The Kowalewski top: a new Lax representation.

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Abstract

The $2 \times 2$ monodromy matrices for the Kowalewski top on the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$ are presented. The corresponding quadratic $R$-matrix structure is the dynamical deformation of the standard $R$-matrix algebras. Some tops and Toda lattices related to the Kowalewski top are discussed.

1 Introduction

The main object for investigation is the Kowalewski top (KT) in the classical and quantum mechanics. In classical mechanics the system under consideration is a special case of motion of a heavy rigid body with fixed point, discovered by Kowalewski in 1889 [1]. It represents a symmetric top in a constant homogeneous field. The principal momenta of inertia relate as $I_1 : I_2 : I_3 = 1 : 1 : 1/2$ and the center of mass located in the equatorial plane. In the body frame, the components of the angular momentum $l_i$ and the Poisson vector $g_i$, $i = 1, 2, 3$ be generators of the Lie algebra $e(3)$ with the Poisson brackets:

$$\{l_i, l_j\} = \varepsilon_{ijk} l_k, \quad \{l_i, g_j\} = \varepsilon_{ijk} g_k,$$

$$\{g_i, g_j\} = 0, \quad i, j, k = 1, 2, 3.$$  \hspace{1cm} (1)

and with the fixed Casimir operators

$$J_2 = (g, g) = a; \quad J_3 = (l, g) = b.$$  \hspace{1cm}

Integrals of motion for the KT are given by

$$J_1 = H = \frac{1}{2} (l_1^2 + l_2^2 + 2 l_3^2) - g_1.$$  \hspace{1cm} (2)

$$J_4 = k_+ k_- = (l_+^2 - 2 g_+)(l_-^2 - 2 g_-),$$

In quantum mechanics the KT has been introduced by Laporte [2] (quasiclassical approach see in [3]).

The KT are generalized to the Kowalewsky gyrostat with the following hamiltonian

$$H = \frac{1}{2} (l_1^2 + l_2^2 + 2 l_3^2 + 2 \gamma l_3) - g_1.$$  \hspace{1cm} (3)
Gyrostat momentum proportional $\gamma$ is perpendicular to the equatorial plane. This system and their counterparts on the Lie algebras $so(4)$ and $so(3, 1)$ have been considered in [4].

The main our aim is in constructing the Lax representations and their quantum counterparts for all these systems.

As a tool for investigations we will apply linear and quadratic $R$-matrix algebras in the quantum and classical inverse scattering method [3]. Let us consider an algebra generated by noncommutative entries of the matrix $T(u)$ satisfying the famous bilinear relation (ternary relation)

$$ R(u - v)T(u) \frac{1}{2} T(v) = \frac{1}{2} T(v)T(u) R(u - v), $$

(4)

or the quaternary relation

$$ R(u - v)T(u) S(u + v) \frac{2}{2} T(v) = \frac{2}{2} T(v)S(u + v) \frac{1}{2} T(u) R(u - v). $$

(5)

where we use the standard notations $\frac{1}{2} T(u) = T(u) \otimes I$, $\frac{2}{2} T(v) = I \otimes T(v)$ and matrices $R(u)$ and $S(u)$ are solutions of the Yang-Baxter equation. For historical reasons these algebras are called the algebras of monodromy matrices. Equations (4) and (5) are called the fundamental commutator relation (FCR) and the reflection equations (RE) [6], respectively. If we consider a simple finite-dimensional Lie algebra $a$ and a $a$-invariant $R$-matrix then the algebra of monodromy matrices [4] after a proper specialization gives the yangian $Y(a)$ introduced by Drinfeld, while algebra of monodromy matrices [4] corresponds to the twisted yangians [4].

We will consider the finite-dimensional irreducible representations of algebras (4-5), which are polynomials on spectral parameter $u$, only. The entries of monodromy matrix $T(u)$ are constructed from the generators of yangian $t_{ij}(u)$ by the rule

$$ T(u) = \sum_{i,j}^{N} t_{ij}(u) \otimes E_{ij} \in Y(a) \otimes \text{End}(\mathbb{C}^N), \quad t_{ij}(u) = \sum_{\alpha} t_{ij}^{\alpha} u^{\alpha}, $$

(6)

where $E_{ij}$ are the standard matrix units. The matrix trace $t(u)$ of the matrix $T(u)$

$$ t(u) = \text{tr} T(u) = \sum_{k=1} T_{kk}(u) $$

(7)

yields a commutative family of operators $J_k$

$$ [t(u), t(v)] = 0, \quad t(u) = \sum_{k} J_k u^k, \quad u, v \in \mathbb{C} $$

(8)

which are integrals of motion of some quantum integrable system.

In the classical limit algebras of monodromy matrices (4) and (5) transform into the quadratic Sklyanin algebras

$$ \{ \frac{1}{2} T(u), \frac{2}{2} T(v) \} = [r(u - v), \frac{1}{2} T(u) \frac{2}{2} T(v)], $$

(9)

$$ \{ \frac{1}{2} T(u), \frac{2}{2} T(v) \} = \left[ r(u - v), \frac{1}{2} T(u) \frac{2}{2} T(v) \right] + $$

(10)

$$ = \frac{1}{2} T(u) s(u + v) \frac{2}{2} T(v) - \frac{2}{2} T(v) s(u + v) \frac{1}{2} T(u). $$

Here matrices $r(u)$ and $s(u)$ are the classical $r$-matrices, $R(u) = 1 + \eta r(u) + O(\eta^2)$ by $\eta \to 0$ and similar for matrix $S(u)$.
If one substitute $T := 1 + \varepsilon L + O(\varepsilon^2)$, $r := \varepsilon r$ and let $\varepsilon \to 0$, then we get the linear $R$-matrix algebra

$$\{L(\lambda), \hat{L}(\mu)\} = [r_{12}(\lambda, \mu) L(\lambda) + [r_{21}(\lambda, \mu), \hat{L}(\mu)],$$

(11)

(see review [8]). We will start with the $4 \times 4$ Lax representation for the KT given by Reyman and Semenov-Tian-Shansky [8], which obeys (11).

Following to a general scheme [5, 8] the Lax pairs

$$\frac{dL(\lambda)}{dt} + [L(\lambda), M(\lambda)] = 0,$$

(12)

for the matrices $L(\lambda)$ and $T(u)$ are constructed by the linear and quadratic algebras (11). Below we fix notations $L(\lambda)$ and $T(u)$ for monodromy matrices, which satisfy to linear (11) and to quadratic [8] algebras of monodromy matrices, respectively.

It is well known, that in classical mechanics some Lax matrices have been proposed for the KT. Complete their list and discussion see in [8]. All these matrices with the spectral parameter are $N \times N$ matrices by $N > 2$. If we want to use the method of separation of variables or to consider the KT in quantum mechanics we can, of course, try to adjust these matrices, for example, applying the experience by Sklyanin [9].

