Z-boson hadronic decay width up to $\mathcal{O}(\alpha_s^4)$-order QCD corrections using the single-scale approach of the principle of maximum conformity

Xu-Dong Huang\textsuperscript{1,a}, Xing-Gang Wu\textsuperscript{1,b} \textsuperscript{,} Xu-Chang Zheng\textsuperscript{1,c}, Qing Yu\textsuperscript{1,d}, Sheng-Quan Wang\textsuperscript{2,e}, Jian-Ming Shen\textsuperscript{3,f}

\textsuperscript{1}Department of Physics, Chongqing University, Chongqing 401331, People’s Republic of China
\textsuperscript{2}Department of Physics, Guizhou Minzu University, Guiyang 550025, People’s Republic of China
\textsuperscript{3}School of Physics and Electronics, Hunan University, Changsha 410082, People’s Republic of China

Abstract

In the paper, we study the properties of the Z-boson hadronic decay width by using the $\mathcal{O}(\alpha_s^4)$-order quantum chromodynamics (QCD) corrections with the help of the principle of maximum conformity (PMC). By using the PMC single-scale approach, we obtain an accurate renormalization scale-and-scheme independent perturbative QCD correction for the $\gamma Z$ boson hadronic decay width, which is independent to any choice of renormalization scale. After applying the PMC, a more convergent pQCD series has been obtained; and the contributions from the unknown $\mathcal{O}(\alpha_s^4)$-order terms are highly suppressed, e.g. conservatively, we have $\Delta \Gamma_{\text{had}}^{Z} = O(\alpha_s^4) \simeq \pm 0.004$ MeV. In combination with the known electro-weak (EW) corrections, QED corrections, EW–QCD mixed corrections, and QED–QCD mixed corrections, our final prediction of the hadronic $Z$ boson width is $\Gamma_{\text{had}}^{Z} = 1744.439^{+3.390}_{-4.33}$ MeV, which agrees with the PDG global fit of experimental measurements, $1744.4 \pm 2.0$ MeV.

1 Introduction

In quantum chromodynamics (QCD), the Z-boson hadronic decay width plays an important role in determining the strong coupling constant ($\alpha_s$). The Z-boson hadronic decay width has been measured by various collaborations at the electron–positron colliders such as LEP and SLC [1,2], which could also be precisely measured in future high luminosity colliders such as the super Z factory [3] or CEPC [4]. Theoretically, the one-loop electroweak (EW) and the mixed EW-QCD contributions to the Z-boson hadronic decay have been investigated in Refs. [5–8], and the two-loop EW contribution has been given in Refs. [9–11]. In large-$m_t$ limit, the higher-loop corrections have been calculated up to $O(\alpha_t^3) [12,13]$, $O(\alpha_s^2 \alpha_t) [14,15]$, and $O(\alpha_s^2 \alpha_s^2) [16–18]$, respectively, where $\alpha_t \equiv y_t^2/4\pi$ with $y_t$ being the top-quark Yukawa coupling constant. The final-state QED radiations have been computed up to $O(\alpha/\alpha_t), O(\alpha/\alpha_s), O(\alpha^2)$ in Ref. [19]. The non-factorizable QCD corrections have been estimated in Refs. [8,20]. The pure perturbative QCD (pQCD) corrections up to $O(\alpha_s^2) [21,22], O(\alpha_s^3) [23–27], O(\alpha_s^4) [28–30]$ have also been performed in the literature. Moreover, the mass corrections to both the vector and axial vector correlators can be found in Refs. [21,22,31–35]. Those achievements give us good opportunities for precise determining of $\alpha_s(M_Z)$, e.g. a recent determination of $\alpha_s$ has been given in Ref. [36].

Following the renormalization group invariance, the physical observable should be independent to theoretical conventions, such as the choices of the renormalization scale and scheme, which is ensured by mutual cancellation of the scale and scheme dependence among different orders for an infinite-order pQCD prediction. However, for a fixed-order pQCD prediction, if the perturbative coefficients and the corresponding $\alpha_s$ do not match properly, the pQCD series may have large scale and scheme ambiguities [37]. Conventionally, the renormalization scale is taken as the “guessed” momentum flow of the process, as well as the one to eliminate the large logs or to minimize the contributions from high-order terms or to achieve the prediction in agreement with the experimental data. Those naive treatment directly causes the mismatching between the strong coupling constant and
its coefficients and resulting in conventional renormalization scale and scheme ambiguities [38,39]. Such guessing treatment decreases the predictive power of pQCD. In fact, predictions based on conventional scale setting are even incorrect for Abelian theory – Quantum Electrodynamics (QED); the renormalization scale of the QED coupling constant can be set unambiguously by using the Gell-Mann-Low method [40].

