On Weyl-covariant channels

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Abstract

Formalism of discrete noncommutative Fourier transform is developed and applied to the study of Weyl-covariant channels. We then extend a result in [7] concerning a bound of the maximal output 2-norm of a Weyl-covariant channel. A class of channels which attain the bound is introduced, for which the multiplicativity of the maximal output 2-norm is proven. Complementary channels are described which share the multiplicativity properties with the Weyl-covariant channels.

1 A noncommutative Fourier transform

A state of finite quantum system is represented by a positive operator \( \rho \) of trace one (density operator) in a Hilbert space \( \mathcal{H} \) of dimensionality \( d \). The set of density operators in \( \mathcal{H} \) is denoted \( \mathfrak{S}(\mathcal{H}) \). A channel \( \Phi \) is a completely positive (CP) trace-preserving (TP) map of the algebra \( \mathfrak{M}(\mathcal{H}) \) of all operators in \( \mathcal{H} \). Although the TP condition is redundant in the context of our results, we shall impose it just for notational convenience.

The maximal output \( p \)-norm of \( \Phi \) is defined as

\[
\nu_p(\Phi) := \sup_{\rho \in \mathfrak{S}(\mathcal{H})} \| \Phi(\rho) \|_p,
\]

where \( \| \|_p \) is the Schatten \( p \)-norm: \( \| \rho \|_p := (\text{tr}|\rho|^p)^{\frac{1}{p}} \).
The current multiplicativity conjecture is that
\[ \nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\nu_p(\Omega), \] (1.2)
for arbitrary channels \( \Phi \) and \( \Omega \), and for \( p \in [1, 2] \). Note that the inequality \( \nu_p(\Phi \otimes \Omega) \geq \nu_p(\Phi)\nu_p(\Omega) \) is straightforward. In this paper we consider the case \( p = 2 \), which is still an open problem (see [3],[8],[10] for some general results in this direction).

Let us choose an orthonormal basis \( \{e_k; k = 0, \ldots, d-1\} \) in \( \mathcal{H} \). Consider the additive cyclic group \( \mathbb{Z}_d \) and define an irreducible projective unitary representation of the group \( Z = \mathbb{Z}_d \oplus \mathbb{Z}_d \) in \( \mathcal{H} \) as
\[ z = (x, y) \mapsto W_z = U^x V^y, \]
where \( x, y \in \mathbb{Z}_d \), and \( U \) and \( V \) are the unitary operators such that
\[ U|e_k\rangle = |e_{k+1(\text{mod} d)}\rangle, \quad V|e_k\rangle = \exp\left(\frac{2\pi ik}{d}\right)|e_k\rangle. \]

The discrete Weyl operators \( W_z \) satisfy relations similar to the canonical commutation relations for Weyl operators on \( Z = \mathbb{R}^s \oplus \mathbb{R}^s \) (see e. g. [4]):
\begin{align*}
W_z W_{z'} &= \exp(i\langle y, x' \rangle)W_{z+z'}; \quad (1.3) \\
W_z W_{z'} &= \exp(i(\langle x', y \rangle - \langle y', x \rangle))W_{z'}W_z; \quad (1.4)
\end{align*}
where \( \langle y, x \rangle := 2\pi yx/d \).

For future use we introduce the duality form on \( Z \)
\[ \langle z', z \rangle := \langle x', x \rangle + \langle y', y \rangle, \]
and the symplectic form
\[ \langle z', Jz \rangle := \langle x', y \rangle - \langle y', x \rangle, \]
where \( J(x, y) := (y, -x) \).

