Kelvin waves of quantized vortex lines in trapped Bose-Einstein condensates

T. P. Simula, T. Mizushima, and K. Machida
Department of Physics, Okayama University, Okayama 700-8530, Japan

We have theoretically investigated Kelvin waves of quantized vortex lines in trapped Bose-Einstein condensates. Counter-rotating perturbation induces an elliptical instability to the initially straight vortex line, driven by a parametric resonance between a quadrupole mode and a pair of Kelvin modes of opposite momenta. Subsequently Kelvin waves rapidly decay to longer wavelengths emitting sound waves in the process. We present a modified Kelvin wave dispersion relation for trapped superfluids and propose a simple method to excite Kelvin waves of specific wave number.

PACS numbers: 03.75.Lm, 67.85.De

In classical hydrodynamics a vortex line is the trajectory obtained by following the local vorticity vector, defined as the curl of the velocity field of the fluid. Wing tip vortices of aircrafts, tornadoes, whirlpools in a flowing water, and cosmic strings provide intuitive mental pictures of vortices. They support a branch of chiral normal modes in which the perturbation propagates along the vortex line and rotates about its unperturbed position therefore distorting the vortex into a helical shape. In 1880, Thomson (Lord Kelvin) derived the dispersion relation for such excitation modes, which are now known as Kelvin waves. These helically vibrating normal modes are objects of fundamental importance in classical hydrodynamics.

Similar structures also exist in superfluids in which vorticity is quantized in units of \( h/m \), where \( h \) is Planck’s constant and \( m \) is the mass of the superfluid particle. Kelvin modes or kelvons of singly quantized rectilinear superfluid vortices have no radial nodes and in axisymmetric systems they are characterized by their angular momentum quantum number \( \ell = -1 \) and momentum \( \hbar k_z \) along the vortex axis. First experimental evidence of Kelvin waves in superfluids was obtained by Hall and they are thought to play an essential role in the formation and decay of superfluid turbulence. In particular, Kelvin wave and Kolmogorov-type cascades are predicted to mediate the energy transfer across length scales in turbulent superfluid systems. Bretin et al. were able to excite and observe Kelvin waves in a single quantum vortex in Bose-Einstein condensates. Their experimental kelvon excitation mechanism was theoretically clarified by Mizushima et al. The ability to address individual vortex lines in these systems opens the possibility to study Kelvin waves and their relation to elementary processes governing superfluid turbulence.

In this Letter we study the microscopic dynamics of Kelvin waves of quantized vortex lines in Bose-Einstein condensates by employing direct simulations of the three-dimensional time-dependent Gross-Pitaevskii equation combined with an analysis of the collective Bogoliubov excitations of the unperturbed state. We treat the dynamics of Kelvin waves fully quantum mechanically in contrast to previously employed phenomenological models such as in Refs. We find counter-rotating quadrupole surface mode (surfon) excitations to decay via Beliaev mechanism into a pair of kelvons of opposite momenta in perfect agreement with Refs. Subsequently the primary kelvons decay into kelvons of longer wavelength by emitting sound waves in the scattering process. This mechanism may help to explain microscopically how superfluid turbulence returns to laminar flow even at zero temperature. We introduce a modification to the semi-classical kelvon dispersion relation extending its validity to trapped superfluids. Finally, we propose an experimentally feasible method to excite kelvons of particular wave number.

Following the experimental procedure of Bretin et al. we consider a Bose-Einstein condensate composed of \( 1.3 \times 10^5 \) \(^{87}\)Rb particles trapped in a harmonic potential whose transverse and axial frequencies are \( \omega_\perp = 2\pi \times 98.5 \) Hz and \( \omega_\parallel = 2\pi \times 11.8 \) Hz, respectively. We model the physics of such system by numerically solving the pure (no additional damping terms included) time-dependent...
Gross-Pitaevskii equation on a parallel computer using the method described in Refs. [19, 20]. We complement the dynamical simulations by directly solving the Bogoliubov-de Gennes eigenvalue equations obtaining the full Kelvin wave dispersion relation [13, 16]. Our initial condition is a near-axisymmetric single-quantum vortex which possess one unit of angular momentum per particle. Such state, shown in Fig. 1(a), is a local energy minimum in the non-rotating trap rather than an absolute ground state which would have zero vorticity.

