Stability of the tree-level vacuum in two Higgs doublet models against charge or CP spontaneous violation

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Abstract. We show that in two Higgs doublet models at tree-level the potential minimum preserving electric charge and CP symmetries, when it exists, is the global one. Furthermore, we derived a very simple condition, involving only the coefficients of the quartic terms of the potential, that guarantees spontaneous CP breaking.

In the Standard Model (SM) of the electroweak interactions the existence of the scalar Higgs doublet is a fundamental piece of the theory. Through it $SU(2)_W \times U(1)_Y$ gauge invariance is broken, the $W^\pm$ and $Z^0$ bosons and the fermions acquire their masses and the renormalisability of the theory is preserved. Despite its importance, the scalar sector of the SM has not yet been directly tested and there is considerable interest in studying its extensions. The simplest of those extensions is the two Higgs doublet model (2HDM). One of the main reasons of interest in this class of models is the possibility of having spontaneous CP violation [1], thus helping to solve the baryogenesis problem [2] (for a review, see [3]). One problem of these models, though, is their immense parameter space - the most general potential that preserves the SM gauge group and does not explicitly break CP has ten independent parameters, and little is known of their allowed values. A similar difficulty afflicts Supersymmetric models, where the parameter space is generally larger. One idea that has been applied to Supersymmetric theories to restrict their allowed parameter space is to use charge and colour breaking (CB) bounds. If a given combination of parameters causes the appearance in the potential of a minimum where charged/coloured fields have vacuum expectation values (vevs), then that combination should be rejected. This appealing idea was introduced by Frère et al [4] and applied, in numerous papers, to several supersymmetric theories [5]. Phenomenological analysis of supersymmetric Higgs masses use this tool to increase the models' predictive power [6]. It is therefore of interest to apply similar techniques to the 2HDM and try to limit its parameter space. The scalars of this theory have no colour quantum numbers but there are charged fields so charge breaking (CB) extrema are in principle possible. Recent work in 2HDM [7], for instance, assumed that the choice of parameters made was such that the scalar potential respected the $U(1)_{em}$ gauge symmetry.

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In reference [8] it was shown that to make sure that there were no stationary points corresponding to charge or CP spontaneous breaking, one had to restrict the parameter space to eight independent parameters. This leads to two independent potentials, stable under renormalisation because they are protected by a $Z_2$ or $U(1)$ global symmetries. It is interesting to stress that these are the usual symmetries introduced to prevent flavour changing neutral currents. In this letter we go back to the study of the tree-level vacuum of the most general 2HDM potential without explicit CP violation, with ten parameters. We prove that for this potential the minimum that preserves CP and $U(1)_{em}$ (from now on named the “Normal minimum”), when it exists, is global. This is a very powerful result, in that it assures that the Normal vacuum is stable, it cannot tunnel to other minima. This letter is structured as follows: we will review the model and the Normal minimum in section 1, then proceed to look at the possibility of charge breaking stationary points, proving there can be no CB minima if the potential has a Normal minimum. Likewise, in section 3 we will prove the equivalent result for spontaneous CP breaking stationary points. In section 4 we will look at the mass matrices of the three types of stationary points.

1 The Normal minimum

We will be working with the most general two Higgs doublet model invariant under the gauge group $SU(2)_W \times U(1)_Y$ that does not explicitly break CP, following the conventions of reference [8]. This model is built with two scalar doublets of hypercharge $Y = 1$,

$$
\Phi_1 = \left( \varphi_1 + i \varphi_2 \right), \quad \Phi_2 = \left( \varphi_3 + i \varphi_4 \right), \quad \Phi_3 = \left( \varphi_6 + i \varphi_8 \right) .
$$

(1)

This numbering of the $\varphi$ fields might seem odd, but it is the most convenient for the latter calculation of the mass matrices. As was shown in [8] this potential has ten independent parameters and may be written as

$$
V = a_1 x_1 + a_2 x_2 + a_3 x_3 + b_{11} x_1^2 + b_{22} x_2^2 + b_{33} x_3^2 + b_{44} x_4^2 + b_{12} x_1 x_2 + b_{13} x_1 x_3 + b_{23} x_2 x_3 ,
$$

