Integrating the rebound effect: accurate predictors for upgrading domestic heating

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One particular definition of the ‘rebound effect’ has won acceptance for its conceptual clarity and mathematical robustness: the energy efficiency elasticity of demand for energy services. This is formulated as a partial differential, and its structure enables transformations with price and energy elasticities. However, when considering heating energy efficiency upgrades of homes, the mathematical predictions can be unreliable because these upgrades involve large changes in efficiency, energy and energy services, whereas differential calculus only holds true for very small changes. This could be one reason why existing estimates of rebound effects are so diverse. This paper shows how this limitation can be remedied, using the German housing stock as a case study. A curve of consumption/efficiency for this stock is derived from empirical studies and, based on the mathematical definition of the rebound effect, a rebound effect relation is derived from this. This curve is then integrated over the likely ranges of energy efficiency upgrades that would correspond to the government's policy of reducing consumption by 80%. The model is found to be mathematically coherent, and suggests energy service rebounds of 28–39% for the German stock as a whole if the 80% goal is achieved.

Keywords: energy efficiency elasticity, energy efficiency upgrades, housing, rebound effect, thermal retrofits, Germany

Introduction

A method is proposed for finding the averaged rebound effects for energy-efficiency upgrades in large samples of existing homes. This paper shows how the mathematical structure of the rebound effect, as commonly defined in the econometric literature, can be extended in order to give coherent results for energy and energy services rebound effects through such upgrades. Without this extension, this is problematic because these upgrades involve very large increases in energy efficiency. The classic formulation of the rebound effect is based on differential calculus, which holds true only for very small changes and therefore fails to account for the very large increases in energy efficiency.

The term ‘rebound effect’ is used in a variety of ways to represent the shortfall in energy savings gains or the increase in energy services consumption that often follow an improvement in energy efficiency. Much of the literature on this phenomenon is imprecise in its definition of the ‘rebound effect’ (e.g. Bonino, Corno, & De Russis, 2012; Deurinck, Saelens, & Roels, 2012; Jakob, Haller, & Marschinski, 2012; Hinnells, 2008). Results of different studies in the same field are often difficult to compare, due to the use of different metrics. For example, in investigating rebound effects for home heating upgrades, various studies compare: the shortfalls in energy savings with expected energy savings (e.g. Haas & Biermayr, 2000; Guerra Santin, Itard, & Visscher, 2009; Jin, 2007); the actual consumption with predicted consumption (e.g. Demanuele, Twedde, & Davies, 2010; Tronchin & Fabbri, 2008); the increase in indoor temperature with the percentage increase in energy efficiency (e.g. Milne & Boardman, 2000); or the increases in other formulations of energy services consumption (e.g. comfort) with the increases in energy efficiency (e.g. Howden-Chapman et al., 2009).

Despite these differences, however, one particular formulation of the ‘rebound effect’ has come to dominate economics-oriented literature, and this formulation has a structure that suggests a quest for high precision. This may be described as the energy efficiency elasticity of demand for energy services, or, more loosely, the infinitesimal proportionate change in the take of energy services as a fraction of an infinitesimal
proportionate change in energy efficiency. Expressed formally, this is:

\[ R_s(S) = \frac{\partial S / S}{\partial e / e} \]

\[ = \frac{\partial S / S}{\partial e / e} \cdot \frac{e}{S} \]

where \( S \) represents the quantity of energy services consumed (e.g. warmth in the home; kilometres travelled; hours using a washing machine); and \( e \) represents the efficiency with which energy is converted into those services. This definition can be traced to Khazzoom (1980) via Berkhout, Muskens, and Velthuijsen (2000). Its mathematical relationships with other variables were outlined by Sorrell and Dimitropoulos (2008). It has become accepted as a standard definition in economics, with many authors at least noting its mathematical basis, even if they then proceed to use a different definition (e.g. Druckman, Chitnis, Sorrell, & Jackson, 2011).

There are two reasons this is structured as a partial differential. Firstly, it allows for the possibility that changes in energy services consumption may occur due to more than one influence at the same time. For example, \( \partial S / \partial e \) may be accompanied by a change in the exogenous price of energy, a shift from colder to warmer winters or a change in the use of a (domestic) room. This paper will not consider the effects of changes in these variables, though it should be possible to incorporate them into an overall value for the rebound effect.

If these variables are controlled for, then there is no reason why \( \partial S / \partial e \) may not be treated as a normal differential, \( dS / de \).

The second reason this is structured as a (partial) differential is that it is designed, as a definition, to deal principally with infinitesimal changes in energy service take \( S \) in response to infinitesimal changes in energy efficiency \( e \). This has two great advantages. One is that it allows for the possibility that \( S \) as a function of \( e \) is non-linear. In such a case, the rebound effect will be different at each point along the curve of services level against energy efficiency, and this can be captured by the differential of the services/efficiency function \( S(e) \), i.e. \( \partial S / \partial e \), multiplied by \( e / S \). As will be shown, this makes it a very useful mathematical structure for dealing with large sample averages of the effects of energy efficiency upgrades on homes, where smooth, differentiable curves of \( S(e) \) are obtainable.

The second advantage is that it enables transformations to be made with functions for price elasticity of demand for energy services, which are usually also non-linear and based on large sample averages (Madlener & Hauertmann, 2011; and see the discussion in Sorrell, 2007; Sorrell & Dimitropoulos, 2008; Sorrell, Dimitropoulos, & Sommerville, 2009).