A correct renormalization scale-setting approach is thus important for achieving an accurate fixed-order pQCD prediction. Many ways have been suggested in the literature, most of them such as the renormalization group improved effective coupling method (FAC) [41,42] and the principle of minimum sensitivity (PMS) [43–45] are designed to find an optimal renormalization scale of the process. On the contrary, the principle of maximum conformality (PMC) [46–50] provides a rigorous idea, whose purpose is not to find an optimal scale, but to determine the effective magnitude of the renormalization group improved due to the elimination of the divergent renormalon terms like $n^{-\beta_0/\alpha_s}$, where the central values are for $\alpha_s=0.1$ MeV. It is noted that the original PMC multi-scale approach [46–50] and single-scale approach are equivalent to each other in sense of perturbative theory [65], but the residual scale dependence emerged in PMC multi-scale method can be greatly suppressed by applying the single-scale approach.

The remaining parts of the paper are organized as follows. In Sect. 2, we will give the detailed PMC treatment for a precise determination of the Z-boson hadronic decay width. In Sect. 3, we will give the numerical results. Section 4 is reserved for a summary.

2 The Z-boson hadronic decay width using the PMC

The hadronic decay width of the Z-boson can be expressed as

$$\Gamma_Z^{\text{had}} = \Gamma_0 R^{\text{nc}} + \Delta \Gamma^{\text{Extra}}_Z,$$  \hspace{1cm} (1)

where the first term stands for the pure pQCD correction with the leading-order (LO) width $\Gamma_0 = \frac{G_F M_Z^2}{2\pi\sqrt{2}}$, and the Fermi coupling constant $G_F = 1.166 \times 10^{-3}\text{GeV}^{-2}$. The second term $\Delta \Gamma^{\text{Extra}}_Z$ contains four less important corrections, i.e.,

$$\Delta \Gamma^{\text{Extra}}_Z = \Delta \Gamma_1 + \Delta \Gamma_2 + \Delta \Gamma_3 + \Delta \Gamma_4$$

$$= -1.577^{+0.183}_{-0.237} + 0.695^{+0.000}_{-0.001} + 6.577^{+0.560}_{-0.560} + 0.609^{+0.061}_{-0.045} (\text{MeV})$$

$$= 6.304^{+0.804}_{-0.847} (\text{MeV}),$$  \hspace{1cm} (2)

where the central values are for $\mu_F = M_Z$, and the errors are for $\mu_F \in [M_Z/2, 2M_Z]$. Here $\Delta \Gamma_1$ is the $b$- and $t$-quark mass corrections to the vector and axial vector correlators [31–35], $\Delta \Gamma_2$ is the quark final-state QED radiation and the mixed QED-QCD correction [19], $\Delta \Gamma_3$ is the electro-weak two-loop corrections and the higher-loop corrections in the large-$m_t$ limit [11], $\Delta \Gamma_4$ is the mixed EW-QCD correction and nonfactorizable QCD correction [6–8,20].

Our main concern is the perturbative QCD corrections to the dominant correlator of the neutral current, which can be divided as the following four parts:

$$R^{\text{nc}} = 3 \left[ \sum_f v_f^2 \nu_{\text{NS}} + \left( \sum_f v_f \right)^2 \nu_s^V + \sum_f \alpha_s^2 r_{\text{NS}}^A + r_s^A \right].$$  \hspace{1cm} (3)

where $v_f = 2I_f - 4q_f s_W$, $a_f = 2I_f$, $q_f$ is the $f$-quark electric charge, $s_W$ is the effective weak mixing angle, and

