Alternating Path and Coloured Clustering

CAI Leizhen* and LEUNG On Yin†

Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, New Territories, Hong Kong SAR, China

July 30, 2018

Abstract

In the Coloured Clustering problem, we wish to colour vertices of an edge coloured graph to produce as many stable edges as possible, i.e., edges with the same colour as their ends. In this paper, we reveal that the problem is in fact a maximum subgraph problem concerning monochromatic subgraphs and alternating paths, and demonstrate the usefulness of such connection in studying these problems.

We obtain a faster algorithm to solve the problem for edge-bicoloured graphs by reducing the problem to a minimum cut problem. On the other hand, we push the NP-completeness of the problem to edge-tricoloured planar bipartite graphs of maximum degree four. Furthermore, we also give FPT algorithms for the problem when we take the numbers of stable edges and unstable edges, respectively, as parameters.

1 Introduction

The following Coloured Clustering problem has been proposed recently by Angel et al. [3] in connection with the classical correlation clustering problem [5]: Compute a vertex colouring of an edge-coloured graph $G$ to produce as many stable edges as possible, i.e., edges with the same colour as their ends. As observed by Ageev and Kononov [1], the problem contains the classical maximum matching problem as a special case as the two problems coincide when all edges have different colours in $G$.

In this paper we will reveal that Coloured Clustering, despite its definition by vertex partition, is in fact the following maximum subgraph problem

---

*Email: lcai@cse.cuhk.edu.hk, Partially supported by CUHK Direct Grant 4055069
†Email: clp01234544@gmail.com
in disguise: find a largest subgraph where every vertex has one colour for its incident edges (Vertex-Monochromatic Subgraph), or, equivalently, delete fewest edges to destroy all alternating paths (Alternating Path Removal). This multiple points of view gives us a better understanding of these problems, and is quite useful in studying them.

We are mainly interested in algorithmic issues of Coloured Clustering, and will consider polynomial-time algorithms, NP-completeness, and also FPT algorithms for the two natural parameters from the subgraph point of view: numbers of edges inside (stable edges) and outside (unstable edges), respectively, the solution subgraph.

1.1 Main results

We now summarize our main results for Coloured Clustering, where \( m \) and \( n \), respectively, are numbers of edges and vertices in \( G \). These results can be translated directly into corresponding results for Vertex-Monochromatic Subgraph and Alternating Path Removal.

- We obtain an \( O(m^{3/2} \log n) \)-time algorithm for edge-bicoloured graphs \( G \) by a reduction to the classical minimum cut problem, which improves the \( O(m^{3/2}n) \)-time algorithm of Angel et al. \[3\] based on independent sets in bipartite graphs. We also give linear-time algorithms for the special case when \( G \) is a complete graph (see §4).

- We push the NP-completeness of the problem to edge-tricoloured planar bipartite graphs of maximum degree four (see §5).

- We derive FPT algorithms for the problem when we take the numbers of stable edges and unstable edges, respectively, as parameter \( k \), which is uncommon for most problems parameterized in this way. Furthermore, we obtain a kernel with at most \( 4k \) vertices and \( 2k^2 + k \) edges for the latter problem (see §6).

1.2 Related work

Both monochromatic subgraphs and alternating paths are at least half-century old, and there is a huge number of papers in the literature dealing with them graph theoretically \[4\] [10]. However, we are not aware of any work on these two subjects that is directly related to the algorithmic problems we study in this paper.

In the literature, papers by Angel et al. \[3\] and Ageev and Kononov \[1\] seem to be the only work that directly study Coloured Clustering. Angel et al. obtain an LP-based \( 1/e^2 \)-approximation algorithm for the problem in
general, which is improved to a 7/23-approximation algorithm by Ageev and Kononov. Angel et al. also give a polynomial-time algorithm for the problem on edge-bicoloured graphs by a reduction to the maximum independent set problem on bipartite graphs, but show the NP-completeness of the problem for edge-tricoloured bipartite graphs.

2 Definitions

An edge-coloured graph $G$ is a simple graph where each edge $e$ has a unique colour $\psi(e) \in \{1, \ldots, t\}$ for some positive integer $t$. We say that $G$ is edge-bicoloured if $t = 2$, and edge-tricoloured if $t = 3$. Unless specified otherwise, we use $m$ and $n$, respectively, for the numbers of edges and vertices of $G$.

