Fast Parallel Frequency Sweeping Algorithms for
Robust $\mathcal{D}$-Stability Margin *

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Abstract

This paper considers the robust $\mathcal{D}$-stability margin problem under polynomial structured real parametric uncertainty. Based on the work of De Gaston and Safonov (1988), we have developed techniques such as, a parallel frequency sweeping strategy, different domain splitting schemes, which significantly reduce the computational complexity and guarantee the convergence.

Keywords: Robust control, Robust $\mathcal{D}$-stability, stability margin, frequency sweeping.

1 Introduction

Robustness of control systems has been one of the central issues in the control community in the last two decades. Most of the research efforts have been devoted to the $\mu$ framework[1, 2, 7, 12, 22] and the Kharitonov framework[3, 11, 19]. One of the well studied robustness analysis problem is the computation of robust stability margin under polynomial structured real parametric uncertainty. A number of different approaches have been proposed in the Kharitonov framework aimed at the nonconservative computation of the robust stability margin. Among these, we recall the geometric programming methods [20], the algorithm based on the Routh table [18], and the domain splitting approach [9, 15, 16] based on the Zero Exclusion Condition[8]. In general, the algorithms in [18, 20] is more efficient than the algorithm in [9]. The main reason is that the algorithms in [18, 20] are essentially based on the Routh-Hurwitz criterion and thus only finite conditions need to be evaluated,

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while the algorithm in [9] is based on the Zero Exclusion Condition and thus a frequency sweeping is essential.

Even though a frequency sweeping is a necessity, an algorithm based on the Zero Exclusion Condition has its particular advantage when dealing with robust $D$-stability problems. For example [3], for high order control systems, a typical specification might be as follows: The closed-loop polynomials should have a pair of “dominant roots” in disks of given radius $\epsilon > 0$ centered at $z_{1,2} = -u \pm jv$, and all remaining roots having real parts less than $-\sigma$ with $\sigma > 0$ (See Figure 1 where $z_1 \in D_1$, $z_2 \in D_2$, $D = D_1 \cup D_2 \cup D_3$). Then, a robust $D$-stability margin problem can be defined as follows: What is the maximum perturbation of plant parameters such that the roots of the closed loop polynomial remain robustly in $D = \{z \in \mathbb{C} : |z - z_1| < \epsilon\} \cup \{z \in \mathbb{C} : |z - z_2| < \epsilon\} \cup \{z \in \mathbb{C} : \Re(z) < \sigma\}$? Since the root region $D$ can be defined as a union of disjoint open subsets with complicated boundary in the complex plane, the robust $D$-stability problems in general cannot be solved by existing results in the $\mu$ framework or the algorithms in [18, 20] which are based on the Routh-Hurwitz criterion. For special cases that $D$ is simply connected and is defined via the Nyquist curve of certain rational polynomials $f(s) = \frac{p(s)}{h(s)}$, the robust $D$-stability problem of $p(s)$ may be reduced to the robust stability problem of polynomial $\hat{p}(s) = p(f(s)).(h(s))^{n_h}$ where $n_h$ is the degree of polynomial $h(s)$ and then the algorithms in [18, 20] may be applied. However, the complexity is increased substantially because the coefficients of $\hat{p}(s)$ may be very complex and the degree of $\hat{p}(s)$ is $n_g$ times of the degree of $p(s)$ where $n_g$ is the degree of polynomial $g(s)$ [17].

![Figure 1: Robust $D$-Stability](image)

The advantage of an algorithm based on the Zero Exclusion Condition is that it can be applied to the robust $D$-stability problem with arbitrary complicated root region $D$. What we only need to do is to verify whether the Zero Exclusion Condition is satisfied for all boundary point of $D$. In situations where the root region $D$ is complicated, applying such algorithms becomes essential. However, the computational complexity can be very high. In particular, the growth of computations is exponential with the number of parameters. It has been shown that these type of problems are
in general NP-hard (see [13] and the reference therein). Moreover, since the frequency search can be discontinuous (as shown in [4]), we usually need to evaluate the Zero Exclusion Condition for many boundary points of $D$ to come up with a reasonably accurate solution. Therefore, there is strong motivation to develop efficient algorithms based on the Zero Exclusion Condition to tackle the robust $D$-stability problems.

The algorithm proposed by De Gaston and Safonov [9] is based on the Zero Exclusion Condition and thus can be applied to the general robust $D$-stability problem. However, there exist two problems.

First of all, it is noted that the convergence of the algorithm in [9] was concluded upon an impractical assumption. That is, a domain can be divided fine enough to converge to a point (see [9] line 40 – 53 of page 156 in the proof of the Convergence Theorem). However, to satisfy this assumption, the computation complexity may be unacceptably high. In this paper, we have shown that it is sufficient to guarantee the convergence in computing the stability margin by guaranteeing that the distance between critical vertices converge to 0. Therefore, it is not necessary to divide a subdomain so many times to collapse it to a point. In contrast, what we need is to make the critical vertices crunch together. Thus, the computation can be reduced greatly. We provide two splitting schemes which guarantee this.

Another problem with the algorithm in [9] is its inefficiency. One main hurdle is its tedious frequency sweeping. Consider a family of uncertain polynomials $p(s, q)$, $q \in \mathcal{Q}$ where $\mathcal{Q}$ is the set of uncertain parameters. Let $k_m(\omega, \mathcal{Q}) = \sup \{k : 0 \notin p(j\omega, k\mathcal{Q})\}$ where $p(j\omega, k\mathcal{Q})$ is the value set associated with frequency $\omega$ and perturbation bound $k$. The algorithms in [9] compute $k_m(\omega, \mathcal{Q})$ exactly for each frequency $\omega$ and compare to find the minimum as the stability margin. To the best of our knowledge, all frequency sweeping techniques in the literature follow this format. In this paper, we investigate a smart frequency sweeping strategy. We compute $k_m$ for $n_r > 1$ frequencies in parallel. Domain splitting is also performed in parallel at each iteration level. Information is exchanged among all subdomains to determine which subdomain for which frequency should be eliminated from further consideration without obtaining the exact value of $k_m$. The stability margin is achieved as the minimum record of the upper bounds of all the subdomains ever generated. The convergence rate is much faster than that of [9].

