Model reduction with pole-zero placement and matching of derivatives

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Abstract

In this paper we consider the model reduction of a large, minimal, linear, time-invariant system of order \( n \) using moment matching techniques. Our goal is to compute an approximation of order \( \nu \ll n \) that matches \( \nu \) moments of the transfer function, has \( \ell \) poles and \( k \) zeros fixed and also matches a number of moments of its derivative. Assuming the original model is known, using a moment matching-based parameterization of the reduced model, we derive explicit linear algebraic constraints to place the desired poles and zeros and to match some moments of the derivative of the transfer function. The corresponding constraints are given by linear systems with the free parameters as unknowns together with solving low order Sylvester equations. Furthermore, since in practice data sets are available rather than the explicit model, we extend these results to the framework of data-driven model reduction. We generalize the Loewner matrices to include the measured data and the imposed pole and derivative constraints as well and use them to compute the approximation that satisfies all the imposed constraints simultaneously through solving again a linear system in the free parameters.

Key words: Moment matching, prescribed poles-zeros-derivatives, data-driven model reduction, Loewner matrices.

1 Introduction

In the control engineering practice of today, since mathematical models of system are increasingly complex and highly dimensional, model reduction is called for to find a low-order approximation. The approximation is suitable for simulation and control if relevant properties, such as certain dynamics (poles and derivative) or the zeros of the given system are preserved.

State-of-the-art. Moment matching-based model reduction techniques stand out as computationally efficient and easy to implement \cite{2}. The models are computed through numerically efficient procedures based on Krylov projections. It has also been shown that a property of the given system can be preserved if all the \( \nu \) matching points are chosen such that a set of \( \nu \) constraints are met. For instance, the problem of finding the reduced order model that minimizes the \( H_2 \) norm of the approximation error has been studied intensively. Hence, in, e.g., \cite{1,11} selecting the mirror images of the \( \nu \) poles of the \( \nu \) order approx-imant as interpolation points where to match \( \nu \) moments of the system and \( \nu \) moments of its first order derivative, yields the model with the lowest \( H_2 \) norm of the approximation error. Lately, for LTI systems, a time-domain Sylvester equation-based approach to moment matching has been taken in \cite{4,15}. The notion of moment is related to the unique solution of a Sylvester equation, see also \cite{9,8}, for earlier results. The time-domain approach yields families of all \( \nu \) order models parametrized in \( \nu \) degrees of freedom, that match a set of \( \nu \) moments of a given \( n \)-th order system at a set of \( \nu \) interpolation points. Imposing constraints on all the \( \nu \) degrees of freedom provide the (subfamilies of) \( \nu \) order models that meet additional desired constraints. For instance, in \cite{4} the \( \nu \) free parameters are selected such that stability and relative degree are preserved, in \cite{15} the \( \nu \) parameters are selected to find the minimal order model. In \cite{14} the model that matches \( 2\nu \) moments as well as the model that matches \( \nu \) moments of the given system and \( \nu \) moments of its first order derivative are computed. Recently, in \cite{19,20}, using optimization algorithms, the the model achieving minimum \( H_2 \) approximation error has been found.

Motivation. From the family of \( \nu \) order models matching \( \nu \) moments, using all the \( \nu \) degrees of freedom yields a (unique) model satisfying only one property, e.g., stability or matching the derivatives to lower the norm of the approximation error. Furthermore, e.g., fixing \( \nu \) poles may not be enough to match desired input-output behaviours
of the system. The zeros of the approximations may differ from those of the original system. In particular, assuming the original system to be of minimum phase, undesired right half-plane zeros may appear in the approximation. Since in control there exist algorithms to place \( \ell < \nu \), see [5], in the context of moment matching-based model reduction, we seek a \( \nu \) order approximation that has \( \ell \) poles, \( k \) zeros and \( \nu - (\ell + k) \) derivatives to match imposed, problem yet unsolved. Moreover, since in practice only sets of measured data of the system are available, we also pose and solve the same problem in the framework of Loewner data matrix-based model reduction.

**Contributions.** We consider an LTI stable, minimal \( n \)-th order system and the family of \( \nu \) order models that match \( \nu \) moments of the given transfer function, parametrized in \( \nu \) degrees of freedom. We determine the model that has \( \ell \) poles, \( k \) zeros and matches \( \nu - (\ell + k) \) derivatives. We first assume that the model of the system is known explicitly. We provide an explicit linear system together with solving low order Sylvester equations that yield the free parameters such that \( \ell \leq \nu \) poles are placed. For a particular canonical form of the interpolation points we write and solve a simpler linear algebraic system (i.e. no need to solve Sylvester equations) yielding the free parameters that place the poles. Also we derive the explicit linear system of algebraic constraints that provide the free parameters such that \( k \leq \nu \) zeros are imposed. Furthermore, we derive an explicit linear system in the free parameters such that \( \nu - (\ell + k) \) moments of the first order derivative of the transfer function are matched. Since it is difficult to yield and manipulate explicit complex and highly dimensional mathematical models, it is practical to use data sets given by measurements performed on the systems. In this case, we solve the problem of finding a reduced order model that matches the data and satisfies the pole and derivatives constraints simultaneously. We hereby exploit and generalize the Loewner matrices presented in [10,18] for model reduction and in, e.g., [17] for control, to include (simultaneous) information about the \( \ell \) prescribed poles and the \( \nu - \ell \) derivatives to be matched by a \( \nu \) order approximation that matches the measured data of the given system. Using the general Loewner matrices, we then compute the model that matches \( \nu \) moments, has \( \ell \) poles fixed and matches \( \nu - \ell \) derivatives of the transfer function of the given system through solving again a linear system in the free parameters.

**Content.** The paper is organized as follows. In Section 2 we briefly recall the time-domain moment matching for linear systems. In Section 3, we give the sets of linear constraints to place certain poles, zeros and match derivatives, respectively. In Section 4, we include all the constraints in the general Loewner matrices and use them to compute the low order approximation that satisfies these constraints. In Section 5 we illustrate the theory on a CD player model. The paper ends with some conclusions.