A vertex $v$ is colourful if its incident edges have at least two different colours, and monochromatic otherwise. A subgraph $H$ of $G$ is vertex-monochromatic if all vertices in $H$ are monochromatic vertices of $H$, and edge-monochromatic if all edges in $H$ have the same colour.

A conflict pair is a pair of adjacent edges of different colours, and an alternating path is a simple path where every pair of consecutive edges forms a conflict pair. The edge-conflict graph of $G$, denoted $X(G)$, is an uncoloured graph where each vertex represents an edge of $G$ and each edge corresponds to a conflict pair in $G$.

A vertex colouring $f$ of $G$ assigns to each vertex $v$ of $G$ a colour $f(v) \in \{1, \ldots, t\}$ for some positive integer $t$. For a vertex colouring $f$ of $G$, an edge $uv$ is stable if its colour $\psi(uv) = f(u) = f(v)$, and unstable otherwise.

Angel et al. have recently proposed the following problem, which is in fact, as we will see shortly, the problem of finding the largest vertex-monochromatic subgraph in $G$ in disguise (see Lemma 3.1).

Coloured Clustering

Input: Edge coloured graph $G$ and positive integer $k$.

Question: Is there a vertex colouring of $G$ that produces at least $k$ stable edges?

The following problem is concerned with purging conflict-pairs (equivalently, alternating paths) by edge deletion, and is the complementary problem of Coloured Clustering (see Corollary 3.2).

Conflict-Pair Removal

Input: Edge coloured graph $G$ and positive integer $k$.

Question: Does $G$ contain at most $k$ edges $E'$ such that $G - E'$ contains no conflict pair?
3 Basic properties

Although COLOURED CLUSTERING is defined by vertex partition (i.e., vertex colouring), it is in fact a maximum subgraph problem in disguise. To see this, we first observe the following equivalent properties for edge-coloured graphs.

**Lemma 3.1** The following statements are equivalent for any edge-coloured graph $G$:

(a). $G$ is vertex-monochromatic.

(b). Every component of $G$ is edge-monochromatic.

(c). $G$ has no alternating path.

(d). $G$ has no conflict pair.

**Proof.** The equivalence between (a) and (b) is obvious, and so is the equivalence between (c) and (d) as a conflict pair is itself an alternating path. Furthermore, it is again obvious that $G$ contains a conflict pair if and only if $G$ has a colourful vertex, and therefore (a) and (d) are equivalent. It follows that the four statements are indeed equivalent.

Observe that for any vertex colouring of $G$, the subgraph formed by stable edges is vertex-monochromatic, and hence COLOURED CLUSTERING is actually equivalent to finding a largest vertex-monochromatic subgraph, which in turn is equivalent to deleting fewest edges to destroy all conflict pairs. This gives us the following complementary relation between COLOURED CLUSTERING and CONFLICT-PAIR REMOVAL.

**Corollary 3.2** There is vertex colouring for $G$ that produces at least $k$ stable edges if and only if $G$ contains at most $m - k$ edges $E'$ such that $G - E'$ has no conflict pair.

The bilateral relation between COLOURED CLUSTERING and CONFLICT-PAIR REMOVAL is akin to that between the classical INDEPENDENT SET and VERTEX COVER. In fact, the former two problems become exactly the latter two in the edge-conflict graph $X(G)$ of $G$ (see Theorem 3.3).

It is very useful to view COLOURED CLUSTERING as an edge deletion problem, instead of a vertex partition problem, which often makes things easier. For instance, it becomes straightforward to obtain the following result of Angel et al. [3] for $X(G)$.

**Theorem 3.3** [Angel et al.3] An edge-coloured graph $G$ admits a vertex colouring that produces at least $k$ stable edges if and only if the edge-conflict graph $X(G)$ of $G$ has an independent set of size at least $k$. 
Proof. By Corollary 3.2 the former statement is equivalent to deleting at most \( m - k \) edge to obtain a graph without conflict pair, which is the same as \( X(G) \) has a vertex cover of size at most \( m - k \), and hence \( X(G) \) has an independent set of size at least \( k \). \( \blacksquare \)

4 Algorithms for edge-bicoloured graphs

Although \textsc{Coloured Clustering} is NP-complete for edge-tricoloured graphs [3], Angel et al. [3] have obtained an \( O(m^{3/2}/n) \)-time algorithm for the problem on edge-bicoloured graphs \( G \) by reducing it to the maximum independent set problem on bipartite graphs \( X(G) \). In this section, we will give a faster \( O(m^{3/2} \log n) \)-time algorithm by considering \textsc{Conflict-Pair Removal}, which leads us to a simple reduction to a minimum cut problem. We also give a linear-time algorithm for the problem on edge-bicoloured complete graphs.

