On study of steady-state dynamics of construction objects on half-infinite foundations using FEM-BEM

M I Kadomtsev and A A Lyapin
Don State Technical University, Gagarina, 1, Rostov-on-Don, 344000, Russia
lyapin.rnd@yandex.ru

Abstract. The paper is devoted to some questions of steady-state processes modeling for half-infinite layered media with surface construction objects using finite and boundary element methods.

1. Introduction

Despite the wide range of practical simulations for civil engineering based on various finite element method software, there still a group of open questions concerned about an object behavior placed on complex foundations such as steady-state harmonic oscillations and wave guiding and interaction with surface placed objects. The modern techniques allowing describing the dynamics of objects placed on half-infinite media are based on coupled usage of finite element method (FEM) and boundary integral equations [1–4]. In particular, in [2] there analyzed the wave fields distribution in a tunnel deeply placed in the soil. The tunnel is simulated using finite elements, the soil – an elastic layered half-space with parallel boundaries. In [3] the stationary behavior of a tape kind foundation in soil is simulated. The soil is represented as double layered half-space. The [4] is devoted to dynamic fractioning of a media with inclusion in term of nonlinear behavior. The inclusion is formed using FEM. Other part is simulated in terms of linear deformations using boundary integral method. The only FEM solution uses the introduction of damping layers, infinite elements or perfectly matched layers approach (PML) [5–7]. It should be mentioned that practice of using PML in such software like ANSYS or COMSOL is limited, mostly, by homogeneous acoustic or elastic bodies. When using, for example in ANSYS, such types of elements like solid185 or 186, the PML option are unavailable for multilayered elements. Therefore this technique is limited also in case of coupling homogeneous layers when analyzing multilayered foundations. The problems in description determined by complexity of dispersion sets of layered structure, that depend both on geometrical and physical parameters of the media, as well as the correct satisfaction of the radiation conditions at the infinity.

2. Problem statement

The coupled behavior of surface object based on an elastic foundation is being studied:

\[ D = \bigcup D_j, \quad j = 0, \ldots, N \]

\[ D_0 \text{ is a half-space } x < 0, \quad D_j = \{ (x, y, z) : \quad x \in (0, h_j) \text{, } y, z \in (-\infty, +\infty) \}, \]  

where \( x \) is the local coordinate of \( j \)-layer, in terms for steady-state oscillations with \( \omega \) frequency under the action of source placed on the object surface (Figure 1).
Figure 1. The object placed on the surface of double layered foundation (N=1).

In this case the reasonable way is to separate of FEM object and the boundary element method (BEM) for the foundation with coupling solutions in along the contact areas S.

The properties of surface object are determined by common matrices of masses, resistances and rigidity $M, C, K$ when solving linear algebraic systems as:

$$
-\omega^2 M - i\omega C + K \cdot u = F
$$

with unknown amplitude vector of nodal displacements $u$ and known everywhere except $S$ the vector of nodal forces $F$.

The boundary integral equation for the foundation could be represented in the following form:

$$
\frac{1}{2} u(r_0) + \int_S T(r_0, r) \cdot u(r) ds = \int_S V(r_0, r) \cdot q(r) ds,
$$

where $u(r), q(r)$ – displacement and stress vectors at $S$; $V(r_0, r), T(r_0, r)$ – matrices of fundamental displacements and stresses caused by concentrated harmonic source placed at the point with radius vector $r_0 = \{x_0, y_0, z_0\}$ and observation point $r$ in multilayered half-space, that account precisely the dispersion properties of the foundation. Physical parameters of linear elastic multilayered foundation are determined by parameter set $E_j, \nu_j, \rho_j, j = 0, ..., N$.

The infinite point requires correct satisfaction of radiation conditions.

3. Solution

The integral equation (2) could be reduced to the system of linear algebraic equations by the boundary element method in respect to the unknown nodal displacements and forces in the way the nodal number and positions coincide with the nodal mesh at $S$ in the matrix equation (1).

The most difficult part of this transition is the procedure of matrix components calculation for the fundamental solutions $V(r_0, r), T(r_0, r)$ in terms of spatial deformations of the media.

