GRADIENT GIBBS MEASURES OF A SOS MODEL ON CAYLEY TREES: 4-PERIODIC BOUNDARY LAWS

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Abstract. For SOS (solid-on-solid) model with external field and with spin values from the set of all integers, on a Cayley tree we give gradient Gibbs measures (GGMs). Such a measure corresponds to a boundary law (a function defined on vertices of Cayley tree) satisfying an infinite system of functional equations. We give several concrete GGMs which correspond to periodic boundary laws.

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1. Introduction and known results

The study of random field \( \xi_x \) from a lattice graph \( L \) (usually \( \mathbb{Z}^d \) or a Cayley tree \( \Gamma^k \)) to a measure space \((E, \mathcal{E})\) is a central component of ergodic theory and statistical physics.

In many classical models from physics (e.g., the Ising model, the Potts model), \( E \) is a finite set (i.e., with a finite underlying measure \( \lambda \)), and \( \xi_x \) has a physical interpretation as the spin of a particle at location \( x \) in a crystal lattice.

Following [1], [2], [6], [7], [8], [9], [10], [11], [14], [15], let us give basic definitions and some known facts related to (gradient) Gibbs measures.

\( \sigma \)-algebra, Hamiltonian. In general, \((E, \mathcal{E})\) is a space with an infinite underlying measure \( \lambda \) (i.e. \( \mathbb{L} \) with counting measure), where \( \mathcal{E} \) is the Borel \( \sigma \)-algebra of \( E \) and \( \xi_x \) usually has a physical interpretation as the spatial position of a particle at location \( x \) in a lattice. In [6] first such models were considered.

The prime examples of unbounded spin systems are harmonic oscillators. Another example is the Ginzburg-Landau interface model; which is obtained from the harmonic oscillators [7], [14].

Denote by \( \Omega \) the set of functions from \( L \) to \( E \), such a function also is called a configuration.

Assume random field \((\xi_x)_{x \in L}\) on \( \Omega \) given as the projection onto the coordinate \( x \in L \):

\[ \xi_x(\omega) = \omega(x) = \omega_x, \quad \omega \in \Omega. \]

For \( \Lambda \subset \mathbb{L} \), denote by \( F_\Lambda \) the smallest \( \sigma \)-algebra with respect to which \( \xi_x \) is measurable for all \( x \in \Lambda \). Write \( T_\Lambda = F_{L \setminus \Lambda} \).

A subset of \( \Omega \), is called a cylinder set if it belongs to \( F_\Lambda \) for some finite set \( \Lambda \subset \mathbb{L} \).

Let \( F \) be the smallest \( \sigma \)-algebra on \( \Omega \) containing the cylinder sets.

Write \( T \) for the tail-\( \sigma \)-algebra, i.e., intersection of \( T_\Lambda \) over all finite subsets \( \Lambda \) of \( L \) the sets in \( T \) are called tail-measurable sets.

Assume that we are given a family of measurable potential functions \( \Phi_\Lambda : \Omega \to \mathbb{R} \cup \{\infty\} \) (one for each finite subset \( \Lambda \) of \( \mathbb{L} \)) each \( \Phi_\Lambda \) is \( F_\Lambda \) measurable.

For each finite subset \( \Lambda \) of \( \mathbb{L} \) define a Hamiltonian:

\[ H_\Lambda(\sigma) = \sum_{\substack{S \subset \mathbb{L} \setminus \Lambda \neq \emptyset \atop S \cap \Lambda \neq \emptyset}} \Phi_S(\sigma), \]

where the sum is taken over finite subsets \( S \).
Gibbs Measures. To define Gibbs measures and gradient Gibbs measures, we will need some additional notation [7], [13].

Let \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\) be general measure spaces.

A function \(\pi : \mathcal{X} \times Y \to [0, \infty]\) is called a probability kernel from \((Y, \mathcal{Y})\) to \((X, \mathcal{X})\) if

1. \(\pi(\cdot | y)\) is a probability measure on \((X, \mathcal{X})\) for each fixed \(y \in Y\), and
2. \(\pi(A | \cdot)\) is \(\mathcal{Y}\)-measurable for each fixed \(A \in \mathcal{X}\).

Such a kernel maps each measure \(\mu\) on \((Y, \mathcal{Y})\) to a measure \(\mu \pi\) on \((X, \mathcal{X})\) by

\[
\mu \pi (A) = \int \pi(A | \cdot) d\mu.
\]

The following is a probability kernel from \((\Omega, T_\Lambda)\) to \((\Omega, \mathcal{F})\):

\[
\gamma_\Lambda(A, \omega) = Z_\Lambda(\omega)^{-1} \int \exp(-H_\Lambda(\sigma_\Lambda \omega_\Lambda))(1_A(\sigma_\Lambda \omega_\Lambda)) \nu^{\otimes \Lambda}(d\sigma_\Lambda),
\]

where \(\nu = \{\nu(i) > 0, i \in E\}\) is a counting measure.

A configuration \(\sigma\) has finite energy if \(\Phi_\Lambda(\sigma) < \infty\) for all finite \(\Lambda\). Moreover, \(\sigma\) is \(\Phi\)-admissible if each \(Z_\Lambda(\sigma)\) is finite and non-zero.

Given a measure \(\mu\) on \((\Omega, \mathcal{F})\), define a new measure \(\mu \gamma_\Lambda\) by

\[
\mu \gamma_\Lambda(A) = \int \gamma_\Lambda(A, \cdot) d\mu.
\]

**Definition 1.** A probability measure \(\mu\) on \((\Omega, \mathcal{F})\) is called a Gibbs measure if \(\mu\) is supported on the set of \(\Phi\)-admissible configurations in \(\Omega\) and for all finite subset \(\Lambda\) we have

\[
\mu \gamma_\Lambda = \mu.
\]

**Gradient Gibbs measure.** For any configuration \(\omega = (\omega(x))_{x \in L} \in E^L\) and edge \(b = (x, y)\) of \(L\), the difference along the edge \(b\) is given by \(\nabla \omega_b = \omega_y - \omega_x\) and \(\nabla \omega\) is called the gradient field of \(\omega\).

The gradient spin variables are now defined by \(\eta_{(x,y)} = \omega_y - \omega_x\) for each \((x, y)\).

The space of gradient configurations denoted by \(\Omega^\nabla\). The measurable structure on the space \(\Omega^\nabla\) is given by \(\sigma\)-algebra

\[
\mathcal{F}^\nabla := \sigma(\{\eta_b \mid b \in L\})
\]

Note that \(\mathcal{F}^\nabla\) is the subset of \(\mathcal{F}\) containing those sets that are invariant under translations \(\omega \to \omega + c\) for \(c \in E\).

