On properties of a deformed Freud weight

Mengkun Zhu∗† and Yang Chen††

1Department of Mathematics, University of Macau, Avenida da Universidade, Taipa, Macau, China

October 8, 2018

Abstract

We study the recurrence coefficients of the monic polynomials \( P_n(z) \) orthogonal with respect to the deformed (also called semi-classical) Freud weight

\[ w_\alpha(x; s, N) = |x|^{\alpha}e^{-N|x^2+s(x^4-x^2)|}, \quad x \in \mathbb{R}, \]

with parameters \( \alpha > -1, N > 0, s \in [0,1] \). We show that the recurrence coefficients \( \beta_n(s) \) satisfy the first discrete Painlevé equation (denoted by dP\( I \)), a differential-difference equation and a second order nonlinear ordinary differential equation (ODE) in \( s \). Here \( n \) is the order of the Hankel matrix generated by \( w_\alpha(x; s, N) \). We describe the asymptotic behavior of the recurrence coefficients in three situations, (i) \( s \to 0, n, N \) finite, (ii) \( n \to \infty, N \) finite, (iii) \( n, N \to \infty \), such that the ratio \( r := \frac{n}{N} \) is bounded away from 0 and closed to 1. We also investigate the existence and uniqueness for the positive solutions of the dP\( I \).

Furthermore, we derive, using the ladder approach, a second order linear ODE satisfied by the polynomials \( P_n(z) \). It is found as \( n \to \infty \), the linear ODE turns to be a biconfluent Heun equation. This paper concludes with the study of the Hankel determinant, \( D_n(s) \), associated with \( w_\alpha(x; s, N) \) when \( n \) tends to infinity.

Key words: Deformed Freud Weight; Unitary Random Matrices; Hankel Determinant; Discrete and Continuous Painlevé Equation; Integrable Systems

1 Introduction

The orthogonal polynomials, recurrence coefficients, eigenvalues and the relevant properties associated with the exponential weights \( e^{-|x|^m}, m \in \mathbb{N} \) as well as

∗Zhu_mengkun@163.com; zhu.mengkun@connect.umac.mo
†yangbrookchen@yahoo.co.uk
Various generalized forms of this weight function on $\mathbb{R}$, have been investigated by many authors, Refs. [3, 5, 9, 12, 13, 17, 19, 29, 34–36].

Let $P_n(z)$ be the monic polynomials of degree $n$ orthogonal with respect to the deformed Freud weight

$$w(x) = w_\alpha(x; s, N) = |x|^\alpha e^{-N[x^4 + s(x^2 - x^4)]}, \quad x \in \mathbb{R}, \quad \alpha > -1, \quad N > 0, \quad s \in [0, 1],$$

(1.1)

where the deformation $s(x^4 - x^2)$, given by $0 \leq s \leq 1$, interpolates between the generalized Gaussian (Hermite) weight ($|x|^\alpha e^{-N x^4}$, $x \in \mathbb{R}$, $\alpha > -1$, $N > 0$) when $s = 0$ and the Freud weight ($|x|^\alpha e^{-N x^2}$, $x \in \mathbb{R}$, $\alpha > -1$, $N > 0$) when $s = 1$.

From the orthogonality condition, given by

$$\int_{\mathbb{R}} P_j(x)P_k(x)w_\alpha(x; s, N)dx = h_j(s; \alpha, N)\delta_{jk}, \quad h_j(s; \alpha, N) > 0, \quad j, k \in \{0, 1, 2, \ldots\},$$

(1.2)

where $\delta_{jk}$ denotes the Kronecker delta, and $h_j$ is the square of $L^2$ norm of the monic polynomial $P_j(x)$, there follows the three term recurrence relation,

$$zP_n(z; s, \alpha, N) = P_{n+1}(z; s, \alpha, N) + \beta_n(s; \alpha, N)P_{n-1}(z; s, \alpha, N), \quad n \geq 0,$$

(1.3)

subject to the initial conditions

$$P_0(z) := 1 \quad \text{and} \quad \beta_0P_{-1}(z) := 0.$$

Multiplying both sides of Eq. (1.3) by $P_{n-1}(z)w_\alpha(z; s, N)$ and integrating this with respect to $z$ on $\mathbb{R}$, which, due to the orthogonality condition (1.2), gives us

$$\beta_n(s) = \frac{1}{h_{n-1}(s)} \int_{\mathbb{R}} zP_n(z)P_{n-1}(z)w_\alpha(z; s, N)dx = \frac{h_n(s)}{h_{n-1}(s)} > 0.$$

It should be pointed out that $P_n(z)$ contains only the terms $z^{n-j}$, $j \leq n$ and even, since our weight function $w_\alpha(x; s, N)$ is even on $\mathbb{R}$. This implies that

$$P_n(-z) = (-1)^n P_n(z) \quad \text{and} \quad P_n(0)P_{n-1}(0) = 0.$$

Then we note the monic polynomials $P_n(z)$, associated with $w_\alpha(x; s, N)$, can be normalized as [11],

$$P_{2j}(z) = z^{2j} + p(2j; s)z^{2j-2} + \cdots + P_{2j}(0),$$

and

$$P_{2j+1}(z) = z^{2j+1} + p(2j + 1; s)z^{2j-1} + \cdots + \text{const.}z$$

$$= z(z^{2j} + p(2j + 1; s)z^{2j-2} + \cdots + \text{const.}).$$

In unitary random matrix theory with the Hermitian ensemble, the joint probability density function of the $n$ eigenvalues $\{x_j\}_{j=1}^n$ is given in [31] by

$$p(x_1, \ldots, x_n) = \prod_{j=1}^n dx_j = \frac{1}{D_n(s)} \frac{1}{n!} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^n w(x_\ell)dx_\ell,$$
where $D_n(s)$ denotes the normalization constant (also called partition function), which reads

$$D_n(s) = \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{\ell=1}^n w(x_{\ell}) \, dx_{\ell},$$

so that

$$\int_{\mathbb{R}^n} p(x_1, \ldots, x_n) \prod_{j=1}^n dx_j = 1.$$

For the problem at hand, the weight function $w(x)$ is given by

$$w(x) = |x|^\alpha e^{-N[x^2+s(x^4-x^2)]}, \quad x \in \mathbb{R}, \quad \alpha > -1, \quad N > 0, \quad s \in [0, 1],$$

and the moments are defined by

$$\mu_j(s; \alpha, N) = \int_{\mathbb{R}} x^j |x|^\alpha e^{-N[x^2+s(x^4-x^2)]} \, dx, \quad j \in \{0, 1, 2, \ldots\}.$$

$D_n(s)$ admits two more alternative representations \[41],

$$D_n(s) := \det (\mu_{j+k})_{j,k=0}^{n-1} = \prod_{j=0}^{n-1} h_j,$$

where $\det (\mu_{j+k})_{j,k=0}^{n-1}$ is the determinant of the Hankel matrix (or moment matrix).

For the deformed Freud weight (1.1), $\mu_0(s; \alpha, N)$ can be evaluated in term of the parabolic cylinder (Hermite-Weber) function $D_\nu(z)$. By definition

$$\mu_0(s; \alpha, N) = \int_{-\infty}^{\infty} |x|^\alpha e^{-N[x^2+s(x^4-x^2)]} \, dx$$

$$= 2 \int_{0}^{\infty} x^\alpha e^{-N[x^2+s(x^4-x^2)]} \, dx$$

$$= \int_{0}^{\infty} y^{\frac{\alpha+1}{2}} e^{-N[y+s(y^2-y)]} \, dy$$

$$= (2Ns)^{-\frac{\alpha+1}{2}} \Gamma \left( \frac{\alpha+1}{2} \right) \exp \left[ \frac{N(1-s)^2}{8s} \right] D_{-\frac{\alpha+1}{2}} \left[ \frac{N(1-s)}{\sqrt{2Ns}} \right],$$

since the parabolic cylinder function $D_\nu(z)$ has the integral representation \[37, §12.5 (i)]

$$D_\nu(z) = \frac{\exp \left( -\frac{z^2}{4} \right)}{\Gamma(-\nu)} \int_{0}^{\infty} t^{-\nu-1} \exp \left( -\frac{t^2}{2} - tz \right) \, dt, \quad \Re \nu < 0.$$

We note that the even moments are

$$\mu_{2n}(s; \alpha, N) = \int_{-\infty}^{\infty} x^{2n} |x|^\alpha e^{-N[x^2+s(x^4-x^2)]} \, dx = \mu_0(s; \alpha + 2n, N), \quad n \in \mathbb{N},$$

3
whilst the odd ones are

\[ \mu_{2n+1}(s; \alpha, N) = \int_{-\infty}^{\infty} x^{2n+1} |x|^\alpha e^{-N[x^2+s(x^4-x^2)]} \, dx = 0, \quad n \in \mathbb{N}. \]

It should be pointed out our recurrence coefficients, moments, \( L^2 \) norm of orthogonal polynomials, Hankel determinant depend on \( s \), also parameters \( N \) and \( \alpha \), but we may not display them unless we have to.

**Remark 1.1.** The dependence of the orthogonal polynomials \( P_n(x; s, \alpha, N) \) on \( s \), \( \alpha \) and \( N \) can be seen from its determinant representation in terms of the moments, or alternatively, from the Heine formula [41, Eq. 2.2.10]

\[
P_n(z; s, \alpha, N) = \frac{1}{D_n(s; \alpha, N)} \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^{n} (z - x_\ell) w_\alpha(x_\ell; s, N) dx_\ell
\]

\[
= \frac{\det \left( \int_{\mathbb{R}} x^{j+k} (z - x) w_\alpha(x; s, N) dx \right)_{j,k=0}^{n-1}}{\det \left( \int_{\mathbb{R}} x^{j+k} w_\alpha(x; s, N) dx \right)_{j,k=0}^{n-1}}.
\]

The remainder of this paper is organized as follows. In Sect. 2 we prove that the recurrence coefficients satisfy the first discrete Painlevé equation (dP-I), a differential-difference equation and a second order ODE. Sects. 3-5 discuss the asymptotic behavior of the recurrence coefficients in three cases, i.e., \( s \to 0 \), \( n, N \) fixed; \( n \to \infty \), \( N \) fixed; \( n, N \to \infty \) but the quantity \( n/N \) fixed. In Sect. 6, we obtain a second order ODE satisfied by the deformed Freud orthogonal polynomials applying the ladder operator approach, as well as a biconfluent Heun equation when \( n \) tends to infinity. Sect. 7 investigates the existence and uniqueness for the solutions of the dP-I. Finally, in Sect. 8, with the aid of the compatibility conditions for the ladder operators, we find the asymptotic expansion of the logarithm of the Hankel determinant associated with our weight.

## 2 Recurrence coefficients of the deformed Freud polynomials

It is known [17, 29, 43] that the recurrence coefficients satisfy the following second order nonlinear difference equation,

**Proposition 2.1.** The recurrence coefficients \( \beta_n(s) \) satisfy the first discrete Painlevé equation, dP-I,

\[
\beta_{n+1}(s) + \beta_n(s) + \beta_{n-1}(s) = \frac{z_n + \gamma(-1)^n}{\beta_n(s)} + \delta,
\]

with

\[
z_n = \frac{2n + \alpha}{8Ns}, \quad \gamma = -\frac{\alpha}{8Ns} \quad \text{and} \quad \delta = \frac{s - 1}{2s}.
\]
Proof. Writing

\[ w(x) = |x|^\alpha w_0(x), \]

where

\[ w_0(x) := e^{-v_0(x)} \quad \text{with} \quad v_0(x) := Nsx^4 + N(1-s)x^2. \]

With the aid of the three term recurrence relation (1.3), repeatedly, we have

\[
\int_{-\infty}^{\infty} [P_n(x)P_{n-1}(x)|x|^\eta] w_0(x)dx = 4Ns \int_{-\infty}^{\infty} x^3P_n(x)P_{n-1}(x)w(x)dx
\]

\[
+ 2N(1-s) \int_{-\infty}^{\infty} xP_n(x)P_{n-1}(x)w(x)dx
\]

\[
= 4Ns\sqrt{\beta_n} (\beta_{n+1} + \beta_n + \beta_{n-1}) + 2N(1-s)\sqrt{\beta_n},
\]

Combining this result and using the identity [43, Eq. (13)],

\[
\int_{-\infty}^{\infty} [P_n(x)P_{n-1}(x)|x|^\eta] \tilde{w}(x)dx = \frac{n + \alpha \Delta_n}{\sqrt{\beta_n}}, \quad \Delta_n = \frac{1 - (-1)^n}{2},
\]

which holds whenever \( \tilde{w}(x) \) is a symmetric weight on the real line, then we get

\[
\frac{n + \alpha \Delta_n}{4Ns} = \beta_n \left[ \beta_{n+1} + \beta_n + \beta_{n-1} + \frac{1-s}{2s} \right]. \tag{2.2}
\]

Remark 2.1. Multiplying Eq. (2.2) by \( s \), we see that \( \beta_n(0; \alpha, N) = \frac{n + \alpha \Delta_n}{2N} \).

