Distance-Increasing Maps of All Length by Simple Mapping Algorithms

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Abstract

Distance-increasing maps from binary vectors to permutations, namely DIMs, are useful for the construction of permutation arrays. While a simple mapping algorithm defining DIMs of even length is known, existing DIMs of odd length are either recursively constructed by merging shorter DIMs or defined by much complicated mapping algorithms. In this paper, DIMs of all length defined by simple mapping algorithms are presented.

Index Terms

distance-increasing maps, distance-preserving maps, Hamming distance, permutation arrays.

I. INTRODUCTION

Let $S_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$. Here we think of a permutation $x = (x_1, x_2, \ldots, x_n)$ as a tuple rather than a map. Let $\mathbb{Z}_2^n$ be the set of all binary vectors of length $n$. Both $S_n$ and $\mathbb{Z}_2^n$ are endowed with the Hamming distance $d$.

A distance-increasing map of length $n$, $n$-DIM for short, is a map $f$ from $\mathbb{Z}_2^n$ to $S_n$ that increases the Hamming distance, that is, $d(f(u), f(v)) > d(u, v)$ for all $u, v \in \mathbb{Z}_2^n$ except when $d(u, v) = n$. DIMs were first studied because of their useful application in constructing permutation arrays. The image of a binary code under a DIM is a permutation array with minimum distance greater than that of the binary code. However, DIMs are also interesting combinatorial objects. It is easy to see by a counting argument that no DIMs of length $< 4$ can possibly exist. First known examples of DIMs are $h_{2m}$ in [1] for $m = 2$ and odd $m \geq 3$. What is interesting with $h_{2m}$ is that they are defined by a very simple mapping algorithm that returns a permutation for a binary vector given as input. Then Chang [2] succeeded in constructing DIMs of all length by recursively merging DIMs of shorter length, beginning with a small set of basic DIMs. The basic DIMs included some maps belonging to the family $h_{2m}$ and a computer-found one. Recently a mapping algorithm as simple as that of $h_{2m}$ defining DIMs of all even length was found [3]. Thus a natural question arose whether there exist DIMs of odd length defined by simple mapping algorithms.

In this paper, we present simple mapping algorithms defining DIMs of all odd length. It turns out that we need separate mapping algorithms for DIMs of length $n \equiv 1 \mod 4$ and DIMs of length $n \equiv 3 \mod 4$. That these algorithms indeed define DIMs is proved by a straightforward method. This method also provides us a proof that the DIMs of even length in [3] indeed defines DIMs. Since the proof is easier and direct than the original proof, we include it in the next section.

We introduce some convenient notations. For a set $S$, $\#S$ denotes the number of elements in the set. For a predicate $P$, $[P]$ gives 1 or 0 if $P$ is true or false, respectively. For integers $u$ and $v$, $\delta(u, v)$ gives 1 if $u = v$ and 0 otherwise.

II. DIMS OF EVEN LENGTH

The following lemma is easy but very useful in subsequent proofs.

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Lemma 1: Let \( a_1, a_2, \ldots, a_n \) be a sequence with \( a_i \) either 0 or 1 and \( n \geq 2 \). Then
\[
\begin{align*}
    a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n \\
    & = \#\{1 \leq i \leq n \mid a_i = 1\} - B(a_i \mid 1 \leq i \leq n) \\
    & \leq \#\{1 \leq i \leq n \mid a_i = 1\} - [a_i = 1 \text{ for some } 1 \leq i \leq n], \\
    & \leq \#\{1 \leq i \leq n \mid a_i = 1\} \\
    & \quad - [a_i = 1 \text{ and } a_j = 0 \text{ for some } 1 \leq i, j \leq n],
\end{align*}
\]
where \( B(a_i \mid 1 \leq i \leq n) \) denotes the number of blocks of consecutive 1’s in the sequence.

Proof: First assertion is obvious if we notice that a block of consecutive 1’s in the sequence contributes \( l - 1 \) to the sum if the length of the block is \( l \). To prove the second assertion, we apply the first to see
\[
\begin{align*}
    & (a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n) + a_n a_1 \\
    & = \#\{1 \leq i \leq n \mid a_i = 1\} - B(a_i \mid 1 \leq i \leq n) + a_n a_1.
\end{align*}
\]
Suppose \( a_n a_1 = 0 \). Then the assertion follows since
\[
B(a_i \mid 1 \leq i \leq n) \geq [a_i = 1 \text{ for some } 1 \leq i \leq n] = [a_i = 1 \text{ and } a_j = 0 \text{ for some } 1 \leq i, j \leq n].
\]
Suppose \( a_n a_1 = 1 \). The claim also follows since
\[
B(a_i \mid 1 \leq i \leq n) > [a_j = 0 \text{ for some } 1 \leq j \leq n] = [a_i = 1 \text{ and } a_j = 0 \text{ for some } 1 \leq i, j \leq n].
\]

We now recall the DIMs of even length given in [3]. Let \( r \geq 2 \). Let \( z_{2r} \) be the map from \( \mathbb{Z}_2^{2r} \) to \( S_{2r} \) defined by

**Mapping algorithm A**

Input: \( u = (u_1, \ldots, u_n) \in \mathbb{Z}_2^n \) (\( n = 2r \))

