Reconstruction and subgaussian operators

Shahar MENDELSON$^1$  Alain PAJOR$^2$
Nicole TOMCZAK-JAEGERMANN$^3$

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Abstract:
We present a randomized method to approximate any vector $v$ from some set $T \subset \mathbb{R}^n$. The data one is given is the set $T$ and $k$ scalar products $(\langle X_i, v \rangle)_{i=1}^k$, where $(X_i)_{i=1}^k$ are i.i.d. isotropic subgaussian random vectors in $\mathbb{R}^n$, and $k \ll n$. We show that with high probability, any $y \in T$ for which $(\langle X_i, y \rangle)_{i=1}^k$ is close to the data vector $(\langle X_i, v \rangle)_{i=1}^k$ will be a good approximation of $v$, and that the degree of approximation is determined by a natural geometric parameter associated with the set $T$.

We also investigate a random method to identify exactly any vector which has a relatively short support using linear subgaussian measurements as above. It turns out that our analysis, when applied to $\{-1, 1\}$-valued vectors with i.i.d. symmetric entries, yields new information on the geometry of faces of random $\{-1, 1\}$-polytope; we show that a $k$-dimensional random $\{-1, 1\}$-polytope with $n$ vertices is $m$-neighborly for very large $m \leq ck / \log(c'n/k)$.

The proofs are based on new estimates on the behavior of the empirical process $\sup_{F \in \mathcal{F}} \left| k^{-1} \sum_{i=1}^k f^2(X_i) - \mathbb{E} f^2 \right|$ when $F$ is a subset of the $L_2$ sphere. The estimates are given in terms of the $\gamma_2$ functional with respect to the $\psi_2$ metric on $F$, and hold both in exponential probability and in expectation.

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$^3$This author holds the Canada Research Chair in Geometric Analysis.
0 Introduction

The aim of this article is to investigate the linear “approximate reconstruction” problem in $\mathbb{R}^n$. In such a problem, one is given a set $T \subset \mathbb{R}^n$ and the goal is to be able to approximate any unknown $v \in T$ using random linear measurements. In other words, one is given the set of values $(\langle X_i, v \rangle)_{i=1}^k$, where $X_1, ..., X_k$ are independent random vectors in $\mathbb{R}^n$ selected according to some probability measure $\mu$. Using this information (and the fact that the unknown vector $v$ belongs to $T$) one has to produce, with very high probability with respect to $\mu^k$, some $t \in T$, such that the Euclidean norm $|t - v| \leq \varepsilon(k)$ for $\varepsilon(k)$ as small as possible. Of course, the random sampling method has to be “universal” in some sense and not tailored to a specific set $T$; and it is natural to expect that the degree of approximation $\varepsilon(k)$ depends on some geometric parameter associated with $T$.

Questions of a similar flavor have been thoroughly studied in nonparametric statistics and statistical learning theory (see, for example, [BBL] and [MT] and references therein). This particular problem has been addressed by several authors (see [CT1, CT2, CT3, RV] for the most recent contributions), in a rather restricted context. First of all, the sets considered were either the unit ball in $\ell_1^n$ or the unit balls in weak $\ell_p^n$ spaces for $0 < p < 1$ - and the proofs of the approximation estimates depended on the choice of those particular sets. Second, the sampling process was done when $X_i$ were distributed according to the Gaussian measure on $\mathbb{R}^n$ or in [CT1] for Fourier ensemble.

In contrast, we present a method which is very general. Our results hold for any set $T \subset \mathbb{R}^n$, and the class of measures that could be used is broad; it contains all probability measures on $\mathbb{R}^n$ which are isotropic and subgaussian, that is, satisfy that for every $y \in \mathbb{R}^n$, $\mathbb{E}|\langle X, y \rangle|^2 = |y|^2$, and the random variables $(X, y)$ are subgaussian with constant $\alpha|y|$ for some $\alpha \geq 1$. (see Definition 2.2 below). This class of measures contains, among others, the Gaussian measure on $\mathbb{R}^n$, the uniform measure on the vertices of the combinatorial cube and the normalized volume measure on various convex, symmetric bodies (e.g. the unit balls of $\ell_p^n$ for $2 \leq p \leq \infty$).

It turns out that the key parameter in the estimate on the degree of approximation $\varepsilon(k)$ is indeed geometric in nature. Moreover, the analysis of the approximation problem is centered around the way the random operator $\Gamma = \sum_{i=1}^k \langle X_i, \cdot \rangle e_i$ (where $(e_i)_{i=1}^k$ is the standard basis in $\mathbb{R}^k$) acts on subsets of the unit sphere.
Our geometric approach, when applied to the sets $T$ considered in $[CT1]$, yields the optimal estimates on $\varepsilon(k)$, and with better probability estimates (of the order of $1 - \exp(-ck)$), even when the sampling is done according to an arbitrary isotropic, subgaussian measure. Moreover, our result is more robust than the one from $[CT1]$ in the following sense. The reconstruction method suggested in $[CT1]$ is to find some $y \in T$ such that for every $1 \leq i \leq k$, $\langle X_i, v \rangle = \langle X_i, y \rangle$, and then to show that such $v$ and $y$ must necessarily be close. From our theorem it is clear that one can choose any $y \in T$ for which $\sum_{i=1}^{k} \langle X_i, y - v \rangle^2$ is relatively small, which is far more stable algorithmically.

For the moment, let us present a simple version of the main result we prove in this direction, and to that end we require the following notation. Let $(g_i)_{i=1}^{n}$ be independent standard Gaussian random variables. Let $T \subset \mathbb{R}^n$ be a star-shaped set (i.e. for every $t \in T$ and $0 \leq \lambda \leq 1$, $\lambda t \in T$) and consider the following geometric parameter

$$\ell_*(T) = \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^{n} g_i t_i \right|$$

which is, up to a factor of the order of $2\sqrt{n}$, the mean width of the body $T$. We now define a more sensitive parameter, which as we will see in this article, is the right parameter to control the error term $\varepsilon(k)$ for a star-shaped set $T$:

$$r_k^*(\theta, T) := \inf \{ \rho > 0 : \rho \geq c \alpha^2 \ell_*(T \cap \rho S^{n-1}) / \theta \sqrt{k} \}.$$

We may now state a version of our main result concerning approximate reconstruction.

**Theorem A** There exist an absolute constant $c_1$ for which the following holds. Let $T$ be a star-shaped subset of $\mathbb{R}^n$. Let $\mu$ be an isotropic, subgaussian measure with constant $\alpha$ on $\mathbb{R}^n$ and set $X_1, ..., X_k$ be independent, selected according to $\mu$. Then, with probability at least $1 - \exp(-c_1 k / \alpha^4)$, every $y, v \in T$ satisfy that

$$|y - v| \leq 2 \left( \frac{1}{k} \sum_{i=1}^{k} (\langle X_i, v \rangle - \langle X_i, y \rangle)^2 \right)^{1/2} + r_k^*(1/2, T - T).$$

The parameter $r_k^*(\theta, T)$ can be estimated for unit ball of classical normed or quasi-normed spaces (see Section 3). In particular, if $T = B_1^n$ then with
the same hypothesis and probability as above, one has

\[ |y - v| \leq 2 \left( \frac{1}{k} \sum_{i=1}^{k} (\langle X_i, v \rangle - \langle X_i, y \rangle)^2 \right)^{1/2} + c\alpha^2 \sqrt{\frac{\log(c\alpha^4n/k)}{k}} \]

where \( c > 0 \) is an absolute constant; this leads to the optimal estimate for \( \varepsilon(k) \) for that set.

The main idea in the proof of this theorem is the fact that excluding a set with exponentially small probability, the random operator \( \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \langle X_i, \cdot \rangle \) is a very good isomorphism on elements of \( T \) whose Euclidean norm is large enough (see Section 2 for more details).

A question of a similar nature we investigate here focuses on “exact reconstruction” of vectors in \( \mathbb{R}^n \) that have a short support. Suppose that \( z \in \mathbb{R}^n \) is in the unit Euclidean ball, and has a relatively short support \( m << n \). The aim is to use a random sampling procedure to identify \( z \) exactly, rather than just to approximate it. The motivation for this problem comes from error correcting codes, in which one has to overcome random noise that corrupts a part of a transmitted signal. The noise is modelled by adding to the transmitted vector the noise vector \( z \). The assumption that the noise does not change many bits in the original signal implies that \( z \) has a short support, and thus, in order to correct the code, one has to identify the noise vector \( z \) exactly. Since error correcting codes are not the main focus of this article, we will not explore this topic further, but rather refer the interested reader to [MS, CT2, RV] and references therein for more information.

In the geometric context we are interested in, the problem has been studied in [CT2, RV], where it was shown that if \( z \) has a short support relative to the dimension \( n \) and the size of the sample \( k \), and if \( \Gamma \) is a \( k \times n \) matrix whose entries are independent, standard Gaussian variables, then with probability at least \( 1 - \exp(-ck) \), the minimization problem

\[ (P) \quad \min \|v\|_{\ell^1} \text{ for } \Gamma v = \Gamma z, \]

has a unique solution, which is \( z \). Thus, solution to this minimization problem will pin-point the “noise vector” \( z \). The idea of using this minimization problem was first suggested in [CDS].

We extend this result to a general random matrix whose rows are \( X_1, \ldots, X_k \), sampled independently according to an isotropic, subgaussian measure.
Theorem B  Let $\Gamma$ be as above. With probability at least $1 - \exp(-c_1 k/\alpha^4)$, any vector $z$ whose support has cardinality less than $c_2 k/\log(c_3 n/k)$ is the unique minimizer of the problem ($P$), where $c_1, c_2, c_3$ are absolute constants.

