HARMONIC VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. The theory of harmonic vector fields on Riemannian manifolds is generalised to pseudo-Riemannian manifolds. Harmonic conformal gradient fields on pseudo-Euclidean hyperquadrics are classified up to congruence, as are harmonic Killing fields on pseudo-Riemannian quadrics. A para-Kähler twisted anti-isometry is be used to correlate harmonic vector fields on the quadrics of neutral signature.

1. Introduction

Attempts to apply the variational theory of harmonic maps [6] to vector fields on Riemannian manifolds $(M, g)$, as a criterion for optimality, foundered at an early stage when it was observed that in the compact case vector fields that are harmonic maps are necessarily parallel [9, 12]. Moreover this remains the case if the vector field is only required to be a harmonic section of the tangent bundle [15]. A more interesting theory [8] emerges in the special case where the vector field has constant length and is required to be a harmonic section of the corresponding sphere sub-bundle of $TM$; unfortunately, its applicability is limited, in the compact case, to manifolds of zero Euler characteristic. In each of these situations, the Riemannian metric $h$ on the codomain—$TM$ or the total space of a sphere sub-bundle—is the Sasaki metric [14] or its restriction, which despite being the most natural choice is ultimately responsible for the rigid behaviour of the energy functional. To alleviate this problem, in [1] the Sasaki metric was replaced by the Cheeger-Gromoll metric [4], embedded as $h_{1,1}$ in a “natural” 2-parameter deformation of $h = h_{0,0}$ through generalised Cheeger-Gromoll metrics on $TM$:

$$\mathcal{CG} = \{h_{p,q} : p, q \in \mathbb{R}\}. \quad (1.1)$$

Although the energy functional behaves no less rigidly with respect to the Cheeger-Gromoll metric [1], other members of $\mathcal{CG}$ are more flexible. In [2], a harmonic vector field on $(M, g)$ was defined to be a harmonic section of $TM$ with respect to the metric $g$ on $M$ and some metric $h_{p,q} \in \mathcal{CG}$ on $TM$, and classifications of harmonic vector fields were obtained for conformal gradient fields and Killing fields on non-flat Riemannian space forms. Typically (but not invariably) a harmonic

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vector field picks a unique element of $\mathcal{CG}$, which, unlike the Cheeger-Gromoll and Sasaki metrics, has variable signature: Riemannian close to the zero section, but Lorentzian further out, with a mild degeneracy on the sphere sub-bundle interfacing the two regions. It should be emphasised that the Riemannian metric $g$ on the base manifold, whose horizontal lift is a component of every generalised Cheeger-Gromoll metric, remains unchanged throughout the deformation (1.1), which is entirely vertical.

In view of the pseudo-Riemannian character of many elements of $\mathcal{CG}$, in this paper we seek a generalisation of the theory of harmonic vector fields to pseudo-Riemannian manifolds $(M, g)$, also referred to as semi-Riemannian manifolds [13]. An initial problem is that when $g$ is not Riemannian the Cheeger-Gromoll metric itself develops a codimension-one singularity, and this phenomenon recurs for many other metrics in $\mathcal{CG}$ (Section 2). Thus even when $M$ is compact the energy functional for vector fields is not in general globally defined, so the variational problem is of necessity entirely local. Despite this, and somewhat remarkably, the singularity in the energy functional is completely resolved at the level of the first variation: the Euler-Lagrange equations for harmonic sections with respect to any $h_{p,q} \in \mathcal{CG}$ are globally defined, and coincide (tensorially) with those obtained in the Riemannian case (Section 3). In Section 5 we extend the classification of harmonic conformal gradient fields on Riemannian space forms obtained in [2] to hyperquadrics of pseudo-Euclidean space (Theorem 5.4), and in Section 6 we examine Killing fields on these spaces, in particular obtaining a condition for a preharmonic Killing field to be harmonic (Theorem 6.4). In Section 7 we show that all Killing fields on the 2-dimensional pseudo-Riemannian quadrics are preharmonic, and complete the classification of harmonic Killing fields in this case: up to congruence there is a unique harmonic Killing field on five of the six metrically distinct quadrics, the exception being the Riemannian sphere (Theorem 7.3). An interesting feature is the existence of a harmonic Killing field on the negative definite pseudo-hyperbolic plane, which is anti-isometrically dual to the Riemannian sphere, illustrating that although harmonic vector fields are invariant under isometry they are not invariant under anti-isometry. We investigate this phenomenon further in Section 8 showing that the combination of an anti-isometry with a para-Kähler twist preserves harmonic vector fields (Proposition 8.1). When applied to the quadrics of neutral signature (quotients of the de Sitter and anti-de Sitter planes), this sets up a correspondence between harmonic Killing fields and harmonic conformal gradient fields, unifying results from Sections 5 and 7.

This paper is based on parts of the first author’s PhD thesis [7].
2. Generalised Cheeger-Gromoll metrics on pseudo-Riemannian vector bundles

A pseudo-Riemannian vector bundle is a vector bundle $\pi: E \to M$ equipped with a linear connection $\nabla$ and holonomy-invariant non-degenerate fibre metric $\langle \cdot, \cdot \rangle$; thus:

$$X(\sigma, \tau) = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle,$$

for all $X \in TM$ and all sections $\sigma, \tau \in \Gamma(E)$, but $\langle \cdot, \cdot \rangle$ is not necessarily positive definite. The canonical example is, of course, the tangent bundle of a pseudo-Riemannian manifold equipped with the Levi-Civita connection. Let $K: TE \to E$ be the associated connection map, and let

$$TE = V \oplus H = \ker(d\pi) \oplus \ker(K)$$

denote the splitting into horizontal and vertical distributions. Recall the characteristic property of the connection map:

$$K(d\sigma(X)) = \nabla_X \sigma. \tag{2.1}$$

Now let $g$ be a pseudo-Riemannian metric on $M$. The familiar construction of the Sasaki metric in the Riemannian case generalises naturally, yielding a pseudo-Riemannian metric on $E$, which we continue to refer to as the Sasaki metric. The construction of the generalised Cheeger-Gromoll metrics in the Riemannian case [1] may also be generalised, as follows. Let $E' \subset E$ be the open dense subset:

$$E' = \{ e \in E : \langle e, e \rangle \neq -1 \},$$

and for each $(p, q) \in \mathbb{R}^2$ define a symmetric $(2, 0)$-tensor $h_{p,q}$ on $E'$ as follows:

$$h_{p,q}(A, B) = g(d\pi(A), d\pi(B)) + \omega^p(e) \left( \langle K(A), K(B) \rangle + q \langle K(A), e \rangle \langle e, K(B) \rangle \right), \tag{2.2}$$

for all $A, B \in T_eE'$ and all $e \in E'$, where $\omega: E' \to \mathbb{R}$ is the smooth function:

