We study both entanglement and the Rényi entropies for 2 dimensional Dirac fermions in
the presence of topological Wilson loops. In the language of $\mathbb{Z}_n$ orbifold theories, the Wilson
loop is interpreted as an electric operator while the orbifold twist operator as a magnetic one.
The entropies depend on the Wilson loops and the corresponding topological phase transitions,
both in the large radius and low temperature limits. This happens because the entropies depend
on the conformal dimensions of the electromagnetic operators.

1 Introduction & Conclusion

This paper is to provide details on the Wilson loop contributions to entanglement and the Rényi
entropies that has been outlined in [1]. It is also a continuation of previous studies [2]. Quantum
entanglement is an important property of quantum theories. While there has been much progress
in the holographic entanglement entropies [3][4][5], exact computations on the field theory side have
been known to be difficult even in 1+1 dimensions [6][7][8][9].

Gauge fields are important tools for investigating the properties of quantum fields. Their time
and space components are chemical potential and current source (actually source of current), respecti-
vely. There have been surprising results regarding entropy in the presence of the background gauge
fields. At zero temperature, they are independent of a finite chemical potential for free fermions
[10][11]. For an infinite system with a finite cut, they also have been shown to be independent of
chemical potential [12]. These surprising, yet scattered, results deserve deeper understanding in a
systematic fashion.

In recent papers [1][2], we have constructed general formulas for entanglement and the Rényi
entropies of 2 dimensional Dirac fermions in the presence of background gauge fields. Some salient
features are the following: First, unlike previous results, we have shown that entropies do depend on chemical potential at zero temperature when chemical potential coincides with one of the energy levels of a quantum system. Second, in the large radius limit, the entropies do not depend on the gauge fields, supporting a recent claim [12] and generalizing it to current source and multi-cuts.

The effects of Wilson loops on entropies has been introduced in [13]. We review this below. The authors of [13] have computed the Rényi entropy on an infinite space. Their Rényi entropy does not have a smooth entanglement entropy limit for $n \to 1$, where $n$ is the number of replica copies.

Here, we generalize the previous results in two different ways. First, we expand our previous results [1][2] by including the contribution of Wilson loops to the entropies. Second, we further develop the results of [13] by accommodating the smooth entanglement entropy limit and the finite size effects. One crucial observation is that the Wilson loop is nothing but the electric operator in the language of $\mathbb{Z}_n$ orbifold theories, while the twist parameter $k/n$ ($k = -(n-1)/2, -(n-3)/2, \cdots , (n-1)/2$ represents the magnetic parameter. The conformal dimension of the electromagnetic operator plays a key role in evaluating the entropies. We employ different normalization factors to have smooth entanglement entropy limits for $n \to 1$. We demonstrate that entanglement entropy depends on the Wilson loops parameter in the large radius limit as well as in the low temperature limit. Figure 1 clearly depicts different elements that play key roles in the entropies. The dotted blue loop with arrows represent the Wilson loops.

The paper is organized as follows. We present the most general formula for the Rényi entropy including the Wilson loops parameter as well as the background gauge fields in §2 following [1]. We also introduce the conformal dimensions of the electromagnetic operators there. In §3 we compute the spin structure independent part of the Rényi entropy. We discuss an alternative normalization that gives us entanglement entropies as smooth limits of the Rényi entropies. In §4 we compute the Rényi and entanglement entropies for the spin structure dependent part. We consider the high temperature entropies and mutual information in separate sections §5 and §6 respectively.

Figure 1: Torus with multiple cuts that have the Wilson loops along with the twist boundary conditions.
2 Generalized entanglement entropy with Wilson loops

We continue the investigation of entanglement entropy for 2 dimensional Dirac fermions by including the Wilson loops around a cut in the 2 dimensional torus. Let us start with known expressions\[1\] for entanglement entropy with \( n \) replica copies in the presence of the background chemical potential \( \mu \) and current source \( J \). For a subsystem \( A \) with a cut stretching between two points \( u \) and \( v \) (with a dimensionless length \((u-v)/2\pi L\)), entanglement entropy is given by

\[
S_n = \frac{1}{1-n} \log \text{Tr}(\rho_A^n) = \frac{1}{1-n} \log \left[ \prod_{k=\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \sigma_k(u)\sigma_{-k}(v) \rangle_{a,b,J,\mu} \right].
\]

Where \( \rho_A \) is a density matrix. Two point functions of the twist operators, \( \sigma_k(u) \) and \( \sigma_{-k}(v) \), are

\[
\langle \sigma_k(u)\sigma_{-k}(v) \rangle_{a,b,J,\mu} = \left| \frac{2\pi i \tau \pi L}{\theta[1/2]((u-v)/2\pi L)} \right|^2 | \frac{\theta^{1/2-a-J}((k u-v)/2\pi L \tau + \tau_1 J + i \tau_2 \mu/\tau)}{\theta^{1/2-a-J}((\tau_1 J + i \tau_2 \mu/\tau))} |^2.
\]

Here we note that the first factor is only a function of the conformal dimension of the twist operator \( k \) and the system size. The second factor depends on the twisted boundary conditions (represented by the parameters \( a, b \), the background gauge fields \( \mu, J \), and the combination of the conformal dimension and the system size \( \frac{k}{n} \frac{u-v}{2\pi L} \)). From now on we use \( \ell_t = u - v \).

Before going further, let us comment on some subtleties of the Rényi entropy\[1\]. It can be written in a slightly different form as \( S_n = \frac{1}{1-n} \left( \log Z[n] - n \log Z[1] \right) \). We have omitted the normalization factor \( Z[1] \) in\[1\], for it usually gives us the numerical value 1 and does not contribute.

Below, we encounter the case with a non-trivial normalization factor, which is crucial to have a smooth limit for entanglement entropy.

Now we generalize entanglement entropy in the presence of the topological Wilson loops \( w \) around the cut connecting two points \( u \) and \( v \). The result is given by\[1\]

\[
S_n^w = \frac{1}{1-n} \left( \log Z[n] - n \log Z[1] \right) = S_n^{w,0} + S_n^{w,\mu;J},
\]

\[
Z[n] = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \sigma_{w,k}(u)\sigma_{-w,-k}(v) \rangle_{a,b,J,\mu}, \quad Z[1] = \langle \sigma_{w,0}(u)\sigma_{-w,0}(v) \rangle_{a,b,J,\mu}.
\]

Here the correlators include the parameters \( w \) and \( k \). Note that the normalization factor has the \( w \) dependence because it is independent of the implementation of the replica trick, which is clear from the discussion around\[3\]. The correlators have been already worked out in\[14\] in the language of ‘electromagnetic’ operators for the \( \mathbb{Z}_n \) orbifold theory compactified on a circle. There, \( k \) is identified as a magnetic charge, while \( w \) as an electric charge. In the context of applying this electromagnetic operator to the entropies, \( k \) has been first identified as the magnetic operator in\[15\] while the parameter \( w \) has been introduced in\[13\]. In\[13\], the Rényi entropy is computed in flat space.

