MEAN-FIELD SPIN GLASS MODELS FROM THE CAVITY–ROST PERSPECTIVE

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ABSTRACT. The Sherrington-Kirkpatrick spin glass model has been studied as a source of insight into the statistical mechanics of systems with highly diversified collections of competing low energy states. The goal of this summary is to present some of the ideas which have emerged in the mathematical study of its free energy. In particular, we highlight the perspective of the cavity dynamics, and the related variational principle. These are expressed in terms of Random Overlap Structures (ROSt), which are used to describe the possible states of the reservoir in the cavity step. The Parisi solution is presented as reflecting the ansatz that it suffices to restrict the variation to hierarchal structures which are discussed here in some detail. While the Parisi solution was proven to be correct, through recent works of F. Guerra and M. Talagrand, the reasons for the effectiveness of the Parisi ansatz still remain to be elucidated. We question whether this could be related to the quasi-stationarity of the special subclass of ROSts given by Ruelle’s hierarchal ‘random probability cascades’ (also known as GREM).

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The Sherrington-Kirkpatrick spin glass model has been studied as a source of insight into statistical mechanics of systems with highly diversified collections of patterns for the minimization of the free energy, or energy. The model is based on a Hamiltonian which incorporates interactions with high levels of frustration and disorder. The goal of this article is to present some of the ideas which have emerged in the study of the SK model, and in particular highlight an approach for the analysis of its free energy influenced by the cavity perspective.

The discussion is organized as follows.

In Section 1 we present the Sherrington-Kirkpatrick model [22], and comment on some of its basic features and puzzles. A more general version of the model is presented in Appendix C. Among the essential features exhibited by these models is the presence of rich diversity of low energy configurations. A proposal for a solution of the SK model was developed in a series of works, driven by the astounding insight of G. Parisi [17]. An essential feature of the proposed solution is the ansatz that at low temperatures the model’s Gibbs states exhibit a hierarchal structure. The Parisi approach was further clarified by Mezard, Parisi and Virasoro [15], and proceeding through Derrida’s REM and GREM calculations [8] which have in turn motivated Ruelle’s construction [20].

In Section 2 we present the cavity perspective and show that it naturally leads to the random overlap structure as the order parameter. An order parameter is a quantity which captures an essential feature of the system, whose determination provides key information on the system’s state. In the ferromagnetic Ising model the role is usually played by the magnetization. Parisi has presented his solution of the SK spin glass model as involving an order parameter which is a monotone function of the unit interval. That, however, presupposes ‘ultrametricity’ or the hierarchal structure discussed below. Without such an assumption, we argue that the natural order parameter is a ROSt.

Section 3 presents interpolation techniques. The considerable recent progress in the mathematical study of the SK model was stimulated, and indeed enabled, by the interpolation argument which was introduced in the work of F. Guerra and F. L. Toninelli [11]. The basic tools are presented here within the context of ROSt.

In Section 4 we discuss a variational formulation of the solution [2], which starts with an extension to general ROSts of the remarkable statement of F. Guerra [10] that Parisi’s ansatz provides a rigorous bound. The extended variational principle is shown to provide the correct answer, but is not computationally effective.

In part 5 we present a hierarchal “random probability cascade” (RPC) model, following closely a construction which was formulated by D. Ruelle. For ROSt within this class, the variational quantity can be presented as a functional defined over monotone functions (equivalently, probability measures) over the unit interval. The Parisi solution can be explained as based on the ansatz that it suffices to restrict the variation to ROSts in that class. The Parisi expression for the free energy was proved by M. Talagrand to be correct [25]. This was established through the criterion which is provided by Guerra’s interpolation bound. However, the notable result still does not fully address the challenge of explaining the reasons for the validity of the Parisi ansatz. In part 6, we comment on that, and on the question whether the reasons for the validity of Parisi’s ansatz could be related to the remarkable quasi-stationarity of the hierarchal RPC under the dynamical process which is naturally associated with the cavity picture [21].
1. THE SHERRINGTON-KIRKPATRICK SPIN GLASS MODEL: BASICS

1.1. Formulation of the model. Spin glass models were formulated in an attempt to provide analyzable and instructive examples of systems with intricate dynamics and equilibrium states of rich structure. A prime example is the Sherrington-Kirkpatrick model [22], whose configurations are described by $N$ spin variables $\{\sigma_i\}_{i=1,\ldots,N}$ taking values $\pm 1$, and interacting via the random Hamiltonian:

$$-H_N(\sigma; \omega, h) = \frac{1}{\sqrt{N}} \sum_{i \neq j} J_{ij}(\omega) \sigma_i \sigma_j + h \sum_i \sigma_i,$$  \hspace{1cm} (1.1)

where $J_{ij}(\omega)$ are iid gaussian random variables with normal distribution, and $h$ is a real parameter.

Since its introduction, the model has attracted considerable discussion and shown itself to contain various surprises. Even before one addresses the complex structure which the model aims to express, one encounters certain basic questions for which the answer is not immediate.

1.2. Comments on the ground state energy. The normalizing factor $N^{-1/2}$, included in eq. (1.1), ensures that the lowest energy

$$E_N(\omega, h) = \min_\sigma H_N(\sigma; \omega, h)$$

is typically of order $N$, even when $h = 0$. However, the fact that this is so requires an argument, since for any a-priori chosen configuration the typical order of magnitude of the energy is only $O(N^{1/2})$ (the distribution of the collection of energies is symmetric under reflection, due to the invariance of the distribution of $\{J\}$ under: $J \rightarrow - J$.) The resolution of this issue is easy. However, some of the next questions are not so simple:

[Q 1.] Is the random variable $E_N(\omega, h)/N$ sharply distributed, for high $N$, and does it converge in distribution to a constant as $N \rightarrow \infty$?

The answer to both questions is "yes" (see [24] and references therein) though for the second question it took considerable time for the answer to be established rigorously. The task was accomplished in a clever and simple argument of Guerra and Toninelli [11].

[Q 2.] Compute, or give an effective way to estimate, the distributional limit (which does exist),

$$E_o \overset{D}{=} \lim_{N \rightarrow \infty} E_N(\omega, h)/N.$$

[Q 3.] Produce an algorithm for determining the energy minimizing configuration, or at least find one for which the resulting energy per volume $H_N(\sigma; \omega)/N$ is close to $E_o$.

It turns out that in order to determine $E_o$, either theoretically or numerically, it is essential to consider the equilibrium states of the model at positive temperatures, which are, of course, of intrinsic interest. Thus, one is led to consider:

- the partition function,

$$Z_N(\beta; \omega, h) = \sum_\sigma e^{-\beta H_N(\sigma; \omega, h)},$$
• the quenched free energy, which is \((-\beta)\) times the following quantity

\[ Q_N(\beta, h) = \mathbb{E}(\log Z_N(\beta; \omega, h)) \]

where \( \mathbb{E}(\cdot) \) represents the average over the random ‘environment’ \( \omega \).

A derivative of the free energy yields the mean value of the energy density in the quenched state:

\[ E_N(\beta, h) := \frac{1}{N} \mathbb{E}(E_N(\beta; \omega, h)) = -\frac{\partial}{\partial \beta} Q_N(\beta, h)/N, \quad (1.2) \]

where \( E_N(\beta; \omega, h) \) is the Gibbs state average energy density

\[ E_N(\beta; \omega, h) = \sum_{\sigma} H_N(\sigma; \omega, h) \frac{e^{-\beta H_N(\sigma; \omega, h)}}{Z_N(\beta; \omega, h)}. \quad (1.3) \]

Standard convexity arguments imply that the mean is, at almost every inverse-temperature \( \beta \), also the typical value of the energy density. More can be said on the so called self averaging property of \( E_N(\beta; \omega, h) \) through the ‘concentration of measure’ principle which is nicely presented in the book of Talagrand \[24\].

1.3. Diversity. Before we turn to the more detailed discussion of the free energy, let us comment on the question concerning explicit algorithms for finding low energy configurations. Two natural algorithms, which are discussed in greater detail in \[1\] for \( h = 0 \), are:

i. The greedy algorithm: \( \sigma_i \) is determined successively, with respect to some order of the indices, by optimizing at each step the sign of the contribution of the new terms. For instance,

\[ \sigma_1 = +1, \quad \text{and for } i = 2, \ldots, N: \quad \sigma_i = -\text{sign} \left\{ \sum_{j=1, \ldots, i-1} J_{ij} \sigma_j \right\}. \quad (1.4) \]

ii. The eigenstate-shadowing algorithm:

\[ \sigma_i = \text{sign} \psi_i(\omega) \quad (1.5) \]

with \( \psi_i(\omega) \) one of the lowest eigenstates of the Hermitian matrix \( J_{ij}(\omega) \), which is sampled with the GOE distribution.

A point to be appreciated here is that while the typical spectrum of the corresponding quadratic form is known, through Wigner’s celebrated semi-circle law, the non-linear problem of determining the minimum restricted to the vertices of the hypercube in \( \mathbb{R}^N \) is, at present, much harder.

The greedy algorithm, which typically yields configurations with \( H_N(\sigma; \omega, 0)/N \approx -0.5319 \) can be easily improved upon, while an improvement over the second one, which typically yields \( H_N(\sigma; \omega, 0)/N \approx -0.6366 \), presents a harder challenge.

It may be noted that both algorithms allow for the construction of many very different configurations with comparable energy. This does not yet prove that such a diversity persists at the bottom of the spectrum, since neither yields the ground state energy per spin, but nevertheless the diversity seen here does offer a hint of the diversity characteristic of the model. Those observations lead one to the fascinating question:

[Q 4.] How much variety is there among the low energy configurations?

By flipping a few spins of the minimizing configuration, one can produce many configurations with energy in the range \( E_N(\omega, h) + O(1) \). However, the question is whether
one finds configurations with energies close to the ground state which are extensively dis-
tinct from each other. For this purpose, the distance between the configurations may be
expressed through their overlap, which is defined as

$$q_{\sigma,\sigma'} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma'_i, \quad (1.6)$$

with $\text{dist}(\sigma,\sigma') := 1 - q_{\sigma,\sigma'}^2$. According to the Parisi picture, at the bottom of the spec-
trum one should find a diverse collection of “competing” configurations, whose energies
and overlaps resemble a RPC process. The description is slightly complicated by the need
to lump the configurations into equivalence classes, according to their mutual overlaps.
Furthermore, the discussion of the ground state is (so far) accessible only after understand-
ing the structure of the positive temperature Gibbs equilibrium states.