Instead of the relation \( W_z^* = W_{-z} \) for the usual Weyl operators, we have
\[ W_z^* = \exp(i\langle y, x \rangle)W_{-z}. \] (1.5)
Moreover,
\[ \text{Tr}W_z W_{z'}^* = d\delta_{z,z'}. \] (1.6)
Consider \( \mathcal{M}(\mathcal{H}) \) as a Hilbert space with the Hilbert-Schmidt inner product. The Weyl operators form an orthogonal basis in \( \mathcal{M}(\mathcal{H}) \). Hence for all \( X \in \mathcal{M}(\mathcal{H}) \)
\[ X = \sum_z f_X(z)W_z, \quad \text{where} \quad f_X(z) = \frac{1}{d}\text{Tr}XW_z^*. \]
The correspondence $X \leftrightarrow f_X(z)$ is a discrete analog of the “noncommutative Fourier transform”, see [4]. It has Parceval-type properties

$$\mathrm{Tr}X = df_X(0); \quad \mathrm{Tr}X^* = d\sum_z |f_X(z)|^2. \quad (1.7)$$

It follows that for a state $\rho \in \mathcal{S}(\mathcal{H})$

$$f_\rho(0) = \frac{1}{d}; \quad |f_\rho(z)| \leq \frac{1}{d} \quad (1.8)$$

and

$$\sum_{z \neq 0} |f_\rho(z)|^2 \leq \frac{d-1}{d^2}. \quad (1.9)$$

Moreover, $\rho$ is a pure state if and only if $\mathrm{Tr}\rho^2 = 1$, which is equivalent to

$$\sum_{z \neq 0} |f_\rho(z)|^2 = \frac{d-1}{d^2}. \quad (1.9)$$

The relation (1.5) implies

$$f_X(z) = \exp(-i\langle y, x \rangle)f_X^*(-z). \quad (1.10)$$

A necessary and sufficient condition for a Hermitian $X \in \mathfrak{M}(\mathcal{H})$ to be positive is: the $d^2 \times d^2$ -matrix

$$[f_X(z' - z) \exp(i\langle y, x - x' \rangle)]_{z,z' \in \mathbb{Z}}$$

is nonnegative definite. The necessity follows from

$$\sum_{z,z'} \overline{c}_z c_{z'} f_X(z' - z) \exp(i\langle y, x - x' \rangle) = \frac{1}{d} \mathrm{Tr}X \left( \sum_z c_z W_z \right)^* \left( \sum_{z'} c_{z'} W_{z'} \right) \geq 0.$$

The proof of sufficiency is similar to that for the case of the “noncommutative Fourier transform”, see [4].

### 2 Multiplicativity for the Weyl-covariant maps and channels

A linear map $\Phi$ of $\mathfrak{M}(\mathcal{H})$ is Weyl-covariant if

$$\Phi(W_z X W_z^*) = W_z \Phi(X) W_z^*$$
for all \( z \in Z \) and \( X \in \mathfrak{M}(\mathcal{H}) \). Inserting \( X = W_{z'} \) we find that \( \Phi(W_{z'}) \) satisfies the same relation (1.4) as \( W_{z'} \), hence \( \Phi(W_{z'})W^*_z \) commute with all \( W_z \). Therefore

\[
\Phi(W_z) = \phi(z)W_z, \quad (2.1)
\]

where \( \phi(z); z \in Z \), is a complex function. By making a normalization, we can always assume that \( \phi(0) = 1 \). The class of such maps we denote \( \mathfrak{M}_1(\mathcal{H}) \). We shall also use the notation \( \|A\|_2 = \sqrt{\text{Tr}A^*A} \).

Defining the Fourier transform

\[
p_\gamma = \frac{1}{d^2} \sum_{z \in Z} \phi(z) \exp(i\langle \gamma, z \rangle),
\]

we have

\[
\phi(z) = \sum_{\gamma \in Z} p_\gamma \exp(-i\langle \gamma, z \rangle),
\]

for \( \gamma = (\alpha, \beta) \in Z \). The relation (1.4) implies that

\[
\Phi(X) = \sum_{\gamma} p_\gamma W_{J\gamma} X W_{J\gamma}^*, \quad (2.2)
\]

There is a simple formula for composition of two Weyl-covariant maps

\[
(\Phi_1 \circ \Phi_2)(\rho) = (\Phi_2 \circ \Phi_1)(\rho) = \sum_{\gamma} p_\gamma W_{J\gamma} \rho W_{J\gamma}^*,
\]

where \( p_\gamma = p_\gamma^{(1)} * p_\gamma^{(2)} \) is the convolution of functions \( p_\gamma^{(1)}, p_\gamma^{(2)} \), defining the maps \( \Phi_1, \Phi_2 \), since the action of the composition on the maps is given by (2.1), where \( \phi(z) = \phi_1(z)\phi_2(z) \).

