TOEPLITZ MATRICES ACTING ON THE $\ell^2$-SPACE OF AN IMPRIMITIVITY BIMODULE

BEATRIZ ABADIE

Abstract. We give a definition of Toeplitz matrix acting on the $\ell^2$-space of an imprimitivity bimodule $X$ over a $C^*$-algebra $A$. We characterize the set of Toeplitz matrices as the closure in a certain topology of the image of the left regular representation of the crossed product $A \rtimes X$.

If $X$ is an imprimitivity bimodule over a $C^*$-algebra $A$, the right Hilbert $A$-module $\ell^2(X) = \bigoplus_{n \in \mathbb{Z}} X \otimes^n$ provides a natural generalization of the Hilbert space $\ell^2(\mathbb{Z})$. The main problem in generalizing the definition of Toeplitz matrix to this setting is that the $i,j$th entry of the matrix associated to an operator acting on $\ell^2(X)$ is an adjointable operator from $X \otimes^j$ to $X \otimes^i$, so one has to make sense of the condition $[T]_{ij} = [T]_{i-1,j-1}$ characterizing classical Toeplitz matrices. However, it seems natural to identify the creation operators $T^j_\eta : X \otimes^j \to X \otimes^i$ and $T^{j-1}_\eta : X \otimes^{j-1} \to X \otimes^{i-1}$, given by tensoring by $\eta \in X \otimes^{i-j}$. By making use of the notion of multiplier of an imprimitivity bimodule discussed by Echterhoff and Raeburn in [ER], we show that there is a unique $A$-bimodule isomorphism from the set of right adjointable maps $L_R(X \otimes^j, X \otimes^i)$ to $L_R(X \otimes^{j-1}, X \otimes^{i-1})$ that takes $T^j_\eta$ to $T^{j-1}_\eta$. This result enables us to define Toeplitz matrices acting on $\ell^2(X)$.

We then turn to the characterization of those operators on $\ell^2(X)$ that are associated to Toeplitz matrices. In order to do that, we consider the crossed product $A \rtimes X$ discussed in [AEE] and its left regular representation $\Lambda$ on $\ell^2(X)$, a canonical representation that agrees with the usual left regular representation for crossed products by an automorphism. Let $\sigma$ be the initial topology on $\mathcal{L}(\ell^2(X))$ induced by the family of seminorms $\mathcal{F} = \{ p_v : v \in X \otimes^j, j \in \mathbb{Z} \}$, where $p_v(T) = \| T(v\delta_j) \|$. Theorem 3.4 characterizes the set of Toeplitz matrices as the $\sigma$-closure of the image of the left regular representation on $\ell^2(X)$ of the crossed product $A \rtimes X$ defined in [AEE].

1. Preliminaries

In this section we briefly expose the background on imprimitivity bimodules and their multipliers. We refer the reader to, for instance, [RW], [La], or [Re] for further results and constructions.

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Let $A$ be a $C^*$-algebra. A right inner product $A$-module is a complex vector space $X$ that is a right $A$-module satisfying the condition
\[ \lambda(xa) = (\lambda x)a = x(\lambda a) \]
for all $\lambda \in \mathbb{C}, x \in X,$ and $a \in A,$
together with a pairing $\langle , \rangle_R : X \times X \to A$ such that

1. $\langle \lambda x + \mu y, z \rangle_R = \lambda \langle x, y \rangle_R + \mu \langle x, z \rangle_R,$
2. $\langle xa, y \rangle_R = \langle x, y \rangle_R a,$
3. $\langle y, x \rangle_R = \langle x, y \rangle_R^*,$
4. $\langle x, x \rangle_R \geq 0,$
5. $\langle x, x \rangle_R = 0$ only if $x = 0.$

A right Hilbert $A$-module consists of an inner product $A$-module $X$ that is complete in the norm
\[ \|x\| = \|\langle x, x \rangle_R\|^{1/2}. \]
A right Hilbert $A$-module $X$ is said to be full if the ideal
\[ \text{span}\{\langle x, y \rangle_R : x, y \in X\} \]
is dense in $A.$

Let $X$ and $Y$ be right Hilbert $A$-modules. A function $T : X \to Y$ is adjointable if there is a function $T^* : Y \to X$ such that
\[ \langle Tx, y \rangle_R = \langle x, T^* y \rangle_R \]
for all $x \in X, y \in Y.$