Interestingly enough, the same analysis yields some information on the geometry of $\{-1,1\}$ random polytopes. Indeed, consider the $k \times n$ matrix $\Gamma$ whose entries are independent symmetric $\{-1,1\}$ valued random variables. Thus, $\Gamma$ is a random operator selected according to the uniform measure on the combinatorial cube $\{-1,1\}^n$, which is an isotropic, subgaussian measure with constant $\alpha = 1$. The columns of $\Gamma$, denoted by $v_1, \ldots, v_n$ are vectors in $\{-1,1\}^k$ and let $K^+ = \text{conv}(v_1, \ldots, v_n)$ be the convex polytope generated by $\Gamma$; $K^+$ is thus called a random $\{-1,1\}$-polytope.

A convex polytope is called $m$-neighborly if any set of less than $m$ of its vertices is the vertex set of a face (see [Z]). The following result yields the surprising fact that a random $\{-1,1\}$-polytope is $m$-neighborly for a relatively large $m$. In particular, it will have the maximal number of $r$-faces for $r \leq m$.

Theorem C  There exist absolute constants $c_1, c_2, c_3$ for which the following holds. For $1 \leq k \leq n$, with probability larger than $1 - \exp(-c_1 k)$, a $k$-dimensional random $\{-1,1\}$ convex polytope with $n$ vertices is $m$-neighborly provided that

$$m \leq \frac{c_2 k}{\log(c_3 n/k)}.$$

The main technical tool we use throughout this article, which is of independent interest, is an estimate of the behavior of the supremum of the empirical process $f \rightarrow Z_f = k^{-1} \sum_{i=1}^{k} f^2(X_i) - 1$ indexed by a subset of the $L_2$ sphere. That is,

$$\sup_{f \in F} \left| \frac{1}{k} \sum_{i=1}^{k} f^2(X_i) - 1 \right|,$$

where $X_1, \ldots, X_k$ are independent, distributed according to the a probability measure $\mu$, and under the assumption that every $f \in F$ satisfies that $\mathbb{E} f^2 = 1$. We assume further that $F$ is a bounded set with respect to the $\psi_2$ norm, defined for a random variable $Y$ by

$$\|Y\|_{\psi_2} = \inf \left\{ u > 0 : \mathbb{E} \exp(Y^2/u^2) \leq 2 \right\}.$$
To formulate the following result, we require the notion of the \( \gamma_p \) functional \([Ta2]\). For a metric space \((T, d)\), an admissible sequence of \(T\) is a collection of subsets of \(T\), \(\{T_s : s \geq 0\}\), such that for every \(s \geq 1\), \(|T_s| = 2^s\) and \(|T_0| = 1\). For \(p = 1, 2\), we define the \( \gamma_p \) functional by

\[
\gamma_p(T, d) = \inf \sup_{t \in T} \sum_{s=0}^{\infty} 2^{s/p} d(t, T_s),
\]

where the infimum is taken with respect to all admissible sequences of \(T\).

**Theorem D** There exist absolute constants \(c_1, c_2, c_3\) and for which the following holds. Let \((\Omega, \mu)\) be a probability space, set \(F\) be a subset of the unit sphere of \(L_2(\mu)\) and assume that \(\text{diam}(F, \| \psi_2 \|) = \alpha\). Then, for any \(\theta > 0\) and \(k \geq 1\) satisfying

\[
c_1 \alpha \gamma_2(F, \| \psi_2 \|) \leq \theta \sqrt{k},
\]

with probability at least \(1 - \exp(-c_2 \theta^2 k / \alpha^4)\),

\[
\sup_{f \in F} |Z_f| \leq \theta.
\]

Moreover, if \(F\) is symmetric, then

\[
\mathbb{E} \sup_{f \in F} |Z_f| \leq c_3 \alpha^2 \gamma_2(F, \| \psi_2 \|) \sqrt{k}.
\]

Theorem D improves a result of a similar flavor from \([KM]\) in two ways. First of all, the bound on the probability is exponential in the sample size \(k\) which was not the case in \([KM]\). Second, we were able to bound the expectation of the supremum of the empirical process using only a \( \gamma_2 \) term. This fact is surprising because the expectation of the supremum of empirical processes is usually controlled by two terms; the first one bounds the subgaussian part of the process and is captured by the \( \gamma_2 \) functional with respect to the underlying metric. The other term is needed to control sub-exponential part of the empirical process, and is bounded by the \( \gamma_1 \) functional with respect to an appropriate metric (see \([Ta2]\) for more information on the connections between the \( \gamma_p \) functionals and empirical processes). Theorem D shows that the expectation of the supremum of \(|Z_f|\) behaves as if \(\{Z_f : f \in F\}\) were a subgaussian process with respect to the \(\psi_2\) metric (although it is not), and this is due to the key fact that all the functions in \(F\) have the same second moment.
We end this introduction with the organization of the article. In Section 1 we present the proof of Theorem D and some of its corollaries we require. In Section 2 we illustrate these results in the case of linear processes which corresponds to linear measurements. In Section 3 we investigate the approximate reconstruction problem for a general set, and in Section 4 we present a proof for the exact reconstruction scheme, with its application to the geometry of random \([-1,1]\)-polytopes.

Throughout this article we will use letters such as \(c, c_1, \ldots\) to denote absolute constants which may change depending on the context. We denote by \(e_i\) the canonical basis of \(\mathbb{R}^n\), by \(|x|\) the Euclidean norm of a vector \(x\) and by \(B_2^n\) the Euclidean unit ball. We also denote by \(|I|\) the cardinality of a finite set \(I\).

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1 Empirical Processes

In this section we present some results in empirical processes that will be central to our analysis. All the results focus on understanding the process \(Z_f = \frac{1}{k} \sum_{i=1}^{k} f^2(X_i) - \mathbb{E} f^2\), where \(k \geq 1\) and \(X_1,...,X_k\) are independent random variables distributed according to a probability measure \(\mu\). In particular, we investigate the behavior of \(\sup_{f \in F} |Z_f|\) in terms of various metric structures on \(F\), and under the key assumption that every \(f \in F\) has the same second moment. The parameters involved are standard in generic chaining type estimates (see [Ta2] for a comprehensive study of this topic).

Recall that the \(\psi_p\) norm of a random variable \(X\) is defined as

\[
\|X\|_{\psi_p} = \inf \{ u > 0 : \mathbb{E} \exp \left( |X|^p / u^p \right) \leq 2 \}.
\]

It is standard to verify (see for example [VW]) that if \(X\) has a bounded \(\psi_2\) norm, then it is subgaussian with parameter \(c\|X\|_{\psi_2}\) for some absolute constant \(c\). More generally, a bounded \(\psi_p\) norm implies that \(X\) has a tail behavior, \(\mathbb{P}(|X| > u)\), of the type \(\exp(-cu^p/\|X\|_{\psi_p})\).

Our first fundamental ingredient is the well known Bernstein’s inequality which we shall use in the form of a \(\psi_1\) estimates ([VW]).
Lemma 1.1 Let $Y_1, \ldots, Y_k$ be independent random variables with zero mean such that for some $b > 0$ and every $i$, $\|Y_i\|_{\psi_1} \leq b$. Then, for any $u > 0$,

$$
P \left( \left| \frac{1}{k} \sum_{i=1}^{k} Y_i \right| > u \right) \leq 2 \exp \left( -ck \min \left( \frac{u}{b}, \frac{u^2}{b^2} \right) \right), \quad (1.1)$$

where $c > 0$ is an absolute constant.

We will be interested in classes of functions $F \subset L_2(\mu)$ bounded in the $\psi_2$-norm; we assume without loss of generality that $F$ is symmetric and we let $\text{diam}(F, \| \cdot \|_{\psi_2}) := 2 \sup_{f \in F} \| f \|_{\psi_2}$. Additionally, in many technical arguments we shall often consider classes $F \subset S_{L_2}$, where $S_{L_2} = \{ f : \| f \|_{L_2} = 1 \}$ is the unit sphere in $L_2(\mu)$.

Let $X_1, X_2, \ldots$ be independent random variables distributed according to $\mu$. Fix $k \geq 1$ and for $f \in F$ define the random variables $Z_f$ and $W_f$ by

$$
Z_f = \frac{1}{k} \sum_{i=1}^{k} f^2(X_i) - \mathbb{E} f^2 \quad \text{and} \quad W_f = \left( \frac{1}{k} \sum_{i=1}^{k} f^2(X_i) \right)^{1/2}.
$$

The first lemma follows easily from Bernstein’s inequality. We state it in the form convenient for further use, and give a proof of one part, for completeness.

Lemma 1.2 There exists an absolute constant $c_1 > 0$ for which the following holds. Let $F \subset S_{L_2}$, $\alpha = \text{diam}(F, \| \cdot \|_{\psi_2})$ and set $k \geq 1$. For every $f, g \in F$ and every $u \geq 2$ we have

$$
P \left( W_{f-g} \geq u \| f - g \|_{\psi_2} \right) \leq 2 \exp(-c_1ku^2).$$

Also, for every $u > 0$,

$$
P \left( |Z_f - Z_g| \geq u \alpha \| f - g \|_{\psi_2} \right) \leq 2 \exp(-c_1k \min(u, u^2)),$$

and

$$
P \left( |Z_f| \geq u \alpha^2 \right) \leq 2 \exp(-c_1k \min(u, u^2)).$$
Proof. We show the standard proof of the first estimate. Other estimates are proved similarly (see e.g., [KM], Lemma 3.2).

