Information Geometry of smooth densities on the Gaussian space: Poincaré inequalities*

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Abstract. We derive bounds for the Orlicz norm of the deviation of a random variable defined on $\mathbb{R}^n$ from its Gaussian mean value. The random variables are assumed to be smooth and the bound itself depends on the Orlicz norm of the gradient. Applications to non-parametric Information Geometry are discussed.

Keywords: Gaussian Poincaré-Wirtinger Inequality · Gaussian Space · Non-parametric Information Geometry · Orlicz Spaces

1 Introduction

In a series of papers [22,12,24,25], the idea of a non-parametric Information Geometry (IG), especially, IG of the Gaussian space has been explored. About the analysis of the Gaussian space, see, for example, [15,19]. This set-up provides a simple way of focusing on a manifold modeled on Banach spaces of smooth densities. Other modeling options are in fact available, for example the global analysis methods of [9], but we prefer to work with assumptions that allow for the use of classical infinite dimensional differential geometry modeled on Banach spaces as in [10].

The present note focuses on technical results about useful differential inequalities and does not consider in detail the applications to Statistics. However, we have in mind two main examples of application. The first one is the statistical estimation method based on Hyvärinen’s divergence,

$$DH(P|Q) = \frac{1}{2} \int |\nabla \log P(x) - \nabla \log Q(x)|^2 P(x) \, dx ,$$

(1)

where $P, Q$ are positive probability densities of the $n$-dimensional Lebesgue space, see in [7,12]. The second one is the Otto’s inner product [20,13], which is defined by

$$\langle f, g \rangle_P = \int \nabla f(x) \cdot \nabla g(x) P(x) \, dx ,$$

(2)

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where $P$ is a probability density and $f, g$ are smooth random variables such that $E_P[f] = E_P[g] = 0$.

We focus on the exponential representation of positive densities $P = p \cdot \gamma = e^{u - K(u)} \cdot \gamma$, where $\gamma$ is the standard Gaussian density. The sufficient statistics $u$ is assumed to belong to an exponential Orlicz space and $\int u(x) \gamma(x) \, dx = 0$. The set of all such couples $(p, u)$ is called statistical bundle. There are other ways to represent positive densities that use deformed exponential functions, $p \propto \exp_{\Lambda}$ and avoid the difficulty of the exponential growth and, for this reason, provide a somehow simpler treatment of smoothness, see [18,17]. We do not discuss here this interesting related approach.

This paper is organized as follows. In section 2 we provide a recap of basic facts about non-parametric IG and discuss the Gaussian case. The results about Poincaré-Wirtinger inequalities are gathered in section 3. This section contains the main contributions of the paper. A collection of simple examples of possible applications concludes the paper.

# 2 Statistical bundle modeled on Orlicz spaces

First, we review below the theory of Orlicz spaces in order to fix convenient notation. The full theory is offered, for example, in [16 Ch. II] and [1 Ch. VII].

## 2.1 Orlicz spaces

In this paper, we will need the following special type of Young function. Cf. [16 §7] for a more general case.

Assume $\phi \in C[0, +\infty]$ is such that: 1) $\phi(0) = 0$; 2) $\phi(u)$ is strictly increasing; 3) $\lim_{u \to +\infty} \phi(u) = +\infty$. The primitive function

$$\Phi(x) = \int_0^x \phi(u) \, du , \quad x \geq 0 ,$$

is strictly convex and will be called a Young function. Cf. [1 §8.2], where $\phi$ is assumed to be right-continuous and non-decreasing.

The inverse function $\psi = \phi^{-1}$ has the same properties 1) to 3) as $\phi$, so that its primitive

$$\Psi(y) = \int_0^y \psi(v) \, dv , \quad y \geq 0 ,$$

is again a Young function. The couple $(\Phi, \Psi)$, is a couple of conjugate Young functions. The relation is symmetric and we write both $\Psi = \Phi_*$ and $\Phi = \Psi_*$. The Young inequality holds true,

$$\Phi(x) + \Psi(y) \geq xy , \quad x, y \geq 0 ,$$

and the Legendre equality holds true,

$$\Phi(x) + \Psi(\phi(x)) = x\phi(x) , \quad x \geq 0 .$$
Here are the specific cases we are going to use:

\[
\Phi(x) = \frac{x^p}{p}, \quad \Psi(y) = \frac{y^q}{q}, \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1 ;
\]

(3)

\[
\exp_2(x) = e^x - 1 - x, \quad (\exp_2)_+(y) = (1 + y) \log(1 + y) - y ;
\]

(4)

\[
(cosh - 1)(x) = \cosh x - 1, \quad (cosh - 1)_+(y) = \int_0^y \sinh^{-1}(v) \, dv ;
\]

(5)

\[
\text{gauss}_2(x) = \exp \left( \frac{1}{2} x^2 \right) - 1 .
\]

(6)