When used mathematically correctly (i.e. for dealing with infinitesimal changes), there is a perfect coherence between two different forms of the rebound effect, namely:

\[ R_s(E) - R_s(S) = -1, \quad \text{where} \quad R_s(E) = \frac{\partial E / E}{\partial e / e} \]

and \( E = \) energy consumption.

Alternatively this can be expressed as:

\[ R_sS - R_s(E) = 1 \]

This means that, for an infinitesimal increase in energy efficiency, a certain proportion of the efficiency increase goes to increase the take of services, while the other proportion goes to reduce the energy consumption. The two proportions add up to 1.0 precisely (i.e. all the energy efficiency increase is accounted for).

The very advantages of this precisely defined formulation of the rebound effect also have disadvantages for dealing with domestic heating energy efficiency upgrades. These upgrades produce very large changes in energy efficiency, often over 150%, which violates the rule of infinitesimal changes (a 60% reduction in energy demand is mathematically equivalent to a 150% increase in energy efficiency). Even for changes of as little as 10% the results begin to become incoherent. As will be shown below, in such cases the two proportions of the energy efficiency gain do not add up to 1.0. Something is lost in the mathematics; not all the energy efficiency gain is accounted for.

This paper offers a way of moving the mathematical structure of this classical formulation of the rebound effect one step forward, in order to overcome this incoherence. Although it could be argued that this is a purely theoretical exercise, there are two reasons why this is important. Firstly, the formulation of the rebound effect outlined above has now become very influential in economics and much policy literature. If it is used, then there are considerable advantages in using it correctly. The vagueness and imprecision of results in many studies (see, for example, the summaries in Yun, Zhang, & Fujiwara, 2013) may not all be due to difficulties in obtaining precise empirical data, but also due to inconsistent or even wrong use of the formulae. Even if there are large spreads and uncertainties in the empirical data, this should not
justify the processing of this data with imprecise or wrongly conceived mathematics.

Secondly, a higher precision of empirical data is possible when dealing with averages of large data sets. Many European countries are committed to reducing the heating energy consumption of their housing stocks by 80% by 2050. Policy-makers may wish to see what magnitude of rebound effects can be expected from the energy-efficiency upgrades that will be required for such a transformation. What proportion of the average energy-efficiency upgrade will be taken, on average, as an increase in energy services, and what proportion will go to reducing energy consumption, if this 80% reduction is attempted? As will be shown below, sufficient data exist on the average heating energy consumption of German households at each specific energy performance rating (EPR) value. These data can enable a first attempt to map the consumption curves that provide the algebraic functions necessary for the rebound effect formulations to be used with more precision. However, in their present form, these formulations are only good for infinitesimal changes, not for the very large energy-efficiency changes that correspond with an 80% reduction in the EPR.

Therefore, this paper offers a way of moving beyond this difficulty. It shows how the rebound effects $R_s(E)$ and $R_0(S)$ can be expressed as functions of the EPR, and how these can be integrated over the full range of the consumption curve, in order to produce mathematically coherent rebound effect values for any magnitude of energy-efficiency upgrade, from any starting point on the EPR scale. The heating energy consumption of the German housing stock is used as a model. Sufficient empirical data are available in this sphere for credible modelling to be achieved.

This is not to claim that these rebound effects will occur in every instance, nor that the model of the heating energy consumption of the housing stock is 100% accurate. What it does show, however, with internal mathematical coherence, is the values of the rebound effects that correspond to these data so modelled. Most importantly, it offers a method that takes the mathematics of the rebound effect a further step forward, so that its definition in terms of infinitesimal changes can become an asset rather than something to be dismissed.

The paper is organized as follows. The next section derives a model of the heating consumption of the German housing stock and uses it to illustrate the incoherence of applying the rebound effect to large increases in heating energy efficiency. The following section provides a series of mathematical steps to produce definite integrals of the rebound effect $R_s(E)$ and $R_0(S)$, which turn out to be infinite series of terms of ever-decreasing magnitude. Then these series are evaluated over any span of energy-efficiency upgrade. The results are then discussed and conclusions drawn.

For clarity of understanding, the following terminology will be used in relation to energy consumption in domestic buildings:

The ‘demand’ or ‘calculated demand’ is the theoretical heating energy consumption that would provide 100% of the energy services necessary for full thermal comfort in the home. It is equivalent to the EPR and is represented by the variable $D$.

The ‘consumption’ or ‘energy consumption’ is the measured or metered heating energy consumption of the current occupants of the dwelling. It is represented by the variable $E$.

Heating energy consumption figures, for both demand and actual consumption, are given in kilowatt-hours per square metre of useful living area per year (kWh/m²·a).

The term ‘heating’ refers to space and water heating combined, though the findings of this study are applicable to either of these separately or both together.

A model for heating energy consumption
Finding a consumption/demand curve
In a study of 36 000 dwellings the German Energy Agency (Deutsche Energie-Agentur – DENA) found the average measured heating energy consumption in the German housing stock to be 30% below the average calculated demand (DENA, 2012, pp. 42–43). Sunikka-Blank and Galvin (2012) brought together existing data sets of over 3700 German dwellings and found a consistent form of relation between specific demand values and the average consumption at each demand value. For any particular value of demand, the average consumption of dwellings with that demand figure followed a mathematically predictable pattern.