Remark 2.2. The nonlinear discrete equation (2.1) can be found from [17, Eq. (23), P. 5], written by Freud, see also [11 §2]. Joshi and Lustri [25] studied the large \( n \) behavior of \( dP_1 \), see also [13].

Proposition 2.2. The recurrence coefficients \( \beta_n(s) \) satisfy the differential-difference equation,

\[
\beta_n' = \frac{\beta_n}{2s} \left[ N(s+1)(\beta_{n+1}(s) - \beta_{n-1}(s)) - 1 \right]. \tag{2.3}
\]

Proof. Differentiating \( h_n(s) \) with respect to \( s \),

\[
\frac{dh_n(s)}{ds} = \frac{d}{ds} \int_{-\infty}^{\infty} P_n^2(x; s)w(x; s)dx
\]

\[
= N \left[ \int_{-\infty}^{\infty} (x^2 - x^4) P_n^2(x; s)w(x; s)dx \right]
\]

\[
= Nh_n(s) \left[ \beta_{n+1} (1 - \beta_{n+1} - \beta_{n+1} - \beta_n) + \beta_n (1 - \beta_{n+1} - \beta_n - \beta_{n-1}) \right]
\]

\[
= Nh_n(s) \left[ \frac{1 + s}{2s} (\beta_{n+1} + \beta_n) - \frac{2n+1}{4Ns} - \frac{\alpha}{4Ns} \right]. \tag{2.4}
\]

Substituting this into

\[
\frac{d\beta_n(s)}{ds} = \beta_n(s) \left[ \frac{h_n'(s)}{h_n(s)} - \frac{h_n'(s)}{h_n(s)} \right],
\]

the result of (2.3) will be obtained.

\[ \square \]
Based on the formulas (2.2) and (2.3), with \( n \) and \( n - 1 \), eliminating the term \( \beta_{n+1} \) and \( \beta_{n-1} \), then we can get a second order ODE for the coefficients \( \beta_n(s) \).

**Lemma 2.1.** The recurrence coefficients \( \beta_n(s) \) satisfy the equation

\[
\beta''_n(s) = \frac{\beta''_n(s)}{2\beta_n(s)} - \frac{2 + s}{s(1 + s)} \beta'_n(s) + \frac{3N^2(1 + s)^2}{8s^2} \beta^3_n(s) + \frac{N^2(1 + s)^2(1 - s)}{4s^3} \beta'_n(s) + \frac{N^2(1 - s)^2(1 + s)^3 - 4s^2(3 - s)}{32s^4(1 + s)} + \frac{N(1 + s)^2}{16s^3} \beta_n(s) - \frac{(1 + s)^2q_n}{128s^4\beta_n(s)},
\]

(2.5)

where the parameters \( p_n \) and \( q_n \) are given by

\[
p_n = \begin{cases} 
n + 2\alpha & \text{even,} \\
n - \alpha & \text{odd,}
\end{cases} \quad \text{and} \quad q_n = \begin{cases} 
n^2 & \text{even,} \\
(n + \alpha)^2 & \text{odd.}
\end{cases}
\]

**Proof.** From Eq. (2.2), we have

\[
\beta_{n-1} = \frac{n + \alpha \Delta_n}{4Ns\beta_n} - \beta_{n+1} - \beta_n - \frac{1 - s}{2s}, 
\]

(2.6)

and

\[
\beta_{n-2} = \frac{n - 1 + \alpha \Delta_{n-1}}{4Ns\beta_{n-1}} - \beta_n - \beta_{n-1} - \frac{1 - s}{2s}. 
\]

(2.7)

From Eq. (2.3), we obtain

\[
\beta_{n+1} = \frac{2s}{N(1 + s)} \beta'_n + \frac{1}{N(1 + s)} + \beta_n. 
\]

(2.8)

Substituting Eq. (2.8) into Eq. (2.6), we get

\[
\beta_{n-1} = -\frac{s}{N(1 + s)} \beta'_n + \frac{n + \alpha \Delta_n}{8Ns\beta_n} - \frac{\beta_n}{2} - \frac{1}{2N(1 + s)} - \frac{1 - s}{4s}. 
\]

(2.9)

Based on the formulas (2.8) and (2.7), we find

\[
\beta_n = \frac{2s}{N(s + 1)} \beta'_{n-1} + \frac{1}{N(1 + s)} + \beta_{n-2}
\]

\[
= \frac{2s}{N(1 + s)} \beta'_{n-1} + \frac{1}{N(1 + s)} + \frac{n - 1 + \alpha_{n-1}}{4Ns\beta_{n-1}} - \beta_n - \beta_{n-1} - \frac{1 - s}{2s}. 
\]

(2.10)

Substituting the expression of \( \beta_{n-1} \), given by Eq. (2.9), with its derivative into Eq. (2.10), one will find the result (2.5) after some simplification.
3 Asymptotics for the recurrence coefficient $\beta_n(s)$ as $s \to 0$

**Theorem 3.1.** Assuming $s \to 0$, the recurrence coefficient $\beta_n(s; \alpha, N)$ satisfying the second order ODE (2.5), has the asymptotic expansion

$$
\beta_n = \frac{n}{2} \left\{ \frac{1}{N} + \frac{N - 3n - 2\alpha}{N^2} s + \frac{18n^2 - 9nN + N^2 + 22n\alpha + 6(-N\alpha + \alpha^2 + 1)}{N^3} s^2 \\
+ \frac{1}{N^4} \left[ N^3 + nN \left( 40n^2 + \frac{15}{2} n - 18N + \frac{85}{2} \right) + 30N - 135n^3 - 162n - \left( 12N^2 - 30nN + 236n^2 - 80n^2N + 100 \right) \alpha + \left( 30N - 126n \right) \alpha^2 - 20\alpha^3 \right] s^3 \\
+ \mathcal{O}(s^4) \right\}, \quad n \text{ even},
$$

whilst

$$
\beta_n = \frac{n}{2} \left\{ \frac{1}{N} + \frac{N - 3n - \alpha}{N^2} s + \frac{18n^2 - 9nN + N^2 + 14n\alpha - 3N\alpha + 2\alpha^2 + 6}{N^3} s^2 \\
+ \frac{1}{N^4} \left[ N^3 + nN \left( 40n^2 + \frac{15}{2} n - 18N + \frac{85}{2} \right) + 30N - 135n^3 - 162n \\
- \left( 6N^2 - 40n^2N + 15nN - \frac{85}{2} N + 169n^2 + 62 \right) \alpha + \left( \frac{15}{2} N - 40nN - 59n \right) \alpha^2 \\
- 5\alpha^3(1 + 8N) \right] s^3 + \mathcal{O}(s^4) \right\}, \quad n \text{ odd}.
$$

**Proof.** We rewrite the Eq. (2.5) as follows:

$$
0 = \frac{1}{s^4} \left[ \frac{N^2(1-s)^2}{32} \beta_n - \frac{(1+s)^2q_n}{128\beta_n} \right] + \frac{1}{s^3} \left[ \frac{N^2(1+s)^2}{4} \beta_n^2 + \frac{N(1+s)^2p_n}{16} \beta_n \right] \\
+ \frac{1}{s^2} \left[ 3N^2(1+s)^2 \beta_n^3 - \frac{3-s}{8(1+s)} \beta_n^4 \right] - \frac{1}{s} (2+s) \beta_n' + \frac{\beta_n^2}{2\beta_n} - \beta_n''.
$$

(3.1)

Considering the coefficient of the term $s^{-4}$ then one finds that

$$
\beta_n \approx \frac{\sqrt{q_n}}{2N(1-s)} = \frac{\sqrt{q_n}}{2N} \left( 1 + s + s^2 + s^3 + \cdots \right).
$$

Then we suppose $\beta_n$ has the following expansion

$$
\beta_n = \frac{\sqrt{q_n}}{2N} \left( \sum_{k=0}^{\infty} c_k s^k \right), \quad s \to 0.
$$
Substituting this into the formula (2.5) with directing calculating, we get the asymptotic expansion of $\beta_n$, reads

$$
\beta_n(s) = \frac{\sqrt{q_n}}{2N} \left[ 1 + \left( \frac{N - p_n - 2\sqrt{q_n}}{N} \right) s + \frac{1}{2N^2} \left( 2N^2 - 6Np_n + 3p_n^2 - 12N\sqrt{q_n} + 16p_n\sqrt{q_n} + 17q_n \right)s^2 + \frac{1}{2N^3} \left( 64N + 2N^3 - 64p_n - 12N^2p_n + 15Np_n^2 - 5p_n^3 - 128\sqrt{q_n} - 24N^2\sqrt{q_n} + 80Np_n\sqrt{q_n} - 48p_n^2\sqrt{q_n} + 85Nq_n - 125p_nq_n - 92q_n^2 \right)s^3 + O(s^4) \right], \ s \to 0.
$$

which completes the proof.

Remark 3.1. For $s = 0$, $\alpha = 0$, $N = 1$, which are the Hermite polynomials orthogonal with the weight function $e^{-x^2}$, one finds $\beta_n = \frac{n}{2}$, see [17, 43].

![Figure 1: Plots of the recurrence coefficients $\beta_n(s; \alpha, N)$, with $\alpha = 12$, $N = 6$, $s$ small.](image)

4 Asymptotics for the recurrence coefficient $\beta_n(s; \alpha, N)$ as $n \to \infty$ with $N$ fixed.

The asymptotic expansion of $\beta_n(s; \alpha, N)$, satisfied by Eq. (2.1), for the special case when $\alpha = 0$ and $s = N = 1$ was studied by Lew and Quarles [26], see also [35, 36]. To analyze the asymptotic properties of $\beta_n(s; \alpha, N)$ as $n \to \infty$. We first give a brief description of Coulomb fluid, see e.g. [7, 8, 10]. The energy of a system of $n$ logarithmically repelling particles on the line confined by an external potential $v(x)$ reads

$$
E(x_1, x_2, \ldots, x_n) = -2 \sum_{1 \leq j < k \leq n} \log |x_j - x_k| + \sum_{j=1}^{n} v(x_j),
$$

The collection particles, for large enough $n$, can be approximated as a continuous fluid with a certain density $\sigma(x)$ supported on a single interval $(a, b) \in \mathbb{R}$, see [16].
This density, \( \sigma(x) \), corresponding to the equilibrium density of the fluid, is obtained by the constrained minimization of the free-energy function, \( F[\sigma] \),

\[
F[\sigma] := \int_a^b \sigma(x)v(x)dx - \int_a^b \int_a^b \sigma(x) \log |x-y|\sigma(y)dxdy.
\]

subject to

\[
\int_a^b \sigma(x)dx = n, \quad \sigma(x) > 0.
\]