Adjointable maps turn out to be bounded linear $A$-module maps. Throughout this work, we will denote with $\mathcal{L}(X, Y)$ the set of right adjointable operators from $X$ to $Y.$

Left Hilbert $A$-modules are defined analogously, by considering left $A$-modules and by replacing conditions 1) and 2) above with

1' $\langle \lambda x + \mu y, z \rangle_L = \lambda \langle x, z \rangle_L + \mu \langle y, z \rangle_L,$
2' $\langle ax, y \rangle_L = a \langle x, y \rangle_L.$

If $X$ and $Y$ are left Hilbert $A$-modules, left adjointable maps from $X$ to $Y$ are defined analogously to the right case. The set of left adjointable operators from $X$ to $Y$ will be denoted by $\mathcal{L}_L(X, Y).$

Let $A$ and $B$ be $C^*$-algebras. An $A - B$ bimodule $X$ is an $A - B$ imprimitivity bimodule if it is a full left Hilbert $A$-module and a full right Hilbert $B$-module such that
\[ \langle x, y \rangle_L z = x \langle y, z \rangle_R, \]
for all $x, y, z \in X.$ There is no ambiguity regarding the norm in $X$ in that case, since (see, for instance, [BMS, Remark 1.9])
\[ \|\langle x, x \rangle_L\| = \|\langle x, x \rangle_R\| \]
for all $x \in X.$
Remark 1.1. As was shown in [BMS, Remark 1.9], if \( X \) is an \( A - B \) imprimitivity bimodule, then

\[
\langle xb, y \rangle_L = \langle x, yb^* \rangle_L \quad \text{and} \quad \langle ax, y \rangle_R = \langle x, a^*y \rangle_R.
\]

for all \( x, y \in X, a \in A, \) and \( b \in B \).

First notice that, if \( z \in X \),

\[
\langle x, yb^* \rangle_L z = x\langle yb^*, z \rangle_R
= x((z, yb^*)_R)^*
= x((z, y)_R b^*)^*
= xb(y, z)_R
= \langle xb, y \rangle_L z.
\]

Thus, if \( a = \langle x, yb^* \rangle_L - \langle xb, y \rangle_L \), then \( az = 0 \) for all \( z \in X \).

Finally,

\[
aa^* = a(\langle yb^*, x \rangle_L - \langle y, xb \rangle_L) = \langle ayb^*, x \rangle_L - \langle ay, xb \rangle_L = 0,
\]

which shows that \( a = 0 \). An analogous reasoning proves the second equation in (1).

The dual of an \( A - B \) imprimitivity bimodule \( X \) was defined in [Rf, 6.17] as the \( B - A \) imprimitivity bimodule \( \tilde{X} \) consisting of the conjugate vector space of \( X \) with the structure given by

\[
\tilde{b}\tilde{x} = \tilde{x}\tilde{b}^*, \bar{a}\tilde{x} = \tilde{a}^*\tilde{x}, \langle \tilde{x}, \tilde{y} \rangle_L = \langle x, y \rangle_R, \langle \tilde{x}, \tilde{y} \rangle_R = \langle x, y \rangle_L,
\]

where \( a \in A, b \in B \), and \( \tilde{x} \) denotes the element \( x \in X \) viewed as an element of the dual bimodule \( \tilde{X} \).

Given an \( A - B \) imprimitivity bimodule \( X \) and a \( B - C \) imprimitivity bimodule \( Y \), the tensor product \( X \otimes Y \) is the \( A - C \) imprimitivity bimodule obtained by completing the left inner product \( A \)-module and right inner product \( C \)-module consisting of the algebraic tensor product \( X \otimes_{B \text{ alg}} Y \) with inner products given on simple tensors by

\[
\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_L = \langle x_1, (y_1, y_2)_L, x_2 \rangle_L \quad \text{and} \quad \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_R = \langle y_1, (x_1, x_2)_R y_2 \rangle_R.
\]

We now recall the notions of multiplier algebra and multiplier bimodule. A multiplier \( m \) of a \( C^* \)-algebra \( A \) is a pair \( m = (L, R) \), where \( L \) and \( R \) are linear maps from \( A \) to itself such that

\[
L(ab) = L(a)b, \quad R(ab) = aR(b), \quad \text{and} \quad aL(b) = R(a)b,
\]

for all \( a, b \in A \). Every \( a \in A \) gives rise to a multiplier \( (L_a, R_a) \), where \( L_a \) and \( R_a \) are left and right multiplication by \( a \), respectively. The set \( M(A) \) of multipliers of \( A \) can be endowed with the structure of a \( C^* \)-algebra [Mu, 2.1.5] in which \( A \) sits as an ideal via the identification mentioned above. Besides (Mu 3.1.8), if \( B \) is a \( C^* \)-algebra containing \( A \) as an ideal, there is a unique homomorphism from \( B \) to \( M(A) \) that is the identity on \( A \).