The map \( \Phi \) is channel if and only if \( \{p_\gamma\} \) is probability distribution on \( Z \), and \( \phi(z) \) – its characteristic function [5]. The relation (2.2) is then the Kraus representation.

Our principal estimate is:

**Theorem 1.** Let \( \Phi \in \mathfrak{M}_1(\mathcal{H}) \) and \( \hat{\rho} \) – an operator in \( \mathcal{H} \otimes K \). Then

\[
\|(\Phi \otimes \text{Id}_K)(\hat{\rho})\|_2^2 \leq \frac{1}{d}(1 - \max_{z \neq 0} |\phi(z)|^2)\|\text{Tr}_\mathcal{H}\hat{\rho}\|_2^2 + \max_{z \neq 0} |\phi(z)|^2 \|\hat{\rho}\|_2^2. \quad (2.3)
\]

**Proof.** Defining \( A_z = \frac{1}{d} \text{Tr}_\mathcal{H}\hat{\rho}(W_z^* \otimes I) \), we have

\[
\hat{\rho} = \sum_z W_z \otimes A_z.
\]
Note that
\[ \|\hat{\rho}\|^2_2 = \text{Tr} \left( \sum_{z'} W^*_{z'} \otimes A^*_{z'} \right) \left( \sum_z W_z \otimes A_z \right) = d \sum_z \text{Tr} A^*_z A_z; \]
\[ \text{Tr}_\mathcal{H} \hat{\rho} = \sum_z \text{Tr} W_z \otimes A_z = d A_0; \]
\[ \|\text{Tr}_\mathcal{H} \hat{\rho}\|^2_2 = d^2 \text{Tr} A^*_0 A_0. \] (2.4)

Next, we have
\[ \| (\Phi \otimes \text{Id}_\mathcal{K})(\hat{\rho}) \|^2_2 = \text{Tr} \left( \sum_{z'} \phi(z')^* W^*_{z'} \otimes A^*_{z'} \right) \left( \sum_z \phi(z) W_z \otimes A_z \right) \]
\[ = d \sum_z |\phi(z)|^2 \text{Tr} A^*_z A_z \]
\[ = d \left( \text{Tr} A^*_0 A_0 + \sum_{z \neq 0} |\phi(z)|^2 \text{Tr} A^*_z A_z \right) \]
\[ \leq d \left( \text{Tr} A^*_0 A_0 + \max_{z \neq 0} |\phi(z)|^2 \sum_{z \neq 0} \text{Tr} A^*_z A_z \right) \] (2.5)
\[ = d \left( 1 - \max_{z \neq 0} |\phi(z)|^2 \right) \text{Tr} A^*_0 A_0 + \max_{z \neq 0} |\phi(z)|^2 \sum_{z} \text{Tr} A^*_z A_z \]
\[ = \frac{1}{d} \left( 1 - \max_{z \neq 0} |\phi(z)|^2 \right) \|\text{Tr}_\mathcal{H} \hat{\rho}\|^2_2 + \max_{z \neq 0} |\phi(z)|^2 \|\hat{\rho}\|^2_2. \]

QED

In the case of one dimensional \( \mathcal{K} \) the bound (2.3) implies the following inequality for channel \( \Phi \) obtained in proposition 9 of [7]:
\[ \text{Tr} \Phi (\rho)^2 \leq \frac{1}{d} \left( 1 + (d - 1) \max_{z \neq 0} |\phi(z)|^2 \right). \]

Moreover, this proposition states that, in the case \( d = 3 \), the equality is attained here for a special pure state \( \rho \). This observation can be substantially generalized (see theorem 3 below).

