A Logspace Solution to the Word and Conjugacy Problem of Generalized Baumslag-Solitar Groups

Armin Weiß
FMI, Universität Stuttgart, Germany

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Abstract
Baumslag-Solitar groups were introduced in 1962 by Baumslag and Solitar as examples for finitely presented non-Hopfian two-generator groups. Since then, they served as examples for a wide range of purposes. As Baumslag-Solitar groups are HNN extensions, there is a natural generalization in terms of graph of groups.

Concerning algorithmic aspects of generalized Baumslag-Solitar groups, several decidability results are known. Indeed, a straightforward application of standard algorithms leads to a polynomial time solution of the word problem (the question whether some word over the generators represents the identity of the group). The conjugacy problem (the question whether two given words represent conjugate group elements) is more complicated; still decidability has been established by Anshel and Stebe for ordinary Baumslag-Solitar groups and for generalized Baumslag-Solitar groups independently by Lockhart and Beeker. However, up to now, no precise complexity estimates have been given.

In this work, we give a LOGSPACE algorithm for both problems. More precisely, we describe a uniform \( \text{TC}^0 \) many-one reduction of the word problem to the word problem of the free group. Then we refine the known techniques for the conjugacy problem and show it is \( \text{AC}^0 \)-Turing-reducible to the word problem of the free group.

Finally, we consider uniform versions (where also the graph of groups is part of the input) of both word and conjugacy problem: while the word problem still is solvable in LOGSPACE, the conjugacy problem becomes EXPSPACE-complete.

Keywords: word problem, conjugacy problem, Baumslag-Solitar group, graph of groups, Logspace

1 Introduction
A Baumslag-Solitar group is a group of the form \( \text{BS}_{p,q} = \langle a, y \mid ya^py^{-1} = a^q \rangle \) for some \( p, q \in \mathbb{Z} \setminus \{0\} \). These groups were introduced in 1962 by Baumslag and Solitar [7] as examples for finitely presented non-Hopfian two-generator groups. They showed that the class of Baumslag-Solitar groups comprises both Hopfian and non-Hopfian groups.
The usual presentation of a Baumslag-Solitar groups is as HNN extension of an infinite cyclic group with one stable letter. The different Baumslag-Solitar groups correspond to the different inclusions of the associated subgroup into the base group. HNN extensions are a special case of fundamental groups of a graph of groups – where the graph consists of exactly one vertex with one attached loop. Thus, there is a natural notion of generalized Baumslag-Solitar group (GBS group) as fundamental group of a graph of groups with infinite cyclic vertex and edge groups – see e.g. [8, 20]. GBS groups were also studied in [32] and characterized as those finitely presented groups of cohomological dimension two which have an infinite cyclic subgroup whose commensurator is the whole group.

Algorithmic problems in group theory were introduced by Max Dehn more than 100 years ago. The two basic problems are the word problem and the conjugacy problem, which are defined as follows: Let $G$ be a finitely generated group.

**Word problem:** On input of some word $w$ written over the generators, decide whether $w = 1$ in $G$.

**Conjugacy problem:** On input of two words $v$ and $w$ written over the generators, decide whether $v$ and $w$ are conjugate, i.e., whether there exists $z \in G$ such that $zvz^{-1} = w$ in $G$.

In recent years, conjugacy played an increasingly important role in non-commutative cryptography, see e.g. [15, 22, 47]. These applications use that it is easy to create elements which are conjugated, but to check whether two given elements are conjugated might be difficult – even if the word problem is easy. In fact, there are groups where the word problem is easy but the conjugacy problem is undecidable [41].

It has been long known that both the word problem and the conjugacy problem in generalized Baumslag-Solitar groups are decidable. Actually, the standard application of Britton reductions leads to a polynomial time algorithm for the word problem (see e.g. [34]). Decidability of the conjugacy problem has been shown by Anshel and Stebe for ordinary Baumslag-Solitar groups [5] and for arbitrary GBS groups independently by Lockhart [36] and Becker [8].

The probably first non-trivial complexity bounds for the word problem have been established by the general theorem by Lipton and Zalcstein [35] resp. Simon [48] that linear groups have word problem in \textsc{logspace} (although linear GBS groups form a small sub-class of all GBS groups). Later, Waack [51] examined the particular GBS group $\langle a, s, t | sas^{-1} = a, tat^{-1} = a^2 \rangle$ as an example of a non-linear group which has word problem in \textsc{logspace}. In order to obtain the \textsc{logspace} bound for the word problem, he used the very special structure of this particular GBS group: the kernel under the canonical map onto the solvable Baumslag-Solitar group $BS_{1,2}$ is a free group.

For solvable GBS groups – which are precisely the Baumslag-Solitar groups $BS_{1,q}$ for $q \in \mathbb{Z}$ – the word problem was shown to be in (non-uniform) $\text{TC}^0$ by Robinson [45] (and also in \textsc{logspace}). Moreover, in [16] it is shown that both the word and the conjugacy problem in $BS_{1,2}$ is in uniform $\text{TC}^0$, indeed. It is straightforward to see that this proof also works for $BS_{1,q}$ for arbitrary $q$, see [52]. The result for the conjugacy problem became possible because of the seminal theorem by Hesse [23, 24] that integer division is in uniform $\text{TC}^0$ – a result which also plays a crucial role in this work.
Apart from these (and some other) special cases, no precise general complexity estimates have been given. In this work, we show that both the word problem and the conjugacy problem of every generalized Baumslag-Solitar group is in LOGSPACE. More precisely, we establish the following results:

**Theorem A.** Let $G$ be a GBS group. There is a uniform $TC^0$ many-one reduction from the word problem of $G$ to the word problem of the free group $F_2$.

Together with the well-known result that linear groups – in particular $F_2$ – have word problem in LOGSPACE [35, 48], this leads to a LOGSPACE algorithm of the word problem. Moreover, in view of [12], Theorem A shows that the word problem of GBS groups is in the complexity class $C=NC^1$ (for a definition see [12]).

**Theorem B.** Let $G$ be a GBS group. The conjugacy problem of $G$ is uniform-$AC^0$-Turing-reducible to the word problem of the free group.

We also consider uniform versions of the word and conjugacy problem (where the GBS group is part of the input – for precise definitions see Section 3.2 and Section 4.3). This leads to the following contrasting theorem:

**Theorem C.**

(i) The uniform word problem for GBS groups is in LOGSPACE. Moreover, if the GBS groups are given as fundamental groups with respect to a spanning tree, the uniform word problem is LOGSPACE-complete.

(ii) The uniform conjugacy problem for GBS groups is EXPSPACE-complete.

The paper is organized as follows: in Section 2, we fix our notation and recall some basic facts on complexity and graphs of groups – the reader who is familiar with these concepts might skip that section and only consult it for clarification. In Section 3, we give the proof of Theorem A, describe how to compute Britton-reduced words, and consider the uniform word problem. Finally, Section 4 deals with the non-uniform and uniform version of the conjugacy problem. Parts of this work are also part of the author’s dissertation [52].

## 2 Preliminaries

### Words.

An **alphabet** is a (finite or infinite) set $\Sigma$; an element $a \in \Sigma$ is called a **letter**. The free monoid over $\Sigma$ is denoted by $\Sigma^*$, its elements are called **words**. The multiplication of the monoid is concatenation of words. The identity element is the empty word $1$. If $w, p, x, q$ are words with $w = pxq$, then we call $x$ a factor of $w$.

### Rewriting systems.

Let $X$ be a set; a **rewriting system** over $X$ is a binary relation $\Rightarrow \subseteq X \times X$. If $(x, y) \in \Rightarrow$, we write $x \Rightarrow y$. The idea of the notation is that $x \Rightarrow y$ indicates that $x$ can be rewritten into $y$ in one step. We denote the reflexive and transitive closure of $\Rightarrow$ by $\Rightarrow^*$; and by $\Leftrightarrow$ its reflexive, transitive, and symmetric closure – it is the smallest equivalence relation such that $x$ and $y$ are in the same class for all $x \Rightarrow y$. 


Rewriting over words. Let $\Sigma$ be an alphabet and $S \subseteq \Sigma^* \times \Sigma^*$ be a set of pairs. This defines a rewriting system $\xrightarrow{S}$ over $\Sigma^*$ by $x \xrightarrow{S} y$ if $x = u\ell v$ and $y = uv$ for some $(\ell, r) \in S$. It is common to denote a rule $(\ell, r) \in S$ by $\ell \rightarrow r$ and we call $S$ itself a rewriting system. Since $\xrightarrow{S}$ is an equivalence relation, we can form the set of equivalence classes $\Sigma^*/S = \{[x] \mid x \in \Sigma^*\}$, where $[x] = \{y \in \Sigma^* \mid x \xrightarrow{S} y\}$. Now, $\Sigma^*/S$ becomes a monoid by $[x] \cdot [y] = [xy]$, and the mapping $x \rightarrow [x]$ yields a canonical homomorphism $\eta : \Sigma^* \rightarrow \Sigma^*/S$. The rewriting system $S$ is called

- confluent if $x \xrightarrow{S} y$ and $x \xrightarrow{S} z$ implies $\exists w : y \xrightarrow{S} w$ and $z \xrightarrow{S} w$,
- terminating if there are no infinite chains $x_0 \xrightarrow{S} x_1 \xrightarrow{S} x_2 \xrightarrow{S} \cdots$.

A rewriting system $S$ is confluent if and only if $x \xrightarrow{S} y$ implies $\exists w : x \xrightarrow{S} w$ and $y \xrightarrow{S} w$ (see [9, 29]). Thus, if $S$ is confluent and terminating, then in every class of $\Sigma^*/S$ there is exactly one element to which no rule of $S$ can be applied.

Groups. We consider a group $G$ together with a surjective homomorphism $\eta : \Sigma^* \rightarrow G$ (a monoid presentation) for some (finite or infinite) alphabet $\Sigma$. In order to keep notation simple, we suppress the homomorphism $\eta$ and consider words also as group elements. We write $w =_G w'$ as a shorthand of $\eta(w) = \eta(w')$ and $w \in_G A$ instead of $\eta(w) \in A$ for $A \subseteq G$ and $w \in \Sigma^*$.

For words (or group elements) $v, w$ we write $v \sim_G w$ to denote conjugacy, i.e., $v \sim_G w$ if and only if there exists some $z \in G$ such that $zvz^{-1} =_G w$. If $H$ is a subgroup of $G$, we write $v \sim_H w$ if there is some $z \in H$ such that $zvz^{-1} =_G w$.

Involutions. An involution on a set $\Sigma$ is a mapping $x \mapsto \overline{x}$ such that $\overline{\overline{x}} = x$. We consider only fixed-point-free involutions, i.e., $x \neq \overline{x}$.

Free groups. Let $\Lambda$ be some alphabet and set $\Sigma = \Lambda \cup \overline{\Lambda}$ where $\overline{\Lambda} = \{\overline{a} \mid a \in \Lambda\}$ is a disjoint copy of $\Lambda$. There is a fixed-point-free involution $\overline{\cdot} : \Sigma \rightarrow \Sigma$ defined by $a \mapsto \overline{a}$ and $\overline{a} \mapsto a$ (i.e., $\overline{\overline{a}} = a$). Consider the confluent and terminating rewriting system of free reductions $S = \{a\overline{a} \rightarrow 1 \mid a \in \Sigma\}$. Some word $w \in \Sigma^*$ is called freely reduced if there is no factor $a\overline{a}$ for any letter $a \in \Sigma$. The rewriting system $S$ defines the free group $F_\Lambda = \Sigma^*/S$. We have $\overline{\overline{a}} = F_\Lambda a^{-1}$ for $a \in \Sigma$. We write $F_2$ as shorthand of $F_{\{a,b\}}$.

Graphs. For the notation of graphs we follow Serre’s book [46]. A graph $Y = (V,E,\iota,\tau,\overline{\cdot})$ is given by the following data: a set of vertices $V = V(Y)$ and a set of edges $E = E(Y)$ together with two mappings $\iota, \tau : E \rightarrow V$ and an involution $e \mapsto \overline{e}$ without fixed points such that $\iota(e) = \tau(\overline{e})$.

An orientation of a graph $Y$ is a subset $D \subseteq E$ such that $E$ is the disjoint union $E = D \cup \overline{D}$. A path with start point $u$ and end point $v$ is a sequence of edges $e_1, \ldots, e_n$ such that $\tau(e_i) = \iota(e_{i+1})$ for all $i$ and $\iota(e_1) = u$ and $\tau(e_n) = v$. A graph is connected if for every pair of vertices there is a path connecting them.
2.2 Complexity

Computation or decision problems are given by functions \( f : \Delta^* \rightarrow \Sigma^* \) for some finite alphabets \( \Delta \) and \( \Sigma \). In case of a decision problem (or formal language) the range of \( f \) is the two element set \( \{0, 1\} \).

LOGSPACE is the class of functions computable by a deterministic Turing machine with working tape bounded logarithmically in the length of the input.

Our result uses the following well-known theorem about linear groups (groups which can be embedded into a matrix group over some field). It was obtained by Lipton and Zalcstein [35] for fields of characteristic 0 and by Simon [48] for other fields.

Theorem 1 ([35, 48]). Linear groups have word problem in LOGSPACE.

