GENERALIZED NONUNIFORM DICHOTOMIES AND LOCAL STABLE MANIFOLDS

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Abstract. We establish the existence of local stable manifolds for semiflows generated by nonlinear perturbations of nonautonomous ordinary linear differential equations in Banach spaces, assuming the existence of a general type of nonuniform dichotomy for the evolution operator that contains the nonuniform exponential and polynomial dichotomies as a very particular case. The family of dichotomies considered allow situations for which the classical Lyapunov exponents are zero. Additionally, we give new examples of application of our stable manifold theorem and study the behavior of the dynamics under perturbations.

1. Introduction

The concept of nonuniform hyperbolicity was introduced by Pesin [10, 11, 12] and generalizes the classical concept of (uniform) hyperbolicity by allowing the rates of expansion and contraction to vary from point to point. For nonuniformly hyperbolic trajectories, Pesin [10] was able to obtain a stable manifold theorem in the finite dimensional setting. Then, in [14] Ruelle gave a proof of this theorem based on the study of perturbations of products of matrices occurring in Oseledec's multiplicative ergodic theorem [8]. Another proof, based on the classical work of Hadamard, was obtained by Pugh and Shub in [13] and uses graph transform techniques. In the infinite dimensional setting, Ruelle [15] proved, following his approach in [14], a stable manifold theorem in Hilbert spaces under some compactness assumptions. For transformations in Banach spaces and under some compactness and invertibility assumptions, Mañé established the existence of stable manifolds in [7] and in [16] Thieullen weakened Mañé’s hypothesis.

In the context of nonautonomous differential equations, stable manifold theorems were also obtained, assuming that the evolution operator have bounds that are nonuniform, more precisely assuming that the evolution operator admits a nonuniform exponential dichotomy, a notion introduced by Barreira and Valls in [1] and inspired both in the classical notion of exponential dichotomy introduced by Perron in [9] and in the notion of nonuniformly hyperbolic trajectory introduced by Pesin in [10, 11, 12]. For more details we refer the reader to the book [2].

Recently it has been addressed the problem of obtaining stable manifolds for perturbations of linear ordinary differential equations assuming the existence of nonuniform dichotomies that are not exponential. Namely, in [5] were obtained
local and global stable manifolds for polynomial dichotomies and in \cite{4} it was proved the existence of global stable manifolds for a generalized type of dichotomy that includes both the polynomial and exponential cases. Also in \cite{3} Barreira and Valls obtained local stable manifolds for perturbations of linear equations assuming a dichotomy that follows growth rates of the form $e^{\rho(t)}$ where $\rho : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is an increasing differentiable function satisfying

$$\lim_{t \to +\infty} \frac{\log t}{\rho(t)} = 0. \quad (1)$$

This definition, in spite of being very general, does not include some of the growth rates considered in this paper, namely, due to (1), this definition do not include for example the polynomial case studied in \cite{5} and the growth rates of Examples 3.4 and 3.5. In the discrete time setting, the existence of global and local stable manifolds for perturbations of some nonuniform polynomial dichotomies was discussed in \cite{6}.

The main objective of this paper is to obtain local stable manifolds for perturbations of nonautonomous linear ordinary differential equations, assuming that the evolution operator associated with the linear equation admits a dichotomy with growth rates given by increasing functions that go to infinity (and therefore more general than the mentioned above). In fact we do not need to assume condition (1) and we allow growth rates given by non-differentiable functions as well as different growth rates in the uniform and in the nonuniform parts of the dichotomy. We also would like to emphasize that the dichotomies considered here include as a particular case the ones considered in \cite{1} and \cite{5} and the theorems proved there, respectively, for nonuniform exponential dichotomies and for nonuniform polynomial dichotomies, are particular cases of the result presented in this paper. We also give new examples of growth rates to which our local stable manifold theorem can be applied. We emphasize that the Lyapunov exponent considered in \cite{2}, for Hilbert spaces, is zero or infinity for most of the dichotomies considered in this paper.

The content of the paper is as follows: in Section 2 we establish the setting, we define the dichotomies and we state the main theorem; in Section 3 we give examples of nonuniform $(\mu, \nu)$-dichotomies for each differentiable growth rates in our family of growth rates and examples of growth rates that verify the conditions of the main theorem; in Section 4 we prove the main theorem; finally, in Section 5 we study how the manifolds obtained vary with the perturbations considered.

2. Main result

Let $X$ be a Banach space and denote by $B(X)$ the space of bounded linear operators acting on $X$. Given a continuous function $A : \mathbb{R}_0^+ \rightarrow B(X)$, we consider the initial value problem

$$v' = A(t)v, \quad v(s) = v_s \quad (2)$$

with $s \geq 0$ and $v_s \in X$. We assume that each solution of (2) is global and denote the evolution operator associated with (2) by $T(t, s)$, i.e., $v(t) = T(t, s)v_s$ for $t \geq 0$.

We say that an increasing function $\mu : \mathbb{R}_0^+ \rightarrow [1, +\infty]$ is a growth rate if $\mu(0) = 1$ and

$$\lim_{t \to +\infty} \mu(t) = +\infty.$$
Let $\mu$ and $\nu$ be growth rates. We say that equation (2) admits a nonuniform $(\mu, \nu)$-dichotomy in $\mathbb{R}_0^+$ if, for each $t \geq 0$, there are projections $P(t)$ such that

$$P(t)T(t, s) = T(t, s)P(s), \quad t, s \geq 0$$

and constants $D \geq 1$, $a < 0 \leq b$ and $\varepsilon \geq 0$ such that, for every $t \geq s \geq 0$,

$$\|T(t, s)P(s)\| \leq D \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)\varepsilon,$$

$$\|T(t, s)^{-1}Q(t)\| \leq D \left( \frac{\mu(t)}{\mu(s)} \right)^{-b} \nu(t)\varepsilon,$$

where $Q(t) = \text{Id} - P(t)$ is the complementary projection. When $\varepsilon = 0$ we say that we have a uniform $(\mu, \nu)$-dichotomy or simply a $(\mu, \nu)$-dichotomy.

For each $t \geq 0$, we define the linear subspaces

$$E(t) = P(t)X \quad \text{and} \quad F(t) = Q(t)X.$$

Without loss of generality, we always identify the spaces $E(t) \times F(t)$ and $E(t) \oplus F(t)$ as the same space and we equip these spaces with the norm given by

$$\|(x, y)\| = \|x\| + \|y\|$$

for each $(x, y) \in E(t) \times F(t)$. The unique solution of (2) can be written in the form

$$v(t) = (U(t, s)\xi, V(t, s)\eta), \quad t \geq s$$

where $v_s = (\xi, \eta) \in E(s) \times F(s)$ and

$$U(t, s) := P(t)T(t, s)P(s) \quad \text{and} \quad V(t, s) := Q(t)T(t, s)Q(s).$$

In this paper we are going to address the problem of obtaining stable manifolds for the nonlinear problem

$$v' = A(t)v + f(t, v), \quad v(0) = v_s$$

when equation (2) admits a nonuniform $(\mu, \nu)$-dichotomy and there are $c > 0$ and $q > 1$ such that the perturbations $f: \mathbb{R}_0^+ \times X \to X$ verify, for $u, v \in X$ and $t \in \mathbb{R}_0^+$, the following conditions

$$f(t, 0) = 0,$$

$$\|f(t, u) - f(t, v)\| \leq c\|u - v\|(\|u\| + \|v\|)^q.$$  \hspace{1cm} (7)

Note that, making $v = 0$ in (7), we have

$$\|f(t, u)\| \leq c\|u\|^{q+1}.$$  \hspace{1cm} (8)

for every $u \in X$.