$$\omega(e) = 1/|1 + \langle e, e \rangle|.$$

If $q = 0$ then $h_{p,q}$ is a pseudo-Riemannian metric on $E'$ with the same signature as the Sasaki metric $h_{0,0}$; otherwise $h_{p,q}$ is of variable signature. More precisely, if $q < 0$ (resp. $q > 0$) then $h_{p,q}$ has the same signature as the Sasaki metric in the region of $E'$ where $\langle e, e \rangle < -1/q$ (resp. $\langle e, e \rangle > -1/q$). Furthermore, for all $q \neq 1$, $h_{p,q}$ degenerates mildly on the sphere bundle:

$$SE(-1/q) = \{ e \in E : \langle e, e \rangle = -1/q \},$$

and if $q < 0$ (resp. $q > 0$) then the index of $h_{p,q}$ increases (resp. decreases) by 1 in the space-like (resp. time-like) region where $\langle e, e \rangle > -1/q$ (resp. $\langle e, e \rangle < -1/q$). The parameters $(p, q)$ are referred to as the metric parameters of the generalised Cheeger-Gromoll metric $h_{p,q}$. If $p \leq 0$ then $h_{p,q}$ extends to $E$, but degenerates drastically to $\pi^*g$ on $SE(-1)$ if $p < 0$. However if $p > 0$ then $h_{p,q}$ becomes singular on $SE(-1)$.
3. Harmonic sections

Let $\sigma$ be a section of $E$, with $\langle \sigma, \sigma \rangle \neq -1$ identically; thus $\sigma^{-1}E' \subset M$ is a non-empty open subset. The local $(p,q)$-energies of $\sigma$ are defined:

$$E_{p,q}(\sigma; U) = \int_U e_{p,q}(\sigma) \text{vol}(g),$$

for all relatively compact open subsets $U \subset \sigma^{-1}E'$, where $e_{p,q}(\sigma): \sigma^{-1}E' \rightarrow \mathbb{R}$ is the $(p,q)$-energy density:

$$e_{p,q}(\sigma) = \frac{1}{2} h_{p,q}(d\sigma, d\sigma).$$

Note that:

$$h_{p,q}(d\sigma, d\sigma) = \sum_i \epsilon_i h_{p,q}(d\sigma(E_i), d\sigma(E_i)),$$

for any $g$-orthonormal local tangent frame $\{E_i\}$ of $M$, where $\epsilon_i = \langle E_i, E_i \rangle = \pm 1$ are the indicator symbols of the frame. It follows from (2.1) and (2.2) that:

$$2e_{p,q}(\sigma) = n + \omega^p(\sigma)(\langle \nabla \sigma, \nabla \sigma \rangle + qg(\nabla F, \nabla F)), \quad (3.1)$$

where $F = \frac{1}{2} \sigma(\sigma)$ and $\nabla F = \text{grad} F$, the pseudo-Riemannian gradient vector field on $M$. We refer to $2F$ as the pseudo-length of $\sigma$.

The vertical/horizontal splitting of $T\sigma$ engenders a decomposition:

$$d\sigma = d^v \sigma + d^h \sigma,$$

and we define the vertical and horizontal $(p,q)$-energy densities by:

$$e_{v,p,q}(\sigma) = \frac{1}{2} h_{v,p,q}(d^v \sigma, d^v \sigma), \quad e_{h,p,q}(\sigma) = \frac{1}{2} h_{h,p,q}(d^h \sigma, d^h \sigma),$$

respectively. Since $V, H$ are $h_{p,q}$-orthogonal:

$$e_{p,q}(\sigma) = e_{v,p,q}(\sigma) + e_{h,p,q}(\sigma),$$

and a brief further inspection of (2.1) and (2.2) reveals that:

$$e_{v,p,q}(\sigma) = \frac{1}{2} \omega^p(\sigma)(\langle \nabla \sigma, \nabla \sigma \rangle + qg(\nabla F, \nabla F)), \quad e_{h,p,q}(\sigma) = n/2.$$

Thus the horizontal $(p,q)$-energy density is globally defined and constant, and:

$$E_{p,q}(\sigma; U) = E_{p,q}^v(\sigma; U) + \frac{n}{2} \text{Vol}(U),$$

where:

$$E_{p,q}^v(\sigma; U) = \int_U e_{v,p,q}(\sigma) \text{vol}(g)$$

is the local vertical $(p,q)$-energy of $\sigma$.

**Definition 3.1.** If $\langle \sigma, \sigma \rangle \neq -1$ identically then $\sigma$ is said to be a $(p,q)$-harmonic section of $E$ if:

$$\frac{d}{dt}_{|t=0} E_{p,q}^v(\sigma_t; U) = 0,$$

for all relatively compact open $U \subset \sigma^{-1}E'$ and all variations $\sigma_t$ supported in $U$, where each $\sigma_t$ is a section of $E$. Note that $\sigma_t(U) \subset E'$ for all sufficiently small $t$. 

The derivation of the Euler-Lagrange equations for this variational problem proceeds in a similar way to the Riemannian case [1], but working in the pseudo-Riemannian environment requires technical vigilance. Given a variation $\sigma_t$ as in Definition 3.1 the variation field $v_t$ is defined, as usual:

$$v_t(x) = \frac{d}{dt}\sigma_t(x).$$

Since $\sigma_t$ is a variation through sections $v_t$ is a lift of $\sigma_t$ into $V$, which may be realised as a section $\rho_t$ by application of the connection map:

$$\rho_t = K \circ v_t.$$

To simplify our main calculation it is convenient to split the first variation into two pieces $V_1$ and $V_2$ as follows:

$$\frac{d}{dt}\bigg|_{t=0} E_{p,q}^\omega(\sigma; U) = \frac{1}{2} \int_U \frac{d}{dt} \bigg|_{t=0} \omega^p(\sigma_t)(\langle \nabla \sigma_t, \nabla \sigma_t \rangle + qg(\nabla F_t, \nabla F_t)) \text{vol}(g)$$
$$+ \frac{1}{2} \int_U \omega^p(\sigma) \frac{d}{dt} \bigg|_{t=0} \langle \nabla \sigma_t, \nabla \sigma_t \rangle + qg(\nabla F_t, \nabla F_t) \text{vol}(g) = V_1 + V_2.$$

We consider each piece in turn, introducing $\alpha = dF \otimes \sigma$, an $E$-valued 1-form on $M$, and denoting $\rho = \rho_0$. The proof of the following result is similar to that of [1]; however note the appearance of an indicator symbol:

$$\epsilon = \frac{1 + 2F}{|1 + 2F|} = \pm 1,$$

the sign of $1 + \langle \sigma, \sigma \rangle$.

**Lemma 3.2.**

1. $\frac{d}{dt}\bigg|_{t=0} \omega^p(\sigma_t) = -2p \epsilon \omega^{p+1}(\sigma)(\sigma, \rho)$.

2. $\frac{d}{dt}\bigg|_{t=0} \langle \nabla \sigma_t, \nabla \sigma_t \rangle = 2\langle \nabla \rho, \nabla \sigma \rangle$.

