Separability of 3-qubits density matrices, related to $l_1$ and $l_2$ norms and to unfolding of tensors into matrices

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Keywords: 3-qubits states, Hilbert-Schmidt (HS) decompositions, separability of density matrices, HOSVD, GHZ and W states

Abstract

We treat separability of 3 (and more) qubits states. Especially we discuss density matrices with maximally disordered subsystems (MDS), by using Hilbert-Schmidt (HS) decompositions, where in the general case these density matrices include 27 HS parameters. By using ‘unfolding methods’, the MDS tensors are converted into matrices and by applying singular values decompositions (SVD) to these matrices the number of the parameters for treating full separability, in the general MDS case, is reduced to 9, and under the condition that the sum of the absolute values of these parameters is not larger than 1, we conclude that the density matrix is fully separable. In order to know if density matrices with MDS are separable, one needs to check with 9 parameters at a time and not with all 27 parameters. We use also Frobenius ($l_2$) norms. For treating bi-separability of 3-qubits MDS density matrices, the 27 HS parameters are divided into 9 triads. If the sum of the nine $l_2$ norms for these triads is not larger than 1, we conclude that the density matrix is bi-separable. We analyze the relations between 3 qubits MDS density matrices and the method of high order singular value decomposition (HOSVD). We demonstrate the use of our methods in examples. For 3-qubits states which are non-MDS the HS decomposition includes up to 63 parameters. If the sum of the absolute values of all the HS parameters is not larger than 1, we conclude that the density matrix is fully separable, and we have explicit expressions for their separability. For the systems of GHZ and W states mixed with white noise we find a simple way to reduce the sum of the absolute values of the HS parameters and get better conditions for their full separability.

1. Introduction

Entanglement of qubits systems is at the core of the quantum computation field. There is much interest in quantum entangled states due to various potential applications that use the quantum properties of such states. The most famous application is the use of quantum systems for a new generation of computers that will be based on principles of quantum computation (QC). Therefore it is of utmost importance to quantify entanglement in such systems and to have a definite criterion when such systems are separable. The definition of full separability of a three-partite system is: A density matrix $\rho$ on Hilbert space $H_A \otimes H_B \otimes H_C$ where $A$, $B$ and $C$ are the three parts of a three-partite system is defined as non-entangled/ separable if there exist density operators $\rho_A^{(i)}$, $\rho_B^{(i)}$, $\rho_C^{(i)}$ and $p_i > 0$ with $\sum_i p_i = 1$ such that: $\rho = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \rho_C^{(i)}$. The interpretation of such definition is that for three-partite separable state these parts are completely independent of each other. In a pictorial description: Assuming 3-qubits fully separable state, we send the 3 qubits to Alice, Charles and Jacob, respectively, which are far from each other. Any measurement made by one of them will not affect the quantum properties of the qubits belonging to others. In the present work we develop sufficient separability conditions for three (and more) qubits. We develop also conditions for bi-separability where in a pictorial description one of the three qubits (e.g., that which belongs to Jacob) is separated from the others two qubits (belonging to Alice and Charles) where the latter two qubits are entangled. In a pictorial description any measurement made by Jacob will not affect the quantum properties of the qubits belonging to Alice and Charles and vice versa.
However, measurement of a qubit belonging to Alice (Charles) can affect the quantum properties of the qubit belonging to Charles (Alice) (obtaining e.g. EPR effects). The quantum analysis of full separability and biseparability of three (and more) qubits is therefore very important both from its theoretical interest and from the quantum computation applications.

In an old paper [1] we analyzed the use of Hilbert-Schmidt (HS) decompositions in relation to information theory. The outer products in the HS decompositions are of Pauli matrices where we relate the correlations of 2-qubits systems to one and two qubits measurements, and the correlations for 3-qubits systems to certain one, two and three-qubits measurements, etc. One advantage of such description is that it is valid for both pure and mixed states. In later works [2–4] we used such decompositions for treating separability and entanglement properties of qubits systems. The separability and entanglements properties of two qubits systems were treated by us in [2]. Our aim in the present article is to develop new methods for treating separability of three qubits (and more) which were not used in our previous works [3, 4]. In our previous work [3] we treated bi-separability problems for 3-qubits systems with maximally disordered subsystems (MDS i.e., density matrices which by tracing over any subsystem it gives the unit matrix [5]. While in the previous work [3] we obtained explicit expressions for bi-separability only for very special MDS cases, we succeed in the present work to give explicit biseparability expressions for any MDS state, including up to 27 parameters (see later in the article equation (44)).

In [4] we have treated several topics which are not treated in the present work (e.g. GHZ diagonal states, Braid states mixed with white noise, qubit and a qudit, etc). In [4] we developed also explicit full separability expressions for 3-qubits MDS states and noticed that such expressions are related to unitary transformation of the $l_1$ norm [6] condition $\sum_{a,b,c} |R_{a,b,c}| \leq 1$, where $R_{a,b,c}$ are the HS parameters of the MDS density matrices. This fact leads us to the idea in the present work that the analysis of three qubits (and more) MDS states becomes complicated due to its tensor expressions which cannot be diagonalized and therefore the methods of ‘unfolding of tensors into matrices’, which have been described in the literature [7–11], should be useful for treating MDS density matrices. We analyze in the present article various properties of the MDS density matrices and extend the analysis to the use of $l_1$ norm and to High Order Singular Values Decompositions (HOSVD). The HOSVD method has been developed in various mathematical works [7–11], but the use of such method to density matrices was not the concern of these authors. We find improvements in the condition for full separability for MDS density matrices by the use of this method. We demonstrate the use of our methods by examples. In the previous work [4] we treated also the separability problem for density matrices which are non-MDS including GHZ and W density matrices mixed with white noise with probability $p$. We found in this work [4] that while for the GHZ case $p \leq 1/5$ is a sufficient and necessary condition for separability the condition $p \leq 1/9$ can be used as a sufficient condition for separability of the W state mixed with white noise. We emphasized in this work [4]: ‘it is not obviously necessary and perhaps may be improved by other separability methods’. In the present work we improve these methods and obtain the condition for full separability of the W state mixed with white noise: $p \leq 1/5$ which is the same as that of GHZ mixed with white noise.

While concurrence is an efficient criterion for entanglement for two-qubits density matrices, in general it is not possible numerically to calculate the concurrence for more than two qubits, especially for the case of 27 MDS parameters (see, e.g., [12] where only lower bounds for quantification of entanglement by concurrence were found). Non-vanishing negativity [13] is a sufficient condition for inseparability of $\rho$ but not necessary. For 3-qubits MDS density matrices (and all such odd n-qubits) we prove in the paper that the eigenvalues of the partial-transpose (PT) are the same as those of $\rho$. Therefore the negativity for these states is zero so that we cannot conclude if it is separable or inseparable. Our work gives sufficient (not necessary) conditions for separability of such density matrices,

It is important to know if a density matrix is separable or entangled. Concurrence of 3-qubits density matrix is very difficult to calculate and it can be done only for special cases. The condition of Negativity $> 0$ is a sufficient condition for entanglement but if Negativity $= 0$ the result is not known. We treat this condition in a converse direction, giving a sufficient condition (not necessary) for separability. In a certain sense these two criterions complement each other. The proofs of our conditions look quite complicated but at the end, the results are quite simple: For getting the final conditions for separability one needs to calculate only the SVD decompositions of $3 \times 3$ and $3 \times 9$ matrices and such calculations are standard mathematical routine.

