Nodal solutions for the weighted biharmonic equation with critical exponential growth

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Abstract

In this paper, we deal with the logarithmic weighted fourth order elliptic equation in the unit disk of $B \subset \mathbb{R}^4$:

$$(P_\lambda) \quad \Delta(w(x)\Delta u) = \lambda f(x, u) \quad \text{in} \quad B, \quad u = \frac{\partial u}{\partial n} \quad \text{on} \quad \partial B,$$

where the nonlinearity $f$ is assumed to have exponential critical growth in view of Adam’s type inequalities. By using the constrained minimization in Nehari set combined with the quantitative deformation lemma and degree theory, we prove the existence of nodal solutions to the problem $(P_\lambda)$.

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1 Introduction

In this paper, we study the following weighted biharmonic problem

\[ (P_{\lambda}) \quad \begin{cases} \Delta(w(x)\Delta u) = \lambda f(x,u) & \text{in } B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \]

where the weight \( w(x) \) is given by

\[ w(x) = (\log \frac{e}{|x|})^{1-\beta}, \beta \in (0,1). \]

\( B \) is the unitary disk in \( \mathbb{R}^4 \). Moreover, the function \( f(x,t) : B \times \mathbb{R} \to \mathbb{R} \) is continuous and behaves like \( \exp\{\alpha t^{1-\beta}\}, \beta \in (0,1) \) as \( t \to +\infty \), for some \( \alpha > 0 \). We also assume that the nonlinearity \( f(x,t) \) satisfy these conditions:

\( (F_1) \) \( f : B \times \mathbb{R} \to \mathbb{R} \) is \( C^1 \) and radial in \( x \).

\( (F_2) \) There exist \( \theta > 2 \) such that we have

\[ 0 < \theta F(x,t) \leq tf(x,t), \forall (x,t) \in B \times \mathbb{R} \setminus \{0\} \]

where

\[ F(x,t) = \int_0^t f(x,s)ds. \]

\( (F_3) \) For each \( x \in B, t \mapsto \frac{f(x,t)}{t} \) is increasing for \( t \in \mathbb{R} \setminus \{0\} \).

\( (F_4) \lim_{t \to 0} \frac{|f(x,t)|}{t} = 0. \)

As an example of such nonlinearity, the function \( f(x,t) = |t|t+|t|t \exp(\alpha|t|^{\gamma}) \) satisfy the conditions \( (F_1), (F_2), (F_3) \) and \( (F_4) \).

In general, the study of fourth order partial differential equations is considered an interesting topic. The interest in studying such equations was stimulated by their applications in micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, see [10, 15, 23]. However many applications are generated by the weighted elliptic problems, such as the study of traveling waves in suspension bridges, radar imaging (see, for example [2, 21]).

Problems without weight \( (w(x) = 1) \), under different assumptions on the nonlinearity \( f \), have been widely studied by many authors. In [3], C. O.
Alves and A. B. Nóbrega established the existence of nodal solution of the following problem

\[
\begin{aligned}
\Delta^2 u = \lambda f(x, u) & \quad \text{in } \Omega \\
0 & = \lambda f(x, u) \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( B = \Delta u \) or \( B = \frac{\partial u}{\partial n} \), where \( \Omega \) is smooth bounded domain of \( \mathbb{R}^N \), \( N \geq 1 \), and the nonlinearity \( f \) is \( C^1 \) function with subcritical polynomial growth, we refer also the reader to [30, 28] and the references therein. In dimension \( N = 2 \), more authors have been discussed problems in second order elliptic equations with different boundary conditions of the following type

\[
(1.2) \quad -\Delta u = f(x, u) \quad \text{in } \Omega \subset \mathbb{R}^2.
\]

We cite the recent work of F. Faraci D. Puglisi, [14] and the references therein. The authors proved, when the nonlinearity \( f \) depending on gradient, the existence of changing-sign solutions by using the theory of invariant sets of descending flow.

In the case the nonlinearity \( f(x, u) \) is positive and has critical exponential growth, equations (1.2) have been investigated in literatures [1, 16, 20, 22]. In dimension \( N \geq 2 \), the critical exponential growth is given by the well known Trudinger-Moser inequality [24, 25]

\[
\sup_{f \in [\alpha |u|^N, \infty]} \int_\Omega e^{\alpha |u|^N} dx < +\infty \quad \text{if and only if} \quad \alpha \leq \alpha_N,
\]

where \( \alpha_N = \omega_{N-1}^{1/N} \) with \( \omega_{N-1} \) is the area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \).

Later, the Trudinger-Moser inequality was improved to weighted inequalities [6, 7]. The influence of the weight in the Sobolev norm was studied as the compact embedding [19].

When the weight is of logarithmic type, Calanchi and Ruf [8] extend the Trudinger-Moser inequality and proved the following results in the weighted Sobolev space

\[
W^{1,N}_{0,rad}(B, \rho) = cl\{u \in C_\infty^{0,rad}(B) \mid \int_B |\nabla u|^N \rho(x) dx < \infty\}.
\]

**Theorem 1.1.** [7]

(i) Let \( \beta \in [0, 1) \) and let \( \rho \) given by \( \rho(x) = \left( \log \frac{1}{|x|} \right)^\beta \), then

\[
\int_B e^{[u]} \gamma dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,rad}(B, \rho), \quad \text{if and only if} \quad \gamma \leq \gamma_{N,\beta} = \frac{N}{(N - 1)(1 - \beta)} = \frac{N'}{1 - \beta}
\]
and
\[\sup_{u \in W^{1,N}_{0,rad}(B,\rho)} \int_B e^{\alpha|u|^N} |\nabla u|^2 \, dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_{N,\beta} = N\left[\frac{1}{\omega_{N-1}}(1-\beta)\right]^{1-\beta}\]

where \(\omega_{N-1}\) is the area of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N\) and \(N'\) is the Hölder conjugate of \(N\).

**(ii)** Let \(\rho\) given by \(\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}\), then
\[\int_B \exp\{e^{\beta|u|^N}\} \, dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,rad}(B,\rho)\]

and
\[\sup_{u \in W^{1,N}_{0,rad}(B,\rho)} \int_B \exp\{\beta e^{\omega_{N-1}|u|^N}\} |\nabla u|^2 \, dx < +\infty \quad \Leftrightarrow \quad \beta \leq N,\]

where \(\omega_{N-1}\) is the area of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N\) and \(N'\) is the Hölder conjugate of \(N\).

These results opened the way to study second order weighted elliptic problems in dimension \(N \geq 2\). We cite the work of Calanchi et al [9], ie the following problem
\[
\begin{cases}
-\nabla.(\nu(x)\nabla u) = f(x,u) & \text{in } B \\
u > 0 & \text{in } B \\
u = 0 & \text{on } \partial B,
\end{cases}
\]

with the weight \(\nu(x) = \log(\frac{e}{|x|})\) and where the function \(f(x,t)\) is continuous in \(B \times \mathbb{R}\) and behaves like \(\exp\{e^{\alpha x^2}\}\) as \(t \to +\infty\), for some \(\alpha > 0\). Also, recently, Deng et al [11] and Zhang [29] studied the following problem
\[
\begin{cases}
-\text{div}\left(\rho(x)|\nabla u|^{N-2}\nabla u\right) + \xi(x)|u|^{N-2}u = f(x,u) & \text{in } B \\
u = 0 & \text{on } \partial B,
\end{cases}
\]

where \(N \geq 2\), the function \(f(x,t)\) is continuous in \(B \times \mathbb{R}\) and behaves like \(\exp\{e^{\alpha x^2}\}\) as \(t \to +\infty\), for some \(\alpha > 0\). The authors proved that there is a non-trivial solution to this problem using Mountain Pass theorem. Also, we mention that Baraket et al [5] studied the following non-autonomous weighted elliptic equations
\[
\begin{cases}
-\text{div}\left(\rho(x)|\nabla u|^{N-2}\nabla u\right) + \xi(x)|u|^{N-2}u = f(x,u) & \text{in } B \\
u > 0 & \text{in } B \\
u = 0 & \text{on } \partial B,
\end{cases}
\]
where $B$ is the unit ball of $\mathbb{R}^N$, $N > 2$, $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t N^{-1}}\}$ as $t \to +\infty$, for some $\alpha > 0$. $\xi : B \to \mathbb{R}$ is a positive continuous function satisfying some conditions. The weight $\rho(x)$ is given by $\rho(x) = (\log \frac{1}{|x|})^{N-1}$.

Let $\Omega \subset \mathbb{R}^4$ be a bounded domain and $w \in L^1(\Omega)$ be a nonnegative function, the weighted Sobolev space is defined as

$$W^{2,2}_0(\Omega, w) = cl\{u \in C_0^\infty(\Omega) \mid \int_\Omega |\Delta u|^2 w(x) dx < \infty\}.$$ 

We will restrict our attention to radial functions and then consider the subspace

$$E = W^{2,2}_{0, rad}(B, w) = cl\{u \in C_0^\infty(B) \mid \int_B |\Delta u|^2 w(x) dx < \infty\},$$ 

equipped with norm

$$\|u\| = \left( \int_B |\Delta u|^2 w(x) dx \right)^{\frac{1}{2}}, \quad w(x) = (\log \frac{e}{|x|})^\beta$$

which comes from the scalar product

$$< u, v > = \int_B \Delta u. \Delta v (\log \frac{e}{|x|})^\beta dx.$$ 

The norm $\|u\| = \left( \int_B |\Delta u|^2 w(x) dx \right)^{\frac{1}{2}}$, and

$$\|u\|_{W^{2,2}_{0, rad}(B, w)} = \left( \int_B u^2 dx + \int_B |\nabla u|^2 dx + \int_B |\Delta u|^2 w(x) dx \right)^{\frac{1}{2}}$$

are equivalent (see Lemma 1).

