Integrated and Differentiated Sequence Spaces

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1 Introduction

The set of all sequences denotes with \( \omega := \mathbb{C}^N := \{ x = (x_k) : x : \mathbb{N} \rightarrow \mathbb{C}, k \rightarrow x_k := x(k) \} \) where \( \mathbb{C} \) denotes the complex field and \( \mathbb{N} \) is the set of positive integers. Each linear subspace of \( \omega \) (with the induced addition and scalar multiplication) is called a sequence space. The following subsets of \( \omega \) are obviously sequence spaces:

- \( \ell_\infty = \{ x = (x_k) \in \omega : \sup_k |x_k| < \infty \} \)
- \( c = \{ x = (x_k) \in \omega : \lim_k x_k \text{ exists} \} \)
- \( c_0 = \{ x = (x_k) \in \omega : \lim_k x_k = 0 \} \)
- \( bs = \{ x = (x_k) \in \omega : \sup_n \left| \sum_{k=1}^n x_k \right| < \infty \} \)
- \( cs = \{ x = (x_k) \in \omega : \left( \sum_{k=1}^n x_k \right) \in c \} \)
- \( \ell_p = \{ x = (x_k) \in \omega : \sum_k |x_k|^p < \infty, \quad 1 \leq p < \infty \} \).

These sequence spaces are Banach spaces with the norms; \( \| x \|_{\ell_\infty} = \sup_k |x_k| \), \( \| x \|_{c_0} = \| x \|_{c_s} = \sup_n \left| \sum_{k=1}^n x_k \right| \) and \( \| x \|_{\ell_p} = (\sum_k |x_k|^p)^{1/p} \) as usual, respectively. Let \( X \) is one of the above mentioned sequence spaces. The concept of integrated and differentiated sequence spaces was employed as

\[
\int X = \{ x = (x_k) \in \omega : (kx_k) \in X \} \quad \text{and} \quad d(X) = \{ x = (x_k) \in \omega : (k^{-1}x_k) \in X \},
\]

in [4].

A coordinate space (or \( K \)-space) is a vector space of numerical sequences, where addition and scalar multiplication

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are defined pointwise. That is, a sequence space $X$ with a linear topology is called a $K$-space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A $BK$-space is a $K$-space, which is also a Banach space with continuous coordinate functionals $p_k(x) = x_k$, $(k = 1, 2, \ldots)$. A $K$-space is called an $FK$-space provided $\lambda$ is a complete linear metric space. An $FK$-space whose topology is normal is called a $BK$-space.

Let $X$ be a $BK$-space. Then $X$ is said to have monotone norm if $\|x^{[m]}\| \geq \|x^{[n]}\|$ for $m > n$ and $\|x\| = \sup \|x^{[n]}\|$. The spaces $c_0$, $c$, $c_0$, $c$, $bs$ have monotone norms.

If a normed sequence space $X$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence of scalars $(\alpha_n)$ such that

$$\lim_{n \to \infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \ldots + \alpha_n b_n)\| = 0$$

then $(b_n)$ is called Schauder basis (or briefly basis) for $X$. The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$, and written as $x = \sum \alpha_k b_k$. An $FK$-space $X$ is said to have $AK$ property, if $\phi \subset X$ and $\{e^k\}$ is a basis for $X$, where $e^k$ is a sequence whose only non-zero term is a 1 in $k^{th}$ place for each $k \in \mathbb{N}$ and $\phi = \text{span} \{e^k\}$, the set of all finitely non-zero sequences. An $FK$-space $X \supset \phi$ is said to have $AB$, if $(x^{[n]})$ is a bounded set in $X$ for each $x \in X$.

By $\mathcal{F}$, we will denote the collection of all finite subsets on $\mathbb{N}$.

Let $X$ and $Y$ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}$, where $k, n \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $X$ into $Y$, and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k) \in X$. The sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $Y$; where

$$(Ax)_n = \sum_k a_{nk} x_k \text{ for each } n \in \mathbb{N}. \quad (1.1)$$

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to $\infty$. By $(X : Y)$, we denote the class of all matrices $A$ such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right side of $(1.1)$ converges for each $n \in \mathbb{N}$ and each $x \in X$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$. A sequence $x$ is said to be $A$-summable to $l$ if $Ax$ converges to $l$ which is called the $A$-limit of $x$.

Let $X$ is a sequence space and $A$ an infinite matrix. The sequence space

$$X_A = \{x = (x_k) : \omega \in \mathcal{F} : Ax \in X\} \quad (1.2)$$

is called the matrix domain of $X$ which is a sequence space. The matrix domains of some particular summability matrices are studied by several authors in literature. In [2], it can be seen the qualified studies related to the matrix domains.

In [3], Başar and Altay have defined the sequence space $bv_p$, which consists of all sequences such that $A$-transforms of them are in $\ell_p$, where $\Delta$ denotes the matrix $\Delta = (\delta_{nk})$

$$\delta_{nk} = \begin{cases} (-1)^{n-k} & (n - 1 \leq k \leq n) \\ 0 & (0 \leq k < n - 1 \text{ or } k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$.

We define the matrices $\Psi = (\psi_{nk})$, $\Gamma = (\gamma_{nk})$, $\Pi = (\pi_{nk})$ and $\Sigma = (\sigma_{nk})$ by

$$\psi_{nk} := \begin{cases} k & (1 \leq k \leq n) \\ 0 & (k > n) \end{cases} \quad (1.3)$$

$$\gamma_{nk} := \begin{cases} k & (n = k) \\ -k & (n - 1 = k) \\ 0 & (\text{other}) \end{cases} \quad (1.4)$$
\( \pi_{nk} := \begin{cases} \frac{1}{k}, & (1 \leq k \leq n) \\ 0, & (k > n) \end{cases} \) \hspace{1cm} (1.5)

\( \sigma_{nk} := \begin{cases} \frac{1}{k}, & (n = k) \\ -\frac{1}{k}, & (n - 1 = k) \\ 0, & (other) \end{cases} \) \hspace{1cm} (1.6)

Now, we can give the matrices \( \Psi^{-1} = (b_{nk}), \Gamma^{-1} = (c_{nk}), \Pi^{-1} = (d_{nk}) \) and \( \Sigma^{-1} = (e_{nk}) \) which are inverse of above matrices, by

\( b_{nk} := \begin{cases} \frac{1}{n}, & (n = k) \\ -\frac{1}{n}, & (n - 1 = k) \\ 0, & (other) \end{cases} \) \hspace{1cm} and \( c_{nk} := \begin{cases} \frac{1}{n}, & (1 \leq k \leq n) \\ 0, & (k > n) \end{cases} \)

\( d_{nk} := \begin{cases} n, & (n = k) \\ -n, & (n - 1 = k) \\ 0, & (other) \end{cases} \) \hspace{1cm} and \( e_{nk} := \begin{cases} n, & (1 \leq k \leq n) \\ 0, & (k > n) \end{cases} \)

The integrated and differentiated sequence spaces which were initiated by Goes and Goes [4]. In the present paper, we study of matrix domains and some properties of the integrated and differentiated sequence spaces. In section 3, we compute the alpha-, beta- and gamma duals of these spaces. Afterward, we characterize matrix classes of these spaces with well-known sequence spaces.