For MDS density matrices, by definition

\[
Tr_A \rho (A, B, C) = (I)_B \otimes (I)_C / 4.
\]  

(1)

Here the 3-qubits are denoted by $A, B, C$, $\rho$ is the density matrix, $I$ denotes the unit $2 \times 2$ matrix, $Tr_A$ represents the trace over qubit $A$ and $\otimes$ denotes the outer product. The density matrix of the 3 qubits system with MDS [4] is given as:
\[ 8 \rho = (I)_A \otimes (I)_B \otimes (I)_C \]
\[ + \sum_{a,b,c=1}^{3} R_{a,b,c}(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C \]
\[ R_{a,b,c} = \text{Tr}(\rho(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C) \]  
(2)

\( (\sigma_a)_A (a = 1, 2, 3) \) are the Pauli matrices for qubit A and similarly for qubits B and C. In the general 3-qubits MDS system 27 HS parameters are included in \( R_{a,b,c} \). Equation (2) represents a special case of the Hilbert-Schmidt (HS) decomposition of 3-qubits density matrices. Choosing different bases for the Pauli matrices of A, B, C, i.e., applying orthogonal transformations to \( (\sigma_1)_A \), \( (\sigma_2)_B \), \( (\sigma_3)_C \), will turn \( R_{a,b,c} \) into, say, \( S_{p,q,r} \).

The crucial point in treating the special case of 3-qubits MDS density matrices by the \( l_1 \) norm is that the condition for separability, given by:

\[ \sum_{a,b,c=1}^{3} |R_{a,b,c}| \leq 1, \]  
(3)

is not invariant under orthogonal transformations and we can improve the condition for separability by using orthogonal transformations which will reduce this ‘separability form’.

Compared to the separability problem of two qubits the separability problem for 3 qubits MDS density matrix becomes very complicated as tensors cannot be diagonalized. In order to overcome the non-diagonalization problem of tensors, methods of ‘unfolding of tensors into matrices’ have been described in the literature [7–11]. These unfolding methods help us in the analysis of 3-qubits MDS systems, but as we are interested in application of such methods to density matrices special unfolding methods for this purpose are developed in the present work. While conditions for entanglement of 3-qubits have been treated by various other authors (see e.g. [14–28]) they have not applied the unfolding methods. Also the other authors have not given explicitly separable forms for the density matrices, which are studied in the present work.

The general idea of getting explicit separability for n-qubits system is that we get a sufficient condition, but it might be not necessary as the condition for separability might be improved by using different methods. This method is different from those methods in which one gets the condition for entanglement [14–23, 25]. Both methods are important and in the present work we concentrate on finding explicit separability forms for n-qubits systems.

We analyze full separability properties of the 3-qubits MDS density matrices given by equation (2) by using unfolding of the tensor \( R_{a,b,c} \). One way of unfolding of the 3-qubits tensor \( R_{a,b,c}(a, b, c = 1, 2, 3) \) relative to A means that you keep the parameter \( a \) fixed as 1 or 2 or 3, (with the corresponding Pauli matrices \( (\sigma_1)_A \), \( (\sigma_2)_A \), or \( (\sigma_3)_A \)) and then the parameters \( R_{1,b,c} \), \( R_{2,b,c} \), or \( R_{3,b,c} \) are considered, respectively, as matrices with \( 3 \times 3 \) dimension. Then, by using the singular-value-decomposition (SVD) [7, 8] for the matrices: \( R_{1,b,c}, R_{2,b,c}, \) and \( R_{3,b,c} \), we get for each of them 3 singular values (SV’s). We show that if the sum of these 9 SV’s is not larger than 1, the density matrix is fully separable and we have an explicitly separable form for the density matrix. Similar unfolding methods can be made relative to \( B \) or \( C \).

In another method we develop fully separable forms for 3 qubits MDS density matrices which are related to Frobenius \( (l_2) \) norms [6] of 9 triads of HS parameters. We show that if the sum of the \( l_2 \) norms of the 9 HS triads is not larger than 1 then the 3-qubits MDS density matrix is fully separable, and we have another explicitly separable form for it.

For 3-qubits the density matrix may not be fully separable but may be bi-separable, i.e., not genuinely entangled [15]. A condition for bi-separability of 3-qubits MDS density matrices is obtained in the present work by the use of one qubit density matrix multiplied by Bell entangled states [2, 29, 30] of the other two qubits. By using this method the 27 HS parameters are divided into 9 triads which are different from those used for full separability. If the sum of the nine \( l_2 \) norms for these different triads is not larger than 1, then we conclude that the density matrix is (at least) bi-separable.

We apply the method of high order singular value decomposition (HOSVD) [7–11] for treatment of sufficient conditions for separability of 3-qubits MDS states. We demonstrate the application of this method in examples.

For the general 3-qubits density matrices, the HS decomposition includes 63 parameters which in a 4 dimensional notation may be written as \( R_{\mu,\nu}^{(i)} \) \( \mu, \nu, \kappa = 0, 1, 2, 3 \). Such terms include products of Pauli matrices \( \sigma_i (i = 1, 2, 3) \), and the unit operator \( \sigma_0 = 1 \). In various actual cases some of these parameters vanish. A sufficient condition for full separability is given by [4]:

\[ 8 \rho = (I)_A \otimes (I)_B \otimes (I)_C \]
\[ + \sum_{a,b,c=1}^{3} R_{a,b,c}(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C \]
\[ R_{a,b,c} = \text{Tr}(\rho(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C) \]  
(2)
\[
\left( \sum_{\rho_{i,j,k}:i,j,k=0}^{3} |R_{\rho_{i,j,k}}| \right) \leq 1; \quad R_{0,0,0} = 1,
\]

(4)

but this condition may be improved. We demonstrate improvements in the condition for full separability by analyzing the system of GHZ state mixed with white noise and of W state mixed with white noise and get better conditions for full separability.

2. Fundamental properties of 3-qubits MDS density matrices

We try to use the Peres-Horodecki (PH) criterion \[31, 32\] to obtain information about the eigenvalues of the MDS density matrix. We show now that the 3-qubits MDS density matrix and its PT have the same eigenvalues. For showing this result we write the 3-qubits MDS density matrix as

\[
\rho = R + \sigma_y R^T.
\]

(8.5)

Here \(R\), given in a short notation, includes all the terms in the summation of equation (2). By performing the full transpose of \(\rho\) into \(\rho^T\), every \(\sigma_y\) in (2) is transformed to \(-\sigma_y\). This transformation does not change the eigenvalues (\(\rho\) and \(\rho^T\) have the same eigenvalues). By a \(180^\circ\) unitary rotation of all qubits around the \(y\) axis the eigenvalues of the density matrix are not changed, but \(\sigma_x \rightarrow -\sigma_x\), \(\sigma_z \rightarrow -\sigma_z\). We denote the resulting density matrix by \(\rho^{TU}\). Here the superscript \(TU\) represents transpose of (2), i.e. of the whole density matrix, plus a unitary transformation. We emphasize that \(\rho^{TU}\) and \(\rho\) have the same eigenvalues. However, since we assumed an odd number of \(\sigma\) we get \(R \rightarrow -R\) by the TU transformation. Hence

\[
8\rho^{TU} = (I) \otimes (I) \otimes I - R.
\]

(6)

On the other hand, the partial transpose plus a \(180^\circ\) rotation around \(y\) for one qubit (say qubit A) also yields

\[
8\rho(PTU; A) = (I) \otimes (I) \otimes I - R.
\]

(7)

We find therefore that \(\rho(PTU; A)\) has the same eigenvalues as \(\rho\). This proof can easily be generalized for any odd number of qubits with MDS, where the eigenvalues of \(\rho\) are equal to the eigenvalues of its PT transformation so that for such systems the PH criterion does not give information about entanglement.

A further conclusion comes from the fact (using the same argument) that for odd \(n\) MDS density matrices the eigenvalues of \((I)^{\otimes n} + R\) are the same as those of \((I)^{\otimes n} - R\); it follows that the eigenvalues of \(\rho\) can be written as

\[
\text{Eigenvalues}(\rho) = \frac{1 \pm \tau_i}{2^n} \quad (|\tau_i| \leq 1).
\]

(8)

The eigenvalues of \(R\) come in pairs \(\pm \tau_i\) for any odd \(n\) MDS density matrix (including the 3-qubits as a special case for \(n = 3\)) and are bounded by \(\frac{1}{2^n}\).