Circuit Complexity. The class \( \text{AC}^0 \) (resp. \( \text{TC}^0 \)) is defined as the class of functions computed by families of circuits of constant depth and polynomial size with unbounded fan-in Boolean gates (and, or, not) (resp. unbounded fan-in Boolean and Majority gates) – the alphabets \( \Delta \) and \( \Sigma \) are encoded over the binary alphabet \( \{0, 1\} \). In the following, we only consider \( D\text{logtime} \)-uniform circuit families and we write \( u\text{AC}^0 \) (resp. \( u\text{TC}^0 \)) as shorthand for \( D\text{logtime} \)-uniform \( \text{AC}^0 \) (resp. \( \text{TC}^0 \)). \( D\text{logtime} \)-uniform means that there is a deterministic Turing machine which decides in time \( O(\log n) \) on input of two gate numbers (given in binary) and the string \( 1^n \) whether there is a wire between the two gates in the \( n \)-input circuit and also decides of which type some gates is. Note that the binary encoding of the gate numbers requires only \( O(\log n) \) bits – thus, the Turing machine is allowed to use time linear in the length of the encodings of the gates. For more details on these definitions we refer to [50].

Reductions. Let \( K \subseteq \Delta^* \) and \( L \subseteq \Sigma^* \) be languages and \( C \) a complexity class. Then \( K \) is called \( C \)-many-one-reducible to \( L \) if there is a \( C \)-computable function \( f : \Delta^* \rightarrow \Sigma^* \) such that \( w \in K \) if and only if \( f(w) \in L \).

A function \( f \) is \( u\text{AC}^0 \)-reducible (or \( u\text{AC}^0 \)-Turing-reducible) to a function \( g \) if there is a \( D\text{logtime} \)-uniform family of \( \text{AC}^0 \) circuits computing \( f \) which, in addition to the Boolean gates, also may use oracle gates for \( g \) (i.e., gates which on input \( x \) output \( g(x) \)). We write \( u\text{AC}^0(F_2) \) for the family of problems which are \( u\text{AC}^0 \)-reducible to the word problem of the free group \( F_2 \).

The Class \( u\text{TC}^0 \) and Arithmetic. Although \( u\text{TC}^0 \) is a very low parallel complexity class, it is still very powerful with respect to arithmetic. By the very definition of \( u\text{AC}^0 \) reducibility, \( \text{MAJORITY} \) is \( u\text{TC}^0 \)-complete. As an immediate consequence, the word problem of \( \mathbb{Z} \) with generators \( \pm 1 \) is also \( u\text{TC}^0 \)-complete (since a sequence over the alphabet \( \{\pm 1\} \) sums up to 0 if and only if there is neither a majority of letters 1 nor of letters \(-1\).

Iterated Addition (resp. Iterated Multiplication) are the following computation problems: On input of \( n \) binary integers \( a_1, \ldots, a_n \) each having \( n \) bits (i.e., the input length is \( N = n^2 \)), compute the binary representation of the sum \( \sum_{i=0}^n a_i \) (resp. product \( \prod_{i=0}^n a_i \)). For Integer Division, the input are two binary \( n \)-bit integers \( a, b \); the binary representation of the integer \( c = \lfloor a/b \rfloor \) has to be computed. The first statement of Theorem 2 is a standard fact, see [50]; the other statements are due to Hesse, [23, 24].
Theorem 2 ([23, 24, 50]). The problems Iterated Addition, Iterated Multiplication, Integer Division are all in $uTC^0$.

We have the following inclusions (note that even $uTC^0 \subseteq P$ is not known to be strict):

$$uTC^0 \subseteq uAC^0(F_2) \subseteq \text{LOGSPACE} \subseteq P.$$  

The first inclusion is because there is a subgroup $\mathbb{Z}$ in $F_2$; the second inclusion is because of Theorem 1.

2.3 Graphs of Groups

Since generalized Baumslag-Solitar groups are defined as fundamental groups of graphs of groups, we give a brief introduction into this topic. Our presentation is a shortened version taken from [17], which in turn is based on Serre’s book [46].

Definition 3 (Graph of Groups). Let $Y = (V(Y), E(Y))$ be a connected graph. A graph of groups $\mathcal{G}$ over $Y$ is given by the following data:

(i) For each vertex $a \in V(Y)$, there is a vertex group $G_a$.

(ii) For each edge $y \in E(Y)$, there is an edge group $G_y$ such that $G_y = G_\tau(y)$.

(iii) For each edge $y \in E(Y)$, there is an injective homomorphism from $G_y$ to $G_\iota(y)$, which is denoted by $c \mapsto c^y$. The image of $G_y$ in $G_\iota(y)$ is denoted by $G_y^\iota$.

In the following, $Y$ is always a finite graph. Since $G_y = G_\tau$, there is also a homomorphism $G_y \to G_\tau(y)$. Thus, for $y \in E(Y)$ with $\iota(y) = a$ and $\tau(y) = b$, there are two isomorphisms and inclusions:

$$G_y \to G_y^\iota \leq G_a, \quad c \mapsto c^y, \quad G_y \to G_y^\iota \leq G_b, \quad c \mapsto c^\tau.$$

The fundamental group of $\mathcal{G}$ can be constructed as subgroup of the larger group $F(\mathcal{G})$: as an (possibly infinite) alphabet we choose a disjoint union

$$\Delta = E(Y) \cup \bigcup_{a \in V(Y)} (G_a \setminus \{1\}),$$

and we define the group

$$F(\mathcal{G}) = \Delta^* / \{ gh = [gh], ye^\tau = c^y \mid a \in V(Y), g, h \in G_a; y \in E(Y), c \in G_y \},$$

where $[gh]$ denotes the element obtained by multiplying $g$ and $h$ in $G_a$ (where $1 \in G_a$ is identified with the empty word).

Let us define subsets of $\Delta^*$ as follows: for $a, b \in V(Y)$, we denote with $\Pi(\mathcal{G}, a, b)$ the set of words where the occurring edges form a path from $a$ to $b$ in $Y$ and the elements of vertex groups between two edges are from the corresponding vertex in the path; more precisely,

$$\Pi(\mathcal{G}, a, b) = \{ g_0 y_1 \cdots y_{n-1} y_n g_n \mid y_i \in E(Y), \iota(y_1) = a, \tau(y_n) = b, \tau(y_i) = \iota(y_{i+1}), g_0 \in G_a, g_i \in G_\iota(y_i) \text{ for all } i \},$$
where again 1 ∈ Ga is identified with the empty word. Moreover, we set
\[ \Pi(G) = \bigcup_{a \in V(Y)} \Pi(G, a, a). \]

In general, the image of \( \Pi(G) \) in \( F(Y) \) is not a group but a so-called groupoid. If \( w = g_0y_1 \cdots y_{n-1}y_n \in \Pi(G) \), then we call \( w \) a \( G \)-factorization of the respective group element in \( F(G) \); by saying this we implicitly require that \( y_i \in E(Y), \tau(y_i) = \iota(y_i), g_i \in G_{\tau(y_i)} \) for all \( i, \tau(y_n) = \iota(y_1) \), and \( g_0 \in G_{\iota(y_1)}. \)

We call \( y_1 \cdots y_n \) the underlying path of \( w \).

For all vertices \( a \in V(Y) \), the image of \( \Pi(G, a, a) \) in \( F(G) \) is a group.

Definition 4.

(i) Let \( a \in V(Y) \). The fundamental group \( \pi_1(G, a) \) of \( G \) with respect to the base point \( a \in V(Y) \) is defined as the image of \( \Pi(G, a, a) \) in \( F(G) \).

(ii) Let \( T \) be a spanning tree of \( Y \) (i.e., a subset of \( E(Y) \) connecting all vertices and not containing any cycles). The fundamental group of \( G \) with respect to \( T \) is defined by
\[ \pi_1(G, T) = F(G)/\{y = 1 \mid y \in T\}. \]

Proposition 5 ([46]). The canonical homomorphism from the subgroup \( \pi_1(G, a) \) of \( F(G) \) to the quotient group \( \pi_1(G, T) \) is an isomorphism. In particular, the two definitions of the fundamental group are independent of the choice of the base point and the spanning tree.

Example 6. Let \( G \) be a graph of groups over the following graph:

\[ \begin{array}{c}
\text{a} \\
\circ \\
\text{y, y}
\end{array} \]

and let \( G_a = \mathbb{Z} = \langle a \rangle \) and \( G_y = G_y = \mathbb{Z} = \langle c \rangle \) and the inclusions given by
\[ c \mapsto a^p \quad \text{and} \quad c \mapsto a^q \]
for some \( p, q \in \mathbb{Z} \setminus \{0\} \). Then the fundamental group \( \pi_1(G, a) \) is the Baumslag-Solitar group
\[ \pi_1(G, a) = BS_{p,q} = \langle a, y \mid ya^py^{-1} = a^q \rangle. \]

Britton Reductions over Graphs of Groups. In [10], Britton reductions were originally defined for HNN extensions. They are given by the rewriting system \( B_G \subseteq \Delta^* \times \Delta^* \) with the following rules (see also [37, Sec. IV.2]):
\[ \begin{align*}
gh & \to [gh] \quad \text{for } a \in V(Y), \ g, h \in G_a \setminus \{1\}, \\
yc & \to c^y \quad \text{for } y \in E(Y), \ c \in G_y.
\end{align*} \]

As \( B_G \) is length-reducing, it is terminating. Furthermore, \( F(G) = \Delta^*/B_G \). A word \( w \in \Delta^* \) is called Britton-reduced if no rule from \( B_G \) can be applied to it.

As \( B_G \) is terminating, there is a Britton-reduced \( \hat{w} \) with \( \hat{w} = F(G)w \) for every \( w \). However, this \( \hat{w} \) might not be unique as \( B_G \) is not confluent in general. Still, the following crucial facts hold:
Lemma 7 (Britton’s Lemma, [10]). Let \( w \in \Delta^* \) be Britton-reduced. If \( w \in F(G) \) \( G_a \), then \( w \) is the empty word or consists of a single letter of \( G_a \). Moreover, if \( w = F(G) 1 \), then \( w = 1 \) (i.e., \( w \) is the empty word).

Lemma 8. If \( v = h_0x_1\cdots x_{n-1}x_nh_n, w = g_0y_1\cdots g_{n-1}y_ng_n \in \Pi(G) \) with \( v = F(G) w \) are Britton-reduced, then \( x_i = y_i \) for all \( i \) and there are \( c_i \in G_{y_i} \) for \( 1 \leq i \leq n \) such that

\[
\begin{align*}
h_0 &= \varepsilon \alpha (c_1^{-1})^{y_1}, \\
h_i &= \varepsilon \beta (c_i^{-1})^{y_{i+1}}, \quad \text{for } 1 \leq i \leq n-1, \text{ and} \\
h_n &= \varepsilon \beta (c_n^{y_n}).
\end{align*}
\]

Using Lemma 7 one obtains a decision procedure for the word problem if the subgroup membership problem of \( G_a^y \) in \( G_a^y \) is decidable, the word problem of \( G_a \) is decidable for some \( a \in V(Y) \), and the isomorphisms \( G_a^y \to G_a^y \) are effectively computable for all \( y \in E(Y) \). However, this does not imply any bound on the complexity. The problem is that – even if all computations can be performed efficiently – the blow up due to the calculations of the isomorphisms \( G_a^y \to G_a^y \) might prevent an efficient solution of the word problem in the fundamental group. An example is the Baumslag group \( G_{1,2} = \langle a, t, b \mid tat^{-1} = a^2, bab^{-1} = t \rangle \), which is an HNN extension of the Baumslag-Solitar group \( BS_{1,2} \). For \( G_{1,2} \), the straightforward algorithm of applying Britton reductions, leads to a non-elementary running time. However, in [42] it is shown that the word problem still can be solved in polynomial time.

For Baumslag-Solitar groups, the straightforward application of Britton reductions yields a polynomial time algorithm if the exponents are stored as binary integers.

Generalized Baumslag-Solitar Groups. A generalized Baumslag-Solitar group (GBS group) is a fundamental group of a finite graph of groups with only infinite cyclic vertex and edge groups. That means a GBS group is completely given by a finite graph \( Y \) and numbers \( \alpha_y, \beta_y \in \mathbb{Z} \setminus \{0\} \) for \( y \in E(Y) \) such that \( \alpha_y = \beta_y \). For \( a \in V(Y) \) we write \( G_a = \langle a \rangle \). Then we have

\[
F(G) = \langle V(Y), E(Y) \mid \gamma y = 1, yb^\alpha \gamma = a^\alpha y, \text{ for } y \in E(Y), a = \alpha(y), b = \tau(y) \rangle
\]

and \( G = \pi_1(G, a) \leq F(G) \) for any \( a \in V(Y) \) as in Definition 4. (Note that \( V(Y) \cup E(Y) \) generates \( F(G) \) as a group, but in general, not as a monoid.)

As we have seen in Example 6, Baumslag-Solitar groups \( BS_{p,q} \) are the special case that \( Y \) consists of one vertex and one loop \( y \) with \( \alpha_y = p, \beta_y = q \).

