String principal bundles and Courant algebroids

Yunhe Sheng, Xiaomeng Xu and Chenchang Zhu

Abstract

Just like Atiyah Lie algebroids encode the infinitesimal symmetries of principal bundles, exact Courant algebroids are believed to encode the infinitesimal symmetries of $S^1$-gerbes. At the same time, transitive Courant algebroids may be viewed as the higher analogue of Atiyah Lie algebroids, and the non-commutative analogue of exact Courant algebroids. In this article, we explore what the “principal bundles” behind transitive Courant algebroids are, and they turn out to be principal 2-bundles of string groups. First, we construct the stack of principal 2-bundles of string groups with connection data. We prove a lifting theorem for the stack of string principal bundles with connections and show the multiplicity of the lifts once they exist. This is a differential geometrical refinement of what is known for string structures by Redden, Waldorf and Stolz-Teichner. We also extend the result of Bressler and Chen-Stiénon-Xu on extension obstruction involving transitive Courant algebroids to the case of transitive Courant algebroids with connections, as a lifting theorem with the description of multiplicity once liftings exist. At the end, we build a morphism between these two stacks. The morphism turns out to be neither injective nor surjective in general, which shows that the process of associating the “higher Atiyah algebroid” loses some information and at the same time, only some special transitive Courant algebroids come from string bundles.

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\textsuperscript{0}Keyword: Courant algebroids, string principal bundles, Pontryagin classes

\textsuperscript{0}MSC: 18B40, 18D35, 46M20, 53D17, 58H05.
1 Introduction

Just like Atiyah Lie algebroids encode the infinitesimal symmetries of G-principal bundles [3, 27], exact Courant algebroids are believed to encode the infinitesimal symmetries of $U(1)$-gerbes (or equivalently $BU(1)$-2-principal bundles) [10, 17, 20, 38, 31]. Transitive Courant algebroids may be viewed as the higher analogue of Atiyah Lie algebroids, and the non-commutative analogue of exact Courant algebroids. In this article, we explore what the “principal bundles” behind transitive Courant algebroids are.

First, we notice that there are topological obstructions for the existence of such transitive Courant algebroids. In [9], Bressler discovered that the obstruction to extend an Atiyah Lie algebroid to a transitive Courant algebroid is given by the first Pontryagin class, we naturally believe that the principal bundle behind a String transitive Courant algebroid is exactly a $\text{Spin}^c$-principal bundle.

Since the topological obstruction for both transitive Courant algebroids and $\text{String}(n)$-principal bundles is provided by the first Pontryagin class, we naturally believe that the principal bundle behind a transitive Courant algebroid is exactly a $\text{String}(n)$-principal bundle.

This belief is also supported by another observation from T-duality. Let us start with two T-dual torus bundles $X, \hat{X}$ over $M$, and matching T-dual $S^1$-gerbes $\mathcal{G} \to X$ and $\hat{\mathcal{G}} \to \hat{X}$, Bouwknegt-Evslin-Mathai [7] and Bunke-Schick [12] proved that the twisted K-theory for the T-dual pairs are isomorphic, that is, there is an isomorphism between twisted K-groups $K^\bullet(X, \mathcal{G}) \cong K^\bullet(\hat{X}, \hat{\mathcal{G}})$. On the level of differential geometric objects, Cavalcanti-Gualtieri [15] proved that the exact Courant algebroid associated to the T-dual $S^1$-gerbes are the same. Now we extend this story $\text{Spin}(n)$-equivariantly. We begin with two T-dual torus bundles $X, \hat{X}$ over $M$, and their matching T-dual string structures $(P, \xi) \to X$ and $(\hat{P}, \hat{\xi}) \to \hat{X}$, where $P$ and $\hat{P}$ are $\text{Spin}(n)$-principal bundles over $X$ and $\hat{X}$ respectively, and $\xi, \hat{\xi}$ are string classes on $P$ and $\hat{P}$ respectively. Leaving alone what the cohomological invariants should be, Baraglia and Hekmati

\[
\begin{array}{ccc}
\text{M} & \xrightarrow{P} & \text{BSpin}(n) \\
\downarrow & & \\
\text{BString}(n)
\end{array}
\]

Thus, if we realise “$\text{String}(n)$-principal bundles” in differential geometry, one may interpret $\frac{1}{2}p_1$ as the lifting obstruction of a $\text{Spin}(n)$-principal bundle to a $\text{String}(n)$-principal bundle. A more familiar fact in this style is that the lifting obstruction of a $\text{SO}(n)$-principal bundle $M \xrightarrow{p} \text{BSO}(n)$ to a $\text{Spin}(n)$-principal bundle $M \xrightarrow{P} \text{BSpin}(n)$ is given by $w_2(P)$. In fact, there is a whole sequence, called the Whitehead tower:

\[
\cdots \to \text{BString}(n) \to \text{BSpin}(n) \to \text{BO}(n),
\]

with obstruction $w_1$, $w_2$ and $\frac{1}{2}p_1$ respectively. Here $w_1$ and $w_2$ are the first and second Stiefel-Whitney classes.

Since the topological obstruction for both transitive Courant algebroids and $\text{String}(n)$-principal bundles is provided by the first Pontryagin class, we naturally believe that the principal bundle behind a transitive Courant algebroid is exactly a $\text{String}(n)$-principal bundle.
showed that, on the level of differential geometric objects, the transitive Courant algebroids associated to both sides are isomorphic.

In this article, we realise \textit{String}(n)-principal bundles and their connections as differential geometric objects by describing the entire (3, 1)-sheaf (or 2-stack) \(\mathcal{B}\text{String}(n)^{p+}\). Then we make the connection between transitive Courant algebroids and string principal bundles explicit and functorial by constructing a morphism between their corresponding stacks.

For this purpose, first we study what a \(\text{String}(n)\)-principal bundle with connection data really is. As we have seen, \(\text{String}(n)\) is a three-connected cover of \(\text{Spin}(n)\), and this forces the model of \(\text{String}(n)\) to be either infinite-dimensional or finite-dimensional however higher (namely being a Lie 2-group)\(^1\). We take the second approach with the model of Schommer-Pries \[35\] for \(\text{String}(n)\). The advantage of this model is that the spaces it involves are all nice finite dimensional manifolds, thus there is no additional analytic difficulty when solving equations or constructing covers; at the same time, this is paid off by algebraic difficulty of chasing through various pages of spectral sequences of cohomological calculation.

First we construct a (3, 1)-presheaf of \(\text{String}(n)\)-principal bundles with connection data \(\mathcal{B}\text{String}(n)^{p}\) and complete it into a (3, 1)-sheaf (or 2-stack) \(\mathcal{B}\text{String}(n)^{p+}\) using the plus construction. This is essentially to build a \(\text{String}(n)\)-principal bundle with a connection from local data and gluing conditions in the fashion of Breen-Messing. Breen and Messing studied connections for gerbes in their original work \[8\]. We also notice that in a recent work \[47\], connections for 2-principal bundles of strict 2-groups are studied both locally and globally. However, the finite-dimensional differential geometric model for \(\text{String}(n)\) is a non-strict Lie 2-group. This forces us to develop our own formula instead of using existing results in literature. It turns out that the glued stack involves descent equations of first Pontryagin class.

In a recent work \[1\], these equations are further studied in a universal setting and proved to be closely related to Kashiwara-Vergne theory and Drinfeld associators.

To justify our construction, we prove directly the lifting theorem that one expects for \(\text{String}(n)\)-principal bundles and provide a comparison to previous string concepts of Stolz-Teichner, Redden and Waldorf respectively in Section 3.3:

\begin{theorem}
Given a \(\text{Spin}(n)\)-principal bundle with a connection \(M \xrightarrow{\bar{P}} \mathcal{B}\text{Spin}(n)\),
\end{theorem}

(i) it lifts to an object in \(M \xrightarrow{P} \mathcal{B}\text{String}(n)^{p+}\),

\[\begin{array}{c}
\text{BString}(n)^{p+} \\
\downarrow \\
M \\
\downarrow \\
\text{BSpin}(n),
\end{array}\]

if and only if \(\frac{1}{2}p_1(\bar{P}) = 0\);

(ii) if a lift in (i) exists, then the isomorphism classes of different lifts form a torsor of the Deligne cohomology group \(H^2(M, U(1) \xrightarrow{\text{dlog}} \Omega^1 \xrightarrow{\text{d}} \Omega^2)\) mod out by a certain subspace \(I\).

It is proved in Theorem \[3.2\] and Theorem \[3.6\].

After this, we build the (2, 1)-sheaf (or 1-stack) of transitive Courant algebroids with connections. We benefit much from \[16\] where transitive Courant algebroids and their gauge transformations are well studied. However, the gauge transformations which preserve the connection data are still needed to be specified. We thus have additional equations in the definition of 1-morphisms (see Eqs. \[33, 35\]). To make the construction mathematically strict, however at the same time avoiding the routine checking of gluing conditions of stacks over several layers, as before, we first construct a (2, 1)-presheaf \(\mathcal{TC}^p\) by simply mapping to the category of standard transitive Courant algebroids with connections and their gauge transformations. Then we complete it to a (2, 1)-sheaf \(\mathcal{TC}^p\) using Nikolaus-Schweigert’s plus construction.

\[1\]There are also models which are both higher and infinite-dimensional.
We prove that the gluing result gives us exactly a transitive Courant algebroid (not necessarily standard) with a connection and its gauge transformations. This in turn justifies our construction of the transitive Courant algebroid stack. There is a subtle difference between our construction of the transitive Courant algebroid stack and the one in [9]. In [9], the stack is directly taken to be a functor mapping to the category of transitive Courant algebroids (not just standard ones), however the checking of gluing conditions seems to be omitted. Also in the language of stacks, a recent work [29] has studied the relation between twisted Courant algebroids and shifted symplectic Lie algebroids, and has further hinted an even higher correspondence of our type involving fivebrane structures.

In the end, we construct a morphism from the (3,1)-sheaf of \text{String}(n)-principal bundles with connections to the (2,1)-sheaf of transitive Courant algebroids with connections for $\text{Spin}(n)$. To achieve this, we only need to build a morphism on the presheaf level since the plus construction is functorial. It turns out that the difficulty of the construction lies on the level of morphisms, that is, to construct the gauge transformation of transitive Courant algebroids associated to that of \text{String}(n)-principal bundles. The formula of the symmetric part of the (3,1)-position in the gauge transformation remains rather mysterious. Ševera suggests us some connection to Alekseev-Malkin-Meinrenken’s theory on group valued moment maps [2]. We reserve it for future investigation. We remark in Appendix A.4 that these gauge transformations are all inner ones noticed by Ševera. Similar results of these inner automorphisms are also studied in [23] in another setting. We further verify that this morphism from the string stack to the Courant stack is neither injective nor surjective. This tells us that the process of associating a “higher Atiyah algebroid” to a string principal 2-bundle loses some information, and, at the same time, not all transitive Courant algebroids come from this process.

2 Preliminaries on prestacks, stacks and the plus construction

Recall that an $(n+1,1)$-presheaf over a category $\mathcal{M}$ is a (higher) functor $\mathcal{M}^{op} \to n\text{Gpd}$ to the higher category of $n$-groupoids, where $n \in \{0,1,2,\ldots\} \cup \{\infty\}$. Here 0-groupoids are interpreted as sets. Therefore, a $(1,1)$-presheaf (or a presheaf) over a category $\mathcal{M}$ is a functor $\mathcal{M}^{op} \to \text{Sets}$ to the category of sets; a $(2,1)$-presheaf over a category $\mathcal{M}$ is a (higher) functor $\mathcal{M}^{op} \to \text{Gpd}$ to the 2-category of (1-)groupoids; and a $(3,1)$-presheaf over a category $\mathcal{M}$ is a (higher) functor $\mathcal{M}^{op} \to 2\text{Gpd}$ to the higher category of 2-groupoids. These are all the cases that will be used in this paper. Then we perform a plus construction (namely a procedure of higher sheafification) to obtain the corresponding sheaves. Sometimes, $(2,1)$-sheaves are also called stacks, and $(3,1)$-sheaves are called 2-stacks. The model we use is as in [28] Section 2]. For technical details, we refer readers to this paper and the references therein.

Here we briefly recall the model we use for 2-groupoids. Our model for a 2-groupoid (in the sense of Duskin and Glenn [19] [24]) is a simplicial set satisfying Kan conditions $\text{Kan}(n,j)$ for all $n \geq 1$ and $0 \leq j \leq n$ and strict Kan conditions $\text{Kanl}(n,j)$ for all $n \geq 3$ and $0 \leq j \leq n$. Readers who are not familiar with Kan conditions may equivalently understand it as a bicategory [6], whose 2-morphisms are invertible and whose 1-morphisms are all invertible up to 2-morphisms. The compositions of 1-morphisms are associative up to an associator, and the associator in turn satisfies a higher coherence condition. For the precise definition, we refer to [43] Definition 5.2, where a semi-strict Lie 2-groupoid is defined. If we equip the object therein with discrete topology, we obtain what a 2-groupoid is. Let us also recall the equivalence between the two different descriptions: if we start with a simplicial set $X_\bullet$ satisfying the above Kan condition, then we take $C_0 = X_0$ on the object level; $C_1 = X_1$ on the 1-morphism level; $C_2 = d_0^{-1}(s_0(X_0))$ on the 2-morphism level, we obtain a bicategory $(C_0, C_1, C_2)$ satisfying required conditions; as for the other direction, we take $X_0 = C_0$, $X_1 = C_1$ and $X_2 = C_1 \times_{C_0} C_1$. For details we refer to [51] Section 4].

Now we shortly recall the process of the plus construction in the case when $\mathcal{M} = \text{Mfd}$ is the category of differential manifolds for our application. Given a $(3,1)$-presheaf $\mathcal{F} : \text{Mfd}^{op} \to 2\text{Gpd}$, the plus construction in [28] gives us a $(3,1)$-sheaf $\mathcal{F}^+ : \text{Mfd}^{op} \to 2\text{Gpd}$. To describe this $(3,1)$-sheaf, we first need to take the
homotopy limit $\text{holim} \mathcal{F}(U(M)_\bullet)$ for an open cover $\{U_i\}$ of $M$ over the Čech simplicial manifold

$$U(M)_\bullet = \sqcup U_i \xleftarrow{\partial_0} \sqcup U_{ij} \xleftarrow{\partial_0} \sqcup U_{ijk} \ldots \ldots \quad (2)$$

Let us describe $\text{holim} \mathcal{F}(U(M)_\bullet)$ explicitly: the result is a 2-groupoid.

- Its object consists of
  - $\text{Ob}_0$ an element $\theta = (\theta_i) \in \mathcal{F}(\sqcup U_i)_0$;
  - $\text{Ob}_1$ an element $g = (g_{ij}) \in \mathcal{F}(\sqcup U_{ij})_1$, which is a 1-morphism $\theta_{|U_{ij}} \xrightarrow{g_{ij}} \theta_{|U_{ij}}$, or equivalently, a 1-morphism $\partial_1^* \theta \Leftrightarrow \partial_0^* \theta$;
  - $\text{Ob}_2$ an element $a = (a_{ijk}) \in \mathcal{F}(\sqcup U_{ijk})_2$, which is a 2-morphism $a : g_{ij} \circ g_{jk} \Leftrightarrow g_{ik}$;
  - $\text{Ob}_3$ pentagon condition for $a$, that is, $(\text{id} \circ h_0 \partial_0^* a \circ \partial_0^* \theta = \alpha(g_{ij}, g_{jk}, g_{kl}) \circ (\partial_1^* a \circ h_0 \text{id}) \circ \partial_1^* \theta)$, where $\alpha(g_{ij}, g_{jk}, g_{kl})$ is the associator $g_{ij} \circ (g_{jk} \circ g_{kl}) \Leftrightarrow (g_{ij} \circ g_{jk}) \circ g_{kl}$, where $\circ h_0$ is the horizontal composition of 2-morphisms.

- A 1-morphism from $(\tilde{\theta}, \tilde{g}, \tilde{a})$ to $(\theta, g, a)$ consists of
  - $\text{1M}_0$ a 1-morphism $A = (A_i) : \theta \leftarrow \tilde{\theta}$ in $\mathcal{F}(\sqcup U_i)$;
  - $\text{1M}_1$ a 2-morphism $f : g \circ \partial_0^* A \Leftrightarrow \partial_1^* A \circ \tilde{g}$ in $\mathcal{F}(\sqcup U_{ij})$;
  - $\text{1M}_2$ a higher coherence condition, $(\text{id} \circ h_0 \partial_0^* a \circ \partial_0^* \theta) \circ (\partial_0^* f \circ h_0 \text{id}) = \partial_1^* f \circ (\text{id} \circ h_0 \partial_0^* \alpha)^{-1}$ of 2-morphisms in $\mathcal{F}(\sqcup U_{ijk})$.

- A 2-morphism from $(\tilde{A}, \tilde{f})$ to $(A, f)$ consists of
  - $\text{2M}_0$ a 2-morphism $\omega : A \leftarrow \tilde{A}$ in $\mathcal{F}(\sqcup U_i)$;
  - $\text{2M}_1$ a higher coherence condition $f \circ (\partial_1^* \omega \circ h_0 \text{id}) = (\text{id} \circ h_0 \partial_0^* \omega \circ h_0 \text{id}) \circ \tilde{f}$.

Then the $(3, 1)$-sheaf $\mathcal{F}^+$ maps $M \in \text{Mfd}$ to the following 2-groupoid:

- $\mathcal{F}^+(M)_0$: an object is a pair $\{U_i, P\}$, where $\{U_i\}$ is a cover of $M$ and $P$ is an object in $\text{holim} \mathcal{F}(U(M)_\bullet)$;
- $\mathcal{F}^+(M)_1$: a 1-morphism from $\{\tilde{U}_i, \tilde{P}\}$ to $\{U_i, P\}$ is a common refinement $\{\tilde{V}_i\}$ of $\{\tilde{U}_i\}$ and $\{U_i\}$, and a 1-morphism $\phi$ in $\text{holim} \mathcal{F}(V(M)_\bullet)$;
- $\mathcal{F}^+(M)_2$: a 2-morphism from $\{\tilde{V}_i, \tilde{\phi}\}$ to $\{V_i, \phi\}$ is a common refinement $\{W_i\}$ of $\{\tilde{V}_i\}$ and $\{V_i\}$, and a 2-morphism $\alpha$ in $\text{holim} \mathcal{F}(W(M)_\bullet)$. Moreover, $\{\{W_i\}, \tilde{\alpha}\}$ and $\{\{W_i\}, \alpha\}$ are identified if $\alpha$ and $\tilde{\alpha}$ are identified on a further common refinement of $\{\tilde{W}_i\}$ and $\{W_i\}$.

The plus construction for $(2, 1)$-presheaves is then a truncation of that of $(3, 1)$-sheaves viewing 2-groupoids as 2-groupoids with identity 2-morphisms. Let us explain it with a nice example.

**Example 2.1.** Given a Lie group $G$ and its Lie algebra $\mathfrak{g}$, there is a $(2, 1)$-presheaf $BG^p : \text{Mfd}^{op} \rightarrow \text{Gpd}$ sending $U \in \text{Mfd}$ to the groupoid whose objects are trivial $G$-principal bundles $U \times G$ together with $\theta \in \Omega^1(U, \mathfrak{g})$ and whose morphisms from $(U \times G, \theta)$ to $(U \times G, \tilde{\theta})$ are gauge transformations $g : U \rightarrow G$ satisfying $\theta - Ad_g \tilde{\theta} = -g^* \theta_{\text{MC}}$; and sending a morphism $U \rightarrow V$ to the functor between the corresponding groupoids induced by pullbacks of principal bundles and differential forms. Here $\theta_{\text{MC}}$ is the right invariant Maurer-Cartan form on $G$. It satisfies the following Maurer-Cartan equation

$$d\theta_{\text{MC}} - \frac{1}{2}[\theta_{\text{MC}}, \theta_{\text{MC}}]_g = 0.$$ 

Let us form the $\text{holim} B^p G(U(M)_\bullet)$ with respect to an open cover $\{U_i\}$ of $M \in \text{Mfd}$. An object in $\text{holim} B^p G(U(M)_\bullet)$ consists of
• $U_i \times G, \theta_i \in \Omega^1(U_i, g)$;

• $g_{ij} : U_{ij} \to G, (U_i \times G, \theta_i) \mapsto (U_j \times G, \theta_j)$, with $\theta_i - \text{ad}_{g_{ij}}\theta_j = -g^\theta_{ij}\theta_{\text{sc}}$;

• compatibility condition $g_{ij} \circ g_{jk} = g_{ik}$ on $U_{ijk}$.

Thus, we see that such an object gives us exactly the local data of a $G$-principal bundle with a connection 1-form. A morphism in $\text{holim } BG^e_p(U(M) \star)$ from $(\theta_i; g_{ij})$ to $(\tilde{\theta}_i; \tilde{g}_{ij})$ consists of

• $g_i : U_i \to G, \theta_i - \text{ad}_{g_i}\tilde{\theta}_i = -g_i^\theta\theta_{\text{sc}}$;

• compatibility condition $g_{ij} \cdot g_j = g_i \cdot \tilde{g}_{ij}$ on $U_{ij}$.