The paper is organized as follows. Section 2 introduces the robust $D$-stability problem and the work of De Gaston and Safonov [9]. Section 3 discusses the Convergence Theorem of [9] and different domain splitting schemes. Section 4 presents our Parallel Frequency Sweeping Algorithm. An illustrative example is given in Section 5 and Section 6 is the conclusion.
2 Preliminary

It is well known that the stability problem of an MIMO system can be reduced to the study of the root location of a related polynomial \[3, 6\]. We consider a family of polynomials \( p(s, q) \) of degree \( n \) whose coefficients \( a_i(q) \) are continuous functions of \( \ell \)-dimensional vector of real uncertain parameters \( q \), each bounded in the interval \([q_i^-, q_i^+]\). More formally, we define

\[
p(s, q) := a_0(q) + a_1(q)s + a_2(q)s^2 + \cdots + a_n(q)s^n
\]

where \( q := (q_1, \cdots, q_\ell) \) and the hypercube \( Q := \{q : q_i^- \leq q_i \leq q_i^+, i = 1, \cdots, \ell\} \) with the nominal parameter \( q^0 \in Q \).

2.1 Robust \( \mathcal{D} \)-stability Margin

**Definition 1** Let \( \mathcal{D} \) be an open region in the complex plane and take \( p(s) \) to be a fixed polynomial. Then \( p(s) \) is said to be \( \mathcal{D} \)-stable if all its roots lie in the region \( \mathcal{D} \).

**Definition 2** A family of polynomials \( \mathcal{P} = \{p(., q) : q \in Q\} \) is said to be robustly \( \mathcal{D} \)-stable if all roots of \( p(s, q) \) lie in \( \mathcal{D} \). For special case when \( \mathcal{D} \) is the open left half plane, \( \mathcal{P} \) is simply said to be robustly stable.

Let \( Q \subseteq Q \). Define value set \( p(z, Q) \subseteq \mathbb{C} \) as follows:

\[
p(z, Q) := \{p(z, q) : q \in Q\}.
\]

Define

\[
kQ := \{q^0 + k(q - q^0) : q \in Q\}.
\]

We first state the Zero Exclusion Condition for uncertain polynomials.

**Theorem 1** \([8]\) The polynomial \( p(s, q) \) is robustly \( \mathcal{D} \)-stable for all \( q \in Q \) if and only if \( p(s, q) \) is stable for some \( q \in Q \) and \( 0 \notin p(z, Q) \) for all \( z \in \partial \mathcal{D} \).

Let \( \mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_N \) be disjoint open subsets of the complex plane and suppose \( \mathcal{P} = \{p(., q) : q \in Q\} \) is a family of polynomials with invariant degree. For each \( q \in Q \) and \( i \in \{1, \cdots, N\} \), let \( n_i(q) \) denote the number of roots of \( p(s, q) \) in \( \mathcal{D}_i \). Finally, assume that \( p(s, q^0) \) has no roots in the boundary of \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \mathcal{D}_N \). Then each of the root indices \( n_i(q) \) remains invariant over \( Q \) if and only if the Zero Exclusion Condition \( 0 \notin p(z, Q) \) is satisfied for all points in \( \partial \mathcal{D} \).

**Definition 3** Suppose \( \mathcal{D} \) is an open subset of the complex plane with boundary \( \partial \mathcal{D} \). Then, given an interval \( I \subseteq \mathbb{R} \), a mapping \( \Phi_D : I \to \partial \mathcal{D} \) is said to be a boundary sweeping function for \( \mathcal{D} \) if \( \Phi_D \) is continuous and onto; i.e., \( \Phi_D \) is continuous and for each point \( z \in \partial \mathcal{D} \), there is some \( \delta \in I \) such that \( \Phi_D(\delta) = z \). The scalar \( \delta \) is called a generalized frequency variable for \( \mathcal{D} \).
Let \( k_m(\delta, Q) := \sup \{ k : 0 \notin p(\Phi_D(\delta), kQ) \} \). The robust \( D \)-stability margin \( k_{\text{max}} \) is given by \( k_{\text{max}} = \inf_{\delta \in I} k_m(\delta, Q) \). In general, when \( D = \bigcup_{l=1}^N D_l \) where \( D_l, l = 1, \cdots, N \) are disjointed open subsets in the complex plane, we can define \( N \) boundary sweeping functions \( \Phi^l_D : I_l \to \partial D_l, l = 1, \cdots, N \) respectively. Then the robust \( D \)-stability margin is given by

\[
 k_{\text{max}} = \min_{l=1,\cdots,N} \inf_{\delta \in I_l} k_m(\delta, Q).
\]

### 2.2 Domain Splitting Algorithms

It is noted that the analysis of robustness under polynomic structured real parametric uncertainty can be converted into a simpler analysis problem dealing with multilinear structured uncertainty \[16, 15]\.

**Definition 4** An uncertain polynomial \( p(s, q) = \sum_{i=0}^n a_i(q)s^i \) is said to have a multi-linear uncertainty structure if each of the coefficient functions \( a_i(q) \) is multi-linear. That is, if all but one component of the vector is fixed, then \( a_i(q) \) is affine linear in the remaining component of \( q \). More generally, \( p(s, q) \) is said to have a polynomic uncertainty structure if each of the coefficient functions \( a_i(q) \) is a multi-variable polynomial in the components of \( q \).

In general, there exists no analytic solution for computing exactly \( k_{\text{max}} \). However, the following Mapping Theorem can be applied to obtain a lower bound for \( k_m(\delta, Q) \) for a family of polynomials of multi-linear uncertainty structure.

**Theorem 2** \( [21] \) Suppose an uncertain polynomial \( p(s, q) \) has a multi-linear uncertainty structure. Then

\[
 \text{conv } p(z, Q) = \text{conv } \{ p(z, q^1), p(z, q^2), \cdots, p(z, q^{2^l}) \}, \quad \forall z \in \partial D
\]

where \( \text{conv} \) denotes the convex hull and \( q^1, \cdots, q^{2^l} \) denotes the \( 2^l \) vertices of the hypercube \( Q \).

Let \( k_l(\delta, Q) := \min \{ k : 0 \in \text{conv } p(\Phi_D(\delta), kQ) \} \). Then by the Mapping Theorem, \( k_l \) is a lower bound, i.e., \( k_l \leq k_m \).

**Definition 5** \( [9] \) Critical vertices are those adjacent extreme points \( M_\alpha, M_\beta \) of \( \text{conv } p(z, k_l Q) \) such that \( 0 \in \text{conv } \{ M_\alpha, M_\beta \} \).