Footnote 2 continued

but the residual scale dependence due to unknown perturbative terms. Such residual scale dependence generally suffer from both the $\alpha_s$-power suppression and the exponential suppression, but could be large due to possibly poor pQCD convergence for the perturbative series of either the PMC scale or the pQCD approximant [52].
$I_f$ is the third component of weak isospin of the left-handed component of $f$. $r_N^V = r_N^A \equiv r_N^S$, and $r_S^A$ stand for the non-singlet, the vector-singlet, and the axial-singlet part, respectively. Those contributions can be further expressed as

$$r_N = 1 + \sum_{i=1}^{n} C_i^{NS} A_i^s, \quad r_V^S = \sum_{i=3}^{n} C_i^{VS} A_i^s, \quad r_S^A = \sum_{i=2}^{n} C_i^{AS} A_i^s,$$

where $A_i^s = \alpha_i/(4\pi)$, and the coefficients of $r_N$, $r_V^S$, and $r_S^A$ can be obtained from Refs. [28–30, 68, 69]. As for $r_S^A$, we adopt conventional scale setting approach to perform our analysis, and numerically, we obtain $\Gamma_{Z}^{had}\lambda_S^{\pm} = [-1.725, -1.685]$ MeV for $\mu_r \in \{M_Z/2, 2M_Z\}$ by using the formulas given by Ref. [28], whose magnitude is quite small in comparison to that of $r_N$, thus fortunately, this approximate treatment will not affect our final conclusions.

The R-ratio can be rewritten as the following perturbative form by using the degeneracy relations [49, 50, 70], i.e.,

$$R^{\text{nc}} = r_0 + r_{1,0} a_s(\mu_r) + (r_2,0 + \beta_0 r_{2,1}) a_s^2(\mu_r) + (r_3,0 + \beta_1 r_{2,1} + 2\beta_0 r_{3,2}) a_s^3(\mu_r) + (r_4,0 + \beta_2 r_{2,1,2} + 2\beta_1 r_{3,1,3} + \frac{5}{2} \beta_1 \beta_0 r_{3,2,2}) a_s^4(\mu_r) + 3\beta_0 a_{4,1}^2 + 3\beta_0^2 a_{4,2}^2 + \beta_1 a_{4,3}^2 A_s^4(\mu_r) + O(a_s^5),$$

where $r_0 = 3(\sum f v_f^2 + \sum f a_f^2)$, and the coefficients $r_{i,j}$ can be obtained from the known coefficients $C_i^{NS}$, $C_i^{VS}$, and $C_i^{AS}$ of $r_N$, $r_V^S$, and $r_S^A$. The coefficients $r_{i,0}$ are $\{\beta_i\}$-independent conformal coefficients, and the $\{\beta_i\}$-dependent non-conformal coefficients $r_{i,j}$ ($j \neq 0$) are generally functions of $\ln \mu_r^2 / M_Z^2$, i.e.,

$$r_{i,j} = \sum_{k=0}^{j} C_{j-k}^i \hat{r}_{i-k,j-k} \ln^k (\mu_r^2 / M_Z^2),$$

where the reduced coefficients $\hat{r}_{i,j}$ are $r_{i,j}|_{\mu_r=M_Z}$, the combination coefficients $C_{j-k}^i = i!/[k!(j-k)!]$. We put the known coefficients $\hat{r}_{i,j}$ up to $O(a_s^6)$-level in the Appendix.

Following the standard PMC single-scale procedures as described in detail in Ref. [65], with the help of RGE, one can determine an effective coupling $\alpha_s(Q_s)$ by absorbing all the non-conformal $\{\beta_i\}$-terms into the running coupling, and the resultant pQCD series becomes the following conformal series,

$$R^{\text{nc}}|_{\text{PMC}} = r_0 + r_{1,0} a_s(Q_s) + r_{2,0} a_s^2(Q_s) + r_{3,0} a_s^3(Q_s) + r_{4,0} a_s^4(Q_s) + O(a_s^5),$$

where $Q_s$ is the PMC scale, which corresponds to the overall effective momentum flow of the process and can be determined up to next-to-next-to-leading log (NNLL) accuracy by using the present known $O(\alpha_s^4)$-order pQCD series; i.e., the $\ln Q_s^2 / M_Z^2$ can be expanded as the following perturbative series,