The choice of the weight and the space $W^{2,2}_{0, rad}(B, w)$ are motivated by the following inequality of Adam’s type.

**Theorem 1.2.** [26] Let $\beta \in (0, 1)$ and let $w$ given by (1.1), then

$$\sup_{u \in W^{2,2}_{0, rad}(B, w), \|u\| \leq 1} \int_B e^{\alpha |u|^{1-\beta}} dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_\beta = 4[8\pi^2(1 - \beta)]^{1-\beta}$$

Let us now state our results. We associate to the problem $(P_\lambda)$ the functional

$$\mathcal{J}_\lambda : E \to \mathbb{R},$$
defined by

\[ \mathcal{J}_\lambda(u) = \frac{1}{2} \int_B |\Delta u|^2 w(x) dx - \lambda \int_B F(x, u) dx, \]

where \( F(x, u) = \int_0^u f(x, t) dt \).

We say that \( u \) is a solution to the problem \((P_\lambda)\), if \( u \) is a weak solution in the following sense.

**Definition 1.1.** We say that a function \( u \in E \) is a solution of the problem \((P_\lambda)\) if

\[ \int_B \Delta u \Delta \varphi w(x) dx = \lambda \int_B f(x, u) \varphi dx, \quad \forall \varphi \in E. \]

It is easy to see that seeking weak solutions of the problem \((P_\lambda)\) is equivalent to find nonzero critical points of the functional \( \mathcal{J}_\lambda \) on \( E \). The energy functional \( \mathcal{J}_\lambda \) is well defined and of class \( C^1 \) since there exist \( a, C > 0 \) positive constants and there exists \( t_1 > 1 \) such for that

\[ |f(x, t)| \leq Ce^{at}, \quad \forall |t| > t_1, \]

whenever the nonlinearity \( f(x, t) \) is critical or subcritical at \(+\infty\).

It is quite clear that finding non trivial weak solutions to the problem \((P_\lambda)\) is equivalent to finding non-zero critical points of the functional \( \mathcal{J}_\lambda \). Moreover, we have

\[ \langle \mathcal{J}_\lambda'(u), \varphi \rangle = \mathcal{J}_\lambda(u) \varphi = \int_B \omega(x) \Delta u \Delta \varphi dx - \lambda \int_B f(x, u) \varphi dx, \quad \varphi \in E. \]

For all \( u \in E \) we write

\[ u = u^+ + u^-, \]

where \( u^+(x) = \max\{u(x), 0\} \), \( u^-(x) = \min\{u(x), 0\} \). We define the Nehari set as

\[ \mathcal{N}_\lambda := \{ u \in E : \langle \mathcal{J}_\lambda'(u), u^+ \rangle = \langle \mathcal{J}_\lambda'(u), u^- \rangle = 0, u^+ \neq 0, u^- \neq 0 \}, \]

It’s easy to verify the following decomposition

\[ \mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(u^+) + \mathcal{J}_\lambda(u^-), \]

and

\[ \langle \mathcal{J}_\lambda'(u), u^+ \rangle = \langle \mathcal{J}_\lambda'(u^+), u^+ \rangle \quad \text{and} \quad \langle \mathcal{J}_\lambda'(u), u^- \rangle = \langle \mathcal{J}_\lambda'(u^-), u^- \rangle \]

We also give the following definitions of the so called nodal solutions and least energy sign-changing solution of problem \((P_\lambda)\).
Definition 1.2. • $v \in E$ is called nodal or sign-changing solution of problem $(P_{\lambda})$ if $v$ is a solution of problem $(P_{\lambda})$ and $v^\pm \neq 0$ a.e in $B$.

• $v \in E$ is called least energy sign-changing solution of problem $(P_{\lambda})$ if $v$ is a sign-changing solution of $(P_{\lambda})$ and

$$J_{\lambda}(v) = \inf \{J_{\lambda}(u) : J'_{\lambda}(u) = 0, u^\pm \neq 0 \text{ a.e in } B\}$$

Influenced by the works cited above, we try to find a minimize of the energy functional $J_{\lambda}$ over the following minimization problem,

$$c_{\lambda} = \inf_{u \in \mathcal{N}_{\lambda}} J_{\lambda}(u)$$

Our approach is based on the Nehari manifold method, which was introduced in [18] and is by now a well-established and useful tool in finding solutions of problems with a variational structure, see [16].

To our best knowledge, there are no result for the nodal solutions to the weighted bilaplacian equation with critical exponential nonlinearity on the weighted Sobolev space $E$.

Now, we give our main results as follows:

For a critical growth nonlinearity, the following result holds.

Theorem 1.3. Assume that $f(x, t)$ has a critical growth at $+\infty$ for some $a_0$ and $(F_1)$, $(F_2)$, $(F_3)$ and $(F_4)$ are satisfied. Then, there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, problem $(P_{\lambda})$ has a least energy nodal (sign-changing) radial solution $v \in \mathcal{N}_{\lambda}$.

This present work is organized as follows: in section 2 we give important preliminaries of weighted Sobolev space and some results for the compactness analysis. In section 3, we give some technical key lemmas, which are used to prove the main result. The section 4 is devoted to prove Theorem 1.3. Finally, we note that a constant $C$ may change from line to another and sometimes we index the constants in order to show how they change.

2 Preliminaries and auxiliary results

2.1 Weighted Lebesgue and Sobolev Spaces setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$, $W^{m,p}(\Omega, w)$, $W_0^{m,p}(\Omega, w)$ and some of their properties that will be used later. Let $S(\Omega)$ be the set of all measurable
real-valued functions defined on $\Omega$ and two measurable functions are considered as the same element if they are equal almost everywhere.

Following Drabek et al. and Kufner in [12], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \{ u : \Omega \to \mathbb{R} \text{ measurable}; \quad \int_{\Omega} w(x)|u|^p \, dx < \infty \}$$

for any real number $1 \leq p < \infty$.

This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \left( \int_{\Omega} w(x)|u|^p \, dx \right)^{\frac{1}{p}}.$$

For $m \geq 2$, let $w$ be a given family of weight functions $w_\tau, |\tau| \leq m$, $w = \{w_\tau(x) \ x \in \Omega, \ |\tau| \leq m\}$.

In [12], the corresponding weighted Sobolev space was defined as

$$W^{m,p}(\Omega, w) = \{ u \in L^p(\Omega), D_\tau u \in L^p(\Omega) \ \forall \ 1 \leq |\tau| \leq m-1, D_\tau u \in L^p(\Omega, w) \ \forall \ |\tau| = m \}$$

endowed with the following norm:

$$\|u\|_{W^{m,p}(\Omega, w)} = \left( \sum_{|\tau| \leq m-1} \int_{\Omega} |D_\tau u|^p \, dx + \sum_{|\tau| = m} \int_{\Omega} |D_\tau u|^p \omega(x) \, dx \right)^{\frac{1}{p}},$$

where $w_\tau = 1$ for all $|\tau| < k$, $w_\tau = \omega$ for all $|\tau| = k$.

If we suppose also that $w(x) \in L^1_{\text{loc}}(\Omega)$, then $C^\infty(\Omega)$ is a subset of $W^{m,p}(\Omega, w)$ and we can introduce the space

$$W^{m,p}_0(\Omega, w)$$

as the closure of $C^\infty(\Omega)$ in $W^{m,p}(\Omega, w)$.

$(L^p(\Omega, w), \| \cdot \|_{p,w})$ and $(W^{m,p}(\Omega, w), \| \cdot \|_{W^{m,p}(\Omega, w)})$ are separable, reflexive Banach spaces provided that $w(x)^{-\frac{m-1}{p}} \in L^1_{\text{loc}}(\Omega)$.

For $w(x) = 1$, one finds the standard Sobolev spaces $W^{m,p}(\Omega)$, $W^{m,p}_0(\Omega)$ and the Lebesgue spaces $L^p(\Omega)$.

Our space setting is

$$E = \{ u \in W^{2,2}_{0,\text{rad}}(B, w) \mid \int_{\Omega} |\Delta u|^2 w(x) \, dx < \infty \}.$$
E is equipped with norm 

\[ \|u\| = \left( \int_B |\Delta u|^2 w(x) dx \right)^{\frac{1}{2}}, \]

which comes from the scalar product

\[ <u, v> = \int_B \Delta u \cdot \Delta v (\log \frac{e}{|x|})^3 dx. \]

We have the following result:

**Lemma 1.** \((E, \|\cdot\|_{W^{2,2}_{0,rad}(B,w)})\) is a Banach space and the norm \(\|\cdot\|\) is equivalent in \(E\) to the norm \(\|\cdot\|_{W^{2,2}_{0,rad}(B,w)}\).