We attempt to understand the role of these two parameters \( k, w \) more systematically by treating them on equal footing. First, let us consider the \( n \) fermions for \( w = 0 \). When the \( m \)-th fermion
Since this additional phase is added uniformly, the diagonal fields have the same effect. The global phase rotation \( \tilde{\psi}_{m+1} = \psi_{m+1} \) and a magnetic parameter \( w, k \) with a conformal dimension \( k \) standard twist operator \( \sigma \). These phase shifts can be handled by the generalized twist operators \( \tilde{\psi}_m = \psi_m + \omega \psi_{m+1} \). The corresponding conformal dimension is given by \( \Delta_w = k^2 / (2n^2) \). The full twist operator is \( \sigma_n = \Pi_{n} \sigma_{k/n} \).

In \[13\], the authors notice that the boundary condition can be further generalized to include a global phase rotation \( \tilde{\psi} \rightarrow e^{i\psi} \). This phase can be added to the boundary condition for \( \tilde{\psi} \) as

\[ \tilde{\psi}_m(e^{2\pi i}(x-u)) = e^{i\omega \tilde{\psi}_{m+1}(x-u)}, \quad \tilde{\psi}_m(e^{2\pi i}(x-v)) = e^{-i\omega \tilde{\psi}_{m-1}(x-v)} \]

(4)

where \( x \) is a complex coordinate. These boundary conditions can be diagonalized by defining new fields

\[ \psi_k = \frac{1}{n} \sum_{m=1}^{n} e^{2\pi imk} \tilde{\psi}_m. \]

(5)

For this new field, the boundary conditions become

\[ \psi_k(e^{2\pi i}(x-u)) = e^{2\pi ik/n} \psi_k(x-u), \quad \psi_k(e^{2\pi i}(x-v)) = e^{-2\pi ik/n} \psi_k(x-v), \]

(6)

where \( k = -(n-1)/2, -(n-3)/2, \ldots, (n-1)/2 \). The phase shift \( e^{2\pi ik/n} \) is generated by the standard twist operator \( \sigma_{k/n} \), which is identified as the magnetic operator is the orbifold theories, with a conformal dimension \( k^2 / (2n^2) \). The full twist operator is \( \sigma_n = \Pi_{n} \sigma_{k/n} \).

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(7)

Since this additional phase is added uniformly, the diagonal fields have the same effect.

\[ \psi_k(e^{2\pi i}(x-u)) = e^{2\pi ik/n+i\omega \psi_k(x-u)}, \quad \psi_k(e^{2\pi i}(x-v)) = e^{-2\pi ik/n-i\omega \psi_k(x-v)}. \]

(8)

These phase shifts can be handled by the generalized twist operators \( \sigma_{w,k} \) with an electric parameter \( w \) and a magnetic parameter \( k \). The corresponding conformal dimension is given by \[1\]

\[ \Delta_{w,k} = \text{conformal dimension} = \frac{1}{2} \left( \frac{k}{n} + \frac{w}{2\pi} + l_k \right)^2. \]

(9)

The constant \( l_k \) stems from the fact that the boundary condition \( \psi \) has an intrinsic ambiguity. This constant is used to minimize the conformal dimension of the twist operator such that \( -1/2 \leq \alpha_{w,k} \leq 1/2 \). We will come back to this point to draw some interesting conclusions below.

The two point functions of the electromagnetic operators \( \sigma_{w,k}(u) \) and \( \sigma_{-w,-k}(v) \), in the presence of \( \mu, J, a \) and \( b \) are given by

\[ \langle \sigma_{w,k}(u) \sigma_{-w,-k}(v) \rangle_{a,b,J,\mu} = \left| \frac{2\pi \eta(\tau)}{\eta(1/2)^3} \right|^2 \left| \frac{\eta_{b-1/2}^{1/2-a-J} \eta_{b-1/2}^{1/2-a-J} \eta_{b-1/2}^{1/2-a-J}}{\eta_{b-1/2}^{1/2-a-J} \eta_{b-1/2}^{1/2-a-J}} \right|^2, \]

(10)

where \( \alpha_{w,k} = \frac{k}{n} + \frac{w}{2\pi} + l_k \). For the rest of the paper, we study the entropies \[3\] of \( \sigma_{w,k}(u) \sigma_{-w,-k}(v) \) by focusing on their dependence on \( w \). Other properties including their dependence on \( \mu, J, a \) and \( b \) have been studied in detail in \[1\].
3 Entanglement entropy independent of spin structures

We compute the spin structure independent entropy $S_n^{w,0}$ in (3) in this section.

$$S_n^{w,0} = \frac{1}{1-n} \left( \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \alpha_{w,k}^2 - n \alpha_{w,0}^2 \right) \log \left[ \frac{2\pi \eta(\tau)^3}{\vartheta_{1/2}^{(1/2)}(\ell_1|\tau)|}\right]^2. \quad (11)$$

$S_n^{w,0}$ is independent of $\mu$ and $J$. We note the second factor $\alpha_{w,0}^2$ comes from the normalization factor $Z[1]$. When $\alpha_{w,k}$ is independent of $w$, the sum over $k$ can be done in a straightforward manner. Now, $\alpha_{w,k}^2$ has an extra dependence on $k$ due to the presence of $l_k$'s, $k = -\frac{n-1}{2}, -\frac{n-3}{2}, \cdots, \frac{n-1}{2}$. Thus we can not perform the summation trivially. The values of $l_k$'s used in [13] caused difficulties in evaluating entanglement entropy from the Rényi entropy. Here we consider the computations more carefully to resolve this issue and to provide a formal way to compute entanglement entropy from Rényi entropy in the presence of the Wilson loop.

As the conformal dimensions of the electromagnetic operators [9] satisfy the condition $-1/2 \leq \alpha_{w,k} \leq 1/2$, the parameters $l_k$'s change their values depending on $w$. For example, all the $l_k$'s vanish for $w < \pi/n$, only $l_{k=(n-1)/2} = -1$ for $\pi/n \leq w < 3\pi/n$, and two of the $l_k$'s are non-zero, $l_{k=(n-1)/2} = -2, l_{k=(n-3)/2} = -1$, for $3\pi/n \leq w < 5\pi/n$, and so on. If $w = 0$, we do not need to introduce the parameter $l_k$. To see its physical consequences in entanglement and the Rényi entropy, it is best to work out a few simple cases. We also discuss for the general case below.

**Entanglement entropy $S_n^{w,0}$ for $w < \pi/n$.**

Let us evaluate entanglement entropy (11) for $w < \pi/n$. We can set $l_k = 0$ for all $k$. Then

$$\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \alpha_{w,k}^2 - n \alpha_{w,0}^2 = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left( \frac{k}{n} + \frac{w}{2\pi} \right)^2 - n \left( \frac{w}{2\pi} \right)^2 = \frac{n^2 - 1}{12n}. \quad (12)$$

Note that entanglement and the Rényi entropies do not depend on the Wilson loop. In fact, for a small electric parameter $w < \pi/n$, the entropies are the same as those without the Wilson loop. We also check that this Rényi entropy has a smooth entanglement entropy limit when we take $n \rightarrow 1$. This result has been reported in [13].