For example, in Jagnow and Wolf (2008), in a study of detached houses and small and large apartments in small and large buildings, with heating systems employing gas, oil and district heating ($n = 200$) the relation could be mapped as:

$$E = 12D^{0.48} - 20$$

Using data from a national survey ($n = 1702$), Loga, Diefenbach, and Born (2011) plotted consumption
and demand for residential buildings of all types with fewer than eight dwellings, and mapped the relation as:

\[ E = \left( \frac{1.3D}{1 + (D/500)} \right) - 0.2D \quad : \quad 100 < D \leq 500 \]

This can be approximated very closely by the relation:

\[ E = 14D^{0.48} - 8 \]

Data from these and other sources (Erhorn, 2007; Kaßner, Wilkens, Wenzel, & Ortjohan, 2010; Knisel & Loga, 2006) were examined and the average value of \( E \) was calculated for each specific value of \( D \) between 20 and 500 kWh/m²a, weighted according to the number of data points in each data set over its respective range. Trial-and-error curve fitting resulted in a relation that mapped very closely to this weighted average, namely:

\[ E = 12D^{0.499} - 29.3 \]  \quad (1)

This is displayed in Figure 1. The error between the weighted average (solid line) and the fitted curve (broken line, labelled ‘index model’ on the legend) averages 1.6 kWh/m²a over the range 100 < \( D < 400 \), but increases to 7.8 kWh/m²a for the lowest value of \( D \), i.e. 20 kWh/m²a, and increases again for \( D > 440 \).

This is not suggested as the definitive mapping of average consumption for each demand value. While most of the data were from randomly selected homes, some of the studies of smaller data sets did not make their selection method clear. Further, the curve fitting follows an idealized mathematical form, and it is possible that the fluctuations in the weighted average line give a more accurate rendering. Nevertheless, the idealized curve is offered as a first attempt at modelling the average heating consumption for each specific value of demand, for at least the range 40 < \( D < 440 \).

Efficiency and energy services

To use the rebound effect formulae, these parameters first need to be interpreted in terms of energy services and energy efficiency. The EPR in Germany, here expressed with the variable \( D \), is the quantity of heating energy needed to make a specific building fully comfortable all year round. The methodology for calculating this quantity is given in a publication of the German Institute of Standards (Deutsche Institut für Normung – DIN) numbered DIN-4180 (DIN, 2003). It takes account of the thermal properties of the building envelope, the heating system, orientation to the sun, heating degree-days, and expected ‘free’ heating from appliances and indoor human activities. According to this standard, to be fully comfortable – i.e. to have 100% energy services – a dwelling must have an indoor temperature of 19°C throughout the whole dwelling and a ventilation rate of 0.7 times the volume of the dwelling per hour. The energy consumption required to achieve this is the EPR, i.e.  \( D \).

Whether this is a good or appropriate definition of 100% energy services is beyond the scope of this paper. The issue is further discussed in other DIN publications, namely DIN 33 403, DIN EN ISO 7730 and DIN 1946. It is assumed in this paper that \( D \) is the quantity of energy that would be required to provide 100% energy services.

Following Giraudet, Guivarch, and Quirion (2012), and Tigchelaar and Menkveld (2011), a further assumption is made here: the level of energy services actually being received is the ratio of actual consumption \( E \) to demand \( D \). For example, a household consuming 110 kWh/m²a in a dwelling with an EPR of 220 kWh/m²a is receiving services \( S \) of 0.5, or 50%. This assumption, too, has its weaknesses, as energy services for home heating may well be seen quite differently by different consumers (some like it hot, some like it cool; some like a breeze, others hate a draft; some like warm air, others like cold air and a hot stove, etc.). Further, it should be noted that household heating behaviour may not be the only reason a dwelling is consuming less than its EPR prior to a thermal retrofit, or more than its EPR after a retrofit. It is also possible that the model for calculating 100% energy services is flawed, or does not fit with the physical thermal characteristics of a particular dwelling. In addition, thermal upgrade measures may be installed badly, such as where there are gaps in insulation, so that post-retrofit EPR values underestimate the true measure. The interaction/interface between users and heating control technologies can be problematic. This

![Figure 1 Curve fitting of index model $E = 12D^{0.499} - 29.3$ to the weighted average of heating consumption/demand for the German housing stock](image-url)
can lead to higher heating energy consumption than the user intends (Galvin, 2013).

As there is no fully comprehensive, quantifiable definition of energy services, the definition mentioned above, namely that $D$ is the quantity of energy required to provide full energy services, is used here. However, there is much scope for technical and socio-technical research to improve upon it.

The definition of energy efficiency $e$ follows from the above. The energy efficiency is inversely proportional to the energy demand, i.e. $e = k/D$. But since $e$ is only used in the form $e/\Delta e$, it is possible to drop the constant $k$ and also assume that a dwelling that requires 1 kWh/m$^2$a to provide 100% energy services is 100% efficient, i.e. for such a dwelling $e = 1$. Given these assumptions it also follows that $S = e \cdot E$.

These assumptions and relations also accord with the reasoning of Sorrell and Dimitropoulos (2008) in their formal, mathematical definitions of the rebound effect. These assumptions are used in the analysis that follows.