(2)

with

$$
x_1 \equiv |\Phi_1|^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2
$$

$$
x_2 \equiv |\Phi_2|^2 = \varphi_5^2 + \varphi_6^2 + \varphi_7^2 + \varphi_8^2
$$

$$
x_3 \equiv Re(\Phi_1^\dagger \Phi_2) = \varphi_1 \varphi_3 + \varphi_2 \varphi_4 + \varphi_5 \varphi_6 + \varphi_7 \varphi_8
$$

$$
x_4 \equiv Im(\Phi_1^\dagger \Phi_2) = \varphi_1 \varphi_4 - \varphi_2 \varphi_3 + \varphi_5 \varphi_8 - \varphi_6 \varphi_7 .
$$

(3)

The $a_i$ parameters have dimensions of squared mass, the $b_{ij}$ parameters are dimensionless, the fields $\varphi_i$ are real functions. Let us introduce a new notation that will be extremely useful: we define a vector $A$ and a square symmetric matrix $B$ as

$$
A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2b_{11} & b_{12} & b_{13} & 0 \\ b_{12} & 2b_{22} & b_{23} & 0 \\ b_{13} & b_{23} & 2b_{33} & 0 \\ 0 & 0 & 0 & 2b_{44} \end{bmatrix} .
$$

(4)

\(^1\)These parameters have well-known relations to the masses and couplings of the physical Higgs particles of the model, see, for instance, [7].
Defining the vector $X = (x_1, x_2, x_3, x_4)$, we can rewrite the potential (2) in the more compact form
\[ V = A^T X + \frac{1}{2} X^T B X. \tag{5} \]

The Normal minimum corresponds to $\varphi_5 = v_1, \varphi_6 = v_2$ and all the remainder $\varphi_i$ equal to zero. This gives, from the above definitions, $x_1 = v_1^2, x_2 = v_2^2, x_3 = v_1 v_2$ and $x_4 = 0$. We can write the relevant minimisation conditions as
\[
\begin{align*}
\frac{\partial V}{\partial v_1} &= 0 \Leftrightarrow \frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_3} = 0 \\
\frac{\partial V}{\partial v_2} &= 0 \Leftrightarrow \frac{\partial V}{\partial x_2} + \frac{\partial V}{\partial x_3} = 0.
\end{align*} \tag{6}
\]

Let us define the vector $V'$, with components $V'_i = \partial V / \partial x_i$, evaluated at the minimum. From the above it is plain that
\[
V' = \begin{bmatrix}
V'_1 \\
V'_2 \\
V'_3 \\
V'_4
\end{bmatrix} = -\frac{V'_3}{2v_1 v_2} \begin{bmatrix}
v_2^2 \\
v_1^2 \\
-2v_1 v_2 \\
0
\end{bmatrix}. \tag{7}
\]

Looking at the expressions for the two first components of $V'$ it is obvious that, regardless of the values of $v_1, v_2$ and $V'_3$, $V'_1$ and $V'_2$ have the same sign. From this point forward we will use $X_N$ to designate the vector $X$ evaluated at the minimum, that is, with components $(v_1^2, v_2^2, v_1 v_2, 0)$. In this notation it is trivial to realize that $X_N^T V' = 0$. Direct analysis of the potential (2) also shows that we can write $V'$ in matrix form as
\[ V' = A + B X_N. \tag{8} \]

The potential (2) is a sum of quadratic and quartic polynomials, let us call them $p_2$ and $p_4$; by performing the sum $\sum_i v_i (\partial V / \partial v_i)$ it is very simple to show that the minimisation conditions imply $2p_2 + 4p_4 = 0$ at the minimum. As such the value of the potential at this stationary point - which we designate by $V_N$ - may be written as:
\[ V_N = \frac{1}{2} A^T X_N = -\frac{1}{2} X_N^T B X_N. \tag{9} \]

We have been speaking of the Normal minimum but the conditions (6) only assure us that the potential has a stationary point. To ensure we are at a minimum we must analyse the second derivatives of $V$ - that is, the matrix of the squared scalar masses - and reject all combinations of parameters $\{a_i, b_{jk}\}$ for which any of the non-zero eigenvalues are negative (this matrix has three zero eigenvalues corresponding to the Goldstone bosons, see section 4). In particular we observe that the squared mass of the charged Higgs is given by $M_{H^\pm}^2 = V'_1 + V'_2$. So the Normal minimum exists if $V'_1 + V'_2 > 0$. Since we have already shown that $V'_1$ and $V'_2$ have the same sign this tells us that both quantities are positive. Therefore we obtain
\[ V'_1 = \left( -\frac{V'_3}{2v_1 v_2} \right) v_2^2 > 0, \tag{10} \]
and so we conclude that the quantity $-V'_3/(2v_1 v_2)$ is positive. This conclusion will be fundamental later on.
2 Charge breaking

