Numerical scheme for stochastic differential equations driven by fractional Brownian motion with $1/4 < H < 1/2$.

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Abstract

In this article, we study a numerical scheme for stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H \in (1/4, 1/2)$. Towards this end, we apply Doss-Sussmann representation of the solution and an approximation of this representation using a first order Taylor expansion. The obtained rate of convergence is $n^{-2H+\rho}$, for $\rho$ small enough.

Key words: Doss-Sussmann representation, fractional Brownian motion, stochastic differential equation, Taylor expansion.

1 Introduction

In this article we are interested in a pathwise approximation of the solution to the stochastic differential equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s) \circ dB_s, \quad t \in [0,T],$$

(1)

where $x \in \mathbb{R}$ and $b, \sigma : \mathbb{R} \to \mathbb{R}$ are measurable functions. The stochastic integral in (1) is understood in the sense of Stratonovich, (see Alós et.al. for details) and $B = \{B_t, t \in [0,T]\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (1/4, 1/2)$. $B$ is a centered Gaussian process with a covariance structure given by

$$\mathbb{E}(B_t B_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}), \quad t \in [0,T].$$

(2)

In (1), the existence and uniqueness for the solution of equation (1) have been established under suitable conditions, which follows from our assumption (see hypothesis (H) in Section 2.1).

Equation (1) has been analyzed by several authors, for different interpretations of stochastic integrals, because of the properties of fractional Brownian motion $B$. Among these properties, we can mention self-similarity, stationary increments, $\rho$-Hölder continuity, for
any \( \rho \in (0, H) \), and the covariance of its increments on intervals decays asymptotically as a negative power of the distance between the intervals. Therefore, equation (1) becomes quite useful in applications in different areas such as physics, biology, finance, etc. (see, e.g., [2, 4, 11]). Hence, it is important to provide approximations to the solution of (1).

For \( H = 1/2 \) (i.e., \( B \) is a Brownian motion), a large number of numerical schemes to approximate the unique solution of (1) has been considered in the literature. The reader can consult Kloeden and Platen [10] (and the references therein), for a complete exposition of this topic. In particular, Talay [16] introduces the Doss-Sussmann transformation [6, 15] in the study of numerical methods to the solution of stochastic differential equations (see Section 2.1 for the definition of this transformation).

For \( H > 1/2 \), numerical schemes for equation (1) have been analyzed by several authors. For instance, we can mention [4, 8, 13] and [14], where the stochastic integrals is interpreted as the extension of the Young integral given in [17] and the forward integral, respectively. It is well-known that these integrals agree with the Stratonovich one under suitable conditions (see Alós and Nualart [3]).

In this paper we are interested in the case \( H < 1/2 \), because numerical schemes for the solution to (1) have been studied only in some particular situations. Namely, Garzón et. al. [7] use the Doss-Sussmann transformation in order to prove the convergence for the Euler scheme associated to (1) by means of an approximation of \( fBm \) via fractional transport processes. In [14], the authors also take advantage of the Doss-Sussmann transformation in order to discuss the Crank-Nicholson method, for \( H \in (1/6, 1/2) \) and \( b \equiv 0 \). Here, they show convergence in law of the error to a random variable, which depends on the solution of the equation and an independent Gaussian random variable. Specifically, the authors state that the rate of convergence of the scheme is of order \( n^{1/2-3H} \). In [12] the authors consider the so-called modified Euler scheme for multidimensional stochastic differential equations driven by \( fBm \) with \( H \in (1/3, 1/2) \). They utilize rough paths techniques in order to obtain the convergence rate of order \( n^{1/2-2H} \). Also, they prove that this rate is sharp. In [5] a numerical scheme for stochastic differential equations driven by a multidimensional \( fBm \) with Hurst parameter greater than 1/3 is introduced. The method is based on a second-order Taylor expansion, where the Lévy area terms are replaced by products of increments of the driving \( fBm \). Here, the order of convergence is \( n^{-(H-\rho)} \), with \( \rho \in (1/3, H) \). In order to get this rate of convergence, the authors use a combination of rough paths techniques and error bounds for the discretization of the Lévy area terms.

In this work we propose an approximation scheme for the solution to (1) with \( H \in (1/4, 1/2) \). To do so, we use a first order Taylor expansion in the Doss-Sussmann representation of the solution. We consider the case \( H \in (1/4, 1/2) \) because it is showed in [3] that the solution of (1) is given by this transformation. However, even in the case \( (0, 1/4) \), our scheme tends to the mentioned transformation. The rate of convergence in this paper is \( n^{-2H+\rho} \), where \( \rho < 2H \) small enough, improving the ones given in [14], [16], [5] and [12]. Also our rate is better than the one obtained in [7] when the \( fBm \) is not approximated by means of fractional transport process. We observe that our method only establishes this rate of convergence for \( H < 1/2 \) because we could only see that the auxiliary inequality [22] below is satisfied in this case. However, the same construction holds for \( H > 1/2 \) (see [14], Proposition 1). In this case, the rate of convergence for the scheme is not the same as the case \( 1/4 < H < 1/2 \). In fact, for \( H > 1/2 \), we only get that the rate of convergence is \( n^{-1+\rho} \) for \( \rho \) small enough.

The paper is organized as follows: In Section 2 we introduce the notations needed in this article. In particular, we explain the Doss-Sussmann-type transformation related to the unique solution to (1). Also, in this section, the scheme is presented and the main result is stated (Theorem 2.2 below). In Section 3 we establish the auxiliary lemmas, which are needed to show, in Section 4, that the main result is true. The proof of the auxiliary lemmas are presented in Section 5. Finally, in the Appendix (Section 6), other auxiliary result is also studied because it is a general result concerning the Taylor expansion for some continuous functions.
2 Preliminaries and main result

In this section, we introduce the basic notions and the framework that we use in this paper. That is, we first describe the Doss-Sussmann transformation given in Doss [6] and Sussmann [15], which is the link between the stochastic and ordinary differential equations (see Alòs et al. [1], or Nourdin and Neuenkirch [14], for fractional Brownian motion case). Then, we provide a numerical method and its rate of convergence for the unique solution of (1). These are the main result of this article (see Theorem 2.2).

2.1 Doss-Sussmann transformation

Henceforth, we consider the stochastic differential equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s) \circ dB_s, \quad t \in [0, T],$$

(3)

where $B = \{B_t : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$, $x \in \mathbb{R}$ and the stochastic integral in (3) is understood in the sense of Stratonovich, which is introduced in [1]. Remember that $B$ is defined in [2]. The coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$ are measurable functions such that

(H) $b \in C^2_b(\mathbb{R})$ and $\sigma \in C^2_b(\mathbb{R})$.

Remark 2.1. By assumption (H), we have, for $z \in \mathbb{R}$,

- $|b(z)| \leq M_1$, $|b'(z)| \leq M_3$ and $|b''(z)| \leq M_6$.
- $|\sigma(z)| \leq M_5$, $|\sigma'(z)| \leq M_2$ and $|\sigma''(z)| \leq M_3$.

We explicitly give these constants so that it will be clear where we use them in our analysis.

Now, we explain the relation between (3) and ordinary differential equations: the so called Doss-Sussmann transformation.

In Alòs et al. (Proposition 6) is proven that the equation (3) has a unique solution of the form

$$X_t = \phi(Y_t, B_t).$$

(4)

The function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is the solution of the ordinary differential equation

$$\frac{\partial \phi}{\partial \beta}(\alpha, \beta) = \sigma(\phi(\alpha, \beta)), \quad \alpha, \beta \in \mathbb{R},$$

$$\phi(\alpha, 0) = \alpha,$$

(5)

and the process $Y$ is the pathwise solution to the equation

$$Y_t = x + \int_0^t \left( \frac{\partial \phi}{\partial \alpha}(Y_s, B_s) \right)^{-1} b(\phi(Y_s, B_s)) ds, \quad t \in [0, T].$$

By Doss [6], we have

$$\frac{\partial \phi}{\partial \alpha}(\alpha, \beta) = \exp \left( \int_0^\beta \sigma'(\phi(\alpha, s)) ds \right),$$

(6)

which implies

$$Y_t = x + \int_0^t \exp \left( - \int_0^B \sigma'(\phi(Y_s, u)) du \right) b(\phi(Y_s, B_s)) ds.$$

(7)

2.2 Numerical Method

In this section, we describe our numerical scheme associated to the unique solution of (3). Towards this end, in Section 2.2.1 we first propose an approximation to the function $\phi$ given in (6), and then, in Section 2.2.2 we approximate the process $Y$. In both sections we suppose that (H) holds.
2.2.1 Approximation of $\phi$

Note that, for $x \in \mathbb{R}$, equation (5) has the form
\[
\phi(x, u) = x + \int_0^u \sigma(\phi(x, s)) ds.
\]

For each $l \in \mathbb{N}$, we take the partition \{${u_i^l, i \in \{-l, \ldots, l\}}$\} of the interval \([-\|B\|_\infty, \|B\|_\infty]\)
given by \(-\|B\|_\infty = u_{-l}^l < \ldots < u_{l-1}^l < u_0^l = 0 < u_1^l < \ldots < u_l^l = \|B\|_\infty\). Here, \(\|B\|_\infty = \sup_{t \in [0, T]} |B(t)|\),
\[
u_{i+1}^l = u_i^l + \frac{\|B\|_\infty}{l} = \frac{(i + 1)\|B\|_\infty}{l}, \quad u_{-(i+1)}^l = u_{-i}^l - \frac{\|B\|_\infty}{l} = -\frac{(i + 1)\|B\|_\infty}{l}.
\]

Let $x \in \mathbb{R}$ be given in (7). Set
\[
M := |x| + T \left( M_1 \exp(M_2 \|B\|_\infty) + \|B\|_{H-\rho} C_3 T^{H-\rho} \right),
\]
where $\rho \in (0, H)$, $\|B\|_{H-\rho}$ is the $(H - \rho)$-Hölder norm of $B$ on $[0, T]$,
\[
C_3 = M_1 M_2 \exp(M_2 \|B\|_\infty) + M_4 \exp(M_2 \|B\|_\infty) M_5 \|B\|_\infty (1 + M_2)
\]
and $M_i, i \in \{1, \ldots, 6\}$ are defined in Remark 2.1.

