ON SOME NEW RESULTS ON ANISOTROPIC SINGULAR PERTURBATIONS OF SECOND ORDER ELLIPTIC OPERATOR

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Abstract. In this article, we deal with some problems involving a class of singularly perturbed elliptic operator. We prove the asymptotic preserving of a general Galerkin method associated to a semilinear problem. We use a particular Galerkin approximation to estimate the convergence rate on the whole domain, for the linear problem. Finally, we study the asymptotic behavior of the semigroup generated.

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1. INTRODUCTION

Anisotropic singular perturbations problems was introduced by Chipot in [1], these problems can model diffusion phenomena when the diffusion parameters become small in certain directions. We refer the reader to [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] for several works on this topic. In this article, we will study some new theoretical aspects which have not been studied before for these problems.

Let us consider the following perturbed elliptic problem

\[ \beta(u_\varepsilon) - \text{div}(A_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \]  

supplemented with the boundary condition

\[ u_\varepsilon = 0 \text{ on } \partial \Omega. \]  

Here, \( \Omega = \omega_1 \times \omega_2 \) where \( \omega_1 \) and \( \omega_2 \) are two bounded open sets of \( \mathbb{R}^q \) and \( \mathbb{R}^{N-q} \), with \( N > q \geq 1 \), and \( f \in L^2(\Omega) \).

We denote by \( x = (x_1, ..., x_N) = (X_1, X_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q} \) i.e. we split the coordinates into two parts. With this notation we set

\[ \nabla = (\partial_{x_1}, ..., \partial_{x_N})^T = \begin{pmatrix} \nabla X_1 \\ \nabla X_2 \end{pmatrix}, \]

where

\[ \nabla X_1 = (\partial_{x_1}, ..., \partial_{x_q})^T \text{ and } \nabla X_2 = (\partial_{x_{q+1}}, ..., \partial_{x_N})^T. \]

The function \( A = (a_{ij})_{1 \leq i,j \leq N} : \Omega \to \mathcal{M}_N(\mathbb{R}) \) satisfies the ellipticity assumptions

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There exists $\lambda > 0$ such that for a.e. $x \in \Omega$
\[ A\xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N. \quad (3) \]

The coefficients of $A$ are bounded that is
\[ a_{ij} \in L^\infty(\Omega) \quad \text{for any } (i,j) \in \{1,2,\ldots,N\}^2. \quad (4) \]

We have decomposed $A$ into four blocks
\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \]
where $A_{11}$, $A_{22}$ are $q \times q$ and $(N - q) \times (N - q)$ matrices respectively. For $\epsilon \in (0,1]$ we have set
\[ A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}. \]

The function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions:

\[ \exists M \geq 0 : \forall s \in \mathbb{R}, |\beta(s)| \leq M (1 + |s|). \quad (6) \]

The weak formulation of the problem (1)-(2) is
\[ \begin{cases} \int_\Omega \beta(u_\epsilon) \varphi dx + \int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in H^1_0(\Omega) \\ u_\epsilon \in H^1_0(\Omega), \end{cases} \quad (7) \]

where the existence and uniqueness is a consequence of the assumptions (3) – (6).

Taking $\epsilon = 0$ in (1) gives
\[ \beta(u) - \text{div}_{X_2} (A_{22} \nabla u) = f \text{ on } \Omega, \quad (8) \]
supplemented with the boundary condition
\[ u(X_1,\cdot) = 0 \text{ in } \partial \omega_2, \text{ for } X_1 \in \omega_1. \quad (9) \]

We introduce the functional space
\[ H^1_0(\Omega;\omega_2) = \left\{ v \in L^2(\Omega) \text{ such that } \nabla_{X_2} v \in L^2(\Omega)^{N-q} \text{ and for a.e. } X_1 \in \omega_1, v(X_1,\cdot) \in H^1_0(\Omega) \right\}, \]
equipped with the norm $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}$. Notice that this norm is equivalent to
\[ \left( \|\cdot\|_{L^2(\Omega)}^2 + \|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}^2 \right)^{1/2}, \]
thanks to Poincaré’s inequality
\[ \|v\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla_{X_2} v\|_{L^2(\Omega)^{N-q}}, \text{ for any } v \in H^1_0(\Omega;\omega_2). \quad (10) \]

One can prove that $H^1_0(\Omega;\omega_2)$ is a Hilbert space. The space $H^1_0(\Omega)$ will be normed by $\|\nabla(\cdot)\|_{L^2(\Omega)^N}$. One can check immediately that the imbedding $H^1_0(\Omega) \hookrightarrow H^1_0(\Omega,\omega_2)$ is continuous.
The weak formulation of the limit problem (8) – (9) is given by

\[
\begin{cases}
\int_{\omega_2} \beta(u(X_1, \cdot)) \psi dX_2 + \int_{\omega_2} A_{22}(X_1, \cdot) \nabla X_2 u(X_1, \cdot) \cdot \nabla X_2 \psi dX_2 \\
u(X_1, \cdot) \in H^1_0(\omega_2), \text{ for a.e. } X_1 \in \omega_1
\end{cases}
\]  

This problem has been studied in [9], and the author proved the following (see Proposition 4 in the above reference)

**Theorem 1.1.** Under assumptions (3), (4), (5) and (6) we have

\[ u_\epsilon \to u \text{ in } L^2(\Omega), \quad \epsilon \nabla X_1 u_\epsilon \to 0 \text{ in } L^2(\Omega)^q \text{ and } \nabla X_2 u_\epsilon \to \nabla X_2 u \text{ in } L^2(\Omega)^{N-q}, \]

where \( u_\epsilon \) is the unique solution of (7) in \( H^1_0(\Omega) \) and \( u \) is the unique solution to (11) in \( H^1_0(\Omega; \omega_2) \).

Notice that for \( \varphi \in H^1_0(\Omega; \omega_2) \), and for a.e \( X_1 \) in \( \omega_1 \) we have \( \varphi(X_1, \cdot) \in H^1_0(\omega_2) \), testing with it in (11) and integrating over \( \omega_1 \) yields

\[ \int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H^1_0(\Omega; \omega_2). \]  

(12)

This paper is organized as follows:

- As a first main result, we will prove the asymptotic preserving of the general Galerkin method for the elliptic problem (1-2). This concept has been introduced by S. Jin in [12] and it could be illustrated by the following commutative diagram

\[
P_{\epsilon,n} \xrightarrow{n \to \infty} P_{\epsilon} \xrightarrow{\epsilon \to 0} P_0
\]

Here, \( P_{\epsilon,n} \) is the Galerkin approximation of the infinite dimensional perturbed problem \( P_{\epsilon} \), and \( P_n \) is the Galerkin approximation of the infinite dimensional limit problem \( P_0 \). We will derive an estimation of the error for a general Galerkin method, and by using a Céa’s type lemmas we prove the asymptotic-preserving of the method.

- As a second main result, we will prove, in the linear case, a new result on the estimation of the global convergence rate, such a result is of the form \( \| \nabla X_2 (u_\epsilon - u) \|_{L^2(\Omega)^{N-q}} \leq C\epsilon \). This estimation is an improvement of the local one proved by Chipot and Guesmia in [3]. Our arguments are founded on the use of a particular Galerkin approximation constructed by a tensor product.

- In section 4 we will give our third main result on the asymptotic behavior of the semigroup generated by the perturbed elliptic operator \( \text{div}(A_{\epsilon} \nabla \cdot) \), and we will give a simple application to linear parabolic problems.

Finally, to make the paper readable, we put some intermediate technical lemmas in the appendix.

2. **Main theorems for the elliptic problem**

**Definition 2.1.** Let \((V_n)\) be a sequence of finite dimensional subspaces of a Hilbert space \(H\). We say that \((V_n)\) approximates \(H\), if for every \(w \in H\).

\[
\inf_{v \in V_n} \|w - v\|_H \to 0 \text{ as } n \to \infty
\]
For a sequence \((V_n)\) of a finite dimensional spaces of \(H^1_0(\Omega)\), and for every \(\epsilon \in (0,1]\) and \(n \in \mathbb{N}\), we denote \(u_{\epsilon,n}\) the unique solution of

\[
\begin{cases}
\int_{\Omega} \beta(u_{\epsilon,n}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon,n} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V_n. \\
u_{\epsilon,n} \in V_n.
\end{cases}
\] (13)

We suppose that

\[
\partial_{x_i} a_{ij} \in L^\infty(\Omega), \partial_{x_i} a_{ij} \in L^\infty(\Omega) \text{ for } i = 1, \ldots, q \text{ and } j = q + 1, \ldots, N.
\] (14)

We have the following

**Theorem 2.2.** Let \(\Omega = \omega_1 \times \omega_2\) where \(\omega_1\) and \(\omega_2\) are two bounded open sets of \(\mathbb{R}^q\) and \(\mathbb{R}^{N-q}\), with \(N > q \geq 1\). Suppose that \(f \in L^2(\Omega)\) and assume (3), (4), (5), (6), and (14). Let \((V_n)\) be a sequence of finite dimensional spaces of \(H^1_0(\Omega)\) which approximates it in the sense of Definition 2.1. Let \((u_{\epsilon,n})\) be the sequence of solutions of (13) then we have:

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} u_{\epsilon,n} = \lim_{n \to \infty} \lim_{\epsilon \to 0} u_{\epsilon,n} = u, \text{ in } H^1_0(\Omega;\omega_2),
\]

where \(u\) is the unique solution of (11) in \(H^1_0(\Omega;\omega_2)\).

Our second result concerns the estimation of the rate of convergence for the continuous problem (7) in the linear case, this result could be seen as a refinement of a result proved in [3]. In the above reference, the authors proved the following interior estimation for the linear problem

For every \(\omega'_1 \subset \subset \omega_1\) open:

\[
\|\nabla X_2(u_\epsilon - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(\epsilon), \text{ and } \|\nabla X_1(u_\epsilon - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(1).
\] (15)

where they have supposed that

\[
\nabla X_1f \in L^2(\Omega)^q,
\]

assumption (14) and

\[
\nabla X_1 A_{22} \in L^\infty(\Omega).
\]

Our contribution consists in extending (15) to the whole domain \(\Omega\), to obtain such a result we take an additional hypothesis on \(A\) and \(f\), namely:

For a.e. \(X_2 \in \omega_2 : f(\cdot, X_2) \in H^1_0(\omega_1)\),

\[
\] (17)

and

\[
\] (18)

**Theorem 2.3.** Let \(\Omega = \omega_1 \times \omega_2\) where \(\omega_1\) and \(\omega_2\) are two bounded open sets of \(\mathbb{R}^q\) and \(\mathbb{R}^{N-q}\), with \(N > q \geq 1\). Suppose that \(\beta = 0\), and let us assume that \(A\) satisfies (3), (4), (14) and (18). Let \(f \in L^2(\Omega)\) such that (16) and (17), then there exists \(C\) depending only on \(f, \lambda, C_{\omega_2}\) and \(A\) such that

\[
\forall \epsilon \in (0,1] : \|\nabla X_2(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \leq C \epsilon,
\]

where \(u_\epsilon\) is the unique solution of (7) in \(H^1_0(\Omega)\) and \(u\) is the unique solution to (11) in \(H^1_0(\Omega;\omega_2)\). Moreover we have

\[
u \in H^1_0(\Omega) \text{ and } \nabla X_1(u_\epsilon - u) \rightharpoonup 0 \text{ weakly in } L^2(\Omega)^q.
\]

The constant \(C\) is of the form \(C_1 \|\nabla X_1 f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)}\) where \(C_1,C_2\) are dependent on \(A, \lambda, C_{\omega_2}\).

The proof of this theorem will be done in two steps. First, we give the proof in the case \(f \in H^1_0(\omega_1) \otimes H^1_0(\omega_2)\), and next that we conclude by a density argument. Let us recall this basic density chain rule, which will be used throughout this article: If \((E, \tau)\) and \((F, \tau')\) are two topological spaces such that \(E \subset F\), and \(E\) is dense in \(F\) and the canonical injection \(E \rightarrow F\) is continuous then, every dense subset in \((E, \tau)\) is also dense in \((F, \tau')\).
Remark 2.4. The hypothesis (17) is necessary to obtain the global boundedness of \( \nabla X_1(u_\epsilon - u) \). We can observe that through this 2d example, we take

\[
A = \text{id}_2, \quad f : (x_1, x_2) \mapsto \cos(x_1) \sin(x_2), \quad \text{and} \quad \Omega = (0, \pi) \times (0, \pi).
\]

In this case, we have \( u(x_1, x_2) = \cos(x_1) \sin(x_2) \). The quantity \( \| \nabla X_1(u_\epsilon - u) \|_{L^2(\Omega)^q} \) could not be bounded. Indeed, if we suppose the converse then according to Theorem 1.1 there exists a subsequence still labeled \( u_\epsilon \) such that \( \nabla X_1(u_\epsilon - u) \rightharpoonup 0 \) weakly in \( L^2(\Omega)^q \), and \( \| \nabla X_1(u_\epsilon - u) \|_{L^2(\Omega)^{N-q}} \rightharpoonup 0 \). Whence \( u \in H^1_0(\Omega) \) which is a contradiction.

Let us finish by giving this remark which will be used later in section 4.

Remark 2.5. Suppose that \( \beta : s \mapsto \mu s \), for some \( \mu > 0 \), and suppose that assumptions of Theorem 2.3 hold, then we have the same results of Theorem 2.3 with the same constants. Assume, in addition, that the block \( A_{12} \) satisfies the following

\[
\partial_{x_1,x_2}^2 a_{ij} \in L^2(\Omega), \quad \text{for} \quad i = 1, \ldots, q, \quad j = q + 1, \ldots, N,
\]

then we have the following

\[
\forall \epsilon \in (0, 1] : \| \nabla X_2(u_\epsilon - u) \|_{L^2(\Omega)} \leq \epsilon \left( C_1' \| \nabla X_1 f \|_{L^2(\Omega)^q} + C_2' \| f \|_{L^2(\Omega)} \right),
\]

where \( C_1', C_2' \) are only dependent on \( A, \lambda, C_\omega \).