This gives us exactly the local data of a gauge transformation preserving connections between the corresponding $G$-principal bundle glued by $g_{ij}$ and $g_i$.

3 (3, 1)-sheaf of string principal bundles

3.1 Finite dimensional model of $\text{String}_p(G)$

In this section, $G$ is a finite dimensional compact Lie group. Let us first recall the finite dimensional model of the Lie 2-group $\text{String}_p(G)$ built in [35] for a given class $p \in H^4(BG \star, \mathbb{Z})$. The idea is to realise $\text{String}_p(G)$ as a $\text{BU}(1)$-central extension of a Lie group $G$

$$\text{BU}(1) \longrightarrow \text{String}_p(G) \longrightarrow G,$$

with the extension class $p \in H^4(BG \star, \mathbb{Z})$. Here $BG \star$ is the simplicial nerve of a Lie group $G$, and $H^*(BG \star, \mathbb{Z})$ denotes the sheaf cohomology of the sheaf of $\mathbb{Z}$-valued functions on $BG \star$. Similarly, we use $H^*(BG \star, U(1))$ and $H^*(BG \star, \mathbb{R})$ to denote the sheaf cohomology of $U(1)$-valued function and $\mathbb{R}$-valued function respectively.

Let us explain a bit the terminology here: a Lie 2-group is a differentiable stack equipped with a group structure (up to homotopy). For example, $\text{BU}(1)$ is an abelian Lie 2-group. Here $\text{BU}(1)$ denotes the stack presented by groupoid $U(1) \Rightarrow pt$. The multiplication $m : \text{BU}(1) \times \text{BU}(1) \to \text{BU}(1)$ is induced by the multiplication of $U(1)$. Notice that since $U(1)$ is abelian, thus the $U(1)$-multiplication is a functor

$$(U(1) \Rightarrow pt) \times (U(1) \Rightarrow pt) \to (U(1) \Rightarrow pt).$$

For more details in the topic of (Lie) 2-groups and examples see e.g. [4] Sect 3.1. In the case that $G = \text{Spin}(n)$, the generator of $H^4(BG \star, \mathbb{Z})$ is given by $\frac{1}{2}p_1$, half of the Pontryagin class. The sheaf cohomology may be calculated by taking a hypercover of $BG \star$ and taking the Čech cohomology, as long as the hypercover is acyclic. This is a classical result. See [18] Proposition 2.4 for a concrete statement. See [22] and [18] Sect. 2 and references therein for definition and properties of the sheaf cohomology for simplicial objects. The cohomology on $BG \star$ is then equivalent to the group cohomology used in [35] originally coming from Segal and Brylinski [36, 37, 11]. In our case, as long as the cover of $BG \star$ on each layer $G^{(\star)}$ is good, namely intersections are contractible, it is acyclic with respect to sheaves in our study.

The short exact sequence of sheaves $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ gives us a long exact sequence of cohomology. Notice that $H^{2+1}(BG \star, \mathbb{R}) = 0$ for compact group $G$. Thus $H^n(BG \star, U(1)) \cong H^{n+1}(BG \star, \mathbb{Z})$ for $n \geq 1$.

To build the finite dimensional model for the Lie 2-group $\text{String}_p(G)$, let us take a good simplicial hypercover $G^{(\star)}$ for $BG \star$ and write the simplicial-Čech double complex whose total cohomology is
We take a representative \((\Theta, \Phi, \eta, 0)\) of \(p \in H^3(BG_\bullet, U(1)) = H^4(BG_\bullet, \mathbb{Z})\), where
\[
\Theta \in C(\sqcup\mathcal{G}_s^{(3)}, U(1)), \quad \Phi \in C(\sqcup\mathcal{G}_p^{(2)}, U(1)), \quad \eta \in C(\sqcup\mathcal{G}_\alpha^{(1)}, U(1)).
\]
The last entry being 0 is implied by the closedness.
To build up a Lie 2-group, we first need to have an underlying Lie groupoid which presents the stack \(\text{String}_n(G)\), and then establish a group structure “up to homotopy” on top of it. Here we follow the convention in [ES, Section 2].

Our underlying Lie groupoid \(\Gamma[\eta]\) is a \(U(1)\)-extension of the Čech groupoid with respect to the cover \(G^{(1)}\), that is \(\sqcup\mathcal{G}_\alpha^{(1)} \times U(1) \Rightarrow \sqcup\mathcal{G}_\alpha^{(1)}\), together with source and target \(s(g_{\alpha,\beta,\gamma}) = g_{\alpha,\gamma}\) and \(t(g_{\alpha,\beta,\gamma}) = g_{\beta}\), multiplication
\[
(g_{\alpha,\beta,\gamma}) (g_{\beta,\gamma,\delta}) = (g_{\alpha,\gamma,\delta} + \eta_{\alpha,\beta,\gamma}(g)),
\]
identity \(e(g_{\alpha}) = (g_{\alpha}, 0)\), and inverse \((g_{\alpha,\beta,\gamma})^{-1} = (g_{\beta,\alpha,\gamma}, -a)\). Here \(\delta(\eta) = 0\) guarantees that the construction gives rise to a Lie groupoid structure.

Now we build the multiplication for the 2-group structure on \(\Gamma[\eta]\), which should be a generalized morphism \(\Gamma[\eta] \times^2 \Gamma(\eta) \Rightarrow \Gamma(\eta)\). We realize the generalized morphism by a span of a Morita morphism and a usual morphism,
\[
\Gamma[\eta] \times^2 \overset{\text{M.E.}}{\longrightarrow} \Gamma^2[\eta] \overset{m}{\longrightarrow} \Gamma(\eta),
\]
where \(\Gamma^2[\eta] = \left(G^{(2)}_{[1]} \times U(1)^{\times 2} \Rightarrow G^{(2)}_{[0]})\right)\) is similarly constructed as \(\Gamma[\eta]\), however, by the pullback Čech cocycle \((d_0^\eta, d_1^\eta) \in C(G^{(2)}_{[2]} \times U(1)^{\times 2})\). Here, and later, \(G^{(j)}_{[i]}\) denotes the disjoint union of \((i + 1)\)-fold intersections of the hypercover \(G^{(j)}\).

The natural projection \(\Gamma^2[\eta] \to \Gamma^{\times 2}[\eta]\) is a Morita morphism, i.e., a morphism that gives arise to a Morita equivalence of Lie groupoids.

The morphism \(\Gamma^2[\eta] \overset{m}{\longrightarrow} \Gamma(\eta)\) is given by
\[
(v_0, v_1, a_0, a_1) \overset{m}{\longrightarrow} (d_1(v_0), d_1(v_1), a_0 + a_1 + \Phi(v_0, v_1)).
\]

It being a groupoid morphism is equivalent to the fact that \(\delta(\eta) + \tilde{\delta}(\Phi) = 0\). However the multiplication is not strictly associative, that is, the following diagram of differentiable stacks commutes up to a 2-
morphism $a$, which is called an associator,

$$\text{String}_p(G) \times 3 \xrightarrow{\text{id} \times m} \text{String}_p(G) \times 2 \xrightarrow{m} \text{String}_p(G).$$

We now find a suitable Lie groupoid presentation of $\text{String}_p(G)^{\times 3}$ so that certain desired morphisms can be written as strict morphisms of Lie groupoids. Notice that there are three maps $m_0, m_1, m_2 : \text{String}_p(G)^{\times 3} \to \text{String}_p(G)$ and $m_3 : \text{String}_p(G)^{\times 2} \to \text{String}_p(G)$ given by strict Lie groupoid morphisms:

$$m_1 : ((w_s, w_t), a_0, a_1, a_2) \mapsto (d_1 d_2(w_s), d_1 d_2(w_t), a_0 + a_1 + a_2 + d_2^* \Phi(w_s, w_t) + d_1^* \Phi(w_s, w_t)),
\quad m_2 : ((w_s, w_t), a_0, a_1, a_2) \mapsto (d_1 d_1(w_s), d_1 d_1(w_t), a_0 + a_1 + a_2 + d_1^* \Phi(w_s, w_t) + d_2^* \Phi(w_s, w_t)).$$

In this model, the associator $a : m_2 \Rightarrow m_1$, is a map $\Gamma^{\text{String}_p(G)_{\times 3}}$.

The naturality condition $m_1(r) a(s(r)) = a(t(r)) m_2(r)$ is equivalent to the equation $\delta(\Phi) = 0$. The pentagon condition for the associator is equivalent to the equation $\delta(\Theta) = 0$.

### 3.2 $(3,1)$-sheaf $\text{BString}(n)_{\mathbb{C}}^{p+}$ of string principal bundles with connection data

Now we take $G = \text{Spin}(n)$ and $p$ to be $\frac{1}{2}p_1 \in H^4(\text{BSpin}(n), \mathbb{Z})$. We denote by $\text{String}(n)$ the corresponding string group $\text{String}_p(G)$.

The $(3,1)$-presheaf of string principal bundles with connections $\text{BString}(n)^{p}_{\mathbb{C}} : \text{Fmd}^{op} \to \text{2Gpd}$ is constructed as follows: first of all, $\text{BString}(n)^{p}_{\mathbb{C}}(U)$ is a 2-groupoid made up by the following data:

- $\text{BString}(n)^{p}_{\mathbb{C}}(U)_0$: an object is a triple $(U \times \text{String}(n) \to U, \theta, B)$ consisting of a locally trivial $\text{String}(n)$-principal bundle $(U \times \text{String}(n) \to U)$ together with a 2-form $B \in \Omega^2(U)$ and an $\mathfrak{so}(n)$-valued 1-form $\theta \in \Omega^1(U, \mathfrak{so}(n))$.

- $\text{BString}(n)^{p}_{\mathbb{C}}(U)_1$: a 1-simplex

$$(U \times \text{String}(n) \to U, \theta_0, B_0) \xleftarrow{\text{ad}_{g_{01}}} (g_{01} : A_{01}, \omega_{01}^2) \xrightarrow{(g_{01}, A_{01}, \omega_{01}^2)} (U \times \text{String}(n) \to U, \theta_1, B_1)$$

consists of a generalized morphism $g_{01} : U \to \text{String}(n)$ given by a bibundle $E_{g_{01}}$, a 1-form $A_{01} \in \Omega^1(U)$ and a 2-form $\omega_{01}^2 \in \Omega^2(U)$ such that

$$B_1 - B_0 = \omega_{01}^2 + dA_{01}, \quad \text{cs}_3(\theta_1) - \text{cs}_3(\theta_0) = d\omega_{01}^2, \quad \theta_0 - \text{ad}_{g_{01}} \theta_1 = -g_{01}^* \theta_{MC},$$

where $\text{cs}_3(\theta)$ is the Chern-Simon 3-form associated to an $\mathfrak{so}(n)$-valued 1-form $\theta$ given by

$$\text{cs}_3(\theta) = (\theta, d\theta) + \frac{1}{3}(\theta, [\theta, \theta]). \quad (4)$$

Here $(-, -)$ is a certain invariant symmetric bilinear form on $\mathfrak{so}(n)$, and $g_{01} : U \xleftarrow{g_{01}} \text{String}(n) \xrightarrow{\text{Spin}(n)} \text{Spin}(n)$ is the composition of $g_{01}$ with the natural projection $\text{String}(n) \xrightarrow{\text{Spin}(n)} \text{Spin}(n)$. Composition is given by the multiplication in $\text{String}(n)$:

$$(g_{01}, A_{01}, \omega_{01}^2) \circ (g_{12}, A_{12}, \omega_{12}^2) = (g_{01} \cdot g_{12}, A_{01} + A_{12}, \omega_{01}^2 + \omega_{12}^2).$$

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• $\text{BString}(n)^p(U)_2$: a 2-simplex with edges $(g_{ij}, A_{ij}, \omega_{ij}^2)$ for $0 \leq i < j \leq 2$, or equivalently (in the model of bicategory), a 2-morphism between $(g_{01}, A_{01}, \omega_{01}^2) \circ (g_{12}, A_{12}, \omega_{12}^2)$ and $(g_{02}, A_{02}, \omega_{02}^2)$, is given by a pair $(f, \omega^1)$ with $f \in C^\infty(U, U(1))$ and $\omega^1 \in \Omega^1(U)$ such that

$$A_{12} - A_{02} + A_{01} = \omega^1 - d\log f, \quad \omega_{12}^2 - \omega_{02}^2 + \omega_{01}^2 = -d\omega^1.$$  

Moreover, $f$ gives rise to an isomorphism of bibundles $g_{01} \cdot g_{12} \simeq g_{02}$.

$$\begin{array}{c}
(\theta_1, B_1) \\
(f, \omega^1) \\
(\theta_0, B_0) \\
(\theta_2, B_2).
\end{array}$$

Given a morphism $U \xrightarrow{\phi} V$, the associated functor $\text{BString}(n)^p(V) \to \text{BString}(n)^p(U)$ is given by pre-compositions and pullbacks of forms.

Now let us look at $\text{holim} \text{BString}(n)^p(U(M)_*)$ for a cover $\{U_i\}$ of $M$, here $U(M)_*$ given in $[\mathcal{D}]$ is the nerve of the Čech groupoid associated to the cover $\{U_i\}$. An object in $\text{holim} \text{BString}(n)^p(U(M)_*)$ consists of

• $U_i \times \text{String}(n) \to U_i$, $B_i \in \Omega^2(U_i)$, $\theta_i \in \Omega^1(U_i, \text{so}(n))$;

• $g_{ij} : U_{ij} \to \text{String}(n)$, $A_{ij} \in \Omega^1(U_{ij})$, $\omega_{ij}^2 \in \Omega^2(U_{ij})$, such that

$$B_j - B_i = dA_{ij} + \omega_{ij}^2, \quad cs_3(\theta_j) - cs_3(\theta_i) = d\omega_{ij}^2, \quad \theta_i - ad_{g_{ij}}\theta_j = -g_{ij}^*\theta_{sc};$$

• $f_{ijk} : U_{ijk} \to U(1)$, $\omega_{ijk}^1 \in \Omega^1(U_{ijk})$, such that $f_{ijk}$ is an isomorphism $g_{ik} \Rightarrow g_{ij} \cdot g_{jk}$, and

$$(\delta A)_{ijk} = \omega_{ijk}^1 - d\log f_{ijk}, \quad (\delta \omega^2)_{ijk} = -d\omega_{ijk}^1.$$  

• a pentagon condition for 2-morphisms indicated by the following diagram,

$$\begin{array}{c}
((g_{ij}, A_{ij}, \omega_{ij}^2) \circ (g_{jk}, A_{jk}, \omega_{jk}^2)) \circ (g_{kl}, A_{kl}, \omega_{kl}^2) \\
(f_{ijk}, \omega_{ijk}^1)
\end{array}$$

$$\begin{array}{c}
\xrightarrow{(id_{\text{String}(n)}, \alpha, 0, 0)} \\
(g_{il}, A_{il}, \omega_{il}^2)
\end{array}$$

$$\begin{array}{c}
((g_{ij}, A_{ij}, \omega_{ij}^2) \circ (g_{jk}, A_{jk}, \omega_{jk}^2)) \circ (g_{kl}, A_{kl}, \omega_{kl}^2) \\
(f_{ijl}, \omega_{ijl}^1)
\end{array}$$

where $\alpha$ is the associator of the string group $\text{String}(n)$, and $\circ_h$ is the horizontal composition of 2-morphisms, noticing that $(g_{ij} \circ g_{jk}) \circ g_{kl}$ and $g_{ik} \circ (g_{jk} \circ g_{kl})$ are composed 1-morphisms $U_{ijk} \xrightarrow{\text{String}(n)^p \times 3} \text{String}(n)$. According to Lemma $[\mathcal{D}]$, the 2-morphism $\text{id} \circ_h a$ is given by $U(1)$-valued functions $F_{ijl} : U_{ijl} \to U(1)$, which converges to a class in the Čech cohomology group $H^3(M, Z)$ determined by the extension class $\frac{1}{2}p_1$. Thus the above diagram says exactly

$$\delta f = F, \quad \delta \omega^1 - \delta d\log f = 0, \quad \delta d\omega^1 = 0.$$  

The latter two equations are implied by $[\mathcal{D}]$.  

2Note that a $U(1)$-valued function on $U$ provides an isomorphism of bibundles from $U$ to $\text{String}(n)$ via the map $BU(1) \to \text{String}(n)$. See also Lemma $[\mathcal{D}]$.  

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A 1-morphism in \( \text{holim} BString(n)^p(U(M)) \) from \((U_i \times \text{String}(n)) \to U_i, \bar{B}_i; \bar{g}_{ij}, \bar{A}_{ij}, \bar{\omega}_{ij}^1; \bar{f}_{ijk}, \bar{\omega}_{ijk}^1)\) to \((U_i \times \text{String}(n)) \to U_i, \theta_i, B_i; g_{ij}, A_{ij}, \omega_{ij}^1; f_{ijk}, \omega_{ijk}^1)\) consists of

- \( g_i : U_i \to \text{String}(n), A_i \in \Omega^1(U_i), \omega_i^2 \in \Omega^2(U_i), \) such that \((g_i, A_i, \omega_i^2) \in BString(n)^p(\cup_i U_i)\) is a 1-morphism between
  \[
  (U_i \times \text{String}(n)) \to U_i, \theta_i, B_i; \]

- \( \bar{B}_i = B_i = \omega_i^2 + dA_i, \omega_{ij}^2 = \text{cs}_3(\bar{\theta}_i) - \text{cs}_3(\theta_i), \theta_i - \text{ad}_g \bar{\theta}_i = -\bar{g}_i^T \theta_M; \)

- an element \((f_{ij}, \omega_{ij}^1) \in BString(n)^p(\cup_j U_{ij})\) which provides a 2-morphism making the following diagram 2-commutative

- a higher coherence condition between 2-morphisms

which gives us the following equation\(^3\)

\[
 f_{ijk} \bar{f}_{ij}^{-1} = (\delta f_{..})_{ijk}, \quad \bar{\omega}_{ijk}^1 = \omega_{ijk}^1 - d \log \bar{f}_{ijk} - \omega_{ijk}^1 + d \log f_{ijk} = - (\delta \omega_{ij}^1)_{ijk} + (\delta \log f_{..})_{ijk},
\]

A 2-morphism in \( \text{holim} BString(n)^p(U(M)) \) from \((\bar{g}_i, \bar{A}_i, \bar{\omega}_i^2; \bar{f}_{ij}, \bar{\omega}_{ij}^1)\) to \((g_i, A_i, \omega_i^2; f_{ijk}, \omega_{ijk}^1)\) consists of

- an element \((f_i, \omega_i^1) \in BString(n)^p(\cup_i U_i)\) from \((\bar{g}_i, \bar{A}_i, \bar{\omega}_i^2; \bar{f}_{ij}, \bar{\omega}_{ij}^1)\) to \((g_i, A_i, \omega_i^2; f_{ijk}, \omega_{ijk}^1)\), that is, \( g_i \triangleleft \bar{g}_i \), and \( A_i - \bar{A}_i = -d \log f_i + \omega_i^1, \omega_i^2 - \bar{\omega}_i^2 = -d \omega_i^1 \); 

- a coherence condition held on \( U_{ij} \),

\[
 (f_{ij}, \omega_{ij}^2) \circ (f_i, \omega_i^1) = (f_j, \omega_j^1) \circ (\bar{f}_{ij}, \bar{\omega}_{ij}^1),
\]

that is\(^4\)

\[
 f_{ij} \bar{f}_{ij}^{-1} = f_j f_i^{-1}, \quad \omega_{ij}^1 - \bar{\omega}_{ij}^1 = (\delta \omega_{ij}^1),
\]

---

\(^3\)These two equations are deduced from the coherence condition between \( g \)'s and \( A \)'s, the one for \( \omega_2 \)'s can be implied by the second equation in [10].

\(^4\)The same as before, the two equations are deduced from the coherence condition between \( g \)'s and \( A \)'s, and the one coming from \( \omega_2 \)'s can be implied by the second equation.
Then \( \text{BString}(n)_{\mathcal{P}}^+ : \text{Mfd}^\mathcal{P} \to 2\text{Gpd} \) is a \((3, 1)\)-sheaf consists of

- \( \text{BString}(n)_{\mathcal{P}}^+(M)_0 \): an object is a pair \( \{(U_i), P_c\} \), where \( \{U_i\} \) is an open cover of \( M \) and \( P_c \) is an element in \( \text{holim} \text{BString}(n)_{\mathcal{P}}(U(M)_\bullet)_0 \);
- \( \text{BString}(n)_{\mathcal{P}}^+(M)_1 \): a 1-morphism between \( \{(U_i), P_c\} \) and \( \{(\tilde{U}_i), \tilde{P}_c\} \) is a pair consisting of a common refinement \( \{V_i\} \) of \( \{U_i\} \) and \( \{U_i\} \) and an element \( \phi_c \in \text{holim} \text{BString}(n)_{\mathcal{P}}(V(M)_\bullet)_1 \);
- \( \text{BString}(n)_{\mathcal{P}}^+(M)_2 \): a 2-morphism between \( \{(V_i), \phi_c\} \) and \( \{(\tilde{V}_i), \tilde{\phi}_c\} \) consists of a common refinement \( \{W_i\} \) of \( \{V_i\} \) and \( \{V_i\} \) and an element \( \alpha_c \in \text{holim} \text{BString}(n)_{\mathcal{P}}(W(M)_\bullet)_2 \). Moreover, \( \{(W_i), \alpha_c\} \) and \( \{(\tilde{W}_i), \tilde{\alpha}_c\} \) are identified if \( \alpha_c \) and \( \tilde{\alpha}_c \) agree on a further common refinement of \( \{W_i\} \) and \( \{\tilde{W}_i\} \).

