Exploration trees and conformal loop ensembles

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Abstract

We construct and study the conformal loop ensembles $\text{CLE}(\kappa)$, defined for $8/3 \leq \kappa \leq 8$, using branching variants of SLE($\kappa$) called exploration trees. The CLE($\kappa$) are random collections of countably many loops in a planar domain that are characterized by certain conformal invariance and Markov properties. We conjecture that they are the scaling limits of various random loop models from statistical physics, including the $O(n)$ loop models.

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# Contents

1 Introduction 3
   1.1 Overview 3
   1.2 Outline 4
   1.3 Planned sequels 5

2 Discrete motivation 6
   2.1 Exploration trees 6
   2.2 SLE definition 10
   2.3 $O(n)$ models and conformal invariance ansatz 11
   2.4 Excursions and renewal times 12
   2.5 Two exploration path variants 13

3 Constructing SLE($\kappa; \rho$) from Bessel and stable processes 14
   3.1 Bessel processes 14
   3.2 Lévy skew stable processes 16
   3.3 Chordal SLE($\kappa; \rho$) 19
   3.4 Radial SLE($\kappa; \rho$) 22

4 Conformal loop ensembles 24
   4.1 Defining loops 24
   4.2 Continuum exploration trees 25
   4.3 Constructing loops from exploration trees 27
   4.4 CLE gasket and conformal radius distribution 30
   4.5 Limiting cases: $\kappa = 8/3$ and $\kappa = 8$ 31

5 Symmetry and uniqueness when $4 < \kappa < 8$ 32
   5.1 Continuity and starting point invariance of CLE 32
   5.2 Uniqueness result for boundary-intersecting CLE 36

6 SLE($\kappa; \rho$) approximations and invariance 41
   6.1 Approximate Bessel processes 41
   6.2 Approximations to chordal SLE($\kappa; \rho$) 44
   6.3 Approximations to radial SLE($\kappa; \kappa - 6$) 45

7 Height functions and other lattices 48
   7.1 Height functions: continuity and monotonicity 48
   7.2 Other lattices and bond percolation 51

8 Open problems 51
1 Introduction

1.1 Overview

Many two dimensional statistical physical models can be interpreted as random collections of disjoint, non-self-intersecting loops in a planar lattice. For example, the loops may be the boundaries between the plus spin and minus spin clusters in an Ising model with spins defined on the faces of a three-regular planar graph.

When the boundary conditions of a random loop model on a simply connected planar domain are set up so that, in addition to the loops, there is one chordal path connecting a pair of boundary points, it is often natural to conjecture (and sometimes possible to prove) that as the grid size gets finer, the law of this random path converges to the law of the chordal Schramm-Loewner evolution SLE(\(\kappa\)) for some \(\kappa > 0\).

Given that such a conjecture holds, it is natural to expect the collection of all loops to have a scaling limit. The primary purpose of this paper is to introduce and study a natural family of candidates for this scaling limit, called the conformal loop ensembles CLE(\(\kappa\)).

The CLE(\(\kappa\)), defined for \(8/3 \leq \kappa \leq 8\), are random collections of loops in a planar domain which look, in some sense, like SLE(\(\kappa\)) locally. At the extremes, CLE(8) almost surely consists of a single space-filling loop, which is the scaling limit of the outer boundary of the free uniform spanning tree (see [S]), and CLE(8/3) almost surely contains no loops at all. When \(8/3 < \kappa < 8\), the collection of loops in a CLE(\(\kappa\)) is almost surely countably infinite. When \(\kappa = 6\), it is equivalent to the random collection of loops described in [2], where it was shown to arise as a scaling limit of the cluster boundaries of site percolation on the triangle lattice. Like SLE(\(\kappa\)), the CLE(\(\kappa\)) loops intersect the boundary of the domain almost surely if and only if \(\kappa > 4\).

We will show that if \(\mathcal{L}\) is any random collection of loops in the closure of a planar domain \(D\) that satisfies certain natural hypotheses related to conformal invariance (and if at least one loop in \(\mathcal{L}\) intersects \(\partial D\) with positive probability) then \(\mathcal{L}\) must be a CLE(\(\kappa\)) for some \(4 < \kappa < 8\) (see Theorem 5.4). In a separate joint paper with Werner, we will prove that the CLE(\(\kappa\)) for \(8/3 \leq \kappa \leq 4\) are the only random ensembles of simple loops in \(D\) that possess certain (somewhat different) conformal symmetries (see Section 1.3 [14]).

A secondary purpose of this paper is to formulate a series of conjectures and open questions related to conformal loop ensembles. For example, we
will conjecture a continuum analog of the FK cluster expansion for Potts models and formulate a precise conjecture about the scaling limit of the $q$-state Potts model for $q \in \{2, 3, 4\}$. We will define height functions for discrete loop ensembles and conjecture a connection with the Gaussian free field. We will conjecture scaling limits for site percolation on certain random graphs, and we will ask about the continuum fields that have CLE loops as level lines.

1.2 Outline

In Section 2, we will discuss random discrete collections of loops on planar graphs. The results in this section (besides the definition of SLE) are not logically necessary for the definition of CLE, but they will clarify our motivation. (The reader who is primarily interested in the continuum may skim this section on a first reading.) For simplicity, we will focus on hexagonal lattice graphs and the so-called $O(n)$ loop models. We will associate to each disjoint collection of non-self-intersecting loops a spanning tree of the graph, called an “exploration tree,” and show that an appropriate class of trees is in one-to-one correspondence with the set of disjoint simple loop ensembles.

In Section 3 we will assemble some basic facts about Bessel processes, Lévy skew stable processes, and SLE($\kappa; \rho$) processes. In particular, we will argue that radial SLE($\kappa; \kappa - 6$) and its variants are the most natural candidates for the limiting laws of a branch of the exploration tree. We will give a one-to-one correspondence between the strictly stable Lévy processes and a family of conformally invariant continuum exploration path models.

Section 4 will use a coupling of SLE($\kappa; \kappa - 6$) processes with different target points to construct a continuum analog of the exploration tree and
use this continuum tree to construct the CLE(κ). This approach to defining loops is related to the one given in [2] for the case κ = 6, which also uses variants of SLE(κ) to “explore” segments of loops, but it is somewhat more canonical in that there are fewer arbitrary choices in the exploration process.

The exploration tree appears to have connections to the Gaussian free field and to percolation on the so-called discrete gaskets (which we discuss as open problems in Section 8). One intriguing point is that the family of conformally invariant exploration tree structures will turn out to have a somewhat different character when κ = 4 and when κ ≠ 4. This is related to the fact that the family of strictly stable Lévy processes corresponding to α = 1 has a different character from the family corresponding to α ≠ 1 (see Section 3).

Section 5 focuses on the non-simple, non-space-filling case 4 < κ < 8 and formulates and proves a uniqueness theorem which says that any random boundary-intersecting loop ensemble which satisfies certain natural hypotheses related to conformal invariance must be a CLE(κ) for some 4 < κ < 8 (Theorem 5.4). Conversely, Theorem 5.4 also states that—if Conjecture 3.11 is true—the CLE(κ) themselves satisfy these hypotheses. (We will not address the analogous questions for κ ≤ 4 because we expect them to be addressed in a subsequent paper [14].)

Section 6 will describe various approximations of SLE(κ; ρ) processes and use them to prove Möbius invariance of SLE(κ; ρ) and other results. The invariance results are similar to those in [12], but there are technical issues that arise when the driving parameters of the Loewner evolutions are not semimartingales; one reasonably simple way around this involves the approximations mentioned above. The approximations also provide the intuition behind some of the conjectures in Section 8.

Section 7 will explore some additional combinatorial constructions in the discrete setting. In particular, we use the “winding number” of the exploration tree to construct a height function for each discrete loop ensemble.

Finally, Section 8 will present a list of conjectures and open problems relevant to CLE.

1.3 Planned sequels

We now mention briefly some work in progress for which we expect this paper to be a prerequisite. The random closed set Γ consisting of points which are not surrounded by a loop in an instance of CLE(κ) is called the CLE(κ) gasket. The physics literature contains many non-rigorous calculations about O(n) model scaling limits that are based on conformal
invariance hypotheses (similar to those of Theorem 5.4 but not always so explicitly formulated). It is natural to interpret these results as predictions about properties of CLE(κ) and CLE(κ) gaskets. In a joint paper with Schramm and Wilson we will compute the probabilistic fractal dimension of Γ (which agrees with a calculation in the physics literature made by Duplantier [3]) and the distribution of the set of conformal radii of the loops surrounding a fixed point in the domain (which agrees with a calculation in the physics literature made by Cardy and Ziff [3]) [13]. These results will be derived from Proposition 4.2 which we prove here, but which was first formulated with Schramm and Wilson as part of this joint project.

In a joint paper with Werner we will show that the sets of outermost loops of the CLE(κ) defined here (i.e., the loops that are not surrounded by any other loops) for $8/3 \leq \kappa \leq 4$ are the only random ensembles $L$ of pairwise disjoint, non-nested simple loops in $D$ with the following natural Markov property: if $B \subset D$ is a deterministic closed set with simply connected complement — and $\tilde{B}$ is defined to be the closure of the set of points surrounded by loops that intersect $B$ — then given $\tilde{B}$, the conditional law of the loops in each component of $D \setminus \tilde{B}$ is the same as the original law of $L$ conformally mapped to that component [14]. We will also show that the set of outermost loops in a CLE(κ) has the same law as the set of loop soup cluster boundaries for a loop soup of intensity $c$ where $c = (3\kappa - 8)(6 - \kappa)/2\kappa$. A form of this statement and a partial proof appear in earlier work by Werner [17].

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2  Discrete motivation

2.1 Exploration trees

Let $\mathcal{H}$ be the infinite hexagonal lattice embedded in $\mathbb{R}^2$. A graph $G$ consisting of the edges and vertices incident to a finite simply connected subset of the hexagonal faces of $\mathcal{H}$ is called a hexagon graph. Let $F$, $E$, and $V$ denote, respectively, the sets of faces, edges, and vertices of $G$.

Let $A$ be an arbitrary subset of $F$; we will refer to the members of $A$ as black hexagons and the members of $F \setminus A$ as white hexagons. Unless oth-
otherwise stated, we will also refer to the hexagons of \( \mathcal{H} \) outside of \( F \) as white. The components of the boundaries of the black clusters form an ensemble of non-self-intersecting loops in \( G \) which do not intersect one another. It is easy to see that this gives a one-to-one correspondence between subsets \( A \subset F \) and collections of disjoint simple loops in \( G \).

Fix a vertex \( v_0 \) on the outer boundary of \( G \) which is incident to only two edges of \( G \), and fix a directed edge \( e_0 \) outside of \( G \), beginning at some vertex \( v_{-1} \) and pointing towards \( v_0 \). For each vertex \( v \) of \( G \), the exploration path \( T_v(A) \) is a directed, non-self-intersecting path \( v_0, v_1, v_2 \ldots \) that ends at \( v \). Each \( v_k \), for \( k \geq 1 \), is chosen in such a way that the sequence \( v_{k-2}, v_{k-1}, v_k \) describes a right turn when the directed edge \( (v_{k-2}, v_{k-1}) \) points to a black face and a left turn if \( (v_{k-2}, v_{k-1}) \) points to a white face unless this choice of \( v_k \) would fail to lie in the same connected component of \( V \setminus \{v_0, v_1, \ldots, v_{k-1}\} \) as \( v \), in which case the path turns the other direction.

The exploration tree \( T(A) \) of \( A \) is the union over all \( v \in V \) of \( T_v(A) \). The reader may check that \( T(A) \) is in fact an out-directed spanning tree of \( G \), rooted at \( v_0 \). Readers familiar with computer algorithms may recognize

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Figure 2.1: A coloring of a hexagon graph and the corresponding exploration tree. The arrow points to the root vertex \( v_0 \).
$T(A)$ as the depth-first search tree of $G$ beginning at $v_0$ under the rule that one searches right first after tracing an edge directed towards a black face and left first after tracing an edge directed towards a white face. See Figure 2.1.

A vertex $v$ is called a **branch point** of $T(A)$ if it has three neighbors in $T(A)$. If $v_1$ and $v_2$ are the immediate descendants of $v$ and $v_3$ is the parent, then we call $v_1$ the **proper descendant** if either the edge $(v_3, v)$ points to a black face and the path $v_3, v, v_1$ is a right turn or $(v_3, v)$ points to a white face and the edge $v_3, v, v_1$ is a left turn. The **proper branch** of $v$ is the subtree of descendants of the proper descendant of $v$.

We say that $v \prec w$ if $v \in T_w(A)$. The reader may easily check that if $v$ and $w$ are adjacent, their exploration paths agree up until the first time one of the two vertices is hit, and thus we have either $v \prec w$ or $w \prec v$. A similar argument implies that $\prec$ always has a unique minimum among the vertices in any connected subset $W$ of $V$. In particular, for each face $f$, there must be some minimal $v$ among the vertices of $f$.

We say a spanning tree is **branch-separated** if every connected subset of $V$ (with respect to adjacency within $G$) has a unique minimal vertex $v$. Roughly speaking, a tree is branch-separated if distinct branches of the tree are disconnected from one another by the path leading from the root to the last common ancestor of the branches. In other words, the tree can only branch at a vertex $v$ if the path from $v_0$ to $v$ passes through one of the three neighbors of $v$ and the remaining two neighbors lie in distinct components of the complement of that path. The above discussion implies that exploration trees are branch separated, and the converse is also true.

**Proposition 2.1.** A spanning tree of $G$ is branch-separated if and only if it is the exploration tree $T(A)$ of some $A \subset F$.

**Proof.** Suppose that $T$ is branch-separated and $f$ is a face of $G$. Let $v$ be the minimal vertex incident to $f$, and let $w$ be the minimal vertex among the remaining vertices of $f$. We say that $f$ is a member of $A$ if the path from $v_0$ to $w$ turns right after hitting $v$. The reader may check that $T = T(A)$.

Since the proof gives us a way to deduce $A$ from $T(A)$, we can also observe the following:

**Proposition 2.2.** The correspondence between subsets of $F$ and branch-separated spanning trees rooted at $v_0$ is one-to-one. In particular, there are exactly $2^{|F|}$ branch-separated spanning trees rooted at $v_0$. 

8
The reader may easily check the following propositions. We will see analogs of these phenomena in the continuum correspondence between loop ensembles and exploration trees. In all of the propositions below, unless otherwise stated, we will assume that all of the faces outside of $G$ are colored white, so that there is a one-to-one correspondence between subsets $A \subset F$ and collections $\{L_1, L_2, \ldots, L_m\}$ of disjoint non-self-intersecting loops in $G$.

**Proposition 2.3.** The exploration tree $T(A)$ contains all but one edge of each of the loops $L_i$. The paths in the tree traverse these loops counterclockwise when black is on the inside and clockwise when black is on the outside.

**Proposition 2.4.** Suppose $v \neq v_0$ lies on the boundary of $G$ and that (instead of the usual all-white coloring) the faces outside of $G$ are colored black if they border edges in the clockwise path $P$ from $v_0$ to $v$ in the boundary of $G$, and white if they border edges in the counterclockwise path. Then the path $T_v(A)$ follows the boundary between a cluster of black hexagons and a cluster of white hexagons.

Proposition 2.4 suggests a simple way to construct the path $T_v(A)$ directly from the loops when $v$ is on the boundary of $G$. (This construction has a natural continuum analog when $4 < \kappa \leq 8$, which is the range for which the CLE($\kappa$) loops intersect the boundary of $D$.) Suppose $v \neq v_0$ lies on the boundary of $G$. Let $M_1, \ldots, M_k$ be the loops among the $\{L_1, \ldots, L_m\}$ that have edges in common with the directed path $P$. Let $I_i$, for $1 \leq i \leq k$, denote the interval of $M_k$ beginning at the first vertex of $P$ contained in $L_i$ and ending at the last vertex in $P$ contained in $L_i$. Let $A_i$ be the directed arc of $M_i$ that starts and ends at the first and last endpoints of $I_i$ and contains no edges of $P$. Let $Q$ be the path whose edge set is the union of the $A_i$ where $i$ ranges over those $i$ for which $I_i$ is maximal (i.e., for which the interval $I_i$ is not contained in any other $I_j$) together with the edges of $P$ that are not contained in any interval $I_i$. Then we have the following:

**Proposition 2.5.** The path $Q$ constructed above is equivalent to $T_v(A)$.

This in turn implies the following:

**Proposition 2.6.** Let $T_{\partial G}(A)$ denote the union of $T_v(A)$ over all $v$ on the boundary of $G$. Then $T_{\partial G}(A)$ determines—and is determined by—the set of the loops which contain boundary edges of $G$. 

9
2.2 SLE definition

We will give a very brief introduction and definition of SLE. There are many excellent SLE surveys (most available on the arXiv) that the reader may consult for more information on SLE and the notion of scaling limit (see, e.g., [18, 7, 6]). Given a real-valued measurable function \( W_t : [0, \infty) \to \mathbb{R} \) and \( z \in \mathbb{H} \), consider the solution of the ODE

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.
\]

(2.1)

When \( W_t \) is continuous, and \( z \) is fixed, this \( g_t(z) \) is a well defined continuous function of \( t \) up until the time \( \inf \{ t : g_t(z) = W_t \} \), at which point it ceases to be well-defined. More generally, if \( W_t \) is merely measurable, then \( g_t(z) \) is still well defined up until the first time \( t \) that the set \( \{ g_s(z) - W_s : s < t \} \) has an accumulation point at zero. In each case, we let \( \tau_z \) be the supremum of the \( t \) for which \( g_t(z) \) is well-defined and write \( K_t := \{ z \in \mathbb{H} : \tau_z \leq t \} \). Then \( K_t \) is a closed set, and \( g_t \) is a conformal map from \( \mathbb{H} \setminus K_t \) to \( \mathbb{H} \). When it exists, we write \( \gamma(t) := \lim_{z \to W_t} g_t^{-1}(z) \) (where the limit is taken over \( z \in \mathbb{H} \)). In the particular case that \( W_t = \sqrt{\kappa} B_t \), where \( B_t \) is Brownian motion, the limit exists almost surely for all \( t \geq 0 \) and \( \gamma : [0, \infty) \to \mathbb{H} \) is a random continuous path, called SLE(\( \kappa \)) [11].