Output: \( x = (x_1, \ldots, x_n) \in S_n \)

begin
\[
(x_1, x_2, \ldots, x_n) \leftarrow (1, 2, \ldots, n);
\]
for \( i \) from 1 to \( r \) do
\[
\text{if } u_{2i-1} = 1 \text{ then } \text{swap}(x_{2i-1}, x_{2i});
\]
for \( i \) from 1 to \( r \) do
\[
\text{if } u_{2i} = 1 \text{ then } \text{swap}(x_{2i}, x_{2i+1});
\]
end

It is shown in [3], as a special case of a more general construction, that \( z_{2r} \) are DIMs of length \( 2r \). We will give a direct proof of this shortly, using a method which we will also apply for DIMs of odd length. But first let us understand what this mapping algorithm does. Let \( u \) be a binary vector. Consider the diagram in Figure 1, which shows the permutation \( (x_1, x_2, \ldots, x_{2r}) \) lying on a big circle with \( x_i = i \) (\( 1 \leq i \leq 2r \)) in small circles. Now each \( u_i \), if it is 1, swaps the two components in the small circles between which \( u_i \) is placed, and this is done in the order
\[
u_1 \rightarrow u_3 \rightarrow \cdots \rightarrow u_{2r-1} \rightarrow u_2 \rightarrow u_4 \rightarrow \cdots \rightarrow u_{2r}.
\]
The resulting permutation is what \( u \) is mapped to under \( z_{2r} \).

**Theorem 2:** Let \( r \geq 2 \). The maps \( z_{2r} \) are DIMs of length \( 2r \).

Proof: Let \( n = 2r \). Let \( u, v \) be two distinct binary vectors, and let \( z_{2r}(u) = x \) and \( z_{2r}(v) = y \). We need to show \( d(x, y) > d(u, v) \) if \( d(u, v) < n \) and \( d(x, y) = n \) if \( d(u, v) = n \). Let
\[
T = \{1 \leq i \leq n \mid x_j = y_j = i \text{ for some } 1 \leq j \leq n\}.
\]
That is, \( i \in T \) if and only if \( i \)'s in \( x \) and \( y \) are at the same position. Clearly \( d(x, y) = n - \#T \) and \( d(u, v) = n - \#\{1 \leq i \leq n \mid u_i = v_i\} \). So we may rephrase our goal as
\[
\#T \leq \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\}
\]
with equality only when the right side is zero, that is, when \( d(u, v) = n \).

For each \( 1 \leq i \leq n \), we can determine precise conditions to have \( i \in T \). To give an example, let \( i = 1 \), and view the diagram in Figure 1. If \( u_1 \neq v_1 \), then it is clear that the position of 1 in \( x \) will never be the same with that of 1 in \( y \) so that \( 1 \notin T \). Suppose \( u_1 = v_1 = 0 \), then 1’s in \( x \) and \( y \) will be at the same position if and only if \( u_{2r} = v_{2r} \). On the other hand, if \( u_1 = v_1 = 1 \), then it is clear that \( 1 \in T \) if and only if \( u_2 = v_2 \). These cases exhaust all possibilities to have \( 1 \in T \). Note that in this reasoning, the order in which swappings take place is crucial. We can express the result as follows:

\[
1 \in T \iff \delta(u_1, 0)\delta(v_1, 0)\delta(u_{2r}, v_{2r}) + \delta(u_1, 1)\delta(v_1, 1)\delta(u_2, v_2) = 1. \tag{3}
\]

In a similar way, we obtain

\[
2 \in T \iff \delta(u_1, 0)\delta(v_1, 0)\delta(u_2, v_2) + \delta(u_1, 1)\delta(v_1, 1)\delta(u_{2r}, v_{2r}) = 1, \tag{4}
\]

\[
\vdots \]

\[
2r - 1 \in T \iff \delta(u_{2r-1}, 0)\delta(v_{2r-1}, 0)\delta(u_{2r-2}, v_{2r-2}) + \delta(u_{2r-1}, 1)\delta(v_{2r-1}, 1)\delta(u_{2r}, v_{2r}) = 1, \tag{5}
\]

\[
2r \in T \iff \delta(u_{2r-1}, 0)\delta(v_{2r-1}, 0)\delta(u_{2r}, v_{2r}) + \delta(u_{2r-1}, 1)\delta(v_{2r-1}, 1)\delta(u_{2r-2}, v_{2r-2}) = 1. \tag{6}
\]

We may simplify by summing the expressions in right side in pairs like

\[
(3) + (4) = \delta(u_1, v_1)\delta(u_2, v_2) + \delta(u_1, v_1)\delta(u_{2r}, v_{2r}),
\]

\[
\vdots
\]

\[
(5) + (6) = \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r-2}, v_{2r-2}) + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r}).
\]

So we obtain

\[
#T = (3) + (4) + \cdots + (5) + (6)
\]

\[
= \delta(u_1, v_1)\delta(u_2, v_2) + \cdots + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r})
\]

\[
+ \delta(u_{2r}, v_{2r})\delta(u_1, v_1)
\]

\[
\leq \{ 1 \leq i \leq n \mid \delta(u_i, v_i) = 1 \}
\]

\[
- [\delta(u_i, v_i) = 1 \text{ and } \delta(u_j, v_j) = 0 \text{ for some } 1 \leq i, j \leq n]
\]

\[
= \{ 1 \leq i \leq n \mid \delta(u_i, v_i) = 1 \}
\]

\[
- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i, j \leq n],
\]

where we used (2) of Lemma 1. Now our claim follows. \(\square\)
III. DIMS OF ODD LENGTH

In this section, we present simple mapping algorithms defining DIMs of odd length. One algorithm defines DIMs of length $4r + 1$ with $r \geq 1$. The other defines DIMs of length $4r - 1$ with $r \geq 2$.

