Quasinormal modes of Kerr-Newman black holes:
coupling of electromagnetic and gravitational perturbations

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We compute numerically the quasinormal modes of Kerr-Newman black holes in the scalar case, for which the perturbation equations are separable. Then we study different approximations to decouple electromagnetic and gravitational perturbations of the Kerr-Newman metric, computing the corresponding quasinormal modes. Our results suggest that the Teukolsky-like equation derived by Dudley and Finley gives a good approximation to the dynamics of a rotating charged black hole for \( Q \lesssim M/2 \). Though insufficient to deal with Kerr-Newman based models of elementary particles, the Dudley-Finley equation should be adequate for astrophysical applications.

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I. INTRODUCTION

The electrovacuum black hole solutions of the Einstein-Maxwell system can be uniquely described by the Kerr-Newman (KN) metric [1], which is the most general among classical black hole solutions. The KN metric is specified by three parameters: the black hole mass \( M \), the charge \( Q \) and the angular momentum per unit mass \( a \equiv J/M \). As long as \( M^2 \geq Q^2 + a^2 \) the KN metric describes a black hole, otherwise it has a naked ring-like singularity. As \( Q \rightarrow 0 \) the KN metric reduces to the rotating Kerr metric, and as \( a \rightarrow 0 \) it reduces to the Reissner-Nordstrøm (RN) metric. Both limits have been studied in great detail [2,3].

As early as 1968 Carter realized that the KN solution has a magnetic dipole moment corresponding to the same g-factor \( g = 2 \) as the electron [4]. This led to the suggestion (recently revisited in [5]: see discussion and references therein) that the KN metric could provide a reasonably adequate model for the external Einstein-Maxwell field of elementary particles [6,7].

For astrophysical black holes \( Q \) is likely to be negligible, electric charge being shorted out by the surrounding plasma [8]. If any charge is present, the no-hair theorem guarantees that charged rotating astrophysical black holes are described by the KN metric. Punsly proposed a gamma-ray burst model based on bipolar outflow from a fast-rotating black hole endowed with a small (by gravitational standards) charge and surrounded by a magnetosphere [9]. Van Putten suggested that hypernovae or black hole-neutron star coalescence may generate rapidly spinning black hole-torus systems, and that the spin energy of the hole could power gamma-ray bursts [10]. In his model the black hole-torus system has a gravitationally weak magnetic field as a result of the remnant flux from the progenitor star (a massive star in hypernovae or a neutron star for a coalescence remnant), and the interplay between rotational effects and magnetic fields may be relevant. Ruffini et al. noticed that pair creation induced by the Heisenberg-Euler-Schwinger vacuum polarization during collapse leading to formation of a charged black hole could explain the energetics of gamma-ray bursts, and proposed a detailed collapse model in a series of papers [11]. More recently, Araya-Gochez suggested that intermittent hyper-accretion produced by magnetorotational instabilities in the accretion disc of a rapidly rotating, newly born black hole may induce resonant excitation of the black hole’s quasinormal modes. His estimates suggest that a 15 \( M_{\odot} \) black hole spinning at \( a \simeq 0.98 M \) and located at 27 Mpc could produce gravitational waves detectable by LIGO II [12].

Both in elementary particle models and in astrophysical applications rotation plays a crucial role. In geometrical units, the typical angular momentum of an electron \( a \sim h/2m_e = 1.93 \times 10^{-34} \) cm, the length associated to the electron charge \( Q = e = 1.38 \times 10^{-34} \) cm and the corresponding mass scale \( M = m_e = 6.76 \times 10^{-56} \) cm, so that \( M \ll Q \ll a \) and the gravitational field is spin-dominated. In fact, the rotation parameter is so large that the KN metric can only model the Einstein-Maxwell field of the electron outside some small radius \( r_0 \) surrounding the ring-like naked singularity. Astrophysical black holes are also expected to be formed in rapid rotation: recent simulations suggest that supermassive stars would form black holes with \( a/M \sim 0.75 \), and many observations are consistent with black holes spinning close to the extremal limit \( a/M \sim 1 \). Reference [13] provides an updated discussion of the observational evidence for black hole spins and of the related uncertainties in present-day astrophysical models. Most black hole models for gamma-ray bursts require rapid rotation [9,10], and the magnetorotational instability considered in [12] is most efficient for large spin parameters, \( a/M \gtrsim 0.9 \).
The KN solution is the only asymptotically flat solution of the Einstein-Maxwell system for which the geodesic and Klein-Gordon equations can be solved by separation of variables [12]. The Dirac equation in the KN metric is also known to be separable [15]. Scalar and Dirac perturbations of a KN black hole can therefore be treated using the same general methods that apply to Kerr black holes. In particular, it is straightforward to compute the quasinormal modes (QNMs) of scalar perturbations of the KN black hole. As far as we know, a complete analysis of the corresponding QNM spectrum is still lacking. The first objective of this paper is to fill this gap, presenting a complete continued-fraction calculation of scalar QNMs of the KN metric for all values of $a$ and $Q$.

Studies of the interplay of electromagnetic (EM) and gravitational fields in the KN metric are plagued by a major technical difficulty: all attempts to decouple the EM and gravitational perturbations of the KN spacetime to date have failed. Section 111 of [2] gives an introduction to this long-standing unsolved problem. Dudley and Finley (10), henceforth DF; see also [17]) make a remarkable study of the separability of linear perturbations of the solutions of the Einstein-Maxwell equations found by Plebański and Demiański [18], which include all vacuum Type D solutions [19]. In their work Dudley and Finley “either keep the geometry fixed and perturb the electric field or, of more interest, keep the electric field fixed and perturb the geometry”. This approach should be appropriate for values of the charge $Q$ at most as large as the perturbations of the spacetime metric (in geometrized units). Dudley and Finley show that a sufficient condition for decoupling is that the spacetime be of Type D, and that the decoupled equations only separate (in Plebański-Demiański coordinates) for perturbing fields of spin $s = 0, 1/2, 1$ and 2.

