Fluctuations of the 2-spin SSK model with magnetic field

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Abstract: We analyze the fluctuation of the free energy, replica overlaps, and overlap with the magnetic fields in the quadratic spherical SK model with a vanishing magnetic field. We identify several different behaviors for these quantities depending on the size of the magnetic field, confirming predictions by Fyodorov-Le Doussal and recent work of Baik, Collins-Wildman, Le Doussal and Wu.

1 Introduction

Zero magnetic field spherical model. In this paper, we study the 2-spin spherical Sherrington-Kirkpatrick (SSK) model with external magnetic field. This is the random Gibbs measure with Hamiltonian given by

\[ H_N(\sigma) := \frac{1}{\sqrt{2N}} \sum_{i \neq j} g_{ij} \sigma_i \sigma_j + hv^T \sigma, \]  

where \( h \in \mathbb{R} \), and \( v \) is a deterministic vector such that \( \|v\|_2^2 = N \),

and the \( g_{ij} \) are iid standard normal random variables. The partition function is given by an integral over the \( N-1 \) dimensional sphere of radius \( \sqrt{N} \) (which we denote by \( S^{N-1} := \{ \sigma \in \mathbb{R}^N : \|\sigma\|_2 = \sqrt{N} \} \)),

\[ Z_{N,\beta,h} = \int_{S^{N-1}} \exp(\beta H_N(\sigma)) \, d\omega_{N-1}(\sigma). \]

Here, \( \omega_{N-1} \) is the uniform measure on \( S^{N-1} \).

The SSK (with zero magnetic field \( h = 0 \)) was introduced by Kosterlitz, Thouless and Jones [13] by analogy with the Sherrington-Kirkpatrick model with Ising spins. The spherical model with quadratic spins which we study here turns out to have a rather different behavior from the SK model with Ising spins or the \( p \)-spin models with \( p \geq 3 \), including spherical models. This is because the quadratic nature of the Hamiltonian and the continuous state space reduce the complexity of the energy landscape, which is not exponential as in other “true” spin glasses. Indeed, spin glasses are commonly characterized by the existence of an exponential (in \( N \)) number of critical points of the Hamiltonian for generic realizations of the disorder, while in the quadratic case on the sphere, this number is at most linear and indeed is \( O(1) \) as the size of the magnetic field increases to order 1. See [2] Theorem 2.8 and Remark 2.11, as well as [11], Section 3. These differences with other spin glasses are also reflected in our methods, which come largely from random matrix theory.

Nevertheless, the model without magnetic field exhibits an interesting phase transition at the critical temperature \( \beta = 1 \), which was already identified by Kosterlitz-Thouless-Jones. The SSK without magnetic was studied by Panchenko and Talagrand in [16]. J. Baik and J. O. Lee [6] used recent results in random matrix theory including the local semicircle law and the method of steepest descent to obtain second order asymptotics for the free energy. They later applied the same methods to analyze a number of variants of the spherical model, including the bipartite SSK model as well as a model including a deterministic Curie-Weiss term in the Hamiltonian. The latter model, although its definition is relatively simple, has an intricate phase diagram comprising three different regimes, which the authors of [7] term ferromagnetic, paramagnetic and spin glass.
Baik and Lee show that in the high temperature phase $\beta < 1$, the fluctuations of the free energy about its asymptotic limit are Gaussian. This is the spherical analog of the central limit theorem by Aizenman-Lebowitz-Frhlich for the free energy in the SK model with Ising spins. In the low temperature regime, however, Baik and Lee found that the free energy fluctuations asymptotically follow the Tracy-Widom GOE distribution, the limiting distribution of the largest eigenvalue of a real symmetric random matrix consisting of Gaussian entries.

Following Baik and Lee’s work, the second author and V. L. Nguyen analyzed the overlap between two independent samples (known as replicas) from the Gibbs measure in the case $h = 0$. They showed that in the high temperature region, the overlap, properly rescaled, is asymptotically normally distributed. The authors of the present paper then extended this result to the low temperature phase, where the asymptotic distribution is no longer Gaussian, but instead given by an explicit distribution which is a function of the Airy$_1$ random point field.

**Model with magnetic field.** In the current work we investigate the effect of a non-zero magnetic field on the fluctuations of the partition function and overlap. In addition to Fyodorov and Le Doussal’s work in physics, which we discuss in more detail below, the closest precedent to the current work in the spin glass literature is the work of Chen and Sen [8], who showed for general even mixed $p$-spin models, $p \geq 2$, the fluctuations of the ground state are Gaussian if $h = c > 0$. In particular for $p = 2$, the Tracy-Widom fluctuations found by Baik and Lee for $h = 0$ disappear.

We study the case $h \sim N^{-\epsilon}$, that is, the magnetic field vanishes as the system grows. As $\alpha > 0$ varies, the nature of the fluctuations of the free energy and overlaps also changes. This situation was studied in some detail in the physics literature by Fyodorov-Le Doussal [11] (see also Dembo-Zeitouni [10]). Their work is formulated in terms of the constrained optimization problem

$$E_{\min}(h) = \min_{\sigma} \left( -H_N(\sigma) \right)$$

subject to the spherical constraint $\|\sigma\| = N$, which corresponds to the ”zero temperature” regime $\beta = 0$.

Fyodorov and Le Doussal identify two distinguished scaling regimes for the magnetic field $h$ in the low temperature phase $\beta > 1$:

1. **Microscopic magnetic field**, $h \sim N^{-1/2}$: this regime is the boundary of the region of ”small” magnetic field, in the sense that the fluctuations of the free energy about its first-order asymptotic limit are not affected by the presence of the magnetic term. These fluctuations coincide with the case $h = 0$ treated by Baik and Lee, and are given by the Tracy-Widom GOE distribution.

2. **Moderate magnetic field**, $h \sim N^{-1/6}$: in this regime the magnetic field modifies the fluctuations asymptotics of the free energy, which are no longer given by the Tracy-Widom distribution. Fyodorov and Le Doussal analyze the tail of the distribution of the minimum in (1.3).

In this paper we obtain several results confirming and extending the predictions of Fyodorov and Le Doussal [11]. In particular, we show the following:

1. For microscopic magnetic field, we confirm that the limiting fluctuations remain the same as in the case $h \neq 0$. Thus the Tracy-Widom fluctuations of the free energy proved by Baik and Lee in the $h = 0$ case persist for values of $h$ up to order $\alpha = \frac{1}{2}$ Nevertheless, we show that the nonzero magnetic field can still be detected at the level of overlaps between replicas, whose distribution does differ from the case $h = 0$.

2. Previous works [10], [11] only consider the far tail of the limiting distribution of the free energy density. We give an expression for the limiting distribution when $0 < \beta < 1$ in terms of quantities from random matrix theory. We also show, somewhat surprisingly, that for values of the magnetic field $h \gg N^{-1/6}$, the fluctuations of the free energy are Gaussian.

3. For larger values of the magnetic field with $\alpha > 1/6$, the fluctuations are Gaussian.
In addition to these results about the fluctuations of the free energy, we also describe the fluctuations of the overlap between two independent samples from the Gibbs measure (“replicas”) as well as the overlap between a sample and the external field. This allows for a more refined description of each of the regimes found by Fyodorov and Le Doussal.

While completing this work we learned that Baik, Collins-Wildman, Le Doussal and Wu [5] have completed a paper discussing various properties of a spherical model with random external field similar to the one we study here. The focus of this work is different from ours; it includes some formal computations at the physics level of rigor, but on the other hand it treats aspects of the model not considered in our work, including various transitions between scaling regimes and the geometry of the Gibbs measure. Some of the results in [5] had previously appeared in Wu’s thesis [20].

We also note that Kivimae studied the fluctuations of the free energy at zero temperature \( \beta = \infty \) in the regime \( h = \mathcal{O}(N^{-1/6}) \), identifying a family of distributions interpolating between Tracy-Widom and Gaussian [12].

1.1 Organization of paper

The remainder of the paper is organized as follows. We will first introduce much of the notation used throughout the paper in Section 2.1, organizing it in a single section for convenient reference. The remainder of Section 2 then contains our main results on the fluctuations of the SSK model. We have split this into three subsections, one for each of the three scaling regimes we consider: Section 2.2 contains the high temperature/large magnetic field regime of Gaussian fluctuations; Section 2.3 the regime of intermediate magnetic field and low temperature; Section 2.4 the regime of microscopic magnetic field and low temperature. In Section 3, we collect the results from random matrix theory that we will use in our paper. Section 4 collects the various contour integral representations we use to prove our results. The remainder of the paper then consists of the proofs of these results. The high temperature or large magnetic field regime is analyzed in Section 5. The regimes of intermediate and microscopic magnetic fields at low temperature are considered in Sections 6 and 7, respectively. Finally, the appendices contain some proofs of auxiliary results.

2 Main results and notation

Our results concern fluctuations of the free energy, the overlap between two independent copies of \( \sigma \in \mathbb{S}^{N-1} \) distributed according to the Gibbs measure associated to the SSK Hamiltonian (“replicas”) and overlap between a replica and the external field \( v \). Our results are different depending on the scaling behavior of the magnetic field and inverse temperature. We therefore organize our results into the three different scaling regimes. The first regime is characterized by Gaussian fluctuations. In this regime, either the temperature is high or the magnetic field does not tend to 0 too quickly – this is quantified by the assumption (2.22) below. The other regimes are the cases of intermediate \( (h \sim N^{-1/6}) \) and microscopic \( (h \sim N^{-1/2}) \) magnetic fields and low temperature. Before stating our results we organize the notation of the paper in the following section for convenient reference.

2.1 Notation

The Hamiltonian of the spherical Sherrington-Kirkpatrick model is

\[
H_N(\sigma) := \frac{1}{\sqrt{2N}} \sum_{i \neq j} g_{ij} \sigma_i \sigma_j + hv^T \sigma,
\]

where

\[
\|v\|_2^2 = N \tag{2.2}
\]

is a fixed, deterministic vector and the \( \{g_{ij}\}_{i<j} \) are a family of iid standard normal random variables. The partition function is,

\[
Z_{N,\beta,h} = \int \exp [\beta H_N(\sigma)] \, d\omega_{N-1}(\sigma) \tag{2.3}
\]
where $d\omega_{N-1}$ is the uniform measure on the $(N - 1)$-sphere of radius $\sqrt{N}$. This is denoted by,

$$S^{N-1} := \{ \sigma \in \mathbb{R}^N : \|\sigma\|_2^2 = N \}. \quad (2.4)$$

Given a Hermitian matrix $A$ we will denote its eigenvalues in decreasing order by $\lambda_i(A)$ and the associated eigenvectors by $u_i(A)$.

We will denote by $M$ the random matrix formed from the disorder variables in the SSK Hamiltonian via

$$M_{ij} = -\frac{1}{\sqrt{2N}}(g_{ij} + g_{ji}), \quad i \neq j$$

and $M_{ii} = 0$. In terms of the SSK Hamiltonian, we have

$$H_N(\sigma) = -\sigma^T M \sigma + hv^T \sigma. \quad (2.6)$$

Up the diagonal being 0, the matrix $M$ is a matrix from the Gaussian Orthogonal Ensemble. In fact, we will often compare the eigenvalues and eigenvectors of $M$ to those of a certain GOE matrix, which we denote by $H$. The off-diagonal elements of $H$ are the same as $M$, and the diagonal entries are given by $H_{ii} = \sqrt{2} g_{ii} N^{-1}$ where $\{g_{ii}\}_{i=1}^N$ are independent standard normal random variables.

Associated with $M$ are the following spectral quantities which describe the limiting density of states of the eigenvalues. The semicircle distribution will be denoted by,

$$\rho_{sc}(E) := \frac{1}{2\pi} \sqrt{4 - E^2} \quad (2.7)$$

with Stieltjes transform given by

$$m_{sc}(z) = \int \frac{\rho_{sc}(E)\,dE}{E - z}. \quad (2.8)$$

The $N$-quantiles of $\rho_{sc}(E)$ are the points defined by

$$\int_{\gamma_i}^{2} \rho_{sc}(E)\,dE = \frac{i}{N}. \quad (2.9)$$

For the matrix $M$ we will denote,

$$m(z) := \frac{1}{N \text{tr}} \frac{1}{M - z} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(M) - z}. \quad (2.10)$$

We will also denote by $m_v(z)$ the following quadratic form in the resolvent,

$$m_v(z) := \frac{1}{N} v^T \frac{1}{M - z} v = \frac{1}{N} \sum_{i=1}^{N} (v^T u_i(M))^2 \quad (2.11)$$

We will denote by $v_i$ the projections of the eigenvectors $u_i(M)$ onto $v$,

$$v_i := v^T u_i(M). \quad (2.12)$$

Note that we will not need to refer to the components of $v$ in the coordinate basis.

At one point we will need to separate the contribution of the largest eigenvalue of $M$ from the quantities $m(z)$ and $m_v(z)$ and so we denote,

$$\tilde{m}(z) := \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(M) - z}, \quad \tilde{m}_v(z) := \frac{1}{N} \sum_{j=2}^{N} (v^T u_j(M))^2 \quad (2.13)$$

Our analysis of the various integrals over the sphere $S^{N-1}$ will proceed via the method of steepest descent. In our application of this method we will have use of the function $G(z)$, which is defined by

$$G(z) := \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i(M)) - \frac{h^2 \beta^2}{N} v^T \frac{1}{M - z} v. \quad (2.14)$$
and its approximation, the function \( g(z) \) defined by
\[
g(z) := \beta z - \int \log(z-x) \rho_{oo}(x) \, dx - h^2 \beta m_{oo}(z).
\] (2.15)

We define here as well the saddles \( \gamma \) and \( \hat{\gamma} \) which are the unique solutions \( \gamma > \lambda_1(M) \) and \( \hat{\gamma} > 2 \) satisfying
\[
G'(\gamma) = 0, \quad g'(\hat{\gamma}) = 0.
\] (2.16)

The parameter
\[
\kappa = \hat{\gamma} - 2 > 0
\] (2.17)
will appear in many error estimates.

We use angular brackets to denote the Gibbs expectation,
\[
\langle f(\sigma) \rangle = \frac{1}{Z_{N,\beta,h}} \int f(\sigma) \exp[\beta H_N(\sigma)] d\omega_{N-1}(\sigma).
\] (2.18)

The notation \( \langle f(\sigma_1, \sigma_2) \rangle \) indicates the expectation with respect to independent copies \( \sigma_1, \sigma_2 \) of the Gibbs expectation.

Given two positive, possibly \( N \)-dependent quantities \( a_N \) and \( b_N \), the notation \( a_N \approx b_N \) means there is a constant \( c > 0 \) so that
\[
a_N \leq b_N \leq c^{-1} a_N.
\] (2.19)

We will say that an event \( \mathcal{F} \) holds with **overwhelming probability** if for any \( D > 0 \) we have \( \mathbb{P}[\mathcal{F}] \geq 1 - N^{-D} \) for all \( N \) large enough. Given a set of events \( \mathcal{F}_i \) depending on a parameter \( i \in I \) we say that the family \( \{\mathcal{F}_i\}_{i \in I} \) holds with overwhelming probability if for any \( D > 0 \) we have for all \( N \) large enough that \( \mathbb{P}[\mathcal{F}_i] \geq 1 - N^{-D} \) for all \( i \in I \).

### 2.2 High temperature or slowly decaying magnetic field (Gaussian regime)

If either the inverse temperature \( \beta \) satisfies \( \beta < 1 \) or if the magnetic field \( h \) is not too small, the fluctuations of the SSK model are in general Gaussian. All of the results in this subsection are proven in Section 5. The following theorem provides an expansion of the free energy in terms of random matrix quantities. Asymptotic fluctuation results from random matrix theory are then used to obtain the asymptotic fluctuations of the thermodynamic quantities of the SSK.

**Proposition 2.1.** Denote by \( \hat{\gamma} \) the unique solution in the region \( \hat{\gamma} > 2 \) to the equation
\[
0 = g'(\hat{\gamma}),
\] (2.20)
where \( g(z) \) was defined in Section 2.1. Denote
\[
\kappa := \sqrt{\hat{\gamma}} - 2
\] (2.21)
We assume that there is a \( \tau > 0 \) so that,
\[
N^{-1/3 + \tau} \leq \frac{h^2 \beta}{|1 - \beta| + \sqrt{h^2 \beta}} + (1 - \beta)_+ = \begin{cases} (1 - \beta) + \sqrt{h^2 \beta}, & 1 \geq \beta \\ \frac{h^2 \beta}{\beta - 1 + \sqrt{h^2 \beta}}, & \beta \geq 1 \end{cases}
\] (2.22)
as well as a constant \( c > 0 \) so that
\[
c < \beta < c^{-1}, \quad h < c^{-1}.
\] (2.23)
Then for any \( D > 0 \) and \( N \) large enough, the following expansion holds with probability at least \( 1 - N^{-D} \):
\[
F_{N,\beta,h} = \frac{G(\hat{\gamma})}{2} + \frac{1}{N} \log \Gamma(N/2) + \frac{1}{N}(N/2 - 1) \log(2/(N\beta)) - \frac{1}{2N} \log(N g''(\hat{\gamma}) \pi)
\]
\[
+ N^{-\tau/10} \mathcal{O} \left( \frac{1}{N} + \frac{h^2 \beta}{N^{1/2} \kappa^{1/4}} \right),
\] (2.24)
where $G$ is as in (2.14). The quantity $\kappa$ has the asymptotics,

$$ \kappa \approx \frac{h^2 \beta}{|1 - \beta| + \sqrt{h^2 \beta}} + (1 - \beta)_+. \quad (2.25) $$

The leading order fluctuations to the free energy can be seen from the expansion (2.24) to be contributed by the quantity $G(\hat{\gamma})$. The quantity $G(\hat{\gamma})$ involves random matrix quantities. The following theorem lists parameter regimes in which the random matrix literature can be applied to determine the asymptotic fluctuations of $G(\hat{\gamma})$. Note that in all cases, the error term in (2.24) is smaller than the fluctuations of $G(\hat{\gamma})$ by a polynomial factor $N^{-\epsilon}$, and so after subtracting the appropriate constant,

$$ C_N := \frac{1}{N} \log \Gamma(N/2) + \frac{1}{N} (N/2 - 1) \log(2/(N\beta)) - \frac{1}{2N} \log(N g''(\hat{\gamma}) \pi), \quad (2.26) $$

the same fluctuation results hold for $F_{N,\beta,h}$. The following theorem is proven in Section 5.1.

**Theorem 2.2.** We have the following results in the various parameter regimes.

1. Assume that $0 < \beta \leq 1$, $h^2 \geq N^\epsilon (N^{-1/2} (1 - \beta)^{1/2} + N^{-2/3})$ and $\beta > c$ for some $c > 0$. Let,

$$ V_N := \frac{\hat{\gamma} + \sqrt{\gamma^2 - 4}}{\sqrt{\gamma + 2}} m_{sc}(\hat{\gamma})^4 \left( m_{sc}(\hat{\gamma})^2 + (1 - N^{-2}\|v\|^2)(1 - m_{sc}(\hat{\gamma})) \right). \quad (2.27) $$

The quantity $V_N$ satisfies $c' \leq V_N \leq (c')^{-1}$ for some $c' > 0$. The random variables,

$$\frac{N^{1/2}\kappa^{1/4}}{h^2 \beta V_N^{1/2}} (G(\hat{\gamma}) - g(\hat{\gamma})) \quad \text{and} \quad 2\frac{N^{1/2}\kappa^{1/4}}{h^2 \beta V_N^{1/2}} (F_{N,\beta,h} - C_N) \quad (2.28)$$

converge to standard normal random variables.

2. Suppose that $c \leq \beta \leq 1 - c$ and $h^2 \leq N^{-1/2-\epsilon}$ for some $\epsilon, c > 0$. Then,

$$ \frac{N}{B_N} (G(\hat{\gamma}) - g(\hat{\gamma}) - A_N), \quad 2\frac{N}{B_N} (F_{N,\beta,h} - g(\hat{\gamma}) - A_N - C_N) \quad (2.29)$$

converge to standard normal random variables where

$$ A_N := \frac{1}{2} \log(1 - \beta^2) - \beta^2 \quad (2.30)$$

and

$$ B_N^2 := -2 \log(1 - \beta^2) - 2\beta^2. \quad (2.31)$$

3. Assume that $C \geq \beta \geq 1$ and $C \geq h^2 \geq N^\tau (N^{-1/3} |1 - \beta| + N^{-2/3})$. Then the results in part 1 hold.

The function $G(\hat{\gamma})$ has two components which fluctuate on different scales. The first is a linear spectral statistic fluctuating on the scale $\sim N^{-1}$ and the second is a quadratic form in the resolvent fluctuating on the scale $h^2 N^{-1/2-\kappa^{-1/4}}$. The gap between the first two parts of the previous theorem (e.g., when $c < \beta < 1 - c$ and $h^2 \sim N^{-1/2}$) represents the regime where these scales are the same; as far as we know a theorem on the joint fluctuations of these two quantities is not available in the random matrix literature, but it would not be difficult to prove given current techniques. However, this is beyond the scope of the current work. However, if one makes either of the following two modifications to the model we are able to obtain a statement about free energy fluctuations in the case $\beta < 1 - c$ and $h^2 = \theta N^{-1/2}$. If either one replaces the SSK Hamiltonian by the simpler model

$$ \tilde{H}_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i,j} g_{ij} \sigma_i \sigma_j + hv^T \sigma \quad (2.32)$$

(note that the diagonal is included) or allow $v$ to be uniformly distributed on the sphere, then we obtain the following.
Theorem 2.3. Assume that $c \leq \beta \leq 1 - c$ and $\beta h^2 = \theta N^{-1/2}$. If the SSK model is replaced by the model above in (2.32) or $v$ is uniformly distributed on the sphere of radius $\sqrt{N}$, then

$$\frac{N}{V_N}(G(\hat{\gamma}) - g(\hat{\gamma}) - E_N), \quad \frac{2N}{V_N}(F_{N,\beta,h} - g(\hat{\gamma}) - E_N - C_N)$$

(2.33)

converge to standard normal random variables, where $E_N$ and $\tilde{V}_N$ are defined in (5.48) below.

The form of $E_N$ and $\tilde{V}_N$ is not particularly enlightening so we defer their definition to when we actually calculate these quantities.

Our next theorem concerns the overlap between a sample $\sigma$ drawn from the Gibbs measure and the external field $v$. We obtain the following result for the Laplace transform of $\sigma \cdot v$.

Theorem 2.4. Assume that (2.22) and (2.23) hold for a $\tau > 0$ and $c > 0$. The following holds for some $c_1 > 0$. For any $D > 0$ and $C > 0$ there is an event with probability at least $1 - N^{-D}$, for $N$ large enough, on which the following estimates hold uniformly for $|\lambda| \leq C$,

$$\log \left( \exp \left[ \lambda N^{-1/2} v \cdot \sigma \right] \right) = \lambda^2 \frac{m_{sc}(\hat{\gamma})}{2 \beta g''(\hat{\gamma})} \left( -m_{sc}(\hat{\gamma}) + 2h^2 \beta (m_{sc}'(\hat{\gamma}))^2 \right) - \lambda \left( N^{-1/2} h v^T (M - \gamma)^{-1} v \right) + O(N^{-c_1})$$

(2.34)

where $\gamma$ is the unique solution $\gamma > \lambda_1(M)$ satisfying $G'(\gamma) = 0$. The coefficient of the quadratic term in $\lambda$ satisfies

$$\frac{m_{sc}'(\hat{\gamma})}{2 \beta g''(\hat{\gamma})} \left( -m_{sc}(\hat{\gamma}) + 2h^2 \beta (m_{sc}'(\hat{\gamma}))^2 \right) = 1.$$ (2.35)

This theorem is proven in Section 5.2. The term linear in $\lambda$ corresponding to the quenched expectation of $N^{-1/2} v \cdot \sigma$ depends on the disorder. Its fluctuations with respect to the disorder random variables are larger than the order 1 fluctuations of $N^{-1/2} v \cdot \sigma$ with respect to the Gibbs measure, and so cannot be replaced by a deterministic quantity in the above statement. Note also that it involves the random saddle $\hat{\gamma}$ instead of its deterministic approximation $\tilde{\gamma}$; the random saddle cannot in general be replaced by $\hat{\gamma}$ as the error is too large to still obtain a statement with an $o(1)$ error in the above estimate.

As can be seen from the following, the linear term is indeed the quenched expectation.

Lemma 2.5. There is a $c_1 > 0$ so that the following holds. For any $D > 0$ with probability at least $1 - N^{-D}$ and $N$ large enough,

$$\langle N^{-1/2} v \cdot \sigma \rangle = N^{-1/2} h v^T (M - \gamma)^{-1} v + O(N^{-c_1}).$$

(2.36)

Finally, we state a theorem for the fluctuations of the overlap between two replicas.

