki varieties consist in a strict sub-space of their space of deformations; therefore, these deformations are generically new objects.

**Remark 1.5.** In the previous definition of Fujiki varieties, we did not emphasize the choice of the terminalization for $S^n/\langle j_\theta(G), S_n \rangle$; this is because this choice will not change the deformation class of the variety as explained by Proposition 2.10.

The deformation class of $S(G)_\theta^{[n]}$ only depends of $\theta$, $n$ and the deformation class of $(S, G)$. To be more precise, if $(S, G)$ and $(S', G')$ are deformation equivalent, given an involution $\theta$ on $G$, we can find an involution $\theta'$ on $G'$ such that $S(G)_\theta^{[n]}$ and $S(G')_{\theta'}^{[n]}$ are deformation equivalent.

It is natural to ask when $S(G)_\theta^{[n]}$ is an irreducible symplectic variety (see Definition 2.9). In particular, note that a primitively symplectic variety with rational singularities and simply connected smooth locus is irreducible symplectic (see Proposition 2.11). In order to study this property, we introduce the following definition.

**Definition 1.6.** An involution $\theta : G \to G$ is said valid if there exist a family of generators $(g_1, ..., g_k)$ of $G$ such that $\theta(g_i) = g_i^{-1}$ for all $i \in \{1, ..., k\}$.

As a consequence of Proposition 1.1 we have the following Corollaries. We distinguish the case $n = 2$ and the case $n \geq 3$.

**Corollary 1.7.** A Fujiki variety $S(G)_\theta^{[2]}$ is an irreducible symplectic variety with simply connected smooth locus if and only if $\theta$ is valid.

**Corollary 1.8.** Let $n \geq 3$ be an integer. All the Fujiki varieties $S(G)_\theta^{[n]}$ are irreducible symplectic varieties with simply connected smooth locus.

These corollaries are proved in Section 3.2.

In some sense the Fujiki construction is fully general.

**Theorem 1.9.** Let $G$ be a finite automorphism group on $S^n$. Let $Y \to S^n/G$ be a terminalization. If $Y$ is an irreducible symplectic variety with simply connected smooth locus, then there exists:

- a K3 surface $\Sigma$,
- a finite symplectic automorphism group $G$ on $\Sigma$ which is abelian when $n \geq 3$,
• a valid involution $\theta$, such that:

$$S^n/G \simeq \Sigma^n / \langle j_0(G), \Sigma_n \rangle.$$  

This theorem is a consequence of Theorem 3.7 and is proven in Section 5.1. Let $\prod S_i$ be the product of K3 surfaces not all isomorphic. Let $G$ be a finite automorphism group on $\prod S_i$. According to [1], Page 10, Remark 2, a terminalization $Y \to \prod S_i/G$ will not be primitively symplectic. Therefore, we see with Theorem 1.9 that the Fujiki construction covers all possible examples obtained from quotients of products of K3 surfaces.

In dimension 4, the Fujiki construction leads to several examples of irreducible symplectic orbifolds.

**Definition 1.10.** A finite symplectic automorphism group $G$ on a K3 surface $S$ will be said admissible if $S/G$ has only singularities of type $A_1$, $A_2$, $A_3$ or $A_5$.

This definition is related to the notion of admissible singularities of Fujiki in [14, Section 7]. Indeed, if $G$ is admissible, then $S^2/\langle j_0(G), \Sigma_2 \rangle$ has admissible singularities in the sense of Fujiki. It corresponds to singularities with a well known orbifold terminalization (see Section 4.2 for more details).

**Theorem 1.11.** All the possible (up to deformation) irreducible symplectic orbifolds in dimension 4 obtained by Fujiki construction with an admissible group $G$ are listed below. We provide their second Betti number $b_2$, the group $G$ used for the construction and their singularities:

| $b_2$ | $G$ | singularities |
|-------|-----|---------------|
| 4     | $\mathbb{A}_1^4$ | $a_2 = 4, a_3 = 6, a_4 = 4, b_6 = 2$ |
| 5     | $C_2 \times \mathbb{A}_4$ | $a_2 = 3, a_3 = 9, a_4 = 3, b_6 = 1$ |
| 5     | $C_2^4 \times C_3$ | $a_2 = 10, a_3 = 15, a_4 = 1, a_8 = 2, b_4 = 1$ |
| 5     | $C_2^3 \times C_6$ | $a_2 = 16, a_3 = 6, a_4 = 4, a_6 = 1, b_4 = 1$ |
| 6     | $C_3 \times \mathbb{S}_3$ | $a_2 = 9, a_3 = 10, a_6 = 1$ |
| 6     | $C_2 \times \mathbb{A}_4$ | $a_2 = 13, a_3 = 6, a_4 = 4, a_6 = 1$ |
| 6     | $C_2^3 \times C_4$ | $a_2 = 10, a_3 = 14, a_4 = 6$ |
| 6     | $C_2^3 \times \mathbb{S}_4$ | $a_2 = 14, a_3 = 15, a_4 = 6$ |
| 6     | $C_2^2 \times \mathbb{S}_3$ | $a_2 = 19, a_3 = 12, a_4 = 6, b_4 = 1$ |
| 7     | $C_2^2 \mathbb{A}_4$ | $a_3 = 12$ |
| 7     | $C_2^2 \times \mathbb{A}_4$ | $a_2 = 12, a_3 = 15, a_4 = 4$ |
| 7     | $C_2^3 \times \mathbb{S}_4$ | $a_2 = 12, a_3 = 3, a_4 = 4$ |
| 7     | $\mathbb{S}_3 \times C_2$ | $a_2 = 20, a_3 = 12, a_4 = 3$ |
| 8     | $C_4$ | $a_2 = 9, a_3 = 6, a_6 = 1$ |
| 8     | $C_2 \times C_6$ | $a_2 = 12, a_3 = 3$ |
| 8     | $C_2^2 \mathbb{A}_3$ | $a_2 = 6$ |
| 8     | $\mathbb{A}_3 \times \mathbb{A}_3$ | $a_2 = 28, a_3 = 12$ |
| 8     | $\mathbb{A}_3 \times \mathbb{S}_4$ | $a_2 = 28, a_3 = 20$ |
| 8     | $\mathbb{S}_4$ | $a_2 = 24, a_3 = 12, a_4 = 3$ |
| 8     | $C_2^3 \times C_4$ | $a_2 = 17, a_4 = 6, b_4 = 1$ |
| 8     | $\mathbb{S}_3 \times C_4$ | $a_2 = 20, a_3 = 15$ |
| 10    | $C_4$ | $a_2 = 10, a_4 = 6$ |
| 10    | $C_2^2 \times C_4$ | $a_2 = 10, a_4 = 6$ |
| 10    | $\mathbb{S}_3$ | $a_2 = 28, a_3 = 12$ |
| 10    | $C_2 \times C_6$ | $a_2 = 12, a_4 = 4$ |
| 10    | $D_0$ | $a_2 = 28, a_3 = 10$ |
| 10    | $C_2 \times \mathbb{S}_4$ | $a_2 = 28, a_3 = 10$ |
| 11    | $C_3$ | $a_3 = 15$ |
| 11    | $D_4$ | $a_2 = 36, a_4 = 3$ |
| 11    | $C_2^2 \times C_2$ | $a_2 = 36, a_4 = 3$ |
| 14    | $C_2^2$ | $a_2 = 36$ |
| 16    | $C_2$ | $a_2 = 28$ |
The previous theorem is a consequence of Theorem 5.6 and Proposition 5.15. The numbers $a_k$ correspond to the number of symplectic cyclic singularities of order $k$, for $k \leq 6$. In dimension 4, there are several possibilities for symplectic cyclic singularities of order 8. In our case there are of analytic type: $C^4 / \langle g \rangle$ with: $g = \text{diag}(\xi_k, \xi_k^{-1}, \xi_k^2, \xi_k^3)$ and their number is denoted by $a_k$. The number $b_k$ corresponds to the number of singular points of the analytic form $C^4 / \langle g, s \rangle$ with:

$$g = \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}) \quad \text{and} \quad s = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$ 

Precisions on the groups names are given in Section 1.5.

There are two main difficulties to prove the previous theorem which are the reasons for the length of this paper. One of the difficulty is to determine the singularities of the orbifolds. The second difficulty is to prove that we have found all the possible valid involutions. For both difficulties, the help of a computer has been required.

**Remark 1.12.** Note that there are three couples of orbifolds that share the same second Betti number and singularities. They are orbifolds constructed with the groups:

- $C_2^2 \cdot C_2$ and $D_4$;
- $C_2^2 \times C_4$ and $C_4$;
- $C_2 \times S_4$ and $D_6$.

These three couples could lead to deformation equivalent orbifolds. However all the other orbifolds are necessarily independent under deformation according to Proposition 2.8. Therefore, we have found at least 29 deformation classes of irreducible symplectic orbifolds.

In higher dimension we have the following results. When $G$ is abelian, there is only one possible valid involution given by $\text{inv}(g) = g^{-1}$ for all $g \in G$. To simplify the notation, when $G$ is abelian, if we do not specify the involution $\theta$, it refers to this unique valid involution $\text{inv}$. In higher dimension, we will see in Lemma 3.6 that without loss of generality, we can assume that $G$ is abelian and $\theta$ valid. We obtain:

**Proposition 1.13.** In higher dimension, all the Fujiki varieties (up to deformation) are given by the 14 following series: $S(C_2)^{[n]}$; $S(C_3)^{[n]}$; $S(C_4^{[2]})^{[n]}$; $S(C_4)^{[n]}$; $S(C_5)^{[n]}$; $S(C_6)^{[n]}$; $S(C_7)^{[n]}$; $S(C_8)^{[n]}$; $S(C_9)^{[n]}$; $S(C_{10})^{[n]}$; $S(C_{12})^{[n]}$. Moreover all these varieties are independent under deformation.

This proposition is proved in Section 3.3. The second Betti numbers of the previous series are provided in Corollary 3.14.

It is not an easy problem to determine if the irreducible symplectic varieties given by Proposition 1.13 are orbifolds since the terminalization are mostly unknown in practice. However in few cases, the terminalizations are easy to get.

**Proposition 1.14.** The varieties $S(C_2^k)^{[n]}$ are irreducible symplectic orbifolds for all $1 \leq k \leq 4$.

This proposition corresponds to Proposition 5.1.

### 1.2 Previous related works

We give an overview on the previous works providing examples of irreducible symplectic varieties. More information can be found in [31].

**Fujiki work**

Fujiki in [14] Theorem 13.1 provides the orbifolds $S(G)^{[2]}$ when $G$ is admissible and abelian. In this paper, we have shown via Corollary 1.7 that these examples are irreducible symplectic orbifolds. Moreover, Fujiki has made mistakes on his computation of the singularities for some of these orbifolds. These mistakes have been corrected in the present paper (see Remark 5.8).
Orbifolds of Nikulin type

The most well known example of IHS orbifolds are the orbifolds of Nikulin type (see [10, Definition 3.1] for the terminology). These orbifolds are deformation equivalent to the Fujiki orbifold $S(C_2)^2$ as explained in [27, Proposition 3.10]. These orbifolds were already known to be irreducible symplectic as explained in [25, Proposition 3.8]. As explained in [27, Proposition 3.12], the Nikulin orbifolds are also deformation equivalent to the Markushevich–Tikhomirov varieties $P^0$ introduced in [23, Definition 3.3]. Their Beauville–Bogomolov lattice has been computed in [24] and their wall divisors are given in [28]. Moreover a compete family of these orbifolds is given in [10].

Quotients of generalized Kummer

Let $K_2(T)$ be a generalized Kummer fourfold constructed from a 2-dimensional torus $T$. The involution $- id$ on $T$ induces a symplectic involution $\iota$ on $K_2(T)$ with a fixed locus given by a K3 surface and 36 isolated points. Let $K' \to K_2(T)/\iota$ be the blow-up in the K3 surface of singularities. The orbifold $K'$ is an irreducible symplectic orbifold as explained in [25, Proposition 3.8]. Moreover, its Beauville–Bogomolov form has been computed in [17]. Since the orbifold $K'$ has second Betti number 8 and 36 singularities of type $a_2$ (see Notation 4.27), we know by Proposition 2.8 that it is not deformation equivalent to one of the examples provided by Theorem 1.11.

Similarly in [13, Section 5.5], we provided the orbifold $K'_3$ obtained as a blow-up of $K_2(T)$ quotiented by an automorphism of order 3. By Proposition 1.11 we know that $K'_3$ is an irreducible symplectic orbifold. However in [13, Section 5.5], it is shown that $K'_3$ has second Betti number 7 and 12 singularities of type $a_3$ (see Notation 4.27). Therefore according to the data of Theorem 1.11 the orbifold $K'_3$ could be deformation equivalent to $S(C_2^3)^2$.

Relative Prymian

Another method to construction irreducible symplectic orbifolds existing in the literature is via relative Prymian (see [23], [1], [22] and [33]). The general idea is the following. Let $(S, H)$ be a polarized K3 surface endowed with an anti-symplectic involution $i$. Let $C \in |H|^i$ be a smooth curve, we can consider the Prym variety $\text{Prym}(C, i|_C)$. The relative Prymian associated to $(S, H)$ and $i$ is a variety $Y$ with a Lagrangian fibration $Y \to |H|^i$ such that the generic fibers are the Prym($C, i|_C$).

The relative Prymian in [24] is deformation equivalent to $S(C_2)^2$ as explained before. However several questions are open when considering the other examples in [11], [22] and [33]. Are they all irreducible symplectic? Are they all orbifold? What are their second Betti numbers? Because of this lack of knowledge, it is not possible to tell if these examples are deformation equivalent or not to one of the examples of the present paper.

Moduli spaces of semi-stable sheaves

In a larger perspective, I also would like to mention the work of Perego and Rapagnetta in [32] who provide series of irreducible symplectic varieties which are not orbifold. These examples are obtained via moduli spaces of semi-stable sheaves on K3 surfaces or abelian surfaces.

1.3 What remains to be explored in dimension 4

Already in dimension 4, many tracks remain to be explore in order to find more examples of irreducible symplectic orbifolds. As mention above an irreducible symplectic orbifold $Y$ can be obtained as a terminalization of a quotient $X/G$ with at least 4 possibilities for $X$.

When $X$ is the product of two K3 surfaces

This is the case considered in the present paper and by Theorem 1.9 it corresponds to the orbifolds obtained via Fujiki construction from a K3 surface $S$, a symplectic group $G$ on $S$ and a valid involution $\theta : G \to G$ (see Definition 1.2). However, in this paper we have explored only the examples with admissible groups $G$ (see Definition 1.10). Therefore according to [36], it remains
47 groups to be studied. The difficulty to study these other cases when \( G \) is not admissible is the lack of known terminalizations. The quotient \( S^2/(j_\theta(G), S_2) \) will have singularities in codimension 2 for which a terminalization is not known. However, our method to determine the valid involutions (see Section 5) and to compute the Betti numbers (see Proposition 3.15 and Section 2.5) will still apply when \( G \) is not admissible.

When \( X \) is a manifold of Kummer type

Let \( T \) be a 2-dimensional torus. Let \( K_2(T) \) be the associated generalized Kummer fourfold. An automorphism group on \( K_2(T) \) is said natural if it is induced by an automorphism on \( T \).

There is a work in progress of Bertini, Capasso, Debarre, Grossi, Mauri and Mazzon (\cite{7}) studying the case of \( X \) being a generalized Kummer fourfold and \( G \) a natural symplectic automorphism group. The orbifolds \( K' \) and \( K'_3 \) mentioned above are examples of this case.

When \( X \) is a manifold of \( K3^{[2]} \)-type

In \cite{15}, the symplectic automorphism groups on such a manifold are classified. Therefore, it could be possible to study these cases.

Let \( S \) be a K3 surface. Let \( S^{[2]} \) be the Hilbert scheme of 2 points on \( S \). As before a natural automorphism group on \( S^{[2]} \) is an automorphism induced by an automorphism on \( S \). Note that the varieties obtained with \( X = S^{[2]} \) and \( G \) a natural symplectic automorphism groups correspond to the Fujiki varieties when \( \theta = \text{id} \). Bertini, Capasso, Debarre, Grossi, Mauri and Mazzon are also studying this case in \cite{7}.

When \( X \) is a 4-dimensional torus

Fujiki in \cite{14} Section 13 and 14, has already partially consider this case. Indeed, we can perform the Fujiki construction with a 2-dimensional torus instead of a K3 surface. Let \( T \) be a 2-dimensional torus, \( G \) a symplectic automorphism group on \( T \) and \( \theta : G \to G \) an involution. We can consider a terminalization \( T(G)_{\theta}^{[2]} \to T^2/(j_\theta(G), S_2) \). In \cite{14} Theorem 13.1, Fujiki studies the examples \( T(G)_{\theta}^{[2]} \) when \( G \) is abelian and \( \theta \) the valid involution on an abelian group. However Proposition 1.1 does not apply in this context; therefore these orbifolds are not necessarily irreducible symplectic. We will show in a future work that some of the \( T(G)_{\theta}^{[2]} \) are irreducible symplectic and some are not.

Relative Prymians

As mentioned above, some irreducible symplectic orbifolds can be obtained via the relative Prymian varieties. There are many possibilities for such constructions varying the K3 surface \( S \), the involution \( i \) and the polarization \( H \). There is a work in progress studying these relative Prymian varieties by Brakkee, Camere, Grossi, Pertusi, Saccà and Viktorova.

To go further

We could even imagine to go further and choose for \( X \) one of our orbifold examples. Especially, it seems possible to classify the symplectic automorphism groups on orbifolds of Nikulin type using the global Torelli theorem (\cite{27} Theorem 1.1)), the knowledge of the wall divisors (\cite{28}) and the recent results on monodromy (\cite{21} Theorem 1.6)).

1.4 Organization of the Paper

The paper is organized as follows. The section 2 is dedicated to some reminders and to the proof of Proposition 1.1. In Sections 3.1 we prove Theorem 1.3. In Section 3.2 Corollaries 1.7 and 1.8 are proved. In Section 3.3 we explain how to compute the second Betti number of a Fujiki variety. In Section 3.4 we study the Fujiki constant of a Fujiki variety and we prove Proposition 1.13. In Section 3.5 we propose a criterion to determine when two involutions on a same group provide bimeromorphic Fujiki varieties. In Section 4 we explain our method to determine the singularities
of a Fujiki variety of dimension 4 when $G$ is admissible. In particular in Section 4.7 we provide a technique to verify the computation of the singularities based on a result of Beckmann and Song [6]. In Section 5 we determine all the valid involutions on an admissible group; in particular we prove Theorem 1.11. Section 6 is dedicated to the proof of Proposition 1.14. Finally in Section A we give the Python programs used for the computations.

1.5 Notation

Notation 1.15. Let $Y$ be a complex space. We denote $Y \reg := Y \setminus \text{Sing} Y$.

Notation 1.16. • We denote by $C_n$ the cyclic group of order $n$.

• We denote by $D_n$ the dihedral group of order $2n$.

• We denote by $S_n$ and $A_n$ the symmetric and the alternating groups of $n$ elements.

• We set $A_{3,3} := S_3 \wr C_2$.

For some groups, the names that we use are not the one used in [60]. We provide the relation between our groups names and the Xiao’s groups names.

| Group name of the paper | Xiao’s group name | SmallGroup |
|-------------------------|------------------|------------|
| $C_2 \times C_4$       | $\Gamma_{2c1}$   | (16,3)     |
| $C_4 \times C_4$       | $\Gamma_{701}$   | (32,6)     |
| $C_2 \times C_2$       | $2^3 C_2$        | (32,27)    |
| $C_2 \times C_4$       | $3^3 C_4$        | (36,9)     |
| $S_3$                  | $S_{3,3}$        | (36,10)    |
| $C_2 \times C_3$       | $4^3 C_3$        | (48,3)     |
| $C_2 \times \mathfrak{A}_4$ | $2^3 (C_2 \times C_6)$ | (48,49) |
| $C_2 \times \mathfrak{A}_4$ | $2^4 C_4$       | (48,50)    |
| $S_3 \times C_2$       | $N_{72}$         | (72,40)    |
| $C_2 \times C_6$       | $2^4 C_6$        | (96,70)    |
| $C_2 \times \mathfrak{S}_3$ | $2^4 D_6$       | (96,227)   |

The group $S_3 \times C_2$ alias $N_{72}$ is one of the maximal symplectic finite group on a K3 surface described by Mukai in [29].

Acknowledgements. I am very grateful to Romain Demelle, Arvid Perego, Martin Schwald and Jieao Song for very useful discussions. I also want to thank Annalisa Grossi, Mirko Mauri and Enrica Mazzon for discussing their oncoming paper and for very interesting comments. This work has been financed by the PRCI SMAGP (ANR-20-CE40-0026-01).

2 Irreducible symplectic varieties

2.1 Definitions

We follow the usual definitions from [2] and [3].

Definition 2.1. Let $X$ be a normal complex analytic space.

• We denote by $\Omega^p_X$ the sheaf of reflexive holomorphic $p$-forms on $X$ given by $i_\ast \Omega^p_{X,\reg}$, where $i : X_\reg \hookrightarrow X$ is the inclusion.

• We call a holomorphic symplectic form on $X$ a closed reflexive 2-form on $X$ which is non-degenerate at each point of $X_\reg$.

• Let $f : \bar{X} \rightarrow X$ be a resolution of singularities. We say that $X$ is a symplectic variety if $X$ admits a symplectic form $\sigma$ such that $f^\ast (\sigma)$ extends to a holomorphic 2-form on $\bar{X}$.
Remark 2.2. By [3], Corollary 1.8], a complex analytic space with only rational singularities is a symplectic variety if and only if it admits a symplectic form.

Definition 2.3. A primitively symplectic variety is a normal compact Kähler symplectic variety with \( h^i(X, \mathcal{O}_X) = 0 \) and \( h^0(X, \Omega_X^{[2]}) = 1 \).

Definition 2.4. Let \( X \) be a compact Kähler complex analytic space with rational singularities. We say that \( X \) is an irreducible holomorphic symplectic (IHS) variety if for all quasi-étale covers \( q : \tilde{X} \rightarrow X \), the algebra \( H^0(\tilde{X}, \Omega_{\tilde{X}}^{[2]}) \) is generated by a holomorphic symplectic form \( \tilde{\sigma} \in H^0(\tilde{X}, \Omega_{\tilde{X}}^{[2]}) \).

Remark 2.5. An irreducible symplectic variety is a primitively symplectic variety; however the contrary is not always true.

2.2 \( \mathbb{Q} \)-factorial and terminal singularities

Definition 2.6. Let \( X \) be a complex space. Let \( Y \) be a \( \mathbb{Q} \)-factorial normal complex space with only terminal singularities. We say that \( Y \) is a terminalization of \( X \) if there exists a crepant proper bimeromorphic morphism \( Y \rightarrow X \).

Theorem 2.7 ([2], Corollary 1.4.3). If \( X \) is an algebraic variety then \( X \) admits a terminalization.

Therefore in an objective of classification, the IHS varieties with \( \mathbb{Q} \)-factorial and terminal singularities are more relevant. Moreover in this case, their moduli spaces are more coherent as shown by the following result.

Proposition 2.8 ([2], Proposition 5.13). Let \( X \) be an IHS varieties with \( \mathbb{Q} \)-factorial and terminal singularities. Then any deformations of \( X \) is locally trivial.

For these reasons, we are going to investigate IHS varieties with \( \mathbb{Q} \)-factorial and terminal singularities in this paper. We recall how to characterized terminal singularities in our case.

Proposition 2.9 ([30], Corollary 1). A symplectic variety has terminal singularities if and only if \( \text{codim} \text{Sing} X \geq 4 \).

Proposition 2.10. Let \( Y \) and \( Y' \) be two primitively symplectic varieties which are two terminalizations of a same projective variety \( Z \). Then \( Y \) and \( Y' \) are deformation equivalent.

Proof. Since \( Y \) and \( Y' \) have terminal singularities, the proof of [24] Lemma 3.2 can be copied word for word to prove that there exists a birational map \( f : Y \dashrightarrow Y' \) which is an isomorphism in codimension 1. Since \( Y \) and \( Y' \) have \( \mathbb{Q} \)-factorial singularities, this implies a well defined an isomorphism map \( f_* : (\text{Pic} Y)_\mathbb{Q} \rightarrow (\text{Pic} Y')_\mathbb{Q} \). Therefore we conclude the proof with [3] Corollary 6.17.

2.3 Fundamental group of the smooth locus

In this paper, we will search for IHS variety with simply connected smooth locus. In this section, we explain why this is relevant and we provide a criterion in the case of terminalizations of quotients.

Proposition 2.11. Let \( X \) be a compact Kähler complex analytic complex space with rational singularities such that \( H^0(X, \Omega_X^{[2]}) = \mathbb{C} \sigma \) with \( \sigma \) a holomorphic symplectic form. If \( \pi_1(\text{reg} X) = 0 \) then \( X \) is an IHS variety.

Proof. Indeed, let \( X \) be as in the statement of the proposition. Let \( \gamma : Y \rightarrow X \) be a quasi-étale cover. By purity of branch loci, \( \gamma \) is étale over the smooth locus \( \text{reg} X \) of \( X \). Since \( \pi_1(\text{reg} X) = 0 \), the cover \( \gamma \) is necessarily trivial. Hence, \( X \) is an IHS variety.

However, there exists IHS varieties with \( \pi_1(\text{reg} X) \neq 0 \).

Proposition 2.12. Let \( X \) be an IHS manifold and \( G \) a non-trivial finite symplectic automorphism group on \( X \). If for all \( g \in G \), we have \( \text{codim} \text{Fix} g \geq 4 \) then \( X/G \) is an IHS variety. Moreover \( \pi_1((X/G)_{\text{reg}}) \neq 0 \).
Proof. By Proposition 2.14, we know that $X/G$ has terminal singularities. Then, the proof of the first statement is identical to the one of [31, Proposition 2.15]. Let $p : X \rightarrow X/G$ be the quotient map. To prove the second statement, we only need to remark that the restriction

$$p : X \rightarrow \left( \bigcup_{g \in G \setminus \{id\}} \text{Fix } g \right) \rightarrow (X/G)_{\text{reg}}$$

is an étale cover. □

In order to obtain irreducible symplectic varieties with simply connected smooth locus, we propose the following criterion. A similar result have been proved independently by Bertini, Capasso, Debarre, Grossi, Mauri and Mazzon ([7]).

**Proposition 2.13.** Let $X$ be a simply connected manifold which admits a nowhere vanishing holomorphic 2-form $\varphi$. Let $G$ be a finite automorphism group on $X$ which respects the holomorphic 2-form $\varphi$ (that is $g^* (\varphi) = \varphi$ for all $g \in G$). Let $Y \rightarrow X/G$ be a terminalization of $X/G$. Let $Y_{\text{reg}} := Y \setminus \text{Sing } Y$. Then $\pi_1(Y_{\text{reg}}) = 0$ if and only if $G$ is generated by automorphisms with fixed locus of codimension 2.

Proof. Let $f : Y \rightarrow X/G$ be a terminalization. Let $Y^o := Y \setminus f^{-1}(f(\text{Sing } Y))$. We consider the restriction $f_{\text{reg}} : Y^o \rightarrow X^o$, with $X^o := X/G \setminus f(\text{Sing } Y)$. Let $P : X \rightarrow X/G$ be the quotient map. We also denote $X' = P^{-1}(X^o)$. The morphism $f_{\text{reg}}$ is a resolution of singularities. Therefore by [20, Theorem 7.8], $\pi_1(Y^o) = \pi_1(X^o)$. By [37, Theorem 3.1], the space $Y_{\text{reg}} \setminus Y^o$ has codimension at least 2 in $Y_{\text{reg}}$, therefore $\pi_1(Y_{\text{reg}}) = \pi_1(Y^o)$. It remains to prove that $\pi_1(X') = 0$ if and only if $G$ is generated by automorphisms with fixed locus of codimension 2.

By Proposition 2.14, we have codim $\text{Sing } Y \geq 4$, in particular codim $f(\text{Sing } Y) \geq 4$. Moreover codim $P^{-1}(f(\text{Sing } Y)) \geq 4$. It follows from [13, Lemma 1.2] that $\pi_1(X') = 0$ if $G$ is generated by automorphisms with fixed locus of codimension 2.

Now we assume that $G$ is not generated by automorphisms with fixed locus of codimension 2 and we will show that $\pi_1(X') \neq 0$. Let $H \subset G$ be the sub-group of $G$ generated by elements with fixed locus of codimension 2. By assumption $H \neq G$. Moreover $H$ is a normal sub-group of $G$. Let $\overline{G} := G/H$. We have $X/G = (X/H)/\overline{G}$. To prove our claim, it is enough to show that $(X'/H) \rightarrow (X'/H)/\overline{G}$ is a non-ramified cover. Let $\overline{\tau} \in \overline{G}$. We show that $\text{Fix } \overline{\tau} \cap X'/H = \emptyset$. Let $Q : X \rightarrow X/H$. Let $\overline{\tau} \in \text{Fix } \overline{\tau}$ and $x \in Q^{-1}(\overline{\tau})$. The sub-group $G_x := \{ g \in G \mid g(x) = x \}$ induces a local action around the point $x$. The point $P(x)$ is a singular point in $X/G$ of analytic type $\mathcal{C}^0 /G_x$, with $n = \dim X$. Since $\overline{\tau} \in \text{Fix } \overline{\tau}$, we have $G_x \not\subseteq H$ and $G_x$ is not generated by automorphisms with fixed locus of codimension 2. It follows by [39, Theorem 1.2] that the space $\mathcal{C}^0 /G_x$ does not admit a crepant resolution. Hence necessarily $P(x) \notin X'$, so $\overline{\tau} \notin X'/H$. □

**Remark 2.14.** We consider the same objects as in the statement of the previous proposition. If $G$ is not generated by automorphisms with fixed locus in codimension 2, it is shown by Bertini, Capasso, Debarre, Grossi, Mauri and Mazzon in [7], that we can always find a terminalization $Y \rightarrow X/G$ such that $Y = Z/H$ with $Z$ a symplectic variety with simply connected smooth locus and $H$ a non trivial group such that codim $\text{Fix } h \geq 4$ for all $h \in H \setminus \{id\}$.

**Remark 2.15.** Searching for examples in an objective of classification, Propositions 2.12, 2.13 and Remark 2.14 show that it is relevant to restrict our attention on the groups $G$ generated by automorphisms with fixed locus in codimension 2 and on irreducible symplectic varieties $Y$ with $\pi_1(Y_{\text{reg}}) = 1$. Moreover, in the orbifold case, the condition $\pi_1(Y_{\text{reg}}) = 1$ is required to be irreducible symplectic (see Definition 2.16 below); this condition allows the uniqueness in the Bogomolov decomposition theorem [11].

### 2.4 Irreducible symplectic orbifolds

As explained in the previous sections 2.2 and 2.3, we want to search for IHS varieties with $\mathbb{Q}$-factorial terminal singularities and a simply connected fixed locus. A particular case are the IHS orbifolds which appear very similar to the IHS manifolds in their behavior (see [25] and [27]).
Definition 2.16. Let $X$ be a compact Kähler complex analytic complex space with a unique, up to scalar, holomorphic symplectic form. We say that $X$ is a primitively symplectic orbifold if:

- $X$ has only quotient singularities;
- $\text{codim} \text{Sing } X \geq 4$.

If in addition $\pi_1(X_{\text{reg}}) = 0$, we say that $X$ is an irreducible symplectic (IHS) orbifold.

Remark 2.17. In the orbifold case, note that the condition $\pi_1(X_{\text{reg}}) = 0$ allows the uniqueness of the Bogomolov decomposition (see \cite{11}).

2.5 Topological invariants of orbifolds in dimension 4

In this section, we recall some results from \cite{13} with some additional computations in order to provide the Betti numbers and the Chern numbers of the examples in Theorem 5.6. This is necessary to apply the verification method described in Section 4.7.

Definition-Proposition 2.18 (\cite{13} Definition-Proposition 2.3]). Let $X$ be an $n$-dimensional complex orbifold. Let $x \in \text{Sing } X$. Then there exist a finite subgroup $G_x$ of $\text{GL}_n(\mathbb{C})$ and an open neighborhood $V_x \subset \mathbb{C}^n$ of the origin $0 \in \mathbb{C}^n$, stable under the action of $G_x$, with $V_x/G_x$ isomorphic to an open neighborhood $U_x$ of $x$, and such that

$$\text{codim} \text{Fix}(g) \geq 2 \text{ for all } g \in G_x \setminus \{\text{id}\}.$$

Such a group $G_x$ is unique up to conjugation. Let $\pi_x : V_x \rightarrow V_x/G_x \simeq U_x$ be a local uniformizing system of $x$, and $G_x$ the regional fundamental group of $X$ at $x$.

Definition 2.19 (locally V-free sheaf). Let $X$ be an orbifold. Let $\mathcal{F}$ be a coherent sheaf on $X$. The sheaf $\mathcal{F}$ is said to be locally $V$-free if for any $x \in X$, there exist a local uniformizing system $(U, V, G, \pi)$, a free coherent sheaf $\mathcal{F}_U$ on $V$, and a $G$-action on $\mathcal{F}_U$ such that $\mathcal{F}_U|_{U_x} \simeq \pi_* (\mathcal{F}_{V_x})$.

Notation 2.20. Let $X$ be an orbifold, $x \in X$ and $\mathcal{F}$ a locally $V$-free sheaf. Let $(U, V, G, \pi)$ be a local uniformizing system of $x$ and let $\mathcal{F}_V$ be a locally free sheaf on $V$ endowed with an action of $G$ as in Definition 2.19. Hence, the fiber of $\mathcal{F}_V$ at $0$ is endowed with an action of $G$, which provides a representation of $G$. We denote by $\rho_{x, \mathcal{F}}$ the representation of $G$ associated with $x$ and $\mathcal{F}$.