Now, we define the function $\phi^l : \mathbb{R}^2 \to \mathbb{R}$ by
\[
\phi^l(z, u) = 0 \quad \text{if} \quad (z, u) \not\in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty];
\]
and, for $k = 1, \ldots, l$,
\[
\phi^l(z, u) = \phi^l(z, u_{k-1}^l) + \int_{u_{k-1}^l}^u \sigma(\phi^l(z, u_{k-1}^l) + (s - u_{k-1}^l)\sigma(\phi^l(z, u_{k-1}^l))) ds,
\]
if $z \in [-M, M]$ and $u \in (u_{k-1}^l, u_k^l]$, with
\[
\phi^l(z, u_0^l) = z, \quad \text{if} \quad z \in [-M, M].
\]
The definition of $\phi^l$ for the case $k = -l, \ldots, 0$ is similar. That is,
\[
\phi^l(z, u) = \phi^l(z, u_k^l) - \int_u^{u_k^l} \sigma(\phi^l(z, u_k^l) + (s - u_k^l)\sigma(\phi^l(z, u_k^l))) ds,
\]
if $z \in [-M, M]$ and $u \in [u_{k-1}^l, u_k^l]$.

Also, we consider the function $\Psi^l : \mathbb{R}^2 \to \mathbb{R}$, which is equal to
\[
\Psi^l(z, u) = 0 \quad \text{if} \quad (z, u) \not\in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty],
\]
and, for $k = 1, \ldots, l$,
\[
\Psi^l(z, u) = \Psi^l(z, u_{k-1}^l) + \int_{u_{k-1}^l}^u (\sigma(\Psi^l(z, u_{k-1}^l)) + \sigma' \sigma(\Psi^l(z, u_{k-1}^l)) (s - u_{k-1}^l)) ds
\]
\[
= \Psi^l(z, u_{k-1}^l) + (u - u_{k-1}^l) \left( \sigma(\Psi^l(z, u_{k-1}^l)) + \sigma' \sigma(\Psi^l(z, u_{k-1}^l)) \left( \frac{u - u_{k-1}^l}{2} \right) \right)
\]
if $z \in [-M, M]$ and $u \in (u_{k-1}^l, u_k^l]$, with
\[
\Psi^l(z, u_0^l) = z, \quad \text{if} \quad z \in [-M, M].
\]
For \( k = -l, \ldots, 0, \) \( \Psi^l \) is introduced as

\[
\Psi^l(z, u) = \Psi^l(z, u_k) - \int_{u_k}^{u_k^l} \left( \sigma \left( \Psi^l(z, u_k^l) \right) + \sigma' \left( \Psi^l(z, u_k^l) \right) (s - u_k^l) \right) ds,
\]

If \( z \in [-M, M] \) and \( u \in [u_{k-1}^l, u_k^l] \). From equation and last equality, it can be seen that \( \Psi^l(z, \cdot) \) is continuous on \([-\|B\|_{\infty}, \|B\|_{\infty}]\).

We remark that the function \( \phi^l \) given in \( (11) \) and \( (13) \) is an auxiliary tool that allows us to use Taylor’s theorem in the analysis of the numerical scheme proposed in this paper (i.e., Theorem \( 2.2 \)). Indeed, the Taylor’s theorem is utilized in Lemma \( 5.2.2 \).

### 2.2.2 Approximation of \( Y \)

Here, we approximate the solution of equation \( (7) \).

For \( l \in \mathbb{N} \), we define the process \( Y^l_t \) as the solution of the following ordinary differential equation, where the existence and uniqueness is guaranteed since the coefficient \( g^l : \mathbb{R}^2 \to \mathbb{R} \) satisfies Lipschitz and linear growth conditions in the second variable (see Remark \( 2.1 \) and Lemma \( 5.3 \)).

\[
Y^l_t = x + \int_0^t g^l (B_s, Y^l_s) ds, \quad Y^l_0 = x, \tag{16}
\]

where

\[
g^l (B_s, Y^l_s) = \exp \left( - \int_0^{B_s} \sigma' (\Psi^l (Y^l_s, u)) du \right) b (\Psi^l (Y^l_s, B_s)). \tag{17}
\]

Now, for \( m \in \mathbb{N} \), we set the partition \( 0 = t_0 < \ldots < t_m = T \) of \([0, T]\) with \( t_{i+1} = t_i + \frac{T}{m} \) and we define the process \( Y^{l,m}_t \) as:

\[
Y^{l,m}_0 = x, \quad Y^{l,m}_t = Y^{l,m}_{t_k} + \int_{t_k}^{t} \left[ g^l (B^m_s, Y^{l,m}_{t_k}) + h^l_1 (B^m_s, Y^{l,m}_{t_k}) (B_s - B^m_s) \right] ds, \tag{18}
\]

for \( t^m_k \leq t < t^m_{k+1} \), where \( h^l_1 (u, z) = \frac{\partial g^l (u, z)}{\partial u} \) and \( g^l \) is given by \( (17) \). So

\[
\frac{\partial g^l (u, z)}{\partial u} = -g^l (u, z) \sigma' (\Psi^l (z, u)) + \exp \left( - \int_0^{u} \sigma' (\Psi^l (z, r)) dr \right) b' (\Psi^l (z, u)) \frac{\partial \Psi^l (z, u)}{\partial u}. \tag{19}
\]

By Remark \( 2.1 \) we can see

\[
|g^l (u, z)| \leq M_1 \exp (M_2 |u|). \tag{20}
\]

Also we have

\[
|h^l_1 (B^m_s, Y^{l,m}_{t_k})| \leq C_3, \tag{21}
\]

where \( C_3 \) is given in \( (9) \). Moreover, assuming that \( (20) \) and \( (21) \) are satisfied, it is not hard to prove by induction that

\[
\sup_{t \in [0, T]} |Y^{n,n}_{t_k}| \leq |x| + T \left( M_1 \exp (M_2 \|B\|_{\infty}) + T^{H-\rho} \|B\|_{H-\rho} C_3 \right) = M.
\]

Finally, in a similar way to Garzón et al. \( [7] \), for \( n \in \mathbb{N} \), we define the approximation \( X^n_t \) of \( X \) by:

\[
X^n_t = \Psi^n (Y^{n,n}_{t_k}, B_t), \tag{22}
\]

where \( \Psi^n \) and \( Y^{n,n}_{t_k} \) are given by \( (15) \) and \( (13) \), respectively.

Now, we are in position to state our main result.
Theorem 2.2. Let (H) be satisfied and $1/4 < H < 1/2$, then

$$|X_t - X^n_t| \leq Cn^{-2(H-\rho)},$$

where $\rho > 0$ is small enough and $C$ is a constant that does not depend on $n$.

Remark 2.3. The constant $C$ has the form

$$C = \exp(2M_2\|B\|_\infty) \left[ C_2 \exp(C_3T) + \frac{M_2^2M_4\|B\|_\infty^3}{6} + \frac{M_2^2M_3\|B\|_\infty^3}{6} \right],$$

with

$$C_1 = (M_4 + M_1M_3\|B\|_\infty) \exp(2M_2\|B\|_\infty + T),$$
$$C_2 = \exp(M_2\|B\|_\infty)(M_4 + M_1M_3\|B\|_\infty)(\frac{M_2^2M_4\|B\|_\infty^3}{6} \exp(M_2\|B\|_\infty) + \frac{M_1M_2^2\|B\|_\infty^3}{6} \exp(2M_2\|B\|_\infty)), $$
$$C_3 = M_1M_2 \exp(2M_2\|B\|_\infty) + M_1 \exp(2M_2\|B\|_\infty) M_5\|B\|_\infty(1 + M_2),$$
$$C_4 = M_4 \exp(2M_2\|B\|_\infty) + C_5T^{H-\rho}\|B\|_{H-\rho},$$
$$C_5 = \exp(M_2\|B\|_\infty)\|M_1M_4\|\|B\|_\infty(1 + M_2)\{M_3M_1M_5 + M_3M_2M_5 + M_3M_4M_5 + M_6M_5\|B\|_\infty(1 + M_2)\} + M_1M_2 + M_4M_5(1 + M_2),$$
$$C_6 = [C_4 \exp(3M_2\|B\|_\infty)(M_4 + M_1M_3\|B\|_\infty)T^{1-(H-\rho)} + (C_5 + C_6)\|B\|_{H-\rho}],$$
$$C_7 = \exp(3M_2\|B\|_\infty)(M_4 + M_1M_3\|B\|_\infty),$$
$$C_8 = M_4M_5 \exp(2M_2\|B\|_\infty)\{M_5 + M_2M_5\|B\|_\infty + M_2M_4\}.$$

Remark 2.4. We choose the constant $M$ because the processes given in (16) and (18), as well as the solution to (4), are bounded by $M$, as it is pointed out in this section.