Writing the unique solution of (5) in the form

$$(x(t, s, v_s), y(t, s, v_s)) \in E(t) \times F(t),$$

problem (5) is equivalent to the following problem

$$x(t) = U(t, s)\xi + \int_s^t U(t, r)f(r, x(r), y(r)) \, dr,$$

$$y(t) = V(t, s)\eta + \int_s^t V(t, r)f(r, x(r), y(r)) \, dr,$$

where $v_s = (\xi, \eta) \in E(s) \times F(s)$. 

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For each \((s, v_s)\) we consider the semiflow
\[
\Psi^\tau(s, v_s) = (s + \tau, x(s + \tau, s, v_s), y(s + \tau, s, v_s)), \quad \tau \geq 0.
\] (11)

Given \(\delta > 0\) and a decreasing function \(\beta: \mathbb{R}_+^0 \to \mathbb{R}_+^0\) we use the following notation
\[
G_{\delta, \beta} = \bigcup_{s \in \mathbb{R}_+^0} \{s\} \times B_{s, \delta, \beta},
\]
where \(B_{s, \delta, \beta}\) is the open ball of \(E(s)\) centered at 0 and with radius \(\delta \beta(s)\).

Let \(X_{\delta, \beta}\) be the space of functions
\[
\phi: G_{\delta, \beta} \to X
\]
such that, for every \((s, \xi), (s, \tilde{\xi}) \in G_{\delta, \beta}\) the following conditions hold
\[
\phi(s, \xi) \in F(s),
\]
\[
\phi(s, 0) = 0,
\]
\[
\|\phi(s, \xi) - \phi(s, \tilde{\xi})\| \leq \|\xi - \tilde{\xi}\|.
\] (14)

By (13), putting \(\tilde{\xi} = 0\) in (12), we immediately conclude that
\[
\|\phi(s, \xi)\| \leq \|\xi\|
\] (15)
for every \((s, \xi) \in G_{\delta, \beta}\). We equip the space \(X_{\delta, \beta}\) with the metric defined by
\[
\|\phi - \psi\|' = \sup \left\{ \frac{\|\phi(s, \xi) - \psi(s, \xi)\|}{\|\xi\|} : (s, \xi) \in G_{\delta, \beta}, \xi \neq 0 \right\}
\] (16)
for every \(\phi, \psi \in X_{\delta, \beta}\). By (15) it follows that, for every \((s, \xi) \in G_{\delta, \beta}\), \(\|\phi(s, \xi)\| \leq \delta \beta(s) \leq \delta \beta(0)\) and this implies that \(X_{\delta, \beta}\) is a complete metric space with the metric given by (16).

We also have to consider the space \(X_{\delta, \beta}^*\) of continuous functions \(\phi: G \to X\), with
\[
G = \bigcup_{s \in \mathbb{R}_+^0} \{s\} \times E(s),
\]
such that \(\phi(s, \xi) \in F(s)\) for every \((s, \xi) \in G\), the restriction \(\phi|_{G_{\delta, \beta}} \in X_{\delta, \beta}\) and
\[
\phi(s, \xi) = \begin{cases} 0 & \text{whenever } \xi \notin B_{s, \delta, \beta}, \\ \phi(s, \xi) & \text{otherwise} \end{cases}
\]
where \(\delta \beta(s) \xi\) whenever \(\xi \notin B_{s, \delta, \beta}\).

Furthermore, since for every \(\phi \in X_{\delta, \beta}\), we have a unique Lipschitz extension of \(\phi\) to \(\bigcup_{t \geq 0} \{t\} \times \overline{B_{t, \delta, \beta}}\), where \(\overline{B_{t, \delta, \beta}}\) is the closure of the ball \(B_{t, \delta, \beta}\), there is a one-to-one correspondence between \(X_{\delta, \beta}\) and \(X_{\delta, \beta}^*\). This one-to-one correspondence allow us to defined a metric in \(X_{\delta, \beta}^*\) using the metric in \(X_{\delta, \beta}\). Namely, this metric can be defined by
\[
\|\phi - \psi\|' = \|\phi - \psi|_{G_{\delta, \beta}}\|',
\] (17)
for every \(\phi, \psi \in X_{\delta, \beta}^*\) and where the right hand side is given by (16). With this metric \(X_{\delta, \beta}^*\) is a complete metric space. Moreover, given \(\phi, \psi \in X_{\delta, \beta}^*\), it follows that
\[
\|\phi(s, \xi) - \psi(s, \xi)\| \leq 2\|\xi - \tilde{\xi}\|\]
(18)
\[
\|\phi(s, \xi) - \psi(s, \xi)\| \leq \|\phi - \psi\|'\|\xi\|
\] (19)
for every \((s, \xi), (s, \tilde{\xi}) \in G\).

For every \(\phi \in X_{\delta, \beta}\) we define the graph
\[
\mathcal{V}_{\phi, \delta, \beta} = \{(s, \xi, \phi(s, \xi)) : (s, \xi) \in G_{\delta, \beta}\}.
\] (20)
We now formulate our stable manifold theorem.

**Theorem 2.1.** Given a Banach space $X$, let $f : \mathbb{R}_0^+ \times X \to X$ be a function satisfying (3) and (4) for some $c > 0$ and $q > 1$. Suppose that equation (2) admits a nonuniform $(\mu, \nu)$-dichotomy in $\mathbb{R}_0^+$ for some growth rates $\mu$ and $\nu$, $D \geq 1$, $a < 0 \leq b$ and $\varepsilon \geq 0$. Assume that

$$\lim_{t \to +\infty} \mu(t)^{a-b} \nu(t)^{\varepsilon} = 0$$

and

$$\int_0^{+\infty} \mu(r)^{a} \nu(r)^{\varepsilon} \, dr \text{ is convergent.}$$

Define the functions $\beta, \tilde{\beta} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

$$\beta(t) = \frac{\mu(t)^a}{\nu(t)^{\varepsilon + 1/q} \left( \int_t^{+\infty} \mu(r)^{aq} \nu(r)^{\varepsilon} \, dr \right)^{1/q}},$$

and $\tilde{\beta}(t) = \beta(t)^{\nu(t)^{-\varepsilon}}$ and suppose that

$$\beta(t) \text{ and } \mu(t)^a \beta(t)^{-1} \text{ are decreasing.}$$

Then, for every $C > D$, choosing $\delta > 0$ sufficiently small, there is a unique $\phi \in \mathcal{X}_{\delta, \beta}$ such that

$$\Psi_{\nu}(\mathcal{V}_{\phi, \delta, \beta}) \subset \mathcal{V}_{\phi, \delta, \beta}$$

for every $\tau \geq 0$, where $\Psi_{\nu}$ is given by (11) and $\mathcal{V}_{\phi, \delta, \beta}$ and $\mathcal{V}_{\phi, \delta, \beta}$ are given by (20).