**Definition 6** \( [9] \) \( m(\alpha, \beta) \) is the number of differing coordinates of two vertices \( q^\alpha, q^\beta \) that are mapped by \( p(z, .) \) to \( M_\alpha, M_\beta \), respectively.

Obviously that it follows from the Mapping Theorem that \( k_l = k_m \) for \( m(\alpha, \beta) = 0, 1 \). For \( m(\alpha, \beta) \geq 2 \), a vertex path is defined as follows.
Definition 7 ([9])

A vertex path is any path between critical vertices $M_{\alpha}, M_{\beta}$ consisting of $m(\alpha, \beta)$ straight-line segments defined by $p(z, \cdot)$ as $q$ progresses from $q^\alpha$ to $q^\beta$ along the edges of the hypercube $Q$.

Define

$$k_u(\delta, Q) := \inf \{ k : \text{At least one of the vertex paths of conv } p(\Phi_D(\delta), kQ) \text{ intercepts the origin} \}.$$ 

It is shown in [9] that $k_u$ is an upper bound, i.e., $k_m \leq k_u$.

In general, it is impractical to compute $k_{\text{max}}$ over all frequencies. The techniques developed in [9, 15, 16] work essentially as follows.

Choose a range of frequency $[\delta_l, \delta_u] \subset I$ and grid it as

$$\delta_j := \delta_l + \frac{(\delta_u - \delta_l)(j - 1)}{n_r n_c}, \quad j = 1, \cdots, n_r n_c$$  \hspace{1cm} (1)

where $n_r \geq 2, \; n_c \geq 1$ are integers. Apply Algorithm 1 to compute an upper bound $k^j_u$ and a lower bound $k^j_l$ for $k_m(\delta_j, Q)$ such that $\frac{k^j_u - k^j_l}{k^j_l} < \epsilon, \; j = 1, \cdots, n_r n_c$. Then an estimate of $k_{\text{max}}$ can be defined as

$$\tilde{k}_{\text{max}} := \min_{j = 1, \cdots, n_r n_c} k_m(\delta_j, Q)$$

which satisfies

$$\min_{j = 1, \cdots, n_r n_c} k^j_l \leq \tilde{k}_{\text{max}} \leq \max_{j = 1, \cdots, n_r n_c} k^j_u$$

with

$$\frac{\max_{j = 1, \cdots, n_r n_c} k^j_u - \min_{j = 1, \cdots, n_r n_c} k^j_l}{\min_{j = 1, \cdots, n_r n_c} k^j_l} < \epsilon.$$

Algorithm 1 [9] — Computing $k_m(\delta, Q)$

- Step 1: Determine lower bound on $k_m$. Designate the initial uncertain parameter domain, the $n$-dimensional hypercube $Q$, as $Q_{11}$.

- Step 2: Determine upper bound on $k_m$.

- Step 3: Iterate to converge lower and upper bounds to $k_m$. Establish an iterative procedure with counter $r = 1, 2, 3, \cdots$. For each iteration perform the following operations on subdomains $Q_{rw}$ where $p$ represents the number of subdomains left in consideration after the $r$th iteration.

  - Step 3-1: Increment $r$, i.e., $r \leftarrow r + 1$.

  - Step 3-2: Make orthogonal cuts midway on the longer edges of each subdomain $Q_{rw}, \; w = 1, \cdots, p$ in order that all edge length ratios remain within a factor of 2 of each other. Designate these two subdomains as $Q_{rw}$ and $Q_{r(w+p)}$. 

• Step 3–3: Obtain $k_{lr} := k_l(\delta, Q_{rw})$ and $k_{lr(w+p)} := k_l(\delta, Q_{r(w+p)})$ via Step 1. (Note: See [9] for handling exceptions).

• Step 3–4: Obtain $k_{ur} := k_u(\delta, Q_{rw})$ and $k_{ur(w+p)} := k_u(\delta, Q_{r(w+p)})$ via Step 1. (Note: See [9] for handling exceptions).

• Step 3–5: Repeat Steps 3–2 to 3–4 for each $w = 1, \cdots, p$.

• Step 3–6: Define $k_{lr} := \min\{k_{lr1}, k_{lr2}, \cdots, k_{lr(2p)}\}$ and $k_{ur} := \min\{k_{ur1}, k_{ur2}, \cdots, k_{ur(2p)}, k_{ur-1}\}$ and $\epsilon_r = \frac{k_{ur} - k_{lr}}{k_{lr}}$. (Note: It is shown in [9] that $k_{lr-1} < k_{lr} < k_{ur} \leq k_{ur-1}$.)

• Step 3–7: Eliminate from further consideration all subdomains $Q_{rw}$, $w = 1, \cdots, 2p$, whose associated $k_{lr} > k_{ur}$. Designate the number so eliminated as $u$ and define a new $p = 2p - u$.

• Step 3–8: Repeat Steps 3–1 to 3–8 until $k_{lr} \rightarrow k_m$. The stop criteria is that $\epsilon_r$ is less than a chosen tolerance $\epsilon > 0$.

Remark 1 In the above conventional frequency sweeping algorithm, the most important mechanism which impacts the efficiency is the elimination of subdomains whose lower bounds are greater than the minimum record of the upper bounds of all subdomains of the frequency being evaluated. This mechanism is implemented in Step 3–7. It has been demonstrated in [14] that, although the theoretical increase in subdomains should be exponential, in practice the growth can be linear due to such mechanism. We would like to note that the elimination processes for different frequencies are independent and hence such independent feature leaves room for a substantial reduction of the growth of subdomains.

3 Domain Splitting and Convergence

One of the most important requirements of an algorithm is on its convergence. For example, for the above algorithm, it is expected that given any tolerance $\epsilon > 0$, the above algorithm stops at finite iteration, i.e., $r < \infty$ for each frequency. In this section, we investigate how a domain splitting can affect the convergence.

3.1 An Impractical Assumption

The convergence of the above algorithm was addressed in [9, 10] and a Convergence Theorem was proposed. However, in the proof of the Convergence Theorem[9], the convergence was concluded upon the assumption that each subdomain converges to a single point by subdivisions (see [9], lines 40 – 53 of page 162). In another paper [16], the convergence was also concluded by assuming that a subdomain is divided fine enough (see the last paragraph in page 767 of [16]). In fact, such an
assumption is in general impractical to be satisfied. This is because the computation is usually very high to divide a domain fine enough to collapse it to a point.