Theorem 2.6. Assume that (2.22) and (2.23) hold for some $\tau > 0$ and $c > 0$. There is a $c_1 > 0$ so that the following holds. For any $D > 0$ and $C > 0$ there is an event with probability at least $1 - N^{-D}$ for $N$ large enough on which the following estimates hold uniformly for $|\lambda| \leq C$,

$$\log \left( \exp \left[ \lambda N^{-1/2} K^{1/4} \sigma^{(1)} \cdot \sigma^{(2)} \right] \right)$$

$$= \frac{\lambda^2}{2 \beta} (m_{sc}'(\hat{\gamma}) K^{1/2} m_{sc}'(\hat{\gamma}) - 2h^2 \beta m_{sc}(\hat{\gamma}) m_{sc}''(\hat{\gamma})) + \lambda N^{-1/2} K^{1/4} h v^T (M - \gamma)^{-2} v + O(N^{-c_1}).$$

(2.37)

This theorem is proven in Section 5.3. For the quenched expectation of the overlap we have

Lemma 2.7. There is a $c_1 > 0$ so that the following holds. For any $D > 0$ with probability at least $1 - N^{-D}$ and $N$ large enough,

$$\frac{K^{1/4}}{N^{1/2}} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle = N^{-1/2} K^{1/4} h v^T (M - \gamma)^{-2} v + O(N^{-c_1}).$$

(2.38)
2.3 Low temperature and intermediate magnetic field

The second scaling regime we find is when the temperature is low ($\beta > 1$) and the magnetic field scales like $h \sim N^{-1/6}$. To be more precise, we assume

$$1 + c \leq \beta \leq c^{-1}, \quad h^2 \beta = \theta N^{-1/3}$$

(2.39)

for a fixed $c > 0$ and fixed $\theta > 0$.

Our first results in this regime concern the free energy fluctuations, which are no longer Gaussian. The following theorem provides an expansion of the free energy in terms of spectral quantities associated to the GOE. It is proven in Section 6.1.

**Theorem 2.8.** Assume that (2.39) holds. For sufficiently small $\varepsilon > 0$ we have that the following estimate holds with probability at least $1 - N^{-\varepsilon}$ for $N$ large enough. We have,

$$N^{2/3} \left( F_{N, \beta, h} - \frac{1}{N} \log(\Gamma(N/2)) + \frac{1}{N} (N/2 - 1) \log(2/(N\beta)) - C_N \right) = \frac{1}{2} X_N + O(N^{-\varepsilon}),$$

(2.40)

where $X_N$ is a random variable that is equal in distribution to a random variable $Y_N$ that satisfies the following. With probability at least $1 - N^{-\varepsilon},$

$$Y_N = N^{2/3}(\beta - 1)(x_b - 2) - \frac{\theta}{N^{2/3}} \sum_{i} \frac{g_i^2}{\mu_i - x_b} + O(N^{-\varepsilon})$$

(2.41)

where the $\{\mu_i\}_{i=1}^N$ are GOE eigenvalues, $\{g_i\}_i$ are independent standard normals and $x_b$ is the smallest solution larger than $\mu_1$ to the equation

$$(\beta - 1) = \frac{\theta}{N^{4/3}} \sum_{i} \frac{g_i^2}{(\mu_i - x_b)^2}.$$

(2.42)

The random variable $X_N$ is constructed from the eigenvalues and eigenvectors of $H$, the GOE matrix associated to $M$ via $H = M + V$ where $V$ is a diagonal matrix of independent Gaussians with variance $1/N$. The proof of the above theorem explicitly constructs $X_N$ (it is of course very similar to $Y_N$ but not stated here for concerns of brevity).

The second result concerns the convergence of the leading order contribution to the free energy. For this statement we let $\{g_i\}_{i=1}^\infty$ be an infinite sequence of iid standard normal random variables, and $\{\chi_i\}_{i=1}^\infty$ be the Airy$_1$ random point field whose first $n$ particles are the limits of the top $n$ rescaled eigenvalues of the GOE, $\{N^{2/3}(\mu_i - 2)\}_{i=1}^n.$

We let $a > 0$ be the unique solution to

$$\beta - 1 = \theta \sum_{i=1}^\infty \frac{g_i^2}{(\chi_i - \chi_1 - a)^2}$$

(2.43)

and $\xi$ be the random variable,

$$\xi := \lim_{n \to \infty} \left( (\beta - 1)(\chi_1 + a) - \theta \left( \sum_{i=1}^n \frac{g_i^2}{\chi_i - \chi_1 - a} + \frac{1}{\pi} \int_0^{(3\pi n/2)^{2/3}} \frac{1}{\sqrt{x}} dx \right) \right).$$

(2.44)

The following theorem is proven in Section 6.2.

**Theorem 2.9.** With $x_b$ as above, the random variable

$$\xi_N := N^{2/3}(\beta - 1)(x_b - 2) - \frac{\theta}{N^{2/3}} \left( \sum_{i=1}^N \frac{g_i^2}{\mu_i - x_b} - N \int \frac{\rho_{sc}(x)}{x - 2} dx \right)$$

(2.45)

converges in distribution to $\xi$ as $N \to \infty$.

For the overlap with the external field, we have the following theorem. This theorem is proven in Section 6.3.
Theorem 2.10. For $|t| \leq C$ we have with probability at least $1 - N^{-\varepsilon}$ and $\varepsilon > 0$ small enough,
\[
\log \langle \exp \left[ \beta^{1/2} t N^{-1/2} v^T \sigma \right] \rangle = \frac{t^2}{2} - \theta^{1/3} (N^{1/3} t) (N^{-1} v^T (M - \gamma)^{-1} v) + \mathcal{O}(N^{-\varepsilon})
\]  
(2.46)

some $\varepsilon > 0$.

For the quenched expectation we have the following.

Theorem 2.11. The random variable
\[
N^{1/3} \left( \frac{1}{N} v^T (M - \gamma)^{-1} v - m_{sc}(2) \right)
\]  
(2.47)
converges in distribution to the random variable,
\[
\lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{g_i^2}{\chi_i - a} + \frac{1}{\pi} \int_{0}^{1} \frac{1}{\sqrt{x}} \text{d}x \right)
\]  
(2.48)
where $a$ is as in (2.43).

For the overlap between two replicas we have the following. It is proven in Section 6.4.

Theorem 2.12. There is a small $c_1 > 0$ so that the following holds. For every $\alpha > 0$ and positive integer $k$, we have for all sufficiently small $\varepsilon > 0$ that there is an $\varepsilon_1 > 0$ such that the following holds on an event of probability at least $1 - N^{-\varepsilon_1}$. We have for all $|t| \leq N^{-\alpha}$,
\[
\log \left\langle \exp \left( t N^{-2/3} \sigma^{(1)} \cdot \sigma^{(2)} \right) \right\rangle = \sum_{j=1}^{k} t^j Z_j + \mathcal{O} \left( N^{-c_1} + |t|^{k+1} N^{\varepsilon} \right)
\]  
(2.49)
where the $Z_j$ are specific functions of the spectral quantities of $M$. On the above event we have $|Z_j| \leq N^\varepsilon$. Additionally, the $Z_j$ jointly converge in distribution to functions of the Airy$_1$ random point field.

We were not able to find a simple form for the $Z_j$; they come from a certain Taylor expansion and involve general sums of products of the Stieltjes transforms of $M$ and the quadratic form $v^T (M - \gamma)^{-k} v$ evaluated at the point $\gamma$ defined above which comes from the steepest descent analysis. However, we are able to show that they converge in distribution to similar quantities of the Airy$_1$ random point field. The latter quantities are constructed by replacing the eigenvalues of $M$ by the particles of the Airy$_1$ random point field and the inner products $v \cdot u_i(M)$ by iid standard normal random variables.

2.4 Low temperature and microscopic magnetic field

The third and final scaling regime we consider is when the temperature is low, $\beta > 1$ and the magnetic field is of order $h \sim N^{-1/2}$. We assume,
\[
1 + c \leq \beta \leq c^{-1}, \quad h^2 \beta = \theta N^{-1},
\]  
(2.50)
for fixed $c > 0$ and $\theta > 0$. The results in this section are proven in Section 7.

Our first results show that for the free energy, the fluctuations coincide with the low temperature result of [6] in that they are governed by the largest eigenvalue of $M$ and are asymptotically Tracy-Widom. This theorem is proven in Section 7.1.

Theorem 2.13. Under the assumption (2.50) we have for all sufficiently small $\varepsilon$ that with probability at least $1 - N^{-\varepsilon_1}$ for some $\varepsilon_1 > 0$ and $N$ large enough that,
\[
\frac{1}{N} \log Z_{N,\beta,h} = \frac{\beta - 1}{2} (\lambda_1(M) - 2) + C_N + \mathcal{O}(N^{-1+\varepsilon}).
\]  
(2.51)
where,
\[
C_N := 2\beta - \int (2 - x) \rho_{sc}(x) \text{d}x + \frac{1}{N} \left( (1 - N/2) \log(\beta) + \frac{N}{2} \log(2\pi) + \frac{1}{2} \log(N) \right).
\]  
(2.52)
Hence, $N^{2/3} (\beta - 2)^{-1} (F_{N,\beta,h} - C_N)$ converges to a $TW_1$ random variable.
The following results concern the overlap between two replicas. They all are proven in Section 7.2. The first concerns the fluctuations of the weights in the Parisi measure. We recall the notation,

\[ v_1 := v^T u_1(M). \] (2.53)

We will later see that \( v_1 \) converges to a standard normal random variable.

**Theorem 2.14.** Suppose that (2.50) holds. For all sufficiently small \( \varepsilon > 0 \) there is an \( \varepsilon_1 > 0 \) so that with probability at least \( 1 - N^{-\varepsilon_1} \) and all \( t > 0 \),

\[ \langle 1_{|N^{-1}\sigma^{(1)} \cdot \sigma^{(2)}| \leq t} \rangle = \frac{1}{2} \pm \frac{1}{2} \tanh^2 \left( \sqrt{v_1^2 \theta(\beta - 1)} \right) + N^\varepsilon \mathcal{O} \left( t + N^{-2/3+\varepsilon} t^{-2} + N^{-1/3} \right). \] (2.54)

We have also that \( v_1 \) converges to a standard normal random variable as \( N \to \infty \), and so we get a convergence in distribution result for the random variable on the LHS for any \( t \) satisfying \( N^{-1/3+3\varepsilon} \leq t \leq N^{-2\varepsilon} \).

We recall our definition

\[ \hat{m}(z) := \frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(M) - z}. \] (2.55)

We will often consider \( \hat{m}(\lambda_1(M)) \) which we will later see satisfies

\[ |\hat{m}(\lambda_1(M)) + 1| \leq N^{-1/3+\varepsilon} \] (2.56)

with probability at least \( 1 - N^{-\varepsilon/10} \) for any sufficiently small \( \varepsilon > 0 \). We have the following result for the quenched expectation of the overlap.

**Theorem 2.15.** For all sufficiently small \( \varepsilon > 0 \) there is an \( \varepsilon_1 > 0 \) so that with probability at least \( 1 - N^{-\varepsilon_1} \),

\[ \frac{1}{N} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle = \frac{\beta + \hat{m}(\lambda_1(M))}{\beta} \left( \tanh \left( \sqrt{v_1^2 \theta(\beta + \hat{m}(\lambda_1(M)))} \right) \right)^2 + \mathcal{O}(N^{-2/3+\varepsilon}). \] (2.57)

In particular, \( N^{-1} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \) converges in distribution to the random variable \( (1-\beta^{-1}) \tanh^2 \sqrt{Z^2 \theta(\beta - 1)} \) where \( Z \) is a standard normal random variable.

In the work [14] we considered the random variable,

\[ \Xi_N := N^{1/3}(\hat{m}(\lambda_1(M)) + 1) \] (2.58)

and showed that it has a limit \( \Xi \) given by,

\[ \Xi := \lim_{n \to \infty} \left( \sum_{i=2}^{n} \frac{1}{\lambda_i - \lambda_1} + \int_{0}^{(\frac{\Delta n}{\pi})^{2/3}} \frac{1}{\pi \sqrt{x}} dx \right) \] (2.59)

where \( \{\lambda_i\}_{i=1}^{\infty} \) are the particles of the Airy random point field.

The random variable \( \Xi_N \) appears in Theorem 2.14 and it would be of interest to obtain a kind of second order fluctuation or conditional result about the quenched expectation of the overlap.

If we Taylor expand the leading order term in (2.57) in \( 1 + \hat{m}(\lambda_1(M)) \) we obtain a convergence statement for a randomly rescaled version of \( N^{-1} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle \) to \( \Xi \) after subtracting and dividing by explicit functions of \( v_1^2 \). However, in our view this is not quite satisfactory as it falls short of a statement of the fluctuations of the overlap conditional on \( v_1^2 \). This is due to the fact that, for the SSK as defined above, the eigenvector projection \( v_1 \) is not stochastically independent from the eigenvalues of \( M \).

However, we are able to obtain a finer result under either of two possible modifications to the model. (These modifications were also introduced above in our discussion of the high temperature and \( h \sim N^{-1/4} \) regime above. We restate them quickly here for the purposes of discussion).
The first modification is if we replace the SSK Hamiltonian by a simpler model,

$$\tilde{H}_N(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i,j} g_{ij} \sigma_i \sigma_j + hv^T \sigma. \tag{2.60}$$

Denote now by $H$ the matrix given by $H_{ij} = (2N)^{-1}(g_{ij} + g_{ji})$. In this case, the same estimate as (2.57) holds with the spectral quantities of $M$ now replaced by those of $H$. In the case of $H$, the eigenvectors and eigenvalues are independent and so we can obtain a statement of the conditional fluctuations of the quenched overlap. The second modification is to simply let $v$ be a vector uniformly distributed on the sphere. In either case we obtain the following.

**Theorem 2.16.** Assume either that the SSK model is replaced by the simpler GOE-associated model (2.60), or that the vector $v$ is uniformly distributed on the $N - 1$ sphere of radius $\sqrt{N}$. Let $\varepsilon > 0$ be sufficiently small, and $F$ a Lipschitz function. Denote by $z$ the random variable $v^T \cdot u_1(H)$ in the case that $M$ is replaced by $H$ or $v^T u_1(M)$ in the case that it is uniformly distributed. There is an event of probability at least $1 - N^{-\varepsilon}$ so that the following holds.

$$\mathbb{E} \left[ F\left(N^{1/3} b^{-1} \left(N^{-1}\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle - a \right) \right) \right] = \mathbb{E}[F(\Xi_N)] + \mathcal{O}(N^{-\varepsilon}), \tag{2.61}$$

where,

$$a := \frac{\beta - 1}{\beta} \tanh^2(Z),$$

$$b := \tanh(Z) \frac{\sinh(Z) \cosh(Z) + Z}{\beta \cosh^2(Z)},$$

$$Z := \sqrt{\beta^2 \theta(\beta - 1)}. \tag{2.62}$$

We have a final result on the fluctuations of the variance.

**Theorem 2.17.** For all sufficiently small $\varepsilon > 0$ there is an $\varepsilon_1 > 0$ so that we have the following estimate with probability at least $1 - N^{-\varepsilon_1}$,

$$\frac{1}{N^2} \left\langle \left( \sigma^{(1)} \cdot \sigma^{(2)} \right) \right\rangle - \left( \frac{1}{N} \left\langle \sigma^{(1)} \cdot \sigma^{(2)} \right\rangle \right)^2 = \frac{(\beta + \tilde{m}(\lambda_1(M)))^2}{\beta^2} \left(1 - \tanh^4 \left( \sqrt{\beta^2 \theta(\beta - 1)} \right) \right) + \mathcal{O}(N^{-2/3 + \varepsilon}), \tag{2.63}$$

and so the random variable on the LHS converges in distribution to the random variable $(1 - \beta^{-1})^2 (1 - \tanh^4 (\sqrt{W^2 \theta(\beta - 1)}))$ where $W$ is a standard normal random variable.

### 3 Results from random matrix theory

In this section we collect some results from random matrix theory. Recall that the matrix $M$ is defined by,

$$M_{ij} = -\frac{1}{\sqrt{2N}} (g_{ij} + g_{ji}), \quad i \neq j \tag{3.1}$$

and $M_{ii} = 0$. The matrix $M$ is, up to the diagonal being 0, a matrix from the Gaussian Orthogonal Ensemble. The semicircle density and its Stieltjes transform are given by,

$$\rho_{sc}(E) := \frac{1}{2\pi} \sqrt{(4 - E^2)_+}, \quad m_{sc}(z) := \int \frac{d\rho_{sc}(E)}{E - z}. \tag{3.2}$$

We denote by $m(z)$ the empirical Stieltjes transform of $M$,

$$m(z) := \frac{1}{N} \text{tr} \left( \frac{1}{M - z} \right). \tag{3.3}$$

For this quantity, we have the following local semicircle law and rigidity result. The statements and proofs can be found in Theorem 2.6 and Theorem 10.3 of [9].
Theorem 3.1. Let $M$ be as above. Let $\omega > 0$. Consider the spectral domain,
\[ S_1 := \{ z = E + i\eta : |E| \leq \omega^{-1}, N^{-1+\omega} \leq \eta \leq \omega^{-1} \}. \tag{3.4} \]
For any $\varepsilon > 0$ and $D > 0$, the following estimate holds for $N$ large enough.
\[ \mathbb{P} \left[ \bigcap_{z \in S_1} \left\{ |m(z) - m_{\text{sc}}(z)| \leq \frac{N^\varepsilon}{N^{\eta}} \right\} \right] \geq 1 - N^{-D}. \tag{3.5} \]
Consider the spectral domain,
\[ S_2 := \{ z = E + i\eta : N^{-2/3+\omega} \leq |E| \leq \omega^{-1}, 0 \leq \eta \leq \omega^{-1} \}. \tag{3.6} \]
For any $\varepsilon > 0$ and $D > 0$, the following estimate holds for $N$ large enough.
\[ \mathbb{P} \left[ \bigcap_{z \in S_2} \left\{ |m(z) - m_{\text{sc}}(z)| \leq \frac{N^\varepsilon}{N(|E| - 2 + \eta)} \right\} \right] \geq 1 - N^{-D}. \tag{3.7} \]
Denote the quantites of the semicircle distribution by $\gamma_i$; they are defined by
\[ \frac{i}{N} = \int_{\gamma_i}^2 d\rho_{\text{sc}}(E) \tag{3.8} \]
and are also called the classical eigenvalue locations. The eigenvalues of $M$ are denoted by
\[ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \]
and are labelled in decreasing order. We have the following rigidity theorem, a statement of which can be found in Theorem 2.9 of [9].

Theorem 3.2. For any $\varepsilon > 0$ and $D > 0$ we have for $N$ large enough,
\[ \mathbb{P} \left[ \bigcap_j \left\{ |\lambda_j - \gamma_j| \leq N^\varepsilon N^{-2/3} \min\{ j^{-1/3}, (N + 1 - j)^{-1/3} \} \right\} \right] \geq 1 - N^{-D}. \tag{3.9} \]

We will also need so-called “isotropic” local law and delocalization results for the matrix $M$. To state them, we resolvent of $M$ by
\[ R(z) = \frac{1}{M - z}. \tag{3.10} \]
The following statements are from Theorems 2.12 and 2.15 of [1].

Theorem 3.3. Let $\omega > 0$ and $S_1$ and $S_2$ be as above. Let $\varepsilon > 0$ and $D > 0$. Let $S^{N-1}$ denote the unit sphere in $\mathbb{R}^N$. For $N$ large enough, we have the following estimates,
\[ \inf_{u,w \in S^{N-1}} \mathbb{P} \left[ \bigcap_{z \in S_1} \left\{ |u^T R(z) w - (u^T w) m_{\text{sc}}(z)| \leq N^\varepsilon \left( \frac{\text{Im} m_{\text{sc}}(z)}{N^{\eta}} + \frac{1}{N^{\eta}} \right) \right\} \right] \geq 1 - N^{-D}. \tag{3.11} \]
and
\[ \inf_{u,w \in S^{N-1}} \mathbb{P} \left[ \bigcap_{z \in S_2} \left\{ |u^T R(z) w - (u^T w) m_{\text{sc}}(z)| \leq N^\varepsilon \left( \frac{\text{Im} m_{\text{sc}}(z)}{N^{\eta}} \right) \right\} \right] \geq 1 - N^{-D}. \tag{3.12} \]
If $u_i$ are the eigenvectors of $M$, then for any $\varepsilon > 0$ and $D > 0$ we have for $N$ large enough,
\[ \inf_{w \in S^{N-1}} \mathbb{P} \left[ \bigcap_j \{ N^{1/2} |w^T u_i| \leq N^\varepsilon \} \right] \geq 1 - N^{-D}. \tag{3.13} \]
The following level repulsion estimate is proven in [14].

Theorem 3.4. Let $\varepsilon > 0$. For $N$ large enough it holds for all $N^{-1/3} < s < 1$ that
\[ \mathbb{P}[|\lambda_1(M) - \lambda_2(M)| \leq sN^{-2/3}] \leq N^\varepsilon s, \quad \mathbb{P}[|\lambda_1(H) - \lambda_2(H)| \leq sN^{-2/3}] \leq N^\varepsilon s. \tag{3.14} \]
3.1 Comparison of $M$ to the GOE

Let $H = M + V$ where $V$ is a diagonal matrix whose entries are iid Gaussians with variance $2/N$, so that $H$ is a matrix from the GOE. The first estimate in the following was proven in [14]. The second estimate is proven in Appendix B.

**Theorem 3.5.** Let $\varepsilon > 0$ and $D > 0$. For $N$ large enough it holds that,

$$
\mathbb{P}\left( \bigcap_{1 \leq j \leq N^{1/10}} \left\{ |\lambda_j(M) - \lambda_j(H)| \leq \frac{N\varepsilon}{N} \right\} \right) \geq 1 - N^{-D}.
$$

(3.15)

Let $u_1(M)$ and $u_1(H)$ be the first eigenvectors of $M$ and $H$. There is a $c > 0$ so that the following estimate holds.

$$
\sup_{v \in S^{N-1}} \mathbb{P}\left[ |(v^T u_1(M))^2 - (v^T u_1(H))^2| \geq N^{-1-c} \right] \leq N^{-c}.
$$

(3.16)

We also have the following lemma, proven in Appendix B.

**Lemma 3.6.** Let $\delta > 0$ and $\omega > 0$. For any $\varepsilon$ and $D > 0$, the following estimates hold for $N$ large enough. First,

$$
\sup_{v \in S^{N-1}} \mathbb{P}\left( \bigcap_{z \in S_1 \cap \{\eta \leq N^{-\delta}\}} \left\{ |v^T (M - z)^{-1} v - v^T (H - z)^{-1} v| \leq N^\varepsilon \left( \frac{1}{\sqrt{N}} + \frac{1}{(N\eta)^2} + \frac{\text{Im}[m_{sc}(z)]}{N\eta} \right) \right\} \right) \geq 1 - N^{-D}.
$$

(3.17)

Second,

$$
\sup_{v \in S^{N-1}} \mathbb{P}\left( \bigcap_{z \in S_2 \cap \{\eta \leq N^{-\delta}\}} \left\{ |v^T (M - z)^{-1} v - v^T (H - z)^{-1} v| \leq N^\varepsilon \left( \frac{1}{\sqrt{N}} + \frac{\text{Im}[m_{sc}(z)]}{N\eta} \right) \right\} \right) \geq 1 - N^{-D}.
$$

(3.18)

3.2 Random matrix fluctuation results

We have the following general result for the linear statistics of symmetric Wigner matrices. An $N \times N$ matrix $W$ is said to be a real symmetric Wigner matrix if upper triangular part consists of independent centered entries such that all entries obey the moment conditions,

$$
\mathbb{E}[|X_{kl}|^k] \leq C_k,
$$

(3.19)

the variance of the off-diagonal entries is $N^{-1}$ and for simplicity we assume that the variance of the diagonal entries equals $w_2/N$ for a fixed constant $w_2$ and that the fourth moment of the off-diagonal entries is $\mathbb{E}[(W_{ij})^4] = S_4 N^{-2}$ for some fixed $S_4$. The following result can be found in [3, 4, 15, 18].

**Theorem 3.7.** Let $f$ be a smooth function. Then,

$$
\frac{1}{V^{1/2}(f)} \left( \text{tr} f(W) - M(f) - N \int f(x) \rho_{sc}(x) dx \right)
$$

(3.20)

converges to a standard normal random variable where,

$$
M(f) = \frac{1}{4} (f(2) + f(-2)) + \frac{1}{2\pi} \int_{-2}^2 f(x) \frac{-1 + (w_2 - 2)(x^2 - 2) + (S_4 - 3)(x^4 - 4x^2 + 1)}{\sqrt{4 - x^2}} dx
$$

(3.21)

and

$$
V(f) = \frac{1}{2\pi^2} \int_{-2}^2 \int_{-2}^2 \left( \frac{f(x) - f(y)}{x - y} \right)^2 \frac{4 - xy}{\sqrt{4 - x^2} \sqrt{4 - y^2}} dx dy
$$

$$
+ (w_2 - 2) \left( \frac{1}{2\pi} \int_{-2}^2 \frac{f(x)}{\sqrt{4 - x^2}} dx \right)^2 + 2(S_4 - 3) \left( \frac{1}{2\pi} \int_{-2}^2 f(x) \frac{x^2 - 2}{\sqrt{4 - x^2}} dx \right)^2.
$$

(3.22)
4 Representation formulas

This section contains the various representation formulas for the free energy and various Gibbs expectations that we will use. They reduce various high-dimensional integrals over the sphere \((N-1)\)-sphere to low dimensional contour integrals. Such representations were first used in \([13]\) and \([6]\) to study the free energy. These representations were extended in \([14,17]\) to study the overlap in the model without magnetic field. Here we extend these representations to allow for a magnetic field in the Hamiltonian.