Notation 2.21. Let $X$ be an orbifold. Let $x \in \text{Sing } X$ be an isolated singularity. We set:

$$s_x := \frac{1}{|G_x|} \left[ 6 \left( \sum_{g \in G_x, g \neq \text{id}} \frac{\text{tr}(\rho_{x,G_x}(g)) - 4}{\det(\rho_{x,T_x}(g) - \text{id})} \right) + |G_x| - 1 \right],$$

with $(U, V, G_x, \pi)$ a local uniformizing system around $x$. We set:

$$S_{0,x} := \frac{1}{|G_x|} \sum_{g \neq \text{id}} \frac{1}{\det(\text{id} - \rho_{x,T_x}(g))}.$$

When $X$ has only isolated singularities, we set:

$$s(X) := \sum_{x \in \text{Sing } X} s_x \quad \text{and} \quad S_0(X) := \sum_{x \in \text{Sing } X} S_{0,x}.$$

Proposition 2.22 (\cite{13} Proposition 3.6]). Let $X$ be a primitively symplectic orbifold of dimension 4. We have:

$$b_4(X) + b_3(X) - 10b_2(X) = 46 + s(X),$$

or equivalently,

$$\chi(X) = 48 + 12b_2(X) - 3b_4(X) + s(X).$$
Remark 2.23. Proposition 2.22 shows that the knowledge of $b_2$, $b_3$ and the singularities is enough to compute the topological Euler characteristic and all the Betti numbers of a 4-dimensional primitively symplectic orbifold ($b_1 = 0$ by [14, Proposition 6.7]).

The invariant $s_x$ can be computed for all kind of isolated singularities $x$. We provide some examples of computations that will be necessary in this paper.

Lemma 2.24 ([13, Example 3.8]). Let $x$ be an isolated singularity of analytic type $\mathbb{C}^4 / \langle g \rangle$, with $g = \text{diag}(\xi_n, \xi^{-1}_n, \xi^{2k}_n, \xi^{-2k}_n)$ and $n \wedge k = 1$. Then

$$s_x = -(n - 1).$$

Lemma 2.25. Let $x$ be an isolated singularity of analytic type $\mathbb{C}^4 / \langle g, \sigma \rangle$, with $g = \text{diag}(\xi^{2k}_n, \xi^{-1}_n, \xi^{2k}_n, \xi^{2k}_n)$ and $\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$.

Then

$$s_x = -(n + 2).$$

Proof. Let $G = \langle g, \sigma \rangle$. The elements of $G \setminus \{\text{id}\}$ can be divided in two sets $A := \{g, g^2, ..., g^{2n-1}\}$ and $B := \{\sigma, \sigma \circ g, ..., \sigma \circ g^{2n-1}\}$. Therefore:

$$s_x := \frac{1}{4n} \left[ \sum_{h \in A} \frac{\text{tr}(\rho_{x,T}(h)) - 4}{\text{det}(\rho_{x,T}(h) - \text{id})} + \sum_{h \in B} \frac{\text{tr}(\rho_{x,T}(h)) - 4}{\text{det}(\rho_{x,T}(h) - \text{id})} + 4n - 1 \right].$$

By [13, (15)]:

$$\sum_{h \in A} \frac{\text{tr}(\rho_{x,T}(h)) - 4}{\text{det}(\rho_{x,T}(h) - \text{id})} = -\frac{4n^2 - 1}{6}.$$

Moreover:

$$\sigma \circ g^k = \begin{pmatrix} 0 & 0 & \xi^{-k}_n & 0 \\ 0 & 0 & 0 & \xi^{2k}_n \\ -\xi^{2k}_n & 0 & 0 & 0 \\ 0 & -\xi^{-k}_n & 0 & 0 \end{pmatrix}.$$

Hence:

$$\text{tr}(\sigma \circ g^k) = 0$$

and

$$\text{det}(\sigma \circ g^k - \text{id}) = 4.$$

We obtain:

$$\sum_{h \in B} \frac{\text{tr}(\rho_{x,T}(h)) - 4}{\text{det}(\rho_{x,T}(h) - \text{id})} = -2n.$$

Therefore:

$$s_x = \frac{1}{4n} \left[ -4n^2 + 1 - 12n + 4n - 1 \right] = -(n + 2).$$

We can also compute the Chern numbers $c_4$ and $c_2^2$ via the Blache–Rieamann–Roch theorem ([9]).

Proposition 2.26 ([9, Theorem 2.14]). Let $X$ be a 4-dimensional compact complex orbifold with only isolated singularities. Then:

$$c_4(X) = \chi(X) - \sum_{x \in \text{Sing} X} \left( 1 - \frac{1}{|G_x|} \right),$$

where $G_x$ is the regional fundamental group of $X$ at $x$. 

11
Proposition 2.27 ([13] (2)). Let $X$ be a primitively symplectic orbifold of dimension 4. Then:

$$c_2(X)^2 = 720 - 240S_0(X) + \frac{c_4(X)}{3}.$$

We provide $S_{0,x}$ for the singularities that will be encountered in this paper.

Lemma 2.28. We consider the isolated singularity $x$ of analytic type $\mathbb{C}^4/G$. We set

$$\sigma := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$ 

(i) When $G = \langle - \text{id} \rangle$, then $S_{0,x} = \frac{1}{27}$.

(ii) When $G = \langle \text{diag}(\xi_3, \xi_3^{-1}, \xi_3^{-1}, \xi_3) \rangle$, then $S_{0,x} = \frac{2}{27}$.

(iii) When $G = \langle \text{diag}(i, -i, -i, i) \rangle$, then $S_{0,x} = \frac{9}{27}$.

(iv) When $G = \langle \text{diag}(\xi_6, \xi_6^{-1}, \xi_6^{-1}, \xi_6) \rangle$, then $S_{0,x} = \frac{329}{864}$.

(v) When $G = \langle \text{diag}(\xi_8, \xi_8^{-1}, \xi_8^3, \xi_8^3) \rangle$, then $S_{0,x} = \frac{41}{27}$.

(vi) When $G = \langle \sigma, \text{diag}(i, -i, -i, i) \rangle$, then $S_{0,x} = \frac{25}{27}$.

(vii) When $G = \langle \sigma, \text{diag}(\xi_6, \xi_6^{-1}, \xi_6^{-1}, \xi_6) \rangle$, then $S_{0,x} = \frac{545}{1728}$.

Proof. (i) We have:

$$S_{0,x} = \frac{1}{2} \times \frac{1}{24} = \frac{1}{25}.$$

(ii) We have:

$$S_{0,x} = \frac{2}{3} \times \frac{1}{(1 - \xi_3)^2 (1 - \xi_3^{-1})^2} = \frac{2}{3} \frac{1}{|1 - \xi_3|^4} = \frac{2}{3} \times \frac{1}{9} = \frac{2}{27}.$$

(iii)

$$S_{0,x} = \frac{1}{4} \left( \frac{1}{24} + \frac{2}{(1 - i)^2 (1 + i)^2} \right) = \frac{1}{4} \left( \frac{1}{24} + \frac{2}{2^2} \right) = \frac{9}{26}.$$ 

(iv) We have:

$$S_{0,x} = \frac{1}{6} \left( \frac{1}{24} + \frac{2}{(1 - \xi_6)^2 (1 - \xi_6^{-1})^2} + \frac{2}{(1 - \xi_3)^2 (1 - \xi_3^{-1})^2} \right) = \frac{1}{6} \left( \frac{1}{24} + \frac{2}{9} \right) = \frac{329}{864}.$$

(v)

$$S_{0,x} = \frac{1}{8} \left( \frac{1}{24} + \frac{2}{(1 - i)^2 (1 + i)^2} + \frac{4}{|1 - \xi_8|^2 |1 - \xi_8^3|^2} \right).$$

However $|1 - \xi_8|^2 = 2 - \sqrt{2}$ and $|1 - \xi_8^3|^2 = 2 + \sqrt{2}$. Hence:

$$S_{0,x} = \frac{1}{8} \left( \frac{1}{16} + \frac{1}{2} + \frac{2}{9} \right) = \frac{41}{27}.$$

(vi) Let $g = \text{diag}(i, -i, -i, i)$ and $G = \langle g, \sigma \rangle$. The elements of $G \setminus \{\text{id}\}$ can be shared in two sets $A := \{g, g^2, g^{-1}\}$ and $B := \{\sigma, \sigma \circ g, \sigma \circ g^2, \sigma \circ g^{-1}\}$. For all $h \in B$, we have $\det(h - \text{id}) = 4$. Hence:

$$S_{0,x} = \frac{1}{8} \left( \frac{4}{4} + \frac{1}{24} + \frac{2}{(1 - i)^2 (1 + i)^2} \right) = \frac{1}{8} \left( 1 + \frac{1}{16} + \frac{1}{2} \right) = \frac{25}{27}.$$ 

12
(vi) Let $g = \text{diag}(\xi_6, \xi_6^{-1}, \xi_6^{-1}, \xi_6)$ and $G = \langle g, \sigma \rangle$. As before, the elements of $G \setminus \{\text{id}\}$ can be shared in two sets $A := \{g, g^2, \ldots, g^4\}$ and $B := \{\sigma, \sigma \circ g, \sigma \circ g^2, \ldots, \sigma \circ g^4\}$. For all $h \in B$, we have $\det(h - \text{id}) = 4$. Hence:

$$S_{0,z} = \frac{1}{12} \left( \frac{6}{4} + \frac{1}{2} + \frac{2}{(1 - \xi_6)^2(1 - \xi_6^{-1})^2} + \frac{2}{(1 - \xi_3)^2(1 - \xi_3^{-1})^2} \right)$$

$$= \frac{1}{12} \left( \frac{6}{4} + \frac{1}{16} + \frac{2}{9} \right) = \frac{545}{1728}.$$

\[ \square \]

3 Quotients of K3 surfaces to the power $n$

3.1 Criterion to be an irreducible symplectic variety

Let $S$ be a K3 surface and $n \geq 2$ and integer. In this section we want to exhibit the finite automorphism groups $\mathcal{G}$ on $S^n$ such that the quotient $S^n/\mathcal{G}$ leads to an irreducible symplectic variety. To simplify the readability, we recall Definition 1.6 and notation (1) from the introduction.

**Notation 3.1.** Let $S$ be a K3 surface. Let $G$ be a finite symplectic automorphism group on $S$. Let $\theta : G \rightarrow G$ be an involution. We denote by $j_\theta : G \rightarrow \text{Aut}(S^n)$ the embedding given by:

$$j_\theta(g)(x_1, x_2, x_3, x_4, \ldots, x_n) = (g(x_1), \theta(g)(x_2), x_3, x_4, \ldots, x_n),$$

that is $j_\theta(g)$ has the diagonal action given by $g$ on the first factor, by $\theta(g)$ on the second factor and by id on the other factors.

**Definition 3.2.** Let $S$ be a K3 surface and $G$ a finite symplectic automorphism group on $S$. Let $\theta : G \rightarrow G$ be an involution. We say that $\theta$ is a valid involution if there exists $(g_1, \ldots, g_k)$ a family of generators of $G$ such that $\theta(g_i) = g_i^{-1}$ for all $i \in \{1, \ldots, k\}$.

**Definition 3.3.** Let $n \geq 2$ be an integer and $S$ a K3 surface. Let $\mathcal{G}$ be a finite subgroup of $\text{Aut}(S^n)$. We say that $\mathcal{G}$ is primitive if we cannot find $\mathcal{H} \subset \mathcal{G}$ a non-trivial normal subgroup and a K3 surface $\Sigma$ such that $\Sigma^n$ is a resolution of $S^n/\mathcal{H}$.

**Definition 3.4.** Let $S$ be a K3 surface and $n \geq 2$ an integer. We call a trivial complex reflexion an element of $\text{Aut}(S)^n$ of the form $(\text{id}, \ldots, \text{id}, g, \text{id}, \ldots, \text{id})$; that is an element which acts trivial on all factors of $S^n$ apart one.

**Lemma 3.5.** Let $S$ be a K3 surface endowed with a symplectic automorphism group $G$. Let $n \geq 3$ be an integer. Let $\theta : G \rightarrow G$ be an involution. The group $(j_\theta(G), \Sigma_n)$ is primitive if and only if it does not contain any trivial complex reflexion different from id.

**Proof.** We set $\mathcal{G} = (j_\theta(G), \Sigma_n)$. We denote by $\mathcal{H}$ the subgroup of $\mathcal{G}$ generated by trivial complex reflexions.

Since $\mathcal{G} \subset \mathcal{G}$ necessarily we have $\mathcal{H} = H^n$ with $H$ a symplectic automorphism group on $S$. Let $\Sigma \rightarrow S/H$ be the K3 surface obtained by crepant resolution. Therefore $\Sigma^n \rightarrow S^n/H^n$ is a resolution. So if $\mathcal{H}$ is not trivial $\mathcal{G}$ is not primitive.

Now we assume that $\mathcal{G}$ is not primitive and we show that $\mathcal{H}$ is not trivial. If $\mathcal{G}$ is not primitive, there exists a non-trivial subgroup $\mathcal{H}' \subset \mathcal{G}$ and a K3 surface $\mathcal{S}$ such that $\Sigma^n \rightarrow S^n/\mathcal{H}'$ is a crepant resolution. We have $h^0(\Omega^2_{S^n}) = h^0(\Omega^2_{\mathcal{S}^n})$. Hence necessarily $\mathcal{H}'$ fixes all the holomorphic 2-form on $S^n$. It implies that $\mathcal{H}' \subset \mathcal{G} \cap \text{Aut}(S^n)$ because an element in $\Sigma_n$ does not fix all the group $H^n(\Omega^2_{\mathcal{S}^n})$. Moreover, by Proposition 2.13, we know that $\mathcal{H}'$ is generated by elements with fixed locus in codimension 2. In Aut$(S^n)$ only the trivial complex reflexions can have a fixed locus in codimension 2. That is $\mathcal{H}'$ is generated by trivial complex reflexions and $\mathcal{H}' \subset \mathcal{H}$.

**Lemma 3.6.** Let $S$ be a K3 surface endowed with a symplectic automorphism group $G$. Let $n \geq 3$ be an integer. Let $\theta : G \rightarrow G$ be an involution. The group $(j_\theta(G), \Sigma_n)$ is primitive if and only if $\theta$ is valid and $G$ is abelian.
Moreover if \( G \) is not abelian or \( \theta \) is not valid, there exists a K3 surface \( \Sigma \), an abelian automorphism group \( G' \) on \( \Sigma \) and a valid involution \( \theta' \) on \( G' \) such that:

\[
S^n/\langle j\omega(G), \mathfrak{G}_n \rangle \simeq \Sigma^n/\langle j\omega(G'), \mathfrak{G}_n \rangle.
\]  

(4)

Proof. Using Lemma 4.1 it is enough to show that \( \mathfrak{G} \) does not contain a trivial complex reflexion different to \( \text{id} \) if and only if \( \theta \) is valid and \( G \) is abelian.

First, we assume that \( G \) is not abelian, then there exists \( g \in G \) and \( h \in G \) such that \( g \circ h \circ g^{-1} \circ h^{-1} \neq \text{id} \). Since \( \mathfrak{G}_n \subseteq \mathfrak{G} \), all the elements of the form \( (\text{id}, \ldots, b, \text{id}, \ldots, \text{id}, b, \ldots) \) are in \( \mathfrak{G} \) with \( b \) in position \( i \) and \( \theta(b) \) in position \( j \), for all \( i \) and \( j \neq i \) in \( \{1, \ldots, n\} \) and \( b \in G \). Therefore, we have:

- \((g, \theta(g), \text{id}, \ldots, \text{id}) \in \mathfrak{G} \);  
- \((h, \text{id}, \theta(h), \ldots, \text{id}) \in \mathfrak{G} \);  
- \((g^{-1}, \theta(g^{-1}), \text{id}, \ldots, \text{id}) \in \mathfrak{G} \);  
- \((h^{-1}, \text{id}, \theta(h^{-1}), \ldots, \text{id}) \in \mathfrak{G} \).

The product of these elements provides:

\[
(g \circ h \circ g^{-1} \circ h^{-1}, \text{id}, \ldots, \text{id}) \in \mathfrak{G}.
\]  

(5)

Now, we assume that \( \theta \) is not valid. Therefore, there exists \( g \in G \) such that \( g \circ \theta(g) \neq \text{id} \). We have:

- \((g, \theta(g), \text{id}, \ldots, \text{id}) \in \mathfrak{G} \);  
- \((\theta(g), \text{id}, g, \ldots, \text{id}) \in \mathfrak{G} \);  
- \((\text{id}, \theta(g^{-1}), g^{-1}, \ldots, \text{id}) \in \mathfrak{G} \).

The product of these elements provides: \((g \circ \theta(g), \text{id}, \ldots, \text{id}) \in \mathfrak{G} \).

Now, we prove the reciprocal; we assume that \( G \) is abelian and \( \theta \) is valid. Let \((g, \text{id}, \ldots, \text{id}) \in \mathfrak{G} \) be a trivial complex reflexion, we are going to show that \( g = \text{id} \). By definition of \( \mathfrak{G} \) with \( \theta \) valid and \( G \) abelian, we can write:

\[
(g, \text{id}, \ldots, \text{id}) = \prod_{1 \leq i < j \leq n} g_{i,j},
\]  

(6)

where \( g_{i,j} \equiv (\text{id}, \ldots, g_{i,j}, \text{id}, \ldots, g_{i,j}^{-1}, \text{id}, \ldots, \text{id}) \) with \( g_{i,j} \in G \) in position \( i \) and \( g_{i,j}^{-1} \) in position \( j \). We set \( g_{j,i} \equiv g_{i,j}^{-1} \). Equation (6) implies that:

\[
g = \prod_{j=2}^{n} g_{1,j};
\]  

(7)

and

\[
\text{id} = \prod_{i=2}^{n} \left( \prod_{1 \leq j \leq n, j \neq i} g_{i,j} \right)
\]  

(8)

Since \( g_{j,i} \equiv g_{i,j}^{-1} \), equation (8) implies that:

\[
\prod_{i=2}^{n} g_{1,i} = \text{id}.
\]

Inverting the previous equation, we obtain that \( g = \text{id} \).

Now, we do not make any assumption on \( G \) and \( \theta \) and we show that there exist a K3 surface \( \Sigma \) endowed with an abelian symplectic automorphism group \( G' \) and a valid involution \( \theta' : G' \rightarrow G' \) such that (4) is verified. Let \([G, G]\) be the commutator subgroup of \( G \). Since \( \mathfrak{G}_n \subseteq \mathfrak{G} \) and by (5), we have \([G, G]_n \subseteq \mathfrak{G} \). Let \( Y \rightarrow S/[G, G] \) be the K3 surface obtained after crepant resolution.
Let \( \overline{\theta} \) be the involution on \( \text{Ab}(G) \) obtained from \( \theta \). Since \( [G,G]^n \) is a normal subgroup of \( G \), the group \( \overline{G} := G/[G,G]^n \) induces an automorphism group on \( Y^n \). Then, we have \( \overline{G}/[G,G]^n = \langle \overline{\rho}(\text{Ab}(G), \mathfrak{S}_n) \rangle \), with \( \text{Ab}(G) \) seen as an automorphism group on \( Y \). Similarly, we consider the group \( H = \langle g \circ \overline{\theta}(g) \mid g \in \text{Ab}(G) \rangle \). We consider \( \Sigma \rightarrow Y/H \) the K3 surface obtained from the crepant resolution. The group \( G' := \text{Ab}(G)/H \) induces an automorphism group on \( \Sigma \) and \( \overline{\theta} \) induces a valid involution \( \theta' \) on \( G' \). By construction the relation \( \ref{correspondence} \) is verified.

**Theorem 3.7.** Let \( n \geq 2 \) be an integer and \( S \) a K3 surface. Let \( \mathcal{G} \) be a finite primitive subgroup of \( \text{Aut}(S^n) \). Let \( Y \) be a terminalization of \( S^n/\mathcal{G} \).

The complex space \( Y \) is an irreducible symplectic variety with \( \pi_1(Y_{reg}) = 0 \) if and only if there exists:

- **G** a symplectic finite automorphism group of \( S \) which is abelian if \( n \geq 3 \) and
- \( \theta : G \rightarrow G \) a valid involution such that:

\[
S^n/\mathcal{G} \simeq S^n/(\langle \rho(G), \mathfrak{S}_n \rangle), \tag{9}
\]

with \( \mathfrak{S}_n \) acting by permutation of the factors.

**Proof.** If \( \ref{correspondence} \) is realized, then \( Y \) is an irreducible symplectic variety with \( \pi_1(Y_{reg}) = 0 \) by Proposition \ref{prop:correspondence}. Indeed, let \((g_1, ..., g_m)\) be a family of generators of \( G \) such that \( \theta(g_i) = g_i^{-1} \) for all \( i \in \{1, ..., m\} \). Then, the group \( \langle \rho(G), \mathfrak{S}_n \rangle \) is generated by the transpositions and by \((1,2) \circ \rho(g_i)\) for \( i \in \{1, ..., m\} \) which fix \( \{ (x, g_1(x), y_3, ..., y_n) \mid (x, y_3, ..., y_n) \in S^{n-1} \} \).

We assume that \( Y \) is an irreducible symplectic variety with \( \pi_1(Y_{reg}) = 0 \) and we are going to prove \( \ref{correspondence} \). We denote by \( pr_i : S^n \rightarrow S \) the \( i \)th projection. As explained in \[ \ref{section:ter} \] Section 3 via the uniqueness of the Bogomolov decomposition theorem, we know that \( \text{Aut}(S^n) \) is given by the natural semi-direct product between \( \text{Aut}(S)^n \) and \( \mathfrak{S}_n \). We denote by \( \text{Aut}_0(S) \) the group of symplectic automorphisms on \( S \).

- **First claim:** \( \mathcal{G} \) is a subgroup of \( \text{Aut}_0(S)^n \times \mathfrak{S}_n \)

We denote \( H^0(\Omega^2_S) = \mathbb{C} \sigma \). To have \( Y \) primitively symplectic, we need that:

\[
H^0(S^n, \Omega^2_{S^n})^\mathcal{G} = \sum_{i=1}^n \lambda_i \text{pr}^*_i(\sigma), \tag{10}
\]

with \( \lambda_i \neq 0 \) for all \( i \). Let \( g \in \text{Aut}(S)^n \cap \mathcal{G} \). We can write \( g = (g_1, ..., g_n) \) with \( g_i \in \text{Aut}(S) \) for all \( i \). We have:

\[
g^*(\sum_{i=1}^n \lambda_i \text{pr}^*_i(\sigma)) = \sum_{i=1}^n \mu_i \lambda_i \text{pr}^*_i(\sigma),
\]

with \( g^*(\sigma) = \mu_i \sigma \). It follows from \[ \ref{primitivity} \] that \( \mu_i = 1 \) for all \( i \). That is \( \mathcal{G} \) is a sub-group of \( \text{Aut}_0(S)^n \times \mathfrak{S}_n \).

- **Second claim:** \( \mathcal{G}/\text{Aut}(S)^n \cap \mathcal{G} \) acts transitively on the \( n \) factors of the product \( S^n \)

We have seen that necessarily, \( \text{Aut}(S)^n \cap \mathcal{G} \) acts trivially on \( H^0(S^n, \Omega^2_{S^n}) \). Hence to respect \[ \ref{primitivity} \], the subgroup \( \mathcal{G}/\text{Aut}(S)^n \cap \mathcal{G} \) has to act transitively on the factors of \( S^n \).

- **Third claim:** \( \mathcal{G} \) does not contain any trivial complex reflexion.

Let \( \mathcal{H} \subset \mathcal{G} \) be the normal subgroup generated by the trivial complex reflexions. By definition of trivial complex reflexion, we have \( \mathcal{H} = \prod_{i=1}^n H_i \), with \( H_i \) finite subgroups of \( \text{Aut}_0(S) \). Then \( S^n/\mathcal{H} = \prod_{i=1}^n S/H_i \). We are going to show that \( S/H_i \simeq S/H_i \) for all \( i \in \{2, ..., n\} \).

By claim 2, there exists \( s_{1,i} \in \mathcal{G} \) that exchanges the first factor of \( S^n \) with the \( i \)th factor. We can write \( s_{1,i} = P \circ (g_1, ..., g_n) \) with \( P \in \mathfrak{S}_n \) such that \( P(i) = 1 \) and \( (g_1, ..., g_n) \in \text{Aut}_0(S)^n \). Let \( h \in H_1 \), we have:

\[
s_{1,i}^{-1} \circ (h, id, ..., id) \circ s_{1,i} = (id, ..., id, g_i^{-1} \circ h \circ g_i, id, ..., id),
\]
Fourth claim: \( \mathcal{G}/(\mathcal{G} \cap \text{Aut}(S)^n) = \mathfrak{S}_n \)

Let \( g \in \text{Aut}(S)^n \cap \mathcal{G} \). Since \( g \) is not a trivial complex reflexion, we can write \( g = (g_1, ..., g_n) \) with at least two non-trivial factors. For simplicity in the notation, we can assume without loss of generality that the factors are \( g_1 \) and \( g_2 \). Therefore \( \text{Fix} \, g \subset \text{Fix} \, g_1 \times \text{Fix} \, g_2 \times S^{n-2} \).

Fifth claim: there exist an automorphism group \( \mathcal{G}' \) such that \( S^n/\mathcal{G} \cong S^n/\mathcal{G}' \) and \( \mathcal{G}' \supset \mathfrak{S}_n \)

We have seen that \( \mathcal{G} \) is an extension \( (\mathcal{G} \cap \text{Aut}(S)^n) : \mathfrak{S}_n \). We are going to prove that we can find \( \mathcal{G} \cong \mathcal{G}' \supset \mathfrak{S}_n \).

Let \( (s_i \circ g_{i,i}) \) be a family of elements of \( \mathcal{G} \) with \( s_i = (j_i, k_i) \) a transposition and \( g_{i,i} \in \mathcal{G} \cap \text{Aut}(S)^n \) with an automorphism \( g_i \in \text{Aut}(S) \) in position \( j_i \), the automorphism \( g_{i, i}^{-1} \) in position \( k_i \) and \( \text{id} \) in all other factors. We have seen in the previous claim that there exists a family \( (s_i \circ g_{i,i}) \) as described before such that \( (s_i) \) generates \( \mathfrak{S}_n \). We choose such a family \( (s_i \circ g_{i,i}) \) minimal; i.e. we cannot find an elements of \( (s_i) \) which is generated by the other elements of the family. We are going to reorder the family \( (s_i \circ g_{i,i}) \) in a convenient way. We keep \( s_1 \circ g_{1,1} \). Then, we define the order recursively as follows. Assume that we have \( (s_i \circ g_{i,i}) \) for \( i \leq m-1 \), we explain how to choose \( s_m \circ g_{m,m} \).

Note that since \( (s_m)_{1 \leq m \leq t} \) generates \( \mathfrak{S}_n \) necessarily we have \( t = n-1 \) and \( \{j_1, k_1, k_2, k_3, ..., k_{n-1}\} = \{1, ..., n\} \). Now, we are going to construct a morphism \( \psi : S^n \to S^n \) such that:

\[
(\psi_{m})_{1 \leq m \leq t} = \mathfrak{S}_n \quad \text{and} \quad j_m \in R_{m-1}, \quad k_m \notin R_{m-1}, \quad \forall \ 2 \leq m \leq t.
\]

We write \( \psi \) diagonally:

\[
\psi = (\psi_1, ..., \psi_n).
\]

We are going to define \( \psi_{k_m} \) recursively. We set \( \psi_{k_1} = g_{k_1}^{-1} \) and \( \psi_{j_1} = \text{id} \). Then, we assume that \( \psi_{k_i} \) has been defined for \( 0 \leq i \leq m - 1 \) and we provide \( \psi_{k_{m}} \). We set \( \psi_{k_{m}} = \psi_{j_{m}} \circ g_{m}^{-1} \).
It remains to prove \( \text{(12)} \). We compute \( \psi \circ s_m \circ g_m \circ \psi^{-1} \): We have:

\[
\psi \circ s_m \circ g_m \circ \psi^{-1} (x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, \psi_{j-1} \circ g_{j-1} \circ \psi_{j-1}^{-1} (x_{k_{j-1}}), x_{j+1}, \ldots, x_{k_{j-1}}-1, \psi_{k_{j-1}-1} \circ g_{k_{j-1}-1} \circ \psi_{k_{j-1}-1}^{-1} (x_{j_{k_{j-1}-1}}), \ldots, x_n).
\]

However, by construction \( \psi_{j-1} \circ g_{j-1} \circ \psi_{j-1}^{-1} = \text{id} \). We set \( G' = \psi \circ G \circ \psi^{-1} \) and we obtain our claim.

- **Sixth claim:** \( G' \cap \text{Aut}(S)^n \) is generated by elementary elements

For simplicity in the notation, an element of the form \( (\text{id}, \ldots, \text{id}, g, \text{id}, \ldots, \text{id}, g^{-1}, \text{id}, \ldots, \text{id}) \) will be called an **elementary element** for the sequel of the proof.

Let \( h \in G' \cap \text{Aut}(S)^n \). As we have seen in the proof of the fourth claim, the group \( G' \) is generated by elements of the form \( s \circ (\text{id}, \ldots, \text{id}, g, \text{id}, \ldots, \text{id}, g^{-1}, \text{id}, \ldots, \text{id}) \), where \( s \) is the transposition \( (i, j) \); \( g \) and \( g^{-1} \) are in position \( i \) and \( j \) respectively. In particular, we can write:

\[
h = \prod s_i \circ (\text{id}, \ldots, \text{id}, g_i, \text{id}, \ldots, \text{id}, g_i^{-1}, \text{id}, \ldots, \text{id}),
\]

with \( s_i \) transpositions. That is \( h = \sigma \circ \alpha \) with \( \sigma \) a permutation and \( \alpha \) the product of elementary elements conjugated by permutations. However, the conjugate by a permutation of an elementary element is an elementary element. Therefore \( h = \sigma \circ \alpha \) with \( \sigma \) a permutation and \( \alpha \) a product of elementary elements. Finally since \( \alpha \in G' \cap \text{Aut}(S)^n \), we have \( \sigma = 0 \) and this prove our claim.

- **Seventh claim:** proof of \( \text{(14)} \)

It remains to show that \( G' \) has the form prescribed by the proposition. Let:

\[
F = \{ g \in \text{Aut}(S) | (g, g^{-1}, \text{id}, \ldots, \text{id}) \in G' \cap \text{Aut}(S)^n \}.
\]

We consider \( G = \langle F \rangle \). Moreover we consider \( \theta : G \to G \) the involution such that \( \theta(g) = g^{-1} \) for all \( g \in F \). We only need to prove that \( \theta \) is a well defined involution to conclude the proof; indeed, with claim 5 and 6, we will have \( G' = \langle j_\theta(G), \sigma_n \rangle \). To prove that \( \theta \) is well defined, it is enough to show that if there exists a dependency relation of elements of \( F \), then it does not lead to any contradiction in the definition of \( \theta \). Let \( h_1 \circ \ldots \circ h_{k-1} = \text{id} \)

with \( h_i \in F \) for all \( i \). This imply:

\[
h_{k-1}^{-1} \circ \ldots \circ h_{1}^{-1} = \text{id}.
\]

We need to show that:

\[
h_{1}^{-1} \circ \ldots \circ h_{k}^{-1} = \text{id}.
\]

We have \( (h_i, h_i^{-1}, \text{id}, \ldots, \text{id}) \in G' \) for all \( i \). Hence:

\[
(h_1 \circ \ldots \circ h_k, h_1^{-1} \circ \ldots \circ h_k^{-1}, \text{id}, \ldots, \text{id}) \in G'.
\]

Therefore multiplying the first factor by \( \text{(13)} \), we obtain:

\[
(\text{id}, h_1^{-1} \circ \ldots \circ h_k^{-1}, \text{id}, \ldots, \text{id}) \in G'.
\]

Hence, by the third claim: \( h_1^{-1} \circ \ldots \circ h_k^{-1} = \text{id} \). This corresponds to \( \text{(14)} \).

- **Last claim:** \( G \) is abelian when \( n \geq 3 \)

Since the group \( G \) is primitive, the group \( G' = \langle j_\theta(G), \sigma_n \rangle \) is also primitive. Therefore by Lemma \( 5.6 \) the group \( G \) is abelian.
Remark 3.8. Let $S$ be a K3 surface, $G$ a finite symplectic automorphism group on $S$ and $θ : G → G$ an involution. Let $G = ⟨jθ(G), S_n⟩$. When $G$ is primitive, we have seen in the previous proof that an element in $⟨jθ(G), S_n⟩$ which has a fixed locus in codimension 2 is of the form:

$$g_{i,j} := s \circ (id,...,id,g,id,...,id,g^{-1},id,...,id),$$

with $s = (i,j)$ the transposition, where $g$ and $g^{-1}$ are in position $i$ and $j$ respectively. Moreover the fixed component in codimension 2 of this automorphism is:

$$\left\{ (y_1,...,y_{i-1},x,y_{i+1},...,y_{j-1},g(x),y_{j+1},...,y_n) \mid (x,y_1,...,y_n) \in S^{n-1} \right\}.$$  

Theorem 1.9 is a direct consequence of Theorem 3.7 and Claim 3 of its proof.

Proof of Theorem 1.9. Let $S$ be a K3 surface. Let $G$ be a finite automorphism group on $S^n$. Let $Y \rightarrow S^n/G$ be a terminalization which is an irreducible symplectic variety with simply connected smooth locus. Let $H \subset G$ be the normal subgroup generated by the trivial complex reflexions. We have seen in Claim 3 of the proof of Theorem 3.7 that there exists a K3 surface $Σ$ such that $Σ^n → S^n/H$ is a crepant resolution. Since $H$ is normal $G/H$ induces an automorphism group on $Σ^n$. Then applying Theorem 3.7 to the couple $(Σ^n, G/H)$ provides Theorem 1.9.

3.2 Fujiki irreducible symplectic varieties

After Theorem 3.7, the definition of Fujiki varieties is very natural; we recall Definition 1.2.