However, the separated equations for KT in the quasiclassical approach look like equations inherent in inverse scattering method with quadratic $R$-matrix relations [3]. Since for the quantization of the KT we prefer to construct a new monodromy matrix for the KT in the $2 \times 2$ auxiliary space. Building such matrix we will use a known matrix in larger auxiliary space, which satisfies a linear $R$-matrix algebra [11].

So, we want to obtain $2 \times 2$ matrix with entries defined on universal enveloping algebra from the $N \times N$ matrix with entries belong to the loop algebra. For this purpose we will use often the geometrical [10] and algebraic [11] connections of the tops on $e(3)$ and the Toda lattices. As a settlement, we oblige to introduce the additive deformations of the basic algebraic relations (4-9) and we obtain a Lax triad for the KT

$$\frac{dL(\lambda)}{dt} + [L(\lambda), M(\lambda)] = N(\lambda),$$

(13)

where matrix $N(\lambda)$ is a traceless matrix. Trace of the matrix $L(\lambda)$ is a generating function of integrals of motion. This situation is analogous to introduction of the dynamical $r$-matrices on loop algebras [12, 13], where for description of the concrete integrable systems in given method we were forced to expand framework of the $R$-matrix formalism to the quality new type of $R$-matrix.

2 Axially symmetric Neumann’s system

Let us recall some results about the Toda lattices in the classical mechanics (see [14, 15]). For the periodic Toda lattices associated with the root system of $A_N$ type the corresponding Lax matrix is given by

$$L(\lambda) = \begin{pmatrix}
p_N & e_{N-1} & 0 & \ldots & 0 & e_{1N}\[1pt]e_{N-1} & p_{N-1} & e_{N-2} & \ldots & 0 & 0\[1pt]\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\[1pt]e_{1N} & 0 & \ldots & 0 & e_1 & e_2\[1pt]
\end{pmatrix},$$

(1)
where \( e_{jn} = e^{(q_j-q_n)/2} \) and \((p_j,q_j)\) are pairs of canonically conjugate variables. Determinant curve of the Lax matrix \(L(\lambda)\) defines by the matrix \(L(\lambda,u) = uI - L(\lambda)\). It is a three diagonal matrix and we can introduce the monodromy matrix in two dimensional auxiliary space \([\mathbb{R}^2]\). It reads

\[
T(u) = T_{N}(u)T_{N-1}(u)\ldots T_{1}(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u),
\]

\[
T_k(u) = \begin{pmatrix} u - p_k & -e^{q_k} \\ e^{-q_k} & 0 \end{pmatrix}.
\]

Entries of \(T(u)\) are the following functions of the minors of \(L(\lambda)\)

\[
A(u) = \det L(\lambda,u)_{\lambda=\infty}, \\
B(u) = -e^{n} \det L^{(N,N)}(\lambda,u), \\
C(u) = e^{g_{N}} \det L^{(1,1)}(\lambda,u), \\
D(u) = \lambda^{2} \det L(\lambda,u)_{\lambda=0},
\]

where \(L^{(j,k)}(\lambda,u)\) means the matrix obtained by removing the \(j\) column and \(k\) row of the matrix \(L(\lambda,u)\). The quantum operator \(T(u)\) is constructed from the classical matrix \([\mathbb{R}^2]\). It obeys the FCR \([\mathbb{R}^2]\) with the rational \(R\)-matrix \(X_{\infty}\), \(XX\) type. For the Toda lattices associated with the Lie algebras of the series \(B_n, C_n\) and \(D_n\), a similar correspondence has been introduced in \([\mathbb{R}^2]\). Such relations can be helpful for the non-three diagonal matrices. For instance, the monodromy matrices for the Toda lattices associated with the root systems of \(D_N\) type have been constructed into this manner.

Let us start with the following Lax matrix for the KT \([\mathbb{R}^2]\)

\[
L(\lambda) = \begin{pmatrix}
\gamma & g_{+}/\lambda & l_{-} & -g_{3}/\lambda \\
-g_{+}/\lambda & -\gamma & g_{3}/\lambda & -l_{+} \\
l_{+} & -g_{3}/\lambda & -2l_{3} + \gamma & -2\lambda - g_{+}/\lambda \\
g_{3}/\lambda & -l_{-} & 2\lambda + g_{+}/\lambda & 2l_{3} - \gamma
\end{pmatrix}, \tag{4}
\]

using natural notations \(l_{\pm} = l_{1} \pm il_{2}, g_{\pm} = g_{1} \pm ig_{2}\).

The Lax representation \([\mathbb{R}^2]\) has been applied by solution of equation of motion in \([\mathbb{R}^2]\). The \(3 \times 3\) matrix \(L^{(1,1)}(\lambda)\) constructed by \(L(\lambda)\) \([\mathbb{R}^2]\), in our notation, describes the Goryachev-Chaplygin top \([\mathbb{R}^2]\). which has also a monodromy matrix on \(2 \times 2\) auxiliary space satisfying the Sklyanin brackets \([\mathbb{R}^2] [\mathbb{R}^2]\).

Motivated by representation \([\mathbb{R}^2]\) we introduce the monodromy matrix \(T_{0}(u)\)

\[
A(u) = \det L^{(1,1),(3,3)}(\lambda,u)_{\lambda=0}, \\
B(u) = i\lambda \det L^{(1,1),(3,4)}(\lambda,u)_{\lambda=0}, \\
C(u) = -i\lambda \det L^{(1,1),(4,3)}(\lambda,u)_{\lambda=0}, \\
D(u) = -\lambda^{2} \det L^{(1,1),(4,4)}(\lambda,u)_{\lambda=0},
\]

\[
T_{0}(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u) = \begin{pmatrix} u^{2} - 2ul_{3} - l_{1}l_{2} & i(u_{+} - g_{3}l_{+}) \\ -il_{1} - g_{3}l_{-} & g_{3}^{2} \end{pmatrix}.
\]

Introduction of the gyrostat parameter \(\gamma\) is equivalent to the shift of the spectral parameter \(u \to (u - \gamma)\) and we put \(\gamma = 0\) for a while.
This matrix corresponds to the axially symmetric Neumann’s system

\[ J_1 = H = l_1^2 + l_2^2 - g_3^2, \quad J_4 = m = l_3, \quad (6) \]

(or to the particular case of the general Lagrange top \([18]\)).