3. The Analysis of a General Galerkin Method

3.1. Preliminaries

Let \( V \subset H^1_0(\Omega) \) be a closed subspace of \( H^1_0(\Omega, \omega_2) \). Notice that \( V \) is closed in \( H^1_0(\Omega) \), thanks to the continuous imbedding \( H^1_0(\Omega) \hookrightarrow H^1_0(\Omega, \omega_2) \). Let \( f \in L^2(\Omega) \), we denote by \( u_{\epsilon,V,f} \) the unique solution of

\[
\left\{ \begin{array}{l}
\int_{\Omega} \beta(u_{\epsilon,V,f}) \varphi dx + \int_{\Omega} A \nabla u_{\epsilon,V,f} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in V \\
u_{\epsilon,V,f} \in V.
\end{array} \right.
\]

(20)

We denote by \( u_{V,f} \) the unique solution of

\[
\left\{ \begin{array}{l}
\int_{\Omega} \beta(u_{V,f}) \varphi dx + \int_{\Omega} A_{22} \nabla X_2 u_{V,f} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in V \\
u_{V,f} \in V.
\end{array} \right.
\]

(21)

The existence of \( u_{\epsilon,V,f} \) follows form the Schauder fixed point theorem. For the existence of \( u_{V,f} \) see Appendix C. The uniqueness, for the two problems, follows immediately from (3) and (5). Now, let us begin by some preliminary lemmas

Lemma 3.1. Under assumptions of Theorem 1.1 and for any \( \epsilon \in (0, 1] \) we have the following bounds

\[
\| \nabla X_2 u_{\epsilon,V,f} \|_{L^2(\Omega)^{N-q}} \leq \frac{C_\omega \| f \|_{L^2(\Omega)}}{\lambda},
\]

(22)

\[
\| \nabla X_1 u_{\epsilon,V,f} \|_{L^2(\Omega)^q} \leq \frac{C_\omega \| f \|_{L^2(\Omega)}}{\epsilon \lambda},
\]

(23)

\[
\| \nabla X_2 u_{V,f} \|_{L^2(\Omega)^{N-q}} \leq \frac{C_\omega \| f \|_{L^2(\Omega)}}{\lambda},
\]

(24)

\[
\| \nabla X_2 u_{f} \|_{L^2(\Omega)^{N-q}} \leq \frac{C_\omega \| f \|_{L^2(\Omega)}}{\lambda},
\]

(25)
\[ \| \beta(u_{e.V,f}) \|_{L^2(\Omega)} \leq M \left( \frac{\| f \|_{L^2(\Omega)}}{\lambda} \right)^{\frac{1}{2}} + \frac{C^2_{\omega_2}}{\lambda}, \] (26)

\[ \| \beta(u_{V,f}) \|_{L^2(\Omega)} \leq M \left( \frac{\| f \|_{L^2(\Omega)}}{\lambda} \right)^{\frac{1}{2}} + \frac{C^2_{\omega_2}}{\lambda}, \] (27)

\[ \| \beta(u_f) \|_{L^2(\Omega)} \leq M \left( \frac{\| f \|_{L^2(\Omega)}}{\lambda} \right), \] (28)

where \( u_f \) denotes the unique solution of (11).

**Proof.** These bounds follow easily from a suitable choice of the test functions, monotonicity and ellipticity assumptions. Let us prove, for example, (25) and (28).

According to Theorem 1.1 one can take \( \varphi = u_f \) in (12), using ellipticity assumption and the fact that \( \int_{\Omega} \beta(u_f)u_f dx \geq 0 \) (thanks to (5)) we obtain

\[ \lambda \int_{\Omega} |\nabla u_f|^2 dx \leq \int_{\Omega} f u_f dx. \]

By the Cauchy-Schwartz inequality and Poincaré’s inequality (10) we obtain (25). Now, using assumption 6 we obtain

\[ |\beta(u_f)|^2 \leq M^2 (1 + |u_f|)^2, \]

integrating over \( \Omega \) and applying Minkowski inequality, (10) and (25) we obtain (28).

Using the above lemma, one can prove the following Céa’s type lemma

**Lemma 3.2.** **Under assumptions of Theorem 1.1 we have**

\[ \| \nabla X_2(u_{V,f} - u_f) \|_{L^2(\Omega)^N} \leq C_{\text{Céa}} \left( \inf_{v \in V} \| \nabla X_2(v - u_f) \|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}, \] (29)

**and for any** \( \epsilon \in (0,1] \)

\[ \| \nabla (u_{e.V,f} - u_{e,f}) \|_{L^2(\Omega)^N} \leq \frac{C'_{\text{Céa}}}{\epsilon^2} \left( \inf_{v \in V} \| \nabla v - \nabla u_f \|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}. \] (30)

where

\[ C_{\text{Céa}} = \left[ 2MC_{\omega_2} \left( |\Omega|^{\frac{1}{2}} + \frac{C^2_{\omega_2}}{\lambda} \| f \|_{L^2(\Omega)} \right) + \| A_{22} \|_{L^\infty(\Omega)} \frac{2C_{\omega_2}}{\lambda} \| f \|_{L^2(\Omega)} \right]^{\frac{1}{2}}, \]

and

\[ C'_{\text{Céa}} = \left[ 2MC_{\Omega} \left( |\Omega|^{\frac{1}{2}} + \frac{C^2_{\omega_2}}{\lambda} \| f \|_{L^2(\Omega)} \right) + \| A \|_{L^\infty(\Omega)} \frac{2C_{\Omega}}{\lambda} \| f \|_{L^2(\Omega)} \right]^{\frac{1}{2}}, \]

here \( C_{\Omega} \) is the Poincaré’s constant of \( \Omega \), and \( u_{e.f} \) is the unique solution of (7).

**Proof.** The proofs of these two inequalities are similar, so let us prove the first one. Using the Galerkin orthogonality one has, for every \( v \in V \):

\[ \int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(u_{V,f} - u_f) dx + \| \nabla X_2(u_{V,f} - u_f) \|_{L^2(\Omega)^N} \]

\[ = \int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(v - u_f) dx + \int_{\Omega} A_{22} \nabla X_2(u_{V,f} - u_f) \cdot \nabla X_2(v - u_f) dx \]
Lemma 3.4. Under assumptions of Theorem 3.2. Error estimates in the Galerkin method

Proof. By subtracting (21) from (20) we get, for every $v \in V$:

$$\int_{\Omega} (\beta(u_{e,V}) - \beta(u_{f})) (u_{e,V} - u_{f}) dx \geq 0,$$

then by Cauchy-Schwarz and Poincaré’s inequalities we derive

$$|\nabla X_2 (u_{e,V} - u_{f})|^2_{L^2(\Omega)^{N-q}} \leq \left[ C_{o_2} \beta (u_{e,V}) - \beta (u_{f}) \right] + \left[ A_{22} \|u_{f}\|_{L^2(\Omega)} \right] \times \|\nabla X_2 (v - u_{f})\|_{L^2(\Omega)^{N-q}}.$$

Now, by using (27), (28) and the triangle inequality we obtain

$$\|\nabla X_2 (u_{e,V} - u_{f})\|^2_{L^2(\Omega)^{N-q}} \leq 2 \left[ M C_{o_2} \left( |\Omega|^\frac{1}{2} + \frac{C_{o_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \left[ A_{22} \|u_{f}\|_{L^2(\Omega)} \right] \times \|\nabla X_2 (v - u_{f})\|_{L^2(\Omega)^{N-q}}.$$

### Remark 3.3
1) If $\beta = 0$ (the linear case) then we have for any $\epsilon \in (0, 1]$

$$\|\nabla u_{e,V,f} - \nabla u_{e,f}\|_{L^2(\Omega)^N} \leq \frac{1}{\lambda} \inf_{v \in V} \|\nabla v - \nabla u_{e,f}\|_{L^2(\Omega)^N}.$$

$$\|\nabla X_2 u_{e,V,f} - \nabla X_2 u_{e,f}\|_{L^2(\Omega)^{N-q}} \leq \frac{1}{\lambda} \inf_{v \in V} \|\nabla X_2 v - \nabla X_2 u_{e,f}\|_{L^2(\Omega)^{N-q}}.$$

2) If $\beta$ is Lipschitz, then we can obtain estimations similar to those of the linear case.

### 3.2. Error estimates in the Galerkin method

**Lemma 3.4.** Under assumptions of Theorem 1.1, suppose in addition that (14) holds. Then we have

$$\|\nabla X_2 (u_{e,V,f} - u_{e,f})\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 \|\nabla X_1 u_{e,V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right)$$

and

$$\|\nabla X_1 (u_{e,V,f} - u_{e,f})\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 \|\nabla X_1 u_{e,V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right)$$

where

$$C_1 = \left( \frac{4(C + C'')}{\lambda} \right)^{\frac{1}{2}} \text{ and } C_2 = \frac{2\sqrt{C''}C_{o_2}}{\lambda^{3/2}}$$

and where $C, C', C''$ are given by (32), (34) and (35). Notice that these constants are independent of $\epsilon, V$ and $f$.

*Proof.* By subtracting (21) from (20) we get, for every $v \in V$:

$$\int_{\Omega} (\beta(u_{e,V}) - \beta(u_{f})) v dx + \epsilon^2 \int_{\Omega} A_{11} \nabla X_1 u_{e,V,f} \cdot \nabla X_1 v dx$$

$$+ \epsilon \int_{\Omega} A_{12} \nabla X_2 u_{e,V,f} \cdot \nabla X_1 v dx + \epsilon \int_{\Omega} A_{21} \nabla X_1 u_{e,V,f} \cdot \nabla X_2 v dx$$

$$+ \int_{\Omega} A_{22} \nabla X_2 (u_{e,V,f} - u_{e,f}) \cdot \nabla X_2 v dx = 0.$$
Testing with $v = u_{e,V,f} - u_{V,f}$ we obtain
\[
\int_{\Omega} (\beta(u_{e,V,f}) - \beta(u_{V,f}))(u_{e,V,f} - u_{V,f})dx + \int_{\Omega} A_{e} \nabla(u_{e,V,f} - u_{V,f}) \cdot \nabla(u_{e,V,f} - u_{V,f})
\]
\[= -\varepsilon^{2} \int_{\Omega} A_{11} \nabla X_{1} u_{V,f} \cdot \nabla X_{1}(u_{e,V,f} - u_{V,f})dx - \varepsilon \int_{\Omega} A_{12} \nabla X_{2} u_{V,f} \cdot \nabla X_{2}(u_{e,V,f} - u_{V,f})dx \]
\[- \varepsilon \int_{\Omega} A_{21} \nabla X_{1} u_{V,f} \cdot \nabla X_{2}(u_{e,V,f} - u_{V,f})dx.
\]
whence, by using (5) and the ellipticity assumption we get
\[
\varepsilon^{2} \lambda \int_{\Omega} |\nabla X_{1}(u_{e,V,f} - u_{V,f})|^{2} dx + \lambda \int_{\Omega} |\nabla X_{2}(u_{e,V,f} - u_{V,f})|^{2} dx \leq \varepsilon^{2} \int_{\Omega} \nabla X_{1}(u_{e,V,f} - u_{V,f}) \cdot \nabla X_{1}(u_{e,V,f} - u_{V,f})dx - \lambda \int_{\Omega} \nabla X_{2}(u_{e,V,f} - u_{V,f}) \cdot \nabla X_{2}(u_{e,V,f} - u_{V,f})dx \]
\[- \lambda \int_{\Omega} A_{21} \nabla X_{1} u_{V,f} \cdot \nabla X_{2}(u_{e,V,f} - u_{V,f})dx.
\]
Estimating the first and the last term of the second member of the above inequality. By using Young’s inequality we obtain,
\[
- \varepsilon^{2} \int_{\Omega} A_{11} \nabla X_{1} u_{V,f} \cdot \nabla X_{1}(u_{e,V,f} - u_{V,f})dx
\]
\[\leq \frac{\varepsilon^{2}}{2} \lambda \int_{\Omega} |\nabla X_{1}(u_{e,V,f} - u_{V,f})|^{2} dx + \frac{\lambda}{2} \int_{\Omega} |\nabla X_{1}(u_{e,V,f} - u_{V,f})|^{2} dx,
\]
and
\[
- \varepsilon \int_{\Omega} A_{21} \nabla X_{1} u_{V,f} \cdot \nabla X_{2}(u_{e,V,f} - u_{V,f})dx
\]
\[\leq \varepsilon \frac{\lambda}{2} \int_{\Omega} |\nabla X_{1} u_{V,f}|^{2} dx + \frac{\lambda}{2} \int_{\Omega} |\nabla X_{2}(u_{e,V,f} - u_{V,f})|^{2} dx,
\]
thus
\[
\frac{\varepsilon^{2}}{2} \lambda \|\nabla X_{1}(u_{e,V,f} - u_{V,f})\|_{L^{2}(\Omega)}^{2} + \frac{\lambda}{2} \|\nabla X_{2}(u_{e,V,f} - u_{V,f})\|_{L^{2}(\Omega)^{N-q}}^{2}
\]
\[\leq C \varepsilon^{2} \int_{\Omega} |\nabla X_{1} u_{V,f}|^{2} dx - \varepsilon \int_{\Omega} A_{12} \nabla X_{2} u_{V,f} \cdot \nabla X_{1}(u_{e,V,f} - u_{V,f})dx,
\]
(31)
where
\[
C = \frac{\|A_{21}\|_{L^{\infty}(\Omega)}^{2} + \|A_{11}\|_{L^{\infty}(\Omega)}^{2}}{2\lambda}.
\]
(32)
Now, we estimate $- \varepsilon \int_{\Omega} A_{12} \nabla X_{1} u_{V,f} \cdot \nabla X_{1}(u_{e,V,f} - u_{V,f})dx$. Since $u_{e,V,f} - u_{V,f} \in H_{1}^{1}(\Omega)$ and $\partial_{x} a_{ij} \in L^{\infty}(\Omega)$, $\partial_{x} a_{ij} \in L^{\infty}(\Omega)$ for $i = 1, \ldots, q$ and $j = q+1, \ldots, N$, then we can show by a simple density argument that for
\[ i = 1, \ldots, q \text{ and } j = q + 1, \ldots, N \quad \partial_{x_k}(a_{ij}(u_{e,V,f} - u_{V,f})) \in L^2(\Omega) \text{ and} \]
\[ \partial_{x_k}(a_{ij}(u_{e,V,f} - u_{V,f})) = (u_{e,V,f} - u_{V,f})\partial_{x_k}a_{ij} + a_{ij}\partial_{x_k}(u_{e,V,f} - u_{V,f}), \text{ for } k = i, j. \]

