Closed-Form Parametric Equation for the Minkowski Sum of \( m \) Ellipsoids in \( \mathbb{R}^N \) and Associated Volume Bounds

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Abstract  
An exact closed-form parametric formula for the Minkowski sum boundary of \( m \) arbitrary ellipsoids in \( N \)-dimensional Euclidean space is given. Expressions for the principal curvatures of these Minkowski sums are derived. These results are then used to obtain volume bounds for the Minkowski sum of ellipsoids in terms of their defining matrices. The lower bounds are sharper than the Brunn-Minkowski inequality.

1 Introduction

The concept of the Minkowski sum of two bodies in \( N \)-dimensional Euclidean space \( \mathbb{R}^N \) is fundamental in the field of convex geometry. When \( N = 3 \), Minkowski sums play important roles in applied fields such as robot motion planning, computational chemistry, and computer graphics [5, 6, 7, 11, 12, 13, 15, 17].

By a body we mean a bounded, connected, open subset of \( \mathbb{R}^N \). Given two such bodies, \( B_1 \) and \( B_2 \), their Minkowski sum is defined by

\[
B_1 + B_2 = \{ x + y \mid x \in B_1, \ y \in B_2 \} .
\]  

(1)

Among all convex bodies with differentiable boundaries, ‘solid ellipsoids’ of the form \( E = \{ Sv : \|v\| < 1 \} \), where \( S \) is a nonsingular \( N \times N \) matrix, constitute a fairly broad, yet simple, class of objects. Solid ellipsoids are convenient since their boundaries have both closed form parametric and implicit descriptions. The ellipsoid \( \partial E \) can be parameterized as

\[
x(\phi) = Su(\phi) \in \mathbb{R}^N ,
\]

(2)

where \( u(\phi) \in \mathbb{S}^{N-1} \) (the unit sphere) is a unit vector, and \( \phi = (\phi_1, ..., \phi_{N-1}) \) are spherical angles (or any local coordinates) that parameterize the sphere.
The corresponding implicit equation of $\partial E$ is

$$\Psi(x) = 1 \text{ where } \Psi(x) = x^T S S^{-1} A = x^T A^{-2} x, \quad A = (SS^T)^{1/2}. \tag{3}$$

We note that $A$ is a positive-definite symmetric matrix, and the ellipsoid $\partial E$ can be reparameterized as

$$x(u(\phi)) = A u(\phi). \tag{4}$$

The symmetric matrix $A$ in (4) is unique, whereas $S$ is not unique.

When using (4), the unit normal $n(\phi)$ to $\partial E$ at $x_{\partial E}(\phi)$ is computed as

$$n(\phi) = \left. \frac{(\nabla \Psi)(x)}{\| (\nabla \Psi)(x) \|} \right|_{x=x(u(\phi))} = \frac{A^{-2} x(u(\phi))}{\| A^{-2} x(u(\phi)) \|} = \frac{A^{-1} u(\phi)}{\| A^{-1} u(\phi) \|}. \tag{5}$$

Since this means that $A n(\phi)$ is a scalar multiple of $u(\phi)$, it is possible to invert the above expression as

$$u(\phi) = \frac{A n(\phi)}{\| A n(\phi) \|}. \tag{6}$$

Combining (4) and (6),

$$x_{\partial E}(n(\phi)) = x(u(\phi)) = \frac{A^2 n(\phi)}{\| A n(\phi) \|}, \tag{7}$$

which confirms the bijectivity of the Gauss map $x \mapsto n$ for ellipsoids. (The Gauss map from the boundary of a convex body is always bijective onto $S^{N-1}$.)

When using (2) with a nonsymmetric matrix $S$, the corresponding equations are

$$n(\phi) = \left. \frac{(\nabla \Psi)(x)}{\| (\nabla \Psi)(x) \|} \right|_{x=x(u(\phi))} = \frac{S^{-T} S^{-1} x(u(\phi))}{\| S^{-T} S^{-1} x(u(\phi)) \|} = \frac{S^{-T} u(\phi)}{\| S^{-T} u(\phi) \|}. \tag{8}$$

and

$$u(\phi) = \frac{S^T n(\phi)}{\| S^T n(\phi) \|}. \tag{9}$$

In [17], the procedure to generate the Minkowski sum boundary for solid ellipsoids $E_1$ and $E_2$ (defined by symmetric matrices $A_1$ and $A_2$) was to morph space so as to compute

$$\partial (A_2^{-1}(E_1 + E_2)) = \partial (A_2^{-1} \cdot E_1 + B^N),$$

where $B^N = \{ x \in \mathbb{R}^N : \| x \| < 1 \}$. Since this corresponds to an external surface offset relative to $\partial (A_2^{-1} \cdot E_1)$ in the direction of the outward normal with a unit distance, then using (8) with $S = A_2^{-1} A_1$ gives the normal to the deformed ellipsoid, and the resulting offset surface is given parametrically by

$$\bar{x}_{1+2}(u) = A_2^{-1} A_1 u + \frac{A_2 A_1^{-1} u}{\| A_2 A_1^{-1} u \|}.$$
Transforming back by multiplying \( \bar{x}_{1+2}(u) \) by \( A_2 \) gives the result in [17]:

\[
x_{1+2}(u) = A_1 u + A_2 \left( \frac{A_2 A_1^{-1} u}{\|A_2 A_1^{-1} u\|} \right).
\] (10)

It would seem from this equation that iterating Minkowski sums would become complicated. Moreover, it is regrettable that

\[
x_{1+2}(u) \neq x_{2+1}(u)
\]
even though \( E_1 + E_2 = E_2 + E_1 \).

This has led us to consider a new way of writing (10). Specifically, substituting (6) (with \( A = A_1 \)) in equation (10) yields

\[
x_{1+2}(\frac{A_1 n}{\|A_1 n\|}) = A_1 \left( \frac{A_1 n}{\|A_1 n\|} \right) + A_2 \left( \frac{A_2 n}{\|A_2 n\|} \right).
\] (11)

Re-parametrizing (11) gives a new parametric formula for \( \partial(E_1 + E_2) \):

\[
x_{\partial(E_1+E_2)}(n) = A_1 \left( \frac{A_1 n}{\|A_1 n\|} \right) + A_2 \left( \frac{A_2 n}{\|A_2 n\|} \right).
\] (12)

The first result of this paper is that this symmetric formula for the Minkowski sum boundary generalizes to \( m \) ellipsoids:

**Theorem 1.1.** Suppose that \( E_1, E_2, \ldots, E_m \) are solid ellipsoids in \( \mathbb{R}^N \) with boundaries given by

\[
\partial E_j = \{ A_j u : u \in S^{N-1} \}, \quad j = 1, \ldots, m,
\]

where the \( A_j \) are symmetric positive-definite \( N \times N \) matrices. Then the boundary of the Minkowski sum \( E_1 + \cdots + E_m \) is given parametrically by

\[
x_{\partial(E_1+\cdots+E_m)}(n) = \sum_{i=1}^{m} \frac{A_i^2 n}{\|A_i n\|}, \quad n \in S^{N-1}.
\] (13)

Furthermore, \( n \) is the (outward-pointing) normal to the boundary of \( E_1 + \cdots + E_m \) at \( x_{\partial(E_1+\cdots+E_m)}(n) \).

