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EQUITABLE COLORINGS OF $K_4$-MINOR-FREE GRAPHS

RÉMI DE JOANNIS DE VERCLOS AND JEAN-SÉBASTIEN SERENI

Abstract. We demonstrate that for every positive integer $\Delta$, every $K_4$-minor-free graph with maximum degree $\Delta$ admits an equitable coloring with $k$ colors where $k \geq \frac{\Delta+3}{2}$. This bound is tight and confirms a conjecture by Zhang and Whu. We do not use the discharging method but rather exploit decomposition trees of $K_4$-minor-free graphs.

1. Introduction

Equitable coloring is an ubiquitous notion. From a combinatorial point of view, it corresponds to a natural variation of usual graph coloring where the color classes are required to all have the same size, plus/minus one vertex. Practically, this is one way to prevent color classes from being very large, which can be useful when using graph coloring for scheduling purposes for instance. Theoretically, equitable colorings were used successfully in a priori unrelated topics, such as probability. Indeed, one of the seminal results regarding equitable colorings is the following theorem, which was established by Hajnal and Szemerédi [2] (the statement was first conjectured by Erdős).

Theorem 1.1 (Hajnal–Szemerédi, 1970). Every graph with maximum degree at most $\Delta$ admits an equitable coloring using $\Delta + 1$ colors.

Theorem 1.1 allowed for a simplified demonstration of the Blow-up lemma — found by Rödl and Ruciński [9]. In addition, this theorem was also used to derive deviation bounds for sums of random variables with some degree of dependence — this was done by Alon and Füredi [1] and by Janson and Ruciński [3]. Let us point out that in 2010, that is, forty years after Theorem 1.1 was proved, a much simpler demonstration was finally found, building on several other related results. More precisely, Kierstead, Kostochka, Mydlarz and Szemerédi [4] managed to find a two-page proof of Theorem 1.1, which also has the advantage to lead to a polynomial-time algorithm that efficiently finds a relevant coloring — contrary to the original argument.

As it turns out, the notion of equitable colorings behaves pretty differently from usual colorings, and it is a challenging task to better comprehend its relation to well-known graph classes. Starting from graphs with bounded maximum degree, it is natural to consider next $d$-degenerate graphs. The following theorem was established by Kostochka and Nakprasit [6], in a more general form.

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\end{itemize}
Theorem 1.2 (Kostochka–Nakprasit, 2003). Let $\Delta$ be an integer greater than 53. If $G$ is a 2-degenerate graph with maximum degree at most $\Delta$, then $G$ is equitably $k$-colorable whenever $k \geq \frac{\Delta+3}{2}$.

Theorem 1.2 partially confirms a conjecture by Zhang and Wu [10, Conjecture 9], (also see [8, Conjecture 6, p. 1209]) that if $\Delta \geq 3$, then every series-parallel graph with maximum degree $\Delta$ admits an equitable $k$-coloring whenever $k \geq \Delta+3$. Indeed, series-parallel graphs are known to be 2-degenerate, so Theorem 1.2 yields that the conjecture is true if $\Delta \geq 54$.

The purpose of our work is to establish the conjecture for all the remaining cases, that is, $\Delta \in \{3, \ldots , 53\}$. (Although, in our proofs we do not use the upper bound on $\Delta$, and simply prove the statement for all $K_4$-minor-free graphs.)

The statement conjectured by Zhang and Wu is actually a strengthening of a result of theirs [10], which establishes that every series-parallel graph with maximum degree $\Delta$ admits an equitable $k$-coloring if $k \geq \Delta$. The conjecture can also be seen as a generalisation of a theorem of Kostochka [5] that every outerplanar with maximum degree $\Delta$ admits an equitable $k$-coloring whenever $k \geq \frac{\Delta+3}{2}$.

It is worth mentioning that Kostochka, Nakprasit and Pemmaraju [7] established (a generalisation of) the following interesting statement.

Theorem 1.3 (Kostochka, Nakprasit & Pemmaraju, 2005). Fix an integer $k \geq 124$. If $G$ is a 2-degenerate graph with maximum degree at most $\frac{1}{2} |V(G)| + 1$, then $G$ admits an equitable $k$-coloring.

Theorem 1.3, however, does not bring us any new information regarding the problem at hands. Indeed, we need to consider graphs with maximum degree $\Delta \leq 53$, while the number of colors needs to be at least 124. Hence for our question the information provided by Theorem 1.3 is already contained in the aforementioned result of Zhang and Wu [10].

As reported earlier, we establish the following.

Theorem 1.4. If $G$ is a $K_4$-minor-free graph with maximum degree $\Delta$, then $G$ admits an equitable $k$-coloring whenever $k \geq \frac{\Delta+3}{2}$.

Contrary to the proof of some of the results mentioned above, we do not rely on discharging, but rather on the structural links between $K_4$-minor-free graphs and two-terminal series-parallel graphs: in particular, our proof heavily relies on a so-called SP-tree. Before proceeding with the proof, we review some folklore properties of $K_4$-minor-free graphs and two-terminal series-parallel graphs and introduce a bit of terminology.

It would be interesting to know whether Theorem 1.4 can be extended to the class of 2-degenerate graphs. A generalisation of this has actually been conjectured in 2003 by Kostochka and Napkrasit [6].

Conjecture 1.5. Fix an integer $\Delta$. If $d \in \{2, \ldots , \Delta\}$ and $G$ is a $d$-degenerate graph with maximum degree at most $\Delta$, then $G$ admits an equitable $k$-coloring whenever $k \geq \frac{\Delta+d+1}{2}$.