Whence

\[-\epsilon \int_{\Omega} A_{12} \nabla x_2 u_{V,f} \cdot \nabla x_1 (u_{e,V,f} - u_{V,f}) dx = -\epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} a_{ij} \partial_{x_k} u_{V,f} \partial_{x_k} (u_{e,V,f} - u_{V,f}) dx \]
\[ = -\epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} \partial_{x_k}(a_{ij}(u_{e,V,f} - u_{V,f})) \partial_{x_k} u_{V,f} dx \]
\[ + \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} (u_{e,V,f} - u_{V,f}) \partial_{x_k} a_{ij} \partial_{x_k} u_{V,f} dx \]
\[ = -\epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} \partial_{x_k}(a_{ij}(u_{e,V,f} - u_{V,f})) \partial_{x_k} u_{V,f} dx \]
\[ + \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} (u_{e,V,f} - u_{V,f}) \partial_{x_k} a_{ij} \partial_{x_k} u_{V,f} dx, \]

where we have used \( \int_{\Omega} \partial_{x_k}(a_{ij}(u_{e,V,f} - u_{V,f})) \partial_{x_k} u_{V,f} dx = \int_{\Omega} \partial_{x_j}(a_{ij}(u_{e,V,f} - u_{V,f})) \partial_{x_k} u_{V,f} dx \) which holds by a simple density argument (recall that \( u_{V,f} \in H^1(\Omega) \)). Therefore

\[-\epsilon \int_{\Omega} A_{12} \nabla x_2 u_{V,f} \cdot \nabla x_1 (u_{e,V,f} - u_{V,f}) dx = -\epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} (u_{e,V,f} - u_{V,f}) \partial_{x_k} a_{ij} \partial_{x_k} u_{V,f} dx \]
\[ \leq \frac{\lambda}{4} \int_{\Omega} |\nabla x_2 (u_{e,V,f} - u_{V,f})|^2 dx + C'' \int_{\Omega} |\nabla x_1 u_{V,f}|^2 dx + C''' \int_{\Omega} |\nabla x_2 u_{V,f}|^2 dx, \]

where

\[ C' = \frac{3 \left[ C_{\omega_k} \max_{1 \leq i, q+1 \leq j \leq N} \| \partial_{x_j} a_{ij} \|_{L^\infty(\Omega)} (N - q) \right]^2 + 3 \left( \max_{1 \leq i, q+1 \leq j \leq N} \| a_{ij} \|_{L^\infty(\Omega)} (N - q) \right)^2}{\lambda}, \]

and

\[ C'' = \frac{3 \left[ q C_{\omega_k} \max_{1 \leq i, q+1 \leq j \leq N} \| \partial_{x_k} a_{ij} \|_{L^\infty(\Omega)} \right]^2}{\lambda}. \]
According to (24) yields
\[
- \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{e,V,f} - u_{V,f}) \, dx \leq \\
\frac{\lambda}{4} \int_{\Omega} |\nabla_{X_2} (u_{e,V,f} - u_{V,f})|^2 \, dx + C^2 \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 \, dx + \epsilon^2 C'' \left( \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right)^2.
\]

Combining (31) and (36) we get
\[
\frac{\epsilon^2 \lambda}{2} \|\nabla_{X_1} (u_{e,V,f} - u_{V,f})\|_{L^2(\Omega)^s}^2 + \frac{\lambda}{4} \|\nabla_{X_2} (u_{e,V,f} - u_{V,f})\|_{L^2(\Omega)^{N-s}}^2 \\
\leq \epsilon^2 \left( (C + C') \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 \, dx + C'' \left( \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right)^2 \right),
\]
and the proof is finished. \(\square\)

Using the triangle inequality, Lemma 3.9 and (29) we obtain the following estimation of the global error between \(u_{e,V,f}\) and \(u_f\)

**Corollary 3.5.** Under assumption of Lemma 3.4 we have for any \(\epsilon \in (0,1]\):
\[
\|\nabla_{X_2} (u_{e,V,f} - u_f)\|_{L^2(\Omega)^{N-s}} \leq \epsilon \left( C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^s} + C_2 \|f\|_{L^2(\Omega)} \right) + C_{\text{cea}} \left( \inf_{v \in V} \|\nabla_{X_2} (v - u_f)\|_{L^2(\Omega)^{N-s}} \right)^{\frac{1}{2}}
\]

We give an important remark which will be used to prove the inequality given in Remark 2.5.

**Remark 3.6.** When \(\beta(s) = \mu s\) for some \(\mu > 0\) and when the bloc \(A_{12}\) satisfies assumption (19) then, by performing some integration by part in the last term of (33) and using the fact that
\[
\|u_{V,f}\|_{L^2(\Omega)} \leq \frac{1}{\mu} \|f\|_{L^2(\Omega)}
\]
we can obtain the following bound
\[
\forall \epsilon \in (0,1] : \|\nabla_{X_2} (u_{e,V,f} - u_f)\|_{L^2(\Omega)} \leq \epsilon \left( C_1' \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^s} + \frac{C_2'}{\mu} \|f\|_{L^2(\Omega)} \right),
\]
where \(C_1', C_2' > 0\) are independent of \(f, V, \mu\) and \(\epsilon\)

### 3.3. Proof of Theorem 2.2

Let \((V_n)\) be a sequence of finite dimensional spaces which approximates \(H^1_0(\Omega)\) in the sense of Definition 2.1. Using the density of \(H^1_0(\Omega)\) in \(H^1_0(\Omega, \omega_2)\) (Lemma A.1, Appendix A), one can check easily that \((V_n)\) approximates \(H^1_0(\Omega, \omega_2)\) in the same sense. Therefore, one has:

For every \(\epsilon \in (0,1] : \inf_{v \in V_n} \|\nabla (v - u_{e,f})\|_{L^2(\Omega)^N} \to 0\) as \(n \to \infty\),
\(\text{(37)}\)
and

\[ \inf_{v \in V_n} \| \nabla X_2 (v - u_f) \|_{L^2(\Omega)^{N-\beta}} \to 0 \text{ as } n \to \infty \]

(38)

According to Lemma 3.4, (29) and (30) we have, for every \( n \in \mathbb{N} \) and \( \epsilon \in (0,1] \):

\[ \| \nabla X_2 (u_{\epsilon,V_n,f} - u_{V_n,f}) \|_{L^2(\Omega)^{N-\beta}} \leq C \left( \inf_{v \in V_n} \| \nabla X_1 u_{V_n,f} \|_{L^2(\Omega)^{\gamma}} + C_2 \| f \|_{L^2(\Omega)} \right), \]

(39)

\[ \| \nabla X_2 (u_{V_n,f} - u_f) \|_{L^2(\Omega)^{N-\beta}} \leq C_{\epsilon,\alpha} \left( \inf_{v \in V_n} \| \nabla X_2 (v - u_f) \|_{L^2(\Omega)^{N-\beta}} \right)^{\frac{1}{2}}, \]

(40)

and

\[ \| \nabla (u_{\epsilon,V_n,f} - u_{\epsilon,f}) \|_{L^2(\Omega)^N} \leq \frac{C_{\epsilon,\alpha}}{\epsilon^2} \left( \inf_{v \in V_n} \| \nabla (v - u_{\epsilon,f}) \|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}, \]

(41)

- Fix \( \epsilon \) and pass to the limit in (41) using (37), one has

\[ u_{\epsilon,V_n,f} \to u_{\epsilon,f} \text{ as } n \to \infty \text{ in } H_0^1(\Omega), \]

whence, the continuous imbedding \( H_0^1(\Omega) \to H_0^1(\Omega,\omega_2) \) gives

\[ u_{\epsilon,V_n,f} \to u_{\epsilon,f} \text{ as } n \to \infty \text{ in } H_0^1(\Omega,\omega_2). \]

Now, passing to the limit as \( \epsilon \to 0 \), using Theorem 1.1, we get

\[ \lim_{\epsilon \to 0} (\lim_n u_{\epsilon,V_n,f}) = u_f \text{ in } H_0^1(\Omega,\omega_2). \]

(42)

- Fix \( n \) and passe to the limit as \( \epsilon \to 0 \) using (39), we get

\[ u_{\epsilon,V_n,f} \to u_{V_n,f} \text{ as } \epsilon \to 0 \text{ in } H_0^1(\Omega,\omega_2). \]

Now, passing to the limit as \( n \to \infty \) in (40) by using (38) we get

\[ \lim_{n \to \infty} (\lim_{\epsilon \to 0} u_{\epsilon,V_n,f}) = u_f \text{ in } H_0^1(\Omega,\omega_2). \]

(43)

Finally, Theorem 2.2 follows from (42) and (43).

### 3.4. Proof of Theorem 2.3

Throughout this subsection we will suppose that \( \beta = 0 \). The key of the proof of Theorem 2.3 is based on the control the quantity \( \| \nabla X_1 u_{V,f} \|_{L^2(\Omega)^\gamma} \) independently of \( V \). In fact, we need the following

**Lemma 3.7.** Let us assume that \( A \) satisfies (3), (4), and that \( A_{22} \) satisfies (18). Let \( V_1 \) and \( V_2 \) be two finite dimensional subspaces of \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively. Let \( f \in V_1 \otimes V_2 \), and let \( u_{V,f} \) be the unique solution in \( V = V_1 \otimes V_2 \) to:

\[ \int_\Omega A_{22}(X_2) \nabla X_2 u_{V,f} \cdot \nabla X_2 vdx = \int_\Omega fvdx, \quad \forall v \in V_1 \otimes V_2. \]

(44)

Then we have

\[ \| \nabla X_1 u_{V,f} \|_{L^2(\Omega)^\gamma} \leq C_3 \| \nabla X_1 f \|_{L^2(\Omega)^\gamma}, \]

where \( C_3 \) is given by \( C_3 = \frac{\sqrt{qC_2}}{\alpha} \).
Proof. The prove is based on the difference quotient method (see for instance [13] page 168). For every \( v = \varphi \otimes \psi \in V_1 \otimes V_2 \), the function \( X_1 \mapsto \int_{\omega_2} A_{22}(X_2) \nabla X_2 u_{V,f}(X_1,X_2) \cdot \nabla X_2 \psi \, dX_2 \) belongs to \( V_1 \). In fact \( u_{V,f} = \sum_{finite} \varphi_i \otimes \psi_i \) and whence \( \int_{\omega_2} A_{22}(X_2) \nabla X_2 u_{V,f} \cdot \nabla X_2 \psi \, dX_2 \) is a linear combination of \( \varphi_i \), thanks to the linearity of the integral. Similarly, the function \( X_1 \mapsto \int_{\omega_2} f(X_1,X_2) \psi \, dX_2 \) belongs to \( V_1 \). Now, from (44) we derive:

\[
\int_{\omega_1} \left( \int_{\omega_2} \{ A_{22}(X_2) \nabla X_2 u_{V,f} \cdot \nabla X_2 \psi - f \, \psi \} \, dX_2 \right) \varphi_1 \, dX_1 = 0,
\]

thus, when \( \varphi \) run through a set of an orthogonal basis of the euclidean space \( V_1 \) equipped with the \( L^2(\omega_1) \)--scalar product, one can deduce that for a.e. \( X_1 \in \omega_1 \):

\[
\int_{\omega_2} A_{22}(X_2) \nabla X_2 u_{V,f}(X_1,X_2) \cdot \nabla X_2 \psi \, dX_2 = \int_{\omega_2} f(X_1,X_2) \psi \, dX_2, \quad \forall \psi \in V_2
\]

Now, fix \( i \in \{1,...,q\} \). Let \( \omega_1' \subset \omega_1 \) open, for any \( 0 < h < d(\omega_1', \partial \omega_1) \) and for any \( (X_1,X_2) \in \omega_1' \times \omega_2 \) we denote \( \tau_h u_{V,f}(x) = u_{V,f}(x_1,...,x_i + h,...,x_q,X_2) \). According to the above equality we get for a.e. \( X_1 \in \omega_1' \) and for every \( \psi \in V_2 \):

\[
\int_{\omega_2} A_{22}(X_2) \nabla X_2 \{ \tau_h u_{V,f}(X_1,X_2) - u_{V,f}(X_1,X_2) \} \nabla X_2 \psi \, dX_2 = \int_{\omega_2} \{ \tau_h f(X_1,X_2) - f(X_1,X_2) \} \psi \, dX_2
\]

For every \( w \in V_1 \otimes V_2 \), and for every \( X_1 \) fixed the function \( w(X_1,\cdot) \) belongs to \( V_2 \), so one can take \( \psi = \tau_h u_{V,f}(X_1,\cdot) - u_{V,f}(X_1,\cdot) \) as a test function in the above equality. Therefore, by using the Cauchy-Schwarz inequality, the ellipticity assumption and Poincaré’s inequality (10) we obtain:

\[
\int_{\omega_2} |\tau_h u_{V,f}(X_1,\cdot) - u_{V,f}(X_1,\cdot)|^2 \, dX_2 \leq \frac{C_{\omega_2}^2}{\lambda^2} \int_{\omega_2} |\tau_h f(X_1,\cdot) - f(X_1,\cdot)|^2 \, dX_2
\]

Now, integrating the above inequality over \( \omega_1' \) yields

\[
\int_{\omega_1' \times \omega_2} |\tau_h u_{V,f} - u_{V,f}|^2 \, dx \leq \frac{C_{\omega_2}^2}{\lambda^2} \int_{\omega_1' \times \omega_2} |\tau_h f - f|^2 \, dx
\]

Since \( \nabla X_1 f \in L^2(\Omega)^q \) then

\[
\int_{\omega_1' \times \omega_2} |\tau_h f - f|^2 \, dx \leq \| \nabla X_1 f \|^2_{L^2(\Omega)^q} h^2
\]

Finally we obtain

\[
\int_{\omega_1' \times \omega_2} \left| \frac{\tau_h u_{V,f} - u_{V,f}}{h} \right|^2 \, dx \leq \frac{C_{\omega_2}^2 \| \nabla X_1 f \|^2_{L^2(\Omega)^q}}{\lambda^2}
\]

Therefore,

\[
\|D_{x_1} u_{V,f}\|_{L^2(\Omega)} \leq \frac{C_{\omega_2}}{\lambda} \| \nabla X_1 f \|_{L^2(\Omega)^q},
\]

and hence

\[
\| \nabla X_1 u_{V,f} \|_{L^2(\Omega)^q} \leq C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q}
\]

with \( C_3 = \frac{\sqrt{C_{\omega_2}}}{\lambda} \). \( \square \)
Remark 3.8. We have a similar result when (44) is replaced by
\[ \mu \int_{\Omega} u_{v,f} v dx + \int_{\Omega} A_{22}(X_2) \nabla X_2 u_{v,f} \cdot \nabla X_2 v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2, \]
where \( \mu > 0 \). In this case we obtain the following
\[ \| \nabla X_1 u_{v,f} \|_{L^2(\Omega)^s} \leq \frac{\sqrt{s}}{\mu} \| \nabla X_1 f \|_{L^2(\Omega)^s}. \]

Now, we can refine the estimations of Lemma 3.4 as follows

Lemma 3.9. Under assumptions of Lemmas 3.4 and 3.7 we have:
\[ \| \nabla X_2 u_{v,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s} + C_2 \| f \|_{L^2(\Omega)} \right) + \frac{\| A_{22} \|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} \| \nabla X_2 v - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}}, \]
and
\[ \| \nabla X_1 u_{v,f} \|_{L^2(\Omega)^s} \leq \frac{1}{\sqrt{s}} \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s} + C_2 \| f \|_{L^2(\Omega)} \right) + C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s}. \]

Proof. We have
\[ \| \nabla X_2 u_{v,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}} \leq \| \nabla X_2 u_{v,f} - \nabla X_2 u_{v,f} \|_{L^2(\Omega)^{N-s}} + \| \nabla X_2 u_{v,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}}. \]
Using Lemma 3.4 and Lemma 3.7 we obtain that
\[ \| \nabla X_2 u_{v,f} - \nabla X_2 u_{v,f} \|_{L^2(\Omega)^{N-s}} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s} + C_2 \| f \|_{L^2(\Omega)} \right), \]
and using Remark 3.3, we deduce
\[ \| \nabla X_2 u_{v,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}} \leq \frac{\| A_{22} \|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} \| \nabla X_2 v - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}}. \]
Using the previous inequalities gives the expected result. The second inequality follows by using the triangle inequality and applying Lemma 3.4 and Lemma 3.7.