$$\ln \frac{Q_s^2}{M_Z^2} = T_0 + T_1 a_s(M_Z) + T_2 a_s^2(M_Z) + O(a_s^3).$$

where

$$T_0 = \frac{\hat{r}_{2,1}}{\hat{r}_{1,0}},$$

$$T_1 = \frac{\beta_0 (\hat{r}_{2,1} - \hat{r}_{1,0}\hat{r}_{3,2})}{\hat{r}_{1,0}^3} + \frac{2(\hat{r}_{2,0}\hat{r}_{2,1} - \hat{r}_{1,0}\hat{r}_{3,1})}{\hat{r}_{1,0}^2},$$

and

$$T_2 = \frac{3\hat{r}_{1,0}(\hat{r}_{2,1} - \hat{r}_{1,0}\hat{r}_{3,2})}{2\hat{r}_{1,0}^3} + \frac{4(\hat{r}_{1,0}\hat{r}_{2,0}\hat{r}_{3,1} - \hat{r}_{2,0}\hat{r}_{3,2} - \hat{r}_{1,0}\hat{r}_{3,4})}{\hat{r}_{1,0}^3} + \frac{\beta_0 (4\hat{r}_{2,1}\hat{r}_{3,1} - \hat{r}_{2,0}\hat{r}_{3,2} - 3\hat{r}_{1,0}\hat{r}_{3,4})}{\hat{r}_{1,0}^3} + \frac{\beta_0^2 (2\hat{r}_{1,0}\hat{r}_{2,0}\hat{r}_{3,1} - \hat{r}_{2,0}\hat{r}_{3,2} - \hat{r}_{1,0}\hat{r}_{4,3})}{\hat{r}_{1,0}^3}.$$

It can be found that $Q_s$ is exactly free of $\mu_r$, and together with the $\mu_r$-independent conformal coefficients $r_{i,0}$, the conventional renormalization scale ambiguity is eliminated. Therefore, the precision of $R^{\text{nc}}$ can be greatly improved by using the PMC. Moreover, the precision of the predictions depend on the perturbative nature of both the $R^{\text{nc}}$ and the $\ln Q_s^2 / M_Z^2$, which shall be numerically analyzed in the following paragraphs.

3 Numerical results

To do the numerical calculation, we adopt the Z-boson mass $M_Z = 91.1876 \pm 0.0021$ GeV and top-quark pole mass $M_t = 172.9$ GeV [71]. We use the four-loop $\alpha_s$-running behavior [52] to analyse the $O(\alpha_s^4)$-order QCD corrections, i.e.,

$$\alpha_s(\mu_r) \approx 1 - \frac{b_1 \ln t}{(\beta_0 t)^2} + \frac{b_2^2 (\ln^2 t - \ln t - 1)}{2(\beta_0 t)^3} + \frac{1}{(\beta_0 t)^4} \left[ \frac{b_3}{2} \left( -3 \ln^3 t + \frac{5}{2} \ln^2 t + 2 \ln t - \frac{1}{2} \right) - 3 b_1 b_2 \ln t + \frac{b_3}{2} \right] + O \left( \frac{1}{(\beta_0 t)^5} \right),$$

Where $t = \ln (\mu_r^2 / \Lambda_{\text{QCD}}^2)$, $b_1 = \beta_1 / \beta_0$, and the $\beta_i (i = 0, 1, 2, 3)$-functions have been calculated in Refs. [72–80]. Taking $\alpha_s(M_Z) = 0.1181$ [71], we obtain $\Lambda_{\text{QCD}}^{(\alpha_s^4=5)} = 209.5$ MeV.
First, by setting all input parameters to be their central values, we present the Z-boson hadronic decay width $\Gamma_Z^{\text{had}}$ up to different known $\alpha_s$-orders under conventional scale-setting approach in Fig. 1. It shows that in agreement with the conventional wisdom, the renormalization scale independence becomes small when we have known more loop terms. For examples, we obtain $\Gamma_Z^{\text{had}} \mid_{\text{Conv.}} = [1744.378, 1744.587]$ MeV for $\mu_r \in [M_Z/2, 2M_Z]$, and $\Gamma_Z^{\text{had}} \mid_{\text{Conv.}} = [1744.378, 1745.008]$ MeV for $\mu_r \in [M_Z/3, 3M_Z]$; e.g., the net scale errors are only $\sim 0.01\%$, and $\sim 0.04\%$, respectively. We should point out that as has been mentioned in the Introduction, such small net scale dependence for the $O(\alpha_s^4)$-order prediction is due to good convergence of the perturbative series, e.g., the relative magnitudes of the $\alpha_s$-terms: $\alpha_s^0$-terms: $\alpha_s^3$-terms: $\alpha_s^4$-terms = 1: 2.9%: -2.2%: -0.4% for the case of $\mu_r = M_Z$; and also due to the cancelation of the scale dependence among different orders. The scale errors for each order term remain unchanged and large, e.g. the $\Gamma_Z^{\text{had}}$ has the following perturbative feature up to $O(\alpha_s^4)$-order: $\Gamma_Z^{\text{had}} \mid_{\text{Conv.}} = 1681.262 + 62.966 \pm 5.925 + 4.268 + 1.802 \pm 4.838 - 1.382 \pm 1.311 - 0.230 \pm 0.055 + 0.275 = 1744.418 \pm 0.040$ (MeV), \hfill (11)