2. The structure of $K_4$-minor-free Graphs

As it turns out, graphs with no $K_4$-minor are strongly related to two-terminal series-parallel graphs. A two-terminal graph is a graph with two distinguished vertices called poles. Two-terminal series-parallel graphs are two-terminal graphs that can be obtained
by the following recursive construction\(^1\). The basic two-terminal series-parallel graph is an edge \(uv\) with the two poles being its end-vertices. For \(i \in \{1, 2\}\), let \(G_i\) be a two-terminal series-parallel graph with poles \(u_i\) and \(v_i\). The graph \(S(G_1, G_2)\) obtained by identifying the vertices \(v_1\) and \(u_2\) is also a two-terminal series-parallel graph and its two poles are the vertices \(u_1\) and \(v_2\). The graph \(S(G_1, G_2)\) obtained in this way is called the serial join of \(G_1\) and \(G_2\). The graph \(P(G_1, G_2)\) obtained by identifying the pairs of vertices \((u_1, u_2)\) and \((v_1, v_2)\); the poles of \(P(G_1, G_2)\) being the identified vertices. Two-terminal series-parallel graphs are precisely those that can be obtained from edges by a series of serial and parallel joins. The decomposition tree corresponding to a two-terminal series-parallel graph \(G\) is not unique. In fact, there is a lot of freedom in its choice as can be seen in the following well-known result.

**Lemma 2.1.** Let \(G\) be a two-terminal series-parallel graph and \(v\) a vertex of \(G\). There exists an SP-decomposition tree such that \(v\) is one of the poles of the graph corresponding to the root of the SP-decomposition tree.

It is also well known that every 2-edge-connected \(K_4\)-minor-free graph is a two-terminal series-parallel graph.

**Lemma 2.2.** Every block of a \(K_4\)-minor-free graph is a two-terminal series-parallel graph.

The set of \(K_4\)-minor-free graphs can also be seen as the closure of two-terminal series-parallel graphs by the spanning subgraph relation.

**Lemma 2.3.** A graph \(G\) has no \(K_4\)-minor if and only if \(G\) is the spanning subgraph of a two-terminal series-parallel graphs.

To see Lemma 2.3, note that spanning subgraphs of two-terminal series-parallel graphs has no \(K_4\)-minor. The reversed direction is deduced by induction on the structure of \(K_4\)-minor-free graph given by Lemma 2.2 using Lemma 2.1.

As a consequence, the \(K_4\)-minor-free graphs are precisely those for which we can choose two poles such that the two-terminal graph obtained can be constructed from the two graphs of size two by a series of serial and parallel joins. The construction of a particular \(K_4\)-minor-free graph \(G\) can thus be encoded by a rooted tree, which is called the SP-decomposition tree of \(G\). Each node of the tree corresponds to a subgraph of \(G\) obtained at a step of the recursive construction of \(G\). The leaves correspond to graphs with only two poles (and no other vertex) that may or may not be connected by an edge. Each inner node of the tree corresponds to either a serial join or to a parallel join. Based on this, there are two types of inner nodes: \(S\)-nodes and \(P\)-nodes. The inner nodes have at least two children: the subgraphs corresponding to their children are joined together by a sequence of serial or parallel joins depending on the type of the node. Since the result of a sequence of serial joins depends on the order in which the serial joins are applied, the children of each inner node are ordered. Without loss of generality, we can assume that the children of a \(P\)-node are \(S\)-nodes and leaves only, and the children of an \(S\) node are \(P\)-nodes and leaves only.

If \(A\) is a two-terminal graph, the vertices of \(A\) distinct from its poles are said to be its inner vertices. The set of inner vertices of \(A\) is \(\text{Inner}(A)\). We define \(\text{width}(A)\), the width of \(A\), to be the number of inner vertices of \(A\), that is, \(\text{width}(A) = |\text{Inner}(A)|\) (note that

\(^1\)We point out that in the literature, such graphs are sometimes called simply 'series-parallel graphs', while this term can also be used to refer to \(K_4\)-minor-free graphs.
width(\(A\)) = |V(\(A\))| − 2). We introduce some terminology for particular two-terminal \(K_4\)-minor-free graphs. A two-terminal graph obtained by a parallel join of several two-edge paths is a diamond. A two-terminal graph obtained by a parallel join of several two-edge paths and an edge is a crystal. Observe that an edge may be seen as a crystal of width 0. If \(i\) is a positive integer, we define \(D(i)\) to be the diamond with width \(i\) and \(C(i)\) to be the crystal with width \(i\). Let \(D'(1)\) be the graph \(K_{1,3}\) with two vertices of degree 1 as poles. For \(i \geq 2\), we define \(D'(i)\) to be the graph obtained by a parallel join of \(D'(1)\) with \(i − 1\) paths of length 2. Let \(C'(i)\) be obtained from \(D'(i)\) by adding an edge between the poles. We let \(P_i\) be the path with \(i\) vertices. If \(G\) is a graph and \(U\) a subset of the vertices of \(G\), we let \(G − U\) be the subgraph of \(G\) induced by the vertices of \(G\) that do not belong to \(U\). For a positive integer \(k\), we take the representatives of \(\mathbb{Z}_k\) to be \(\{1, \ldots, k\}\), rather than the more common \(\{0, \ldots, k − 1\}\). An equitable \(k\)-coloring of a graph \(G\) is a mapping \(\alpha : V(\(G\)) \to \mathbb{Z}_k\) such that \(|\alpha^{-1}(\{i\})|\) and \(|\alpha^{-1}(\{j\})|\) differ by at most one for every \((i, j) \in \mathbb{Z}_k^2\).

The next lemma is a simple but useful remark about common neighbors of the poles of a \(K_4\)-minor-free graph.