Similarly, we define

\[
\mathcal{T}_\Lambda^\nabla = \mathcal{T}_\Lambda \cap \mathcal{F}^\nabla, \quad \mathcal{F}_\Lambda^\nabla = \mathcal{F}_\Lambda \cap \mathcal{F}^\nabla.
\]

Let \(\Phi\) be a translation invariant gradient potential. Since, given any \(\Lambda \in \mathcal{F}^\nabla\), the kernels \(\gamma_\Lambda^\Phi(A, \omega)\) are \(\mathcal{F}^\nabla\)-measurable functions of \(\omega\), it follows that the kernel sends a given measure \(\mu\) on \((\Omega, \mathcal{F}^\nabla)\) to another measure \(\mu \gamma_\Lambda^\Phi\) on \((\Omega, \mathcal{F}^\nabla)\).

**Definition 2.** A measure \(\mu\) on \((\Omega, \mathcal{F}^\nabla)\) is called a gradient Gibbs measure if for all finite subset \(\Lambda\) we have

\[
\mu \gamma_\Lambda^\Phi = \mu.
\]

Note that, if \(\mu\) is a Gibbs measure on \((\Omega, \mathcal{F})\), then its restriction to \(\mathcal{F}^\nabla\) is a gradient Gibbs measure.

A gradient Gibbs measure is said to be localized or smooth if it arises as the restriction of a Gibbs measure in this way. Otherwise, it is non-localized or rough.

It is known [7], [5, Theorem 8.19.] that many natural Gibbs measures on \(\mathbb{Z}^d\) are rough when \(d \in \{1, 2\}\).

**Construction of gradient Gibbs measure on Cayley trees.** Following [11] we consider models where spin-configuration \(\omega\) is a function from the vertices of the Cayley tree \(\Gamma^k = (V, \vec{L})\) to the set \(E = \mathbb{Z}\), where \(V\) is the set of vertices and \(\vec{L}\) is the set of
oriented edges (bonds) of the tree (see Chapter 1 of [13] for properties of the Cayley tree).

For nearest-neighbor (n.n.) interaction potential $\Phi = (\Phi_b)_b$, where $b = \langle x, y \rangle$ is an edge, define symmetric transfer matrices $Q_b$ by

$$Q_b(\omega_b) = e^{-\left(\Phi_b(\omega_b) + |\partial x|^{-1}\Phi_x(\omega_x) + |\partial y|^{-1}\Phi_y(\omega_y)\right)},$$

where $\partial x$ is the set of all nearest-neighbors of $x$ and $|S|$ denotes the number of elements of the set $S$.

Define the Markov (Gibbsian) specification as

$$\gamma^\Phi_{\Lambda}(\sigma_\Lambda = \omega_\Lambda|\omega) = (Z^\Phi_{\Lambda})(\omega)^{-1} \prod_{b \in \Lambda \neq \emptyset} Q_b(\omega_b).$$

If for any bond $b = \langle x, y \rangle$ the transfer operator $Q_b(\omega_b)$ is a function of gradient spin variable $\zeta_b = \omega_y - \omega_x$ then the underlying potential $\Phi$ is called a gradient interaction potential.

Boundary laws (see [15]) which allow to describe the set $\mathcal{G}(\gamma)$ of all Gibbs measures (that are Markov chains on trees).

**Definition 3.** A family of vectors $\{l_{xy}\}_{(x,y) \in L}$ with $l_{xy} = (l_{xy}(i) : i \in \mathbb{Z}) \in (0, \infty)^{\mathbb{Z}}$ is called a boundary law for the transfer operators $\{Q_b\}_{b \in L}$ if for each $\langle x, y \rangle \in \tilde{L}$ there exists a constant $c_{xy} > 0$ such that the consistency equation

$$l_{xy}(i) = c_{xy} \prod_{z \in \partial x \setminus \{y\}} \sum_{j \in \mathbb{Z}} Q_{xz}(i, j) l_{xz}(j)$$

holds for every $i \in \mathbb{Z}$.

A boundary law is called $q$-periodic if $l_{xy}(i + q) = l_{xy}(i)$ for every oriented edge $\langle x, y \rangle \in \tilde{L}$ and each $i \in \mathbb{Z}$.

It is known that there is a one-to-one correspondence between boundary laws and tree-indexed Markov chains if the boundary laws are normalisable in the sense of Zachary [15]:

**Definition 4.** A boundary law $l$ is said to be normalisable if and only if

$$\sum_{i \in \mathbb{Z}} \left( \prod_{z \in \partial x \setminus \{j\}} \sum_{j \in \mathbb{Z}} Q_{xz}(i, j) l_{xz}(j) \right) < \infty$$

at any $x \in V$.

For any $\Lambda \subset V$ we define its outer boundary as

$$\partial \Lambda := \{ x \notin \Lambda : \langle x, y \rangle \text{ for some } y \in \Lambda \}.$$ 

The correspondence now reads the following:

**Theorem 1.** [13] For any Markov specification $\gamma$ with associated family of transfer matrices $(Q_b)_{b \in L}$ we have

1. Each normalisable boundary law $(l_{xy})_{x,y}$ for $(Q_b)_{b \in L}$ defines a unique tree-indexed Markov chain $\mu \in \mathcal{G}(\gamma)$ via the equation given for any connected set $\Lambda \subset V$

$$\mu(\sigma_{\Lambda \cup \partial \Lambda} = \omega_{\Lambda \cup \partial \Lambda}) = (Z_\Lambda)^{-1} \prod_{y \in \partial \Lambda} l_{y\Lambda}(\omega_y) \prod_{b \in \Lambda \neq \emptyset} Q_b(\omega_b),$$

where for any $y \in \partial \Lambda$, $y_\Lambda$ denotes the unique n.n. of $y$ in $\Lambda$.

2. Conversely, every tree-indexed Markov chain $\mu \in \mathcal{G}(\gamma)$ admits a representation of the form [15] in terms of a normalisable boundary law (unique up to a constant positive factor).
The Markov chain $\mu$ defined in (1.4) has the transition probabilities

$$P_{xy}(i,j) = \mu(\sigma_y = j \mid \sigma_x = i) = \frac{l_{yx}(j)Q_{yx}(j,i)}{\sum_{s} l_{yx}(s)Q_{yx}(s,i)}.$$  

(1.5)

The expressions (1.5) may exist even in situations where the underlying boundary law $(l_{xy})_{x,y}$ is not normalisable. However, the Markov chain given by (1.5), in general, does not have an invariant probability measure. Therefore in [8], [9], [10], [11] some non-normalisable boundary laws are used to give gradient Gibbs measures.

Now we give some results of above mentioned papers. Consider a model on Cayley tree $\Gamma^k = (V, L)$, where the spin takes values in the set of all integer numbers $\mathbb{Z}$. The set of all configurations is $\Omega := \mathbb{Z}^V$.

For $\Lambda \subset V$, fix a site $w \in \Lambda$. If the boundary law $l$ is assumed to be $q$-periodic, then take $s \in \mathbb{Z}_q = \{0, 1, \ldots, q - 1\}$ and define probability measure $\nu_{w,s}$ on $\mathbb{Z}^{\{b \in L \mid b \subset \Lambda\}}$ by

$$\nu_{w,s}(\eta_{\Lambda \cup \partial\Lambda} = \zeta_{\Lambda \cup \partial\Lambda}) =$$

$$Z_{w,s}^\Lambda \prod_{y \in \partial\Lambda} l_{y^w}(T_q(s + \sum_{b \in \Gamma(w,y)} \zeta_b)) \prod_{b \cap \Lambda \neq \emptyset} Q_b(\zeta_b),$$

where $Z_{w,s}^\Lambda$ is a normalization constant, $\Gamma(w,y)$ is the unique path from $w$ to $y$ and $T_q : \mathbb{Z} \to \mathbb{Z}_q$ denotes the coset projection.

