Algebra of symmetry operators for Klein-Gordon-Fock equation

V.V. Obukhov

Tomsk State Pedagogical University, 60 Kievskaya St., Tomsk, 634041, Russia.
e.mail: obukhov@tspu.edu.ru.
Tomsk State University of Control Systems and Radio Electronics, 36, Lenin Avenue, Tomsk, 634050, Russia

Abstract

All external electromagnetic fields in which the Klein-Gordon-Fock equation admits the first-order symmetry operators are found, provided that in the space-time \( V_4 \) a group of motion \( G_3 \) acts simply transitively on a non-null subspace of transitivity \( V_3 \). It is shown that in the case of a Riemannian space \( V_n \), in which the group \( G_r \) acts simply transitively, the algebra of symmetry operators of the \( n \)-dimensional Klein-Gordon-Fock equation in an external admissible electromagnetic field coincides with the algebra of operators of the group \( G_r \).

Keyword: Klein-Gordon-Fock equation, algebra of symmetry operators, theory of symmetry, separation of variables, linear partial differential equations.

1 Introduction

The Klein-Gordon-Fock equation describes the dynamics of a charged massive scalar particle interacting with an electromagnetic field. Recently, interest in the Klein-Gordon-Fock equation has grown significantly due to attempts to solve the problem of dark matter in the framework of the scalar-tensor theory. For the successful construction of a realistic model, it is especially important to have, if not an exact solution of the basic equation, then at least a reliable approximate solution. This possibility was provided by the method of complete separation of variables, which makes it possible to reduce the original equations in partial variables to systems of ordinary differential equations. In this case, obtaining approximate solutions is not a problem, while traditional approaches in the framework of perturbation theory may turn out to be ineffective. We should note that when finding all known exact solutions of the gravitational field equations (including self-consistent ones) on certain stages, there is always a separation of variables. The symmetry of the classical and quantum one-particle motion equations is closely related to the symmetry of the space itself. If space admits sets of geometric objects consisting of the Killing fields, then the motion equations also have symmetry operators defined by the same sets. From a physical point of view, sets of three geometric objects are of particular interest. In the Stackel spaces, such sets are called complete. They consist of mutually commuting Killing...
vector and tensor fields. Homogeneous spaces are other interesting types of spaces with three geometric objects. In them, the group of motion \( G_3 \) acts simply transitively on a non-null hypersurface of transitivity. The theory of complete separation of variables (or the theory of the Stackel spaces) is a consequence of the theory of symmetry. The complete separation of variables in the classical, and under certain conditions, in the quantum motion equations for a test particle is possible only in the Stackel space. The Stackel spaces were named after Paul Stackel, who gave the first example of such space [1]. Besides Stackel, useful and applicable contributions to the construction of the theory were made by Levi-Civita [2], Yarov-Yarovoy [3], and Shapovalov [4]-[6]. V. V. Shapovalov proved the main theorem of the theory of Stackel spaces. The theorem makes it possible to carry out an invariant partition of the of the Stackel metrics set into equivalence classes. This made it possible to solve classification problems separately for each type of space (to list all nonequivalent metrics and electromagnetic potentials).

In paper [7], the theory was generalized to the case of complex privileged coordinate systems. A more detailed description of the theory and a fairly detailed bibliography can be found in the works [8]-[10]. Shapovalov’s theorem is of great theoretical and applied relevance. In particular, the theorem made it possible to obtain and systematize all cases of complete separation of variables in one-particle equations of classical and quantum mechanics in flat space-time. A large number of papers have been devoted to the theory of complete separation of variables since its inception. Nevertheless, it can be considered complete only for the free Hamilton-Jacobi equation. After the publication of articles [10] - [13] the classification problem of complete separation of variables for the Hamilton-Jacobi equation in an external electromagnetic field has been solved. However, even for the free Klein-Gordon-Fock equation, the problem of constructing and classifying the Stackel spaces is still topical, since the equations that define, according to Shapovalov’s theorem, the necessary and sufficient conditions for the complete separation of variables have not yet been solved in a general case. Moreover, they have not been solved for the free Dirac-Fock equation, as well as for all quantum equations of motion in the external fields of a gauge nature. Only isolated results have been obtained. For example, the problem of complete separation of variables in the Klein-Gordon-Fock equation has been solved for the Einstein spaces and for vacuum solutions of the Einstein equations (see. [14]-[17]). In the papers [18]-[20] intersections of sets of homogeneous spaces and the Stackel spaces have been considered. Due to the high level of symmetry of the Stackel spaces, they remain interesting objects for research in various branches of theoretical and mathematical physics. An important direction is associated with the study of geometry and physics in spaces and fields that admit complete separation of variables in quantum equations of motion (see for, example, [21]-[25]).

Solutions of scalar equations are widely used in cosmology, including in the study of the of dark matter and dark energy problem (see, for example, [26].) Methods of the symmetry theory of are used to justify the choice of models of the extended gravity cosmology (see, for example, [27]), and to find realistic models - [28], [29], etc.

Let us note one more feature of the Stackel spaces. The presence vector and tensor Killing fields in space-time allows separating variables in the Einstein equation, since in a privileged coordinate system the metric contains only functions, arbitrarily depending on one of the non-privileged variables. The fact that the metric is given with the indicated arbitrariness makes it possible to reduce the problem of integrating field equations and equations of motion to the problem of solving functional equations and, as a consequence,
to the problem of integrating systems of ordinary differential equations. The same arbitrary nature is peculiar for space-time manifold $V_4$, when a group of motions $G_3$ acts simply transitive on a subspace $V_3$.

The gravitational equations for the space-time manifolds with such groups also admit the separation of variables, and the solution of the field equations is reduced to the classification of the energy-momentum tensor of matter admitted by a given geometry. The methods of the complete separation of variables theory in these spaces are generally inapplicable. For them, it is possible to use the method of linear partial differential equations integration, developed in [30], using non-commutative algebras of symmetry operators. The method made it possible to significantly expand the classification of external fields and Riemannian manifolds that admit the existence of exact solutions of the Klein-Gordon-Fock equation, and served as the foundation for the study of quantum effects in homogeneous spaces. In particular, in [31]-[33], a complete classification of spaces admitting a simply transitive action of the motions groups $G_4$ was obtained, provided that the Klein-Gordon-Fock equation is exactly solved by non-commutative integration methods. In [34] - [38], a similar problem was solved for Dirac-Fock equation.

In this article, a complete classification of admissible electromagnetic fields is carried out for the case when the groups of motions $G_3$ act simply transitively on the nonzero subspace $V_3$ of the space-time manifold $V_4$. All the corresponding electromagnetic fields for such groups are found. By admissible we mean fields for which the Klein-Gordon-Fock equation admits symmetry operators.

Let us note the following circumstance. As it was already noted, all external electromagnetic fields in which the Hamilton-Jacobi equation admits complete separation of variables for the test charge were found [10]-[13]. Thus, the problem under consideration can be viewed as an extension of the work, the final goal of which is to classify all admissible external electromagnetic fields, both with respect to the action of the symmetry operators of the complete set and with respect to the action of the operators of the group $G_r$.

2 Conditions for the existence of symmetry operators

Consider a Riemannian space $V_n$, in which a group $G_r$ acts simply transitively on a subspace $V_r$. Coordinate indexes of variables in the local coordinate system $[u^i]$ of the space $V_n$ will be denoted as follows: $i, j, k, l = 0, 1, \ldots n - 1$. The transitivity subspace where the group $G_r$ acts is given by the system of equations:

$$\psi^p(u^i) = \text{const}, \quad (p, q = 0, \ldots, n - r - 1).$$

In what follows, it is assumed that $\psi^p = u^p$. Then the transitivity surface is given by the condition $u^p = \text{const}$. The local coordinate system in the subspace $V_r$ will be denoted $[u^\alpha]$. Indexes $\alpha, \beta, \gamma, \sigma, \tau$ range from $n - r$ to $n - 1$.