Given a Young function \( \Phi \), and a probability measure \( \mu \), the Orlicz space \( L_\Phi(\mu) \) is the Banach space whose closed unit ball is \( \{ f \in L^0(\mu) \mid \int \Phi(|f|) \, d\mu \leq 1 \} \). That is,

\[
\|f\|_{L_\Phi(\mu)} \leq \alpha \quad \text{if, and only if,} \quad \int \Phi(\alpha^{-1} |f|) \, d\mu \leq 1 .
\]

(7)

From the Young inequality, it holds

\[
\int |uv| \, d\mu \leq \int \Phi(|u|) \, d\mu + \int \Phi_*(|v|) \, d\mu .
\]

This provides a separating duality \( \langle u, v \rangle = \int uv \, d\mu \) of \( L^\Phi(\mu) \) and \( L^{\Phi_*(\mu)} \) such that

\[
\langle u, v \rangle_\mu \leq 2 \|u\|_{L_\Phi(\mu)} \|v\|_{L_{\Phi_*(\mu)}} .
\]

From the conjugation between \( \Phi \) and \( \Psi \), an equivalent norm can be defined, namely,

\[
\|f\|_{L_\Phi^*(\mu)}^* = \sup \left\{ \langle f, g \rangle_\mu \mid \|f\|_{L_\Phi(\mu)} \leq 1 \right\}
\]

(8)

Domination relation between Young functions imply continuous injection properties for the corresponding Orlicz spaces. We say that \( \Phi_2 \) \textit{eventually dominates} \( \Phi_1 \), written \( \Phi_1 \prec \Phi_2 \), if there is a constant \( \kappa \) such that \( \Phi_1(x) \leq \Phi_2(\kappa x) \) for all \( x \) larger than some \( \bar{x} \). As, in our case, \( \mu \) is a probability measure, the continuous embedding \( L_{\Phi_2}(\mu) \to L_{\Phi_1}(\mu) \) holds if, and only if, \( \Phi_1 \prec \Phi_2 \). See a proof in \cite[Th. 8.2]{1}. If \( \Phi_1 \prec \Phi_2 \), then \( (\Phi_2)_* \prec (\Phi_1)_* \). With reference to our examples eqs. (4) and (5), we see that \( \exp_2 \) and \( (cosh - 1) \) are equivalent. They both are eventually dominated by \( \text{gauss}_2 \) and eventually dominate all powers.

A special case occurs when there exists a function \( \Phi \) such that \( \Phi(ax) \leq C(a)\Phi(ax) \) for all \( a \geq 0 \). This is true, for example, for a power function and in the case of the functions \( (\exp_2)_* \) and \( (cosh - 1)_* \). In such a case, the dual couple is a couple of reflexive Banach spaces and bounded functions are a dense set.

The spaces corresponding to case (3) are ordinary Lebesgue spaces. The cases eqs. (4) and (5) provides isomorphic spaces \( L_{(cosh - 1)}(\mu) \leftrightarrow L_{\exp_2}(\mu) \) are of special interest because they provide the model spaces for our non-parametric IG, see section 2.3 below.
Moreover, a function $f$ belongs to such spaces if, and only if, it is sub-exponential, that is, there exist constants $C_1, C_2 > 0$ such that
\[
\mathbb{P}_\mu (|f| \geq t) \leq C_1 \exp \{-C_2 t\}, \quad t \geq 0.
\]

Sub-exponential random variable are of special interest in applications because they admit an explicit exponential bounds in the Law of Large Numbers. Random variables whose square is sub-exponential are called sub-gaussian. They belong to the Orlicz space associated to our case \cite{6}. There is a large literature on this subject, see, for example, \cite{12,30,28}.

We are going to use a further notation. For each Young function $\Phi$, the function $\tilde{\Phi}(x) = \Phi(x^2)$ is again a Young function such that $\|f\|_{L_{\Phi}(\mu)} \leq \lambda$ if, and only if, $\|f^2\|_{L_{\tilde{\Phi}}(\mu)} \leq \lambda^2$. We denote the resulting space by $L_{\tilde{\Phi}}^2(\mu)$. With reference to our examples, it holds $L_{\text{gauss}}(\mu) = L_{(\cosh^{-1})^2}(\mu)$.

As an application of this notation, consider that for each increasing convex $\Phi$ it holds $E(fg) \leq \tilde{\Phi}(f^2 + g^2)/2 \leq (\tilde{\Phi}(f^2) + \phi(g^2))/2$. It follows that when the $L_{\tilde{\Phi}}^2(\mu)$-norm of $f$ and of $g$ is bounded by one, the $L_{\tilde{\Phi}}^2(\mu)$-norm of $f, g$, and $fg$, is bounded by one. The need to control the product of two random variables in $L_{(\cosh^{-1})^2}(\mu)$ appears, for example, in the study of the covariant derivatives of the statistical bundle, see \cite{6,13,20}.