Upon minimization \([42]\), the equilibrium density \( \sigma(x) \) is found to satisfy the integral equation,

\[
L := v(x) - 2 \int_a^b \log |x-y|\sigma(y)dy, \quad x \in [a,b],
\]

where \( L \) is the Lagrange multiplier. The derivative of this equation over \( x \) gives rise to a singular integral equation

\[
v'(x) - 2 \text{p.v.} \int_a^b \frac{\sigma(y)}{x-y}dy = 0, \quad x \in [a,b].
\]

where p.v. denotes the Cauchy principal value. Based on the theory of singular integral equations \([33]\), we find

\[
\sigma(x) = \frac{1}{2\pi^2} \sqrt{\frac{b-x}{x-a}} \text{p.v.} \int_a^b \frac{v'(y)}{y-x} \sqrt{\frac{y-a}{b-y}}dy.
\]

Thus the normalization, \( \int_a^b \sigma(x)dx = n \), becomes

\[
\frac{1}{2\pi} \int_a^b \sqrt{\frac{y-a}{b-y}}v'(y)dy = n.
\]

with a supplementary condition, see \([7, 8]\),

\[
\int_a^b \frac{v'(x)}{\sqrt{(b-x)(x-a)}} = 0.
\]

Consequently, the normalization condition becomes

\[
\int_a^b \frac{xv'(x)}{\sqrt{(b-x)(x-a)}} = 2\pi n.
\]

(4.3)

For the weight at hand, the equilibrium density \( \sigma(x) \) supported on \((-b, b)\), and

\[
v(x) = - \log w_\alpha(x; s) = -\alpha \log |x| + Nsx^4 + N(1-s)x^2,
\]

we find,
Lemma 4.1. For sufficiently large \( n \), with the parameters \( N \) and \( \alpha \) finite, we have

\[
b^2 \simeq \frac{2\sqrt{3}}{3} N^{-\frac{1}{2}} s^{-\frac{1}{2}} n^\frac{1}{2}.
\] (4.5)

Proof. It follows from Eq. (4.4) that

\[
v'(x) = -\text{p.v.} \frac{\alpha}{x} + 4Ns x^3 + 2N(1-s)x.
\]

Putting \( a = -b \), we find from Eq. (4.3) that

\[
2\pi n = \lim_{\epsilon \to 0^+} \left( \int_{-\epsilon}^{-b} -\alpha + 4Ns x^4 + 2N(1-s)x^2 \frac{dx}{\sqrt{b^2 - x^2}} \right)
\]

\[
= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{b} \frac{-\alpha + 4Ns y^2 + 2N(1-s)y}{\sqrt{(b^2 - y)y}} dy
\]

\[
= -\alpha \pi + \frac{3\pi}{2} Ns b^4 + \pi N(1-s)b^2.
\] (4.6)

Since \( b^2 > 0 \), we have

\[
b^2 = \frac{s - 1 + \sqrt{(1-s)^2 + 6s \left(\frac{2n+\alpha}{N}\right)}}{3s}
\]

\[
\simeq \frac{2\sqrt{3}}{3} N^{-\frac{1}{2}} s^{-\frac{1}{2}} n^\frac{1}{2}, \quad n \to \infty.
\] (4.7)

Lemma 4.2. A upper bound for the recurrence coefficient \( \beta_n(s; \alpha, N) \) with respect to the weight \( w_\alpha(x; s, N) \) is given by

\[
\beta_n < \frac{n^\frac{1}{2}}{2\sqrt{Ns}} + s - \frac{1}{4s} + \frac{\alpha \Delta_n + \frac{N(1-s)^2}{4s}}{4\sqrt{Ns}} n^\frac{1}{2} + \sum_{j=2}^{\infty} \frac{b_j}{\sqrt{Ns}} \left[ \frac{\alpha \Delta_n + \frac{N(1-s)^2}{4s}}{n^j} \right]^j,
\] (4.8)

where

\[
b_j = \frac{(-1)^{j-1}(2j-3)!}{2^{j-1}j!(j-2)!}.
\]

Proof. From Eq. (2.1), we obtain

\[
\beta_n^2 + \frac{1-s}{2s} \beta_n + \beta_n (\beta_{n+1} + \beta_{n-1}) = \frac{n + \alpha \Delta_n}{4Ns},
\]

it follows that

\[
\beta_n^2 + \frac{1-s}{2s} \beta_n < \frac{n + \alpha \Delta_n}{4Ns},
\]

due to \( \beta_n = \frac{h_n}{h_{n-1}} > 0 \), where

\[
h_n(s; \alpha, N) = \int_{-\infty}^{\infty} P_n^2(x) w_\alpha(x; s, N) dx.
\]
So

\[
0 < \beta_n < -\frac{1-s}{4s} + \sqrt{\left(\frac{1-s}{4s}\right)^2 + \frac{n + \alpha \Delta_n}{4N_s}}
\]

\[
= \frac{s - 1}{4s} + \frac{n^{\frac{1}{2}}}{2\sqrt{N}s} \left[1 + \frac{\alpha \Delta_n + \frac{N(1-s)^2}{4s}}{n}\right]^{\frac{1}{2}}
\]

\[
= \frac{n^{\frac{1}{2}}}{2\sqrt{N}s} + \frac{s - 1}{4s} + \frac{\alpha \Delta_n + \frac{N(1-s)^2}{4s}}{4\sqrt{N}s} n^{-\frac{1}{2}} + \sum_{j=2}^{\infty} \frac{b_j}{\sqrt{N}s} \left[\alpha \Delta_n + \frac{N(1-s)^2}{4s}\right]^j n^{-\frac{j}{2}},
\]

where

\[
b_j = \frac{(-1)^{j-1}(2j-3)!!}{2^{j+1}j!} = \frac{(-1)^{j-1}(2j-3)!}{2^{2j-1}j!(j-2)!},
\]

which completes the proof.

In the following, we provide the asymptotic expansion of \(\beta_n(s; \alpha, N)\) in Eq. \((1.3)\) as \(n \to \infty\) with \(N\) fixed, for \(s \in [0, 1], \alpha > -1\).

**Theorem 4.1.** Let \(n \to \infty\) with \(N\) fixed, the recurrence coefficient \(\beta_n(s; \alpha, N)\) associated with monic deformed Freud polynomials satisfying the nonlinear discrete equation \((2.7)\), i.e.

\[
\beta_n \left(\beta_{n+1} + \beta_n + \beta_{n-1} + \frac{1-s}{2s}\right) = \frac{n + \alpha \Delta_n}{4N_s}, \quad \Delta_n = \frac{1 - (-1)^n}{2}, \quad (4.9)
\]

has the asymptotic expansion

\[
\beta_n = \frac{1}{2\sqrt{3}s} \sqrt{\frac{n}{N}} - \frac{1-s}{12s} + \frac{(1-s)^2}{48\sqrt{3}s^2} \sqrt{\frac{N}{n}} + \frac{1}{48\sqrt{3Ns}} - \frac{N^2(1-s)^4}{2304\sqrt{3}s^2} n^{-\frac{3}{2}} - 1 - \frac{s}{288s} n^{-2} + \frac{N^2(1-s)^6 - 144\sqrt{N}s^2(1-s)^2}{55296\sqrt{3}s^3} n^{-\frac{5}{2}} + \frac{N(1-s)^3}{1728s^2} n^{-3} (4.10)
\]

whilst

\[
\beta_n = \frac{1}{2\sqrt{3}s} \sqrt{\frac{n}{N}} - \frac{1-s}{12s} + \left[\sqrt{\frac{N(1-s)^2}{48\sqrt{3}s^2}} + \frac{\alpha}{4\sqrt{3Ns}}\right] \sqrt{\frac{1}{n}} + \left\{-\frac{[N(1-s)^2 + 12s\alpha]^2}{2304\sqrt{3Ns}^2} + \frac{1}{48\sqrt{3Ns}} \right\} n^{-\frac{1}{2}} - \frac{1-s}{288s} n^{-2} + \Psi(s)n^{-\frac{5}{2}} + \frac{N(1-s)^3 + 12s(1-s)\alpha}{1728s^2} n^{-3} + O\left(n^{-\frac{7}{2}}\right), \quad n \text{ even},
\]

\[
(4.11)
\]

with

\[
\Psi(s) = \frac{N^2(1-s)^6 + 36N^2(1-s)^4s\alpha + 144\sqrt{N}(1-s)^2s^2(3\alpha^2 - 1) + 1728s^3\alpha(\alpha^2 - 1)}{55296\sqrt{3}s^3}.
\]
Proof. In fact, Lemma 4.1 gives the first term in the asymptotic expansion of 
$\beta_n(s; \alpha, N)$, since

$$\beta_n \simeq \left( \frac{b - a}{4} \right)^2 \simeq \frac{1}{2\sqrt{3Ns}} n^{\frac{1}{2}}, \ a = -b.$$ 

Hence it seems reasonable to assume $\beta_n(s; \alpha, N)$ has the expansion of the form

$$\beta_n = \frac{n^{\frac{1}{2}}}{2\sqrt{3Ns}} \left( 1 + \sum_{k=1}^{\infty} d_k n^{-\frac{k}{2}} \right), \ n \to \infty. \quad (4.12)$$

Replacing $n$ by $n \pm 1$ in Eq. (4.12), we obtain

$$\beta_{n \pm 1} = \frac{(n \pm 1)^{\frac{1}{2}}}{2\sqrt{3Ns}} \left[ 1 + \sum_{k=1}^{\infty} d_k (n \pm 1)^{-\frac{k}{2}} \right]$$

$$= \frac{n^{\frac{1}{2}} (1 \pm \frac{1}{n})^{\frac{1}{2}}}{2\sqrt{3Ns}} \left[ 1 + \sum_{k=1}^{\infty} d_k n^{-\frac{k}{2}} \left( 1 \pm \frac{1}{n} \right)^{-\frac{k}{2}} \right]$$

$$= \frac{n^{\frac{1}{2}}}{2\sqrt{3Ns}} \left[ 1 + \frac{d_1}{n^2} + \frac{2d_2 \pm 1}{2n} + \frac{d_3}{n^2} + \frac{8d_4 \mp 4d_2 - 1}{8n^2} + \frac{d_5 \mp d_3}{n^2} + \frac{16d_6 \mp 24d_4 + 6d_2 \pm 1}{16n^2} + \frac{d_7 \mp 2d_5 + d_3}{n^2} + O\left(n^{-4}\right) \right], \quad (4.13)$$

by doing an asymptotic expansion in the above equation. Substituting Eqs. (4.12) and (4.13) into Eq. (4.9), followed by comparing the corresponding coefficients on both sides, we find

$$d_1 = -\frac{\sqrt{N}(1 - s)}{2\sqrt{3}s}, \ d_2 = \frac{N(1 - s)^2}{24s} + \frac{\alpha \Delta_N}{2}, \ d_3 = 0,$$

$$d_4 = \frac{1}{24} - \frac{[N(1-s)^2 + 12s\alpha \Delta_n]^2}{1152s^2}, \ d_5 = -\frac{\sqrt{N}(1 - s)}{48\sqrt{3}s},$$

$$d_6 = \frac{[N(1-s)^2 + 12s\alpha \Delta_n][N(1-s)^2 - 12s + 12s\alpha \Delta_n][N(1-s)^2 + 12s + 12s\alpha \Delta_n]}{27648s^3},$$

$$d_7 = \frac{\sqrt{N}(1 - s)[N(1-s)^2 + 12s\alpha \Delta_n]}{288\sqrt{3}s^2}. \quad (4.14)$$

Hence, we obtain the asymptotic expansions (4.10) and (4.11), followed by some computation.

**Corollary 4.1.** Assume that $\beta_n(s; \alpha, N)$ satisfying Eq. (4.9). Then for $s \in [0, 1]$, $\alpha > -1$, $N > 0$:

(i) the sequence $\left\{ \frac{\beta_n(s; \alpha, N)}{n^{\frac{1}{2}}} \right\}_{n=1}^{\infty}$ is bounded;

(ii) $\lim_{n \to \infty} \frac{\beta_n(s; \alpha, N)}{\sqrt{n}} = \frac{1}{2\sqrt{3Ns}}$. 