A. DIMs of length $4r + 1$

Let $n = 4r + 1$ with $r \geq 1$. Let $z_{4r+1}$ be the map from $\mathbb{Z}_2^{4r+1}$ to $S_{4r+1}$ defined by

**Mapping algorithm B**

**Input:** $u = (u_1, \ldots, u_n) \in \mathbb{Z}_2^n$ ($n = 4r + 1$)

**Output:** $x = (x_1, \ldots, x_n) \in S_n$

**begin**

$(x_1, x_2, \ldots, x_n) \leftarrow (1, 2, \ldots, n)$;

for $i$ from 1 to $2r$ do

if $u_{2i-1} = 1$ then swap($x_{2i-1}, x_{2i}$);

if $u_n = 1$ then swap($x_n, x_1$);

if $u_{2i} = 1$ then swap($x_{2i}, x_{2i+1}$);

end

**end**

We give an intuitive description of what the algorithm does. Let $u$ be a binary vector given as input. The diagram in Figure 2 shows the permutation initialized with $x_i = i$ for $1 \leq i \leq n$. Note that there are $2r + 1$ components on the left big circle and $2r + 2$ components on the right big circle and two components $x_1$ and $x_{2r+1}$ are shared between the two circles. Notice that there is a new variable $u'_n$ placed between $x_1 = 1$ and $x_{2r+1} = 2r + 1$. The variable $u'_n$ is simply set to the value of $u_n$. That is, $u'_n = u_n$. Then swappings take place in the order

$$u_1 \rightarrow u_3 \rightarrow \cdots \rightarrow u_{2r-1} \rightarrow u_{2r+1} \rightarrow \cdots \rightarrow u_{n-2} \rightarrow u_n \rightarrow u'_n \rightarrow u_2 \rightarrow \cdots \rightarrow u_{2r-2} \rightarrow u_{2r} \rightarrow u_{2r+2} \rightarrow \cdots \rightarrow u_{n-1}.$$ 

Now $u$ is mapped to the resulting permutation.

**Theorem 3:** Let $r \geq 1$. The maps $z_{4r+1}$ are DIMs of length $4r + 1$.

**Proof:** Let $n = 4r + 1$. Let $z_{4r+1}(u) = x$ and $z_{4r+1}(v) = y$ for two distinct binary vectors $u$ and $v$. Let

$$T = \{1 \leq i \leq n \mid x_j = y_j = i \text{ for some } 1 \leq j \leq n\}.$$ 

The proof proceeds in the same manner with that about DIMs of even length. So our goal is to show

$$\#T \leq \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\}$$

with equality only when the right side is zero.

It is a bit more complicated to determine the conditions to have $i \in T$. So we again give an example with $i = 1$. In the following reasoning, we must keep in mind the swapping order of $z_{4r+1}$. If $u_1 \neq v_1$, then the position of

![Fig. 2. Swapping order of $z_{4r+1}$](attachment:image.png)
1 in \( x \) will never be the same with that in \( y \), that is \( 1 \notin T \). Suppose \( u_1 = v_1 = 1 \), then 1’s in \( x \) and \( y \) will be at the same position if and only if \( u_2 = v_2 \). If \( u_1 = v_1 = 0 \), then we need to consider \( u_n \) and \( v_n \). If \( u_n \neq v_n \), then \( 1 \notin T \). If \( u_n = v_n = 1 \), then \( 1 \notin T \) if and only if \( u_{n-1} = v_{n-1} \). If \( u_n = v_n = 0 \), then we again consider \( u'_n \) and \( v'_n \). If \( u'_n \neq v'_n \), then \( 1 \notin T \). If \( u'_n = v'_n = 0 \), then \( 1 \in T \). If \( u'_n = v'_n = 1 \), then we still consider \( u_{2r} \) and \( v_{2r} \) to see that \( 1 \in T \) if and only if \( u_{2r} = v_{2r} \). We express the result formally as

\[
1 \in T \iff \delta(u_1, 0) \delta(v_1, 0) \delta(u_2, v_2) + \delta(u_1, 0) \delta(v_1, 0) \delta(u_n, 0) \delta(u'_n, 0) \delta(v'_n, 0) + \delta(u_1, 0) \delta(v_1, 0) \delta(u_{n-1}, 1) \delta(v_{n-1}, 1) \delta(u_{2r}, v_{2r}) + \delta(u_1, 1) \delta(v_1, 1) \delta(u_{2r}, v_{2r}) = 1.
\]