### 2. The Cavity Perspective

#### 2.1. The incremental free energy.

It is convenient to present the pressure $P_N(\beta, h) := Q_N(\beta, h)/N$ as a sum of incre-
ments, which describe the effect of the gradual increase in the system’s size, starting from

$$P_N(\beta, h) := \frac{1}{N} \mathbb{E}(\ln Z_N) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left( \ln \frac{Z_{n+1}}{Z_n} \right). \quad (2.1)$$

The sequence $P_N(\beta, h)$ converges if and only if the sequence of increments is Cesaro-
convergent, in which case:

$$P(\beta, h) := \lim_{N \to \infty} P_N(\beta, h) = \langle c- \rangle \lim_{N \to \infty} \mathbb{E}\left( \ln \frac{Z_{N+1}}{Z_N} \right). \quad (2.2)$$

For an intuitive description of the incremental term let us describe the configuration of
a large reservoir of $N$ spins by the symbol $\alpha = (\sigma_1, \ldots, \sigma_N)$, and let the next spin be
denoted $\hat{\sigma} = \sigma_{N+1}$. Then

$$Z_{N+1} \frac{Z_N}{Z_{N+1}} = \frac{\sum_{\alpha, \hat{\sigma}} e^{-\beta H_{N+1}(\alpha, \hat{\sigma}; \omega)}}{\sum_{\alpha} e^{-\beta H_N(\alpha; \omega)}}. \quad (2.3)$$

We would like to cast the ratio as the effect of the addition of the single spin $\hat{\sigma}$ to a reservoir
whose state is described by $\alpha$, and is governed by the Hamiltonian $H_N(\alpha, \omega)$ . First,
however, one needs to deal with a minor inconvenience: as we go from size $N$ to $N+1$,
the interaction in $\alpha$ diminishes because of the change in the normalizing factor

$$\frac{1}{\sqrt{N}} J_{ij} \rightarrow \frac{1}{\sqrt{N+1}} \hat{J}_{ij}. \quad (2.4)$$

To address this, we rewrite the interaction $H_N(\alpha)$ in a form which will allow a natural
subtraction:

$$-H_N(\alpha) = \frac{1}{\sqrt{N+1}} \sum_{i<j} J_{ij} \alpha_i \alpha_j + \frac{h}{\sqrt{N}} \cdot \alpha + \frac{1}{\sqrt{N(N+1)}} \sum_{i<j} \tilde{J}_{ij} \alpha_i \alpha_j, \quad (2.5)$$

with $\tilde{J}_{ij}$ are independent normal Gaussians, and $h$ is the vector with all components equal
$h$. This is to be compared with

$$-H_{N+1}(\alpha, \hat{\sigma}) = \frac{1}{\sqrt{N+1}} \sum_{i<j} J_{ij} \alpha_i \alpha_j + \frac{\hat{h}}{\sqrt{N+1}} \cdot \alpha + \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N} J_{i,N+1} \alpha_i \hat{\sigma} + h \hat{\sigma}, \quad (2.6)$$

where $\hat{\sigma}$ is the new spin and $\hat{h}$ is the new field.
For brevity, let us denote the two terms which appear above as independent additions to $H_N(\alpha)$, as the Gaussian random variables:

$$
\kappa_\alpha = \frac{1}{\sqrt{N(N+1)}} \sum_{i<j} J_{ij} \alpha_i \alpha_j, \quad V_\alpha = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N} J_{i,N+1} \alpha_i .
$$

Thus, we get

$$
E(\ln \frac{Z_{N+1}}{Z_N}) = E(\ln \frac{\sum_{\alpha,\hat{\sigma}} \xi_\alpha e^{\beta(V_\alpha \hat{\sigma} + h \cdot \hat{\sigma})}}{\sum_{\alpha} \xi_\alpha e^{\beta \kappa_\alpha}}),
$$

where $\{\xi_\alpha\}$ are the weights.

Equation (2.8) expresses the incremental contribution to the free energy in terms of the mean free energy of a particle added to a reservoir whose internal state is described by $\alpha$, corrected by an inverse-fugacity term ($\kappa$). The latter may be thought of as the free energy of a 'place holder', or a vacancy, for the cavity into which the $(N+1)$st particle is added.

2.2. The cavity dynamics. One may note that the addition of a particle to the reservoir of $N$ particles has an effect on the state of the reservoir. For $N \gg 1$, the value of the added spin, $\hat{\sigma}$, does not affect significantly the field which would exist for the next increment in $N$, the direct contribution being only of the order $O(1/\sqrt{N})$. Hence, for the next addition of a particle we may continue to regard the state of the reservoir as given by just the configuration $\alpha$. However, the weight of the configuration (which is still to be normalized to yield its probability) undergoes the change:

$$
\xi_\alpha \mapsto \xi_\alpha \sum_{\hat{\sigma}=\pm 1} e^{\beta \left[ \frac{1}{\sqrt{N+1}} \sum_{i<j} J_{ij} \alpha_i \alpha_j + h \cdot \hat{\sigma} \right]}.
$$

Equation (2.8) expresses the incremental contribution to the free energy in terms of the mean free energy of a particle added to a reservoir whose internal state is described by $\alpha$, corrected by an inverse-fugacity term ($\kappa$). The latter may be thought of as the free energy of a 'place holder', or a vacancy, for the cavity into which the $(N+1)$st particle is added.

2.3. Random Overlap Structures. The state of the reservoir is relevant in so far as it correlates with the cavity field $V_\alpha$ and fugacity variable $\kappa_\alpha$. In order to keep track of just the relevant information, it is natural to introduce the following concept of a random overlap structure [2]. The definition is somewhat tentative, as we do not address here the possibility that the a continuum of states will be needed for the reservoir, in the limit $N \to \infty$. (One may envision an extension of the definition, but that will require addressing some technical issues.) Instead, we consider the case that states of the reservoir form just a countable collection, which we order by the weights. Even this simple concept allows to formulate variational bounds, and in fact even capture Parisi’s ansatz.

**Definition 2.1.** A random overlap structure (ROSt) is a probability space $(\Omega, \mu)$ over which there are defined: i. a monotone nondecreasing sequence $\{\xi_n(\omega)\}$, and ii. an $N \times N$ matrix $\{q_{n,n'}(\omega)\}$ such that for $\mu$ a.e. $\omega$

1. $\xi_n(\omega) \geq 0$ and $0 < \sum_\omega \xi(\omega) n < \infty$;
2. $q_{n,n'}(\omega)$ corresponds to a real, positive semidefinite form;
3. $q_{n,n}(\omega) = 1$ for all $n \in \mathbb{N}$ (which implies $|q_{n,n'}| \leq 1$).
Here, for clarity of the concept, we label the states of the reservoir not by \( \alpha \), as above, but by \( n \in \mathbb{N} \). However, as we shall see below, in the presence of the additional structure the somewhat vaguer notation will be convenient. We shall not change the symbol for the weights but rather just tacitly assume that the sequence \( \{ \xi_n \} \) is ordered, whereas \( \{ \xi_\alpha \} \) is just a collection of the weights attached to an index which may have some additional structure, as will be encountered below.

### 2.4. The incremental free energy functional.

In the above discussion, we presented the cavity dynamics as the process of adding a single spin. But one can also add directly \( M \) spins. To describe the effect of that, one may associate with each state of the ROSt new independent families of centered Gaussian random variables \( \{ \eta_i^{\alpha} \}_{i=1,\ldots,M} \) and \( \kappa^{\alpha} \) with the covariances

\[
\mathbb{E}(\eta_i^{\alpha} \eta_{i'}^{\alpha'}) = \delta_{i,j} q^{\alpha,\alpha'} \quad \text{and} \quad \mathbb{E}(\kappa^{\alpha} \kappa^{\alpha'}) = \frac{1}{2} q^{2}_{\alpha,\alpha'} .
\]

For the added \( M \)-spin configuration \( \{ \sigma_i \}_{i=1,\ldots,M} \) we define

\[
V^{\alpha,\sigma} = \sum_{i=1}^{M} \eta_i^{\alpha} \sigma_i .
\]

Motivated by the above consideration of the incremental free energy in case the ROSt is just the SK system of \( N \) particle, we define the more general ROSt functional:

\[
G_M(\mu) \overset{\text{def}}{=} \frac{1}{M} \mathbb{E} \left( \ln \left( \sum_{\alpha,\sigma} \xi_{\alpha} e^{\beta(V_{\alpha,\sigma} + h \cdot \sigma)} \right) \right) .
\]

To ensure that this functional is well defined, let us note:

#### Lemma 2.2.

For any configuration of the ROSt,

\[
\ln \left( \frac{\sum_{\alpha} \xi_{\alpha} e^{\beta \sqrt{M} \kappa^{\alpha}}}{\sum_{\alpha} \xi_{\alpha}} \right) \quad \text{and} \quad \ln \left( \frac{\sum_{\alpha,\sigma} \xi_{\alpha} e^{\beta(V_{\alpha,\sigma} + h \cdot \sigma)}}{\sum_{\alpha} \xi_{\alpha}} \right)
\]

are integrable with respect to the Gaussian measure averages over \( \kappa^{\alpha} \) and \( V_{\alpha,\sigma} \) (denoted below by \( \mathbb{E}_{\kappa,V} \)).

#### Proof.

To estimate the mean of the absolute value, it is convenient to use the identity,

\[
|X| = X + 2|X|_{+}, \text{ for } X \in \mathbb{R}.
\]

Applying the Jensen’s inequality to the average over \( \alpha \) we get

\[
- \ln \sum_{\alpha} \frac{\xi_{\alpha} e^{\beta \sqrt{M} \kappa^{\alpha}}}{\sum_{\alpha} \xi_{\alpha}} \leq \frac{\sum_{\alpha} \xi_{\alpha} \beta \sqrt{M} \kappa^{\alpha}}{\sum_{\alpha} \xi_{\alpha}} .
\]

With another application of the Jensen inequality, this time to the average over the Gaussian variables, \( \mathbb{E}_{\kappa}(\ln Q) \leq \ln \mathbb{E}_{\kappa}(Q) \), we get

\[
\mathbb{E}_{\kappa} \left( \ln \left( \frac{\sum_{\alpha} \xi_{\alpha} e^{\beta \sqrt{M} \kappa^{\alpha}}}{\sum_{\alpha} \xi_{\alpha}} \right) \right) \leq \ln \sum_{\alpha} \frac{\xi_{\alpha} \mathbb{E}(e^{\beta \sqrt{M} \kappa^{\alpha}})}{\sum_{\alpha} \xi_{\alpha}} + \frac{2 \sum_{\alpha} \xi_{\alpha} e^{\beta \sqrt{M} \kappa^{\alpha}}}{\sum_{\alpha} \xi_{\alpha}} \mathbb{E}_{\kappa}(\kappa^{\alpha}) \leq \frac{\beta^2 M}{4} + \beta \sqrt{2M} < \infty .
\]

Similar bounds apply to the second quantity in (2.14). \( \square \)

It should be clear from the above discussion and some elementary estimates, as the one given below, that in case the ROSt is just the system of \( N >> M \) particles with the Gibbs equilibrium state corresponding to the SK interaction \( (\mu_N) \),

\[
G_M(\mu_N) = \frac{1}{M} \mathbb{E}(\log[Z_{N+M}(\beta,h)/Z_N(\beta,h)]) + O(1/M).
\]
However, rather surprisingly, it turns out that quite generally the ROSt functional provides an upper bound:

**Theorem 2.3** (AS, a generalization of Guerra’s bound [10]). For any ROSt:

\[
\frac{1}{M} \mathbb{E}(\ln Z_M) \leq G_M(\mu) + o(1),
\]

(2.18)

where \( o(1) \) vanishes for \( M \to \infty \).

Furthermore, one gets the following expression for the difference:

\[
G_M(\mu) - \frac{1}{M} \mathbb{E}(\ln Z_M) = \beta^2 \int_0^1 dt \mathbb{E}_t^{(2)}((q_{\sigma,\sigma'} - q_{\alpha,\alpha'})^2).
\]

(2.19)

Here \( \mathbb{E}_t^{(2)}(\cdot) \) is a double replica average which is defined in Section 4, where this proposition is proved as part of Theorem 4.1.

