Tensor Decompositions for Identifying Directed Graph Topologies and Tracking Dynamic Networks

Yanning Shen, Student Member, IEEE, Brian Baingana, Member, IEEE, and Georgios B. Giannakis, Fellow, IEEE

Abstract—Directed networks are pervasive both in nature and engineered systems, often underlying the complex behavior observed in biological systems, microblogs and social interactions over the web, as well as global financial markets. Since their structures are often unobservable, in order to facilitate network analytics, one generally resorts to approaches capitalizing on measurable nodal processes to infer the unknown topology. Structural equation models (SEMs) are capable of incorporating exogenous inputs to resolve inherent directional ambiguities. However, conventional SEMs assume full knowledge of exogenous inputs, which may not be readily available in some practical settings. The present paper advocates a novel SEM-based topology inference approach that entails factorization of a three-way tensor, constructed from the observed nodal data, using the well-known parallel factor (PARAFAC) decomposition. It turns out that second-order piecewise stationary statistics of exogenous variables suffice to identify the hidden topology. Capitalizing on the uniqueness properties inherent to high-order tensor factorizations, it is shown that topology identification is possible under reasonably mild conditions. In addition, to facilitate real-time operation and inference of time-varying networks, an adaptive (PARAFAC) tensor decomposition scheme which tracks the topology-revealing tensor factors is developed. Extensive tests on simulated and real stock quote data demonstrate the merits of the novel tensor-based approach.

Index Terms—Structural equation models, CANDECOMP/PARAFAC (CP) decomposition, network topology inference.

I. INTRODUCTION

The study of networks and network phenomena has recently emerged as a major catalyst for collectively understanding the behavior of complex systems [8], [15], [26]. Such systems are ubiquitous, and commonly arise in both natural and man-made settings. For example, online interactions over the web are commonly facilitated through social networks such as Facebook and Twitter, while sophisticated brain functions are the result of vast interactions within complex neuronal networks; see e.g., [27] and references therein. Other networks naturally emerge in settings as diverse as financial markets, genomics and proteomics, power grids, and transportation systems, to name just a few.

While some of these networks are directly observable, due to e.g., presence of physical or engineered links between nodes, most complex networks have hidden topologies, which must first be inferred in order to conduct meaningful network analytics [15, Ch. 7]; see also [11], [21], [25]. Prominent among these are SEMs, a family of statistical approaches for causal (a.k.a., path) analysis in complex systems, with several applications specifically tailored to graph topology inference; see e.g., [3], [6], [9]. In a nutshell, SEMs capture the relationship between observed nodal processes or measurements, and the unknown causal network. The key contribution of SEMs is two-fold: a) they are conceptually simple, often resorting to tractable linear models; and b) SEMs explicitly account for the role played by exogenous or confounding inputs in observed nodal processes, which turn out to be critical in resolving directional ambiguities [4].

In settings where measurement of exogenous inputs is costly or impractical, contemporary SEMs are quite limited with regard to unique identification of hidden network topologies. For example, in financial networks comprising stocks as nodes and their interdependencies as links, publicly-traded stock prices (endogenous) are known to depend on stock purchases (exogenous) by investors, whose details are often unknown to the public for privacy reasons. On the other hand, each publicly-traded company may broadcast monthly statistical summaries of purchases of its stock. Assuming that such statistical information is known or obtainable, the present paper advocates novel approaches that capitalize on factorization of carefully constructed tensors, or multi-modal arrays. As demonstrated later, inference of the network topology is shown possible under reasonable conditions, using only correlation information of the exogenous inputs. The crux of our novel framework lies in positing that exogenous inputs exhibit piecewise-stationary correlations, from which three-way tensors are constructed using a special instance of SEMs.

By leveraging the well-known parallel factor (PARAFAC) tensor decomposition [16], it is shown that edge connectivity information is captured through one of the factors, while identifiability of the network topology is guaranteed due to uniqueness of the factorization. Interestingly, casting the problem as tensor decomposition also opens up opportunities to blindly estimate both the unknown topology and local correlation matrices of the exogenous inputs; see also [29], [30]. PARAFAC decomposition is a powerful tool for multilinear algebra introduced by [13], and its merits have been permeated within diverse application domains [31], e.g., wireless communications [32], blind source separation [18], [22], as well as community detection on graphs [1], [24]. The present paper broadens these well-documented merits to tasks involving network topology inference. Numerical tests on simulated and real data corroborate the efficacy of the novel approach.

Since most real-world networks are time-varying, the advo-
cated tensor-based approach is accordingly extended to track
topology changes. Moreover, nodal data are often acquired
in real-time streams, rendering batch inference algorithms
impractical. Toward satisfying the dual need to mitigate batch
computational overhead, and track dynamic topologies, an
online variant of the novel algorithm is developed. Motivated
by the adaptive PARAFAC decomposition [20], [23], a novel
real-time estimator is put forth to track the topology-revealing
tensor factors, using second-order statistics of the exogenous
inputs.

To place this work in context, several prior studies have
focused on tracking time-varying networks from nodal pro-
cesses. For example, dynamic information diffusion networks
were tracked via maximum likelihood estimators in [12], while
a sparse piecewise stationary graphical model was put forth
to track undirected networks in [2]. Dynamic SEMs were
also advocated for inference of dynamic and directed cascade
networks in [3]. More recent work in [34] resorted to hidden
Markov models (HMMs) to track diffusion links.

PARAFAC decompositions have previously been advocated
in e.g., blind source separation (BSS) tasks, which separate
source signals from their mixed observations; see e.g., [18],
[22]. It is worth mentioning at the outset that tensor-based
SEM approaches posit that each node may be the recipient of
multiple inputs. This amounts to relaxing the restriction on
node degree, allowing it to take values 

\[ a_{ij} \]

while and 

\[ b_{ij} \]

respectively. The identity matrix will be denoted by 

\[ I \]

will stand for matrix transposition, and maximum eigenvalue,
(\matrices{column vectors}, while operators
in Section V. Finally, results of corroborating numerical tests
will leverage the flexibility of tensors to capture more complex
mining. For instance, several community detection approaches
also recently been adopted in network analytics and graph
identification. Identifiability conditions developed in this paper
must exploit special properties inherent to SEMs; and ii) the
by factors obtained from the tensor decomposition, and one
BSS, namely: i) network topologies are not directly revealed
processes. For example, dynamic information diffusion networks
were tracked via maximum likelihood estimators in [12], while
topology changes. Moreover, nodal data are often acquired
in e.g., [14] for details. In this case, the noisy version
\[ y_{jt} = y_{it} + b_{ij}x_{it} + c_{jt} \]

where \( A_{ij} = a_{ij} \), and \( c_{jt} \) denotes an “error” term that
captures unmodeled dynamics. The coefficients \( \{a_{ij}\} \)
are unknown, and \( a_{ij} \neq 0 \) signifies that a directed
edge from \( j \) to \( i \) is present. Collecting nodal measurements
\( y_{1t} = [y_{11}, \ldots, y_{1N}]^T \), and \( x_{1t} = [x_{11}, \ldots, x_{1N}]^T \)
per slot \( t \), and temporarily assuming that \( e_{jt} = 0 \), the noise-free version
of (1) can be compactly written as

\[ y_{t} = A_{it} + B_{it} \]

where \( A_{ii} = 0 \) and \( B := \text{Diag}(b_{11}, \ldots, b_{NN}) \) denotes a diagonal
coefficient matrix.