Note that we can use (13) to define \( x_{\partial(E_1+\cdots+E_m)}(n) \) for all vectors \( n \in \mathbb{R}^N \), so that we have

\[
x_{\partial(E_1+\cdots+E_m)}(cn) = x_{\partial(E_1+\cdots+E_m)}(n) \quad \forall c \in \mathbb{R}_{>0} \forall n \in \mathbb{R}^N.
\] (14)

We then obtain a formula for the principal curvatures of the Minkowski sum boundary:
Theorem 1.2. For solid ellipsoids $E_1, \ldots, E_m$ in $\mathbb{R}^N$ given by symmetric positive-definite matrices $A_1, \ldots, A_m$ as in Theorem 1.1, we let

$$C(n) \equiv \sum_{j=1}^{m} \frac{A_j^2}{\|A_j n\|^2} - \frac{A_j^2 n n^T A_j^2}{\|A_j n\|^4}, \quad n \in S^{N-1}. \quad (15)$$

Then $C(n)$ is positive semidefinite of rank $N - 1$, and the principal curvatures of the Minkowski sum boundary $\partial (E_1 + \cdots + E_m)$ at $x_{\partial (E_1 + \cdots + E_m)}(n)$ are the reciprocals of the positive eigenvalues of $C(n)$.

Indeed, we show below that $C(n) n = 0$ and that the self-adjoint operator $C(n)$ is positive-definite on the hyperplane orthogonal to $n$.

### 1.1 Proof of Theorem 1.1

The result holds for $m = 1$ by (7). Using the method of induction, we assume that the formula in (13) holds for $x_{\partial (E_1 + \cdots + E_m)}(n)$ and seek to show that it holds for $x_{\partial (E_1 + \cdots + E_{m+1})}(n)$. As in the proof of (10), we shall apply $A_{m+1}^{-1}$ to the whole scene and compute a normal offset.

The fact that $n$ is the outward-pointing unit normal to $x_{\partial (E_1 + \cdots + E_m)}(n)$ also follows by an elementary computation. Specifically, if $n = n(\phi)$ where $\phi = (\phi_1, \ldots, \phi_{N-1})$ are the spherical angles parametrizing the unit sphere (or are any local coordinates on the unit sphere), then a basis for the tangent hyperplane at the point $x_{\partial (E_1 + \cdots + E_m)}(n)$ on this surface will be the $N - 1$ tangent vectors $\partial x_{\partial (E_1 + \cdots + E_m)}(n(\phi_1, \ldots, \phi_{N-1}))/\partial \phi_j$. These tangent vectors will be a sum over $i$ of

$$\frac{\partial}{\partial \phi_j} \left( \frac{A_i^2 n}{\|A_i n\|^2} \right) = C(A_i, n) \frac{\partial n}{\partial \phi_j} \quad (16)$$

where

$$C(A_i, n) \equiv \frac{A_i^2}{\|A_i n\|^2} - \frac{A_i^2 n n^T A_i^2}{\|A_i n\|^4} = [C(A_i, n)]^T. \quad (17)$$

We note that for $v \in \mathbb{R}^N$, we have

$$v^T C(A_i, n) v = \frac{\|A_i n\|^2 \|A_i v\|^2 - [A_i v]^T (A_i n)^2}{\|A_i n\|^4} \geq 0, \quad (18)$$

with equality if and only if $v = c n$. It follows that

$$[C(A_i, n)] n = 0. \quad (19)$$

Thus

$$n^T \frac{\partial}{\partial \phi_j} \left( \frac{A_i^2 n}{\|A_i n\|^2} \right) = n^T C(A_i, n) \frac{\partial n}{\partial \phi_j} = 0,$$

and consequently

$$n^T \frac{\partial x_{\partial (E_1 + \cdots + E_{m+1})}(n(\phi_1, \ldots, \phi_{N-1}))}{\partial \phi_j} = 0. \quad (20)$$
Hence, $n$ is the normal to $x_{\partial(E_1 + \cdots + E_m)}(n)$. It follows that $A_{m+1} n$ is normal to $A_{m+1}^{-1} x_{\partial(E_1 + \cdots + E_m)}(n)$. Therefore, the Minkowski sum of $A_{m+1}^{-1} x_{\partial(E_1 + \cdots + E_m)}(n)$ with the unit ball $B^N = A_{m+1}^{-1} : E_{m+1}$ has boundary given parametrically as

$$x_{\partial A_{m+1}^{-1}(E_1 + \cdots + E_{m+1})}(n) = A_{m+1}^{-1} x_{\partial(E_1 + \cdots + E_m)}(n) + \frac{A_{m+1} n}{\|A_{m+1} n\|}.$$ 

The boundary of the Minkowski sum of $m+1$ ellipsoids is then

$$x_{\partial(E_1 + \cdots + E_{m+1})}(n) = A_{m+1} x_{\partial A_{m+1}^{-1}(E_1 + \cdots + E_{m+1})}(n) = \sum_{i=1}^{m+1} \frac{A_i^2 n}{\|A_i n\|},$$ 

which is exactly the formula in (13) with $m \rightarrow m + 1$.