Furthermore, given $s \geq 0$, we have

$$\|\Psi_{t-s}(p_{s, \xi}) - \Psi_{t-s}(p_{s, \xi})\| \leq 2C \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s) \|\xi - \bar{\xi}\|$$

for every $t \geq s$ and $\xi, \bar{\xi} \in B_{\delta, \beta}$, where $p_{s, \xi} = (s, \xi, \phi(s, \xi))$.

3. Examples

We start with an example of a nonautonomous linear equation that admits a nonuniform $(\mu, \nu)$-dichotomy with arbitrary differentiable growth rates $\mu$ and $\nu$.

**Example 3.1.** Let $\varepsilon > 0$ and $a < 0 \leq b$. Put $\omega = \varepsilon/2$ and let $\mu, \nu$ be arbitrary differentiable growth rates. The differential equation in $\mathbb{R}^2$ given by

$$\begin{cases}
    u' = \left( \frac{a \mu'(t)}{\mu(t)} + \frac{\omega \nu'(t)}{\nu(t)} \right) (\cos t - 1) - \omega \log \nu(t) \sin t \, u \\
    v' = \left( \frac{b \mu'(t)}{\mu(t)} - \frac{\omega \nu'(t)}{\nu(t)} \right) (\cos t + 1) + \omega \log \mu(t) \sin t \, v
\end{cases}$$

(26)

has the following evolution operator

$$T(t, s)(u, v) = (U(t, s)u, V(t, s)v),$$

where

$$U(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^a e^{\omega \log \nu(t)(\cos t - 1) - \omega \log \nu(s)(\cos s - 1)},$$

$$V(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^b e^{-\omega \log \nu(t)(\cos t - 1) + \omega \log \nu(s)(\cos s - 1)}.$$
Using the projections \( P(t) : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( P(t)(u, v) = (u, 0) \) we have

\[
\|T(t, s)P(s)\| = |U(t, s)| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^\varepsilon
\]

\[
\|T(t, s)^{-1}Q(s)\| = |V(t, s)^{-1}| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^{-b} \nu(t)^\varepsilon
\]

and thus (20) admits a \((\mu, \nu)\)-dichotomy. Moreover, if \( t = 2k\pi \) and \( s = (2k - 1)\pi, k \in \mathbb{N} \), then

\[
U(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^\varepsilon
\]

and this ensures us that the nonuniform part can not be removed.

Now we will give examples of application of Theorem 2.1.

**Example 3.2.** If \( \mu(t) = \nu(t) = e^t \), we get the local stable manifold theorem obtained by Barreira and Valls in [1]. In fact, in this case condition (21) becomes \( a + \varepsilon < b \) and condition (22) becomes \( aq + \varepsilon < 0 \). The function \( \beta \) is given by

\[
\beta(t) = |aq + \varepsilon|^{1/q} e^{- (1 + 2/q) t}
\]

which is a decreasing function and

\[
\mu(t) \beta(t)^{-1} = |aq + \varepsilon|^{1/q} e^{(a + \varepsilon(1 + 2/q)) t}
\]

is decreasing if \( a + \varepsilon(1 + 2/q) < 0 \).

**Example 3.3.** Making \( \mu(t) = \nu(t) = 1 + t \), we get the local stable manifold theorem obtained by the present authors in [3]. In this case, condition (21) becomes \( a + \varepsilon < b \) and condition (22) becomes \( aq + \varepsilon + 1 < 0 \). Moreover,

\[
\beta(t) = |aq + \varepsilon + 1|^{1/q} (1 + t)^{-(\varepsilon(1 + 2/q) + 1/q) t}
\]

is a decreasing function and

\[
\mu(t) \beta(t)^{-1} = |aq + \varepsilon + 1|^{1/q} (1 + t)^{(a + \varepsilon(1 + 2/q) + 1/q) t}
\]

is a decreasing function if \( a + \varepsilon(1 + 2/q) + 1/q < 0 \).

In the next examples, we consider new growth rates for which Theorem 2.1 holds. Recall that Example 3.1 allows us to construct an example of a differential equation whose evolution operator has a dichotomy with the growth rates given.

**Example 3.4.** Consider

\[
\mu(t) = (1 + t)(1 + \log(1 + t))^\lambda \quad \text{and} \quad \nu(t) = 1 + \log(1 + t)
\]

with \( \lambda > 0 \). Then

\[
\lim_{t \to +\infty} \mu(t)^{a - b} \nu(t)^\varepsilon = \lim_{t \to +\infty} (1 + t)^{a - b}(1 + \log(1 + t))^{(a - b)\lambda + \varepsilon} = 0
\]

and

\[
\int_0^{+\infty} \mu(t)^aq \nu(t)^\varepsilon dt = \int_0^{+\infty} (1 + t)^aq (1 + \log(1 + t))^{aq\lambda + \varepsilon} dt
\]

is convergent if \( aq < -1 \) or \( aq = -1 \) and \( \varepsilon - \lambda < -1 \). When \( aq = -1 \) and \( \varepsilon - \lambda < -1 \) we have that

\[
\beta(t) = (\lambda - \varepsilon - 1)^{1/q}(t + 1)^{-1/q}(1 + \log(1 + t))^{-(\varepsilon(1 + 2/q) - 1)/q}
\]
is a decreasing function and

\[ \mu(t)^a \beta(t)^{-1} = (\lambda - \varepsilon - 1)^{-1/q} (1 + \log(1 + t))^{\varepsilon(1+2/q)+1/q-\lambda/q} \]

is decreasing if \( \varepsilon(1 + 2/q) + 1/q - \lambda/q < 0 \). Hence, taking into account that \( \varepsilon(1 + 2/q) + 1/q - \lambda/q < 0 \) implies that \( \varepsilon - \lambda < -1 \), for these growth rates we have a local stable manifold theorem if \( \alpha q = -1 \) and \( \varepsilon(1 + 2/q) + 1/q - \lambda/q < 0 \).