### 2.2 Calculus of the Gaussian space

From now on, our base probability space is the Gaussian probability space $(\mathbb{R}^n, \gamma)$, $\gamma(z) = (2\pi)^{n/2} \exp \left(-\frac{|z|^2}{2}\right)$. We will use a few simple facts about the analysis of the Gaussian space, see \cite{14} Ch. V.

Let us denote by $C^k_{\text{poly}}(\mathbb{R}^n)$, $k = 0, 1, \ldots$, the vector space of functions which are differentiable up to order $k$ and which are bounded, together with all derivatives, by a polynomial. This class of functions is dense in $L^2(\gamma)$. For each couple $f, g \in C^1_{\text{poly}}(\mathbb{R}^n)$, we have
\[
\int f(x) \partial_i g(x) \gamma(x) \, dx = \int \delta_i f(x) g(x) \gamma(x) \, dx,
\]
where the divergence operator $\delta_i$ is defined by $\delta_i f(x) = x_i f(x) - \partial_i f(x)$. Multidimensional notations will be used, for example,
\[
\int \nabla f(x) \cdot \nabla g(x) \gamma(x) \, dx = \int f(x) \delta \cdot \nabla g(x) \gamma(x) \, dx, \quad f, g \in C^2_{\text{poly}}(\mathbb{R}^n),
\]
with $\delta \cdot \nabla g(x) = x \cdot \nabla g(x) - \Delta g(x)$.

For example, in this notation, the divergence of eq. \cite{11} with $P = p \cdot \gamma$, $Q = q \cdot \gamma$, and $p, q \in C^2_{\text{poly}}(\mathbb{R}^n)$, becomes
\[
\frac{1}{2} \int \nabla \log \frac{p(x)}{q(x)} \cdot \nabla \log \frac{p(x)}{q(x)} p(x) \gamma(x) \, dx =
\]
\[
\frac{1}{2} \int \log \frac{p(x)}{q(x)} \delta \cdot \nabla \log \left( \frac{p(x)}{q(x)} p(x) \right) \gamma(x) \, dx.
\]
The inner product eq. (2) becomes, with $P = p \cdot \gamma$ and $f, g, p \in C^2_{\text{poly}}(\mathbb{R}^n)$,

$$\int \nabla f(x) \cdot \nabla g(x) \ p(x) \ \gamma(x) \ dx = \int f(x) \delta \cdot \nabla (g(x)p(x)) \ \gamma(x) \ dx .$$

Hermite polynomials $H_\alpha = \delta^\alpha$ provide an orthogonal basis for $L^2(\gamma)$ such that $\partial_i H_\alpha = \alpha_i H_\alpha - e_i$. In turn, this provides a way to prove that there is a closure of both operator $\partial_i$ and $\delta_i$ on a domain which is an Hilbert subspace of $L^2(\gamma)$. Moreover, the closure of $\partial_i$ is the infinitesimal generator of the translation operator. See the full theory in [15,3] and the applications to IG in [12,25].

2.3 Exponential bundle

We refer to [23,25] for the definition of maximal exponential manifold $\mathcal{E}(\gamma)$, and of statistical bundle $S\mathcal{E}(\gamma)$. Below we report the results that are necessary in the context of the present paper.

A key result is the proof of the following statement of necessary and sufficient conditions, see [5] and [27, Th. 4.7].

**Proposition 1.** For all $p, q \in \mathcal{E}(\gamma)$ it holds $q = e^{u-K_p(u)} \cdot p$, where $u \in L_{(\cosh-1)}(\gamma)$, $E_p[u] = 0$, and $u$ belongs to the interior of the proper domain of the convex function $K_p$. This property is equivalent to any of the following:

1. $p$ and $q$ are connected by an open exponential arc;
2. $L_{(\cosh-1)}(p) = L_{(\cosh-1)}(q)$ and the norms are equivalent;
3. $p/q \in \cup_{a>1} L^a(q)$ and $q/p \in \cup_{a>1} L^a(p)$.

Item 2 ensures that all the fibers of the statistical bundle, namely $S_p \mathcal{E}(\gamma)$, $p \in \mathcal{E}(\gamma)$, are isomorphic. Item 3 gives a explicit description of the exponential manifold. For example, let $p$ be a positive probability density with respect to $\gamma$, and take $q = 1$ and $a = 2$. Then sufficient conditions for $p \in \mathcal{E}(\gamma)$ are

$$\int p(x)^2 \ \gamma(x) \ dx < \infty \ \text{and} \ \int \frac{1}{p(x)} \ \gamma(x) \ dx < \infty .$$

It is interesting to note that there is, so to say, a bound above and a bound below.

