Yang-Mills theory
from non-critical string

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Abstract

The correspondence of the non-critical string theory and the Yang-Mills theory is examined according to the recent Polyakov’s proposal, and two possible solutions of the bulk equations are addressed near the fixed points of the pure Yang-Mills theory: (i) One solution asymptotically approaches to the AdS space at the ultraviolet limit where the conformally invariant field theory is realized. (ii) The second one approaches to the flat space in both the infrared and the ultraviolet limits. The area law of the Wilson-loop and the asymptotic freedom with logarithmic behaviour are seen in the respective limit.
1 Introduction

Inspired by the conjecture [1, 2, 3] that the conformal symmetric super Yang-Mills theory is dual to the type IIB supergravity in $AdS_5 \times S^5$, several quantities of the Yang-Mills theory have been calculated in terms of the string theory with D3-branes. There are two directions to extend this duality to the non-supersymmetric gauge theory. One way is to compactify one dimension to obtain a finite temperature gauge theory [4], and we get a non-supersymmetric theory. The running coupling constant is obtained by regarding the radius of the compactification as the cut-off. Although this formulation yields a successful result for the area law of the Wilson-loop, the difficulty appears in taking the continuum limit where the temperature is taken as infinite. Since the coupling constant in this formalism is proportional to the temperature, then the tension-parameter of the QCD string exceeds the Planck mass.

An alternative approach has been proposed by [5] in terms of the non-critical string theory which is constructed from the supersymmetric Liouville theory. In this theory, the space-time fermions are removed by taking the GSO projection as in the type 0 string theory [6, 7]. As a result, two kinds of R-R fields and also the tachyon field appear in the bulk space other than the usual bosons of the type II theory. While in the world volume of the D-brane, the tachyon can be removed by imposing the GSO projection on the open string sector. Then only the Yang-Mills field and some scalar fields exist in the D-brane world volume, and the mass scale of the gauge theory is given by the Liouville field.

This formulation however suffers from a possible instability due to the bulk tachyon when the dimension is higher than two. This could be avoided by considering an appropriate potential [8] which leads to the condensation of the tachyon or a curved space-time [10]. An interesting idea of the resolution for this problem has recently been given by considering the couplings of the R-R fields and the tachyon [11, 12]. Non-perturbative approaches would however be necessary to obtain a satisfactory resolution of this problem since we must know the correct functional form of the action depending on the tachyon to solve the problem of its vacuum.

While the idea of [5] was extended to type IIB [13, 14, 15, 16, 28, 18] and type 0B [11, 12, 19, 20, 21] models by regarding the fifth coordinate of $AdS_5$ as the energy-scale of the gauge theory. In type IIB model, asymptotic
freedom of the gauge coupling has been shown by adding the axion field to the effective supergravity action [13], but it does not show the logarithmic decreasing of the gauge-coupling. On the other hand, the asymptotic freedom with the logarithmic behavior was seen in the case of the type 0 model, but it seems to be difficult to connect ultraviolet solution and the favorite infrared solution [20].

As indicated in [5], the non-critical string theory also has a solution corresponding to the conformal invariant gauge theory. Around this fixed point, some approaches to solving the renormalization group equations have been given in the scheme of the non-critical string theory [22, 23, 24, 25], but more considerations would be needed since the asymptotic freedom and other important properties of the gauge theory are still obscure.

The purpose of this paper is to study furthermore the running coupling constant and the Wilson loop of the gauge theory in terms of the non-critical string theory, in which it is possible to study the pure Yang-Mills theory. Here we point out that there are two kinds of flows of the renormalization group equations in the pure Yang-Mills theory. They are specified by two types of asymptotic forms of the bulk solutions, i.e. the AdS space and the flat space. The latter type solution shows the asymptotic freedom with logarithmic decreasing for the coupling constant in the ultraviolet region and the area law of the Wilson-loop in the infrared limit. These behaviours are shown by imposing some conditions on the tachyon potential and the couplings of the tachyon to the R-R fields. But both kinds of solutions are characterized by the different expectation value of the tachyon since the conditions imposed on the potential on this point are not compatible.

In section two we give the gravitational equations to be solved and the conformal invariant interacting solution. The running behaviours from this fixed point are discussed in the section three, and we find that the above fixed point is the ultraviolet one smoothly connected to the solution given in type II theory. In section four, we show another solution with the asymptotic freedom near the ultraviolet fixed point. The gauge coupling decreases logarithmically with the energy scale, and this solution would be connected to the one given in the infrared limit. From this solution, we can see the area law of the Wilson-loop. In the final section, the concluding remarks are given.
2 The gravitational equations and conformal fixed point

The effective action of the non-critical string theory, whose boundary represents the gauge theory, should include R-R $p+1$ potential $A_{p+1}$ other than the usual NS-NS fields. And we expect that $N D_{p+1}$-branes are stacked on the boundary to make the U(N) gauge theory there.

