EQUIVARIANT GROMOV-WITTEN INVARIANTS OF ALGEBRAIC GKM MANIFOLDS

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ABSTRACT. An algebraic GKM manifold is a non-singular algebraic variety equipped with an algebraic action of an algebraic torus, with only finitely many torus fixed points and finitely many one-dimensional orbits. In this expository article, we use virtual localization to express equivariant Gromov-Witten invariants of any algebraic GKM manifold (which is not necessarily compact) in terms of Hodge integrals over moduli stacks of stable curves and the GKM graph of the GKM manifold.

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1. INTRODUCTION

In this paper, we work over $\mathbb{C}$.

1.1. Gromov-Witten invariants of a smooth projective variety. Let $X$ be a smooth projective variety. Gromov-Witten invariants of $X$ are virtual counts of parametrized
algebraic curves of $X$. More precisely, let $\overline{M}_{g,n}(X,\beta)$ be the Kontsevich’s moduli space of $n$-pointed, genus $g$, degree $\beta$ stable maps to $X$, where $\beta \in H_2(X,\beta)$ is an effective curve class. It is a proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$d^{\text{vir}} = \int_\beta c_1(T_X) + (\dim X - 3)(1 - g) + n,$$

where $\int$ stands for the pairing between the (rational) homology and cohomology. There is a virtual fundamental class $[31,4,2]$ [127]

$$\langle \overline{M}_{g,n}(X,\beta) \rangle^{\text{vir}} \in A_{d^{\text{vir}}} (\overline{M}_{g,n}(X,\beta); \mathbb{Q})$$

which can also be viewed as an element in $H_{2d^{\text{vir}}} (\overline{M}_{g,n}(X,\beta); \mathbb{Q})$. The virtual fundamental class defines a $\mathbb{Q}$-linear map

$$\int_{\langle \overline{M}_{g,n}(X,\beta) \rangle^{\text{vir}}} : H^*(\overline{M}_{g,n}(X,\beta); \mathbb{Q}) \to \mathbb{Q}.$$ 

For $i = 1, \ldots, n$, let $\text{ev}_i : \overline{M}_{g,n}(X,\beta) \to X$ be the evaluation map at the $i$-th marked point. Genus $g$, degree $\beta$ descendant Gromov-Witten invariants of $X$ are defined by

$$\langle \tau_1(\gamma_1), \ldots, \tau_n(\gamma_n) \rangle_{g,\beta}^X : = \int_{\langle \overline{M}_{g,n}(X,\beta) \rangle^{\text{vir}}} \prod_{i=1}^n (\text{ev}_i^* \gamma_i \cup \psi_i^{a_i}) \in \mathbb{Q}$$

where $\gamma_i \in H^*(X; \mathbb{Q})$, $a_i \in \mathbb{Z}_{\geq 0}$, and $\psi_i \in H^2(\overline{M}_{g,n}(X,\beta); \mathbb{Q})$ are $\psi$-classes (to be defined in Section 3.4). If $\gamma_i \in H^k(X; \mathbb{Q})$ then the (2) vanishes unless

$$\sum_{i=1}^n (d_i + 2a_i - 2) = 2(\int_\beta c_1(T_X) + (\dim X - 3)(1 - g)).$$

If $X$ is a smooth algebraic variety which is not projective, then $\overline{M}_{g,n}(X,\beta)$ is usually not proper, so $[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}$ is not defined. If $X$ is not projective but $\overline{M}_{g,n}(X,\beta)$ is proper for some particular $g, n, \beta$, then $[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}$ exists and the invariants in (2) are defined for such $g, n, \beta$.

1.2. Equivariant Gromov-Witten invariants and virtual localization. Suppose that $T = (\mathbb{C}^*)^m$ acts algebraically on $X$. Then $T$ acts on $\overline{M}_{g,n}(X,\beta)$. There exists a $T$-equivariant perfect obstruction theory, and a $T$-equivariant virtual fundamental class

$$[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}_T \in A_{d^{\text{vir}}(\overline{M}_{g,n}(X,\beta); \mathbb{Q})}.$$ 

Let $R_T := H_T^*(\{\text{point}\}; \mathbb{Q}) = H^*(BT; \mathbb{Q}) = \mathbb{Q}[u_1, \ldots, u_m]$ be the $T$-equivariant cohomology of a point, where $u_i \in H^2_T(BT; \mathbb{Q})$. There is an $R_T$-linear map

$$\int_{\langle \overline{M}_{g,n}(X,\beta) \rangle^{\text{vir}}_T} : H^*_T(\overline{M}_{g,n}(X,\beta); \mathbb{Q}) \to H^*_T(\{\text{point}\}; \mathbb{Q}) = R_T.$$ 

Genus $g$, degree $\beta$ $T$-equivariant descendant Gromov-Witten invariants of $X$ are defined by

$$\langle \tau_1(\gamma_1^T), \ldots, \tau_n(\gamma_n^T) \rangle_{g,\beta}^X : = \int_{\langle \overline{M}_{g,n}(X,\beta) \rangle^{\text{vir}}_T} \prod_{i=1}^n (\text{ev}_i^* \gamma_i^T \cup (\psi_i^T)^{a_i}) \in R_T,$$
where $\gamma_i^T \in H^*_T(X;\mathbb{Q})$, and $\psi_i^T \in H^2(T(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}))$ is a $T$-equivariant lift of $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$. If $\gamma_i^T \in H^2_T(X;\mathbb{Q})$ then
\[
\langle \tau_{a_1}(\gamma_1^T), \ldots, \tau_{a_n}(\gamma_n^T) \rangle_{X^T}^{\mathbb{Q}} \in H^2_{T-}\langle d+2a_i-2 \rangle^{2d_{\text{vir}}} \{ \text{point} \};\mathbb{Q} \rangle.
\]
In particular, (3) vanishes if $\sum_{i=1}^n (d_i + 2a_i - 2) < 2d_{\text{vir}}$.

The $T$-equivariant Gromov-Witten invariants (2) are related to the Gromov-Witten invariants (1) as follows. Let $\gamma_i$ be the image of $\gamma_i^T$ under the ring homomorphism $H^*_T(X;\mathbb{Q}) \to H^*(X;\mathbb{Q})$. Then
\[
\langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n) \rangle_{X^T}^{\mathbb{Q}} = \langle \tau_{a_1}(\gamma_1^T), \ldots, \tau_{a_n}(\gamma_n^T) \rangle_{X^T}^{\mathbb{Q}}|_{u_1=\ldots=u_m=0}.
\]

The torus fixed part of the restriction of the $T$-equivariant perfect obstruction theory to the $T$ fixed substack $\overline{\mathcal{M}}_{g,n}(X,\beta)^T \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$ defines a perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$ and a virtual class $[\overline{\mathcal{M}}_{g,n}(X,\beta)^T]_{vir}^{\mathbb{Q}} \in A^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$.

The torus moving part defines the virtual normal bundle $N^{\text{vir}}$ of the inclusion $\overline{\mathcal{M}}_{g,n}(X,\beta)^T \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$. By localization of virtual fundamental class [18, 3], the RHS of (3) is equal to
\[
\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)^T]_{vir}} \prod_{i=1}^n i^*\left( e^v_T(\gamma_i^T \cup (\psi_i^T)^{a_i}) \right) e_T(N^{\text{vir}})
\]
where $i : \overline{\mathcal{M}}_{g,n}(X,\beta)^T \hookrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)$ is the inclusion, and $e_T(N^{\text{vir}})$ is the $T$-equivariant Euler class of $N^{\text{vir}}$. It is known that $e_T(N^{\text{vir}})$ is invertible in $H^*_T(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \otimes R_T$, where $Q_T = \mathbb{Q}[u_1, \ldots, u_m]$ is the fractional field of $R_T = \mathbb{Q}[u_1, \ldots, u_m]$.

Suppose that $X$ is non-compact, and for some $g, n, \beta$, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not proper but $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$ is. Then the RHS of (3) is not defined, but (4) is. In this case, define $T$-equivariant Gromov-Witten invariants $\langle \tau_{a_1}(\gamma_1^T), \ldots, \tau_{a_n}(\gamma_n^T) \rangle_{X^T}^{\mathbb{Q}}$ by (4), which is an element in $Q(u_1, \ldots, u_m)$ instead of $Q[u_1, \ldots, u_m]$.

1.3. Algebraic GKM manifolds and their Gromov-Witten invariants. In this paper, an algebraic GKM manifold, named after Goresky-Kottwitz-MacPherson, is a non-singular algebraic variety equipped with an algebraic action of $T = (\mathbb{C}^*)^m$, such that there are finitely many torus fixed points and finitely many one-dimensional orbits. Examples of algebraic GKM manifolds include toric manifolds, Grassmannians, flag manifolds, etc.

If $X$ is an algebraic GKM manifold then each connected component of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is, up to some quasi-finite map, a product of moduli stacks of pointed stable curves, and the RHS of (4) can be expressed in terms Hodge integrals on moduli stacks of pointed stable curves. This algorithm was first described by Kontsevich for genus zero Gromov-Witten invariants of $\mathbb{P}^r$ in 1994 [29], before the construction of virtual fundamental class and the proof of virtual localization. The moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ of genus zero stable maps to $\mathbb{P}^r$ are proper smooth DM stacks, so there exists a fundamental class $[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)] \in H_{2}((\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d);\mathbb{Q})$, and one may apply the classical Atiyah-Bott localization formula (1) in this case. In [18], T. Graber and R. Pandharipande used their virtual localization formula to derive an explicit formula for all genus Gromov-Witten invariants of $\mathbb{P}^r$. (See also K. Behrend [3, Section 4.]) H. Spielberg derived a formula of genus zero Gromov-Witten invariants of toric manifolds in his thesis [44]. Localization computations
of all genus equivariant Gromov-Witten invariants of toric manifolds can be found in [33]. The main purpose of this paper is to provide details of the virtual localization calculations of all genus equivariant Gromov-Witten invariants for general algebraic GKM manifolds described on page 20-21 of [17].

