Equivalence of Dirichlet and Neumann problems for the Laplace operator in elliptical and doubly-connected regions

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Abstract

The present paper provides an equivalence of the solutions of Dirichlet and Neumann problems for the Laplace operator in doubly-connected as well as elliptical regions, in the sense that solving one of these problems leads by an explicit formula to a solution of the other problem. In addition, sufficient condition for continuous extension of higher-order partial derivatives of the solutions of the Neumann problem, to the closure of the region where they are defined, is given.

Key words: Dirichlet problem, Neumann problem, Laplace operator, harmonic functions, analytic functions.

1 Introduction

The Dirichlet and the Neumann problems are fundamental in the theory of differential equations. Recently the connection between these problems was investigated and it was shown that in the case of the Laplace operator (and other differential operators satisfying certain homogeneity conditions) there is a connection between these problems, in the sense that solving one of these problems leads by an explicit formula to a solution of the other problem. The planar domains taken into consideration in these articles were simply connected. In the present paper the author shows that a similar connection between the Dirichlet and Neumann problems holds in the case of planar doubly-connected regions. Although the elliptical regions are obviously planar simply-connected regions for which, as specified above, the connection between the two problems has been explicitly provided, the conformal mapping on which this connection relies on is cumbersome thus making the representation of the Neumann problem in terms of the solution of the Dirichlet problem somehow redundant for a direct application. This issue is fixed in the paper at hand by considering another approach for obtaining the desired connection. This approach is based on the Joukowsky transform.

The structure of the paper is as follows. In Section 2 the notation is established and some preparations are made. In Section 3 the author presents the first main result (Theorem 1), with an equivalent formulation in terms of cartesian coordinates (Theorem 4). In addition, an extension of it to the case of general planar doubly-connected regions is given using conformal maps (Theorem 6). Its second part presents the connection between the Neumann and the Dirichlet problems in the case of elliptical regions. Section 4 draws some final conclusions.
2 Preliminaries

2.1 Notations

Denote by $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$ the unit disk in $\mathbb{C}$, by $\hat{\mathbb{U}}$ the punctured unit disk, by $C_r$ the circle centered in origin of radius $r$, and the annulus with radii $0 \leq r_1 < r_2$ by $A_{r_1, r_2} = \{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$, respectively. Furthermore if $\rho > 1$ let $E_\rho$ be the interior of the ellipse given by

$$
\frac{4x^2}{(\rho+\rho^{-1})^2} + \frac{4y^2}{(\rho-\rho^{-1})^2} = 1,
$$

and if $\theta \in (-\pi, \pi]$ let $H_\theta$ be the hyperbola described by

$$
\frac{x^2}{\cos^2 \theta} - \frac{y^2}{\sin^2 \theta} = 1
$$

in which case $H_\theta = \{ H_\theta : \theta \in (-\pi, \pi] \}$. In addition for any region $\Omega$, $C^1(\Omega)$ will stand for the set of all functions $h \in C^1(\Omega)$ for which the gradient $\nabla h$ can be continuously extended to $\Omega$, and $N(\Omega)$ will stand for the set of all functions $h \in C^1(\Omega)$ for which the (outward) normal derivative exists and is finite at all points of $\partial \Omega$. Furthermore $\partial \Omega \in C^{m,\alpha}$, for some $0 < \alpha < 1$, if there exists a parametrization of $\partial \Omega$ for which the $m$th-order derivative exists at any point and it is $\alpha$-Holder continuous. Finally a real-valued function $\Phi$ defined on the boundary of $\Omega$ is said to belong to $C^{n,\alpha}(\partial \Omega)$ if for any $C^{m,\alpha}$ parametrization $\Gamma$ of $\partial \Omega$, $m \geq n$, the function $\Phi \circ \Gamma$ belongs to $C^{n,\alpha}$. Throughout the paper the author will switch between the complex and the $\mathbb{R}^2$ notations, depending on the context to discriminate between them. For example if $u$ is a harmonic function defined on some region containing the point $(\cos \theta, \sin \theta)$ then $\frac{\partial u}{\partial e^{i\theta}}(e^{i\theta})$ is a shorthand representing the directional derivative of $u$ in the direction of the vector $[\cos \theta, \sin \theta]^T$, evaluated in $(\cos \theta, \sin \theta)$.

2.2 Preliminary aspects

If $D \subset \mathbb{C}$ is a bounded (smooth) region, consider the Dirichlet and the Neumann problems for the Laplace operator in $D$

$$
\begin{align*}
\Delta u &= 0 \quad \text{in } D \\
u \cdot \nabla u &= g \quad \text{on } \partial D,
\end{align*}
$$

and

$$
\begin{align*}
\Delta U &= 0 \quad \text{in } D \\
\frac{\partial U}{\partial \nu} &= f \quad \text{on } \partial D,
\end{align*}
$$

where $\nu$ is the outward unit normal to the boundary of $D$. In the particular case when $D = A_{r_1, r_2}$, $r_1 > 0$, we have

$$
\nu(z) = \begin{cases} 
\frac{z}{r_2}, & \text{if } |z| = r_2, \\
-\frac{z}{r_1}, & \text{if } |z| = r_1.
\end{cases}
$$

By a (classical) solution of the Dirichlet/Neumann problems above it is understood a function $u \in C^2(D) \cap C^0(\bar{D})$, respectively $U \in C^2(D) \cap N(D)$, which satisfies (1), respectively (2).

Remark 1. Let \( D \) be either \( A_{r_1,r_2} \), \( r_1 > 0 \), or \( E_p \), \( p > 1 \).

Using the maximum principle for harmonic functions (see, e.g. [1, Theorem 2.2.4]), it can be seen that for continuous boundary data \( g \) the Dirichlet problem (1) has a unique solution. Also if \( f \) is a continuous function satisfying \( \int_{\partial D} f d\sigma = 0 \) then it can be proved that the Neumann problem (2) always has a solution, which is unique up to additive constants.

The existence of solutions of the Dirichlet and the Neumann problems in the case of the punctured disk \( A_{0,r_2} \) requires special attention. As shown by Zaremba’s example, for continuous boundary data \( g \) and \( r_1 = 0 \), the Dirichlet problem (1) has a solution iff \( g(0) = \frac{1}{2\pi r_2} \int_0^{2\pi} g(r_2 e^{i\theta}) d\theta \). Also, for continuous boundary data \( f \) and \( r_1 = 0 \), the boundary condition at the origin of the Neumann problem (2) should be ignored (the exterior normal to \( \partial A_{0,r_2} \) at the origin cannot be properly defined), and a solution of (2) satisfying the boundary condition just on \( \partial A_{0,r_2} \setminus \{0\} \) exists iff \( \int_0^{2\pi} f(r_2 e^{i\theta}) d\theta = 0 \) (this is a direct consequence of Theorem 1 and Proposition 1 below; also see Definition 2 and Remark 4).

When \( D = A_{r_1,r_2} \), due to the radial symmetry of the region, it is natural to consider polar coordinates \((r,\theta)\), defined by \( r = |z| \) and \( \theta = \arg(z) \in [-\pi, \pi) \) for \( z \in A_{r_1,r_2} \).

The link between the cartesian and polar coordinates formulation of the Dirichlet and Neumann problems (1) – (2) when \( D = A_{r_1,r_2} \) is given by the following proposition.

Proposition 1. If \( w \in C^2(A_{r_1,r_2}) \) satisfies \( \Delta w = 0 \) in \( A_{r_1,r_2} \), then the function \( \hat{w} : (r_1,r_2) \times \mathbb{R} \to \mathbb{R} \) defined by \( \hat{w}(r,\theta) = w(re^{i\theta}) \) is \( 2\pi \)-periodic in the second variable, has continuous second order partial derivatives and satisfies

\[
\hat{w}_{rr} + \frac{1}{r} \hat{w}_r + \frac{1}{r^2} \hat{w}_{\theta\theta} = 0 \quad \text{in} \quad (r_1,r_2) \times \mathbb{R}. \tag{4}
\]

Conversely, if the function \( \hat{w} : (r_1,r_2) \times \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic in the second variable, has continuous second order partial derivatives and satisfies (4), then the function \( w : A_{r_1,r_2} \to \mathbb{R} \) defined by \( w(z) = \hat{w}(|z|, \arg(z)) \) belongs to \( C^2(A_{r_1,r_2}) \) and satisfies \( \Delta w = 0 \) in \( A_{r_1,r_2} \).

Moreover, \( w \) has a continuous extension to \( \overline{A_{r_1,r_2}} \) iff \( \hat{w} \) has a continuous extension to \( [r_1,r_2] \times \mathbb{R} \), and in this case

\[
w(re^{i\theta}) = \hat{w}(r,\theta) \quad \text{for} \quad (r,\theta) \in [r_1,r_2] \times \mathbb{R}.
\]

Also \( w \) has (outer) normal derivative at a point \( re^{i\theta} \in \partial A_{r_1,r_2} \) iff \( \hat{w} \) has partial derivative with respect to the first variable at the point \( (r,\theta) \in \{r_1,r_2\} \times \mathbb{R} \), and in this case

\[
\frac{\partial w}{\partial \nu}(re^{i\theta}) = \begin{cases} \hat{w}_r(r,\theta), & \text{if} \ r = r_2, \\ -\hat{w}_r(r,\theta), & \text{if} \ r = r_1. \end{cases} \tag{5}
\]

Finally \( w \in C^1(\overline{A_{r_1,r_2}}) \) iff \( \hat{w} \in C^1([r_1,r_2] \times \mathbb{R}) \).
Proof. The direct implication is immediate. For the converse, by using the $2\pi$-periodicity of $\hat{w}$ in the second variable and the fact that it has continuous second order partial derivatives, lengthy computations show that $w \in C^2(A_{r_1,r_2})$. Also, it is not difficult to check that
\[
\Delta w(z) = \hat{w}_{rr}(|z|, \arg(z)) + \frac{1}{|z|} \hat{w}_r(|z|, \arg(z)) + \frac{1}{|z|^2} \hat{w}_{\theta\theta}(|z|, \arg(z)) = 0, \ z \in A_{r_1,r_2},
\]
where the last equality follows by using hypothesis (4).

The fact that $w$ has a continuous extension to the boundary of the domain iff $\hat{w}$ does is immediate.

Next notice that for any $\theta \in \mathbb{R}$ the corresponding directional derivatives are given by:
\[
\begin{align*}
\frac{\partial w}{\partial r}(re^{i\theta}) &= \lim_{t \to 0} \frac{w(r(cos\theta, sin\theta) + t(cos\theta, sin\theta)) - w(r(cos\theta, sin\theta))}{t} = \hat{w}_r(r, \theta), \ r \in (r_1,r_2) \\
\frac{\partial w}{\partial \theta}(r_2e^{i\theta}) &= \lim_{t \to 0} \frac{w(r_2(cos\theta, sin\theta) + t(cos\theta, sin\theta)) - w(r_2(cos\theta, sin\theta))}{t} = \hat{w}_r(r_2, \theta), \\
\frac{\partial w}{\partial \theta}(r_1e^{i\theta}) &= -\lim_{t \to 0} \frac{w(r_1(cos\theta, sin\theta) + t(cos\theta, sin\theta)) - w(r_1(cos\theta, sin\theta))}{t} = -\hat{w}_r(r_1, \theta).
\end{align*}
\]

For the last claim lengthy computations show that
\[
\begin{align*}
wx(re^{i\theta}) &= -\frac{r \sin \theta}{r^2} \hat{w}_\theta(r, \theta) + \frac{r \cos \theta}{r^2} \hat{w}_r(r, \theta) \quad \text{if} \quad \cos \theta < 0, \ \sin \theta \neq 0, \\
w_y(re^{i\theta}) &= \frac{r \cos \theta}{r^2} \hat{w}_\theta(r, \theta) + \frac{r \sin \theta}{r^2} \hat{w}_r(r, \theta) \quad \text{if} \quad \cos \theta < 0, \ \sin \theta \neq 0, \\
w_x(re^{-i\pi}) &= -\hat{w}_r(-r,-\pi), \\
w_y(re^{-i\pi}) &= -\hat{w}_\theta(-r,-\pi).
\end{align*}
\] Combining equations (6) with the fact that arg(·) is harmonic in $\mathbb{C} \setminus \{z : \Re(z) < 0, \ \Im(z) = 0\}$ (and hence arg(·) $\in C^2(\mathbb{C} \setminus \{z : \Re(z) < 0, \ \Im(z) = 0\}$) the conclusion follows. This ends the proof. \hfill \Box

What is more is that the above proposition shows that in the case of annuli one can reformulate the Dirichlet and the Neumann problems (1) – (2) in polar coordinates as follows: find $u = u(r, \theta) \in C^2((r_1,r_2) \times \mathbb{R}) \cap C^0([r_1,r_2] \times \mathbb{R})$ which is $2\pi$-periodic in the second variable and satisfies
\[
\begin{align*}
&u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{in} \ (r_1,r_2) \times \mathbb{R}, \\
u = \varphi \quad \text{on} \ \{r_1,r_2\} \times \mathbb{R},
\end{align*}
\]
respectively find $U = U(r, \theta) \in C^2((r_1,r_2) \times \mathbb{R}) \cap C^0([r_1,r_2] \times \mathbb{R})$ which is $2\pi$-periodic in the second variable and satisfies
\[
\begin{align*}
&U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad \text{in} \ (r_1,r_2) \times \mathbb{R}, \\
U_r = \phi \quad \text{on} \ \{r_1,r_2\} \times \mathbb{R},
\end{align*}
\]
and the boundary data \( \varphi, \phi : \{r_1, r_2\} \times \mathbb{R} \) is related to the boundary data \( f, g : \partial A_{r_1, r_2} \to \mathbb{R} \) in (1) – (2) by

\[
\varphi(r, \theta) = g(re^{i\theta}) \quad \text{and} \quad \phi(r, \theta) = \begin{cases} f(re^{i\theta}) & \text{if } r = r_2, \\ -f(re^{i\theta}) & \text{if } r = r_1, \end{cases}
\]

and we note that in particular the functions \( \varphi, \phi \) are \( 2\pi \)-periodic in the second variable.