**Notation.** \( \mathbb{R} \) is the set of real numbers and \( \mathbb{C} \) is the set of complex numbers. \( \mathbb{C}^0 \) is the set of complex numbers with zero real part and \( \mathbb{C}^- \) denotes the set of complex numbers with negative real part. If \( A \) is a real matrix, then \( A^T \) is the transpose. \( \sigma(A) \) is the set of eigenvalues of \( A \).

## 2 Preliminaries

In this section we briefly review the computation of the family of \( \nu \) order models matching \( \nu \) moments of a stable LTI system. For more details see e.g., [4,15].

### 2.1 Time-domain moment matching for linear systems

Consider a single input-single output (SISO) linear time-invariant (LTI) minimal system

\[
\Sigma: \quad \dot{x} = Ax + Bu, \quad y = Cx, \quad (1)
\]

with the state \( x \in \mathbb{R}^n \), the input \( u \in \mathbb{R} \) and the output \( y \in \mathbb{R} \). The transfer function of (1) is

\[
K(s) = C(sI - A)^{-1}B, \quad K: \mathbb{C} \to \mathbb{C}. \quad (2)
\]

Throughout the rest of the paper we assume that the system (1) is stable, that is \( \sigma(A) \subseteq \mathbb{C}^- \). For the sake of clarity we consider the SISO case. However the results can be extended to the multiple input-multiple output case, as explained in Section 2.2. Moreover, for the sake of clarity, without loss of generality, throughout the rest of the paper, we consider real quantities.

Assume that (1) is a minimal realization of the transfer function \( K(s) \). The moments of (2) are defined as follows. 

**Definition 1** [2,4] The \( k \)-moment of system (1) with the transfer function \( K \) as in (2) at \( s_1 \in \mathbb{R} \) is defined by

\[
\eta_k(s_1) = (-1)^{k/2} \frac{1}{k!} \left[ d^k K(s)/ds^k \right]_{s=s_1} \in \mathbb{R}.
\]

Let \( s_i \in \mathbb{R} \setminus \sigma(A), \ i = 0 : l, \ l \geq 0 \) be a set of real numbers. Take \( j_i \geq 0 \) such that \( \sum_{i=0}^l (j_i - 1) = \nu \). For each \( i \), let \( \eta_0(s_i), \ldots, \eta_{j_i}(s_i) \) denote the \( \nu \) moments of order \( j_i + 1 \) of (1) at the given points \( s_i \). For the sake of clarity, we will drop the order of the moment, unless it is required explicitly. Let \( S \in \mathbb{R}^n \times \nu \), with the spectrum \( \sigma(S) = \{ s_i \mid i = 0 : l \} \) such that \( \sigma(S) \cap \sigma(A) = \emptyset \). Let \( L \in \mathbb{R}^{1 \times \nu} \), such that the pair \( (L,S) \) is observable. Denote by \( \Pi \in \mathbb{R}^{n \times \nu} \) be the solution of the Sylvester equation

\[
A \Pi + BL = \Pi S. \quad (3)
\]

Furthermore, since the system is minimal, assuming that \( \sigma(A) \cap \sigma(S) = \emptyset \), then \( \Pi \) is the unique solution of the equation (3) and rank \( \Pi = \nu \), see e.g. [6]. Then, the moments of (1) are characterised as follows

**Proposition 1** [4] The \( \nu \) moments \( \eta_0(s_i), \ldots, \eta_{j_i}(s_i), i = 1 : l \) of system (1) at \( s(s) \) are in one-to-one relation with the elements of the matrix \( C \Pi \).

Consider the LTI system

\[
\dot{\xi} = F \xi + Gu, \quad \psi = H \xi,
\]

1. By one-to-one relation between a set of moments and the elements of a matrix, we mean that the moments are uniquely determined by the elements of the matrix.
with $F \in \mathbb{R}^{v \times v}$, $G \in \mathbb{R}^v$ and $H \in \mathbb{R}^{p \times v}$, and the corresponding transfer function

$$K_G(s) = H(sI - F)^{-1}G.$$  

Let $\tilde{h}_0(s_1), ..., \tilde{h}_i(s_i)$ denote the first $i + 1$ moments of $K_G$ at $s_i$. Then, we define:

**Definition 2 (Moment matching)** [13] A system $K_G$ matches $\nu$ moments of a given system $K$ at $\{s_0, ..., s_i\}$, if $\tilde{h}_k(s_i) = \tilde{h}_k(s_i)$, for all $k = 0 : j_i, i = 0 : l$.

The following result gives necessary and sufficient conditions for a low-order system to achieve moment matching.

**Proposition 2** [4] Fix $S \in \mathbb{R}^{v \times v}$ and $L \in \mathbb{R}^{l \times v}$, such that the pair $(L, S)$ is observable and $\sigma(S) \cap \sigma(A) = \emptyset$. Furthermore, assume that $\sigma(F) \cap \sigma(S) = \emptyset$. Then, the reduced system $K_G$ matches the moments of (1) at $\sigma(S)$ if and only if $HP = CPI$, where the matrix $P \in \mathbb{R}^{v \times v}$ is the unique solution of the Sylvester equation $FP + GL = PS$.

The results are directly extended to the MIMO case as in, e.g., [15]. Consider a MIMO system (1), with input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$ and the transfer function $K(s) \in \mathbb{C}^{p \times m}$. Without loss of generality, let $S \in \mathbb{R}^{v \times v}$ with $\sigma(S) = \{s_1, ..., s_p\}$ and $L = [\beta_1 \, \beta_2 \, ... \, \beta_\nu] \in \mathbb{R}^{m \times \nu}$, $l_i \in \mathbb{R}^m, i = 1 : \nu$, be such that the pair $(L, S)$ is observable. Let $\Pi \in \mathbb{R}^{n \times n}$ be the unique solution of the Sylvester equation (3) and $\Upsilon_D$ be the unique solution of the Sylvester equation

$$S_D\Upsilon_D = \Upsilon_D A + RC,$$  

with $R = L_2$.