In the above algorithm, the criterion adopted in splitting a domain is that “make orthogonal cut midway on the larger edges”. It is also addressed in [10, 16] that a splitting of a domain should be made in a way guaranteeing two critical vertices remained in different subdomains. In general, there is more than one way to satisfy these two criteria. We would like to note that, in general, a splitting scheme which just consists of these two criteria is not sufficient to obtain a sequence of lower bounds (or upper bounds) converging to $k_m$ or a sequence of subdomains converging to a single point in $\mathbb{R}^l$.

For example, consider a hypercube $Q = \{q \in \mathbb{R}^5 : q_i \in [0,1], i = 1,\ldots,5\}$. Let $Q_1 = Q$. Based on the above two criteria, $Q_r$ can be splitted as $Q_{r+1}$ and $Q'_r$ with $k_m(\delta, Q_{r+1}) = k_m(\delta, Q)$ at the $r$-th splitting, $r = 1, 2, \ldots$. We cannot exclude the possibility that there exists $r_c < \infty$ such that the following are true.

- Critical vertices differ in coordinates $q_1, q_2, q_3, q_4$ for $r > r_c$.
- Coordinates $q_1, q_2$ are cut in round robin order for $r > r_c$.

Finally, we will end up with a degenerate hypercube, with $q_1, q_2, q_5$ being constants and $q_3, q_4$ varying within intervals, i.e., a planar “box”. Because $q_3, q_4$ can vary in intervals, it is possible that there is a gap between the upper bound $k_u$ and the lower bound $k_l$, i.e., $\exists \nu > 0$ such that $k_u - k_l > \nu$.

Therefore, it is important to raise the following question:

What kind of splitting guarantees the convergence?

### 3.2 Guaranteed Critical Vertices Distance Convergence

Consider a hypercube $Q = \{q \in \mathbb{R}^\ell : q_i \in [q^-_i, q^+_i], i = 1,\ldots,\ell\}$. Define a sequence (finite or infinite) of domains $\{Q_r\}$ iteratively as follows.

- **Step 1**: Let $Q_1 = Q$. Let $r = 1$.
- **Step 2**: If the critical vertices of domain $Q_r$, denoted by $q^{\alpha_r}$ and $q^{\beta_r}$, differ in no more than one coordinate then the iteration process is terminated, otherwise choose $i_\ast \in U_r := \{i : q_{i_\ast}^{\alpha_r} \neq q_{i_\ast}^{\beta_r}\}$ and designate either $\{q \in Q_r : \frac{q_{i_\ast}^{\alpha_r} + q_{i_\ast}^{\beta_r}}{2} \leq q_{i_\ast} \leq \max\{q_{i_\ast}^{\alpha_r}, q_{i_\ast}^{\beta_r}\}\}$ or $\{q \in Q_r : \min\{q_{i_\ast}^{\alpha_r}, q_{i_\ast}^{\beta_r}\} \leq q_{i_\ast} \leq \frac{q_{i_\ast}^{\alpha_r} + q_{i_\ast}^{\beta_r}}{2}\}$ as $Q_{r+1}$.
- **Step 3**: Set $r = r + 1$. Go to Step 2.

In general, there are more than one way to choose $i_\ast \in U_r$. Let $Q_r = \{q \in \mathbb{R}^\ell : q_i \in [q^-_{i,r}, q^+_{i,r}], i = 1,\ldots,\ell\}$, we can define a splitting scheme as follows.
**Definition 8** A maximal-cut is a partition of $Q_r$ as above by choosing $i_* \in U_r$ such that

$$q_{i*,r}^+ - q_{i*,r}^- = \max_{i \in U_r} q_{i,r}^+ - q_{i,r}^-.$$

Another splitting scheme adopted in [10] was that the cut should be made over the coordinate that has been subdivided the least number of times. More formally, we define a fair-cut scheme as follows.

**Definition 9** A fair-cut is a partition of $Q_r$ as above by choosing $i_* \in U_r$ such that

$$\frac{q_{i*,r}^+ - q_{i*,r}^-}{q_{i*,r}^+ - q_{i*,r}^-} = \min_{i \in U_r} \frac{q_{i,r}^+ - q_{i,r}^-}{q_{i,r}^+ - q_{i,r}^-}.$$

Now we discuss the properties of the above two domain splitting schemes.

**Theorem 3** Let $\{Q_r\}$ be a sequence of domains generated as above by applying the maximal-cut scheme in each splitting. Then, we have that either $\{Q_r\}$ is a finite sequence, i.e., $\exists r_0 < \infty$ such that the critical vertices of $Q_{r_0}$ differ in no more than one coordinates, or $\{Q_r\}$ is an infinite sequence such that

$$\lim_{r \to \infty} \|p(\Phi_D(\delta), q_{\alpha^r}) - p(\Phi_D(\delta), q_{\beta^r})\| = 0$$

and

$$\lim_{r \to \infty} k_l(\delta, Q_r) = \lim_{r \to \infty} k_u(\delta, Q_r) = \lim_{r \to \infty} k_m(\delta, Q_r).$$

Moreover, the same result follows if $\{Q_r\}$ is a sequence of domains generated as above by applying the fair-cut scheme in each splitting.

**Proof.** We only need to consider the case that $\{Q_r\}$ is an infinite sequence. Decompose the coordinates index set $I = \{1, \cdots, \ell\}$ as $I = I_f \cup I_\infty$ where

$$I_f = \{i \in I : [q_i^-, q_i^+] \text{ is divided finite many times}\}$$

and

$$I_\infty = \{i \in I : [q_i^-, q_i^+] \text{ is divided infinite many times}\}.$$

Obviously, $\lim_{r \to \infty} \|q_{\alpha^r}^+ - q_{\beta^r}^-\| = 0$ for the case that $I_f = \phi$. We only need to consider the case that $I_f \neq \phi$, $I_\infty \neq \phi$. Note that $\exists r_1 > 0$ such that $q_{i,r}^+ = q_{i,r_1}^+$, $q_{i,r}^- = q_{i,r_1}^-$, $\forall i \in I_f$, $\forall r \geq r_1$. Define

$$\zeta = \min_{i \in I_f} q_{i,r_1}^+ - q_{i,r_1}^-.$$

Then

$$\min_{i \in I_f} q_{i,r}^+ - q_{i,r}^- = \zeta > 0, \forall r \geq r_1.$$
Note that \( \exists r_2 > 0 \) such that
\[
q_{i,r}^+ - q_{i,r}^- < \zeta, \quad \forall i \in I_\infty, \quad \forall r \geq r_2.
\]