**Proof.** The Sobolev weighted space \((E, \|\cdot\|_{W^{2,2}_{0,rad}(B,w)})\) is a normed linear space. In order to prove that it is a Banach space, let \(\{u_n\}\) be a Cauchy sequence that

\[ \|u_n - u_m\|_{W^{2,2}_{0,rad}(B,w)} \to 0 \quad \text{as} \quad n, m \to +\infty. \]

Therefore \(\{u_n\}\) is also a Cauchy sequence in \((W^{2,2}_{0,rad}(B,w), \|\cdot\|_{W^{2,2}_{0,rad}(B,w)})\).

By the completeness of the last space, there exists \(u \in W^{2,2}_{0,rad}(B,w)\) such that

\[ (2.1) \quad \|u_n - u\|_{W^{2,2}_{0,rad}(B,w)} \to 0 \quad \text{as} \quad n \to +\infty. \]

Since \(\|u\|^2_{W^{1,2}_{0,rad}(B)} = \int_B |\nabla u|^2 dx\), then

\[ \|u\|_{W^{1,2}_{0,rad}(B)} \leq \|u\|_{W^{2,2}_{0,rad}(B,w)}, \]

for all \(u \in E\), the sequence \(\{u_n\}\) is also a Cauchy sequence in \((W^{1,2}_{0,rad}(B), \|\cdot\|_{W^{1,2}_{0,rad}(B)})\).

By the completeness of \((W^{1,2}_{0,rad}(B), \|\cdot\|_{W^{1,2}_{0,rad}(B)})\) there exists \(v \in W^{1,2}_{0,rad}(B)\) such that

\[ (2.2) \quad \|u_n - v\|_{W^{1,2}_{0,rad}(B)} \to 0 \quad \text{as} \quad n \to +\infty \]

Since \(u \in W^{2,2}_{0,rad}(B,w), u \in W^{1,2}_{0,rad}(B)\) and by (2.1) we obtain

\[ (2.3) \quad \|u_n - u\|_{W^{1,2}_{0,rad}(B)} \to 0 \quad \text{as} \quad n \to +\infty, \]

and from (2.2), (2.3), we have

\[ \|u - v\|_{W^{1,2}_{0,rad}(B)} \leq \|u_n - u\|_{W^{1,2}_{0,rad}(B)} + \|u_n - v\|_{W^{1,2}_{0,rad}(B)} \to 0 \quad \text{as} \quad n \to +\infty, \]
so \( u = v \) a.e in \( B \), \( u \in E \) and satisfies
\[
\| u_n - u \|_{W^{2,2}_{0,rad}(B, w)} \to 0 \quad \text{as} \quad n \to +\infty.
\]

Now we prove that \( \| \cdot \| \) is equivalent to \( \| \cdot \|_{W^{2,2}_{0,rad}(B, w)} \) in \( E \).
\[
\| u \|_{W^{2,2}_{0}}^2 = \| u \|_2^2 + \| \nabla u \|_2^2 + \int_B |\Delta u|^2 w(x) dx.
\]

For all \( u \in W^{2,2}_{0,rad}(B) \), we have
\[
\| u \|_2^2 = \int_B |\Delta u|^2 w(x) dx \leq \| u \|_2^2 + \| \nabla u \|_2^2 + \int_B |\Delta u|^2 w(x) dx
\]

On the other hand, for all \( u \in W^{2,2}_{0,rad}(B, w) \), by Poincaré inequality,
\[
\| u \|_2^2 \leq C \| \nabla u \|_2^2,
\]
and using the Green formula we get
\[
\int_B \nabla u \nabla u = - \int_B u \Delta u + \int_{\partial B} u \frac{\partial u}{\partial n} \leq \int_B |u \Delta u|.
\]

By Young inequality, we get for all \( \varepsilon > 0 \)
\[
\left| \int_B u \Delta u \right| \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 + \frac{\varepsilon}{2} \int_B |u|^2,
\]

Again, by the Poincaré inequality and using the fact that \( w(x) \geq 1 \), for all \( x \in B \), we get
\[
\int_B \nabla^2 u u dx \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 dx + \frac{\varepsilon}{2} \int_B |\nabla u|^2 dx \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 w(x) dx + \frac{\varepsilon}{2} C^2 \int_B |\nabla u|^2 dx.
\]

Hence
\[
(1 - \frac{\varepsilon}{2} C^2) \int_B |\nabla u|^2 dx \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 w(x) dx,
\]

which implies that
\[
(2.4) \quad \| \nabla u \|_2^2 \leq C \int_B |\Delta u|^2 w(x) dx
\]

and it is easy to conclude. \( \square \)
2.2 Compactness analysis

In this section, we will present a number of technical Lemmas for our future use. We begin by the radial Lemma.

**Lemma 2.** [7] Let $u$ be a radially symmetric function in $C^1_0(B)$. Then, we have

$$|u(x)| \leq \frac{1}{2\sqrt{2\pi}} \left( \frac{1}{\sqrt{1-\beta}} \right) \parallel u \parallel^2.$$ 

It follows that the embedding $E \hookrightarrow L^q(B)$ is continuous for all $q \geq 1$, and that there exists a constant $C > 0$ such that $\parallel u \parallel_{N^q} \leq C \parallel u \parallel$, for all $u \in E$. Moreover, the embedding $E \hookrightarrow L^q(B)$ is compact for all $q \geq N$. In the sequel, we give an important compactness result of Lions type.

**Lemma 3.** Let $(u_k)_k$ be a sequence in $E$. Suppose that, $\parallel u_k \parallel = 1$, $u_k \rightharpoonup u$ weakly in $E$, $u_k(x) \to u(x)$ a.e $x \in B$, and $u \not\equiv 0$. Then

$$\sup_k \int_B e^{p \alpha \beta |u_k|^\gamma} dx < +\infty,$$

where $\alpha \beta = 4[8\pi^2(1-\beta)]^{1-\beta}$, for all $1 < p < U(u)$ where $U(u)$ is given by:

$$U(u) := \begin{cases} 
\frac{1}{(1-\parallel u \parallel^2)^\frac{1}{2}} & \text{if } \parallel u \parallel < 1 \\
+\infty & \text{if } \parallel u \parallel = 1
\end{cases}$$

**Proof**

Since $\parallel u \parallel \leq \lim_k \parallel u_k \parallel = 1$, we will split the evidence into two cases.

**Case 1 : $\parallel u \parallel < 1.$** We assume by contradiction for some $p_1 < U(u)$, we have

$$\sup_k \int_B \exp(\alpha \beta p_1 u_k^\gamma) dx = +\infty.$$

Set

$$B_L^k = \{ x \in B : u_k(x) \geq L \}$$

where $L$ is a constant that we will choose later. Let $v_k = u_k - L$. we have

$$\text{(2.5) } (1+a)^q \leq (1+\varepsilon)a^q + (1- \frac{1}{(1+\varepsilon)^{q-1}})^{1-q}, \quad \forall a \geq 0, \quad \forall \varepsilon > 0 \quad \forall q > 1.$$

So, using (3.2), we get

$$\text{(2.6) } |u_k|^\gamma = |u_k - L + L|^\gamma$$

$$\leq (|u_k - L| + |L|)^\gamma$$

$$\leq (1+\varepsilon)|u_k - L|^\gamma + (1- \frac{1}{(1+\varepsilon)^{q-1}})^{1-\gamma}|L|^\gamma$$

$$\leq (1+\varepsilon)v_k^\gamma + C(\varepsilon, \gamma)L^\gamma.$$
We have

\[ \int_B \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx = \int_{B_L^k} \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx + \int_{B \setminus B_L^k} \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx \]
\[ \leq \int_{B_L^k} \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx \]
\[ + c \exp (\alpha_{\beta} p_1 L) \]
\[ \leq \int_{B_L^k} \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx + c(L, \gamma, |B|), \]

and then

\[ \sup_k \int_{B_L^k} \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx = \infty. \]

By (3.2) we have

\[ \int_{B_L^k} \exp (\alpha_{\beta} p_1 u_k^\gamma) \, dx \leq \exp (\alpha_{\beta} p_1 C(\varepsilon, \gamma) L) \]
\[ \times \int_{B_L^k} \exp ((1 + \varepsilon) \alpha_{\beta} p_1 v_k^\gamma) \, dx. \]

Since, \( p_1 < U(u) \), there exists \( \bar{p}_1 \) such that \( \bar{p}_1 = (1 + \varepsilon)p_1 < U(u) \). Thus

(2.7) \[ \sup_k \int_{B_L^k} \exp (\bar{p}_1 \alpha_{\beta} v_k^\gamma) \, dx = \infty \]

Now, we define

\[ T_L(u) = \min \{ L, u \} \text{ and } T_L(u) = u - T_L(u) \]

and choose \( L \) such that

(2.8) \[ \frac{1 - \|u\|^2}{1 - \|T_L u\|^2} > \left( \frac{\bar{p}_1}{U(u)} \right)^{\frac{2}{\gamma}}. \]