Then we start from the following action,

$$S_D = \frac{1}{2\kappa^2} \int dx^D \sqrt{|g|} \left\{ e^{-2\Phi} \left( R - 4(\nabla \Phi)^2 + (\nabla T)^2 + V(T) + c \right) + \frac{1}{2(p + 2)!} f(T) F_{p+2}^2 \right\},$$

(1)

where $c = -(10-D)/2\alpha'$, and $F_{p+2} = dA_{p+1}$ is the field strength of $A_{p+1}$. The total dimension $D$ includes the Liouville direction, which is denoted by $r$. The tachyon potential is represented by $V(T)$, and $f(T)$ denotes the couplings between the tachyon and the R-R field investigated in [11]. We consider their simple forms in the next section. It should be noticed that there is a problem of the stability of the tachyon in extending the non-critical string theory to the higher dimension ($D > 2$). In order to evade this problem, we assume the condensation of the tachyon and/or the resultant curved space which includes the asymptotically AdS space. In the case of AdS space, the tachyon can be stabilized even if it has a small negative mass-squared [10]. These points would be cleared by using the exact forms of $V(T)$ and $f(T)$, but we do not know them. So the problem of the stability of the tachyon is open here.

The equations of motion are written as

$$R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \Phi = -\nabla_\mu T \nabla_\nu T + e^{2\Phi} f(T) T^A_{\mu\nu},$$

(2)

$$4 \nabla_\mu \Phi \nabla^\mu \Phi - 2 \nabla^2 \Phi = \frac{D - 2d - 2}{4(p + 2)!} e^{2\Phi} f(T) F_{p+2}^2 + V_c(T),$$

(3)

$$\nabla^2 T - 2\nabla_\mu \Phi \nabla^\mu T = \frac{1}{2} V_c''(T) + \frac{1}{4(p + 2)!} e^{2\Phi} f'(T) F_{p+2}^2$$

(4)

Although four kinds of the fundamental D-brane can be considered [11], we here consider only the electric type of R-R charge.
\[ \partial_\mu (\sqrt{|g|} f(T) F^{\mu \nu_1 \cdots \nu_{p+1}}) = 0, \]  
\[ (5) \]

where \( V_c(T) = c + V(T) \) and

\[ T^{\mu \nu}_A = -\frac{1}{2(p+1)!} \left( F^{\mu \nu_1 \cdots \nu_{p+1}} F_{\nu \nu_1 \cdots \nu_{p+1}} - \frac{g_{\mu \nu}}{2(p+2)} F^{\nu_1 \cdots \nu_{p+2}} F_{\nu_1 \cdots \nu_{p+2}} \right). \]  
\[ (6) \]

The advantage of the non-critical string scheme is to be able to avoid the extra dimensions of the bulk space. These freedoms are corresponding to the adjoint scalars in the gauge theory, and they are not necessary to see the characteristic features of the Yang-Mills theory. So the dimension \( D \) is set as \( D = p + 2 \) to consider the case of the pure Yang-Mills theory hereafter. But the extra dimensions seem to be essential to obtain an asymptotic free solution given in the type 0 model. We comment on this point in the proceeding sections.

We solve the above equations according to the following ansatz;

\[ ds^2 = e^{2A(r)} \eta_{\mu \nu} dx^\mu dx^\nu + e^{2B(r)} dr^2 \]  
\[ (7) \]

\[ \Phi \equiv \Phi(r), \quad T \equiv T(r) \quad \text{and} \quad A_{01 \cdots p} = -e^c(r), \]  
\[ (8) \]

where \( x^\mu, \mu = 0 \sim p \), denote the space-time coordinates, and \( r \) represents the Liouville direction. Here \( r \) plays the role of the energy scale for the \( p + 1 \)-dimensional gauge theory [5]. The equation (3) is solved as

\[ \partial_r e^{c(r)} = \frac{N}{f(T)} e^{dA+B} \]  
\[ (9) \]

where \( d = p + 1 \) and \( N \) denotes the number of the p-brane. Then the remaining equations (2) and (3) are rewritten as,

\[ -\ddot{A} \dot{B} + \dot{A} \ddot{A} + 2\dot{A} \dot{\Phi} = e^{2B+2\Phi} \frac{N^2}{4f(T)}, \]  
\[ (10) \]

\[ d(\ddot{A} + \dot{A}^2 - \dot{A} \dot{B}) - 2(\ddot{\Phi} - \dot{B} \dot{\Phi}) = e^{2B+2\Phi} \frac{N^2}{4f(T)} - \dot{T}^2, \]  
\[ (11) \]

\[ 2(\dddot{\Phi} + \Phi(\ddot{A} - \dot{B})) - 4\dot{\Phi}^2 = e^{2B+2\Phi} \frac{(d + 1)N^2}{4f(T)} + e^{2B} V_c(T) \]  
\[ (12) \]

\[ \ddot{T} + (d\dot{A} - \dot{B}) \dot{T} - 2\ddot{\Phi} \dot{\Phi} = -\frac{f'(T)}{2f^2(T)} N^2 e^{2B+2\Phi} + \frac{1}{2} e^{2B} V'_c(T) \]  
\[ (13) \]
where \( \dot{\cdot} \) denotes the derivative with respect to \( r \).