In [ER, Definition 1.1], Echterhoff and Raeburn define the notion of the multiplier bimodule of an imprimitivity bimodule. A multiplier of an
A–B imprimitivity bimodule $Y$ consists of a pair $m = (m_A, m_B)$, where $m_A : A \rightarrow Y$ is $A$-linear, $m_B : B \rightarrow Y$ is $B$-linear, and

$$m_A(a)b = am_B(b),$$

for all $a \in A$, $b \in B$.

Like in the case of $C^*$-algebras, $y \in Y$ can be viewed as the multiplier $(y_A, y_B)$, where $y_A(a) = ay$ and $y_B(b) = yb$, for all $a \in A$ and $b \in B$. By means of this identification, the set $M(Y)$ of multipliers of $Y$ can be made into an $A$–$B$ bimodule, by setting

$$am = m_A(a) \text{ and } mb = m_B(b),$$

for all $a \in A$, $b \in B$, and $m = (m_A, m_B) \in M(Y)$.

Proposition 1.2 in [ER] characterizes $M(Y)$ up to isomorphism as the $A$–$B$ bimodule satisfying the following two properties:

1. $M(Y)$ contains a copy of the bimodule $Y$ such that $AM(Y) \subset Y$ and $M(Y)B \subset Y$.
2. If $M$ is an $A$–$B$ bimodule satisfying (1), then there is a unique $A$–$B$ bimodule homomorphism from $M$ to $M(Y)$ that is the identity on $Y$.

2. Adjointable maps as multipliers

Let $Y$ and $Z$ be an $A$–$B$ and a $B$–$C$ imprimitivity bimodule, respectively. For $y_0 \in Y$ and $z_0 \in Z$, we denote by $T_{y_0}^Z \in \mathcal{L}_R(Z, Y \otimes Z)$ and $R_{z_0}^Y \in \mathcal{L}_L(Y, Y \otimes Z)$ the creation operators defined by

$$T_{y_0}^Z(z) = y_0 \otimes z \text{ and } R_{z_0}^Y(y) = y \otimes z_0.$$  

It is well known, and easy to check, that the maps $y \mapsto T_{y_0}^Z$ and $z \mapsto R_{z_0}^Y$ are isometric, and that

$$(T_{y_0}^Z)^*(y \otimes z) = (y_0, y)_R z \text{ and } (R_{z_0}^Y)^*(y \otimes z) = y(z, z_0)_L,$$

for all $y, y_0 \in Y$ and $z, z_0 \in Z$.

We will also be making use of the equation

$$\tag{3} (R_{z_0}^Y)^*(\eta) \otimes w = \eta(z, w)_R,$$

for all $z, w \in Z$ and $\eta \in Y \otimes Z$.

By virtue of the continuity of both sides in Equation 3 it suffices to check it for $\eta$ in the algebraic tensor product $Y \otimes_{alg} Z$.

If $\eta = \sum_i y_i \otimes z_i$, then

$$\tag{3} (R_{z_0}^Y)^*(\eta) \otimes w = \sum_i y_i (z_i, z)_L \otimes w = \sum_i y_i \otimes (z_i, z)_L w =$$

$$= \sum_i y_i \otimes z_i (z, w)_R = \eta(z, w)_R.$$
Notation 2.1. Let $Y$ and $Z$ be an $A-C$ and a $B-C$ imprimitivity bimodule, respectively. Throughout this work, we will view $L_R(Z,Y)$ as an $A-B$ bimodule for the actions

$$(a \cdot \phi)(z) = a\phi(z) \quad \text{and} \quad (\phi \cdot b)(z) = \phi(bz),$$

for $a \in A$, $b \in B$, $z \in Z$, and $\phi \in L_R(Z,Y)$.