**Remark 2.** The compatibility condition \( \int_{\partial A_{r_1, r_2}} f \, d\sigma = 0 \) for the existence of a solution of the Neumann problem \( (2) \) in cartesian coordinates becomes, in polar coordinates, the following:

\[
\int_{0}^{2\pi} r_1 \phi(r_1, \theta) \, d\theta = \int_{0}^{2\pi} r_2 \phi(r_2, \theta) \, d\theta.
\] (9)

### 3 Main results

#### 3.1 The annulus and doubly-connected regions

With the above preparations, the main result of this section is presented below.

**Theorem 1.** Let \( 0 < r_1 < r_2 \) and assume \( \phi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( 2\pi \)-periodic in the second variable, and satisfies the compatibility condition

\[
\int_{0}^{2\pi} r_1 \phi(r_1, \theta) \, d\theta = \int_{0}^{2\pi} r_2 \phi(r_2, \theta) \, d\theta.
\]

If \( u \) is the solution of the Dirichlet problem \( (7) \) with \( \varphi(r, \theta) = r\phi(r, \theta) \) on \( \{r_1, r_2\} \times \mathbb{R} \), then for any \( (r, \theta) \in [r_1, r_2] \times \mathbb{R} \),

\[
U(r, \theta) = \int_{\frac{\sqrt{r_1 r_2}}{r}}^{\frac{\sqrt{r_1 r_2}}{r}} \int_{0}^{\theta} \left( C - \int_{0}^{t} u_r(\sqrt{r_1 r_2}, \tau) \, d\tau \right) \, dt,
\] (10)

where

\[
C = \frac{\sqrt{r_1 r_2}}{2\pi} \int_{0}^{\frac{\sqrt{r_1 r_2}}{r}} u_r(\sqrt{r_1 r_2}, \tau) \, d\tau \, d\tau,
\] (11)

is the solution of the Neumann problem \( (8) \) satisfying \( U(\sqrt{r_1 r_2}, 0) = 0 \). In addition \( U_r \) can be continuously extended to \( [r_1, r_2] \times \mathbb{R} \).

Conversely, if \( \varphi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( 2\pi \)-periodic in the second variable, and satisfies \( \int_{0}^{2\pi} \varphi(r_1, \theta) \, d\theta = \int_{0}^{2\pi} \varphi(r_2, \theta) \, d\theta \), and if \( U \) is any solution of the Neumann problem \( (8) \) with \( \phi(r, \theta) = \frac{\varphi(r, \theta)}{r} \) for \( (r, \theta) \in \{r_1, r_2\} \times \mathbb{R} \), then

\[
u(r, \theta) = ru(r, \theta), \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R},
\]

is the solution of the Dirichlet problem \( (7) \).
Step 1. Show that $\phi \rho$ where $\phi$ is the desired solution of the Neumann problem (8) on $A_{\alpha,a}$ with boundary data $\phi (r, \theta) = \begin{cases} f (r e^{i \theta}) & \text{if } r = a, \\ -f (r e^{i \theta}) & \text{if } r = \frac{1}{a}, \end{cases}$ where $u$ is the solution of the Dirichlet problem (7) with boundary data $\varphi (r, \theta) = r \phi (r, \theta)$ on $\{ 1, a \} \times \mathbb{R}$.

Proof. Let me first consider $r_2 = \frac{1}{r_1} = a > 1$, in which case the problem reduces to showing that the function

$$U (r, \theta) = \int_0^1 \frac{u (r \rho, \theta)}{\rho} d\rho + \int_0^\theta \left( C - \int_0^t u_r (1, \tau) d\tau \right) dt, \quad (12)$$

is the desired solution of the Neumann problem (8) on $A_{\alpha,a}$ with boundary data $\phi (r, \theta) = \begin{cases} f (r e^{i \theta}) & \text{if } r = a, \\ -f (r e^{i \theta}) & \text{if } r = \frac{1}{a}, \end{cases}$ where $u$ is the solution of the Dirichlet problem (7) it follows by Proposition 1 that $U (r, \theta) = \frac{u (r, \theta)}{r}$, $U_r (r, \theta) = \frac{u_r (r, \theta)}{r}$, $U_r (a, \theta) = \frac{u (a, \theta)}{\alpha} = \phi (a, \theta)$, $U_r (\frac{1}{a}, \theta) = \frac{u (\frac{1}{a}, \theta)}{\alpha} = \phi (\frac{1}{a}, \theta)$.

Define $W : A_{\alpha,a} \to \mathbb{R}$, $W (z) := u (|z|, \arg (z))$. Since $u$ is the solution of the Dirichlet problem (7) it follows by Proposition 1 that $W (r e^{i \theta}) = \varphi (r, \theta)$, $\forall (r, \theta) \in \{ \frac{1}{a}, a \}$.

Then $\exists \alpha, \beta \in \mathbb{R}$ such that $\int_0^{2\pi} W (r e^{i \theta}) d\theta = \alpha \log r + \beta, \forall r \in \left[ \frac{1}{a}, a \right]$ (see Chapter 4). But then $-\alpha \log a + \beta = \int_{C_{\frac{1}{a}}} W \left( \frac{1}{a} e^{i \theta} \right) d\theta = \int_0^{2\pi} u \left( \frac{1}{a}, \theta \right) d\theta = \frac{2\pi}{a} \phi \left( \frac{1}{a}, \theta \right) d\theta = \frac{2\pi}{0} a \phi (a, \theta) d\theta = \int_{C_{a}} W \left( a e^{i \theta} \right) = \alpha \log a + \beta$ which implies that $\alpha = 0$.

To sum up $\int_{C_r} W \left( r e^{i \theta} \right) d\theta = \int_0^{2\pi} u (r, \theta) d\theta$ is a constant function of $r$. Taking the derivative it follows that $\int_0^{2\pi} u (r, \theta) d\theta = 0$. Since $1 \in \left( \frac{1}{a}, a \right)$, an application of the Dominant Convergence Theorem together with the above identity concludes this part of the proof.
Step 2. Show that \( U(r, \theta + 2\pi) = U(r, \theta), \ (r, \theta) \in \left(\frac{1}{a}, a\right) \times \mathbb{R} \).

Compute \( U(r, \theta + 2\pi) = \int_0^{\theta + 2\pi} \frac{u(r, \theta + 2\pi)}{\rho} d\rho - \int_0^\theta \int u_r(1, \tau) d\tau dt + C(\theta + 2\pi) = \frac{1}{\pi} \int \frac{u(r, \theta)}{\rho} d\rho - \int_0^\theta \int u_r(1, \tau) d\tau dt + 2\pi C. \)

Since \( u(r, \theta) = \lim_{t \to 0}^{t + 2\pi} \frac{u(r, \theta) - u(r, \theta + 2\pi)}{r - r_0} \) it follows that \( u_r(r_0, \theta + 2\pi) = \lim_{r \to r_0} \frac{u(r, \theta) - u(r, \theta + 2\pi)}{r - r_0} = u_r(r_0, \theta) \) \( \forall (r_0, \theta). \) Thus \( u_r(1, \cdot) \) is 2\( \pi \)-periodic and so \( \int_0^t u_r(1, \tau) d\tau = 2\pi \int_0^t u_r(1, \tau) d\tau \) \( \text{Step 1.} \) Consequently it follows that the function \( t \to \int_0^t u_r(1, \tau) d\tau \) is 2\( \pi \)-periodic. This shows in turn that

\[
\int_0^\theta \int u_r(1, \tau) d\tau dt = 2\pi \int_0^\theta u_r(1, \tau) d\tau dt = 2\pi C \quad \text{which concludes this part of the proof.}
\]

Step 3. Show that \( U \) satisfies (4) in \( \left(\frac{1}{a}, a\right) \times \mathbb{R} \).

Indeed, using Leibniz–Newton formula

\[
\begin{align*}
U_r(r, \theta) &= \frac{u(1, \theta)}{r} + \frac{1}{r} \int u_r(r, \rho, \theta) d\rho,
U_{rr}(r, \theta) &= -\frac{u(1, \theta)}{r^2} + \frac{u_r(1, \theta)}{r^2} + \frac{1}{r^2} \rho u_{rr}(r, \rho, \theta) d\rho,
U_{\theta\theta}(r, \theta) &= -u_r(1, \theta) + \frac{1}{r^2} \int u_{\theta\theta}(r, \rho, \theta) d\rho,
\end{align*}
\]

and thus \( U_{rr}(r, \theta) + \frac{1}{r} U_r(r, \theta) + \frac{1}{r^2} U_{\theta\theta}(r, \theta) = \int \rho \left( u_{rr}(r, \rho, \theta) + \frac{1}{r} u_r(r, \rho, \theta) + \frac{1}{r^2} u_{\theta\theta}(r, \rho, \theta) \right) d\rho, \) where the quantity in the right-hand side is identically 0 since \( u \) verifies relation (4).

Step 4. Show that the derivative of \( U \) with respect to the first argument exists, is finite, and equals \( \phi(r, \theta), \) at all points \( (r, \theta) \in \{r_1, r_2\} \times \mathbb{R}. \)

Indeed \( \lim_{r \to a^+} \frac{U(r, \theta) - U(a, \theta)}{r - a} = \phi(a, \theta), \) and likewise \( \lim_{r \to a^+} \frac{U(r, \theta) - U \left( \frac{1}{a}, \theta \right)}{r - \frac{1}{a}} = \phi \left( \frac{1}{a}, \theta \right). \)

This completes the proof of the first part in the case \( r_2 = a > 1 > \frac{1}{a} = r_1. \)

For the general case \( 0 < r_1 < r_2 \) define \( \lambda = \frac{1}{\sqrt{r_1 r_2}}, \ a = \sqrt{\frac{r_2}{r_1}} \) and let \( \tilde{u} \) be the solution of the Dirichlet problem (7) on \( A_{\frac{1}{a}, a} \) with boundary data \( \tilde{\varphi}(r, \theta) = \varphi \left( \frac{1}{a}, \theta \right) = \frac{\lambda}{r} \varphi \left( \frac{1}{a}, \theta \right) \) \( \forall (r, \theta) \in \left(\frac{1}{a}, a\right) \times \mathbb{R}. \) By the previous part the function \( \tilde{U}(r, \theta) = \int_0^\lambda \frac{\tilde{u}(r, \rho, \theta)}{\rho} d\rho + \int_0^\theta \left( C - \int_0^t \tilde{u}_r(1, \tau) d\tau \right) dt \) is the solution of the
Neumann problem \([8]\) with boundary data \(\tilde{\phi}(r, \theta) = \frac{\phi(r, \theta)}{r}\) on \(\{\frac{a}{2}, a\} \times \mathbb{R}\) which satisfies \(\tilde{U}(1, 0) = 0\).

Consequently defining \(U(R, \theta) = \tilde{U}(\lambda R, \theta), (R, \theta) \in (r_1, r_2) \times \mathbb{R}\), it follows that \(\frac{\partial}{\partial R}U(R, \theta) = \lambda \tilde{U}_r(\lambda R, \theta)\) from where \(\frac{\partial U}{\partial R}(r_2, \theta) = \lambda \tilde{U}_r(a, \theta) = \lambda \frac{\partial }{\partial R}(\frac{1}{a}, \theta) = \phi(r_2, \theta)\) and also \(\frac{\partial U}{\partial R}(r_1, \theta) = \lambda \tilde{U}_r(\frac{1}{a}, \theta) = \lambda \frac{\partial }{\partial R}(\frac{1}{a}, \theta) = \phi(r_1, \theta)\). In addition notice that equation (4) is fulfilled for \(U\) on \((r_1, r_2) \times \mathbb{R}\), and since \(U(\sqrt{r_1 r_2}, 0) = \tilde{U}(1, 0) = 0\), the proof of the first part is completed.

Using the first part of the Theorem, the proof of the second part is trivial. \(\square\)

If an additional assumption on the smoothness of \(\phi\) is added, one can strengthen the result in Theorem [1]. To see that this is the case, define a G-strong solution of the Neumann problem \([8]\) to be a function \(U \in C^2((r_1, r_2) \times \mathbb{R}) \cap C^1([r_1, r_2] \times \mathbb{R})\) satisfying relations \([8]\). In the same way, \(U\) will be called a G-strong solution of the Neumann problem \([2]\) if \(U \in C^2(D) \cap C^1(D)\) and satisfies \([2]\), \(D = A_{r_1; r_2}\).