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$$S_D\Upsilon_D = \Upsilon_D A + RC,$$  

with $R = L_2$. We assume that the pair $(S_D, R)$ is controllable such that $\text{rank} \, \Upsilon_D = \mu$. Then the moments of $K'(s)$ at $\sigma(S_\nu)$ are given by the elements of the matrix $\Upsilon_D \Pi \in \mathbb{R}^{n \times v}$, see [14]. To this end, define

$$\Sigma' : \dot{x} = Ax + Bu, \; \dot{z} = Az + x, \; y = -Cz,$$

where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$, as in Figure 1. Note that the transfer function of $\Sigma'$ is $K'(s)$, with $K$ from (2). Introducing $\Sigma'$ to the signal generator

$$\dot{w} = S\omega, \; \theta = L\omega, \; \omega(0) \neq 0, \; \omega \in \mathbb{R}^v$$  

by $u = \theta$ and to the generalized signal generator

$$\dot{\omega} = S_D\omega + Rw, \; d = \omega + \Upsilon_D z, \; \omega(0) = 0, \; \omega(t) \in \mathbb{R}^v,$$

by $w = y$, where $\Upsilon_D$ is the unique solution of (6) and $R = L_2$, yields the output signal $d(t)$, see Figure (1). The next result shows the relevant properties of signal $d$.

**Proposition 3** [14] Consider the interconnection of system $\Sigma'$ with the signal generators (7) and (8), defined by $u = \theta$ and $v = y$, respectively. Then the output signal $d(t)$ satisfies the equation

$$\dot{d} = S_Dd + \Upsilon_D\omega + \Upsilon_D e^A(t_0 - \Pi\omega_0),$$  

if and only if $\Pi$ is the unique solution of equation (3) and $\Upsilon_D$ is the unique solution of equation (6).

### 2.2 Time-domain moment matching for MIMO systems

The results are directly extended to the MIMO case as in, e.g., [15]. Consider a MIMO system (1), with input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$ and the transfer function $K(s) \in \mathbb{C}^{p \times m}$. Without loss of generality, let $S \in \mathbb{R}^{v \times v}$ with $\sigma(S) = \{s_1, ..., s_p\}$ and $L = [\beta_1 \, \beta_2 \, ... \, \beta_\nu] \in \mathbb{R}^{m \times \nu}$, $l_i \in \mathbb{R}^m, i = 1 : \nu$, be such that the pair $(L, S)$ is observable. Let $\Pi \in \mathbb{R}^{n \times n}$ be the unique solution of the Sylvester equation (3). Then, the moments $\eta(s_i) = K(s_i)\beta_i, \; \eta(s_i), \; i = 1, ..., \nu$ of at $\{s_1, ..., s_p\} \subset \sigma(S)$ are in one-to-one relation with $\Pi$. Then, the moment matching from Definition 2 for MIMO systems is equivalent to satisfying the right tangential interpolation conditions as in [7], i.e., $K(s_i)\beta_i = K_G(s_i)\beta_i, \; i = 1 : \nu$, with $K_G$ given by (5).

The relations also hold for any $s_i$ with multiplicity $j_i$. It immediately follows that a family of reduced order MIMO models that achieve moment matching in the sense of satisfying the tangential interpolation conditions is given by $\Sigma_G$ described by the equation (4). Hence, without loss of
Assuming (11) holds, we get
\[ 0 = C \]
Since assuming \( H \) holds, we write
\[ v \]
Hence, by (10), we write
\[ v \]
In this section we derive linear relations parametrized in \( G \) yielding the subfamily of \( \nu \) order models that preserve \( \ell \) poles, \( k \) zeros of the given system and matches the derivatives of \( K(s) \) as in (2) at \( \nu - (\ell + k) \) points.

### 3 Model reduction with pole-zero placement and matching of derivatives

In this section we place \( \ell \) poles of the reduced order, for example in some of the poles of the original system, by properly selecting \( G \). Consider an LTI system (1) and the class of reduced \( \nu \) order models \( \Sigma_G \) from (4) that match \( \nu \) moments of (1) at \( \sigma(S) \). Let \( \lambda_i \in \mathbb{R}, i = 1 : \ell \), \( \ell \leq \nu \) be such that \( \lambda_i \notin \sigma(S) \). Then \( \lambda_i \) are poles of \( \Sigma_G \) if \( \det(\lambda I - S + GL) = 0 \), \( i = 1 : \ell \). To this end, let \( Q_p \in \mathbb{R}^{\ell \times \ell} \) be a matrix such that \( \sigma(Q_p) = \{\lambda_1, \ldots, \lambda_\ell\} \). Furthermore, consider \( C_p \in \mathbb{R}^{1 \times n} \) such that \( C_p \Pi = 0 \), where \( \Pi \) solves (3), and let \( \Upsilon_p \in \mathbb{R}^{\ell \times n} \) be the unique solution of the Sylvester equation
\[
Q_p \Upsilon_p = \Upsilon_p A + R_p C_p, \tag{10}
\]
with \( R_p \in \mathbb{R}^{\ell} \) any matrix such that the pair \( (Q_p, R_p) \) is controllable. Hence \( \text{rank}(\Upsilon_p) = \ell \), see, e.g., [6]. The next result imposes linear constraints on \( G \) such that the reduced model \( \Sigma_G \) has \( \ell \) poles at \( \{\lambda_1, \ldots, \lambda_\ell\} \).

**Theorem 1** Let \( \Sigma_G \) as in (4) be a \( \nu \) order model that matches the moments of (1) at \( \sigma(S) \). Furthermore, let \( \Upsilon_p \in \mathbb{R}^{\ell \times n} \) be the unique solution of the Sylvester equation (10) and assume that \( \text{rank}(\Upsilon_p \Pi) = \ell \). If \( G \) is a solution of the equation
\[
\Upsilon_p \Pi G = \Upsilon_p B, \tag{11}
\]
then \( \sigma(Q_p) = \{\lambda_1, \ldots, \lambda_\ell\} \subseteq \sigma(S - GL) \).