4.1 Faster algorithm

One bottleneck of the algorithm of Angel et al. lies in the size of the edge-conflict graph \( X(G) \) which contains \( O(m) \) vertices and \( O(mn) \) edges. Here we use a different approach of reduction to construct a digraph \( G' \) with only \( O(m) \) vertices and edges, and then solve an equivalent minimum cut problem on \( G' \) to solve our problem.

Let \( G = (V,E) \) be an edge-bicoloured graph with colours \{1, 2\}, and consider \textsc{Conflict-Pair Removal}. Our idea is to transform every conflict pair in \( G \) into an \((s,t)\)-path in a digraph \( G' \) with source \( s \) and sink \( t \). For this purpose, we construct digraph \( G' \) from \( G \) as follows (see Figure 1 for an example of the construction):

1. Take graph \( G \) and add two new vertices — source \( s \) and sink \( t \).

2. For each edge \( v_iv_j \) of \( G \), create a new vertex \( v_{ij} \) to represent edge \( v_iv_j \). If \( v_iv_j \) has colour 1 then replace it by two edges \( v_{ij}v_i \) and \( v_{ij}v_j \) and add edge \( sv_{ij} \). Otherwise replace the edge by two edges \( v_{ij}v_i \) and \( v_jv_{ij} \) and add edge \( v_{ij}t \).

An \((s,t)\)-cut in \( G' \) is a set of edges whose deletion disconnects sink \( t \) from source \( s \). Let \( v_{ij}v_{ij'} \) and \( v_{ij}v_{ij'}' \) be an arbitrary conflict pair of \( G \). Without loss of generality, we may assume that \( v_iv_j \) has colour 1 and \( v_{ij}v_{ij'} \) has colour 2. By the construction of \( G' \), there is a unique \((s,t)\)-path

\[
P(j,i,j') = sv_{ij}, v_{ij}v_i, v_{ij}v_{ij'}, v_{ij'}t
\]

in \( G' \) that goes through vertices \( v_{ij} \) and \( v_{ij'} \). For convenience, we refer to edges \( sv_{ij} \) and \( v_{ij}t \) as \textit{external edges} and the other two edges as \textit{middle edges}. Edges
Figure 1: Digraph $G'$ from graph $G$, where shaded vertices in $G'$ correspond to edges in $G$ and thick edges indicate corresponding solution edges.

$v_iv_j$ and $v_iv_j'$ of $G$ correspond to external edges $sv_{ij}$ and $v_{ij}'t$, respectively, in $G'$. We also call an $(s,t)$-cut a normal cut if the cut contains no middle edge of any $P(j,i,j')$.

**Lemma 4.1** Let $E'$ be a set of edges in $G$. Then $G - E'$ contains no conflict pair if and only if corresponding edges of $E'$ in $G'$ form a normal $(s,t)$-cut of $G'$.

**Proof.** There is a one-to-one correspondence between conflict pair $v_iv_j$ and $v_iv_j'$ in $G$ and external edges $sv_{ij}$ and $v_{ij}'t$ in $G'$ in such a way that the conflict pair is destroyed if and only if the $(s,t)$-path $P(j,i,j')$ is disconnected. This clearly implies the lemma.

The above lemma enables us to solve **CONFLICT-PAIR REMOVAL** on edge-bicoloured graphs by reducing it to the minimum cut problem, which yields a faster algorithm.

**Theorem 4.2** **CONFLICT-PAIR REMOVAL** for edge-bicoloured graphs $G$ can be solved in $O(m^{3/2} \log n)$.

**Proof.** By Lemma 4.1 we can reduce our problem on $G$ to the minimum normal $(s,t)$-cut problem on digraph $G'$. Observe that for any $(s,t)$-cut, we can always replace a middle edge by an external edge without increasing the size of the cut. Therefore we need only solve the minimum $(s,t)$-cut problem on digraph $G'$, which can be accomplished by the maximum flow algorithm of Goldberg and Rao [9].