Definition 3.9. Let $S$ be a projective K3 surface and $G$ a finite symplectic automorphism group on $S$. Let $θ : G → G$ be an involution. Let $n ∈ \mathbb{N} \setminus \{0,1\}$. We denote by

$$S(G)^{[n]} → S^n/⟨jθ(G), S_n⟩,$$

a terminalization of $S^n/⟨jθ(G), S_n⟩$ and we call it the Fujiki variety of dimension $2n$ associated to $(S,G,θ)$.

Remark 3.10. According to Lemma 3.6 when $n ≥ 3$, we can assume without loss of generalities that $G$ is abelian and $θ$ is valid.

Notation 3.11. When $G$ is abelian, there is only one valid involution which is $inv(g) = g^{-1}$. To simplify the notation, when $G$ is abelian, we denote $S(G)^{[n]}$ instead of $S(G)^{[n]}_{inv}$.

Proposition 3.12. A Fujiki variety is a primitively symplectic variety.

Proof. Let $Y = S(G)^{[n]}_θ$ be a Fujiki variety. We only have to verify that $h^1(Y, O_Y) = 0$. Since $S^n$ is simply connected, according to [13] Lemma 1.2, this is also the case for $S^n/⟨jθ(G), S_n⟩$. Since $Y$ and $S^n/⟨jθ(G), S_n⟩$ have rational singularities, the result follows.

We are ready to prove Corollary 1.7 and 1.8.

Proof of Corollary 1.7 If $θ$ is valid, $S(G)^{[n]}_θ$ is an irreducible symplectic variety with simply connected smooth locus by Theorem 3.7.

Now, we assume that $S(G)^{[n]}_θ$ is an irreducible symplectic variety with simply connected smooth locus. We set $G = ⟨jθ(G), S_2⟩$. We apply Claim 7 of the proof of Theorem 3.7 to $G$. Therefore, we can find $G'$ an automorphism group on $S$ and $θ'$ a valid involution on $G'$ such that $G = ⟨jθ'(G'), S_2⟩$. Since $G ∩ \text{Aut}(S^n) = jθ(G)$, necessarily, we obtain $G = G'$ and $θ = θ'$.

Proof of Corollary 1.8 By Lemma 3.6 we can assume that $G$ is abelian and $θ$ valid. Then the result follows directly from Theorem 3.7.

18
3.3 Second Betti numbers

Proposition 3.13. Let $S$ be a $K3$ surface, $G$ an finite abelian symplectic automorphism group on $S$ and $\theta : G \to G$ a valid involution. Let $n \geq 3$ be an integer. Then:

$$b_2 \left(S(G)_g^n\right) = \dim_{\mathbb{C}} H^2(S, \mathbb{C})^G + 1.$$  

Proof. Let $G = \langle j G, \Sigma_n \rangle$. We set $\pi : S^n \to S^n/G$. According to Remark 3.3, the sub-variety of $S^n$ of codimension 2 fixed by an element of $G$ are of the form:

$$D_{i,j,g} := \{ (y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_j-1, g(x), y_{j+1}, \ldots, y_n) | (x, y_1, \ldots, y_n) \in S^{n-1} \},$$

with $g \in G$ such that $\theta(g) = g^{-1}$. We are going to prove that $G$ acts transitively on this set of varieties; that is all these varieties have the same image by $\pi$. For this purpose, we prove that there exists $f \in G$ such that $f(D_{i,j,g}) = D_{1,2,id}$ for all $i, j$ and $g \in G$ such that $\theta(g) = g^{-1}$. Indeed, we can choose $f = h \circ s$, with $s$ a permutation which exchanges $i$ with 1 and $j$ with 2 and:

$$h = (id, g^{-1}, g, id, \ldots, id).$$

Hence, we obtain that $\text{Sing}(S^n/G)$ contains only one irreducible component of codimension 2 that we denote by $D$. Moreover each variety $D_{i,j,g}$ is globally fixed by only one non trivial element in $G$ which is $g_{i,j}$ (see the notation in Remark 3.3). The automorphism $g_{i,j}$ is an involution; therefore a generic point in $D$ corresponds to a singularities of analytic type $\mathbb{C}^{2n-2} \times (\mathbb{C}^2 / \pm \text{id})$. Since the singularities in codimension 4 are terminal (see Proposition 2.9), necessarily a terminalization of $S^n/G$ will have only one exceptional divisor. To conclude the proof, we only have to remark that $\dim_{\mathbb{C}} H^2(S^n/G, \mathbb{C}) = \dim_{\mathbb{C}} H^2(S^n, \mathbb{C})^G$ and by the Künneth formula: $\dim_{\mathbb{C}} H^2(S^n, \mathbb{C})^G = \dim_{\mathbb{C}} H^2(S, \mathbb{C})^G$.

When $n \geq 3$, according to Lemma 3.6, there are relatively few possibilities for a Fujiki variety. According to 3.1 and excluding $K3^{[n]}$, there are 14 series. We provide their second Betti number. The group names are given in Section 1.5.

Corollary 3.14. We have the following second Betti numbers:

$$b_2 \left(S(C_2)^{[n]}\right) = 15; \quad b_2 \left(S(C_3)^{[n]}\right) = 11; \quad b_2 \left(S(C_4)^{[n]}\right) = 9;$$

$$b_2 \left(S(C_4)^{[n]}\right) = 7; \quad b_2 \left(S(C_6)^{[n]}\right) = 7; \quad b_2 \left(S(C_7)^{[n]}\right) = 5; \quad b_2 \left(S(C_8)^{[n]}\right) = 9;$$

$$b_2 \left(S(C_2 \times C_4)^{[n]}\right) = 7; \quad b_2 \left(S(C_6)^{[n]}\right) = 5; \quad b_2 \left(S(C_8)^{[n]}\right) = 7; \quad b_2 \left(S(C_2 \times C_6)^{[n]}\right) = 5;$$

$$b_2 \left(S(C_2)^{[n]}\right) = 8; \quad b_2 \left(S(C_4)^{[n]}\right) = 5.$$

When $n = 2$, the situation is much richer.

Proposition 3.15. Let $S$ be a $K3$ surface and $G$ a finite symplectic automorphism group on $S$. Let $\theta : G \to G$ be an involution (not necessarily valid). Let $F := \{ g \in G | \theta(g) = g^{-1} \}$. The group $G$ acts on $F$ via the action $g \cdot h = \theta(g) \circ h \circ g^{-1}$. We denote by $F/G$ the orbits of this action. Then:

$$b_2 \left(S(G)^{[2]}_g\right) = \dim_{\mathbb{C}} H^2(S, \mathbb{C})^G + \# (F/G).$$

Proof. The proof is similar to the one of Proposition 3.13. As before, we have:

$$b_2 \left(S(G)^{[2]}_g\right) = \dim_{\mathbb{C}} H^2(S, \mathbb{C})^G + R,$$

with $R$ the number of irreducible components of $\text{Sing}S^2/G$ in codimension 2. We are going to verify that $R = \# (F/G)$. As we have seen in Remark 1.5, a fixed surface by an element of $G$ is given by:

$$D_g := \{ (x, g(x)) | x \in S \},$$
for some \( g \in F \). Hence the set of fixed surfaces can be identified with \( F \). Let \( h \in G \), then \( h(x, g(x)) = (h(x), \theta(h) \circ g(x)) \). If we set \( y = h(x) \), we see that:

\[
h(D_g) = D_{\theta(h) \circ g h^{-1}}.
\]

Let \( s_0 \) be the involution which permutes the two factors of \( S^2 \). We have \( s_0(D_g) = D_{g^{-1}} = g(D_g) \).

Hence the orbits under the action of \( G \) on \( F \) correspond to the orbits under the action of \( G \) as stated in the statement of the proposition. Let \( \pi : S^2 \to S^2/G \) be the quotient map; since two fixed surfaces have the same image by \( \pi \) if and only if there are in a same orbit under the action of \( G \), we obtain that \( R = \#(F/G) \).

\[ \square \]

### 3.4 Fujiki relation

Let \( S \) be a K3 surface and \( G \) a finite symplectic automorphism group on \( S \). Let \( \theta : G \to G \) be an involution. Let \( n \geq 2 \) be an integer. Let \( \mathcal{G} = \langle j_\theta(G), \mathfrak{S}_n \rangle \). We assume that \( G \) is abelian and \( \theta \) valid when \( n \geq 3 \). Let \( \pi : S^n \to S^n/G \) be the quotient map and \( r : (\mathcal{G}G)^{[n]} \to S^n/G \) be a terminalization. We consider the following injection:

\[
\epsilon : H^2(S, \mathbb{Z})^G \to H^2(S^n/G, \mathbb{Z}), \alpha \mapsto \pi_*(\alpha \otimes 1 \otimes ... \otimes 1).
\]

**Lemma 3.16.** We have:

\[ |\mathcal{G}| = |G|^{n-1} n!. \]

**Proof.** First, note that \( \mathcal{G} \) does not contain any trivial complex reflexion different from \( \text{id} \). It is immediate when \( n = 2 \) and a consequence of Lemmas 5.5 and 5.6 when \( n \geq 3 \). We have \( \mathcal{G} = (G^n \cap \mathcal{G}) \times \mathcal{G}_n \). Let \( g = (g_1, ..., g_n) \in G^n \cap \mathcal{G} \), its first \( n-1 \) factors \( g_i \) can be chosen freely in \( G \). However, the last factor \( g_n \) is determined by the \( n-1 \) first factors because \( \mathcal{G} \) does not contain any trivial complex reflexion different from \( \text{id} \). Therefore \( \#(G^n \cap \mathcal{G}) = \#(G^{n-1}) \). The result follows.

We recall the following well known result.

**Proposition 3.17 ([19] Section 3.4).** Let \( X \) be a primitive symplectic variety of dimension \( 2n \). There exists a non-degenerate quadratic form \( q_X \) on \( H^2(X, \mathbb{Z}) \) and a positive rational number \( C_X \) such that:

\[ \alpha^{2n} = C_X q_X(\alpha)^n, \]

for all \( \alpha \in H^2(X, \mathbb{Z}) \).

**Remark 3.18.** The form \( q_X \) is called the Beauville–Bogomolov form and \( C_X \) is called the Fujiki constant.

In the case of Fujiki variety, we obtain the following result.

**Proposition 3.19.** Let \( q_{S(G)^{[n]}}^{[n]} \) and \( C_{S(G)^{[n]}}^{[n]} \) be respectively the Beauville–Bogomolov quadratic form and the Fujiki constant of \( S(G)^{[n]} \). Then:

\[ C_{S(G)^{[n]}}^{[n]}(\pi^*(\epsilon(\alpha)))^n = \frac{(2n)! |G|^n}{n!2^n} \left( |G|^{2n-3} (n-1)!^2 (\alpha^2) \right)^n, \]

for all \( \alpha \in H^2(S, \mathbb{Z}) \).

**Proof.** Let \( \alpha \in H^2(S, \mathbb{Z})^G \). We denote \( \tilde{\alpha} = \sum_{i=1}^{n} \pi^*_i(\alpha) \). By [20] Lemma 3.6] and Lemma 3.16:

\[ \pi_*(\tilde{\alpha})^{2n} = \left( |G|^{n-1} n! \right)^{2n-1} \pi_* (\alpha^{2n}). \]

We have \( \pi_*(\sum_{i=1}^{n} \pi^*_i(\alpha)) = \sum_{i=1}^{n} \pi_*(\pi^*_i(\alpha)) = n \epsilon(\alpha) \); therefore:

\[ n^{2n} \epsilon(\alpha)^{2n} = \left( |G|^{n-1} n! \right)^{2n-1} \frac{(2n)!}{2^n} (\epsilon(\alpha))^{2n}. \]

\[ 20 \]
We obtain:

\[ \varepsilon(\alpha)^{2n} = \frac{(2n)!}{n^2} \left( |G|^{n-1} (n-1)! \right)^{2n-1} \]

So:

\[ \varepsilon(\alpha)^{2n} = \frac{(2n)! |G|}{n^2} \left( |G|^{2n-3} (n-1)! (\alpha^2) \right)^n. \]

Finally, the Fujiki relation (Proposition 3.17) gives the result.

**Remark 3.20.** If \( G \) is trivial, the morphism \( i : H^2(S, \mathbb{Z}) \to H^2(S[n], \mathbb{Z}) \) introduced in [3] is given by \( \epsilon \frac{\pi^*}{(n-1)!} \). Hence, we obtain the well known Fujiki constant of \( S[n] \). If \( G \) is not trivial, it is not clear if \( \epsilon \frac{\pi^*}{(n-1)!} \) is integral.

Actually, when our variety come from a partial resolution of a quotient, we can always provide a similar result as Proposition 3.19 (it will be used in Section 4.7).

**Proposition 3.21.** Let \( X \) be a primitively symplectic variety of dimension \( 2n \). Let \( H \) be a finite symplectic automorphism group on \( X \). Let \( r : Y \to X/H \) be a proper bimeromorphic morphism such that \( Y \) is a primitively symplectic variety. Let \( \pi : X \to X/H \) be the quotient map. Then:

\[ C_Y q_Y (r^* \pi_*(\alpha))^n = |H|^{2n-1} C_X q_X (\alpha)^n. \]

**Proof.** Let \( \alpha \in H^2(X, \mathbb{Z})^H \). By [26] Lemma 3.6, we have:

\[ \pi_*(\alpha)^{2n} = |H|^{2n-1} \pi_*(\alpha^{2n}). \quad (15) \]

By Proposition 3.17 we have:

\[ r^* \pi_*(\alpha)^{2n} = C_Y q_Y (r^* \pi_*(\alpha))^n \text{ and } \alpha^{2n} = C_X q_X (\alpha)^n. \]

Therefore (15) becomes:

\[ C_Y q_Y (r^* \pi_*(\alpha))^n = |H|^{2n-1} C_X q_X (\alpha)^n. \]

\[ \square \]

Now, we are ready to prove Proposition 1.13.

**Proof of Proposition 1.13.** As we have seen in Lemma 3.19 when \( n \geq 3 \), the Fujiki varieties are given by 14 series corresponding to the 14 possible abelian groups on a K3 surface. It remains to show that these 14 series are independent under deformation.

We recall from Corollary 3.13 that a Fujiki variety in dimension higher or equal to 6 is irreducible symplectic; moreover by construction it has \( \mathbb{Q} \)-factorial and terminal singularities. Therefore by Proposition 2.2 if two such varieties are deformation equivalent they have the same Fujiki constant and the same second Betti number. Using Corollary 3.11 we are going to compare the Fujiki constants of all the varieties with the same second Betti number. If two varieties \( S(H_1)[n] \) and \( S(H_2)[n] \) would have the same Fujiki constant, Proposition 3.19 will provide that:

\[ \frac{|H_1| \left( |H_1|^{2n-3} \right)^n}{q_{S(H_1)[n]}(r^*(\epsilon(\alpha)))^n} = \frac{|H_2| \left( |H_2|^{2n-3} \right)^n}{q_{S(H_2)[n]}(r^*(\epsilon(\alpha)))^n}, \]

for some \( \alpha \in H^2(S, \mathbb{Z}) \) such that \( \alpha^2 \neq 0 \). Hence \( \left( \frac{|H_1|}{|H_2|} \right)^{1/n} \) is a rational number. According to Corollary 3.14 all the fractions \( \left| \frac{H_1}{H_2} \right| \) that we have to consider are:

\[ 3, 4, 1, 2, 5, 6, 8, 9, 2, 3, 8, 7, 7, 7, 2, 3. \]

However, for any \( n \geq 3 \), the \( n \)th roots of all the previous numbers are irrational.

\[ \square \]
3.5 Some bimeromorphisms in dimension 4

Two different involutions \( \theta \) on a given group can lead to two bimeromorphic Fujiki varieties.

**Definition 3.22.** Let \( G \) be a finite symplectic automorphism group on a K3 surface \( S \). Let \( \theta_1 \) and \( \theta_2 \) be two involutions on \( G \). We say that \( \theta_1 \) and \( \theta_2 \) are equivalent if there is an isomorphism

\[
S^2 / \langle \theta_1(G), \mathfrak{S}_2 \rangle \to S^2 / \langle \theta_2(G), \mathfrak{S}_2 \rangle.
\]

**Remark 3.23.** Let \( G \) be a finite symplectic automorphism group on a K3 surface \( S \). According to Proposition 2.10, if \( \theta_1 \) and \( \theta_2 \) are equivalent the varieties \( S(G)^2_\theta \) and \( S(G)^2 \) are equivalent by deformation.

The following results will be essential to prove Theorem 1.11 in Section 5.

**Proposition 3.24.** Let \( G \) be a finite symplectic automorphisms group on a K3 surface \( S \). Let \( \bar{G} \) be another automorphism group on \( S \) (not necessarily symplectic nor finite) such that \( G \subset \bar{G} \). Let \( \theta_1 \) and \( \theta_2 \) be two involutions on \( G \). Let \( \mathcal{G}_i = \langle \theta_i(G), \mathfrak{S}_2 \rangle \), with \( i \in \{1, 2\} \). We assume that there exists \( h_1 \in \bar{G} \) and \( h_2 \in \bar{G} \) such that:

(i) \( h_1 \) and \( h_2 \) acts on \( G \) by conjugation;

(ii) \( \theta_2(h_1 \circ h_2^{-1}) = h_2 \circ h_1^{-1} \);

(iii) \( h_1 \circ h_2^{-1} \in G \);

(iv) \( \theta_2(h_1 \circ g \circ h_1^{-1}) = h_2 \circ \theta_1(g) \circ h_2^{-1} \) for all \( g \in G \).

Then \( \theta_1 \) and \( \theta_2 \) are equivalent.

**Proof.** We consider the following isomorphism:

\[
\Phi : \quad S^2 \xrightarrow{\Phi} S^2 \quad (x, y) \mapsto (h_1(x), h_2(y))
\]

and we show that it leads to an isomorphism between the quotients. Let \( (x, y) \in S^2 \) and \( g \in G \), we need to show that \( \Phi(x, y), \Phi(y, x) \) and \( \Phi(g(x), \theta_1(g)(y)) \) are in the same orbit under the action of \( \mathcal{G}_2 \). We have:

- \( \Phi(x, y) = (h_1(x), h_2(y)) \);
- \( \Phi(y, x) = (h_1(y), h_2(x)) \);
- \( \Phi(g(x), \theta_1(g)(y)) = (h_1 \circ g(x), h_2 \circ \theta_1(g)(y)) \).

Let \( \mathfrak{S}_2 = \langle s_0 \rangle \). We obtain:

\[
h_1 \circ h_2^{-1} \circ s_0 \Phi(y, x) = h_1 \circ h_2^{-1}(h_2(x), h_1(y)) = (h_1(x), \theta_2(h_1 \circ h_2^{-1}) \circ h_1(y)) = (h_1(x), h_2(y)).
\]

Similarly, we need to show that \( (h_1(x), h_2(y)) \) and \( (h_1 \circ g(x), h_2 \circ \theta_1(g)(y)) \) are in the same orbit under the action of \( \mathcal{G}_2 \). If we consider the action of \( h_1 \circ g \circ h_1^{-1} \) on \( (h_1(x), h_2(y)) \), we only need to verify that:

\[
\theta_2(h_1 \circ g \circ h_1^{-1}) \circ h_2 = h_2 \circ \theta_1(g).
\]

That is:

\[
\theta_2(h_1 \circ g \circ h_1^{-1}) = h_2 \circ \theta_1(g) \circ h_2^{-1},
\]

which is exactly our assumption. \( \square \)

With the previous proposition, we can find many equivalent involutions. We provide an example.

**Corollary 3.25.** Let \( G \) be a finite symplectic automorphism group on a K3 surface \( S \). We assume that \( G \) has a trivial center. Then all inner involutions of \( G \) are equivalent.

**Proof.** Let \( \theta_1 \) and \( \theta_2 \) be two inner involutions on \( G \). There exists \( f_1 \) and \( f_2 \) of order 2 such that \( \theta_1(g) = f_1 \circ g \circ f_1 \) and \( \theta_2(g) = f_2 \circ g \circ f_2 \) for all \( g \in G \). We choose \( h_2 = f_2 \circ f_1 \) and \( h_1 = \text{id} \) and we conclude with Proposition 3.24. \( \square \)
4 Computing the singularities in dimension 4

4.1 Overview of the section

Let $S$ be a K3 surface, $G$ a symplectic automorphism group on $S$ and $\theta$ an involution of $G$ (not necessarily valid). Following the notation of Section A.2 we set $G = \langle j_0(G), S_2 \rangle$.

In this section, we propose a method to compute the singularities of $S(G)^1_2$ when the quotient $S/G$ only has singularities of type $A1$, $A2$, $A3$ and $A5$; when $S/G$ has other kind of singularities, a terminalization of $S^2/G$ is not known. The objective is to characterize the singularities of $S(G)^1_2$ uniquely in term of $G$ and $\theta$. The number of singularities will be obtained in term of the cardinals of some subsets of $G$. These cardinals can then be computed via a computer program (see Section A.3).

The section is organized as follows. In Section 4.2, we recall the terminalizations of some singularities of $S^2/G$ that have been provided by Fujiki in [14] Section 7. In Section 4.3, we provide the notion of specific fixed points of an automorphism; this notion will be fundamental to determined the singularities of $S(G)^1_2$. In Sections 4.4 and 4.5, we study respectively the specific fixed points for the action of $G$ on $S$ and for the action of $G$ on $S \times S$. We deduce in Section 4.6 the singularities of $S(G)^1_2$. Since the computation of singularities is quite technical, we propose a method to verify the results in Section 4.7; in particular, this method reveals mistakes of computation in [13] and [14] (a correction will be provided in Section 5).

4.2 Some known local terminalizations of symplectic singularities in dimension 4

In [13] Section 7, Fujiki considers several kinds of quotient singularities that he calls $N_k$ and $M_k$ for $k \in \{2, 3, 4, 6\}$. Fujiki call these singularities admissible singularities. They are defined as follows:

$$N_k = \mathbb{C}^4 / \langle g_k, s_0 \rangle \quad \text{and} \quad M_k = \mathbb{C}^4 / \langle h_k, s_0 \rangle,$$

with

$$g_k = \text{diag}(\xi_k, \xi_k^{-1}, \xi_k^{-1}, \xi_k), \quad h_k = \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}) \quad \text{and} \quad s_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. $$

Remark 4.1. For $k > 2$, note that $\langle h_k, s_0 \rangle$ is commutative, however $\langle g_k, s_0 \rangle$ is not.

Proposition 4.2 ([14], Section 7). There exists terminalizations $\tilde{N}_k$ and $\tilde{M}_k$ of $N_k$ and $M_k$ receptively such that:

- $\tilde{N}_2 = \tilde{M}_2$, $\tilde{N}_3$ and $\tilde{N}_4$ are smooth;
- $\tilde{N}_6$ has one isolated singularities of analytic type $\mathbb{C}^4 / - \text{id}$;
- $\tilde{M}_3$ has 2 isolated singularities of analytic type $\mathbb{C}^4 / g_3$;
- $\tilde{M}_4$ has 4 isolated singularities of analytic type $\mathbb{C}^4 / - \text{id}$;
- $\tilde{M}_6$ has 4 isolated singularities of analytic type $\mathbb{C}^4 / g_3$.

We refer to [14] Section 7 for the concrete construction of these terminalizations.

4.3 Definitions, hypothesis and notations

Definition 4.3. Fix a set $X$ and a group $G$ acting on $X$. Let $x \in X$ and $G_x$ be its stabilizer under the $G$-action. We say that $x$ is a specific fixed point of $g$ (related to the action of $G$ on $X$) if:

$$G_x = \langle g \rangle.$$
Let $G$ be an automorphism group on a K3 surface $S$. We will consider the specific fixed points only in the two following configurations:

- $G$ acts on $S$;
- $G = \langle j_0(G), \mathfrak{A}_2 \rangle$ acts on $S \times S$ (see Notation 4.4 for the definition of $j_0$).

**Notation 4.4.** To avoid too complicated notation, when we consider the action of the group $G$ on $S \times S$, the subgroup $j_0(G)$ will be denoted by $G$ and an element $j_0(g)$ will be denoted by $g$.

We recall Definition 1.10 from the introduction.

**Definition 4.5.** A finite symplectic automorphism group $G$ on a K3 surface $S$ will be said admissible if $S/G$ has only singularities of type $A_1$, $A_2$, $A_3$ or $A_5$.

When $G$ is admissible, we will see that $S^2 / \langle j_0(G), \mathfrak{A}_2 \rangle$ will have only codimension 2 singularities of analytic type $N_k$ or $M_k$ with $k \in \{2, 3, 4, 6\}$. For this reason, in order to determine the terminalization of $S^2 / G$ using Proposition 4.2, we make the following assumption.

**Hypothesis 4.6.** In all this section, we assume that $G$ is an admissible group. In particular, $G$ only contains automorphisms of order 2, 3, 4 or 6.

**Definition 4.7.** A surface in $S \times S$ is call a fixed surface if there exists an element in $G$ which fixes all its points.

**Notation 4.8.** Let $g \in G \setminus \{\text{id}\}$. We set:

- $\text{SpFix}_S g$ the set of specific fixed points of $g$ for the action of $G$ on $S$;
- $n(g) := \# \text{SpFix}_S g$;
- $\text{SpFix}_{S \times S} g$ the set of specific fixed points of $g$ for the action of $G$ on $S \times S$;
- $F := \{ g \in G | \ \theta(g) = g^{-1} \}$;
- $s_0$ the automorphism on $S \times S$ which exchange the two copies of $S$: $s_0(x, y) = (y, x)$;
- $Z := S \times S / G$;
- $\Sigma := \{ (x, s(x)) | \ x \in S, \ s \in G \}$.

- Let $g \in G$ and $x \in \text{SpFix}_S g$. We set $\Sigma_{g,x} := \{ (x, y) \in \Sigma | \ y \in \text{SpFix}_S \theta(g) \}$.

**Definition 4.9.** Let $h_1$ and $h_2$ in $G \setminus \{\text{id}\}$. A point $(x, y) \in S \times S$ will be said of type $(h_1, h_2)$ is $x \in \text{SpFix}_S h_1$ and $y \in \text{SpFix}_S h_2$.

### 4.4 Specific fixed points for the action on $S$

In this section we study the specific fixed points of an automorphism living in $G$ acting on $S$. We recall that $G$ is assumed to be admissible in Hypothesis 4.6.

**Lemma 4.10.** Let $g \in G$. If $g$ has order 4 (resp. has order 6), then $g$ has 4 (resp. 2) specific fixed points.

**Proof.** For instance by [13, Chapter 15 section 1], we know that an automorphism of order 4 on a K3 surface has 4 fixed points and an automorphism of order 6 has 2 fixed points. Moreover if such point was not specific, it would contradict our Hypothesis 4.6 on $G$.

**Lemma 4.11.** Let $g \in G$ be an automorphism of order 3. Let $k_0(g)$ be the number of different cyclic sub-groups of $G$ of order 6 which contain $g$. Then $6 - 2k_0(g)$ is a non-negative integer which corresponds to the number of specific fixed point of $g$. 

24
Proof. Let \((g_i)_{i \in \{1, \ldots, k_6(g)\}}\) be the cyclic groups of order 6 containing \(g\). By Lemma 4.10, we know that each automorphism \(g_i\) has 2 specific fixed points. Since \(g^2_i = g\), we have:

\[
\text{Fix } g \supset \bigcup_{i=1}^{k_6(g)} \text{Fix } g_i,
\]

the union being disjoint because of Hypothesis 4.14. Note that \# Fix \(g = 6\) by [16] Chapter 15 section 1. Let \(x \in \text{Fix } g \setminus \bigcup_{i=1}^{k_6(g)} \text{Fix } g_i\); it remains to show that \(x\) is a specific fixed point of \(g\). If it is not the case then \(|G_x| > 3\). By hypothesis 4.14, it follows that \(|G_x| = 6\) and \(G_x\) is cyclic. Therefore, there exists \(h \in G\) such that \(G_x = \langle h \rangle\). Moreover since \(g \in G_x\), we have \(h^2 = g\) or \(h^{-2} = g\). Hence \(h\) or \(h^{-1}\) is one of the \(g_i\), \(i \in \{1, \ldots, k_6(g)\}\). This is a contradiction since \(x \in \text{Fix } g \setminus \bigcup_{i=1}^{k_6(g)} \text{Fix } g_i\).

Lemma 4.12. Let \(g \in G\) be an automorphism of order 2. Let \(k_6(g)\) (resp. \(k_4(g)\)) be the number of different cyclic subgroups of \(G\) of order 6 (resp. 4) which contain \(g\). Then \(8 - 2k_6(g) - 4k_4(g)\) is a non-negative integer which corresponds to the number of specific fixed point of \(g\).

Proof. The proof is identical to the one of Lemma 4.11.

4.5 Specific fixed points for the action on \(S \times S\)

4.5.1 Preliminaries

The objective of this section is to determine the specific fixed points of the elements of \(G\). First, we highlight that there are two different kind of fixed points on \(S \times S\). We recall Remark 3.8 in dimension 4.

Lemma 4.13. The set of fixed surfaces in \(S \times S\) is in bijection with \(F\). A fixed surface in \(S \times S\) is given by:

\[
\{ (x, g(x)) \mid x \in S \},
\]

with \(g \in F\) and it is fixed by \(s_0 \circ g\).

We can be slightly more general.

Lemma 4.14. Let \(g \in G\). A point \((x, y) \in S \times S\) is fixed by \(s_0 \circ g\) if and only if \(y = g(x)\) and \(x\) is a fixed point of \(\theta(g) \circ g\).

Proof. We have:

\[
s_0 \circ g(x, y) = (\theta(g)(y), g(x)) = (x, y).
\]

That is:

\[
y = g(x) \text{ and } x = \theta(g)(y).
\]

Hence \(x\) is a fixed point of \(\theta(g) \circ g\).

By the two previous lemmas, we want to study separately the fixed points of the form \((x, s(x))\) for some \(s \in G\). Indeed, these fixed points will have more complicated stabilizers; moreover some of them will lead to resolved singularities according to Proposition 4.2 and Lemma 4.13. First, we are going to characterize these fixed points. Let \(g \in G \setminus \{\text{id}\}\) and \(x \in \text{SpFix } g\); the set \(\Sigma_{g,x}\) defined in Notation 4.8 corresponds to fixed points of \(g\) of the form \((x, s(x))\) for some \(s \in G\). We are going to characterize \(\Sigma_{g,x}\) only in term of \(G\) and \(\theta\).

Lemma 4.15. Let \(g \in G \setminus \{\text{id}\}\). Let:

\[
\mathcal{S}_g := \{ s \in G \mid s^{-1} \circ \theta(g) \circ s = g \text{ or } s^{-1} \circ \theta(g) \circ s = g^{-1} \}.
\]

There exists a bijection:

\[
\mathcal{S}_g / \langle g \rangle \to \Sigma_{g,x},
\]

where \(\mathcal{S}_g / \langle g \rangle\) are the orbits of \(\mathcal{S}_g\) under the right multiplication action of \(\langle g \rangle\).
Proof. Let \( g \in G \setminus \{ \text{id} \} \). We consider \( x \in \text{SpFix}_S g \) and \( y \in \text{SpFix}_S \theta(g) \). We want to know when there exists \( s \in G \) such that \( y = s(x) \). That is:

\[
\theta(g)(s(x)) = s(x).
\]

That is:

\[
s^{-1} \circ \theta(g) \circ s(x) = x.
\]

Since \( x \in \text{SpFix}_g \), this is possible if and only if \( s^{-1} \circ \theta(g) \circ s \in \langle g \rangle \). In fact, since \( s^{-1} \circ \theta(g) \circ s \) and \( g \) have the same order, this possible if and only if \( s^{-1} \circ \theta(g) \circ s = g \) or \( = g^{-1} \). Hence we have a surjective map:

\[
S_g \to \Sigma_{g,x} : s \mapsto (x, s(x)).
\]

If \( s \) and \( s' \) are in \( S_g \), it can be that \( s(x) = s'(x) \). This means that \( x \) is a fixed point of \( s'^{-1} \circ s \). Again, this is possible if and only if \( s'^{-1} \circ s \in \langle g \rangle \). Then, by taking the orbit of \( S_g \) under the right multiplication action of \( \langle g \rangle \), we obtain the lemma.

Remark 4.16. Let \( g \in G \setminus \{ \text{id} \} \) and \( x \in \text{SpFix}_g \). We have seen that \( (x, s(x)) \) is fixed by \( g \) if and only if \( s \in S_g \).

Remark 4.17. We will also need the following set:

\[
S_{g_1,g_2} := \{ s \in G | s^{-1} \circ \theta(g_2) \circ s = g_1 \text{ or } s^{-1} \circ \theta(g_2) \circ s = g_1^{-1} \},
\]

for some \( g_1 \) and \( g_2 \) in \( G \setminus \{ \text{id} \} \). Similarly, \( S_{g_1,g_2} / \langle g_1 \rangle \) corresponds to fixed points of the form \( (x, s(x)) \) with \( x \in \text{SpFix}_{g_1} \), \( s \in G \) and \( s(x) \) fixed by \( \theta(g_2) \).

These points will be encountered in the following circumstance. We assume that \( g_1 \neq g_2 \) and there exists \( g \in G \setminus \{ \text{id} \} \) such that \( \langle g_1 \rangle \cap \langle g_2 \rangle = \langle g \rangle \). Then \( (x, s(x)) \) will not be a fixed point of \( g_1 \) nor of \( g_2 \) but a fixed point of \( g \). These points will have to be taken in account as fixed points of \( g \).

This situation can append when \( g \) has order 2 or 3.

In the sequel, we distinguish the specific fixed points which are in \( \Sigma \) and the ones which are not.