We claim that

\[ \lim \sup_k \int_{B_L^k} \omega(x) |\Delta v_k|^2 \, dx < \left( \frac{1}{\bar{p}_1} \right)^{\frac{2}{\gamma}}. \]

If this is not the case, then up to a subsequence, we get

\[ \int_{B_L^k} \omega(x) |\Delta v_k|^2 \, dx = \int_B \omega(x) |\Delta T_L u_k|^2 \, dx \geq \left( \frac{1}{\bar{p}_1} \right)^{\frac{2}{\gamma}} + o_k(1). \]

Thus,
\[(\frac{1}{\bar{p}_1})^{\frac{2}{\gamma}} + \int_B \omega(x) |\Delta T^Lu_k|^2 \, dx + o_k(1) \leq \int_B \omega(x) |\Delta T^Lu_k|^2 \, dx + \int_{B \setminus B^k_L} \omega(x) |\Delta u_k|^2 \, dx \]

\[= \int_{B^k_L} \omega(x) |\Delta u_k|^2 \, dx + \int_{B^k_L} \omega(x) |\Delta u_k|^2 \, dx = 1.\]

For \(L > 0\) fixed, \(T^Lu_k\) is also bounded in \(E\). Hence, up to a subsequence, \(T^Lu_k \to T^Lu\) in \(E\) and \(T^Lu_k \to T^Lu\) almost everywhere in \(B\). By the lower semicontinuity of the norm in \(E\) and the above inequality, we have

\[\bar{p}_1 \geq \frac{1}{(1 - \lim \inf_{k \to +\infty} \|T^Lu_k\|^2)^{\frac{2}{\gamma}}} \geq \frac{1}{(1 - \|T^Lu\|^2)^{\frac{2}{\gamma}}},\]

combining with \((2.8)\), we obtain

\[\bar{p}_1 \geq \frac{1}{(1 - \|T^Lu\|^2)^{\frac{2}{\gamma}}} \geq \frac{\bar{p}_1}{U(u)} \cdot \frac{1}{(1 - \|T^Lu\|^2)^{\frac{2}{\gamma}}} = \bar{p}_1,\]

which is a contradiction. Therefore

\[\lim \sup_k \int_{B^k_L} \omega(x) |\Delta v_k|^2 \, dx < \left(\frac{1}{\bar{p}_1}\right)^{\frac{2}{\gamma}}.\]

By the Adam’s inequality \((1.4)\), we deduce that

\[\sup_k \int_{B^k_L} \exp(\bar{p}_1 \beta \alpha v_k^\gamma) \, dx < \infty\]

which is also a contradiction. The proof is finished in this case.

**Case 2: \(\|u\| = 1\).** We can then proceed as in case 1 and obtain

\[\sup_k \int_{B^k_L} \exp(\bar{p}_1 \beta \alpha v_k^\gamma) \, dx = \infty\]

where \(\bar{p}_1 = (1 + \varepsilon)p_1\). Then we have

\[\lim \sup_k \int_{B^k_L} \omega(x) |\Delta v_k|^2 \, dx = \lim \sup_k \int_B \omega(x) |\Delta T^Lu_k|^2 \, dx \geq \left(\frac{1}{\bar{p}_1}\right)^{\frac{2}{\gamma}}\]

thus,

\[\|T^Lu\|^2 \leq \lim \inf_k \int_B \omega(x) |\Delta T^Lu_k|^2 \, dx \leq 1 - \lim \sup_k \int_B \omega(x) |\Delta T^Lu_k|^2 \, dx \leq 1 - \left(\frac{1}{\bar{p}_1}\right)^{\frac{2}{\gamma}}.\]
On the other hand, since \( \|u\| = 1 \), we can take \( L \) large enough such that

\[
\|T^L u\|^2 > 1 - \frac{1}{3} \left( \frac{1}{p_1} \right)^{\frac{2}{3}}
\]

which is a contradiction, and the proof is complete in this case. This complete the proof. A second important Lemma.

**Lemma 4.** [16] Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( f: \overline{\Omega} \times \mathbb{R} \) be a continuous function. Let \( \{u_n\}_n \) be a sequence in \( L^1(\Omega) \) converging to \( u \) in \( L^1(\Omega) \). Assume that \( f(x, u_n) \) and \( f(x, u) \) are also in \( L^1(\Omega) \). If

\[
\int_{\Omega} |f(x, u_n)u_n|dx \leq C,
\]

where \( C \) is a positive constant, then

\[
f(x, u_n) \rightarrow f(x, u) \text{ in } L^1(\Omega).
\]

### 3 Some technical lemmas

In the following we assume, unless otherwise stated, that the function \( f \) satisfies the conditions \((F_1)\) to \((F_4)\). Let \( u \) in \( E \) with \( u^\pm \neq \nabla \) a.e. in the ball \( B \), and we define the function \( \Upsilon_u : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) and mapping \( L_u : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2 \) as

\[
(3.1) \quad \Upsilon_u(p, q) = J_\lambda(pu^+ + qu^-),
\]

and

\[
(3.2) \quad L_u(p, q) = \langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle, \langle J'_\lambda(pu^+ + qu^-), qu^- \rangle
\]

**Lemma 5.** (i) For each \( u \) in \( E \) with \( u^+ \neq 0 \) and \( u^- \neq 0 \), there exists an unique couple \( (p_u, q_u) \in (0, \infty) \times (0, \infty) \) such that \( p_u u^+ + q_u u^- \in \mathcal{N}_\lambda \).

In particular, the set \( \mathcal{N}_\lambda \) is nonempty.

(ii) For all \( p, q \geq 0 \) with \( (p, q) \neq (p_u, q_u) \), we have

\[
J_\lambda(pu^+ + qu^-) < J_\lambda(p_u u^+ + q_u u^-).
\]

Proof. (i)

Since \( f \) is subcritical or critical, and From \((F_1)\) and \((F_4)\), for all \( \varepsilon > 0 \), there exists a positive constant \( C_1 = C_1(\varepsilon) \) such that

\[
(3.3) \quad f(x, t)t \leq \varepsilon|t|^2 + C_1|t|^s \exp(\alpha|t|) \text{ for all } \alpha > \alpha_0, s > 2.
\]
Nodal solutions for the weighted biharmonic equation with critical exponential growth

Now, given \( u \in E \) fixed with \( u^+ \neq 0 \) and \( u^- \neq 0 \). From (3.3), for all \( \varepsilon > 0 \), we have

\[
\langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle = \langle J_\lambda(pu^+), pu^+ \rangle \\
= \|pu^+\|^2 - \lambda \int_B f(x, pu^+)pu^+dx \\
\geq \|pu^+\|^2 - \lambda \varepsilon \int_B |pu^+|^2dx - \lambda C_1 \int_B |pu^+|^\gamma \exp(\alpha p|u^+|^\gamma)dx
\]

Using the Hölder inequality, with \( a, a' > 1 \) such that \( \frac{1}{a} + \frac{1}{a'} = 1 \), and Lemma 2, we get

\[
\langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle \geq \|pu^+\|^2 - \lambda \varepsilon C_2 \|pu^+\|^2 - \lambda C_1 \left( \int_B |pu^+|^{a'}dx \right)^{\frac{1}{a'}} \left( \int_B \exp(\alpha p|u^+|^\gamma)dx \right)^{\frac{1}{a}} \\
\geq (1 - \varepsilon C_2 - \lambda \varepsilon C_1) \|pu^+\|^2 - \lambda C_1 \left( \int_B \exp (\alpha a\|pu^+\|^{\gamma}(|u^+|_{u^+}^{\gamma})dx \right)^{\frac{1}{a}} C_3 \|pu^+\|^s
\]

By (1.4), the last integral is finite provided \( p > 0 \) is chosen small enough such that \( \alpha a \|pu^+\|^{\gamma} \leq \alpha_{N, \beta} \). Then,

\[
(3.4) \quad \langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle \geq (1 - \varepsilon C_2 - \lambda \varepsilon C_1) \|pu^+\|^2 - \lambda C_4 \|pu^+\|^s
\]

holds. Choosing \( \varepsilon > 0 \) such that \( 1 - \varepsilon C_2 - \lambda \varepsilon C_1 > 0 \) and for all \( q > 0 \) and \( s > N \), we get \( \langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle > 0 \). In the similar way, it can be proved that \( \langle J'_\lambda(pu^+ + qu^-), pu^- \rangle > 0 \) for \( q > 0 \) small enough and all \( p > 0 \). Therefore, it is quite easy to state that there exists \( t_1 > 0 \) such that

\[
(3.5) \quad \langle J'_\lambda(t_1u^+ + qu^-), t_1u^+ \rangle > 0, \quad \langle J'_\lambda(pu^+ + t_1u^-), t_1u^- \rangle > 0 \quad \text{for all} \; p, q > 0.
\]

From \((F_3)\), we can derive that there exists \( C_5, C_6 > 0 \) such that

\[
(3.6) \quad F(x, t) \geq C_5 |t|^\theta - C_6.
\]

Now, choose \( p = t_2^* > t_1 \) with \( t_2^* \) large enough. Then, by using (3.3), (3.6), we get

\[
\langle J'_\lambda(t_2^*u^+ + qu^-), t_2^*u^+ \rangle = \langle J'_\lambda(t_2^*u^+), t_2^*u^+ \rangle \\
\leq \|t_2^*u^+\|^2 - \lambda \int_B C_5 |t_2^*u^+|^\theta dx + \lambda C_6 |B| \\
\leq 0,
\]
for \( q \in [t_1, t_2] \). Also, we can choose \( q = t_2^* > t_1 \) with \( t_2^* \) large enough and then
\[
\langle \mathcal{J}'_\lambda(t_2^*u^+ + t_2^*u^-), t_2^*u^+ \rangle < 0 \text{ holds for } p \in [t_1, t_2].
\]

Therefore, if \( t_2 > t_2^* \) is large enough, then we obtain that
\[
\mathcal{J}'_\lambda(t_2u^+ + qu^-), t_2u^+ < 0 \quad \text{and} \quad \langle \mathcal{J}'_\lambda(pu^+ + t_2u^-), t_2u^- \rangle < 0 \text{ for all } p, q \in [t_1, t_2].
\]

Joining (3.5) and (3.7) with Miranda’s Theorem [4], there exists at least a couple of points \((p_u, q_u) \in (0, \infty) \times (0, \infty)\) such that \( L_u(p_u, q_u) = (0, 0) \), i.e., \( p_uu^+ + q_uu^- \in \mathcal{N}_\lambda \).

