COUNTING FIXED POINTS FREE VECTOR FIELDS ON $\mathbb{B}^2$

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Abstract. The purpose of this elementary note is to count stationary points free general position vector fields on the unit disk $\mathbb{B}^2$, depending on the number of touch points between the flow lines and the boundary. Of course, we are counting the number of classes under a natural equivalence relation on such fields. It turns out that the number of such objects (diagrams) may be calculated via some formula involving Catalan numbers, moreover, the corresponding sequence may be identified as sequence A275607 from OEIS (the On-line Encyclopedia of Integer Sequences). Catalan numbers appear while counting various objects of geometric nature. Finally, an algorithm for finding all such diagrams under consideration, is proposed. It is based on enumerating all independent vertex sets of some highly symmetric graph of geometric origin.

1. Preliminary

A generic (smooth) vector field without stationary points in the disk $\mathbb{B}^2$ may be represented by a simple diagram as depicted at Fig. 1. Here we are identifying the vector field with the induced flow on the disk, so we shall classify fields by investigating the corresponding flow diagrams.

Let us start with some simple observations about general position fields in $\mathbb{B}^2$:

a) A flow line is touching the boundary $\mathbb{S}^1$ in at most 1 point.

b) The number of touch points between all the flow lines and the boundary is finite.

c) Suppose there are $m$ touch points between the flow lines and $\mathbb{S}^1$, yet there exist $m+2$ other exceptional points on the boundary where the flow lines diagram is formed locally by “semi-circles”. We shall refer to latter points as $a$-nodes, while the touch points between level lines and the boundary will be referred to as $b$-nodes. For example, at Fig. 1 the total number of nodes is 10 and we have 6 $a$-nodes (1,2,4,5,6,8) and 4 $b$-nodes (3,7,9,10). (In some sense, $a$-nodes may be thought of as “sources”, while $b$-nodes - as “sinks”.) Now, the main observation is that

$$\#(a\text{-nodes}) - \#(b\text{-nodes}) = 2.$$

This equality becomes evident after drawing several examples by hand, but also may be proved rigorously either combinatorially, or by counting the degree of the vector field along the boundary $\mathbb{S}^1$ (it should equal 0, since the field is stationary points free). Note that the total number of nodes $2m + 2$ is even.

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In view of the above remarks, we may now settle the frame for counting fields. We have some fixed set $A$ of $n$ points on the circle $S^1$, where $n$ is even

$$A = \{p_1, \ldots, p_n\}.$$ 

We shall call points $p_i$ nodes. (Without loss of generality we may suppose $p_i$ to be the vertices of a fixed regular $n$-gon.) Then we are considering the class $\mathcal{V}_n$ of vector fields such that any field $v \in \mathcal{V}_n$ has $\frac{n}{2} + 1$ $a$-nodes and $\frac{n}{2} - 1$ $b$-nodes, all of them lying in set $A$. Now we define some natural equivalence relation in $\mathcal{V}_n$:

$v_1 \sim v_2$ if there is an orientation preserving diffeomorphism $\varphi : \mathbb{B}^2 \to \mathbb{B}^2$ such that $\varphi|_A = \text{id}$ (the nodes are fixed) and $\varphi$ is transforming the flow lines of $v_1$ onto those of $v_2$.

Let $D_n = \tilde{\mathcal{V}}_n$ be the corresponding quotient set. It is clear that the equivalence classes are finite in number and may be visualized by simple diagrams. Set

$$T_n = |D_n|,$$

so, $T_n$ is the number of diagrams with $n$ nodes.

The main purpose of this article is to find the numbers $T_n$. We shall prove that

$$T_n = 3^{\frac{n}{2} - 2} \left( C_{\frac{n}{2}} + 2C_{\frac{n}{2} - 1} \right),$$

(1.1)

where $C_m$ is the $m$-th Catalan number (cf. [1], [2], [3])

$$C_m = \frac{(2m)!}{(m+1)!m!}.$$
Setting \( n = 2k \), we get a more compact formula
\[
T_{2k} = 3^{k-2} \left( C_k + 2C_{k-1} \right).
\] (1.2)