Recall that \(d\omega_{N-1}\) is the uniform measure on the \(N-1\) sphere of radius \(\sqrt{N}\), which we denote by \(\mathbb{S}^{N-1}\). Under \(d\omega_{N-1}\), the sphere has volume

\[
|\mathbb{S}^{N-1}| = \int d\omega_{N-1}(\sigma) = \frac{2\pi^{N/2}}{\Gamma(N/2)} N^{N-1/2}. \tag{4.1}
\]

**Proposition 4.1.** Let \(M\) be a real symmetric matrix, \(v\) a vector and \(\beta, \lambda > 0\). Then,

\[
\int \exp\left[\frac{\beta}{2} \sigma^T M \sigma + \lambda v^T \sigma \right] d\omega_{N-1}(\sigma) = \frac{\beta^{N/2}}{(2\pi)^{N/2}} \int_{\gamma-i\infty}^{\gamma+i\infty} \exp\left[\frac{N}{2} G_o(z,v,\lambda,\beta)\right] dz \tag{4.2}
\]

where,

\[
G_o(z,v,\lambda,\beta) := \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i(M)) - \frac{\lambda^2}{N\beta} v^T \frac{1}{M - z} v \tag{4.3}
\]

and \(\gamma > \lambda_1(M)\). Furthermore,

\[
\int \int \exp\left[\frac{\beta}{2} \sigma_1^T M \sigma_1 + \frac{\beta}{2} \sigma_2^T M \sigma_2 + t \sigma_1^T \sigma_2 + \lambda v^T (\sigma_1 + \sigma_2) \right] d\omega_{N-1}(\sigma_1) d\omega_{N-1}(\sigma_2) = \frac{\beta^2 N}{(2\pi)^N} \int \int \exp\left[\frac{N}{2} G_d(z,w,v,t,\lambda,\beta)\right] dz dw, \tag{4.4}
\]

where

\[
G_d = \beta(z + w) - \frac{1}{N} \sum_{i=1}^{N} \log(\beta^2 (z - \lambda_i)(w - \lambda_i) - t^2) - \frac{\lambda^2}{N} \sum_{i=1}^{N} (v^T u_i(M))^2 \frac{\beta(2\lambda_i - z - w) - 2t}{\beta^2(\lambda_i - w)(\lambda_i - z) - t^2}. \tag{4.5}
\]

**Proof.** Fix \(z \in \mathbb{C}\) with \(\text{Re}[z] > \lambda_1(M)\) and consider the Gaussian integral over \(\mathbb{R}^N\),

\[
\int_{\mathbb{R}^N} \exp\left[\frac{\beta}{2} x^T (M - z) x + \lambda v^T x \right] dx = \left(\frac{2\pi \beta}{\beta}\right)^{N/2} \exp\left[-\frac{\lambda^2}{2\beta} \frac{v^T (M - z)^{-1} v}{2} - \frac{1}{2} \sum_i \log(z - \lambda_i)\right]. \tag{4.6}
\]

On the other hand,

\[
\int_{\mathbb{R}^N} \exp\left[\frac{\beta}{2} x^T (M - z) x + \lambda v^T x \right] dx = \int_0^\infty \exp\left[-\frac{\beta z}{2}\right] r^{N-1} J(r) dr, \tag{4.7}
\]

where

\[
J(r) := \int_{|\sigma| = 1} \exp\left[\frac{\beta}{2} r^2 \sigma^T M \sigma + r \lambda v^T \sigma \right] d\sigma, \tag{4.8}
\]

and \(d\sigma\) is uniform measure over the \(N-1\) sphere of radius 1. Note that the LHS of (4.2) equals \(N^{N-1/2} J(\sqrt{N})\). Making a change of variable \(u = \frac{\beta}{2} r^2\), we see that

\[
\int_0^\infty \exp\left[-\frac{\beta z}{2} r^2\right] r^{N-1} J(r) dr = \int_0^\infty \exp[-zu] \frac{1}{\beta} \left(\frac{2u}{\beta}\right)^{\frac{N-2}{2}} J((2u/\beta)^{1/2}) du. \tag{4.9}
\]

The RHS is the Laplace transform of a function that equals the LHS of (4.2) at \(u = \frac{\beta}{2} N\) times the constant \(N^{-1/2}\beta^{-1}\). The result (4.2) follows from the Laplace inversion formula.
For (4.4), we begin with considering for fixed \( z, w \in \mathbb{C} \) and \( \text{Re}[z], \text{Re}[w] > \lambda_1(M) \), the following Gaussian integral

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \exp \left[ \frac{\beta}{2} x^T (M - z)x + \frac{\beta}{2} y^T (M - w)y + tx^T y + \lambda v^T (x + y) \right] \, dx \, dy 
\]

\[
= (2\pi)^N \exp \left[ -\sum_{i=1}^{N} \frac{(v^T u_i)^2}{2} \right] \frac{1}{\beta^2} \sum_{i=1}^{N} \log(\beta^2 (z - \lambda_i)(w - \lambda_i) - t^2) \right] . \tag{4.10}
\]

On the other hand, considering this integral in polar coordinates and making the same change of variables as above we have,

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \exp \left[ \frac{\beta}{2} x^T (M - z)x + \frac{\beta}{2} y^T (M - w)y + tx^T y + \lambda v^T (x + y) \right] \, dx \, dy 
\]

\[
= \int_{0}^{\infty} \int_{0}^{\infty} \exp \left[ -\frac{\beta}{2} \frac{z^T z}{r^2} - \frac{\beta}{2} \frac{w^T w}{r^2} \right] (r_1 r_2)^{N-1} \mathcal{J}(r_1, r_2) \, dr_1 \, dr_2 
= \int_{0}^{\infty} \int_{0}^{\infty} \exp \left[ -zu_1 - wu_2 \right] \frac{1}{\beta^2} \left( \frac{4u_1 u_2}{\beta^2} \right) \mathcal{J}(2u_1^2, 2u_2^2) \, du_1 \, du_2 \tag{4.11}
\]

where

\[
\mathcal{J}(r_1, r_2) := \int_{|x_1| \leq r_1 \leq |x_2| \leq r_2 = 1} \exp \left[ \frac{\beta}{2} \frac{r_1^2 r_2^2 M \sigma_1 + \beta}{\beta^2} \frac{r_1^2 r_2^2 M \sigma_2}{\beta^2} + \frac{1}{r_1^2 r_2^2} \mathcal{J}(G_u(z, v, \lambda, \beta) + G_o(w, v, \lambda, \beta)) \right] \, d\sigma_1 \, d\sigma_2 \tag{4.12}
\]

We recognize the last line of (4.11) as the multidimensional Laplace transform of a function that equals the LHS of (4.4) at \( u_1 = u_2 = \frac{1}{\beta} N \) times the constant \( N^{-1} \beta^{-2} \). The result follows.

**Proposition 4.2.** Let \( M \) be a real symmetric matrix, \( v \) a vector, and \( \beta, \lambda > 0 \). Then,

\[
\int (\sigma(1) \cdot \sigma(2))^2 \exp \left[ \frac{\beta}{2} (\sigma(1)^T M \sigma(1) + \sigma(2)^T M \sigma(2)) + \lambda v^T (\sigma(1) + \sigma(2)) \right] \, d\sigma(1) \, d\sigma(2)
\]

\[
= \frac{\beta^2 N}{(2\pi)^N} \mathcal{F} \left[ \frac{\beta}{2} \frac{r_1^2 r_2^2 M \sigma_1 + \beta}{\beta^2} \frac{r_1^2 r_2^2 M \sigma_2}{\beta^2} + \frac{1}{r_1^2 r_2^2} \mathcal{J}(G_u(z, v, \lambda, \beta) + G_o(w, v, \lambda, \beta)) \right] \, dz \, dw \tag{4.13}
\]

where \( \mathcal{F} \) is a vertical line in the complex plane lying to the right of \( \lambda_1(M) \) and the function \( G_o \) is as in the previous proposition. Additionally,

\[
\int (\sigma(1) \cdot \sigma(2))^2 \exp \left[ \frac{\beta}{2} (\sigma(1)^T M \sigma(1) + \sigma(2)^T M \sigma(2)) + \lambda v^T (\sigma(1) + \sigma(2)) \right] \, d\sigma(1) \, d\sigma(2)
\]

\[
= \frac{\beta^2 N}{(2\pi)^N} \mathcal{F} \left[ \frac{\beta}{2} \frac{r_1^2 r_2^2 M \sigma_1 + \beta}{\beta^2} \frac{r_1^2 r_2^2 M \sigma_2}{\beta^2} + \frac{1}{r_1^2 r_2^2} \mathcal{J}(G_u(z, v, \lambda, \beta) + G_o(w, v, \lambda, \beta)) \right] \, dz \, dw \tag{4.14}
\]

**Proof.** We first consider the Gaussian integral

\[
\int_{\mathbb{R}^N} x \cdot y \exp \left[ \frac{\beta}{2} (x^T (M - z)x + y^T (M - w)y) + \lambda v^T (x + y) \right] \, dy \exp \left[ -\frac{\lambda^2}{2\beta} (w^T (M - z)^{-1} v + v^T (M - w)^{-1} v) \right]
\]

\[
= \frac{\beta^2}{\beta^2} \frac{1}{(M - z)(M - w)} v \, \exp \left[ -\frac{\lambda^2}{2\beta} (w^T (M - z)^{-1} v + v^T (M - w)^{-1} v) \right] \, dy \tag{4.15}
\]

15
On the other hand,

\[
\int_{\mathbb{R}^{2N}} x \cdot y \exp \left[ \frac{\beta}{2} (x^T (M - z)x + y^T (M - w)y + \lambda v^T (x + y) \right] \\
= \int_0^\infty \int_0^\infty \exp \left[ \frac{-\beta z^2}{2} - \frac{\beta w^2}{2} \right] (rs)^{N-1} J_1(r,s)drds
\]  \hspace{1cm} (4.16)

where,

\[ J_1(r, s) = \int_{||\sigma|| = ||\omega|| = 1} (rs \omega \cdot \sigma) \exp \left[ \frac{\beta}{2} (r^2 \sigma M \sigma + s^2 \omega M \omega + \lambda v^T (r \sigma + s \omega) \right] d\sigma d\omega. \hspace{1cm} (4.17) \]

The quantity on the line (4.13) that we want to calculate equals \( N^{N-1} J_1(\sqrt{N}, \sqrt{N}) \). Proceeding as in the previous proposition yields the claimed representation.

For the second representation we begin with the Gaussian integral,

\[
\int_{\mathbb{R}^{2N}} (x \cdot y)^2 \exp \left[ \frac{\beta}{2} (x^T (M - z)x + y^T (M - w)y + \lambda v^T (x + y) \right] \\
= \left\{ \frac{1}{\beta^2} \sum_i \frac{1}{(\lambda_i - z)(\lambda_i - w)} - \frac{\lambda^2}{\beta^3} v^T \left( \frac{1}{(M - z)^2(M - w)} + \frac{1}{(M - w)^2(M - z)} \right) v \\
+ \frac{\lambda^4}{\beta^4} \left( v^T \frac{1}{(M - w)(M - z)} v \right)^2 \right\} \\
\times (2\pi)^N \exp \left[ \frac{\lambda^2}{2\beta} (v^T (M - z)^{-1} v + v^T (M - w)^{-1}v) - \frac{1}{2} \sum_i \log(z - \lambda_i) + \log(w - \lambda_i) \right] \hspace{1cm} (4.18)
\]

We have also that,

\[
\int_{\mathbb{R}^{2N}} (x \cdot y)^2 \exp \left[ \frac{\beta}{2} (x^T (M - z)x + y^T (M - w)y + \lambda v^T (x + y) \right] \\
= \int_0^\infty \int_0^\infty \exp \left[ \frac{-\beta z^2}{2} - \frac{\beta w^2}{2} \right] (rs)^{N-1} J_2(r,s)drds
\]  \hspace{1cm} (4.19)

where,

\[ J_2(r, 2) = \int_{||\sigma|| = ||\omega|| = 1} (rs \omega \cdot \sigma)^2 \exp \left[ \frac{\beta}{2} (r^2 \sigma M \sigma + s^2 \omega M \omega + \lambda v^T (r \sigma + s \omega) \right] d\sigma d\omega. \hspace{1cm} (4.20) \]

Note that \( N^{N-1} J_2(\sqrt{N}, \sqrt{N}) \) is the quantity we want to calculate. We proceed as before.

By the same methodology as the previous proposition, we derive the following after some tedious calculations which we omit for brevity.

**Proposition 4.3.** Denote the functions,

\[
F_1(z, w) = \frac{1}{\beta^2} \sum_i \frac{1}{(\lambda_i - z)(\lambda_i - w)} \hspace{1cm} F_2(z, w) = \frac{\lambda^2}{\beta^2} \sum_i \frac{v_i^2}{(\lambda_i - z)(\lambda_i - w)}. \hspace{1cm} (4.21)
\]

The following holds.

\[
\int \left( (\sigma^{(1)} \cdot \sigma^{(2)})^4 \exp \left[ \frac{\beta}{2} \left( (\sigma^{(1)})^T M \sigma^{(1)} + (\sigma^{(2)})^T M \sigma^{(2)} \right) + \lambda v^T (\sigma^{(1)} + \sigma^{(2)}) \right] \right] d\sigma^{(1)} d\sigma^{(2)} \\
= \frac{\beta^2 N}{(2\pi i)^4} \left( \frac{2\pi}{\beta} \right)^N \int dz dw \exp \left[ \frac{N}{2} \left( G_o(z, v, \lambda, \beta) + G_o(w, v, \lambda, \beta) \right) \right] \times \left\{ 6\beta^{-2} F_{1zw} + 3(F_1)^2 + 3\beta^{-2} (F_{2z} + F_{2w})^2 + 41\beta^{-2} F_2 F_{2zw} + 6\beta^{-2} F_{2z} F_{2w} + 6F_2 (-\beta^{-1} F_{2z} + \beta^{-1} F_{2w} + F_1) \right\} \\
- 6\beta^{-3} (F_{2zzw} + F_{2wzw}) - 6\beta^{-1} F_1 (F_{2z} + F_{2w}) \hspace{1cm} (4.22)
\]
5 Gaussian regime

In this section we will investigate the fluctuations of the SSK model in the regime of Gaussian fluctuations, i.e., in which either the temperature is high or the magnetic field is not decaying too quickly. This section contains all the proofs of the results of Section 2.2.

5.1 Free energy fluctuations

In this section we consider the free energy fluctuations. Recall that the free energy is the quantity,

\[ F_{N, \beta, h} = \frac{1}{N} \log Z_{N, \beta, h}, \quad Z_{N, \beta, h} = \int \exp \left[ \beta H_N(\sigma) \right] \frac{d\omega_{N-1}(\sigma)}{|\mathcal{S}|}. \]  

(5.1)

Recall that the Hamiltonian may be written in terms of the matrix \( M \)

\[ \beta H_N(\sigma) = \frac{\beta}{2} \sigma^T M \sigma + \beta h v^T \sigma = \frac{\beta}{2} \sigma^T M \sigma + \sqrt{\beta \theta} v^T \sigma, \]  

(5.2)

where we defined \( \theta = h^2 \beta \). We will assume that for some \( C, c > 0 \) and \( \tau > 0, \)

\[ C \geq (1 - \beta)_+ + \frac{\theta}{|1 - \beta| + \sqrt{\theta}} \geq N^{-1/3 + \tau}, \quad c \leq \beta \leq C, \quad \theta \leq C. \]  

(5.3)

Throughout Section 5 we will abbreviate the eigenvalues of \( M \) as

\[ \lambda_i := \lambda_i(M). \]  

(5.4)

Recall they are arranged in decreasing order. From Proposition 4.1, we see that the free energy can be expressed in terms of a contour integral involving the function,

\[ G(z) = \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i) - \frac{\theta}{N} v^T \frac{1}{M - z} v. \]  

(5.5)

For \( E > 2, \)

\[ G'(E) = \beta + m(E) - \theta N^{-1} v^T (M - z)^{-2} v \]

\[ = \beta + m_{sc}(E) - \theta m'_{sc}(E) + N^c \mathcal{O} \left( \frac{1}{N(E - 2)} + \frac{\theta}{N^{1/2}(E - 2)^{5/4}} \right) \]  

(5.6)

which holds with overwhelming probability as long as \( E - 2 > N^{-2/3 + \epsilon} \), using Theorems 3.1 and 3.3. Since \( G'(E) \) is monotonic, is positive for large \( E \) and tends to \(-\infty\) as \( E \) tends to \( \lambda_1 \) from the right, we see that the equation

\[ G'(\gamma) = 0 \]  

(5.7)

has a unique solution for \( E > \lambda_1 \). We first seek to establish preliminary estimates on the location of the saddle \( \gamma \). Recall our definition of the function \( g(z) \) in Section 2.1 so that,

\[ g'(z) = \beta + m_{sc}(z) - \theta m'_{sc}(z). \]  

(5.8)

Note \( g(z) \) is an approximation to \( G(z) \). We define the deterministic saddle \( \gamma \) by

\[ g'(\gamma) = 0. \]  

(5.9)

Using the fact that

\[ 1 + m_{sc}(\gamma) \approx \sqrt{\gamma - 2}, \quad m'_{sc}(\gamma) \approx (\gamma - 2)^{-1/2}, \]  

(5.10)

a routine calculation using the quadratic formula shows that

\[ \sqrt{\gamma - 2} = (1 - \beta)_+ + \frac{\theta}{|1 - \beta| + \sqrt{\theta}} = \begin{cases} (1 - \beta) + \sqrt{\theta}, & 1 \geq \beta \\ \frac{\theta}{|\beta - 1| + \sqrt{\theta}}, & \beta \geq 1 \end{cases} \]  

(5.11)
Hence, under the assumption (5.3),
\[ \hat{\gamma} - 2 \geq N^{-2/3 + \tau}, \]  
for \( N \) large enough. Next, observe for \( E - 2 \geq N^{-2/3 + \varepsilon} \), we have with overwhelming probability,
\[
G''(E) = m'(E) - 2\theta N^{-1} v^T(M - z)^{-3} v \\
= m''_{w_2}(E) - 2\theta m''_{w_2}(E) + N^\varepsilon O \left( \frac{1}{N(E - 2)^2} + \frac{\theta}{N^{1/2}(E - 2)^{9/4}} \right) \\
\approx \frac{1}{\sqrt{E - 2}} + \frac{\theta}{(E - 2)^{3/2}}. 
\]  
(5.13)

Note that the condition that \( E - 2 \geq N^{-2/3 + \varepsilon} \) ensures that the error term in the second to last expression is smaller than the final line appearing above, justifying the inequalities implicit in the last line (our definition of \( \approx \) appears in Section 2.1). For \( E - 2 \geq N^{-2/3 + \varepsilon} \) we have,
\[
\frac{|g'(E) - G'(E)|}{|g''(E)|} \leq N^{\varepsilon'} \frac{1}{1 + \theta/(E - 2)} \left( \frac{1}{N\sqrt{E - 2}} + \frac{\theta}{N^{1/2}(E - 2)^{3/4}} \right) \leq C N^{\varepsilon'} N^{-\tau/10} 
\]  
(5.14)

with overwhelming probability for any \( \varepsilon, \varepsilon' \). Since \( \hat{\gamma} - 2 \geq N^{-2/3 + \tau} \), and \( g'(E) \) and \( G'(E) \) are monotonic, we can use this estimate with, e.g., \( \varepsilon = \tau/10 \) and \( \varepsilon' < \tau/100 \) to conclude the following.

**Lemma 5.1.** Assume that (5.3) holds. Then, for \( N \) large enough, \( \hat{\gamma} - 2 \geq N^{-2/3 + \tau} \) and with overwhelming probability, the random saddle \( \gamma \) satisfies \( \gamma - 2 \geq N^{-2/3 + \tau} \) and
\[
|\hat{\gamma} - \gamma| \leq N^{\varepsilon'} \frac{1}{1 + \theta(\hat{\gamma} - 2)^{-1}} \left( \frac{1}{N\sqrt{\hat{\gamma} - 2}} + \frac{\theta}{N^{1/2}(\hat{\gamma} - 2)^{3/4}} \right) \leq N^{\varepsilon'} N^{-2/3 - \tau/10}, 
\]  
(5.15)

for any \( \varepsilon' > 0 \).

Given the above, the following notation will prove useful; we define \( \kappa \) to be the distance of \( \hat{\gamma} \) to the spectral edge:
\[
\kappa := \hat{\gamma} - 2. 
\]  
(5.16)

With these preparations, we implement the method of steepest descent in the following proposition. Proposition 2.1 follows.

**Proposition 5.2.** Assume that (5.3) holds. Then, with overwhelming probability,
\[
F_{N, \beta, h} = \frac{G(\gamma)}{2} + \frac{1}{N} \log \Gamma(N/2) + \frac{1}{N} \log (2/(N\beta)) - \frac{1}{2N} \log (N G''(\gamma) \pi) + O(N^{-1 - \tau/10}) \\
= \frac{G(\hat{\gamma})}{2} + \frac{1}{N} \log \Gamma(N/2) + \frac{1}{N} \log (2/(N\beta)) - \frac{1}{2N} \log (N g''(\hat{\gamma}) \pi) \\
+ N^{-\tau/10} O \left( \frac{1}{N} + \frac{\theta}{N^{1/2} \kappa^{1/4}} \right) 
\]  
(5.17)

**Proof.** Denoting the saddle point (5.7) by \( \gamma \), using Proposition 4.1 we are led to investigate the contour integral,
\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \exp \left[ \frac{N}{2}(G(z) - G(\gamma)) \right] \, dz. 
\]  
(5.18)

We decompose the contour into two components, \( \Gamma_1 \cup \Gamma_2 \) where,
\[
\Gamma_1 := \{ \gamma \pm it, |t| \leq N^{\varepsilon} \}, \quad \Gamma_2 := \{ \gamma \pm it, |t| > N^{\varepsilon} \}. 
\]  
(5.19)

The contribution from \( \Gamma_2 \) is exponentially small, and is handled via similar calculations to those appearing in Lemma 5.4 of [6].

Via Taylor expansion and the estimates of Theorems 3.1 and 3.3, we have for any \( |t| \leq 10 \), with overwhelming probability,
\[
G(\gamma + it) - G(\gamma) = -\frac{t^2}{2} G''(\gamma) + O \left( |t^3| \kappa^{-3/2} (1 + \theta \kappa^{-1}) \right). 
\]  
(5.20)
Fixing $\alpha > 0$ sufficiently small, we will use this expansion when
\[
|t| \leq t_\alpha := \frac{N^\alpha}{N^{1/2}} |G''(\gamma)|^{-1/2} \leq \frac{N^\alpha}{N^{1/2}} \kappa^{1/4} (1 + \theta/\kappa)^{-1/2}, \tag{5.21}
\]
where the final inequalities hold with overwhelming probability. Then, for $|t| \leq t_\alpha$, the error term in (5.20) is
\[
|t^3| \kappa^{-3/2} (1 + \theta \kappa) \leq \frac{1}{N} \frac{N^3}{N^{1/2} \kappa^{3/4}} \frac{1}{(1 + \theta/\kappa)^{1/2}} \leq N^{-1 - \tau/10} \tag{5.22}
\]
for $\alpha < \tau/100$. Hence,
\[
\int_{\gamma-i\alpha}^{\gamma+i\alpha} \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] dz = \int_{-t_\alpha}^{t_\alpha} \exp \left[ -\frac{N}{4} G''(\gamma) t^2 \right] \left( 1 + O(N^{-\tau/10}) \right) \, dt = i^{\gamma} \sqrt{\frac{4\pi}{NG''(\gamma)}} \left( 1 + O(N^{-\tau/10}) \right) \tag{5.23}
\]
Using also that $\bar{\partial}_y \text{Re}[G(E + i\eta)] = -\text{Im}[G'(E + i\eta)] < 0$ for $E > \lambda_1$ and the fact that
\[
\text{Re}[G(E + it_\alpha) - G(\gamma)] \approx -i^2 G''(\gamma) \approx -N^{\alpha-1} \tag{5.24}
\]
we see that the contribution of the contour integral from $t_\alpha < |t| < N^\varepsilon$ is $O(e^{-CN^\varepsilon})$. Summarizing, this proves that
\[
|S^{N-1} |Z_{N,\beta,h} = \frac{\beta N^{1/2}}{2\pi} \left( \frac{2\pi}{\beta} \right)^{N/2} \exp \left[ \frac{N}{2} G(\gamma) \right] \sqrt{\frac{4\pi}{NG''(\gamma)}} \left( 1 + O(N^{-\tau/10}) \right), \tag{5.25}
\]
with overwhelming probability. Hence, with overwhelming probability,
\[
\frac{1}{N} \log Z_{N,\beta,h} = \frac{G(\gamma)}{2} + \frac{1}{N} \log \Gamma(N/2) + \frac{1}{N} (N/2 - 1) \log \log(2/(N\beta)) - \frac{1}{2N} \log(NG''(\gamma)\pi) + O(N^{-1 - \tau/10}), \tag{5.26}
\]
which proves the first claim of the proposition. Now, with overwhelming probability, for any $\varepsilon > 0$,
\[
G(\gamma) - G(\hat{\gamma}) = O(G''(\gamma)(\gamma - \hat{\gamma})) = \frac{N^\varepsilon}{\kappa^{1/2} (1 + \theta/\kappa)} \mathcal{O} \left( \frac{1}{N^{2\kappa}} + \frac{\theta^2}{N^{\kappa/2}} \right) = N^{-\tau/10} \mathcal{O} \left( \frac{1}{N} + \frac{\theta}{N^{1/2} \kappa^{1/4}} \right), \tag{5.27}
\]
and
\[
\log(G''(\gamma)) = \log(g''(\hat{\gamma})) + O(N^{-\tau/10}). \tag{5.28}
\]
Hence,
\[
\frac{1}{N} \log Z_{N,\beta,h} = \frac{G(\gamma)}{2} + \frac{1}{N} \log \Gamma(N/2) + \frac{1}{N} (N/2 - 1) \log \log(2/(N\beta)) - \frac{1}{2N} \log(Ng''(\hat{\gamma})\pi) + N^{-\tau/10} \mathcal{O} \left( \frac{1}{N} + \frac{\theta}{N^{1/2} \kappa^{1/4}} \right), \tag{5.29}
\]
which is the final claim of the proposition.