Proposition 1.3 in [ER] identifies the multiplier bimodule $M(Y)$ of an $A-B$ imprimitivity bimodule $Y$ with $L_R(B,Y)$, with the $A-B$ bimodule structure established in Notation 2.1 and the copy of $Y$ obtained via the map $y \mapsto T^B_y \in L_R(B,Y)$, where $T^B_y(b) = yb$ for $b \in B$ and $y \in Y$. The following theorem generalizes that result, which follows when one takes $Z = B$.

Theorem 2.2. Let $Y$ and $Z$ be an $A-B$ and a $B-C$ imprimitivity bimodule, respectively. Then the $A-B$ bimodule $L_R(Z,Y \otimes Z)$, provided with the copy of $Y$ given by $T^Y_y = \{T^Z_{y^*} : y \in Y\}$, is isomorphic to $M(Y)$.

Proof. Let $M$ denote the $A-B$ bimodule $L_R(Z,Y \otimes Z)$. First note that the map $y \mapsto T^Z_{y^*}$ is an $A-B$ bimodule homomorphism:

$$(T^Z_{y^*} \cdot b)(z) = T^Z_y(bz) = y \otimes bz = zb \otimes y = T^Z_{yb}(z),$$

and

$$(a \cdot T^Z_y)(z) = aT^Z_y(z) = ay \otimes z = T^Z_{ay}(z),$$

for all $a \in A$, $b \in B$, $y \in Y$, and $z \in Z$.

Therefore, we can identify the $A-B$ bimodule $Y$ with the closed $A-B$ sub-bimodule $T^Y_y$ of $M$.

We now show that $AM \subset T^Y_y$. Let $a = \langle u \otimes v, u' \otimes v' \rangle_L$, where $u, u' \in Y$ and $v, v' \in Z$.

Then

$$(a \cdot \phi)(z) = \langle u \otimes v, u' \otimes v' \rangle_L \phi(z)$$

$$= u \otimes v \langle u' \otimes v', \phi(z) \rangle_R$$

$$= u \otimes v \langle \phi^*(v' \otimes v), z \rangle_R$$

$$= u \otimes \langle v, \phi^*(u' \otimes v') \rangle_R z$$

$$= T^Z_{a}(z,\phi^*(u' \otimes v'))L(z),$$

for all $z \in Z$. Therefore, $a \cdot \phi \in T^Y_y$.

It follows that the set $A_0 := \{a \in A : a \cdot \phi \in T^Y_y\}$ is dense in $A$. Since $T^Y_y$ is closed in $M$ and the action of $A$ on $M$ is continuous, we conclude that $A_0 = A$.

We next show that $MB \subset T^Y_y$. Let $b = \langle u, v \rangle_L$, with $u, v \in Z$, and let $\phi \in M$. If $z \in Z$, then

$$(\phi \cdot b)(z) = \phi(\langle u, v \rangle_L z) = \phi(u \langle v, z \rangle_R) = \phi(u) \langle v, z \rangle_R.$$
If \( \phi(u) = \sum_{i=1}^{n} y_i \otimes z_i \), for \( y_i \in Y, z_i \in Z, i = 1, \cdots, n \), then, by the equation above,

\[
(\phi \cdot b)(z) = \sum_{i=1}^{n} y_i \otimes z_i \langle v, z \rangle_R = T^Z_{\sum_i y_i \langle z_i, v \rangle_L}(z) = T^Z_{\langle \sum_i y_i \otimes z_i \rangle}(z),
\]

where \( R^Y_y \) is as in Equation (2).

We now show that

\[
\phi \cdot b = T^Z_{\langle R^Y_y \rangle^* \langle \phi(u) \rangle}(\phi(u))
\]

if \( b = \langle u, v \rangle_L \), for \( u, v \in Z \). Let \( \eta_k \) be a sequence in the algebraic tensor product \( Y \otimes_{alg} Z \) converging to \( \phi(u) \). Then, as above,

\[
(\phi \cdot b)(z) = \phi(u) \langle v, z \rangle_R = \lim_{k \to \infty} \eta_k \langle v, z \rangle_R = \lim_{k \to \infty} T^Z_{\langle R^Y_y \rangle^* \langle \eta_k \rangle}(z) = T^Z_{\langle R^Y_y \rangle^* \langle \phi(u) \rangle}(z).
\]

We have thus shown that \( \phi \cdot \langle u, v \rangle_L \in T_Y \), for all \( u, v \in Z \). Now, a reasoning similar to that above shows that \( (\phi \cdot b) \in T_Y \) for all \( b \in B \).