$\square$
Remark 4.1. Putting $s = 1$, $\alpha = 0$, $N = 1$, the classical result, obtained by Lew and Quarles [26] for the Freud weight $e^{-x^4}$, is recovered, i.e.

\[
\lim_{n \to \infty} \frac{\beta_n(1; 0, 1)}{\sqrt{n}} = \frac{1}{2\sqrt{3}}.
\]

Corollary 4.2. For $s \in [0, 1]$, $\alpha > -1$, $N > 0$, the recurrence coefficients $\beta_n(s; \alpha, N)$ in Eq. (4.9) satisfy

\[
\frac{\beta_{n+1}(s; \alpha, N)}{\beta_n(s; \alpha, N)} = 1 + O(n^{-1}), \quad n \to \infty,
\]

\[
\frac{\beta_n(s; \alpha, N)}{a^n} = \frac{1}{4} + O(n^{-1}), \quad n \to \infty.
\]

where $a_n$ is sometimes called the Mhaskar-Rakhmanov-Saff number [33, 40], the unique positive root of the equation

\[
\mu = 2\pi \int_0^1 \frac{a_n y Q'(a_n y)}{\sqrt{1 - y^2}} dy,
\]

with $Q(x) = Nsx^4 + N(1 - s)x^2$.

Proof. See [14, Thm. 2.1]. \(\Box\)

5 Asymptotics for the recurrence coefficient $\beta_n(s; \alpha, N)$ as $n \to \infty$, $N \to \infty$ with $\frac{n}{N}$ fixed.

In Section 2 we have derived that the recurrence coefficient $\beta_n(s; \alpha, N)$ satisfies the second order ODE (2.5). Let $0 < r_0 \leq r := \frac{n}{N} \leq 1$, i.e. the quantity $\frac{n}{N}$ is bounded away from 0 and close to 1. Substituting $n = Nr$ into Eq. (2.5) gives,

\[
\left[\frac{3(1 + s)^2}{8s^2} \beta_n^3(s) + \frac{(1 + s)^2(1 - s)}{4s^3} \beta_n^2(s) + \frac{(1 + s)^2(1 - 2s + 2rs + s^2)}{32s^4} \beta_n(s)\right]
- \frac{(1 + s)^2 r^2}{128s^4 \beta_n(s)} N^2 + \left[\frac{\alpha(1 + s)^2}{8s^3} \beta_n(s)\right] N - \beta_n''(s) + \frac{\beta_n'(s)}{2\beta_n(s)} - \frac{2 + s}{s(1 + s)} \beta_n'(s)
- \frac{3 - s}{8s^2(1 + s)} \beta_n(s) = 0, \quad n \text{ is even},
\]

and

\[
\left[\frac{3(1 + s)^2}{8s^2} \beta_n^3(s) + \frac{(1 + s)^2(1 - s)}{4s^3} \beta_n^2(s) + \frac{(1 + s)^2(1 - 2s + 2rs + s^2)}{32s^4} \beta_n(s)\right]
- \frac{(1 + s)^2 r^2}{128s^4 \beta_n(s)} N^2 + \left[\frac{\alpha(1 + s)^2}{16s^3} \beta_n(s) + \frac{r\alpha(1 + s)^2}{64s^4 \beta_n(s)}\right] N - \beta_n''(s) + \frac{\beta_n'(s)}{2\beta_n(s)}
- \frac{2 + s}{s(1 + s)} \beta_n'(s) - \frac{3 - s}{8s^2(1 + s)} \beta_n(s) - \frac{\alpha^2(1 + s)^2}{128s^4 \beta_n(s)} = 0, \quad n \text{ is odd},
\]

13
which imply that
\[
\frac{3(1 + s)^2}{8s^2} \beta_n^2(s) + \frac{(1 + s)^2(1 - s)}{4s^3} \beta_n^2(s) + \frac{(1 + s)^2(1 - 2s + 2rs + s^2)}{32s^4} \beta_n(s) \\
- \frac{(1 + s)^2r^2}{128s^4 \beta_n(s)} = 0, \quad N \to \infty.
\] (5.3)

Solving this equation, we get the solution
\[
\beta_n(s) = \frac{s - 1 + \sqrt{1 - 2s + 12rs + s^2}}{12s},
\] (5.4)

which is consistent with the result (4.7) since
\[
\beta_n(s) \simeq \frac{b^2}{4}, \quad N \to \infty.
\] (5.5)

Therefore, we assume that \(\beta_n(s)\) has the asymptotic expansion
\[
\beta_n(s) = a_0(s) + \sum_{k=1}^{\infty} \frac{a_k(s)}{N^k}, \quad N \to \infty.
\] (5.6)

where
\[
a_0(s) := \frac{s - 1 + \sqrt{1 - 2s + 12rs + s^2}}{12s}.
\]

Let \(g(s) := \sqrt{1 - 2s + 12rs + s^2}\) and \(f(s) := 1 - 2s - 4rs + s^2\). Then substituting the asymptotic expansion Eq. (5.6) into Eqs. (5.1) and (5.2), respectively, we obtain
\[
a_1(s) = \begin{cases} 
\frac{|s-1+g(s)|}{4(s-1)g(s)-2g^2(s)}, & n \text{ even}, \\
\frac{a_{1-s}}{f(s)g(s)} - \frac{4asr}{f(s)g(s)}, & n \text{ odd},
\end{cases}
\]

and, for \(n\) is even,
\[
a_2(s) = \frac{s(s-1)}{2g^2(s)} + \frac{s(4-3\alpha^2)}{8g^4(s)} + \frac{3\alpha^2 s(s-1)}{2f^2(s)} + \frac{15\alpha^2 s + 3\alpha^2 s^2(12r + 5s - 10)}{8f(s)g(s)}
\]
else, if \(n\) is odd,
\[
a_2(s) = \frac{s(s-1)}{2g^4(s)} + \frac{s(4-3\alpha^2)}{8g^3(s)} + \frac{\alpha^2 s(1-s)}{2f^2(s)} - \frac{3\alpha^2 s}{8f(s)g(s)}.
\]

**Theorem 5.1.** As \(n, N \to \infty\), with the quantity \(\frac{s}{N}\) is bounded away from 0 and close to 1, the recurrence coefficients have the asymptotic expansion
\[
\beta_n(s) = \frac{s - 1 + g(s)}{12s} + \frac{|s-1+g(s)|}{4(s-1)g(s)-2g^2(s)} \frac{1}{N} + \frac{s(s-1)}{2g^4(s)} + \frac{s(4-3\alpha^2)}{8g^3(s)} \\
+ \frac{3\alpha^2 s(s-1)}{2f^2(s)} + \frac{15\alpha^2 s + 3\alpha^2 s^2(12r + 5s - 10)}{8f(s)g(s)} \frac{1}{N^2} + O(N^{-3}), \quad n \text{ even},
\]
whilst
\[
\beta_n(s) = \frac{s - 1 + g(s)}{12s} + \left[ \frac{\alpha(1 - s)}{2f(s)} - \frac{4\alpha sr}{f(s)g(s)} \right] \frac{1}{N} + \left[ \frac{s(s - 1)}{2g^4(s)} + \frac{s(4 - 3\alpha^2)}{8g^3(s)} \right] \frac{1}{N^2} + O(N^{-3}) ,
\]
with
\[
g(s) = \sqrt{1 - 2s + 12rs + s^2} \quad \text{and} \quad f(s) = 1 - 2s - 4rs + s^2.
\]

Remark 5.1. Higher order corrections can be calculated systematically in Mathematica, but will be quite cumbersome. Hence we only give the expansion terms up to order $N^{-2}$.

6 The deformed Freud polynomials

Based on an extension of the ladder operators technique developed in [6], Chen and Feigin [5] studied the weight $\tilde{w}(x)|x - t|^K$, $x, t, K \in \mathbb{R}$, for any smooth reference weight $\tilde{w}(x)$. They showed that when $\tilde{w}(x)$ is the Gaussian (Hermite) weight $(e^{-x^2}, x \in \mathbb{R})$, the recurrence coefficients satisfy a particular two-parameter Painlevé IV equation. Filipuk et al [19] found that the recurrence coefficients for the Freud weight $|x|^{2\alpha + 1} e^{-x^4 + tx^2}$, $x, t \in \mathbb{R}, \alpha > -1$ are related to the solutions of the Painlevé IV and the first discrete Painlevé equation. In [12] [13], Clarkson et al gave a systematic study on Freud weight $|x|^{2\alpha + 1} e^{-x^4 + tx^2}$, and some generalized work for [5]. In the following theorem, we give the differential-difference equation (i.e., lowering operator) satisfied by the deformed Freud orthogonal polynomials associated with the weight $w_n(x; s, N)$ given in Eq. (1.1).

Lemma 6.1. The monic orthogonal polynomials $P_n(z; s, \alpha, N)$ with respect to the deformed Freud weight $w_n(x; s, N)$ on $\mathbb{R}$ satisfy the differential-difference recurrence relation
\[
P_n'(z) = \beta_n(s)A_n(z)P_{n-1}(z) - B_n(z)P_n(z), \quad (6.1)
\]
where
\[
A_n(z) := \frac{1}{h_n} \int_{-\infty}^{\infty} \frac{v_0'(y) - v_0'(y)}{z - y} P_n^2(y)w(y) dy,
\]
\[
B_n(z) := \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} \frac{v_0'(y) - v_0'(y)}{z - y} P_n(y)P_{n-1}(y)w(y) dy + \alpha \frac{(1 - (-1)^n)}{2z},
\]
with
\[
v_0(x) = Nsx^4 + N(1 - s)x^2, \quad x \in \mathbb{R}, \ s \in [0, 1], \ N > 0. \quad (6.2)
\]

Proof. Since the derivative of $P_n(z)$ is a polynomial of degree $n - 1$ in $z$, hence $P_n'(z)$ can be given by
\[
P_n'(z) = \sum_{j=0}^{n-1} c_{n,j} P_j(z). \quad (6.3)
\]
Using the orthogonality relations, and integrating by parts, we have

\[ c_{n,j} = \frac{1}{h_j} \int_{-\infty}^{\infty} P'_n(y)P_j(y)w(y)dy = \frac{1}{h_j} \int_{-\infty}^{\infty} P_n(y)P_j(y) \left[ v'_0(y) - \frac{\alpha}{y} \right] w(y)dy, \quad (6.4) \]

where

\[ v_0(y) := Nsy^4 + N(1-s)y^2. \quad (6.5) \]

Substituting Eq. (6.4) into Eq. (6.3) and summing over \( j \) applying the Christoffel-Darboux formula [23, Theorem 2.2.2],

\[ \sum_{j=0}^{n-1} \frac{P_j(z)P_j(y)}{h_j} = \frac{P_n(z)P_{n-1}(y) - P_n(y)P_{n-1}(z)}{(z-y)h_{n-1}}, \]

we have

\[ P'_n(z) = \int_{-\infty}^{\infty} P_n(y) \sum_{j=0}^{n-1} \frac{P_j(z)P_j(y)}{h_j} \left[ v'_0(y) - \frac{\alpha}{y} \right] w(y)dy \]

\[ = \int_{-\infty}^{\infty} P_n(y) \sum_{j=0}^{n-1} \frac{P_j(z)P_j(y)}{h_j} [v'_0(y) - v'_0(z)] w(y)dy \]

\[ - \alpha \int_{-\infty}^{\infty} \frac{P_n(y)}{y} \sum_{j=0}^{n-1} \frac{P_j(z)P_j(y)}{h_j} w(y)dy \]

\[ = - \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} \left[ P_n(z)P_n(y)P_{n-1}(y) - P^2_n(y)P_{n-1}(z) \right] \frac{v'_0(z) - v'_0(y)}{z-y} w(y)dy \]

\[ - \frac{\alpha}{z} \int_{-\infty}^{\infty} \left[ ((z-y) + y)P_n(y) \left[ \sum_{j=0}^{n-1} \frac{P_j(z)P_j(y)}{h_j} \right] \right] w(y)dy \]

\[ = \frac{P_{n-1}(z)}{h_{n-1}} \int_{-\infty}^{\infty} P^2_n(y) \frac{v'_0(z) - v'_0(y)}{z-y} w(y)dy - \frac{\alpha P_n(z)}{z} \int_{-\infty}^{\infty} \frac{P_n(y)P_{n-1}(y)}{y} w(y)dy \]

\[ - \frac{P_n(z)}{h_{n-1}} \int_{-\infty}^{\infty} P_n(y)P_{n-1}(y) \frac{v'_0(z) - v'_0(y)}{z-y} w(y)dy. \quad (6.6) \]

Furthermore, by an inductive argument based on the three term recurrence relation Eq. (1.3) with the initial conditions

\[ P_0(z) := 1, \quad \text{and} \quad \beta_0P_{-1}(z) = 0, \quad (6.7) \]
we obtain
\[
\frac{1}{h_{n-1}} \int_{-\infty}^{\infty} \frac{P_n(y)P_{n-1}(y)}{y} w(y)dy = \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} \frac{[yP_{n-1}(y) - \beta_{n-1}P_{n-2}(y)] P_{n-1}(y)}{y} w(y)dy
\]
\[
= \begin{cases} 1, & n = 1, \\ 1 - \frac{1}{h_{n-2}} \int_{-\infty}^{\infty} \frac{P_{n-1}(y)P_{n-2}(y)}{y} w(y)dy, & n \geq 2, \\ 0, & n = 2, \\ \frac{1}{h_{n-3}} \int_{-\infty}^{\infty} \frac{P_{n-2}(y)P_{n-3}(y)}{y} w(y)dy, & n \geq 3, \\ \vdots \\ 0, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}
\]
(6.8)

Therefore, (bearing in mind \(\beta_n = h_n/h_{n-1}\)) the result is derived by Eqs. (6.6) and (6.8) immediately.