Similar reasonings give

\[
2 \in T \iff \delta(u_2, v_2) + \delta(u_1, 1) \delta(v_1, 1) \delta(u_n, 0) \delta(v_n, 0) \delta(u'_n, 0) \delta(v'_n, 0) + \delta(u_1, 1) \delta(v_1, 1) \delta(u_{n-1}, 1) \delta(v_{n-1}, 1) \delta(u_{2r}, v_{2r}) + \delta(u_1, 1) \delta(v_1, 1) \delta(u_{2r}, v_{2r}) = 1. \tag{7}
\]

\[
3 \in T \iff \delta(u_3, 0) \delta(v_3, 0) \delta(u_2, v_2) + \delta(u_3, 1) \delta(v_3, 1) \delta(u_{2r}, v_{2r}) = 1. \tag{9}
\]

\[
4 \in T \iff \delta(u_4, v_4) + \delta(u_3, 1) \delta(v_3, 1) \delta(u_{2r}, v_{2r}) = 1. \tag{10}
\]

\[
\vdots
\]

\[
2r - 1 \in T \iff \delta(u_{2r-1}, 0) \delta(v_{2r-1}, 0) \delta(u_{2r-2}, v_{2r-2}) + \delta(u_{2r-1}, 1) \delta(v_{2r-1}, 1) \delta(u_{2r}, v_{2r}) = 1. \tag{11}
\]

\[
2r \in T \iff \delta(u_{2r}, v_{2r}) + \delta(u_{2r-1}, 1) \delta(v_{2r-1}, 1) \delta(u_{2r-2}, v_{2r-2}) = 1. \tag{12}
\]

\[
2r + 1 \in T \iff \delta(u_{2r+1}, 0) \delta(v_{2r+1}, 0) \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, 0) \delta(v_{2r+1}, 0) \delta(u'_n, 0) \delta(v'_n, 0) \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, 1) \delta(v_{2r+1}, 1) \delta(u_{2r+2}, v_{2r+2}) = 1. \tag{13}
\]

\[
2r + 2 \in T \iff \delta(u_{2r+2}, v_{2r+2}) + \delta(u_{2r+1}, 1) \delta(v_{2r+1}, 1) \delta(u'_n, 0) \delta(v'_n, 0) \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, 1) \delta(v_{2r+1}, 1) \delta(u'_n, 1) \delta(v'_n, 1) = 1. \tag{14}
\]

\[
2r + 3 \in T \iff \delta(u_{2r+3}, 0) \delta(v_{2r+3}, 0) \delta(u_{2r+2}, v_{2r+2}) + \delta(u_{2r+3}, 1) \delta(v_{2r+3}, 1) \delta(u_{2r+4}, v_{2r+4}) = 1. \tag{15}
\]

\[
\vdots
\]
\[ n - 1 \in T \]
\[ \iff \delta(u_{n-2}, 0)\delta(v_{n-2}, 0)\delta(u_{n-1}, v_{n-1}) + \delta(u_{n-2}, 1)\delta(v_{n-2}, 1)\delta(u_{n-3}, v_{n-3}) = 1, \quad (16) \]
\[ n \in T \]
\[ \iff \delta(u_n, 0)\delta(v_n, 0)\delta(u_{n-1}, v_{n-1}) + \delta(u_n, 1)\delta(v_n, 1)\delta(u_n', 0)\delta(v_n', 0) + \delta(u_n, 1)\delta(v_n, 1)\delta(u_n', 1)\delta(u_{2r}, v_{2r}) = 1. \quad (17) \]

Note that (9), (10), \ldots, (11), (12) and (15), \ldots, (16) do not appear if \( r = 1 \).
Now \#T equals the total of sums in the right sides of (7)–(17). The sum is simplified greatly if we first compute in groups. Thus

\[
\begin{align*}
(7) + (8) &= \delta(u_1, v_1)\delta(v_2, v_2) + \delta(u_1, v_1)\delta(u_n, 0)\delta(v_n, 0) \\
&\quad + \delta(u_1, v_1)\delta(u_{n-1}, v_{n-1})\delta(u_n, 1)\delta(v_n, 1), \\
(9) + \cdots + (12) &= \delta(u_2, v_2)\delta(u_3, v_3) + \delta(u_3, v_3)\delta(u_4, v_4) + \cdots \\
&\quad + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r}), \\
(13) + (14) &= \delta(u_{2r+1}, v_{2r+1})\delta(u_{2r+2}, v_{2r+2}) \\
&\quad + \delta(u_{2r+1}, v_{2r+1})\delta(u_{2r}, v_{2r})\delta(u_n, 0)\delta(v_n, 0) \\
&\quad + \delta(u_{2r+1}, v_{2r+1})\delta(u_n, 1)\delta(v_n, 1), \\
(15) + \cdots + (16) &= \delta(u_{2r+2}, v_{2r+2})\delta(u_{2r+3}, v_{2r+3}) + \cdots \\
&\quad + \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1}), \\
(17) &= \delta(u_{n-1}, v_{n-1})\delta(u_n, 0)\delta(v_n, 0) \\
&\quad + \delta(u_{2r}, v_{2r})\delta(u_n, 1)\delta(v_n, 1).
\end{align*}
\]

Summing all, \#T equals

\[
\begin{align*}
&\delta(u_1, v_1)\delta(v_2, v_2) + \cdots + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r}) \\
&\quad + \delta(u_{2r+1}, v_{2r+1})\delta(u_{2r+2}, v_{2r+2}) + \cdots + \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1}) \\
&\quad + \delta(u_1, v_1)\delta(u_n, 0)\delta(v_n, 0) + \delta(u_1, v_1)\delta(u_{n-1}, v_{n-1})\delta(u_n, 1)\delta(v_n, 1) \\
&\quad + \delta(u_{2r}, v_{2r})\delta(u_{2r+1}, v_{2r+1})\delta(u_n, 0)\delta(v_n, 0) \\
&\quad + \delta(u_{2r+1}, v_{2r+1})\delta(u_n, 1)\delta(v_n, 1) \\
&\quad + \delta(u_{n-1}, v_{n-1})\delta(u_n, 0)\delta(v_n, 0) + \delta(u_{2r}, v_{2r})\delta(u_n, 1)\delta(v_n, 1).
\end{align*}
\]