**Proof.** Let \( \lambda \in \sigma(Q_p) \). Then there exists the (left) eigenvector \( v \in \mathbb{R}^\nu \), \( v \neq 0 \), such that \( v^T (\lambda I - Q_p) = 0 \). Post multiplying with \( \Upsilon_p \Pi \) yields
\[
v^T (\lambda \Upsilon_p \Pi - Q_p \Upsilon_p \Pi) = 0.
\]
Hence, by (10), we write
\[
v^T (\lambda \Upsilon_p \Pi - \Upsilon_p \Pi G + R_p C_p \Pi) = 0.
\]
Since assuming \( C_p \Pi = 0 \) leads to \( v^T (\lambda \Upsilon_p \Pi - \Upsilon_p \Pi G) = 0 \), using (3) further yields
\[
v^T (\lambda \Upsilon_p \Pi - \Upsilon_p \Pi S + \Upsilon_p BL) = 0.
\]
Assuming (11) holds, we get
\[
v^T \Upsilon_p (\lambda I - S + GL) = 0.
\]
Since we assume that \( \text{rank}(\Upsilon_p \Pi) = \ell \), then \( (\Upsilon_p \Pi)^T v = 0 \) if and only if \( v = 0 \). Hence, \( \lambda \in \sigma(S - GL) \) with the (left) eigenvector \( (\Upsilon_p \Pi)^T v \) and the claim follows. \( \square \)

**Remark 1** If \( \ell = \nu \) and \( \Upsilon_p \Pi \) is assumed invertible, then
\[
\sigma(S - GL) = \sigma(Q_p), \tag{12}
\]
if and only if
\[
G = (\Upsilon_p \Pi)^{-1} \Upsilon_p B.
\]

Note that \( \Upsilon_p \) and \( \Pi \) can be easily computed explicitly using, e.g., Krylov projections and a coordinate transformation, to avoid solving any Sylvester equation. Furthermore, a sufficient condition to satisfy (11) is to select \( G \) as a solution of the matrix equation \( \Pi G = B \). Hence, postmultiplying equation (10) with \( \Pi \) yields \( Q_p \Upsilon_p \Pi = \Upsilon_p \Pi G \). Using equation (3) one immediately gets \( \Upsilon_p \Pi G \Pi = \Upsilon_p \Pi G \). Moreover, if \( \Upsilon_p \Pi \) is assumed invertible, then the \( \nu \) order model \( G \) with \( \nu \) such that \( \Pi G \) is written equivalently as
\[
(\Upsilon_p \Pi)^{-1} \Upsilon_p \Pi G \Pi = S - GL, \quad G = (\Upsilon_p \Pi)^{-1} \Upsilon_p B.
\]

When \( S \) and \( L \) are chosen in canonical form and zero-order moments are considered, (11) can be replaced by a simpler system of linear algebraic constraints on the parameter \( G \in \mathbb{R}^\nu \) without employing Sylvester equations.

**Proposition 4** Let \( S = \text{diag}(s_1, \ldots, s_\nu) \) and \( L = \{1 \ldots 1\} \in \mathbb{R}^{1 \times \nu} \). Then \( \{\lambda_1, \ldots, \lambda_\ell\} \) are poles of \( K_G(s) \) as in (5) if and only if \( G \in \mathbb{R}^\nu \) is the solution of the linear system
\[
1 + LD_k^{-1}G = 0, \quad \forall k = 1 : \ell, \tag{13}
\]
with \( D_k = \text{diag}(\theta_{k_1}, \ldots, \theta_{k_\nu}) \), where \( \theta_{ki} = \lambda_k - s_i \), \( i = 1 : \nu \) and \( k = 1 : \ell \). \( \square \)

**Proof.** Note that \( \lambda_i \in \mathbb{C} \) is a pole of \( K_G(s) \) from (5) if \( \det(\lambda I - S + GL) = 0 \). Explicitly writing the determinant yields the equivalent equation
\[
\begin{vmatrix}
\theta_{k_1} + g_1 & g_1 & \ldots & g_1 \\
g_2 & \theta_{k_2} + g_2 & \ldots & g_2 \\
\vdots & \vdots & \ddots & \vdots \\
g_{\nu} & g_{\nu} & \ldots & \theta_{k_\nu} + g_\nu
\end{vmatrix} = 0,
\]
and equivalently, in matrix form
\[
\det(D_k + GL) = 0, \tag{13}
\]
where \( D_k = \text{diag}(\theta_{k_1}, \ldots, \theta_{k_\nu}) \), for each \( k = 1 : \ell \). Using the well-known Sherman-Morrison-Woodbury formula [12], the claim follows immediately. \( \square \)

**Remark 2** Theorem 1 yields only a sufficient condition on \( G \) such that \( \ell \) of the poles of \( \Sigma_G \) are fixed, when \( S, L \) and \( Q_p \) are arbitrary matrices such that the pair \( (L, S) \) is observable and the pair \( (Q_p, R_p) \) is controllable, whereas Proposition 4 gives a necessary and sufficient condition to place \( \ell \) poles when \( S, L \) and \( Q_p \) are in the canonical forms.
Stable approximations  Consider the families of approximations $\Sigma_G$ described by the equations (4) and the problem of finding $G$ such that the reduced order system is asymptotically stable. The goal is achieved by selecting $G$ such that $\sigma(S - GL) \cap \sigma(A) = \emptyset$ and $\sigma(S - GL) = \{\lambda_1, \ldots, \lambda_p\} \subset \mathbb{C}^-$. Note that, by the observability of the pair $(L, S)$, there exists a unique matrix $G$ such that this condition holds, see [4] for more details.