4.5.2 Specific fixed points which are not in \( \Sigma \)

Lemma 4.18. Let \( g \in G \setminus \{ \text{id} \} \). Let \( t(g) := \#(S_g / \langle g \rangle) \). If \( g_1 \) and \( g_2 \) are in \( G \setminus \{ \text{id} \} \), we also set \( t(g_1,g_2) := \#(S_{g_1,g_2} / \langle g_1 \rangle) \). The number \( N(g) \) of specific fixed points of \( g \) for the action on \( S \times S \) which are not of the form \( (x, s(x)) \) for some \( s \in G \) are obtained as follows:

- If \( O(g) = 6 \), \( N(g) = 2(2 - t(g)) \).
- If \( O(g) = 4 \), \( N(g) = 4(4 - t(g)) \).
- If \( O(g) = 3 \), let \( (g_i)_{i \in [1,...,k_0(g)]} \) be the automorphisms (modulo inverse) such that \( g_i^2 = g \). Then:
  \[
  N(g) = (6 - 2k_0(g))(6 + 2k_0(g) - t(g)) + \sum_{i,j,i \neq j} 2(2 - t(g_i,g_j)).
  \]
- If \( O(g) = 2 \), let \( (g_i)_{i \in [1,...,k_0(g)]} \) be the automorphisms (modulo inverse) such that \( g_i^3 = g \) and \( (h_i)_{i \in [1,...,k_0(g)]} \) be the automorphisms (modulo inverse) such that \( h_i^2 = g \). Then:
  \[
  N(g) = (8 - 2k_0(g) - 4k_4(g))(8 + 2k_0(g) + 4k_4(g) - t(g)) + \sum_{i,j,i \neq j} 4(2 - t(g_i,g_j)) + \sum_{i,j,i \neq j} 16k_0(g)k_4(g).
  \]

Proof. We recall from Definition 4.7 that a point \( (x, y) \in S \times S \) will be said of type \((h_1, h_2)\) if \( x \in \text{SpFix}_{h_1} \) and \( y \in \text{SpFix}_{h_2} \). If \( g \) is of order 4 or 6, there is only one possibility: a specific fixed point of \( g \) for the action on \( S \times S \) will be of type \((g, \theta(g))\).

By Lemma 4.15, there is a bijection between \( \{ s(x) \mid s \in G \} \cap \text{SpFix} \theta(g) \) and \( S_g / \langle g \rangle \). Therefore, we have \( n(g) \) choices for \( x \) and \( n(g) - t(g) \) choices for \( y \). This gives the result for \( g \) of order 4 and 6.

Now, we assume that \( O(g) = 3 \). There are several cases for the specific fixed points of \( g \) for the action on \( S \times S \):
The number of specific fixed points for each type are given by:

(1) points of type \((g, \theta(g))\);

(2) points of type \((g, \theta(g_i))\) or \((g_i, \theta(g))\);

(3) points of type \((g, \theta(g_j))\), with \(i \neq j\).

The type (1) provides \((6 - 2k_0(g))(6 - 2k_0(g) - t(g))\) specific fixed points. Each \(\theta(g_i)\) or \(g_i\) has 2 fixed points, hence the type (2) provides \(k_0(g) \times 2 \times (6 - k_0(g))\) fixed points. For the type (3), we can choose \(x \in \text{SpFix}_1\), then we need to choose \(y \in \text{SpFix}_\theta(g_j)\) such that \(y\) cannot be written \(s(x)\) for some \(s \in G\). Hence, we have 2 choices for \(x\) and then \((2 - t(g_i, g_j))\) choices for \(y\). Hence there are \(\sum_{i,j,i \neq j} 2(2 - t(g_i, g_j))\) fixed points of type (3). Then, the result follows for \(O(g) = 3\).

Similarly, when \(O(g) = 2\), there are several cases:

(1) points of type \((g, \theta(g))\);

(2) points of type \((g, \theta(g_i))\) or \((g_i, \theta(g))\);

(3) points of type \((g, \theta(h_i))\) or \((h_i, \theta(g))\);

(4) points of type \((g_i, \theta(g_j))\), with \(i \neq j\);

(5) points of type \((h_i, \theta(h_j))\), with \(i \neq j\);

(6) points of type \((h_i, \theta(g_j))\) or \((g_i, \theta(h_j))\).

The number of specific fixed points for each type are given by:

(1) \((6 - 2k_0(g) - 4k_1(g))(6 - 2k_0(g) - 4k_1(g) - t(g))\);

(2) \(4k_0(g)(6 - 2k_0(g) - 4k_1(g))\);

(3) \(8k_1(g)(6 - 2k_0(g) - 4k_1(g))\);

(4) \(\sum_{i,j,i \neq j} 2(2 - t(g_i, g_j))\);

(5) \(\sum_{i,j,i \neq j} 4(4 - t(h_i, h_j))\);

(6) \(4k_1(g) \times 2k_0(g) + 2k_0(g) \times 4k_1(g) = 16k_0(g)k_1(g)\).

\(\square\)

4.5.3 Fixed points in \(\Sigma\)

It remains to study fixed points of the form \((x, s(x))\) for \(s \in G\).

**Lemma 4.19.** Let \(x \in S\) and \(s \in G\) such that there exists \(h_1\) and \(h_2\) in \(G \setminus \{\text{id}\}\) with \(x \in \text{SpFix} h_1\) and \(s(x) \in \text{SpFix} h_2\). Then \(h_1\) and \(h_2\) have the same order.

**Proof.** Since \(h_2\) fixes \(s(x)\) then \(s^{-1} \circ h_2 \circ s\) fixes \(x\). Hence \(s^{-1} \circ h_2 \circ s \in \langle h_1 \rangle\). That is \(O(h_2) \leq O(h_1)\). Similarly since \(h_1\) fixes \(x\), the element \(s \circ h_1 \circ s^{-1}\) fixes \(s(x)\). Therefore, \(s \circ h_1 \circ s^{-1} \in \langle h_2 \rangle\). That is \(O(h_1) \leq O(h_2)\). \(\square\)

Then, there are two possible types for fixed points of the form \((x, s(x)):\)

- points of type \((g, \theta(g))\), that \(x \in \text{SpFix} g\) and \(s(x) \in \text{SpFix} \theta(g)\), for some \(g \in G \setminus \{\text{id}\}\);
- points of type \((g_i, \theta(g_j))\), that \(x \in \text{SpFix} g_i\) and \(s(x) \in \text{SpFix} \theta(g_j)\), with \(g_i\) and \(g_j\) two different automorphisms of the same order such that there exists \(g \in G \setminus \{\text{id}\}\) with \(g \in \langle g_i \rangle\) and \(g \in \langle g_j \rangle\). In this case \(g\) has order 2 or 3 and \(g_i\) has order 4 or 6.

The other types of fixed points encountered in the proof of Lemma 4.18 are impossible by Lemma 4.19.

The two possible type of fixed points are treated by the following lemma.
Lemma 4.20. Let $g \in G \setminus \{\text{id}\}$. Let $g_i$ and $g_j$ be two automorphisms of the same order such that $\langle g_i \rangle \cap \langle g_j \rangle = \langle g \rangle$ (eventually with $g_i = g_j = g$). Let $s \in S_{g_i,g_j}$ and $x \in \text{SpFix}_{g_i}$. We set:

$$H_{g_i,s} := \left\{ s_0 \circ h \mid (s^{-1} \circ h) \circ (\theta(h) \circ s) \in \langle g_i \rangle^2 \right\}.$$

Then the stabilizer of $(x,s(x))$ is $(H_{g_i,s},g)$.

**Proof.** The elements in $G \setminus \{\text{id}\}$ can be divided in two sets: $G \setminus \{\text{id}\}$ and $s_0 G := \{ s_0 \circ h \mid h \in G \}$. We consider these two sets separately. Let $x \in \text{SpFix}_{g_i}$ and $s \in S_{g_i,g_j}$. To start, we consider $\alpha \in G \setminus \{\text{id}\}$ such that $\alpha$ fixes $(x,s(x))$. This is equivalent to:

$$(\alpha(x), \theta(\alpha) \circ s(x)) = (x,s(x)) \iff \alpha(x) = x \text{ and } s^{-1} \circ \theta(\alpha) \circ s(x) = x.$$ 

Therefore:

$$\alpha \in \langle g_i \rangle, \text{ and } s^{-1} \circ \theta(\alpha) \circ s \in \langle g_i \rangle.$$ 

By (10), this is equivalent to:

$$\alpha \in \langle g_i \rangle \cap \langle g_j \rangle = \langle g \rangle.$$ 

It remains to compute $\langle H_{g_i,s},g \rangle$ in case (i). Let $h \in G$ such that $s_0 \circ h$ fixes $(x,s(x))$. It means that:

$$\theta(h) \circ s(x), h(x)) = (x,s(x)).$$ 

This is equivalent to

$$\theta(h) \circ s(x) = x \text{ and } s^{-1} \circ h(x) = x.$$ 

Since $x \in \text{SpFix}_{g_i}$, it is equivalent to $s_0 \circ h \in H_{g_i,s}$.

**Remark 4.21.** If $g_i \neq g_j$, we have described a fixed point of type $(g_i, \theta(g_j))$. If $g_i = g_j$, we have described a fixed point of type $(g_i, \theta(g_i))$.

Lemma 4.22. Let $g \in G \setminus \{\text{id}\}$. Let $g_i$, $g_j$, $s$ and $H_{g_i,s}$ as in Lemma 4.20. There are only two possibilities for the group $\langle H_{g_i,s},g \rangle$.

(i) if there exists $h = s \circ g_k^t$ with $k \in \mathbb{N}$ such that $\theta(h) \circ h \in \langle g \rangle$, then $\langle H_{g_i,s},g \rangle = \langle g, s_0 \circ h \rangle$.

(ii) if there is not an element $h = s \circ g_k^t$ with $k \in \mathbb{N}$ such that $\theta(h) \circ h \in \langle g \rangle$, then $\langle H_{g_i,s},g \rangle = \langle g \rangle$.

**Proof.** We show that in case (i) the set $H_{g_i,s}$ is not empty and in case (ii) the set $H_{g_i,s}$ is empty. First, we assume that $H_{g_i,s}$ is not empty. Let $s_0 \circ h \in H_{g_i,s}$, then $(s_0 \circ h)^2$ fixes $(x,s(x))$ (with $x \in \text{SpFix}_{g_i}$). However, $(s_0 \circ h)^2 = \theta(h) \circ h$. Hence $\theta(h) \circ h$ fixes $x$, hence

$$\theta(h) \circ h \in \langle g_i \rangle. \quad (17)$$

Moreover, $\theta(\theta(h) \circ h) = h \circ \theta(h)$ fixes $s(x) \in \text{SpFix}_{g_j} \theta(g_j)$. Hence $h \circ \theta(h) \in \langle \theta(g_j) \rangle$, that is: $\theta(h) \circ h \in \langle g_j \rangle$. Combined with (17), we have:

$$\theta(h) \circ h \in \langle g \rangle. \quad (18)$$

Moreover by definition of $H_{g_i,s}$, we have $h = s \circ g_k^t$ for some $k \in \mathbb{N}$. We have shown that if $H_{g_i,s}$ is not empty then there exists an element $h = s \circ g_k^t$, with $k \in \mathbb{N}$ and $\theta(h) \circ h \in \langle g \rangle$. By contraposition, we obtain that if there is not such an element $h$, then $H_{g_i,s}$ is empty and we are in case (ii).

Let $h \in G$ which respects the hypothesis of case (i), then $s_0 \circ h \in H_{g_i,s}$. Indeed, $s^{-1} \circ h \in \langle g_i \rangle$ and $\theta(h) \circ s = g^t \circ h^{-1} \circ s = g^t \circ g_k^{-t}$ for some $t \in \mathbb{N}$. It remains to show that $(H_{g_i,s},g) \subset \langle g, s_0 \circ h \rangle$. Let $s_0 \circ h' \in H_{g_i,s}$ be another element. As before, we have $s_0 \circ h \circ s_0 \circ h' = \theta(h) \circ h'$ which fixes $(x,s(x))$. Then $\theta(h) \circ h' \in \langle g_i \rangle \cap \langle g_j \rangle = \langle g \rangle$. Hence there is $m \in \mathbb{N}$ such that:

$$\theta(h) \circ h' = g^m.$$ 

By (18), we obtain:

$$g^k \circ h^{-1} \circ h' = g^m.$$ 

So:

$$h' = h \circ g^{m-k}.$$ 

□
With the next lemma, we verify that we have investigated all fixed points by an element of $G \setminus \{\text{id}\}$.

**Lemma 4.23.** Let $s \in G \setminus \{\text{id}\}$ such that $s_0 \circ s$ has at least one fixed points $(x, y) \in S^2$. Then there are two possibilities.

(i) $s \in F$ and $\langle s_0 \circ s \rangle$ is the stabilizer of $(x, y)$.

(ii) There exists $g \in G \setminus \{\text{id}\}$ and $g_i \in G \setminus \{\text{id}\}$ such that $\langle H_{g_i, s}, g \rangle$ is the stabilizer of $(x, y)$ (eventually with $g_i = g$).

**Proof.** By Lemma 4.14 we have $y = s(x)$ and $x$ is a fixed point of $\theta(s)$ and $s$. Then, there are two possibilities.

(ii) There exists $(g_i, g_j, g) \in G \setminus \{\text{id}\}^3$ such that $x \in \text{SpFix}g_i$ and $s \in S_{g_i, g_j}$ with $\langle g \rangle = \langle g_i \rangle \cap \langle g_j \rangle$ (eventually with $g = g_i = g_j$). This is the situation of Lemma 4.20.

(i) The previous situation does not occur.

We show that if we do not have (ii), then $s \in F$ and $\langle s_0 \circ s \rangle$ is the stabilizer of $(x, y)$. If (ii) does not occur it can be for three different reasons.

(a) There is no element $g_i \in G \setminus \{\text{id}\}$ such that $x \in \text{SpFix}g_i$;

(b) There is no element $g_j \in G \setminus \{\text{id}\}$ such that $s \in S_{g_i, g_j}$;

(c) $\langle g_i \rangle \cap \langle g_j \rangle = \{\text{id}\}$.

We show that each of these cases implies (i), i.e. the stabilizer of $(x, y)$ is $\langle s_0 \circ s \rangle$ for $s \in F$. In case (a), $x$ is not fixed by a non-trivial element, however $\theta(s)$ fixes $x$, so $\theta(s) \circ s = \text{id}$. Moreover, if $(x, y)$ is fixed by an element of $G \setminus \{\text{id}\}$, it is necessarily an element of the form $s_0 \circ h$, with $h \in G$. However, in this case, we have $y = h(x)$. So $s(x) = h(x)$; therefore again by hypothesis on $x, s = h$.

Case (b) means that we cannot find any element $g_j \in G \setminus \{\text{id}\}$ such that $s(x) \in \text{SpFix}\theta(g_j)$. In particular, $s(x)$ is not fixed by an element in $G \setminus \{\text{id}\}$. But, this implies that there is no element in $G \setminus \{\text{id}\}$ that fixes $x$ and we are back to the case (a).

In case (c), we cannot find $g \in G \setminus \{\text{id}\}$ which fixes $(x, s(x))$. Hence $(s_0 \circ s)^2 = \theta(s) \circ s = \text{id}$. That is:

$$\theta(s) = s^{-1}$$

Moreover if there exists $h \in G$ such that $s_0 \circ h$ fixes $(x, s(x))$, then $s_0 \circ s \circ s_0 \circ h = \theta(s) \circ h = \text{id}$.

By (19), we obtain $s^{-1} \circ h = \text{id}$; that is $h = s$.

**Remark 4.24.** This last lemma shows that we have investigated all possible fixed points by an element of $G$. Indeed, the elements in $G \setminus \{\text{id}\}$ can be divided in two sets $G \setminus \{\text{id}\}$ and $s_0G := \{s_0 \circ g \mid g \in G\}$. The fixed points by an element of $G \setminus \{\text{id}\}$ can be in $\Sigma$ or not; the one which are not in $\Sigma$ are investigated in Lemma 4.18, the one in $\Sigma$ are investigated in Lemma 4.20. Lemma 4.23 show that the points fixed by an element in $s_0G$ are described by the Lemma 4.20 or are generic points of a fixed surface (see Lemma 4.13).

To summarize, we can divide the fixed points by an element of $G \setminus \text{id}$ in three categories:

1. The fixed points which are not in $\Sigma$; they are only fixed by automorphisms in $G \setminus \text{id}$.
2. The non-generic fixed points in $\Sigma$ (case (ii) of Lemma 4.23). Their stabilizers are described by Lemmas 4.20 and 4.22.
3. The fixed points which are generic points of a fixed surfaces (see Lemma 4.13). They correspond to case (i) of Lemma 4.23.
4.5.4 The possible stabilizers

In this section we investigate which groups the stabilizers \( \langle s_0 \circ h, g \rangle \) of Lemma 4.22 can be. We set

\[
T_m := \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\xi_m & 0 & 0 & 0 \\
0 & \xi_m^{-1} & 0 & 0
\end{pmatrix},
\]

where \( \xi_m \) is a \( m \)-root of the unity with \( m \in \mathbb{N}^* \).

**Lemma 4.25.** Let \( g \in G \setminus \{id\} \) and \( s \in G \) such that \( G_{(x,s(x))} := \langle s_0 \circ h, g \rangle \) is the stabilizer of a point \((x,s(x)) \in S^2\) as in Lemma 4.22 Then:

\[
g \circ (s_0 \circ h) = (s_0 \circ h) \circ g \text{ or } g \circ (s_0 \circ h) = (s_0 \circ h) \circ g^{-1}.
\]

**Proof.** According to Lemmas 4.22 and 4.23 there exists \( g_i \) and \( g_j \) in \( G \) such that \( s \in \mathcal{S}_{g_i,g_j} \) and \( \langle g_i \rangle \cap \langle g_j \rangle = \langle g \rangle \); in particular, there exists \( t \in \mathbb{N} \) such that \( g = g_i^g \). Moreover, there exists \( k \in \mathbb{N} \) such that \( h = s \circ g_i^k \). It follows that:

\[
(s_0 \circ h) \circ g = s_0 \circ s \circ g_i^k \circ g_j^t = s_0 \circ s \circ g_i^t \circ g_j^k.
\]

By (16):

\[
s_0 \circ s \circ g_i^t \circ g_j^k = s_0 \circ \theta(g_j)^{\pm t} \circ s \circ g_i^k = g^{\pm t} \circ s_0 \circ s \circ g_i^k.
\]

However \( g_i \) and \( g_j \) has the same order; hence \( g_j^i = g^{\pm 1} \). This concludes the proof. \( \square \)

**Lemma 4.26.** Let \( g \in G \setminus \{id\} \) and \( s \in G \) such that \( G_{(x,s(x))} := \langle s_0 \circ h, g \rangle \) is the stabilizer of a point \((x,s(x)) \in S^2\) as in Lemma 4.22

(i) If \( G_{x,s(x)} \) is abelian, then the local action of \( G_{x,s(x)} \) around \((x,s(x))\) corresponds to the action of the group

\[
\langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}), T_1 \rangle \text{ or } \langle T_1 \rangle,
\]

on \( \mathbb{C}^4 \), around 0, with \( k \in \{2,3,4,6\} \).

(ii) If \( G_{x,s(x)} \) is non-abelian, then the local action of \( G_{x,s(x)} \) around \((x,s(x))\) corresponds to the action of the group

\[
\langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}), T_1 \rangle \text{ or } \langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}), T_2 \rangle,
\]

on \( \mathbb{C}^4 \), around 0, with respectively \( k \in \{3,4,6\} \) or \( k \in \{4,6\} \).

**Proof.** The automorphism \( g \) only has isolated fixed points and is of order 2, 3, 4 or 6. Hence, in a good base of \( \mathbb{C}^4 \), its local action around \((x,s(x))\) can be written:

\[
Q_k := \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}),
\]

for \( k \in \{2,3,4,6\} \).

The automorphism \( s_0 \circ h \) when it acts on \( S^2 \) exchanges the K3 surfaces. Therefore, the local action of \( s_0 \circ h \) around \((x,s(x))\) has to be of the form:

\[
H := \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix},
\]

with \( A \) and \( B \) in \( \text{SU}(2) \). Moreover, by (15), we recall that:

\[
(s_0 \circ h)^2 \in \langle g \rangle.
\]

So \( H^2 = \text{diag}(\xi_m, \xi_m^{-1}, \xi_m, \xi_m^{-1}) \) with \( m \in \{1,2,3,4,6\} \) and \( m \) divides \( k \). We obtain that:

\[
AB = BA = \text{diag}(\xi_m, \xi_m^{-1}).
\]

So \( H^2 = \text{diag}(\xi_m, \xi_m^{-1}, \xi_m, \xi_m^{-1}) \) with \( m \in \{1,2,3,4,6\} \) and \( m \) divides \( k \). We obtain that:

\[
AB = BA = \text{diag}(\xi_m, \xi_m^{-1}).
\]
We denote by $I_2$ the identity matrix in dimension 2. If $k = 2$, $Q_k$ is identical for all basis of $\mathbb{C}^4$; therefore, we can take $A = \text{id}$ in a convenient basis of $\mathbb{C}^4$. Hence $H = T_1$ or $H = T_2$. We have found the two cases of (i) when $k = 2$.

Now, we assume that $k \neq 2$ for all the sequel of the proof. We can write

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

So, we have:

$$A^{-1} = \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}.$$  

By (22), we have $B = \text{diag}(\xi_m, \xi_m^{-1})A^{-1}$. Therefore $\text{diag}(\xi_m, \xi_m^{-1})A = A\text{diag}(\xi_m, \xi_m^{-1})$ provides that:

$$a_2(\xi_m^{-1} - \xi_m) = 0 \quad \text{and} \quad a_3(\xi_m^{-1} - \xi_m) = 0. \quad (23)$$

These equations have two different behaviors if $m \in \{1, 2\}$ or if $m \in \{3, 4, 6\}$.

- **First case** $m \in \{1, 2\}$:
  
  In this case, we obtain from (22):
  
  $$A = \pm B^{-1}.$$ 

  Let $\mathcal{G} := \text{diag}(\xi_k, \xi_k^{-1})$. By Lemma 4.25, we know that:
  
  $$Q_k H = H Q_k^{\pm 1}.$$ 

  It follows:
  
  $$A \mathcal{G} = \mathcal{G}^{\pm 1}A.$$ 

  If $A \mathcal{G} = \mathcal{G} A$, then $\xi_k a_2 = \xi_k^{-1} a_2$ and $\xi_k a_3 = \xi_k^{-1} a_3$; that is $a_2 = a_3 = 0$. Therefore in a convenient basis of $\mathbb{C}^4$, we can have $A = \text{id}$ keeping $Q_k = \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1})$. We obtain $H = T_1$ or $H = T_2$. We have found the two possible groups:

  (a) $\langle T_1, \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}) \rangle$ which is the left case of (i)

  (b) $\langle T_2, \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}) \rangle$ with $k \in \{4, 6\}$.

  We will see that case (b) also corresponds to one of the groups of (i). If $k = 4$, then we can replace $T_2$ by:

  $$T_2 \times \text{diag}(i, -i, i, -i) = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

  which is $T_1$ after changing the base of $\mathbb{C}^4$ by $(ie_1, -ie_2, e_3, e_4)$. If $k = 6$, then $T_2$ can be replaced by:

  $$T_2 \times \text{diag}(\xi_3, \xi_3^{-1}, \xi_3, \xi_3^{-1}) = \begin{pmatrix} 0 & 0 & \xi_3 & 0 \\ 0 & 0 & 0 & \xi_3^{-1} \\ -\xi_3 & 0 & 0 & 0 \\ 0 & -\xi_3^{-1} & 0 & 0 \end{pmatrix},$$

  which is $T_6$ after changing the base of $\mathbb{C}^4$ by $(\xi_3 e_1, \xi_3^{-1} e_2, e_3, e_4)$.

  If $A \mathcal{G} = \mathcal{G}^{-1} A$, then $a_1 = a_4 = 0$. Denoting $(e_1, e_2, e_3, e_4)$ the canonical basis of $\mathbb{C}^4$. We can exchange $e_3$ and $e_4$ and obtain a new basis $(e_1', e_2', e_3', e_4')$. In this new basis: $Q_k = \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1})$ and $A = \text{diag}(a_2, a_3)$. Then, after replacing $e_3'$ by $e_3'/a_2$ and $e_4'$ by $e_4'/a_3$, we obtain $H = T_1$ or $H = T_2$. This corresponds to case (ii).

- **Second case** $m \in \{3, 4, 6\}$:
  
  In this case (23) provides that $a_2 = a_3 = 0$. Then taking $(e_1, e_2, e_3/a_1, e_4/a_2)$ for the basis of $\mathbb{C}^4$, we can set that $A = I_2$ and we obtain by (22) that $H = T_m$. To fit with case (i), it remains to show that we can always take $m = 1$ or $m = k$. 

31
If \( k \in \{3\} \), there is nothing to prove since \( m \) divides \( k \).

If \( k = 4 \), there are 3 possibilities for \( m \) which are 1, 2 or 4. If \( m = 1 \) or \( m = 4 \), there is nothing to prove. If \( m = 2 \), as before \( T_2 \) can be replaced by \( T_2 \times \text{diag}(i, -i, i, -i) \) which is \( T_1 \) after a change of basis which keeps \( Q_k \) unchanged.

If \( k = 6 \), the are 4 possibilities for \( m \) which are 1, 2, 3 or 6. The cases \( m = 1 \) and \( m = 6 \) are clear. As before \( T_3 \) can be replaced by \( T_3 \times \text{diag}(\xi_0, \xi_0^{-1}, \xi_0, \xi_0^{-1}) \) which is \( T_1 \) in a good basis and \( T_2 \) can be replaced by \( T_2 \times \text{diag}(\xi_3, \xi_3^{-1}, \xi_3, \xi_3^{-1}) \) which is \( T_6 \) in a good basis. 

\[ \square \]

### 4.6 Singularities of \( S(G)^{[2]} \)

**Notation 4.27.** Let \( X \) be a given orbifold. We denote as follows the singularities of \( X \).

- We denote by \( a_k(X) \) the number of singularities of analytic type
  \[ \mathbb{C}^4 / \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}) , \]
  where \( k \in \{2, 3, 4, 6\} \). We also say that these singularities are of type \( a_k \).
- For \( k \in \{4, 6\} \), we denote by \( a_{2k}(X) \) the number of singularities of analytic type
  \[ \mathbb{C}^4 / T_k , \]
  where \( T_k \) is defined in \( \text{[27]} \). We also say that these singularities are of type \( a_{2k} \).
- For \( k \in \{4, 6\} \), we denote by \( b_k(X) \) the number of singularities of analytic type
  \[ \mathbb{C}^4 / \langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}), T_2 \rangle , \]
  with \( T_2 \) which is defined in \( \text{[27]} \). We also say that these singularities are of type \( b_k \).

When there is no ambiguity on \( X \), we simply denote \( a_k \) and \( b_k \) instead of \( a_k(X) \) and \( b_k(X) \).

We consider the elements of order \( k \) of a group \( G \).

**Notation 4.28.** On \( G \), we consider the bijection \( \text{inv}(g) = g^{-1} \). The group \( \langle \text{inv} \rangle \) acts on \( G \). For \( k \in \{2, 3, 4, 6\} \), we denote:

\[ \mathcal{O}(G)_k := \{ g \in G \mid \mathcal{O}(g) = k \} / \langle \text{inv} \rangle , \]
where \( \mathcal{O}(g) \) is the order of \( g \). In other words, in \( \mathcal{O}(G)_k \) we have identified \( g \) and \( g^{-1} \).

**Remark 4.29.** An element \( g \in G \) and its inverse \( g^{-1} \) will have the same specific fixed points. So these fixed points have to be counted only once for \( g \) or for \( g^{-1} \). It is why the sets \( \mathcal{O}(G)_k \) have been introduced.

According to Lemma \( \text{[415]} \), \( S_g \) corresponds to fixed point of \( g \) of the form \( (x, s(x)) \). In the previous section, we have seen that the stabilizer of \( (x, s(x)) \) can be different groups. The objective of the next notation is to make a partition of \( S_g \) with each subsets corresponding to one of the possible stabilizers. All the notation are explained in Remark \( \text{[431]} \).

**Notation 4.30.** Let \( g \in G \setminus \{\text{id}\} \).

- Let \( S_g \) be the set defined in Lemma \( \text{[4.15]} \). According to Lemma \( \text{[4.22]} \), the set \( S_g \) can be divided in two subsets:
  \[ S_g^+ := \{ s \in S_g \mid \exists \alpha \in \langle g \rangle , \theta(s \circ \alpha) \circ s \circ \alpha \in \langle g \rangle \} \text{ and } S_g^- := S_g \setminus S_g^+ . \]
  The set \( S_g^+ \) can also be divided in two subsets:
  \[ S_g^{+c} := \{ s \in S_g^+ \mid g \text{ and } s_0 \circ s \text{ commute} \} \text{ and } S_g^{+nc} := \{ s \in S_g^+ \mid g \text{ and } s_0 \circ s \text{ do not commute} \} . \]
  We set \( S_g^\bullet := S_g^+ / \langle g \rangle \) the orbits under the right multiplication action, with "\( \bullet \)" which can be any of the previous decorations.

32
• We consider the same notation for $S_{g, s}$, where $g, s$ are two different automorphisms in $G$ such that $\langle g \rangle \cap \langle s \rangle = \langle g \rangle$.

$$S_{g, s}^+ := \left\{ s \in S_{g, s} \mid \exists \alpha \in \langle g \rangle, \; \theta(s \circ \alpha) \circ s \circ \alpha \in \langle g \rangle \right\} \quad \text{and} \quad S_{g, s}^- := S_{g, s} \setminus S_{g, s}^+.$$  

Also:

$$S_{g, s}^{+, c} := \left\{ s \in S_{g, s}^+ \mid g \text{ and } s \circ 0 \circ s \text{ commute} \right\} \quad \text{and} \quad S_{g, s}^{+, nc} := \left\{ s \in S_{g, s}^+ \mid g \text{ and } s_0 \circ s \text{ do not commute} \right\}.$$  

As before, we denote $\xi_{g, s}$ which can be any of the previous decorations.

• We set

$$\mathcal{F}_g := \left\{ s \circ \alpha \mid s \in F, \; \alpha \in \langle g \rangle \right\}$$  

and $\mathcal{F}_g := F / \langle g \rangle$ the orbits under the right multiplication action. We recall that $F$ is defined in Notation 4.8.

Remark 4.31. Let $(x, s(x)) \in \Sigma$.

• By Lemma 4.15, $(x, s(x))$ is fixed by $g \in G$ and of type $(g, \theta(g))$ (resp. $(g, \theta(g))$) if and only if $s \in S_g$ (resp. $s \in S_{g, s}$).

• Moreover the stabilizer of $(x, s(x))$ is as in case (i) of Lemma 4.22 if and only if $s \in S_g^+$ (resp. $s \in S_{g, s}^+$). Then, the stabilizer of $(x, s(x))$ is as in case (ii) of Lemma 4.22 if and only if $s \in S_g^-$ (resp. $s \in S_{g, s}^-$).

• When $s \in S_g^+$ (resp. $s \in S_{g, s}^+$), accordingly to Lemma 4.26, there are four possibilities for the stabilizer of $(x, s(x))$. These four possibilities correspond to the following four subsets of $S_g^+$ (resp. $s \in S_{g, s}^+$):

$$S_g^{+, c} \cap \mathcal{F}_g, \quad S_g^{+, nc} \cap \mathcal{F}_g, \quad S_g^{+, c} \setminus \mathcal{F}_g, \quad \text{and} \quad S_g^{+, nc} \setminus \mathcal{F}_g$$

(resp. $S_{g, s}^{+, c} \cap \mathcal{F}_{g, s}, \quad S_{g, s}^{+, nc} \cap \mathcal{F}_{g, s}, \quad S_{g, s}^{+, c} \setminus \mathcal{F}_{g, s}, \quad \text{and} \quad S_{g, s}^{+, nc} \setminus \mathcal{F}_{g, s}$).

Proposition 4.32. Let $S$ be a K3 surface and $G$ an finite admissible symplectic automorphism group on $S$. Let $\theta$ be an involution on $G$ (not necessarily valid). Then $S(G)^{[2]}$ has only singularities of type $a_k$ and $b_m$ for $k \in \{2, 3, 4, 6, 8, 12\}$ and $m \in \{4, 6\}$.

Proof. Let $(x, y) \in S^2$. If $(x, y) \notin \Sigma$, then by Lemma 1.14 the stabilizer of $(x, y)$ is given by $\langle g \rangle$ with $g$ of order 2, 3, 4 or 6. If $(x, y) \in \Sigma$, by Lemmas 1.24 and 1.22 the stabilizer of $(x, y)$ can only be $\langle g \rangle$ or $\langle g, s_0 \circ s \rangle$, with $g$ of order 2, 3, 4 or 6 and $s \in G$. Therefore we obtain that the singularities of $Z = S^2 / G$ can be of type $a_k$ for $k \in \{2, 3, 4, 6\}$ or of type $C^4 / G(x, y)$ with $G(x, y)$ described in Lemma 1.26.

If $G(x, y) = \langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k, \xi_k^{-1}), T_1 \rangle$ or $G(x, y) = \langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k^{-1}, \xi_k), T_1 \rangle$ with $k \in \{2, 3, 4, 6\}$, the quotient $C^4 / G(x, y)$ is a singularity of type $M_k$ or $N_k$ respectively. According to Proposition 4.2 these singularities induce on $S(G)^{[2]}$ only singularities of type $a_2$ or $a_3$.

The case when $G(x, y) = \langle \text{diag}(\xi_k, \xi_k^{-1}, \xi_k^{-1}, \xi_k), T_2 \rangle$ for $k \in \{4, 6\}$ provides the singularities of type $b_3$.

The case $G(x, y) = \langle T_k \rangle$ for $k \in \{2, 3, 4, 6\}$ provides the singularities of type $a_2 k$. When $k \in \{2, 3\}$ the singularities $C^4 / \text{diag}(\xi_{2k}, \xi_{2k}^{-1}, \xi_{2k}, \xi_{2k})$ and $C^4 / T_k$ are equivalent; however this is not the case when $k \in \{4, 6\}$.