At $t = 0$ we suddenly switch on an elliptical rotating perturbation

$$V_{\text{pert}}(\mathbf{r}, t) = \frac{1}{2} \epsilon m \omega^2 \left[(x^2 - y^2) \cos(2\Omega t) + 2xy \sin(2\Omega t)\right],$$

where $\epsilon = 0.025$, and $m$ is the mass of the atom. The frequency $\Omega = -0.6\omega_\perp$ is resonant with the counter-rotating (with respect to the superfluid flow) $\ell = -2$ quadrupole surfon of the unperturbed condensate [14]. This resonant external drive transfers population from the initial vortex state to the counter-rotating quadrupole surface thus changing the total orbital angular momentum of the system. When the perturbation is switched off at $t = 35$ ms, the $z$-component of the angular momentum has been reduced from $L_z/N = 1\hbar$ to 0.47$\hbar$. From there on $L_z$ is a constant of motion since the bare trap potential is cylindrically symmetric. In contrast, if the system is continuously rotated, the growing surface instability leads to (anti)vortex nucleation as in nonrotating systems [21, 22, 23, 24, 25] and eventually $L_z$ changes its sign. If $\Omega$ is held constant the angular momentum transfer is limited since the perturbation phase of resonance when the quadrupole surfon becomes frequency shifted due to the changing angular momentum.

The vortex line stays straight until $t = 64$ ms at which stage the quadrupolar condensate density modulation in the plane perpendicular to the vortex axis induces squeezing in the vicinity of the vortex core. The superflow then undergoes an elliptical instability, the vortex line length stretches and its shape deforms from straight to sinusoidal, oscillating in the plane of the elliptical deformation as illustrated in Fig. 2(b). The elliptical instability explains how the two-dimensional perturbation yields three-dimensional flow instability, and it is driven by a parametric resonance between the $\ell = -2$ surfon and a pair of $\ell = -1$ kelvons with opposite axial momenta. The elliptical instability mechanism has been reviewed by Kerswell in the context of classical hydrodynamics [26]. As the instability grows the shape of the vortex changes from plane-sinusoidal to helical as shown in Fig. 2(c). Both the vortex core radius and the amplitude of these Kelvin waves are larger near the ends of the condensate, explained by the lower local particle density in those regions. These Kelvin waves decay rapidly via phonon emission into kelvons of longer wave-length. The phonons are visible as ripples on the condensate surface in Fig. 1. After the primary kelvons have decayed the system is left in a bent vortex state, Fig. 1(d), and the final configuration is determined by the amount of angular momentum left in the system after the external perturbation is switched off. If, instead, the co-rotating quadrupole surfon is resonantly populated using $\Omega = 0.8\omega_\perp$ in Eq. 1, the vortex remains straight for all times and the $\ell = +2$ surfon population is undamped as is verified by our simulations. In this case the conserved angular momentum of the system after the 35 ms excitation is $L_z/N = 1.51\hbar$ and the condensate density oscillates with the characteristic quadrupole frequency $f_{\ell = +2} = 102$ Hz.

To further quantify the kelvon excitation and decay process, we have located the planar position $(x_v, y_v)$ of the vortex phase singularity as functions of $z$ and $t$. Fourier transform $F$ of the complex signal $\Theta(z) = \arctan(y_v/x_v) + i\arctan(x_v/y_v)$ reveals clear peaks at wave vectors corresponding to the kelvon excitations present in the system. Figure 2(a) shows $|F[\Theta(z)]|^2$ as a function of time for the $\ell = -2$ initial perturbation. There emerges a strong signal at $|k_z| = 0.9\mu m^{-1}$ which shifts rapidly around 150 ms to $|k_z| = 0.2\mu m^{-1}$. Figure 2(b) shows the normalized quadrupole moment $Q(t) = \int xyn(r) \, dr$, where $n(r)$ is the condensate density, of the system as function of time for the $\ell = -2$ (solid line) and $\ell = +2$ (dashed line) perturbations. The signal for counter-rotated case is strongly damped due to the resonant surfon-kelvon coupling while the co-rotated
The axial trap potential. Nevertheless, we may assign a well-defined momentum \( \hbar k_z \) for the Bogoliubov modes. The two diamonds near the kelvon dispersion are the initial and final kelvon modes, shown respectively in Fig.3(b) and Fig.3(d), whose wave vectors and energies (rotation frequencies) are extracted from the time-dependent simulations. The sign of the kelvon frequency determines the sense of rotation of the vortex in the laboratory frame. In Fig.4 we have plotted the Bogoliubov \( u_k(x, z) \) modes for a few of the lowest collective even (left) and odd (right) \( z \)-parity kelvon excitations. The numbers on top and bottom of each frame indicate the excitation frequency and momentum, respectively, of the mode and correspond to those in Fig.3. All kelvons are highly localized within the vortex core.