For the running time of the algorithm, we first note that $G'$ contains $N = m + n + 2$ vertices and $M = 3m$ edges, and can be constructed in $O(m + n)$
time. Since every edge of $G'$ has capacity 1, Goldberg and Rao’s algorithm takes $O(\min(N^{2/3}, M^{1/2}) M \log N^2/M)$ time, which gives us $O(m^{3/2} \log n)$ time as $M, N = O(m)$.

**Corollary 4.3** Coloured Clustering for edge-bicoloured graphs $G$ can be solved in $O(m^{3/2} \log n)$ time.

### 4.2 Complete graphs

We now turn to the special case of Coloured Clustering when $G = (V, E)$ is an edge-bicoloured complete graph, and present a linear-time algorithm. Let $f$ be a vertex-2-colouring of $G$ that colours vertices $V_1$ by colour 1 and vertices $V_2$ by colour 2. For a vertex $v$, let $d_1(v)$ be the number of edges of colour 1 incident with $v$. Let $m_1$ be the number of edges with colour 1. We can completely determine the number of stable edges produced by $f$ as follows.

**Lemma 4.4** For a vertex-2-colouring $f$ of an edge-bicoloured complete graph $G$, the number $S_f$ of stable edges produced by $f$ equals

$$\sum_{v \in V_1} d_1(v) + \left( \frac{|V_2|}{2} \right) - m_1.$$

**Proof.** Let $A$ and $B$ be numbers of edges of colour 1 in $G[V_1]$ and $G[V_2]$ respectively. By the definition of stable edges, we have

$$S_f = A + \left( \frac{|V_2|}{2} \right) - B$$

as $G[V_2]$ is a complete graph. On the other hand, $B = m_1 - C$, where $C$ is the number of edges of colour 1 covered by vertices $V_1$. Therefore

$$S_f = A + \left( \frac{|V_2|}{2} \right) + C - m_1,$$

and the lemma follows from the fact that $A + C = \sum_{v \in V_1} d_1(v)$.

With the formula in the above lemma, we can easily and efficiently solve Coloured Clustering for edge-bicoloured complete graphs.

**Corollary 4.5** Coloured Clustering can be solve in $O(n^2)$ for edge-bicoloured complete graphs.

**Proof.** From Lemma 4.4 we see that once we fix the size of $V_1$ to be $k$, $S_f$ is maximized when we choose $k$ vertices $v$ with largest $d_1(v)$ as vertices in $V_1$. 

Therefore we can compute the maximum value of $S_f$ for each $0 \leq k \leq n$, and find an optimal vertex-2-colouring for $G$. The whole process clearly takes $O(n^2)$ time as we can first sort vertices according to $d_1(v)$.

We can also use a similar idea to solve Coloured Clustering in $O(n^2)$ time for edge-bicoloured complete bipartite graphs, which will appear in our full paper.

## 5 NP-completeness

Angel et al. [3] have shown the NP-completeness of Coloured Clustering for edge-tricoloured bipartite graphs. In this section, we further push the intractability of the problem to edge-tricoloured planar bipartite graphs of bounded degree. Recall that a vertex colouring is proper if the two ends of every edge receive different colours.

**Theorem 5.1** Coloured Clustering is NP-complete for edge-tricoloured planar bipartite graphs of maximum degree four.

**Proof.** Garey, Johnson and Stockmeyer [8] proved the NP-completeness of Independent Set on cubic planar graphs, and we give a reduction from this restricted case of Independent Set to our problem. For an arbitrary cubic planar graph $G = (V,E)$ with $V = \{v_1, \ldots, v_n\}$, we construct an edge-tricoloured planar bipartite graph $G' = (V', E')$ of maximum degree four as follows:

1. Compute a proper vertex 3-colouring $\psi$ of $G$.

2. For each edge $v_iv_j \in E$, subdivide it by a new vertex $v_{ij}$ (i.e., replace edge $v_iv_j$ by two edges $v_iv_{ij}$ and $v_{ij}v_j$), and colour edges $v_iv_{ij}$ and $v_{ij}v_j$ by $\psi(v_i)$ and $\psi(v_j)$ respectively.

3. For each vertex $v_i \in V$, add a new vertex $v_i^*$ and edge $v_iv_i^*$, and colour edge $v_i^*v_i$ by a colour in $\{1, 2, 3\}$ different from $\psi(v_i)$.