3 Preliminary lemmas

In this section, we stated the auxiliary tools that we need in order to prove our main result Theorem 2.2. The first four lemmas are related to the apriori estimates of $\phi$. We recall you that the constants $M_i$, $i \in \{1, \ldots, 6\}$ are introduced in Remark 2.1.

Lemma 3.1. Let $\phi$ and $\phi^i$ be given by (9) and (15), respectively. Then, Hypothesis (H) implies that, for $(z, u) \in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty]$, we have

$$|\phi(z, u) - \phi^i(z, u)| \leq \frac{M_2^2M_5\|B\|_\infty^3}{61^2} \exp(2M_2\|B\|_\infty),$$

Lemma 3.2. Let $\phi^i$ and $\Psi^i$ be given by (18) and (13), respectively. Then, Hypothesis (H) implies

$$|\phi^i(z, u) - \Psi^i(z, u)| \leq \frac{M_1M_2^2\|B\|_\infty^3}{61^2} \exp(2M_2\|B\|_\infty),$$

for $(z, u) \in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty]$.

Lemma 3.3. Let $\Psi^i$ be introduced in (13) and Hypothesis (H) hold. Then, for $(z_1, z_2, u) \in [-M, M]^2 \times [-\|B\|_\infty, \|B\|_\infty]$

$$|\Psi^i(z_1, u) - \Psi^i(z_2, u)| \leq |z_1 - z_2| \exp(2M_2\|B\|_\infty).$$

Lemma 3.4. Let $\phi^i$ be given in (17). Then, under Hypothesis (H),

$$|\phi^i(z_1, u) - \phi^i(z_2, u)| \leq |z_1 - z_2| \exp(2M_2\|B\|_\infty),$$

for $(z_1, z_2, u) \in [-M, M]^2 \times [-\|B\|_\infty, \|B\|_\infty]$.

Now we proceed to state the lemmas referred to the estimates on $Y^n - Y$. 

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Lemma 3.5. Assume that Hypothesis (H) is satisfied. Let \( Y \) and \( Y^n \) be given in (3) and (10), respectively. Then, for \( t \in [0, T] \),

\[
|Y_t - Y^n_t| \leq \exp(C_1 T) \frac{C_2}{n^2},
\]

where

\[
C_1 = (M_4 + M_1 M_3 \|B\|_\infty) \exp(M_2(\|B\|_\infty + T))
\]

and

\[
C_2 = \exp(M_2 \|B\|_\infty)(M_4 + M_1 M_3 \|B\|_\infty)
\times \left( \frac{TM_2^2 M_5 \|B\|_\infty^3}{6} \exp(M_2 \|B\|_\infty) + \frac{TM_3 M_2^2 \|B\|_\infty^3}{6} \exp(2M_2 \|B\|_\infty) \right),
\]

Lemma 3.6. Let \( Y^{n,n} \) be defined in (15). Then Hypothesis (H) implies, for \( s \in (t^n_k, t^n_{k+1}] \),

\[
|Y_s^{n,n} - Y^{n,n}_t| \leq C_4 (s - t^n_k),
\]

where \( C_4 = M_1 \exp(M_2 \|B\|_\infty) + C_3 T^{H-\rho} \|B\|_{H-\rho} \) and

\[
C_3 = M_1 M_2 \exp(M_2 \|B\|_\infty) + M_4 \exp(M_2 \|B\|_\infty) M_5 \|B\|_\infty (1 + M_2).
\]

Lemma 3.7. Suppose that Hypothesis (H) holds. Let \( Y^n \) and \( Y^{n,n} \) be given in (10) and (15), respectively. Then,

\[
|Y^n_t - Y^{n,n}_t| \leq C_6 T \left( \frac{T}{n} \right)^{2(H-\rho)} \exp(C_7 T), \quad t \in [0, T],
\]

where \( 0 < \rho < H \),

\[
C_5 = \exp(M_2 \|B\|_\infty) \left[ \|B\|_\infty(1 + M_2) (M_4 M_1 M_3 + M_2 M_4 M_5 + M_6 M_5 \|B\|_\infty(1 + M_2))
\right.
\]

\[
\left. + M_1 M_2 + M_4 M_5 (1 + M_2) \right],
\]

\[
C_6 = \left[ C_4 \exp(3M_2 \|B\|_\infty) [M_4 + M_1 M_3 \|B\|_\infty] T^{1-2(H-\rho)} + (C_5 + C_6) \|B\|_{H-\rho} \right],
\]

with \( C_4 \) given in Lemma 3.6 and

\[
C_7 = \exp(3M_2 \|B\|_\infty) [M_4 + M_1 M_3 \|B\|_\infty],
\]

\[
C_8 = M_4 M_2 \exp(M_2 \|B\|_\infty) [(M_5 + M_3 M_2) \|B\|_\infty + M_5 M_2].
\]

4 Convergence of the Scheme: Proof of Theorem 2.2

We are now ready to prove the main result of this article, which gives a theoretical bound on the speed of convergence for \( X^n \) defined in (22). Remember that the constants \( M_i, i \in \{1, \ldots, 6\} \), are given in Remark 2.1.

Proof. By (4) and (22), we have, for \( t \in [0, T] \),

\[
|X_t - X^n_t| \leq H_1(t) + H_2(t) + H_3(t),
\]

where

\[
H_1(t) = |\phi(Y_t, B_t) - \phi^n(Y^n_t, B_t)|
\]

\[
H_2(t) = |\phi^n(Y^n_t, B_t) - \phi^n(Y^{n,n}_t, B_t)|
\]

\[
H_3(t) = |\phi^n(Y^{n,n}_t, B_t) - \Psi^n(Y^{n,n}_t, B_t)|.
\]

Now we proceed to obtain estimates of \( H_1, H_2 \) and \( H_3 \). By property (6), we get
\[ H_1(t) \leq |\phi(Y_t, B_t) - \phi(Y^n_t, B_t)| + |\phi(Y^n_t, B_t) - \phi^n(Y^n_t, B_t)| \]
\[ \leq \exp(M_2\|B\|_\infty) |Y_t - Y^n_t| + |\phi(Y^n_t, B_t) - \phi^n(Y^n_t, B_t)|. \]

Therefore, by Lemmas 3.1 and 3.3

\[ H_1(t) \leq \exp(M_2\|B\|_\infty) \exp(C_1T) \frac{C_2}{n^2} + \frac{M_2^2M_5\|B\|_\infty^3}{6n^2} \exp(M_2\|B\|_\infty). \quad (26) \]

Also Lemmas 3.4 and 3.7 yield

\[ H_2(t) \leq \exp(2M_2\|B\|_\infty)|Y^n_t - Y^n_{t,n}| \]
\[ \leq \exp(2M_2\|B\|_\infty)C_5T \left(\frac{T}{n}\right)^{2(H-\rho)} \exp(C_7T). \quad (27) \]

For \( H_3 \), we use Lemma 3.2 So

\[ H_3(t) \leq \frac{M_3M_2^2\|B\|_\infty^3}{6n^2} \exp(2M_2\|B\|_\infty). \quad (28) \]

Finally, from (26) to (28), we have

\[ |X_t - X^n_t| \leq Cn^{-2(H-\rho)}, \]

which shows that Theorem 2.2 holds. \( \square \)

5 Proofs of preliminary lemmas

Here, we provide the proofs of Lemmas 3.1 to 3.7. First, we will prove, by induction, that the statements of Lemmas 3.1 to 3.4 hold for all \( k = 1, 2, \ldots, l \) and \( u \in (u_{k-1}^l, u_k^l) \). We will consider for simplicity the case \( u > 0 \), the other case can be treated similarly.

**Proof of Lemma 3.1**

Proof. Let \( z \in [-M, M] \). We will prove by induction that, for all \( k \in \{1, \ldots, l\} \) and \( u \in (u_{k-1}^l, u_k^l) \), we have

\[ |\phi(z, u) - \phi^l(z, u)| \leq \frac{M_2^2M_5\|B\|_\infty^3}{6l^3} \tilde{C}_k, \quad (29) \]

where \( \tilde{C}_k = \exp(M_2(u_k^l - u_0^l)) + \ldots + \exp(M_2(u_{k-1}^l - u_{k-1}^l)) \). As a consequence we obtain the global bound

\[ |\phi(z, u) - \phi^l(z, u)| \leq \frac{M_2^2M_5\|B\|_\infty^3}{6l^2} \exp(M_2\|B\|_\infty), \quad (30) \]

where \( M_2 \) and \( M_5 \) are constants independent of \( k \) and they are given in Remark 2.1.