More generally, suppose that \( \gamma : [0, \infty) \to \overline{\mathbb{H}} \) is any continuous path such that for every \( t \geq 0 \) the value \( \gamma(t) \) lies on the boundary of \( \mathbb{H} \setminus \gamma(0, t] \). For every \( t \in [0, T] \), there is a unique conformal homeomorphism \( g_t : \mathbb{H} \setminus \gamma[0, t] \) which satisfies the so-called hydrodynamic normalization at infinity

\[
\lim_{z \to \infty} g_t(z) - z = 0.
\]

The limit

\[
\text{cap}_\infty(\gamma[0, t]) := \lim_{z \to \infty} z(g_t(z) - z)/2
\]

is real and monotone non-decreasing in \( t \). It is called the (half plane) capacity of \( \gamma[0, t] \) from \( \infty \), or just capacity, for short. When \( \text{cap}_\infty(\gamma[0, t]) \) is also continuous in \( t \), it is natural to reparameterize \( \gamma \) so that \( \text{cap}_\infty(\gamma[0, t]) = t \). Loewner’s theorem states that if \( \gamma \) is a simple path, then the maps \( g_t \) satisfy (2.1) with \( W(t) = g_t(\gamma(t)) \) (where \( g_t : \mathbb{H} \setminus \gamma[0, t] \to \mathbb{H} \) is extended continuously to the point \( \gamma(t) \)). We can now formulate Schramm’s characterization of SLE:

**Theorem 2.7.** The SLE(\( \kappa \)) for \( \kappa \geq 0 \) are the only random continuous (when parameterized by capacity) paths \( \gamma : [0, \infty) \to \overline{\mathbb{H}} \) with the following so-called...
conformal Markov property. Fix $T \in \mathbb{R}$. Then given $\gamma$ on the set $[0, T]$, the conditional law of $\gamma$ on the set $[T, \infty)$ is (up to a time change) the image of the original law of $\gamma$ under the conformal map $g_T^{-1}$ (extended continuously to $\overline{\mathbb{H}}$). Moreover, the law of $\gamma$ is (up to a rescaling of time) invariant under conformal automorphisms of $\mathbb{H}$ that fix 0 and $\infty$ (i.e., the maps $z \rightarrow az$ for $a > 0$).

When $\gamma$ is an SLE($\kappa$), the conformal Markov property continues to hold when $T$ is replaced by an arbitrary stopping time.

It is not hard to see why this is true: the continuity of $\gamma$ together with the conformal Markov property implies the continuity of $W_t$. The scale invariance implies that the law of $W_t$ is the same as the law of $a^{-1/2}W_{at}$ when $a > 0$. The conformal Markov property implies stationarity and independence of increments $W_{t_1} - W_{t_2}$ and $W_{s_1} - W_{s_2}$ when $(s_1, s_2)$ and $(t_1, t_2)$ do not overlap. These properties together imply that $W_t$ must be a constant multiple of Brownian motion.

2.3 $O(n)$ models and conformal invariance ansatz

Let $G$ be a hexagon graph. In the so-called $O(n)$ loop model, one samples a collection of disjoint non-self-intersecting loops in $G$, where each such collection has probability proportional to $n^N x^L$ where $N$ is the number of loops, $L$ is the total number of edges in the loops, and $n$ and $x$ are fixed positive constants. This can be written as $\exp[\mathcal{E}]$ where $\mathcal{E} := N \log n + L \log x$ is a Hamiltonian on the space of loop configurations. Equivalently, we may color all hexagons outside of $F$ white and then sample a subset $A \subset F$ with probability proportional to $n^{N(A)} x^{L(A)}$ where $N(A)$ is the number of black plus the number of white clusters and $L(A)$ is the total number of edges in the loops. If $n = 1$, then this is simply the Ising model. It is ferromagnetic when $x < 1$, anti-ferromagnetic when $x > 1$, and independent Bernoulli percolation when $x = 1$.

We remark (see Figure 2.1) that for every loop there is exactly one edge in that loop that is not in $T(A)$; hence $N(A) = |E_L(A) \setminus T(A)|$, where $E_L(A)$ is the set of edges that lie in a loop, and $L(A) = |E_L|$. Thus, we may interpret the $O(n)$ model as a random pair $(E_L, T) = (E_L(A), T(A))$, with a Hamiltonian that is a linear combination of $|E_L \setminus T|$ and $|E_L|$. (In fact, any linear combination of the four quantities $|E_L \cup T|$, $|E_L \setminus T|$, $|T \setminus E_L|$, and $|E_L \cap T|$ can be written this way, up to an additive constant, since $|T|$ and $|E|$ are both fixed.)

A natural variant of the $O(n)$ model is the following. Fix vertices $a$ and $b$ on the boundary of $G$ and suppose that the hexagons outside of $F$ are
colored in such a way that they are white whenever they are incident to an edge in one arc of the boundary of \( G \) with endpoints \( a \) and \( b \), and black whenever they are incident to an edge of the complementary arc. In this case, given any choice of \( A \), let \( P \) be the path of edges in \( F \) that lie on the boundary between the cluster of black hexagons that includes the black boundary arc and the cluster of white hexagons that includes the white boundary arc. We let \( L \) be the total number of edges in \( E \) separating black hexagons from white hexagons (including those edges in \( P \)) and let \( N \) be the total number of loops formed by these edges (not counting the path \( P \)), and as before sample a subset \( A \subset F \) with probability proportional to \( n^{N(A)} x^{L(A)} \). We call this a \textbf{one-chordal-arc \( O(n) \) model}.

In light of conformal invariance hypotheses from the physics literature it is natural to conjecture that as the mesh size gets finer, the path \( P \) in the one-chordal-arc \( O(n) \) model converges in law to a random path which satisfies a conformal Markov property—and hence, by Theorem 2.7, is \( \text{SLE}(\kappa) \) for some \( \kappa \). Neinhuis and Kager [6], following work by Duplantier, Neinhuis, and others (which uses the so-called “Coulomb gas method”), have gone further and conjectured precise values for \( \kappa \): namely, they conjecture that the scaling limit of \( P \) is

1. \( \text{SLE}(\kappa) \), where \( n = -2 \cos(4\pi/\kappa) \) and \( 4 \leq \kappa \leq 8 \), if \( 0 < n \leq 2 \) and \( x > x_c \),

2. \( \text{SLE}(\kappa) \), where \( n = -2 \cos(4\pi/\kappa) \) and \( 8/3 \leq \kappa \leq 4 \), if \( 0 < n \leq 2 \) and \( x = x_c \),

3. a straight line (or a shortest length path from \( a \) to \( b \), if the domain is not convex) if either \( x < x_c \) or \( n > 2 \),

where \( x_c = [2 + (2 - n)^{1/2}]^{-1/2} \). Note that \( x_c \) increases monotonically as \( n \) increases from 0 to 2, and for each \( n \in (0,2) \), the equation \( n = -2 \cos(4\pi/\kappa) \) has two solutions, one in \((8/3,4)\) and one in \((4,8)\). A precise version of this conjecture has been proved in the case \( x_c = 1/2 \) and \( n = 1 \) (which corresponds to critical Bernoulli site percolation) [2, 15].

\section*{2.4 Excursions and renewal times}

We now describe one way to construct loops from trees—a continuum version of which will be used in Section 4.3 to define the \( \text{CLE}(\kappa) \). Fix a vertex \( v \) of \( G \). For each \( k \), let \( K_k \) be the set of hexagons whose colors are determined by the first \( k \) vertices \( v_1, v_2, \ldots v_k \) of the exploration path \( T_v(A) \). Let \( G_k \) be the connected component of the set of faces of \( G \) in the complement of \( K_k \)
that contains a face incident to \( v \). The reader may observe that for each \( k \), the conditional law of the coloring of the faces within \( G_k \), given the colors of the faces determined by \( v_1, \ldots, v_k \), is given by either an \( O(n) \) model (i.e., the hexagons on the boundary of \( G_k \) are all one color) or a one-chordal-arc \( O(n) \) model (i.e., boundary conditions are given by one white and one black arc). Let \( 0 = k_0, k_1, k_2, \ldots \) be the values of \( k \) for which the boundary of \( G_k \) is monochromatic. We refer to these \( k_i \) as renewal times and to the paths between these times consisting of edges that separate black and white faces as excursions. The reader may observe (see Figure 2.1) that each excursion traces part of a loop. If we extend the excursion to a longer directed path in \( T(A) \) in such a way that the path turns in the proper direction at each branch point of \( T(V) \), then this extended path will trace out the remainder of the loop (except for one edge).

### 2.5 Two exploration path variants

We now describe two variants of the exploration path. In later sections we will use continuum analogs of these variants to define CLE(\( \kappa \)) when \( \kappa \leq 4 \).

In the exploration tree described above, the orientations of the outermost loops are all clockwise, and the sequence of nested loops surrounding a single face strictly alternates between clockwise and counterclockwise.

**Variant 1:** Fix a parameter \( \beta \in [-1, 1] \) and independently orient each outermost loop clockwise with probability \( \frac{1-\beta}{2} \) and counterclockwise with probability \( \frac{1+\beta}{2} \). Inductively, we define orientations for the remaining loops, orienting the loop the same way as the smallest (in terms of enclosed area) loop that surrounds it with probability \( \frac{1-\beta}{2} \) and the opposite way with probability \( \frac{1+\beta}{2} \). We also independently assign an orientation to each isolated vertex (i.e., a vertex which does not lie in any loop), orienting the vertex the same way as the smallest loop that surrounds it (or clockwise if the vertex is not surrounded by a loop) with probability \( \frac{1-\beta}{2} \). The case \( \beta = 0 \) is particularly natural, since in this case all loops and vertices are oriented independently with fair coins.

Given an oriented collection of loops and vertices, we can define an oriented exploration tree as follows: Fix a vertex \( v_0 \) on the outer boundary of \( G \) which is incident to only two edges of \( G \), and fix a directed edge \( e_0 \) outside of \( G \), beginning at some vertex \( v_{-1} \) and pointing towards \( v_0 \). For each vertex \( v \) of \( G \), the oriented exploration path \( T_v(A) \) is a directed, non-self-intersecting path \( v_0, v_1, v_2, \ldots \) that ends at \( v \). Each \( v_k \) is chosen in
such a way that the sequence \( v_{k-2}, v_{k-1}, v_k \) describes a left turn if \( v_{k-1} \) is a counterclockwise isolated vertex, a right turn if \( v_{k-1} \) is a clockwise isolated vertex, and a turn that causes \( (v_{k-1}, v_k) \) to be an edge of a loop oriented in the direction of the loop if \( v \) is not isolated — unless this choice of \( v_k \) would fail to lie in the same connected component of \( V \setminus \{v_0, v_1, \ldots, v_{k-1}\} \) as \( v \), in which case the path turns the other direction. The reader may check that the union of the \( T_v(A) \) is a spanning tree. We call this the oriented exploration tree. Note that our original definition of exploration tree corresponds to the case \( \beta = 1 \), while the mirror image (i.e., the model with the roles of black and white reversed) corresponds to \( \beta = -1 \).

**Variant 2:** To motivate this variant, we note that we will eventually want to conjecture the existence of a scaling limit for the individual exploration tree paths \( T_v(A) \) which has a continuous Loewner evolution. However, when \( \kappa \leq 4 \) and \( \beta = 1 \), we cannot expect the scaling limit of \( T_v(A) \) to have a continuous Loewner evolution when \( v \) is on the boundary of \( G \), since we expect there to be no macroscopic loops that hit the boundary; in this case, in light of Proposition 2.4, we would expect \( T_v \) to tend to a path that traces the left boundary of the domain from \( v_0 \) to \( v \).

In the \( O(n) \) model setting, recall from the previous section that given the values \( v_0, v_1, \ldots, v_k \) up to a renewal time \( k \), the law of the remainder of the path \( T_v(A) \) is that of an exploration path from \( v_k \) to \( v \) in a new hexagon graph \( G_k \) with a new starting point \( v_k \). Now suppose that we generate \( T_v(A) \) as above except that at each renewal time \( k \), we “shift” this new starting point by some constant-order number of edges to the right along the boundary of \( G_k \) (adding these edges to \( T_v(A) \)) before continuing to grow \( T_v(A) \) according to the usual rules. Then we get a variant of \( T_v(A) \) which we might expect (if the shift sizes are chosen properly) to have a meaningful scaling limit. We will make these notions more precise in the continuum setting, where the shifts will be replaced with a local time Lévy compensation used to make \( \text{SLE}(\kappa; \kappa - 6) \) well-defined when \( \frac{8}{3} < \kappa \leq 4 \).

### 3 Constructing \( \text{SLE}(\kappa; \rho) \) from Bessel and stable processes

#### 3.1 Bessel processes

In this section we define Bessel processes and state some standard facts that will be useful in defining conformal loop ensembles. See Chapter XI
of [10], including Exercises 1.25 and 1.26. The square Bessel process of dimension $\delta > 0$, written $\text{BESQ}^\delta$, is the unique strong solution $Z_t$ to the SDE:

$$Z_t = Z_0 + 2 \int_0^t \sqrt{Z_s} dB_s + \delta t,$$

(3.1)

where $Z_0$ is a fixed initial value and $B_t$ is standard Brownian motion. (Recall that a strong solution to (3.1), as defined e.g. in [10], is a coupling of $Z_t$ and $B_t$ in which (3.1) almost surely holds for all $t$ and $Z_t$ is adapted to the filtration generated by $B_t$—i.e., the law of $Z_t$ is non-anticipative.)

The Bessel process of dimension $\delta$, written $\text{BES}^\delta$, is the process $X_t = \sqrt{Z_t}$, where $Z_t$ is a $\text{BESQ}^\delta$. We sometimes write $\text{BES}^\delta_x$ for the $\text{BES}^\delta$ process started at $X_0 = x \geq 0$.

**Proposition 3.1.** If $\delta \geq 2$, and $X_t$ is a $\text{BES}^\delta$, then almost surely $X_t > 0$ for all $t > 0$. If $0 < \delta < 2$, then $X_t$ almost surely assumes the value zero on a non-empty random set with zero Lebesgue measure.

**Proposition 3.2.** For all $\delta > 0$, the Bessel processes $X_t$ are invariant under Brownian scaling. That is, given a constant $c > 0$, the process $c^{-1/2} X_{ct}$ has the same law as $X_t$.

**Proposition 3.3.** If $\delta > 1$ and $X_t$ is a $\text{BES}^\delta$ process, then $X_t$ is a semimartingale and a strong solution to the SDE

$$X_t = X_0 + B_t + \frac{\delta - 1}{2} \int_0^t X_s^{-1} ds.$$  

(3.2)

If $\delta \leq 1$ and $X_0 > 0$, then (3.2) still holds up until the first $t$ for which $X_t = 0$, but for all larger $t$, the integral $\int_0^t X_s^{-1} ds$ is infinite (so that (3.2) cannot hold). If $\delta > 0$ and $X_t$ is any continuous process adapted to the filtration generated by $B_t$ which is instantaneous reflecting at zero (i.e., the Lebesgue measure of $\{ t : X_t = 0 \}$ is almost surely zero) and almost surely satisfies

$$\frac{\partial}{\partial t} (X_t - B_t) = \frac{\delta - 1}{2} X_t^{-1}$$

whenever $X_t \neq 0$, then the law of $|X_t|$ is that of a $\text{BES}^\delta$ process.

**Proposition 3.4.** When $\delta = 1$ and $X_t$ is a $\text{BES}^\delta$, the process $X_t$ has the law of the absolute value of a standard Brownian motion. It also satisfies the equation
\[ X_t = B_t + \frac{1}{2} l_0^t \tag{3.3} \]

where \( l_0 \) is the local time of \( X_t \) at 0, where the local time \( l_t^x \) is defined to be the almost surely continuous function of \( x \) and \( t \) for which

\[ \int_0^t f(X_s)ds = \int_0^\infty f(x)l_t^x dx, \]

for all \( t > 0 \) and measurable functions \( f \).

**Proposition 3.5.** When \( \delta \in (0, 1) \), there almost surely exists a family of occupation densities \( l_t^x \) for a Bessel process \( X_t \) that are continuous in \( x \) and \( t \) such that for each \( t > 0 \) and measurable \( f \), we have

\[ \int_0^t f(X_s)ds = \int_0^\infty f(x)l_t^x x^{\delta-1} dx. \]

For \( t > 0 \) and \( \delta \in (0, 1) \), the integral \( \int_0^t X_s^{-1}ds \) is almost surely infinite. However, the so called principal value

\[ \text{P.V.} \int_0^t X_s^{-1}ds := \int_0^\infty x^{\delta-2}(l_t^x - l_0^t)ds \]

is almost surely finite for all \( t \), and satisfies

\[ X_t = X_0 + B_t + \frac{\delta - 1}{2} \int_0^t X_s^{-1}ds. \tag{3.4} \]

In light of (3.4) we may take \( \text{P.V.} \int_0^t X_s^{-1}ds = \frac{2}{\delta-1}(X_t - X_0 - B_t) \) as an alternate definition for the principal value when \( \delta \in (0, 1) \) (when the coupling of \( X_t \) and \( B_t \) is given). It is clear from (3.4) that this integral is also a process that satisfies Brownian scaling.

### 3.2 Lévy skew stable processes

We now review some basic facts about Lévy skew stable distributions and their connection to Bessel processes and skew Bessel processes. They are not hard to derive directly, but the reader may see, e.g., [9, 10, 4, 16] and the references therein for more details and many additional results. The Lévy skew stable probability distribution is the Fourier transform of its characteristic function \( \phi \), defined as follows. Fix parameters \( c > 0, \beta \in [-1, 1], \mu \in \mathbb{R} \) and \( \alpha \in (0, 2) \). Then
\( \phi(\lambda) = \exp \left[ i\lambda \mu - |\epsilon \lambda|^\alpha (1 - i\beta \text{sign}(\lambda) \Phi) \right], \quad (3.5) \)

where \( \Phi = \tan(\pi \alpha/2) \) if \( \alpha \neq 1 \), and \( \Phi = - (2/\pi) \log |\epsilon| \) for \( \alpha = 1 \). When \( \alpha \in (0, 2) \), the corresponding Lévy measure is

\[
\Lambda(d\eta) = \frac{c\alpha}{\Gamma(1 - \alpha)} \eta^{-\alpha - 1} d\eta \quad (t > 0),
\]

if \( \beta = 1 \) and more generally the measure \( \Lambda_\beta \) defined by

\[
\Lambda_\beta(A) := \frac{1 + \beta}{2} \Lambda(A) + \frac{1 - \beta}{2} \Lambda(-A),
\]

for measurable subsets \( A \) of \( \mathbb{R} \).

For each choice of parameters \( \alpha, \mu, \beta \) as above and constant \( b > 0 \), there is a corresponding stable process \( S_t \) with independent, stationary increments, such that for each fixed \( t \), the law of \( S_t \) is given by the Lévy skew stable distribution with parameter \( c \) where \( c^\alpha = bt \). We denote this process by \( S(\alpha, \beta, \mu, b) \). It is supported on the positive reals if and only if \( \mu \geq 0 \), \( \beta = 1 \), and \( \alpha < 1 \). The jump discontinuities in \( S_t \) have sizes that are distributed as a Poissonian point process sampled from the corresponding Lévy measure. When \( b = 1 \), this means in particular that the expected number of jump discontinuities in \( S_t \) whose size lies in a set \( A \) that occur between time \( s_1 > 0 \) and time \( s_2 > s_1 \) is given by \((s_2 - s_1)\Lambda_\beta(A)\). When \( \beta = 1 \) the jumps are almost surely all positive; when \( \beta = -1 \) they are almost surely all negative.

The following is easy to derive from the Markov and scaling properties of Bessel processes:

**Proposition 3.6.** Let \( l = l(t) = l_0 t^{\delta/2} \) be the zero local time of a Bessel process \( X_t \) with parameter \( \delta \in (0, 1) \cup (1, 2) \). Then \( t(l) \) and \( P.V. \int_0^{t(l)} X_s ds \) are both Lévy skew stable processes indexed by \( l \) with parameters \( \beta = 1 \), \( \mu = 0 \), \( \alpha = 1 - \delta/2 \) and \( \alpha = 2 - \delta \) respectively, and some positive \( b \). When \( \delta \in (0, 2) \), the zero set of a Bessel of dimension \( \delta \) is the range of a non-decreasing stable process (a.k.a. stable subordinator) with parameter \( \alpha = 1 - \delta/2 \).