**Lemma 2.4.** If \(H\) is a \(K_4\)-minor-free graph with poles \(a\) and \(b\), then \(N_H(a) \cap N_H(b)\) is an independent set of \(H\).

**Proof.** We prove by induction on the number of vertices of the SP-decomposition tree of \(H\) that no two vertices in \(N_H(a) \cap N_H(b)\) belong to a same component of \(H \setminus \{a, b\}\).

- The statement is trivial if the SP-tree has only one node, that is if \(H\) has two vertices.
- If \(H\) is the series join of \(H_1\) and \(H_2\), then the only possible common neighbor of \(a\) and \(b\) is the common pole of \(H_1\) and \(H_2\). The statement is therefore true in this case also.
- If \(H\) is the parallel join of \(H_1\) and \(H_2\), then let \(x\) and \(y\) be two common neighbors of \(a\) and \(b\). Either \(x\) and \(y\) belong to \(H_i\) for some \(i \in \{1, 2\}\), in which case the result follows from the induction hypothesis applied on \(H_i\); or \(x\) and \(y\) are in different components of \(H \setminus \{a, b\}\).

\\(\square\\)

Let \(T\) be an SP-decomposition tree (of a \(K_4\)-minor-free graph), and \(n\) be a node of \(T\) representing the subgraph \(H\) with poles \(a\) and \(b\). Assume that \(H − \{a, b\}\) has \(m\) components \(C^1, \ldots, C^m\). The node \(n\) is in normal form if \(m ≤ 1\) (i.e. \(H − \{a, b\}\) either is connected or has no vertex at all), or if \(n\) is a parallel node with children \(H^1, \ldots, H^m\) plus the edge \(ab\) if \(ab \in E(\(H\))\), where \(H^i\) is the subgraph of \(H\) induced by \(C^i \cup \{a, b\}\) from which we remove the edge \(ab\) if it is present. The tree \(T\) is in normal form if every node of \(T\) is in normal form.

**Lemma 2.5.** If \(G\) is a \(K_4\)-minor-free graph, then \(G\) admits a construction tree \(T\) in normal form.

**Proof.** As a \(K_4\)-minor-free graph, \(G\) has two vertices \(a\) and \(b\) and an SP-decomposition tree \(T\) that represents the two-terminal graph \(G\) with poles \(a\) and \(b\). Note that we may assume that \(T\) is a binary tree (where P-nodes and S-nodes may not alternate).

To prove the lemma, we describe an inductive procedure that transform the (binary) SP-decomposition tree \(T\) into an SP-decomposition tree \(T'\) in normal form that represents the same graph \(G\). Assume that this procedure exists for trees with fewer nodes than \(T\). If \(n\) is a
leaf, then \( G \) has two vertices and further \( V(G) - \{a,b\} \) is empty, so \( n \) is in normal form indeed. So we now suppose that \( n \) has two children representing the graphs \( G_1 \) and \( G_2 \), respectively. By induction, for each \( i \in \{1,2\} \) there is a tree \( T_i \) in normal form that represents \( G_i \). We distinguish two cases depending on the type of the root \( n \) of \( T \).

- Suppose that \( n \) is a P-node, so \( G = P(G_1, G_2) \). Let \( C_i^1, \ldots, C_i^{m_i} \) be the components of \( G_i - \{a,b\} \), and note that \( m_i \) is a positive integer. If \( m_i = 1 \), then we set \( H_i^1 := G_i \). If \( m_i \geq 2 \), then according to the definition of normal forms the graph \( G_i \) is encoded in \( T_i \) by the parallel join of \( H_i^1, \ldots, H_i^{m_i} \), plus possibly the edge \( ab \). (We recall that it means that each graph \( H_i^j \) is the subgraph of \( G_i \) induced by \( C_i^j \cup \{a,b\} \) from which the edge \( ab \) is deleted if it is present.) The sought SP-decomposition tree \( T' \) is then obtained by making a new P-node \( n \) the parent node of each of the SP-decomposition trees representing \( H_1^1, \ldots, H_1^{m_1}, H_2^1, \ldots, H_2^{m_2} \) (each of them in normal form), and, possibly, of a leaf representing an edge if \( ab \in E(G) \).

- Suppose that \( n \) is an S-node, so \( G = S(G_1, G_2) \). First note that \( ab \notin E(G) \). Let \( c \) be the common pole of \( G_1 \) and \( G_2 \). Let \( C_1^1, \ldots, C_1^{k_1} \) be the components of \( G_1 - \{a,c\} \) that contain a neighbor of \( c \) and let \( C_1^{k_1+1}, \ldots, C_1^{m_1} \) be the other components of \( G_1 - \{a,c\} \). We define analogously the components \( C_2^1, \ldots, C_2^{m_2} \) and the index \( k_2 \) with respect to \( G_2 - \{b,c\} \). For each \( j \in \{1, \ldots, m_i\} \), we define \( H_i^j \) to be the subgraph of \( G \) corresponding to the component \( C_i^j \) of \( G_1 - \{a,c\} \) as in the definition of normal forms. The graphs \( H_1^1, \ldots, H_1^{m_1}, H_2^1, \ldots, H_2^{m_2} \) are defined analogously with respect to \( G_2 - \{b,c\} \).