There is a summation within the specified limits of index change on repeated upper and lower indexes. The subject of our study are the conditions for the existence of the algebra of first order symmetric operators (integrals of motion) of the classical and quantum motion equations for a charged scalar test particle in an external electromagnetic field.
2.1 Hamilton-Jacobi equation.

Let us consider the Hamilton-Jacobi equation for a charged test particle in an external electromagnetic field with the potential $A_i$:

$$g^{ij}P_iP_j = m, \quad P_i = p_i + A_i, \quad p_i = \frac{\partial \varphi}{\partial x^i}. \quad (1)$$

It is commonly known that the first order integrals of motion of the free Hamilton-Jacobi equation are given by Killing vector fields $\xi^i$ and have the following form

$$Y_\alpha = \xi^i p_i. \quad (2)$$

Let us show that in case if the equation $(1)$ has $r$ independent first-order integrals of motion, these integrals have the form of $(2)$. We will try to find the solution of the motion integrals in the form:

$$\hat{Y}_\alpha = \zeta^i p_i. \quad (3)$$

The equation $(1)$ allows the motion integrals of the form $(3)$ if $H$ and $\hat{Y}_\alpha$ commute with respect to the Poisson brackets:

$$[H, \hat{Y}_\alpha]_P = \frac{\partial H}{\partial p_i} \frac{\partial \hat{Y}_\alpha}{\partial x^i} - \frac{\partial \hat{Y}_\alpha}{\partial x^i} \frac{\partial H}{\partial p_i} = (g^{ij}\zeta^j_{\alpha,i} + g^{il}\zeta^j_{\alpha,l} - g^{ij}\xi^l_{\alpha,l})P_iP_j + 2g^{ia}(\zeta^j_{\alpha} F_{ji} + (\zeta^j_{\alpha} A_{\beta})_i)P_\sigma = 0. \quad (4)$$

The functions $\zeta^j_{\alpha}$ satisfy the Killing equations:

$$g^{il}\zeta^j_{\alpha,l} + g^{ij}\zeta^l_{\alpha,j} - g^{ij}\zeta^j_{\alpha} = 0,$$

and therefore they coincide with the Killing vector field:

$$\zeta^j_{\alpha} = \xi^j_{\alpha}. \quad (5)$$

The coefficients before $P_iP_j$ in the equations $(4)$ must vanish. Therefore, from the equations $(4)$ it follows:

$$(\xi^j_{\alpha} A_j)_{,i} = \xi^j_{\alpha} F_{ij}. \quad (6)$$

In contrast to the free Hamilton-Jacobi equation, the equation $(1)$ in the general case has no integrals of motion. The system of equations $(5)$ defines the set of admissible electromagnetic fields. In these fields, the equation $(1)$ has $r$ first-order integrals of motion defined by the group $G_r$. The transitivity subspace where the group $G_r$ acts is given by the system of equations: We consider the following subsystem of the system $(5)$:

$$(\xi^j_{\alpha} A_j)_{,p} = \xi^j_{\alpha} F_{pj}. \quad (6)$$

Since in the chosen coordinate system $\xi^j_{\alpha,p} = 0$, from the equations $(5)$ it follows:

$$\xi^\beta_{\alpha} A_{p,\beta} = 0 \quad \rightarrow \quad A_p = A_p(u^q). \quad (7)$$

Thus, the components $A_p$ can be made zero by the gradient transformation of the potential. Further, we will select the calibration of the potential in exactly this way:

$$A_p = 0.$$
Then from (5) it follows:

\[(\xi^\beta A_\beta)_\gamma = \xi^\beta F_{\gamma\beta}.\]  

We prove, that (8) is compatible. Indeed the system (8) can be present in the form:

\[\hat{Y}_\alpha A_\beta = C^\gamma_{\alpha\beta} A_\gamma, \quad A_\beta = \xi^\alpha_{\beta} A_\alpha,\]  

Then the compatibility conditions can be transformed as follows:

\[(\delta^\sigma_\alpha [\hat{Y}_\beta \hat{Y}_\gamma] - C^\sigma_{\alpha\beta} \hat{Y}_\beta + C^\sigma_{\beta\alpha} \hat{Y}_\gamma) A_\sigma = (C^\sigma_{\alpha\beta} C^\sigma_{\gamma\tau} + C^\sigma_{\beta\gamma} C^\sigma_{\alpha\tau} + C^\sigma_{\gamma\alpha} C^\sigma_{\beta\tau}) A_\sigma = 0.\]

(The Bianchi identity is used.)

In [33] it was proved that the Hamilton-Jacobi and the Klein-Gordon-Fock equations admit a symmetry operator of the form

\[\hat{Y}_\alpha = \xi^i_\alpha (\hat{p}_i + A_i) + \gamma_\alpha.\]

if and only if the electromagnetic field satisfies the system of equations

\[\gamma_\alpha, i = \xi^J_\alpha F_{ji}.\]

By comparing the equations (5) and (6) one can show that:

\[\gamma_\alpha = -\xi^\beta_\alpha A_\beta.\]

Therefore the operator (10) takes the form (2).

### 2.2 Klein-Gordon-Fock equation

Let us consider the Klein-Gordon-Fock equation:

\[\hat{H} \varphi = (g^{ij} \hat{P}_i \hat{P}_j) \varphi = m \varphi, \quad \hat{P}_j = -i \hat{\nabla}_i + A_i.\]

\[\hat{\nabla}_i -\text{operator of the covariant derivative, with metric-compatible connectivity, corresponding to the operator of the partial derivative } - \quad \partial_i = i \hat{\nabla}_i \quad \text{with respect to the coordinate } u_i; \quad \varphi \quad \text{is a field of a scalar particle with mass } m.\]

We denote the Laplace-Beltrami operator as \(\hat{H}_0:\)

\[\hat{H}_0 = g^{ij} \hat{\nabla}_i \hat{\nabla}_j.\]

Then the operator \(\hat{H}\) can be presented as:

\[\hat{H} = -\hat{H}_0 + \hat{H}_c. \quad \hat{H}_c = 2A^i \hat{p}_i - i(\hat{\nabla}_i A^i) + A_i A_i.\]

Let us prove the

Statement.

Algebra of symmetry operators of the Klein-Gordon-Fock equation coincides with the Lie algebra of the group \(G_F\). That is the Klein-Gordon-Fock operator commutes with the operators

\[\hat{Y}_\alpha = \xi^i_\alpha \hat{p}_i,\]
for the admissible external electromagnetic field.

Indeed the commutator of operators $\hat{H}$ and $\hat{p}_i$ has the form:

$$[\hat{H}, \hat{Y}_i] = [\hat{H}_0, \hat{Y}_i] + [\hat{H}_e, \hat{Y}_i] = 0.$$  

As it is known,

$$[\hat{H}_0, \hat{Y}_i] = 0 \to g_{ij} \hat{\partial}_i \hat{\partial}_j + (g_{ij} + g_{ij} \chi_{ij} \hat{\partial}_i) = 0.$$  

(16)

We denoted here: $2 \chi_i = -g_{kl}g_{ij} \chi_{kl,i}$. Since the functions $\xi_i$ satisfy the Killing equations, from (16) we get the condition:

$$g_{il} \xi_{kl} - (g_{il} + g_{il} \chi_{il}) \xi_i = 0.$$  

(17)

Using the consequences from the Killing equations we get:

$$g^{il} \xi_{kl} + (g^{il} + g^{il} \chi_{il}) \xi_i = (g^{il} \chi_{il}) \xi_i = 0.$$  

(18)

The equation (17) is a consequence of the equations (18) and the Killing equations, because:

$$\xi_{i}^l + \chi_{i} \xi_{i}^l = 0.$$  

(18)

Let us consider the equation (19). Using the condition (18) the first term can be presented as:

$$\xi_{i}^l A^l_{i} \chi_{i}^l - \xi_{i}^l A^l_{i} \chi_{i}^l = (A^l_{i} \chi_{i}^l \xi_{i}^l + \chi_{i} \xi_{i}^l A^l_{i}) - \xi_{i}^l A^l_{i} \chi_{i} = 0.$$  

(19)

The equation (19) is a consequence of the equations (18) and the Killing equations, because:

$$\xi_{i}^l A^l_{i} \chi_{i}^l + \xi_{i}^l A^l_{i} \chi_{i} = (A^l_{i} \chi_{i}^l \xi_{i}^l + \chi_{i} \xi_{i}^l A^l_{i}) - \xi_{i}^l A^l_{i} \chi_{i} = 0.$$  

(20)

Thus only one condition (18) remains. The algebra exists, if and only if the admissible electromagnetic field exists.
The **Statement** is proved.