**Remark:** 1. There is an interesting similarity, but also contrast on which we comment next, between Theorem 2.3 and the Gibbs variational principle. For an arbitrary Hamiltonian \( H(\sigma) \), and the initial probability measure \( \rho_0(d\sigma) \), any probability distribution on the spins, \( \mu(d\sigma) \), yields a variational lower bound for the logarithm of the partition function \( Z = \sum_{\sigma} e^{\beta H(\sigma)} \):

\[
\ln Z \geq S(\mu|\rho_0) - \beta \mu(H),
\]

(2.20)

where \( S(\mu|\rho_0) \) is the relative entropy of \( \mu \) with respect to \( \rho_0 \), and \( \mu(H) \) is the expectation value of \( H \) with respect to \( \mu \). The inequality is saturated (for a finite system) if and only if \( \mu \) is the Gibbs equilibrium state \( \rho(\sigma) = \frac{e^{-\beta H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}} \rho_0(\sigma) \).

2. It is thus curious that the ROSt variational principle yields upper bounds on the quenched free energy, whereas the usual Gibbs variational estimate yields lower bounds. We owe to Anton Bovier the interesting observation that this change may be related to one of the puzzles encountered in Parisi’s original argument. There, in the replica calculation the usual role of minima and maxima are reversed due to the change of sign in \( n(n-1) \) when \( n \to 0 \).

3. As would be explained below, restricting the variational bound to the hierarchal ROSt, RPC, one obtains the result of Guerra [10] that Parisi’s solution provides an upper bound on the pressure (lower bound on the free energy).

### 3. Interpolation Arguments

#### 3.1. A Gaussian Differentiation Formula

The derivation of the variational principle rests on the following differentiation formula.

**Lemma 3.1.** Let \( \Gamma \) be a finite index set and \( \{X_\gamma\}_{\gamma \in \Gamma} \) be a sequence of centered, gaussian random variables whose correlations depend on a parameter \( t \in (0,1) \):

\[
\mathbb{E}_t (X_{\gamma} X_{\gamma'}) = C_{\gamma,\gamma'}(t),
\]

(3.1)

with \( C_{\gamma,\gamma'}(t) \) differentiable in \( t \) and uniformly positive \( (C_{\gamma,\gamma'}(t) \geq \varepsilon \mathbb{I}) \) as a quadratic form.

Then, for any function \( \psi : \mathbb{R}^{\dim \Gamma} \to \mathbb{R} \) with continuous second partial derivatives that are polynomially bounded:

\[
\frac{d}{dt} \mathbb{E}_t (\psi((X_\gamma))) = \frac{d}{dt} \left( \sum_{\gamma,\gamma'} C_{\gamma,\gamma'}(t) \mathbb{E}_t \left( \frac{\partial^2 \psi}{\partial X_\gamma \partial X_{\gamma'}} \right) \right).
\]

(3.2)
For polynomial functions $\psi$ the differentiation formula can be obtained rather directly from Wick’s rule \[23\], or through the integration by parts formula for gaussian random variables. In appendix A we present a proof based on the Fourier transform representation. The statement can be further extended to functions whose second derivatives increase slower than any inverse gaussian.

In our applications, we will be differentiating functions of a specific form. For this reason, we state:

**Corollary 3.2.** Let $\{X_\gamma\}_{\gamma \in \Gamma}$ be a collection of Gaussian random variables as in Lemma 3.1, with

\[
\sup_t \sup_{\gamma, \gamma' \in \Gamma} \left| \frac{d}{dt} C_{\gamma, \gamma'}(t) \right| < \infty ,
\]

(3.3)

and $\{\xi_\gamma\}_{\gamma \in \Gamma}$ a summable sequence of positive numbers. Let

\[
\psi(\{X_\gamma\}) := \ln \left( \sum_{\gamma \in \Gamma} \xi_\gamma e^{-\beta X_\gamma} \right),
\]

(3.4)

with some $\beta > 0$. Then, for any $0 \leq t_1 < t_2 \leq 1$

\[
E_{t_2} (\psi (\{X_\gamma\})) - E_{t_1} (\psi (\{X_\gamma\})) = \beta^2 \frac{1}{2} \int_{t_1}^{t_2} \left[ E^{(1)}_{t} \left( \frac{d}{dt} C_{\gamma, \gamma'}(t) \right) - E^{(2)}_{t} \left( \frac{d}{dt} C_{\gamma, \gamma'}(t) \right) \right] dt ,
\]

(3.5)

where $E_t^{(n)}$ represent the “weighted replica averages”, which are defined, for bounded functions $f : \Gamma^n \to \mathbb{R}$, by

\[
E_t^{(n)} (f(\gamma_1, \ldots, \gamma_n)) := E \left( \sum_{\gamma_1, \ldots, \gamma_n} f(\gamma_1, \ldots, \gamma_n) \prod_{i=1}^n \zeta_t(\gamma_i) \right),
\]

(3.6)

with

\[
\zeta_t(\gamma) := \frac{\xi_\gamma e^{-\beta X_\gamma}}{\sum_{\gamma'} \xi_{\gamma'} e^{-\beta X_{\gamma'}}}.
\]

(3.7)

**Proof.** For $\Gamma$ a finite set, the statement is a direct application of (3.2). For infinite $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ let $\Gamma_n = \{\gamma_1, \ldots, \gamma_n\}$. Then, as just stated,

\[
\frac{d}{dt} E \left( \ln \sum_{\gamma \in \Gamma_n} \xi_\gamma e^{-\beta X_\gamma} \right) = \beta^2 \frac{1}{2} \left[ E^{(1)}_{n, t} \left( \frac{d}{dt} C_{\gamma, \gamma'}(t) \right) - E^{(2)}_{n, t} \left( \frac{d}{dt} C_{\gamma, \gamma'}(t) \right) \right] ,
\]

(3.8)

where the annealed multi-replica measures $E^{(1)}_{n, t}$ and $E^{(2)}_{n, t}$ are with respect to the random discrete measure generated by the finite sequence

\[
\zeta_{n, t}(\gamma) := \frac{\xi_\gamma e^{-\beta X_\gamma}}{\sum_{\gamma' \in \Gamma_n} \xi_{\gamma'} e^{-\beta X_{\gamma'}}} = \frac{\zeta_t(\gamma)}{\sum_{\gamma' \in \Gamma_n} \zeta_t(\gamma')},
\]

(3.9)

for $\gamma \in \Gamma_n$. Thus, the statement holds for the finite subsets $\Gamma_n$. As $n \to \infty$ the random measures determined by $\zeta_{n, t}$ converge to the random measure determined by $\zeta_t$, e.g., in the total variation norm. The claimed (3.5) then follows using the integrated version of (3.8), (3.3), and the bounded convergence theorem. \[\square\]
Remarks: 1. The subscript $t$ above indicates that these averages depend on the external parameter $t$ through the weights $\mathbf{A}$. 
2. The derivative of $\psi$ separates into two crucial terms. In many applications, the term involving the single replica average, $\mathbb{E}^{(1)}(\psi)$, will vanish because the variance of $X_\gamma$ (i.e., the diagonal term) will remain constant with respect to the interpolation parameter $t$. For such cases, one sees that if the off diagonal terms $C_{\gamma',\gamma}(t)$ only decrease with $t$, then the function $\mathbb{E}(\psi)$ increases in $t$. Stated differently: the average goes up when the variables $X_\gamma$ become less correlated.
3. Lemma 3.1 and Corollary 3.2 are related to Slepian’s inequality, c.f. [13].

While various interesting conclusions follow from monotonicity alone, it helps to go beyond that. Following is a useful bound.

**Corollary 3.3.** Suppose $\{X_\gamma\}$ and $\{Y_\gamma\}$ are two independent sequences of centered gaussian random variables. Suppose that $\psi$ is as in Corollary 3.2 Then

$$\left|\mathbb{E}(\psi(X_\gamma)) - \mathbb{E}(\psi(Y_\gamma))\right| \leq \beta^2 \max_{\gamma,\gamma'} \left|\mathbb{E}(X_\gamma X_{\gamma'}) - \mathbb{E}(Y_\gamma Y_{\gamma'})\right|$$  \hspace{1cm} (3.10)

**Proof.** Consider the Gaussian family $\{Z_\gamma\}$ with covariance

$$C_{\gamma,\gamma'}(t) = t \mathbb{E}(X_\gamma X_{\gamma'}) + (1 - t) \mathbb{E}(Y_\gamma Y_{\gamma'}) .$$  \hspace{1cm} (3.11)

By Corollary 3.2 one obtains a formula for the derivative of $\mathbb{E}(\psi(\{Z_\gamma\}))$, which can be bounded by the right-hand-side of (3.10) at every $t \in (0, 1)$.

We note that if the variances of $X_\gamma$ and $Y_\gamma$ are equal for each $\gamma$ then $\beta^2$ can be replaced by $\beta^2/2$ in (3.10).

### 3.2. GT interpolation and sub-additivity of the free energy.

The gaussian differentiation formula (3.2) permits a quick derivation of the fundamental result of Guerra and Toninelli [11] proving the existence of the free energy for the SK model.

In order to state their result it is useful to include extra diagonal terms in the Hamiltonian. These have a vanishingly small effect in the $N \to \infty$ limit, but allow for the simplest statement of the theorem.

$$-H_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j + h \cdot \sigma ,$$  \hspace{1cm} (3.12)

where the $J_{ij}$ are i.i.d. $N(0, 1)$ random variables. This changes the covariance matrix entries by an amount of order $O(1/N)$. Therefore, by Corollary 3.3 it does not affect $P(\beta, h)$. Henceforth, all $P_N(\beta, h)$, etc., are defined relative to this Hamiltonian. Alternatively, one can define a centered Gaussian process $\{K_N(\sigma), \sigma \in \{+1, -1\}^N\}$ with covariance

$$\mathbb{E}(K_N(\sigma)K_N(\sigma')) = \frac{N}{2} g_{\sigma,\sigma'}^2 ,$$  \hspace{1cm} (3.13)

and then $-H_N(\sigma) = K_N(\sigma) + h \cdot \sigma$.

The first application of the interpolation is the super-additivity of the quenched free energy

$$Q_N(\beta, h) := \mathbb{E}(\ln[Z_N(\beta, h)]) .$$  \hspace{1cm} (3.14)

**Theorem 3.4.** (Guerra-Toninelli [11]) For any $N, M \in \mathbb{N}$,

$$Q_N(\beta, h) + Q_M(\beta, h) \leq Q_{N+M}(\beta, h).$$  \hspace{1cm} (3.15)
for any \( \alpha, \alpha' \) and \( \sigma, \sigma' \):}

\[
q_{\gamma, \gamma'} = \frac{N}{N + M} q_{\alpha, \alpha'} + \frac{M}{N + M} q_{\sigma, \sigma'}.
\]

Convexity of the function \( f(q) = q^2 \), allows to conclude that

\[
\frac{d}{dt} C_{\gamma, \gamma'}(t) \leq 0,
\]

and therefore, \( \mathbb{E}(\psi(t)) \) is increasing by (3.18). This completes the proof. \( \square \)

Theorem 3.4 immediately implies the existence of the thermodynamic limit:

**Corollary 3.5.** i) For any \( \beta \) and \( h \),

\[
P(\beta, h) := \lim_{N \to \infty} P_N(\beta, h)
\]

exists. Furthermore, defining \( \mathcal{P}_N(\beta, h; \omega) = \frac{1}{N} \ln Z_N(\beta, h; \omega) \),

\[
\lim_{N \to \infty} \mathcal{P}_N(\beta, h; \omega) = P(\beta, h),
\]

where the limit is in distribution.

ii) The pressure may also be represented as

\[
P(\beta, h) = \lim_{M \to \infty} \lim_{N \to \infty} \frac{1}{M} \mathbb{E} \left( \ln \left[ \frac{Z_{N+M}(\beta, h)}{Z_N(\beta, h)} \right] \right),
\]

for any \( \beta \) and \( h \).
It should be noted here that prior to the GT argument it was known that the fluctuations of \( P_N(\beta, h; \omega) \) are of diminishing size as \( N \to \infty \), a fact which can be deduced by either martingale methods \[18\] or a concentration of measure argument \[24\]. The ‘monotonicity of the interpolation’ argument \[11\] adds the last missing step, which is the convergence of the sequence \( P_N(\beta, h) \).