**Remark 3.1.** For simplicity, we call an element \( P_c \in \text{holim} \text{BString}(n)_{\mathcal{P}}(U(M)_\bullet)_0 \) a string data. The construction of our string (pre)sheaf works for a general compact Lie group \( G \) with the extension class \( \mathcal{P} \) a multiple of \( p_1 \), as long as we adjust the coefficient in the front of the Chern-Simons 3-form by this multiple also.

### 3.3 Lifting Theorem and Comparison

As we state in the introduction, there are already several ways to grasp the concept of string structure. Redden’s string class is probably the most accessible and concise, while Waldorf’s method includes concepts involving differential forms, such as Courant algebroids, descent equations, and Deligne cohomology. We found it also much easier in this language to connect with concepts involving differential forms, such as Courant algebroids, descent equations, and Deligne cohomology. We may always find fine enough good covers, this direct comparison to previous methods might be a wrong approach to see the nature of the problem since both sides (especially our side) involves heavy machinery. We remark (Remark 3.7 3.9 3.10 3.11) carefully on links of these concepts, and focus ourselves on the proof of lifting theorem from the viewpoint of Stolz-Teichner for justificiation.

Let \( BG_c \) be the \((2, 1)\)-sheaf of \( G \)-principal bundles with connections. We take the model \( BG_c^p \) for it. Thus comparing the construction of \( BString(n)_{\mathcal{P}}^+ \) with Example 2.1, we see that there is a natural projection \( BString(n)_{\mathcal{P}}^+ \longrightarrow BSpin(n)_c \), by forgetting higher data. More precisely, given an object

\[
P_c = (\sqcup U_i; U_i \times \text{String}(n) \to U_i, \theta_i, B_i; g_{ij}, A_{ij}, \omega_{ij}^2; f_{ijk}, \omega_{ijk}^1) \in \text{BString}(n)_{\mathcal{P}}^+,
\]

where \( (\theta_i, B_i; g_{ij}, A_{ij}, \omega_{ij}^2; f_{ijk}, \omega_{ijk}^1) \) satisfies equations \( \square, \Box, \) and \( \square, \) since the isomorphism of bibundles between \( g_{ij} : g_{jk} \rightleftharpoons g_{ik} \) is given by a \( U(1) \)-function \( f_{ijk} \), the projected \( Spin(n) \)-valued function, \( g_{ij} : U_{ij} \to Spin(n) \), satisfies strict cocycle condition \( g_{ij} \cdot g_{jk} = g_{ik} \). This gives us a \( Spin(n) \)-principal bundle \( \tilde{P} \). Furthermore, equation \( \square \) implies that \( \theta_i \) provides a connection on \( \tilde{P} \).

**Theorem 3.2.** An object \( M \xrightarrow{P_c} BSpin(n)_c \) in \( BSpin(n)_c \) over a fine enough good cover \( \{U_i\} \) of \( M \) lifts to an object in \( M \xrightarrow{\tilde{P}_c} \text{BString}(n)_{\mathcal{P}}^+ \) in \( \text{BString}(n)_{\mathcal{P}}^+ \)

\[
\begin{array}{ccc}
M & \xrightarrow{P_c} & BSpin(n)_c \\
\downarrow & & \\
\text{BString}(n)_{\mathcal{P}}^+ & \xrightarrow{\tilde{P}_c} & BSpin(n)_c,
\end{array}
\]

if and only if \( \frac{1}{2}p_1(\tilde{P}) = 0 \), where \( \tilde{P} \) is the \( Spin(n) \)-principal bundle that \( \tilde{P}_c \) glues to.

**Remark 3.3.** This theorem says that as long as a good cover is fine enough, the obstruction for a \( Spin(n) \) data \( P_c \) to lift to a \( \text{String}(n) \) data is \( \frac{1}{2}p_1(\tilde{P}_c) \). Since we may always find fine enough good covers, this theorem justifies that our construction \( \text{BString}(n)_{\mathcal{P}}^+ \) is indeed reasonable to be called the \((3, 1)\)-pre sheaf of \( \text{String}(n) \)-principal bundles.
We first prove some technical lemmas:

**Lemma 3.4.** Given a function \( \tilde{g} : U \to G \), one may always lift it to a morphism \( g : U \to \text{String}(G) \) if \( \tilde{g}^{-1}(G^1_\alpha) = U \times \tilde{g}_{G,pr} G^1_\alpha \)'s are contractible, where \( pr : \sqcup G^1_\alpha \to G \) is the covering map.

**Proof.** A morphism \( g : U \to \text{String}(G) \) is given by a bibundle \( E \) which is a \( \sqcup G^1_\alpha \times U(1) \to \sqcup G^1_\alpha \) principal bundle over \( U \). We know that the underlying morphism \( \tilde{g} : U \to G \) of \( g \) is given by the bibundle \( U \times \tilde{g}_{G,pr} \sqcup G^1_\alpha \), and \( E \) is an \( U(1) \)-bundle over \( U \). Since \( \tilde{g}^{-1}(G^1_\alpha) \)'s are contractible, \( E \cong U \times \tilde{g}_{G,pr} \sqcup G^1_\alpha \times U(1) \) as manifolds. Suppose that the action is given by

\[
(x, g_{\alpha}, a) \cdot (g_{\alpha}, a) := (x, g_{\beta}, a + a' + \lambda_{\alpha\beta}(x)), \quad \text{for } (x, g_{\alpha}, a) \in E, \text{ and } (g_{\alpha}, a') \in G^1_{\alpha\beta} \times U(1),
\]

for a certain function \( \lambda_{\alpha\beta} : U \to U(1) \). The associativity of the action is equivalent to the fact that \( (\delta \lambda)_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma} (\tilde{g}(x)) \). But \( H^2(U \times \tilde{g}_{G,pr} \sqcup G^1_\alpha, U(1)) = 0 \), thus 2-cocycle \( \tilde{g}^* \eta \) is always exact. Therefore we may always find such \( \lambda \).

We endow a simplicial nerve \( \mathcal{G}\bullet \) of \( BG \)–the nerve of \( G \). Suppose that the extension class \( p \in H^3(BG, U(1)) \cong H^4(BG \bullet, \mathbb{Z}) \) is represented by the \( U(1) \)-valued 3-cocycle \((\Theta, \Phi, \eta, 0)\) supported on this cover. The last entry being 0 is implied by the closedness. Notice that \( \tilde{g}_{ij} : U_{ij} \to G \) extends to a simplicial morphism \( \tilde{g}_\bullet \) from the simplicial nerve \( \mathcal{G}\bullet \) of the covering groupoid \( \sqcup U_{ij} \to \sqcup U_i \) to \( BG \bullet \). On each simplicial level \( U(M)_k = \sqcup U_{ij...i_k} \), we endow it with the pullback cover of the one on \( B_k \) pulled back by \( \tilde{g}_{ij} \). We may always start with a fine enough cover \( \{ U_i \} \) so that \( \tilde{g}_{ij}(U_i) \) is either entirely in \( G_\alpha \) or does not intersect \( G_\alpha \). Thus we may assume that both \( \{ U_i \} \) and the pullback covers are good. Then the simplicial-Čech double complex (11) calculates the cohomology \( H^\bullet(U(M)_\bullet, U(1)) \cong H^\bullet(M, U(1)) \).

We denote the pullback cocycle \( \tilde{g}_\bullet^* (\Theta, \Phi, \eta, 0) \) by \((\tilde{\Theta}, \tilde{\Phi}, \tilde{\eta}, 0)\) and it is a cocycle in double complex (11) representing a class \( p(P_c) \in H^3(M, \mathbb{U}(1)) \).

\[
\begin{array}{cccc}
\delta & \delta & \delta & \delta \\
C(\sqcup U_{ijkl}, U(1)) & C(\sqcup U_{ijkl}, U(1)) & \cdots & \\
\delta & \delta & \delta & \delta \\
C(\sqcup U_{ij}, U(1)) & C(\sqcup U_{ij}, U(1)) & \cdots & \\
\delta & \delta & \delta & \delta \\
C(\sqcup U_{ij}, U(1)) & C(\sqcup U_{ij}, U(1)) & \cdots & \\
\delta & \delta & \delta & \delta \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}
\tag{11}
\]

**Lemma 3.5.** The 2-morphism \( (\text{id} \circ \alpha, a, 0, 0) \) in diagram (7) is given by \( U(1) \)-valued functions \( F_{ijkl} : U_{ijkl} \to U(1) \). Moreover, if the cover is fine enough, \( F_{ijkl} \)'s give rise to a cocycle representing \( \tilde{\eta}_1(P_c) \in H^3(M, \mathbb{U}(1)) \).

**Proof.** We continue to use the notation and a fine enough cover \( \{ U_i \} \) given just before this lemma. For us now \( G = \text{Spin}(n) \). Since \( \tilde{H}^2(\sqcup U_{ij}, U(1)) = 0 \), \( (\delta \tilde{\eta}) = 0 \) implies that \( \tilde{\eta} = -\delta \lambda \). We continue this tic-tac-toe procedure, since \( \tilde{H}^2(\sqcup U_{ijkl}, U(1)) = \tilde{H}^2(\sqcup U_{ijkl}, U(1)) = 0 \), we have

\[
\tilde{\eta} = -\delta \lambda, \quad \tilde{\Phi} = \delta \lambda + \delta \varphi, \quad \tilde{\Theta} = \delta \varphi + F,
\tag{12}
\]
and $F$ is a function $U_{ijkl} \to U(1)$. A calculation shows that the bibundle of $(g_{ij} \circ g_{jk}) \circ g_{kl}$ is

$$U_{ijkl} \times_{G^3} \sqcup G^3_s \times (U(1))^3 \times \sqcup G_\alpha \sqcup G_\beta \times U(1)/ \sqcup G^3_{s,t} \times (U(1))^3,$$

where the action is given by \[ (x_{ijkl}, w_s, a_1, a_2, a_3, g_{\alpha, \beta}, a) \cdot (w_{s,t}, a'_1, a'_2, a'_3) \]

$$= (x_{ijkl}, w_t, a_1 + a'_1 + \lambda_{\alpha, \gamma}(x_{ij}), a_2 + a'_2 + \lambda_{\alpha, \gamma}(x_{jk}), a_3 + a'_3 + \lambda_{\alpha, \gamma}(x_{kl}), g_{\gamma, \beta},$$

$$a - a'_1 - a'_2 - a'_3 - d^2_{ijkl} \Phi(w_{s,t}) - d^2_{ijkl} \Phi(w_{s,t}) - \eta_{\alpha, \beta, \gamma}(x_{ij})).$$

This bibundle is isomorphic to the bibundle $U_{ijkl} \times_G \sqcup G_\alpha \times U(1)$ through

$$[(x_{ijkl}, w_s, a_1, a_2, a_3, g_{\alpha, \beta}, a)] \mapsto [(x_{ijkl}, g_{\beta}, a + a_1 + a_2 + a_3 + \varphi_p(v_0) + \varphi_p(v_2) + \lambda_{\alpha, \beta}(x_{ijkl}))],$$

where the right action of $\sqcup G_{\alpha, \beta} \times U(1) \Rightarrow \sqcup G_\alpha$ is given by

$$(x_{ijkl}, g_{\beta}, a) \cdot (g_{\beta', \beta}, a') = (x_{ijkl}, g_{\beta'}, a + a' + \lambda_{\beta, \beta'}(x)).$$

The morphism $\psi$ is well-defined thanks to the second equation in (12), and it is a bibundle isomorphism thanks to the first equation in (12).

The bibundle of $(g_{ij} \circ (g_{jk} \circ g_{kl})$ is given by exactly the same form but the quotient is given by a different action,

$$(x_{ijkl}, w_s, a_1, a_2, a_3, g_{\alpha, \beta}, a) \cdot (w_{s,t}, a'_1, a'_2, a'_3)$$

$$= (x_{ijkl}, w_t, a_1 + a'_1 + \lambda_{\alpha, \gamma}(x_{ij}), a_2 + a'_2 + \lambda_{\alpha, \gamma}(x_{jk}), a_3 + a'_3 + \lambda_{\alpha, \gamma}(x_{kl}), g_{\gamma, \beta},$$

$$a - a'_1 - a'_2 - a'_3 - d^2_{ijkl} \Phi(w_{s,t}) - d^2_{ijkl} \Phi(w_{s,t}) - \eta_{\alpha, \beta, \gamma}(x_{ij})).$$

Similarly, this bibundle is also isomorphic to the same bibundle $U_{ijkl} \times_G \sqcup G_\alpha \times U(1)$ through

$$[(x_{ijkl}, w_s, a_1, a_2, a_3, g_{\alpha, \beta}, a)] \mapsto [(x_{ijkl}, g_{\beta}, a + a_1 + a_2 + a_3 + \varphi_p(v_1) + \varphi_p(v_3) + \lambda_{\alpha, \beta}(x_{ijkl}))].$$

The 2-morphism $\text{id} \circ \eta_{\alpha}$ is then to add $\Theta$ on the last $U(1)$ component, and is explicitly given by

$$(x_{ijkl}, w_s, a_1, a_2, a_3, g_{\alpha, \beta}, a) \mapsto (x_{ijkl}, w_s, a_1, a_2, a_3, g_{\alpha, \beta}, a - \Theta(w_s)),$$

before quotient. This map is equivariant with respect to the above two actions because $\delta \Phi = \delta \Theta$, thus it descends to the quotients. Under the isomorphism $\psi$ and $\psi'$, this 2-morphism is then given by

$$\psi'(\text{id} \circ \eta_{\alpha} a(\psi^{-1}(x_{ijkl}, g_{\beta}, a))) = (x_{ijkl}, g_{\beta}, a + F_{ijkl}(x)),$$

guaranteed by the last equation of (12). Moreover, $[[F, 0, 0, 0]] = [[\Theta, \Phi, \eta, 0]] = \frac{1}{2} p_1(T_c) \in \tilde{H}^3_{\bullet}(M, U(1))$. Thus $F$ is a representative of $\frac{1}{2} p_1(T_c)$. \[ \square \]

Now we are ready to prove Theorem 3.2

**Proof.** If there is a lifting object $P_c \in B\text{String}(n)_c$ over an object $M \to B\text{Spin}(n)_c$ over $\{U_i\}$ in $B\text{Spin}_c$, we write

$$P_c = (\sqcup U_i ; U_i \times \text{String}_c(n)) \to U_i, \theta_i, B_i; g_{ij}, A_{ij}, \omega^2_{ij}; f_{ijk}, \omega^1_{ijk}),$$

where $(\theta_i, B_i; g_{ij}, A_{ij}, \omega^2_{ij}; f_{ijk}, \omega^1_{ijk})$ satisfies equations (6), (6) and (8). Then (8) and Lemma 3.3 implies that $\frac{1}{2} p_1(T_c) = 0$.\footnote{Recall that $\{U_i\}$ is fine enough so that $g_{ij}(U_{ij})$ is either entirely in $G_\alpha$ or does not intersect $G_\alpha$. Then by Lemma 3.4 one may always take trivial bibundles for $g_{ij}$, therefore their various composites.}

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For the other direction, we first do some preparation: given an object $M \xrightarrow{\varphi} \mathbf{BSpin}(n)_e$ over a good cover $\{U_i\}$ of $M$ in $\mathbf{BSpin}(n)_c$, we take a $U(1)$-cocycle $F$ representing $\frac{1}{2} p_1(\varphi) \in \bar{H}^3_{U_e}(M, U(1)) = H^3(M, U(1))$. Let
\begin{equation}
D_m = U(1) \xrightarrow{d \log} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^m
\end{equation}
be the Deligne sheaf of depth $m$. Recall that the Deligne cohomology $H^\bullet(M, D_3)$ is then the limit of the total cohomology of the following double complex taking over all covers $\{U_i\}$ of $M$,
\begin{equation}
\begin{pmatrix}
C(U(M)_*, U(1)) & \xrightarrow{d \log} \Omega^1(U(M)_*) & \xrightarrow{d} \Omega^2(U(M)_*) & \xrightarrow{d} \Omega^3(U(M)_*) & \delta
\end{pmatrix}
\end{equation}
Then the general theory of Deligne cohomology tells us that there is a surjective morphism $H^\bullet(M, D_3) \xrightarrow{\pi} H^\bullet(M, U(1))$ given by forgetting the part of differential forms of a Deligne cocycle. Moreover, there is a morphism $H^3(M, D_3) \xrightarrow{d} \Omega^4(M)$ to closed 4-forms by $[\{F, \omega^1, \omega^2, \omega^3\}] \mapsto d\omega^3$. Here notice that $\omega^3 = (\omega^3_i)$ is made up by local 3-forms on each $U_i$. However, $d\omega^3$ glue together to a closed global 4-form, which we denote by $d\omega^3$, and it is independent of the choice of the Deligne cocycle. The above two morphisms fit into the following commutative diagram:

\begin{equation}
\begin{array}{ccc}
H^3(M, D_3) & \xrightarrow{d} & \Omega^4(M) \\
\downarrow \pi & & \downarrow \delta
\\
H^3(M, U(1)) & \xrightarrow{\delta} & H^4(M, \mathbb{Z})
\end{array}
\end{equation}

Since the natural morphism $H^3(M, D_3) \rightarrow H^3(M, U(1))$ is surjective, we lift $F$ to a Deligne cocycle $(F, \omega^1, \omega^2, \omega^3)$, that is
\begin{equation}
\delta \omega^1 - d \log F = 0, \quad \delta \omega^2 + d \omega^1 = 0, \quad \delta \omega^3 - d \omega^2 = 0.
\end{equation}
Now we adjust $\omega^3$ to be $\text{cs}_3(\theta_i)$, where $\theta_i$ is the connection data with respect to the cover $\{U_i\}$ of $\varphi$. Both $d\omega^3$ and $d\text{cs}_3(\theta_i)$ give to closed global 4-forms, and both represent the deRham classes $\frac{\partial}{\partial t}$ of the natural morphism $\pi: \Omega(U) \rightarrow \Omega(U)$ of Lemma 3.4, where $\gamma_i \in \Omega^3(M)$ and $\beta_i \in \Omega^2(U_i)$. Then it is easy to verify that the Deligne class $(F, \omega^1, \omega^2 + \delta \beta, \text{cs}_3(\theta_i))$ is a lift of $F$. Thus we can begin with a Deligne cocycle $(F, \omega^1, \omega^2, \omega^3)$ with $\omega^3 = \text{cs}_3(\theta_i)$.

Now we construct a lift of $\varphi$ with respect to the cover $\{U_i\}$ under the condition $\frac{1}{2} p_1(\varphi) = 0$. Fix a good cover $G^{\bullet}(\cdot)$ of $\mathbf{BSpin}(n)_e$ as in the construction of $\mathbf{String}(n)$ in Subsection 3.1. Refining $\{U_i\}$ if necessary, we may assume that $\hat{g}_{ij}(U_{ij})$ either falls entirely into $G^{(1)}_n$ or does not intersect $G^{(0)}_n$. Then the condition in Lemma 3.4 is naturally fulfilled. Thus, there is no obstruction to lift the transition functions $\hat{g}_{ij}$ for $\varphi$ to $g_{ij}: U_{ij} \rightarrow \mathbf{String}(n)$. Since $\frac{1}{2} p_1(\varphi) = 0$, we may take a primitive $f$ of $F$, that is $F = \delta f$. Since $d \log F = \delta \omega^1$, $\omega^1 = \delta A + d \log f$ for some $A = (A_{ij}) \in \Omega^1(U_{ij})$. We continue such a tic-tac-toe process, and find

\begin{equation}
F = \delta f, \quad \omega^1 = \delta A + d \log f, \quad \omega^2 = \delta B - dA,
\end{equation}
for $f = (f_{ijk}) \in U(1)(\cup U_{ijk})$, $A = (A_{ij}) \in \Omega^1(\cup U_{ij})$, and $B = (B_i) \in \Omega^2(\cup U_i)$. Both $g_{ij} \circ g_{jk}$ and $g_{ik}$ are morphisms from $U_{ijk}$ to $\mathbf{String}(n)$. They are presented by isomorphic fibrebundles from the discrete groupoid $U_{ijk} \rightarrow U_{ijk} \rightarrow \Gamma[\eta]$ with a similar construction to that of $\psi$ in Lemma 3.5. Then as in the proof of Lemma 3.4, we see that a $U(1)$-valued function $f_{ijk}$ on $U_{ijk}$ serves as a 2-morphism $g_{jk} \Rightarrow g_{ij} \circ g_{jk}$ because $U(1) \rightarrow pt$ is a subgroupoid of $\Gamma[\eta]$ and sits in the center of it. Thus, $(\cup U_i; U_i \times \mathbf{String}(n) \rightarrow U_i, \theta_i, B_i; g_{ij}, A_{ij}, \omega^3_{ij}; f_{ijk}, \omega^3_{ijk})$ is a lift of $\varphi$.}

As shown in [44], if a $\mathbf{Spin}(n)$-principal bundle $\varphi$ admits string classes then the possible choices of the string classes form a torsor of $H^3(M, \mathbb{Z})$. Then later in [40], the author further showed that for a fixed
Chern-Simons 2-gerbe over \( \bar{P} \), the choices of trivialisations modding out isomorphisms correspond exactly to string classes on \( \bar{P} \).