**Theorem 2.** [11] Let $(l_{xy})_{x,y \in L}$ be any $q$-periodic boundary law to some gradient interaction potential. Fix any site $w \in V$ and any class label $s \in \mathbb{Z}_q$. Then

$$\nu_{w,s}(\eta_{\Lambda \cup \partial\Lambda} = \zeta_{\Lambda \cup \partial\Lambda}) =$$

$$Z_{w,s}^\Lambda \prod_{y \in \partial\Lambda} l_{y^w}(T_q(s + \sum_{b \in \Gamma(w,y)} \zeta_b)) \prod_{b \cap \Lambda \neq \emptyset} Q_b(\zeta_b),$$

(1.6)

gives a consistent family of probability measures on the gradient space $\Omega^\Lambda$. Here $\Lambda$ with $w \in \Lambda \subset V$ is any finite connected set, $\zeta_{\Lambda \cup \partial\Lambda} \in \mathbb{Z}^{\{b \in L \mid b \subset (\Lambda \cup \partial\Lambda)\}}$ and $Z_{w,s}^\Lambda$ is a normalization constant.

The measures $\nu_{w,s}$ will be called pinned gradient measures.

If $q$-periodic boundary law and the underlying potential are translation invariant then it is possible to obtain probability measure $\nu$ on the gradient space by mixing the pinned gradient measures:

**Theorem 3.** [11] Let a $q$-periodic boundary law $l$ and its gradient interaction potential are translation invariant. Let $\Lambda \subset V$ be any finite connected set and let $w \in \Lambda$ be any vertex. Then the measure $\nu$ with marginals given by

$$\nu(\eta_{\Lambda \cup \partial\Lambda} = \zeta_{\Lambda \cup \partial\Lambda}) = Z_{\Lambda} \left(\sum_{s \in \mathbb{Z}_q} \prod_{y \in \partial\Lambda} l(s + \sum_{b \in \Gamma(w,y)} \zeta_b) \prod_{b \cap \Lambda \neq \emptyset} Q(\zeta_b),\right)$$

(1.7)

where $Z_{\Lambda}$ is a normalisation constant, defines a translation invariant gradient Gibbs measure on $\Omega^\Lambda$.

**SOS model.** The (formal) Hamiltonian of the SOS model is

$$H(\omega) = -J \sum_{(x,y) \in L} |\omega_x - \omega_y|, \ \omega \in \Omega,$$

(1.8)

where $J \in \mathbb{R}^+$ is a constant.

In [8], using Theorem 3 some gradient Gibbs measures are found.

Let $\beta > 0$ be inverse temperature and $\theta := \exp(-J\beta) < 1$. The transfer operator $Q$ then reads $Q(i - j) = \theta^{|i-j|}$ for any $i, j \in \mathbb{Z}$, and a translation invariant boundary law,
denoted by \( z \), is any positive function on \( \mathbb{Z} \) solving the consistency equation, whose values we will denote by \( z_i \) instead of \( z(i) \). By definition of the boundary law it is only unique up to multiplication with any positive prefactor. Hence we may choose this constant in a way such that we have \( z_0 = 1 \).

Set \( Z_0 := \mathbb{Z} \setminus \{0\} \). Then the boundary law equation (for translation-invariant case, i.e. \( l_b \equiv l \), for all \( b \in L \)) reads

\[
z_i = \left( \theta^{|i|} + \frac{\sum_{j \in Z_0} \theta^{|i-j|} z_j}{1 + \sum_{j \in Z_0} \theta^{|j|} z_j} \right)^k, \quad i \in Z_0.
\] (1.9)

Let \( z(\theta) = (z_i = z_i(\theta), i \in Z_0) \) be a solution to (1.9).

Proposition 1. [8] If \( z_0 = 1 \) (i.e. \( u_0 = 1 \)) then the equation (1.9) is equivalent to the following

\[
u_i^k = \frac{u_{i-1} + u_{i+1} - \tau u_i}{u_{i-1} + u_{i+1} - \tau}, \quad i \in \mathbb{Z},
\] (1.10)

where \( \tau = \theta^{-1} + \theta \).

In general, solutions of (1.10) are not known. But in class of periodic solutions, some results are obtained. The following theorem is proved for \( k = 2 \) and 4-periodic boundary laws:

Theorem 4. [8] For the SOS model (1.8) on the binary tree (i.e. \( k = 2 \)) with parameter \( \tau = \theta + \theta^{-1} \) the following assertions hold

1. If \( \tau \leq 4 \) then there is precisely one GGM associated to a 4-periodic boundary law.
2. If \( 4 < \tau \leq 6 \) then there are precisely two GGMs.
3. If \( 6 < \tau < 2 + 2\sqrt{5} \) then there are precisely three GGMs.
4. If \( \tau \geq 2 + 2\sqrt{5} \) then there are precisely four such measures.

The following theorem is proved for any \( k \geq 2 \) and 3-periodic boundary laws.

Denote \( \tau_0 := \frac{2k + 1}{k - 1} \).

Theorem 5. [8] For the SOS-model on the \( k \)-regular tree, \( k \geq 2 \), with parameter \( \tau \) there is \( \tau_c \) such that \( 0 < \tau_c < \tau_0 \) and the following holds:

1. If \( \tau < \tau_c \) then there is no any GGM corresponding to a nontrivial 3-periodic boundary.
2. At \( \tau = \tau_c \) there is a unique GGM corresponding to a nontrivial 3-periodic boundary law.
3. For \( \tau > \tau_c, \tau \neq \tau_0 \) (resp. \( \tau = \tau_0 \)) there are exactly two such (resp. one) GGMs.

The GGMs described above are all different from the GGMs mentioned in Theorem 4.

General case. Assume that the transfer operator \( \{Q_b\}_{b \in L} \) defined in (1.11), is summable, i.e.

\[
\sum_{i \in \mathbb{Z}} Q_b(i) < \infty \quad \text{for all} \quad b \in L.
\]

The following is the main result of [9]:

Theorem 6. For any summable \( Q \) and any degree \( k \geq 2 \) there is a finite period \( q_0(k) \) such that for all \( q \geq q_0(k) \) there are GGMs of period \( q \) which are not translation invariant.

Moreover, in [10] the authors provided general conditions in terms of the relevant \( p \)-norms of the associated transfer operator \( Q \) which ensure the existence of a countable family of proper Gibbs measures. The existence of delocalized GGMs is proved, under
natural conditions on \( Q \). This implies coexistence of both types of measures for large classes of models including the SOS-model, and heavy-tailed models arising for instance for potentials of logarithmic growth.