**Proof of Corollary 3.5** The results claimed in \[3.24\] and \[3.26\] are simple consequences of \[3.15\]: namely if \( \{Q_N\}_{N \in \mathbb{N}} \) is a super-additive sequence, then the following limit exists and may be calculated incrementally

\[
\lim_{N \to \infty} \frac{Q_N}{N} = \lim_{M \to \infty} \liminf_{N \to \infty} \frac{Q_{N+M} - Q_N}{M},
\]

see Lemma B.1 below.

The convergence \[3.25\] follows from \[3.24\] since, as mentioned above, the range of the probability distribution of \( \frac{1}{N} \ln Z_N(\beta, h; \omega) \) narrows as \( N \to \infty \) - a fact proven in \[18, 24\]. □

**Remarks:** 1. While (i) recovers the Guerra and Toninelli result \[11\], (ii) is an observation which was useful in the proof of the variational principle \[2\].

2. The reader is cautioned that the super-additivity of the quenched pressure, and the particular direction for its monotonicity under the process of ‘amalgamation’ in which two blocks are interpolated into a single system, is not a thermodynamic principle akin to the Gibbs-phenomenon. For the Curie-Weiss model the inequality in \[3.15\] is reversed (as a simple calculation will show).

### 4. The ROSt variational principle

For convenience, let us remind ourselves that the functional representing the increase in the free energy due to the incorporation of \( M \) spins \( \sigma \in \{+1, -1\}^M \) into a ROSt \( \mu \) whose configurations are described by \( (\{\xi_\alpha\}_\alpha, \{q_{\alpha, \alpha'}\}) \) is

\[
G_M(\beta, h; \mu) = \frac{1}{M} \mathbb{E} \left( \ln \frac{\sum_{\alpha, \sigma} \xi_\alpha e^{\beta(V_{\alpha, \sigma} + 2)} \sqrt{M\kappa_{\alpha}}}{\sum_{\alpha} \xi_\alpha e^{\beta\sqrt{M\kappa_{\alpha}}}} \right).
\]

(4.1)

with \( \{\kappa_\alpha, V_{\alpha, \sigma}\} \) Gaussian random variables of covariance,

\[
\mathbb{E}(\kappa_\alpha \kappa_{\alpha'}) = \frac{1}{2} q_{\alpha, \alpha'},
\]

(4.2)

\[
\mathbb{E}(V_{\alpha, \sigma} V_{\alpha', \sigma'}) = M q_{\alpha, \alpha'} q_{\sigma, \sigma'}.
\]

(4.3)

**Theorem 4.1.** 1. For any \( N \) and any ROSt \( \mu \)

\[
\frac{1}{N} \mathbb{E}(\ln Z_N(\beta, h)) \leq G_N(\beta, h; \mu)
\]

(4.4)

with

\[
G_N(\beta, h; \mu) - \frac{1}{N} \mathbb{E}(\ln Z_N(\beta, h)) = \frac{\beta^2}{2} \int_0^1 dt \mathbb{E}_t^{(2)}((q_{\alpha, \alpha'} - q_{\sigma, \sigma'})^2).
\]

(4.5)

2. The pressure is given by

\[
P(\beta, h) = \lim_{M \to \infty} \inf_{\mu: \text{ROSt}} G_M(\beta, h; \mu),
\]

(4.6)

where the limit \( M \to \infty \) also equals the supremum over \( M \).
The replica expectation $\mathbb{E}_t^{(2)}$ is just as in Corollary 3.2 with respect to the interpolating Gaussian process with covariance

$$
\mathbb{E}(X_\gamma X_{\gamma'}) = C_{\gamma,\gamma'}(t) = \frac{N}{2}[(1-t)(q_{\alpha,\alpha'}^2 + q_{\sigma,\sigma'}^2) + 2t q_{\sigma,\sigma'} q_{\alpha,\alpha'}],
$$

(4.7)

where $\gamma = (\alpha, \sigma)$.

**Proof.** Part 1.: The argument is a slight modification of the interpolation scheme described in Theorem 3.4. Here we consider a system composed of a finite block of spins $\sigma$, whose interactions are determined by the SK model, and a reservoir of configurations $\alpha$, whose overlaps are governed by a ROSt $\mu$. Again, we interpolate between a decoupled state of the system and a state in which some interactions are allowed. The interpolating Hamiltonian is

$$
-H_N(\alpha, \sigma; t) := \sqrt{1-t} \left[ K_N(\sigma) + \sqrt{N} \kappa_\alpha \right] + \sqrt{t} V_{\alpha,\sigma} + h \cdot \gamma,
$$

(4.8)

where the random couplings in $K_N(\sigma)$, $\kappa_\alpha$, and $V_{\alpha,\sigma}$, defined in (3.13), (4.2), and (4.3) respectively, are each drawn independently.

The function

$$
\tilde{\psi}(t) := \frac{1}{N} \ln \left[ \sum_\alpha \xi_\alpha e^{\beta H_N(\alpha; t)} \right],
$$

(4.9)

is easily seen to satisfy

$$
\mathbb{E}(\tilde{\psi}(0)) = P_N(\beta, h) \quad \text{and} \quad \mathbb{E}(\tilde{\psi}(1)) = G_N(\beta, h; \mu),
$$

(4.10)

where $\mathbb{E}(\cdot)$ stands for integration with respect to all random variables appearing in (4.9).

Our differentiation formula (3.5) applies again. Letting $\gamma$ now denote pairs $\gamma = (\alpha, \sigma)$, we have $-H_N(\gamma; t) = X_\gamma + h \cdot \gamma$ where a direct calculation shows that

$$
\frac{d}{dt} C_{\gamma,\gamma'}(t) = -\frac{N}{2} (q_{\alpha,\alpha'} - q_{\sigma,\sigma'})^2.
$$

(4.11)

The covariance derivative vanishes for $\gamma = \gamma'$, since $q_{\alpha,\alpha} = q_{\sigma,\sigma} = 1$; as we saw, already that and the definite sign in (4.11) imply monotonicity. The full statement, (4.5), follows by (3.5) and the fundamental theorem of calculus.

Part 2.: We now note that there exists a sequence of ROSts $\mu_{NK}^S$ for which

$$
G_M(\beta, h; \mu_{NK}^S) = \frac{1}{M} \mathbb{E} \left( \ln \left[ Z_{N+M}(\beta, h) \right] \right) + o\left( \frac{M}{N} \right).
$$

(4.12)

To see that, it suffices to consider the example which has motivated the concept, namely the case when the ROSt $\mu_{NK}^S$ is provided by another SK systems of $N$ particles, with $N >> M$.

Adapting (3.3) to an increment by $M$ we get

$$
\mathbb{E} \left( \ln \frac{Z_{N+1}}{Z_N} \right) = \mathbb{E} \left( \ln \frac{\sum_\alpha \xi_\alpha e^{\beta(V_{\alpha,\sigma} + h \cdot \sigma)}}{\sum_\alpha \xi_\alpha e^{\beta \kappa_\alpha}} \right),
$$

(4.13)

where $\{\xi_\alpha\}$ are the weights

$$
\xi_\alpha = \exp \left( \beta \left[ \frac{1}{\sqrt{N+M}} \sum_{i<j}^N J_{ij} \alpha_i \alpha_j + h \cdot \alpha \right] \right).
$$

(4.14)
The quantities $V_{\alpha,\sigma}$ and $\kappa_{\alpha}$ are Gaussian variables whose covariance differs from the corresponding factors in the desired variational quantity by the factor of $\frac{N}{N+M} = 1 - \frac{M}{N} + O\left(\frac{M}{N}\right)^2$. Applying Corollary 3.3, one may determine that

$$\left|G_M(\beta, h; \mu^{S_N}) - \frac{1}{M} \mathbb{E} \left( \ln \left( \frac{Z_{N+M}(\beta, h)}{Z_N(\beta, h)} \right) \right) \right| = O\left(\frac{M}{N}\right),$$

from which (4.12) follows.

Combining this result with Corollary 3.5 part (ii) gives part (2) of Theorem 4.1. □

5. Hierarchical Random Probability Cascades (RPC)

In his commentary on the story of Oedipus, Andre Gide brought up the observation that there exist universally valid answers, which are applicable to many questions.¹ A “universal answer”, in the form of a hierarchal structure which appears to play a key role in various complex systems, has emerged also in the study of spin-glass models.

In this section, we describe a family of ROSts each of which is endowed with a remarkable property: quasi-stationarity under a class of time evolutions which includes the cavity dynamics of Section 2.2. An intriguing and relevant question is whether the class of examples discussed here includes all the ROSts which exhibit a robust version of quasi-stationarity. Before explaining the question, or conjecture, let us present the “random energy model” and its hierarchal extension. Both were introduced by Ruelle, as the point processes capturing the $N \to \infty$ limit of Derrida’s finite model calculations, and called the REM (for random energy model) and the GREM (for a generalized random energy model). Seeking a descriptive term we shall refer to these as the hierarchal “random probability cascades” (RPC).

5.1. The Random Energy Model (REM). The basic building block for the hierarchal probability cascades is the REM, or REM$_x$ to be specific, which is the Poisson point process on $[0, \infty)$ with density given by $\rho_x(d\xi) = -d\xi^{-x}$. Here $x$ is a parameter ranging over $(0, 1)$, the minus sign is to ensure that the measure is positive, and each configuration, drawn according to the REM$_x$, is represented by a sequence of non-negative numbers denoted by $\{\xi_\alpha(\omega)\}$. Denoting the occupation number of a Borel set $A \subset [0, \infty)$ by

$$N_A(\omega) := \# \{ \alpha : \xi_\alpha(\omega) \in A \},$$

what is stated above means that for the REM$_x$:

i. the occupation numbers of disjoint sets form independent random variables,  
ii. the distribution of the occupation number is Poissonian:

$$\mathbb{P} \left( N_A(\omega) = k \right) = \frac{\rho(A)^k}{k!} e^{-\rho(A)},$$

with $\rho(A)$ the mean value:

$$\rho_x(A) \equiv \mathbb{E} \left( N_A(\omega) \right) = -\int_A d\xi^{-x}.$$  

The REM$_x$ process also appears in extreme value theory; in some probability references it is denoted PD$(x, 0)$, (19).

¹In the case of Oedipus, “I/man”, points towards the answer to the two questions which Oedipus faced at turning points in his life, the one posed by the Sphinx and the other on which years later he has sought the advice of Tiresias. A. Gide: “Oedipe” (1931).
By (5.3), for any $\varepsilon > 0$
\[ \mathbb{E}(N_{[\varepsilon,\infty)}(\omega)) = \frac{1}{\varepsilon^x}. \] (5.4)
It readily follows that with probability one it is possible to re-label its points in descending order, i.e., write $\{\xi_\alpha(\omega)\} = \{\xi_n(\omega)\}_{n=1}^\infty$ where
\[ \xi_1(\omega) > \xi_2(\omega) > \cdots. \] (5.5)
Furthermore, one has:

**Theorem 5.1.** Let $0 < x < 1$, then with respect to the point process REM$_x$, almost surely:

i. \[ n^{1/x} \xi_n(\omega) \to 1 \quad (n \to \infty). \] (5.6)

ii. the following sum converges if and only if $v > x$:
\[ \sum_n (\xi_n(\omega))^v < \infty, \] (5.7)

iii. the partition function $Z(\omega) := \sum_n \xi_n(\omega)$ is almost surely finite, with an infinitely divisible distribution, satisfying the addition law: $Z \sim 2^{-1/x}(Z + Z')$ where $Z'$ is an iid copy of $Z$;

iv. the $u$-moment of $Z$ is finite if and only if $u < x$:
\[ \mathbb{E}(Z^u) < \infty. \] (5.8)

where $\mathbb{E}$ represents the expectation value over REM$_x$.

