Application of a fundamental solution to a problem operator for modeling vibrations of elastically supported load-bearing structures

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Abstract The work is devoted to the construction of effective computational algorithms for modeling vibrations of elastically supported load-bearing systems. An approach is developed based on the representation of a mathematical model in the form of partial differential equations for generalized functions. With this approach, the presence of elements with a very high stiffness in the design made it necessary to carry out the calculation with a very small time step. To build a computational algorithm, a fundamental solution to the problem operator is used, which allows one to obtain a solution at the next time step for each point of the spatial grid independently. The need to carry out the calculation with a very small time step arises only in nodes that coincide with the location of structural elements with very high stiffness. The proposed approach allows simulating impact forces and instantaneous change of boundary conditions. The possibility of using the model with an absolutely rigid restraining support as a limit transition from a model of elastic bond of high rigidity was researched on.

1. Introduction
General scheme of approximate algorithms for solving functional equations \( Au = f \) consists in approximating the operator \( A \) sequence of statements \( A_n \) and solving a sequence of equations \( A_n u = f_n \). Moreover, it is necessary to approximate not only the operator \( A \), but also boundary conditions (initial conditions, conditions at external and / or internal borders). For example, when implementing projection-grid algorithms, it is required to subordinate basis functions to boundary conditions. This is a difficult task, especially in the case of many spatial variables.

In [1-3], a method was proposed for constructing an approximate solution in which the original problem is associated with an equation for generalized functions, the solution of which will be the original problem. Representation of the problem in the form of an equation for generalized functions allows us to formulate the problem and construct an approximate solution in the presence of point sources.
Subsequently, for hyperbolic systems of linear differential equations of the first order \[4\], this approach was developed for the case when it is possible to construct a fundamental solution to the operator of the problem for generalized functions. Using the example of equations of transverse vibrations of an elastic thin-walled rod, the method of applying a fundamental solution to a problem operator to construct an approximate solution of the initial-boundary-value problem of this equation is described.

2. Formulation of the problem
To solve a number of engineering problems, it is necessary to predict the dynamic behavior of a variety of structurally non-linear systems. These are systems with one-way connections, turning connections on and off, mechanisms with gaps, structures with elastic bonds pre-compressed by their own weight or with stroke limiters, etc. To model the behavior of such systems in the mathematical model, partial differential equations with non-linear, non-smooth boundary conditions are used. This includes the floating bridges of the continuous system with additional restrictive supports. Between the span and the supports there are vertical gaps that can close when the moving load is skipped.

Let us consider structural-nonlinear vibrations of an elastically supported carrier system using the span of a floating bridge with restrictive rigid supports as an example. The problem is considered in a flat setting. As a mechanical model, we accept the model of an elastic rod with free ends $2l$ and linear density $\mu$ and full mass $M$, driven by a combination of concentrated forces from dynamic tire pressures $R_s(t, x)$, efforts to interact with floating supports $S_s(t, x)$, $s = 1 \ldots N$, transmitted through elastic bonds with stiffness coefficients and forces of interaction with transitional parts of the bridge $P_p(t, x), p = 1, 2$, are time-varying. When the gaps are closed on the elastic rod, additional forces begin to act from the side of the restrictive supports $Q_q(t, x), q = 1, 2$, which are modeled by elastic bonds with stiffness coefficients $\eta_q \gg \eta_s$.

The following notation is used:
$E$ - modulus of elasticity of the span material,
$J$ - span moment of inertia.

Following the theory of bending-torsional vibrations of a thin-walled elastic rod described in \[5\] and introducing the designation $a = \frac{EJ}{\mu}$, a mathematical model can be written in the form of an initial-boundary value problem for a partial differential equation with constant coefficients

$$\begin{align*}
\frac{\partial^2 u}{\partial t^2} + a \frac{\partial^4 u}{\partial x^4} &= -g + \frac{1}{\mu} \sum_{f} R_f, \quad x \in [-l, l], \quad t \in [0, T] \\
\sum_{f} R_f &= \sum_{s} S_s + \sum_{q} Q_q + \sum_{p} P_p + \sum_{r} R_r,
\end{align*}$$

with initial data

$$u(t = 0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(t = 0, x) = u_1(x)$$

and conditions at the border $\Gamma: x = \mp l$

$$\frac{\partial^2 u}{\partial x^2}(t, -l) = 0, \quad \frac{\partial^2 u}{\partial x^2}(t, l) = 0.$$  

The right-hand side of equation (1) is assumed to be sufficiently smooth so that the problem is uniquely solvable. We associate this problem with the equation for generalized functions.
3. The equation for generalized functions

In what follows, the concepts of the theory of generalized functions are used, as set forth, for example, in [6].