**Lemma 6.2.** \(A_n(z)\) and \(B_n(z)\) defined by Lemma 6.1 satisfy the relation:
\[
A_n(z) = \frac{v_0'(z)}{z} + B_n(z) + B_{n+1}(z) - \frac{\alpha}{z^2}. \tag{6.9}
\]

**Proof.** Be the definition of \(A_n(z)\), we rewrite it as
\[
A_n(z) = \frac{1}{zh_n} \left\{ \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z - y} yP_n^2(y)w(y)dy + \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z - y} P_n^2(y)w(y)dy \right\}
\]
\[
= \frac{1}{zh_n} \left\{ \int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(y)}{z - y} [P_{n+1}(y) + \beta_n P_{n-1}(y)] P_n(y)w(y)dy + v_0(z)h_n \right. \\
- \int_{-\infty}^{\infty} P_n^2(y) \left[ \frac{\alpha}{y} w(y) - w'(y) \right] dy \\
\left. = \frac{v_0'(z)}{z} + \frac{1}{z} \left\{ B_{n+1}(z) - \frac{\alpha}{2z} \left[ 1 - (-1)^{n+1} \right] + B_n(z) - \frac{\alpha}{2z} \left[ 1 - (-1)^n \right] \right\} \right. \\
= \frac{v_0'(z)}{z} + \frac{B_n(z)}{z} + \frac{B_{n+1}(z)}{z} - \frac{\alpha}{z^2},
\]
which completes the proof. \(\square\)

**Theorem 6.1.** The monic orthogonal polynomials \(P_n(z; s, \alpha, N)\) with respect to deformed Freud weight \(1.1\) satisfy the linear second order ODE:
\[
P_n''(z) + R_n(z)P_n'(z) + Q_n(z)P_n(z) = 0, \tag{6.10}
\]
where
\[
R_n(z) = -4Ns z^3 - 2N(1-s)z + \frac{\alpha}{z} - \frac{2z}{z^2 + \frac{1}{2s} + \beta_n + \beta_{n+1}}, \tag{6.11}
\]
\[
Q_n(z) = 4Ns \beta_n \left[ 1 + \alpha(-1)^n \right] + 16N^2 s^2 \beta_n \left[ \frac{1-s}{2s} + \beta_n + \beta_{n-1} \right] \left[ \frac{1-s}{2s} + \beta_n + \beta_{n+1} \right] \\
- \alpha \left[ 1 - (-1)^n \right] \left[ N(1-s) + \frac{1}{2z^2} \right] - \frac{8Ns \beta_n z^2 + \alpha \left[ 1 - (-1)^n \right]}{z^2 + \frac{1}{2s} + \beta_n + \beta_{n+1}} + 4nNs z^2. \tag{6.12}
\]
Proof. From the differential-difference equation (6.1), we have the following type of raising operator,

\[ P'_{n-1}(z) = -B_{n-1}(z)P_{n-1}(z) + A_{n-1}(z)\beta_{n-1}(s)P_{n-2}(z) \]

\[ = -B_{n-1}(z)P_{n-1}(z) + zA_{n-1}(z)P_{n-1}(z) - A_{n-1}(z)P_n(z) \]

Using the relation between \( A_n(z) \) and \( B_n(z) \), see Lemma 6.2 we get

\[ P'_{n-1}(z) = -B_{n-1}(z)P_{n-1}(z) + z \left[ \frac{v_0'(z)}{z} + \frac{B_{n-1}(z) + B_n(z)}{z} - \frac{\alpha}{z^2} \right] P_{n-1}(z) - A_{n-1}(z)P_n(z) \]

\[ = \left[ v_0'(z) + B_n(z) - \frac{\alpha}{z} \right] P_{n-1}(z) - A_{n-1}(z)P_n(z). \] (6.13)

Differentiating both sides of Eq. (6.1) with respect to \( z \) we obtain

\[ P''(z) = -B'_n(z)P_n(z) - B_n(z)P'_n(z) + \beta_n(s)A'_n(z)P_{n-1}(z) + \beta_n(s)A_n(z)P'_{n-1}(z). \]

Substituting Eq. (6.13) into above, we obtain

\[ P''(z) = -B'_n(z)P_n(z) - B_n(z)P'_n(z) - \beta_n(s)A_n(z)A_{n-1}(z)P_n(z) \]

\[ + \left\{ \beta_n(s)A'_n(z) + \beta_n(s)A_n(z) \left[ v_0'(z) + B_n(z) - \frac{\alpha}{z} \right] \right\} P_{n-1}(z). \] (6.14)

Eliminating \( P_{n-1}(z) \) from Eqs. (6.1) and (6.14), we find

\[ P''_n(z) + R_n(z)P'_n(z) + Q_n(z)P_n(z) = 0, \] (6.15)

where

\[ R_n(z) := -\frac{A'_n(z)}{A_n(z)} - v_0'(z) + \frac{\alpha}{z}, \] (6.16)

\[ Q_n(z) := B'_n(z) - \frac{A'_n(z)B_n(z)}{A_n(z)} - \left[ v_0'(z) + B_n(z) - \frac{\alpha}{z} \right] B_n(z) + \beta_nA_n(z)A_{n-1}(z). \] (6.17)

Furthermore,

\[ A_n(z) = \frac{4Ns}{h_n} \int_{\infty}^{\infty} (z^2 + zy + y^2) P_n^2(y)w(y)dy + \frac{2N(1-s)}{h_n} \int_{-\infty}^{\infty} P_n^2(y)w(y)dy \]

\[ = 4NsN + \frac{4Ns}{h_n} \int_{\infty}^{\infty} [P_{n+1}(y) + \beta_nP_{n-1}(y)] [P_{n+1}(y) + \beta_nP_{n-1}(y)] w(y)dy \]

\[ + \frac{4Ns}{h_n} \int_{-\infty}^{\infty} [P_{n+1}(y) + \beta_nP_{n-1}(y)] P_n(y)w(y)dy + 2N(1-s) \]

\[ = 4NsN^2 + 4Ns(\beta_{n+1} + \beta_n) + 2N(1-s), \] (6.18)
since
\[
\frac{v_0'(z) - v_0(y)}{z - y} = 4Ns \left( z^2 + zy + y^2 \right) + 2N(1 - s).
\]

Similarly, we get
\[
B_n(z) = 4Ns\beta_n z + \frac{\alpha}{2z} \left[ 1 - (-1)^n \right].
\]

Substituting Eqs. (6.5) and (6.18) into Eq. (6.16), we obtain
\[
R_n(z) = -4Ns z^3 - 2N(1 - s)z + \frac{\alpha}{z} - \frac{2z}{z^2 + 1 - s} + \beta_n + \beta_{n+1}.
\]

Substituting Eqs. (6.5), (6.18) and (6.19) into Eq. (6.17), we get
\[
Q_n(z) = 4Ns\beta_n \left[ 1 + \alpha \left( -1 \right)^n \right] + 16N^2s^2\beta_n \left[ \frac{1 - s}{2s} + \beta_n + \beta_{n-1} \right] \left[ \frac{1 - s}{2s} + \beta_n + \beta_{n+1} \right]
\]
\[
- \alpha \left[ 1 - (-1)^n \right] N(1 - s) + \frac{1}{2z^2} - \frac{8Ns\beta_n z^2 + \alpha \left[ 1 - (-1)^n \right]}{z^2 + 1 - s} + \beta_n + \beta_{n+1}
\]
\[
+ Ns^2 \left\{ 16Ns\beta_n \left( \beta_{n+1} + \beta_n + \beta_{n-1} \right) - 2\alpha \left[ 1 - (-1)^n \right] + 8N(1 - s)\beta_n \right\}.
\]

The proof will be completed by using Eq. (4.9) to instead the last term of the above formula.

Since from Theorem 4.1 we have
\[
\beta_n(s) = \frac{1}{2\sqrt{3Ns}} n^{\frac{3}{2}} + O(1), \quad n \to \infty,
\]

it follows from Eq. (6.11) and Eq. (6.12), as \( n \to \infty \), that
\[
R_n(z) = -4Ns z^3 - 2N(1 - s)z + \frac{\alpha}{z} + O \left( n^{-\frac{1}{2}} \right),
\]
\[
Q_n(z) = \left( \frac{4Ns^2n^\frac{3}{2}}{3} \right)^{\frac{3}{2}} + O(n).
\]

Now we reconsider the Eq. (6.10), with denoting \( P_n(z) \) by \( \tilde{P}_n(z) \) as \( n \to \infty \),
\[
\tilde{P}_n''(z) - \left[ 4Ns z^3 + 2N(1 - s)z - \frac{\alpha}{z} \right] \tilde{P}_n'(z) + \left[ \frac{4(Ns)^\frac{3}{2}n}{3} \right]^{\frac{3}{2}} \tilde{P}_n(z) = 0. \tag{6.20}
\]

Let
\[
\kappa := (Ns)^\frac{3}{2}n, \tag{6.21}
\]
where \( Ns \to 0^+, \quad n \to \infty \), such that \( \kappa \) fixed. We find the Eq. (6.20) is equivalent to the biconfluent Heun equation (BHE) cf. [88 §31.12, 31.12.3]
\[
u''(x) + \left( \frac{\gamma}{x} + \delta + \epsilon \right) u'(x) + \left( \eta - \frac{\rho}{x} \right) u(x) = 0 \tag{6.22}
\]
through the transformation

\[ \tilde{P}_n(z; s, \alpha) = u(x; \gamma, \delta, \eta, \rho), \quad x = \sqrt{2}z^2, \]

with parameters

\[ \gamma = -\frac{1}{2} - \frac{\alpha}{2}, \quad \delta = \frac{\sqrt{2}N(1-s)}{2}, \quad \eta = 0, \quad \text{and} \quad \rho = -\frac{\sqrt{6}}{9}\kappa^{\frac{3}{2}}. \tag{6.23} \]

**Remark 6.1.** The BHE is widely encountered in contemporary mathematics and physics research. For example, many Schrödinger equations can be solved applying the BHE, and in atomic and nuclear physics, the BHE frequently appears in studying the motion of quantum particles in 1-, 2- or 3-dimensional confinement potentials, see [39]. The solutions of the BHE attract many authors, see e.g. [4, 18, 24]. Based on [24], we show that Eq. (6.22), with parameters given in Eq. (6.23), has the solutions in terms of the Hermite functions, reads \((i = \sqrt{-1})\)

\[ u(x) = k_1 \sum_{j=0}^{\infty} c_j H_{2j-\alpha-1} \left( i \left( \frac{x}{\sqrt{2}} + \frac{N(1-s)}{2} \right) \right) + k_2 \sum_{j=0}^{\infty} c_j H_{2j-\alpha-1} \left( -i \left( \frac{x}{\sqrt{2}} + \frac{N(1-s)}{2} \right) \right) \tag{6.24} \]

where \(k_1\) and \(k_2\) are constants, whilst the coefficients \(c_n\) satisfied a three term recurrence relation:

\[ L_j c_j + M_{j-1} c_{j-1} + T_{j-2} c_{j-2} = 0, \]

with initial conditions \(c_{-2} = c_{-1} = 0\), and here

\[ L_j = -ij(2j - \alpha - 1), \]
\[ M_j = \mp \left[ \frac{9N(1-s)(2j - \alpha - 1) - 4\sqrt{3}\kappa^{\frac{3}{2}}}{18\sqrt{2}} \right], \]
\[ T_j = -\frac{i2j - \alpha - 1}{2\sqrt{2}}, \]

where the signs \(\mp\) in \(M_j\) corresponding to \(\pm i\) in Eq. (6.24). See details in [24].