We say that an \( S(\alpha, \beta, \mu, b) \) process is **strictly stable** if altering \( b \) (i.e., rescaling time by a constant factor) has the same effect on the law of the process as multiplying the process by a deterministic constant. The following is not hard to derive from (3.5):

**Proposition 3.7.** Fix \( \alpha \in (0, 2) \), \( \beta \in [-1, 1] \), and \( \mu \in \mathbb{R} \). Then the Lévy skew stable process \( S(\alpha, \beta, \mu, b) \) is strictly stable if and only if one of the following holds:
1. $\alpha \neq 1, \mu = 0$.

2. $\alpha = 1, \beta = 0$.

We remark that it is easy to see from (3.5) that one can obtain the characteristic function corresponding to $\alpha = 1, \beta = 0$ and $\mu \neq 0$ as a uniform limit of the characteristic functions corresponding to $\alpha \neq 0, \mu = 0$ and $\beta \neq 0$ if one takes $\beta \to 0$ at an appropriate rate as $\alpha \to 1$.

The skew Bessel process with dimension $\delta$ and skew parameter $\beta$ is a continuous process $X_t$ for which the law of $|X_t|$ is that of the Bessel process of the corresponding dimension, but for each excursion of $|X_t|$ (i.e., each connected component of $\{t : |X_t| > 0\}$) we toss an independent coin and make $X_t$ positive on that excursion with probability $\frac{1+\delta^2}{2}$ and negative with probability $\frac{1-\delta^2}{2}$. When $\beta = 0$, the resulting process is called the symmetric Bessel process. Now we define a process $Y_t$ that will play the same role as the principal value $P.V. \int_0^t X_s ds$ when $\beta \neq 1$.

**Proposition 3.8.** Let $l = l(t) = l_0 t$ be the zero local time of a skew Bessel process $X_t$ with parameters $\beta \in [-1,1]$ and $\delta \in (0,2)$. If either $\delta \neq 1$ or $\delta = 1$ and $\beta = 0$, then $X_t$ can be coupled with a continuous process $Y_t$ such that

1. The pair $(X_t, Y_t)$ is adapted to the filtration generated by the Brownian motion $B_t$.

2. $Y'_t = X_t^{-1}$ on the set $\{t : X_t \neq 0\}$, almost surely.

3. The law of $(X_t, Y_t)$ is invariant under Brownian scaling.

4. If $T$ is a stopping time of $(X_t, Y_t)$ for which $X_T = 0$ almost surely, then $T$ is a renewal time in that conditioned on $T$, the law of $(X_{T+t}, Y_{T+t} - Y_T)$ (for $t \geq 0$) is the same as the original law of $(X_t, Y_t)$.

In any such coupling, $Y_{t(l)}$ is an $S(2-\delta, \beta, \mu, b)$ process indexed by the local time parameter $l$, where $b$ is as given in Proposition 3.6 (for the case $\beta = 1$) and $\mu = 0$ unless $\delta = 1$. The law of $(X_t, Y_t)$ is uniquely determined by the properties above (together with the parameter $\mu$, in the case $\delta = 1$).

It is not difficult to derive Proposition 3.8 from Proposition 3.6 when $\delta \neq 1$. We sketch the construction of $Y_t$ as follows. To construct $Y_t$, it is sufficient to determine the values of $Y_t$ on the set $\{t : X_t = 0\}$, since the other values for $Y_t$ may then be obtained by integrating $X_t^{-1}$ on each excursion. Thus it is enough to determine the process $Y_{t(l)}$ as a function of $l$. When
$\beta = 1$, the jump discontinuities in $Y_{t(t)}$ (each of which corresponds to the integral of $X_t^{-1}$ over an interval on which $X_t \neq 0$) are distributed according to the Poisson process derived from the Lévy measure $\Lambda$ described above; swapping the sign of a $\frac{1-\beta}{2}$ fraction of these jumps corresponds to replacing $\Lambda$ with $\Lambda_{\beta}$ where

$$\Lambda_{\beta}(A) = \frac{1+\beta}{2}\Lambda(A) + \frac{1-\beta}{2}\Lambda(-A),$$

which in turn corresponds to changing the skew parameter of the process $Y_{t(t)}$ from 1 to $\beta$. When $\delta = 1$, $\mu = 0$, and $\beta = 0$, the process $(X_t, Y_t)$ may be obtained as a limit as $\delta \to 1$ of the $(X_t, Y_t)$ couplings with $\beta = 0$ and $\mu = 0$. Adding a non-zero value for $\mu$ amounts to replacing $Y_t$ with $Y_t + \mu t_0$.

### 3.3 Chordal SLE($\kappa; \rho$)

Fix a constant $\rho \in \mathbb{R}$. Write $\delta = 1 + \frac{2(\rho+2)}{\kappa}$. Suppose $\delta > 0$ and $\delta \neq 1$ (i.e., $\rho \neq -2$). Let $X_t$ be a BES$^\delta_x$, and let $O_t$ and $W_t$ be given by

$$O_t = -2\kappa^{-1/2} \text{P.V.} \int_0^t X_s^{-1} ds$$
$$W_t = O_t + \sqrt{\kappa} X_t$$

where initial values $W_t = \sqrt{\kappa} x$ and $O_t < W_t$ are given. We then define SLE($\kappa; \rho$) to be the growing family of closed sets $K_t$ determined by the Loewner evolution with the driving parameter $W_t$ as given above. (See the definition of SLE, Section 2.2)

More generally, let $\beta \in [-1, 1]$, $\kappa > 0$, $\mu \in \mathbb{R}$, and $\rho \in \mathbb{R}$ be given and define $\delta = 1 + \frac{2(\rho+2)}{\kappa}$ as above. If either $\delta \in (0, 1) \cup (1, 2)$ and $\mu = 0$ or $\delta = 1$ and $\beta = 0$, then we define **skew SLE($\kappa; \rho$) with parameters** $\beta$ **and** $\mu$, which we denote SLE$^\mu_\beta(\kappa; \rho)$, the same way as we defined SLE($\kappa; \rho$) above except that we replace P.V. $\int_0^t X_s^{-1}$ with the process $Y_t$ of Proposition 3.8. In other words, we begin with the pair $(X_t, Y_t)$ from Proposition 3.8 (with some initial values $X_0$ and $Y_0$ fixed) and define $O_t = -2\kappa^{-1/2} Y_t$ and $W_t = O_t + \sqrt{\kappa} X_t$. Note that SLE($\kappa; \rho$) is equivalent to SLE$^\mu_\beta(\kappa; \rho)$ with $\beta = 1$, $\mu = 0$. We will assume $O_0 = W_0 = 0$ when we don’t specify otherwise.

Given a domain $D$ with marked boundary points $a$ and $b$, a chordal SLE($\kappa$) from $a$ to $b$ in $D$ is the image of chordal SLE from 0 to $\infty$ in $\mathbb{H}$ (as
defined above) under any conformal map taking $a$ to 0 and $b$ to $\infty$. We will be particularly interested in chordal SLE($\kappa; \kappa - 6$) because of the following:

**Proposition 3.9.** *In the chordal SLE context, fix $W_0 = 0$ and $O_0 = a$ for some non-zero $a \in \mathbb{R}$. Then an SLE($\kappa; \kappa - 6$) (from 0 to $\infty$) in $\mathbb{H}$, with these initial values — stopped at the first time $W_t = O_t$ — has the same law (up to time change) as a chordal SLE($\kappa$) in $\mathbb{H}$ from 0 to $a$ (which we may view as a random path $\gamma$) — stopped the first time $t$ for which $a$ and $\infty$ fail to lie on the boundary of the same connected component of $\mathbb{H} \setminus \gamma([0, t])$.

The reader may view the following as a reason to expect SLE($\kappa; \kappa - 6$) to be the scaling limit of the $T_v(A)$ described in Section 2 when $A$ is the coloring corresponding to an $O(n)$ model and $v$ is a boundary vertex.

**Proposition 3.10.** *Suppose $K_t$ is a random Loewner evolution in $\mathbb{R}$, driven by some continuous $W_t$, and write $O_t = g_t(\inf\{K_t \cap \mathbb{R}\})$. Suppose that $K_t$ satisfies the following:

1. **Scale Invariance:** for any positive constant $c$, the law of the process $t \rightarrow K_t$ is the same as that of the process $t \rightarrow cK_t$ up to time parameterization.

2. **Renewal property:** given $W_t$ up to any stopping time $T$ for which $O_T = W_T$ almost surely, the conditional law of the process $W_{T+t} - W_T$, for $t \geq 0$, is the same as the original law of $W_t$.

3. **Conformal Markov property:** given $W_t$ up to any fixed time $T$, the conditional law of the growth process $g_T K_{T+t} - W_{T+t}$ — up to time $\inf\{t : t \geq 0, O_{T+t} = W_{T+t}\}$ — is the same as that of an ordinary chordal SLE($\kappa$) in $\mathbb{H}$ from $W_T$ to $O_T$ up to that time. (Here $g_T K_{T+t}$ is a subset of $\mathbb{H}$ but the closure is taken in $\overline{\mathbb{H}}$.)

Then $K_t$ is an SLE($\kappa; \kappa - 6$) for some $\kappa > 4$. Conversely, the three properties above hold more generally when $W_t$ and $O_t$ are as in the definition of the SLE$_\beta^\mu(\kappa; \kappa - 6)$ process with $\kappa > 8/3$, $\mu \in \mathbb{R}$, and $\beta \in [-1, 1]$ (provided $\beta = 0$ if $\kappa = 4$ and $\mu = 0$ if $\kappa \neq 4$), although in this generality it is no longer the case that $O_t = g_t(\inf\{K_t \cap \mathbb{R}\})$. The conformal Markov property holds when $T$ is replaced with an arbitrary stopping time $T$ for which $O_T \neq W_T$ almost surely.

**Proof.** First we claim that $X_t = \kappa^{-1/2}(W_t - O_t)$ is a Bessel process with $\delta = 1 + \frac{2(\rho+2)}{\kappa}$, where $\rho = \kappa - 6$. It follows from Proposition 3.9 that when
$X_t > 0$, it evolves according to the SDE for this process. Since $X_t$ is almost surely positive, it is sufficient by Proposition 3.3 to show that the Lebesgue measure of the set of times $t$ for which $W_t = g_t(\inf\{K_t \cap \mathbb{R}\})$ is almost surely zero. In fact, we claim that this is true for any continuous Loewner evolution $W_t$.

To see this, fix $\epsilon > 0$ and let $A_t$ be the largest integer multiple of $\epsilon$ less than $\inf\{K_t \cap \mathbb{R}\}$. The process $\tilde{O}_t = g_t(A_t)$ evolves differentiably according to (2.1) except at discrete times when it jumps by discrete amounts to the left, and the monotonicity of (2.1) implies that $|\tilde{O}_t - O_t| \leq \epsilon$ for all $t$. Now, $(W_t - O_t) < \epsilon$ implies $W_t - O_t < 2\epsilon$. Let $T$ be the first time for which $O_t \leq C$, for some constant $C < 0$. Since $\tilde{O}_0 = -\epsilon$ and $\tilde{O}_T \geq C - \epsilon$, it follows from (2.1) that the Lebesgue measure of $\{t : 0 \leq t \leq T, |W_t - O_t| < \epsilon\}$ is less than or equal to $\epsilon C$. Since this holds for any $\epsilon$, it in particular implies that $\{t : 0 \leq t \leq T, W_t = O_t\}$ has Lebesgue measure zero. Since this holds for any $C$, it proves the claim.

Second, the reader may easily check that $X_t = \kappa^{-1/2}(W_t - O_t)$ and $Y_t = -2\sqrt{\kappa}O_t$ satisfy the hypotheses of Proposition 3.8; the fact that $Y_t' = X_t^{-1}$ on the set $\{t : X_t \neq 0\}$, almost surely, follows from Proposition 3.9 and the conformal Markov property, while the other properties are consequences of the scale invariance and renewal assumptions. It follows that the (O_t,W_t) pair arising in some SLE_\beta(\kappa; \rho). The fact that O_t is almost surely non-decreasing implies that \beta = 1 and \delta > 1; hence \kappa > 4.

The concluding two sentences of Proposition 3.10 are immediate from Propositions 3.9 and 3.8.

An ordinary SLE starting from a boundary point a and ending at a boundary point b on a planar domain is believed to have the same law (up to a time change) as an SLE starting at b and ending at a [11]. (This must be the case for $8/3 < \kappa < 8$ if SLE(\kappa) is the scaling limit of the one-boundary-arc O(n) models discussed in Section 2.3) However, this invariance does not readily follow from the definition of SLE. The following conjecture will turn out to be relevant to the study of conformal loop ensembles:

**Conjecture 3.11.** Fix $4 < \kappa < 8$ (and $\mu = 0$, $\beta = 1$). Then the processes SLE(\kappa) and chordal SLE(\kappa; \kappa - 6) are both almost surely continuous paths. The laws of these paths—up to direction of parameterization—are invariant under anticonformal automorphisms of D that swap the endpoints of the paths.

The fact that SLE(\kappa) is continuous appears in [11], but the proof has never been extended to SLE(\kappa; \rho) processes.
3.4 Radial SLE($\kappa; \rho$)

We will now introduce radial SLE($\kappa; \rho$). Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Radial SLE($\kappa$) is a random path from a boundary point of $\mathbb{D}$ to the center of $\mathbb{D}$ defined the same way as chordal SLE($\kappa$) except that instead of (2.1) we use the ODE

$$\partial_t g_t(z) = \Psi(W_t, g_t(z)),$$

where $W_t$ is a point on the unit circle and (following notation from [12])

$$\Psi(w, z) := -\frac{z + w}{z - w}.$$

In this case SLE($\kappa$) is defined by taking $W_t = e^{i\sqrt{\kappa}B_t}$ where $B_t$ is a standard Brownian motion. Equivalently, $W_t$ is the solution to the SDE

$$dW_t = (-\kappa/2)W_t dt + i\sqrt{\kappa}W_t dB_t.$$

Given any measurable driving function $W_t : [0, \infty) \to \partial\mathbb{D}$, we let $\tau_z$ be the supremum of the $t$ for which $g_t(z)$ is well-defined and write $K_t := \{z \in \mathbb{D} : \tau_z \leq t\}$. Then $g_t$ is a conformal map from $\mathbb{D} \setminus K_t$ to $\mathbb{D}$. When it exists, we write $\gamma(t) := \lim_{z \to W_t} g_t^{-1}(z)$, as in the chordal case. (Recall that in the setting of radial SLE($\kappa$), $\gamma$ exists and is continuous almost surely for all $\kappa \geq 0$ [11].) Note that $t$ represents not the half-plane capacity of $K_t$ (as in the chordal case) but $-1$ times the log of the conformal radius of $\mathbb{D} \setminus K_t$ viewed from zero (i.e., $t = \log |g_t'(0)|$).

If we take $O_t$ to be another point on the unit circle, we can define radial SLE($\kappa; \rho$) — at least up until the first time $O_t$ and $W_t$ collide — by taking

$$dO_t = \Psi(W_t, O_t) dt$$

and

$$dW_t = (-\kappa/2)W_t dt + i\sqrt{\kappa}W_t dB_t + \frac{\rho}{2} \bar{\Psi}(O_t, W_t) dt,$$

where

$$\bar{\Psi}(z, w) := \frac{\Psi(z, w) + \Psi(\overline{z}^{-1}, w)}{2}.$$

Given initial values $O_0 \neq W_0$ the solution to this SDE exists uniquely up until the first time $W_t = O_t$; see [12], which also proves the following analog of Proposition 3.9

**Proposition 3.12.** Fix $W_0 = 1$ and $O_0 = a$ to be distinct points on the unit circle $\partial\mathbb{D}$. Then a radial SLE($\kappa; \kappa - 6$) in $\mathbb{D}$ (from 1 to 0) — stopped
the first time $W_t = O_t$ — has the same law (up to time change) as a chordal SLE($\kappa$) path $\gamma$ in $\mathbb{D}$ from 1 to $a$ — stopped the first time $t$ for which $a$ fails to lie on the component of $\mathbb{H} \setminus \gamma([0, t])$ containing 0.

We can extend the definition of radial SLE($\kappa$; $\kappa - 6$) beyond times for which $W_t = O_t$ by mapping the corresponding chordal SLE($\kappa$; $\kappa - 6$) into the unit disc. If $O_t$ and $W_t$ are continuous processes on $\partial \mathbb{D}$, then let $\hat{O}_t$ denote the lifting of $\text{arg}(W_t - O_t)$ to a continuous function on $\mathbb{R}$. We will prove the following in Section 6:

**Proposition 3.13.** Fix $\kappa \in (8/3, 8)$, $\beta \in [-1, 1]$, and $\mu \in \mathbb{R}$ (such that $\beta = 0$ if $\kappa = 4$ and $\mu = 0$ if $\kappa \neq 4$). Then there exists a unique continuous Markovian diffusion on pairs $(W_t, \hat{O}_t)$ with the following property: when the initial values $(W_0, \hat{O}_0)$ are such that $\hat{O}_0 = 2k\pi$ for some integer $k$ and $\psi$ is any conformal map (if $k$ is even) or an anti-conformal map (if $k$ is odd) from $\mathbb{D}$ to $\mathbb{H}$ for which $\psi(W_0) = 0$, the image of the corresponding radial Loewner evolution $K_t$ under $\psi$—up until the first time $t$ that $\psi^{-1}(\infty) \in K_t$—is given by chordal SLE$^\mu_\beta(\kappa; \kappa - 6)$ with initial values $W_0 = O_0 = 0$ (up until that time).

We then define radial SLE$^\mu_\beta(\kappa; \kappa - 6)$ (in $\mathbb{D}$ with target 0) to be the radial Loewner evolution driven by the $W_t$ from Proposition 3.13 When not specified otherwise, we take initial values to be $W_0 = 1$ and $\hat{O}_0 = 0$ (so that $O_t = 1$). We can then define radial SLE$^\mu_\beta(\kappa; \kappa - 6)$ in any other domain $D$ with a fixed target $z \in D$ to be the image of the radial SLE$^\mu_{\beta}(\kappa; \kappa - 6)$ defined above under a conformal map that sends 0 to $z$ and $W_0$ and $O_0$ to the appropriate initial values on the boundary of $D$. The following is an immediate consequence of Proposition 3.13 (and the fact that the choice of $\phi$ in the statement was arbitrary).

**Proposition 3.14.** For any $\kappa > 8/3$, $\mu \in \mathbb{R}$, and $\beta \in [-1, 1]$ (where $\beta = 0$ if $\kappa = 4$, $\mu = 0$ if $\kappa \neq 4$) the law of radial SLE$^\mu_\beta(\kappa; \kappa - 6)$ is target invariant. That is, if we fix initial values $W_0 = O_0 = 1$ and $\hat{O}_0 = 0$ and we fix distinct points $a, b \in \mathbb{D}$, then the law of SLE$^\mu_\beta(\kappa; \kappa - 6)$ targeted at $a$ and the law of SLE$^\mu_\beta(\kappa; \kappa - 6)$ targeted at $b$ — both defined up to supremum of the set of times $t$ for which $a \notin K_t$ and $b \notin K_t$ — are the same (up to a time change).
4 Conformal loop ensembles

4.1 Defining loops

Before defining CLE, we need to define a suitable space of loops. (See also [1, 2] where similar spaces of loops and σ-algebras on loop ensembles are introduced.) A simple loop is a subset of \( \mathbb{C} \) which is homeomorphic to the circle \( S^1 \). Equivalently, we identify a simple loop with a homeomorphism from \( S^1 \) to a subset of \( \mathbb{C} \), modulo monotone reparameterization.