Let \( \tilde{u}(t,x) \) - the solution of the problem (1), (2), (3). Define it by zero outside the interval \([-l,l]\) and with \( t < 0 \). We leave the previous designation for the function formed \( u(t,x) \): \[
\begin{align*}
  u(t,x) &= \begin{cases} 
    \tilde{u}(t,x), & \text{if } x \in [-l,l] \text{ and } t \geq 0 \\
    0, & \text{if } x \notin [-l,l] \text{ or } t < 0
  \end{cases}
\end{align*}
\]

We also introduce the notation \( v_0(t) = u(t,-l), \ v_1(t) = \frac{\partial u}{\partial x}(t,-l), \ v_2(t) = \frac{\partial^2 u}{\partial x^2}(t,-l), \)

\( w_0(t) = u(t,l), \ w_1(t) = \frac{\partial u}{\partial x}(t,l), \ w_2(t) = \frac{\partial^2 u}{\partial x^2}(t,l). \)

We show that \( u(t,x) \), considered as a generalized function of \( D' \), satisfies the equation

\[
\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = -g + \sum_{\mu} \frac{\mu}{\mu} + u_0 \cdot \delta(t) + u_1 \cdot \delta(t) \ldots
\]

\[
\ldots + a^2 (v_0 \delta''(x + l) + v_1 \delta''(x + l) + v_2 \delta'(x + l) - w_0 \delta''(x - l) - w_1 \delta''(x - l) - w_2 \delta'(x - l))
\]

Indeed, for all \( \varphi(t,x) \in D \) the right equality

\[
\int_{-l}^{l} \int_{0}^{\infty} \frac{\partial^2 \varphi}{\partial t^2} dt dx = \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial^2 u}{\partial t^2} \varphi dt dx + \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial \varphi}{\partial t} \varphi dt dx - \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial u}{\partial t} \varphi dt dx .
\]

Given the initial conditions (2), we obtain

\[
\int_{-l}^{l} \int_{0}^{\infty} \frac{\partial^2 \varphi}{\partial t^2} dt dx = \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial^2 u}{\partial t^2} \varphi dt dx - \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial u}{\partial t}(0,x) \varphi dx + \int_{-l}^{l} \int_{0}^{\infty} u_0 \varphi(0,x) dx .
\]

Similarly

\[
\int_{-l}^{l} \int_{0}^{\infty} \frac{\partial^4 \varphi}{\partial x^4} dt dx = \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial^4 u}{\partial x^4} \varphi dt dx + \int_{-l}^{l} \int_{0}^{\infty} \frac{\partial u}{\partial x}(0,x) \varphi dx + \int_{-l}^{l} \int_{0}^{\infty} u_0 \varphi(0,x) dx .
\]

Given the boundary conditions (3) and the introduced notation, we obtain (4).

3.1 Point Source Modeling

On the right side of equation (4), each of the terms \( F_f(t,x) \) makes sense of the force acting per unit length. Total acting force \( F_f(t) = \int_{-l}^{l} F_f(t,x) dx \). Function \( F_f(t,x) \) is not equal to zero only within a certain spot. Let it be a segment centered at a point \( x_f \), \( 2\varepsilon \) long. The result of the action of a generalized function \( F_f(t,x) \in D \) to arbitrary function \( \varphi(t,x) \in D' \) in this case equals

\[
F_f \varphi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_f \varphi dx dt .
\]
We can assume that the size of this spot $2\varepsilon$ is negligible compared to the characteristic size of the problem. We proceed in equation (6) to the limit for $\varepsilon \to 0$, while maintaining the value of the total effective force $F'_{f}$

$$F'_{f}(t)\varphi = \int_{0}^{\infty}F'_{f}(t,x)dxdt = F'_{f}(t)\cdot \delta(x-x_{j}), \varphi .$$

As a result, equation (4) takes the form

$$\frac{\partial^{2}u}{\partial t^{2}} + a^{2}\frac{\partial^{4}u}{\partial x^{4}} = -g + \frac{1}{\mu} \sum_{j}F'_{f}(t)\delta(x-x_{j}) + u_{0} \cdot \delta'(t) + u_{1} \cdot \delta(t) + \ldots + a^{2}v_{0} \cdot \delta''(x+l) + a^{2}v_{1} \cdot \delta''(x+l) - a^{2}w_{0} \cdot \delta''(x-l) - a^{2}w_{1} \cdot \delta''(x-l).$$