1.4. Outline. In Section 2 we define algebraic GKM manifolds and their GKM graphs, following [19, 20]. In Section 3 we give a brief review of Gromov-Witten theory. In Section 4 we compute all genus equivariant descendant Gromov-Witten invariants of an arbitrary algebraic GKM manifold by virtual localization. Most of Section 4 is straightforward generalization of the \( P^r \) case discussed in [29] (genus 0) and [18, Section 4], [3, Section 4] (higher genus); see also [21, Chapter 27].

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2. Algebraic GKM manifolds

In this section, we review the geometry of algebraic GKM manifolds, following [19], and introduce the GKM graph associated to an algebraic GKM manifold, following [20]. The GKM graph in this paper can be non-compact since we consider algebraic GKM manifolds which are not necessarily compact. In Section 4 we will see that the GKM graph contains all the information needed for computing Gromov-Witten invariants and equivariant Gromov-Witten invariants of the GKM manifold.

2.1. Basic notation. Let \( X \) be a non-singular algebraic variety of dimension \( r \). We say \( X \) is an algebraic GKM manifold if it is equipped with an algebraic action of a complex algebraic torus \( T = (\mathbb{C}^\ast)^m \) with only finitely many torus fixed points and finitely many one-dimensional orbits.

Let \( N = \text{Hom}(\mathbb{C}^\ast, T) \cong \mathbb{Z}^m \) be the lattice of 1-parameter subgroups of \( T \), and let \( M = \text{Hom}(T, \mathbb{C}^\ast) \) be the lattice of irreducible characters of \( T \). Then \( M = \text{Hom}(N, \mathbb{Z}) \) is the dual lattice of \( N \). Let \( N_\mathbb{R} = N \otimes \mathbb{R} \) and \( M_\mathbb{R} = M \otimes \mathbb{R} \), so that they are dual real vector spaces of dimension \( k \). Let \( K = U(1)^m \) be the maximal compact subgroup of \( T \). Then \( N_\mathbb{R} \) can be canonically identified with the Lie algebra of \( K \). Let \( N_\mathbb{Q} = N \otimes \mathbb{Q} \) and let \( M_\mathbb{Q} = M \otimes \mathbb{Q} \). Then \( M_\mathbb{Q} \) can be canonically identified with \( H^2_T(\text{point}; \mathbb{Q}) \).

We make the following assumption on \( X \).

Assumption 5. (1) The set \( X^T \) of \( T \) fixed points in \( X \) is non-empty.

(2) The closure of a one-dimensional orbit is either a complex projective line \( \mathbb{P}^1 \) or a complex affine line \( \mathbb{C} \).

Note that (1) and (2) hold when \( X \) is proper. Indeed, if \( X \) is proper then the closure of any one-dimensional orbit is \( \mathbb{P}^1 \).

Example 6. If \( X \) is a non-singular toric variety defined by a finite fan, then \( X \) is an algebraic GKM manifold.
Example 7 (The Grassmannian $Gr(k, m)$). Let $Gr(k, m)$ be the set of $k$-dimensional linear subspace of $\mathbb{C}^m$. It is a nonsingular projective variety of dimension $k(m - k)$. Let $T = (\mathbb{C}^*)^m$ act on $\mathbb{C}^m$ by

$$(t_1, \ldots, t_m) \cdot (z_1, \ldots, z_m) = (t_1z_1, \ldots, t_mz_m).$$

Given $t \in T$, let $\phi_t : \mathbb{C}^m \to \mathbb{C}^m$ be defined by $\phi_t(z) = t \cdot z$. Let $T$ act on $Gr(k, m)$ by $t \cdot V = \phi_t(V)$, where $V$ is a $k$-dimensional linear subspace of $\mathbb{C}^m$. Given $J \subset \{1, \ldots, m\}$, let $f^c := \{1, \ldots, m\} \setminus J$, and define

$$C^J := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : z_i = 0 \text{ if } i \in f^c \} \cong \mathbb{C}^{|J|}.$$

Note that $\phi_t(C^J) = C^J$ for any $t \in T, J \subset \{1, \ldots, n\}$.

The torus-fixed points in $Gr(k, m)$ are

$$Gr(k, m)^T = \{C^J : J \subset \{1, \ldots, n\}, |J| = k\}.$$

So there are $\binom{m}{k}$ torus-fixed points in $G(k, m)$.

Let $C^J$ and $C^{J'}$ be distinct $T$-fixed points in $Gr(k, m)$. Then $C^J \cap C^{J'} = C^{J \cap J'}$. There is a torus-fixed line connecting $C^J$ and $C^{J'}$ if and only of $|J \cap J'| = k - 1$. In this case, $|J \cup J'| = k + 1$. The $T$-fixed lines in $G(k, m)$ are

$$\{\ell_{l,K} : I \subset K \subset \{1, \ldots, m\}, |I| = k - 1, |K| = k + 1\}$$

where

$$\ell_{l,K} = \{V \in G(k, m) : C^J \subset V \subset \mathbb{C}^k \} \cong \mathbb{P}^1.$$

Suppose that $I \subset K \subset \{1, \ldots, m\}$, and $|I| = k - 1, |K| = k + 1$. Then $K = I \cup \{j_1, j_2\}$, where $j_1, j_2 \in f^c$. So there are $(m-1)(m-k+1)$ torus-fixed lines in $G(k, m)$.

2.2. GKM graph. Let $X$ be an algebraic GKM manifold of dimension $r$, so that $T = (\mathbb{C}^*)^m$ acts algebraically on $X$.

Following [20], we define a graph $Y$ as follows. Let $V(Y)$ (resp. $E(Y)$) denote the set of vertices (resp. edges) in $Y$.

1. (Vertices) We assign a vertex $\sigma$ to each torus fixed point $p_\sigma$ in $X$.
2. (Edges) We assign an edge $e$ to each one-dimensional $O_e$ in $X$. Let $\ell_e$ be the closure of $O_e$.
3. (Flags) The set of flags in the graph $Y$ is given by

$$F(Y) = \{(e, \sigma) \in E(Y) \times F(Y) : \sigma \in e\} = \{(e, \sigma) \in E(Y) \times F(Y) : p_\sigma \in \ell_e\}.$$

The Assumption [5] can be rephrased in terms of the graph $Y$.

Assumption 8.

1. $V(Y)$ is non-empty.
2. Each edge in $E(Y)$ contains at least one vertex.

Let $E(Y)_c = \{e \in E(Y) : \ell_e \cong \mathbb{P}^1\}$ be the set of compact edges in $Y$. Note that $E(Y)_c = E(Y)$ if $X$ is proper.

Given a vertex $\sigma \in V(Y)$, we denote by $E_\sigma$ the set of edges containing $\sigma$, i.e. $E_\sigma := \{e \in E : (e, \sigma) \in F(Y)\}$. Then $|E_\sigma| = r$ for all $\sigma \in V(Y)$, so $Y$ is an $r$-valent graph.

Given a flag $(e, \sigma) \in F(Y)$, let $w(e, \sigma) \in M = \text{Hom}(T, \mathbb{C}^*)$ be the weight of $T$-action on $T_{p_\sigma}O_e$, the tangent line to $\ell_e$ at the fixed point $p_\sigma$, namely

$$w(e, \sigma) := c^1(T_{p_\sigma}O_e) \in H^2(T; \mathbb{C}) \cong M.$$

This gives rise to a map $w : F(Y) \to M$ satisfying the following properties.
The formal completion $\hat{\Upsilon}$ can be reconstructed from the graph curve is a connected algebraic curve $X$ complex algebraic tori, then $T$ and $x$ dered marked points.

Example 9 (Gr$(r, m)$). The GKM graph of Gr$(r, m)$ is a $k(m - k)$-valent graph $Y$ such that

$$
V(Y) = \{\sigma_j : J \subset \{1, \ldots, n\}, |J| = k\}
$$

$$
E(Y) = E(Y)_c = \{\sigma_{I,K} : I \subset K \subset \{1, \ldots, m\}, |I| = k - 1, |K| = k + 1\},
$$

$$
F(Y) = \{\{(\epsilon_{I,K}, \sigma_I) \in E(Y) \times V(Y) : I \subset J \subset K\}
$$

$$
\omega(\epsilon_{I,K}, \sigma_{I,J_{1,2}}) = -\omega(\epsilon_{I,K}, \sigma_{J_{1,2}}) = u_{j_2} - u_{j_1}, \quad j_1, j_2 \in I^c, K = I \cup \{j_1, j_2\}.
$$

We define the 1-skeleton of $X$ to be the union of 1-dimensional orbit closures:

$$
X^1 := \bigcup_{\epsilon \in E(Y)} \ell_\epsilon.
$$

The formal completion $\hat{X}$ of $X$ along the 1-skeleton $X^1$, together with the $T$-action, can be reconstructed from the graph $Y$ and $\omega : F(Y) \to M$. We call $(Y, \omega)$ the GKM graph of $X$ with the $T$-action. If $\rho : T' \to T$ is a homomorphism between complex algebraic tori, then $T'$ acts on $X$ by $t' \cdot x = \rho(t') \cdot x$, where $t' \in T', \rho(t') \in T, x \in X$. The GKM graph of $X$ with this $T'$-action is given by $(Y, \rho^* \circ \omega)$, where $\rho^* : M = \text{Hom}(T, C^*) \to \text{Hom}(T', C^*)$.