### Incoherence of the rebound effect

According to the German Energy Agency, the average heating energy demand in German dwellings is around 220 kWh/m$^2$a (DENA, 2012). Consider a thermal upgrade that produces an 80% reduction in heating energy demand in an ‘average’ dwelling. Using equation (1) above, this gives point-to-point change in heating energy efficiency in which:

\[
D_1 = 220 \implies E_1 = 147.43;
D_2 = 44 \implies E_2 = 50.00
\]

Applying the above rules to these figures for $S$ and $E$ gives:

\[
S_1 = \frac{147.73}{220} = 0.6715; \quad S_2 = \frac{50.00}{44} = 1.1364
\]

\[
e_1 = \frac{1}{D_1} = \frac{1}{220} = 0.00455; \quad e_2 = \frac{1}{D_2} = \frac{1}{44} = 0.0227
\]

Consider first the energy services rebound effect $R_e(S)$

\[
R_e(S) = \frac{\Delta S}{S} \cdot \frac{e}{\Delta e} = \frac{1.1364 - 0.6715}{0.6715} \cdot \frac{0.00455}{0.0227 - 0.00455} = 0.1736
\]

This implies that 17.36% of the energy efficiency upgrade has gone to increasing the take of energy services. The expectation is that the remaining 82.64% has gone to reducing energy consumption, in order to satisfy the relation $R_e(E) - R_e(S) = -1$.

However, when considering $R_e(E)$ it is found:

\[
R_e(E) = \frac{\Delta E}{E} \cdot \frac{e}{\Delta e} = \frac{50 - 147.73}{147.73} \cdot \frac{0.00455}{0.0227 - 0.00455} = -0.1658
\]

This implies that only 16.58% of the energy efficiency increase has gone to reduce energy consumption, and that the remaining 83.42% should have gone to increasing the take of energy services.

The discrepancy between these is very large:

\[
R_e(E) - R_e(S) = -0.1658 - 0.1736 = -0.3394
\]

which is a long way from $-1$. A full 66.06% of the energy efficiency increase has simply disappeared. In terms of the classic rebound effect definition these results are incoherent and impossible to interpret. This is because the methodology violates the mathematics of that definition: the changes that come through a typical energy efficiency upgrade are large, not infinitesimally small.

It may be thought possible to dispense with the $e/\Delta e$ term altogether and simply say that energy services have increased by $(1.1364 - 0.6715)/0.6715 = 69\%$, while energy consumption has reduced by $(147.73 - 50.0)/147.73 = 66\%$. However, this does not describe the ‘rebound effect’, as it omits the essential parameter of the increase in energy efficiency as the driver of the changes in energy services and energy consumption.

A more sophisticated set of mathematical tools is needed to compute the rebound effect, a partial differential, in a large one-off change, and obtain coherent results. This issue is addressed below.

### The rebound effect as a definite integral

**Method**

This method involves four steps, in addition to having derived equation (1) above from empirical sources. Firstly, equations are needed in $e$ for the change in services $S$ and for the change in consumption $E$ along the range of likely changes in $e$ due to an energy efficiency upgrade. Specifically, equations are needed for $S = f(e)$ and $E = g(e)$.

Secondly, these need to be differentiated and the results multiplied by $S/E$ and $e/E$ respectively, to give curvilinear relations for $R_e(S)$ and $R_e(E)$.
Thirdly, \( R_e(E) \) and \( R_a(S) \) need to be transformed into functions in \( D \), the energy demand. This will give the rebound effect at every point along the demand curve. For every value of \( D \) along the consumption/demand curve, it will say what portion of an (infinitesimal) energy efficiency increase will go to increasing energy services, and what portion will go to reducing energy consumption.

Fourthly, the functions in \( D \) which have been obtained for \( R_a(E) \) and \( R_a(S) \) need to be integrated. This allows the definite integrals to be found for each of these, for the range over which \( D \) changes due to an energy efficiency upgrade. Dividing the definite integrals by the range of change in \( D \) will give the true averages for \( R_a(E) \) and \( R_a(S) \) over this range. If the mathematics is correct, then the relation \( R_a(E) - R_a(S) = -1 \) will hold true for the numerical results, and the results will be coherent.

**Steps prior to the integration**

Consider equation (1), \( E = 12D^{0.499} - 29.3 \). Substituting \( \varepsilon = 1/D \) in this gives:

\[
E = 12\varepsilon^{-0.499} - 29.3
\]  
(2)

Hence:

\[
R_a(E) = \frac{\partial E}{\partial \varepsilon} \cdot \varepsilon = -5.988\varepsilon^{-0.499}
\]  
(3)

Substituting \( D = 1/\varepsilon \) in this gives an equation for the energy rebound effect as a function of \( D \):

\[
R_a(E) = \frac{-5.988D^{0.499}}{12D^{0.499} - 29.3}
\]  
(4)

This is the relation that needs to be integrated. Before doing so, it can be simplified to:

\[
R_a(E) = \frac{(4.983D^{-0.499} - 2.004)^{-1}}
\]

(5)

Or more generally:

\[
R_a(E) = (aD^b - c)^{-1}
\]  
(6)

The general form of the energy consumption/demand relation, equation (1), can be expressed as:

\[
E = P \cdot D^Q - T
\]  
(7)

Hence in equation (6):

\[
a = \frac{T}{P \cdot Q} \quad b = -Q \quad c = \frac{-1}{Q}
\]  
(8, 9, 10)

These transformations will enable the use of the integration and programming method described below to derive the integrated rebound effects for any heating consumption/demand curve of the form \( E = P \cdot D^Q - T \). It was shown above that curves of this form fit well with all the existing data sets. If fuller data sets come to light in future, then the method described below will work provided the relation between demand and the average heating consumption for each demand value can be represented by this general mathematical form. If not, the new form will have to be processed by the methodology above according to its own rules of differentiation, and the methodology below according to its rules of integration.