We see that inside an object \( P_c \in \text{BString}(n) \), the determining information, is a covering \( \{ U_i \} \) together with \( (\theta_i, B_i; g_{ij}, A_{ij}, f_{ijk}) \), other terms \( (F_{ijkl}, \omega^1_{ij}, \omega^2_{ij}, \omega^3_{ij} = cs_3(\theta_i)) \) are determined by these terms through \( \text{(16)} \). These terms \((F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, \omega^3_{ijk} = cs_3(\theta_i))\) representing a refinement of \( \frac{1}{2}\mathbb{P}_1(\bar{P}_c) \in H^3(M, U(1)) \), may be viewed as the information for a given Chern-Simons 2-gerbe over \( \bar{P}_c \) and its connections. Thus, after adding connection data inside, we may ask ourselves again how many string data lift \( \bar{P}_c \in \text{BSpin}(n) \), if we fix a choice of the cocycle \((F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, \omega^3_{ijk} = cs_3(\theta_i))\).

The Deligne cohomology group \( H^2(M, D_2) \) may be viewed as a refinement of \( H^2(M, U(1)) \cong H^3(M, \mathbb{Z}) \), where \( D_2 \) is the Deligne sheaf defined in \( \text{(13)} \). We have the following diagram (see also \( \text{(14)} \))

\[
\begin{array}{cccc}
\ker \pi_2 \cap \ker d & \rightarrow & H^2(M, D_3) & = \ker d \\
\downarrow & & \downarrow \pi_3 & \rightarrow \pi_3 \\
\ker \pi_2 & \rightarrow & H^2(M, D_2) & \rightarrow \Omega^3_{cl}(M)
\end{array}
\]

Since we have

\[
H^3(M, \mathbb{Z}) = H^2(M, D_3) / \ker \pi_3 \cong H^2(M, D_2) / \ker \pi_3 \rightarrow H^3(M, \mathbb{Z}) = H^2(M, D_2) / \ker \pi_2,
\]

and \( H^3(M, \mathbb{Z}) \) may be viewed naturally both as a subgroup and a quotient of our group, we show that different lifts of \( \bar{P}_c \) modding out isomorphisms is a torsor of \( H^2(M, D_2) / \ker \pi_3 \) in the next theorem.

**Theorem 3.6.** Given an object \( M \xrightarrow{P} \text{BSpin}(n) \), over a good cover \( \{ U_i \} \) of \( M \) in \( \text{BSpin}(n) \), and a fixed Deligne cocycle \((F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i))\) representing a refinement of \( \frac{1}{2}\mathbb{P}_1(\bar{P}_c) \in H^3(M, U(1)) \), let us denote the set of all possible \( P_c \)'s lifting \( \bar{P}_c \) with fixed \((F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i))\) by \( S(F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i)) \). Then the Deligne cohomology group \( H^2(M, D_2) \) acts on

\[
S(F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i)) / \text{1-morphisms},
\]

This action descends to the quotient \( H^2(M, D_2) / \ker \pi_3 \) and makes \( S(F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i)) / \text{1-morphisms} \) a \( (H^2(M, D_2) / \ker \pi_3) \)-torsor.

**Proof.** Given a cocycle \((f^h, A^h, B^h)\) representing an element in \( H^2(M, D_2) \), it acts on \( S(F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i)) \) by

\[
(U_i, \theta_i, B_i; g_{ij}, A_{ij}, f_{ijk}) \mapsto (U_i, \theta_i, B_i + B^h_i; g_{ij}, A_{ij} + A^h_{ij}, f_{ijk} + f^h_{ijk}).
\]

If \((f^h, A^h, B^h) = D(\varphi, \alpha)\), then the above two elements in \( S(F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i)) \) are connected by an isomorphism \( (-1, -\alpha, 0; \varphi, 0) \). Thus \( \text{(17)} \) gives rise to an action of \( H^2(M, D_2) \) on \( S(F_{ijkl}, \omega^1_{ijk}, \omega^2_{ijk}, cs_3(\theta_i)) / \text{1-morphisms} \).

Now we prove that the action of a cocycle \((f^h, A^h, B^h)\) induces an isomorphism if and only if \([f^h, A^h, B^h] \in \ker \pi_2 \cap \ker d\). This then will complete the proof of our statement.

First of all, the isomorphism induced by \((f^h, A^h, B^h)\) is possibly given through another finer cover \( \{ V_i \} \). However as we may always pull back our cocycle to \( V_i \), we might as well assume that \( V_i = U_i \). By a direct calculation, \((f^h, A^h, B^h)\) induces an isomorphism \((1, A_i, \omega^2_i; f_{ij}, \omega^1_{ij})\), if and only if

\[
B^h_i = \omega^2_i + dA_i, \quad d\omega^2_i = 0, \quad A^h_{ij} = \delta(A_{ij}) - \omega^1_{ij} + d\log f_{ij}, \quad \delta \omega^2_i = -d\omega^1_{ij}, \quad (\delta f_{ij})_{ijk} = (f^h_{ijk})^{-1}.
\]
Thus one direction is clear. If \((f^h, A^h, B^h)\) induces an isomorphism, then \(dB_i^h = 0\) and \(f^h\) is a coboundary, which exactly shows that \(d((f^h, A^h, B^h)) = 0\) and \(\pi_2(([f^h, A^h, B^h])) = 0\) respectively.

For the other direction, if \(dB_i^h = 0\) and \(f^i_{ijk} = (\delta f_{..})_{ijk}\) for some \(f_{..}\), then \((1, 0, B_i^h; f_{ij}, -A_{ij}^h + d\log f_{ij})\) gives us a desired isomorphism.

**Remark 3.7.** If we forget the connection data inside a string data \(P_c\) and only remember \((U_i; g_{ij}; f_{ijk})\), then the action of \(H^2(M, D_2)\) simplifies to that of \(H^3(M, \mathbb{Z})\) through \(\pi_2\). Thus we recover the structure of the torsor in [44] for the string structures over a Spin\((n)\)-principal bundle via the projection \(H^2(M, D_2)/\ker\pi_3 \to H^3(M, \mathbb{Z})\).

For a string data \(P_c = (U_i \times \text{String}(n), \theta_i, B_i; g_{ij}, A_{ij}; f_{ijk})\), we see that \(\delta(\text{cs}_3(\theta_i) - dB_i) = 0\) by (3), thus \(\{\text{cs}_3(\theta_i) - dB_i\}\) give rise to a global 3-form \(H\) on \(M\). We define \(H\) to be the curvature of \(P_c\). By (3), we see that isomorphic string data over the same \(P_c\) has the same curvature. Notice that when \((f^h, A^h, B^h)\) acts on \(P_c\), the curvature is changed by \(dB_i^h\) which glue to a global closed 3-form \(d((f^h, A^h, B^h)) \in \Omega^3(M)\). Thus we have the following corollary,

**Corollary 3.8.** Given an object \(M \to \mathcal{P} \to \text{BSpin}(n)\), over \(M\), and a Deligne cocycle \((F_{ijk}, \omega^1_{ijk}, \omega^2_{ij}, \text{cs}_3(\theta_i))\), the curvatures of all possible \(P_c\)'s lifting \(\mathcal{P}\) with fixed \((F_{ijk}, \omega^1_{ijk}, \omega^2_{ij}, \text{cs}_3(\theta_i))\) form a torsor of \(\text{im }\delta\).

**Remark 3.9.** We notice the following commutative diagram

\[
\begin{array}{ccc}
H^2(M, D_2) & \xrightarrow{\pi_2} & H^3(M, \mathbb{Z}) \\
\downarrow d & & \downarrow \otimes \mathbb{R} \\
\Omega^3_{cl}(M) & \xrightarrow{} & H^3(M, \mathbb{R}).
\end{array}
\]

One may interpret \(\text{im }\delta\) as “integral” forms. Certainly, \(\delta\) is not always surjective.

**Remark 3.10.** We conjecture that when we glue the local data of a string data, we will obtain an \(S^1\)-gerbe over the underlying \(\text{Spin}(n)\)-principal bundle \(\mathcal{P}\). This gives us the access to Redden’s string class. For this problem, a possible way is to apply the descent for \(n\)-bundles of Wolfson [44] Theorem 5.7) to realize the gluing process. The difficulty would lie further on gluing differential forms to obtain an integral form which presents the string class. Notice that already in Redden’s thesis, there was no explicit formula to adjust to an integral form for the string class. We thus leave it for future investigation.

**Remark 3.11.** To compare thoroughly with Waldorf’s method using trivialization of Chern-Simons 2-gerbe is neither a simple task. Both definitions use heavy machineries, one with bundle gerbe theory, and the other with stack theory. Nevertheless, some traces of equivalence are rather visible. In [46], Definition 2.2.1], if we take \(Y\) to be \(\sqcup_i U_i\), then bundle gerbe \(S\) characterised by a closed 3-form, over each \(U_i\), corresponds to 2-form \(B_i\) in a string data, since a closed form is locally exact. Similarly the bundle gerbe \(\mathcal{P}\) corresponds to \(\omega^1_{ij}\), isomorphism \(A\) corresponds to \(A_{ij}\), \(\mathcal{M}\) corresponds to \(\omega^2_{ij}\), and \(\sigma\) corresponds to \(f_{ijk}\). Then various coherence conditions correspond to our descent equations for differential forms.

4\(^{\text{Here we thank Konrad Waldorf for very helpful conversations.}}\) (2, 1)-sheaf \(\text{T}\mathcal{C}^{\mu+}_c\) of transitive Courant algebroids with connections

The notion of a Courant algebroid was introduced in [26]. See also [32, 33, 34, 41] for various other aspects of Courant algebroids.
Definition 4.1. A Courant algebroid is a vector bundle $C$ together with a bundle map $\rho : C \to TM$, a nondegenerate symmetric bilinear form $\langle -, - \rangle$, and an operation $\llbracket - , - \rrbracket : \Gamma(C) \times \Gamma(C) \to \Gamma(C)$ such that for all $e_1, e_2, e_3 \in \Gamma(C)$, the following axioms hold:

(i) $(\Gamma(C), \llbracket - , - \rrbracket)$ is a Leibniz algebra;

(ii) $\langle \llbracket e_1, e_1 \rrbracket, e_2 \rangle = \frac{1}{2} \rho(e_2) \langle e_1, e_1 \rangle$;

(iii) $\rho(e_1) \langle e_2, e_3 \rangle = \langle \llbracket e_1, e_2 \rrbracket, e_3 \rangle + \langle e_2, \llbracket e_1, e_3 \rrbracket \rangle$.

A Courant algebroid $(C, \llbracket - , - \rrbracket, \langle -, - \rangle)$ is called transitive if $\rho$ is surjective, that is, $\text{im} \rho = TM$. A transitive Courant algebroid is an extension of a transitive Lie algebroid. However, not every transitive Lie algebroid $A$ admits such a Courant extension. The obstruction is given by the first Pontryagin class [16, Proposition 2.2]. Here the bracket $\llbracket - , - \rrbracket$ is a Leibniz algebra, whose fiber is isomorphic to a quadratic Lie algebra $(\mathfrak{g}, \langle -, - \rangle^\mathfrak{g})$. We will also use $(\langle -, - \rangle^\mathfrak{g})$ to denote the fiberwise metric on $\mathcal{G}$. A connection of a transitive Courant algebroid $C$ consists of the following data:

- an isotropic splitting $s : TM \to C$ of the short exact sequence (21);
- a splitting $\sigma_s : \mathcal{G} \to \ker \rho$ of the short exact sequence (22) that is orthogonal to $s(TM)$ in $C$, i.e. $\langle s(X), \sigma_s(a) \rangle = 0$ for all $X \in \Gamma(TM)$ and $a \in \Gamma(\mathcal{G})$.

In [16], the authors show that splittings $s$ and $\sigma_s$ always exist. A connection gives rise to an isomorphism $C \cong TM \oplus \mathcal{G} \oplus T^*M$ between vector bundles. Transferring the Courant algebroid structure on $C$ to $TM \oplus \mathcal{G} \oplus T^*M$, we obtain the transitive Courant algebroid $(TM \oplus \mathcal{G} \oplus T^*M, \llbracket - , - \rrbracket^T_{\nabla, R, H}, \langle -, - \rangle^T, \text{pr}_{TM})$, which is determined by a connection $\nabla$ on $\mathcal{G}$, a 2-form $R \in \Omega^2(M, \mathcal{G})$ and a 3-form $H \in \Omega^3(M)$, which obey a set of conditions given in [16, Proposition 2.2]. Here the bracket $\llbracket -, - \rrbracket^T_{\nabla, R, H}$ and the pairing $\langle -, - \rangle^T$ are defined by

\[
\llbracket X + a + \xi, Y + b + \eta \rrbracket^T_{\nabla, R, H} = [X, Y] + \nabla_X b - \nabla_Y a + [a, b]_{\mathcal{G}} + R(X, Y) + L_X \eta - i_Y d\xi + P(a, b) - 2Q(X, b) + 2Q(Y, a) + H(X, Y),
\]

\[
\langle X + a + \xi, Y + b + \eta \rangle^T = \frac{1}{2} (\xi(Y) + \eta(X)) + (a, b)^{\mathcal{G}},
\]

where $P : \Gamma(\mathcal{G}) \otimes \Gamma(\mathcal{G}) \to \Omega^1(M)$ and $Q : \mathfrak{X}(M) \otimes \Gamma(\mathcal{G}) \to \Omega^3(M)$ are given by

\[
P(a, b)(Y) = 2(b, \nabla Y a)^{\mathcal{G}},
\]

\[
Q(X, a)(Y) = (a, R(X, Y))^{\mathcal{G}}.
\]
In particular, if \( G \) is the trivial bundle \( M \times g \) and the connection is given by \( \nabla_Xa = X(a) \), we obtain the **standard transitive Courant algebroid** structure on \( TM \oplus (M \times g) \oplus T^*M \) with the Courant bracket given by
\[
[X + a + \xi, Y + b + \eta]^T_S = [X, Y] + X(b) - Y(a) + [a, b]_g + L_X \eta - i_Y d\xi + \mathcal{P}(a, b),
\]
where \( \mathcal{P} : \Gamma(M \times g) \oplus \Gamma(M \times g) \to \Omega^1(M) \) is given by
\[
\mathcal{P}(a, b)(Y) = 2(b, Y(a))^g.
\]
For simplicity, for an object \( U \in \text{Mfd} \), we write
\[
\mathcal{T}_g U := TU \oplus (U \times g) \oplus T^*U.
\]

According to [16, Proposition 2.7], automorphisms of the standard transitive Courant algebroid are given as follows.

**Corollary 4.2.** An automorphism of the standard transitive Courant algebroid \( (\mathcal{T}_g M, [\cdot, \cdot]^T_S, \langle \cdot, \cdot \rangle^T, \text{pr}_T M) \),
where \([\cdot, \cdot]^T_S\) and \( \langle \cdot, \cdot \rangle^T \) are given by [27] and [28] respectively, is of the form
\[
\begin{pmatrix}
1 & \phi & \tau \\
\phi & \tau & 0 \\
\beta & -2\phi^* \tau & 1
\end{pmatrix},
\]
where \( \tau \) is an orthogonal automorphism of the bundle of quadratic Lie algebras \( M \times g \) and \( \beta : TM \to M \times g \) and \( \beta : TM \to T^*M \) are bundle maps satisfying the following compatibility conditions:
\[
\frac{1}{2}(\beta(X)(Y) + \beta(Y)(X)) + (\phi(X), \phi(Y))^g = 0, \quad \text{(28)}
\]
\[
\tau(X(b)) - X(\tau(b)) - [\phi(X), \tau(b)]_g = 0, \quad \text{(29)}
\]
\[
d\phi + \frac{1}{2}[\phi, \phi]^g = 0, \quad \text{(30)}
\]
\[
L_X(\beta(Y)) - i_Y d(\beta(X)) - \beta([X, Y]) + \mathcal{P}(\phi(X), \phi(Y)) = 0. \quad \text{(31)}
\]

Here \( \phi^* : M \times g \to T^*M \) is defined by
\[
\phi^*(a)(X) = (a, \phi(X))^g, \quad \forall a \in \Gamma(M \times g), X \in \Gamma(TM).
\]

There is a \((2, 1)\)-presheaf of transitive Courant algebroids with connections \( \text{TC}^p : \text{Mfd}^{op} \to \text{Gpd} \), where \( \text{Mfd}^{op} \) is the opposite category of \( \text{Mfd} \), and \( \text{Gpd} \) is the 2-category of (discrete) groupoids and groupoid morphisms.

For an object \( U \in \text{Mfd} \), the groupoid \( \text{TC}^p(U) \) is made up by the following data:

- \( \text{TC}^p(U)_0 \): an object is a 6-tuple \((\mathcal{T}_g U, [\cdot, \cdot]^T_S, \langle \cdot, \cdot \rangle^T, \text{pr}_T U, \theta, B)\), where \( \theta \in \Omega^1(U, g) \) is a \( g \)-valued 1-form, \( B \in \Omega^2(U) \) is a 2-form, and \([\cdot, \cdot]^T_S\) and \( \langle \cdot, \cdot \rangle^T \) are given by [27] and [28] respectively. We will simply denote an object by \((\mathcal{T}_g U, \theta, B)\) in the sequel.
- \( \text{TC}^p(U)_1 \): a 1-morphism from \((\mathcal{T}_g U, \tilde{\theta}, \tilde{B})\) to \((\mathcal{T}_g U, \theta, B)\) is an automorphism of the standard transitive Courant algebroid \((\mathcal{T}_g U, [\cdot, \cdot]^T_S, \langle \cdot, \cdot \rangle^T, \text{pr}_T U)\) given by the matrix
\[
\begin{pmatrix}
1 & \phi & \tau \\
\phi & \tau & 0 \\
\beta & -2\phi^* \tau & 1
\end{pmatrix}
\]
such that
\[
\theta(X) = \tau\tilde{\theta}(X) + \phi(X), \quad \text{(33)}
\]
\[
i_X(\tilde{B} - B) = \beta(X) + (\theta, \theta(X))^g - (\tilde{\theta}, \tilde{\theta}(X))^g - 2\phi^* \tau(\tilde{\theta}(X)), \quad \text{(34)}
\]
\[
cs_3(\tilde{\theta}) - cs_3(\theta) = d(\tilde{B} - B). \quad \text{(35)}
\]

Here the Chern-Simon 3-form \( cs_3(\theta) \) of \( \theta \) is a 3-form on \( U \) defined by [41] using bilinear form \( \langle \cdot, \cdot \rangle^g \).

The composition of 1-morphisms is simply the matrix multiplication.
Then for a morphism $\varphi : U \to V$ in $\text{Mfd}$, the associated functor $\text{TC}^p_c(\varphi) : \text{TC}^p_c(V) \to \text{TC}^p_c(U)$ is induced by pulling back forms.

Take an open cover $\{U_i\}$ of $M \in \text{Mfd}$. An object in $\text{holim} \text{TC}^p_c(U(M)\bullet)$ consists of

- an object $\sqcup(T_\varphi U_i, \theta_i, B_i)$ in $\text{TC}^p_c(\sqcup U_i)_0$,
- a 1-morphism $\Lambda_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ \phi_{ij} & \tau_{ij} & 0 \\ \beta_{ij} & -2\phi^*_{ij} \tau_{ij} & 1 \end{pmatrix}$ in $\text{TC}^p_c(\sqcup U_{ij})_1$ from $(T_\varphi U_{ij}, \theta_{ij}, B_{ij}|_{U_{ij}})$ to $(T_\varphi U_{ij}, \theta_{ij}|_{U_{ij}}, B_{ij}|_{U_{ij}})$.