3. GROMOV-WITTEN THEORY

In this section, we give a brief review of Gromov-Witten theory and equivariant Gromov-Witten theory.

3.1. Moduli of stable curves and Hodge integrals. An $n$-pointed, genus $g$ prestable curve is a connected algebraic curve $C$ of arithmetic genus $g$ together with $n$ ordered marked points $x_1, \ldots, x_n \in C$, where $C$ has at most nodal singularities, and $x_1, \ldots, x_n$ are distinct smooth points. An $n$-pointed, genus $g$ prestable curve $(C, x_1, \ldots, x_n)$ is stable if its automorphism group is finite, or equivalently,

$$
\text{Hom}_{\overline{O}_C}(\Omega_C(x_1 + \cdots + x_n), \overline{O}_C) = 0.
$$

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of $n$-pointed, genus $g$ stable curves, where $n, g$ are non-negative integers. We assume that $2g - 2 + n > 0$, so that $\overline{\mathcal{M}}_{g,n}$ is nonempty. Then $\overline{\mathcal{M}}_{g,n}$ is a proper smooth Deligne-Mumford stack of dimension $3g - 3 + n$.
The tangent space of $\overline{M}_{g,n}$ at a moduli point $[(C, x_1, \ldots, x_n)] \in \overline{M}_{g,n}$ is given by

$$\text{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C(x_1 + \cdots + x_n), \mathcal{O}_C).$$

Since $\overline{M}_{g,n}$ is a proper Deligne-Mumford stack, we may define

$$\int_{\overline{M}_{g,n}}: A^*(\overline{M}_{g,n}; \mathbb{Q}) \to \mathbb{Q}.$$

We now introduce some classes in $A^*(\overline{M}_{g,n})$. There is a forgetful morphism $\pi: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ given by forgetting the $(n+1)$-th marked point (and contracting the unstable irreducible component if there is one):

$$[(C, x_1, \ldots, x_n, x_{n+1})] \mapsto [(C^{st}, x_1, \ldots, x_n)]$$

where $(C^{st}, x_1, \ldots, x_n)$ is the stabilization of the prestable curve $(C, x_1, \ldots, x_n)$. $\pi: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ can be identified with the universal curve over $\overline{M}_{g,n}$.

- $(\lambda)$ classes Let $\omega_\pi$ be the relative dualizing sheaf of $\pi: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$. The Hodge bundle $E = \pi_* \omega_\pi$ is a rank $g$ vector bundle over $\overline{M}_{g,n}$ whose fiber over the moduli point $[(C, x_1, \ldots, x_n)] \in \overline{M}_{g,n}$ is $H^0(C, \omega_C)$, the space of sections of the dualizing sheaf $\omega_C$ of the curve $C$. The $\lambda$ classes are defined by

$$\lambda_j = c_j(E) \in A^j(\overline{M}_{g,n}; \mathbb{Q}).$$

- $(\psi)$ classes The $i$-th marked point $x_i$ gives rise a section $s_i: \overline{M}_{g,n} \to \overline{M}_{g,n+1}$ of the universal curve. Let $L_i = s_i^* \omega_\pi$ be the line bundle over $\overline{M}_{g,n}$ whose fiber over the moduli point $[(C, x_1, \ldots, x_n)] \in \overline{M}_{g,n}$ is the cotangent line $T^*_x C$ of $C$ at $x_i$. The $\psi$ classes are defined by

$$\psi_i = c_1(L_i) \in A^1(\overline{M}_{g,n}; \mathbb{Q}).$$

Hodge integrals are top intersection numbers of $\lambda$ classes and $\psi$ classes:

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \in \mathbb{Q}.\quad (10)$$

By definition, (10) is zero unless

$$a_1 + \cdots + a_n + k_1 + 2k_2 + \cdots + gk_g = 3g - 3 + n.$$

Using Mumford’s Grothendieck-Riemann-Roch calculations in [37], Faber proved, in [11], that general Hodge integrals can be uniquely reconstructed from the $\psi$ integrals (also known as descendant integrals):

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.\quad (11)$$

The descendant integrals can be computed recursively by Witten’s conjecture which asserts that the $\psi$ integrals (11) satisfy a system of differential equations known as the KdV equations [46]. The KdV equations and the string equation determine all the $\psi$ integrals (11) from the initial value $\int_{\overline{M}_{0,3}} 1 = 1$. For example, from the initial value $\int_{\overline{M}_{0,3}} 1 = 1$ and the string equation, one can derive the following formula of genus 0 descendant integrals:

$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \frac{(n - 3)!}{a_1! \cdots a_n!}.\quad (12)$$
where $a_1 + \cdots + a_n = n - 3$ [29 Section 3.3.2].

The Witten’s conjecture was first proved by Kontsevich in [28]. By now, Witten’s conjecture has been reproved many times (Okounkov-Pandharipande [38], Mirzakhani [35], Kim-Liu [24], Kazarian-Lando [23], Chen-Li-Liu [6], Kazarian [22], Mulase-Zhang [36], etc.).

### 3.2. Moduli of stable maps.

Let $X$ be a nonsingular projective or quasi-projective variety, and let $\beta \in H_2(X; \mathbb{Z})$. An $n$-pointed, genus $g$, degree $\beta$ prestable map to $X$ is a morphism $f : (C, x_1, \ldots, x_n) \to X$, where $(C, x_1, \ldots, x_n)$ is an $n$-pointed, genus $g$ prestable curve, and $f_*[C] = \beta$. Two prestable maps

$$f : (C, x_1, \ldots, x_n) \to X, \quad f' : (C', x'_1, \ldots, x'_n) \to X$$

are isomorphic if there exists an isomorphism $\phi : (C, x_1, \ldots, x_n) \to (C', x'_1, \ldots, x'_n)$ of $n$-pointed prestable curves such that $f = f' \circ \phi$. A prestable map $f : (C, x_1, \ldots, x_n) \to X$ is stable if its automorphism group is finite. The notion of stable maps was introduced by Kontsevich [29].

The moduli space $\overline{M}_{g,n}(X, \beta)$ of $n$-pointed, genus $g$, degree $\beta$ stable maps to $X$ is a Deligne-Mumford stack which is proper when $X$ is projective [5].

### 3.3. Obstruction theory and virtual fundamental classes.

The tangent space $T^1$ and the obstruction space $T^2$ at a moduli point $[f : (C, x_1, \ldots, x_n) \to X] \in \overline{M}_{g,n}(X, \beta)$ fit in the tangent-obstruction exact sequence:

$$0 \to \text{Ext}^0_{\mathcal{O}_C}(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C) \to H^0(C, f^*T_X) \to T^1$$

$$\to \text{Ext}^1_{\mathcal{O}_C}(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C) \to H^1(C, f^*T_X) \to T^2 \to 0$$

(13)

where

- $\text{Ext}^0_{\mathcal{O}_C}(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C)$ is the space of infinitesimal automorphisms of the domain $(C, x_1, \ldots, x_n)$,
- $\text{Ext}^1_{\mathcal{O}_C}(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C)$ is the space of infinitesimal deformations of the domain $(C, x_1, \ldots, x_n)$,
- $H^0(C, f^*T_X)$ is the space of infinitesimal deformations of the map $f$, and
- $H^1(C, f^*T_X)$ is the space of obstructions to deforming the map $f$.

$T^1$ and $T^2$ form sheaves $T^1$ and $T^2$ on the moduli space $\overline{M}_{g,n}(X, \beta)$.

Let $X$ be a nonsingular projective variety. We say $X$ is convex if $H^1(C, f^*T_X) = 0$ for all genus 0 stable maps $f$. Projective spaces $\mathbb{P}^n$, or more generally, generalized flag varieties $G/P$, are examples of convex varieties. When $X$ is convex and $g = 0$, the obstruction sheaf $T^2 = 0$, and the moduli space $\overline{M}_{0,n}(X, \beta)$ is a smooth Deligne-Mumford stack.

In general, $\overline{M}_{g,n}(X, \beta)$ is a singular Deligne-Mumford stack equipped with a perfect obstruction theory: there is a two term complex of locally free sheaves $E \to F$ on $\overline{M}_{g,n}(X, \beta)$ such that

$$0 \to T^1 \to F^\vee \to E^\vee \to T^2 \to 0$$

is an exact sequence of sheaves. (See [4] for the complete definition of a perfect obstruction theory.) The virtual dimension $d^\text{vir}$ of $\overline{M}_{g,n}(X, \beta)$ is the rank of the virtual
tangent bundle $T^{\text{vir}} = F^v - E^v$.

(14) $d^{\text{vir}} = \int_{\beta} c_1(T_X) + (\dim X - 3)(1 - g) + n$

Suppose that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is proper. (Recall that if $X$ is projective then $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is proper for any $g$, $n$, $\beta$.) Then there is a virtual fundamental class

$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \in A_{g,n}(\overline{\mathcal{M}}_{g,n}(X, \beta); \mathbb{Q}).$

The virtual fundamental class has been constructed by Li-Tian [31], Behrend-Fantechi [4] in algebraic Gromov-Witten theory. The virtual fundamental class allows us to define

$\int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} : A^*(\overline{\mathcal{M}}_{g,n}(X, \beta)) \to \mathbb{Q}, \quad \alpha \mapsto \deg(\alpha \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}).$

3.4. **Gromov-Witten invariants.** Let $X$ be a nonsingular projective variety. Gromov-Witten invariants are rational numbers defined by applying

$\int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} : A^*(\overline{\mathcal{M}}_{g,n}(X, \beta)) \to \mathbb{Q}$

to certain classes in $A^*(\overline{\mathcal{M}}_{g,n}(X, \beta))$.