\subsection*{5.1.1 Gaussian fluctuations in a few regimes}

First we consider the high temperature regime $\beta \leq 1$. In this regime, the assumption (5.3) is equivalent up to constants to
\[
C \geq (1 - \beta) + \sqrt{\theta} \geq N^{-1/3 + \tau}. \tag{5.30}
\]
The expansion of Proposition 5.2 shows that the fluctuations of the free energy are determined by the fluctuations of $G(\hat{\gamma})$ which has two fluctuating terms, a linear test statistic which fluctuates on the scale $N^{-1}$ and a quadratic form in the resolvent which has fluctuations on the order
\[
\theta N^{-1/2} \kappa^{-1/4} \approx \theta N^{-1/2} ((1 - \beta) + \sqrt{\theta})^{-1/2}.
\]
Note that in the high temperature regime, the assumption \(5.3\) and the requirement that the order of the fluctuations of the resolvent term are polynomially larger than the linear statistics term is equivalent to assuming,
\[
\theta^2 \geq N^{\varepsilon_1 - 1} (1 - \beta) + \sqrt{\theta}
\]  
(5.31)
for an \(\varepsilon_1 > 0\). Hence, the entire regime of parameters in the high temperature case in which we expect the resolvent fluctuations to dominate is encapsulated in the following theorem, which follows from Proposition 5.2 and Theorem B.1.

**Theorem 5.3.** Assume that \(\beta \leq 1\) and that \(\theta^2 \geq N^{\varepsilon_1 - 1} (1 - \beta) + \sqrt{\theta}\) for some \(\varepsilon > 0\). Then the leading order fluctuations of the free energy are given by the quantity
\[
-\frac{\theta}{N} \mathcal{J} \left. \frac{1}{M - \hat{\gamma}} \right|_v.
\]
(5.32)
We have that,
\[
\frac{N^{1/2} \kappa^{1/4}}{\theta V_{1/2}^N} (G(\hat{\gamma}) - g(\hat{\gamma}))
\]
(5.33)
converges to a standard normal random variable, where
\[
V_N = \hat{\gamma} + \frac{\sqrt{\gamma^2 - 4}}{\sqrt{\gamma + 2}} m_{sc}(\hat{\gamma})^4 \left( m_{sc}^2(\hat{\gamma}) + (1 - N^{-2} \|v\|_4^4)(1 - m_{sc}^2(\hat{\gamma})) \right)
\]
(5.34)
is bounded away from 0 and above by a constant.

The complementary regime in the high temperature case is when \(\theta^2 \ll N^{\varepsilon_1 - 1} (1 - \beta) + \sqrt{\theta}\) which up to polynomial factors is, up to the assumption \(5.3\), equivalent to \((1 - \beta) \geq N^{\varepsilon}(\theta^2 N + N^{-1/3})\). However, due to lacking a suitable theorem on linear statistics near the spectral edge of the GOE, we assume that the deterministic approximation \(\hat{\gamma}\) to the saddle satisfies \(\kappa \geq c\), which is then equivalent to \((1 - \beta) \geq c\). So we assume \(\theta \leq N^{-1/2 - \varepsilon}\) for some \(\varepsilon > 0\). In this case, define \(\hat{\gamma}_2\) as the solution of \(\beta \hat{\gamma}_2 + m_{sc}(\hat{\gamma}) = 0\), i.e., the same as \(\hat{\gamma}\) but with \(\theta = 0\). Then \(|\hat{\gamma} - \hat{\gamma}_2| \leq CN^{-1/2 - \varepsilon}\) and one can see that \(|G(\hat{\gamma}) - G(\hat{\gamma}_2)| \leq CN^{-1-\varepsilon}\) with overwhelming probability.

Hence, we have the following theorem, which follows from [6].

**Theorem 5.4.** In the high temperature regime when \(c \leq \beta \leq 1 - c\) for \(c > 0\) and \(\theta \leq N^{-1/2 - \varepsilon}\) for an \(\varepsilon > 0\) we have that the leading order fluctuations of the free energy are given by the linear statistic,
\[
\frac{1}{N} \sum_{i=1}^{N} \log(\hat{\gamma}_2 - \lambda_i)
\]
(5.35)
which are order \(N^{-1}\). We have that,
\[
\frac{N}{B_N} (G(\hat{\gamma}_2) - g(\hat{\gamma}_2) - A_N)
\]
(5.36)
converges to a standard normal random variable where,
\[
A_N := \frac{1}{2} \log(1 - \beta^2) - \beta^2
\]
(5.37)
and
\[
B_N^2 := -2 \log(1 - \beta^2) - 2\beta^2.
\]
(5.38)

In the low temperature regime, the assumption \(5.3\) is equivalent to \(\theta \gg N^{-1/3}(\beta - 1) + \sqrt{\theta}\). The fluctuations from the resolvent term are larger than the linear statistics term when \(\theta((1 - \beta) + \sqrt{\theta}) \gg N^{-1}\). This always holds due to the fact that the assumption \(5.3\) implies that \(\theta \gg N^{-2/3}\). Hence we have the following.
Theorem 5.5. If (5.3) holds in the low temperature case $\beta \geq 1$, then the free energy fluctuations are as in Theorem 5.3.

Theorem 2.2 follows from the above statements.

Proof of Theorem 2.3. In light of the above, we need only examine the fluctuations of $G(\hat{\gamma}) - g(\hat{\gamma})$. Under the assumptions of the Theorem this quantity has the same distribution as,

$$G(\hat{\gamma}) - g(\hat{\gamma}) \overset{d}{=} - \frac{1}{N} \sum_i \log(\hat{\gamma} - \lambda_i) + \int \log(\hat{\gamma} - x) \rho_{sc}(x) dx + \frac{\theta}{N^{1/2}} m_{sc}(\hat{\gamma}) - \frac{\theta}{||n||_2^2 N^{1/2}} \sum_i \frac{n_i^2}{\lambda_i - \hat{\gamma}}$$  \hspace{1cm} (5.39)

where $\{n_i\}_{i=1}^N$ is a vector of iid standard normal random variables, and $\lambda_i$ are the eigenvalues of the matrix made from the coefficients $g_{ij}$ as usual. In particular, the vector $n$ is independent from the eigenvalues $\lambda_i$. From [6] we have that the random variable,

$$\frac{N}{v_n} \left( - \frac{1}{N} \sum_i \log(\hat{\gamma} - \lambda_i) + \int \log(\hat{\gamma} - x) \rho_{sc}(x) dx + e_n \right)$$

(5.40)

converges to a standard normal random variable where,

$$e_n = \frac{1}{4} (\log(\hat{\gamma} - 2) - \log(\hat{\gamma} + 2)) - \frac{\tau_0}{2} + \tau_2 (w_2 - 2)$$

$$v_n = \log \left( \frac{(\hat{\gamma} + \sqrt{\hat{\gamma}^2 - 4})^2}{4(\hat{\gamma}^2 - 4)} \right) + \tau_1^2 (w_2 - 2)$$

(5.41)

where $w_2 = 0$ in the case where we replaced $v$ by a uniform vector and $w_2 = 2$ in the other case. The coefficients $\tau_i$ are

$$\tau_0 = \log(\hat{\gamma} + \sqrt{\hat{\gamma}^2 - 4}) - \log(2)$$

$$\tau_1 = \frac{1}{2} \sqrt{\hat{\gamma}^2 - 4} - \frac{\hat{\gamma}}{2}$$

$$\tau_2 = \frac{\hat{\gamma}}{4} \sqrt{\hat{\gamma}^2 - 4} - \frac{\hat{\gamma}^2}{4} + \frac{1}{2}.$$  \hspace{1cm} (5.42)

For the other term, we easily see that

$$\frac{\theta}{||n||_2^2 N^{1/2}} \sum_i \frac{n_i^2}{\lambda_i - \hat{\gamma}} = \frac{\theta}{||n||_2^2 N^{1/2}} \sum_i \frac{n_i^2}{\lambda_i - \hat{\gamma}} + \frac{\theta}{||n||_2^2 N^{1/2}} N(m(\hat{\gamma}) - m_{sc}(\hat{\gamma})) + \frac{\theta}{||n||_2^2 N^{1/2}} Nm_{sc}(\hat{\gamma})$$

$$= \frac{\theta}{N N^{1/2}} \sum_i \frac{n_i^2}{\lambda_i - \hat{\gamma}} + \frac{\theta}{N^{1/2}} m_{sc}(\hat{\gamma}) + \frac{\theta m_{sc}(\hat{\gamma})}{N^{1/2}} (1 - N^{-1} ||n||_2^2) + O\left( N^{\varepsilon-3/2} \right)$$

$$= \frac{\theta}{N N^{1/2}} \sum_i (n_i^2 - 1) \left( \frac{1}{\lambda_i - \hat{\gamma}} - m_{sc}(\hat{\gamma}) \right) + \frac{\theta}{N^{1/2}} m_{sc}(\hat{\gamma}) + O\left( N^{\varepsilon-3/2} \right)$$  \hspace{1cm} (5.43)

for any $\varepsilon > 0$ with overwhelming probability. Note we used the local law and the fact that

$$N/||n||_2^2 = 1 - (N^{-1} ||n||_2^2 - 1) + O\left( N^{\varepsilon-1/2} \right) = 1 + O\left( N^{\varepsilon-1/2} \right)$$  \hspace{1cm} (5.44)

which holds with overwhelming probability. Since the $n_i$ are independent of the $\lambda_i$ we have that the conditional variance

$$\text{Var} \left( \frac{1}{N^{1/2}} \sum_i (n_i^2 - 1) \left( \frac{1}{\lambda_i - \hat{\gamma}} - \frac{1}{\gamma_i - \hat{\gamma}} \right) \mid \{\lambda_i\}_{i=1}^N \right) \leq C \frac{N^\varepsilon}{N}$$  \hspace{1cm} (5.45)

with overwhelming probability by rigidity. Hence,

$$\frac{\theta}{N N^{1/2}} \sum_i (n_i^2 - 1) \left( \frac{1}{\lambda_i - \hat{\gamma}} - m_{sc}(\hat{\gamma}) \right) = \frac{\theta}{N N^{1/2}} \sum_i (n_i^2 - 1) \left( \frac{1}{\gamma_i - \hat{\gamma}} - m_{sc}(\hat{\gamma}) \right) + O(N^\varepsilon-3/2)$$  \hspace{1cm} (5.46)
with overwhelming probability. When multiplied by $N$, the quantity on the LHS is a centered Gaussian random variable with variance converging to,

$$\tilde{\nu} := \int \left( \frac{1}{(x-\hat{\gamma})^2} - m_{sc}(\hat{\gamma}) \right)^2 \rho_{sc}(x) dx.$$  \hfill (5.47)

Hence, we conclude the proof of the theorem and find the formulas for $\tilde{V}_N$ and $E_N$ to be,

$$\tilde{V}_N = \tilde{\nu} + v_n, \quad E_N = -e_n,$$  \hfill (5.48)

where these quantities have been defined during this proof.

5.2 Overlap with external field

This section will prove Theorem 2.4, which will involve the calculation of the Laplace transform,

$$\langle e^{\lambda v^T \sigma} \rangle.$$  \hfill (5.49)

The natural scale for $\lambda$ turns out to be

$$|\lambda| \leq C N^{-1/2}.$$  \hfill (5.50)

We will continue to assume (5.3). The analysis builds on that of the previous subsection and so we will continue to use the notation introduced there. For example, we will continue to use the functions $G(z), g(z)$ and the saddles $\hat{\gamma}, \gamma$ in (5.7) and (5.9), as well as the parameter $\kappa$ (5.16). From Proposition 4.1 we see that the Laplace transform (5.49) of the overlap with the external field can be represented as a ratio of two contour integrals, one involving the function $G(z)$ as above, and the second involving the function,

$$G_u(z) := G(z) - u N^{-1} v^T (M - z)^{-1} v.$$  \hfill (5.51)

where the parameter $u$ is defined as

$$u = 2 h \lambda + \lambda^2 \beta^{-1} = 2 \lambda (\theta / \beta)^{1/2} + \lambda^2 \beta^{-1}.$$  \hfill (5.52)

Note that under (5.3) and (5.50) we have,

$$|u| \leq N^{-\tau/100}(1 + \theta / \kappa) \kappa.$$  \hfill (5.53)

We extend our expansion Proposition 5.2 to the contour integral with $G_u(z)$. The estimate (5.53) implies that in the low temperature regime we have $|u| \ll \theta$ (because $\kappa \leq C \theta$ here) and in the high temperature regime that $|u| \ll \kappa$ (as $\theta \leq C \kappa$ here) and so (5.3) will be satisfied for $\theta$ replaced by $\theta + u$. The point of this discussion is then that we can apply the expansion obtained in the previous subsection to calculate the Laplace transform. In what follows we seek to expand the quantities coming from the saddle point analysis applied to $G_u$ around those coming from $G$.

We now define

$$g_u(z) = g(z) - u m_{sc}(z)$$  \hfill (5.54)

and $\hat{\gamma}_u$ by

$$g'_u(\hat{\gamma}_u) = 0.$$  \hfill (5.55)

We have,

$$g'(\hat{\gamma}_u) - g'(\hat{\gamma}) = u m'_{sc}(\hat{\gamma}_u).$$  \hfill (5.56)

Taylor expanding this relation, we find,

$$(\hat{\gamma}_u - \hat{\gamma}) \simeq \frac{u}{1 + \theta / \kappa}.$$  \hfill (5.57)
Substituting this estimate back into the Taylor expansion of \((5.56)\) we find,

\[
(\tilde{\gamma}_u - \tilde{\gamma}) = \frac{u m'_{sc}(\tilde{\gamma})}{g''(\tilde{\gamma})} - \frac{1}{2g''(\tilde{\gamma})}g''(\tilde{\gamma})(\tilde{\gamma}_u - \tilde{\gamma})^2 + u \frac{m''_{sc}(\tilde{\gamma})}{g''(\tilde{\gamma})}(\tilde{\gamma}_u - \tilde{\gamma}) + \mathcal{O} ((1 + \theta/\kappa)^{-3}u^3\kappa^{-2})
\]

\[
= \frac{u m'_{sc}(\tilde{\gamma})}{g''(\tilde{\gamma})} - \frac{1}{2g''(\tilde{\gamma})}u^2 m'_{sc}(\tilde{\gamma})^2 + u^2 \frac{m''_{sc}(\tilde{\gamma})m_{sc}(\tilde{\gamma})}{(g''(\tilde{\gamma}))^2} + \mathcal{O} ((1 + \theta/\kappa)^{-3}u^3\kappa^{-2}).
\]

(5.58)

Defining then \(\gamma_u\) by

\[
G_u(\gamma_u) = 0
\]

(5.59)
a similar calculation to that above gives the following estimates with overwhelming probability. First,

\[
(\gamma_u - \gamma) = \frac{u}{1 + \theta/\kappa}.
\]

(5.60)

Second:

\[
(\gamma_u - \gamma) = \frac{u N^{-1}v^T(M - \gamma)^{-2}v}{G''(\gamma)} - \frac{G''''(\gamma)}{2G''(\gamma)} u^2 (N^{-1}v^T(M - \gamma)^{-2}v)^2 + u^2 \frac{N^{-1}v^T(M - \gamma)^{-2}v}{(G''(\gamma))^2} + N^\varepsilon \mathcal{O} ((1 + \theta/\kappa)^{-3}u^3\kappa^{-2}),
\]

(5.61)

for any \(\varepsilon > 0\). With these preparations, we can begin the proof of the following.

**Theorem 5.6.** Suppose that \(|t| \leq C\) and that \((5.3)\) holds. Then there is an \(\varepsilon > 0\) so that with overwhelming probability,

\[
\log \left( \langle e^{tN^{-1/2}v^T\sigma} \rangle \right) = t^2 A_N + t B_N + \mathcal{O}(N^{-\varepsilon})
\]

(5.62)

where,

\[
A_N := \frac{1}{\beta} \left( \frac{\theta m'(\tilde{\gamma})^2}{g''(\tilde{\gamma})} - \frac{1}{2} m_{sc}(\tilde{\gamma}) \right),
\]

(5.63)

and

\[
B_N := -N^{1/2}\theta^{1/2} (N^{-1}v^T(M - \gamma)^{-1}v)^{3/2} \beta^{-1/2}.
\]

(5.64)

**Proof.** Proposition 5.2 implies that with overwhelming probability,

\[
\log \langle e^{\lambda N^{-1/2}v^T\sigma} \rangle = \frac{N}{2} (G_u(\gamma_u) - G(\gamma)) - \frac{1}{2} \log(G''(\gamma)/G''(\gamma_u)) + \mathcal{O}(N^{-\varepsilon/10}).
\]

(5.65)

Using (5.60) we find that with overwhelming probability,

\[
\frac{N}{2} (G_u(\gamma_u) - G(\gamma)) = \frac{N G''(\gamma)}{4}(\gamma_u - \gamma)^2 - \frac{N u}{2} v^T(M - \gamma_u)^{-1}v + N^\varepsilon \mathcal{O} \left( N|u|^3\kappa^{-3/2}(1 + \theta/\kappa)^{-2} \right)
\]

(5.66)

Applying (5.61) we find that with overwhelming probability,

\[
\frac{N G''(\gamma)}{4}(\gamma_u - \gamma)^2 = Nu^2 \frac{1}{4} (N^{-1}v^T(M - \gamma)^{-2}v)^2 + N^\varepsilon \mathcal{O} \left( |u|^3 N\kappa^{-3/2}(1 + \theta/\kappa)^{-2} \right).
\]

(5.67)
We also calculate the linear contribution,

\[
\frac{N}{2}uN^{-1}v^T(M - \gamma u)^{-1}v = \frac{N}{2}uN^{-1}v^T(M - \gamma)^{-1}v + \frac{N}{2}u(\gamma_\alpha - \gamma)u^T(M - \gamma)^{-2}v \\
+ N^\varepsilon \mathcal{O}\left(|u|^3 N \kappa^{-3/2}(1 + \theta/\kappa)^{-2}\right) \\
= \frac{N}{2}uN^{-1}v^T(M - \gamma)^{-1}v + \frac{N}{2}u^2 (N^{-1}u^T(M - \gamma)^{-2}v)^2 \\
+ N^\varepsilon \mathcal{O}\left(|u|^3 N \kappa^{-3/2}(1 + \theta/\kappa)^{-2}\right)
\]  

(5.68)

We put these two calculations together and substitute back \( u = 2\lambda(\theta/\beta)^{1/2} + \lambda^2 \beta^{-1} \), to find that with overwhelming probability,

\[
\frac{N}{2}G(\gamma_\alpha) - G(\gamma) = \frac{N}{2}uN^{-1}v^T(M - \gamma_\alpha)^{-1}v \\
= \lambda^2 \frac{N}{\beta}(\frac{\theta (N^{-1}u^T(M - \gamma)^{-2}v)^2}{G'(\gamma)} - \frac{1}{2}(N^{-1}u^T(M - \gamma)^{-1}v)) \\
+ \mathcal{O}\left(|u|^3 N \kappa^{-3/2}(1 + \theta/\kappa)^{-2}\right) + \mathcal{O}\left(N(|\lambda|^3 \theta^{1/2} + \lambda^4)\kappa^{-1/2}(1 + \theta/\kappa)^{-1}\right). 
\]  

(5.69)

Note that,

\[
\log(G'(\gamma)) - \log(G'_{\beta}(\gamma)) = \mathcal{O}\left(\frac{|u|}{\kappa(1 + \theta/\kappa)}\right).
\]  

(5.70)

We now let \( \lambda = tN^{-1/2} \) for \( |t| \leq C \) to see that the error terms in (5.69) and (5.70) are \( \mathcal{O}(N^{-c}) \) for some \( c > 0 \). Finally, using Lemma 5.1, Theorems 3.1 and 3.3 we see,

\[
|\left(-\frac{\theta (N^{-1}u^T(M - \gamma)^{-2}v)^2}{G'(\gamma)} - \frac{1}{2}(N^{-1}u^T(M - \gamma)^{-1}v)\right) - \left(-\frac{\theta(m_{\beta}(\gamma))^2}{\beta^\varepsilon(\gamma)} - \frac{1}{2}m_{\beta}(\gamma)\right)| \leq N^{-c}
\]  

(5.71)

with overwhelming probability for some \( c > 0 \). This yields the claim. \( \square \)

In order to conclude Theorem 2.4 we calculate the coefficient corresponding to the variance.

\[
A_N = -\left(\frac{\theta(m_{\beta}(\gamma))^2}{\beta^\varepsilon(\gamma)} - \frac{1}{2}m_{\beta}(\gamma)\right) \\
= m_{\beta}(\gamma)\left(-m_{\beta}(\gamma) + \theta m_{\beta}(\gamma) m_{\beta}(\gamma)/m_{\beta}(\gamma) - 2\theta m_{\beta}(\gamma)\right) \\
= m_{\beta}(\gamma)\left(-m_{\beta}(\gamma) + \theta m_{\beta}(\gamma) (1 - m_{\beta}(\gamma)) - 2\theta m_{\beta}(\gamma)\right) \\
= m_{\beta}(\gamma)\left(-m_{\beta}(\gamma) + \theta m_{\beta}(\gamma) m_{\beta}(\gamma)/m_{\beta}(\gamma)\right) \\
= m_{\beta}(\gamma)\left(-m_{\beta}(\gamma) + \theta m_{\beta}(\gamma) m_{\beta}(\gamma)\right)
\]  

(5.72)

In the third line we used the identity \( m_{\beta}(\gamma) m_{\beta}(\gamma) = 2(m_{\beta}(\gamma))^2/(1 - m_{\beta}(\gamma)) \), and in the last line the identity \( m_{\beta}(\gamma) = m_{\beta}(\gamma)/(1 - m_{\beta}(\gamma)) \). For the last line, the prefactor is of order \((1 + \theta/\kappa)^{-1}\) and the term in the brackets is of order \((1 + \theta/\kappa)^{-1}\).

**Proof of Lemma 2.5.** Let \( X := N^{-1/2}v \cdot \sigma - B_N \), with \( B_N \) as in Theorem 5.6. We have, on the event of that theorem,

\[
\frac{1}{t} \left(\langle e^{tX} \rangle - 1\right) = \frac{1}{t} \left(e^{2A_N + \mathcal{O}(N^{-c_1})} - 1\right) = \mathcal{O}(t^{-1}N^{-c} + t),
\]  

(5.73)

for \(|t| \leq 1\). On the other hand,

\[
\frac{1}{t} \left(\langle e^{tX} \rangle - 1\right) = \langle X \rangle + t \mathcal{O}(\langle X^2 e^{t|X|} \rangle).
\]  

(5.74)

We choose \( t = N^{-c_1/2} \). With this choice, \( \langle X^2 e^{t|X|} \rangle \leq C \langle e^X + e^{-X} \rangle \leq C' \).

(5.75)

We conclude the proof. \( \square \)
5.3 Overlap between two replicas

In this section, we consider the asymptotic fluctuations of the overlap,

\[ R_{12} = \frac{1}{N} \sigma^{(1)} \cdot \sigma^{(2)}, \tag{5.76} \]

and prove Theorem 2.6. As in the previous subsections, we write:

\[ \beta H_N(\sigma) = \frac{\beta}{2} \sigma^T M \sigma + \sqrt{\beta} \theta v^T \sigma. \tag{5.77} \]

We again assume that

\[ C \geq (1 - \beta) + \frac{\theta}{|1 - \beta| + \sqrt{\theta}} \geq N^{-1/3 + \tau}, \quad \theta \leq C, \quad c \leq \beta \leq C. \tag{5.78} \]

From Proposition 4.1 we have,

\[ \langle \exp \left[ \beta t \sigma^{(1)} \cdot \sigma^{(2)} \right] \rangle = \frac{\int_{\Gamma} \exp \frac{N}{2} G_1(z,w) dw}{\int_{\Gamma} \exp \frac{N}{2} G(z) dz} \tag{5.79} \]

where \( \Gamma \) is a vertical line in the complex plane lying to the right of the spectrum of \( M \). Here the function \( G(z) \) is as above, and

\[ G_1(z,w) = \beta(z + w) - \frac{1}{N} \sum_{i=1}^{N} \log((z - \lambda_i)(w - \lambda_i) - t^2) - \frac{\theta}{N} \sum_{i=1}^{N} v_i^2 \frac{(2\lambda_i - z - w) - 2t}{(\lambda_i - w)(\lambda_i - z) - t^2}. \tag{5.80} \]

Here, we recall the notation

\[ v_i = v^T u_i(M), \tag{5.81} \]

for the projection of the vector \( v \) onto the \( i \)th eigenvector of \( M \). We will use this notation for the remainder of the section. We will again use the notation \( g(z) \) as well as \( \gamma \) and \( \hat{\gamma} \) for the saddles and the parameter \( \kappa \) as in the previous subsections. We first prove the following lemma, which provides an expansion of \( G_1 \) around a point \( \gamma_1 \). The point \( \gamma_1 \) will eventually be chosen to be an approximate saddle for \( G_1(z,w) \).