**Theorem 2.** Let \(0 < r_1 < r_2\) and assume \(\phi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R}\) is \(2\pi\)-periodic in the second variable, satisfies the compatibility condition \(\int_0^{2\pi} r_1 \phi(r_1, \theta) \, d\theta = 0\), \(\int_0^{2\pi} r_2 \phi(r_2, \theta) \, d\theta\), and in addition \(\phi(r, \cdot) \in C^{2,\alpha}(\mathbb{R}) \forall r \in \{r_1, r_2\},\) for some \(0 < \alpha < 1\).

If \(u\) is the solution of the Dirichlet problem \([7]\) with \(\varphi(r, \theta) = r \phi(r, \theta)\) on \(\{r_1, r_2\} \times \mathbb{R}\), then

\[
U(r, \theta) = \int_{\sqrt{r_1 r_2}}^{1} \frac{u(r \rho, \theta)}{\rho} \, d\rho + \sqrt{r_1 r_2} \int_0^\theta \left( C - \int_0^t u_r(\sqrt{r_1 r_2}, \tau) \, d\tau \right) \, dt,
\]

where

\[
C = \frac{\sqrt{r_1 r_2}}{2\pi} \int_0^t \int_0^{2\pi} u_r(\sqrt{r_1 r_2}, \tau) \, d\tau \, dt,
\]

is the G-strong solution of the Neumann problem \([8]\) satisfying \(U(\sqrt{r_1 r_2}, 0) = 0\).

Conversely, if \(\varphi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R}\) is continuous, \(2\pi\)-periodic in the second variable, and satisfies \(\int_0^{2\pi} \varphi(r_1, \theta) \, d\theta = \int_0^{2\pi} \varphi(r_2, \theta) \, d\theta\), and if \(U\) is any (G-strong) solution of the Neumann problem \([8]\) with \(\phi(r, \theta) = \frac{\varphi(r, \theta)}{r}\) for \((r, \theta) \in \{r_1, r_2\} \times \mathbb{R}\), then

\[
u(r, \theta) = rU_r(r, \theta), \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R},
\]

is the solution of the Dirichlet problem \([7]\).
Proof. In the light of Theorem 1 it will only have to be shown that \( U_0 \) can be continuously extended to \([r_1, r_2] \times \mathbb{R}\). Also it is sufficient to consider the case \( r_2 = a = \frac{1}{r_1} > 1 \). The proof will be based on the well-known Fourier series-solution for the Dirichlet problem in \( A_{\frac{1}{r_1}, a} \) with boundary data \( \varphi(r, \theta) = r \Phi(r, \theta), \ r \in \left[ \frac{1}{a}, a \right] \).

To this end let \( u \) be the solution of the Dirichlet problem \( \Phi \) with boundary data \( \varphi \).

Since \( \phi(r, \cdot) \in C^{2, \alpha}(\mathbb{R}) \) it follows that \( \varphi(r, \cdot) \in C^{2, \alpha}(\mathbb{R}) \) as well and we have
\[
\begin{cases}
\varphi(r_1, \theta) = a_0^{(1)} + \sum_{k=1}^{\infty} a_k^{(1)} \cos k \theta + b_k^{(1)} \sin k \theta, \\
\varphi(r_2, \theta) = a_0^{(2)} + \sum_{k=1}^{\infty} a_k^{(2)} \cos k \theta + b_k^{(2)} \sin k \theta,
\end{cases}
\]
where \( \max \{a_k^{(1)}, a_k^{(2)}, b_k^{(1)}, b_k^{(2)}\} \leq \frac{\mathcal{K}}{k^{2+\alpha}} \) for some \( \mathcal{K} > 0, k \in \mathbb{N}^* \). Consequently if \( A, B \) as well as \( C_k, D_k, E_k, G_k, k \in \mathbb{N}^* \), satisfy the following system of equations
\[
\begin{align*}
A - B \log a &= a_0^{(1)}, \\
A + a \log a &= a_0^{(2)}, \\
C_k a_k^{(1)} + D_k a_k^{(2)} &= a_k^{(1)}, \\
C_k a_k^{(1)} + D_k a_k^{(2)} &= a_k^{(2)}, \\
E_k a_k^{(1)} + G_k a_k^{(1)} &= b_k^{(1)}, \\
E_k a_k^{(1)} + G_k a_k^{(1)} &= b_k^{(2)},
\end{align*}
\]
it can be easily deduced that

\[
\begin{align*}
|C_k r^k + D_k r^{-k}| &\leq \left( |a_k^{(1)}| + |a_k^{(2)}| \right) \leq \frac{2\mathcal{K}}{k^{2+\alpha}}, \\
|E_k r^k + G_k r^{-k}| &\leq \left( |b_k^{(1)}| + |b_k^{(2)}| \right) \leq \frac{2\mathcal{K}}{k^{2+\alpha}}.
\end{align*}
\]

For any \( r \in \left[ \frac{1}{a}, a \right] \), and also if \( k \) is large enough \( \max \{|C_k r^k|, |D_k r^{-k}|, |E_k r^k|, |G_k r^{-k}| \} \leq 4 \left( \frac{\mathcal{K}}{a} \right)^k \left( \frac{1}{a^k} \right) \). Using now the last inequality it is easy to prove that
\[
\begin{align*}
u(r, \theta) &= A + B \log r + \sum_{k=1}^{\infty} \left( C_k r^k + D_k r^{-k} \right) \cos k \theta + \left( E_k r^k + G_k r^{-k} \right) \sin k \theta, \\
u_\theta(r, \theta) &= A + B \log r \\
&\quad + \sum_{k=1}^{\infty} \left[ - \left( kC_k r^k + kD_k r^{-k} \right) \sin k \theta + \left( kE_k r^k + kG_k r^{-k} \right) \cos k \theta \right].
\end{align*}
\]

Using them again it can be inferred that \( u_\theta \) can be continuously extended to \( \left[ \frac{1}{a}, a \right] \times \mathbb{R} \). But then it follows that \( u_\theta \) is uniformly bounded on \( \left[ \frac{1}{a}, a \right] \times \left[ -\pi, \pi \right] \), and by periodicity also on \( \left[ \frac{1}{a}, a \right] \times \mathbb{R} \). So it can be argued that
\[
U_\theta(r, \theta) = \frac{1}{\sqrt{\pi} r_1 r_2} \int_0^{\theta} \frac{u_\theta(r, \rho)}{\rho} \, d\rho + \sqrt{r_1 r_2} \mathcal{C} + \int_0^{\theta} u_\theta(\sqrt{r_1 r_2} t) \, dt \text{ can be continuously extended to } \left[ \frac{1}{a}, a \right] \times \mathbb{R}. \text{ This completes the proof.} \]
Some corollaries and remarks will be provided in what follows.

First notice that for \( r_1 \downarrow 0 \) and \( r_2 = 1 \), the region \( A_{r_1,r_2} \) becomes the punctured unit disk \( A_{0,1} = \{ z \in C : 0 < |z| < 1 \} = \hat{U} \). If \( w : \hat{U} \to \mathbb{R} \) is a harmonic function having a finite limit at the origin (an isolated boundary point of the domain), then it is known that \( w \) can be extended by continuity at the origin, and the resulting function is harmonic in \( U \).

If \( w \) has a continuous extension to \( \hat{U} \), with boundary values \( w(0) \equiv \phi(0, \cdot) \) (constant function of \( \theta \in \mathbb{R} \)) and \( w(e^{i\theta}) = \phi(1, \theta), \theta \in \mathbb{R} \), then the condition

\[
\frac{1}{2\pi} \int_0^{2\pi} \phi(0, \theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \, d\theta
\]

in Theorem 1 is a necessary condition for the solvability of the Dirichlet problem in \( \hat{U} \) with continuous boundary data \( \phi \), and it shows that

\[
w(0) \equiv \phi(0, \cdot) = 1
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \, d\theta \equiv \phi(0, \cdot).
\]

Subtracting a constant if necessary (i.e. considering \( w - w(0) \) instead of \( w \)), without loss of generality it can be assumed that \( w(0) = 0 \), or equivalently

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \, d\theta \equiv \phi(0, \cdot).
\]

The above discussion shows that in the case of the punctured disk \( \hat{U} = A_{0,1} \), the Dirichlet problem \([7]\) has a unique solution for continuous boundary data \( \phi \) under the hypothesis

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \, d\theta \equiv \phi(0, \cdot)
\]

(which implies \( w(0) = 0 \)), which coincides with the solution of the Dirichlet problem in the whole unit disk \( U \) with boundary data \( \phi(1, \cdot) \) on \( \partial U \), and thus under these hypotheses one can simply ignore the boundary condition at the origin (isolated boundary point of \( \hat{U} \)).

Similarly, for continuous boundary data \( \phi \) satisfying

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \, d\theta,
\]


\[
\phi(0, \theta) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt, \theta \in \mathbb{R},
\]

the Neumann problem \([8]\) has a unique solution such that \( U(0, \cdot) \equiv 0 \), which coincides with the solution of the Neumann problem in the whole unit disk \( U \) that vanishes for \( |z| = 0 \) and has boundary data \( \phi(1, \cdot) \) on \( \partial U \). Indeed let \( U_0 \) be the solution of the Neumann problem in \( U \) with boundary data \( \Phi_0(z) = \phi(1, \arg(z)) \) satisfying \( U_0(0) = 0 \). By applying

\([6, \text{Theorem 1}]\) it follows that \( U_0(z) = \int_0^{\frac{\pi}{2}} \frac{u_0(\rho z)}{\rho} \, d\rho \), where \( u_0 \) is the solution of the Dirichlet problem in \( U \) with boundary data \( \varphi_0 = \Phi_0 \). But then denoting \( \hat{U}_0(r, \theta) = U_0(\rho e^{i\theta}), (r, \theta) \in [0, 1] \times \mathbb{R} \), it follows by Proposition 1 that \( \hat{U}_0 \) is 2\( \pi \)-periodic in the second variable, has continuous second order partial derivatives, and satisfies equation \([4]\) in \( (0, 1) \times \mathbb{R} \), and in addition it has finite partial derivative with respect to the first variable at any point \( (1, \theta_0) \), \( \theta_0 \in \mathbb{R} \). Moreover
Remark 3. As we have already remarked, Definition 1 is equivalent to the Dirichlet problem, in a consistent way, for the punctured disk as well. This fact well as the Dominant Convergence Theorem, that \( \frac{\partial u_0}{\partial r}(0) = \frac{1}{\pi} \int_0^{2\pi} \Phi_0(e^{it}) \cos t \, dt = \frac{1}{\pi} \int_0^{2\pi} \Phi_0(e^{it}) \sin t \, dt \), which finally gives \( \frac{\partial u_0}{\partial r}(0, \theta) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt \).

To sum up it can be concluded that

\[
\frac{\partial U_0}{\partial r}(r, \theta) = \begin{cases} \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt, & \text{if } r = 0, \\ \phi(1, \theta), & \text{if } r = 1. \end{cases}
\]

With this preamble I shall introduce the following two definitions, with the convention that in both of them \( D = A_{0,1} \).

**Definition 1.** If \( \varphi : \mathbb{R} \to \mathbb{R} \) is continuous, \( 2\pi \)-periodic, and satisfies \( \int_0^{2\pi} \varphi(\theta) \, d\theta = 0 \), then the Dirichlet problem in polar coordinates for \( D \) consists in finding \( u = u(r, \theta) \in C^2((0,1) \times \mathbb{R}) \cap C^0([0,1] \times \mathbb{R}) \) which is \( 2\pi \)-periodic in the second variable and satisfies

\[
\begin{align*}
    u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0 & \text{in } (0,1) \times \mathbb{R}, \\
    u(1, \cdot) &= \varphi(\cdot), \\
    u(0, \cdot) &= 0.
\end{align*}
\]

(16)

**Definition 2.** If \( \phi : \{0;1\} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( 2\pi \)-periodic in the second argument, and satisfies \( \int_0^{2\pi} \phi(1, \theta) \, d\theta = 0 \) as well as \( \phi(0, \theta) = \frac{1}{\pi} \int_0^{2\pi} \cos(t - \theta) \phi(1, t) \, dt \), then the Neumann problem in polar coordinates for \( D \) consists in finding \( U \in C^2((0,1) \times \mathbb{R}) \cap N((0,1) \times \mathbb{R}) \) which is \( 2\pi \)-periodic in the second variable, and satisfies

\[
\begin{align*}
    U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} &= 0 & \text{in } (0,1) \times \mathbb{R}, \\
    U_r(0, \cdot) &= \phi \quad \text{in } \{0;1\} \times \mathbb{R}, \\
    U(0, \cdot) &= 0.
\end{align*}
\]

(17)

**Remark 3.** As we have already remarked, Definition 1 is equivalent to the Dirichlet problem (1) for \( D = \hat{U} \) and boundary data \( g(z) = \varphi(\arg(z)) \) on \( \partial \hat{U} \).

**Remark 4.** Definition 2, instead, comes with a novelty which allows us to formulate this problem, in a consistent way, for the punctured disk as well. This fact
is in contrast with the (classical) Neumann problem where the (outward) normal derivative at \( \{0\} \) can not be defined.

In addition it reveals that if \( \hat{U} \) is the solution of the Neumann problem \([17]\) on \( A_{0;1} \), then \( \hat{U} \) is just the representation in polar coordinates of the (classical) solution \( U \) to the Neumann problem \([2]\) on \( \mathbb{U} \), with boundary data \( f(z) = \phi(1, \arg(z)) \) on \( \partial \mathbb{U} \) and \( U(0) = 0 \).