**Theorem 2.** Let \( \Phi \in \mathcal{W}_1(\mathcal{H}) \) be such that
\[ |\phi(z)| \leq 1; \quad z \in Z, \] (2.6)
and
\[ \nu_2(\Phi) = \frac{1}{\sqrt{d}} \left( 1 + (d - 1) \max_{z \neq 0} |\phi(z)|^2 \right)^{\frac{1}{2}}, \] (2.7)
then the multiplicativity of the maximal output 2-norm holds for $\Phi \otimes \Omega$, where $\Omega$ is an arbitrary CP map.

Proof. We have

$$\|\text{Tr}_H(\text{Id}_H \otimes \Omega)(\hat{\rho})\|_2 = \|\Omega(\text{Tr}_H\hat{\rho})\|_2 \leq \nu_2(\Omega)$$

$$\|(\text{Id}_H \otimes \Omega)(\hat{\rho})\|_2 \leq \nu_2(\text{Id}_H \otimes \Omega) = \nu_2(\Omega),$$

where the last equality follows from \[1\]. Replacing $\hat{\rho}$ by $(\text{Id}_H \otimes \Omega)(\hat{\rho})$ in Theorem 1 and using (2.6) gives

$$\|\Phi \otimes \Omega)(\hat{\rho})\|_2 \leq \left(1 + (d-1) \max_{z \neq 0} |\phi(z)|^2 \right) (\nu_2(\Omega))^2.$$  \hspace{1cm} (2.8)

Therefore by (2.7)

$$\nu_2(\Phi \otimes \Omega) \leq \nu_2(\Phi) \nu_2(\Omega).$$

QED

Define the set of optimizers of $|\phi(z)|$ for $z \neq 0$

$$E_{\text{max}} := \{z : z \neq 0, z = \arg \max_{z \neq 0} |\phi(z)|\}.$$

For a unit vector $|\psi\rangle \in \mathcal{H}$ consider the subset of $Z$ defined as

$$\mathcal{G}_\psi := \{z : |\psi\rangle \text{ is an eigenvector of } W_z\}.$$ \hspace{1cm} (2.10)

By (1.3) $\mathcal{G}_\psi$ is a subgroup of $Z$ and $|\mathcal{G}_\psi| \leq d$ as we shall see from the proof of theorem 3.

**Theorem 3.** Let $d$ be arbitrary. A necessary condition for the equality (2.7) is $|E_{\text{max}}| \geq d - 1$. A sufficient condition is that there is a subgroup $\mathcal{G}_\psi \subseteq Z$ such that $|\mathcal{G}_\psi| = d$ and

$$\mathcal{G}_\psi \setminus \{0\} \subseteq E_{\text{max}}.$$

Proof. If (2.7) holds then there exists a pure state $\rho$ such that equality holds in (2.5) with $A_z = f_\rho(z)$. This implies $\mathcal{N} := \{z : z \neq 0, f_\rho(z) \neq 0\} \subseteq E_{\text{max}}$. Hence the necessity follows from (1.8) and (1.9).

Let $|\psi\rangle$ be a common eigenvector for the unitaries $W_z; z \in \mathcal{G}_\psi$, with eigenvalues $c_z$ of modulus 1, and let us show first that $|\mathcal{G}_\psi| \leq d$. If $|\mathcal{G}_\psi| \geq d$, then the operator

$$X = \frac{1}{d} \left(I + \sum_{z \in \mathcal{L}} c_z W_z\right),$$ \hspace{1cm} (2.11)
where $\mathcal{L}$ is any subset of $\mathcal{G}_\psi \setminus \{0\}$, such that $|\mathcal{L}| = d - 1$, satisfies $X|\psi\rangle = |\psi\rangle$, and $\text{Tr}X^*X = 1$ by (1.7). This can be only the case if $X = \rho_0 = |\psi\rangle\langle\psi|$. Then it follows: 1) $|\mathcal{G}_\psi| = d$, for otherwise the operator $\rho_0$ would have several different decompositions (2.11) corresponding to different subsets $\mathcal{L}$; 2) under the assumptions of the theorem

