THE DEGREE OF COMMUTATIVITY AND LAMPLIGHTER GROUPS

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Abstract. The degree of commutativity of a group $G$ measures the probability of choosing two elements in $G$ which commute. There are many results studying this for finite groups. In [AMV], this was generalised to infinite groups. In this note, we compute the degree of commutativity for wreath products of the form $\mathbb{Z} \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ where $F$ is any finite group.

1. Introduction

Let $F$ be a finite group. Then degree of commutativity of $F$, denoted $dc(F)$, is the probability of choosing two elements in $F$ which commute i.e.

$$dc(F) := \frac{|\{(a, b) \in F^2 : ab = ba\}|}{|F|^2}.$$  

This definition was generalised to infinite groups in [AMV] in the following way. Let $G$ be a finitely generated group and let $S$ be a finite generating set for $G$. Let $|g|_S$ denote the length of $g$ with respect to the generating set $S$ i.e. the infimum of all word lengths of words in $S$ which represent $g$. For any $n \in \mathbb{N}$, let $B_S(n) := \{g \in G : |g|_S \leq n\}$, the ball of radius $n$ in the Cayley graph of $G$ with respect to the generating set $S$. Then the degree of commutativity of $G$ with respect to $S$, as defined in [AMV], is

$$\limsup_{n \to \infty} \frac{|\{(a, b) \in B_S(n)^2 : ab = ba\}|}{|B_S(n)|^2}$$

and is denoted by $dc_S(G)$. They also pose an intriguing conjecture.

Conjecture. [AMV] Conj. 1.6 Let $G$ be a finitely generated group, and let $S$ be a finite generating set for $G$. Then: (i) $dc_S(G) > 0$ if and only if $G$ is virtually abelian; and (ii) $dc_S(G) > 5/8$ if and only if $G$ is abelian.

They verify this conjecture for hyperbolic groups and groups of polynomial growth (for an introduction to the growth of groups, see [Gri91]). In this note we will investigate the conjecture for groups which are wreath products.

Perhaps the best known example of an infinite wreath product are the lamplighter groups $C \wr \mathbb{Z}$ where $C$ is cyclic. In this note we investigate such groups and from this also obtain a result for groups of the form $F \wr \mathbb{Z}$ where $F$ is finite. Such groups are sensible to investigate with respect to the conjecture since they have exponential growth and yet all elements in the base of $C \wr \mathbb{Z}$ commute. We obtain the following results.

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Theorem 1. Let $G = C \wr \mathbb{Z}$, where $C$ is a non-trivial cyclic group. Let $S$ be a generating set for $G$ of size 2, consisting of a generator of $\mathbb{Z}$ and a generator of $C_i$ for some $i \in \mathbb{Z}$. Then $dc_S(G) = 0$.

Theorem 2. Let $F$ be finite group. Then there is a finite generating set $S$ of $F \wr \mathbb{Z}$ such that $dc_S(F \wr \mathbb{Z}) = 0$.

Note that the groups of Theorem 2 include the first known examples of non-residually finite groups with degree of commutativity 0, since it is currently open as to whether there exists a non-residually finite hyperbolic group. We now introduce wreath products from an algebraic viewpoint, but will provide intuition (using permutations) below.

**Definition.** Given groups $G$ and $H$, the unrestricted wreath product of $G$ and $H$ has elements consisting of an element $h \in H$ and a function $f : H \to G$. Let $B'$ be the set of all such functions. If $f_1, f_2 \in B'$, then $(f_1 \times f_2)(h) := f_1(h) \cdot f_2(h)$ for all $h \in H$, where $\cdot$ denotes the binary operation of $G$. Moreover if $k \in H$ then $k^{-1}(f(h))k := f(hk^{-1})$ for all $h \in H$. This is equal to the semidirect product $B' \rtimes H$. The restricted wreath product, denoted $G \wr H$, is defined analogously as the semidirect product $B \rtimes H$ where $H$ is the head of $G \wr H$ and $B$, the base of $G \wr H$, is the subgroup of $B'$ consisting of functions with finite support i.e. functions $f \in B'$ such that $f(h) \neq 1$ for only finitely many $h$. Since the base is a direct sum of $|H|$ copies of $G$, for any $h \in H$ let $G_h$ denote the copy of $G$ corresponding to $h$.