Let $ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ be the evaluation at the $i$-th marked point: $ev_i$ sends $[f : (C, x_1, \ldots, x_n) \to X] \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ to $f(x_i) \in X$. Given $\gamma_1, \ldots, \gamma_n \in A^*(X)$, define

(15) $\langle \gamma_1, \ldots, \gamma_n \rangle_{g, \beta}^X = \int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} ev_1^* \gamma_1 \cup \cdots \cup ev_n^* \gamma_n \in \mathbb{Q}.$

These are known as the primary Gromov-Witten invariants of $X$. More generally, we may also view $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}$ as a class in $H_{2d}(\overline{\mathcal{M}}_{g,n}(X, \beta))$. Then (15) is defined for ordinary cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(X)$, including odd cohomology classes which do not come from $A^*(\overline{\mathcal{M}}_{g,n}(X, \beta))$.

Let $\pi : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ be the universal curve. For $i = 1, \ldots, n$, let $s_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n+1}(X, \beta)$ be the section which corresponds to the $i$-th marked point. Let $\omega_\pi \to \overline{\mathcal{M}}_{g,n+1}(X, \beta)$ be the relative dualizing sheaf of $\pi$, and let $L_i = s_i^* \omega_\pi$ be the line bundle over $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber at the moduli point $[f : (C, x_1, \ldots, x_n) \to X] \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the cotangent line $T_{x_i}^* C$ at the $i$-th marked point $x_i$. The $\psi$-classes are defined to be

$\psi_i := c_1(L_i) \in A^1(\overline{\mathcal{M}}_{g,n}(X, \beta)), \quad i = 1, \ldots, n.$

We use the same notation $\psi_i$ to denote the corresponding classes in the ordinary cohomology group $H^2(\overline{\mathcal{M}}_{g,n}(X, \beta))$.

Genus $g$, degree $\beta$ descendant Gromov-Witten invariants of $X$ are defined by

(16) $\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g, \beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} ev_1^* \gamma_1 \cup \psi_1^{a_1} \cup \cdots \cup ev_n^* \gamma_n \cup \psi_n^{a_n} \in \mathbb{Q}.$

Suppose that $\gamma_i \in H^{d_i}(X)$. Then (16) is zero unless

(17) $\sum_{i=1}^n (d_i + 2a_i - 2) = 2 \left( \int_{\beta} c_1(T_X) + (\dim X - 3)(1 - g) \right).$
3.5. Equivariant Gromov-Witten invariants. Let \( X \) be a non-singular projective or quasi-projective algebraic variety, equipped with an algebraic action of \( T = (\mathbb{C}^*)^m \). Then \( T \) acts on \( \overline{M}_{g,n}(X,\beta) \) by

\[
t \cdot [f : (C, x_1, \ldots, x_n) \to X] \mapsto [t \cdot f : (C, x_1, \ldots, x_n) \to X]
\]

where \((t \cdot f)(z) = t \cdot f(z), z \in \mathbb{C}\). The evaluation maps \( \text{ev}_i : \overline{M}_{g,n}(X,\beta) \to X \) are \( T \)-equivariant and induce \( \text{ev}_i^* : A_T^*(X;\mathbb{Q}) \to A_T^*(\overline{M}_{g,n}(X,\beta);\mathbb{Q}) \).

Suppose that \( \overline{M}_{g,n}(X,\beta) \) is proper, so that there are virtual fundamental classes \( \overline{M}_{g,n}(X,\beta) \) \( \text{vir} \) \( A_{d_1}^*(\overline{M}_{g,n}(X,\beta);\mathbb{Q}), \overline{M}_{g,n}(X,\beta) \) \( \text{vir} \) \( A_{d_2}^*(\overline{M}_{g,n}(X,\beta);\mathbb{Q}), \) where

\[
d_{\text{vir}} = \int_{\beta} c_1(TX) + (r - 3)(1 - g) + n.
\]

Given \( \gamma_i \in A_{d_i}^*(X;\mathbb{Q}) = H_{2d_i}^2(X;\mathbb{Q}) \) and \( a_i \in \mathbb{Z}_{\geq 0} \), define \( \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_X^{g,\beta} \) as in Section 5.4

\[
\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_X^{g,\beta} = \int_{\overline{M}_{g,n}(X,\beta) \text{vir}} \prod_{i=1}^n (\text{ev}_i^* \gamma_i \cup \psi_i^{a_i}) \in \mathbb{Q}.
\]

By definition, (18) is zero unless \( \sum_{i=1}^n d_i = d_{\text{vir}} \). In this case,

\[
\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_X^{g,\beta} = \int_{\overline{M}_{g,n}(X,\beta) \text{vir}} \prod_{i=1}^n (\text{ev}_i^* \gamma_i \cup (\psi_i^T)^{a_i})
\]

where \( \gamma_i^T \in A^d(T^*)^d(X) \) is any \( T \)-equivariant lift of \( \gamma_i \in A_{d_i}^*(X) \), and \( \psi_i^T \in A_T^*(\overline{M}_{g,n}(X,\beta)) \) is any \( T \)-equivariant lift of \( \psi_i \in A^1(\overline{M}_{g,n}(X,\beta)) \).

In this paper, we fix a choice of \( \psi_i^T \) as follows. A stable map \( f : (C, x_1, \ldots, x_n) \to X \) induces \( C \)-linear maps \( T_x C \to T_{f(x)}X \) for \( i = 1, \ldots, n \). This gives rise to \( L_i \to \text{ev}_i^* TX \). The \( T \)-action on \( X \) induces a \( T \)-action on \( TX \), so that \( TX \) is a \( T \)-equivariant vector bundle over \( X \), and \( \text{ev}_i^* TX \) is a \( T \)-equivariant vector bundle over \( \overline{M}_{g,n}(X,\beta) \). Let \( T \) act on \( L_i \) such that \( L_i^\vee \to \text{ev}_i^* TX \) is \( T \)-equivariant, and define

\[
\psi_i^T = c_1^T(L_i) \in A_T^*(\overline{M}_{g,n}(X,\beta)), \quad i = 1, \ldots, n.
\]

Then \( \psi_i^T \) is a \( T \)-equivariant lift of \( \psi_i = c_1(L_i) \in A^1(\overline{M}_{g,n}(X,\beta)) \).

Given \( \gamma_i^T \in A^d(T^*)^d(X) \), we define genus \( g \), degree \( \beta \) \( T \)-equivariant descendant Gromov-Witten invariants

\[
\langle \tau_{a_1}(\gamma_1^T) \cdots \tau_{a_n}(\gamma_n^T) \rangle_X^{g,\beta} := \int_{\overline{M}_{g,n}(X,\beta) \text{vir}} \prod_{i=1}^n (\text{ev}_i^* \gamma_i \cup (\psi_i^T)^{a_i})
\]

\[
\in \mathbb{Q}[u_1, \ldots, u_m](\sum_{i=1}^n d_i - d_{\text{vir}}).
\]

where \( \mathbb{Q}[u_1, \ldots, u_m](k) \) is the space of degree \( k \) homogeneous polynomials in \( u_1, \ldots, u_m \) with rational coefficients. In particular,

\[
\langle \tau_{a_1}(\gamma_1^T) \cdots \tau_{a_n}(\gamma_n^T) \rangle_X^{g,\beta} = \begin{cases} 0, & \sum_{i=1}^n d_i < d_{\text{vir}}, \\ \langle \tau_{a_1}(\gamma_1), \cdots, \tau_{a_n}(\gamma_n) \rangle_X^{g,\beta} \in \mathbb{Q}, & \sum_{i=1}^n d_i = d_{\text{vir}}. \end{cases}
\]

where \( \gamma_i \in A_{d_i}^d(X;\mathbb{Q}) \) is the image of \( \gamma_i^T \) under \( A^d_T(X;\mathbb{Q}) \to A^d(X;\mathbb{Q}) \).
Let $\overline{\mathcal{M}}_{g,n}(X,\beta)^T \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$ be the substack of $T$-fixed points, and let $i : \overline{\mathcal{M}}_{g,n}(X,\beta)^T \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ be the inclusion. Let $N^\text{vir}$ be the virtual normal bundle of substack $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$; in general, $N^\text{vir}$ has different ranks on different connected components of $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$. By virtual localization,

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^\text{vir}} \prod_{i=1}^n \left( \text{ev}_i^* \gamma_i^T \cup (\psi_i^T)^{a_i} \right) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)^T]^\text{vir}} i^* \prod_{i=1}^n \left( \text{ev}_i^* \gamma_i^T \cup (\psi_i^T)^{a_i} \right) / e^i(N^\text{vir}).$$

If $\overline{\mathcal{M}}_{g,n}(X,\beta)^T$ is proper but $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not, we define

$$\langle \tau_{a_1}(\gamma_1^T), \ldots, \tau_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{X_T} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^\text{vir}} \prod_{i=1}^n \left( \text{ev}_i^* \gamma_i^T \cup (\psi_i^T)^{a_i} \right) / e^i(N^\text{vir}) \in \mathbb{Q}r.$$  

When $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not proper, the right hand side of (22) is a rational function (instead of a polynomial) in $u_1, \ldots, u_n$. It can be nonzero when $\sum d_i \neq d^\text{vir}$, and does not have a nonequivariant limit (obtained by setting $u_i = 0$) in general. 

### 4. Virtual Localization

In this section, we compute all genus equivariant descendant Gromov-Witten invariants of any algebraic GKM manifold by virtual localization. This generalizes the toric case in Section 5.

Let $X$ be an algebraic GKM manifold of dimension $r$, with an algebraic action of $T = (\mathbb{C}^*)^m$.