Let \( \mathcal{C} \) be the set of homeomorphisms from \( S^1 \) to subsets of \( \mathbb{C} \) and let \( \overline{\mathcal{C}} \) denote the closure of \( \mathcal{C} \) with respect to the \( L^\infty \) metric. A quasisimple loop is an element of \( \mathcal{C} \), modulo monotone reparameterization. A quasisimple loop may intersect itself, but it cannot “cross” itself transversely. Clearly, the winding number of a quasisimple loop \( L \) around a point \( z \in \mathbb{C} \setminus \eta(S^1) \) is equal to zero or \( \pm 1 \) (since this is true of simple loops and remains true under uniform limits provided that the limit fails to intersect \( z \)). In the latter case, we say that \( z \) is surrounded by \( L \).

We define the distance between quasisimple loops \( L_1 \) and \( L_2 \) by

\[
d(L_1, L_2) = \inf ||\zeta_1 - \zeta_2||_\infty,
\]

where the infimum is taken over pairs \( \zeta_1 : S^1 \to \mathbb{R}^2 \) and \( \zeta_2 : S^1 \to \mathbb{R}^2 \) of parameterizations of \( L_1 \) and \( L_2 \).

Let \( D \) be a bounded domain. Then the set \( \Omega_D \) of all discrete subsets of the set of quasisimple loops in \( \overline{D} \) is a metric space under the Hausdorff metric induced by the metric \( d(\cdot, \cdot) \) described above. Denote by \( \mathcal{F}_D \) the Borel σ-algebra on \( \Omega_D \). Most natural functions of \( \Omega_D \) (such as the number of loops completely surrounding a fixed disc, or the number of loops intersecting two disjoint open sets, or the event that the outermost loop surrounding a fixed point is a simple loop) can be shown to be \( \mathcal{F}_D \) measurable.

The most natural definition of a random loop ensemble is a random variable whose law is a probability measure on \( (\Omega_D, \mathcal{F}_D) \). Unfortunately, when defining CLE(\( \kappa \)) in arbitrary domains, it may not be enough to consider quasisimple loops. For example, consider the case that \( D \) is a non-Jordan domain such as the square \( (0, 2) \times (0, 2) \) minus the set \( \{n^{-1} : n \in \mathbb{Z}_+\} \times (0, 1) \). When \( \kappa = 8 \), we expect CLE(\( \kappa \)) to be a single space filling loop. However, it is clear that any loop in the closure of \( D \) which is space filling in \( D \) cannot be a continuous closed curve (since its \( y \) coordinate must oscillate between 0 and 1 infinitely many times). Even when the original domain is a Jordan domain and \( \kappa < 8 \), one may worry a priori that as we construct loops through an exploration process we may create domains which are no longer Jordan.
domains. It will therefore be convenient to slightly expand our definition of loops.

A pinned loop in $\mathbb{H}$ is a quasisimple loop $L$ in $\mathbb{H}$ which intersects the origin at 0 and has the property that if $\eta : S^1 \to \mathbb{H}$ is a parameterization of the loop then $\eta^{-1}\mathbb{H}$ is dense. Roughly speaking, we now wish to define a “conformal loop” in $\mathbb{C}$ to be the image of a pinned loop under a conformal map $\phi$ from $\mathbb{H}$ to another domain $D \subset \mathbb{C}$. Note that the parameterization $\phi \circ \eta$ of this image is well-defined on an open dense subset $U_0$ of $S^1$ but may not extend continuously to all of $S^1$. If $U$ is the set of all points $s$ in $S^1$ for which $\phi \circ \eta$ can be extended to a continuous function in a neighborhood of $s$, then it is clear that $U$ is the largest open set to which $\phi \circ \eta$ can be continuously extended. Formally, a conformal loop is a map $\zeta$ from a dense open subset $U$ of $S^1$ into $\mathbb{C}$ (modulo monotone reparameterization) such that

1. There exists a conformal map $\psi$ from $\mathbb{H}$ to a superset of $\zeta(U)$ with the property that $\psi^{-1} \circ \zeta$ is the restriction to $U$ of a function $\eta \in \overline{\mathbb{C}}$ with $\eta^{-1}\mathbb{H} \subset U$.

2. The set $U$ is maximal, i.e., that there exists no open proper superset $U'$ of $U$ such that $\zeta$ can be extended to a continuous function of $U'$.

If $L = (\zeta, U)$ is a conformal loop, we will sometimes abuse notation slightly and use $L$ to denote the corresponding set $\zeta(U) \subset \mathbb{C}$ of points on the loop. A point in $z \in \mathbb{C} \setminus L$ is surrounded by $L$ if $g(z)$ is surrounded by $\eta$, where $\eta$ is as described above. The reader may check that this definition is the same for every $g$, and that the set of $z$ surrounded by $L$ is a union of bounded connected components of $\mathbb{C} \setminus L$.

Our initial approach to defining CLE($\kappa$) will be to define a coupling of SLE$^\mu_\beta(\kappa; \kappa - 6)$ processes called a continuum exploration tree and to show that this process almost surely determines a countable collection of conformal loops. We will then show in Section 5 that—if Conjecture 3.11 holds—these loops are almost surely quasisimple and the laws of the CLE($\kappa$) may be equivalently described as measures on $(\Omega_D, \mathcal{F}_D)$.

### 4.2 Continuum exploration trees

Fix initial values $W_0 = O_0 = 1 \in \partial \mathbb{D}$. Then Proposition 3.14 implies that an SLE$^\mu_\beta(\kappa; \kappa - 6)$ targeted at $a_1 \in \mathbb{D}$ and an SLE$^\mu_\beta(\kappa; \kappa - 6)$ targeted at another point $a_2 \in \mathbb{D}$ can be coupled in such a way that the corresponding growth processes $K_t^{a_1}$ and $K_t^{a_2}$ agree (after a time change) up to the first
time $t$ at which $a_1$ and $a_2$ are separated (i.e., $a_1 \in K_t^{a_2}$ and $a_2 \in K_t^{a_1}$) and evolve independently of one another after that time. In other words, we may construct this coupling by first sampling an $\text{SLE}_{\beta}^{\mu}(\kappa; \kappa - 6)$ process $K_t^{a_1}$ targeted at $a_1$, and then sampling an $\text{SLE}_{\beta}^{\mu}(\kappa; \kappa - 6)$ process $K_t^{a_2}$ targeted at $a_2$ conditioned on it agreeing with the first sample (up to time change) until the first time that $a_1$ and $a_2$ are separated.

(We remark that if it were known that radial $\text{SLE}_{\beta}^{\mu}(\kappa; \kappa - 6)$ were a continuous path $\gamma$, then the separation time would be the smallest $t$ for which $a_1$ and $a_2$ lie in distinct components of $D \setminus \gamma([0, t])$. We use instead the language of growth processes because we have not proved that radial $\text{SLE}_{\beta}^{\mu}(\kappa; \kappa - 6)$ is a continuous path; however, if we had a proof of Conjecture 3.11, then the continuity of radial $\text{SLE}_{\beta}^{\mu}(\kappa; \kappa - 6)$ would be an easy consequence, and the above construction would describe a random path that “branches” at the point $\gamma(t)$.)

We can define a similar coupling inductively for sets $a_1, \ldots, a_k$ with $k > 2$ as follows. To sample from such a coupling, first we choose the growth processes $K_t^{a_j}$ for all $j < k$ and then conditioned on these processes, we choose $K_t^{a_k}$ conditioned on having it agree with each $K_t^{a_j}$ (after a time change) up until the first time $a_k$ is separated from $a_j$. We can interpret this as an $\text{SLE}_{\beta}^{\mu}(\kappa; \rho)$ process that “branches” at each of the finitely many times that a pair $a_i$ and $a_j$ becomes separated for the first time.

In fact, we may repeat this procedure infinitely many times — for some countable dense sequence $a_1, a_2, a_3, \ldots$ of points in $D$ — to obtain a coupling of $\text{SLE}_{\beta}^{\mu}(\kappa; \kappa - 6)$ growth processes $K_t^{a_i}$ from 1 to $a_i$. (Formally, if $\Omega^{a_i}$ is the space of continuous driving parameters $W$ for growth processes targeted at $a_i$ and $\mathcal{F}^{a_i}$ is the smallest $\sigma$-algebra which makes $W_t$ measurable for each fixed $t$, then this coupling is a random variable in $\prod \Omega^{a_i}$ whose law is measurable with respect to the product $\sigma$-algebra generated by the $\mathcal{F}^{a_i}$.) Note that for every $i \neq j$, the $K_t^{a_i}$ and $K_t^{a_j}$ agree almost surely (after a time change) until the first time $t$ that $a_i$ and $a_j$ are separated (i.e., $a_i \in K_t^{a_j}$ and $a_j \in K_t^{a_i}$), after which they evolve independently. We may view this collection of $K_t^{a_i}$ as a single random growth process which branches at each of the countably many times that some $a_i$ becomes separated from some $a_j$ for the first time.

Given a family of growth process $K_t^{a_i}$ (one for each $i \geq 1$) chosen from such a coupling, we may almost surely uniquely (up to time change) define, for each point $z \in \bar{D}$, a growth process $K_t^z$ such that for each $a_i$, the processes $K_t^{a_i}$ and $K_t^z$ agree (after a time change) until the first time $a_i$ and $z$ are separated. It is not hard to see that the joint law of the processes $K_t^z$ (now
defined for all $z \in \mathbb{D}$) is independent of our choice of countable dense set $\{a_i\}$.

The complete collection of processes $K^z_t$, for $z \in \mathbb{D}$, is called branching SLE($\kappa; \kappa - 6$), or the continuum exploration tree.

4.3 Constructing loops from exploration trees

The $K^z_t$ are analogous to the exploration paths $T_v(A)$ that we defined in Section 2.1 in the discrete setting. (Recall Figure 2.1) This section will describe an algorithm for constructing a family of conformal loops from a continuum exploration tree. The algorithm is motivated by the discrete picture, in particular Section 2.4.

Recall that in the discrete setting, the paths $T_v(A)$ trace part (but not necessarily all) of each loop that surrounds $v$. The reader may observe (recalling Section 2.4) that if $T_v(A)$ begins to trace such a loop at a vertex $w$ at step $m$, then it will continue to trace that loop until the first time $n$ that $v$ and $w$ are separated by the path drawn thus far (i.e., $w$ fails to lie on the boundary of $G_n$). The portion of $T_v(A)$ between $m$ and $n$ is an arc of a loop. The $m$ and $n$ are successive renewal times (as defined in Section 2.4), and the corresponding graphs $G_m$ and $G_n$ have monochromatic boundaries of opposite colors (as defined in Section 2.4). We will now make the analogous construction in the continuum setting.

Fix $z \in \mathbb{D}$ (we may assume $z = 0$, applying a conformal map otherwise). Write $\theta_t = \arg W_t - \arg O_t$, where the branch of arg is chosen so that $\theta_t$ is continuous in $t$ and $\theta_0 = 0$. Since $\rho = \kappa - 6$, using the definition for SLE($\kappa; \rho$) given in 3.4, we see that the difference $\theta_t = \arg W_t - \arg O_t$ is a real-valued process that evolves according to the diffusion

$$d\theta_t = \frac{\kappa - 4}{2} \cot(\theta_t/2) dt + \sqrt{\kappa} dB_t$$

in between those times $t$ for which $\theta_t$ is an integer multiple of $2\pi$.

We will now construct a sequence $L^z_j$ of nested conformal loops surrounding $z$, such that each $L^z_{j+1}$ is surrounded by $L^z_j$. First, we call a time $t$ a loop closure time if it is the first time that $\theta_t$ hits a particular integer multiple of $2\pi$ after the last previous time $s$ that it hit a different multiple. (Having $\theta_t$ change from an odd to an even multiple of $2\pi$ corresponds to having the monochromatic boundary of $G_m$ and $G_n$ be of opposite colors in the discrete setting, as discussed above.) Denote by $t^z_j$ the $j$th such time and by $s^z_j$ the corresponding value of $s$ (which correspond to $m$ and $n$ in the discrete setting). Denote by $A^z_j$ the component of $\mathbb{D} \setminus K_t$ that contains $z$. 

27
Figure 4.1: The left figure is a schematic drawing of $\gamma_j^z$ when $4 < \kappa < 8$. The black dot which does not lie on the path is $\phi(z)$. The continuous path $\gamma_j^z$ begins at 0 (at time $s_j^z$) and ends when it reaches the first point $\gamma_j^z(t_j^z)$ (also shown as a black dot), where $t_j^z$ is the first time that $\phi(z)$ is separated from the origin. (For discrete intuition, imagine that $\mathbb{R}$ is made of white hexagons, and the path $\gamma_j^z$ has black hexagons on its left side and white hexagons on its right side.) Conditioned on this $\gamma_j^z$, the law of the remainder of the loop $\phi(L_j^z)$ is that of an SLE($\kappa$) in $\mathbb{H} \setminus \gamma_j^z([s_j^z, t_j^z])$ started at $\gamma_j^z(t_j^z)$ and ending at 0. The middle figure is a schematic drawing of $\gamma_j^z'$ for a different value $z'$ (the black dot which does not lie on the path); in this case, $\gamma_j^z'$ is an extension of the path $\gamma_j^z$. The figure on the right indicates the union of all such extensions. This is the complete loop $\phi(L_j^z)$, a quasisimple closed loop in $\mathbb{H}$ which begins and ends at 0.

We will see that the process $K_t$ between times $s_j^z$ and $t_j^z$ traces part of a conformal loop. To begin to describe that loop, let $\phi : \mathbb{D} \setminus K_{s_j^z} \to \mathbb{H}$ be the composition of $g_{s_j^z}$ with a conformal map from $\mathbb{D}$ to $\mathbb{H}$ that sends $W_{s_j^z}$ to zero.

We now claim that for all $t \in [s_j^z, t_j^z]$, the set $\mathbb{H} \setminus \phi K_t$ can be written, almost surely, as the unbounded component of $\zeta([s_j^z, t])$ where $\zeta : [s_j^z, t_j^z] \to \mathbb{H}$ is a continuous path which extends continuously to its endpoints. To see this, note that the corresponding SLE($\kappa$) in a Jordan domain is almost surely continuous by [11]. For each fixed $s \in (s_j, t_j)$, the law of the evolution of $K_t$ after time $s$ is that of an SLE($\kappa$). If $h_s$ is a conformal map from $\mathbb{H} \setminus \phi K_s$ to $\mathbb{H}$ (with the hydrodynamic normalization at infinity) then this implies that the growth of $h_s \phi K_t$ is indeed given by a continuous function $\zeta_s$ on $[s, t_j]$. We can then set $\zeta(t) = h_s^{-1}(\zeta_s(t))$ whenever $h_s^{-1}(\zeta_s(t))$ is well defined. Since the $h_s$ converge uniformly to the identity, the claim follows from the fact that the uniform limit of continuous functions is continuous. We denote by $\gamma_j^z$ the path $\zeta$ described above for a fixed choice of $z$ and $j$.

When $\kappa \leq 4$, we know that chordal SLE($\kappa$) is a simple path which
extends continuously to its endpoints and does intersect the boundary of the domain it is defined on except at these endpoints; from this we may conclude that the second endpoint of \( \gamma_j^z \) is the same as the first endpoint, so that \( \gamma_j^z \) is in fact a simple closed curve. We then define \( L_j^z \) to be the image under \( \phi^{-1} \) of \( \gamma_j^z \). Then \( L_j^z \) is a conformal loop by definition.

When \( \kappa > 4 \), the path \( \gamma_j^z \) stops the first time that \( z \) fails to lie on the component of \( \mathbb{H} \setminus \gamma_j^z \) that contains the first endpoint of \( \gamma_j^z \) on its boundary. We do not expect \( \gamma_j^z \) to be a closed loop. However, we may consider some \( z' \in \mathbb{D} \setminus \bigcup_{s < s'} K_{s'}^{z} \). Then it is not hard to see that either \( \gamma_j^{z'} \) starts at a different time from \( \gamma_j^z \) or \( \gamma_j^z \) is (up to time change) a proper sub-arc of \( \gamma_j^{z'} \); moreover, the union of the \( \gamma_j^{z'} \) for which the latter holds is a quasisimple closed loop. Given \( \gamma_j^z \), its law is given by a chordal SLE(\( \kappa \)) (in the appropriate component of \( \mathbb{H} \setminus \gamma_j^z \)) from the last endpoint of \( \gamma_j^z \) to the first endpoint of \( \gamma_j^z \). Now we define \( L_j^z \) to be the image under \( \phi^{-1} \) of this loop. Again, \( L_j^z \) is a conformal loop by definition.

To get a more intuitive understanding of the relationship between loops and trees, recall that in the discrete setting, at each branch point that lies on a loop we have a notion of a “proper” branch (which continues to follow the loop) and an “improper” branch (which does not continue to follow the loop). In the continuum setting, we can define branch points analogously; given a pair \( a_i \neq a_j \), and given \( K_t^{a_i} \) and \( K_t^{a_j} \) time changed so that they agree up until the first time (call it \( T \)) for which \( a_i \) and \( a_j \) are separated, we say that \( K_T \) is the branch set of \( a_i \) and \( a_j \), and \( g_T^{-1} W_t \) (when it exists—i.e., when \( g_T^{-1} \) extends continuously \( W_t \)) is their branch point. When \( \kappa \leq 4 \), it is clear that the exploration tree cannot branch in the middle of tracing a loop (since SLE(\( \kappa \)) is simple in this case). However, when \( \kappa > 4 \), the tree can branch while tracing the boundary of a loop; for each of the countably many branch points of this form, there is a proper branch (which continues to trace the boundary of the loop) and an improper branch (which turns into a region whose boundary is part of the loop).

When \( \kappa \in (8/3, 8) \), \( \beta \in [-1, 1] \), and \( \mu \in \mathbb{R} \) (with \( \beta = 0 \) if \( \kappa = 4 \) and \( \mu = 0 \) otherwise), we define \( \text{CLE}^\mu_\beta(\kappa) \) to be the collection of loops of the form \( L_j^z \) described above. We will show in Section 5.1 that if Conjecture 3.11 holds and \( \kappa \in (4, 8) \), then the law of the collection of loops \( \text{CLE}^\mu_\beta(\kappa) \) is independent of \( \beta \) and \( \mu \) — and can thus be denoted \( \text{CLE}(\kappa) \) (see Proposition 5.1). (A similar fact for \( \kappa \leq 4 \) will appear in [14].) In the absence of a complete proof of this fact, we will (somewhat arbitrarily) declare \( \text{CLE}(\kappa) \) to be the ensemble of loops corresponding to \( \mu = 0 \) and either \( \beta = 1 \) if \( \kappa \in (4, 8) \), or \( \beta = 0 \) if \( \kappa \in (8/3, 4] \). The following is immediate from Proposition 3.14 (and
does not rely on Conjecture 3.11):

**Proposition 4.1.** Fix \( \kappa \in (8/3, 8) \), \( \beta \in [-1, 1] \), and \( \mu \in \mathbb{R} \) (with \( \beta = 0 \) if \( \kappa = 4 \) and \( \mu = 0 \) otherwise). Then the law of \( \text{CLE}_\beta^\mu(\kappa) \) in a simply connected domain \( D \) is invariant under conformal automorphisms of \( D \) that fix the starting point of the exploration tree.