2. 4-PERIODIC BOUNDARY LAWS FOR \( k \geq 2 \)

In this section our goal is to find solutions of (1.10) which have the form

\[
\begin{align*}
    u_n &= \begin{cases} 
        1, & \text{if } n = 2m, \\
        a, & \text{if } n = 4m - 1, \quad m \in \mathbb{Z} \\
        b, & \text{if } n = 4m + 1,
    \end{cases}
\end{align*}
\]

(2.1)

where \( a \) and \( b \) some positive numbers.

Then from (1.10) for \( a \) and \( b \) we get the following system of equations

\[
\begin{align*}
    (a + b - \tau)b^k + \tau b - 2 &= 0 \\
    (a + b - \tau)a^k + \tau a - 2 &= 0.
\end{align*}
\]

(2.2)

The case \( k = 2 \) is fully analyzed in \([8]\) and the following is proved

Proposition 2. For \( k = 2 \) the periodic solutions of the form (2.1) (i.e. solutions of the system (2.2)) depend on the parameter \( \tau = 2 \cosh(\beta) \) in the following way.

1. If \( \tau \leq 4 \) then there is a unique solution.
2. If \( 4 < \tau \leq 6 \) then there are exactly two solutions.
3. If \( 6 < \tau < 2 + 2\sqrt{5} \) then there are exactly four solutions.
4. If \( \tau \geq 2 + 2\sqrt{5} \) then there are exactly five solutions.

where explicit formula of each solution is found.

Now we reduce the system (2.2) to a polynomial equation with one unknown \( a \). To do this from the first (resp. second) equation of (2.2) find \( a \) (resp. \( b \)):

\[
\begin{align*}
    a &= f(b) := \tau - b + (2 - \tau b)b^{-k} \\
    b &= f(a).
\end{align*}
\]

(2.3)

Thus the system (2.2) is reduced to

\[
    a = f(f(a)).
\]

(2.4)

Note that solutions of \( a = f(a) \) are solutions to (2.4) too. It is easy to see that \( a = f(a) \) is equivalent to

\[
    Q(a) := 2a^{k+1} - \tau a^k + \tau a - 2 = 0
\]

(2.5)

The equation (2.5) has the solution \( a = 1 \) independently of the parameters \( (\tau, k) \). Dividing both sides of (2.5) by \( a - 1 \) we get

\[
2a^k + (2 - \tau)(a^{k-1} + a^{k-2} + \cdots + a) + 2 = 0.
\]

(2.6)

The following lemma gives the number of solutions to equation (2.6) (compare with Lemma 4.7 in \([8]\)):

Lemma 1. For each \( k \geq 2 \), there is exactly one critical value of \( \tau \), i.e., \( \tau_c = \tau_c(k) := 2 \cdot \frac{k+1}{k+1} \), such that

1. if \( \tau < \tau_c \) then (2.6) has no positive solution;
2. if \( \tau = \tau_c \) then the equation has a unique solution \( a = 1 \);
3. if \( \tau > \tau_c \), then it has exactly two solutions (both different from 1);
Proof. From (2.6) we get
\[ \tau = \psi_k(a) := 2 + \frac{2(a^k + 1)}{a^{k-1} + a^{k-2} + \ldots + a}. \]
We have \( \psi_k(a) > 2, a > 0 \) and \( \psi_k'(a) = 0 \) is equivalent to
\[ \sum_{j=1}^{k-1} (k - j)a^{k-j-1} - \sum_{j=1}^{k-1} ja^{j-1} = 0. \] (2.7)
The last polynomial equation has exactly one positive solution, because signs of its coefficients changed only one time, and at \( a = 0 \) it is negative, i.e. -1 and at \( a = +\infty \) it is positive. Moreover, this unique solution is \( a = 1 \), because putting \( a = 1 \) in (2.7) we get
\[ \sum_{j=1}^{k-1} (k - j) - \sum_{j=1}^{k-1} j = \sum_{j=1}^{k-1} k - 2 \sum_{j=1}^{k-1} j = k(k - 1) - 2 \cdot \frac{k(k - 1)}{2} = 0. \]
Thus \( \psi_k(a) \) has unique minimum at \( a = 1 \), and \( \lim_{a \to 0} \psi_k(a) = \lim_{a \to +\infty} \psi_k(a) = +\infty \). Consequently,
\[ \tau_c = \tau_c(k) = \min_{a > 0} \psi_k(a) = \psi_k(1) = 2 \cdot \frac{k + 1}{k - 1}. \]
These properties of \( \psi_k(a) \) completes the proof. \( \square \)

Remark 1. The equation (2.6) was also considered in [3]. Lemma 3 improves their result (see Theorem 5.2 of [3]), because we found explicit formula for the critical value \( \tau_c \).

Now we want to find solutions of (2.3) which are different from solutions of \( a = f(a) \) (i.e., \( Q(a) = 0 \)).

By simple calculations the equation (2.3) can be rewritten as
\[ P(a) := (2 - \tau a)[2 - \tau a + a^k - a^{k+1}]^k - a^k \tau a^{k+1} + (2 - \tau^2)a^k + \tau^2 a - 2\tau = 0. \] (2.8)
Recall that \( Q(a) \) divides \( P(a) \). Now we shall find \( \frac{P(a)}{Q(a)} \). It is easy to see that \( P(a) \) can be written as
\[ P(a) = (2 - \tau a)[a^{k+1} - Q(a)]^k - a^k \tau a^{k+1} + a^k \tau k Q(a) \]
\[ = (2 - \tau a) \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} a^{j+1} a^{k-j} - a^k \tau a^{k+1} + a^k \tau k Q(a) \]
\[ = (2 - \tau a) \sum_{j=0}^{k-1} (-1)^{j} \binom{k}{j} a^{j+1} Q^{k-j}(a) + (2 - \tau a)a^k - a^{k+1} (2 - \tau a) - a^k \tau a^k Q(a) \]
\[ = (2 - \tau a) \left( \sum_{j=0}^{k-1} (-1)^{j} \binom{k}{j} a^{j+1} Q^{k-j}(a) + a^k (2 - \tau a) - a^k \tau a^k Q(a) \right) \]
\[ = Q(a) \left( (2 - \tau a) \sum_{j=0}^{k-1} (-1)^{j} \binom{k}{j} a^{j+1} Q^{k-j}(a) + a^k (2 - \tau a) - a^k \tau a^k Q(a) \right) \].
Consequently, the equation (2.3) in case \( Q(a) \neq 0 \) is reduced to
\[ U(a) := \tau a^k - (2 - \tau a) \sum_{j=0}^{k-1} (-1)^{j} \binom{k}{j} a^{j+1} Q^{k-j-1}(a) = 0. \] (2.9)
But \( U(a) \) may have some roots coinciding with roots of \( Q(a) \). This is result of the following lemma.
Lemma 2. Let \( a = \hat{a} \) be a root of \( Q(a) \). Then \( \hat{a} \) is a root of \( U(a) \) iff \( \hat{a} = \frac{2k}{\tau(k-1)} \) and \( \tau \) satisfies
\[
(k-1)^{k}r^{k+1} - (k-1)2^{k-1}k^{k-2} + (2k)^{k+1} = 0.
\]
(2.10)

Proof. For the root \( \hat{a} \) we have \( Q(\hat{a}) = 0 \). Therefore from (2.9), i.e. \( U(\hat{a}) = 0 \), we get
\[
\tau\hat{a}^2 + (2 - \tau\hat{a})\hat{a}^{k-1} = 0 \iff \tau\hat{a} + (2 - \tau\hat{a})k = 0 \iff \hat{a} = \frac{2k}{\tau(k-1)}.
\]
Then \( Q(\hat{a}) = Q\left(\frac{2k}{\tau(k-1)}\right) = 0 \) gives (2.10).

\[\square\]

Lemma 3. For each fixed \( k \geq 2 \) the equation (2.10) has exactly two solutions \( \tau_1(k) = \frac{2k}{k-1} \) and \( \tau_2(k) > \frac{2(k+1)}{k-1} \).