According to the definition of normal forms, either \( H_i^j \) is empty, so \( n \) is in normal form, hence so is the tree \( T' \). This concludes the proof.

\[ \Box \]

3. Reductions

We note that the statement of Theorem 1.4 is true if \( k \leq 2 \), since then \( \Delta \in \{0,1\} \). So from now on we assume that \( k \geq 3 \). We fix a minimal counter-example \((G,k)\), where \( k \geq \lceil \Delta(G)+3 \rceil \), along with an SP tree-decomposition \( T \) of \( G \) with every node in normal form (Lemma 2.5 ensures that this is possible). It follows that \( k < |V(G)| \), as any graph \( H \) admits an equitable \( t \)-coloring if \( t \geq |V(H)| \). We may also assume that \( G \) is connected. As a consequence, every component of a subgraph of \( G \) with poles \( a \) and \( b \) that is represented by a subtree of \( T \) contains \( a \) or \( b \). A subtree \( T' \) of \( T \) is a construction subtree if \( T' \) is rooted at a node \( r \) of \( T \) and \( T' - \{r\} \) consists of at least two subtrees of \( T - \{r\} \) containing children of \( r \) such that if \( r \) is an \( S \)-nodes, then all these children are consecutive around \( r \) in \( T \).

Throughout this section, each time a coloring \( c \) is obtained by induction (or, equivalently, by a minimality argument), we assume the colors to be ordered increasingly, that is, such that \( |\alpha^{-1}\{\{i\}\}| \leq |\alpha^{-1}\{\{j\}\}| \) for every two colors \( i \) and \( j \) with \( i < j \). (This condition implies
that if we consider a $k$-coloring $\alpha$ of an $n'$-vertex graph with $n' < k$, then the colors used by $c$ are precisely $k, k - 1, \ldots, k - n'$, each being used exactly once.)

**Lemma 3.1.** The graph $G$ has no construction subtree representing a subgraph $C(k - 1)$ or $D(k - 1)$.

*Proof.* Suppose, on the contrary, that $H$ is such a subgraph of $G$. Let $a$ and $b$ be the poles and $v_1, \ldots, v_t$ the inner vertices of $H$. Let $F$ be the graph constructed from $G$ by contracting $V(H)$ to a vertex $c$, removing parallel edges and loops when they occur. Note that $F$ has no $K_4$-minor. In addition, $d_F(c) \leq d_G(a) + d_G(b) - 2(k - 1) \leq 2k - 4$. By the minimality of $G$, there is an equitable $k$-coloring $\alpha$ of $F$. Define $\alpha'(v) := \alpha(v)$ for $v \in V \setminus V(H)$. Note that $\alpha'$ is a partial proper coloring of $G$, that is, a proper coloring defined on a subset of $V(G)$. To finish the proof, it suffices to extend $\alpha'$ to a proper coloring of $G$ such that the multisets $\{\alpha'(a), \alpha'(b), \alpha'(v_1), \ldots, \alpha'(v_k)\}$ and $\{\alpha(c), 1, \ldots, k\}$ are equal. (Note that in this latter multiset one color has multiplicity two — namely $\alpha(c)$ — and $k - 1$ colors have multiplicity one.) We now distinguish two cases.

- If $ab \notin E$, then we set $\alpha'(a) := \alpha(b) := \alpha(c)$ and we color $v_1, \ldots, v_k$ using all the elements of the set $\{1, \ldots, k\} \setminus \{\alpha(c)\}$.
- If $ab \in E$, then $a$ has at most $\Delta - k \leq k - 3$ colored neighbors. So $a$ can be properly colored with a color $\alpha'(a)$ different from $\alpha(c)$. Similarly, $b$ has at most $k - 2$ colored neighbors (including $a$), so $b$ can be properly colored with a color $\alpha'(b)$ different from $\alpha(c)$ (and from $\alpha'(a)$). Now, we color $v_1, \ldots, v_k$ using the elements of the multiset $\{\alpha(c), \alpha(c), 1, \ldots, k\} \setminus \{\alpha'(a), \alpha'(b)\}$, with the corresponding multiplicities. □

**Corollary 3.2.** For every integer $t \geq k - 1$, the graph $G$ has no construction subtree representing a subgraph $C(t)$ or $D(t)$.

*Proof.* Assume otherwise that $H$ is such a subgraph of $G$. Let $a$ and $b$ be the poles of $H$. Let $n$ be the root of the construction subtree that represents $H$. Since $n$ is in normal form and $H - \{a, b\}$ is an independent set of size $t$, the node $n$ is a parallel node with at least $t$ children representing a path $P_3$ with end vertices $a$ and $b$ (the node $n$ may have other children as well). Choosing $n$ as a root along with $k - 1$ of the children of $n$ representing a $P_3$ yields a construction subtree of $T$ that represents $D(k - 1)$, which contradicts Lemma 3.1. □

**Lemma 3.3.** If a construction subtree of $T$ represents a graph $H$ with $1 \leq \width(H) \leq k$, then $\inner(H)$ is dominated by a pole of $H$ unless $\width(H) \geq 2$ and $H \in \{C'(t), D'(t)\}$, where $t = \width(H) - 1$.

*Proof.* Assume that each of the poles $a$ and $b$ of $H$ has a non-neighbor in $\inner(H)$, which we name $a'$ and $b'$, respectively. Note that it is possible to ensure that $a' \neq b'$ unless $\inner(H) \setminus N(a) = \inner(H) \setminus N(b) = \{a'\}$. In this latter case, since each component of $H$ contains $a$ or $b$ as reported earlier, we deduce that $H$ is connected. It then follows from Lemma 2.4 that $H$ is equal to either $C'(t)$ or $D'(t)$, with $t = \width(H) - 1 \geq 1$.