3 Bounding the Orlicz norm with the Orlicz norm of the gradient

We discuss now inequalities related to the classical Gauss-Poincaré inequality,

$$\int \left( f(x) - \int f(y) \ \gamma(y) \ dy \right)^2 \ \gamma(x) \ dx \leq \int |\nabla f(x)|^2 \ \gamma(x) \ dx , \quad (9)$$

where $f \in C^1_{\text{poly}}(\mathbb{R}^n)$. A proof is given, for example, in [19, § 1.4] and will follow as a particular case in an inequality to be proved below.
In terms of norms, the inequality above is equivalent to \( \| f - \bar{f} \|_{L^2(\gamma)} \leq \| \nabla f \|_{L^2(\gamma)} \), where \( \bar{f} = \int f(y) \gamma(y) \, dy \). One can check whether the constant 1 is optimal, by taking \( f(x) = \sum_i x_i \) and observing that the two sides both take the value \( \sqrt{n} \).

This is an example of differential inequality of high interest. For example, if \( p \in C^2_{\text{poly}} \) is a probability density with respect to \( \gamma \), then the \( \chi^2 \)-divergence of \( P = p \cdot \gamma \) from \( \gamma \) is bounded as follows.

\[
D_{\chi^2}(P|\gamma) = \int (p(x) - 1)^2 \gamma(x) \, dx \leq \int |\nabla p(x)|^2 \gamma(x) \, dx = \int \delta \cdot \nabla p(x) \, p(x) \, \gamma(x) \, dx ,
\]

where \( \delta \cdot \nabla p(x) = x \cdot \nabla p(x) - \Delta p(x) \). As \( \int \delta \cdot \nabla p(x) \, \gamma(x) \, dx = 0 \), the RHS is equal to

\[
\int \delta \cdot \nabla p(x)(p(x) - 1) \, \gamma(x) \, dx \leq \frac{1}{2} \int (\delta \cdot \nabla p(x))^2 \, \gamma(x) \, dx + \frac{1}{2} \int (p(x) - 1)^2 \, \gamma(x) \, dx ,
\]

so that, in conclusion,

\[
\int (p(x) - 1)^2 \, \gamma(x) \, dx \leq \int (\delta \cdot \nabla p(x))^2 \, \gamma(x) \, dx .
\]

### 3.1 Ornstein-Uhlenbeck semi-group

The method of proof relies on use of the Ornstein-Uhlenbeck semi-group which is defined on each \( C^k_{\text{poly}}(\mathbb{R}^n) \), \( k = 0, 1, \ldots \), by the Mehler formula

\[
P_tf(x) = \int f(e^{-t}x + \sqrt{1-e^{-2t}}y) \, \gamma(y) \, dy , \quad t \geq 0 , \quad f \in C^k_{\text{poly}}(\mathbb{R}^n) , \quad (10)
\]

see [14, V-1.5] and [19 § 1.3]. Notice that \( P_0f = f \) and \( P_\infty f = \bar{f} \).

If \( X, Y \) are independent standard Gaussian random variables in \( \mathbb{R}^n \), then

\[
X_t = e^{-t}X + \sqrt{1-e^{-2t}}Y , \quad Y_t = \sqrt{1-e^{-2t}}X - e^{-t}Y , \quad (11)
\]

are independent standard Gaussian random variables for all \( t \geq 0 \). It is well known, and easily checked, that the infinitesimal generator of the Ornstein-Uhlenbeck semi-group is \( -\delta \cdot \nabla \), that is, for each \( f \in C^2_{\text{poly}}(\mathbb{R}^n) \), it holds

\[
\frac{d}{dt}P_tf(x) = \int \nabla f(e^{-t}x + \sqrt{1-e^{-2t}}y) \cdot \left(-e^{-t}x + \frac{e^{-2t}}{1-e^{-2t}}y\right) \, \gamma(y) \, dy \quad (12)
\]

\[
= -\left(\delta \cdot \nabla\right)P_tf(x) \quad (13)
\]

\[
= -P_t(\delta \cdot \nabla)f(x) . \quad (14)
\]
See [14, V.1.5].

These computations are well known in stochastic calculus, see, for example [8, § 5.6]. In fact, because of eq. (14), the function \( p(x, t) = P_t p(x) \) is a solution of the equation

\[
\frac{\partial}{\partial t} p(x, t) + \Delta p(x, t) - x \cdot \nabla p(x, t) = 0 \ , \quad p(x, 0) = p(x) ,
\]

which is the Kolmogorov equation for diffusion \( dX_t = -X_t + \sqrt{2}dW_t \). Similarly, the function \( u(x) = \int_0^\infty e^{-t} P_t f(x) \ dt \) is a solution of the equation

\[
\delta \cdot \nabla u(x) + u(x) = f(x) \ .
\] (15)

By the change of variable eq. (11) and Jensen's inequality it easily follows that for each convex function \( \Phi \) it holds

\[
\int \Phi (p_t f(x)) \gamma(x) \ dx \leq \int \Phi(f(x)) \gamma(x) \ dx \ .
\] (16)

That is, for all \( t \geq 0 \), the mapping \( f \mapsto P_t f \) is non-expansive for the norm of each Orlicz space \( L_\Phi(\gamma) \).

We will discuss now a first set of inequalities that involves convexity and differentiation as it is in eq. (9). This set depends on the following proposition.

