Abstract

A multidimensional gravitational model on the manifold $M = M_0 \times \prod_{i=1}^{n} M_i$, where $M_i$ are Einstein spaces ($i \geq 1$), is studied. For $N_0 = \dim M_0 > 2$ the $\sigma$ model representation is considered and it is shown that the corresponding Euclidean Toda-like system does not satisfy the Adler-van-Moerbeke criterion. For $M_0 = \mathbb{R}^{N_0}$, $N_0 = 3, 4, 6$ (and the total dimension $D = \dim M = 11, 10, 11$, respectively) nonsingular spherically symmetric solutions to vacuum Einstein equations are obtained and their generalizations to arbitrary signatures are considered. It is proved that for a non-Euclidean signature the Riemann tensor squared of the solutions diverges on certain hypersurfaces in $\mathbb{R}^{N_0}$.
1 Introduction

Our paper is devoted to studying a model of multidimensional gravity considered previously in Refs. [1–3] (see also [24–27]). This model contains “our space” $M_0$ of dimension $N_0$ and a set of internal Einstein spaces $M_1, \ldots, M_n$. All scale factors of $M_i$ are supposed to be functions on $M_0$. For physical applications $N_0 \leq 4$ (e.g. $N_0 = 1, 2$ corresponds to cosmology and axial symmetry, respectively).

On the classical level the model is equivalent to some tensor-multiscalar theory and may be also treated as a generalization of the standard Brans-Dicke theory with the parameter $\omega = 1/N' - 1$, where $N'$ is the total internal space dimension.

It should be noted that scalar-tensor theories are rather popular now (see, for example [4–8]).

For $N_0 = 1$ we get a multidimensional cosmological model considered by many authors [10–42]. This model contains “our space” $M_0 \times M_1 \times \ldots \times M_n$, for ten-dimensional superstring gravity [50] and two spectively. (We thus obtain as well one exact solution $\sigma$ with the topology $M_0 \times M_1 \rightarrow \mathbb{R}$, which is more explicit manner than in [1, 3]. In Sec. 4 three integrable non-trivial families of solutions were obtained for a cosmological model satisfying the equation

$R_{m,n}[g^i] = \lambda_i g^i_{m,n}$,

where

$g^0 = g^0_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ (2.3)

is a metric on the manifold $M_0$ and $g^i$ is a metric on $M_i$ satisfying the equation

$R_{m,n}[g^i] = \lambda_i g^i_{m,n}$,

$m_i, n_i = 1, \ldots, N_i$; $\lambda_i = \text{const.}$, $i = 1, \ldots, n$. Thus $(M_i, g^i)$ are Einstein spaces. The functions $\gamma, \phi^i : M_0 \rightarrow \mathbb{R}$ are smooth.

Remark 1. It is more correct to write (2.2) as

$g = \exp[2\gamma(x)] g^0 + \sum_{i=1}^{n} \exp[2\phi^i(x)] g^i$

where we denote by $\hat{g}^a = p_\alpha^a g^a$ the pullback of the metric $g^a$ to the manifold $M$ by the canonical projection: $p_\alpha : M \rightarrow M_\alpha$, $\alpha = 0, \ldots, n$. In what follows all "hats" over metrics will be omitted.

Here we are interested in exact solutions to the Einstein equations with a cosmological constant

$R_{MN}[g] - \frac{1}{2} g_{MN} R[g] = -\Lambda g_{MN}$

(2.5)

for the metric (2.2) defined on the manifold (2.1). The set of equations (2.5) is equivalent to

$R_{MN}[g] = \frac{2\Lambda}{D - 2} g_{MN}$,

(2.6)

where $D = \sum_{k=0}^{n} N_k = \dim M$ is the dimension of the manifold (2.1). $N_k = \dim M_k$, $k = 0, \ldots, n$. Eqs. (2.5) are the field equations corresponding to the action

$S = S[g] = \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|g|} [R[g] - 2\Lambda] + S_{GH}$

(2.7)
where we denote \(|g| = |\det(g_{MN})|\): \(S_{GH}\) is the standard Gibbons-Hawking boundary term \([13]\). This term is essential for a quantum treatment of the problem.

The nonzero Ricci tensor components for the metric (2.2) are (see the Appendix)

\[
R_{\mu\nu}[g] = R_{\mu\nu}[g^0] + g^0_{\mu\nu}[\Delta_0\gamma + (2 - N_0)(\partial\gamma)^2 - \partial\gamma \sum_{j=1}^n N_j \phi^j] + (2 - N_0)(\partial\gamma)^2 \sum_{i=1}^n N_i (\partial\phi)^2 \\
- (N_0 - 2)(\partial\gamma)^2 - (\partial f)^2 - 2\Delta_0(f + \gamma)
\]

where \(f = f(\gamma, \phi) = (N_0 - 2)\gamma + \sum_{j=1}^n N_j \phi^j\). (2.10)

Using (2.8) and (2.9), it is not difficult to verify that the field equations (2.5) (or, equivalently, (2.6)) may be obtained as the equations of motion corresponding to \(g^0\). The scalar curvature for (2.2) is

\[
R[g] = \sum_{i=1}^n e^{-2\phi_i} R[g^i] + e^{-2\gamma} \left\{ \left[ R[g^0] - \sum_{i=1}^n N_i (\partial\phi)^2 \right] + (N_0 - 2)(\partial\gamma)^2 - (\partial f)^2 - 2\Delta_0(f + \gamma) \right\}
\]

where \(f = f(\gamma, \phi) = (N_0 - 2)\gamma + \sum_{j=1}^n N_j \phi^j\). (2.11)