It may be useful to provide some of the intuition used when thinking about lamplighter groups i.e. groups of the form $C \wr \mathbb{Z}$ where $C$ is cyclic. Each of these groups acts naturally on the corresponding set $C \times \mathbb{Z}$. We shall picture $C$ as addition modulo $n$ if $|C| = n$ and as $\mathbb{Z}$ otherwise. Hence $C = \{0, 1, \ldots, n-1\}$ or $C = \mathbb{Z}$.

A well used generating set is $\{a_0, t\}$ where supp$(a_0) = \{(0, 0), (1, 0), \ldots, (n-1, 0)\}$ and supp$(t) = C \times \mathbb{Z}$ with $t : (m, n) \to (m, n + 1)$ for all $m \in C$ and $n \in \mathbb{Z}$. In the case where $|C| = 2$, the base of $C \wr \mathbb{Z}$ can be thought of as a countable collection of street lamps, with each lamp having an ‘off’ or ‘on’ setting. If $2 < |C| < \infty$, then we can consider each ‘lamp’ to have a finite number of settings (possibly corresponding to different levels of brightness). In the case of $\mathbb{Z} \wr \mathbb{Z}$, the base can be thought of as lamps, where each lamp has an associated ‘voltage’ which takes a value in $\mathbb{Z}$. Although this intuition will not be taken any further, it can also be seen to apply to subgroups of $\mathbb{R} \wr \mathbb{R}$.

**Remark.** In the case where $G$ is finite, it is well known that

$$dc(G) = \frac{\# \text{ conjugacy classes of } G}{|G|}.$$ 

One could therefore define the degree of commutativity for any finitely generated infinite group with respect to a finite generating set $S$ to be

$$\limsup_{n \to \infty} \frac{\# \text{ conjugacy classes in } S(n)}{|S(n)|}.$$ 

Such a limit may not be a real limit. Note that this definition includes the conjugacy growth function of $G$, which was introduced in [Bab88] and studied, for example, in [GS10] and [HO13].

Two questions then present themselves.
Question 1. With this definition for degree of commutativity, does the conjecture above (from [AMV]) hold?

Question 2. Does this definition for the degree of commutativity coincide with (1) above?

The author is unaware of such questions having been posed before, and these questions are not discussed further in this note.

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2. Proving Theorem 1

The key result we shall draw upon is the following. For the group $G = H \wr \mathbb{Z}$ we shall use the base of $H \wr \mathbb{Z}$ as the set $N$.

Lemma 2.1. [AMV, Lem. 3.1] Let $G$ be a finitely generated group, and let $S$ be a finite generating system for $G$. Suppose that there exists a subset $N \subseteq G$ satisfying the following conditions:

i) $N$ is $S$-negligible, i.e. $\lim_{n \to \infty} |N \cap B_S(n)| = 0$;

ii) $\lim_{n \to \infty} \frac{|C_G(g) \cap B_S(n)|}{|B_S(n)|} = 0$ uniformly in $g \in G \setminus N$.

Then, $dc_S(G) = 0$.

Remark. Throughout we will restrict ourselves to generating sets which are the union of a generator of $\mathbb{Z}$ and a generating set for $G_i$ for some fixed $i \in \mathbb{Z}$.

2.1. Proving that groups $C_i \mathbb{Z}$ satisfy (ii) of Lemma 2.1

This is the simpler of the two conditions to prove for such groups. We first introduce the translation lengths of a group. For more discussions on these, see [Con97] and the references therein.

Definition 2.2. Let $G$ be a finitely generated group with finite generating set $S$ and let $g \in G$. Then $\tau_S(g) := \limsup_{n \to \infty} \frac{|g^n|_S}{n}$, the translation length of $g$. Let $F(G)$ denote the set of non-torsion elements in $G$. If there is a finite generating set $S'$ of $G$ such that $\{\tau_{S'}(g) : g \in F(G)\}$ is uniformly bounded away from 0, then we say that $G$ is translation discrete. If a group is translation discrete with respect to one finite generating set, it is translation discrete with respect to all generating sets (see [Con98, Lem. 2.6.1])

We shall use the following.

Lemma 2.3. Let $G$ be finitely generated, $S$ a finite generating set for $G$, and $|B_S(n)| \geq f(n)$ for all $n \in \mathbb{N}$, where $f$ is a polynomial of degree 2. Let $N \subseteq G$. If (i) $C_G(g)$ is cyclic for all $g \in G \setminus N$; and (ii) the translation lengths of $G$ are uniformly bounded away from 0, then $\lim_{n \to \infty} \frac{|C_G(g) \cap B_S(n)|}{|B_S(n)|} = 0$ uniformly in $g \in G \setminus N$.