We now assume that $a' \neq b'$, which yields to a contradiction. Indeed, let $F$ be the graph $G - \inner(H)$ to which we add the edge $ab$ if it is not already present. By the minimality of $G$ there is an equitable $k$-coloring $\alpha$ of $F$. To obtain a contradiction, it suffices to extend $\alpha$ to a proper coloring of $G$ such that $\{\alpha(v) \mid v \in \inner(H)\}$ equals $\{1, \ldots, \width(H)\}$. (We recall that the colors are increasingly ordered.)
To do so, we define $\alpha(a') := \alpha(a)$ if $\alpha(a) \leq \text{width}(H)$ and $\alpha(b') := \alpha(b)$ if $\alpha(b) \leq \text{width}(H)$ and we arbitrarily assign the colors of $\{1, \ldots, \text{width}(H)\} \setminus \{\alpha(a), \alpha(b)\}$ to the non-colored vertices, each color being assigned once. □

Our next statement is a direct consequence of Lemma 3.3.

**Corollary 3.4.** If a construction subtree of $T$ represents a graph $H$ with $\text{width}(H) \leq k$, then the subgraph induced by $\text{Inner}(H)$ is a forest.

**Proof.** The statement is clear if $H \in \{C'(t), D'(t)\}$ for some integer $t$, so by Lemma 3.3 we can assume that $\text{Inner}(H)$ is dominated by a pole $a$ of $H$. Then $\text{Inner}(H)$ induces an acyclic graph, as otherwise $\text{Inner}(H) \cup \{a\}$ would induce a subgraph of $G$ containing a subdivision of $K_4$. □

**Lemma 3.5.** Let $H$ be a graph with poles $a$ and $b$ represented by a construction subtree of $T$ and assume that $\text{width}(H) = k - 1$. Then $d_H(a) + d_H(b) \leq 2k - 4$.

**Proof.** Assume on the contrary that $d_H(a) + d_H(b) \geq 2k - 3$. Let $F$ be the graph obtained from $G$ by contracting $H$ into one vertex $c$, again removing parallel edges and loops when they occur. In other words, we set $V(F) := (V(G) \setminus V(H)) \cup \{c\}$ and $N_F(v) := N_G(v)$ for $v \in V(G) \setminus V(H)$ while $N_F(c) := (N_G(a) \cup N_G(b)) \cap V(F)$. By our assumption, $d_G(c) \leq d_G(a) - d_H(a) + d_G(b) - d_H(b) \leq 2\Delta - (2k - 3) \leq \Delta$. Consequently, $F$ is a $K_4$-minor-free graph with maximum degree at most $\Delta$. By the minimality of $G$ there is an equitable $k$-coloring $\alpha$ of $F$. To obtain an equitable colouring of $G$, it suffices to extend $\alpha$ to $V(G)$ in such a way that the multisets $\{\alpha(v) | v \in V(H)\}$ and $\{\alpha(c), 1, \ldots, k\}$ are equal. We note that Corollary 3.4 yields that $\text{Inner}(H)$ induces an acyclic graph. We distinguish three cases.

- If $ab \notin E(G)$ then we define $\alpha(a) := \alpha(b) := \alpha(c)$ and we arbitrarily distribute all the colors in $\{1, \ldots, k\} \setminus \{\alpha(c)\}$ to the vertices in $\text{Inner}(H)$.
- If $ab \in E(G)$ and $a$ has a non-neighbor $a' \in \text{Inner}(H)$, then by Lemma 3.3, it follows that either $b$ dominates $\text{Inner}(H)$ or $H = C'(k-2)$. In both cases, we know that $b$ has at least $k-2$ neighbors in $\text{Inner}(H)$. It follows that $b$ has at most $\Delta - (k-2) \leq k-1$ neighbors outside of $\text{Inner}(H)$, including $a$. We define $\alpha(a) := \alpha(a') := \alpha(c)$. By the preceding remark it is possible to properly color $b$ with a color $\alpha(b)$ (so in particular $\alpha(a) \neq \alpha(b)$). To finish the coloring, we assign arbitrarily all the colors in $\{1, \ldots, k\} \setminus \{\alpha(a), \alpha(b)\}$ to the vertices in $\text{Inner}(H) \setminus \{a'\}$.
- If both $a$ and $b$ dominate $\text{Inner}(H)$, then by Lemma 2.4 we know that $H = C(k-1)$, which does not occur by Lemma 3.1. □

**Lemma 3.6.** If $H$ is a graph represented by a construction subtree of $G$, then $\text{width}(H) \neq k - 1$.

**Proof.** Assume otherwise that there is such a graph $H$ with $\text{width}(H) = k - 1$. By Lemma 3.3, we may assume that a pole $a$ of $H$ has at least $k-2$ neighbors in $\text{Inner}(H)$. Let $b$ be the other pole of $H$. By Lemma 3.5, we have $d_H(b) \leq 2k - 4 - d_H(a) \leq k - 2$. It follows that $b$ has a non-neighbor $b'$ in $\text{Inner}(H)$. By the minimality of $G$, the graph $F := G - (\text{Inner}(H) \cup \{a\})$ has an equitable $k$-coloring $\alpha$. To finish the proof, it suffices to extend $\alpha$ to $V(G)$ in such a way that $\{\alpha(v) | v \in \text{Inner}(H) \cup \{a\}\}$ equals $\{1, \ldots, k\}$. Since $a$ has at most $k-1$ colored neighbors, it is possible to properly color $a$ with a color $\alpha(a)$. We set $\alpha(b') := \alpha(b)$ unless
α(a) = α(b). Then we arbitrarily color the (k − 1 or k − 2) non-colored vertices using all the
(k − 1 or k − 2) colors in \{1, \ldots, k\} \setminus \{α(a), α(b)\}.

**Corollary 3.7.** If \( H \) is a graph represented by a construction subtree of \( G \), then \( H \notin \{C'(k − 1), D'(k − 1)\} \).