The integrals of motion of the Hamilton-Jacobi and the Klein-Gordon-Fock equations exist for the same electromagnetic fields and have the same form. In order to find the admissible electromagnetic fields, it is necessary to investigate the compatibility conditions of the system (6) having the form:

\[ \gamma_{\sigma,\alpha i} - \gamma_{\sigma,\alpha a} = \xi^\beta F_{i\alpha,\beta} + \xi^\beta_{\sigma,i} F_{\beta\alpha} - \xi^\beta_{\sigma,a} F_{\beta i} = 0. \]

The system (21) must be supplemented with the Bianchi identities. One can use the solutions \( F_{i\alpha} \) to find the potential of the admissible electromagnetic field by integrating the compatible system of equations:

\[ A_{\alpha,\beta} - A_{\beta,\alpha} = F_{\beta\alpha}. \]

We follow A. Z. Petrov [39]. This book contains all metrics of space-time manifolds in which the groups \( G_3(N) \) act. We will hold to the notation accepted in this book with exception - the non-ignored variables will be denoted by \( u^0 \). In addition for convenience we will use the notation \( u^i = u_i \). Functions that depend only one the variables \( u^0 = u_0 \) are denoted by lowercase Greek letters with a single right subscript. Examples: \( \alpha_0 = \alpha_0(u^0) \): \( \xi^\alpha = \xi^\alpha_0(u^0) \). Constants are denoted by lowercase Latin letters with the *tilde* symbol. Derivatives with respect to the variables \( u_0 \) are denoted by dots. Example: \( \dot{\alpha}_0 = \partial\alpha/\partial u_0 \).

### 3 Solvable groups \( G_3(N) \).

According to the Bianchi classification, depending on the set of structural constants \( C_{\alpha,\beta} \), there are 9 types of groups \( G_3(N) \). The first seven groups are solvable. Let us list them.

**Solvable groups**:

- \( G_3(I) \) : \( C_{\alpha,\beta} = 0; \)
- \( G_3(II) \) : \( C_{12}^\alpha = 0, C_{13}^\alpha = 0, C_{23}^\alpha = \delta_1^\alpha; \)
- \( G_3(III) \) : \( C_{12}^\alpha = 0, C_{13}^\alpha = \delta_1^\alpha, C_{23}^\alpha = 0; \)
- \( G_3(IV) \) : \( C_{12}^\alpha = 0, C_{13}^\alpha = \delta_1^\alpha, C_{23}^\alpha = \delta_1^\alpha + \delta_2^\alpha; \)
- \( G_3(V) \) : \( C_{12}^\alpha = 0, C_{13}^\alpha = \delta_1^\alpha, C_{23}^\alpha = \delta_2^\alpha; \)
- \( G_3(VI) \) : \( C_{12}^\alpha = 0, C_{13}^\alpha = \delta_1^\alpha, C_{23}^\alpha = q\delta_2^\alpha. \) \( (q \neq 0, 1); \)
- \( G_3(VII) \) : \( C_{12}^\alpha = 0, C_{13}^\alpha = \delta_1^\alpha, C_{23}^\alpha = 2\delta_2^\alpha \cos\alpha, \quad \alpha = \text{const.} \)

The groups \( G_3(I) - G_3(VII) \) contain the Abelian subgroup with the Killings vectors \( \xi_1 = \delta_1^\alpha, \xi_2 = \delta_2^\alpha \). Therefore from (21) and from the Bianchi identities, it follows:

\[ F_{ij,1} = F_{ij,2} = F_{12,3} = F_{12,0} = 0, \quad F_{13,0} + F_{01,3} = 0, \quad F_{23,0} + F_{02,3} = 0. \]

For the groups \( G(I) - G(VI) \) the functions \( \xi_3^\alpha, \xi_{3,i}^\alpha \) can be presented as the general formula:

\[ \xi_3^\alpha = (ku_1 + \varepsilon u_2)\delta_1^\alpha + nu_2\delta_2^\alpha - \delta_3^\alpha, \quad \xi_{3,i}^\alpha = (k\delta_{1i} + \varepsilon\delta_{2i})\delta_2^\alpha + n\delta_{1,2}\delta_2^\alpha. \]
Parameters: \( k, \varepsilon, n \) for each number \( N \) take values:

- \( N = I \) \( \rightarrow k = n = \varepsilon = 0 \),
- \( N = II \) \( \rightarrow k = n = 0, \varepsilon = 1 \),
- \( N = III \) \( \rightarrow k = 1, n = \varepsilon = 0 \),
- \( N = IV \) \( \rightarrow k = 1, n = \varepsilon = 1 \),
- \( N = V \) \( \rightarrow k = n = 1, \varepsilon = 0 \),
- \( N = VI \) \( \rightarrow k = 1, n = 2, \varepsilon = 0 \).

The equation system (21) together with the Bianchi identities has the form:

\[
\begin{align*}
F_{01,3} = k F_{01}, & \quad F_{02,3} = \varepsilon F_{01} + n F_{02}, & \quad F_{03,3} = 0, \\
F_{13,3} = k F_{13}, & \quad F_{12,3} = (k + n) F_{12}, & \quad F_{23,3} = \varepsilon F_{13} + n F_{23}, \\
F_{12,0} = F_{12,3} = 0, & \quad F_{01,3} + F_{13,0} = 0, & \quad F_{02,3} + F_{23,0} = 0.
\end{align*}
\]  

(25)

For the group \( G(VII) \) the functions \( \xi^\alpha, \xi^\alpha_{3,i} \) can be presented as following:

\[
\begin{align*}
\xi^\alpha_{3} &= -u_2 \delta^\alpha_1 + (2u_2 \cos \alpha + u_1) \delta^\alpha_2 + \delta^\alpha_3, \\
\xi^\alpha_{3,i} &= -\delta^\alpha_2 \delta^\alpha_1 + (2\delta^\alpha_2 \cos \alpha + \delta^\alpha_1) \delta^\alpha_2.
\end{align*}
\]  

(26)

The formulas (26) differ from ones listed in the book [39] in regard to the exchange:

\( q = 2 \cos \alpha, (\alpha = \text{const}) \).

Let us consider the equations (25) for each of the groups \( G_3(I) - G_3(VI) \).

### 3.1 The group \( G_3(I) \).

In this case \( \xi^\alpha_3 = \delta^\alpha_3 \). From (25) it follows:

\[
F_{ij,3} = 0, \quad F_{ij,0} = 0 \rightarrow F_{i0} = \beta_{i0}(u_0).
\]  

(27)

Potential \( A_i \) is derived from the system of differential equation:

\[
A_{j,i} - A_{j,i} = F_{ij},
\]  

(28)

and has the form:

\[
A_{\alpha} = \beta_{\alpha} + \tilde{c}_{\alpha\beta} u^\beta, \quad \tilde{c}_{\alpha\beta} = -\tilde{c}_{\beta\alpha}.
\]

Using the equations (21), one can show that \( \tilde{c}_{\alpha\beta} = 0 \).

