Non Perturbative Check of N=2, D=4 Heterotic/Type II Duality

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We review here some of the checks of string-string duality between N=2, D=4 Heterotic and Type II models. The heterotic low energy field theory is reproduced on the type II side. It is also shown to be a generalization in the string context of the rigid N = 2 Yang-Mills theory of Seiberg and Witten which is exactly known. The non perturbative information of this rigid theory is then recovered on the type II side. This talk is based on a work done in collaboration with I. Antoniadis.

1. INTRODUCTION

We would like here to review some of the arguments for “String-String Dualities.” Let us first precise what we mean by this. Any string theory is defined in perturbation theory. In particular, it is characterized by a Fock space, whose states have their dynamics dictated by string loop calculations. The physics described by such models is perfectly consistent at any energy scale, but is not complete in the sense that it does not take into account non perturbative phenomena such as space-time instantons which are present in field theory. One aim should then be to try to define at the string level what non perturbative could mean, but this direction is perhaps out of reach as far as we do not have a second quantized formulation of string theory. Consequently, each string theory should be viewed has the perturbative part of some exact physics. Now, it would be miraculous if all these supposed exact physics where the same. Firstly, this would be a strong sign that with string theory, we go in the right way to find a unifying complete description of physics. And, by the way, since all these string perturbation theories are different, they could describe different perturbative regimes of the underlying exact theory. In particular, it would become possible to describe non perturbative effects in the context of a particular string model, simply by studying other perturbative string models. This equivalence of all these models is the so-called String-String Duality.

Various checks of this scenario are available. Some of them, which we review here, are explicit and quantitative; they are based on the following ideas. If one chooses two string theories whose spectra have some common part, the interactions of this common spectrum must be the same. Then, it is consistent to say that the spectra which are not present in each model are fully non perturbative states. As an exemple, we will consider here a class of D = 4, N = 2 type II and heterotic models which have exactly the same perturbative spectrum. More precisely, we will review the exemple of ref. Other ones can be found in refs. Notice that some type I realizations of cousin models are also given in. An important point to note is that N = 2 space-time supersymmetry implies that the moduli space of such a theory is splitted in a product of two manifolds, one for the scalars of the vector multiplets and the other for the scalars of the neutral hypermultiplets. The sigma models based on these manifolds describe the low energy Lagrangians for the vector multiplets and the neutral hypermultiplets, respectively. All these states are in fact the complete massless spectrum. For type II A compactifications on Calabi-Yau threefolds with Hodge numbers h₁₁ and h₁₂, we have h₁₁ vector
multiplets and the graviphoton which give rise to the abelian gauge group $U(1)^{h_{12}+1}$. Also, there are $h_{12} + 1$ neutral hypermultiplets where the 1 counts for the type II dilaton. Now, as we said before, the vector multiplets are totally decoupled to these hypermultiplets and in particular they do not see at all the type II coupling constant which sits in the dilaton. Consequently, the vector multiplets sigma model is not renormalized at all both perturbatively and non perturbatively. Note that this sigma model is renormalized in $\alpha'/R$ by world-sheet instantons, where $R$ characterizes the size of the Calabi-Yau. However, the $(2,2)$ underlying superconformal field theory can be realized by a type II B compactification on the so-called mirror Calabi-Yau. Consequently, this theory describes the same target space physics. However, the vectormultiplet scalars parametrize the complex structure i.e. the shape of this mirror manifold and are then independant of its volume. Therefore, their sigma model can be determined in the mirror Calabi-Yau large volume limit $R^* \to +\infty$ where $\alpha'/R^* = 0$ and no world-sheet correction survive \cite{10,11}. The final type II result obtained by classical considerations on the world-sheet and string loop expension is presented in Section 2 in a particular exemple.