The potential has only three types of non-trivial stationary points: the Normal one; a charge-breaking stationary point, where one of the charged fields \( \varphi \) has a non-zero vev; and a CP-breaking stationary point where one of the imaginary neutral component fields has a non-zero vev. In this section we will deal with charge breaking (CB), specifically, the configuration where the fields that have vevs are \( \varphi_5 = v'_1 \), \( \varphi_6 = v'_2 \) and \( \varphi_3 = \alpha \). The last vev breaks charge conservation and would give mass to the photon. We can calculate the derivatives of the potential with respect to \( v'_1 \), \( v'_2 \) and \( \alpha \) explicitly and arrive at the conclusion that there is always a solution for \( \alpha = 0 \), which corresponds to the Normal extremum. It is however easier to deal with the results if one uses the matrix notation. We define the vector \( Y \) to have components \((x_1, x_2, x_3, x_4)\) evaluated at the CB stationary point, that is,

\[
Y = \begin{bmatrix}
v'_1^2 \\
v'_2^2 + \alpha^2 \\
v'_1 v'_2 \\
0
\end{bmatrix}.
\]  

Unlike the Normal case we now have three independent variables, this allows us to write the stationarity conditions as

\[
\frac{\partial V}{\partial X}\bigg|_{X=Y} = 0 \iff A + BY = 0.
\]  

Hence, as long as the matrix \( B \) is invertible, the solution \( Y \) is such that

\[
Y = -B^{-1}A.
\]  

We observe that the CB vevs are given by a linear equation, which means that this solution, if it exists, is unique. This is unlike the Normal case, where the minimisation conditions lead to a system of two coupled cubic equations and as such can in principle produce multiple solutions. Notice however that the CB solution does not always exist even if \( B \) is invertible - the first two components of the vector \( Y \) must necessarily be positive, and not all choices of \( A \) and \( B \) matrices will give such a result in eq. (13). Finally, the same reasoning that led us to eq. (13) can be applied to this stationary point and we may write the value of the potential, \( V_{CB} \), as

\[
V_{CB} = \frac{1}{2} A^T Y = -\frac{1}{2} Y^T BY.
\]  

We now have all the ingredients necessary to show that, if the Normal minimum exists, it is always deeper than the CB stationary point. We look again at equation and use equation to write

\[
V' = -BY + BX_N.
\]  

Remembering that \( X_N^T V' = 0 \) we obtain

\[
-X_N^T BY + X_N^T BX_N = 0
\]  

and thus

\[
X_N^T BY = X_N^T BX_N = -2V_N,
\]  

the last step arising from equation. We now calculate the quantity \( Y^T V' \),

\[
Y^T V' = -Y^T BY + Y^T BX_N.
\]
Because the matrix $B$ is symmetric we have $Y^T BX_N = X_N^T BY = -2V_N$ and, from equation (14), $Y^T BY = -2V_{CB}$, so

$$V_{CB} - V_N = \frac{1}{2} Y^T V'. \quad (19)$$

The product $Y^T V'$ can be explicitly written as (from (7) and (11))

$$Y^T V' = \left( -\frac{V'_3}{2v_1 v_2} \right) \left[ v'_1 v'_2 + (v'_2 + \alpha^2) v'_1 - 2 v'_1 v'_2 v_1 v_2 \right]$$

$$= \left( -\frac{V'_3}{2v_1 v_2} \right) (v'_1 v_2 - v'_2 v_1)^2 + \alpha^2 v'_1^2 \quad . \quad (20)$$

At the end of section 1 we have demonstrated that the quantity $-V'_3/(2v_1 v_2)$ is positive so we conclude that

$$V_{CB} - V_N > 0 \quad , \quad (21)$$

which is to say, the Normal minimum is always deeper than the CB stationary point. So we conclude that the Normal minimum is perfectly stable and can never tunnel to a charge-breaking vacuum. In fact, as we will see in section 4, the matrix $B$ determines the character of the CB stationary point - if it is positive definite, so is the matrix of the squared masses at the CB stationary point. Multiplying both sides of eq. (15) by $B^{-1}$ first and then by $V'^T$, it is simple to rewrite $Y^T V'$ as

$$Y^T V' = -V'^T B^{-1} V' \quad . \quad (22)$$

We have shown that when the Normal minimum exists $V'^T B^{-1} V'$ is negative. This implies that the matrix $B^{-1}$ is not positive definite. So $B$ is also not positive definite. We will prove in the appendix that its first entry, $2b_{11}$, is necessarily positive to prevent the potential from being unbounded from below. Hence $B$ cannot be negative definite either. Therefore, the CB stationary point is necessarily a saddle point.