### 3.2 Group \( G_3(II) \).

In this case \( k = n = 0, \varepsilon = 1 \) and the system of equation (25) have the form:

\[
\begin{align*}
F_{12,3} &= F_{12,0} = F_{13,3} = F_{03,3} = 0, \\
F_{02,3} &= F_{01}, & \quad F_{23,0} + F_{02,3} = 0, & \quad F_{23,3} + F_{13} = 0, \\
F_{13,0} + F_{01,3} = 0, & \quad F_{23,0} + F_{02,3} = 0.
\end{align*}
\]  

(29)

From the first equations if the system (29) it follows:

\[
F_{12} = \tilde{a}, \quad F_{03} = \gamma_0, \quad F_{13} = \tilde{b}.
\]
By placing these equations in the remaining equations of the system (30) we get:

\[ F_{01} = \dot{\alpha}_0, \quad F_{02} = \dot{\alpha}_0 u_3 + \dot{\beta}_0, \quad F_{23} = -\alpha_0 + 2\bar{b} u_3. \]  

(30)

A particular solution of the system (28) in the selected gauge has the form:

\[ A_1 = \alpha_0 - 2\bar{b} u_3, \quad A_2 = \alpha_0 u_3 + \beta_0 + \bar{a} u_1 - \bar{b} u_3^2, \quad A_3 = \gamma_0. \]  

(31)

Using the equations (9), one can show that \( \bar{a} = \bar{b} = 0 \).

### 3.3 Group \( G_3(III) \).

In this case \( k = 0, \quad n = \varepsilon = 0 \) and the systems of equations (25) have the form:

\[
\begin{aligned}
F_{03,3} &= F_{02,3} = F_{23,3} = F_{12,3} = F_{12,0} = 0; \\
F_{01,3} &= F_{01}, \quad F_{12,3} = F_{12}, \quad F_{13,3} = F_{13}; \\
F_{23,0} + F_{02,3} &= 0, \quad F_{13,0} + F_{01,3} = 0.
\end{aligned}
\]  

(32)

Hence we find the functions \( F_{ij} \):

\[ F_{01} = \dot{\alpha}_0 \exp u_3, \quad F_{02} = \dot{\beta}_0, \quad F_{03} = \dot{\gamma}_0, \]

\[ F_{12} = 0, \quad F_{13} = -\alpha_0 \exp u_3, \quad F_{23} = \bar{a}. \]

A particular solution of the system (28) in the selected gauge has the form:

\[ A_1 = \alpha_0 \exp u_3, \quad A_2 = \beta_0, \quad A_3 = \gamma_0 + \bar{a} u_2. \]  

(33)

Using the equations (9), one can show that \( \bar{a} = 0 \).

### 3.4 Group \( G_3(IV) \).

In this case \( k = n = \varepsilon = 1 \) and the systems of equations (25) have the form:

\[
\begin{aligned}
F_{01,3} &= F_{01}, \quad F_{02,3} = F_{01} + F_{02}, \quad F_{03,3} = 0, \\
F_{12,3} &= 2F_{12}, \quad F_{13,3} = F_{13}, \quad F_{23,3} = F_{23} + F_{13} = 0, \\
F_{13,0} + F_{01,3} &= 0, \quad F_{23,0} + F_{02,3} = 0, \quad F_{01,3} + F_{13,0} = 0, \quad F_{12,3} = 0.
\end{aligned}
\]  

(34)

From here it follows:

\[ F_{01} = \dot{\alpha}_0 \exp u_3, \quad F_{02} = (\dot{\alpha}_0 u_3 + \dot{\beta}_0) \exp u_3, \quad F_{03} = \dot{\gamma}_0, \]

\[ F_{12} = 0, \quad F_{13} = -\alpha_0 \exp u_3, \quad F_{23} = -(\alpha_0 u_3^2 + \alpha_0 + \beta_0) \exp u_3. \]

A particular solution of the system (28) in the selected gauge has the form:

\[ A_1 = \alpha_0 \exp u_3, \quad A_2 = (\alpha_0 u_3 + \beta_0) \exp u_3, \quad A_3 = \gamma_0. \]  

(35)
3.5 Group $G_3(V)$.

In this case $k = n = 1$, $\varepsilon = 0$, and the systems of equations (25) have the form:

$$
\begin{align*}
F_{01,3} &= F_{01}, \quad F_{02,3} = F_{02}, \quad F_{03,3} = 0; \\
F_{12,3} &= F_{12}, \quad F_{13,3} = F_{13}, \quad F_{23,3} = F_{23}; \\
F_{13,0} + F_{01,3} &= 0, \quad F_{23,0} + F_{02,3} = 0, \quad F_{12,3} = F_{12,0} = 0.
\end{align*}
$$

(36)

From here it follows:

$$
F_{01} = \alpha_0 \exp u_3, \quad F_{02} = \beta_0 \exp u_3, \quad F_{03} = \gamma_0,
$$

$$
F_{12} = 0, \quad F_{13} = -\alpha_0 \exp u_3, \quad F_{23} = -\beta_0 \exp u_3.
$$

A particular solution of the system (28) in the selected gauge has the form:

$$
A_1 = \alpha_0 \exp u_3, \quad A_2 = \beta_0 \exp u_3, \quad A_3 = \gamma_0.
$$

(37)

3.6 Group $G_3(VI)$.

In this case $k = 1$, $n = 2$, $\varepsilon = 0$ and the systems of equations (25) have the form:

$$
\begin{align*}
F_{01,3} &= F_{01}, \quad F_{02,3} = 2F_{02}, \quad F_{03,3} = 0; \\
F_{12,3} &= 3F_{12}, \quad F_{13,3} = F_{13}, \quad F_{23,3} = 2F_{23}; \\
F_{12,0} &= F_{12,3} = 0, \quad F_{13,0} + F_{01,3} = 0, \quad F_{23,0} + F_{02,3} = 0.
\end{align*}
$$

(38)

From here it follows:

$$
F_{01} = \alpha_0 \exp u_3, \quad F_{02} = \beta_0 \exp 2u_3, \quad F_{03} = \gamma_0,
$$

$$
F_{12} = 0, \quad F_{13} = -\alpha_0 \exp u_3, \quad F_{23} = -2\beta_0 \exp 2u_3.
$$

A particular solution of the system (26) in the selected gauge has the form:

$$
A_1 = \alpha_0 \exp u_3, \quad A_2 = \beta_0 \exp 2u_3, \quad A_3 = \gamma_0.
$$

(39)

3.7 Group $G_3(VII)$.

In this case the relations (24) occur, and the systems of equations (21) can be presented as:

$$
F_{\alpha,3} + \delta_{\alpha 1} F_{2\alpha} - \delta_{\alpha 1} F_{2\alpha} + \delta_{2\alpha} (2F_{2\alpha} \cos \alpha - F_{1\alpha}) - \delta_{2\alpha} (2F_{2\alpha} \cos \alpha - F_{1\alpha}) = 0.
$$

Hence, using the Bianchi identities as well, we obtain the following system of equations:

$$
\begin{align*}
F_{01,3} + F_{02} &= 0, \quad F_{02,3} + 2\cos \alpha F_{02} - F_{01} = 0, \quad F_{03,3} = 0; \\
F_{12} &= 0, \quad F_{13,3} + F_{23} = 0, \quad F_{23,3} + 2\cos \alpha F_{23} - F_{13} = 0, \\
F_{13,0} + F_{01,3} &= 0, \quad F_{23,0} + F_{02,3} = 0.
\end{align*}
$$

(40)

First of all, let us find the functions $F_{13}$, $F_{02}$:

$$
F_{13} = (\alpha_0 \sin (u_3 \sin \alpha) + \mu_0 \cos (u_3 \cos \alpha)) \exp (-u_3 \cos \alpha),
$$

$$
F_{02} = (\alpha_0 \sin (u_3 \sin \alpha) + \beta_0 \cos (u_3 \cos \alpha)) \exp (-u_3 \cos \alpha),
$$