$$
\text{Tr}\Phi(\rho_0)^*\Phi(\rho_0) = d \left( \frac{1}{d^2} + \sum_{z \in \mathcal{G}_\psi \setminus \{0\}} |\phi(z)|^2 \frac{|c_z|^2}{d^2} \right) = \frac{1}{d} \left( 1 + (d - 1) \max_{z \neq 0} |\phi(z)|^2 \right).
$$

QED (2.12)

A subset $\mathcal{F} \subseteq Z = \mathbb{Z}_d \oplus \mathbb{Z}_d$ will be called degenerate if the symplectic form vanishes on $\mathcal{F}$:

$$
\langle z', Jz \rangle = 0, \quad z, z' \in \mathcal{F}.
$$

A subgroup of $Z$ generated by $\mathcal{F}$ is again a degenerate subset. Let $\mathcal{F}$ be degenerate, then the operators $W_z; z \in \mathcal{F}$, all commute by (1.4) and hence have common eigenvector(s). We conclude that $\mathcal{F} \subseteq \mathcal{G}_\psi$ for some $\psi$, hence $|\mathcal{F}| \leq d$, and if the equality holds, then $\mathcal{F}$ is a (maximal degenerate) subgroup of $Z$.

**Examples**

1) Consider the cyclic subgroup generated by an element $z \in Z$

$$
\mathcal{G}(z) := \{kz : k = 0, 1, \ldots, d - 1\}.
$$

This subgroup is degenerate and $|\mathcal{G}(z)| = d$ in the case where $z = (\alpha, \beta)$ and $\alpha, \beta, d$ have no common nontrivial divisor, in particular if $d$ is prime.

2) Assume $d = p_1p_2$, where $p_1, p_2$ are primes, then the subgroup generated by two elements $(p_1, 0)$ and $(0, p_2)$ is a maximal degenerate noncyclic subgroup.

**Corollary.** If there is a maximal degenerate subgroup $\mathcal{G} \subseteq \mathcal{E}_{\text{max}} \cup \{0\}$, then (2.7) holds.

**Examples**

1) As noticed in [7], the condition of Theorem 3 always holds if $d = 3$ and $\Phi$ is a channel. By using the fact that $2z_0 = -z_0$ in case $d = 3$, our Theorem 2 implies the multiplicativity of 2-norm in case $|\phi(z)| = |\phi(-z)|$, e.g. the map $\Phi$ is hermitian.

2) Any unital qubit ($d = 2$) channel is unitarily equivalent to the form

$$
\Phi(\rho) = \sum_{\gamma} p_\gamma \sigma_\gamma \rho \sigma_\gamma,
$$

where $\gamma = 0, x, y, z$ and $\sigma_\gamma$ are the Pauli matrices (see e.g. [9]). But in the case $d = 2$ the discrete Weyl operators are

$$
W_{00} = I = \sigma_0, \quad W_{01} = V = \sigma_z, \quad W_{10} = U = \sigma_x, \quad W_{11} = UV = -i\sigma_y.
$$
Thus any unital qubit channel is covariant with respect to the projective representation of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by these discrete Weyl operators.

For any $z \neq 0$ the cyclic group $\mathcal{G}(z)$ consists of two elements $\{0, z\}$. Hence the assumption of the corollary is always satisfied for the unital qubit channels. More generally, it holds for arbitrary qubit map $\Phi \in \mathfrak{W}_1(\mathcal{H})$.