Using (2.8) and (2.9), it is not difficult to verify that the field equations (2.5) (or, equivalently, (2.6)) may be obtained as the equations of motion corresponding to the action

\[
S[\gamma^0, \gamma, \phi] = \frac{1}{2\kappa_0^2} \int_M d^nx \sqrt{|g^0|} e^{(\gamma, \phi)} \left\{ R[g^0] - \sum_{i=1}^n N_i (\partial\phi)^2 - (N_0 - 2)(\partial\gamma)^2 - (\partial f)^2 - 2\Delta_0(f + \gamma) \right\}
\]

where \(|g^0| = |\det(g^0_{\mu\nu})|\) and similar notations are applied to the metrics \(g^i\), \(i = 1, \ldots, n\). For finite internal space volumes (e.g. compact \(M_i\))

\[
V_i = \int_{M_i} d^{n_i}x \sqrt{|g^i|} \quad < +\infty,
\]

the action (2.12) coincides with the action (2.7), i.e.

\[
S[\gamma^0, \gamma, \phi] = S[g],
\]

where \(g\) is defined by the relation (2.2) and

\[
\kappa^2 = \kappa_0^2 \prod_{i=1}^n V_i.
\]

This may be readily verified using the following relation for the scalar curvature (2.10):

\[
R[g] = \sum_{i=1}^n e^{-2\phi_i} R[g^i] + e^{-2\gamma} \left\{ R[g^0] - \sum_{i=1}^n N_i (\partial\phi)^2 \right\}
- (N_0 - 2)(\partial\gamma)^2 - (\partial f)^2 - 2\Delta_0(f + \gamma) + R_B \right\},
\]

where

\[
R_B = (1/\sqrt{|g^0|}) e^{-f} \partial\mu [-2 e^f \sqrt{|g^0|} g^{0\mu} \partial_\nu (f + \gamma)]
\]

gives rise to the Gibbons-Hawking boundary term

\[
S_{GH} = \frac{1}{2\kappa^2} \int_M d^Dx \sqrt{|g|} \left\{ -e^{-2\gamma} R_B \right\}.
\]

\[3\] The non-exceptional case \(N_0 \neq 2\)

In order to simplify the action (2.12), we use, as in \([1]\) for \(N_0 \neq 2\), the gauge

\[
\gamma = \gamma_0(\phi) = \frac{1}{2 - N_0} \sum_{i=1}^n N_i \phi^i.
\]

(3.1)

It means that \(f = f(\gamma_0, \phi) = 0\), or the conformal Einstein-Pauli frame is used. Evidently this frame does not exist for \(N_0 = 2\). For the cosmological case \(N_0 = 1, g^0 = -dt \otimes dt\), and (3.1) corresponds to the harmonic-time gauge \([20]\). From (3.1) we get

\[
S_0[g^0, \phi] = S_\sigma[g^0, \gamma_0, \phi] = \frac{1}{2\kappa_0^2} \int_{M_0} d^nx \sqrt{|g^0|} \left\{ R[g^0] - G_{ij} g^{0\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - 2V(\phi) \right\},
\]

where

\[
G_{ij} = N_i \delta_{ij} + \frac{N_i N_j}{N_0 - 2}
\]

are the components of the “midisuperspace” (or target space) metric on \(R^n\)

\[
G = G_{ij} d\phi^i \otimes d\phi^j
\]

and

\[
V = V(\phi) = \Lambda e^{2\gamma_0(\phi)} - \frac{1}{2} \sum_{i=1}^n \lambda_i N_i e^{-2\phi^i + 2\gamma_0(\phi)}
\]

is the potential. (Here we corrected a misprint in Eq. (11) from \([1]\)). Thus, we are led to the action of a self-gravitating \(\sigma\) model with a flat target space \((R^n, G)\) (3.4) and a self-interaction described by the potential (3.5).
For $N_0 = 1$, $g^0 = -dt \otimes dt$ the action (3.2) coincides with the well-known cosmological one \[^{24}\]. In this case the minisuperspace metric (3.3) is pseudo-Euclidean \[^{27, 29}\].

**Remark 2.** We note that in the infinite-dimensional case $n = \infty$ the potential (3.5) is well-defined if the following restrictions are imposed:

$$
\sum_{i=1}^{n} |\alpha_i| N_i < +\infty, \quad \sum_{i=1}^{n} N_i |\phi^i| < +\infty. \quad (3.6)
$$

In the case $N_0 = 1, \phi = (\phi^i)$ belongs to a Banach space with $l_1$-norm \[^{32}\].

### 3.1. The case $N_0 > 2$

For $N_0 > 2$ the midisuperspace metric (3.3) is Euclidean. The potential (3.5) may be rewritten as

$$
V(\phi) = \sum_{\alpha=0}^{n} A_\alpha \exp[u_\alpha^0 \phi^\alpha], \quad (3.7)
$$

including the cosmological constant and the curvature terms, where $A_0 = \Lambda, A_j = -\frac{1}{2}\Delta_j N_j$ and

$$
u_i^0 = \frac{2N_i}{2-N_0}, \quad u_i^j = 2\left(-\delta_i^j + \frac{N_i}{2-N_0}\right), \quad (3.8)
$$

$i, j = 1, \ldots, n$. Thus the potential (3.5) has a Toda-like form.

Let

$$
\langle u, v \rangle_s = G^{ij} u_i v_j \quad (3.9)
$$

be a quadratic form on $\mathbb{R}^n$. Here

$$
G^{ij} = \frac{\delta_{ij}}{N_i} + \frac{1}{2-D} \quad (3.10)
$$

are components of the matrix inverse to the matrix $(G_{ij})$ in (3.3). For the vectors (3.8) $u^\alpha = (u_\alpha^0) \in \mathbb{R}^n, \alpha = 0, \ldots, n$, we get the following relations:

$$
\langle u^0, u^0 \rangle_s = \frac{4(D-N_0)}{(N_0-2)(D-2)}, \quad (3.11)
$$

$$
\langle u^0, u^j \rangle_s = \frac{4}{(N_0-2)}, \quad (3.12)
$$

$$
\langle u^i, u^j \rangle_s = 4\left(\delta_{ij} + \frac{1}{N_0-2}\right), \quad (3.13)
$$

$i, j = 1, \ldots, n$.