Recalling (15), the matrix $C(n)$ of Theorem 1.2 is given by

$$C(n) = \sum_{i=1}^{m} C(A_i, n), \quad \text{for } n \in S^{N-1},$$

and it follows from (19) that

$$C(n)n = 0. \quad \text{(22)}$$

### 1.2 Properties of the Symmetric Formula Under Linear Transformation

We shall use the notation

$$E_A \doteq \{ x \in \mathbb{R}^N : x^T A^{-2} x < 1 \},$$

when $A$ is a symmetric positive-definite $N \times N$ matrix; equivalently, the boundary of $E_A$ is given implicitly by (41) or (7). If $S \in \text{GL}(N)$ is applied to (23), we obtain

$$S \cdot E_A = \{ x \in \mathbb{R}^N : x^T S^{-T} A^{-2} S^{-1} x < 1 \} = \{ x \in \mathbb{R}^N : x^T \tilde{A}^{-2} x < 1 \},$$

where

$$\tilde{A} = (SA^2S^T)^{1/2}.$$  

Hence,

$$x_{\partial(S \cdot E_A)}(n) = x_{\partial E_A}(n) = x_{\partial E_{(SA^2S^T)^{1/2}}}(n) = \frac{SA^2S^T n}{\sqrt{n^TSA^2S^T n}}. \quad \text{(25)}$$

Consequently the symmetric parametric Minkowski sum formula has the property that

$$x_{\partial S \cdot (E_{A_1} + E_{A_2})}(n) = x_{\partial E_{(SA_1^2S^T)^{1/2}}}(n) + x_{\partial E_{(SA_2^2S^T)^{1/2}}}(n).$$
The \( C \) matrix transforms as
\[
C \left( (SA_i^2S^T)^{1/2}, n \right) = \frac{1}{\|S^T n\|} S C \left( A_i, \frac{S^T n}{\|S^T n\|} \right) S^T, \tag{26}
\]
and (21) inherits this property. If we let \( n \in \mathbb{R}^N \) be a vector of arbitrary length in definition (17), then we have
\[
C \left( (SA_i^2S^T)^{1/2}, n \right) = S C (A_i, S^T n) S^T. \tag{27}
\]

2 Differential Geometry of Minkowski Sum Boundaries: Proof of Theorem 1.2

As with any parametric hyper-surface, the elements of the metric tensor for the parameterized Minkowski sum boundary in \( \mathbb{R}^N \) can be computed in coordinates (e.g., in terms of \( N-1 \) spherical-coordinates) as
\[
g_{ij} = \frac{\partial x}{\partial \phi_i} \cdot \frac{\partial x}{\partial \phi_j}, \tag{28}
\]
where
\[
x = x_{\theta(E_1+\cdots+E_m)}(n(\phi)). \tag{29}
\]
The elements of the second fundamental form can be computed as
\[
l_{ij} = -n \cdot \frac{\partial^2 x}{\partial \phi_i \partial \phi_j}. \tag{30}
\]
Then with \( G = [g_{ij}] \) and \( L = [l_{ij}] \), the \( N-1 \) principal curvatures are obtained as the eigenvalues of the matrix \( G^{-1}L \). As will be seen below, the parametric form (13) for the Minkowski sum simplifies this computation.

Combining (13), (16) and (21),
\[
\frac{\partial x}{\partial \phi_i} = C(n) \frac{\partial n}{\partial \phi_i}, \tag{31}
\]
and so
\[
g_{ij} = \frac{\partial n^T}{\partial \phi_i} C^T(n)C(n) \frac{\partial n}{\partial \phi_j} = \frac{\partial n^T}{\partial \phi_i} C(n)^2 \frac{\partial n}{\partial \phi_j}, \tag{32}
\]
where we have used the symmetry of \( C \). The whole metric tensor can then be written as
\[
G = J_{EN-1}^T C^2 J_{EN-1} \tag{33}
\]
where
\[
J_{EN-1} = \left[ \frac{\partial n}{\partial \phi_1}, \frac{\partial n}{\partial \phi_2}, \ldots, \frac{\partial n}{\partial \phi_{N-1}} \right] \tag{34}
\]
is the Jacobian for the sphere such that its metric tensor is
\[
G_{EN-1} = J_{EN-1}^T J_{EN-1}. \tag{35}
\]
Note that since each $\partial n/\partial \phi_i$ is tangent to $S^{N-1}$, we have
\[ J_{S^{N-1}}^T n = 0. \] (36)

In the above equations and in the following, all matrices are functions of $n$.

Let $\{v^1(n),\ldots,v^{N-1}(n)\}$ be a (moving) orthonormal basis for the tangent hyperplane to the boundary hypersurface at $x(n)$; this hyperplane is orthogonal to the null direction $n$ of $C$. Since $C$ is self-adjoint (symmetric), the range of $C$ is also spanned by the $v^i(n)$. Thus we can express $C$ in terms of these basis vectors. Explicitly, we define $N \times (N - 1)$ matrix
\[ M = \begin{bmatrix} v^1(n) & v^2(n) & \cdots & v^{N-1}(n) \end{bmatrix}, \] (37)
which has the properties
\[ M^T M = I_{N-1}, \quad MM^T + nn^T = I_N, \quad \text{and} \quad n^T M = 0. \] (38)

Then
\[ \tilde{C} = M^T C M \] (39)
is the matrix of the operator $C$ on the tangent hyperplane with respect to the orthonormal basis $\{v^1(n),\ldots,v^{N-1}(n)\}$.

Moreover, we can write
\[ \tilde{J}_{S^{N-1}} = M^T J_{S^{N-1}}. \] (40)

It then follows from (36) and (38) that
\[ J_{S^{N-1}} = (MM^T + nn^T)J_{S^{N-1}} = MJ_{S^{N-1}}, \] (41)
and therefore by (35) and (38),
\[ G_{S^{N-1}} = J_{S^{N-1}}^T J_{S^{N-1}}. \] (42)

By (19) and (38)–(39),
\[ \tilde{C}^2 = M^T C (MM^T + nn^T) C M = M^T C^2 M, \]
and hence
\[ G = J_{S^{N-1}}^T \tilde{C}^2 J_{S^{N-1}} \text{ and } G_{S^{N-1}} = J_{S^{N-1}}^T J_{S^{N-1}}. \] (43)

Thus
\[ \det G = \det(\tilde{C})^2 \det G_{S^{N-1}}. \]

If the $\phi_i$ are the standard spherical angles on $S^{N-1}$, then the tangent vectors $\partial n/\partial \phi_i$ at $x(n)$ are orthogonal and we could choose
\[ v^i(n) = \left\| \frac{\partial n}{\partial \phi_i} \right\|^{-1} \frac{\partial n}{\partial \phi_i}, \quad 1 \leq j \leq N - 1. \] (44)
With the choice \( J_{SN-1} \) and \( G_{SN-1} \) are diagonal matrices:

\[
J_{SN-1} = \text{diag} \left( \left\| \frac{\partial \mathbf{n}}{\partial \phi_1} \right\|, \ldots, \left\| \frac{\partial \mathbf{n}}{\partial \phi_{N-1}} \right\| \right),
\]

\[
G_{SN-1} = \text{diag} \left( \left\| \frac{\partial \mathbf{n}}{\partial \phi_1} \right\|^2, \ldots, \left\| \frac{\partial \mathbf{n}}{\partial \phi_{N-1}} \right\|^2 \right).
\]