3. $\frac{d}{dt}\bigg|_{t=0} g(\nabla F_t, \nabla F_t) = 2\langle \alpha, \nabla \rho \rangle + 2\langle \nabla \nabla F \sigma, \rho \rangle$.

**Proposition 3.3.** The pieces of the first variation are:

$$V_1 = -pe \int_U \omega^{p+1}(\sigma)(\sigma, \rho)(\langle \nabla \sigma, \nabla \sigma \rangle + qg(\nabla F, \nabla F)) \text{vol}(g),$$

$$V_2 = \int_U \omega^p(\sigma)(\langle \nabla \sigma + q\alpha, \nabla \rho \rangle + q(\nabla \nabla F \sigma, \rho) \text{vol}(g).$$

We now recall that for a $E$-valued 1-form $\beta$ and smooth function $f: M \to \mathbb{R}$:

$$\nabla^*(f \beta) = f \nabla^* \beta - \beta(\nabla f),$$

where $\nabla^* \beta = -\text{tr} \nabla \beta$, the pseudo-Riemannian codifferential.
Proposition 3.4. The codifferential of $\gamma = \omega^p(\sigma)(\nabla \sigma + q \alpha)$ is:
\[
\nabla^* \gamma = \omega^p(\sigma)(\nabla^* \nabla \sigma + q((\Delta F)\sigma - \nabla \nabla F)\sigma) + 2p \epsilon \omega^{p+1}(\sigma)(\nabla \nabla F\sigma + q(\nabla F, \nabla F)\sigma),
\]
where $\nabla^2 = -\text{tr} \nabla^2$ is the rough Laplacian, and $\Delta F = -\text{div} \nabla F$ is the pseudo-Riemannian Laplace-Beltrami operator.

Proof. Take $\beta = \nabla \sigma + q \alpha$ and $f = \omega^p(\sigma)$ in $(3.2)$. Then:
\[
\nabla f = -2p \epsilon \omega^{p+1}(\sigma) \nabla \nabla F,
\]
hence:
\[
\nabla^* \gamma = \nabla^* (f \beta) = \omega^p(\sigma)(\nabla^* \nabla \sigma + q \nabla^* \alpha) + 2p \epsilon \omega^{p+1}(\sigma)(\nabla \nabla F\sigma + q(\nabla F, \nabla F)\sigma).
\]
Finally note that:
\[
\nabla^* \alpha = (\Delta F)\sigma - \nabla \nabla F\sigma.
\]

We are now in a position to derive the Euler-Lagrange equations for $(p, q)$-harmonic sections.

Theorem 3.5. The section $\sigma \in \Gamma(E)$ is a $(p, q)$-harmonic section if and only if:
\[
\tau_{p, q}(\sigma) = 0,
\]
where:
\[
\tau_{p, q}(\sigma) = T_p(\sigma) - \phi_{p, q}(\sigma)\sigma,
\]
with:
\[
T_p(\sigma) = (1 + 2F)\nabla^* \nabla \sigma + 2p \nabla \nabla F\sigma,
\]
\[
\phi_{p, q}(\sigma) = p(\nabla \nabla \sigma, \nabla \sigma) - pq g(\nabla F, \nabla F) - q(1 + 2F)\Delta F.
\]

Proof. By Proposition 3.3,
\[
V_1 = -p \int_M \epsilon \omega^{p+1}(\sigma) \langle (\nabla \nabla \sigma, \nabla \sigma) + qg(\nabla F, \nabla F)\sigma, \rho \rangle \text{vol}(g),
\]
\[
V_2 = \int_M \langle \nabla^* \gamma + q \omega^p(\sigma) \nabla \nabla F\sigma, \rho \rangle \text{vol}(g),
\]
where we have used integration by parts to rewrite $V_2$ in divergence form. Now by Proposition 3.4 after a cancellation of terms:
\[
V_2 = \int_M \epsilon \omega^{p+1}(\sigma) \langle \epsilon |1 + 2F| (\nabla^* \nabla \sigma + q(\Delta F)\sigma) + 2p(\nabla \nabla F\sigma + q(\nabla F, \nabla F)\sigma), \rho \rangle \text{vol}(g).
\]
Therefore:
\[
V_1 + V_2 = \int_M \epsilon \omega^{p+1}(\sigma) \langle \tau_{p, q}(\sigma), \rho \rangle \text{vol}(g),
\]
noting that:
\[
\epsilon |1 + 2F| = 1 + 2F.
\]
The result now follows from pseudo-Riemannian non-degeneracy in $L^2_{\text{loc}}$: if $\xi$ is a section of the pseudo-Riemannian vector bundle $E \to M$ and $\int_U \langle \xi, \rho \rangle \text{vol}(g) = 0$ for all relatively compact open $U \subset M$ and $\rho \in \Gamma(E)$ with support in $U$, then $\xi = 0$. □
Remarks 3.6.
(1) The Euler-Lagrange equations resolve the singularity in the vertical \((p, q)\)-energy functional: they are valid on all of \(M\), not just on \(\sigma^{-1}E'\).
(2) If \(\langle \sigma, \sigma \rangle \equiv k \neq -1\) then the Euler-Lagrange equations are:
\[(1 + k)\nabla^* \nabla \sigma = p(\nabla \sigma, \nabla \sigma)\sigma.\]
If \(k \neq 0\) and \(p = 1 + 1/k\) then this is the equation for \(\sigma\) to be a harmonic section of the sphere bundle \(S\mathcal{E}(k)\) equipped with the restriction of the Sasaki metric. Thus, for all \(k \neq -1, 0\), \(\sigma\) is a harmonic section of \(S\mathcal{E}(k) \to M\) if and only if \(\sigma\) is \((p, q)\)-harmonic for \(p = 1 + 1/k\) and all \(q\).
(3) If \(\langle \sigma, \sigma \rangle \equiv -1\), i.e. \(\sigma^{-1}E' = \emptyset\), then \(T_p(\sigma) = 0\) and \(\phi_{p, q}(\sigma) = p(\nabla \sigma, \nabla \sigma)\). We therefore say that \(\sigma\) is \((0, q)\)-harmonic for all \(q \in \mathbb{R}\).
(4) If \(\sigma\) is parallel then \(\sigma\) is \((p, q)\)-harmonic for all \((p, q)\).

The following definition generalises that of \([2]\).

Definition 3.7. A section \(\sigma\) of a pseudo-Riemannian vector bundle \(\pi: \mathcal{E} \to M\) is \(p\)-preharmonic if \(T_p(\sigma)\) is pointwise collinear with \(\sigma\), and preharmonic if \(\sigma\) is \(p\)-preharmonic for all \(p\).

Preharmonicity means:
(1) There exists a smooth function \(\nu: M \to \mathbb{R}\) such that \(\nabla^* \nabla \sigma = \nu \sigma\); for example if \(\sigma\) is an eigenfunction of the rough Laplacian.
(2) There exists a smooth function \(\zeta: M \to \mathbb{R}\) such that \(\nabla_{\nabla F} \sigma = \zeta \sigma\).
We refer to \(\zeta\) as the spinmaker of \(\sigma\), as in \([2]\). The following result is a direct generalisation of the Riemannian case used in \([2]\).