3.2 Zero placement linear constraints

Consider a system (1) and the family of $\nu$ order models $\Sigma_G$ that approximate (1) by matching $\nu$ moments, for all $G \in \mathbb{R}^\nu$. Let $z_1, \ldots, z_k \in \mathbb{R}, \ k \leq \nu$. By, e.g., [4,16,13], there exists a subfamily of models $\Sigma_G$, with the property that the set of zeros of each model contains $z_1, \ldots, z_k$. Equivalently, there exists $G$ such that

$$\det \left[ z_iI - S \ G \right]_{C \Pi, 0} = 0, \quad i = 1 : k. \tag{14}$$

Now, let $G = [g_1 \ g_2 \ \ldots \ g_\nu]^T \in \mathbb{R}^\nu$. Then, it follows immediately that condition (14) is equivalent to a system of $k$ equations with $\nu$ unknowns $g_1, \ldots, g_\nu$, given by

$$(-1)^\nu [g_1 \zeta_1(z_1) + g_2 \zeta_2(z_2) + \cdots + (-1)^\nu g_\nu \zeta_\nu(z_1)] = 0,$$

$$(-1)^\nu [g_1 \zeta_1(z_2) + g_2 \zeta_2(z_2) + \cdots + (-1)^\nu g_\nu \zeta_\nu(z_2)] = 0,$$

$$(-1)^\nu [g_1 \zeta_1(z_k) + g_2 \zeta_2(z_k) + \cdots + (-1)^\nu g_\nu \zeta_\nu(z_k)] = 0,$$

with $\zeta_j(s)$ polynomials of degree $\nu - 1$, $j = 1 : \nu$. Similarly, when $S$ and $L$ are chosen in canonical form, the previous system of linear equations can be replaced by a simpler system of linear algebraic constraints on the parameter $G \in \mathbb{R}^\nu$ which does not require computations of polynomials of degree $\nu - 1$.

**Proposition 5** Let $S = \text{diag}(s_1, \ldots, s_\nu)$, $L = [1 \ \ldots \ 1]$ and explicitly write the moments $C \Pi = [\eta_1 \ \ldots \ \eta_\nu]$. Then $\Sigma_G$ as in (4) is a model which has $\{z_1, \ldots, z_k\}$ among the zeros of the transfer function $K_G(s)$ given by (5), if and only if the elements of the matrix $G = [g_1 \ g_2 \ \ldots \ g_\nu]^T$ satisfy the linear equations

$$\sum_{i=1}^\nu \frac{\eta_i}{\gamma_{ji}} g_i = 0, \quad j = 1 : k, \tag{15}$$

where $\gamma_{ji} = z_j - s_i, \ i = 1 : \nu, \ j = 1 : k$.

**PROOF.** Note that $\{z_1, \ldots, z_k\}$ are zeros of $K_G(s)$ if and only if (14) is satisfied, i.e.,

$$\begin{bmatrix} \gamma_{j1} & 0 & 0 & \cdots & 0 & g_1 \\ 0 & \gamma_{j2} & 0 & \cdots & 0 & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{j\nu} & g_\nu \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_\nu & 0 \end{bmatrix} = 0,$$

$$\gamma_{ji} = z_j - s_i, \ i = 1 : \nu, \ j = 1 : k.$$

First, note that $\gamma_{ji} \neq 0$ for all $i, j$. Successively decomposing the determinant by the last column and computing the resulting minors performing row decomposition yields

$$\sum_{i=1}^\nu \eta_i g_i \prod_{l=1\,;l\neq i}^\nu \gamma_{jl} = 0, \quad j = 1 : k.$$

Then, dividing by $\prod_{l=1\,;l\neq i}^\nu \gamma_{jl} \neq 0$ leads to the claim. $\square$

We again observe that Proposition 5 gives a necessary and sufficient condition to place $k$ zeros when $S$ and $L$ are in the canonical forms.

3.3 Matching derivatives as linear constraints

In this section we derive a set of linear relations parametrized in the matrix $G \in \mathbb{R}^\nu$ yielding the subfamily of models that match $\nu$ moments of $K(s)$ and $\mu < \nu$ moments of $K'(s)$. Without loss of generality, let $S = \text{diag}(S_1, S_D), S_D \in \mathbb{R}^{\mu \times \mu}$. We impose matching properties at the first order derivative of $K(s)$ in the sense of matching the relation defining signal $d(t)$ in Proposition 3. The explicit computation of the derivatives is not required. Consider a model $\Sigma_G$ as in (4) with the transfer function $K_G(s)$ given by (5). Simple calculations yield the state-space representation of $K_G'(s)$ as [14]

$$\Sigma_G': \dot{\chi} = (S - GL)\chi + Gu, \ \chi = (S - GL)\chi + \xi, \ \eta = -C\Pi\chi,$$

with $\chi(t) \in \mathbb{R}^\nu$. Considering the interconnection of $\Sigma'$ to the signal generators $\dot{\omega} = S\omega, \ \dot{\theta} = L\omega, \ \omega(0) \neq 0, \ \omega \in \mathbb{R}^\nu$ by $u = \theta$ and $\dot{\omega} = S_D\dot{\omega} + R\omega, \ d = \dot{\omega} + T_D\dot{\omega}, \ \omega(0) = 0, \ \sigma(t) \in \mathbb{R}^\mu$, by $v = \eta$, respectively, yields the output

$$\zeta(t) = \sigma(t) + P\chi(t).$$

We say that the moments of $K_G'(s)$ matches the moments of $K'(s)$ at $\sigma(S_D)$ if the dynamics of $\zeta(t)$ are similar to the dynamics of $d(t)$ from (9), i.e.,

$$\dot{\zeta} = S_D\zeta + T_D\Pi\chi,$$

with $T_D$ the solution of (6) and $\Pi$ the solution of (3). The next result presents the selection of $G \in \mathbb{R}^\nu$ such that $K_G$ matches $\nu$ moments of $K$ at $\sigma(S)$ and $K_G'$ matches $\mu$ moments of $K'$ at $\sigma(S_D)$.
Theorem 2 Let $\Pi$ be the unique solution of equation (3) and $T_D$ be the unique solution of equation (6). Consider a model $\Sigma_G$ as in (4) with the transfer function $K_G(s)$ as in (5). Then $\mu$ moments of $K'_G$ match $\mu$ moments of $K'$ at $\sigma(Q) \subset \sigma(S)$, if and only if

$$\Upsilon_D II G = \Upsilon_D B.$$  \hspace{1cm} (16)