It is clear that $G'$ is an edge-tricoloured planar bipartite graphs of maximum degree four. By Brooks’ Theorem, every cubic graph except $K_4$ admits a proper vertex 3-colouring, and we can use an algorithm of Lovász [11] to compute a proper vertex 3-colouring of a cubic graph in linear time. Therefore, the above construction of $G'$ takes polynomial time. We claim that $G$ has an independent set of size $k$ if and only if $G'$ admits a vertex colouring that produces $k + |E|$ stable edges.

Suppose that $G$ contains an independent set $I$ of size $k$. We define a vertex-3-colouring $f$ of $G'$ as follows:
1. For each vertex $v_i^* \in V^*$, set $f(v_i^*)$ to be the colour of edge $v_i v_i^*$.

2. For each vertex $v_i \in V$, set $f(v_i)$ to be the colour of edge $v_i v_i^*$ if $v_i \in I$ and $f(v_i) = \psi(v_i)$ otherwise.

3. For each vertex $v_{ij}$, set $f(v_{ij})$ to be $\psi(v_i)$ if $v_i \not\in I$ and $\psi(v_j)$ otherwise.

Clearly, $f$ produces $k$ stable edges $v_i v_i^*$ after Step 2. For any vertex $v_{ij}$, since $I$ contains at most one of $v_i$ and $v_j$, exactly one of $v_i v_{ij}$ and $v_{ij} v_j$ becomes a stable edge after Step 3. Therefore $f$ produces $k + |E|$ stable edges for $G'$.

Conversely, call each edge $v_i v_i^*$ an outside edge, and let $f'$ be a vertex 3-colouring of $G'$ that produces $k + |E|$ stable edges and also minimizes the number of outside edges among these stable edges. Let

$$I = \{v_i : \text{edge } v_i v_i^* \text{ is stable}\}.$$

For every vertex $v_{ij}$ in $G'$, since edges $v_i v_{ij}$ and $v_{ij} v_j$ have different colours, at most one of these two edges is a stable edge for any vertex colouring of $G'$. It follows that at least $k$ stable edges are formed by outside edges and hence $I$ contains at least $k$ vertices.

We claim that $I$ is an independent set of $G$. Suppose to the contrary that for some vertices $v_i, v_j \in I$, $v_i v_j$ is an edge of $G$. First we note that amongst all edges incident with $v_i$, $v_i v_i^*$ is the only stable edge under $f'$ as $f'(v_i) \neq \psi(v_i)$, and similar situation holds for all edges incident with $v_j$. In particular, neither $v_i v_{ij}$ nor $v_{ij} v_j$ is a stable edge. We now recolour both vertices $v_i$ and $v_{ij}$ by the colour of edge $v_i v_{ij}$ (note that $v_{ij}$ may have received that colour already under $f'$) to obtain a new vertex 3-colouring $f''$ (see Figure 2 for an example of the situation).

![Figure 2: (a) Situation under vertex colouring $f'$. (b) Situation after recolouring vertices $v_i$ and $v_{ij}$. Stable edges are indicated by thick edges.](image)

Comparing with colouring $f'$, this new colouring $f''$ reduces one stable edge (namely, edge $v_i v_i^*$), but produces a new stable edge $v_i v_i^*$ (and probably also other new stable edges). Therefore $f''$ produces at least $k + |E|$ stable edges.
that contains one less outside edges than $f'$, contradicting the choice of $f'$. This contradiction implies that $I$ is indeed an independent set of $G$ with at least $k$ vertices, and hence the theorem holds.

**Corollary 5.2** Conflict-Pair Removal is NP-complete for edge-tricoloured planar bipartite graphs of maximum degree four.

6 FPT algorithms

We now turn to the parameterized complexity of Coloured Clustering, and give FPT algorithms for the problem with respect to both the number of stable edges and the number of unstable edges as parameter $k$. This is quite interesting as it is uncommon for a problem to admit FPT algorithms both ways when parameterized in this manner.

6.1 Stable edges

First we take the number of stable edges produced by a vertex colouring of $G$ as parameter $k$, and use Coloured Cluster $[k]$ to denote this parameterized problem. We will give an FPT algorithm that uses random partition in the spirit of the colour coding method of Alon, Yuster and Zwick [2], which implies an FPT algorithm for Independent Set $[k]$ in edge-conflict graphs. Note that if the number $t$ of colours in $G$ is a constant, then the problem is trivially solved in FPT time as it contains a trivial kernel with at most $kt$ edges and hence $2kt$ vertices. Also note that the problem is not as easy as it looks, for it contains the maximum matching problem as a special case when all edges have different colours.