The singularities of types $a_8, a_{12}, b_4$ and $b_6$ are appearing only in one configuration; they are the rarest and then the simplest to determine.
Proposition 4.33. Let $G$ be a finite admissible symplectic automorphism group action on a $K3$ surface $S$. Let $a_k$ and $b_k$ be the number of singularities of $S(G)^{[2]}_g$ as defined in Notation 4.27. We have:

\[ a_8 = \frac{16}{|G|} \sum_{g \in \mathcal{O}(G)_4} \#(\mathcal{S}^+,^c_g \setminus \mathcal{F}_g); \]  
\[ a_{12} = \frac{12}{|G|} \sum_{g \in \mathcal{O}(G)_6} \#(\mathcal{S}^+,^c_g \setminus \mathcal{F}_g); \]  
\[ b_4 = \frac{16}{|G|} \sum_{g \in \mathcal{O}(G)_4} \#(\mathcal{S}^+,nc_g \setminus \mathcal{F}_g); \]  
\[ b_6 = \frac{12}{|G|} \sum_{g \in \mathcal{O}(G)_6} \#(\mathcal{S}^+,nc_g \setminus \mathcal{F}_g). \]

Proof. A singularity of type $a_8$ comes from a fixed point with a stabilizer which is cyclic of order 8. According to Section 4.5 and Remark 4.24, this can only occur in the right side of case (i) of Lemma 4.26 with $\mathcal{O}(g) = 4$. Therefore, a singularity of type $a_8$ comes from a fixed point $(x,y)$ with a stabilizer $\langle g, s_0 \circ s \rangle$ which verifies the following conditions:

(i) $g$ has order 4;

(ii) the fixed point if of the form $(x,s(x))$ with $x \in \text{SpFix}g$ and $s \in \mathcal{S}^+_g$;

(iii) $g$ and $s_0 \circ s$ commutes;

(iv) $x \notin \mathcal{F}_g$.

According to Lemma 4.22, the condition (ii) is needed to have a stabilizer of the form $\langle g, s_0 \circ s \rangle$ rather than $(g)$. The condition (iii) is needed to have an abelian stabilizer. The condition (iv) is needed to choose the right side of Lemma 4.26 (i) rather than the left side.

Now, we count how many points with such a stabilizer there are in $S^2$. These points are of the form $(x,s(x))$ with $x \in \text{SpFix}g$ and $s \in \mathcal{S}^+_g \setminus \mathcal{F}_g$. We recall that if $s' = s \circ g'$ then $s'(x) = s(x)$. Therefore by Lemma 4.10, there are $4 \times \#(\mathcal{S}^+,^c_g \setminus \mathcal{F}_g)$ such fixed points. Let $\text{Fix}_8$ be the set of points in $S^2$ with a stabilizer which is a cyclic group of order 8. Then:

\[ \# \text{Fix}_8 = 4 \sum_{g \in \mathcal{O}(G)_4} \#(\mathcal{S}^+,^c_g \setminus \mathcal{F}_g). \]

The group $G$ acts on $\text{Fix}_8$. By definition of $\text{Fix}_8$, the orbits of this action have cardinal $\frac{|G|}{8} = \frac{|G|}{4}$; each of these orbits provide a singularity of type $a_8$. We obtain:

\[ a_8 = 4 \times \frac{4}{|G|} \sum_{g \in \mathcal{O}(G)_4} \#(\mathcal{S}^+,^c_g \setminus \mathcal{F}_g). \]

The proofs of 26, 27 and 28 are identical with other conditions on the stabilizer.

Remark 4.34. Actually, considering the list of all possible $G$ and $\theta$ on a K3 surface, the singularity $a_{12}$ will never be encountered, but this could not have been predicted only from Hypothesis 4.16 (see Section 5).

The next singularities, slightly more frequent, are the ones of type $a_4$ and $a_6$. We recall that $t(g)$ is defined in Lemma 4.18.

Proposition 4.35. Let $G$ be a finite admissible symplectic automorphism group acting on a K3 surface $S$. Let $a_k$ be the number of singularities of $S(G)^{[2]}_g$ as defined in Notation 4.27. We have:

\[ a_6 = \frac{3}{|G|} \sum_{g \in \mathcal{O}(G)_3} (6 - 2k_b(g))\#(\mathcal{S}^+_g \setminus \mathcal{F}_g) + \frac{6}{|G|} \sum_{g \in \mathcal{O}(G)_3} \sum_{1 \leq i < j \leq k_b(g)} \#(\mathcal{S}^+,^c_{g_i,g_j} \setminus \mathcal{F}_{g_i}) \]
\[ + \frac{6}{|G|} \sum_{g \in \mathcal{O}(G)_6} \#(\mathcal{S}^+_g) + \frac{6}{|G|} \sum_{g \in \mathcal{O}(G)_6} 2 - t(g), \]

34
where for each \( g \in \mathcal{O}(G)_3 \), the \((g_i)_{i \in \{1, \ldots, k_6(g)\}}\) are generators of the different cyclic groups of order 6 containing \( g \). Similarly:

\[
a_4 = \frac{2}{|G|} \sum_{g \in \mathcal{O}(G)_4} (8 - 2k_6(g) - 4k_4(g)) \#(\Sigma_g^+ \setminus \mathcal{F}_g) + \frac{4}{|G|} \sum_{g \in \mathcal{O}(G)_2} \sum_{1 \leq i \neq j \leq k_6(g)} \#(\Sigma_{g_i, g_j}^+ \setminus \mathcal{F}_{g_i})
\]

\[
+ \frac{8}{|G|} \sum_{g \in \mathcal{O}(G)_2} \sum_{1 \leq i \neq j \leq k_6(g)} \#(\Sigma^+_{g_i, g_j} \setminus \mathcal{F}_{g_i}) + \frac{8}{|G|} \sum_{g \in \mathcal{O}(G)_4} \#(\Sigma_g^+) + \frac{8}{|G|} \sum_{g \in \mathcal{O}(G)_4} 4 - t(g),
\]

where for each \( g \in \mathcal{O}(G)_2 \), the \((g_i)_{i \in \{1, \ldots, k_6(g)\}}\) are generators of the different cyclic groups of order 6 containing \( g \) and the \((h_i)_{i \in \{1, \ldots, k_6(g)\}}\) are generators of the different cyclic groups of order 4 containing \( g \).

**Proof.** We prove the proposition for \( a_4 \), the proof is identical for \( a_6 \). The singularities of type \( a_4 \) can appear from three different configurations of fixed points:

(a) a fixed point in \( \Sigma \) with a cyclic stabilizer \( \langle g, s_0 \circ h \rangle \) with \( g \in G \) of order 2 and \((s_0 \circ h)^2 \in \langle g \rangle \). This is the case (i) of Lemma 4.22 and the right side of Lemma 4.26 (i).

(b) a fixed point in \( \Sigma \) with a stabilizer \( \langle g \rangle \) with \( g \in G \) of order 4. This is the case (ii) of Lemma 4.22.

(c) a fixed point which is not in \( \Sigma \) fixed by an automorphism \( g \in G \) of order 4. This is described in Lemma 4.18.

We are going to count all the fixed points from these three configurations.

(c) The simplest configuration are the one given by Lemma 4.18. Let \( g \) be an automorphism of order 4, then \( g \) has \( 4(4 - t(g)) \) specific fixed points which are not in \( \Sigma \). Moreover the stabilizer of such a fixed point has order 4; so the action of \( G \) on this set of points has orbits of cardinal \( \frac{|G|}{4} = \frac{|G|}{3} \). Therefore, the contribution to the singularities of these kind of fixed points is:

\[
\frac{2}{|G|} \sum_{g \in \mathcal{O}(G)_4} 4(4 - t(g)) = \frac{8}{|G|} \sum_{g \in \mathcal{O}(G)_4} (4 - t(g)).
\]

(b) These fixed points are points that can be written \((x, s(x)) \) with \( s \in \mathcal{S}_g^+ \) and \( x \) fixed by an automorphism of order 4. Therefore there are 4 choices for \( x \) by Lemma 4.10 and \#(\Sigma_g^+) \) choices for \( s(x) \) according to Lemma 4.22 (ii). As before the orbits of the action of \( G \) on this set of points have cardinal \( \frac{|G|}{3} \). Hence the contribution of these fixed points to the singularities is:

\[
\frac{2}{|G|} \sum_{g \in \mathcal{O}(G)_4} 4 \times \#(\Sigma_g^+) = \frac{8}{|G|} \sum_{g \in \mathcal{O}(G)_4} \#(\Sigma_g^+).
\]

(a) These fixed points have a stabilizer \( \langle g, s_0 \circ h \rangle \) as described in Lemma 4.22 (i) and the right side of Lemma 4.26 (i), with \( g \) of order 2. These fixed points can be of three different types:

(I) the type \( \langle g, \theta(g) \rangle \);

(II) the type \( \langle g_i, \theta(g_j) \rangle \) with \( g_i \) and \( g_j \) of order 6 such that \( \langle g_i \rangle \cap \langle g_j \rangle = \langle g \rangle \);

(III) the type \( \langle h_i, \theta(h_j) \rangle \) with \( h_i \) and \( h_j \) of order 4 such that \( \langle h_i \rangle \cap \langle h_j \rangle = \langle g \rangle \).

We count the number of fixed points for each types.

(I) Let \((x, s(x))\) be a fixed point of type \( (g, \theta(g)) \) with stabilizer \( \langle g, s_0 \circ h \rangle \). There are \( 8 - k_6(g) - k_4(g) \) choices for \( x \) according to Lemma 4.12 and there are \#(\Sigma_g^+ \setminus \mathcal{F}_g) \) choices for \( s(x) \). Therefore, these fixed points contribute to

\[
\frac{2}{|G|} \sum_{g \in \mathcal{O}(G)_2} (8 - 2k_6(g) - 4k_4(g)) \#(\Sigma_g^+ \setminus \mathcal{F}_g)
\]

singularities of type \( a_4 \).
Moreover, we have:

\[ \frac{2}{|G|} \sum_{g \in O(G)_{13}} \sum_{1 \leq i \neq j \leq k_{0}(g)} 2 \times \#(\mathcal{S}_{g_i,g_j}^{r} \cap \mathcal{F}_{g_j}) = \frac{4}{|G|} \sum_{g \in O(G)_{21}} \sum_{1 \leq i \neq j \leq k_{0}(g)} \#(\mathcal{S}_{g_i,g_j}^{r} \cap \mathcal{F}_{g_j}). \]

(III) In this case, the computation is identical to the one in case (II).

Finally, it remains to count the singularities of type \(a_2\) and \(a_3\) which are the most frequent.

**Proposition 4.36.** Let \(G\) be a finite admissible symplectic automorphism group action on a K3 surface \(S\). Let \(a_3\) be the number of singularities of \(S(G)^{[2]}\) as defined in Notation 4.37. For each \(g \in G\), let \(N(g)\) be the numbers provided in Lemma 4.18. We have:

\[
a_3 = \frac{3}{2 |G|} \sum_{g \in O(G)_{13}} N(g) + \frac{3}{2 |G|} \sum_{g \in O(G)_{21}} (6 - 2k_{0}(g)) \#(\mathcal{S}_{g}^{r}) + \frac{3}{|G|} \sum_{g \in O(G)_{21}} \sum_{1 \leq i \neq j \leq k_{0}(g)} \#(\mathcal{S}_{g_i,g_j}^{r})
\]
\[+ \frac{6}{|G|} \sum_{g \in O(G)_{13}} (6 - 2k_{0}(g)) \#(\mathcal{S}_{g}^{r,c} \cap \mathcal{F}_{g}) + \frac{12}{|G|} \sum_{g \in O(G)_{21}} \sum_{1 \leq i \neq j \leq k_{0}(g)} \#(\mathcal{S}_{g_i,g_j}^{r,c} \cap \mathcal{F}_{g}).
\]
\[+ \frac{48}{|G|} \sum_{g \in O(G)_{6}} \#(\mathcal{S}_{g}^{r,c} \cap \mathcal{F}_{g}).
\]

Moreover, we have:

\[
a_2 = \frac{1}{|G|} \sum_{g \in O(G)_{21}} N(g) + \frac{1}{|G|} \sum_{g \in O(G)_{21}} (8 - 2k_{0}(g) - 4k_{4}(g)) \#(\mathcal{S}_{g}^{r}) + \frac{2}{|G|} \sum_{g \in O(G)_{21}} \sum_{1 \leq i \neq j \leq k_{0}(g)} \#(\mathcal{S}_{g_i,g_j}^{r})
\]
\[+ \frac{4}{|G|} \sum_{g \in O(G)_{21}} \sum_{1 \leq i \neq j \leq k_{0}(g)} \#(\mathcal{S}_{g_i,g_j}^{r,c} \cap \mathcal{F}_{g}) + \frac{12}{|G|} \sum_{g \in O(G)_{6}} \#(\mathcal{S}_{g}^{r,c} \cap \mathcal{F}_{g}) + \frac{64}{|G|} \sum_{g \in O(G)_{4}} \#(\mathcal{S}_{g}^{r,c} \cap \mathcal{F}_{g}).
\]

**Proof.** The computation is slightly different for \(a_3\) and \(a_2\), so we are going to prove the two equations.

**Computation of \(a_3\):**

The singularities of type \(a_3\) can appear from three different configurations of fixed points.

(a) The fixed points which are not in \(\Sigma\) and are fixed by an automorphism of order 3. There are described in Lemma 4.18.

(b) The fixed points which are in \(\Sigma\) with a stabilizer of order 3. This is the case of Lemma 4.22 when \(g\) has order 3.

(c) It can also be singularities remaining after a partial resolution as described in Proposition 4.22.

We count the singularities for each of these configurations.

(a) According to Lemma 4.18, the number of fixed points in this configuration is \(N(g)\) for each \(g \in O(G)_{13}\). Hence, it provides the following contribution to the singularities:

\[
\frac{3}{2 |G|} \sum_{g \in O(G)_{13}} N(g).
\]

(b) The fixed points of this configuration are in \(\Sigma\), so can be written \((x, s(x))\). They can be of two different types:
(I) type \((g, \theta(g))\);

(II) type \((g_i, \theta(g_j))\), with \(g_i\) and \(g_j\) of order 6.

Since the stabilizer of these fixed points is of order 3, we are in the case (ii) of Lemma 4.22.

Therefore, in case (I), there are \(6 - 2k_0(g)\) choices for \(x\) according to Lemma 4.11 and \(#(S_g)\) choices for \(s(x)\). We obtain the following contribution to the singularities:

\[
\frac{3}{2|G|} \sum_{g \in O(G)_3} (6 - 2k_0(g))#(S_g).
\]

Similarly, in case (II), we have 2 choices for \(x\) according to Lemma 4.10 and \(#(S_{g_i, g_j})\) choices for \(s(x)\). We obtain the following contribution to the singularities:

\[
\frac{3}{|G|} \sum_{g \in O(G)_3} \sum_{1 \leq i \neq j \leq k_0(g)} #(S_{g_i, g_j}).
\]

(c) The singularities in this case are obtained as remaining singularities of terminalizations (see Proposition 4.12). Therefore, they come from fixed points contained in a fixed surface. These fixed points can be written \((x, s(x))\) with \(s \in F\) (see Lemma 4.13). If the stabilizer of \((x, s(x))\) is given by \((s_0 \circ s)\) then the singularities induced by \((x, s(x))\) is resolved by a terminalization.

Hence by Lemma 4.23 a fixed point \((x, s(x))\) with \(s \in F\) can induce singularities on \(S(G)^2\) if and only if there exists \(g, g_i, g_j\) in \(G \setminus \{id\}\) such that \(s \in S_{g, g_i, g_j}\) (eventually with \(g = g_i = g_j\)).

According to Proposition 4.2 there are two configurations that can lead to singularities of type \(a_3\) on the terminalization:

(I) when \(g\) has order 3 and the stabilizer of \((x, s(x))\) is abelian;

(II) when \(g\) has order 6 and the stabilizer of \((x, s(x))\) is abelian.

We count the singularities induced by these two previous configurations. We stat by considering the case (II) which is slightly simplest.

(II) Since \(g\) has order 6, by Lemma 4.10 there are 2 choices for \(x\). Since \(s \in F_g\) and the stabilizer of \((x, s(x))\) is abelian, by Lemma 4.10 there are \(#S^+_g \cap F_g\) choices for \(s(x)\).

Moreover according to Proposition 4.2, the terminalization provides 4 singularities of type \(a_3\). Note also that the stabilizer of \((x, s(x))\) has order 2 \(\times 6 = 12\). We obtain that the contribution to the singularities in this case is:

\[
\frac{12}{2|G|} \sum_{g \in O(G)_3} 4 \times 2 \times #(S^+_g \cap F_g) = \frac{48}{|G|} \sum_{g \in O(G)_3} #(S^+_g \cap F_g).
\]

(I) In this case \(g\) has order 3. Therefore, the stabilizer of \((x, s(x))\) has order 2 \(\times 3 = 6\).

Moreover, \((x, s(x))\) can be of type \((g, \theta(g))\) or \((g_i, \theta(g_j))\) with \(\langle g_i \rangle \cap \langle g_j \rangle = \langle g \rangle\) and \(g_i\) and \(g_j\) of order 6.

When \((x, s(x))\) is of type \((g, \theta(g))\), applying the same results as before (Lemmas 4.11 4.15 and Proposition 4.2), there are \(6 - 2k_0(g)\) choices for \(x\) and \(#S^+_{g, g_i} \cap F_g\) choices for \(s(x)\). Moreover, the terminalization in this case provide 2 singularities of type \(a_3\). We obtain the following contribution to the singularities:

\[
\frac{6}{2|G|} \sum_{g \in O(G)_3} 2 \times (6 - 2k_0(g))#(S^+_{g, g_i} \cap F_g) = \frac{6}{|G|} \sum_{g \in O(G)_3} (6 - 2k_0(g))#(S^+_{g, g_i} \cap F_g).
\]

When \((x, s(x))\) is of type \((g_i, \theta(g_j))\), there are 2 choices for \(x\) and \(#S^+_{g_i, g_j} \cap F_{g_i}\) choices for \(s(x)\). Hence, we have the following contribution to the singularities:

\[
\frac{6}{2|G|} \sum_{g \in O(G)_3} \sum_{1 \leq i \neq j \leq k_0(g)} 2 \times 2 \times #(S^+_{g_i, g_j} \cap F_{g_i}) = \frac{12}{|G|} \sum_{g \in O(G)_3} \sum_{1 \leq i \neq j \leq k_0(g)} #(S^+_{g_i, g_j} \cap F_{g_i}).
\]
Computation of $a_2$:

As before, the singularities of type $a_2$ can appear from three different configurations of fixed points.

(a) The fixed points which are not in $\Sigma$ and are fixed by an automorphism of order 2. There are described in Lemma 4.18.

(b) The fixed points which are in $\Sigma$ with a stabilizer of order 2. This is the case of Lemma 4.22 (ii) when $g$ has order 2.

(c) It can also be singularities remaining after a partial resolution as described in Proposition 4.2.

The computation in case (a) follows as before from Lemma 4.18. The case (b) is also similar apart that the fixed points $(x, s(x))$ can be of three different types: $(g, \theta(g))$, with $g$ of order 2, $(g_i, \theta(g_j))$ with $g_i, g_j$ of order 6 and $(h_i, \theta(h_j))$ with $h_i, h_j$ of order 4. Then the proof is identical to the proof for $a_3$ replacing Lemma 4.11 by Lemma 4.12.

However, the case (c) is slightly different for $a_3$ and for $a_2$. Case (c) comes from fixed points $(x, s(x)) \in \Sigma$ with a stabilizer $\langle g, s_0 \circ s \rangle$ and $s \in F$ (see Lemma 4.13). According to Proposition 4.2, the terminalizations which leave singularities of type $a_2$ are the following:

(I) When $\langle g, s_0 \circ s \rangle$ is non-abelian and $g$ has order 6.

(II) When $\langle g, s_0 \circ s \rangle$ is abelian and $g$ has order 4.

We count the singularities provided by these two configurations.

(I) In this case $g$ has order 6. Hence there are 2 choices for $x$ by Lemma 4.10. Since the stabilizer is non-abelian, there are $\#(\mathcal{S}^+_{g,nc} \cap \mathcal{F}_g)$ choices for $s(x)$ (see Lemma 4.15). Moreover according to Proposition 4.2, the terminalization provides one singularity of type $a_2$. Note also that the stabilizer of $(x, s(x))$ has order $2 \times 6 = 12$. We obtain that the contribution to the singularities in this case is:

$$\frac{12}{2 |G|} \sum_{g \in \mathcal{O}(G)_6} 2 \times \#(\mathcal{S}^+_{g,nc} \cap \mathcal{F}_g) = \frac{12}{|G|} \sum_{g \in \mathcal{O}(G)_6} \#(\mathcal{S}^+_{g,nc} \cap \mathcal{F}_g).$$

(II) In this case $g$ has order 4. Hence, there are 4 choices for $x$ according to Lemma 4.10. Since the stabilizer is abelian, there are $\#(\mathcal{S}^+_{g,c} \cap \mathcal{F}_g)$ choices for $s(x)$. By Proposition 4.2, the terminalization provides 4 singularities of type $a_2$. In this case the stabilizer of $(x, s(x))$ has order $2 \times 4 = 8$. We obtain that the contribution to the singularities in this case is:

$$\frac{8}{2 |G|} \sum_{g \in \mathcal{O}(G)_4} 4 \times 4 \times \#(\mathcal{S}^+_{g,c} \cap \mathcal{F}_g) = \frac{64}{|G|} \sum_{g \in \mathcal{O}(G)_4} \#(\mathcal{S}^+_{g,c} \cap \mathcal{F}_g).$$

4.7 A practical method to verify the computations of the singularities

This section come from a discussion with Song Jieao (see [14]).

Proposition 4.37. Let $X$ be a primitively symplectic orbifold. Let $C_X$ be its Fujiki constant (see Section 3.4). Then, the number

$$\sqrt{\frac{(7c_2(X)^2 - 4c_4(X))C_X}{15}}$$

is a rational number.
Proof. Let $X$ be a primitively symplectic orbifold. By [25, Lemma 4.6], there exists a rational number $C(c_2)$ such that:

$$c_2(X) \cdot \alpha^2 = C(c_2) q_X(\alpha),$$

for all $\alpha \in H^2(X, \mathbb{C})$. By [6, Corollary 2.6 and Section 3], we have:

$$7c_2(X)^2 - 4c_4(X) = \frac{15C(c_2)^2}{C_X}.$$ (29)

Then:

$$C(c_2) = \sqrt{\frac{(7c_2(X)^2 - 4c_4(X))C_X}{15}}$$ (30)

which is a rational number.

Corollary 4.38. Let $S$ be a K3 surface and $G$ a symplectic automorphism group on $S$. Let $X = S(G)^[2]$. Then:

$$\sqrt{|G|(7c_2(X)^2 - 4c_4(X))}$$

is a rational number.

Proof. By Proposition 3.19, we have:

$$C_X q_X(r^*r(\alpha))^2 = 3|G|(|G|\alpha^2)^2,$$ (31)

where $r$ and $\epsilon$ are defined in Section 3.4 and $\alpha \in H^2(S, \mathbb{C})$. Combining (31) with Proposition 4.37, we obtain our result.

Remark 4.39. In practice, we use the results of Section 2.5 to compute $\sqrt{|G|(7c_2(X)^2 - 4c_4(X))}$ (see Section 5); then it can be expressed in term of singularities and the Betti numbers $b_2, b_3$. Therefore Proposition 4.37 can be used to verify if the computation of the singularities is correct.

For this purpose, we set:

Notation 4.40. $C(c_2) := \sqrt{\frac{|G|(7c_2(X)^2 - 4c_4(X))}{b}}$.

We provide an example of use of Proposition 4.37. The following error have been noticed and corrected in [6].

Erratum 4.41. Let $K_2(T)$ be a generalized Kummer fourfold endowed with $g_4$ a natural symplectic automorphism of order 4 (coming from an automorphism on the torus). Let $K'_4$ be a terminalization of the quotient $K_2(T)/g_4$.

The computation of the singularities of $K'_4$ in [13, Sections 5.4] is incorrect. The correct singularities are 8 singularities of analytic type $\mathbb{C}^4 / g_4$ and 30 singularities of analytic type $\mathbb{C}^4 / -\text{id}$, with $g_4 = \text{diag}(i, -i, -i, i)$.

Proof. Let $b_2$ (resp. $a_2$) be the number of singularities of analytic type $\mathbb{C}^4 / -\text{id}$ (resp. $\mathbb{C}^4 / g_4$). In [13, Sections 5.4], it is given:

$b_2(K'_4) = 6$, $b_3(K'_4)$, $a_2 = 45$, and $a_4 = 2$.

We show that this result leads to $\sqrt{\frac{(7c_2(X)^2 - 4c_4(X))C_X}{b}}$ irrational. By Lemma 2.24, we have:

$$s(K'_4) = -45 - 3 \times 2 = -51.$$ 

By Proposition 2.22 we obtain:

$$\chi(K'_4) = 48 + 12 \times 6 - 51 = 69.$$ 

By Proposition 2.26

$$c_4(K'_4) = 69 - \frac{45}{2} - \frac{2 \times 3}{4} = 45.$$ 

39
By Lemma 2.28
\[ S_0(K'_4) = \frac{45}{2^5} + \frac{18}{2^6} = \frac{27}{16}. \]

Therefore by Proposition 2.27
\[ c_2(K'_4)^2 = 720 - \frac{240 \times 27}{16} + 15 = 330. \]

Since \( C_{K'_4}(I) = 3^2 \), by Proposition 5.21
\[ C_{K'_4} = y^2, \]

with \( y \) a rational number. Therefore:
\[ \frac{(7c_2(K'_4)^2 - 4c_4(K'_4))C_{K'_4}}{15} = \frac{(7 \times 330 - 4 \times 45)y^2}{15} = 142y^2. \]

By Proposition 4.37 \( \sqrt{142} \) is a rational number; this leads to a contradiction. It proves that the computation of the singularities of \( K'_4 \) in [13, Sections 5.4] is incorrect. The correct computation is provided in [6, Appendix A].

Remark 4.42. Let \( D_6 \) be the dihedral group of order 6. With the same method, we find that the computations of the singularities of \( S(D_6)^{[2]} \) in [13, Section 5.7] and of \( S(C_4)^{[2]} \), \( S(C_2 \times C_3)^{[2]} \), \( S(C_2 \times C_6)^{[2]} \) in [14, Section 13] are incorrect. The correct computations are provided in Section 5.

5 Several examples in dimension 4

5.1 Overview of the section

The purpose of this section is to list all the examples of deformation classes of IHS orbifolds \( S(G)^{[2]} \) such that \((S, G)\) verifies Hypothesis 4.6. The main difficulty is to find all the possible valid involutions \( \theta \) for each possible groups \( G \). To do so we express all groups \( G \) as a subgroup of a symmetric group \( \mathfrak{S}_n \) and the involutions on \( G \) are obtained via conjugation by an element of order 2 in \( \mathfrak{S}_n \) (see Lemma 5.1).

The possible groups \( G \) are classified in [36], we will follow this list. Then, all the invariants of \( S(G)^{[2]} \) are computing using the results of the previous sections.

The Section is organized as follows. We first provide the main results: Lemma 5.1 where all classes of valid involutions are described and Theorem 5.6 which provides all our examples of orbifolds in dimension 4. In Section 5.3, we explain how the Python programs are used to prove our main results. Section 5.4 is devoted to the proof of Lemma 5.1. Finally in Section 5.5, we show that some of the orbifolds obtained in Theorem 5.6 are deformation equivalent.

5.2 Main results

Every groups \( G \) are embedded in a permutation group \( \mathfrak{S}_n \); this embedding is described by giving a family of generators of \( G \) in \( \mathfrak{S}_n \). Then the involutions on \( G \) are described via an element of order 2 in \( \mathfrak{S}_n \); the involution is obtained by the action by conjugation of this element on \( G \). For instance \( \mathfrak{A}_4 \) is embedded in \( \mathfrak{S}_4 \) via the family of generators \( \{(0, 1, 2), (1, 2, 3)\} \). Then the transposition \((1, 2)\) induced an involution on \( \mathfrak{A}_4 \): \( \theta(g) = (1, 2) \circ g \circ (1, 2) \).

We recall that the equivalent relation between involutions on a finite symplectic group \( G \) is stated in Definition 3.22.
Lemma 5.1. Let $G$ be a finite admissible non-abelian symplectic automorphism group on a K3 surface. The list of equivalent classes of valid involutions on $G$ is given by the following representatives:

| $G$          | $n$ | Embedding in $S_n$ | classes of valid involutions                       |
|-------------|-----|--------------------|----------------------------------------------------|
| $S_3$       | 3   | trivial            | $\text{id}$                                        |
| $D_4$       | 4   | $(0,1,2,3);(0,3)(1,2)$ | $\text{id}$                                        |
| $A_4$       | 4   | $(0,1,2);(1,2,3)$   | $(1,2)$                                            |
| $D_6$       | 6   | $(0,1,2,3,4,5);(0,5)(1,4)(2,3)$ | $\text{id}$                                        |
| $C_2 \times D_4$ | 6   | $(0,1,2,3);(0,3)(1,2);(4,5)$ | $\text{id}$                                        |
| $C_2^2 \times C_4$ | 7   | $(0,1,2,7)(3,4,5,6);(0,4)(2,6)$ | $(0,6)(2,4)$                                        |
| $A_{3,3}$   | 9   | $(0,1,8)(2,3,4)(5,6,7);(0,1)(2,5)(3,7)(4,6);(3,6)(1,4,7)(2,5,8)$ | $\text{id and } (0,5)(1,2)(3,7)$                   |
| $C_3 \times S_3$ | 6   | $(0,1,2);(0,1);(3,4,5)$ | $(3,4)$                                            |
| $S_4$       | 4   | trivial            | $\text{id}$                                        |
| $S_2 \times A_4$ | 6   | $(0,1,2);(1,2,3);(4,5)$ | $(1,2)$                                            |
| $C_2^2 \times C_2$ | 8   | $(0,1);(2,3);(4,5);(6,7);(0,4)(1,5)(2,6)(3,7)$ | $\text{id and } (2,6)(3,7)$                         |
| $C_2^2 \times C_2$ | 8   | $(0,1,2,7)(3,4,5,6);(1,5)(2,6)$ | $(1,3)(5,7)$                                        |
| $C_2^3 \times C_4$ | 6   | $(0,3,4,1)(2,5);(1,3,5);(0,4)(1,3)$ | $(0,2)$                                            |
| $C_3 \times A_4$ | 7   | $(0,1,2);(1,2,3);(4,5,6)$ | $(1,2)(4,6)$                                        |
| $S_2 \times S_3$ | 6   | trivial            | $\text{id}$                                        |
| $C_2^2 \times A_4$ | 12  | $(0,9)(2,11)(3,6)(5,8);(0,6)(2,8)(3,9)(5,11);(0,4,8)(1,5,9)(2,6,10)(3,7,11)$ | $(0,4)(1,9)(3,7)(6,10)$                             |
| $C_2^4 \times C_4$ | 12  | $(0,9,6,3)(2,5,8,11);(0,4,8)(1,5,9)(2,6,10)(3,7,11)$ | $(1,8)(2,7)(4,11)(5,10)$                            |
| $S_3 \times S_3$ | 6   | trivial            | $\text{id}$                                        |
| $C_2^2 \times A_4$ | 8   | $(0,1,2)(1,2,3)(4,5)(6,7)$ | $(1,2)$                                            |
| $S_3 \times C_2$ | 8   | $(0,1)(2,4)(5,6);(0,1,8)(2,3,4)(5,6,7);(0,7)(2,5,6)(1,4,7)(2,5,8)$ | $\text{id}$                                        |
| $C_2^2 \times S_3$ | 8   | $(0,7)(1,2)(0,1,2)(4,5,6);(0,4)(1,6)(2,5,3,7)$ | $\text{id and } (1,7)(3,5)$                         |
| $C_2 \times C_6$ | 8   | $(0,2)(1,7)(3,5)(4,6);(3,5)(4,6);(0,7)(1,2)(3,4)(5,6);(0,4)(1,5)(2,6)(3,7)$ | $(0,1)(4,5)$                                        |
| $A_4^2$     | 8   | $(0,1,2);(1,2,3);(4,5,6);(5,6,7)$ | $(1,2)(5,6)$                                        |

The proof of Lemma 5.1 will be given in Section 5.4. The group names are explained in Section 4.

Remark 5.2. Note that when $G$ is abelian, there is only one valid involution which is given by the inverse: $g \mapsto g^{-1}$.

Remark 5.3. We remark in Lemma 5.1 that for most of the groups, there is only one class of valid involutions. There are two classes only for the groups $A_{3,3}$, $C_2 \times C_2$ and $C_2^2 \times S_3$. Moreover, when there are two classes of valid involutions, one is always the class of id.

Notation 5.4. As a consequence of Remark 5.3 and to simplify the notation, we adopt the following convention. When there is only one class $\theta$ of valid involutions on the group $G$, the orbifold $S(G)^{[2]}_\theta$ will be denoted by $S(G)^{[2]}$. When there are two classes of valid involutions on $G$ and one of them is the class of id, the two associated orbifolds will be denoted by $S(G)^{[2]}_\text{id}$ and $S(G)^{[2]}_{\text{id}}$.

We deduce from Lemma 5.1 all the possible classes of Fujiki orbifolds obtained from an admissible group $G$. In particular, we provide the singularities and the Betti numbers of these orbifolds. We start with the third Betti number.

Proposition 5.5. Let $S$ be a K3 surface and $G$ a finite admissible symplectic automorphism group on $S$. Then:

\[ b_3 \left( S(G)^{[2]}_\theta \right) = 0. \]

Proof. According to Lemma 4.13 the fixed surfaces by the action of $G$ on $S^2$ are given by

\[ \{(x,g(x))| \ x \in S\}, \]
with $g \in F$. Therefore all the fixed surfaces are isomorphic to $S$; hence with a trivial third Betti number. Then, we conclude the proof by applying [14, Lemma 7.11]. Note that "admissible singularities" in [14] are the singularities described in Section A.2.