This implies that

$$\theta_i(X) = \tau_{ij} \phi_j(X) + \phi_{ij}(X), \quad \beta_{ij} - 2\phi^*_{ij} \tau_{ij} \beta_{kj} = 0$$

$$\text{cs}_3(\theta_j) - \text{cs}_3(\theta_i) = d(B_j - B_i).$$

- compatibility conditions $\Lambda_{ij} \Lambda_{jk} = \Lambda_{ik}$ on $U_{ijk}$, which are equivalent to the following equations

$$\phi_{ij} + \tau_{ij} \phi_{kj} = \phi_{ik}, \quad \beta_{ij} = \beta_{ik},$$

$$\tau_{ij} \tau_{kj} = \tau_{ik}, \quad -2\phi^*_{ij} \tau_{ij} \tau_{kj} - 2\phi^*_{ik} \tau_{ik} = -2\phi^*_{ij} \tau_{ij} \phi_{kj} + \beta_{ij}.$$  

**Definition 4.3.** An object $C_c = \bigcup(T_\varphi U_i, [-, -]_T, \langle - , - \rangle_T, \theta_i, \tau_{ij}, \beta_{ij})$ in $\text{holim} \text{TC}^p_c(U(M)\bullet)$ is called a transitive Courant data. For simplicity, we also denote $C_c$ by $(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})$ when there is no confusion.

Note that $\text{holim} \text{TC}^p_c(U(M)\bullet)$ of the $(2, 1)$-presheaf $\text{TC}^p_c$ might be empty. When it is not, we may describe the objects and morphisms by the following two propositions. We then describe the condition for $\text{holim} \text{TC}^p_c(U(M)\bullet)$ to be non-empty.

**Proposition 4.4.** A transitive Courant data $C_c = (\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})$ gives rise to a transitive Courant algebroid $(C, [-, -], \langle - , - \rangle, \rho)$ with a connection.

**Proof.** Since $\Lambda_{ij} \Lambda_{jk} = \Lambda_{ik}$, $T_\varphi U_i$'s glue to a vector bundle $C$. Since $\Lambda_{ij}$ preserves the standard bracket $[[-, -]]_T$ and the standard pairing $\langle - , - \rangle_T$ on $T_\varphi U_i$, we have a well-defined bracket $[[-, -]]$ and a non-degenerate symmetric bilinear form $\langle - , - \rangle$ on $\Gamma(C)$. Clearly, we have the following exact sequence of vector bundles:

$$0 \to \ker \rho \to C \xrightarrow{\rho} TM \to 0,$$

where $\rho$ is induced by the projection $T_\varphi U_i \to TU_i$.

The fact that $\Lambda_{ij}$ preserves the standard bracket $[[-, -]]_T$ and the standard pairing $\langle - , - \rangle_T$ also implies that Axioms (i)-(iii) in Definition 4.1 are satisfied. Therefore, $(C, [-, -], \langle - , - \rangle, \rho)$ is a transitive Courant algebroid.

On $U_i$, consider the splitting $s_i : TU_i \to C|_{U_i}$ given by

$$s_i(X) = X + \theta_i(X) - (\theta_i, \theta_i(X))_{\theta} - i_X B_i.$$  

(39)

Straightforward calculation shows that $\langle s_i(X), s_i(Y) \rangle_T = 0$. Thus, the splitting $s_i$ is isotropic. Eqs. (36) and (37) implies that $\Lambda_{ij}s_j(X) = s_i(X)$. Thus, we have a globally well-defined isotropic splitting $s : TM \to C$.

Furthermore, $\coprod U_i \times \mathfrak{g}$ and the transition function $\tau_{ij}$ give us a Lie algebra bundle $\mathcal{G}$, and there is a short exact sequence

$$0 \to T^*M \to \ker \rho \xrightarrow{\rho'} \mathcal{G} \to 0.$$  

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where \( \rho' \) is induced by the projection \( (U_i \times g) \oplus T^*U_i \to U_i \times g \). Consider the splitting \( \sigma_{s_i} : \mathcal{G}|_{U_i} \to (\ker \rho)|_{U_i} \) given by

\[
\sigma_{s_i}(a) = a - 2(\theta_i, a)^g.
\]

Then \( \langle s_i(X), \sigma_{s_i}(a) \rangle^T = 0 \). Thus, \( \sigma_{s_i} \) is orthogonal to \( s \). By (36), we have

\[
\begin{pmatrix}
\tau_{ij} & 0 \\
-2\phi_i^j \tau_{ij} & 1
\end{pmatrix}
\begin{pmatrix}
\frac{a}{2} \\
-2(\theta_j, a)^g
\end{pmatrix} =
\begin{pmatrix}
-2\phi_i^j \tau_{ij}(a) \\
-2(\theta_i, \tau_{ij}(a))^g
\end{pmatrix},
\]

which implies that we have a globally well-defined splitting \( \sigma_s : \mathcal{G} \to \ker \rho \) that orthogonal to the splitting \( s \).

**Remark 4.5.** In Appendix A.5, we write down the explicit formula for the glued transitive Courant algebroid in the form provided in [16].

A 1-morphism in \( \text{holim} \mathbb{TC}^c(\mathbb{U}(M) \bullet) \) from \( \langle \tilde{\theta}, \tilde{B}; \tilde{\phi}_{ij}, \tilde{\tau}_i, \tilde{\beta}_i \rangle \) to \( \langle \theta, B; \phi_{ij}, \tau_i, \beta_i \rangle \) consists of a 1-morphism

\[
\begin{pmatrix}
\phi_i & \tau_i & 0 \\
0 & \beta_i & -2\phi_i^j \tau_{ij}
\end{pmatrix}
\]

in \( \text{holim} \mathbb{TC}^c(\mathbb{U}_{ij}) \) from \( \sqcup \mathbb{T}_gU_i, \tilde{\theta}, \tilde{B}_i \) to \( \sqcup \mathbb{T}_gU_i, \theta, B_i \), which satisfies

\[
\Lambda_{ij}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \phi_i & \tau_i \\
0 & \beta_i & -2\phi_i^j \tau_{ij}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \phi_i & \tau_i \\
0 & \beta_i & -2\phi_i^j \tau_{ij}
\end{pmatrix}
\]

**Proposition 4.6.** A 1-morphism in \( \text{holim} \mathbb{TC}^c(\mathbb{U}(M) \bullet) \) gives rise to a Courant algebroid isomorphism preserving connections.

**Proof.** The proof is similar to that of Proposition A.3. Eq. (11) implies that the local morphisms glue together to a global morphism \( \mathfrak{B} \) between the gluing results of two Courant data. Eqs. (33) and (34) imply that \( \mathfrak{B}s_i(X) = s_i(X) \) and \( \mathfrak{B}\sigma_s^- = \sigma_s \). Thus \( \mathfrak{B} \) preserves connections.

By Proposition A.3 and Proposition 4.3, after the plus construction, we arrive at the \((2, 1)\)-sheaf \( \mathbb{TC}^c_+ \) of transitive Courant algebroids with connections, where the 1-morphism are the isomorphisms of Courant algebroid preserving connections. See [23] for the notion of morphisms (not necessary isomorphisms) of Courant algebroids. However, in our case, all our sheaves (stacks) are functors to (higher) groupoids, that is, 1-morphisms are always isomorphisms.

Obviously, there is a projection \( \text{pr} \) from the \((2, 1)\)-presheaf \( \mathbb{TC}^c_+ \) to the \((2, 1)\)-presheaf \( \mathbb{TL}^c_+ \), which sends a transitive Courant data \( \left( \sqcup (\mathbb{T}_gU_i, [\cdot, -]_S^T, \langle \cdot, - \rangle^T, \text{pr}_{TU_i}, \theta, B_i); \phi_{ij}, \tau_{ij}, \beta_i \right) \) to the transitive Lie data \( \left( \sqcup (TU_i \oplus (U_i \times g), [-, -]_S^T, \text{pr}_{TU_i}, \theta, B_i); \phi_{ij}, \tau_{ij} \right) \) (see Definition A.2), and behaves in a similar obvious way on the level of morphisms. After plus construction, we arrive at a projection \( \mathbb{TC}^{c+}_+ \text{pr} \to \mathbb{TL}^{c+}_+ \). Here \( \mathbb{TL}^c_+ \) and \( \mathbb{TL}^{c+}_+ \) are the \((2, 1)\)-presheaf and \((2, 1)\)-sheaf of transitive Lie algebroids respectively. See Appendix A.1 for more details.

Now we fix a transitive Lie data \( \mathbb{A}_c = \left( \sqcup (TU_i \oplus (U_i \times g), [-, -]_S^T, \text{pr}_{TU_i}, \theta, B_i); \phi_{ij}, \tau_{ij} \right) \). We simply denote it by \( (\theta; \phi_{ij}, \tau_{ij}) \).

**Lemma 4.7.** Define \( R_i : \wedge^2 \Gamma(TU_i) \to \Gamma(U_i \times g) \) by

\[
R_i = d\theta_i + \frac{1}{2} [\theta_i, \theta_i]_g.
\]

Then \( R_i \)'s glue to a globally well-defined curvature \( R : \wedge^2 TM \to \mathcal{G} \). We call \( R \) the curvature of our transitive Lie data.

**Proof.** We need to show \( \tau_{ij} R_j = R_i \). We bring (30) inside the expression. Then the result follows from Eqs. (29) and (30).
Lemma 4.10. There exists a lift $A_c$ of the transitive Lie data $A_c$ is defined through the curvature $R$,

$$p_1(A_c) := (R, R)^\mathfrak{g}.$$  \hspace{1cm} (43)

Theorem 4.9. Given an object $M \xrightarrow{A_c} TL^p_c$ in $TL^p_c$ over a good cover $\{U_i\}$ of $M$,

(i) there exists a lift $C_c$ that fits the diagram

\[
\begin{array}{c}
M \\
\downarrow_{A_c} \\
\downarrow_{TL^p_c}
\end{array}
\quad \begin{array}{c}
\text{TC}^p_c
\end{array}
\]  \hspace{1cm} (44)

if and only if $p_1(A_c) = 0$;

(ii) if a lift exists, then the space of lifts up to isomorphisms forms a torsor of $\Omega^3_c(M)$.

The proof of this theorem is given in the following two lemmas.

Lemma 4.10. There exists a lift $C_c$ of $A_c$ in diagram (44) if and only if

$$p_1(A_c) = 0.$$ 

Proof. First, assume that $C_c = \left( \cup (TgU_i, [-,-]^T, \langle -,- \rangle^T, \text{pr}_{TU_i}, \theta_i, B_i); \phi_{ij}, \tau_{ij}, \beta_{ij} \right)$ is a lift of $A_c$, then set $H_i := cs_3(\theta_i) - dB_i$ on each $U_i$. By (38), we have $cs_3(\theta_i) - dB_i = cs_3(\theta_j) - dB_j$, which implies that $H_i = H_j$ on $U_{ij}$. Thus, $H_i$'s glue to a global 3-form $H$. We call $H$ the curvature of $C_c$. Since $H|_{U_i} = cs_3(\theta_i) - dB_i$, and $cs_3(\theta_i) = (\theta_i, d\theta_i)^g + \frac{1}{3}([\theta_i, \theta_i])^g$, we have

$$dH|_{U_i} = d((\theta_i, d\theta_i)^g + \frac{1}{3}([\theta_i, \theta_i])^g) = (d\theta_i, d\theta_i)^g + (d\theta_i, [\theta_i, \theta_i])^g.$$ 

Here we use the fact that $(-,-)^g$ is adjoint invariant. Another direct computation gives

$$(R_i, R_i)^g = (d\theta_i + \frac{1}{2}[\theta_i, \theta_i]_g, d\theta_i + \frac{1}{2}[\theta_i, \theta_i]_g)^g = (d\theta_i, d\theta_i)^g + (d\theta_i, [\theta_i, \theta_i])^g.$$ 

Therefore, we have $(R, R)^g = dH$, i.e. $p_1(A_c) = 0$.

On the other hand, given $A_c = \left( \cup (TU_i \oplus (U_i \times g), [-,-]^T, \text{pr}_{TU_i}, \theta_i); \phi_{ij}, \tau_{ij} \right)$ and assuming $p_1(A_c) = 0$, let $H \in \Omega^3(M)$ be such that $(R, R)^g = dH$. Since the cover is good, we may come up with 2-forms $B_i \in \Omega^2(U_i)$ satisfying

$$H|_{U_i} = cs_3(\theta_i) - dB_i.$$ 

Let $\beta_{ij} : TU_{ij} \longrightarrow T^*U_{ij}$ be the bundle map uniquely determined by (41). Then one checks that

$$\left( \cup (\mathbb{T}gU_i, [-,-]^T, \langle -,- \rangle^T, \text{pr}_{TU_i}, \theta_i); \phi_{ij}, \tau_{ij}, \beta_{ij} \right)$$

is a Courant data whose image under the projection $\text{TC}^p_c \xrightarrow{pr} TL^p_c$ coincides with $A_c$. \textbf{□}

Now assume that there exists a lift of $A_c = \left( \cup (TU_i \oplus (U_i \times g), [-,-]^T, \text{pr}_{TU_i}, \theta_i); \phi_{ij}, \tau_{ij} \right) \in TL^p_c(M)$, where again $A_c$ is over a good cover of $M$. We denote the fiber category of $\text{pr}$ over $A_c$ by $S_{A_c}$. Then the space of lifts up to isomorphisms is the set of equivalent classes $S_{A_c}/\sim$-morphisms. We define an action
of $\Omega^3(M)$ on this space as follows. For all $h \in \Omega^3(M)$, assume that $h|_{U_i} = dB^h_i$. Define the action of $h$ on $S_{A_1}/1$-morphisms by

$$h \cdot [(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})] = [(\theta_i, B_i + B^h_i; \phi_{ij}, \tau_{ij}, \beta_{ij} + B^h_j - B^h_i)].$$

(45)

Here $[\cdot]$ denotes the isomorphism class in the quotient $S_{A_1}/1$-morphisms. Now we prove that the above action is well defined.

- It does not depend on the choice of $\{B^h_i\}$. In fact, if $h|_{U_i} = dB^h_i$, we let $B_i = \bar{B}^h_i - B^h_i$, which is closed. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ B_i & 0 & 1 \end{pmatrix}$$

gives rise to an isomorphism from $((\theta_i, B_i + \bar{B}^h_i; \phi_{ij}, \tau_{ij}, \beta_{ij} + \bar{B}^h_j - \bar{B}^h_i))$ to $(\theta_i, B_i + B^h_i; \phi_{ij}, \tau_{ij}, \beta_{ij} + B^h_j - B^h_i)$.

- It does not depend on the choice of a representative $(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})$ of an isomorphism class. The isomorphism is possibly given through another finer cover $\{V_i\}$. However as we may always pull back our data to $V_i$, we might as well assume that $V_i = U_i$. Assume that we have a 1-morphism $\Lambda$ from a transitive Courant data $(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij}) \in S_{A_1}$ to a transitive Courant data $(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij}) \in S_{A_1}$. Then by (35), one checks that $\Lambda$ is a 1-morphism from $(\theta_i, B_i + B^h_i; \phi_{ij}, \tau_{ij}, \beta_{ij} + B^h_j - B^h_i)$ to $(\theta_i, B_i + B^h_i; \phi_{ij}, \tau_{ij}, \beta_{ij} + B^h_j - B^h_i)$.

**Lemma 4.11.** With the above notations, $\Omega^3(M)$ acts on $S_{A_1}/1$-morphisms freely and transitively. Thus, $S_{A_1}/1$-morphisms is an $\Omega^3(M)$-torsor.

**Proof.** We first show that the action of $\Omega^3(M)$ is free. If $h \cdot [(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})]$ is isomorphic to $[(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})]$, by (35), we deduce that $dB^h_i = 0$, which implies that $h = 0$.

Then we show that the action is transitive. For two objects $(\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij})$ and $(\theta_i, B'_i; \phi_{ij}, \tau_{ij}, \beta'_{ij})$ in $S_{A_1}$, let $B^h_i = B'_i - B_i$. Since both $\{c_{s3}(\theta_i) - dB_i\}$ and $\{c_{s3}(\theta_i) - dB'_i\}$ can be glued to a global 3-form, we deduce that $\{dB^h_i\}$ can be glued to a global closed 3-form $h$. By (37), we deduce that

$$\begin{align*}
\iota_X (B'_i - B_i) &= \beta'_{ij} + (\theta_i, \theta_i(X))^g - (\theta_i, \theta_j(X))^g - 2\phi^g_{ij} \tau_{ij}(\theta_j(X)), \\
\iota_X (B_j - B_i) &= \beta_{ij} + (\theta_i, \theta_i(X))^g - (\theta_i, \theta_j(X))^g - 2\phi^g_{ij} \tau_{ij}(\theta_j(X)),
\end{align*}$$

which implies that

$$\beta'_{ij} = \beta_{ij} + B^h_j - B^h_i.$$ 

Thus, $h \cdot (\theta_i, B_i; \phi_{ij}, \tau_{ij}, \beta_{ij}) = (\theta_i, B'_i; \phi_{ij}, \tau_{ij}, \beta'_{ij})$. It finishes the proof. $\blacksquare$

**Corollary 4.12.** Given an object $M \xrightarrow{\pi_1} TL^+_e$ with $p_1(A_e) = 0$, the curvatures of all possible lifts form an $\Omega^3(M)$-torsor.

**Remark 4.13.** Note that objects in the category $TC^+_e(M)$ are transitive Courant algebroids with connections and 1-morphisms in $TC^+_e(M)$ are isomorphisms between transitive Courant algebroids that preserve connections. Thus, obviously there is a forgetful functor $F$ from the category $TC^+_e(M)$ to the usual category of transitive Courant algebroids $TC(M)$. Similarly, there is also a forgetful functor from the category $TL^+_e(M)$ to the usual category of transitive Lie algebroids $TL(M)$ over $M$. Let $S_A$ denote the fiber of the projection $pr : TC(M) \longrightarrow TL(M)$. Then we have the following diagram:

$$
\begin{array}{ccc}
S_{A_e} & \xrightarrow{F} & TC^+_e(M) \\
\downarrow & & \downarrow \text{pr} \\
S_A & \xrightarrow{F} & TC(M) \\
\downarrow & & \downarrow \text{pr} \\
& & TL(M)
\end{array}
$$
In [10], the isomorphism classes in $S_A$ are classified for a quadratic transitive Lie algebroid $A \in \mathbb{T}L(M)$ with vanishing first Pontryagin class. Compared to Lemma [4, 17] the classification results are not the same since our morphisms need to preserve connections. More precisely, our space is a torsor of $\Omega^3(M)$ which surjectively maps to the group $H^3(M, \mathbb{R})/I$ of which their space is a torsor. Here $I$ is a certain subspace of $H^3(M, \mathbb{R})$ which comes from automorphisms of the Lie algebroid $A$. In our case, such automorphisms do not show up because our $\Omega^3(M)$-action [13] fixes the underlying transitive Lie data.

Example 4.14 (action Courant algebroids). A quadratic Lie algebra $(\mathfrak{g}, [-,-], (\cdot, -)^\mathfrak{g})$ gives rise to a string Lie 2-algebra [22 Sect. 2] and may be viewed as a Courant algebroid over a point. Extending this idea to a general base manifold $M$, the authors in [25] construct a natural example of Courant algebroids as follows: given an action $\rho : \mathfrak{g} \to X(M)$ whose stabilizer at each point on $M$ is a coisotropic subspace of $\mathfrak{g}$, there is an action Courant algebroid $(M \times \mathfrak{g}, \|\cdot,\cdot\|, (\cdot, \cdot), \rho)$, where the anchor is given by the action $\rho$, the bilinear form $(\cdot, \cdot)$ is the pointwise pairing induced by $(\cdot, \cdot)^\mathfrak{g}$ on $\mathfrak{g}$, and the Courant bracket on the sections of the vector bundle $M \times \mathfrak{g} \to M$ is given by

$$[X, Y] = [X, Y] + L_{\rho(X)}Y - L_{\rho(Y)}X + \rho^*(dX, Y), \quad \forall X, Y \in C^\infty(M, \mathfrak{g}).$$

Now we study this example in the special case of homogeneous spaces. Take $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ equipped with the nondegenerate bilinear form $(A, B)^\mathfrak{g} = \text{tr}(AB)$, and let $\mathfrak{t}_{\geq 0}, \mathfrak{t}_0$, and $\mathfrak{t}$ be the Lie subalgebras of non-strict upper triangular, diagonal, and strict upper triangular matrices respectively. Let $K = \text{GL}_n(\mathbb{C})$, and $K_{\geq 0}, K_0$, and $K$ be the matrix groups corresponding to $\mathfrak{t}_{\geq 0}, \mathfrak{t}_0$, and $\mathfrak{t}$ respectively. Take $M$ to be the homogeneous space $K/K_{\geq 0}$. In this case, the anchor $\rho$ is surjective and it follows that the action Courant algebroid $M \times \mathfrak{g}$ is a transitive Courant algebroid. Thus, after choosing a connection, we have a split of Courant algebroid $M \times \mathfrak{g} \cong TM \oplus \mathcal{G} \oplus T^*M$, with the Courant bracket and the pairing defined in [23] and [24].

