Cohomology of the Schrödinger-Virasoro conformal algebra

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Abstract: Both the basic cohomology groups and the reduced cohomology groups of the Schrödinger-Virasoro conformal algebra with trivial coefficients are completely determined.

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1 Introduction

Lie conformal algebra, which was introduced by Kac in [6, 7], gives an axiomatic description of the operator product expansion (or rather its Fourier transform) of chiral fields in conformal field theory (see [2]). It has been shown that the theory of Lie conformal algebras has close connections to vertex algebras, infinite-dimensional Lie algebras satisfying the locality property in [8] such as affine Kac-Moody algebras, the Virasoro algebra, and Hamiltonian formalism in the theory of nonlinear evolution equations (see [1, 5, 7, 10, 11, 14, 15]). The general structure theory, representation theory of Lie conformal algebras were systematically developed in [4, 5].

In this paper, we focus on the cohomology theory of Lie conformal algebras. The general cohomology theory of conformal algebras with coefficients in an arbitrary conformal module was developed in [3], where explicit computations of cohomology groups for the Virasoro conformal algebra and current conformal algebra were given. Some low dimensional cohomology groups of the general Lie conformal algebras $gc_N$ were studied in [9]. All cohomology groups of the Heisenberg-Virasoro conformal algebra with trivial coefficients were determined in [13]. It was also shown in [3] that the basic cohomology groups of Lie conformal algebra are naturally isomorphic to those of its annihilation Lie algebra. However, the arbitrary dimensional cohomology groups of a Lie algebra, especially of a Lie algebra with infinite dimension, are very difficult to compute. So the the cohomology theory of Lie conformal algebra actually gives an efficient method to determine the cohomology groups of certain Lie algebras. In this paper, we will compute the cohomology of the Schrödinger-Virasoro conformal algebra (see Definition 2.2), which was first introduced in [11] and has a close relationship with the well-known Schrödinger-Virasoro Lie algebra. Our methods may be useful to determine higher cohomology groups of some infinite dimensional Lie algebras. This is the main motivation to present our work.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions, notations, and related known results about Lie conformal algebras. In Section 3, we determine the basic cohomology groups of the Schrödinger-Virasoro conformal algebra with coefficients in its trivial module $C_a$. In Section 4, we compute the reduced cohomology groups of the Schrödinger-Virasoro conformal algebra with coefficients in its module $C_a$. As a byproduct, we also compute the reduced cohomology groups with coefficients in $M_{a,\beta}$ in case $\beta \neq 0$. Our main results are summarized in Theorems 3.11, 4.1, 4.4, 4.5.

Throughout this paper, we use notations $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ to represent the set of complex numbers, integers and nonnegative integers, respectively. In addition, all vector spaces and tensor products are over $\mathbb{C}$. In the absence of ambiguity, we abbreviate $\otimes_\mathbb{C}$ to $\otimes$.

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2 Preliminaries

In this section, we recall some basic definitions, notations and related results about Lie conformal algebras for later use. For a detailed description, one can refer to [3, 13].

2.1 Lie conformal algebra

Definition 2.1 ([5]) A Lie conformal algebra $\mathcal{A}$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-bilinear map $\mathcal{A} \otimes \mathcal{A} \to \mathbb{C}[\lambda] \otimes \mathcal{A}$, $a \otimes b \mapsto [a, b]$ subject to the following relations $(a, b, c \in \mathcal{A})$:

\[
\begin{align*}
[\partial a, b] &= -\lambda [a, b], \\
[a, \partial b] &= (\partial + \lambda)[a, b], \\
[a, b] &= -[b, -\lambda \partial a], \\
[a, b, c] &= [[a, b]_{\lambda+\mu} + [b, a]_\mu].
\end{align*}
\]

The Lie conformal algebra of our study is the so-called Schrödinger-Virasoro conformal algebra introduced in [11]. And its definition is given in the following.

Definition 2.2 The Schrödinger-Virasoro conformal algebra is a finite Lie conformal algebra $SV = \mathbb{C}[\partial] L \oplus \mathbb{C}[\partial] M \oplus \mathbb{C}[\partial] Y$, endowed with the following nontrivial $\lambda$-brackets

\[
\begin{align*}
[L_L L] &= (\partial + 2\lambda)L, \\
[Y_L Y] &= (\partial + 3\lambda)Y, \\
[L_L Y] &= (\partial + 2\lambda)M, \\
[L_L M] &= (\partial + \lambda)M, \\
[M_L L] &= \lambda M.
\end{align*}
\]

The Schrödinger-Virasoro conformal algebra $SV$ contains the Virasoro conformal algebra $Vir = \mathbb{C}[\partial] L$, whose representation theory and cohomology theory were investigated in [4] and [3], respectively. Also, $SV$ contains the Heisenberg-Virasoro conformal algebra $HV = \mathbb{C}[\partial] L \oplus \mathbb{C}[\partial] M$, whose representation theory and cohomology theory were investigated in [12] and [13], respectively.