#### 4.1. Torus fixed points and graph notation.

In this subsection, we describe the $T$-fixed points in $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Following Kontsevich [29], given a stable map $f : (C, x_1, \ldots, x_n) \to X$ such that

$$[f : (C, x_1, \ldots, x_n) \to X] \in \overline{\mathcal{M}}_{g,n}(X,\beta)^T,$$

we will associate a decorated graph $\Gamma$.

We first give a formal definition.

**Definition 23.** A decorated graph $\tilde{\Gamma} = (\Gamma, \tilde{f}, \tilde{d}, \tilde{g}, \tilde{s})$ for $n$-pointed, genus $g$, degree $\beta$ stable maps to $X$ consists of the following data.

1. $\Gamma$ is a compact, connected 1 dimensional CW complex. We denote the set of vertices (resp. edges) in $\Gamma$ by $V(\Gamma)$ (resp. $E(\Gamma)$). Let

$$F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) \mid v \in e\}$$

be the set of flags in $\Gamma$.

2. The label map $\tilde{f} : V(\Gamma) \cup E(\Gamma) \to V(Y) \cup E(Y)_c$ sends a vertex $v \in V(\Gamma)$ to a vertex $\sigma_v \in V(Y)$, and sends an edge $e \in E(\Gamma)$ to an edge $e_e \in E(Y)_c$. Moreover, $\tilde{f}$ defines a map from the graph $\Gamma$ to the graph $Y$: if $(e, v)$ is a flag in $\Gamma$ then $(e_v, \sigma_v)$ is a flag in $Y$.

3. The degree map $\tilde{d} : E(\Gamma) \to \mathbb{Z}_{>0}$ sends an edge $e \in E(\Gamma)$ to a positive integer $d_e$.

4. The genus map $\tilde{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ sends a vertex $v \in V(\Gamma)$ to a non-negative integer $g_v$.

5. The marking map $\tilde{s} : \{1, 2, \ldots, n\} \to V(\Gamma)$ is defined if $n > 0$.

The above maps satisfy the following two constraints:
(i) (topology of the domain) \[ \sum_{v \in V(\Gamma)} g_v + |E(\Gamma)| - |V(\Gamma)| + 1 = g. \]

(ii) (topology of the map) \[ \sum_{e \in E(\Gamma)} d_e[\ell_e] = \beta. \]

Let \( G_{g,n}(X, \beta) \) be the set of all decorated graphs \( \Gamma = (\Gamma, f, \tilde{d}, \tilde{g}, \tilde{s}) \) satisfying the above constraints.

We now describe the geometry and combinatorics of a stable map \( f : (C, x_1, \ldots, x_n) \to X \) which represents a \( T \)-fixed point in \( \overline{M}_{g,n}(X, \beta) \).

For any \( t \in T \), there exists an automorphism \( \phi_t : (C, x_1, \ldots, x_n) \to (C', x_1, \ldots, x_n) \) such that \( t \cdot f(z) = f \circ \phi_t(z) \) for any \( z \in C \). Let \( C' \) be an irreducible component of \( C \), and let \( f' = f|_{C'} : C' \to X \). There are two possibilities:

Case 1: \( f' \) is a constant map, and \( f(C') = \{ p_\sigma \} \), where \( p_\sigma \) is a fixed point in \( X \) associated to some \( \sigma \in V(Y) \).

Case 2: \( C' \cong \mathbb{P}^1 \) and \( f(C') = \ell_e \), where \( \ell_e \) is a \( T \)-invariant \( \mathbb{P}^1 \) in \( X \) associated to some \( e \in E(Y)_c \).

We define a decorated graph \( \Gamma \) associated to \( f : (C, x_1, \ldots, x_n) \to X \) as follows.

1. (Vertices) We assign a vertex \( v \) to each connected component \( C_v \) of \( f^{-1}(X) \).
   a. (label) \( f(C_v) = \{ p_\sigma \} \) for some \( \sigma \in V(Y) \); we define \( \tilde{f}(v) = \sigma_v = \sigma \).
   b. (genus) \( C_v \) is a curve or a point. If \( C_v \) is a curve then we define \( \tilde{g}(v) = g_v \) to be the arithmetic genus of \( C_v \); if \( C_v \) is a point then we define \( \tilde{g}(v) = g_v = 0 \).
   c. (marking) For \( i = 1, \ldots, n \), define \( \tilde{s}(i) = v \) if \( x_i \in C_v \).

2. (Edges) For any \( e \in E(Y) \), let \( O_e \cong C^* \) be the 1-dimensional orbit whose closure is \( \ell_e \). Then \( X^1 \setminus X^T = \bigsqcup_{e \in E(Y)} O_e \) where the right hand side is a disjoint union of connected components. We assign an edge \( e \) to each connected component \( O_e \cong C^* \) of \( f^{-1}(X^1 \setminus X^T) \).
   a. (label) Let \( C_e \cong \mathbb{P}^1 \) be the closure of \( O_e \). Then \( f(C_e) = \ell_e \) for some \( e \in E(Y)_c \); we define \( \tilde{f}(e) = \epsilon_e = e \).
   b. (degree) We define \( \tilde{d}(e) = d_e \) to be the degree of the map \( f|_{C_e} : C_e \cong \mathbb{P}^1 \to \ell_e \cong \mathbb{P}^1 \).

3. (Flags) The set of flags in the graph \( \Gamma \) is defined by \( F(\Gamma) = \{ (e, v) \in E(\Gamma) \times V(\Gamma) \mid C_e \cap C_v \neq \emptyset \} \).

The above (1), (2), (3) define a decorated graph \( \Gamma = (\Gamma, f, \tilde{d}, \tilde{g}, \tilde{s}) \) satisfying the constraints (i) and (ii) in Definition 22. Therefore \( \Gamma \in G_{g,n}(X, \beta) \). This gives a map from \( \overline{M}_{g,n}(X, \beta)^T \) to the discrete set \( G_{g,n}(X, \beta) \). Let \( F_\Gamma \subset \overline{M}_{g,n}(X, \beta)^T \) denote the preimage of \( \Gamma \). Then \( \overline{M}_{g,n}(X, \beta)^T = \bigsqcup_{\Gamma \in G_{g,n}(X, \beta)} F_\Gamma \) where the right hand side is a disjoint union of connected components. We next describe the fixed locus \( F_\Gamma \) associated to each decorated graph \( \Gamma \in G_{g,n}(X, \beta) \). For later convenience, we introduce some definitions.
Definition 24. Given a vertex \( v \in V(\Gamma) \), we define
\[
E_v = \{ e \in E(\Gamma) \mid (e, v) \in F(\Gamma) \},
\]
the set of edges emanating from \( v \), and define \( S_v = s^{-1}(v) \subset \{1, \ldots, n\} \). The valency of \( v \) is given by \( \text{val}(v) = |E_v| \). Let \( n_v = |S_v| \) be the number of marked points contained in \( C_v \). We say a vertex is stable if \( 2g_v - 2 + \text{val}(v) + n_v > 0 \). Let \( V^S(\Gamma) \) be the set of stable vertices in \( V(\Gamma) \). There are three types of unstable vertices:
\[
\begin{align*}
V^1(\Gamma) &= \{ v \in V(\Gamma) \mid g_v = 0, \text{val}(v) = 1, n_v = 0 \}, \\
V^{1,1}(\Gamma) &= \{ v \in V(\Gamma) \mid g_v = 0, \text{val}(v) = n_v = 1 \}, \\
V^2(\Gamma) &= \{ v \in V(\Gamma) \mid g_v = 0, \text{val}(v) = 2, n_v = 0 \}.
\end{align*}
\]
Then \( V(\Gamma) \) is the disjoint union of \( V^1(\Gamma) \), \( V^{1,1}(\Gamma) \), \( V^2(\Gamma) \), and \( V^S(\Gamma) \).

The set of stable flags is defined to be
\[
F^S(\Gamma) = \{ (e, v) \in F(\Gamma) \mid v \in V^S(\Gamma) \}.
\]

Given a decorated graph \( \vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s}) \), the curves \( C_e \) and the maps \( f|_{C_e} : C_e \to \ell_e, \subset X \) are determined by \( \vec{\Gamma} \). If \( v \notin V^S(\Gamma) \) then \( C_v \) is a point. If \( v \in V^S(\Gamma) \) then \( C_v \) is a curve, and \( y(e, v) := C_e \cap C_v \) is a node of \( C \) for \( e \in E_v \).
\[
(C_v, \{ y(e, v) \mid e \in E_v \} \cup \{ x_i \mid i \in S_v \})
\]
is a \( (\text{val}(v) + n_v) \)-pointed, genus \( g_v \) curve, which represents a point in \( \overline{M}_{g_v, \text{val}(v) + n_v} \).

We call this moduli space \( \overline{M}_{g_v, E_v \cup S_v} \) instead of \( \overline{M}_{g_v, \text{val}(v) + n_v} \) because we would like to label the marked points on \( C_v \) by \( E_v \cup S_v \) instead of \( \{1, 2, \ldots, \text{val}(v) + n_v\} \).

Then
\[
\mathcal{M}_\vec{\Gamma} = \prod_{v \in V^S(\Gamma)} \overline{M}_{g_v, E_v \cup S_v}.
\]

The automorphism group \( A_\vec{\Gamma} \) for any point \( [f : (C, x_1, \ldots, x_n) \to X] \in F_\vec{\Gamma} \) fits in the following short exact sequence of groups:
\[
1 \to \prod_{e \in E(\Gamma)} \mathbb{Z}_{d_e} \to A_\vec{\Gamma} \to \text{Aut}(\vec{\Gamma}) \to 1
\]
where \( \mathbb{Z}_{d_e} \) is the automorphism group of the degree \( d_e \) morphism
\[
f|_{C_e} : C_e \cong \mathbb{P}^1 \to \ell_e, \cong \mathbb{P}^1,
\]
and \( \text{Aut}(\vec{\Gamma}) \) is the automorphism group of the decorated graph \( \vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s}) \).