We now treat three different cases. For the case \( u_n = 0 \) and \( v_n = 0 \):
\[
\#T = \delta(u_1, v_1)\delta(u_2, v_2) + \cdots + \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1}) \\
&\quad + \delta(u_1, v_1) + \delta(u_{n-1}, v_{n-1}) \\
\leq &\#\{1 \leq i \leq n - 1 \mid \delta(u_i, v_i) = 1\} - B(\delta(u_i, v_i) \mid 1 \leq i \leq n - 1) \\
&\quad + \delta(u_1, v_1) + \delta(u_{n-1}, v_{n-1}) \\
= &\#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - 1 - B(\delta(u_i, v_i) \mid 1 \leq i \leq n - 1) \\
&\quad + \delta(u_1, v_1) + \delta(u_{n-1}, v_{n-1}) \\
\leq &\#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - 1,
\]
where (1) of Lemma 1 is used. The last inequality is valid since
\[
B(\delta(u_i, v_i) \mid 1 \leq i \leq n - 1) \geq \delta(u_1, v_1) + \delta(u_{n-1}, v_{n-1}),
\]
which is itself easy to verify. If at most one of \( \delta(u_1, v_1) = 1 \) and \( \delta(u_{n-1}, v_{n-1}) = 1 \) is true, then the inequality clearly holds. If \( \delta(u_1, v_1) = \delta(u_{n-1}, v_{n-1}) = 1 \), then there must be some \( 1 < j < n - 1 \) such that \( \delta(u_j, v_j) = 0 \) because \( u \) and \( v \) are distinct binary vectors, and hence \( B(\delta(u_i, v_i) \mid 1 \leq i \leq n - 1) \geq 2 \).
For the case $u_n = 1$ and $v_n = 1$:

$$\#T = \delta(u_1, v_1)\delta(u_2, v_2) + \cdots + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r})$$

$$+ \delta(u_{2r+1}, v_{2r+1})\delta(u_{2r+2}, v_{2r+2}) + \cdots$$

$$+ \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1})$$

$$+ \delta(u_1, v_1)\delta(u_{n-1}, v_{n-1}) + \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, v_{2r+1})$$

$$\leq \#\{1 \leq i \leq 2r \mid \delta(u_i, v_i) = 1\} - B(\delta(u_i, v_i) \mid 1 \leq i \leq 2r)$$

$$+ \#\{2r + 1 \leq i \leq n - 1 \mid \delta(u_i, v_i) = 1\} - B(\delta(u_i, v_i) \mid 2r + 1 \leq i \leq n - 1)$$

$$+ \delta(u_1, v_1)\delta(u_{n-1}, v_{n-1}) + \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, v_{2r+1})$$

$$\leq \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - 1 - B(\delta(u_i, v_i) \mid 1 \leq i \leq 2r)$$

$$- B(\delta(u_i, v_i) \mid 2r + 1 \leq i \leq n - 1)$$

$$+ \delta(u_1, v_1)\delta(u_{n-1}, v_{n-1}) + \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, v_{2r+1})$$

$$\leq \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - 1.$$  

The last inequality is valid because

$$B(\delta(u_i, v_i) \mid 1 \leq i \leq 2r) + B(\delta(u_i, v_i) \mid 2r + 1 \leq i \leq n - 1)$$

$$\geq \delta(u_1, v_1)\delta(u_{n-1}, v_{n-1}) + \delta(u_{2r}, v_{2r}) + \delta(u_{2r+1}, v_{2r+1}),$$

which is easily verified in a similar way as above.

For the case $u_n \neq v_n$:

$$\#T = \delta(u_1, v_1)\delta(u_2, v_2) + \cdots + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r})$$

$$+ \delta(u_{2r+1}, v_{2r+1})\delta(u_{2r+2}, v_{2r+2}) + \cdots$$

$$+ \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1})$$

$$\leq \#\{1 \leq i \leq 2r \mid \delta(u_i, v_i) = 1\} - B(\delta(u_i, v_i) \mid 1 \leq i \leq 2r)$$

$$+ \#\{2r + 1 \leq i \leq n - 1 \mid \delta(u_i, v_i) = 1\} - B(\delta(u_i, v_i) \mid 2r + 1 \leq i \leq n - 1)$$

$$= \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - B(\delta(u_i, v_i) \mid 1 \leq i \leq 2r)$$

$$- B(\delta(u_i, v_i) \mid 2r + 1 \leq i \leq n - 1)$$

$$\leq \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\}$$

$$- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq n],$$

where the last inequality is easily verified.