10
By placing them into the system’s equations (40), we find the relation between the functions \(\alpha_0, \beta_0, \nu_0, \mu_0\):

\[
\nu_0 = \alpha_0, \quad \mu_0 = \beta_0
\]

following which, from the relations:

\[
F_{02} = -F_{01,3}, \quad F_{23} = -F_{13,3}, \quad F_{03} = 0,
\]

we define the functions \(F_{01}, F_{23}, F_{03}\):

\[
F_{23} = (\alpha_0 \sin (-\alpha + u_3 \sin \alpha) + \beta_0 \cos (-\alpha + u_3 \cos \alpha)) \exp (-u_3 \cos \alpha),
\]

\[
F_{01} = (\dot{\alpha}_0 \sin (\alpha + u_3 \sin \alpha) + \dot{\beta}_0 \cos (\alpha + u_3 \cos \alpha)) \exp (-u_3 \cos \alpha), \quad F_{03} = \gamma_0
\]

A particular solution of the system (28) in the selected gauge has the form:

\[
A_1 = (\alpha_0 \sin (\alpha + u_3 \sin \alpha) + \beta_0 \cos (\alpha + u_3 \cos \alpha)) \exp (-u_3 \cos \alpha), \quad A_2 = (\alpha_0 \sin (u_3 \sin \alpha) + \beta_0 \cos (u_3 \cos \alpha)) \exp (-u_3 \cos \alpha), \quad A_3 = \gamma_0.
\] (41)

4 Insolvable groups \(G_3(N)\).

Unsolvable groups \(G_3(VIII)\) and \(G_3(IX)\) do not contain the Abelian subgroups and have more complex algebraic structures:

\[
G_3(VIII) : \quad C^{\alpha}_{12} = \delta^\alpha_1, \quad C^{\alpha}_{13} = 2\delta^\alpha_2, \quad C^{\alpha}_{23} = -\delta^\alpha_3.
\]

\[
G_3(IX) : \quad C^{\alpha}_{12} = \delta^\alpha_3, \quad C^{\alpha}_{13} = -\delta^\alpha_2, \quad C^{\alpha}_{23} = \delta^\alpha_1.
\]

For both structures the Killing vector field \(\xi^1\) has the form:

\[
\xi^1 = \delta^2_2.
\]

Therefore from the system (21) when \(\sigma = 1\) it follows:

\[
F_{ij,2} = 0.
\]

Taking this condition into account, we consider the remaining equations of the system.

4.1 Group \(G_3(VIII)\).

Functions \(\xi^\alpha_2, \xi^\alpha_3\) and their derivatives have the form:

\[
\begin{cases}
\xi^\alpha_2 = u_2\delta^\alpha_2 + \delta^\alpha_3, & \xi^\alpha_{2,i} = \delta_{i2}\delta^\alpha_2, \\
\xi^\alpha_3 = \delta^\alpha_i \exp u_3 + \delta^\alpha_2 u_3^2 + 2u_2\delta^\alpha_3, & \xi^\alpha_{3,i} = \delta_{3i}\delta^\alpha_1 \exp u_3 + 2(u_2\delta^\alpha_2 + \delta^\alpha_3)\delta_{2i}.
\end{cases}
\] (42)

Because \(F_{ij,2} = 0\), the system of equations (21) splits into two

\[
\begin{cases}
F_{\alpha,\alpha} + (\delta_{3i}F_{1\alpha} - \delta_{3\alpha}F_{1i}) + 2(\delta_{2i}2F_{3\alpha} - \delta_{2\alpha}F_{3i}) \exp -u_3 = 0; \\
F_{\alpha,\beta} + \delta_{1i}F_{2\alpha} - \delta_{1\alpha}F_{2i} + \delta_{2i}(2F_{2\alpha} \cos \alpha - F_{1\alpha}) - \delta_{2\alpha}(2F_{2i} \cos \alpha - F_{1i}) = 0.
\end{cases}
\] (43)

Hence, using the Bianchi identities as well, we obtain the following system of equations:
From the system (47) we get:

\[
\begin{align*}
F_{12,3} + F_{12} &= 0, & F_{23,3} + F_{23} &= 0, & F_{02,3} + F_{02} &= 0; \\
F_{02,1} + 2F_{03} \exp -u_3 &= 0, & F_{03,1} + F_{01} &= 0; \\
F_{12,1} + 2F_{13} \exp -u_3 &= 0 & F_{23,1} + F_{12} &= 0; \\
F_{01,3} = F_{01,1} &= F_{13,3} = F_{13,1} = F_{03,3} = 0; \\
F_{12,0} + F_{02,1} &= 0, & F_{01,3} = F_{03,1} &= 0, \\
F_{02,3} + F_{23,0} &= 0, & F_{12,3,0} + F_{23,1} &= 0. \\
\end{align*}
\]

(44)

By integrating this system we get:

\[
\begin{align*}
F_{01} &= \alpha_0, & F_{02} &= (\alpha_0 u_1^2 + 2\beta_0 u_1 + \gamma_0) \exp -u_3, & F_{03} &= -(\alpha u_1 + \beta_0), \\
F_{12} &= 2(\alpha_0 u_1 + \beta_0) \exp -u_3, & F_{13} &= -\alpha_0, & F_{23} &= (\alpha_0 u_1^2 + 2\beta_0 u_1 + \gamma_0) \exp -u_3,
\end{align*}
\]

A solution of the system (20) in the selected gauge has the form:

\[
A_1 = \alpha_0, & A_2 = (\alpha_0 u_1^2 + 2\beta_0 u_1 + \gamma_0) \exp -u_3, & A_3 &= -(\alpha_0 u_1 + \beta_0).
\]

4.2 Group $G_3(I X)$.

Functions $\xi^a_2$, $\xi^a_3$ and their derivatives have the form:

\[
\begin{align*}
\xi^a_2 &= \delta^a_0 \cos u_2 + (\delta^a_3 - \delta^a_2 \cos u_1) \frac{\sin u_2}{\sin u_1}, & \xi^a_3 &= \frac{\partial \xi^a_2}{\partial u_2}; \\
\xi^a_2, \xi^a_3 &= \sin u_2(-\delta_2 \delta_1^a + \frac{A_{11} u_1^2}{\sin^2 u_2}(\delta_2^a - \delta_3^a \cos u_1)) + (\delta^a_3 - \delta^a_2 \cos u_1) \delta_2^a \cos u_2 \sin^2 u_1.
\end{align*}
\]

(45)

Because $F_{ij,2} = 0$, the system of equations (21) splits into two subsystems:

\[
\begin{align*}
\begin{cases}
F_{1\alpha,1} \sin u_1 + \delta_{2i}(F_{3\alpha} - F_{2\alpha} \cos u_1) - \delta_{2\alpha}(F_{3i} - F_{2i} \cos u_1) = 0; \\
F_{1\alpha,3} \sin u_1 + (\delta_{2\alpha} F_{1i} - \delta_{2i} F_{1\alpha}) \sin^2 u_1 + \delta_{1i}(F_{2\alpha} - F_{3\alpha} \cos u_1) - \delta_{1\alpha}(F_{2i} - F_{3i} \cos u_1) = 0;
\end{cases}
\end{align*}
\]

(46)

Hence, using the Bianchi identities as well, we obtain the following systems of equations:

\[
\begin{align*}
\begin{cases}
F_{01,1} &= F_{03,1} = F_{13,1} = F_{03,3} = 0, \\
F_{01,3} \sin u_1 + F_{02} - \cos u_1 F_{03} &= 0; \\
F_{02,3} - F_{01} \sin u_1 &= 0; \\
F_{12,3} \sin u_1 + F_{23} \cos u_1 &= 0; \\
F_{13,3} \sin u_1 + F_{23} &= 0; \\
F_{23,3} - F_{13} \sin u_1 &= 0.
\end{cases}
\end{align*}
\]