**Proposition 2.** For all \( \Phi: \mathbb{R}^n \) convex and all \( f \in C^1_{poly}(\mathbb{R}^n) \), it holds

\[
\int \Phi \left( f(x) - \int f(y) \gamma(y) \ dy \right) \gamma(x) \ dx \leq \int \int \Phi \left( \frac{\pi}{2} \nabla f(x) \cdot y \right) \gamma(x) \gamma(y) \ dxdy = \frac{1}{\sqrt{2\pi}} \int \int \Phi \left( \frac{\pi}{2} |\nabla f(x)| z \right) e^{-z^2/2} \gamma(x) \ dzdx = \int \bar{\Phi} (|\nabla f(x)|) \gamma(x) \ dx \ ,
\] (17)

where \( \bar{\Phi} \) is the convex function defined by

\[
\bar{\Phi}(a) = \int \Phi \left( \frac{\pi}{2} az \right) \gamma(z) \ dz \ .
\] (18)

**Proof.** It follows from eqs. (10) and (12) that

\[
f(x) - \int f(y) \gamma(y) \ dy = P_0 f(x) - P_\infty f(x) = - \int_0^\infty \frac{d}{dt} P_t f(x) \ dt = \frac{\pi}{2} \int_0^\infty p(t) \ dt \int \left( \nabla f(e^{-t} x + \sqrt{1 - e^{-2t}} y) \cdot \left( \sqrt{1 - e^{-2t}} x - e^{-t} y \right) \gamma(y) \ dy ,
\]

where \( p(t) = \frac{2}{\pi} \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \) is a probability density on \( t \geq 0 \). After that, the application of Jensen inequality and the change of variable (11), gives eq. (17). See more details in [25].
The arguments used here differ from those used, for example, in [19], which are based on the equation for the infinitesimal generator eqs. (13) and (14). We will come to that point later. Notice that we can take $\Phi(s) = s^2$ and derive a Poincaré inequality with a non-optimal constant $> 1$.

We can prove now a set of inequalities of the Poincaré type. The first example is $\Phi(s) = e^s$. In such a case, the equation for the moment generating function of the Gaussian distribution gives

$$\Phi(a) = \int \exp \left( \frac{\pi}{2} a z \right) \gamma(z) \, dz = \exp \left( \frac{\pi^2 a^2}{8} \right),$$

so that the inequality eq. (17) becomes

$$\int \exp \left( f(x) - f(0) \right) \gamma(x) \, dx \leq \int \exp \left( \frac{\pi^2}{8} |\nabla f(x)|^2 \right) \gamma(x) \, dx . \quad (19)$$

More clearly, we can change $f$ to $\frac{2\kappa}{\pi} f$ and write

$$\int \exp \left( \frac{2\kappa}{\pi} (f(x) - f(0)) \right) \gamma(x) \, dx \leq \int \exp \left( \frac{\kappa^2}{2} |\nabla f(x)|^2 \right) \gamma(x) \, dx = \frac{1}{\sqrt{2\pi}} \int \exp \left( -\frac{1}{2} \left( |x|^2 - \kappa^2 |\nabla f(x)|^2 \right) \right) \, dx . \quad (20)$$

The inequality above is non-trivial only if the RHS is bounded, that is

$$|\nabla f(x)| < \kappa^{-1} |x| , \quad x \in \mathbb{R}^n , \quad (21)$$

that is, the function $f$ is Lipschitz. We have found that $f \in C^1(\mathbb{R}^n)$ and globally Lipschitz implies that $f$ is sub-exponential in the Gaussian space.

The first case of bound for Orlicz norms we consider is the Lebesgue norm, $\Phi(t) = s^{2p}, p > 1/2$. In such a case,

$$\tilde{\Phi}(a) = \left( \frac{\pi}{2} \right)^{2p} m(2p) \, a^{2p} ,$$

where $m(2p)$ is the $2p$-moment of the standard Gaussian distribution. It follows that

$$\left\| f - \int f(y) \gamma(y) \, dy \right\|_{L^{2p}(\gamma)} \leq \frac{\pi}{2} (m(2p))^{1/2p} \|f\|_{L^{2p}(\gamma)} ,$$

Notice that in the case $p = 1$ we recover the Gauss-Poincaré inequality with a different, not optimal, constant.