First for \( k = 1 \), let \( 0 = u_0^l < u \leq u_1^l \), then [3], [11], the Lipschitz condition on \( \sigma \) (with constant \( M_2 \)) and the fact that \( \phi(z, u_0^l) = \phi^l(z, u_0^l) = z \) imply

\[ |\phi(z, u) - \phi^l(z, u)| \leq M_2 \int_{u_0^l}^u |\phi(z, s) - z - (s - u_0^l)\sigma(z)| \, ds \]
\[ \leq M_2 \int_{u_0^l}^u |\phi(z, s) - \phi^l(z, s)| \, ds + M_2 \int_{u_0^l}^u |\phi^l(z, s) - z - (s - u_0^l)\sigma(z)| \, ds \]
\[ = M_2 \int_{u_0^l}^u |\phi(z, s) - \phi^l(z, s)| \, ds + M_2 \Omega_0^l. \quad (31) \]
Next, we bound the term $I_0^i$,

$$I_0^i = \int_{u_0^i}^u |\phi^i(z, s) - z - (s - u_0^i)\sigma(z)| \, ds = \int_{u_0^i}^u \left| \phi^i(z, s) - z - \int_{u_0^i}^s \sigma(z) \, dr \right| \, ds.$$ 

From (11), the Lipschitz condition and the bound on $\sigma$, we get

$$I_0^i = \int_{u_0^i}^u \left| \int_{u_0^i}^s \sigma(z + (r - u_0^i)\sigma(z)) \, dr - \int_{u_0^i}^s \sigma(z) \, dr \right| \, ds \leq \int_{u_0^i}^u \left| \sigma(z + (r - u_0^i)\sigma(z)) - \sigma(z) \right| \, dr \, ds \leq M_2 \int_{u_0^i}^u \left( r - u_0^i \right) |\sigma(z)| \, dr \, ds \leq \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3}. \quad (32)$$

Therefore by (11), (32) and the Gronwall lemma we obtain

$$|\phi(z) - \phi^i(z, u)| \leq \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3} \exp \left( M_2 (u_i^i - u_0^i) \right), \quad \text{for } u \in (0, u_i^i].$$

Now, consider an index $k \in \{2, \ldots, l\}$. Our induction assumption is that (29) is true for $u \in (u_{k-1}^i, u_k^i]$. We shall now propagate the induction, that is prove that the inequality is also true for its successor, $k+1$. We will thus study (29) for $u \in (u_k^i, u_{k+1}^i]$. Following (8), (11) and our induction hypothesis we establish

$$|\phi(z, u) - \phi^i(z, u)| \leq \left| \phi(z, u_k^i) - \phi^i(z, u_k^i) \right| + \int_{u_k^i}^u \left| \sigma(\phi(z, s)) - \sigma(\phi^i(z, u_k^i) + (s - u_k^i)\sigma(\phi^i(z, u_k^i))) \right| \, ds,$$

$$\leq \tilde{C}_k \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3} + \int_{u_k^i}^u \left| \sigma(\phi(z, s)) - \sigma(\phi^i(z, u_k^i) + (s - u_k^i)\sigma(\phi^i(z, u_k^i))) \right| \, ds. \quad (33)$$

From Lipschitz condition on $\sigma$,

$$|\phi(z, u) - \phi^i(z, u)| \leq \tilde{C}_k \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3} + M_2 \int_{u_k^i}^u |\phi(z, s) - \phi^i(z, s)| \, ds + M_2 \int_{u_k^i}^u |\phi(z, s) - \phi^i(z, s)| \, ds + M_2 I_k^i, \quad (33)$$

where $\tilde{C}_k = \exp \left( M_2 (u_k^i - u_{k-1}^i) \right) + \ldots + \exp \left( M_2 (u_k^i - u_{k-1}^i) \right)$.

Now, we analyze the term $I_k^i$, given in equation (33). From (11), the Lipschitz condition and the bound on $\sigma$ we obtain

$$I_k^i \leq \int_{u_k^i}^u \int_{u_k^i}^s \left| \sigma(\phi(z, u_k^i) + (r - u_k^i)\sigma(\phi(z, u_k^i))) - \sigma(\phi^i(z, u_k^i)) \right| \, dr \, ds \leq \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3}. \quad (34)$$

Therefore inequalities (33) and (34) yield

$$|\phi(z, u) - \phi^i(z, u)| \leq \tilde{C}_k \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3} + \frac{M_2 M_5 \|B\|_\infty^3}{6 l^3} + M_2 \int_{u_k^i}^u |\phi(z, s) - \phi^i(z, s)| \, ds.$$
Thus, the Gronwall lemma allows us to establish

\[ |\phi(z, u) - \phi'(z, u)| \leq (\tilde{C}_k + 1) \left( \frac{M_2^3 M_5^3 B}{6l^3} \right) \exp(M_2(u_k^{i+1} - u_k^i)) \]

\[ = (\exp(M_2(u_k^i - u_0^i)) + \ldots + \exp(M_2(u_k^i - u_{k-1}^i))) + 1 \exp(M_2(u_k^i - u_k^i) + \frac{M_2^3 M_5^3 B}{6l^3} \exp(M_2 B^{3\infty}) \big) \]

\[ = \tilde{C}_k + 1 + \frac{M_2^3 M_5^3 B}{6l^3}, \]

which shows that (29) is satisfied for \( k + 1 \).

Now, we prove that (29) is true. For all \((z, u) \in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty]\), there is some \( k \in \{1, \ldots, l\} \) such that \( u_k^i < u \leq u_k^i \) and by the previous calculations

\[ |\phi(z, u) - \phi'(z, u)| \leq \frac{M_2^3 M_5^3 B}{6l^3} \exp(M_2 B^{3\infty}), \]

proving the lemma. \( \Box \)

**Proof of Lemma 3.2**

**Proof.** As in the proof of Lemma 3.1 we will prove by induction that, for all \( k \in \{1, \ldots, l\} \) and \( u \in (u_k^{i-1}, u_k^i) \), we have

\[ |\phi'(z, u) - \Psi'(z, u)| \leq \frac{M_1^3 M_2^3 B^{3\infty}}{6l^3} k (1 + A_1)^k, \quad (35) \]

with \( A_1 = 1 + k \left( \frac{M_4^3 \|B\|_\infty}{k^3} \right)^2 \). Hence,

\[ |\phi'(z, u) - \Psi'(z, u)| \leq \frac{M_3^3 M_4^3 [\|B\|_{\infty}^2]}{6l^3} \exp(2M_2 \|B\|_{\infty}). \quad (36) \]

where \( M_2, M_3 \) and \( M_4 \) are constants independent on \( k \) and are given in Remark 2.1.

We first assume that \( k = 1 \). If \( 0 \leq u_0^i < u \leq u_0^i \) and from equalities (11) to (15) we obtain that

\[ |\phi'(z, u) - \Psi'(z, u)| \leq \int_{u_0^i}^u \left[ (\phi'(z, u_0^i) + (s - u_0^i) \sigma (\phi'(z, u_0^i))) \right] \]

\[ = \left[ \sigma (\Psi'(z, u_0^i)) + \sigma' (\Psi'(z, u_0^i)) (s - u_0^i) \right] ds \]

\[ = \int_{u_0^i}^u \left( \sigma (z + (s - u_0^i) \sigma (z)) - \sigma (z) - \sigma'(z) (s - u_0^i) \sigma (z)) \right) ds. \]

By Taylor’s theorem there exists a point \( \theta \in (\inf \{z, z + (s - u_0^i) \sigma (z)\}, \sup \{z, z + (s - u_0^i) \sigma (z)\}) \) such that

\[ |\phi'(z, u) - \Psi'(z, u)| \leq \int_{u_0^i}^u \left( \frac{\sigma''(\theta)}{2} (s - u_0^i)^2 |\sigma(z)|^2 \right) ds \]

\[ \leq \frac{M_3^3 M_4^3 \|B\|_{\infty}^2}{6l^3} 1 (1 + A_1)^1. \]

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Thus (35) holds for any \( u \in (u_{k-1}, u_k) \). We will thus study (35) for \( u \in (u_k, u_{k+1}) \). Following equations (11) to (15) and our induction hypothesis, we get

\[
|\phi^l(z, u) - \Psi^l(z, u)| \leq |\phi^l(z, u_k) - \Psi^l(z, u_k)| + \int_{u_k}^{u} |\phi^l(z, u) - \phi^l(z, u_k) + (s - u_k)\phi^l(z, u_k)|\,ds \\
- [\sigma(\Psi^l(z, u_k)) + \sigma'(\Psi^l(z, u_k)) (s - u_k)\sigma(\Psi^l(z, u_k))] \leq \frac{M_3M_5^3\|B\|_3^3}{6l^3} k(1 + A_l)^k \\
+ \int_{u_k}^{u} |\sigma(\phi^l(z, u) - \phi^l(z, u_k) + (s - u_k)\phi^l(z, u_k)) - \sigma[\Psi^l(z, u_k) + (s - u_k)\sigma(\Psi^l(z, u_k))]|\,ds \\
+ \int_{u_k}^{u} |\sigma(\Psi^l(z, u_k) + (s - u_k)\sigma(\Psi^l(z, u_k))) - \sigma(\Psi^l(z, u_k)) + \sigma'(\Psi^l(z, u_k)) (s - u_k)|\,ds \\
= \frac{M_3M_5^3\|B\|_3^3}{6l^3} k(1 + A_l)^k + J_1^k + J_2^k.
\]