**Proof.** i. On the scale of $t \equiv \xi^{-x}$, REM$_x$ is a Poisson process of fixed density ($= 1$). Let $N(t; \omega) := N_{(t^{-1}, \infty)}(\omega)$, and let $t(n; \omega)$ be the inverse function. Then, by the Law of Large Numbers (or the ergodic theorem), $N(t; \omega)/t \to 1$, almost surely (for $t \to \infty$). This can be rewritten as $t(n; \omega)/n \to 1$, which implies (5.6). (Estimates on the deviations can be deduced using the law of the iterated logarithm.)

ii. The a.s. finiteness statement (5.7) can be deduced from (5.6), or alternatively by splitting from the sum the finite (almost surely) collection of terms with $\xi > 1$, and noting that the main term is then of finite mean.

iii. The divisibility law for the distribution of $Z(\omega)$ is a direct consequence of the divisibility of the Poisson point process.

iv. A simple device which facilitates the derivation of (5.8) is the bound, for $0 \leq u \leq 1$:
\[ Z(\omega)^u \leq Z_{[0,1]}(\omega)^u + \sum_{\xi_n \in (1,\infty)} \xi_n^u(\omega) \] (5.9)
where $Z_{[0,1]}$ is the contribution due to $\xi_n \in (0, 1]$. With the help of the Hölder inequality, at $p = 1/u$, applied to the first term, (5.8) can be deduced by a direct calculation. □

5.1.1. **Quasi-Stationarity of the REM.** Among the more compelling attributes of the REM point processes, is their quasi-stationarity under the dynamics which correspond to increments through independent factors.

The time evolution can be described through a sequence of steps applied to a configuration $\{\xi_n(\omega)\}$ generated according to a REM$_x$ process. First, the points of the configuration are labeled in descending order $\{\xi_n\} = \{\xi_n\}_{n=1}^\infty$ as described in (5.5). Here we have omitted the dependence of the sequence on the randomness $\omega$, and we will continue to do so, where convenient, in the following. Next, a non-negative sequence of iid random variables $\{\gamma_n\}_{n=1}^\infty$ is drawn independently of $\{\xi_n\}$, with probability distribution $g(d\gamma)$. A
new configuration is obtained by multiplying $\xi_n$ by the random weights $\gamma_n$. To retain the monotonicity which is assumed in our notation, the resulting configuration is relabeled in descending order, and it therefore takes the form

$$\tilde{\xi}_n := \gamma_{\pi(n)} \xi_{\pi(n)} \quad (5.10)$$

where $\pi$ is the appropriate permutation. We also denote

$$\tilde{\gamma}_n := \gamma_{\pi(n)}. \quad (5.11)$$

Thus, while $\gamma_n$ is the factor by which $\xi_n$ is multiplied “going forward” in time, $\tilde{\gamma}_n$ is the factor by which $\tilde{\xi}_n$ was increased in the last step.

**Theorem 5.2.** For any $x \in (0, 1)$ and a probability distribution $g(d\gamma)$ of finite moment: $\langle \gamma^x \rangle := \int \gamma^x g(d\gamma) < \infty$, there is a constant $K$ so that the REM$_x$ distribution is stationary under the time evolution produced by the random factors $\{\gamma_n\}$, as described above, corrected by the factor

$$K = \langle \gamma^x \rangle^{1/x}, \quad (5.12)$$

in the sense that:

$$\{K^{-1} \tilde{\xi}_n \} \overset{D}{=} \{\xi_n\}. \quad (5.13)$$

Furthermore, the past increments $\{\tilde{\gamma}_n(\omega)\}$ form a sequence of iid random variables with the modified probability distribution

$$\tilde{g}(d\tilde{\gamma}) = \frac{\tilde{\gamma}^x g(d\tilde{\gamma})}{\langle \gamma^x \rangle} \quad (5.14)$$

which are also independent of $\{\tilde{\xi}_n\}$.

The last statement may appear paradoxical: you start with a sequence of the iid random variables $\{\gamma_n\}$, reshuffle them a bit, producing the permuted sequence $\{\tilde{\gamma}_n\}$, and the result is a sequence of iid variables with a different distribution! This would certainly not be possible for any finite collection of random variables, but it is apparently possible in the infinite setting due to the existence of a bottomless reservoir.

**Theorem 5.2** The proof can be obtained through the moment generating functionals, or alternatively the observation that the joint distribution of the collection $\{(\xi_n, \gamma_n)\}$ corresponds to the Poisson process in $\mathbb{R}_+ \times \mathbb{R}_+$ with the density: $-d\xi^{-x} g(d\gamma)$. The collection of points $\{\tilde{(\xi_n, \gamma_n)}\}$ also forms a Poisson process, since its occupation numbers for disjoint regions of $\mathbb{R}_+ \times \mathbb{R}_+$ are independent, and have the density $-d(\tilde{\xi}/\tilde{\gamma})^{-x} g(d\tilde{\gamma})$. It helps to write this density so that it becomes a probability measure in the second variable:

$$-d(\tilde{\xi}/\tilde{\gamma})^{-x} g(d\tilde{\gamma}) = -d \left( \frac{\tilde{\xi}}{\langle \gamma^x \rangle^{1/x}} \right)^{-x} \times \frac{\tilde{\gamma}^x g(d\tilde{\gamma})}{\langle \gamma^x \rangle}. \quad (5.15)$$

The fact that the second factor on the right hand side is a probability measure which does not depend on $\xi$ allows to quickly read from the above the statements which are asserted in the Theorem. □

The above argument is discussed a bit more explicitly in [21]. **Theorem 5.2** states that each of the REM$_x$ processes is invariant under the stochastic evolution up to a deterministic correction. A general result of Liggett [14] implies that such invariance in fact singles out
this class of processes. A strengthening of this statement was obtained in the work of Ruz-
maikina and Aizenman [21]. For our applications, it suffices to know that the distribution of the relative weights is stationary, in the sense that:

\[ \left\{ \frac{\xi_n}{Z} \right\} \overset{\text{D}}{=} \left\{ \frac{\tilde{\xi}_n}{\tilde{Z}} \right\}, \quad (5.16) \]

where the partition function \( Z \), resp. \( \tilde{Z} \), are as introduced in Theorem 5.1 (iii). In ref. [21] this property was termed “quasi-stationarity”, and it was shown there that, under certain limitations on the point process and the distribution of the independent weights, just this property limits the point process to REM \( x \) at some value of the parameter \( x \in (0, 1) \).

5.2. The Random Probability Cascades (RPC). The REM point processes were used by D. Ruelle as building blocks for a hierarchal process which capture the results of Derrida's calculations involving the large \( N \) limit of the free energy in the so-called Generalized Random Energy Models. In line with Parisi’s fundamental insight concerning the SK spin-glass model, the parameter for the construction is a monotone function \( x(q) \) taking \([0, 1]\) into itself. Convenient examples, and approximations, are provided by piecewise constant functions. For each \( k \in \mathbb{N} \), a piecewise constant right-continuous function \( x(q) \) is specified by a pair of monotone sequences

\[ 0 = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} = 1, \]
\[ 0 = q_0 < q_1 < q_2 < \cdots < q_k < q_{k+1} = 1, \quad (5.17) \]

in particular,

\[ x(q) := \sum_{i=0}^{k} x_i \chi[q_i, q_{i+1})(q) \quad (5.18) \]

with \( x(1) = 1 \).

Following is the hierarchal construction parametrized by this data.

i. Start with a REM \( x_1 \) process whose points are symbolically labeled as \( \{\xi^{(1)}_{\alpha_1}\} \). Here the subscript \( \alpha_1 \) is intended to represent a label which just identifies the points; not their respective ordering. (If absolutely desired, \( \alpha_1 \) could be regarded as taking values in a random subset of the line.)

ii. Next, for each \( \alpha_1 \), we generate a REM \( x_2 \) process whose points are designated \( \xi^{(2)}_{\alpha_2; \alpha_1} \). The processes corresponding to different values of \( \alpha_1 \) are chosen independently.

iii. The construction is iterated up to \( n = k \). At the \( n \)-th step, independent versions of the REM \( x_n \) process are generated for each of the distinct values of the “address” \( (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \), and the resulting points are designated as \( \xi^{(n)}_{\alpha_n; \alpha_1, \ldots, \alpha_{n-1}} \).

The construction yields a hierarchal family of addresses of the form

\[ \alpha = (\alpha_1, \ldots, \alpha_k). \quad (5.19) \]

With each value of \( \alpha \), we associate

\[ \xi_{\alpha} := \prod_{n=1}^{k} \xi^{(n)}_{\alpha_n; \alpha_1, \ldots, \alpha_{n-1}}. \quad (5.20) \]

The result of the above construction is a point process whose configurations consist of the collection \( \xi(\omega) := \{\xi_{\alpha}\} \), where \( \omega - \) which is omitted on the right hand side- represents all the randomness which enters the above construction. (Specifically, all the above choices
can be represented by functions defined over a probability space whose points are denoted by \( \omega \).

The hierarchal addresses, which play a role in the explicit construction, can ipso-facto be replaced by the more generic ROSI notation, for which the information is expressed through the overlap kernel, which here is defined as:

\[
q_{\alpha, \alpha'} \equiv q_{n(\alpha, \alpha')} \quad n(\alpha, \alpha') := \max\{j : j \leq k, (\alpha_1, ..., \alpha_j) = (\alpha'_1, ..., \alpha'_j)\}.
\] (5.21)

An overlap kernel corresponds to a hierarchal address if and only if the condition:

\[
q_{\alpha, \alpha'} \leq r
\]
is transitive for each real \( r \). The condition can equivalently be expressed as “ultrametricity” \[15\] of the distance function \( \text{dist}(\alpha, \alpha') := 1 - q_{\alpha, \alpha'} \).

Let us note that for \( \mathbb{P}^{(2)} \) – the probability measure associated to \( \mathbb{E}^{(2)} \) – a calculation yields \[20\]

\[
\mathbb{P}^{(2)}(q_{\alpha, \alpha'} \geq q) = x(q).
\]

As a direct consequence of Theorem 5.1 and Theorem 5.2 one has:

**Theorem 5.3.** For \( k \geq 2 \), and \( 0 < x_1 < ... < x_k < 1 \), the partition function \( Z = \sum_{\alpha=(\alpha_1, ..., \alpha_k)} \xi_{\alpha} \) is almost surely finite and in distribution satisfies:

\[
Z \overset{\mathbb{P}}{=} Z_{x_1} \prod_{n=2}^{k} \mathbb{E}\left((Z_{x_{n-1}})^{x_{n-1}}\right)^{1/x_{n-1}}
\] (5.22)

where \( Z_{x_n} \) is a random variable having the distribution of a partition function under \( \text{REM}_{x_n} \). In particular,

\[
\mathbb{E}(\log Z) < \infty.
\] (5.23)

The above construction yields a process whose configurations consist of the pair:

\[
(\{\xi_{\alpha}(\omega)\}_{\alpha}, \{q_{\alpha, \alpha'}(\omega)\}_{\alpha, \alpha'})
\]

of: i. a point subset of \([0, \infty)\), and ii. an overlap kernel, which conveys the genealogical information. Our main interest will concern the system of normalized weights, along with the overlaps, i.e.,

\[
(\{\xi_{\alpha}(\omega)/Z(\omega)\}, \{q_{\alpha, \alpha'}(\omega)\}_{\alpha, \alpha'})
\] (5.25)

We refer to this process as the Random Probability Cascade.