Proof. It is easy to see that \( \tau = \frac{2k}{k-1} \) is a solution to (2.10), for any \( k \geq 2 \). Moreover, the corresponding \( \hat{a} \) is 1.

Denote \( L = (k-1)\tau - 2k \) then from (2.10) we get
\[
0 = (L + 2k)^{k+1} - 2^{k-1}k^{k}(L + 2k)^2 + (k-1)(2k)^{k+1}
\]
\[
= \sum_{j=0}^{k+1} \binom{k+1}{j}L^{k+1-j}(2k)^{j} - 2^{k-1}k^{k}L^2 - (2k)^{k+1}L - 2^{k+1}k^{k+2} + (k-1)(2k)^{k+1}
\]
\[
= \sum_{j=0}^{k} \binom{k+1}{j}L^{k+1-j}(2k)^{j} + (2k)^{k+1} - 2^{k-1}k^{k}L^2 - (2k)^{k+1}L - 2^{k+1}k^{k+2} + (k-1)(2k)^{k+1}
\]
\[
= L \left( \sum_{j=0}^{k} \binom{k+1}{j}L^{k-j}(2k)^{j} - 2^{k-1}k^{k}L - (2k)^{k+1} \right)
\]
\[
= L \left( \sum_{j=0}^{k} \binom{k+1}{j}L^{k-j}(2k)^{j} + \frac{k(k-1)}{2}L - (k-1)(2k)^{k} \right).
\]
Thus \( L = 0 \) (i.e. \( \tau = \tau_1(k) \)) or
\[
\sum_{j=0}^{k-2} \binom{k+1}{j}L^{k-j}(2k)^{j} + \frac{k(k-1)}{2}L - (k-1)(2k)^{k} = 0.
\]
(2.11)

By Lemma 1 we know that \( Q(a) = 0 \) has a solution different from 1 iff \( \tau > \tau_c = \frac{2(k+1)}{k-1} \).

Therefore \( L = (k-1)\tau - 2k > (k-1) \cdot \frac{2(k+1)}{k-1} - 2k = 2 > 0 \). The polynomial equation (2.11) has exactly one positive solution, denoted by \( L^* \), because signs of its coefficients changed only one time. Moreover, we have \( L^* > 2 \), because at \( L = 2 \) the LHS of (2.11) is negative:
\[
2^k[(k+1)^{k+1} - (3k+1)k^k] < 0
\]
and at \( L = +\infty \) it is positive. Thus \( \tau_2(k) = \frac{L^*+2k}{k-1} \).

\[\square\]

Example 1. For \( k = 2, 3, 4 \) we have
\[
\tau_1(2) = 4, \quad \tau_2(2) = 2 + 2\sqrt{5}.
\]
\[
\tau_1(3) = 3, \quad \tau_2(3) = 3\sqrt{2}.
\]
\[
\tau_1(4) = 8/3, \quad \tau_2(4) \approx 3.497.
\]
Remark 2. It seems impossible to solve equation (2.9) for each \( k \geq 2 \). But one can use numerical methods to give some its solutions for concrete values of parameters. Therefore, one can try to solve it for small values of \( k \). In [8] the case \( k = 2 \) is fully analyzed. Below we shall consider the case \( k = 3 \). Cases \( k \geq 4 \) remains open.

Case \( k = 3 \). In this case the equation (2.9) has the form

\[
g(a, \tau) := (\tau^2 + 2)a^8 - (\tau^2 + 2)\tau a^7 + \tau^2 a^6 + 2(\tau^2 + 2)\tau a^5 \\
- 4(\tau^2 + 1)a^4 - (\tau^2 - 8)\tau a^3 + 6\tau^2 a^2 - 12\tau a + 8 = 0.
\]

It is well known (see [12], p.28) that the number of positive roots of the polynomial \((2.12)\) does not exceed the number of sign changes of its coefficients. Since \( \tau > 2 \), the number of positive roots of the polynomial \((2.12)\) is at most 6. Numerical analysis shows that for some values of \( \tau \) there are exactly 6 solutions (see Fig.1). Indeed, rewrite \((2.12)\) as

\[
a = Y(a, \tau) := \frac{(\tau^2 + 2)\tau a^7 + 4(\tau^2 + 1)a^4 + (\tau^2 - 8)\tau a^3 + 13\tau a - 8}{(\tau^2 + 2)a^3 + \tau^2 a^2 + 2(\tau^2 + 2)\tau a^4 + 6\tau^2 a + \tau}.
\]

Note that, for fixed \( \tau > 2 \), the function \( Y(a, \tau) \) is continuous with respect to both arguments \( a > 0, \tau > 2 \) and is a bounded function. Moreover, \( Y(0, \tau) = -\frac{8}{\tau} \) and \( Y(+\infty, \tau) = \tau \).

Remark 3. For \( k = 3 \) one can explicitly find all positive solutions of \((2.3)\), i.e.

\[
a_1 = 1, \quad 0 < a_2 = \frac{1}{4}(\tau - \sqrt{\tau^2 - 16}) < 1, \quad 0 < a_3 = \frac{1}{4}(\tau + \sqrt{\tau^2 - 16}) > 1.
\]

Moreover, \( a_2 a_3 = 1 \).

In case \( a \neq b \), we do not know explicit solutions of \((2.12)\). Since \( b = f(a) \) may be negative for some positive solutions \( a \), we have to check positivity of \( b \) for each positive \( a \). To avoid this difficulty, in case \( k = 3 \) and \( a \neq b \), we solve \((2.2)\) as follows.

We rewrite the system of equations \((2.2)\) for the case \( k = 3 \).

\[
\begin{cases}
(a + b - \tau)a^3 + \tau a - 2 = 0; \\
(a + b - \tau)b^3 + \tau b - 2 = 0.
\end{cases}
\]
Lemma 4. For the system (2.13) there are critical values \( \tau_{cr}^{(1)} \approx 3.13039 \) and \( \tau_{cr}^{(2)} \approx 4.01009 \) of \( \tau \) such that the following assertions hold

1. If \( \tau \in [2, \tau_{cr}^{(1)}] \) then there is no any solution.
2. If \( \tau \in (\tau_{cr}^{(1)}, \tau_{cr}^{(2)}) \) then there is precisely one solution.
3. If \( \tau \in (\tau_{cr}^{(2)}, \infty) \) then there are precisely two solution to (2.13).