Computation of the second fundamental form simplifies as a consequence of the symmetric parameterization of the Minkowski sum boundary. Specifically,

\[
\frac{\partial^2 \mathbf{x}}{\partial \phi_k \partial \phi_j} = \frac{\partial C}{\partial \phi_j} \frac{\partial \mathbf{n}}{\partial \phi_k} + C \frac{\partial^2 \mathbf{n}}{\partial \phi_k \partial \phi_j},
\]

simplifies due to the structure of \( C \) by observing that since \( C \mathbf{n} = 0 \), then differentiating gives

\[
\frac{\partial C}{\partial \phi_j} \mathbf{n} + C \frac{\partial \mathbf{n}}{\partial \phi_j} = 0.
\]

This allows us to write

\[
l_{kj} = \frac{\partial \mathbf{n}^T}{\partial \phi_j} C \frac{\partial \mathbf{n}}{\partial \phi_k},
\]

or

\[
L = J_{SN-1}^T C J_{SN-1}.
\]

As with \( G \), this can be expressed in terms of square matrices by using (36) and (38) to obtain

\[
L = \tilde{J}_{SN-1}^T \tilde{C} \tilde{J}_{SN-1}.
\]

The principal curvatures \( \kappa_i = \kappa_i(x) \) of the \( m \)-fold Minkowski sum boundary can then be computed as

\[
\kappa_i = \lambda_i(G^{-1} L) = \lambda_i \left( (J_{SN-1}^{-1} \tilde{C}^{-2} \tilde{J}_{SN-1}^T)(J_{SN-1}^T \tilde{C} \tilde{J}_{SN-1}) \right),
\]

where \( \lambda_i \) denotes the \( i \)-th eigenvalue. After cancelation and using the fact that eigenvalues are invariant under similarity transformations, we obtain

\[
\kappa_i(x(n)) = \lambda_i(\tilde{C}(n)^{-1}) = \frac{1}{\lambda_i(C(n))}, \quad i = 1, \ldots, N - 1.
\]

The last equality follows from the fact that the eigenvalues of the \( (N-1) \times (N-1) \) matrix \( \tilde{C} \) are the same as those for the \( N \times N \) matrix \( C \), except for the single zero eigenvalue \( \lambda_N(C) = 0 \). This completes the proof of Theorem 1.2.

The \( \tilde{C} \) matrix is also useful for computing integrals over the boundary of ellipsoidal sums. Invariant integration on the sphere is defined by

\[
\int_{S^{N-1}} f(\mathbf{n}) d\sigma_{N-1}(\mathbf{n}) = \int f(\mathbf{n}(\phi)) \left[ \det G_{SN-1}(\phi) \right]^{1/2} d\phi_1 \cdots d\phi_{N-1},
\]
where $\sigma_{N-1}$ is volume measure on $\mathbb{S}^{N-1}$. By (43), integration over the boundary of the Minkowski sum $\Sigma = E_1 + \cdots + E_m$

$$\int_{\partial\Sigma} f(x) \, d\mathrm{vol}(x) = \int f(x(n(\phi))) \det G(n(\phi))^{1/2} d\phi_1 \cdots d\phi_{N-1},$$

can be computed as

$$\int_{\partial\Sigma} f(x) \, d\mathrm{vol}(x) = \int_{\mathbb{S}^{N-1}} f(x(n)) \det \tilde{C}(n) \, d\sigma_{N-1}(n). \quad (53)$$

This equation will be used together with Steiner’s formula in Section 4 to compute the volumes of Minkowski sums of ellipsoids.

## 3 Volume Bounds Using Bounding Ellipsoids

Given inner and outer ellipsoidal bounds of the form

$$E_{A_{\text{inner}}} \subseteq \sum_i E_i \subseteq E_{A_{\text{outer}}},$$

and noting that $\text{Vol}(E_A) = \text{Vol}(\mathbb{B}^N) \det A$, where $\text{Vol}(\mathbb{B}^N) = \frac{\pi^{N/2}}{\Gamma(1/2)}$ is the volume of the unit $N$-ball, we have the obvious volume bounds

$$\text{Vol}(\mathbb{B}^N) \det (A_{\text{inner}}) \leq \text{Vol} \left( \sum_i E_i \right) \leq \text{Vol}(\mathbb{B}^N) \det (A_{\text{outer}}). \quad (54)$$

In the following, we review some formulas for $A_{\text{inner}}$ and $A_{\text{outer}}$ which we apply to (54), and which can be used to obtain further volume estimates in Section 4.

### 3.1 Optimal Lower Bounds for Minkowski Sums of Two Ellipsoids

An ellipsoid can be fit inside the Minkowski sum $E_1 + \cdots + E_m$ of the solid ellipsoids $E_j$ of Theorem 1.1 by the following argument. Consider a solid ellipsoid, $E_{\text{inner}}$, defined by $x^T A_{\text{inner}}^{-1} x < 1$. Recalling (7), we can parameterize $\partial E_{\text{inner}}$ by its normal $n$ as

$$x_{\partial E_{\text{inner}}}(n) = \frac{A_{\text{inner}}^{-1} n}{\|A_{\text{inner}} n\|}, \quad (55)$$

where $A_{\text{inner}}$ is symmetric, positive-definite.

The containment condition $E_{\text{inner}} \subseteq E_1 + \cdots + E_m$ can be written as the inequality

$$n^T x_{\partial E_{\text{inner}}}(n) \leq n^T x_{\partial (E_1 + E_2 + \cdots + E_m)}(n), \quad (56)$$

for all $n \in \mathbb{S}^{N-1}$. By (12) and (55), we then obtain the general condition

$$E_{\text{inner}} \subseteq E_1 + \cdots + E_m \iff \|A_{\text{inner}} v\| \leq \sum_{j=1}^m \|A_j v\|, \forall v \in \mathbb{R}^N. \quad (57)$$
Hence by the triangle inequality, the matrix

$$A_{\text{sum}} = \sum_{i=1}^{m} A_i$$  \hfill (58)

satisfies these conditions and hence \(E_{A_{\text{sum}}} \subset E_1 + \cdots + E_m\).