The universal property of \( M(Y) \) and the identification of \( M(Y) \) with \( \mathcal{L}_R(B, Y) \) mentioned above yield now an \( A \rightarrow B \) bimodule homomorphism

\[ J : M \rightarrow M(Y) \]

such that \( J(T^Z_y) = T^B_y \) for all \( y \in Y \).

Let \( H : M(Y) \rightarrow M \) be defined by

\[ [H(\phi)](bz) = \phi(b) \otimes z, \]

for all \( \phi \in M(Y), b \in B, \) and \( z \in Z \). Notice that the definition above makes sense, since \( H(\phi) \) is the composition of \( \phi \otimes \text{id}_Z \) and the canonical isomorphism between \( Z \) and \( B \otimes Z \).

Besides, \( H \) is an \( A \rightarrow B \) bimodule homomorphism:

\[ [H(a \cdot \phi)](bz) = (a \cdot \phi)(b) \otimes z = a \phi(b) \otimes z = [a \cdot H(\phi)](z), \]

and

\[ [H(\phi \cdot c)](bz) = (\phi \cdot c)(b) \otimes z = \phi(cb) \otimes z = [H(\phi) \cdot c](bz), \]

for all \( \phi \in M(Y), a \in A, b, c \in B, \) and \( z \in Z \).

We now show that \( H = J^{-1} \). In fact, we have that

\[ [H(T^B_y)](bz) = T^B_y(b) \otimes z = yb \otimes z = y \otimes bz = T^Z_y(bz), \]

for all \( y \in Y, b \in B, \) and \( z \in Z \). That is, \( H(T^B_y) = T^Z_y \) for all \( y \in Y \).

It now follows that \( JH : M(Y) \rightarrow M(Y) \) is an \( A \rightarrow B \) bimodule homomorphism that is the identity on \( Y \). We conclude from the universal property of \( M(Y) \) that \( JH = \text{Id}_{M(Y)} \).
Finally, we prove that $HJ = \text{Id}_M$. First recall that, by Equation (5),

$$
\phi \cdot b = \sum_{i=1}^{n} T_{(R Y)}^{*}(\phi(u)),
$$

if $\phi \in M$ and $b = \langle u, v \rangle_l$, where $u, v \in Z$.

Therefore, if $c \in B$, then

$$(6) \quad (J \phi)(bc) = ([J \phi] \cdot b)(c) = [J(\phi \cdot b)](c) = \sum_{i=1}^{n} T_{(R Y)}^{*}(\phi(u))(c).$$

Then, by Equation (3), for $c \in B$ and $z \in Z$,

$$(HJ \phi)(bcz) = (J \phi)(bcz) \otimes z = T_{(R Y)}^{*}(\phi(u))(c) \otimes z = \phi(u)(v, cz)_R = \phi(\langle u, v \rangle_l cz) = \phi(bc z).$$

for all $c \in B$ and $z \in Z$.

A standard continuity argument completes now the proof.

□

3. Toeplitz matrices

Let $X$ be an imprimitivity bimodule over a $C^*$-algebra $A$. In this section we make use of Theorem 2.2 in order to define Toeplitz matrices acting on $\ell^2(X)$. We then describe Toeplitz matrices in terms of the left regular representation of the crossed product $A \rtimes X$ discussed in [AEE].

As usual, if $k < 0$, $X^{\otimes k}$ denotes the Hilbert $C^*$-bimodule $X^{\otimes -k}$, $\tilde{X}$ being the dual bimodule defined in [Rf]. If $\eta \in X^{\otimes k}$, we denote by $T_\eta^n$ the operator $T_\eta^{X^{\otimes n}} \in \mathcal{L}_R(X^{\otimes n}, X^{\otimes n+k})$, where we make the usual identifications of $a \otimes x$ with $ax$, $x \otimes a$ with $xa$, $\tilde{x} \otimes y$ with $\langle x, y \rangle_R$, and $x \otimes \tilde{y}$ with $\langle x, y \rangle_l$.