PROOF. We first prove the necessity. Since $\zeta = \omega + P\chi$, then $\dot{\zeta} = \dot{\omega} + P\dot{\chi}$. The moments of $\Sigma_G$ match the moments of $\Sigma'$ at $\sigma(S_D)$ if $\dot{\zeta} = S_D\omega + P_D\chi$, hence, since $\dot{\omega} = S_D\omega + Rw$ and $w = \eta$, where $\eta$ is the output of $\Sigma_G$, we write

$$S_D\omega - RC\Pi\chi + P(S - GL)\chi + P\xi = S_D\omega + S_D P\chi + T_D\Pi\xi,$$

for all $\xi$ and $\chi$. Then, $P = T_D II P$ and $PS - SDP = RC\Pi + PGL$. Equivalently, $\Re T_D II P - T_D II S = \Upsilon_D II G + RC\Pi$. Hence $T_D II G = S_D II P - T_D II S - RC\Pi$. By (6), $Q T_D II = (T_D A + RC)II$. Then,

$$T_D II G = T_D II S - T_D A II.$$

By (3), $T_D A II = T_D (II S - BL)$ and then the claim follows. Since the sufficiency uses similar arguments, the proof is omitted. \hfill $\square$

Remark 3 Note that if $\mu = \nu$, then all the $\nu$ derivatives of $K_G(s)$ are matched at $\sigma(S_D) = \sigma(S)$, by selecting

$$G = (T_D II)^{-1} T_D B,$$

where the matrix $T_D II \in \mathbb{R}^{n \times \nu}$ is assumed invertible, see, e.g., [14].

Remark 4 Consider a $n$-th order system (1) with the matrices $A, B, C$ that describe the model, given explicitly (with the transfer function $K$). Furthermore, $\Sigma_G$ (with the transfer function $K_G$) define a family of $\nu$ order models that match $\nu$ moments of (1) at $s_1, \ldots, s_{\nu}$, parametrized in $G = [g_1 \ldots g_\nu]^T \in \mathbb{R}^\nu$, where $S$ and $L$ are in canonical form. Let $\{\lambda_1, \ldots, \lambda_\nu\} \subseteq \mathbb{R} \setminus \sigma(S)$ and $\{z_1, \ldots, z_{\nu}\} \subseteq \mathbb{R}$, $\ell = \nu - \ell < \nu$. Collecting the linear constraints (12), (15) and (16) yield the following system of linear equations in the unknowns $g_1, \ldots, g_\nu$:

$$\begin{cases}
1 + L D_k^{-1} G = 0, & k = 1 : \ell, \\
\sum_{i=1}^\nu \frac{\partial}{\partial s_i} g_i = 0, & j = 1 : k,
\end{cases}
\hspace{1cm} (17)$$

with $D_k = \text{diag}(\theta \ell_{k1}, \ldots, \theta \ell_{kn})$, $\theta \ell_{ki} = \lambda_k - s_i, i = 1 : \nu, k = 1 : \ell; \gamma_{ij} = z_j - s_i, i = 1 : \nu, j = 1 : k$.

$T_D II G = T_D B$,

with $K_G(s_j) = K(s_j)$, $j = (\ell + k) + 1 : \nu$.

However, in practice, the matrices $A, B, C$ of (1) are not known explicitly, but data measurements are available.

This motivates the extension of our results to the case of data-driven model order reduction using the Loewner matrices presented in [18]. We generalize the Loewner matrices to yield the reduced order model $\Sigma_G$ as in (4) that matches $\nu$ moments of (1), has $\ell$ poles prescribed and matches $\nu - \ell$ derivatives of the transfer function $K$.

4 Loewner matrices-based model reduction with pole placement and matching of derivatives

In this section we compute a general version of the Loewner matrices given in [18], to contain information about the input-output data to match, about fixing desired poles and about the derivatives to be matched. We further determine the approximation $\Sigma_G$, as in (4), that matches $\nu$ moments of (1) and matches the pole and derivative data as well.

Consider the LTI system (1) with the transfer function $K$ as in (2) and the sets of points $s_1, s_2, \ldots, s_{\ell}, s_{\ell+1}, \ldots, s_{\nu}$, not eigenvalues of the matrix $A$, and $\{\lambda_1, \ldots, \lambda_{\ell}\}$, with $\ell \leq \nu$. To the points $s_j$ and $\lambda_j$ correspond the data information $K(s_j)$ and $K(\lambda_j)$, respectively. Construct the generalized Loewner and shifted Loewner matrices, respectively,

$$\begin{align}
\tilde{\mathbf{L}}_{ij} &= \begin{cases}
\frac{K(\lambda_i) - K(s_j)}{\lambda_i - s_j}, & i, j = 1 : \ell, \\
\frac{K(s_i) - K(s_j)}{s_i - s_j}, & i \neq j = \ell + 1 : \nu, \\
K'(s_i), & i = \ell + 1 : \nu,
\end{cases} \\
\sigma \tilde{\mathbf{L}}_{ij} &= \begin{cases}
\frac{\lambda_i K(\lambda_j) - s_j K(s_j)}{\lambda_i - s_j}, & i = 1 : \ell, \\
\frac{s_i K(s_i) - s_j K(s_j)}{s_i - s_j}, & i \neq j = \ell + 1 : \nu,
\end{cases}
\end{align}$$  \hspace{1cm} (18a, 18b)

Note that these matrices are constructed from the data to match $\nu$ moments at $s_1, s_2, \ldots, s_{\ell}, s_{\ell+1}, \ldots, s_{\nu}$, place $\ell$ poles at $\{\lambda_1, \ldots, \lambda_{\ell}\}$ and match $\nu - \ell$ derivatives of $K$ at $s_{\ell+1}, \ldots, s_{\nu}$.