Mashhoon [21] first presented arguments in favor of the stability of the KN metric. Instead of explicitly computing QNMs, he used an approximate argument (originally due to Goebel [21] and reviewed in the Appendix): in this sense his stability proof is not fully convincing. Mashhoon’s analysis is based on perturbations of test null rays in the unstable circular orbit of a KN black hole, and is strictly valid only in the eikonal limit $l \gg 1$. Here we show by a direct calculation that Mashhoon’s claim is correct, and that his predictions are surprisingly accurate even for small values of $l$. At least for scalar perturbations our investigation of the QNM spectrum can be considered conclusive, because for $s = 0$ the DF equation is exact.

One of us (KK) used the DF equation to compute the fundamental gravitational QNM using WKB methods [22]. The main problem of this approach is not computational, but physical. The WKB approximation is reasonably accurate for all values of $a$ and $Q$ (as we show comparing results to a continued-fraction calculation in Table I). However, the DF equation is derived under a number of mathematical assumptions, and it is only approximately valid for gravitational and EM perturbations of KN black holes.

A purpose of this paper is to clarify the physical range of validity of the DF equation and the physical meaning of their approximations. We first notice that the $a \to 0$ limit of the DF equation does not yield any of the two Schrödinger-like equations describing coupled EM-gravitational perturbations of the RN black hole. Then we consider metric perturbations of the RN black hole freezing EM perturbations (or, vice versa, EM perturbations freezing the metric), and we compute the associated QNMs. We show that for values of the charge $Q \lesssim M/2$ the results are in good (but not exact) agreement with the DF equation. In other words, the assumptions behind the separability conditions leading to the DF equation are not equivalent to simply freezing EM (or gravitational) perturbations. Nonetheless, the DF equation yields EM and gravitational QNMs in good quantitative agreement with results for the coupled EM-gravitational RN perturbation equations when $Q \lesssim M/2$. Furthermore, the qualitative behavior of gravitational and EM QNMs as functions of charge and angular momentum is very similar to the results of our “exact” calculations for scalar perturbations. Our calculations suggest that the DF equation provides a reasonable approximation of the EM and gravitational dynamics of a KN black hole, at least for the gravitationally small values of $Q$ expected in astrophysical applications.

The plan of the paper is as follows. In Sec. II we present the DF equation and we compute the corresponding QNM frequencies using Leaver’s continued fraction technique [23]. Our calculation is “exact” (in the sense that the perturbation equations involve no approximations) for $s = 0$: as far as we know, it represents the first example of numerical evidence for the stability of the KN metric to scalar perturbations. For EM and gravitational perturbations the RN ($a \to 0$) limit of the DF equation does not yield the standard RN perturbation equations, which are summarized for completeness in Sec. III. To clarify the meaning (and the limits of applicability) of the DF equation for EM and gravitational perturbations, in Sec. IV we consider approximate versions of the RN perturbation equations. To our knowledge, the “frozen” picture we present in this Section has never been studied before. To conclude we summarize our findings, discuss their meaning for concrete applications in astrophysics and elementary particle models, and list some open problems for future research. In the Appendix, for completeness, we summarize results of Ref. [21] on perturbed equatorial circular orbits of null rays in the KN metric and their relation with the QNMs in the eikonal limit.
Then we use the corresponding eigenvalue to look for zeros of the continued fraction as a function of \(\omega\), the angular separation constant \(E\) used in [22] is related to Leaver’s \(A_{lm}\) by \(E = A_{lm} + s(s+1)\). Correcting a typo in Eq. (5) of [22] (the factor \(4i s K' \) should read \(2i s K'\)), the DF equation reads

\[
\Delta^{-s} \frac{d}{dr} \left[ \Delta^{s+1} \frac{dR}{dr} \right] + \frac{1}{\Delta} \left[ K^2 - is \frac{d\Delta}{dr} K + \Delta \left( 2is \frac{dK}{dr} - \lambda \right) \right] R = 0,
\]

(2)

where \(K \equiv (r^2 + a^2)\omega - am\), \(\Delta \equiv r^2 - a^2 + Q^2\) and \(\lambda = A_{lm} + (\omega)^2 - 2am\omega\). The parameter \(s = 0, -1, -2\) for scalar, EM and gravitational perturbations respectively, and \(a\) is the Kerr rotation parameter \((0 \leq a < 1/2)\). In the Schwarzschild limit \(a \to 0\) the angular separation constant \(A_{lm} \to l(l+1) - s(s+1)\). The DF equation was also studied by Detweiler and Ove [21]. It is exact only when the spin \(s = 0\) or when we set the charge \(Q = 0\) (and then it reduces to the standard Teukolsky equation for Kerr black holes [23]).

Let us consider the boundary conditions of Eq. (2) at the horizon. The horizon radius \(r_+ = (1 + b)/2\), where \(b \equiv \sqrt{1 - 4(a^2 + Q^2)}\). From the indicial equation it follows that the ingoing solution at the horizon is such that

\[
R_{lm} \sim (r - r_+)^{-\alpha_+} \text{ as } r \to r_+,
\]

(3)

where \(\alpha_+ = [\omega(r_+ - Q^2) - am]/b\). Similarly, imposing purely outgoing radiation at infinity we find

\[
R_{lm} \sim r^{-2s-1+i\omega} e^{i\omega r} \text{ as } r \to \infty.
\]

(4)

Both these boundary conditions on the radial equation (2) and the corresponding regularity conditions on the angular equation can be cast as three-term continued fraction relations of the form

\[
0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \ldots}}.
\]

(5)

The coefficients of the radial continued fraction are:

\[
\alpha_j = j^2 + (2 - 2i\sigma_+ - s)j + \{ -32[\omega a^3 m + ib^2 \sigma_+] + 16[ib^2 s \sigma_+ + samb + \omega^2 a^4 + a^2 m^2 + a^2 \omega b + b^2 - b^2 \sigma_+ - am\omega b - is \omega a^2 b - sb^2] + 8[ib^2 a^2 \omega^2 + a^2 \omega^2 - am\omega - ib^2 \omega - b^2 \omega b] + 4[\omega b + \omega b^3 - ib^3 \omega - is \omega b] + \omega^2 (b^4 + 6b^2 + 1) \}/(16b^2),
\]