Now we will show the uniqueness of the couple \((p_u, q_u)\). Indeed, it is sufficient to show that if \( u \in \mathcal{N}_\lambda \) and \( p_0u^+ + q_0u^- \in \mathcal{N}_\lambda \) with \( p_0 > 0 \) and \( q_0 > 0 \), then \((p_0, q_0) = (1, 1)\). Let us assume that \( u \in \mathcal{N}_\lambda \) and \( p_0u^+ + q_0u^- \in \mathcal{N}_\lambda \). We will then get \( \langle \mathcal{J}'_\lambda(p_0u^+ + q_0u^-), p_0u^+ \rangle = 0, \langle \mathcal{J}'_\lambda(p_0u^+ + q_0u^-), p_0u^- \rangle = 0, \) and \( \langle \mathcal{J}'_\lambda(u), u^+ \rangle = 0 \), that is,
\[
\begin{align*}
\|p_0u^+\|^2 &= \lambda \int_B f(x, p_0u^+)p_0u^+ \, dx \\
\|q_0u^-\|^2 &= \lambda \int_B f(x, q_0u^-)q_0u^- \, dx \\
\|u^+\|^2 &= \lambda \int_B f(x, u^+)u^+ \, dx \\
\|u^-\|^2 &= \lambda \int_B f(x, u^-)u^- \, dx
\end{align*}
\]

Combining (3.8) and (3.10), we deduce that
\[
0 = \lambda \int_B \frac{f(x, p_0u^+)p_0u^+}{p_0^2} \, dx - \lambda \int_B f(x, u^+)u^+ \, dx.
\]

It follows from \((F_4)\) that \( t \mapsto \frac{f(x, t)}{t} \) is increasing for \( t > 0 \), which implies that \( p_0 = 1 \). We can also show, using \((F_4), (3.9)\) and \((3.11)\), that \( q_0 = 1 \).

This completes the proof of (i).

(ii) To prove (ii), it is sufficient to show that \((p_u, q_u)\) is the unique maximum point of \( \Upsilon_u \in (0, \infty) \times (0, \infty) \). From (3.7), (3.8) and \( \theta > 2 \), we have
\[
\begin{align*}
\Upsilon_u(p, q) &= \mathcal{J}_\lambda(pu^+ + qu^-) \\
&= \frac{1}{2}\|pu^+ + qu^-\|^2 - \lambda \int_B F(x, pu^+ + qu^-) \, dx \\
&\leq \frac{p^2}{2}\|u^+\|^2 + \frac{q^2}{2}\|u^-\|^2 - \lambda C_5p^\theta \int_B |u^+|^\theta dx - \lambda C_5q^\theta \int_B |u^-|^\theta dx + \lambda C_6|B|
\end{align*}
\]
which implies that \( \lim\limits_{|(p,q)| \to \infty} \Upsilon_u(p,q) = -\infty \). Hence, it suffices to see that the maximum point of \( \Upsilon_u \) cannot be realized on the boundary of \([0, \infty) \times [0, \infty)\).

We argue by contradiction and assume that \((0, q)\) with \(q \geq 0\) is a maximum point of \(\Upsilon_u\). Then from (3.5), we have

\[
p \frac{d}{dp} [J_\lambda(pu^+ + qu^-)] = \langle J_\lambda'(pu^+) , pu^+ \rangle > 0,
\]

for small \(p > 0\), which means that \(\Upsilon_u\) is increasing with respect to \(p\) if \(p > 0\) is small enough. This gives a contradiction. We can similarly deduce that \(\Upsilon_u\) cannot realize its global maximum on \((p, 0)\) with \(p \geq 0\).

**Lemma 6.** For any \(u \in E\) with \(u^+ \neq 0\) and \(u^- \neq 0\), such that \(\langle J_\lambda'(pu^+, pu^+) \rangle \leq 0\), the unique maximum point \((p_u, q_u)\) of \(\Upsilon_u\) on \([0, \infty) \times [0, \infty)\) belongs to \((0, 1] \times (0, 1]\).

Proof. Here we will only prove that \(0 < p_u \leq 1\). The proof of \(0 < q_u \leq 1\) is similar. Since \(pu^+, q_uu^- \in \mathcal{N}_\lambda\), we have that

\[
\|pu^+\|^2 = \lambda \int_B f(x, pu^+) u^+ dx
\]

Moreover, by \(\langle J_\lambda'(pu^+, pu^+) \rangle \leq 0\), we have that

\[
\|u^+\|^2 \leq \lambda \int_B f(x, u^+) u^+ dx.
\]

Combining (3.12) and (3.13), it follows that

\[
\int_B f(x, u^+) u^+ dx \geq \int_B \frac{f(x, pu^+) p_u u^+}{p_u^2} dx.
\]

Now, we suppose, by contradiction, that \(p_u > 1\). By \((F_3)\), \(t \mapsto \frac{f(x, t)}{t}\) is increasing for \(t > 0\), which contradicts inequality (3.14). Therefore, \(0 < p_u \leq 1\).

**Lemma 7.** For all \(u \in \mathcal{N}_\lambda\),

(i) there exists \(\kappa > 0\) such that

\[\|u^+\|, \|u^-\| \geq \kappa;\]

(ii) \(J_\lambda(u) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2\)

Proof. (i) We argue by contradiction. Suppose that there exists a sequence \(\{u_n^+\} \subset \mathcal{N}_\lambda\) such that \(u_n^+ \to 0\) in \(E\). Since \(\{u_n\} \subset \mathcal{N}_\lambda\), then
\begin{equation}
\langle J'_\lambda(u_n), u_n^+ \rangle = 0. \text{ Hence, it follows from (3.3), (3.4) and the radial Lemma 2 that}
\end{equation}

\begin{equation}
\|u_n^+\|^2 = \lambda \int_B f(u_n^+)u_n^+dx
\leq \epsilon \lambda \int_B |u_n^+|^2dx + \lambda C_1 \int_B |u_n^+|^s \exp(\alpha |u_n^+|^\gamma)dx
\leq \epsilon C_6\|u_n^+\|^2 + \lambda C_1 \int_B |u_n^+|^s \exp(\alpha |u_n^+|^\gamma)dx
\end{equation}

Let \(a > 1\) with \(\frac{1}{a} + \frac{1}{a'} = 1\). Since \(u_n^+ \to 0\) in \(E\), for \(n\) large enough, we get \(\|u_n^+\| \leq (\frac{\alpha\beta}{\alpha a})^\frac{1}{\gamma}\). From Hölder inequality, (1.4) and again the radial Lemma 2, we have

\begin{align*}
\int_B |u_n^+|^s \exp(\alpha |u_n^+|^\gamma)dx &\leq \left( \int_B |u_n^+|^{sa'}dx \right)^{\frac{1}{a'}} \left( \int_B \exp(\alpha a\|u^+\|^\gamma(\frac{|u^+_n|}{\|u^+_n\|^\gamma})dx \right)^{\frac{1}{a}} \\
&\leq C_7 \left( \int_B |u_n^+|^{sa'}dx \right)^{\frac{1}{a'}} \leq C_8 \|u_n^+\|^s
\end{align*}

Combining (3.15) with the last inequality, for \(n\) large enough, we obtain

\begin{equation}
\|u_n^+\|^2 \leq \lambda \epsilon C_6\|u_n^+\|^2 + \lambda C_8\|u_n^+\|^s
\end{equation}

Choose suitable \(\epsilon > 0\) such that \(1 - \lambda \epsilon C_6 > 0\). Since \(2 < s\), then (3.16) contradicts the fact that \(u_n^+ \to 0\) in \(E\).