The separability problem can also be related to Frobenius \((l_2)\) norms which are given by the square root of sums of squared HS parameters \[7, 8\]. Let us prove the following relation for a 3 qubits MDS density matrix:

\[
\sum_{a,b,c=1}^{3} R_{a,b,c}^2 \leq 1.
\]

(9)

We note first that

\[
\text{Tr}[(8\rho)^2] = 8 + 8 \sum_{a,b,c=1}^{3} R_{a,b,c}^2.
\]

(10)

On the other hand

\[
\text{Tr}[(8\rho)^2] = 64 \sum_{i=1}^{8} \lambda_i^2.
\]

(11)

Here, \(\lambda_i\) are the 8 eigenvalues of \(\rho\). Since \(0 \leq \lambda_i \leq \frac{1}{4}\) (recalling that the 8 \(\tau_i\) come in 4 pairs \(\pm |\tau_i|\), as given by (8)) we write:

\[
\lambda_i = \frac{1}{8} + q_i; \quad q_i = \frac{\tau_i}{8}; \quad |q_i| \leq \frac{1}{8}; \quad \sum_{i=1}^{8} q_i = 0.
\]

(12)
Hence

\[
\sum_{i=1}^{8} \lambda_i^2 = \sum_{i=1}^{8} \left( \frac{1}{8} + q_i \right)^2 \geq \frac{8}{64} \left( \frac{1}{4} \right).
\]

\[+ \sum_{i=1}^{8} (q_i^2) \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\]

(13)

By using (10–13), we get (9). Equation (9) may be generalized to any MDS density matrix with odd-\(n\). Note that the equality in (9) holds only if \(|n| = 1\), i.e.

\[\lambda_i = \frac{1}{4} \ (i = 1, 2, 3, 4)\; \quad \lambda_j = 0 \ (j = 5, 6, 7, 8).
\]

(14)

According to (9), a necessary condition for (2) to be a density matrix is that the Frobenius norm of the sum of the 27 parameters, represented by the left side of (9), should not be larger than 1.

3. Unfolding of the 3-qubits MDS tensor \(R_{a,b,c}\) into matrices

In this section we describe unfolding processes by which tensors are unfolded into matrices. Such processes have been described in the literature [7–11] but the use of such unfolding processes becomes different in the present paper as we relate the analysis to density matrices, which was not the concern of the other works [7–11].

A certain unfolding of a tensor \(R_{a,b,c}\) with dimension \(n_1 \times n_2 \times n_3\) is obtained by assembling the \(R\)’s entries into a matrix with dimension \(N_1 \times N_2 = n_1 \times n_2 \times n_1\) [7–11]. We denote the unfolded matrix of \(R_{a,b,c}\), relative to qubit A, as \(R_{(b,c)}(a) = R_{(1)},\) and the matrix \(R_{(1)}\) is arranged so that \(R_{1,b,c}\) are inserted in the first row, \(R_{2,b,c}\) are inserted in the second row and \(R_{3,b,c}\) in the third row. The indices \((b,c) = 1, 1; 1, 2; 1, 3; 2, 1; 2, 2; 2, 3; 3, 1; 3, 2; 3, 3\) are inserted into the 1, 2, …, 9 columns, respectively. By using this unfolding process \(R_{a,b,c}\) is unfolded into:

\[
R_{(1)} = R_{(a)(b,c)} = \begin{cases}
R_{111}, R_{112}, R_{113}, R_{121}, R_{122}, R_{123}, R_{131}, R_{132}, R_{133} \\
R_{211}, R_{212}, R_{213}, R_{221}, R_{222}, R_{223}, R_{231}, R_{232}, R_{233} \\
R_{311}, R_{312}, R_{313}, R_{321}, R_{322}, R_{323}, R_{331}, R_{332}, R_{333}
\end{cases}.
\]

(15)

One should note that \(R_{(a)(b,c)}\) is a \(3 \times 9\) matrix while the matrices \(R_{(a)(b,c)}\) (without the brackets around \((b,c)\)) related to the discussions about Pauli matrices in the next section, represent \(3 \times 3\) matrices. The elements of the \(3 \times 3\) matrix \(R_{(a)(b,c)}\) are composed of the \(a\)’th row elements of \(R_{(a)(b,c)}\).

The tensor \(R_{a,b,c}\) can be unfolded relative to qubit B by exchanging \(R_{(a)(b,c)}\) into \(R_{(b,c)}(a)\) so that for \((b) = 1, 2, 3\) the entries \(R_{a,b,c}\) are inserted in the first, second and third row, respectively, and the entries related to \((a,c) = 1, 1; 1, 2; 1, 3; 2, 1; 2, 2; 2, 3; 3, 1; 3, 2; 3, 3\) are inserted into the 1, 2, …, 9 columns, respectively. In a similar way the tensor \(R_{a,b,c}\) can be unfolded relative to qubit C [7–11].

4. Explicitly separable forms for 3-qubits MDS density matrices related to the \(l_1\) norm

A fully separable-like form for the density matrix (2) related to the \(l_1\) norm can be given as:

\[
8 \rho_{ABC} = (1/4) \sum_{a,b,c=1}^{3} \left| R_{a,b,c} \right|^2 \left\{ \left[ (I)_A + (\sigma_a)_A \right] \otimes \left[ (I)_B - (\sigma_b)_B \right] \otimes \left[ (I)_C + \text{sign}(R_{a,b,c}(\sigma_c)) \right] \right\} + \left\{ \left[ (I)_A + (\sigma_a)_A \right] \otimes \left[ (I)_B + (\sigma_b)_B \right] \otimes \left[ (I)_C + \text{sign}(R_{a,b,c}(\sigma_c)) \right] \right\} + \left\{ \left[ (I)_A - (\sigma_a)_A \right] \otimes \left[ (I)_B - (\sigma_b)_B \right] \otimes \left[ (I)_C + \text{sign}(R_{a,b,c}(\sigma_c)) \right] \right\} + \left\{ \left[ (I)_A - (\sigma_a)_A \right] \otimes \left[ (I)_B + (\sigma_b)_B \right] \otimes \left[ (I)_C - \text{sign}(R_{a,b,c}(\sigma_c)) \right] \right\}.
\]

(16)

Here

\[
\text{sign}(R_{a,b,c}) = \begin{cases}
+1 & \text{for } R_{a,b,c} > 0 \\
-1 & \text{for } R_{a,b,c} < 0
\end{cases}.
\]

(17)

Each expression in the curly brackets of (16) represents a pure state density matrix multiplied by 2. We get according to (16) that a sufficient condition for full separability is given by the relation (3) given in the introduction, i.e. \(\sum_{a,b,c=1}^{3} \left| R_{a,b,c} \right| \leq 1\). This seems the simplest sufficient condition for full separability but it is
not necessary and may be improved. However before that we would like to explain how we obtain (16) and how it can be generalized to any n-qubit MDS system.

For each \( R_{a,b,c} \) in (16) expanding the products in the curly brackets which multiply \( |R_{a,b,c}\rangle \), one is left with the product of the unit operators and the relevant MDS terms (all other terms obtained in this expansion cancel out). A similar construction can be made for any number of qubits, say \( n \). The number of products in the curly bracket will be \( 2^{n-1} \). We demonstrate the construction for 3-qubits \( A, B, C, D, E \) by starting with the product:

\[
\left\{ \{I\}_A + (\sigma_a)_A \} \otimes \{I\}_B + (\sigma_b)_B \} \otimes \{I\}_C + (\sigma_c)_C \}
\times \otimes \{I\}_D + (\sigma_d)_D \} \otimes \{I\}_E + \text{sign}(R_{\text{abcde}})(\sigma_e)_E \}
\]

(18)

Multiply (18) by \( |R_{a,b,c,d,e}\rangle \) and then by expanding (18) we get the products of the \( \sigma \)'s corresponding to the MDS term of \( |R_{a,b,c,d,e}\rangle \), but we get also products which are not MDS (in addition to a product of the unit matrices multiplied by \( |R_{a,b,c,d,e}\rangle \)). In order to avoid the wrong terms, change an even number of pluses into minuses for all products in (18), e.g.

\[
\left\{ \{I\}_A - (\sigma_a)_A \} \otimes \{I\}_B - (\sigma_b)_B \} \otimes \{I\}_C + (\sigma_c)_C \}
\times \otimes \{I\}_D + (\sigma_d)_D \} \otimes \{I\}_E + \text{sign}(R_{\text{abcde}})(\sigma_e)_E \}
\]

(19)

and add all the terms in analogy to (16).