**Example 3.5.** Let

\[ \mu(t) = (1 + t)(1 + \log(1 + t))(1 + \log(1 + \log(1 + t)))^\lambda \]

and

\[ \nu(t) = 1 + \log(1 + \log(1 + t)) \]

with \( \lambda > 0 \). Then

\[
\begin{align*}
\lim_{t \to +\infty} \mu(t)^a \nu(t)^\varepsilon &= \lim_{t \to +\infty} (1 + t)^{a-b}(1 + \log(1 + t))^{a-b}(1 + \log(1 + \log(1 + t)))^{(a-b)\lambda + \varepsilon} \\
&= 0
\end{align*}
\]

and

\[
\int_{0}^{+\infty} \mu(t)^a \nu(t)^\varepsilon \, dt = \int_{0}^{+\infty} (1 + t)^{aq}(1 + \log(1 + t))^{aq(1 + \log(1 + \log(1 + t)))^{a+\lambda + \varepsilon}} \, dt
\]

is convergent if \( \alpha q < -1 \) or \( \alpha q = -1 \land \varepsilon - \lambda < -1 \). When \( \alpha q = -1 \) and \( \varepsilon - \lambda < -1 \)

we have that

\[
\beta(t) = (\lambda - \varepsilon - 1)^{-1/q}(1 + \log(1 + t))^{-1/q}(1 + \log(1 + \log(1 + t)))^{-\varepsilon(1+2/q)-1/q}
\]

is a decreasing function and

\[
\mu(t)^a \beta(t)^{-1} = (\lambda - \varepsilon - 1)^{-1/q}(1 + \log(1 + \log(1 + t)))^{\varepsilon(1+2/q)+1/q-\lambda/q}
\]

is decreasing if \( \varepsilon(1 + 2/q) + 1/q - \lambda/q < 0 \). Hence, if \( \alpha q = -1 \) and \( \varepsilon(1 + 2/q) + 1/q - \lambda/q < 0 \) we have again a local stable manifold theorem.

## 4. Proof of Theorem 2.1

From (49) and (10) we conclude that, to prove (25), we must have

\[
x(t, \xi) = U(t, s)\xi + \int_{s}^{t} U(t, r)f(r, x(r, \xi), \phi(r, x(r, \xi))) \, dr,
\]

(27)

and

\[
\phi(t, x(t, \xi)) = V(t, s)\phi(s, \xi) + \int_{s}^{t} V(t, r)f(r, x(r, \xi), \phi(r, x(r, \xi))) \, dr
\]

(28)

for every \( s \geq 0, t \geq s \) and \( \xi \in B_{s, \delta, \beta} \). We are going to prove that (46) and (48) hold using the Banach fixed point theorem.

Thus, let \( B_{s, \delta, \beta} \) be the space of functions

\[ x : [s, +\infty] \times B_{s, \delta, \beta} \to X \]

such that, for every \( t \geq s \) and \( \xi, \tilde{\xi} \in B_{s, \delta, \beta} \), we have

\[
x(t, \xi) \in E(t),
\]

(29)

\[
x(s, \xi) = \xi, \ x(t, 0) = 0,
\]

(30)

\[
\|x(t, \xi) - x(t, \tilde{\xi})\| \leq C \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^\varepsilon \|\xi - \tilde{\xi}\|,
\]

(31)
Making $\xi = 0$ in \((31)\) we obtain the following estimate
\[
\|x(t, \xi)\| \leq C \left(\frac{\mu(t)}{\mu(s)}\right)^a v(s)^\gamma \|\xi\| \leq C\delta \left(\frac{\mu(t)}{\mu(s)}\right)^a v(s)^\gamma \beta(s).
\] (32)

We equip $\mathcal{B}_{s, \delta, \beta}$ with the metric defined by
\[
\|x - y\| = \sup \left\{ \frac{\|x(t, \xi) - y(t, \xi)\|}{\|\xi\|} \left(\frac{\mu(t)}{\mu(s)}\right)^{-a} v(s)^{-\gamma}: t \geq s, \xi \in \mathcal{B}_{s, \delta, \beta} \setminus \{0\} \right\}
\] (33)
for every $x, y \in \mathcal{B}_{s, \delta, \beta}$. With this metric $\mathcal{B}_{s, \delta, \beta}$ is a complete metric space.

Given $x \in \mathcal{B}_{s, \delta, \beta}$ and $\phi \in X_{s, \delta, \beta}^*$ we use the following notation
\[
\phi_x(r, \xi) = \phi(r, x(r, \xi)) \text{ and } f_{x, \phi}(r, \xi) = f(t, x(r, \xi), \phi_x(r, \xi)).
\]

**Lemma 4.1.** For every $\phi \in X_{s, \delta, \beta}^*$, choosing $\delta > 0$ sufficiently small, there is one and only one $x = x_\phi \in \mathcal{B}_{s, \delta, \beta}$ such that
\[
x(t, \xi) = U(t, s)\xi + \int_s^t U(t, r)f_{x, \phi}(r, \xi) \, dr
\] (34)
for every $t \geq s$ and $\xi \in \mathcal{B}_{s, \delta, \beta}$. Moreover, choosing $\delta > 0$ sufficiently small, we have
\[
\|x_\phi(t, \xi) - x_\psi(t, \xi)\| \leq C \left(\frac{\mu(t)}{\mu(s)}\right)^a \|\xi\| \cdot \|\phi - \psi\|
\] (35)
for every $\phi, \psi \in X_{s, \delta, \beta}^*$, every $t \geq s$ and every $\xi \in \mathcal{B}_{s, \delta, \beta}$.

**Proof.** In $\mathcal{B}_{s, \delta, \beta}$ we define an operator $J = J_\phi$ by
\[
(Jx)(t, \xi) = U(t, s)\xi + \int_s^t U(t, r)f_{x, \phi}(r, \xi) \, dr.
\]
Obviously, $(Jx)(s, \xi) = \xi$ for every $\xi \in \mathcal{B}_{s, \delta, \beta}$ and from \((30), (13)\) and \((6)\) it follows that $(Jx)(t, 0) = 0$ for every $t \geq s$. Moreover, $Jx$ satisfies \((29)\) for every $t \geq s$ and every $\xi \in \mathcal{B}_{s, \delta, \beta}$.