The next corollary shows that Theorem \([1]\) is a generalization of the main result in \([6]\) (actually its first part is exactly Theorem 1 in \([6]\) when the unit ball has dimension 2). This will show, in particular, that the theory presented so far is a more powerful tool in \( \mathbb{R}^2 \) which embeds the main result in \([6]\) as a particular case.

**Corollary 1.** Assume \( f : \partial \mathbb{U} \rightarrow \mathbb{R} \) is continuous and satisfies \( \int_0^{2\pi} f \, d\theta = 0 \). If \( u \) is the solution of the Dirichlet problem \([1]\) on \( \mathbb{U} \) with boundary data \( g(z) = f(z) \) on \( \partial \mathbb{U} \), then

\[
U(z) = \int_0^1 \frac{u(\rho z)}{\rho} \, d\rho, \quad z \in \mathbb{U}
\]

is the solution of the Neumann problem \([2]\), satisfying \( U(0) = 0 \).

Conversely if \( g : \partial \mathbb{U} \rightarrow \mathbb{R} \) is a continuous function satisfying \( \int_0^{2\pi} g \, d\theta = 0 \) and if \( U \) is any solution of the Neumann problem \([2]\) on \( D = \mathbb{U} \) with boundary data \( f = g \), then the solution \( u \) of the Dirichlet problem \([1]\) on \( \mathbb{U} \) with boundary data \( g \) is given by

\[
u(r e^{i\theta}) = r \left( U_x(r e^{i\theta}) \cos \theta + U_y(r e^{i\theta}) \sin \theta \right) =: r \frac{\partial U}{\partial e^{i\theta}}(r e^{i\theta}), \quad r \in \mathbb{U}.
\]

**Proof.** Define \( r_n = r_1(n) = \frac{1}{n^2} \), \( A_n = A_{r_n;1} \), \( n \in \mathbb{N} \setminus \{0, 1\} \). On \( A_n \) let \( u_n(\cdot) \) be the solution of the Dirichlet problem \([1]\) with boundary data \( g_n = u \) on \( \partial A_n \). By the uniqueness of the solution of the Dirichlet problem it follows that \( u_n = u \) on \( A_n \). Consequently define

\[
U_n(r e^{i\theta}) = \int_0^1 \frac{u_n(r e^{i\theta})}{r} \, d\rho + \frac{1}{n} \int_0^\theta \left( C_n - \int_0^t \frac{\partial u_n}{\partial e^{i\tau}} \left( \frac{e^{i\tau}}{n} \right) \, d\tau \right) dt, \quad r e^{i\theta} \in A_n, \quad \text{where} \quad C_n = \frac{1}{2\pi n} \int_0^{2\pi} \int_0^1 \frac{\partial u_n}{\partial e^{i\tau}} \left( \frac{e^{i\tau}}{n} \right) \, d\tau \, dt.
\]

Let now \( K \subset A_{0;1} \) be any compact set. Hence \( \exists N_K \in \mathbb{N} \setminus \{0, 1\} \) such that \( \forall n \geq N_K \Rightarrow K \subset A_n \). So choose any \( n \geq N_K \) and any \( p \in \mathbb{N}^* \) and observe that

\[
|U_{n+p}(r e^{i\theta}) - U_n(r e^{i\theta})| \leq \int_0^1 \frac{u(r e^{i\theta})}{r} \, d\rho + \frac{1}{n+p} \int_0^\theta \left( C_{n+p} - \int_0^t \frac{\partial u}{\partial e^{i\tau}} \left( \frac{e^{i\tau}}{n+p} \right) \, d\tau \right) dt - \frac{1}{n} \int_0^\theta \left( C_n - \int_0^t \frac{\partial u}{\partial e^{i\tau}} \left( \frac{e^{i\tau}}{n} \right) \, d\tau \right) dt.
\]
To evaluate the first term, notice first that since \(u \in C^0(U) \cap C^1(U)\) we have
\[
\lim_{\rho \to 0} \frac{u(\rho e^{i\theta})}{\rho} = u_x(0)r \cos \theta + u_y(0)r \sin \theta = r \frac{\partial u}{\partial e^{i\theta}}(0) \forall \ re^{i\theta} \in \mathbb{U}.
\]
Hence \(\exists M_1 > 0\) such that
\[
\left| \frac{u(\rho e^{i\theta})}{\rho} \right| \leq M_1 \forall \rho \in [0,1], \forall \ re^{i\theta} \in \mathbb{U}.
\]
On the other hand, since \(0 \notin K\) it follows that there exists \(\delta_K > 0\) such that \(d(0, K) = \delta_K\). The last two observations in turn imply that
\[
\int_{(1/n+p)} \left| \frac{u(\rho e^{i\theta})}{\rho} \right| d\rho \leq \frac{p M_1}{\delta_k n(n+p)}, \forall \ re^{i\theta} \in K, \forall n \geq N_K, \forall p \in \mathbb{N}^*.
\]
Next since \(u \in C^1(U)\) it can be concluded that \(\nabla u\) is bounded on, say, \(|z| \leq 2/3\) which in turn shows that one can choose \(M_2 > 0\) for which
\[
\left| \nabla u \left( \frac{\rho e^{i\theta}}{n} \right) \right| \leq M_2, \forall r \in \mathbb{R}. \quad \text{So} \quad |C_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 M_2 \ d\tau dr = \frac{\pi M_2}{n}, \forall n \in \mathbb{N} \setminus \{0,1\}.
\]
Finally, putting everything together it follows that \(|U_{n+p}(r e^{i\theta}) - U_n(r e^{i\theta})| \leq \left| \int_{(1/n+p)} \frac{\rho M_1}{\delta_k n(n+p)} + \theta \pi M_2 \left( \frac{1}{n^2} - \frac{1}{(n+p)^2} \right) + \theta^2 M_2 n \right|, \forall \ re^{i\theta} \in K, \forall n \geq N_K.
\]
Since we can consider without loss of generality that \(\theta \in [-\pi, \pi]\), it follows that the sequence of harmonic functions \(\{U_n\}_{n=2}^\infty\) is uniformly Cauchy on \(K\), and hence on any compact subset of \(A_{0;1}\). Furthermore it is easy to see that
\[
\lim_{n \to \infty} U_n(z) = \frac{1}{0} \frac{u(\rho z)}{\rho} d\rho = U(z) \text{ on } A_{0;1}.
\]
Hence \(U\) is harmonic on \(A_{0;1}\). In addition, using the Dominant Convergence Theorem it follows that \(\lim U(x) = 0\).

This shows that \(U\) can be (uniquely) extended to a harmonic function on the whole unit disk, which I shall also denote for brevity \(U\). It is not difficult to check that \(U\) can actually be extended by continuity to the whole \(\overline{U}\).

Finally
\[
\frac{\partial U}{\partial e^{i\theta}} (e^{i\theta}) = \lim_{\epsilon \to 0} \frac{U(e^{i\theta} + \epsilon e^{i\theta}) - U(e^{i\theta})}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_1^{1+\epsilon} \frac{u(\rho e^{i\theta})}{\rho} \ d\rho = u(e^{i\theta}) = g(e^{i\theta}), \forall \theta \in \mathbb{R}.
\]

For the second part denote \(\hat{U}(r, \theta) = U(r e^{i\theta})\), \(\hat{u}(r, \theta) = u(r e^{i\theta})\), \(r e^{i\theta} \in \mathbb{U}\), where one can choose \(U(0) = 0\). Using the first part \(\hat{U}(r, \theta) = \int_0^r \frac{\hat{u}(\rho r, \theta)}{\rho} \ d\rho = \int_0^r \frac{\hat{u}((\rho, \theta))}{\rho} \ d\rho, \ re^{i\theta} \in \mathbb{U}\). Taking the derivative with respect to the first argument one obtains \(\hat{U}_r(r, \theta) = \hat{u}(r, \theta) = r \hat{U}(r, \theta)\), for any \(r \in (0;1)\). Since \(\hat{U}_r(r, \theta) = \frac{\partial U}{\partial e^{i\theta}} (r e^{i\theta})\) the conclusion follows. \(\Box\)

In the particular case when the boundary data is symmetric, the result in Theorem 1 has the following simplified form.
Theorem 3. Let \(0 < r_1 < r_2\) and assume \(\phi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R}\) is continuous, \(2\pi\)-periodic in the second argument, verifies the Dirichlet conditions as a function of \(\theta\), and satisfies \(r_1 \phi(r_1, \theta) = r_2 \phi(r_2, \theta)\) for \(\theta \in \mathbb{R}\). If \(u\) is the solution of the Dirichlet problem \([7]\) with boundary data \(\varphi(r, \theta) = r \phi(r, \theta)\) on \(\{r_1, r_2\} \times \mathbb{R}\), then

\[
U(r, \theta) = \int_{\sqrt{r_1^2 - r^2}}^{r} \frac{u(r, \theta, \rho)}{\rho} d\rho, \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R}
\]

is the solution of the Neumann problem \([3]\) satisfying \(U(\sqrt{r_1 r_2}, \theta) \equiv 0\).

Conversely, if \(\varphi : \{r_1, r_2\} \times \mathbb{R} \to \mathbb{R}\) is continuous, \(2\pi\)-periodic in the second variable, and satisfies \(\varphi(r_1, \theta) = \varphi(r_2, \theta)\) for \(\theta \in \mathbb{R}\), and if \(U\) is the solution of the Neumann problem \([3]\) with boundary data \(\phi(r, \theta) = \frac{\varphi(r, \theta)}{r}\) on \(\{r_1, r_2\} \times \mathbb{R}\), then

\[
u(r, \theta) = rU_r(r, \theta), \quad (r, \theta) \in [r_1, r_2] \times \mathbb{R},
\]

is the solution of the Dirichlet problem \([7]\).

Proof. It will be shown that under the additional hypothesis \(r_1 \phi(r_1, \theta) = r_2 \phi(r_2, \theta), \theta \in \mathbb{R}\) one has \(u(r, \theta) = u\left(\frac{r_1 r_2}{r}, \theta\right)\ \forall (r, \theta) \in [r_1, r_2] \times \mathbb{R}\) from where it follows by derivation with respect to the first argument that \(u_r(r, \theta) \equiv -\frac{r_1 r_2}{r} u_r\left(\frac{r_1 r_2}{r}, \theta\right)\) and taking \(r = \sqrt{r_1 r_2}\) it follows that \(u_r(\sqrt{r_1 r_2}, \theta) \equiv -u_r(\sqrt{r_1 r_2}, \theta)\) which implies that \(u_r(\sqrt{r_1 r_2}, \theta) = 0 \ \forall \ \theta \in \mathbb{R}\), and so \(U\) will have the desired expression.

Notice that it is enough to prove the result for the special case \(r_2 = a, \ r_1 = a^{-1}\), for the general case follows from this one by scalarization, in the same way it was done in the proof of Theorem \([1]\). Hence it can be assumed without loss of generality that \(r_2 = a, \ r_1 = a^{-1}, \ a > 1\). Writing again the Fourier expansions for \(\varphi(r_2, \cdot) \equiv \varphi(r_1, \cdot)\) it is obtained \(\varphi(r_2, \theta) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) = \varphi(r_1, \theta) \ \forall \ \theta \in \mathbb{R}\). But then the solution of the Dirichlet problem \([7]\) on \((\frac{1}{a}, a) \times \mathbb{R}\) with boundary data \(\varphi\) is given by

\[
u(r, \theta) = A + B \log r + \sum_{k=1}^{\infty} \left[(C_k r^k + D_k r^{-k}) \cos k\theta + (E_k r^k + G_k r^{-k}) \sin k\theta\right],
\]

where

\[
\begin{cases}
A - B \log a = a_0, \\
A + B \log a = a_0, \\
C_k a^{-k} + D_k a^k = a_k, \\
C_k a^k + D_k a^{-k} = a_k, \\
E_k a^{-k} + G_k a^k = b_k, \\
E_k a^k + G_k a^{-k} = b_k,
\end{cases}
\]

\(A = a_0, \ B = 0, \ C_k = D_k = \frac{a_k}{a^k + a^{-k}}, \ E_k = G_k = \frac{b_k}{a^k + a^{-k}}, \ k \in \mathbb{N}_0\), and hence

\[
u(r, \theta) = a_0 + \sum_{k=1}^{\infty} \left[\frac{a_k}{a^k + a^{-k}} (a_k \cos k\theta + b_k \sin k\theta)\right], \ (r, \theta) \in \left[\frac{1}{a}, a\right] \times \mathbb{R},
\]

which gives \(u(\cdot, \theta) \equiv u\left(\frac{1}{a}, \theta\right)\) and concludes the proof. \(\square\)
Combining Theorem 1, Theorem 2, and Proposition 1, one obtains

**Theorem 4.** Let \( f : \partial A_{r_1} \to \mathbb{R} \) be a continuous function satisfying \( \int \frac{f}{r} \, ds = 0 \).