Note that with \( B \) diagonal, (1) implicitly assumes that each
node is associated with a single exogenous input. In fact, it is
possible to generalize (1) to settings where a single exogenous
input may be applied to several nodes, or where a single
node may be the recipient of multiple inputs. This amounts
to relaxing the restriction on \( B \), allowing it to take values
from the set of non-diagonal square matrices. In addition,
in more general SEMs \( x_{1t} \) and \( y_{1t} \) are indirectly observed
latent variables, each adhering to measurement models, namely,
\( y_{jt} = C_{ij}x_{1t} + \delta_{jt} \) and \( u_{jt} = C_{ij}x_{1t} + \delta_{jt} \), with corresponding
noise terms \( \delta_{jt} \) and \( \delta_{jt} \); see e.g., [14] for details. In this case,
the noisy version \( (y_{jt} = A_{ij}x_{1t} + B_{jt} + e_{jt}) \) of (2) is often
referred to as the structural model. This paper deals with
problems where \( x_{1t} \) and \( y_{1t} \) are directly observable, and there
is no extra measurement model. The problem statement can
now be formally stated as follows.

**II. PRELIMINARIES AND PROBLEM STATEMENT**

Consider a network \( G(V, E) \) that comprises \( N \) nodes,
with its topology captured by an unknown adjacency matrix \( A \in \mathbb{R}^{N \times N} \). Let \( a_{ij} \) denote entry \((i, j)\) of \( A \), which is nonzero
only if there is an edge between nodes \( i \) and \( j \); see Figure [1]
It will generally be assumed that \( G \) is a directed graph, that is
(\( A \neq A^T \)).

Suppose the network abstracts a complex system with
measurable inputs and an observable output process that
propagates over the network following directed links. Let \( x_{it} \)
denote the input to node \( i \) at slot \( t \), and \( y_{it} \) the \( t \)-th observation
of the propagating process measured at node \( i \). In the context
of brain networks, \( y_{it} \) could represent the \( t \)-th time sample
of an electroencephalogram (EEG), or functional magnetic
resonance imaging (fMRI) measurement at region \( i \), while \( x_{it} \)
could be a controlled stimulus that affects a specific region of
the brain. In social networks (e.g., Twitter or Facebook) over
which information diffuses, \( y_{it} \) could represent the timestamp
when subscriber \( i \) tweeted or shared a viral story, while \( x_{it} \)
could measure their level of interest in such stories.

In general, SEMs postulate that \( y_{it} \) depends on two classes
of variables, namely: i) measurements of the diffusing process \( \{y_{jt}\}_{j \neq i} \) (a.k.a. endogenous variables); and ii) external inputs
\( x_{it} \) (a.k.a. exogenous variables). Most contemporary SEM
approaches posit that \( y_{it} \) depends linearly on both \( \{y_{jt}\}_{j \neq i} \)
and \( x_{it} \) that is,

\[ y_{it} = \sum_{j 
eq i} a_{ij} y_{jt} + b_{ij} x_{it} + e_{it} \] (1)

where \( A_{ij} := a_{ij} \), and \( e_{it} \) denotes an “error” term that
captures unmodeled dynamics. The coefficients \( \{a_{ij}\} \)
and \( \{b_{ij}\} \) are unknown, and \( a_{ij} \neq 0 \) signifies that a directed
edge from \( j \) to \( i \) is present. Collecting nodal measurements
\( y_{1t} := [y_{11}, \ldots, y_{1N}]^T \), and \( x_{1t} := [x_{11}, \ldots, x_{1N}]^T \)
per slot \( t \), and temporarily assuming that \( e_{jt} = 0 \), the noise-free version
of (1) can be compactly written as

\[ y_{t} = A_{it} + B_{it} \]

(2)
Problem statement: Given \( \{ y_t, x_t \}_{t=1}^T \), the goal is to recover the underlying directed network topology \( A \).

III. A TENSOR FACTORIZATION APPROACH

Building upon (1), this section puts forth a novel tensor factorization approach to unveil the hidden network topology. To this end, the following assumptions are adopted.

\((as0)\) Exogenous data \( \{ x_t^{(m)} \} \) are piecewise-stationary over time segments \( t \in [\tau_m, \tau_{m+1} - 1] \), \( m = 1, \ldots, M + 1 \), each with a fixed correlation matrix \( R_x^m := \mathbb{E}[x_t^{(m)}(x_t^{(m)})^\top] \).

\((as1)\) Entries of \( x_t \) are zero mean and uncorrelated per \( t \); that is, \( \mathbb{E}\{x_{it}x_{jt}\} = 0, \forall i \neq j \).

\((as2)\) Matrix \( (I - A) \) is invertible; and

\((as3)\) Matrix \( B \) is diagonal with nonzero diagonal entries.

Under \((as0)\) and \((as2)\), it is possible to rewrite (2) as

\[
y_t = (I - A)^{-1}Bx_t = Ax_t
\]

where \( A := (I - A)^{-1}B \), and superscript \((m)\) has been dropped with the understanding that \( t \) stays within one segment, and thus (3) holds \( \forall m \). The per segment correlation matrix \( R_y^m := \mathbb{E}[y_t y_t^\top] \) is thus given by (cf. (3))

\[
R_y^m = AR_x^m A^\top, \quad t \in [\tau_m, \tau_{m+1} - 1]. \tag{4}
\]

Under (as1), one can express (4) as the weighted sum of rank-one matrices as

\[
R_y^m = A \text{Diag}(\rho_{m}), \quad A^\top = \sum_{i=1}^N \rho_{m_i} a_i a_i^\top \tag{5}
\]

where \( a_i \) denotes the \( i \)-th column of \( A \), and \( \rho_{m_i} := [\rho_{m_1}, \ldots, \rho_{m_N}]^\top \), with \( \rho_{m_i} := \mathbb{E}[x_{it}] \), for \( t \in [\tau_m, \tau_{m+1} - 1] \).

Consider the three-way tensor \( R_y^m \in \mathbb{R}^{N \times N \times M} \), constructed by setting the \( m \)-th slice \( R_{y,\ldots,m}^m = R_y^m \). Letting \( \alpha_{ij} \) denote the \((j, k, l)\) entry of the tensor outer product \( a_i \circ b_j \circ c_k \), it turns out that \( R_y^m \) can be written as (see also Figure 2)

\[
R_y^m = \sum_{i=1}^N \alpha_i \circ a_i \circ r_i^m \tag{6}
\]

with entry \((j, k, l)\) given by

\[
[R_y^m]_{jkl} = \sum_{i=1}^N \alpha_{ij} \alpha_{ik} r_i^m \tag{7}
\]

where \( r_i^m := [\rho_{m_1}, \ldots, \rho_{m_N}]^\top \). Interestingly, (6) amounts to the so-called partial symmetric PARAFAC decomposition of \( R_y^m \) into factor matrices \( A, \hat{A}, \) and \( R_x^m := [r_1^m \ldots r_N^m] \in \mathbb{R}^{M \times N} \); see e.g., [10]. Although \( R_y^m \) is generally unknown, it can be readily estimated using sample averaging as

\[
\hat{R}_y^m = \frac{1}{\tau_{m+1} - \tau_m} \sum_{t=\tau_m}^{\tau_{m+1}-1} y_t y_t^\top, \quad m = 1, \ldots, M \tag{8}
\]

from endogenous measurements.

The present paper relies on this three-way tensor constructed from second-order statistics of the nodal measurements, and leverages the uniqueness properties inherent to PARAFAC decompositions to identify the hidden network topology; see e.g., [17] for key uniqueness results. Indeed, a number of standard PARAFAC decomposition algorithms can be adopted to estimate \( A \); e.g., via alternating least-squares (ALS) iterations. Under reasonable conditions, it will be possible to recover \( A \), once \( \hat{A} \) has been found. The next proposition formally states the sufficient conditions required to uniquely identify \( A \), after determining \( \hat{A} \) from the PARAFAC decomposition.