We claim that \( \mathcal{U}_r \cap I_f = \emptyset, \quad \forall r > \max\{r_1, r_2\} \). In fact, if this is not the case, then \( \exists i_* \in \mathcal{U}_r \cap I_f \) such that
\[
q_{i_*}^+ - q_{i_*}^- = \max_{i \in \mathcal{U}_r} q_{i,r}^+ - q_{i,r}^- \geq \zeta
\]
because \( q_{i,r}^+ - q_{i,r}^- < \zeta, \quad \forall i \in \mathcal{U}_r \cap I_\infty \). It follows that \( Q_r \) is split as \( Q_{r+1} \) and \( Q'_{r+1} \) by dividing interval \( [q_{i_*}^-, q_{i_*}^+] \), which contradicts to \( q_{i,r}^+ = q_{i,r_1}, \quad q_{i,r}^- = q_{i,r_1}, \quad \forall i \in I_f, \quad \forall r \geq r_1 \). Thus, the claim is true and it follows that
\[
||q^{\alpha_r} - q^{\beta_r}||^2 = \sum_{i \in I_\infty} (q_{i,r}^{\alpha_r} - q_{i,r}^{\beta_r})^2 \leq \sum_{i \in I_\infty} (q_{i,r-1}^+ - q_{i,r-1}^-)^2, \quad \forall r > \max\{r_1, r_2\}.
\]

Therefore, \( \lim_{r \to \infty} ||q^{\alpha_r} - q^{\beta_r}|| = 0 \). Since \( p(z, q) \) is a continuous function of \( q \), it follows that \( \lim_{r \to \infty} ||p(\Phi_D(\delta), q^{\alpha_r}) - p(\Phi_D(\delta), q^{\beta_r})|| = 0 \). By the definition of \( k_l \) and \( k_u \), we have
\[
\lim_{r \to \infty} k_l(\delta, Q_r) = \lim_{r \to \infty} k_u(\delta, Q_r) = \lim_{r \to \infty} k_m(\delta, Q_r).
\]

Similarly, to show that the same result follows if \( \{Q_r\} \) is a sequence of domains generated as above by applying the fair-cut scheme in each splitting, we only need to consider the case that \( I_f \neq \emptyset, \quad I_\infty \neq \emptyset \). Note that \( \exists r_3 > 0 \) such that \( q_{i,r}^+ = q_{i,r_1}, \quad q_{i,r}^- = q_{i,r_3}, \quad \forall i \in I_f, \quad \forall r \geq r_3 \). Define
\[
n_s = \max_{i \in I_f} \frac{q_{i}^+ - q_{i^-}}{q_{i,r_3}^+ - q_{i,r_3^-}}.
\]
Then
\[
\max_{i \in I_f} \frac{q_{i}^+ - q_{i^-}}{q_{i,r}^+ - q_{i,r^-}} = n_s < \infty, \quad \forall r \geq r_3.
\]

Note that \( \exists r_4 > 0 \) such that
\[
\frac{q_{i}^+ - q_{i^-}}{q_{i,r}^+ - q_{i,r^-}} > n_s, \quad \forall i \in I_\infty, \quad \forall r \geq r_4.
\]

We claim that \( \mathcal{U}_r \cap I_f = \emptyset, \quad \forall r > \max\{r_3, r_4\} \). In fact, if this is not the case, then \( \exists i_* \in \mathcal{U}_r \cap I_f \) such that
\[
\frac{q_{i,r}^+ - q_{i,r}^-}{q_{i,r}^+ - q_{i,r}^-} = \min_{i \in \mathcal{U}_r} \frac{q_{i}^+ - q_{i^-}}{q_{i,r}^+ - q_{i,r}^-} \leq n_s
\]
because \( \frac{q_{i,r}^+ - q_{i,r}^-}{q_{i,r}^+ - q_{i,r}^-} > n_s, \quad \forall i \in \mathcal{U}_r \cap I_\infty \). It follows that \( Q_r \) is split as \( Q_{r+1} \) and \( Q'_{r+1} \) by dividing interval \( [q_{i,r}^-, q_{i,r}^+] \), which contradicts to \( q_{i,r}^+ = q_{i,r_1}, \quad q_{i,r}^- = q_{i,r_3}, \quad \forall i \in I_f, \quad \forall r \geq r_3 \). Thus, the claim is true and it follows that
\[
||q^{\alpha_r} - q^{\beta_r}||^2 = \sum_{i \in I_\infty} (q_{i,r}^{\alpha_r} - q_{i,r}^{\beta_r})^2 \leq \sum_{i \in I_\infty} (q_{i,r-1}^+ - q_{i,r-1}^-)^2, \quad \forall r > \max\{r_3, r_4\}.
\]
Therefore, by the same argument as in the maximal-cut schemes, the result follows.

\[ \square \]

**Remark 2** From the proof of the theorem we can see that both domain splitting schemes guarantee 
\[ \|q^{\alpha} - q^{\beta}\| \to 0 \] while allow \( Q_r \to Q_\infty \) where \( Q_\infty \) is not a single point in \( \mathbb{R}^\ell \). Clearly, to make a subdomain converge to a single point requires much more computational effort than to make \( \|q^{\alpha} - q^{\beta}\| \to 0 \). As we can see later, \( \|q^{\alpha} - q^{\beta}\| \to 0 \) leads to the existence of a sequence of lower bounds (or upper bounds) converging to \( k_m \). Therefore, an algorithm based on the maximal-cut (or fair-cut) splitting scheme will reduce much computational effort in computing \( k_m \) than other algorithms based on making subdomains converge to a single point in \( \mathbb{R}^\ell \). From the proof, we can also see that the convergence will not follow if the domain splitting is made along the larger but not the largest edges of each subdomain. It was remarked in [10] that a fair-cut avoids the problem of getting into very narrow and long subdomains which can decrease the convergence speed. From the proof, we can see that it plays a role much more than affecting the speed of convergence. It is a sufficient condition to the existence of a sequence of lower bounds (or upper bounds) converging to \( k_m \). We would like to point out that the maximal-cut scheme has better worst case convergence behavior than that of the fair-cut scheme.