### 3.2. The Adler-van-Moerbeke criterion

For a fixed metric $g^0$ the action (3.2) coincides with the action of a Euclidean Toda-like system, i.e. a dynamical (physical) system with the potential in the form of a sum of exponents depending on linear combinations of coordinates (fields). For Toda-like systems in the dimension $N_0 = 1$ \[^{47, 49}\] (with the appropriate number of exponents) we know that the integrable cases (open and closed Toda lattices) occur when the vectors $u^\alpha$ in the exponents correspond to roots of an appropriate finite-dimensional semisimple Lie algebra or an infinite-dimensional affine Lie algebra.

This situation may be described by the so-called Adler-van-Moerbeke criterion \[^{44}\]. Here we formally extend this criterion to the case $N_0 > 1$ and apply it to our model with a fixed metric $g^0$.

When all $A_\alpha \neq 0$ in (3.5) and the vectors $u^\alpha$ satisfy the Adler-van-Moerbeke criterion \[^{44}\],

$$
K_{\alpha\beta} = \frac{2\langle u^\alpha, u^\beta \rangle_s}{\langle u^\alpha, u^\alpha \rangle_s} = \hat{C}_{\alpha\beta}, \quad (3.14)
$$

$\alpha, \beta = 0, \ldots, n$, where $\hat{C} = (\hat{C}_{\alpha\beta})$ is the Cartan matrix corresponding to some affine Lie algebra $\mathcal{G}$ \[^{45}\], then the considered Toda-like system (3.2) with fixed $g^0$ is equivalent to an $N_0$-dimensional closed Toda lattice on $(M_0, g^0)$ corresponding to $\mathcal{G}$.

When $\Lambda = 0, \lambda_i \neq 0, i = 1, \ldots, n, n \geq 2$ and

$$
K_{ij} = C_{ij}, \quad (3.15)
$$

$i, j = 1, \ldots, n$, where $C = (C_{ij})$ is the Cartan matrix corresponding to some semisimple Lie algebra $G$ of rank $n$, then the Toda-like system (3.2) with fixed $g^0$ is equivalent to an $N_0$-dimensional open Toda lattice on $(M_0, g^0)$ corresponding to $\mathcal{G}$.

Now, we show that the relations (3.14) and (3.15) are not satisfied for $N_i \in \mathbb{N}$ ($N_i > 1$, since $\lambda_i \neq 0$), $i = 1, \ldots, n$, $n \geq 2$. Indeed, from (3.13) we get

$$
K_{ij} = \frac{2\left[\delta_{ij}/N_j + 1/(N_j-2)\right]}{1/N_j + 1/(N_j-2)} > 0, \quad (3.16)
$$

It follows from (3.16) that the relation (3.15) is never satisfied for $N_i \in \mathbb{N}$, since

$$
C_{ij} = -n_{ij}, \quad n_{ij} \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}, \quad (3.17)
$$

for $i \neq j$ ($n_{ij} = 0, 1, 2, 3$). For the same reason \[^{18}\] the relation (3.14) is never satisfied for positive integers $N_j$ and $n \geq 1$ (see (3.12)). Thus, the model under consideration (3.2) (with fixed $g^0$) is not equivalent to an $N_0$-dimensional (closed or open) Toda lattice (when the number of nonzero terms in the potential (3.5) is greater than one) and seems to be a rather nontrivial object of non-linear analysis.

**Remark 3.** If we consider (at least formally) the model (3.2) with $\Lambda = 0$ and complex dimensions $N_j, j = 1, \ldots, n$, obeying the restriction

$$
\det(G_{ij}) = N_1 \ldots N_n \frac{2-D}{2-N_0} \neq 0, \quad (3.18)
$$

then we find the following solution of (3.15): $n = 2$,

$$
\{N_1, N_2\} = \left\{\frac{1}{3}(2-N_0), \frac{k}{k+2}(2-N_0)\right\}, \quad (3.19)
$$

$k = 1, 2, 3$, corresponding to the Lie algebras $a_2 = sl(3), b_2 = so(5)$ and $g_2$, respectively. (The cosmological case $N_0 = 1$ was considered earlier in Ref. \[^{22}\]. For $N_0 = 1, k = 1$ see also Ref. \[^{23}\].)
3.3. Diagonalization

The case $N_0 > 2$. Let us diagonalize the midisupermetric. This may be useful for quantization of the $\sigma$ model under study. For $N_0 > 2$ the midisupermetric may be diagonalized by the linear transformation

$$\varphi^a = S_i^a \phi^i,$$

(3.20)

where

$$S_i^a \delta S_j^b = G_{ij},$$

(3.21)

$a, b = 1, \ldots, n$; $i, j = 1, \ldots, n$. Then Eq. (3.4) reads:

$$G = \delta_{ab} \varphi^a \otimes d\varphi^b.$$  

(3.22)

An example of diagonalization (3.20), (3.21) is

$$\varphi^1 = q^{-1} \sum_{i=1}^n N_i \phi^i,$$

(3.23)

$$\varphi^b = \left[ N_{b-1} / (S_{b-1}) \right]^{1/2} \sum_{j=b}^n N_j (\phi^j - \delta^{b-1})$$

(3.24)

\[ \text{where} \quad q = q(N_0, D) \equiv \left[ \frac{(D - N_0)(N_0 - 2)}{(D - 2)} \right]^{1/2}, \quad \Sigma_a = \sum_{j=a}^n N_j. \]

(3.25)

Consider a more general class of the diagonalization (3.20) satisfying (3.23) or, equivalently,

$$S_i^a = q^{-1} N_i,$$

(3.26)

Let us introduce

$$S^a = (S_i^a) \in \mathbb{R}^n,$$

(3.27)

$a = 1, \ldots, n$. The relation (3.21) is equivalent to

$$S_i^a G^{ij} S_j^b = \langle S^a, S^b \rangle = \delta^{ab}.$$  

(3.28)

For $a, b = 1$ the relation (3.28) is satisfied identically due to (3.25) and (3.26) (see also (3.8), (3.11)). For $b > 1$

$$0 = \langle S^1, S^b \rangle = q^{-1} N_i G^{ij} S_j^b = q^{-1} \frac{2 - N_0}{2 - D} \sum_{j=1}^n S_j^b,$$