There is a morphism \( i_{\vec{\Gamma}} : \mathcal{M}_\vec{\Gamma} \to \overline{M}_{g, n}(X, \beta) \) whose image is the fixed locus \( F_\vec{\Gamma} \) associated to \( \vec{\Gamma} \in G_{g, n}(X, \beta) \). The morphism \( i_{\vec{\Gamma}} \) induces an isomorphism \( [\mathcal{M}_\vec{\Gamma} / A_\vec{\Gamma}] \cong F_\vec{\Gamma} \).

4.2. Virtual tangent and normal bundles. Given a decorated graph \( \vec{\Gamma} \in G_{g, n}(X, \beta) \) and a stable map \( f : (C, x_1, \ldots, x_n) \to X \) which represents a point in the fixed locus \( F_{\vec{\Gamma}} \) associated to \( \vec{\Gamma} \), let
\[
\begin{align*}
B_1 &= \text{Hom}(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C), \\
B_2 &= H^0(C, f^*TX) \\
B_4 &= \text{Ext}^1(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C), \\
B_5 &= H^1(C, f^*TX)
\end{align*}
\]
We have the following exact sequences:

\[(25) \quad 0 \to B_1^f \to B_2^f \to T^1 f \to B_4^f \to B_5^f \to T^2 f \to 0\]

\[(26) \quad 0 \to B_1^m \to B_2^m \to T^1 m \to B_4^m \to B_5^m \to T^2 m \to 0\]

The irreducible components of $C$ are

\[\{C_v \mid v \in V^S(\Gamma)\} \cup \{C_e \mid e \in E(\Gamma)\}.\]

The nodes of $C$ are

\[\{y_v = C_v \mid v \in V^2(\Gamma)\} \cup \{(e, v) \mid (e, v) \in F^S(\Gamma)\}\]

4.2.1. Automorphisms of the domain. Given any $(e, v) \in F(\Gamma)$, let $y(e, v) = C_e \cap C_v$, and define

\[w_{(e, v)} := e^T(T_y(e, v)C_e) = \frac{w(e, v)}{d_e} \in H^2_T(y(e, v); \mathbb{Q}) = M \otimes \mathbb{Z} \mathbb{Q}.\]

We have

\[B_1^f = \bigoplus_{\gamma \in \gamma(\Gamma)} \text{Hom}(\Omega_{\gamma, v}(y(e, v) + y(e, v')), \mathcal{O}_{\gamma})\]

\[= \bigoplus_{\gamma \in \gamma(\Gamma)} H^0(C_e, T_{\gamma}(-y(e, v) - y(e, v'))\]

\[B_1^m = \bigoplus_{y \in V^1(\Gamma), (e, v) \in F(\Gamma)} y_{(e, v)}C_e\]

4.2.2. Deformations of the domain. Given any $v \in V^S(\Gamma)$, define a divisor $x_v$ of $C_v$ by

\[x_v = \sum_{i \in S_v} x_i + \sum_{e \in E_v} y(e, v).\]

Then

\[B_4^f = \bigoplus_{v \in V^S(\Gamma)} \text{Ext}^1(\Omega_{v, C_v}, \mathcal{O}_C) = \bigoplus_{v \in V^S(\Gamma)} T_{\gamma} v E_v \cup S_v\]

\[B_4^m = \bigoplus_{y \in V^2(\Gamma), E_v = \{e, v\}} T_y v C_e \otimes T_y v C_{e'} \bigoplus_{y(\Gamma) \in F(\Gamma)} T_y(e, v)C_v \otimes T_y(e, v)C_{e'}\]

where

\[e^T(T_y v C_e \otimes T_y v C_{e'}) = w_{(e, v)} + w_{(e', v)}, \quad v \in V^2(\Gamma)\]

\[e^T(T_y(e, v)C_v \otimes T_y(e, v)C_e) = w_{(e, v)} - \psi_{(e, v)}, \quad v \in V^S(\Gamma)\]
4.2.3. **Unifying stable and unstable vertices.** From the discussion in Section 4.2.1 and Section 4.2.2,

\[
\frac{e^T(B_m)}{e^T(B_m^4)} = \prod_{v \in V^1(\Gamma)} \frac{w(e,v)}{\prod_{v \in V^2(\Gamma), E_v = (e,e')}} \frac{1}{w(e,v) + w(e',v)} 
\]

(27)

\[
\cdots \prod_{v \in V^3(\Gamma)} \prod_{e \in E_v} \left( w(e,v) - \psi(e,v) \right).
\]

Recall that

\[
\mathcal{M}_{\Gamma} = \prod_{v \in V(\Gamma)} \mathcal{M}_{\mathcal{G}_v, E_v \cup S_v}.
\]

To unify the stable and unstable vertices, we use the following convention for the empty sets \(\mathcal{M}_{0,1}\) and \(\mathcal{M}_{0,2}\). Let \(w_1, w_2\) be formal variables.

(i) \(\mathcal{M}_{0,1}\) is a \(-2\) dimensional space, and

\[
\int_{\mathcal{M}_{0,1}} \frac{1}{w_1 - \psi_1} = w_1.
\]

(ii) \(\mathcal{M}_{0,2}\) is a \(-1\) dimensional space, and

\[
\int_{\mathcal{M}_{0,2}} \frac{1}{w_1 - \psi_1} = \int_{\mathcal{M}_{0,2}} (w_1 - \psi_1)(w_2 - \psi_2) = \frac{1}{w_1 + w_2}.
\]

(iii) \(\mathcal{M}_{\Gamma} = \prod_{v \in V(\Gamma)} \mathcal{M}_{\mathcal{G}_v, E_v \cup S_v}\).

With the above conventions (i), (ii), (iii), we may rewrite (27) as

\[
\frac{e^T(B_m)}{e^T(B_m^4)} = \prod_{v \in V(\Gamma)} \prod_{e \in E_v} \left( w(e,v) - \psi(e,v) \right).
\]

The following lemma shows that the conventions (i) and (ii) are consistent with the stable case \(\mathcal{M}_{0,n}, n \geq 3\).

**Lemma 32.** For any positive integer \(n\) and formal variables \(w_1, \ldots, w_n\), we have

(a) \(\int_{\mathcal{M}_{0,n}} \prod_{i=1}^n (w_i - \psi_i) = \frac{1}{w_1 \cdots w_n} \left( \frac{1}{w_1} + \cdots + \frac{1}{w_n} \right)^{n-3} \).

(b) \(\int_{\mathcal{M}_{0,n}} \frac{1}{w_1 - \psi_1} = w_1^{2-n} \).

**Proof.** (a) The cases \(n = 1\) and \(n = 2\) follow from the definitions (28) and (29), respectively. For \(n \geq 3\), we have

\[
\int_{\mathcal{M}_{0,n}} \prod_{i=1}^n (w_i - \psi_i) = \frac{1}{w_1 \cdots w_n} \int_{\mathcal{M}_{0,n}} \prod_{i=1}^n (1 - \psi_i) = \frac{1}{w_1 \cdots w_n} \sum_{a_1 + \cdots + a_n = n-3} w_1^{-a_1} \cdots w_n^{-a_n} \int_{\mathcal{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}
\]

where

\[
\int_{\mathcal{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \frac{(n-3)!}{a_1! \cdots a_n!}.
\]
So
\[ \int_{\mathcal{M}_{0,n}} \frac{1}{w_1 - \psi_1} = \frac{1}{w_1} \cdot \cdots \cdot \frac{1}{w_n} \left( \frac{1}{w_1} + \cdots + \frac{1}{w_n} \right)^{n-3}. \]

(b) The cases \( n = 1 \) and \( n = 2 \) follow from the definitions \((28)\) and \((30)\), respectively. For \( n \geq 3 \), we have
\[ \int_{\mathcal{M}_{0,n}} \frac{1}{w_1 - \psi_1} = \frac{1}{w_1} \int_{\mathcal{M}_{0,n}} \frac{1}{w_1 - \psi_1} = \frac{1}{w_4} \left( \frac{1}{w_1} ight)^{2-n} = \frac{w^{2-n}}{w_1^n}. \]

\[ \square \]

4.2.4. Deformation of the map. Consider the normalization sequence
\[ 0 \to \mathcal{O}_C \to \bigoplus_{v \in V^2(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{C_e} \]
\[ \to \bigoplus_{v \in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v) \in F^2(\Gamma)} \mathcal{O}_{y(e,v)} \to 0. \]

We twist the above short exact sequence of sheaves by \( f^* TX \). The resulting short exact sequence gives rise a long exact sequence of cohomology groups
\[ 0 \to B_2 \to \bigoplus_{v \in V^2(\Gamma)} H^0(C_v) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e) \]
\[ \to \bigoplus_{v \in V^2(\Gamma)} T_{f(y_v)} X \oplus \bigoplus_{(e,v) \in F^2(\Gamma)} T_{f(y(e,v))} X \]
\[ \to B_5 \to \bigoplus_{v \in V^2(\Gamma)} H^1(C_v) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e) \to 0. \]

where
\[ H^i(C_v) = H^i(C_v, (f|_{C_v})^*TX) \cong H^i(C_v, \mathcal{O}_{C_v}) \otimes T_{p_v} X, \]
\[ H^i(C_e) = H^i(C_e, (f|_{C_e})^*TX) \]
for \( i = 0, 1 \). We have
\[ H^0(C_v) = T_{p_v} X \]
\[ H^1(C_v) = H^0(C_v, \omega_{C_v})^\vee \otimes T_{p_v} X. \]