Clearly,

\[ E W^2_{f-g} = \frac{1}{k} \mathbb{E} \sum_{i=1}^{k} (f - g)^2(X_i) = \mathbb{E}(f - g)^2(X_1) = \|f - g\|_{L^2}^2. \]

Applying Bernstein’s inequality it follows that for \( t > 0 \),

\[ \mathbb{P} \left( |W^2_{f-g} - \|f - g\|_{L^2}^2| \geq t \right) \leq 2 \exp \left( -ck \min \left( \frac{t}{\|f - g\|^2_{\psi_1}}, \left( \frac{t}{\|f - g\|^2_{\psi_1}} \right)^2 \right) \right). \]

Since \( \|h^2\|_{\psi_1} = \|h\|_{\psi_2}^2 \) for every function \( h \), then letting \( t = (u^2 - 1)\|f - g\|_{\psi_2}^2 \),

\[ \mathbb{P} \left( W^2_{f-g} \geq u^2\|f - g\|_{\psi_2}^2 \right) \leq \mathbb{P} \left( W^2_{f-g} - \|f - g\|_{L^2}^2 \geq (u^2 - 1)\|f - g\|_{\psi_2}^2 \right) \leq 2 \exp \left( -ck \min(u^2/2, u^4/4) \right), \]

as promised. \( \square \)

Now we return to one of the basic notions used in this paper, that of the \( \gamma_2 \)-functional. Let \((T, d)\) be a metric space. Recall that an admissible sequence of \( T \) is a collection of subsets of \( T \), \( \{T_s : s \geq 0\} \), such that for every \( s \geq 1 \), \( |T_s| = 2^s \) and \( |T_0| = 1 \).

**Definition 1.3** For a metric space \((T, d)\) and \( p = 1, 2 \), define

\[ \gamma_p(T, d) = \inf \sup_{t \in T} \sum_{s=0}^{\infty} 2^{s/p} d(t, T_s), \]

where the infimum is taken with respect to all admissible sequences of \( T \). In cases where the metric is clear from the context, we will denote the \( \gamma_p \) functional by \( \gamma_p(T) \).

Set \( \pi_s : T \to T_s \) to be a metric projection function onto \( T_s \), that is, for every \( t \in T \), \( \pi_s(t) \) is a nearest element to \( t \) in \( T_s \) with respect to the metric \( d \). It
is easy to verify by the triangle inequality that for every admissible sequence and every \( t \in T \),

\[
\sum_{s=0}^{\infty} 2^{s/2} d(\pi_{s+1}(t), \pi_s(t)) \leq (1 + 1/\sqrt{2}) \sum_{s=0}^{\infty} 2^{s/2} d(T, T_s)
\]

and that \( \text{diam}(T, d) \leq 2\gamma_2(T, d) \).

We say that a set \( F \) is star-shaped if the fact that \( f \in F \) implies that \( \lambda f \in F \) for every \( 0 \leq \lambda \leq 1 \).

The next Theorem shows that excluding a set with exponentially small probability, \( W_f \) is close to being an isometry in the \( L_2(\mu) \) sense for functions in \( F \) that have a relatively large norm.

**Theorem 1.4** There exist absolute constants \( c, \bar{c} > 0 \) for which the following holds. Let \( F \subset L_2(\mu) \) be star-shaped, \( \alpha = \text{diam}(F, \| \psi_2 \|) \) and \( k \geq 1 \). For any \( 0 < \theta < 1 \), with probability at least \( 1 - \exp(-\bar{c}\theta^2 k/\alpha^4) \), then for all \( f \in F \) satisfying \( \mathbb{E} f^2 \geq r^*_k(\theta)^2 \), we have

\[
(1 - \theta) \mathbb{E} f^2 \leq W^2_f \leq (1 + \theta) \mathbb{E} f^2,
\]

where

\[
r^*_k(\theta) = r^*_k(\theta, F) := \inf \left\{ \rho > 0 : \rho \geq c \alpha \frac{\gamma_2(F \cap \rho S_{L_2}, \| \psi_2 \|)}{\theta \sqrt{k}} \right\}.
\]

The two-sided inequality (1.2) is intimately related to an estimate on \( \sup_{f \in F} |Z_f| \), which, in turn, is based on two ingredients. The first one shows, in the language of the standard chaining approach, that one can control the "end parts" of all chains. Its proof is essentially the same as Lemma 2.3 from [KM].

**Lemma 1.5** There exists an absolute constant \( C \) for which the following holds. Let \( F \subset S_{L_2} \), \( \alpha = \text{diam}(F, \| \psi_2 \|) \) and \( k \geq 1 \). There is \( F' \subset F \) such that \( |F'| \leq 4^k \) and with probability at least \( 1 - \exp(-k) \), we have, for every \( f \in F \),

\[
W_{f-\pi_{F'}(f)} \leq C \gamma_2(F, \| \psi_2 \|)/\sqrt{k},
\]

where \( \pi_{F'}(f) \) is a nearest point to \( f \) in \( F' \) with respect to the \( \psi_2 \) metric.

**Proof.** Let \( \{F_s : s \geq 0\} \) be an “almost optimal” admissible sequence of \( F \). Then for every \( f \in F \),

\[
\sum_{s=0}^{\infty} 2^{s/2} \| \pi_{s+1}(f) - \pi_s(f) \|_{\psi_2} \leq 2\gamma_2(F, \| \psi_2 \|).
\]
Let $s_0$ be the minimal integer such that $2^{s_0} > k$, and let $F' = F_{s_0}$. Then $|F'| \leq 2^{2k} = 4^k$. Write
\[ f - \pi_{s_0}(f) = \sum_{s=s_0}^{\infty} (\pi_{s+1}(f) - \pi_s(f)). \]
Since $W$ is sub-additive then
\[ W_{f - \pi_{s_0}(f)} \leq \sum_{s=s_0}^{\infty} W_{\pi_{s+1}(f) - \pi_s(f)}. \]
For any $f \in F$, $s \geq s_0$ and $\xi \geq 2$, noting that $2^s > k$, it follows by Lemma 1.2 that
\[ \mathbb{P} \left( W_{\pi_{s+1}(f) - \pi_s(f)} \geq \xi \frac{2^{s/2}}{\sqrt{k}} \| \pi_{s+1}(f) - \pi_s(f) \|_{\psi_2} \right) \leq 2 \exp(-c_1 \xi^2 2^s). \quad (1.5) \]
Since $|F_s| \leq 2^{2^s}$, there are at most $2^{2^{s+2}}$ pairs of $\pi_{s+1}(f)$ and $\pi_s(f)$. Thus, for every $s \geq s_0$, the probability of the event from (1.5) holding for some $f \in F$ is less than or equal to $2^{2^{s+2}} \cdot 2 \exp(-c_1 \xi^2 2^s) \leq \exp(2^{s+3} - c_\xi^2 2^s)$, which, for $\xi \geq \xi_0 := \max(4/\sqrt{c_1}, 2)$, does not exceed $\exp(-c_1 \xi^2 2^{s-1})$.

Combining these estimates together it follows that
\[ W_{f - \pi_{s_0}(f)} \leq \xi \sum_{s=s_0}^{\infty} \frac{2^{s/2}}{\sqrt{k}} \| \pi_{s+1}(f) - \pi_s(f) \|_{\psi_2} \leq 2 \xi \frac{\gamma_2(F, \| \|_{\psi_2})}{\sqrt{k}}, \]
outside a set of probability
\[ \sum_{s=s_0}^{\infty} \exp(-c_1 \xi^2 2^{s-1}) \leq \exp(-c_1 \xi^2 2^{s_0}/4). \]
We complete the proof setting, for example $\xi = \max(\xi_0, 2/\sqrt{c_1})$ and recalling that $2^{s_0} > k$.

**Remark 1.6** The proof of the lemma shows that there exist absolute constants $c', c'' > 0$ such that for every $\xi \geq c'$,
\[ \mathbb{P} \left( \sup_{f \in F} W_{f - \pi(f)} \geq \xi \frac{\gamma_2(F, \| \|_{\psi_2})}{\sqrt{k}} \right) \leq \exp(-c'' \xi^2 k), \]
a fact which will be used later.
The next lemma estimates the supremum \( \sup_{f \in F'} |Z_f| \), where the supremum is taken over a subset \( F' \) of \( F \) of a relatively small cardinality, or in other words, over the “beginning part” of a chain. However, in order to get an exponential in \( k \) estimates on probability, we require a separate argument (generic chaining) for the “middle part” of a chain while for the “very beginning” it is sufficient to use a standard concentration estimate.

**Lemma 1.7** There exist absolute constants \( C \) and \( c'' > 0 \) for which the following holds. Let \( F \subset S_{L_2} \) and \( \alpha = \text{diam}(F, \| \psi_2 \|) \). Let \( k \geq 1 \) and \( F' \subset F \) such that \( |F'| \leq 4^k \). Then for every \( w > 0 \),

\[
\sup_{f \in F'} |Z_f| \leq C \frac{\gamma_2(F,\| \psi_2 \|)}{\sqrt{k}} + \alpha^2 w,
\]

with probability larger than or equal to \( 1 - 3 \exp(-c'' k \min(w, w^2)) \).