(6)

\[
\beta_j = -2j^2 + 2[\omega(b+1) + 2i\sigma_+ - s]j + \{ 32[2 \omega a^3 m + 16[ib^2 \sigma_+ + sb^2 \sigma_+ + b^2 \sigma_+ - a^2 m^2 - \omega^2 a^4] + 8[ib^2 a^2 \omega^2 + a^2 \omega^2 - am\omega - sb^2 - a^2 \omega^2 - b^2 - b^2 A_{lm} + \omega^2 (3b^4 + 6b^2 - 1) \}/(8b^2),
\]

\[
\gamma_j = j^2 - 2i[\omega + \sigma_+] - s]j - \{ 32[2 \omega a^3 m + 16[ib^2 \sigma_+ + sb^2 \sigma_+ + b^2 \sigma_+ - a^2 m^2 - \omega^2 a^4] + 10\omega b^2 + 8[ib^2 a^2 \omega^2 + a^2 \omega^2 - b^2 - b^2 A_{lm} + \omega^2 (3b^4 + 6b^2 - 1) \}/(16b^2).
\]

(7)

and those of the angular continued fraction can be found in [23]. To find QNM frequencies we first fix the values of \(a\), \(\ell\), \(m\) and \(\omega\), and find the angular separation constant \(A_{lm}(\omega)\) looking for zeros of the angular continued fraction. Then we use the corresponding eigenvalue to look for zeros of the radial continued fraction as a function of \(\omega\). The \(n\)-th quasinormal frequency is (numerically) the most stable root of the \(n\)-th inversion of the continued-fraction relation [19], i.e., it is the root of

\[
\beta_n = \frac{\alpha_{n-1} \gamma_n - \alpha_{n-2} \gamma_{n-1}}{\beta_{n-1} - \beta_{n-2}} \ldots \frac{\alpha_0 \gamma_1}{\beta_1 - \beta_2} \ldots = \frac{\alpha_n \gamma_{n+1} - \alpha_{n+1} \gamma_{n+2}}{\beta_{n+1} - \beta_{n+2}} \ldots, \quad (n = 1, 2, \ldots).
\]
TABLE I: Fundamental scalar QNM of the KN metric with $l = m = 0$ for selected values of the charge and angular momentum.

| $a$ | $Q = 0$ | $Q = 0.1$ | $Q = 0.2$ | $Q = 0.3$ | $Q = 0.4$ |
|-----|---------|---------|---------|---------|---------|
| 0   | (0.220910,-0.209791) | (0.222479,-0.210122) | (0.227471,-0.210998) | (0.236910,-0.211849) | (0.253168,-0.210094) |
| 0.1 | (0.221535,-0.209025) | (0.223121,-0.209319) | (0.228169,-0.210067) | (0.237712,-0.210596) | (0.254019,-0.207839) |
| 0.2 | (0.223398,-0.206506) | (0.225033,-0.206673) | (0.230230,-0.206961) | (0.239994,-0.206302) | (0.255542,-0.199555) |
| 0.3 | (0.226422,-0.201397) | (0.228021,-0.201267) | (0.233297,-0.200439) | (0.242568,-0.196687) | -          |
| 0.4 | (0.229074,-0.191402) | (0.230486,-0.190541) | (0.234035,-0.186969) | -          | -          |

The infinite continued fraction appearing in equation (7) can be summed “bottom to top” starting from some large truncation index $N$. Nollert [26] has shown that the convergence of the procedure improves if such a sum is started using a series expansion for the “rest” of the continued fraction, $R_N$, defined by the equation

$$R_N = \frac{\gamma N + 1}{\beta N + 1 - \alpha N + 1 R_{N+1}}.$$

(8)

The series expansion reads

$$R_N = \sum_{k=0}^{\infty} C_k N^{-k/2},$$

(9)

where the first few coefficients have the same form as in the Kerr case [30], except for the charge-dependent correction in $b$: $C_0 = -1$, $C_1 = \pm \sqrt{-2i \omega b}$, $C_2 = [3/4 + i \omega (b + 1) - s]^{-1}$. In Table I we give numerical results for the fundamental scalar QNM with $l = m = 0$ as a function of charge and angular momentum. Dashed entries in this and the following Tables correspond to combinations of $Q$ and $a$ for which $Q^2 + a^2 \geq M^2$.

Fig. 1 shows trajectories described by selected scalar QNMs in the complex plane. In each panel, the thick black line corresponds to modes of a RN (possibly charged but non-rotating) black hole. As $Q$ increases the RN mode moves counterclockwise in the complex frequency plane; open circles on the thick line mark increasing values of the charge ($Q = 0$, 0.05, 0.1, ...). For fixed values of $Q$ (namely $Q = 0$, 0.1, 0.2, 0.3, 0.4) we plot the KN QNM trajectories as we increase $a$: the results are the five thin lines branching from the RN limit. Open circles on these curves mark increasing values of the angular momentum ($a = 0$, 0.05, 0.1, ...). The observed spiraling behavior (as a function of both charge $Q$ and angular momentum $a$) confirms and extends results presented in [27, 28, 29, 30, 31, 32, 33] and summarized in [34]. The bottom-right panel shows trajectories of the fundamental scalar mode with $l = 2$, $m = 0$. As we increase $l$ the counterclockwise bending (that for $l = 0$ is very pronounced even for the lowest overtones) only shows up at higher overtone indices ($n \sim 10$ for $l = 2$).

When $l > 0$, and for any value of the charge $0 \leq Q < 1/2$, rotation induces a Zeeman-like splitting of the modes. In Fig. 2 we show the splitting of the real part of the frequency of the fundamental scalar QNM with $l = 2$ for three representative values of the charge (from left to right: $Q = 0$, 0.2, 0.4). For Kerr black holes, the lowest-lying modes with $l = m$ tend to approach the critical frequency for superradiance $m \Omega$ (where $\Omega$ is the rotational velocity of the black hole horizon) in the extremal limit. Detweiler [35] first presented analytical arguments to explain this clustering of modes at the superradiant frequency in the extremal limit. The implicit assumptions in his argument have recently been re-examined in [30, 37]. In the general KN case the extremal frequencies for modes with $l = |m|$ have analytically been computed by Mashhoon [21]. We quote his result, which is in excellent agreement with our numerical calculations (see Fig. 3 below), at the end of the Appendix.