Theorem 3.8. Let \(\sigma\) be a preharmonic section of a pseudo-Riemannian vector bundle over a pseudo-Riemannian manifold. Then \(\sigma\) is a \((p, q)\)-harmonic section if and only if:
\[(p + q + 2qF)\Delta F + 2p(1 + qF)\zeta + (1 + 2(1 - p)F)\nu = 0.\]

Proof. This follows from Proposition 3.5 and the Weitzenböck identity:
\[\langle \nabla^* \nabla \sigma, \sigma \rangle = \langle \nabla \sigma, \nabla \sigma \rangle + \Delta F; \quad (3.3)\]
which continues to hold in the pseudo-Riemannian case. \(\square\)

4. Harmonic vector fields and pseudo-Riemannian hyperquadrics

Henceforward we consider the situation \(\mathcal{E} = TM\) where \((M, g)\) is a pseudo-Riemannian manifold, with \(\nabla\) the Levi-Civita connection and \(\langle , \rangle = g\), the pseudometric on \(M\). In this case, sections of \(\mathcal{E}\) are of course vector fields on \(M\).
Definition 4.1. A vector field $\sigma$ on $(M, g)$ is said to be a harmonic vector field if $\sigma$ is a $(p, q)$-harmonic section of the tangent bundle, for some $(p, q)$. In this case $(p, q)$ are said to be metric parameters for $\sigma$.

The metric parameters for a harmonic vector field need not be unique. This was already known in the Riemannian case [2], even for vector fields of non-constant length, and we will exhibit further non-Riemannian examples in Theorem 5.4.

The natural action of the isometry group of $(M, g)$ on vector fields is via the push-forward:

$$(\varphi \sigma)(x) = d\varphi (\sigma(\varphi^{-1}(x))),$$

for all isometries $\varphi$. The vector fields $\sigma$ and $\varphi \sigma$ are then said to be congruent. As in the Riemannian case, harmonic vector fields are determined up to congruence.

Theorem 4.2. Let $\sigma$ be a harmonic vector field on the pseudo-Riemannian manifold $(M, g)$, and let $\varphi$ be an isometry of $(M, g)$. Then $\varphi \sigma$ is also harmonic, with the same metric parameters.

Proof. Pseudo-Riemannian isometries are totally geodesic: $\nabla d\varphi = 0$. It then follows from Theorem 3.5 that:

$$\tau_{p,q}(\varphi \sigma) = \varphi \tau_{p,q}(\sigma).$$

□

Remark 4.3. Although harmonic vector fields are invariant under isometry, in general they are not invariant under simple rescalings $\sigma \mapsto c \sigma$, $c \in \mathbb{R}$; examples of this were already noted in [1]. Consequently, when solving the Euler-Lagrange equations scale factors cannot be neglected.

Recall that a space form is a simply-connected complete pseudo-Riemannian manifold of constant sectional curvature, and two space forms are isometric if and only if they have the same dimension, index and sectional curvature [13, Proposition 8.23]. For computational and geometric purposes we work with hyperquadric models, which in some cases are only locally isometric to the corresponding space form. Let $\mathbb{R}^{n+1}_u$ denote pseudo-Euclidean space of index $u$, with inner product:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_{n+1-u} y_{n+1-u} - \cdots - x_{n+1} y_{n+1},$$

(4.1)

and let $Q: \mathbb{R}^{n+1}_u \rightarrow \mathbb{R}$ be the associated quadratic form $Q(x) = \langle x, x \rangle$.

Definition 4.4. The pseudo-sphere (resp. pseudo-hyperbolic space) of dimension $n$, index $v$ and radius $r$ is the hyperquadric:

$$S^n_v(r) = \{ x \in \mathbb{R}^{n+1}_v : Q(x) = r^2 \} \quad \text{(resp. } H^n_v(r) = \{ x \in \mathbb{R}^{n+1}_{v+1} : Q(x) = -r^2 \}),$$

equipped with the induced metric. The sectional curvature is $1/r^2$ (resp. $-1/r^2$).
Recall that a diffeomorphism $\varphi: (M, g) \to (N, h)$ of pseudo-Riemannian manifolds is an anti-isometry if:

$$h(d\varphi(X), d\varphi(Y)) = -g(X, Y), \quad \text{for all } X, Y \in \Gamma(TM).$$

Note that for two pseudo-Riemannian $n$-manifolds to be anti-isometric the sum of their indices must equal $n$. The pseudo-Euclidean anti-isometry:

$$\varphi: \mathbb{R}^{n+1}_v \to \mathbb{R}^{n+1}_{n-v}; \quad \varphi(x_1, \ldots, x_{n+1}) = (x_{n+1-v}, \ldots, x_{n+1}, x_1, \ldots, x_n - v)$$

carries $S^n_v(r)$ anti-isometrically onto $H^n_{n-v}(r)$ and vice-versa; its restriction is the canonical anti-isometry between these two hyperquadrics. In pseudo-Riemannian geometry anti-isometric spaces are often considered to be identical. However, although anti-isometries are totally geodesic, from the viewpoint of harmonic vector fields they are not so natural. More precisely, if $\sigma$ is a harmonic vector field on $(M, g)$ and $\varphi: (M, g) \to (N, h)$ is an anti-isometry then the push-forward $\varphi_* \sigma$ need not be a harmonic vector field on $(N, h)$. This is essentially because the term $1 + 2F$ in the Euler-Lagrange equations (Theorem 3.5) is not invariant; a concrete example is given in Section 5 (see Example 5.6).

### 5. Harmonic conformal gradient fields

The construction of conformal gradient fields on Riemannian space forms generalises to pseudo-Riemannian hyperquadrics. Let $M = S^n_v(1)$ or $M = H^n_{n-v}(1)$, and let $V$ denote the ambient pseudo-Euclidean space with inner product $\langle \cdot, \cdot \rangle$ as defined in (4.1). Note that the equation of the hyperquadric is $\langle x, x \rangle = \epsilon$ where $\epsilon = \pm 1$ is the curvature. Let $a \in V$ have pseudo-length

$$\mu = \langle a, a \rangle,$$

and let $\alpha: M \to \mathbb{R}$ be the restriction of the covector field metrically dual to $a$:

$$\alpha(x) = \langle x, a \rangle, \quad \text{for all } x \in M.$$

Then the conformal gradient field $\sigma$ on $M$ with pole $a$ is:

$$\sigma = \text{grad} \alpha = \nabla \alpha,$$

where the gradient is, of course, that intrinsic to the hyperquadric. We now record some relevant properties of pseudo-Riemannian conformal gradient fields, computations of which are essentially identical to those given in [2, Section 3].