\(\text{(ii) Given } u \in \mathcal{N}_\lambda, \text{ by the definition of } \mathcal{N}_\lambda \text{ and } (F_3) \text{ we obtain}
\begin{align*}
J_\lambda(u) = J_\lambda(u) - \frac{1}{\theta} \langle J'_\lambda(u), u \rangle \\
= \frac{1}{2} \|u_n\|^2 + \lambda \left( \int_B \frac{1}{\theta} f(x, u)u - F(x, u)dx \right) \\
\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2
\end{align*}
\end{equation}

Lemma 7 implies that \(J_\lambda(u) > 0\) for all \(u \in \mathcal{N}_\lambda\). As a consequence, \(J_\lambda\) is bounded by below in \(\mathcal{N}_\lambda\), and therefore \(c_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u)\) is well-defined.

The following lemma deals with the asymptotic property of \(c_\lambda\).

\textbf{Lemma 8.} Let \(c_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u)\), then \(\lim_{\lambda \to \infty} c_\lambda = 0\)

\begin{proof}
Let us Fix \(u \in E\) with \(u^+ = 0\). Then, by Lemma 5, there exists a point pair \((p_\lambda, q_\lambda)\) such that \(p_\lambda u^+ + q_\lambda u^- \in \mathcal{N}_\lambda\) for each \(\lambda > 0\). Let \(\mathcal{T}_u\) be the set defined by

\(\mathcal{T}_u := \{(p_\lambda, q_\lambda) \in [0, \infty) \times [0, \infty) : L_u(p_\lambda, q_\lambda) = (0, 0), \lambda > 0\}\),
where \( L_u \) is given by (3.2).

Since \( p_\lambda u^+ + q_\lambda u^- \in N_\lambda \), by assumption (F2), (3.7) and (3.8), we have

\[
p_\lambda^2 \|u^+\|^2 + q_\lambda^2 \|u^-\|^2 = \lambda \int_B f(x, p_\lambda u^+ + q_\lambda u^-)(p_\lambda u^+ + q_\lambda u^-)dx
\geq \lambda \theta C_5 p_\lambda^\theta \int_B |u^+|^\theta dx + \lambda \theta C_5 q_\lambda^\theta \int_B |u^-|^\theta dx - \lambda \theta C_6 |B|.
\]

Since \( \theta > 2 \), the set \( T_u \) is bounded. Therefore, if \( \{\lambda_n\} \subset (0, \infty) \) satisfies \( \lambda_n \to \infty \) as \( n \to \infty \), then up to subsequence, there exists \( \bar{p}, \bar{q} > 0 \), such that \( p_{\lambda_n} \to \bar{p} \) and \( q_{\lambda_n} \to \bar{q} \).

We claim that \( \bar{p} = \bar{q} = 0 \). We proceed by contradiction and suppose that \( \bar{p} > 0 \) and \( \bar{q} > 0 \). For each \( n \in \mathbb{N} \), \( p_{\lambda_n} u^+ + q_{\lambda_n} u^- \in N_{\lambda_n} \). So,

\[
(3.17) \quad \|p_{\lambda_n} u^+ + q_{\lambda_n} u^-\|^2 = \lambda_n \int_B f(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)dx.
\]

It should be noted that \( p_{\lambda_n} u^+ \to \bar{p} u^+ \) and \( q_{\lambda_n} u^- \to \bar{q} u^- \) in \( E \).

On one hand, \( \lambda_n \to 0 \) as \( n \to \infty \) and \( \{p_{\lambda_n} u^+ + q_{\lambda_n} u^-\} \) is bounded in \( E \). On the other hand, from (3.17), we have

\[
\int_B |\Delta(\bar{p} u^+ + \bar{q} u^-)|^2 dx = \left( \lim_{n \to \infty} \lambda_n \right) \lim_{n \to \infty} \int_B f(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)dx
\]

which is impossible.

Thus, \( \bar{p} = \bar{q} = 0 \), so, \( p_{\lambda_n} \to 0 \) and \( q_{\lambda_n} \to 0 \) as \( n \to \infty \). Finally, by (F2) and (3.17), we have

\[ 0 \leq c_\lambda = \inf_{\mathcal{N}_\lambda} \mathcal{J}_\lambda (u) \leq \mathcal{J}_\lambda (p_{\lambda_n} u^+ + q_{\lambda_n} u^-) \to 0. \]

Consequently, \( c_\lambda \to 0 \) as \( \lambda \to \infty \).

**Lemma 9.** If \( u_0 \in \mathcal{N}_\lambda \) satisfies \( \mathcal{J}_\lambda (u_0) = c_\lambda \), then \( \mathcal{J}'_\lambda (u_0) = 0 \).

**Proof.** We proceed by contradiction. We assume that \( \mathcal{J}'_\lambda (u_0) \neq 0 \). By the continuity of \( \mathcal{J}'_\lambda \), there exists \( \iota, \delta \geq 0 \) such that

\[
(3.18) \quad \|\mathcal{J}'_\lambda (v)\|_{E^*} \geq \iota \text{ for all } \|v - u_0\| \leq 3\delta.
\]

Choose \( \tau \in (0, \min\left\{ \frac{1}{4}, \frac{\delta}{4\|u_0\|} \right\}) \). Let \( D = (1 - \tau, 1 + \tau) \times (1 - \tau, 1 + \tau) \) and define \( g : D \to E \), by

\[ g(\rho, \vartheta) = \rho u_0^+ + \vartheta u_0^-, (\rho, \vartheta) \in D \]
By virtue of $u_0 \in \mathcal{N}_\lambda$, $\mathcal{J}_\lambda(u_0) = c_\lambda$ and Lemma 5, it is easy to see that

$$
(3.19) \quad \bar{c}_\lambda := \max_{\partial D} \mathcal{J}_\lambda \circ g < c_\lambda.
$$

Let $\epsilon := \min\{\frac{c_\lambda - c_0}{3}, \frac{c_\lambda}{8}\}$, $S_r := B(u_0, r), r \geq 0$ and $\mathcal{J}_\lambda^{a} := \mathcal{J}_\lambda^{-1}([-\infty, a])$. According to the Quantitative Deformation Lemma [[27], Lemma 2.3], there exists a deformation $\eta \in C([0, 1] \times g(D), E)$ such that:

1. $\eta(1, v) = v$, if $v \not\in \mathcal{J}_\lambda^{-1}([c_\lambda - 2\epsilon, c_\lambda + 2\epsilon]) \cap S_{2\delta}$

2. $\eta(1, \mathcal{J}_\lambda^{c_\lambda + \epsilon} \cap S_{\delta}) \subset \mathcal{J}_\lambda^{c_\lambda - \epsilon}$,

3. $\mathcal{J}_\lambda(\eta(1, v)) \leq \mathcal{J}_\lambda(v)$, for all $v \in E$.

By lemma 5 (ii), we have $\mathcal{J}_\lambda(g(\rho, \vartheta)) \leq c_\lambda$. In addition, we have,

$$
\|g(s, t) - u_0\| = \|(\rho-1)u_0^+ + (\vartheta-1)u_0^-\| \leq |\rho-1|\|u_0^+\| + |\vartheta-1|\|u_0^-\| \leq 2\tau\|u_0\|,
$$

then $g(\rho, \vartheta) \in S_{\delta}$ for $(\rho, \vartheta) \in \bar{D}$. Therefore, it follows from (2) that

$$
(3.20) \quad \max_{(\rho, \vartheta) \in D} \mathcal{J}_\lambda(\eta(1, g(\rho, \vartheta))) \leq c_\lambda - \epsilon.
$$

In the following, we prove that $\eta(1, g(D)) \cap \mathcal{N}_\lambda$ is nonempty. And in this case it contradicts (3.20) due to the definition of $c_\lambda$. To do this, we first define

$$
\tilde{g}(\rho, \vartheta) := \eta(1, g(\rho, \vartheta)),
$$

$$
\Upsilon_0(\rho, \vartheta) = (\langle \mathcal{J}_\lambda'(g(\rho, \vartheta)), u_0^+ \rangle, \langle \mathcal{J}_\lambda'(g(\rho, \vartheta)), u_0^- \rangle)
$$

$$
= (\langle \mathcal{J}_\lambda'(\rho u_0^+ + \vartheta u_0^-), u_0^+ \rangle, \langle \mathcal{J}_\lambda'(\rho u_0^+ + \vartheta u_0^-), u_0^- \rangle)
$$

$$
:= (\varphi^1_{u_0}(\rho, \vartheta), \varphi^2_{u_0}(\rho, \vartheta))
$$

and

$$
\Upsilon_1(\rho, \vartheta) := \bigg(\frac{1}{\rho}\langle \mathcal{J}_\lambda'(\tilde{g}(\rho, \vartheta)), (\tilde{g}(\rho, \vartheta))^+ \rangle, \frac{1}{\vartheta}\langle \mathcal{J}_\lambda'(\tilde{g}(\rho, \vartheta)), (\tilde{g}(\rho, \vartheta))^+ \rangle\bigg).
$$