Our idea is to randomly partition vertices of $G$ into $k$ parts $V_1,\ldots,V_k$ in a hope that a $k$-solution consists of $k_i$ stable edges, where $\sum_{i=1}^{k} k_i = k$, in each $G[V_i]$. Indeed we have a good chance to succeed in this way.

**Algorithm** Coloured-Clustering$[k]$

Randomly partition vertices $V$ of $G$ into $V_1,\ldots,V_k$.

Compute the most frequently used colour $c_i$ for each $G[V_i]$.

Colour all vertices in $V_i$ by $c_i$.

**Lemma 6.1** For any yes-instance of Coloured Cluster$[k]$, the vertex colouring constructed by Algorithm Coloured-Clustering$[k]$ has probability at least $k^{-2k}$ to produce at least $k$ stable edges.
Proof. Consider a vertex colouring of $G$ that produces at least $k$ stable edges $E'$, which clearly have at most $k$ different colours. Let $E'_i$ be edges in $E'$ of colour $c_i$ and let $k_i = |E'_i|$ for $1 \leq i \leq k$. We estimate the probability that all edges of $E'_i$ lie in $G[V_i]$. A vertex has probability $k^{-1}$ to be in $V_i$, and hence the above event happens with probability at least $k^{-2k}$ as $E'_i$ contains at most $2k_i$ vertices. It follows that, with probability at least

$$k - \sum_{i=1}^{k} 2k_i = k^{-2k},$$

all edges of each $E'_i$ lie entirely inside $G[V_i]$. Therefore each $G[V_i]$ contains at least $k_i$ edges of same colour $c_i$, which can be made stable by colouring all vertices in $G[V_i]$ by colour $c_i$. It follows that the algorithm produces at least $k$ stable edges with probability $k^{-2k}$.

The algorithm runs in $O(k^{2k}(m+n))$ expected time, and can be made into a deterministic FPT algorithm by standard derandomization with a family of perfect hashing functions.

Theorem 6.2 Coloured Cluster$[k]$ can be solved in FPT time.

6.2 Unstable edges

Now we take the number of unstable edges in a vertex colouring as parameter $k$, and use Conflict-Pair Removal$[k]$ to denote this parameterized problem. By a result of Angel et al. [3], the problem is equivalent to finding a vertex cover of size $k$ in the edge-conflict graph $X(G)$ of $G$, and hence admits an FPT algorithm by transforming it to the $k$-vertex cover problem in $X(G)$. However the time for the transformation takes $O(mn)$ time as $X(G)$ contains $O(m)$ vertices and $O(mn)$ edges, and the total time for the algorithm takes $O(mn + 1.2783^k)$ time. Here we combine kernelization with weighted vertex cover to obtain an improved algorithm with running time $O(m + n + 1.2783^k)$.

To start with, we construct in linear time the following edge-coloured weighted graph $G^*$, called condensed graph, by representing monochromatic vertices of one colour by a single vertex, and then parallel edges between two vertices by a single weighted edge. See Figure 3 for an example of the construction.

**Step 1.** For each colour $c$, contract all monochromatic vertices of colour $c$ into a single vertex $v_c$.

**Step 2.** For each pair of adjacent vertices, if there is only one edge between them, then set the weight of the edge to 1, otherwise replace all parallel edges between them by a single edge of the same colour $c$ and set its weight to be the number of replaced parallel edges.

---

1 All such parallel edges have the same colour as they correspond to edges between a vertex and monochromatic vertices of the same colour.
Figure 3: (a) Edge-coloured graph $G$ where each dashed ellipse indicates monochromatic vertices of same colour. (b) The condensed graph $G^*$ of $G$ where an edge of weight more than 1 has its weight as the superscript of its colour.

It turns out that the clustering problem on $G$ is equivalent to a weighted version of the problem on the condensed graph $G^*$.

**Lemma 6.3** Graph $G$ has at most $k$ unstable edges if and only if $G^*$ has unstable edges of total weight at most $k$.

**Proof.** By the construction of $G^*$, we have the following correspondence between edges in $G$ and $G^*$: every edge in $G$ between two colourful vertices remains so in $G^*$, and for any colourful vertex $v$, all edges between $v$ and monochromatic vertices of colour $c$ correspond to edge $vv_c$ in $G^*$. Also all monochromatic vertices in $G^*$ form an independent set.