One should notice that in such expansion we get 16 rows instead of the 4 rows used in (16) and the changes in normalization conditions should be taken into account: \( 8\rho_{ABC} \rightarrow 32\rho_{ABCDE} \); \( \frac{1}{4} \rightarrow \frac{1}{16} \). The condition for full separability of (3) is changed to

\[
\left( \sum_{a,b,c,d,e=1}^{3} |R_{a,b,c,d,e}\rangle \right) \leq 1
\]

(20)

The 3-qubits products of equation (3) can be written (say relative to qubit \( A \)) as

\[
\sum_{a,b,c=1}^{3} R_{a,b,c}(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C
\]

\[
= \sum_{a=1}^{3} (\sigma_a)_A \otimes \sum_{b,c=1}^{3} R^{(a)}_{b,c} (\sigma_b)_B \otimes (\sigma_c)_C
\]

(21)

Notice that here \( R^{(a)}_{b,c} (a = 1, 2, 3) \) denotes a \( 3 \times 3 \) matrix. We note that the matrices \( R^{(a)}_{b,c} \) are used here for the purpose of analyzing separability properties of the density matrix (2). We use now transformations which reduce the 27 HS parameters, \( R_{a,b,c} \), to 9 parameters with smaller \( l_1 \) norm. Performing the SVD [7, 8] on the matrices \( R^{(a)}_{b,c} \) in (21) we get:

\[
\sum_{a=1}^{3} (\sigma_a)_A \otimes \sum_{b,c=1}^{3} R^{(a)}_{b,c} (\sigma_b)_B \otimes (\sigma_c)_C
\]

\[
= \sum_{a=1}^{3} (\sigma_a)_A \otimes \sum_{i=1}^{3} R^{(a)}_{i} (\sigma_i)^{a}_A \otimes (\sigma_i)^{a}_C.
\]

(22)

Here we used the SVD relation

\[
\sum_{b,c} U^{(a)}_{b,i} R^{(a)}_{b,c} V^{(a)}_{c,i} = \delta_{i,b} R^{(a)}_{b,c},
\]

(23)

where \( R^{(a)}_{b,c} \) are the SV of \( R^{(a)}_{b,c} \), \( U^{(a)}_{b,i} \) and \( V^{(a)}_{c,i} \) are \( 3 \times 3 \) real orthogonal matrices, and \( (\sigma_i)^{a}_A = \sum_{b=1}^{3} U^{(a)}_{b,i} (\sigma_b)_B \), etc. Taking absolute values in (23) we get

\[
|R^{(a)}_{b,c}| \leq \sum_{i=1}^{3} |U^{(a)}_{i,b}||V^{(a)}_{i,c}||R^{(a)}_{i,b,c}|
\]

(24)

Performing the summation over \( i \) we get

\[
\sum_{i=1}^{3} |R^{(a)}_{b,c}| \leq \sum_{b,c=1}^{3} \sum_{i=1}^{3} |U^{(a)}_{i,b}||R^{(a)}_{b,c}||V^{(a)}_{i,c}|
\]

(25)
\[ |U^{(i)}_{a,b}|, \text{ for a certain } b, \text{ and } |V^{(a)}_{c,d}|, \text{ for a certain } c, \text{ are unit vectors so that we get} \]
\[ \sum_{i=1}^{3} |U^{(i)}_{a,b}| |V^{(a)}_{c,d}| \leq 1. \]  \hspace{1cm} (26)

Substituting (26) into (25) we get the relation
\[ \sum_{i=1}^{3} |R^{(a)}_{i}| \leq \sum_{b,c=1}^{3} |R^{(a)}_{b,c}|. \]  \hspace{1cm} (27)

We find that the sum of the SV's absolute values \( \sum_{i=1}^{3} |R^{(a)}_{i}| \) is smaller or equal to the \( l_1 \) norm of the matrix \( R^{(a)}_{b,c} \). Since usually (26) is a strict inequality we expect a corresponding improvement in the sufficient condition.

Using the right hand side of (22) in (21) and using the general criterion (3) we find that under the condition (relative to A),
\[ \sum_{i,a=1}^{3} |R^{(a)}_{i}| \leq 1, \]  \hspace{1cm} (28)

an explicitly fully separable form for the 3-qubits MDS density matrix is obtained. In a similar way by using this procedure relative to B or C one gets, respectively,
\[ \sum_{i,b=1}^{3} |R^{(b)}_{i}| \leq 1; \quad \sum_{i,c=1}^{3} |R^{(c)}_{i}| \leq 1. \]  \hspace{1cm} (29)

One can choose the optimal condition for explicit full separability from the three conditions given by (28) and (29).

We demonstrate the present method, for improving the condition for separability by decreasing the \( l_1 \) norm, in the following example:

**Example 1: MDS density matrix with random 27 \( R_{a,b,c} \) parameters**

We have chosen, at random, 27 \( R_{a,b,c} \) parameters which are inserted in the unfolded matrix \( R_{(1)} = R^{(a)}_{(b,c)} \) of (15) as:
\[
R_{(1)} = \begin{pmatrix}
0.0607, & 0.012, & -0.0369, & 0.0216, & 0.0697, & 0.0952, & 0.0912, & -0.0323, & 0.0344, \\
0.0892, & 0.0489, & 0.0643, & 0.0377, & -0.0451, & 0.0433, & -0.0632, & 0.0381, & -0.0675, \\
0.0415, & 0.0305, & 0.0438, & 0.0425, & 0.0322, & 0.0671, & 0.0283, & 0.0514, & -0, & 0673
\end{pmatrix}
\]  \hspace{1cm} (30)

By substituting these values in the density matrix (2) and performing all multiplications of Pauli matrices we arrive at the density matrix which has the eigenvalues:
\[ \begin{align*}
\lambda_1 &= 0.553179; \\
\lambda_2 &= 1.446821; \\
\lambda_3 &= 1.353291; \\
\lambda_4 &= 0.646709; \\
\lambda_5 &= 1.121965; \\
\lambda_6 &= 0.878035; \\
\lambda_7 &= 1.062880; \\
\lambda_8 &= 0.937120.
\end{align*} \]  \hspace{1cm} (31)

We get 4 pairs of eigenvalues, with the relations:
\[ \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = \lambda_5 + \lambda_6 = \lambda_7 + \lambda_8 = 1/4. \]  \hspace{1cm} (32)

This result is in agreement with (8).

As a sufficient condition (3) for separability, we get for the \( R_{a,b,c} \) parameters of (30):
\[ \left( \sum_{a,b,c=1}^{3} |R_{a,b,c}| \right) = 1.356 > 1. \]  \hspace{1cm} (33)

This simple criterion fails to show full separability, but by using the SVD for the matrices \( R^{(a)}_{b,c} \) as developed in (22–28) we can use the relation: \( \sum_{i,a=1}^{3} |R^{(a)}_{i}| \leq 1 \) as a sufficient condition for full separability. For this purpose we calculate the SV’s, \( R^{(a)}_{i}(a, i = 1, 2, 3) \) of the matrices \( R^{(a)}_{b,c} \) and then we get:
\[ \sum_{i,a=1}^{3} |R^{(a)}_{i}| = 0.7199 < 1. \]  \hspace{1cm} (34)
Here the value 0.7199 has been obtained by the sum of the 9 SV's of $R^{(a)}_{i}$ $(i = 1, 2, 3)$. While in the above analysis the 27 parameters $R_{a,b,c}$ were related to the 3 matrices $R^{(a)}_{b,c}$ $(a = 1, 2, 3)$ similar analysis can be made if they will be related to $R^{(b)}_{a,c}$ or $R^{(c)}_{a,b}$. One can choose the optimal condition for explicit full separability from these 3 possibilities.