(3.29)

or, equivalently,

$$0 = \sum_{j=1}^n S_j^b.$$  

(3.30)

Here we use the relation

$$G^{ij} N_j = \frac{2 - N_0}{2 - D}.$$  

(3.31)

For $\hat{a}, \hat{b} > 1$ we get from (3.30)

$$\delta^{\hat{a}\hat{b}} = \langle S^{\hat{a}}, S^{\hat{b}} \rangle = \left( \frac{\delta_{ij}}{N_i} + \frac{1}{2 - D} \right) S_i^{\hat{a}} S_j^{\hat{b}} = \delta_{\hat{a}\hat{b}} S_i^{\hat{a}} S_j^{\hat{b}}$$  

(3.32)

or, equivalently,

$$\sum_{i=1}^n N_i S_i^{\hat{a}} S_i^{\hat{b}} = \delta^{\hat{a}\hat{b}}.$$  

(3.33)

Thus, when the condition (3.26) is imposed, the relation (3.21) is equivalent to the set of relations (3.30), (3.33). It is not difficult to verify that these relations are satisfied for $(S_i^a)$ from (3.24). For the inverse matrix we get from (3.28)

$$\hat{S}_a^i = G^{ij} S_j^b \delta_{ba} = G^{ij} S_j^a$$

(3.34)

and, hence, (see (3.26) and (3.31))

$$\hat{S}_a^i = G^{ij} S_j^1 = q^{-1} \frac{2 - N_0}{2 - D} = \frac{q}{D - N_0}.$$  

(3.35)

From the relation

$$\hat{S}_a^i G_{ij} \hat{S}_b^j = \delta_{ab}$$

(3.36)

(following from (3.28)) and Eqs. (3.10), (3.35), (3.36) we get

$$\sum_{j=1}^n N_j \hat{S}_b^j = 0, \quad \sum_{i=1}^n N_i \hat{S}_a^i \hat{S}_b^i = \delta_{ab},$$

(3.37)

$\hat{a}, \hat{b} > 1$. Here we have used the relation

$$\sum_{i=1}^n G_{ij} = N_j \frac{D - 2}{N_0 - 2}.$$  

(3.38)

In the new variables (3.20) satisfying (3.26) the action (3.2) reads:

$$S = \frac{1}{2 \kappa_0^2} \int_{M_0} d^{N_0} \sqrt{|g^0|} \left\{ R[g^0] - \frac{\lambda}{\kappa_0^2} g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b - 2V \right\}.$$  

(3.39)

where

$$V = \sum_{\alpha=0}^n A_\alpha \exp[\hat{\varphi}_\alpha^a \varphi^a].$$  

(3.40)

Here the following notation is used:

$$\hat{u}_a = S^a_i u_i.$$  

(3.41)

It follows from (3.35) that

$$\hat{u}_1 = \hat{S}_a^i u_i = \frac{q}{D - N_0} \sum_{i=1}^n u_i.$$  

(3.42)

For the vectors (3.8), corresponding to the $\Lambda$-term and the curvature components, respectively, we have

$$\hat{u}_0^0 = \frac{2q}{2 - N_0}, \quad \hat{u}_1^1 = -2q^{-1},$$  

(3.43)
\[ j = 1, \ldots, n. \] We denote \( \vec{u}_s = (\vec{u}_2, \ldots, \vec{u}_n) \). Then \( \vec{u}_s^0 = 0 \) (see (3.37)) and
\[ \vec{u}_s^i \vec{u}_s^j = \langle u^i, u^j \rangle_s + 4q^{-2} = 4\left( \frac{\delta_{ij}}{N_s} + \frac{1}{N_0 - D} \right), \quad (3.44) \]
i, j = 1, \ldots, n (see (3.13), (3.43)). Thus the potential (3.40) (see (3.5)) may be written as
\[ V = \Lambda \exp \left[ \frac{2q\varphi^i}{2 - N_0} \right] + \exp(-2q^{-1}\varphi^i) V_s(\vec{\varphi}_s), \quad (3.45) \]
where
\[ V_s(\vec{\varphi}_s) = \sum_{i=1}^{n} (-\frac{1}{2}\lambda_i N_i) \exp(\vec{u}_s^i \vec{\varphi}_s), \quad (3.46) \]
\( \vec{\varphi}_s = (\varphi_2, \ldots, \varphi_n) \) and the vectors \( \vec{u}_s^i \in \mathbb{R}^{n-1} \) satisfy the relations (3.44).

**The cosmological case** \( N_0 = 1 \). In the cosmological case \( M_0 = \mathbb{R} \), \( g^0 = \delta_{\alpha\beta} \) (2.27) satisfies (2.28) (\( N(t) > 0 \) is the lapse function) for the metric (2.2) (3.27) reads [29]:
\[ g = -e^{2\gamma(t)} N^2(t) dt \otimes dt + \sum_{i=1}^{n} e^{2\phi_i(z)} g^i, \quad (3.47) \]
the action (3.2) reads [29]:
\[ S = S[N, \phi] = \frac{1}{\kappa_0^2} \int dt N \left\{ \frac{1}{2} N^{-2} \sum_{i,j} G_{ij} \phi^i \phi^j - V(\phi) \right\}, \quad (3.48) \]
where
\[ G_{ij} = N_i \delta_{ij} - N_i N_j \quad (3.49) \]
are components of a pseudo-Euclidean minisuperspace metric on \( \mathbb{R}^n \) and the potential \( V \) is defined in (3.5).