2.2 Conformal module

Definition 2.3 ([4]) A conformal module $V$ over a Lie conformal algebra $\mathcal{A}$ is a $\mathbb{C}[\partial]$-module equipped with a $\mathbb{C}$-bilinear map $\mathcal{A} \otimes V \to V[\lambda]$, $a \otimes v \mapsto a \lambda v$, satisfying the following relations for any $a, b \in \mathcal{A}$, $v \in V$,

\[
\begin{align*}
&a \lambda (b \mu v) - b \mu (a \lambda v) = [a, b]_{\lambda+\mu} v, \\
&(\partial a) \lambda v - b a \lambda v = -\lambda a \lambda v, \\
&a \lambda (\partial v) = (\partial + \lambda) a \lambda v.
\end{align*}
\]

Example 2.4 Let $\mathcal{A}$ be an arbitrary Lie conformal algebra and $a \in \mathbb{C}$. Then $\mathcal{A}$ admits a family of 1-dimensional modules $\mathbb{C}_a$ defined by

\[\mathbb{C}_a = \mathbb{C}, \quad \partial v = a v, \quad A_a v = 0, \quad \forall v \in \mathbb{C}_a.\]

And we abbreviate $\mathbb{C}_0$ to $\mathbb{C}$ in the sequel. It is easy to check that the modules $\mathbb{C}_a$ with $a \in \mathbb{C}$ exhaust all trivial irreducible $\mathcal{A}$-modules.

The classification of all finite irreducible nontrivial $SV$-modules was obtained in [12].

Proposition 2.5 All free nontrivial $SV$-modules of rank one over $\mathbb{C}[\partial]$ are as follows $(\alpha, \beta \in \mathbb{C})$:

\[M_{\alpha, \beta} = \mathbb{C}[\partial] v, \quad L_{\lambda} v = (\partial + \alpha \lambda + \beta) v, \quad Y_{\lambda} v = M_{\lambda} v = 0.\]

Moreover, the module $M_{\alpha, \beta}$ is irreducible if and only if $\alpha$ is non-zero. The module $M_{0,\beta}$ contains a unique nontrivial submodule $(\partial + \beta) M_{0, \beta}$ isomorphic to $M_{1, \beta}$. The modules $M_{\alpha, \beta}$ with $\alpha \neq 0$ exhaust all finite irreducible nontrivial $SV$-modules.
2.3 Basic cohomology

Definition 2.6 ([3]) An n-cochain \((n \in \mathbb{Z}_+)\) of a Lie conformal algebra \(A\) with coefficients in an \(A\)-module \(V\) is a \(\mathbb{C}\)-linear map

\[
\gamma : A^\otimes \rightarrow V[\lambda_1, \cdots, \lambda_n], \quad a_1 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_1, \cdots, \lambda_n}(a_1, \cdots, a_n)
\]
satisfying the following conditions:

1. \(\gamma_{\lambda_1, \cdots, \lambda_n}(a_1, \cdots, \partial a_1, \cdots, a_n) = -\lambda_1 \gamma_{\lambda_1, \cdots, \lambda_n}(a_1, \cdots, a_n)\) (conformal antilinearity),
2. \(\gamma\) is skew-symmetric with respect to simultaneous permutations of \(a_i\)'s and \(\lambda_i\)'s (skew-symmetry).

Denote by \(\tilde{C}^n(A, V)\) the set of all n-cochains. The differential \(d_n\) of an n-cochain \(\gamma\) is defined as follows:

\[
(d_n \gamma)_{\lambda_1, \cdots, \lambda_{n+1}}(a_1, \cdots, a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} a_{i1} \gamma_{\lambda_1, \cdots, \lambda_i, \cdots, \lambda_{n+1}}(a_1, \cdots, \hat{a}_i, \cdots, a_{n+1})
\]

\[
+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \gamma_{\lambda_1, \cdots, \lambda_i, \cdots, \lambda_j, \cdots, \lambda_{n+1}}([a_{i1} a_{j1}], a_1, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots, a_{n+1}), \tag{2.4}
\]

where \(\gamma\) is linearly extended over the polynomials in \(\lambda_i\).

It was shown in [3] that the operator \(d\) preserves the space of cochains and \(d^2 = 0\). Thus the cochains of a Lie conformal algebra \(A\) with coefficients in an \(A\)-module \(V\) form a complex, called the basic complex:

\[
\cdots \rightarrow \tilde{C}^{n-1}(A, V) \xrightarrow{d_{n-1}} \tilde{C}^n(A, V) \xrightarrow{d_n} \tilde{C}^{n+1}(A, V) \rightarrow \cdots \tag{2.5}
\]

The related cohomology is called the basic cohomology of the Lie conformal algebra \(A\) with coefficients in its module \(V\) and denoted by \(\tilde{H}^q(A, V), q \in \mathbb{Z}_+,\) more details see the following definition.

Definition 2.7 An element \(\gamma\) in \(\tilde{C}^q(A, V)\) is called a q-cocycle if \(d(\gamma) = 0\); a q-coboundary if there exists a \((q-1)\)-cochain \(\phi \in \tilde{C}^{q-1}(A, V)\) such that \(\gamma = d(\phi)\). Two cochains \(\gamma_1\) and \(\gamma_2\) are called equivalent if \(\gamma_1 - \gamma_2\) is a coboundary.

Denote by \(\tilde{D}^q(A, V)\) and \(\tilde{B}^q(A, V)\) the spaces of q-cocycles and q-boundaries, respectively. Then, we can obtain that

\[
\tilde{H}^q(A, V) = \tilde{D}^q(A, V)/\tilde{B}^q(A, V) = \{\text{equivalent classes of q-cocycles}\}.\]