In $N=2$ rigid quantum field theory, the analysis of Seiberg and Witten \cite{12} showed how the knowledge of the singularities on a moduli space which are interpreted as infrared divergences can be used to determine the full exact Lagrangian. Therefore it is natural to extend this idea in the context of strings: if two models have some singularities in common in their respective moduli spaces, we expect that the sigma models based on these manifold should be the same at least locally around these singularities and therefore the underlying string theories should be dual. Now, in particular, the type II moduli space we consider here have a point from which two branches of singularities emerge \cite{13}. This point is similar to the classical singularity of the $N=2$, $SU(2)$ pure Yang-Mills rigid theory \cite{12} where we recover the $SU(2)$ gauge group for zero VEV of the Higgs field. In perturbation theory, this singularity remains and is then splitted in two points when the non perturbative effects are taken into account. If this analogy is valid, a heterotic model which generalizes this rigid situation has a chance to be dual to the type II original theory. Such a model can be obtained by a compactification on $K_3 \times T^2$ in order to recover $N=2$ supersymmetry. Since the heterotic dilaton sits now in one of the vector multiplets, their sigma model has perturbative and non perturbative corrections. The latter are unknown, and the former will be determined in Section 3 for a particular model, following the method introduced in \cite{14,15}. The result will agree with the one found on the type II side and will be the first check of the equivalence of the two models.

In Section 4, we will derive the heterotic perturbative duality group. By this, we mean the set of T-duality transformations of the one-loop prepotential (which is in fact the full perturbation theory due to a $N=2$ non renormalization theorem) and axionic shift. When put in matrix representation, we will recover the perturbative monodromy generator of the rigid Yang-Mills theory. In this context, monodromy means transformation of an original theory to another wich describes the same physics.

In Section 5, we will find that the perturbative heterotic duality group is part of the exact type II monodromy group. This will be our second check of the equivalence of the two theories. Finally, we would like to give a non perturbative argument for this string-string duality. Even if non perturbative effects are not defined in the context of string theory, the rigid limit of the low energy limit of the heterotic model is exactly known. Moreover, this information is totally equivalent to the knowledge of the second monodromy generator of the rigid theory \cite{12}. Therefore, if we identify this matrix in the exact type II monodromy group, we will end up with the result that the space-time instanton corrections of the heterotic theory which survive in the flat limit are present in the type II A side where they originate from world-sheet instanton corrections! As a final result, we will see that the type II monodromy group contains a generator whose interpretation on the heterotic side corresponds roughly speaking to an exchange of the coupling constant with the radius of compactification on the torus.
2. THE TYPE II MODEL

Let us consider the rank-3 type II superstring compactification on the Calabi-Yau manifold $X_8$ with Betti numbers $b_{1,1} = 2$ and $b_{1,2} = 86$, giving rise to 2 vector multiplets (besides the graviphoton) and $86 + 1$ hypermultiplets including the dilaton. This manifold is defined as a hypersurface of degree 8 in the weighted projective space $\mathbb{WCP}^4[1, 1, 2, 2, 2]$. The scalars of the two vector multiplets parametrize the Kähler structure of $X_8^*$ or, equivalently, the complex structure of the mirror $X_8^*$ defined as the zero locus of the polynomial

$$z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 - 8\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^4 z_2^4,$$

modded by a discrete group $\mathbb{Z}_4$. Its complex structure is then conveniently described in terms of $x = -2\phi/(8\psi)^{1/4}$ and $y = 1/(2\phi)^{1/2}$.

The Yukawa couplings are worked out in an explicit form in refs. [15,16]. As follows from $N = 2$ special geometry, there is a preferred coordinate system in which they are given as triple derivatives of an analytic prepotential [16,17]. The corresponding special coordinates $t_1$ and $t_2$ are given by the inverted mirror map,

$$x(t_1, t_2) = \frac{1}{h(t_1)}(1 + \mathcal{O}(t_1, q_2)),$$

$$y(t_1, t_2) = q_2 l(t_1)(1 + \mathcal{O}(t_1, q_2)),$$

where $q_2 = e^{2\pi i t_1}$, $q_2 = e^{2\pi i t_2}$ and $\mathcal{O}(t_1, q_2)$ denotes terms which are at least of order one in $q_2$ and order zero in $q_1$. The function $l(t_1) = 1 + \mathcal{O}(q_2)$, while $h(t_1) = 1/q_1 + 104 + \mathcal{O}(q_2)$ will be given below. The prepotential can then be expressed as

$$\mathcal{Y} = -2t_1^2 t_2 - \frac{4}{3} t_1^3 + P(t_1, t_2) + g(t_1) + g^{NP}(t_1, t_2),$$

where $P(t_1, t_2)$ is a quadratic polynomial in $t_1$ and $t_2$ whose coefficients are integration constants, and

$$g(t_1) = \frac{1}{8\pi i} \sum_{n \geq 1} \frac{y_n}{n^3} g_n^n,$$

$$g^{NP}(t_1, t_2) = \sum_{n \geq 1, m \geq 0} g_{nm}^{NP} q_2^n q_1^m$$

for $y_n$ and $g_{nm}^{NP}$ known integer coefficients related to world-sheet instanton numbers.