5. Full separability for 3-qubits MDS density matrices related to the $l_2$ norm

A fully separable-like form for the density matrix (2) related to $l_2$ norms can be given as:

$$8 \rho_{AbC} = \frac{1}{4} \sum_{a,b=1}^{3} \sum_{c=1}^{3} R_{a,b,c} \left[ \mathbb{I} + \mathbb{I}_2 \right] + \left\{ \left[ (I)_A + (\sigma_a)_A \right] \otimes \left[ (I)_B + (\sigma_b)_B \right] \right\} \otimes \left\{ (I)_C + \left( \frac{(R_{a,b,1}(\sigma_a) + R_{a,b,2}(\sigma_b) + R_{a,b,3}(\sigma_c))}{\sqrt{R_{a,b,1}^2 + R_{a,b,2}^2 + R_{a,b,3}^2}} \right) \right\} \right. \\
\left. \times \left\{ \left[ (I)_A - (\sigma_a)_A \right] \otimes \left[ (I)_B - (\sigma_b)_B \right] \right\} \otimes \left\{ (I)_C - \left( \frac{(R_{a,b,1}(\sigma_a) + R_{a,b,2}(\sigma_b) + R_{a,b,3}(\sigma_c))}{\sqrt{R_{a,b,1}^2 + R_{a,b,2}^2 + R_{a,b,3}^2}} \right) \right\} \right\}. \tag{35}$$

Each term in the curly brackets of (35) represents a pure state density matrix multiplied by 2. It is straightforward to show that the complicated separable form of (35) is reduced to the density matrix (2) by manipulating all the cross products in this equation. According to (35) a sufficient condition for full separability is given by

$$\sum_{a,b=1}^{3} \left( \sum_{c=1}^{3} R_{a,b,c} \right)^{1/2} \leq 1. \tag{36}$$

In (36), $\left( \sum_{c=1}^{3} R_{a,b,c} \right)^{1/2}$ for certain $a, b$ values, represents the $l_2$ norm of a triad of HS parameters $(c = 1, 2, 3)$. If the sum of the $l_2$ norms over 9 triads of HS parameters $(a, b = 1, 2, 3)$ is not larger than 1 we can conclude that the density matrix (2) is fully separable and we have the explicit form (35) for its full separability. While the explicit form (16) of the density matrix (2) gives the sufficient condition for full separability

$$\left( \sum_{a,b,c=1}^{3} |R_{a,b,c}| \right) \leq 1,$$

related to the $l_1$ norm, the explicit form (35) of the density matrix (2) gives the sufficient condition for full separability related to the $l_2$ norm by (36) where the left side of this equation includes sum of 9 $l_2$ norms of triads. Here again the use of (36) can be generalized to any n-qubit system. Using the criterion (36) for full separability we get for the above example (represented by the matrix $R_{1(1)}$ of (30)):

$$\sum_{a,b=1}^{3} \left( \sum_{c=1}^{3} R_{a,b,c} \right)^{1/2} = 0.829456 < 1. \tag{37}$$

We find that the density matrix in the above example can be presented by the explicitly separable form of equation (35).

6. Bi-separability for 3-qubits MDS density matrices

In our previous work [2] we treated bi-separability of 3-qubits MDS density matrix by using one qubit density matrix multiplied by entangled Bell states [2, 29, 30], of the other two qubits, but we treated only very special cases. In the present section we generalize the analysis to any 3-qubits MDS density matrix including up to 27 HS parameters.

Let us show the bi-separability obtained for the following simple 3-qubits MDS density matrix:

$$8 \rho_1 = (I)_A \otimes (I)_B \otimes (I)_C \tag{38}$$

$$+ R_{111}(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C \tag{38}$$

$$+ R_{222}(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C \tag{38}$$

$$+ R_{333}(\sigma_a)_A \otimes (\sigma_b)_B \otimes (\sigma_c)_C \tag{38}$$
The 8 eigenvalues of this density matrix are given by

\[
\begin{align*}
\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = (1/8) \left[ 1 + \sqrt{\sum_{i=1}^{3} R_{ii}^2} \right]; \\
\lambda_5 &= \lambda_6 = \lambda_7 = \lambda_8 = (1/8) \left[ 1 - \sqrt{\sum_{i=1}^{3} R_{ii}^2} \right].
\end{align*}
\]

(39)

Hence it is a density matrix when \( R_{ii} \) are within the unit sphere, i.e. when

\[
\sum_{i=1}^{3} R_{ii}^2 \leq 1.2.
\]

(40)

Explicit bi-separable expression for the density matrix (38) is given by:

\[
\rho_1 = \begin{cases}
(I_{A} - R_{111}(\sigma_x)_A + R_{222}(\sigma_y)_A + R_{333}(\sigma_z)_A) \otimes |\Phi^{(-)}\rangle_{BC}\langle\Phi^{(-)}|_{BC} & \\
+ [I_{A} + R_{111}(\sigma_x)_A - R_{222}(\sigma_y)_A + R_{333}(\sigma_z)_A] \otimes |\Phi^{(+)}\rangle_{BC}\langle\Phi^{(+)}|_{BC} & \\
+ [I_{A} + R_{111}(\sigma_x)_A + R_{222}(\sigma_y)_A - R_{333}(\sigma_z)_A] \otimes |\Psi^{(+)}\rangle_{BC}\langle\Psi^{(+)}|_{BC} & \\
+ [I_{A} - R_{111}(\sigma_x)_A - R_{222}(\sigma_y)_A - R_{333}(\sigma_z)_A] \otimes |\Psi^{(-)}\rangle_{BC}\langle\Psi^{(-)}|_{BC} &
\end{cases}
\]

(41)

Here \( |\Phi^{(-)}\rangle_{BC}, |\Phi^{(+)}\rangle_{BC} \) and \( |\Psi^{(+)}\rangle_{BC} \) are the Bell states \([29, 30]\) of the qubits pair \( B \) and \( C \) expanded in terms of Pauli matrices as:

\[
\begin{align*}
4 |\Phi^{(-)}\rangle_{BC}\langle\Phi^{(-)}|_{BC} &= \begin{bmatrix}
(I_B \otimes (I_C - (\sigma_x)_B \otimes (\sigma_x)_C) \\
+ (\sigma_y)_B \otimes (\sigma_y)_C \\
+ (\sigma_z)_B \otimes (\sigma_z)_C
\end{bmatrix}; \\
4 |\Phi^{(+)}\rangle_{BC}\langle\Phi^{(+)}|_{BC} &= \begin{bmatrix}
(I_B \otimes (I_C + (\sigma_x)_B \otimes (\sigma_x)_C) \\
- (\sigma_y)_B \otimes (\sigma_y)_C \\
+ (\sigma_z)_B \otimes (\sigma_z)_C
\end{bmatrix}; \\
4 |\Psi^{(+)}\rangle_{BC}\langle\Psi^{(+)}|_{BC} &= \begin{bmatrix}
(I_B \otimes (I_C + (\sigma_x)_B \otimes (\sigma_x)_C) \\
+ (\sigma_y)_B \otimes (\sigma_y)_C \\
- (\sigma_z)_B \otimes (\sigma_z)_C
\end{bmatrix}; \\
4 |\Psi^{(-)}\rangle_{BC}\langle\Psi^{(-)}|_{BC} &= \begin{bmatrix}
(I_B \otimes (I_C - (\sigma_x)_B \otimes (\sigma_x)_C) \\
- (\sigma_y)_B \otimes (\sigma_y)_C \\
- (\sigma_z)_B \otimes (\sigma_z)_C
\end{bmatrix}.
\end{align*}
\]

(42)

Equation (41), is a bi-separable density matrix, under the condition

\[
\sqrt{R_{111}^2 + R_{222}^2 + R_{333}^2} \leq 1,
\]

(43)

which is equivalent to the condition (40) for \( \rho \) of equation (38) to be a density matrix.