In Table II we consider gravitational perturbations with $l = 2$, $m = 0$. To assess the reliability of WKB methods in the KN case we compare results from Leaver’s continued fraction approach (which can be considered exact, within the given numerical accuracy) with the third order WKB technique used in [22], building on previous results in [38, 39]. The agreement for the fundamental gravitational QNM with $l = 2$, $m = 0$ is excellent, typical errors being smaller than one part in a thousand. The WKB approach systematically underestimates the real parts and overestimates the imaginary parts. Third order WKB methods become slightly less reliable for large rotation parameter $a$: at fixed $a$, their accuracy depends very weakly on $Q$. Table III shows results for the $l = m = 2$ “barlike” mode, which is generally believed to be dominant in the quasinormal ringing following gravitational collapse to a rotating black hole [40]. Perhaps the most important outcome of our analysis is a null result: we could not find any unstable mode. Our quasinormal mode calculation confirms qualitatively Mashhoon’s arguments in favor of the stability of the KN metric.
FIG. 1: Left to right and top to bottom, the first three panels show trajectories in the complex plane of the first three scalar QNMs of the KN black hole for $l = m = 0$. The thick black curve corresponds to $a = 0$ (RN limit). The thin curves are obtained increasing $a$ from zero to the extremal limit for fixed values of the charge ($Q = 0$, $0.1$, $0.2$, $0.3$, $0.4$ respectively). Open circles (when present) mark selected values of the charge ($Q = 0$, $0.05$, $0.1$ . . . along the thick RN line) and angular momentum ($a = 0$, $0.05$, $0.1$ . . . along the thin lines). Bottom right panel: same plot for the fundamental mode with $s = 0$ and $l = 2$. Thin lines are now trajectories of the $m = 0$ KN modes for $Q = 0$, $0.1$, $0.2$, $0.3$, $0.4$. For clarity we do not show modes with other values of $m$.

FIG. 2: Real part of the fundamental scalar QNM of a KN black hole for $l = 2$ and different values of $m$. Curves from top to bottom refer to $m = 2, 1, 0, -1, -2$. The three panels correspond to different values of $Q$, as indicated. For $Q \neq 0$, the vertical line marks the extremal limit.
FIG. 3: Real part (top) and imaginary part (bottom) of the fundamental $l = 2$ gravitational QNM of a KN black hole. Solid lines refer to $m = 2$, dashed lines to $m = -2$. Different panels correspond to different values of the charge, as indicated. For $Q \neq 0$, the vertical line marks the extremal limit. Thick (black) lines are numerical results from the DF equation. Thin (red) lines are obtained from Mashhoon’s arguments, which are strictly valid only as $l \to \infty$ (see Appendix). Considering that we are looking at modes with $l = 2$ the agreement of Mashhoon’s predictions with numerical results is quite impressive, particularly for the real part of the fundamental QNM frequency and for large values of $Q$ and $a$.

TABLE II: Comparison of QNM frequencies of the DF equation obtained by the continued fraction method (first row for each value of $a$) with third-order WKB results from [22] (second row). Numbers refer to the fundamental gravitational QNM with $l = 2$, $m = 0$.

| $a$ | $Q = 0$ | $Q = 0.1$ | $Q = 0.2$ | $Q = 0.3$ | $Q = 0.4$ |
|-----|---------|-----------|-----------|-----------|-----------|
| 0   | (0.747343, -0.177925) | (0.753643, -0.178602) | (0.773902, -0.180620) | (0.813278, -0.183793) | (0.886468, -0.186603) |
|     | (0.7463, -0.1784) | (0.7530, -0.1792) | (0.7734, -0.1812) | (0.8128, -0.1844) | (0.8860, -0.1870) |
| 0.1 | (0.750248, -0.177401) | (0.756664, -0.178053) | (0.777325, -0.179977) | (0.817630, -0.182905) | (0.893230, -0.184893) |
|     | (0.7496, -0.1780) | (0.7560, -0.1786) | (0.7835, -0.1804) | (0.8172, -0.1834) | (0.8928, -0.1854) |
| 0.2 | (0.759363, -0.175653) | (0.766159, -0.176211) | (0.788148, -0.177790) | (0.831579, -0.179789) | (0.915681, -0.178292) |
|     | (0.7586, -0.1762) | (0.7654, -0.1766) | (0.7874, -0.1782) | (0.8310, -0.1802) | (0.9150, -0.1788) |
| 0.275 | (0.771072, -0.173158) | (0.778394, -0.173563) | (0.802246, -0.174561) | (0.850222, -0.174875) | (0.947708, -0.165369) |
|     | (0.7702, -0.1736) | (0.7776, -0.1740) | (0.8014, -0.1750) | (0.8494, -0.1754) | (0.9470, -0.1660) |
| 0.3 | (0.776108, -0.171989) | (0.783669, -0.172315) | (0.808382, -0.173003) | (0.858524, -0.172355) | - |
|     | (0.7752, -0.1724) | (0.7828, -0.1728) | (0.8074, -0.1734) | (0.8578, -0.1728) | - |
| 0.4 | (0.803835, -0.164313) | (0.812886, -0.163977) | (0.843068, -0.161921) | - | - |
|     | (0.8026, -0.1648) | (0.8118, -0.1646) | (0.8410, -0.1626) | - | - |
| 0.45 | (0.824009, -0.156965) | (0.834319, -0.155756) | (0.869094, -0.149647) | - | - |
|     | (0.8228, -0.1576) | (0.8330, -0.1564) | (0.8680, -0.1506) | - | - |
| 0.48 | (0.838981, -0.150055) | (0.850244, -0.147806) | - | - | - |
|     | (0.8378, -0.1510) | (0.8490, -0.1488) | - | - | - |
The equations of time dependence, EM and axial gravitational perturbations of the RN metric are described by two coupled wave equations. Using tensor spherical harmonics to separate the angular dependence and a Fourier decomposition to get rid of the radial dependence, we arrive at a system of coupled equations for the radial perturbations.

where $\Delta = r^2 - 2m/r - Q^2$ is the location of the inner (Cauchy) and outer (event) horizons of the RN metric. This is no surprise: for the reasons discussed in the Introduction we expect the DF equation to be a good approximation only for small values of $Q$. Below we will show that this is indeed the case, and give a quantitative meaning to the above statement.