The monodromy matrix \(T_0(u)\) \([3]\) has been introduced before in \([1]\) by using of an isomorphism of universal enveloping algebras, which exist in the one-parameter subset of orbits \(O (J_2 = (g,g) = a^2 \text{ and } J_3 = (l,g) = 0)\) only. At the level \(J_3 = (l,g) = 0\) matrix \(T_0(u)\) \([3]\) obeys the Sklyanin brackets \([3]\) with the rational \(R\)-matrix of the \(XX\) type \([5]\)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = 2i \frac{u}{u-v} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i = \frac{\eta}{u-v} P, \quad (7)
\]

where \(P\) is a permutation operator of auxiliary spaces and \(\eta = 2i\).

At \(J_3 = (l,g) = 0\) the spectral invariants of the Lax matrix \(T_0(u)\) are the generating functions of the integrals of motion \((6)\) and of the Casimir operators \((1)\), respectively:

\[
t_0(u) \equiv \text{tr} T_0(u) = u^2 - 2uJ_4 - J_1, \quad (8)
\]

\[
\Delta_0(u) \equiv \det T_0(u) = u^2 J_2. \quad (9)
\]

So, matrix \(T_0(u)\) \([3]\) describes a completely integrable system in the one-parameter subset of orbits \(O (J_2 = (g,g) = a^2 \text{ and } J_3 = (l,g) = 0)\) in \(e(3)^*\).

However, in contrast with \([11]\) the matrix \(T_0(u)\) \([5]\) was obtained by the matrix \(L(\lambda)\) \([4]\) defined on a whole phase space and the Lagrange top is a complete integrable system on the general orbits \(O (J_2 = (g,g) = a^2 \text{ and } J_3 = (l,g) = b)\) in \(e(3)^*\) \([18]\). Therefore, at the next section, we go to investigate the matrix \(T_0(u)\).

3 Deformation of the Sklyanin brackets

According to \([12, 13]\) we can introduce an additive deformations of the algebras of monodromy matrices. One simple deformation of the matrix \(T_0(u)\) has been considered in \([11, 12]\)

\[
T_1(u) = T_0(u) + \begin{pmatrix}
\mu & 0 \\
\frac{g_3}{\theta} & 0 \\
0 & 0
\end{pmatrix} = T_0(u) + \mu \begin{pmatrix}
D^{-1} & 0 \\
0 & 0
\end{pmatrix}, \quad \mu \in \mathbb{R}. \quad (1)
\]

The corresponding new hamiltonian reads \(H^{new} = H^{old} + \mu g_3^2\), where \(H^{old}\) is a hamiltonian \([6]\). Matrix \(T_0(u)\) \([3]\) and modified matrix \(T_1(u)\) \([1]\) obey the same quadratic \(R\)-matrix algebra \([3]\). The main advantage of this deformation is an alteration of the hamiltonian without an alteration of the \(R\)-matrix algebra.

Let us introduce two matrices

\[
F(u) = 2u(l,g)g_3^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2uJ_3 \begin{pmatrix} D^{-1/2} & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
T(u) = T_0(u) + F(u). \quad (2)
\]

Matrix \(T(u)\) is an additive deformation of the matrix \(T_0(u)\) and they are coupled by certain deformation of the Sklyanin brackets \([6]\). In the Sec. \(7\) the similar dynamical deformations will be considered for the Toda lattice associated with the Lie algebra \(G_2\).
Theorem 1 For an arbitrary magnitude of the Casimir operator \( J_3 = (l, g) \) matrices \( T_0(u) \) and \( T(u) \) obey the following relations

\[
\begin{align*}
\left\{ \mathcal{T}_0(u), \mathcal{T}_0(v) \right\} &= \left[ r(u - v), \mathcal{T}(u) \mathcal{T}(v) \right] = \\
&= \left[ r(u - v), \mathcal{T}(u) \mathcal{T}_0(v) \right] + W(u, v, l_j, g_j), \quad \text{(3)} \\
\left\{ \mathcal{T}(u), \mathcal{T}(v) \right\} &= \left[ r(u - v), \mathcal{T}(u) \mathcal{T}(v) \right] + \\
&+ \frac{1}{T(u)} s_1 \mathcal{T}(v) - \frac{1}{T(v)} s_2 \mathcal{T}(u) + \frac{2}{T(u)} s_1 \mathcal{T}(v) - \frac{2}{T(v)} s_1 \mathcal{T}(u). \quad \text{(4)}
\end{align*}
\]

The corresponding matrices have the form

\[
\begin{align*}
W &= \left[ r(u - v), \mathcal{T}(u) \mathcal{T}_0(v) + \mathcal{T}(u) \mathcal{T}_0(v) \right], \\
r &= \frac{\eta P}{u - v} = \frac{2i}{u - v} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i, \quad \eta = 2i, \\
s_1 &= \frac{-\eta u l(g)}{4g^3(u - v)} (I + \sigma_3) \otimes (I - \sigma_3) = \frac{-\eta u J_3}{g^3(u - v)} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
s_2(u, v) &= P s_1(v, u) P, \quad \text{(7)}
\end{align*}
\]

where \( \sigma_i \) are Pauli matrices and \( P \) is a permutation operator of auxiliary spaces.

The proof is a direct but lengthy computation.

According by \([3]\) the Lax representation is constructed by using deformed \( R \)-matrix structure

\[
\mathcal{T}_0(u) = \{ H, T(u) \} = [M(u), T(u)]. \quad \text{(8)}
\]

Matrix \( M(u) \) is derived from the algebraic relations \([3]\) using the definition of the hamiltonian

\[
H = \Phi_v [\text{tr} T_0(v)] = \Phi_v [l_0(v)] \quad \text{with} \quad \Phi_v[z(v)] \equiv z(v)|_{v=0}. \quad \text{(9)}
\]

This matrix equal to

\[
M(u) = -\Phi_v \text{tr}_2 [I \otimes T(v) \cdot r(u - v)]
\]

\[
= \left[ \frac{-\eta}{u - v} \begin{pmatrix} A(v) - D(v) & B(v) \\
C(v) & D(v) - A(v) \end{pmatrix} \right]_{v=0}. \quad \text{(10)}
\]

Here \( A, B, C \) and \( D \) are entries of the \( T(u) \) \([3]\) and \( \text{tr}_2 \) means trace in the second auxiliary space. More precisely,

\[
\mathcal{T}_0(u) = [M(u), T(u)] = [M(u), T_0(u)] + N, \quad \text{with} \quad M(u) = -4J_3 \begin{pmatrix} 0 & -l_+ \\
l_- & 0 \end{pmatrix}.
\]

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It is either a Lax pair at the level \( J_3 = 0 \) or a Lax triad for an arbitrary magnitude of \( J_3 \). This Lax representation is a compatibility condition for the following linear problems

\[
T(u) \varphi(u) - \lambda \varphi(u) = \psi(u),
\]

\[
\frac{d\varphi(u)}{dt} + M(u) \varphi(u) = 0, \quad \frac{d\psi(u)}{dt} + M(u) \psi(u) = N(u) \varphi(u),
\]

where \( \varphi \) is so-called Baker-Akhiezer function at the level \( J_3 = (l, g) = 0 \).