### 4.4 CLE gasket and conformal radius distribution

The closure \( \Gamma \) of the union \( \cup L_z^1 \) of all “outermost” loops is called the CLE gasket. It is a random closed set. It is not hard to see that conditioned on \( \Gamma \), the law of the non-outermost loops is given by an independent CLE in each component of the complement of \( \Gamma \). When \( \kappa \leq 4 \), the loops are the boundaries of the gasket. Various properties of \( \Gamma \) will be investigated in [13]. As a prelude to [13], we describe one such property here.

Recall that when \( \rho = \kappa - 6 \), the difference \( \theta_t = \log W_t - \log O_t \), in radial coordinates, evolves according to the diffusion (4.1) in between those times \( t \) for which \( \theta_t = 0 \) or \( \theta_t = 2\pi \). Let \( \mathbb{D} \) be the unit disc and let \( A \) be the component of the complement of the gasket that contains the origin—i.e., \( A = \mathbb{D} \setminus K_t^z \), where \( t = t_z^1 \). The conformal radius of \( A \) (viewed from the origin) is defined to be \( |f'(0)|^{-1} \) where \( f \) is any conformal map from \( A \) to \( \mathbb{D} \) that fixes the origin. Let \( T \) be \(-1\) times the log of the conformal radius of \( A \). By the construction using branching SLE(\( \kappa; \kappa - 6 \)), it is clear that \( T \) is equal to the time in a SLE(\( \kappa; \kappa - 6 \)) evolution that \( \theta_t \) first hits \( \pm 2\pi \) when \( \theta_0 = 0 \).

The probability density for the law of \( T \) will be explicitly computed in [13]. For now, we offer the following:

**Proposition 4.2.** There is a unique adapted (to filtration generated by the Brownian motion \( B_t \)) and almost surely continuous random process \( R_t \) on the interval \([0, 2\pi]\) that is instantaneously reflecting at its endpoints (i.e., the total amount of time spent at 0 or 2\( \pi \) almost surely has Lebesgue measure zero), satisfies \( R_0 = 0 \), and—in between times that \( R_t \) hits the boundary—evolves according to the SDE (4.1). The law of \( T \) is equivalent to the law of \( \inf\{t : R_t = 2\pi\} \).

**Proof.** To prove the existence of a process \( R_t \) with the properties described above, we will show that \( |\theta_t| \) (where \( \theta_t \) is as defined from the appropriate radial SLE\( _\beta^\mu(\kappa; \rho) \) as in Section 4.3), has these properties. We already observed in Section 4.3 that the evolution of \( |\theta_t| \) in between times \([0, 2\pi]\) is given by (4.1); the fact that \( |\theta_t| \) is adapted and instantaneously reflecting
can be seen by changing coordinates to the setting of chordal SLE($\kappa; \rho$) in the half plane (Proposition 3.13) and then recalling that Bessel processes of dimension $\delta > 1$ are instantaneously reflecting (Proposition 3.1).

Next, if $R_t$ is any process with these properties, we may approximate it by the process $R'_{\epsilon t}$ that evolves according to the SDE (4.1) except that it immediately jumps to $\epsilon$ each time the process hits zero. Clearly, for each fixed $a \in (0, 2\pi]$, the law of $\inf\{t : R'_{\epsilon t} = a\}$ is stochastically decreasing in $\epsilon$ (for $\epsilon < a$) and converges to the law of $\inf\{t : R_t = a\}$ as $\epsilon \to 0$ (since $R_t$ is instantaneously reflecting). From this it is not hard to show that the $R'_{\epsilon t}$ converge in law (with respect to uniform topology on compact intervals) to $R_t$, and hence the law of $R_t$ is uniquely determined. The fact that the law of $T$ is equivalent to the law of $\inf\{t : R_t = 2\pi\}$ is now immediate from the way that the loops were defined (in terms of $\theta_t$) in Section 4.3.

4.5 Limiting cases: $\kappa = 8/3$ and $\kappa = 8$

As $\kappa \to 8/3$ from above, the dimension $\delta$ of the corresponding Bessel $X_t$ tends to zero; the process $X_t$ itself converges weakly to zero (with respect to the uniform topology on compact intervals of time) as $\delta \to 0$.

From this and results of Section 4.4 it is not hard to see that as $\kappa \to 8/3$ from above and $z \in D$ is fixed, the conformal radius of the interior of each $L_j^z$, viewed from $z$, tends to zero in probability. We thus define CLE(8/3) to be the loop ensemble which almost surely contains no loops. (Alternatively, we could define every point in $D$ to be its own loop.)

We remark that in the chordal SLE setting (recall the definitions in Section 3.3), as $\kappa$ tends to 8/3 from above, the $X_t$ becomes more tightly concentrated around zero, so that in the limit we have

$$W_t = \frac{4}{(2\rho + 2)/\kappa}B_t = 2\sqrt{\kappa}/(\kappa - 4)B_t = \sqrt{6}B_t,$$

and hence, the driving parameter of the SLE($\kappa; \kappa - 6$) process converges weakly to that of SLE(6) as $\kappa$ tends to 8/3. It is somewhat natural that (as the loops become very small), the law of the exploration path should converge to a process with the locality property that SLE(6) has (given that renewal times — i.e., times when $X_t = 0$ — come with increasing frequency as $\kappa \to 8/3$).

Next, we observe that when $\kappa \geq 8$ and $D$ is a Jordan domain, the process SLE($\kappa; \kappa - 6$) may be viewed as a space-filling path $\gamma$ that starts and ends at the origin. To see this, note that in the half plane formulation we may first condition on $W_t$ and $O_t$ up until a small stopping time $s$ for which
$W_s \neq O_s$ almost surely. Conditioned on $W_t$ and $O_t$ up until this time, the law of the remainder of the process is given by an SLE($\kappa$) (by Proposition 3.9), which is almost surely space filling when $\kappa > 8$ [11]. Since we may take $s$ as small as we like, it follows from [11] that the process is a continuous path (when $D$ is a Jordan domain) starting and ending at the same point.

We can therefore define CLE($\kappa$) to be this single space-filling loop when $\kappa = 8$. This loop can be approximated by choosing two points $a$ and $b$ very close together on the boundary of $\mathbb{H}$ and drawing an SLE($\kappa$) from $a$ to $b$. It follows from [8] that CLE($\kappa$) is the scaling limit of the path that traces the boundary of the free uniform spanning tree (see [8] for a precise definition of this paths).

In fact, we could define CLE($\kappa$) analogously as a single space-filling loop for any $\kappa > 8$. When $\kappa > 8$, however, we do not expect the law of this loop to be invariant under all conformal and anticonformal automorphisms of $D$ (even though it is invariant under the conformal automorphisms of $D$ that fix the starting point). It is clear that the law of CLE($\kappa$) is invariant under all conformal and anticonformal automorphisms of $D$ when $\kappa = 8$ (because of the uniform spanning tree interpretation) and when $\kappa = 8/3$ (trivially). We will discuss analogous invariance questions for the case $8/3 < \kappa < 8$ in Section 5.

5 Symmetry and uniqueness when $4 < \kappa < 8$

5.1 Continuity and starting point invariance of CLE

This section will derive some consequences of Conjecture 3.11 that apply when $4 < \kappa < 8$. Analogous questions for $8/3 \leq \kappa \leq 4$ will be dealt with in a subsequent paper ([14]). First, we define boundary branching SLE($\kappa; \kappa - 6$) to be a coupling of chordal SLE($\kappa; \kappa - 6$) processes targeted at each point in a countable dense set of boundary points of $\mathbb{D}$ (instead of radial SLE($\kappa; \kappa - 6$) processes targeted at interior points of $\mathbb{D}$). We may view this as a subset of the full branching SLE($\kappa; \kappa - 6$) tree, in which branching is only allowed to occur at points on the boundary of $\mathbb{D}$. Recall that the discrete analog of this tree traced all of the loops that hit the boundary of the hexagonal graph (Proposition 2.6).

In the continuum, it is also not hard to see that each of the conformal loops traced by boundary branching SLE($\kappa; \kappa - 6$) intersects the boundary of $\mathbb{D}$. Let $\chi$ be the closure of the set of points on loops traced by this process. This $\chi$ is a random closed subset of the CLE($\kappa$) gasket. By construction, conditioned on $\chi$, the law of the remaining loops is given by an indepen-
dent CLE(κ) in each component of the complement of χ, with appropriate starting points. We will use this fact to prove the following:

**Proposition 5.1.** Fix $4 < \kappa < 8$, $\beta \in [-1, 1]$, and $\mu = 0$. Suppose that Conjecture 3.11 holds for $\kappa$. Then the loops in a CLE$^\mu(\kappa)$ are almost surely quasisimple loops whenever $D$ is a Jordan domain. Moreover, the law of CLE$^\mu(\kappa)$ is invariant under conformal and anticonformal automorphisms of $D$ and this law is independent of $\beta$.

**Proof.** It can be seen from Proposition 3.13 that if chordal SLE($\kappa; \kappa - 6$) is a continuous path when it is defined in a Jordan domain, then radial SLE($\kappa; \kappa - 6$) is also; it then follows that the loops in CLE($\kappa$) are almost surely quasisimple, since in this case $\partial(D \setminus K^{a_j}_t)$ can be traced by a continuous curve, for every $j$ and $t$, almost surely, and hence any conformal map from $\mathbb{H}$ to $D \setminus K^{a_j}_t$ extends continuously to $\mathbb{R}$. This also means that the processes $K^\varepsilon_t$ are the hulls corresponding to actual continuous paths $\gamma^\varepsilon(t)$ almost surely (i.e., for each fixed $z$, $D \setminus K^\varepsilon_t$ is almost surely the component of $D \setminus \gamma^\varepsilon(0, t]$ which contains $z$, for all $t \geq 0$).

Now, to prove starting point independence, fix distinct points $a$ and $b$ on the boundary of $D$ and let $\psi$ be an anti-conformal map (i.e., a composition of a conformal map and a reflection) from $D$ to $D$ such that $\psi(a) = b$ and $\psi(b) = a$. It will clearly be enough to show that the law of the CLE($\kappa$) is invariant under such a map $\psi$, since the group of conformal and anticonformal automorphisms of $D$ is generated by functions $\psi$ of this form. In fact, it will suffice if we can show that the law of the set of loops in $\chi$ (i.e., the loops traced by the boundary-branching subtree of the exploration tree) have a law which is invariant under such a $\psi$, since the law of all of the loops may be generated inductively (as discussed above) from the law of $\chi$.

Let $T$ be an exploration tree $T$ started at $a$. We define a subtree $T^{a,b}$ of the boundary-branching exploration tree which includes only the exploration path from $a$ to $b$ (i.e., the process $K^\varepsilon_t$, which is a continuous path if Conjecture 3.11 holds) together with all paths obtained by proper branches off of $T^{a,b}$. In other words, this is the smallest subtree of the exploration tree which traces all of the loops which are partially traced by the exploration path from $a$ to $b$. See Figure 5.1. (In the discrete analog, given a set $A$ and boundary vertices $a = v_0$ and $b = v$, this would be the smallest subtree of $T(A)$ that contains $T_v(A)$ together with all but one edge of every loop that $T_v(A)$ intersects.)

First, we claim that we can couple two instances $T^{a,b}_1$ and $T^{a,b}_2$ of the exploration-tree-valued random variable $T^{a,b}$ in such a way that the set $A_1$ of quasisimple loops traced by $T_1^{a,b}$ is the same as the set $\psi A_2$, where $A_2$ is
Figure 5.1: The left figure is a schematic diagram of an \( \text{SLE}(\kappa; \kappa - 6) \) \( \gamma \) from the right boundary point \( a \) to the left boundary point \( b \), where \( 4 < \kappa < 8 \). In this stylized drawing, the path hits the clockwise arc from \( a \) to \( b \) at two points (instead of infinitely many). These two points separate the arc into three segments. On the right, an \( \text{SLE}(\kappa) \) is drawn in each of the three regions of \( \mathbb{D} \setminus \gamma \) that have these segments as boundaries. The starting point for the \( \text{SLE}(\kappa) \) is the left endpoint of the segment and the terminal point is the right endpoint. This creates three quasisimple loops. For discrete intuition, one can imagine white hexagons along \( \partial \mathbb{D} \) and the outside of each of the three loops and black hexagons along the inside of each of three loops—so each component of the interior of the complement of the loops drawn has a monochromatic boundary. The exploration tree started at \( a \) traces each of these loops in the counterclockwise direction.

The set of loops traced by \( T_{a,b}^{a,b} \), almost surely. To construct this coupling, we first arrange so that the exploration path \( \gamma \) from \( a \) to \( b \) in \( T_{a,b}^{a,b} \) is the image under \( \psi \) of the exploration path from \( a \) to \( b \) in \( T_{a,b}^{a,b} \) (which we can do if \( \text{SLE}(\kappa; \kappa - 6) \) has time reversal symmetry, since the law of \( \gamma \) is that of \( \text{SLE}(\kappa; \kappa - 6) \)).

Next, let \( D \) be a component of \( \mathbb{D} \setminus \gamma \) that includes a segment of \( \partial D \) in its boundary; let \( y_1 \) and \( y_2 \) be the first and last points on \( \partial D \) hit by the path \( \gamma \); and let \( \hat{\gamma} \) be the segment of \( \gamma \) which starts and ends at these points. If \( \kappa > 4 \), then \( \hat{\gamma} \) is part of a loop of both \( \phi \mathcal{A}_1 \) and \( \mathcal{A}_2 \); in one of the trees \( \psi T_1 \) and \( T_2 \), the remainder of the loop (obtained by following the proper branches in the exploration process) is given by an \( \text{SLE}(\kappa) \) from \( y_1 \) to \( y_2 \). In the other tree, it is an \( \text{SLE}(\kappa) \) from \( y_2 \) to \( y_1 \) in the same domain. We couple the \( T_1 \) so that these two paths are equivalent up to time parameterization
(which we can do if SLE($\kappa$) has time reversal symmetry). Then $\psi A_1 = A_2$ almost surely.

Now, let $D$ be a component of the interior of $\mathbb{D}$ minus the union of the loops in $A_1$ such that $D$ includes a segment of $\partial D$ in its boundary, and let $y_1$ and $y_2$ be the first and last points of that segment $\partial D$. Then the law of the remainder of the boundary branching tree for $L_2$ within the set $D$ is the image under an anticonformal map $\psi_D$ from $D$ to itself (that maps $y_2$ to $y_1$ and $y_1$ to $y_2$) of the law for $L_1$. We can then apply the same coupling we constructed above, where we replace $D$ with $D$ and the pair $(a,b)$ with $(y_1,y_2)$. It is not hard to see that by repeating this process for different segments of the boundary of $\mathbb{D}$, we obtain a coupling of the boundary-branching exploration trees with the property that the loop ensembles $B_1$ and $\phi B_2$ generated by those trees agree almost surely.

Finally, we need to show the independence of $\beta$. Consider a single exploration path $K_\vec{z}$ of the exploration tree. (Assume $z = 0$.) Each stopping time $T$ for which $O_T = W_T$ is a renewal time in the sense that conditioned on $K_\vec{z}$, the law of the loops in $D \setminus K_\vec{z}$ is given by a CLE($\kappa$) in $D \setminus K_\vec{z}$. Now, if we replace the exploration tree of this ensemble within $D \setminus K_\vec{z}$ by its image under an anticonformal map, this will not change the law of the corresponding loop ensemble.

Let $\hat{O}_t$ denote the lifting of $\text{arg}(W_t - O_t)$ to a continuous function on $\mathbb{R}$. Recall (Proposition 3.13) that radial SLE($\kappa$; $\kappa - 6$) is defined so that if initial values $(W_0, \hat{O}_0)$ are given where $\hat{O}_0 = 2k\pi$ for some integer $k$, and if $\psi$ is a conformal map (if $k$ is even) or an anti-conformal map (if $k$ is odd) from $\mathbb{D}$ to $\mathbb{H}$ for which $\psi(W_0) = 0$, then the image of the corresponding radial Loewner evolution $K_t$ under $\psi$—up until the first time $t$ that $\psi^{-1}(\infty)$ fails to lie on the boundary of $D \setminus K_t$—is given by chordal SLE$_\kappa^{\mu}(\kappa; \kappa - 6)$ with initial values $W_0 = O_0 = 0$.

Fix $n > 0$. One natural choice of stopping time $T = T_j$ (defined for each $j > 0$) is the first time $t$ such the area of $D \setminus K_\vec{z}_j$ is less than $j/n$ and $O_t = W_t$. At each such stopping time we may toss an independent coin with parameter $(\beta + 1)/2$. With probability $(\beta + 1)/2$, we continue the exploration tree with the original orientation (if $\hat{O}_0$ is an even multiple of $2\pi$) or the opposite orientation (if $\hat{O}_0$ is an odd multiple of $2\pi$). An exploration path thus defined targeted at $z_1$ can be coupled with an exploration path thus defined targeted at $z_2$ so that the two agree (after a time change) up until the first time that $z_1$ and $z_2$ are separated (since the definition of the stopping time makes no reference to choice of target point). Coupling together paths of this form targeted at the dense set $\{a_j\}$, we obtain a variant of the exploration tree which generates a set of loops with the same
law as CLE(κ). As n → ∞, it is not hard to see that each exploration path in a tree thus defined converges (in law with respect to uniform topology on compact time intervals) to \( \text{SLE}_\beta^\mu(\kappa; \kappa - 6) \) (with \( \mu = 0 \)). Since for each n, the law of the loops is the same, the limit gives a coupling of the law of the loops in a CLE(κ) with those of a branching \( \text{SLE}_\beta^\mu(\kappa; \kappa - 6) \) process, which completes the proof. \( \square \)

5.2 Uniqueness result for boundary-intersecting CLE

In this section we will focus on the case \( 4 < \kappa < 8 \). We will argue that if the scaling limit of an \( O(n) \) models exists and satisfies a few basic properties and conformal symmetries, then it must be a CLE(κ) for some \( \kappa > 4 \), at least when the model parameters are in the range for which the limiting loops hit the boundary. In some sense we have already done this—namely, we proved Proposition 3.10 which suggests that, under reasonable conformal invariance hypotheses, the scaling limit of the (non-oriented) exploration tree paths should be SLE(κ; \( \kappa - 6 \))—and hence the scaling limit of the exploration tree should be branching SLE(κ; \( \kappa - 6 \)). Our definition of CLE(κ) was derived from that assumption.

Nonetheless, it is natural to wonder whether there are nice loop ensembles which do not naturally arise from exploration trees in the same way. Assuming Conjecture 3.11 we will give a more complete axiomatic characterization of CLE(κ). Recall that a loop is outermost if it is not surrounded by any other loop.

