Deformed versus undeformed cat states encoding qubit

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We study the possibility of exploiting superpositions of coherent states to encode qubit. A comparison between the use of deformed and undeformed bosonic algebra is made in connection with the amplitude damping errors.

I. INTRODUCTION

Controlling quantum coherence is one of the most fundamental issues in modern information processing\textsuperscript{[1]}. The most popular solution in the field of quantum information are quantum error correction codes\textsuperscript{[2]} and error avoiding codes\textsuperscript{[3]}, both based on encoding the state into carefully selected subspaces of a larger Hilbert space involving ancillary systems. The main limitation of these strategies for combating decoherence is the large amount of extra space resources required\textsuperscript{[4]}; in particular, if fault tolerant error correction is also considered, the number of ancillary qubits enormously increases. For this reason, other alternative approaches which do not require any ancillary resources have been pursued\textsuperscript{[5]}.

In quantum information theory logical states are encoded as two orthogonal pure states\textsuperscript{[1]}. The simplest example is provided by a single two-level system. However, there is no fundamental reasons to restrict oneself to physical system with two dimensional Hilbert space for the encoding. It may be more convenient to encode logical states as a superposition over a large number of basis states.

On the other hand, while the coupling with the environment is fixed, we are free to choose how we encode the qubits, hence the choice of the basis for the logical encoding may change the error introduced.

In this paper we study the qubit encoding in the superposition of coherent states of a bosonic mode. This latter will be considered from the generic approach of deformed algebra\textsuperscript{[6]}, showing that deformation can be profitably used to reduce amplitude damping errors.

II. CAT STATES ENCODING QUBIT

Let us introduce a $f$-coherent state defined\textsuperscript{[6]} as the eigenstate of the annihilation operator of a $f$-deformed bosonic field $A = a \sqrt{f(a^\dagger a)}$, where $f$ is an operator-valued function of the number operator (here it is assumed Hermitian and real) and $a$ being the annihilation operator of the undeformed field. In general, $f$ can be made dependent on continuous parameters, in such a way that, for given particular values, the usual algebra is recovered. The $f$-coherent state can be written as

$$|\zeta, f\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{|n|!}} |n\rangle, \quad \mathcal{N} = [\exp_f(\zeta^2)]^{-1/2},$$

where we have considered the amplitude $\zeta \in \mathbb{R}$, and we have introduced

$$\exp_f(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{|n|!},$$

$$|n|! \equiv [nf(n)] \times [(n-1)f(n-1)] \times \ldots \times [f(2)] \times [f(1)] \times [f(0)].$$

The function $\exp_f$ is a deformed version of the usual exponential function. They become coincident when $f$ is the identity. Notice that $\exp_f(x) \exp_f(y) \neq \exp_f(x+y)$, i.e. we have a non-extensive exponential which can be found in many physical problems\textsuperscript{[8]}.

Let us now consider the superpositions\textsuperscript{[6]}

$$|\Phi_+\rangle = \mathcal{N}_+ (|\zeta, f\rangle + | - \zeta, f\rangle), \quad \mathcal{N}_+ = [2 + 2N^2 \exp_f(-\zeta^2)]^{-1/2},$$

$$|\Phi_-\rangle = \mathcal{N}_- (|\zeta, f\rangle - | - \zeta, f\rangle), \quad \mathcal{N}_- = [2 - 2N^2 \exp_f(-\zeta^2)]^{-1/2}.$$
These states represent a generalization of the well known even and odd cat states \[^{10}\]
and reduce to them whenever \(f \to 1\).

Since \(|\Phi_+\rangle\) and \(|\Phi_-\rangle\) are orthogonal, we are led to the following logical encoding for a single qubit \[^{11}\]
\[
|\overline{0}\rangle \equiv |\Phi_+\rangle = N_+ ((\zeta, f) + | - \zeta, f \rangle),
\]
\[
|\overline{1}\rangle \equiv |\Phi_-\rangle = N_- ((\zeta, f) - | - \zeta, f \rangle).
\]

In case of no deformation this reduces to the encoding procedure proposed in Ref. \[^{12}\]. In such a case \(N_+\) and \(N_-\) tend to become equal as soon as \(|\zeta| > 1\). Instead, in the general case, their difference drastically depends on the field deformation. This can be evaluated by introducing the parameter
\[
\Delta = \frac{|N_+ - N_-|}{\min[N_+, N_-]},
\]
which represents the relative error done by assuming \(N_+ = N_-\). \(\Delta\) plays an important role in the qubit operations as we shall see.