Taking into account that the first Catalan numbers for \( m = 0, 1, 2, \ldots \) are
\[
1, 1, 2, 5, 14, 42, 132, 429, \ldots,
\]

it is not difficult to calculate the first terms of sequence \( T_{2k} \) for \( k \geq 1 \):
\[
1, 4, 27, 216, 1890, 17496, 168399, \ldots,
\]

so, \( T_2 = 1 \), \( T_4 = 4 \), \( T_6 = 27 \), \( T_8 = 216 \), etc. A quick check in OEIS’s database suggests that highly likely \( \{T_{2k}\} \) is in fact sequence A275607 (see [4]). We show

Figure 2. \( T_2 = 1, T_4 = 4 \).
Figure 3. \( T_6 = 3.9 = 27 \) (rotation at \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \)).

this later by comparing formula (1.1) with the explicit expression defining A275607. The diagrams adjusting that \( T_2 = 1, T_4 = 4 \) and \( T_6 = 27 \) are depicted at Fig. 2 and Fig. 3, respectively. In the latter figure one has to perform rotations at angle \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \) to get all 3.9 = 27 solutions. All 27 diagrams are different, which becomes evident after examining the position of the vertical punctured line after rotation.

Note that the numbers \( T_n \) are counting vector fields only \textit{combinatorially}, not \textit{topologically}. The latter problem seems to be much harder and will be discussed at the end.

2. IN IDENTITY WITH CATALAN NUMBERS

The Catalan numbers may be defined by the basic recurrence relation

\[
C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}. \tag{2.1}
\]

Iterating the above identity, it is not difficult to obtain some relation involving triple products \( C_i C_j C_k \).

**Proposition 1.** For \( n \geq 2 \) the following identity holds true

\[
\sum_{i+j+k=n-1} C_i C_j C_k = C_{n+1} - C_n. \tag{2.2}
\]
where the sum is taken over all triples \((i, j, k)\) such that \(i + j + k = n - 1\) and \(i, j, k \geq 0\).

For example, for \(n = 3\) the above formula gives

\[
C_0C_1C_1 + C_1C_0C_1 + C_1C_1C_0 + C_0C_0C_2 + C_0C_2C_0 + C_2C_0C_0 = C_4 - C_3.
\]

After substitution we get

\[
1 + 1 + 1 + 2 + 2 + 2 = 14 - 5,
\]

which is, of course, true.

To prove the general case, it suffices to write the basic relation \((2.1)\) in the form

\[
C_{n+1} - C_n = \sum_{i=1}^{n} C_iC_{n-i} = \sum_{i=1}^{n} \left( \sum_{j=0}^{i-1} C_jC_{i-j-1} \right) C_{n-i}
\]

and to notice that the last expression is exactly the left-hand side of \((2.2)\).

3. PROOF OF THE MAIN FORMULA

Let \(D_n^0 \subset D_n\) be the set of \((n\)-nodes\) diagrams with a fixed node \((say\ p_1)\) which is prescribed to be an \(a\)-node. The following proposition is counting the number of such field classes.

**Lemma 1.** \(|D_n^0| = 3^{n-1} - 1 = 3^{n-1} \frac{(n-2)!}{(\frac{n}{2})!(\frac{n}{2})!}\)

**Proof.** Set \(\alpha_n = |D_n^0|\). Note that there are exactly 3 situations realizing elements from \(D_n^0\), these are depicted at Fig. 4. This may be seen by following the flow lines going out of node \(p_1\) and notice that they have to lean on a touching line \(\lambda\) (for \(n \geq 4\)). Then, according to the position of this line with respect to \(p_1\), there are 3 topologically different possibilities represented at Fig. 4. The above touching line is separating the disk into 3 regions, say \(A, B, C\), we shall suppose that \(C\) is containing node \(p_1\). Note that in region \(C\) the flow is very simple and topologically unique (a sort of a flow-box), so \(C\) will not take part in the calculation.

Observe furthermore that the restriction of the field diagram under consideration on regions \(A\) and \(B\) may be interpreted as some elements from \(D_k^0\) and \(D_{n-k}^0\), respectively, where \(k\) is even and \(2 \leq k \leq n - 2\). This may be seen as follows: we may split \(B^2\) along the touching line \(\lambda\) into 3 disjoint pieces and then to shrink the part of \(\lambda\) lying on each one, as shown at Fig. 5. In such a way we get 3 smaller disks \(A_0, B_0, C_0\) with a new born node on each one that occurs to be an \(a\)-node (source). This node will be set to be the distinguished one from the definition of \(D_n^0\). So, the induced fields on \(A_0, B_0, C_0\) are in fact elements from \(D_k^0, D_{n-k}^0\) and \(D_2^0\), respectively. Of course, \(D_2^0\) is a 1-point set, so it doesn’t take part in the calculation.