**Entanglement entropy $S_n^{w,0}$ for $\pi/n \leq w < 3\pi/n$.**

For $\pi/n \leq w < 3\pi/n$, the Rényi entropy experiences a transition that requires one of the $l_k$ to be non-zero, $l_{(n-1)/2} = -1$. Then the parenthesis of (11) can be evaluated naively as

$$\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left( \frac{k}{n} + \frac{w}{2\pi} \right)^2 - n \left( \frac{w}{2\pi} \right)^2 = \frac{n^2 - 1}{12n} - \frac{w}{\pi} + \frac{1}{n}. \quad (13)$$

This seems to be fine for the Rényi entropy. What happens to the entanglement entropy? The limit $n \rightarrow 1$ is not well defined, and we end up with a singular entanglement entropy. This difficulty has been discussed in [13].
After looking back more carefully, we realize that it is unclear how to define $\alpha_{w,0}$ in (11). In fact, $\alpha_{w,0}$ can be chosen differently for different topological sectors. By reminding that one of the motivation of the Rényi entropy is to evaluate entanglement entropy, we choose

$$\alpha_{w,0}^2 = \lim_{n \to 1} \left( \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \alpha_{w,k}^2 \right). \tag{14}$$

With this choice, the Rényi entropy has a smooth limit for entanglement entropy.

With this in mind, we first evaluate $\sum_k \alpha_{w,k}^2$ and use (14) to compute the normalization factor.

$$\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \alpha_{w,k}^2 = \frac{n^2 - 1}{12n} + n\left( \frac{w}{2\pi} \right)^2 - \frac{w}{\pi} + \frac{1}{n}, \quad \alpha_{w,0}^2 = \left( \frac{w}{2\pi} \right)^2 - \frac{w}{\pi} + 1. \tag{15}$$

The entropies are

$$S_{n,0}^w = -\left( \frac{n + 1}{12n} + \frac{w}{\pi} - \frac{n + 1}{n} \right) \log \left| \frac{2\pi \eta(\tau)^3}{\vartheta_{1/2}^2 \left( \frac{L}{2\pi L} \right)} \right|^2. \tag{16}$$

Entanglement entropy can be obtained by taking $n \to 1$! Note the entropies do depend on the Wilson loops parameter $w$. Due to the different normalization we take, the results for the Rényi and entanglement entropies are different than those of [13].

**Entanglement entropy** $S_{n,0}^w$ for $3\pi/n \leq w < 5\pi/n$.

We consider the case for $3\pi/n \leq w < 5\pi/n$. The Rényi entropy experiences another transition that requires two of the $l_k$’s to be non-zero, $l_{(n-3)/2} = -1$ and $l_{(n-1)/2} = -2$.[1] Thus,

$$\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \alpha_{w,k}^2 = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left( \frac{k + w}{2\pi} \right)^2 - \frac{w}{\pi} - 1 - \frac{w}{2\pi} \left( \frac{n - 3}{2n} + \frac{w}{2\pi} \right)^2 + \frac{w}{2\pi} \left( \frac{n - 1}{2n} + \frac{w}{2\pi} \right)^2 - \frac{w}{2\pi} \left( \frac{n - 1}{2n} + \frac{w}{2\pi} \right)^2 \tag{17}$$

$$= \frac{n^2 - 1}{12n} + n\left( \frac{w}{2\pi} \right)^2 - \frac{3w}{\pi} + \frac{2n + 5}{n}.$$

With the new normalization (14),

$$\alpha_{w,0}^2 = \left( \frac{w}{2\pi} \right)^2 - \frac{3w}{\pi} + 7. \tag{18}$$

The entropies are

$$S_{n,0}^w = -\left( \frac{n + 1}{12n} + \frac{3w}{\pi} - \frac{7n + 5}{n} \right) \log \left| \frac{2\pi \eta(\tau)^3}{\vartheta_{1/2}^2 \left( \frac{L}{2\pi L} \right)} \right|^2. \tag{19}$$

---

[1] There seem to be more than one possible choice for the parameters $l_k$’s, that might be still consistent with the parameter $w$ for large $n$. All these cases except we consider in the main body excludes $n = 1$ case. Thus, requiring to have entanglement entropy limit uniquely determine the $l_k$’s and thus the Rényi entropy.
In this section, we compute $S_{n}^{w,0}$ for general case, $(2p-1)\pi/n \leq w < (2p+1)\pi/n$.

We present a general case with $p$ non-zero values for $l_k$’s. The Rényi entropy experiences $p$ transitions. Their values are $l_{(n-(2p-1))/2} = -1$, $l_{(n-(2p-3))/2} = -2$, $\cdots$, $l_{(n-1)/2} = -p$. Thus,

$$
\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{a_{w,k}^2}{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left( \frac{k + \frac{w}{2\pi}}{n} \right)^2 + \left( \frac{n - (2p - 1)}{2n} + \frac{w}{2\pi} - 1 \right)^2 - \left( \frac{n - (2p - 1)}{2n} + \frac{w}{2\pi} \right)^2 + \cdots + \left( \frac{n - 1}{2n} + \frac{w}{2\pi} - p \right)^2 - \left( \frac{n - 1}{2n} + \frac{w}{2\pi} \right)^2}
$$

$$
= n^2 - \frac{1}{12n} + n \left( \frac{w}{2\pi} \right)^2 - \frac{p(p+1)}{2} \left( \frac{w}{\pi} + 1 \right) + \frac{n + 1}{6} \frac{p(p+1)(2p+1)}{}.
$$

With the new normalization [14], $a_{w,0}^2 = \left( \frac{w}{2\pi} \right)^2 - \frac{p(p+1)}{2} \left( \frac{w}{\pi} + 1 \right) + \frac{p(p+1)(2p+1)}{3}$. The entropies are

$$
S_{n}^{w,0} = -\left( \frac{n + 1}{12n} + \frac{p(p+1)}{2} \left( \frac{w}{\pi} + 1 \right) - \frac{2n + 1}{n} \frac{p(p+1)(2p+1)}{6} \right) \log \left| \frac{2\pi\eta^3}{\vartheta_{1/2}^{1/2}(\frac{\ell_t}{2\pi L}|\tau)} \right|^2.
$$

Let us summarize some important observations from this result. First, in the presence of the Wilson loops, the extra phase factors, $l_k$’s, have important roles to have proper conformal dimensions for the electromagnetic operators. Due to this, there are different topological sectors. These topological sectors can be distinguished by the Rényi entropy and/or entanglement entropy. Second, this Rényi entropy has a smooth and well defined entanglement entropy limit contrary to the previous results [13]. This happens because we employ different normalization factors for different topological sectors. This entanglement entropy is plotted in figure 2. Third, the high temperature limit can be computed as in [2]. The front factor depending on $w$ and $p$ is the same as in (21) with the other parts unmodified.

Finally, the Rényi and entanglement entropies have non-zero contributions when we take the infinite volume limit $\frac{\ell_t}{2\pi L} \to 0$ because of the theta function $\vartheta_{1/2}^{1/2}(\frac{\ell_t}{2\pi L}|\tau) = \theta_1(\frac{\ell_t}{2\pi L}|\tau) \propto \sin(\frac{\ell_t}{2\pi L})$.