By Theorem 2.2 there is a unique $A - A$ bimodule isomorphism

$$
\alpha^{n,m} : \mathcal{L}_R(X^{\otimes n}, X^{\otimes m}) \to \mathcal{L}_R(X^{\otimes n-1}, X^{\otimes m-1})
$$

such that $\alpha^{n,m}(T_\eta^n) = T_\eta^{n-1}$, for all $n, m \in \mathbb{Z}$ and $\eta \in X^{\otimes m-n}$.

We denote by $\ell^2(X)$ the right Hilbert $C^*$-module over $A$ given by

$$
\ell^2(X) = \bigoplus_{k=-\infty}^{+\infty} X^{\otimes k},
$$

and by $\mathcal{L}(\ell^2(X))$ the space of right adjointable operators on $\ell^2(X)$.

An operator $T \in \mathcal{L}(\ell^2(X))$ can be represented by a matrix $[T]$, where $[T]_{ij} \in \mathcal{L}_R(X^{\otimes j}, X^{\otimes i})$ is given by

$$
[T]_{ij} = \Pi_i T E_j,
$$

where $\Pi_i$ are the projections in $\ell^2(X)$. The product at $\Pi_i$ of operators is given by

$$
[T]_{ij} = \Pi_i T E_j,
$$

and the product of operators is given by

$$
[T]_{ij} \cdot [T']_{jk} = [T]_{ij} [T']_{jk} = \Pi_i T E_j \cdot \Pi_j T' E_k = \Pi_i T E_j T' E_k.
$$
for the usual maps $E_k : X^\otimes k \to \ell^2(X)$ and $\Pi_k : \ell^2(X) \to X^\otimes k$, defined by $E_k(u) = u \cdot k$ and $\Pi_k f = f(k)$, for all $k \in \mathbb{Z}$.

The automorphisms $\alpha^{n,m}$ defined above yield a natural definition of Toeplitz matrix.

**Definition 3.1.** Let $T \in \mathcal{L}(\ell^2(X))$. The matrix $[T]$ is said to be a Toeplitz matrix if $\alpha^{1,1}([T]_{ij}) = [T]_{i-1,j-1}$ for all $i, j \in \mathbb{Z}$.

**Example 3.2. Classical Toeplitz matrices**

Let $X = \mathbb{C}$ be the $\mathbb{C} - \mathbb{C}$ imprimitivity bimodule obtained by letting $\mathbb{C}$ act on itself with left and right multiplication, with inner products
\[
\langle \lambda, \mu \rangle_L = \lambda \overline{\mu} \quad \text{and} \quad \langle \lambda, \mu \rangle_R = \overline{\lambda \mu}.
\]
Since conjugation identifies the $\mathbb{C} - \mathbb{C}$ imprimitivity bimodules $C$ and $\overline{C}$, $\ell^2(X)$ is the usual Hilbert space $\ell^2(\mathbb{Z})$.

Besides, $\mathcal{L}(X^\otimes n, X^\otimes m) \simeq L(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$ consists of left multiplication by complex numbers, and $\alpha^{n,m}$ is the identity for all $n, m \in \mathbb{Z}$. It follows that the matrix $[T]$ associated to an operator $T \in \mathcal{L}(\ell^2(X))$ is a Toeplitz matrix if and only if $[T]_{ij} = [T]_{i-1,j-1}$ for all $i, j \in \mathbb{Z}$. That is, if and only if $[T]$, viewed as a an operator acting on the Hilbert space $\ell^2(\mathbb{Z})$, is a Toeplitz matrix in the classical sense.

**Example 3.3.** Given an $A - A$ imprimitivity bimodule $X$, the crossed product $A \rtimes X$ was defined in [AEE Definition 2.4]. Theorem 2.9 in [AEE] shows that $A \rtimes X$ is the cross-sectional $C^*$-algebra of a Fell bundle with fibers $\{X^\otimes n : n \in \mathbb{Z}\}$. It follows from [FD VIII.16.12] that $A \rtimes X$ acts on $\ell^2(X)$ via the representation $\Lambda$ (the left regular representation, following the terminology of [FX 2.3]) given by
\[
[\Lambda_f(\eta)](l) = \sum_{k \in \mathbb{Z}} f(l - k) \otimes \eta(k),
\]
for all $\eta \in \ell^2(X)$, $l \in \mathbb{Z}$, and all compactly supported cross-sections $f \in A \rtimes X$.