Lemma 34. Let \( \sigma \in V(Y) \), so that \( p_\sigma \) is a \( T \)-fixed point in \( X \). Define
\[ w(\sigma) = e^T(T_{p_\sigma} X) \in H_2^T(\text{point}; \mathbb{Q}) \]
\[ h(\sigma, g) = \frac{e^T(E^\vee \otimes T_{p_\sigma} X)}{e^T(T_{p_\sigma} X)} \in H_2^{2g-1}(N_{g,n}; \mathbb{Q}). \]

Then
\[ w(\sigma) = \prod_{e \in E(\sigma)} w(e, \sigma). \]
\[ h(\sigma, g) = \prod_{e \in E(\sigma)} \Lambda_{g}^\vee(w(e, \sigma)) \]
\[ \Lambda_{g}^\vee(u) = \sum_{i=0}^{g} (-1)^i \lambda_i u^i. \]
Proof. $T_{p_v}X = \bigoplus_{e \in E_v} T_{p_v} \ell_e$, where $e^T(T_{p_v} \ell_e) = w(e, \sigma)$. So

$$e^T(T_{p_v}) = \prod_{e \in E_v} w(e, \sigma),$$

$$\frac{e^T(E^\vee \otimes T_{p_v} \ell_e)}{e^T(T_{p_v} \ell_e)} = \prod_{e \in E_v} \frac{e^T(E^\vee \otimes T_{p_v} \ell_e)}{w(e, \sigma)},$$

where

$$e^T(E^\vee \otimes T_{p_v} \ell_e) = \sum_{i=0}^g (-1)^i c_i(E) e^T(T_{p_v} \ell_e)^{g-i} = \sum_{i=0}^g (-1)^i \lambda_i w(e, \sigma)^{g-i}.$$

The map $B_1 \to B_2$ sends $H^0(C_\epsilon, T_{C_\epsilon}(-y(e, v) - y(e', v)))$ isomorphically to $H^0(C_\epsilon, (f|_{C_\epsilon})^* T_{C_\epsilon})$, the fixed part of $H^0(C_\epsilon, (f|_{C_\epsilon})^* T_{C_\epsilon})$.

Lemma 37. Given $d \in \mathbb{Z}_{\geq 0}$ and $e \in E(Y)_c$, define $\sigma, \sigma', e_i, e'_j, a_i$ as in Section 2.2 and let $f_d : \mathbb{P}^1 \to \ell_e \cong \mathbb{P}^1$ be the unique degree $d$ map totally ramified over the two $T$-fixed points $p_v$ and $p_{v'}$ in $\ell_e$. Define

$$h(e, d) = \frac{e^T(H^1(\mathbb{P}^1, f_d^* TX)^m)}{e^T(H^0(\mathbb{P}^1, f_d^* TX)^m)}.$$

Then

$$h(e, d) = \frac{(-1)^d d 2d}{(d!)^2 \lambda w(e, \sigma) 2d} \prod_{i=1}^{r-1} b(w(e, \sigma), w(e_i, \sigma), da_i).$$

where

$$b(u, w, a) = \begin{cases} \prod_{j=0}^g (w - ju)^{-1}, & a \in \mathbb{Z}, a \geq 0, \\ \prod_{j=1}^{d-1} (w + ju), & a \in \mathbb{Z}, a < 0. \end{cases}$$

Proof. We use the notation in Section 2.2. We have

$$N_{l_e/X} = L_1 \oplus \cdots \oplus L_{r-1}.$$ 

The weights of $T$-actions on $(L_i)_{p_v}$ and $(L_i)_{p_v}$ are $w(e_i, \sigma)$ and $w(e_i, \sigma) - a_i w(e, \sigma)$, respectively. The weights of $T$-actions on $T_0 \mathbb{P}^1$, $T_\infty \mathbb{P}^1$, $(f_d^* L_i)_0$, $(f_d^* L_i)_\infty$ are $u := w(e, \sigma) - u$, $w_i := w(e_i, \sigma)$, $w_i - da_i u$, respectively. By Example 33 Example 19,

$$\text{ch}_T(H^0(\mathbb{P}^1, f_d^* L_i) - H^1(\mathbb{P}^1, f_d^* L_i)) = \begin{cases} \sum_{j=0}^{d} e^{w_i - ju}, & a_i \geq 0, \\ \sum_{j=1}^{d-1} e^{w_i + ju}, & a_i < 0. \end{cases}$$

Note that $w_i + ju$ is nonzero for any $j \in \mathbb{Z}$ since $w_i$ and $u$ are linearly independent for $i = 1, \ldots, n - 1$. So

$$\frac{e^T(H^1(\mathbb{P}^1, f_d^* L_i))}{e^T(H^0(\mathbb{P}^1, f_d^* L_i))} = \frac{e^T(H^1(\mathbb{P}^1, f_d^* L_i))^m}{e^T(H^0(\mathbb{P}^1, f_d^* L_i))^m} = b(u, w_i, da_i),$$

where $b(u, w, a)$ is defined by (39). By Example 33 Example 19,

$$\text{ch}_T(H^0(\mathbb{P}^1, f_d^* T_{\ell_e}) - H^1(\mathbb{P}^1, f_d^* T_{\ell_e})) = \sum_{j=0}^{d} e^{ju} = 1 + \sum_{j=1}^{d} e^{jw(e, \sigma)/d + e^{-jw(e, \sigma)/d}}.$$
We conclude that

\[ e^T(H^1(\mathbb{P}^1, f_d^*T\ell_e)^m) = \prod_{j=1}^d \frac{-d^2}{(d!)^2 w(e, \sigma)^{2d}} \] 

Therefore,

\[ e^T(H^1(\mathbb{P}^1, f_d^*TX)^m) = e^T(H^0(\mathbb{P}^1, f_d^*TX)^m) = \prod_{i=1}^{r-1} e^T(H^1(\mathbb{P}^1, f_d^*L_i)^m) \]

\[ = \frac{(-1)^d d^{2d}}{(d!)^2 w(e, \sigma)^{2d}} \prod_{i=1}^{r-1} b(\frac{w(e, \sigma)}{d}, w(e, \sigma), da_i) . \]

Finally, \( f(y_v) = p_{\tau_v} = f(y(e, v)) \), and

\[ e^T(T_{p_{\tau_v}}X) = w(\tau_v). \]

From the above discussion, we conclude that

\[ e^T(B_2^m) = \prod_{v \in V^2(\Gamma)} w(\sigma_v) \cdot \prod_{(e, v) \in E^2(\Gamma)} w(\sigma_v) \cdot \prod_{v \in V(\Gamma)} h(\sigma_v, g_v) \cdot \prod_{(e, \sigma_v) \in E(\Gamma)} h(e_v, d_e) \]

\[ \prod_{v \in V(\Gamma)} (h(\sigma_v, g_v) \cdot w(\sigma_v)^\text{val}(v)) \cdot \prod_{e \in E(\Gamma)} h(e_v, d_e) \]

where \( w(\sigma), h(\sigma, g) \), and \( h(e, d) \) are defined by (35), (36), (38), respectively.

4.3. Contribution from each graph.

4.3.1. Virtual tangent bundle. We have \( B_4^f = B_2^f, B_3^f = 0 \). So

\[ T^1f = B_2^f = \bigoplus_{v \in V^2(\Gamma)} T_{p_{\tau_v}} \mathcal{M}_{g_v, E_v \cup S_{\tau_v}} \]

\[ T^2f = 0. \]

We conclude that

\[ \left[ \prod_{v \in V^2(\Gamma)} \mathcal{M}_{g_v, E_v \cup S_{\tau_v}} \right]_{\text{vir}} = \prod_{v \in V^2(\Gamma)} [\mathcal{M}_{g_v, E_v \cup S_{\tau_v}}]. \]

4.3.2. Virtual normal bundle. Let \( N_{\mathcal{F}_1}^{\text{vir}} \) be the pull back of the virtual normal bundle of \( \mathcal{F}_1 \) in \( \mathcal{M}_{g, \beta}(X, \beta) \) under \( i_{\mathcal{F}_1} : \mathcal{M}_{\mathcal{F}_1} \to \mathcal{F}_1 \). Then

\[ \frac{1}{e^T(N_{\mathcal{F}_1}^{\text{vir}})} = e^T(B_2^m) e^T(B_2^m) = \prod_{v \in V(\Gamma)} h(\sigma_v, g_v) \cdot w(\sigma_v)^\text{val}(v) \cdot \prod_{e \in E(\Gamma)} h(e_v, d_e) \]

4.3.3. Integrand. Given \( \tau \in V(Y) \), let

\[ i_{\tau}^\tau : A^\tau_+(X) \to A^\tau_+(p_{\tau}) = Q[u_1, \ldots, u_t] \]

be induced by the inclusion \( i_{\tau} : p_{\tau} \to X \). Then

\[ i_{\tau}^\tau = \prod_{i=1}^N \left( \text{ev}_{\tau}^\tau, i_{\tau}^\tau \right) \]