Using Lemma 5.1 and the programs of Section A.1 we obtain the following theorem. For each orbifold, we provide the second Betti number $b_2$, the singularities, the forth Betti number $b_4$, the Euler characteristic $\chi$, the Chern numbers $c_3, c_2^2$ and $C(c_2)$ defined in Notation 4.30. The different type of singularities are defined in Notation 4.27.

**Theorem 5.6.** The different Fujiki orbifolds $S(G)_{[2]}$ (modulo deformation), obtained from a finite admissible symplectic group $G$ on a K3 surface $S$, are listed below.

| $G, \theta$ | $b_2$ | singularities | $b_4$ | $\chi$ | $c_4$ | $c_2^2$ | $C(c_2)$ |
|-------------|------|---------------|------|-------|------|-------|---------|
| $C_2$       | 16   | $a_2 = 28$    | 178  | 212   | 198  | 576   | 36      |
| $C_3$       | 11   | $a_3 = 15$    | 126  | 150   | 140  | 500   | 42      |
| $C_2^2$     | 14   | $a_2 = 36$    | 150  | 180   | 162  | 504   | 48      |
| $C_4$       | 10   | $a_2 = 10, a_4 = 6$ | 118 | 140   | 130.5 | 486   | 48      |
| $E_3$       | 10   | $a_2 = 28, a_3 = 12$ | 94  | 116   | 94   | 328   | 48      |
| $C_6$       | 8    | $a_2 = 9, a_3 = 3, \theta = 1$ | 80  | 118   | 320  | 137   | 60      |
| $C_2^2$     | 16   | $a_2 = 28$    | 178  | 212   | 198  | 576   | 72      |
| $D_4$       | 14   | $a_2 = 36, a_4 = 3$ | 111 | 135   | 114.75 | 387  | 60      |
| $C_2 \times C_4$ | 10  | $a_2 = 12, a_4 = 4$ | 122 | 144   | 135  | 540   | 72      |
| $C_2^2$     | 7    | $a_3 = 12$    | 92   | 108   | 100  | 540   | 78      |
| $A_4$       | 7    | $a_2 = 12, a_3 = 15, a_4 = 4$ | 62  | 78    | 59   | 248   | 60      |
| $D_6$       | 10   | $a_2 = 28, a_3 = 10$ | 98  | 120   | 114  | 720   | 72      |
| $C_2 \times C_6$ | 8    | $a_2 = 12, a_3 = 3$ | 108 | 126   | 118  | 616   | 96      |
| $C_2 \times D_4$ | 14   | $a_2 = 36$    | 150  | 180   | 162  | 504   | 96      |
| $C_2^2 \times C_4$ | 10  | $a_2 = 10, a_4 = 6$ | 118 | 140   | 130.5 | 486  | 96      |
| $C_2$       | 8    | $a_2 = 6$     | 120  | 138   | 135  | 720   | 72      |
| $A_{3,3}, id$ | 8    | $a_2 = 28, a_3 = 12$ | 74  | 92    | 70   | 320   | 60      |
| $A_{3,3}, \not{id}$ | 8   | $a_2 = 28, a_3 = 20$ | 58  | 76    | 32   | 320   | 60      |
| $C_2 \times D_3$ | 6    | $a_2 = 9, a_3 = 10, a_6 = 1$ | 72  | 86    | 74   | 408   | 60      |
| $C_2 \times A_4$ | 5    | $a_2 = 24, a_3 = 12, a_4 = 3, a_6 = 1$ | 69  | 87    | 64.75 | 247   | 72      |
| $C_2 \times C_2, id$ | 16  | $a_2 = 28$    | 178  | 212   | 198  | 576   | 144     |
| $C_2^2 \times C_2, \not{id}$ | 14  | $a_2 = 36, a_4 = 3$ | 111 | 155   | 114.75 | 387  | 120     |
| $C_2 \times C_4$ | 8    | $a_2 = 17, a_4 = 6, b_4 = 1$ | 87  | 105   | 124  | 373.5 | 120     |
| $C_2 \times A_4$ | 6    | $a_2 = 10, a_3 = 14, a_4 = 6$ | 50  | 64    | 74   | 320   | 96      |
| $C_2 \times C_2 \times A_4$ | 5  | $a_2 = 3, a_3 = 9, a_3 = 3, b_4 = 1$ | 61  | 73    | 81   | 320   | 132     |
| $C_2^2$     | 14   | $a_2 = 20, a_3 = 15$ | 76  | 94    | 74   | 328   | 120     |
| $C_2^2 \times A_4$ | 6    | $a_2 = 14, a_3 = 15, a_4 = 6$ | 44  | 58    | 36.5 | 158   | 96      |
| $C_2 \times C_3$ | 5    | $a_2 = 10, a_3 = 15, a_4 = 1, a_3 = 2, b_4 = 1$ | 35  | 47    | 244  | 153.5 | 96      |
| $C_2 \times C_3$ | 5    | $a_2 = 10, a_3 = 15, a_4 = 1, a_3 = 2, b_4 = 1$ | 35  | 47    | 244  | 153.5 | 96      |
| $C_2 \times A_3$ | 10   | $a_2 = 28, a_3 = 10$ | 98  | 120   | 114  | 720   | 144     |
| $C_2^2 \times A_4$ | 7    | $a_2 = 12, a_3 = 3, a_4 = 4$ | 86  | 102   | 91   | 472   | 168     |
| $C_2^2 \times C_2$ | 7    | $a_2 = 20, a_3 = 12, a_4 = 3$ | 63  | 79    | 275  | 156    |
| $C_2^2 \times A_3, id$ | 10  | $a_2 = 28, a_3 = 12$ | 94  | 116   | 94   | 328   | 192     |
| $C_2^2 \times A_3, \not{id}$ | 6   | $a_2 = 19, a_3 = 12, a_4 = 6, b_4 = 1$ | 41  | 55    | 244  | 125.5 | 120     |
| $C_2^2 \times C_6$ | 5    | $a_2 = 16, a_3 = 6, a_4 = 4, a_6 = 1, b_4 = 1$ | 47  | 59    | 1535 | 125.5 | 168     |
| $A_3$       | 4    | $a_2 = 4, a_3 = 6, a_4 = 4, b_6 = 2$ | 48  | 58    | 1535 | 240    |

**Proof.** We explain formally how to compute the data of the theorem and we explain how to use the Python program in Section 5.3.

**Second Betti number**

The second Betti number is computed directly from Proposition 5.15 and Program A.5. Program A.5 provides the factor $\#(F/G)$ of Proposition 5.15. The factors $\text{rk} H^2(S, \mathbb{Z})^G$ are found in 36.
for each $G$.

The singularities

The singularities are obtained from Propositions 4.33, 4.35, 4.36 and Program A.6. According to Proposition 4.32, the singularities of $S(G)^{[2]}_b$ are expressed in terms of $a_2, a_3, a_4, a_6, a_8, b_4$ and $b_6$ (in practice, we realize that the singularities of type $a_{12}$ never appear; so for simplicity in the computation, we omit it).

The fourth Betti number and the Euler characteristic

We set $b_4(X) = 46 + 10b_2(X) + s(X)$ and $\chi(X) = 48 + 12b_2(X) + s(X)$. Then by Definition of $s(X)$ (see Notation 2.21) and Lemmas 2.24, 2.25

$$b_4(X) = 46 + 10b_2(X) - 2a_2 - 3a_4 - 5a_6 - 7a_8 - 4b_4 - 5b_6$$

and:

$$\chi(X) = 48 + 12b_2(X) - 2a_2 - 3a_4 - 5a_6 - 7a_8 - 4b_4 - 5b_6.$$ (32)

The Chern numbers $c_4$ and $c_2^2$

By Proposition 2.26

$$c_4(X) = \chi(X) - \frac{a_2}{2} - \frac{2a_3}{3} - \frac{3a_4}{4} - \frac{5a_6}{6} - \frac{7a_8}{8} - \frac{7b_4}{8} - \frac{11b_6}{12}. \quad (34)$$

Moreover by Proposition 2.26

$$c_2(X)^2 = 720 + \frac{c_4(X)}{3} - 240 \left( \frac{a_2}{2^5} + \frac{2a_3}{2^7} + \frac{3a_4}{2^6} + \frac{9a_6}{864} + \frac{41a_8}{2^7} + \frac{25b_4}{2^7} + \frac{545b_6}{1728} \right). \quad (35)$$

Remark 5.7. The orbifold $S(C^4_2)^{[2]}$ has been omitted since it is deformation equivalent to $S^{[2]}$ as explained in [14, Proposition 14.5] (see Proposition 5.11).

Remark 5.8. Note that Theorem 5.6 corrects some mistakes related to the singularities in [14, Theorem 13.1]. Indeed, the singularities of $S(C_6)^{[2]}$, $S(C_2 \times C_3)^{[2]}$, $S(C_2 \times C_6)^{[2]}$ and $S(C^4_2)^{[2]}$ provided in [14, Theorem 13.1] are incorrect. If we compute $\mathcal{O}(c_2)$ with the data of [14, Theorem 13.1], we find respectively:

$$\sqrt{3720}, \ 16\sqrt{19}, \ 36\sqrt{7}, \text{ and } 8\sqrt{210},$$

which are irrational numbers.

As a consequence of Proposition 5.8, we have the following proposition.

Proposition 5.9. The Betti numbers and the singularities of an irreducible symplectic orbifold are invariant under deformation.

Remark 5.10. Therefore, we know that two orbifolds $S(G_1)^{[2]}_b$ and $S(G_2)^{[2]}_b$ are not deformation equivalent when their second Betti number or their singularities are different. It follows that Theorem 5.6 provides at least 29 deformation classes of irreducible symplectic orbifolds.

In Section 5.10, we are also going to show that some of the orbifolds listed in Theorem 5.6 are deformation equivalent.

Remark 5.11. Let $S$ be a K3 surface endowed with a finite admissible symplectic automorphism group $G$. All the results of Section 4 and all the methods described in this section still applied to $S(G)^{[2]}_b$ when the involution $\theta$ is not valid. Therefore, we could also provide all the examples of Fujiki primitively symplectic orbifolds obtained from a finite admissible symplectic automorphism group $G$ endowed with a non-valid involution.
5.3 Use of the programs

In this section, we will describe all the methods to study the orbifolds $S(G)^{(2)}$ for a given $G$ using our Python programs. In particular, we will provide the Python commands for using the programs in Section A. To help the understanding, we will illustrate the explanations by studying a precise example of group. We will consider the group $C_2^2 \rtimes C_4$, denoted in [36] by $\Gamma_{2c1}$, and also called SmallGroup(16,3). Since the method is similar for all groups, the other examples in Section 5.4 will be less detailed.

Embedding in a permutation group

All our programs are based on permutation groups. Therefore a first step is to find a permutation representation for each groups. Each group will be given via a list of generators in a permutation group. To do so we use the two data bases [12] and [34]. For instance, in [12], we find that the group $C_2^2 \rtimes C_4$ has a permutation representation with 8 elements which is $8T_{10}$. Then in [34], we can find the following generators for $8T_{10}$:

\[
C_{2p2C4} = \{\text{Permutation}(0,1,2,7)(3,4,5,6), \text{Permutation}(7)(0,4)(2,6)\}
\]

Finding the equivalence classes of valid involutions: first method

This method is efficient when the group is small enough.

A valid involution can also be seen via a set of generators. Indeed if $\theta$ is a valid involution, there exists a set $\{g_1, ..., g_k\}$ of generators such that $\theta(g_i) = g_i^{-1}$ for all $i \in \{1, ..., k\}$. Therefore the involution $\theta$ is fully defined by the set $\{g_1, ..., g_k\}$. Program A.2 is based on this remark. In particular, it uses a direct consequence of the Tarski irredudant Basis Theorem.

**Definition 5.12.** Let $G$ be a finite group. Let $B \subset G$ be a subset. We say that $B$ is a basis of $G$ if:

(i) $\langle B \rangle = G$ and;

(ii) none of the proper subsets of $B$ generate $G$.

To shorten, a basis is a minimal set of generators.

**Theorem 5.13** (Tarski irredundant Basis Theorem). Let $G$ be a finite group. Let $d(G)$ and $m(G)$ be respectively the the smallest and the largest cardinality for a basis of $G$. Then for all $d(G) \leq k \leq m(G)$, there exists a basis of $G$ of cardinality $k$.

Program A.2 follows several steps:

- We find all the family of generators of the group $G$ using Theorem 5.13. This is the function `generators`

- Among these families of generators we select the ones that provide a well defined involution on $G$. We also eliminate the redundancy: if two family of generators provide the same involution, we consider only one of them. There are the functions `bijection` and `involutions`

- Finally, we find the valid involutions which are equivalent. This is the function `classes_valid_involution2`

based on Proposition 3.24.
For instance with the group $C_2^2 \rtimes C_4$. We have:

```python
>>> P=involutions(C2p2C4)
>>> P
[[Permutation(0, 1, 2, 7)(3, 4, 5, 6), Permutation(7)(0, 4)(2, 6)], [Permutation(0, 3, 6, 1)(2, 5, 4, 7), Permutation(7)(0, 4)(2, 6)]]
```

We find two different valid involutions $\theta_1$ and $\theta_2$ each given by a family of generators. Then, we apply the function `classes_valid_involution2` with the group $\tilde{G} = G$ (see Proposition 3.24). The program will search two elements $h_1$ and $h_2$ in $\tilde{G} = G$ that verify the conditions of Proposition 3.24. The function `classes_valid_involution2` takes as first argument the list of valid involution that we want to test, as second argument the generators of $G$ and as third argument the generators of $\tilde{G}$.

```python
>>> classes_valid_involution2(P,C2p2C4,C2p2C4)
[[Permutation(0, 1, 2, 7)(3, 4, 5, 6), Permutation(7)(0, 4)(2, 6)]]
```

We obtain only one class of valid involution. In Section 5.24 when we do not provide any precisions, it means that applying the function `classes_valid_involution2` with $\tilde{G} = G$ is enough to conclude; in most of the cases, it will be so.

### Finding the equivalence classes of valid involutions: second method

The next method is the most efficient when the group $G$ is with a trivial center and with a small enough $\text{Aut}(G)$. The objective of this method is to see involution as conjugation by an element of order 2. We have the following exact sequence:

$$
0 \longrightarrow Z(G) \longrightarrow G \xrightarrow{\Phi} \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 0,
$$

with $\Phi(g)(h) = ghg^{-1}$. Assume that $Z(G) = \{\text{id}\}$ and we can find a permutation representation such that: $G \hookrightarrow \text{Aut}(G) \hookrightarrow \mathfrak{S}_n$. Then all involutions on $G$ can be expressed by the conjugation by an element of $\mathfrak{S}_n$. Indeed, let $f \in \text{Aut}(G)$, then $f \circ \Phi(g) \circ f^{-1} = \Phi(f(g))$. Based on these remarks, Program A.1 is constructed as follows.

- First, we consider all the elements of order 2 in $\mathfrak{S}_n$ and we select the ones which correspond to valid involutions on $G$ (if we know a family of generators of $\text{Aut}(G)$ we can only consider the elements of order 2 in $\text{Aut}(G)$). This is given by the function:

  ```python
  valid_involution
  ```

- Second, we find the valid involutions which are equivalent using the same method, used for Program A.2, based on Prop 3.23. This is given by the function:

  ```python
  classes_valid_involution
  ```

This method is not very well adapted to the group $C_2^2 \rtimes C_4$ because it is a group with a non-trivial center. Indeed, if a group $G$ as a center there is a risk to miss involutions with a non-trivial action on $Z(G)$. However in the case of $C_2^2 \rtimes C_4$, we already know that we only have one class of valid involution; therefore we can still apply this method in order to have a simpler expression for our valid involution. The function `valid_involution` takes as first argument a family of generators of $G$ and as second argument the integer $n$ which specify that $G$ is embedded in $\mathfrak{S}_n$. The function `classes_valid_involution` take as first argument a family of generators of $G$, as second argument a family of generators of $\tilde{G}$, as third argument the list of involutions that we want to test and as fifth argument the integer $n$. We apply the function with $\tilde{G} = G = C_2^2 \rtimes C_4$ and we find one class of valid involutions.

```python
>>> P=valid_involutions(C2p2C4,8)
```
This second method is very well adapted for groups as $\mathfrak{S}_n$ or $\mathfrak{A}_n$, since:

$$\text{Aut}(\mathfrak{S}_n) = \mathfrak{S}_n \text{ and } \text{Aut}(\mathfrak{A}_n) = \mathfrak{S}_n,$$

for $n \notin \{1, 2, 6\}$. This second method can also be used for some groups with non-trivial center using the following trick.

**Proposition 5.14.** We assume that $G = A \times G'$ with $A$ a finite abelian group and $G'$ a finite group. We denote by $\text{inv}_A$ the unique valid involution on $A$. Then all valid involutions on $G$ can be written $\text{inv}_A \times \theta'$, where $\theta'$ is a valid involution on $G'$.

**Proof.** To simplify the notation, we identify $A$ with $A \times \{\text{id}\}$ and $G'$ with $\{\text{id}\} \times G'$. Let $\theta$ be a valid involution on $G$. Therefore, there exists a family of generators $(f_1, \ldots, f_k)$ such that $\theta(f_j) = f_j^{-1}$ for all $j \in \{1, \ldots, k\}$. All this generators can be written $f_j = a_j f'_j$ with $a_j \in A$ and $f'_j \in G'$. Let $a \in A$; we can write $a$ as a product of elements of $(f_1, \ldots, f_k)$:

$$a = \prod f_j = \prod a_j \prod f'_j = \prod a_j,$$

because, necessarily $\prod f'_j = \text{id}$. Moreover:

$$\theta(a) = \prod f_j^{-1} = \prod a_j^{-1} \prod f'_j^{-1} = \prod a_j^{-1} = a^{-1}.$$

Hence $\theta|_A = \text{inv}_A$. Necessarily, $(f'_1, \ldots, f'_k)$ is a family of generators of $G'$. Furthermore, we have:

$$a_j^{-1} f_j^{-1} = f_j^{-1} = \theta(f_j) = \theta(a_j) \theta(f'_j) = a_j^{-1} \theta(f'_j).$$

That is $\theta(f'_j) = f_j^{-1}$. Hence $\theta|_{G'}$ is a valid involution. \(\square\)

**Finding the second Betti number**

To find the second Betti number, we use Program A.5 based on Proposition 8.15. Program A.5 provides the number of exceptional divisors of $S(G)/[2] \to S \times S/\langle j_0(G), \mathfrak{S}_2 \rangle$, which is the factor $\#(F/G)$ in Proposition 4.15 (see also Section 3.2 for the notation). Then $\text{rk } H^2(S, \mathbb{Z})G$ is found in \[36\].

```python
>>> T=Permutation(7)(0, 6)(2, 4)
>>> div(C2p2C4,T)
5
```

In \[36\], we find that $\text{rk } H^2(S, \mathbb{Z})C_2^2 \rtimes C_4 = 5$. Hence, we obtain:

$$b_2 \left( S(C_2^2 \rtimes C_4)/[2] \right) = 5 + 5 = 10.$$

**Finding the singularities**

To find the singularities we use Program A.6. The function "singularities" return a table $[a_2, a_3, a_4, a_6, a_8, b_4, b_6]$.

```python
>>> singularities(C2p2C4,T)
[10,0,6,0,0,0,0]
```

This means that $S(C_2^2 \rtimes C_4)/[2]$ has 10 singularities of type $C_4 / -\text{id}$ and 6 of type $C_4 / \text{diag}(i, -i, i, -i)$.

**Finding the other data**

We obtain the remaining data as explained in the proof of Theorem 5.6. Since, it is easy computation, we do not provide a program for that. When $G = C_2^2 \rtimes C_4$, we obtain:

$$b_4 = 118, \; \chi = 140, \; c_4 = 130.5, \; c_2^2 = 486 \text{ and } C(c_2) = 96.$$
5.4 Proof of Lemma 5.1

The case of the group $G = C_2^2 \rtimes C_4$ has already been treated in Section 5.3.

When $G = D_4$, $D_6$ or $C_2 \times D_4$

We apply Program A.2 to find that all valid involutions are equivalent to $\text{id}$.

When $G = S_3$ or $S_4$

Since $S_3$ or $S_4$ can be generated by transpositions, $\text{id}$ is valid. Moreover, by Corollary 3.25 it is the only valid involution modulo equivalence.

When $G = C_3 \times S_3$, $C_2 \times S_4$, $C_2 \times A_4$, $C_3 \times A_4$ or $C_2^2 \times A_4$

The proofs for all these groups are identical so we provide the proof only for $C_3 \times S_3$. According to Proposition 5.14 we know that all valid involutions on $C_3 \times S_3$ will be of the form $\text{inv}_{C_3 \times \theta'}$, with $\theta'$ a valid involution on $S_3$. In particular, they can all be obtained by conjugation by an element in $C_3 \times S_3 \to S_6$. Then, we can apply Program A.1 to see that there is only one class of valid involution.

When $G = S_2^3$

The group $\text{Aut}(S_2^3)$ is given by $S_3 \rtimes C_2$; that is the group generated by the automorphisms on each of the two factors $S_3$ and by an automorphism which exchange the two factors $S_3$. Therefore, we have an embedding:

$$S_3^3 \to \text{Aut}(S_3^3) \to S_6.$$ 

Hence, we can apply Program A.1 as explained in Section 5.3 to find that there is only one class of valid involution on $S_3^3$. Moreover, since $S_3^3$ is generated by transpositions, $\text{id}$ is a valid involution.

When $G = C_3^2 \times C_4$

Note that $C_3^2 \rtimes C_4$ is the faithful semi-direct product between $C_3^2$ and $C_4$. According to the data bases [12] and [34], the group $C_3^2 \times C_4$ can be embedded in $S_8$ via the following generators.

$C_2^pC_4 = \{\text{Permutation}(0, 1, 2, 7)(3, 4, 5, 6), \text{Permutation}(7)(1, 5)(2, 6)\}$

We apply Program A.2 and we find one class of valid involution. We can also apply Program A.1 to obtain a valid involution via conjugation by an element of $S_8$; we find:

Permutation(1, 3)(5, 7)

When $G = C_3^2 \times C_4$

We consider $C_3^2 \rtimes C_4$, the faithful semi-direct product between $C_3^2$ and $C_4$. According to the data bases [12] and [34], the group $C_3^2 \times C_4$ can be embedded in $S_8$ via the following generators.

$C_3^pC_4 = \{\text{Permutation}(0, 3, 4, 1)(2, 5), \text{Permutation}(1, 3, 5), \text{Permutation}(5)(0, 4)(1, 3)\}$

As before, we apply Program A.2 and we find one class of valid involution. For more convenience, an example of valid involution via conjugation by an element of $S_8$ can be found using Program A.1.

Permutation(5)(0, 2)
When $G = S_3 \wr C_2$

We consider $S_3 \wr C_2$ the wreath product of $S_3$ and $C_2$. In [12], we can find that $\text{Aut}(S_3 \wr C_2) = \text{AΓL}_1(F_9)$. Considering also [34], we find that this group can be embedded in $S_9$ via the following generators.

\[
\text{AutN72} = [\text{Permutation}(8)(0,5,3,4,1,2,7,6), \text{Permutation}(8)(0,1)(2,4)(5,6), \text{Permutation}(0,1,8)(2,3,4)(5,6,7), \text{Permutation}(0,3,6)(1,4,7)(2,5,8)]
\]

Moreover, [12] and [34] also provide an embedding of $S_3 \wr C_2$ in $S_9$ given by the following generators.

\[
\text{N72} = [\text{Permutation}(8)(0,1)(2,4)(5,6), \text{Permutation}(8)(0,7)(1,3)(4,6), \text{Permutation}(0,1,8)(2,3,4)(5,6,7), \text{Permutation}(0,3,6)(1,4,7)(2,5,8)]
\]

We can verify with Python that these two embedding in $S_9$ are compatible, i.e. we have:

\[
S_3 \wr C_2 \hookrightarrow \text{Aut}(S_3 \wr C_2) \hookrightarrow S_9.
\]

>>> G=PermutationGroup(AutN72)
>>> H=PermutationGroup(N72)
>>> H.is_normal(G)
True

Hence, we can use Program A.1 as explained in Section 5.3 to verify that there is only one class of valid involution on $S_3 \wr C_2$. Moreover, since $S_3 \wr C_2$ is generated by elements of order 2, $id$ is a valid involution.

When $G = C_4^4 \rtimes C_6$

We consider $\text{SmallGroup}(96, 70)$ which is a faithful semi-direct product between $C_4^4$ and $C_6$.

The proof is similar to the one for $S_3 \wr C_2$; using Python, [12] and [34], we can find an embedding:

\[
C_4^4 \rtimes C_6 \hookrightarrow \text{Aut}(C_4^4 \rtimes C_6) \hookrightarrow S_{12},
\]

with $C_4^4 \rtimes C_6$ normal in $\text{Aut}(C_4^4 \rtimes C_6)$.

\[
\text{C2p4C6} = [\text{Permutation}(0,2,4,6,8,10)(1,3,5,7,9,11), \text{Permutation}(0,5,11,6)(1,7,2,8)(3,9)(4,10)]
\]

\[
\text{AutC2p4C6} = [\text{Permutation}(0,11)(1,2), \text{Permutation}(0,2,4,6,8,10)(1,3,5,7,9,11), \text{Permutation}(0,10)(1,7)(2,8)(3,5)(4,6)(9,11)]
\]

Then, we use Program A.1 as explained in Section 5.3 to show that there is only one class of valid involution on $C_4^4 \rtimes C_6$. When we know that there is only one class of valid involution, it can be more convenient to express a valid involution as conjugation by an element of $S_8$ instead of $S_{12}$. The group $C_4^4 \rtimes C_6$ can be embedded in $S_8$ with the following generators:

\[
\text{C2p4C6} = [\text{Permutation}(0,2)(1,7)(3,5)(4,6), \text{Permutation}(7)(3,5)(4,6), \text{Permutation}(0,7)(1,2)(3,4)(5,6), \text{Permutation}(7)(0,1,2)(3,5,4), \text{Permutation}(0,4)(1,5)(2,6)(3,7)]
\]

Then, a valid involution is given by the conjugation action of:

\[
\text{Permutation}(7)(0, 1)(4, 5)
\]

When $G = \mathfrak{A}_4^2$

The proof is very similar to the one for $\mathfrak{A}_4^3$. The group $\text{Aut}(\mathfrak{A}_4^2)$ is given by $\mathfrak{A}_4 \wr C_2$; that is the group generated by the automorphisms on each of the two factors $\mathfrak{A}_4$ and by an automorphism

48
which exchange the two factors $\mathfrak{A}_4$. Therefore, we have an embedding:

$$\mathfrak{A}_4^2 \hookrightarrow \text{Aut}(\mathfrak{A}_4^2) \hookrightarrow S_8.$$  

Hence, we can apply Program A.1 as explained in Section 5.3 to find that there is only one class of valid involution on $\mathfrak{A}_4^2$. The program provides the following representative.

\[
\text{Permutation}(1, 2)(5, 6)
\]

**When $G = \mathfrak{A}_{3,3}$**

This is the first case which is more complicated. The group $\text{Aut}(\mathfrak{A}_{3,3})$ is given by $\text{AGL}_2(\mathbb{F}_3)$. According to [12] and [34], we can find an embedding

$$\mathfrak{A}_{3,3} \hookrightarrow \text{Aut}(\mathfrak{A}_{3,3}) \hookrightarrow S_9,$$

obtained with the following generators:

\[
\begin{align*}
\mathfrak{A}_{33} &= \{\text{Permutation}(0,1,8)(2,3,4)(5,6,7), \text{Permutation}(8)(0,1)(2,5)(3,7)(4,6), \\
&\quad \text{Permutation}(0,3,6)(1,4,7)(2,5,8)\}, \\
\text{Aut}\mathfrak{A}_{33} &= \{\text{Permutation}(0,5,3,4,1,2,7,6), \text{Permutation}(0,1,8)(2,3,4)(5,6,7), \\
&\quad \text{Permutation}(2,3,4)(5,7,6), \text{Permutation}(0,3,6)(1,4,7)(2,5,8)\}.
\end{align*}
\]

If we apply Program A.2 with $\tilde{G} = G = \mathfrak{A}_{3,3}$, we find 7 possible classes of valid involutions (see Proposition 3.24 for the notation and Section 5.3 for explanations on the programs).

\[
\begin{align*}
&\text{[Permutation}(0, 2, 7)(1, 3, 5)(4, 6, 8), \text{Permutation}(0, 1, 8)(2, 3, 4)(5, 6, 7), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)], \\
&\text{[Permutation}(0, 1, 8)(2, 3, 4)(5, 6, 7), \text{Permutation}(1, 8)(2, 7)(3, 6)(4, 5), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)], \\
&\text{[Permutation}(8)(0, 1)(2, 5)(3, 7)(4, 6), \text{Permutation}(0, 1, 8)(2, 3, 4)(5, 6, 7), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)], \\
&\text{[Permutation}(0, 2)(1, 4)(3, 8)(5, 6), \text{Permutation}(0, 1, 8)(2, 3, 4)(5, 6, 7), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)], \\
&\text{[Permutation}(0, 2, 7)(1, 3, 5)(4, 6, 8), \text{Permutation}(1, 8)(2, 7)(3, 6)(4, 5), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)], \\
&\text{[Permutation}(0, 2, 7)(1, 3, 5)(4, 6, 8), \text{Permutation}(0, 2)(1, 4)(3, 8)(5, 6), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)], \\
&\text{[Permutation}(0, 3, 6)(1, 4, 7)(2, 5, 8), \text{Permutation}(1, 8)(2, 7)(3, 6)(4, 5), \\
&\quad \text{Permutation}(0, 5)(1, 7)(2, 3)(6, 8)]\end{align*}
\]

If we apply again Program A.2 with $\tilde{G} = \text{Aut}(\mathfrak{A}_{3,3})$, we find only two classes of valid involutions.

However, we do not know if $\text{Aut}(\mathfrak{A}_{3,3}) = \text{AGL}_2(\mathbb{F}_3)$ is an automorphism group on a K3 surface. So we cannot choose $\tilde{G} = \text{Aut}(\mathfrak{A}_{3,3})$ a priori. To solve this problem, we will use Program A.3. Let $(\theta_j)_{j \in \{1, \ldots, 7\}}$ be the 7 valid involutions that we find before. With Program A.3 we can find 5 such embeddings for $2 \leq j \leq 6$:

$$\mathfrak{A}_{3,3} \hookrightarrow H_j \hookrightarrow \text{Aut}(\mathfrak{A}_{3,3}),$$

such that $|H_j| = 36$ and applying Proposition 3.24 with $\tilde{G} = H_j$ provide $\theta_j \sim \theta_{j+1}$. Necessarily $H_j$ can only be $S_{23}$ or $C_2^3 \rtimes C_4$ which are both automorphism group on a K3 surface (the only groups of order 36 containing $\mathfrak{A}_{3,3}$ according to [12]). We provide the Python command to find $H_2$.

```python
>>> A=[Permutation(0, 1, 8)(2, 3, 4)(5, 6, 7), Permutation(1, 8)(2, 7)(3, 6)(4, 5), \\
    Permutation(0, 5)(1, 7)(2, 3)(6, 8)]
>>> B=[Permutation(8)(0, 1)(2, 5)(3, 7)(4, 6), Permutation(0, 1, 8)(2, 3, 4)(5, 6, 7), \\
    Permutation(0, 5)(1, 7)(2, 3)(6, 8)]
>>> test(A33bis,AutA33,A,B,36)
Permutation(0, 4, 8, 3, 1, 2)(5, 6, 7)
```
The program return the missing element to generate $H_2$ from $\mathfrak{A}_{3,3}$. That is $H_2$ is generated by:

$$H_2=\{\text{Permutation}(0,1,8)(2,3,4)(5,6,7), \text{Permutation}(8)(0,1)(2,5)(3,7)(4,6), \text{Permutation}(0,3,6)(1,4,7)(2,5,8), \text{Permutation}(0,4,8,3,1,2)(5,6,7)\}$$

after this process, it remains potential two classes of valid involutions. We can see that they are not equivalent by computing the singularities of the associated orbifolds. Since the singularities are different, these two involutions are not equivalent. Since $\mathfrak{A}_{3,3}$ is generated by elements of order $2$, one of the classes is the class of $\text{id}$. For more convenience, we can also use Program A.1, to find a representative of the other class as conjugation by an element of $\mathfrak{S}_9$:

$$\text{Permutation}(8)(0, 5)(1, 2)(3, 7)$$

**When $G = C_2^2 \wr C_2$**

We consider $C_2^2 \wr C_2$ the wreath product of $C_2^2$ with $C_2$.