where the central values are for $\mu_r = M_Z$, and the errors are obtained by varying $\mu_r \in [M_Z/2, 2M_Z]$. It shows that the absolute scale errors are 16\% 495\%, 131\%, and 143\% for the $\alpha_s$-terms, $\alpha_s^2$-terms, $\alpha_s^3$-terms, and $\alpha_s^4$-terms, respectively; and there do have large scale cancellations among different orders.

Second, we present the Z-boson hadronic decay width $\Gamma_Z^{\text{had}}$ up to different known $\alpha_s$-orders under the PMC scale-setting approach in Fig. 2. At the $O(\alpha_s)$-order level, the perturbative series of $\Gamma_Z^{\text{had}}$ does not have $\{\beta_i\}$-terms to fix the $\alpha_s$ value, thus the prediction of $\Gamma_Z^{\text{had}} \mid_{\text{PMC}}$ is the same as the conventional one. The PMC starts to work at the $O(\alpha_s^2)$ and higher order levels. It shows that after applying the PMC, the pQCD convergence can be greatly improved, e.g., the relative magnitudes of the $\alpha_s$-terms: $\alpha_s^0$-terms: $\alpha_s^3$-terms: $\alpha_s^4$-terms of the pQCD series becomes 1: 4.34\%: -0.49\% by applying the PMC scale-setting to the perturbative series up to $O(\alpha_s^2)$, whose PMC scale $Q_s = 113.0$ GeV is fixed up to the NLL accuracy. The relative magnitudes of the $\alpha_s$-terms: $\alpha_s^0$-terms: $\alpha_s^3$-terms: $\alpha_s^4$-terms of the PMC series becomes 1: 4.33\%: -0.49\%: -0.01\% by applying the PMC up to $O(\alpha_s^4)$, whose PMC scale $Q_s = 114.9$ GeV is fixed up to NNLL accuracy. And there is no renormalization scale dependence for $\Gamma_Z^{\text{had}}$ at any fixed order, i.e., $\Gamma_Z^{\text{had}} \mid_{\text{PMC}} = 1681.262 + 60.838 + 2.634 - 0.299 + 0.004 = 1744.439$ (MeV), \hfill (12)

where each perturbative terms and the net total decay width are unchanged for any choice of $\mu_r$. This behavior is consistent with that of the previous PMC multi-scale approach analysis on $R^{\text{tot}}$ [81]. The PMC single scale $Q_s$ is an effective scale which effectively replaces the individual PMC scales introduced in the PMC multi-scale approach in the sense of a mean value theorem, which can be regarded as the overall effective momentum flow of the process; it shows stability and convergence with increasing order in pQCD via the pQCD approximates. More explicitly, we obtain $Q_s = 114.9$...
GeV \sim 1.3 M_Z$, which can be fixed up to NNLL accuracy by using the present known $O(\alpha_s^4)$-order pQCD series, i.e.,

\[
\ln \frac{Q_s^2}{M_Z^2} = 0.2249 + 21.7363a_s(M_Z) + 376.287a_s^2(M_Z) \\
= 0.2249 + 0.2043 + 0.0332. \quad (13)
\]

One may observe that the relative magnitudes of each order terms in $Q_s$ perturbative series are $1 : 91\% : 15\%$, which also shows a good convergence behavior.