**Lemma 5.7.** Assume,

\[ t^2 \leq \log(N) \frac{\kappa^{1/2}}{N} \tag{5.82} \]

and let \( \gamma_1 \in \mathbb{R} \) satisfy \( |\gamma_1 - \gamma| \leq \kappa/\log(N) \). Let \( \delta > 0 \) be sufficiently small. For \( z \) and \( w \) satisfying

\[ |z - \gamma_1| + |w - \gamma_1| \leq N^\delta \frac{\kappa^{1/4}}{N^{1/2}(1 + \theta/\kappa)^{1/2}}, \tag{5.83} \]

we have with overwhelming probability,

\[
G_1(z,w) = \beta(z + w) - N^{-1} \sum_{i=1}^{N} \log((\lambda_i - \gamma_1) (\lambda_i - w)) - \theta N^{-1} [v^T (M - z + t)^{-1} v + v^T (M - w + t)^{-1} v] \\
+ t^2 m_N'(\gamma_1) \\
+ N^\varepsilon O(t^4 \kappa^{-5/2} + t^2 \kappa^{-5/4} N^{\delta-1/2}) \\
+ N^\varepsilon O(\theta \kappa^{-2} N^{-1} N^{2\delta}(1 + \theta/\kappa)^{-1} |t| + t^2 \kappa^{-1} + N^\delta \kappa^{1/4} N^{-1/2}). \tag{5.84}
\]

for any \( \varepsilon > 0 \).

**Proof.** First we note that by taking \( \delta > 0 \) sufficiently small we can assume that the LHS of (5.83) is smaller than \( N^{-\varepsilon} \kappa \) for some small \( \varepsilon > 0 \). This, combined with our assumption that \( |\gamma_1 - \gamma| \leq \kappa/\log(N) \) implies that with overwhelming probability that Re\([z]\) and Re\([w]\) will both be larger than
2 + N^{-2/3+\tau/2}. This allows for the use of the rigidity and local estimates of Theorems 3.1 and 3.2 and the Taylor expansion of quantities appearing in denominators below.

We start by rewriting $G_1(z, w)$ as follows.

$$G_1(z, w) = \beta(z + w) - \frac{1}{N} \sum_{i=1}^{N} \log((z - \lambda_i)(w - \lambda_i))$$

$$- \frac{1}{N} \sum_{i=1}^{N} \log(1 - t^2/((z - \lambda_i)(w - \lambda_i)))$$

$$- \theta \sum_{i=1}^{N} \frac{(2\lambda_i - z - w)}{\beta^2(\lambda_i - w)(\lambda_i - z) - t^2}$$

$$+ 2t \frac{\theta^3}{N} \sum_{i=1}^{N} \frac{1}{\beta^2(\lambda_i - w)(\lambda_i - z) - t^2}.$$  

(5.85)

(5.86)

(5.87)

(5.88)

First,

$$- \frac{1}{N} \sum_{i=1}^{N} \log(1 - t^2/((z - \lambda_i)(w - \lambda_i))) = t^2 \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(z - \lambda_i)(w - \lambda_i)} + O(t^4\kappa^{-5/2})$$

$$= t^2 m_N'(\gamma_1) + O(t^4\kappa^{-5/2} + t^2\kappa^{-5/4}N^{\delta_1-1/2}).$$  

(5.89)

We next turn to (5.87). We have the expansion,

$$
\frac{\lambda_i - z}{(\lambda_i - w)(\lambda_i - z) - t^2} = \frac{\lambda_i - \gamma_1}{(\lambda_i - \gamma_1)^2 - t^2} + (\gamma_1 - w) \left( - \frac{(\lambda_i - \gamma_1)^2}{((\lambda_i - \gamma_1)^2 - t^2)^2} \right) 
+ (\gamma_1 - z) \left( \frac{1}{(\lambda_i - \gamma_1)^2 - t^2} - \frac{1}{((\lambda_i - \gamma_1)^2 - t^2)^2} \right) 
+ (\gamma_1 - z)^2 \left( \frac{(\lambda_i - \gamma_1)^3}{((\lambda_i - \gamma_1)^2 - t^2)^3} - \frac{(\lambda_i - \gamma_1)}{((\lambda_i - \gamma_1)^2 - t^2)^2} \right) 
+ (\gamma_1 - w)^2 \left( \frac{(\lambda_i - \gamma_1)^3}{((\lambda_i - \gamma_1)^2 - t^2)^3} - \frac{(\gamma_1 - \gamma_1)}{((\lambda_i - \gamma_1)^2 - t^2)^2} \right) 
+ 2(\gamma_1 - z)(\gamma_1 - w) \left( \frac{(\lambda_i - \gamma_1)^3}{((\lambda_i - \gamma_1)^2 - t^2)^3} - \frac{1}{((\lambda_i - \gamma_1)^2 - t^2)^2} \right) 
+ O((|z - \gamma_1|^3 + |w - \gamma_1|^3)|\lambda_i - \gamma_1|^{-4}),
$$

(5.90)

as well as

$$\frac{1}{\lambda_i - z} = \frac{1}{\lambda_i - \gamma_1} + (z - \gamma_1) \frac{1}{(\lambda_i - \gamma_1)^2} + (z - \gamma_1)^2 \frac{1}{(\lambda_i - \gamma_1)^3} + O(|z - \gamma_1|^3|\lambda_i - \gamma_1|^{-4}).$$  

(5.91)

Using this as well as Theorems 3.2 and 3.3 (the latter to estimate $|v_i|^2 \leq N^\epsilon$ with overwhelming probability) we find that with overwhelming probability,

$$- \theta \sum_{i=1}^{N} \frac{v_i^2}{\lambda_i - \gamma_1} \left( \frac{2(\lambda_i - z - w)}{(\lambda_i - w)(\lambda_i - z) - t^2} \right) + \theta \sum_{i=1}^{N} \frac{v_i^2}{\lambda_i - z} \left( \frac{1}{(\lambda_i - z) - t^2} + \frac{1}{\lambda_i - w} \right)$$

$$= 2\theta \sum_{i=1}^{N} \frac{v_i^2}{\lambda_i - \gamma_1} \left( \frac{1}{(\lambda_i - \gamma_1)^2} - \frac{\lambda_i - \gamma_1}{(\lambda_i - \gamma_1)^2 - t^2} \right)$$

$$+ [(\gamma_1 - z) + (\gamma_1 - w)] \left( \frac{\theta}{N} \sum_{i=1}^{N} \frac{v_i^2}{\lambda_i - \gamma_1} \left( \frac{2(\lambda_i - \gamma_1)^2}{((\lambda_i - \gamma_1)^2 - t^2)^2} - \frac{1}{(\lambda_i - \gamma_1)^2 - t^2} - \frac{1}{(\lambda_i - \gamma_1)^2} \right) \right)$$

$$+ N^\epsilon \mathcal{O} \left( \theta \kappa^{-2} N^{-1} N^{2\delta_1} (1 + \theta/\kappa)^{-1} (t^2 \kappa^{-1} + N^{\delta_1/4} N^{-1/2}) \right).$$  

(5.92)
Note that the terms quadratic in \((z - \gamma_1)\) and \((w - \gamma_1)\) were estimated using the cancellation between the above two expansions which gives the error terms that have the \(t^2\) terms above. Similarly, we use

\[
\frac{1}{(\lambda_i - w)(\lambda_i - z) - t^2} = \frac{1}{(\lambda_i - \gamma_1)^2 - t^2} - (\gamma_1 - z) + (\gamma_1 - w)) \frac{\lambda_i - \gamma_1}{((\lambda_i - \gamma_1)^2 - t^2)^2} + N^2\mathcal{O} \left( |z - \gamma_1|^2 + |w - \gamma_1|^2 |\lambda_i - \gamma_1|^4 \right)
\]

(5.93)

to find,

\[
2t \frac{\theta}{N} \sum_{i=1}^{N} v_i^2 \frac{1}{(\lambda_i - w)(\lambda_i - z) - t^2} = \frac{2t \theta}{N} \sum_{i=1}^{N} v_i^2 \frac{1}{(\lambda_i - \gamma_1)^2 - t^2} + \left[ (\gamma_1 - z) + (\gamma_1 - w) \right] \left( \frac{-2t}{N} \sum_{i=1}^{N} v_i^2 \frac{\lambda_i - \gamma_1}{((\lambda_i - \gamma_1)^2 - t^2)^2} \right) + O\left( N^{2\varepsilon} |t| N^{-1} \theta (1 + \theta/\kappa)^{-1} \kappa^{-2} \right).
\]

(5.94)

Note,

\[
\frac{2(\lambda_i - \gamma_1)^2 - t(\lambda_i - \gamma_1)}{((\lambda_i - \gamma_1)^2 - t^2)^2} - \frac{1}{(\lambda_i - \gamma_1)^2 - t^2} = \frac{1}{(\lambda_i - \gamma_1 + t)^2}.
\]

(5.95)

We have so far arrived at the following expansion which holds with overwhelming probability,

\[
G_1(z, w) = G(z) + G(w) + \left[ (\gamma_1 - z) + (\gamma_1 - w) \right] \left( -\theta N^{-1} v^T (M - \gamma_1)^{-2} v + \theta N^{-1} v^T (M - \gamma_1 + t)^{-2} v \right) + t^2 \kappa N^2 (\gamma_1) + 2\theta N^{-1} v^T (M - \gamma_1)^{-1} v - 2\theta N^{-1} v^T (M - \gamma_1 + t)^{-1} v + N^2 \mathcal{O}(t^4 \kappa^{-5/2} + t^2 \kappa^{-5/4} N^3 \kappa^{-1/2}) + N^2 \mathcal{O}(\kappa^{-2} N^{-1} t^2 (1 + \theta/\kappa)^{-1} (|t| + t^2 \kappa^{-1} + N^2 \delta^1/4 N^{-1/2})).
\]

(5.96)

Now, using

\[
\theta N^{-1} \left\{ -v^T (M - z)^{-1} v + (\gamma_1 - z) (v^T (M - \gamma_1 + t)^{-2} v - v^T (M - \gamma_1)^{-2} v) + v^T (M - \gamma_1)^{-1} - v^T (M - \gamma_1 + t)^{-1} v \right\}
\]

\[
= -\theta N^{-1} v^T (M + t - z)^{-1} v + N^2 \mathcal{O}(\kappa^{-2} N^{-1} t^2 (1 + \theta/\kappa)^{-1} (t^2 \kappa^{-1} + N^2 \delta^1/4 N^{-1/2})
\]

(5.97)

we conclude the claim.

The above lemma prompts us to investigate the function

\[
G_2(z) := \beta z - \frac{1}{N} \sum_{i} \log(z - \lambda_i) - \theta N^{-1} v^T (M - z + t)^{-1} v.
\]

(5.98)

We define \(\gamma_1\) by

\[
\beta + m(\gamma_1) = N^{-1} \theta v^T (M - \gamma_1 + t)^{-2} v.
\]

(5.99)

From the equation,

\[
G'(\gamma_1) - G'(\gamma) = \theta N^{-1} v^T (M - \gamma_1 + t)^{-2} v - v^T (M - \gamma_1)^{-2} v,
\]

(5.100)

we see that with overwhelming probability we have, for any \(\varepsilon > 0\),

\[
|\gamma_1 - \gamma| \leq C \theta \kappa^{-1} (1 + \theta/\kappa)^{-1} \leq C t, \quad \gamma_1 - \gamma = \frac{-2t \theta N^{-1} v^T (M - \gamma_1)^{-3} v}{G''(\gamma)} + N^2 \mathcal{O} \left( t^2 / \kappa \right).
\]

(5.101)

From the equation

\[
G'(\gamma_1 - t) - G'(\gamma) = m_N(\gamma_1 - t) - m_N(\gamma_1),
\]

(5.102)
we see that
\[ |γ_1 - t - γ| \leq C t (1 + θ/κ)^{-1} \]  
and
\[
γ_1 - t - γ = -t m_N'(γ_1 - t)/G''(γ) + O(t^2(1 + θ/κ)^{-2}κ^{-1})
\]
\[ = -t m_N'(γ)/G''(γ) + O(t^2(1 + θ/κ)^{-2}κ^{-1}). \]  

Note that by expanding \( γ_1 - t - γ \) instead of \( γ_1 - γ \) we have picked up some extra factors of \( (1 + θ/κ)^{-1} \) which will prove useful later. We are now ready to prove the following.

**Proposition 5.8.** With overwhelming probability, we have for a \( c > 0 \), and \( t \) satisfying,
\[ t^2 \leq \log(N)\sqrt{κN}^{-1} \]  
the expansion,
\[
\log(\langle e^{tβσ^{(1)} - σ^{(2)}} \rangle) = \frac{N t^2}{2} m'(γ) (1 - θ(2N^{-1}v^T(M - γ)^{-3}v)(G''(γ)^{-1}) + tθv^T(M - γ)^{-2}v
\]
\[ + O(N^{-c}). \]  

**Proof.** With \( z = γ_1 + is_1 \) and \( w = γ_1 + is_2 \) and \( |s_i| \leq N^5κ^{1/4}N^{-1/2}(1 + θ/κ)^{1/2} \) we see that,
\[
\frac{N}{2}(G_1(z, w) - 2G_2(γ_1) - t^2 m_N'(γ_1)) = -N \frac{G_2''(γ_1)}{4}(s_1^2 + s_2^2) + O(N^{-c}) \]  
for some \( c > 0 \), taking \( δ > 0 \) sufficiently small, with overwhelming probability. Note that since by the above estimates and assumptions on \( \theta \) we see, \( G_2''(γ_1) \) we have that,
\[
G_2''(γ_1) = \frac{1}{κ^{1/2}}(1 + θ/κ). \]  
Therefore, a straightforward modification of the saddle point analysis presented in Proposition 5.2 gives,
\[
\log(\langle e^{tβσ^{(1)} - σ^{(2)}} \rangle) = N(G_2(γ_1) - G(γ)) + \frac{N t^2 m'(γ_1)}{2}
\]
\[ + \log(G_2''(γ_1)) - \log(G''(γ)) + O(N^{-c}). \]  
for some \( c > 0 \), with overwhelming probability. From our expansions of \( γ_1 \) in terms of \( γ \) and \( t \) it is easy to see that
\[
|t^2 m'(γ_1)) - t^2 m'(γ)| + |log(G_2''(γ_1)) - log(G''(γ))| \leq N^{-c}. \]  
Therefore we have the following estimate with overwhelming probability,
\[
\log(\langle e^{tβσ^{(1)} - σ^{(2)}} \rangle) = N(G_2(γ_1) - G(γ)) + \frac{N t^2 m'(γ)}{2} + O(N^{-c}). \]  
Now,
\[
G_2(γ_1) - G(γ) = m(γ)(γ_1 - γ) - θN^{-1}v^T(M - γ)^{-2}v(γ_1 - γ - t)
\]
\[ + \frac{m'(γ)}{2}(γ_1 - γ)^2 - θN^{-1}v^T(M - γ)^{-3}v(γ_1 - γ - t)^2
\]
\[ + O(|t|^3κ^{-3/2}). \]  
Using \( G'(γ) = 0 \) we see,
\[
m_N(γ)(γ_1 - γ) - θN^{-1}v^T(M - γ)^{-2}v(γ_1 - γ - t) = tθN^{-1}v^T(M - γ)^{-2}v. \]
Applying now (5.101) and (5.104) we have with overwhelming probability,

\[
\frac{m'(\gamma)}{2} (\gamma_1 - \gamma)^2 - \theta N^{-1} v^T (M - \gamma)^{-3} v (\gamma_1 - \gamma - t)^2
\]

\[
= - \frac{m'(\gamma) \theta 2 N^{-1} v^T (M - \gamma)^{-3} v^2}{2 G''(\gamma)} + N^\varepsilon O(\|\mu\|^3 N^{-3/2}).
\] (5.114)

Hence,

\[
\log \langle e^{\beta (\sigma_{(1)} - \sigma_{(2)})} \rangle = \frac{N \ell^2}{2} m'(\gamma) (1 - \theta (2 N^{-1} v^T (M - \gamma)^{-3} v) G''(\gamma)^{-1}) + t \theta v^T (M - \gamma)^{-2} v \\
+ O(N^{-\varepsilon}).
\] (5.115)

with overwhelming probability, for some \( c > 0 \).

From Theorems 3.1 and 3.3 as well as Lemma 5.1 we see that,

\[
\left| m'(\gamma) (1 - \theta (2 N^{-1} v^T (M - \gamma)^{-3} v) G''(\gamma)^{-1}) - m'_{sc}(\gamma) \right| \leq N^{-c}
\] (5.116)

for some \( c > 0 \) with overwhelming probability. Theorem 2.6 follows from this and the previous Proposition.

Lemma 2.7 follows in a similar fashion to Lemma 2.5.

6  \( h = O(N^{-1/6}) \) and \( \beta > 1 \)

In this section we will prove the results in Section 2.3. We assume that the parameters \( \beta \) and \( h \) satisfy,

\[
1 + c \leq \beta \leq c^{-1}, \quad h^2 \beta = \theta N^{-1/3}
\] (6.1)

for a fixed \( c > 0 \) and fixed \( \theta > 0 \).

6.1 Free energy

In this section we investigate the free energy and prove Theorem 2.8. From our choice of \( h^2 \beta = \theta N^{-1/3} \) and Proposition 4.1 we are led to investigate the function,

\[
G(z) = \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i) - \frac{\theta}{N^{4/3}} v^T \frac{1}{M - z} v.
\] (6.2)

We recall the definition of \( H = M + V \) with \( V \) iid Gaussians of variance \( 2/N \) so that \( H \) is a GOE matrix. We define the quantities \( \gamma, \hat{\gamma}_M \) and \( \hat{\gamma}_H \) by,

\[
\beta + m_N(\gamma) = \frac{\theta}{N^{4/3}} v^T \frac{1}{(M - \gamma)^2} v,
\] (6.3)

\[
\beta - 1 = \frac{\theta}{N^{4/3}} v^T \frac{1}{(M - \gamma_M)^2} v,
\] (6.4)

\[
\beta - 1 = \frac{\theta}{N^{4/3}} v^T \frac{1}{(H - \hat{\gamma}_H)^2} v.
\] (6.5)

From Theorem 3.5 and the fact that \( u_1(H) \) is uniformly distributed on the \( N - 1 \) sphere, we have the following lemma.

**Lemma 6.1.** For all sufficiently small \( \varepsilon > 0 \), we have,

\[
\mathbb{P}[(v^T u_1(M))^2 \leq N^{-\varepsilon}] \leq N^{-\varepsilon/2}.
\] (6.6)

We now define an event \( \mathcal{F}_{\varepsilon_1} \) on which a number of estimates for the eigenvalues and eigenvectors of \( M \) and \( H \) hold.
Definition 6.2. Let \( \varepsilon_1 > 0 \) be sufficiently small. We define \( \mathcal{F}_{\varepsilon_1} \) to be the event that all of the following estimates hold. First, we assume that following the level repulsion estimates hold,

\[
|\lambda_1(M) - \lambda_2(M)| \geq N^{-2/3-\varepsilon_1/100}, \quad |\lambda_1(H) - \lambda_2(H)| \geq N^{-2/3-\varepsilon_1/100}.
\]  

(6.7)

We assume the following lower bound for the projection of the largest eigenvectors of \( M \) and \( H \) onto \( v \),

\[
(v^T u_1(M))^2 \geq N^{-\varepsilon_1/100}, \quad (v^T u_1(H))^2 \geq N^{-\varepsilon_1/100}.
\]  

(6.8)

We assume that the rigidity estimates of Theorem 3.2 hold with \( \varepsilon = \varepsilon_1/(10^6) \), for the eigenvalues of \( M \) and \( H \). We also assume the delocalization bounds,

\[
\sup_i (v^T u_i(M))^2 + (v^T u_i(H))^2 \leq N^{\varepsilon_1/10^6}.
\]  

(6.9)

We assume that the events of Lemma 3.6 hold with \( \omega = \varepsilon_1/10^6 \) in the definitions of \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) and \( \delta = 1/1000 \) and \( \varepsilon = \varepsilon_1/10^6 \). With the same choice of parameters we also assume both isotropic estimates of Theorem 3.3 hold with \( u, w = N^{-1/2}v \). We assume also that the events of Theorem 3.5 hold with \( \varepsilon = \varepsilon_1/10^6 \) in the first estimate and \( c = \varepsilon_1/1000 \) in the second. \qed

From all of the results in Section 3 we know that \( \mathcal{F}_{\varepsilon_1} \) holds with probability at least \( \mathbb{P}[\mathcal{F}_{\varepsilon_1}] \geq 1 - N^{-c_\varepsilon \varepsilon_1} \) for some \( c_\varepsilon > 0 \):

Lemma 6.3. There is a small \( c_\varepsilon > 0 \) so that for sufficiently small \( \varepsilon_1 > 0 \) that \( \mathcal{F}_{\varepsilon_1} \) holds with probability at least \( 1 - N^{-c_\varepsilon \varepsilon_1} \).

Let us write,

\[
\gamma = \lambda_1(M) + \frac{s}{N^{2/3}}, \quad \hat{\gamma}_M = \lambda_1(M) + \frac{s}{N^{2/3}}, \quad \hat{\gamma}_H = \lambda_1(M) + \frac{\hat{s}_H}{N^{2/3}}.
\]  

(6.10)

For these quantities we now establish some preliminary estimates.

Lemma 6.4. For all sufficiently small \( \varepsilon_1 > 0 \) we have on the event \( \mathcal{F}_{\varepsilon_1} \) that,

\[
N^{-\varepsilon_1/100} \leq \hat{s}_M, \hat{s}_H, s \leq N^{\varepsilon_1/100}, \quad |s - \hat{s}_M| \leq N^{\varepsilon_1/3-1/3}.
\]  

(6.11)

There is moreover a small \( c_1 > 0 \) so that if \( \varepsilon_1 > 0 \) is sufficiently small then on \( \mathcal{F}_{\varepsilon_1} \),

\[
|\hat{s}_M - \hat{s}_H| \leq N^{-c_1}, \quad |\hat{\gamma}_M - \hat{\gamma}_H| \leq N^{-2/3-c_1}.
\]  

(6.12)

Proof. We have,

\[
\beta - 1 \geq \theta (v^T u_1(M))^2 \frac{1}{\hat{s}_M^{4/3}},
\]  

(6.13)

and so on the event \( \mathcal{F}_{\varepsilon_1}, \hat{s}_M \geq cN^{-\varepsilon_1/100} \). The same estimates also clearly hold for \( s \) and \( \hat{s}_H \) (for \( \hat{s}_H \) we also that the estimates (3.15) hold on \( \mathcal{F}_{\varepsilon_1} \)). For an upper bound we use the fact that the delocalization estimates hold for \( v^T u_i(M) \) to see that,

\[
\frac{\theta}{N^{4/3}} |v^T| \frac{1}{(M - \hat{\gamma}_M)^2} v \leq \theta N^{\varepsilon_1} N^{\varepsilon_1/1000} (\hat{s}_M)^{-2} + \sum_{i > N^{\varepsilon_1/1000}} \frac{N^{2/3} \lambda_i(M) - \hat{\gamma}_M}{(M - \hat{\gamma}_M)^2} N^{\varepsilon_1/10^6}
\]  

(6.14)

for any \( \varepsilon > 0 \). Taking \( \varepsilon = \varepsilon_1/1000 \), we see that by the rigidity estimates, the second term is \( \mathcal{O}(N^{-\varepsilon_1/3+\varepsilon_1/10^6}) \). We therefore get the inequality,

\[
\beta - 1 \geq \frac{2\theta N^{\varepsilon_1/1000} (\hat{s}_M)^{-2}}{2}
\]  

(6.15)

and so \( (\hat{s}_M) \leq N^{\varepsilon} \) on an event of overwhelming probability. A similar conclusion holds for \( s \) and \( \hat{s}_H \). On \( \mathcal{F}_{\varepsilon_1} \) we may now assume that \( N^{-\varepsilon_1/100} \leq s \leq N^{\varepsilon_1/100} \) and the same for \( \hat{s}_M \) and \( \hat{s}_H \). In the region \( N^{-2/3-\varepsilon_1/50} \leq E \leq N^{-2/3+\varepsilon_1/50} \), we see that

\[
N^{-\varepsilon_1/10} \leq \frac{1}{N^{2/3}} \int dE \frac{\theta}{N^{4/3}} |v^T| \frac{1}{(M - \hat{\gamma}_M - E)^2} v \leq N^{\varepsilon_1/10}.
\]  

(6.16)
From this, we see that
\[ N^{-1/10} |s - \hat{s}_M| \leq \left| \frac{1}{N} v^T \frac{1}{(M - \gamma)^2} v - \frac{1}{N} v^T \frac{1}{(M - \hat{\gamma}_M)^2} v \right| \leq N^{\varepsilon_1/10} |s - \hat{s}_M| \quad (6.17) \]

From the fact that the rigidity and level repulsion estimates are assumed to hold on $\mathcal{F}_{\varepsilon_1}$, we have that the estimate
\[ |m_N(\gamma) + 1| \leq N^{\varepsilon_1/50 - 1/3} \]
holds on $\mathcal{F}_{\varepsilon_1}$. This implies,
\[ |s - \hat{s}_M| \leq N^{\varepsilon_1/5 - 1/3}. \quad (6.19) \]

We now turn to proving the estimate on the difference $\hat{s}_M - \hat{s}_M$. Consider now $E = \lambda_1(M) + uN^{-2/3}$ for $N^{-2\varepsilon_2} \leq u \leq N^{2\varepsilon_2}$. Let $\varepsilon_2 = 10^{-6}$. Let $\eta = N^{-2/3 - 1/1000}$. We begin by rewriting,
\[ N^{-4/3} v^T (H - E)^{-2} v - N^{-4/3} v^T (M - E)^{-2} v \quad (6.20) \]
\[ = N^{-4/3} v^T (H - E)^{-2} v - N^{-4/3} v^T (H - (E + \ii \eta))^2 v \quad (6.21) \]
\[ + N^{-4/3} v^T (M - (E + \ii \eta))^2 v - N^{-4/3} v^T (M - E)^{-2} v \quad (6.22) \]
\[ + N^{-4/3} v^T (H - (E + \ii \eta))^2 v - N^{-4/3} v^T (M - (E + \ii \eta))^2 v \quad (6.23) \]