**Proof.** Let \( (F_s)_{s=0}^{\infty} \) be an almost optimal admissible sequence of \( F' \), set \( s_0 \) to be the minimal integer such that \( 2^{s_0} > 2k \) and fix \( s_1 \leq s_0 \) to be determined later. Since \( |F'| \leq 4^k \), it follows that \( F_s = F' \) for every \( s \geq s_0 \), and that

\[
Z_f - Z_{\pi s_1}(f) = \sum_{s=s_1+1}^{s_0} (Z_{\pi s}(f) - Z_{\pi s-1}(f)).
\]

By Lemma 1.2, for every \( f \in F' \), \( 1 \leq s \leq s_0 \) and \( u > 0 \),

\[
\mathbb{P}( |Z_{\pi s}(f) - Z_{\pi s-1}(f)| \geq u \alpha \sqrt{\frac{2^s}{k}} \| \pi s+1(x) - \pi s(x) \|_{\psi_2} ) \\
\leq 2 \exp(-c_1 \min((u \sqrt{2^s/k}), (u \sqrt{2^s/k})^2)) \\
\leq 2 \exp(-c_1 \min(u, u^2) 2^{s-2}).
\]

(For the latter inequality observe that if \( s \leq s_0 \) then \( 2^s/k \leq 4 \), and thus \( \min((u \sqrt{2^s/k}), (u \sqrt{2^s/k})^2) \geq \min(u, u^2) 2^s/(4k) \).

Taking \( u \) large enough (for example, \( u = 2^5/c_1 \) will suffice) we may ensure that

\[
\sum_{s=s_1+1}^{s_0} 2^{2s+2} \exp(-c_1 u 2^{s-2}) \leq \sum_{s=s_1+1}^{s_0} \exp(-2^{s+3}) \leq \exp(-2^{s_1}).
\]
Therefore, since there are at most $2^{2s_1}$ possible pairs of $\pi_{s+1}(f)$ and $\pi_s(f)$, there is a set of probability at most $\exp(-2s_1)$ such that outside this set we have

$$\sup_{f \in F'} |Z_f - Z_{\pi_{s+1}(f)}| \leq \frac{\alpha u}{\sqrt{k}} \sum_{s=s_1}^{s_0} 2^{s/2} \|\pi_{s+1}(x) - \pi_s(x)\|_2 \leq c' \frac{\gamma_2(F)}{\sqrt{k}}.$$ 

Denote $F_{s_1}$ by $F''$ and observe that $|F''| \leq 2^{2s_1}$. Thus the later estimate implies

$$\sup_{f \in F'} |Z_f| \leq c' \frac{\gamma_2(F)}{\sqrt{k}} + \sup_{g \in F''} |Z_g|.$$ 

Applying Lemma 1.2 for every $w > 0$ we get

$$\mathbb{P} (|Z_g| \geq \alpha^2 w) \leq 2 \exp(-c_1 k \min(w, w^2)).$$

Thus, given $w > 0$, choose $s_1 \leq s_0$ to be the largest integer such that $2^{s_1} < c_1 k \min(w, w^2)/2$. Therefore, outside a set of probability less than or equal to $|F''| 2 \exp(-c_1 k \min(w, w^2)) \leq \exp(-c_1 k \min(w, w^2)/2)$ we have $|Z_g| \leq \alpha^2 w$ for all $g \in F''$. To conclude, outside a set of probability $3 \exp(-c_1 k \min(w, w^2)/2)$,

$$\sup_{f \in F''} |Z_f| \leq c' \alpha \frac{\gamma_2(F)}{\sqrt{k}} + \alpha^2 w,$$

as required. \hfill \blacksquare

**Proof of Theorem 1.4.** Fix an arbitrary $\rho > 0$, and for the purpose of this proof we let $F(\rho) = F/\rho \cap S_{L_2}$, where $F/\rho = \{f/\rho : f \in F\}$.

Our first and main aim is to estimate $\sup_{f \in F(\rho)} |Z_f|$ on a set of probability close to 1.

Fix $u, w > 0$ to be determined later. Let $F' \subset F(\rho)$ be as Lemma 1.3 with $|F'| \leq 4^k$. For every $f \in F(\rho)$ denote by $\pi(f) = \pi_{F'}(f)$ a closest point to $f$ with respect to the $\psi_2$ metric on $F(\rho)$. By writing $f = (f - \pi(f)) + \pi(f)$, it is evident that

$$|Z_f| \leq W^2_{f - \pi(f)} + 2W_{f - \pi(f)} W_{\pi(f)} + |Z_{\pi(f)}|,$$

and thus,

$$\sup_{f \in F(\rho)} |Z_f| \leq \sup_{f \in F(\rho)} W^2_{f - \pi(f)} + 2 \sup_{f \in F(\rho)} W_{f - \pi(f)} \sup_{g \in F'} W_g + \sup_{g \in F'} |Z_g|. \quad (1.6)$$

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Applying Lemma 1.5, the first term in (1.6) is estimated using the fact that
\[ \sup_{f \in F(\rho)} W_{f-\pi(f)} \leq C \frac{\gamma_2(F(\rho), \| \psi_2 \|)}{\sqrt{k}}, \]
with probability at least 1 – exp(–k).

For every \( f \in F(\rho) \) and every \( u > 0 \) we have
\[ \mathbb{P}\{W_f \geq 1 + u\alpha^2\} \leq \mathbb{P}\{W_f^2 \geq 1 + u\alpha^2\} \leq \mathbb{P}\{|Z_f| \geq u\alpha^2\} \]
and, by Lemma 1.2, the latter probability is at most \( 2 \exp(-ck \min(u, u^2)) \), where \( c > 0 \) is an absolute constant.

Combining these two estimates with Lemma 1.7 and substituting into (1.6), \( \sup_{f \in F(\rho)} |Z_f| \) is upper bounded by
\[
C^2 \frac{\gamma_2(F(\rho), \| \psi_2 \|)^2}{k} + C \frac{\gamma_2(F(\rho), \| \psi_2 \|)}{\sqrt{k}} (1 + u\alpha^2) \\
+C'' \frac{\alpha \gamma_2(F(\rho), \| \psi_2 \|)}{\sqrt{k}} + \alpha^2 w, \tag{1.7}
\]
with probability at least 1 – \( 2e^{-k} - 2e^{-ck \min(u, u^2)} - 3e^{-ck \min(w, w^2)} \).

Given \( 0 < \theta < 1 \) we want the condition \( \sup_{f \in F(\rho)} |Z_f| \leq \theta \) to be satisfied with probability close to 1. This can be achieved by imposing suitable conditions on the parameters involved. Namely, if \( u = 1/\alpha^2 < 1 \) and if \( \rho > 0 \) and \( w > 0 \) satisfy
\[
\tilde{C} \alpha \frac{\gamma_2(F(\rho), \| \psi_2 \|)}{\sqrt{k}} \leq \theta/4, \quad C'' \alpha^2 w \leq \theta/4, \tag{1.8}
\]
where \( \tilde{C} = \max(2C, C'') \), then each of the last three terms in (1.7) is less than or equal to \( \theta/4 \), and the first term is less than or equal to \( (\theta/4)^2 \).

In order to ensure that (1.8) holds, we let \( w = \min(1, \theta/(4C''\alpha^2)) \). The above discussion shows that as long as \( \rho \) satisfies
\[
4 \tilde{C} \alpha \frac{\gamma_2(F(\rho), \| \psi_2 \|)}{\theta \sqrt{k}} \leq 1, \tag{1.9}
\]
then \( \sup_{f \in F(\rho)} |Z_f| \leq \theta \) on a set of measure larger than or equal to \( 1 - 7e^{-ck\theta^2/\alpha^4} \), where \( c > 0 \) is an absolute constant. Hence, whenever \( \rho \) satisfies
then (1.2) holds for all \( f \in F(\rho) \). Finally, note that \( \gamma_2(F(\rho), \| \| \psi_2) = (1/\rho)\gamma_2(F \cap \rho S_{L^2}, \| \| \psi_2) \), and thus (1.9) is equivalent to the inequality in the definition of \( r_k^*(\theta) \).

To conclude the proof, for a fixed \( 0 < \theta < 1 \) set \( r = r_k^*(\theta) \), with \( c = 4 \tilde{C} \) being the constant from (1.9). Note that if \( X_1, \ldots, X_k \) satisfy (1.2) for all \( f \in F(r) \) then, since \( F \) is star-shaped, the homogeneity of this condition implies that the same holds for all \( f \in F \) with \( E f^2 \geq r^2 \), as claimed.

Let us note two consequences for the supremum of the process \( Z_f \), which is of independent interest.

**Corollary 1.8** There exist absolute constants \( C', c' > 0 \) for which the following holds. Let \( F \subset S_{L^2} \), \( \alpha = \text{diam}(F, \| \| \psi_2) \) and \( k \geq 1 \). With probability at least \( 1 - \exp (-c' \gamma_2^2(F, \| \| \psi_2)/\alpha^3) \) one has

\[
\sup_{f \in F} |Z_f| \leq C' \alpha \max \left( \frac{\gamma_2(F, \| \| \psi_2)}{\sqrt{k}}, \frac{\gamma_2^2(F, \| \| \psi_2)}{k} \right).
\]

Moreover, if \( F \) is symmetric,

\[
\mathbb{E} \sup_{f \in F} |Z_f| \leq C' \alpha \max \left( \frac{\gamma_2(F, \| \| \psi_2)}{\sqrt{k}}, \frac{\gamma_2^2(F, \| \| \psi_2)}{k} \right).
\]

**Proof.** This follows from the proof of Theorem 1.4 with \( \rho = 1 \). More precisely, the first part is a direct consequence of (1.7).