Here

$$\sum_{f}^{2}\delta'(x-x_{j}) = \sum_{s}S_{s}^{*} \cdot \delta(x-x_{s}) + \sum_{q}Q_{q}^{*} \cdot \delta(x-x_{q}) + \sum_{p}P_{p}^{*} \cdot \delta(x-x_{p}) + \sum_{r}R_{r}^{*} \cdot \delta(x-x_{r}(t))$$

$$S_{s}^{*} = \frac{Mg}{N} - \eta_{s}u(t,x_{s}) - \text{total force transmitted to the span from the side of the } s\text{-th intermediate support},$$

$$\eta_{s} \text{ - rigidity of this support.}$$

$$Q_{q}^{*} = -\theta(u^{*} - u(t,x_{q}))\eta_{q}(u(t,x_{q}) - u^{*}) - \text{total force transmitted to the span from the side of the } q\text{-th intermediate support.}$$

$$P_{p}^{*}(t) \cdot \delta(x-x_{p}(t)) \text{ - total force transmitted to the span from the side of the } p\text{-th transitional part, which is in contact with the span at the point } x_{p} \text{.}$$

$$R_{r}^{*}(t) \cdot \delta(x-x_{r}(t)) \text{ - total force transmitted to the span from the } r\text{-th mobile load, which at the moment of time } t \text{ is in contact with the span at the point } x_{r}(t) \text{.}$$

Note that the forces from the intermediate pores and restrictive supports $S_{s}^{*}$ and $Q_{q}^{*}$ depend on the deformation profile of the superstructure at the points of contact of these supports with the superstructure $u(t,x_{s})$ and $u(t,x_{q})$. And efforts from transitional parts and moving load do not depend on the deformation profile of the span.

### 3.2 Fundamental decision

Build a fundamental solution $\mathcal{E}(t,x)$ of an operator of equation (7). According to [10], the fundamental solution is a generalized function satisfying the equation

$$\frac{\partial^{2}\mathcal{E}}{\partial t^{2}} + a^{2}\frac{\partial^{4}\mathcal{E}}{\partial x^{4}} = \delta(t,x),$$
Denote by $\hat{\mathcal{E}}(t,\xi) = F_s \mathcal{E}$ - Fourier transform $\mathcal{E} t,x$ by spatial variables. We perform the Fourier transform of the last equation in spatial variables. Given that $F_x \left[ \frac{\partial \mathcal{E}}{\partial x} \right] = -i \xi F_x \mathcal{E}$, for the generalized function we obtain the equation

$$\frac{\partial^2 \hat{\mathcal{E}}}{\partial t^2} - a^2 \xi^2 \hat{\mathcal{E}} = \delta(t),$$

The solution to this equation has the form

$$\hat{\mathcal{E}} = \theta(t) \frac{\sin a \xi t}{a \xi},$$

where $\theta(t)$ - Heaviside function

$$\theta(t) = \begin{cases} 1, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0 \end{cases}$$

Following the expansion of the sine in a power series,

$$\hat{\mathcal{E}} = \theta(t) \frac{\sin a \xi t}{a \xi} + \frac{1}{6a} (\xi^2 - \xi^4)(at)^3 + O((at)^5)$$

Performing the inverse Fourier transform, we obtain

$$\mathcal{E}(t,x) = \frac{1}{2a} \theta(at - |x|) - \frac{1}{6a} \delta''''(x) + \delta'''(x)\quad at^3 + O(at^5).$$

The solution to problem (7) is defined as the convolution of the fundamental solution with the right-hand side. According to [6]

$$V(t,x) = \frac{1}{2a} \theta(at - |x|) \ast g = \frac{g}{2a} \int_0^{t-at(x-ct)} \int_0^{t-at(x+c\tau)} d\xi d\tau = \frac{g}{2a} at^2$$

$$V^{(1)}(t,x) = \frac{1}{2a} \theta(at - |x|) \ast u_0, \delta'(t) = \frac{\theta(t)}{2} (u_0(x + at) + u_0(x - at))$$

$$V^{(2)}(t,x) = \frac{1}{2a} \theta(at - |x|) \ast u_1, \delta(t) = \frac{\theta(t)}{2a} \int_{t-at}^{t+at} u_1(\xi)d\xi.$$
In [4], lemma was proved, which states that if $p(t)$ is a locally integrable function and $p(t) = 0$, then for $a \neq 0$

$$\theta \cdot t \cdot t - \frac{\theta \cdot t - \frac{x - at}{a}}{} = \frac{1}{a} \bigg[ \theta \bigg( \frac{x}{a} \bigg) \bigg] \bigg[ t - \frac{x}{a} \bigg].$$

Then

$$\frac{\partial(at - \frac{|x|}{a})}{2a} \cdot \theta(t - \frac{|x|}{a}) = \frac{\partial \theta(at - \frac{|x|}{a})}{2a^2} \cdot \theta(-x) - \theta(x) \cdot p \left[ t - \frac{|x|}{a} \right] \quad (12)$$