The next observation is that the 3 cases from Fig. 4 are identical from computational point of view. Therefore for numbers \(\alpha_n = |D_n^0|\) we get the following recurrence relation

\[
\alpha_n = 3 (\alpha_2\alpha_{n-2} + \alpha_4\alpha_{n-4} + \cdots + \alpha_{n-4}\alpha_4 + \alpha_{n-2}\alpha_2).
\]

Now it is not difficult to prove the lemma by induction. Suppose the formula is true for each \(k \leq n - 2\). Clearly, for \(n = 2\) it is true, as \(\alpha_2 = 1\). Then, after
substitution in the right-hand side of the recurrence relation, one gets
\[ \alpha_n = 3^{n - 1} \left( C_0 C_{n/2 - 2} + C_1 C_{n/2 - 3} + \cdots + C_{n/2 - 3} C_1 + C_{n/2 - 2} C_0 \right) = 3^{n - 1} C_{n/2 - 1}, \]
where in the last equality we make use of the basic relation (2.1) about Catalan numbers.

The lemma is proved.\( \square \)

Note that \( \alpha_n \) is in fact sequence A005159 from OEIS.

We are now ready to prove the main formula about arbitrary diagram \( s \).

**Theorem 1.** Let \( T_n = |D_n| \). Then \( T_n = 3^{n - 2} \left( C_{n/2} + 2C_{n/2 - 1} \right) \).

**Proof.** Take some \( v \in D_n \) and consider node \( p_1 \). There are two cases:

1) \( p_1 \) is an \( a \)-node
2) \( p_1 \) is a \( b \)-node.

Clearly, \( T_n \) is a sum of cases 1) and 2). In case 1) the number of such fields equals \( \alpha_n \) and is counted by **Lemma 1**.

Suppose now that \( p_1 \) is a \( b \)-node and \( \lambda \) is the touching line passing through it (see Fig. 6). Then \( \lambda \) is separating the disk into some regions \( A, B, C \). Now, splitting \( \mathbb{B}^2 \) along \( \lambda \) and reasoning exactly as in the proof of **Lemma 1**, it is clear that for this particular choice of the touching line \( \lambda \) we get a summand in case 2) of the form

![Figure 4. 3 cases for regions A, B, C, the flow on C is trivial.](image)
\[ \alpha_k \alpha_l \alpha_m \] where \( k + l + m = n + 2 \) for some even numbers \( k, l, m \geq 2 \). Taking into account these remarks we have
\[ T_n = \alpha_n + \sum_{k+l+m=n+2} \alpha_k \alpha_l \alpha_m, \]
where the sum runs over all triples of even numbers \( k, l, m \geq 2 \) for which \( k + l + m = n + 2 \). We now refer to Lemma 1 to calculate
\[ \alpha_k \alpha_l \alpha_m = 3^k - 1 C_{\frac{k}{2} - 1} C_{\frac{l}{2} - 1} C_{\frac{m}{2} - 1} = 3^{k-2} C_{\frac{k}{2} - 1} C_{\frac{l}{2} - 1} C_{\frac{m}{2} - 1}. \]
Furthermore, it is not difficult to see that by Proposition 1
\[ \sum_{k+l+m=n+2} C_{\frac{k}{2} - 1} C_{\frac{l}{2} - 1} C_{\frac{m}{2} - 1} = C_{\frac{n}{2}} - C_{\frac{n}{2} - 1}, \]
so, finally we have
\[ T_n = 3^\frac{n}{2} - 1 C_{\frac{n}{2} - 1} + 3^\frac{n}{2} - 2 (C_{\frac{n}{2}} - C_{\frac{n}{2} - 1}). \]
Figure 6. The case when $p_1$ is a $b$-node.

and after simplification one gets

$$T_n = 3^{\frac{n}{2} - 2} \left( C_{\frac{n}{2}} + 2C_{\frac{n}{2} - 1} \right).$$

The theorem is proved.

\[\square\]

**Remark 1.** It follows from the proof of the theorem that each touching line $\lambda$ takes part into exactly $3^{\frac{n}{2} - 2} C_{\frac{k}{2} - 1} C_{\frac{l}{2} - 1} C_{\frac{m}{2} - 1}$ diagrams from $D_n$. Here $k, l, m$ depend on the size of regions $A, B, C$.