To treat bi-separability of the general case of MDS density matrix given by equation (2) (up to 27 MDS terms), we can divide \( \sum_{a,b,c=1}^{3} R_{ab,c}(\sigma_a)_b \otimes (\sigma_b)_c \) into 9 groups of triads. Starting with (38) and (41) we can apply 8 transformations to the Pauli matrices of each qubit, obtaining 8 triads with corresponding HS parameters. Each triad may be treated as in (41, 42). Together with (41) they include the 27 .. parameters. The relevant triads are indicated in equation (44) below. Therefore the sufficient condition for bi-separability of equation (2) becomes that the sum of the Frobenius norms of the 9 (at most) triads of MDS-parameters is not larger than 1. Such condition is sufficient for bi-separability but the sufficient condition for bi-separability may perhaps be improved by other methods.

The final conclusion from this analysis is that a sufficient condition for bi-separability of a 3-qubits MDS density matrix is that the sum of \( l_2 \) norms over 9 triads of HS parameters is not larger than 1, i.e.,
\[
\sqrt{R_{11}^2 + R_{22}^2 + R_{33}^2} + \sqrt{R_{12}^2 + R_{31}^2 + R_{23}^2} \\
+ \sqrt{R_{13}^2 + R_{23}^2 + R_{31}^2} + \sqrt{R_{12}^2 + R_{32}^2 + R_{23}^2} \\
+ \sqrt{R_{13}^2 + R_{21}^2 + R_{33}^2} + \sqrt{R_{12}^2 + R_{32}^2 + R_{21}^2} \\
+ \sqrt{R_{13}^2 + R_{21}^2 + R_{32}^2} + \sqrt{R_{12}^2 + R_{31}^2 + R_{23}^2} \\
+ \sqrt{R_{11}^2 + R_{22}^2 + R_{33}^2} \leq 1.
\]

By substituting the 27 HS parameters of (30) in (44) we get the condition for bi-separability:

\[0.7042 < 1,\]

which is better than the conditions for full separability: 0.7199 < 1 (in (34) and 0.8295 < 1 (in (37)). By increasing the HS parameters we can arrive at a state where the condition for bi-separability is satisfied but not the conditions for full separability.

7. Full separability of 3-qubits MDS density matrices improved by the use of the high order singular value decomposition (HOSVD)

It is interesting to see how the present methods can be extended by relating them to the method of high order singular value decomposition (HOSVD) [7–11].

In the HOSVD method we use the SVD for the matrices \(R_{(1)}\), \(R_{(2)}\), and \(R_{(3)}\) (given, respectively, by (15) for \(R_{(1)}\) and similar expressions for \(R_{(2)}\) and \(R_{(3)}\):

\[
R_{(1)} = U_1 \Sigma_1 V_1^T; \quad R_{(2)} = U_2 \Sigma_2 V_2^T; \quad R_{(3)} = U_3 \Sigma_3 V_3^T.
\]

Here, the subscripts 1, 2, 3 refer to the unfolded matrices: \(R_{(1)}\), \(R_{(2)}\), \(R_{(3)}\), respectively. The orthogonal matrices \(U_1\), \(U_2\), \(U_3\) are of \(3 \times 9\) dimension. The singular matrices \(\Sigma_1\), \(\Sigma_2\), \(\Sigma_3\) are of \(3 \times 9\) dimension and the matrices \(V_1\), \(V_2\), \(V_3\) are of \(9 \times 9\) dimension. The matrices \(\Sigma_i\) are given by

\[
\Sigma_i = \begin{pmatrix}
s_1(R_{(i)}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_2(R_{(i)}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_3(R_{(i)}) & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix},
\]

\(s_1(R_{(i)})\), \(s_2(R_{(i)})\), \(s_3(R_{(i)})\) are the three singular values of \(R_{(i)}\) \((i = 1, 2, 3)\). The tensor \(R_{a,b,c}\) is related to the core tensor \(S\) by the transformation [7–11],

\[
S_{p,q,r} = \sum_{a=1}^{3} \sum_{b=1}^{3} \sum_{c=1}^{3} R_{a,b,c} U_1(a,p)^{T} U_2(b,q)^{T} U_3(c,q)^{T}.
\]

While direct calculation of the core tensor \(S\) (which includes the above Tucker products [7–11]) is quite complicated, for our purpose of calculating sufficient conditions for full separability, it is enough to calculate \(S_{(1)}\) which is the unfolding of the core tensor \(S\) relative to qubit \(A\), and is given by [7–11]

\[
S_{(1)} \equiv S^{(p)}_{(q,r)} = \sum_{1} V_1(U_3 \otimes U_2).
\]

The unfolded matrix \(S_{(1)}\) has various special properties [7–11] including orthogonality between its rows. Also

\[
\sum_{q,r=1}^{3} (S^{(p)}_{(q,r)})^2 \quad (p = 1, 2, 3) \text{ are equal, respectively, to the singular values, } s_1(R_{(1)}), s_2(R_{(1)}), s_3(R_{(1)}), \text{ of } \Sigma_1.
\]

Applying the orthogonal transformations, \(U_1^T(1)\), \(U_2^T(2)\), \(U_3^T(3)\) to \(\tilde{\sigma}_a\), \(\tilde{\sigma}_b\), \(\tilde{\sigma}_c\), respectively, means choosing new bases for the Pauli matrices: \(\tilde{\sigma}_a\), \(\tilde{\sigma}_b\), \(\tilde{\sigma}_c\). In terms of these, equation (2) becomes

\[
\delta\rho = (I)_A \otimes (I)_B \otimes (I)_C \\
+ \sum_{p,q,r=1}^{3} S_{p,q,r} (\tilde{\sigma}_a)_A \otimes (\tilde{\sigma}_b)_B \otimes (\tilde{\sigma}_c)_C.
\]

All the various previous formulas may be written in terms of \(S_{p,q,r}\) instead of \(R_{a,b,c}\).

We demonstrate now the use of the HOSVD method, for improving the condition for full separability, in the following examples.

Example 2: MDS density matrix with 27 equal HS parameters

For this case \(R_{a,b,c} = \alpha\) and the simplest condition for full separability (3) yields: \(\alpha \leq 1/27\). The simplest condition for full separability according to the \(l_2\) norm (36)) yields: \(\alpha \leq 1/9\sqrt{3}\).

This is also the condition for bi-separability (44).
The unfolding of the 3-qubits density matrix relative to qubit $A$ can be written as

$$8\rho = (I)_A \otimes (I)_B \otimes (I)_C,$$

$$+ \sum_{a=1}^{3} (\sigma_a)_A \otimes \sum_{b,c=1}^{3} R_{b,c}^{(a)} (\sigma_b)_B \otimes (\sigma_c)_C;$$

$$R_{b,c}^{(a)} \equiv R_{a,b,c} = \alpha. \quad (51)$$

Then the 3 matrices $R_{b,c}^{(a)}$ are given by

$$R_{b,c}^{(a)} = \left( \begin{array}{lll} \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{array} \right); \quad a = 1, 2, 3. \quad (52)$$

By calculating the SV's, of $R_{b,c}^{(a)}$ of (52), we get

$$R_1^{(a)} = 3\alpha; \quad R_2^{(a)} = R_3^{(a)} = 0; \quad a = 1, 2, 3. \quad (53)$$

Then we have

$$\sum_{a=1}^{3} \sum_{b,c=1}^{3} R_{b,c}^{(a)} = 9\alpha. \quad (54)$$

Therefore a sufficient condition for full separability is now given by

$$\alpha \leq 1/9. \quad (55)$$

The eigenvalues of the density matrix $\rho$ in this example are given by:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1 + 3\sqrt{3} \alpha}{8};$$

$$\lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \frac{1 - 3\sqrt{3} \alpha}{8}. \quad (56)$$

We have a density matrix under the condition: $3\sqrt{3} \ |\alpha| \leq 1$. In the region: $1/9 < \alpha \leq 1/3\sqrt{3}$ (55) for full separability does not hold. We will show now that this condition can be greatly improved by the use of HOSVD, so that in the whole region that we have a density matrix it is fully separable.