(47)

\[
\begin{align*}
\begin{cases}
F_{02,1} \sin u_1 - \cos u_1 F_{02} + F_{03} &= 0; \\
F_{12,1} \sin u_1 - \cos u_1 F_{12} + F_{13} &= 0; \\
F_{23,1} \sin u_1 - F_{23} \cos u_1 &= 0;
\end{cases}
\end{align*}
\]

(48)

\[
\begin{align*}
\begin{cases}
F_{12,3} + F_{23,1} &= 0; \\
F_{12,0} &= F_{02,1}; \\
F_{01,3} + F_{13,0} &= 0; \\
F_{02,3} + F_{23,0} &= 0.
\end{cases}
\end{align*}
\]

(49)

From the system (47) we get:

\[
\begin{align*}
F_{13} &= a_1 \sin u_3 + b_1 \cos u_3, & F_{23} &= -\sin u_1(a_1 \cos u_3 - b_1 \sin u_3), \\
F_{12} &= \cos u_1(a_1 \sin u_3 + b_1 \cos u_3) + c_1,
\end{align*}
\]

12
where \(a_1, b_1 c_1\) functions of variables \(u_0, u_1\). Using the Bianchi identities \(49\) and rest equations of the system \((47), (48)\), we get:

\[
a_1 = \alpha_0 \quad b_1 = \beta_0, \quad c_1 = 0.
\]

Final solution can be present in the form:

\[
\begin{align*}
F_{01} &= \gamma_0 + (\dot{\alpha}_0 \cos u_3 - \dot{\beta}_0 \sin u_3), \\
F_{02} &= (\dot{\alpha}_0 \sin u_3 + \dot{\beta}_0 \cos u_3) \sin u_1, \\
F_{03} &= 0, \\
F_{12} &= (\alpha_0 \sin u_3 + \beta_0 \cos u_3) \cos u_1, \\
F_{13} &= \alpha_0 \sin u_3 + \beta_0 \cos u_3, \\
F_{23} &= (-\alpha_0 \cos u_3 + \beta_0 \sin u_3) \sin u_1.
\end{align*}
\]

(50)

A particular solution of the system \((28)\) in the selected gauge has the form:

\[
A_3 = 0, \quad A_1 = \gamma_0 + (\alpha_0 \cos u_3 - \beta_0 \sin u_3), \quad A_2 = (\alpha_0 \sin u_3 + \beta_0 \cos u_3) \sin u_1.
\]

(51)

5 Conclusion.

In conclusion, we note some ways for using the obtained results.

1. The considered metrics define homogeneous spaces, due to which the results are of interest in cosmology, especially when studying the processes occurring in the early stages of the Universe evolution.

2. The found external admissible fields, due to the special symmetry of homogeneous spaces, make it possible to construct interaction models of the axion field with the electromagnetic field, which is of interest when studying the problem of dark matter.

3. The results can be used to obtain exact self-consistent solutions in the General Theory of Relativity, in the scalar-tensor theory of gravity, in the Vaidya problem, as well as in the integration of field equations in other gravitational theories.

4. The results can be used to construct a theory of non-commutative integration of quantum motion equations in a strong gravitational field in the presence of fields of a gauge nature.

6 Appendix

For the sake of convenience, we present all the obtained results. For each group, the metric, electromagnetic potential, and integrals of motion are given. The metrics were found in \(39\). We follow the notation used in this book. All functions \(a_{\alpha\beta}\) depend only on the variable \(u^0\):

\[
a_{\alpha\beta} = a_{\alpha\beta}(u^0).
\]

6.1 Group \(G_3(I)\).

1. Metrics:

\[
ds^2 = a_{\alpha\beta}du^\alpha du^\beta + edu^0, \quad e^2 = 1.
\]
2. Potential of the admissible electromagnetic field:

\[ A_0 = 0, \quad A_\alpha = \alpha \alpha, \quad \alpha = \alpha (u^0). \]

3. Integrals of motion:

\[ \hat{Y}_\alpha = \hat{p}_\alpha. \]

6.2 Group \( G_3(II) \).

1. Metrics:

\[
ds^2 = du^2 a_{11} + 2 du^1 du^2 (a_{12} + a_{11} u^3) + 2 du^1 du^3 a_{13} + \]
\[
du^2 (a_{22} + 2 a_{12} u^3 + a_{11} u^3) + 2 du^2 du^3 (a_{23} + a_{13} u^3) + \]
\[
+ du^3 a_{33} + e du^3, \quad e^2 = 1.
\]

2. Potential:

\[ A_0 = 0, \quad A_1 = \alpha_0, \quad A_2 = \alpha_0 u_3 + \beta_0, \quad A_3 = \gamma_0. \]

3. Integrals of motion:

\[ \hat{Y}_1 = \hat{p}_1, \quad \hat{Y}_2 = \hat{p}_2, \quad \hat{Y}_3 = u^2 \hat{p}_1 - \hat{p}_3. \]

6.3 Group \( G_3(III) \).

1. Metrics:

\[
ds^2 = du^2 a_{11} \exp 2 u^3 + 2 du^1 du^2 a_{12} \exp u^3 + 2 du^1 du^3 a_{13} \exp u^3 + \]
\[
2 du^2 du^3 a_{23} + du^2 a_{22} + du^3 a_{33} + e du^3, \quad e^2 = 1.
\]

2. Potential of the admissible electromagnetic field:

\[ A_0 = 0, \quad A_1 = \alpha_0 \exp u^3, \quad A_2 = \beta_0, \quad A_3 = \gamma_0. \]

3. Integrals of motion:

\[ \hat{Y}_1 = \hat{p}_1, \quad \hat{Y}_2 = \hat{p}_2, \quad \hat{Y}_3 = u^2 \hat{p}_1 - \hat{p}_3 \]

6.4 Group \( G_3(IV) \).

1. Metrics:

\[
ds^2 = du^2 a_{11} \exp 2 u^3 + 2 du^1 du^2 (a_{12} + a_{11} u^3) \exp 2 u^3 + 2 du^1 du^3 a_{13} \exp u^3 + \]
\[
2 du^2 du^3 (a_{23} + a_{13} u^3) \exp u^3 + du^2 (a_{22} + 2 a_{12} u^3 + a_{11} u^3) \exp 2 u^3 \]
\[
+ du^3 a_{33} + e du^3, \quad e^2 = 1.
\]

2. Potential of the admissible electromagnetic field:

\[ A_0 = 0, \quad A_1 = \alpha_0 \exp u^3, \quad A_2 = (\alpha_0 u^3 + \beta_0) \exp u^3, \quad A_3 = \gamma_0. \]

3. Integrals of motion:

\[ \hat{Y}_1 = \hat{p}_1, \quad \hat{Y}_2 = \hat{p}_2, \quad \hat{Y}_3 = (u^2 + u^1) \hat{p}_1 + u^2 \hat{p}_2 - \hat{p}_3. \]
6.5 Group $G_3(V)$.