Moreover, a simple calculation, shows that

$$
\frac{\varphi^1_{u_0}(\rho, \vartheta)}{\partial \rho} \bigg|_{(1,1)} = \|u_0^+\|^2 - \lambda \int_B f'(x, u_0^+) |u_0^+|^2 dx
$$

$$
= \lambda \int_B f(u_0^+) u_0^+ dx - \lambda \int_B f'(x, u_0^+) |u_0^+|^2 dx
$$

and

$$
\frac{\varphi^1_{u_0}(\rho, \vartheta)}{\partial \vartheta} \bigg|_{(1,1)} = 0.
$$
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In the same manner,
\[ \frac{\varphi^2_{u_0}(\rho, \vartheta)}{\partial \rho} \bigg|_{(1,1)} = 0 \]
and
\[ \frac{\varphi^2_{u_0}(\rho, \vartheta)}{\partial \vartheta} \bigg|_{(1,1)} = \lambda \int_B f(x, u_0^-) u_0^- \, dx - \lambda \int_B f'(x, u_0^-) |u_0^-|^2 \, dx \]

Let
\[ J = \begin{pmatrix}
\varphi^1_{u_0}(\rho, \vartheta) \bigg|_{(1,1)} & \varphi^2_{u_0}(\rho, \vartheta) \bigg|_{(1,1)} \\
\varphi^1_{u_0}(\rho, \vartheta) \bigg|_{(1,1)} & \varphi^2_{u_0}(\rho, \vartheta) \bigg|_{(1,1)}
\end{pmatrix}. \]

Then we have \( \det J \neq 0 \). Therefore, the point \((0, 1)\) is the unique isolated zero of the \( C^1 \) function \( \Upsilon_0 \). By using the Brouwer’s degree in \( \mathbb{R}^2 \), we deduce that \( \deg(\Upsilon_0, D, 0) = 1 \).

Now, it follows from (3.20) and (1) that \( g(\rho, \vartheta) = \bar{g}(\rho, \vartheta) \) on \( \partial D \). For the boundary dependence of Brouwer’s degree (see [13], Theorem 4.5), there holds \( \deg(\Upsilon_1, D, 0) = \deg(\Upsilon_0, D, 0) = 1 \). Therefore, there exists some \((\bar{\rho}, \bar{\vartheta}) \in D\) such that
\[ \eta(1, g(\bar{\rho}, \bar{\vartheta})) \in N_\lambda. \]

This finish the proof of the Lemma.

**Lemma 10.** If \( \nu \) is a least energy sign-changing solution of problem \((P_\lambda)\), then \( \nu \) has exactly two nodal domains

**Proof.** Assume by contradiction that \( \nu = \nu_1 + \nu_2 + \nu_3 \) satisfies

\[ \nu_i \neq 0, i = 1, 2, 3, \nu_1 \geq 0, \nu_2 \leq 0, \text{ a.e. in } B \]
\[ B_1 \cap B_2 = \emptyset, B_1 := \{ x \in B : \nu_1(x) > 0 \}, B_2 := \{ x \in B : \nu_2(x) < 0 \} \]
\[ \nu_1 \bigg|_{B \setminus B_1 \cup B_2} = \nu_2 \bigg|_{B \setminus B_2 \cup B_1} = \nu_3 \bigg|_{B_1 \cup B_2} = 0, \]
and
\[ \langle \mathcal{J}'(\nu), \nu_i \rangle = 0 \text{ for } i = 1, 2, 3, \]

Let \( \nu = \nu_1 + \nu_2 \) and it is easy to see that \( \nu^+ = \nu_1, \nu^- = \nu_2 \) and \( \nu^\pm \neq 0 \).

From Lemma (5), it follows that there exists a unique couple \((p_\nu, q_\nu) \in [0, \infty) \times [0, \infty)\) such that \( p_\nu \nu_1 + q_\nu \nu_2 \in N_\lambda \). So, \( \mathcal{J}_\lambda(p_\nu \nu_1 + q_\nu \nu_2) \geq c_\lambda \).

Moreover, using (3.21), we obtain that \( \langle \mathcal{J}'(\nu), \nu^\pm \rangle = 0 \). Then, by Lemma
we have $0 < p_\nu, q_\nu \leq 1$.

Now, combining (3.21), $(F_3)$ and $(F_4)$, we have that

$$0 = \frac{1}{\theta} \langle J'_\lambda(v), v_3 \rangle = \frac{1}{\theta} \langle J'_\lambda(v_3), v_3 \rangle$$

$$< J'_\lambda(v_3) ,$$

and

$$c_\lambda \leq J_\lambda(p_\nu v_1 + q_\nu v_2)$$

$$= J_\lambda(p_\nu v_1 + q_\nu v_2) - \frac{1}{\theta} \langle J'_\lambda(p_\nu v_1 + q_\nu v_2), p_\nu v_1 + q_\nu v_2 \rangle$$

$$= \left( \frac{1}{2} - \frac{1}{\theta} \right) p_\nu^2 \|v_1\|^2 + \left( \frac{1}{2} - \frac{1}{\theta} \right) q_\nu^2 \|v_2\|^2$$

$$+ \lambda \int_B \left[ \frac{1}{\theta} f(x, p_\nu v_1)(p_\nu v_1) - F(x, p_\nu v_2) \right] dx + \lambda \int_B \left[ \frac{1}{\theta} f(x, q_\nu v_1)(q_\nu v_2) - F(x, q_\nu v_2) \right] dx$$

$$\leq J_\lambda(v_1 + v_2) - \frac{1}{\theta} \langle J'_\lambda(v_1 + v_2), v_1 + v_2 \rangle$$

$$= J_\lambda(v_1 + v_2) + \frac{1}{\theta} \langle J'_\lambda(v), v_3 \rangle$$

$$< J_\lambda(v_1 + v_2) + J_\lambda(v_3) = J_\lambda(v) = c_\lambda,$$

which is a contradiction. Therefore, $v_3 = 0$ and $v$ has exactly two nodal domains.

### 4 Proof of Theorem

**Lemma 11.** There exists $\lambda^* > 0$ such that if $\lambda \geq \lambda^*$, and $\{v_n\} \subset N_\lambda$ is a minimizing sequence for $c_\lambda$, then there exists some $v \in N_\lambda$ such that $J_\lambda(v) = c_\lambda$.

**Proof.** Let $\{v_n\} \subset N_\lambda$ be a sequence such that $\lim_{n \to \infty} J_\lambda(v_n) = c_\lambda$. We have

$$J_\lambda(v_n) \to c_\lambda \quad \text{and} \quad \langle J'_\lambda(v_n), \varphi \rangle \to 0, \forall \varphi \in E$$

that is

$$J_\lambda(v_n) = \frac{1}{2} \|v_n\|^2 - \int_B F(x, v_n) dx \to c_\lambda, \quad n \to +\infty$$

(4.1)

and

$$|\langle J'_\lambda(v_n), \varphi \rangle| = \left| \int_B \omega(x) \nabla v_n \cdot \Delta \varphi dx - \int_B f(x, v_n) \varphi dx \right| \leq \varepsilon_n \|\varphi\|,$$

(4.2)
for all $\varphi \in E$, where $\varepsilon_n \to 0$, as $n \to +\infty$.

By lemma 7, $v_n$ is bounded in $E$. Furthermore, we have from (4.2) and $(F_2)$, that

\begin{equation}
0 < \int_B f(x, u_n) u_n \leq C
\end{equation}

and

\begin{equation}
0 < \int_B F(x, u_n) \leq C.
\end{equation}

Since by Lemma 6, we have

\begin{equation}
f(x, u_n) \to f(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty,
\end{equation}

then, it follows from $(H_2)$ and the generalized Lebesgue dominated convergence Theorem that

\begin{equation}
F(x, u_n) \to F(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty.
\end{equation}

We have that, up to a subsequence,

\begin{equation}
\begin{aligned}
v_n &\to v \text{ in } E, \\
v_n &\to v \text{ in } L^t(B) \text{ for } t \in [1, \infty), \\
v_n &\to v \text{ a.e. in } B, \\
v_n^\pm &\to v^\pm \text{ in } E, \\
v_n^\pm &\to v^\pm \text{ in } L^t(B) \text{ for } t \in [1, \infty), \\
v_n^\pm &\to v^\pm \text{ a.e. in } B
\end{aligned}
\end{equation}

for some $v \in E$.

Noticing that, according to lemma 8, there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, we get

$$c_\lambda < \frac{1}{2} \left( \frac{\alpha_\beta}{\alpha_0} \right)^2.$$

In the sequel, the results that are valid for $v_n$ and $v$, are also valid for $v_n^\pm$ and $v^\pm$. Next, we are going to make some Claims.