Third, it is helpful to predict the magnitude of the “unknown” higher-order pQCD corrections. The renormalization scale independent PMC series is helpful for such purpose. Because the PMC series has a good perturbative convergence, e.g., the magnitude of $O(\alpha_s^4)$-order term is only 0.01% of $O(\alpha_s)$-order term, it is reasonable to take the magnitude of the last known term $\pm|\Delta_{4,0}^\text{PMC}(Q_s)|$ as a conservative prediction of the uncalculated higher-order terms [39]. By further taking the variation of $\Delta Q_s \simeq -1.9 \text{ GeV}$, which is the difference between the NLL and NNLL PMC scales, as the magnitude of its unknown NNLL term into consideration, we obtain

\[
\Delta \Gamma_{Z}^\text{had} \simeq 0.004 \text{ (MeV)}. \quad (14)
\]

Finally, after eliminating the renormalization scale uncertainty by applying the PMC, we still have uncertainties from the $\alpha_s$ fixed-point error $\Delta \alpha_s(M_Z)$ and the $Z$-boson mass error $\Delta M_Z$. As for the $\alpha_s$ fixed-point error, by using $\Delta \alpha_s(M_Z) = 0.0011$ [71] together with the four-loop $\alpha_s^\text{running}$ behavior, we obtain $\Lambda_{\text{QCD}}^{n_f=5} = 209.5^{+13.2}_{-12.6} \text{ MeV}$ and

\[
\Delta \Gamma_{Z}^\text{had} \Delta \alpha_s(M_Z) \simeq 0.574 \text{ (MeV)}. \quad (15)
\]

And for the error of $Z$-boson mass $\Delta M_Z = \pm 0.0021 \text{ GeV}$, we obtain

\[
\Delta \Gamma_{Z}^\text{had} \Delta M_Z \simeq 0.120 \text{ (MeV)}. \quad (16)
\]

Here, when discussing one uncertainty, the other input parameters shall be set as their central values.

As a whole, the squared average of the above mentioned three errors leads to a net error, $\pm 0.586 \text{ MeV}$, to the PMC prediction of the total decay width $\Gamma_{Z}^\text{had}$, among which the magnitude of $\Delta \alpha_s(M_Z)$ dominates the error sources. Thus more precise measurements on the reference point $\alpha_s(M_Z)$ is important for a more precise pQCD prediction.

4 Summary

Under conventional scale-setting approach, the fixed-order scale-setting ambiguity could be softened by including enough higher-order loop terms due to large cancelation among different orders; for the present considered decay width up to $O(\alpha_s^4)$-order, the net scale uncertainty is $\pm (0.169_{-0.040}) \text{ MeV}$ for $\mu_r \in [M_Z/2, 2M_Z]$; and by further including the mentioned other error sources, we have $\Gamma_{Z}^\text{had} \text{ Conv.} = 1744.418^{+1.595}_{-1.621} \text{ (MeV)}$.

In the paper, we have presented an accurate prediction on the $Z$-boson hadronic decay width by applying the PMC single-scale approach to eliminate the conventional renormalization scale ambiguity. We obtain

\[
\Gamma_{Z}^\text{had} |_{\text{PMC}} = 1744.439^{+1.390}_{-1.433} \text{ (MeV)}. \quad (17)
\]

where the errors are the sum of two parts, one is the squared average of those from $\Delta \alpha_s(M_Z), \Delta M_Z$, and the uncalculated higher-order terms, another is the error from $\Delta \Gamma_{Z}^\text{extra}$ as given in Eq. (2). After applying the PMC single-scale approach, the pQCD series becomes scale independent and more convergent, thus a reliable pQCD prediction can be achieved. Due to the perturbative terms have been known up to enough high-orders, the predictions under the PMC and conventional scale-setting approaches are consistent with each other. We present the PMC prediction of the $Z$-boson hadronic decay width in Fig. 3, where the experimental data are presented as a comparison. The PMC prediction agrees with the PDG global fit of the experimental measurements. Thus, one obtains optimal fixed-order predictions for the $Z$-boson hadronic decay width by applying the PMC, enabling high precision test of the Standard Model.