This is different from the other part, $S_{n}^{w,\mu,J}$, that depends on the spin structure. We consider this case in the coming sections. This is clearly noted in [1].

## 4 Entanglement entropy depending on spin structures

In this section, we compute $S_{n}^{w,\mu,J}$ given in [3] and [10] with emphasis on the Wilson loop parameter $w$, which has been called as an electric parameter [14]. As mentioned in the previous section, determining normalization factors is not straightforward. We compute the Rényi entropy with this
in mind.

\[
S_{n,\mu,J}^{\omega} = \frac{1}{1-n} \left[ \log \tilde{Z}_s[n] - n \log \tilde{Z}_s[1] \right],
\]

\[
\tilde{Z}_s[n] = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left| \frac{\vartheta^{[1/2-a-J]}(\tfrac{k}{2\pi}, \omega, k + \tau_1 + i\tau_2 \mu|\tau)}{\vartheta^{[1/2-a-J]}(\tau_1 + i\tau_2 \mu|\tau)} \right|^2, \quad \tilde{Z}_s[1] = \lim_{n \to 1} \tilde{Z}_s[n].
\]

Where \( \tilde{Z}_s[n] \) is a part of \( Z[n] \) that depends on the spin structures.

We note the \( \omega \) dependence of the Rényi and entanglement entropies on a torus has not been studied in the literature. The Jacobi theta functions in (22) have \( k \) dependences in the argument. Obtaining closed form by summing over \( k \) with full generality is not feasible. The case with \( \omega = 0 \) has been throughly studied in various limits, such as the zero temperature limit and the large radius limit, in [2]. There we showed novel and interesting results. For more details, refer to [2].

4.1 Entanglement entropy \( S_{n,\mu,J}^{\omega} \) with \( \mu = J = 0 \).

We choose \( \mu = J = 0 \) to illustrate the physical effects of the Wilson loops in the entropies for the anti-periodic fermions. Thus, we set \( a = b = \frac{1}{2} \). We compute the entropies using (22).

\[
\log \tilde{Z}_s[n] = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \frac{\vartheta(\tfrac{k}{2\pi})}{\vartheta(0|\tau)} \right|^2.
\]

Similar computations have been presented in [2]. Thus we quote the results for various limits by emphasizing the role of \( l_k \)’s.
Low temperature limit

First, we consider $w < \pi/n$ in the low temperature limit $\beta = 2\pi \tau_2 \to \infty$. We find

$$
\log \tilde{Z}_s[n] = 4 \sum_{k=-\infty}^{n-1} \sum_{l,m=1}^{\infty} \frac{(-1)^{l-1} \cos(\alpha(lm-1/2))}{l} \left[ \cos \left( \frac{\ell_lk}{L} \frac{k}{n} + \frac{w}{2\pi} + l_k \right) - 1 \right]
$$

$$
= 4 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \sinh \left( \frac{\ell_l}{2} \right) \frac{\cos(\ell_l/2)}{\cosh(l\beta) - \cos(l\alpha)} \left[ \cos \left( \frac{\ell_l}{2\pi L} w \right) \sin \left( \frac{\ell_l}{2\pi L} \right) - n \right].
$$

(25)

Where we expand $i \beta(z|\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^{m-1/2})(1 + y^{-1}q^{m-1/2})$ in the low temperature limit, $|q| = |e^{2\pi i \tau}| \ll 1$ for general $y = e^{2\pi iz}$. The parameter $\alpha = 2\pi \tau_1$ is different from $\alpha_{w,k}$. The sums over $k$ and $m$ can be done independently. We choose the normalization factor to ensure the existence of the smooth limit for entanglement entropy.

$$
\log \tilde{Z}_s[1] = 4 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \sinh \left( \frac{\ell_l}{2} \right) \frac{\cos(\ell_l/2)}{\cosh(l\beta) - \cos(l\alpha)} \left[ \cos \left( \frac{\ell_l}{2\pi L} w \right) - 1 \right].
$$

(26)

Thus, the Rényi entropy becomes

$$
S_n^{w,\mu=0,J=0} = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \sinh \left( \frac{\ell_l}{2} \right) \frac{\cos(\ell_l/2)}{\cosh(l\beta) - \cos(l\alpha)} \cos \left( \frac{\ell_l}{2\pi L} w \right) \frac{4}{1-n} \left[ \sin \left( \frac{\ell_l}{2\pi L} \right) - n \right].
$$

(27)

Entanglement entropy is

$$
S_n^{w,\mu=0,J=0} \to 1 = 4 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \sinh \left( \frac{\ell_l}{2} \right) \frac{\cos(\ell_l/2)}{\cosh(l\beta) - \cos(l\alpha)} \cos \left( \frac{\ell_l}{2\pi L} w \right) \left[ 1 - \frac{\ell_l}{2L} \cot \left( \frac{\ell_l}{2L} \right) \right].
$$

(28)

Note that the entropies depend on the Wilson loop parameter $w$ as $\cos \left( \frac{\ell_l}{2\pi L} w \right)$ for small $w$.

Let us quote the results for a general case $(2p - 1)\pi/n \leq w < (2p + 1)\pi/n$ with an integer $p$. There are extra terms, similar to those in the previous section, in addition to the terms (25).

$$
\log \tilde{Z}_s[n] = 4 \sum_{l,m=1}^{\infty} \frac{(-1)^{l-1} \cos(\alpha(lm-1/2))}{l} \left[ \sum_{k=-\infty}^{n-1} \cos \left( \frac{\ell_lk}{L} \frac{k}{n} + \frac{w}{2\pi} + l_k \right) - n \right]
$$

$$
+ \sum_{r=1}^{p} \left\{ \cos \left( \frac{\ell_l}{L} \left[ n-(2p-2r+1) \right] + \frac{w}{2\pi} - r \right) - \cos \left( \frac{\ell_l}{L} \left[ n-(2p-2r+1) \right] + \frac{w}{2\pi} \right) \right\}.
$$

(29)

The first line is the same as (25) and so does the entropies. The second line contains all the new terms due to the topological transitions and the sum over $r$ gives

$$
\left\{ \cos \left( \frac{\ell_l}{2\pi L} (wn-(n+1)p\pi) \right) \sin \left( \frac{\ell_lp(n-1)}{2Ln} \right) \sin \left( \frac{\ell_l(n-1)p}{2Ln} \right) - \cos \left( \frac{\ell_l}{2\pi L} (wn+(n-p)\pi) \right) \sin \left( \frac{\ell_lp}{2Ln} \right) \frac{\ell_l}{2\pi L} \right\}.
$$
Thus log $\tilde{Z}_s[n]$ is the combination of (25) and this result. The corresponding normalization factor can be obtained by taking $n \to 1$ limit.