### 3 CP breaking

Besides the Normal minimum or the CB stationary point, we have another possible stationary point, one that spontaneously breaks CP conservation. In this case the fields which have non-zero vevs are $\varphi_5 = \varphi'_1$, $\varphi_6 = \varphi'_2$ and $\varphi_7 = \delta$ - this last one breaks CP. The variables $x_i$, at this stationary point, are $x_1 = \varphi'_1^2 + \delta^2$, $x_2 = \varphi'_2^2$, $x_3 = \varphi'_1 \varphi'_2$ and $x_4 = -\varphi'_2 \delta$. We see that

$$x_1^2 = x_1 x_2 - x_3^2$$

and as such the potential for this field configuration can be written as

$$V = a_1 x_1 + a_2 x_2 + a_3 x_3 + b_{11} x_1^2 + b_{22} x_2^2 + (b_{33} - b_{44}) x_3^2 + (b_{12} + b_{44}) x_1 x_2 + b_{13} x_1 x_3 + b_{23} x_2 x_3 \quad . \quad (23)$$

Defining the vector $Z$ and the square, symmetric matrix $B_{CP}$ to be

$$Z = \begin{bmatrix} \varphi'_1^2 + \delta^2 \\ \varphi'_2^2 \\ \varphi'_1 \varphi'_2 \\ 0 \end{bmatrix}, \quad B_{CP} = \begin{bmatrix} 2b_{11} & b_{12} + b_{44} & b_{13} & 0 \\ b_{12} + b_{44} & 2b_{22} & b_{23} & 0 \\ b_{13} & b_{23} & 2(b_{33} - b_{44}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad (24)$$

we see that the potential at this stationary point may be written in terms of $\{\varphi'_1, \varphi'_2, \delta\}$ as

$$V_{CP} = A^T Z + \frac{1}{2} Z^T B_{CP} Z \quad . \quad (25)$$
Then we are exactly in the conditions of the previous section: the CP breaking vevs are given by the equation

\[ Z = -B_{CP}^{-1}A , \]  
(26)

the value of the potential at this stationary point is given by

\[ V_{CP} = \frac{1}{2} A^T Z = -\frac{1}{2} Z^T B_{CP} Z , \]
(27)

and repeating the steps of the previous section we find

\[ V_{CP} - V_N = -\frac{1}{2} V'^T B_{CP}^{-1} V' = \frac{1}{2} Z^T V' . \]  
(28)

Calculating \( Z^T V' \) explicitly we find

\[ Z^T V' = \left(-\frac{V'_1}{2v_1v_2}\right) \left[(v'_1 v_2 - v'_2 v_1)^2 + \delta^2 v_2^2\right] \]
(29)

and so we conclude that

\[ V_{CP} - V_N > 0 . \]  
(30)

That is, once we have found a Normal minimum, it is always deeper than the CP-breaking stationary point. Again, we have proved that \( B_{CP} \) is not positive definite and, since its first entry is positive (see the appendix), it cannot be negative definite. As before \( B_{CP} \) determines the nature of the squared mass matrix at this stationary point (see section 4), which means that if a Normal minimum exists, the CP stationary point is necessarily a saddle point.

### 4 Mass matrices

To determine the nature of the stationary points one must analyse the second derivatives of the potential, which is to say, the scalar squared mass matrices. They are given by

\[ \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} = \frac{\partial V}{\partial x_l} \frac{\partial^2 x_l}{\partial \varphi_i \partial \varphi_j} + \frac{\partial^2 V}{\partial x_l \partial x_m} \frac{\partial x_l}{\partial \varphi_i} \frac{\partial x_m}{\partial \varphi_j} . \]
(31)