Remark 3.10. When \( \beta(s) = \mu s \), for a certain \( \mu > 0 \), we obtain, by combining Remarks 3.6 and 3.8, the bound
\[ \| \nabla X_2 (u_{v,f} - u_{v,f}) \|_{L^2(\Omega)} \leq \frac{\mu}{\sqrt{s}} \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s} + C_2 \| f \|_{L^2(\Omega)} \right). \]

Now, we are able to give the first convergence result

Lemma 3.11. Suppose that assumptions of Lemmas 3.4 and 3.7 hold. Let \( f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2) \). Then we have, for any \( \epsilon \in (0,1] \):
\[ \| \nabla X_2 u_{v,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-s}} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s} + C_2 \| f \|_{L^2(\Omega)} \right), \]
and
\[ \| \nabla X_1 u_{v,f} \|_{L^2(\Omega)^s} \leq \frac{1}{\sqrt{s}} \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s} + C_2 \| f \|_{L^2(\Omega)} \right) + C_3 \| \nabla X_1 f \|_{L^2(\Omega)^s}. \]
Proof. Let \( (V_n^{(1)})_{n \geq 0} \) and \( (V_n^{(2)})_{n \geq 0} \) be two nondecreasing sequences of finite dimensional subspaces of \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively, whose the union of each one is dense in the corresponding space and such that \( f \in V_0^{(1)} \otimes V_0^{(2)} \), such a sequence always exits. Indeed, let \( \{e_0^{(1)}\}_{i \in \mathbb{N}} \) and \( \{e_1^{(2)}\}_{j \in \mathbb{N}} \) be a Hilbert basis of \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively, we know that \( \bigcup_{n \geq 0} \text{span}(e_0^{(1)}, ..., e_n^{(1)}) \) and \( \bigcup_{n \geq 0} \text{span}(e_0^{(2)}, ..., e_n^{(2)}) \) are dense in \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively, in the other hand we have \( f = \sum_{i=0}^{m} f_i^{(1)} \times f_i^{(2)} \) for some \( m \in \mathbb{N} \) and \( f_i^{(1)} \in H_0^1(\omega_1) \), \( f_i^{(2)} \in H_0^1(\omega_2) \) for \( i = 0, ..., m \). Then we set, for every \( n \in \mathbb{N} \):

\[
V_n^{(1)} := \text{span}(e_0^{(1)}, ..., e_n^{(1)}, f_0^{(1)}, ..., f_m^{(1)}),
\]

\[
V_n^{(2)} := \text{span}(e_0^{(2)}, ..., e_n^{(2)}, f_0^{(2)}, ..., f_m^{(2)}).
\]

Now, since \( f \) belongs to each \( V_n^{(1)} \otimes V_n^{(2)} \) then according to Lemma 3.9 with \( V_n := V_n^{(1)} \otimes V_n^{(2)} \) one has, for every \( \epsilon \in (0, 1], n \in \mathbb{N} \):

\[
\| \nabla X_2 u_{\epsilon,V_n} f - \nabla X_2 u_f \|_{L^2(\Omega)^N} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)} \right) + \inf_{v \in V_n} \| \nabla X_2 v - \nabla X_2 u_f \|_{L^2(\Omega)^N}.
\]

According to Corollary A.5 in Appendix A \( \bigcup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)}) \) is dense in \( H_0^1(\Omega) \). Using the fact that the sequence \( (V_n)_{n \geq 0} \) is nondecreasing then we obtain that

\[
\forall \epsilon \in (0, 1]: \lim_{n \to \infty} \inf_{v \in V_n} \| \nabla v - \nabla u_{\epsilon,f} \|_{L^2(\Omega)^N} = 0,
\]

and therefore, by using (30) we get

\[
\forall \epsilon \in (0, 1]: \lim_{n \to \infty} \| \nabla u_{\epsilon,V_n,f} - \nabla u_{\epsilon,f} \|_{L^2(\Omega)^N} = 0,
\]

and thus

\[
\forall \epsilon \in (0, 1]: \lim_{n \to \infty} \| \nabla X_2 u_{\epsilon,V_n,f} - \nabla X_2 u_{\epsilon,f} \|_{L^2(\Omega)^N} = 0, \text{ and } \lim_{n \to \infty} \| \nabla X_1 u_{\epsilon,V_n,f} - \nabla X_1 u_{\epsilon,f} \|_{L^2(\Omega)^q} = 0.
\]

Using the fact that \( H_0^1(\Omega) \) is dense in \( H_0^1(\Omega, \omega_2) \) (Lemma A.1, Appendix A) and that the imbedding \( H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2) \) is continuous then \( \bigcup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)}) \) is dense in \( H_0^1(\Omega, \omega_2) \). Using the fact that the sequence \( (V_n)_{n \geq 0} \) is nondecreasing then we obtain that

\[
\lim_{n \to \infty} \inf_{v \in V_n} \| \nabla X_2 v - \nabla X_2 u_f \|_{L^2(\Omega)^N} = 0.
\]

Then the passage to the limit as \( n \to \infty \) in the above inequality gives

\[
\forall \epsilon \in (0, 1]: \| \nabla X_2 u_{\epsilon,f} \|_{L^2(\Omega)^N} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)} \right).
\]

Finally, by using the second inequality of Lemma 3.9 we get

\[
\forall \epsilon \in (0, 1]: \| \nabla X_1 u_{\epsilon,V_n,f} \|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)} \right) + C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q},
\]

and the passage to limit as \( n \to \infty \) shows the second estimation of the lemma. □
Now, we are able to give the proof of Theorem 2.3. Let us introduce the space
\[ H^1_0(\Omega; \omega_1) = \{ v \in L^2(\Omega) \text{ such that } \nabla X_1 v \in L^2(\Omega)^q \text{ and for a.e. } X_2 \in \omega_2, v(\cdot, X_2) \in H^1_0(\omega_1) \}, \]
normed by the Hilbert norm\[ \| v \|_{L^2(\Omega)^q} \] for any \( v \in H^1_0(\Omega; \omega_1) \)

(45)

Let \( f \in L^2(\Omega) \) such that (16) and (17), thus \( f \in H^1_0(\Omega; \omega_1) \). According to Lemma A.3 of Appendix A \( H^1_0(\omega_1) \otimes H^1_0(\omega_2) \) is dense in \( H^1_0(\Omega) \), and according to Remark A.2 of Appendix A \( H^1_0(\Omega) \) is dense in \( H^1_0(\Omega; \omega_1) \), then it follows that \( H^1_0(\omega_1) \otimes H^1_0(\omega_2) \) is dense in \( H^1_0(\Omega; \omega_1) \), thanks to the continuous imbedding \( H^1_0(\Omega) \rightarrow H^1_0(\Omega; \omega_1) \). Therefore, for \( \delta > 0 \) there exists \( g_\delta \in H^1_0(\omega_1) \otimes H^1_0(\omega_2) \) such that

\[ \| \nabla X_1 (f - g_\delta) \|_{L^2(\Omega)^q} \leq \delta. \] (46)

Let \( u_{e, g_\delta} \) be the unique solution of (7) with \( f \) replaced by \( g_\delta \). Testing with \( u_{e,f} - u_{e, g_\delta} \) in the difference of weak formulations

\[ \int\Omega A_e \nabla (u_{e,f} - u_{e, g_\delta}) \cdot \nabla \varphi dx = \int\Omega (f - g_\delta) \varphi dx, \forall \varphi \in H^1_0(\Omega), \]
we obtain

\[ \| \nabla X_2 u_{e,f} - \nabla X_2 u_{e, g_\delta} \|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda \epsilon} \delta, \quad \text{and} \quad \| \nabla X_1 u_{e,f} - \nabla X_1 u_{e, g_\delta} \|_{L^2(\Omega)^q} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda \epsilon} \delta, \]

where we have used the ellipticity assumption, Poincaré’s inequalities (10), (45) and (46). By a passage to the limit as \( \epsilon \to 0 \) in the first above inequality, using Theorem 1.1, we get

\[ \| \nabla X_2 u_{e,f} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta. \]

Applying Lemma 3.11 on \( u_{e, g_\delta} \) and \( u_{g_\delta} \) we obtain

\[ \| \nabla X_2 u_{e, g_\delta} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 g_\delta \|_{L^2(\Omega)^q} + C_2 \| g_\delta \|_{L^2(\Omega)} \right), \]

and from (46) we derive

\[ \| \nabla X_2 u_{e, g_\delta} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 (\| \nabla X_1 f \|_{L^2(\Omega)^q} + \delta) + C_2 \| g_\delta \|_{L^2(\Omega)} \right). \]

Notice that \( \| g_\delta \|_{L^2(\Omega)} \rightarrow \| f \|_{L^2(\Omega)} \) as \( \delta \to 0 \), thanks to (46) and Poincaré’s inequality (45). Finally the triangle inequality gives

\[ \| \nabla X_2 u_{e,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-q}} \leq \| \nabla X_2 u_{e,f} - \nabla X_2 u_{e, g_\delta} \|_{L^2(\Omega)^{N-q}} + \| \nabla X_2 u_{e, g_\delta} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)^{N-q}} + \| \nabla X_2 u_{g_\delta} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-q}} \]

\[ \leq \epsilon \left( C_1 C_3 (\| \nabla X_1 f \|_{L^2(\Omega)^q} + \delta) + C_2 \| g_\delta \|_{L^2(\Omega)} \right) + \frac{2 C_{\omega_1} C_{\omega_2}}{\lambda \epsilon} \delta. \]

Passing to the limit as \( \delta \to 0 \) we obtain

\[ \| \nabla X_2 u_{e,f} - \nabla X_2 u_f \|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)} \right), \]
which is the estimation given in Theorem 2.3.

For the estimation in the first direction, we have

\[
\left\| \nabla X_1 u_e, f \right\|_{L^2(\Omega)^q} \leq \left\| \nabla X_1 u_{e, f} - \nabla X_1 u_{e, g_3} \right\|_{L^2(\Omega)^q} + \left\| \nabla X_1 u_{e, g_3} \right\|_{L^2(\Omega)^q}
\]

\[
\leq \frac{C_1 C_3}{\lambda \epsilon} \delta + \frac{1}{\sqrt{2}} \left( C_1 C_3 \left\| \nabla X_1 g_3 \right\|_{L^2(\Omega)^q} + C_2 \left\| g_3 \right\|_{L^2(\Omega)^q} \right) + C_3 \left\| \nabla X_1 g_3 \right\|_{L^2(\Omega)^q},
\]

where we have applied the triangle inequality and Lemma 3.11. Passing to the limit as \( \delta \to 0 \), by using (46) we obtain

\[
\left\| \nabla X_1 u_{e, f} \right\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 C_3 \left\| \nabla X_1 f \right\|_{L^2(\Omega)^q} + C_2 \left\| f \right\|_{L^2(\Omega)^q} \right) + C_3 \left\| \nabla X_1 f \right\|_{L^2(\Omega)^q}.
\]

Hence, by a passage to the limit in \( L^2(\Omega) \) \( - weak \) as \( \epsilon \to 0 \), up to a subsequence, we show that \( u_f \) belongs to \( H^1_0(\Omega) \), and by a contradiction argument, using the metrisability (for the weak topology) of weakly closed bounded subsets in separable Hilbert spaces, one can show that the global sequence \( (\nabla X_1 u_{e, f})_\epsilon \) converges weakly to \( \nabla X_1 u_f \) in \( L^2(\Omega)^q \), and this completes the proof of Theorem 2.3.

Remark 3.12. In the case \( \beta(s) = \mu s \) with \( \mu > 0 \) We repeat the same arguments of this subsection by using Remark 3.10 and we obtain the inequality of Remark 2.5.