Proposition 1: If \( a_{ij} = 0, b_{ij} \neq 0 \forall j \), \( b_{ij} = 0 \forall i \neq j \), and \( A \) is invertible, then \( \hat{A} \) can be uniquely expressed in terms of \( A \) as \( \hat{A} = I - (\text{Diag}(A^{-1}))^{-1}A^{-1} \).

Proof: See Appendix A.

Regarding the decomposition in (6), one can make the following important observations: (i) \( \text{rank}(R_y^m) = N \); (ii) two factors of \( R_y^m \) are identical; and (iii) exogenous inputs \( \{ x_t \}_{t=1}^T \) are generally accessible, and can be readily tuned to satisfy piecewise stationarity along with the additional conditions necessary to guarantee identifiability of \( A \).

To quantify accessibility in (iii), one can consider \( R_y^m \) known a priori, where \( \Omega \) denotes the index set of the available entries of \( R_x \), i.e., \( R_{y,\Omega}^m = r_{ij} \) for \( (i, j) \in \Omega \). Given noisy tensor data, these considerations (i)–(iii) prompt the next criterion for obtaining the wanted factors

\[
(\hat{Z}_1, \hat{Z}_2, \hat{Z}_3) = \arg\min_{Z_1, Z_2, Z_3} \left\| R_y^m - \sum_{n=1}^N z_{1n} \circ z_{2n} \circ z_{3n} \right\|_F^2
\]

s.t. \( Z_1 = Z_2, \quad [Z_{3}]_{i,j} = [R_{\Omega}]_{i,j}, \forall (i, j) \in \Omega \) \hspace{1cm} (P1)

where \( z_{im} \) denotes the \( n \)-th column of matrix \( Z_i \). Note that (P1) can be solved via partially symmetric PARAFAC decomposition, even when noise is present, using e.g., the individual differences in multidimensional scaling [7]. Upon obtaining the estimated factors \( \hat{Z}_1, \hat{Z}_2 \) and \( \hat{Z}_3 \), matrix \( \hat{A} \) can be found
Remark 1: The PARAFAC decomposition generally assumes no prior knowledge about \( \mathbf{R}^x \); that is, \( \Omega = \emptyset \) in (P1). In principle, one can estimate the topology even without correlation information of the exogenous inputs. Interestingly, this amounts to blindly estimating the topology and exogenous correlation matrices, which is of considerable merit when measurement of external inputs is impossible, or rather costly.

IV. IDENTIFIABILITY ISSUES

Although casting network topology identification task as a tensor decomposition problem leads to enhanced flexibility, one has to contend with identifiability issues common to both matrix and tensor factorizations. In order to establish identifiability conditions for \( \mathbf{A} \) and \( \mathbf{B} \), this section will first explore conditions under which \( \mathbf{A} \) is uniquely identifiable. To this end, a couple of definitions are in order.

Definition 1. The Kruskal rank of a matrix \( \mathbf{Z} \in \mathbb{R}^{N \times M} \) (denoted hereafter as \( \text{kr}(\mathbf{Z}) \)) is defined as the maximum number \( k \) such that any combination of \( k \) columns of \( \mathbf{Z} \) constitutes a full rank submatrix.

Definition 2. Essential uniqueness of a tensor factorization refers to uniqueness up to scaling and permutation ambiguity.

Algorithm 1 Topology inference via tensor decomposition

**Input:** \( \mathbf{R}^y_{t_m}, \{ \mathbf{y}_t \}, M, \eta \)

**S1.** Tensor construction:

Set \( m \)-th frontal slice of \( \mathbf{R}^y \in \mathbb{R}^{N \times N \times M} \) to

\[
\mathbf{R}^y_{m} = \frac{1}{\tau_{m+1} - \tau_{m}} \sum_{\tau_{m}+1}^{\tau_{m+1}} \mathbf{y}_t \mathbf{y}_t^\top, \quad m = 1, \ldots, M
\]

**S2.** PARAFAC decomposition:

Solve \((P1)\) to find \((\mathbf{Z}_1, \hat{\mathbf{Z}}_2, \hat{\mathbf{Z}}_3)\)

**S3.** SEM estimates for topology inference:

\[
\hat{\mathbf{A}} = \mathbf{Z}_1
\]

**S4.** Edge identification:

\[
[\hat{\mathbf{A}}]_{ij} \neq 0 \text{ if } [\hat{\mathbf{A}}]_{ij} > \eta, \text{ otherwise } [\hat{\mathbf{A}}]_{ij} = 0, \forall (i,j)
\]

as (cf. Proposition 1)

\[
\hat{\mathbf{A}} = \mathbf{Z}_1
\]

\[
\hat{\mathbf{A}} = \mathbf{I} - \left( \text{Diag}(\hat{\mathbf{A}}^{-1}) \right)^{-1} \hat{\mathbf{A}}^{-1}
\]

With Definitions 1 and 2 in mind, consider PARAFAC decomposition for a three way tensor \( \mathbf{P} = (\mathbf{U}, \mathbf{V}, \mathbf{W}) \). Theorem 1 establishes sufficient conditions for essential uniqueness of the tensor decomposition; see \cite{33} and \cite{17} for further details and a proof of the theorem.

Theorem 1: Let \((\mathbf{U}, \mathbf{V}, \mathbf{W})\) denote the PARAFAC factors obtained by decomposing a three-way tensor \( \mathbf{P} \) into \( K \) rank-one tensors. If Kruskal’s condition holds, namely,

\[
\text{kr}(\mathbf{U}) + \text{kr}(\mathbf{V}) + \text{kr}(\mathbf{W}) \geq 2K + 2
\]

and there exists an alternative set of matrices \((\hat{\mathbf{U}}, \hat{\mathbf{V}}, \hat{\mathbf{W}})\) constituting a PARAFAC decomposition of \( \mathbf{P} \), then there exists a permutation matrix \( \mathbf{\Pi} \), and diagonal scaling matrices \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \), such that \( \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 = \mathbf{I}, \hat{\mathbf{U}} = \mathbf{\Pi} \mathbf{A}_1, \hat{\mathbf{V}} = \mathbf{\Pi} \mathbf{A}_2, \hat{\mathbf{W}} = \mathbf{\Pi} \mathbf{A}_3 \).

Proof: See \cite{33} for a general proof with complex entries.

As a prerequisite to identification of \( \mathbf{A} \), the following proposition establishes essential uniqueness of \( \mathbf{A} \), based on the tensor-based interpretation advocated in the prequel.

Proposition 2: If \( \text{kr}(\mathbf{R}^x) \geq 2 \), then \( \mathbf{A} := (\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \) is uniquely identifiable up to a scaling and permutation ambiguity via PARAFAC decomposition of \( \mathbf{R}^y \).

Proof: Upon recognizing that \( \text{rank}(\mathbf{R}^y) = N \) from \cite{6}, in order for \((11)\) to hold, we need

\[
2\text{kr}(\mathbf{A}) + \text{kr}(\mathbf{R}^x) \geq 2N + 2
\]

Under (as2) and (as3), matrices \((\mathbf{I} - \mathbf{A})\) and \( \mathbf{B} \) are invertible, which implies that \( \mathbf{A} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \) is invertible, and hence \( \text{kr}(\mathbf{A}) = N \). From \((12)\), essential uniqueness can thus be guaranteed as long as \( \text{kr}(\mathbf{R}^x) \geq 2 \), which completes the proof.