Via the data bases [12] and [34], we can find the following embeddings with $C_2^4 \ltimes C_6$ the semi-direct product encounter earlier:

$$C_2^2 \wr C_2 \hookrightarrow C_2^4 \ltimes C_6 \hookrightarrow \mathfrak{S}_8.$$  

It is given by the following generators:

$$\text{C2p4C2}=\text{PermutationGroup}([\text{Permutation}(0,2)(1,7)(3,5)(4,6), \text{Permutation}(7)(3,4)(5,6), \text{Permutation}(0,7)(1,2)(3,4)(5,6), \text{Permutation}(0,4)(1,5)(2,6)(3,7)])$$

$$\text{C2p4C6}=\text{PermutationGroup}([\text{Permutation}(0,2)(1,7)(3,5)(4,6), \text{Permutation}(7)(3,5)(4,6), \text{Permutation}(0,7)(1,2)(3,4)(5,6), \text{Permutation}(7)(0,1,2)(3,5,4), \text{Permutation}(0,4)(1,5)(2,6)(3,7)])$$

Then, we apply Proposition 3.24 via Program A.2, with $\tilde{G} = C_2^4 \ltimes C_6$ and we find at most two equivalent classes for valid involutions on $C_2^2 \wr C_2$. We compute the singularities for the two associated Fujiki orbifolds; since we find different singularities, the two valid involutions cannot be equivalent. Since $C_2^2 \wr C_2$ is generated by element of order $2$, one of the two classes of valid involution has $\text{id}$ as a representative. It can be convenient to find a representative of the other valid involution class via conjugation by an element of $\mathfrak{S}_8$. To do so, we can also consider a simpler embedding of $C_2^2 \wr C_2$ in $\mathfrak{S}_8$ given by the following generators:

$$\text{C2p4C2}=[\text{Permutation}(7)(0,1), \text{Permutation}(7)(2,3), \text{Permutation}(7)(4,5), \text{Permutation}(6,7), \text{Permutation}(0,4)(1,5)(2,6)(3,7)]$$

Then using Program A.1, we obtain the other valid involution class via conjugation by element of $\mathfrak{S}_8$:

$$\text{Permutation}(2, 6)(3, 7)$$

**When $G = C_2^4 \rtimes C_3$**

We consider $C_2^4 \rtimes C_3$ which is the faithful semi-direct product between $C_2^4$ and $C_3$.

In this case, if we try to apply Proposition 3.24 with $\tilde{G} = G$, we will find several valid involutions that could still be equivalent. Therefore, we need to consider a bigger $\tilde{G}$. We are going to take for $\tilde{G}$ the group $\tilde{F}$ defined in [29] Section 2, n°5] acting on the Fermat quartic and containing $C_2^4 \rtimes C_3$.

This group is generated by the automorphisms of order $4$: $(x, y, z, t) \mapsto (i^a x, i^b y, i^c z, i^d t)$, with $a, b, c, d \in C_4$ and by the permutations of the coordinates. This group can be embedding in $\mathfrak{S}_{64}$ via its natural action on the set:

$$\{ (i^a : i^b : i^c : 1) \mid (a, b, c) \in C_4^3 \}.$$
We refer to Program A.4 for the realization in Python. We consider an embedding of $C_2^4 \rtimes C_3$ in $\tilde{F}$ (see Program A.4 for an example). Then, we apply Proposition 3.24 via Program A.2 with $\tilde{G} = \tilde{F}$ to find only one class of valid involutions on $C_2^4 \rtimes C_3$.

To find a convenient representative of valid involution on $C_2^4 \rtimes C_3$, it is simpler to embed $C_2^4 \rtimes C_3$ in $S_{12}$. We find some possible generators in [12] with [34]:

$$C_2^4C_3 = \{\text{Permutation}(0,9,6,3,2,5,8,11), \text{Permutation}(0,4,8,1,5,9,2,6,10,3,7,11)\}$$

Then a valid involution can be obtained via conjugation by the following element:

$$\text{Permutation}(1, 8)(2, 7)(4, 11)(5, 10)$$

When $G = C_2^2 \rtimes A_4$

We consider $C_2^2 \rtimes A_4$ the non-trivial semi-direct product between $C_2^2$ and $A_4$.

The proof is similar to the one for $C_2^2 \rtimes C_2$. The group $C_2^2 \rtimes A_4$ can be embedded in $S_{16}$ via the following generators:

$$C_2p2A_4 = \{\text{Permutation}(0,8)(1,9)(2,10)(3,11)(4,14)(5,15)(6,12)(7,13), \text{Permutation}(0,11,14)(1,9,13)(2,8,15)(3,10,12)(4,6,5), \text{Permutation}(0,10)(1,11)(2,8)(3,9)(4,12)(5,13)(6,14)(7,15)\}$$

If we apply Proposition 3.24 via Program A.2 taking $\tilde{G} = G$, we obtain 10 valid involutions ($\theta_i$). To show that there is only one class of valid involutions, we can proceed as follows. We consider the group $(C_2^2 \rtimes A_4) \rtimes A_5$ which can be embedded in $S_{16}$ easily; it is generated by $C_2^2 \rtimes A_4$ and the additional 3 following elements of order 2:

$$T_1 = \text{Permutation}(0,1)(11,9)(14,13)(2,3)(8,10)(15,12), \ T_2 = \text{Permutation}(0,2)(11,8)(14,15)(1,3)(9,10)(13,12), \ T_3 = \text{Permutation}(0,4)(11,6)(14,5)(2,13)(9,15)(1,8)$$

We obtain:

$$C_2^2 \rtimes A_4 \hookrightarrow (C_2^2 \rtimes A_4) \rtimes A_5 \hookrightarrow S_{16}.$$ We have the idea to consider this group because from [12], we know that $\text{Out}(C_2^2 \rtimes A_4) = S_5$; therefore the previous group is easy to construct and close to $\text{Aut}(C_2^2 \rtimes A_4)$.

We cannot apply Proposition 3.24 directly with $\tilde{G} = (C_2^2 \rtimes A_4) \rtimes A_5$ because we do not know if it is an automorphism group on a K3 surface. However, we know by [36], that $C_2^4 \rtimes C_6$ (group already considered before) is a symplectic automorphism group on a K3 surface. Then, we can apply Program A.3 to find 9 embeddings:

$$C_2^2 \rtimes A_4 \hookrightarrow H_i \hookrightarrow (C_2^2 \rtimes A_4) \rtimes A_5,$$

with $H_i \simeq C_2^4 \rtimes C_6$. Via Program A.3, each of these embeddings can be chosen such that applying Proposition 3.24 with $\tilde{G} = H_i$ shows that $\theta_1$ is equivalent to $\theta_i$. Finally, we obtain only one class of valid involution.

The group $C_2^2 \rtimes A_4$ can also be embedded in $S_{12}$ via the following generators:

$$C_2p2A_4 = \{\text{Permutation}(0,9)(2,11)(3,6)(5,8), \text{Permutation}(0,6)(2,8)(3,9)(5,11), \text{Permutation}(0,4,8)(1,5,9)(2,6,10)(3,7,11)\}$$

For simplicity, we can express a valid involution on $C_2^2 \rtimes A_4$ as conjugation by an element of $S_{12}$ using Program A.1:

$$\text{Permutation}(11)(0, 4)(1, 9)(3, 7)(6, 10)$$
When \( G = C_2^4 \rtimes S_3 \)

We consider the group \( \text{SmallGroup}(96, 227) \); it is a semi-direct product of \( C_2^4 \) with \( S_3 \). We will use the same idea as developed in the proof for \( C_2^2 \rtimes C_2 \). According to the data base \([12]\) and \([34]\), we can embed \( C_2^4 \rtimes S_3 \) in \( S_8 \) as follows:

\[
C_2^4 S_3 = \langle \text{Permutation}(0,7)(1,2), \text{Permutation}(7)(0,1,2)(4,5,6), \text{Permutation}(0,4)(1,6)(2,5)(3,7) \rangle
\]

We know by \([12]\) that \(|\text{Aut}(C_2^4 \rtimes S_3)| = 576\). Moreover, since \( S_8 \) is small enough, we can verify via the function “The_Automorphisms” of Program A.1 that \( \text{Aut}(C_2^4 \rtimes S_3) \hookrightarrow S_8 \). The function “The_Automorphisms” will consider every element in \( S_8 \) and will keep the elements which provide an automorphism on \( C_2^4 \rtimes S_3 \) via conjugation. The function takes as first argument a list of generators of \( G \) and as second argument a list of elements to test (see Section A.1 for more explanations on the program):

```python
>>> S8=PermutationGroup([Permutation(0,1,2,3,4,5,6,7),Permutation(0,1)])
>>> M=S8._elements

>>> len(The_Automorphisms(C2p4S3,M))
576

>>> The_Automorphisms_trivial(C2p4S3,M)
[Permutation(7)]
```

The function "The_Automorphisms_trivial" gives all the elements of \( S_8 \) which commute with all element of \( C_2^4 \rtimes S_3 \). In our case, there is only the identity; therefore, we have proved that \( \text{Aut}(C_2^4 \rtimes S_3) \subset S_8 \).

Since \( C_2^4 \rtimes S_3 \) has a trivial center, we obtain: \( C_2^4 \rtimes S_3 \hookrightarrow \text{Aut}(C_2^4 \rtimes S_3) \hookrightarrow S_8 \). Therefore all valid involutions of \( C_2^4 \rtimes S_3 \) can be obtained with Program A.1.

If we apply Proposition 3.24 with \( \tilde{G} = C_2^4 \rtimes S_3 \), we find 4 involutions \((\theta_i)_{i \in \{1,2,3,4\}}\). We consider the same method used for the previous groups to show that \( \tilde{G} = C_2^4 \rtimes S_3 \) has 2 classes of valid involutions. Via Program A.3, we find two embeddings:

\[
C_2^4 \rtimes S_3 \hookrightarrow H_j \hookrightarrow S_8,
\]

with \( H_j \) of order 192 and such that using Proposition 3.24 with \( \tilde{G} = H_j \) provides \( \theta_2 \sim \theta_j \), \( j \in \{3,4\} \). We choose the groups \( H_j \) with trivial center, then the \( H_j \) can only be isomorphic to \( H_{192} = C_2^4 \rtimes D_6 \) according to \([12]\); moreover this group is a symplectic group on a K3 surface as explained in [29, Section 2, n°8]. This shows that there are at most two equivalent classes of valid involution on \( C_2^4 \rtimes S_3 \) given by \( \text{id} \) and:

\[
\text{Permutation}(1, 7)(3, 5)
\]

5.5 Deformation equivalent orbifolds

The next proposition is a generalization of an idea of Fujiki [14, Proposition 14.5].

**Proposition 5.15.** The following couples of orbifolds are deformation equivalent:

(i) \( S(C_2^4)[2] \sim S[2] \);

(ii) \( S(C_2^2 \rtimes C_2)[2] \sim S(C_2)[2] \);

(iii) \( S(C_4^2 \rtimes S_3)[2] \sim S(S_3)[2] \);

(iv) \( S(C_2)[2] \sim S(C_2^2)[2] \);

(v) \( S(C_2 \times D_4)[2] \sim S(C_2^2)[2] \).
\textbf{Proof.} The general idea of the proof is to find some bimeromorphisms between the orbifolds listed in the proposition and then to apply Proposition 2.10. The bimeromorphisms will be obtained as follows. Let $S \to T/\sim$ be a Kummer surface with $T$ a torus. Assume that we want to show that two orbifolds $S(j_{G})_{2}$ and $S(\bar{G})_{2}$ are deformation equivalent. Using Proposition 2.10, it is enough to find a bimeromorphism between $S^{2}/(j_{0}(G_{1}), S_{2})$ and $S^{2}/(j_{0}(G_{2}), S_{2})$ (see Notation 4 for the definition of $j_{0}$). Such a bimeromorphism will be obtained via an isogeny of $T^{2}$. We denote $G_{i} := (j_{0}(G_{i}), S_{2})$ for all $i \in \{1, 2\}$. The objective of the proof will be to find two groups $\bar{G}_{1}$ and $\bar{G}_{2}$ acting on $T^{2}$ such that there exists a bimeromorphism:

$$T^{2}/\bar{G}_{1} \longrightarrow T^{2}/\bar{G}_{2},$$

with the groups $\bar{G}_{i}$ which are given by extensions of $\langle (\sim \text{id}, \sim \text{id}) \rangle$ by $G_{i}$:

$$\langle (\sim \text{id}, \sim \text{id}) \rangle : G_{i}$$

which induce the groups $\bar{G}_{i}$ on $S^{2}$, for $i \in \{1, 2\}$. Let $g$ and $h$ be two automorphisms on $T$, we denote by $(g, h)$ the induced diagonal action on $T^{2}$. To simplify the expression, we set $I := \langle (\sim \text{id}, \sim \text{id}) \rangle$.

The isogeny used to obtain the different bimeromorphisms will be always the same:

$$f: \quad T^{2} \quad \longrightarrow \quad T^{2} \quad \text{via} \quad (x, y) \quad \longrightarrow \quad (x + y, x - y).$$

Note that:

$$\text{Ker } f = \{ (a, a) \mid a \in T[2] \} \simeq C_{4}^{2}.$$ (36)

To finish the proof, we only have to provide the groups $\bar{G}_{1}$ and $\bar{G}_{2}$ in the different cases. We denote by $s_{0}$ the automorphism of $T^{2}$ which exchanges the two factors $(s_{0}(x, y) = (y, x))$.

(i) In this case, $G_{2} = \langle s_{0} \rangle$ and $G_{1} = \langle s_{0} \rangle \times \{ t(a, a) \mid a \in T[2] \}$, with $t(a, a)$ the translation by $(a, a)$. Let $H$ be a sub-group of $\text{Aut}(T^{2})$, we denote $f_{*}(H) := \{ f \circ h \mid h \in H \}$. Since \eqref{iso}, note that:

$$f_{*}(I \times G_{1}) = I \times G_{2}.$$ Therefore, $f$ induces an isomorphism:

$$T^{2} \quad \xrightarrow{\bar{G}_{1}} \quad T^{2} \quad \xrightarrow{\bar{G}_{2}} \quad \text{via} \quad \frac{T^{2}}{I \times \bar{G}_{1}} \quad \frac{T^{2}}{I \times \bar{G}_{2}}.$$ Moreover, the group $\bar{G}_{2} = I \times G_{2}$ induces the group $S_{2}$ on $S^{2}$ and the group $\bar{G}_{1} = I \times G_{1}$ induces the group $\langle j_{\text{id}}(C_{4}^{2}), S_{2} \rangle$.

(ii) Since the method is identical as (i), we only provide the groups $\bar{G}_{1}$ and $\bar{G}_{2}$. In this case, we assume that $T = E \times E$, with $E$ an elliptic curve. We consider $s_{2} \in \text{Aut}(T)$ given by the matrix:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ Note that $s_{2}^{2} = \sim \text{id}$. Then, we take $\bar{G}_{2} = (I, (s_{2}, s_{2}), s_{0})$ and $\bar{G}_{1} = \{ \{ t(a, a) \mid a \in T[2] \} \times (I, (s_{2}, s_{2})) \} \times \langle s_{0} \rangle$.

The group $\bar{G}_{2}$ is an extension of $I$ and $\langle (s_{2}, s_{2}), s_{0} \rangle / (\sim \text{id}, \sim \text{id})$ and the group $\bar{G}_{2}$ is an extension of $I$ and $\langle \{ t(a, a) \mid a \in T[2] \} \times \langle (s_{2}, s_{2}) \rangle / (\sim \text{id}, \sim \text{id}) \} \times \langle s_{0} \rangle$. Therefore, we have $\bar{G}_{2} \simeq C_{2} \times \langle s_{0} \rangle$ and $\bar{G}_{1} \simeq \langle C_{2}^{2} \times C_{2} \times \langle s_{0} \rangle \rangle$.

(iii) In this case, we assume that $T = E_{3} \times E_{3}$, with $E_{3} := \mathbb{C} / \langle 1, \xi_{3} \rangle$, where $\xi_{3}$ is a third root of the unity. We denote by $\rho$ the automorphism on $T$ with the diagonal matrix $\text{diag}(\xi_{3}, \xi_{3}^{-1})$. Then, we take $\bar{G}_{2} = \langle (s_{2}, s_{2}), (\rho, \rho), I, s_{0} \rangle$ and $\bar{G}_{1} = \{ \{ t(a, a) \mid a \in T[2] \} \times (I, (s_{2}, s_{2}), (\rho, \rho)) \} \times \langle s_{0} \rangle$.
(iv) Let $a \in T[2]$. We take $\mathcal{G}_2 = \langle I, t_{(a,a)}, s_0 \rangle$ and $\mathcal{G}_1 = \{ t_{(b,b)} \mid b \in T[2] \} \times \langle I, s_0, t_{(a,a)} \rangle$. Note that $f \circ t_{(a,a)} = t_{(a,a)}$. Moreover the quotient $T^2/\mathcal{G}_1$ can be written differently as follows:

$$T^2/\mathcal{G}_1 = T^2/\mathcal{G}_1'$$

with $T' := T / \langle t_{(a,a)} \rangle$ and $\mathcal{G}_1' = \mathcal{G}_1 / \langle t_{(a,a)}, t_{(0,a)} \rangle$. We have $\mathcal{G}_1' \simeq I \times C_2^3 \times \langle s'_0 \rangle$, with $s'_0$ which exchanges the two factors of $T^2$; then $f$ provides a bimeromorphism:

$$S^2/\langle j_{id}(C_2^3), \mathcal{G}_2 \rangle \rightarrow S^2/\langle j_{id}(C_2), \mathcal{G}_2 \rangle,$$

where $S'$ is the Kummer surface obtained from $T'$ and $S$ the Kummer surface obtained from $T$. We conclude as usual with Proposition 2.10.

(v) The method is very similar to (iv). The next bimeromorphism will be obtained as before via $f$ and a quotient torus $T'$ with an isomorphism:

$$T'^2/\mathcal{G}_1' \rightarrow T^2/\mathcal{G}_2.$$

So we directly provide $T'$, $\mathcal{G}_1'$ and $\mathcal{G}_2$. We assume that $T = E \times E$ with $E = \mathbb{C} / \langle 1, x \rangle$ an elliptic curve. Then we consider $\mathcal{G}_2 = \langle (s_2, s_2), I, s_0, t_{(\frac{1}{2}, \frac{1}{2}, 0, 0)} \rangle$ and

$$\mathcal{G}_1 = \{ t_{(a,a)} \mid a \in T[2] \} \times \langle (s_2, s_2), I, t_{(\frac{1}{2}, \frac{1}{2}, 0, 0)}, t_{(0, 0, \frac{1}{2}, \frac{1}{2})} \rangle \times \langle s_0 \rangle.$$

Note that $f \circ t_{(\frac{1}{2}, \frac{1}{2}, 0, 0)} = f \circ t_{(0, 0, \frac{1}{2}, \frac{1}{2})} = t_{(\frac{1}{2}, \frac{1}{2}, 0, 0)}$. Then we consider $T' = T / \langle t_{(\frac{1}{2}, \frac{1}{2})} \rangle$ and $\mathcal{G}_1' = \mathcal{G}_1 / \langle t_{(\frac{1}{2}, 0, 0, 0)}, t_{(0, 0, \frac{1}{2}, \frac{1}{2})} \rangle$. It remains to verify that $\mathcal{G}_1'$ is the relevant group. Let $g \in \mathcal{G}_1$, we denote its image in $\mathcal{G}_1'$ by $\overline{g}$. The group $\mathcal{G}_1'$ is generated by $\mathcal{T}$, $\mathcal{S}$, $(s_2, s_2) \circ \overline{t_{(\frac{1}{2}, 0, 0, 0)}}$, $\overline{t_{(\frac{1}{2}, 0, 0, 0)}}$ and $\overline{t_{(\frac{1}{2}, 0, 0, 0)}}$. Note that $(s_2, s_2) \circ \overline{t_{(\frac{1}{2}, 0, 0, 0)}}^2 = (-\text{id}, -\text{id}) \circ \overline{t_{(\frac{1}{2}, 0, 0, 0)}}$. The group $\langle s_2 \circ \overline{t_{(\frac{1}{2}, 0, 0, 0)}}, t_{(\frac{1}{2}, 0, 0, 0)} \rangle / \text{id} \simeq D_4$ because

$$(s_2 \circ \overline{t_{(\frac{1}{2}, 0, 0, 0)}})^{-1} = -\overline{t_{(\frac{1}{2}, 0, 0, 0)}} \circ s_2.$$

Moreover $\overline{t_{(\frac{1}{2}, 0, 0, 0)}}$ commutes with $(s_2, s_2)$ since we have quotiented $T$ by $t_{(\frac{1}{2}, \frac{1}{2})}$. Therefore, we have:

$$\mathcal{G}_1' = \mathcal{G}_1' / \mathcal{T} \simeq D_4 \times C_2 \times \langle s_0 \rangle.$$

Remark 5.16. Note that we still have three couples of orbifolds with common Betti numbers and singularities. These couples are:

- $S(C_2^2 \times C_2_2)_{\text{id}}^{[2]}$ and $S(D_4)[2]$;
- $S(C_2^2 \times C_4)_{\text{id}}^{[2]}$ and $S(C_4)[2]$;
- $S(C_2 \times S_4)_{\text{id}}^{[2]}$ and $S(D_6)[2]$.

These couples could be pairs of deformation equivalent orbifolds; however, the previous method do not apply for these couples.

6 Some examples in dimension 6

In dimension hither than 4, the terminalizations are unknown. However, in few cases, it is still possible to conclude.

Proposition 6.1. The varieties $S(C_2^3)_{[k]}$ for $1 \leq k \leq 4$ are irreducible symplectic orbifolds.
Proof. We recall that $S(C_2^k)^{[3]}$ is obtained as a terminalization of the quotient $S^3/G$, with

$$G = \langle \{ (t, t, \text{id}) | t \in C_2^k \} \cup \mathcal{S}_3 \rangle.$$  

By Theorem 5.1, we know that $S(C_2^k)^{[3]}$ is an irreducible symplectic varieties with a simply connected smooth locus. Moreover by Propositions 2.10 and 2.8, we know that if one terminalization of $S^3/G$ is orbifold, then all the terminalizations are orbifold. Therefore, it only remains to find one terminalization that make $S(C_2^k)^{[3]}$ an orbifold.

We denote $\pi: S^3 \to S^3/G$ the quotient map. The group $G$ has cardinal $3 \times 2^{k+1}$. We denote by $O$ the subset of $S^3$ which contains the elements with an orbit under the action of $G$ of a cardinal strictly smaller than $3 \times 2^{k+1}$. To find the terminalization of $S^3/G$, we have first to find the irreducible components of $O$ of dimension 4. We denote by $O_4$ the union of all these components. According to Remark 5.3, these components are of the form:

$$\{ (x, t(x)) | x \in S \} \times S \text{ or } \{ (x, x) | x \in S \} \times S,$$

with $t \in C_2^k \setminus \text{id}$. The points of these components will be fixed by a subgroup of $G$. Up to permutation of the factors, the different possibilities are the following. We recall that a point in $S$ can only be fixed by one unique involution.

(i) generic point of $O_4$; they are fixed by an automorphism of order 2: a transposition or the composition of a transposition with an involution.

(ii) The points of the form $(x, x, y)$ with $x \in \text{Fix} \ t$, $t \in C_2^k \setminus \text{id}$ and $y$ a generic point in $S$. These points are fixed by $\langle (1, 2), (t, t, \text{id}) \rangle$.

(iii) The points of the form $(x, t(x), t'(x))$ with $t$ and $t'$ two different involutions $C_2^k$ (possibly $\text{id}$) and $x$ a generic point in $S$. These points are fixed by $\langle (1, 2) \circ (t, t, \text{id}), (1, 3) \circ (t', \text{id}, t') \rangle$.

(iv) The points of the form $(x, x, y)$ with $x \in \text{Fix} \ t$, $t \in C_2^k \setminus \text{id}$ and $y \in \text{Fix} \ t$ but $y$ is not in the orbit of $x$ under the action of $C_2^k$. These points are fixed by $\langle (1, 2), (t, t, \text{id}), (t, \text{id}, t), (\text{id}, t, t) \rangle$.

(v) The points of the form $(x, x, t'(x))$ with $x \in \text{Fix} \ t$, $t$ and $t'$ two different involutions in $C_2^k \setminus \text{id}$. These points are fixed by $\langle (1, 2), (t, t, \text{id}), (t, \text{id}, t), (\text{id}, t, t), (2, 3) \circ (\text{id}, t', t') \rangle$.

(vi) The points of the form $(x, x, x)$ with $x \in \text{Fix} \ t$ and $t \in C_2^k \setminus \text{id}$. These points are fixed by $\langle \mathcal{S}_3, (t, t, \text{id}), (t, \text{id}, t), (\text{id}, t, t) \rangle$.

The image by $\pi$ of the previous points are of the following analytic types:

(i) the type $(\mathbb{C}^2)^3 / (\text{id}, \text{id}, \text{id})$; a blow-up resolves crepantly these singularities.

(ii) The type $(\mathbb{C}^2)^3 / \langle (\text{id}, \text{id}, \text{id}), (1, 2) \rangle$. This quotient corresponds to $(\mathbb{C}^2)^3 / \langle (\text{id}, \text{id}, \text{id}), (1, 2) \rangle \times \mathbb{C}^2$. Then, a crepant resolution is provided by Proposition 4.2.

(iii) The type $(\mathbb{C}^2)^3 / \mathcal{S}_3$. A crepant resolution is obtained by the Hilbert scheme of 3 points on $\mathbb{C}^2$.

(iv) Let $B = \langle (\text{id}, \text{id}, \text{id}), (1, 2), (\text{id}, \text{id}, \text{id}, \text{id}), (\text{id}, \text{id}, \text{id}, \text{id}), \text{id}, \text{id}, \text{id}, \text{id}) \rangle$.

In this case, the singular point is of analytic type $(\mathbb{C}^2)^3 / B$. The group $A = \langle (\text{id}, \text{id}, \text{id}), (1, 2) \rangle$ is normal in $B$. Then, we can first quotient by $A$. By Proposition 4.2 we have a crepant resolution $(\mathbb{C}^2)^3 / A \to (\mathbb{C}^2)^3 / A$. Then the involutions $(\text{id}, \text{id}, \text{id})$ and $(\text{id}, \text{id}, \text{id})$ induce involutions on $(\mathbb{C}^2)^3 / A$ with a fixed locus in codimension 4. We keep the same notation for the induced involutions on $(\mathbb{C}^2)^3 / A$. Then $(\mathbb{C}^2)^3 / A / \langle (\text{id}, \text{id}, \text{id}), (\text{id}, \text{id}, \text{id}), (\text{id}, \text{id}, \text{id}) \rangle$ has singularities of codimension 4. By Proposition 2.9 we have found a terminalization.
(v) and (vi) The both cases (v) and (vi) are of analytic type \((C^2)^3 / \langle (-\text{id}, -\text{id}, \text{id}), S_3 \rangle\). The sequel of the proof is dedicated to this case.

There is a natural embedding of \((C^2)^3\) in \((C^2)^4\) given by \((x, y, z) \mapsto (x, y, z, -x - y - z)\). Therefore there is a natural action of \(S_4\) on \((C^2)^3\) via the previous embedding. For instance the permutation \((1, 2)(3, 4)\) acts on \((C^2)^3\) by sending \((x, y, z)\) to \((y, x, -x - y - z)\). We are going to show that cases (v) and (vi) are analytically equivalent to \((C^2)^3 / S_4\). This will end the proof, since this quotient has a crepant resolution given by the Kummer resolution. We consider the following change of variables:

\[
\begin{align*}
\varphi: \quad (C^2)^3 & \rightarrow (C^2)^3 \\
(x, y, z) & \mapsto \left(\frac{y + z - x}{2}, \frac{x + z - y}{2}, \frac{x + y - z}{2}\right).
\end{align*}
\]

Via this change of variables the action of \(S_3\) is unchanged. Moreover, we have:

\[
\begin{align*}
\varphi \circ (-\text{id}, -\text{id}, \text{id})(x, y, z) &= \left(\frac{-y + z + x}{2}, \frac{-x + z + y}{2}, \frac{-x - y - z}{2}\right) \\
&= \left(\frac{-y + z + x}{2}, \frac{-x + z + y}{2}, \frac{-x + y + z}{2} - \frac{x - y + z}{2} - \frac{x + y - z}{2}\right).
\end{align*}
\]

Therefore, the action of \((-\text{id}, -\text{id}, \text{id})\) is now given by the following matrix:

\[
\begin{pmatrix}
0 & 1 & -1 \\
1 & 0 & -1 \\
0 & 0 & -1
\end{pmatrix}
\]

This corresponds to the action of \((1, 2)(3, 4)\). It follows that the action of the group \(\langle (-\text{id}, -\text{id}, \text{id}), S_3 \rangle\) after the change of variables corresponds to the action of the group \(((1, 2)(3, 4), S_3) = S_4\). 

\textbf{Remark 6.2.} According to Corollary 3.14, the previous proposition provides 6-dimensional irreducible symplectic orbifolds with second Betti number 15, 11, 9 and 8 respectively.
A Appendix: programs in Python

A.1 Involutions

This program corresponds to the method 2 for finding classes of valid involutions described in Section 5.3. We start with this method which is simpler to code.

Program A.1.

```python
from sympy.combinatorics.perm_groups import PermutationGroup
from sympy.combinatorics.permutations import Permutation
from itertools import combinations
from sympy.combinatorics.named_groups import SymmetricGroup

#theta is defined via conjugation by an element T.

def theta(M, T):
    return T*M*(T**-1)

#The group G is obtained as a subgroup of S_n generated by a list of elements L.
#The next function runs in a list P and returns the elements in P which provide
#automorphisms on G by conjugation. These automorphisms are put in list A.

def The_Automorphisms(L, P):
    l = len(L)
    G = PermutationGroup(L)
    A = []
    k = len(P)
    for j in range(0, k):
        a = True
        for m in range(0, l):
            a = a*(P[j]*L[m]*P[j]**-1 in G)  # We test that all generators of G is sent in G
            # by conjugation by P[j].
        A = A + [P[j]]
    return A

#The next function finds the elements in P which provide trivial automorphism on G;
#they are put in list A. This is important to express the automorphism group of G.

def The_Automorphisms_trivial(L, P):
    l = len(L)
    G = PermutationGroup(L)
    A = []
    k = len(P)
    for j in range(0, k):
        N = []
        for m in range(0, l):
            N = N + [P[j]*L[m]*P[j]**-1]  # We compute the image of the generators
        if N == L:
            A = A + [P[j]]  # We check if the action on the generators is trivial.
    return A

#For a given theta, the next function determines the set F defined in Notation 3.2.
#It corresponds to the elements g in G which are sent to g^{-1} by theta.
```
def surfaces_fixes(L,T):
    #T defines theta by conjugation.
    G=PermutationGroup(L)
    M=G._elements
    n=len(M)
    F=[]
    for i in range(0,n):
        if theta(M[i],T)==M[i]**-1:
            #We check the condition to be in F
            F=F+[M[i]]
    return F

#The next function determines the valid involutions obtained by conjugation after
#an embedding of G in the permutation group S_n.

def valid_involution(L,n):
    Pv=[]
    G=PermutationGroup(L)
    #Definition of G.
    K=SymmetricGroup(n)
    #G is embedded in K=S_n.
    k=int(n/2)
    #We are going to construct all the order 2 elements in K.
    for i in range(0,k+1):
        #i is half the length of the support of an order 2 element.
        T=Permutation(n-1)
        for j in range(0,2*i,2):
            #Construction of an order 2 element of length 2i.
            T=T*Permutation(j,j+1)
        R=list(K.conjugacy_class(T))
        #R contains all the order 2 elements of length 2i.
        R=The_Automorphisms(L,R)
        #We restrict to the order 2 elements which provide
        for j in range(0,len(R)):
            F=surfaces_fixes(L,R[j])
            H=PermutationGroup(F)
            #We test if the involution is valid.
            if H.order()==G.order():
                Pv=Pv+[R[j]]
        #The valid involutions are put in the list Pv.
    return Pv

#The next function provides the list modulo equivalence of valid involutions obtained
#by conjugation via elements in S_n. It is based on Proposition 3.24.
#N is a list of generators for the group G tilde in Proposition 3.24.
#We start with some valid involutions in P. Each time that the assumptions of
#Proposition 3.24 are verified for two involutions, we keep only one of the both.
#The result will be given in the list Pv.

def classes_valid_involution(L,N,P,n):
    Gtilde=PermutationGroup(N)
    G=PermutationGroup(L)
    M=Gtilde._elements
    P2=[]
    k=len(L)
    if L!=N:
        M=The_Automorphisms(L,M)
    while P1=[]:
        #We only consider the elements of Gtilde
        #which act on G by conjugation.
        M=The_Automorphisms(L,M)
        #The elements h1 and h2 of Proposition 3.24
        #will be searched in the list M.
        while P1=[]:
            I=P[0]
            P.remove(P[0])
            LI=[I]
            a=True
            j=0
            for i in range(0,k):
                LI=LI+[theta(L[i],I)]
                a=v checked True and j<len(P2):

58
m1=0
while a==True and m1<len(M):
    #The elements M[m1] and M[m2] correspond to
    #the elements h_1 and h_2 of Proposition 3.24.
    m2=0
    while a==True and m2<len(M):
        #We test the condition of Proposition 3.24 for
        #theta_I and theta_{P2[j]}.
        LP=[]
        Q=M[m1]*M[m2]**-1
        if theta(Q,I)==Q**-1 and (Q in G):
            for l in range(0,k):
                R=theta(M[m1],I)
                LP=LP+[R**-1*M[m2]*theta(L[l],P2[j])*M[m2]**-1*R]
                a=a*(LP!=LL)
        m2=m2+1
        m1=m1+1
    j=j+1
if a:
    #If the conditions of Proposition 3.24
    P2=P2+[I]
    #are never verified, we add I to the list P2.
return P2

This program corresponds to the method 1 for finding classes of valid involutions described in
Section 5.3.