Before considering the problem of the renormalization group flow of the gauge theory, we give the anti-de Sitter solution as a conformal invariant fixed point since it is the starting point of our study. It can be found by solving the above equations by assuming that \( \Phi \) and \( T \) are constant, \( i.e. \) independent on \( r \), and taking the following ansatz,

\[
e^A = \left( \frac{r}{r_0} \right)^\gamma, \quad e^B = \left( \frac{r}{r_0} \right)^b, \tag{14}\]

where \( r_0 \) denotes a scale parameter which measures the scalar curvature (see (18)). After a small calculation, we get

\[
b = -1, \tag{15}\]

and

\[
V_c(T_0) + \frac{d + 1}{4f(T_0)} \lambda_0^2 = 0, \quad \gamma^2 = \frac{r_0^2}{4df(T_0)} \lambda_0^2, \tag{16}\]

\[
V_c'(T_0) - \frac{f'(T_0)}{f^2(T_0)} \lambda_0^2 = 0, \tag{17}\]

where \( \lambda_0 = Ne^{\Phi_0} \) is the t’Hooft coupling constant. Here we notice that the scale parameter \( r_0 \) is related to the scalar curvature \( R \) as

\[
R \equiv g^{\mu\nu} R_{\mu\nu} = (1 + d) d \frac{\gamma^2}{r_0^2}, \tag{18}\]

which is derived from Eqs. (14) and (15).

From Eqs. (14) \( \sim \) (17), we can determine \( T_0, \lambda_0 \) and \( \gamma \) if the explicit forms of \( V_c(T) \) and \( f(T) \) are given. As an example, we solve them by using the following simple forms

\[
V_c(T) = -\frac{9 - d}{2} - \frac{T^2}{2}, \quad f(T) = 1 + T + \frac{T^2}{2}, \tag{19}\]

where we take as \( \alpha' = 1 \), and the above forms are obtained from the expansion of the effective action near \( T = 0 \) [11]. Then the solutions are given as

\[
\lambda_0 = 1.17, \quad T_0 = -0.814, \quad \gamma = 0.753. \tag{20}\]
We can see the deviation of the value of $\gamma$ from one, which is obtained in the type IIB model.

Next, we briefly comment on the formulation based on the thermalization, where the Hawking temperature $T_H$ is introduced as a cut-off parameter of the gauge theory according to [4]. The equations are solved by the following ansatz with the thermal deformation $h(r)$,

$$ ds^2 = -e^{2A_0(r)}dx^0dx^0 + e^{2A(r)}\delta_{ij}dx^idx^j + e^{2B(r)}dr^2 $$

(21)

where $i, j = 1 \sim 3$ and

$$ e^{A_0} = h(r)\left(\frac{r}{r_0}\right)^{\gamma_0}, \quad e^A = \left(\frac{r}{r_0}\right)^{\gamma}, \quad e^B = h^{-1}(r)\left(\frac{r}{r_0}\right)^{b}.$$  \hspace{1cm} (22)

After a calculation, we find the solution eqs.(15), (16), (17) and

$$ h(r) = 1 - \left(\frac{r_1}{r}\right)^{d\gamma}, $$

(23)

where $r_1$ denotes a new scale parameter. And $T_H$ can be derived from the completeness of the metric as follows,

$$ T_H = \frac{d\gamma}{4\pi r_0^{\gamma+1} r_1^{\gamma}}. $$

(24)

In this construction, we can see the well-known thermodynamical properties of the Yang-Mills gas with the temperature $T_H$, but it is difficult to connect this solution to the asymptotic free running coupling constant, $\lambda_p$, of $p$-dimensional gauge theory. It is given as,

$$ \lambda_p = \lambda_0 T_H. $$

(25)

Since $\lambda_0$ is a constant and $T_H \to \infty$ in the ultra-violet limit, then $\lambda_p$ grows with $T_H$. So it seems necessary to consider other formulation of the conformal breaking. As discussed below, one direction is to solve the equations by considering the $r$ dependence of the fields $\Phi$ and $T$, which are regarded as the running coupling constants.
3 Asymptotic CFT solution