The fundamental solutions for multilayered half-space could be represented as [8] by using the double integral Fourier transformation along the coordinates $(y, z)$:

$$
V_{jk}(r_0, r) = \frac{a}{4\pi^2 \mu} \int \int \exp(-i\alpha(y-y_0)-i\beta(z-z_0)) V_{jk}^*(x, x_0, \alpha, \beta) d\alpha d\beta
$$
\[ T_{jk}(r_0, r) = \frac{1}{4\pi^2} \int_{\Gamma_{1,2}} \exp(-i\alpha(y-y_0) - i\beta(z-z_0))\mathcal{F}_{jk}^* (x, x_0, \alpha, \beta) d\alpha d\beta, \quad j, k = 1, 2, 3. \]

Here, contours \( \Gamma_j \) are determined in accordance with the principle of limiting absorption [9], which takes into account the distribution of the real poles of the integrands in the representation (3). The numerical procedure for calculating the integrals in (3) is quite laborious. Therefore, we can make the transition to the polar coordinate system by introducing new variables:
\[
\alpha = u \cdot \cos \varphi, \quad \beta = u \cdot \sin \varphi, \quad u \in \Gamma, \quad \varphi \in [0, 2\pi).
\]

The contour \( \Gamma \) is the ray \([0, +\infty)\), deviating into the lower half-plane of the parameter \( u \) in the area of the positive roots of the integrand denominator.

We also introduce a polar coordinate system associated with the position of the harmonic point in the plane \( Oyz: \)
\[
y - y_0 = r \cdot \cos \psi, \quad z - z_0 = r \cdot \sin \psi, \quad r \in [0, +\infty), \psi \in [0, 2\pi)
\]

In this way we have:
\[
V_{jk}^* (r_0, r) = \frac{a}{4\pi^2} \int_{\Gamma} 2\pi \exp(-iur \cos(\varphi - \psi))\mathcal{F}_{jk}^* (x, x_0, u \cos \varphi, u \sin \varphi) ud\varphi du
\]
\[
T_{jk}^* (r_0, r) = \frac{a}{4\pi^2} \int_{\Gamma} 2\pi \exp(-iur \cos(\varphi - \psi))\mathcal{F}_{jk}^* (x, x_0, u \cos \varphi, u \sin \varphi) ud\varphi du.
\]

For the field refracted from planar boundaries of layered media \( x = 0 \) and \( x = h_j \) there could be written as:
\[
\mathcal{V}_{jk}^* (x, x_0, u \cos \varphi, u \sin \varphi) = \sum_{m=1}^{3} K_{km} (x, u, \varphi) \mathcal{X}_m^{(j)} (u \cos \varphi, u \sin \varphi) - \text{displacements},
\]
\[
\mathcal{T}_{jk}^* (x, x_0, u \cos \varphi, u \sin \varphi) = \sum_{m=1}^{3} L_{km} (x, u, \varphi) \mathcal{X}_m^{(j)} (u \cos \varphi, u \sin \varphi) - \text{stresses}
\]
of the plane with \( e_x \) normal.

Here, \( \mathcal{X}_m^{(j)} (u \cos \varphi, u \sin \varphi) \) are Fourier transformations for additional stress vectors, arising along corresponding boundaries \( x = 0 \) and \( x = h_j \), received from the system of algebraic equations in terms of satisfaction the coupling conditions of neighbor layers and day-light surface. The corresponding analysis is demonstrated that:
\[
\mathcal{X}_1^{(1)} (u \cos \varphi, u \sin \varphi) = F_{11} (u), \quad \mathcal{X}_2^{(1)} (u \cos \varphi, u \sin \varphi) = F_{12} (u) \cos \varphi, \quad \mathcal{X}_3^{(1)} (u \cos \varphi, u \sin \varphi) = F_{13} (u) \sin \varphi,
\]
\[
\mathcal{X}_1^{(2)} (u \cos \varphi, u \sin \varphi) = F_{21} (u) \cos \varphi, \quad \mathcal{X}_2^{(2)} (u \cos \varphi, u \sin \varphi) = F_{22} (u) + F_{22}^{(2)} (u) \cos 2\varphi, \quad \mathcal{X}_3^{(2)} (u \cos \varphi, u \sin \varphi) = F_{23} (u) \sin 2\varphi,
\]
\[
\mathcal{X}_1^{(3)} (u \cos \varphi, u \sin \varphi) = F_{31} (u) \sin \varphi, \quad \mathcal{X}_2^{(3)} (u \cos \varphi, u \sin \varphi) = F_{32} (u) \sin 2\varphi, \quad \mathcal{X}_3^{(3)} (u \cos \varphi, u \sin \varphi) = F_{33}^{(1)} (u) + F_{33}^{(2)} (u) \cos 2\varphi.
\]