Now the three cases combine to give the required result. \(\square\)

**B. DIMs of length \(4r - 1\)**

We now turn to DIMs of length \(4r - 1\). Much of the material in this subsection goes parallel with that of the previous subsection. So we omit some repetitive details. Let \(n = 4r - 1\) with \(r \geq 2\). Let \(\delta_{4r-1}\) be the map from \(\mathbb{Z}_2^{4r-1}\) to \(S_{4r-1}\) defined by

**Mapping algorithm C**

**Input:** \(u = (u_1, \ldots, u_n) \in \mathbb{Z}_2^n \quad (n = 4r - 1)\)

**Output:** \(x = (x_1, \ldots, x_n) \in S_n\)

**begin**

\[(x_1, x_2, \ldots, x_n) \leftarrow (1, 2, \ldots, n);\]

for \(i\) from 1 to \(2r - 1\) do

if \(u_{2i-1} = 1\) then swap\((x_{2i-1}, x_{2i});\)

if \(u_n = 1\) then swap\((x_n, x_{2r});\)
if $u_{2r} = 1$ then swap$(x_1, x_{2r})$;
for $i$ from $1$ to $2r - 1$ do
  if $u_{2i} = 1$ then swap$(x_{2i}, x_{2i+1})$;
end

We again give an intuitive description of what the algorithm does. Let \( u \) be a binary vector given as input. The diagram in Figure 3 shows the permutation initialized with \( x_i = i \) for \( 1 \leq i \leq 4r - 1 \). Note that there are \( 2r \) elements on the left big circle and \( 2r \) elements on the right big circle and that \( x_{2r} = 2r \) is shared between the two. Notice that there is a new variable \( u'_{2r} \) between \( x_1 = 1 \) and \( x_{2r} = 2r \). The variable \( u'_{2r} \) is set to the value of \( u_{2r} \) so that \( u'_{2r} = u_{2r} \). Then swapping take place in the order

\[
u_1 \rightarrow u_3 \rightarrow \cdots \rightarrow u_{2r-1} \rightarrow u_{2r+1} \rightarrow \cdots \rightarrow u_{n-2} \rightarrow u_n \rightarrow u'_{2r} \rightarrow u_2 \rightarrow \cdots \rightarrow u_{2r-2} \rightarrow u_{2r} \rightarrow u_{2r+2} \rightarrow \cdots \rightarrow u_{n-1}.
\]

Then \( u \) is mapped to the resulting permutation under \( z_{4r-1} \).

\[\text{Fig. 3. Swapping order of } z_{4r-1}\]

**Theorem 4:** Let \( r \geq 2 \). The maps \( z_{4r-1} \) are DIMs of length \( 4r - 1 \).

**Proof:** Let \( n = 4r - 1 \). Let \( z_{4r-1}(u) = x \) and \( z_{4r-1}(v) = y \) for two distinct binary vectors \( u \) and \( v \). Let

\[T = \{1 \leq i \leq n \mid x_j = y_j = i \text{ for some } 1 \leq j \leq n\}.
\]

Again our goal is to show

\[\#T \leq \#\{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\}
\]

with equality only when the right side is zero.

The conditions to have \( i \in T \) for \( 1 \leq i \leq n \) are as follows.

\[1 \in T \quad \iff \quad \delta(u_1, 0)\delta(v_1, 0)\delta(u'_{2r}, 0)\delta(v'_{2r}, 0) + \delta(u_1, 0)\delta(v_1, 0)\delta(u'_{2r}, 1)\delta(v'_{2r}, 1)\delta(u_{2r}, v_{2r}) + \delta(u_1, 1)\delta(v_1, 1)\delta(u_2, v_2) = 1, \tag{18}\]

\[2 \in T \quad \iff \quad \delta(u_1, 0)\delta(v_1, 0)\delta(u_2, v_2) + \delta(u_1, 1)\delta(v_1, 1)\delta(u'_{2r}, 0)\delta(v'_{2r}, 0) + \delta(u_1, 1)\delta(v_1, 1)\delta(u'_{2r}, 1)\delta(v'_{2r}, 1)\delta(u_{2r}, v_{2r}) = 1, \tag{19}\]

\[3 \in T \quad \iff \quad \delta(u_3, 0)\delta(v_3, 0)\delta(u_2, v_2) + \delta(u_3, 1)\delta(v_3, 1)\delta(u_4, v_4) = 1, \tag{20}\]