First, we give the solution whose ultraviolet limit is given by the AdS solution given in the previous section with constant $\Phi$ and $T$. Here the $r$-dependence of $\Phi$ and $T$ are turned on, and the equations are solved by adopting the ansatz $a(r)$ for the metric and the following form of the solution,

$$A(r) = \gamma \ln(r/r_0) + a(r), \quad B(r) = -\ln(r/r_0) + b(r),$$

$$\Phi(r) = \Phi_0 + \phi(r), \quad T(r) = T_0 + t(r),$$

where $\gamma$, $\Phi_0$ and $T_0$ are determined by the equations (15), (16) and (17). The deviations, $\{a(r), b(r), \phi(r), t(r)\}$, from the AdS limit are obtained by solving Eqs. (10) $\sim$ (13). Since it is difficult to find an exact solution, we solve the equations by expanding the deviations in power series of $(r_0/r)^\alpha$ as

$$\chi^i(r) = \sum_n \chi^i_n (r_0/r)^{n\alpha},$$

where the deviations defined in (27) are denoted by a vector $\chi^i(r) \equiv (a(r), b(r), \phi(r), t(r))$. For example, the first component represents $a(r) = \sum_n a_n (r_0/r)^{n\alpha}$. The coefficients $\chi^i_n (\ a_n \ etc.)$ and the index $\alpha$ are determined by solving the equations. Here we notice two points: (i) The reason why $\alpha = 1$ is not considered is that the index $\alpha$ corresponds to the lowest order coefficient of the $\beta$-function of the gauge coupling as seen in Eq. (35). In general this coefficient is determined by the dynamical property of the system. (ii) The expansion form (28) can be used for both the ultraviolet and infrared regions for the positive and negative value of $\alpha$ respectively.

The solution of the lowest order equations is equivalent to Eqs. (16) and (17). From the first order equations of $O(r^{-\alpha})$, the following condition is obtained if there is a non-trivial solution,

$$[\alpha(\alpha - d\gamma)]^2 + \alpha(\alpha - d\gamma)(\frac{1}{2}V(T_0)r_0^2 - f_2)$$

$$- \frac{1}{2}V(T_0)r_0^2f_2 + V'(T_0)r_0^2f_1 = 0,$$

where

$$f_1 = \frac{1}{4}V(T_0)r_0^2(V'(T_0)/V(T_0) + f'(T_0)/f(T_0)) .$$
Here the Eq. (29) comes from the Eqs. (12) and (13) as a condition that \( \phi_1 \) and \( t_1 \) are not zero. If we take \( \phi_1 = t_1 = 0 \), then \( a_1 = b_1 = 0 \) are obtained from Eqs. (10) and (11). And this is the trivial solution. Since \( T_0 \) and \( \gamma \) are determined from Eqs. (16) and (17) if \( V(T) \) and \( f(T) \) are given, then the value of \( \alpha \) is given by solving Eq. (29) for non-trivial solutions. We notice here that Eq. (29) generally has four solutions for \( \alpha \), but there is no principle to choose one of them at present. Here, we adopt the solution which is smoothly continued to the one of IIB model given at \( D = 10 \) and \( d = 4 \). In this case, the right hand side (r.h.s.) of Eq. (3) is zero, then the corresponding solution is obtained from Eq. (12) by equating its r.h.s. to be zero. From the condition that \( \phi_1 \neq 0 \), we obtain

\[
\alpha = d\gamma. \tag{32}
\]

In type IIB model [14], \( \gamma = 1 \) and \( \alpha = 4 \) has been obtained. In the non-critical case, we take the above form of Eq. (32) as the solution for \( \alpha \), but the value of \( \gamma \) is different from the one of the type IIB case. The value of \( \gamma \) is given by the solution of the zeroth order equations, which depends on the form of \( V_c(T) \) and \( f(T) \). Their simple form are given in the previous section, but we must here impose a condition to them such that we can find the solution (32). Substituting this solution into (24), the next condition is obtained

\[
\left( \frac{V_c'(T_0)}{V_c(T_0)} \right) \left( \frac{f'(T_0)}{f(T_0)} \right) + \frac{f'(T_0)}{f(T_0)} = \frac{V_c''(T_0)}{V_c'(T_0)} - \frac{f''(T_0)}{f'(T_0)} + 2 \frac{f'(T_0)}{f(T_0)}. \tag{33}
\]

It is easily seen that the simple form of \( V_c(T) \) and \( f(T) \) given in the previous section (Eq. (19)) give two real solutions \( \alpha = 3.43 \) and \( -0.416 \) with \( T_0 = -0.814 \), but \( d\gamma = 3.01 \) so these solutions do not satisfy the above condition. In order to get the solution \( \alpha = d\gamma \), we must find other form of (19) such that they satisfy Eq. (33). It is straightforward to determine the coefficients of the expansions by solving the equations if the explicit forms of \( V_c(T) \) and \( f(T) \) are given. But to find such functions is out of our present work, so it would be addressed in a separate paper.