Proof. This argument can be found within the proof of [AMV] Thm. 1.7. From (ii), there exists a constant $\lambda \in \mathbb{R}$ such that $\tau_S(g) \geq 1/\lambda$ for all $g \in G$. 

Let \( h \in G \). By (i), \( C_G(h) = \langle g \rangle \) for some \( g \in G \). We now consider how \( C_G(h) \cap \mathbb{B}_S(n) \) grows with respect to \( n \). If \( g^k \in C_G(h) \cap \mathbb{B}_S(n) \), then \( |g^k|_s \leq n \) and \( |g^k|_s \geq |k|\tau_S(g) \geq |k|/\lambda \). Thus \( |k| \leq \lambda n \) and

\[
|C_G(h) \cap \mathbb{B}_S(n)| \leq 2\lambda n + 1.
\]

Hence, since \( \mathbb{B}_S(n) \) grows faster than any linear function, the claim follows. \( \Box \)

We must therefore show the two conditions in this lemma are satisfied. Note that they are independent of the choice of finite generating set used.

**Definition 2.4.** Let \( A \) denote the base of \( G = H \wr \mathbb{Z} \) where \( H \) is a finitely generated group. If \( g \in A \), then \( g = \prod_{i \in I} g_i \) where \( I \) is a finite subset of \( \mathbb{Z} \) and \( g_i \in H_i \) for each \( i \in I \). Now \( g_{\min} := \inf \{ I \} \) and \( g_{\max} := \sup \{ I \} \), the infimum and supremum of \( I \), respectively.

**Lemma 2.5.** Let \( G := H \wr \mathbb{Z} \) and let \( A \) denote the base of \( G \). If \( g \in A \), then \( C_G(g) \leq A \) (and if \( H \) is abelian, then \( C_G(g) = A \)). If \( g \in G \setminus A \), then \( C_G(g) \) is cyclic.

**Proof.** The first claim is clear. For the second, let \( g \in G \setminus A \), so that \( g = wt_k \) for some \( w \in A \) and \( k \in \mathbb{Z} \setminus \{0\} \). Now, for any \( v \in A \),

\[
v^{-1}wt_kv = wt_k
\]

\[
\Leftrightarrow v^{-1}wt_kvt^{-k} = w
\]

\[
\Leftrightarrow t^kvt^{-k} = w^{-1}vw
\]

and so, if \( v \) is non-trivial, then \( (w^{-1}vw)_{\min} > (t^kvt^{-k})_{\min} \) and so \( v \notin C_G(w^k) \). Now assume that \( vt^\alpha \in C_G(w^k) \). If \( v't^\alpha \in C_G(w^k) \), then \( v't^\alpha(v't^\alpha)^{-1} = v'v^{-1} \) and so by (2), \( v'v^{-1} = 1 \). Thus for each \( s \in \mathbb{Z} \) such that \( vt^s \in C_G(w^k) \), there is no \( v' \neq v \) such that \( v't^s \in C_G(w^k) \). Now assume that \( \alpha \) is the smallest positive integer such that there exists a \( v \in A \) with \( vt^\alpha \in C_G(w^k) \). If, for some \( \beta \in \mathbb{Z} \) there is a \( u \in A \) such that \( ut^\beta \in C_G(w^k) \), then, by the division algorithm, \( \beta = n\alpha \) for some \( n \in \mathbb{Z} \). Thus \( w^\beta = (w^\alpha)^n \) since for each \( s \in \mathbb{Z} \) there is at most one \( v \in A \) such that \( vt^s \in C_G(w^k) \). \( \Box \)

**Lemma 2.6.** Let \( G = H \wr \mathbb{Z} \) where \( H \) is a finitely generated group and let \( A \) denote the base of \( G \). Then \( \{ \tau_S(g) : g \in G \setminus A \} \) is uniformly bounded away from 0 i.e. \( G \) is translation discrete.

**Proof.** Let \( S_H \) denote a finite generating set for \( H_0 \). We work with the generating set \( S := S_H \cup \{ t \} \) of \( G \).