2.4 Reduced cohomology

Moreover, one can define a (left) \(\mathbb{C}[\partial]\)-module structure on \(\tilde{C}^n(A, V)\) by

\[
(\partial \gamma)_{\lambda_1, \cdots, \lambda_n}(a_1, \cdots, a_n) = (\partial_V + \sum_{i=1}^n \lambda_i) \gamma_{\lambda_1, \cdots, \lambda_n}(a_1, \cdots, a_n),
\]

where \(\partial_V\) denotes the action of \(\partial\) on \(V\). Then \(d \partial = \partial d\) and

\[
\cdots \rightarrow \partial\tilde{C}^{n-1}(A, V) \xrightarrow{d_{n-1}} \partial\tilde{C}^n(A, V) \xrightarrow{d_n} \partial\tilde{C}^{n+1}(A, V) \rightarrow \cdots \tag{2.6}
\]
forms a subcomplex of the basic complex. The quotient complex

\[ \cdots \rightarrow \frac{\mathcal{C}^{n-1}(A,V)}{\partial \mathcal{C}^{n-1}(A,V)} \xrightarrow{d_{n-1}} \frac{\mathcal{C}^n(A,V)}{\partial \mathcal{C}^n(A,V)} \xrightarrow{d_n} \frac{\mathcal{C}^{n+1}(A,V)}{\partial \mathcal{C}^{n+1}(A,V)} \rightarrow \cdots \]  

(2.7)
is called the reduced complex. And its cohomology is called the reduced cohomology of the Lie conformal algebra $A$ with coefficients in $V$ and denoted by $H^q(A,V)$, $q \in \mathbb{Z}_+$. 

**Remark 2.8** The basic cohomology $\tilde{H}^q(A,V)$ is naturally a $\mathbb{C}[\partial]$-module, whereas the reduced cohomology $H^q(A,V)$ is a complex vector space.

The exact sequence $0 \rightarrow \partial \tilde{C}^• \xrightarrow{i} \tilde{C}^• \xrightarrow{p} C^• \rightarrow 0$ gives a long exact sequence of the cohomology groups:

\[ 0 \rightarrow H^0(\partial \tilde{C}^•) \xrightarrow{i_0} \tilde{H}^0(A,V) \xrightarrow{p_0} H^0(A,V) \rightarrow \] 
\[ H^1(\partial \tilde{C}^•) \xrightarrow{i_1} \tilde{H}^1(A,V) \xrightarrow{p_1} H^1(A,V) \rightarrow \] 
\[ H^2(\partial \tilde{C}^•) \xrightarrow{i_2} \tilde{H}^2(A,V) \xrightarrow{p_2} H^2(A,V) \rightarrow \cdots . \]  

(2.8)

**Proposition 2.9** ([3]) In degrees $\geq 1$, the complexes $\tilde{C}^•$ and $\partial \tilde{C}^•$ are isomorphic under the map

\[ \tilde{C}^• \rightarrow \partial \tilde{C}^•, \quad \gamma \mapsto \partial \cdot \gamma. \]  

(2.9)

Therefore, $H^q(\partial \tilde{C}^•) \cong \tilde{H}^q(A,V)$ for $q \geq 1$.

**Remark 2.10** In the long exact sequence (2.8), the maps $H^q(\partial \tilde{C}^•) \rightarrow \tilde{H}^q(A,V)$ induced by the embedding $\partial \tilde{C}^• \subset \tilde{C}^•$ are not isomorphisms.

## 3 Basic cohomology of $SV$ with trivial coefficients

In this section, we will compute the basic cohomology groups of $SV$ with coefficients in its trivial module $\mathbb{C}_a$. Since $\tilde{H}^q(SV, \mathbb{C}_a) \cong \tilde{H}^q(SV, \mathbb{C})$ for any $a \in \mathbb{C}$ ([3]), we only need to compute $\tilde{H}^q(SV, \mathbb{C})$. In this case, by (2.4), the differential $d_n$ of a $n$-cochain $\gamma$ is given as follows:

\[ (d_n \gamma)_{\lambda_1, \ldots, \lambda_{n+1}}(a_1, \ldots, a_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \gamma_{\lambda_i, \lambda_j, \lambda_1, \ldots, \lambda_i, \lambda_j, \lambda_{n+1}} ([a_i, a_j], a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots, a_{n+1}). \]

**Lemma 3.1** $\tilde{H}^0(SV, \mathbb{C}) = H^0(SV, \mathbb{C}) = \mathbb{C}$.

**Proof.** For any $\gamma \in \tilde{C}^0(SV, \mathbb{C}) = \mathbb{C}$, $(d_0 \gamma)_{\lambda}(a) = a_\lambda \gamma = 0$ for $a \in SV$. This means $\tilde{D}^0(SV, \mathbb{C}) = \mathbb{C}$ and $\tilde{B}^0(SV, \mathbb{C}) = 0$. Thus $\tilde{H}^0(SV, \mathbb{C}) = \mathbb{C}$. Moreover, $H^0(SV, \mathbb{C}) = \mathbb{C}$ since $\partial \mathbb{C} = 0$. \hfill $\square$

Let $\gamma \in \tilde{C}^q(H^\nu, \mathbb{C})$ with $q > 0$. By Definition 2.6, $\gamma$ is determined by its value on $a_1 \otimes \cdots \otimes a_q$ with $a_i \in \{L, Y, M\}$. Since $\gamma$ is skew-symmetric, we can always assume that the first $k$ variables are $L$, the middle $l$ variables are $Y$ and the last $m$ variables are $M$ in $\gamma_{\lambda_1, \ldots, \lambda_q}(a_1, \ldots, a_q)$. Thus we can regard $\gamma_{\lambda_1, \ldots, \lambda_q}(a_1, \ldots, a_q)$ as a polynomial in $\lambda_1, \ldots, \lambda_q$, which is skew-symmetric.
in $\lambda_1, \cdots, \lambda_k$, in $\lambda_{k+1}, \cdots, \lambda_{k+l}$, and in $\lambda_{k+l+1}, \cdots, \lambda_{k+l+m}$, respectively, where $q = k + l + m$. Therefore, $\gamma_{\lambda_1, \cdots, \lambda_q}(a_1, \cdots, a_q)$ is divisible by

$$\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \prod_{1 \leq i < j \leq l} (\lambda_{k+i} - \lambda_{k+j}) \prod_{1 \leq i < j \leq m} (\lambda_{k+l+i} - \lambda_{k+l+j}),$$

whose degree is $\binom{k}{2} + \binom{l}{2} + \binom{m}{2}$.