The cases $\Phi(a) = a^{2p}$ are special in that we can use the in the proof the multiplicative property $\Phi(ab) = \Phi(a)\Phi(b)$. The argument generalizes to the case where the convex function $\Phi$ is a Young function whose increase is controlled.
through a function $C$, $\Phi(uv) \leq C(u)\Phi(v)$, and, moreover, such that there exists a $\kappa > 0$ for which
\[
\int C\left(\frac{\pi}{2} \kappa u\right) \gamma(u) \, du \leq 1, \tag{22}
\]
then eq. (18) becomes
\[
\tilde{\Phi}(\kappa a) = \int \Phi\left(\frac{\pi}{2} \kappa az\right) \gamma(z) \, dz \leq \int C\left(\frac{\pi}{2} \kappa z\right) \gamma(z) \, dz \, \Phi(a) \leq \Phi(a).
\]
By using this bound in eq. (17), we get
\[
\int \Phi\left(\kappa \left(f(x) - \int f(y) \gamma(y) \, dy\right)\right) \gamma(x) \, dx \leq \int \Phi(|\nabla f(x)|) \gamma(x) \, dx.
\]
Assume now that $\|\nabla f\|_{L_{\Phi}(\gamma)} \leq 1$ so that the LHS does not exceed 1. Then
\[
\kappa \|\nabla f\|_{L_{\Phi}(\gamma)} \leq 1, \text{ which in turn implies the inequality}
\]
\[
\|\nabla f\|_{L_{\Phi}(\gamma)} \leq \kappa^{-1} \|\nabla f\|_{L_{\Phi}(\gamma)}. \tag{23}
\]
For example, for $(\exp_2)_{*}(y) = (1+y)\log(1+y)-y$ we can take $C(u) = |u|\vee|u|^2$ and we want a $\kappa > 0$ such that
\[
\int \left(\frac{\pi}{2} \kappa |u|\right) \vee \left(\frac{\pi}{2} \kappa |u|^2\right) \gamma(u) \, du \leq 1.
\]
Such a $\kappa$ exists because $C$ is $\gamma$-integrable, continous, and $C(0) = 0$. For example, as $C(u) \leq u + u^2$, $u \geq 0$, we have
\[
\int C\left(\frac{\pi}{2} \kappa u\right) \gamma(u) \, du = 2 \int_{0}^{\infty} C\left(\frac{\pi}{2} \kappa u\right) \gamma(u) \, du \leq \pi \kappa \int_{0}^{\infty} u \gamma(u) \, du + \frac{\pi^2}{2} \kappa^2 \int_{0}^{\infty} u^2 \gamma(u) \, du = \sqrt{\frac{\pi^2}{2} \kappa + \frac{\pi^2}{4} \kappa^2}
\]
and we can take $k > 0$ satisfying $\sqrt{\frac{\pi^2}{2} \kappa + \frac{\pi^2}{4} \kappa^2} = 1$.

For us, it is of special interest the case of the Young function $\Phi = \cosh - 1$ for which the a bound as above does not exists. Instead, we use eq. (20) with $\kappa$ and $-\kappa$ to get
\[
\int (\cosh - 1) \left(\frac{2\kappa}{\pi} (f(x) - f)\right) \gamma(x) \, dx \leq \int \text{gauss}_2 (\kappa |\nabla f(x)|) \gamma(x) \, dx. \tag{24}
\]
Now, if $\kappa = \frac{\|\nabla f\|_{L^{\exp 2}(\gamma)}}{\|f \circ \sigma\|_{L^{\cosh -1}(\gamma)}}$, then the LHS is smaller or equal then 1, and hence $2\kappa/\pi \|f - \overline{f}\|_{L^{\cosh -1}(\gamma)} \leq 1$. It follows that

$$\|f - \overline{f}\|_{L^{\cosh -1}(\gamma)} \leq \frac{\pi}{2} \|\nabla f\|_{L^{\text{gauss}}}(\gamma).$$

(25)

Our last case of this series is the Young function $\text{gauss}_2(x) = \exp \left(\frac{1}{2} |x|^2\right) - 1$. Assume $f \in C^0_{\text{poly}}(\mathbb{R}^n) \cap L^{\text{gauss}}_2(\gamma)$, that is, there exists a constant $\lambda > 0$ such that

$$\int \text{gauss}_2(\lambda^{-1} f(x)) \gamma(x) \, dx =$$

$$\frac{1}{\sqrt{2\pi}} \int \exp \left( -\frac{1}{2} \left( |x|^2 - \lambda^{-2} f(x)^2 \right) \right) \, dx - 1 < +\infty.$$

This holds if, and only if, $|x|^2 > \lambda^{-2} |f(x)|^2$, $x \in \mathbb{R}^n$, that is, $f$ is bounded by a linear function with coefficient $\lambda > \sup_x |f(x)| / |x|$. The case does not seem to be of our interest.

In fact, if we compute $\text{gauss}_2(\lambda^{-1} a)$, we find

$$\int \exp \left( \frac{\pi}{2} \kappa a z \right) \gamma(z) \, dz =$$

$$\frac{1}{\sqrt{2\pi}} \int \exp \left( \frac{1}{2} \left( 1 - \left( \frac{\pi}{2} \right)^2 \kappa a^2 \right) z^2 \right) \, dz - 1 =$$

$$\left( 1 - \left( \frac{\pi}{2} \right)^2 \kappa^2 a^2 \right)^{-1/2} - 1$$

if the argument of $(\cdot)^{-1/2}$ is positive, $+\infty$ otherwise. This function does not belong to the class of Young function we are considering here and would require a special study. The inequality we obtain is

$$\int \text{gauss}_2(\kappa f(x)) \gamma(x) \, dx \leq \int \left( 1 - \left( \frac{\pi}{2} \right)^2 \kappa^2 |\nabla f(x)|^2 \right)^{-1/2} \gamma(x) \, dx$$

In the following proposition we give a summary of the inequalities proved so far. We do not care in this paper to define explicitly the relevant Gauss-Sobolev spaces, but this could be done in the spirit of [25].