The above form of the prepotential suggests that this type II model has a heterotic dual realization by identifying $t_2$ with the heterotic dilaton $S$ and $t_1$ with the single heterotic modulus $T$ [3]. In fact, the general rank-3 heterotic prepotential $F$ takes the form

$$F = -2ST^2 + f(T) + f^{NP}(T, S),$$

where the first two terms of the right hand side correspond to the classical and one-loop contributions, respectively. $f^{NP}$ denotes the non-perturbative corrections

$$f^{NP}(T, S) = \sum_{n \geq 1} f_n^{NP} q_2^n$$

where $q_2 = e^{2\pi i S}$ and $\langle S \rangle = \frac{\theta}{\pi} + \frac{\psi}{2\pi}$ in terms of the $\theta$-angle and the four-dimensional string coupling constant $g_s$.

This identification is motivated by the fact that in the weak coupling limit $q_2 \to 0$ (or equivalently $y \to 0$), the discriminant locus of the mirror Calabi-Yau $X_8^*$

$$\Delta = \{(1 - 2^8 x)^2 - 2^{18} x^2 y\{1 - 4y\} = 0 \}$$

becomes a perfect square, in complete analogy with the situation in the rigid SU(2) $N = 2$ supersymmetric Yang-Mills theory where the classical singular SU(2) point splits into two separate branches in the quantum theory [12]. The first factor in eq. (8) defines the conifold singularity while the second factor gives rise to an isolated singularity $y = 1/4$ corresponding to the infinite strong coupling limit $S \to 0$ in the heterotic theory [13].

The guide to construct the heterotic realization is to first find the classical $T$-duality group [1,12]. This can be determined by the set of transformations which identify all solutions of the conifold singularity in the weak coupling limit $y = 0, x = 1/256$ in terms of the special coordinate $T$,

$$h(T) = 256 \quad \text{and} \quad S = i\infty,$$

where $h$ is defined in eq. (8). In fact $h$ was found to be (up to an additive constant 104) the Haup-
modul for $\Gamma_0(2)_+$ which should play the role of the heterotic $T$-duality group,
\[ h(T) = \frac{[\theta_3^4(T) + \theta_4^4(T)]^4}{16 [\eta(T)\eta(2T)]^3}, \]
where $\theta_{3,4}$ are the Jacobi $\theta$-functions and $\eta$ the Dedekind function.

$\Gamma_0(2)_+$ is the group of modular transformations
\[ T \rightarrow \frac{aT + b}{cT + d} = MT \quad ; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
with $a, b, c, d$ integers, generated by any two of the transformations
\[ W : \quad T \rightarrow -\frac{1}{2T}, \quad U : \quad T \rightarrow T + 1, \quad V : \quad T \rightarrow -\frac{2T + 1}{2T}. \]

$W$ and $V$ generate the little groups of the fixed points $i/\sqrt{2}$ and $(i - 1)/2$ which are of order 2 and 4, respectively. $b$ is a bijection between the $\Gamma_0(2)_+$ fundamental domain and the Riemann sphere with three punctures such that $h(i/\sqrt{2}) = 256$, $h((i - 1)/2) = 0$ and $h(i\infty) = \infty$. Finally the solutions of the equation (11) for the conifold singularity in the perturbative limit consist in the images of $i/\sqrt{2}$ under the action of $\Gamma_0(2)_+$, which shows also that this group is the modular one for $T$.