4. Perturbations of semigroups of linear operators

4.1. Preliminaries

For the standard basic theory of semigroups of bounded linear operators, we refer the reader to [14]. Let us begin by some reminders. Let \( E \) be a real Banach space. An unbounded linear operator \( A : D(A) \subset E \to E \) is said to be closed if for every sequence \( (x_n) \) of \( D(A) \) such that \( (x_n) \) and \( (Ax_n) \) converge in \( E \), we have \( \lim x_n \in D(A) \) and \( \lim A(x_n) = A(\lim x_n) \). An operator is said to be densely defined on \( E \) if its domain \( D(A) \) is dense in \( E \). Let \( \mu \in \mathbb{R} \) we said that \( \mu \) belongs to the resolvent set of \( A \) if \( \mu I - A : D(A) \to E \) is one-to-one and onto and such that \( R_\mu = (\mu I - A)^{-1} : E \to D(A) \subset E \) is a bounded operator on \( E \). Notice that \( R_\mu \) and \( A \) commute on \( D(A) \), that is \( \forall x \in D(A) : R_\mu Ax = AR_\mu x \). Let \( A \) be a densely defined closed operator. The bounded operator

\[
A_\mu = \mu A(\mu I - A)^{-1} = \mu AR_\mu = \mu^2 R_\mu - \mu I,
\]

is called the Yosida approximation of \( A \). We check immediately that \( A_\mu \) and \( A \) commute on \( D(A) \) that is for every \( x \in D(A) \) we have \( A_\mu x \in D(A) \) and \( AA_\mu x = A_\mu Ax \). Furthermore, since \( A \) is closed then \( e^{tA_\mu} \) and \( A \) commute on \( D(A) \) that is

\[
\forall t \in \mathbb{R}, \forall x \in D(A), e^{tA_\mu} x \in D(A), \quad (47)
\]

and

\[
A e^{tA_\mu} x = e^{tA_\mu} Ax = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A_\mu)^k Ax,
\]

indeed, we can check by induction that if \( x \in D(A) \) then \( (A_\mu)^k x \in D(A) \), and that \( (A_\mu)^k \) and \( A \) commute on \( D(A) \). Recall also that if \( (\mu I - A)^{-1} \) exists for \( \mu > 0 \) and such that \( \left\| (\mu I - A)^{-1} \right\| \leq \frac{1}{\mu} \) then

\[
\forall t \geq 0 : \left\| e^{tA_\mu} \right\| = \left\| e^{tu^2 R_\mu} \right\| \times \left\| e^{-\mu t} \right\| \leq e^{t\mu^2 \left\| R_\mu \right\|} \times e^{-\mu t} \leq 1,
\]

where \( \left\| \cdot \right\| \) is the operator norm of \( L(E) \). A \( C_0 \) semigroup of bounded linear operators on \( E \) is a family of bounded operators \( (S(t))_{t \geq 0} \) of \( L(E) \) such that: \( S(0) = I \), for every \( t, s \geq 0 : S(t + s) = S(t)S(s) \), and for every \( x \in E : \| S(t)x - x \|_E \to 0 \) as \( t \to 0 \). \( (S(t))_{t \geq 0} \) is called a semigroup of contractions if for every \( t \geq 0 : \| S(t)x \|_E \leq 1 \). Now, let us recall the well-known Hill-Yosida theorem in its Hilbertian (real) version: An unbounded operator \( A \) is the infinitesimal generator of a \( C_0 \) semigroup of contraction \( (S(t))_{t \geq 0} \) if and only
if \( A \) is maximal dissipative, that is when \( \mu I - A \) is surjective for every \( \mu > 0 \) and for every \( x \in D(A) : \langle Ax, x \rangle \leq 0 \). Recall that, in this case \( D(A) \) is dense and \( A \) is closed whose the resolvent set contains \( ]0, +\infty[ \). Furthermore, for every \( t \geq 0 \), \( e^{tA_x} \) converges, in the strong operator topology, to \( S(t) \), as \( \mu \to +\infty \) that is
\[
\forall x \in E : e^{tA_x}x \to S(t)x \text{ in } E \text{ as } \mu \to +\infty.
\]

Let \( \Omega \) as in the introduction. The basic Hilbert space in the sequel is \( E = L^2(\Omega) \). For any \( \epsilon \in (0, 1] \), we introduce the operator \( A_\epsilon \) acting on \( L^2(\Omega) \) and given by the formula
\[
A_\epsilon u = \text{div}(A_\epsilon \nabla u),
\]
where \( A_\epsilon \) is given as in the introduction of this paper. The domain of \( A_\epsilon \) is given by
\[
D(A_\epsilon) = \{ u \in H^1_0(\Omega) \mid \text{div}(A_\epsilon \nabla u) \in L^2(\Omega) \},
\]
where \( \text{div}(A_\epsilon \nabla u) \in L^2(\Omega) \) is taken in the distributional sense. Now, we introduce the operator \( A_0 \) defined on
\[
D(A_0) = \{ u \in H^1_0(\Omega; \omega_2) \mid \text{div}_{X_2}(A_{22} \nabla_{X_2} u) \in L^2(\Omega) \},
\]
by the formula
\[
A_0 u = \text{div}_{X_2}(A_{22} \nabla_{X_2} u).
\]
We check immediately, by using assumptions (3 – 4), that \( A_\epsilon \) and \( A_0 \) are maximal dissipative and therefore, they are the infinitesimal generators of a \( C_0 \) semigroups of contractions on \( L^2(\Omega) \), denoted \( (S_\epsilon(t))_{t \geq 0} \) and \( (S_0(t))_{t \geq 0} \) respectively. For \( \mu > 0 \) we denote by \( R_{\epsilon, \mu} \) the resolvent of \( A_\epsilon \). Similarly, we denote by \( R_{0, \mu} \) the resolvent of \( A_0 \).

For \( f \in L^2(\Omega) \), we denote \( u_{\epsilon, \mu} \) the unique solution in \( H^1_0(\Omega) \) to
\[
\mu \int \Omega u_{\epsilon, \mu} \varphi dx + \int \Omega A_\epsilon \nabla u_{\epsilon, \mu} \cdot \nabla \varphi dx = \int \Omega f \varphi dx, \forall \varphi \in H^1_0(\Omega),
\]
we have \( R_{\epsilon, \mu} f = u_{\epsilon, \mu} \) and \( \| R_{\epsilon, \mu} \| \leq \frac{1}{\mu} \), where \( \| \cdot \| \) is the operator norm of \( L(L^2(\Omega)) \). Similarly, let \( u_{0, \mu} \) be the unique solution in \( H^1_0(\Omega; \omega_2) \) to
\[
\mu \int \Omega u_{0, \mu} \varphi dx + \int \Omega A_{22} \nabla_{X_2} u_{0, \mu} \cdot \nabla \varphi dx = \int \Omega f \varphi dx, \forall \varphi \in H^1_0(\Omega; \omega_2), \tag{48}
\]
we have \( R_{0, \mu} f = u_{0, \mu} \) and \( \| R_{0, \mu} \| \leq \frac{1}{\mu} \). According to Remark 2.5, we have the following

**Lemma 4.1.** Assume (3), (4), (14), (18) and (19). Let \( f \in H^1_0(\Omega; \omega_1) \), there exists \( C_{A, \Omega} > 0 \) only depending on \( A \) and \( \Omega \), such that:
\[
\forall \epsilon \in (0, 1], \forall \mu > 0 : \| R_{\epsilon, \mu} f - R_{0, \mu} f \|_{L^2(\Omega)} \leq C_{A, \Omega} \times \epsilon \times \frac{1}{\mu} \times \left( \| \nabla_{X_2} f \|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)} \right). \tag{49}
\]

### 4.2. The asymptotic behavior of the perturbed semigroup

In this subsection, we study the relationship between the semigroups \( (S_\epsilon(t))_{t \geq 0} \) and \( (S_0(t))_{t \geq 0} \). We will assume that
\[
A \text{ is lipschitz on } \Omega. \tag{50}
\]
Notice that (50) shows that, for any \( \epsilon \in (0, 1] : \)
\[
H^1_0(\Omega) \cap H^2(\Omega) \subset D(A_0) \cap D(A_\epsilon).
\]
Remark also that (50) implies (14). Now, we can give the main theorem of this section.
Theorem 4.2. Assume that $\Omega = \omega_1 \times \omega_2$ is a bounded domain of $\mathbb{R}^q \times \mathbb{R}^{N-q}$. Assume (3–4), (18), (19) and (50) then for every $\varrho \in L^2(\Omega)$ and $T \geq 0$ we have:

$$\sup_{t \in [0,T]} \| S_\varrho(t) g - S_\varrho(t) g \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0.$$ 

In particular, for $\varrho \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2))$ we have:

$$\sup_{t \in [0,T]} \| S_\varrho(t) g - S_\varrho(t) g\|_{L^2(\Omega)} \leq C_{g,\Omega} \times T \times \epsilon.$$

Let us begin by this important lemma

Lemma 4.3. Suppose that assumptions of Theorem 4.2 hold. Let $\varrho \in H^1_0(\Omega) \cap D(A_0)$ such that

$$\text{div} X_1(A_{11} \nabla X_1 \varrho), \text{div} X_1(A_{12} \nabla X_2 \varrho), \text{div} X_2(A_{21} \nabla X_1 \varrho) \in L^2(\Omega),$$

and $A_0 \varrho \in H^1_0(\Omega; \omega_1)$. Then, there exists a constant $C_{f,A_\Omega} > 0$ such that for every $\mu > 0$, $\epsilon \in (0,1]$ we have:

$$\| A_{\epsilon,\mu} \varrho - A_{0,\mu} \varrho \|_{L^2(\Omega)} \leq C_{f,A_\Omega} \times \epsilon,$$

where $A_{\epsilon,\mu}$ and $A_{0,\mu}$ are the Yosida approximations of $A_{\epsilon}$ and $A_0$ respectively and

$$C_{f,A_\Omega} = \| \text{div} X_1(A_{11} \nabla X_1 \varrho) \|_{L^2(\Omega)} + \| \text{div} X_1(A_{12} \nabla X_2 \varrho) \|_{L^2(\Omega)}$$

$$+ \| \text{div} X_2(A_{21} \nabla X_1 \varrho) \|_{L^2(\Omega)} + C_{A_\Omega} \left( \| \nabla X_1 A_0 \varrho \|_{L^2(\Omega)} + \| A_0 \varrho \|_{L^2(\Omega)} \right).$$

Proof. Let $\epsilon \in (0,1]$ and $\mu > 0$. The bounded operators $A_{\epsilon,\mu}$, $A_{0,\mu}$ of $L(L^2(\Omega))$ are given by:

$$A_{\epsilon,\mu} = \mu A_{\epsilon} R_{\epsilon,\mu} \text{ and } A_{0,\mu} = \mu A_0 R_{0,\mu}$$

Now, under the above hypothesis we obtain that $\varrho \in D(A_{\epsilon}) \cap D(A_0)$, and

$$\| A_{\epsilon,\mu} \varrho - A_{0,\mu} \varrho \|_{L^2(\Omega)} = \mu \| R_{\epsilon,\mu} A_{\epsilon} \varrho - R_{0,\mu} A_0 \varrho \|_{L^2(\Omega)} = \mu \| R_{\epsilon,\mu} A_{\epsilon} \varrho - R_{0,\mu} A_0 \varrho \|_{L^2(\Omega)}$$

$$\leq \mu \| R_{\epsilon,\mu} A_{\epsilon} \varrho - R_{0,\mu} A_0 \varrho \|_{L^2(\Omega)} + \mu \| R_{\epsilon,\mu} A_{\epsilon} \varrho - R_{0,\mu} A_0 \varrho \|_{L^2(\Omega)}$$

$$\leq \mu \| R_{\epsilon,\mu} \| \times \| A_{\epsilon} \varrho - A_0 \varrho \|_{L^2(\Omega)} + \mu \| R_{\epsilon,\mu} A_{\epsilon} \varrho - R_{0,\mu} A_0 \varrho \|_{L^2(\Omega)}.$$

Since $A_0 \varrho \in H^1_0(\Omega; \omega_1)$ by hypothesis, then by using (49) (where we replace $\varrho$ by $A_0 \varrho$) and the fact that $\| R_{\epsilon,\mu} \| \leq \frac{1}{\mu}$ we obtain

$$\| A_{\epsilon,\mu} \varrho - A_{0,\mu} \varrho \|_{L^2(\Omega)} \leq \| A_{\epsilon} \varrho - A_0 \varrho \|_{L^2(\Omega)} + \epsilon C_{A_\Omega} \left( \| \nabla X_1 A_0 \varrho \|_{L^2(\Omega)} + \| A_0 \varrho \|_{L^2(\Omega)} \right)$$

$$= \epsilon \left( + \| \text{div} X_1(A_{11} \nabla X_1 \varrho) \|_{L^2(\Omega)} + \| \text{div} X_1(A_{12} \nabla X_2 \varrho) \|_{L^2(\Omega)} + \| \text{div} X_2(A_{21} \nabla X_1 \varrho) \|_{L^2(\Omega)} + C_{A_\Omega} \left( \| \nabla X_1 A_0 \varrho \|_{L^2(\Omega)} + \| A_0 \varrho \|_{L^2(\Omega)} \right) \right)$$

$$\leq C_{f,A_\Omega} \times \epsilon.$$

where we have used

$$A_{\epsilon} \varrho - A_0 \varrho = \epsilon A_{\epsilon} (A_{11} \nabla X_1 \varrho) + \epsilon A_{\epsilon} (A_{12} \nabla X_2 \varrho) + \epsilon A_{\epsilon} (A_{21} \nabla X_1 \varrho),$$

and the proof of the lemma is finished. \qed
Lemma 4.4. Under assumptions of Theorem 4.2, we have for any $g \in \left( H_0^1 \cap H^2(\omega_1) \right) \otimes \left( H_0^1 \cap H^2(\omega_2) \right)$:

$$\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \left\| e^{t\mathcal{A}_\epsilon, \mu} g - e^{t\mathcal{A}_0, \mu} g \right\|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times t \times \epsilon,$$

where $C_{g, A, \Omega}$ is independent of $\mu$ and $\epsilon$.

Proof. Let $\mu > 0$ and $t \geq 0$ and $\epsilon \in (0, 1]$ , we have

\[ e^{t\mathcal{A}_0, \mu} - e^{t\mathcal{A}_\epsilon, \mu} = \int_0^t \frac{d}{ds} \left( e^{(t-s)\mathcal{A}_\epsilon, \mu} e^{s\mathcal{A}_0, \mu} \right) ds = \int_0^t e^{(t-s)\mathcal{A}_\epsilon, \mu} (\mathcal{A}_0, \mu - \mathcal{A}_\epsilon, \mu) e^{s\mathcal{A}_0, \mu} ds. \]

Hence for $g \in L^2(\Omega)$ we have

\[ \left\| e^{t\mathcal{A}_\epsilon, \mu} g - e^{t\mathcal{A}_0, \mu} g \right\|_{L^2(\Omega)} \leq \int_0^t \left\| \mathcal{A}_0, \mu e^{s\mathcal{A}_0, \mu} g - \mathcal{A}_\epsilon, \mu e^{s\mathcal{A}_0, \mu} g \right\|_{L^2(\Omega)} ds \tag{51} \]

where have used the fact that $\left\| e^{(t-s)\mathcal{A}_\epsilon, \mu} \right\| \leq 1$, since $t - s \geq 0$.