The unfolded matrices: $R_{(1)}$ of (15) (and equal ones for $R_{(2)}$, $R_{(3)}$) are given in the present example by

$$R_{(1)} = R_{(2)} = R_{(3)} = \left( \begin{array}{lllllllll} \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \end{array} \right). \quad (57)$$

The high order transformed matrix $S_{(1)}$ is calculated by equation (49) where $\sum V_i$ and $V_i$ are obtained by the SVD of $R_{(1)}$, $U_3$ and $U_2$ are calculated by the SVD of $R_{(3)}$ and $R_{(2)}$, respectively.

After straightforward calculations we get for this example

$$S_{(1)} = S_{(2)} = S_{(3)} = \left( \begin{array}{llllllllll} 3\sqrt{3} \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (58)$$

So, the condition for full separability is $3\sqrt{3} \ |\alpha| < 1$, which is equivalent to the condition for the present example to be a density matrix.

**Example 3: Three-qubits MDS density matrices with 27 HS parameters**

Let us assume that we have 3-qubits MDS density matrix with the following unfolding matrix $R_{(1)}$ of equation (15) relative to qubit $A$:

$$R_{(1)} = R_{(a,b,c)} = \left( \begin{array}{llllllllll} 0.08923, 0.01764, -0.05057, \\ 0.03175, 0.10246, 0.13994, 0.13406, \\ -0.04753, 0.05057 \\ 0.13112, 0.07203, 0.09452, 0.05542, \\ -0.06615, 0.06351, \\ -0.09291, 0.05601, -0.09923 \\ 0.06101, 0.04484, 0.06413, 0.06248, \\ 0.04733, 0.09863, 0.04165, 0.07556, -0.09893 \end{array} \right). \quad (59)$$

We would like to show that the sufficient condition for full separability is improved by using the HOSVD, following the relation $S_{(1)} = \sum V_i (U_3 \otimes U_2)$ by which we change $R_{(1)}$ into $S_{(1)}$, $\sum V_i$ are calculated by the SVD of the unfolded matrix $R_{(1)}$ while $U_2$ and $U_3$ are calculated by the SVD of the unfolded matrices $R_{(2)}$ and $R_{(3)}$, respectively. A straightforward calculation gives
One should notice that the $p^{th}$ row ($p = 1, 2, 3$) of equation (60) includes 9 terms where each 3 of them are inserted in the first, second and third row of the matrix $S^{(p)}_{g,r}$ respectively. By calculating the singular values of (60) we get: Sum of singular values $= 0.997 < 1$ so that the condition of HOSVD for full separability is satisfied. On the other hand we get:

$$\sum_{i=1}^{3} |R_{0,i}| = 1.0583 > 1, \sum_{i=1}^{3} \sum_{k=1}^{3} (R_{a,b,c})^{1/2} = 1.2193 > 1$$

so that, respectively, the $l_1$ norm condition (equation (28)) and the $l_2$ norm condition (equation (36)) are not satisfied.

We find that the use of HOSVD for obtaining $S_{0,i}$ leads to sufficient condition for full separability, in the above two examples, while the original matrix $R_{g,r}$ does not give a sufficient condition for full separability. The HOSVD is found to be useful for improving the condition for full separability in the above two examples, but we would like to emphasize that the HOSVD condition gives only sufficient condition (but not necessary).

8. Explicitly separable forms for GHZ state and W state mixed with white noise

An arbitrary 3-qubits density matrix can be written as

$$8\rho_{ABC} = \sum_{\mu,\nu,\kappa=0}^{3} R_{\mu,\nu,\kappa}(\sigma_{\mu}) \otimes (\sigma_{\nu}) \otimes (\sigma_{\kappa});$$

$$R_{0,0,0} = 1; \quad \sigma_{0} = I. \quad \text{(61)}$$

The hermiticity of $\rho$ is equivalent to the condition that the HS parameters $R_{\mu,\nu,\kappa}$ are real. In the general case (61) includes 63 HS parameters, but some of these parameters may vanish. The simplest condition for full separability of the density matrix (61) is given by [4]

$$\sum_{\mu,\nu,\kappa=0}^{3} |R_{\mu,\nu,\kappa}| \leq 1. \quad \text{(62)}$$

But, usually this condition can be improved very much.

In the previous work [4] we treated the separability problem for density matrices which are non-MDS, including GHZ and W density matrices mixed with white noise with a probability $p$. We found in this work that while for the GHZ case $p \leq 1/5$ is sufficient and necessary condition for separability the condition $p \leq 1/9$ can be used as a sufficient condition for separability of the W state mixed with white noise. We improve such calculations in the present section showing that the condition $p \leq 1/5$ is sufficient for separability of the W state mixed with white noise which is similar to that of GHZ state.

Let us treat, separately, the system of GHZ state mixed with white noise and that of W state mixed with white noise.

A GHZ state is:

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}[|0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C + |1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C].$$

The HS decomposition of the density matrix $|\psi\rangle \langle \psi|_{GHZ}$ is given by [4]:

$$8\rho(\text{GHZ}) = (I)_{A} \otimes (I)_{B} \otimes (I)_{C}$$

$$+ (\sigma_{a})_{A} \otimes (\sigma_{a})_{B} \otimes (\sigma_{a})_{C} + (I)_{A} \otimes (\sigma_{b})_{B} \otimes (\sigma_{b})_{C}$$

$$+ (\sigma_{b})_{A} \otimes (I)_{B} \otimes (\sigma_{b})_{C}$$

$$+ (\sigma_{c})_{A} \otimes (I)_{B} \otimes (\sigma_{c})_{C}$$

$$+ (\sigma_{a})_{A} \otimes (\sigma_{a})_{B} \otimes (\sigma_{b})_{C}$$

$$- (\sigma_{a})_{A} \otimes (\sigma_{b})_{B} \otimes (\sigma_{a})_{C} - (\sigma_{a})_{A} \otimes (\sigma_{b})_{B} \otimes (\sigma_{b})_{C} - (\sigma_{a})_{A} \otimes (\sigma_{b})_{B} \otimes (\sigma_{c})_{C} + (\sigma_{a})_{A} \otimes (\sigma_{c})_{B} \otimes (\sigma_{b})_{C} + (\sigma_{a})_{A} \otimes (\sigma_{c})_{B} \otimes (\sigma_{c})_{C}$$

$$+ (\sigma_{a})_{A} \otimes (\sigma_{c})_{B} \otimes (\sigma_{c})_{C}.$$ \quad \text{(64)}

This density matrix with probability $p$ mixed with white noise is given by

$$\rho(\text{GHZ; mixed}) = p\rho(\text{GHZ}) + (1 - p)(I)_{A} \otimes (I)_{B} \otimes (I)_{C}. \quad \text{(65)}$$
Inserting (64) into (65), we get:

\[
8\rho(\text{GHZ; mixed}) = p \left\{ (I)_A \otimes (I)_B \otimes (I)_C + (\sigma_x)_A \otimes (\sigma_x)_B \otimes (\sigma_x)_C \\
+ (I)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (\sigma_z)_A \otimes (I)_B \otimes (\sigma_z)_C \\
+ (I)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (I)_A \otimes (I)_B \otimes (I)_C \right\} \\
+ (1 - p)(I)_A \otimes (I)_B \otimes (I)_C.
\]  (66)