**Proof.** Assume otherwise that \( H \) is such a graph, with poles \( a \) and \( b \), and represented by a construction subtree of \( G \) with root \( n \). Since \( n \) is in normal form and \( H − \{a, b\} \) is disconnected, the node \( n \) is a parallel node with a children representing a star \( K_{1,3} \) and (at least) \( k − 2 \) children each representing a path \( P_3 \) with end-vertices \( a \) and \( b \) (the node \( n \) may have further children). It follows that \( T \) has a construction subtree of \( G \) rooted on \( n \) representing \( D'(k − 2) \), which has width \( k − 1 \). This contradicts Lemma 3.6.

**Lemma 3.8.** If \( H \) is a graph represented by a construction subtree of \( G \), then \( \text{width}(H) \neq k \).

**Proof.** Suppose, on the contrary, that \( H \) is such a graph with width \( k \). Let \( a \) and \( b \) be the poles of \( H \). By Lemmas 3.2 and 3.7, we know that \( H \notin \{C(k), C'(k − 1), D(k), D'(k − 1)\} \). It now follows from Lemma 3.3, that \( a \) dominates \( \text{Inner}(H) \). Then \( b \) has a non-neighbor \( b' \in \text{Inner}(H) \), for otherwise \( b \) also would dominate \( \text{Inner}(H) \), so Lemma 2.4 would imply that \( H \in \{C(k), D(k)\} \).

Let \( F \) be the graph \( G − \text{Inner}(H) \) to which we add the edge \( ab \) if it is not already present. By the minimality of \( F \) there is an equitable \( k \)-coloring \( α \) of \( F \). To finish the proof, it suffices to deduce a proper coloring \( α' \) of \( G \) that equals \( α \) on \( V(G) \setminus (\text{Inner}(H) \cup \{a\}) \) and such that the multisets \( \{α'(u) \mid u \in \text{Inner}(H) \cup \{a\} \} \) and \( \{1, \ldots, k\} \cup \{α(a)\} \) are equal. We distinguish two cases depending on the value of \( k \).

- **Case 1:** \( k \geq 4 \). Since \( a \) has \( k \) neighbors in \( \text{Inner}(H) \), the vertex \( a \) has at most \( Δ − k \leq k − 3 \) colored neighbors, so we can properly recolor \( a \) with a color \( α'(a) \) different from both \( α(a) \) and \( α(b) \). By Corollary 3.4, \( \text{Inner}(H) \) is a forest and we know that \( |\text{Inner}(H) \setminus \{b'\}| = k − 1 \geq 3 \), so there is an independent set \( A \subset \text{Inner}(H) \setminus \{b'\} \) of size 2. To complete the coloring, we assign \( α(b) \) to \( b' \) and \( α(a) \) to the vertices in \( A \) and we distribute arbitrarily the colors in \( \{1, \ldots, k\} \setminus \{α'(a), α(a), α(b)\} \) to the non-colored vertices.

- **Case 2:** \( k = 3 \). Since \( a \) dominates a set of size \( k \), it holds that \( k \leq Δ \leq 2k − 3 \), so \( k = 3 = Δ \). Moreover, it also follows that \( ab \notin E \). As a consequence of Corollary 3.4, the set \( \text{Inner}(H) \) contains two non-adjacent vertices \( v_1 \) and \( v_2 \). Let \( u \) be the third vertex in \( \text{Inner}(H) \), so \( \text{Inner}(H) = \{v_1, v_2, u\} \). We define \( α'(v_i) := α(b) \), we set \( α'(v_i) := α(a) \) for \( i \in \{1, 2\} \) and we attribute to \( u \) the third color, that is the one in \( \{1, 2, 3\} \setminus \{α(a), α(b)\} \).

In both cases, we obtain an equitable \( k \)-coloring of \( G \), a contradiction.

Our last two lemmas rely on the following observation.

**Observation 3.9.** Let \( m \) be a positive integer and let \( λ_1, \ldots, λ_m \in \{1, 2\} \). If \( A_1 \) and \( A_2 \) are two subsets of the vertices of a graph \( G \) that has no edge between \( A_1 \) and \( A_2 \), then the vertices in \( A_1 \cup A_2 \) can be properly colored using the colors \( 1, \ldots, m \) with respective multiplicities \( λ_1, \ldots, λ_m \) whenever \( \sum_{j=1}^{m} λ_j = |A_1| + |A_2| \) and \( |A_i| \leq m \) for \( i \in \{1, 2\} \).
Proof. For \( s \in \{1, 2\} \), set \( m_s := \{ i \in \{1, \ldots, m\} \mid \lambda_i = s \} \). We know that \( |A_1| + |A_2| = m_1 + 2m_2 = m + m_2 \). We deduce that \( A_1 \leq m_2 \) and \( A_2 \leq m_2 \). This ensures that the following greedy procedure is valid. For every color \( i \) with \( \lambda_i = 2 \), we color one vertex in \( A_1 \) and one vertex in \( A_2 \) with \( i \). After that, it remains to assign arbitrarily the \( m_1 \) colors of multiplicity 1 to the \( m_1 \) non-colored vertices. \( \square \)

Lemma 3.10. Let \( H := P(H_1, H_2) \) be a graph represented by a construction subtree of \( T \). Assume that width\((H_i) \leq k - 2 \) for \( i \in \{1, 2\} \). Then width\((H) \leq k - 2 \).

Proof. We proceed by contradiction. Let \( H \) be a minimal counter-example. By Lemmas 3.6 and 3.8, we know that width\((H) = k + \mu \) for some positive integer \( \mu \).