From the delocalization estimates that hold on $\mathcal{F}_{\varepsilon_1}$ we have that $|v^T u(H)| + |v^T u(M)| \leq N^{2\varepsilon_1/10^6}$ and we can assume $\varepsilon_1 < \varepsilon_2/100$. Using the rigidity estimates that hold on $\mathcal{F}_{\varepsilon_1}$ we see that the term (6.21) can estimated as,
\[ \left| N^{-4/3} v^T (H - E)^{-2} v - N^{-4/3} v^T (H - \ii \eta)^{-2} v \right| \leq C \eta N^{\varepsilon_1/10} \sum_{i=1}^{N} \frac{1}{N^4 \epsilon_1 |\lambda_i^2(H) - E^2|} \leq C \eta N^{10\varepsilon_2} N^{2/3}, \quad (6.24) \]
for all $E$ specified above. The term (6.22) is similar. For (6.23) we have the estimates (3.17) assumed to hold on $\mathcal{F}_{\varepsilon_1}$ which gives an estimate of (using the Cauchy-Riemann formula),
\[ \left| N^{-4/3} v^T (H - \ii \eta)^{-2} v - N^{-4/3} v^T (M - \ii \eta)^{-2} v \right| \leq \frac{1}{\eta N^{1/3}} \left( \frac{1}{N^{1/2}} + \frac{1}{N^{2/3}} + \frac{N^{-1/3}}{N \eta} \right) \quad (6.25) \]

Note we used $\text{Im}[m_{sc}] \leq C \tau (E - \epsilon)^{1/2} + \eta^{1/2}$. Under our assumptions on $\varepsilon_2$ and $\eta$ we see that both (6.25) and (6.27) are $O(N^{-\alpha})$ for some small $\alpha > 0$, on the event $\mathcal{F}_{\varepsilon_1}$. We can then write,
\[ \left| N^{-4/3} v^T (M - \hat{\gamma}_H)^{-2} v - N^{-4/3} v^T (M - \hat{\gamma}_M)^{-2} v \right| = \left| N^{-4/3} v^T (M - \hat{\gamma}_H)^{-2} v - N^{-4/3} v^T (M - \hat{\gamma}_M)^{-2} v \right| \quad (6.28) \]

On $\mathcal{F}_{\varepsilon_1}$ we can assume that $|\lambda_1(M) - \lambda_1(H)| \leq N^{-5/6}$ and that $N^{-\varepsilon} \leq \hat{s}_M, \hat{s}_H \leq N^\varepsilon$, with $\varepsilon < \varepsilon_2/2$, with $\varepsilon_2$ as above, and $\varepsilon < \alpha/100$, where $\alpha$ is as above. As we have seen, the LHS of (6.28) is $O(N^{-\alpha})$ on $\mathcal{F}_{\varepsilon_1}$. From the inequality which holds for $a, b > \lambda_1(M)$,
\[ |v^T (M - a)^{-2} v - v^T (M - b)^{-2} v| = |b - a| |v^T [(M - a)^{-2} (M - b)^{-1} + (M - a)^{-1} (M - b)^{-2}]| \geq |b - a| |v^T u_1(M)| |(\lambda_1 - a)^{-2} (\lambda_1 - b)^{-1}|, \quad (6.29) \]
we find
\[ \left| N^{-4/3} v^T (M - \hat{\gamma}_H)^{-2} v - N^{-4/3} v^T (M - \hat{\gamma}_M)^{-2} v \right| \leq N^{-\varepsilon_1} |\hat{s}_M - \hat{s}_H|. \quad (6.30) \]

On the other hand, we assumed that the LHS of (6.28) is less than $N^{-\alpha}$, and $\varepsilon_1 < \alpha/100$. Hence, we have that there is some $c_1 > 0$ so that $|\hat{s}_M - \hat{s}_H| \leq N^{-c_1}$ on the event $\mathcal{F}_{\varepsilon_1}$. With these preliminary estimates on the saddle locations out of the way, we turn to the remainder of the proof. The method of steepest descent will allow us to derive an expansion for the free energy.
in terms of the saddle $\gamma$; our estimates above then allow us to replace this by quantities involving only the spectral quantities of $H$. The convergence of this latter system will be carried out in the next section.

For all sufficiently small $\varepsilon_1 > 0$, we have on the event $\mathcal{F}_{\varepsilon_1}$ that, due to the level repulsion, rigidity and estimate of Lemma 6.1 that,

$$N^{2k/3-\varepsilon_1/500} \leq |G^{(k+1)}(\gamma)| \leq N^{2k/3+\varepsilon_1/500}, \quad k = 1, 2$$

(6.31)

and the expansion,

$$G(\gamma + it) - G(\gamma) = -\frac{G''(\gamma)}{2} t^2 + \mathcal{O}(N^{-1-1/10})$$

(6.32)

for $|t| \leq N^{-5/6+1/100}$. A straightforward modification of the steepest descent argument in Proposition 4.1 gives the following.

**Proposition 6.5.** Assume that (6.1) holds. There is a small $c_1 > 0$ so that for sufficiently small $\varepsilon_1 > 0$ we have that on the event $\mathcal{F}_{\varepsilon_1}$, the following expansion for the free energy holds

$$\frac{1}{N} \log Z_{N,\beta,h} = \frac{1}{2} G(\gamma) + \frac{1}{N} \log(\Gamma(N/2)) + \frac{1}{N}(N/2 - 1) \log(2/(N\beta)) - \frac{1}{2N} \log(NG''(\gamma)\pi) + \mathcal{O}(N^{-1-c_1}),$$

(6.33)

where $\gamma$ is the saddle defined by (6.3).

We expect the fluctuations of the first term, $G(\gamma)$ to be $\mathcal{O}(N^{-2/3})$, so we will ignore the $N^{-1}$ log term in examining fluctuations. We now replace the quantities that we obtained from the saddle point analysis with spectral quantities involving only the GOE matrix $H$; the latter will be easier to analyze later.

**Lemma 6.6.** There is a small $c_1 > 0$ so that for sufficiently small $\varepsilon_1 > 0$, on the event $\mathcal{F}_{\varepsilon_1}$ it holds that,

$$N^{2/3} \left( F_{N,\beta,h} - \frac{1}{N} \log(\Gamma(N/2)) + \frac{1}{N}(N/2 - 1) \log(2/(N\beta)) - C_N \right)$$

$$= \frac{1}{2} \left( (\beta - 1)N^{2/3}(\hat{\gamma}_H - 2) - \frac{\theta}{N^{2/3}} \hat{v}^T(H - \hat{\gamma}_H)^{-1}v \right) + \mathcal{O}(N^{-c_1}).$$

(6.34)

**Proof.** Using the rigidity estimates of the event $\mathcal{F}_{\varepsilon_1}$ and the estimates of Lemma 6.4 we first have,

$$\beta(\gamma - 2) - \frac{1}{N} \sum_{i=1}^N \log(\gamma - \lambda_i) = (\beta - 1)(\hat{\gamma}_H - 2) + C_N + \mathcal{O}(N^{-2/3-c})$$

(6.35)

for some $c > 0$. Here, $C_N$ is the constant

$$C_N := - \int_{-2}^2 \log(2-x) d\rho_{sc}(x).$$

(6.36)

We estimate

$$N^{-4/3} |\hat{v}^T(H - \hat{\gamma}_H)^{-1}v - \hat{v}^T(M - \gamma)^{-1}v| \leq N^{-4/3} |\hat{v}^T(H - \hat{\gamma}_H)^{-1}v - \hat{v}^T(M - \hat{\gamma}_H)^{-1}v|$$

(6.37)

$$+ N^{-4/3} |\hat{v}^T(H - \hat{\gamma}_H)^{-1}v - \hat{v}^T(H - \gamma)^{-1}v|$$

(6.38)

The term (6.37) is similar to (6.20) with $E = \hat{\gamma}_H$ and by the same proof of our estimate of (6.20) we find that (6.37) is $\mathcal{O}(N^{-2/3-c})$ for some $c > 0$. The same estimate holds for (6.38) given the second estimate of (6.12). Therefore, on $\mathcal{F}_{\varepsilon_1}$

$$N^{-4/3} |\hat{v}^T(H - \hat{\gamma}_H)^{-1}v - \hat{v}^T(M - \gamma)^{-1}v| \leq N^{-2/3-c}.$$
This concludes the proof.

This completes our expansion of the free energy in terms of GOE quantities. We now prove Theorem 2.8.

**Proof of Theorem 2.8.** The random variable \( X_N \) is the first term on the RHS of (6.34). The eigenvectors and eigenvalues of the GOE matrix \( H \) are independent, and the eigenvector matrix is uniformly distributed on the orthogonal group. Hence, the quantities \( \{(u^T u_i(H))^2\}_{i=1}^N \) have the same joint distribution as \( \{N g_i^2/\sum_j g_j^2\}_{i=1}^N \) where the \( \{g_i\}_{i=1}^N \) are iid standard normal random variables. Let now \( \{\mu_i\}_{i=1}^N \) be a vector of GOE eigenvalues in decreasing order independent of the \( \{g_i\}_{i=1}^N \).

Let \( x_a \) be the solution to

\[
\beta - 1 = \frac{\theta}{N^{4/3}} \frac{1}{N^{-1}} \sum_{i=1}^N g_i^2 \sum_{i=1}^N (\mu_i - x_a)^2.
\]

Then the main term \( X_N \) in the expansion of the free energy has the same distribution as,

\[
(\beta - 1)N^{2/3}(\gamma_H - 2) - \frac{\theta}{N^{2/3}} u^T (H - \gamma_H) - \frac{1}{\theta} \frac{\mu}{N^{2/3}} \sum_{i=1}^N g_i^2 \sum_{i=1}^N (\mu_i - x_a).
\]

This is the random variable \( Y_N \) in the theorem statement. Consider now the alternative system where \( x_b \) is the solution to

\[
\beta - 1 = \frac{\theta}{N^{4/3}} \sum_{i=1}^N g_i^2 \frac{(\mu_i - x_b)^2}{(\mu_i - x_b)^2}.
\]

and the quantity,

\[
(\beta - 1)N^{2/3}(x_b - 2) - \frac{\theta}{N^{2/3}} \sum_{i=1}^N g_i^2 \sum_{i=1}^N (\mu_i - x_b).
\]

Now we have that

\[
N^{-1} \sum_{i=1}^N g_i^2 = 1 + \mathcal{O}(N^{-1/2 + \epsilon}), \quad \sup_i |g_i| \leq N^\epsilon
\]

with overwhelming probability for any \( \epsilon > 0 \). On the event that the above estimates as well as the rigidity estimates hold for \( \mu_i \) with sufficiently small \( \epsilon > 0 \) that,

\[
\left| \frac{\theta}{N^{4/3}} \sum_{i=1}^N g_i^2 \frac{(\mu_i - x_a)^2}{(\mu_i - x_a)^2} - \frac{\theta}{N^{4/3}} \frac{1}{N^{-1}} \sum_{i=1}^N g_i^2 \sum_{i=1}^N (\mu_i - E)^2 \right| \leq N^{3\epsilon + 10\epsilon} N^{-1/2}
\]

for \( E \) satisfying \( N^{-2/3} \leq E - \mu_1 \). Hence, similar arguments leading our estimates (6.12) yield,

\[
|x_a - x_b| \leq N^{-5/6}
\]

with probability at least \( 1 - N^{-\epsilon} \) for sufficiently small \( \epsilon > 0 \). Furthermore, we write,

\[
\frac{1}{N^{-1} \sum_{i=1}^N g_i^2} \sum_{i=1}^N (\mu_i - x_a) - \frac{1}{N^{2/3}} \sum_{i=1}^N (\mu_i - x_b) = \frac{\theta}{N^{2/3}} \sum_{i=1}^N (\mu_i - x_a) - \frac{\theta}{N^{2/3}} \sum_{i=1}^N (\mu_i - x_b) + (1 - N^{-1} \sum_{i=1}^N g_i^2) \frac{\theta}{N^{2/3}} \sum_{i=1}^N (\mu_i - x_a).
\]

With probability at least \( 1 - N^{-\epsilon} \) the first line is \( \mathcal{O}(|x_a - x_b|N^{2/3 + 10\epsilon}) \) and the second line is \( \mathcal{O}(N^{1/3 + \epsilon} N^{-1/2}) \). This concludes the proof. \( \square \)
6.2 Convergence of the saddle system

In this section we prove the following theorem, which implies Theorem 2.9. In the course of the proof we also obtain that the quantities $\mu$ and $\Xi$ in the statement of Theorem 2.9 are well-defined.

Theorem 6.7. Let $\{\mu_i\}_{i=1}^N$ be the eigenvalues of a GOE matrix. Let $\{g_i\}_{i=1}^N$ be iid Gaussian random variables independent from $\{\mu_i\}_{i=1}^N$. Let $x_b$ be the unique solution s.t. $x_b > \mu_1$ to,

$$\beta - 1 = \frac{\theta}{N^{4/3}} \sum_{i=1}^N \frac{g_i^2}{(\mu_i - x_b)^2}.$$  (6.49)

Then $x_b$ and the random variable,

$$N^{2/3}(x_b - 2) + \frac{\theta}{N^{2/3}} \sum_{i=1}^N \left( \frac{g_i^2}{(\mu_i - x_b)} - \frac{1}{(\gamma_i - 2)} \right)$$  (6.50)

converge to the corresponding quantities for the Airy$_1$ random point field.

We will use the following lemma, a consequence of [14].

Lemma 6.8. There is a $c_0 > 0$ so that the following statement holds. Let $\varepsilon > 0$. Then there is a $K > 0$ depending on $\varepsilon$ so that for $N$ large enough,

$$\mathbb{P} \left[ \bigcap_{j=K}^N \left\{ c_0 j^{2/3} \leq N^{2/3}(\mu_j - 2) \leq c_0^{-1} j^{2/3} \right\} \right] \geq 1 - \varepsilon.$$  (6.51)

We define the random variable $s$ by

$$x_b = \mu_1 + N^{-2/3} s.$$  (6.52)

We first prove the following.

Lemma 6.9. The random variables $s$ and $s^{-1}$ are tight.

Proof. For a lower bound,

$$\beta - 1 \geq \frac{\theta g_1^2}{s^2}.$$  (6.53)

For an upper bound, we have on the event of Lemma 6.8 for all $k_0 > K$,

$$\frac{\beta - 1}{\theta} \leq \frac{1}{s} \sum_{j=1}^{k_0} g_j^2 + C \sum_{j>k_0} \frac{g_j^2}{j^{4/3}}$$  (6.54)

for some $C > 0$ independent of $k_0$. For $k_0$ sufficiently large, there is an event of probability at least $1 - \varepsilon$ on which the second term is less than $\frac{\beta - 1}{\theta}$, by Markov’s inequality. Hence, there is an event of probability at least $1 - 2\varepsilon$ on which,

$$s^2 \leq \frac{2\theta}{\beta - 1} \sum_{j=1}^{k_0} g_j^2.$$  (6.55)

This yields the claim. \(\square\)

We now define $s_n$ to be the unique positive solution to,

$$\beta - 1 = \frac{\theta}{N^{4/3}} \sum_{i=1}^n \frac{g_i^2}{(s_n N^{-2/3} + \mu_1 - \mu_i)^2}.$$  (6.56)

Lemma 6.10. Let $\varepsilon > 0$. There is an $n_0$, depending on $\varepsilon$ so that for all $N$ large enough, so that there is an event $\mathcal{F}_{n_0}$ on which the following estimate for all $n \geq n_0$,

$$|s - s_n| \leq \varepsilon$$  (6.57)

and $\mathbb{P}[\mathcal{F}_{n_0}] \geq 1 - \varepsilon$. 

34
Proof. First note that \( s_n \leq s \). We have the estimates,
\[
\sum_{j > n} \frac{g_j^2}{N^{4/3}(\mu_j - \mu_1)^2} \geq \sum_{j > n} \frac{g_j^2}{N^{4/3}(\mu_j - \mu_1 - sN^{-2/3})^2}
\]
\[
= \sum_{j=1}^{n} \frac{g_j^2}{(\mu_j - \mu_1 - sN^{-2/3})^2} - \sum_{j=1}^{n} \frac{g_j^2}{(\mu_j - \mu_1 - sN^{-2/3})^2}
\]
\[
= (s - s_n) \sum_{j=1}^{n} \frac{g_j^2}{N^{8/3}(\mu_j - \mu_1 - sN^{-2/3})^2(\mu_j - \mu_1 - sN^{-2/3})^2}
\]
\[
\geq (s - s_n) \frac{g_j^2}{s^2}.
\]
(6.58)
Due to the previous lemma, for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) on which \( g_j^2/s^3 \geq \delta \) with probability \( 1 - \varepsilon \). Hence, with probability at least \( 1 - \varepsilon \), we have that for all \( n \),
\[
(s - s_n) \leq \frac{1}{\delta} \sum_{j = n}^{N} \frac{g_j^2}{N^{4/3}(\mu_j - \mu_1)^2}.
\]
Let \( K \) be so that the event of Lemma 6.8 holds with probability at least \( 1 - \varepsilon \). As long as \( n \geq K \), we have that there is an event with probability at least \( 1 - \varepsilon \) on which,
\[
(s - s_n) \leq \frac{C}{\delta} \sum_{j = K}^{N} \frac{g_j^2}{s^{4/3}}.
\]
(6.60)
Note that this inequality holds for every \( n > K \) and \( C, \delta \) are independent of \( K \). Now if \( K \) is sufficiently large then by Markov’s inequality with probability at least \( 1 - \varepsilon \) the term on the RHS is less than \( \varepsilon \).

Lemma 6.11. Let \( \varepsilon > 0 \). There is an \( n_1 \), so that for \( n \geq n_1 \) we have that there is an event \( \mathcal{F}_n \) which holds with probability at least \( 1 - \varepsilon \) on which,
\[
\frac{1}{N^{2/3}} \left| \sum_{j=2}^{N} \left( \frac{g_j^2}{(\mu_j - \mu_1 - sN^{-2/3})^2} - \frac{1}{\gamma_j - 2} \right) \right| \leq \varepsilon,
\]
(6.61)
and
\[
|g_j^2/s - g_k^2/s_k| \leq \varepsilon.
\]
(6.62)
The same statement holds with the \( g_j^2 \) replaced by 1 everywhere.

Proof. The second estimate is an easy consequence of our previous estimates. For the first estimate, we first let \( n_0, C_0 > 0, \delta, c_0 \) and \( \mathcal{F} \) be the event with \( P[\mathcal{F}] \geq 1 - \varepsilon \) on which,
\[
c_0j^{2/3} \leq N^{2/3}|\mu_j - \mu_1| \leq c_0^{-1}j^{2/3}, \quad j \geq n_0
\]
(6.63)
and also
\[
|\mu_j - 2|N^{2/3} \leq C_0
\]
(6.64)
and \( \delta \leq s \leq \delta^{-1} \). Note that \( c_0 \) does not depend on \( \varepsilon \) but the other parameters do. For \( n \geq n_0 \), we estimate
\[
\frac{1}{N^{2/3}} \left| \sum_{j>n} \frac{g_j^2}{(\mu_j - \mu_1 - sN^{-2/3})^2} \right| 1_{\mathcal{F}}
\]
\[
\leq \sum_{j>n} C|g_j|^2 |\mu_j - 2|N^{2/3} \leq c_0j^{2/3} + \delta^{-1} + C_0
\]
\[
+ \frac{1}{N^{2/3}} \sum_{j>n} \left| \frac{g_j^2 - 1}{(\gamma_j - 1)} \right|.
\]
(6.65)
The second moment of the last term is bounded by $Cn^{-1/3}$. The first term has, due to \cite{[14]}, expectation bounded by $Cn^{-1/5}$. So if $n > n_1$, with $n_1$ sufficiently large, than these terms are less than $\varepsilon$ with probability at least $1 - \varepsilon$.

Now, fix $\varepsilon_1 > 0$. By taking $n_1$ sufficiently large, we may assume that for all $n > n_1$ that $|s - s_n| < \varepsilon_1$ with probability at least $1 - \varepsilon$. Assume $\varepsilon_1 < \delta/2$. Call this event $F_1$. Let $C_2$ be a constant so that $\sum_{j=1}^{\infty} g_j^2 \leq C_2$ with probability at least $1 - \varepsilon$ and call this event $F_2$. Note that the constant $C_2$ depends only on $n_0$ and not on $\varepsilon_1$ or $n_1$. On $F \cap F_1 \cap F_2$,

$$
\frac{1}{N^{2/3}} \left| \sum_{j=2}^{n} g_j^2 \left( \frac{1}{\mu_j - \mu_1 - sN^{-2/3} - \frac{1}{\mu_j - \mu_1 - sN^{-2/3}} \right) \right| 
\leq |s - s_n| 10\delta^{-2} C_2 + \varepsilon_1 \sum_{j=n_0+1}^{n} \frac{g_j^2}{c_j^{1/3}} 
\leq \varepsilon_1 \left( 10\delta^{-2} C_2 + \sum_{j=n_0+1}^{n} \frac{g_j^2}{c_j^{1/3}} \right) \quad (6.66)
$$

By Markov’s inequality, there is an $M > 0$ so that the second term in the brackets on the last line is less than $M$ with probability at least $1 - \varepsilon$ (this $M$ does not depend on $\varepsilon_1$ or $n$). Hence, by taking $\varepsilon_1$ small enough depending on $C_2$, $\delta$ and $M$, we get that this is less than $\varepsilon$.

**Proof of Theorem 6.7.** Denote by $y_n$ the random variable

$$
y_n := N^{2/3} (\mu_1 + s_n N^{-2/3} - 2) + \frac{\theta}{N^{2/3}} \sum_{i=1}^{n} \left( \frac{g_i^2}{\chi_i - \chi - 1 - \gamma_i - 1} \right). \quad (6.67)
$$

Note that $y_N$ is the random variable whose convergence we are interested in. For any bounded Lipschitz function $F$ and any $\varepsilon > 0$ we see that there is an $n$ so that for all $N$ large enough,

$$
|\mathbb{E}[F(y_N)] - \mathbb{E}[F(y_n)]| \leq \varepsilon. \quad (6.68)
$$

Since $g_i^2 > 0$ almost surely, and that the GOE eigenvalues are a.s. distinct, we see that $s_n$ and $y_n$ are continuous functions of $\{\mu_i\}_{i=1}^{n}$. Let $\{\chi_i\}_{i=1}^{\infty}$ be the Airy$_1$ random point field, normalized so that the joint limit of $\{N^{2/3} (\mu_i - 2)\}_{i=1}^{n}$ is the first $n$ particles of the Airy$_1$ random point field.

If $a_n$ denotes the solution to

$$
\beta - 1 = \theta \sum_{i=1}^{n} \frac{g_i^2}{(\chi_i - \chi - 1 - a_n)^2} \quad (6.69)
$$

and $z_n$ the random variable,

$$
z_n := \chi + a_n + \theta \sum_{i=1}^{n} \left( \frac{g_i^2}{\chi_i - \chi - 1 - a_n} + \frac{1}{(\pi i/3)^{2/3}} \right). \quad (6.70)
$$

We have by the joint convergence of the $n$ largest GOE eigenvalues to the Airy$_1$ random point field,

$$
\lim_{N \to \infty} \mathbb{E}[F(y_n)] = \mathbb{E}[F(z_n)]. \quad (6.71)
$$

It remains to prove that,

$$
\lim_{n \to \infty} \mathbb{E}[F(z_n)] = \mathbb{E}[F(z_\infty)]. \quad (6.72)
$$

The arguments are almost identical to the arguments involving the analogous statements for $y_n$ and $y_N$. Indeed, all the estimates for the GOE eigenvalues $\mu_i$ we used have direct analogues for the Airy$_1$ random point field – this is proven in \cite{[14]}.

The only substantial difference is that one must truncate the analogue of (6.65) using the Kolmogorov Three Series lemma, as it is an infinite random sum that is not absolutely summable. Note that these arguments also show that the random variables $a_\infty$ and $z_\infty$ are well defined.

The same method as above gives a proof of the following generalization.
Proposition 6.12. Let \( \varepsilon > 0 \), \( k \geq 2 \) an integer, and let \( s_n \) be as above. There is an \( n_1 > 0 \) so that for all \( n > n_1 \) there is an event \( \mathcal{F}_n \) with probability at least \( 1 - \varepsilon \) so that

\[
\frac{1}{N^{2k/3}} \left| \sum_{i=1}^{N} \frac{g_j^2}{(\mu_j - \mu_1 - sN^{-2/3})^k} - \sum_{i=1}^{n} \frac{g_j^2}{(\mu_j - \mu_1 - s_nN^{-2/3})^k} \right| \leq \varepsilon. \quad (6.73)
\]

The same statement holds with the \( g_j^2 \) replaced by 1.

Additionally, we have the following.