**Remark 2.** It might be of some interest to consider the following specification of the problem:

Find the number of diagrams from class $D_n$ with prescribed distribution of $a$- and $b$-nodes. Clearly, it suffices to fix the places of $b$-nodes which may be done by $\binom{n}{n/2} - 1$ ways. Note that the result sensitively depends on the initial distribution of the nodes. For example, if $n = 6$ we have to choose a pair of $b$-nodes and one checks that there are 3 possible results for the corresponding number: $(1, 2) \rightarrow 1$, $(1, 3) \rightarrow 2$, $(1, 4) \rightarrow 3$. For greater values of $n$ the problem seems much harder.

4. **Sequence A275607**

As we mentioned at the beginning, the calculation of the first terms of sequence $T_{2k}$, $k = 1, 2, \ldots$ gives

1, 4, 27, 216, 1890, 17496, 168399, \ldots

which coincide with the first terms of sequence A275607 from OEIS. An explicit formula for A275607 was found by Charles R. Greathouse (see [4])

$$a(n) = \frac{2.3^n(n + 1)}{2n^2 + n} \binom{2n + 1}{n - 1}.$$

It may be shown that $T_{2n+2} = a(n)$, or

$$3^{n-1} (C_{n+1} + 2C_n) = \frac{2.3^n(n + 1)}{2n^2 + n} \binom{2n + 1}{n - 1}.$$
identifying in such a way our sequence $T_{2k}$ with A275607. By the way, Wolfram Alpha gives the following equality

$$3^{n-1}(C_{n+1} + 2C_n) = \frac{2.4^n(n + 1)\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n + 3)},$$

and since the right-hand side is the original definition of A275607, this is another way to identify $T_{2k}$.

Taking into account that the (ordinary) generating function for Catalan numbers is $\frac{1-\sqrt{1-4x}}{2x}$, it is not difficult to find a generating function for sequence $T_{2k}$:

$$T(x) = \frac{(1 + 6x)(1 - \sqrt{1 - 12x})}{54x}.$$

Indeed, a calculation with Wolfram Alpha gives

$$\frac{(1 + 6x)(1 - \sqrt{1 - 12x})}{54x} = \frac{1}{9} + x + 4x^2 + 27x^3 + 216x^4 + 1890x^5 + 17496x^6 + \ldots,$$

so the coefficient of $x^k$ equals $T_{2k}$.

5. HOW TO FIND ALL DIAGRAMS WITH $n$ NODES

The knowledge of numbers $T_n$ doesn’t tell us how to find the corresponding diagrams with $n$ nodes. Even for $n = 8$ the attempt to find these by hand seems counterproductive, as their number is 216.

Here we propose an algorithm for this task, which has two steps:

1) constructing some graph $\Gamma_n$ whose maximal independent vertex sets are in 1-to-1 correspondence with the $n$ nodes diagrams

2) finding all such independent sets in $\Gamma_n$ and then tracing back all diagrams with $n$ nodes $D_n$.

The productivity of this algorithm depends on the algorithm selected for enumerating the maximal independent sets in $\Gamma_n$. Note that there might be a significant improvement of the known general algorithms, in view of the fact that all the maximal independent sets of $\Gamma_n$ surely have one and the same size $= \frac{n}{2} - 1$. Moreover, the number of all such sets is known a priori and should equal number $T_n$. Another advantage is the observation that the graph $\Gamma_n$ has a rich group of symmetries, so each maximal independent set automatically produces a series of such sets in $\Gamma_n$.

1. Construction of graph $\Gamma_n$

a) The vertices

Roughly speaking, the vertices of $\Gamma_n$ are the virtual touching lines $\lambda$ in the disk and two $\lambda, \lambda'$ are connected by an edge if and only if they are, in some sense, linked in $\mathbb{R}^2$.