The Kelvin wave dispersion relation involving Bessel functions is derived for an infinitely long classical vortex and its long wavelength limit is often applied to superfluid vortices by simply replacing the classical circulation by its quantum mechanical counterpart. Guided by our numerical experiments for \( N \) in the range \( (1-50) \times 10^4 \), we propose a modification to such semi-classical dispersion relation extending its validity to the finite and inhomogeneous systems:

\[
\omega(k_0 + k_z) \approx \omega_0 + \frac{\hbar k^2}{2m} \log \left( \frac{1}{|r_z k_z|} \right), \quad |r_z k_z| \ll 1
\]
the density. As shown in Fig. 3, Eq. (2) (solid curve) is in an excellent agreement with the Bogoliubov spectrum (except for the pure bending modes). We have also plotted Eq. (2) using \(\omega_0 = k_0 = 0\) and \(r_c = \xi = 0.3\mu m\) (dashed curve) showing the failure of the original kelvon dispersion relation. We emphasize that the usual density dependent healing length \(\xi\) provides poor estimate for the constant \(r_c\) for all values of \(N\) studied here.

Here we propose a simple method to excite kelvons of specific wave number on demand. We may still make use of the parametric surfon-kelvon resonance by simply shifting the whole kelvon dispersion curve with respect to the quadrupole frequency. This is achieved by adding a Gaussian pinning potential

\[
V_{\text{pin}}(x, y, z) = \frac{V_0 \sigma_0^2}{\sigma(z)^2} \exp\left(-\frac{2[x^2 + y^2]}{\sigma(z)^2}\right) \tag{3}
\]

where \(\sigma(z) = \sigma_0 \sqrt{1 + (z/z_R)^2}\) and we have chosen \(\sigma_0 = 2\mu m\) and a Rayleigh range of \(z_R = 20\mu m\). The vortex pinning effectively modifies the local potential at the vortex core \(21, 22\). The strength of the applied potential \(V_0\) may be used to control the wave vector of the kelvon resonant with the quadrupole surfon. Since the pinning potential is localized in the centre of the trap it does not radically affect the resonant quadrupole surfon frequency itself. To demonstrate the feasibility of this method we have performed simulations using different values for \(V_0\) in Eq. (3) the only difference to the previously described excitation method being the addition of the pinning potential at \(t = 0\). Setting \(V_0 = \{8.0, 6.0, 4.0, 2.0, -2.0, -4.0\}h\omega_{\perp}\) results in an initial kelvon excitation with \(|k_z| = \{0.2, 0.4, 0.7, 0.7, 0.9, 1.0\} \mu m^{-1}\), respectively. Experimentally, this method could be realized using tightly focused optical potentials allowing full control over the kelvon excitation process.

In conclusion, we have theoretically studied the microscopic excitation and decay mechanism of superfluid Kelvin waves in trapped Bose-Einstein condensates. Our results provide direct verification for the experimental observations of Bretin et al. \(14\). We interpreted the coupling between the external perturbation and the excitation of Kelvin waves in terms of an elliptical instability mechanism driven by a parametric resonance between the counter-rotating quadrupole surfon and a pair of kelvons of opposite moments. Subsequently, the Kelvin waves were found to decay to longer wavelength excitations via phonon emission. We presented a modification to the Kelvin wave dispersion relation for inhomogeneous superfluids. Finally, we proposed an experimental method to excite Kelvin waves of any wave number. We have shown that the pure kelvon decay provides a powerful damping mechanism in superfluid systems even at zero temperature. This may be viewed as an elementary decay channel for superfluid turbulence, the higher order nonlinear processes involving vortex (self) reconnections.

Our methods also enable us to go beyond the in-plane Tkachenko modes \(31, 32, 33, 34\) and to study the full dispersion relation of Kelvin-Tkachenko collective modes in three-dimensional vortex lattices.

We would like to thank N. Hayashi for discussions. This work was supported by the Japan Society for the Promotion of Science (JSPS).

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