Let \( a \) and \( b \) be the poles of \( H \). We first prove that every component \( U \) of \( H - \{a, b\} \) has at least \( \mu + 2 \) vertices. Indeed, since the root \( n \) of the construction subtree representing \( H \) is in normal form, the node \( n \) is a parallel node and the subgraph induced by \( \{a, b\} \), from which we remove the edge \( ab \) if it is present, is represented by a children of \( n \), so \( H' := H - U \) is represented by a construction subtree of \( T \). If moreover \( |U| \leq \mu + 1 \), then \( H' \) has width at least \( k - 1 \), thereby contradicting the minimality of \( H \). In particular, width\((H_i) \geq \mu + 2 \geq 3 \) for \( i \in \{1, 2\} \), so \( k \geq 5 \).

Assume for the time being that neither \( a \) nor \( b \) dominates Inner\((H)\). By the remark above and Lemma 3.3, we know that each of Inner\((H_1)\) and Inner\((H_2)\) is dominated by either \( a \) or \( b \). Consequently, we may assume that \( a \) dominates Inner\((H_1)\) but not Inner\((H_2)\) and \( b \) dominates Inner\((H_2)\) but not Inner\((H_1)\). Let \( u_1 \in \text{Inner}(H_1) \) and \( u_2 \in \text{Inner}(H_2) \) be non-neighbors of \( b \) and \( a \), respectively. We distinguish two cases depending on the value of \( \mu \).

First case: \( \mu \leq 2 \). Let \( F \) be the graph \( G - \text{Inner}(H) \) to which we add a crystal \( C(\mu) \) with poles \( a \) and \( b \). Let \( v_1, \ldots, v_\mu \) be the inner vertices of this new crystal. Note that \( |V(G)| - |V(F)| = k \). Since \( d_H(a) \geq \text{width}(H_1) \geq \mu + 2 \) and similarly \( d_H(b) \geq \mu + 2 \), the graph \( F \) has maximum degree at most \( \Delta \).

By the minimality of \( G \) there is an equitable \( k \)-coloring \( \alpha \) of \( F \). Note that the restriction of \( \alpha \) to \( V(G) \setminus \text{Inner}(H) \) is also a proper partial coloring of \( G \). To equitably color \( G \), it suffices to extend this partial coloring to a proper coloring \( \beta \) of \( G \) such that the multiset \( \{ \beta(v) \mid v \in \text{Inner}(H) \} \) equals the multiset \( C := \{1, \ldots, k\} \cup \{ \alpha(v_i) \mid 1 \leq i \leq \mu \} \).

The colors \( \alpha(a) \) and \( \alpha(b) \) both have multiplicity exactly 1 in \( C \) and the maximal multiplicity in \( C \) is at most \( \mu + 1 \). We set \( \beta(u_1) := \alpha(b) \) and \( \beta(u_2) := \alpha(a) \).

If the maximal multiplicity in \( C \) is 2, then Observation 3.9 ensures that we can properly assign the \( k - 2 \) remaining colors since each of \( H_1 \) and \( H_2 \) has at most \( k - 3 \) non-colored vertices. This yields an equitable \( k \)-coloring of \( G \), which is a contradiction.

If the maximal multiplicity in \( C \) is 3, then \( \mu = 2 \), so width\((H_1) \geq k + 2 \geq 4 \). It follows then that \( \text{Inner}(H_1) \setminus \{u_1\} \) contains an independent set \( \{v_1, v_2\} \) of size 2. Indeed, otherwise \( \text{Inner}(H_1) \setminus \{u_1\} \) would be a clique of size at least 3, which with \( a \) or \( b \) would induce a copy of \( K_4 \) in \( G \). Let \( v_3 \) be a vertex in \( \text{Inner}(H_2) \setminus \{u_2\} \). We color \( v_1, v_2 \) and \( v_3 \) with the (unique) color of multiplicity 3 in \( C \). Again, observation 3.9 ensures that we can properly assign the \( k - 3 \) remaining colors since each of \( H_1 \) and \( H_2 \) has at most \( k - 4 \) non-colored vertices.

Second case: \( \mu \geq 3 \). Let \( F \) be the graph \( G - \text{Inner}(H) \) to which we add the edge \( ab \) if it is not already present. By the minimality of \( G \) there is an equitable \( k \)-coloring \( \alpha \) of \( F \). To equitably color \( G \), it suffices to extend \( \alpha \) to a proper coloring of \( G \) such that the multiset \( \{ \alpha(v) \mid v \in \text{Inner}(H) \} \) equals the multiset \( C := \{1, \ldots, k, 1, \ldots, \mu \} \).
As \( \mu \geq 3 \), every component of \( H - \{a, b\} \) has at least \( \mu + 2 \geq 5 \) vertices. Consequently, \( a \) has two non-adjacent non-neighbors \( w_2 \) and \( w'_2 \) in \( \text{Inner}(H_2) \). To see this, consider a component \( U \) of \( \text{Inner}(H_2) \). By Lemma 2.4, the set \( U \) contains only one neighbor of \( a \). It follows that \( |U \setminus N(a)| \geq 3 \), which gives the announced property since by Corollary 3.4 the set \( U \) induces a tree in \( G \). One proves similarly that \( b \) has two non-adjacent non-neighbors \( w_1 \) and \( w'_1 \) in \( \text{Inner}(H_1) \). We set \( \alpha(w_2) := \alpha(a) \), \( \alpha(w_1) := \alpha(b) \) and if necessary \( \alpha(w'_2) := \alpha(a) \) and/or \( \alpha(w'_1) := \alpha(b) \). After this, each of \( H_1 \) and \( H_2 \) has at most \( k - 3 \) non-colored vertices. By Observation 3.9, we can extend this coloring using the \( k - 2 \) remaining colors in \( C \).