Therefore,
\[
[\Lambda_f]_{ij} = T^j_{f(i-j)},
\]
and $[\Lambda_f]$ is a Toeplitz matrix.

**Theorem 3.4.** Let $\sigma$ be the initial topology on $\mathcal{L}(\ell^2(X))$ induced by the family of seminorms $\mathcal{F} = \{p_v : v \in X^\otimes j, j \in \mathbb{Z}\}$, where $p_v(T) = \|T(v\delta_j)\|$, and let $\Lambda$ the left regular representation defined in Example 3.3.

If $R \in B(\ell^2(X))$, then $[R]$ is a Toeplitz matrix if and only if $R \in \overline{\Lambda(A \rtimes Z)}'$.

**Proof.** Let $R \in \overline{\Lambda(A \rtimes Z)}'$. Since the set of compactly supported cross-sections is dense in the norm in $A \rtimes X$, we may assume that $R$ is the $\sigma$-limit of a net $\{\Lambda_{f_d}\}$, where $f_d$ is a compactly supported cross-section in $A \rtimes X$ for all $d$. We first assume that $[R]_{ij} = T^j_{f_d(i-j)}$ for all $i, j \in \mathbb{Z}$ and $\eta_{ij} \in X^\otimes i - j$. 

Then, if \( v \in X^{\otimes j} \),

\[
[R]_{ij}(v) = [R(v\delta_j)](i) = \lim_d [\Lambda f_d(v\delta_j)](i) = \lim_d f_d(i - j) \otimes v.
\]

Therefore,

\[
\eta_{ij} \otimes v = \lim_d f_d(i - j) \otimes v,
\]

for all \( i, j \in \mathbb{Z} \) and \( v \in X^{\otimes j} \).

Now, if \( v \in X^{\otimes j} \) and \( w \in X \), then

\[
[R]_{i+1,j+1}(v \otimes w) = \lim_d f_d(i - j) \otimes v \otimes w = \eta_{ij} \otimes v \otimes w = T_{\eta_{ij}}^{i+1}(v \otimes w).
\]

It follows that \( \alpha^{i+1,j+1}([R]_{i+1,j+1}) = [R]_{ij} \) for all \( i, j \in \mathbb{Z} \). Consequently, \( [R] \) is a Toeplitz matrix.

In the general case, since \( R \cdot a \) is as above for all \( a \in A \), then, for all \( i, j \in \mathbb{Z} \),

\[
\alpha^{i,j}([R]_{ij} \cdot a) = \alpha^{i,j}([R \cdot a]_{ij}) = [R \cdot a]_{i-1,j-1} = [R]_{i-1,j-1} \cdot a.
\]

Since the maps \( \alpha^{i,j} \) are \( A - A \) bimodule homomorphisms, this shows that

\[
\alpha^{i,j}([R]_{ij} \cdot a) = [R]_{i-1,j-1} \cdot a.
\]

The result now follows from the fact that, if \( \{e_\lambda\} \) is an approximate identity of \( A \), then \( S(v) = \lim_\lambda (S \cdot e_\lambda)(v) \) for all \( S \in \mathcal{L}(X^{\otimes i}, X^{\otimes i}) \), \( v \in X^{\otimes j} \), and \( i, j \in \mathbb{Z} \).

We now turn to the converse statement. Let \( [R] \) be a Toeplitz matrix. Assume first that \( [R] \) is such that

(7) \quad for all \( k \in \mathbb{Z} \) there exists \( u_k \in X^{\otimes k} \) such that \( [R]_{ij} = T_{u(i-j)}^j \).

Set \( u = \sum_k u_k \delta_k \). If \( \{e_\lambda\} \) is an approximate identity of \( A \), then, since \( u e_\lambda = R(e_\lambda \delta_0), u e_\lambda \in \ell^2(X) \) and \( \|ue_\lambda\| \leq \|R\| \) for all \( \lambda \). This implies that \( u \in \ell^2(X) \).

Now, given \( N \in \mathbb{N} \), let \( f_N \in A \rtimes \mathbb{Z} \) be defined by \( f_N = \sum_{|k| \leq N} u_k \delta_k \). We next show that \( \Lambda f_N \) converges to \( R \) in the topology \( \sigma \).