For the “moreover part” first use (1.6) for expectations, estimate the middle term by Cauchy-Schwarz inequality and note that \( W^2_g \leq 1 + Z_g \) for all \( g \in F' \) to yield that in order to estimate \( \mathbb{E} \sup_{f \in F} |Z_f| \) it suffice to bound

\[
\mathbb{E} \sup_{f \in F'} |Z_f| \quad \text{and} \quad \mathbb{E} \sup_{f \in F} W^2_{f - \pi(f)}.
\]

For simplicity denote \( \gamma_2(F, \| \| \psi_2) \) by \( \gamma_2(F) \) and let us begin with the second term. Applying Remark 1.6 and setting \( G = \{ f - \pi(f) : f \in F \} \) and \( u = c' \gamma_2(F)/\sqrt{k} \), where \( c' \) is the constant from the remark, we obtain

\[
\int_0^\infty \mathbb{P} \left( \sup_{g \in G} W^2_g \geq t \right) dt \leq u^2 + \int_{u^2}^\infty \mathbb{P} \left( \sup_{g \in G} W^2_g \geq t \right) dt \\
\leq u^2 + u^2 \int_1^\infty \exp(-c''vk).dv
\]
where the last inequality follows by changing the integration variable to \( t = u^2 v \). This implies that

\[
\mathbb{E} \sup_{f \in F} W_2^2 \pi(f) \leq C' \gamma_2^2(F, \| \psi_2 \|_k),
\]

for some absolute constant \( C' > 1 \).

Next, we have to bound \( \mathbb{E} \sup_{f \in F'} |Z_f| \), and to that end we use Lemma 1.7. Setting \( u = 2C\alpha \gamma_2(F)/\sqrt{k} \), and then changing the integration variable to \( t = u/2 + \alpha^2 w \), it is evident that

\[
\int_0^\infty \mathbb{P} \left( \sup_{f \in F'} |Z_f| \geq t \right) dt \leq u + \int_u^\infty \mathbb{P} \left( \sup_{f \in F'} |Z_f| \geq t \right) dt
\]

\[
= u + \alpha^2 \int_{u/2\alpha^2}^\infty \mathbb{P} \left( \sup_{f \in F'} |Z_f| \geq \frac{u}{2} + \alpha^2 w \right) dw
\]

\[
\leq u + 3\alpha^2 \int_{u/2\alpha^2}^\infty \exp \left( -c'' k \min(w^2, w) \right) dw.
\]

Changing variables in the last integral \( w = ru/2\alpha^2 \) and using the fact that \( \gamma_2(F) \geq 1 \) for a symmetric set \( F \) in the unit sphere, the last expression is bounded above by

\[
u + (3/2)u \int_1^\infty \exp \left( -c'' \min \left( \frac{r}{2\alpha^2}, \left( \frac{r}{2\alpha^2} \right)^2 \right) \right) dr = C'u,
\]

where \( C > 0 \) is an absolute constant.

2 Subgaussian Operators

We now illustrate the general result of Section 1 in the case of linear processes, which was the topic that motivated our study. The processes correspond then to random matrices with rows distributed according to measures on \( \mathbb{R}^n \) satisfying some natural geometric conditions. Our result imply concentration estimates for related random subgaussian operators, which eventually lead to the desired reconstruction results for linear measurements for general sets.

The fundamental result that allows us to pass from the purely metric statement of the previous section to the geometric result we present below.
follows from Talagrand’s lower bound on the expectation of the supremum of a Gaussian process in terms of $\gamma_2$ of the indexing set. To present our result, the starting point is the fundamental definition of the $\ell_*$-functional (which is in fact the so-called $\ell$-functional of a polar set).

**Definition 2.1** Let $T \subset \mathbb{R}^n$ and let $g_1, ..., g_n$ be independent standard Gaussian random variables. Denote by $\ell_*(T) = \mathbb{E} \sup_{t \in T} |\sum_{i=1}^n g_it_i|$, where $t = (t_i)_{i=1}^n \in \mathbb{R}^n$.

There is a close connection between the $\ell_*$- and $\gamma_2$- functionals given by the majorizing measure Theorem. Let $\{G_t : t \in T\}$ be a Gaussian process indexed by a set $T$, and for every $s, t \in T$, let $d^2(s, t) = \mathbb{E}|G_s - G_t|^2$. Then

$$c_2 \gamma_2(T, d) \leq \mathbb{E} \sup_{t \in T} |G_t| \leq c_3 \gamma_2(T, d),$$

where $c_2, c_3 > 0$ are absolute constants. The upper bound is due to Fernique and the lower bound was established by Talagrand. The proof of both parts and the most recent survey on the topic can be found in [Ta2].

In particular, if $T \subset \mathbb{R}^n$ and $G_t = \sum g_it_i$, then $d(s, t) = |s - t|$, and thus

$$c_2 \gamma_2(T, | |) \leq \ell_*(T) \leq c_3 \gamma_2(T, | |), \quad (2.1)$$

**Definition 2.2** A probability measure $\mu$ on $\mathbb{R}^n$ is called isotropic if for every $y \in \mathbb{R}^n$, $\mathbb{E}|\langle X, y \rangle|^2 = |y|^2$, where $X$ is distributed according to $\mu$.

A measure $\mu$ satisfies a $\psi_2$ condition with a constant $\alpha$ if for every $y \in \mathbb{R}^n$,

$$\|\langle X, y \rangle\|_{\psi_2} \leq \alpha|y|.$$

A subgaussian or $\psi_2$ operator is a random operator of the form

$$\Gamma = \sum_{i=1}^k \langle X_i, \cdot \rangle e_i \quad (2.2)$$

where the $X_i$ are distributed according to an isotropic $\psi_2$ measure.

Perhaps the most important example of an isotropic $\psi_2$ probability measure on $\mathbb{R}^n$ with a bounded constant other than the Gaussian measure is the uniform measure on $\{-1, 1\}^n$. Naturally, if $X$ is distributed according to a general isotropic $\psi_2$ measure then the coordinates of $X$ need no longer be independent. For example, the normalized Lebesgue measure on an appropriate multiple of the unit ball in $\ell_p^n$ for $2 \leq p \leq \infty$ is an isotropic $\psi_2$ measure.
with a constant independent of $n$ and $p$. For more details on such measures see [MP].

For a set $T \subset \mathbb{R}^n$ and $\rho > 0$ let

$$T_\rho = T \cap \rho S^{n-1}. \quad (2.3)$$

The next result shows that given $T \subset \mathbb{R}^n$, subgaussian operators are very close to being an isometry on the subset of elements of $T$ which have a “large enough” norm.

**Theorem 2.3** There exist absolute constants $c, \bar{c} > 0$ for which the following holds. Let $T \subset \mathbb{R}^n$ be a star-shaped set. Let $\mu$ be an isotropic $\psi_2$ probability measure with constant $\alpha \geq 1$. Let $k \geq 1$, and $X_1, \ldots, X_k$ be independent, distributed according to $\mu$ and define $\Gamma$ by (2.2). For $0 < \theta < 1$, with probability at least $1 - \exp(-\bar{c}^2 k/\alpha^4)$, then for all $x \in T$ such that $|x| \geq r_k^*(\theta)$, we have

$$(1 - \theta)|x|^2 \leq \frac{|\Gamma x|^2}{k} \leq (1 + \theta)|x|^2, \quad (2.4)$$

where

$$r_k^*(\theta) = r_k^*(\theta, T) := \inf \left\{ \rho > 0 : \rho \geq c \alpha^2 \ell_*(T_\rho) / (\theta \sqrt{k}) \right\}. \quad (2.5)$$

In particular, with the same probability, every $x \in T$ satisfies

$$|x|^2 \leq \max \left\{ (1 - \theta)^{-1} |\Gamma x|^2 / k, \ r_k^*(\theta)^2 \right\}. \quad (2.6)$$

**Proof.** We use Theorem 1.4 for the set of functions $F$ consisting of linear functionals of the form $f = f_x = \langle \cdot, x \rangle$, for $x \in T$. By the isotropicity of $\mu$, $\|f\|_{L_2} = |x|$ for $f = f_x \in F$. Also, since $\mu$ is $\psi_2$ with constant $\alpha$ then it follows by (2.1) that for all $\rho > 0$,

$$\gamma_2(F \cap \rho S_{L_2}, \| \psi_2 \) \leq \alpha \gamma_2(F \cap \rho S_{L_2}, \| \psi_2) \leq (\alpha/c) \ell_*(T_\rho),$$

as promised.

**Remark 2.4** It is clear from the proof of Theorem 1.4 that the upper estimates in (2.4) hold for $\theta \geq 1$ as well, with appropriate probability estimates and a modified expression for $r_k^*$ in (2.5). Note that of course in this case the lower estimate in (2.4) became vacuous. The same remark is valid for the estimate in (1.2) as well.

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The last result immediately leads to an estimate for the diameter of random sections of a set $T$ in $\mathbb{R}^n$, given by kernels of random operators $\Gamma$, and which is a $\psi_2$-counterpart of the main result from [PT1] (see also [PT2]).

**Corollary 2.5** There exist absolute constants $\tilde{c}, \tilde{c}' > 0$ for which the following holds. Let $T \subset \mathbb{R}^n$ be a star-shaped set and let $\mu$ be an isotropic $\psi_2$ probability measure with constant $\alpha \geq 1$. Set $k \geq 1$, put $X_1, \ldots, X_k$ to be independent, distributed according to $\mu$ and define $\Gamma$ by (2.2). Then, with probability at least $1 - \exp(-\tilde{c}k/\alpha^4)$,

$$\text{diam}(\ker \Gamma \cap T) \leq r^*_k(T),$$

where

$$r^*_k = r^*_k(T) := \inf\left\{ \rho > 0 : \rho \geq \tilde{c}' \alpha^2 \ell_*(T_{\rho}/\sqrt{k}) \right\}.$$  \hspace{1cm} (2.6)

Moreover, with the same probability, $\text{diam}(\ker \Gamma \cap T) \leq \tilde{c}' \alpha^2 \ell_*(T)/\sqrt{k}$.