It also follows from convolution differentiation formulas that

$$\frac{1}{6a} \left( \delta''(x) + \delta'''(x) \right) (at)^{3} \cdot \theta'(-x) = \frac{1}{2} \left( \delta''(x) + \delta'''(x) \right) (at)^{3}$$

$$\delta''(x) + \delta'''(x) \right) (at)^{3} \cdot \theta'(-x) = \frac{1}{6a} \left( \delta''(x) + \delta'''(x) \right) (at)^{3}$$

Consider the point $(t, x)$ such that $\theta(at - \frac{|x - x_{j}|}{a}) = 0$ for all $f$ and $\theta(at - \frac{|x \pm l|}{a}) = 0$. In accordance with (11) and (12) at this point $E^* \cdot \delta'(x - x_{j}) = 0$ for all $f$ and $E^* \cdot \delta''(x \pm l) = 0$, $E^* \cdot \delta'(x \pm l) = 0$.

Then, as follows from the formulae (8) - (13)

$$u(\Delta t, x) = \left( \frac{g}{a^2} + u_0''(x) + u_0'''(x) \right) \left( \frac{\Delta t}{2} \right)^2 + \sum_{\tau} \theta \left( \frac{at - \frac{|x - x_{\tau}|}{a}}{\frac{\Delta t}{2}} \right) \Delta \tau \bigg[ \int_{0}^{\tau} \mathcal{R} d\tau \bigg]$$

$$\frac{\partial \delta'}{\partial t} (\Delta t, x) = \left( \frac{g}{a^2} + au_0''(x) + au_0'''(x) \right) a \Delta t + \sum_{\tau} \theta \left( \frac{at - \frac{|x - x_{\tau}|}{a}}{\frac{\Delta t}{2}} \right) \Delta \tau \bigg[ \int_{0}^{\tau} \mathcal{R} d\tau \bigg]$$

Consider the points $(t, x)$ in the vicinity of the point $x_{j}$, corresponding to one of the intermediate supports $S_{x_{j}}$, such that $\theta(at - \frac{|x - x_{j}|}{a}) = 1$, and far enough from the border $\theta(at - \frac{|x \pm l|}{a}) = 0$ and other intermediate supports $\theta(at - \frac{|x - x_{\pm l}|}{a}) = 1$. For such points $E^* \cdot \delta''(x \pm l) = 0$, $E^* \cdot \delta'(x \pm l) = 0$.

Then up to $\mathcal{O} \left( \frac{at}{2} \right)^3$

$$u(t, x) = \left( \frac{g}{a^2} + u_0''(x) + u_0'''(x) \right) \left( \frac{\Delta t}{2} \right)^2 + \frac{1}{2a \mu} \bigg[ \int_{0}^{\tau} \mathcal{R} d\tau \bigg]$$

$$\frac{\partial \delta'}{\partial t} (\Delta t, x) = \left( \frac{g}{a^2} + au_0''(x) + au_0'''(x) \right) a \Delta t + \sum_{\tau} \Delta \tau \bigg[ \int_{0}^{\tau} \mathcal{R} d\tau \bigg]$$

$$\frac{\partial \delta'}{\partial t} (\Delta t, x) = \left( \frac{g}{a^2} + au_0''(x) + au_0'''(x) \right) a \Delta t + \sum_{\tau} \Delta \tau \bigg[ \int_{0}^{\tau} \mathcal{R} d\tau \bigg]$$

Proceeding in (15) to the limit as $x \to x_{j}$, $x \to x_{j}$, we get
\[
\begin{align*}
    &u(t,x) = -\left(\frac{g}{a^2} + u_0^{\prime\prime}(x_0) + u_0^{\prime\prime\prime}(x_0)\right)\left(\frac{\alpha t^2}{2} + \frac{1}{2a\mu} \int_0^\infty \sum_r \theta(t-z-x_r) \int_0^{|z-x_r|} R_\mu \, d\tau \right) \\
    &+ u_0(x_0 + \alpha t) + u_0(x_0 - \alpha t) + \frac{1}{2a} \int_{x_0 - \alpha t}^{x_0 + \alpha t} u_i(\xi) \, d\xi \\
\end{align*}
\]

(16)

We differentiate equality (16) with respect to \( t \) and take into account that \( S_2^* = \frac{M\gamma}{N} - \eta_k u(t,x) \), we get the Cauchy problem for the function \( u_k(t) = u(t,x) \)

\[
\begin{align*}
    &\frac{\partial u_k}{\partial t} + \frac{\eta_k}{a \mu} u_k = -\left(\frac{g}{a} + au_0^{\prime\prime}(x_0) + au_0^{\prime\prime\prime}(x_0)\right)at + \frac{M\gamma}{2a\mu}N \cdots \\
    &+ \sum_r \theta(at-|z-x_r|) \frac{a}{2a\mu} \cdots \\
    &+ a \frac{\partial u_0}{\partial x}(x_0 + at) - \frac{\partial u_0}{\partial x}(x_0 - at) + u_0(x_0 + at) + u_0(x_0 - at) \\
    &u_k(t = 0) = u_0(x_0) \\
\end{align*}
\]

(17)

We solve this Cauchy problem and define \( u(\Delta t,x) \) and \( \frac{\partial u}{\partial t}(\Delta t,x) \).