**Remark 6.2.** Note that if we make the transformation \(\tilde{P}_n(z) = \hat{P}_n(y)\) in Eq. (6.20), with \(y = \left(\frac{4\kappa}{N}\right)^{\frac{3}{2}} z\), then in the limit as \(n \to \infty\) \((Ns \to 0, \text{ with } \kappa \text{ fixed})\), we obtain

\[ \hat{P}_n''(y) + \frac{\alpha}{y} \hat{P}_n'(y) + \hat{P}_n(y) = 0, \]

which has the solution

\[ \hat{P}_n(y) = y^{1/2} \left[ c_1 J_{1-\alpha}(y) + c_2 Y_{1-\alpha}(y) \right], \]

where \(J_n(x), Y_n(x)\) are corresponding with the Bessel function of the first and second kind, respectively. Hence the polynomials \(P_n(z)\) might have Mehler-Heine type asymptotic formulae.
7 Existence uniqueness of positive solutions

In the paper \[1\], Alsulami et al. discussed the existence and uniqueness of a positive solution for the nonhomogeneous nonlinear second order difference equations of the form,

\[
\ell_n = x_n (\sigma_{n,1} x_{n+1} + \sigma_{n,0} x_n + \sigma_{n,-1} \beta_{n-1}) + \kappa_n x_n
\] (7.1)

with the initial conditions \(x_0 \in \mathbb{R}, x_1 \geq 0\), whilst \(\kappa_n \in \mathbb{R}, \sigma_{n,j} > 0, j \in \{0, \pm 1\}\), or \(\{\sigma_{n,0} > 0, \sigma_{n,-1} \geq 0, \sigma_{n,1} \geq 0\}\). Following the results presented in \[1\], the solution of Eq. (4.9) is obtained.

**Theorem 7.1.** For \(\alpha > -1\) and \(\beta_0 = 0\), there exists a unique positive \(\beta_1\), restricted by

\[
\beta_1(s; \alpha, N) = \mu_2(s; \alpha, N) = \mu_0(s; \alpha + 2, N) = \frac{\alpha + 1}{2\sqrt{2N}s} D_{-\alpha+3} \left( \frac{N(1-s)}{\sqrt{2Ns}} \right) > 0, \quad (7.2)
\]

such that the sequence \(\{\beta_n(s; \alpha, N)\}\) satisfied by

\[
\beta_n \left( \beta_{n+1} + \beta_{n} + \frac{1 - s}{2s} \right) = \frac{n + \alpha \Delta_n}{4Ns}, \quad \Delta_n = \frac{1 - (-1)^n}{2}, \quad (7.3)
\]

is positive and the solution increases.

**Proof.** Obviously, Eq. (7.3) is a special case of Eq. (7.1) with

\[
\sigma_{n,j} = 1, \quad j \in \{0, \pm 1\}, \quad \kappa_n = \frac{1 - s}{2s}, \quad \ell_n = \frac{n + \alpha \Delta_n}{4Ns},
\]

where \(\Delta_n = \frac{1 - (-1)^n}{2}\). The existence of \(\beta_1(s; \alpha, N) > 0\) such that the sequence \(\{\beta_n(s; \alpha, N)\}\) is positive follows directly from \[1\] Thm. 4.1 and the uniqueness of the solution of Eq. (7.3) follows immediately from \[1\] Thm. 5.2.

\[\]

Figure 2: Plots of \(\beta_1(s; \alpha, 1), s \in [0, 1]\) with various \(\alpha\).
Figure 3: Plots of $\beta_1(s; 1, N)$, $s \in [0, 1]$ with various $N$.

Figure 4: Plots of the recurrence coefficients $\beta_n(s; \alpha, 1)$, in given initial condition (7.2) with various choices of $(s; \alpha)$, and $N = 1$.

Remark 7.1. The figures are plotted by Mathematica taking 500 digits precision.
Figure 5: Plots of the recurrence coefficients $\beta_n \left( \frac{1}{2}; 3, N \right)$, in given initial condition (7.2) with various choices of $N$, and $s = \frac{1}{2}, \alpha = 3$.

Figure 6: Plots of the recurrence coefficients $\beta_n \left( \frac{1}{2}; 3, 1 \right)$, with initial conditions $\tilde{\beta}_0 \left( \frac{1}{2}; 3, 1 \right) = 0$, $\tilde{\beta}_1 \left( \frac{1}{2}; 3, 1 \right) = \beta_1 \left( \frac{1}{2}; 3, 1 \right) \pm \varepsilon, \varepsilon \in \{10^{-1}, 10^{-3}, 10^{-5}\}$. 
Figure 7: Plots of the recurrence coefficients $\beta_n \left(\frac{1}{2}; 3, 1\right)$, with initial conditions $\tilde{\beta}_0 \left(\frac{1}{2}; 3, 1\right) = \pm 10^{-10}$, $\tilde{\beta}_1 \left(\frac{1}{2}; 3, 1\right) = \beta_1 \left(\frac{1}{2}; 3, 1\right)$.

8 The Hankel determinant

The well-known supplementary conditions $(S_1)$, $(S_2)$ and the “sum rule” $(S'_2)$ for the ladder operators, satisfied by the functions $A_n(z)$ and $B_n(z)$, continue to hold by a slight modification. These are,

$$B_{n+1}(z) + B_n(z) = zA_n(z) - v'_0(z) + \frac{\alpha}{z}, \quad (S_1)$$
$$1 + z[B_{n+1}(z) - B_n(z)] = \beta_{n+1}(s)A_{n+1}(z) - \beta_n(s)A_{n-1}(z), \quad (S_2)$$

and

$$B_n^2(z) + \left[v'_0(z) - \frac{\alpha}{z}\right]B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n(s)A_n(z)A_{n-1}(z). \quad (S'_2)$$

where $v_0(z) = Nsz^4 + N(1 - s)z^2$, $z \in \mathbb{R}$, $s \in [0, 1]$. $A_n(z)$ and $B_n(z)$ have been given in Eq. (6.18) and Eq. (6.19), which read

$$A_n(z) = 4Ns^2 + 4Ns(\beta_{n+1} + \beta_n) + 2N(1 - s), \quad (8.1)$$

and

$$B_n(z) = 4Ns\beta_n z + \frac{\alpha}{z} \Delta_n, \text{ with } \Delta_n = \left[1 - (-1)^n\right]\frac{1}{2}. \quad (8.2)$$

The three identities $(S_1)$, $(S_2)$ and $(S'_2)$ were also stated in Magnus [28]. Based on this and the formula (1.4), we first display the relation between $D_n(s; \alpha, N)$ and $\beta_n(s; \alpha, N)$ in the next lemma.

Lemma 8.1. The logarithmic derivative of the Hankel determinant can be expressed as

$$\frac{d}{ds} \log D_n(s) = \frac{N(s^2 - 1)(n + \alpha\Delta_n) - 2sn(n + \alpha)}{8s^2} + \frac{(1 + s)(n + \alpha\Delta_n)^2}{32s^2\beta_n}$$
$$+ \frac{N^2(1 - s^2)^2}{8s^2(1 + s)} + 2N^2(1 + s)^2(n + 2\alpha - 3\alpha\Delta_n) - 4s^2\beta_n$$
$$+ \frac{N^2(1 - s^2)}{2s}\beta_n^2 + \frac{1}{2}N^2(1 + s)\beta_n^3 - \frac{2s}{1 + s}\left(\frac{s\beta_n^2}{\beta_n} + \beta_n'\right). \quad (8.3)$$
Proof. It follows from Eq. (1.4) that,
\[
\log D_n(s) = \sum_{j=0}^{n-1} \log h_j(s),
\] (8.4)
we find by taking the derivative of Eq. (8.4) with respect to \(s\),
\[
\frac{D'_n(s)}{D_n(s)} = \sum_{j=0}^{n-1} \frac{h'_j(s)}{h_j(s)} = \frac{N(1 + s)}{2s} \sum_{j=0}^{n-1} (\beta_{j+1} + \beta_j) - \frac{n^2}{4s} - \frac{n\alpha}{4s}.
\]

Here we have used the identity (2.4), which gives
\[
\frac{h'_n(s)}{h_n(s)} = \frac{N(1 + s)}{2s} (\beta_{n+1} + \beta_n) - \frac{2n + 1}{4s} - \frac{\alpha}{4s}.
\]
Substituting Eqs. (8.1), (8.2) and \(v_0(x) = Nsx^4 + N(1 - s)x^2\) into \(S'_2\), we obtain
\[
[4nNs + 16N^2s^2\beta_n^2 + 4Ns\alpha\Delta_n + 8N^2s(1 - s)\beta_n] z^2 + 4Ns\alpha\beta_n(2\Delta_n - 1) + 2N(1 - s)\alpha\Delta_n + 2nN(1 - s) + 4Ns \sum_{j=0}^{n-1} (\beta_{j+1} + \beta_j) = 16N^2s^2\beta_n \left[ \beta_{n+1} + 2\beta_n + \beta_{n-1} + \frac{1 - s}{s} \right] z^2 + 16N^2s^2\beta_n \left[ \beta_{n+1} + \beta_n + \frac{1 - s}{2s} \right] \left[ \beta_n + \beta_{n-1} + \frac{1 - s}{2s} \right].
\]

Equating coefficients on \(z\) gives
\[
\beta_n \left( \beta_{n+1} + \beta_n + \beta_{n-1} + \frac{1 - s}{2s} \right) = \frac{n + \alpha\Delta_n}{4Ns},
\] (8.5)
\[
\sum_{j=0}^{n-1} (\beta_{j+1} + \beta_j) = 4Ns\beta_n \left[ \beta_{n+1} + \beta_n + \frac{1 - s}{2s} \right] \left[ \beta_n + \beta_{n-1} + \frac{1 - s}{2s} \right] + \alpha\beta_n(1 - 2\Delta_n) - \frac{1 - s}{2s} (n + \alpha\Delta_n). \tag{8.6}
\]

Hence we have
\[
\frac{d}{ds} \log D_n(s) = 2N^2(1 + s)\beta_n \left( \beta_{n+1} + \beta_n + \frac{1 - s}{2s} \right) \left( \beta_n + \beta_{n-1} + \frac{1 - s}{2s} \right) + \frac{N(1 + s)}{2s} \alpha\beta_n(1 - 2\Delta_n) - \frac{N(1 - s)^2}{4s^2} (n + \alpha\Delta_n) - \frac{n(n + \alpha)}{4s}. \tag{8.7}
\]

From the Eq. (8.5), we find
\[
\beta_n \left( \beta_{n+1} + \beta_n + \frac{1 - s}{2s} \right) = \frac{n + \alpha\Delta_n}{4Ns} - \beta_n\beta_{n-1}.
\]

Substituting this into Eq. (8.7) and eliminating \(\beta_{n-1}\) by Eq. (2.9), the result is obtained.
Let $N \to \infty$, $n \to \infty$ but the term $n/N$ tends to a fixed number $r$, which is bounded away from 0 and close to 1. If we substitute the asymptotic expansion of $\beta_n(s)$, given by Theorem 5.1, into the right hand of Eq. (8.3), we find

$$\frac{D'_n(s)}{D_n(s)} = A(s)N^2 + B(s)N + C(s) + \frac{D(s)}{N} + \mathcal{O}(N^{-2}), \; N \to \infty, \tag{8.8}$$

where the constants $A(s), B(s), C(s)$ and $D(s)$ can be found in Appendix B.