The Gaussian case (that is, when $\mu$ is the standard Gaussian measure on $\mathbb{R}^n$), although not explicitly stated in [PT1], follows immediately from the proof in that paper. The parameter $\inf\{ \rho > 0 : \ell_*(T \cap \rho B_2^n) \leq C\rho \sqrt{k} \}$ was introduced in [PT2].

A version of Corollary 2.5 for random $\pm 1$-vectors follows from the result in [A], as observed in [MP].

**Proof of Corollary 2.5.** Applying Theorem 2.3 with $\theta = 1/2$, say, we get that if $x \in T$ and $|x| \geq r^*_k(1/2)$ then $\Gamma x \neq 0$. Thus for $x \in \ker \Gamma \cap T$ we have $|x| \leq r^*_k(1/2)$ and the first conclusion follows by adjusting the constants.

Observe that since the function $\ell_*(T_{\rho})/\rho = \ell_*((1/\rho)T \cap S^{n-1})$ is decreasing in $\rho$ then $r^*_k$ actually satisfies the equality in the defining formula (2.6). Combining this and the obvious estimate $\ell_*(T_{\rho}) \leq \ell_*(T)$, concludes the “moreover” part.

Finally, let us note a special case of Theorem 2.3 for subsets of the sphere.

**Corollary 2.6** Let $T \subset S^{n-1}$ and let $\mu$, $\alpha$, $k$, $X_i$, $\Gamma$ and $\theta$ be the same as in Theorem 2.3. As long as $k$ satisfies $k \geq (c' \alpha^4/\theta^2)\ell_*(T)^2$, then with probability at least $1 - \exp(-\tilde{c} \theta^2 k/\alpha^4)$, for all $x \in T$,

$$1 - \theta \leq \frac{|\Gamma x|}{\sqrt{k}} \leq 1 + \theta,$$ \hspace{1cm} (2.7)

where $c, \tilde{c} > 0$ are absolute constants.
Proof. Let $c, \bar{c} > 0$ be the constants from Theorem 2.3. Observe that the condition on $k$, with $c' = c^2$, is equivalent to $r_k^*(\theta, \bar{T}) \leq 1$, where $\bar{T} = \{\lambda x : x \in T, 0 \leq \lambda \leq 1\}$. Then (2.7) immediately follows from (2.4).

3 Approximate reconstruction

Next, we show how one can apply Theorem 2.3 to reconstruct any fixed $v \in T$ for any set $T \subset \mathbb{R}^n$, where the data at hand are linear subgaussian measurements of the form $\langle X_i, v \rangle$.

The reconstruction algorithm we choose is as follows: for a fixed $\varepsilon > 0$, find some $t \in T$ such that

$$\left( \frac{1}{k} \sum_{i=1}^{k} (\langle X_i, v \rangle - \langle X_i, t \rangle)^2 \right)^{1/2} \leq \varepsilon.$$ 

The fact that we only need to find $t$ for which $(\langle X_i, t \rangle)_{i=1}^{k}$ is close to $(\langle X_i, v \rangle)_{i=1}^{k}$ rather than equal to it, is very important algorithmically because it is a far simpler problem.

Let us show why such an algorithm can be used to solve the approximate reconstruction problem.

Consider $\bar{T} = \{\lambda(t - s) : t, s \in T, 0 \leq \lambda \leq 1\}$ and observe that by Theorem 2.3 for every $0 < \theta < 1$, with high probability, every such $t \in T$ satisfies that

$$|t - v| \leq \frac{\varepsilon}{1 - \theta} + r_k^*(\theta, \bar{T}).$$

Hence, to bound the reconstruction error, one needs to estimate $r_k^*(\theta, \bar{T})$. Of course, if $T$ happens to be convex and symmetric then $\bar{T} \subset 2T$ which is star-shaped and thus

$$|t - v| \leq \frac{\varepsilon}{1 - \theta} + r_k^*(\theta, 2T).$$

In a more general case, when $T$ is symmetric and quasi-convex with constant $a \geq 1$, (i.e., $T + T \subset 2aT$ and $T$ is star-shaped), then

$$|t - v| \leq \frac{\varepsilon}{1 - \theta} + r_k^*(\theta, aT).$$

Therefore, in the quasi-convex case, the ability to approximate any point in $T$ using this kind of random sampling depends on the expectation of the
The supremum of a Gaussian process indexed by the intersection of \( T \) and a sphere of a radius \( \rho \) as a function of the radius. For a general set \( T \), the reconstruction error is controlled by the behavior of the expectation of the supremum of the Gaussian process indexed by the intersection of \( T \) with spheres of radius \( \rho \), and this function of \( \rho \) is just the modulus of continuity of the Gaussian process indexed by the set \( \{ \lambda t : 0 \leq \lambda \leq 1, \ t \in T \} \) (i.e., the expectation of the supremum of the Gaussian process indexed by the set \( \{ \lambda(t-s) : 0 \leq \lambda \leq 1, \ t,s \in T, \ |t-s| = \rho \} \)).

The parameters \( r^*_k(\theta, T) \) can be estimated for the unit ball of classical normed or quasi-normed spaces. The two examples we consider here are the unit ball in \( \ell_1^n \), denoted by \( B_{1,n} \), and the unit balls in the \( \ell_p^n \) spaces \( \ell_p^n \) for \( 0 < p < 1 \), denoted by \( B_{p,n} \). Recall that \( B_{p,n} \) is the set of all \( x = (x_i)_{i=1}^n \in \mathbb{R}^n \) such that the cardinality \( |\{i : |x_i| \geq s\}| \leq s^{-p} \) for all \( s > 0 \), and observe that \( B_{p,n} \) is a quasi convex body with constant \( a = 2^{1/p} \). Let us mention that there is nothing “magical” about the examples we consider here. Those are simply the cases considered in [CT1, RV].

In order to bound \( r^*_k \) for these sets we shall use the approach from [GLMP], and combine it with Theorem 2.3 to recover and extend the results from [CT1, RV].

**Theorem 3.1** There is an absolute constant \( \bar{c} \) for which the following holds. Let \( 1 \leq k \leq n \) and \( 0 < \theta < 1 \), and set \( \varepsilon > 0 \). Let \( \mu \) be an isotropic \( \psi_2 \) probability measure on \( \mathbb{R}^n \) with constant \( \alpha \), and let \( X_1, \ldots, X_k \) be independent, distributed according to \( \mu \). For any \( 0 < p < 1 \), with probability at least \( 1 - \exp(-c\theta^2 k/\alpha^4) \), if \( v, y \in B_{p,n} \) satisfy that \( (\sum_{i=1}^k (X_i, v - y)^2 / k)^{1/2} \leq \varepsilon \), then

\[
|y - v| \leq \frac{\varepsilon}{1 - \theta} + 2^{1/p+1} \left( \frac{1}{p} - 1 \right)^{-1} \left( C_{\alpha, \theta} \frac{\log(C_{\alpha, \theta} n/k)}{k} \right)^{1/p-1/2},
\]

where \( C_{\alpha, \theta} = c\alpha^4/\theta^2 \) and \( c > 0 \) is an absolute constant.

If \( v, y \in B_{1,n} \) satisfy the same assumption then with the same probability estimate,

\[
|y - v| \leq \frac{\varepsilon}{1 - \theta} + \left( C_{\alpha, \theta} \frac{\log(C_{\alpha, \theta} n/k)}{k} \right)^{1/2}.
\]

To prove Theorem 3.1 we require the following elementary fact.
Lemma 3.2 Let $0 < p < 1$ and $1 \leq m \leq n$. Then, for every $x \in \mathbb{R}^n$,

$$\sup_{z \in B_{p,\infty} \cap \rho B_2^n} \langle x, z \rangle \leq 2\rho \left( \sum_{i=1}^{m} x_i^* \right)^{1/2},$$

where $\rho = (1/p - 1)^{-1} m^{1/2 - 1/p}$ and $(x_i^*)_{i=1}^m$ is a non-increasing rearrangement of $(|x_i|)_{i=1}^n$.

Moreover,

$$\sup_{z \in B_{1,\infty} \cap \rho B_2^n} \langle x, z \rangle \leq 2\rho \left( \sum_{i=1}^{m} x_i^2 \right)^{1/2},$$

with $\rho = 1/\sqrt{m}$.

Proof. We will present a proof for the case $0 < p < 1$. The case of $B_1^n$ is similar.

Recall a well known fact that for two sequences of positive numbers $a_i, b_i$ such that $a_1 \geq a_2 \geq \ldots$, the sum $\sum a_i b_{\pi(i)}$ is maximal over all permutations $\pi$ of the index set, if $b_{\pi(1)} \geq b_{\pi(2)} \geq \ldots$. It follows that, for any $\rho > 0, m \geq 1$ and $z \in B_{p,\infty} \cap \rho B_2^n$,

$$\langle x, z \rangle \leq \rho \left( \sum_{i=1}^{m} x_i^* \right)^{1/2} + \sum_{i>m} x_i^* \frac{1}{i^{1/p}} \leq \rho \left( \sum_{i=1}^{m} x_i^2 \right)^{1/2} \left( \frac{1}{\sqrt{m}} \sum_{i>m} \frac{1}{i^{1/p}} \right) \leq \rho \left( \sum_{i=1}^{m} x_i^2 \right)^{1/2} \left( \rho + \left( \frac{1}{p} - 1 \right)^{-1} \frac{1}{m^{1/p - 1/2}} \right).$$

By the definition of $\rho$, this completes the proof.