If \( u \) is the solution of the Dirichlet problem \( \mathbf{1} \) with boundary data

\[
 g(z) = \begin{cases} 
 r_2 f(z) & \text{if } |z| = r_2, \\
 -r_1 f(z) & \text{if } |z| = r_1, 
\end{cases}
\]

then the solution \( U \) of the Neumann problem \( \mathbf{2} \) with boundary data \( f \) satisfying \( U(\sqrt{r_1 r_2}) = 0 \) is given by

\[
 U(re^{i\theta}) = \frac{1}{\sqrt{r_1 r_2}} \int_0^{2\pi} \frac{u(re^{i\theta})}{\rho} \, d\rho + \sqrt{r_1 r_2} \int_0^t \left( C - \int_0^t \frac{\partial u}{\partial e^{i\tau}}(\sqrt{r_1 r_2} e^{i\tau}) \, d\tau \right) \, dt, \quad e^{i\theta} \in A_{r_1 r_2},
\]

where

\[
 C = \frac{\sqrt{r_1 r_2}}{2\pi} \int_0^{2\pi} \int_0^t \frac{\partial u}{\partial e^{i\tau}}(\sqrt{r_1 r_2} e^{i\tau}) \, d\tau \, dt.
\]

If in addition \( f \in C^{2,\alpha}(\partial A_{r_1 r_2}) \), then \( U \) given in \( \mathbf{21} \) is a G-strong solution of the Neumann problem \( \mathbf{2} \).

Conversely, if \( g : \partial D \to \mathbb{R} \) is a continuous function satisfying

\[
 \int_0^{2\pi} g(r_2 e^{i\theta}) \, d\theta = \int_0^{2\pi} g(r_1 e^{i\theta}) \, d\theta,
\]

and if \( U \) is any (G-strong) solution of the Neumann problem \( \mathbf{2} \) with boundary data \( f(z) = \begin{cases} 
 g(z) & \text{if } |z| = r_2, \\
 -g(z) & \text{if } |z| = r_1, 
\end{cases} \)

then \( u(re^{i\theta}) = r \left[ U_x(re^{i\theta}) \cos \theta + U_y(re^{i\theta}) \sin \theta \right] = r \frac{\partial U}{\partial e^{i\theta}}(re^{i\theta}), \quad re^{i\theta} \in A_{r_1 r_2}, \)

is the solution of the Dirichlet problem \( \mathbf{1} \) with boundary data \( g \).

**Remark 5.** The constant \( C \) appearing in \( \mathbf{22} \) has a nice interpretation. Indeed it can be shown that

\[
 C = \frac{1}{2\pi} \int_{C_{\sqrt{r_1 r_2}}} v_0(e^{i\theta}) \, d\theta,
\]

where \( v_0 \) is the conjugate harmonic of \( u \) satisfying \( v_0(\sqrt{r_1 r_2}) = 0 \).

Using the conformal invariance of harmonic functions and Theorem 4, an important general result is obtained. Before stating it, some preparations are needed. First let \( D \subset \mathbb{C} \) be some smooth doubly connected region, whose boundary consists of two bounded Jordan curves which are the images of two regular bijective parametrizations \( \Gamma_i, \; i \in \{1, 2\} \). It will be assumed that \( \Gamma_1 \) corresponds to the inner contour. Following the approach in Chapter 6 let \( \omega_1 \) be the harmonic measure of \( \{\Gamma_1\} \) with respect to the region \( D \), and define \( \alpha_1 = \int_{\{\Gamma_1\}} \frac{d\omega_1}{\partial n} \, ds \). Consequently define \( \omega = \lambda_1 \omega_1 \), where \( \lambda_1 = \frac{2\pi}{\alpha_1} \), \( p = \frac{\partial \omega}{\partial \xi} - i \frac{\partial \omega}{\partial \eta} \), \( q = \int_{w_0} p \) and finally

\[
 G = e^q,
\]
where \( w_0 \) is an arbitrary fixed point in \( D \).

**Lemma 1.** Assume \( \partial D \in C^{2,\alpha} \). Then \( G \) defined above has the following properties:

1. \( G \) is well defined on \( \overline{D} \);
2. \( G(\overline{D}) = \overline{A_{1;e^{\lambda_1}}} \) and the mapping is one-to-one. In addition \( G(\{\Gamma_2\}) = C_1 \) and \( G(\{\Gamma_1\}) = C_{e^{\lambda_1}} \), respectively;
3. \( G \) is a conformal representation of \( D \) on \( A_{1;e^{\lambda_1}} \);
4. If \( F = G^{-1} \) then the limit \( \lim_{z \to z^*} \frac{F(z) - F(z)}{z - z^*} = F'(z^*) \) exists at all points \( z^* \in A_{1;e^{\lambda_1}} \), and it can be continuously extended to \( A_{1;e^{\lambda_1}} \);
5. The limit \( \lim_{z \to z^*, z \in \overline{A_{1;e^{\lambda_1}}}} \frac{F'(z^*) - F'(z)}{z^* - z} = F''(z^*) \) exists at all points \( z^* \in A_{1;e^{\lambda_1}} \), and it can be continuously extended to \( A_{1;e^{\lambda_1}} \).

**Proof.** For the proof of i. – iii. see [3, Chapter 6, Theorem 10]. For point iv. notice first that the assumption \( \partial D \in C^{2,\alpha} \) implies (using Kellogg's Theorem) that \( \nabla \omega \) can be continuously extended to \( \overline{D} \). Consequently \( G \) extends continuously to \( \overline{D} \). Using this aspect and the conformity of \( F \) in \( A_{1;e^{\lambda_1}} \) it is easy to show that \( F \) can be extended continuously to \( \overline{A_{1;e^{\lambda_1}}} \). The next step is to evaluate the limit

\[
\lim_{w \to w^*} \frac{G(w^*) - G(w)}{w^* - w}
\]

when \( w^* \in \partial D \) and \( w \in \overline{D} \). To this end it is helpful to notice that one may assume without loss of generality that the points \( w_0, w, w^* \) always belong to the same rectifiable curve. With this observation in mind it is easy to show that \( \lim_{w \to w^*} \frac{G(w^*) - G(w)}{w^* - w} = p(w^*)G(w^*) \), \( w^* \in D \). Hence one can extend the derivative of \( G \) by continuity to \( \overline{D} \) by setting \( G'(w) := p(w)G(w) \) if \( w \in \partial D \). Finally compute

\[
\lim_{z \to z^*} \frac{F(z) - F(z^*)}{z - z^*} = \lim_{w \to F(z^*)} \frac{1}{\frac{G(F(z^*) - G(w))}{G'(F(z^*))}} = \frac{1}{G'(F(z^*))}, \quad z^* \in \partial A_{1;e^{\lambda_1}}.
\]

In order to conclude, it only remains to prove that \( G'_{\partial D} \) does not vanish at any point. So assume there exists \( w^* \in \partial D \) such that \( G'(w^*) = 0 \), and one may assume without loss of generality that this point belongs to the exterior contour (for the case when \( w^* \) belongs to the inner contour, the reasoning is the same). If \( \Gamma_2(t^*) = w^* \) for some \( t^* \in \mathbb{R} \) then, on one hand, \( \frac{d}{d t^*} \{G(\Gamma_2(t))\}_{t=t^*} = 0 \). On the other hand, since \( G \) is a one-to-one map of \( \partial D \) on \( \partial A_{1;e^{\lambda_1}} \) it turns out (eventually using a continuity argument) that \( \arg(G(\Gamma_2(t))) \) is a strictly monotonic function, which in turn implies that \( \frac{d}{d t} \{\arg(G(\Gamma_2(t)))\} \) has constant sign (and is thus non-vanishing at any point). But then

\[
\left\{ \frac{d}{d t} \{G(\Gamma_2(t))\}_{t=t^*} = iG'(\Gamma_2(t^*)) \right\}
\]

\( t = t^* \neq 0 \) which is a contradiction.

For the last point fix any arbitrary \( z^* \in \partial A_{1;e^{\lambda_1}} \) and denoting \( w = F(z) \), \( w^* = F(z^*) \) compute

\[
\lim_{z \to z^*, z \in \overline{A_{1;e^{\lambda_2}}}} \frac{F(z) - F(z^*)}{z - z^*} = \frac{1}{G'(w^*)} \lim_{w \to w^*, w \in \overline{D}} \frac{1}{G'(w)} \frac{G'(w^*) - G'(w)}{G(w) - G(w^*)} =
\]
so that in addition the ratio \( F_B \) was continuously extended to \( \overline{D} \). Let \( B_{w^*} \) be a simply-connected, relatively open (with respect to \( D \)) neighborhood of \( w^* \), and let \( w_0^* \) be any point in \( B_{w^*} \) which will be chosen later on. It is easy to see that the function \( \int w^* \) extends continuously to \( \partial D \) (use the Cauchy-Riemann equations as well as Kellogg's Theorem for \( \omega_1 \)), it follows that \( \int w^* \) can actually be extended by continuity to \( B_{w^*} \) (this argument was actually used in point iv. when \( F^* \) was continuously extended to \( \overline{D} \)). Assume for simplicity that the line segments with edges \((w_0^*, w), (w_0^*, w^*)\) are in \( B_{w^*} \) (otherwise one may consider several line segments with intermediate points \( w_0^*, w_1^*, \ldots, w_k^* \) and suppose that \( w_0^* \) was chosen so that in addition the ratio \( \| w_0^* - w \| \) stays bounded as \( w \to w^* \). Consequently

\[
\lim_{w \to w^*, \ w \in \ov{D}} \left( p(w^*) - p(w) \right) w \in \ov{D} = \lim_{w \to w^*, \ w \in \ov{D}} \left( p(w^*) - p(w) \right) w \in \ov{D} = \lim_{w \to w^* \ w \in \ov{D}} \left[ \left( p(w^*) - p(w_0^*) \right) w \right.
\]

where in the last expression, using the integral representation for \( p \), as well as the Dominant Convergence Theorem, the first term converges to \( \frac{\partial^2 \omega}{\partial x \partial y}(w^*) - i \frac{\partial^2 \omega}{\partial y \partial y}(w^*) \) and the second term converges to 0. In conclusion \( \lim_{w \to w^*, \ w \in \ov{D}} \left( p(w^*) - p(w) \right) w \in \ov{D} = \int_0^1 \frac{\partial^2 \omega}{\partial y \partial y}(w^*) - i \frac{\partial^2 \omega}{\partial x \partial y}(w^*) \not= 0 \).

\( p'(w^*) \). Returning we can finally compute the desired limit as

\[
\lim_{z \to z^*, \ z \in A_{1,e^{\lambda_1}}} \frac{F'(z) - F'(z^*)}{z - z^*} = -\frac{1}{\sqrt[\lambda]{\sum_{e^{\lambda_1}}}} \lim_{w \to w^*, \ w \in \ov{D}} \left[ \frac{G'(w^*) - G'(w)}{w - w^*} \right] = -z^* \left[ p^2(F'(z^*)) + p'(F'(z^*)) \right] (F'(z^*))^3 =: F''(z^*). \]

This expression obviously holds for all the points \( z^* \in A_{1,e^{\lambda_1}} \) as well, and it is thus seen that \( F'' \) can be continuously extended to \( A_{1,e^{\lambda_1}} \). This ends the proof.

**Theorem 5.** Let \( \partial D \in C^{2,\alpha} \), \( 0 < \alpha < 1 \), and in addition suppose that \( \Phi \in C^{2,\alpha}(\partial D) \) satisfies the compatibility condition \( \int \Phi \cdot \partial \sigma = 0 \). If \( \Phi \) is any solution of the Neumann problem \((\partial D) \) with boundary data \( \Phi \) then \( \nabla U \) can be continuously extended to \( \overline{D} \).