Note that essential uniqueness is not sufficient for identification of the hidden network topology, due to the inherent permutation and scaling ambiguities. To this end, we will subsequently pursue identifiability conditions for settings where \( \mathbf{R}^x \) may be fully, or partially available, or even completely unavailable on a case-by-case basis.

A. Identifiability with fully known \( \mathbf{R}^x \)

First, we will explore identifiability of the topology when \( \mathbf{R}^x \) is completely known, while highlighting the importance of information about exogenous inputs \( \{\mathbf{x}_t\} \).

Theorem 2: If \( \mathbf{x}_t \) and \( \mathbf{y}_t \) obey the SEM in \((2)\), for all \( t = 1, \ldots, N \), with \( \mathbf{A} \) and \( \mathbf{B} \) satisfying (as2) and (as3), respectively, and if \( \mathbf{R}^x \) is known and satisfies \( \text{kr}(\mathbf{R}^x) \geq 2 \), then \( \mathbf{A} \) can be uniquely identified via Algorithm 1.

Proof: Suppose there is an alternative triplet \( (\mathbf{A}', \mathbf{A}', \mathbf{R}'^x) \), also decomposing \( \mathbf{R}^y \) into \( N \) rank-one tensors in \((P1)\). Theorem 1 asserts that there is a permutation matrix \( \mathbf{\Pi} \), and diagonal scaling matrices \( \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\} \) so that

\[
\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 = \mathbf{I}
\]

and

\[
\mathbf{A}' = \mathbf{A} \mathbf{\Pi} \mathbf{A}_1
\]

\[
\mathbf{A}' = \mathbf{A} \mathbf{\Pi} \mathbf{A}_2
\]

\[
\mathbf{R}'^x = \mathbf{R}^x \mathbf{\Pi} \mathbf{A}_3
\]
B. Identifiability with partially known $\mathbf{R}^x$

The last subsection assumed that second-order statistics of $x_t$ were available for all time slots $m = 1, \ldots, M$. However, ample empirical evidence suggests that such information may not be fully available at times. For instance, not all statistics of the stock prices may be available to a given investor in financial markets over time. In brain connectivity studies, one may only have explicit knowledge about exogenous variables in some experimental settings, but not others. Such limitations motivate the analysis of identifiability in settings where one only has access to partial information about second-order statistics of exogenous inputs; that is, $\mathbf{R}^x$ contains misses.

In order to capture the partial availability of $\mathbf{R}^x$, suppose $\Omega_i$ denotes set of indices corresponding to known entries per column $i$ of $\mathbf{R}^x$. Furthermore, let $r^x_i$ denote a sub-vector of $\mathbf{r}^x$, whose entries are indexed by $\Omega_i \cup \Omega_j$ (recall that $r^x_i$ denotes the $i$-th column of $\mathbf{R}^x$). Based on these definitions, the next theorem establishes identifiability conditions for settings where $\mathbf{R}^x$ is only partially available.

**Theorem 3**: If $\mathbf{r}^x_i$ and $\mathbf{r}^x_j$ are linearly independent for any $i \neq j$, then the network adjacency matrix $\mathbf{A}$ can be uniquely identified via Algorithm 1.

**Proof**: Suppose there exists an alternative PARAFAC solution $(\mathbf{A}, \mathbf{\hat{A}}, \mathbf{\hat{R}}^x)$ that also decomposes $\mathbf{R}^y$ into $N$ rank-one tensors (cf. S2 in Algorithm 1). According to Theorem 1, there exists a permutation matrix $\mathbf{\hat{\Pi}}$ and diagonal scaling matrices $\{\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2, \hat{\mathbf{A}}_3\}$ such that

$$\hat{\mathbf{A}}_1 \hat{\mathbf{A}}_2 \hat{\mathbf{A}}_3 = \mathbf{I} \quad \text{(18)}$$

Next, substituting (16b) into (14a), and letting $\mathbf{A} = \mathbf{A}_1 = \mathbf{A}_2$, one obtains

$$\mathbf{A}' = \mathbf{A} \mathbf{A}.$$ (17)

**Lemma 2**: If the PARAFAC solution obtained in S3 of Algorithm 1 satisfies $\mathbf{A} = \mathbf{A} \mathbf{A}$, then $\mathbf{A}$ can be uniquely identified; that is, $\hat{\mathbf{A}} = \mathbf{A}$.

**Proof**: See Appendix C.

Combining Lemma 2 with (17) completes the proof of Theorem 2.

### B. Identifiability with partially known $\mathbf{R}^y$

The last subsection assumed that second-order statistics of $x_t$ were available for all time slots $m = 1, \ldots, M$. However, ample empirical evidence suggests that such information may not be fully available at times. For instance, not all statistics of the stock prices may be available to a given investor in financial markets over time. In brain connectivity studies, one may only have explicit knowledge about exogenous variables in some experimental settings, but not others. Such limitations motivate the analysis of identifiability in settings where one only has access to partial information about second-order statistics of exogenous inputs; that is, $\mathbf{R}^x$ contains misses.

In order to capture the partial availability of $\mathbf{R}^x$, suppose $\Omega_i$ denotes set of indices corresponding to known entries per column $i$ of $\mathbf{R}^x$. Furthermore, let $r^y_i$ denote a sub-vector of $\mathbf{r}^y$, whose entries are indexed by $\Omega_i \cup \Omega_j$ (recall that $r^y_i$ denotes the $i$-th column of $\mathbf{R}^y$). Based on these definitions, the next theorem establishes identifiability conditions for settings where $\mathbf{R}^y$ is only partially available.

**Theorem 2**: If $\mathbf{r}^y_i$ and $\mathbf{r}^y_j$ are linearly independent for any $i \neq j$, then the network adjacency matrix $\mathbf{A}$ can be uniquely identified via Algorithm 1.

**Proof**: Suppose there exists an alternative PARAFAC solution $(\mathbf{A}, \mathbf{\hat{A}}, \mathbf{\hat{R}}^x)$ that also decomposes $\mathbf{R}^y$ into $N$ rank-one tensors (cf. S2 in Algorithm 1). According to Theorem 1, there exists a permutation matrix $\mathbf{\hat{\Pi}}$ and diagonal scaling matrices $\{\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2, \hat{\mathbf{A}}_3\}$ such that

$$\hat{\mathbf{A}}_1 \hat{\mathbf{A}}_2 \hat{\mathbf{A}}_3 = \mathbf{I} \quad \text{(18)}$$

Next, substituting (16b) into (14a), and letting $\mathbf{A} = \mathbf{A}_1 = \mathbf{A}_2$, one obtains

$$\mathbf{A}' = \mathbf{A} \mathbf{A}.$$ (17)

**Lemma 2**: If the PARAFAC solution obtained in S3 of Algorithm 1 satisfies $\mathbf{A} = \mathbf{A} \mathbf{A}$, then $\mathbf{A}$ can be uniquely identified; that is, $\hat{\mathbf{A}} = \mathbf{A}$.

**Proof**: See Appendix C.

Combining Lemma 2 with (17) completes the proof of Theorem 2.
vocated so far focuses on settings where the network topology does not vary with time. The rest of the paper goes beyond this assumption, and explores scenarios where the link structure may even evolve over time, with the ultimate goal of tracking the network topology, possibly in real time.

V. TRACKING DYNAMIC NETWORK TOPOLOGIES

It has hitherto been taken for granted that all past data are available, and the developed tensor-based approaches will operate in batch mode. In fact, Algorithm 1 is conducted entirely offline, with \( \mathbf{R}_m^y \) obtained or computed a priori. However, practical constraints often render it impossible to operate in batch mode; for instance, nodal data in large-scale networks (e.g., modern social media and the web) can only be acquired in real-time streams since any attempts to store such data for batch processing will quickly overwhelm operators.