2 The integrated and differentiated sequence spaces

In [4], the integrated sequence space \( \int bv \) is given. In this section, we will obtain to matrix domains of the sequence space \( \ell_1 \) by using the new matrices in Section 1. We will show that the integrated and differentiated spaces are Banach spaces, \( BK \)–spaces, norm isomorphic to \( \ell_1 \), separable and these spaces have \( AK \)–property. The spaces \( \int bv \) and \( \int \ell_1 \) have monotone norms and so the spaces \( \int bv \) and \( \int \ell_1 \) have \( AB \)–property. Also we give the Schauder basis of the spaces \( \int bv \) and \( d(bv) \).

Firstly, we give definition of new sequence spaces:

The integrated spaces defined by

\[
\int \ell_1 := \left\{ x = (x_k) \in \omega : \sum_k |kx_k| < \infty \right\}
\]

\[
\int bv := \left\{ x = (x_k) \in \omega : \sum_k |kx_k - (k-1)x_{k-1}| < \infty \right\}
\]

and the differentiated spaces defined by

\[
d(\ell_1) := \left\{ x = (x_k) \in \omega : \sum_k \frac{1}{k} |x_k| < \infty \right\}
\]

\[
d(bv) := \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{k} x_k - \frac{1}{k-1} x_{k-1} \right| < \infty \right\}
\]
Consider the notation (1.2) and the matrices (1.3), (1.4), (1.5) and (1.6). From here, we can redefine the spaces \( f \ell_1 \), \( \text{f} \text{bv}, d(\ell_1) \) and \( d(\text{bv}) \) by

\[
(\ell_1)_\Psi = \int \ell_1, \quad (\ell_1)_\Gamma = \int \text{bv}, \quad (\ell_1)_\Pi = d(\ell_1), \quad (\ell_1)_\Sigma = d(\text{bv}).
\]

Let \( x = (x_k) \in f \ell_1 \). The \( \Psi \)-transform of a sequence \( x = (x_k) \) is defined by

\[
y_n := (\Psi x)_n = \sum_{k=1}^{n} |x_k| \quad (n \in \mathbb{N}) \tag{2.7}
\]
where \( \Psi \) is defined by (1.3).

Let \( x = (x_k) \in f \text{bv} \) and \( \Delta x_k = x_k - x_{k-1} \). The \( \Gamma \)-transform of a sequence \( x = (x_k) \) is defined by

\[
y_k := (\Gamma x)_k = \left\{ \begin{array}{ll}
x_1 & , \quad k = 1 \\
\Delta(kx_k) & , \quad k \geq 2
\end{array} \right. \tag{2.8}
\]
where \( \Gamma \) is defined by (1.4).

Let \( x = (x_k) \in d(\ell_1) \). The \( \Pi \)-transform of a sequence \( x = (x_k) \) is defined by

\[
y_n := (\Pi x)_n = \sum_{k=1}^{n} \frac{1}{k} |x_k| \quad (n \in \mathbb{N}) \tag{2.9}
\]
where \( \Pi \) is defined by (1.5).

Let \( x = (x_k) \in d(\text{bv}) \) and \( \Delta x_k = x_k - x_{k-1} \). The \( \Sigma \)-transform of a sequence \( x = (x_k) \) is defined by

\[
y_k := (\Sigma x)_k = \left\{ \begin{array}{ll}
x_1 & , \quad k = 1 \\
\Delta(k^{-1}x_k) & , \quad k \geq 2
\end{array} \right. \tag{2.10}
\]
where \( \Sigma \) is defined by (1.6).

**Theorem 2.1.** The spaces \( f \ell_1 \) and \( d(\ell_1) \) are Banach spaces with the norms \( \|x\|_{f \ell_1} = \sum_{k=1}^{n} |x_k| \) and \( \|x\|_{d(\ell_1)} = \sum_{k=1}^{n} |k^{-1}x_k| \), respectively.

**Proof.** It is easily checked that \( \|x\|_{f \ell_1} = \sum_{k=1}^{n} |x_k| \) defines a norm on the space \( f \ell_1 \). Now we show that the space \( f \ell_1 \) is complete.

Let us take any Cauchy sequence \( (x') \) in the space \( f \ell_1 \), where \( (x') = (x'^{(i)}_0, x'^{(i)}_1, x'^{(i)}_2, \ldots) \). Then, for a given \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) depending upon \( \varepsilon \) such that

\[
\|x^i - x'^{i}\|_{f \ell_1} < \varepsilon \tag{2.11}
\]
for all \( i, j \geq n_0(\varepsilon) \). By using the definition of the Cauchy sequence, we have, for each fixed \( k \in \mathbb{N} \) that,

\[
\|x^i - x'^{i}\|_{f \ell_1} = \sum_{k} |(\Psi x^i)_k - (\Psi x'^{i})_k| < \varepsilon
\]
for all \( i, j \geq n_0(\varepsilon) \), which leads us to the fact that \((\Psi x^i)_0, (\Psi x^i)_1, (\Psi x^i)_2, \ldots\) is a Cauchy sequence of complex numbers for every fixed \( k \in \mathbb{N} \). Since \( \mathbb{C} \) is complete, it converges, that is, \((\Psi x^i)_k \to (\Psi x)_k\) as \( i \to \infty \). Using these infinitely many limits \((\Psi x)_0, (\Psi x)_1, (\Psi x)_2, \ldots\), we define the sequence \((\Psi x)_0, (\Psi x)_1, (\Psi x)_2, \ldots\). From (2.11) for each \( m \in \mathbb{N} \) and \( i, j \geq n_0(\varepsilon) \)

\[
\sum_{k=1}^{m} |(\Psi x^i)_k - (\Psi x'^{i})_k| < \varepsilon. \tag{2.12}
\]
Take any $i \geq n_0(\varepsilon)$. First let $j \to \infty$ in (2.12) and $m \to \infty$, we obtain $\|x' - x'\|_{\ell^1} \leq \varepsilon$. Finally taking $\varepsilon = 1$ in (2.12) and letting $i \geq n_0(1)$, we have

$$\|x\|_{\ell^1} \leq \|x'\|_{\ell^1} + \|x' - x\|_{\ell^1} \leq 1 + \|x'\|_{\ell^1}$$

which implies that $x \in \ell^1$. Since $\|x - x'\|_{\ell^1} \leq \varepsilon$ for all $i \geq n_0(\varepsilon)$ it follows that $x' \to x$ as $i \to \infty$. Therefore, we have shown that the space $\ell^1$ is complete.