When \(E_{\text{inner}}\) is contained in the Minkowski sum \(\Sigma \doteq E_1 + \cdots + E_m\), a boundary point \(x_{\partial E_{\text{inner}}}(n) \in \partial E_{\text{inner}}\) is also in the boundary of \(\Sigma\) if and only if equality holds in (56); i.e.,

$$\|A_{\text{inner}} n\| = \sum_{j=1}^{m} \|A_j n\|.$$  \hfill (59)

If (59) holds, then \(n\) is also the unit normal to \(\Sigma\) at \(x(n)\).

For the case \(m = 2\), the inner ellipsoid \(E_{\text{sum}} = E_{A_{\text{sum}}}\) will contact the boundary of the Minkowski sum at \(2N\) (or more) points. Indeed, let \(v_1, \ldots, v_N\) be eigenvectors of \(A_1^{-1} A_2\) (which is diagonalizable since it is a conjugate of \(A_1^{-1/2} A_2 A_1^{-1/2}\)) with eigenvalues \(\lambda_1, \ldots, \lambda_N\), respectively. Then \(A_2 v_j = \lambda_j A_1 v_j\), and hence by formula (13) of Theorem 1.1,

$$x_{\partial (E_1 + E_2)}(v_j) = (1 + \lambda_j) \frac{A_2^2 v_j}{\|A_1^2 v_j\|} = \frac{A_{\text{sum}}^2 v_j}{\|A_{\text{sum}}^2 v_j\|} = x_{\partial E_{\text{sum}}}(v_j).$$

Therefore (when \(m = 2\)), \(E_{\text{sum}}\) contacts the boundary of \(E_1 + E_2\) at the \(2N\) points \(\pm x_{\partial E_{\text{sum}}}(v_j)\). Having these \(2N\) contacts, \(E_{\text{sum}}\) is a good lower bound for the Minkowski sum of two ellipsoids. However, except in special cases where \(E_1\) and \(E_2\) have the same principal axes (i.e., \(A_1\) commutes with \(A_2\)), \(E_{\text{sum}}\) will not coincide with the maximal volume inner ellipsoid described in Theorem 3.5 and Lemma 3.7 below.

For the general case of the Minkowski sum \(\Sigma\) of three or more solid ellipsoids, \(\partial E_{\text{sum}} \cap \partial \Sigma = \emptyset\). Indeed, if none of the eigenvectors of \(A_1^{-1} A_2\) are eigenvectors of \(A_1^{-1} A_3\), then

$$\|A_{\text{sum}} n\| < \sum_{j=1}^{m} \|A_j n\|$$

for all vectors \(n\), and hence \(E_{\text{sum}}\) can be dilated and remain inside \(\Sigma\), and thus \(E_{\text{sum}}\) will not have maximal volume. However, there will be contact points in special cases when the ellipsoids share the same semi-axes. (See Remark 3.3 below.

To determine if an inner ellipsoid has maximal volume, one can apply the following result of F. John [10] (see also [2]):

**Theorem 3.1.** (John [10]) Let \(K\) be a convex body that is symmetric about \(0\). A solid ellipsoid \(E = \{Av: \|v\| < 1\}\) contained in \(K\) has maximal volume among all ellipsoids contained in \(K\) if and only if there exist points \(x_1, \cdots, x_k\)
\((k \geq N)\) in \(\partial E \cap \partial K\) and constants \(c_1, \ldots, c_k\) such that

\[
\sum_{j=1}^{k} c_j (y^T A^{-1} x_j) A^{-1} x_j = y, \tag{60}
\]

for all \(y \in \mathbb{R}^N\). Furthermore \(E\) is unique.

The solid ellipsoid \(E\) is called the Löwner–John ellipsoid. Note that the vectors \(A^{-1} x_j\) lie in the unit sphere.

**Example 3.2.** Condition \(60\) holds in the following cases:

- \(k = N\), the \(A^{-1} x_j\) are orthonormal, and \(c_j = 1\) for \(j = 1, \ldots, N\).
- \(N = 2\), \(k \geq 3\), \(A^{-1} x_j = (\cos(2\pi j/k), \sin(2\pi j/k))\) and \(c_j = 2/k\) for \(0 \leq j \leq k - 1\).

**Remark 3.3.** For example, if the \(A_i\) are diagonal positive-definite matrices, then \(\ref{59}\) holds for \(n = e_j\) (the standard basis vectors), and the conditions in Theorem \(3.1\) are met with \(x_j = A_{\text{sum}} e_j\).

A formula for the maximal volume inner ellipsoid of the Minkowski sum \(E_1 + E_2\) was given by Chernousko \([5]\) (see also \([11]\)). Chernousko’s formula (equation \(61\) below) can be described in terms of the operator geometric mean:

**Definition 3.4.** Let \(P, Q\) be positive-definite symmetric matrices. The geometric mean \(P \# Q\) of \(P\) and \(Q\) is given by

\[
P \# Q = P^{1/2} \left( P^{-1/2} Q P^{-1/2} \right)^{1/2} P^{1/2}.
\]

We note that \(P \# Q = Q \# P\), and

\[
P \# Q = P^{1/2} Q^{1/2} \iff PQ = QP.
\]

(See \([3]\).) The geometric mean \(P \# Q\) can also be interpreted as the midpoint of the geodesic from \(P\) to \(Q\) in the Riemannian metric on \(\text{GL}(N)\) \([1]\).