If in (16) the integral \( \int_0^{\infty} S_2^* \, d\tau \) replaced by quadrature according to the trapezoid formula, then to calculate \( u(\Delta t,x) \) and \( \frac{\partial u}{\partial t}(\Delta t,x) \) up to terms \( O\left((at)^2\right) \) we get the expression

\[
\begin{align*}
    &u(\Delta t,x) = \frac{1}{1 + \frac{\eta_k}{4a^2\mu} \Delta t} \left\{ \frac{Mg\Delta t}{2a^2\mu N} - \eta_k u_0(x_0) \Delta t + \sum_r \theta(\Delta t-|z-x_r|) \int_0^{\infty} R_\mu \, d\tau \right\} \\
    &+ u_0(x_0 + \Delta t) + u_0(x_0 - \Delta t) + \frac{1}{2a} \int_{x_0 - \Delta t}^{x_0 + \Delta t} u_i(\xi) \, d\xi \\
\end{align*}
\]

(18)

Similar to the above with respect to the intermediate supports, for restrictive supports, given that \( Q_k^* = -\theta(u^* - u(t,x)) \eta_k (u^* - u(t,x)) \), we get the Cauchy problem for the function \( u_k(t) = u(t,x) \)
\[
\frac{\partial u}{\partial t} + \frac{\eta_u}{2a\mu} \theta(u^* - u_q)(u^* - u_q) = \left( \frac{g}{a} + au''_0(x_q) + au'''_0(x_q) \right) \Delta t + \left[ \frac{2}{2a\mu} \right] \sum_l \left| \frac{x_q - x_l}{a} \right| \delta \left( \Delta t - \frac{|x_q - x_l|}{a} \right) \ldots \]
\[
+ \sum_l \theta \left( \Delta t - \frac{|x_q - x_l|}{a} \right) \frac{\Re^*_l(t)}{2a\mu} \ldots
\]
\[
+ \frac{\partial u_0}{\partial \xi} (x_q + at) - \frac{\partial u_0}{\partial \xi} (x_q - at) + \frac{u_0(x_q + at) + u_0(x_q - at)}{2}
\]
\[
u_q(t = 0) = u_0(x_q)
\]

The difference from the case of intermediate supports is that the stiffness coefficients of the limit supports significantly exceed the stiffness coefficients of the intermediate supports \( \eta_q \gg \eta_l \). Also, we believe that the place of contact of the left transitional part with the span coincides with the position of the left restrictive support. In the same way, the place of contact of the right transitional part with the span coincides with the position of the right restrictive support. We solve this Cauchy problem and define \( u(t', x_q) \) and \( \frac{\partial u}{\partial t}(t', x_q) \).

Until the span comes into contact with the restrictive support dynamics \( u_q(t) \) described by an equation similar to the equation (14)
\[
u_q(t) = -\left( \frac{g}{a} + u''_0(x_q) + u'''_0(x_q) \right) \Delta t + \frac{\partial u_0}{\partial \xi} (x_q + at) - \frac{\partial u_0}{\partial \xi} (x_q - at) + \frac{u_0(x_q + at) + u_0(x_q - at)}{2}
\]
\[
u_q(t = 0) = u_0(x_q)
\]

Moment \( t^* \) span contact with restrictive support \( u_q(t^*) = u^* \) can be determined by solving the quadratic equation in the absence of an external load or using an iterative procedure in the presence of an external load.

After contact with the restrictive support, due to the very large stiffness of the restrictive support and therefore the extremely short time of interaction with the support, the dynamics \( u_q(t) \) described by the Cauchy problem
\[
\frac{\partial u_q}{\partial t} - \frac{\eta_u}{2a\mu} (u^* - u_q) = u_1
\]
\[
u_q(t = t^*) = u^*
\]

Here \( u_1 = \frac{\partial u_q}{\partial t} \) - the speed of the point of contact of the span with the support at the moment of contact \( t^* \). The solution to problem (21) has the form
\[
u_q(t) = \frac{2a\mu}{\eta_q} u_1 e^{-\frac{\eta_u}{2a\mu}(t - t^*)} + u^* - \frac{2a\mu}{\eta_q} u_1
\]
\[
\frac{\partial u_q}{\partial t}(t) = -u_q e^{-\frac{\eta_u}{2a\mu}(t - t^*)}
\]
In the limit \( \eta_q \to \infty \) at the moment of separation of the span from the restrictive support \( u_q = u' \) and the speed of the point of contact of the span with the support, \( \frac{\partial u_q}{\partial t} = 0 \).