By \((7)\) and \((8)\) it follows for every $r \geq s$ and every $\xi \in \mathcal{B}_{s, \delta, \beta}$ that
\[
\|f_{x, \phi}(r, \xi) - f_{x, \phi}(r, \tilde{\xi})\| \leq c \left(\|x(r, \xi) - x(r, \tilde{\xi})\| + \|\phi_x(r, \xi) - \phi_x(r, \tilde{\xi})\|\right) \times \left(\|x(r, \xi)\| + \|\phi_x(r, \xi)\| + \|x(r, \xi)\|\right)^q
\]
\[
\leq 3^{q+1}c \|x(r, \xi) - x(r, \tilde{\xi})\| \left(\|x(r, \xi)\| + \|x(r, \tilde{\xi})\|\right)^q
\]
and by \((31)\) and \((32)\) we get the following estimate
\[
\|f_{x, \phi}(r, \xi) - f_{x, \phi}(r, \tilde{\xi})\| \leq 2^{q+1}cC^{q+1}\delta^q \left(\frac{\mu(r)}{\mu(s)}\right)^{aq+a} v(s)^{q+1} \beta(s)^q \|\xi - \tilde{\xi}\|. \quad (36)
\]
From \((33)\), the last estimate and because by \((23)\) we have
\[
\mu(s)^{-aq}v(s)^{q+1}\beta(s)^q \int_s^{+\infty} \mu(r)^{aq}v(r)^q \, dr = 1,
\] (37)
we obtain the following estimate
\[
\int_s^t \| U(t, r) \| \cdot \| f_{x, \phi}(r, \xi) - f_{x, \phi}(r, \bar{\xi}) \| \, dr \\
\leq 2^{q+1} c C^{q+1} D^q \varepsilon a \left( \frac{\mu(t)}{\mu(s)} \right) \mu(s)^{-a \varepsilon} \nu(s)^{q(1+q)} \beta(s)^q \| \xi - \bar{\xi} \| \\
\leq 2^{q+1} c C^{q+1} D^q \varepsilon a \left( \frac{\mu(t)}{\mu(s)} \right) \| \xi - \bar{\xi} \|
\]
and using again (3) it follows that
\[
\| (J_\phi x)(t, \xi) - (J_\phi x)(t, \bar{\xi}) \| \\
\leq \| U(t, s) \| \| \xi - \bar{\xi} \| + \int_s^t \| U(t, r) \| \cdot \| f_{x, \phi}(r, \xi) - f_{x, \phi}(r, \bar{\xi}) \| \, dr \\
\leq (D + 2^{q+1} c C^{q+1} D^q) \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^{q} \| \xi - \bar{\xi} \|.
\]
Therefore, since \( C > D \), choosing \( \delta > 0 \) sufficiently small we have
\[
\| (J_\phi x)(t, \xi) - (J_\phi x)(t, \bar{\xi}) \| \leq C \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^{q} \| \xi - \bar{\xi} \|,
\]
and this implies the inclusion \( J(\mathcal{B}_{s, \delta, \beta}) \subseteq \mathcal{B}_{s, \delta, \beta} \).

Now we are going to prove that if \( \delta \) is sufficiently small \( J_\phi \) is a contraction. Given \( x, y \in \mathcal{B}_{s, \delta, \beta} \), from (7) and (32) we have for every \( r \geq s \) and every \( \xi \in \mathcal{B}_{s, \delta, \beta} \)
\[
\| f_{x, \phi}(r, \xi) - f_{y, \phi}(r, \xi) \| \leq c \| x(r, \xi) - y(r, \xi) \| + \| \phi_x(r, \xi) - \phi_y(r, \xi) \| \times \\
\| \| f_{x, \phi}(r, \xi) \| + \| \phi_x(r, \xi) \| + \| y(r, \xi) \| + \| \phi_y(r, \xi) \| \| )^q \\
\leq 3^{q+1} c \| x(r, \xi) - y(r, \xi) \| + \| y(r, \xi) \| \| x(r, \xi) \| + \| y(r, \xi) \| \| ^q \\
\leq 2^{q+1} c C^{q+1} \varepsilon a \left( \frac{\mu(r)}{\mu(s)} \right)^q \nu(s)^{q(1+q)} \beta(s)^q \| x(r, \xi) - y(r, \xi) \| \\
\leq 2^{q+1} c C^{q+1} \varepsilon a \left( \frac{\mu(r)}{\mu(s)} \right)^q \nu(s)^{q(1+q)} \beta(s)^q \| x - y \| \| \xi \|.
\]
and by (3) and (37) it follows that
\[
\| (J_\phi x)(t, \xi) - (J_\phi y)(t, \xi) \| \\
\leq \int_s^t \| U(t, r) \| \| f_{x, \phi}(r, \xi) - f_{y, \phi}(r, \xi) \| \, dr \\
\leq 2^{q+1} c C^{q+1} \varepsilon a \left( \frac{\mu(t)}{\mu(s)} \right)^q \mu(s)^{-a \varepsilon} \nu(s)^{q(1+q)} \beta(s)^q \| x - y \| \| \xi \| \\
\leq 2^{q+1} c C^{q+1} \varepsilon a \left( \frac{\mu(t)}{\mu(s)} \right)^q \| x - y \| \| \xi \|.
\]
Therefore
\[
\| J_\phi x - J_\phi y \| \leq 2^{q+1} c C^{q+1} D^q \| x - y \|.
\]
and choosing \( \delta > 0 \) sufficiently small, \( J_\phi \) is a contraction. By the Banach fixed point theorem, \( J_\phi \) has a unique fixed point \( x_\phi \in \mathcal{B}_{s, \delta, \beta} \) and \( x_\phi \) verifies (34).
Now we will prove (35). Let \(\phi, \psi \in X^s_{\delta, \beta}\) and let \(z \in B_{s, \delta, \beta}\) be defined by \(z(t, \xi) = U(t, s)\xi\) for every \(t \geq s\) and every \(\xi \in B_{s, \delta, \beta}\). Putting \(z_1 = y_1 = z\) and \(y_{n+1} = J_\phi y_n\) and \(z_{n+1} = J_\psi z_n\) for each \(n \in \mathbb{N}\), we have
\[
\|x_\phi - x_\psi\|' = \lim_{n \to \infty} \|y_n - z_n\|'.
\]
Hence, to prove (35) it is enough to prove that for each \(n \in \mathbb{N}\) we have
\[
\|y_n(t, \xi) - z_n(t, \xi)\| \leq C \left( \frac{\mu(t)}{\mu(s)} \right)^a \|\phi - \psi\|'
\]
for every \(t \geq s\) and every \(\xi \in B_{s, \delta, \beta}\). We are going to prove (38) by mathematical induction on \(n\). For \(n = 1\) there is nothing to prove. Suppose that (38) is true for \(n\). Then from (7), (13), (22), (19) and (38) we have, since \(\nu(s) \geq 1\),
\[
\|f_{y_n, \phi}(r, \xi) - f_{z_n, \psi}(r, \xi)\|
\leq c (\|y_n(r, \xi) - z_n(r, \xi)\| + \|\phi_{y_n}(r, \xi) - \psi_{z_n}(r, \xi)\|) \times
\leq 3^n c (\|y_n(r, \xi) - z_n(r, \xi)\| + \|\phi_{y_n}(r, \xi)\| + \|\psi_{z_n}(r, \xi)\|) \leq \|y_n(r, \xi)\| + \|z_n(r, \xi)\|)^q
\leq 3^n c (\|y_n(r, \xi) - z_n(r, \xi)\| + \|\phi_{y_n}(r, \xi)\| + \|\psi_{z_n}(r, \xi)\|) \left[ 2C \delta \left( \frac{\mu(r)}{\mu(s)} \right)^a \nu(s)^{q+1} \beta(s)^q \|\phi - \psi\|' \right]^{q}
\leq 2^{q+2} 3^n c C^{q+1} \delta^{q} \left( \frac{\mu(r)}{\mu(s)} \right)^a \nu(s)^{q+1} \beta(s)^q \|\phi - \psi\|'
\]
and this implies that
\[
\|y_{n+1}(t, \xi) - z_{n+1}(t, \xi)\|
\leq \int_s^t \|U(t, r)\| \|f_{y_n, \phi}(r, \xi) - f_{z_n, \psi}(r, \xi)\| dr
\leq K \left( \frac{\mu(t)}{\mu(s)} \right)^a \mu(s)^{-aq} \nu(s)^{q+1} \beta(s)^q \|\phi - \psi\|' \int_s^t \mu(r)^a \nu(r)^{q+1} \beta(s)^q \|\phi - \psi\|' \|\phi - \psi\|' \|\phi - \psi\|'
\leq K \left( \frac{\mu(t)}{\mu(s)} \right)^a \|\phi - \psi\|'
\]
with \(K = 2^{q+2} 3^n c C^{q+1} D \delta^{q}\). Choosing \(\delta > 0\) such that \(\delta^{q} < \frac{1}{2^{q+2} 3^n c C^{q+1} D}\), we have
\[
\|y_{n+1}(t, \xi) - z_{n+1}(t, \xi)\| \leq C \left( \frac{\mu(t)}{\mu(s)} \right)^a \|\phi - \psi\|'.
\]
Therefore (38) is true for every \(n \in \mathbb{N}\) and this completes the proof of the lemma.