A further parameter which characterizes the cat states (hence our encoded qubit) in equation (6) is the separation between the two superposed states \[^{13}\]. This distance can be written as
\[
d \equiv \langle \zeta, f | (a + a^\dagger) | \zeta, f \rangle
\]
\[
= N^2 \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{|n|} f!} \left\{ \frac{\sqrt{n} \zeta^{(n-1)}}{\sqrt{|n-1|} f!} + \frac{\sqrt{n+1} \zeta^{(n+1)}}{\sqrt{|n+1|} f!} \right\}.
\]

For the case \(f \to 1\), we know \[^{14}\] that the amplitude damping take place on a time scale inversely proportional to \(d\). Thus, for a given \(\zeta\), the possibility to change \(d\) through a suitable algebraic deformation (see, e.g., Ref. \[^{15}\]) would be very important to prevent errors on the encoded qubit.

Among the infinite possible choices of \(f\) we are going to consider
\[
f(n) = \frac{L^1_n(\xi^2)}{(n + 1)L^0_n(\xi^2)}, \quad \xi \in \mathbb{R},
\]
which we name \(L\)-deformation, since \(L^m_n\) indicates the associate Laguerre polynomial. Such type of deformation arises in ion trapped systems (e.g., when an ion is bichromatically driven far from the Lamb-Dicke regime) \[^{17}\], then it could be accessible just in systems actually used for experimental quantum information (see e.g., \[^{16}\]).

In figure 1a we show the behaviour of \(\Delta\) as function of parameter \(\xi\), while in figure 1b we have plotted the distance \(d\) as function of parameter \(\xi\). From these figures results the possibility to have a distance \(d\) smaller than the undeformed case still maintaining \(\Delta \approx 0\) when \(|\zeta| > 1\).
III. AMPLITUDE DAMPING ERRORS

The amplitude damping errors on the qubit can be ascribed to a dissipative interaction with an environment. This can be described (in interaction picture) by the following master equation of the Lindblad form

$$\dot{\rho} = \gamma a\rho a^\dagger - \frac{\gamma}{2} \{a^\dagger a, \rho\} ,$$

where $\gamma$ is the damping rate, and we have set the bath temperature equal to zero. The decoherence effect on the state $\rho(0) = |\Psi^\pm\rangle\langle\Psi^\pm|$ can be described in the following way

$$\rho(t) = \sum_{k=0}^{\infty} \Upsilon_k(t)\rho(0)\Upsilon_k^\dagger(t) ,$$

where

$$\Upsilon_k(t) = \sum_{n=k}^{\infty} \sqrt{\binom{n}{k}} [\eta(t)]^{(n-k)/2} [1 - \eta(t)]^{k/2} |n-k\rangle\langle n| ,$$

with $\eta(t) = e^{-\gamma t}$.

In Ref. the survival of the quantum coherence in deformed cat states has been shown. Here, the robustness of deformed cat states (qubit) against dissipative decoherence can be seen by considering the fidelity

$$\mathcal{F}(t) = \text{Tr} \{\rho(t)\rho(0)\} ,$$

which tell us to what extent the evolved state remains faithful to the initial one. Starting from $\rho(0) = |\Phi^\pm\rangle\langle\Phi^\pm|$ we get...
\[
\mathcal{F}_{\pm}(t) = N^4N^4 \sum_{k=0}^{\infty} \sum_{n,m=k}^{\infty} \sqrt{\binom{n}{k}} \binom{m}{k} [\eta(t)]^{(n+m)/2-k} \left[ 1 - \eta(t) \right]^k 
\]
\[
\times \frac{(\zeta)^n \pm (-\zeta)^n}{\sqrt{|n|!}} \times \frac{(\zeta)^m \pm (-\zeta)^m}{\sqrt{|m|!}} \times \frac{(\zeta)^{n-k} \pm (-\zeta)^{n-k}}{\sqrt{|n-k|!}} \times \frac{(\zeta)^{m-k} \pm (-\zeta)^{m-k}}{\sqrt{|m-k|!}}.
\]

(16)

In Fig. 2 we compare (in time) the fidelity of an undeformed cat state (dashed line) with that of a deformed one (solid line). We see the possibility to improve the fidelity by introducing an algebraic deformation on the bosonic mode. This is essentially due to the reduced effective distance between the two superposed states and to the fact that the latter (once deformed) are no longer eigenstates (nor near eigenstates) of the irreversible operator appearing in Eq. (12) [18]. It is also worth noting the asymmetry between \( F_+ \) and \( F_- \) which shows that the state encoded on the even cat is more robust than that encoded on the odd. The used values of parameter \( \xi \) guarantees that also \( \Delta \approx 0 \).