**Remark:** The last step in the hierarchal construction should correspond to \( \text{REM}_{x_{k+1}} \) at \( x_{k+1} = 1 \), which may be seen as problematic since for \( x = 1 \) the normalization \( Z \) diverges. Nevertheless, for \( x > 1 \), the normalized average is well defined for all the quantities of interest. For simplicity of the presentation we shall not stress this point here, and approach the value \( x = 1 \) only as a limit.

**5.3. Quasi-stationarity of RPC.** The hierarchal RPC inherits and broadens the remarkable quasi-stationarity property of the REM processes. In the context of RPC, the dynamics allow also correlated evolution of the point configuration. The construction of the evolution is similar to that considered for the REM model, except that the random factors \( \gamma_n \) are now of the form:

\[
\gamma_n = e^{\psi(\eta_n)}
\] (5.26)

with \( \{\eta_n\} \) a collection of Gaussian random variables of covariance

\[
\mathbb{E}(\eta_n \eta_{n'}) = q_{\alpha(n), \alpha(n')} \, ,
\] (5.27)

where \( \alpha = \alpha(n) \) is the inverse of the bijection \( n = n(\alpha) \).

Unlike the previous case, the dynamics are now correlated. The correlations between the increments of the “competing” points are determined through the overlap function, but...
are not affected by the relative ranking of their position on the line, which changes in the course of the time evolution.

It is important to note that the covariance condition (5.27) is satisfiable, i.e., the hierarchical kernel \( q_n(\omega, \omega') \) is always positive definite. To see that, it is useful to construct an auxiliary genealogical tree for which the ultrametric kernel coincides with the value of \( q \) at which the ancestral lines of \( \omega \) and \( \omega' \) split. A Gaussian process with the covariance (5.27) is obtained by associating with each \( \omega \) the integral of white noise along the branches of the tree, in a path leading from the root to \( \omega \), with the covariance \( \mathbb{E}((d\eta)^2) = dq \). Furthermore, by restricting the white noise integral to only \( q \in [0, t] \), one obtains a family of Gaussian variables with an extra parameter \( \eta_n(t) \), with the covariance:

\[
\mathbb{E}(\eta_n(t)\eta_{n'}(t)) = \min\{t, q_n(\omega), q_n(\omega')\}.
\]

A convenient explicit representation is obtained by presenting the Gaussian variables \( \eta_n(\omega) \) as sums of mutually independent terms, which in the algorithm described above correspond to the integrals of white noise over distinct segment of the genealogical tree:

\[
\eta_n(\omega) = \sum_{i=1}^{k} \sqrt{q_{i+1} - q_i} Z_{i, \omega},
\]

where \( Z_{i, \omega} \) are normal Gaussian variables with the covariance

\[
\mathbb{E}(Z_{i, \omega} Z_{j, \omega'}) = \delta_{i,j} I[q_n(\omega) \geq q_{j+1}].
\]

For a simple statement of the quasi-stationarity, a relevant class of function is defined by the Lipschitz norm:

\[
\|\psi\|_{Lip} := \sup_{x,y} \frac{|\psi(x) - \psi(y)|}{|x - y|}.
\]

**Theorem 5.4.** Under the dynamics described above, for any \( \psi \) of bounded Lipschitz norm, the configuration which results from the above dynamics has the same distribution as the process obtained by multiplying \( \xi_n(\omega) \) by a constant, \( e^{\psi_0} \):

\[
\{ e^{\psi(\xi_n(\omega))} \xi_n(\omega) \} \overset{D}{=} \{ e^{\psi_0} \xi_n(\omega) \},
\]

with \( \psi_0 \) described below. In particular, the partition function satisfies

\[
Z \overset{D}{=} e^{\psi_0} Z.
\]

and the process is quasi-stationary, in the sense that the distribution of the relative weights \( \{ \xi_n(\omega)/Z(\omega) \} \) is stationary, satisfying the appropriate version of eq. (5.16).

**Proof.** This statement can be obtained by a direct iteration of the quasi-stationarity property of the REM processes which are used in the construction of the RPC. It is convenient to define the partial quantities, for any \( j = 1, \ldots, k \):

\[
\xi_{n}^{(j)} := \prod_{n=1}^{j} \xi_{\alpha_n, \omega_1, \ldots, \alpha_{n-1}} \quad \text{and} \quad \eta_{n}^{(j)} := \sum_{i=0}^{j} \sqrt{q_{i+1} - q_i} Z_{i, \omega}.
\]

Conditioning on the collection of variables \( \eta_{n}^{(k-1)}(\omega) \) and \( \xi_{n}^{(k-1)}(\omega) \) and for each \( \alpha_1, \ldots, \alpha_{k-1} \), let us consider the evolution for the corresponding subtree which corresponds to multiplication by

\[
\gamma_{k, \omega} := e^{\psi(\eta_{n}^{(k-1)}(\omega) + \sqrt{q_{k+1} - q_k} Z_k(\omega))} \xi_{n}^{(k-1)}(\omega).
\]
By Theorem 5.2, for each subfamily corresponding to a specified $\alpha_1, ..., \alpha_{k-1}$:

$$\left\{ e^{\psi_0(\eta_0)} \xi_{\eta_0} \right\} \overset{D}{=} \left\{ (\gamma_k^{1/\alpha_k})^{1/\alpha_k} \xi_{\alpha_k; \alpha_1, ..., \alpha_{k-1}} \right\}, \tag{5.36}$$

where $\langle \cdot \rangle$ represents integration with respect to the variables $Z_{k, \eta}$.

The above procedure of conditioning and averaging may be iterated. Starting from:

$$\psi_k(y) \equiv \psi, \text{ and denoting by } E_z(\cdot) \text{ the average over the normal gaussian random variable } z,$$

we define recursively for $j = k \rightarrow 0$:

$$\psi_j(y) := \frac{1}{x_j} \ln \left[ E_z \left( e^{\psi_{j+1}(y + \sqrt{q_{j+1}} - q_j z)} \right) \right], \tag{5.37}$$

It is easy to check that under the Lipschitz condition on $\psi$ the iteration step is well defined, and, furthermore, the Lipschitz norm does not increase under the mapping $\psi_j(\cdot) \mapsto \psi_{j-1}(\cdot)$. One obtains

$$\left\{ e^{\psi(\eta_0)} \xi_{\eta_0} \right\} \overset{D}{=} \cdots \left\{ e^{\psi_1(\eta_0)} \xi_{\eta_0} \right\} \overset{D}{=} \left\{ e^{\psi_0} \xi_{\eta_0} \right\}. \tag{5.38}$$

In this sequence, the deterministic quantity appearing in (5.32) is

$$\psi_0 := \ln \left[ E_z \left( e^{\psi_1(\sqrt{q_1} z)} \right) \right]. \tag{5.39}$$

The deterministic value of $\psi_0$ can be alternatively characterized through the solution of a specific partial differential equation. Namely, consider functions of two variables $f = f(q, y)$ which satisfy, for $t \in [0, 1]$

$$\frac{\partial f}{\partial q} + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial y^2} + x(q) \left( \frac{\partial f}{\partial y} \right)^2 \right] = 0, \tag{5.40}$$

with the $t = 1$ boundary condition:

$$f(1, y) = \ln [\cosh (\beta(y + h))]. \tag{5.41}$$

One may note that the function $x(q)$ enters here as a parameter for the partial differential equation. Going backward in time, the equation is particularly simple to solve over intervals where $x(q)$ is constant. Using the Cole-Hopf transformation, on which more is said next, the solution is provided by the iterative procedure which is described in the above proof. From this perspective, the value of $\psi_0$ corresponds to $\psi_0 = f(0, 0; x)$. We shall now expand on this point.

### 5.4. Quasi-stationarity of RPC in terms of the Parisi equation.

An alternative perspective on Theorem 5.4 is provided by a continuous time version of the quasi-stationarity. As it turns out, equation (5.40), which plays a key role in the Parisi solution, appears also as a Martingale condition for the cavity dynamics with respect to the RPC hierarchal ROSt.

For a given function $\psi$ consider the two parameter function $f(t, y) = f(t, y; x)$, which satisfies the boundary conditions:

$$f(1, y) = \psi(y + h). \tag{5.42}$$

and the partial differential equation (5.40), which is to be solved from $q = 1$ down to $q = 0$.

Theorem 5.4 admits the following extension, about which we learned from D. Ruelle. For simplicity it is implicitly assumed here that the function is suitably differentiable and bounded. Upon closer analysis, it suffices to assume the Lipschitz condition, as in Theorem 5.4.
Theorem 5.5. Let \( f(t, y) \) be a function satisfying (5.40). For configurations of the hierarchal RPC which correspond to a piecewise constant function \( x(q) \), let:

\[
\xi_\alpha(t; \omega) := e^{f(t, \eta(t; \omega))} \xi_\alpha(\omega),
\]

Then the probability distribution of the ROST configuration \( \xi_\alpha(t; \omega) \) is independent of \( t \). In particular, it coincides with that of \( e^{\psi_0} \xi_\alpha(\omega) \) where the deterministic factor is

\[
\psi_0 = f(0, 0)
\]

The statement can be proved along the lines of the above proof of Theorem 5.4, or in terms of stochastic PDE and Ito’s formula. We refer the reader for further details on the latter perspective to [3].

Over intervals of constant \( x(q) \) the differential equation can be solved through the convolution of the function \( e^{f(q, \eta)/x(q)} \) with suitable Gaussian measures. This is a slight variation of the well-known Cole-Hopf transform familiar in the context of nonlinear integrable PDE’s.

In the special case of \( x(\cdot) \) constant over the entire interval \((0, 1)\) the RPC is really a REM. In this situation, one readily verifies that the solution of (5.40) derived through the Cole-Hopf transformation, starting with the boundary conditions (5.42), at \( t = 1 \), is exactly what one would obtain using (5.13). For piecewise constant \( x(q) \) this argument can be employed in steps, to again conclude that the PDE formulation matches with the results of an iteration of Theorem 5.2, i.e., (5.38). Subdividing the intervals into short segments the statement can also be easily understood from the perspective of Ito’s formula, as is discussed more explicitly in [3].

The formulation of the solution in terms of the differential equation has the advantage of being well defined even when the piecewise constant \( x(q) \) is replaced by a continuous function. For the existence of the continuum limit it is imperative to restrict the attention to the ROST given by the normalized weights, as in (5.25).

Let us now return to the spin glass model for whose solution the above plays a key role.

6. Relation with the Parisi solution

6.1. The Parisi formula. The partial differential equation, (5.40) has made its appearance in the work of Parisi on the SK model, in the context of rather different considerations. Without reviewing here Parisi’s approach, and his hierarchal ansatz for replica symmetry breaking, let us present the resulting conjecture for the free energy, a.k.a. the ‘Parisi solution’.