To see the efficiency of the maximal-cut (or the fair-cut) splitting scheme, it is helpful to compare the image of the last subdomain resulted from the the maximal-cut (or the fair-cut) splitting scheme and the image of the last subdomain obtained by the finely subdivision. The situation is shown in Figure 2.

![Figure 2: The Image of the Last Subdomain](image)

4 Parallel Frequency Sweeping Algorithm

In this section we shall investigate a new frequency sweeping structure.
4.1 The Main Root of Inefficiency

To the best of our knowledge, no effort in the existing literature has been devoted to exploit a smart frequency sweeping strategy. Existing techniques are basically as follows: Choose and grid a range of frequency. Then calculate exactly $k_m$ for each gridded frequency. Finally, compare to find the minimum $k_m$ and return it as an estimate of $k_{\text{max}}$.

For complicated root region $D$, the number of frequencies to be evaluated for $k_m$ would be substantial in order to obtain a reasonably good estimate for $k_{\text{max}}$. Even the computation of $k_m$ for each frequency is very efficient, the overall complexity is still very high, because we need to evaluate $k_m$ for many frequencies.

Thus for the sake of efficiency, it is natural to conceive a smart frequency sweeping strategy. More specifically, we would raise the following question, 

*Is it possible to obtain the stability margin $k_{\text{max}}$ without tightly bounding $k_m(\delta_j, Q)$ for each frequency $\delta_j$?*

The following section is devoted to answering this question.

4.2 Parallel Frequency Sweeping Algorithm

Consider the same set of gridded frequencies $\delta_j$, $j = 1, \cdots, n_r n_c$ defined by (1) and relabel them as

$$\delta_{ij} := \delta_l + \frac{(\delta_u - \delta_l)(i - 1)}{n_r} + \frac{(\delta_u - \delta_l)(j - 1)}{n_r n_c}, \quad i = 1, \cdots, n_r, \quad j = 1, \cdots, n_c.$$

We are now in a position to present our Parallel Frequency Sweeping Algorithm as follows.

**Algorithm 2 — Parallel Frequency Sweeping Algorithm**

- **Step 1:** Initialize. Set $j = 1$. Set $\hat{k} = \infty$. Set tolerance $\epsilon > 0$. Set maximal iteration number $IT$.

- **Step 2:** Update $\hat{k}$ and record the number of iterations $r(j)$ for frequency $\delta_{ij}$ by the following steps.

  - Step 2–1: Let $U_{ij} = \{Q_k\} = Q$, $i = 1, \cdots, n_r$. Set $r = 1$.
  - Step 2–2: If $r = IT + 1$ or $U_{ij}$ is empty for any $i \in \{1, \cdots, n_r\}$ then record $r(j) = r$ and go to Step 3, else do the following for all $i$ such that $U_{ij}$ is not empty.

    * Choose $Q$ to be any element of $U_{ij}$ with
      $$k_l(\delta_{ij}, Q) = \min_{Q_k \in U_{ij}} k_l(\delta_{ij}, Q_k).$$

    * Partition $Q$ into $Q_a$ and $Q_b$ by applying a maximal-cut.

    * Remove $Q$ from $U_{ij}$.
\* Update
\[ \hat{k} = \min\{\hat{k}, k_u(\delta_{ij}, Q_a), k_u(\delta_{ij}, Q_b)\}. \] (2)

\* Add any \( Q \in \{Q_a, Q_b\} \) with two or more critical vertices to \( \mathcal{U}_{ij} \).

\* Remove from \( \mathcal{U}_{ij} \) any \( Q \) with
\[ 0 \notin \text{conv} \left( p\left( \Phi_D(\delta_{ij}), \frac{\hat{k}}{1+\epsilon} Q \right) \right) \]. (3)

\( - \) Step 2-3: Set \( r = r + 1 \) and go to Step 2-2.

● Step 3: If \( j = n_c \) then STOP, else set \( j = j + 1 \) and go to Step 2.

In Algorithm 2, \( n_r \) branches of frequency sweeping are performed in parallel with starting frequencies \( \delta_{i1}, i = 1, \ldots, n_r \) and step size \( \frac{\delta_u - \delta_l}{n_r n_c} \). Each branch of frequency sweeping is not independent, they exchange information. The information is applied to determine the subdomains to be eliminated from further consideration and to update \( \hat{k} \). Finally, \( \hat{k} \) is returned as the robust stability margin.

Algorithm 2 is visualized in the following Figure 3.

Figure 3: A Picture of Parallel Frequency Sweeping Algorithm. \( N = 3, \; K = 4. \)

**Remark 3** As we can see from Step 2-2, there are two mechanisms which contribute to the efficiency of the Parallel Frequency Sweeping Algorithm. First, any subdomain \( Q \) that satisfies condition (3), which is equivalent to
\[ k_l(\delta_{ij}, Q) < \frac{\hat{k}}{1+\epsilon}, \] (4)
will never be partitioned again and thus can be eliminated from further consideration. Second, any subdomain \( Q \) with critical vertices differing in no more than one coordinates will never be partitioned again and thus can be eliminated from further consideration.

We would like to note that the proposed Parallel Frequency Sweeping provides substantial improvement on efficiency than the algorithms in [9]. This can be explained by the significant relaxation of
the condition for eliminating a subdomain from consideration. By (2), (3) and (4) we can see that \( \hat{k} \) is the minimum record of the upper bounds among all subdomains evaluated (no matter it belongs to the same frequency or not) and is contracted to \( \hat{k}_{1+\epsilon} \). In contrast, in algorithms of \([9]\) the minimum record of upper bounds is obtained for the frequency being considered only. Therefore, the condition for eliminating a subdomain from consideration is much looser than its counterpart of algorithms in \([9]\). Consequently, such a significant relaxation results in a substantial decrease of the number of subdomains needed to be evaluated.

**Remark 4** In Algorithm 2, at each iteration, only the domain with the smallest lower bound is partitioned. This mechanism differs from that of Algorithm 1 in which all domains are partitioned and thus the number of domains increases rapidly. We can see that Algorithm 2 effectively controls the growth of the number of subdomains and thus is much efficient than the conventional algorithm.

**Remark 5** It is important to note that Algorithm 2 involves only one CPU processor. It is fundamentally different from the parallel algorithms which involves more than one CPU processor.