We comment on the meaning of \( \alpha \) from the viewpoint of the Yang-Mills theory. The solutions of \( \alpha = d\gamma > 0 \) can be regarded as the non-critical
string version of the one given in the type IIB model [14]. As in [14], we can see the renormalization group equation of the gauge coupling constant, $\lambda = N e^\Phi = Ng^2_{YM}$. It is expanded as

$$\lambda = \lambda_0 \left( 1 + \phi \left( \frac{r_0}{r} \right)^\alpha + \cdots \right), \quad (34)$$

then we obtain from the definition of the $\beta$-function, $\beta(\lambda) \equiv r d\lambda/dr$, the following result:

$$\beta'(\lambda_0) = -\alpha. \quad (35)$$

From this, we can see that the AdS limit is the ultraviolet (infrared) fixed point for $\alpha > 0$ ($\alpha < 0$). Although the result is dependent on the form of $V_c(T)$ and $f(T)$, it is expected that the AdS is the ultraviolet fixed point if it is smoothly connected to the type IIB model.

Nextly we consider the quark-antiquark potential ($U_{Q\bar{Q}}$) with a distance $L$ between them. The potential can be obtained by estimating the Wilson-loop according to [29]. The minimized Nambu-Goto action, $S_m$, and the distance $L$ can be obtained by using the approximate solution with the first order corrections as

$$S_m = \frac{\tau}{2\pi} E^{1/2} \frac{r_0}{4\gamma} [B(-1/4, 1/2) + (a_1 + b_1)E^{-\alpha/2\gamma}B(-1/4 + \alpha/4\gamma, 1/2)], \quad (36)$$

$$L = E^{-1/2} \frac{r_0}{4\gamma} [B(3/4, 1/2) + (b_1 - 3a_1)E^{-\alpha/2\gamma}B(3/4 + \alpha/4\gamma, 1/2)]. \quad (37)$$

Here $B(a, b)$ denotes the beta-function and $E$ is an integral constant. And $\tau$ is the length in the time-direction of the Wilson loop. The above formula are obtained from the approximate solution of the order $O((r_0/r)^\alpha)$, so we write the resultant potential derived from them as the following approximate form,

$$U_{Q\bar{Q}} = \frac{1}{2\pi} c_3 \frac{r_0}{L} (1 + c_2(\frac{L}{c_1 r_0})^{\alpha/\gamma} + \cdots), \quad (38)$$

where

$$c_1 = \frac{1}{2\gamma} B(\frac{3}{4}, \frac{1}{2}), \quad c_3 = \frac{c_1}{4\gamma} B(-\frac{1}{4}, \frac{1}{2}), \quad (39)$$

$$c_2 = (b_1 - 3a_1) \frac{B(\frac{3}{4} + \frac{\alpha}{4\gamma}, \frac{1}{2})}{B(\frac{1}{2}, \frac{1}{2})} + (a_1 + b_1) \frac{B(-\frac{1}{4} + \frac{\alpha}{4\gamma}, \frac{1}{2})}{B(-\frac{1}{4}, \frac{1}{2})}. \quad (40)$$
For \( c_3 > 0 \), Coulomb attraction can be seen near the fixed point as in type IIB model. The next order correction in \( U_{QQ} \) would deviate from the result of [14] since \( \alpha/\gamma \neq 4 \) is expected.

The solutions obtained here can be regarded as fluctuations around the AdS space near its limit, and they are the renormalizable one. Then the solutions represent the renormalization group flow of the theory, which is existing on the boundary of AdS, with non-zero expectation value of some operator in the theory [20, 27, 28]. This point was discussed also for type II theory in the first paper of [13]. The CFT on the boundary is the \( N = 4 \) super Yang-Mills theory in the case of type IIB, but it would not be a supersymmetric theory in our case since our model is based on the type 0 theory in non-critical dimension. In any case, the solutions considered in this section would represent the running behaviour of the parameters in a different phase from that we are searching for. In the next section, we discuss the more desirable solution which shows the asymptotic freedom at ultraviolet region.

### 4 Asymptotic free solution

Here we investigate the solutions which leads to the asymptotic freedom of the gauge coupling constant, i.e. it decreases logarithmically with the energy scale. The equations in the ultraviolet region (large \( r \)) are solved in terms of the following expansions for \( \chi^i \) according to [20, 12],

\[
a(r) = a_0 \ln y + a_1 \frac{1}{y} (\ln y + a_{10}) + a_2 \frac{1}{y^2} (\ln^2 y + a_{21} \ln y + a_{20}) + \cdots, \tag{41}
\]

\[
b(r) = b_0 \ln y + b_1 \frac{1}{y} (\ln y + b_{10}) + b_2 \frac{1}{y^2} (\ln^2 y + b_{21} \ln y + b_{20}) + \cdots, \tag{42}
\]

\[
\phi(r) = \phi_0 \ln y + \phi_1 \frac{1}{y} \ln y + \phi_2 \frac{1}{y^2} \ln^2 y + \cdots, \tag{43}
\]