Acknowledgements This work is partly supported by the Chongqing Graduate Research and Innovation Foundation under Grant no. CYB19065 and no. ystd1912, the National Natural Science Foundation of...
China under Grant no. 11625520, no. 11905056, no. 11975187, no. 11947406, and the Fundamental Research Funds for the Central Universities under Grant No.2020CQJQY-Z003.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: All the figures and the numerical predictions can be derived from the formulas presented in the paper, so the data do not need to be deposited.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Funded by SCOAP3.

Appendix: The PMC reduced perturbative coefficients \( \hat{r}_{i,j} \)

In this appendix, we give the required PMC reduced coefficients \( \hat{r}_{i,j} \) for the perturbative series of the Z-boson hadronic decay width, which can be obtained from Refs. [28–30, 68, 69] with proper transformations, i.e.,

\[
\hat{r}_{1,0} = 9 \gamma_1^{NS} \left( \sum_f a_f^2 + \sum_f v_f^2 \right),
\]

\[
\hat{r}_{2,0} = 4 \left[ 9 \gamma_2^{NS} \left( \sum_f a_f^2 + \sum_f v_f^2 \right) - 37 + 12 \ln \frac{M_Z^2}{M_t^2} \right],
\]

\[
\hat{r}_{2,1} = 9 \Pi_1^{NS} \left( \sum_f a_f^2 + \sum_f v_f^2 \right),
\]

\[
\hat{r}_{3,0} = 144 \left[ \gamma_3^{NS} + \frac{\gamma_3^S}{\sum_f q_f^2} \left( \sum_f a_f^2 + \sum_f v_f^2 \right) \right] \left( \sum_f v_f^2 \right) + \left( \sum_f v_f^2 \right)^2 \left( \frac{440}{9} - \frac{320 \zeta_3}{3} \right) + \frac{2144}{3} \ln \frac{M_Z^2}{M_t^2} + 368 \ln \frac{M_Z^2}{M_t^2} + 192 \zeta_3 + 368 \pi^2 - \frac{40600}{9}.
\]

\[
\hat{r}_{3,1} = 36 \Pi_2^{NS} \left( \sum_f a_f^2 + \sum_f v_f^2 \right),
\]

\[
\hat{r}_{3,2} = -3 \pi^2 \gamma_1^{NS} \left( \sum_f a_f^2 + \sum_f v_f^2 \right).
\]

References

1. S. Schael et al., ALEPH and DELPHI and L3 and OPAL and SLD collaborations and LEP electroweak working group and SLD electroweak group and SLD heavy flavour group. Phys. Rep. 427, 257 (2006)

2. J. Alcaraz et al. [ALEPH and DELPHI and L3 and OPAL Collaborations and LEP Electroweak Working Group], arXiv:0712.0929 [hep-ex]