**Remark 6** The speed of computation depends on the choice of integers \( n_r \) and \( n_c \). When the total number of frequencies is fixed (i.e., \( n_r n_c \) is constant), the number of branches of frequency sweeping \( n_r \) should not be too small. Small \( n_r \) will hinder the improvement of efficiency benefited from the parallelism. However, very large \( n_r \) will not result in optimal performance either.

In addition to the novel frequency sweeping strategy, another character of Algorithm 2 is that there is no tolerance criteria directly forced on the final result, however, the final result falls into tolerance automatically.

**Theorem 4** Suppose that the maximum iteration number \( IT = \infty \). For arbitrary tolerance \( \epsilon > 0 \), Algorithm 2 stops with a finite number of domain splittings for each \( j \), i.e., \( r(j) < \infty \), \( \forall j \). Moreover, the final \( \hat{k} \) satisfies

\[
0 \leq \frac{\hat{k} - \hat{k}_{\text{max}}}{\hat{k}_{\text{max}}} < \epsilon.
\]

**Proof.** We first show the final \( \hat{k} \geq \hat{k}_{\text{max}} \). Let \( k_u \) be the upper bound of domain \( Q \subseteq Q \) which ever appeared during the execution of Algorithm 2. Let \( \delta \) be the associated frequency of \( Q \). Note that \( 0 \in p(\Phi_D(\delta), k_u Q) \subset p(\Phi_D(\delta), k_u Q) \) and thus \( k_u \geq \hat{k}_{\text{max}} \). Note that the final \( \hat{k} \) is the minimum record of all such \( k_u \)'s, thus \( \hat{k} \geq \hat{k}_{\text{max}} \).

We next need to show that Algorithm 2 stops with a finite iteration number \( r(j) \) for each \( j \). Suppose \( \exists j \) such that \( r \) goes to \( \infty \). Then \( \exists \delta_{ij} \) such that \( r \) goes to \( \infty \). Therefore, we can construct a sequence of nested domains \( \{Q_r\} \) such that \( Q_1 \supset Q_2 \supset \cdots \supset Q_r \supset Q_{r+1} \supset \cdots \). Thus by Theorem 3 we have that \( \forall \epsilon > 0, \exists r_0 < \infty \) such that \( k_u(\delta_{ij}, Q_r) - k_l(\delta_{ij}, Q_r) < \frac{\epsilon}{1+\epsilon} \hat{k}_{\text{max}}, \forall r \geq r_0 \).
Thus \( \min\{\hat{k}, k_u(\delta_{ij}, Q_{r_0})\} - k_l(\delta_{ij}, Q_{r_0}) < \frac{\epsilon}{1 + \epsilon} \hat{k}_{\text{max}} \). Note that \( k_l(\delta_{ij}, Q_{r_0+1}) \geq k_l(\delta_{ij}, Q_{r_0}) \) because \( Q_{r_0+1} \subset Q_{r_0} \). Also note that \( \hat{k} \) never increases, thus we have \( \hat{k} - k_l(\delta_{ij}, Q_{r_0+1}) < \frac{\epsilon}{1 + \epsilon} \hat{k}_{\text{max}} \). Note that \( \hat{k} \geq \hat{k}_{\text{max}} \), we have

\[
\hat{k} - k_l(\delta_{ij}, Q_{r_0+1}) < \frac{\epsilon}{1 + \epsilon} \hat{k} \implies \frac{\hat{k}}{1 + \epsilon} < k_l(\delta_{ij}, Q_{r_0+1}),
\]

which implies that \( 0 \notin \text{conv} \left( p \left( \Phi_D(\delta_{ij}), \frac{\hat{k}}{1 + \epsilon} Q_{r_0+1} \right) \right) \). Therefore, by Algorithm 2 \( Q_{r_0+1} \) will not be splitted. This is a contradiction. So, we have shown that Algorithm 2 stops with a finite number of domain splittings for each \( j \).

Note that \( \exists \delta_{ij} \) such that \( k_m(\delta_{ij}, Q) = \hat{k}_{\text{max}} \). Moreover, \( \exists q^* \in Q \) such that \( p(\Phi_D(\delta_{ij}), \hat{k}_{\text{max}} q^*) = 0 \). Since Algorithm 2 stops with a finite number of domain splittings for each \( j \), we have that all subdomains ever generated are finally eliminated from consideration. Thus, there must exists \( Q^* \) which contains \( q^* \) be eliminated from consideration at a certain level of splitting.

Assume that \( \hat{k} = \bar{k} \) when \( Q^* \) is eliminated from consideration. Then either \( 0 \notin p(\Phi_D(\delta_{ij}), \frac{\hat{k}}{1 + \epsilon} Q^*) \) or the critical vertices of \( Q^* \) differ in no more than one coordinates. If the first case is true, then by \( p(\Phi_D(\delta_{ij}), \hat{k}_{\text{max}} q^*) = 0 \) and \( q^* \in Q^* \), we have that \( 0 \in p(\Phi_D(\delta_{ij}), \hat{k}_{\text{max}} Q^*) \). Thus by \( 0 \notin p(\Phi_D(\delta_{ij}), \frac{\hat{k}}{1 + \epsilon} Q^*) \), we have \( \frac{\hat{k}}{1 + \epsilon} < \hat{k}_{\text{max}} \). Obviously, the final \( \hat{k} \leq \bar{k} \) and thus

\[
\frac{\hat{k}}{1 + \epsilon} < \hat{k}_{\text{max}} \implies \frac{\hat{k} - \hat{k}_{\text{max}}}{\hat{k}_{\text{max}}} < \epsilon.
\]

If the latter case is true, then we have

\[
k_l(\delta_{ij}, Q^*) = k_u(\delta_{ij}, Q^*) = \hat{k}_{\text{max}} = \hat{k}.
\]

The proof is thus completed.

\[ \square \]

**Remark 7** From the proof, we can see that the existence of a sequence of upper bounds converging to \( \hat{k}_{\text{max}} \) is due to the convergence of the distance of critical vertices.

**Remark 8** By specifying \( \epsilon \) in Algorithm 2, we can obtain an estimate \( \hat{k} \) such that \( 0 \leq \frac{\hat{k} - \hat{k}_{\text{max}}}{k_{\text{max}}} < \epsilon \). In this sense, we can say that Algorithm 2 provides a global solution for searching \( \hat{k}_{\text{max}} \). However, it should be noted that it is not necessary a global solution for the exact robust \( D \)-stability margin \( k_{\text{max}} \). This is because it is impossible to search the whole range of the generalized frequency \( \delta \). It is only feasible to perform the search over a set of discrete values of \( \delta \).