Suppose for simplicity that the set of nodes is a regular $n$-gon and the nodes are clockwise marked by integers

$$A = \{1, 2, \ldots, n\}.$$

Then the vertices of $\Gamma_n$ are all triples of nodes $(a, p, q)$ such that

1) $a, p, q \in A, a \neq p, q$ and all 3 nodes have the same parity, e.g. (1, 3, 7), (8, 2, 6)

2) the triple $(a, p, q)$ is clockwise oriented on $S^1$, in case $a, p, q$ are different

3) all triples of type $(a, p, p)$, where $a$ and $p$ are different and have the same parity, belong to $\Gamma_n$, e.g. (2, 4, 4), (4, 2, 2), (7, 3, 3).
Figure 7. Vertices of $\Gamma_n$ of type $(a,p,q)$ and $(a,p,p)$.

It turns out that the touching lines of a given diagram from $D_n$ may be coded by such triples as follows:
- $a$ is the node where line $\lambda$ is touching $S^1$
- $p$ and $q$ are the nodes closest to $\lambda$ which lie in the component of $B^2 \setminus \lambda$ containing the whole $\lambda$ in its boundary (see Fig.7-1).

Note that the case $(a,p,p)$ is not excluded and corresponds to the situation from Fig.7-2. Another important observation is that the parity of $a, p, q$ is the same, as there is an odd number of nodes between $a$ and either of $p, q$. In such a way, any touching line is uniquely identified by some triple $(a,p,q) \in \Gamma_n$.

Furthermore, it is not difficult to calculate the number of vertices of $\Gamma_n$:

$$|\Gamma_n| = \frac{n^2(n-2)}{8}.$$

For example $|\Gamma_4| = 4$, $|\Gamma_6| = 18$, that may be checked by hand.

b) The edges

Now we have to define when two vertices of $\Gamma_n$ are connected by an edge. We shall proceed in an equivalent way by defining when two vertices are not connected by an edge. So, we shall define the complementary graph $\overline{\Gamma_n}$ rather than $\Gamma_n$. The cause for this approach is that graph $\Gamma_n$ turns out to be quite dense, so $\overline{\Gamma_n}$ should be sparse and thus easier to describe (depict). Then, of course, each invariant of $\Gamma_n$ transfers to a dual invariant of $\overline{\Gamma_n}$ and vice versa.

It is convenient to consider addition in the circular set $A$ modulo $n$, e.g. $n+1 = 1$.

Hereafter we describe the situations when two vertices of $\overline{\Gamma_n}$ are connected by an edge. Let $v_1 = (a_1, p_1, q_1), v_2 = (a_2, p_2, q_2)$ be two vertices of $\overline{\Gamma_n}$.

1a) If $a_1 = a_2$, then $v_1$ and $v_2$ are not connected in $\overline{\Gamma_n}$.
1b) $(a,b,b)$ and $(b,a,a)$ are also not connected in $\overline{\Gamma_n}$ (see Fig.8-2)
2) If the nodes $a_i, p_i, q_i$ are all different, then
2a) $v_1$ and $v_2$ are connected if $Co(v_1) \cap Co(v_2) = \emptyset$, where $Co(v)$ is the convex hull of the nodes of $v$, see Fig.9-1.
2b) In case $Co(v_1) \cap Co(v_2) \neq \emptyset$, then $v_1$ and $v_2$ are not connected in $\overline{\Gamma_n}$ (and thus connected in $\Gamma_n$, see Fig.8-1).

E.g. $(1,3,5)$ and $(8,6,7)$ are connected, while $(1,3,5)$ and $(8,2,4)$ are not.
3) The following pairs are connected:
   3a) \((a, p + 1, q)\) and \((b, q + 1, p)\), where \(a\) and \(b\) have different parity, all 6 nodes involved are different and segment \([a, b]\) is separating \([p, p + 1]\) from \([q, q + 1]\). E.g. \((1, 3, 7)\) and \((6, 8, 2)\). This situation is depicted on Fig.9-2
   3b) \((a, p, b)\) and \((b, p, x)\) where \(a\) and \(b\) have the same parity, e.g. \((2, 4, 8)\) and \((8, 4, 6)\), \((1, 3, 5)\) and \((5, 3, 3)\); see Fig.10-1
   3c) \((a, b, p)\) and \((b, x, p)\) where \(a\) and \(b\) have the same parity, e.g. \((2, 6, 8)\) and \((6, 8, 8)\); see Fig.10-2.