\(\square\)

**Lemma 4.2.** If \(\delta > 0\) is sufficiently small and \(\phi \in X^s_{\delta, \beta}\), denoting by \(x_\phi\) the unique function given by Lemma 4.1, the following properties hold:

a) if the identity
\[
\phi_{x_\phi}(t, \xi) = V(t, s)\phi(s, \xi) + \int_s^t V(t, r) f_{x_\phi, \phi}(r, \xi) dr
\]
holds for every \(s \geq 0\), \(t \geq s\) and \(\xi \in B_{s, \delta, \beta}\), then
\[
\phi(s, \xi) = - \int_s^{+\infty} V(r, s)^{-1} f_{x_\phi, \phi}(r, \xi) dr
\]
for every $s \geq 0$ and every $\xi \in B_{s,\delta,\beta}$;

b) if (40) holds for every $s \geq 0$ and every $\xi \in B_{s,\delta,\beta}$, then (59) holds for every $s \geq 0$ and every $\xi \in B_{s,\delta,\beta}$.

Proof. First we prove that the integral in (40) is convergent. From (8) and (32) we have

$$\|f_{x_0,\phi}(r,\xi)\| \leq c \left( \|x_0(r,\xi)\| + \|\phi_x(r,\xi)\| \right)^q + 1 \leq 3^{q+1}c \|x_0(r,\xi)\|^q + 1 \leq 3^{q+1}c C^{q+1}\delta^{q+1} \left( \frac{\mu(r)}{\mu(s)} \right)^a \nu(s)^c \beta(s)^q + 1$$

and, since $a - b < 0$, this implies that

$$\int_s^{+\infty} \|V(r,s)^{-1}\| \|f_{x_0,\phi}(r,\xi)\| dr \leq 3^{q+1}c C^{q+1} D \delta^{q+1} \mu(s)^{-aq + a + b} \nu(s)^c \beta(s)^q + 1 \int_s^{+\infty} \mu(r)^{aq + a - b} \nu(r)^c dr$$

$$\leq 3^{q+1}c C^{q+1} D \delta^{q+1} \mu(s)^{-aq} \nu(s)^c \beta(s)^q + 1 \int_s^{+\infty} \mu(r)^{aq} \nu(r)^c dr$$

and, by (22), the integral is convergent.

If (39) is true for every $s \geq 0$, $s \geq 0$ and $\xi \in B_{s,\delta,\beta}$ then

$$\phi(s, \xi) = V(t,s)^{-1} \phi_{x_0}(t, \xi) - \int_s^t V(r,s)^{-1} f_{x_0,\phi}(r, \xi) dr \quad \text{(41)}$$

From (41), (32) and (21) we have

$$\lim_{t \to +\infty} \|V(t,s)^{-1} \phi_{x_0}(t, \xi)\| \leq \lim_{t \to +\infty} D \left( \frac{\mu(t)}{\mu(s)} \right)^{-b} \nu(t)^c 2C \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^c \|\xi\| = 0.$$
replacing \((s, \xi)\) by \((t, x_\phi(t, \xi))\) in (12) we have
\[
\phi(t, x_\phi(t, \xi)) = - \int_{t}^{+\infty} V(r, t)^{-1} f(F_{r-t}(t, x_\phi(t, \xi)), \phi(F_{r-t}(t, x_\phi(t, \xi)))) \, dr
\]
\[
= - \int_{t}^{+\infty} V(r, t)^{-1} f(r, x_\phi(r, \xi), \phi(r, x_\phi(r, \xi))) \, dr
\]
\[
= - \int_{t}^{+\infty} V(r, t)^{-1} f_{x_\phi, \phi}(r, \xi) \, dr.
\]
Then, using the fact that \(V(t, s)V(r, s)^{-1} = V(t, r)\), we have by (10)
\[
V(t, s)\phi(s, \xi) = - \int_{s}^{+\infty} V(t, s)V(r, s)^{-1} f_{x_\phi, \phi}(r, \xi) \, dr
\]
and this is equivalent to
\[
V(t, s)\phi(s, \xi) + \int_{s}^{t} V(t, r)f_{x_\phi, \phi}(r, \xi) \, dr = - \int_{t}^{+\infty} V(r, t)^{-1} f_{x_\phi, \phi}(r, \xi) \, dr
\]
\[
= \phi(t, x_\phi(t, \xi)).
\]
This finishes the proof of the lemma. \(\square\)

**Lemma 4.3.** Choosing \(\delta > 0\) sufficiently small, there is a unique \(\phi \in \mathcal{X}^{*}_{\delta, \beta}\) such that (10) holds for every \(s \geq 0\) and every \(\xi \in E(s)\).

**Proof.** Define in \(\mathcal{X}^{*}_{\delta, \beta}\) the operator \(\Phi\) by
\[
(\Phi \phi)(s, \xi) = - \int_{s}^{+\infty} V(r, s)^{-1} f_{x_\phi, \phi}(r, \xi) \, dr
\]
for every \((s, \xi) \in G_{\delta, \beta}\) (and this can be extended uniquely by continuity to the closure of \(G_{\delta, \beta}\)) and by
\[
(\Phi \phi)(s, \xi) = (\Phi \phi) \left( s, \frac{\delta \beta(s)\xi}{\|\xi\|} \right).
\]
for every \((s, \xi) \notin G_{\delta, \beta}\). It follows immediately from the definition of \(\Phi\) that \(\Phi \phi\) satisfies (12). Furthermore, from (4), (36) and (37), we have \((\Phi \phi)(s, 0) = 0\) for every \(s \geq 0\).