Proof. We add the first and second equations of (2.13), i.e.,

\[
(a + b - \tau)(a^3 + b^3) + \tau(a + b) - 4 = 0.
\]

If \( a + b - \tau = 0 \) then there is one solution to (2.13), i.e., \( a = b = \frac{2}{\tau} \). We have to find \( (a, b), a \neq b \) solutions, so we can suppose \( a + b - \tau \neq 0 \). Consequently,

\[
a^3 + b^3 = \frac{4 - \tau(a + b)}{a + b - \tau}.
\]

From the last equality, one gets

\[
3ab = (a + b)^2 + \frac{\tau(a + b) - 4}{(a + b)(a + b - \tau)}. \tag{2.14}
\]

Now, we subtract the second equation of (2.13) from the first one. Then

\[
(a + b - \tau)(a^3 - b^3) + \tau(a - b) = 0.
\]

Since \( a \neq b \), both sides can be divided by \( a - b \) and we obtain the following

\[
ab = (a + b)^2 + \frac{\tau}{a + b - \tau}. \tag{2.15}
\]

Let \( a + b = x \). Then by (2.14) and (2.15), we have

\[
2x^2 + \frac{3\tau}{x - \tau} = \frac{\tau x - 4}{x(x - \tau)}.
\]

The last equation can be written as

\[
x^4 - \tau x^3 + \tau x + 2 = 0. \tag{2.16}
\]

By Ferrari’s method for solving a quartic equation, the equation (2.16) can be written as

\[
\left(x^2 - \frac{\tau}{2} x + \frac{c(\tau)}{2}\right)^2 - \left[\left(\frac{\tau^2}{4} + c(\tau)\right)x^2 - \left(\frac{\tau c(\tau)}{2} + \tau\right)x + \left(\frac{c^2(\tau)}{4} - 2\right)\right] = 0, \tag{2.17}
\]

where \( c(\tau) \) is a real root of the following polynomial

\[
P(z) := z^3 - (\tau^2 + 8)z - 3\tau^2 = 0.
\]

Denote

\[
F(\tau) = -\frac{(\tau^2 + 8)}{3}, \quad R(\tau) = \frac{3\tau^2}{2}.
\]

To find a certain view of \( c(\tau) \), we shall find a real root of \( P(z) \). By Cardano’s formula, roots of \( P(z) \) are:

\[
z_1(\tau) = S(\tau) + T(\tau),
\]

\[
z_2(\tau) = -\frac{S(\tau) + T(\tau)}{2} + i\sqrt{3} \frac{1}{2}(S(\tau) - T(\tau)),
\]

\[
z_3(\tau) = -\frac{S(\tau) + T(\tau)}{2} - i\sqrt{3} \frac{1}{2}(S(\tau) - T(\tau)),
\]

where

\[
S(\tau) = \sqrt[3]{R(\tau) + \sqrt{F^3(\tau) + R^2(\tau)}}, \quad T(\tau) = \sqrt[3]{R(\tau) - \sqrt{F^3(\tau) + R^2(\tau)}}.
\]

It’s known that the expression \( D(\tau) = F^3(\tau) + R^2(\tau) \) is called the discriminant of the equation.
If $D(\tau) > 0$, then one root is real and two are complex conjugates.
If $D(\tau) = 0$, then all roots are real, and at least two are equal.
If $D(\tau) < 0$, then all roots are real and unequal.

We have
\[ D(\tau) = F^3(\tau) + R^2(\tau) = -\frac{1}{27} \tau^6 + \frac{49}{36} \tau^4 - \frac{64}{9} \tau^2 - \frac{512}{27}. \]

Note that there are two positive solutions of $D(\tau)$ on $[2, +\infty)$ i.e., $\tau_1 \approx 2.994$ and $\tau_2 \approx 5.45$ and $D(\tau) < 0$ for $\tau \in [2, \tau_1) \cup (\tau_2, +\infty)$ and $D(\tau) > 0$ for $\tau \in (\tau_1, \tau_2)$. For all cases, $S(\tau) + T(\tau)$ is a real solution and we can choose $c(\tau)$ as $S(\tau) + T(\tau)$. In addition, $c(\tau) = S(\tau) + T(\tau) > 0$. The equation (2.17) can be written as
\[
\left( x^2 + \left( \sqrt{\frac{\tau^2}{4} + c(\tau) - \frac{\tau}{2}} \right) x + \frac{c(\tau)}{2} - \sqrt{\frac{c^2(\tau)}{4} - 2} \right) \times \left( x^2 - \left( \sqrt{\frac{\tau^2}{4} + c(\tau) + \frac{\tau}{2}} \right) x + \frac{c(\tau)}{2} + \sqrt{\frac{c^2(\tau)}{4} - 2} \right) = 0.
\]

It’s easy to check that
\[
x^2 + \left( \sqrt{\frac{\tau^2}{4} + c(\tau) - \frac{\tau}{2}} \right) x + \frac{c(\tau)}{2} - \sqrt{\frac{c^2(\tau)}{4} - 2} > 0, \quad x \in \mathbb{R}.
\]

Hence, from the last equality we obtain
\[
x_{1,2}(\tau) = \frac{1}{2} \left[ \sqrt{\frac{\tau^2}{2} + c(\tau) + \frac{\tau}{2}} \pm \sqrt{\left( \sqrt{\frac{\tau^2}{2} + c(\tau) + \frac{\tau}{2}} \right)^2 - 2 \left( c(\tau) + \sqrt{c^2(\tau) - 8} \right) \right].
\]

From the equation (2.15)
\[
ab = x_i^2(\tau) + \frac{\tau}{x_i^2(\tau) - \tau}, \quad i \in \{1, 2\}.
\]

Namely,
\[
a(x_i(\tau) - a) = x_i^2(\tau) + \frac{\tau}{x_i^2(\tau) - \tau}.
\]

Consequently,
\[
a_{1,2}^{(i)}(\tau) = \frac{x_i(\tau) \pm \sqrt{x_i^2(\tau) - 4 \left( x_i^2(\tau) + \frac{\tau}{x_i(\tau) - \tau} \right)}}{2}.
\]

From numerical analysis,
\[
x_i^2(\tau) - 4 \left( x_i^2(\tau) + \frac{\tau}{x_i(\tau) - \tau} \right) < 0, \quad \tau \in [\tau_1, +\infty).
\]

Also,
\[
x_2^2(\tau) - 4 \left( x_2^2(\tau) + \frac{\tau}{x_2(\tau) - \tau} \right) \geq 0, \quad \tau \in [\tau_{cr}^{(1)}, +\infty), \quad \tau_{cr}^{(1)} \approx 3.13039.
\]