If \( g \in G \setminus A \), then \( g = wt_k \) where \( w \in A \) and \( t \in \mathbb{Z} \setminus \{0\} \). Thus for any \( n \in \mathbb{N} \),

\[
|g^n|_S \geq |k|n \geq n \text{ and so } \tau_S(g) \geq 1.
\]

Let \( H \) be finitely generated with \( \tau_S(H) \subseteq \mathbb{N} \cup \{0\} \) for some finite generating set \( S \). Then one can prove, with \( S' \) as a finite generating set consisting of the generating set \( S \) for \( H_0 \) and a generator of the head of \( H \wr \mathbb{Z} \), that \( \tau_{S'}(H \wr \mathbb{Z}) = \mathbb{N} \cup \{0\} \) and that \( \tau_{S'}^{-1}(0) \) is equal to \( \{ w \in \bigoplus_{i \in I} H_i : |I| \text{ is a finite subset of } \mathbb{Z} \text{ and } w \text{ is torsion} \} \).

Moreover, if we drop the condition on the translation lengths of \( H \) and let \( A \) denote the base of \( H \wr \mathbb{Z} \), then \( \tau_{S'}(H \wr \mathbb{Z} \setminus A) = \mathbb{N} \).
2.2. **Proving that groups $C \wr \mathbb{Z}$ satisfy (i) of Lemma 2.1.**

The author is unaware of how to show that the negligibility of a set is independent of the generating set used. When working with groups of exponential growth, it seems that the ‘density’ of a set $A \subset G$ may depend on the choice of generating set. Here, density is thought of as the number
\[
\limsup_{n \to \infty} \frac{|N \cap B_S(n)|}{|B_S(n)|}
\]
(so that a set is negligible if and only if it has density 0). Note that if the negligibility of a set is independent of the finite generating set used, then the results that follow would apply to any finite generating set.

**Remark.** We shall work with the generating set $\langle a_0, t \rangle$ where $a_0$ is a generator of $C_0$ and $t$ is a generator of $\mathbb{Z}$. The arguments also work for $C_i$ for any $i \in \mathbb{Z}$.

Essentially we reduce counting the number of elements in the base in $B_S(n)$ to known results regarding the number of possible compositions of a number.

**Definition 2.7.** A multiset, denoted $[\ldots]$, is a collection of objects where repeats are allowed e.g. $[1, 2, 2, 3, 5]$. An ordered multiset, denoted $[\ldots]_{\text{ord}}$, is a multiset with a given ordering. Thus $[1, 2, 2, 3, 5]_{\text{ord}} \neq [1, 2, 3, 2, 5]_{\text{ord}}$.

**Definition 2.8.** Let $n \in \mathbb{N}$. Then a composition of $n$ is an ordered collection of natural numbers that sum to $n$. Thus there is a natural correspondence between compositions of $n$ and ordered multisets whose elements lie in $\mathbb{N}$ and sum to $n$. A weak composition of $n$ is a collection of non-negative integers that sum to $n$. There is a natural correspondence between weak compositions of $n$ and ordered multisets whose elements lie in $\mathbb{N} \cup \{0\}$ and sum to $n$.

The following are well known.

**Lemma 2.9.** Let $n \in \mathbb{N}$. Then the number of compositions of $n$ is $2^{n-1}$.

**Proof.** We consider a multiset with elements in $\mathbb{N}$, which sum to $n$, and where each box either represents a plus or a comma.

$[1\Box 1\Box 1 \ldots 1\Box 1]_{\text{ord}}$

Now, for each box a choice of a comma or a plus provides a unique ordered multiset consisting of elements in $\mathbb{N}$.

**Lemma 2.10.** Let $n \in \mathbb{N}$. Then the number of weak compositions of $n$ into exactly $k$ parts is given by the binomial coefficient
\[
\binom{n + k - 1}{k - 1}.
\]

**Proof.** From the previous proof the number of compositions of $n$ into exactly $k$ parts is given by the number of ways of placing exactly $k - 1$ commas into $n - 1$ boxes i.e.
\[
\binom{n - 1}{k - 1}.
\]

Now, each composition of $n + k$ into $k$ parts can be thought of as a weak composition of $n$ into $k$ parts by mapping $k$ element multisets which sum to $n + k$ and consist of natural numbers to $k$ element multisets which sum to $n$ and consist of non-negative integers i.e. the map $[m_1, m_2, \ldots, m_k]_{\text{ord}} \mapsto [m_1 - 1, m_2 - 1, \ldots, m_k - 1]_{\text{ord}}$. 

\[\square\]
We are now ready to prove our first theorem.