1. Metrics:

$$ds^2 = du^2 a_{11} \exp 2u^3 + 2du^2 a_{12} \exp 2u^3 + 2du^3 a_{13} \exp u^3 +$$

$$2du^2 a_{23} \exp 3u^3 + du^2 a_{22} \exp 2u^3$$

$$+ du^3 a_{33} + e du^{32}, \quad e^2 = 1.$$

2. Potential of the admissible electromagnetic field:

$$A_0 = 0, \quad A_1 = \alpha_0 \exp u^3, \quad A_2 = \beta_0 \exp u^3, \quad A_3 = \gamma_0.$$

3. Integrals of motion:

$$\dot{Y}_1 = \hat{p}_1, \quad \dot{Y}_2 = \hat{p}_2, \quad \dot{Y}_3 = u^1 \hat{p}_1 + u^2 \hat{p}_2 - \hat{p}_3.$$

6.6 Group $G_3(VI)$.

1. Metrics:

$$ds^2 = du^2 a_{11} \exp 2u^3 + 2du^3 a_{12} \exp 3u^3 + 2du^3 a_{13} \exp u^3 +$$

$$2du^2 a_{23} \exp 2u^3 + du^{32} a_{22} \exp 4u^3$$

$$+ du^3 a_{33} + e du^{32}, \quad e^2 = 1.$$

2. Potential of the admissible electromagnetic field:

$$A_0 = 0, \quad A_1 = \alpha_0 \exp u^3, \quad A_2 = \beta_0 \exp 2u^3, \quad A_3 = \gamma_0.$$

3. Integrals of motion:

$$\dot{Y}_1 = \hat{p}_1, \quad \dot{Y}_2 = \hat{p}_2, \quad \dot{Y}_3 = u^1 \hat{p}_1 + 2u^2 \hat{p}_2 - \hat{p}_3.$$

6.7 Group $G_3(VII)$.

1. Metrics:

$$ds^2 = du^2 [a_{11} + a_{12} \cos (2u^3 \sin \alpha) + a_{22} \sin (2u^3 \sin \alpha)] \exp (2u^3 \cos \alpha) + 2du^3 [a_{11} \cos \alpha + (a_{12} \cos \alpha +$$

$$+ a_{22} \sin \alpha) \cos (2u^3 \sin \alpha) + (a_{22} \cos \alpha - a_{12} \sin \alpha) \sin (2u^3 \sin \alpha)] \exp (2u^3 \cos \alpha) +$$

$$du^2 [(a_{11} + (a_{12} \cos 2\alpha + a_{22} \sin 2\alpha) \cos (2u^3 \sin \alpha) + (a_{22} \cos 2\alpha - a_{12} \sin 2\alpha) \sin (2u^3 \sin \alpha)] \exp (2u^3 \cos \alpha) +$$

$$2du^3 [(a_{11} \cos \alpha - a_{22} \sin \alpha) \cos (u^3 \sin \alpha) + (a_{22} \sin \alpha + a_{12} \cos \alpha) \sin (u^3 \sin \alpha)] \exp (u^3 \cos \alpha) +$$

$$2du^3 (a_{23} \sin (2u^3 \sin \alpha) + a_{13} \cos (2u^3 \sin \alpha)] \exp (u^3 \cos \alpha) + du^3 a_{33} + ed u^3, \quad e^2 = 1.$$

2. Potential of the admissible electromagnetic field:

$$A_0 = 0, \quad A_1 = (\alpha_0 \sin (\alpha + u^3 \sin \alpha) + \beta_0 \cos (\alpha + u^3 \cos \alpha)) \exp (-u^3 \cos \alpha),$$

$$A_2 = (\alpha_0 \sin (u^3 \sin \alpha) + \beta_0 \cos (u^3 \cos \alpha)) \exp (-u^3 \cos \alpha), \quad A_3 = \gamma_0.$$

3. Integrals of motion:

$$\dot{Y}_1 = \hat{p}_1, \quad \dot{Y}_2 = \hat{p}_2, \quad \dot{Y}_3 = -u^2 \hat{p}_1 + (2u^2 \cos \alpha + u^1) \hat{p}_2 + \hat{p}_3.$$
6.8 Group $G_3(VIII)$.  
1. Metrics: 
\[
ds^2 = du^2 a_{11} + 2du^1 du^2(a_{11} u^1 - 2a_{13} u^1 + a_{12}) \exp -u^3 + 2du^1 du^3(a_{13} - a_{11} u^1) +
\]
\[
du^2 [a_{22} - 4a_{23} u^1 + 2(a_{12} + 2a_{33})u_1^2 - 4a_{13} u^1 + a_{11} u_1^4] \exp -2u^3
\]
\[
+ 2du^2 du^3 |a_{23} - (a_{12} + 2a_{33})u_1 + 3a_{13} u^1 - a_{11} u^3| \exp -u^3
\]
\[
+ 2du^2 2(a_{11} u^1 - 2a_{13} u^1 + a_{33}) + e du^2, \quad e^2 = 1.
\]
2. Potential of the admissible electromagnetic field: 
\[
A_0 = 0, \quad A_1 = \alpha_0, \quad A_2 = (\alpha_0 u^2 + 2\beta_0 u^1 + \gamma_0) \exp -u^3, \quad A_3 = -(\alpha_0 u^1 + \beta_0).
\]
3. Integrals of motion: 
\[
\hat{Y}_1 = \hat{p}_2, \quad \hat{Y}_2 = u^2 \hat{p}_2 + \hat{p}_3, \quad \hat{Y}_3 = \hat{p}_1 \exp u^3 + \hat{p}_2 u^2 + 2u^2 \hat{p}_3
\]

6.9 Group $G_3(IX)$.  
1. Metrics: 
\[
ds^2 = du^1{}^2 [a_{11} - (2a_{12} \cos 2u^3 + a_{22} \sin 2u^3)] + 2du^1 du^3((a_{13} \cos u^3 - a_{12} \sin u^3) +
\]
\[
+ 2du^1 du^2 [(a_{13} \cos u^3 - a_{23} \sin u^3) \cos u^1 + (a_{12} \cos 2u^3 - a_{22} \sin 2u^3) \sin u^1]
\]
\[
+ du^2 [a_{33} \cos u^3 + (a_{13} \cos u^3 + a_{12} \sin u^3) \sin u^1 + (a_{12} \sin 2u^3 + a_{22} \cos 2u^3 + a_{11}) \sin u^1]
\]
\[
2du^2 du^3 (a_{33} \cos u_1 + (a_{23} \cos u^3 + a_{13} \sin u^3) \sin u^1) + du^2 a_{33} + e du^2.
\]
2. Potential of the admissible electromagnetic field: 
\[
A_0 = A_3 = 0, \quad A_1 = (\alpha_0 \cos u^3 - \beta_0 \sin u^3), \quad A_2 = (\alpha_0 \sin u^3 + \beta_0 \cos u^3) \sin u^1.
\]
3. Integrals of motion: 
\[
\hat{Y}_1 = \hat{p}_2, \quad \hat{Y}_2 = \hat{p}_1 \cos u^2 + (\hat{p}_3 - \hat{p}_2 \cos u^1) \frac{\sin u^2}{\sin u^1}, \quad \hat{Y}_3 = -\hat{p}_1 \sin u^2 + (\hat{p}_3 - \hat{p}_2 \cos u^1) \frac{\cos u^2}{\sin u^1}.
\]

This work was supported by Ministry of Science and High Education of Russian Federation, project FEWF-2020-0003.

References

[1] Stackel. Uber die integration der Hamiltonschen differentialgleichung mittels separation der variablen. Math. Ann. 1897, 49, (145-147 pp.);
[2] Levi-Civita T. Sulla Integrazione Della Equazione Di Hamilton-Jacobi Per Separazione Di Variabili. Math. Ann. 1904, 59, (383-397 pp.);
[3] Jarov-Jrovoy M.S. Integration of Hamilton-Jacobi equation by complete separation of variables method. J. Appl. Math. Mech. 1963, 27, No 6, (173-219 pp.);
[4] Shapovalov V.N., Symmetry of motion equations of free particle in riemannian space. Sov. Phys. J. 1975, 18, (1650-1654pp.); doi.org/10.1007/BF00892779;

[5] Shapovalov V.N., Symmetry and separation of variables in the Hamilton-Jacobi equation. Sov. Phys. J. 1978, 21, (1124-1132pp.). doi: 10.1007/BF00894560;

[6] Shapovalov V.N., Stackel’s spaces. Sib. Math. J. 1979, 20, (1117-1130pp.); doi: org/10.1007/BF00971844;

[7] Bagrov V. G., Obukhov V.V. Complete separation of variables in the free Hamilton-Jacobi equation, Theor. Math. Phys. 1993, 97, 2, (1275-1289pp.); doi: 10.1007/BF01016874;