In a similar way, we can prove that the space $d(\ell^1)$ is a Banach space.

**Theorem 2.2.** The spaces $\ell^1$ and $d(\ell^1)$ are BK-spaces with the norms $\|x\|_{\ell^1} = \sum_k |kx_k|$ and $\|x\|_{d(\ell^1)} = \sum_k |k^{-1}x_k|$, respectively.

**Proof.** Let $x = (x_k) \in \ell^1$. We define $f_k(x) = x_k$ for all $k \in \mathbb{N}$. Then, we have

$$\|x\|_{\ell^1} = 1|x_1| + 2|x_2| + 3|x_3| + \cdots + k|x_k| + \cdots$$

Hence $k|x_k| \leq \|x\|_{\ell^1} \Rightarrow |x_k| \leq K\|x\|_{\ell^1} \Rightarrow \|f_k(x)\| = K\|x\|_{\ell^1}$. Thus, $f_k$ is a continuous linear functional for each $k$. Then $\ell^1$ is a BK-space. In a similar way, we can prove that the space $d(\ell^1)$ is a BK-space.

**Lemma 2.1.** [4] The space $\ell^1$ is a BK-space with the norm $\|x\|_{\ell^1} = \sum_k |\Delta(kx_k)|$.

**Theorem 2.3.** The space $d(\ell^1)$ is a BK-space with the norm $\|x\|_{d(\ell^1)} = \sum_k |\Delta(k^{-1}x_k)|$.

**Proof.** Since $d(\ell^1) = [\ell^1]_{\Sigma}$ holds, $\ell^1$ is a BK-space with the norm $\|x\|_{\ell^1}$ and the matrix $\Sigma$ is a triangle matrix. Theorem 4.3.2 of Wilansky[6] gives the fact that the space $d(\ell^1)$ is a BK-space.

**Theorem 2.4.** The following statements hold:

(i) The spaces $\ell^1$ and $d(\ell^1)$ have AK-property.

(ii) The spaces $\ell^1$ and $d(\ell^1)$ have AK-property.

**Proof.** The fact that of the space $\ell^1$ has AK-property was given by Goes[4]. Then, we will only prove that the space $d(\ell^1)$ has AK-property in (ii). Let $x = (x_k) \in d(\ell^1)$ and $x^{[n]} = \{x_1, x_2, \ldots, x_n, 0, 0, \cdots\}$. Hence,

$$x - x^{[n]} = \{0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots\} \Rightarrow \|x - x^{[n]}\|_{d(\ell^1)} = \|0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots\|$$

and since $x \in d(\ell^1)$,

$$\|x - x^{[n]}\|_{d(\ell^1)} = \sum_{k \geq n+1} |\Delta \left( \frac{1}{k} x_k \right) | \to 0 \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \lim_{n \to \infty} \|x - x^{[n]}\|_{d(\ell^1)} = 0 \Rightarrow x^{[n]} \to x \quad \text{as} \quad n \to \infty \quad \text{in} \quad d(\ell^1).$$

Then the space $d(\ell^1)$ has AK-property.

**Theorem 2.5.** The spaces $\ell^1$, $\ell^1$, $d(\ell^1)$ and $d(\ell^1)$ are norm isomorphic to $\ell^1$.

**Proof.** We must show that a linear bijection between the spaces $\ell^1$ and $\ell^1$ exists. Consider the transformation $T$ defined, with the notation (2.8), from $\ell^1$ to $\ell^1$ by $x \to y = Tx$. The linearity of $T$ is clear. Also, it is trivial that $x = \theta$ whenever $Tx = \theta$ and therefore, $T$ is injective.

Let $y \in \ell^1$ and define the sequence $x = (x_k)$ by $x_k = \frac{1}{k} \sum_{j=1}^k y_j$. Then

$$\|x\|_{\ell^1} = \sum_k |\Delta(kx_k)| = \sum_k \left| \frac{1}{k} \sum_{j=1}^{k} y_j - (k - 1) \frac{1}{k - 1} \sum_{j=1}^{k-1} y_j \right| = \sum_k |y_k| = \|y\|_{\ell^1} < \infty.$$
Then, we have that \( x \in \mathcal{f} b v \). So, \( T \) is surjective and norm preserving. Hence \( T \) is a linear bijection. It shows that the space \( \mathcal{f} b v \) is norm isomorphic to \( \ell_1 \).

Similarly, using the notations (2.7), (2.9) and (2.10), we can define the transformation \( U, V, S \) from \( \ell_1, d(\ell_1), d(bv) \) to \( \ell_1 \) by \( x \rightarrow y = Ux, x \rightarrow y = Vx, x \rightarrow y = Sx \), respectively. And also, if we choose the sequences \( x = (x_k) \) by \( x_k = k^{-1}\Delta y_k \) for \( \ell_1 \), by \( x_k = k^j\Sigma_{j=1}^{\infty}y_j \) for \( d(\ell_1) \), by \( x_k = k^j\Sigma_{j=1}^{\infty}y_j \) for \( d bv \) while \( y \in \ell_1 \), then we obtain \( \ell_1, d(\ell_1) \) and \( d(bv) \) are norm isomorphic to \( \ell_1 \) with the norms.

**Theorem 2.6.** The spaces \( \mathcal{f} b v \) and \( \ell_1 \) have monotone norm.

**Proof.** Let \( x = (x_k) \in \mathcal{f} b v \). We define the norms \( \|x\|_{\mathcal{f} b v} = \sum_{k=1}^{n} |\Delta(kx_k)| \) and \( \|x[n]\|_{\mathcal{f} b v} = \sum_{k=1}^{m} |\Delta(kx_k)| \), for all \( x \in \mathcal{f} b v \).