(40)

\[ i_{\tau}^\tau \cdot \bigcup_{v \in V^1(E)} \left( \prod_{i \in S_{\tau v}} i_{\tau v}^{-1} \gamma_i^{\tau} (-w(e, v)^{\partial_i}) \right) \]

\[ \bigcup_{v \in V^2(\Gamma)} \left( \prod_{i \in E_v} i_{\tau v}^{-1} \gamma_i^{\tau} \prod_{e \in E_v} \psi(e, \tau) \right) \]
To unify the stable vertices in $V^S(\Gamma)$ and the unstable vertices in $V^{1,1}(\Gamma)$, we use the following convention: for $a \in \mathbb{Z}_{\geq 0},$

\begin{equation}
\int_{\mathcal{M}_{0,2}} \frac{\psi_2^a}{w_1 - \psi_1} = (-w_1)^a. \tag{41}
\end{equation}

In particular, (30) is obtained by setting $a = 0$. With the convention (41), we may rewrite (40) as

\begin{equation}
\left[ i^* \prod_{i=1}^n \left( \text{ev}_i^* \gamma_i^T \cup (\psi_i^T)^{a_i} \right) \right] = \prod_{v \in V(\Gamma)} \left( \prod_{i \in S_v} i_{v,i}^* \gamma_i^T \prod_{e \in E_v} \psi_{e,v}^{a_i} \right). \tag{42}
\end{equation}

The following lemma shows that the convention (41) is consistent with the stable case $\mathcal{M}_{0,n}, n \geq 3.$

**Lemma 43.** Let $n, a$ be integers, $n \geq 2, a \geq 0.$ Then

\begin{equation}
\int_{\mathcal{M}_{0,n}} \frac{\psi_2^a}{w_1 - \psi_1} = \begin{cases} 
\prod_{i=0}^{a-1} \frac{(n - 3 - i)}{a!} w_1^{a + 2 - a} & \text{if } n = 2 \text{ or } 0 \leq a \leq n - 3, \\
0 & \text{otherwise.}
\end{cases} \tag{43}
\end{equation}

**Proof.** The case $n = 2$ follows from (41). For $n \geq 3,$

\begin{align*}
\int_{\mathcal{M}_{0,n}} \frac{\psi_2^a}{w_1 - \psi_1} &= \frac{1}{w_1} \int_{\mathcal{M}_{0,n}} \frac{\psi_2^a}{1 - \frac{\psi_1}{w_1}} = w_1^{a + 2 - n} \int_{\mathcal{M}_{0,n}} \psi_2^{n - 3 - a}\psi_2^a \\
&= w_1^{a + 2 - n} \frac{(n - 3)!}{(n - 3 - a)!a!} \frac{\prod_{i=0}^{a-1} (n - 3 - i)}{a!} w_1^{a + 2 - n}.
\end{align*}

\hfill \square

### 4.3.4. Integral

The contribution of

\begin{equation}
\int_{\mathcal{M}_{0,n}(X, \beta)[\text{Vir}, \Gamma]} i^* \prod_{i=1}^n \left( \text{ev}_i^* \gamma_i^T \cup (\psi_i^T)^{a_i} \right) e^\gamma(N^{\text{vir}})
\end{equation}

from the fixed locus $\mathcal{F}_F$ is given by

\begin{equation}
\frac{1}{|A_{\Gamma}|} \prod_{e \in E(\Gamma)} \text{h}(\sigma_e, d_e) \prod_{v \in V(\Gamma)} \left( w(\sigma_v)^{\text{val}(v)} \prod_{i \in S_v} i_{v,i}^* \gamma_i^T \right) \cdot \prod_{v \in V(\Gamma)} \int_{\mathcal{M}_{0,n}(E^v \cup S_v)} \frac{\text{h}(\sigma_v, \xi_v \cdot \prod_{e \in E_v} \psi_{e,v}^{a_i})}{\prod_{e \in E_v} (w(\xi_e) - \psi_{e,v})}
\end{equation}

where $|A_{\Gamma}| = |\text{Aut}(\Gamma)| \cdot \prod_{e \in E(\Gamma)} d_e.$

### 4.4. Sum over graphs

Summing over the contribution from each graph $\Gamma$ given in Section 4.3.4 above, we obtain the following formula.
Theorem 44.

\[
\langle \tau_{a_1}(\gamma_1^T) \cdots \tau_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{X_T}
= \sum_{\bar{\Gamma} \in G_{g,n}(X,\beta)} \frac{1}{|\text{Aut}(\bar{\Gamma})|} \prod_{e \in E(\Gamma)} \frac{h(e_\nu, d_e)}{d_e} \prod_{v \in V(\Gamma)} \left( w(\sigma_v)^{\text{val}(v)} \prod_{i \in S_v} i_{v_i}^{\gamma_i^T} \right)
\cdot \frac{1}{\prod_{v \in V(\Gamma)} \prod_{i \in S_v} \psi_i^{\beta_i}}
\]

where \(h(e, d), w(\sigma), h(\sigma, g)\) are given by (38), (35), (36), respectively, and we have the following convention for the \(v \notin V^S(\Gamma)\):

\[
\int_{\mathcal{M}_{0,1}^{g,\bar{\nu}}(w_1 - \psi_1)} = w_1, \quad \int_{\mathcal{M}_{0,2}^{g,\bar{\nu}}(w_1 - \psi_1)(w_2 - \psi_2)} = \frac{1}{w_1 + w_2},
\]

\[
\int_{\mathcal{M}_{0,2}^{g,\bar{\nu}}(w_1 - \psi_1)} = (-w_1)^a, \quad a \in \mathbb{Z}_{\geq 0}.
\]

Given \(g \in \mathbb{Z}_{\geq 0}, r\) weights \(\bar{\omega} = \{w_1, \ldots, w_r\}\), \(r\) partitions \(\bar{\mu} = \{\mu^1, \ldots, \mu^r\}\), and \(a_1, \ldots, a_k \in \mathbb{Z}\), let \(\ell(\mu^i)\) be the length of \(\mu^i\), and let \(\ell(\bar{\mu}) = \sum_{i=1}^r \ell(\mu^i)\). We define

\[
\langle \tau_{a_1}, \ldots, \tau_{a_k} \rangle_{g,\bar{\mu},\bar{\omega}} = \int_{\mathcal{M}_{g,\ell(\bar{\mu}) + k}^{\bar{\omega}}} \prod_{i=1}^r \left( \sum_{\gamma_i = 1}^{\mu^i - 1} \frac{1}{\left( \frac{w_i}{\psi_i} \right)^{\beta_i}} \right) \prod_{b=1}^k \psi_i^{\beta_i}.
\]

Given \(v \in V(\Gamma)\), define \(\bar{\omega}(v) = \{w(e, \sigma_v) \mid (e, \sigma_v) \in F(\bar{\Gamma})\}\). Given \(v \in V(\Gamma)\), and \(e \in E(\bar{\Gamma})\), let \(\mu^{e,\sigma}\) be a (possibly empty) partition defined by \(\{d_e \mid e \in E(\bar{\Gamma})\} = e\},

and define \(\bar{\mu}(v) = \{\mu^{e,\sigma} \mid (e, \sigma_v) \in F(\bar{\Gamma})\}\). Then (45) can be rewritten as

\[
\langle \tau_{a_1}(\gamma_1^T) \cdots \tau_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{X_T}
= \sum_{\bar{\Gamma} \in G_{g,n}(X,\beta)} \frac{1}{|\text{Aut}(\bar{\Gamma})|} \prod_{e \in E(\Gamma)} \frac{h(e_\nu, d_e)}{d_e} \prod_{v \in V(\Gamma)} \left( w(\sigma_v)^{\text{val}(v)} \prod_{i \in S_v} i_{v_i}^{\gamma_i^T} \right)
\cdot \prod_{v \in V(\Gamma)} \prod_{i \in S_v} \psi_i^{\beta_i}.
\]

Recall that

\[
g = \sum_{v \in V(\Gamma)} g_v + |E(\Gamma)| - |V(\Gamma)| + 1
\]

so

\[
2g - 2 = \sum_{v \in V(\Gamma)} (2g_v - 2 + \text{val}(v)).
\]

Given \(\bar{\Gamma} = (\Gamma, \bar{f}, \bar{d}, \bar{g}, \bar{\omega})\), let \(\bar{\Gamma}' = (\Gamma, f, d, \bar{g}, \bar{\omega})\) be the decorated graph obtained by forgetting the genus map. Let \(G_{n}(X, \beta) = \{\bar{\Gamma}' \mid \bar{\Gamma}' \in \bigcup_{g \geq 0} G_{g,n}(X, \beta)\}\). Define

\[
\langle \tau_{a_1}(\gamma_1^T), \ldots, \tau_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{X_T} = \sum_{g \geq 0} u^{2g - 2} \langle \tau_{a_1}(\gamma_1^T), \ldots, \tau_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{X_T}
\]

(48)

\[
\langle \tau_{a_1}, \ldots, \tau_{a_k} \rangle_{g,\beta,\bar{\omega}} = \sum_{g \geq 0} u^{2g - 2 + \ell(\bar{\mu})} \langle \tau_{a_1}, \ldots, \tau_{a_k} \rangle_{g,\bar{\mu},\bar{\omega}}.
\]

Then we have the following formula for the generating function (47).
Theorem 49.

\[
\left\{ \tau_{a_1} (\gamma_{T_1}) \cdots \tau_{a_n} (\gamma_{T_n}) \mid u \right\}_{\beta}^{X_T} = \sum_{\mathbf{\Gamma}' \in G_\beta(X, \beta)} \frac{1}{|\text{Aut}(\mathbf{\Gamma})|} \prod_{e \in E(\mathbf{\Gamma})} \frac{\mathbf{h}(\epsilon_e, d_e)}{d_e} \cdot \prod_{v \in V(\mathbf{\Gamma})} \left( \prod_{i \in S_v} \epsilon_i^T \right) \left( \prod_{j \in S_v} \tau_{a_j} \mid u \right) \bar{\mu}(v), \bar{\nu}(v). \]

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