**Theorem 3.5.** (Chernousko \([5]\)) Let \(A, B\) be symmetric positive-definite \(N \times N\) matrices, and let \(E_A, E_B\) be given by \(\ref{23}\). Then the Löwner–John ellipsoid (maximal volume inner ellipsoid) \(E_{\text{John}}\) for the Minkowski sum \(E_A + E_B\) is given by

\[
E_{\text{John}} = \{ F(A, B) v : \|v\| < 1 \}, \quad F(A, B) = \left[ A^2 + 2 A^2 \# B^2 + B^2 \right]^{1/2}. \tag{61}
\]

**Remark 3.6.** We note that

\[
E_{\text{John}} = E_{\text{sum}} \iff F(A, B) = A + B \iff AB = BA \iff (AB) = (AB)^T,
\]

which occurs when \(E_A\) and \(E_B\) share all their axes.
We provide here a short proof of Theorem 3.5 using Theorem 3.1

Lemma 3.7. Let \( A, B \) be symmetric positive-definite \( N \times N \) matrices. Then the John ellipsoid (maximal volume inner ellipsoid) for the Minkowski sum \( E_A + E_B \) is given by

\[
E_{\text{John}} = \{ Sv : \|v\| < 1 \}, \quad S = A \left[ I + (A^{-1}B^2A^{-1})^{1/2} \right].
\]  

(62)

Proof. We first consider the case where \( A = I \). To show that \( E_{\text{John}} \) is defined by \( A \sum = I + B \) in this case, we let \( n_1, \ldots, n_N \) be orthonormal eigenvectors of \( B \) with eigenvalues \( \lambda_1, \ldots, \lambda_N \), respectively. Then

\[
\|A_{\sum} n_j\| = 1 + \lambda_j = \|In_j\| + \|Bn_j\|,
\]

and thus by (59),

\[
x_{\partial E_{\sum}}(n_j) = \frac{A_{\sum}^2 n_j}{\|A_{\sum} n_j\|} = (1 + \lambda_j) n_j \in \partial E_{\sum} \cap \partial (E_A + E_B),
\]

where \( E_{\sum} = \{ A_{\sum} u : \|u\| < 1 \} \). Then \( A_{\sum}^{-1} x_{\partial E_{\sum}}(n_j) = n_j \), and therefore John’s condition (60) (with \( x_j = x_{\partial E_{\sum}}(n_j) \), \( c_j = 1 \), and \( A \) replaced by \( A_{\sum} \)) is satisfied for the ellipsoid \( E_{\sum} \).

For the general case, we apply the linear transformation \( A^{-1} \), so that we have

\[
A^{-1} \cdot E_A = \mathbb{B}^N, \quad A^{-1} \cdot E_B = \{ Bv : \|v\| < 1 \},
\]

where

\[
\tilde{B} \doteq [(A^{-1}B)(A^{-1}B)^T]^{1/2} = (A^{-1}B^2A^{-1})^{1/2}.
\]

(63)

Since \( A^{-1} \cdot E_{\text{John}} \) is the John ellipsoid for \( A^{-1} \cdot E_A + A^{-1} \cdot E_B \), we have

\[
A^{-1} \cdot E_{\text{John}} = \{ (I + \tilde{B})v : \|v\| < 1 \}.
\]

Left-multiplying by \( A \) and applying (63), we obtain (62).

To complete the proof of Theorem 3.5, we have by (3)–(4),

\[
F(A, B) = (SS^T)^{1/2} = \left[ A^2 + 2A(A^{-1}B^2A^{-1})^{1/2}A + B^2 \right]^{1/2}.
\]

(64)

Example 3.8. Figure 1 illustrates Theorem 3.5 using the matrices

\[
A = \begin{pmatrix} 5 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}.
\]

In the figure, the ellipses \( \partial E_A, \partial E_B \subset \mathbb{R}^2 \) are black, the Minkowski sum boundary \( \partial (E_A + E_B) \) is green, \( \partial E_{\sum} = \partial E_A + \partial E_B \) is blue, and \( \partial E_{\text{John}} \) is red.

The area of \( E_{\text{John}} \) is approximately 113.14, whereas \( E_{\sum} \) has area \( \approx 108.38 \).
Remark 3.9. Kurzhanski–Vályi [11] gives a family of inner ellipsoids $E_{\hat{S}}$, with

$$\hat{S}^2 = S^{-1} \left[ (SA^2S)^{\frac{1}{2}} + (SB^2S)^{\frac{1}{2}} \right]^2 S^{-1},$$

where $S$ is any symmetric positive-definite matrix. The union of the ellipsoids $E_{\hat{S}}$ covers the entire Minkowski sum $E_A + E_B$. If $S = A^{-1}$ or $S = B^{-1}$, then $\hat{S} = F(A, B)$.

3.2 Comparison with Brunn-Minkowski

The Brunn-Minkowski inequality states that

$$|\text{Vol}(K_1 + K_2)|^{1/N} \geq |\text{Vol}(K_1)|^{1/N} + |\text{Vol}(K_2)|^{1/N}. $$

(If $K_1, K_2$ are both convex and have positive volume, then equality holds if and only if they are homothetic, i.e., $K_1 = cK_2$.) For ellipsoidal bodies $E_1$ and $E_2$, the first inequality of (54) gives a sharper inequality than Brunn-Minkowski:

Theorem 3.10. Let $E_i = A_i \cdot B^N$, $i = 1, 2$, be ellipsoids, where $A_1, A_2$ are positive-definite symmetric matrices. Then

$$[\text{Vol}(E_1 + E_2)]^{1/N} \geq \text{Vol}(B^N)^{1/N} \det (F(A_1, A_2))^{1/N} \geq \text{Vol}(B^N)^{1/N} \det (A_1 + A_2)^{1/N} \geq \text{Vol}(E_1)^{1/N} + \text{Vol}(E_2)^{1/N},$$

where $F(A, B)$ is given by (61) (or equivalently by (64)).
Proof. The first two inequalities of the theorem follow from the optimality of John’s ellipsoid and (54). The Minkowski inequality for determinants yields

\[
\text{Vol}(\mathbb{B}^N)^{1/N} \det (A_1 + A_2)^{1/N} \geq \text{Vol}(\mathbb{B}^N)^{1/N} \left[ (\det A_1)^{1/N} + (\det A_2)^{1/N} \right] \\
= \text{Vol}(E_1)^{1/N} + \text{Vol}(E_2)^{1/N}.
\]

Consider for example, the case when \( N = 2 \) and \( E_1, E_2 \) are degenerate ellipses with \( A_1 = \text{diag}[a_1, 0] \) and \( A_2 = \text{diag}[0, a_2] \). Each has zero area, but the Minkowski sum will be a \((2a_1) \times (2a_2)\) rectangle. In this case, Brunn-Minkowski gives \( 4a_1a_2 > 0 \) while Theorem 3.10 gives \( 4a_1a_2 > \pi a_1a_2 \).