$$\log \tilde{Z}_s[1] = 4 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \frac{\sinh(l\beta/2) \cos(l\alpha/2)}{\cosh(l\beta) - \cos(l\alpha)} \left\{ \cos \left( \frac{\ell_l}{2\pi L} w \right) - 1 \right\} + p \cos \left[ \frac{\ell_l(w-2p\pi)}{2\pi L} \right] - \cos \left[ \frac{\ell_l(w+(1-p)\pi)}{2\pi L} \right] \frac{\sin(\ell_l p)}{\sin(\ell_l/2)} \right\} . \quad (30)$$

The Rényi entropy can be obtained by combining these two computations together. We present the explicit form for entanglement entropy.

$$S_{n \to 1}^\omega = 4 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \frac{\sinh(l\beta/2) \cos(l\alpha/2)}{\cosh(l\beta) - \cos(l\alpha)} \left\{ \cos \left( \frac{\ell_l}{2\pi L} w \right) \left[ 1 - \frac{\ell_l}{2L} \cot \left( \frac{\ell_l}{2L} \right) \right] \right\} + p \cos \left[ \frac{\ell_l(w-2p\pi)}{2\pi L} \right] - \cos \left[ \frac{\ell_l(w+(1-p)\pi)}{2\pi L} \right] \frac{\sin(\ell_l p)}{\sin(\ell_l/2)} \right\} . \quad (31)$$

The first line in the final result is the same as the case for $w < \pi/n$. The last line is an additional contribution for the topological phase transitions with non-zero values of the $l_k$’s. We can check that the last line vanishes for $p = 0$.

The parameter $p$ has been introduced to take into account of the topological transitions when the value of $w$ increases. Our entropies (28) and (31) show interesting behaviors. As we increase the Wilson loop parameter $w$ (and $p$) accordingly, the entropies reveal oscillatory as well as linear behaviors (31). We present some typical plots of entanglement entropy for $l = 1$ in the unit of $4 \frac{\sinh(l\beta/2) \cos(l\alpha/2)}{\cosh(l\beta) - \cos(l\alpha)}$ in figures 3 and 4.
Figure 4: The left plot is entanglement entropy \( S_{n \to 1}^\omega \) for \( l = 1 \) with \( \ell_t/L = 0.75 \) (large subsystem size) for \( 0 \leq w < 401 \pi \). The inset plot for \( 0 \leq w < 25 \pi \) shows the first plateau clearly. The right figure is an enlarged plot for \( 79 \pi \leq w < 121 \pi \) with clear demonstration of typical behavior of the entropy.

**Large radius limit**

The large radius limit can be conveniently evaluated with a simple observation. The theta function can be written in a slightly different form as \( \vartheta_3(z|\tau) = \prod_{m=1}^{\infty} (1 - q^m)(1 + q^{2m-1} + 2\cos(2\pi z)q^{m-1/2}) \), where \( q = e^{2\pi i \tau} \). Instead of expanding \( q \), we expand a small factor \( \delta z \) in \( z_1 = z_2 + \delta z \) with \( \delta z \propto \ell_t/L \). Thus,\[
\left| \frac{(1-q^m)(1+y_1q^{m-1/2})(1+y_2^{-1}q^{m-1/2})}{(1-q^m)(1+y_2q^{m-1/2})(1+y_1^{-1}q^{m-1/2})} \right|^2 = \left| \frac{1 + q^{2m-1} + 2\cos(2\pi z_1)q^{m-1/2}}{1 + q^{2m-1} + 2\cos(2\pi z_2)q^{m-1/2}} \right|^2 = 1 + O \left( \frac{\ell_t^2}{4\pi^2 L^2} \right)^2.
\]

This shows that the \( w \) dependence of the spin dependent entropies vanishes as fast as \( \frac{\ell_t^2}{4\pi^2 L^2} \), meaning that \( w \) dependence vanishes at zero temperature.

Actual computation of \( S_{w,0,J=0}^{n,L\to\infty} \) can be done by observing \( z_1 = \delta z = \frac{\ell_t}{2\pi L} \left[ k + \frac{w}{2\pi} + l_k \right] \ll 1, z_2 = 0 \).

\[
S_{n,L\to\infty}^{w,\mu=0,J=0} = \left( \frac{n+1}{12n} + \frac{p(p+1)}{2} \left[ \frac{w}{\pi} + 1 \right] - \frac{2n+1}{n} \right) \frac{p+1}{2} (2p+1) \left. \right|_{z_1 = \delta z = \frac{\ell_t}{2\pi L} \left[ k + \frac{w}{2\pi} + l_k \right] \ll 1, z_2 = 0} \times \frac{\ell_t^2}{4L^2} \left[ 1 + \tanh^2 \left( \frac{\beta \mu}{2} \right) + \sum_{r=1}^{p} \frac{4 + 4 \cosh(\beta \mu) \cosh(\beta r)}{\cosh(\beta \mu) + \cosh(\beta r)} \right].
\]

Thus we confirm the spin structure dependent entropies vanish at the large radius limit. We note that this result for the large radius limit applies for all the spin dependent entropies regardless of the parameters, \( w, \mu, J, a \) and \( b \). Thus, we do not consider the large radius limit anymore.

Importantly, of course, there exist finite and dominant contributions in the large radius limit from the spin structure independent part \( S_{n}^{w,0} \) that we have mentioned in the previous section and also given in [1].
4.2 Entanglement entropy $S_{n,\mu,J}^\omega$ with $\mu,J$.

Here we present the physical effects of the Wilson loops in the entropies of the anti-periodic fermions in the low temperature limit, $\beta = 2\pi T_2 \to \infty$. Thus $a + J = b = 1/2$. Similar computations were presented in [2]. We compute $\log \hat{Z}_s[n] = \sum_{l,m=1}^{n-1} \frac{2(1)e^{\beta \mu}}{l e^{\beta(m-1/2)}} \left[ \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \left\{ \cos \left( \frac{\ell l}{L} \alpha_{w,k} + \alpha(l - m + \frac{1}{2}) \right) \right\} \right] ^2 \left[ 1 - \frac{\ell l}{2L} \cot \left( \frac{\ell l}{2L} \right) \right]$. 

$$S_{n,\mu,J}^\omega = 2 \sum_{l,m=1}^{\infty} \frac{(-1)^l e^{\beta \mu}}{l e^{\beta(m-1/2)}} \left\{ \cos \left( \alpha(l - m + \frac{1}{2}) + \frac{\ell l}{2L} w \right) \right\} \left[ 1 - \frac{\ell l}{2L} \cot \left( \frac{\ell l}{2L} \right) \right]$$

$$+ p \cos \left( \alpha(l - m + \frac{1}{2}) + \frac{\ell l (w-2p\pi)}{2L} \right) \sin \left( \alpha(l - m + \frac{1}{2}) + \frac{\ell l (w-2p\pi)}{2L} \right) \left\{ e^{\beta \mu} \to e^{-\beta \mu} \right\} \left\{ \alpha(l - m + \frac{1}{2}) \to \alpha(l - m + \frac{1}{2}) \right\}.$$
Figure 5: Left: This 3 dimensional plot is entanglement entropy \( S_{l,m}^{\ell / L} \) for \( l = 1, m = 1 \) with \( \ell / L = 0.5 \) for \( 0 \leq w < 21\pi \) and \( 0 < \alpha J < 2\pi \) in the unit of \( e^{\beta(\mu - 1/2)} \). Note that the entropy develops discontinuity for \( w = (2r - 1)\pi \) with an integer \( r \). Different strips \( (2r - 1)\pi < w < (2r + 1)\pi \) have developed interesting twists along the range \( 0 < \alpha J < \alpha_J \) depending on the parameter \( r \). This is the case even when \( \mu \neq 0 \). When \( \beta \to 0 \), the entropy develops further dependence on the parameter \( \alpha = 2\pi \tau_1 \). To demonstrate this, we plot the same function with \( \alpha = 0 \) in the right plot, where the entropy is continuous as we change \( w \).