\[4 \in T \quad \iff \quad \delta(u_3, 0)\delta(v_3, 0)\delta(u_4, v_4) + \delta(u_3, 1)\delta(v_3, 1)\delta(u_2, v_2) = 1, \tag{21}\]
\[2r - 2 \in T\]
\[\quad \iff \delta(u_{2r-3}, 0)\delta(v_{2r-3}, 0)\delta(u_{2r-2}, v_{2r-2})\]
\[+ \delta(u_{2r-3}, 1)\delta(v_{2r-3}, 1)\delta(u_{2r-1}, v_{2r-1}) = 1,\] (22)
\[2r - 1 \in T\]
\[\quad \iff \delta(u_{2r-1}, 0)\delta(v_{2r-1}, 0)\delta(u_{2r-2}, v_{2r-2})\]
\[+ \delta(u_{2r-1}, 1)\delta(v_{2r-1}, 1)\delta(u_{2r-0}, 0)\delta(v_{2r-1}, 1)\delta(u_{2r-2}, v_{2r-1})\]
\[+ \delta(u_{2r-1}, 1)\delta(v_{2r-1}, 1)\delta(u_{2r-0}, 0)\delta(v_{2r-1}, 1)\delta(u_{2r-2}, v_{2r-1}) = 1.\] (23)
\[2r \in T\]
\[\quad \iff \delta(u_{2r+1}, 0)\delta(v_{2r+1}, 0)\delta(u_{2r+2}, v_{2r+2})\]
\[+ \delta(u_{2r+1}, 1)\delta(v_{2r+1}, 1)\delta(u_{2r+2}, v_{2r+2}) = 1,\] (24)
\[2r + 1 \in T\]
\[\quad \iff \delta(u_{2r+2}, 0)\delta(v_{2r+2}, 0)\delta(u_{2r+3}, v_{2r+3})\]
\[+ \delta(u_{2r+2}, 1)\delta(v_{2r+2}, 1)\delta(u_{2r+3}, v_{2r+3}) = 1,\] (25)
\[2r + 2 \in T\]
\[\quad \iff \delta(u_{2r+3}, 0)\delta(v_{2r+3}, 0)\delta(u_{2r+4}, v_{2r+4})\]
\[+ \delta(u_{2r+3}, 1)\delta(v_{2r+3}, 1)\delta(u_{2r+4}, v_{2r+4}) = 1,\] (26)
\[2r + 3 \in T\]
\[\quad \iff \delta(u_{2r+4}, 0)\delta(v_{2r+4}, 0)\delta(u_{2r+5}, v_{2r+5})\]
\[+ \delta(u_{2r+4}, 1)\delta(v_{2r+4}, 1)\delta(u_{2r+5}, v_{2r+5}) = 1,\] (27)
\[\vdots\]
\[n - 2 \in T\]
\[\quad \iff \delta(u_{n-2}, 0)\delta(v_{n-2}, 0)\delta(u_{n-3}, v_{n-3})\]
\[+ \delta(u_{n-2}, 1)\delta(v_{n-2}, 1)\delta(u_{n-1}, v_{n-1}) = 1,\] (28)
\[n - 1 \in T\]
\[\quad \iff \delta(u_{n-2}, 0)\delta(v_{n-2}, 0)\delta(u_{n-1}, v_{n-1})\]
\[+ \delta(u_{n-2}, 1)\delta(v_{n-2}, 1)\delta(u_{n-3}, v_{n-3}) = 1,\] (29)
\[n \in T\]
\[\quad \iff \delta(u_{n}, 0)\delta(v_{n}, 0)\delta(u_{n-1}, v_{n-1})\]
\[+ \delta(u_{n}, 1)\delta(v_{n}, 1)\delta(u_{n-2}, v_{n-1})\delta(v_{2r}, 1) = 1.\] (30)

Note that (20), (21), \ldots, (22) and (26), (27), \ldots, (28) do not appear if \(r = 2\). We simplify by computing the sums in groups:

\[(18) + (19) = \delta(u_1, v_1)\delta(v_2, v_2) + \delta(u_1, v_1)\delta(v_2, v_2),\]
\[(20) + \cdots + (22) = \delta(u_2, v_2)\delta(u_3, v_3) + \delta(u_3, v_3)\delta(u_4, v_4) + \cdots + \delta(u_{2r-3}, v_{2r-3})\delta(u_{2r-2}, v_{2r-2}),\]
\[(23) + (24) = \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r-2}, v_{2r-2}) + \delta(u_{2r-1}, v_{2r-1})\delta(u_{2r}, v_{2r})\delta(u_n, 0)\delta(v_n, 0)\]
cases are trivial. Suppose this case. Then it suffices to note that
where the last inequality follows from

We treat three separate cases. For the case $u_{n-1} = T$

Thus we obtain

We treat three separate cases. For the case $u_{n-1} = 0$ and $v_{n-1} = 0$:

where the last inequality follows from

which can be verified as follows. We only consider the case that $\delta(u_{2r-1}, v_{2r-1}) = \delta(u_{n-1}, v_{n-1}) = 1$ as the other cases are trivial. Suppose this case. Then it suffices to note that $\delta(u_{2r}, v_{2r}) = 0$ implies that

$[\delta(u_i, v_i) = 1 and \delta(u_j, v_j) = 0 for some 1 \leq i, j \leq 2r]$

$+ [\delta(u_i, v_i) = 1 for some 2r \leq i \leq n - 1] + \delta(u_{2r}, v_{2r})$

$\geq \delta(u_{2r-1}, v_{2r-1})\delta(u_{n-1}, v_{n-1}) + \delta(u_{n-1}, v_{n-1}),$

$[\delta(u_i, v_i) = 1 and \delta(u_j, v_j) = 0 for some 1 \leq i, j \leq 2r] = 1.$
For the case $u_n = 1$ and $v_n = 1$:

$$
\# T = \delta(u_1, v_1)\delta(u_2, v_2) + \cdots \\
+ \delta(u_{2r-2}, v_{2r-2})\delta(u_{2r-1}, v_{2r-1}) \\
+ \delta(u_{2r}, v_{2r})\delta(u_{2r+1}, v_{2r+1}) + \cdots \\
+ \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1}) \\
+ \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) + \delta(u_{2r}, v_{2r}) \\
\leq \# \{1 \leq i \leq 2r - 1 \mid \delta(u_i, v_i) = 1\} \\
- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq 2r - 1] \\
+ \# \{2r \leq i \leq n - 1 \mid \delta(u_i, v_i) = 1\} \\
- [\delta(u_i, v_i) = 1 \text{ for some } 2r \leq i \leq n - 1] \\
+ \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) + \delta(u_{2r}, v_{2r}) \\
= \# \{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - 1 \\
- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq 2r - 1] \\
- [\delta(u_i, v_i) = 1 \text{ for some } 2r \leq i \leq n - 1] \\
+ \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) + \delta(u_{2r}, v_{2r}) \\
\leq \# \{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} - 1,
$$

where the last inequality follows from

$$
[\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq 2r - 1] \\
+ [\delta(u_i, v_i) = 1 \text{ for some } 2r \leq i \leq n - 1] \\
\geq \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) + \delta(u_{2r}, v_{2r}),
$$

which is easy to verify.