**Proof.** Set \( r_1 = 1 \), \( r_2 = e^{\lambda_1} \) and let \( F : A_{r_1,r_2} \to D \) be the conformal map given in Lemma 1. Without loss of generality assume \( U(F(\sqrt{r_1}r_2)) = 0 \). Define \( e_j : [0,1] \to C_{r_j}, e_j(t) = r_je^{2\pi it} \), \( j \in \{ 1; 2 \} \), and also \( f : \partial A_{r_1,r_2} \to \mathbb{R} \), \( f := (\Phi \circ F) | F' \). Since \( 0 \not\in A_{r_1,r_2} \), it follows that \( f \in C^2(\partial A_{r_1,r_2}) \) and consequently we can compute \( \frac{d^2}{dt} f(e_j(t)) = \frac{d^2}{dt} \Phi (F(e_j(t))) \cdot |F'(e_j(t))| + 2 \frac{d}{dt} \Phi (F(e_j(t))) \cdot \frac{d}{dt} |F'(e_j(t))| \) which is obviously \( \alpha \)-Holder continuous. In conclusion \( f \in C^{2,\alpha}(\partial A_{r_1,r_2}) \). Consequently let \( V \) be the solution of the Neumann problem \((\partial D) \) on \( A_{r_1,r_2} \) with boundary data \( f \), satisfying \( V(\sqrt{r_1}r_2) = 0 \).
(using direct computations together with the assumption $\int_{\partial D} \Phi \, d\sigma = 0$ it is not difficult to show that \( f \) satisfies the compatibility condition $\int_{\partial A_{r_1,r_2}} f \, ds = 0$). Then according to Theorem 4 it follows that \( V \) is a G-strong solution of the Neumann problem (2). Consequently \( \nabla V \) can be continuously extended to \( A_{r_1,r_2} \) (and, as before, its continuous extension will be denoted \( \nabla V \) as well). Now set \( W := V \circ G \) which shows that \( W \) is harmonic in \( D \) and furthermore \( W(F(\sqrt{r_1r_2})) = 0 \). Also taking the partial derivatives of \( W \) with respect to \( \xi \) and \( \eta \) it follows that for any \( w \in \partial D \)

\[
\frac{\partial W}{\partial \xi}(w) = \Re \left( \frac{\nabla V(G(w))}{F'(G(w))} \right),
\]

\[
\frac{\partial W}{\partial \eta}(w) = -\Im \left( \frac{\nabla V(G(w))}{F'(G(w))} \right).
\]

Hence \( \nabla W = \frac{\nabla V \circ G}{F \circ G} \) on \( D \) which proves, together with point iv of Lemma 1, that \( \nabla W \) extends continuously to \( \bar{D} \). Consequently it follows by Lagrange’s Theorem that \( \frac{\partial W}{\partial \nu}(w) = \langle \nabla W(w); \nu(w) \rangle \), \( w \in \partial D \), where \( \nu \) is the (outward) normal derivative at \( \partial D \) and it is given by

\[
\nu(F(z)) = \begin{cases} 
\frac{zF'(z)}{r_2|F'(z)|} & \text{if } |G(w)| = r_2, \\
-\frac{zF'(z)}{r_1|F'(z)|} & \text{if } |G(w)| = r_1.
\end{cases}
\]

Returning \( \frac{\partial W}{\partial \nu}(w) = \langle \nabla W(w); \nu(w) \rangle = \frac{f \circ G}{|F'(G(w))|} = \Phi, \ \forall \ w \in \partial D \). To sum up \( W \) is harmonic in \( D \), has boundary data \( \Phi \), and satisfies \( W(F(\sqrt{r_1r_2})) = 0 \). So \( W - U = \text{constant} \). Since \( \nabla W \) can be continuously extended to \( \bar{D} \), the same is obviously true for \( U \). This completes the proof. \( \square \)

**Remark 6.** The above result guarantees the continuous extension of the gradient of a solution \( U \) of the Neumann problem (2) under the additional assumption that \( \partial D \in C^{2,\alpha} \), \( \Phi \in C^{2,\alpha}(\partial D) \). Using the ideas in the proof of Theorem 4 as well as Lemma 4, one can deduce that the second-order partial derivatives of \( U \) can also be extended by continuity to \( \bar{D} \). Furthermore, a similar reasoning shows that when the assumption is strengthened to \( \partial D \in C^{m,\alpha} \), \( \Phi \in C^{n,\alpha}(\partial D) \), \( m \geq n \), the \( n \)-order partial derivatives of \( U \) can be continuously extended to \( \bar{D} \).

**Theorem 6.** Set \( r_1 = 1, \ r_2 = e^{\lambda_1} \) and assume \( \Phi \in C^0(\partial D) \) satisfies the compatibility condition $\int_{\partial D} \Phi \, d\sigma = 0$. If \( F : A_{r_1,r_2} \rightarrow D \) is the conformal map given in Lemma 4, and if the Neumann problem (2) with boundary data \( f = \Phi \) has a
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G-strong solution $U$, satisfying $U(F^{-1}(\sqrt{r_1 r_2})) = 0$, then $U$ is given by

$$U(w) = \frac{1}{\sqrt{r_1 r_2}} \int_{|F^{-1}(w)|}^{\frac{2\pi}{r_1 r_2}} u(F(\rho F^{-1}(w))) d\rho$$

\[ + \sqrt{r_1 r_2} \int_0^{\frac{2\pi}{r_1 r_2}} \left[ \hat{C} - R \left( \int_0^t \nabla u(F(\sqrt{r_1 r_2} e^{i\tau})) e^{i\tau} F'(\sqrt{r_1 r_2} e^{i\tau}) d\tau \right) \right] dt, \]

where

$$\hat{C} = \frac{\sqrt{r_1 r_2}}{2\pi} \int_0^{2\pi} R \left( \int_0^t \nabla u(F(\sqrt{r_1 r_2} e^{i\tau})) e^{i\tau} F'(\sqrt{r_1 r_2} e^{i\tau}) d\tau \right) dt,$$

and where $u$ is the solution of the Dirichlet problem [1] with boundary data

$$\varphi(w) = \begin{cases} r_2 \Phi(w)|F'(F^{-1}(w))|, & \text{if } |F^{-1}(w)| = r_2, \\ -r_1 \Phi(w)|F'(F^{-1}(w))|, & \text{if } |F^{-1}(w)| = r_1. \end{cases}$$

Conversely if $\partial D \in C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\partial D)$ satisfies $\int_0^{2\pi} \varphi(F(r_2 e^{i\theta})) d\theta = \int_0^{2\pi} \varphi(F(r_1 e^{i\theta})) d\theta$, and if $U$ is any G-strong solution of the Neumann problem [2]

with boundary data $\hat{\Phi}(w) = \begin{cases} \varphi(w) \frac{1}{r_2(F'(F^{-1}(w))) |F'(F^{-1}(w))|}, & \text{if } |F^{-1}(w)| = r_2, \\ \varphi(w) \frac{1}{r_1(F'(F^{-1}(w))) |F'(F^{-1}(w))|}, & \text{if } |F^{-1}(w)| = r_1, \end{cases}$

$$u(w) = \frac{\langle \nabla U(w); \nabla \omega_1(w) \rangle}{\lambda_1 \|
abla \omega_1(w)\|^2}, \quad w \in D$$

is the solution of the Dirichlet problem [1] with boundary data $g = \varphi$.

Proof. For brevity the following notations will be adopted $V = U \circ F$, $z = x + iy$, $z \in A_{r_1, r_2}$, $w = \xi + i\eta$, $w \in \overline{D}$, $\nu : \partial D \to \mathbb{R}$ is the outward normal derivative at $D$, and finally $n : \partial A_{r_1, r_2} \to \mathbb{R}$ is the outward normal derivative at $A_{r_1, r_2}$.

We have $\Phi(w) = \langle \nabla U(w), \nu(w) \rangle$ on $\partial D$ and notice that

$$\nu(w) = \nu(F(z)) = \begin{cases} \frac{z F'(z)}{r_2|F'(z)|}, & \text{if } |z| = r_2, \\ -\frac{z F'(z)}{r_1|F'(z)|}, & \text{if } |z| = r_1. \end{cases}$$

Hence

$$\Phi(w) = R \left( \nabla U(w) \nu(w) \right), \quad \forall w \in \partial D.$$
cessively
\[
\frac{\partial}{\partial x}(U(F(z))) = \frac{\partial U}{\partial \xi}(w) \frac{\partial \xi}{\partial x}(z) + \frac{\partial U}{\partial \eta}(w) \frac{\partial \eta}{\partial x}(z) \\
= \langle \nabla U(w), F'(z) \rangle = \mathcal{R}\left(\nabla U(w)F'(z)\right),
\]
\[
\frac{\partial}{\partial y}(U(F(z))) = \frac{\partial U}{\partial \xi}(w) \frac{\partial \xi}{\partial y}(z) + \frac{\partial U}{\partial \eta}(w) \frac{\partial \eta}{\partial y}(z)
\]
\[= -\frac{\partial U}{\partial \xi}(w) \frac{\partial \eta}{\partial x}(z) - \frac{\partial U}{\partial \eta}(w) \frac{\partial \xi}{\partial x}(z) = -\mathcal{I}\left(\nabla U(w)F'(z)\right).
\]

In conclusion \(\nabla V(z) = \mathcal{R}\left(\nabla U(w)F'(z)\right) - i \mathcal{I}\left(\nabla U(w)F'(z)\right), z \in A_{r_1, r_2},\) from where it follows by a continuity argument that
\[
\nabla V(z) = \nabla U(w)F'(z), \ w = F(z), \ z \in A_{r_1, r_2}.
\]

But then \(\frac{\partial V}{\partial n}(z) = \langle \nabla V(z), \frac{z}{|z|} \rangle = \mathcal{R}\left(\nabla U(w)F'(z)\right) = |F'(z)|\mathcal{R}\left(\nabla U(w)\Phi(w)\right),\) and similarly \(\frac{\partial V}{\partial n}(z) = \langle \nabla V(z), -\frac{z}{|z|} \rangle = \mathcal{R}\left(\nabla U(w)F'(z)\right) = |F'(z)|\mathcal{R}\left(\nabla U(w)\nu(w)\right) = |F'(z)|\Phi(w).\) To sum up
\[
\frac{\partial V}{\partial n}(z) = \Phi(F(z))|F'(z)|, \ \forall z \in \partial A_{r_1, r_2}.
\]

But then defining \(\Phi_V : \partial A_{r_1, r_2} \rightarrow \mathbb{R}, \ \Phi_V = \frac{\partial V}{\partial n},\) since \(V\) is harmonic in \(A_{r_1, r_2}\) and \(\nabla V\) can be extended by continuity to \(A_{r_1, r_2},\) it follows that \(V\) is a \(G\)-strong solution of the Neumann problem \(2\) on \(A_{r_1, r_2}\) with boundary data \(\Phi_V\) (it can be checked by direct computations, using relation \(27),\) that indeed \(\int_{\partial A_{r_1, r_2}} \Phi_V \ d\sigma = 0).\) In addition \(V(\sqrt{r_1 r_2}) = U(F(\sqrt{r_1 r_2})) = 0.\) So applying Theorem 4 for \(\Phi_V\)

\[
V(z) = \frac{1}{\sqrt{r_1 r_2}} \int_{\sqrt{r_1 r_2}}^{\sqrt{r_1 r_2}} \left( C - \int_{0}^{t} v_x(\sqrt{r_1 r_2} e^{i \tau}) \cos \tau + v_y(\sqrt{r_1 r_2} e^{i \tau}) \sin \tau d\tau \right) dt,
\]
where \(C = \frac{\sqrt{r_1 r_2}}{2\pi} \int_{0}^{2\pi} \int_{0}^{t} v_x(\sqrt{r_1 r_2} e^{i \tau}) \cos \tau + v_y(\sqrt{r_1 r_2} e^{i \tau}) \sin \tau d\tau dt,\) and where \(v\) is the solution of the Dirichlet problem \(1\) on \(A_{r_1, r_2}\) with boundary data
\[
\varphi_V(z) = \begin{cases} 2\Phi_V(z) & \text{if } |z| = r_2, \\ -r_1 \Phi_V(z) & \text{if } |z| = r_1. \end{cases}
\]
Consequently, if \(u = v \circ G\) then \(u\) is harmonic in \(D,\) extends continuously to \(\bar{D},\) and has continuous boundary data \(\varphi_V \circ G,\) which
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3.2 Elliptical regions

Let \( J : \mathbb{C}^* \to \mathbb{C}, \ J(w) := \frac{1}{2} \left( w + \frac{1}{w} \right) \) be the Joukowsky transform, and define \( J_+ = J_{\left| w \in \mathbb{D} e^{i\theta} : \theta \in (-\pi, 0) \right]}, \ T_+ = J_+^{-1}, \ T_- = J_{\left| w \in \mathbb{D} e^{i\theta} : \theta \in [-\pi, 0) \right]}, \ T_- = J_-^{-1}. \) Throughout this section, the argument of a complex number will be defined as taking values in \([-\pi, \pi]\) and if \( z \) is any complex number then its square root will be defined as

\[
\sqrt{z} = \begin{cases} 
|z| e^{i\frac{\arg(z)}{2}} & \text{if } z \neq 0, \\
0 & \text{if } z = 0.
\end{cases}
\]

**Remark 7.** It is not difficult to notice that the set \( \mathcal{H} \) as defined in Sec. 2.1 is actually the set of all hyperbolas orthogonal to the family of confocal ellipses having foci \( \{(-1, 0), (1, 0)\} \). This aspect will play an essential role in the proof of Theorem 7.
Lemma 2. \( T_+ \) has the following properties

1. it is well defined on the whole \( \mathbb{C} \);

2. it is analytic in \( \mathbb{C} \setminus [-1,1] \) with nonzero derivative;

3. for any point \( \xi \in [-1,1] \) and any sequence \( \{z_n\}_{n=1}^{\infty} \) satisfying
   
   i. there exist \( H_\theta \in H \) for which \( z_n \in H_\theta \cap \{ \Im(z) > 0 \} \ \forall \ n \in \mathbb{N}^* \),
   
   ii. \( z_n \to \xi \),
   
   one has \( \lim_{n \to \infty} T_+(z_n) = T_+(\xi) \);

4. \( T_+(z) = z + \sqrt{z^2-1} \), \( z \in \mathbb{C} \).

Proof. 1. It is not difficult to see that \( J_+ \left( U^c \setminus \{ e^{i\theta} : \theta \in (-\pi,0) \} \right) = \mathbb{C} \) and that \( J_+ \) is invertible there (see for instance [3, Chapter 4.2]). This property was actually used in the definition of \( T_+ \).