In fact, if \( v \in X^{\otimes j} \), then

\[
(R - \Lambda f_N)(v\delta_j) = \sum_{|k| > N} (u_k \otimes v) \delta_{k+j}.
\]
Therefore,

\[ \| (R - \Lambda f_N)(v\delta_j) \|^2 = \| \sum_{|k| > N} \langle u_k \otimes v, u_k \otimes v \rangle_R \| \]
\[ = \| \sum_{|k| > N} \langle v, \langle u_k, u_k \rangle_R \rangle_R \| \]
\[ = \| \sum_{|k| > N} \langle v, \langle u_k, u_k \rangle_R \rangle_R \| \]
\[ \leq \| \sum_{|k| > N} \langle u_k, u_k \rangle_R \| \| v \|^2 < \epsilon \]

from some \( N \) on.

For the general case, let \( \{ e_\lambda \} \) be an approximate identity of \( A \). For each \( \lambda \), \( [R \cdot e_\lambda] \) is a Toeplitz matrix satisfying (7). Thus, for each \( \lambda \), there is a sequence \( \{ f_{N,\lambda} \} \) of compactly supported functions in \( A \times X \) such that

\[ \lim_{N} \Lambda f_{N,\lambda}(v\delta_j) = R(e_\lambda v\delta_j) \]

for all \( j \in \mathbb{Z} \) and \( v \in X^{\otimes j} \).

Given \( \epsilon > 0 \) and \( v_i \in X^{\otimes j_i} \) for \( i = 1, \ldots, k \), let \( \lambda_0 \) be such that

\[ \| v_i - e_{\lambda_0} v_i \| < \epsilon/\| R \|| \]

for all \( i = 1, \ldots, k \).

Now choose \( N_0 \) so that

\[ \| \Lambda f_{N_0,\lambda_0}(v_i\delta_j) - R(e_{\lambda_0} v_i \delta_j) \| < \epsilon/2 \]

for \( i = 1, \ldots, k \).

Then

\[ \| (R - \Lambda f_{N_0,\lambda_0}) (v_i\delta_j) \| \leq \| R ((v_i - e_{\lambda_0} v_i) \delta_j) \| + \| R (e_{\lambda_0} v_i \delta_j) - \Lambda f_{N_0,\lambda_0} (v_i \delta_j) \| \]
\[ < \epsilon, \]

for all \( i = 1, \ldots, k \). \( \square \)

REFERENCES

[AEE] Abadie, B.; Eilers, S.; Exel, R. Morita equivalence for crossed products by Hilbert \( C^* \)-bimodules, Transactions of the American Mathematical Society, Vol. 350, No. 8, (1998), pp. 3043-3054.

[ER] Echterhoff, S.; Raeburn, I. Multipliers of imprimitivity bimodules and Morita equivalence of crossed products, Math. Scand., 76, (1995), pp. 289-309.

[BMS] Brown, L.; Mingo, J. and Shen, N. Quasi-multipliers and embeddings of Hilbert \( C^* \)-bimodules, Canadian Journal of Mathematics, 46 (6), pp. 1150-1174.

[Ex1] Exel, R. Amenability for Fell bundles, J. Reine Angew. Math. 492 (1997), 41-73.

[FD] Fell, J.M.G. and Doran, R.S. Representations of \( * \)-Algebras, Locally Compact Groups, and Banach \( * \)-Algebraic Bundles, Pure and Applied Mathematics, vols. 125-126, Academic Press, 1988.

[La] Lance, C. Hilbert \( C^* \)-modules. A toolkit for operator algebrasists, London Mathematical Society Lecture Notes Series, 210, 1995, Cambridge University Press.

[Mu] Murphy, G. \( C^* \)-Algebras and Operator Theory, 1990, Academic Press.

[RW] Raeburn, I.; Williams, D. Morita equivalence and continuous-trace \( C^* \)-algebras, Mathematics Surveys and Monographs, Vol. 60, 1998, American Mathematical Society.
TOEPLITZ MATRICES ACTING ON THE $\ell^2$-SPACE OF AN IMPRIMITIVITY BIMODULE

[Ref] Rieffel, M. *Induced representations of $C^*$-algebras*, Advances in Mathematics, *13*, 2, (1974), 176-257.

Centro de Matemática. Facultad de Ciencias. Iguá 4225, CP 11 400, Montevideo, Uruguay.

Email address: abadie@cmat.edu.uy