Let us consider the diagonalization
\[ \varphi^a = S^a_i \phi^i, \quad S^a_i \eta_{ab} S^b_j = G_{ij}, \quad (3.50) \]
\((\eta_{ab}) = \text{diag}(-1,1, \ldots, 1))\), \( a, b = 0, \ldots, n - 1; \ i, j = 1, \ldots, n \) satisfying Eq. (3.26) with \( q \) from (3.25) (\( N_0 = 1 \)). Just as before, it may be shown that in the new variables \( \varphi^a \) the action (3.48) has the form
\[ S = S[N, \phi] = \frac{1}{\kappa_0^2} \int dt N \left\{ \frac{1}{2} N^{-2} \sum_{i,j} \eta_{ab} \dot{\varphi}^a \dot{\varphi}^b - V(\phi) \right\} \quad (3.51) \]
with the potential (3.5) rewritten in the new variables
\[ V = \Lambda \exp[2q\varphi^0] + \exp(2q^{-1}\varphi^0) V_s(\vec{\varphi}_s), \quad (3.52) \]
where \( V_s(\vec{\varphi}_s) \) is defined in (3.46), the vectors \( \vec{u}_s^i \in \mathbb{R}^{n-1} \) satisfy the relations (3.44) with \( N_0 = 1 \), and \( \vec{\varphi}_s = (\varphi_1, \ldots, \varphi_{n-1}) \).

**4 Exact solutions**

Here we consider the metric (2.2) defined on the manifold (2.1) with the relations (2.4) and
\[ M_0 = \mathbb{R}^{N_0}, \quad g^0 = \sum_{a=1}^{N_0} dx^a \otimes dx^a, \quad (4.1) \]
assuming \( N_0 > 2 \). Thus the \( N_0 \)-dimensional section of the metric (2.2) is conformally flat. One of the simplest Ansätze (2.2) is the following:
\[ \gamma = \alpha_0 u(|x|^2), \quad \phi^i = \alpha_i u(|x|^2) + \beta_i, \quad (4.2) \]
where \( \alpha_0, \alpha_i, \beta_i \) are constants, \( i = 1, \ldots, n \), and \( |x|^2 = \sum_{a=1}^{N_0} (x^a)^2 \). We are interested in spherically symmetric solutions to the Einstein equations (2.5) with \( \Lambda = 0 \) governed by the function \( u = u(z) \) and the parameters \( \alpha_0, \beta_i \). The field equations
\[ R_{MN}[g] = 0 \quad (4.3) \]
for the metric (2.2) satisfying (4.1) and (4.2), are equivalent to the following set of equations:
\[ A \equiv -\alpha_0 (4zu'' + 2N_0 u') \]
\[ + 4\alpha_0 \dot{\alpha} z(u')^2 + 2\dot{\alpha} u' = 0, \quad (4.4) \]
\[ B \equiv \dot{\alpha} u'' + [(N_0 - 2)\alpha_0^2 \]
\[ + 2 \alpha_0 \sum_{j=1}^{N_0} N_j \alpha_j - \sum_{j=1}^{N_0} N_j \alpha_j^2 (u')^2 = 0, \quad (4.5) \]
\[ C_i \equiv \lambda_i - \alpha_i e^{2(\alpha_i - \alpha_0) u + 2\beta_i} \times \]
\[ [4zu'' + 2N_0 u' - 4\dot{\alpha} z(u')^2] = 0, \quad (4.6) \]
i = 1, \ldots, n. Here \( u' = du/dz, \ u'' = d^2u/dz^2 \) and
\[ \dot{\alpha} = (2 - N_0) \alpha_0 - \sum_{j=1}^{N_0} N_j \alpha_j. \quad (4.7) \]

The reduction of (4.3) to Eqs. (4.4)-(4.6) takes place due to the following representation for the Ricci tensor components (2.8) and (2.9) in our case (4.2):
\[ R_{ab}[g] = A \delta_{ab} + 4B x^a x^b, \quad (4.8) \]
\[ R_{m,n}[g] = C_i g_{m,n}, \quad (4.9) \]
a, b = 1, \ldots, N_0; \ i = 1, \ldots, n.

Here we adopt the following Ansatz for the function \( u(z) \) from (4.2):
\[ u(z) = \ln(C + z), \quad (4.10) \]
where \( C \) is a constant. Under the substitution (4.10) Eq. (4.4) is satisfied identically if
\[ \dot{\alpha} = -1, \quad \alpha_0 = -1/N_0. \quad (4.11) \]
(We note that \( u'' = -(u')^2 \). For \( C \neq 0 \), (4.4) implies (4.11.).) Then, (4.4) and (4.5) read:
\[ \sum_{j=1}^{n} N_j \alpha_j = 2 - \frac{2}{N_0}, \quad (4.12) \]
\[ \sum_{j=1}^{n} N_j \alpha_j^2 = \frac{(N_0 - 1)(N_0 - 2)}{N_0^2}. \quad (4.13) \]
Eqs. (4.6) are equivalent to the relations
\[ 2(\alpha_0 - \alpha_i) = -1, \quad 2N_0\alpha_i e^{2\beta_i} = \lambda_i, \quad (4.14) \]
i = 1, \ldots, n. From (4.11) and (4.14) we obtain
\[ \alpha_i = \frac{1}{2} - \frac{1}{N_0}, \quad e^{2\beta_i} = \frac{\lambda_i}{N_0 - 2} \neq 0. \quad (4.15) \]

A substitution of (4.15) into (4.12), (4.13) gives the following Diophantus equation for the dimensions \( N_v \):
\[ \sum_{j=1}^{n} N_j = \frac{4(N_0 - 1)}{N_0 - 2}. \quad (4.16) \]

Eq. (4.16) has the solutions
\[ \sum_{j=1}^{n} N_j = 8, 6, 5 \quad \text{for} \quad N_0 = 3, 4, 6, \quad \text{(4.17)} \]
respectively. From (2.2), (4.1), (4.2), (4.10), (4.11) and (4.15) we obtain the metric
\[ g = \left[C + |x|^2\right]^{-2/N_0} \left[ \sum_{a=1}^{N_0} dx^a \otimes dx^a + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i \right] \quad (4.18) \]
defined on the manifold
\[ M = R^{N_0}_C \times M_1 \times \ldots \times M_n, \quad (4.19) \]
where
\[ R^{N_0}_C = \{ x \in R^{N_0} : C + |x|^2 > 0 \} \subset R^{N_0} \quad (4.20) \]
is an open domain in \( R^{N_0}, \ C \in R \). The metric (4.18) describes, for \( N_0 = 3, 4, 6 \), three families of spherically symmetric (\( O(N_0) \)-symmetric) solutions to the vacuum Einstein equations (4.3) with \( n \) internal Einstein spaces of nonzero curvature (\( M_i, g^i \)) (2.4). It follows from (4.16), (4.17) that
\[ D = N_0 + \sum_{j=1}^{n} N_j = \frac{N_0^2}{N_0 - 2} + 2 = 11, 10, 11, \quad (4.21) \]
\[ n \leq n_0 = 4, 3, 2 \quad (4.22) \]
for \( N_0 = 3, 4, 6 \), respectively.