From the Lipschitz condition on \( \sigma \), and our induction hypothesis

\[
J_1^k \leq M_2 \int_{u_k}^{u} |\phi^l(z, u_k) - \Psi^l(z, u_k)|\,ds + M_2 \int_{u_k}^{u} (s - u_k) |\phi^l(z, u_k) - \Psi^l(z, u_k)|\,ds \\
\leq \frac{M_2M_3M_5^3\|B\|_3^3}{6l^3} k(1 + A_l)^k (u - u_k) + \frac{M_2^2M_3M_5^3\|B\|_3^3}{12l^3} k(1 + A_l)^k (u - u_k)^2.
\]

By Taylor’s theorem there exists a point \( \theta_k \in (\inf\{\Psi^l(z, u_k), \Psi^l(z, u_k) + (s - u_k)\sigma(\Psi^l(z, u_k))\}, \sup\{\Psi^l(z, u_k), \Psi^l(z, u_k) + (s - u_k)\sigma(\Psi^l(z, u_k))\}) \) such that

\[
J_2^k \leq \int_{u_k}^{u} \frac{\sigma''(\theta_k)}{2} |\sigma(\Psi^l(z, u_k))|^2 (s - u_k)^2\,ds \leq \frac{M_3M_5^2}{6} (u - u_k)^3.
\]

Therefore

\[
|\phi^l(z, u) - \Psi^l(z, u)| \leq \frac{M_3M_5^3\|B\|_3^3}{6l^3} k(1 + A_l)^k + \frac{M_2M_3M_5^3\|B\|_3^3}{6l^3} k(1 + A_l)^k (u - u_k) \\
+ \frac{M_2^2M_3M_5^3\|B\|_3^3}{12l^3} k(1 + A_l)^k (u - u_k)^2 + \frac{M_2M_3^2}{6} (u - u_k)^3.
\]

Since \((u - u_k)^k \leq \frac{\|B\|_3^3}{l} \) we obtain

\[
|\phi^l(z, u) - \Psi^l(z, u)| \leq \frac{M_3M_5^3\|B\|_3^3}{6l^3} \left[ k(1 + A_l)^k + \frac{M_2\|B\|_3^3}{l} k(1 + A_l)^k + \frac{M_2\|B\|_3^3}{2l^2} k(1 + A_l)^k + 1 \right].
\]

Since \( 1 < (1 + A_l)^{k+1} \), then

\[
|\phi^l(z, u) - \Psi^l(z, u)| \leq \frac{M_3M_5^3\|B\|_3^3}{6l^3} \left[ k(1 + A_l)^{k+1} + 1 \right] \\
\leq \frac{M_3M_5^3\|B\|_3^3}{6l^3} \left[ k(1 + A_l)^{k+1} + (1 + A_l)^{k+1} \right] = \frac{M_3M_5^3\|B\|_3^3}{6l^3} (k + 1)(1 + A_l)^{k+1}.
\]

Thus (35) holds for any \( k \in \{1, \ldots, l\} \).
Finally, we see that (36) is satisfied. For all \((z, u) \in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty]\), there is some \(k \in \{1, \ldots, l\}\) such that \(u^j_{k-1} < u \leq u^j_k\) and by (35),

\[
|\phi(z, u) - \Psi^j(z, u)| \leq \frac{M_3 M_5^2 \|B\|_\infty^3}{6l^3} k (1 + A_1)^k \\
\leq \frac{M_3 M_5^2 \|B\|_\infty^3}{6l^2} (1 + A_1)^k \\
\leq \frac{M_3 M_5^2 \|B\|_\infty^3}{6l^2} \exp(2M_2 \|B\|_\infty).
\]

Thus, the proof is complete. \(\square\)

**Proof of Lemma 3.3**

*Proof*. We will prove by induction that, for all \(k \in \{1, 2, \ldots, l\}\) and \(u \in (u^j_{k-1}, u^j_k]\), we have

\[
|\Psi^j(z_1, u) - \Psi^j(z_2, u)| \leq |z_1 - z_2| \prod_{j=1}^k \left[ 1 + M_2 (u^j_j - u^j_{j-1}) + [M_2^2 + M_3 M_5] \frac{(u^j_j - u^j_{j-1})^2}{2} \right].
\]

(37)

Furthermore, for all \(k \in \mathbb{N}\) we have obtained a global bound

\[
|\Psi^j(z_1, u) - \Psi^j(z_2, u)| \leq |z_1 - z_2| \exp(2M_2 \|B\|_\infty).
\]

In a similar way as in previous lemmas, if \(0 = u^j_0 < u \leq u^j_1\), then by equation (15) and the fact that \(\Psi^j(z, u^j_0) = z\), we have

\[
|\Psi^j(z_1, u) - \Psi^j(z_2, u)| \leq |z_1 - z_2| \left[ 1 + M_2 (u^j_1 - u^j_0) + (M_2^2 + M_3 M_5) \frac{(u^j_1 - u^j_0)^2}{2} \right].
\]

Then for \(k = 1\) (37) is satisfied. Now, consider that (37) is true for \(k\). Then, we will prove that the inequality is true for its successor, \(k + 1\). For that, we will study (37) for \(u \in (u^j_k, u^j_{k+1}]\).

Following (16), Lipschitz condition and hypothesis on the second derivative of \(\sigma\), we have

\[
|\Psi^j(z_1, u) - \Psi^j(z_2, u)| \\
\leq |\Psi^j(z_1, u^j_k) - \Psi^j(z_2, u^j_k)| + M_2 \int_{u^j_k}^u |\Psi^j(z_1, u^j_k) - \Psi^j(z_2, u^j_k)| \, ds \\
+ (M_2^2 + M_3 M_5) \int_{u^j_k}^u |\Psi^j(z_1, u^j_k) - \Psi^j(z_2, u^j_k)| (s - u^j_k) \, ds \\
= |\Psi^j(z_1, u^j_k) - \Psi^j(z_2, u^j_k)| \left( 1 + M_2 (u^j_k - u^j_k) + (M_2^2 + M_3 M_5) \frac{(u^j_k - u^j_k)^2}{2} \right).
\]

Consequently, from our induction hypothesis, we get

\[
|\Psi^j(z_1, u) - \Psi^j(z_2, u)| \leq |z_1 - z_2| \prod_{j=1}^{k+1} \left[ 1 + M_2 (u^j_j - u^j_{j-1}) + [M_2^2 + M_3 M_5] \frac{(u^j_j - u^j_{j-1})^2}{2} \right] \\
\times \left( 1 + M_2 (u^j_{k+1} - u^j_k) + [M_2^2 + M_3 M_5] \frac{(u^j_{k+1} - u^j_k)^2}{2} \right) \\
= |z_1 - z_2| \prod_{j=1}^{k+1} \left( 1 + M_2 (u^j_j - u^j_{j-1}) + [M_2^2 + M_3 M_5] \frac{(u^j_j - u^j_{j-1})^2}{2} \right),
\]

which implies that (37) is satisfied.
Now, for all \((z, u) \in [-M, M] \times [-\|B\|_\infty, \|B\|_\infty]\), there is some \(k \in \{1, \ldots, l\}\) such that \(u_{k-1}^l < u \leq u_k^l\) and from (37)

\[
|\Psi^l(z_1, u) - \Psi^l(z_2, u)| \leq |z_1 - z_2| \prod_{j=1}^k \left(1 + M_2(u_j^l - u_{j-1}^l) + M_2^2 M_3 \frac{(u_j^l - u_{j-1}^l)^2}{2}\right)
\]

\[
\leq |z_1 - z_2| \left[1 + \frac{2M_2\|B\|_\infty}{l}\right]
\]

where the last inequality is due to the fact that for \(l\) large enough \(\frac{M_2\|B\|_\infty + M_3 M_5\|B\|_\infty / M_2}{2l} < 1\) and \(1 + \frac{2M_2\|B\|_\infty}{l} < \exp(2M_2\|B\|_\infty)\). Thus the proof is complete.

\[\square\]

**Proof of Lemma 3.4**

**Proof.** We will prove by induction that, for all \(k \in \{1, \ldots, l\}\) and \(u \in (u_{k-1}^l, u_k^l)\), we have

\[
|\phi^l(z_1, u) - \phi^l(z_2, u)| \leq |z_1 - z_2| \prod_{j=1}^k \left[1 + M_2(u_j^l - u_{j-1}^l) + M_2^2 \frac{(u_j^l - u_{j-1}^l)^2}{2}\right]. \tag{38}
\]

As a consequence, for all \(k \in \mathbb{N}\),

\[
|\phi^l(z_1, u) - \phi^l(z_2, u)| \leq |z_1 - z_2| \exp(2M_2\|B\|_\infty),
\]

is true.