Consider the set of elements in the unit ball with “short support”, defined by

$$U_m = \{ x \in S^{n-1} : |\{ i : x_i \neq 0 \}| \leq m \}.$$

Note that Lemma 3.2 combined with a duality argument implies that for every $1 \leq m \leq n$ and every $I \subset \{1, \ldots, n\}$ with $|I| \leq m$,

$$\sqrt{|I|} B_1^n \cap S^{n-1} \subset 2 \text{conv} \ U_m \cap S^{n-1}. \quad (3.1)$$
The next step is to bound the expectation of the supremum of the Gaussian process indexed by \( U_m \).

**Lemma 3.3** There exist an absolute constant \( c \) such that for every \( 1 \leq m \leq n \),

\[
\ell_*(\text{conv} \ U_m) \leq c \sqrt{m \log(cn/m)}.
\]

**Proof.** Recall that for every \( 1 \leq m \leq n \), there is a set \( \Lambda_m \) of cardinality at most \( 5^m \) such that \( B_2^m \subset 2 \text{ conv } \Lambda_m \) (for example, a successive approximation shows that we may take as \( \Lambda_m \) an \( 1/2 \)-net in \( B_2^m \)). Hence there is a subset of \( B_2^m \) of cardinality at most \( 5^m \binom{n}{m} \) such that \( U_m \subset 2 \text{ conv } \Lambda \). It is well known (see for example [LT]) that for every \( T \subset B_n^2 \),

\[
\ell_*(\text{conv} \ T) = \ell_*(T) \leq c \sqrt{\log(|T|)},
\]

and thus,

\[
\ell_*(\text{conv} \ U_m) \leq c \sqrt{\log \left( 5^m \binom{n}{m} \right)},
\]

from which the claim follows. \( \blacksquare \)

Finally, we are ready to estimate \( r^*_{k}(\theta, B_{p,\infty}^n) \) and \( r^*_{k}(\theta, B_1^n) \).

**Lemma 3.4** There exists an absolute constant \( c \) such that for any \( 0 < p < 1 \) and \( 1 \leq k \leq n \),

\[
r^*_{k}(\theta, B_{p,\infty}^n) \leq c \left( \frac{1}{p} - 1 \right)^{-1} \left( \frac{\log(cn\alpha^4/\theta^2 k)}{\theta^2 k/\alpha^4} \right)^{1/p-1/2}
\]

and

\[
r^*_{k}(\theta, B_1^n) \leq c \left( \frac{\log(cn\alpha^4/\theta^2 k)}{\theta^2 k/\alpha^4} \right)^{1/2}.
\]

**Proof.** Again, we present a proof for \( B_{p,\infty}^n \), while the treatment of \( B_1^n \) is similar and thus omitted.

Let \( 0 < p < 1 \) and \( 1 \leq k \leq n \), and set \( 1 \leq m \leq n \) to be determined later. Clearly,

\[
\left( \sum_{i=1}^{m} x_i^* \right)^{1/2} = \sup_{y \in \mathcal{U}_m} \langle x, y \rangle,
\]

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and thus, by Lemma 3.2, 
\[ \| \ell_\ast(B_{b,\infty}^n \cap \rho B_2^n) \| \leq 2\rho \ell_\ast(U_m), \]
where \( \rho = (1/p - 1)^{-1} m^{1/2 - 1/p} \). From the definition of \( r_k^\ast(\theta) \) in Theorem 2.3, it suffices to determine \( m \) (and thus \( \rho \)) such that
\[ c \alpha^2 \ell_\ast(U_m) \leq \theta \sqrt{k}, \]
which by Lemma 3.3 comes to \( c \alpha^2 \sqrt{m \log (cn/m)} \leq \theta \sqrt{k} \) for some other numerical constant \( c \). It is standard to verify that the last inequality holds true provided that
\[ m \leq c \frac{\theta^2 k / \alpha^4}{\log (cn \alpha^4 / \theta^2 k)}, \]
and thus
\[ r_k^\ast(\theta, B_{p,\infty}^n) \leq c \left( \frac{1}{p} - 1 \right)^{-1} \left( \frac{\log (cn \alpha^4 / \theta^2 k)}{\theta^2 k / \alpha^4} \right)^{1/p - 1/2}. \]

Proof of Corollary 3.1. The proof follows immediately from Theorem 2.3 and Lemma 3.4.

4 Exact reconstruction

Let us consider the following problem from the error correcting code theory. A linear code is given by an \( n \times (n - k) \) real matrix \( A \). Thus, a vector \( x \in \mathbb{R}^{n-k} \) generates the vector \( Ax \in \mathbb{R}^n \). Suppose that \( Ax \) is corrupted by a noise vector \( z \in \mathbb{R}^n \) and the assumption we make is that \( z \) is sparse, that is, has a short support, which we denote by \( \text{supp}(z) = \{ i : z_i \neq 0 \} \). The problem is to reconstruct \( x \) from the data, which is the noisy output \( y = Ax + z \).

For this purpose, consider a \( k \times n \) matrix \( \Gamma \) such that \( \Gamma A = 0 \). Thus \( \Gamma z = \Gamma y \) and correcting the noise is reduced to identifying the sparse vector \( z \) (rather than approximating it) from the data \( \Gamma z \) - which is the problem we focus on here.

In this context, a linear programming approach called the basis pursuit algorithm, was recently shown to be relevant for this goal [CDS]. This method is the following minimization problem
\[ (P) \quad \min \| t \|_{\ell_1}, \quad \Gamma t = \Gamma z \]
(and recall that the $\ell_1$-norm is defined by $\|t\|_{\ell_1} = \sum_{i=1}^{n} |t_i|$ for any $t = (t_i)_{i=1}^{n} \in \mathbb{R}^n$).

For the analysis of the reconstruction of sparse vectors by this basis pursuit algorithm, we refer to [CDS] and the recent papers [CT2, CT3].

In this section, we show that if $\Gamma$ is an isotropic $\psi_2$ matrix then with high probability, for any vector $z$ whose support has size less than $Ck \log(cn/k)$ (for some absolute constant $C$), the problem $(P)$ above has a unique solution that is equal to $z$. It means that such random matrices can be used to reconstruct any sparse vector, as long as the size of the support is not too large. This extends the recent result proved in [CT2] and [RV] for Gaussian matrices.

**Theorem 4.1** There exist absolute constants $c, C$ and $\bar{c}$ for which the following holds. Let $\mu$ be an isotropic $\psi_2$ probability measure with constant $\alpha \geq 1$. For $1 \leq k \leq n$, set $X_1, \ldots, X_k$ to be independent, distributed according to $\mu$ and let $\Gamma = \sum_{i=1}^{k} \langle X_i, \cdot \rangle e_i$. Then with probability at least $1 - \exp(-\bar{c}k/\alpha^4)$,

$$|\text{supp}(z)| \leq \frac{Ck}{\alpha^4 \log(cn^4/k)}$$

is the unique minimizer of the problem

$$\text{(P)} \quad \min_{t \in \mathbb{R}^n} \|t\|_{\ell_1}, \quad \Gamma t = \Gamma z.$$

The proof of Theorem 4.1 is based on a scheme of the proof of [CT2], but is simpler and more general, as it holds for an arbitrary isotropic $\psi_2$ random matrix.

Let us remark that problem (P) is equivalent to the following one

$$\text{(P')} \quad \min_{t \in \mathbb{R}^n} \|y - At\|_{\ell_1}$$

where $\Gamma A = 0$. Thus we obtain the reconstruction result:

**Corollary 4.2** Let $A$ be a $n \times (n-k)$ matrix. Set $\Gamma$ to be a $k \times n$ matrix that satisfies the conclusion of the previous Theorem with the constants $c$ and $C$, and for which $\Gamma A = 0$. For any $x \in \mathbb{R}^{n-k}$ and any $y = Ax + z$, if $|\text{supp}(z)| \leq \frac{Ck}{\log(cn/k)}$, then $x$ is the unique minimizer of the problem

$$\min_{t \in \mathbb{R}^n} \|y - At\|_{\ell_1}.$$
The condition $\Gamma A = 0$ means that the range of $A$ is a subspace of the kernel of $\Gamma$. Due to the rotation invariance of a Gaussian matrix (from both sides), the range and the kernel are random elements of the Grassmann manifold of the corresponding dimensions. Therefore, random Gaussian matrices $A$ and $\Gamma$ satisfy the conclusion of Corollary 4.2.

4.1 Proof of Theorem 4.1

As in [CT2], the proof consists of finding a simple condition for a fixed matrix $\Gamma$ to satisfy the conclusion of our Theorem. We then apply a result from the previous section to show that random matrices satisfy this condition.

The first step is to provide some criteria which ensure that the problem $\left(P\right)$ has a unique solution as specified in Theorem 4.1. This convex optimization problem can be represented as a linear programming problem. Indeed, let $z \in \mathbb{R}^n$ and set

$$I^+ = \{ i : z_i > 0 \}, \quad I^- = \{ i : z_i < 0 \}, \quad I = I^+ \cup I^- \quad \text{(4.1)}$$

Denote by $C$ the cone of constraint

$$C = \{ t \in \mathbb{R}^n : \sum_{i \in I^+} t_i - \sum_{i \in I^-} t_i + \sum_{i \in I^c} |t_i| \leq 0 \}$$

corresponding to the $\ell_1$-norm.