**Proposition 5.1.** Let $\sigma$ be a conformal gradient field on $M$, with pole $a$. Then for all $X, Y \in TM$:

1. $\sigma(x) = a - \epsilon \alpha(x)x$.
2. $2F = \langle \sigma, \sigma \rangle = \mu - \epsilon \alpha^2$.
3. $\nabla_X \sigma = -\epsilon \alpha X$.
4. $\nabla^2_{X,Y} \sigma = -\epsilon \langle \sigma, X \rangle Y$. 

Remarks 5.2.
(1) By Proposition 5.1(1), if \( \varphi: S^a_n(1) \to H^a_n(1) \) is the canonical anti-isometry, and \( \sigma \) is a conformal gradient field on \( S^a_n(1) \) with pole vector \( a \), then \( \varphi \sigma \) is a conformal gradient field on \( H^a_n(1) \) with pole vector \( \varphi(a) \).
(2) It follows from Proposition 5.1(2) that \( \sigma(x) \) is null if and only if \( \epsilon \mu > 0 \) and \( x = \pm a/\sqrt{|\mu|} \). However then \( \sigma(x) = 0 \) by (1). Therefore \( \sigma \) is either space-like or time-like, although it is not possible to discern which from the signs of \( \mu \) and \( \epsilon \). If \( \epsilon \mu < 0 \) then \( \sigma \) has no zeros.

Proposition 5.3. If \( \sigma \) is a conformal gradient field then \( \sigma \) is preharmonic, with:
\[ \nu = \epsilon, \quad \zeta = \epsilon(\mu - 2F) \]

Proof. We calculate:
\[ \nabla^*\nabla \sigma = -\text{tr} \nabla^2 \sigma = -\sum_i \epsilon_i \nabla^2 E_i \sigma \]
\[ = \sum_i \epsilon_i \langle \sigma, E_i \rangle E_i, \quad \text{by Proposition 5.1(4)} \]
\[ = \epsilon \sigma, \]

hence \( \nu = \epsilon \). Furthermore:
\[ \nabla F = -\epsilon \alpha \sum_i \epsilon_i \langle \sigma, E_i \rangle E_i = -\epsilon \alpha \sigma. \quad (5.1) \]

Therefore by Proposition 5.1(3):
\[ \nabla F \sigma = -\epsilon \alpha \nabla F = \alpha^2 \sigma = \epsilon(\mu - 2F)\sigma, \]

hence \( \zeta = \epsilon(\mu - 2F) \).

Theorem 5.4. Let \( \sigma \) be a conformal gradient field on a pseudo-Riemannian hyperquadric, with pole \( a \). If \( \mu = \langle a, a \rangle \geq 0 \) then \( \sigma \) is \((p,q)\)-harmonic if and only if:
\[ n > 2, \quad \mu = 1/(n - 2), \quad p = n + 1, \quad q = 2 - n. \]

If \( \mu < 0 \) then \( \sigma \) is \((p,q)\)-harmonic if and only if \( \mu = -1 \) and:
\[ p = n + 1, \quad q = \frac{1 + n - n^2}{n}; \]
or:
\[ n > 2, \quad p = 1/(2 - n), \quad q = 0. \]

Proof. Since \( \sigma \) is preharmonic the harmonic equations simplify to those of Proposition 3.8 with \( \nu = \epsilon \) and \( \zeta = \epsilon(\mu - 2F) \). By Proposition 5.1 and (5.1) the Laplacian of \( F \) is:
\[ \Delta F = -\text{div} \nabla F = \epsilon \langle \sigma, \sigma \rangle - n \alpha^2 = 2\epsilon F(1 + n) - \epsilon n\mu. \]

Therefore the harmonic equations reduce to the following polynomial in \( F \):
\[ 0 = (p + q + 2qF)(2(1 + n)F - n\mu) + 2p(1 + qF)(\mu - 2F) + 1 + 2(1 - p)F. \]

This is in fact the same polynomial that appears in the Riemannian case [2, Theorem 3.2], and the analysis proceeds in the same way. \( \square \)
It is interesting to note that Theorem 5.4 does not depend on the curvature of the hyperquadric. However it does depend on the index of the ambient space: if this is strictly positive (resp. negative) definite then necessarily $\mu > 0$ (resp. $\mu < 0$). It should also be noted that although the harmonic conformal gradient fields are metrically unique if $\mu > 0$, if $\mu < 0$ and $n > 2$ there are two sets of metric parameters. However if $n = 2$ the metric parameters are unique, and equal to $(3, -1/2)$ for all quadrics other than the Riemannian 2-sphere.

Finally we note that the harmonic conformal gradient fields are uniquely determined up to congruence by the pseudo-length of the pole vector.

**Theorem 5.5.** The congruence class of a conformal gradient field on a pseudo-Riemannian hyperquadric is determined by $\mu = \langle a, a \rangle$, where $a$ is the pole vector of the field.

**Proof.** Let $\sigma, \tilde{\sigma}$ be conformal gradient fields with pole vectors $a, \tilde{a}$ respectively, such that $\mu = \tilde{\mu}$. There exists an ambient isometry $\Phi \in O^{++}(n + 1, u)$, where $u$ is the index of $\nabla$, such that $\Phi(a) = \tilde{a}$. The function $\tilde{\alpha}$ is:

$$\tilde{\alpha}(x) = \langle \tilde{a}, x \rangle = \langle \Phi(a), x \rangle = \langle a, \Phi^{-1}(x) \rangle;$$

thus:

$$\tilde{\alpha} = \alpha \circ \Phi^{-1}.$$

For all $X \in T_x M$:

$$\langle \nabla \tilde{\alpha}, X \rangle = d\tilde{\alpha}(X) = d\alpha(d\Phi^{-1}(X)) = \langle \nabla \alpha, d\Phi^{-1}(X) \rangle = \langle \nabla \alpha, \Phi^{-1}(X) \rangle = \langle \Phi(\nabla \alpha), X \rangle = \langle d\Phi(\nabla \alpha), X \rangle,$$

where $\nabla \alpha$ is evaluated at $\Phi^{-1}(x)$. Therefore:

$$\tilde{\sigma}(x) = \nabla \tilde{\alpha}(x) = d\Phi(\nabla \alpha(\Phi^{-1}(x))) = d\Phi \circ \sigma \circ \Phi^{-1}(x).$$

Hence $\tilde{\sigma} = \varphi \sigma$ where $\varphi = \Phi|_M$. $\square$

**Example 5.6.** Consider $M = H_2^3$, the negative definite 2-sphere. Then the conformal gradient field with pole vector $(0, 0, 1)$ is $(3, -1/2)$-harmonic. This vector field has two zeros, at $\pm(0, 0, 1)$, and up to congruence it is the unique harmonic conformal gradient field. This is a contrast to the positive definite 2-sphere $S_0^2$, which has no harmonic conformal gradient fields. Furthermore $H_2^3$ and $S_0^2$ are anti-isometric, the canonical anti-isometry being the identity map, and the push-forward of $\sigma$ to $S_0^2$ is also a conformal gradient field (Remarks 5.2), illustrating that the Euler-Lagrange equations for harmonic vector fields are not invariant under anti-isometry.
6. Preharmonic Killing fields on pseudo-Riemannian hyperquadrics