3 The Word Problem

In [45], Robinson showed that the word problem of non-cyclic free groups is \( NC^1 \)-hard. Hence, for non-solvable GBS groups, we cannot expect the word or conjugacy problem to be in \( uTC^0 \) since they contain a free group of rank two. For ordinary Baumslag-Solitar groups, the word problem has recently been shown to be in \( NC^2 \) [31]. In the author’s dissertation [52], this is improved to \( LOGDCFL \) – which means that it is \( LOGSPACE \)-reducible to a deterministic
context-free language. Here we aim for a LOGSPACE algorithm – or, more precisely, for a $uTC^0$ many-one reduction to the word problem of the free group $F_2$.

Let $G = \pi_1(\mathcal{G}, a)$ be a fixed GBS group given by a graph $Y$ and numbers $\alpha_y, \beta_y \in \mathbb{Z} \setminus \{0\}$ for $y \in E(Y)$ and $a \in V(Y)$. Our alphabet is $\Delta = E(Y) \cup \{a^k \mid a \in V(Y), k \in \mathbb{Z}\}$ – for simplicity we allow $k = 0$ and identify the letter $a^0$ with the empty word. We say that a word or $\mathcal{G}$-factorization $w$ is represented in binary if the numbers $k$ are written as binary integers (using a variable number of bits) – in the following we always assume this binary representation.

It turns out to be more convenient to work outside of $G$ and to consider arbitrary $\mathcal{G}$-factorizations $w \in \Pi(\mathcal{G})$. Recall that a $\mathcal{G}$-factorization of some group element is a word

$$w = a_0^k y_1 a_1^{k_1} \cdots y_n a_n^{k_n}$$

with $a_i = \tau(y_i) = \iota(y_{i+1})$ for $0 < i < n$, $a_n = \tau(y_n) = \iota(y_1)$, and $k_i \in \mathbb{Z}$.

In the following, we always write $a_i$ as shorthand of $\iota(y_{i+1})$.

**Lemma 9.** Let $w = a_0^k y_1 a_1^{k_1} \cdots y_n a_n^{k_n} \in \Pi(\mathcal{G})$. If $w \in F(\mathcal{G}) \langle a_0 \rangle$, then we have $w =_F a_0^k$ for

$$k = \sum_{\nu=0}^n k_{\nu} \prod_{\mu=1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}$$

where $\alpha_{\mu} = \alpha_{y_{\mu}}$ and $\beta_{\mu} = \beta_{y_{\mu}}$ for $1 \leq \mu \leq n$.

**Proof.** If $w = a_0^k$, then the formula is obviously correct. Hence, let $n > 0$. Then by Lemma 7, all the edges $y_i$ can be cancelled by Britton reductions. In particular, we can find some $1 < i \leq n$ such that $w = a_0^k y_1 y_i y_i' w''$ with $y_1 = \overline{y}_1$ and $w' = a_1^k y_2 \cdots a_{i-1} y_{i-1} \in F(\mathcal{G}) \langle a_1 \rangle$ and $w'' = a_i^k y_{i+1} \cdots a_n^k \in F(\mathcal{G}) \langle a_i \rangle = \langle a_0 \rangle$.

By induction, we have $w' =_F a_1^k w'$ and $w'' =_F a_0^k$ where

$$k' = \sum_{\nu=1}^{i-1} k_{\nu} \prod_{\mu=2}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}$$

and

$$k'' = \sum_{\nu=0}^n k_{\nu} \prod_{\mu=1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}.$$

Since $y_1 y_i' \in F(\mathcal{G}) \langle a_0 \rangle$, we have $y_1 y_i' =_F a_0^{\frac{\alpha_{i+1}}{\beta_i}} k'$ and $\prod_{\mu=1}^{i-1} \frac{\alpha_{\mu}}{\beta_{\mu}} = 1$. Hence,

$$k = k_0 + \frac{\alpha_{i+1}}{\beta_i} k' + k'' = \sum_{\nu=0}^n k_{\nu} \prod_{\mu=1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}.$$

\[\square\]

For the rest of this section, we let $w = a_0^k y_1 a_1^{k_1} \cdots y_n a_n^{k_n}$ be a $\mathcal{G}$-factorization given in binary. For $0 \leq i \leq j \leq n$, we define

$$w_{i,j} = a_1^k y_i a_i^{k_i+1} \cdots y_j a_j^{k_j}$$

and

$$k_{i,j} = \sum_{\nu=0}^{j-1} k_{\nu} \prod_{\mu=1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}} \in \mathbb{Q} \quad (1)$$
analogously to $k$ in Lemma 9 where again $\alpha_{\mu} = \alpha_{y_{\mu}}$ and $\beta_{\mu} = \beta_{y_{\mu}}$ for $1 \leq \mu \leq n$. Note that we do not assume that $a_{i_1}^{k_1} a_{i_{1+1}}^{k_{1+1}} \cdots a_{i_j}^{k_j}$ lies in $(\alpha_i)$ – yet the numbers $k_{i,j}$ will play an important role in what follows. In particular, with the notation of Lemma 9, we have $k = k_{0,n}$. Moreover, by Lemma 9, we have

**Lemma 10.**

$w_{i,j} \in F(G) (\alpha_i)$ if and only if $w_{i,j} = F(G) a_i^{k_{i,j}}$.

**Lemma 11.** The numbers $k_{i,j}$ (as fractions of binary integers) can be computed by a uniform family of $\text{TC}^0$ circuits – even if the numbers $\alpha_y, \beta_y$ are part of input.

**Proof.** Iterated Addition and Iterated Multiplication are in $\text{uTC}^0$, see Theorem 2; hence, the rational numbers $k_{i,j}$ can be computed in $\text{uTC}^0$ according to (1). Be aware that we do not require that the fractions are reduced. □

Now, pick some orientation $D \subseteq E(Y)$ of the edges (for every pair $y, \overline{y}$ choose exactly one of them to be in $D$). Consider the canonical map $p : G \rightarrow \mathbb{Z}^D$ onto the abelianization of the subgroup generated by the edges, which is defined by $a \mapsto 0$ for $a \in V(Y)$ and $y \mapsto -e_y, \overline{y} \mapsto -e_y$ for $y \in D$ (where $e_y$ is the unit vector having 1 at position $y$ and 0 otherwise). With other words $p$ counts the exponents of the edges. Consider the following observations:

- If $w = F(G) 1$, then every edge $y$ in $w$ can be canceled with some $\overline{y}$ by Britton reductions.

- Consider a factor $y \overline{y}$ for some word $v$. If $y$ cancels with $\overline{y}$, then necessarily we have $p(v) = 0$, i.e., all edges occurring in between have exponent sum zero.

Now, the idea is to introduce colors and assign them to the letters $y_i$ such that $y_i$ and $\overline{y}_j$ get the same color only if they potentially might cancel. In order to do so, we start by defining a relation $\sim_c \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ and set

\[
i \sim_c j \text{ if and only if } y_i = \overline{y}_j \text{ and } \begin{cases} p(w_{i,j-1}) = 0 \text{ and } k_{i,j-1} \in \beta_i \mathbb{Z} \quad \text{if } i < j, \\ p(w_{j,i-1}) = 0 \text{ and } k_{j,i-1} \in \beta_j \mathbb{Z} \quad \text{if } j < i. \end{cases}
\]

Thus, $\sim_c$ is symmetric and we have $i \not\sim_c i$ for all $i$. Informally speaking, we have $i \sim_c j$ if and only if everything in between vanishes in the abelian quotient $\mathbb{Z}^D$ and $y_i$ and $\overline{y}_j$ cancel given that everything in between cancels to something in $(\alpha_i)$ (the latter is a consequence of Lemma 10).

**Lemma 12.** If $i \sim_c \ell$, $\ell \sim_c m$, and $m \sim_c j$, then also $i \sim_c j$.

**Proof.** If two of the indices $i, j, \ell, m$ coincide, what can be the case only if $i = m$ or $\ell = j$, we are done. Otherwise, we have to show that $y_i = \overline{y}_j \Rightarrow p(w_{i,j-1}) = 0$, and $k_{i,j-1} \in \beta_i \mathbb{Z}$ (resp. $p(w_{j,i-1}) = 0$ and $k_{j,i-1} \in \beta_j \mathbb{Z}$ for $j < i$). We have $y_i = \overline{y}_\ell = y_m = \overline{y}_j$.

In order to see the other two conditions, we put the indices $i, j, \ell, m$ in ascending order. That means we fix $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ such that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{i, j, \ell, m\}$. There are three situations to consider, as depicted in Figure 1:

(i) $y_{\lambda_1} = y_{\lambda_2}$ and $y_{\lambda_3} = y_{\lambda_4} = \overline{y}_{\lambda_1}$,

(ii) $y_{\lambda_1} = y_{\lambda_2}$ and $y_{\lambda_3} = y_{\lambda_4} = \overline{y}_{\lambda_1}$,
Thus, since three of these vectors are zero, so is the fourth (i.e., we have shown where $\rho$ and $\rho$ are 

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subseteq k$$

(iii) $y_{\lambda_1} = y_{\lambda_4}$ and $y_{\lambda_2} = y_{\lambda_3} = \overline{y}_{\lambda_1}$.

All these cases have in common that there are exactly four pairings $\{\lambda_i, \lambda_j\}$ with $y_{\lambda_i} = \overline{y}_{\lambda_j}$, and these four pairings correspond to the four pairings $\{i, \ell\}$, $\{\ell, m\}$, $\{m, j\}$, and $\{i, j\}$. In each case, the conditions $\rho(w_{\lambda_i}, \lambda_i - 1) = 0$ and $k_{\lambda_i, \lambda_i - 1} \in \beta_{\lambda_i} \mathbb{Z}$ hold for three of the $\{\lambda_i, \lambda_j\}$, and we have to show it for the fourth.

In case (i), we have

$$\rho(w_{\lambda_1}, \lambda_4 - 1) = \rho(w_{\lambda_1}, \lambda_3 - 1) + \rho(w_{\lambda_2}, \lambda_4 - 1) - \rho(w_{\lambda_2}, \lambda_3 - 1).$$

Thus, since three of these vectors are zero, so is the fourth (i.e., we have shown that $\rho(w_{\lambda_i}, \lambda_i - 1) = 0$ resp. $\rho(w_{\lambda_i}, \lambda_i - 1) = 0$). In particular, we have $\rho(w_{\lambda_1}, \lambda_2) = \rho(w_{\lambda_1}, \lambda_4 - 1) - \rho(w_{\lambda_2}, \lambda_4 - 1) = 0$. Hence,

$$\prod_{\mu = \lambda_1 + 1}^{\lambda_2} \frac{\alpha_\mu}{\beta_\mu} = \prod_{y \in D} \left( \frac{\alpha_y}{\beta_y} \right)^{\rho(w_{\lambda_1}, \lambda_2)_y} = 1,$$

where $\rho(w_{\lambda_1}, \lambda_2)_y$ denotes the component of the vector belonging to $y$ (recall $D \subseteq E(Y)$ is the orientation) – the first equality is because $\frac{\alpha_y}{\beta_y} = \left( \frac{\alpha_{\overline{y}}}{\beta_{\overline{y}}} \right)^{-1}$, and $\rho(w_{\lambda_1}, \lambda_2)_y$ simply counts the number of occurrences of $y$ (positive) and $\overline{y}$ (negative) in $w_{\lambda_1, \lambda_2}$. It follows that

$$k_{\lambda_1, \lambda_4 - 1} = \sum_{\nu = \lambda_1}^{\lambda_2 - 1} \prod_{\mu = \lambda_1 + 1}^{\nu} \frac{\alpha_\mu}{\beta_\mu}$$

$$= \sum_{\nu = \lambda_1}^{\lambda_2 - 1} k_{\nu} \cdot \prod_{\mu = \lambda_1 + 1}^{\nu} \frac{\alpha_\mu}{\beta_\mu} + \sum_{\nu = \lambda_2}^{\lambda_4 - 1} k_{\nu} \cdot \prod_{\mu = \lambda_1 + 1}^{\nu} \frac{\alpha_\mu}{\beta_\mu} - \sum_{\nu = \lambda_2}^{\lambda_3 - 1} k_{\nu} \cdot \prod_{\mu = \lambda_1 + 1}^{\nu} \frac{\alpha_\mu}{\beta_\mu}$$

$$= k_{\lambda_1, \lambda_3 - 1} + k_{\lambda_2, \lambda_4 - 1} - k_{\lambda_2, \lambda_3 - 1}.$$
Hence, since three of them are in $\beta_{\lambda^*}Z = \beta_{\lambda^*}Z$, so is the fourth.