Following [3], we define an operator $\tau : \tilde{C}^q(\mathcal{H}_V, \mathbb{C}) \to \tilde{C}^{q-1}(\mathcal{H}_V, \mathbb{C})$ by

$$(\tau \gamma)_{\lambda_1, \cdots, \lambda_{q-1}, a_1, \cdots, a_{q-1}} = (-1)^{q-1} \frac{\partial}{\partial \lambda} \gamma_{\lambda_1, \cdots, \lambda_{q-1}, \lambda}(a_1, \cdots, a_{q-1}, L)|_{\lambda = 0}.$$  

(3.1)

By direct computations (referring to [13]), we have

$$((d \tau + \tau d)\gamma)_{\lambda_1, \cdots, \lambda_q}(a_1, \cdots, a_q)$$

$$= (-1)^q \frac{\partial}{\partial \lambda} \sum_{i=1}^q (-1)^{i+q+1} \gamma_{\lambda_1, \cdots, \lambda_i, \lambda_1, \cdots, \lambda_q}(a_{i+1}, \cdots, a_{i+q-1}, a_i, \cdots, a_q)|_{\lambda = 0}$$

$$= \frac{\partial}{\partial \lambda} \sum_{i=1}^q \gamma_{\lambda_1, \cdots, \lambda_i, \lambda_1, \cdots, \lambda_q}(a_{i+1}, \cdots, a_{i+1}, a_i, \cdots, a_q)|_{\lambda = 0}$$

$$= \frac{\partial}{\partial \lambda} \sum_{i=1}^q (\lambda_i - \lambda) \gamma_{\lambda_1, \cdots, \lambda_i, \lambda_1, \cdots, \lambda_q}(a_{i+1}, \cdots, a_{i+1}, a_i, \cdots, a_q)|_{\lambda = 0}$$

$$+ \frac{\partial}{\partial \lambda} \sum_{i=k+1}^{k+l} (\lambda_i - \frac{1}{2}) \gamma_{\lambda_1, \cdots, \lambda_i, \lambda_1, \cdots, \lambda_q}(a_{i+1}, \cdots, a_{i+1}, a_i, \cdots, a_q)|_{\lambda = 0}$$

$$+ \frac{\partial}{\partial \lambda} \sum_{i=k+l+1}^{k+l+m} \lambda_i \gamma_{\lambda_1, \cdots, \lambda_i, \lambda_1, \cdots, \lambda_q}(a_{i+1}, \cdots, a_{i+1}, a_i, \cdots, a_q)|_{\lambda = 0}$$

$$= (\text{deg} \gamma - k - \frac{l}{2}) \gamma,$$  

(3.2)

where $\text{deg} \gamma$ is the total degree of $\gamma$ in $\lambda_1, \cdots, \lambda_q$. Therefore, if a $q$-cycyle $\gamma$ satisfies $\text{deg} \gamma \neq k + \frac{l}{2}$, it must be a coboundary. Only those homogeneous cochains whose degree as a polynomial is equal to $k + \frac{l}{2}$ contribute to the cohomology of $\tilde{C}^*(\mathcal{S}V, \mathbb{C})$.

Consider the quadratic inequality

$$\binom{k}{2} + \binom{l}{2} + \binom{m}{2} \leq k + \frac{l}{2}.$$  

(3.3)

**Lemma 3.2** All non-negative integral solutions satisfying $k + \frac{l}{2} \in \mathbb{Z}_+$ of the inequality (3.3) are listed in the following table:

| $q = k + l + m$ | $(k, l, m)$ | $\binom{k}{2} + \binom{l}{2} + \binom{m}{2}$ | $\text{deg} \gamma = k + \frac{l}{2}$ |
|-----------------|-------------|------------------------------------------|----------------------------------|
| 0               | $(0, 0, 0)$ | 0                                        | 0                                |
| 1               | $(0, 0, 1)$ | 0                                        | 0                                |
|                 | $(1, 0, 0)$ | 0                                        | 1                                |
Proof. We should consider \((k, l, m) = (1, 0, 0)\) and \((0, 0, 1)\). Let \(\gamma\) be a 1-cocycle. Assume \(\gamma_\lambda(L) = a \lambda, \gamma_\lambda(Y) = 0\) and \(\gamma_\lambda(M) = b\). Then \(d\gamma_{\lambda,\lambda_2}(L, L) = -a(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) = 0\) and \(d\gamma_{\lambda_1,\lambda_2}(L, M) = b\lambda_2 = 0\), implying \(\gamma = 0\).

\(\Box\)

Lemma 3.5 \(\tilde{H}^2(SV, \mathbb{C}) = 0\).