Now, we suppose that $g \in \left( H_0^1 \cap H^2(\omega_1) \right) \otimes \left( H_0^1 \cap H^2(\omega_2) \right)$ \ (remark that $g \in D(A_0)$) and for $s \geq 0$ we set

\[ f_g := e^{s\mathcal{A}_0, \mu} g \]

We can prove that $f_g$ satisfies the same hypothesis satisfied by the function $f$ of Lemma 4.3 and moreover, for every $i, j = 1, ..., q$ we have:

\[ \left\| D_{x, x, j} f_g \right\|_{L^2(\Omega)} = \left\| D_{x, x, j} g \right\|_{L^2(\Omega)}, \quad \left\| D_{x, i} f_g \right\|_{L^2(\Omega)} = \left\| D_{x, i} g \right\|_{L^2(\Omega)}, \]

and

\[ \left\| (\mathcal{A}_0 f_g) \right\|_{L^2(\Omega)} \leq \left\| \mathcal{A}_0 g \right\|_{L^2(\Omega)}, \quad \left\| D_{x, i} (\mathcal{A}_0 f_g) \right\|_{L^2(\Omega)} \leq \left\| D_{x, i} (\mathcal{A}_0 g) \right\|_{L^2(\Omega)}, \]

and for every $i, j = 1, ..., q; \ j = q + 1, ..., N$ we have:

\[ \left\| D_{x, i} f_g \right\|_{L^2(\Omega)} \leq \frac{1}{\lambda} \left\| \mathcal{A}_0 g \right\|_{L^2(\Omega)} \left\| g \right\|_{L^2(\Omega)} \quad \text{and} \quad \left\| D_{x, i, j} f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| D_{x, i} \mathcal{A}_0 g \right\|_{L^2(\Omega)} \left\| D_{x, j} g \right\|_{L^2(\Omega)}. \]

The proof of these assertions follows from the identity $e^{s\mathcal{A}_0, \mu} (g_1 \otimes g_2) = g_1 \otimes e^{s\mathcal{A}_0, \mu} g_2$ \ (see Appendix B).

Applying Lemma 4.3, and using the above inequalities with (50) we get

\[ \left\| \mathcal{A}_0, \mu e^{s\mathcal{A}_0, \mu} g - \mathcal{A}_\epsilon, \mu e^{s\mathcal{A}_0, \mu} g \right\|_{L^2(\Omega)} \leq \epsilon \left( \left\| \text{div}_X (A_{11} \nabla_X f_g) \right\|_{L^2(\Omega)} + \left\| \text{div}_X (A_{12} \nabla_X f_g) \right\|_{L^2(\Omega)} + \left\| \text{div}_X (A_{13} \nabla_X f_g) \right\|_{L^2(\Omega)} + \left\| \text{div}_X (A_{14} \nabla_X f_g) \right\|_{L^2(\Omega)} \right) \]

\[ \leq C_{g, A, \Omega} \times \epsilon. \]

Notice that $C_{g, A, \Omega}$ does not depend in $s$, $\epsilon$ and $\mu$. Finally, integrating the above inequality in $s$ over $[0, t]$ and by using (51) we get the desired result. \qed

Now, we are able to prove Theorem 4.2. First we prove the case when $g \in \left( H_0^1 \cap H^2(\omega_1) \right) \otimes \left( H_0^1 \cap H^2(\omega_2) \right)$ and we conclude by a density argument. So let $g$ as mentioned above, by Lemma 4.4 we have

\[ \forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \left\| e^{t\mathcal{A}_\epsilon, \mu} g - e^{t\mathcal{A}_0, \mu} g \right\|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times t \times \epsilon. \tag{52} \]
Therefore, by passing to the limit in (52) as \( \mu \to +\infty \) we get (see the preliminaries, the abstract part)
\[
\forall t \geq 0, \forall \epsilon \in (0, 1] : \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon,
\]
whence for \( T \geq 0 \) fixed we obtain
\[
\forall \epsilon \in (0, 1] : \sup_{t \in [0,T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times T \times \epsilon.
\] (53)

Whence
\[
\sup_{t \in [0,T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0.
\] (54)

Now, let \( g \in L^2(\Omega) \) and let \( \delta > 0 \), by density there exists \( g_\delta \in \left( H^1_0 \cap H^2(\omega_1) \right) \otimes \left( H^1_0 \cap H^2(\omega_2) \right) \) such that
\[
\| g - g_\delta \|_{L^2(\Omega)} \leq \frac{\delta}{4}.
\]

According to (54) there exists \( \epsilon_\delta > 0 \) such that
\[
\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0,T]} \| S_\epsilon(t)g_\delta - S_0(t)g_\delta \|_{L^2(\Omega)} \leq \frac{\delta}{2}.
\]

Whence, by the triangle inequality we get
\[
\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0,T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq \frac{\delta}{2} + \sup_{t \in [0,T]} \left( \| S_\epsilon(t)g \| + \| S_0(t)g \| \right) \times \| g_\delta - g \|_{L^2(\Omega)}.
\]

Using the fact that the semigroups \( (S_\epsilon(t))_{t \geq 0} \) and \( (S_0(t))_{t \geq 0} \) are of contraction, we get
\[
\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0,T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq \delta.
\]

So, \( \sup_{t \in [0,T]} \| S_\epsilon(t)f - S_0(t)f \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0 \). The second assertion of the theorem is given by (53) and the proof of the theorem is completed.

### 4.3. An application to a linear parabolic equation

Theorem 4.2 gives an opening for the study of anisotropic singular perturbations of evolution partial differential equations from the semigroup’s point of view. In this subsection we just give a simple and short application to the linear parabolic equation
\[
\frac{\partial u_\epsilon}{\partial t} - \text{div}(A_\epsilon \nabla u_\epsilon) = 0,
\] (55)
supplemented with the boundary and the initial conditions
\[
\begin{align*}
 u_\epsilon(t, \cdot) &= 0 \text{ in } \partial \Omega \text{ for } t \in (0, +\infty) \\
 u_\epsilon(0, \cdot) &= u_{\epsilon,0}.
\end{align*}
\] (56) (57)

The limit problem is
\[
\frac{\partial u}{\partial t} - \text{div}_{X_2}(A_{22} \nabla_{X_2} u) = 0,
\] (58)
supplemented with the boundary and the initial conditions
\[
\begin{align*}
 u(t, \cdot) &= 0 \text{ in } \omega_1 \times \partial \omega_2 \text{ for } t \in (0, +\infty) \\
 u(0, \cdot) &= u_0.
\end{align*}
\] (59) (60)
The operator form of (55) – (57) and (58) – (60) reads

\[ \frac{d u_\epsilon}{d t} - A_\epsilon u_\epsilon = 0, \text{ with } u_\epsilon(0) = u_{\epsilon,0}, \]  

and

\[ \frac{d u}{d t} - A_0 u = 0, \text{ with } u(0) = u_0. \]

Suppose that \( u_0 \in D(A_0) \) and \( u_{\epsilon,0} \in D(A_\epsilon) \). Assume that (3), (4) and then it follows that (61), (62) have a unique classical solutions

\[ u_\epsilon \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(A_\epsilon)), \]

and

\[ u \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(A_0)). \]

We have the following convergence result.

**Theorem 4.5.** Suppose that \( u_0 \in D(A_0) \) and \( u_{\epsilon,0} \in D(A_\epsilon) \) such that \( u_{\epsilon,0} \to u_0 \) in \( L^2(\Omega) \), then under assumptions of Theorem 4.2, we have for any \( T \geq 0 \):

\[ \sup_{t \in [0,T]} \| u_\epsilon(t) - u(t) \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0. \]  

Moreover, if \( u_{\epsilon,0} \) and \( u_0 \) are in \( H^2(\Omega) \) and such that \((u_\epsilon,0)\) is bounded in \( H^2(\Omega) \) and \( \| \nabla X_\epsilon(u_\epsilon,0-u_0) \|_{L^2(\Omega)} \to 0 \), \( \| \nabla^2 X_\epsilon(u_\epsilon,0-u_0) \|_{L^2(\Omega)} \to 0 \) as \( \epsilon \to 0 \), then we have

\[ \sup_{t \in [0,T]} \left\| \frac{d}{d t}(u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} \to 0. \]

**Proof.** It is well known that the solutions \( u_\epsilon, u \) are given by

\[ u_\epsilon(t) = S_\epsilon(t)u_{\epsilon,0} \text{ and } u_0(t) = S_0(t)u_0, \text{ for every } t \geq 0. \]

Let \( T \geq 0 \), we have

\[ \sup_{t \in [0,T]} \| u_\epsilon(t) - u(t) \|_{L^2(\Omega)} \leq \sup_{t \in [0,T]} \| S_\epsilon(t)u_{\epsilon,0} - S_\epsilon(t)u_0 \|_{L^2(\Omega)} + \sup_{t \in [0,T]} \| S_\epsilon(t)u_0 - S_0(t)u_0 \|_{L^2(\Omega)} \]

\[ \leq \| u_{\epsilon,0} - u_0 \|_{L^2(\Omega)} + \sup_{t \in [0,T]} \| S_\epsilon(t)u_0 - S_0(t)u_0 \|_{L^2(\Omega)}. \]

Passing to the limit as \( \epsilon \to 0 \) by using Theorem 4.2, we get \( \sup_{t \in [0,T]} \| u_\epsilon(t) - u(t) \|_{L^2(\Omega)} \to 0. \)

For the second affirmation, we have

\[ \left\| \frac{d}{d t}(u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} \leq \| S_\epsilon(t)A_\epsilon u_{\epsilon,0} - S_\epsilon(t)A_0 u_0 \|_{L^2(\Omega)} \]

\[ \leq \| A_\epsilon u_{\epsilon,0} - A_0 u_0 \|_{L^2(\Omega)} + \sup_{t \in [0,T]} \| S_\epsilon(t)A_0 u_0 - S_0(t)A_0 u_0 \|_{L^2(\Omega)}. \]

As \( (u_{\epsilon,0}) \) is bounded in \( H^2(\Omega), u_0 \in H^2(\Omega) \) and \( \| \nabla X_\epsilon(u_{\epsilon,0}-u_0) \|_{L^2(\Omega)} \to 0, \| \nabla^2 X_\epsilon(u_{\epsilon,0}-u_0) \|_{L^2(\Omega)} \to 0 \) as \( \epsilon \to 0 \), then by using (50) we get immediately \( \| A_\epsilon u_{\epsilon,0} - A_0 u_0 \|_{L^2(\Omega)} \to 0 \) as \( \epsilon \to 0 \), and we conclude by applying Theorem 4.2. \( \square \)
Remark 4.6. Consider the nonhomogeneous parabolic equations associated to (55) and (58) with second member f(t, x). Suppose that f is regular enough, for example f ∈ Lip([0, T]; L^2(Ω)), then the associated classical solutions u_ε and u exist and they are unique. In this case, we have the same convergence result (63). The proof follows immediately from the use of the following integral representation formulas of the solutions

\[ u_\varepsilon(t) = S_\varepsilon(t)u_{\varepsilon,0} + \int_0^t S_\varepsilon(t-r)f(r)dr, \]

\[ u(t) = S_0(t)u_0 + \int_0^t S_0(t-r)f(r)dr, \]

Theorem 4.2, and Lebesgue’s theorem.

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Appendix A. Density lemmas

Let ω_1 and ω_2 be two open bounded subsets of \( \mathbb{R}^q \) and \( \mathbb{R}^{N-q} \) respectively. Recall that

\[ H^1_0(Ω; ω_2) = \{ u \in L^2(Ω) \mid ∇X_2u \in L^2(Ω), \text{a.e.} X_1 \in ω_1, u(X_1, \cdot) \in H^1_0(ω_2) \}, \]

normed by \( \| ∇X_2(\cdot) \|_{L^2(Ω)} \).

We have the following

**Lemma A.1.** The space \( H^1_0(Ω) \) is dense in \( H^1_0(Ω; ω_2) \).

**Proof.** Let \( u \in H^1_0(Ω; ω_2) \) fixed. Let \( l \) be the linear form defined on \( H^1_0(Ω) \) by

\[ ∀φ \in H^1_0(Ω) : l(φ) = \int_Ω ∇X_2u \cdot ∇X_2φ dx \]

\( l \) is continuous on \( H^1_0(Ω) \), indeed we have

\[ ∀φ \in H^1_0(Ω) : |l(φ)| ≤ \| ∇X_2u \|_{L^2(Ω)} \| ∇X_2φ \|_{L^2(Ω)} \]

and then,

\[ ∀φ \in H^1_0(Ω) : |l(φ)| ≤ \| ∇X_2u \|_{L^2(Ω)} \| ∇φ \|_{L^2(Ω)} . \]

For every \( n \in \mathbb{N}^* \), we denote \( u_n \) the unique solution of

\[ \left\{ \begin{array}{l}
\frac{1}{n} \int_Ω ∇X_1u_n \cdot ∇X_1φ dx + \int_Ω ∇X_2u_n \cdot ∇X_2φ dx = l(φ), \forall φ \in H^1_0(Ω)
\end{array} \right. \]

(64)

where the existence and uniqueness follows from the Lax-Milgram theorem. Testing with \( u_n \) in (64) we get, for every \( n \in \mathbb{N}^* \)

\[ \frac{1}{n^2} \int_Ω |∇X_1u_n|^2 dx + \int_Ω |∇X_2u_n|^2 dx ≤ \| ∇X_2u \|_{L^2(Ω)} \| ∇X_2u_n \|_{L^2(Ω)} ; \]

then, we deduce that

\[ ∀n \in \mathbb{N}^* : \| ∇X_2u_n \|_{L^2(Ω)} ≤ \| ∇X_2u \|_{L^2(Ω)} , \]

(65)

and

\[ ∀n \in \mathbb{N}^* : \frac{1}{n} \| ∇X_1u_n \|_{L^2(Ω)} ≤ \| ∇X_2u \|_{L^2(Ω)} . \]

(66)
Using (65) and Poincaré’s inequality we obtain:
\[ \forall n \in \mathbb{N}^* : \|u_n\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla X_2 u\|_{L^2(\Omega)}. \] (67)

Reflexivity of \( L^2(\Omega) \) shows that there exists, \( u_\infty, u'_\infty, u''_\infty \in L^2(\Omega) \) and a subsequence still labeled \( (u_n) \) such that
\[ u_n \rightharpoonup u_\infty, \nabla X_2 u_n \rightharpoonup u'_\infty \text{ and } \frac{1}{n} \nabla X_1 u_n \rightharpoonup u''_\infty \text{ in } L^2(\Omega) \text{ weakly}. \]

Since the derivation on \( D'(\Omega) \) is continuous we get
\[ u_n \rightharpoonup u_\infty, \nabla X_2 u_n \rightharpoonup \nabla X_2 u_\infty \text{ and } \frac{1}{n} \nabla X_1 u_n \rightharpoonup 0 \text{ in } L^2(\Omega) \text{ weakly}. \] (68)

**1) we have** \( u_\infty \in H^1_0(\Omega; \omega_2) : \)

By the Mazur Lemma, there exists a sequence \( (U_n) \) of convex combinations of \( \{u_n\} \) such that
\[ \nabla X_2 U_n \rightharpoonup \nabla X_2 u_\infty \text{ in } L^2(\Omega) \text{ strongly}, \] (69)
then by the Lebesgue theorem there exists a subsequence \( (U_{n_k}) \) such that:
\[ \text{For a.e. } X_1 \in \omega_1 : \nabla X_2 U_{n_k}(X_1, \cdot) \rightharpoonup \nabla X_2 u_\infty(X_1, \cdot) \text{ in } L^2(\omega_2) \text{ strongly}. \] (70)

Now, since \( (U_{n_k}) \in H^1_0(\Omega)^N \) then
\[ \text{For a.e.} X_1 \in \omega_1 : (U_{n_k}(X_1, \cdot)) \in H^1_0(\omega_2)^N. \] (71)

Combining (70) and (71) we deduce:
\[ \text{For a.e. } X_1 \in \omega_1, \ u_\infty(X_1, \cdot) \in H^1_0(\omega_2), \]
and the proof of \( u_\infty \in H^1_0(\Omega; \omega_2) \) is finished.