Now let \( \sigma \) be a Killing field on a pseudo-Riemannian hyperquadric \( M \) of curvature \( \epsilon = \pm 1 \). Then \( \sigma \) is the restriction to \( M \) of a unique skew-symmetric linear transformation \( A: V \to V \). Thus if \( A \) has matrix \((a_{ij})\) with respect to an orthonormal frame of \( V \) then:

\[
a_{ij} = -\epsilon_i \epsilon_j a_{ji},
\]

(6.1)

where the \( \epsilon_i \) are the indicator symbols of the frame. It follows from the pseudo-Riemannian Gauss formula \([13]\) that for all \( X \in T_x M \) and all \( x \in M \):

\[
\nabla_X \sigma = A(X) - \epsilon \langle A(X), x \rangle x,
\]

(6.2)

where the point \( x \) is regarded as a unit normal. Note that since \( A \) is skew-symmetric so is \( A^3 \), whose restriction therefore defines a vector field \( \hat{\sigma} \) on \( M \).

Lemma 6.1. If \( \sigma \) is a Killing field on a pseudo-Riemannian hyperquadric of curvature \( \epsilon \) then:

\[
\nabla_{\nabla_F} \sigma = -\hat{\sigma} - 2\epsilon F \sigma.
\]

Proof. Since \( 2F = \langle \sigma, \sigma \rangle \) we have:

\[
\nabla F = \sum_i \epsilon_i \langle \nabla E_i, \sigma \rangle E_i = \sum_i \epsilon_i \langle A(E_i), A(x) \rangle E_i,
\]

by (6.2)

\[
= -\sum_i \epsilon_i \langle E_i, A^2(x) \rangle E_i = -A^2(x) - \epsilon \langle \sigma, \sigma \rangle x.
\]

Therefore by (6.2) again:

\[
\nabla_{\nabla_F} \sigma = A(\nabla F) - \epsilon \langle A(\nabla F), x \rangle x = -A^3(x) - 2\epsilon F \sigma,
\]

since \( \langle A^3(x), x \rangle = 0 \). \( \square \)

Proposition 6.2. A Killing field \( \sigma \) on a pseudo-Riemannian hyperquadric of curvature \( \epsilon \) is preharmonic if and only if \( A^3 = \lambda A \) for some \( \lambda \in \mathbb{R} \), in which case the spinnaker is:

\[
\zeta = -(\lambda + 2\epsilon F).
\]

Proof. Since \( \sigma \) is a Killing field \([15]\):

\[
\nabla^* \nabla \sigma = \text{Ric}(\sigma) = \epsilon(n - 1)\sigma.
\]

(6.3)

Therefore (Definition \([37]\) \( \sigma \) is preharmonic if and only if for all \( x \in M \):

\[
A^3(x) - \lambda(x) A(x) = 0,
\]

for some smooth function \( \lambda: M \to \mathbb{R} \), by Lemma \([6.1]\). Differentiating this equation yields:

\[
A^3(X) - \lambda(x) A(X) = d\lambda(X) A(x),
\]

for all \( X \in T_x M \). Since \( x \) may be regarded as a unit normal to \( M \), it follows from these two equations that for each \( x \in M \) the linear map \( A^3 - \lambda(x) A \) has rank at most one. However non-trivial skew-symmetric transformations of pseudo-Euclidean space have rank at least two by \([6.1]\). Therefore \( A^3 - \lambda(x) A = 0 \), hence \( d\lambda(X) A(x) = 0 \). Since \( \ker(A) \) is a subspace of codimension at least 2, whose
intersection with $M$ is a submanifold of dimension at most $n - 1$, it follows that $d\lambda = 0$ on an open dense subset and hence by continuity everywhere. Since $M$ is connected, $\lambda$ is constant. The expression for $\zeta$ then follows from Lemma 6.1. □

Finally, we calculate the Laplacian of the pseudo-length of a Killing field.

**Lemma 6.3.** If $\sigma$ is a Killing field on a hyperquadric $M$ of curvature $\epsilon$ then:  
$$\Delta F = 2\epsilon(n + 1)F - \langle A, A \rangle,$$
where $\langle A, A \rangle$ is the ambient pseudo-Euclidean inner product.

**Proof.** From the Weitzenböck formula (3.3) and (6.3):

\begin{align*}
\Delta F &= \langle \nabla^* \nabla \sigma, \sigma \rangle - \langle \nabla \sigma, \nabla \sigma \rangle \\
&= 2\epsilon(n - 1)F - \langle \nabla \sigma, \nabla \sigma \rangle.
\end{align*}

By (6.2):

\begin{align*}
\langle \nabla \sigma, \nabla \sigma \rangle &= \sum_i \epsilon_i \langle \nabla_{E_i} \sigma, \nabla_{E_i} \sigma \rangle \\
&= \sum_i \epsilon_i (\langle A(E_i), A(E_i) \rangle - \epsilon \langle A(E_i), x \rangle^2) \\
&= \langle A, A \rangle - \epsilon \langle A(x), A(x) \rangle - \epsilon \sum_i \epsilon_i \langle \sigma(x), E_i \rangle^2 \\
&= \langle A, A \rangle - 2\epsilon \langle \sigma, \sigma \rangle. \quad \Box
\end{align*}

Combining Proposition 6.2 and Lemma 6.3 with Proposition 3.8 yields the following criterion for a preharmonic Killing field to be harmonic.

**Theorem 6.4.** Let $\sigma$ be a preharmonic Killing field on pseudo-Riemannian hyperquadric of curvature $\epsilon$. Then $\sigma$ is $(p, q)$-harmonic if and only if:

\[
0 = \epsilon(n + 1 - p)q(2F)^2 + (\epsilon(n - 1 + (n + 1)q) - pq\lambda - q\langle A, A \rangle)(2F) + \epsilon(n - 1 - 2p\lambda - (p + q)\langle A, A \rangle).
\]

We will see that in the 2-dimensional case all Killing fields are preharmonic.

7. Harmonic Killing fields on pseudo-Riemannian quadrics

We recall that there are six pseudo-Riemannian quadrics, namely:

- The Riemannian 2-sphere and its anti-isometric counterpart, which are spheres in $\mathbb{R}^3_0$ and $\mathbb{R}^3_3$ respectively.
- The hyperbolic plane and its anti-isometric counterpart, which are connected components of hyperboloids of two sheets in $\mathbb{R}^3_1$ and $\mathbb{R}^3_2$ respectively.
- The neutral quadrics, $S^2_1$ and $H^2_1$, which are hyperboloids of one sheet in $\mathbb{R}^3_1$ and $\mathbb{R}^3_2$ respectively.