Proof. We only need to consider \((k, l, m) = (0, 2, 0), (1, 0, 1)\) and \((2, 0, 0)\). Let \(\gamma\) be a 2-cocycle. Assume \(\gamma_{\lambda_1,\lambda_2}(L, L) = a(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2), \gamma_{\lambda_1,\lambda_2}(L, M) = b\lambda_1 + c\lambda_2\) and \(\gamma_{\lambda_1,\lambda_2}(Y, Y) = e(\lambda_1 - \lambda_2)\) for some \(a, b, c, e \in \mathbb{C}\). Let \(\phi\) be a 1-cocycle defined by \(\phi_\lambda(L) = a \lambda, \phi_\lambda(Y) = 0, \phi_\lambda(M) = e\). Replacing \(\gamma\) by \(\gamma + d\phi\), we can assume that \(\gamma_{\lambda_1,\lambda_2}(L, L) = 0\) and \(\gamma_{\lambda_1,\lambda_2}(Y, Y) = 0\). Then by

\[d\gamma_{\lambda_1,\lambda_2,\lambda_3}(L, Y, Y) = -\gamma_{\lambda_2+\lambda_3,\lambda_1}(\partial_2 2\lambda_2) = (\lambda_2 - \lambda_3)(b\lambda_1 + c\lambda_2 + c\lambda_3) = 0,\]

we have \(\gamma = 0\).

\(\Box\)

Lemma 3.6 \(\tilde{H}^3(SV, \mathbb{C}) = \mathbb{C}\Phi\), where \(\Phi\) is defined by

\[\Phi_{\lambda_1,\lambda_2,\lambda_3}(L, L, L) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3).\]

In particular, \(\text{dim } \tilde{H}^3(SV, \mathbb{C}) = 1\).

Proof. We should consider \((k, l, m) = (3, 0, 0), (2, 0, 1), (1, 2, 0), (1, 0, 2)\) and \((0, 2, 1)\). Let \(\gamma\) be an arbitrary 3-cocycle. Then we can assume

\[
\begin{align*}
\gamma_{\lambda_1,\lambda_2,\lambda_3}(L, L, L) &= a(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3), \\
\gamma_{\lambda_1,\lambda_2,\lambda_3}(L, L, M) &= (\lambda_1 - \lambda_2)(b\lambda_1 + b\lambda_2 + c\lambda_3), \\
\gamma_{\lambda_1,\lambda_2,\lambda_3}(L, Y, Y) &= (\lambda_2 - \lambda_3)(e\lambda_1 + f\lambda_2 + f\lambda_3), \\
\gamma_{\lambda_1,\lambda_2,\lambda_3}(L, M, M) &= g(\lambda_2 - \lambda_3), \\
\gamma_{\lambda_1,\lambda_2,\lambda_3}(Y, Y, M) &= h(\lambda_1 - \lambda_2),
\end{align*}
\]
where $a, b, c, e, f, g, h \in \mathbb{C}$. In the following, we try to determine these parameters.

Let $\varphi$ be a 2-cochain defined by $\varphi_{\lambda_1, \lambda_2}(L, M) = b\lambda_1$ and $\varphi_{\lambda_1, \lambda_3}(Y, Y) = -f(\lambda_1 - \lambda_2)$. Then
\[
(d\varphi)_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = (\lambda_1 - \lambda_2)(-b\lambda_1 - b\lambda_2 - b\lambda_3),
\]
\[
(d\varphi)_{\lambda_1, \lambda_2, \lambda_3}(L, Y, Y) = (\lambda_2 - \lambda_3)(b\lambda_1 - f\lambda_2 - f\lambda_3).
\]
Replacing $\gamma$ by $\gamma + d\varphi$, we can assume $b = f = 0$. Consequently, $\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = c(\lambda_1 - \lambda_2)\lambda_3$ and $\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, Y, Y) = e(\lambda_2 - \lambda_3)\lambda_1$ for some $c, e \in \mathbb{C}$. Then by direct computations, we have
\[
(d\gamma)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, Y, Y) = -(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(e\lambda_1 + e\lambda_2 + (c + e)\lambda_3 + (c + e)\lambda_4) = 0,
\]
which implies $c = e = 0$.

Similarly, we can obtain that
\[
(d\gamma)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, Y, Y, M) = (\lambda_2 - \lambda_3)((g + h)\lambda_2 + (g + h)\lambda_3 + (h - g)\lambda_4) = 0,
\]
which implies $g = h = 0$.

By the cohomology theory of the Virasoro conformal algebra in [4], we know that the cochain defined by $\Phi_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$ is a cocycle, but not a coboundary. Thus we have $\tilde{H}^3(S\mathcal{V}, \mathbb{C}) = \mathbb{C}\Phi$. \hfill $\Box$

**Remark 3.7** The above skew-symmetric function $\Phi : S\mathcal{V} \otimes S\mathcal{V} \otimes S\mathcal{V} \to \mathbb{C}[\lambda_1, \lambda_2, \lambda_3]$ has values $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$ on $L \otimes L \otimes L$ and $0$ on others.

**Lemma 3.8** $\tilde{H}^4(S\mathcal{V}, \mathbb{C}) = 0$.

**Proof.** For $q = 4$, we should consider $(k, l, m) = (3, 0, 1), (2, 2, 0), (2, 0, 2)$ and $(1, 2, 1)$. Let $\gamma$ be an arbitrary 4-cocycle. Then we can assume $\gamma$ is defined by
\[
\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, M) = a(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3),
\]
\[
\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) = b(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4),
\]
\[
\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, Y, Y) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(e\lambda_1 + e\lambda_2 + e\lambda_3 + e\lambda_4),
\]
\[
\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, Y, Y, M) = (\lambda_2 - \lambda_3)(f\lambda_1 + g\lambda_2 + g\lambda_3 + h\lambda_4),
\]
where $a, b, c, e, f, g, h \in \mathbb{C}$.