Equally important is the observation that most real-world networks evolve over time, namely, new edges and nodes may appear, while others become obsolete during the observation period. Consequently, even if a batch approach were to overcome challenges due to the sheer scale of the data, networks (e.g., modern social media and the web) can only be revealed that the updates admit recursive forms as follows

\[
y_t = \mathbf{A}_m y_t + \mathbf{B}_m x_t + e_t
\]

per \( m = 1, \ldots, M \), with \( e_t \) similarly capturing unmodeled dynamics, while coefficients \( \{a_{ij}^m\} \) and \( \{(b_{ij}^m)\} \) are unknown. With \( y_t, x_t \), and \( e_t \) previously defined, \( \text{vec} \) can be written in vector form as

\[
\mathbf{y}_t = \mathbf{A}_m \mathbf{y}_t + \mathbf{B}_m \mathbf{x}_t + \mathbf{e}_t
\]

where \( \mathbf{A}_m \) and \( \mathbf{B}_m = \text{Diag}(b_{11}^m, \ldots, b_{NN}^m) \). Based on \( \mathbf{m} \), we will develop an algorithm to track \( \{\mathbf{A}_m, \mathbf{B}_m\}_{m=1}^M \) using measured endogenous variables, and the sequence of correlation matrices \( \{\mathbf{R}_m^y\}_{m=1}^M \).

Key to the novel topology tracking algorithm is recognizing that the tensor-based approach of Section III can be extended to settings where the network exhibits piecewise-invariant topology variations. To this end, define \( \mathbf{A}_m := (\mathbf{I} - \mathbf{A}_m)^{-1} \mathbf{B}_m \), and consider a tensor with the \( m \)-th slice

\[
\mathbf{R}_m^y = \mathbf{A}_m \mathbf{R}_m^x \mathbf{A}_m^T, \quad t \in [\tau_m, \tau_{m+1} - 1]
\]

sequentially appended at \( t = \tau_{m+1} \), for \( m = 1, \ldots, M \); see also (5) and Figure 3. Allowing \( \mathbf{R}_m^y \) to grow sequentially along one mode is well motivated for real-time operation, where data may be acquired in a streaming manner. In this case, unveiling the evolving network topology calls for approaches that are capable of tracking tensor factors. In fact, the topology tracking algorithm developed next builds upon a prior sequential tensor factorization approach, namely, PARAFAC via recursive least-squares tracking (PARAFAC-RLST); see e.g., [23] for details.

B. Exponentially-weighted least-squares estimator

Let \( \mathbf{R}_m^- := \text{vec}(\mathbf{R}_m^y) \) denote the vectorization of \( \mathbf{R}_m^y \), and note that \( \mathbf{R}_m^- \) can be written as \( \mathbf{R}_m^- = \mathbf{H}_m \rho_m^r \), where \( \mathbf{H}_m := \mathbf{A}_m \mathbf{A}_m^T \) is an \( N^2 \times N \) matrix, and \( \rho_m^r \) is defined after (5).

To track \( \mathbf{H}_m \), we advocate an exponentially-weighted least-squares estimator, namely,

\[
\mathbf{H}_m = \arg \min_{\mathbf{H}} \sum_{l=1}^{m} \beta^{m-l} \| \mathbf{R}_l^- - \mathbf{H} \rho_l^r \|_2^2
\]

for \( m = 1, \ldots, M \), where \( \beta \in (0, 1) \) denotes a forgetting factor, which facilitates tracking topology changes by down-weighting past data when \( \beta < 1 \).

Letting \( f_m(\mathbf{H}) := \sum_{l=1}^{m} \beta^{m-l} \| \mathbf{R}_l^- - \mathbf{H} \rho_l^r \|_2^2 \) denote the cost function per segment \( m \), and taking the gradient with respect to \( \mathbf{H} \), one obtains

\[
\nabla f_m(\mathbf{H}) = 2 \sum_{l=1}^{m} \beta^{m-l} (\mathbf{R}_l^- - \mathbf{H} \rho_l^r) (\rho_l^r)^\top.
\]

Setting \( \nabla f_m(\mathbf{H}) = 0 \), and solving for \( \mathbf{H}_m \) yields

\[
\mathbf{H}_m = \mathbf{Q}_m \mathbf{P}_m^{-1}
\]

where \( \mathbf{Q}_m := \sum_{l=1}^{m} \beta^{m-l} (\mathbf{R}_l^- (\rho_l^r)^\top) \) and \( \mathbf{P}_m := \sum_{l=1}^{m} \beta^{m-l} (\rho_l^r)^\top (\rho_l^r)^\top \). Further inspection of \( \mathbf{P}_m \) and \( \mathbf{Q}_m \) reveals that the updates admit recursive forms as follows

\[
\mathbf{P}_m := \beta \mathbf{P}_{m-1} + \rho_m^r (\rho_m^r)^\top
\]

\[
\mathbf{Q}_m := \beta \mathbf{Q}_{m-1} + \rho_m^r (\rho_m^r)^\top.
\]

Moreover, letting \( \mathbf{W}_m := \mathbf{P}_m^{-1}, \) one can resort to the matrix inversion lemma to recursively compute inverses as

\[
\mathbf{W}_m = \beta^{-1} \left[ \mathbf{W}_{m-1} - \frac{\mathbf{W}_{m-1} \rho_m^r (\rho_m^r)^\top \mathbf{W}_{m-1}}{\beta + (\rho_m^r)^\top \mathbf{W}_{m-1} \rho_m^r} \right].
\]

It is worth pointing out that the simple recursive updates lead to a markedly reduced computational burden, while only requiring fixed memory storage costs.
Algorithm 2: Tensor-based network topology tracking

Input: \( \{p^x_m\}_{m=1}^M \), \( \{y_t\} \), \( \beta, W_0, Q_0 = 0 \), \( \eta \)
for \( m = 1, \ldots, M \) do

1. Tensor formation
   Set frontal slice \( m \) of \( R^y \) to \( R^y_m \), as in (6).

2. Variable updates:
   \( Q_m := \beta Q_{m-1} + \hat{F} \)(\( \rho^x_m \))\(^T \)
   Update \( W_m \) via (31)
   Update \( \alpha_{im} \) via (33), for \( i = 1, \ldots, N \)

3. SEM estimates for topology tracking:
   Estimate \( A_m \) via (34).

Return \( \hat{A}_m \).

for Edge identification:

\[ \hat{A}_{m} \] \[ ij \neq 0 \text{ if } [\hat{A}_m]_{ij} > \eta, \text{otherwise } [\hat{A}_m]_{ij} = 0, \forall (i, j) \]

Once \( H_m \) is estimated, \( A_m := [\alpha_{1m}, \ldots, \alpha_{Nm}] \) can be recovered by recalling that the \( t \)th column of \( H_m \) is given by

\[ h_{im} = \alpha_{im} \otimes \alpha_{im} = \text{vec}(\alpha_{im} \alpha_{im}^\top). \]

Recognizing that \( \hat{H}_{im} := \alpha_{im} \alpha_{im}^\top \) is a rank one matrix, \( \alpha_{im} \) can be estimated via the leading eigenvector of \( \hat{H}_{im} \), namely

\[ \hat{\alpha}_{im} \approx \hat{\lambda}_{\text{max}}(\hat{H}_{im}) \hat{v}_{\text{max}}(\hat{H}_{im}). \]

where the eigen-pair \( \{\hat{\lambda}_{\text{max}}(\hat{H}_{im}), \hat{v}_{\text{max}}(\hat{H}_{im})\} \) denotes the leading eigenvalue of \( \hat{H}_{im} \), and its corresponding eigenvector, both obtainable via the power iteration [10]. This is carried out per column of \( \hat{A}_m \) to obtain \( \hat{A}_m := [\hat{\alpha}_{1m}, \ldots, \hat{\alpha}_{Nm}] \), while \( A_m \) can be estimated as (cf. Proposition 1)

\[ \hat{A}_m = I - \left( \text{Diag}(\hat{A}_m) \right)^{-1} \hat{A}_m. \]

Algorithm 2 lists the steps involved in tracking evolving network topologies via the scheme advocated in this section.