The other cases follow with the same arguments: in case (ii) we have

$$\rho(w_{\lambda_1,\lambda_2-1}) = \rho(w_{\lambda_1,\lambda_2-1}) + \rho(y_{\lambda_1}) + \rho(w_{\lambda_2,\lambda_3-1}) + \rho(y_{\lambda_3}) + \rho(w_{\lambda_3,\lambda_4-1})$$

$$= \rho(w_{\lambda_1,\lambda_2-1}) + \rho(w_{\lambda_3,\lambda_4-1})$$

because $y_{\lambda_2} = y_{\lambda_1}$, what again implies that all of them are zero. Like in the first case, we have $\prod_{\mu=\lambda_1+1}^{\lambda_2} \frac{\alpha_{\mu}}{\beta_{\mu}} = \frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}}$ (because $\rho(w_{\lambda_1,\lambda_2-1}) = 0$) and $\prod_{\mu=\lambda_1+1}^{\lambda_3} \frac{\alpha_{\mu}}{\beta_{\mu}} = 1$ (because $\rho(w_{\lambda_1,\lambda_3}) = \rho(w_{\lambda_1,\lambda_4-1}) - \rho(w_{\lambda_3,\lambda_4-1}) = 0$). It follows that

$$k_{\lambda_1,\lambda_4-1} = \sum_{\nu=\lambda_1}^{\lambda_2-1} k_{\nu} \cdot \prod_{\mu=\lambda_1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}$$

$$= \sum_{\nu=\lambda_1}^{\lambda_2-1} k_{\nu} \cdot \prod_{\mu=\lambda_1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}} + \frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}} \cdot \sum_{\nu=\lambda_1}^{\lambda_2-1} k_{\nu} \cdot \prod_{\mu=\lambda_1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}} + \sum_{\nu=\lambda_3}^{\lambda_4-1} k_{\nu} \cdot \prod_{\mu=\lambda_3+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}$$

$$= k_{\lambda_1,\lambda_2-1} + \frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}} \cdot k_{\lambda_2,\lambda_3-1} + k_{\lambda_3,\lambda_4-1}.$$

Since $y_{\lambda_2} = y_{\lambda_3} = y_{\lambda_4}$, we have $\alpha_{\lambda_2} = \beta_{\lambda_1}$ and $\beta_{\lambda_2} = \beta_{\lambda_1}$. That means we have $\frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}} \cdot k_{\lambda_2,\lambda_3-1} \in \beta_{\lambda_1}Z$ if and only if $k_{\lambda_2,\lambda_3-1} \in \beta_{\lambda_1}Z$, and $k_{\lambda_3,\lambda_4-1} \in \beta_{\lambda_1}Z$ if and only if $k_{\lambda_3,\lambda_4-1} \in \beta_{\lambda_1}Z$. Thus, since for three of the $k_{\lambda,\lambda'}$ we have $k_{\lambda,\lambda'} \in \beta_{\lambda}Z$, this is true also for the fourth.

Finally, in case (iii), because of $y_{\lambda_2} = y_{\lambda_3}$, we have

$$\rho(w_{\lambda_1,\lambda_3-1}) = \rho(w_{\lambda_2,\lambda_4-1}) = \rho(w_{\lambda_1,\lambda_3}) = \rho(w_{\lambda_2,\lambda_4-1}).$$

Therefore, they are all 0. As before, $\rho(w_{\lambda_1,\lambda_2-1}) = 0$ implies that $\prod_{\mu=\lambda_1+1}^{\lambda_2-1} \frac{\alpha_{\mu}}{\beta_{\mu}} = 1$ and $\rho(w_{\lambda_2,\lambda_3}) = 0$ implies that $\prod_{\mu=\lambda_2+1}^{\lambda_3} \frac{\alpha_{\mu}}{\beta_{\mu}} = 1$. Thus, we have

$$k_{\lambda_1,\lambda_3-1} - k_{\lambda_1,\lambda_2-1} = \sum_{\nu=\lambda_1}^{\lambda_2-1} k_{\nu} \cdot \prod_{\mu=\lambda_1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}} - \sum_{\nu=\lambda_1}^{\lambda_3-1} k_{\nu} \cdot \prod_{\mu=\lambda_1+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}$$

$$= \frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}} \cdot \sum_{\nu=\lambda_2}^{\lambda_3-1} k_{\nu} \cdot \prod_{\mu=\lambda_2+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}}$$

$$= \frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}} \left( \sum_{\nu=\lambda_2}^{\lambda_3-1} k_{\nu} \cdot \prod_{\mu=\lambda_2+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}} - \sum_{\nu=\lambda_3}^{\lambda_4-1} k_{\nu} \cdot \prod_{\mu=\lambda_3+1}^{\nu} \frac{\alpha_{\mu}}{\beta_{\mu}} \right)$$

$$= \frac{\alpha_{\lambda_2}}{\beta_{\lambda_2}} \cdot (k_{\lambda_2,\lambda_4-1} - k_{\lambda_3,\lambda_4-1})$$

with $\alpha_{\lambda_2} = \beta_{\lambda_1}$ and $\beta_{\lambda_2} = \beta_{\lambda_1}$. So, again since for three of the $k_{\lambda,\lambda'}$ we have $k_{\lambda,\lambda'} \in \beta_{\lambda}Z$, this is true also for the fourth.

Now, we define a new relation $\sim \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ as $i \sim j$ if and only if there is some $\ell$ with $i \sim \ell$ and $\ell \sim j$. Moreover, we set $i \sim i$ for all $i$. 

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Lemma 13. \( \approx \) is an equivalence relation.

**Proof.** By definition, \( \approx \) is reflexive. Because \( \sim_c \) is symmetric, \( \approx \) is also symmetric. Transitivity follows from Lemma 12.

Denote by \( \Sigma_w = \{ [i] \mid i \in \{1, \ldots, n\} \} \) the set of equivalence classes of \( \approx \). For \( [i] \in \Sigma_w \) define \( [i] = [j] \) if \( i \sim_c j \) – if no such \( j \) exists, we add a new element \( [i] \) to \( \Sigma_w \). From the definition of \( \approx \) it follows that \( \approx \) is well-defined. Moreover, we have \([i] = [i] \) and \([i] \neq [i] \) for all \( [i] \in \Sigma_w \). In particular, \( \Sigma_w \) is an alphabet with fixed-point-free involution. We can think of each class \( [i] \bigcup [i] \) as a color assigned to the edges \( y_i \).

From the definition of \( \sim_c \) and Lemma 10 it is clear that only edges with the same color can cancel. Let \( \Lambda_w \subseteq \Sigma_w \) such that \( \Sigma_w = \Lambda_w \cup \Sigma_w \) as a disjoint union, i.e., for every pair \( [i], [j] \) exactly one of them is in \( \Lambda_w \). Then we have \( \Sigma^*_w / \{ [i] = [i] \bigcup [i] = 1 \} = F_{\Lambda_w} \).

We define \( \mathcal{C}(w) = [1] \cdots [n], \quad \mathcal{C}(w, j) = [i + 1] \cdots [j]. \)

**Lemma 14.** \( w_{i, j} \in F(G) \langle a_i \rangle \) if and only if \( \mathcal{C}(w_{i, j}) = F(\Lambda_w) \).

Before we prove Lemma 14, we present an example and some consequences.

**Example 15.** Consider the group \( BS_{2, 3} \) and the word \( w = yaya^3ya^3ya^2ya^3y. \)

Then we have
\[
\mathcal{C}(w) = [1] [2] [3] [4] [5] [6] [7] [8] = [1] [2] [3] [2] [1] [1] [1] = F_{\Lambda_w} 1.
\]

Indeed, consider for example the factor \( ya^3y. \) As \( k_{3, 3} = 3 \in 3\mathbb{Z} \), it follows that \( 3 \sim_c 4 \) and thus \( [4] = [3] \); however, \( 2 \not\sim_c 3 \) since \( k_{2, 2} = 1 \not\in 2\mathbb{Z} \), see Figure 2. By Lemma 14, we know that \( w \in \langle a \rangle \).

![Figure 2: \( \rho(w) \) and \( \mathcal{C}(w) \) depicted graphically – each color represents one \( [i] \bigcup [i] \).](image)

As immediate consequences of Britton’s Lemma, Lemma 9, and Lemma 14, we obtain:

**Corollary 16.** \( w = F(G) 1 \) if and only if \( \mathcal{C}(w) = F(\Lambda_w) 1 \) and \( k_{0, n} = 0. \)

**Corollary 17.** For \( w = a_0^{k_0} y_1 a_1^{k_1} \cdots y_n a_n^{k_n}, \) let \( [i_1] \cdots [i_j] \in \Sigma_w^* \) be freely reduced with \( \mathcal{C}(w) = [1] \cdots [n] = F_{\Lambda_w} [i_1] \cdots [i_j]. \) Then the \( G \)-factorization
\[
\tilde{w} = a_0^{k_{0, i_1 - 1}} y_{i_1} a_1^{k_{1, i_2 - 1}} \cdots y_{i_j} a_n^{k_{j, n}}
\]
is Britton-reduced and \( w = F(G) \tilde{w}. \)
Thus, we obtain

\[ k \text{ we have } \]

with

\[ \text{we have } \]

\[ \text{factorization } w \]

\[ \text{problem for } G \]

\[ \text{reduced and we can write it in the form } \]

\[ \text{by Lemma 10, } w \]

\[ \text{with } i + 1 \sim \ell. \]

By induction, we know that \( i + 1 \mid C(w_{i+1, \ell-1}) \) \( C(w_{\ell,j}) \) \( F(\Lambda_\omega) \) 1. Thus, we obtain

\[ C(w_{i,j}) = [i + 1] C(w_{i+1, \ell-1}) [\ell] C(w_{\ell,j}) = F(\Lambda_\omega) 1. \]

For the other direction let \( C(w_{i,j}) = F(\Lambda_\omega) 1. \) Then \( C(w_{i,j}) \) is not freely reduced and we can write it in the form

\[ C(w_{i,j}) = [i + 1] C(w_{i+1, \ell-1}) [\ell] C(w_{\ell,j}) \]

for some \( \ell \) with \( [i + 1] = [\ell] \) and \( C(w_{i+1, \ell-1}) = F(\Lambda_\omega) C(w_{\ell,j}) = F(\Lambda_\omega) 1. \)

By induction, we know that \( w_{i+1, \ell-1} \in F(G) \{a_{i+1}\} \) and \( w_{\ell,j} \in F(G) \{a_\ell\}; \) thus, by Lemma 10, \( w_{i+1, \ell-1} = F(G) a_{i+1}^{k_{i+1, \ell-1}} \) and \( w_{\ell,j} = F(G) a_{\ell}^{k_{\ell,j}}. \) Since \( [i + 1] \sim_\ell [\ell] \), we have \( y_{i+1} = \overline{y}_\ell \) and \( k_{i+1, \ell-1} \in \beta_{i+1} \mathbb{Z}. \) As, in particular, \( a_i = a_{\ell} \), we obtain

\[ w_{i,j} = a_{\ell}^{k_{\ell,j}} \]

\[ y_{i+1} \]

\[ w_{\ell,j} \]

\[ = F(G) a_{\ell}^{k_{\ell,j}} y_{i+1} a_{i+1}^{k_{i+1, \ell-1}} y_{\ell} a_{\ell}^{k_{\ell,j}} \]

\[ = a_{\ell}^{k_{\ell,j}} a_{i+1}^{\beta_{i+1} k_{i+1, \ell-1}} a_{\ell}^{k_{\ell,j}} \in \langle a_i \rangle. \]

\[ \square \]

Now, we are ready to describe a \( \text{uTC}^0 \)-many-one reduction of the word problem for \( G \)-factorizations to the free group \( F_2 = \langle a, b \rangle. \) The input is a \( G \)-factorization \( w \), the output some word in \( \overline{w} \in \{ a,\overline{a}, b,\overline{b} \}^* \) such that \( w = F(G) 1 \) if and only if \( \overline{w} = F_2 1. \) The circuit computes the following steps:

**Algorithm 18.**

(i) Compute \( k_{0,n}. \) If \( k_{0,n} \neq 0 \), then output \( a \) (or some arbitrary other non-
identity element of \( F_2 \)).

(ii) Otherwise, compute and output an encoding of \( C(w) \) in \( F_2 \) as follows:

(a) For all pairs \( i < j \) check independently in parallel whether \( i \sim_\ell j \) in \( \text{uTC}^0: \)

1. check whether \( y_i = \overline{y}_j \),
2. compute \( \rho(w_{i,j-1}) \) and check whether \( \rho(w_{i,j-1}) = 0 \),
3. compute \( k_{i,j-1} \), check whether \( k_{i,j-1} \in \mathbb{Z} \) and, if yes, whether \( \beta_i \mid k_{i,j-1}. \)

If all points hold, then \( i \sim_\ell j \), otherwise not.
(b) For every index \( i \) compute in parallel the smallest \( j \) with \( j \in [i] \cup [\overline{i}] \) as representative of \([i]\) – depending on whether \( j \in [i] \) or \( j \in [\overline{i}] \) the corresponding output is \( b^j \) or \( b^{\overline{j}} \).

(c) Concatenate all output words of the previous step.

By Lemma 11 and Hesse’s result Theorem 2, step (i) and (ii) (a) can be computed in \( \text{uTC}^0 \). Steps (ii) (b) and (ii) (c) are straightforward in \( \text{uTC}^0 \). Indeed, the smallest \( j \in [i] \cup [\overline{i}] \) satisfies the first order formula

\[
\left( i = j \lor i \sim_C j \lor \bigvee_k (i \sim_C k \land k \sim_C j) \right) \land \bigwedge_{k<j} \neg \left( i \sim_C k \lor \bigvee_\ell (i \sim_C \ell \land \ell \sim_C k) \right),
\]

which describes an \( \text{uAC}^0 \) circuit in the obvious way (see [6] for the general correspondence between circuits and formulas). Step (ii) (c) can be seen as the application of a homomorphism of free monoids, what can be done in \( \text{uTC}^0 \) (see [33]). Thus, we have established a \( \text{uTC}^0 \) many-one reduction to the word problem of \( F_2 \).