We now prove that there exists a suitable connection such that the split form on the right hand side is a standard transitive Courant algebroid. Firstly, following [50] Proposition 3.9 the underlying transitive Lie algebroid $A = M \times \mathfrak{g}/(\ker \rho)^\perp$ is the Atiyah Lie algebroid associated to the $K_0 = (\mathbb{C}^n)^n$ principal bundle $K/K_+ \to M$. However, this principal bundle is trivial. This can be seen as follows. Note that $M$ is isomorphic to the flag variety $F_n := \{E_\bullet = (E_1 \subsetneq E_2 \subsetneq \cdots \subset E_n = \mathbb{C}^n) \mid \dim E_i = i\}$. Let $U_i$ be the tautological $i$-dimensional vector bundle over $M$, whose fiber over a point (a flag $E_\bullet$) is the vector space $E_i$ of the flag. These form a filtration $0 = U_0 \subset U_1 \subsetneq \cdots \subset U_n = F_n = \mathbb{C}^n$. If we consider the standard representation of $K_0 = \text{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$, then the associated vector bundle of the $K_0$ principal bundle $K_+ \to M$ is isomorphic to $\oplus_{i=1}^n U_i$. Here $L_i := U_i/U_{i-1}$ is a line bundle for each $0 < i \leq n$. Therefore the triviality of $K/K_+ \to M$ follows from the fact that the associated bundle $\oplus_{i=1}^n L_i \cong U_n$ is trivial.

Upon choosing a global trivialization, we can take the natural connection on the principal bundle $K/K_+ \to M$, which in turn induces a split of the Atiyah Lie algebroid $A \cong TM \oplus (M \times \mathfrak{t}_0)$ with connection $\nabla_X(a) = X(a)$. Let $TM \oplus (M \times \mathfrak{t}_0) \oplus T^*M$ be the corresponding split of $M \times \mathfrak{g}$. By Proposition A.9 and the fact that the curvature $R$ of the natural connection is 0, the Courant bracket on the split form is the standard Courant bracket up to a 3-form $H \in \Omega^3(M)$ satisfying $dH = (R, R)^{\mathfrak{g}_0} = 0$. Following from Borel’s result, as a complete flag variety, $M$ has vanishing odd cohomology. Thus we may assume $H = dB$ for a certain 2-form $B$, and perform a $B$-field transformation for $TM \oplus (M \times \mathfrak{t}_0) \oplus T^*M, X + a + \xi \mapsto X + a + \xi + i_X B$, and arrive at the standard bracket. Thus composing these two steps, we obtain an isomorphism from $M \times \mathfrak{g}$ to the standard transitive Courant algebroid.

Notice that the $K_0$ principal bundle $K/K_+ \to M$ is not necessarily trivial for general $K$. For example, when $K = \text{SL}_n(\mathbb{C})$, the similar construction will give us nontrivial $K_0 = (\mathbb{C}^n)^{n-1}$ principal bundle. Since $K_0$ is abelian, the Cartan 3-form on it is 0, thus the basic gerbe on it is a trivial gerbe. Nevertheless, we shall not expect string groups to be trivial. Therefore, there will be different features for abelian counterpart of string structure. We leave it for future discussion.

\footnote{We thank very much Eckhard Meinrenken for sharing this example with us.}
5 Morphism from the string sheaf to the transitive Courant sheaf

Having constructed the sheaves of $\text{String}(n)$-principal bundles and transitive Courant algebroids with connections, we show that there is a canonical morphism between them. On the level of objects without connections, one could build the correspondence between string structures and transitive Courant algebroids using the reduction method as in [5,13]. Nevertheless, to obtain a functor, it is convenient to use our language. Throughout this section, we take the presheaf of transitive Courant algebroids with connections, we show that there is a canonical morphism between them. On the level of objects with-
Lemma 5.4. Let \((g_{\theta_1}, A_{\theta_1}, \omega^2_{\theta_1})\) be a 1-morphism from \((U \times \text{String}(n) \to U, \theta_1, B_1)\) to \((U \times \text{String}(n) \to U, \theta_0, B_0)\). Then \(\Lambda_{\theta_1}\) given in Theorem 5.7 is a 1-morphism in \(\mathcal{T}_c^\theta\) from \((\mathcal{T}_c^\theta U, \theta_1, B_1)\) to \((\mathcal{T}_c^\theta U, \theta_0, B_0)\).

Proof. By definition, we first need to show that \(\Lambda_{\theta_1}\) is indeed an automorphism of the standard transitive Courant algebroid \((\mathcal{T}_c^\theta U, \{\cdot, \cdot\}_S, (-, -)^T, \text{pr}_{TU})\). That is to prove the entries of the vector bundle map

\[
\Lambda_{\theta_1} = \begin{pmatrix}
1 & \theta_0 - \text{ad}_{\bar{g}_{\theta_1}}\theta_1 & 0 \\
\bar{g}^\ast_{\theta_1} & 0 & 0 \\
0 & 2(\bar{g}^\ast_{\theta_1} \theta_{MC})_g & 1
\end{pmatrix}
\]

satisfy the identities (28)-(31). Note that \((g_{\theta_1} : U \to \text{String}(n), A_{\theta_1} \in \Omega^1(U), \omega^2_{\theta_1} \in \Omega^2(U))\) gives rise to a 1-morphism,

\[
(U \times \text{String}(n) \to U, \theta_0, B_0) \xrightarrow{\langle g_{\theta_1}, A_{\theta_1}, \omega^2_{\theta_1} \rangle} (U \times \text{String}(n) \to U, \theta_1, B_1),
\]

if and only if

\[
dA_{\theta_1} = \omega^2 + B_1 - B_0, \quad d\omega^2_{\theta_1} = c_3(\theta_1) - c_3(\theta_0), \quad \theta_0 - \text{ad}_{\bar{g}_{\theta_1}}\theta_1 = -\bar{g}^\ast\theta_{MC},
\]

(49)

where \(\bar{g}_{\theta_1} : U \to G\) is the underlying morphism of \(g_{\theta_1} : U \to \text{String}(n)\).

The symmetric part of \(\beta_{\theta_1} : TU \to T^*U\) is \(-\bar{g}^\ast_1 \theta_{MC} \circ \bar{g}^\ast_1 \theta_{MC}\), which we denote by \(\beta_{\theta_1}^{sym}\). Therefore,

\[
\frac{1}{2}(\beta_{\theta_1}(X)(Y) + \beta_{\theta_1}(Y)(X)) = \beta_{\theta_1}^{sym}(X)(Y) = -(\bar{g}^\ast_1 \theta_{MC}(X), -\bar{g}^\ast_1 \theta_{MC}(Y))^g,
\]

which implies that (28) holds.

By (47) and (48), we deduce that (29) and (30) hold.

The skewsymmetric part of \(\beta_{\theta_1}\) is \(-\bar{g}^\ast_1 \theta_{MC}\), which we denote by \(\beta_{\theta_1}^{skew}\). Obviously, we have

\[
\langle L_X \beta_{\theta_1}^{skew}(Y) - i_Y d\beta_{\theta_1}^{skew}(X) - \beta_{\theta_1}^{skew}([X,Y]), Z \rangle = d\beta_{\theta_1}^{skew}(X, Y, Z).
\]

Furthermore, we have

\[
c_3(\theta_1) - c_3(\theta_0) = d(\theta_0, \bar{g}^\ast_1 \theta_{MC}) + \frac{1}{6}g^\ast_1 \mathfrak{c},
\]

(50)

where \(\mathfrak{c} \in \Omega^3(G)\) defined by

\[
\frac{1}{6} \mathfrak{c}(\dot{a}, \dot{b}, \dot{c}) = (\theta_{MC}(\dot{a}), \theta_{MC}(\dot{b}), \theta_{MC}(\dot{c}))^g = (a, [b, c]^g)^g, \quad \forall a, b, c \in \mathfrak{g}.
\]

Here \(\dot{a}, \dot{b}, \dot{c}\) are right invariant vector fields on \(G\).

By (19) and (20), we obtain

\[
d\beta_{\theta_1}^{skew} = c_3(\theta_1) - c_3(\theta_0) - d(\bar{g}^\ast_1 \theta_{MC}, \theta)^g = -\frac{1}{6}g^\ast_1 \mathfrak{c}.
\]

(51)

On the other hand, by straightforward computations, we have

\[
\langle L_X \beta_{\theta_1}^{sym}(Y) - i_Y d\beta_{\theta_1}^{sym}(X) - \beta_{\theta_1}^{sym}([X,Y]) + \mathcal{P}(\bar{g}^\ast_1 \theta_{MC}, \bar{g}^\ast_1 \theta_{MC}), Z \rangle = \bar{g}^\ast_1 \mathfrak{c}(X, Y, Z),
\]

which implies that (31) holds.

Finally, (19) implies that the conditions in (33)-(35) are satisfied. Thus \(\Lambda_{\theta_1}\) is a 1-morphism in \(\mathcal{T}_c^\theta\).

It finishes the proof. \(\square\)

Recall that the model we use for a 2-groupoid is a simplicial set satisfying Kan conditions \(\text{Kan}(n, j)\) for all \(n \geq 1\) and \(0 \leq j \leq n\) and strict Kan conditions \(\text{Kan}(n, j)\) for all \(n \geq 3\) and \(0 \leq j \leq n\). Lemma 5.4 verifies that \(\Phi(U) : \text{BString}(n)^\xi(U)_1 \to \mathcal{T}_c^\theta(U)_1\) is well-defined. The following lemma will verify that the map \(\Phi(U) : \text{BString}(n)^\xi(U)_2 \to \mathcal{T}_c^\theta(U)_2\) on 2-simplices commutes with the face maps.
**Lemma 5.5.** Given any 2-morphism \((f, \omega)\) between \(T_i \circ T_{i_2}\) and \(T_{i_2}\), where \(T_i := (g_{ij}, A_{ij}, \omega_{ij}^2)_{0 \leq i < j \leq 2} \in \text{BString}(n)_{\text{p}}(U)\), are 1-morphisms between the objects \((U \times \text{String}(n) \to U, \theta_i, B_i)_{i=1,2,3} \in \text{BString}(n)_{\text{p}}(U)\), the corresponding images \(\Lambda_{i_{12}} := \Phi(T_{i_{12}}) = \Phi(g_{ij}, A_{ij}, \omega_{ij}^2)_{0 \leq i < j \leq 2}\) under the morphism \(\Phi\) satisfy the condition \(\Lambda_{i_0} \Lambda_{i_{12}} = \Lambda_{i_2}\), i.e. we have the following commutative diagram:

\[
\begin{array}{ccc}
(\theta_1, B_1) & \xrightarrow{\lambda_{i_0}} & (\theta_0, B_0) \\
\downarrow & & \downarrow \\
(\theta_0, B_0) & \xrightarrow{\lambda_{i_2}} & (\theta_2, B_2)
\end{array}
\]

**Proof.** By \(g_{i_0} g_{i_2} = g_{i_2}\), we have

\[
\Lambda_{i_0} \Lambda_{i_{12}} = \begin{pmatrix}
1 & -\bar{g}_{i_0}^* \theta_{MC} & 0 \\
-\bar{g}_{i_0}^* \theta_{MC} & \beta_{i_0} + 2(\bar{g}_{i_0} \theta_{MC})^* \circ \text{ad}_{\bar{g}_{i_0}} & 0 \\
\beta_{i_0} + 2(\bar{g}_{i_0} \theta_{MC})^* \circ \text{ad}_{\bar{g}_{i_0}} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \beta_{i_2} + 2(\bar{g}_{i_2} \theta_{MC})^* \circ \text{ad}_{\bar{g}_{i_2}} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

where

\[
D_{i_2} = 2(\bar{g}_{i_2} \theta_{MC})^* \circ \text{ad}_{\bar{g}_{i_0}} \circ \text{ad}_{\bar{g}_{i_2}} + 2(\bar{g}_{i_2} \theta_{MC})^* \circ \text{ad}_{\bar{g}_{i_2}}
\]

and

\[
D_{i_3} = \beta_{i_0} + 2(\bar{g}_{i_3} \theta_{MC})^* \circ \text{ad}_{\bar{g}_{i_0}} \circ \bar{g}_{i_2} \theta_{MC} + \beta_{i_2}
\]

Hence, we have \(\Lambda_{i_0} \Lambda_{i_{12}} = \Lambda_{i_0} \Lambda_{i_{12}} = \Lambda_{i_2}\). □

**Proof of Theorem 5.1.** Recall that given any manifold \(U\), a morphism between the 2-groupoids \(\text{BString}(n)_{\text{p}}(U)\) and \(\text{TC}^p_{\text{p}}(U)\) is a simplicial morphism of the underlying simplicial sets. Lemma 5.4 and 5.5 verify that \(\Phi(U) : \text{BString}(n)_{\text{p}}(U) \to \text{TC}^p_{\text{p}}(U)\) is indeed a morphism of the underlying simplicial sets. Hence we only need to prove the naturality of the map \(\Phi\). To do this, let us assume that \(r_{V,U} : V \to U\) is a smooth map between two manifolds \(V\) and \(U\). Then \(r_{V,U}\) induces maps

- \(\text{TC}^p_{\text{p}}(r_{V,U}) : \text{TC}^p_{\text{p}}(U) \to \text{TC}^p_{\text{p}}(V)\) given by

\[
(T_{i_0} U, \theta, B) \mapsto (T_{i_0} V, r_{V,U}^* \theta, r_{V,U}^* B),
\]

- \(\text{BString}(n)_{\text{p}}(r_{V,U}) : \text{BString}(n)_{\text{p}}(U) \to \text{BString}(n)_{\text{p}}(V)\) given by

\[
((U, \theta, B); g, A, \omega^2; f, \omega^1) \mapsto (V, r_{V,U}^* \theta; r_{V,U}^* B; r_{V,U}^* g, r_{V,U}^* A, r_{V,U}^* (\omega^2); r_{V,U}^* f, r_{V,U}^* (\omega^1)).
\]
Here \((U, \theta, B)\) stands for an element \((U \times \text{String}(n) \to U, \theta, B) \in \mathbf{BString}(n)_{c}^{p}(U)\). It is then straightforward to obtain the following commutative diagram

\[
\begin{array}{ccc}
\mathbf{BString}(n)_{c}^{p}(U) & \xrightarrow{\Phi(U)} & \mathbf{TC}_{c}^{p}(U) \\
\mathbf{BString}(n)_{c}^{p}(U_{r}, V) & \xrightarrow{\Phi(V)} & \mathbf{TC}_{c}^{p}(U_{r}, V),
\end{array}
\]

which shows the naturality of the map \(\Phi\). Thus \(\Phi\) is morphism from \(\mathbf{BString}(n)^{p}_{c}\) to \(\mathbf{TC}^{p}_{c}\). 

As a corollary, we have proven the desired result,

**Corollary 5.6.** There is a natural morphism \(\Phi\) from the \((3,1)\)-sheaf \(\mathbf{BString}(n)^{p+}_{c}\) to the \((2,1)\)-sheaf \(\mathbf{TC}^{p+}_{c}\).

By the discussion above, the morphism \(\Phi\) can be described explicitly as follows. Given any manifold \(M\), the morphism \(\Phi: \mathbf{BString}(n)^{p+}_{c}(M) \to \mathbf{TC}^{p+}_{c}(M)\) is given on the 0-, 1- and 2-simplices respectively by

- on 0-simplices

\[
\Phi(\{U_{i}\}, P_{c}) = \left(\{U_{i}\}, (\sqcup_{i} (T_{g} U_{i}, \theta_{i}, B_{i}); -\bar{g}_{ij}^* \theta_{MC}, \text{ad}_{\bar{g}_{ij}}, \beta_{ij})\right)
\]

where \(\{U_{i}\}\) is an open cover of \(M\) and \(P_{c} := (\sqcup (U_{i} \times \text{String}(n) \to U_{i}, \theta_{i}, B_{i}; g_{ij}, A_{ij}, \omega_{ij}^{2}, f_{ijk}, \omega_{ijk}^{1}))\) is string data, \(\bar{g}_{ij}: U_{ij} \to G\) is the underlying morphism of \(g_{ij}\), and \(\beta_{ij}\) is given by

\[
\beta_{ij} = -(\bar{g}_{ij}^* \theta_{MC}, \theta_{i})^{g} + \bar{d}A_{ij} + \omega_{ij}^{2} - (\bar{g}_{ij}^* \theta_{MC})^{g} \circ \bar{g}_{ij}^{*} \theta_{MC}.
\]

- on 1-simplices

\[
\Phi(\{V_{i}\}, \phi_{c}) = (T_{g} V_{i}, \Lambda_{i}),
\]

where \(\{V_{i}\}\) is a common refinement of \(\{U_{i}\}\), \(\{\tilde{U}_{i}\}\), and \(\phi_{c} := (g_{i} : V_{i} \to \text{String}(n), A_{i}, \omega_{i}^{2})\) provides a 1-morphism between \((\{U_{i}\}, P_{c})\) and \((\{U_{i}\}, \tilde{P}_{c})\). \(\Lambda_{i}\) is the 1-morphism from \((T_{g} V_{i}, \theta_{i}, B_{i})\) to \((T_{g} V_{i}, \tilde{\theta}_{i}, B_{i})\) defined as before

\[
\Lambda_{i} = \begin{pmatrix} 1 & 0 & 0 \\ -\bar{g}_{i}^* \theta_{MC} & \text{ad}_{\bar{g}_{i}} & 0 \\ \beta_{i} & 2(\bar{g}_{i}^* \theta_{MC})^{g} & 1 \end{pmatrix},
\]

where \(\bar{g}_{i}: V_{i} \to G\) is the underlying morphism of \(g_{i}\) and \(\beta_{i} : TV_{i} \to T^{*} V_{i}\) is given by

\[
\beta_{i} = -(\bar{g}_{i}^* \theta_{MC}, \theta_{i})^{g} + \bar{d}A_{i} + \omega_{i}^{2} - (\bar{g}_{i}^* \theta_{MC})^{g} \circ \bar{g}_{i}^* \theta_{MC};
\]

- on 2-simplices

\[
\Phi(\{W_{i}\}, \alpha_{c}) = 1,
\]

where \(\{W_{i}\}\) is a common refinement of \(\{V_{i}\}\), \(\{\tilde{V}_{i}\}\), and \(\alpha_{c}\) provides a 2-morphism between \((\{V_{i}\}, \phi_{c})\) and \((\{\tilde{V}_{i}\}, \tilde{\phi}_{c})\).

As an object in \(\mathbf{TC}^{p+}_{c}(M)\) glues to a Courant algebroid by the discussion in Appendix A.3, let us describe explicitly the Courant algebroid with a connection associated to a \(\text{String}(n)\)-principal bundle with connection data on a manifold \(M\).