2. Indeed choose any point \( z \in \mathbb{C} \setminus [-1,1] \) and let \( w = T_+(z) \). Since \( T_+ \left( \mathbb{C} \setminus [-1,1] \right) = \{ \zeta \in \mathbb{C} : |\zeta| > 1 \} \) is an open set and \( J_+'(w) = J'(w) \neq 0 \) it follows that there is an open neighborhood \( U_w \) of \( w \) in \( \{ \zeta \in \mathbb{C} : |\zeta| > 1 \} \) such that \( J_+'(u) \neq 0 \ \forall u \in U_w \). Define \( V_z = J_+(U_w) \). Then \( V_z \) is an open subset of \( \mathbb{C} \) and we have \( T_+'(l) = \frac{1}{J_+'(T_+(l))} \in \mathbb{C}^* \ \forall l \in V_z \).

3. As \( z_n \in H_\theta \cap \{ \Im(z) > 0 \} \) it follows that
   
   \( z_n = \frac{1}{2} \left( \rho_n + \frac{1}{\rho_n} \right) \cos \theta + \frac{i}{2} \left( \rho_n - \frac{1}{\rho_n} \right) \sin \theta \) for some \( \rho_n > 1 \) and some \( \theta \in [0,\pi] \). On the other hand since \( z_n \to \xi \in [-1,1] \) it follows that \( \rho_n \to 1 \) and thus \( \xi = \cos \theta \). One can also deduce that \( T_+(z_n) = \rho_n e^{i\theta} \) and thus \( \lim_{n \to \infty} T_+(z_n) = e^{i\theta} \). But \( J_+(e^{i\theta}) = \frac{1}{2} \left( e^{i\theta} + e^{-i\theta} \right) = \cos \theta = \xi \) and so \( T_+(z_n) \to e^{i\theta} = T_+(\xi) \).

4. See [3, Chapter 3].

When \( D = E_\rho, \rho > 1 \), a G-strong solution of the Neumann problem (2) is defined to be a function \( U \in C^2 \cap C^1(\overline{D}) \) satisfying relations (2).

Remark 8. When an additional assumption on the smoothness of \( f \) is added, it can be proved that the Neumann problem (2) on \( D = E_\rho, \rho > 1 \), actually has a G-strong solution, which is obviously unique up to an additive constant. Indeed if \( \Phi \in C^{2,\alpha}(\partial\overline{U}) \) satisfies the compatibility condition \( \int_{\partial\overline{U}} \Phi \sigma = 0 \), then using Corollary 1 and Kellogg’s Theorem it can be easily deduced that the Neumann problem (2) on \( D = \overline{U} \) with boundary data \( \Phi \) has a G-strong solution. Combining this with [3]
Theorem 3.6. It follows that the Neumann problem (2) on \( D = E_\rho, \rho > 1 \), with boundary data \( f \) has a \( G \)-strong solution provided it is assumed in addition that \( f \) belongs to \( C^{2,\alpha}(\partial E_\rho) \). Similarly as specified for the case of doubly-connected regions, when the assumption is strengthened to \( f \in C^{n,\alpha}(\partial E_\rho) \) the \( n \)-th order partial derivatives of \( U \) can be continuously extended to \( \overline{D} \). Furthermore it is easy to show that they are actually \( \alpha \)-Holder continuous by first showing this result on \( U \) using Kellogg’s Theorem as well as Corollary [7] and then the corresponding Riemann (conformal) map to transfer the property on \( E_\rho \).

Theorem 7. Assume that the Neumann problem (2) on \( E_\rho, \rho > 1 \), with boundary data \( f \) has a \( G \)-strong solution \( U \) satisfying \( U(1) = 0 \). Then, letting \( R(z)e^{i\Theta(z)} := T_+(z), \Theta(z) \in (-\pi, \pi] \), one has

\[
U(z) = \int_0^{\frac{1}{\rho} \left| \frac{1}{z} \right|} \frac{u(tT_+(z))}{t} \, dt - \Theta(z) \int_0^{\frac{1}{\rho} \left| \frac{1}{z} \right|} \frac{\partial u}{\partial e^{i\tau}} \left(e^{i\tau}\right) \, d\tau dt, \quad z \in E_\rho, \tag{29}
\]

where \( u \) is the solution of the Dirichlet problem on \( \{ \frac{1}{\rho} < \left| w \right| < \rho \} \) with boundary data

\[
\varphi_- := \frac{f \circ J}{\rho \left| T_- \circ J \right|} \text{ on } C_{\frac{1}{\rho}},
\]

\[
\varphi_+ := \frac{f \circ J}{\left| T_+ \circ J \right|} \text{ on } C_\rho.
\]

Remark 9. The proof of Theorem [7] provides, in addition, a nice interpretation of the second term in the right-hand side of the equality (29). Indeed

\[
\Theta(z) \int_0^{\frac{1}{\rho} \left| \frac{1}{z} \right|} \frac{\partial u}{\partial e^{i\tau}} \left(e^{i\tau}\right) \, d\tau dt = -U(z^*_0(z)), \quad z \in E_\rho \setminus [-1, 1], \tag{30}
\]

where \( z_0^*(z) \) is the intersection of the (unique) hyperbola \( H_\theta \) containing \( z \) with the line segment \([-1, 1]\).

Proof. Let \( U : E_\rho \to \mathbb{R} \) be as in the statement of the theorem. Define \( V = U \circ J_{\frac{1}{\rho} \left| \frac{1}{z} \right|} \) and notice that \( V \) thus obtained is harmonic on \( A_{\frac{1}{\rho}, \rho} \) and also

\[
\frac{\partial V}{\partial \nu}(w) = V_{\xi}(w) \frac{\Re(w)}{\rho} + V_{\eta}(w) \frac{\Im(w)}{\rho}, \quad \forall \ w \in C_\rho, \text{ where } w =: \xi + i\eta \text{ and } z =: x + iy.
\]

Defining

\[
\omega(w) = V_{\xi}(w) - iV_{\eta}(w) = \nabla V(w), \quad \{ \frac{1}{\rho} < \left| w \right| < \rho \},
\]

one obtains alternatively:

\[
\frac{\partial V}{\partial \nu}(w) = \Re \left( \frac{\omega(w)}{\rho} \right), \quad \forall \ w \in C_\rho.
\]
Using the Cauchy-Riemann equations together with the harmonicity of $V$ it follows that $\omega$ is analytic on $\{ \frac{1}{\rho} < |w| < \rho \}$. Furthermore $\frac{\partial V}{\partial \nu}(w) = \Re \left( \frac{\omega(w)w}{\rho} \right)$, $w \in C_{\rho}$. On the other hand let $G$ be an analytic function such that $U = \Re(G)$ on $E_{\rho}$ (which is always possible since $U$ is harmonic and $E_{\rho}$ is a simply connected region). But then setting $F = G \circ J_{\frac{\Lambda_{\rho}}{\pi}}$, one obtains $V = \Re(F)$. Consequently it follows that $F' = \omega$ on $\{ \frac{1}{\rho} < |w| < \rho \}$, which gives $F(w) = F(w_0) + \int_{w_0}^{w} \omega(o) \, do$, or equivalently

$$G(J(w)) = G(J(w_0)) + \int_{w_0}^{w} \omega(o) \, do,$$

(31)

where $w_0 = w_0(w)$ is to be specified later on.

Notice now that any $o \in \{ 1 < |w| < \rho \}$ is of the form $T_+(\lambda)$ for some $\lambda \in E_{\rho} \setminus [-1, 1]$, and hence according to Lemma 2, point (2) $\lambda' = \frac{2\lambda^{2}(\lambda)}{\lambda^{2}(\lambda) - 1}$ and consequently

$$G(z) = G(z_0) + \int_{z_0}^{z} \omega(T_+(\lambda)) \lambda'(\lambda) d\lambda = G(z_0) + 2 \int_{z_0}^{z} \omega(T_+(\lambda)) \frac{T_+^{2}(\lambda)}{T_+(\lambda) - 1} d\lambda,$$

where $z_0 := J(w_0)$.

I shall divide the rest of the proof into several steps.

Step 1. Denote $w = T_+(z)$ for any $w \in A_{1,\rho}$ and consequently define the following curve

$$\gamma^w_\epsilon(t) = \begin{cases} wt, & t \in \left[ \frac{1+\epsilon}{|w|}, 1 \right]; \\
(1+\epsilon) \exp \left( i \arg(w) \frac{t-\epsilon}{|w|} \right), & t \in \left[ \epsilon, \frac{1+\epsilon}{|w|} \right]; \\
1+\epsilon, & t \in \left[ \frac{\epsilon}{2}, \epsilon \right) \end{cases}$$

from where it follows that

$$\gamma^w_\epsilon(t) = \begin{cases} \frac{i \arg(w)}{|w|} (1+\epsilon) \exp \left( i \arg(w) \frac{t-\epsilon}{|w|} \right), & t \in \left( \epsilon, \frac{1+\epsilon}{|w|} \right); \text{ for any} \\
1, & t \in \left( \frac{\epsilon}{2}, \epsilon \right) \end{cases}$$

$w \in \{ 1 < |w| \leq \rho \}$ and any $\epsilon > 0$ small enough.

Now define $\lambda^z_\epsilon(t) = J_+ (\gamma^w_\epsilon(t))$ which gives

$$\dot{\lambda}^z_\epsilon(t) = \frac{\gamma^w_\epsilon(t)}{T_+^{\prime}(\lambda^z_\epsilon(t))}.$$
Also \( \lim_{\epsilon \to 0} G \left( J_+ \left( 1 + \frac{\epsilon}{2} \right) \right) = \lim_{\epsilon \to 0} G \left( J \left( 1 + \frac{\epsilon}{2} \right) \right) = G(0) \). To this end setting \( z_0 = z_0(\epsilon) := J_+ \left( 1 + \frac{\epsilon}{2} \right) \) in \( (31) \), one obtains

\[
G(z) = G \left( J_+ \left( 1 + \frac{\epsilon}{2} \right) \right) + \frac{1}{2} \int_{\frac{1+\epsilon}{|w|}}^{1+\epsilon} \frac{tw \omega(tw)}{t} \, dt + \frac{1}{2} \int \omega(1 + t) \, dt
\]

\[
+ i \frac{\arg(w)}{|w|} - \frac{i}{\epsilon} \int \omega(\gamma^w(t)) \gamma^w(t) \, dt, \quad \epsilon > 0.
\]

Setting \( \epsilon \to 0 \) it follows by the use of Dominant Convergence Theorem that

\[
G(z) = G(1) + \frac{1}{2} \int \frac{tT_+(z) \omega(tT_+(z))}{t} \, dt + iR(z) \Theta(z) \int_0^1 \omega(\gamma^w(t)) \gamma^w(t) \, dt,
\]

where \( \gamma^w = \lim_{\epsilon \to 0} \gamma^w_\epsilon \). Taking the real part in the equation above, it follows that

\[
U(z) = \frac{1}{\pi} \int \frac{\Re \left[ tT_+(z) \omega(tT_+(z)) \right]}{t} \, dt - \frac{1}{\pi} \int_0^1 \Im \left[ \omega(\gamma^w(t)) \gamma^w(t) \right] \, dt,
\]

where the normalization \( U(1) = 0 \) was used.

**Step 2.** Link the first integral term in \( (32) \) to the solution of some Dirichlet problem on \( A^w_{\rho} \).

To do so I shall first evaluate the corresponding boundary values on \( C_\rho \) using the function \( T_+ \) and some family of curves \( \Gamma^w_{+,\epsilon} \), and second the corresponding boundary values on \( C_{\rho} \) using the function \( T_- \) and some family of curves \( \Gamma^w_{-,\epsilon} \), respectively.

Define \( \Gamma^w_{\epsilon,\epsilon} : \left[ \frac{1}{|w|} + \epsilon, 1 \right] \to \{ 1 < |w| \leq \rho \} \), \( \Gamma^w_{\epsilon,\epsilon}(t) = tw \) for any \( w \in \{ 1 < |w| \leq \rho \} \) and any \( \epsilon > 0 \) small enough. Thus \( \Gamma^w_{\epsilon,\epsilon}(t) = w \) and defining \( \Lambda^z_{\epsilon,\epsilon}(t) = T_+(\Gamma^w_{\epsilon,\epsilon}(t)) \) it follows that the image of \( \Lambda^z_{\epsilon,\epsilon} \) is a branch of some hyperbola in \( \mathcal{H} \) orthogonal to \( \partial \mathcal{E}_\rho \) which approaches some \( z^*_0 = z^*_0(\epsilon) \leq \rho \) as \( \epsilon \to 0 \). Also notice that \( \Lambda^z_{\epsilon,\epsilon}(t) = J'_+(\Gamma^w_{\epsilon,\epsilon}(t))w, \) and hence

\[
\frac{d}{dt} \left( \Lambda^z_{\epsilon,\epsilon}(t) \right) = \frac{T_+(z)}{T'_+(\Lambda^z_{\epsilon,\epsilon}(t))}.
\]