Hence, we have only two cases $a \in \{a_1^{(2)}(\tau), a_2^{(2)}(\tau)\}$. When $a_1^{(2)}(\tau)$ (resp $a_2^{(2)}(\tau)$) belongs to the interval $(0, x_2(\tau))$ then $b_1^{(2)}(\tau)$ (resp. $b_2^{(2)}(\tau)$) is also positive. Again we use numerical analysis and obtain the following results: if $\tau \in [2, \tau_{cr}^{(1)}]$, then $a_2^{(2)}(\tau) > x_2(\tau)$, i.e., the equation (2.13) has no any positive solution such that $a \neq b$. Also, for all $\tau \in (\tau_{cr}^{(1)}, \tau_{cr}^{(2)})$ we have $a_2^{(2)}(\tau) < x_2(\tau)$, i.e., the equation (2.13) has exactly one positive solution with
Let \( \omega \in (\tau_{cr}^{(2)}, +\infty) \), then \( a_1^{(2)}(\tau) < x_2(\tau) \) and in this case the equation \((2.13)\) has exactly two positive solutions with \( a \neq b \), where \( \tau_{cr}^{(2)} \approx 4,01009 \). \(\square\)

Denote by \( N_k(\tau) \) the number of positive roots of \((2.13)\). Then (for \( k = 3 \)) by Lemma 1 (where \( \tau_c(3) = 4 \), Lemma 3, Lemma 4) we obtain the following formula

\[
N_3(\tau) = \begin{cases} 
1, & \text{if } \tau \in (2, \tau_{cr}^{(1)}] \\
2, & \text{if } \tau \in (\tau_{cr}^{(1)}, 4] \\
3, & \text{if } \tau \in (4, \tau_{cr}^{(2)}] \cup \{3\sqrt{2}\} \\
4, & \text{if } \tau \in (\tau_{cr}^{(2)}, +\infty) \setminus \{3\sqrt{2}\}, 
\end{cases}
\]

(2.18)

where \( \tau_{cr}^{(1)} \approx 3.13039, \tau_{cr}^{(2)} \approx 4,01009 \).

In \([8]\) some statements on identifiability of GGM with respect to the class of boundary laws are proven. In particular, for 4-periodic case the following is known.

**Lemma 5.** \([8]\) Consider any 4-periodic boundary law constructed by \( a, b \) given in \((2.1)\) and denote the associated GGM by \( \nu^{(a,b)} \). Let \((a_1, b_1), (a_2, b_2)\) be two such boundary laws. If \( \nu^{(a_1,b_1)} = \nu^{(a_2,b_2)} \) then necessarily

\[
(a_1 + b_1)a_2 + b_2 = 4.
\]

Based on formula \((2.18)\), Remark 3 and Lemma 5 we conclude the following

**Theorem 7.** For the SOS model \((1.3)\) on the Cayley tree of order \( k = 3 \) there are critical values \( \tau_{cr}^{(1)} \approx 3.13039, \tau_{cr}^{(2)} \approx 4,01009 \) such that the following assertions hold

1. If \( \tau \leq \tau_{cr}^{(1)} \) then there is precisely one GGM associated to a boundary law of the type \((2.1)\).
2. If \( \tau \in (\tau_{cr}^{(1)}, 4] \) then there are precisely two such GGMs.
3. If \( \tau \in (4, \tau_{cr}^{(2)}] \cup \{3\sqrt{2}\} \) then there are at most three such GGMs.
4. If \( \tau \in (\tau_{cr}^{(2)}, +\infty) \setminus \{3\sqrt{2}\} \) then there are at most four such measures associated to boundary laws of the type \((2.1)\).

3. SOS Model with an External Field

### 3.1. The boundary law equation in case of non-zero external field.

In this section for \( \sigma_n : x \in V_n \mapsto \sigma(x) \in \mathbb{Z} \), consider Hamiltonian of SOS model with external field \( \Phi : \mathbb{Z} \to \mathbb{R} \), i.e.,

\[
H(\sigma_n) = -J \sum_{(x,y) : x,y \in V_n} |\sigma(x) - \sigma(y)| + \sum_{x \in V_n} \Phi(\sigma(x)).
\]

Then the equation for translation-invariant boundary laws has the following form

\[
z_i = \frac{h(i)}{h(0)} \left( \frac{\theta^{|i|} + \sum_{j \in \mathbb{Z}_0} \theta^{|i-j|} z_j}{1 + \sum_{j \in \mathbb{Z}_0} \theta^{|i-j|} z_j} \right)^k, \quad i \in \mathbb{Z}_0.
\]

(3.1)

Note that this equation coincides with \((1.9)\) for \( h(i)/h(0) \equiv 1 \).

Let \( z(\theta) = (z_i = z_i(\theta) > 0, i \in \mathbb{Z}_0) \) be a solution to \((3.1)\). Denote

\[
l_i \equiv l_i(\theta) = -\sum_{j=-\infty}^{-1} \theta^{|i-j|} z_j, \quad r_i \equiv r_i(\theta) = \sum_{j=1}^{\infty} \theta^{|i-j|} z_j, \quad i \in \mathbb{Z}_0.
\]

(3.2)

It is clear that each \( l_i \) and \( r_i \) can be a finite positive number or \(+\infty\).
Lemma 6. For each $i \in \mathbb{Z}_0$ we have
  - $l_i < +\infty$ if and only if $l_0 < +\infty$;
  - $r_i < +\infty$ if and only if $r_0 < +\infty$.

Proposition 3. Assume $h(0) = 1$. A vector $z = (z_i, i \in \mathbb{Z})$, with $z_0 = 1$, is a solution to (3.1) if and only if for $s_i = k\sqrt{z_i h(i)}$ the following holds
  \[ h(i) s_i^k = \frac{s_{i-1} + s_{i+1} - \tau s_i}{s_i + s_{i-1} - \tau}, \quad i \in \mathbb{Z}, \]  \tag{3.3} \]
where $\tau = \theta^{-1} + \theta = 2 \cosh(\beta)$.

Proof. (cf. with the proof of Proposition 4.3 of [8]). Take some $C > 0$ and denote
  \[ v_i = C \cdot h^{1/k}(i) \left( \theta^{|i|} + \sum_{j \in \mathbb{Z}_0} \theta^{|i-j|} z_j \right), \quad i \in \mathbb{Z}. \]

Then from (3.1) we get $z_i = \left( \frac{v_i}{v_0} \right)^k$ and consequently,
  \[ \left( \frac{v_i}{v_0} \right)^k = \frac{h(i)}{h(0)} \left( \theta^{|i|} + \sum_{j \in \mathbb{Z}_0} \theta^{|i-j|} \left( \frac{v_j}{v_0} \right)^k \right)^k, \quad i \in \mathbb{Z}_0. \]  \tag{3.4} \]

From (3.4) we obtain
  \[ v_i = C \cdot \sqrt[k]{h(i)} \left( \sum_{j=1}^{+\infty} \theta^j v_{i-j}^k + \sum_{j=1}^{+\infty} \theta^j v_{i+j}^k \right), \quad i \in \mathbb{Z}. \]

By the last equality we get
  \[ \frac{v_i-1}{\sqrt[k]{h(i-1)}} + \frac{v_i+1}{\sqrt[k]{h(i+1)}} - \tau \cdot \frac{v_i}{\sqrt[k]{h(i)}} = C \cdot \left( \theta - \frac{1}{\theta} \right) v_i^k. \]