Introducing the ansatz of hierarchal pattern of replica symmetry breaking - a concept for which the reader is referred to [17, 15] - Parisi has introduced the idea that the order parameter for the SK model is a monotone function, \( x : [0, 1] \to [0, 1] \). Somewhat analogously to the much simpler case of the Curie Weiss mean field ferromagnetic model, the value of the order parameter can be characterized through either self consistency, based on the cavity analysis of the cavity dynamics (discussed in Chapters 4 and 5 of [15]), or through a variational principle. That has led Parisi to investigate solutions \( f = f(q, y) \) of the partial differential equation

\[
\frac{\partial f}{\partial q} + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial y^2} + x(q) \left( \frac{\partial f}{\partial y} \right)^2 \right] = 0,
\]

subject to the boundary condition

\[
f(1, y) = \ln \cosh (\beta(y + h)) .
\]
The resulting value of \( f(0, 0) \equiv f(0, 0; x) \) is incorporated in the Parisi functional, which is defined as:

\[
P[x] := \ln[2] + f(0, 0; x) - \frac{\beta^2}{2} \int_0^1 q(x) \, dq.
\]

The end result is Parisi’s proposal that:

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}(Z_N) = \inf_{x(\cdot)} P[x] := G_{\text{Parisi}}.
\]

where the infimum is over monotone functions of the unit interval with values in \([0, 1]\).

The remarkable arguments of Parisi are still beyond mathematical analysis, but its main conclusion is now known to be correct.

In a surprising development, F. Guerra \[10\] proved:

**Lemma 6.1** (Guerra variational principle),

\[
\inf_{x(\cdot)} P[x] \leq G_{\text{Parisi}}.
\]

The analysis, which employs an interpolation argument, yields also a criterion for the saturation of the inequality. The statement was given a different form in our work \[2\]: the variational principle was generalized into infimum of the functional \( G(\beta, h; \mu) \) over ROSt’s \((\mu)\), and it was shown that in that generality the infimum yields the correct value (Theorem 4.1). Independently of that, M. Talagrand \[25\] has proven that the Parisi conjecture is correct. The proof employs the criterion provided by Guerra’s analysis, and insights supported by a heavy dosage of calculus.

We shall now show how Guerra’s variational principle is incorporated in the ROSt bound, (4.4) of Theorem 4.1.

6.2. **The free energy of Hierarchal ROSts.** For the hierarchal RPC, the calculation of the ROSt functional \( G_M(\beta, h; \mu) \equiv G_M(\mu) \) is greatly facilitated by their quasi-stationarity property. We shall now demonstrate that the free energy functional corresponding to the RPC of a given function \( x(q) \) is independent of \( M \) and coincides with Parisi’s functional \( P[x] \), of (6.3).

The ROSt free energy functional, which is defined in (4.1), can be written as

\[
G_M(\mu) = G^{(1)}_M(\mu) - G^{(2)}_M(\mu)
\]

with

\[
G^{(1)}_M(\mu) = \frac{1}{M} \mathbb{E} \left( \ln \left[ \frac{\sum_{\alpha,\sigma} \xi_\alpha e^{\beta(V_{\alpha,\sigma} + h \cdot \sigma)}}{\sum_\alpha \xi_\alpha} \right] \right)
\]

and

\[
G^{(2)}_M(\mu) = \frac{1}{M} \mathbb{E} \left( \ln \left[ \frac{\sum_{\alpha,\sigma} \xi_\alpha e^{\beta \sqrt{M} \varepsilon_\alpha}}{\sum_\alpha \xi_\alpha} \right] \right).
\]

**Lemma 6.2.** Let \( \mu \) be a ROSt having weights generated by an RPC with parameter \( x = (x_1, \ldots, x_n) \) and overlap function \( q \). Then for any \( M \in \mathbb{N} \):

\[
G^{(1)}_M(\mu) = \ln[2] + f(0, 0; x),
\]

and

\[
G^{(2)}_M(\mu) = \frac{\beta^2}{2} \int_0^1 q(x) \, dq.
\]
In particular, the free energy functional coincides with the Parisi functional at $x$, i.e.,

$$G_M(\beta, h; \mu) = P[x].$$  \hfill (6.11)

**Proof.** Summing over the spins $\sigma$, we cast $G_M^{(1)}$ in the form

$$G_M^{(1)}(\mu) = \ln[2] + \frac{1}{M}\mathbb{E}\left(\ln \left[\frac{\sum_\alpha \xi_\alpha \prod_{i=1}^M e^{\psi(\eta_{i,\alpha})}}{\sum_\alpha \xi_\alpha}\right]\right),$$  \hfill (6.12)

where

$$\psi(\eta_{i,\alpha}) := \ln[cosh(\beta(\eta_{i,\alpha} + h))],$$  \hfill (6.13)

and $\eta_{i,\alpha}$ are Gaussian variables with the covariance:

$$\mathbb{E}(\eta_{i,\alpha} \eta_{i',\alpha'}) = \delta_{i,i'} q_{\alpha,\alpha'}. \hfill (6.14)$$

The quasi-stationarity of the ROSt readily implies that the contributions of the independent factors $e^{\psi(\eta_{i,\alpha})}$ of the right hand side in (6.12) factorizes, and thus $G_M^{(1)}$ is independent of $M$. Furthermore, by Theorem 5.5, we see that

$$\left\{\xi_\alpha \prod_{i=1}^M e^{\psi(\eta_{i,\alpha})}\right\} \overset{D}{=} \left\{e^{Mf(0,0;\mu)} \xi_\alpha\right\}. \hfill (6.15)$$

This proves (6.9).

We calculate $G_M^{(2)}$ by interpolation: For any $t \in [0,1]$, define the function

$$F(t) := \frac{1}{M}\mathbb{E}\left(\ln \left[\frac{\sum_\alpha \xi_\alpha e^{\frac{\beta}{M}\sqrt{q_{\alpha,\alpha'}}}}{\sum_\alpha \xi_\alpha}\right]\right).$$  \hfill (6.16)

Note that

$$F(1) = G_M^{(2)}(\mu) \quad \text{and} \quad F(0) = 0. \hfill (6.17)$$

Using Lemma 3.3, we see that

$$F'(t) = \frac{\beta^2}{2} \left(\frac{1}{2} - \mathbb{E}_q^{(2)}\left(\frac{q_{\alpha,\alpha'}}{2}\right)\right) = \frac{\beta^2}{2} \mathbb{E}_q^{(2)}\left(\int_0^1 q \, dq\right). \hfill (6.18)$$

$$= \frac{\beta^2}{2} \int_0^1 \mathbb{P}^{(2)}(q_{\alpha,\alpha'} \leq q) \, dq = \frac{\beta^2}{2} \int_0^1 x(q) \, dq. \hfill (6.19)$$

In (6.18) we have used quasi-stationarity of $\xi_\alpha$ to remove the dependence on $t$. Equation (6.10) follows through the integration of $F'(t)$. □

### 6.3. An Open Problem: Explaining the validity of the Parisi ansatz.

As was mentioned above, it is now a Theorem, proven by M. Talagrand [25], that Parisi’s ansatz indeed yields the correct solution for the free energy of the SK model. However, it still seems reasonable to say that an “explanation” of the reasons for the validity of the Parisi ansatz continues to present an open challenge. Could RPC’s be the only ‘robustly’ quasi-stationary ROSI’s, and could the validity of Parisi’s ansatz be explained by that? Can one formulate some other fundamental reason for the validity of the Parisi calculation? Given the versatility of the applications of the Parisi approach, it may be of interest to shed more light on any of these questions.
APPENDIX A. THE GAUSSIAN DIFFERENTIATION LEMMA

Lemma A.1. Let \( X_t \in \mathbb{R}^n, t \in (0, 1) \), be a vector-valued Gaussian process, with covariance \( C_t \) which is continuously differentiable. Suppose that \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) is twice continuously differentiable and compactly supported. Then
\[
\frac{d}{dt} \mathbb{E}[\psi(X_t)] = \frac{1}{2} \mathbb{E}
\left[
\langle \nabla, \hat{C}_t \nabla \rangle \psi \right](X_t)
\].
(A.1)

Proof. The joint density function for \( X_t \) is
\[
\rho_t(x) = \frac{\exp \left( -\frac{1}{2} \langle x, C_t^{-1} x \rangle \right)}{\sqrt{\det(2\pi C_t)}}.
\] (A.2)

In terms of the Fourier transform, \( \hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i \langle k, x \rangle} f(x) \, dx \),
\[
\mathbb{E}[\psi(X_t)] := \int_{\mathbb{R}^n} \psi(x) \rho_t(x) \, dx = \int_{\mathbb{R}^n} \hat{\psi}(k) \hat{\rho}_t(k) \, dk,
\] (by Plancherel theorem). Since \( \hat{\rho}_t(k) = \exp(-2\pi^2 \langle k, C_t k \rangle) \), a direct calculation shows
\[
\frac{d}{dt} \mathbb{E}[\psi(X_t)] = -2\pi^2 \int_{\mathbb{R}^n} \langle k, \hat{C}_t k \rangle \hat{\psi}(k) \hat{\rho}_t(k) \, dk.
\] (A.4)

But, since \( (\nabla \psi)^\sim(k) = 2\pi i k \hat{\psi}(k) \), we see that
\[
-2\pi^2 \langle k, \hat{C}_t k \rangle \hat{\psi}(k) = \frac{1}{2} \left( (\nabla, \hat{C}_t \nabla)^\sim \right)(k).
\] (A.5)

So, by Plancherel’s theorem again,
\[
\frac{d}{dt} \mathbb{E}[\psi(X_t)] = \frac{1}{2} \int_{\mathbb{R}^n} \left( (\nabla, \hat{C}_t \nabla)^\sim \right)(k) \hat{\rho}_t(k) \, dk = \frac{1}{2} \mathbb{E}
\left[
(\nabla, \hat{C}_t \nabla)^\sim \right](X_t)
\].
(A.6)

□

We need the following extension of this result to a wider class of functions \( \psi \), which is enabled by a density argument.

Corollary A.2. Let \( X_t \) be as in Lemma A.1. Suppose \( \psi \in C^2(\mathbb{R}^n) \) and \( \nabla \psi, \nabla^2 \psi \in L^1(\mathbb{R}^n, \rho_t) \) for every \( t \in (0, 1) \). Also suppose that
\[
\left( t \mapsto \mathbb{E} \left[ |\psi(X_t)| + \|\nabla \psi(X_t)\| + \|\nabla^2 \psi(X_t)\| \right] \right) \in L^1_{\text{loc}}((0, 1)).
\]

Then \( \mathbb{E}[\psi(X_t)] \) is absolutely continuous and \( (A.1) \) holds for almost all \( t \in (0, 1) \).