3. J.P. Ma and Z.X. Zhang (The super Z-factory group), Sci. China Phys. Mech. Astron. 53, 1947 (2010)

4. J.B. Guimaraes da Costa et al. [CEPC Study Group], arXiv:1811.10545 [hep-ex]

5. A.A. Akhundov, D.Y. Bardin, T. Riemann, Nucl. Phys. B 276, 1 (1986)

6. V.A. Novikov, L.B. Okun, M.I. Vysotsky, Nucl. Phys. B 397, 35 (1993)

7. V.A. Novikov, L.B. Okun, M.I. Vysotsky, Phys. Lett. B 320, 388 (1994)

8. A. Czarnecki, J.H. Kuhn, Phys. Rev. Lett. 77, 3955 (1996)

9. A. Freitas, J. High Energy Phys. 04, 070 (2014)

10. A. Freitas, Phys. Lett. B 730, 50 (2014)

11. I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, J. Usovitsch, Phys. Lett. B 783, 86 (2018)
12. L. Avdeev, J. Fleischer, S. Mikhailov, O. Tarasov, Phys. Lett. B 336, 560 (1994)
13. K.G. Chetyrkin, J.H. Kuhn, M. Steinhauser, Phys. Lett. B 351, 331 (1995)
14. M. Faisst, J.H. Kuhn, T. Seidensticker, O. Veretin, Nucl. Phys. B 665, 649 (2003)
15. J.J. van der Bij, K.G. Chetyrkin, M. Faisst, G. Jikia, T. Seidensticker, Phys. Lett. B 498, 156 (2001)
16. Y. Schroder, M. Steinhauser, Phys. Lett. B 622, 124 (2005)
17. K.G. Chetyrkin, M. Faisst, J.H. Kuhn, P. Maierhofer, C. Sturm, Phys. Rev. Lett. 97, 102003 (2006)
18. R. Boguszal, M. Czakon, Nucl. Phys. B 755, 221 (2006)
19. A.L. Kataev, Phys. Lett. B 287, 209 (1992)
20. R. Harlander, T. Seidensticker, M. Steinhauser, Phys. Lett. B 426, 125 (1998)
21. B.A. Kniehl, J.H. Kuhn, Nucl. Phys. B 329, 547 (1990)
22. B.A. Kniehl, J.H. Kuhn, Phys. Lett. B 224, 229 (1989)
23. S.G. Gorishnii, A.L. Kataev, S.A. Larin, Phys. Lett. B 259, 144 (1991)
24. L.R. Surguladze, M.A. Samuel, Phys. Rev. Lett. 66, 560 (1991)
25. S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, Phys. Lett. B 320, 159 (1994)
26. K.G. Chetyrkin, J.H. Kuhn, Phys. Lett. B 308, 127 (1993)
27. K.G. Chetyrkin, O.V. Tarasov, Phys. Lett. B 327, 114 (1994)
28. P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, J. Rittinger, PoS RADCOR 2011, 030 (2011)
29. P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, Phys. Rev. Lett. 101, 012002 (2008)
30. P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, J. Rittinger, Phys. Rev. Lett. 108, 222003 (2012)
31. K.G. Chetyrkin, J.H. Kuhn, A. Kwiatkowski, Phys. Rep. 277, 189 (1996)
32. K.G. Chetyrkin, R.V. Harlander, J.H. Kuhn, Nucl. Phys. B 586, 56 (2000)
33. P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, Nucl. Phys. Proc. Suppl. 135, 243 (2004)
34. K.G. Chetyrkin, Phys. Lett. B 307, 169 (1993)
35. S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, Nucl. Phys. B 438, 278 (1995)
36. D. D’Enterria, PoS ALPHAS 2019, 008 (2019)
37. S.J. Brodsky, X.G. Wu, Phys. Rev. D 86, 054018 (2012)
38. X.G. Wu, S.J. Brodsky, M. Mojaza, Prog. Part. Nucl. Phys. 72, 44 (2013)
39. X.G. Wu, Y. Ma, S.Q. Wang, H.B. Fu, H.H. Ma, S.J. Brodsky, M. Mojaza, Rep. Prog. Phys. 78, 126201 (2015)
40. M. Gell-Mann, F.E. Low, Phys. Rev. 95, 1300 (1954)
41. G. Grunberg, Phys. Lett. B 95, 70 (1980)
42. G. Grunberg, Phys. Rev. D 29, 2315 (1984)
43. P.M. Stevenson, Phys. Lett. B 100, 61 (1981)
44. P.M. Stevenson, Nucl. Phys. B 203, 472 (1982)
45. S.J. Brodsky, X.G. Wu, Phys. Rev. D 85, 034038 (2012)
46. S.J. Brodsky, L. Di Giustino, Phys. Rev. D 86, 085026 (2012)
47. S.J. Brodsky, X.G. Wu, Phys. Rev. Lett. 109, 042002 (2012)
48. M. Mojaza, S.J. Brodsky, X.G. Wu, Phys. Rev. Lett. 110, 192001 (2013)
49. S.J. Brodsky, M. Mojaza, X.G. Wu, Phys. Rev. D 89, 014027 (2014)
50. X.G. Wu, J.M. Shen, B.L. Du, S.J. Brodsky, Phys. Rev. D 97, 094030 (2018)
51. X.G. Wu, J.M. Shen, B.L. Du, X.D. Huang, S.Q. Wang, S.J. Brodsky, Prog. Part. Nucl. Phys. 108, 103706 (2019)
52. D. Boito, M. Jamin, R. Miravitllas, Phys. Rev. Lett. 117, 152001 (2016)
53. M. Beneke, V.M. Braun, Phys. Lett. B 348, 513 (1995)
54. M. Neubert, Phys. Rev. D 51, 5924 (1995)
55. M. Beneke, Phys. Rep. 317, 1 (1999)