Proceeding further observe that

\[
\omega \left( \Lambda^z_{\epsilon,\epsilon}(t) \right) T'_+(\Lambda^z_{\epsilon,\epsilon}(t)) \Lambda^z_{\epsilon,\epsilon}(t) = \omega \left( T_+(\Lambda^z_{\epsilon,\epsilon}(t)) \right) w = \omega(tw) w, \quad (33)
\]
and also
\[ \omega \left( \Lambda^z_{+,\epsilon}(t) \right) T'_+(\Lambda^z_{+,\epsilon}(t)) \hat{\Lambda}^z_{+,\epsilon}(t) = \nabla U(\Lambda^z_{+,\epsilon}(t)) \hat{\Lambda}^z_{+,\epsilon}(t), \] (34)
both (33) and (34) being true for any \( \epsilon > 0 \), \( w \in \{ 1 < |w| < \rho \} \), \( t \in \left[ \frac{1}{|w|} + \epsilon, 1 \right] \), where in the derivation of (34) the following important observation was made
\[ \nabla U(z) = \omega(T_+(z)) \cdot T'_+(z), \forall z \in E_{\rho} \setminus [-1, 1]. \] (35)
Combining now (33) and (34) it follows that
\[ \mathcal{R} \left[ \omega(tw) w \right] = \langle \nabla U(\Lambda^z_{+,\epsilon}(t)); \hat{\Lambda}^z_{+,\epsilon}(t) \rangle, \]
\( \epsilon > 0 \), \( w \in \{ 1 < |w| \leq \rho \} \), \( t \in \left[ \frac{1}{|w|} + \epsilon, 1 \right] \) and choosing any \( z^* \in \partial E_{\rho} \)
\[ \mathcal{R} \left[ \omega(T_+(z^*)) T_+(z^*) \right] = \langle \nabla U(z^*); \hat{\Lambda}^z_{+,\epsilon}(1) \rangle, \] (36)
where \( \hat{\Lambda}^z_{+,\epsilon}(1) \) is thus the outward normal derivative in \( z^* \) at \( \partial E_{\rho} \). Consequently compute \( |\hat{\Lambda}^z_{+,\epsilon}(1)| = \frac{|T_+(z^*)|}{|T'_+(z^*)|} = |\frac{\rho}{T'_+(z^*)}| \) which shows, using relation (36), that
\[ \mathcal{R} \left[ \omega(T_+(z^*)) T_+(z^*) \right] = \langle \nabla U(z^*); \frac{\hat{\Lambda}^z_{+,\epsilon}(1)}{|\hat{\Lambda}^z_{+,\epsilon}(1)|} \rangle = f(z^*) \frac{\rho}{T'_+(z^*)}, \]
or equivalently
\[ \mathcal{R} \left[ \omega(w^*_+) w^*_+ \right] = \frac{\rho f(J_+(w^*_+))}{|T'_+(J_+(w^*_+))|} = \frac{\rho f(J_+(w^*_+))}{|J'_+(w^*_+)|}, w^*_+ \in C_{\rho}, \] (37)
where \( w^*_+ := T_+(z^*) \).

On the other hand notice that \( U(z) = V(T_- (z)) \forall z \in E_{\rho} \) which gives (exactly as it was done for the previous case)
\[ \nabla U(z) = \omega(T_- (z)) T'_-(z), \forall z \in E_{\rho} \setminus [-1, 1]. \]
In the same way define \( \Gamma^w_{-,\epsilon} : \left[ 1, \frac{1}{|w|+\epsilon} \right] \rightarrow \left\{ \frac{1}{\rho} \leq |w| < 1 \right\} \), \( \Gamma^w_{-,\epsilon}(t) = tw \) for any \( w \in \left\{ \frac{1}{\rho} \leq |w| < 1 \right\} \) and any \( \epsilon > 0 \) small enough. Thus \( \hat{\Gamma}^w_{-,\epsilon}(t) = w \) and by defining \( \Lambda^z_{-,\epsilon}(t) := J_-(\Gamma^w_{-,\epsilon}(t)) \) it follows that the image of \( \Lambda^z_{-,\epsilon} \) is also a branch of some hyperbola in \( \mathcal{H} \) orthogonal to \( \partial E_{\rho} \) which approaches some \( z^*_0 = z^*_0(\epsilon) \in [-1, 1] \) as \( \epsilon \to 0 \). Also notice that \( \hat{\Lambda}^z_{-,\epsilon}(t) = J'_-(\Gamma^w_{-,\epsilon}(t)) w \) and hence
\[ \frac{d}{dt} \left( \Lambda^z_{-,\epsilon}(t) \right) = \frac{T_-(z)}{T'_-(\Lambda^z_{-,\epsilon}(t))}. \]
But then one obtains, similarly as for (33) and (34)
\[
\omega \left( T_\epsilon' (\Lambda_{\epsilon}^z(t)) \right) T_\epsilon' (\Lambda_{\epsilon}^z(t)) \hat{\Lambda}_{\epsilon}^z(t) = \omega \left( T_\epsilon (\Lambda_{\epsilon}^z(t)) \right) w = \omega (tw) w,
\]
and also
\[
\omega \left( T_\epsilon' (\Lambda_{\epsilon}^z(t)) \right) T_\epsilon' (\Lambda_{\epsilon}^z(t)) \hat{\Lambda}_{\epsilon}^z(t) = \nabla U(\Lambda_{\epsilon}^z(t)) \hat{\Lambda}_{\epsilon}^z(t),
\]
both (38) and (39) being true for any \( \epsilon > 0, \ w \in \{ \frac{1}{\rho}, \ l \leq |w| < 1 \} \) and any \( t \in \left[ \frac{1}{|w|}, 1 \right] \), respectively. Combining (38) and (39) it follows that
\[
\Re \left[ \omega (tw) w \right] = \langle \nabla U(\Lambda_{\epsilon}^z(t)); \hat{\Lambda}_{\epsilon}^z(t) \rangle,
\]
\( \epsilon > 0, \ w \in \{ \frac{1}{\rho}, \ l \leq |w| < 1 \}, \ t \in \left[ \frac{1}{|w|}, 1 \right] \) and choosing any \( z^* \in \partial E_\rho \)
\[
\Re \left[ \omega (T_\epsilon (z^*)) T_\epsilon (z^*) \right] = \langle \nabla U(z^*); \hat{\Lambda}_{\epsilon}^z(1) \rangle, \quad (40)
\]
where \( \hat{\Lambda}_{\epsilon}^z(1) \) is thus the outward normal derivative in \( z^* \) at \( \partial E_\rho \). Consequently compute \( |\hat{\Lambda}_{\epsilon}^z(1)| = \frac{T_\epsilon(z^*)}{|T_\epsilon(z^*)|} = \frac{1}{\rho |T_\epsilon(z^*)|} \) which shows, using relation (40), that
\[
\Re \left[ \omega (T_\epsilon (z^*)) T_\epsilon (z^*) \right] = \langle \nabla U(z^*); \frac{\hat{\Lambda}_{\epsilon}^z(1)}{|\hat{\Lambda}_{\epsilon}^z(1)|} \rangle = \frac{f(z^*)}{\rho |T_\epsilon(z^*)|}, \quad z^* \in \partial E_\rho,
\]
or equivalently
\[
\Re \left[ \omega (w^* w^*) \right] = \frac{f \left( J_-(w^*) \right)}{\rho |T_\epsilon (J_-(w^*))|} = \frac{f(J_-(w^*)) |J'_-(w^*)|}{\rho}, \quad w^* \in C_{\rho}, \quad (41)
\]
where \( w^* := T_\epsilon (z^*) \).

Finally, using equation (32), equations (37) and (41) together with the analyticity of \( w\omega(w) \) on \( A_{\rho}^1 \), the continuity (and hence boundedness) of \( u(w) := \Re \left[ w\omega(w) \right] \) on \( A_{\rho}^{-1} \), and the uniqueness of the solution to the \( \epsilon \) Dirichlet problem, it follows that \( u \) is the solution of the Dirichlet problem on \( A_{\rho}^1 \) with boundary data \( \varphi_- \circ J \) on \( C_{\rho}^- \) and \( \varphi_+ \circ J \) on \( C_{\rho}^+ \), respectively.

To sum up it has been shown so far that
\[
U(z) = \int_{\frac{1}{\rho}}^{\frac{1}{\rho^2}} \frac{u(tT_\epsilon(z))}{t} dt - R(z) \Theta(z) \int_{0}^{\frac{1}{\rho^2}} \Im \left[ \omega (\gamma^w(t)) \gamma^w(t) \right] dt, \quad (42)
\]
where $u$ is the solution of the Dirichlet problem on $\{\frac{1}{\rho} < |w| < \rho\}$ with boundary values

\[ \varphi_- = \frac{f \circ J}{\rho |T_+ \circ J|} \text{ on } C_{\frac{1}{\rho}}, \]
\[ \varphi_+ = \rho \frac{f \circ J}{|T_+ \circ J|} \text{ on } C_{\rho}. \]

Step 3. Link the second integral in (42) to $u$.

To do so notice that since $w\omega(w)$ is an analytic function on $A_{\frac{1}{\rho}}$, which extends continuously to $A_{\frac{1}{\rho}}^1$, it follows that $\Im[w\omega(w)]$ is an harmonic function on $A_{\frac{1}{\rho}}$ which extends continuously to $A_{\frac{1}{\rho}}^1$. Letting $w\omega(w) =: u(w) + iv(w)$ it follows that $v$ is a harmonic function on $A_{\frac{1}{\rho}}$ which extends continuously to $A_{\frac{1}{\rho}}^1$ and the idea is to determine $v$ from $u$.

Using (35)

\[ \omega(T_+(z)) = \frac{\nabla U(z)}{T_+(z)}, \quad z \in E_{\rho} \setminus [-1, 1]. \]

Define the sequence $z_n = 1 + \frac{1}{n}$ for any $n$ large enough so that $z_n \in E_{\rho}$. Then $T_+(z_n) \in A_{1,\rho} \cap \mathbb{R}_+$ and by point (3) of Lemma 2 it follows that $T_+(z_n) \to T_+(1) = 1$. In addition $T_+(z_n)$ is well defined by point (2) of the same lemma and using point (4) of the same result it follows that $T_+(z_n) = 1 + \frac{z_n}{\sqrt{z_n^2 - 1}}$, which shows that $|T_+(z_n)| \geq \frac{|z_n|}{|\sqrt{z_n^2 - 1}|} - 1 > \frac{1}{|\sqrt{z_n^2 - 1}|} - 1$ for $n$ large. Since $\frac{1}{\sqrt{z_n^2 - 1}} \to 0$ it follows that $|T_+(z_n)| \geq \frac{1}{2\sqrt{z_n^2 - 1}} \to \infty$.

But $U \in C^1(E_{\rho})$ and since $(1, 0) \in E_{\rho}$, $E_{\rho}$ open, it follows that there is some neighborhood of $(1, 0)$ contained in $E_{\rho}$ on which $|\nabla U| \leq M$, for some $M > 0$. Consequently, it follows that $\exists N \in \mathbb{N}^*$ such that $\forall n \geq N$ one has $|\omega(T_+(z_n))| = \frac{\nabla U(z_n)}{|T_+(z_n)|} \leq \frac{M}{|T_+(z_n)|} \to 0.$

To sum up $\omega(w_n) \to 0$ as $w_n \to 1$, $w_n \in A_{1,\rho} \cap \mathbb{R}_+$. Since $\omega$ is continuous on $A_{\frac{1}{\rho}} \supset A_{1,\rho}$ we have $\omega(1) = \lim_{w_n \to 1, w_n \in A_{1,\rho} \cap \mathbb{R}_+} \omega(w_n) = 0$. This gives

\[ v(1) = 0. \]

Since $u$ and $v$ are conjugate-harmonic functions and $v(1) = 0$, it follows that one can precisely determine $v$ solely from $u$. Indeed using the Cauchy-Riemann equations it follows that $v(a, b) = \int_{\gamma} dv = \int_0^{\arg(a+ib)} \frac{\partial u}{\partial \theta}(e^{i\theta}) \, d\theta$

whenever $a + ib \in \gamma$, where $\gamma$ is considered to be the curve $e^{it}$ for $t \in [0, \arg(a + ib)]$. Hence it follows that
Equivalence of Dirichlet and Neumann problems in annuli

\[ \frac{1}{R(z)} \int_0^1 \Im \left[ \omega (\gamma^w(t)) \gamma^w(t) \right] dt = \frac{1}{R(z)} tR(z)\Theta(z) \int_0^1 \frac{\partial u}{\partial e^{i\tau}} \left( e^{i\tau} \right) d\tau dt, \]

so combining equation (42) with equation (43) and using a change of variable, Theorem 7 is proved for any \( z \in E_\rho \setminus [-1,1] \).

If \( z \in [-1,1] \) choose any sequence \( \{z_n\}_{n=1}^{\infty} \) as in point (3) of Lemma 2 such that \( z_n \to z \). Then using the same point of Lemma 2, a continuity argument for \( U \), the Dominant Convergence Theorem, as well as the boundedness of \( \frac{\partial u}{\partial e^{i\tau}} \) on \( C_1 \) the proof is completed.

\[ \square \]

4 Conclusions

The paper provides an equivalence between the solutions of Neumann and Dirichlet problems in the case of doubly-connected regions (Theorem 6). This equivalence is expressed by the fact that solving any of these two problems leads by an analytic formula to an explicit solution of the other problem. For elliptical regions, a solution of the Neumann problem is provided in terms of the solution of a certain Dirichlet problem (Theorem 7).

In addition to these equivalences, the theory developed in this paper provides sufficient conditions for continuous extension of higher-order partial derivatives of the solutions of the Neumann problem (2) (Remark 6 for the case of doubly-connected regions and Remark 8 for the case of elliptical regions). These conditions and results are very much in the spirit of those stated in the well-known Kellogg’s Theorem where the problem of continuous extension of higher-order derivatives for the solution of the Dirichlet problem was investigated.
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