Now consider the points \((t, x)\) in a neighborhood of the right boundary of the region such that \( \theta(at - |x - l|) = 1 \), and far enough from the left border \( \theta(at - |x + l|) = 0 \) and intermediate and restrictive supports \( \theta(at - |x - x_q|) = 0 \) and \( \theta(at - |x - x_q|) = 0 \). Then in the limit at \( x \to l \) on the left and considering that \( u(t, x) \to w_0(t), \ \frac{\partial u}{\partial x}(t, x) \to w'_0(t), \ \frac{\partial^2 u}{\partial x^2}(t, x) \to w''_0(t), \ \frac{\partial^3 u}{\partial x^3}(t, x) \to w'''_0(t) = 0 \), and provided that the initial data is approximated by piecewise polynomial functions of the second order, we obtain

\[
w_0 = -\frac{t^2}{2} - \frac{1}{2a^2} \frac{\partial^2 w_0}{\partial t^2} - \frac{1}{2a} \frac{\partial w_0}{\partial t} + \frac{u_0(l-at)}{2} + \frac{1}{2a} \int_{l-at}^l u_0(\xi)d\xi \tag{23}
\]

\[
w_0' = \frac{1}{2a^3} \frac{\partial^3 w_0}{\partial t^3} - \frac{1}{2a^2} \frac{\partial^2 w_0}{\partial t^2} + \frac{1}{2} \frac{\partial u_0}{\partial x}(l-at) - \frac{u_0(l-at)}{2a} \tag{24}
\]

\[
0 = -\frac{1}{2a^2} \frac{\partial^2 w_0}{\partial t^2} - \frac{1}{2a^3} \frac{\partial^3 w_0}{\partial t^3} + \frac{1}{2} \frac{\partial^2 u_0}{\partial x^2}(l-at) - \frac{1}{2a} \frac{\partial u_0}{\partial x}(l-at) \tag{25}
\]

Differentiate in time equation (24), multiply it by \( \frac{1}{a} \) and subtract the equation from it (25)

\[
\frac{\partial w_0'}{\partial t} = -a \frac{\partial^2 u_0}{\partial x^2}(l-at) + \frac{\partial u_0}{\partial x}(l-at) \tag{26}
\]

Substitute the resulting expression for \( \frac{\partial w_0'}{\partial t} \) into the equation (23). As a result, to determine \( w_0(t) \) we get the Cauchy problem

\[
\frac{d^2 w_0}{dt^2} + 2a^2 w_0 = -a^2 g \frac{\partial^2 u_0}{\partial x^3}(l-at) + a^2 \frac{\partial^2 u_0}{dx^2}(l-at) + \frac{u_0(\xi)}{\partial x}(l-at) - a \int_{l-at}^l u_0(\xi)d\xi
\]

\[
w_0(t = 0) = u_0(l), \ \frac{dw_0}{dt}(t=0) = u'_0(l) \tag{27}
\]

We solve this Cauchy problem and define \( u(\Delta t, l) = \frac{w_0}{\Delta t} \) and \( \frac{\partial u}{\partial t}(\Delta t, l) = \frac{\partial w_0}{\partial t}(\Delta t) \).

Exactly the same for determining \( v_0(t) \), we get the Cauchy problem

\[
\frac{d^2 v_0}{dt^2} + 2a^2 v_0 = -a^2 g \frac{\partial^2 u_0}{dx^2}(-l+at) + a^2 \frac{\partial^2 u_0}{dx^2}(-l+at) + \frac{u_0(\xi)}{\partial x}(-l+at) + a \int_{-l}^{-l-at} u_0(\xi)d\xi
\]

\[
v_0(t = 0) = u_0(-l), \ \frac{dv_0}{dt}(t=0) = u'_0(-l) \tag{28}
\]

Having decided which we define \( u(\Delta t, -l) = \frac{v_0}{\Delta t} \) and \( \frac{\partial u}{\partial t}(\Delta t, -l) = \frac{\partial v_0}{\partial t}(\Delta t) \).

Now we have everything we need to build a computational algorithm for finding an approximate solution to the initial-boundary-value problem (1) - (3).
4. Computational algorithm and numerical experiments

We cut a segment \(-l, l\) by nodes \(x_p, p = 1: P\) on \(P - 1\) intervals. At the same time, we require that the position of each intermediate and each restrictive support coincide with one of the nodes \(x_p\). Denote by \(h_p = x_{p+1} - x_p, p = 1: P - 1\) of these intervals.