Let

$$S = \text{diag}(s_1, \ldots, s_{\ell}, s_{\ell+1}, \ldots, s_{\nu}) = \text{diag}(S_1, S_D),$$  \hspace{1cm} (19)

with $S_D = \text{diag}(s_{\ell+1}, \ldots, s_{\nu})$ and let $L = [1 \ldots 1] = [L_1 \ L_2] \in \mathbb{R}^{\nu \times \nu}, L_2 \in \mathbb{R}^{(\nu - \ell) \times (\nu - \ell)}$. Furthermore, let

$$Q = \text{diag}(\lambda_1, \ldots, \lambda_\nu, s_{\ell+1}, \ldots, s_{\nu}) = \text{diag}(Q_P, S_D),$$  \hspace{1cm} (20)

with $Q_P = \text{diag}(\lambda_1, \ldots, \lambda_{\ell})$. Let $\Pi$ be the solution of the Sylvester equation (3), $A\Pi + BL = II$. Furthermore, construct $\Upsilon = [\Upsilon_P \ U_D]^T \in \mathbb{R}^{\nu \times \nu}$, where $\Upsilon_P$ is the unique solution of the Sylvester equation (10), $Q \ U_P = T_P A + R_P C_P$, where $C_P \in \mathbb{R}^{1 \times n}$ such that $C \Pi = 0$ and $T_D$ is the unique solution of the Sylvester equation (6), $S_D \ U_D = T_D A + RC$, where $R = L^2_2$. Note that, in matrix form, $\Upsilon$ is the unique solution of the Sylvester equation

$$Q \ U = T A + R(C_P, C),$$  \hspace{1cm} (21)
where \( R(C_P, C) = [(R_P C_P)^T (RC)^T]^T \). We now give a result stating that the Loewner matrices (18) can be written directly in terms of \( \Upsilon \) and \( \Pi \) and that they are solutions of Sylvester equations.

**Theorem 3** Consider the Loewner matrices from (18) and the matrices \( S \) and \( Q \) defined by (19) and (20). Let \( \Pi \) be the unique solution of (3) and \( \Upsilon \) be the unique solution of (21). Consider the following statements.

1. \( L \) is defined by equation (18a).
2. \( L = -\Upsilon \Pi \) and satisfies the Sylvester equation \( LS - QL = RCP\Pi - \Upsilon BL \).
3. \( \sigma L \) is defined by equation (18b).
4. \( \sigma L = S - (\Upsilon \Pi)^{-1} \Upsilon BL \) and satisfies the Sylvester equation \( \sigma LL - Q\sigma L = RC\Pi S - Q\Upsilon BL \).

Then (1) \( \iff \) (2) and (3) \( \iff \) (4).

**Proof.** We first prove statement (1) \( \iff \) (2). Note that (18a) can be equivalently written as

\[
\Lambda_i L_s - L_s j = C(\Lambda_i I - A)^{-1} B - C(s_i I - A)^{-1} B, \quad (22)
\]

for all \( i, j = 1 : \ell \) and for all \( i \neq j = 1 : \nu, i \neq j \). Furthermore, equivalently,

\[
L_{ij} = C(s_i I - A)^{-1} B - C(\Lambda_j I - A)^{-1} B
\]

\[
= C[(s_i I - A)^{-1} - (\Lambda_j I - A)^{-1}] B
\]

\[
= C(s_i I - A)^{-1} \left[ (\Lambda_j I - A) - s_i I + A \right] (\Lambda_j I - A)^{-1} B, \quad s_i - \Lambda_j
\]

for all \( i, j = 1 : \ell \) and for all \( i \neq j = 1 : \nu, i \neq j \). Hence, we get

\[
L_{ij} = -C(s_i I - A)^{-1} (\Lambda_j I - A)^{-1} B, \quad s_i - \Lambda_j
\]

for all \( i, j = 1 : \ell \) and for all \( i \neq j = 1 : \nu, i \neq j \). Moreover, from (18a) we get

\[
L_{ii} = -K'(s_i) = C(s_i I - A)^{-2} B
\]

\[
= C(s_i I - A)^{-1} (s_i I - A)^{-1} B, \quad i = 1 : \nu.
\]

Denote by \( Y_i = C(s_i I - A)^{-1} \) and \( \Pi_j = (\Lambda_j I - A)^{-1} B \), \( i = 1 : \nu \). It is straightforward that \( \Upsilon = [\Upsilon_1^T \ \Upsilon_2^T \ \ldots \ \Upsilon_{\nu}^T]^T \in \mathbb{C}^{\nu \times \nu} \) and \( \Pi = [\Pi_1 \ \Pi_2 \ \ldots \ \Pi_{\nu}] \in \mathbb{C}^{\nu \times \nu} \) are the unique solutions of the Sylvester equations (21) and (3), respectively. Hence, \( L_{ij} \) as in (18a) can be written equivalently as

\[
L_{ij} = -\Upsilon_i \Pi_j, \quad \forall i, j = 1 : \ell \text{ and } \forall i, j = 1 : \nu, i \neq j\]

\[
L_{ii} = -\Upsilon_i \Pi_i, \quad \forall i = 1 : \nu.
\]

Furthermore, writing (22) for each \( i, j \) yields the claim. The arguments for \( \sigma L \) are similar, hence omitted. \( \square \)

**Theorem 4** Consider the system (1), and the matrices \( S \) and \( Q \) from (19) and (20), respectively. Let \( \Upsilon \) the unique solution of (21) and \( \Pi \) the unique solution of the equation (3). Consider the family of models \( \Sigma_G \) described by the equations (4) with the transfer function (5). Then, for \( G \) from (4) satisfying the relation

\[
G = -L^{-1} \Upsilon B, \quad (23)
\]

with \( L \) given by (18a) assumed invertible, the model \( \Sigma_G \) matches \( \nu \) moments of (1) at \( \sigma(S) = \{s_1, \ldots, s_\ell\} \), has \( \ell \) poles placed at \( \{\lambda_1, \ldots, \lambda_\ell\} \subset \sigma(Q) \) and matches \( \nu - \ell \) derivatives of \( K(s) \) at \( \{s_{\ell+1}, \ldots, s_\nu\} \subset \sigma(S) \). Explicitly, the \( \nu \) order model as in (5), with \( G \) given by (23) is the Loewner approximation of \( K \), i.e.,

\[
K_{-L^{-1} \Upsilon B}(s) = \Pi(\sigma L - s L)^{-1} \Upsilon B. \quad (24)
\]