**Proposition 3.** There exists constants $C_1$, $C_2(p)$, $C_3$ such that for all $f \in C_{\text{poly}}^1(\mathbb{R}^n)$ the following inequalities hold:

$$\left\| f - f(y) \gamma(y) \right\|_{L^{1}(\gamma)} \leq C_1 \|\nabla f\|_{L^{1}_{\text{cosh}}}(\gamma).$$

(26)

$$\left\| f - f(y) \gamma(y) \right\|_{L^p(\gamma)} \leq C_2(p) \|\nabla f\|_{L^p(\gamma)}, \quad p > 1/2.$$  

(27)

$$\left\| f - f(y) \gamma(y) \right\|_{L^{2p}(\gamma)} \leq C_3 \|\nabla f\|_{L^{2p}(\gamma)}.$$  

(28)
3.2 Generator of the Ornstein-Uhlenbeck semi-group

We consider now a further set of inequalities which are based on the use of infinitesimal generator $-\delta \cdot \nabla$ of the Ornstein-Uhlenbeck semigroup, see eqs. (13) and (14). Compare, for example, [19, § 1.3.7].

We have, for all $f \in C^2_{poly}(\mathbb{R}^n)$, that

$$f(x) - \bar{f} = -\int_0^\infty \frac{d}{dt} P_t f(x) \, dt = \int_0^\infty \delta \cdot \nabla P_t f(x) \, dt .$$

(29)

Note that

$$\nabla P_t f(x) = \nabla \int f(e^{-t}x + \sqrt{1-e^{-2t}}y) \gamma(y) \, dy = e^{-t} \int \nabla f(e^{-t}x + \sqrt{1-e^{-2t}}y) \gamma(y) \, dy = e^{-t} P_t \nabla f(x) ,$$

so that

$$P_t \delta \cdot \nabla f(x) = \delta \cdot \nabla P_t f(x) = e^{-t} \delta \cdot P_t \nabla f(x) .$$

Now, eq. (29) becomes

$$f(x) - \bar{f} = \int_0^\infty e^{-t} \delta \cdot P_t \nabla f(x) \, dt .$$

(30)

As

$$\int \delta \cdot \nabla f(x) \gamma(x) \, dx = 0 ,$$

the covariance of $f, g \in C^0_{poly}(\mathbb{R}^n)$ is

$$\text{Cov}_\gamma(f, g) = \int (f(x) - \bar{f}) g(x) \gamma(x) \, dx = \int (f(x) - \bar{f}) (g(x) - \bar{g}) \gamma(x) \, dx .$$

It follows that for all $f, g \in C^2_{poly}(\mathbb{R}^n)$ we derive from eq. (30)

$$\text{Cov}_\gamma(f, g) = \int_0^\infty e^{-t} \int P_t \nabla f(x) \cdot \nabla g(x) \gamma(x) \, dx \, dt .$$

(31)

We repeat here the result of in [25, Prop. 5]. Let $|\cdot|_1$ and $|\cdot|_2$ be two norms on $\mathbb{R}^n$, such that $|x \cdot y| \leq |x|_1 |y|_2$. For a Young function $\Phi$, consider the norm of $L_\Phi(\gamma)$ and the conjugate space endowed with the dual norm,

$$\|f\|_{L_\Phi(\gamma)} = \sup \left\{ \int f g \, \gamma \left| \int \Phi(g) \, \gamma \leq 1 \right. \right\} .$$

In the reference above, it is proved the following inequality that includes the standard Poincaré case when $\Phi(u) = u^2/2$.

**Proposition 4.** Given a Young function $\Phi$ and norms $|\cdot|_1, |\cdot|_2$ on $\mathbb{R}^n$ such that $x \cdot y \leq |x|_1 |y|_2$, $x, y \in \mathbb{R}^n$, for all $f, g \in C^1_{poly}(\mathbb{R}^n)$, it holds

$$|\text{Cov}_M(f, g)| \leq \|\nabla f\|_{L_\Phi(\gamma)} \|\nabla g\|_{L_\Phi(\gamma)} .$$
4 Discussion and conclusions

We have collected here a list of possible applications of the information geometry of the Gaussian space that has been introduced in [12][25] and further developed in the present paper.