For \( n < m \),

\[
\|x[n]\|_{\mathcal{f} b v} = \sum_{k=1}^{n} |\Delta(kx_k)| \leq \sum_{k=1}^{m} |\Delta(kx_k)| = \|x[m]\|_{\mathcal{f} b v},
\]

that is,

\[
\|x[n]\|_{\mathcal{f} b v} \geq \|x[m]\|_{\mathcal{f} b v}. \tag{2.13}
\]

The sequence \( \{\|x[n]\|\} \) is a monotonically increasing sequence and bounded above.

\[
\sup\|x[n]\|_{\mathcal{f} b v} = \sup \left( \sum_{k=1}^{n} |\Delta(kx_k)| \right) = \left( \sum_{k=1}^{n} |\Delta(kx_k)| \right) = \|x\|_{\mathcal{f} b v}. \tag{2.14}
\]

From (2.13) and (2.14), it follows that the space \( \mathcal{f} b v \) has the monotone norm.

In a similar, we can obtain that the space \( \mathcal{f} \ell_1 \) has the monotone norm.

**Lemma 2.2 ([6], Theorem 10.3.12, ).** Any space with a monotone norm has \( AB \).

From Theorem 2.6 and Lemma 2.2, we can give following corollary:

**Corollary 2.1.** The spaces \( \mathcal{f} b v \) and \( \mathcal{f} \ell_1 \) have \( AB \).

Since the isomorphisms \( T \) and \( S \), defined in the proof of Theorem 2.5 are surjective, the inverse image of the basis \( \{a[k]\}_{k \in N} \) of the space \( \ell_1 \) is the basis of the spaces \( \mathcal{f} b v \) and \( d(bv) \). Therefore, we have the following:

**Theorem 2.7.** The following statements hold:

(i) Define a sequence \( t[k] := \{t^{(k)}_n\}_{n \in N} \) of elements of the space \( \mathcal{f} b v \) for every fixed \( k \in N \) by

\[
t^{(k)}_n = \begin{cases} 
1/k & \text{if } n \geq k \\
0 & \text{if } n < k 
\end{cases}
\]

Then the sequence \( \{t[k]\}_{k \in N} \) is a basis for the spaces \( \mathcal{f} b v \) and if we choose \( E_k = (Ax) \) for all \( k \in N \), where the matrix \( A \) defined by (1.4), then any \( x \in \mathcal{f} b v \) has a unique representation of the form

\[
x := \sum_{k} E_k t^{(k)}.
\]

(ii) Define a sequence \( s[k] := \{s^{(k)}_n\}_{n \in N} \) of elements of the space \( d(bv) \) for every fixed \( k \in N \) by

\[
s^{(k)}_n = \begin{cases} 
k & \text{if } n \geq k \\
0 & \text{if } n < k 
\end{cases}
\]
Then the sequence \( \{x^{(k)}\}_{k \in \mathbb{N}} \) is a basis for the space \( d(bv) \) and if we choose \( F_k = (Bx)_k \) for all \( k \in \mathbb{N} \), where the matrix \( B \) defined by (1.6), then any \( x \in d(bv) \) has a unique representation of the form

\[
x := \sum_k F_k s^{(k)}.
\]

The result follows from fact that if a space has a Schauder basis, then it is separable. Hence, we can give following corollary:

**Corollary 2.2.** The spaces \( f \) bv and \( d(bv) \) are separable.

### 3 The \( \alpha-, \beta- \) and \( \gamma- \) Duals of the integrated and differentiated sequence spaces

In this section, we state and prove the theorems determining the \( \alpha-, \beta- \) and \( \gamma- \) duals of the sequence spaces \( f \) bv and \( d(bv) \).

Let \( x \) and \( y \) be sequences, \( X \) and \( Y \) be subsets of \( \omega \) and \( A = (a_{nk})_{n,k=0}^{\infty} \) be an infinite matrix of complex numbers. We write \( xy = (x_\omega y_\omega)_{\omega=0}^{\infty} \) for the multiplier space of \( X \) and \( Y \). In the special cases of \( Y = \{\ell_1, cs, bs\} \), we write \( x^\omega = x_{\omega} \) and \( x^Y = x_{\omega} \) for all \( x \in X \) for the \( \alpha- \) dual, \( \beta- \) dual, \( \gamma- \) dual of \( X \). By \( A_n = (a_{nk})_{k=0}^{\infty} \) we denote the sequence in the \( n \)-th row of \( A \), and we write \( A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \) for all \( n \) and \( x \).

**Lemma 3.1.** [1, Theorem 2.1] Let \( \lambda, \mu \) be the BK-spaces and \( B^U_\mu = (b_{nk}) \) be defined via the sequence \( \alpha = (a_k) \in \mu \) and triangle matrix \( U = (u_{nk}) \) by

\[
b_{nk} = \sum_{j=k}^{n} \alpha_j u_{nj} v_{jk}
\]

for all \( k, n \in \mathbb{N} \). Then, the inclusion \( \mu \lambda^U \subset \lambda^U \) holds if and only if the matrix \( B^U_\mu = U D_{\alpha} U^{-1} \) is in the classes \( (\lambda : \lambda) \), where \( D_{\alpha} \) is the diagonal matrix defined by \( [D_{\alpha}]_{mn} = a_{0n} \) for all \( n \in \mathbb{N} \).

**Lemma 3.2.** [1, Theorem 3.1] Let \( B^U_\mu = (b_{nk}) \) be defined via a sequence \( a = (a_k) \in \omega \) and inverse of the triangle matrix \( U = (u_{nk}) \) by

\[
b_{nk} = \sum_{j=k}^{n} a_j v_{jk}
\]

for all \( k, n \in \mathbb{N} \). Then,

\[
\lambda^B_\mu = \{a = (a_k) \in \omega : B^U \in (\lambda : c)\}.
\]

and

\[
\lambda^\gamma_\mu = \{a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty)\}.
\]

**Lemma 3.3.** Let \( A = (a_{nk}) \) be an infinite matrix. Then, the following statements hold:

(i) \( A \in (\ell_1 : \ell_\infty) \) if and only if

\[
\sup_{k,n \in \mathbb{N}} |a_{nk}| < \infty. \tag{3.15}
\]

(ii) \( A \in (\ell_1 : c) \) if and only if (3.15) holds, and there are \( a_\infty, c \in \mathbb{C} \) such that

\[
\lim_{n \to \infty} a_{nk} = a_k \quad \text{for each} \quad k \in \mathbb{N}. \tag{3.16}
\]
(iii) \( A \in \{ \ell_1 : \ell_1 \} \) if and only if

\[
\sup_{k \in \mathbb{N}} \sum_{n} |a_{nk}| < \infty. \tag{3.17}
\]