Therefore integrating from $s = 0$ to $s = 1$, and substituting the constants given in Appendix B, we have

**Lemma 8.2.** For any $\alpha > -1$, as $n \to \infty$ with $n = rN$, $r$ fixed, we have

$$\log \frac{D_n(1)}{D_n(0)} = \int_0^1 \frac{d}{ds} \log D_n(s) \, ds = \frac{(3 - 2 \log 3 - 2 \log r)}{8} n^2 + \frac{\alpha (-1 + \log 3 + \log r)}{4} n + \frac{(3\alpha^2 - 1) \log 2}{12} - \frac{\alpha^2 \log 3}{4}$$

$$+ \frac{\alpha (\alpha^2 + 3)}{48} \frac{1}{n} + \mathcal{O}(n^{-2}), \quad n \text{ even},$$

and

$$\log \frac{D_n(1)}{D_n(0)} = \frac{(3 - 2 \log 3 - 2 \log r)}{8} n^2 - \frac{\alpha (-1 + \log 3 + \log r)}{4} n + \frac{(3\alpha^2 - 1) \log 2}{12}$$

$$- \frac{\alpha^2 \log 3}{4} + \frac{\alpha (\alpha^2 - 3)}{48} \frac{1}{n} + \mathcal{O}(n^{-2}), \quad n \text{ odd}. \tag{8.9}$$

**Proof.** We calculate the integrals in Eq. (8.8) with the aid of Mathematica,

$$\log \frac{D_n(1)}{D_n(0)} = \int_0^1 \frac{d}{ds} \log D_n(s) \, ds = N^2 \int_0^1 A(s) \, ds + N \int_0^1 B(s) \, ds + \int_0^1 C(s) \, ds + \frac{1}{N} \int_0^1 D(s) \, ds + \mathcal{O}(N^{-2})$$

$$= \frac{r^2 (3 - 2 \log 3 - 2 \log r)}{8} N^2 + \frac{r \alpha (-1 + \log 3 + \log r)}{4} N + \frac{(3\alpha^2 - 1) \log 2}{12} - \frac{\alpha^2 \log 3}{4}$$

$$+ \frac{\alpha (\alpha^2 + 3)}{48} \frac{1}{N} + \mathcal{O}(N^{-2})$$

$$= \frac{(3 - 2 \log 3 - 2 \log r)}{8} n^2 + \frac{\alpha (-1 + \log 3 + \log r)}{4} n + \frac{(3\alpha^2 - 1) \log 2}{12} - \frac{\alpha^2 \log 3}{4}$$

$$+ \frac{\alpha (\alpha^2 + 3)}{48} \frac{1}{n} + \mathcal{O}(n^{-2}), \quad n \text{ even},$$

Similarly, we get the result for the case $n$ is odd. \square
Mehta and Normand [32] gives

\[
D_n(0) = \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{1 \leq k < m \leq n} (x_k - x_j)^2 \prod_{\ell=1}^{n} |x_{\ell}|^{\alpha} e^{-N x_{\ell}^2} dx_{\ell}
\]

\[
= \frac{(2\pi)^{\frac{n}{2}}}{(2N)^{\frac{n}{2}} N^{\frac{\alpha n}{2}}} \prod_{j=1}^{n} \frac{\Gamma \left( \frac{\alpha + 1}{2} + \left[ \frac{j}{2} \right] \right)}{\Gamma \left( \frac{1}{2} + \left[ \frac{j}{2} \right] \right)} j!,
\]

where \([n]\) denotes the integer part of \(n\).

Alternatively, when \(n\) is even, Han and Chen [20], see also [15], gives the expression of \(D_n(0)\) as

\[
D_n(0) = \frac{(2\pi)^{\frac{n}{2}}}{(2N)^{\frac{n}{2}} N^{\frac{\alpha n}{2}}} \prod_{j=1}^{n} \frac{G \left( \frac{\alpha + 3}{2} \right) G \left( \frac{\alpha + 1}{2} \right)}{G \left( \frac{n + 1}{2} \right) G \left( \frac{n + 3}{2} \right)}
\]

(8.11)

where \(G\) is the Barnes \(G\)-function, that satisfies the functional relation \(G(z+1) = \Gamma(z)G(z)\), with \(G(0) = 1\), see Voros [44]. It is well-known that the Barnes \(G\)-function has the asymptotic expansion:

\[
\log G(z+1) \sim \frac{z^2}{4} + z \log \Gamma(z+1) - \left[ \frac{z(z+1)}{2} + \frac{1}{12} \right] \log z - \log A
\]

\[+ \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k+1}}\]  

as \(z \to \infty\) with \(|\arg z| < \pi\). Here \(B_n\) are Bernoulli numbers and

\[
A = \exp \left( \frac{1}{12} - \zeta'(-1) \right)
\]

is the Glaisher-Kinkelin constant, \(A = 1.2824271291 \ldots\), where \(\zeta(x)\) is the Riemann zeta function, see for example [38, E.q. 5.17.5]. Then substituting Eq. (8.12) into Eq. (8.11), we obtain,

\[
\log D_n(0) = \frac{2 \log r - 2 \log 2 - 3}{4} n^2 + \left[ \log(2\pi) - \frac{\alpha (1 + \log 2 - \log r)}{2} \right] n
\]

\[+ \frac{3\alpha^2 - 1}{12} \log n - \log A + \frac{1 + 6\alpha \log(2\pi) - 3\alpha^2 \log 2}{12}
\]

\[+ \log \frac{G \left( \frac{3}{2} \right) G \left( \frac{1}{2} \right)}{G \left( \frac{n+3}{2} \right) G \left( \frac{n+1}{2} \right)} + \frac{\alpha^2 + \alpha}{12n} + O \left( n^{-2} \right), \ n \text{ even.}
\]

(8.13)

A similar calculation gives

\[
D_n(0) = \frac{(2\pi)^{\frac{n}{2}}}{(2N)^{\frac{n}{2}} N^{\frac{\alpha n}{2}}} \prod_{j=1}^{n} \frac{G \left( \frac{n + 2}{2} \right) G \left( \frac{n + 1}{2} \right)}{G \left( \frac{n + 3}{2} \right) G \left( \frac{n + 1}{2} \right)}
\]

(8.13)

\[
\left[ G \left( \frac{n + 2}{2} \right) \right]^2, \ n \text{ odd.}
\]
which has the asymptotic expansion
\[
\log D_n(0) = \frac{2 \log r - 2 \log 2 - 3}{4} n^2 + \left[ \log(2\pi) - \frac{\alpha (1 + \log 2 - \log r)}{2} \right] n \\
+ \frac{3\alpha^2 - 1}{12} \log n - \log A + \frac{1 + 6\alpha \log(2\pi) - 3\alpha^2 \log 2}{12} \\
+ \log \frac{G(\frac{3}{2}) G(\frac{1}{2})}{G(\frac{\alpha+3}{2}) G(\frac{\alpha+1}{2})} + \frac{\alpha^3 - 2\alpha}{12n} + O(n^{-2}), \quad n \text{ odd.}
\] (8.14)

Consequently, we get the asymptotics expansion for the Hankel determinant $D_n(1)$.

\[\text{Theorem 8.1.} \quad \text{As } r = \frac{n}{N} \text{ tends to 1, we have}\]
\[
\log D_n(1) = \frac{2 \log r - \log 144 - 3}{8} n^2 + \frac{1}{4} \left[ 3\alpha \log r + 4 \log(2\pi) - \alpha \left( 3 + \log \frac{4}{3} \right) \right] n \\
+ \frac{3\alpha^2 - 1}{12} \log n + \frac{1}{12} \left[ 1 - \log 2 + 6\alpha \log(2\pi) - 3\alpha^2 \log 3 \right] - \log A \\
+ \log \frac{G(\frac{3}{2}) G(\frac{1}{2})}{G(\frac{\alpha+3}{2}) G(\frac{\alpha+1}{2})} + \frac{\alpha (5\alpha^2 + 7)}{48} + O(n^{-2}), \quad n \text{ even,}
\] (8.15)

\[\text{whilst}\]
\[
\log D_n(1) = \frac{2 \log r - \log 144 - 3}{8} n^2 + \frac{1}{4} \left[ \alpha \log r + 4 \log(2\pi) - \alpha - \alpha \log 12 \right] n \\
+ \frac{3\alpha^2 - 1}{12} \log n + \frac{1}{12} \left[ 1 - \log 2 + 6\alpha \log(2\pi) - 3\alpha^2 \log 3 \right] - \log A \\
+ \log \frac{G(\frac{3}{2}) G(\frac{1}{2})}{G(\frac{\alpha+3}{2}) G(\frac{\alpha+1}{2})} + \frac{\alpha (5\alpha^2 - 11)}{48} + O(n^{-2}), \quad n \text{ odd.}
\] (8.16)

**Proof.** The result is obtained by direct computation based on Eqs. (8.13) and (8.14) and Lemma 8.2. \qed

9 **Acknowledgements**

The financial support of the Macau Science and Technology Development Fund under grant number FDCT 130/2014/A3 and FDCT 023/2017/A1 are gratefully acknowledged. We would also like to thank the University of Macau for generous support: MYRG 2014-00011 FST, MYRG 2014-00004 FST.

10 **Appendix A**

\[
dP_x \quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + \gamma(-1)^n}{x_n} + \delta.
\]
11 Appendix B

Here we list the coefficients of the Eq. (8.8).

For \( A(s) \), letting \( \tilde{g}(s) := s - 1 + g(s) \), where \( g(s) = \sqrt{1 - 2s + 12rs + s^2} \), then we have

\[
A(s) = \frac{3r^2(1 + s)}{8s \tilde{g}(s)} - \frac{r(1 + 2rs - s^2)}{8s^2} + \frac{(1 + s)(1 - 2s + 2rs + s^2) \tilde{g}(s)}{96s^3} + \frac{(1 - s^2) \tilde{g}^2(s)}{288s^3} + \frac{(1 + s) \tilde{g}^3(s)}{3456s^3}, \quad n \text{ even or odd.}
\]

For \( B(s) \), defining

\[
B_0(s) := \frac{\alpha}{6s^2 g(s) \tilde{g}^2(s)} \left\{ 1 - g(s) + s(6r - 1)(2g(s) - 3) + s^2 \left[ 2 + (6r - 18r^2) (g(s) - 4) \right] + s^3(6r - 2)(g(s) - 1) + s^4(g(s) + 12r - 3) + s^5 \right\},
\]

then we obtain \( B(s) = -B_0(s) \) if \( n \) is even and \( B(s) = B_0(s) \) if \( n \) is odd.