The dual of the above type II model should be a rank-3 $N = 2$ heterotic vacuum with one vector multiplet modulus $T$ besides the dilaton $S$, and 87 hypermultiplets. The method of constructing such models is described in ref. [3], although the exact $K_3$ compactification which leads to this particular spectrum is not presently known. Here we concentrate on the $T^2$ lattice part which is relevant for understanding the moduli space of vector multiplets. A heterotic dual should have a one-loop correction to its prepotential $f(T)$ equal to the type II quantity $g(t_1)$ which originates from world-sheet instantons [3]. In order to determine such a one-loop contribution $f(T)$ which is actually the complete perturbative correction due to a $N = 2$ non renormalization theorem based on analyticity and the invariance under the axionic shift. In $N = 2$ supergravity, the Kähler metric takes the form
\[ K_{T\bar{T}} = K^{(0)}_{T\bar{T}} \left[ 1 + \frac{2i}{S - S} I + \cdots \right], \]
where the tree-level metric $K^{(0)}_{T\bar{T}} = -2/(T - \bar{T})$ and $I(T, \bar{T})$ is given by
\[ I = \frac{i}{8} \left( \partial_T - \frac{2}{T - T} \right) \left( \partial_T - \frac{4}{T - T} \right) f + c.c. \]

Now, $I$ can be computed by a one loop string calculation of an amplitude involving the antisymmetric tensor, using the method of ref. [20]. The result must be invariant under modular transformations on $T$. From the expression (14), one can easily deduce the fifth derivative of the one loop prepotential, $f^{(5)}$, and see that it is a modular function of weight 6. An expression for $f(T)$ up to a quartic polynomial is then
\[ f(T) = \int_{I_0}^T \frac{(T - T')^4}{4!} f^{(5)}(T') dT', \]
where $I_0$ is an arbitrary point. The path of integration should not cross any singularity of $f^{(5)}$, while the result of the integral depends on the homology class of such paths. Different choices of homology classes of paths change $f$ by quartic polynomials. Moreover under a modular transformation (11), $f$ does not transform covariantly. Using its integral representation (13), we see that it has a weight $-4$ up to an addition of a quartic polynomial, $\mathcal{M}(T)$,
\[ f(MT) = (\det M)^2 (cT + d)^{-4} [f(T) + \mathcal{M}(T)]. \]

In order to leave the physical metric invariant, one can see that the polynomials $\mathcal{M}(T)$ are quartic with real coefficients.

Finally, in order to guarantee modular invariance of the full effective action, the dilaton should also transform. Imposing the general condition that modular transformations may be compensated by Kähler transformations, one has to redefine the dilaton $S$ to
\[ S - \frac{c}{2} \left( \frac{f_T + \mathcal{M}_T}{cT + d} + c^2 \frac{f + \mathcal{M}}{(cT + d)^2} + \frac{\mathcal{M}_{TT}}{12} + \lambda_M, \right. \]
where the additive real constant $\lambda_M$ correspond to the perturbative heterotic symmetry of the axion shift.

The above discussion applies to any rank-3 $N = 2$ heterotic string compactification. We now specialize to the dual candidate of the type II model described in Section 2, for which the classical duality group is $\Gamma_0(2)$. Furthermore, the order 2 fixed point in its fundamental domain, $T = i/\sqrt{2}$, which was found from the conifold singularity in the type II model, should correspond in the heterotic realization to the perturbative $SU(2)$ enhanced symmetry point. Imposing these two constraints, it is possible to find an explicit heterotic model \[\text{[1]}\]. Its one-loop Kähler metric involves

$$I = \sum_{\ell=1}^{6} \frac{d^2 \tau}{3/2} \tilde{C}_\ell(\tau) \partial_\tau \left( \tau^{1/2} \hat{Z} \right)$$

$$Z = \sum_{pL,pR \in \Gamma_\ell} e^{\pi i \tau |p_\ell|^2} e^{-\pi i \tau p_\ell^2},$$

where the sum over $\ell$ extends over six lattices denoted by $\Gamma$ which parametrise the left and right momenta

$$p_L = \frac{i \sqrt{2}}{T - \bar{T}} (n_1 + n_2 \bar{T}^2 + 2m\bar{T})$$

$$p_R = \frac{i \sqrt{2}}{\bar{T} - T} (n_1 + n_2 T\bar{T} + m(T + \bar{T})).$$

$\tilde{C}_\ell$’s are $T$-independent modular functions with well-defined transformation properties dictated by modular invariance of the integrand. Actually $\tilde{C}_\ell$ is the trace of $(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} / \bar{\eta}$ in the Ramond sectors of the corresponding remaining conformal blocks. Finally, the integral is over the Teichmüller parameter of the world-sheet torus inside its fundamental domain. A corresponding expression of the fifth derivative of the one loop prepotential $f^{(5)}(T)$ can then be obtained. A study of its singularities and the knowledge of its weight six under $\Gamma_0(2)$ modular transformations can be used to get the expression

$$f^{(5)}(T) = \frac{64}{\pi} \left( \frac{h_T}{h - h \left( \frac{i}{\sqrt{2}} \right)} \right)^3 5h + 3h \left( \frac{i}{\sqrt{2}} \right) / h^2.$$