Let $\phi$ be a 3-cochain defined by $\phi_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = (\lambda_1 - \lambda_2)\lambda_3$. Then by direct computations, we have
\[
(d\phi)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, Y, Y) = -(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_3 + \lambda_4),
\]
\[
(d\phi)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, M) = 0.
\]
Let $\psi$ be a 3-cochain defined by $\psi_{\lambda_1, \lambda_2, \lambda_3}(L, Y, Y) = \lambda_1(\lambda_2 - \lambda_3)$. Then by direct computations, we have
\[
(d\psi)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, Y, Y) = -(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4).
\]
Replacing $\gamma$ by $\gamma + (e - c)(d\phi) + c(d\psi)$, we can assume $c = e = 0$ and $\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, Y, Y) = 0$. Then by
\[
(d\gamma)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, L, Y, Y) = a(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5) = 0,
\]
we deduce that $a = 0$. Therefore, $\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, L, M) = 0$.

Similarly, let $\Phi$ be a 3-cochain defined by $\Phi_{\lambda_1,\lambda_2,\lambda_3}(L, M, M) = \lambda_2 - \lambda_3$. Then by direct computations, we have

$$(d\Phi)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, Y, Y, M) = (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_4),$$

$$(d\Phi)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, M, M) = 0.$$

Let $\Psi$ be a 3-cochain defined by $\Psi_{\lambda_1,\lambda_2,\lambda_3}(Y, Y, M) = \lambda_1 - \lambda_2$. Then by direct computations, we have

$$(d\Psi)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, Y, Y, M) = (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 + \lambda_4).$$

Replacing $\gamma$ by $\gamma - \frac{q-h}{2}(d\Phi) - \frac{q+h}{2}(d\Psi)$, we can assume $q = h = 0$.

Now, we can assume

$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, M, M) = b(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)$,

$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, Y, Y, M) = f(\lambda_2 - \lambda_3)$,

where $b, f \in \mathbb{C}$. Then by

$$(d\gamma)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) = b(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(-\lambda_3 - \lambda_4 + \lambda_5) - f(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5),$$

we deduce that $b = f = 0$. Thus we have $\gamma = 0$. \hfill $\square$

**Lemma 3.9** $\tilde{H}^5(SY, \mathbb{C}) = \mathbb{C}L \oplus \mathbb{C}\Psi$, where

$$L_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\lambda_5$$

and

$$\Psi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, Y, Y, M, M) = (\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5).$$

In particular, $\dim \tilde{H}^5(SY, \mathbb{C}) = 2$.

**Proof.** For $q = 5$, we should consider $(k, l, m) = (3, 2, 0), (2, 2, 1)$ and $(1, 2, 2)$. Let $\gamma$ be an arbitrary 5-cocycle. Then we can assume that $\gamma$ is defined by

$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, L, Y, Y) = a(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5)$,

$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(b\lambda_1 + b\lambda_2 + c\lambda_3 + c\lambda_4 + e\lambda_5)$,

$\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, Y, Y, M, M) = f(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5)$,

where $a, b, c, e, f \in \mathbb{C}$.

Let $\phi$ be a 4-cochain defined by

$$\phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, L, M) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3).$$

Then $d\phi$ is given by

$$(d\phi)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, L, Y, Y) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5),$$

$$(d\phi)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, L, M, M) = 0.$$

Replacing $\gamma$ by $\gamma - a(d\phi)$, we have $\gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, L, Y, Y) = 0$. 

\hfill 8
Define
\[ \psi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, L, M, M) = (c - b)(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4), \]
\[ \psi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, Y, Y, M) = b\lambda_1(\lambda_2 - \lambda_3). \]

Then
\[ (d\psi)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) = -(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(b\lambda_1 + b\lambda_2 + c\lambda_3 + c\lambda_4 + (2b - c)\lambda_5). \]

Replacing \( \gamma \) by \( \gamma - d(\psi) \), we can assume
\[ \gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) = e(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\lambda_5. \]

Then by
\[ (d\gamma)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, L, Y, M, M) = 0. \]

Define \( \Lambda_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, L, Y, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\lambda_5 \), which is a cocycle, but not a coboundary.

Similarly, we can show that a cochain defined by \( \Psi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(L, Y, Y, M, M) = (\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5) \) is a cocycle, but not a coboundary. \( \square \)

**Lemma 3.10** \( \tilde{H}^6(SV, \mathbb{C}) = \mathbb{C}\Omega \oplus \mathbb{C}\Theta \), where
\[ \Omega_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5) \]
and
\[ \Theta_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, Y, Y, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6). \]

In particular, \( \dim \tilde{H}^6(SV, \mathbb{C}) = 2 \).

*Proof.* For \( q = 6 \), we should consider \( (k, l, m) = (3, 2, 1) \) and \((2, 2, 2)\).

First, let \( \gamma \) be the 6-cochain defined by
\[ \gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5). \]

It can be only a coboundary of \( \phi \), defined by
\[ \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) \quad \text{and} \quad \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, L, M, M). \]

But by the proof of Lemma 3.9, \( \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) \) is a cocycle. Furthermore, the degree of \( \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, M, M, M) \) is at least 4, which implies \( \gamma \) is not a coboundary. It is straightforward to check \( d\gamma = 0 \).

Second, let \( \gamma \) be the 6-cochain defined by
\[ \gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, Y, Y, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6). \]

It can be only a coboundary of \( \phi \), defined by
\[ \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, Y, Y, M, M) \quad \text{and} \quad \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, M, M, M). \]

As shown above, we can obtain that \( \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) \) is a cocycle and the degree of \( \phi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, M, M, M) \) is at least 4. Thus \( \gamma \) is not a coboundary. Moreover, the degree of a 7-cochain is at least 5 on \( (L, L, L, Y, Y, M, M) \) and at least 7 on \( (L, L, Y, Y, Y, Y, M) \). Then we deduce that \( d\gamma = 0 \).