\[
t(r) = t_1 \frac{1}{y} + t_2 \frac{1}{y^2} (\ln y + t_{20}) + \cdots, \tag{44}
\]

where \( y = \ln(r/r_0) \), and we assume \( \phi_0 < 0 \), which is required from the asymptotic freedom.
While the equations (10) ∼ (13) are rewritten by using $\chi^i$ as follows:

$$\ddot{a} + \dot{a}(\dot{a} - \dot{b} - 2\dot{\phi}) + \gamma(2\dot{d} - \dot{b} - 2\dot{\phi}) = -d\gamma^2 + \frac{r_0^2 \lambda_0^2}{4f(T)} e^{2(b+\phi)}, \quad (45)$$

$$d[\ddot{a} + \gamma(2\dot{a} - \dot{b}) + \dot{a}(\dot{a} - \dot{b})] - 2(\ddot{\phi} - \dot{b}\dot{\phi}) + \dot{t}^2 = -d\gamma^2 + \frac{r_0^2 \lambda_0^2}{4f(T)} e^{2(b+\phi)}, \quad (46)$$

$$\ddot{\phi} + \dot{\phi}(\dot{d} - \dot{b} - 2\dot{\phi}) + d\gamma\dot{\phi} = (d+1)\frac{r_0^2 \lambda_0^2}{8f(T)} e^{2(b+\phi)} + \frac{r_0^2 V_c(T)}{2} e^{2b}, \quad (47)$$

$$\ddot{t} + \dot{t}(\dot{d} - \dot{b} - 2\dot{\phi}) + d\gamma\dot{t} = -\frac{r_0^2 \lambda_0^2 f'(T)}{2 f^2(T)} e^{2(b+\phi)} + \frac{r_0^2 V'_c(T)}{2} e^{2b}, \quad (48)$$

where the dot denotes the derivative with respect to $y$.

For this formulation, we firstly find

$$\gamma = 0. \quad (49)$$

This is seen as follows. From Eqs. (17) and (16), we must require the relation, $\bar{b}_0 = -\bar{\phi}_0 > 0$, to obtain the result $\gamma \neq 0$. If we use this relation in Eqs. (17) and (18), then we obtain $V_c(T_0) = V'_c(T_0) = 0$. Then $\lambda_0 = 0$ is derived from the same equations, but this result is not compatible with $\gamma \neq 0$ from Eqs. (15) and (16). Then we are led to the result (19). This result implies that the bulk space is asymptotically not the AdS but the flat space as seen from (18). This seems to be reasonable since the flux of the R-R field, which was the essential factor to derive the AdS space, is suppressed in the equations to be solved by the factor $e^{2b}$ near the ultraviolet limit due to the asymptotic freedom. And the flux term contributes to the higher orders of equations to reform the flat space to a curved one.

We notice however that there are two ways to get an asymptotic free solution which is compatible with the asymptotic AdS bulk-space. One way is to consider the extra dimension with the following metric,

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} dr^2 + e^{2K(r)} \hat{g}_{ab} dx^a dx^b, \quad (50)$$

where $\hat{g}_{ab}$ denotes (D-d-1)-dimensional sphere. In this case, the solution for the RR-field (9) is altered and the factor $e^{2(b+\phi)}$ of the R-R flux term in the equations is changed to $e^{2(b+\phi-|D-d-1|K)}$. Then the R-R term in the lowest order of equations can be retained even in the asymptotic-free case since the
energy dependence of $\phi$ can be cancelled by $b$ and $K$. This was explicitly seen in the Einstein frame of the type-0 model [20]. As a result, one can get an asymptotic AdS solution with the asymptotic freedom. However this is restricted to the case of $D = 10$ for $p = 3$ due to the consistency of the equations, which demands the condition, $D - 2p - 4 = 0$. Then this type of solutions are excluded for the pure Yang-Mills theory, in which $D = 5$ for $p = 3$, and also for $D < 10$ with the extra spaces $x^a$.

The second way is to consider a special form of $f(T)$, which could cancel the logarithmic energy dependence of $e^{2(b+\phi)}$. However this idea seems to be unreasonable since the tachyon should largely deviate from $T_0$ at large $y$ due to the factor $\ln y$ which is necessary to cancel the same factor in $b + \phi$. As a result, one expects that the bulk system would become unstable there.

Then it is the most probable case to consider the suppression of the RR term in the ultraviolet limit. This implies that the asymptotic bulk space in the non-critical string theory is not the AdS but the flat space which is dual to the asymptotic-free Yang-Mills theory. Then we firstly solve the equations by neglecting the R-R term in the lowest order. While the order of this R-R term is determined by solving the equations of lower order series where this term is still neglected.