4.1. Nonsingular solutions

For \( C > 0 \), \( R^{N_0}_C = R^{N_0} \) and the metric (4.18) describes spherically symmetric nonsingular solutions to the Einstein equations defined on the manifold
\[ R^{N_0} \times M_1 \times \ldots \times M_n. \quad (4.23) \]
(It should be stressed that the \( N_0 \)-dimensional part of the metric (4.18) has Euclidean signature.) A special case of this solution with \( N_0 = 6, \ n = 1, \ N_1 = 5 \) was recently considered in [42].

4.2. Exceptional solutions

Let us consider the solution (4.18) with \( C = 0 \). It can be written as follows:
\[ g = dp \otimes dp + \rho^2 g_*, \quad \rho = \alpha^{-1}|x|^\alpha \quad (4.24) \]
where \( \alpha = 1 - 2/N_0 \) and
\[ g_* = \alpha^2 \left[g(S^{N_0-1}) + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i \right] \quad (4.25) \]
is the Einstein metric on the manifold
\[ M_* = S^{N_0-1} \times M_1 \times \ldots \times M_n. \quad (4.26) \]
Here \( g(S^{N_0-1}) \) is the canonical metric on an \( (N_0-1) \)-dimensional sphere \( S^{N_0-1} \). The metric \( g_* \) in (4.24) satisfies the relation
\[ \text{Ric} \left[ g_* \right] = (D - 2)g_*, \quad (4.27) \]
where \( \text{Ric} \left[ g_* \right] \) is the Ricci tensor corresponding to \( g_* \) and \( D = \text{dim} M \). The metric (4.24) is defined on the manifold \( R_+ \times M_* \) (see Remark 1) and is non-flat, as may be verified using the relations (6.2)-(6.4) from the Appendix. The \( N_0 \)-dimensional section of the metric is also non-flat (due to ”deficit” of the spherical angle). Since the solution (4.24) is an attractor for (4.18) as \( |x| \to \infty \), we see that the metric (4.18) and its \( N_0 \)-dimensional section have non-flat asymptotics.

4.3. Solutions with arbitrary signature

The solution (4.18) may be considered as a special case of the following solutions with arbitrary signature of “our” space:
\[ g = \left[C + \eta_{ab}x^a x^b\right]^{-2/N_0} \left[ \eta_{ab}dx^a \otimes dx^b + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i \right] \quad (4.28) \]
\[ + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i \quad (4.28) \]
Here
\[ \eta = (\eta_{ab}) = \text{diag}(w_1, \ldots, w_{N_0}), \quad w_a = \pm 1. \quad (4.29) \]
The metric (4.28) is defined on the manifold
\[ M = R^{N_0}_{C, \eta} \times M_1 \times \ldots \times M_n, \quad (4.30) \]
where
\[ R^{N_0}_{C, \eta} = \{ x \in R^{N_0} : C + \eta_{ab}x^a x^b > 0 \} \subset R^{N_0} \quad (4.31) \]
is supposed to be non-empty (i.e. the case when \( C < 0 \) and all \( w_a = -1 \) in (4.29) is excluded). The metric (4.28) satisfies the vacuum Einstein equations (4.3). It may be obtained from (4.18) by a Wick-type rotation, i.e. we write \( x^a = w^a a^{1/2} x^a, \ w_a > 0 \), in (4.18) and then perform an analytical continuation in \( w_a \).
Proposition 1. The Riemann tensor squared for the metric (4.28) has the form
\[ I[g] = R_{MNQP}[g] R^{MNQP}[g] = (C + x^2)^{-2-2\alpha} (\bar{I}_1 + \bar{I}_2), \] (4.32)
where
\[ \bar{I}_1 = (\alpha - 1) (N_0 - 1) [16 C^2 + 2 (N_0 - 2) (2C + (\alpha + 1)x^2)], \] (4.33)
\[ \bar{I}_2 = -4\alpha^2 N (N_0 - 2) x^2 (C + x^2) + (C + x^2)^2 \sum_{i=1}^{n} \left( \frac{N_0 - 2}{\lambda_i} \right)^2 I[g^i] + 2\alpha^4 N (N_0 - 1) (x^2)^2 + 4\alpha^2 N (N_0 - 1) (\alpha x^2 + C)^2; \] (4.34)
here \( \alpha = 1 - 2/N_0, x^2 = \eta_{ab} x^a x^b, N = \sum_{j=1}^{n} N_j \) and \( I[g^i] \) is the Riemann tensor squared for the metric \( g^i \).

Proof. Eqs. (4.32)-(4.34) may be obtained using the formula (6.10) from the Appendix. But a simpler way is to calculate first the Riemann tensor squared in the Euclidean case \( \eta_{ab} = \delta_{ab} \),

\[ g = [C + r^2]^\gamma \left\{ \frac{dr \otimes dr + r^2 d\Omega_{N_0-1}^2}{C + r^2} + \sum_{i=1}^{n} \frac{\lambda_i}{N_0 - 2} g^i \right\}, \] (4.35)
where \( r^2 = \delta_{ab} x^a x^b \) and \( d\Omega_{N_0-1}^2 = g(S^{N_0-1}) \) is the metric on \( S^{N_0-1} \), using the “cosmological” relation (6.15) from the Appendix, and then perform the Wick rotation \( r^2 \rightarrow \eta_{ab} x^a x^b \).