By multiplication of functions \( K_{km} (x, u, \varphi) \) and \( \mathcal{X}_m^{(j)} \) in (5), according to (4), the functions of \( \varphi \) angle for the fundamental displacements as well as stresses could be represented as:
\[ \tilde{V}_{11}^* (x, x_0, u \cos \varphi, u \sin \varphi) = G_{11} (x, x_0, u), \quad \tilde{V}_{12}^* = G_{12} (x, x_0, u) \cos \varphi, \]
\[ \tilde{V}_{13}^* = G_{13} (x, x_0, u) \sin \varphi, \quad \tilde{V}_{21}^* = G_{21} (x, x_0, u) \cos \varphi, \]
\[ \tilde{V}_{22}^* = G_{22} (x, x_0, u) + G_{22}^{(2)} (x, x_0, u) \cos 2\varphi, \quad \tilde{V}_{23}^* = G_{23} (x, x_0, u) \sin 2\varphi, \]
\[ \tilde{V}_{31}^* = G_{31} (x, x_0, u) \sin \varphi, \quad \tilde{V}_{32}^* = G_{32} (x, x_0, u) \sin 2\varphi, \]
\[ \tilde{V}_{33}^* = G_{33}^{(1)} (x, x_0, u) + G_{33}^{(2)} (x, x_0, u) \cos 2\varphi. \]

In this way, the calculation of \( \varphi \) integral could be performed analytically. For this there could be used the well-known result [10]:
\[
\int_0^{2\pi} \exp \left[ -iur \cos (\varphi - \psi) \right] \frac{\cos n\varphi}{\sin n\varphi} d\varphi = 2\pi \exp \left( -\frac{i\pi}{2} \right) J_n (ur) \left\{ \cos n\psi \sin n\varphi \right\}, \quad n = 0,1,2.
\]
\[ J_n (x) \] - Bessel function of the 1-st kind.

As a result, the integral representation in the form of Fourier-Bessel integral could be written for every component of fundamental solutions for multilayered media:
\[
V_{jk}^* (r_0, r) = \frac{a}{2\pi \mu} \exp \left( -\frac{i\pi}{2} \right) \left\{ \cos n\psi \sin n\varphi \right\} \int G_{jk} (x, x_0, u) J_n (ur) \exp (2i\psi) du \quad (6)
\]

It is required to separate the functions under integral on the parts \( G_{jj}^{(1)} (x, x_0, u) \) and \( G_{jj}^{(2)} (x, x_0, u) \) for the diagonal components of the matrix of fundamental displacements \( V_{jj}^* (r_0, r), \quad j = 2,3. \) To do that, use could be made of the particular values of this functions for \( \varphi = 0 \) and \( \varphi = \pi/4. \)

Consequently,
\[ G_{jj}^{(1)} (x, x_0, u) = \tilde{V}_{11}^* \left( x, x_0, u \frac{\sqrt{2}}{2}, u \frac{\sqrt{2}}{2} \right), \]
\[ G_{jj}^{(2)} (x, x_0, u) = \tilde{V}_{11}^* \left( x, x_0, u, 0 \right) - \tilde{V}_{11}^* \left( x, x_0, u \frac{\sqrt{2}}{2}, u \frac{\sqrt{2}}{2} \right). \]

The 8-node Gauss quadrature formulas with adaptive stepping and estimations for the values at infinity were chosen for numerical realization of the integral (7) after the transformations.

4. Result discussion
It should be mentioned, that simulation results of steady-state object vibrations placed on multilayered foundation are depend significantly on the relationship between the elastic and geometric parameters of layered medium. In particular, one can distinguish between cases of a "hard" and "soft" layer on a half-space.