On the other hand, there is no reason to suppress $V_c(T)$ in the lowest order of equations, and the situation depends on the value of $\bar{b}_0$. There are two possibilities, i.e. (i) $\bar{b}_0 = -1$ and (ii) $\bar{b}_0 < -1$. In the first case, $V_c(T)$ should be retained. And $V_c(T)$ is also neglected in the second case, but it is seen in this case that we can not determine the value of $\bar{b}_0$ and $\bar{\phi}_0$ by the lower order of equations at least up to $O(y^{-3})$. Then we cosider the former case, $\bar{b}_0 = -1$, hereafter.

In this case, the potential term $V_c(T)$ is included in the lowest order equations of (12) and (13). There are two possible solutions $\bar{\phi}_0 = -1/2$ and $\bar{\phi}_0 = -1$. Here we adopt the latter solution since it leads to the correct index of the logarithmic decreasing of the gauge coupling, i.e. $g^2_{YM} \propto 1/y$. This means that the R-R term appears in the equations of order $O(1/y^4)$ for the first time. Then the followings are obtained by solving the equations up to the order of $O(1/y^4)$:

\[
a(r) = \frac{\lambda^2_0}{4} \frac{1}{y^2} \ln y + \cdots, \quad b(r) = -\ln y - \frac{\lambda^2_0}{2} \frac{1}{y^2} (\ln^2 y + \ln y) + \cdots, \quad (51)
\]
\[ \phi(r) = -\ln y - \frac{\bar{\lambda}_0^2}{4} \frac{1}{y^2} \ln^2 y + \cdots, \quad t(r) = O\left(\frac{1}{y^3}\right), \quad (52) \]

and
\[ r_0^2 V_c(T_0) = -4, \quad V'_c(T_0) = 0, \quad (53) \]

where \( \bar{\lambda}_0^2 = \lambda_0^2 / f(T_0) \). The coefficients depending on \( \bar{\lambda}_0^2 \) are coming from equations of order \( O(1/y^4) \).

Other coefficients are given by solving the equations of higher orders, but we do not do it here. For the solution up to this order, we can make the following remarks: (i) The asymptotic bulk geometry is the flat space. This is seen by changing the Liouville coordinate \( r \) to \( \rho \equiv \ln \ln r \), and also from Eq. (18) for \( \gamma = 0 \). (ii) The Yang-Mills coupling constant, which is defined by \( g_{YM}^2 = e^\Phi \), decreases like \( g_{YM}^2 \propto (\ln r)^{-1} \) at large \( r \). And this index \(-1\) is the expected value from the perturbative calculation in the Yang-Mills theory. But there is no term like \( y^{-1} \ln y \) in \( \phi(r) \), so the two-loop order correction for \( g_{YM}^2 \) is lacking in this result. (iii) The value of the potential at \( T_0 \) is restricted through the conditions in (53). We can see that the latter condition \( V'_c(T_0) = 0 \) is not compatible with the condition required in the section 3 to obtain the AdS fixed point. Then the asymptotic free solution would represent the different phase of the gauge theory from the one given in the previous section.

Related to the second remark, we notice that the expected value of index is not obtained in the type 0B model. But this problem is resolved in 0B model by considering the effective coupling-constant obtained by evaluating the Wilson-Loop [20] near the ultraviolet region.

In our model, the leading part of the \( \bar{Q}Q \) potential can be estimated by using the lowest order of solution:
\[ A(r) = 0, \quad B(r) = -\ln(r/r_0) + b_0 \ln \ln(r/r_0). \quad (54) \]

According to the usual analysis, we obtain
\[ U_{\bar{Q}Q} = S_0 \sqrt{1 + \left(\frac{L}{L_0}\right)^2}, \quad (55) \]

where
\[ S_0 = \frac{\tau}{2\pi} L_0, \quad L_0 = r_0 \int_\epsilon^\infty dy y^{b_0}. \quad (56) \]
Here we introduced an infrared cutoff $\epsilon$ since the approximate formula (54) is useful for large $r$ (large $y$). Since we are considering the ultraviolet region, then (55) would be useful at small $L$ and it should be expanded as follows,

$$U_{Q\bar{Q}} \sim S_0 \left[ 1 + \frac{1}{2} \left( \frac{L}{L_0} \right)^2 \right].$$

(57)

This represents the harmonic type of potential, and the Coulomb potential does not appear. The reason why we can not see the $1/L$ term is that our solution is not the asymptotic AdS one, which leads to a conformal field theory as a fixed point and the Coulomb potential. While, a Coulomb potential with a logarithmic correction was found in the type 0B model. This is because of that the bulk solution is asymptotically AdS in this case. But this kind of behaviour can not be seen in the pure Yang-Mills theory because the bulk space is asymptotically flat. So we can not discuss the problem of the index of the logarithmic behaviour of the coupling constant as in the type 0B model.

Although Eq. (57) is valid at small $L$, this potential implies the quark confinement. In fact, the Eq. (55) shows the linear potential at large $L$. So we examine whether the similar potential can be obtained also in the infrared region.