Remark 3 (Initialization): Matrix \( P_m \) in (29) is rank deficient when \( m \leq N \), rendering the update in (28) impossible. This can be addressed by setting \( W_0 = P_0^{-1} = a I \), for a very large constant \( a \) (e.g., \( a = 10^5 \)). Since \( P_m^{-1} \) is a variance estimate of \( H_m \), this initialization amounts to placing little confidence in the initial values. Matrix \( Q_0 \) is initialized as an all-zero matrix.

VI. NUMERICAL TESTS

In order to assess the effectiveness of the novel algorithms, this section presents test results from experiments conducted on both simulated and real network data. Consideration was given to scenarios involving both static and dynamic networks.

A. Tests on static simulated networks

Data generation. A Kronecker random graph comprising \( N = 64 \) nodes was generated from a prescribed “seed matrix”

\[ S_0 := \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \]

in order to obtain a binary-valued \( 64 \times 64 \) matrix via repeated Kronecker products, namely \( S = S_0 \otimes S_0 \otimes S_0 \); see also [19]. Using the binary matrix \( S \) to describe the zero and nonzero entries of the topology, the Kronecker graph with adjacency matrix \( A \) was then constructed by randomly sampling each entry from a uniform distribution with \( a_{ij} \sim \text{Unif}(0.2s_{ij}, 0.5s_{ij}) \). To generate synthetic endogenous measurements, the observation horizon was set to \( T = ML \) time-slots, which were partitioned into \( M \) windows of fixed length \( L \), using pre-selected boundaries \( \{\tau_m\}_{m=1}^{M+1} \) with \( \tau_1 = 1 \) and \( L := \tau_{m+1} - \tau_m \), for several values of \( L \) and \( M \). Per \( t \in [\tau_m, \tau_{m+1} - 1] \), exogenous inputs were sampled as \( x_t \sim N(0, 10^{-2}I) \), \( y_t \) was generated using the SEM, that is, \( y_t = (I - \Lambda)^{-1}(Bx_t + e_t) \), where \( B \) is a diagonal matrix with \( B_{jj} \) drawn uniformly from the interval [2,3].

In order to conduct PARAFAC decompositions, an implementation in the open source Tensorlab 3.0 toolbox was adopted [35]. Upon running Algorithm 1 an edge was declared present if the estimate \( \hat{a}_{ij} \) was found to exceed a prescribed threshold. The threshold was selected to yield the lowest edge identification error rate (EIER), which is defined as

\[ \text{EIER} := \frac{||S - \hat{S}||_0}{N(N - 1)} \times 100\% \] (35)

with the operator \( || \cdot ||_0 \) denoting the number of nonzero entries of its argument. Matrix \( S \in \{0,1\}^{N \times N} \) denotes the ground-truth binary edge indicator matrix, while \( \hat{S} \) denotes its estimate obtained by the novel scheme.

Experiments were run for different values of \( M \), and error plots were generated using EIER values averaged over 500 independent runs.

Results. Figure 4 depicts actual and inferred adjacency matrices, resulting from one realization of Algorithm 1 for \( M \in \{10, 20\} \), with \( L = 1,000 \) per experiment. As shown in the plot, fewer edges are erroneously identified as the number of windows \( M \) increases. This is not really surprising because the probability that the condition in Theorem 2 is satisfied will improve with larger \( M \). Figures 5 plots EIER values against \( L \), averaged over 500 independent runs of Algorithm 1 for \( M = 10 \) and \( M = 20 \).

Figure 5(a) plots the observed error performance over several window lengths (\( L \)), when \( R^x \) is fully available, whereas Figure 5(b) was obtained after random omission of entries in \( R^x \) with probability 0.5. On the other hand, Figure 5(c) depicts performance in the completely blind case, that is, \( \Omega = \emptyset \). In all three scenarios, there is a general increase in edge identification accuracy with \( L \), since wider window lengths yield improved estimates of the correlation matrices per window.
Not surprisingly, the semi-blind topology inference approach in Section IVB outperforms the completely blind alternative ($\Omega = \emptyset$), since one presumably has more prior information available. On the other hand, in the completely blind case, Algorithm 1 still results in a reliable estimate of the network topology with low edge identification error.

In several real-world applications, exogenous variables are often unavailable or costly to measure, hence performance benchmarks for the developed algorithm in such blind settings are of considerable interest. To facilitate further assessment of the stability of the novel algorithm when operating in blind scenarios, an extended experiment was carried out as follows. Per experiment trial, an unweighted Erdős-Rényi graph model, as detailed in the previous subsection. Edge weights in the initial non-zero support of $A_0$ were varied over time windows, following two edge-variation patterns: p1) $a_{ij}^{m} = a_{ij}^{0} + 0.1\sin(0.01m)$, for $m = 1, \ldots, 200$; and p2) $a_{ij}^{m} = 0$ with probability 0.2 at the 50th and 100th time windows. For $L = 500$, $L = 2,000$, and $L = 3,000$, exogenous measurements were simulated over time slots, partitioned into $M$ windows of fixed length $L$. The window boundaries were preselected as $\{\tau_m\}_{m=1}^{M+1}$, with $\tau_1 = 1$ and $L := \tau_{m+1} - \tau_m$. For $t \in [\tau_m, \tau_{m+1} - 1]$, exogenous inputs were sampled as $x_t \sim N(0, \sigma_i^2 I)$, with $\{\gamma_m\}_{m=1}^{M}$ set to $M$ distinct values. With $e_t$ sampled i.i.d. from $N(0, 10^{-2} I)$, $y_t$ was similarly generated using the SEM, that is, $y_t = (I - A_m)^{-1}(Bx_t + e_t)$, where $[B]_{ij} \sim \mathcal{U}[2,3]$.

**Results.** Algorithm 2 was run on the simulated data using $\beta = 0.999$, with an edge declared present if $\hat{a}_{ij}$ exceeded the inherent permutation ambiguity.

**B. Simulated piecewise-constant network**

**Data generation.** An initial 64-node network was generated with adjacency matrix $A_0$ via the Kronecker random graph model, as detailed in the previous subsection. Edge weights in the initial non-zero support of $A_0$ were varied over time windows, following two edge-variation patterns: p1) $a_{ij}^{m} = a_{ij}^{0} + 0.1\sin(0.01m)$, for $m = 1, \ldots, 200$; and p2) $a_{ij}^{m} = 0$ with probability 0.2 at the 50th and 100th time windows. For $L = 500$, $L = 2,000$, and $L = 3,000$, endogenous measurements were simulated over time slots, partitioned into $M$ windows of fixed length $L$. The window boundaries were preselected as $\{\tau_m\}_{m=1}^{M+1}$, with $\tau_1 = 1$ and $L := \tau_{m+1} - \tau_m$. For $t \in [\tau_m, \tau_{m+1} - 1]$, exogenous inputs were sampled as $x_t \sim N(0, \sigma_i^2 I)$, with $\{\gamma_m\}_{m=1}^{M}$ set to $M$ distinct values. With $e_t$ sampled i.i.d. from $N(0, 10^{-2} I)$, $y_t$ was similarly generated using the SEM, that is, $y_t = (I - A_m)^{-1}(Bx_t + e_t)$, where $[B]_{ij} \sim \mathcal{U}[2,3]$.