Now the invariants are trace of \( T_0(u) \), which remains a generating function of the integrals of motion \( \mathbb{M} \), and determinant of \( T(u) \):

\[
t_0(u) = \text{tr} \ T_0(u) = u^2 - 2uw - H, \quad \Delta(u) = \text{det} \ T(u) = u^2 J_2.
\]

The dual determinant of \( T_0(u) \) and trace of \( T(u) \) are the dynamical variables:

\[
\Delta_0(u) = \det T_0(u) = \Delta(u) - 2J_3 g_3 u, \quad t(u) = \text{tr} \ T(u) = t_0(u) + \frac{2J_3 u}{g_3}.
\]

So, we obtained some starting point for a machinery of the inverse scattering method. Below we introduce the quantum counterpart of the presented deformation and consider a Lax representation for the Goryachev-Chaplygin top. Then we will try to apply the standard scheme related to the reflection equations \( \mathbb{R} \).

4 Quantum axially symmetric Neumann’s system

Let variables \( l_i, g_i, \ i = 1, 2, 3 \) be generators of the Lie algebra \( \mathfrak{e}(3) \) with commutator relations:

\[
[l_i, l_j] = -i\eta \varepsilon_{ijk} l_k, \quad [l_i, g_j] = -i\eta \varepsilon_{ijk} g_k, \quad [g_i, g_j] = 0, \quad i, j = 1, 2, 3.
\]

The quantum operator \( T_0(u) \) related to a classical monodromy matrix \( \mathbb{R} \) has the form

\[
T_0(u) = \begin{pmatrix}
  u^2 - 2l_3 u - l_2^2 - \frac{1}{4} & i(g_+ u - \frac{1}{2}(g_3, l_+)) \\
  i(g_- u - \frac{1}{2}(g_3, l_-)) & g_3^2
\end{pmatrix},
\]

here braces \( \{,\} \) mean an anticommutator. Operator \( T_0(u) \) \( \mathbb{R} \) at the level \( J_3 = (l, g) = 0 \) obeys the FCR \( \mathbb{R} \) with the \( R \)-matrix of the XXX type \( R(u) = u + i\eta P \) \( \mathbb{R} \).

Introduce two additional matrices

\[
F(u) = \frac{(u - i\eta)J_3}{g_3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T(u) = T_0(u) + F(u).
\]

Below the spectral parameter \( u \) in the \( F(u) \) is always shifted by the constant \( i\eta \).

Theorem 2 By the arbitrary values of the Casimir operator \( J_3 = (l, g) \) operator \( T_0(u) \) \( \mathbb{R} \) obeys the following deformation of the FCR \( \mathbb{R} \)

\[
R(u - v) \frac{1}{T_0(u)} T_0(v) - \frac{1}{T_0(v)} T_0(u) R(u - v) = W(u, v, l_j, g_j)
\]

\[
W(u, v, l_j, g_j) = \left[ \frac{1}{F(u)T_0(v)} + \frac{1}{T_0(u)F(v)} \right] R(u - v),
\]

where \( [\, \, \, \, ] \) stands for a matrix commutator.
The proof is a straightforward calculation.

The deformed FCR (3) assumes another forms:

\[
R(u - v) \left( \frac{1}{T_0} T_0^2 + \frac{1}{T_0} T_0 \right) = \left( \frac{1}{T_0} T_0 + \frac{1}{T_0} T_0 \right) R(u - v),
\]

\[
R(u - v) \left( \frac{1}{T_0} T_0^2 - \frac{1}{T_0} T_0 \right) R(u - v) = \left( \frac{1}{T_0} T_0 + \frac{1}{T_0} T_0 \right) R(u - v),
\]

(for the sake of brevity we have omitted the arguments \( u, v \) in the last two formulas).

Generating function of the quantum integrals of motions is the trace of \( T_0(u) \)

\[
t(u) = \text{tr} T_0(u), \quad [t(u), t(v)] = 0.
\]

The quantum determinant \( \det_q T(u) \) is the central element now

\[
\Delta(u) \equiv \det_q T(u) = (u^2 - 1/4) J_2.
\]

By using the deformation of the FCR (3) the quantum Kowalewski top will considered in the Sec. 8.

5 The Goryachev-Chaplygin top

The axially symmetric Neumann’s system related to the two-particle Toda lattice associated to the root system \( \mathcal{A}_2 \). It is well known that the Goryachev-Chaplygin top (GCT) related to the three-particle Toda lattice associated to the root system \( \mathcal{A}_3 \). Now we present relations between the corresponding Lax representations.

The Goryachev-Chaplygin top (GCT) represents a symmetric top in a constant homogeneous field with the principal momenta of inertia satisfying \( I_1 : I_2 : I_3 = 1 : 1 : 1/4 \) and the center of mass located in equatorial plane. Hamiltonian of the GCT is

\[
J_1 = H = \frac{1}{2} \left( l_1^2 + l_2^2 + 4l_3^2 \right) - g_1.
\]

It is completely integrable in the one-parameter subset of orbits \( \mathcal{O} \) \( J_2 = (g, g) = a^2 \) and \( J_3 = (l, g) = 0 \) in \( e(3)^* \).

The GCT has been investigated in quantum inverse scattering method by Sklyanin and generalized in [19]. It was a starting point for these investigations.

For construction of the monodromy matrix for the GCT we recall basic results of [20].

**Lemma 1** Let the monodromy matrix \( T(u) \in Y(g) \otimes \text{End}(\mathbb{C}^2) \) obeys the FCR with \( R \)-matrix of the XXX type. If exist such element \( K \in Y(g) \) that

\[
[K(u), A(u)] = [K(u), D(u)] = 0,
\]

\[
[K(u), B(u)] = \eta B(u), \quad [K(u), C(u)] = -\eta C(u),
\]

then the monodromy matrix

\[
T_1(u) = \begin{pmatrix}
\frac{u - p + K}{\beta e^{iq}}
\gamma e^{-iq}
\end{pmatrix}
\begin{pmatrix}
A(u) & e^{iq} B(u) \\
e^{-iq} C(u) & D(u)
\end{pmatrix} \in Y(g \oplus w) \otimes \text{End}(\mathbb{C}^2),
\]

where generators of \( w \) are \([p, q] = -i\eta \) and \( \beta, \gamma \in \mathbb{C} \), obeys the FCR as well.
The proof consists of in the fact that the two matrices in the product \( \mathcal{T} \) obey the FCR \( \mathcal{E} \) with one \( R \)-matrix and their entries mutually commute.