Extending the notation of section 1 we define \( V'_i = \partial V/\partial x_i \) evaluated at any of the three stationary points that we are considering. The first term in the above equation is written as an 8 \( \times \) 8 matrix of the form

\[ [M^2_i] = \begin{bmatrix} M^2_{11} & 0 \\ 0 & M^2_{12} \end{bmatrix} \]
(32)

where \( M^2_{11} \) and \( M^2_{12} \) are 4 \( \times \) 4 matrices given by

\[ M^2_{11} = \begin{bmatrix} 2V'_1 & 0 & V'_3 & V'_4 \\ 0 & 2V'_1 & -V'_4 & V'_3 \\ V'_3 & -V'_4 & 2V'_2 & 0 \\ V'_4 & V'_3 & 0 & 2V'_2 \end{bmatrix}, \quad M^2_{12} = \begin{bmatrix} 2V'_1 & V'_3 & 0 & V'_4 \\ V'_3 & 2V'_2 & -V'_4 & 0 \\ 0 & -V'_4 & 2V'_1 & V'_3 \\ V'_4 & 0 & V'_3 & 2V'_2 \end{bmatrix} . \]  
(33)

In the second term of eq. 31, the derivative \( \partial^2 V/\partial x_l \partial x_m \) is clearly the matrix element \( B_{lm} \) of the matrix \( B \) defined in eq. 4. As for the derivatives \( \partial x_l/\partial \varphi_j \), they form a 4 \( \times \) 8 matrix which we call \( C \), given by

\[ [C] = \begin{bmatrix} 0 & 0 & 0 & 0 & 2\varphi_5 & 0 & 2\varphi_7 & 0 \\ 0 & 0 & 2\varphi_3 & 0 & 0 & 2\varphi_6 & 0 & 0 \\ \varphi_3 & 0 & 0 & \varphi_6 & \varphi_5 & 0 & \varphi_7 \\ 0 & -\varphi_3 & 0 & 0 & -\varphi_7 & -\varphi_6 & \varphi_5 \end{bmatrix} , \]  
(34)
This matrix is evaluated at each of the different stationary points which is why only the fields \( \varphi_3, \varphi_5, \varphi_6, \varphi_7 \) appear, the rest are always zero at the stationary points. Then the scalar squared mass matrix becomes

\[
[M^2] = \frac{1}{2} \left( [M_1^2] + C^T B C \right),
\]

where the factor of 1/2 in the left hand-side is due to the fields \( \varphi \) being real. Let us now analyse this matrix for each of the three cases.

- The Normal stationary point

From section 2 we see that \( V'_4 = 0 \) and \( V'_3 = 4 V'_1 V'_2 \). In this case only \( \varphi_5 = v_1 \) and \( \varphi_6 = v_2 \) are non zero, so that the first four columns of the matrix \( C \) are zeros. As a consequence the matrix \( C^T B C \) is block diagonal, with the first diagonal \( 4 \times 4 \) block composed of zeros, that is,

\[
C^T B C = \begin{bmatrix} 0 & 0 & \ast & \ast \\ 0 & B' & \ast & \ast \end{bmatrix},
\]

where \( B' \) is a \( 4 \times 4 \) matrix. In our notation this means that the matrix \( M_{11}^2/2 \) is now the mass matrix of the charged Higgs, and it is very simple to see that it has two zero eigenvalues and a doubly degenerate eigenvalue given by \( V'_1 + V'_2 \). The matrices \( M_{12}^2 \) and \( B' \) are also block-diagonal, with two \( 2 \times 2 \) non-zero matrices along the diagonal. One of these is the mass matrix of the pseudo-scalar sector, given by

\[
\begin{bmatrix} V'_1 + b_{44} v_2^2 & V'_3 - b_{44} v_1 v_2 \\ V'_3 - b_{44} v_1 v_2 & V'_2 + b_{44} v_2^2 \end{bmatrix}.
\]

This matrix has a zero eigenvalue and another one equal to \( V'_1 + V'_2 + b_{44} (v_1^2 + v_2^2) \). The mass matrix for the CP-even scalar sector is given by

\[
[M_{12}^2] = \begin{bmatrix} V'_1 + H_1 & V'_3 + H_3 \\ V'_3 + H_3 & V'_2 + H_2 \end{bmatrix},
\]

with

\[
\begin{align*}
H_1 &= 4 b_{11} v_1^2 + 2 b_{13} v_1 v_2 + b_{33} v_2^2 \\
H_2 &= 4 b_{22} v_2^2 + 2 b_{23} v_1 v_2 + b_{33} v_1^2 \\
H_3 &= (2 b_{12} + b_{33}) v_1 v_2 + b_{13} v_1^2 + b_{23} v_2^2.
\end{align*}
\]