5 High temperature limit

In this section we consider the high temperature limit of the entropies. As mentioned, spin structure independent entropies still have interesting dependences on \( w \) similar to (21). Here we focus on the spin structure dependent entropies. For the anti-periodic fermions, we can use the S-duality formula for the theta-function, \( \vartheta_3(z|\tau) = (-i\tau)^{-1/2}e^{-\pi i z^2/\tau} \vartheta_3(z/\tau| -1/\tau) \). In general, the formula (22) can be rewritten as

\[
S_{w,\mu,J}^{\ell,H} = \frac{1}{1-n} \left[ \log \tilde{Z}_s^H[n] - n \log \tilde{Z}_s^H[1] \right],
\]

where

\[
\tilde{Z}_s^H[n] = \prod_{k=-n}^{n-1} \left| e^{-i\pi/2} \left( \frac{\ell_1}{2\pi L} + \tau_1 J + i\tau_2 \mu \right)^2 \frac{\vartheta[1/2-a-J](\frac{\ell_1}{2\pi L} + \tau_1 J + i\tau_2 \mu | -1/\tau)}{\vartheta[1/2-a-J](\tau_1 J + i\tau_2 \mu | -1/\tau)} \right|^2, \quad (38)
\]

and \( \tilde{Z}_s^H[1] = \lim_{n \to 1} \tilde{Z}_s^H[n] \).

We noticed previously that there are two different limits depending on \( \alpha = 2\pi \tau_1 \). The two high temperature limits with \( \alpha = 0 \) and \( \alpha \neq 0 \) are different and the order of taking \( \beta \to 0 \) with these do not commute. We consider a simpler case \( \alpha = 0 \) first.

**Entanglement entropy** \( S_{w,\mu,J}^{\ell,H} \) with \( \alpha = 0 \).

Let us focus on the anti-periodic fermions for \( \alpha = 0, \tau = i\tau_2 = i\frac{2\pi}{\beta} \). To see the physical properties of \( w \) clearly, we set \( \mu = J = 0 \). The first two exponential factors in (38) can be done directly by referring to the previous computation (20). For the general case with \( p \) transitions,
\[
\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| e^{-\frac{2\pi^2}{\beta}(\frac{\ell}{2\pi} \left[ \frac{k}{n} + \frac{w}{2\pi} + k \right])} \right|^2 = -\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\ell^2}{\beta L^2} \left( \frac{k}{n} + \frac{w}{2\pi} + k \right)^2 \
+ \frac{\ell^2}{\beta L^2} \left( \frac{n^2 - 1}{12n} + n \left( \frac{w}{2\pi} \right)^2 - \frac{p(p+1)}{2} \left( \frac{w}{\pi} + 1 \right) + \frac{n+1}{n} \frac{p(p+1)(2p+1)}{6} \right). \tag{39}
\]

For the second theta functions of \([38]\) with \(p\) transitions, we also have
\[
\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \frac{\vartheta_3(\frac{\ell}{2\pi} \frac{aw,k}{\frac{1}{\tau}} \frac{1}{\vartheta_3(0|\frac{1}{\tau})})}{\vartheta_3(0|\frac{1}{\tau})} \right|^2 = \sum_{l=1}^{\infty} \frac{2(-1)^{l-1}}{l \sinh(\frac{2\pi l}{\beta})} \left[ \cosh \left( \frac{\pi l}{\beta L} \right) \sinh \left( \frac{\pi l}{\beta L} \right) - n \right. \\
+ \frac{\sinh(\frac{w}{2\pi} + \frac{1}{2} - \frac{p}{2}) - \sinh(\frac{w}{2\pi} + \frac{1}{2})}{\sinh(\frac{1}{2\pi})} +\frac{\sinh(\frac{w}{2\pi} - p) - \sinh(\frac{w}{2\pi} - \frac{p}{2})}{\sinh(\frac{1}{2\pi})} \left. \right] \tag{40}
\]

log \(Z_s^H[n]\) can be obtained by combining these two computations, \([39]\) and \([40]\). By computing the normalization factor log \(Z_s^H[1]\), we can get the Rényi entropy.

We write down entanglement entropy for the anti-periodic fermions.
\[
S_{n \to 1,H}^{w,\mu=0,J=0} = \frac{\ell^2}{\beta L^2} \left( \frac{1}{3} + \frac{p(p+1)w}{\pi} - 2p^2(p+1) \right) \tag{41}
+ \sum_{l=1}^{\infty} \frac{2(-1)^{l-1}}{l \sinh(\frac{2\pi l}{\beta})} \left[ \cosh \left( \frac{\pi l}{\beta L} \right) \left[ 1 - \frac{\pi \ell l}{\beta L} \cot \left( \frac{\pi \ell l}{\beta L} \right) \right] \\
+ p \cosh(p - \frac{w}{2\pi}) + \frac{1}{\sinh(\frac{1}{2})} \left( \sinh \left( \frac{w}{2\pi} + \frac{1}{2} - p \right) - \sinh \left( \frac{w}{2\pi} + \frac{1}{2} \right) \right) \right]. \tag{42}
\]

There are two different contributions to the entanglement entropy for \(\alpha = 0\) in the high temperature limit. The terms in the first line \([41]\) dominate over the rest of the terms for small \(w\). The contributions \([42]\) that contain all the topological transitions are actually dominant for large \(w\). The former is due to \(\frac{1}{\beta}\) as \(\beta \to 0\), while the latter is due to the last term \(- \sinh(\frac{w}{2\pi} + \frac{1}{2})\). This is quite an interesting result and is depicted in the figure [6].

Let us briefly mention the entropies in the presence of chemical potential and current source, \(\mu \neq 0\) and \(J \neq 0\). It turns out that the result is almost the same as \([41]\) and \([42]\). For the first contribution \([41]\), we have the same result due to the complex conjugate even in the presence of \(\mu, J\). For the second contribution \([42]\), we need to multiply \(\cos(2\pi \mu l)\) inside the \(l\) summation. This is due to the simple \(\mu\) dependence. Note that there is no explicit \(J\) dependence because \(J\) always enter with the combination \(\alpha J\), which is removed by setting \(\alpha = 0\). Nevertheless, the \(J\) dependence is hidden in the \(a + J = 1/2\) condition for the anti-periodic fermions.