Theorem 6.13. For any \( k_0 \geq 2 \) we have the joint convergence of the random variables,

\[
\left\{ \frac{1}{N^{2k/3}} \sum_{i=1}^{N} \frac{g_j^2}{(\mu_i - \mu_1 - sN^{-2/3})^k}, \frac{1}{N^{2k/3}} \sum_{i=1}^{N} \frac{1}{(\mu_i - \mu_1 - sN^{-2/3})^k} \right\}_{k=2}^{k_0}
\]

and

\[
\left\{ s, \frac{1}{N^{2/3}} \sum_{i=1}^{N} \frac{g_j^2}{\mu_i - \mu_1 - sN^{-2/3}} - \frac{1}{\gamma_i - 2}, \frac{1}{N^{2/3}} \sum_{i=1}^{N} \frac{1}{\mu_i - \mu_1 - sN^{-2/3}} - \frac{1}{\gamma_i - 2} \right\}
\]

to the corresponding quantities of the \( \text{Airy}_1 \) random point field.

6.3 Overlap with external field

In this short section we prove Theorem 2.10. We continue to consider the regime of parameters \( \beta > 1 \) and \( h^2/\beta = \theta N^{-1/3} \) for a fixed \( \theta > 0 \). Recall also our definition of the event \( \mathcal{F}_{\varepsilon_1} \) in Definition 6.2. We consider now,

\[
\log \langle \exp \left[ \beta^{1/2} t N^{-1/6} v^T \sigma \right] \rangle \quad (6.76)
\]

which is then a ratio of contour integrals, involving the function \( G(z) \) as above and

\[
G_u(z) := G(z) - uN^{-4/3}v^T(M - z)^{-1}v, \quad (6.77)
\]

where

\[
u = 2t\theta^{1/2} + t^2. \quad (6.78)
\]

Note that due to our choice of scaling, the function \( G_u(z) \) differs slightly from the definition in Section 5. We assume that

\[
|u| \leq N^{-\alpha} \quad (6.79)
\]

for some \( \alpha > 0 \). Let

\[
G_u'(\gamma_u) = 0, \quad G'(\gamma) = 0. \quad (6.80)
\]

Then we see that on the event \( \mathcal{F}_{\varepsilon_1} \) that for sufficiently small \( \varepsilon_1 > 0 \), we have

\[
|\gamma_u - \gamma| \leq N^{\varepsilon_1}uN^{-2/3}. \quad (6.81)
\]

As a consequence of Proposition 6.5 we obtain the following.

Lemma 6.14. There is a \( c_1 > 0 \) so that the following holds on \( \mathcal{F}_{\varepsilon_1} \) for sufficiently small \( \varepsilon_1 > 0 \). For all \( |u| \leq N^{-\alpha} \),

\[
\log \langle \exp \left[ \beta^{1/2} t N^{-1/6} v^T \sigma \right] \rangle = \frac{N}{2} (G_u(\gamma_u) - G(\gamma)) - \log G_u'(\gamma_u) + \log G'(\gamma) + O(N^{-c_1}). \quad (6.82)
\]
We now turn to estimating the quantities appearing in the above lemma. First we have by Taylor expansion that on $\mathcal{F}_{\varepsilon_1}$,
\[
\gamma_u - \gamma = \frac{u}{G''(\gamma)} N^{-4/3} v^T (M - \gamma)^{-2} v + N^{\varepsilon_1/10} O(u^2 N^{-2/3}),
\]
(6.83)
where we used the lower bounds $\gamma, \gamma_1 \geq \lambda_1(M) + N^{-2/3 - \varepsilon_1/100}$ which hold on $\mathcal{F}_{\varepsilon_1}$ as well as the delocalization and rigidity estimates. Furthermore we have on $\mathcal{F}_{\varepsilon_1}$,
\[
N(G_1(\gamma_1) - G(\gamma)) = \frac{1}{2} N G''(\gamma)(\gamma_1 - \gamma)^2 - u N^{-1/3} v^T (M - \gamma)^{-1} v - u(\gamma_1 - \gamma) N^{-1/3} v^T (M - \gamma)^{-2} v + N^{\varepsilon_1/5} O\left(N^{1/3} |u|^3\right)
\]
\[
= - \frac{1}{2} \left( N^{1/3} u^2 \right) \frac{(N^{-4/3} v^T (M - \gamma)^{-2} v)^2}{N^{-2/3} G''(\gamma)} - u N^{-1/3} v^T (M - \gamma)^{-1} v + N^{\varepsilon_1} O\left(N^{1/3} |u|^3\right).
\]
(6.84)
Note also that
\[
|G''(\gamma_u) - G''(\gamma)| \leq N^{2/3 + \varepsilon_1/10} |u|,
\]
(6.85)
on $\mathcal{F}_{\varepsilon_1}$.

Hence we see that on $\mathcal{F}_{\varepsilon_1}$
\[
\log \langle \exp \left[ \beta^{1/2} t N^{-1/6} v^T \sigma \right] \rangle = - \frac{1}{4} \left( N^{1/3} u^2 \right) \frac{(N^{-4/3} v^T (M - \gamma)^{-2} v)^2}{N^{-2/3} G''(\gamma)}
\]
\[- \frac{1}{2} u N^{-1/3} v^T (M - \gamma)^{-1} v + N^{\varepsilon_1} O\left(N^{1/3} |u|^3 + |u|\right).
\]
(6.86)
Now, on $\mathcal{F}_{\varepsilon_1}$,
\[
\frac{1}{4} \left( N^{1/3} u^2 \right) \frac{(N^{-4/3} v^T (M - \gamma)^{-2} v)^2}{N^{-2/3} G''(\gamma)} - \frac{1}{2} u N^{-1/3} v^T (M - \gamma)^{-1} v
\]
\[- \frac{1}{2} \left( t^2 N^{2/3} \right) N^{-1} v^T (M - \gamma)^{-1} v - \beta^{1/2} t N^{-1/3} v^T (M - \gamma)^{-1} v + O(N^{\varepsilon_1} N^{1/3} t^2).
\]
(6.87)
We have therefore proven the following lemma.

**Lemma 6.15.** There is a small $c_1 > 0$ so that the following holds on $\mathcal{F}_{\varepsilon_1}$ for sufficiently small $\varepsilon_1 > 0$.

For all $|t| \leq C$, we have
\[
\log \langle \exp \left[ \beta^{1/2} t N^{-1/2} v^T \sigma \right] \rangle = - \frac{1}{2} \left( t^2 N^{1} \right) v^T (M - \gamma)^{-1} v - \theta^{1/2} t N^{1/3} N^{-1} v^T (M - \gamma)^{-1} v + O(N^{-c_1} t^2).
\]
(6.88)

For small $\varepsilon_2 > 0$ we have, on the event $\mathcal{F}_{\varepsilon_1}$ for $\varepsilon_1 < \varepsilon_2$,
\[
\frac{1}{N} v^T (M - \gamma)^{-1} v = \frac{1}{N} v^T (M - \gamma - N^{\varepsilon_2 - 2/3} - 1) v + O\left(N^{\varepsilon_2 + \varepsilon_1 - 1/3}\right)
\]
\[
= m_{sc} (\gamma + N^{\varepsilon_2 - 2/3}) + O\left(N^{\varepsilon_1 + \varepsilon_2 - 1/3}\right)
\]
\[
= - 1 + O(N^{-c}),
\]
(6.89)
for some small $c > 0$. Note that in passing to the second line we used the fact that the isotropic estimates of Theorem 3.3 are assumed to hold on $\mathcal{F}_{\varepsilon_1}$. This together with the previous lemma proves Theorem 2.10.
6.4 Overlap between two replicas

In this section we will prove Theorem 2.12. We continue to consider,

$$\beta H_N(\sigma) = \frac{\beta}{2} \sigma^T M\sigma + \sqrt{\beta} \theta N^{-1/6} \nu^T \sigma.$$  \hfill (6.90)

We fix $\alpha > 0$ and assume that

$$|t| \leq N^{-\alpha}.$$  \hfill (6.91)

According to our representation formulas,

$$\left\langle \exp \left[ \beta N^{-2/3} t \sigma^{(1)} \cdot \sigma^{(2)} \right] \right\rangle = \int_{F} \exp \frac{N}{2} G_1(z, w) \, dz \, dw$$  \hfill (6.92)

where,

$$G_1(z, w) = \beta(z + w) - \frac{1}{N} \sum_{i=1}^{N} \log((z - \lambda_i)(w - \lambda_i) - N^{-4/3} t^2) - \frac{\theta}{N^{4/3}} \sum_{i=1}^{N} v_i^2 \frac{2\lambda_i - z - w - 2t N^{-2/3}}{(\lambda_i - w)(\lambda_i - z) - N^{-4/3} t^2}.$$  \hfill (6.93)

We define $\gamma_1$ to be the largest solution to

$$\partial_z G_1(\gamma_1, \gamma_1) = 0.$$  \hfill (6.94)

Calculating the derivative, we see that this is the solution to the equation

$$\beta + \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i - \gamma_1}{((\lambda_i - \gamma_1)^2 - N^{-4/3} t^2)} = \frac{\theta}{N^{4/3}} \sum_{i=1}^{N} v_i^2 \frac{1}{(\lambda_i - \gamma_1 + N^{-2/3} t)^2}.$$  \hfill (6.95)

The limit of the LHS as $\gamma_1 \to \lambda_1 + N^{-2/3}|t|$ from above is $-\infty$ whereas the limit on the RHS is either $+\infty$ or a positive number. As $\gamma_1 \to 0$, the LHS goes to $\beta$ and the RHS goes to $0$, so $\gamma_1 > \lambda_1 + N^{-2/3}|t|$. For $\gamma_1 > \lambda_1 + N^{-2/3}|t|$, both sides of the equation are monotonic functions of $\gamma_1$. Furthermore, the LHS is less than $\beta$ for such $\gamma_1$ and the RHS is larger than

$$\frac{\theta}{N^{4/3}} \sum_{i=1}^{N} v_i^2 \frac{1}{(\lambda_i - \gamma_1 + N^{-2/3} t)^2} \geq \frac{\theta v_i^2}{(N^{2/3} \lambda_i - \gamma_1 + N^{-2/3} t)^2}.$$  \hfill (6.96)

Hence, on the event $F_{\varepsilon_1}$ we have that, $N^{2/3} (\gamma_1 - (\lambda_1 + N^{-2/3} t)) \geq N^{-\varepsilon_1/100}$. An upper bound can be proven via a similar argument to Lemma 6.4. Hence, we have the following.

**Lemma 6.16.** Assume that $|t| \leq N^{-\alpha}$. For $\varepsilon_1$ sufficiently small depending on $\alpha$, we have on $F_{\varepsilon_1}$ that

$$N^{-\varepsilon_1/100} \leq N^{2/3} (\gamma_1 - \lambda_1) \leq N^{\varepsilon_1/100}.$$  \hfill (6.97)

Using the above lower bound above and the rigidity estimates, we see that on $F_{\varepsilon_1}$,

$$\left| 1 + \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i - \gamma_1}{((\lambda_i - \gamma_1)^2 - N^{-4/3} t^2)} \right| \leq N^{-1/3 + \varepsilon_1/10}.$$  \hfill (6.98)

Therefore, in a similar fashion to the second estimates of Lemma 6.4, we see that on $F_{\varepsilon_1}$,

$$|\gamma - (\gamma_1 - N^{-2/3} t)| \leq N^{-1 + \varepsilon_1}.$$  \hfill (6.99)

The Hessian of $G$ at $\gamma_1$ has diagonal elements equal to,

$$\partial_z^2 G_1(\gamma_1, \gamma_1) = \frac{1}{N} \sum_{i=1}^{N} \frac{(\lambda_i - \gamma_1)^2}{((\lambda_i - \gamma_1)^2 - N^{-4/3} t^2)^2}$$

$$-2 \frac{\theta}{N^{4/3}} \sum_{i=1}^{N} v_i^2 \frac{(\lambda_i - \gamma_1)}{((\lambda_i - \gamma_1)^2 - N^{-4/3} t^2)(\lambda_i - \gamma_1 + N^{-2/3} t)^2}.$$  \hfill (6.100)
Lemma 6.17. Let $0.01 > \alpha > 0$. Let $G_1(z,w)$ be as above. There is an $\varepsilon_1 > 0$ depending on $\alpha$ so on the event $\mathcal{F}_{\varepsilon_1}$ we have the following. Assume $|t| \leq N^{-\alpha}$ and $z = \gamma_1 + iu$ and $w = \gamma_1 + iv$. Assume $u \geq N^{-2/3-\alpha/100}$. Then the following estimate holds.

$$\text{Re}[G_1(z,w) - G_1(\gamma_1,\gamma_1)] \leq -N^{-2/3-1/10} - c_1\mathbb{1}_{(u > N^{1/100})} \log(u/N^{1/100}).$$

Proof. First when $|v| \leq N^{-2/3-\alpha/100}$ and $u = N^{-2/3-\alpha/100}$ we see that for $\varepsilon_1 > 0$ sufficiently small that on $\mathcal{F}_{\varepsilon_1}$ we have by Taylor expansion,

$$\text{Re}[G_1(z,w) - G_1(\gamma_1,\gamma_1)] \leq -N^{2/3-\varepsilon_1}(u^2 + v^2) + \mathcal{O}(N^{4/3+\varepsilon_1}(|u|^3 + |v|^3)) \leq -N^{-2/3-1/10}.$$  

By direct calculation we see that on $\mathcal{F}_{\varepsilon_1}$,

$$\partial_z G_1(z,w) = \partial_z G(z) + N^\varepsilon \mathcal{O} \left( \frac{t^2}{NN^{4/3}} \sum_i \frac{1}{|z - \lambda_i|^2|w - \lambda_i|} \right) + N^\varepsilon \mathcal{O} \left( \frac{t}{N^{6/3}} \sum_i \frac{1}{|\lambda_i - z|^2|\lambda_i - w|} \right).$$

For $u \geq N^{-2/3-\alpha/2}$ we have $|t N^{-2/3}/(\lambda_i - z)| \leq N^{-\alpha/2}$. So with $\varepsilon_1 > 0$ small enough we see that the error terms are $\mathcal{O}(N^{-\alpha/4})$. On the other hand, if $N^{-2/3-\alpha/100} \leq u \leq N^{2/3+\alpha/100}$,

$$\text{Im}[\partial_z G(z)] \geq \frac{1}{N^{4/3}} \frac{u^2(\gamma_1 - \lambda_1)}{((\lambda_1 - \gamma_1)^2 + u^2)^2} \geq N^{-\alpha/10},$$

on $\mathcal{F}_{\varepsilon_1}$ as long as $\varepsilon_1$ is sufficiently small. Note we used (6.8). Hence, we see that on $\mathcal{F}_{\varepsilon_1}$ we have that $\text{Im}[\partial_z G_1(z,w)] \geq 0$ for $|v| \leq N^{-2/3-\alpha/100}$ and $N^{-2/3-\alpha/100} < u < N^{-2/3+\alpha/100}$ and so we conclude the estimate of the Lemma for this range of $z$, $w$. For $u > N^{-2/3+\alpha/100}$ we see from the fact that the estimates of Theorem 3.3 for $u, w = N^{-1/2}$ are assumed to hold on $\mathcal{F}_{\varepsilon_1}$ as well as the rigidity estimates that,

$$\text{Im}[\partial_z G(z)] = N^{-1/3} \text{Im}[m_{sc}'(z)] + \mathcal{O}(N^{-\alpha/1000}(N^{2/3}u)^{-1})$$

on $\mathcal{F}_{\varepsilon_1}$ for $\varepsilon_1 > 0$ sufficiently small. A straightforward calculation gives, for $u \leq 1/10$,

$$\text{Im}[m_{sc}'(z)] \geq \frac{c}{u^{1/2}}.$$
as long as $|\gamma_1 - 2| \leq N^{2/3+\alpha/10000}$. We see also that the errors in (6.107) are $O(N^{-\alpha/4}(N^{2/3}u)^{-1})$. Hence, we see that for $N^{2/3+\alpha/10} \leq u \leq 1/10$ that $\text{Im} [\hat{e}_u G_1(z, w)] \geq 0$ for this range of $u$ as well. For $u > 1/10$, we see that the errors in (6.107) are $O(N^{-1/10}|u|^2)$ and that $\text{Im} [\hat{e}_u G(z)] \geq c/u$. So we have proven the required estimates as long as $|v| \leq N^{-2/3-\alpha/100}$.

Reversing the roles of $z$ and $w$ above we see that on $\mathcal{F}_{\varepsilon_1}$,

$$\text{Im} [\hat{e}_u G_1(z, w)] \geq 0$$

(6.111)

for any $u \geq N^{-2/3-\alpha/100}$ and $v \geq N^{-2/3-\alpha/100}$. This completes the proof.

The following is an easy consequence of the previous lemma and the method of steepest descent.

**Proposition 6.18.** There is a $c_1 > 0$ so that the following holds. Let $\alpha > 0$. Then there is an $\varepsilon_1 > 0$ depending on $\alpha$ so that on $\mathcal{F}_{\varepsilon_1}$,

$$\log (\exp[\beta N^{-2/3} t \sigma^{(1)} \cdot \sigma^{(2)}]) = \frac{N}{2} (G_1(\gamma_1, \gamma_1) - 2G(\gamma))$$

$$+ \log \det \nabla^2 G_1(\gamma_1, \gamma_1) - 2 \log G''(\gamma) + O(N^{-c_1}).$$

(6.112)

We now prove the following.

**Proposition 6.19.** There is a $c_1 > 0$ so that the following holds. Let $\alpha > 0$. Then on the event $\mathcal{F}_{\varepsilon_1}$ for $\varepsilon_1 > 0$ sufficiently small depending on $\alpha > 0$ we have,

$$\log (\exp[\beta N^{-2/3} t \sigma^{(1)} \cdot \sigma^{(2)}]) = \frac{N}{2} \left(2\beta t N^{-2/3} - \frac{1}{N} \sum_{i=1}^{N} \log(1 + 2t N^{-2/3}(\gamma - \lambda_i)^{-1})\right)$$

$$+ \log \left(1 - \frac{1}{\theta N^{-1/3} m''(\gamma)} \left[\theta N^{-1/3} m''(\gamma) - \frac{2\theta}{N^{1/3}} \sum_i v_i^2 \left(\frac{1}{\lambda_i - \gamma - 2t N^{-2/3}}(\lambda_i - \gamma)^2\right)\right] + O(N^{-c_1})\right)$$

(6.113)

**Proof.** Now from (6.99) we have from a second order Taylor expansion that on $\mathcal{F}_{\varepsilon_1}$ for $\varepsilon_1 > 0$ sufficiently small,

$$G_1(\gamma_1, \gamma_1) = G_1(\gamma + N^{-2/3} t, \gamma + N^{-2/3} t) + O\left(N^{-7/6}\right).$$

(6.114)

We have the equality,

$$G_1(\gamma + t N^{-2/3}, \gamma + t N^{-2/3}) - 2G(\gamma) = 2\beta t N^{-2/3} - \frac{1}{N} \sum_{i=1}^{N} \log(1 + 2t N^{-2/3}(\gamma - \lambda_i)^{-1})$$

(6.115)

We begin by calculating the determinant of the Hessian. We have,

$$\det \nabla^2 G_1(\gamma + N^{-2/3} t, \gamma + N^{-2/3} t) = \left\{\frac{1}{N} \sum_{i=1}^{N} (\lambda_i - \gamma)^2 - 2t N^{-2/3} (\lambda_i - \gamma)^{-1}\right\}^{-2}$$

$$+ \left\{\frac{1}{N} \sum_{i=1}^{N} \frac{(\lambda_i - \gamma)^2 + 2 N^{-4/3} t^2 - 2t N^{-2/3} (\lambda_i - \gamma)^2}{((\lambda_i - \gamma)^2 - 2t N^{-2/3} (\lambda_i - \gamma)^2)} - \frac{2\theta}{N^{1/3}} \sum_i v_i^2 \left(\frac{1}{\lambda_i - \gamma - 2t N^{-2/3} t}(\lambda_i - \gamma)^2\right)\right\}.$$

(6.116)

For the first factor, we have on $\mathcal{F}_{\varepsilon_1}$,

$$\frac{1}{N} \sum_{i=1}^{N} \frac{(\lambda_i - \gamma)^2}{(\lambda_i - \gamma)^2 - 2t N^{-2/3} t(\lambda_i - \gamma)} - \frac{2\theta}{N^{1/3}} \sum_i v_i^2 \left(\frac{1}{\lambda_i - \gamma - 2t N^{-2/3} t}(\lambda_i - \gamma)^2\right) = G''(\gamma)(1 + O(N^{-1/3 + \varepsilon_1/10})).$$

(6.117)

Recall our notation,

$$m_v(z) := \frac{1}{N} v^T \frac{1}{M - z} v.$$

(6.118)
so that
\[ G''(\gamma) = -\theta N^{-1/3}m''_\nu(\gamma) + O(N^{1/3+\varepsilon_1/10}). \] (6.119)

Then,
\[
\frac{1}{N} \sum_i \frac{(\lambda_i - \gamma)^2 + 2N^{-4/3}t^2 - 2tN^{-2/3}(\lambda_i - \gamma)^2}{((\lambda_i - \gamma)^2 - 2tN^{-2/3}(\lambda_i - \gamma))^2} - \frac{2\theta}{N^{4/3}} \sum_i v_i^2 \left( \frac{1}{(\lambda_i - \gamma - 2tN^{-2/3})(\lambda_i - \gamma)^2} \right) = G''(\gamma) + \left[ \theta N^{-1/3}m''_\nu(\gamma) - \frac{2\theta}{N^{4/3}} \sum_i v_i^2 \left( \frac{1}{(\lambda_i - \gamma - 2tN^{-2/3})(\lambda_i - \gamma)^2} \right) \right] + O(N^{1/3+\varepsilon_1/10}). \] (6.120)

This proves the claim. □

From Proposition 6.19 we see that there is a small \( c_1 > 0 \) so that for every \( \alpha > 0 \) and \( k > 0 \) we have for \( \varepsilon_1 > 0 \) sufficiently small depending on \( \alpha, k \) that on \( \mathcal{F}_{\varepsilon_1} \),
\[
\log\langle \exp \left[ \beta N^{-2/3}t o(1) \cdot \sigma^{(2)} \right] \rangle \\
= t \left( \beta N^{-1/3} + \frac{1}{N^{2/3}} \sum_{i=1}^N \frac{2}{\lambda_i - \gamma} - \frac{2\theta}{3} N^{-5/3}m''_\nu(\gamma) \right) \\
+ \sum_{j=2}^k t^j (X_j + Y_j) + O \left( \mu_1^{k+1} N^{2k\varepsilon_1} + N^{-c_1} \right),
\] (6.121)
where,
\[
X_j := \frac{1}{N^{2j/3}} \frac{2^j}{j!} Nm^{(j-1)}(\gamma) \\
Y_j := \frac{d}{ds} \log \left( 1 - \frac{1}{\theta N^{-1/3}m''_\nu(\gamma)} \left[ \theta N^{-1/3}m''_\nu(\gamma) - \frac{2\theta}{N^{4/3}} \sum_i v_i^2 \left( \frac{1}{(\lambda_i - \gamma - 2tN^{-2/3})(\lambda_i - \gamma)^2} \right) \right] \right) \bigg|_{s=0}.
\] (6.122)

Now, \( Y_j \) is a sum of terms which each are products of the form,
\[
\frac{N^{-1/3}m^{(j)}_\nu(\gamma)}{N^{-1}m'_\nu(\gamma)}.
\] (6.123)

So for any \( \varepsilon > 0 \) there is a \( \varepsilon_2 > 0 \) so that each term \( X_j \) and \( Y_j \) is \( O(N^{\varepsilon}) \), with probability at least \( 1 - N^{-\varepsilon_2} \). Note that
\[
N^{-\varepsilon} \leq N^{-1}m'_\nu(\gamma) \leq N^{\varepsilon}
\] (6.124)
with the same probability with probability at least \( 1 - N^{\varepsilon/100} \). We recall the definition of \( H \), the GOE matrix associated to \( M \) via \( H = M + V \), and \( V \) is a diagonal matrix of iid centered normal random variables with variance \( 2/N \), as well as the definition of \( \hat{\gamma}_H \) in (6.3). From Lemma 6.4 and the estimates in its proof, we have the following.

**Lemma 6.20.** There is a small \( c_1 > 0 \) so that the following holds on \( \mathcal{F}_{\varepsilon_1} \) for all sufficiently small \( \varepsilon_1 > 0 \). First,
\[
|\gamma - \hat{\gamma}_H| \leq N^{-2/3-c_1}.
\] (6.125)

Second, for any \( k \) and \( \varepsilon > 0 \) we have, for any \( N^{-2/3-\varepsilon} \leq E - \lambda_1(M) \leq N^{-2/3+\varepsilon} \),
\[
\frac{1}{N^{2k/3}} \left( \left| \operatorname{tr} \frac{1}{(M - E)^k} - \operatorname{tr} \frac{1}{(H - E)^k} \right| + \left| v^T (M - E)^{-k} v - v^T (H - E)^{-k} v \right| \right) \leq N^{-c_1+2k\varepsilon}.
\] (6.126)

With this lemma we see that we can replace all the quantities in the coefficients in our Taylor series (6.121) by the corresponding quantities involving \( H \) and \( \hat{\gamma}_H \), at a cost of \( O(N^{-c_1/2}) \) on the event \( \mathcal{F}_{\varepsilon_1} \) where now \( \varepsilon_1 > 0 \) must be taken small enough depending on the order of the Taylor expansion. Note that the quantities in the denominator of the \( Y_j \)'s are only products of \( N^{-1}m'_\nu(\gamma) \) for which we have the lower bound (6.124), so we do not have any small denominator difficulties.