Note that in none of the above steps the actual graph played a role – only the numbers \( \alpha_y, \beta_y \) were used. This is because, up to now, we assumed that the input is already given as \( G \)-factorization. But also the transformation of elements of \( \pi_1(G, T) \) into \( G \)-factorizations can be done in \( \text{uTC}^0 \) as we see in the next theorem, which proves Theorem A.

**Theorem 19.** Let \( G = \pi_1(G, a) \cong \pi_1(G, T) \) be a GBS group with graph \( Y \) and \( \Delta = E(Y) \cup \{ a^k \mid a \in V(Y), k \in \mathbb{Z} \} \). There is a many-one reduction computed by a uniform family of \( \text{TC}^0 \)-circuits from each of the problems

1. given a word \( w \in \Delta^* \), decide whether \( w \) is a \( G \)-factorization and, if so, decide whether \( w =_{F(G)} 1 \).
2. given a word \( w \in \Delta^* \), decide whether \( w =_{\pi_1(G, T)} 1 \),

   to the word problem of the free group \( F_2 \). In particular, the word problem of \( G \) is in \( \text{LOGSPACE} \).

**Proof.** In order to decide whether \( w \) is a \( G \)-factorization, one simply needs to verify whether \( w \) is of the form \( a_0^{k_0} y_1 a_1^{k_1} \cdots y_n a_n^{k_n} \) and then check whether \( a_0 = a_n \) and \( a_{i-1} = \iota(y_i) \) and \( \tau(y_i) = a_i \) for all \( 1 \leq i \leq n \). This can be done in \( \text{uAC}^0 \).

Then it remains to apply Algorithm 18 – which we already have seen to be in \( \text{uTC}^0 \).

For (ii), one needs to compute the isomorphism \( \pi_1(G, T) \to \pi_1(G, a) \). For \( a, b \in V(Y) \) let \( T[a, b] \) denote the unique path from \( a \) to \( b \) in the spanning tree \( T \). We read \( T[a, b] \) as a group element. To compute the isomorphism, every letter \( y \in E(Y) \) has to be replaced by the word \( T[a, \iota(y)] y T[\tau(y), a] \) and every letter \( b^k \) with \( b \in V(Y), k \in \mathbb{Z} \) by \( T[a, b] b^k T[b, a] \). This means we apply a homomorphism of free monoids, what can be done in \( \text{uTC}^0 \) (see [33]). Moreover, this replacement produces a \( G \)-factorization as output. \( \square \)

### 3.1 Computing Britton-reduced words

Before we consider the problem of computing Britton-reduced words, we focus on the analog problem in free groups, the computation of freely reduced words.
Since already in the solution of the word problem free groups of arbitrary rank were appearing, we consider the alphabet \( \Lambda \) as part of the input and assume that it is properly encoded over the binary alphabet \( \{0, 1\} \). In particular, we assume that the involution \( \Lambda \cup \overline{\Lambda} \to \Lambda \cup \overline{\Lambda} \) can be computed in \( \text{uAC}^0 \) – e. g. by a bit-flip.

**Proposition 20.** The following problem is \( \text{uAC}^0 \)-reducible to the word problem of \( F_2 \): given a finite alphabet \( \Lambda \) and a word \( w \in (\Lambda \cup \overline{\Lambda})^* \), compute a freely reduced word \( \hat{w} \in (\Lambda \cup \overline{\Lambda})^* \) with \( \hat{w} =_{F(\Lambda)} w \).

**Proof.** We follow a similar approach as for the solution of the word problem of GBS groups. For \( w = w_1 \cdots w_n \) with \( w_i \in \Lambda \cup \overline{\Lambda} \), we set \( w_{i,j} = w_{i+1} \cdots w_j \). We define an equivalence relation \( \approx_{F(\Lambda)} \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\} \) by

\[
i \approx_{F(\Lambda)} j \text{ if and only if } w_i = w_j \text{ and } \begin{cases} w_{i,j} =_{F(\Lambda)} 1 & \text{if } i < j, \\ w_{j,i} =_{F(\Lambda)} 1 & \text{if } j < i. \end{cases}
\]

By using the embedding of \( F(\Lambda) \) into \( F_2 \), it can be checked in \( \text{uAC}^0(F_2) \) for all pairs \( i, j \) whether \( i \approx_{F(\Lambda)} j \). Furthermore, let us define a partial map

\[
\tau : \{1, \ldots, n\}/_{\approx_{F(\Lambda)}} \to \{1, \ldots, n\}/_{\approx_{F(\Lambda)}}
\]

\[
[i] \mapsto [\overline{i}] = [j] \text{ if there is some } j \text{ with } w_i = \overline{w}_j \text{ and } w_{i,j-1} =_{F(\Lambda)} 1 \text{ (resp. } w_{j,i-1} =_{F(\Lambda)} 1).\]

To see that this map is well defined, we have to verify two points:

(i) that the map \( i \mapsto [\overline{i}] \) is well-defined;

(ii) that \( [i] = [\overline{j}] \) if \( i \approx_{F(\Lambda)} j \).

For the first point, consider \( i < j < k \) with \( w_i = \overline{w}_j \), \( w_{i,j-1} =_{F(\Lambda)} 1 \) and \( w_j = \overline{w}_k \). Then we have \( w_j = w_i \) and \( w_j = w_k \) with \( w_{j,k} =_{F(\Lambda)} (w_{j-1,j} w_j)^{-1} w_{i,k-1} w_k =_{F(\Lambda)} 1 \) – hence, \( j \approx_{F(\Lambda)} k \). Likewise all other orderings of \( i, j, k \) can be dealt with; hence, the image \( [i] \) is uniquely defined for each \( i \).

For the second point, let \( i \approx_{F(\Lambda)} j \) and \( k \in [j] \) with \( i < j < k \). Then we have \( w_k = w_j \) and \( w_{i,k-1} =_{F(\Lambda)} w_i \). Again all other orderings of \( i, j, k \) follow the same way.

Since \( [i] = [\overline{i}] \) for all \( i \), we have a well-defined partial involution \( \tau \). In the following, if \( [\overline{i}] \) is not defined, we consider it to be the empty set.

When looking at the indices in \( [i] \cup [\overline{i}] \) in ascending order, indices from \( [i] \) and indices from \( [\overline{i}] \) always alternate. This is because if \( w_{i,j} = 1 \), then there must be some \( k \in \{i + 1, \ldots, j - 1\} \) such that \( w_k \) cancels with \( w_j \) by free reductions. In particular, \( w_k = \overline{w}_j \) and \( w_{k,j-1} = 1 \). Thus, \( k = [\overline{j}] \).

Therefore, we have \( ||i|| - ||\overline{i}|| \leq 1 \) for all \( i \). Moreover, if two equivalence classes \( [i] \) and \( [\overline{i}] \) have the same number of members, then all corresponding letters \( w_j \) for \( j \in [i] \cup [\overline{i}] \) can be canceled by free reductions. On the other hand, if \( ||i|| - ||\overline{i}|| = 1 \), then after any sequence of free reductions, there remains still one letter \( w_j \) for some \( j \in [i] \cup [\overline{i}] \) which cannot be canceled. This is because a letter \( w_i \) can only cancel with a letter \( w_j \) if \( w_i = \overline{w}_j \) and \( w_{i,j-1} =_{F(\Lambda)} 1 \) (resp. \( w_{j,i-1} =_{F(\Lambda)} 1 \)) – with other words, \( w_i \) can only cancel with letters \( w_j \) for \( j \in [\overline{i}] \).
Thus, for each $i$ with $|i| - |\overline{i}| = 1$, denote by $j_{|i|}$ the maximal index in $[i]$. Now, the freely reduced word $\hat{w}$ consists of exactly those $w_j$ with $j = j_{|i|}$ for some $i$. All other letters are deleted. Apart from the computation of $\approx_{\delta}$ and $\tau$, everything can be done in $uTC^0$ (with the same arguments as steps (ii) (b) and (ii) (c) of Algorithm 18); hence, the whole procedure is in $uAC^0(F_2)$. \hfill \Box

**Corollary 21.** The following problems are $uAC^0$-reducible to the word problem of the free group $F_2$:

(i) given a word $w \in \Delta^*$, decide whether $w$ is a $\mathcal{G}$-factorization and, if so, compute a Britton-reduced $\mathcal{G}$-factorization $\hat{w}$ with $\hat{w} =_{F(\mathcal{G})} w$.

(ii) given a word $w \in \Delta^*$, compute a Britton-reduced $\mathcal{G}$-factorization $\hat{w}$ with $\hat{w} =_{\pi(\mathcal{G}, T)} w$.

Moreover, the number of bits required for $\hat{w}$ is linear in the number of bits of $w$.

**Proof.** As in the proof of Theorem 19 it can be checked in $uAC^0$ whether $w$ is a $\mathcal{G}$-factorization (resp. a $\mathcal{G}$-factorization can be computed from $w$ via the isomorphism $\pi_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, a)$ in $uTC^0$). Thus, we can assume that $w$ is a $\mathcal{G}$-factorization.

We can compute $C(w) \in \Lambda^*_w$ by step (ii) of Algorithm 18 in $uTC^0$ (or more precisely, a proper encoding of $C(w)$ over an alphabet of fixed size). By Proposition 20, a Britton-reduced word $\hat{C}(w) = [i_1] \cdots [i_j] \in \Lambda^*_w$ can be computed in $uAC^0(F_2)$. By Corollary 17, the desired output is $\hat{w} = a_0^{k_0}a_1^{k_1} \cdots a_n^{k_n}$ as before, it can be computed from $[i_1] \cdots [i_j]$ in $uTC^0$. According to (1), the number of bits of $k_{i,j}$ is linear in $j - i + \max \{\log |k_{i,j}| \mid \nu \in \{i, \ldots, j\}\}$. Thus, the number of bits of $\hat{w}$ is linear in the number of bits of $w$. \hfill \Box

### 3.2 Uniform versions of the word problem

It is not obvious what the uniform version of the word problem of GBS groups is (i.e., a version of the word problem where the group is part of the input). Indeed, there are different ways how to define a uniform version of the word problem – and they lead to slightly different complexity bounds. We consider a uniform version of Theorem 19 (i) and a uniform version of Theorem 19 (ii).

In the uniform versions we assume that the graph of groups is given in a proper encoding. For instance we assume that the encoding consists of the numbers $|V(Y)|$ and $|E(Y)|$ and a list of tuples $(y, \iota(y), \tau(y), \alpha_y, \beta_y, \overline{y})$ for the edges. Here $y, \overline{y} \in \{0, \ldots, |E(Y)| - 1\}$ and $\iota(y), \tau(y) \in \{0, \ldots, |V(Y)| - 1\}$ and all numbers (also the $\alpha_y, \beta_y$) are encoded as binary integers using the same number of bits for all $y$. The graph of groups also defines the alphabet $\Delta = E(Y) \cup \{a^k \mid a \in V(Y), k \in \mathbb{Z}\}$. Recall that the integer exponents $k$ are represented in binary using a variable number of bits.

We say an encoding is *valid*, if all tuples are properly formed, for every edge $y$, there is an inverse edge $\overline{y}$ satisfying $\iota(y) = \tau(\overline{y})$ and $\alpha_y = \beta_{\overline{y}}$, and the graph is connected.

**Corollary 22.** The following problem is $uTC^0$-many-one-reducible to the word problem of $F_2$. Input: a valid encoding of a graph of groups $\mathcal{G}$ and a word $w \in \Delta^*$. Decide whether $w$ is a $\mathcal{G}$-factorization and, if so, decide whether $w =_{F(\mathcal{G})} 1$. 

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Note that we need the promise in Corollary 22 that the input is a valid encoding of a graph of groups. Indeed, it cannot be checked whether the graph is connected in $uTC^0$ unless $uTC^0 = \text{LOGSPACE}$ (by [14], already connectivity for forests is LOGSPACE-complete with respect to $\text{NC}^1$-reductions – the reduction is actually a $u\text{AC}^0$ reduction, see also [30]).

On the other hand, by the seminal paper by Reingold [44], connectivity of undirected graphs can be checked in LOGSPACE. Hence, as the other points can be easily verified in $u\text{AC}^0$, it can be checked in LOGSPACE whether an encoding of a graph of groups is valid.

Proof of Corollary 22. We need to verify two things, namely, that for some word $w \in \Delta^*$ it can be checked in $uTC^0$ whether it is a $G$-factorization and, second, that Algorithm 18 is still in $uTC^0$ also if the graph is part of the input.

For the first point, it only needs to be checked whether $w$ is of the form $a_0^{\alpha_0} y_1 a_1^{\beta_1} \cdots y_n a_n^{\alpha_n}$ and, if so, whether $a_0 = a_n$ and $\iota(y_i) = a_i$ and $\tau(y_i) = a_{i+1}$ for all $i$. This can be done in $u\text{AC}^0$.