**The integration**

From equation (5) it was seen that the energy rebound effect curve \( R_a(E) \) has the form:

\[
R_a(E) = (ax^b + c)^{-1}
\]

Note that \( x \) is used here rather than \( D \) as it makes the expressions visually easier to follow through the steps of integration in Appendix 1 in the Supplementary data online.

Let \( I = \int (ax^b + c)^{-1} \, dx \)

where \( x \) = calculated heating demand

Using:

\[
\int udv = uv - \int vdu
\]

and first setting \( u = (ax^b + c)^{-1} \) and \( dv = 1 \cdot dx \) it is possible to integrate successively by parts, as detailed in Appendix 1. The result is an infinite series:

\[
I = x(ax^b + c)^{-1} + \frac{abx^{b-1}}{b+1} \cdot (ax^b + c)^{-2}
\]

\[
+ \frac{2a^2b^2x^{2b+1}}{(b+1)(2b+1)} \cdot (ax^b + c)^{-3}
\]

\[
+ \frac{6a^3b^3x^{3b+1}}{(b+1)(2b+1)(3b+1)} \cdot (ax^b + c)^{-4} \ldots \text{etc.}
\]

(11)

This can alternatively be expressed as:

\[
I = \sum_{n=1}^{\infty} \frac{(n-1)!(ab)^{n-1}x^{(n-1)b+1} + (ax^b + c)^{-n}}{[nb+1][n-1+b+1][n-2+b+1]\ldots[0+b+1]}
\]

(12)

Although this is an infinite series, it will be evident that the magnitude of the terms reduces rapidly for the values of \( a, b \) and \( c \) that can be encountered in the equation for \( R_a(E) \). In calculating \( I \) for various values of these variables and for high values of demand, terms beyond the 30th, and in some cases the 15th, were smaller than could be registered by a computer. The number of terms of significant value in the infinite series increased to 170 for demands lower than 28 kWh/m²a, but the series was always a diminishing one.

715
The algorithm (12) was coded as a computer program in Visual Basic, as in this language code can easily be added to produce Excel spreadsheets with printouts of results, from which graphical displays can be produced. The program calculated the definite integrals for values of \( D = x \) corresponding to pre- and post-energy efficiency upgrade values of the demand. It subtracted the latter from the former and divided the result by the difference between the two values of \( D \) to give the average value of \( R_e(E) \) over the span of the upgrade. Expressed formally this is:

\[
R_e(E)\bigg|_{D_1}^{D_2} = \frac{\int_{D_2}^{D_1} R_e(S) dD}{D_1 - D_2}
\]

**Proving the results are coherent**

The results may be deemed coherent if and only if \( R_e(E) - R_e(S) = -1 \) in all cases. The expression for \( R_e(S) \) as a function of \( D \) was found for the case where equation (2) holds, i.e. beginning with equation (2):

\[
E = 12e^{-0.499} - 29.3
\]

Substituting \( S = e \cdot E \) gives:

\[
S = 12e^{0.501} - 29.3 \quad (13)
\]

This gives the energy services rebound effect:

\[
R_e(S) = \frac{\partial S}{\partial e} \cdot e = \frac{6.012e^{0.501} - 29.3e}{12e^{0.501} - 29.3e}
\]

Substituting \( e = 1/D \) gives:

\[
R_e(S) = \frac{6.012D^{0.499} - 29.3}{12D^{0.499} - 29.3} \quad (14)
\]

At first this function appears more difficult to integrate than that for \( R_e(E) \), but two steps make this easier. Firstly, in Appendix 2 in the Supplementary data online it is proven that for these particular expressions of \( R_e(S) \) and \( R_e(E) \), the relation holds true that:

\[
R_e(E) - R_e(S) = -1
\]

Secondly, based on this, in Appendix 3 in the Supplementary data online it is proven that:

\[
\frac{\int_{D_1}^{D_2} R_e(S) dD}{D_1 - D_2} = \frac{\int_{D_1}^{D_2} I_{\frac{D_1}{D_2}}}{D_1 - D_2} + 1
\]

i.e.

\[
\frac{\int_{D_1}^{D_2} R_e(S) dD}{D_1 - D_2} = \frac{\int_{D_1}^{D_2} R_e(E) dD}{D_1 - D_2} + 1
\]

This proves coherence between the rebound effects for energy and for energy services averaged over any range of heating demand. To calculate the averaged energy services rebound effect, 1 is simply added to the energy rebound effect.

It could be asked why a cumbersome, difficult-to-integrate function is used, when a rough approximation to the curve in Figure 1 would be given by a function that integrates in one step, such as:

\[
E = AD^B
\]

Appendix 4 in the Supplementary data online shows how functions of this form produce a rebound effect result that is simply a constant equal to \( 1 - B \). Such functions are very useful for approximate or good-enough comparisons of rebound effects, where a trade-off has to be made between computational time and level of precision. However, in dealing with just one data set, such as the German housing stock, a case can be made for a higher level of precision.