Note that the quadrics of index 0 and 2 are in fact space forms, whereas the neutral quadrics are not.
Proposition 7.1. Let $\sigma$ be a Killing field on a pseudo-Riemannian quadric of curvature $\epsilon$, whose representing matrix $A$ with respect to an orthonormal frame of $\mathbb{R}^3_u$ is:

$$A = \begin{pmatrix} 0 & a & b \\ -\epsilon_1 \epsilon_2 a & 0 & c \\ -\epsilon_1 \epsilon_3 b & -\epsilon_2 \epsilon_3 c & 0 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$ and $\epsilon_1, \epsilon_2, \epsilon_3$ are the indicator symbols of the frame. Then $\sigma$ is preharmonic, and:

$$\lambda = -\epsilon_1 \epsilon_2 a^2 - \epsilon_1 \epsilon_3 b^2 - \epsilon_2 \epsilon_3 c^2.$$

Proof. By Proposition 6.2 it suffices to calculate $A^3$ and compare it with $A$. □

Theorem 7.2. Let $\sigma$ be a Killing field on a pseudo-Riemannian quadric of curvature $\epsilon$. Then $\sigma$ is $(p, q)$-harmonic if and only if:

$$p = 3, \quad q = -1/2, \quad \lambda = \epsilon.$$

Proof. Consider first:

$$\langle A, A \rangle = \sum_i \epsilon_i \langle A(e_i), A(e_i) \rangle = \sum_{i,j} \epsilon_i \epsilon_j a_{ij}^2 = -2\lambda,$$

by Proposition 7.1. Therefore, since $\sigma$ is preharmonic, by Theorem 6.4 $\sigma$ is $(p, q)$-harmonic if and only if:

$$0 = \epsilon(3-p)q(2F)^2 + (\epsilon(3q+1) + (2-p)q\lambda)(2F) + 2q\lambda + \epsilon.$$

The leading coefficient of this polynomial in $F$ vanishes if and only if $p = 3$ or $q = 0$; however if $q = 0$ the linear term cannot vanish. When $p = 3$ the remaining equations reduce to:

$$\epsilon(3q+1) - q\lambda = 0 = \epsilon + 2q\lambda,$$

which yield the stated values of $q$ and $\lambda$. □

We note that for the Riemannian 2-sphere $\lambda < 0$, so Theorem 7.2 precludes the existence of harmonic Killing fields, as already observed in [2]. Comparison of Proposition 7.1 and Theorem 7.2 shows that harmonic Killing fields on each of the remaining pseudo-Riemannian quadrics form a 2-dimensional sub-manifold of the 3-dimensional Lie algebra of all Killing fields. However we will show that this sub-manifold is actually a single congruence class; thus up to congruence there is a unique harmonic Killing field. In fact we will show that the congruence class of a Killing field on a pseudo-Riemannian quadric is determined by $\lambda$. This was already observed for $S^2_0$ and $H^2_0$ in [2], from which it may be deduced also for $H^2_2$ and $S^2_2$, since the space of Killing fields and its congruence structure is preserved by the canonical anti-isometry, leaving only the neutral quadrics. It suffices to consider $H^2_1$, and we first establish the qualitative behaviour of Killing fields in this case.
Proposition 7.3. The fixed points of a Killing field on $H^2$ are categorised by $\lambda = a^2 + b^2 - c^2$. The Killing field has:

1. no fixed points if $\lambda < 0$;
2. two ideal fixed points, one on each component of the boundary at infinity, if $\lambda = 0$;
3. two fixed points if $\lambda > 0$.

Proof. Let $C \subset \mathbb{R}^3$ be the cylinder:

$$C = \{(x, y, z) : -1 < x < 1, \ y^2 + z^2 = 1\},$$

and project $H^2$ onto $C$ along rays through the origin. This gives a map:

$$\psi : H^2 \rightarrow C; \ \psi(x, y, z) = \frac{1}{\sqrt{1 + x^2}}(x, y, z),$$

with differential:

$$d\psi(x, y, z)(u, v, w) = \frac{1}{(1 + x^2)^{3/2}}(u, -xyu + v(1 + x^2), -xzu - w(1 + x^2)).$$

The inverse map is:

$$\psi^{-1}(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{\sqrt{1 - \bar{x}^2}}(\bar{x}, \bar{y}, \bar{z}) = (x, y, z).$$

The components of $\sigma(x)$ are:

$$u = ay + bz, \ \ v = ax + cz, \ \ w = bx - cy.$$ 

Therefore the projection $\tilde{\sigma}$ of $\sigma$ to the cylinder is:

$$\tilde{\sigma} = d\psi(x, y, z)(u, v, w) = ((a\bar{y} + b\bar{z})(1 - \bar{x}^2), -\bar{x}y(a\bar{y} + b\bar{z}) + (a\bar{x} + c\bar{z}), -\bar{x}z(a\bar{y} + b\bar{z}) - (b\bar{x} - c\bar{y})).$$

Note that $\tilde{\sigma}$ extends smoothly across $\partial C$; ie. when $\bar{x} = \pm 1$. Then $\tilde{\sigma}(\bar{x}, \bar{y}, \bar{z}) = 0$ for $(\bar{x}, \bar{y}, \bar{z}) \in C \cup \partial C$ if and only if:

$$0 = (a\bar{y} + b\bar{z})(1 - \bar{x}^2),$$

$$0 = -\bar{x}y(a\bar{y} + b\bar{z}) + (a\bar{x} + c\bar{z}),$$

$$0 = -\bar{x}z(a\bar{y} + b\bar{z}) - (b\bar{x} - c\bar{y}).$$

If $\lambda = 0$ then there are two solutions, namely $\pm(1, b/c, -a/c) \in \partial C$, one on each component. If $\lambda > 0$ then there are two solutions:

$$\pm\left(\frac{c}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, \frac{-a}{\sqrt{a^2 + b^2}}\right) \in C,$$

which correspond to:

$$\pm\left(\frac{c}{\sqrt{a^2 + b^2 - c^2}}, \frac{b}{\sqrt{a^2 + b^2 - c^2}}, \frac{-a}{\sqrt{a^2 + b^2 - c^2}}\right) \in H^2.$$ 

Finally if $\lambda < 0$ then there are no solutions. \qed
Proposition 7.4. Let $\sigma$ be a Killing field on $H^2_1$. If $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, respectively, then $\sigma$ is congruent to the Killing field with matrix, respectively:

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \sqrt{-\lambda} \\
0 & -\sqrt{-\lambda} & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & \sqrt{\lambda} & 0 \\
\sqrt{\lambda} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Proof. We prove the case $\lambda < 0$; the other cases are similar. Consider the infinitesimal isometry of $H^2_1$ with matrix:

\[
\begin{pmatrix}
0 & \alpha & \beta \\
\alpha & 0 & 0 \\
\beta & 0 & 0
\end{pmatrix},
\]

where $\alpha = b/\sqrt{a^2 + b^2}$ and $\beta = -a/\sqrt{a^2 + b^2}$. The flow is the restriction to $H^2_1$ of the following linear flow on $\mathbb{R}^3_2$:

\[
\Phi_t = \begin{pmatrix}
\cosh t & \beta \sinh t & \alpha \sinh t \\
\beta \sinh t & \alpha^2 - \beta^2 \cosh t & -\alpha \beta (1 - \cosh t) \\
\alpha \sinh t & -\alpha \beta (1 - \cosh t) & \beta^2 + \alpha^2 \cosh t
\end{pmatrix}.
\]