**Lemma 5.2.** Fix \( 4 < \kappa < 8 \) and suppose that Conjecture 3.11 holds for that \( \kappa \). Let \( D \) be a Jordan domain. Let the boundary branching exploration tree and CLE(κ) be coupled as in the previous section. Then the loops traced by the boundary branching exploration tree are almost surely the only outermost loops in the CLE(κ) which intersect the boundary of \( \mathbb{D} \).

The proof makes use of the following simple fact, which we will prove in Section 6.

**Proposition 5.3.** Let \( Z_T \) be number of times that a Bessel process \( X_t \) of dimension \( \delta > 1 \) crosses the interval \([0, \epsilon]\) from bottom to top between time 0 and time \( T \). Then \( \lim_{\epsilon \to 0} \epsilon E Z_T = 0 \).

**Proof of Lemma 5.2** By Proposition 5.1 we have (assuming that Conjecture 3.11 holds) that the law of CLE(κ) is invariant under the choice of starting point of the exploration tree. At a renewal time of an exploration path (when \( O_t = W_t \)), the conditional law of the loops in \( \mathbb{H} \setminus K_t \) is that of a
CLE(κ) in \( \mathbb{H} \setminus K_t \). We can therefore shift the starting point of an exploration path at these renewal times without changing the law of ensemble of loops generated at the end. (We may think of this as exploring the same loop ensemble beginning at a different point.)

We now define a process \((O'_t, W'_t)\) — an invariant of our usual \((O_t, W_t)\) — that involves shifts of this form. This process begins by evolving according to the diffusion describing an SLE(κ; ρ) process until time \( t_1 = \inf\{t : -\epsilon \in K'_t\} \), where \( K'_t \) is the Loewner hull generated by \((O'_t, W'_t)\). (Since we are assuming SLE(κ; ρ) is continuous, there is a continuous γ corresponding to \( K \) up to this point, and \( t_1 \) is the first time this path hits \((-\infty, -\epsilon]\).) At this point the pair \((O'_t, W'_t)\) jumps \( \epsilon \) units to the right (producing a discontinuity in γ). Inductively, we let \( t_{k+1} \) be the first time \( t \) after time \( t_k \) that \( K_t \) hits \((-\infty, 0)\) (i.e., the corresponding continuous segment of γ beginning at time \( t \) hits a point in \((-\infty, 0]\)). At each \( t_k \), the process \((O'_t, W'_t)\) jumps \( \epsilon \) units to the right and continues to evolve according to a diffusion SLE(κ; ρ) process until it jumps again at \( t_{k+1} \). Thus, we can couple an \((O'_t, W'_t)\) with the driving parameters \((O_t, W_t)\) of an ordinary SLE(κ; ρ) process in such a way that \((O'_t, W'_t) = (O_t, W_t) + \epsilon(Y_t, Y_t)\) where \( Y_t = \inf\{k : t_k \geq t\} \). Note also that \( O'_{t_k} = W'_{t_k} \) for each \( k \geq 1 \) almost surely.

Let \( Z_T \) be the number of times that \(|W_t - O_t| \) hits \( \epsilon \) for the first time after the last time it hit 0 before time \( t = T \). (In other words, \( Z_T \) is the number of upward crossings of the interval \([0, \epsilon]\) before time \( T \).) Now, for each \( t_k \), the probability \( q \) that \(|W'_t - O'_t| \) reaches the value \( \epsilon \) before time \( t_{k+1} \) is independent of \( \epsilon \) and of the process \((W'_t, O'_t)\) up to time \( t_k \) (by scale invariance and the Markov property). Thus, the expected number of \( t_k \)'s occurring before time \( T \) which are followed by such an upward crossing (at any point before \( t_{k+1} \), which may occur after time \( T \)) is exactly \( qEY_T \). We conclude that \( qEY_T \leq EZ_T + 1 \). Proposition 5.3 implies that \( \lim_{t \to 0} EY_T = 0 \) and hence \( \lim_{t \to 0} EZ_T = 0 \).

Now, fix \( a > 0 \) and consider an SLE(κ; ρ) stopped at \( T = \inf\{t : -a \in K'_t\} \). We may analogously define \( T' = \int\{t : -a \in K'_t\} \). Now consider a coupling of \((O_t, W_t)\) and \((O'_t, W'_t)\) processes (one for each choice of \( \epsilon = 1/n \)) in which the loops partially traced by the corresponding Loewner evolutions belong to the same instance of CLE(κ). Clearly, in such a coupling the loops traced by \( K_t \) up until time \( T \) all touch the interval \((-a, 0)\). However, the set \( K'_{T'} \) contains all the loops that intersect \((-a, 0)\) (and possibly many more loops). (To see this, note that the corresponding γ' hits the interval \((-a, 0)\) only finitely many times, and after each such time there is a discontinuous jump to the right before the process starts again.) The law of \( T' \) therefore stochastically dominates that of \( T \) (since capacity is an increasing function.
of sets); since \( \epsilon Y_T \to 0 \) in probability as \( \epsilon \to 0 \), it is not hard to see that \( T' - T \to 0 \) in probability as \( \epsilon \to 0 \).

Now we claim that almost surely there is no loop in \( \mathbb{H} \setminus K_T \) that intersects the interval \([-a, 0]\). If there were such a loop with positive probability, then if we explored the same loop process with the \((Q'_t, W'_t)\) pair for some fixed \( \epsilon \), as described above, it would have to hit the loop before time \( T' \) for all \( \epsilon \), and thus \( \lim_{\epsilon \to 0} T' \) would be strictly larger than \( T \). Since \( T' \geq T \) almost surely, and \( T' \to T \) in law as \( \epsilon \to 0 \), we must have \( T' \to T \) almost surely as \( \epsilon \to 0 \).

Since each component of \( \mathbb{D} \setminus \chi \) can be obtained as a union of \( K_t \) processes of the form described above, we see that there are almost surely no loops in any such component that intersect \( \partial \mathbb{D} \).

For simplicity, Theorem 5.4 will focus only on the law of the set \( \mathcal{L} \) of outermost loops in a \( \text{CLE}(\kappa) \). (Once this law is known, the law of the entire loop ensemble can be determined inductively.)

**Theorem 5.4.** Suppose that \( \mathcal{L} \) is a random countably infinite ensemble of quasisimple non-nested loops on \( \mathbb{H} \) (formally, a probability measure on \((\Omega_D, \mathcal{F}_D)\), as defined in Section 5.1). If Conjecture 3.11 holds for some \( \kappa \in (4, 8) \) and \( \mathcal{L} \) is the set of outermost loops of a \( \text{CLE}(\kappa) \), then \( \mathcal{L} \) has the following properties:

1. **Conformal invariance:** The law of \( \mathcal{L} \) is invariant under conformal automorphisms of \( \mathbb{H} \).

2. **Boundary intersection:** The set \( \mathcal{L}_\partial \) of loops of \( \mathcal{L} \) that intersect \( \mathbb{R} \) is almost surely non-empty, and almost surely no loop in \( \mathcal{L}_\partial \) hits any single point in \( \mathbb{R} \) more than once.

3. **Local finiteness:** Given an interval \([a, b]\) and an open set \( A \subset \mathbb{H} \) whose closure is disjoint from \([a, b]\), there are almost surely only finitely many loops which intersect both \( A \) and the interval \([a, b]\).

4. **Conformal Markov property:** Given \( a \), \( b \), and \( A \) as in the previous item, let \( x \) be the right-most point in \([a, b]\) that lies in one of the (finitely many) loops \( L \) that intersects both \( A \) and \([a, b]\). Let \( J \) be the counterclockwise arc of \( L \) which begins at \( x \) and ends at the first \( y \) at which it hits \( \partial A \). Then \( J \) almost surely does not intersect \([a, x]\). Given \( J \) and the collection \( \mathcal{L}_{(x,b)} \) of all loops that intersect \((x,b)\), the conditional law of the counterclockwise arc of \( L \) from \( y \) to \( x \) is given by an \( \text{SLE}(\kappa) \) from \( y \) to \( x \) in the component of

\[
\mathbb{D} \setminus \bigcup \{L : L \in \mathcal{L}_{(x,b)}\} \cup J
\]
that has \([a,x]\) as part of its boundary.

5. **Renewal property:** Conditioned on the set \(\mathcal{L}_{[a,b]}\) of loops of \(\mathcal{L}\) intersecting an interval \([a,b]\) the law of the remaining loops in \(\mathcal{L}\) is given by a product of independent random loop ensembles in the (non-loop-surrounded) components of \(\mathbb{D} \setminus \bigcup_{a,b} \mathcal{L}_{[a,b]}\), each of which has the same law as the original law of \(\mathcal{L}\) conformally mapped to that component.

Conversely (whether Conjecture 3.11 holds or not), if \(\mathcal{L}\) is any random countably infinite collection of quasisimple loops with the properties listed above, then it must be a CLE(\(\kappa\)) for some \(4 < \kappa < 8\).

Of course, if one wishes to avoid making explicit reference to SLE(\(\kappa\)) in the conformal Markov property described above, one can replace the requirement that the path from \(y\) to \(x\) is an SLE(\(\kappa\)) with the requirement that this random path satisfies the hypotheses Theorem 2.7.

**Proof.** We begin by showing that if Conjecture 3.11 holds for \(\kappa\) with \(4 < \kappa < 8\), then CLE(\(\kappa\)) has the listed properties. Proposition 5.1 implies 1, and 2 follows from easily the fact that SLE(\(\kappa\)) hits the boundary and hits each point on the boundary at most once. (The latter fact is an easy consequence of continuity and time reversal symmetry of SLE(\(\kappa\)) and the fact that SLE(\(\kappa\)) hits each predetermined boundary point with probability zero. If \(\gamma\) is a path from \(0\) to \(\infty\) in \(\mathbb{H}\) that comes from a continuous Loewner evolution and \(\gamma\) hits some \(a \in \mathbb{R} \setminus \{0\}\) at two distinct times and hits another boundary point in \(\mathbb{R}\) of the same sign in between these two times, then the image of \(\gamma\) under the inversion \(z \to 1/\overline{z}\) cannot be a path that comes from a continuous Loewner evolution. Moreover, for each fixed rational \(t\), the probability that \(\gamma\) hits the last boundary point that it hit before time \(t\) for a second time after time \(t\) is zero. Hence, the probability that \(\gamma\) hits a point in \(\mathbb{R}\) at distinct times without hitting another point in \(\mathbb{R}\) in between those times is also zero.)

In the context of 3, the hypothesis that SLE(\(\kappa;\kappa - 6\)) is continuous implies that a chordal SLE(\(\kappa;\kappa - 6\)) from \(a\) to \(b\) can have at most finitely many excursions away from the interval \([a,b]\) that intersect \(A\). This together with Lemma 5.2 implies 3. Then 4 and 5 follow immediately from the conformal Markov property and renewal properties of SLE(\(\kappa;\rho\)) described in Proposition 3.10.

Now we proceed to the converse. Suppose that \(\mathcal{L}\) is a random ensemble of non-nested quasisimple loops with all of the properties listed above. Then by conformal invariance, the probability that \(\mathcal{L}\) contains a loop intersecting an interval of \(\partial \mathbb{R}\) is the same for all intervals. Since this probability must
approach one as the length of the interval approaches $\infty$, we conclude that $\mathcal{L}$ contains a loop intersecting each open interval of $\partial D$ with probability one.

Next, we will use a continuum analog of the construction of Propositions 2.4 and 2.5 to construct a path from 0 to $\infty$ as follows.

Let $M_1, M_2, \ldots$ be an enumeration of the loops in $\mathcal{L}$ that intersect the negative real axis. For each $i \geq 1$, let $I_i$ be the interval

$$(\inf L_i \cap (-\infty, 0), \sup L_i \cap (-\infty, 0)).$$

When the loop $L_i$ is counterclockwise oriented, let $A_i$ be the portion of the loop $L_i$ that starts at $\sup I_i$ and ends at $\inf I_i$. Suppose that every $I_i$ is contained in some maximal $I_j$. Then we may consider the concatenation of the $A_i$, where $i$ ranges over those $i$ for which $I_i$ is maximal (i.e., for which the interval $I_i$ is not contained in any distinct $I_j$). More precisely, consider any map from $(-\infty, 0]$ to $\mathbb{H}$ that maps each interval $I_i$ to the corresponding loop segment $A_i$. This is a map defined on an open dense subset $\mathbb{R}$ to $\partial \mathbb{H}$, and local finiteness implies that it extends continuously to a map from $[-\infty, 0]$ to $\partial \mathbb{H}$. Let $Q_t$ be a parameterization of this path in the opposite direction (so that $Q_0 = 0$).

We claim that the law of $Q_t$ must be that of SLE($\kappa; \kappa - 6$) for some $\kappa > 4$ when it is parameterized by capacity. To prove this, it is enough to verify the hypotheses of Proposition 3.10.

Let $t_a$ be the first time $t$ for which $Q_t \in (-\infty, a]$. The renewal property implies that conditioned on $Q_t$ up to time $t_a$, the law of the remainder of $Q_t$ is the same as the original law of $Q_t$ after a conformal map $g_t : \mathbb{H} \setminus Q_t([0, t_a]) \to \mathbb{H}$ that fixes $Q_{t_a}$ and $\infty$. In particular, the path $Q_t$, for $t \geq t_a$, remains in the closure of the infinite component of $\mathbb{H} \setminus Q_t([0, t_a]) \to \mathbb{H}$ almost surely. Now, the conformal Markov property of Proposition 3.10 follows immediately from the conformal Markov property cited here provided the stopping time is of the form $T = \inf\{t : Q_t \in A\}$, where $A \subset \mathbb{H}$ is an open set whose closure does not contain 0. If $T'$ is any stopping time such that almost surely $T' > T$ and $Q_{T'}$ lies on the same $A_i$ as $Q_T$ almost surely, then it follows from Theorem 2.7. The general result follows by noting that for any stopping time $T''$ we can find a sequence of stopping times of the form $T'$ that converge to $T''$ from below almost surely.

Finally, the conformal Markov properties implies that the Loewner evolution $W_t$ corresponding to $Q_t$ is continuous at all $t$ for which $Q_t \not\in \mathbb{R}$. That this holds for general $t$ follows easily from local finiteness. We have now proved all the hypotheses of Proposition 3.10.

By conformal invariance, the analogously defined $Q_t$—targeted at another point in $\mathbb{R}$ instead of $\infty$—will also have the law of SLE($\kappa; \kappa - 6$) tar-
geted at that point. If we consider a countable dense set of the boundary of \( \mathbb{H} \), then we may define the union of the corresponding maps \( Q \) targeted at these points to be a continuum exploration tree, which is a form of branching SLE\((\kappa; \kappa - 6)\); the fact that after the process branches the branches evolve independently is immediate from the renewal property; it follows that the law of the tree is the same as the one given in Section 4.3. The reader may now check that the boundary-intersecting loops of \( L \) can be almost surely recovered from this tree by applying the algorithm of Section 4.3. By the renewal property, the law of the boundary-intersecting loops determines the law of \( L \).

\[ \square \]

6 SLE\((\kappa; \rho)\) approximations and invariance

The authors in [12] proved a number of invariance properties and coordinate changes for SLE\((\kappa; \rho)\) started with \( O_0 \neq W_0 \) (so that \( X_0 \neq 0 \)) and stopped at the first time \( t \) such that \( O_t = W_t \) (when \( X_0 = 0 \)). The main purpose of this section is to prove Proposition 3.13, which is essentially an extension of the analogous result in [12] to times beyond the first time that \( O_t = W_t \).

In order to prove this (without repeating all of the calculations in [12]), we begin by proving Proposition 6.3, which shows that SLE\(_\mu^\beta\)(\(\kappa; \rho\)) can be approximated by processes in which \( O_t \) and \( W_t \) are instantly pushed apart by a small fixed amount (which depends on \( \mu, \beta, \) and \( \kappa \)) each time they collide. Proposition 6.3 may also give the reader more intuition about what the SLE\(_\mu^\beta\)(\(\kappa; \rho\)) processes are. (Some of the conjectures presented in Section 8 are based on this intuition.) Most of the following exposition will focus on the case that \( \mu = 0, \beta = 1, \) and \( \kappa \neq 4; \) the more general case will follow as a consequence of this case.

6.1 Approximate Bessel processes

Fix \( \epsilon > 0 \). Then we define an \( \epsilon \)-BES\(_\delta_\epsilon\) process \( X_t^\epsilon \) to be a Markov process beginning at some initial value \( X_0^\epsilon = x \) that evolves according to (3.2) except that each time it hits zero it immediately jumps to \( \epsilon \) and continues. We may thus write

\[
X_t^\epsilon = X_0^\epsilon + \int_0^t \frac{\delta - 1}{2X_s^\epsilon} ds + B_t + J_t^\epsilon
\]

(6.1)

where \( J_t^\epsilon \) is \( \epsilon \) times the number of \( \epsilon \)-jump discontinuities in \( X_t^\epsilon \) up to and including time \( t \). (Note that if a jump occurs at \( t \), then we write \( X_t^\epsilon = \epsilon \), so the process is upper semicontinuous.) More generally, a randomly
**Jumping** $\epsilon$-BES$_x^\delta$ process is a process in which the jump sizes are random but the size of the jumps are almost surely less than $\epsilon$, and the jump sizes are adapted to the filtration generated by $B_t$. We denote the size of the $i$th jump by $\epsilon_i$ and the time by $t_i$ and write

$$J_\epsilon^t = \sum_{t_i \leq t} \epsilon_i.$$  

For simplicity, we also require that for each $t > 0$ the set $\{i : t_i \leq t\}$ is almost surely finite. In particular, this implies that the set of jump times $t_i$ is almost surely a discrete set.

**Proposition 6.1.** For each $\epsilon > 0$, let $X_\epsilon$ denote any randomly jumping $\epsilon$-BES$_x^\delta$ process. As $\epsilon \to 0$, the $X_\epsilon^t$ converge in law to a BES$_x^\delta$ with respect to the $L^\infty$ metric on a fixed interval $[0, T]$, with $T > 0$. We also have almost surely, as $\epsilon \to 0$,

1. $J_\epsilon^T \to 0$ if $\delta > 1$.
2. $J_\epsilon^T \to l_0^T$ if $\delta = 1$.
3. $J_\epsilon^T \to \infty$ if $0 < \delta < 1$.
4. $J_\epsilon^{t^2} := \sum_{t_i \leq T} \epsilon_i^2 \to 0$ for all $\delta > 0$.

**Proof.** We will deduce all of these results from the existence of a continuous Bessel process $X_t$ of dimension $\delta$, which is a strong solution to (3.2) away from zero, satisfies Brownian scaling, and almost surely hits zero on a set of zero Lebesgue measure (Propositions 3.1 and 3.2).