Clearly, graph \(\Gamma_4\) is a 4-clique, while \(\Gamma_6\) may be characterized by the observation that its complementary graph \(\overline{\Gamma_6}\) is a disjoint sum of 3 copies of the Kuratowski graph \(K_{3,3}\). This becomes evident after examination of Fig. 3. Note that \(\overline{\Gamma_6} = K_{3,3} \sqcup K_{3,3} \sqcup K_{3,3}\) is triangle free and has 27 edges, which illustrates Theorem 2, since \(T_6 = 27\).

It is not difficult to show that \(\Gamma_n\) is connected for \(n \geq 6\), while \(\overline{\Gamma_n}\) is connected for \(n \geq 8\). Also, only \(\Gamma_4\) and \(\Gamma_6\) are regular, the others are not. For example, \(\Gamma_8\) has two types of vertices - of degree 6 and 15.
The importance of graph $\Gamma_n$ rely on the observation that each diagram from $D_n$ corresponds to a maximal independent set of vertices of $\Gamma_n$ and vice versa. Then, as explained above, it is convenient to pass to the complementary graph $\overline{\Gamma_n}$ and to notice that the maximal independent sets in $\Gamma_n$ correspond to maximal cliques in $\overline{\Gamma_n}$.

Theorem 2. The maximal cliques of graph $\Gamma_n$ are in 1-to-1 correspondence with the diagrams from $D_n$. All the maximal cliques have one and the same size $= \frac{n}{2} - 1$. Their number is $T_n = 3^{\frac{n}{2} - 2} \left( C_{\frac{n}{2}} + 2C_{\frac{n}{2} - 1} \right)$, where $C_m$ is the m-th Catalan number.

The proof follows easily from the above considerations.

Then one may refer to the known algorithms for finding the maximal cliques of a given graph, e.g. the Bron-Kerbosch algorithm (see [5]). However, we wouldn’t recommend the general algorithms in our case. For example, the worst-case running time of the Bron-Kerbosch algorithm is $O \left( 3^{\frac{n}{2}} \right)$, where $N$ is the number of vertices, which applied to $\overline{\Gamma_n}$ gives $O \left( 3^{\frac{n}{2}} \right)$ as an a priori estimate (very slow). Hereafter we shall list the most important properties of $\Gamma_n$ and $\overline{\Gamma_n}$ that might suggest some much better algorithm.

1. The graphs $\Gamma_n$ and $\overline{\Gamma_n}$ are connected for $n \geq 8$.
2. $|\Gamma_n| = |\overline{\Gamma_n}| = \frac{n^2(n-2)}{8}$.
3. All the maximal cliques of $\overline{\Gamma_n}$ have one and the same size $= \frac{n}{2} - 1$.
4. The number of all maximal cliques of $\overline{\Gamma_n}$ is $3^{\frac{n}{2} - 2} \left( C_{\frac{n}{2}} + 2C_{\frac{n}{2} - 1} \right)$.
5. Each vertex of $\overline{\Gamma_n}$ takes part into exactly $3^{\frac{n}{2} - 2}C_{\frac{n}{2} - 1}C_{\frac{n}{2} - 1}C_{\frac{n}{2} - 1}$ maximal cliques, where $k, l, m$ depend on the geometric line representing the vertex.
6. $\Gamma_n$ and $\overline{\Gamma_n}$ are not regular for $n \geq 8$, but the degree of each vertex may be calculated considering the position of the touching line corresponding to the vertex.
7. The group $G_n = Z_n \oplus Z_2^2$ is acting in a natural way in both $\Gamma_n$ and $\overline{\Gamma_n}$. This is explained in the next section.

6. COUNTING FIELDS TOPOLOGICALLY

We shall try in this final section to settle the general problem for counting stationary points free vector fields in $\mathbb{B}^2$ from topological point of view. Note that in
our previous counting scheme one and the same topological picture appears many times, see for example Fig. 3. So, it is a good point to eliminate multiple pictures and then to count topologically different diagrams. Let us say from the beginning that we won’t do this here. This problem seems to be much harder than the combinatorial one solved in Section 3. Anyway, we shall describe formally the topology equivalence classes of fields, reducing these to a quite concrete object. The problem is to find its cardinality.