**Theorem 3.1.** \((fbv)^a = \alpha_1\), where \( \alpha_1 := \{ a = (a_k) \in \omega : \sup_{k,N \in \mathbb{N}} \sum_{k \in K} |\sum_{n \in N} a_{nk}| < \infty \}\)

**Proof.** Let us take any \( a = (a_k) \in \omega \) and consider the equality

\[
a_{\omega} x_n = \sum_{k=1}^{n} \frac{a_k}{n} y_k = (F y)_n \quad (n \in \mathbb{N}). \tag{3.18}
\]

where \( F = (f_{nk}) \) is defined by

\[
f_{nk} = \begin{cases} a_n/n & (1 \leq k \leq n) \\ 0 & (k > n) \end{cases}
\]

for all \( k, n \in \mathbb{N} \). It follows from (3.18) with Lemma 3.3 (iii) that \( ax = (a_n x_n) \in \ell_1 \) whenever \( x = (x_k) \in fbv \) if and only if \( F y \in \ell_1 \) whenever \( y \in \ell_1 \). This means that \( a = (a_n) \in (fbv)^a \) whenever \( x = (x_n) \in fbv \) if and only if \( F \in (fbv : \ell_1) \). This gives the result that \((fbv)^a = \alpha_1\). \( \Box \)

The following theorem can be proved as similar way in proof of Theorem 3.1, so we omit details.

**Theorem 3.2.** \((dv)^a = \alpha_2\), where \( \alpha_2 := \{ a = (a_k) \in \omega : \sup_{k,N \in \mathbb{N}} \sum_{k \in K} |\sum_{n \in N} a_{nk}| < \infty \}\)

**Theorem 3.3.** \((f \ell_1)^a = \alpha_3\), where \( \alpha_3 := \{ a = (a_k) \in \omega : \sup_{k \in \mathbb{N}} \sum_{n} |a_{nk}| < \infty \}\)

**Theorem 3.4.** \((d \ell_1)^a = \alpha_4\), where \( \alpha_4 := \{ a = (a_k) \in \omega : \sup_{k \in \mathbb{N}} |\sum_{n} a_{nk}| < \infty \}\)

**Theorem 3.5.** \((fbv)^\beta = \beta_1 \cap cs\), where \( \beta_1 := \{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} |\sum_{j=1}^{n} a_j| < \infty \}\)

**Proof.** Consider the equation

\[
\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left( \frac{1}{k} \sum_{j=1}^{k} y_j \right) = \sum_{k=1}^{n} \left( \frac{n}{k} \sum_{j=k}^{n} a_j \right) y_k = (G y)_n \tag{3.19}
\]

where \( G = (g_{nk}) \) is defined by

\[
g_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{a_j}{j} & (0 \leq k \leq n) \\ \frac{a_n}{n - k + 1} & (k > n) \end{cases}
\]

for all \( n, k \in \mathbb{N} \). Then we deduce from Lemma 3.3 (ii) with (3.19) that \( ax = (a_n x_k) \in cs \) whenever \( x = (x_k) \in fbv \) if and only if \( G y \in c \) whenever \( y = (y_k) \in \ell_1 \). We obtain from Lemma 3.1 and Lemma 3.2, the result that \( a = (a_k) \in (fbv)^\beta \) if and only if \( G \in (\ell_1 : c) \), which is what we wished to prove. \( \Box \)

**Lemma 3.4.** \([d(c)]^\beta = bv \Rightarrow [d(cs)]^\beta = fbv\)

From Theorem 3.5 and Lemma 3.4, we have,

**Theorem 3.6.** \((bv)^\beta = cs \Rightarrow [d(bv)]^\beta = fc\)

**Theorem 3.7.** Let \( a = (a_k) \in \omega \), the matrix \( H = (h_{nk}) \) by

\[
h_{nk} := \begin{cases} \frac{a_k}{k} - \frac{a_{k+1}}{k+1} & (k \leq n) \\ \frac{a_n}{n} & (k = n) \\ 0 & (k > n) \end{cases} \tag{3.20}
\]
and define the sets

\[ \beta_2 := \left\{ a = (a_k) \in \omega : \sup_n \sum_k |h_{nk}| < \infty \right\} \]

\[ \beta_3 := \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} h_{nk} \text{ exists for each } k \in \mathbb{N} \right\}. \]

Then \((f \ell_1)^\beta = \beta_2 \cap \beta_3\).

Proof. Consider the equation

\[ \sum_{k=1}^n a_kx_k = \sum_{k=1}^n \frac{a_k}{k} (y_k - y_{k-1}) = \sum_{k=1}^{n-1} \left( \frac{a_k}{k} - \frac{a_{k+1}}{k+1} \right) y_k + \frac{a_n}{n} y_n = (Hy)_n, \tag{3.21} \]

where \(H = (h_{nk})\) is defined by (3.20). Then we deduce from Lemma 3.3 (ii) with (3.21) that \(ax = (a_kx_k) \in cs\) whenever \(x = (x_k) \in f \ell_1\) if and only if \(Hy \in c\) whenever \(y = (y_k) \in \ell_1\). Therefore, we derive from (3.15) and (3.16) that \((f \ell_1)^\beta = \beta_2 \cap \beta_3\). \(\square\)

**Theorem 3.8.** Let \(a = (a_k) \in \omega\), the matrix \(P = (p_{nk})\) by

\[ p_{nk} := \begin{cases} 
ka_k - (k+1)a_{k+1} & , \quad (k \leq n) \\
n \omega & , \quad (k = n) \\
0 & , \quad (k > n)
\end{cases} \]

and define the sets

\[ \beta_4 := \left\{ a = (a_k) \in \omega : \sup_n \sum_k |p_{nk}| < \infty \right\} \]

\[ \beta_5 := \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} p_{nk} \text{ exists for each } k \in \mathbb{N} \right\}. \]

Then \((d(\ell_1))^\beta = \beta_4 \cap \beta_5\).

**Theorem 3.9.** \([f \omega]^\gamma = \beta_1\)

*Proof.* This can be obtained by analogy with the proof of Theorem 3.5 with Lemma 3.3 (i) instead of Lemma 3.3 (ii). So we omit the details. \(\square\)

**Theorem 3.10.** \([d(bs)]^\gamma = f \omega\), where \(d(bs) = \{x = (x_k) \in \omega : (k^{-1}x_k) \in bs\}\).

**Theorem 3.11.** \([f \ell_1]^\gamma := \beta_2\).

**Theorem 3.12.** \([d(\ell_1)]^\gamma := \beta_4\).

4 Matrix Mappings on the integrated and differentiated sequence spaces

In this section, we characterize some matrix transformations on the spaces \(f \omega\) and \(d(\omega)\).