Definition of the monodromy matrix \( T_1(u) \) \( \mathcal{T} \) can be rewritten in other form

\[
T_1(u) = \begin{pmatrix} e^{-\frac{i\mu}{2}} & 0 \\ 0 & e^{\frac{i\mu}{2}} \end{pmatrix} \begin{pmatrix} u - p + K & \beta \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} e^{\frac{i\mu}{2}} & 0 \\ 0 & e^{-\frac{i\mu}{2}} \end{pmatrix},
\]

which we can consider as some kind of gauge transformation of standard rule in quantum inverse scattering method \( \mathcal{E} \).

**Lemma 2** Let the matrix \( T(u) \in Y(\mathfrak{g}) \otimes \text{End}(\mathbb{C}^2) \) is a finite-dimensional irreducible representation of the algebra of monodromy matrices \( \mathcal{T} \), which is polynomial of spectral parameter \( u \). If the entries of the matrix \( T(u) \) have the following asymptotic behavior

\[
A(u) = u^N - a_1 u^{N-1} + a_2 u^{N-2} + \ldots, \quad B(u) = b_1 u^{N-1} + \ldots,
\]

\[
C(u) = c_1 u^{N-1} + \ldots, \quad D(u) = d_1 u^{N-2} + \ldots,
\]

then the element \( K = a_1 \) obeys the conditions of a Lemma 1. Representations \( T(u) \) and \( T_1(u) \) \( \mathcal{T} \) are related to the integrable systems with the following integrals of motion

\[
t(u) = \text{tr} T(u) = u^N - a_1 u^{N-1} + a_2 + d_1 u^{N-2} + \ldots,
\]

\[
t_1(u) = \text{tr} T_1(u) = u^{N+1} - pu^N + (a_2 - a_1^2 - pa_1 + \beta c_1 + \gamma b_1) u^{N-1} + \ldots,
\]

where operator \( p \) has a continuous spectra.

For the proof we have to substitute the asymptotes \( \mathcal{E} \) to FCR \( \mathcal{E} \).

These Lemma’s have been introduced to \( R \)-matrices of XXX and XXZ types in \( \mathcal{T} \) by considering the classical and relativistic Toda lattices in the Jacoby systems of coordinates. By using these Lemma’s and the matrix \( T_0(u) \) \( \mathcal{T} \) we obtain the new monodromy matrix \( T_1(u) \) with entries

\[
A(u) = (u - p + 2\gamma) \left( u^2 - 2t_3 u - t_1^2 - t_2^2 - 1/4 - \frac{\mu^2 - 1/4}{g_3^2} \right) + i\beta \left[ ug_+ - \{g_3, l_- \}/2 \right],
\]

\[
B(u) = e^{i\gamma} \left[ i(u - p + 2\gamma) (ug_+ - \{g_3, l_+ \}/2) + \beta g_3 \right],
\]

\[
C(u) = e^{-i\gamma} \left[ u^2 - 2t_3 u - t_1^2 - t_2^2 - 1/4 - \frac{\mu^2 - 1/4}{g_3^2} \right],
\]

\[
D(u) = i\gamma \left( ug_+ - \{g_3, l_+ \}/2 \right).
\]

Matrix \( T_1(u) \) at the level \( \mu = 1/2 \) has been introduced in \( \mathcal{E} \) and generalized for an arbitrary magnitude of \( \mu \) in \( \mathcal{E} \).

6 **The Kowalewski-Chaplygin-Goryachev top and reflection equation**

Consider representations \( U(u) \) of twisted yangians related to the reflection equation \( \mathcal{E} \). The algebra of monodromy matrices \( \mathcal{T} \) and its classical counterpart \( \mathcal{E} \) have two important
The symbol $T^i$ means a transposition matrix and symbols $T^a$ and $T^i$ related to antipod map and to involution map, respectively, in theory of quantum groups \cite{3}.

In the classical mechanics monodromy matrix $U(u)$ related to reflection equation can be constructed as

$$U(u) = T_+(u)T_-(u),$$

with the matrices $T_\pm$ defined by

$$T_-(u) = T_1(u)K_-(u)T_1^t(u),$$

$$T_+(u) = T_2(u)K_+(u)T_2^a(u).$$

Here matrices $T_j(u)$, $j = 1, 2$ obey the Sklyanin brackets \cite{4} with some matrix $r(u)$, $K_\pm(u)$ are known solutions to reflection equation \cite{14} with same matrix $r(u)$ and $s(u + v) = r(u + v)$. The matrices $T_\pm(u)$ \cite{2} obey the classical reflection equation as well \cite{3}.

Consider a simple $\mathbb{C}$–number solutions of RE \cite{3} $K_\pm(u)$, which correspond to various boundary conditions for integrable systems \cite{3}. For the rational $R$-matrix of $XXX$ type matrices $K_\pm(u)$ are given by

$$K_-(u, \alpha_1, \beta_1, \gamma_1) = \begin{pmatrix} \alpha_1 + u\beta_1 & u \\ u\gamma_1 & \alpha_1 - u\beta_1 \end{pmatrix},$$

$$K_+(u, \alpha_2, \beta_2, \gamma_2) = \begin{pmatrix} -\alpha_2 + u\beta_2 & -u\gamma_2 \\ -u & -\alpha_2 - u\beta_2 \end{pmatrix}.$$  

For instance, the boundary matrices $K_\pm$ \cite{3} are applied to construct the monodromy matrices for the Toda lattices associated to the $B_n$ and $C_n$ root systems \cite{3}.

By taking matrix $T_1(u - \gamma)$ \cite{1} with the gyrostat parameter $\gamma$ and matrices $K_\pm(u)$ \cite{3} we construct the new monodromy matrix $U(u)$ as \cite{3} ($T_2(u) = I$). At the level $J_3 = 0$ we get the completely integrable system with integrals of motion defined by the trace of $U(u)$. The hamiltonian of this system is equal to

$$H = l_+l_- + 2l_3^2 - i(\alpha_1 g_+ - \alpha_2 g_-) + \frac{1}{2}(\gamma_1 g_+^2 - \gamma_2 g_-^2) + \frac{\mu}{g_3^2}$$

$$-ig_3(\beta_1 l_+ - \beta_2 l_-) - 2il_3(\beta_2 g_+ - \beta_1 g_-) + \gamma(2l_3 + i\beta_1 g_+ - i\beta_2 g_-),$$

If $\gamma = \beta_1 = \beta_2 = 0$ integrable system with the hamiltonian \cite{3} can be identified with the Kowalewski-Chaplygin-Goryachev top \cite{11}.

In quantum mechanics operators $T_\pm(u)$ are

$$T_-(u) = T(u - \gamma)K_-(u - \eta/2)\sigma_2T^t(-u - \gamma)\sigma_2,$$

$$T_+(u) = K_+(u + \eta/2).$$

Here operators $T(u)$ and $K_\pm(u)$ are representations of monodromy matrix algebras related to FCR \cite{6} and to RE \cite{3} with quantum matrices $R(u)$ and $S(u + v) = R(u + v - \eta)$, respectively.