If $\cosh(t_0) = -c/\sqrt{-\lambda}$ then:

\[
\Phi_{-t_0} A \Phi_{t_0} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \sqrt{-\lambda} \\
0 & -\sqrt{-\lambda} & 0
\end{pmatrix},
\]

which yields the desired congruence. \qed

Theorem 7.5. Let $M$ be a pseudo-Riemannian quadric of curvature $\epsilon = \pm 1$, other than the Riemannian 2-sphere. Then up to congruence there exists a unique harmonic Killing field $\sigma$ on $M$, which is the restriction of one of the following matrices:

\[
\begin{pmatrix}
0 & 1 & 0 \\
\epsilon & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & \epsilon & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

according as $M = S^2_2$ or $H^2_0$, $M = S^2_1$ or $H^2_1$ or $M = H^2_0$, respectively. In all cases the metric parameters of $\sigma$ are $(3, -1/2)$.

8. Para-Kähler twisted anti-isometries

We recall [5] that an almost para-Hermitian structure on a pseudo-Riemannian manifold $(M, g)$ is a skew-symmetric $(1, 1)$ tensor field $J$ satisfying $J^2 = 1$. The existence of such a structure forces $(M, g)$ to be of even dimension and neutral signature. If in addition $\nabla J = 0$ then $J$ is para-Kähler. Because almost para-Hermitian structures are anti-isometric:

$$g(JX, JY) = -g(X, Y),$$
a para-Kähler twisted harmonic vector field need not be harmonic; see Example 8.3 below. However combining an anti-isometry \( \varphi \) with a para-Kähler twist \( J \) rectifies this problem, for both \( \varphi \) and \( J \).

**Proposition 8.1.** Let \((M, g, J)\) be a para-Kähler manifold and \( \varphi : (M, g) \to (N, h) \) an anti-isometry. If \( \sigma \) is a harmonic vector field on \((M, g)\) then \( \varphi \cdot (J\sigma) \) is a harmonic vector field on \((N, h)\), with the same metric parameters.

**Proof.** Abbreviating \( \tilde{\sigma} = \varphi \cdot (J\sigma) \), we have:

\[
h(\tilde{\sigma}, \tilde{\sigma}) = h(\varphi \cdot (J\sigma), \varphi \cdot (J\sigma)) = \varphi \cdot (J\sigma) = g(\sigma, \sigma).
\]

Thus \( \tilde{F} = F \). All remaining terms in the equations of Theorem 3.5 are invariant, since \( d\varphi \) and \( J \) are parallel, and we conclude that:

\[
\tau_{p,q}(\tilde{\sigma}) = \varphi \cdot (J\tau_{p,q}(\sigma)).
\]

\[\square\]

**Remark 8.2.** A similar result holds if \( \varphi \) is an anti-isometry into a para-Kähler manifold: if \( \sigma \) is a harmonic vector field on the domain then \( J\varphi \cdot \sigma \) is a harmonic vector field on the codomain.

We recall also that a vector field \( \sigma \) on \((M, g)\) is said to be closed conformal if \( \sigma \) is conformal and its metrically dual 1-form is closed [3]. By [10, 11] closed conformal vector fields are characterised by the following generalisation of Proposition 5.1 (4):

\[
\nabla_X \sigma = \psi \sigma,
\] (8.1)

for some smooth function \( \psi : M \to \mathbb{R} \), where necessarily \( n\psi = \text{div} \sigma \).

**Proposition 8.3.** Let \( \sigma \) be a closed conformal vector field on a para-Kähler manifold \((M, g, J)\). Then \( J\sigma \) is a Killing field.

**Proof.** Since \( J \) is para-Kähler:

\[
g(\nabla_X (J\sigma), Y) + g(X, \nabla_Y (J\sigma)) = g(\nabla_X \sigma, Y) + g(X, J \nabla_Y \sigma)
\]

\[
= g(J(\psi X), Y) + g(X, J(\psi Y)), \text{ by (8.1)}
\]

\[
= -\psi g(X, JY) + \psi g(X, JY) = 0.
\]

Hence \( J\sigma \) is Killing. \[\square\]

Every oriented 2-dimensional pseudo-Riemannian manifold of neutral signature admits a unique para-Kähler structure that is compatible with the orientation in the following sense. The null vectors \( L \subset TM \) may be written \( L = L_1 \cup L_2 \) where \( L_1, L_2 \subset TM \) are distinct line sub-bundles, labelled such that if \((A, B)\) is a positively oriented local tangent frame with \( A \in L_1 \) and \( B \in L_2 \) then \( A + B \) is space-like (which implies \( A - B \) is time-like). Then define:

\[
J A = A, \quad J B = -B.
\]
It is easily checked that $J$ is para-Kähler. In particular, if $M$ is a neutral quadric then it follows from Proposition 8.3 that $\sigma \mapsto J\sigma$ yields a linear involutive isomorphism between the Killing and conformal gradient fields on $M$, since both spaces have the same dimension (namely, 3). Hence by Proposition 8.1 if $\varphi$ is the canonical anti-isometry from $H^2_1$ to $S^2_1$ then $\sigma \mapsto \varphi(J\sigma)$ yields a bijection between the unique congruence class of harmonic conformal gradient fields (resp. Killing fields) on $H^2_1$ and the congruence class of harmonic Killing fields (resp. conformal gradient fields) on $S^2_1$. These classes are also bijectively equivalent via the correspondence of Remark 8.2, using the para-Kähler structure of $S^2_1$. However since $\varphi$ is paraholomorphic the two bijections are in fact the same.

**Example 8.4.** As an explicit example, let $\sigma$ be the conformal gradient field on $H^2_1$ with pole vector $(0, 0, 1)$, which is harmonic by Theorem 5.4. Then:

$$(J\sigma)(x, y, z) = (y, x, 0),$$

which although Killing (Proposition 7.4), with the same zeros $(0, 0, \pm 1)$ as $\sigma$, is not harmonic (Theorem 7.5): indeed, the harmonic Killing fields on $H^2_1$ have no fixed points. However the push-forward of $J\sigma$ to $S^2_1$ under the anti-isometry:

$$\varphi(x, y, z) = (z, x, y)$$

is the vector field:

$$(x, y, z) \mapsto d\varphi((J\sigma)(y, z, x)) = d\varphi(z, y, 0) = (0, z, y),$$

which by Theorem 7.5 is harmonic, with zeros $(\pm 1, 0, 0)$.

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