### III. REISSNER-NORDSTRÖM: THE STANDARD TREATMENT

Here we briefly summarize the computational procedure for RN black holes ($a = 0$). More details can be found in Refs. [24, 26] and especially [41]. This Section is primarily intended to establish notation for the “frozen” approximation to be introduced in Section IV.

It is well known that polar perturbations of the RN black hole can be obtained from the axial perturbations by a Chandrasekhar transformation [2]; in particular, the QNM spectra of polar and axial perturbations are the same. For this reason in the following we only consider axial perturbations, that we denote by a superscript (a). Let us introduce a tortoise coordinate $r_*$ by the usual relation

$$\frac{dr}{dr_*} = \frac{\Delta}{r^2},$$

where $\Delta = r^2 - r + Q^2$ (recall that in our units $2M = 1$ and $0 \leq Q < 1/2$). Explicitly, the tortoise coordinate can be written as

$$r_* = r + \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2}{r_+ - r_-} \ln(r - r_-),$$

where $r_{\pm} = (1 \pm \sqrt{1 - 4Q^2})/2$ is the location of the inner (Cauchy) and outer (event) horizons of the RN metric. Using tensor spherical harmonics to separate the angular dependence and a Fourier decomposition to get rid of the time dependence, EM and axial gravitational perturbations of the RN metric are described by two coupled wave equations:

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) H_2^{(-)} = \frac{\Delta}{r^3} \left[ l(l+1)r - \frac{3}{2} \frac{4Q^2}{r} \right] H_2^{(-)} - \frac{3}{2} H_2^{(+)} + 2Q \sqrt{(l-1)(l+2)} H_1^{(-)},$$

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) H_1^{(-)} = \frac{\Delta}{r^3} \left[ l(l+1)r - \frac{3}{2} \frac{4Q^2}{r} \right] H_1^{(-)} + \frac{3}{2} H_1^{(+)} + 2Q \sqrt{(l-1)(l+2)} H_2^{(-)},$$

where $\omega^2 = \omega^2 + \omega^2$ and $l = m = 2$.

### TABLE III: QNM frequencies of the DF equation obtained by the continued fraction method. Numbers refer to the fundamental gravitational QNM with $l = m = 2$.

| $a$ | $Q = 0$ | $Q = 0.1$ | $Q = 0.2$ | $Q = 0.3$ | $Q = 0.4$ |
|-----|---------|---------|---------|---------|---------|
| 0.0 | (0.747343-0.177925) | (0.753643-0.178602) | (0.773902-0.180620) | (0.813278-0.183793) | (0.886468-0.186603) |
| 0.1 | (0.804291-0.176622) | (0.812131-0.177253) | (0.837615-0.179080) | (0.888479-0.181653) | (0.989234-0.181863) |
| 0.2 | (0.879684-0.173764) | (0.890062-0.174238) | (0.924394-0.175451) | (0.996178-0.176132) | (1.158394-0.170666) |
| 0.257 | (0.956470-0.169604) | (0.970151-0.169774) | (1.016478-0.169756) | (1.120103-0.166278) | (1.432934-0.122011) |
| 0.3 | (0.988090-0.167530) | (1.003384-0.167519) | (1.055787-0.166749) | (1.177413-0.160442) | - |
| 0.4 | (1.172034-0.151259) | (1.200883-0.149228) | (1.311304-0.138796) | - | - |
| 0.45 | (1.343229-0.129738) | (1.395907-0.123109) | (1.678651-0.074535) | - | - |
| 0.48 | (1.535348-0.098867) | (1.653330-0.077666) | - | - | - |

Fig. 3 shows that the agreement is more than qualitative. In fact his argument (which is based on the eikonal approximation, and is expected to be accurate only for $l \gg 1$) captures almost perfectly the behavior of the fundamental gravitational QNMs with $l = |m| = 2$ as functions of charge $Q$ and angular momentum $a$. This agreement is particularly good for the real part of the frequencies, and it can be exploited for “quick and dirty” estimates of QNM excitation in astrophysical scenarios (see e.g. [12]).

A major problem of the DF equation is that it only provides an approximation to the problem of coupled EM-gravitational perturbations of the KN black hole. One can easily check that the limit $a \to 0$ (first line of Table III) we do not recover the QNM frequencies of the RN metric. This is no surprise: for the reasons discussed in the Introduction we expect the DF equation to be a good approximation only for small values of $Q$. Below we will show that this is indeed the case, and give a quantitative meaning to the above statement.
where $H_2^{(-)}$ corresponds to perturbations of the gravitational field and $H_1^{(-)}$ to perturbations of the EM field. The usual procedure is to decouple the system (12) to obtain Schrödinger-like equations of the form
\[
\left(\frac{d^2}{dr^2} + \omega^2\right) Z_i^{(-)} = V_i^{(-)} Z_i^{(-)},
\]
where
\[
V_i^{(-)} = \frac{\Delta}{r^5} \left[ l(l+1)r - q_j + \frac{4Q^2}{r} \right], \quad (i, j = 1, 2, \ i \neq j),
\]
and
\[
q_1 = \left[ 3 + \sqrt{9 + 16Q^2(l-1)(l+2)} \right]/2, \quad q_2 = \left[ 3 - \sqrt{9 + 16Q^2(l-1)(l+2)} \right]/2.
\]
The price to pay is that, unlike $H_1^{(-)}$ and $H_2^{(-)}$, the decoupled radial functions $Z_1^{(-)}$ and $Z_2^{(-)}$ are not simply degrees of freedom of the EM and gravitational field for any $Q \neq 0$. Only in the limit $Q = 0$ do the potentials $V_1^{(-)}$ and $V_2^{(-)}$ describe, respectively, purely EM and axial–gravitational perturbations of a Schwarzschild black hole. The radial equations (13) are solved by a series expansion of the form
\[
Z_i^{(-)} = \frac{r}{r} e^{-2\omega r} \left( r_+ - r_- \right)^{-2\omega} \left( r - r_- \right)^{1 + i\omega u} u_{+}^{1 - i\omega r} u_{-}^{2 - i\omega r} / (r_+ - r_-)^{\infty} \sum_{j=0} a_j u^j,
\]
where $u = (r - r_+) / (r - r_-)$ and the coefficients $a_j$ are determined by a four-term recursion relation. The problem can be reduced to a three-term recursion relation of the form (15) using a Gaussian elimination step (41). Then we can use standard techniques to determine the quasinormal frequencies. The convergence of the summation can be improved using Nollert’s expansion (49) for the rest. The first few coefficients of Nollert’s series can be found in (30).