Let us know \(u_p(t) = u(t, x_p)\) and \(v_p(t) = \frac{\partial u}{\partial t}(t, x_p)\) - the value of the solution and its time derivative in nodes \(x_p\) at time \(t\). We will connect with each node \(x_p\) 2nd order interpolation polynomial \(U_p(x, u_p(t))\), constructed by the values of the solution at time \(t\) in this node \(u_p(t)\) and in two nearby nodes. Similarly, with each node \(x_p\) we connect the 2nd order interpolation polynomial \(V_p(x, v_p(t))\), constructed by the values of the time derivative at a time \(t\) in this node \(v_p(t)\) and in two nearby nodes. Further, where this does not cause confusion, the interpolation polynomials will be written as \(U_p(x)\) and \(V_p(x)\), omitting their explicit dependence on \(u_p(t)\) and \(v_p(t)\).

We calculate \(u_p(t + \Delta t) = u(t + \Delta t, x_p)\) and \(v_p(t + \Delta t) = \frac{\partial u}{\partial t}(t + \Delta t, x_p)\) - the value of the solution and its first time derivative at the same nodes at a time \(t + \Delta t\), where \(\Delta t = \frac{h}{a}\).

Solution in nodes \(x_p\), which do not match any left border \(x_p = -l\), neither with the right border \(x_p = l\) and do not coincide with the position of any intermediate support \(x_p = x_s\) and no restrictive support \(x_p = x_q\) calculated in accordance with formulas (14). The solution at the node coinciding with the left boundary or with the right boundary is calculated by solving the Cauchy problem (28) or (27), respectively, on the interval \(0, \Delta t\). And if the knot \(x_p\) coincides with the position of the intermediate support, we calculate the solution in this node, solving the Cauchy problem (18) in one interval or another on the interval \(0, \Delta t\).

For node \(x_q\), coinciding with the restrictive support solution \(u(t + \Delta t, x_q)\) and \(\frac{\partial u}{\partial t}(t + \Delta t, x_q)\) at this node we calculate, solving the Cauchy problem (19) on the interval \(0, \Delta t\) or using asymptotic expressions for \(\eta_q \rightarrow \infty\), which significantly reduces the time required for calculations. To do this, we calculate the solution in the node \(x_q\) at time \(t + \Delta t\) according to formulae (20), corresponding to the situation when the span does not come in contact with the restrictive support. If the position of the span is above the limit support \(u_q, t + \Delta t > u^*\), then we proceed to the calculation at the next time interval.

If \(u_q, t + \Delta t \leq u^*\), then we assume \(u(t + \Delta t, x_q) = u^*\) and \(\frac{\partial u}{\partial t}(t + \Delta t, x_q) = 0\)

We approximate the values of the solution and its time derivative on the previous time layer, at time \(t\) by the polynomials \(U_p(x)\) and \(V_p(x)\).

A number of numerical experiments were carried out with the constructed computational model, including those with a moving active load at various parameters of this load. In these experiments, a calculation scheme was used with the parameters described in detail in [7-10] and further used in [1-3]. These experiments showed, even in a single-processor implementation, the high performance of a computational model based on this algorithm. The calculation time is significantly less than the time required to calculate similar scenarios by other methods. Including probably the most effective of them.
- the modified Galerkin method, on piecewise linear basis functions developed by the authors in earlier works [3]. The analysis showed that the presence of elements with very high stiffness in the elastic-supported load-bearing structure (for example, limiting or intermediate supports) leads to the need to numerically solve the Cauchy problem for the Galerkin method ODE system with a very small time step. In the algorithm proposed in this paper, such a need arises only in nodes that coincide with the location of structural elements with high rigidity. In other nodes, the solution is calculated by explicit formulas with a significantly large time step.

![Figure 1](image.png)

**Figure 1.** The relationship between the logarithm of the mean square and the logarithm of the stiffness coefficient.

Below are the results of numerical experiments in which the question was solved: “Is the model proposed above with an absolutely rigid restrictive support the limiting case of models in which the restrictive supports were represented by springs with high stiffness?” For this, the deformed state of the rod was calculated using a model with absolutely elastic restrictive supports. At the same time, a calculation was performed with the same parameters and initial conditions for a model with restrictive supports in the form of springs of high stiffness at 5 values of elastic bond stiffness. At a fixed point in time over 20 sections of the elastic rod, the average square of the difference between the solutions obtained for models with an absolutely elastic restrictive support and an elastic support of large but finite stiffness is calculated. The result shown in figure 1 confirms the convergence of the solution using models with elastic restrictive supports to the solution of the model with an absolutely elastic support with an increase in the rigidity of elastic bonds.