- The CB stationary point

The mass matrix in this case is very simple given that every \( V'_i \) is zero, which means \( M_{11}^2 = 0 \). The matrix \( C \) now has only one column of zeros but it is very easy to see that three other columns are linearly dependent. With judicious operations on the lines and columns of the matrix \( C \) we manage to set to zero its first four columns and therefore write

\[
[M^2] = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & C'^T B C' \end{bmatrix}
\]
where $C'$ is a $4 \times 4$ matrix. It is then obvious that $M^2$ has four zero eigenvalues, exactly those we would expect having broken the $U(1)_{em}$ symmetry. The non-zero eigenvalues are then those of the matrix

$$[M_{CB}^2] = \frac{1}{2} C'^T B C'$$

and this expression implies that $[M_{CB}^2]$ is positive definite if and only if $B$ is positive definite. If $B$ is negative or semi-positive definite $[M_{CB}^2]$ is likewise defined. This justifies our assertion in section 2 that the CB stationary points are saddle points.

With further operations on lines and columns it is still possible to simplify $[M_{CB}^2]$; one of its eigenvalues is $b_{44}(v_1^2 + v_2^2 + \alpha^2)$, the remaining three are those of the following $3 \times 3$ matrix,

$$\begin{bmatrix}
CB_{11} & CB_{12} & CB_{13} \\
CB_{21} & CB_{22} & CB_{23} \\
CB_{31} & CB_{32} & CB_{33}
\end{bmatrix},$$

with

$$
\begin{align*}
CB_{11} &= (b_{13} + b_{23}) v_1' v_2' + b_{33} (v_1'^2 + v_2'^2 + \alpha^2) \\
CB_{12} &= b_{33} v_1' v_2' + b_{23} (v_2'^2 + \alpha^2) \\
CB_{13} &= b_{33} v_2'^2 + b_{13} v_1' v_2' \\
CB_{21} &= b_{23} (v_1'^2 + v_2'^2 + \alpha^2) + (4b_{22} + 2b_{12}) v_1' v_2' \\
CB_{22} &= b_{23} v_1' v_2' + 4b_{22} (v_2'^2 + \alpha^2) \\
CB_{23} &= b_{23} v_2'^2 + 2b_{12} v_1' v_2' \\
CB_{31} &= (4b_{11} + 2b_{12}) v_1'^2 + b_{13} \frac{v_1'}{v_2'} (v_1'^2 + v_2'^2 + \alpha^2) \\
CB_{32} &= b_{13} v_1'^2 + 2b_{12} \frac{v_1'}{v_2'} (v_2'^2 + \alpha^2) \\
CB_{33} &= b_{13} v_1' v_2' + 4b_{11} v_1'^2.
\end{align*}
$$

- The CP stationary point

Again the matrix $C$ has the first four columns equal to zero, which means that the matrix $B$ only has an impact on the lower right $4 \times 4$ corner of $[M^2]$. From the expressions of section 3 it is easy to obtain $V_1' = -b_{44} v_2'^2$, $V_2' = -b_{44} (v_1'^2 + \delta^2)$, $V_3' = 2b_{44} v_1' v_2'$ and $V_4' = -2b_{44} v_2' \delta$. Once again the matrix $M_2^2$ is the squared mass matrix of the charged Higgs sector, it has two zero eigenvalues and a doubly degenerate non-zero one, given by $-b_{44} (v_1'^2 + v_2'^2 + \delta^2)$. The $\delta$ vev now causes mixing between the CP-even and odd scalar fields so that we are left with a $4 \times 4$ symmetric matrix, $M_{CP}^2$, whose entries are

$$
\begin{align*}
M_{CP}^2(1,1) &= 4b_{11} v_1'^2 + 2b_{13} v_1' v_2' + (b_{33} - b_{44}) v_2'^2 \\
M_{CP}^2(1,2) &= b_{13} v_1'^2 + (b_{33} + b_{44} + 2b_{12}) v_1' v_2' + b_{23} v_2'^2 \\
M_{CP}^2(1,3) &= (b_{13} v_2'^2 + 4b_{11} v_1') \delta \\
M_{CP}^2(1,4) &= [b_{13} v_1' + (b_{33} - b_{44}) v_2'] \delta \\
M_{CP}^2(2,2) &= (b_{33} - b_{44}) v_1'^2 + 2b_{23} v_1' v_2' + 4b_{22} v_2'^2 \\
M_{CP}^2(2,3) &= [b_{13} v_1' + 2(b_{12} + b_{44}) v_2'] \delta \\
M_{CP}^2(2,4) &= [b_{23} v_2' + (b_{33} - b_{44}) v_1'] \delta
\end{align*}$$
\[
M^2_{CP}(3,3) = 4 b_{11} \delta^2 \\
M^2_{CP}(3,4) = b_{13} \delta^2 \\
M^2_{CP}(4,4) = (b_{33} - b_{44}) \delta^2 .
\]