We shall write throughout for brevity that

\[ \tilde{a}_{nk} := \sum_{j=k}^\infty a_{nj}, \quad \tilde{b}_{nk} := \sum_{j=k}^\infty ja_{nj}, \quad \tilde{c}_{nk} := \frac{a_{nk}}{k} - \frac{a_{n,k+1}}{k+1}, \quad \tilde{d}_{nk} := ka_{nk} - (k+1)a_{n,k+1} \]

\[ \tilde{a}_{nk} := nb_{nk} - (n-1)b_{n-1,k}, \quad \tilde{b}_{nk} := \frac{1}{n}b_{nk} - \frac{1}{n-1}b_{n-1,k}, \quad \tilde{c}_{nk} := \sum_{j=1}^n j b_{jk}, \quad \tilde{d}_{nk} := \sum_{j=1}^n \frac{1}{j} b_{jk} \]

for all \(k, n \in \mathbb{N}\).
Lemma 4.1. [1] Let $X$, $Y$ be any two sequence spaces, $A$ be an infinite matrix and $U$ a triangle matrix. Then, $A \in (X : Y)$ if and only if $UA \in (X : Y)$.

Theorem 4.1. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $R = (r_{nk})$ are connected with the relation
\[
r_{nk} = \tilde{a}_{nk} \quad (4.22)
\]
for all $k, n \in \mathbb{N}$ and $Y$ is any given sequence space. Then, $A \in (f \, bv : Y)$ if and only if $(a_{nk})_{k \in \mathbb{N}} \in (f \, bv)^\beta$ for all $n \in \mathbb{N}$ and $R \in (\ell_1 : Y)$.

Proof. Let $Y$ be any given sequence space. Suppose that (4.22) holds between $A = (a_{nk})$ and $R = (r_{nk})$, and take into account that the spaces $f \, bv$ and $\ell_1$ are norm isomorphic.

Let $A \in (f \, bv : Y)$ and take any $y = (y_k) \in \ell_1$. Then $\Gamma R$ exists and $(a_{nk})_{k \in \mathbb{N}} \in (f \, bv)^\beta$ which yields that (4.22) is necessary and $(r_{nk})_{k \in \mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, $Ry$ exists for each $y \in \ell_1$ and thus
\[
\sum_k r_{nk} y_k = \sum_n a_{nk} x_k \quad \text{for all } n \in \mathbb{N},
\]
we obtain that $Ry = Ax$ which leads us to the consequence $R \in (\ell_1 : Y)$.

Conversely, let $(a_{nk})_{k \in \mathbb{N}} \in (f \, bv)^\beta$ for each $n \in \mathbb{N}$ and $R \in (\ell_1 : Y)$ hold, and take any $x = (x_k) \in f \, bv$. Then, $Ax$ exists. Therefore, we obtain from the equality
\[
\sum_{k=1}^m a_{nk} x_k = \sum_{k=1}^m a_{nk} \left[ \frac{1}{k} \sum_{j=1}^k y_j \right] = \sum_{k=1}^m \left( \sum_{j=k}^n \frac{a_{nj}}{j} \right) y_k \quad \text{for all } m, n \in \mathbb{N}
\]
as $m \to \infty$ that $Ax = Ry$ and this shows that $A \in (f \, bv : Y)$. This completes the proof.

Following theorems can be proved as in Theorem 4.1.

Theorem 4.2. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $S = (s_{nk})$ are connected with the relation
\[
g_{nk} = \tilde{b}_{nk}
\]
for all $k, n \in \mathbb{N}$ and $Y$ is any given sequence space. Then, $A \in (d(bv) : Y)$ if and only if $(a_{nk})_{k \in \mathbb{N}} \in (d(bv))^\beta$ for all $n \in \mathbb{N}$ and $S \in (\ell_1 : Y)$.

Theorem 4.3. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $T = (t_{nk})$ are connected with the relation
\[
t_{nk} = \tilde{c}_{nk}
\]
for all $k, n \in \mathbb{N}$ and $Y$ is any given sequence space. Then, $A \in (f \, \ell_1 : Y)$ if and only if $(a_{nk})_{k \in \mathbb{N}} \in (f \, \ell_1)^\beta$ for all $n \in \mathbb{N}$ and $T \in (\ell_1 : Y)$.

Theorem 4.4. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $U = (u_{nk})$ are connected with the relation
\[
u_{nk} = \tilde{d}_{nk}
\]
for all $k, n \in \mathbb{N}$ and $Y$ is any given sequence space. Then, $A \in (d(\ell_1) : Y)$ if and only if $(a_{nk})_{k \in \mathbb{N}} \in (d(\ell_1))^\beta$ for all $n \in \mathbb{N}$ and $U \in (\ell_1 : Y)$.

Theorem 4.5. Suppose that the entries of the infinite matrices $B = (b_{nk})$ and $V = (v_{nk})$ are connected with the relation
\[
v_{nk} = \tilde{a}_{nk}
\]
for all $k, n \in \mathbb{N}$ and $Y$ is any given sequence space. Then, $B \in (Y : f \, bv)$ if and only if $V \in (Y : \ell_1)$. 

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Proof. Let \( z = (z_k) \in Y \) and consider the following equality
\[
\sum_{k=1}^{m} a_{nk} z_k = \sum_{k=1}^{m} (na_{nk} - (n - 1)a_{n-1,k}) z_k \quad \text{for all, } m,n \in \mathbb{N}
\]
which yields that as \( m \to \infty \) that \( \{Vz\}_n = \{\Gamma(Bz)\}_n \) for all \( n \in \mathbb{N} \). Therefore, one can observe from here that \( Bz \in \int bv \) whenever \( z \in Y \) if and only if \( Vz \in \ell_1 \) whenever \( z \in Y \).

Following theorems can be proved as in Theorem 4.5.

**Theorem 4.6.** Suppose that the entries of the infinite matrices \( B = (b_{nk}) \) and \( W = (w_{nk}) \) are connected with the relation
\[
w_{nk} = \hat{b}_{nk}
\]
for all \( k,n \in \mathbb{N} \) and \( Y \) is any given sequence space. Then, \( B \in (Y : d(bv)) \) if and only if \( W \in (Y : \ell_1) \).