Finally, we discuss the periodic fermions that has \(\vartheta_2\) instead of \(\vartheta_3\). Thus we can compute the high temperature limit of the periodic fermions by studying the S-dual formula for \(\vartheta_2\) which is
\[
\vartheta_2(z|\tau) = (-i\tau)^{-1/2} e^{-\pi i z^2/\tau} \vartheta_4(z/\tau) - 1/\tau.
\]
Now, the difference between \(\vartheta_3\) and \(\vartheta_4\) is a couple of sign. The resulting difference in entanglement entropy is one factor in \([42]\). Instead of \(\sum_{l=1}^{\infty} (-1)^{l-1}\), you can use \(\sum_{l=1}^{\infty} \) without the factor \((-1)^{l-1}\). We have looked in \(l = 1\) case specifically to find that the figure [6] still holds for the periodic fermions with \(\alpha = 0\).
Figure 6: Top: Entanglement entropy $S_{n \to 1, H}$ given in equations (41) and (42) for the parameters $\ell_t/L = 0.25$, $\beta = 0.1$. The entropy for $0 < w < 27$ is dominated by the first contribution (41) and rather different from the second contribution (42) that dominates over the range $w > 27$.

**Entanglement entropy** $S_{n, H}^{w, \mu, J}$ with $\alpha \neq 0$.

For the high temperature limit, we can ignore the $\beta$ dependence in $\tau$ for $2\pi\tau = 2\pi\tau_1 + i2\pi\tau_2 = \alpha + i\beta \to \alpha$. Then the entropy formula for the anti-periodic fermion goes

$$\log \tilde{Z}_H^n[n] = \sum_{k=-n/2}^{n/2-1} \log \left| \frac{e^{-\frac{\beta 2\pi^2 \alpha}{\alpha^2}} \left( e^{\frac{\beta \alpha \alpha \alpha \omega \beta}{2\alpha} + \frac{\beta \omega}{2\alpha} \right) \beta_3 \left( \frac{\beta \alpha}{2\alpha} + \frac{\beta \omega}{2\alpha} \right)}{e^{-\frac{\beta 2\pi^2 \alpha}{\alpha^2} \left( \frac{\beta \omega}{2\alpha} \right) \beta_3 \left( \frac{\beta \omega}{2\alpha} \right)}} \right|^2$$

$$= 2 \sum_{m=1}^{\infty} \sum_{k=-n/2}^{n/2-1} \log \left( \frac{\cos(2\pi J + \frac{\beta \omega}{\alpha} \frac{k}{\alpha} + \frac{\beta \omega}{\alpha} \frac{\omega}{\alpha}) + \cos\left( \frac{4\pi^2 \alpha}{\alpha^2} [m - \frac{1}{2}] \right)}{\cos(2\pi J) + \cos\left( \frac{4\pi^2 \alpha}{\alpha^2} [m - \frac{1}{2}] \right)} \right), \quad (43)$$

where we expand the theta functions with index $m$ and use $(1 + e^{ia+ib})(1 + e^{ia-ib})(1 + e^{-ia+ib})(1 + e^{-ia-ib}) = 4(\cos a + \cos b)^2$. For this case, we can not do the $k$ sum. We need to rely on some other methods than analytical approach. It will be interesting to look into this further.
6 Mutual Information

The Mutual information is useful to measure the entanglement between two intervals, and \( A \) and \( B \) of length \( \ell_A \) and \( \ell_B \) separated by \( \ell_C \). The mutual information is finite and free of UV divergences. In the context of chemical potential and current source, it has been considered to include \( \mu \) and \( J \) in \([11][1][2]\). Here, we generalize this further to include the Wilson loop parameter \( w \) with the assumption that the Wilson loop parameters are the same for the sub-spaces denoted by \( A \) and \( B \). Similar to the entropies, the mutual information factories into two different contributions.

\[
I_n^w(A, B) = I_n^{w,0}(A, B) + I_n^{w,\mu,J}(A, B). \tag{44}
\]

The first contribution is generalized to include the Wilson loops contribution as

\[
I_n^{w,0}(A, B) = \frac{1}{1-n}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \alpha_{w,k}^2 - \alpha_{w,0}^2 \log \left| \frac{\vartheta[1/2](\ell_A + \ell_B + \ell_C + \ell_{\mu} + \ell_J + \ell_{\tau})}{\vartheta[1/2](\ell_A + \ell_B + \ell_C + \ell_{\mu} + \ell_J + \ell_{\tau})} \right|^2. \tag{45}
\]

Where \( \alpha_{w,k} = \frac{k}{n} + \frac{w}{2\pi} + l_k \). This sum for the front fact has been evaluated \([21]\) to give us \( \frac{n+1}{12\pi} + \frac{n}{n+1}(\frac{\pi}{2} + 1) - 2n^{-1}p(p+1)(2p+1) \). The rest of the factor has been already studied.

The second contribution has the dependences on current source \( J \), chemical potential \( \mu \) and the Wilson loops parameter \( w \). We can write it as \( I_n^{w,\mu,J} = \log \tilde{Z}_s^{I}[n] - n\tilde{Z}_s^{I}[1] \)

\[
\tilde{Z}_s^{I}[n] = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left| \frac{\vartheta[1/2-a-J](\ell_A + \ell_B + \ell_{\mu} + \ell_J + i\tau_2\mu + \ell_{\tau})}{\vartheta[1/2-a-J](\ell_A + \ell_B + \ell_{\mu} + \ell_J + i\tau_2\mu + \ell_{\tau})} \right|^2. \tag{46}
\]

Various properties of this mutual information has been studied previously. Let us consider a low temperature limit: \( \beta \to \infty \) for the anti-periodic fermions with \( b = 1/2, a + J = 1/2 \). We also include the \( p \) transitions for the range of the Wilson loop parameter \( (2p-1)\pi/n \leq w < (2p+1)\pi/n \).

\[
\log \tilde{Z}_s^{I}[n] = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \left| \frac{\vartheta_3(\ell_A + \ell_B + \ell_{\mu} + \ell_J + i\tau_2\mu + \ell_{\tau})}{\vartheta_3(\ell_A + \ell_B + \ell_{\mu} + \ell_J + i\tau_2\mu + \ell_{\tau})} \right|^2. \tag{47}
\]

Upon examining \([47]\), we observe that

\[
\log \left| \frac{\vartheta_3(\ell_A \cdots)}{\vartheta_3(0 \cdots)} \right|^2\frac{\vartheta_3(\ell_B \cdots)}{\vartheta_3(0 \cdots)} \frac{\vartheta_3(\ell_A + \ell_B \cdots)}{\vartheta_3(0 \cdots)} \right|^2 \tag{48}
\]

The computations with two Theta functions has been carried out in various places in this paper, for example \([36]\). Now, the answer is sum of these expressions for \( \ell_A, \ell_B \) and \( \ell_A + \ell_B \), which is lengthy. We do not explicitly write them here. We note that the spin dependent mutual information is independent of the separation \( \ell_C \) between the two entanglement regions \( \ell_A \) and \( \ell_B \). The full answer for the mutual information is the sum \([45]\) and \([46]\) for the anti-periodic fermions.
We would like to have some further remarks on the mutual information. First, let us consider the zero temperature limit. The explicit computations for two theta functions \( \text{[48]} \) has been done in the section \([4,2]\) for the anti-periodic fermions, and the final entanglement expression is given in \([36]\) for \(-1/2 < \mu < 1/2\). One needs care for the range of \( \mu \) for the low temperature limit as mentioned previously. When \( \mu \) is half an integer, there is a finite contribution at zero temperature limit. This value of \( \mu \) has been identified as the energy level of the Dirac fermion on a circle. Thus entanglement entropy can be useful to identify energy levels of quantum theories \([1,2]\). For other values of \( \mu \), refer to \([2]\). We also check that the result reduces to the previous computation, equation \((4.5)\) in \([2]\), when \( w = 0 \).