First, we will construct a coupling of the process $X_\epsilon^t$ with a BES$_x^\delta$ process $X_t$ in such a way that that the two processes agree when certain intervals of time are excised from latter. We will use $X$ use it to construct $X_\epsilon^t$ as follows. First, set $X_\epsilon^0 = X^0$ for $t \in [0, t_1)$, where $t_1$ is the time at which $X_t$ first hits zero. (Note that $t_1 = 0$ and the interval is empty if $x = 0$.) At this point we sample $\epsilon_1$ from the law of $\epsilon_1$ (in the $X^\epsilon$ process) conditioned on our choice of $X_\epsilon^t$ up to time $t_1$. Now we inductively define times $t_i$ and $s_i$ as follows. Let $s_0 = 0$ and let $s_i$ be such that $t_i + s_i$ is equal to the first time $t$ after $t_i + s_{i-1}$ for which $X_t = \epsilon_i$. Then we define $X_\epsilon^t = X_{t+s_i}$ for $t \in [t_i, t_{i+1})$, where $t_{i+1}$ is the first time $t > t_i$ for which $X_{t+s_i} = 0$. Then we choose $\epsilon_{i+1}$ from the law of $\epsilon_{i+1}$ (in the $X^\epsilon$ process) conditioned on our choice of $X_\epsilon^t$ up to time $t_{i+1}$ and continue.

We may think of $X_\epsilon^t$ as being obtained from $X_t$ by “skipping” the intervals of time $[t_i + s_{i-1}, t_i + s_i)$. On each such interval, $X_t = 0$ when $t$ is the
left endpoint, \( X_t = \epsilon_i \) when \( t \) is the right endpoint, and \( 0 \leq X_t < \epsilon_i \) for other times \( t \) in the interval. Hence the skipped time during finite interval \([0, T]\) is a subset of the set of times \( t \) for which \( X_t \leq \epsilon \). The total measure of the latter set tends to zero as \( \epsilon \to 0 \), so it is clear that \( X^\epsilon \) and \( X \) agree on \([0, T]\) up to translation of time by an amount that tends to zero as \( \epsilon \to 0 \). Since \( X \) is almost surely continuous (and hence uniformly continuous on \([0, T']\) for any fixed \( T' > T \)), this implies that in the couplings above \( X^\epsilon \to X \)

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Now \( \mathbf{a} \) follows immediately from the uniform convergence of \( X^\epsilon \) to \( X \) on \([0, T]\) together with convergence of the other terms (besides \( J^\epsilon_t \)) on the right hand side of (6.1) to the corresponding terms in (3.2). We get \( \mathbf{b} \) by a similar argument and the fact (from Proposition 3.3) that the integral on the right hands side of (6.1) tends to \(-\infty \) as \( \epsilon \to 0 \). When \( \delta = 1 \), this integral is equal to zero, so the \( \mathbf{c} \) follows from (6.1) and (3.3).

Next, let \( t_a \) be the first time \( t \) for which \( J^\epsilon_t \geq a \) for some fixed \( a > 0 \) and observe that \( \mathbf{d} \) will follow if we can show that \( t_a \to \infty \) in probability, as \( \epsilon \to 0 \), for each \( a > 0 \). Let \( A \) be a random variable whose law is that of the first time that a BES_\delta \( X \) hits zero when \( x = 1 \). Let \( m \) and \( v \) be the mean and variance of \( \max(A, 1) \). Write \( b_i = \max(t_{i+1} - t_i, \epsilon_i^2) - m\epsilon_i^2 \). Given \( \epsilon_i \), the law of \( b_i \) is that of \( \epsilon^2 \max(A, 1) - m \) and has zero mean and variance \( \epsilon_i^4 v \). Thus

\[
M_t = \sum_{i: t_i \leq t} b_i
\]

is a martingale.

The variance of \( M_{t_a} \) is \( O(\epsilon^2) \). Hence \( \sum_{i: t_i \leq t_a} \max(t_i - t_{i+1}, \epsilon_i^2) - \sum m\epsilon_i^2 \) tends to zero in probability as \( \epsilon \to 0 \), which implies

\[
\sum_{i: t_i \leq t_a} \max(t_i - t_{i+1}, \epsilon_i^2) \to ma
\]

in probability. The left hand side is bounded by \( N_{t_a} \) where \( N_t \) is the Lebesgue measure of the set of times in \([0, t]\) that are at most \( \epsilon \) from a time \( s \) for which \( X_s = 0 \). Since \( N_t \) tends to zero in probability for each fixed \( t \), as \( \epsilon \to 0 \), we must have \( t_a \to \infty \) in probability.

**Proof of Proposition 5.3.** This is immediate from the fact that when \( X_t \) and \( X^\epsilon_t \) are coupled as in the proof above, we have \( J^\epsilon_t \geq \epsilon Z_t \) almost surely for all \( t \).
6.2 Approximations to chordal SLE(\(\kappa; \rho\))

Let \(O_t\) and \(W_t\) be the parameters of an SLE(\(\kappa; \rho\)) processes. We will first consider the case that \(\rho < -2\), so that \(\delta < 1\), and we assume also that \(\delta > 0\). Take \(X_t^\epsilon\) to be a randomly jumping \(\epsilon\) BES\(\delta\). Then we may write

\[
X_t^\epsilon = \int_0^t \frac{\delta - 1}{2X_s^\epsilon} ds + B_t + J_t^\epsilon
\]

\[
Y_t^\epsilon := \frac{2}{\delta - 1}(X_t^\epsilon - B_t) = \int_0^t (X_s^\epsilon)^{-1} ds + \frac{2}{\delta - 1}J_t^\epsilon
\]

By Proposition 6.1 and (3.4), the processes \(X^\epsilon\) and \(Y^\epsilon\) converge in law to \(X\) and \(Y\) as \(\epsilon \to 0\) (with respect to the \(L^\infty\) metric on finite intervals), where \(X\) is a BES\(\delta\) and \(Y_t = \text{P.V.} \int_0^t X_s^{-1} ds\). This implies that the following converge (in law with respect to \(L^\infty\) metric on compact intervals) to \(O_t\) and \(W_t\) as \(\epsilon \to 0\):

\[
O_t^\epsilon := -\frac{2}{\sqrt{\kappa}}Y_t^\epsilon = -\frac{2}{\sqrt{\kappa}} \int_0^t (X_s^\epsilon)^{-1} ds + \frac{-4}{\sqrt{\kappa}(\delta - 1)} J_t^\epsilon
\]

\[
W_t^\epsilon := O_t^\epsilon + \sqrt{\kappa}X_t^\epsilon = \left(-\frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}(\delta - 1)}{2}\right) \int_0^t (X_s^\epsilon)^{-1} ds + \left(-\frac{4}{\sqrt{\kappa}(\delta - 1)} + \sqrt{\kappa}\right) J_t^\epsilon - \sqrt{\kappa}B_t
\]

Equivalently we may write

\[
\begin{pmatrix}
O_t^\epsilon \\
W_t^\epsilon
\end{pmatrix} = \begin{pmatrix}
o_1 & o_2 & o_3 \\
\frac{\sqrt{\kappa}}{\rho + 2} & \frac{\sqrt{\kappa}}{\rho + 2} & 0
\end{pmatrix} \begin{pmatrix}
\int_0^t (X_s^\epsilon)^{-1} ds \\
J_t^\epsilon \\
B_t
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
o_1 & o_2 & o_3 \\
\frac{\sqrt{\kappa}}{\rho + 2} & \frac{\sqrt{\kappa}}{\rho + 2} & 0
\end{pmatrix} = \begin{pmatrix}
\frac{-2}{\sqrt{\kappa}} & \frac{-2\sqrt{\kappa}}{\rho + 2} & 0 \\
\frac{-2\sqrt{\kappa}}{\rho + 2} + \sqrt{\kappa} & -\sqrt{\kappa} & -\sqrt{\kappa}
\end{pmatrix}.
\]

When we substitute \(\delta = 1 + \frac{2(\rho + 2)}{\kappa}\) this becomes

\[
\begin{pmatrix}
\frac{-2}{\sqrt{\kappa}} & \frac{-2\sqrt{\kappa}}{\rho + 2} & 0 \\
\frac{-2\sqrt{\kappa}}{\rho + 2} + \sqrt{\kappa} & -\sqrt{\kappa} & -\sqrt{\kappa}
\end{pmatrix} = \sqrt{\kappa} \begin{pmatrix}
\frac{-2}{\rho} & \frac{-2}{\rho + 2} & 0 \\
\frac{-2\sqrt{\kappa}}{\rho + 2} & -\sqrt{\kappa} & -1
\end{pmatrix}. \tag{6.2}
\]
The values $O_s^\epsilon \leq W_s^\epsilon$ evolve as they would in an ordinary $\text{SLE}(\kappa; \rho)$ process up until the first time $t$ for which $W_t^\epsilon = O_t^\epsilon$. At the $i$th time that this happens, the value $W_t^\epsilon$ jumps by $\hat{\epsilon}_i := w_2 \epsilon_i = \frac{\sqrt{\rho}}{\rho + 2} \epsilon_i$ and $O_t^\epsilon$ jumps by the amount $\hat{\epsilon}_i := o_2 \epsilon_i = -\frac{2}{\rho} \epsilon_i$. The jump in $W_t^\epsilon$ corresponds to a discontinuity in the path $\gamma$ that the Loewner evolution describes. Equivalently, a jump in $W_t^\epsilon$ (when time is parameterized by capacity) corresponds to having $\gamma$ “trace the boundary of $K_t$” until the image of its tip under $g_t$ has moved $\hat{\epsilon}_i$ units to the right; we also have $O_t^\epsilon$ move along the boundary of $K_t$ until its image under $g_t$ has moved $\hat{\epsilon}_i = \frac{2}{\rho} \epsilon_i$ units to the right. (Since $\rho < -2$, we have $\hat{\epsilon}_i < \epsilon_i$.)

The above yields a construction of an $\epsilon$ approximation of an $\text{SLE}(\kappa; \rho)$ process defined for all time in terms of $\text{SLE}(\kappa; \rho)$ processes starting at $W_0 \neq O_0$ and stopped the first time $W_t = O_t$. Observe that the ratio fixed $\hat{\epsilon}_i / \epsilon_i$ in the above construction is canonical since $J_t^\epsilon \to \infty$ as $\epsilon \to 0$ when $\rho < -2$, by Proposition 6.1. If we fix any other ratio for $\hat{\epsilon}_i / \epsilon_i$ and take $\hat{\epsilon}_i \to 0$, then $|W_t^\epsilon|$ will converge in law to $\infty$ instead of to a continuous process $W$.

If $\rho \geq -2$ (so that $\delta \geq 1$), we define $O_t^\delta$ and $W_t^\delta$ as in (6.2) but with $w_2 = 0$ and $o_2 = -\sqrt{\kappa}$. Thus, as before, the values $O_s^\delta \leq W_s^\delta$ evolve as they would in an ordinary $\text{SLE}(\kappa; \rho)$ process up until the first time $t$ for which $W_t^\delta = O_t^\delta$. However, at that point, the value $O_t^\delta$ jumps by $\hat{\epsilon} := \sqrt{\kappa} \epsilon$ and $W_t^\delta$ does not jump. Thus $W_t^\delta$ is almost surely continuous. Because, by Proposition 6.1, $J_t^\epsilon \to 0$ in this setting, the process still converges to $\text{SLE}(\kappa; \rho)$ in the limit:

**Proposition 6.2.** In both the $0 < \delta < 1$ and the $\delta \geq 1$ settings discussed above, the processes $W_t^\epsilon$ converge in law (with respect to the $L^\infty$ metric on any compact interval $[0, T]$) to the driving parameter of $\text{SLE}(\kappa; \rho)$ as $\epsilon \to 0$.

### 6.3 Approximations to radial $\text{SLE}(\kappa; \kappa - 6)$

We can define an approximation to radial $\text{SLE}(\kappa; \kappa - 6)$ the same way we did before in the chordal case in Section 6.2 except that the jumps are along the unit circle instead of the real line. That is, if $\kappa < 4$ (so that $\rho < -2$), then at the $i$th time $W_t^\epsilon = O_t^\epsilon$ the value $\arg W_t^\epsilon$ (instead of the value $W_t^\epsilon$) jumps by $\hat{\epsilon}_i := \mathcal{E} w_2 \epsilon_i$ and $\arg O_t^\epsilon$ jumps by the amount $\hat{\epsilon}_i := \mathcal{E} o_2 \epsilon_i$ where $\mathcal{E} \in \{-1, 1\}$ is 1 if $W_t$ collided with $O_t$ on the clockwise side of $O_t$ and $-1$ if $W_t$ collided with $O_t$ on the counterclockwise side of $O_t$. When $\kappa > 4$ (so that $\rho > -2$), the value $O_t^\epsilon$ jumps by $\hat{\epsilon} := \mathcal{E} \sqrt{\kappa} \epsilon$ units to the left and $W_t^\epsilon$ does not jump.
Proposition 6.3. Let \( W^\epsilon \) and \( O^\epsilon \) be the processes discussed above, where initial values \( a \in \partial \mathbb{D} \) for \( W^\epsilon_0 \) and \( b \in \partial \mathbb{D} \) for \( O^\epsilon_0 \) are given. As \( \epsilon \to 0 \), the \( W^\epsilon_t \) described above converges in law (with respect to the \( L^\infty \) metric on fixed compact intervals of time) to a random process \( W_t \). Moreover if \( \psi \) is a conformal map from \( \mathbb{D} \) to \( \mathbb{H} \), then the image \( K_t \) of the corresponding Loewner evolution \( K_t \) under \( \psi \) is (up to a time change) a growing family of closed sets given by a Loewner evolution whose driving parameter converges in law (with respect to the \( L^\infty \) norm on intervals of the form \([0,T]\)) where \( T \) is any bounded stopping time that satisfies \( T < \overline{T} := \inf \{ t : \psi(0) \in \overline{K}_t \} \) almost surely) to the driving parameter of SLE(\( \kappa; \kappa - 6 \)) with initial values \( W_0 = \psi(a) \) and \( O_0 = \psi(b) \).

Proof. We will prove the latter statement first. Let \( \phi_0 : \mathbb{D} \to \mathbb{H} \) be the conformal map given by \( \phi_0(z) = \frac{2i}{z+1} - i \). This satisfies \( \phi_0(1) = 0 \). Given \( a, b \in \mathbb{R} \), define \( \phi_{a,b}(z) \) by \( \phi_{a,b}(z) = a\phi_0(e^{ib}z) - a\phi_0(e^{ib}) \). We also have \( \phi_{a,b}(1) = 0 \), and \( a \) and \( b \) parameterize the set of conformal maps from \( \mathbb{D} \) to \( \mathbb{H} \) with this property.

Take \( K_s = \phi K_s \). In this proof we will use \( s \) to denote the time of the radial process and define \( t = t(s) \) be the half-plane capacity of \( K_s \subset \mathbb{H} \). (We also write \( s(t) \) for the choice of \( s \) for which \( t = t(s) \).) Denote by \( g_s : \mathbb{D} \setminus K_s \to \mathbb{D} \) the Loewner conformal map at time \( s \) in the disc, and by \( \hat{g}_t : \mathbb{H} \setminus \overline{K}_{t(s)} \to \mathbb{H} \) the conformal Loewner map in the half-plane with the hydrodynamic normalization (i.e., \( \lim_{|z| \to \infty} |g_t(z) - z| = 0 \)).

Write \( \psi_t = \hat{g}_t \phi g_s^{-1} \). Then \( \psi_t \) is a conformal map from \( \mathbb{D} \) to \( \mathbb{H} \). We can extend this map to the boundaries and write \( \tilde{W}^t = \phi_t W^\epsilon_t \), and \( \tilde{O}^t = \psi_t O^\epsilon_t \). Write \( \tilde{f}_t = g_{s(t)}/W_{s(t)} \) and \( \tilde{f}_t = \hat{g}_t - \tilde{W}_t \). These maps are normalized to send the tip of \( K_t \) to 1 and \( 0 \), respectively. Write \( \tilde{\psi}_t = \tilde{f}_t \phi \tilde{f}_t^{-1} \).

This \( \tilde{\psi}_t : \mathbb{D} \to \mathbb{H} \) is a map that evolves, as \( t \) grows, within the two parameter family of conformal maps from \( \mathbb{D} \) to \( \mathbb{H} \) that map 1 to 0. It can be described by the pair \( a = a_t \) and \( b = b_t \) as described above. It is not hard to work out the SDE describing the time evolution of \( a_t \) and \( b_t \) (which is similar to what is done in [12]). For our purposes, it is enough to observe that they vary continuously in \( t \) (which is immediate from the fact that the \( g_t \) vary continuously with \( t \)). Fix positive real constants \( c, \), \( C \), and \( d < \pi \) and let \( T_{c,d} \) be the first time that \( W_t = \hat{O}_t \) after the time \( \inf \{ t : t = C \text{ or } a_t = c \text{ or } |b_t| = d \} \).

Let \( F_t = F_{a_t,b_t}(\theta) = \phi_{a_t,b_t}(e^{i\theta}) \) be the real-valued extension of \( \phi_{a_t,b_t} \) to \( \partial \mathbb{D} \) (parameterized by the real angle parameter \( \theta \)). Clearly, each of the derivatives of \( F_t \) at \( \theta = 0 \) is uniformly bounded up to time \( T_{c,d} \).

By Proposition 3.12, \( W^\epsilon_t \) and \( O^\epsilon_t \) initially evolve (up to a time change)
According to the rule of an SLE(κ) from \(W_0^\epsilon\) to \(O_0^\epsilon\). However, whenever \(\tilde{W}_t\) and \(\tilde{O}_t\) collide there are jumps in both \(\tilde{W}_t\) and \(\tilde{O}_t\) of size \(F_t(\tilde{\epsilon}_i)\) and \(F_t(\hat{\epsilon}_i)\). Because of the uniform bounds on the derivatives of \(F_t\), the sizes of the \(i\)th jumps of \(W_0^\epsilon\) and \(O_0^\epsilon\) are \(\tilde{\epsilon}_i\) and \(\hat{\epsilon}_i\) (for \(\epsilon_i = F_t(\epsilon_i)\)) plus an error which is \(O(\epsilon^2)\). Letting \(\epsilon\) tend to zero, the fact that convergence holds on the interval \([0, T_{c,d,C}]\) follows immediately from Proposition 6.1. The latter proposition statement then follows from the fact that \(a_t = \pi < \pi\) and \(b_t < \infty\) up until time \(T\), and hence given any bounded stopping time \(T\) almost surely less than \(T\), we can choose \(c, C\), and \(d\) large enough so that the probability that \(T > T_{c,D,C}\) is arbitrarily close to zero.

To prove the first statement in the proposition, we first note that the proof above implies the convergence in law of \(W_0^\epsilon\), at least up to some positive stopping time \(T\) for which \(O_T = W_T\) almost surely, to some limiting process (namely, the driving parameter of the \(\phi\) pre-image of chordal SLE(\(\kappa; \kappa - 6\))). By the Markovian property of the pair \((W_t, O_t)\), and the fact that these stopping times are renewal times, this convergence holds for a stopping time whose law is an independent sum of \(k\) stopping times of this form. The first statement then follows by taking \(k \to \infty\) and noting that the probability that such a sum is less than any fixed constant tends to zero in \(k\).

We have now essentially proved Proposition 3.13.

**Proof of Proposition 3.13.** When \(\kappa \neq 4\) and \(\beta = 1\) and \(\mu = 0\), this existence of the process is immediate from Proposition 6.3, and the uniqueness is trivial. When \(\delta \neq 1\), and \(\beta\) is general, similar arguments to those in Section 6.2 can be used to approximate SLE(\(\kappa; \kappa - 6\)) with randomly jumping processes, to define branching analogs of skew SLE(\(\kappa; \kappa - 6\)), and to show that these processes are invariant under Mobius transformations of the domain that fix the starting point. The only difference is that in the \(\epsilon\) approximations, each time \(\tilde{W}_t\) and \(\tilde{O}_t\) collide, instead of always adding the appropriate \(\tilde{\epsilon}_i\) to \(\tilde{W}_t\) and \(\hat{\epsilon}_i\) to \(\tilde{O}_t\), we add these values with probability \(\frac{1-\beta}{2}\) and subtract them with probability \(\frac{1-\beta}{2}\).