Proposition 2. For the metric (4.28) with a non-Euclidean signature \( (\eta_{ab}) \neq (\delta_{ab}) \) and \( C \neq 0 \)
\[ R_{MNQP}[g] R^{MNQP}[g] \rightarrow +\infty \] (4.36)
as \( C + \eta_{ab} x^a x^b \rightarrow +0 \).

Proof. From (4.32)-(4.34) we obtain
\[ R_{MNQP}[g] R^{MNQP}[g] \sim A_1 [C + \eta_{ab} x^a x^b]^{-2-2\alpha}, \] (4.37)
where
\[ A_1 = (\alpha - 1)(N_0 - 1) C^2 [16 + 2(N_0 - 2)(1 - \alpha)^2] + 2N \alpha^2 C^2 [2 + (N_0 - 1) \alpha^2 + 2(N_0 - 1) - \alpha^2] > 0. \] (4.38)

Then (4.36) follows from (4.37), (4.38) and \( \alpha > 0 \).

The curvature-splitting trick. The solution (4.28) with \( n \) internal spaces may be obtained from the one with \( n = 1 \) by so-called “curvature-splitting” trick. Let us consider a set of \( k \) Einstein manifolds \( (M_i, h^i) \) of nonzero curvature, i.e.
\[ \text{Ric} (h^i) = \mu_i h^i, \] (4.46)
where \( \mu_i \neq 0 \) is a real constant, \( i = 1, \ldots, k \). Let \( \mu \neq 0 \) be a real number. Then
\[ h = \sum_{i=1}^{k} \frac{\mu_i}{\mu} h^i. \] (4.47)
is an Einstein metric, (correctly) defined on
\[ \mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_k \]
and satisfying
\[ \text{Ric} (h) = \mu h. \]
Indeed,
\[ \text{Ric} (h) = \sum_{i=1}^{k} \text{Ric} \left( \frac{\mu_i}{\mu} h^i \right) = \sum_{i=1}^{k} \text{Ric}(h^i) = \sum_{i=1}^{k} \mu_i h^i = \mu h. \]
(Here we have simplified the notations according to case the action (2.12) reads (we put here
\[ = \text{Ric} \left( \frac{\mu}{\mu} h \right) \]
\[ \text{Indeed,} \]
\[ \text{Ric} (h) = \sum_{i=1}^{k} \text{Ric} \left( \frac{\mu_i}{\mu} h^i \right) \]
\[ = \sum_{i=1}^{k} \text{Ric}(h^i) = \sum_{i=1}^{k} \mu_i h^i = \mu h. \]
(4.50)

5 The case \( N_0 = 2 \)
Consider now the exceptional case \( N_0 = 2 \). In this case the action (2.12) reads (we put here \( \kappa_0^2 = 1 \))
\[ S = S_{\sigma}[g^0, \gamma, \phi] \]
\[ = \frac{1}{2} \int_{M_0} d^2x \sqrt{|g^0|} \exp \left( \sum_{i=1}^{n} N_i \phi^i \right) \left\{ R[g^0] \right. \]
\[ - \tilde{G}_{ij}(\partial \phi^i)(\partial \phi^j)g^{0\mu\nu} + 2(\partial \gamma) \sum_{j=1}^{n} N_j \partial \phi^j \]
\[ + \sum_{i=1}^{n} \lambda_i N_i e^{-2\phi^i+2\gamma} - 2\Lambda e^{2\gamma} \right\}. \]
(5.1)
where \( \tilde{G}_{ij} \) is the cosmological minisuperspace metric (3.49). From (5.1) we see that the minisuperspace metric crucially depends upon the choice of \( \gamma \). For \( \gamma = 0 \) we get from (5.1) the action with a conformally flat minisuperspace metric of pseudo-Euclidean signature
\[ S = \frac{1}{2} \int_{M_0} d^2x \sqrt{|g^0|} \exp \left( \sum_{i=1}^{n} N_i \phi^i \right) \left\{ R[g^0] \right. \]
\[ - \tilde{G}_{ij}(\partial \phi^i)(\partial \phi^j)g^{0\mu\nu} + \sum_{i=1}^{n} \lambda_i N_i e^{-2\phi^i} - 2\Lambda \right\}. \]
(5.2)
Another choice of the conformal frame parameter
\[ \gamma = -\frac{1}{2} \sum_{i=1}^{n} N_i \phi^i \]
leads us to the action
\[ S = \frac{1}{2} \int_{M_0} d^2x \sqrt{|g^0|} \exp \left( \sum_{i=1}^{n} N_i \phi^i \right) \left\{ R[g^0] \right. \]
\[ - \sum_{i=1}^{n} N_i(\partial_\mu \phi^i)(\partial_\nu \phi^i)g^{0\mu\nu} \]
\[ + \left( \sum_{i=1}^{n} \lambda_i N_i e^{-2\phi^i} - 2\Lambda \right) \exp \left( - \sum_{i=1}^{n} N_i \phi^i \right) \right\}. \]
(5.4)
with a Euclidean conformally flat minisuperspace metric. Note that in Ref. [3] the action (5.2) was reduced to a “string-like” form (for \( n = 1 \) see, for example, [13]).