**Theorem 1**  Let $G = C \wr Z$ and $S = \langle a, t \rangle$ be a generating set for $G$ with $a \in C_i$ (for some $i \in \mathbb{Z}$) and $t \in Z$. Then the base $A$ of $G$ is negligible in $G$.

**Proof.** Fix an $n \in \mathbb{N}$. Our aim is to produce a bound for $|B_S(n) \cap A|$. For discussions on normal forms for elements of $C \wr Z$, see [CT05]. Let $k \leq n/2$. If $|g| = n$ and $g_{\min} \geq 0$, then there is a word of length $n$ of the form below which represents $g$

\[ w^{(0)} t^{-1} w^{(1)} t^{-1} \ldots w^{(k-1)} t^{-1} w^{(k)} t w^{(k+1)} t \ldots w^{(2k)} \]

where, for each $i \in \{0, 1, \ldots, 2l\}$, $w^{(i)} = a^{d_i}$ for some $d_i \in \mathbb{Z}$. Now, any word $g \in |B_S(n) \cap A|$ with $g_{\min} \geq 0$ can be expressed in the form (3) and must satisfy

\[ \sum_{i=0}^{2k} |w^{(i)}|_{\{a\}} \leq n - 2k. \]

We now justify why it is sufficient to look at only those $g \in A$ with $g_{\min} \geq 0$. Let $A_s := \{ g \in A : g_{\min} \geq s \}$. By conjugating a word of the form (3) by $t^{-s}$, we have, for any $s \in \mathbb{Z} \setminus \mathbb{N}$, that

\[ |B_S(n) \cap (A_s \setminus A_{s+1})| \leq |B_S(n) \cap A_0|. \]

Also, for any $s \leq -n$, $|B_S(n) \cap (A_s \setminus A_{s+1})| = 0$. Thus

\[ |B_S(n) \cap A| \leq \left( \bigcup_{-n \leq s \leq -1} |B_S(n) \cap (A_s \setminus A_{s+1})| \right) \cup |B_S(n) \cap A_0| \]

\[ \leq (n + 1)|B_S(n) \cap A_0|. \]

Since groups of the form $C \wr Z$ (where $C$ is a non-trivial cyclic group) are of exponential growth, producing a bound for $|B_S(n) \cap A_0|$ will be sufficient to bound $|B_S(n) \cap A_0|$.

In (3), the words $\{w^{(j)} : j = k + 1, k + 2, \ldots, 2k\}$ are redundant since

\[ w^{(0)} t^{-1} w^{(1)} t^{-1} \ldots w^{(k-1)} t^{-1} w^{(k)} t w^{(k+1)} t \ldots w^{(2k)} = w^{(0)} w^{(2k)} t^{-1} w^{(1)} w^{(2k-1)} t^{-1} \ldots w^{(k-1)} w^{(k+1)} t^{-1} w^{(k)} t^k. \]

Thus any word $g \in |B_S(n) \cap A|$ with $g_{\min} \geq 0$ can be expressed in the form

\[ w^{(0)} t^{-1} w^{(1)} t^{-1} \ldots w^{(k-1)} t^{-1} w^{(k)} t^k \]

where, for each $i \in \{0, 1, \ldots, k\}$, $w^{(i)} = a^{d_i}$, for some $d_i \in \mathbb{Z}$.

From [BT15], the growth of $C \wr Z$ with our generating set is greater than $2^n$ if $|C| \geq 3$ and is $\frac{1 + \sqrt{5}}{2}$ if $|C| = 2$.

We first work with $|C| = 2$. In this case each $w^{(i)}$ has length 0 or 1. Thus, for each $k$, there are at most $2^{k+1}$ choices for the values of $\{w^{(i)} : i = 0, 1, \ldots, k\}$. Hence the size of $|B_S(n) \cap A_0|$ is bounded by

\[ \sum_{j=0}^{n/2} 2^{j+1} \leq 4 \cdot (\sqrt{2})^n \leq 4 \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^n \]

and so the base of $C_2 \wr Z$ is negligible.