Given \((s, \xi), (s, \bar{\xi}) \in G_{\delta, \beta}\), from (4), (36) and (37), we have
\[
\|(\Phi \phi)(s, \xi) - (\Phi \phi)(s, \bar{\xi})\|
\]
\[
\leq \int_{s}^{+\infty} \|V(r, s)^{-1}\| \cdot \|f_{x_\phi, \phi}(r, \xi) - f_{x_\phi, \phi}(r, \bar{\xi})\| \, dr
\]
\[
\leq 2^{q+1} C^{q+1} D^{q+1} \mu(s)^{-aq-a+b} \nu(s)^{q+1} \beta(s)^{q+1} \|\xi - \bar{\xi}\| \int_{s}^{+\infty} \mu(r)^{aq-a-b} \nu(r)^{q+1} \, dr
\]
\[
\leq 2^{q+1} C^{q+1} D^{q+1} \mu(s)^{-aq-a+b} \nu(s)^{q+1} \beta(s)^{q+1} \|\xi - \bar{\xi}\| \int_{s}^{+\infty} \mu(r)^{aq} \nu(r)^{q+1} \, dr
\]
\[
= 2^{q+1} C^{q+1} D^{q+1} \|\xi - \bar{\xi}\|.
\]
Choosing \(\delta\) sufficiently small we get
\[
\|(\Phi \phi)(s, \xi) - (\Phi \phi)(s, \bar{\xi})\| \leq \|\xi - \bar{\xi}\|.
\]
Therefore \(\Phi \left( \mathcal{X}^{*}_{\delta, \beta} \right) \subseteq \mathcal{X}^{*}_{\delta, \beta}\).
Now we prove that $\Phi$ is a contraction in $\mathcal{X}_{\delta,\beta}^s$ with the metric given by $\| \cdot \|$. Let $\phi, \psi \in \mathcal{X}_{\delta,\beta}^s$ and $s \geq 0$. Then, since for every $r \geq s$ and every $\xi \in B_{s,\delta,\beta}$ we have

\[
\| f_{x_0,\phi}(r, \xi) - f_{x_0,\psi}(r, \xi) \| \\
\leq c \left( \| x_0(r, \xi) - x_0(r, \xi) \| + \| \phi_{x_0}(r, \xi) - \psi_{x_0}(r, \xi) \| \right) \times \\
\times \left( \| x_0(r, \xi) \| + \| x_0(r, \xi) \| + \| \phi_{x_0}(r, \xi) \| + \| \psi_{x_0}(r, \xi) \| \right)^q \\
\leq 3^q (3\| x_0(r, \xi) - x_0(r, \xi) \| + \| \phi_{x_0}(r, \xi) - \psi_{x_0}(r, \xi) \|) \left( \| x_0(r, \xi) \| + \| x_0(r, \xi) \| \right)^q \\
\leq 3^q (3\| x_0(r, \xi) - x_0(r, \xi) \| + \| \phi - \psi \| \| x_0(r, \xi) \|) \left[ 2C\delta \left( \frac{\mu(r)}{\mu(s)} \right)^{a \varphi(s) \beta(s)} \right]^q
\]

it follows from (35) that

\[
\| f_{x_0,\phi}(r, \xi) - f_{x_0,\psi}(r, \xi) \| \\
\leq 2^{q+2}3^q C^{q+1} \delta q \left( \frac{\mu(r)}{\mu(s)} \right)^{a \varphi(s) \beta(s)} \| x_0 \| \| \phi - \psi \|
\]

for every $r \geq s$ and every $\xi \in B_{s,\delta,\beta}$. From (4), (21) and (22) we have

\[
\| (\Phi\phi)(s, \xi) - (\Phi\psi)(s, \xi) \| \\
\leq \int_s^{+\infty} \| V(r, s)^{-1} \| \| f_{x_0,\phi}(r, \xi) - f_{x_0,\psi}(r, \xi) \| \, dr \\
\leq K \mu(s)^{-a \varphi - b \nu(s)} \varphi^{q+1} \beta(s)^q \| x_0 \| \| \phi - \psi \| \| x_0 \| \| \phi - \psi \| \int_s^{+\infty} \mu(r)^{a \varphi - \nu(r)} \, dr \\
= K \| x_0 \| \| \phi - \psi \| \| x_0 \| \| \phi - \psi \|
\]

for every $s \geq 0$ and every $\xi \in B_{s,\delta,\beta}$ and with $K = 2^{q+2}3^q C^{q+1} D \delta q$. Therefore, for every $\phi, \psi \in \mathcal{X}_{\delta,\beta}^s$, we obtain

\[
\| (\Phi\phi)(s, \xi) - (\Phi\psi)(s, \xi) \| \leq 2^{q+2}3^q C^{q+1} D \delta q \| x_0 \| \| \phi - \psi \|
\]

and choosing $\delta > 0$ sufficiently small, $\Phi$ is a contraction on $\mathcal{X}_{\delta,\beta}^s$.

By the Banach fixed point theorem, $\Phi$ has a unique fixed point and this fixed point verifies (10) for every $s \geq 0$ and every $\xi \in E(s)$. \hfill $\square$

Now the proof of Theorem 2.4 follows easily.

Proof of Theorem 2.7. For each $\phi \in \mathcal{X}_{\delta,\beta}^s$, using Lemma 4.1 there is a unique function $x_0$ in $B_{s,\delta,\beta}$ satisfying (27). By Lemma 4.2 solving (28) is equivalent to solve (10), and from Lemma 4.3 there is a unique solution of (10). Hence, choosing $\delta > 0$ sufficiently small, the existence of a stable manifold is established.
Moreover, for every $s \geq 0$, every $t \geq s$ and every $\xi, \bar{\xi} \in B_{s, \delta, \beta}$, it follows from (14) and (31) that

$$
\|\Psi_{t-s}(p_{s, \xi}) - \Psi_{t-s}(p_{s, \xi})\| = \| (t, x_\phi(t, \xi), \phi_{x_\phi}(t, \xi)) - (t, x_\phi(t, \bar{\xi}), \phi_{x_\phi}(t, \bar{\xi})) \|
\leq \|x_\phi(t, \xi) - x_\phi(t, \bar{\xi})\| + \|\phi_{x_\phi}(t, \xi) - \phi_{x_\phi}(t, \bar{\xi})\|
\leq 2\|x_\phi(t, \xi) - x_\phi(t, \bar{\xi})\|
\leq 2C \left( \frac{\mu(t)}{\mu(s)} \right)^a \nu(s)^\tau \|\xi - \bar{\xi}\|
$$

and the theorem is proved. \hfill \square

5. Behavior under perturbations

In this section we assume that equation (2) admits a $(\mu, \nu)$-dichotomy for some $D \geq 1$, $a < 0 \leq b$ and $\varepsilon > 0$. Given $c > 0$ and $q > 1$, let $\mathcal{P}_{c,q}$ be the class of all perturbations $f : [0, +\infty) \times X \to X$ that verify conditions (6) and (7) with the given $c$ and $q$. In $\mathcal{P}_{c,q}$ we can define a metric by

$$
\|f - \bar{f}\|' = \sup \left\{ \frac{\|f(t, u) - \bar{f}(t, u)\|}{\|u\|^{q+1}} : t \geq 0, u \in X \setminus \{0\} \right\},
$$

for every $f, \bar{f} \in \mathcal{P}_{c,q}$.