3) The $d$-depolarizing channel

$$
\Phi(\rho) = \lambda \rho + (1 - \lambda) \text{Tr}\rho \frac{1}{d} I
$$

is unitarily covariant, hence Weyl-covariant. For this channel $\phi(z) = \lambda$ for $z \neq 0$, hence $\mathcal{E}_{\text{max}} = Z \setminus \{0\}$ and the assumption of the corollary is trivially satisfied. Moreover, the conclusion holds for the map $\Phi$ with arbitrary $\lambda \in \mathbb{C}, |\lambda| \leq 1$.

4) Let $\mathcal{G}$ be a subgroup of order $d$ and define a function on $Z$:

$$
\phi(z) = \begin{cases} 
1, & z = 0 \\
 a + b, & z \in \mathcal{G} \setminus \{0\} \\
b, & z \notin \mathcal{G}
\end{cases}
$$

where $a, b$ are complex numbers to be restricted later. By the Fourier transform,

$$
p_\gamma = \frac{1}{d^2} \sum_z \phi(z) \exp(i\langle \gamma, z \rangle)
$$

for a subgroup $\Gamma \subseteq \mathbb{Z}$. To see this, for each $\gamma \notin \Gamma^\perp$ choose $\bar{z} \in Z$ such that $\langle \gamma, \bar{z} \rangle \neq 0$. Factor $\Gamma$ by $\mathcal{G}(\bar{z})$ of the order, say, $N$. Then the sum over each coset is 0:

$$
\sum_{k=0}^{N-1} \exp(i\langle \gamma, z + k\bar{z} \rangle) = \exp(i\langle \gamma, z \rangle) \sum_{k=0}^{N-1} \exp(i\langle \gamma, \bar{z} \rangle k) = 0.
$$

A direct calculation shows that the Weyl-covariant map $\Phi$ defined by the function (2.13) can be written as

$$
\Phi(\rho) = a\Psi(\rho) + b\rho + (1 - a - b)\text{Tr}\rho \frac{I}{d}.
$$
where

\[ \Psi(\rho) = \frac{1}{d} \sum_{z \in G} W_{Jz} \rho W_{Jz}^*. \] (2.18)

If the group \( G \) is maximal degenerate, then \( G^\perp = JG \). To see this, take \( J(x', y') = (y', -x') \in G^\perp \). This implies \( \langle y', x \rangle + \langle -x', y \rangle = 0 \) for all \( (x, y) \in G \). Since \( G \) is maximal degenerate we have \( G^\perp \subseteq JG \). The inverse inclusion is obvious. Thus (2.18) takes the form

\[ \Psi(\rho) = \frac{1}{d} \sum_{z \in G} W_z \rho W_z^*. \] (2.19)

Assuming \( |b| \leq 1, |a + b| \leq 1 \) gives the condition (2.6). Moreover, if \( a, b \) satisfy the condition \( |a + b| \geq |b| \), the map (2.17) has the property in the corollary, giving another case for which the multiplicativity of 2-norm holds.

If \( G \) is maximal cyclic then \( \Psi \) is a “completely dephasing channel”:

\[ \Psi(\rho) = \sum_{j=1}^{d} |h_j\rangle \langle h_j| \rho |h_j\rangle \langle h_j|, \] (2.20)

where \( \{h_j\} \) is the orthonormal basis of the commuting operators \( \{W_z; z \in G\} \) as we shall show in a moment. The sum is an expectation onto Abelian subalgebra of operators diagonal in basis \( \{|h_k\}\). In this case the condition \( |a + b| \geq |b| \) becomes redundant. In fact defining the Weyl operators relative to the new basis \( \{|h_k\}\), we have the relation (2.19), where \( G = \{k(0, 1) : k = 0, \ldots, d - 1\} \). Then \( G^\perp = \{l(1, 0) : l = 0, \ldots, d - 1\} \), and

\[ |a + b| \geq |b| \Rightarrow G \setminus \{0\} \subseteq \mathcal{E}_{\text{max}} \]

\[ |a + b| \leq |b| \Rightarrow G^\perp \setminus \{0\} \subseteq \mathcal{E}_{\text{max}} \]

so that the condition of the corollary is always fulfilled.