For the case $u_n \neq v_n$:

$$
\# T = \delta(u_1, v_1)\delta(u_2, v_2) + \cdots \\
+ \delta(u_{2r-2}, v_{2r-2})\delta(u_{2r-1}, v_{2r-1}) \\
+ \delta(u_{2r}, v_{2r})\delta(u_{2r+1}, v_{2r+1}) + \cdots \\
+ \delta(u_{n-2}, v_{n-2})\delta(u_{n-1}, v_{n-1}) \\
+ \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) \\
\leq \# \{1 \leq i \leq 2r - 1 \mid \delta(u_i, v_i) = 1\} \\
- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq 2r - 1] \\
+ \# \{2r \leq i \leq n - 1 \mid \delta(u_i, v_i) = 1\} \\
- [\delta(u_i, v_i) = 1 \text{ for some } 2r \leq i \leq n - 1] \\
+ \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) \\
= \# \{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} \\
- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq 2r - 1] \\
- [\delta(u_i, v_i) = 1 \text{ for some } 2r \leq i \leq n - 1] \\
+ \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) \\
\leq \# \{1 \leq i \leq n \mid \delta(u_i, v_i) = 1\} \\
- [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq n],
$$

where the last inequality follows since

$$
[\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq 2r - 1] \\
+ [\delta(u_i, v_i) = 1 \text{ for some } 2r \leq i \leq n - 1] \\
\geq \delta(u_1, v_1)\delta(u_{2r}, v_{2r}) + [\delta(u_i, v_i) = 1 \text{ for some } 1 \leq i \leq n],$$
which is easy to verify.

Combining the three cases, we get the required result.

IV. EXAMPLES

For a given DIM $f$ of length $n$, the square matrix $[D_{ij}]$ of size $n$ whose component $D_{ij}$ is the number of unordered pairs of binary vectors $u, v \in \mathbb{Z}_2^n$ such that $d(u, v) = i$ and $d(f(u), f(v)) = j$ is useful to see how much the map $f$ is distance-increasing. Let us call the matrix the distance expansion table of the DIM.

Here we exhibit distance expansion tables of DIMs $z_n$ of length $n = 5, 6, 7, 8, 9, 11$ defined by our mapping algorithms given in previous sections. The DIMs $z_5$ and $z_9$ whose distance expansion tables are in Figures 4 and 8 are defined by the mapping algorithm $B$. Note that $z_5$ is the shortest DIM defined by the algorithm $B$. Likewise, DIMs $z_7$ and $z_{11}$ whose distance expansion tables are in Figures 5 and 9 are defined by the mapping algorithm $C$. The DIM $z_7$ is the shortest DIM defined by the algorithm $C$.

It is easy to see, as claimed in [3], that $z_{2r}$ are actually equivalent to $h_{2m}$ from [1] when $r = m$ is odd or $r = m = 2$. So $z_6$ and $h_6$ have the same distance expansion table shown in Figure 5. The distance expansion table of $z_8$ in Figure 7 first appeared in [3].

After these concrete examples, we conclude that we now have simple mapping algorithms defining DIMs of all length.

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| 0 1024 0 0 0 0 0 0 |
|-------------------|
| 0 0 1024 2560 0 0 0 0 |
| 0 0 0 1024 4096 2048 0 0 |
| 0 0 0 0 1024 4608 3072 256 |
| 0 0 0 0 0 1024 4096 2048 |
| 0 0 0 0 0 0 1024 |
| 0 0 0 0 0 0 0 128 |

Fig. 7. Distance expansion table of $z_8$

| 0 2048 256 0 0 0 0 0 0 |
|------------------------|
| 0 0 1792 6400 1024 0 0 0 0 |
| 0 0 0 1536 10240 8704 1024 0 0 |
| 0 0 0 0 1280 12800 13824 4352 0 |
| 0 0 0 0 0 1536 12544 14848 3328 |
| 0 0 0 0 0 0 1792 11008 8704 |
| 0 0 0 0 0 0 0 2048 7168 |
| 0 0 0 0 0 0 0 0 2304 |
| 0 0 0 0 0 0 0 0 256 |

Fig. 8. Distance expansion table of $z_9$

| 0 10240 1024 0 0 0 0 0 0 |
|--------------------------|
| 0 0 9728 39936 6656 0 0 0 0 |
| 0 0 0 9216 67584 78848 13312 0 0 |
| 0 0 0 0 8704 86016 158208 76800 8192 |
| 0 0 0 0 0 8192 97280 206848 137216 |
| 0 0 0 0 0 0 8704 97280 214528 |
| 0 0 0 0 0 0 9216 90112 168960 |
| 0 0 0 0 0 0 0 9728 73728 |
| 0 0 0 0 0 0 0 0 10240 |
| 0 0 0 0 0 0 0 0 1024 |

Fig. 9. Distance expansion table of $z_{11}$