Note that if $|t|$ is small enough then $\|z+t\|_{\ell_1} = \sum_{i \in I^+} (z_i+t_i) - \sum_{i \in I^-} (z_i+t_i) + \sum_{i \in I^c} |t_i|$. Thus, the solution of $\left(P\right)$ is unique and equals to $z$ if and only if

$$\ker \Gamma \cap C = \{ 0 \} \quad \text{(4.2)}$$

By the Hahn-Banach separation Theorem, the latter is equivalent to the existence of a linear form $\tilde{w} \in \mathbb{R}^n$ vanishing on $\ker \Gamma$ and positive on $C \setminus \{ 0 \}$.

After appropriate normalization, it is easy to check that such an $\tilde{w}$ satisfies that $\tilde{w} = \sum_{i=1}^k \alpha_i X_i$, $\langle \tilde{w}, e_i \rangle = 1$ for all $i \in I^+$, $\langle \tilde{w}, e_i \rangle = -1$ for all $i \in I^-$, and $|\langle \tilde{w}, e_i \rangle| < 1$ for all $i \in I^c$. Setting $w = \sum_{i=1}^k \alpha_i e_i$ and noticing that $\langle \tilde{w}, e_i \rangle = \langle w, \Gamma e_i \rangle$ we arrive at the following criterion.

**Lemma 4.3** Let $\Gamma$ be a $k \times n$ matrix and $z \in \mathbb{R}^n$. With the notation (4.1), the problem

$$(P) \quad \min \| t \|_{\ell_1}, \quad \Gamma t = \Gamma z$$

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has a unique solution which equals to $z$, if and only if there exists $w \in \mathbb{R}^k$ such that
\[
\forall i \in I^+ \langle w, \Gamma e_i \rangle = 1, \quad \forall i \in I^- \langle w, \Gamma e_i \rangle = -1, \quad \forall i \in I^c |\langle w, \Gamma e_i \rangle| < 1.
\]

The second preliminary result we require follows from Corollary 2.6 and the estimates of the previous section.

**Theorem 4.4** There exist absolute constants $c, C$ and $\bar{c}$ for which the following holds. Let $\mu, \alpha, k$ and $\Gamma$ be as in Theorem 4.1. Then, for every $0 < \theta < 1$, with probability at least $1 - \exp(-\bar{c} \theta^2 k/\alpha^4)$, every $x \in 2 \text{conv } U_{4m} \cap S^{n-1}$ satisfies that
\[
(1 - \theta)|x|^2 \leq \frac{|\Gamma x|^2}{k} \leq (1 + \theta)|x|^2, \tag{4.3}
\]
provided that
\[
m \leq C\frac{\theta^2 k/\alpha^4}{\log(c \alpha^4/\theta^2 k)}.
\]

**Proof.** Applying Corollary 2.6 to $T = 2 \text{conv } U_{4m} \cap S^{n-1}$, we only have to check that $k \geq (c' \alpha^4/\theta^2) \ell_*(T)^2$, which from Lemma 3.3 reduces to verifying that $k \geq (c' \alpha^4/\theta^2) cm \log(cn/m)$. The conclusion now follows from the same computation as in the proof of Lemma 3.4.

**Proof of Theorem 4.1** Observe that if $t \in C \cap S^{n-1}$ then $\|t\|_{t_1} \leq 2 \sum_{i \in I} |t_i| \leq 2\sqrt{|I|}$, where $I$ is the support of $z$. Hence,
\[
C \cap S^{n-1} \subset \sqrt{|I|} B^n_1 \cap S^{n-1}.
\]

This inclusion and condition (4.2) clearly imply that if $\Gamma$ does not vanish on any point of $\sqrt{|I|} B^n_1 \cap S^{n-1}$, then the solution of $(P)$ is unique and equals to $z$. By (3.1) we have
\[
\sqrt{4|I|} B^n_1 \cap S^{n-1} \subset 2 \text{conv } U_{4m} \cap S^{n-1}.
\]

Therefore, if $\Gamma$ does not vanish on any point of $2 \text{conv } U_{4m} \cap S^{n-1}$ then $z$ is the unique solution of $(P)$. Applying Theorem 4.4, the lower bound in (4.3) shows that indeed, $\Gamma$ does not vanish on any point of the required set, provided that
\[
m \leq Ck \frac{\alpha^4 \log(c \alpha^4/k)}{k}
\]
for some suitable constants $c$ and $C$. 

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4.2 The geometry of faces of random polytopes

Next, we investigate the geometry of random polytopes. Let $\Gamma$ be a $k \times n$ isotropic $\psi_2$ matrix. For $1 \leq i \leq n$ let $v_i = \Gamma(e_i)$ be the vector columns of the matrix $\Gamma$ and set $K^+(\Gamma)$ (resp., $K(\Gamma)$) to be the convex hull (resp., the symmetric convex hull) of these vectors.

In this situation, the random model that makes sense is when $X = (x_i)_{i=1}^n$, where $(x_i)_{i=1}^n$ are independent, identically distributed random variables for which $\mathbb{E}|x_i|^2 = 1$ and $\|x_i\|_{\psi_2} \leq \alpha$. It is standard to verify that in this case $X = (x_i)_{i=1}^n$ is an isotropic $\psi_2$ vector with constant $\alpha$, and moreover, each vertex of the polytope is given by $v_i = (x_{i,j})^k_{j=1}$.

A polytope is called $m$-neighborly if any set of less than $m$ vertices is the vertex set of a face. In the symmetric setting, we will say that a symmetric polytope is $m$-symmetric-neighborly if any set of less than $m$ vertices containing no-opposite pairs, is the vertex set of a face.

The condition of Lemma 4.3 may be reformulated by saying that the set $\{v_i : i \in I^+\} \cup \{-v_i : i \in I^-\}$ is the vertex set of a face of the polytope $K(\Gamma)$. Thus, the condition for the exact reconstruction using the basis pursuit method for any vector $z$ with $|\text{supp}(z)| \leq m$ may be reformulated as a geometric property of the polytope $K(\Gamma)$ (see [CT2, RV]): namely, that for all disjoint subsets $I^+$ and $I^-$ of $\{1, \ldots, n\}$ such that $|I^+| + |I^-| \leq m$, the set $\{v_i : i \in I^+\} \cup \{-v_i : i \in I^-\}$ is the vertex set of a face of the polytope $K(\Gamma)$. That is, $K(\Gamma)$ is $m$-symmetric-neighborly. A similar analysis may be done in the non-symmetric case, for $K^+(\Gamma)$, where now $I^-$ is empty.

Lemma 4.5 Let $\Gamma$, $K(\Gamma)$ and $K^+(\Gamma)$ be as above. Then the problem

$$(P) \quad \min \|t\|_{\ell_1}, \quad \Gamma t = \Gamma z$$

has a unique solution which equals to $z$ for any vector $z$ (resp., $z \geq 0$) such that $|\text{supp}(z)| \leq m$, if and only if $K(\Gamma)$ (resp., $K^+(\Gamma)$) is $m$-symmetric-neighborly (resp., $m$-neighborly).

Applying Theorem 4.1, we obtain

Theorem 4.6 There exist absolute constants $c, C$ and $\bar{c}$ for which the following holds. Let $\mu$ be an isotropic $\psi_2$ probability measure with constant $\alpha \geq 1$ and let $k$ and $\Gamma$ be as above. Then, with probability at least $1 - \exp(-\bar{c}k/\alpha^4)$,
the polytopes $K^+(\Gamma)$ and $K(\Gamma)$ are $m$-neighbory and $m$-symmetric-neighborly, respectively, for every $m$ satisfying

$$m \leq \frac{Ck}{\alpha^4 \log(cn^4/k)}.$$

The statement of Theorem 4.6 for $K(\Gamma)$ and for a Gaussian matrix $\Gamma$ is the main result of [RV]. However, a striking fact is that the same results holds for a random $\{-1,1\}$-matrix. In such a case, $K^+(\Gamma)$ is the convex hull of $n$ random vertices of the discrete cube $\{-1,+1\}^k$, also known as a random $\{-1,1\}$-polytope. With high probability, every $(m-1)$-dimensional face of $K^+(\Gamma)$ is a simplex and there are $\binom{n}{m}$ such faces, for $m \leq Ck/\log(cn^4/k)$.

**Remark 4.7** Let us mention some related results about random $\{-1,1\}$-polytopes. A result of [BP] states that for such polytopes, the number of facets, which are the $k-1$-dimensional faces, may be super-exponential in the dimension $k$, for an appropriate choice of the number $n$ of vertices. Denote by $f_q(K^+(\Gamma))$ the number of $q$-dimensional faces of the polytope $K^+(\Gamma)$. The quantitative estimate in [BP] was recently improved in [GGM] where it is shown that there are positive constants $a, b$ such that for $k^a \leq n \leq \exp(bk)$, one has $\mathbb{E} f_{k-1}(K^+(\Gamma)) \geq (\ln n / \ln k)^{k/2}$. For lower dimensional faces a threshold of $f_q(K^+(\Gamma))/\binom{n}{q+1}$ was established in [K].

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S. Mendelson Centre for Mathematics and its Applications, The Australian National University, Canberra, ACT 0200, Australia

shahar.mendelson@anu.edu.au

A. Pajor Laboratoire d’Analyse et Mathématiques Appliquées, Université de Marne-la-Vallée, 5 boulevard Descartes, Champs sur Marne, 77454 Marne-la-Vallée, Cedex 2, France

alain.pajor@univ-mlv.fr

N. Tomczak-Jaegermann Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

nicole@ellpspace.math.ualberta.ca