Thus, we describe the eight parameters family ($\alpha_j, \beta_j, \gamma_j$ with $j = 1, 2; \mu, \gamma$) of completely integrable at the level $J_3 = (l,g) = 0$ systems on the Lie algebra $\mathfrak{e}(3)$ in quantum inverse scattering method (for comparison see \cite{21}).
7  Toda lattice associated to the Lie algebra $G_2$

The family of tops introduced in the previous section can be associated with the two-particle Toda lattices related to the root system $\mathcal{BC}_2$. The remaining non-trivial two-root system is $G_2$. We hope that consideration of the monodromy matrix in the two dimensional auxiliary space for the corresponding Toda lattice gives us some background for a search of the non-standard Lax representation for the KT.

The group $G_2$ is of rank two and dimension 14 and it has two simple roots $\alpha_1$ and $\alpha_2$. The Weyl group of $G_2$ is the permutation group of order 3 with inversion, generated by $\tau_1$ and $\tau_2$,

$$\tau_1 : (a_1, a_2) \rightarrow (-a_1, a_2 - 3a_1), \quad \tau_2 : (a_1, a_2) \rightarrow (-a_1, a_2 - 3a_1).$$

The root system is easier to describe in the standard basis in a large space $\mathbb{R}^3$. We will use tree pairs of canonically conjugate variables $(q_j, p_j)$ with a linear constraint $\sum q_j = \sum p_j = 0$.

The non-constrained system can be obtained by using the following canonical transformation

$$q_1 \rightarrow \sqrt{3}q_1 + \frac{q_3}{3}, \quad q_2 \rightarrow -2q_2 + \frac{q_3}{3}, \quad q_3 \rightarrow -\sqrt{3}q_1 + q_2 + \frac{q_3}{3},$$

$$p_1 \rightarrow \frac{p_1}{2\sqrt{3}} + \frac{p_2}{6} + p_3, \quad p_2 \rightarrow -\frac{p_2}{3} + p_3, \quad p_3 \rightarrow -\frac{p_1}{2\sqrt{3}} + \frac{p_2}{6} + p_3,$$

which transforms the corresponding Hamiltonian to the natural form.

According by [14] the Lax representation is equal to

$$\mathcal{L} = \begin{pmatrix} -\beta_3 & -\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -\beta_2 & -\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -\beta_1 & -2\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \beta_2 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \beta_3 & 0 \end{pmatrix}$$ (2)

where $\beta_1 = p_3 - p_1$, $\beta_2 = p_1 - p_2$, $\beta_3 = p_3 - p_2$,

$$\alpha_1 = e^{q_1} = e^{q_1-q_2}, \quad \alpha_2 = 3e^{q_2} = 3e^{-2q_1+q_2+q_3},$$

It is a three diagonal matrix and, therefore, we can easy obtain the corresponding monodromy matrix in the two dimensional auxiliary space $\mathcal{L}^\sigma$.

Let us define three matrices $L_j(u)$

$$L_1 = \begin{pmatrix} u - p_1 - p_2 & -e^{q_1-q_2} \\ e^{-q_1+q_2} & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} u - p_3 - p_1 & -3e^{q_3-q_1} \\ 3e^{-q_3+q_1} & 0 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} u - p_3 - p_2 & -\frac{1}{3}e^{q_3-q_2} \\ \frac{1}{3}e^{-q_1+q_2} & 0 \end{pmatrix},$$ (3)

and two boundary matrices $T_{\pm}(u)$

$$K_- = \begin{pmatrix} 2 & u \\ 0 & 2 \end{pmatrix}, \quad K_+ = \begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix}, \quad T_- = L_1 K_- L_1^\sigma, \quad T_+ = L_3^\sigma K_+ L_3.$$ (4)
The monodromy matrix for the open Toda lattice associated to the algebra $G_2$ is equal to

\[
U(u) = T_+(u)L_2(u)T_-(u)L_2^\sigma(u),
\]

\[
\text{tr } U(u) = -u \det (\mathcal{L} - uI) = -u^8 + h_1u^6 + h_2u^4 + h_3u^2,
\]

where $h_j$ are integrals of motion. This form for matrix $U(u)$ is a more symmetrical form for the Weyl group (I). For the affine algebra $G_2^{(1)}$ we have to substitute new matrix $T_+(u)$ into (5)

\[
\tilde{T}_+ = \begin{pmatrix}
\frac{(u - p)e^q + (u + p)e^{-q}}{3} & -\frac{1}{9}(e^q - e^{-q})^2 \\
-(u^2 - p^2) & \frac{(u + p)e^q + (u - p)e^{-q}}{3}
\end{pmatrix},
\]

where $p = p_3 - p_2$ and $q = q_3 - q_2$.

The Poisson brackets relations for the matrices $L_k(u)$ (3) have the following polylinear form

\[
\{L_j(u), L_k(v)\} = \delta_{jk}[r(u - v), L_j(u)L_k(v)] +
\]

\[
+ \ (1 - \delta_{jk})(-1)^{2j-k} \left( [r_1, L_j(u)] + [r_2, L_k(v)] \right),
\]

\[
j \geq k, \ j, k = 1, 2, 3.
\]

here $r(u - v)$ is a standard $R$-matrix of the $XXX$ type (5) and the independent from spectral parameter matrices $r_{1,2}$ are given by

\[
r_1 = -\frac{1}{4}(I - \sigma_3) \otimes (I + \sigma_3), \quad r_2 = Pr_1P,
\]

(in comparison with (4)). Locally $(j = k)$ these Poisson brackets relations are the standard Sklyanin brackets and, therefore, matrices $T_\pm(u)$ (4) obey the standard non-dynamical RE (5).

By using this polylinear algebra and factorization (7) the basic property of trace of a monodromy matrix $U(u)$

\[
\{t(u), t(v)\} = 0, \quad t(u) = \text{tr } U(u),
\]

can be easy proved.