**Theorem 5** Suppose that the maximum iteration number \( IT < \infty \) and that Algorithm 2 stops with \( r(j) < IT, \forall j \). Then the final \( \hat{k} \) satisfies

\[
0 \leq \frac{\hat{k} - \hat{k}_{\text{max}}}{k_{\text{max}}} < \epsilon.
\]
Proof. Since Algorithm 2 stops with $r(j) < IT$, $\forall j$, we can conclude that all subdomains ever generated are finally eliminated from consideration. Thus the result follows from similar argument as for Theorem 4.

5 An Illustrative Example

Our computational experience shows that Algorithm 2 provides a significant improvement upon conventional algorithms for most control problems. Moreover, the improvement depends on the problems and can be arbitrarily good. To illustrate, we consider an example with $\mathcal{D}$ chosen to be the open left half plane. The applications to the problems with complicated $\mathcal{D}$ are in exactly the same spirit.

The state space equation of the linear system is as follows:

$$
\dot{x} = A(q)x + Bu \\
y = Cx
$$

where

$$
A(q) = \begin{bmatrix}
-1 - 0.5q_1 & -10 & -1 & 10 \\
-0.5 & -1 & 1 & 0.5 \\
0.5 & -4 & -1 & -10 \\
-10 & 0.5 & 0 & -2.5 - 1.5q_2 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
2 \\
1 \\
1 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0.5 & 1 & 0.5 \\
\end{bmatrix}
$$

with uncertain parameter $q \in \mathcal{Q} = [0, 1] \times [0, 3] \subset \mathbb{R}^2$. We obtained a polynomial for this system as $p(s, q) = \det(sI - A(q))$ which has a multilinear structure.

The upper bound and lower bound of $k_m$ on $\mathcal{Q}$ is shown in Figures 4 – 7. We can see that for most of the frequencies the upper bounds and lower bounds are far apart and thus the importance of domain splitting is obvious.

To compute $k_{\text{max}}$, we uniformly grid frequency band $[0.01, 15.01]$ and obtain 1,500 grid frequencies as

$$
\omega_j = 0.01j, \quad j = 1, \ldots, 1500.
$$

In Algorithm 2, we choose the relative error $\epsilon = 0.01$ and $n_r = 30$, $n_c = 50$. The 1,500 frequencies are regrouped as

$$
\omega_{ij} = 0.01 + 0.2(i - 1) + 0.01(j - 1), \quad i = 1, \ldots, 30; \quad j = 1, \ldots, 50.
$$

We ran the program in a Sun Spark work-station. The running time is about 80 seconds. The total number of domains evaluated is 1,570. We obtained $\hat{k} = 1.4384$ which is achieved at frequency...
\( \omega_{20,18} = 9.78 \). By Theorem 4, we can conclude that

\[
0 \leq \frac{\hat{k} - \tilde{k}_{\text{max}}}{\tilde{k}_{\text{max}}} < \epsilon = 0.01.
\]

To compare the performance of the conventional algorithm with that of Algorithm 2, it is fair to choose the tolerance \( \epsilon_r = 0.01 \) in Algorithm 1. We also ran the program in the same work-station. The running time is about 9 hours. The total number of domains evaluated is 64,813. We obtained \( \hat{k} = 1.4380 \) which is achieved at frequency \( \omega_{9783} = 9.783 \). Therefore, Algorithm 2 has a speed-up of 400 over the conventional algorithm. Moreover, the number of domains evaluated in Algorithm 2 is only a small fraction (which is \( \frac{1570}{64813} \approx 0.0242 \)) of that of the conventional algorithm. The number of domains evaluated in Algorithm 2 and the conventional one for each frequency is shown respectively in Figure 8 and Figure 9.

We can see that Algorithm 2 provides much superior performance than the conventional algorithms. The improvement comes from the characteristic eliminating mechanisms in Algorithm 2. More formally, we describe the eliminating process in Algorithm 2 as follows.

Let \( U \) be a record of the global upper bound achieved by frequency \( \omega \). Let \( Q_{ij} \subseteq Q \) be a domain associated with frequency \( \omega_{ij} \). When \( Q_{ij} \) is eliminated, i.e., \( k_t(\omega_{ij}, Q_{ij}) > \frac{U}{1+\epsilon} \) is satisfied, there are only three cases as follows.
Figure 5: $k_m$ upper bound and lower bound on $Q$. The upper bound is plotted in dashed line and the lower bound is plotted in solid line.

- Case (i): $\omega_{ij} < \omega$. We call the elimination as Backward Pruning.
- Case (ii): $\omega_{ij} > \omega$. We call the elimination as Forward Pruning.
- Case (iii): $\omega_{ij} = \omega$. We call the elimination as Present Pruning.

All the above three types of pruning processes play important roles in Algorithm 2. However, there is only Present Pruning in the conventional algorithm. Therefore, Algorithm 2 has a much powerful pruning mechanism and is much more efficient.

In this example, we have 24 records which are shown in Figure 10. The effectiveness of the three types of pruning processes are shown respectively in Figures 12, 13 and 14.

6 Conclusion

We have developed techniques such as, a parallel frequency sweeping strategy, different domain splitting schemes, which significantly reduce the computational complexity and guarantee the convergence. Our computational experience shows that Algorithm 2 provides a substantial improvement on efficiency in comparison with the conventional algorithms.
Figure 6: $k_m$ upper bound and lower bound on $Q$. The upper bound is plotted in dashed line and the lower bound is plotted in solid line.

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Figure 10: Evolution of the global upper bound. The $y$-coordinate represents the record value and the $x$-coordinate represents the frequency achieving it. Two consecutive records are connected by dashed line.
Figure 11: Evolution of the global upper bound. The $y$-coordinate represents the record value and the $x$-coordinate represents the frequency achieving it. Two consequent records are connected by dashed line.
Figure 12: Backward Pruning. The $y$-coordinates represents the number of domains eliminated by the record as Case (i). The $x$-coordinate represents the record index.
Figure 13: Forward Pruning. The $y$-coordinates represents the number of domains eliminated by the record as Case (ii). The $x$-coordinate represents the record index.
Figure 14: Present Pruning. The $y$-coordinates represents the number of domains eliminated by the record as Case (iii). The $x$-coordinate represents the record index.