**Theorem 4.7.** Suppose that the entries of the infinite matrices \( B = (b_{nk}) \) and \( Z = (z_{nk}) \) are connected with the relation
\[
z_{nk} = \hat{c}_{nk}
\]
for all \( k,n \in \mathbb{N} \) and \( Y \) is any given sequence space. Then, \( B \in (Y : d(\ell_1)) \) if and only if \( Z \in (Y : \ell_1) \).

**Theorem 4.8.** Suppose that the entries of the infinite matrices \( B = (b_{nk}) \) and \( M = (m_{nk}) \) are connected with the relation
\[
m_{nk} = \hat{d}_{nk}
\]
for all \( k,n \in \mathbb{N} \) and \( Y \) is any given sequence space. Then, \( B \in (Y : d(\ell_1)) \) if and only if \( M \in (Y : \ell_1) \).

**Lemma 4.2.** The following statements hold:

(i) \( A \in (\ell_1 : bs) \) if and only if
\[
\sup_{k,m \in \mathbb{N}} \left| \sum_{n=0}^{m} a_{nk} \right| < \infty.
\]

(ii) \( A \in (\ell_1 : cs) \) if and only if (4.23) holds, and
\[
\sum_{n} a_{nk} \text{ convergent for each } k \in \mathbb{N}.
\]

(iii) \( A \in (\ell_1 : c_0s) \) if and only if (4.23) holds, and
\[
\sum_{n} a_{nk} = 0 \text{ for each } k \in \mathbb{N}.
\]

**Lemma 4.3.** The following statements hold:

(i) \( A \in (\ell_\infty : \ell_1) = (c : \ell_1) = (c_0 : \ell_1) \) if and only if
\[
\sup_{N,K \in \mathbb{N}} \left| \sum_{n \in N} \sum_{k \in K} a_{nk} \right| < \infty
\]
(ii) \( A \in (bs : \ell_1) \) if and only if
\[
\lim_{k} a_{nk} = 0 \quad \text{for each } n \in \mathbb{N}. \tag{4.27}
\]
\[
\sup_{N,K \in \mathcal{D}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k+1}) \right| < \infty \tag{4.28}
\]
(iii) \( A \in (cs : \ell_1) \) if and only if
\[
\sup_{N,K \in \mathcal{D}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k-1}) \right| < \infty \tag{4.29}
\]
(iv) \( A \in (c_0s : \ell_1) \) if and only if (4.28) holds.

Now, we can give the following results:

**Corollary 4.1.** The following statements hold:

(i) \( A = (a_{nk}) \in (\{ b_{v} : \ell_{\infty} \}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ \{ b_{v} \} \}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (\{ b_{v} : c \}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ \{ b_{v} \} \}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \).

(iii) \( A \in (\{ b_{v} : c_0 \}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ \{ b_{v} \} \}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( \alpha_k = 0 \) as \( \tilde{a}_{nk} \) instead of \( a_{nk} \).

(iv) \( A = (a_{nk}) \in (\{ b_{v} : bs \}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ \{ b_{v} \} \}^\beta \) for all \( n \in \mathbb{N} \) and (4.23) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \).

(v) \( A = (a_{nk}) \in (\{ b_{v} : cs \}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ \{ b_{v} \} \}^\beta \) for all \( n \in \mathbb{N} \) and (4.23), (4.24) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \).

(vi) \( A = (a_{nk}) \in (\{ b_{v} : c_0s \}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ \{ b_{v} \} \}^\beta \) for all \( n \in \mathbb{N} \) and (4.23), (4.25) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \).

**Corollary 4.2.** The following statements hold:

(i) \( A = (a_{nk}) \in (d(bv) : \ell_{\infty}) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ d(bv) \}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) holds with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (d(bv) : c) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ d(bv) \}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(iii) \( A \in (d(bv) : c_0) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ d(bv) \}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( \alpha_k = 0 \) as \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(iv) \( A = (a_{nk}) \in (d(bv) : bs) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ d(bv) \}^\beta \) for all \( n \in \mathbb{N} \) and (4.23) holds with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(v) \( A = (a_{nk}) \in (d(bv) : cs) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ d(bv) \}^\beta \) for all \( n \in \mathbb{N} \) and (4.23), (4.24) hold with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(vi) \( A = (a_{nk}) \in (d(bv) : c_0s) \) if and only if \( \{ a_{nk} \}_{k \in \mathbb{N}} \in \{ d(bv) \}^\beta \) for all \( n \in \mathbb{N} \) and (4.23), (4.25) hold with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

**Corollary 4.3.** The following statements hold:
(i) \( A = (a_{nk}) \in (\ell_1 : c_0) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_1\}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) holds with \( c_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (\ell_1 : c) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_1\}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( c_{nk} \) instead of \( a_{nk} \).

(iii) \( A \in (\ell_1 : c_0) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_1\}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( a_k = 0 \) as \( c_{nk} \) instead of \( a_{nk} \).

(iv) \( A = (a_{nk}) \in (\ell_1 : bs) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_1\}^\beta \) for all \( n \in \mathbb{N} \) and (4.23) holds with \( c_{nk} \) instead of \( a_{nk} \).

(v) \( A = (a_{nk}) \in (\ell_1 : cs) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_1\}^\beta \) for all \( n \in \mathbb{N} \) and (4.23), (4.24) hold with \( c_{nk} \) instead of \( a_{nk} \).

Corollary 4.4. The following statements hold:

(i) \( A = (a_{nk}) \in (d(\ell_1) : \ell_\infty) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{d(\ell_1)\}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) holds with \( d_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (d(\ell_1) : c) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{d(\ell_1)\}^\beta \) for all \( n \in \mathbb{N} \) and (3.15) and (3.16) hold with \( d_{nk} \) instead of \( a_{nk} \).

Corollary 4.5. We have:

(i) \( A = (a_{nk}) \in (\ell_\infty : \langle 1 \rangle) = (c : \langle 1 \rangle) \) if and only if (4.26) hold with \( d_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (bs : \langle 1 \rangle) \) if and only if (4.27) and (4.28) hold with \( d_{nk} \) instead of \( a_{nk} \).

(iii) \( A = (a_{nk}) \in (cs : \langle 1 \rangle) \) if and only if (4.29) holds with \( d_{nk} \) instead of \( a_{nk} \).

(iv) \( A = (a_{nk}) \in (c_0s : \langle 1 \rangle) \) if and only if (4.28) holds with \( d_{nk} \) instead of \( a_{nk} \).