4.1 Sub-exponential random variables

Let \( f \in C^2_{\text{poly}}(\mathbb{R}^n) \) be a random variable of the Gaussian space. Assume moreover that \( f \) is globally Lipschitz, that is,

\[
|\nabla f(x)| \leq \|f\|_{\text{Lip}(\mathbb{R}^n)} |x|
\]

where \( \|f\|_{\text{Lip}(\mathbb{R}^n)} \) is the Lipschitz semi-norm, that is, the best constant. It follows from eq. (24) that \( f \in L(\cosh^{-1})(\gamma) \) and the norm admits a computable bound.

If \( p \) is any probability density of the maximal exponential model of \( \gamma \), that is, it is connected to 1 by an open exponential arc, then proposition implies that \( f \in L(\cosh^{-1})(p) \), that is, \( f \) is sub-exponential under the distribution \( P = p \cdot \gamma \). If the sequence \( (X_n)_{n=1}^{\infty} \) is independent and with distribution \( p \cdot \gamma \), then the sequence of sample means will converge,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(X_j) = \int f(x) \ p(x) \ \gamma(x) \ dx ,
\]

with an exponential bound on the tail probability. See, for example, [29][§2.8].

4.2 Hyvärinen divergence

Here we adapt [21] to the Gaussian case. Consider the Hyvärinen divergence of eq. (1) in the Gaussian case, that is, \( P = p \cdot \gamma \) and \( Q = q \cdot \gamma \). As a function of \( q \) is of the form

\[
H(q) = \frac{1}{2} \int |\nabla \log p(x)|^2 p(x) \ \gamma(x) \ dx + \frac{1}{2} \int |\nabla \log q(x)|^2 p(x) \ \gamma(x) \ dx - \int \nabla \log p(x) \cdot \nabla \log q(x) \ p(x) \ \gamma(x) \ dx ,
\]

where the first term does not depend on \( q \) and the second term is an expectation with respect to \( p \cdot \gamma \). As \( \nabla \log p = p^{-1} \nabla p \), the third term equals

\[
- \int \delta \cdot \nabla q(x) \ p(x) \ \gamma(x) \ dx ,
\]

hence it is again a \( p \)-expectation. To minimize the Hyvärinen divergence we must minimize the \( p \)-expected value of the local score

\[
S(q, x) = \frac{1}{2} |\nabla \log q(x)|^2 - \delta \cdot \nabla \log q(x)
\]
If $p$ and $q$ belong to the maximal exponential model of $\gamma$, then $q = e^{u - K(u)}$ with $u \in L_{(\cosh - 1)}(\gamma)$ and $\int u(x) \gamma(x) \, dx = 0$. The local score becomes $\frac{1}{2} |\nabla u|^2 - \delta \cdot \nabla u$. To compute the $p$-expected value of the score with an independent sample of $p \cdot \gamma$ we have interest to assume that the score is in $L_{(\cosh - 1)}(\gamma)$, because this assumption implies the good convergence of the empirical means for all $p$, as it was explained in the section above.

Assume, for example, $\nabla u \in L_{(\cosh - 1)}(\gamma)$. This implies directly $|\nabla u|^2 \in L_{(\cosh - 1)}(\gamma)$. Moreover, we need to assume that the $L_{(\cosh - 1)}(\gamma)$-norm of $\delta \cdot \nabla u$ is finite. Under such assumptions it seems reasonable to hope that the minimization on a suitable model of the sample expectation of the Hyvärinen score is consistent.

### 4.3 Otto’s metric

Let $P = p \cdot \gamma$ with $p$ in the maximal exponential model of $\gamma$. Let $f$ and $g$ be in the $p$-fiber of the statistical manifold, that is, $f, g \in L_{(\cosh - 1)}(p) = L_{(\cosh - 1)}(\gamma)$ and $\int f(x) \gamma(x) \, dx = \int g(x) \gamma(x) \, dx = 0$. The Otto’s inner product \cite{Amari98} becomes

$$\int \nabla f(x) \cdot \nabla g(x) \, p(x) \gamma(x) \, dx = \int f(x) \delta \cdot (p(x) \nabla g(x)) \gamma(x) \, dx.$$

The LHS is well defined and regular if we assume $\nabla f, \nabla g \in L_{(\cosh - 1)}^2(\gamma)$, because, in such a case, $|\nabla f|^2, |\nabla g|^2 \in L_{(\cosh - 1)}(\gamma) = L_{(\cosh - 1)}(p)$. The RHS provides the representation of the inner product in the inner product defined in $L_{(\cosh - 1)}(\gamma)$. Note that the mapping $g \mapsto \delta \cdot (p \nabla g)$ is 1-to-1 if $g$ is restricted by $\int g(x)p(x) \gamma(x) \, dx = 0$. The inverse of this mapping provides the natural gradient of the Otto’s inner product in the sense of \cite{Amari98}.

### 4.4 Conclusion

In this paper we have derived bounds of the Orlicz norms of interest in IG based on the Orlicz norm of the gradient. The schematic examples above provide, in our opinion, a motivation for further study of this approach. As a bottom line, it should be noted that there is a large literature on Sobolev spaces with weight that we have, regrettably, not used here. Its study would surely provide more precise and deep results than those presented here.

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