6 Appendix

6.1. Riemann tensor
Here we consider the metric
\[ g = g^0 + \sum_{i=1}^{n} e^{2\phi^i(x)}g^i. \]
(6.1)
defined on the manifold (2.1), where the metrics \( g^0 \) and \( g^i \) are defined on \( M_0 \) and \( M_i \) respectively, \( i = 1, \ldots, n \). The nonzero components of the Riemann tensor corresponding to (6.1) are
\[ R_{\mu\nu\rho\sigma}[g] = R_{\mu\nu\rho\sigma}[g^0]. \]
(6.2)
\[ R_{\mu m,n}[g] = -R_{m,\nu\mu}[g] = -R_{\mu m,n,\nu}[g]. \]
(6.3)
\[ R_{m,\nu\mu}[g] = -e^{2\phi^j}g_{m,n,i}[\nabla_\mu [\nabla^0 \phi^i]](\partial_\nu \phi^j) \]
\[ + (\partial_\nu \phi^j)(\partial_\rho \phi^i) \]
\[ = \delta_{ik}\delta_{jl}R_{m,n,p,i,j}[g] \]
\[ + e^{2\phi^i+2\phi^j}g^{0\mu\nu}(\partial_\mu \phi^j)(\partial_\nu \phi^i) \]
\[ + e^{2\phi^j}g_{00}^i(\partial_\nu \phi^j)(\partial_\rho \phi^i)g_{m,n,i}^j \]
\[ - \delta_{ik}\delta_{jl}R_{m,n,p,i,j}[g]. \]
(6.4)
where the indices \( \mu, \nu, \rho, \sigma \) correspond to \( M_0, m_i, n_i, p_i, q_i \) to \( M_i \); \( i, j, k, l = 1, \ldots, n \), \( \nabla[g^0] \) is a covariant derivative with respect to \( g^0 \).

The relations (6.2)-(6.4) may be obtained from the following relations for the nonzero components of the Christofel-Schwarz symbols:
\[ \Gamma^\nu_{\mu\rho}[g] = \Gamma^\nu_{\mu\rho}[g^0], \]
(6.5)
\[ \Gamma^m_{n,\nu\mu}[g] = \Gamma^m_{n,\nu\mu}[g] = \delta_{mn}\partial_\nu \phi^i, \]
(6.6)
\[ \Gamma^\mu_{m,n,i}[g] = -g^{0\mu\nu}(\partial_\nu \phi^j)g_{m,n,i}^j, \]
(6.7)
\[ \Gamma^m_{n,p,i}[g] = \Gamma^m_{n,p,i}[g^i]. \]
(6.8)

6.2. Riemann tensor squared.
We denote the squared Riemann tensor by
\[ I[g] \equiv R_{MNPQ}[g]R^{MNPQ}[g]. \]
(6.9)
As follows from Eqs. (6.2)-(6.4), for the metric (6.1)
\[ I[g] = I[g^0] + \sum_{i=1}^{n} \left\{ e^{-4\phi^i}I[g^0] - 4e^{-2\phi^i}U[g^0, \phi^i]R[g^0] \right. \]
\[ - 2N_i U^2[g^0, \phi^i] + 4N_i V[g^0, \phi^i] \}
\[ + \sum_{i,j=1}^{n} 2N_i N_j [g^{0\mu\nu}(\partial_\mu \phi^i)(\partial_\nu \phi^j)]^2, \]
(6.10)
where $R[g^i]$ is the scalar curvature of $g^i$ and $N_i = \dim M_i$, $i = 1, \ldots, n$. In (6.10)

$$
U[g, \phi] = g^{MN} (\partial_M \phi) \partial_N \phi,
$$

$$
V[g, \phi] = g^{M_1 N_1} g^{M_2 N_2} \times \nabla_M (\partial_M \phi) (\partial_M \phi) \partial_N \phi)/\nabla_N (\partial_N \phi) (\partial_N \phi),
$$

(6.11)

$$
(6.12)
$$

where $\nabla = \nabla[g]$ is a covariant derivative with respect to $g$.

### 6.3. The cosmological case

Consider now the special case of (6.10) with $M_0 = (t_1, t_2)$, $t_1 < t_2$. Thus we consider the metric

$$
g_c = -B(t)dt \otimes dt + \sum_{i=1}^n A_i(t)g^i,
$$

(6.13)

defined on the manifold

$$
M = (t_1, t_2) \times M_1 \times \ldots \times M_n.
$$

(6.14)

From (6.11) we obtain the Riemann tensor squared for the metric (6.13)

$$
I[g_c] = \sum_{i=1}^n \left\{ A_i^{-1}I[g^i] + A_i^{-3}B^{-1} \dot{A}_i^2 R[g^i] \right\} - \frac{1}{8} N_i B^{-2} A_i^{-4} \dot{A}_i^4 + \frac{1}{4} N_i B^{-2} (2A_i^{-1} \dot{A}_i - B^{-1} \dot{B}_i A_i^{-1} \dot{A}_i - A_i^{-2} \dot{A}_i^2)^2)
$$

$$
+ \frac{1}{8} B^{-2} \left\{ \sum_{i=1}^n N_i (A_i^{-1} \dot{A}_i)^2 \right\}^2.
$$

(6.15)

### 6.4. Conformal transformation

We present for convenience the well-known relations

$$
e^{-2\gamma} R_{\mu\nu\rho\sigma} [e^{2\gamma} g^0] = R_{\mu\nu\rho\sigma} [g^0] + Y_{\nu\rho} g_{\mu\sigma} - Y_{\mu\rho} g_{\nu\sigma} - Y_{\nu\sigma} g_{\mu\rho} + Y_{\mu\sigma} g_{\nu\rho},
$$

(6.16)

$$
R_{\mu\nu} [e^{2\gamma} g^0] = R_{\mu\nu} [g^0] + (2N_0) Y_{\mu\nu} - g_{\mu\nu} g^{\rho\sigma} Y_{\rho\sigma},
$$

(6.17)

$$
\Delta [e^{2\gamma} g^0] = e^{-2\gamma} \left\{ \Delta_0 + (N_0 - 2) g^{00} \right\} (\partial_\mu \gamma) \partial_\nu.
$$

(6.18)

where, as in Subsec. 2.1, the metric $g^0$ is defined on $M_0$, $\dim M_0 = N_0$, $\Delta_0$ is the Laplace-Beltrami operator on $M_0$ and

$$
Y_{\mu\nu} = \gamma_{\mu\nu} - \gamma_\mu \gamma_\nu + \frac{1}{2} \gamma_{\mu\nu} \gamma^\rho \gamma^\rho.
$$

(6.19)

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