A first non-trivial check of the proposed duality between this heterotic model and the type II compactification described in Section 2, is the comparison of the two corresponding prepotentials $F$ and $Y$ of eqs. \[\text{13}\] and \[\text{14}\]. Indeed, the identification $t_1 = T$ and $t_2 = \bar{T}$ should imply

$$f^{(5)} = g^{(5)} = 4\pi^2 \sum_{n \geq 1} n^2 y_n q_n^a,$$

which we verified up to the fifth order in the $q_i$ expansion using the numerical values for the coefficients $y_n$’s entering in the expression \[\text{13}\] and given in ref. \[\text{10}\].

3. PERTURBATIVE HETEROTIC DUALITY GROUP

In this section, we wish to determine the perturbative duality group of the present heterotic model in order to compare it with the analogous one of the rigid theory of Seiberg and Witten. If the latter is real the flat limit of the low energy limit of the former, there should be some agreement between the two groups.

As we already mentioned, at the classical level, the $T$-duality group of the heterotic model is $\Gamma_0(2) \subset \Gamma_0(2) \subset \Gamma_0(2) \subset \Gamma_0(2)$, which is generated by the transformations $W$ and $V$ defined in eq. \[\text{12}\]. These generators obey the relations

$$W^2 = V^4 = 1 \quad UVW = 1.$$

The order 2 generator $W$ corresponds to the Weyl reflection of the mixed gauge group at the enhanced symmetry point $T = i/\sqrt{2}$ \[\text{21}\]. Away from this point, $SU(2)$ is spontaneously broken by the vacuum expectation value, $a$, of a Higgs field along the flat direction of the scalar potential, with the identification $a \propto (T - i/\sqrt{2})/(T + i/\sqrt{2})$ near the non-abelian point. Thus, the $\mathbb{Z}_2$ transformation $T \rightarrow WT = -1/(2T)$ acts on $a$ as the parity $a \rightarrow -a$ which remains unbroken after the Higgs phenomenon.

At the quantum level, because of the singularity at $T = i/\sqrt{2}$, when moving a point around the singularity, the one loop prepotential $f$ transforms as:
coefficients, and redefining the dilaton as

$$\mathcal{M}_{i/\sqrt{2}}(T) = -4 \left( T - \frac{i}{\sqrt{2}} \right)^2 \left( T + \frac{i}{\sqrt{2}} \right)^2.$$  

The polynomial $\mathcal{M}_{i/\sqrt{2}}$ can be determined by expanding the expression \[20\] around $T = i/\sqrt{2}$. The T-duality group $\Gamma_6(2)_+$ is then modified at one loop level to a new one $G$ those relations are

$$W^2 = M_{i/\sqrt{2}} \quad V^4 = 1 \quad UVW = 1, \quad (24)$$

where $M_{i/\sqrt{2}}$ stands for the perturbative nontrivial monodromy.

One can now use the modified group relations \[24\] to determine the transformation properties of $f$ under the generators $W$ and $V$. This amounts to determine the corresponding quartic polynomials $\mathcal{M}$ entering in eq. \[16\], which we will denote by $\mathcal{M}_W$ and $\mathcal{M}_V$ respectively. The result is the following:

$$\mathcal{M}_W = -4(1+w_0)T^4+2w_1T^3-2T^2+w_1T+w_0 \quad (25)$$

$$\mathcal{M}_V = \sum_{n=0}^{4} v_n T^n \quad (26)$$

with

$$v_0 = 1 + w_0, \quad v_4 = v_3 - \frac{4}{3} v_2 + 2v_1 - 4 v_0. \quad (27)$$

$$\mathcal{M}_U = -\mathcal{M}_W(T+1)-4(T+1)^3 \mathcal{M}_V \left( \frac{-1}{2T+2} \right) \quad (28)$$

The five independent parameters $w_{0,1}$ and $v_{1,2,3}$ entering in the polynomials $\mathcal{M}_W$ and $\mathcal{M}_V$ can be chosen arbitrarily using the freedom to add to $f$ a quartic polynomial $P(T)$ with real coefficients, and redefining the dilaton as $S \to S + \mathcal{P}_{TT}/12$.