If \(0 = u_0 < u \leq u_1\), then by equations (10) and (11) and the fact that \(\phi^l(z, u_0) = z\), we have

\[
|\phi^l(z_1, u) - \phi^l(z_2, u)| \leq |z_1 - z_2| \left[1 + M_2(u_1^l - u_0^l) + M_2^2 \frac{(u_1^l - u_0^l)^2}{2}\right].
\]

Therefore for \(k = 1\) (38) is satisfied. Now, let (38) be true until \(k\). Therefore, it remains to prove that this inequality is true for its successor, \(k + 1\). For that, we choose \(u \in (u_k^l, u_{k+1}^l)\).

Using (11) and Lipschitz condition on \(\sigma\) again, we can write

\[
|\phi^l(z_1, u_k) - \phi^l(z_2, u_k)|
\]

\[
\leq |\phi^l(z_1, u_k) - \phi^l(z_2, u_k)| + M_2 \int_{u_k}^u |\phi^l(z_1, u_k)| - |\phi^l(z_2, u_k)| |ds
\]

\[
+ M_2^2 \int_{u_k}^u |\phi^l(z_1, u_k) - \phi^l(z_2, u_k)| (s - u_k) |ds
\]

\[
= |\phi^l(z_1, u_k) - \phi^l(z_2, u_k)| \left(1 + M_2(u - u_k^l) + M_2^2 \frac{(u - u_k^l)^2}{2}\right).
\]

Our induction hypothesis leads us to

\[
|\phi^l(z_1, u) - \phi^l(z_2, u)| \leq |z_1 - z_2| \prod_{j=1}^{k+1} \left[1 + M_2(u_j^l - u_{j-1}^l) + M_2^2 \frac{(u_j^l - u_{j-1}^l)^2}{2}\right].
\]

Therefore, (38) for any \(k \in \{1, \ldots, l\}\).
Now, for all \( u \in [-\|B\|_{\infty}, \|B\|_{\infty}] \), there is some \( k \in \{1, \ldots, l\} \) such that \( u_{k-1}^l < u \leq u_k^l \) and, by (38),
\[
|\phi^l(z_1, u) - \phi^l(z_2, u)| \leq |z_1 - z_2| \prod_{j=1}^{k} \left( 1 + M_2(u_j^l - u_{j-1}^l) + M_2^2 \frac{(u_j^l - u_{j-1}^l)^2}{2} \right)
\]
\[
\leq |z_1 - z_2| \left[ 1 + \frac{2M_2 \|B\|_{\infty}}{l} \right]^l \leq |z_1 - z_2| \exp(2M_2 \|B\|_{\infty}),
\]
where the last inequality is due by the fact that for \( l \) large enough \( \frac{M_2}{l} < 1 \) and \( \left( 1 + \frac{2M_2 \|B\|_{\infty}}{l} \right)^l < \exp(2M_2 \|B\|_{\infty}) \). Therefore (23) is satisfied and the proof is complete. \( \square \)

**Proof of Lemma 3.5**

*Proof.* By equations (7) and (16), we have, for \( t \in [0, T] \),
\[
|Y_t - Y^n_t| \leq \int_0^t \left| \exp \left( - \int_0^{B_s} \sigma'(\phi(Y_s, u)) \, du \right) b(\phi(Y_s, B_s)) \right| ds
\]
\[
- \exp \left( - \int_0^{B_s} \sigma'(\Psi^n(Y^n_s, u)) \, du \right) b(\Psi^n(Y^n_s, B_s)) \right| ds
\]
\[
\leq K_1 + K_2,
\]
with
\[
K_1 = \int_0^t \left| \exp \left( - \int_0^{B_s} \sigma'(\phi(Y_s, u)) \, du \right) b(\phi(Y_s, B_s)) - b(\Psi^n(Y^n_s, B_s)) \right| ds,
\]
and
\[
K_2 = \int_0^t \left| \exp \left( - \int_0^{B_s} \sigma'(\phi(Y_s, u)) \, du \right) - \exp \left( - \int_0^{B_s} \sigma'(\Psi^n(Y^n_s, u)) \, du \right) \right| b(\Psi^n(Y^n_s, B_s)) \right| ds.
\]
Therefore by (19), the Lipschitz properties on \( b \) and \( \sigma \), and Lemmas 3.1 and 3.2, we obtain
\[
K_1 \leq M_4 \exp(M_2 \|B\|_{\infty}) \int_0^t \left| \phi(Y_s, B_s) - \Psi^n(Y^n_s, B_s) \right| ds
\]
\[
\leq M_4 \exp(M_2 \|B\|_{\infty}) \left( \int_0^t \left| \phi(Y_s, B_s) - \phi(Y^n_s, B_s) \right| ds + \int_0^t \left| \phi(Y^n_s, B_s) - \phi^n(Y^n_s, B_s) \right| ds \right.
\]
\[
+ \int_0^t \left| \phi^n(Y^n_s, B_s) - \Psi^n(Y^n_s, B_s) \right| ds \right)
\]
\[
\leq M_4 \exp(M_2 \|B\|_{\infty}) \left( \int_0^t \exp(M_2 \|B\|_{\infty})|Y_s - Y^n_s| ds \right.
\]
\[
+ \frac{TM_2^2M_5^3 \|B\|_{\infty}^3}{6n^2} \exp(M_2 \|B\|_{\infty}) + \frac{TM_3M_5^2 \|B\|_{\infty}^3}{6n^2} \exp(2M_2 \|B\|_{\infty}) \right).
\]
Now, by the mean value theorem, we get
\[
K_2 \leq M_1 M_3 \exp(M_2 \|B\|_{\infty}) \int_0^t \int_0^{[B_s]} \left| \phi(Y_s, u) - \Psi^n(Y^n_s, u) \right| du \, ds.
\]
Hence, proceeding as in \( K_1 \), we obtain
\[
K_2 \leq M_1 M_3 \exp(M_2 \|B\|_{\infty}) \|B\|_{\infty} \left( \int_0^t \exp(M_2 \|B\|_{\infty})|Y_s - Y^n_s| ds \right.
\]
\[
+ \frac{TM_2^2M_5^3 \|B\|_{\infty}^3}{6n^2} \exp(M_2 \|B\|_{\infty}) + \frac{TM_3M_5^2 \|B\|_{\infty}^3}{6n^2} \exp(2M_2 \|B\|_{\infty}) \right).
\]
Finally, the desired result is achieved by direct application of the Gronwall lemma.

**Proof of Lemma 3.6**

Proof. Recall that $h_1^n(z, u) = \frac{\partial g^n(z, u)}{\partial z}$. Then, by equations (10) to (18) we obtain that

$$|Y_t^n - Y_t^{n,n}| \leq \int_{t_k^n}^{t} \left| g^n(B_{t_k^n}, Y_{t_k^n}^n, n, n) \right| \left( B_u - B_{t_k^n} \right) \, du$$

$$\leq M_1 \exp(M_2\|B\|_\infty) (s - t_k^n) + \left| h_1^n(B_{t_k^n}, Y_{t_k^n}^n) \right| \int_{t_k^n}^{s} \left| B_u - B_{t_k^n} \right| \, du$$

$$\leq M_1 \exp(M_2\|B\|_\infty) (s - t_k^n) + \|B\|_{H-\rho} \exp(C_3(s - t_k^n)^{1+H-\rho})$$

$$\leq C_4(s - t_k^n),$$

where

$$|h_1^n(B_{t_k^n}, Y_{t_k^n}^n)| \leq M_1 M_2 \exp(M_2\|B\|_\infty) + M_4 \exp(M_2\|B\|_\infty) M_5 \|B\|_\infty (1 + M_2) = C_3,$$

and

$$C_4 = M_1 \exp(M_2\|B\|_\infty) + C_3 T^{H-\rho} \|B\|_{H-\rho}.$$}

The specific computation of the bound of the term $h_1^n(z, u)$ is left in the Appendix (Section 4).

**Proof of Lemma 3.7**

Proof. Let $n \in \mathbb{N}$ be fixed. We will prove Lemma 3.7 by induction on $k$ again. That is, for every $k \in \{1, \ldots, n\}$ and $t \in (t_{k-1}, t_k)$, we have

$$|Y_t^n - Y_t^{n,n}| \leq C_0 \sum_{j=1}^{k} (t_j^n - t_{j-1}^n)^{1+2(H-\rho)} \exp(C_7(t_j^n - t_{j-1}^n)),$$

(39)

here $0 < \rho < H$. As a consequence, for all $k \in \{1, \ldots, n\}$ we obtained the global bound

$$|Y_t^n - Y_t^{n,n}| \leq C_0 T^{1+2(H-\rho)} \exp(C_7 T) \frac{1}{n^{2(H-\rho)}},$$

where $C_0$ and $C_7$ are given in (24) and (29), respectively.