Let us show that (2.19) is the same as the completely dephasing channel (2.20). Let \( G = \{kz_0 : k = 0, 1, \ldots, d - 1\} \), then we have

\[ \Psi(\rho) = \frac{1}{d} \sum_{k=0}^{d-1} W_{kz_0} \rho W_{kz_0}^* = \frac{1}{d} \sum_{k=0}^{d-1} (W_{z_0})^k \rho (W_{z_0}^*)^k. \] (2.21)

Let

\[ W_{z_0} |h_j\rangle = c_j |h_j\rangle; \quad c_j = \exp \left( \frac{2\pi i}{d} \alpha_j \right), \] (2.22)

where all \( \alpha_j \) must be different mod \( d \), for otherwise two different pure states emerging from the corresponding eigenvectors would have the same representations (2.11). Hence we can assume that \( \alpha_j = j + \alpha_0; j = 0, 1, \ldots, d - 1 \). Therefore
we have
\[
\Psi(|h_m\rangle\langle h_n|) = \frac{1}{d} \sum_{k=0}^{d-1} (W_{z_0})^k |h_m\rangle\langle h_n|(W_{z_0}^*)^k
\]
\[
= \frac{1}{d} \sum_{k=0}^{d-1} \exp \left( \frac{2\pi i}{d} (m - n)k \right) |h_m\rangle\langle h_n|
\]
\[
= \begin{cases} |h_m\rangle\langle h_m| & m = n \\ 0 & m \neq n \end{cases}.
\]
(2.23)

Finally consider the case where (2.17) is channel. If the point \((a,b) \in \mathbb{R}^2\) is in the triangle, defined by the corners \((0,1), (-1/(d-1),0), (d/(d-1),-1/(d-1))\) (see Figure), the function \(p_\gamma\) is nonnegative for all \(\gamma\) and defines the Weyl-covariant channel (2.2). The condition \(|a+b| \geq |b|\) then amounts to \(a(a+2b) \geq 0\) (this corresponds to the shaded area on the Figure). In the case of the channel (2.17) with \(\Psi\) given by (2.20) it becomes redundant. The multiplicativity of 2-norm in this case follows also from a general result in [10]. To investigate this case further in terms of the additivity of the minimal output entropy and the multiplicativity for \(p \in [1, +\infty]\), see [2].

3 Complementary channels

The relation between a channel and its complementary [6] (conjugate [7]) was investigated in these papers to show that the multiplicativity of the original channel implies that of the complementary channel. Suppose the original channel is given by the Kraus representation

\[
\Phi(\rho) = \sum_{\alpha=1}^{d_C} W_\alpha \rho W_\alpha^*, \quad W_\alpha : \mathcal{H}_A \to \mathcal{H}_B
\]

and the complementary channel by

\[
\tilde{\Phi}(\rho) = \sum_{t=1}^{d_B} \tilde{W}_t \rho \tilde{W}_t^*, \quad \tilde{W}_t : \mathcal{H}_A \to \mathcal{H}_C.
\]

Here \(d_B = \dim \mathcal{H}_B\) and \(d_C = \dim \mathcal{H}_C\). Then [6]

\[
\langle \tilde{e}_\alpha | \tilde{W}_t \rangle = \langle e_t | W_\alpha \rangle,
\]

where \(\{e_t\}_t\) is an orthonormal basis in \(\mathcal{H}_B\) and \(\{\tilde{e}_\alpha\}_\alpha\) in \(\mathcal{H}_C\).
In this section we compute the complementary of a Weyl-covariant channel. This was also derived in [7] but we give somewhat more explicit form by using a different method. In this section we use, for convenience, different notations for the Weyl-covariant channel

\[ \Phi(\rho) = \sum_{x,y=1}^{d} \lambda_{x,y}^2 U^x V^y \rho (U^x V^y)^* . \]