**IV. FREEZING REISSNER-NORDSTRÖM: AN APPROXIMATE DECOUPLING**

If we are interested in the oscillations of an astrophysical black hole we can usually assume the black hole charge (in geometrized units) to be small, $Q \ll M$. In this limit it is reasonable to ignore the coupling between the EM field and the metric. In the following we study QNM frequencies of the RN black hole ignoring the mutual effect of the EM field on the metric and vice versa. We simply decouple the system (12) setting to zero the EM (gravitational) perturbations $H_1^{(-)}$ ($H_2^{(-)}$). The resulting equations are:
\[
\left(\frac{d^2}{dr^2} + \omega^2 - V_i^* \right) H_i^{(-)} = 0, \quad (i = 1, 2).
\]
The gravitational potential when we freeze EM perturbations, $H_2^{(-)} = 0$, is
\[
V_2^* = \frac{\Delta}{r^5} \left[ l(l+1)r - 3 + \frac{4Q^2}{r} \right].
\]
$V_2^*$ reduces to the standard RW potential for $Q = 0$. The EM potential freezing metric perturbations, $H_2^{(-)} = 0$, becomes
\[
V_1^* = \frac{\Delta}{r^5} \left[ l(l+1)r + \frac{4Q^2}{r} \right].
\]
Similarly to $V_2^*$, $V_1^*$ reduces to the potential for EM perturbations of Schwarzschild black holes when $Q = 0$. The two potentials can be written in the compact form
\[
V_i^* = \frac{\Delta}{r^5} \left[ l(l+1)r + (1 - s^2) + \frac{4Q^2}{r} \right], \quad (i = |s| = 1, 2).
\]
TABLE IV: Scalar, EM and gravitational QNMs of the RN black hole. The first line gives third order WKB results, and the second line sixth order WKB results for the decoupled potentials \( WKB3 \) and \( WKB6 \). Sixth order WKB results have been kindly provided by Vitor Cardoso. The third line gives Leaver’s results for the DF equation in the limit \( a \to 0 \). In the EM and gravitational cases, the fourth line gives results from Leaver’s method applied to the coupled EM-gravitational potentials \( V_i^{(-)} \) \((i = 1, 2)\).

|                | \( l = 0 \)                  | \( l = 1 \)                  | \( l = 2 \)                  | \( l = 3 \)                  | \( l = 4 \)                  |
|----------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| Scalar, \( l = 0 \) | \( Q = 0 \) | \( Q = 0.1 \) | \( Q = 0.2 \) | \( Q = 0.3 \) | \( Q = 0.4 \) |
| \( WKB3, V_0 \) | (0.209290, -0.230390) | (0.210984, -0.230440) | (0.216361, -0.230392) | (0.226237, -0.229299) | (0.249493, -0.223521) |
| \( WKB6, V_0 \) | (0.209399, -0.201628) | (0.222946, -0.202043) | (0.227572, -0.203102) | (0.237028, -0.204762) | (0.251727, -0.207292) |
| Leaver, DF | (0.220910, -0.209791) | (0.222479, -0.210122) | (0.227471, -0.210998) | (0.236910, -0.211849) | (0.253168, -0.210094) |

|                | \( l = 2 \)                  | \( l = 1 \)                  | \( l = 2 \)                  | \( l = 3 \)                  | \( l = 4 \)                  |
|----------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| EM, \( l = 1 \) | \( Q = 0 \) | \( Q = 0.1 \) | \( Q = 0.2 \) | \( Q = 0.3 \) | \( Q = 0.4 \) |
| \( WKB3, V_1^* \) | (0.966422, -0.193610) | (0.973006, -0.194022) | (0.993994, -0.195186) | (1.033982, -0.196707) | (1.105339, -0.196689) |
| \( WKB6, V_1^* \) | (0.972841, -0.193532) | (0.973859, -0.193948) | (0.994823, -0.195120) | (1.034772, -0.196664) | (1.106104, -0.196689) |
| Leaver, DF | (0.972898, -0.193518) | (0.987736, -0.193635) | (0.994862, -0.195107) | (1.034775, -0.196649) | (1.106105, -0.196675) |

|                | \( l = 2 \)                  | \( l = 1 \)                  | \( l = 2 \)                  | \( l = 3 \)                  | \( l = 4 \)                  |
|----------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| EM, \( l = 2 \) | \( Q = 0 \) | \( Q = 0.1 \) | \( Q = 0.2 \) | \( Q = 0.3 \) | \( Q = 0.4 \) |
| \( WKB3, V_2^* \) | (0.941740, -0.186212) | (0.948236, -0.186976) | (0.519336, -0.189280) | (0.561324, -0.193009) | (0.643293, -0.196606) |
| \( WKB6, V_2^* \) | (0.949838, -0.185274) | (0.952809, -0.186118) | (0.523726, -0.188648) | (0.565304, -0.192902) | (0.646616, -0.197253) |
| Leaver, DF | (0.946527, -0.184975) | (0.500367, -0.185468) | (0.516275, -0.186892) | (0.536404, -0.188917) | (0.579729, -0.189708) |
| Leaver, DF | (0.946527, -0.184975) | (0.500367, -0.185468) | (0.516275, -0.186892) | (0.536404, -0.188917) | (0.579729, -0.189708) |