For \( C(s) \), we first define

\[
c_0(s) := \frac{1 + s}{6s^2 g^2(1 + 2g - 2s(1 - 6r + g) + s^2)},
\]

\[
c_1(s) := -1 + 6s - 3rs - 15s^2 + 12rs^2 + 72r^2 s^2 + 20s^3 - 18rs^3 - 144r^2 s^3 - 432r^3 s^3 - 15s^4 + 12rs^4 + 72r^2 s^4 + 6s^5 - 3rs^5 - s^6,
\]

\[
c_2(s) := g \left( 1 - 5s - 3rs + 10s^2 + 9rs^2 - 36r^2 s^2 - 10s^3 - 9rs^3 + 36r^2 s^3 + 5s^4 + 3rs^4 - s^5 \right),
\]

\[
c_3(s) := g \left( -3 + 15s - 27rs - 30s^2 + 81rs^2 + 108r^2 s^2 + 30s^3 - 81rs^3 - 108r^2 s^3 - 15s^4 + 27rs^4 + 3s^5 \right) \alpha^2,
\]

\[
c_4(s) := (3 - 18s + 45rs + 45s^2 - 180rs^2 - 60s^3 + 270rs^3 - 1296r^3 s^3 + 45s^4 - 180rs^4 - 18s^5 + 45rs^5 + 3s^6) \alpha^2,
\]

then we find that \( C(s) = c_0(s) [c_1(s) + c_2(s) + c_3(s) + c_4(s)] \), either \( n \) is even or odd.

For \( D(s) \), we get \( D(s) := d_0(s) [d_1(s) + d_2(s)] \) when \( n \) is even, with

\[
d_0(s) := -\frac{(1 + s) \alpha}{g^2(1 + 2g - 2s(1 - 6r + g) + s^2)^3},
\]

\[
d_1(s) := g \left( -3 + 15s - 15rs - 30s^2 + 45rs^2 - 36r^2 s^2 + 30s^3 - 45rs^3 + 36r^2 s^3 - 15s^4 + 15rs^4 + 3s^5 - \alpha^2 + 5s \alpha^2 - 15rs \alpha^2 - 10s^2 \alpha^2 + 45rs^2 \alpha^2 + 36r^2 s^2 \alpha^2 + 10s^3 \alpha^2 - 45rs^3 \alpha^2 - 36r^2 s^3 \alpha^2 - 5s^4 \alpha^2 + 15rs^4 \alpha^2 + s^5 \alpha^2 \right),
\]

\[
d_2(s) := 3 - 18s + 33rs + 45s^2 - 132rs^2 + 72r^2 s^2 - 60s^3 + 198rs^3 - 144r^2 s^3 - 2160r^3 s^3 + 45s^4 - 132rs^4 + 72r^2 s^4 - 18s^5 + 33rs^5 + 3s^6 + \alpha^2 - 6s \alpha^2 + 21rs \alpha^2 + 15s^2 \alpha^2 - 84rs^2 \alpha^2 + 36r^2 s^2 \alpha^2 - 20s^3 \alpha^2 + 126rs^3 \alpha^2 - 72r^2 s^3 \alpha^2 - 864r^3 s^3 \alpha^2 + 15s^4 \alpha^2 - 84rs^4 \alpha^2 + 36r^2 s^4 \alpha^2 - 6s^5 \alpha^2 + 21rs^5 \alpha^2 + s^6 \alpha^2.
\]
else if \( n \) is odd, we have
\[
D(s) := \hat{d}_0(s) \left[ \hat{d}_1(s) + \hat{d}_2(s) \right],
\]
where
\[
\hat{d}_0(s) := \frac{\alpha}{4f^2g^5}, \quad \text{with} \quad f = 1 - 2s - 4rs + s^2,
\]
\[
\hat{d}_1(s) := g(1 - 4s + 56r + 5s^2 - 112rs^2 + 272r^2s^2 - 5s^4 + 112rs^4 - 272r^2s^4 + 4s^5
\]
\[
- 56rs^5 - s^6 - \alpha^2 + 4sr\alpha^2 - 24rs^2\alpha^2 + 48r^2s^3\alpha^2 - 144r^2s^2\alpha^2 + 5s^4\alpha^2
\]
\[
- 48s^4\alpha^2 + 144r^2s^3\alpha^2 - 4s^5\alpha^2 + s^6\alpha^2),
\]
\[
\hat{d}_2(s) := -2 + 10s - 64rs - 18s^2 + 192rs^2 - 544r^2s^2 + 10s^3 - 128rs^3 + 544r^2s^3 - 768r^3s^3
\]
\[
+ 10s^4 - 128rs^4 + 544r^2s^4 - 768r^3s^4 - 18s^5 + 192rs^5 - 544r^2s^5 + 10s^6 - 64rs^6
\]
\[
- 2s^7 + \alpha^2 - 5sr\alpha^2 + 32sr\alpha^2 + 9s^2\alpha^2 - 96rs^2\alpha^2 + 272r^2s^3\alpha^2 - 5s^3\alpha^2 + 64rs^3\alpha^2
\]
\[
- 272r^2s^3\alpha^2 + 384r^3s^3\alpha^2 - 5s^4\alpha^2 + 64rs^4\alpha^2 - 272r^2s^4\alpha^2 + 384r^3s^4\alpha^2 + 9s^5\alpha^2
\]
\[
- 96rs^5\alpha^2 + 272r^2s^5\alpha^2 - 5s^6\alpha^2 + 32rs^6\alpha^2 + s^7\alpha^2.
\]

**References**

[1] S. M. Alsulami, P. Nevai, J. Szabados, and W. V. Assche, A family of nonlinear difference equations: existence uniqueness and asymptotic behaviour of positive solutions. J. Approx. Theory, 193 (2015), 39-55.

[2] E. Basor, Y. Chen and T. Ehrhardt, Painlevé V and time-dependent Jacobi polynomials, J. Phys. A: Math. Theor. 43 (2010), 015204.

[3] P. Bleher and A. R. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, Ann. of Math. 150 (1999), 185-266.

[4] E. S. Cheb-Terrab, Solutions for the general, confluent and biconfluent Heun equations and their connection with Abel equations, J. Phys. A: Math. Gen. 37 (2004), 9923-9949.

[5] Y. Chen and M. V. Feigin, Painlevé IV and degenerate Gaussian Unitary Ensembles, J. Phys. A: Math. Gen., 39 (2006), 12381-12393.

[6] Y. Chen and M. E. H. Ismail, Ladder operators and differential equations for orthogonal polynomials, J. Phys. A: Math. Gen. 30 (1997), 7817-7829.

[7] Y. Chen and M. E. H. Ismail, Thermodynamic relations of the Hermitian matrix ensembles, J. Phys. A: Math. Gen. 30 (1997), 6633-6654.

[8] Y. Chen and N. Lawrence, On the linear statistics of Hermitian random matrices, J. Phys. A: Math. Gen. 31 (1998), 1141-1152.

[9] Y. Chen and D. S. Lubinsky, Smallest eigenvalues of Hankel matrices for the exponential weights, J. Math. Anal. Appl. 293 (2004), 476-495.
[10] Y. Chen and M. R. Mckay, Coulomb fluid, Painlevé transcendents and the information theory of MIMO systems, IEEE Trans. Inf. Theory, 58 (2002), 4594-4634.

[11] T. S. Chihara, An introduction to Orthogonal Polynomials, Dover Publications, INC., New York, 1978.

[12] P. A. Clarkson, K. Jordaan and A. Kelil, A generalized Freud weight, Stud. Appl. Math. 136 (2016), 288-320.

[13] P. A. Clarkson and K. Jordaan, Properties of generalized Freud polynomials, J. Approx. Theory, 225 (2018), 148-175.

[14] S. B. Damelin, Asymptotics of recurrence coefficients for orthogonal polynomials on the line - Magnus’s method revisited, Math. Comp., 73 (2004), 191-209.

[15] A. Deaño and N. J. Simm, On the probability of positive-definiteness in the gGUE via semi-classical Laguerre polynomials, J. Approx. Theor. 220 (2017), 44-59.

[16] F. J. Dyson, Statistical theory of the energy levels of complex systems I-III, J. Math. Phys. 3 (1962), 140-175.

[17] G. Freud, On the coefficients in the recursion formulae of orthogonal polynomials, Proc. R. Irish Acad. Sect. A 76 (1976), pp. 1-6.

[18] E. M. Ferreira and J. Sesma, Global solutions of the biconfluent Heun equation, Numer. Algor. 71 (2016), 797-809.

[19] G. Filipuk, W. van Assche and L. Zhang, The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation, J. Phys. A: Math. Theor. 45 (2012), 205201.

[20] P. Han and Y. Chen, The recurrence coefficients of a semi-classical Laguerre polynomials and the large $n$ asymptotics of the associated Hankel determinant, Random Matrices: Theory and Applications, 6 (2017), 1740002.

[21] E. Hendriksen and H. van Rossum, Semiclassical orthogonal polynomials, in: C. Brezinski, A. Draux, A. P. Magnus, P. Maroni and A. Ronveaux (Eds.), Orthogonal Polynomials and Applications, in: Lect. Notes Math., vol. 1171, Springer-Verlag, Berlin, 1985, pp. 354-361.

[22] M. E. H. Ismail, An electrostatic mode for zeros of general orthogonal polynomials, Pacific J. Math. 193 (2000), 355-369.

[23] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Encyclopedia of Mathematics and its Applications, Vol. 98, Cambridge University Press, Cambridge, 2005.
[24] T. A. Ishkhanyan and A. M. Ishkhanyan, Solutions of the bi-confluent Heun equations in terms of the Hermite functions, Annals of Physics, 383 (2017), 79-91.

[25] N. Joshi and C. J. Lustri, Stokes phenomena in discrete Painlevé I, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 471 (2015), 20140874.

[26] J. S. Lew and D. A. Quarles, Nonnegative solutions of a nonlinear recurrence. J. Approx. Theory. 38 (1983), 357-379.

[27] A. P. Magnus, On Freud’s equations for the exponential weights, J. Approx. Theory, 46 (1986), 65-99.

[28] A. P. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials. J. Comput. Appl. Math. 57 (1995), 215-237.

[29] A. P. Magnus, Freud’s equations for orthogonal polynomials as discrete Painlevé equations Symmetries and Integrability of Difference Equations (London Mathematical Society Lecture Note Series vol 225) (Cambridge: Cambridge University Press), 1999, pp. 228−243.

[30] P. Maroni, Prolégomènes à l’étude des polynômes orthogonaux semi-classiques, Ann. Mat. Pura Appl. (4) 149 (1987), 165-184.

[31] M. L. Mehta, Random Matrices 3rd ed., Elsevier (Singapore) Pte Ltd., Singapore, 2006.

[32] M. L. Mehta and J. M. Normand, Probability density of the determinant of a random Hermitian matrix, J. Phys. A: Math. Gen. 31 (1998), 5377-5391.

[33] S. G. Mikhlin, Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology, 2nd rev. ed., Pergamon Press, New York, 1964.

[34] H. N. Mhaskar and E. B. Saff, Extremal problems for polynomials with exponential weights, Trans. Amer. Math. Soc., 285 (1984), 204-234.

[35] P. Nevai, Orthogonal polynomials associated with \( \exp(-x^4) \), in: Z. Ditzian, A. Meir, S. D. Riemenschneider and A. Sharma (Eds.), Second Edmonton Conference on Approximation Theory, in: CMS Conf. Proc., vol. 3, Amer. Math. Soc., Providence, RI, 1983, pp. 263-285.

[36] S. Noschese and L. Pasquini, On the nonnegative solution of a Freud three-term recurrence, J. Approx. Theory, 99 (1999), 54-67.

[37] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), DLMF Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
[38] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[39] A. Ronveaux, Heun’s Differential Equations, Oxford University Press, London, 1995.

[40] E. A. Rakhmanov, On asymptotic properties of polynomials orthogonal on the real axis, Math. Sb. (N. S.), 119 (1982), 163-203.

[41] G. Szegő, *Orthogonal Polynomials*, AMS, New York, 1939.

[42] M. Tsuji, *Potential Theory in Modern Function Theory*. Tokyo, Japan: Maruzen, 1959.

[43] W. Van Assche, Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials, in: S. Elaydi, J. Cushing, R. Lasser, V. Papageorgiou, A. Ruffing and W. Van Assche(Eds.), *Difference Equations, Special Functions and Orthogonal Polynomials*, World Scientific, Hackensack, NJ, 2007, pp. 687–725.

[44] A. Voros, Spectral Functions, Special Functions and the Selberg Zeta Function, Commun. Math. Phys. 110 (1987), 439-465.