This matrix has one zero eigenvalue which gives a total of three Goldstone bosons, as was to be expected. The most interesting aspect of the CP mass matrix lies in the value of the squared charged mass, here proportional to \(- b_{44}\), whereas in the CB case it was proportional to \(+ b_{44}\). This means that we can never have, for the same choice of \(b_{44}\), a CB minimum and a CP one, as was concluded in ref. [8]. If we define the matrix \(C''\) to be the restriction of matrix \(C\) to its last four columns it is trivial to show that

\[
[M^2_{CP}] = \frac{1}{2} C''^T B_{CP} C'' .
\]

So, just like the CB case, the \(B_{CP}\) matrix determines the nature of the stationary point: if it is positive definite - and \(b_{44} < 0\), due to the charged Higgs eigenvalue - the stationary point is a minimum. If not, it is necessarily a saddle point.

5 Conclusions

In this paper we have shown a remarkable result: that the two Higgs doublet model is “protected” against electric charge or CP spontaneous breaking. In other words, if the model has a minimum preserving \(U(1)_{em}\) and CP, that minimum is global. In this way, there is absolutely no possibility of tunneling to deeper minima, and, for instance, the masslessness of the photon is guaranteed in these models. We also obtained a simple relation telling us that the only case where charge breaking might occur is when the matrix \(B\) is positive definite. This situation implies that no Normal minima exists. For the future, when one is scanning the parameter space of the 2HDM one can safely exclude, from the beginning, the combinations of \(b_{ij}\) parameters that give rise to a positive defined \(B\).

Charge breaking would be disastrous but there is considerable interest, from cosmologists to particle physicists, in models with the possibility of spontaneous CP violation. We have determined that this cannot happen for those ranges of parameters that lead to Normal minima. However, we have also established a very precise condition for spontaneous CP breaking to occur: CP is spontaneously broken if and only \(V'^T B^{-1}_{CP} V' > 0\). In these circumstances the 2HDM no longer has a Normal minimum. A simpler condition for spontaneous CP violation is to demand that the matrix \(B_{CP}\) be positive definite - this condition, together with \(b_{44} < 0\), guarantees that the CP stationary point, when it exists, is a minimum. We can also guarantee that if we find a CP minimum, that too is safe against charge breaking. This arises from the fact that a CP minimum requires \(b_{44} < 0\) which automatically implies the matrix \(B\) is not positive defined. Then we will have \(V_{CB} - V_N > 0\) and \(V_{CP} - V_N < 0\). Therefore, \(V_{CB} - V_{CP} > 0\) and the CP minimum is safe against charge breaking.

Let us also stress that our conclusions are absolutely general, independent of particular values of the parameters of the theory, obviously. They hold for any of the more restricted models considered in ref. [8]. It is simple to recover the conditions presented in that reference to avoid CP minima by analysing the matrix \(B_{CP}\). We remark that the Higgs potential of the Supersymmetric Standard Model (SSM) is also included in the potentials we studied - in fact, it corresponds to the case \(b_{11} = b_{22} = -b_{12}/2 = M_Z^2/(2v^2)\), \(b_{33} = b_{44} = 2 M_W^2/v^2\) and the...

\[\text{In reference [9] CP violating quantities involving only the Higgs sector were derived in models with explicit CP violation.}\]
remaining $b$ parameters set to zero, following the conventions of ref. [10]. So we could conclude that at tree-level, the Supersymmetric Higgs potential is safe against charge of CP violation, though this would not preclude charge, colour or CP breaking arising from other scalar fields present in those models. However, we must be cautious: it has been shown [11] that one-loop contributions to the minimisation of the potential have an enormous impact on charge breaking bounds in Supersymmetric models. Also, it was recently shown [12] that unless one performs a full one-loop calculation (both for the potential and the vevs, in both the CB potential and the “normal” one) the bounds one obtains can be overestimated. Therefore, we urge caution in applying these conclusions to the SSM. Nevertheless one would expect the one-loop contributions to be much less important in the non-supersymmetric 2HDM due to the much smaller particle content of the latter theory.