First for $k = 1$ and $t \in (t_0^n, t_1^n)$, equations (10) to (18) imply

$$|Y_t^n - Y_t^{n,n}| \leq \int_{t_0^n}^{t} \left[ g^n(B_s, Y_s^n) - g^n(B_{t_0^n}, x) + h_1^n(B_{t_0^n}, x)(B_s - B_{t_0^n}) \right] \, ds$$

$$\leq F_1 + F_2 + F_3,$$

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where

\[ F_1 = \int_{t_0}^{t} |g^n(B_s, Y^n_s) - g^n(B_s, Y^{n,n}_s)| \, ds \]

\[ F_2 = \int_{t_0}^{t} |g^n(B_s, Y^{n,n}_s) - g^n(B_s, Y^{n,n}_{t_0})| \, ds \]

and

\[ F_3 = \int_{t_0}^{t} \left| g^n(B_s, Y^{n,n}_{t_0}) - \left[ g^n(B_{t_0}^{n,n}, Y^{n,n}_{t_0}) + h^n_s(B_{t_0}^{n,n}, Y^{n,n}_{t_0})(B_s - B_{t_0}^{n,n}) \right] \right| \, ds . \]

Equality (17) and the triangle inequality allow us to write

\[ F_1 = \int_{t_0}^{t} \exp \left( - \int_{0}^{B_s} \sigma'(\Psi^n(Y^n, r)) \, dr \right) b(\Psi^n(Y^n, B_s)) \]

\[ - \exp \left( - \int_{0}^{B_s} \sigma'(\Psi^n(Y^{n,n}, r)) \, dr \right) b(\Psi^n(Y^{n,n}, B_s)) \, ds \]

\[ \leq \int_{t_0}^{t} \exp \left( - \int_{0}^{B_s} \sigma'(\Psi^n(Y^n, r)) \, dr \right) \left| b(\Psi^n(Y^n, B_s)) - b(\Psi^n(Y^{n,n}, B_s)) \right| \, ds \]

\[ + \int_{t_0}^{t} \exp \left( - \int_{0}^{B_s} \sigma'(\Psi^n(Y^n, r)) \, dr \right) - \exp \left( - \int_{0}^{B_s} \sigma'(\Psi^n(Y^{n,n}, r)) \, dr \right) \]

\[ \times \left| b(\Psi^n(Y^{n,n}, B_s)) \right| \, ds . \]

Therefore, the Lipschitz property on \( b \) and the mean value theorem yield

\[ F_1 \leq M_4 \exp(M_2\|B\|_\infty) \int_{t_0}^{t} |\Psi^n(Y^n, B_s) - \Psi^n(Y^{n,n}, B_s)| \, ds , \]

\[ + M_1 M_3 \exp(M_2\|B\|_\infty) \int_{0}^{B_s} |\Psi^n(Y^n, r) - \Psi^n(Y^{n,n}, r)| \, dr \, ds . \]

Consequently, Lemma 5.3 lead us to

\[ F_1 \leq M_4 \exp(3M_2\|B\|_\infty) \int_{t_0}^{t} |Y^n_s - Y^{n,n}_s| \, ds \]

\[ + M_1 M_3 \exp(3M_2\|B\|_\infty)\|B\|_\infty \int_{t_0}^{t} |Y^n_s - Y^{n,n}_s| \, ds \]

\[ = \exp(3M_2\|B\|_\infty) [M_4 + M_1 M_3\|B\|_\infty] \int_{t_0}^{t} |Y^n_s - Y^{n,n}_s| \, ds . \]

Proceeding similarly as in \( F_1 \),

\[ F_2 \leq M_4 \exp(M_2\|B\|_\infty) \int_{t_0}^{t} \left| \Psi^n(Y^{n,n}, B_s) - \Psi^n(Y^{n,n}_{t_0}, B_s) \right| \, ds , \]

\[ + M_1 M_3 \exp(M_2\|B\|_\infty) \int_{t_0}^{t} \left| \Psi^n(Y^{n,n}, r) - \Psi^n(Y^{n,n}_{t_0}, r) \right| \, dr \, ds \]

\[ \leq \exp(3M_2\|B\|_\infty) [M_4 + M_1 M_3\|B\|_\infty] \int_{t_0}^{t} |Y^{n,n}_s - Y^{n,n}_{t_0}| \, ds . \]

Hence, using Lemma 3.6 we can establish

\[ F_2 \leq C_4 \exp(3M_2\|B\|_\infty) [M_4 + M_1 M_3\|B\|_\infty] \frac{(t - t_0)^2}{2} . \]
Now, we deal with $F_3$. From Lemma 6.1 (Section 6),

$$F_3 \leq C_5 \int_{t_0}^{t} (B_s - B_{t_0})^2 ds + \int_{t_0}^{t} \left| \sum_{j=1}^{n} 1_{\{B_s \in (u_{j-1}, u_{j})\}} \sum_{k=1}^{j} (B_s - u_k^n) \Delta_{j+k} g^{n'} \right| ds$$

$$+ \int_{t_0}^{t} \left| \sum_{j=-n+1}^{0} 1_{\{B_s \in (u_{j-1}, u_{j})\}} \sum_{k=-j}^{0} (B_s - u_k^n) \Delta_{j+k} g^{n'} \right| ds,$$  \hspace{1cm} (40)

where

$$\Delta_{j+k} g^{n'} = \frac{\partial g^n(u_{j+k}^+, Y_{t_0}^{n,n})}{\partial u} - \frac{\partial g^n(u_{j+k}^-, Y_{t_0}^{n,n})}{\partial u}$$

and

$$C_5 = \exp(M_2 \|B\|_{\infty}) \left[ \|B\|_{\infty}(1 + M_2) (M_1 M_1 M_5 + M_2 M_4 M_5 + M_6 M_5 \|B\|_{\infty}(1 + M_2)) + M_1 M_2 + M_4 M_5 (1 + M_2) \right].$$

Note that (19) implies

$$\left| \Delta_{j+k} g^{n'} \right| \leq M_4 \exp(M_2 \|B\|_{\infty}) \left| \frac{\partial \Psi^n(Y_{t_0}^{n,n}, u_{j+k}^+)}{\partial u} - \frac{\partial \Psi^n(Y_{t_0}^{n,n}, u_{j+k}^-)}{\partial u} \right|$$

$$= M_4 \exp(M_2 \|B\|_{\infty}) \left| \sigma \left( \Psi^n(Y_{t_0}^{n,n}, u_{j+k}) \right) - \sigma \left( \Psi^n(Y_{t_0}^{n,n}, u_{j+k-1}) \right) \right|$$

$$\leq M_4 M_2 \exp(M_2 \|B\|_{\infty}) \left| \Psi^n(Y_{t_0}^{n,n}, u_{j+k}) - \Psi^n(Y_{t_0}^{n,n}, u_{j+k-1}) \right|$$

$$+ M_5 M_2 (u_{j+k} - u_{j+k-1}) \left| (M_5 + M_5 M_2) \frac{(u_j - u_{j-1})^2}{2} \right|$$

$$+ M_5 M_2 (u_{j+k} - u_{j+k-1}) \left| (M_5 + M_5 M_2) \|B\|_{\infty} \right|$$

$$\leq M_4 M_2 \exp(M_2 \|B\|_{\infty}) (u_j - u_{j-1}) \left| (M_5 + M_5 M_2) \|B\|_{\infty} \right|$$

$$+ M_5 M_2$$

$$= C_8 (u_j - u_{j-1}).$$

Hence, (40) implies

$$F_3 \leq (C_5 + C_8) \int_{t_0}^{t} (B_s - B_{t_0})^2 ds \leq (C_5 + C_8) \|B\|_{H^{-\rho}} \frac{(t - t_0^n)^{1+2(H-\rho)}}{2}.$$

Since $H < 1/2$, then the previous estimations for $F_1, F_2$ and $F_3$ give

$$|Y^n_t - Y^{n,n}_t| \leq$$

$$\exp(3M_2 \|B\|_{\infty})[M_4 + M_1 M_3] \|B\|_{\infty} \int_{t_0}^{t} |Y^n_s - Y^{n,n}_s| ds$$

$$+ \left[ C_4 \exp(3M_2 \|B\|_{\infty})[M_4 + M_1 M_3] \|B\|_{\infty} \right] T^{1-2(H-\rho)}$$

$$+(C_5 + C_8) \|B\|_{H^{-\rho}} \frac{(t - t_0^n)^{1+2(H-\rho)}}{2}$$

$$= C_6 (t - t_0^n)^{1+2(H-\rho)} + C_7 \int_{t_0}^{t} |Y^n_s - Y^{n,n}_s| ds.$$

Then by the Gronwall lemma and $t \in (t_0^n, t_1^n]$, we conclude

$$|Y^n_t - Y^{n,n}_t| \leq C_6 \exp(C_7(t_1^n - t_0^n))(t_1^n - t_0^n)^{1+2(H-\rho)}.$$
Now we show that (39) is true for $k + 1$ if it holds for $k$. So we choose $t \in (t^n_k, t^n_{k+1}]$. Towards this end, we proceed as in the case $k = 1$:

$$
|Y^n_t - Y^n_{t_k}| \leq |Y^n_{t_k} - Y^n_{t^n_k}| + \left| \int_{t_k}^{t} \left( g^n (B_s, Y^n_s) - g^n (B^n_{t_k}, Y^n_{t^n_k}) - h^n_1 (B^n_{t_k}, Y^n_{t^n_k}) (B_s - B^n_{t_k}) \right) ds \right| \\
\leq C_6 \sum_{j=1}^{k} (t^n_j - t^n_{j-1})^{1+2(H-\rho)} \exp(C_7(t^n_{j-1} - t^n_j)) + C_6 (t^n_{k+1} - t^n_k)^{1+2(H-\rho)} \\
+ C_7 \int_{t^n_k}^{t} |Y^n_s - Y^n_{t^n_k}| ds.
$$