Program A.2.

from sympy.combinatorics.perm_groups import PermutationGroup
from sympy.combinatorics.permutations import Permutation
from itertools import combinations
from Program8_1 import The_Automorphisms
import copy
import math

#Given B a family of generators, the next function is a recursive construction of all the
#elements of the group G by product of elements of B. To obtained the involution
#theta which sends each element of B to its inverse, we need to know how every elements
#of G is written as a product of elements of B. E will be the list of elements of G
#already constructed; we start the program with E=[\id] and at each steps we produce
#elements obtained from products of elements of E and elements of B.

def tab(B,E):
    G=PermutationGroup(B)
    N=G._elements
    b=len(B)
    B2=[G.identity]+B
    if set(E)==set(N):
        return E
    else:
        e=len(E)
        PP=[]
        for i in range(0,e):
            for j in range(0,b+1):
                #We obtain a new list of elements PP by multiplying
                PP=PP+[E[i]*B2[j]]
        #all elements of E by all elements of B2.
        return tab(B,PP)

#The involution theta defined previously sends all elements of B to their inverse.
#To know the image of an element x of G by theta, we use the previous function tab
#which decomposes x in a product of elements of B. The argument R of the function
#will be taken equal to tab(B,[\id]). We need this argument to avoid recomputing
#tab(B,[\id]) each time that we need to use theta. The program is constructed as follows.
#Let x be an element of G. We search x in R. Then i the index of x in R will tell how x
#decomposes as a product of elements of B. It is obtained by writing i in base len(B)+1.
#The Python function "%" provides the rest of the Euclidian division and the function "//" 
#provides the quotient.
def theta(R,B):
    G=PermutationGroup(B)
    B=[G.identity]+B
    b=len(B)
    a=int(math.log(len(R),b)) # a corresponds to the number of multiplications that we need to obtain all elements of G.
    i=R.index(x) #We find the element x in the list R and knowing its index i, we can deduce its decomposition in elements of B.
    T=G.identity
    for j in range(0,a):
        T=(B[i%len(B)])^{-1}*T #We take the inverses of the elements that compose x.
        i=i//len(B)
    return T
    return f
    #The group G is given by a family of generators L. The next function returns all possible basis of G, it is based on the Tarski Theorem (see Definition 5.12 and Theorem 5.13).

def generators(L):
    G=PermutationGroup(L)
    N=G._elements
    N.remove(G.identity)
    m=len(N)
    N=[] #N will contain the basis that we have already found.
    a=True #a will be the test that tells the program when to stop.
    k=2 #K will be the number of elements of a basis.
    while a:
        A=list(combinations(M,k))
        r=len(A) #If N is not empty, the program stops apart if we find a larger basis (see below).
        a=False
        O=copy.copy(N) #O will contain the basis that we already got of length smaller than or equal to k-1.
        for i in range(0,r):
            b=True
            for j in range(0,len(O)):
                b=b*(not (O[j].issubset(set(A[i])))) #We verify that A[i] does not strictly contain a basis.
            if b:
                H=PermutationGroup(list(A[i]))
                if G.order()==H.order(): #If moreover A[i] generates G, then it is a basis.
                    O=O+[set(A[i])]
                    a=True
                    #If b has been True at least once, we need to continue the program, so a becomes True, else, a stay False and the program will stop, according to Theorem 5.13 (there is no larger basis).
                    k=k+1
                    N2=[]
                    n=len(N)
                    for i in range(0,n):
                        N2=N2+[list(N[i])]
                    return N2 #Here for practical reasons, we just write the basis as list instead of set.

    #The next function keeps only the basis that define a bijection from G to G. To verify that we have a bijection, we only need to check that the image of theta has the same cardinal as G. We also eliminate redundancy; if two basis give the same bijection, we keep only one of them.

def bijections(L):
    A=generators(L)
    G=PermutationGroup(L)
    N=G._elements
    m=len(N)
    N2=[] #A2 will be the list of bijections
AA2=[]
#AA2 is the list of the images of
while A1=[]:
    I=A[0]
    RI=tab(I,[H.identity])
    A.remove(A[0])
    M3=[]
    for i in range(0,m):
        M3=M3+[theta(RI,I)(M[i])]
    if set(M3)==set(M):
        a=True
        j=0
        while a==True and j<len(AA2):
            a=a*(M3!=AA2[j])
            j=j+1
        if a:
            A2=A2+[I]
            return
        A2=A2+[M3]
    AA2=AA2+[M3]

#The next function keeps the morphisms among the bijections. They are put in the list A2.
def involutions(L):
    A=bijections(L)
    G=PermutationGroup(L)
    H=PermutationGroup(1)
    N=G._elements
    M=H._elements
    C=list(combinations(M,2))
    A2=[]
    #We consider C the set of all combinations of 2 elements (x,y). We will verify that
    while A!=[]:
        I=A[0]
        RI=tab(I,[G.identity])
        A.remove(A[0])
        a=True
        j=0
        while a==True and j<len(C):
            a=a*(theta(RI,I)(C[j][0]*C[j][1])==theta(RI,I)(C[j][0])*theta(RI,I)(C[j][1]))
            j=j+1
        if a:
            A2=A2+[I]
    return A2

#The next function is identical to "class_valid_involution" of Program 8.1.
#However, since theta has been modified, we need to rewrite the function accordingly.
def classes_valid_involution2(P,L,N):
    H=PermutationGroup(N)
    M=The_Automorphisms(L,M)
    while P!=[]:
        I=P[0]
        RI=tab(I,[G.identity])
        P.remove(P[0])
        a=True
        j=0
        while a==True and j<len(P2):
            RP2=tab(P2[jj],[G.identity])
            LL=[]
            return
for i in range(0,k):
    LL=LL+[\theta(RP2,P2[j])(L[i])]

m1=0
while a==True and m1<len(M):
    m2=0
    while a==True and m2<len(M):
        LP=[]
        Q=M[m1]*M[m2]**-1
        if (Q in G):
            if \theta(RI,I)(Q)==Q**-1:
                for l in range(0,k):
                    LP=LP+[M[m2]**-1*\theta(RI,I)(M[m1]*L[l]*M[m1]**-1)*M[m2]]
                a=a*(LP!=LL)
        m2=m2+1
    m1=m1+1
j=j+1
if a:
P2=P2+[I]
return P2

The next program is a tool to find the group $\tilde{G}$ of Proposition 3.24. We start with $H \supset G$ an over-group of $G$, $\theta_1$ and $\theta_2$ two valid involutions on $G$. The objective of the program is to find $\tilde{G}$ such that:

- $G \subset \tilde{G} \subset H$;
- $\tilde{G}$ is an automorphism group on a K3 surface;
- we can find $h_1$ and $h_2$ in $\tilde{G}$ that verified the assumptions of Proposition 3.24 given $\theta_1$ and $\theta_2$ equivalent.

Keeping the notation of Proposition 3.24 for more efficiency, the program will be running on $h_1 \circ h_2^{-1}$ and $h_2$ instead of $h_1$ and $h_2$. Indeed, since $h_1 \circ h_2^{-1}$ has to be in $G$, this will lead to less operations than searching for $h_1$ and $h_2$ that could be in $H$.

Program A.3.

```python
from sympy.combinatorics.perm_groups import PermutationGroup
from sympy.combinatorics.permutations import Permutation
from Program8_2 import tab
from Program8_2 import theta

#L is a family of generators of G, N a family of generators of H,
#A and B are the two valid involutions that we want to compare
#(they are given by a basis of G). The variable r is the cardinal
#that we want for the group G tilde. The next program will construct the list P.
#Let P[i] be an element of the list P; the group G tilde will be generated by G and P[i].

def test(L,N,A,B,r):
    n=len(L)
    H=PermutationGroup(N)
    G=PermutationGroup(L)
    N=G._elements
    m=len(N)
    h2=H._elements
    m=len(h2)
    P=[]
    Ra=tab(A,[H.identity])
    Rb=tab(B,[H.identity])
    HH=[]
    for k in range(0,n):
        if \theta(Rb,B)(N[k])==N[k]**-1:
            HH=HH+[k]
```
\[ HH=HH+[H[k]] \]  # as in Proposition 3.24.

\[
f = \text{len}(HH)
\]

\[
\text{for } i \text{ in range}(0, m):
\]

\[
\text{for } j \text{ in range}(0, f):
\]

\[
h1=HH[j] \cdot h2[i]
\]

\[
b = \text{True}
\]

\[
\text{for } x \text{ in range}(0, l):
\]

\[
# \text{ we search for } h1 \text{ and } h2 \text{ that}
\]

\[
\text{left} = \theta(R_b, B)(h1 \cdot L[x] \cdot h1^{-1}) \text{ verified Proposition 3.24.}
\]

\[
\text{right} = h2[i] \cdot \theta(R_a, A)(L[x]) \cdot h2[i]^{-1}
\]

\[
b = b \cdot (\text{left} == \text{right})
\]

\[
\text{if } b:
\]

\[
K = \text{PermutationGroup}(L+[h2[i]]) \text{ We construct } G \text{ tilde.}
\]

\[
\text{if } (K.\text{order}()==r):
\]

\[
P = P + [h2[i]]
\]

\[
return P
\]

The next program is a realization of the group \( \tilde{F} \) from [29] Section 2, n°5 as a permutation subgroup of \( \mathfrak{S}_{64} \) (see Section 5.4). We recall that the group \( \tilde{F} \) is the group of automorphisms of the Fermat quartic. It is generated by the automorphisms of order 4: \( (x, y, z, t) \mapsto (i^ax, i^by, i^cz, i^dt) \) with \( a, b, c, d \in \mathbb{C}_4 \) and the permutations of variables. The group has a faithful natural action on

\[
E = \{ (i^a : i^b : i^c : 1) | (a, b, c) \in \mathbb{C}_4^3 \}
\]

We number the set \( E \) in base 4 with 1 which corresponds to 0, i to 1, -1 to 2 and -i to 3. For instance \( (1 : i : -i : -1) \) is the number 310 in base 4. In the next program the element \( u \) corresponds to the morphism \( (x, y, z, t) \mapsto (ix, iy, iz, it) \) and the element \( w \) to the morphism \( (x, y, z, t) \mapsto (x, iy, iz, it) \); the element \( t1 \) corresponds to the morphism \( (x, y, z, t) \mapsto (y, x, iz, it) \) and the element \( t3 \) to the morphism \( (x, y, z, t) \mapsto (x, y, it, tz) \). These six previous elements generate \( \tilde{F} \) as a sub-group of \( \mathfrak{S}_{64} \).

Program A.4.

```python
from sympy.combinatorics.perm_groups import PermutationGroup
from sympy.combinatorics.permuations import Permutation

def F():
    u=Permutation(63)
    v=Permutation(63)
    w=Permutation(63)
    tl1=Permutation(63)
    tl2=Permutation(63)
    tl3=Permutation(63)
    for i in range(0,61,4):
        u=u*Permutation(i, i+1, i+2, i+3)
    for j in range(0,4):
        for k in range(0,4):
            v=v*Permutation(k+16*j, k+16*j+4, k+16*j+8, k+16*j+12)
    for r in range(0,4):
        for s in range(0,4):
            w=w*Permutation(r+4*s, r+4*s+16, r+4*s+32, r+4*s+48)
    for x in range(0,4):
        for y in range(0,4):
            for z in range(y+1,4):
                tl1=tl1*Permutation(y+4*z+16*x, z+4*y+16*x)
                tl2=tl2*Permutation(z+4*y+16*x, z+4*y+16*x)
    L=list(range(16,64))
    while L!=[]:
        e=L[0]//16
        p=(L[0]%16)//4
        q=((L[0]%16)%4
```

63
\[ t_3 = t_3 \cdot \text{Permutation}(q \cdot 4 \cdot p \cdot 16 \cdot e, ((q-e) \cdot 4) \cdot ((p-e) \cdot 4) + 16 \cdot (4-e)) \]

\[ \text{L remove} \text{L[0]} \]

\[ \text{L remove} \text{(((q-e) \cdot 4) \cdot ((p-e) \cdot 4) + 16 \cdot (4-e))} \]

\[ \text{return} \{u, v, w, t1, t2, t3\} \]

We propose an embedding of \( C_4^2 \rtimes C_3 \) in \( \tilde{F} \) via the following generators (needed in Section 5.4):

\[
C_{4p2C3} = \langle \text{Permutation}(0, 25, 37)(1, 26, 38)(2, 27, 39)(3, 24, 36)(4, 41, 16)(5, 42, 17)(6, 43, 18)(7, 40, 19)(8, 57, 15)(9, 58, 12)(10, 59, 13)(11, 56, 14)(28, 52, 62)(29, 53, 63)(30, 54, 60)(31, 55, 61)(32, 51, 45)(33, 48, 46)(34, 49, 47)(35, 50, 44), \\
\text{Permutation}(0, 7, 40, 47)(1, 50, 41, 26)(2, 45, 42, 5)(3, 24, 43, 48)(4, 23, 44, 63)(6, 61, 46, 21)(8, 39, 32, 15)(9, 18, 33, 58)(10, 13, 34, 37)(11, 56, 35, 16)(12, 55, 36, 31)(14, 29, 38, 53)(17, 54, 57, 30)(19, 28, 59, 52)(20, 27, 60, 51)(22, 49, 62, 25) \rangle
\]

### A.2 Second Betti number

The next program is used to compute the second Betti number and is based on Proposition 3.15. The next program is used to compute the second Betti number and is based on Proposition 3.15. The list \( N \) will be the list of representatives of orbits for the action of \( G \) on \( F \). In the program, the list \( F \) is initially given by all the elements \( g \in G \) such that \( \theta(g) = g^{-1} \).

**Program A.5.**

```python
from sympy.combinatorics.perm_groups import PermutationGroup
from sympy.combinatorics.permutations import Permutation
from Program8_1 import theta
from Program8_1 import surfaces_fixes

def div (L,T):
    G = PermutationGroup(L)
    N = G._elements
    n = len(N)
    F = surfaces_fixes(L,T)
    N = []
    while F != []:
        b = F[0]
        N = N + [b]  # We consider b as a representative
        for i in range(0,n):
            a = theta(M[i],T)*b*(M[i]**-1)  # We compute the orbit of b
            if a in F:
                F.remove(a)  # We remove all the elements of
        return len(N)
```

### A.3 The singularities

The next program provides the singularities and is based on Section 4. The objective is to compute the formulas of Propositions 4.33, 4.35 and 4.36.

**Program A.6.**

```python
from sympy.combinatorics.perm_groups import PermutationGroup
from sympy.combinatorics.permutations import Permutation
from Program8_1 import surfaces_fixes
from Program8_1 import theta

# The next function will be used to compute the numbers k_6 and k_4 from Lemmas 4.11
# and 4.12. The arguments of the function are n which can be 6 or 4, L a list of generators
# of G and g an element of G. The list K will contain all elements g_i (modulo inverse)
# of order n such that g is in <g_i>.
```
def k(n,L,g):
    G=PermutationGroup(L)
    N=G._elements
    K=[]
    m=len(N)
    for i in range(0,m):
        if N[i].order()==n and not (N[i]**-1 in K): #We consider only the elements modulo inverse.
            H=PermutationGroup([N[i],g])
            if H.is_cyclic: #We test if g is in <g_i>.
                K=K+[N[i]]
    return K

#The next function computes the set of orbits S_{g1,g2}/<g1> defined in Remark 4.17.
#It is denoted Sg in the program. The list L is a family of generators of G.
#T is a permutation which induces theta by conjugation; g1 and g2 are two elements in G.

def S(L,T,g1,g2):
    G=PermutationGroup(L)
    N=G._elements
    Sg=[]
    o=g1.order()
    while N!=[]:
        s=N[0]
        if s**-1*theta(g2,T)*s==g1 or s**-1*theta(g2,T)*s==g1**-1:
            Sg=Sg+[s] #We test the condition to be in Sg (see Remark 4.17).
        for i in range(0,o):
            if s*g1**i in N: #We only consider a representative for each orbit under the action of <g1>.
                N.remove(s*g1**i)
    return Sg

#For convenient reason, we set the function t for the cardinal of Sg as in Lemma 4.18.

def t(L,T,g1,g2):
    return len(S(L,T,g1,g2))

#The next function returns the number of specific fixed points on S^2 which are not of the form (x,s(x)) as explained in Lemma 4.18. We just follow the formuлас given #in Lemma 4.18.

def N(L,g,T):
    tl=t(L,T,g,g) #tl is t(g) in Lemma 4.18.
    o=g.order()
    G=PermutationGroup(L)
    N=G._elements
    if o==6:
        return 2*(2-tl)
    if o==4:
        return 4*(4-tl)
    if o==3:
        K=k(6,L,g)
        k6=len(K)
        K=0
        for j in range(0,k6):
            K=K+2*(2-t(L,T,K[j],K[j]))+2*(2-t(L,T,K[j],K[j]))
I=(6-2*k6)*(6+2*k6-tl)+R
return I
if o==2:
    K6=k(6,L,g)
    K4=k(4,L,g)
    k6=len(K6)
    k4=len(K4)
    R6=0
    R4=0
    for i in range(0,k6):
        for j in range(i+1,k6):
            R6=R6+2*(2-t(L,T,K6[i],K6[j]))+2*(2-t(L,T,K6[j],K6[i]))
    for i in range(0,k4):
        for j in range(i+1,k4):
            R4=R4+4*(4-t(L,T,K4[i],K4[j]))+4*(4-t(L,T,K4[j],K4[i]))
    I=(8-2*k6-4*k4)*(8+2*k6+4*k4-tl)+R6+R4+16*k6*k4
    return I

# Now the idea is to provide all the sets given in (24) to compute the formulas of Proposition 4.33, 4.35 and 4.36. The list S in argument of the function will be $S_{g1,g2}$; it is an argument to avoid to recompute $S_{g1,g2}$ at each uses. The next function computes the lists $S_{g1,g2}^+$ and $S_{g1,g2}^-$ from Notation 4.30. The argument "plus" can be True or False. The function returns $S_{g1,g2}^+$ if plus==True and returns $S_{g1,g2}^-$ if plus==False.

def S_PlusMinus(T,g1,g,S,plus):
o=g1.order()
Spus=[]
Sminus=[]
H=PermutationGroup([g])
for i in range(0,len(S)):
a=True
j=0
s=S[i]
while j<o and a==True:
    u=s*g1**j
    if theta(u,T)*u in H:
        a=False
        Spus=Spus+[S[i]]
    j=j+1
if a==True:
    if plus==True:
        return Spus
    else:
        return Sminus

# The list S in argument of the next function will be $S_{g1,g2}^+$.
# The next function computes the lists $S_{g1,g2}^{+,c}$ and $S_{g1,g2}^{+,nc}$ from Notation 4.30. The argument "c" can be True or False. The function returns $S_{g1,g2}^{+,c}$ if c==True and returns $S_{g1,g2}^{+,nc}$ if c==False.

def S_c(T,S,g,c):
    Sc=[]
    Snc=[]
    for i in range(0,len(S)):
        if theta(g,T)*S[i]==S[i]*g:
            # We check the condition to be in $S_{g1,g2}^{+,c}$.
            a=True
            j=1
            while j<o and a==True:
                u=s*g1**j
                if theta(u,T)*u in H:
                    a=False
                    Sc=Sc+[S[i]]
                j=j+1
            if a==True:
                if c==True:
                    return Sc
                else:
                    return Snc

    # The list S in argument of the next function will be $S_{g1,g2}^{+,c}$.
    # The next function computes the lists $S_{g1,g2}^{+,c}$ and $S_{g1,g2}^{+,nc}$ from Notation 4.30. The argument "c" can be True or False. The function returns $S_{g1,g2}^{+,c}$ if c==True and returns $S_{g1,g2}^{+,nc}$ if c==False.

def S_c(T,S,g,c):
    Sc=[]
    Snc=[]
    for i in range(0,len(S)):
        if theta(g,T)*S[i]==S[i]*g:
            # We check the condition to be in $S_{g1,g2}^{+,c}$.
```python
def SF(T, S, F, g1, f):
    Sf = []
    Snf = []
    o = g1.order()
    for i in range(0, len(S)):
        a = True
        j = 0
        s = S[i]
        while j < o and a == True:
            v = s * (g1**j)
            if v in F:
                a = False
                Sf = Sf + [S[i]]
            j = j + 1
        if a == True:
            Snf = Snf + [S[i]]
        if f == True:
            return Sf
        else:
            return Snf
```

As usual, the argument $F$ of the function will be the list of elements sent to their inverses by $\theta$. The argument $S$ of the next function will be $S_{(g_1, g_2)^{+}, c}$ or $S_{(g_1, g_2)^{+}, nc}$. Then the function will return $S$ intersected with $F_g$ or $S$ setminus $F_g$. $F_g$ is defined in Notation 4.30. The argument "f" can be True or False. The function returns $S$ intersected with $F_g$ if $f$ = True and returns $S$ minus $F_g$ if $f$ = False.

```python
# The argument $Sp$ of the next functions a8, B4 and B6 will be $S_g^{+}$. The argument n is the cardinal of $G$.
# We just follows the formulas from Proposition 4.33. The next functions provide the contribution to the singularities of one element $g$.

def a8(L, T, g, Sp, F, n):
    if g.order() == 4:
        return 16 * len(SF(T, S_c(T, Sp, g, True), F, g, False)) / n
    else:
        return 0

def B4(L, T, g, Sp, F, n):
    if g.order() == 4:
        return 16 * len(SF(T, S_c(T, Sp, g, False), F, g, False)) / n
    else:
        return 0

def B6(L, T, g, Sp, F, n):
    if g.order() == 6:
        return 12 * len(SF(T, S_c(T, Sp, g, False), F, g, False)) / n
    else:
        return 0
```

67
The argument $St$ of the next functions will be $Sg$ and the argument $Sp$ will be $Sg^\ast$.
We follow the formulas from Propositions 4.34 and 4.35.

```python
def a6(L,T,g,St,Sp,F,n):
    I=0
    if g.order()==6:
        I=I+6*len(S_PlusMinus(T,g,g,St,False))/n
        I=I+3*N(L,g,T)/n
        return I
    elif g.order()==3:
        K=k(6,L,g)
        k6=len(K)
        I=3*(6-2*k6)*len(SF(T,S_c(T,Sp,g,True),F,g,False))/n
        for i in range(0,k6):
            for j in range(i+1,k6):
                Sij=S_PlusMinus(T,K[i],g,S(L,T,K[i],K[j]),True)  # We compute $S_{K[i],K[j]}^\ast$
                Sji=S_PlusMinus(T,K[j],g,S(L,T,K[j],K[i]),True)
                I=I+6*len(SF(T,S_c(T,Sij,g,True),F,K[i],False))/n
                I=I+6*len(SF(T,S_c(T,Sji,g,True),F,K[j],False))/n
        return I
    else:
        return 0
def a4(L,T,g,St,Sp,F,n):
    I=0
    if g.order()==4:
        I=I+8*len(S_PlusMinus(T,g,g,St,False))/n
        I=I+2*N(L,g,T)/n
        return I
    elif g.order()==2:
        K6=k(6,L,g)
        K4=k(4,L,g)
        k6=len(K6)
        k4=len(K4)
        I=2*(8-2*k6-4*k4)*len(SF(T,S_c(T,Sp,g,True),F,g,False))/n
        for i in range(0,k6):
            for j in range(i+1,k6):
                Sij=S_PlusMinus(T,K6[i],g,S(L,T,K6[i],K6[j]),True)
                Sji=S_PlusMinus(T,K6[j],g,S(L,T,K6[j],K6[i]),True)
                I=I+4*len(SF(T,S_c(T,Sij,g,True),F,K6[i],False))/n
                I=I+4*len(SF(T,S_c(T,Sji,g,True),F,K6[j],False))/n
        for i in range(0,k4):
            for j in range(i+1,k4):
                Sij=S_PlusMinus(T,K4[i],g,S(L,T,K4[i],K4[j]),True)
                Sji=S_PlusMinus(T,K4[j],g,S(L,T,K4[j],K4[i]),True)
                I=I+8*len(SF(T,S_c(T,Sij,g,True),F,K4[i],False))/n
                I=I+8*len(SF(T,S_c(T,Sji,g,True),F,K4[j],False))/n
        return I
    else:
        return 0
def a3(L,T,g,St,Sp,F,n):
    I=0
    if g.order()==6:
        I=48*len(SF(T,S_c(T,Sp,g,True),F,g,True))/n
        return I
    elif g.order()==3:
        K=k(6,L,g)
        k6=len(K)
        I=3*N(L,g,T)/(2*n)
```
\[ I = I + 3(6 - 2k6) \cdot \text{len}(\text{S\_PlusMinus}(T, g, g, St, False))/(2n) \]

\[ I = I + 6(6 - 2k6) \cdot \text{len}(\text{SF}(T, S_c(T, Sp, g, True), F, g, True))/n \]

for i in range(0, k6):
    for j in range(i+1, k6):
        \[ S_{ij} = S(L, T, K6[i], K6[j]) \]
        \[ S_{ji} = S(L, T, K6[j], K6[i]) \]
        \[ S_{plusij} = S_{\text{PlusMinus}}(T, K6[i], g, S_{ij}, True) \]
        \[ S_{plusji} = S_{\text{PlusMinus}}(T, K6[j], g, S_{ji}, True) \]
        \[ I = I + 12 \cdot \text{len}(\text{SF}(T, S_c(T, S_{\text{\_PlusMinus}}(T, K6[i], g, S_{ij}, True), F, g, True))/n \]
        \[ I = I + 12 \cdot \text{len}(\text{SF}(T, S_c(T, S_{\text{\_PlusMinus}}(T, K6[j], g, S_{ji}, True), F, g, True))/n \]
        \[ I = I + 3 \cdot \text{len}(\text{S\_PlusMinus}(T, K6[i], g, S_{ij}, False))/n \]
        \[ I = I + 3 \cdot \text{len}(\text{S\_PlusMinus}(T, K6[j], g, S_{ji}, False))/n \]

return I

else:
    return 0

def a2(L, T, g, St, Sp, F, n):
    I = 0
    if g.order() == 6:
        I = 12 \cdot \text{len}(\text{SF}(T, S_c(T, Sp, g, False), F, g, True))/n
        return I
    elif g.order() == 4:
        I = 64 \cdot \text{len}(\text{SF}(T, S_c(T, Sp, g, True), F, g, True))/n
        return I
    elif g.order() == 2:
        K6 = k(6, L, g)
        K4 = k(4, L, g)
        k6 = \text{len}(K6)
        k4 = \text{len}(K4)
        I = N(L, g, T)/n
        I = I + (8 - 2k6 - 4k4) \cdot \text{len}(\text{S\_PlusMinus}(T, g, g, St, False))/n
        for i in range(0, k6):
            for j in range(i+1, k6):
                \[ S_{ij} = S(L, T, K6[i], K6[j]) \]
                \[ S_{ji} = S(L, T, K6[j], K6[i]) \]
                \[ I = I + 2 \cdot \text{len}(\text{S\_PlusMinus}(T, K6[i], g, S_{ij}, False))/n \]
                \[ I = I + 2 \cdot \text{len}(\text{S\_PlusMinus}(T, K6[j], g, S_{ji}, False))/n \]
        for i in range(0, k4):
            for j in range(i+1, k4):
                \[ S_{ij} = S(L, T, K4[i], K4[j]) \]
                \[ S_{ji} = S(L, T, K4[j], K4[i]) \]
                \[ I = I + 4 \cdot \text{len}(\text{S\_PlusMinus}(T, K4[i], g, S_{ij}, False))/n \]
                \[ I = I + 4 \cdot \text{len}(\text{S\_PlusMinus}(T, K4[j], g, S_{ji}, False))/n \]
        return I
    else:
        return 0

# We combine all our previous computations in a global function
# which provides all the singularities. It returns the list
# a=[a2, a3, a4, a6, a8, b4, b6] (see Notation 4.27).

def singularities(L, T):
    G = PermutationGroup(L)
    n = G.order()
    N = G._elements
    a = [0, 0, 0, 0, 0, 0, 0]
    N.remove(G.identity)
    F = surfaces.fixes(L, T)
    while N != []:  
        g = N[0]  
        # We are going to compute the contribution  
        # to the singularities of each g in G.
        #
        69
\begin{verbatim}
St=S(L,T,g,g)  # St is the list Sg
Sp=S_PlusMinus(T,g,g,St,True)  # Sp is the list Sg^+
a[0]=a[0]+a2(L,T,g,St,Sp,F,n)
a[1]=a[1]+a3(L,T,g,St,Sp,F,n)
a[2]=a[2]+a4(L,T,g,St,Sp,F,n)
a[3]=a[3]+a6(L,T,g,St,Sp,F,n)
a[4]=a[4]+a8(L,T,g,Sp,F,n)
a[5]=a[5]+B4(L,T,g,Sp,F,n)
a[6]=a[6]+B6(L,T,g,Sp,F,n)
M.remove(g)
if g.order()！=2:
    M.remove(g**-1)  # we remove g**-1 to avoid counting the
    # singularities twice.
return a
\end{verbatim}

References

[1] E. Arbarello, G. Saccà, and A. Ferretti, Relative Prym varieties associated to the double cover of an Enriques surface, J. Differential Geom. 100 (2015), no. 2, 191-250.

[2] B. Bakker, H. Guenancia, and C. Lehn, Algebraic approximation and the decomposition theorem for Kähler Calabi–Yau varieties [arXiv:2012.00441]

[3] B. Bakker and C. Lehn, The global moduli theory of symplectic varieties, [arXiv:1812.09748]

[4] A. Beauville, Some remarks on Kähler manifolds with c1=0. Classification of algebraic and analytic manifolds (Katata, 1983), 1-26.

[5] A. Beauville, Variétés kähleriennes dont la première classe de chern est nulle, J. Differential geometry, 18 (1983) 755-782.

[6] T. Beckmann and J. Song, Second Chern class and Fujiki constants of hyperkähler manifolds. [arXiv:2201.07767]

[7] V. Bertini, A. Capasso, O. Debarre, A. Grossi, M. Mauri and E. Masson, Terminalizations of quotients of induced symplectic automorphisms on K3^[n] or generalized Kummer varieties. In preparation.

[8] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type. J. Amer. Math. Soc., 23 (2): 405-468, (2010).

[9] R. Blache, Chern classes and Hirzebruch–Riemann–Roch theorem for coherent sheaves on complex projective orbifolds with isolated singularities, Math. Z. 222, 7-57 (1996).

[10] C. Camere, A. Garbagnati, G. Kapustka and M. Kapustka, Projective models of Nikulin orbifolds, [arXiv:2104.09234]

[11] F. Campana, Orbifolds à première classe de Chern nulle. The Fano Conference, 339-351, Univ. Torino, Turin, (2004).

[12] T. Dokchitser, GroupNames, https://people.maths.bris.ac.uk/ matyd/GroupNames/

[13] L. Fu and M. Menet, On the Betti numbers of compact holomorphic symplectic orbifolds of dimension four. Math. Z. 299, No. 1-2, 203-231 (2021).

[14] A. Fujiki, On Primitively Symplectic Compact Kähler V-manifolds of Dimension Four. Classification of algebraic and analytic manifolds (Katata, 1983), 71-250.

[15] G. Höhn and G. Mason, Finite groups of symplectic automorphisms of Hyperkähler manifolds of type K3. Bull. Inst. Math., Acad. Sin. (N.S.) 14, No. 2, 189-264 (2019).

[16] D. Huybrechts, Lectures on K3 surfaces, Camb. Stud. Adv. Math. 158 (2016).
[17] S. Kapfer, G. Menet, *Integral cohomology of the generalized Kummer fourfold*, Algebraic Geometry, 5 (5) (2018) 523-567.

[18] S. Kebekus and C. Schnell, *Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities*, J. Am. Math. Soc. 34, No. 2, 315-368 (2021).

[19] T. Kirschner, *Period Mappings with Applications to Symplectic Complex Spaces*, Lecture Notes in Mathematics 2140, Springer. Version [arXiv:1210.4197](https://arxiv.org/abs/1210.4197).

[20] J. Kollár, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. math. 113, 177-215 (1993).

[21] C. Lehn, G. Mongardi, and G. Pacienza, *The Morrison–Kawamata cone conjecture for singular symplectic varieties*, arXiv:2207.14754.

[22] T. Matteini, *A singular symplectic variety of dimension 6 with a Lagrangian Prymian fibration*, Manuscripta Math. 149 (2016), no. 1-2, 131-151.

[23] D. Markushevich, A.S. Tikhomirov, *New symplectic V-manifolds of dimension four via the relative compactified Prymian*, International Journal of Mathematics, Vol. 18, No. 10 (2007) 1187-1224.

[24] G. Menet, *Beauville–Bogomolov lattice for a singular symplectic variety of dimension 4*, Journal of pure and apply algebra (2014), 1455-1495.

[25] G. Menet, *Global Torelli theorem for irreducible symplectic orbifolds*, Journal de Mathématiques pures et appliquées, 137 (2020), no.9, 213-237.

[26] G. Menet, *On the integral cohomology of quotients of manifolds by cyclic groups*, Journal de Mathématiques pures et appliquées, 119, (2018), no.9, 280-325.

[27] G. Menet, U. Riess, *On the Kähler cone of irreducible symplectic orbifolds*, [arXiv:2009.04873](https://arxiv.org/abs/2009.04873).

[28] G. Menet, U. Riess, *Wall divisors on irreducible symplectic orbifolds of Nikulin-type*, [arXiv:2209.03667](https://arxiv.org/abs/2209.03667).

[29] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math., 94 (1988), 183-221.

[30] Y. Namikawa, *A note on symplectic singularities*, [arXiv:math/0101028](https://arxiv.org/abs/math/0101028).

[31] A. Perego, *Examples of irreducible symplectic varieties*, Colombo, Elisabetta (ed.) et al., Birational geometry and moduli spaces. Collected papers presented at the INdAM workshop, Rome, Italy, June 11-15, 2018. Cham: Springer. Springer INdAM Ser. 39, 151-172 (2020).

[32] A. Perego and A. Rapagnetta, *The moduli spaces of sheaves on K3 surfaces are irreducible symplectic varieties*, [arXiv:1802.01182](https://arxiv.org/abs/1802.01182).

[33] C. Shen, *Lagrangian fibrations by Prym varieties* PhD thesis (2020).

[34] The LMFDB Collaboration, *The L-functions and modular forms database*, https://www.lmfdb.org/GaloisGroup/

[35] M. Verbitsky, *Holomorphic symplectic geometry and orbifold singularities*, Asian J. Math. 4, No. 3, 553-563 (2000).

[36] G. Xiao, *Galois covers between K3 surfaces*, Annales de l’institut Fourier, tome 46, no 1 (1996), p. 73-88.

[37] R. Yamagishi, *On smoothness of minimal models of quotient singularities by finite subgroups of SL_n(C)*, Glasg. Math. J. 60, No. 3, 603-634 (2018).
Grégoire MENET
Laboratoire Paul Painlevé
59 655 Villeneuve d’Ascq cedex,
gregoire.menet@univ-lille.fr