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Appendix: bounds on the $b$-parameters

Requiring that the potential [24] be bounded from below in all possible directions imposes some interesting bounds on the $b$ parameters. These bounds are obtained studying particular field directions and requiring that the quartic terms’ limit as the fields go to infinity be positive or at most zero. For instance, we can choose a direction such that all variables except $x_1$ are zero, and $x_1 \to \infty$ (for instance, by choosing $\varphi_1 \to \infty$ and all remaining $\varphi_i$ equal to zero). Then along this direction the potential goes to infinity as $b_{11} \varphi_1^4$, and if we want this limit to be positive or zero, we must demand that $b_{11} \ge 0$. Likewise, by choosing $\varphi_3 \to \infty$ and all others zero, for instance, we would obtain the condition $b_{22} \ge 0$. We cannot obtain a similar condition for $b_{33}$ and $b_{44}$ because we can never find a field direction with $x_{3,4} \to \infty$ without having $x_{1,2}$ diverging as well. Let us now consider the direction $\varphi_1, \varphi_5 \to \infty$ and all others set to zero. Only $x_1$ and $x_2$ are non-zero and the potential is reduced to the terms $b_{11} \varphi_1^4 + b_{22} \varphi_5^4 + b_{12} \varphi_1^2 \varphi_5^2$. By choosing polar coordinates such that $\varphi_1 = r \cos \theta$ and $\varphi_2 = r \sin \theta$, the potential will always be greater or equal to zero at infinity along this direction if, for any $0 \le \theta \le \pi/2$,

$$b_{11} \cos^2 \theta + b_{22} \sin^2 \theta + b_{12} \sin \theta \cos \theta \ge 0 \ .$$  \hspace{1cm} (47)

A simple minimisation in $\theta$ shows that this occurs as long as

$$b_{12} \ge -2\sqrt{b_{11} b_{22}} \ .$$  \hspace{1cm} (48)

Likewise, if we choose the direction $\varphi_1, \varphi_4 \to \infty$ and again use polar coordinates, the boundedness from below condition translates into

$$b_{11} \cos^2 \varphi_1 + b_{22} \sin^2 \varphi_2 + (b_{12} + b_{44}) \sin \theta \cos \theta \ge 0 \ .$$  \hspace{1cm} (49)

and so the condition we obtain is similar to the previous one,

$$b_{12} + b_{44} \ge -2\sqrt{b_{11} b_{22}} \ .$$  \hspace{1cm} (50)

With the direction $\varphi_1, \varphi_3 \to \infty$ we can study what happens with the $b_{3i}$ parameters, the condition we obtain is (making $\varphi_1 = r \cos \theta$ and $\varphi_3 = r \sin \theta$ so that now there are no restrictions on the value of $\theta$)

$$b_{11} \cos^4 \theta + b_{22} \sin^4 \theta + (b_{12} + b_{33}) \sin^2 \theta \cos^2 \theta + b_{13} \cos^3 \theta \sin \theta + b_{23} \sin^3 \theta \cos \theta \ge 0 \ .$$  \hspace{1cm} (51)
Since this condition has to hold for any value of $\theta$ we can make the change $\theta \rightarrow -\theta$ and obtain
\[ |b_{13} \cos^3 \theta \sin \theta + b_{23} \sin^3 \theta \cos \theta| \leq b_{11} \cos^4 \theta + b_{22} \sin^4 \theta + (b_{12} + b_{33}) \sin^2 \theta \cos^2 \theta . \] (52)
So, we conclude that, for any $\theta$, we have
\[ (b_{11} \cos^4 \theta + b_{22} \sin^4 \theta + (b_{12} + b_{33}) \sin^2 \theta \cos^2 \theta \geq 0 \] (53)
and we obtain a condition similar to (48) and (50),
\[ b_{12} + b_{33} \geq -2\sqrt{b_{11}b_{22}} . \] (54)
Finally, making $\theta = \pi/4$ in eq. (52), we obtain a more manageable bound on $b_{13}$ and $b_{23}$,
\[ |b_{13} + b_{23}| \leq b_{11} + b_{22} + b_{12} + b_{33} . \] (55)

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