In figure 2 the deformation profile of an elastically supported load-bearing structure at a fixed time of 2.4 sec, calculated according to the model with elastic restrictive supports with high but finite stiffness at the indicated 5 values of elastic bond is shown. The nature of the convergence of the solutions obtained using the model with an elastic bounding support of large, but finite stiffness, to the solution of the model with an absolutely elastic bounding support with an increase in the stiffness coefficient of elastic supports can be determined using this figure. The greatest difference between the solutions is observed at the locations of the restrictive supports. But with an increase in the stiffness coefficient of elastic supports, this difference monotonously decreases. Thus, we can consider convergence not only in the Euclidean norm, but also in the uniform norm or, in other words, uniform convergence.
Figure 2. Convergence of the solution when using a model with elastic restrictive supports to the solution of a model with absolutely elastic support.

5. Conclusion
Using an example of an initial-boundary-value problem for an equation that describes structurally nonlinear vibrations of a span bridge with intermediate supports and restrictive rigid supports, an algorithm for finding an approximate solution is proposed. Unlike traditional algorithms related to the approximation of differential operators, this algorithm is based on the approximation of a fundamental solution to the problem operator.

The proposed algorithm is based on a reformulation of the initial-boundary value problem in the form of an equation for generalized functions. Boundary and initial conditions include equations for generalized functions. Using the technique of generalized functions allows solving problems with point and even instantaneous sources.

The implementation of the algorithm is associated with the construction of a fundamental solution to the problem operator. If it is not possible to accurately obtain the fundamental solution, it is necessary to construct the approximation of the fundamental solution. The accuracy of this approximation should be consistent with the order of approximation of the solution with respect to spatial variables.

Conducted computational experiments have shown, even in a single-processor implementation, high performance computing model based on this algorithm. The calculation time is significantly less than the time required to calculate similar scenarios by other methods.

A model of absolutely rigid restrictive support is proposed. It is shown that such a model is the limiting case of models with restrictive supports of large, but finite stiffness. It should be noted that the presence in the design of restrictive supports with great rigidity, and even absolutely rigid supports, does not lead to an increase in the calculation time, unlike other algorithms for solving the considered class of initial-boundary value problems.

The calculation of the solution on the next time layer for each node of the spatial grid is performed independently. Therefore, the algorithm has a high potential for parallelization and can be effectively implemented on multiprocessor and / or multicore computing systems.

References
[1] Gridnev S Yu and Skalko Yu I 2017 Numerical analysis of nonlinear oscillations of an elastically supported deformable system with limit supports at the ends Marine Intelligent Technologies vol 3 (St. Petersburg) 4 (38) 37 [in Russian]
[2] Gridnev S Yu, Skalko Yu I, Ravodin I V and Yanaeva V V 2018 Simulation of vibrations of a continuously elastic supported rod with varying boundary conditions under the action of a movable MATEC Web of Conferences 196 01053 (2018) XXVII RSP Seminar Theoretical Foundation of Civil Engineering https://doi.org/10.1051/matecconf/201819601053

[3] Gridnev S Yu, Skalko Yu I, Minaeva N V and Yanaeva V V 2019 Comparative analysis of models of limiting supports in the study of structurally nonlinear oscillations of elastically supported bar from mobile load IOP Conf. Series: Journal of Physics: Conf. Series 1203 012030 IOP Publishing doi: 10.1088/1742-6596/1203/1/012030

[4] Skalko Yu I, Gridnev S Yu and Yagfarova A 2019 Modeling the focusing of the energy of elastic waves in block-fractured medium during the long-term vibration action Proc. Int. Scient. Conf. Energy Management of Municipal Facilities and Sustainable Energy Technologies EMMFT 2018. Advances in Intelligent Systems and Computing vol 2 eds. Murgul V and Pasetti M (Springer Nature Switzerland AG) 983 534

[5] Vlasov V Z 1963 Thin-walled elastic rods. The principles of constructing a general technical theory of shells Selected Works vol 2 (Publishing House of the USSR Academy of Sciences) p 507 [in Russian]

[6] Vladimirov V S 1979 Generalized functions in mathematical physics (Moscow: Science vol 4) p 379 [in Russian]

[7] Ufimtsev E, Voronina M 2016 Procedia Engineering Research of total mechanical energy of steel roof truss during structurally nonlinear oscillations 150 1891

[8] Feigin M I 1994 Forced oscillations of systems with discontinuous nonlinearities (Moscow: Science) p 285 [in Russian]

[9] Lukashevich A A 2011 The solution of contact problems for elastic systems with one-way connections by the method of step-by-step analysis (Dissertation for the degree of Doctor of Technical Sciences, Tomsk) p 283 [in Russian]

[10] Laursen T A, Chawla V 1997 Int. J. for Num. Meth. Eng Design of energy conserving algorithms for frictionless dynamic contact problems 40 863