Theorem 3.10 generalizes to the sum of \( m \) ellipsoids, although the bounds become looser. For example, given three ellipsoids \( E_1, E_2, E_3 \), defined by \( A_1, A_2, A_3 \), a lower bound on the Minkowski sum determinant can be obtained recursively by considering the three positive matrices

\[
A' = F(F(A_1, A_2), A_3), \quad A'' = F(F(A_1, A_3), A_2), \quad A''' = F(F(A_3, A_2), A_1),
\]

each defining an inner ellipsoid for \( E_1 + E_2 + E_3 \). Then

\[
\left[ \text{Vol}(E_1 + E_2 + E_3) \right]^{1/N} \geq \text{Vol}(\mathbb{B}^N)^{1/N} \max \{ \det A', \det A'', \det A''' \}^{1/N} \\
\geq \text{Vol}(\mathbb{B}^N)^{1/N} \det (A_1 + A_2 + A_3)^{1/N} \\
\geq \text{Vol}(E_1)^{1/N} + \text{Vol}(E_2)^{1/N} + \text{Vol}(E_3)^{1/N}.
\]

3.3 Upper Bounds for Minkowski Sums of Ellipsoids

A family of solid ellipsoids containing the Minkowski sum \( E_1 + \cdots + E_m \) is given as follows (see [7, 11, 16]):

\[
E_{\gamma}^{\text{outer}} = \{ A_\gamma v : \|v\| < 1 \}, \quad A_\gamma^2 = \sum_{i=1}^{m} \gamma_i A_i^2,
\]

for \( \gamma = (\gamma_1, \ldots, \gamma_m) \) with

\[
\gamma_i > 0 \quad \text{and} \quad \sum_{i=1}^{m} \gamma_i = 1.
\]

By the method of Section 3.1, \( E_1 + \cdots + E_m \subset E_{\gamma}^{\text{outer}} \) if and only if

\[
\sum_{i=1}^{m} \| A_i u \| \leq \| A_\gamma u \|, \quad \forall u \in \mathbb{R}^N.
\]
In fact, if $\gamma_i > 0$ for $i = 1, \ldots, m$, then by the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{m} \|A_iu\| = \sum_{i=1}^{m} \gamma_i^{-1/2} \|\gamma_i^{1/2}A_iu\| \leq \left( \sum_{i=1}^{m} \gamma_i^{-1} \right)^{1/2} \left( \sum_{i=1}^{m} \|\gamma_i^{1/2}A_iu\|^2 \right)^{1/2}. $$

Imposing the constraint

$$\sum_{i=1}^{m} \gamma_i^{-1} = 1$$

and observing that

$$\left( \sum_{i=1}^{m} \|\gamma_i^{1/2}A_iu\|^2 \right)^{1/2} = \left( \sum_{i=1}^{m} u^T(\gamma_i A_i^2)u \right)^{1/2} = (u^T A_i^2 u)^{1/2} = \|A_iu\|$$

reproduces the well-known constraint (67), and hence $E_1 + \cdots + E_m \subset E_{\text{outer}}^{\gamma}$ when $\gamma$ satisfies (66).

In the case when $m = 2$, the outer ellipsoid defined by (65)–(66) has minimal volume when

$$\gamma_1 = 1 + \beta^{-1}, \quad \gamma_2 = 1 + \beta, \quad (68)$$

where $\beta$ is the (unique) positive solution of the equation

$$\sum_{j=1}^{N} \frac{1 - \beta^2 \lambda_j^2 (A_1^{-1} A_2)}{1 + \beta \lambda_j^2 (A_1^{-1} A_2)} = 0; \quad (69)$$

see [9, 11, 16]. Thus (54) provides the upper bound on the volume

$$\text{Vol}(E_1 + E_2) \leq \text{Vol}(B^N) \det(\gamma_1 A_1^2 + \gamma_2 A_2^2)^{1/2}, \quad (70)$$

where $\gamma_1, \gamma_2$ are given by (68), (69). When $A_2 = RA_1 R^T$, with $R$ being a rotation matrix, symmetry yields $\beta = 1$ and thus $\gamma_1 = \gamma_2 = 2$.

In some contexts where rapid computations are required, alternative choices can be made with good effect such as

$$\beta' = \sqrt{\frac{\text{tr}(A_1^2)}{\text{tr}(A_2^2)}} \quad \Rightarrow \quad \gamma_1' = 1 + \sqrt{\frac{\text{tr}(A_2^2)}{\text{tr}(A_1^2)}} \quad \text{and} \quad \gamma_2' = 1 + \sqrt{\frac{\text{tr}(A_1^2)}{\text{tr}(A_2^2)}}. \quad (71)$$

The choice in (71) gives the enclosing ellipsoid that minimizes the sum of squared semi-axes lengths [11 Lemma 2.5.2]. A heuristic choice motivated by (71) for the case of an $m$-fold Minkowski sum is

$$\gamma_i' = \sum_{j=1}^{m} \sqrt{\frac{\text{tr}(A_j^2)}}{\text{tr}(A_i^2)}. \quad (72)$$

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It can be shown that the minimal volume ellipsoid of the form enclosing an \( m \)-fold Minkowski sum is defined by \([9, 11, 12]\)

\[
A(l) = \left( \sum_{j=1}^m \|A_jl\| \right)^{\frac{1}{2}} \left( \sum_{i=1}^m \frac{A_i^2}{\|A_i\|} \right)^{\frac{1}{2}}
\]

(73)

where \( l \) is the value of \( u \in S^{N-1} \) that minimizes \( \det A(u) \). That is, the optimal choice for \( \gamma_i \) is defined by

\[
\frac{1}{\gamma_i} = \frac{\|A_i\|}{\sum_{j=1}^m \|A_j\|}.
\]

(74)

An algorithm for finding \( l \) so that \( A(l) \) defines the minimal volume ellipsoid is given in \([9]\).

4 Bounds on Volume Using Steiner’s Formula

Given a convex body \( K \subset \mathbb{R}^N \), Steiner’s Formula gives the volume of the offset body as

\[
\text{Vol}(K + rB^N) = \sum_{j=0}^N \binom{N}{j} W_j(K) r^j,
\]

(75)

where the quantities \( W_j(K) \) are the quermassintegrals of \( K \). (See \([4, 8, 11]\).) In particular, Steiner’s Formula for the area of the Minkowski sum of a 2D convex body \( K \) with a disk of radius \( r \) is

\[
\mathcal{A}(K + rB^2) = \mathcal{A}(K) + rL(\partial K) + \frac{r^2}{2} \mathcal{K}(\partial K)
\]

where \( \mathcal{A}(K) \) is the area of \( K \), \( L(\partial K) \) is the length of the boundary (i.e., its perimeter), and

\[
\mathcal{K}(\partial K) = 2\pi
\]

is the integral of curvature around the boundary. Consequently,

\[
\mathcal{A}(K + rB^2) = \mathcal{A}(K) + rL(\partial K) + \mathcal{A}(rB^2).
\]