It turns out that the answer to the question is closely related to the action of group $G_n = \mathbb{Z}_n \oplus \mathbb{Z}_2^n$ on $\Gamma_n$.

We say that group $G$ is acting on graph $\Gamma$, if $G$ is acting on the vertex set of $\Gamma$ and is incidence preserving. It is geometrically evident that $\mathbb{Z}_n$ is acting in $\Gamma_n$ and $\Gamma_n$ by rotations in the set of nodes at angle $2\pi/n$, while $\mathbb{Z}_2^n$ is acting by symmetries about the lines containing pairs of opposite nodes $(k, k + \frac{n}{2})$, as well as pairs of type $(k + \frac{n+1}{2}, k + \frac{n+1}{2})$ (recall that addition is mod $n$). Let now $\Delta_n$ be the set of all maximal cliques in $\Gamma_n$, then the action of $G_n$ on $\Gamma_n$ transfers to an action on $\Delta_n$. Note that this action is not free and even has fixed points. Each element of $\Delta_n$ is periodic and its period is a divisor of $2n$. The action of the $\mathbb{Z}_n$-factor of $G_n$ cannot have involutory or fixed elements (for $n \geq 4$), while, of course, each element of $\Delta_n$ is involutory under $\mathbb{Z}_2^n$. One advantage, we may take profit of, is the following simple observation:

If $C \in \Delta_n$ is a maximal clique of $\Gamma_n$, then all the sets $GC = \{gC \mid g \in G_n\}$ are maximal cliques as well. This may provide us with many different diagrams from $D_n$ for free, in the best case with $2n$ new diagrams.

Topological equivalence of stationary points free vector fields in $\mathbb{B}^2$ is defined in a natural way:

If $v_1 \sim v_2$ if there is a diffeomorphism $\varphi : \mathbb{B}^2 \to \mathbb{B}^2$ which is transforming the flow lines of $v_1$ onto those of $v_2$. (Here $\varphi$ is allowed to be orientation reversing.)

Note that if $v_1 \sim v_2$, then surely $v_1$ and $v_2$ both have $\frac{n}{2} + 1$-nodes and $\frac{n}{2} - 1$ b-nodes on $S^1$ for some even $n$. Let $R_n$ denote the set of equivalence classes of such fields. Clearly, the problem is to find its cardinality $|R_n|$.\s

**Proposition 2.** Let $\Delta_n$ be the set of maximal cliques of graph $\Gamma_n$ and the group $G_n = \mathbb{Z}_n \oplus \mathbb{Z}_2^n$ is acting on it as described. Let $\Delta_n|G_n$ be the orbit space under this action. Then $|\Delta_n|_{G_n} = |R_n|$.

**Remark 3.** If in the definition of equivalent fields we restrict ourselves to orientation preserving diffeomorphisms, it suffices to consider only the $\mathbb{Z}_n$-action on $\Delta_n$. Namely, if $R_n^0 \subset R_n$ are the equivalence classes defined by such diffeomorphisms, then $|\Delta_n|_{\mathbb{Z}_n} = |R_n^0|$.

The above proposition provides us with some “brute force” algorithm for finding $|R_n|$ and $|R_n^0|$.

Obviously, $|R_4| = |R_4^0| = 1$. For $n = 6$ things are equally easy; see Fig.11 to justify that $|R_6| = 4$, $|R_6^0| = 6$.

It is clear that the above problem has a local variant:
Given some maximal clique \( C \in \Delta_n \) (= diagram from \( D_n \)), then what is the cardinality of its orbit \( \{ gC \mid g \in G_n \} \)?

7. Final remarks

Let us finally list some natural problems, that might trace a route for further investigations.

1) Find the number of topological diagrams \( |R_n| \) and \( |R^0_n| \) (see Section 6). Find an algorithm for generating these diagrams.

2) Find some “good” algorithm for generating diagrams from \( D_n \) based on the high symmetry of graph \( \Gamma_n \).

3) Enlarge the class of vector fields under consideration by admitting multiple touches between a flow line and the boundary.

4) Consider the class of vector fields with non degenerate singularities inside the disk and solve appropriate calculation problems in this class.

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