Corollary 4.6. We have:

(i) \( A = (a_{nk}) \in (\ell_\infty : d(\ell_1)) = (c : d(\ell_1)) = (c_0 : d(\ell_1)) \) if and only if (4.26) hold with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(ii) \( A = (a_{nk}) \in (bs : d(\ell_1)) \) if and only if (4.27) and (4.28) hold with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(iii) \( A = (a_{nk}) \in (cs : d(\ell_1)) \) if and only if (4.29) holds with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

(iv) \( A = (a_{nk}) \in (c_0s : d(\ell_1)) \) if and only if (4.28) holds with \( \tilde{b}_{nk} \) instead of \( a_{nk} \).

Corollary 4.7. We have:

(i) \( A = (a_{nk}) \in (\ell_\infty : c_0) = (c : c_0) = (c_0 : c_0) \) if and only if (4.26) hold with \( \tilde{c}_{nk} \) instead of \( a_{nk} \).
In this paper, we proved that the integrated and differentiated sequence spaces are Banach spaces, $BK$ (integrated rate space) present results. If we can choose different sequence spaces for the space $Y$ numbers.

(iii) $A = (a_{nk}) \in (c_{bs} : \ell_1)$ if and only if (4.28) holds with $\widehat{a}_{nk}$ instead of $a_{nk}$.

Corollary 4.8. We have:

(i) $A = (a_{nk}) \in (\ell_{bs} : d(\ell_1)) = (c : d(\ell_1)) = (c_0 : d(\ell_1))$ if and only if (4.26) hold with $\widehat{a}_{nk}$ instead of $a_{nk}$.

(ii) $A = (a_{nk}) \in (bs : d(\ell_1))$ if and only if (4.27) and (4.28) hold with $\widehat{a}_{nk}$ instead of $a_{nk}$.

(iii) $A = (a_{nk}) \in (cs : d(\ell_1))$ if and only if (4.29) holds with $\widehat{a}_{nk}$ instead of $a_{nk}$.

(iv) $A = (a_{nk}) \in (c_{bs} : d(\ell_1))$ if and only if (4.28) holds with $\widehat{a}_{nk}$ instead of $a_{nk}$.

5 Conclusion

Goes and Goes [4] introduced the integrated and differentiated sequence spaces. Subramanian et.al. [5] gave the integrated rate space $\int \ell_\pi$ and studied some properties of this space. And they also characterized the matrix classes $(f_{\ell_\pi} : Y)$, where $Y = \{\ell_{as}, c_{as}, b_{as}, b_{ps}, b_{pv}, b_{sv}, b_{sp}, d_{as}, d_{ps}, d_{pv}, d_{sv}, d_{sp}, e_{as}, e_{ps}, e_{pv}, e_{sv}, e_{sp}, f_{as}, f_{ps}, f_{pv}, f_{sv}, f_{sp}, g_{as}, g_{ps}, g_{pv}, g_{sv}, g_{sp}, h_{as}, h_{ps}, h_{pv}, h_{sv}, h_{sp}, i_{as}, i_{ps}, i_{pv}, i_{sv}, i_{sp}, j_{as}, j_{ps}, j_{pv}, j_{sv}, j_{sp}, k_{as}, k_{ps}, k_{pv}, k_{sv}, k_{sp}, l_{as}, l_{ps}, l_{pv}, l_{sv}, l_{sp}, m_{as}, m_{ps}, m_{pv}, m_{sv}, m_{sp}, n_{as}, n_{ps}, n_{pv}, n_{sv}, n_{sp}, o_{as}, o_{ps}, o_{pv}, o_{sv}, o_{sp}, p_{as}, p_{ps}, p_{pv}, p_{sv}, p_{sp}, q_{as}, q_{ps}, q_{pv}, q_{sv}, q_{sp}, r_{as}, r_{ps}, r_{pv}, r_{sv}, r_{sp}, s_{as}, s_{ps}, s_{pv}, s_{sv}, s_{sp}, t_{as}, t_{ps}, t_{pv}, t_{sv}, t_{sp}, u_{as}, u_{ps}, u_{pv}, u_{sv}, u_{sp}, v_{as}, v_{ps}, v_{pv}, v_{sv}, v_{sp}, w_{as}, w_{ps}, w_{pv}, w_{sv}, w_{sp}, x_{as}, x_{ps}, x_{pv}, x_{sv}, x_{sp}, y_{as}, y_{ps}, y_{pv}, y_{sv}, y_{sp}, z_{as}, z_{ps}, z_{pv}, z_{sv}, z_{sp}\}$. There are no studies on differentiated sequence spaces. In this paper, we proved that the integrated and differentiated sequence spaces are Banach spaces, $BK$—spaces and norm isomorphic to $\ell_\pi$. We showed that these spaces have $AK$— and $AB$— properties and also the spaces $\int bv, \int \ell_\pi$ have monotone norm. Also we define the Schauder basis of the spaces $\int bv, d(bv)$ and a result of the fact that the spaces $\int bv, d(bv)$ are separable. We compute the alpha-, beta- and gamma-duals of these spaces. For $Y = \{\ell_{as}, c_{bs}, b_{as}, c_{as}, c_{ps}\}$, we characterize matrix classes $(f_{bv} : Y), (f_{\ell_1} : Y), (d(\ell_1) : Y), (d(bv) : Y)$ and $(Y : f_{bv}), (Y : f_{\ell_1}), (cs : d(\ell_1)), (Y : d(bv))$ in the last section. We should note from now on that the investigation of the domain of some particular limitation matrices, namely Cesaro means of order one, Euler means of order $r$, Riesz means, Nrlund means, the double band matrix $B(r, s)$, the triple band matrix $B(r, s)$, etc., in the spaces $\int bv$ and $d(bv)$ will lead us to new results which are not comparable with the present results. If we can choose different sequence spaces for the space $Y$, it can study new matrix characterizations of $(f_{bv} : Y), (f_{\ell_1} : Y), (d(\ell_1) : Y), (d(bv) : Y)$ and $(Y : f_{bv}), (Y : f_{\ell_1}), (cs : d(\ell_1)), (Y : d(bv))$. Also the spaces $\int bv$ and $d(bv)$ can be defined by an index $p$ and paranormed sequence spaces as $p = (p_k)$ is a sequence of strictly positive numbers.

Acknowledgements

I would like to express thanks to Scientific Projects Coordination Unit of Istanbul University which for supportive of this work. Project Number 34465

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