**Claim 1.** By the definition of the weak convergence, we get $\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$. Then, passing to the limit in (4.2) and using (4.4),we obtain that $v$ is a weak solution of the problem (1.1) that is

$$\int_B (w(x) \Delta u \Delta \varphi) \, dx = \int_B f(x, u) \varphi \, dx, \quad \text{for all } \varphi \in E.$$
Claim 2. $v^+ \neq 0$ and $v^- \neq 0$. Suppose, by contradiction, $v^+ = 0$. Therefore, $$\int_B F(x, v_n)dx \to 0$$ and consequently we get
\begin{equation}
\frac{1}{2}\|v_n\|^2 \to c_\lambda < \frac{1}{2}(\frac{\alpha_\beta}{\alpha_0})^{\frac{2}{\gamma}}.
\end{equation}

First, we claim that there exists $q > 1$ such that
\begin{equation}
\int_B |f(x, v_n)|^q dx \leq C.
\end{equation}

By (4.2), we have
\begin{equation*}
\left|\|v_n\|^2 - \int_B f(x, v_n)v_n dx\right| \leq C\varepsilon_n.
\end{equation*}
So
\begin{equation*}
\|v_n\|^2 \leq C\varepsilon_n + \left(\int_B |f(x, v_n)|^q dx\right)^{\frac{1}{q}}\left(\int_B |v_n|^q' dx\right)^{\frac{1}{q'}},
\end{equation*}
where $q'$ is the conjugate of $q$. Since $(v_n)$ converge to 0 in $L^q(B)$
\begin{equation*}
\lim_{n \to +\infty} \|v_n\|^2 = 0.
\end{equation*}
According to Lemma 7, this result cannot occur. Now for the proof of the claim (4.9), since $f$ has critical growth, for every $\varepsilon > 0$ and $q > 1$ there exists $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\varepsilon$, we have
\begin{equation*}
|f(x, t)|^q \leq Ce^{\alpha_0(\varepsilon + 1)t^\gamma}.
\end{equation*}
Consequently,
\begin{equation*}
\int_B |f(x, v_n)|^q dx = \int_{\{|v_n| \leq t_\varepsilon\}} |f(x, v_n)|^q dx + \int_{\{|v_n| > t_\varepsilon\}} |f(x, v_n)|^q dx \leq \frac{\pi^2}{2} \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C \int_B e^{\alpha_0(\varepsilon + 1)|v_n|^\gamma} dx.
\end{equation*}
Since $2c_\lambda < \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}$, there exists $\eta \in (0, \frac{1}{2})$ such that $2c_\lambda = (1 - \eta)\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}$.
On the other hand, $\|v_n\|^{\gamma} \to (2c_\lambda)^{\frac{\gamma}{2}}$, so there exists $n_\eta > 0$ such that for all $n \geq n_\eta$, we get $\|v_n\|^{\gamma} \leq (1 - \eta)\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}$. Therefore,
\begin{equation*}
\alpha_0(1 + \varepsilon)(\frac{|v_n|}{\|v_n\|})^{\gamma}\|v_n\|^{\gamma} \leq (1 + \varepsilon)(1 - \eta)\alpha_\beta.
\end{equation*}
We choose $\varepsilon > 0$ small enough to get
\begin{equation*}
\alpha_0(1 + \varepsilon)\|v_n\|^{\gamma} \leq \alpha_\beta.
\end{equation*}
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So, the second integral is uniformly bounded in view of (1.4) and the claim is proved.

Since \((v_n)\) is bounded, up to a subsequence, we can assume that \(\|v_n\| \to \rho > 0\). We affirm that \(J_\lambda(v) = c_\lambda\). Indeed, by \((F_2)\) and claim 2, we have

\[
(4.10) \quad J_\lambda(v) = \frac{1}{2} \int_B [f(x,v)v - 2F(x,v)]dx \geq 0.
\]

Now, using the lower semi continuity of the norm and (4.5), we get,

\[
J_\lambda(v) \leq \frac{1}{2} \liminf_{n \to +\infty} \|v_n\|^2 - \int_B F(x,v)dx = c_\lambda.
\]

Suppose that

\[
J_\lambda(v) < c_\lambda.
\]

Then

\[
(4.11) \quad \|v\|^2 < \rho^2.
\]

In addition,

\[
(4.12) \quad \frac{1}{2} \lim_{n \to +\infty} \|v_n\|^2 = (c_\lambda + \int_B F(x,v)dx),
\]

which means that

\[
\rho^2 = 2\left(c_\lambda + \int_B F(x,v)dx\right).
\]

Set

\[
u_n = \frac{v_n}{\|v_n\|} \quad \text{and} \quad u = \frac{v}{\rho}.
\]

We have \(\|u_n\| = 1\), \(u_n \to u\) in \(E\), \(u \neq 0\) and \(\|u\| < 1\). So, by Lemma 3, we get

\[
\sup_n \int_B e^{p \frac{\alpha_N}{\beta}\|u_n\|^\gamma}dx < +\infty,
\]

provided \(1 < p < \left(1 - \|u\|^2\right)^{-\frac{\gamma}{2}}\).

By (4.5) and (4.12), we have the following equality

\[
2c_\lambda - 2J_\lambda(v) = \rho^2 - \|v\|^2.
\]

From (4.10) and the last equality, we obtain

\[
(4.13) \quad \rho^2 \leq 2c_\lambda + \|v\|^2 < \left(\frac{\alpha_\beta}{\alpha_0}\right)^\frac{2}{\gamma} + \|v\|^2.
\]
Since
\[ \rho^{\gamma} = \left( \frac{\rho^2 - \|v\|^2}{1 - \|u\|^2} \right)^{\frac{1}{(1-\beta)}}, \]
we deduce from (4.13) that
\[ (4.14) \quad \rho^{\gamma} < \left( \frac{\left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}}{1 - \|u\|^2} \right)^{\frac{1}{(1-\beta)}}. \]

On one hand, we have this estimate \( \int_B |f(x, \upsilon_n)|^q dx < C \). Indeed, since \( f \) has critical growth, for every \( \varepsilon > 0 \) and \( q > 1 \) there exists \( t_\varepsilon > 0 \) and \( C > 0 \) such that for all \( |t| \geq t_\varepsilon \), we have
\[ |f(x, t)|^q \leq Ce^{\alpha_0(\varepsilon + 1)t^{\gamma}}. \]
So,
\[
\int_B |f(x, \upsilon_n)|^q dx = \int_{\{|\upsilon_n| \leq t_\varepsilon\}} |f(x, \upsilon_n)|^q dx + \int_{\{|\upsilon_n| > t_\varepsilon\}} |f(x, \upsilon_n)|^q dx \leq \frac{\pi^2}{2} \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C \int_B e^{\alpha_0(\varepsilon + 1)|\upsilon_n|^{\gamma}} dx \leq C \int_B e^{\alpha_0(1+\varepsilon)|\upsilon_n|^{\gamma} \frac{\|\upsilon_n\|^{\gamma}}{\|\upsilon_n\|^{\gamma}}} dx \leq C,
\]
provided \( \alpha_0(1 + \varepsilon)^{\|\upsilon_n\|^{\gamma}} \leq p \alpha_\beta \) and \( 1 < p < U(u) = (1 - \|u\|^2)^{\frac{q}{2}} \).

From (4.14), there exists \( \delta \in (0, \frac{1}{2}) \) such that \( \rho^{\gamma} = (1 - 2\delta)\left( \frac{\alpha_\beta}{\alpha_0} \right)^{\frac{2}{\gamma}} (\frac{1}{1 - \|u\|^2})^{\frac{1}{(1-\beta)}} \).

Since \( \lim_{n \to +\infty} \|\upsilon_n\|^{\gamma} = \rho^{\gamma} \), then, for \( n \) large enough
\[ \alpha_0(1 + \varepsilon)^{\|\upsilon_n\|^{\gamma}} \leq (1 + \varepsilon)(1 - \delta) \quad \alpha_\beta \left( \frac{1}{1 - \|u\|^2} \right)^{\frac{\gamma}{2}}. \]

We choose \( \varepsilon > 0 \) small enough such that \( (1 + \varepsilon)(1 - \delta) < 1 \) which implies that
\[ \alpha_0(1 + \varepsilon)^{\|\upsilon_n\|^{\gamma}} < \alpha_\beta \left( \frac{1}{1 - \|u\|^2} \right)^{\frac{\gamma}{2}}. \]

Hence, the sequence \( (f(x, \upsilon_n)) \) is bounded in \( L^q, \; q > 1 \).

Using the Hölder inequality, we deduce that
\[
\left| \int_B f(x, \upsilon_n)(\upsilon_n - v) dx \right| \leq \left( \int_B |f(x, \upsilon_n)|^{q} dx \right)^{\frac{1}{q}} \left( \int_B |\upsilon_n - v|^{q'} dx \right)^{\frac{1}{q'}} \leq C \left( \int_B |\upsilon_n - v|^{q'} dx \right)^{\frac{1}{q'}} \to 0 \quad \text{as} \; n \to +\infty,
\]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \).

Since \( \langle J'_\lambda(v_n), (v_n - v) \rangle = o_n(1) \), it follows that
\[
\int_B (\omega(x) \Delta v_n (\Delta v_n - \nabla v)) dx \to 0.
\]

On the other side,
\[
\int_B \omega(x) \Delta v_n (\Delta v_n - \Delta v) dx = \|v_n\|^2 - \int_B \omega(x) \Delta v_n \Delta v dx.
\]

Passing to the limit in the last equality, we get
\[
\rho^2 - \|v\|^2 = 0,
\]
therefore \( \|v\| = \rho \). This is in contradiction with (4.9). Therefore, \( J_\lambda(v) = c_{\lambda} \). By Claim 1, \( J'_\lambda(v) = 0 \) and by Claim 2, \( v \neq 0 \).

**Proof of Theorem 1.3.** From Lemma 10 and Lemma 11, we deduce that \( v \) is a least energy sign-changing solution for problem \((P_\lambda)\) with exactly two nodal domains.

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