Proof. Let \( \eta : \mathbb{R}^n \rightarrow \mathbb{R} \) be any smooth function, with compact support, such that \( 0 \leq \eta \leq 1 \) and such that \( \eta(0) = 1 \). Define
\[
\psi_{\varepsilon}(x) = \eta(\varepsilon x) \psi(x),
\]
for each \( \varepsilon > 0 \). So \( \psi_{\varepsilon} \) is twice continuously differentiable, and with compact support. Also, \( \psi_{\varepsilon} \rightarrow \psi \) and \( \nabla^2 \psi_{\varepsilon} \rightarrow \nabla^2 \psi \), pointwise, as \( \varepsilon \rightarrow 0 \). Finally, we know that \( |\psi_{\varepsilon}(x)| \leq |\psi(x)| \) for all \( x \in \mathbb{R}^n \), and
\[
\|\nabla^2 \psi_{\varepsilon}(x)\| \leq K \left( |\psi(x)| + \|\nabla \psi(x)\| + \|\nabla^2 \psi(x)\| \right),
\] (A.7)
for some constant \( K < \infty \). (The constant depends only on the sup norm of \( \|\nabla \eta(x)\| \) and \( \|\nabla^2 \eta(x)\| \).)
By Lemma A.1, integrating,
\[ E[\psi_\varepsilon(X_t)]_{t_1}^{t_2} = \frac{1}{2} \int_{t_1}^{t_2} E \left[ \left( \langle \nabla, \dot{C}_t \nabla \rangle \psi_\varepsilon \right)(X_t) \right] \, dt, \]
for each \( t_1, t_2 \in (0, 1) \) and all \( \varepsilon > 0 \). By the dominated convergence theorem,
\[ \lim_{\varepsilon \downarrow 0} E[\psi_\varepsilon(X_t)] = E[\psi(X_t)] \]
for every \( t \in (0, 1) \). In particular, it is true at \( t = t_1 \) and \( t = t_2 \). Similarly, by the dominated convergence theorem
\[ \lim_{\varepsilon \downarrow 0} E \left[ \left( \langle \nabla, \dot{C}_t \nabla \rangle \psi_\varepsilon \right)(X_t) \right] = E \left[ \left( \langle \nabla, \dot{C}_t \nabla \rangle \psi \right)(X_t) \right], \]
for every \( t \in [t_1, t_2] \). But, moreover, the integral of the upper bound in (A.7), integrated against \( \rho_t \), is a function of \( t \) which is locally integrable, by our hypothesis. Therefore, we can apply the DCT to the \( t \)-integral, itself, to determine
\[ \lim_{\varepsilon \downarrow 0} \int_{t_1}^{t_2} E \left[ \left( \langle \nabla, \dot{C}_t \nabla \rangle \psi_\varepsilon \right)(X_t) \right] \, dt = \int_{t_1}^{t_2} E \left[ \left( \langle \nabla, \dot{C}_t \nabla \rangle \psi \right)(X_t) \right] \, dt. \]
So
\[ E[\psi(X_t)]_{t_1}^{t_2} = \frac{1}{2} \int_{t_1}^{t_2} E \left[ \left( \langle \nabla, \dot{C}_t \nabla \rangle \psi \right)(X_t) \right] \, dt. \]
Since this is true for every \( t_1, t_2 \in (0, 1) \), Lebesgue’s differentiation theorem implies the corollary. □

APPENDIX B. LIMITS FOR SUPER-ADDITIVE SEQUENCES

In the proof of Theorem 4.1 we made use of the following known statement. For completeness we present its proof.

Lemma B.1. Let \( \{Q_N\}_{N \in \mathbb{N}} \) be a super-additive sequence of real numbers, in the sense that for any \( N, M \in \mathbb{N} \),
\[ Q_N + Q_M \leq Q_{N+M}. \] (B.1)
Then the following limit exists, with value in \( \mathbb{R} \cup \{\infty\} \), and satisfies
\[ \lim_{N \to \infty} \frac{Q_N}{N} = \sup_{N} \frac{Q_N}{N}. \] (B.2)
Moreover,
\[ \lim_{N \to \infty} \frac{Q_N}{N} = \lim_{M \to \infty} \liminf_{N \to \infty} \frac{Q_{N+M} - Q_N}{M}. \] (B.3)
Proof. Let \( M \in \mathbb{N} \). For any integer \( N > M \), one may write \( N = n \cdot M + k \) with \( 1 \leq k < M \), and by super-additivity,
\[ \liminf_{N \to \infty} \frac{Q_N}{N} \geq \frac{Q_M}{M}. \] (B.4)
Thus
\[ \limsup_{N \to \infty} \frac{Q_N}{N} \leq \sup_{M} \frac{Q_M}{M} \leq \liminf_{N \to \infty} \frac{Q_N}{N}, \] (B.5)
from which (B.2) follows.

The proof of (B.3) follows by demonstrating two inequalities. An immediate consequence of (B.1), is
\[ \frac{Q_M}{M} \leq \liminf_{N \to \infty} \frac{Q_{N+M} - Q_N}{M}, \] (B.6)
and a lower bound, which is part of the claim \[\text{(B.3)}\], follows from the fact that the limit in \[\text{(B.2)}\] exists.

The matching upper bound may be obtained by noting that for any \(n \in \mathbb{N}\)
\[
Q_{nM+N} - Q_N = \frac{\sum_{j=1}^{n} Q_{jM+N} - Q_{(j-1)M+N}}{nM+N},
\]
and for each \(j = 1, \ldots, n\)
\[
Q_{jM+N} - Q_{(j-1)M+N} \geq \inf_{k \geq N} [Q_{kM} - Q_k].
\]
Inserting \[\text{(B.8)}\] into \[\text{(B.7)}\], taking \(n \to \infty\), and then the supremum over \(N\), we arrive at
\[
\lim_{n \to \infty} Q_n \geq \lim_{N \to \infty} \frac{Q_{nM+N} - Q_N}{M},
\]
which completes the proof of \[\text{(B.3)}\]. □

APPENDIX C. GENERAL INTERACTIONS

In this appendix, we will illustrate that the results provided in the main text for the SK Hamiltonian have a simple analogue for more general Hamiltonians. As was done in \[2\], we will demonstrate that our analysis also holds for models of the type
\[
H_N(\sigma, h) := -K_N(\sigma) - h \cdot \sigma,
\]
where the interaction term \(K_N(\sigma)\) is now taken to be a centered Gaussian process, indexed by the spins \(\sigma\), with the covariance
\[
\mathbb{E}(K_N(\sigma) K_N(\sigma')) = \frac{N}{2} f(q_{\sigma,\sigma'}).
\]
Here, for convenience, \(f\) is written as a function of the spin overlap. We will assume that \(f\) is a positive power series; i.e.,
\[
f(q) := \sum_{r=1}^{\infty} |a_r|^2 q^r \text{ on } [-1, 1]
\]
with the normalization \(\sum_{r=1}^{\infty} |a_r|^2 = 1\). An explicit realization of such an interaction \(K_N\) is given in terms of the multi-spin interaction:
\[
K_N(\sigma) = \sqrt{\frac{N}{2}} \sum_{r=1}^{\infty} \frac{a_r}{N^{r/2}} \sum_{i_1, \ldots, i_r=1}^{N} J_{i_1 \ldots i_r} \sigma_{i_1} \cdots \sigma_{i_r},
\]
where \(J := \{ J_{i_1 \ldots i_r} \}\) is a family of independent normal Gaussian variables.

For the results discussed here, we further assume that \(f\) is convex on \([-1, 1]\). The importance of such a condition has been recognized in the literature, e.g. in \[12\] convexity was used to prove convergence for the free energy density, in the limit \(N \to \infty\). Derrida’s 
\(p\)-spin models \[8\] are obtained by the special choices \(f(q) = q^p\) for \(p \in \mathbb{N}\), and for these convexity holds if \(p \in 2\mathbb{N}\). In particular, setting \(p = 2\), one recovers the SK model, except that in contrast to \[11\] the tensor in \[\text{(C.3)}\] need not be symmetric. For convenience we also include here diagonal terms, but these do not affect the results.

In the analysis of the free energy it is convenient to first assume that the second derivative of \(f\) is continuous up to the boundary, and then use continuity arguments for an extension of the results. We proceed under this additional assumption.

The analogue of Corollary \[5\] and Theorem \[4\] hold for Hamiltonians defined with the Gaussian interactions given by \[\text{(C.2)}\].
Theorem C.1. For any $\beta$ and $h$, define $P_N(\beta, h) = \frac{1}{N} Q_N(\beta, h)$ relative the Hamiltonian given by (C.1). Then,

$$P(\beta, h) := \lim_{N \to \infty} P_N(\beta, h),$$

(C.4)

exists, and moreover,

$$\lim_{N \to \infty} P_N(\beta, h; \omega) = P(\beta, h),$$

(C.5)

almost surely.

Proof. With the very same interpolation scheme (3.17) and (3.18), excepting that the random variables are now defined via (C.2), one derives

$$\frac{d}{dt} C_{\tau, \tau'}(t) = \frac{N + M}{2} f(q_{\tau, \tau'}) - \frac{N}{2} f(q_{\alpha, \alpha'}) - \frac{M}{2} f(q_{\sigma, \sigma'}),$$

(C.6)

in place of (3.21). Superadditivity, as before, follows from the convexity $f$ and (3.22). □

For the Hamiltonian given by (C.1), one may also develop a cavity perspective by performing the change in free-energy analysis as described in Sections 2 and 3. Using the definition of the interactions (C.2), the covariance of a system of $N + M$ spins $\gamma = (\alpha, \sigma)$ is given by

$$E(K_{N+M}(\gamma)K_{N+M}(\gamma')) = \frac{N + M}{2} f(q_{\gamma, \gamma'}),$$

(C.7)

where we have adopted the notation used in Section 2. To first order, the overlap of the combined system may be expressed in terms of the overlaps within the two blocks as

$$q_{\gamma, \gamma'} = q_{\alpha, \alpha'} + (q_{\sigma, \sigma'} - q_{\alpha, \alpha'}) \frac{M}{N} + O\left(\left(\frac{M}{N}\right)^2\right),$$

(C.8)

see equation (3.22). Taylor expansion of the function $f$, again to first order, yields

$$\frac{N + M}{2} f(q_{\gamma, \gamma'}) - \frac{N}{2} f(q_{\alpha, \alpha'}) = - \frac{M}{2} \phi(q_{\alpha, \alpha'}) + \frac{M}{2} q_{\sigma, \sigma'} f'(q_{\alpha, \alpha'}) + O\left(\frac{M^2}{N}\right),$$

(C.9)

where

$$\phi(q) := q f'(q) - f(q).$$

(C.10)

Now, given a ROSt $\mu$, one may define two sets of independent, centered gaussian random variables ${\kappa_\alpha}$ and $\{V_{\alpha, \sigma}\}$, which are attuned to the more general Hamiltonian (C.1). As indicated by (C.9), these random variables are defined by prescribing their covariances as follows:

$$E(\kappa_\alpha \kappa_{\alpha'}) = \frac{\phi(q_{\alpha, \alpha'})}{2},$$

(C.11)

where $\phi$ is as defined in (C.10), and

$$E(V_{\alpha, \sigma} V_{\alpha', \sigma'}) = \frac{M}{2} f'(q_{\alpha, \alpha'}) q_{\sigma, \sigma'},$$

(C.12)

(the positivity of the covariance can be concluded from the representation (3.3), (3.2)). Correspondingly, a free energy functional, analogous to (4.1), may be defined as

$$G_M(\beta, h; \mu) = \frac{1}{M} E \left( \ln \left[ \frac{\sum_{\alpha, \sigma} \xi_\alpha e^{\beta (V_{\alpha, \sigma} + h \cdot \sigma)}}{\sum_{\alpha} \xi_\alpha e^{\beta \sqrt{M} \kappa_\alpha}} \right] \right).$$

(C.13)

With these new definitions, one may derive a variational principle analogous to Theorem 4.1. Moreover, as in Theorem 4.1, ROSts formed by $N$ particle systems with the Hamiltonian (C.1) may be used to demonstrate that the inequality actually saturates. Through an adaptation of the methods discussed above one can prove:
Theorem C.2. Let $\beta \geq 0$ and $h \in \mathbb{R}$.

i) For any $M \in \mathbb{N}$,

$$P_M(\beta, h) \leq \inf_{\mu: \text{RSt}} G_M(\beta, h; \mu). \quad (C.14)$$

ii) The pressure of the system corresponding to (C.1) may be realized through:

$$P(\beta, h) = \lim_{M \to \infty} \inf_{\mu: \text{RSt}} G_M(\beta, h; \mu). \quad (C.15)$$

For further discussion the reader is referred to [2].

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