Figure 2 demonstrates the example of amplitude-frequency characteristics for vertical displacements of point on the lower boundary of the surface object placed on a two-layer foundation in the frequency range before 200 Hz. Line 1 is the results of coupled usage of FEM and BEM, 2 is for the damping layer technique applied, line 3 is the PML-layers consideration.

The similar dependence for a "soft" layer based on a half-space, characterized by the presence of surface waves of the Rayleigh type, are shown in Figure 3.

The oscillations of the curve near the monotonic line 3 indicate some errors of the method used in the FEM simulation. In particular, such an error is enhanced by the presence of surface waves at the points of the dispersion curves with the maximum group velocity.

4
Figure 2. "Hard" layer. $E_1=3200$ MPa, $E_0=100$ MPa.

Figure 3. "Soft" layer. $E_1=200$ MPa, $E_0=1000$ MPa.

The oscillations of the curve near the monotonic line 3 indicate some errors of the method used in the FEM simulation. In particular, such an error is enhanced by the presence of surface waves at the points of the dispersion curves with the maximum group velocity.

In particular, the calculations were carried out with precision control when creating finite element mesh: reducing the size of the elements until the convergence of results is achieved, and also applying adaptive partitioning options. The thickness of the PML layer varied from 5 to 10 finite elements. The outer boundaries of the two-layer half-space were removed at the distance of about 25 surface layer thickness or in such way, so the oscillations observed at least 2 wavelengths. We note that the problems of convergence for the PML technique were considered, for example in [11]. Improving
convergence is possible using adaptive FEM [12–14]. However, this significantly increases the calculation time and in some cases is unacceptable, since it is based on a posteriori estimation of the accuracy of the result obtained.

5. Conclusion
In this paper, the problem for the simulation of harmonic oscillations of a surface object placed on a multilayer substrate using the FEM-BEM method in a spatial statement is considered. The corresponding kernels of the integral representations for the wave fields using BEM are reduced to the form of single contour integrals.

Comparison of the simulation results for displacements near the foundation of the object with the PML method is performed. The best numerical stability of the hybrid FEM-BEM method is shown in comparison with PML for layered media of different structures.

The procedure for the coupled usage of FEM and BEM can be transferred to the fluid-saturated media that determine the properties of the substrate, and also applies to the case of the presence of backward waves in layered media.

Acknowledgments
The work is financially supported by the Russian Foundation for Base Research (project 18-01-00715-a).

References
[1] Kadomtsev M I, Lyapin A A and Seleznev M G 2010 Struct. Mech. Anal. Constr. 3 61–4
[2] Gupta S, Hussein M F M, Degrande G, Hunt H E M and Clouteau D 2007 Soil Dyn. Earthquake Eng. 27 608–24
[3] Spyraitsa C C and Xub C 2003 Soil Dyn. Earthquake Eng. 23 383–89
[4] Mobasher M E and Waisman H 2016 Int. J. Numer. Meth. Eng. 105 8 599–19
[5] Hastings F D, Schneider J B and Broschat S L 1996 J. Acoust. Soc. Amer. 100 3061–9
[6] Appelo D and Kreiss G 2006 J. Comput. Phys. 215 2 642–60
[7] Basu U 2009 Int. J. Num. Meth. Eng. 77 2 151–76
[8] Babeshko V A, Glushkov E V and Zinchenko Z F 1989 Dynamics of Nonhomogeneous Linear Elastic Medium (Moscow: Nauka) p 344
[9] Vorovich I A and Babeshko V A 1979 Mixed Dynamic Problems of The Theory of Elasticity for Non-classical Domains (Moscow: Nauka) p 320
[10] Prudnikov A P, Brychkov Yu A and Marichev O I 1998 Integrals & Series VI: Elem. Fun. (New York: Gordon and Breach Science Publishers)
[11] Chen Z, Xiang X and Zhang X 2016 Math. Comp. 85 2687–714
[12] Jiang X, Li P, Li J and Zheng W 2017 ESAIM: M2AN 51 5 2017–047
[13] Chen Z and Wu H 2003 SIAM J. Numer. Anal. 41 799–26
[14] Chen Z and Liu X 2005 SIAM J. Numer. Anal. 43 645–71