For the case where $|C| > 2$, we shall use Lemma 2.10. Our aim is to show that $|(B_S(n) \setminus B_S(n-1)) \cap A_0|$ is bounded by a function which has growth rate $2^n$ (since this will mean that $|B_S(n) \cap A_0|$ is also bounded by a function which has growth rate
$2^n).$ Fix an $k \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}$. We note that all such elements can be represented by a word of the form (4), where $|w(i)| < n - 2k$ for all $i \in \{0, \ldots, k \}$. Each word is in bijection with a multiset

$$[u(0), v(0), u(1), v(1), \ldots, u(k-1), v(k-1), u(k), v(k)]_{\text{ord}}$$

where $u(i), v(i) \in \mathbb{N} \cup \{0\}$ and the condition that $u(i)v(i) = 0$ for all $i$. This is therefore bounded by the number of weak compositions of $n - 2k$ into $2k + 2$ parts. From Lemma 2.10 this is equal to

$$\left( \frac{n - 2k + 2k + 2 - 1}{2k + 2 - 1} \right) = \left( \frac{n + 1}{2k + 1} \right).$$

Now we sum over all viable $k$:

$$\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{n + 1}{2k + 1} \right) \leq \sum_{j=0}^{n+1} \left( \frac{n + 1}{j} \right) = 2^n.$$

Hence $|(\mathcal{B}_S(n) \setminus \mathcal{B}_S(n - 1)) \cap A| \leq (n + 1)(\mathcal{B}_S(n) \setminus \mathcal{B}_S(n - 1)) \cap A_0| \leq (n + 1) \cdot 2^n$, and so is negligible in $C \cap \mathbb{Z}$. \hfill \Box

Note that from this proof it immediately follows that $(C_2 \times C_2) \wr \mathbb{Z}$, with the generating set consisting of two generators of $(C_2 \times C_2)_0$ and a generator of the head, has degree of commutativity 0.

**Theorem 2** Let $G := F \wr \mathbb{Z}$ where $F$ is a non-trivial finite group. Then there exists a generating set $S$ such that $dc_S(G) = 0$.

**Proof.** Let $|F| = m > 1$ and let $A$ denote the base of $G$. Then $A := \bigoplus_{i \in \mathbb{Z}} F_i$ where $F_i = F$ for each $i \in \mathbb{Z}$. Let $S$ denote the generating set consisting of the non-trivial elements of $F_0$ and a generator $t$ of the head of $G$. From Section 2.2 we need only show that the base of $G$ is negligible in $G$.

First we produce a lower bound on the growth of $G$. Consider words of the form

$$w_1tw_2tw_3 \ldots tw_k t^\epsilon$$

where, for each $i$, $w_i \in S$ and $\epsilon \in \{0, 1\}$. There are $m^k$ such words (since $|S| = m$) and so $|\mathcal{B}_S(n)| \geq |\mathcal{B}_S(n) \setminus \mathcal{B}_S(n - 1)| \geq m^{[n/2]}$.

We now produce an upper bound on the growth of $A$, the base of $G$. As with the previous proof, we produce an upper bound for words $g \in A \cap (\mathcal{B}_S(n) \setminus \mathcal{B}_S(n - 1))$ with $g_{\min} \geq 0$. Such words are of the form

$$w_0 t^{-1}w_1 t^{-1}w_2 t^{-1} \ldots w_{k-1} t^{-1}w_k t^k$$

where each $w_i$ is either trivial or in $S \setminus \{t\}$ and $\left\lfloor \frac{n-1}{n-2k} \right\rfloor \leq k \leq n - 1$ since there must be at least 1 non-trivial $w_i$ and at most $\left\lfloor \frac{n-1}{n-2k} \right\rfloor$ non-trivial $w_i$. This produces the bound

$$\sum_{k=\left\lfloor \frac{n-1}{n-2k} \right\rfloor}^{n-1} \left( \frac{k + 1}{n - 2k} \right) (m - 1)^{n-2k}$$. 
since, for each $k$, $n - 2k$ of the $\{w_i \mid i = 0, \ldots, k\}$ may be chosen from $S \setminus \{t\}$ and the other $w_i$ are trivial. Now
\[
\sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} \binom{k + 1}{n - 2k} (m - 1)^{n-2k} \leq \sum_{k=\lceil \frac{n}{2} \rceil}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n}{n - 2k} (m - 1)^{n-2k} \\
\leq (m - 1)^2 \cdot \sum_{j=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{j} (m - 1)^j \\
\leq (m - 1)^2 \cdot (m - 1 + 1)^{\lfloor n/3 \rfloor}
\]
and so the base of $G$ is negligible in $G$. □

We end by posing a question. This seems natural in the context of Theorem 1 and Theorem 2.

**Question 3.** Given a finitely generated group $H$, is the base of $G := H \wr \mathbb{Z}$ negligible in $G$? Moreover, what if $\mathbb{Z}$ is replaced with another finitely generated infinite group?

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