If we assume that some representation $U(u)$ associated to the root system $G_2$ can not be expanded on the simplest factors as above, then we must introduce a more complicated dynamical $R$-matrix structure. Consider two matrices $T^{(1,2)}(u)$ defined by

\[
T^{(1)}(u) = L_2(u)L_1(u), \quad T^{(2)}(u) = L_3(u)L_2(u)L_1(u).
\]

Their Poisson brackets relations are calculated from the polylinear algebra (3) and we can prove that we can not close these relations at the quadratic $R$-matrix algebra by using only the $\mathbb{C}$-number $R$-matrices. These Poisson brackets relations have the form of the deformed Sklyanin brackets (4) with the dynamical matrices $s_k = \alpha_k(p, q)r_k$, \quad $k = 1, 2$ (7). Here expressions for the dynamical coefficients $\alpha_k(p, q)$ are simply recovered from (6).
Theorem 3

Matrices $T^{(j)}$ have one common property for their traces

$$t^{(j)}(u) = \text{tr} T^{(j)}(u) = u^n + h_1 u^{n-1} + h_2 u^{n-2} + h_3, \quad j = 1, 2, n = j + 1,$$

$$\{t^{(1)}(u), t^{(1)}(v)\} = u - v, \quad \{t^{(2)}(u), t^{(2)}(v)\} = (u - v)(uv + u + v + 1),$$

$$\{h_i, h_k\} = 1, \quad i > k, \quad i, k = 1, 2, 3,$$

and, of course, these traces do not the generating functions of integrals of motion.

The corresponding matrices $T^{(j)}(u) = T^{(j)}K_T T^{(j)}{\sigma}$ obey the dynamical deformations of the classical reflection equation (9)

$$\left\{ T_{-}, T_{-}^{-1/2} \right\} = \left[ T_{-} r(u - v) T_{-}^{-1/2} + \frac{1}{r} T_{-} r(u + v) T_{-}^{-1/2} - \frac{1}{r} T_{-} r(u + v) T_{-}^{-1/2} + W(u, v, p_j, q_j) \right], \quad (9)$$

where $W(u, v, p_j, q_j)$ is matrix-function of spectral parameters and of dynamical variables. We choose the simplest form for the dynamical deformations of the RE (9). Of course, matrix $W$ can be presented as the various combinations of $T^{(j)}$ and proper dynamical $R$-matrices. May be, these combinations will more deeply reflect a structure of the Weyl group of $G_2$ (9).

We have to emphasize here that matrices $T^{(j)}_-$ relate to at most then another factorization of the monodromy matrix $U(u)$ (5)

$$U(u) = T_{+}(u) T_{-}^{(1)}(u) = K_{+}(u) T_{-}^{(2)}(u).$$

Recall that the monodromy matrix $T_0(u)$ (3) for the Neumann’s system is closely connected to the matrix $T(u)$ (2) for the Toda lattice. However, the structure of the phase spaces are different (5). Therefore, for the Toda lattice $2 \times 2$ matrix $T(u)$ are factorized on the one-particles matrices (2). The corresponding matrix $T_0(u)$ in $e(3)^*$ can not be expanded on the simplest factors. Then for the some integrable system associated with the exceptional algebra $G_2$ we could get the monodromy matrix $U(u)$ without the simplest expansion as (3).

8 The Lax triad for the Kowalewski top

Motivated by the previous example we will look the such non-factorable monodromy matrix for the KT. Let boundary matrices $K_{\pm}$ be

$$K_{+}(u) = \begin{pmatrix} -\alpha_2 & 0 \\ -u & -\alpha_2 \end{pmatrix}, \quad \text{and} \quad K_{-}(u) = \begin{pmatrix} \alpha_1 & u \\ 0 & \alpha_1 \end{pmatrix}, \quad (1)$$

(in comparison with (2)).

The monodromy matrix $U(u) = K_{+} T_0(u) K_{-} T_0^{i}(u) = K_{+} T_{-}$, with $T_0(u)$ given by (5), corresponds to the KT in the one-parameter subset of orbits $O$ $(J_2 = (g, g) = a^2$ and $J_3 = (l, g) = 0)$. Now we try to take up such additive deformation that the deformed monodromy matrix is described the KT on whole phase space $J_3 \neq 0$. Notice, that for the deformed Sklyanin brackets (3) the maps $T(u) \to T^{\sigma}(u)$ (4) are the automorphisms as well.

Theorem 3 For the Kowalewski top on the Lie algebra $e(3)$ with integrals of motion

$$J_1 = H = (l_+ l_- + 2l_3^2) - i(\alpha_1 g_+ + \alpha_2 g_-),$$

$$J_2 = (g, g) = a^2,$$

$$J_3 = (l, g),$$

$$J_4 = k_+ k_- = (l_3^2 + 2i\alpha_2 g_+)(l_3^2 + 2i\alpha_1 g_-), \quad (2)$$

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the monodromy matrix $U(u)$ is given by

$$U(u) = K_+(u)T_-(u).$$  \hfill (3)

Here

$$T_-(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & A(-u) \end{pmatrix} = T_0(u)K_-(u)T_0^i(u) + G_- =$$

$$= T_0(u)K_-(u)T_0^i(u) + uJ_3 \begin{pmatrix} il_-u + \alpha_1 g_3 & 2i\alpha_1 l_+ \\ 0 & il_-u - \alpha_1 g_3 \end{pmatrix}.$$  \hfill (4)

More precisely,

$$A(u) = -iu(u_3 g_- + (g_3 l_- - 2l_3 g_-)u^2 + (-ia_2 \alpha_1 + g_+ k_-)u - g_3 l_+ k_-),$$

$$B(u) = u^5 - 2(l_+ l_- + 2l_3^2 - i\alpha_1 g_+)u^3 + l_+^2 k_- u,$$

$$C(u) = -u^3 g_- + g_3^2 k_- u$$

Matrix $U(u)$ \hfill (3) obeys the Lax representation in the form

$$\frac{dU(u)}{dt} = [K_+, M_-] + K_+ N_-,$$  \hfill (5)

$$M_- = 2i \begin{pmatrix} -ig_- & -1 \\ 0 & ig_- \end{pmatrix},$$  \hfill (6)

$$N_- = uJ_3 \begin{pmatrix} 3l_-u^2 - l_+ k_- & 0 \\ 2ig_3 k_- & -3l_-u^2 + l_+ k_- \end{pmatrix}. $$  \hfill (7)

Trace of monodromy matrix $U(u)$ is a generating functions of integrals of motion

$$t(u) = -u^6 + 2u^4J_1 - u^2(J_4 + 2\alpha_1 \alpha_2 J_2),$$  \hfill (8)

Matrix $T_-(u)$ has a typical to RE \hfill (10) symmetry property $T^i_-(u) = T_-(u)$, which relates to the involutions on the phase space. It reflects the corresponding symmetry of the Jacobian of spectral curve defined by the 4 x 4 matrix $L(\lambda)$ \hfill (10).