Since

\[
\mathcal{A}(E_{A_1} + E_{A_2}) = (\det A_2) \mathcal{A}(A_2^{-1} E_{A_1} + B^2),
\]

Steiner’s formula can be used to compute \( \mathcal{A}(E_{A_1} + E_{A_2}) \) exactly as

\[
\mathcal{A}(E_{A_1} + E_{A_2}) = \mathcal{A}(E_{A_1}) + (\det A_2) \mathcal{L}(\partial(A_2^{-1} E_{A_1})) + \mathcal{A}(E_{A_2})
\]

\[
= \pi (\det A_1 + \det A_2) + 4\lambda_1 (\det A_2) E \left( \frac{1 - \lambda_2/\lambda_1^2}{1/2} \right),
\]

where \( \lambda_1, \lambda_2 \) are the eigenvalues of \( A_2^{-1} A_1 \) and \( E(x) \doteq \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 \theta} \, d\theta \) is an elliptic integral.
Letting $K = A_3^{-1}(E_{A_1} + E_{A_2})$, it is possible to repeat the same procedure as above to compute the exact area $A(E_{A_1} + E_{A_2} + E_{A_3})$ as

$$A(E_{A_1} + E_{A_2} + E_{A_3}) = A(E_{A_1} + E_{A_2}) + (\det A_3)\mathcal{L}(\partial(A_3^{-1}E_{A_1} + A_3^{-1}E_{A_2})) + A(E_{A_3}).$$

This is exactly computable by letting $K_i = A_3^{-1}E_{A_i}$ and using the general fact that

$$\mathcal{L}(\partial(K_1 + K_2)) = \mathcal{L}(\partial K_1) + \mathcal{L}(\partial K_2).$$

This leads to a recursive algorithm to exactly compute the area of an $m$-fold Minkowski sum of ellipses in the plane.

In higher dimensions, we can take a similar approach to tightly bound the volume of Minkowski sums, but the approach will no longer give an exact equality for $m > 2$ ellipsoids. This is now demonstrated in the 3D case.

Steiner’s Formula for the volume of the Minkowski sum of a 3D convex body with a ball of radius $r$ is

$$\text{Vol}(K + rB^3) = \text{Vol}(K) + rA(\partial K) + r^2M(\partial K) + \frac{r^3}{3}K(\partial K),$$

where $\text{Vol}(K)$ is the volume of $K$, $A(\partial K)$ is the surface area of the boundary $\partial K$, and $M(\partial K)$ and $K(\partial K)$ are respectively the integral of mean and Gaussian curvature over the whole boundary $\partial K$. From the Gauss-Bonnet Theorem,

$$K(\partial K) = 2\pi \chi(\partial K) = 4\pi.$$

Consequently

$$\text{Vol}(K + rB^3) = \text{Vol}(K) + rA(\partial K) + r^2M(\partial K) + \text{Vol}(rB^3).$$

Therefore, for the sum of a pair of ellipsoids

$$\text{Vol}(E_{A_1} + E_{A_2}) = (\det A_2)\text{Vol}(A_2^{-1}E_{A_1} + B^2),$$

the same approach as in the planar case can be used to exactly compute

$$\text{Vol}(E_{A_1} + E_{A_2}) = \text{Vol}(E_{A_1}) + (\det A_2)A(\partial(A_2^{-1}E_{A_1})) + (\det A_2)M(\partial(A_2^{-1}E_{A_1})) + \text{Vol}(E_{A_2}).$$

The quantities in this formula can be computed using (53). In particular,

$$A(\partial E_A) = \int_{n \in S^2} \det \tilde{C}(n) \, d\sigma_2(n).$$

Furthermore, by Theorem 1.2,

$$M(\partial E_A) = \frac{1}{2} \int_{S^2} \text{tr} \left[ \tilde{C}(n)^{-1} \right] \det \tilde{C}(n) \, d\sigma_2(n)$$

$$= \frac{1}{2} \int_{S^2} \text{tr}[\tilde{C}(n)] \, d\sigma_2(n) = \frac{1}{2} \int_{S^2} \text{tr}[C(n)] \, d\sigma_2(n).$$

(76)
From commutativity of the Minkowski sum, it must be that

\[
    (\det A_2)A(\partial(A_2^{-1}E_{A_1})) + (\det A_2)M(\partial(A_2^{-1}E_{A_1})) = (\det A_1)A(\partial(A_1^{-1}E_{A_2})) + (\det A_1)M(\partial(A_1^{-1}E_{A_2})).
\]

If it is assumed that \(A\) and \(M\) are exactly computable for ellipsoids, then the above provides a closed-form formula for \(\text{Vol}(E_{A_1} + E_{A_2})\).

Following the same logic as in the planar case,

\[
    \text{Vol}(E_{A_1} + E_{A_2} + E_{A_3}) = \text{Vol}(E_{A_1} + E_{A_2}) + (\det A_3)A(\partial(A_3^{-1}E_{A_1} + A_3^{-1}E_{A_2})) + (\det A_3)M(\partial(A_3^{-1}E_{A_1} + A_3^{-1}E_{A_2})) + \text{Vol}(E_{A_3}). \quad (77)
\]

And \(\text{Vol}(E_{A_1} + E_{A_2})\) is fed forward from the previous step. Moreover, by the additivity of \(W_2\) in 3D (see [8, 14] or by (76)),

\[
    M(\partial(K_1 + K_2)) = M(\partial K_1) + M(\partial K_2),
\]

but no such equality exists for surface area. Since it is known that \(A(\partial K) = W_1(K)\) is the average area of the orthogonal projections of \(K\) onto planes in \(\mathbb{R}^3\) [13], it follows that if \(E_{\text{inner}} \subseteq K \subseteq E_{\text{outer}}\) then

\[
    A(\partial E_{\text{inner}}) \leq A(\partial K) \leq A(\partial E_{\text{outer}}). \quad (78)
\]

This can be used together with the ellipsoidal bounds in Section 3 to provide volume bounds in (77). The same reasoning can be applied in higher dimensions using Steiner’s Formula (75), where the result of Theorem 1.2 continues to be applicable.

5 Conclusions

This paper presents an exact closed-form parametric expression for the boundary of the Minkowski sum of \(m\) ellipsoids in \(\mathbb{R}^N\) and uses this result to obtain formulas for its principal curvatures and to give bounds on the enclosed volume.

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