Therefore, using the Gronwall lemma again and $t \in (t^n_k, t^n_{k+1}]$

$$
|Y^n_t - Y^n_{t^n_k}| \leq \left[ C_6 \sum_{j=1}^{k} (t^n_j - t^n_{j-1})^{1+2(H-\rho)} \exp(C_7(t^n_{j-1} - t^n_j)) + C_6 (t^n_{k+1} - t^n_k)^{1+2(H-\rho)} \right] \exp(C_7(t^n_{k+1} - t^n_k))
$$

$$
= C_6 \sum_{j=1}^{k} (t^n_j - t^n_{j-1})^{1+2(H-\rho)} \exp(C_7(t^n_{j-1} - t^n_j)) + C_6 (t^n_{k+1} - t^n_k)^{1+2(H-\rho)} \exp(C_7(t^n_{k+1} - t^n_k))
$$

$$
= C_6 \sum_{j=1}^{k+1} (t^n_j - t^n_{j-1})^{1+2(H-\rho)} \exp(C_7(t^n_{j-1} - t^n_j)).
$$

Therefore (39) is true for any $k \leq n$.

Finally, for all $t \in [0, T]$, there exists $k \in \{1, \ldots, n\}$ such that $t^n_{k-1} < t \leq t^n_k$. Thus (39) implies

$$
|Y^n_t - Y^n_{t^n_k}| \leq C_6 \left( \frac{T}{n} \right)^{1+2(H-\rho)} k \exp(C_7(t^n_k - t^n_0))
$$

$$
\leq C_6 T \left( \frac{T}{n} \right)^{2(H-\rho)} \exp(C_7 T),
$$

and the proof is complete. \hfill \Box

6 Appendix

Here, we consider the following useful result for the analysis of the convergence of the scheme.

**Lemma 6.1.** Let $\{u^j_l\}$ be a partition of the interval $[-R, R]$ given by $-R = u^j_{-l} < \ldots < u^j_{-1} < u^j_0 = 0 < u^j_1 < \ldots < u^j_l = R$ and $f : [-R, R] \to \mathbb{R}$ a $C^2([u^j_l, u^j_{l+1}])$-function for each $j \in \{-l, \ldots, l-1\}$. Also let $f'$ be continuous on $[-R, R]$, $C$ a constant such that

$$
\sup_{j \in \{-l, \ldots, l-1\}} \| f'' \|_{\infty, [u^j_l, u^j_{l+1}]} = C < \infty,
$$

and $x \in (u^j_l, u^j_{l+1}]$ and $y \in (u^j_{j+k}, u^j_{j+k+1}]$. Then,

$$
|f(y) - f(x) - f'(x)(y - x)| \leq \frac{C}{2} (y - x)^2 + \sum_{p=1}^{k} \Delta_{j+p} f'(y - u_{j+p}), \quad (41)
$$

where

$$
\Delta_{j+p} f' = |f'(u_{j+p}+) - f'(u_{j+p}^-)|.
$$
Proof. We will prove that (41) holds via induction on $k$.

We start our induction with $k = 1$. That is, we consider two consecutive intervals. If $x \in (u_j, u_{j+1}]$ and $y \in (u_{j+1}, u_{j+2}]$. Then,

\[
|f(y) - f(x)| - f'(x+y)(y-x) |
\leq |f(y) - f(u_{j+1}) - f'(u_{j+1}+)(y-u_{j+1})|
+ |f(u_{j+1}) + f'(u_{j+1}+)(y-u_{j+1}) - f(x) - f'(x+y)(y-x)|
\leq \frac{C}{2}(y-u_{j+1})^2 + |f(u_{j+1}) - f(x) - f'(x+y)(u_{j+1} - x)|
+ |f'(u_{j+1}+ - f'(x+y)(y-u_{j+1})|
\leq \frac{C}{2}(y-u_{j+1})^2 + \frac{C}{2}(u_{j+1} - x)^2
+ |f'(u_{j+1}+ - f'(u_{j+1})| + |f'(u_{j+1} - f'(x+y)| (y-u_{j+1})
= \frac{C}{2}(y-u_{j+1})^2 + \frac{C}{2}(u_{j+1} - x)^2 + (y-u_{j+1})(\Delta_{j+1}f')
+ C(u_{j+1} - x)(y-u_{j+1})
= \frac{C}{2}(y-x)^2 + (\Delta_{j+1}f')(y-u_{j+1}).
\]

It means, (41) holds for $k = 1$.

It remains to prove that the inequality (41) is true for its successor, $k + 1$ assuming that until $k$ is satisfied. To do so, choose $x \in [u_j, u_{j+1}]$ and $y \in [u_{j+k+1}, u_{j+k+2}]$. Then,

\[
|f(y) - f(x)| - f'(x+y)(y-x) |
\leq |f(y) - f(u_{j+k+1}) - f'(u_{j+k+1}+)(y-u_{j+k+1})|
+ |f(u_{j+k+1} + f'(u_{j+k+1}+)(y-u_{j+k+1}) - f(x) - f'(x+y)(y-x)|
\leq \frac{C}{2}(y-u_{j+k+1})^2 + |f(u_{j+k+1}) - f(x) - f'(x+y)(u_{j+k+1} - x)|
+ |f'(u_{j+k+1}+ - f'(x+y)(y-u_{j+k+1})|
\leq \frac{C}{2}(y-u_{j+k+1})^2 + \frac{C}{2}(u_{j+k+1} - x)^2
+ \sum_{p=1}^{k}(u_{j+k+1} - u_{j+p})\Delta_{j+p}f' + C(u_{j+k+1} - u_{j+k})
+ \sum_{p=1}^{k}\Delta_{j+p}f' + C(u_{j+k} - x) (y-u_{j+k+1})
\leq \frac{C}{2}(y-x)^2 + \sum_{p=1}^{k}(y-u_{j+p})\Delta_{j+p}f' + (y-u_{j+k+1})\Delta_{j+k+1}f'.
\]

Therefore, (41) is satisfied for $k + 1$ and the proof is complete. 

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References

[1] E. Alòs, J.A. León, and D. Nualart. Stochastic Stratonovich calculus fBm for fractional Brownian motion with Hurst parameter less than 1/2. *Taiwanese J. Math.*, 5(3):609–632, 2001.

[2] E. Alòs, J.A. León, and J. Vives. On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility. *Finance and Stochastics*, 11(4):571–589, 2007.

[3] E. Alòs and D. Nualart. Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastic Reports*, 75(3):129–152, 2003.

[4] H. Araya, J.A. León, and S. Torres. On local linearization method for Stochastic Differential Equations driven by fractional Brownian motion. *Preprint*, 2018+.

[5] A. Deya, A. Neuenkirch, and S. Tindel. A Milstein-type scheme without Lévy area terms for SDEs driven by fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, 48(2):518–550, 2012.

[6] H. Doss. Liens entre équations différentielles stochastiques et ordinaires. *Ann. Inst. H. Poincaré Sect. B(N.S)*, 13:99–125, 1977.

[7] J. Garzón, L.G. Gorostiza, and J.A. León. Approximations of fractional stochastic differential equations by means of transport processes. *Communications on Stochastic Analysis*, 3(5):433–456, 2011.

[8] Y. Hu, Y. Liu, and D. Nualart. Rate of convergence and asymptotic error distribution of Euler approximation schemes for fractional diffusion. *The Annals of Applied Probability.*, 26(2):1147–1207, 2016.

[9] I. Kaj and M.S. Taqqu. *Convergence to Fractional Brownian Motion and to the Telecom Process: the Integral Representation Approach*, pages 383–427. Birkhäuser Basel, Basel, 2008.

[10] P. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1992.

[11] C. Klüppelberg and C. Kühl. Fractional Brownian motion as a weak limit of Poisson shot noise processes with applications to finance. *Stochastic Processes and their Applications*, 113(2):333 – 351, 2004.

[12] Y. Liu and S. Tindel. First-order Euler scheme for SDEs driven by fractional Brownian motions: the rough case. *Preprint*, 2017+.

[13] Y. Mishura and G. Shevchenko. The rate of convergence for Euler approximations of solutions of stochastic differential equations driven by fractional Brownian motion. *Stochastics An International Journal of Probability and Stochastic Processes*, 80(5):489–511, 2008.

[14] I. Nourdin and A. Neuenkirch. Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion. *J. Theor. Probab.*, 20(4):871–899, 2007.

[15] H.J. Sussmann. On the Gap Between Deterministic and Stochastic Ordinary Differential Equations. *Ann. Probab.*, 6(1):19–41, 02 1978.
[16] D. Talay. Resolution trajectorielle et analyse numérique des équations différentielles stochastiques. *Stochastics, 9*(4):275–306, 1983.

[17] M. Zähle. Integration with respect to fractal functions and stochastic calculus I. *Probab. Theory Relat. Fields, 111*:333–374, 1998.