**2) we have** \( u_\infty = u : \)

Passing to the limit in (64) by using (68) we obtain
\[ \int_{\omega_2} \nabla X_2 u_\infty \cdot \nabla X_2 \varphi dX_2 = \int_{\omega_2} \nabla X_2 u \cdot \nabla X_2 \varphi dX_2, \forall \varphi \in H^1_0(\Omega). \] (72)

For every \( \varphi_1 \in H^1_0(\omega_1) \) and \( \varphi_2 \in H^1_0(\omega_2) \) taking, \( \varphi = \varphi_1 \otimes \varphi_2 \) in (72) we obtain, for a.e. \( X_1 \in \omega_1 \)
\[ \int_{\omega_2} \nabla X_2 u_\infty(X_1, \cdot) \cdot \nabla X_2 \varphi_2 dX_2 = \int_{\omega_2} \nabla X_2 u(X_1, \cdot) \cdot \nabla X_2 \varphi_2 dX_2, \forall \varphi_2 \in H^1_0(\omega_2). \]

For a.e. \( X_1 \in \omega_1 \), taking \( \varphi_2 = u_\infty(X_1, \cdot) - u(X_1, \cdot) \), which belong to \( H^1_0(\omega_2) \), in the above equality yields:
\[ \int_{\omega_2} |\nabla X_2 (u_\infty(X_1, \cdot) - u(X_1, \cdot))|^2 dX_2 = 0. \]

Integrating over \( \omega_1 \) we deduce
\[ \int_{\Omega} |\nabla X_2 (u_\infty - u)|^2 dx = 0. \]

and finally since \( \|\nabla X_2 (\cdot)\|_{L^2(\Omega)} \) is a norm on \( H^1_0(\Omega; \omega_2) \) we get,
\[ u_\infty = u. \] (73)
combining (69) and (73) we get the desired result.

Remark A.2. By symmetry, \( H_0^1(\Omega) \) is dense in the space

\[
H_0^1(\Omega; \omega_1) = \left\{ u \in L^2(\Omega) \mid \nabla X_1 u \in L^2(\Omega), \text{ and for a.e. } X_2 \in \omega_2, u(\cdot, X_2) \in H_0^1(\omega_1) \right\},
\]

normed by \( \| \nabla X_1 \cdot \|_{L^2(\Omega)} \).

Lemma A.3. The space \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) is dense in \( H_0^1(\Omega) \).

Proof. For a functions \( \varphi : \omega_1 \to \mathbb{R}, \psi : \omega_2 \to \mathbb{R} \) we denote by \( \varphi \otimes \psi \) the function defined on \( \Omega \) by \( (\varphi \otimes \psi)(X_1, X_2) = \varphi(X_1) \times \psi(X_2) \), the tensor product \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) is the vector space generated by the elements of the form \( \varphi \otimes \psi \) with \( \varphi \) and \( \psi \) in \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively.

It is well known that \( D(\omega_1) \otimes D(\omega_2) \) is dense in \( D(\omega_1 \times \omega_2) \), here \( D(\omega_1 \times \omega_2) \) is equipped with its natural topology (the inductive limit topology). It is clear that the injection of \( D(\omega_1 \times \omega_2) \) in \( H_0^1(\omega_1 \times \omega_2) \) is continuous, thanks to the inequality

\[
\forall u \in D(\Omega) : \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq \sqrt{N \times \text{mes}(\Omega)} \times \left( \max_{1 \leq i \leq N} \sup_{\text{Support}(u)} |\partial_x u| \right).
\]

Hence, by the density chain rule we obtain the density of \( D(\omega_1) \otimes D(\omega_2) \) in \( H_0^1(\Omega) \), and finally since \( D(\omega_1) \otimes D(\omega_2) \subset H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) we get the desired result. \( \square \)

Lemma A.4. Let \( (V_n^{(1)}) \) and \( (V_n^{(2)}) \) be two sequences of subspaces (not necessarily of finite dimension) of \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively. If \( \cup V_n^{(1)} \) and \( \cup V_n^{(2)} \) are dense in \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively, then \( \bigcup_{n,m} (V_n^{(1)} \otimes V_m^{(2)}) \) is dense in \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) for the induced topology of \( H_0^1(\Omega) \). In particular, if \( (V_n^{(1)}) \) and \( (V_n^{(2)}) \) are nondecreasing then \( \bigcup_n (V_n^{(1)} \otimes V_n^{(2)}) \) is dense in \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \).

Proof. Let’s start by a useful inequality. For \( u \otimes v \) in \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) we have:

\[
\| u \otimes v \|_{H_0^1(\Omega)}^2 = \int_{\Omega} |\nabla X_1 (u \otimes v) |^2 \, dx + \int_{\Omega} |\nabla X_2 (u \otimes v) |^2 \, dx \\
= \left( \int_{\omega_2} v^2 \, dX_2 \right) \times \left( \int_{\omega_1} |\nabla X_1 u |^2 \, dX_1 \right) \\
+ \left( \int_{\omega_1} u^2 \, dX_1 \right) \times \left( \int_{\omega_2} |\nabla X_2 v |^2 \, dX_2 \right) \\
\leq C \| u \|_{H_0^1(\omega_1)}^2 \times \| v \|_{H_0^1(\omega_2)}^2, \tag{74}
\]

where we have used Fubini’s theorem and Poincaré’s inequality, here \( C = C_{\omega_1}^2 + C_{\omega_2}^2 > 0 \). Now, fix \( \eta > 0 \) and let \( \varphi \otimes \psi \in H_0^1(\omega_1) \otimes H_0^1(\omega_2) \), by density of \( \cup V_n^{(1)} \) in \( H_0^1(\omega_1) \) there exists \( n \in \mathbb{N} \) and \( \varphi_n \in V_n^{(1)} \) such that:

\[
\| \varphi \|_{H_0^1(\omega_2)} \times \| \varphi_n - \varphi \|_{H_0^1(\omega_1)} \leq \frac{\eta}{2\sqrt{C}}.
\]

Similarly by density of \( \cup V_n^{(2)} \) in \( H_0^1(\omega_2) \), there exits \( m \in \mathbb{N} \) (which depends on \( n \) and \( \psi \)) and \( \psi_m \in V_m^{(2)} \) such that

\[
\| \varphi_n \|_{H_0^1(\omega_1)} \times \| \psi_m - \psi \|_{H_0^1(\omega_2)} \leq \frac{\eta}{2\sqrt{C}}.
\]
Whence, by using the triangle inequality and (74) we obtain
\[ \| \varphi \otimes \psi - \varphi_n \otimes \psi_m \|_{H_0^1(\Omega)} \leq \eta. \]  
(75)

Now, since every element of \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) could be written as \( \sum_{i=1}^l \varphi_i \otimes \psi_i \), then by using the triangle inequality and using (75) with \( \eta \) replaced by \( \frac{\eta}{4} \), one gets the desired result. \( \square \)

**Corollary A.5.** \( \text{vect} \left( \bigcup_n \left( V_n^{(1)}(\omega_1) \otimes V_n^{(2)}(\omega_2) \right) \right) \) is dense in \( H_0^1(\Omega) \). In particular, if \( (V_n^{(1)}) \) and \( (V_n^{(2)}) \) are nondecreasing then \( \bigcup_n \left( V_n^{(1)} \otimes V_n^{(2)} \right) \) is dense in \( H_0^1(\Omega) \).

### Appendix B. Semigroup

**Lemma B.1.** Assume (3 - 4), (18) and let \( f_1 \in L^2(\omega_1), f_2 \in L^2(\omega_2) \) then for every \( \mu > 0 \) we have
\[ R_{0, \mu}(f_1 \otimes f_2) = f_1 \otimes (R_{0, \mu}f_2). \]

Notice that \( R_{0, \mu}f_2 \in H_0^1(\omega_2) \). Moreover, we have
\[ \mathcal{A}_{0, \mu}(f_1 \otimes f_2) = f_1 \otimes (\mathcal{A}_{0, \mu}f_2). \]

Notice also that \( \mathcal{A}_{0, \mu}f_2 \in L^2(\omega_2) \). Here, \( \mathcal{A}_{0, \mu} \) is the Yosida approximation of \( \mathcal{A}_0 \) that is \( \mathcal{A}_{0, \mu} = \mu \mathcal{A}_0 R_{0, \mu} \).

**Proof.** Let \( v_2 \in H_0^1(\omega_2) \) be the unique solution in \( H_0^1(\omega_2) \) to
\[ \mu \int \omega_2 v_2 \varphi dX_2 + \int \omega_2 A_{22}(X_2) \nabla X_2 v_2 \cdot \nabla X_2 \varphi dX_2 = \int \omega_2 f_2 \varphi dX_2, \forall \varphi \in H_0^1(\omega_2). \]
(76)

Let \( \varphi \in H_0^1(\Omega; \omega_2) \), then \( \varphi(X_1, \cdot) \in H_0^1(\omega_2) \) for a.e. \( X_1 \in \omega_1 \). Let \( f_1 \in L^2(\omega_1) \), multiplying (76) by \( f_1 \), testing in (76) with \( \varphi(X_1, \cdot) \) and integrating over \( \omega_1 \) yields
\[ \mu \int \omega_1 f_1 v_2 \varphi dX_1 + \int \omega_1 A_{22}(X_2) \nabla X_2 (f_1 v_2) \cdot \nabla X_2 \varphi dX_1 = \int \omega_1 f_1 f_2 \varphi dX. \]

It is clear that \( f_1 v_2 \in H_0^1(\Omega; \omega_2) \) whence, \( R_{0, \mu}(f_1 \otimes f_2) = f_1 \otimes v_2 \), in particular when \( f_1 = 1 \) we have \( R_{0, \mu}(f_2) = v_2 \), and therefore \( R_{0, \mu}(f_1 \otimes f_2) = f_1 \otimes R_{0, \mu}(f_2) \). The second assertion follows immediately from the first one, in fact
\[ \mathcal{A}_{0, \mu}(f_1 \otimes f_2) = \mu \mathcal{A}_0 R_{0, \mu}(f_1 \otimes f_2) = \mu \mathcal{A}_0 (f_1 \otimes R_{0, \mu}f_2). \]

We have \( R_{0, \mu}f_2 \in D(\mathcal{A}_0) \cap H_0^1(\omega_2) \) then by (18) (the operator \( \mathcal{A}_0 \) does not depend on the \( X_1 \) direction), we get
\[ \mathcal{A}_0(f_1 \otimes R_{0, \mu}f_2) = f_1 \otimes \mathcal{A}_0(R_{0, \mu}f_2), \]
notice that \( \mathcal{A}_0(R_{0, \mu}f_2) \in L^2(\omega_2) \). Finally we get
\[ \mathcal{A}_{0, \mu}(f_1 \otimes f_2) = \mu f_1 \otimes \mathcal{A}_0(R_{0, \mu}f_2) = f_1 \otimes \mathcal{A}_{0, \mu}(f_2). \]
\( \square \)

Now, for \( s \geq 0, \mu > 0 \) and any \( g \in L^2(\Omega) \), we denote \( f_g := e^{s \mathcal{A}_{0, \mu}} g \).
Lemma B.2. Assume (3–4), (18). Let \( g = g_1 \otimes g_2 \in L^2(\omega_1) \otimes L^2(\omega_2) \) then for \( s \geq 0, \mu > 0 \) we have:

\[
f_g = g_1 \otimes e^{sA_{0,\mu}} g_2.
\]

Notice that \( e^{sA_{0,\mu}} g_2 \in L^2(\omega_2) \).

Proof. we have

\[
f_g = e^{sA_{0,\mu}} g = \sum_{k=0}^{\infty} \frac{s^k}{k!} A_{0,\mu}^k g,
\]

where the series converges in \( L^2(\Omega) \). By an immediate induction we get by using B.1

\[
\forall k \in \mathbb{N} : A_{0,\mu}^k g = g_1 \otimes A_{0,\mu}^k g_2,
\]

with \( A_{0,\mu}^k g_2 \in L^2(\omega_2) \) for every \( k \in \mathbb{N} \), and the Lemma follows. \( \square \)

Lemma B.3. Assume (3–4), (18). Let \( g \in H^2(\omega_1) \otimes L^2(\omega_2) \) then for \( s, \mu > 0, i, j = 1, \ldots, q \) we have:

\[
D_{x,i}^2 f_g, D_{x,j}^2 f_g \in L^2(\Omega),
\]

such that

\[
D_{x,i}^2 f_g = e^{sA_{0,\mu}} (D_{x,i}^2 g), \quad D_{x,j}^2 f_g = e^{sA_{0,\mu}} (D_{x,j} g).
\]

and:

\[
\| D_{x,i}^2 f_g \|_{L^2(\Omega)} \leq \| D_{x,i}^2 g \|_{L^2(\Omega)}, \quad \| D_{x,j}^2 f_g \|_{L^2(\Omega)} \leq \| D_{x,j} g \|_{L^2(\Omega)}.
\]

Proof. 1) Suppose the simple case when \( g = g_1 \otimes g_2 \). So let \( g = g_1 \otimes g_2 \in H^2(\omega_1) \otimes L^2(\omega_2) \) and let us prove assertions (77). By Lemma B.2 we get

\[
f_g = g_1 \otimes e^{sA_{0,\mu}} (g_2),
\]

with \( e^{sA_{0,\mu}} g_2 \in L^2(\omega_2) \). Hence, for \( i, j = 1, \ldots, q \) we have \( D_{x,i}^2 f_g \in L^2(\Omega) \) and

\[
D_{x,i}^2 f_g = \left( D_{x,i}^2 g_1 \right) \otimes e^{sA_{0,\mu}} g_2,
\]

and applying B.2 we get

\[
D_{x,i}^2 f_g = e^{sA_{0,\mu}} (D_{x,i}^2 g).
\]

Similarly we get \( D_{x,j}^2 f_g = e^{sA_{0,\mu}} (D_{x,j} g) \), and assertion (77) follows.