This equality by the notation $s_i = \sqrt[k]{\frac{s_i}{h(i)}}$ gives (3.3). Conversely, from (3.3) one gets (3.1). \hfill \square

3.2. A case of 4-periodic external field for $k = 2$. Here we shall find solutions of (3.3) for external field
  \[ h(i) = \begin{cases} 1, & \text{if } i = 2m, \\ h_1, & \text{if } i = 4m - 1, \quad m \in \mathbb{Z} \\ h_2, & \text{if } i = 4m + 1, \end{cases} \]  \tag{3.5} \]
where $h_1$ and $h_2$ are some positive numbers and a solution which has the form
  \[ u_n = \begin{cases} 1, & \text{if } n = 2m, \\ a, & \text{if } n = 4m - 1, \quad m \in \mathbb{Z} \\ b, & \text{if } n = 4m + 1, \end{cases} \]  \tag{3.6} \]
where $a$ and $b$ some positive numbers.

Then from (3.3) for $a$ and $b$ we get the following system of equations
  \[ (a + b - \tau) h_1 a^k + \tau b - 2 = 0 \]
  \[ (a + b - \tau) h_1 b^k + \tau a - 2 = 0. \]  \tag{3.7} \]
For simplicity we consider the case \( k = 2 \) and \( h_1 = h_2 = h \). In this case, subtracting from the first equation of the system (3.7) the second one we get

\[(b - a)[a - b] = 0.
\]

This gives three possible cases:

1. **Case \( a = b \)**. In this case from the first equation of (3.7) we get

\[2ha^3 - h\tau a^2 + \tau a - 2 = 0. \quad (3.9)\]

**Lemma 7.** Any real solution of (3.9) is positive.

**Proof.** Since \( h > 0 \), \( \tau > 2h \), if \( a \leq 0 \) then LHS of (3.9) is strictly negative. \( \square \)

Rewrite the cubic equation (3.9) as

\[a^3 + \alpha a^2 + \beta a + \gamma = 0\]

where \( \alpha = -\frac{\tau}{2}, \beta = \frac{\tau}{2h}, \gamma = -\frac{1}{h}. \)

Let

\[p = \beta - \frac{\alpha^2}{3} \quad \text{and} \quad q = \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma\]

Then the discriminant \( \Delta \) of the cubic equation is

\[\Delta = \frac{q^2}{4} + \frac{p^3}{27}. \]

**Lemma 8.** The following assertions hold

- **Case \( \Delta > 0 \)**. In this case there is only one positive real solution. It is

\[a_1 = \left(-\frac{q}{2} + \sqrt[3]{\Delta}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt[3]{\Delta}\right)^{\frac{1}{3}} + \frac{\tau}{6} \]

- **Case \( \Delta = 0 \)**. In this case there are two positive real solutions. These roots are

\[a_1 = -2\left(\frac{q}{2}\right)^{\frac{1}{3}} + \frac{\tau}{6} \quad \text{and} \quad a_2 = a_3 = \left(\frac{q}{2}\right)^{\frac{1}{3}} + \frac{\tau}{6}\]

- **Case \( \Delta < 0 \)**. In this case \(-p > 0\) and there are three positive real solutions:

\[a_1 = \left(\frac{-q}{2} + \sqrt{-p}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{2\sqrt{p}(\sqrt{-p})}{3}\right)\right)\right) + \frac{\tau}{6}\]

\[a_2 = \left(\frac{-q}{2} + \sqrt{-p}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{2\sqrt{p}(\sqrt{-p})}{3}\right)\right)\right) + \frac{\tau}{6}\]

\[a_3 = \left(\frac{-q}{2} + \sqrt{-p}\cos\left(\frac{1}{3}\sin^{-1}\left(\frac{2\sqrt{p}(\sqrt{-p})}{3}\right)\right)\right) + \frac{\tau}{6}\]

**Proof.** The conditions of existence of real solutions are well-known\(^1\). The positivity of each solution follows from Lemma 7. \( \square \)

**Case \( a + b = \frac{1}{2h}(h\tau + \sqrt{h\tau(h\tau - 4)}) \).** In this case from the second equation of (3.7) we get

\[(h\tau - \sqrt{h\tau(h\tau - 4)})a^2 - 2\tau a + 4 = 0.\]

Note that this equation has only positive real solutions. Moreover, one can see that if

\[(\tau, h) \in A := \left\{ (\tau, h) \in \mathbb{R}^2_+ : h \geq \frac{\tau^2}{4(\tau^2 - 8)}, \quad \text{if} \quad 2\sqrt{2} < \tau < 4 \right\} \]

\[\frac{\tau}{4}, \quad \text{if} \quad \tau \geq 4\]

\(^1\)https://en.wikipedia.org/wiki/Cubic_equation
then the quadratic equation has the following positive solutions

\[ a_4 = \frac{\tau - \sqrt{\tau^2 - 4h\tau + 4\sqrt{h\tau(h\tau - 4)}}}{h\tau - \sqrt{h\tau(h\tau - 4)}} \quad \text{and} \quad a_5 = \frac{\tau + \sqrt{\tau^2 - 4h\tau + 4\sqrt{h\tau(h\tau - 4)}}}{h\tau - \sqrt{h\tau(h\tau - 4)}}. \]

Using (3.8) we get \( b_4 = a_5 \) and \( b_5 = a_4 \).

Case \( a + b = \frac{1}{2\tau}(h\tau - \sqrt{h\tau(h\tau - 4)}) \). In this case we obtain

\[ (h\tau + \sqrt{h\tau(h\tau - 4)})a^2 - 2\tau a + 4 = 0 \]

which for

\[ (\tau, h) \in B := \left\{ (\tau, h) \in \mathbb{R}^+_\tau : \tau \geq 4, \frac{4}{\tau} \leq h \leq \frac{\tau^3}{8(\tau^2 - 4)} \right\} \]

has the following solutions

\[ a_6 = \frac{\tau - \sqrt{\tau^2 - 4\tau h - 4\sqrt{h\tau(h\tau - 4)}}}{h\tau + \sqrt{h\tau(h\tau - 4)}} \quad \text{and} \quad a_7 = \frac{\tau + \sqrt{\tau^2 - 4\tau h - 4\sqrt{h\tau(h\tau - 4)}}}{h\tau + \sqrt{h\tau(h\tau - 4)}}. \]

Using (3.8) we get \( b_6 = a_7 \) and \( b_7 = a_6 \). Clearly all of these solutions are positive.

Denote by \( \mu_i \) the gradient Gibbs measure corresponding to solution \( (a_i, b_i), i = 1, 2, \ldots, 7 \).

Thus depending on the values \( (\tau, h) \) related to \( \Delta \) and sets \( A, B \) we have the following result.

**Theorem 8.** For the SOS model with 4-periodic external field there are up to seven 4-periodic gradient Gibbs measures \( \mu_i, i = 1, \ldots, 7 \).

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