Algorithm 18 is almost independent of the graph. Indeed, there is only the lookup of the numbers $\alpha_y, \beta_y$, the check whether $y = \overline{x}$ for edges $x, y$, and the choice of the orientation $D$ in order to compute the homomorphism $\rho$. The first two points are straightforward and also the last point is no difficulty: for a pair $y, \overline{y}$, simply choose the one with smaller index in the list coding the graph to be in $D$.

The uniform version of Theorem 19 (ii) is not so immediate. The difficulty lies in the computation of the paths $T[a, b]$. This problem is complete for LOGSPACE under NC$^1$ reductions [14] (and indeed under $u\text{AC}^0$ reductions as remarked above). Thus, together with the computation of the isomorphism $\pi_1(G, T) \rightarrow \pi_1(G, a)$, the algorithm of Theorem 19 (ii) is no longer a $uTC^0$ many-one reduction (or at least it is not known whether it is). Still, we can prove the following result, which together with Corollary 22 yields the proof for the first part of Theorem C.

Corollary 23. The following problem is complete for LOGSPACE under $u\text{AC}^0$ reductions: Given a (valid) encoding of graph of groups $G$ with a spanning tree $T$ (given as list of edges) and a word $w \in \Delta^*$. Decide whether $w = \pi_1(G, T)$.

Note that the question whether $w = \pi_1(G, T)$ depends on the spanning tree $T$. For instance assume that $w = y_1$ consists of a single edge. Then we have $w = \pi_1(G, T)$ if and only if $y_1$ is part of the spanning tree. Therefore, we require $T$ to be part of the input, although by [43] (together with [44]), for a given graph a spanning tree can be computed in LOGSPACE.

Proof. Since apart from the computation of the isomorphism $\pi_1(G, T) \rightarrow \pi_1(G, a)$ (which is in LOGSPACE by [14]), we are in the same situation as in Corollary 22, it remains to prove the hardness part.
We reduce the following special version of Undirected Forest Accessibility (see [14]) to our problem. The problem receives an undirected forest (i.e., an acyclic graph) \( \Gamma \) with precisely two connected components and three vertices \( s, t, u \in V(\Gamma) \) as input such that \( t \) and \( u \) are in two different connected components \( \) – we may assume that the graph is given as list of tuples \( (y, \iota(y), \tau(y), \overline{y}) \) representing edges where each tuple has the same bit-length. The question is whether \( s \) and \( t \) are connected by a path.

In order to obtain an instance for the uniform word problem of GBS groups \((\mathcal{G}, T, w)\), we take the input forest \( \Gamma \) and assign to every edge \( \alpha \) component of \( \Gamma \) in which \( s \) lies. In particular, \( s = \pi_1(\mathcal{G}, T) \) if and only if \( s \) and \( t \) are connected by some path in \( \Gamma \).

Now, \( \pi_1(\mathcal{G}, T) \) is isomorphic to the amalgamated product \((t) \ast_{y_2u_2} \langle u \rangle\) and we have either \( s = \pi_1(\mathcal{G}, T) t \) or \( s = \pi_1(\mathcal{G}, T) u \) – depending on the connected component of \( \Gamma \) in which \( s \) lies.

4 The Conjugacy Problem

Decidability of conjugacy in Baumslag-Solitar groups was established by Anshel and Stebe [5]. In [1] this was generalized to the special case of GBS groups where the graph \( \mathcal{Y} \) consists of only one vertex (i.e., an HNN extension with several stable letters). Later in [26], Horadam showed that the conjugacy problem is decidable in GBS groups if there is some constant \( c \in \mathbb{Z} \) with \( \alpha_y = c \) for all \( y \in E(\mathcal{Y}) \). In [27], this was further generalized to some other class of GBS groups which contains the linear GBS groups (with the generalization that they also considered infinite graphs); in [36], Lockhart gave a solution for all GBS groups. Finally, in [8], Beeker independently gave a solution of the conjugacy problem in all GBS groups.

Before we start with the solution of the conjugacy problem in GBS groups, we recall some general facts about conjugacy in fundamental groups of graphs of groups.

4.1 Conjugacy and Graphs of Groups

Let \( \mathcal{G} \) again be an arbitrary graph of groups with graph \( \mathcal{Y} \) and \( a \in V(\mathcal{Y}) \).

Lemma 24. Let \( g, h \in \pi_1(\mathcal{G}, a) \leq F(\mathcal{G}) \). If \( g \sim_{F(\mathcal{G})} h \), then already \( g \sim_{\pi_1(\mathcal{G}, a)} h \).

Proof. Let \( \varphi : F(\mathcal{G}) \to \pi_1(\mathcal{G}, T) \) be the projection and \( \psi : \pi_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, a) \) be the canonical isomorphism. If \( z \in F(\mathcal{G}) \) is a conjugator, then \( \psi(\varphi(z)) \in \pi_1(\mathcal{G}, a) \) is also a conjugator.

By Lemma 24, instead of testing conjugacy in the fundamental group \( \pi_1(\mathcal{G}, a) \), we can test it in the larger group \( F(\mathcal{G}) \). This simplifies the algorithms substantially because for \( \mathcal{G} \)-factorizations in \( F(\mathcal{G}) \) there is good notion of cyclically Britton-reduced elements.

Let \( w = g_0g_1 \cdots g_n g_0 \in \Pi(\mathcal{G}) \). We say that \( v \) is a cyclic permutation of \( w \) if there are \( u, u' \in \Delta^* \) such that \( w = uu'v \) and \( v = u'u \). A word \( w \in \Delta^* \) is called
cyclically Britton-reduced if every cyclic permutation of \( w \) is Britton-reduced. That means \( w \) is cyclically Britton-reduced if and only if \( uw \) is Britton-reduced or \( w \in G_a \) for some \( a \in V(Y) \). The following lemma provides a tool to compute cyclically Britton-reduced \( G \)-factorizations.

**Lemma 25.** Let \( w = g_0y_1g_1 \cdots y_ng_n \in \Pi(G) \) with \( n \geq 1 \) be Britton-reduced. Then for

\[
y_{[n/2+1]}g_{[n/2+1]} \cdots y_ny_0y_1g_1 \cdots y_{[n/2]}g_{[n/2]} \xrightarrow{B_G} \hat{w},
\]

if \( \hat{w} \) is Britton-reduced, then \( \hat{w} \) is cyclically Britton-reduced and \( w \sim \hat{w} \).

**Proof.** It is clear that \( w \sim \hat{w} \). If \( \hat{w} \) does not contain any \( y \in E(Y) \), we are done. In the other case, we have to show that \( \hat{w}w \) is Britton-reduced. When computing \( \hat{w} \), Britton reductions may only occur in the middle; thus, we know that \( y_{[n/2+1]} \) is still present in \( \hat{w} \). If \( \hat{w}w \) is not Britton-reduced, then the occurrence of \( y_{[n/2+1]} \) in the second factor \( \hat{w} \) must cancel with something in the first factor. This can be either \( y_{[n/2+1]} \) or \( y_{[n/2]} \) depending on whether \( y_{[n/2]} \) has been canceled when computing \( \hat{w} \). However, the first case would mean that \( y_{[n/2+1]} \) is self-inverse; the second case is a contradiction to the assumption that \( w \) was Britton-reduced.

Let \( C \) denote the union of all \( G_a^y \). The following result is due to Horadam [25]; it is the main tool for deciding the conjugacy problem. For amalgamated products, it first appeared in [38]; the special case for HNN extensions is known as Collins’ Lemma [13] – see also [37, Thm. IV.2.5].

**Theorem 26** (Conjugacy Criterion, [25]). Let \( w \in \Pi(G) \) be cyclically Britton-reduced. Then one of the following cases holds:

(i) There is some \( a \in V(Y) \) with \( w \in G_a \) (\( w \) is called elliptic).

(a) If \( w \sim_{F(G)} c \) for some \( c \in C \), then there exists a sequence of elements \( c = c_0c_1 \cdots c_m \in C \) such that \( c_m \sim_{G_a} w \) and for every \( i \) there is some \( b_i \in \Delta \) with \( c_i = \Delta b_i c_{i-1}^{-1} b_i \).

(b) If \( w \) is not conjugate to any \( c \in C \) and \( w \sim_{F(G)} v \) for some cyclically Britton-reduced \( v \), then \( v \in G_a \) and \( v \sim_{G_a} w \).

(ii) We have \( w \not\in G_a \) for any \( a \in V(Y) \) (\( w \) is called hyperbolic), i.e., \( w \) has the form \( w = y_1g_1 \cdots y_ng_n \) with \( n \geq 1 \). If \( w \) is conjugate to a cyclically Britton-reduced \( G \)-factorization \( v = x_1h_1 \cdots x_mh_m \), then \( m = n \) and there are \( i \in \{1, \ldots, n\} \) and \( c \in G_a^y \subseteq C \) such that

\[
v = F(G)c_yg_1 \cdots y_ny_0y_1g_1 \cdots y_{i-1}g_{i-1} c^{-1},
\]

i.e., \( w \) can be transformed into \( v \) by a cyclic permutation followed by a conjugation with an element of \( C \).

### 4.2 Conjugacy in GBS groups

The input for the conjugacy problem are two words \( v, w \in \Delta^* \). As we have seen in the proof of Theorem 19, we may assume that \( v \) and \( w \) are either words
representing group elements of the fundamental groups with respect to some spanning tree $\pi_1(G, T)$ or $G$-factorizations of elements of $\pi_1(G, a)$. In view of Theorem 26, a first step towards the solution of the conjugacy problem is the computation of cyclically Britton-reduced $G$-factorizations. By Corollary 21 we can compute Britton-reduced $G$-factorizations in $uAC^0(F_2)$. Thus, by Lemma 25, also cyclically Britton-reduced $G$-factorizations can be computed in $uAC^0(F_2)$.

Before we start to examine conjugacy, we need a technical lemma:

**Lemma 27.** Let $P$ be some fixed finite set of prime numbers. The following problem is solvable in $uTC^0$: Given $c_i, d_i \in \mathbb{Z}$ (in binary) for $i = 0, \ldots, n$ such that $d_i$ has only prime factors in $P$. Decide whether the system of congruences

$$x \equiv c_i \mod d_i$$

for $i = 0, \ldots, n$ has a solution.

**Proof.** Since $P$ is finite, the following can be done for all $p \in P$ in parallel. Considering only powers of $p$, the system of congruences transforms into

$$x \equiv c_i \mod p^{e_i}$$

where $e_i$ is maximal such that $p^{e_i}$ divides $d_i$. Such $e_i$ can be determined in $uTC^0$ by checking whether $p^{e_i}$ divides $d_i$ for all $0 \leq e \leq \log |d_i|$ in parallel using Theorem 2 for Integer Division. If there is some $i \neq j$ with $e_i \leq e_j$ and $c_i \not\equiv c_j \mod p^{e_i}$, then (2) obviously does not have a solution. Again this can be checked in parallel for all pairs $i, j$. If there is no such pair $i \neq j$, (2) is equivalent to a single congruence $x \equiv c \mod p^e$ where $e = \max_i \{0, \ldots, n\} e_i$ and $c = c_i$ for the respective $i$.

If (2) has a solution for all $p \in P$, then there is a solution for the original congruence by the Chinese Remainder Theorem.

**Proposition 28.** The following problem is in $uTC^0$: Given two cyclically Britton-reduced hyperbolic $G$-factorizations $v, w \in \Pi(G)$ in binary representation, decide whether $v \sim_{F(G)} w$.

**Proof.** By assumption, we are in case (ii) of the Conjugacy Criterion, Theorem 26. Let

$$v = y_1a_1^{k_1} \cdots y_na_n^{k_n}$$

be a $G$-factorization.

By Theorem 26 (ii) we know that if $v$ and $w$ are conjugate, then the underlying path $y_1 \cdots y_n$ of $v$ is a cyclic permutation of the underlying path of $w$. Since in $uTC^0$ all these cyclic permutation can be checked in parallel, we may assume that $w$ is of the form

$$w = y_1d_1^{l_1} \cdots y_n^{l_n}.$$
Like in [26], these equations imply that it is decidable whether \(v\) and \(w\) are conjugate. As we aim for a good complexity bound, we have to take a closer look. By solving these equations for \(x_{i+1}\), we obtain

\[
x_1 = \frac{x}{\alpha_1},
\]

\[
x_{i+1} = \frac{k_i - \ell_i + \beta_i x_i}{\alpha_{i+1}} \quad \text{for } i = 1, \ldots, n - 1,
\]

\[
x = k_n - \ell_n + \beta_n x_n.
\]

By induction follows

\[
x_i = \frac{1}{\alpha_i} \left( x \cdot \prod_{\mu=1}^{i-1} \frac{\beta_{\mu}}{\alpha_{\mu}} + \sum_{\nu=1}^{i-1} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{i-1} \frac{\beta_{\mu}}{\alpha_{\mu}} \right) \quad \text{for } i = 1, \ldots, n,
\]

(3)

and the last equation becomes

\[
x = k_n - \ell_n + x \cdot \prod_{\mu=1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}} + \sum_{\nu=1}^{n-1} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}}.
\]

(4)

We distinguish two cases:

First, assume that (4) has a unique solution. Then, the rational values \(x_i\) are also determined uniquely and we have \(v \sim F(G) w\) if and only if \(x\) and the \(x_i\) are all integers. In this case, we have \(\prod_{\mu=1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}} \neq 1\) and

\[
x = \frac{\sum_{\nu=1}^{n} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}}}{1 - \prod_{\mu=1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}}}.
\]

All occurring numbers are rationals; hence, they can be represented as fractions of binary integers. Since Iterated Multiplication is in \(\text{uTC}^0\) (Theorem 2), the products can be computed. A common denominator for the sums can be computed by Iterated Multiplication, again. Thus, calculating the quotient is just Iterated Addition (Theorem 2). Let \(c, d, e, f \in \mathbb{Z}\) be such that

\[
\frac{c}{d} = \sum_{\nu=1}^{n} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}} \quad \text{and} \quad \frac{e}{f} = 1 - \prod_{\mu=1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}}.
\]

In case \(x = \frac{c}{d}\) is an integer, we can determine this by applying Hesse’s circuit for Integer Division (Theorem 2) to \(cf\) and \(de\). If \(x\) is not an integer, we can notice that by multiplying the result of the division with \(de\); if the result is not \(cf\), there is no \(x\) with \(a^x v a^{-x} = F(G) w\).