We can proceed the analysis with the same formalism in the infrared region. First, the variable $y$ is changed to $y = -\ln r$ and the region of large $y$ (small $r$) is considered. Then, Eqs. (45) $\sim$ (48) used in the ultraviolet region are also useful in the infrared region if we change the sign of the single derivative terms like $\dot{a} \rightarrow -\dot{a}$. Then we can use the same expansion form of $\chi^i$, (41) $\sim$ (44), for the infrared region.

In solving the equations, we firstly consider the R-R term as in the ultraviolet limit. This term is proportional to $e^{2(b+\phi)}$, so we expect that it increases in the infrared region since $e^\Phi$ becomes large. Here we search for a solution which asymptotically approaches to the flat space also in the infrared limit, i.e. $\gamma = 0$. Because this solution could lead to a favourable $Q\bar{Q}$ potential. In order to obtain this type of solution, we suppose that $e^\Phi \rightarrow e^{\Phi_0}$ in the infrared limit and $b_0 < -1$. Namely, the R-R term is suppressed in the lowest order equations also in the infrared limit, and the gauge coupling approaches a large but finite-constant value. Due to these restrictions, we are led to the solution

$$a_0 = 0, \quad \phi_0 = 0,$$

(58)
by solving the lowest order equations of $O(1/y^2)$.

From the next order of equations, we obtain $b_0 = -2$. So the R-R term and the potential of $T$ appears in the equations of order $O(1/y^4)$ for the first time. Up to this order, we obtain

$$a(r) = O\left(\frac{1}{y}\right), \quad b(r) = -2 \ln y + O\left(\frac{\ln y}{y}\right), \quad \phi(r) = O\left(\frac{\ln y}{y^2}\right), \quad t(r) = -2 \frac{f'(T_0)}{f(T_0)} \frac{1}{y^2} + O\left(1/y^3\right),$$

(59)

and

$$\lambda_0^2 = 8 f(T_0).$$

(60)

The coefficients of the above terms written as $O(\ldots)$ can not be obtained by the parameters given in the equations up to this order, but they are taken to be non-zero values. The last result (61) is obtained by using Eq. (54) obtained in the ultraviolet region by assuming that the two solutions are connected smoothly. This implies that the coupling constant at the infrared fixed point is given by the coupling of the tachyon to the R-R fields.

For this solution, we obtain the same $Q\bar{Q}$ potential with the one obtained in the ultraviolet region, Eq. (55). It should be used for large $L$ in this case, and we can see the linear potential,

$$U_{Q\bar{Q}} \sim \frac{\tau}{2\pi} L.$$  

(62)

From this, the QCD string tension is obtained as $1/2\pi$ in the unit of $\alpha' = 1$. This is the expected behavior for the Yang-Mills theory in the infrared region. It is important to find a solution which connects the solutions given in the two limits, but it is open here.

5 Conclusions

By extending the idea of the AdS/CFT correspondence, we have examined the equations of the effective action of non-critical string theory as the renormalization group equations of the Yang-Mills theory. The analysis is restricted to the case of $D = 5$, where $D$ is the bulk dimension, to see the properties of the pure Yang-Mills theory. In this model, the unknown functions of the tachyon are included, so we need their explicit form if we want
the quantitative and definite results. However, we could get several qualitative features of the Yan-Mills theory from the asymptotic solutions of the non-critical string. Two kinds of ultraviolet fixed points have been found; (i) One is the asymptotic free fixed point with the logarithmic decreasing of the gauge coupling constant with respect to the energy scale. The index of the lowest term of the gauge coupling is consistent with the Yang-Mills theory, but the two loop correction term is lacking. Applying this scheme to the small energy scale limit, the solution near the infrared fixed point is also found. (ii) Another is the one of a conformal field theory limit corresponding to the AdS bulk space, which is related to the solution of type IIB model if we impose an appropriate condition on the potential \( V_c(T) \) and the coupling to the R-R field \( f(T) \) of the tachyon field \( T \) at \( T = T_0 \).

The second solution approaches to the AdS near the fixed point, and Coulombnic \( Q\bar{Q} \) potential is seen from the estimation of the Wilson-loop by using the asymptotic solution. This solution can be considered as the variation of the one given in the type IIB model. On the other hand, the asymptotic form of the first type solution is the flat space, and this leads to the harmonic oscillator type \( Q\bar{Q} \) potential in the ultraviolet limit. And the linear potential is obtained in the infrared region, where the gauge coupling approaches to a constant fixed point. Then it would be probable that there is an exact solution which smoothly connect these two limit solutions. It is an open problem to find such solution here.

The two ultraviolet fixed points of type (i) and (ii) are realized at different value of \( T_0 \), which is independent on the energy scale, so these points could not be connected by one solution of our equations presented here. New elements or new dynamical speculations would be needed to connect them. This would be an interesting problem of the pure Yang-Mills theory.

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