From now on, we assume that \( a \) dominates \( \text{Inner}(H) \). Set \( F := G - (\text{Inner}(H) \cup \{a\}) \). By the minimality of \( G \) there is an equitable \( k \)-coloring \( \alpha \) of \( F \). To equitably color \( G \), it suffices to extend \( \alpha \) to a proper coloring of \( G \) such that the multiset \( \{\alpha(v) \mid v \in \text{Inner}(H) \cup \{a\}\} \) equals the multiset \( C := \{1, \ldots, k + \mu + 1\} \), where integers are reduced modulo \( k \). Note that \( k + \mu + 1 \leq \text{width}(H_1) + \text{width}(H_2) + 1 < 2k \) so every color has multiplicity either 1 or 2 in \( C \).

The vertex \( a \) has at most \( k - 3 - \mu \) colored neighbors. There are \( k - 1 - \mu \) colors with multiplicity one in \( C \). Consequently, it is possible to color \( a \) with a color of multiplicity one that is different from \( \alpha(b) \).

We now place the color \( \alpha(b) \). We know that \( \text{width}(H) \geq k + \mu \geq 6 \). By Lemma 2.4, and since each component of \( H \setminus \{a, b\} \) has size at least \( \mu + 2 \geq 3 \), the vertex \( b \) has at least one non-neighbor in each of \( H_1 \) and \( H_2 \). We color a number of these non-neighbors equal to the multiplicity of \( \alpha(b) \) in \( C \) (which is either 1 or 2) using the color \( \alpha(b) \). Observation 3.9 then ensures that we can obtain an equitable coloring with the \( k - 2 \) remaining colors. \( \square \)

**Lemma 3.11.** Let \( H := S(H_1, H_2) \) be a graph represented by a construction subtree of \( T \). Assume that \( \text{width}(H_i) \leq k - 2 \) for \( i \in \{1, 2\} \). Then \( \text{width}(H) \leq k - 2 \).

**Proof.** Suppose, on the contrary, that \( H \) contradicts the statement. Subject to this, we choose \( H \) to have as few vertices as possible. We may assume that \( \text{width}(H) \geq k + 1 \) by Lemmas 3.6 and 3.8. Let \( b \) be the common pole of \( H_1 \) and \( H_2 \) and let \( a \) and \( c \) be the other poles of \( H_1 \) and \( H_2 \), respectively.

**Case 1:** For each \( i \in \{1, 2\} \), the subgraph of \( G \) induced by \( \text{Inner}(H_i) \) contains an independent set \( \{u^1_i, u^2_i\} \) of size 2.

Let \( F \) be the graph \( G - \text{Inner}(H) \) to which we add the edge \( ac \) if it is not already present. By the minimality of \( G \) there is an equitable \( k \)-coloring \( \alpha \) of \( F \), which we aim to extend to \( G \) such that the multiset \( \{\alpha(v) \mid v \in \text{Inner}(H)\} \) equals the multiset \( C := \{1, \ldots, \text{width}(H)\} \), where each integer is reduced modulo \( k \).

We know that \( \text{width}(H) \leq \text{width}(H_1) + \text{width}(H_2) + 1 \leq 2k - 3 \). It follows that there is a color \( \gamma \in \{1, \ldots, k\} \setminus \{\alpha(a), \alpha(c)\} \) of multiplicity one in \( C \). We set \( \alpha(b) := \gamma \), \( \alpha(u^1_1) := \alpha(b) \), \( \alpha(u^2_1) := \alpha(a) \) and if necessary \( \alpha(u^1_2) := \alpha(b) \) and/or \( \alpha(u^2_2) := \alpha(a) \). For each \( i \in \{1, 2\} \), the subgraph \( H_i \) has at most \( k - 3 \) non-colored vertices left, so by Observation 3.9 it is possible to extend the coloring using the \( k - 3 \) remaining colors with the corresponding multiplicities.

**Case 2:** \( \text{Inner}(H_1) \) induces a clique.

We know that \( \text{width}(H_1) \geq \text{width}(H) - \text{width}(H_2) - 1 \geq k + 1 - (k - 2) - 1 \geq 2 \).

By Corollary 3.4, \( \text{Inner}(H_1) \) is a forest, so \( \text{width}(H_1) = 2 \). It forces moreover \( \text{width}(H_2) \) to be \( k - 2 \). This in particular implies that \( k \geq 4 \). Observe that the minimality of \( H \) ensures that each of the poles \( a \) and \( c \) has at least two neighbors in \( H \).
Let $d$ and $e$ be the inner vertices of $H_1$. We define $F$ to be the graph $G - (\text{Inner}(H_1) \cup \text{Inner}(H_2))$ to which we add the edges $ab$, $bc$ and $ac$ if not already present. Note that the graph thus obtained still has maximum degree at most $\Delta$. By the minimality of $G$ there is an equitable $k$-coloring $\alpha$ of $F$.

It remains to deduce an equitable $k$-coloring of $G$. To do so, we recolor $b$ with a color $\gamma$ different from $\alpha(a)$, from $\alpha(b)$ and from $\alpha(c)$, which is possible as $k \geq 4$. Next we color $d$ with $\alpha(b)$ and $e$ with $\alpha(c)$. It now suffices to distribute arbitrarily the colors in $\{1, \ldots, k\} \setminus \{\gamma, \alpha(c)\}$ to the vertices in $\text{Inner}(H_2)$. 

We are now ready to conclude.

Proof of Theorem 1.4. A direct induction on the tree $T$ using Lemmas 3.10 and 3.11 shows that $G$ has at most $k - 2$ inner vertices. This contradicts our assumption that $|V(G)| > k$, thereby finishing the proof of Theorem 1.4. 

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