**Results**

**Case of attempted 80% reduction**

First, the results are calculated for an upgrade that reduces the demand by 80\%, from the average German demand of 220 kWh/m\(^2\)a. This is an interesting case because it relates to the stated aim of the German government to be achieved by 2050. The results for the energy rebound effect are displayed in Figure 2. Each point along the curve, starting at the right end, gives the energy rebound effect for an upgrade from \( D = 220 \) kWh/m\(^2\)a to the post-upgrade value of \( D \). Moving to the left shows the energy rebound effects for successively higher upgrades, all starting from \( D = 220 \) kWh/m\(^2\)a.

This shows, for example, that an upgrade that reduces demand from 220 to 200 kWh/m\(^2\)a gives an energy rebound effect of \(-0.6008\), meaning that 60\% of the energy efficiency increase goes to reduce energy...
consumption while 40% goes to increase energy service take. For an upgrade from 220 to 100 kWh/m²a the figures are 62.24% and 37.76%. For an upgrade from 220 to 44 kWh/m²a (an 80% reduction in demand) the figures are 65.3% and 34.7%. (Note that this curve does not show rebound effects for upgrades starting at lower demand figures than 220.) These figures are coherent, in that if one starts from the energy services rebound effect then the same results are obtained.

To illustrate the difference this integral-based approach makes to precision, Figure 3 gives the curve from Figure 2, labelled (a) in Figure 3, along with two others: (b) the single-point rebound effect, i.e. the traditional formulation of the rebound effect for infinitesimal changes at all points along the curve, and (c) the point-to-point rebound effect, i.e. using \( R_{e}(E) = (\Delta E/E) \cdot (e/\Delta e) \) over the range 220 to 44 kWh/m²a.

Figure 3 illustrates that the non-integrated, single point rebound effect (b) shows significantly higher (numerically more negative) reductions in energy consumption as the demand diminishes. This is of little practical use, in itself, as nobody in the real world is likely to be interested in infinitesimal improvements in energy efficiency. It can lead to a mistake, however. It would be a mistake to think that an upgrade over the full span of, for example, 220–44 kWh/m²a will produce rebound effects for large changes in \( E \) and \( e \), gives the same result as \( (\Delta E/E) \cdot (e/\Delta e) \). The error is considerable: \(-0.1654\) compared with \(-0.6530\). The error becomes progressively larger with higher magnitudes of energy efficiency upgrade. Further, starting such a calculation with the energy services rebound gives equally incoherent results.

Returning to the integrated rebound effect calculations, note that reducing the demand by 80% will not reduce consumption by 80%. Instead it reduces consumption from 147.7 to 50.0 kWh/m²a, a reduction of 66%. For an 80% reduction in actual average consumption, the average post-upgrade consumption would need to be just under 30 kWh/m²a. Inverting equation (1) to calculate \( D \) from this value of \( E \) shows that the average post-upgrade demand would have to be just under 25 kWh/m²a to achieve a real 80% reduction in consumption. The computer programme as coded can integrate for values of \( D \) as low as 28 kWh/m²a, but below that level over 170 terms are needed to achieve convergence of the infinite series, and at this point the factor (n − 1!), i.e. 169!, is too large for the computer to calculate. For an upgrade from a demand of 220 to 28 kWh/m²a the energy rebound effect is −0.6667, and the trend is for it to be increasing sharply. Therefore, an expected value would be around −0.7 for an upgrade from 220 to 25 kWh/m²a.

An incidental issue, however, is the question of how plausible it is to maintain that a reduction to an average demand of 25 kWh/m²a is possible for the German housing stock (for a discussion, see Galvin, 2010; Jakob, 2006).

Low- and high-consuming households

For the consumption/demand curve given in equation (1) the energy rebound effect for upgrades from a demand of 220 kWh/m²a to any demand level down to 44 kWh are given in Figure 2, as noted above. As explained there, this curve is the best estimate available for how average consumption varies with demand. However, not all households consume the average heating energy for their dwelling’s demand. Therefore, it is interesting to ask what magnitude of rebound effects could be expected from households with other consumption/demand curves. Greller, Schröder, Hundt, Mundry, & Papert (2010) show that although there is a wide range of actual consumption \( E \) for each specific demand value \( D \), the shape and spread of the distribution of \( E \) at each value of \( D \) is fairly consistent (see especially Greller et al., 2010, p. 2, figure 1).

Hence a range of consumption/demand curves can be considered, with values of \( E \) in various proportions to each other.

To begin, consider ‘low-consuming’ households, which consume half the national average for any specified
demand. These may be low-income households (Milne & Boardman, 2000), but not necessarily households in extremely thermally leaky homes). The consumption/demand relation is therefore:

\[ E = 6D^{0.499} - 14.65 \]  

Hence \( P = 6, Q = 0.499, T = 14.65 \), so that from equations (8–10):

\[ a = 4.893 \quad b = -0.499 \quad c = -2.004 \]

This gives the identical energy rebound effect equation as that for the average household, i.e.

\[ R_1(E) = (4.983D^{-0.499} - 2.004)^{-1} \]

This non-intuitive result is due to the fact that, while these homes reduce less energy in absolute terms after an upgrade than ‘average’ homes do, the quantity of energy they consumed prior to the upgrade was also less, in the same proportion.