[8] Benenti S., Separability in Riemannian Manifolds, SIGMA. 2016, 12, 013, (1-21 pp.); doi.org/10.3842/SIGMA.2016.013;

[9] W. Miller. Symmetry And Separation Of Variables. Cambridge University Press: Cambridge. 1984, (318 p.p.);

[10] Obukhov V.V. Hamilton-Jacobi equation for a charged test particle in the Stackel space of type (2.0). Symmetry, 2020, 12, (1289-1291 p.p.); arXiv:2007.09492 [gr-qc];

[11] Obukhov V.V. Integration of the Hamilton-Jacobi and Maxwell equations for Diagonal metrics. Russ. Phys. J. 2020, 63, N 7, (33-35 pp.); (Izv. Vuz. Fiz. 2020, 63, 7, (21)); doi:10.1007/s11182-020-02169-2;

[12] Obukhov V.V. Hamilton-Jacobi equation for a charged test particle in the Stackel space of type (2.1). Int.J.Geom.Meth.Mod.Phys. /bf 2020, 17, N 14, 2050186; doi: 10.1142/S0219887820501868;

[13] Obukhov V.V. Separation of variables in Hamilton-Jacobi and Klein-Gordon-Fock equations for a charged test particle in the stackel spaces of type (1.1). Int.J.Geom.Meth.Mod.Phys. 2021, 18, 3, (2150036); doi:10.1142/S0219887821500365, arXiv:2012.02548 gr-qc;

[14] Bagrov V. G. and Obukhov V. V. Separation of variables for the Klein-Gordon equation in special staeckel space-times, Class. Quant. Grav., 1990, 7, (19-25); doi: 10.1088/0264-9381/7/1/008;

[15] Bagrov V.G., Obukhov V.V., Shapovalov A.V. Special Stackel electrovac spacetimes. Pramana J. Phys. 1986, 26, 2, (93-108pp.); doi.org/10.1007/BF02847629;

[16] Bagrov V.G., Obukhov V.V. Classes of exact solutions of the Einstein-Maxwell equations, Ann. der Phys.. 1983, B 40, H 4/5, (181-188 pp.); doi:10.1002/andp.19834950402;

[17] Carter B. New family of Einstein spaces. Phys.Lett. 1968, A.25, No 9, (399-400 pp.); doi.org/10.1016/0375-9601(68)90240-5;

[18] E. Osetrin and K. Osetrin, Pure radiation in space-time models that admit integration of the eikonal equation by the separation of variables method. J. Math. Phys. 2017, 58, 11, 112504;

[19] K.E. Osetrin, A.E. Filippov, E.R. Osetrin. The spacetime models with dust matter that admit separation of variables in Hamilton-Jacobi equations of a test particle. Modern Physics Letters A. 2016,31, 3, (410);

[20] K.Osetrin and E.Osetrin. Shapovalov wave-like spacetimes.Symmetry. 2020, 12, 1372;
[21] Maharaj S.D., Goswami R., Chervon S. V. and Nikolaev A. V. Exact solutions for scalar field cosmology in f(R) gravity. *Modern Physics Letters A* Vol. 2017, 32, 30, 1750164 (18 pp.); doi.org/10.1142/S0217732317501644;

[22] Rajaratnam K., McLenaghan R.G., and Valero C. Orthogonal separation of the Hamilton-Jacobi equation on spaces of constant curvature. *SIGMA*, 2016, *em* 12, N 117, (30 pp.); doi.org/10.3842/SIGMA.2016.117;

[23] Rajaratnam K., McLenaghan R.G., Classification of Hamilton-Jacobi separation in orthogonal coordinates with diagonal curvature, *J. Math. Phys.* 2014, 55, 083521, (16 pp.); doi:10.1063/1.4893335;

[24] McLenaghan R. G., Rastelli G. and Valero C. Complete separability of the Hamilton-Jacobi equation for the charged particle orbits in a Lienard-Wiehert field *em J. Math. Phys. /bf 2020, 61*, (122903); doi.org/10.1063/5.0030305;

[25] Gray F., Houri T., Kubiznak D. and Yasui Y. Symmetry operators for the conformal wave equation. ArXiv 2101.06700v1 [gr - qc] 17 Jan 2021;

[26] Bamba K., S. Capozziello S., Nojiri S. and Odintsov S.D. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests, *Astrophys. Space Sci.* 2012, *em* 342, 342, (155 pp.); doi: 10.1007/s10509-012-1181-8;

[27] Capozziello S., De Laurentis M., Odintsov D. Hamiltonian dynamics and Noether symmetries in extended gravity cosmology. *Eur. Phys. J.* 2012, *C* 72, 2068 (22 pp.); doi: 10.1140/epjc/s10052-012-2068-0;

[28] Makarenko A. N., Obukhov V. V. Exact solutions in modified gravity models. *Entropy*. 2012, *em* 14, N 7, (1140-1153); doi: 10.3390/e14071140;

[29] Akdemir A.O., Butt S.I., Nadeem M., Ragusa M.A. New General Variants of Chebyshev Type Inequalities via Generalized Fractional Integral Operators. *Mathematics*. 2021, 9, N 2; (art.n. 122);

[30] Shapovalov, A.V., Shirikov I.V. Noncommutative integration method for linear partial differential equations. functional algebras and dimensional reduction. *Theoret. And Math. Phys.* 1996, *106:1*, (1-10 pp.);

[31] Magazev A. A., Shirikov I. V., Yu. A. Yurevich Yu. A. Integrable magnetic geodesic flows on Lie groups. *Theor. and math. phys.* 2008, *156*, N 2, (1127-1140); doi:10.4213/tmf6240;

[32] Magazev A. A. Constructing a complete integral of the hamilton-jacobi equation on pseudo-riemannian spaces with simply transitive groups of motions *Mathematical Physics Analysis and Geometry*. 2021, *24*(2):11; doi: 10.1007/s11040-021-09385-3;

[33] Magazev A.A. Integrating Klein-Gordon-Fock equations in an extremal electromagnetic field on Lie groups. *Theor. and Math.Phys.,* 2012, *173:3*, (1654-1667); arxiv.org/abs/1406.5698;

[34] Shapovalov A. V., Breev A. I. Symmetry operators and separation of variables in the (2 + 1)-dimensional Dirac equation with external electromagnetic field, *Int.J.Geom.Meth.Mod.Phys.* 2018, *em* 15, N 5, (1850085), (26 pp.); arXiv: math-ph/1709.04644;

[35] Shapovalov A. V., Breev A. I. The Dirac equation in an external electromagnetic field: symmetry algebra and exact integration, XXIII International Conference on Integrable Systems and Quantum Symmetries (ISQS-23) (Prague, 23-27
June 2015), *Journal of Physics: Conference Series*, **2016**, 670, (012015, 12 pp.); [http://iopscience.iop.org/1742-6596/670/1/012015](http://iopscience.iop.org/1742-6596/670/1/012015)

[36] Breev A. I. Shapovalov A. V. Noncommutative. Integrability of the Klein-Gordon and Dirac equation in (2+1)-dimensional spacetime, *Russ. Phys. J.* **2017**, 59, N 11, (1956-1961); link.springer.com/article/10.1007/s11182-017-1001-2;

[37] Shapovalov A. V., Breev, Non-commutative integration of the Dirac equation in homogeneous spaces, *Symmetry*, **2020**, 12, N 11, (1867); doi.org/10.3390/sym12111867, arXiv: math-ph/2011.06401;

[38] Makarenko A.N., Obukhov V.V., Osetrin K.E. Integrability of Einstein-Weyl equations for spatially homogeneous models of type III by Bianchi. *Russ. Phys. J.* **2002**, 45, 1, (49-55); doi: 10.1023/A:1016045704207;

[39] Petrov A. Z. Einstein Spaces, *Oxford*, **1969**. (*Russian Original Published By Nauka, Moscow, 1951*).