Let \( e_t \) be a row vector with \( t \)-th entry 1 and others 0. Then

\[ U^x = \begin{pmatrix} e_{1-x} \\ \vdots \\ e_{t-x} \\ \vdots \\ e_{d-x} \end{pmatrix} ; \quad V^y = \text{diag} \left[ \exp \left( \frac{2\pi i}{d} y \right), \ldots, \exp \left( \frac{2\pi i}{d} ty \right), \ldots, 1 \right] . \]

Hence

\[ U^x V^y = \begin{pmatrix} \exp \left( \frac{2\pi i}{d} (1-x) y \right) e_{1-x} \\ \vdots \\ \exp \left( \frac{2\pi i}{d} (t-x) y \right) e_{t-x} \\ \vdots \\ \exp \left( \frac{2\pi i}{d} (d-x) y \right) e_{d-x} \end{pmatrix} . \]

Therefore reordering the Kraus operators we have

\[ \tilde{W}_t = \begin{pmatrix} \tilde{W}_t^{(1)} \\ \vdots \\ \tilde{W}_t^{(s)} \\ \vdots \\ \tilde{W}_t^{(d)} \end{pmatrix} . \]
Here

\[ \tilde{W}_t^{(s)} = \begin{pmatrix}
  \lambda_{d,1-s} \exp \left( \frac{2\pi i}{d} t (1-s) \right) e_t \\
  \vdots \\
  \lambda_{1,u,1-s} \exp \left( \frac{2\pi i}{d} (t-1+u)(1-s) \right) e_{t-1+u} \\
  \vdots \\
  \lambda_{1,1-s} \exp \left( \frac{2\pi i}{d} (t-1)(1-s) \right) e_{t-1}
\end{pmatrix} \]

\[ = \begin{pmatrix}
  \lambda_{d,1-s} e_t \\
  \vdots \\
  \lambda_{1,1-s} e_{t-1}
\end{pmatrix} \begin{pmatrix}
  \exp \left( \frac{2\pi i}{d} 1 \cdot (1-s) \right) & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & \exp \left( \frac{2\pi i}{d} d (1-s) \right)
\end{pmatrix}
\]

\[ = \begin{pmatrix}
  \lambda_{d,1-s} & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & \lambda_{1,1-s}
\end{pmatrix} U^{1-t} V^{1-s}
\]

\[ = D_s U^{1-t} V^{1-s}, \]

where

\[ D_s = \begin{pmatrix}
  \lambda_{d,1-s} & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & \lambda_{1,1-s}
\end{pmatrix}, \]

which is a diagonal matrix defined for each \( s \). Then we have

\[ \tilde{W}_t = \begin{pmatrix}
  D_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & D_d
\end{pmatrix} \begin{pmatrix}
  U^{1-t} V^d \\
  \vdots \\
  U^{1-t} V
\end{pmatrix}. \]

In general, a channel of the form

\[ \Phi(\rho) = \sum_{k=1}^{N} A_k \rho A_k^* \]

can be rewritten as

\[ \Phi(\rho) = (A_1, \ldots, A_N) (I \otimes \rho) \begin{pmatrix}
  A_1^* \\
  \vdots \\
  A_N^*
\end{pmatrix}. \]

Therefore the complementary channel of the Weyl covariant channel can be written as

\[ \tilde{\Phi}(\rho) = \tilde{W}(I \otimes \rho) \tilde{W}^*, \]

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where
\[
\tilde{W} = \begin{pmatrix}
D_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & D_d
\end{pmatrix}
\begin{pmatrix}
U^d V^d & \cdots & U^1 V^d \\
\vdots & \ddots & \vdots \\
U^d V^1 & \cdots & U^1 V^1
\end{pmatrix}.
\]

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A : (0, 1); \quad B : \left(\frac{d}{d-1}, -\frac{1}{d-1}\right); \quad E : \left(-\frac{1}{d-1}, 0\right);

AB : a + b = 1; \quad AE : a(d - 1) - b = -1;

BE : a(d - 1) + b(d^2 - 1) = -1; \quad CF : a + 2b = 0