If \(x\) is an integer, the numbers \(x_i\) can be computed in \(\text{uTC}^0\) with the same technique, and it can be checked whether \(x_i \in \mathbb{Z}\) for all \(i\). Thus, we are done with the case that (4) has a unique solution.

In the second case, we have \(\prod_{\mu=1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}} = 1\). Then (4) is equivalent to

\[
k_n - \ell_n + \sum_{\nu=1}^{n-1} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{n} \frac{\beta_{\mu}}{\alpha_{\mu}} = 0.
\]

Again, this equality can be checked in \(\text{uTC}^0\) as before. If the equality does not hold, then there is no \(x\) with \(a^x v a^{-x} = F(G) w\). Otherwise, by (3), we have
\[ a^x v a^{-x} = F(g) w \text{ for } x \in \mathbb{Z} \text{ if and only if } \]
\[
x_i = \frac{1}{\alpha_i} \left( x \cdot \prod_{\mu=1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} + \sum_{\nu=1}^{i-1} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} \right) \in \mathbb{Z} \quad \text{for all } i \in \{1, \ldots, n\}.
\]

By solving for \( x \), we obtain
\[
x \in \mathbb{Z} \cap \bigcap_{i=1}^{n} \left( \alpha_i \cdot \prod_{\mu=1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} \right) \cdot \left( -\frac{1}{\alpha_i} \sum_{\nu=1}^{i-1} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} + \mathbb{Z} \right). \tag{5}
\]

Let \( M \in \mathbb{Z} \) be the product of the denominators of all terms in this intersection and \( c_i, d_i \in \mathbb{Z} \) such that
\[
\frac{c_i}{M} = -\left( \prod_{\mu=1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} \right) \cdot \sum_{\nu=1}^{i-1} (k_{\nu} - \ell_{\nu}) \cdot \prod_{\mu=\nu+1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} \quad \text{and}
\]
\[
\frac{d_i}{M} = \alpha_i \cdot \prod_{\mu=1}^{i-1} \beta_{\mu}\frac{\alpha_{\mu}}{\beta_{\mu}} \quad \text{for } i = 1, \ldots, n.
\]

In addition, we set \( c_0 = 0 \) and \( d_0 = M \). Now, \( (5) \) is equivalent to
\[
x \in \frac{1}{M} (c_i + d_i \mathbb{Z}) \quad \text{for } i = 0, \ldots, n.
\]

We substitute \( Mx \) by \( z \). Because of the choice of \( c_0 \) and \( d_0 \), the existence of an integer solution \( x \) is equivalent to the system of congruences
\[
z \equiv c_i \mod d_i \quad \text{for } i = 0, \ldots, n \tag{6}
\]

having a solution. Let \( P \) be the finite set of prime divisors of the \( \alpha_y \) and \( \beta_y \) for \( y \in E(Y) \). As \( M \) as well as the \( d_i \)s are products of the \( \alpha_y \) and \( \beta_y \), they have only prime factors in \( P \). Furthermore, as before, the numbers \( c_i, d_i, M \) and \( M \) can be computed in \( uTC^0 \). By Lemma 27, it can be checked in \( uTC^0 \) whether \( (6) \) has a solution.

Before we examine the conjugacy problem for elliptic elements in GBS group, we consider the special case of Baumslag-Solitar groups \( BS_{p,q} \), where the solution is straightforward.

**Proposition 29.** The following problem is in \( uTC^0 \): Given \( v = a^k \) and \( w = a^\ell \) with \( k, \ell \in \mathbb{Z} \) given in binary, decide whether \( v \sim_{BS_{p,q}} w \).

**Proof.** We are in case (i)a or (i)b of the Conjugacy Criterion, Theorem 26. Since a conjugation with \( a \) has no effect and a conjugation with \( y^{\pm 1} \) multiplies the exponent by \( \frac{q}{p} \) resp. \( \frac{p}{q} \), we have
\[
\alpha^k \sim \alpha^\ell \iff \exists j \in \mathbb{Z} \text{ such that } k \cdot \left( \frac{q}{p} \right)^j = \ell \quad \text{and} \quad \begin{cases} k \in p\mathbb{Z}, \ell \in q\mathbb{Z}, & \text{if } j > 0, \\ k \in q\mathbb{Z}, \ell \in p\mathbb{Z}, & \text{if } j < 0. \end{cases}
\]

Since we have \( |j| \leq \log_{|q/p|} \max \{|k|,|\ell|\} \) if such \( j \) exists, only polynomially many (in the input size) values for \( j \) need to be tested, what can be done in parallel. As Iterated Multiplication and Integer Division are in \( uTC^0 \) ([23, 24], see Theorem 2), we have concluded the proof of Proposition 29. \( \square \)
Thus, for ordinary Baumslag-Solitar groups, we have solved the conjugacy problem completely by combining Corollary 21 with Proposition 28 and Proposition 29.

For arbitrary GBS groups, it remains to examine elliptic elements (cases (i)a and (i)b of Theorem 26). We follow the ideas of Anshel [1, 2, 3] in order to describe a uAC0(F2) solution to the conjugacy problem in this case.

Let \( v = a^k, \quad w = b^\ell \) for some \( a, b \in V(Y), \quad k, \ell \in \mathbb{Z} \). By Theorem 26, we know that \( v \sim_{F(G)} w \) if and only if there is some \( z = a_0 y_1 a_1^k \cdots y_n a_n^{k_n} \in \Pi(G, b, a) \) such that \( za^{k_0} z^{-1} = F(G) w \) and

\[
y_i a_i^{k_i} \cdots y_n a_n^{k_n} a^k \cdot a_n^{-k_n} \cdots a_1^{-k_1} \in G_{y_i} \quad \text{for all } i.
\]

Since a conjugation with \( a_i \) has no effect on elements of \( G_{a_i} = \langle a_i \rangle \), we may assume that \( z = y_1 \cdots y_n \) if \( v \) and \( w \) are conjugate.

Let \( \mathcal{P} = \{ p_1, \ldots, p_m \} \) as before be the set of prime divisors occurring in the \( \alpha_y \) for \( y \in E(Y) \). Here and in what follows, we treat \(-1\) as a prime number. Let

\[
k = r_k \cdot \prod_{i=1}^m p_i^{e_i(k)}, \quad \ell = r_\ell \cdot \prod_{i=1}^m p_i^{e_i(\ell)},
\]

such that \( r_k, r_\ell > 0 \) are not divisible by any \( p \in \mathcal{P} \setminus \{-1\} \). The numbers \( r_k, r_\ell \) and the exponents \( e_i(k), e_i(\ell) \) can be computed in uTC0 as before by checking for all \( p \in \mathcal{P} \) and \( e \leq \log |k| \) (or, more precisely, for all \( e \) at most the number of bits used to represent \( k \)) in parallel whether \( p^e \) divides \( k \) using Hesse’s uTC0 circuit for INTEGER DIVISION, Theorem 2 (for \( p = -1 \), it has to be checked whether \( k > 0 \) – and likewise for \( \ell \). If \( v \sim_{F(G)} w \), then \( r_k = r_\ell \). Hence, all the information it remains to consider is given by the the vectors \( (e_1(k), \ldots, e_m(k)), (e_1(\ell), \ldots, e_m(\ell)) \in \mathbb{N}^m \) and the vertices \( a, b \in V(Y) \). In order to code also the vertices as vectors, we consider vectors in \( \mathbb{N}^m \times \mathbb{N}^V(Y) \) where a vertex \( a \) is encoded by the unit vector \( \vec{u}_a \in \mathbb{N}^V(Y) \) (which has a 1 at position \( a \) and 0 otherwise).

Let us define an equivalence relation on \( \mathbb{N}^m \times \mathbb{N}^V(Y) \) which reflects conjugacy in \( F(G) \). For \( \vec{e} = (e_1, \ldots, e_m, \vec{u}_a), \quad \vec{f} = (f_1, \ldots, f_m, \vec{u}_b) \in \mathbb{N}^m \times \mathbb{N}^V(Y) \) with arbitrary \( (e_1, \ldots, e_m), (f_1, \ldots, f_m) \in \mathbb{N}^m \) and \( a, b \in V(Y) \), we define \( \vec{e} \sim \vec{f} \) if

\[
a \Pi_{1 \leq i \leq m} p_i^{e_i} \sim_{F(G)} b \Pi_{1 \leq i \leq m} p_i^{f_i};
\]

for \( \vec{e} = (e_1, \ldots, e_m, \vec{e}), \quad \vec{f} = (f_1, \ldots, f_m, \vec{f}) \in \mathbb{N}^m \times \mathbb{N}^V(Y) \) with \( \vec{e}^* \) and \( \vec{f}^* \) not being zero nor a unit vector, we define \( \vec{e} \sim \vec{f} \) regardless what the \( e_i, f_i \) are. As an immediate consequence of this definition, we have

**Lemma 30.** Let \( a, b \in V(Y), \quad k, \ell \in \mathbb{Z} \). Then \( a^k \sim_{F(G)} b^\ell \) if and only if \( r_k = r_\ell \) and \( (e_1(k), \ldots, e_m(k), \vec{u}_a) \sim (e_1(\ell), \ldots, e_m(\ell), \vec{u}_b) \).

The numbers \( e_i(k), e_i(\ell) \) of (7) are bounded by a linear function in the input size. In particular, we have a uTC0-many-one reduction from the question whether \( a^k \sim_{F(G)} b^\ell \) to the question whether \( (e_1(k), \ldots, e_m(k), \vec{u}_a) \sim (e_1(\ell), \ldots, e_m(\ell), \vec{u}_b) \) where the numbers \( e_i(k), e_i(\ell) \) are represented in unary. Thus, we aim for a uAC0(F2) circuit to decide whether \( \vec{e} \sim \vec{f} \) for vectors \( \vec{e}, \vec{f} \in \mathbb{N}^m \times \mathbb{N}^V(Y) \). This can be achieved by using the following crucial observation, which is another immediate consequence of the definition of \( \sim \).
Lemma 31. If $\vec{e} \sim \vec{f}$, then also $\vec{e} + \vec{g} \sim \vec{f} + \vec{g}$ for all $g \in \mathbb{N}^m \times \mathbb{N}^{V(Y)}$. In particular, $\sim$ defines a congruence on $\mathbb{N}^m \times \mathbb{N}^{V(Y)}$.

Thus, $(\mathbb{N}^m \times \mathbb{N}^{V(Y)})/\sim$ is a commutative monoid and it remains to solve the word problem of this monoid. Malcev [39] and Emelichev [19] showed that the word problem for finitely generated commutative monoids is decidable – even if the congruence is part of the input.

In [18, Thm. II], Eilenberg and Schützenberger showed that every congruence on $\mathbb{N}^M$ is a semilinear subset of $\mathbb{N}^M \times \mathbb{N}^M$ (this follows also from the results [49], that congruences are definable by Presburger formulas, and [21], that Presburger definable sets are semilinear – for definition of all these notions we refer to the respective papers). In [28, Thm. 1], Ibarra, Jiang, Chang, and Ravikumar showed that membership in a fixed semilinear set can be decided in uniform $\text{NC}^1$. As the word problem of $F_2$ is hard for uniform $\text{NC}^1$ under $u\text{AC}^0$ reductions [45], this means that for every fixed congruence $\sim \subseteq \mathbb{N}^M \times \mathbb{N}^M$, on input of $u, v \in \mathbb{N}^M$, it can be decided in $u\text{AC}^0(F_2)$ whether $u \sim v$.

Thus, by Lemma 30, it can be decided in $u\text{AC}^0(F_2)$ whether $a^k \sim_{F(G)} b^\ell$ for $a, b \in V(Y), k, \ell \in \mathbb{Z}$. Now, we can combine this result with Corollary 21 (calculation of Britton-reduced $G$-factorizations) and Proposition 28 (solution to conjugacy in the hyperbolic case) and we obtain a proof of the main result on conjugacy, Theorem B.

Theorem 32. Let $G$ be a generalized Baumslag-Solitar group. Then the conjugacy problem of $G$ is in $u\text{AC}^0(F_2)$.