An explicitly separable-like form for \(8\rho(\text{GHZ; mixed})\) is given by:

\[
8\rho(\text{GHZ; mixed}) = \\
\frac{p}{4} \left[ (I)_A \otimes (I)_B \otimes (I)_C + (\sigma_x)_A \otimes (\sigma_x)_B \otimes (I - \sigma_z)_C \\
+ (I)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (\sigma_z)_A \otimes (I)_B \otimes (\sigma_z)_C \\
+ (I)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C \\
+ (I)_A \otimes (I)_B \otimes (I)_C \right] \\
+ \frac{p}{2} \left[ (I)_A \otimes (I)_B \otimes (I)_C \\
+ (I)_A \otimes (\sigma_z)_B \otimes (I + \sigma_z)_C \\
+ (\sigma_z)_A \otimes (I)_B \otimes (I + \sigma_z)_C \\
+ (I)_A \otimes (\sigma_z)_B \otimes (I + \sigma_z)_C \\
+ (\sigma_z)_A \otimes (\sigma_z)_B \otimes (I + \sigma_z)_C \\
+ (I)_A \otimes (I)_B \otimes (I - \sigma_z)_C \right] \\
+ (1 - 5p)(I)_A \otimes (I)_B \otimes (I)_C.
\]  (67)

Therefore a sufficient condition for full separability is given as

\[
5p \leq 1.
\]  (68)

An explicitly separable form for the density matrix (64) was given in [33]. The relation (68) has been discussed in various works [25, 27], showing that this condition is both sufficient and necessary for full separability.

We treat now the sufficient condition for full separability for \(W\) state with probability \(p\) mixed with white noise.

The \(W\) state is given by:

\[
|\psi\rangle_W = \frac{1}{\sqrt{3}}[|0\rangle_A \otimes |0\rangle_B \otimes |1\rangle_C + |0\rangle_A \otimes |1\rangle_B \otimes |0\rangle_C \\
+ |1\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C].
\]  (69)
The HS decomposition of the density matrix $|\Psi\rangle\langle\Psi|_W$ is given by:

$$3 \cdot 8\rho(W) = 2(\sigma_y)_A \otimes (I)_B \otimes (\sigma_y)_C$$

$$+ 2(I)_A \otimes (\sigma_y)_B \otimes (\sigma_y)_C + 2(\sigma_y)_A \otimes (\sigma_y)_B \otimes (I)_C$$

$$+ (I + \sigma_z)_A \otimes (I + \sigma_z)_B \otimes (I + \sigma_z)_C$$

$$+ (I + \sigma_z)_A \otimes (I + \sigma_z)_B \otimes (I + \sigma_z)_C$$

$$+ (I + \sigma_z)_A \otimes (I - \sigma_z)_B \otimes (I - \sigma_z)_C$$

$$+ (I - \sigma_z)_A \otimes (I + \sigma_z)_B \otimes (I + \sigma_z)_C$$

$$+ 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C + 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C$$

$$+ 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C + 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C$$

$$+ 2(\sigma_z)_A \otimes (I)_B \otimes (\sigma_z)_C + 2(I)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C$$

$$+ 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C$$

$$+ 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C + 2(\sigma_z)_A \otimes (\sigma_z)_B \otimes (\sigma_z)_C$$

(70)

The $W$ state with probability $p$ mixed with white noise is given by

$$8\rho(W; \text{mixed}) = p8\rho(W) + (1 - p)(I)_A \otimes (I)_B \otimes (I)_C.$$

(71)

Then we get:

$$8\rho(W; \text{mixed}) = (1 - p)(I)_A \otimes (I)_B \otimes (I)_C$$

$$+ \frac{p}{3}(I + \sigma_z)_A \otimes (I + \sigma_z)_B \otimes (I - \sigma_z)_C$$

$$+ \frac{p}{3}(I + \sigma_z)_A \otimes (I - \sigma_z)_B \otimes (I + \sigma_z)_C$$

$$+ \frac{p}{3}(I + \sigma_z)_A \otimes (I + \sigma_z)_B \otimes (I + \sigma_z)_C$$

$$+ \frac{2}{3}p(\sigma_y)_A \otimes (I + \sigma_z)_B \otimes (\sigma_y)_C$$

$$+ \frac{2}{3}p(\sigma_z)_A \otimes (I + \sigma_z)_B \otimes (\sigma_y)_C$$

$$+ \frac{2}{3}p(I + \sigma_z)_A \otimes (I + \sigma_z)_B \otimes (\sigma_y)_C$$

$$+ \frac{2}{3}p(\sigma_z)_A \otimes (\sigma_z)_B \otimes (I + \sigma_z)_C$$

$$+ \frac{2}{3}p(\sigma_y)_A \otimes (\sigma_z)_B \otimes (I + \sigma_z)_C$$

(72)

Except for the first and second row of (72), the terms in the rows 3, 4, 5 need to be written as outer products of density matrices of qubits $A, B, C$. As an example it is easy to see that

$$\frac{2p}{3}(\sigma_y)_A \otimes (I + \sigma_z)_B \otimes (\sigma_y)_C$$

$$= \frac{2p}{3} \left[ \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{array} \right]$$

(73)

Taking this into account, we obtain that

$$8\rho(W; \text{mixed}) = (1 - 5p)(I)_A \otimes (I)_B \otimes (I)_C$$

$$+ 8\sum_i p_i(\rho_i)_A \otimes (\rho_i)_B \otimes (\rho_i)_C; \quad \sum_i p_i = 5p,$$

(74)

where $(\rho_i)_A \otimes (\rho_i)_B \otimes (\rho_i)_C$ represent outer products of density matrices of qubits $A, B, C$, where not all $\rho_i$ are equal to the unit matrix $I$.

The sufficient condition for full separability of $\rho(W; \text{mixed})$ is then obtained as:

$$5p \leq 1,$$

(75)
which is similar to that of GHZ mixed with white noise. By using the PT transformation, we find [27] that under the condition $p > 3/(3 + 8\sqrt{2}) \approx 0.209589$ the density matrix of $W$ state mixed with white noise is not fully separable. Under the condition $p < 0.2$ this density matrix is fully separable so only in a very small region the full separability problem is not clarified.

9. Conclusions

We have shown that the PH and Negativity criterions are inconclusive for 3-qubits MDS states as these density matrices and their PT have the same eigenvalues. These eigenvalues come in 4 pairs where the sum of eigenvalues for each pair is 1/4. These results can be generalized to any odd-$n$ MDS density matrix where in the general case the sum of eigenvalues in one pair is given by $1/2^{n-1}$.

The main results for MDS density matrices can be summarized as follows.

Tensors related to 3-qubits can be converted to matrices: $R_{11}$ given by equation (15) and similar ones for $R_{12}$ and $R_{13}$. A fully separable-like form for the 3-qubits MDS density matrix related to the $l_1$ norm is given by (16). This equation may be generalized to any n-qubit density matrix. By using unitary transformations of the $l_1$ norm for the MDS 3-qubits matrices the condition for full separability is given by (28) or (29). Such conditions give much better criterions for full separability relative to that of (3). A fully separable form for the 3-qubits MDS density matrix related to the $l_1$ norm is given by equation (35). This separable form leads to full separability condition given by equation (36). This separable form can be generalized to any n-qubit MDS density matrix. Bi-separability of 3-qubits MDS density matrix was obtained in the present work by using one qubit density matrix multiplied by entangled Bell states [2, 29, 30] of the other two qubits. Previous conditions for special bi-separability cases [4] were generalized to any 3-qubits MDS state given by equation (44) including up to 27 HS parameters. HOSVD method for obtaining the condition for full separability has been analyzed and its use was demonstrated in examples.

For GHZ and $W$ states mixed with white noise with probability $p$ we found explicitly separable forms for their density matrix showing by improving previous calculations [4] that for both cases the density matrix is fully separable under the condition $5p \leq 1$.

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