As mentioned above, we can obtain the main result of this section as follows.
Theorem 3.11 The dimension of $\tilde{H}^q(SV, \mathbb{C})$ is given by

$$\dim \tilde{H}^q(SV, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, 3, \\ 2 & \text{if } q = 5, 6, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\tilde{H}^q(SV, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } q = 0, \\ \mathbb{C}(\Phi) & \text{if } q = 3, \\ \mathbb{C}(\Lambda) \oplus \mathbb{C}(\Psi) & \text{if } q = 5, \\ \mathbb{C}(\Omega) \oplus \mathbb{C}(\Theta) & \text{if } q = 6, \\ 0 & \text{otherwise,} \end{cases}$$

(3.4)

where

$$\Phi_{\lambda_1,\lambda_2,\lambda_3}(L, L, L) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3),$$

$$\Lambda_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\lambda_5,$$

$$\Psi_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5}(L, Y, Y, M, M) = (\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5),$$

$$\Omega_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5),$$

$$\Theta_{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6}(L, L, Y, Y, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6).$$

Proof. It follows directly from the previous Lemmas.

Remark 3.12 The corresponding annihilation algebra of $SV$ is

$$\text{Lie}(SV)^+ = \sum_{m \geq -1} \mathbb{C}L_m + \sum_{n \geq 0} \mathbb{C}M_n + \sum_{p \in \frac{1}{2} + \mathbb{Z}} \mathbb{C}Y_p,$$

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, M_n] = -nM_{m+n},$$

$$[Y_p, Y_q] = (p - q)M_{p+q}, \quad [L_m, Y_p] = \left(\frac{m}{2} - p\right)Y_{m+p},$$

which is a 'half part' of the Schrödinger-Virasoro Lie algebra ([11]). So by [3], the dimension of all the cohomology groups of $\text{Lie}(SV)^+$ are given by

$$\dim H^q(\text{Lie}(SV)^+, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, 3, \\ 2 & \text{if } q = 5, 6, \\ 0 & \text{otherwise.} \end{cases}$$

4 Reduced cohomology of $SV$ with coefficients in its module

In this section, we compute the reduced cohomology groups of $SV$ with coefficients in its trivial module $\mathbb{C}_a$ and with coefficients in $M_{\alpha, \beta}$ in the case of $\beta \neq 0$.

4.1 Computation of $H^q(SV, \mathbb{C})$

Theorem 4.1 The dimension of $H^q(SV, \mathbb{C})$ are given by

$$\dim H^q(SV, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, 2, 3, \\ 2 & \text{if } q = 4, 6, \\ 4 & \text{if } q = 5, \\ 0 & \text{otherwise.} \end{cases}$$
Proof. By Proposition 2.9, the map $\gamma \mapsto \partial \cdot \gamma$ gives an isomorphism such that $\tilde{H}^q(SV, \mathbb{C}) \cong H^q(\partial \tilde{C}^\bullet)$ for all $q \geq 1$. Therefore, we can obtain the following result immediately by the discussion of Section 3.1.

$$H^q(\partial \tilde{C}^\bullet) = \begin{cases} \mathbb{C}(\partial \Phi) & \text{if } q = 3, \\ \mathbb{C}(\partial \Lambda) \oplus \mathbb{C}(\partial \Psi) & \text{if } q = 5, \\ \mathbb{C}(\partial \Omega) \oplus \mathbb{C}(\partial \Theta) & \text{if } q = 6, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

Similar to the discussions in Section 2, we can obtain the following long exact sequence of cohomology groups:

$$\cdots \rightarrow H^q(\partial \tilde{C}^\bullet) \xrightarrow{i_q} \tilde{H}^q(SV, \mathbb{C}) \xrightarrow{p_q} H^q(SV, \mathbb{C}) \xrightarrow{w_q} H^{q+1}(\partial \tilde{C}^\bullet) \rightarrow \cdots \quad (4.2)$$

where $i_q, p_q$ are induced by $i, p$ in (2.8) respectively and $w_q$ is the $q$-th connecting homomorphism. Take $\partial \gamma \in H^q(\partial \tilde{C}^\bullet)$ with a nonzero element $\gamma \in \tilde{H}^q(SV, \mathbb{C})$ of degree $k + \frac{l}{2}$, we can obtain that $i_q(\partial \gamma) = \partial \gamma \in \tilde{H}^q(SV, \mathbb{C})$. Since $\partial \gamma = (\sum \lambda_i)\gamma$, we have

$$\deg(\partial \gamma) = \deg(\gamma) + 1 = k + \frac{l}{2} + 1.$$ 

Thus, $\partial \gamma = 0 \in \tilde{H}^q(SV, \mathbb{C})$ (Here we use the property $a = 0$). Thus, the image of $i_q$ is zero. Then the long exact sequence (4.2) splits into the following short exact sequence immediately:

$$0 \rightarrow \tilde{H}^q(SV, \mathbb{C}) \xrightarrow{p_q} H^q(SV, \mathbb{C}) \xrightarrow{w_q} H^{q+1}(\partial \tilde{C}^\bullet) \rightarrow 0. \quad (4.3)$$

for all $q \geq 1$. Thus, we have

$$\dim H^q(SV, \mathbb{C}) = \dim \tilde{H}^q(SV, \mathbb{C}) + \dim H^{q+1}(\partial \tilde{C}^\bullet)$$

$$= \dim \tilde{H}^q(SV, \mathbb{C}) + \dim \tilde{H}^{q+1}(SV, \mathbb{C}) \quad (4.4)$$

for all $q \geq 1$. Then the result follows. \hfill \square

Remark 4.2 It was shown in [11] that there is a unique nontrivial extension of $SV$ by a 1-dimensional center. This coincides with our result $\dim H^2(SV, \mathbb{C}) = 1$.