Better results could be obtained by exploring other regions of the parameter \( \xi \), however, this requires noticeable computer resources.

![Figure 2](image)

**FIG. 2.** Fidelity \( F_+ \) (a) and \( F_- \) (b) as function of dimensionless time \( \gamma t \), for \( \zeta^2 = 3 \). The dashed line represents the non-deformed case while the solid lines refer to \( L \)-deformed case. The used value of \( \xi \) corresponds to that indicated by the arrows in figure 8.

### IV. LOGICAL OPERATIONS

A logical encoding is useless if we cannot implement one and two qubit operations on the encoded states. We now show Hamiltonians suitable to perform the fundamental logical operations by generalizing the arguments of Ref. [12].

For what concern the single qubit rotation we can construct the Hamiltonian generating such transformation by simply using a driving term, that is

\[
H_R = \beta A^\dagger + \beta^* A = \beta \sqrt{f(a^\dagger a)}a^\dagger + \beta^* a \sqrt{f(a^\dagger a)},
\]

(17)
with $\beta$ the complex driven amplitude. The time evolution under such Hamiltonian, for small value of $|\beta|t$, can be described by using the split operator method

$$U \approx \exp \left[-i\beta t A^\dagger \right] \exp \left[-i\beta^* t A \right].$$

Then, for a sufficiently large value of $\zeta$, for which it is also $\Delta \approx 0$, we have

$$U|0\rangle \approx \cos \theta |0\rangle - i \sin \theta |1\rangle ,$$

$$U|1\rangle \approx \sin \theta |0\rangle - i \cos \theta |1\rangle ,$$

where $\theta = 2\zeta \beta t$ and we assumed $\beta \in \mathbb{R}$. Equation (19) is equivalent to the Hadamard transform provided to have $\theta = \pi/4$. Notice that the operation (19), hence Hadamard transform, is possible just when $N_+ \approx N_-$. For what concern the two-qubit gate, the simplest way to realize a universal gate is to employ the following Hamiltonian

$$H_{CP} = \chi g^{-1}(A^\dagger A)g^{-1}(B^\dagger B) = \chi a^\dagger ab^\dagger ,$$

where $b, b^\dagger (B, B^\dagger)$ are the undeformed (deformed) ladder operators of the second mode. Furthermore $g(x) = xf(x)$. If the interaction time is such that $\chi t = \pi$, we have

$$e^{-iH_{CP}t} |0\rangle |0\rangle = |0\rangle |0\rangle ,$$

$$e^{-iH_{CP}t} |0\rangle |1\rangle = |0\rangle |1\rangle ,$$

$$e^{-iH_{CP}t} |1\rangle |0\rangle = -|1\rangle |0\rangle ,$$

The above two-qubit gate represents a conditional phase shift, and can be easily understood by reminding that $|0\rangle$ contains only even bosonic and $|1\rangle$ only odd bosonic number.

Here, we do not deal the question of how to implement the above interactions, but we simply postulate their existence. On the other hand, an arbitrary unitary transformation can be built up efficiently, to any desired precision, by using elementary interactions [22], realizable in several systems [3]. In particular, there already exist proposals to engineer any Hamiltonian for a trapped ion [23].

V. CONCLUSION

In conclusion, we have shown that quantum information can be suitably encoded in cat states. Moreover, the deformed version of such states offers the possibility of reducing amplitude damping errors. The introduction of algebraic field deformation to reduce the decoherence effects of qubit is reminiscent of “passive” strategy like “error avoiding codes” [3]. However, in this case the whole Hilbert space is exploited without waste of any degrees of freedom.

Beside trapped systems showing $L$-deformed states [17], one can look for other type of realistic deformations leading analogous effects. Potential candidates could be Bose-Einstein condensates where the requirement of particle number conserving leads to a modification of the field algebra [24].

Finally, the studied problem constitutes a building block of a more general and intriguing problem concerning the group theoretical approach to field deformation [25] and decoherence [26] (e.g., find a suitable deformation such that $F_{\pm}$ remains approximately one) which we plan to deal in a future.

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1 This form of the split operator method is only accurate to first order in $|\beta|t$, since it ignores higher order terms involving the commutator of $A$ and $A^\dagger$. However, this method is superior to expanding $U$ to first order in $|\beta|t$ because it evolves the state unitarily. Higher-order split operator methods also exist [21].
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