Given a \(\text{String}(n)\) data

\[
P_{c} = (\sqcup U_{i} \times \text{String}(n) \to \sqcup U_{i}, \theta_{i}, B_{i}; g_{ij}, A_{ij}, \omega_{ij}^{2}, f_{ijk}, \omega_{ijk}^{1})
\]

27
over a cover \( \{ U_i \} \) of \( M \) in \( \text{BString}(n)_E^+ \), the corresponding transitive Courant algebroid with a connection under the morphism \( \Phi : \text{BString}(n)_E^+(M) \to \text{TC}^+_E(M) \) is

\[
E \cong (TM \oplus \mathcal{G} \oplus T^*M, \llbracket -,- \rrbracket^T_{\nabla,R,H}, \langle -,- \rangle^T_{\pi}, \text{pr}_{TM}),
\]
equipped with bracket \( \llbracket -,- \rrbracket^T_{\nabla,R,H} \) and pairing \( \langle -,- \rangle^T_{\pi} \) as in formula (23), (24) with the global 2-form \( R \in \Omega^2(M, \mathcal{G}) \), the connection \( \nabla : \Gamma(TM) \otimes \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}) \) and the global 3-form \( H \in \Omega^3(M) \) defined on each \( U_i \) respectively by

\[
\nabla_X a := X(a) + [\theta_i(X), a]_g, \quad \forall X \in \Gamma(TU_i), a \in \Gamma(U_i \times \mathfrak{g}),
\]

\[
R_i := \frac{1}{2}[\theta_i, \theta_i]_g,
\]

\[
H_i := cs_3(\theta_i) - dB_i.
\]

### 5.2 Property of the morphism \( \Phi : \text{BString}(n)^{p+}_E \to \text{TC}^+_E \)

We denote the (3,1)-presheaf of U(1)-gerbes (or BU(1)-principal bundles) with connection data by \( \text{BBU}(1)^p_E \). Then, induced by the morphism \( BU(1) \to \text{String}(n) \), there is a morphism \( \text{BBU}(1)^p_E \to \text{BString}(n)^p_E \) given by

\[
(U \times BU(1), B \in \Omega^2(U); L : U \to BU(1), A \in \Omega^1(U); a : U \to U(1)) \mapsto (U \times \text{String}(n), B, \theta = 0; g = \iota \circ L, A, \omega = 0; a, \omega^1 = 0).
\]

On the side of algebroids, we have similar constructions. Let us denote by \( \text{EC}^p_E \) the (2,1)-presheaf of exact Courant algebroids (see Appendix A.2 for the definition and notations). There is a morphism on the presheaf level, \( \text{EC}^p_E \to \text{TC}^+_E \), given by

\[
(TU, \llbracket -,- \rrbracket^E_S, \langle -,- \rangle^E, \text{pr}_{TU}, B; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \mapsto \bigg( T_{gU}, \llbracket -,- \rrbracket^T_S, \langle -,- \rangle^T, \text{pr}_{TU}, 0, B; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bigg).
\]

Let us denote by \( \text{TC}^+_E \) the (2,1)-presheaf of transitive Lie algebroids (see Appendix A.1). Similar to the morphism \( \text{BString}(n)^p_E \bar{\to} \text{BSpin}(n)^p_E \) described before Theorem 3.2, we have a morphism \( \text{TC}^+_E \to \text{TL}^+_E \) defined also by forgetting some data,

\[
\bigg( T_{gU}, \llbracket -,- \rrbracket^T_S, \langle -,- \rangle^T, \text{pr}_{TU}; \theta, B; \begin{pmatrix} 1 & 0 \\ \phi & \tau \\ \beta & -2\phi^* \circ \tau \end{pmatrix} \bigg) \mapsto \bigg( TU \oplus (U \times \mathfrak{g}), \llbracket -,- \rrbracket^T_S, \text{pr}_{TU}, \theta; \begin{pmatrix} 1 \\ 0 \\ \phi \end{pmatrix} \bigg).
\]

Then we can also construct morphisms \( \Phi_{U(1)} : \text{BBU}(1)_E \to \text{EC}^p_E \) and \( \Phi_{\text{Spin}(n)} : \text{BSpin}(n)_E \to \text{TL}^+_E \), for \( \mathfrak{g} = \mathfrak{so}(n) \). The constructions are essentially given in [38] [27], and here we spell it out in our setting. On the level of objects, they are given by

\[
\Phi_{U(1)} : \{ U_i \times BU(1), B_i; L_{ij}, A_{ij}; a_{ijk} \} \mapsto \{ TU_i \oplus T^*U_i, B_i; \begin{pmatrix} 1 & 0 \\ dA_{ij} \end{pmatrix} \},
\]

\[
\Phi_{\text{Spin}(n)} : \{ U_i \times \text{Spin}(n), \theta_i; g_{ij} \} \mapsto \{ TU_i \oplus (U_i \times \mathfrak{so}(n)), \llbracket -,- \rrbracket^T_S, \text{pr}_{TU}, \theta; \begin{pmatrix} 1 & 0 \\ -g_{ij}^{\theta} \theta_{ijc} \\ \text{ad}_{g_{ij}} \end{pmatrix} \}, \tag{52}
\]

and on the level of morphisms, they are given by corresponding pullbacks and pre-compositions.
Thus, we have the following commutative diagram to connect the principal bundle side and the algebroid side,

\[
\begin{array}{c}
\text{BBU}(1) \quad \Phi_{U(1)}(1) \\
\downarrow \quad \downarrow \\
\text{BString}(n) \quad \Phi \\
\downarrow \quad \downarrow \\
\text{BSpin}(n) \quad \Phi_{\text{Spin}(n)}(n) \\
\end{array}
\]

(53)

Now we show that \( \Phi_{U(1)} \) is not injective in general and this implies that \( \Phi \) is not injective. And \( \Phi_{\text{Spin}(n)} \) is not surjective thus \( \Phi \) can not be surjective. Moreover, even on the fibre of the image of \( \Phi_{\text{Spin}(n)} \), \( \Phi \) can not be surjective in general.

**Lemma 5.7.** When \( H^3(M, \mathbb{Z}) \) has torsion, \( \Phi_{U(1)} \) is not injective on the level of objects and not fully faithful.

**Proof.** We take a cocycle \( a_{ijk} \) representing a torsion element in \( H^3(M, \mathbb{Z}) \), lifting it to a Deligne cocycle \((a_{ijk}^0, A_{ij}^0, B_{ij}^0)\), then \( dB_{ij}^0 \) glues to an exact 3-form. We now show that such a Deligne cocycle may always be adjusted by an exact one to \((a_{ijk}^0, A_{ij}^0, B_{ij}^0|_{U_i})\) for a global 2-form \( B^0 \) and some closed \( A_{ij}^0 \). Since \( dB_{ij}^0 \) glues to an exact 3-form, there is a global 2-form \( B^0 \) such that \( B_{ij}^0 = B|_{U_i} + dA_{ij}^0 \). Then adjusting the original Deligne cocycle by \( D(1, A^0) \) will fulfill our aim.

Therefore we might as well assume that we lift \( a_{ijk} \) to a Deligne cocycle \((a_{ijk}^0, A_{ij}^0, B_{ij}^0|_{U_i})\) satisfying \( dA_{ij}^0 = 0 \). Then clearly the image of the two objects \((U_i \times BU(1), B_i; L_{ij}, A_{ij}; a_{ijk})\) \( \in \text{BBU}(1) \) and \((U_i \times BU(1), B_i + B_{ij}^0; L_{ij}, A_{ij} + A_{ij}^0; a_{ijk} + a_{ijk}^0)\) in \( \text{BBU}(1) \) are the same. However, it is clear that there exists no morphism between these two objects because \( a_{ijk}^0 \) is not exact. ■

**Lemma 5.8.** In general, the map \( \Phi \) is not injective on the level of objects and not fully faithful.

**Proof.** We take the two different gerbes with connection data constructed in Lemma 5.7 \( G_1 \) and \( G_2 \), which maps to the same object under \( \Phi_{U(1)} \). Then we see that \( \mathcal{B}(G_1) \) and \( \mathcal{B}(G_2) \) are non-isomorphic string data but mapping to the same Courant data on the right hand side. ■

**Lemma 5.9.** The map \( \Phi \) preserves curvatures.

**Proof.** It is clear from the definition of curvatures on both sides. ■

**Lemma 5.10.** The map \( \Phi \) is not essentially surjective in general.

**Proof.** The map \( \Phi_{\text{Spin}(n)} \) is not essentially surjective because there are non-integrable transitive Lie algebroids. To show \( \Phi \) is not essentially surjective, we need to find a non-integrable transitive Lie algebroid \( A \) whose \( p_1(A) = 0 \). Notice that integrability is a property preserved by isomorphisms of Lie algebroids.

We take \( M = \mathbb{R}^3 - \{p_1\} - \{p_2\} \) where \( p_1, p_2 \in \mathbb{R}^3 \) are two different points and \( A = TM \times \mathbb{R} \), with the following Lie bracket

\[
[(X, f), (Y, g)] = [X, Y] + X(f) - Y(g) + \omega(X, Y).
\]

Here \( \omega = \iota^*_j \text{pr}_1^* \omega_a + \sqrt{2} \iota^*_2 \text{pr}_2^* \omega_a \) with \( \iota_j : M \rightarrow \mathbb{R}^3 - p_j, \text{pr}_j : \mathbb{R}^3 - p_j \rightarrow S^2 \) for \( j = 1, 2 \), and \( \omega_a \) the area form on \( S^2 \). Then the period \( \{f, \omega, \gamma \in \pi_2(M)\} \) of \( \omega \) is dense in \( \mathbb{R} \). Therefore, \( A \) is not integrable by [8]. On the other hand, it is clear \( p_1(A) = 0 \) because there is no non-trivial 4-forms on \( M \). ■

**Lemma 5.11.** On the fibre of the image of \( \Phi_{\text{Spin}(n)} \), \( \Phi \) is in general not essentially surjective.

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8We thank very much Pavol Ševera for pointing out this example to the third author.
Proof. As pointed out in Lemma 5.9, \( \Phi \) preserves the curvature. If we fix an object \( M \xrightarrow{\rho} B\text{Spin}(n) \) and look at all possible lifts of \( \rho \), we see that curvatures for these lifts form a torsor of \( \text{im} \, d \) as in Corollary 5.8. Now on the Courant side, fixing the underlying \( \text{Spin}(n) \)-principal bundle and its connection data means that we consider all possible Courant lifts over a fixed Atiyah Lie algebroid and its connection data. By Corollary 4.12, the set of curvatures for such Courant lifts is a torsor of \( \Omega^3_{\text{cl}}(M) \). As pointed out in Remark 3.9, \( H^2(M, D_2) \xrightarrow{\sim} \Omega^3_{\text{cl}}(M) \) is not surjective in general. Since curvatures are preserved under isomorphisms, we see that \( \Phi \) can not be essentially surjective in general even on the fibre of the image of \( \Phi_{\text{Spin}(n)} \). □

A Appendix

A.1 (2, 1)-sheaf \( \mathcal{T}_c \) of transitive Lie algebroids with connections

Definition A.1. A Lie algebroid structure on a vector bundle \( A \longrightarrow M \) is a pair that consists of a Lie algebra structure \([-, -]\) on the section space \( \Gamma(A) \) and a bundle map \( \rho : A \longrightarrow TM \), called the anchor, such that the following relation is satisfied:

\[
[X, fY] = f[X, Y] + \rho(X)(f)Y, \quad \forall \, X, Y \in \Gamma(A), \, f \in C^\infty(M).
\]

A Lie algebroid \( A \) is called transitive if \( \rho \) is surjective, i.e. \( \text{im} \, \rho = TM \). Denote by \( \mathcal{G} = \ker \rho \). Then \( \mathcal{G} \) is a bundle of Lie algebras, whose fibre is isomorphic to a Lie algebra \((\mathfrak{g}, [-, -]_\mathfrak{g})\). We have the following short exact sequence:

\[
0 \longrightarrow \mathcal{G} \longrightarrow A \xrightarrow{\rho} TM \longrightarrow 0.
\]

A splitting \( s : TM \longrightarrow A \) gives rise to a connection \( \nabla \) on \( \mathcal{G} \) by

\[
\nabla_X a = [s(X), a], \quad \forall \, X \in \Gamma(A), \, a \in \Gamma(\mathcal{G}).
\]

Thus we call such a splitting \( s : TM \longrightarrow A \) a connection of \( A \). Connections always exist by partition of unity. Thus, after picking a connection, we have \( A \cong TM \oplus \mathcal{G} \), and the induced bracket on \( TM \oplus \mathcal{G} \) is

\[
[X + a, Y + b]_\mathcal{G} = [X, Y] + \nabla_X b - \nabla_Y a + [a, b]_\mathfrak{g} + R(X, Y), \quad \forall \, X, Y \in \Gamma(TM), \, a, b \in \Gamma(\mathcal{G}),
\]

(54)

where \( R(X, Y) = [s(X), s(Y)] - s([X, Y]) \) is the curvature of the connection \( s \). In other words, a transitive Lie algebroid with a connection is always isomorphic to \( (TM \oplus \mathcal{G}, [-, -]_\mathcal{G}, \rho = \text{pr}_{TM}) \) and the isomorphism depends on the choice of the connection.

In particular, if \( \mathcal{G} = M \times \mathfrak{g} \) is a trivial bundle and the connection \( \nabla \) is given by the flat connection \( \nabla_X b = X(b) \), we obtain the standard bracket

\[
[X + a, Y + b]_\mathcal{G}^T = [X, Y] + X(b) - Y(a) + [a, b]_\mathfrak{g}.
\]

(55)

An automorphism of the standard transitive Lie algebroid is given by a matrix \( \begin{pmatrix} 1 & 0 \\ \phi & \tau \end{pmatrix} \), where \( \tau : M \longrightarrow \text{Aut}(\mathfrak{g}) \) and \( \phi \in \Omega^1(M, \mathfrak{g}) \) satisfy

\[
\phi([X, Y]) = X(\phi(Y)) - Y(\phi(X)) + [\phi(X), \phi(Y)]_{\mathfrak{g}},
\]

\[
\tau([a, b]_{\mathfrak{g}}) = [\tau(a), \tau(b)]_{\mathfrak{g}},
\]

\[
\tau(X(b)) = X(\tau(b)) + [\phi(X), \tau(b)]_{\mathfrak{g}}.
\]

There is a \((2, 1)\)-presheaf of transitive Lie algebroids with connections \( \mathcal{T}_c^p : \text{Mfd}^{\text{op}} \rightarrow \text{Gpd} \), where \( \text{Mfd}^{\text{op}} \) is the opposite category of \( \text{Mfd} \), and \( \text{Gpd} \) is the 2-category of (discrete) groupoids and groupoid morphisms.

For an object \( U \in \text{Mfd} \), the groupoid \( \mathcal{T}_c^p(U) \) is made up by the following data:
\textbf{Definition A.2.} We call an object \( (\sqcup (TU_i \oplus (U_i \times g), \theta_i), \phi_{ij}, \tau_{ij}) \) in \( \text{holim} \mathcal{L}^p(U(M)_*) \) a transitive Lie data.

Given a transitive Lie data \( (\sqcup (TU_i \oplus (U_i \times g), \theta_i), \phi_{ij}, \tau_{ij}) \), since \( \Lambda_{ij} = \begin{pmatrix} 1 & 0 \\ \phi_{ij} & \tau_{ij} \end{pmatrix} \) satisfies the cocycle condition \( \Lambda_{ij} \Lambda_{jk} = \Lambda_{ik} \), we can glue \( TU_i \oplus (U_i \times g) \)'s and obtain a vector bundle

\[ A = \coprod TU_i \oplus (U_i \times g) / \sim, \]

where the equivalence relation \( \sim \) is given by

\[ X + a \sim Y + b \iff \begin{pmatrix} Y \\ b \end{pmatrix} = \Lambda_{ij} \begin{pmatrix} X \\ a \end{pmatrix}, \quad \forall X + a \in TU_j \oplus (U_j \times g), \; Y + b \in TU_i \oplus (U_i \times g). \]

\textbf{Proposition A.3.} A transitive Lie data \( (\sqcup (TU_i \oplus (U_i \times g), \theta_i), \phi_{ij}, \tau_{ij}) \) gives rise to a transitive Lie algebroid \((A, [-,-], \rho)\) with a connection \( s : TM \rightarrow A\).

\textbf{Proof.} Obviously, the vector bundle \( A \) fits the following short exact sequence:

\[ 0 \rightarrow G \rightarrow A \xrightarrow{\rho} TM \rightarrow 0, \]

where \( G \) denotes the Lie algebra bundle obtained from the transition function \( \tau_{ij} \) and \( \rho \) is induced by the projection \( TU_i \oplus (U_i \times g) \rightarrow TU_i \).
Since $\Lambda_{ij}$ preserves the Lie bracket $[-,-][S]$, there is a well-defined Lie bracket $[-,-]$ on $\Gamma(A)$. Then we obtain a Lie algebroid $(A, [-,-], \rho)$.

On $U_i$, consider the splitting $s_i : TU_i \rightarrow A|U_i$ given by
$$s_i(X) = X + \theta_i(X).$$

By (59), we have $\Lambda_{ij}s_i(X) = s_i(X)$, which implies that we have a global splitting $s : TM \rightarrow A$. \[\blacksquare\]

A 1-morphism from $\vartheta((TU_i \oplus (U_i \times g), [-,-]_S, \bar{\theta}_i, \bar{\phi}_ij, \bar{\tau}_ij))$ to $\vartheta((TU_i \oplus (U_i \times g), [-,-]_S, \theta_i, \phi_ij, \tau_ij))$ in $\text{holim}_{\text{TL}}^p(U(M)_\bullet)$ consists of a 1-morphism $\left(\begin{array}{cc}1 & 0 \\ \phi_i & \tau_i \end{array}\right)$ from $\vartheta((TU_i \oplus (U_i \times g), \bar{\theta}_i, \bar{\phi}_ij, \bar{\tau}_ij))$ to $\vartheta((TU_i \oplus (U_i \times g), [-,-]_S, \theta_i, \phi_ij, \tau_ij))$ in $\text{TL}^p(U_i)_1$, which satisfies
$$\Lambda_{ij} \left(\begin{array}{cc}1 & 0 \\ \phi_i & \tau_i \end{array}\right) = \left(\begin{array}{cc}1 & 0 \\ \phi_i & \tau_i \end{array}\right) \bar{\Lambda}_{ij}. \quad (61)$$

We have

**Proposition A.4.** A 1-morphism in $\text{holim}_{\text{TL}}^p(U(M)_\bullet)$ gives rise to a Lie algebroid isomorphism preserving connections.

**Proof.** Denote by $(\bar{A}, [-,-], \bar{\rho})$ (respectively $(A, [-,-], \rho)$) the transitive Lie algebroid with the connection $\bar{s} : TM \rightarrow \bar{A}$ (respectively $s : TM \rightarrow A$) obtained from the object $\left(\vartheta((TU_i \oplus (U_i \times g), \bar{\theta}_i, \bar{\phi}_ij, \bar{\tau}_ij)\right)$ (respectively $\left(\vartheta((TU_i \oplus (U_i \times g), \theta_i, \phi_ij, \tau_ij)\right)$) in $\text{holim}_{\text{TL}}^p(U(M)_\bullet)$. Thanks to (61), a 1-morphism $\left(\begin{array}{cc}1 & 0 \\ \phi_i & \tau_i \end{array}\right)$ from $\vartheta((TU_i \oplus (U_i \times g), \bar{\theta}_i, \bar{\phi}_ij, \bar{\tau}_ij))$ to $\vartheta((TU_i \oplus (U_i \times g), [-,-]_S, \theta_i, \phi_ij, \tau_ij)$ glues to a bundle map which gives rise to a Lie algebroid isomorphism between $(\bar{A}, [-,-], \bar{\rho})$ and $(A, [-,-], \rho)$.

Furthermore, we have
$$\left(\begin{array}{cc}1 & 0 \\ \phi_i & \tau_i \end{array}\right) \bar{s}_i(X) = X + \phi_i(X) + \tau_i \bar{\theta}_i(X) = X + \theta_i(X) = s_i(X),$$
which implies that the connections are also preserved. \[\blacksquare\]

**Remark A.5.** The above way to glue a transitive Lie algebroid is essentially the same as the one given by Mackenzie [27].

By Proposition A.3 and A.4, it is not hard to see that after the plus construction we arrive at a $(2,1)$-sheaf $\text{TL}^p_c$ which maps to the category of transitive Lie algebroids with connections essentially surjectively and fully faithfully.

### A.2 $(2,1)$-sheaf $\text{EC}^p_c$ of exact Courant algebroids with connections

The **standard Courant algebroid** is $(TM \oplus T^*M, [[-, -]]_S, \langle -,- \rangle^E, \text{pr}_{TM})$, where $[[-, -]]_S$ is the standard Dorfman bracket given by
$$[[X + \xi, Y + \eta]]^E = [X,Y] + L_X \eta - L_Y \xi,$$
and $\langle -,- \rangle^E$ is the canonical symmetric bilinear form given by
$$\langle X + \xi, Y + \eta \rangle^E = \frac{1}{2} \langle \xi(Y) + \eta(X) \rangle.$$

A Courant algebroid $C$ is called **exact** if we have the following short exact sequence
$$0 \rightarrow T^*M \xrightarrow{\rho^*} C \xrightarrow{\rho} TM \rightarrow 0.$$
A **connection** of an exact Courant algebroid $C$ is an isotropic splitting\(^9\) $s : TM \to C$. As before, connections always exist. By choosing a connection $s : TM \to C$, the vector bundle $C$ is isomorphic to $TM \oplus T^*M$. Then transferring the Courant algebroid structure on $C$ to that on $TM \oplus T^*M$, we obtain the Courant algebroid $(TM \oplus T^*M, [-, -]_h, \langle -, - \rangle^E, \text{pr}_TM)$, where the nondegenerate symmetric pairing $\langle -, - \rangle^E$ is given by (63) and the bracket $[-, -]_h^E$ is given by

$$[X + \xi, Y + \eta]_h^E = [X + \xi, Y + \eta]_S^E + i_Y i_X h.$$  

(64)

Here $h \in \Omega^3_3(M)$, defined by $h(X, Y) = [s(X), s(Y)] - s[X, Y]$, is the curvature of the connection $s$. In \[41\], the authors show that exact Courant algebroids over $M$ are classified by $H^3(M, \mathbb{R})$.

Now we construct the $(2, 1)$-presheaf of exact Courant algebroids with connections over the category of (differential) manifolds $\text{Mfd}$. For simplicity, for an object $U \in \text{Mfd}$, we write $TU := TU \oplus T^*U$.

There is a $(2, 1)$-presheaf of exact Courant algebroids with connections $\text{EC}^p_c : \text{Mfd}^{op} \to \text{Gpd}$, where $\text{Mfd}^{op}$ is the opposite category of $\text{Mfd}$, and $\text{Gpd}$ is the 2-category of (set theoretical) groupoids and groupoid morphisms.