|                | \( l = 2 \)                  | \( l = 1 \)                  | \( l = 2 \)                  | \( l = 3 \)                  | \( l = 4 \)                  |
|----------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| Gravitational, \( l = 2 \) | \( Q = 0 \) | \( Q = 0.1 \) | \( Q = 0.2 \) | \( Q = 0.3 \) | \( Q = 0.4 \) |
| \( WKB3, V_2^* \) | (0.768324, -0.178435) | (0.751578, -0.178831) | (0.768355, -0.179955) | (0.800439, -0.181460) | (0.858116, -0.181608) |
| \( WKB6, V_2^* \) | (0.747239, -0.177781) | (0.752520, -0.178204) | (0.769379, -0.179415) | (0.801591, -0.181106) | (0.859434, -0.181619) |
| Leaver, DF | (0.747343, -0.177925) | (0.753643, -0.178602) | (0.773092, -0.180620) | (0.813278, -0.183793) | (0.886468, -0.186603) |
| Leaver, \( V_i^{(-)} \) | (0.747343, -0.177925) | (0.749489, -0.185805) | (0.523843, -0.188312) | (0.565513, -0.192408) | (0.646987, -0.196545) |

For completeness we also consider scalar perturbations of the RN black hole: in this case no approximations are involved, and the potential reads \( V_0 = \frac{\Delta}{r^l} \left[ l(l+1)r + 1 - \frac{2Q^2}{r} \right] \). \( (21) \)

In Table IV we present the fundamental scalar, EM and gravitational QNM frequency for five selected values of the charge. From top to bottom we consider scalar perturbations with \( l = 0 \) and \( l = 2 \), EM perturbations with \( l = 1 \) and \( l = 2 \) and gravitational perturbations with \( l = 2 \).

We computed QNMs of the potentials \( WKB3 \) and \( WKB6 \) using a third order WKB expansion \( (14) \), then we checked convergence using a sixth order WKB expansion \( (15) \). Sixth order results have been kindly provided by Vitor Cardoso. WKB results from the third (sixth) order expansion are given in the rows marked \( WKB3 \) (WKB6). “Leaver, DF” means that QNM frequencies have been computed using continued fractions and the \( a = 0 \) limit of the DF equation \( (2) \) with the appropriate value of \( s \). Finally, “Leaver, \( V_i^{(-)} \)” means we applied Leaver’s continued fraction method to the coupled EM-gravitational system \( (18) \); the results are the “true” oscillation frequencies of the RN black hole \( (30, 41) \). Some comments are in order:

i) The sixth order WKB results can be considered reliable. Even in the case in which we expect the worst convergence (scalar perturbations with \( l = 0 \)) a sixth order WKB expansion agrees very well with results from the DF equation (which in this case, we stress it again, is exact). For scalar modes with \( l = 2 \) the agreement between the sixth order WKB and the results from the DF equation is quite astonishing. This is really a double check: not only it proves that WKB results can be considered reliable, it also shows (by a completely independent calculation) that the scalar QNMs obtained in Sec. 11 have the correct limit as \( a \to 0 \).
ii) The perfect agreement between decoupled and coupled EM perturbations with $l = 1$ is no surprise. Mathematically, a quick inspection of Eq. (13) with $i = 1$ reveals that, since for $l = 1$ we have $q_2 = 0$ for any value of $Q$, EM perturbations with $l = 1$ are always decoupled from gravitational perturbations. Of course, physically this decoupling is due to the nonradiative character of dipolar gravitational fields (the first radiative multipole being the quadrupole, $l = 2$). Notice however that for EM fields and $l = 1$ QNMs of the DF equation disagree with the other two approaches, the deviations increasing for large values of $Q$. This is evidence that when we consider the DF equation with $s = −1$ we are not really “killing” gravitational perturbations. However, to a good approximation, QNM frequencies of the DF equation for $s = −1$ are very close to frequencies obtained freezing metric perturbations in RN when we consider small values of the charge, say $Q \lesssim M/2$. This is also true for EM QNM frequencies with $l = 2$, in which case EM perturbations are coupled with gravitational perturbations.

iii) Inspection of gravitational QNMs with $l = 2$ confirms the above conclusions: when we consider the DF equation with $s = −2$ we are not really “killing” EM perturbations. However, to a good approximation, QNM frequencies of the DF equation for $s = −2$ are very close to those obtained freezing EM perturbations in RN when we consider small values of the charge, say $Q \lesssim M/2$.

iv) For RN black holes with $Q \lesssim M/2$ the DF equation provides the correct RN oscillation frequencies of the full coupled EM-gravitational perturbations system within about 1 %. We can reasonably expect our results to have the same level of accuracy for rotating, KN black holes, at least when $Q \lesssim M/2$ and $a$ is not too large.

A. Asymptotic modes of “frozen” RN black holes

Highly damped black hole QNMs received considerable attention recently, due to a conjectured relation with quantum gravity. Motl and Neitzke used a monodromy calculation to compute asymptotic QNM frequencies of RN black holes. The key element of their calculation is the leading asymptotic behavior of the potential close to the origin. They showed that, since the leading term of both potentials in (12) as $r \to 0$ is

$$V_i \sim \frac{j^2 - 1}{4r_s^2}, \quad (i = 1, 2),$$

with $j = 5/3$, asymptotic QNMs are given by the implicit formula

$$e^{\beta \omega} + 2 + 3e^{-\beta_I \omega} = 0,$$

where $\beta$ and $\beta_I$ are the Hawking temperatures of the outer and inner RN horizons, respectively. The “frozen” potentials have the same leading-order behavior:

$$V_i^* \sim \frac{4Q^4}{r_s^6} \sim \frac{4}{9r_s^2}, \quad (i = 1, 2),$$

where we used the fact that $r_s \sim r^3/(3Q^2)$ as $r \to 0$. Therefore, if we kill EM (or gravitational) perturbations keeping the charge in the background, the asymptotic QNMs are still given by (23). In this sense, the wild oscillations of asymptotic QNM frequencies as a function of $Q$ are not induced by the EM-gravitational coupling. In fact, the presence of the charge term in the potential (and not the coupling of EM-gravitational perturbations) is the reason for the different topology of the Stokes lines in the monodromy calculation.