2) Now, let \( g \in H^2(\omega_1) \otimes L^2(\omega_2) \), since \( g \) is a finite sum of elements of the form \( g_1 \otimes g_2 \), then by linearity we get

\[
D_{x,i}^2 f_g = e^{sA_{0,\mu}} (D_{x,i}^2 g), \quad D_{x,j}^2 f_g = e^{sA_{0,\mu}} (D_{x,j} g),
\]

therefore

\[
\| D_{x,i}^2 f_g \|_{L^2(\Omega)} \leq \| e^{sA_{0,\mu}} \| \| D_{x,i}^2 g \|_{L^2(\Omega)} \leq \| D_{x,i}^2 g \|_{L^2(\Omega)},
\]

and similarly we obtain the second inequality of (78). \( \square \)

Lemma B.4. Assume (3–4), (18) and (50). Let \( g \in \left( H^1_0 \cap H^2(\omega_1) \right) \otimes \left( H^1_0 \cap H^2(\omega_2) \right) \) then, for \( s \geq 0, \mu > 0 \) :

\[
f_g \in D(A_0), \quad A_0 f_g \in H^1_0(\Omega; \omega_1), \quad \text{and} \quad D_{x_i} (A_0 f_g) = e^{sA_{0,\mu}} (D_{x_i} A_0 g),
\]

\[
\| (A_0 f_g) \|_{L^2(\Omega)} \leq \| A_0 g \|_{L^2(\Omega)} \quad \text{and} \quad \| D_{x_i} (A_0 f_g) \|_{L^2(\Omega)} \leq \| D_{x_i} A_0 g \|_{L^2(\Omega)}, \quad i = 1, \ldots, q,
\]

Proof. 1) Suppose \( g = g_1 \otimes g_2 \in \left( H^1_0 \cap H^2(\omega_1) \right) \otimes \left( H^1_0 \cap H^2(\omega_2) \right) \) and let us prove (79). Since \( g \in D(A_0) \), thanks to (50), then \( f_g = e^{sA_{0,\mu}} g \in D(A_0) \) and \( A_0 f_g = e^{sA_{0,\mu}} A_0 g \) (thanks to (47)). Now, we have

\[
A_0 f_g = A_0 (e^{sA_{0,\mu}} g) = A_0 (g_1 \otimes e^{sA_{0,\mu}} g_2).
\]

Notice that, \( g_2 \in D(A_0) \), thanks to (50) then \( e^{sA_{0,\mu}} g_2 \in D(A_0) \) (thanks to (47)), hence

\[
A_0 f_g = g_1 A_0 e^{sA_{0,\mu}} g_2.
\]
where we have used the fact that $A_0$ is independent of the $X_1$ – direction. Using the fact that $e^{sA_0 \mu}$ and $A_0$ commute on $D(A_0)$, we get

$$A_0 f_g = g_1 e^{sA_0 \mu} A_0 g_2.$$  

Now, we have $A_0 g_2 \in L^2(\omega_2)$ then $e^{A_0 \mu} A_0 g_2 \in L^2(\omega_2)$ (thanks to Lemma B.2), however $g_1 \in H_0^1(\omega_1)$, then

$$A_0 f_g \in H_0^1(\Omega; \omega_1).$$  

Whence, for $i = 1, \ldots, q$ we have

$$D_{x_i}(A_0 f_g) = D_{x_i} g_1 \otimes e^{sA_0 \mu} A_0 g_2,$$

and hence by B.2 we get

$$D_{x_i}(A_0 f_g) = e^{sA_0 \mu} (D_{x_i} g_1 \otimes A_0 g_2) = e^{sA_0 \mu} (D_{x_i} A_0 g).$$

(Remark that $D_{x_i} A_0 g \in L^2(\Omega)$ since $g_1 \in H_0^1(\omega_1)$ and $A_0 g_2 \in L^2(\omega_2)$). 2) Now, for a general $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, assertion (79) follows by linearity. Finally, we show (80). We have

$$\|(A_0 f_g)_{L^2(\Omega)} = \|e^{sA_0 \mu} (A_0 g)\|_{L^2(\Omega)} \leq \|e^{sA_0 \mu} \|_{L^2(\Omega)} \|A_0 g\|_{L^2(\Omega)} \leq \|A_0 g\|_{L^2(\Omega)}.$$  

For $i = 1, \ldots, q$ we get

$$\|D_{x_i}(A_0 f_g)\|_{L^2(\Omega)} = \|e^{sA_0 \mu} (D_{x_i} A_0 g)\|_{L^2(\Omega)} \|A_0 g\|_{L^2(\Omega)} \leq \|e^{sA_0 \mu} \|_{L^2(\Omega)} \|D_{x_i} A_0 g\|_{L^2(\Omega)} \leq \|D_{x_i} A_0 g\|_{L^2(\Omega)}.$$

\[ \square \]

**Lemma B.5.** Assume (3 – 4), (18) and (50). Let $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ then, for $s \geq 0, \mu > 0, i = 1, \ldots, q, \ j = q + 1, \ldots, N$ we have $D_{x_i} f_g, D_{x_i} f_g \in L^2(\Omega)$ and

$$\|D_{x_i} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|A_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}, \quad \|D_{x_i} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|D_{x_i} A_0 g\|_{L^2(\Omega)} \|D_{x_i} g\|_{L^2(\Omega)}.$$  

(81)

**Proof.** 1) Let us show the first inequality of (81). Suppose $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$. Notice that $g \in D(A_0)$, thanks to (50) then according to (47) we have $f_g \in D(A_0) \subset H_0^1(\Omega; \omega_2)$, i.e. for $j \in \{q + 1, \ldots, N\}$ fixed, we have $D_{x_j} f_g \in L^2(\Omega)$, and

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|A_0 f_g\|_{L^2(\Omega)} \|f_g\|_{L^2(\Omega)} \leq \frac{1}{\lambda} \|A_0 g\|_{L^2(\Omega)} \|f_g\|_{L^2(\Omega)}.$$  

We have, $\|A_0 f_g\|_{L^2(\Omega)} = \|A_0 e^{sA_0 \mu} g\|_{L^2(\Omega)} = \|e^{sA_0 \mu} A_0 g\|_{L^2(\Omega)} \leq \|A_0 g\|_{L^2(\Omega)}$, and $\|f_g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$, therefore

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|A_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$  

2) Now, let $1 \leq i \leq q$ fixed, then according to Lemma B.3 we have $D_{x_i} f_g = e^{sA_0 \mu} (D_{x_i} g)$, notice that $D_{x_i} g = D_{x_i} g_1 \otimes g_2 \in D(A_0)$ and hence, $D_{x_i} f_g \in D(A_0)$, in particular $D_{x_i} f_g \in H_0^1(\Omega; \omega_2)$, and for $q + 1 \leq j \leq N$
we have
\[ \left\| D_{x,x}^2 f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| -A_0 D_x, f_g, D_x, f_g \right\|_{L^2(\Omega)} \]
\[ \leq \frac{1}{\lambda} \left\| A_0 D_x, f_g \right\|_{L^2(\Omega)} \left\| D_x, f_g \right\|_{L^2(\Omega)} . \]

We have,
\[ \left\| A_0 D_x, f_g \right\|_{L^2(\Omega)} = \left\| e^{\xi A_0} (D_x, g) \right\|_{L^2(\Omega)} = \left\| e^{\xi A_0} (A_0 D_x, g) \right\|_{L^2(\Omega)} \]
\[ \leq \left\| (D_x, A_0 g) \right\|_{L^2(\Omega)} , \]
according to (78) we have \( \left\| D_x, f_g \right\|_{L^2(\Omega)} \leq \left\| D_x, g \right\|_{L^2(\Omega)} \), finally we obtain
\[ \left\| D_{x,x}^2 f_g \right\|_{L^2(\Omega)} \leq \frac{1}{\lambda} \left\| (D_x, A_0 g) \right\|_{L^2(\Omega)} \left\| D_x, g \right\|_{L^2(\Omega)} . \]

\[ \square \]

**Lemma B.6.** Under assumptions of Lemma B.5, we have for \( g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2)) \):
\[ f_g \in H_0^1(\Omega) \cap D(A_0), \]
(82)
and
\[ \text{div}_{X_1}(A_{11} \nabla_{X_1} f), \text{div}_{X_1}(A_{12} \nabla_{X_2} f), \text{div}_{X_2}(A_{21} \nabla_{X_1} f) \in L^2(\Omega). \]
(83)

**Proof.** Let us prove (82). In Lemma B.4 we proved that \( f_g \in D(A_0) \). Let us show that \( f_g \in H_0^1(\Omega) \). Suppose the simple case \( g = g_1 \otimes g_2 \), we have \( f_g = g_1 \otimes e^{^\xi A_0} g_2 \). Since \( g_2 \in D(A_0) \), then \( e^{^\xi A_0} g_2 \in D(A_0) \), in particular we have \( e^{^\xi A_0} g_2 \in H_0^1(\Omega; \omega_2) \) however, according to Lemma B.2 \( e^{^\xi A_0} g_2 \in L^2(\omega_2) \), hence \( e^{^\xi A_0} g_2 \in H_0^1(\omega_2) \), finally as \( g_1 \in H_0^1(\omega_1) \) we get \( f_g \in H_0^1(\Omega) \). For a general \( g \) in the tensor product space, the proof follows by linearity.

Now, let us show (83). According to Lemmas B.3, B.5 all these derivatives \( D_x, f_g, D_{x,x}^2 f_g \) for \( 1 \leq i, j \leq q \), and \( D_x, f_g, D_{x,x}^2 f_g \) for \( 1 \leq i \leq q, q + 1 \leq j \leq N \) belong to \( L^2(\Omega) \). Whence, combining that with (50) we get (83).

**Appendix C. Existence Theorems**

Let \( V \subset H_0^1(\Omega) \) a subspace. We consider also the problem
\[ \left\{ \begin{array}{l}
\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\
\end{array} \right. \]
u \( \in V \).
(84)
with \( A_{22} \) and \( \beta \) as in the introduction.

**Proposition C.1.** If \( V \) is closed in \( H_0^1(\Omega; \omega_2) \) then there exists a solution to (84).

**Proof.** We consider the perturbed problem
\[ \left\{ \begin{array}{l}
\int_{\Omega} \beta(u_\epsilon) \varphi dx + \int_{\Omega} \tilde{A}_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\
\end{array} \right. \]
u_\epsilon \( \in V \).
(85)
with
\[ \tilde{A}_\epsilon = \begin{pmatrix} e^2 I_q & 0 \\
0 & A_{22} \end{pmatrix} \]
The space $V$ is closed in $H^1_0(\Omega)$ and $\tilde{A}_\epsilon$ is bounded and coercive, then by the Schauder fixed point theorem there exists a solution $u_\epsilon$ to (85). This solution is unique in $V$ thanks to monotonicity and coercivity of $\tilde{A}_\epsilon$. Testing with $u_\epsilon$ we obtain

$$\epsilon \|\nabla X_1 u_\epsilon\|_{L^2(\Omega)} \cdot \|\nabla X_2 u_\epsilon\|_{L^2(\Omega)} \cdot \|u_\epsilon\|_{L^2(\Omega)} \leq C,$$

where $C$ is independent of $\epsilon$, we have used that $\int_\Omega \beta(u_\epsilon) u_\epsilon \, dx \geq 0$ (thanks to monotonicity of $\beta$ and $\beta(0) = 0$). And we also have

$$\|\beta(u_\epsilon)\|_{L^2(\Omega)} \leq M(\|\Omega\|^4 + C),$$

so there exists $v \in L^2(\Omega)$, $u \in L^2(\Omega)$, $\nabla X_2 u \in L^2(\Omega)$ and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that

$$\beta(u_{\epsilon_k}) \rightharpoonup v, \; \epsilon_k \nabla X_1 u_{\epsilon_k} \rightarrow 0, \; \nabla X_2 u_{\epsilon_k} \rightarrow \nabla X_2 u, \; u_{\epsilon_k} \rightharpoonup u \text{ in } L^2(\Omega)-\text{weak} \quad (86)$$

Passing to the limit in the weak formulation of (85) we get

$$\int_\Omega v \varphi \, dx + \int_\Omega A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi \, dx = \int_\Omega f \varphi \, dx, \forall \varphi \in V \quad (87)$$

Take $\varphi = u_{\epsilon_k}$ in the previous equality and passing to the limit we get

$$\int_\Omega v u \, dx + \int_\Omega A_{22} \nabla X_2 u \cdot \nabla X_2 u \, dx = \int_\Omega f u \, dx \quad (88)$$

Let us consider the quantity

$$0 \leq I_k = \int_\Omega \epsilon^2 |\nabla X_1 u_{\epsilon_k}|^2 \, dx + \int_\Omega A_{22} \nabla X_1(u_{\epsilon_k} - u) \cdot \nabla X_1(u_{\epsilon_k} - u)$$

$$+ \int_\Omega (\beta(u_{\epsilon_k}) - \beta(u))(u_{\epsilon_k} - u) \, dx$$

$$= \int_\Omega f u_{\epsilon_k} \, dx - \int_\Omega A_{22} \nabla X_2 u_{\epsilon_k} \cdot \nabla X_2 u_{\epsilon_k} \, dx - \int_\Omega A_{22} \nabla X_2 u \cdot \nabla X_2 u_{\epsilon_k} \, dx$$

$$+ \int_\Omega f u \, dx - \int_\Omega v u \, dx - \int_\Omega \beta(u_{\epsilon_k}) \, dx$$

$$- \int_\Omega \beta(u_{\epsilon_k}) \, dx + \int_\Omega \beta(u) \, dx$$

Remark that this quantity is positive thanks to the ellipticity and monotonicity assumptions. Passing to the limit as $k \rightarrow \infty$ using (86), (88) we get

$$\lim I_k = 0$$

And finally the ellipticity assumption shows that

$$\|\epsilon_k \nabla X_1 u_{\epsilon_k}\|_{L^2(\Omega)} \cdot \|u_{\epsilon_k} - u\|_{L^2(\Omega)} \cdot \|\nabla X_2(u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0 \quad (89)$$

and therefore,

$$\beta(u_{\epsilon_k}) \rightarrow \beta(u) \text{ strongly in } L^2$$

Whence (87) becomes

$$\int_\Omega \beta(u) \varphi \, dx + \int_\Omega A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi \, dx = \int_\Omega f \varphi \, dx, \forall \varphi \in V$$

$$\|\nabla X_2(u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0$$

shows that $u \in H^1_0(\Omega; \omega_2)$, and therefore since $V$ is closed in $H^1_0(\Omega; \omega_2)$ then $u \in V$. $$\square$$
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