The computation for the periodic fermions can be done similarly. The result is the same as the spin structure independent part, while the spin structure dependent part has some modifications, \( m - 1/2 \rightarrow m \) and additional term in the summation over the index \( l \). This new contribution is from the cos factor in the \( \vartheta_2 \) for the periodic fermions. For more details, refer to \([2]\). For the large radius limit, the spin dependent part \( I_n^{w,\mu,J} \) of the mutual information also vanishes, while \( I_n^{w,0}(A,B) \) given in \((45)\) survives.

Finally, we comment on the high temperature limit \( \beta \rightarrow 0 \). The general formulas \((45)\) and \((46)\) can be rewritten by using the modular transformation. We first consider \((45)\) that is independent of \( \mu, J \).

\[
I_{n,H}^{w,0}(A,B) = \frac{1}{1-\eta} \left( \sum_{k=-n}^{n-1} \alpha_w^{2,k} - \alpha_w^{2,0} \right) \times \log \left| \frac{e^{-\frac{\pi}{\tau} \left( \frac{A+\frac{B+C}{2}}{2\pi L} \right)^2} - e^{-\frac{\pi}{\tau} \left( \frac{C}{2\pi} \right)^2}}{e^{-\frac{\pi}{\tau} \left( \frac{A+\frac{B+C}{2}}{2\pi L} \right)^2} - e^{-\frac{\pi}{\tau} \left( \frac{B+C}{2\pi} \right)^2}} \right|^2 \right|^2. (49)
\]

Here we choose the normalization factor so that there exist \( n \rightarrow 1 \) limit. As mentioned, the high temperature limit is sensitive to \( \alpha = 2\pi \tau_1 \). For \( \alpha \neq 0 \), we compute it by using \( \frac{1}{\tau} = \frac{2\pi}{\alpha + \beta} \rightarrow \frac{2\pi}{\alpha} \). Thus \( I_{n,H}^{w,0}(A,B) \) gives us for \( \alpha \neq 0 \)

\[
I_{n,H}^{w,0}(\alpha \neq 0) = -C_{n,w} \left\{ 2 \log \left( \frac{\sin(\pi \frac{\ell_A + \ell_B + \ell_C}{L_n}) \sin(\pi \frac{\ell_C}{L_n})}{\sin(\pi \frac{\ell_A + \ell_B + \ell_C}{L_n}) \sin(\pi \frac{\ell_B + \ell_C}{L_n})} \right) \right\} (50)
+ 2 \sum_{m=1}^{\infty} \log \left\{ \frac{\cos(2\pi \frac{\ell_A + \ell_B + \ell_C}{L_n}) - \cos(4\pi^2 \frac{m}{\alpha})}{\cos(2\pi \frac{\ell_A + \ell_C}{L_n}) - \cos(4\pi^2 \frac{m}{\alpha})} \right\} \left\{ \frac{\cos(2\pi \frac{\ell_B + \ell_C}{L_n}) - \cos(4\pi^2 \frac{m}{\alpha})}{\cos(2\pi \frac{\ell_B + \ell_C}{L_n}) - \cos(4\pi^2 \frac{m}{\alpha})} \right\}. \]

Where \( C_{n,w} = \frac{n+1}{2n} + \frac{w(p+1)}{2} \left( \frac{w}{\pi} + 1 \right) - \frac{2n+1}{n} \frac{p(p+1)(2p+1)}{6} \) is the coefficient we obtained in \([21]\). We can compute the mutual information for \( \alpha = 0 \) with \( \frac{1}{\tau} = -i \frac{2\pi}{\alpha} \). There is an additional term due to the change for sine function to the hyperbolic sine in the exponential factor in \((49)\). Thus we get

\[
I_{n,H}^{w,0}(\alpha = 0) = -C_{n,w} \left\{ - \frac{2}{\beta} \frac{\ell_A \ell_B}{L_n^2} + 2 \log \left( \frac{\sinh(\pi \frac{\ell_A + \ell_B + \ell_C}{L_n}) \sinh(\pi \frac{\ell_C}{L_n})}{\sinh(\pi \frac{\ell_A + \ell_B + \ell_C}{L_n}) \sinh(\pi \frac{\ell_B + \ell_C}{L_n})} \right) \right\} \left\{ \frac{\cos(2\pi \frac{\ell_A + \ell_B + \ell_C}{L_n}) - \cosh(4\pi^2 \frac{m}{\beta})}{\cosh(2\pi \frac{\ell_A + \ell_C}{L_n}) - \cosh(4\pi^2 \frac{m}{\beta})} \right\} \left\{ \frac{\cos(2\pi \frac{\ell_B + \ell_C}{L_n}) - \cosh(4\pi^2 \frac{m}{\beta})}{\cosh(2\pi \frac{\ell_B + \ell_C}{L_n}) - \cosh(4\pi^2 \frac{m}{\beta})} \right\}. (51)
\]

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These results for the case \( w = 0 \) have been reported previously [2].

The chemical potential and current source dependent mutual information has further contributions. We consider
\[
I_{w,\mu,J}^{n,H} = \frac{1}{4\pi} \left( \tilde{I}_H[n] - n\tilde{I}_H[1] \right).
\]

The four exponential factors yield the value 1 for \( \alpha \neq 0 \) and \( \beta \to 0 \) because they have imaginary exponents. For \( \alpha = 0, \beta \) is important even though we take \( \beta \to 0 \) limit. The exponential factors contribute to the mutual information \( I_{w,\mu,J}^{n,H} \) in the \( n \to 1 \) limit.

\[
-C_{n,p,w} \frac{2}{\beta} \frac{\ell_A\ell_B}{L^2}, \quad \text{for} \quad \alpha = 0.
\]

For the other four Theta functions in (52), we consider the anti-periodic fermions \( a + J = b = 1/2 \) with \( \alpha = 0, \beta \to 0 \). We have four different \( \theta_3 \) that can be read off by using the trick (48). Thus we have the final result that has three different sets of the results (41) and (42) with \( \ell_t \) replaced by \( \ell_A, \ell_B \) and \( \ell_A + \ell_B \) (with relative - sign for the latter). The result for \( \alpha \neq 0 \) is similar to (43) with two additional factors with different \( \ell \)'s. Periodic fermions have similar results with the modifications \( m-1/2 \to m \).

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