**Results.** Algorithm 2 was run on the simulated data using $\beta = 0.999$, with an edge declared present if $\hat{a}_{ij}$ exceeded the inherent permutation ambiguity.

**Fig. 5: EIER for different window lengths, with: a) $\Omega = \{(i, j)|i = 1, \ldots, N, j = 1, \ldots, M\}$; b) 50% misses in $R^x$; c) $\Omega = \emptyset$.**

**Fig. 6: Performance in blind scenario: a) EIER; b) Success rate.**
[21x195]E-MSE
[21x195]E-MSE
[28x41]10
[28x41]10
[28x110]10
[28x139]10
[28x237]10
[29x188]EIER (%)
[31x286]10
[31x315]10
[31x384]10
[36x288]10
[36x338]12
[36x386]14
[40x46]-4
[40x46]-5
[40x95]-4
[40x144]-3
[40x183]-2
[40x242]-1
[42x44]0
[42x44]0
[42x92]2
[42x92]2
[42x142]4
[42x190]6
[42x240]8
[49x218]by
[49x218]m
[49x230]obtained by running Algorithm 2 during the window indexed
[49x242]crease in value, but gracefully return to lower values. Figure 9
[49x266]where the edge support is known to change, error metrics in-
[49x277]evolution of the network remarkably well. During windows
[49x301]100
[49x313]∥
[49x337]η
[49x349]set to yield the lowest EIER. Algorithm per-
[49x337]0 20 40 60 80 100
[49x349]0 50 100 150 200
[49x349]0 50 100 150 200
[57x-1305]60
[57x-410]30
[57x-111]20
[59x289]As shown by both Figures 7 and 8, Algorithm 2 tracks the
[62x384]m
[63x312]ˆ
[92x313]m
[95x313]F
[104x313]2
[104x313]N
[120x313]. In addition, both error metrics
[133x129]2016
[133x129]to September
[136x129]30
[155x313]1))
[168x313]1
[172x313]). In addition, both error metrics
[200x19]m
[200x19]m
[200x19]m
[200x19]m
[206x129]2016
[206x129]days in
[210x604]m
[213x604]m
[219x335]Pattern 2 (L = 500)
[219x350]Pattern 2 (L = 2,000)
[219x366]Pattern 2 (L = 3,000)
[220x335]Pattern 1 (L = 500)
[220x350]Pattern 1 (L = 2,000)
[220x366]Pattern 1 (L = 3,000)
[221x335]Pattern 2 (L = 500)
[221x350]Pattern 2 (L = 2,000)
[221x366]Pattern 2 (L = 3,000)
[222x337]Pattern 1 (L = 500)
[222x352]Pattern 1 (L = 2,000)
[222x368]Pattern 1 (L = 3,000)
[227x851]m
[240x129]1
[245x129],
[250x129]prices were obtained as time series for dates ranging from
[256x153]depicted by Figure 11, with (a) representing a majority vote
[270x349]strate that when the network has a few nodes, there is a high
[312x452]privy to the public, hence \( \Omega = \emptyset \). Furthermore, it was observed
[312x479]the centered time series. Furthermore, money invested in the
[312x491]centered to have zero mean; see Figure 10 for a plot of
[312x515)\) variate time series were adopted as endogenous variables
[312x530]Results.
[312x530]For this set of experiments, the combined multi-
[312x554]changes in share prices for Microsoft, Intel, and sometimes
[312x566]instance, a significant drop in Intel stock prices often signals
[312x580]that most stock prices tend to exhibit steady quarterly trends
[312x592]rising (falling), and the window length was consequently
[312x604]set to \( L = 100 \) for all tests. Algorithm \( \Pi \) was then run with
[312x606]\( \Omega = \emptyset \), and \( M = 12 \) to infer the causal dependencies between
[312x609]the selected stock prices.

According to the discussion in Section \( \text{IV} \) there is no
guarantee of identifiability in the completely blind setting.
Fortunately, the simulated tests depicted by Figure \( \text{IX} \) demonstrate
that when the network has a few nodes, there is a high
probability of successful recovery of the true network in the
presence of noise. Based on this empirical observation, it is
reasonable to expect that if only a few stocks are selected,
then many trials will yield the true network upon running
Algorithm \( \Pi \) with random initializations. To this end, 100
independent runs of Algorithm \( \Pi \) were done with random
initializations, and it turned out that most estimates yielded
the same support for \( A \), with very slight variations in actual values
of its entries. Consequently, a simple scheme was adopted to
infer the network topology from the ensemble of estimates.
Unique topologies based on the support of \( A \) for the 100
realizations were enumerated, and a majority voting scheme
was adopted to reach consensus on the final topology. The
most frequent network topologies from the experiments are
depicted by Figure \( \text{IX} \) with (a) representing a majority vote
of 92 out of 100, while (b) was the result inferred from 68
experiments. The figure shows very strong dependencies in
the first group of technology companies, while the second
plot shows stronger inter-dependencies between Macy’s and
Nordstrom than the others. Interestingly, both Macy’s and
Nordstrom are well-known “brick-and-mortar” retailers and
competitors. The stronger dependence between them seems
to agree with the expectation that changes in the price of one
would be expected to indirectly impact the other.

C. Tests on real networks

Data description. To conduct tests on real-world networks,
historical stock price data were downloaded through a free
Yahoo application program interface (API). Historical closing
prices were obtained as time series for dates ranging from
December 23, 2011 to September 30, 2016 (1,200 days in
total). The stock time series were grouped into two clusters,
namely: a) large technology companies (Exxon-Mobil, Intel,
Microsoft, Yahoo, and General Electric), and b) online and
brick-and-mortar retailers (Bon-Ton, E-bay, Macy’s, and Nord-
strom). Choices of which stocks were classified under the
two groups were based on prior knowledge of historical inter-
dependencies existing among them in financial markets. For
instance, a significant drop in Intel stock prices often signals
changes in share prices for Microsoft, Intel, and sometimes
General Electric.

Results. For this set of experiments, the combined multi-
variate time series were adopted as endogenous variables
\( \{y_{i}\}_{t=1}^{200} \), after a pre-processing step in which they were
centered to have zero mean; see Figure 10 for a plot of
the centered time series. Furthermore, money invested in the
stocks constitutes exogenous inputs \( \{x_{i}\}_{t=1}^{200} \), which are
not known in this case, since such information is generally not
privy to the public, hence \( \Omega = \emptyset \). Furthermore, it was observed
that most stock prices tend to exhibit steady quarterly trends
(rising or falling), and the window length was consequently
set to \( L = 100 \) for all tests. Algorithm \( \Pi \) was then run with
\( \Omega = \emptyset \), and \( M = 12 \) to infer the causal dependencies between
the selected stock prices.

As shown by both Figures \( \text{VII} \) and \( \text{VIII} \) Algorithm \( \Pi \) tracks the
evolution of the network remarkably well. During windows
where the edge support is known to change, error metrics
increase in value, but gracefully return to lower values. Figure \( \text{IX} \)
depicts heatmaps of actual and inferred adjacency matrices,
obtained by running Algorithm \( \Pi \) during the window indexed
by \( m = 200 \) for scenario p2).

Fig. 7: EIER vs. \( m \) for: (a) Scenario p1; and (b) Scenario p2.

Fig. 8: MSE vs \( m \) for: a) Scenario p1; b) Scenario p2.

Fig. 9: Actual and inferred networks at \( m = 200 \).