For calculation of the corresponding Poisson structure we express $T_-(u)$ through matrices $T_0(u)$ and $F(u)$ \hfill (10), which have the known Poisson structure \hfill (10). Assume

$$s_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{I + \sigma_3}{2}, \quad s_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{I - \sigma_3}{2},$$  \hfill (9)

then the monodromy matrix $T_-(u)$ \hfill (10) is equal to

$$T_-(u) = T_0(u - \gamma)K_-(u)T_0^i(u - \gamma) +$$

$$+ \left[ T_0(u - \gamma)s_b + \frac{1}{2}s_b(T_0(u - \gamma) - T_0^i(u + \gamma) \sigma_2)s_a \right] K_-(u)F^i(u - \gamma)$$

$$+ F(u - \gamma)K_-(u) \left[ s_a T_0^i(u - \gamma) + \frac{1}{2}s_b(T_0^i(u - \gamma) - T_0(u + \gamma))s_a \right].$$
Here
\[ T_0^1(u - \gamma) = \sigma_2 T_0^1(-u - \gamma) \sigma_2 = \sigma_2 T_0^1(-(u + \gamma)) \sigma_2 \]
and we introduce a shift of the spectral parameter \( u \rightarrow u - \gamma \) for description of the Kowalewski gyrostat. Then the trace of \( U(u) \) is a generating function of integrals of motion for the Kowalewski gyrostat
\[
t(u) = -u^6 + 2u^4(J_1 + \gamma^2) - u^2 \left( J_4 + 2\alpha_1 \alpha_2 J_2 - \gamma^2(2J_4 + \gamma^2) \right) - 2\gamma \alpha_1 \alpha_2 J_2 ,
\]
\[
J_1 = H = (l_+ l_- + 2l_3^2 + 2\gamma l_3) - i(\alpha_1 g_+ + \alpha_2 g_-),
\]
\[
J_4 = k_+ k_- - 4\gamma (l_3 + \gamma) l_+ l_- - 4i\gamma g_3 (\alpha_1 l_+ + \alpha_2 l_-).
\]

Next by using factorization (10) and the deformed Sklyanin brackets (3), we can prove that matrix \( T_-(u) \) obeys the following deformations of the classical reflection equation
\[
\left\{ \begin{array}{c}
1 \\
2 \\
T_-, T_-
\end{array} \right\} = \left[ r(u-v), \begin{array}{c} 1 \\
2 \\
T_-, T_-
\end{array} \right] + \begin{array}{c} 1 \\
2 \\
T_-, T_-
\end{array} r(u+v) \begin{array}{c} 2 \\
1 \\
T_-, T_-
\end{array} - \begin{array}{c} 2 \\
1 \\
T_-, T_-
\end{array} r(u+v) \begin{array}{c} 1 \\
2 \\
T_-, T_-
\end{array} + W(u, v, l_j, g_j). \quad (11)
\]

For the KT (\( \gamma = 0 \)) the dynamical matrix \( W(u, v, l_j, g_j) \) is given by
\[
W(u, v, l_j, g_j) = uv J_3 \begin{pmatrix}
a(u, v) & b(u, v) & -b(v, u) & 0 \\
c(u, v) & a(u, -v) & d(u, v) & -b(-v, u) \\
-c(v, u) & -d(v, u) & a(-u, v) & b(-u, v) \\
0 & -c(-v, u) & c(-u, v) & a(-u, -v)
\end{pmatrix}, \quad (12)
\]
where
\[
a(u, v) = ik_- \left( g_3(u^2 - v^2) - 2g_4 l_-(u - v) \right), \\
b(u, v) = 6l_- u^2 v^2 - 2l_+ k_-(u^2 - 4l_3 u + l_+ l_- + v^2), \\
c(u, v) = 2g_3 k_- (g_3 l_- - 2ug_-), \\
d(u, v) = 4ig_3 k_- (u^2 + l_+ l_-).
\]

Dynamical matrix \( W(u, v, l_j, g_j) \) can be expressed in the terms of the matrices \( T_0, F \) and \( R \) by using the representation (10) and the deformed Sklyanin brackets (3) as well.

The Lax representation (5) are constructed from this deformed R-matrix brackets (11) according to (3), as for the Neumann’s system (8). The Lax triad has an additive freedom for the matrix \( M \), we always can pass from the Lax triad \((U, M, N)\) to triad \((U, M + M_1, N + [L, M_1])\). Our choice (3) is fixed by the Lax pair at the level \( J_3 = (l, g) = 0 \), which relates to a pure reflection equations (11).

We understand, that we present a few artificial construction in comparison with the construction of the Lax representation on loop algebras by Reyman and Semenov-Tian-Shansky (8). However, we have some new positive properties of the proposed Lax representation.

Under the following transformation of universal enveloping algebras and of definition of matrix \( T_- (u) \)
\[
k_- \rightarrow k_- \pm 1/4 \quad \text{and} \quad T_- (u) \rightarrow T_- (u) \mp i k_- u^2 \sigma_3/4, \quad (14)
\]
the new monodromy matrix \( U(u) = K_+(u) T_- (u) \) describes the Kowalewski top on the Lie algebras \( so(4) \) and \( so(3, 1) \) with integrals of motion introduced in (8).

Monodromy matrix for the quantum Kowalewski top can be obtained according to a general scheme (8). Substituting into definition of monodromy matrix (11), quantum operators
$T_0(u)$ and $F(u)$ \cite{2} and boundary matrices $K_{\pm}(u \pm \eta/2)$, we obtain the quantum $U$-operator for the Kowalewski top. It means that trace of this $U$-operator is a generating function of true integrals of motion in the quantum mechanics

$$
t(u) \equiv \text{tr } U(u), \quad [t(u), t(v)] = 0,
$$

$$
J_1 = H = l_1^2 + l_2^2 + 2l_3^2 - i\alpha_1 g_+ - i\alpha_2 g_-, 
$$

$$
J_4 = \frac{1}{2}\{k_+, k_-\} + 2\eta^2\{l_+, l_-\}. \quad (15)
$$

Here operators $k_{\pm}$ have been defined in \cite{2} and braces $\{,\}$ mean anticommutator of quantum operators. Operator $T_-(u)$ obeys the deformation of quantum reflection equation

$$
R(u - v)^{1/2} T_-(u) R(u + v - \eta)^{1/2} T_-(v) - 2 T_-(v) R(u + v - \eta) T_-(u) R(u - v) = W. 
$$

Here quantum matrix $W(u, v, l_j, g_j)$ is obtained by application of the deformed FCR \cite{2} to the operator $T_-(u)$ defined by \cite{1}.

9 Conclusions

We present a monodromy matrix on $2 \times 2$ auxiliary space for classical and quantum Kowalewski top. This matrix relates with the additively deformed reflection equation \cite{11}. Deformations $W(u, v, l_j, g_j)$ \cite{12} depends on the spectral parameters and the dynamical variables. The dynamical deformation of quadratic $R$-matrix algebras can be considered as analog of dynamical $r$-matrices for linear $R$-matrix algebras \cite{12, 13}.

However this complication of inverse scattering method will be justified if it allows to describe a quite wide set of integrable systems, as for linear dynamical $R$-matrix, or to solve completely such nontrivial system as the Kowalewski top. We hope to obtain such confirmations in the forthcoming publications.

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