Now suppose that $G$ has a vertex colouring $f$ that produces $k$ unstable edges. Without loss of generality, we may assume that every monochromatic vertex $v$ in $G$ has its own colour as $f(v)$ since this will not increase unstable edges. For this $f$, we have a natural vertex colouring $f^*$ for $G^*$: the colour of each vertex retains its colour under $f$. It is obvious that an edge in $G^*$ between two colourful vertices is an unstable edge under $f^*$ if and only if it is an unstable edge in $G$ under $f$. For edges $E_c(v)$ in $G$ between a colourful vertex $v$ and monochromatic vertices with colour $c$, either all edges in $E_c(v)$ are stable or all are unstable as $f$ colours all these monochromatic vertices by colour $c$, implying that all edges in $E_c(v)$ are unstable under $f$ if and only if $vv_c$ is unstable under $f^*$. Therefore $f^*$ produces unstable edges of total weight $k$ in $G^*$.

Conversely, suppose that $G^*$ contains a set $U$ of unstable edges of total weight $k$, and let $U'$ be the corresponding $k$ edges in $G$. Clearly $G-U'$ contains no conflict pair, and hence $G$ has at most $k$ unstable edges. 

\[\blacksquare\]
Further to the above lemma, $G^*$ can be regarded as a kernel as its size is bounded by a function of $k$ whenever $G$ has at most $k$ unstable edges. Note that the bounds in the following lemma are tight.

**Lemma 6.4** If $G$ has at most $k$ unstable edges, then $G^*$ has at most $4k$ vertices and $2k^2 + k$ edges.

**Proof.** Let $[C, M]$ be the cut that partitions the vertices of $G$ into colourful vertices $C$ and monochromatic vertices $M$. Let $A$ be the set of unstable edges inside $G[C]$, and $B$ the set of unstable edges across the cut. Observe that each edge of $A$ is incident with at most two vertices of $C$, and each edge of $B$ is incident with one vertex of $C$. Furthermore, every colourful vertex is incident with at least one unstable edge. Therefore $|C| \leq 2|A| + |B| \leq 2k$.

Now consider the condensed graph $G^*$, and note that the cut $[C, M]$ corresponds to the cut $[C, M^*]$ for monochromatic vertices $M^*$ of $G^*$. Furthermore, $A$ consists of unstable edges inside $G^*[C]$, and $B$ corresponds to unstable edges $B^*$ across $[C, M^*]$ and $|B^*| \leq |B|$.

In $G^*$, every vertex in $C$ is incident with at most one stable edge in $[C, M^*]$. Therefore $[C, M^*]$ contains at most

$$|C| + |B^*| \leq (2|A| + |B|) + |B^*| \leq 2(|A| + |B|) = 2k$$

edges, and hence $M^*$ contains at most $2k$ vertices. It follows that $G^*$ contains at most $4k$ vertices, and at most $2k^2 + k$ edges as $G^*[M^*]$ is edgeless.

With Lemma 6.3 and Lemma 6.4 in hand, we obtain the following FPT algorithm for **CONFLICT-PAIR REMOVAL**$[k]$.

**Algorithm** Conflict-Pair-Removal$[k]$

Construct the condensed graph $G^*$ from $G$;

if $G^*$ contains more than $4k$ vertices or $2k^2 + k$ edges

then return “No” and **halt**;

Construct the edge-conflict $X(G^*)$ of $G^*$;

if $X(G^*)$ has a vertex cover of weight at most $k$

then return “Yes”

else return “No”.

**Theorem 6.5** **CONFLICT-PAIR REMOVAL**$[k]$ can be solved in $O(m+n+1.2783^k)$ time.

**Proof.** The correctness of the algorithm follows from Lemma 6.3 and Lemma 6.4, and we analyze the running time of the algorithm. The construction of the
condensed graph \( G^* \) clearly takes \( O(m + n) \) time, and the construction of the edge-conflict graph \( X(G^*) \) takes \( O(k^3) \) time as \( G^* \) contains \( O(k) \) vertices and \( O(k^2) \) edges. Note that \( X(G^*) \) contains \( O(k^2) \) vertices and \( O(k^3) \) edges. Since it takes \( O(kn + 1.2783^k) \) to solve the weighted vertex cover problem \([7, 12]\), it takes \( O(k^3 + 1.2783^k) = O(1.2783^k) \) to solve the problem for \( G^* \), and hence the overall time is \( O(m + n + 1.2783^k) \).