Finally, the full perturbative symmetry group is the direct product of $G$ with the real constant dilaton shift,

$$D : \quad S \to S + \lambda. \quad (29)$$

The symplectic structure of $N = 2$ supergravity implies that all symmetry transformations of the effective low energy theory must be contained in the symplectic group $Sp(6, \mathbb{R})$ \[16\] which is broken to $Sp(6, \mathbb{Z})$ by quantum effects. It is then convenient to introduce a field basis where all transformations act linearly. When we obtain a matrix representation of the perturbative heterotic duality group, we can compare it to the rigid case one. We define three homogeneous coordinates $X^I$, $I = 0, 1, 2$ by

$$T = \frac{X^1}{X^0} \quad \text{and} \quad S = \frac{X^2}{X^0}. \quad (30)$$

In terms of these variables, the prepotential is a homogeneous function of degree 2

$$F(X^0, X^1, X^2) = (X^0)^2 F \left( \frac{X^1}{X^0}, \frac{X^2}{X^0} \right), \quad (31)$$

and the Kähler potential takes the form

$$K = - \ln[i(X^I F_I - X^I \mathcal{F}_I)] \quad \text{with} \quad \mathcal{F}_I = \frac{\partial F}{\partial X^I}. \quad (32)$$

In this way, all symmetries must act in the basis $(F_I, X^I)$ as symplectic transformations which leave the Kähler potential \[22\] manifestly invariant. Their symplectic action on the homogeneous basis is uniquely defined from the corresponding transformations of the fields $T, S$ and the prepotential $F$. Now, any choice of the parameters $\lambda_{W,V,U}, w_{0,1}$ and $v_{1,2,3}$ entering eq. \[17\], \[20\] and \[24\] are allowed, but a suitable one can be used to simplify the resulting representation of the generators $W, V$ and $D$. In particular, when written in some particular basis $(\mathcal{F}_I, \tilde{X}^I)$, the matrix $\tilde{W}$ which represents the generator $W$ whose classical part corresponds to the Weyl reflection of the $SU(2)$ enhanced gauge symmetry coincides with the perturbative monodromy $M_\infty$ of the rigid theory \[12\]:

$$\tilde{W} = \tilde{M}_\infty = \left( \begin{array}{cccccc} -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \quad (33)$$

Near the enhanced symmetry point, $\tilde{X}^0 \sim 2i(T - i/\sqrt{2})$ and the subspace $(\mathcal{F}_0, X^0)$ is associated to the rigid supersymmetric theory. The generator
\( \hat{W} \) acts non-trivially only on this subspace and its action represented by the corresponding 2 × 2 submatrix can be identified with \( M_\infty \) of ref. \cite{12}. This result completes the check that the perturbative heterotic model considered here is really a string generalisation of the rigid model of Seiberg and Witten.

4. EXACT TYPE II MONODROMY GROUP

Our first task is to identify the generators of the perturbative heterotic duality group derived in the previous section as elements of the type II monodromy group. The latter was worked out in ref. \cite{10} and was shown to be a subgroup of \( Sp(6, \mathbb{Z}) \), generated by 3 elements denoted by \( A, T \) and \( B \). The element \( A \) generates an exact \( \mathbb{Z}_8 \) symmetry and satisfies \( A^8 = 1 \). The other two generators \( T \) and \( B \) are associated to the monodromies around the conifold and the strong coupling loci, respectively, described by the discriminant \( \mathcal{P} \), and they are subject to some group relations given in ref. \cite{10}.

Firstly, by considering the large complex structure limit, two independent (mutually commuting) translations were identified, acting on the special coordinates \( t_1, t_2 \) of eq. \( (3) \):

\[
S_1 = (AT)^{-2} : t_1 \to t_1 + 1 \\
S_2 = (ATB)^{-1} : t_2 \to t_2 + 1.
\]

From the identification with the heterotic variables \( t_1 = T \) and \( t_2 = S \), one concludes that in a suitable basis one should have:

\[
S_1 = U \\
S_2 = D|_{\lambda=1}.
\]

The one loop heterotic prepotential \( f \) is shifted by the polynomial \( \mathcal{M}_U \) of eq. \( (28) \) under the action of \( U \), while it remains inert under \( D \). These transformations allow us to determine the integration constants \( P(t_1, t_2) \) entering in the expression of the type II prepotential \( \mathcal{P} \) for which the identification \( (35) \) happens to be valid.