V. CONCLUSIONS

In this paper we presented the first continued-fraction calculation of QNMs of the KN black hole based on the approximate perturbation equation derived by Dudley and Finley. For scalar perturbations their equation is exact; for EM and gravitational perturbations, it provides a good approximation for values of the charge $Q \lesssim M/2$. We found no evidence for instabilities, extending a previous analysis by Mashhoon, who presented arguments in favor of the stability of KN black holes in the eikonal approximation. To understand the meaning of the DF approximation we analysed in detail the zero-rotation (RN) limit. We computed QNMs for “frozen” (purely EM or gravitational) perturbations of RN black holes and compared results with the zero-rotation limit of the DF equation. We found that the DF equation is equivalent to frozen EM (gravitational) perturbations of RN only when $Q \lesssim M/2$. In this regime, and for $a = 0$, the DF equation provides the correct QNMs of the full coupled EM-gravitational...
perturbations system within about 1 %. We expect our results to have the same level of accuracy also for rotating black holes, at least when \( Q \lesssim M/2 \) and \( a \) is not too large.

To confirm this expectation we need a more complete analysis of the KN perturbation problem. Some suggestions to decouple EM and gravitational perturbations of KN black holes were given in \[49\] and, more recently, in \[51\]. Chitre \[51\] found a separable equation for rotating black holes with small values of \( Q \). His treatment was later extended by Lee \[52\] (see also \[53\]). As far as we know, no attempt has been made to compute the corresponding QNMs. Results obtained from Lee’s small-\( Q \) expansion should not deviate much from ours, since the DF equation gives accurate QNM frequencies in the limit \( Q \ll M \).

Our study shows that the DF equation can be used to provide reliable estimates of the QNM frequencies of slightly charged, rotating BHs. Mashhoon’s approximate treatment, summarized in the Appendix, captures the essential physics and gives very good estimates for the \( l = |n| = 2 \) fundamental QNM as a function of both \( Q \) and \( a \) (see Fig. 3). These results could be useful to investigate gravitational wave emission in coincidence with black hole models of gamma-ray bursts. Proposed models to date include bipolar outflow from a fast-rotating, slightly charged KN black hole surrounded by a magnetosphere \[9\], black hole-torus systems \[10\], pair creation due to vacuum polarization in gravitational collapse to a charged black hole \[11\] and hyper-accretion scenarios \[12\].

An interesting extension of our work concerns the study of Dirac perturbations of the KN metric. In this case separability is not an issue \[13\]. Pekeris and collaborators considered the nucleus as a Kerr-Newman source \[7\]. Assuming the angular momentum of the source to be the intrinsic spin angular momentum of the nucleus they found a remarkable result: the Dirac equation in the KN background predicts the hyperfine splitting observed in muonium, positronium and hydrogen to within the uncertainty in the respective QED corrections, except for an unaccounted factor of 2. Their analysis is worth revisiting, considering also that a very complete mathematical study of the Dirac equation in the KN spacetime appeared after Pekeris’ work \[54\]. We hope to return to this problem in the future.

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APPENDIX A: PERTURBED ORBITS OF NULL RAYS AND THE EIKONAL LIMIT

In this Appendix we summarize Mashhoon’s analysis of the perturbations of unstable circular photon orbits in the KN spacetime \[20\]. His physical model is closely related to (but slightly different from) Goebel’s \[21\]. The key idea is that, since QNMs represent a general property of the spacetime, we can find them using the time evolution of any convenient perturbation. In particular, Mashhoon considers an aggregate of massless particles in the unstable equatorial circular orbit of a KN black hole. The radius \( r_0 \) of this orbit is given by the roots of

\[
\begin{align*}
    r_0^2 - 3r_0/2 + 2Q^2 \pm 2a(r_0/2 - Q^2)^{1/2} &= 0. \\
    \end{align*}
\]

The upper sign refers to corotating and the lower sign to counterrotating orbits, and only solutions with \( r_0 \geq r_+ \) are acceptable. The corresponding orbital frequencies are

\[ \omega_\pm = \frac{1}{a \pm r_0^2(r_0/2 - Q^2)^{-1/2}}. \]

According to Mashhoon’s model, the real part of QNM frequencies in the eikonal limit is proportional to the frequency of the perturbed bundle of null rays that escape to infinity, \( \omega_\pm \), and the imaginary part is proportional to

\[ \gamma = \left| \omega_\pm \right| \frac{4\Delta(r_0)[3(2r_0)^{-1} - 4Q^2r_0^{-2}]^{1/2}}{r_0(2r_0 - 1)}. \]

Perturbed null rays corotating (counterrotating) with the black hole correspond to \( l = \pm m \), so Mashhoon’s analysis predicts QNM frequencies

\[ \omega^{(M)} = (\pm i\omega_\pm, -(n - 1/2)\gamma), \]

where the \(-1/2\) in the imaginary part accounts for the fact that we count modes starting from \( n = 1 \), following Leaver’s convention \[22\]. In the Schwarzschild limit Mashhoon’s model predicts \( \omega_\pm = \gamma = 2/(3\sqrt{3}) \simeq 0.384900 \), in
agreement with the leading-order large-\(l\) expansion from the WKB approximation (Eq. (3.1) in [38]):

\[ \omega = \left( \frac{2l}{3\sqrt{3}}, \frac{2(n-1/2)}{3\sqrt{3}} \right). \]  

(A5)

In the extremal limit we can eliminate (say) \(Q\) in favour of \(a\) in Eq. (A1) and solve analytically for the extremal frequencies \(\omega_{\text{ext}}\). For corotating orbits the result is \(\omega_{\text{ext}} = (2 - 3a)^{-1}\) for \(0 \leq a \leq 1/4\), and \(\omega_{\text{ext}} = 4a(1 + 4a^2)^{-1}\) for \(1/4 \leq a \leq 1/2\). For counterrotating orbits, \(\omega_{\text{ext}} = -(2 + 3a)^{-1}\) in the whole range \(0 \leq a \leq 1/2\).

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