Similarly, high-consuming households that consume, say, twice as much heating fuel as the average for any specific demand will also show the same rebound effect as average and fuel-poor households. Of course, their absolute level of consumption will be twice as high after an upgrade as the average, just as the absolute level for a fuel-thrifty household will be half the average. This will be of interest to policy actors seeking to reduce consumption in upgraded homes – they might try to persuade high-consuming householders to consume less. In this sense the rebound effect calculation method offered here could assist policy-makers to anticipate possible post-upgrade social interventions.

Nevertheless, it must also be emphasized that people do not always behave consistently with mathematical modelling. Occupants might use heat unpredictably after a large energy efficiency upgrade. But the mathematics can help social scientists identify which post-upgrade behaviours might best be explained by factors that lie outside of what the rebound effect parameters cover. For example, if a previously high-consuming household (as defined above) consumes significantly less than twice the new demand figure after an upgrade, the analysis here suggests that this change requires a social explanation.

**Cases with different pre-upgrade heating energy demands**

Although there is no difference in the rebound effect for high-consuming, average and low-consuming households as defined above, it should be noted that there is a difference for the same cohort but with different pre-upgrade demand levels. Rebound effects were shown above for a range of depths of upgrade, all starting from a demand of 220 kWh/m²a. In this section, energy rebound effects will be shown along the same consumption/demand curve, but starting with (a) 420 kWh/m²a, (b) 220 kWh/m²a, as above, and (c) 100 kWh/m²a. In all cases these dwellings are upgraded to a demand of 44 kWh/m²a. These are displayed in Figure 4.

As seen in Figure 4, upgrading a dwelling with a higher pre-upgrade heating demand produces a smaller (less negative) energy rebound effect than for a dwelling with a lower pre-upgrade demand: its energy services rebound effect is significantly higher. A larger portion of the energy efficiency improvement goes to increasing energy services, than for dwellings on the same consumption/demand curve that have lower pre-upgrade demand.

For example, consider a dwelling with a pre-upgrade demand of 440 kWh/m²a that is upgraded only as far as a new demand of 220 kWh/m²a. This will bring it to the same position on the demand curve as a dwelling that was always at 220 kWh/m²a (the two dwellings will now both have the same \( E \) and \( D \)), but the first dwelling will have undergone a rebound effect in order to get there. Likewise, when all three dwellings end up with a demand of 44 kWh/m²a, their energy rebound effects are \(-0.61\), \(-0.65\) and \(-0.72\) respectively. Expressed in terms of energy services take, their rebound effects are 39%, 35% and 28% respectively. This is the case even though they all end up consuming the same quantity of heating energy.

This is the sense in which higher rebound effects can be expected from upgrades of thermally inferior homes. Boardman (2012) and Milne and Boardman (2000)
maintain that ‘fuel poverty’ is characterized by such homes. However, this need not deter state investment to upgrade these homes, as the social dividend of improved health and well-being can be balanced alongside the environmental dividend of at least a modicum of energy savings (Howden-Chapman et al., 2009, 2012; Jenkins, Nordhaus, & Shellenberer, 2011).

This is not exclusively an issue of low-income households having larger rebounds, as it is often popularly claimed. Rather, higher rebound effects are to be expected from upgrades of thermally worse dwellings, regardless of who lives in them or how heavily they habitually consume.

Cases with steeper consumption/demand curves

Finally, the case is considered of households whose consumption/demand ratio more nearly approximates the line $E = D$, i.e. the index of $D$ is closer to 1 than to 0.5. One such model is given by the relation:

$$ E = 5D^{0.7} - 8 $$

(17)

The consumption/demand curve for this is displayed alongside that for the ‘average’ model, together with the line $E = D$, in Figure 5.

For this relation, the values are $a = 2.286$, $b = -0.7$, $c = 1.429$, giving the energy rebound curve:

$$ R_4(E) = (2.286D^{-0.7} - 1.429)^{-1} $$

(18)

The energy rebound effect $R_4(E)$ for an upgrade of such a dwelling from 220 kWh/m$^2$a to a range of demand levels down to 44 kWh/m$^2$a is displayed alongside that of the average model, in Figure 6.

This shows that for the high-consuming household, a significantly greater proportion of the energy efficiency gain goes to reduction of energy consumption, than for the average case. The nearer the index of $D$ is to 1.0, i.e. the nearer consumption/demand curve approximates the line $E = D$, the more the energy-efficiency gain is translated directly into gains in energy saving. However, in all the empirically derived models of consumption in residential buildings known to the author, the index of $D$ is well under 0.6, and generally close to 0.5. Recent studies of non-residential buildings, however, show indices of $D$ closer to 1.0 (IWU, 2014; Oschatz, Jagnow, & Wolff, 2014).

Hence the high-consuming case could prove to be of interest to studies of rebound effects in non-residential buildings. It illustrates that if more accurate empirical estimates of the consumption/demand curve are produced in the future, the higher the index of $D$, then the lower the energy services rebound effect will be and the higher the proportion of the energy efficiency increase is that goes to reducing energy.