(a) (b)
Fig. 10: Plot of the two groups of stock prices over the observation duration with zero-mean centering: a) technology companies; and b) online and “brick-and-mortar” retailers. The stock ticker symbol for each company is shown in the legend (in parentheses).

Fig. 11: Visualization of network topologies inferred from the stock price time series, depicting: a) technology companies; and b) online and “brick-and-mortar” retailers. Notice the stronger dependencies between the two competing “brick-and-mortar” retailers, Macy’s (MCY) and Nordstrom (NDM).

VII. CONCLUSIONS

This paper put forth a novel approach for inference of network topologies from the statistics of nodal processes. Leveraging SEMs, the network topology inference task was reformulated as a constrained PARAFAC tensor decomposition. Recognizing the inherent uniqueness challenges, conditions under which the network can be uniquely identified were derived. Unlike conventional SEMs, which require exact information of the exogenous inputs in order to guarantee identifiability, it was proven that the novel tensor-based approach is capable of uniquely identifying the network topology only with partial information of the second-order statistics of nodal exogenous inputs.

The framework was further extended to facilitate real-time sequential estimation of the network topology by developing a novel topology tracking algorithm. An exponentially weighted least-squares estimator was advocated for the topology tracking problem, making it possible to efficiently solve the problem “on the fly.” To assess the effectiveness of the novel approaches, extensive numerical tests were conducted on both simulated data and historical stock prices of several publicly-traded corporations.

In order to broaden the scope of this work, there are several intriguing directions for future investigation, namely: a) developing algorithms that are capable of exploiting prior knowledge pertaining to the network structure e.g., edge sparsity or power law degree distributions; and b) distributed implementation of the novel algorithms, which is well-motivated,
especially when dealing with large-scale networks.

APPENDIX

A. Proof of Proposition 1
Since diagonal entries of $\mathbf{A}$ are all zero, and $\mathbf{B}^{-1}$ is a diagonal matrix with nonzero entries, $\mathbf{A}$ is invertible; that is,

$$\mathbf{A}^{-1} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A}).$$

(36)

Clearly, the diagonal entries of $\mathbf{A}^{-1}$ coincide with those of $\mathbf{B}^{-1}$, which implies that

$$\mathbf{B} = (\text{Diag}(\mathbf{A}^{-1}))^{-1}. $$

(37)

Recognizing that $\mathbf{BA}^{-1} = \mathbf{I} - \mathbf{A}$, one can write

$$\mathbf{A} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1} = \mathbf{I} - (\text{Diag}(\mathbf{A}^{-1}))^{-1}\mathbf{A}^{-1}$$

(38)

which completes the proof.

B. Proof of Lemma 2
First, note that (15) can be written as

$$\mathbf{R}^x - \mathbf{R}^z \mathbf{\Pi} \mathbf{A}_3 = \mathbf{0}_{M \times N}$$

(39)

and recall that $\mathbf{\Pi}$ is a permutation matrix; hence, each constituent column in $\mathbf{\Pi}$ comprises zeros with the exception of a single entry set to one. Letting $\pi_{ij}$ denote the $(i,j)$-th entry of $\mathbf{\Pi}$, assume without loss of generality that $\pi_{ij} = 1$ and $\pi_{kj} = 0$, $\forall k \neq i$. Consequently, with $\mathbf{p}_j \in \mathbb{R}^N$ representing column $j$ of $\mathbf{P} := \mathbf{\Pi} \mathbf{A}_3$, one can equivalently write

$$\mathbf{p}_j = [0, \ldots, 0, \pi_{ij} \lambda_j, 0, \ldots, 0]^\top$$

(40)

where $\lambda_j \neq 0$ denotes the $j$-th diagonal entry of $\mathbf{A}_3$. Extracting the $j$-th column on both sides of (39), namely,

$$\mathbf{r}_j^z - \mathbf{R}^z \mathbf{p}_j = \mathbf{0}_{M \times 1}$$

(41)

and combining (40) and (41), one obtains

$$\mathbf{r}_j^z = \pi_{ij} \lambda_j \mathbf{r}_j^x. $$

(42)

When $i \neq j$, (42) implies that $\mathbf{r}_j^x$ and $\mathbf{r}_j^z$ are linearly dependent, which contradicts the condition $\mathbf{kr}(\mathbf{R}^z) \geq 2$ in Lemma 1. Hence, for (42) to hold for some nonzero $\lambda_j$, it is necessary that $i = j$, which is equivalent to requiring $\pi_{jj} = 1$ and $\lambda_j = 1$. Since this holds for any $j$, one deduces that

$$\mathbf{\Pi} = \mathbf{I}, \quad \mathbf{A}_3 = \mathbf{I}. $$

(43)

C. Proof of Lemma 3

Recalling from Algorithm 1 that

$$\hat{\mathbf{A}} = \mathbf{I} - \left(\text{Diag}(\hat{\mathbf{A}}^{-1})\right)^{-1}\hat{\mathbf{A}}^{-1},$$

and substituting $\hat{\mathbf{A}} = \mathbf{A}\hat{\mathbf{A}}$, one obtains

$$\hat{\mathbf{A}} = \mathbf{I} - \left(\text{Diag}[(\mathbf{A}\hat{\mathbf{A}})^{-1}]\right)^{-1}(\mathbf{A}\hat{\mathbf{A}})^{-1}$$

$$= \mathbf{I} - \left(\text{Diag}(\mathbf{A}^{-1})\right)^{-1} \mathbf{AA}^{-1}\hat{\mathbf{A}}^{-1}$$

$$= \mathbf{I} - \left(\text{Diag}(\mathbf{A}^{-1})\right)^{-1} \hat{\mathbf{A}}^{-1}. $$

(44)

Comparing with Proposition 1 it is clear that $\hat{\mathbf{A}} = \mathbf{A}$, which concludes the proof.

D. Proof of Lemma 4

First, assume without loss of generality that column $j$ of the permutation matrix $\mathbf{\Pi}$ satisfies $\pi_{ij} = 1$ and $\pi_{kj} = 0$, $\forall k \neq i$, with $\pi_{ij}$ denoting entry $(i,j)$ of $\mathbf{\Pi}$. Since $\mathbf{p}_j \in \mathbb{R}^N$, the $j$-th column of $\mathbf{P} := \mathbf{\Pi}\mathbf{A}_3$ can be written as

$$\tilde{\mathbf{p}}_j := [0, \ldots, 0, \pi_{ij} \lambda_j, 0, \ldots, 0]^\top$$

(45)

with $\lambda_j \neq 0$ representing the $j$-th diagonal entry of $\mathbf{A}_3$. Extracting entries indexed by $\Omega_i \cup \Omega_j$ in column $j$ on both sides of (20), one has

$$\tilde{\mathbf{r}}_j = \pi_{ij} \lambda_j \tilde{\mathbf{r}}_i $$

(46)

and assuming that $i \neq j$, (46) implies that $\tilde{\mathbf{r}}_j$ and $\tilde{\mathbf{r}}_i$ are linearly dependent, which contradicts the condition in Theorem 2. As a result, for (46) to hold true for some nonzero $\lambda_j$, it is necessary that $i = j$, which is equivalent to having $\pi_{jj} = 1$ and $\lambda_j = 1$. Recognizing that this holds for any $j$, one arrives at

$$\tilde{\mathbf{P}} = \mathbf{I}, \quad \hat{\mathbf{A}}_3 = \mathbf{I}. $$

(47)

ACKNOWLEDGMENT

The work of this paper was initiated in a project of a course taught by Prof. N. D. Sidiropoulos, and the authors would like to thank him for his feedback.

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