The purpose of this section is to see how the manifolds in Theorem 2.1 vary with the perturbations. To do this we consider two perturbation $f, \bar{f} \in \mathcal{P}_{c,q}$ and the functions $\phi$ and $\bar{\phi}$ given by Theorem 2.1 when we perturb equation (2) with $f$ and $\bar{f}$, respectively, and we compare the distance between $\phi$ and $\bar{\phi}$ in the metric given by (16) with the distance between $f$ and $\bar{f}$ in the metric given by (43).

**Theorem 5.1.** Let $c > 0$ and $q > 1$. Suppose that equation (2) admits a $(\mu, \nu)$-dichotomy for some $D \geq 1$, $a < 0 \leq b$ and $\varepsilon > 0$ and that the hypothesis of Theorem 2.1 are satisfied. Then, choosing for $\delta > 0$ sufficiently small, there exists $K > 0$ such that

$$
\|\phi - \bar{\phi}\|' \leq K\|f - \bar{f}\|'
$$

for every $f, \bar{f} \in \mathcal{P}_{c,q}$, where $\phi, \bar{\phi} \in X_{s, \delta, \beta}$ are the functions given by Theorem 2.1, corresponding to the perturbations $f$ and $\bar{f}$, respectively.

**Proof.** Let $(s, \xi) \in G_{s, \delta, \beta}$. From (40) we obtain

$$
\|\phi(s, \xi) - \bar{\phi}(s, \xi)\| \leq \int_s^{+\infty} \|V(r, s)^{-1}\| \cdot \|f_{x_\phi}(r, \xi) - \bar{f}_{x_\phi}(r, \xi)\| dr,
$$

where $\phi, \bar{\phi} \in X_{s, \delta, \beta}$ are the functions given by Theorem 2.1, corresponding to the perturbations $f$ and $\bar{f}$, respectively.
where \(x_\phi, x_\phi^0\) are the functions given by Lemma 4.1 associated with \((f, \phi)\) and \((\bar{f}, \bar{\phi})\), respectively. By (38), (18), (7), (19), (32) and (33) we have for \(r \geq s\)
\[
\|f_{x_\phi, \phi}(r, \xi) - \bar{f}_{x_\phi, \phi}(r, \xi)\|
\leq \|f_{x_\phi, \phi}(r, \xi) - \bar{f}_{x_\phi, \phi}(r, \xi)\| + \|\bar{f}_{x_\phi, \phi}(r, \xi) - \bar{f}_{x_\phi, \phi}(r, \xi)\|
\leq 3^{q+1}\|f - \bar{f}'\|\|x_\phi(r, \xi)\|^{q+1} + 3^{q+1}c\|x_\phi(r, \xi) - x_\phi^0(r, \xi)\| \|x_\phi(r, \xi)\| + \|x_\phi^0(r, \xi)\|^{q+1}
\]
\[
+ 3^q\|\bar{f}'\|\|x_\phi(r, \xi)\| \|x_\phi(r, \xi)\| \|x_\phi^0(r, \xi)\|^{q+1}
\]
\[
\leq 3^{q+1}C^{q+1}\delta^q\|f - \bar{f}'\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q
\]
\[
+ 2^q3^{q+1}cC^q\delta^q\|x_\phi - x_\phi^0\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q
\]
\[
+ 2^q3^{q+1}C^q\delta^q\|\bar{f}'\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q
\]
(45)

and using (37), the last estimate, (44) and taking into account that \(a - b < 0\), we get
\[
\|f_{x_\phi, \phi}(r, \xi) - \bar{f}_{x_\phi, \phi}(r, \xi)\|
\leq 3^{q+1}C^{q+1}\delta^q\|f - \bar{f}'\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q
\]
\[
+ 2^q3^{q+1}cC^q\delta^q\|x_\phi - x_\phi^0\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q
\]
\[
+ 2^q3^{q+1}C^q\delta^q\|\bar{f}'\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q
\]
(46)

Now, we will estimate \(\|x_\phi - x_\phi^0\|\). By (27), (45) and (3) we obtain for every \(t \geq s\)
\[
\left(\frac{\mu(t)}{\mu(s)}\right)^{-a} \|\nu(s)\|^{-\xi} \|x_\phi(t, \xi) - x_\phi^0(t, \xi)\|
\leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-a} \|\nu(s)\|^{-\xi} \int_s^t \|U(t, r)\| \|f_{x_\phi, \phi}(r, \xi) - \bar{f}_{x_\phi, \phi}(r, \xi)\| dr
\leq 3^{q+1}C^{q+1}\delta^q\|f - \bar{f}'\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q \int_s^t \mu(r)^a\|\nu(r)\|^\xi dr
\]
\[
+ 2^q3^{q+1}cC^q\delta^q\|x_\phi - x_\phi^0\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q \int_s^t \mu(r)^a\|\nu(r)\|^\xi dr
\]
\[
+ 2^q3^{q+1}C^q\delta^q\|\bar{f}'\|\|\xi\| \|\mu(r)\|^{q+1}\|\nu(s)\|^{q+1}\beta(s)^q \int_s^t \mu(r)^a\|\nu(r)\|^\xi dr
\]
(47)
and using (37) this implies that
\[
\|x_\phi - x_{\bar{\phi}}\|' \leq 3^{r+1}C^q D\delta^q \|f - \bar{f}\|' + 2^q 3^{r+1}cC^q D\delta^q \|x_\phi - x_{\bar{\phi}}\|' + 2^q 3^{r}cC^q D\delta^q \|\phi - \bar{\phi}\|'.
\]
Thus, for \(\delta > 0\) such that \(2^q 3^{r+1}cC^q D\delta^q < 1/2\) we have
\[
\|x_\phi - x_{\bar{\phi}}\|' \leq 2 \cdot 3^{r+1}C^q D\delta^q \|f - \bar{f}\|' + 2^q 3^{r+1}cC^q D\delta^q \|\phi - \bar{\phi}\|'.
\]
It follows from the last estimate, (46) and \(2^q 3^{r+1}cC^q D\delta^q < 1/2\) that
\[
\|\phi(s,\xi) - \bar{\phi}(s,\xi)\|' \leq 2 \cdot 3^{r+1}C^q D\delta^q \|f - \bar{f}\|' \|\xi\| + 2^q 3^r C^q D\delta^q \|\phi - \bar{\phi}\|' \|\xi\|
\]
for every \((s,\xi) \in G_{\delta,\beta}\). Hence we get
\[
\|\phi - \bar{\phi}\|' \leq 2 \cdot 3^{r+1}C^q D\delta^q \|f - \bar{f}\|' + 2^q 3^r C^q D\delta^q \|\phi - \bar{\phi}\|'
\]
and if we choose \(\delta > 0\) sufficiently small such that \(2^q 3^r C^q D\delta^q < 1/2\) we obtain
\[
\|\phi - \bar{\phi}\|' \leq 4 \cdot 3^{r+1}C^q D\delta^q \|f - \bar{f}\|'
\]
and this proves the theorem. \(\square\)

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