An important point to stress is that the ranges of actual and calculated consumption used in this analysis, though large, are quite typical for housing stocks. Perhaps surprisingly, empirical studies show there can be large rebound effects even in dwellings that were relatively thermally efficient before being thermally upgraded (e.g. Galvin, 2013; Hong, Oreszczyn, & Ridley, 2006). There may be a limit to how warm a household will heat their home, but other factors, such as window opening and leaving the heating on...
The energy services rebound effect

The term ‘rebound effect’ usually denotes a shortfall in energy savings and/or an increase in energy service take following an energy efficiency upgrade. The term is often used imprecisely in academic literature. However, formulations of the concept have clustered around a precise mathematical definition of the rebound effect, as the energy efficiency elasticity of energy services, with its correlate the energy efficiency elasticity of energy consumption. This conceptual precision has enabled some degree of stability and interchangeability between different sets of empirically derived results.

This definition is structured, necessarily, as a partial differential. In its simplest form it only holds true for infinitesimal changes in energy consumption and energy service take associated with infinitesimal changes in energy efficiency. This makes its results internally inconsistent and incoherent for large changes, of the type seen in energy-efficiency upgrades of existing homes.

This paper has shown how this problem can be solved for these upgrades, using five steps. Firstly, a relation of the type \( E = f(D) \) is obtained, from empirical studies, for the average energy consumption \( E \) for each specific value of heating demand \( D \). Secondly, this is transformed into a relation of the type \( E = f(\varepsilon) \), where \( \varepsilon \) is the heating efficiency of the dwelling. Thirdly, this relation is differentiated and the result multiplied by \( \varepsilon/E \), to give the energy rebound effect relation \( R_e(E) = (\partial E/\partial \varepsilon) \cdot (\varepsilon/E) \). This is then transformed into a function in \( D \), to give a precise figure for the energy rebound effect at any point along the consumption/demand curve. Finally, this function is integrated, and the definite integral between pre- and post-upgrade values of \( D \) is calculated and divided by the difference between the two values of \( D \), to give the precise energy rebound effect for the entire upgrade.

The energy services rebound effect \( R_e(S) \) can be calculated from this result simply by adding 1.0 to it. The results are mathematically coherent and consistent with the properties of curvilinear functions. They indicate precisely what rebound effect would ensue if consumers behaved according to the consumption/demand model developed from empirical studies.

For the average German dwelling, with a demand of 220 kWh/m²a, reducing the demand by an average of 80% would lead to an energy efficiency elasticity of energy consumption of \(-0.653\), meaning that 65.3% of the energy efficiency improvement would go to reducing energy consumption, while 34.7% would go to increasing the take of energy services: a rebound effect of 34.7%. An 80% reduction in energy consumption would not be achieved, as energy consumption would reduce, not from 220 to 44 kWh/m²a, but from 148 to 50 kWh/m²a, a reduction of 66.2%. To achieve an actual reduction in consumption of 80% this dwelling would have to be upgraded from its present demand of 220 kWh/m²a to 25 kWh/m²a. Here 70% of the energy efficiency gain would go to reducing energy consumption and the remaining 30% to increasing the take of energy services.

Rebound effects will be higher for dwellings with high pre-upgrade demand, than for those with lower pre-upgrade demand, regardless of the depth of the upgrade. For dwellings with the national average consumption at each specific demand value, i.e., situated on the consumption/demand curve \( E = 12D^{0.499} - 20.3 \), energy rebound effects (elasticities of energy consumption) will range from around \(-0.61\) to \(-0.72\) when these dwellings are upgraded to a demand of 44 kWh/m²a, depending on their pre-upgrade demand. This means the energy services rebound effect will range from 39% to 28%. The mathematics indicate that this is the range the German government needs to consider, in its aim to reduce consumption by 80%.

Low- and high-consuming homes produce the same rebound effect as homes with average consumption for each specific demand value. The cases that bring greater rebound effects are those with the highest pre-upgrade demand, as outlined above. Some of these are large, old, detached or semi-detached homes with few occupants per m² of living area. Many are the dwellings of ‘fuel-poor’ households and are simply very thermally inadequate. Despite possibly larger rebound effects there is countervailing social value in upgrading them.

Different constellations of rebound effects would occur for households that follow consumption/demand curves with indexes of \( D \) that differ from 0.499. The author has not yet found a large data set of residential buildings that gives an index which deviates far from this value, but newly emerging studies of consumption in non-residential buildings suggest values closer to 1.0. This would imply lower rebound effects, but from a higher energy consumption baseline.

These findings could be useful for policy-makers attempting to estimate the likely energy savings from programmes to upgrade national housing stock, e.g., to reduce heating energy consumption by 80%.

Nevertheless, it must be emphasized that these results are reliable only mathematically, given the model of...
consumption/demand derived empirically. For example, if a country were to succeed in reducing the heating energy consumption of its housing stock by 80%, this would be a massive socio-technical transformation, and might result in cultural shifts that lead to unforeseen changes in consumption patterns. The actual effects of upgrades can only be known empirically, from household-by-household investigation. However, it is suggested here that such investigation would benefit from knowing what the mathematically calculated effects of an upgrade are, so as to be better able to identify rogue or non-neutral shifts in consumption habits. In any case, if a potentially precise mathematical tool exists for processing empirical data, nothing will be lost by taking that tool to a higher level of precision, even if the data are themselves subject to imprecision and uncertainty.

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