4.3 The Uniform Conjugacy Problem

In Section 3.2, we have seen that the uniform version of the word problem for GBS groups was essentially as difficult as the word problem for a fixed GBS group. For conjugacy this picture changes dramatically. Like for the word problem in Section 3.2, the uniform conjugacy problem for GBS groups receives as input a graph of groups $G$ consisting of a finite graph $Y$ and numbers $\alpha_y, \beta_y \in \mathbb{Z}[0]$ for $y \in E(Y)$ and two $G$-factorizations $v, w \in \Delta^*$, where as before $\Delta = E(Y) \cup \{a^k \mid a \in V(Y), k \in \mathbb{Z}\}$. The question is whether $v \sim_{F(G)} w$ (what by Lemma 24 is equivalent to conjugacy in the fundamental group with respect to a base point).

In [4], Anshel and McAloon considered a special (more difficult) variant of the uniform conjugacy problem; they showed that the so-called finite special equality problem for some GBS groups is decidable but not primitive recursive. However, they did not consider the uniform conjugacy problem. By following the ideas for the non-uniform case (which themselves are based on Anshel’s work [1, 2, 3]), we obtain a precise complexity estimate for the uniform conjugacy problem.

Theorem 33. The uniform conjugacy problem for GBS groups is $\text{EXPSPACE}$-complete – even if the numbers $\alpha_y, \beta_y$ are given in unary.

This concludes the proof of Theorem C. The proof of Theorem 33 is an application of the next theorem by Cardoza, Lipton and Meyer [11] resp. Mayr and Meyer [40].

Theorem 34 ([11, 40]). The uniform word problem for finitely presented commutative semigroups is $\text{EXPSPACE}$-complete.
Proof of Theorem 33. For the hardness part, we give a LOGSPACE reduction from the uniform word problem of f. g. commutative semigroups to the uniform conjugacy problem for GBS groups. W.l.o.g. we only consider commutative monoids. Let \( m \in \mathbb{N}, e, f \in \mathbb{N}^m, (r_i, s_i) \in \{1, \ldots, n\} \) with \( r_i, s_i \in \mathbb{N}^m \) be some instance for the uniform word problem of commutative monoids (i.e., the question is whether \( e \sim f \) for the smallest congruence \( \sim \) satisfying \( r_i \sim s_i \) for all \( i \)).

We construct an instance for the uniform conjugacy problem as follows: The graph \( Y \) consists of a single vertex \( a \); for all \( i \in \{1, \ldots, n\} \) there is a pair of edges \( y_i, y_i' \in E(Y) \). Let \( P = \{p_1, \ldots, p_m\} \) be the set of the first \( m \) prime numbers. The numbers \( p_j \) can be computed in LOGSPACE since each of them requires a logarithmic (in \( m \)) number of bits, only (by the prime number theorem there are enough primes). Now, for every relator \( (r_i, s_i) \), we define \( \alpha_{y_i} = \prod_{j=1}^{m} p_j^{(r_{i1})_j} \) and \( \beta_{y_i} = \prod_{j=1}^{m} p_j^{(s_{i1})_j} \), where \( (r_{i1})_j \) denotes the \( j \)th component of the vector \( r_i \), and \( k = \prod_{j=1}^{m} p_j^{e_j} \) and \( \ell = \prod_{j=1}^{m} p_j^{f_j} \). According to the proof in [40], we may assume that all the vectors \( e, f, r_i \) and \( s_i \) (for all \( i \)) have at most four non-zero entries and these non-zero entries are at most 2. Thus, the results \( k, \ell, \alpha_{y_i}, \) and \( \beta_{y_i} \) are bounded polynomially in the input length and they can be written down in unary on the output tape. In particular, the products can be computed in LOGSPACE. Now we have \( a^k \sim_{F(G)} a^\ell \) if and only if \( e \sim f \).

It remains to show that the uniform conjugacy problem is in EXPSPACE. The two input words for an instance of the uniform conjugacy problem for GBS groups can be cyclically Britton-reduced as in Corollary 21. Note, however, that the linear bound on the size of the cyclically Britton-reduced words does not hold anymore. Still the size remains bounded polynomially.

The algorithm of Proposition 28 can be executed in polynomial time even if the graph of groups is part of the input. This gives a polynomial time bound for hyperbolic elements. However, we do not know a better bound as the proof of Proposition 28 involves a computation of greatest common divisors (or prime factorizations) of the numbers \( \alpha_{y_i}, \beta_{y_i} \). For elliptic elements, by Lemma 30, we obtain an instance of the uniform word problem of commutative semigroups, which is in EXPSPACE. □

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References

[1] M. Anshel. The conjugacy problem for HNN groups and the word problem for commutative semigroups. Proc. Amer. Math. Soc., 61(2):223–224, 1976.

[2] M. Anshel. Conjugate powers in HNN groups. Proc. Amer. Math. Soc., 54:19–23, 1976.
[3] M. Anshel. Decision problems for HNN groups and vector addition systems. *Math. Comput.*, 30(133):154–156, 1976.

[4] M. Anshel and K. McAloon. Reducibilities among decision problems for HNN groups, vector addition systems and subsystems of Peano arithmetic. *Proc. Amer. Math. Soc.*, 89(3):425–429, 1983.

[5] M. Anshel and P. Stebe. The solvability of the conjugacy problem for certain HNN groups. *Bull. Amer. Math. Soc.*, 80:266–270, 1974.

[6] D. A. M. Barrington, N. Immerman, and H. Straubing. On uniformity within NC$^1$. *J. Comput. Syst. Sci.*, 41(3):274–306, 1990.

[7] G. Baumslag and D. Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68:199–201, 1962.

[8] B. Beeker. *Problèmes géométriques et algorithmiques dans des graphes de groupes*. PhD thesis, Université de Caen Basse-Normandie, 2011.

[9] R. Book and F. Otto. *String-Rewriting Systems*. Springer-Verlag, 1993.

[10] J. L. Britton. The word problem. *Ann. of Math.*, 77:16–32, 1963.

[11] E. W. Cardoza, R. J. Lipton, and A. R. Meyer. Exponential space complete problems for Petri nets and commutative semigroups: preliminary report. In *Eighth Annual ACM Symposium on Theory of Computing (Hershey, Pa., 1976)*, pages 50–54. Assoc. Comput. Mach., New York, 1976.

[12] H. Caussinus, P. McKenzie, D. Thérien, and H. Vollmer. Nondeterministic NC$^3$ computation. *J. Comput. Syst. Sci.*, 57(2):200–212, 1998.

[13] D. J. Collins. On embedding groups and the conjugacy problem. *J. London Math. Soc. (2)*, 1:674–682, 1969.

[14] S. A. Cook and P. McKenzie. Problems complete for deterministic logarithmic space. *J. Algorithms*, 8(3):385–394, 1987.

[15] M. J. Craven and H. C. Jimbo. Evolutionary algorithm solution of the multiple conjugacy search problem in groups, and its applications to cryptography. *Groups Complexity Cryptology*, 4:135–165, 2012.

[16] V. Diekert, A. G. Myasnikov, and A. Weiß. Conjugacy in Baumslag’s Group, Generic Case Complexity, and Division in Power Circuits. In A. Pardo and A. Viola, editors, *Latin American Theoretical Informatics Symposium*, volume 8392 of *LNCS*, pages 1–12. Springer, 2014.

[17] V. Diekert and A. Weiß. Context-Free Groups and Bass-Serre Theory. *ArXiv e-prints*, 2013.

[18] S. Eilenberg and M. P. Schützenberger. Rational sets in commutative monoids. *Journal of Algebra*, 13:173–191, 1969.

[19] V. A. Emelichev. Commutative semigroups with one defining relation. *Shuya Gosudarstvennyi Pedagogicheskii Institut Uchenye Zapiski*, 6:227–242, 1958.
[20] M. Forester. On uniqueness of JSJ decompositions of finitely generated groups. *Comment. Math. Helv.*, 78(4):740–751, 2003.

[21] S. Ginsburg and E. H. Spanier. Semigroups, Presburger formulas and languages. *Pacific Journal of Mathematics*, 16:285–296, 1966.

[22] D. Grigoriev and V. Shpilrain. Authentication from matrix conjugation. *Groups Complexity Cryptology*, 1:199–205, 2009.

[23] W. Hesse. Division is in uniform $TC^0$. In F. Orejas, P. G. Spirakis, and J. van Leeuwen, editors, *ICALP*, volume 2076 of *Lecture Notes in Computer Science*, pages 104–114. Springer, 2001.

[24] W. Hesse, E. Allender, and D. A. M. Barrington. Uniform constant-depth threshold circuits for division and iterated multiplication. *Journal of Computer and System Sciences*, 65:695–716, 2002.

[25] K. J. Horadam. The word problem and related results for graph product groups. *Proc. American Mathematical Society*, 82:407–408, 1981.

[26] K. J. Horadam. The conjugacy problem for graph products with central cyclic edge groups. *Proc. Amer. Math. Soc.*, 91(3):345–350, 1984.

[27] K. J. Horadam and G. E. Farr. The conjugacy problem for HNN extensions with infinite cyclic associated groups. *Proc. Amer. Math. Soc.*, 120(4):1009–1015, 1994.

[28] O. H. Ibarra, T. Jiang, J. H. Chang, and B. Ravikumar. Some classes of languages in $NC^1$. *Inform. and Comput.*, 90(1):86–106, 1991.

[29] M. Jantzen. *Confluent String Rewriting*, volume 14 of EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1988.

[30] B. Jenner, K.-J. Lange, and P. McKenzie. Tree isomorphism and some other complete problems for deterministic logspace. publication #1059, DIRO, Université de Montréal, 1997.

[31] J. Kausch. Private conversation, 2013.

[32] P. H. Kropholler. Baumslag-Solitar groups and some other groups of cohomological dimension two. *Comment. Math. Helv.*, 65(4):547–558, 1990.

[33] K. Lange and P. McKenzie. On the complexity of free monoid morphisms. In K. Chwa and O. H. Ibarra, editors, *Algorithms and Computation, 9th International Symposium, ISAAC ’98, Proceedings*, volume 1533 of *Lecture Notes in Computer Science*, pages 247–256. Springer, 1998.

[34] J. Laun. *Solving algorithmic problems in Baumslag-Solitar groups and their extensions using data compression*. Dissertation, Institut für Formale Methoden der Informatik, Universität Stuttgart, 2012.

[35] R. J. Lipton and Y. Zalcstein. Word problems solvable in logspace. *J. ACM*, 24:522–526, 1977.

[36] J. M. Lockhart. The conjugacy problem for graph products with infinite cyclic edge groups. *Proc. Amer. Math. Soc.*, 114(3):603–606, 1992.
[37] R. Lyndon and P. Schupp. *Combinatorial Group Theory*. Classics in Mathematics. Springer, 2001. First edition 1977.

[38] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory*. Interscience Publishers (New York), 1966. Reprint of the 2nd edition (1976): 2004.

[39] A. I. Malcev. On homomorphisms of finite groups. *Ivano Gosudarstvennyi Pedagogicheskii Institut Uchenye Zapiski*, 18:49–60, 1958.

[40] E. W. Mayr and A. R. Meyer. The complexity of the word problems for commutative semigroups and polynomial ideals. *Advances in Math.*, 46:305–329, 1982.

[41] C. F. Miller III. *On group-theoretic decision problems and their classification*, vol. 68 of *Annals of Mathematics Studies*. Princeton University Press, 1971.

[42] A. G. Myasnikov, A. Ushakov, and D. W. Won. The Word Problem in the Baumslag group with a non-elementary Dehn function is polynomial time decidable. *Journal of Algebra*, 345:324–342, 2011.

[43] N. Nisan and A. Ta-Shma. Symmetric logspace is closed under complement. In F. T. Leighton and A. Borodin, editors, *Proceedings of the Twenty-Seventh Annual ACM Symposium on Theory of Computing*, 29 May-1 June 1995, Las Vegas, Nevada, USA, pages 140–146. ACM, 1995.

[44] O. Reingold. Undirected connectivity in log-space. *J. ACM*, 55(4), 2008.

[45] D. Robinson. *Parallel Algorithms for Group Word Problems*. PhD thesis, University of California, San Diego, 1993.

[46] J.-P. Serre. *Arbres, amalgames, SL_{2}*. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[47] V. Shpilrain and G. Zapata. Combinatorial group theory and public key cryptography. *Applicable Algebra in Engineering, Communication and Computing*, 17:291–302, 2006.

[48] H.-U. Simon. Word problems for groups and contextfree recognition. In *Proceedings of Fundamentals of Computation Theory (FCT’79), Berlin/Wendisch-Rietz (GDR)*, pages 417–422. Akademie-Verlag, 1979.

[49] M. A. Ta˘ ıclin. Algorithmic problems for commutative semigroups. *Dokl. Akad. Nauk SSSR*, 178:786–789, 1968.

[50] H. Vollmer. *Introduction to Circuit Complexity*. Springer, Berlin, 1999.

[51] S. Waack. Tape complexity of word problems. In F. Gécseg, editor, *Proceedings of Fundamentals of Computation Theory (FCT’81)*, volume 117 of *Lecture Notes in Computer Science*, pages 467–471. Springer, 1981.

[52] A. Weiß. *On the Complexity of Conjugacy in Amalgamated Products and HNN Extensions*. Dissertation, Institut für Formale Methoden der Informatik, Universität Stuttgart, 2015.