It is not difficult to check by (4.3) that the basis of $H^q(SV, \mathbb{C})$ can be obtained by combining the images of a basis of $\tilde{H}^q(SV, \mathbb{C})$ with the pre-images of a basis of $\tilde{H}^{q+1}(SV, \mathbb{C})$. Let $\gamma$ be a nonzero $(q + 1)$-coycle of degree $k + \frac{l}{2}$ such that $\partial \gamma \in H^{q+1}(\partial \tilde{C}^\bullet)$. By (3.2), we can obtain that

$$d(\tau(\partial \gamma)) = (d\tau + \tau d)(\partial \gamma) = (\deg(\partial \gamma) - k - \frac{l}{2})(\partial \gamma) = ((k + \frac{l}{2} + 1) - k - \frac{l}{2})(\partial \gamma) = \partial \gamma. \quad (4.5)$$

Thus, the pre-image $w_q^{-1}(\partial \gamma)$ of $\partial \gamma$ under the connecting homomorphism $w_q$ is $\tau(\partial \gamma)$, i.e., $w_q^{-1}(\partial \gamma) = \tau(\partial \gamma)$.

Due to the above discussions, we can obtain a specific system to calculate the basis of $H^q(SV, \mathbb{C})$ so that we can describe the structure of $H^q(SV, \mathbb{C})$ more clearly.
Corollary 4.3

\[ H^q(SV, \mathbb{C}) = \begin{cases} 
\mathbb{C} & \text{if } q = 0, \\
\mathbb{C}(\Phi) & \text{if } q = 2, \\
\mathbb{C}(\Phi) & \text{if } q = 3, \\
\mathbb{C}(\Lambda) \oplus \mathbb{C}(\Psi) & \text{if } q = 4, \\
\mathbb{C}(\Lambda) \oplus \mathbb{C}(\Psi) \oplus \mathbb{C}(\Omega) \oplus \mathbb{C}(\Theta) & \text{if } q = 5, \\
\mathbb{C}(\Omega) \oplus \mathbb{C}(\Theta) & \text{if } q = 6, \\
0 & \text{otherwise}, 
\end{cases} \tag{4.6} \]

where \( \bar{X} = \tau(\partial X) \) and \( X \in \{\Phi, \Lambda, \Psi, \Omega, \Theta\} \) as shown in Theorem 3.11. More specifically,

\[
\Phi_{\lambda_1, \lambda_2}(L, L) = -\lambda_1^3 + \lambda_2^3,
\]

\[
\Lambda_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, Y, Y, M) = (\lambda_2 + \lambda_3 + \lambda_4)(\lambda_2 - \lambda_3)\lambda_4,
\]

\[
\Psi_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(Y, Y, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4),
\]

\[
\Omega_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, Y, Y, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)),
\]

\[
\Theta_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, Y, Y, M, M) = (\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5)(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5).
\]

Proof. It follows directly from the Theorem 3.11, (4.4) and (4.5). About the concrete expression of \( \Phi, \Lambda, \Psi, \Omega, \Theta \), we just take \( \Phi \) for example, others can be proved similarly.

\[
\Phi_{\lambda_1, \lambda_2}(L, L) = (\tau(\partial \Phi))_{\lambda_1, \lambda_2}(L, L) = (-1)^2 \frac{\partial}{\partial \lambda}(\partial \Phi)_{\lambda_1, \lambda_2, \lambda}(L, L)|_{\lambda = 0} = \frac{\partial}{\partial \lambda}(\lambda_1 + \lambda_2 + \lambda) \Phi_{\lambda_1, \lambda_2, \lambda}(L, L)|_{\lambda = 0} = \frac{\partial}{\partial \lambda}(\lambda_1 + \lambda_2 + \lambda)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda)(\lambda_2 - \lambda)|_{\lambda = 0} = -\lambda_1^3 + \lambda_2^3.
\]

\[
\square
\]

4.2 Computation of \( H^q(SV, \mathbb{C}_a) \) if \( a \neq 0 \)

Theorem 4.4 For any \( q \in \mathbb{Z}_+ \), \( H^q(SV, \mathbb{C}_a) = 0 \) if \( a \neq 0 \).

Proof. Similar to the proof of Lemma 3.2 in [13], we can define an operator \( \tau_2 : \tilde{C}^q(SV, \mathbb{C}_a) \rightarrow \tilde{C}^{q-1}(SV, \mathbb{C}_a) \) by

\[
(\tau_2 \gamma)_{\lambda_1, \ldots, \lambda_{q-1}}(a_1, \ldots, a_{q-1}) = (-1)^{q-1}\gamma_{\lambda_1, \ldots, \lambda_{q-1}, \lambda}(a_1, \ldots, a_{q-1}, L)|_{\lambda = 0}. \tag{4.7}
\]

Then

\[
(d \tau_2 + \tau_2 d) \gamma \equiv -a \gamma \pmod{\partial \tilde{C}^q(SV, \mathbb{C}_a)},
\]

which implies \( H^q(SV, \mathbb{C}_a) = 0 \) for all \( q \geq 0 \) if \( a \neq 0 \). \square

A similar argument shows

Theorem 4.5 \( H^q(SV, M_{a,b}) = 0 \) if \( \beta \neq 0 \).
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