Secondly, the fact that the generator \( A \) is of order 8 suggests that the order 4 generator \( V \)

\[
\hat{V} = \hat{M}_1
\]

should be identified with a conjugation of \( A^2 \) (or its inverse). Indeed it is easy to verify that the matrix representation of \( V \) and \( A^2 \) are identical. Thus, we have shown that the perturbative dualities generated by \( U \) and \( V \), together with the quantized dilaton shift \( D|_{\lambda=1} \), form a subgroup of the type II symmetries. Using eqs. \( (34), (35) \) and the group relations \( (24) \) one has:

\[
U = (AT)^{-2}, \\
V = A^2, \\
W = A^{-1}TAT, \\
D|_{\lambda=1} = (ATB)^{-1}.
\]

This is an additional perturbative check of the equivalence of the heterotic and type II models considered here.

Our next task is to give a non perturbative argument for this equivalence. The heterotic model is not totally unknown at the exact level since the flat limit of its low energy limit is just the theory of Seiberg and Witten which is completely known. This non perturbative information happens to be equivalent to the knowledge of the exact monodromy group in the rigid case \cite{12}. Therefore, our aim is now to identify the quantum monodromy group \( \Gamma(2) \) of the \( SU(2) \) rigid field theory as a subgroup of the exact monodromy group of the type II theory. The rigid group is generated by two elements, \( M_\infty \) and \( M_1 \), which satisfy the relation \( (2) \):

\[
M_\infty = M_1 M_{-1},
\]

where \( M_\infty \) is the perturbative monodromy, while \( M_1 \) and \( M_{-1} \) correspond to the monodromies around the points where dyonic hypermultiplets become massless and they are conjugate to each other. These properties can be used as a guide for the identification \( (23) \). We have shown in the previous section that \( M_\infty \) coincides with the generator \( W \) of the heterotic duality group. A simple inspection of its form \( (33) \) suggests that \( M_1 \) should be identified with \( T \) (or its conjugate \( A^{-1}T A \)). Indeed one can easily verify that in the basis used in eq. \( (33) \) the generator \( T \) takes the form:

\[
\hat{T} = \hat{M}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
One sees that in analogy with $\hat{W}$, $\hat{T}$ acts non-trivially only on the subspace of the rigid supersymmetric theory ($\tilde{F}_0, \tilde{X}^0$), and its action represented by the corresponding $2 \times 2$ submatrix coincides with $M_1$ of ref. \[12\]. Thus, we have shown that the non perturbative monodromies of the $SU(2)$ rigid field theory form a $\Gamma(2)$ subgroup of the type II monodromy group,

$$M_\infty = W, \ M_1 = T, \ M_{-1} = (AT)^{-1}T(AT).$$ \[39\]

Finally, let us note that the heterotic $T$-duality group extended with the quantized dilaton shift and the non perturbative monodromy of the rigid field theory \[39\] form a subgroup of the exact type II symmetries generated, for instance, by the elements $A^2, A^{-1}TA, A^{-1}B,$ and $T$. One needs just to introduce the “square root” of the perturbative generator $V = A^2$, or equivalently $B,$ to recover the full type II group. The generator $B$ is related to the monodromy around the non perturbative singularity $y = 1/4$ corresponding to infinite coupling, where new massless dyonic hypermultiplets appear with charges under the “dilaton” $U(1)$. This singular line which is not present in the rigid theory is a new stringy phenomenon related to the dilaton and seems to be a generic feature of string vacua. From the matrix representation of the generator $B$, one finds that it corresponds to the transformation,

$$T \leftrightarrow S'$$ \[40\]

where $S' \equiv S + T - 1$ should be the correct identification of the heterotic dilaton based on the physical requirement that the transformation $B$ preserves the positivity of the imaginary part of the dilaton, \textit{i.e.} of the inverse square of the coupling constant $\hat{\phi}$.

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