**Remark 5** Consider the LTI system (1) with the transfer function \( K \) as in (2) and the sets of points \( \{s_1, s_2, \ldots, s_\ell, s_{\ell+1}, \ldots, s_\nu\} \), not eigenvalues of the matrix \( A \), and \( \{\lambda_1, \ldots, \lambda_\ell\} \), with \( \ell \leq \nu \). Assume that the system is characterized through data sets, in the sense that to each point \( s_i \) and \( \lambda_i \) corresponds the data information \( K(s_i) \) and \( K(\lambda_i) \), respectively, \( i = 1 : \ell \) and \( j = 1 : \nu \). Construct the generalized Loewner and shifted Loewner matrices as in (18). Then, the reduced order model \( \Sigma_G \) given by (24), satisfies the constraints

\[
\Sigma_{-L^{-1} \Upsilon B} \text{ has } \ell \text{ poles at } \lambda_1, \ldots, \lambda_\ell,
\]

\[
K'_{-L^{-1} \Upsilon B}(s_j) = K'(s_j), \quad j = 1 : \nu.
\]

Note that (24) is built only using the available data, i.e.,

\[
\text{the matrix } \Pi \text{ comes from the data } K(s_i), j = 1 : \nu;
\]

\[
\text{the matrix } \Upsilon B \text{ comes from the data } K(\lambda_i), i = 1 : \ell
\]

\[
\text{and } K'(s_j), j = 1 : \nu.
\]

Furthermore let \( \Sigma_G \) (with the transfer function \( K_G \)) define a family of \( \nu \) order models that match the \( \nu \) data of (1), \( K(s_j) \) at \( \{s_1, \ldots, s_\ell\} \), parametrized in \( G = [g_1 \ldots g_\nu]^T \in \mathbb{R}^\nu \). Then, the approximation (24) is obtained for \( G \) as in (23) built from the Loewner matrices and the data \( K(\lambda_i), i = 1 : \ell \) and \( K'(s_j), j = 1 : \nu \).

In general, for any non-derogatory matrices \( Q \) and \( S \), with \( R \) and \( L \) such that the pair \( (L, S) \) is observable and the pair \( (Q, R) \) is controllable, the matrix \( \hat{\Upsilon} = -\Upsilon \Pi \), with \( \Upsilon \) and \( \Pi \) the unique solutions of (21) and (3), respectively, satisfies the properties of a Loewner matrix.

**Theorem 5** Consider system (1). Let \( S \in \mathbb{R}^{\nu \times \nu} \) be any matrix with \( \sigma(S) = \{s_1, s_2, \ldots, s_\ell, s_{\ell+1}, \ldots, s_\nu\} \) not poles of (2) and \( L \in \mathbb{R}^{1 \times \nu} \) such that the pair \( (L, S) \) is observable. Also let \( Q \in \mathbb{R}^{\nu \times \nu} \) be any matrix with \( \sigma(Q) = \{\lambda_1, \lambda_2, \ldots, \lambda_\nu\} \), not poles of (2). Furthermore, let \( \Pi \) be the unique solution of the Sylvester equation (3), and \( \Upsilon \) be the unique solution of (21). Then, the matrices

\[
\hat{\Upsilon} = -\Upsilon \Pi, \quad (25a)
\]

\[
\hat{\sigma L} = S - (\Upsilon \Pi)^{-1} \Upsilon BL \quad (25b)
\]

satisfy the equations

\[
\hat{L} S - Q \hat{L} = L^T \hat{C} \Pi - \Upsilon BL, \quad (26a)
\]

\[
\hat{\sigma L} S - Q \hat{\sigma L} = L^T C \Pi S - Q \Upsilon BL. \quad (26b)
\]
The converse also holds, i.e., any matrices satisfying (26) are given by (25) and, furthermore

\[ \hat{L} = T_Q^{-1}LT_S, \]

where \( T_Q \in \mathbb{R}^{\nu \times \nu} \) is such that

\[ T_Q D T_Q^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_\ell, s, \ell+1, \ldots, s_\nu) \]

and \( T_S \in \mathbb{R}^{\nu \times \nu} \) is such that

\[ T_S D T_S^{-1} = \text{diag}(s_1, \ldots, s, s, \ell+1, \ldots, s_\nu). \]

**Proof.** Premultiplying (3) with \( \Upsilon \) yields \( \Upsilon A \Pi + \Upsilon B L = \Upsilon B \Pi S \). By (21), \( \Upsilon A = Q \Upsilon - R(C, C) \). Hence

\[
(Q \Upsilon - R(C, C)) \Pi + \Upsilon B L = \Upsilon B \Pi S
\]

which is equivalent to the Sylvester equation satisfied by \( \hat{L} = -\Upsilon S \) in [18, equation (12)]. It follows that \( \Upsilon S - (\Upsilon S)(\Upsilon S)^{-1} \Upsilon L S) = (Q - R(C, C)) \Pi (\Upsilon S)^{-1} \Pi (\Upsilon L S) \). Hence, one can write

\[
Q \hat{L} + R(C, C) \Pi = -S + (\Upsilon S)^{-1} \Upsilon L S.
\]

In the sequel we compute the matrices \( G \) that yield the approximations \( \Sigma_G \) (with the transfer function \( K_G \)) of order \( \nu \) which

- has \( \ell \) poles at \( \lambda_1, \ldots, \lambda_\ell \),
- has \( k \) zeros at \( z_1, \ldots, z_k \),
- satisfies the property that the derivatives of \( K_G \) match the derivatives of \( K \) at \( s_1, s_2, \ldots, s_\nu \).

We compute \( G \) for \( \nu = 3, 6, 12 \), for different values of \( \ell \) and \( k \). We compare the results of the proposed method with the \( \nu \) order balanced truncation approximation \( K_{\text{BT}} \) and the \( \nu \) order Iterative Rational Krylov Algorithm approximation, \( K_{\text{IRKA}} \). The results of the simulations are presented in Table 1, discussed below.

- The set of interpolation points is chosen *arbitrarily* in the complex plane. It contains zero for DC-gain preservation.
6 Conclusions

In this paper we have computed a low order approximation that matches the moments of a given large LTI system, has some poles and zeros fixed and matches a number of derivatives. We have presented explicit linear algebraic constraints for the placement of the desired poles and zeros and for matching the moments of some derivatives of the system. However, since in practice data sets are usually available rather than the model, we have extended the results to the framework of Loewner matrices to include the data and the imposed pole and derivative constraints, yielding the approximation that satisfies the imposed constraints simultaneously.

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