When \(\kappa = 4\), \(\beta = 0\), and \(\mu \in \mathbb{R}\), we obtain laws of the driving parameters for chordal SLE(\(\kappa; \kappa - 6\)) as weak limits of the laws of the corresponding processes for \(\kappa \neq 4\) (note the convergence of the corresponding Lévy processes described in Section 3.2). We may thus obtain laws for radial SLE(\(\kappa; \kappa - 6\)) as the corresponding limits of the laws for radial SLE(\(\kappa; \kappa - 6\)) as \(\kappa \to 4\). In all cases, the uniqueness is trivial once existence is shown.

We briefly remark that when \(\kappa \neq 4\) and \(\mu = \beta = 0\), it is natural to modify
the approximation of SLE($\kappa; \kappa - 6$) so that instead of adding some $w_2 \epsilon_i$ to $W_t^\epsilon$ and $o_2 \epsilon_i$ to $O_t^\epsilon$ at a jump time we leave $W_t^\epsilon$ fixed and add $(o_2 - w_2) \epsilon_i$ to $O_t^\epsilon$. This amounts to shifting the whole process by $\pm w_2 \epsilon_i$. The variance of the sum of these random shifts is the expected sum of the squares of the $w_2 \epsilon_i$, and it tends to zero as $\epsilon \to 0$, by Proposition 6.1. It is then natural to rescale and replace $(o_2 - w_2) \epsilon_i$ with $\epsilon_i$ — so that at each jump, $O_t^\epsilon$ jumps by $\pm \epsilon_i$. This definition makes sense when $\kappa = 4$ as well; in this case, the interested reader may check that one can obtain SLE$_\beta^\mu(\kappa; \kappa - 6)$ for general $\mu$ as a limit by replacing $(\pm \epsilon_i)$ with $(\mu \epsilon - \epsilon_i, \mu \epsilon + \epsilon_i)$.

7 Height functions and other lattices

7.1 Height functions: continuity and monotonicity

We now return to the discrete setting of Section 2. That is, we let $G$ be a hexagon graph with a set $F$ of hexagonal faces and a fixed vertex $v_0$ on its boundary and a directed edge $e_0$ of the hexagonal lattice pointing to $v_0$ from outside of $G$. Each coloring $A$ of the faces in $F$ determines an exploration tree which is the union of exploration paths $T_v(A)$ over all vertices $v$ in $G$.

Let $f_0$ be the face in $F$ that is incident to $v_0$. Now, given a subset $A$ of $F$, we define a height function $h_A : F \to \mathbb{Z}$, defined up to additive constant, by writing $h_A(f) - h_A(f_0)$ to be the number of left turns minus the number of right turns taken by the sequence of edges in $T_v(A)$, where $v$ is the minimal vertex on $F$ in $T(A)$ (i.e., the first vertex of $F$ hit by the exploration tree). We refer to the value $\frac{2\pi}{\kappa} h_A(f)$ as the winding number of the face $f$. Its value modulo $2\pi$ determines the angle of the edge of $T(A)$ that points to the $\prec$-minimal vertex of $f$.

In Section 8, we will make a few conjectures involving these height function converges to the multiple of the Gaussian free field when $h_A(f)$. We derive two simple combinatorial properties of the height function here.

Proposition 7.1. If $f_1$ and $f_2$ are faces that share an edge, then $|h_A(f_1) - h_A(f_2)| \leq 6$.

Proof. Let $v$ be the minimal vertex of $f_1$ and $w$ the minimal vertex of $f_2$ in the $\prec$ ordering. The proposition follows by examining the possible positions of $v$ and $w$. \qed

Proposition 7.2. If $h_A(f_0)$ is fixed, then the height function $h_A$ is a decreasing function of $A$, i.e., $A \subset B$ implies $h_A \geq h_B$. 

48
At first glance, the direction of the proposition may be surprising: it says that if we add faces to $A$ (so that we increase the number of left turns at places where $T(A)$ hits faces for the first time) then the height function goes down (i.e., we decrease number of left turns minus right turns in $T(A)$ before the first time it hits a given face). To get the right intuition, consider the case that $G$ is the graph shown in Figure 2.1 and $A = F$; in this case, by turning left each time we hit a hexagon for the first time, we force $T(A)$ to be a single path that hugs the outer boundary and spirals clockwise inward from $v_0$. When $T(A)$ hits a hexagon for the first time, the winding number is at its lowest possible value.

**Proof.** It is enough to prove that $h_A(f) \geq h_B(f)$ when $B \setminus A$ consists of a single face $f'$. Clearly, $T(A)$ and $T(B)$ will have the same minimal vertex $v'$ on $f'$, so we may as well remove the path from $v_0$ to $v'$ and assume without loss of generality that $f' = f_0$, in which case the result is a consequence of Lemma 7.3 (below).

Sometimes it will be useful to compare $h_A$ with the height function $h_A'$ corresponding to a modified exploration tree that begins at a boundary vertex $v'_0 \neq v_0$ instead of at $v_0$. To make this comparison, we may choose the additive constant $h_A(f_0')$ to be $h_A(f_0)$ plus the number of left turns minus the number of right turns in a path from the $e_0$ pointing to $v$ to the $e_0'$ pointing to $v_0$ that doesn’t intersect the interior of $G$ and makes a partial revolution around $G$ in the clockwise direction. We refer to such a change as a partial clockwise rotation of the model. We define counterclockwise rotations analogously.

**Lemma 7.3.** If $A$ is fixed, and we replace $v_0$ with another vertex $v'_0$ on the boundary and define $h_A'$ accordingly via a counterclockwise rotation, then $h_A' \geq h_A$.

**Proof.** We aim to prove $h'_A(f) \geq h_A(f)$ for some given face $f \in F$. To this end, let $C$ be the largest cluster of either white or black faces which contains $f$ in its interior. (For example, if $f$ is the lone black face surrounded by white faces in the center of Figure 2.1, then $C$ is large white cluster that has five hexagons on the boundary of $G$.) Write $\partial C$ for the set of faces of $C$ which are incident to a hexagon in the unbounded component of $H \setminus C$. If $f \notin \partial C$, then let $\bar{C}$ be the component of $F \setminus C$ containing $f$. Let $P$ be the path in $T(A)$ connecting $v_0$ to the first vertex at which $T(A)$ hits $f$. If $v_0$ does not lie on $C$, then let $D$ be the cluster—with color opposite to that of $C$—which borders $C$ and lies in the component of $F \setminus C$ along which $v_0$ lies.
It is not hard to see that after $P$ hits $D$, it will follow the outer contour of $D$ counterclockwise (if $D$ is black) or clockwise (if $D$ is white) until it first hits $C$; thus, the first time $P$ hits $C$ will be at the first boundary vertex of $C$ that lies counterclockwise (if $D$ is black) or clockwise (if $D$ is white) of $v_0$ along the boundary of $G$.

Either way, once $P$ hits that vertex, it traces the complete contour of $C$ once (in a direction depending on the color of $C$)—hitting all but one edge of the contour—before turning to hit a face in $\hat{C}$ (if $f \not\in \partial C$). Since the position at which $P$ first hits $C$ (measured either clockwise or counterclockwise around the boundary of $C$) is monotone in the position of $v_0$ (measured either clockwise or counterclockwise around the boundary of $G$), we lose no generality in assuming that $C = G$, and if $f \in \partial C$, the result easily follows. If $f \not\in \partial C$, then the position at which $P$ first hits $\hat{C}$ (measured either clockwise or counterclockwise around the boundary of $\hat{C}$) is also monotone in the position of $v_0$, so it is now enough to prove the result for $G = \hat{C}$. The lemma follows by induction on the size of $C$.

Proposition 7.2 implies that if we sample $A$ according to any measure that satisfies the FKG inequality (i.e., increasing functions of $A$ are not negatively correlated), then the random height function $h_A$ also satisfies the FKG inequality. In particular, this is the case if we sample $A$ using Bernoulli percolation or a ferromagnetic Ising model.

Figure 7.1: A grid graph with nine vertices and twelve edges for bond percolation (left) and a corresponding three-regular graph (right). Each percolation configuration of the left graph (i.e., a subset of the edges) corresponds to a coloring of the octagons (one for each edge) in the right graph. The squares are given a fixed coloring in the pattern shown.
7.2 Other lattices and bond percolation

The construction of the exploration tree and height functions in the previous sections works for any periodic, three-regular planar, periodic graph. If the graph is not three-regular, we can make it three-regular by replacing each degree $d > 3$ vertex with a $d$-gon, and then coloring all of these $d$-gons white (i.e., the $d$-gons are not allowed to be subsets of $A$, since they were not present in our original graph).

In fact, we can also define loop ensembles and exploration trees corresponding to instances of bond percolation on a periodic lattice $G$. First form a three-regular graph $G'$ whose faces correspond to the edges, vertices, and faces of $G$; then deterministically color those faces corresponding to vertices of $G$ black and those faces corresponding to faces of $G$ white. Figure 7.1 illustrates this construction for a small grid graph.

8 Open problems

The following two questions might be closely related; it is not clear which of the two will be easier to address first. (Recall Section 2.3 for $O(n)$ model definitions.) When presenting this and other conjectures about scaling limits, we will not specify the desired topology of convergence, since part of the problem may be determining which topology is most natural and tractable.

**Problem 8.1.** Are the scaling limits of the $O(n)$ models actually given by CLE$(\kappa)$, where $\kappa$ is as given in Section 2.3? Do the height functions of these models have scaling limits given by a multiple of the Gaussian free field, with some boundary conditions? We conjecture that the answer to the second question is yes whenever $\kappa > 4$ and $\beta = 1$.

Next, suppose that $8/3 < \kappa \leq 4$. Given an instance $\mathcal{L}$ of CLE$(\kappa)$, we may choose an orientation for each loop. We define a function $h_k(z)$ to be the number of loops in the set $\{L_1^z, \ldots, L_k^z\}$ that are oriented counterclockwise around $z$ minus the number that are oriented clockwise.

**Problem 8.2.** What can be said about $h = \lim_{i \to \infty} h_i$? We expect this convergence to hold in the space of distributions, i.e., the limit of $\int_D \phi(z) h_i(z) dz$ should exist almost surely for each smooth function $\phi$ on $D$. When $\kappa = 4$, the random distribution $h$ should be a multiple of the Gaussian free field. What about the other values of $\kappa$? Is there a natural description of these fields that does not involve SLE? Can we make sense of the expectation of
\[ \int_D \phi(z) h(z) dz \] when \( h \) has piecewise constant boundary conditions? Is the set of loops completely determined by the distribution \( h \) almost surely?

In principle, our construction of CLE(\( \kappa \)) in terms of branching SLE(\( \kappa; \rho \)) should allow one to compute multi-point correlation functions for the fields mentioned above, but it is not clear whether this can be done explicitly (or how enlightening the answer will be). Next, one may try to generalize Conjecture 3.11 as follows:

**Problem 8.3.** Fix a domain \( D \) with boundary points \( a \) and \( b \). For what values of \( \kappa, \rho, \mu \), and skew constant \( \beta \) is it the case that SLE(\( \mu, \beta \); \( \kappa; \rho \)) is a continuous path almost surely? When is the law of an SLE(\( \mu, \beta \); \( \kappa; \rho \)) from \( a \) to \( b \) in \( D \) the same (up to parameterization) as the law of its image under an anti-conformal map of \( D \) that maps \( b \) to \( a \) and \( a \) to \( b \)?

As discussed in Section 5, a proof of Conjecture 3.11 would in particular yield a proof that the CLE(\( \kappa \)) loops are almost surely continuous when \( D \) is a Jordan domain and \( 4 < \kappa < 8 \). One might expect that even in more general domains, the CLE(\( \kappa \)) are almost surely continuous for these values of \( \kappa \). Some loops intersect the boundary of \( D \) almost surely, but it may be the case that the “bad” boundary points of \( D \) are rare enough that the loops are unlikely to intersect \( \partial D \) at those boundary points.

**Problem 8.4.** Let \( D \) be an arbitrary simply connected planar domain. Are all of the loops in a CLE(\( \kappa \)) almost surely continuous in this case when \( 4 < \kappa < 8 \)?

If a path \( \gamma \) chosen from SLE(\( \kappa; \kappa - 6 \)) is almost surely continuous, and \( \kappa \leq 4 \), then it is natural to define the trunk of \( \gamma \) by \( \{ \gamma(t) : O_t = W_t \} \). Our intuitive picture of SLE(\( \kappa; \rho \)) is that it consists of the trunk together with a pairwise disjoint collection of loops of the CLE, each of which is rooted at a single point on the trunk. If the skew constant \( \beta \) is 1, then we expect all of the loops to lie to one side of the trunk. Otherwise, we expect there to be loops on both sides of the trunk, where the fraction of loops which lie on one side or the other is determined by \( \beta \).

**Problem 8.5.** For what values of \( \kappa, \rho \), and skew constant \( \beta \) is it the case that the trunk of an SLE(\( \kappa; \kappa - 6 \)) is almost surely continuous? Is the trunk also an SLE(\( \kappa; \rho \)) process? We conjecture that when \( \beta = 0 \), the trunk has the law of an SLE(\( \kappa', \kappa' - 6 \), \( \kappa' - 6 \)) process, where \( \kappa' = 16/\kappa \).
Given a disjoint simple loop ensemble in a planar graph, we define the \textbf{discrete gasket} to be the graph obtained by deleting every vertex (plus incident edges) that is surrounded by a loop. See Figure \ref{fig:discrete_gasket}. All of the vertices in the discrete gasket have degree two or three—hence if we consider a coloring of the faces of the discrete gasket, we can draw an exploration tree corresponding to that coloring. A path in this exploration tree has “faces” on either side of it that correspond to loops of the $O(n)$ model. The trunk of an $\text{SLE}_\beta^\mu(\kappa; \kappa - 6)$, with $\mu = \beta = 0$, has $\text{CLE}(\kappa)$ loops on either side of it. This suggests the following question:

\textbf{Problem 8.6.} \textit{Consider independent Bernoulli percolation with $p = 1/2$ on the faces of a gasket derived from a critical $O(n)$ model (whose scaling limit we expect to be $\text{CLE}(\kappa)$ for some $8/3 < \kappa \leq 4$). What is the scaling limit of the set of cluster boundaries of this percolation? What is the scaling limit of a branch of the exploration tree (say, from one fixed boundary vertex $a$ to another boundary vertex $b$)? We conjecture that its law is the same as that of the trunk of a branching $\text{SLE}_\beta^\mu(\kappa; \kappa - 6)$ process with $\beta = \mu = 0$.}

A modification of the above is that we independently color each face comprised of multiple hexagons black with probability $p$ and independently
color each face made up of exactly one hexagon black with some probability \( p' \).

**Problem 8.7.** For each given value of \( p \), is there a unique \( p' \) for which the scaling limit of the exploration tree of this coloring is given by the trunk of branching SLE\(_\beta\)(\( \kappa; \kappa - 6 \)) with \( \beta = 2p - 1 \)? What can be said about Ising and \( O(n) \) models on the discrete gasket?

The intuition is that the \( p' \) weight of the small hexagons generates the Lévy compensation which was necessary to make SLE\(_\beta\)(\( \kappa; \kappa - 6 \)) well defined when \( \beta \neq 0 \).

The **FK cluster model** corresponding to an expansion of the \( q \)-state Potts models, may be viewed as a random subset of the edges of a planar graph (i.e., a random instance of non-independent bond percolation; see, e.g., \([6]\) for details), where the probability of a set of edges is proportional to \((e^\beta - 1)^b q^c\) where \( \beta \) is some constant, \( b \) is the number of edges, and \( c \) is the number of connected components of the subgraph of the original graph containing those edges (and all of the original vertices). As discussed in section 7.2, each such subset determines a collection of loops, and it is commonly conjectured that for a critical value \( \beta_c \) this set of loops has a non-trivial scaling limit. Following an analogous conjecture for SLE(\( \kappa \)) given in \([6]\), we ask the following:

**Problem 8.8.** When \( 0 < q \leq 4 \) and \( \beta = \beta_c \), is the scaling limit of the set of loops corresponding to the critical FK clusters on a planar lattice given by CLE(\( \kappa \)) where \( q = 2 + 2 \cos(8\pi/\kappa) \), and \( 4 \leq \kappa \leq 8 \)? Is this true in both the case of free boundary conditions (where all subgraphs are allowed) and wired boundary conditions (where all boundary edges are deterministically included in each subgraph—but the outer boundary of the outermost cluster is not counted as a loop)? (The two are equivalent for self-dual graphs like \( \mathbb{Z}^2 \). \([6]\))

Let \( \mathcal{L}_j \) be the set of \( j \)-th nested loops in an instance of CLE(\( \kappa \)) (i.e., the set of loops of the form \( L_j^z \)). In the case of free boundary conditions, we may define a **continuum FK cluster** \( C \) to be the set of points on or surrounded by a loop \( L \in \mathcal{L}_j \), where \( j \) is odd, minus the set of all points surrounded by loops of the form \( L_{j+1}^z \). Each continuum FK cluster is a random closed set. In the case wired boundary conditions, a continuum FK cluster is the set of points on or surrounded by a loop \( L \in \mathcal{L}_j \), where \( j \) is even (where we formally define \( \mathcal{L}_0 \) to consist of the single loop given by the boundary of the domain), minus the set of all points surrounded by loops of the form \( L_{j+1}^z \).
In the discrete setting, one way to sample from the $q$-state Potts model is to first sample a collection of FK clusters according to the model described above and then assign one of the $q$ spins (uniformly at random and independently) to each cluster (assigning all of the vertices in that cluster the corresponding spin). (Free and wired boundary conditions in the FK cluster model corresponding to free and constant-spin boundary conditions in the corresponding Potts model.) We now seek to define a continuum analog of this construction. In the continuum setting, we can also uniformly and independently assign one of the $q$-states to each continuum FK cluster. We then define a **continuum spin cluster** to be a connected component of the set of continuum FK clusters of a given spin (with two continuum FK clusters considered adjacent if their intersection is non-empty).

**Problem 8.9.** We conjecture that the macroscopic same-spin clusters in the $q$-state Potts models for $q \in \{2, 3, 4\}$ have scaling limits given by the continuum spin clusters described above.

Even for non-integer $1 < q \leq 4$, we can define the “outermost spin cluster” in the wired case to be the cluster of FK clusters consisting of those whose spins are the “same as the outermost cluster,” where each cluster is assigned to have the same spin as the outermost cluster with probability $1/q$. In addition to discrete questions like Problem 8.9 we can now ask a purely continuum question.

**Problem 8.10.** In the case of wired boundary conditions, is the law of the outermost continuum spin cluster corresponding to CLE($\kappa$) (for $\kappa \in [4, 6)$, $q \in (1, 4]$) given by the CLE($\kappa'$) gasket for $\kappa' = 16/\kappa$?

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55
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