7 Concluding remarks

We have revealed that COLOURED CLUSTERING, a vertex partition problem, is in fact subgraph problems VERTEX-MONOCHROMATIC SUBGRAPH and ALTERNATING PATH REMOVAL in disguise, and demonstrated the usefulness of this multiple points of view in studying these problems. Indeed, our improved algorithm for edge-bicoloured graphs and FPT algorithms for general edge-coloured graphs have benefited a lot from the perspective of CONFLICT-PAIR REMOVAL.

We now briefly discuss a few open problems in the language of monochromatic subgraphs and alternating paths for readers to ponder.

**Question 1.** For edge-bicoloured graphs, is there a faster algorithm for deleting fewest edges to obtain a vertex-monochromatic subgraph?

There seems to be a good chance to solve the problem faster than our algorithm, and one possible approach is to reduce the number of vertices in the reduction to minimum cut from the current \( O(m) \) to \( O(n) \).

**Question 2.** For CONFLICT-PAIR REMOVAL on general edge-coloured graphs, is there an \( r \)-approximation algorithm for some constant \( r < 2 \)?

The problem admits a simple 2-approximation algorithm through its connection with VERTEX COVER, and seems easier than the latter problem. It is possible that we can do better for the problem, perhaps through ILP relaxation.

**Question 3.** For edge-coloured graphs, does the problem of finding a vertex-monochromatic subgraph with at least \( k \) edges admit a polynomial kernel?

The above problem is appealing for its connection with the classical maximum matching problem. On one hand, we may use a maximum matching of \( G \) as a starting point for a polynomial kernel; and on the other hand if we can obtain a polynomial kernel of \( G \) in \( o(m\sqrt{n}) \) time, we may use the kernel to speed up maximum matching algorithms. Of course, it may be the case that the problem admits no polynomial kernel unless \( \text{NP} \subseteq \text{coNP/poly} \).

**Question 4.** For edge-coloured graphs, is there an FPT algorithm for the
problem of destroying all alternating cycles by deleting at most $k$ edges?

Although the definition of the problem resembles that of Alternating Path Removal, the problem seems much more difficult as the problem does not have a finite forbidden structure like conflict pair for the latter problem. We note that the problem is NP-complete by a simple reduction from Vertex Cover.

References

[1] Ageev, A., and Kononov, A., Improved Approximations for the Max $k$-Colored Clustering Problem. In: International Workshop on Approximation and Online Algorithms 2014, LNCS 8952 (pp. 1-10), 2015.

[2] Alon, N., Yuster, R., and Zwick, U., Color-coding, Journal of the ACM 42:844–856, 1995.

[3] Angel, E., Bampis, E., Kononov, A., Paparas, D., Pountourakis, E., and Zissimopoulos, V., Clustering on $k$-edge-colored graphs. Discrete Applied Mathematics, 211:15-22, 2016.

[4] Bang-Jensen, J., and Gutin, G., Alternating cycles and paths in edge-coloured multigraphs: a survey. Discrete Mathematics, 165:39-60, 1997.

[5] Bansal, N., Blum, A., and Chawla, S., Correlation clustering. Machine Learning, 56(1-3):89-113, 2004.

[6] Cai, L., Fixed-parameter tractability of graph modification problems for hereditary properties. Information Processing Letters, 58(4):157–206, 1996.

[7] Chen, J., Kanj, I. A., and Xia, G., Improved upper bounds for vertex cover. Theoretical Computer Science, 411(40-42):3736-3756, 2010.

[8] Garey, M. R., Johnson, D. S., and Stockmeyer, L., Some simplified NP-complete graph problems. Theoretical computer science, 1(3):237-267, 1976.

[9] Goldberg, A. V., and Rao, S., Beyond the flow decomposition barrier. Journal of the ACM (JACM), 45(5):783-797, 1998.

[10] Kano, M., and Li, X., Monochromatic and heterochromatic subgraphs in edge-colored graphs-a survey. Graphs and Combinatorics, 24(4):237-263, 2008.

[11] L. Lovász, Three short proofs in graph theory. J. Combin. Theory Ser. B, 19:269-271, 1975.

[12] Niedermeier, R., and Rossmanith, P., On efficient fixed-parameter algorithms for weighted vertex cover. Journal of Algorithms, 47(2):63-77, 2003.