For an object $U \in \text{Mfd}$, the groupoid $\text{EC}^p_c(U)$ is made up by the following data:

- **$\text{EC}^p_c(U)_0$**: an object is a quintuple $(TU, [-, -]^E_S, \langle -, - \rangle^E, \text{pr}_{TU}, B)$, where $B \in \Omega^2(U)$ is a 2-form, $\langle -, - \rangle^E$ and $\langle -, - \rangle^E$ are given by (62) and (63) respectively. We will simply denote an object by $(TU, B)$.

- **$\text{EC}^p_c(U)_1$**: a 1-morphism from $(TU, \tilde{B})$ to $(TU, B)$ is a bundle automorphism of $TU$ given by the matrix $\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$, where $B \in \Omega^2(U)$ is a closed 2-form such that $\tilde{B} - B = B$. This matrix preserves the standard Courant bracket $[-, -]_S^E$ and the pairing $\langle -, - \rangle^E$. The composition of 1-morphisms is simply the matrix multiplication.

Then for a morphism $\varphi : U \to V$ in $\text{Mfd}$, as in the case of Lie algebroids, the associated functor $\text{EC}^p_c(\varphi) : \text{EC}^p_c(V) \to \text{EC}^p_c(U)$ is induced by pulling back forms.

Take an open cover $\{U_i\}$ of $M \in \text{Mfd}$. An object in $\text{holim} \text{EC}^p_c(U(M)_\bullet)$ consists of

- an object $\sqcup(U_i, B_i)$ in $\text{EC}^p_c(\sqcup U_i)_0$,

- $\Lambda_{ij} = \begin{pmatrix} 1 & 0 \\ B_{ij} & 1 \end{pmatrix} \in \text{EC}^p_c(\sqcup U_{ij})_1$ which is a 1-morphism from $(TU_{ij}, B_j|_{U_{ij}})$ to $(TU_{ij}, B_i|_{U_{ij}})$, therefore satisfying

$$B_j|_{U_{ij}} - B_i|_{U_{ij}} = B_{ij}.$$  

- compatibility conditions $\Lambda_{ij} \Lambda_{jk} = \Lambda_{ik}$ on $U_{ijk}$ which automatically holds.

The plus construction gives us a $(2, 1)$-sheaf $\text{EC}^p_c$. For a manifold $M$, an object of $\text{EC}^p_c(M)$ consists of a cover $\{U_i\}$ and a $U(M)$-equivariant object of $\text{EC}^p_c$ described above. Naturally we ask what the above data glues to. It turns out that the gluing result is an exact Courant algebroid with a connection. The gluing procedure is the same as the one given in [41]. To be self-contained, we give the result using the language of this paper.

**Proposition A.6.** An object $\left(\sqcup(TU_i, B_i), B_{ij}\right)$ in $\text{holim} \text{EC}^p_c(U(M)_\bullet)$ gives rise to an exact Courant algebroid $(C, [-, -], \langle -, - \rangle, \rho)$ with a connection $s : TM \to C$.

\(^9\)A splitting $s : TM \to C$ is called isotropic if the image of $s$ is an isotropic subbundle, i.e. $(s(X), s(Y)) = 0$, for all $X, Y \in \Gamma(TM)$. 

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Proof. Given an object \( \left( \sqcup (TU_i, B_i), B_{ij} \right) \) in \( \text{holim} \mathbb{E}C^\ell_c(U(M)_\bullet) \), as before, the cocycle condition \( \Lambda_{ij} \Lambda_{jk} = \Lambda_{ik} \) implies that \( TU_i \)'s glue to a vector bundle \( C \) via transition matrices \( \Lambda_{ij} \)'s. Since \( \Lambda_{ij} \) preserves the standard bracket \( [-,-]_S \), we have a well-defined bracket \( [-,-]^E \) on \( \Gamma(C) \). Furthermore, since \( \Lambda_{ij} \) also preserves the standard pairing \( \langle -,- \rangle^E \) on \( TU_i \), we obtain a global nondegenerate symmetric bilinear form \( \langle -,- \rangle \) on \( C \). Obviously, \( C \) fits the following exact sequence of vector bundles,

\[
0 \longrightarrow T^*M \xrightarrow{\rho^*} C \xrightarrow{\rho} TM \longrightarrow 0,
\]

where \( \rho \) is induced by the projection \( TU_i \longrightarrow TU_i \).

Also, by the facts that \( \Lambda_{ij} \) preserves the standard bracket \( [-,-]_S^E \) and the standard pairing \( \langle -,- \rangle^E \), Axioms (i)-(iii) in Definition 4.1 are satisfied. Therefore, \( (C, [-,-], \langle -,- \rangle, \rho) \) is an exact Courant algebroid.

The 2-forms \( \{B_i\} \) induce an isotropic splitting \( s: TM \longrightarrow C \) via

\[
s(X) = X - i_X B_i, \quad X \in U_i.
\]

Note that the definition of \( s \) does not depend on choices of \( U_i \). In fact, if \( X \in U_i \cap U_j \), it is straightforward to see that \( X - i_X B_i \sim X - i_X B_j \). \( \blacksquare \)

In \( \text{holim} \mathbb{E}C^\ell_c(U(M)_\bullet) \), a 1-morphism from an object \( \left( \sqcup (TU_i, \tilde{B}_i), \tilde{B}_{ij} \right) \) to another object \( \left( \sqcup (TU_i, B_i), B_{ij} \right) \) consists of a 1-morphism \( \Lambda_{ij} \left( \begin{array}{cc} 1 & 0 \\ B_j & 1 \end{array} \right) \) from \( \sqcup (TU_i, \tilde{B}_i) \) to \( \sqcup (TU_i, B_i) \) in \( \mathbb{E}C^\ell_c(\sqcup U_i)_1 \), which satisfies

\[
\Lambda_{ij} \left( \begin{array}{cc} 1 & 0 \\ B_j & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ B_i & 1 \end{array} \right) \Lambda_{ij}.
\] (65)

Then we have

Proposition A.7. A 1-morphism in \( \text{holim} \mathbb{E}C^\ell_c(U(M)_\bullet) \) gives rise to an exact Courant algebroid isomorphism preserving connections.

Proof. The proof is similar to that of Proposition A.4. Eq. (65) is the important information which implies the gluing result. The fact that the bundle map \( \mathcal{B} \) also preserves the connection, namely \( \mathcal{B}(\tilde{s}(X)) = s(X) \), follows from the following calculation,

\[
\mathcal{B}(\tilde{s}(X)) = X - i_X \tilde{B}_i + B_i = X - i_X B_i = s(X), \quad \forall X \in \Gamma(TM).
\]

The proof is finished. \( \blacksquare \)

Similar to the case of transitive Lie algebroids, after the plus construction, we arrive at the \((2,1)\)-sheaf \( \mathbb{E}C^\ell_c^+ \) of exact Courant algebroids with connections.

### A.3 Gluing transitive Courant algebroids via local data

In this subsection, we give the explicit formula for the transitive Courant algebroid glued by pieces of standard transitive Courant algebroids in a transitive Courant data. This also shows how we may obtain the bracket of a general transitive Courant algebroid from the standard one. As in Proposition A.4, given a transitive Courant data \( C_c \), there is a corresponding transitive Courant algebroid \( (C, [-,-], \langle -,- \rangle, \rho) \). Using the two splittings \( s \) and \( \sigma_c \) given in [33] and [40], we obtain an isomorphism \( S: TM \oplus \mathcal{G} \oplus T^*M \longrightarrow C \) given by

\[
S(X + a + \xi) = s(X) + \sigma_c(a) + \xi.
\]

(66)

Recall that locally, \( S_i = S|_{U_i} : TU_i \times (U_i \times \mathfrak{g}) \times T^*U_i \longrightarrow C|_{U_i} \) and its inverse are given by

\[
S_i(X + a + \xi) = X + \theta_i(X) - (\theta_i, \theta_i(X))^g - i_X B_i + a - 2(\theta_i, a)^g + \xi,
\]

\[
S_i^{-1}(X + a + \xi) = X - \theta_i(X) - (\theta_i, \theta_i(X))^g + i_X B_i + a + 2(\theta_i, a)^g + \xi.
\]
Having $TM \oplus \mathcal{G} \oplus T^*M$ equipped with the pairing given by (24), a straightforward computation shows that $S$ preserves the pairing.

**Lemma A.8.** Define $\nabla^i : \Gamma(TU_i) \otimes \Gamma(U_i \times \mathfrak{g}) \longrightarrow \Gamma(U_i \times \mathfrak{g})$ by

$$\nabla^i_X a = X(a) + [\theta_i(X), a]_\mathfrak{g}, \quad \forall X \in \Gamma(TU_i), a \in \Gamma(U_i \times \mathfrak{g}).$$

(67)

Then, we have

$$\tau_{ij} \nabla^j_X a = \nabla^j_X \tau_{ij} a, \quad \forall X \in \Gamma(TU_{ij}), a \in \Gamma(U_{ij} \times \mathfrak{g}).$$

Thus, by gluing $\nabla^i$, we obtain a globally well-defined connection $\nabla : \Gamma(TM) \otimes \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{G})$.

**Proof.** By (29) and (30), we have

$$\tau_{ij} \nabla^j_X a - \nabla^j_X \tau_{ij} a = \tau_{ij}(X(a) + [\theta_j(X), a]_\mathfrak{g}) - X(\tau_{ij} a) - [\theta_i(X), \tau_{ij} a]_\mathfrak{g}
= \tau_{ij}(X(a)) + [\tau_{ij} \theta_j(X), \tau_{ij} a]_\mathfrak{g} - X(\tau_{ij} a) - [\tau_{ij} \theta_j(X) + \phi_{ij}(X), \tau_{ij} a]_\mathfrak{g}
= 0.$$

The proof is finished.

Now we see that given a Courant data $C_c$, we have a connection $\nabla$, a curvature $R$ of the underlining Lie data $A_c$ given in Lemma 4.7 and a curvature $H$ of $C_c$. Thus $TM \oplus \mathcal{G} \oplus T^*M$ may be equipped with a transitive Courant algebroid structure with the Courant bracket $[-,-]_{\nabla,R,H}$ given as in (23).

**Proposition A.9.** The morphism $S$ in (66) is an isomorphism of Courant algebroids.

**Proof.** We pull back the bracket on $\Gamma(C)$ to $\Gamma(TM \oplus \mathcal{G} \oplus T^*M)$ via $S$ and denote it by $[-,-]_{\text{ind}}$. The only nontrivial thing to check is that $[-,-]_{\text{ind}} = [-,-]_{\nabla,R,H}^T$. For all $a, b \in \Gamma(U_i \times \mathfrak{g})$, we have

$$[a,b]_{\text{ind}} = S^{-1}_i [S_i(a), S_i(b)]^T = S^{-1}_i [a - 2(\theta_i,a)\mathfrak{g}, b - 2(\theta_i,b)\mathfrak{g}]^T = S^{-1}_i [(a,b)\mathfrak{g} + \mathcal{P}(a,b))
= [a,b]_\mathfrak{g} + \mathcal{P}(a,b) + 2(\theta_i,[a,b]_\mathfrak{g})\mathfrak{g}.$$  

By (67) and Lemma A.8, we have

$$\mathcal{P}(a,b) + 2(\theta_i,[a,b]_\mathfrak{g})\mathfrak{g}(Y) = 2(b,Y(a))\mathfrak{g} + 2(\theta_i(Y),[a,b]_\mathfrak{g})\mathfrak{g}
= 2(b,Y(a) + [\theta_i(Y),a]_\mathfrak{g})\mathfrak{g} = 2(b,\nabla^i_Y a)\mathfrak{g}
= \mathcal{P}(a,b)(Y),$$

which implies that

$$[a,b]_{\text{ind}} = [a,b]_\mathfrak{g} + \mathcal{P}(a,b).$$

For all $X \in \Gamma(TU_i), b \in \Gamma(U_i \times \mathfrak{g})$, we have

$$[X,b]_{\text{ind}} = S^{-1}_i [S_i(X), S_i(b)]^T = S^{-1}_i [X + \theta_i(X) - (\theta_i,\theta_i(X))\mathfrak{g} - i_X B_i, b - 2(\theta_i,b)\mathfrak{g}]^T
= S^{-1}_i (X(b) - 2L_X(\theta_i,b)\mathfrak{g} + [\theta_i(X),b]_\mathfrak{g} + \mathcal{P}(\theta_i(X),b))
= X(b) + [\theta_i(X),b]_\mathfrak{g} - 2L_X(\theta_i,b)\mathfrak{g} + \mathcal{P}(\theta_i(X),b) + 2(\theta_i,X(b) + [\theta_i(X),b]_\mathfrak{g})\mathfrak{g}.$$  

By (67) and

$$\left(-2L_X(\theta_i,b)\mathfrak{g} + \mathcal{P}(\theta_i(X),b) + 2(\theta_i,X(b) + [\theta_i(X),b]_\mathfrak{g})\mathfrak{g}\right)(Y) = -2(R(X,Y),b)\mathfrak{g} = -2Q(X,b)(Y),$$

we get

$$[X,b]_{\text{ind}} = \nabla_X b - 2Q(X,b).$$

(69)
Similarly, we have

\[ [a, Y]_{\text{ind}} = 2Q(Y, a) - \nabla_Y a. \]  

(70)

For all \( X, Y \in \Gamma(TU_i) \), we have

\[ [X, Y]_{\text{ind}} = S_i^{-1} [S_i(X), S_i(Y)]^T_S \]

\[ = S_i^{-1} \left( [X + \theta_i(X) - (\theta_i, \theta_i(X))^\theta - i_X B_i, Y + \theta_i(Y) - (\theta_i, \theta_i(Y))^\theta - i_Y B_i] \right) \]

\[ = S_i^{-1} \left( [X, Y] + X(\theta_i(Y)) - Y(\theta_i(X)) + [\theta_i(X), \theta_i(Y)]_g \right) \]

\[ - L_X(\theta_i, \theta_i(Y))^g + L_Y(\theta_i, \theta_i(X))^g + \mathcal{P}(\theta_i(X), \theta_i(Y)) - L_X i_Y B_i + i_Y d i_X B_i \]

\[ = [X, Y] - \theta_i([X, Y]) + X(\theta_i(Y)) - Y(\theta_i(X)) + [\theta_i(X), \theta_i(Y)]_g \]

\[ - L_X i_Y B_i + i_Y d i_X B_i + i_{[X,Y]} B_i + \Xi, \]

where

\[ \Xi = -L_X(\theta_i, \theta_i(Y))^g + L_Y(\theta_i, \theta_i(X))^g + \mathcal{P}(\theta_i(X), \theta_i(Y)) - (\theta_i, \theta_i([X, Y]))^g \]

\[ + 2(\theta_i, X(\theta_i(Y)) - Y(\theta_i(X)) + [\theta_i(X), \theta_i(Y)]_g)^g. \]

Obviously, we have

\[ -\theta_i([X, Y]) + X(\theta_i(Y)) - Y(\theta_i(X)) + [\theta_i(X), \theta_i(Y)]_g = d\theta_i(X, Y) + [\theta_i(X), \theta_i(Y)]_g = R_i(X, Y), \]

\[ -L_X i_Y B_i + i_Y d i_X B_i + i_{[X,Y]} B_i = -d B_i(X, Y, \cdot). \]

Furthermore, we have

\[ \Xi(Z) = 2(\theta_i(Y), Z\theta_i(X))^g - X(\theta_i(Z), \theta_i(Y))^g + (\theta_i[X, Z], \theta_i(Y))^g + d(\theta_i, \theta_i(X))^g(y, Z) \]

\[ - (\theta_i(Z), \theta_i[X, Y])^g + 2(\theta_i(Z), X\theta_i(Y)) - Y(\theta_i(X)) + [\theta_i(X), \theta_i(Y)]_g)^g \]

\[ = (\theta_i, d\theta_i)^g + \frac{1}{3} (\theta_i, [\theta_i, \theta_i]^g) \]

\[ = (\theta_i, d\theta_i)^g + \frac{1}{3} (\theta_i, [\theta_i, \theta_i]^g) (X, Y, Z) \]

\[ = c_{s_3}(\theta_i)(X, Y, Z). \]

Therefore, by Lemma 4.7 and the fact that \(-d B_i + c_{s_3}(\theta_i)\) can be glued to a global 3-form \(H\), we have

\[ [X, Y]_{\text{ind}} = [X, Y] + R(X, Y) + \left( -d B_i + c_{s_3}(\theta_i) \right)(X, Y, \cdot) = [X, Y] + R(X, Y) + H(X, Y, \cdot). \]  

(71)

Furthermore, it is straightforward to obtain that

\[ [X, \eta]_{\text{ind}} = L_X \eta, \quad [\xi, Y]_{\text{ind}} = -i_Y d \xi, \quad [a, \eta]_{\text{ind}} = 0, \quad [\eta, b]_{\text{ind}} = 0, \quad [\xi, \eta]_{\text{ind}} = 0. \]  

(72)

By (65)–(72), we deduce that the induced bracket \([\cdot, \cdot]_{\text{ind}}\) is exactly given by (55), i.e.

\[ [\cdot, \cdot]_{\text{ind}} = [\cdot, \cdot]^T_{\nabla, R, H}. \]

The proof is finished. 

A.4 Inner automorphisms of transitive Courant algebroids

In this subsection, we prove that the automorphisms that appeared in Proposition 5.1 are inner automorphisms of the standard transitive Courant algebroid \((T_g U, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot), \cdot, \cdot, \cdot, \cdot, \cdot, \cdot)\). In his letter to Weinstein [8], Ševera claimed that the inner automorphism group \(\text{Inn}(U)\) of the standard transitive
Courant algebroid over $U$ is an extension of the group of $G$-valued function $C^\infty(U, G)$ by closed 2-forms $\Omega^2_{cl}(U)$,\footnote{\begin{equation}
\tag{73}
\Omega^2_{cl}(U) \to \text{Inn}(U) \to C^\infty(U, G).
\end{equation}}
More precisely, an inner automorphism is a pair $(g, \omega)$, where $g$ is a $G$-valued function and $\omega \in \Omega^2_{cl}(U)$, such that
\begin{equation}
\tag{74}
d\omega + g^*C = 0,
\end{equation}
where $C = \frac{1}{6}([\theta_{MC}, [\theta_{MC}, \theta_{MC}]], \theta_{MC})$. The group structure is given by
\begin{align*}
(g_1, \omega_1)(g_2, \omega_2) &= (g_1g_2, \omega_1 + \omega_2 + (g_1^*\theta_{MC}, \text{ad}_{g_1}g_2^*\theta_{MC})^g), \\
(g, \omega)^{-1} &= (g^{-1}, -\omega).
\end{align*}

Now we give the corresponding matrix form of an inner automorphism. The matrix corresponding to $(g, \omega)$ is given by
\begin{equation}
\tag{75}
\Psi = \begin{pmatrix}
1 & 0 & 0 \\
-\theta_{MC} & 0 & 0 \\
\omega - (g^*\theta_{MC})^* \circ g^*\theta_{MC} & 2(g^*\theta_{MC})^* \circ \text{ad}_g & 1
\end{pmatrix}.
\end{equation}

**Proposition A.10.** $\Psi$ given above is an automorphism of the standard transitive Courant algebroid.

**Proof.** It is straightforward to see that (28)-(30) hold. For all $X, Y, Z \in \Gamma(TU)$, we have
\begin{align*}
\langle L_X\omega(Y) - \frac{1}{2}C_{X,Y} \omega(X) - \omega([X,Y]), Z \rangle &= d\omega(X,Y,Z).
\end{align*}
Denote by $\beta^\text{sym} = -(g^*\theta_{MC})^* \circ g^*\theta_{MC}$. By straightforward computations, we have
\begin{align*}
\langle L_X\beta^\text{sym}(Y) - \frac{1}{2}C_{X,Y} \beta^\text{sym}(X) - \beta^\text{sym}([X,Y]), Z \rangle &= g^*C(X,Y,Z).
\end{align*}
By (74), we deduce that (31) holds. Thus $\Psi$ given above is an automorphism. \qed

**Corollary A.11.** $\Lambda_{01}$ given in Proposition 5.1 is an inner automorphism of the standard transitive Courant algebroid.

**Acknowledgement**
The last author thanks much Giovanni Felder for patiently going through complicated calculations together and pointing out correct directions for the most technical part of this article during her visit to ETH. The last two authors thank Anton Alekseev and Pavol Ševera for many inspiring conversations. We warmly thank Olivier Brahic, Zhuo Chen, Domenico Fiorenza, Fei Han, Kirill Mackenzie, Eckhard Meinrenken, Ralf Meyer, Thomas Nikolaus, Dmitry Roytenberg, Thomas Schick, Chris Schommer-Pries, Urs Schreiber, Konrad Waldorf and Marco Zambon for very helpful discussion. Y. Sheng is supported by NSFC (11471139) and NSF of Jilin Province (20170101050JC); X. Xu was partially supported by the SNSF grants P2GEP2-165118 and NCCR SwissMAP; C. Zhu is supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.

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