From this observation and Theorem 6.13 we conclude Theorem 2.12.
7 \( h = \mathcal{O}(N^{-1/2}) \text{ and } \beta > 1 \)

We now consider the regime of low temperature and very small magnetic field. We assume that,

\[
C \geq \beta \geq 1 + c, \quad h^2 \beta = N^{-1/2} \theta,
\]

for a fixed \( \theta \in \mathbb{R} \). In this section we prove the results of Section 2.4.

7.1 Free energy

In this section we will examine the free energy, and prove Theorem 2.13. We will also develop estimates which will be used to study the overlap in the next subsection.

\[
G(z) = \beta z - \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i) - \frac{\theta}{N^2} v^T \frac{1}{M - z} v.
\]

Eventually we are going to expand \( G(z) \) around a saddle \( \gamma \), and apply the method of steepest descent. The saddle will turn out to be distance \( \mathcal{O}(N^{-1}) \) from \( \lambda_1 \).

For our analysis we will again use the event \( \mathcal{F}_{\varepsilon_1} \) that was defined in Definition 6.2. Fix a small \( \kappa > 0 \) satisfying at least \( \kappa < \frac{1}{100} \). The portion of the steepest descent contour that will contribute to the integral lies in the region

\[
\mathcal{D}_\kappa := \{ z : N^{-1+\kappa} + \lambda_1 > \text{Re}[z] > \lambda_1 - N^{-1+\kappa}, \text{Im}[z] \leq N^{-1+2\kappa} \}.
\]

The portion not lying in \( \mathcal{D}_\kappa \) will have an exponentially small contribution. We first analyze the behavior of \( G(z) \) inside \( \mathcal{D}_\kappa \). For \( z \in \mathcal{D}_\kappa \) we have the following expansion that holds on \( \mathcal{F}_{\varepsilon_1} \), using the level repulsion, rigidity and delocalization estimates:

\[
G'_N(z) = (\beta - 1) + \frac{1}{N(\lambda_1 - z)} - \frac{\theta v_1^2}{N^2(\lambda_1 - z)^2} + \mathcal{O}(N^{-1/3} + \varepsilon_1/10). \tag{7.4}
\]

This prompts us to define \( \gamma = \beta_1 + \varepsilon_1/N \), leading to a quadratic equation for \( \varepsilon_1/2 \): \( \beta - 1 = \frac{1}{\varepsilon_1/2} + \frac{\theta v_1^2}{\varepsilon_1/2} \), which has a positive solution,

\[
\varepsilon_1 = 1 + \sqrt{1 + 4(\beta - 1)\theta v_1^2}.
\]

Note the estimate

\[
\varepsilon_1 = 1 + |v_1|.
\]

Note that our notation has changed slightly compared to previous sections: \( \gamma \) is not defined as a solution to the equation \( G'(x) = 0 \). Instead \( \gamma \) solves an approximation to the saddle point equation which, while random, is more tractable than the full equation. Note that \( \varepsilon_1/2 \) is random through its dependence on the quantity \( v_1^2 \). We use the same notation \( \varepsilon_1/2 \) as in the paper [14], where \( \theta = 0 \) (so that in that paper \( \varepsilon_1/2 \) was deterministic).

We also introduce the following notation

\[
\hat{m}(z) := \frac{1}{N} \sum_{i=2}^{N} \frac{1}{\lambda_i(M) - z}, \quad \hat{m}_v(z) := \frac{1}{N} \sum_{i=2}^{N} \frac{v_i^2}{\lambda_i(M) - z}, \tag{7.8}
\]

i.e., we separate the contribution of the largest eigenvalue from \( m \) and \( m_v \).

For \( |z| \leq N^{-1+\kappa} \) the following expansion holds on \( \mathcal{F}_{\varepsilon_1} \) for sufficiently small \( \varepsilon_1 > 0 \),

\[
G_N(z + \gamma) - G(\gamma) = (\beta + \hat{m}_N(\gamma))z - \frac{1}{N} \log(1 + Nz/c_\beta) + \frac{z^2}{2} \hat{m}'_N(\gamma) + \frac{v_1^2\theta}{Nc_\beta} \left( \frac{1}{1 + Nz/c_\beta} - 1 \right) + \frac{z\theta}{N} \hat{m}'_v(\gamma) + \mathcal{O}(N^{-2+3\kappa+\varepsilon_1/10}). \tag{7.9}
\]
Making the change of variable $u = Nz/c_\beta$ and multiplying the above function by $N$ we see that we should examine the function

$$f(u) := c_\beta(\beta - 1)u - \log(1 + u) + \frac{\theta \nu^2}{c_\beta} \left( \frac{1}{1 + u} - 1 \right). \tag{7.10}$$

Define

$$B + 1 = c_\beta(\beta - 1) = \frac{\theta \nu^2}{c_\beta} + 1 \tag{7.11}$$

so that, (note $B > 0$)

$$f(u) = (B + 1)u - \log(1 + u) + B \left( \frac{1}{1 + u} - 1 \right). \tag{7.12}$$

We denote $u = E + i \eta$ and look for solutions to

$$\eta \left( B + 1 - \frac{B}{(1 + E)^2 + \eta^2} \right) = \arg ((1 + E) + i \eta) \tag{7.13}$$

for $E < 0$. Note that this is the steepest descent contour for $f(z)$, i.e. $\text{Im}[f(z)] = 0$. We will consider $\eta$ as a function of $E$ and look to find estimates on $\eta(E)$. We first want to check that $\eta(E)$ is in fact well-defined. Accordingly, we first prove the following lemma which shows that $\eta(E)$ is well-defined and derives some basic properties.

**Lemma 7.1.** For $E < 0$ we define the function $\eta(E)$ as follows. First, $\eta(0) = 0$ and,

$$\eta(-1) = \frac{\pi}{2} + \sqrt{\frac{x^2}{4} + 4B(B + 1)} \over 2(B + 1). \tag{7.14}$$

For $-1 < E < 0$ there is a unique solution to the equation

$$\eta \left( B + 1 - \frac{B}{(1 + E)^2 + \eta^2} \right) = \arg ((1 + E) + i \eta) \tag{7.15}$$

satisfying $0 < \eta < \eta(-1)$. We define $\eta(E)$ to be this solution. Then $\eta(E)$ is strictly decreasing on $-1 < E < 0$ and is smooth. For $E < -1$ there is a unique solution of (7.15) on the domain $\eta > \eta_*(E)$ where

$$\eta_*(E) := \inf \{ \eta > 0 : (1 + E)^2 + \eta^2 > B(1 + B)^{-1} \} \tag{7.16}$$

The function $\eta(E)$ is smooth for $E < -1$.

**Proof.** At $E = -1$ we obtain a quadratic equation for $\eta$ (the RHS simplifies to $\frac{\pi}{2}$ for $\eta > 0$) and see that the only positive solution to (7.13) is given by

$$\eta(-1) = \frac{\frac{\pi}{2} + \sqrt{\frac{x^2}{4} + 4B(B + 1)}}{2(B + 1)}. \tag{7.17}$$

At $E = 0$ we see that $\eta(0) = 0$ is a solution to (7.13). Now,

$$\frac{d}{d\eta} \left( \eta(B + 1) - \eta B \frac{1 + \eta^2}{1 + \eta^2} - \arg(1 + i \eta) \right) = \frac{(B + 1)\eta^2}{1 + \eta^2} + \frac{2\eta^2 B}{(1 + \eta^2)^2} > 0, \tag{7.18}$$

and so this is the unique solution to (7.13). To simplify discussion, define

$$F_E(\eta) := \eta \left( B + 1 - \frac{B}{(1 + E)^2 + \eta^2} \right) - \arg ((1 + E) + i \eta) \tag{7.19}$$

with the convention that $F_E(0) = -\frac{\pi}{2}$. Note also $F_{-1}(\eta) > 0$. We see that on the vertical half-line in the complex plane given by

$$\{ z = -1 + i \eta : \eta > 0 \} \tag{7.20}$$
that $F_{-1}(\eta) < 0$ for $\eta < \eta(-1)$ and $F_{-1}(\eta) > 0$ for $\eta > \eta(-1)$. Moreover, $F_0(\eta) > 0$ for $\eta > 0$. Consider now the horizontal line segments $-1 < E < 0$ at fixed height $0 < \eta < \eta(-1)$. We have,

$$\frac{d}{dE} F_E(\eta) = \frac{2B\eta(1+E)}{(1+E)^2 + \eta^2} + \frac{\eta}{(1+E)^2 + \eta^2} > 0. \quad (7.21)$$

So for fixed $0 < \eta < \eta(-1)$ we have that the function $E \mapsto F_E(\eta)$ is strictly increasing, and satisfies $F_{-1}(\eta) < 0$ and $F_0(\eta) > 0$, so for each $\eta$ in between $0 < \eta < \eta(0)$ there is some $-1 < E < 0$ s.t. $F_E(\eta) = 0$. We will see later that this function $\eta \mapsto E$ is invertible and so we can define $E \mapsto \eta(E)$ as its inverse.

We now consider $E < -1$. Define first $\eta_*(E)$ by,

$$\eta_*(E) := \inf \{\eta > 0 : (1+E)^2 + \eta^2 > B(1+B)^{-1}\} \quad (7.22)$$

Note that $F_E(\eta_*(E)) < 0$ whether or not $\eta_*(E)$ is 0 or strictly positive. Next,

$$F'_E(\eta) = (B+1) - \frac{B}{(1+E)^2 + \eta^2} + \frac{2\eta B}{((1+E)^2 + \eta^2)^2} - \frac{1+E}{(1+E)^2 + \eta^2} \quad (7.23)$$

For $\eta > \eta_*(E)$, the sum of the first two terms is a positive quantity and so $F'_E(\eta) > 0$ for $\eta > \eta_*(E)$. Moreover, $F_E(\infty) = \infty$ so we find a unique solution in the domain $\eta > \eta_*(E)$ for each $E$. Here, we see also by the implicit function theorem that $E \mapsto \eta(E)$ is a smooth function.

We return now to the region $-1 < E < 0$, and consider

$$f'(u) = B + 1 - \frac{1}{1+u} - \frac{B}{(1+u)^2}. \quad (7.24)$$

We claim that if $u$ lies on the contour that we have described, then $\text{Re}[f'(u)] > 0$. In the region $-1 < E < 0$ this shows a strictly monotonic relation between the $E$ and $\eta(E)$ constructed above, fulfilling our above promise to show that the relation $\eta \mapsto E$ is invertible. Moreover, this shows that $E \mapsto \eta(E)$ is smooth. Let $1+u = x+iy$. We have,

$$\text{Re}[f'(u)] = B + 1 - \frac{x}{x^2 + y^2} - \frac{B}{x^2 + y^2} + 2B\frac{y^2}{(x^2 + y^2)^2} \quad (7.25)$$

Now, on the contour we have that $B + 1 - B/(x^2 + y^2) = y^{-1} \arctan(y/x)$. Hence, on the contour we have

$$\text{Re}[f'(u)] = y^{-1} \arctan(y/x) - \frac{x}{x^2 + y^2} + 2B\frac{y^2}{(x^2 + y^2)^2}$$

$$= \frac{1}{y} \left( \arctan(y/x) - \frac{y/x}{1 + (y/x)^2} \right) + 2B\frac{y^2}{(x^2 + y^2)^2}. \quad (7.26)$$

We claim that for $w > 0$, that the function $w \mapsto \arctan(w) - w/(1 + w^2)$ is strictly positive. At $w = 0$ the function is 0. Its derivative is $2w^2(1 + w^2)^{-2} > 0$, which proves this claim. We therefore conclude that $\text{Re}[f'(u)] > 0$ on the contour.

We require some estimates on the steepest descent contour which are summarized in the following.

**Lemma 7.2.** There is a $c > 0$ so that if $-c < E < 0$ we have,

$$\eta(E) = \sqrt{-E + \frac{2B}{1 + 3B} (1 + O(|E|))} \quad (7.27)$$

For any $c > 0$ there is a $c' > 0$ so that $\eta(E) \geq c'$ if $-1 < E < -c$. There is a $C > 0$ so that $\eta(E) \leq C$ for all $E < 0$. Finally for $E < -1$ we have,

$$\eta(E) \geq \frac{\pi}{2(B+1)}. \quad (7.28)$$

Note that the constants above do not depend on $B$. 

45
Proof. We first examine the behavior of the contour near \( E = 0 \). For \( |u| < \frac{1}{2} \), we can write
\[
f(u) = \frac{1}{2} u^2 (1 + 2B) - \frac{\eta^3}{6} (2 + 6B) + f_1(u)
\] (7.29)
where \( f_1(u) \) is a function satisfying \( |\text{Im}[f_1(u)]| \leq C(1 + B)|E|^3 + |E|\eta^2 + \eta^4 \). Hence, we find the equation,
\[
\eta^2 (1 + 3B) = -E(1 + 2B) + (1 + 3B)E^2 + (1 + B)\mathcal{O}(|E|) + \eta^4.
\] (7.30)
Hence,
\[
\eta(E) = \sqrt{-E\frac{1 + 2B}{1 + 3B} (1 + \mathcal{O}(|E|))}
\] (7.31)
for \( |E| \leq c \), some \( c > 0 \).

Due to the monotonicity of \( E \rightarrow \eta(E) \) between \(-1 < E < 0\) we see from the above that for any \( c > 0 \), there is a \( c' > 0 \) such that \( \eta(E) > c' \) if \(-1 < E < -c\). Note also that in this range, \( \eta(E) \leq \eta(-1) \leq C \). Now for \( E < -1 \) we have,
\[
\eta \geq \frac{\pi}{2(B+1)}.
\] (7.32)
For an upper bound, we first have
\[
\pi \geq \eta(1 + B - \frac{B}{\eta^2})
\] (7.33)
If \( \eta^2 > 10 \), then we see that \( \eta \leq C\pi/(1 + B) \). So, we have \( \eta(E) \leq C \) for \( E < -1 \). Hence, we have the lemma.

In order to control the function \( G \) on the steepest descent contour we establish the following estimates on \( \text{Re}[f'] \).

**Lemma 7.3.** Let \( c > 0 \). Then there is a \( c' > 0 \) and \( C' > 0 \) so that for \(-1 < E < -c \) we have,
\[
\text{Re}[f'(E + i\eta)] > c'(1 + B)
\] (7.34)
and
\[
\text{Re}[f'(E + i\eta)] > c'
\] (7.35)
for \( E < -1 \). We have \( |\text{Im}[f'(E + i\eta)]| \leq C \) as long as \( E < -c \).

**Proof.** Letting \( 1 + u = x + iy \) we have as above
\[
\text{Re}[f'(u)] = \frac{1}{y} \left( \arctan(y/x) - \frac{y/x}{1 + (y/x)^2} \right) + \frac{2B y^2}{(x^2 + y^2)^2}
\] (7.36)
Now suppose that \(-1 < E < -c\). Then as above \( \eta(E) > c' \) for some \( c' > 0 \). This implies that \( y/x > c' \). Therefore, we have
\[
\left( \arctan(y/x) - \frac{y/x}{1 + (y/x)^2} \right) \geq c''
\] (7.37)
for some \( c'' > 0 \). Therefore, \( \text{Re}[f'(u)] \geq c''(1 + B) \) for some \( c'' \). For \( E < -1 \) we have,
\[
\text{Re}[f'(u)] > \frac{\pi}{2y} > c_1
\] (7.38)
for some \( c_1 > 0 \). Now,
\[
\text{Im}[f'(u)] = \frac{y}{x^2 + y^2} + \frac{B x y}{(x^2 + y^2)^2}
\] (7.39)
and we see \( |\text{Im}[f'(u)]| \leq C(B + 1) \) for \( E < -c \). Hence we conclude the proof of the lemma.

Now we define the contour \( \Gamma_\kappa \) by,
\[
\Gamma_\kappa := \{ z = (E + i\eta(E))c_3 N^{-1} : 0 \leq -E \leq N^\kappa \}.
\] (7.40)
Finally, with our estimates on \( f \) established, we can obtain the following estimates on \( G \) along the steepest descent contour.
Lemma 7.4. Let $\kappa > 0$. For sufficiently small $\varepsilon_1 > 0$ the following holds on $F_{\varepsilon_1}$ for $N$ large enough. For any $c > 0$ there is a $c' > 0$ we have for $\eta \in \Gamma_{\kappa}$,

$$N\text{Re}[G(z + \gamma) - G(\gamma)] \leq -Nc_\beta^{-1}c'(\text{Re}[z] - cc_\beta N^{-1})1_{\{\text{Re}[z] \leq -cc_\beta N^{-1}\}} + N^{-1/3 + \varepsilon_1}. \quad (7.41)$$

Moreover, there are $C > 0$ and $c'' > 0$ so that on the line $z = -N^\kappa + i\eta$ we have the estimate,

$$N\text{Re}[G(z + \gamma) - G(\gamma)] \leq -c''N^\kappa c_\beta^{-1} - 1_{\{\eta > C + 1\}}Nc'' \log(\eta). \quad (7.42)$$

Proof. We see that for $|u| \leq N^\kappa$,

$$G(uc_\beta/N + \gamma) - G(\gamma) = \frac{1}{N}f(u) + \mathcal{O}(N^{-1/3 - 1/\kappa + \varepsilon_1/10}), \quad (7.43)$$

by Taylor expansion on $F_{\varepsilon_1}$. Recall the equations,

$$\frac{d\eta}{dE} = -\frac{\text{Im}[f']}{\text{Re}[f']}, \quad \frac{d}{dE}\text{Re}[f(E, \eta(E))] = \text{Re}[f'] + \frac{\text{Im}[f']^2}{\text{Re}[f']}. \quad (7.44)$$

Hence, by integrating the estimates of Lemma 7.3 we find that for $-N^\kappa < E < 0$ and any $c > 0$ there is a $c' > 0$ such that on the event $F_{\varepsilon_1}$,

$$N\text{Re}[G((E + i\eta(E))c_\beta/N + \gamma) - G(\gamma)] \leq -c'(|E| - c)1_{\{E \leq -c\}} + N^{-1/3 + \kappa + \varepsilon}. \quad (7.45)$$

This proves the first part of the lemma. We now control $G$ on the vertical line $E = -N^\kappa$ and $\eta \geq \eta(-N^\kappa)$. We calculate,

$$\frac{d}{d\eta}\text{Re}[f(E + i\eta)] = -\text{Im}[f'] = -\frac{\eta}{(1 + E)^2 + \eta^2} \left( 1 + \frac{B(1 + E)}{(1 + E)^2 + \eta^2} \right). \quad (7.46)$$

As long as $|B| \leq N^{\kappa/2}$, which we can guarantee by taking $\varepsilon_1 > 0$ sufficiently small then the fraction in the brackets above is $o(1)$ and so in this case $-\text{Im}[f'] < 0$. Therefore, for $N^\kappa > \eta > \eta(-N^\kappa)$ we have,

$$N\text{Re}[G((-N^\kappa + i\eta)c_\beta/N + \gamma) - G(\gamma)] \leq -cN^\kappa \quad (7.47)$$

for some $c > 0$. Note that we also clearly have $\text{Im}[f'] \leq C$ when $|B| \leq N^{\kappa/2}$ and since $\eta(-N^\kappa) \leq C$, the estimate (7.47) holds also for $0 \leq \eta \leq \eta(-N^\kappa)$.

Now, we calculate

$$\frac{d}{d\eta}\text{Re}[G(E + i\eta + \gamma)] = -\text{Im}[m_N(z)] - \theta N^{-2}\text{Im}[v^T(M - z)^{-2}v] \leq -\frac{1}{N} \sum_{i=1}^{N} \frac{\eta}{|\lambda_i - z|^2} \frac{|v_i|^2 N^{-1}\theta}{|\lambda_i - z|^2}. \quad (7.48)$$

Hence as long as $\eta \geq N^{-1+\kappa}$ for any $\kappa > 0$ this will be negative as long as $|v_i|^2 \leq N^{\kappa/2}$ which holds on $F_{\varepsilon_1}$ by our choice of $\varepsilon_1 > 0$. Hence the estimate (7.47) holds for any $\eta \geq \eta(-N^\kappa)$.

Now for any $|E| \leq 5$ and $\eta \geq 10$ we have,

$$\text{Im}[G'(E + i\eta)] \geq \text{Im}[m_N(E + i\eta)] - \frac{C}{\eta^2} \geq \frac{c}{\eta} - \frac{C}{\eta^2} \quad (7.49)$$

where in the first inequality we used the fact that $||v||_2^2 = N$.

This yields the claim. \hfill \Box

Note that during the proof we established the following.

Lemma 7.5. For any $c > 0$ there is a $c' > 0$ so that for $\eta \in \Gamma_{\kappa}$,

$$\text{Re}[f(zN/cb)] \leq -Nc_\beta^{-1}c'(\text{Re}[z] - cc_\beta N^{-1})1_{\{\text{Re}[z] \leq -cc_\beta N^{-1}\}}. \quad (7.50)$$
We denote by $\Gamma_\kappa$ the contour
\[
\tilde{\Gamma}_\kappa = \Gamma_\kappa \cup \{z = -N^\kappa + i\eta, \eta < \eta(-N^\kappa)\}.
\] (7.51)
We can now calculate the free energy. Using our representation formula we have,
\[
\frac{1}{N} \log Z_{N, \beta, \kappa} = \frac{1}{N} \left( (1 - N/2) \log(\beta) + \frac{N}{2} \log(2\pi) + \frac{1}{2} \log(N) \right) \\
+ \log \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp \left[ \frac{N}{2} G(z) \right] dz,
\] (7.52)
where $\gamma$ is as above. In the following we apply the method of steepest descent to calculate the above contour integral.

**Proposition 7.6.** For $\kappa > 0$ and all sufficiently small $\varepsilon_1 > 0$ it holds on $\mathcal{F}_{\varepsilon_1}$ that,
\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] dz \\
= \int_{\Gamma} \exp \left[ \frac{\beta + \tilde{m}_N(\gamma)}{2} c_\beta u - \frac{1}{2} \log(u) + \frac{v_1^2 \theta}{2c_\beta} \right] \\
\times \exp \left[ -c_\beta (\beta + \tilde{m}_N(\gamma))/2 - \frac{v_1^2 \theta}{2c_\beta} \right] \left( 1 + \frac{c_\beta^2}{4N} (u - 1)^2 \tilde{m}_N'(\gamma) - \theta c_\beta \frac{u - 1}{2N} \tilde{m}_\theta'(\gamma) \right) \frac{d\mu_{\beta}}{N} + \mathcal{O}(N^{-2+4\kappa+\varepsilon_1})
\] (7.53)
where $\Gamma$ is a keyhole contour circling $u = 0$ and continuing above and below the real axis to $-\infty$.

**Proof.** Using Lemma 7.4 we can first replace the contour consisting of a vertical line in the complex plane by $\tilde{\Gamma}_\kappa$ at the cost of an exponential error,
\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] dz = \int_{\Gamma_\kappa} \exp \left[ \frac{N}{2} (G(z + \gamma) - G(\gamma)) \right] dz + \mathcal{O}(e^{-N^\varepsilon}),
\] (7.54)
on the event $\mathcal{F}_{\varepsilon_1}$ for some $c > 0$. Via Taylor expansions we have on $\mathcal{F}_{\varepsilon_1}$,
\[
\int_{\Gamma_\kappa} \exp \left[ \frac{N}{2} (G(z + \gamma) - G(\gamma)) \right] dz \\
= \int_{\Gamma_\kappa} \exp \left[ \frac{N}{2} (\beta + \tilde{m}_N(\gamma) + \frac{Nz^2}{4} \tilde{m}_N'(\gamma) + \frac{v_1^2 \theta}{2c_\beta} \left( \frac{1}{1 + Nz/c_\beta} - 1 \right) - \frac{z\theta}{2} \tilde{m}_\theta'(\gamma) \right] dz \\
+ \mathcal{O}(N^{-2+4\kappa+\varepsilon_1})
\] (7.55)
Now for $E \leq -N^\kappa/N$ we see that
\[
\text{Re} \left\{ \frac{N}{2} (\beta + \tilde{m}_N(\gamma)) z - \frac{1}{2} \log \left( 1 + \frac{Nz}{c_\beta} \right) + \frac{v_1^2 \theta}{2c_\beta} \left( \frac{1}{1 + \frac{Nz}{c_\beta}} - 1 \right) \right\} \leq -cN|E|
\] (7.56)
and so if we define $\Gamma$ as a keyhole contour around $z = -c_\beta/N$, we see we can replace $\tilde{\Gamma}_\kappa$ by $\Gamma$ at an error of $\mathcal{O}(e^{-N^{\kappa/2}})$. Making the change of variable $u = 1 + Nz/c_\beta$ (so that $\Gamma$ is the image of $\Gamma$ under
this change of variable) we see that,
\[
\int_{\gamma-i\infty}^{\gamma+i\infty} \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] \, dz \\
= \int_G \exp \left[ \frac{\beta + \tilde{m}_N(\gamma)}{2} c_\beta u - \frac{1}{2} \log(u) + \frac{v^2 \theta}{2c_\beta} \right] \times \exp \left[ -c_\beta (\beta + \tilde{m}_N(\gamma))/2 - \frac{v^2 \theta}{2c_\beta} \right] \left( 1 + \frac{c^2}{4N} (u - 1)^2 \tilde{m}'_N(\gamma) - \theta c_\beta \frac{u - 1}{2N} \tilde{m}'_P(\gamma) \right) \frac{du c_\beta}{N} + \mathcal{O}(-2 + \epsilon). 
\]
(7.57)
This yields the first claim. The second is obtained in a similar fashion, but by dropping the polynomial terms in (7.55).

We need the following representation for Bessel functions.

**Lemma 7.7.** Let $\tilde{\Gamma}$ be a keyhole contour encircling $z = 0$ and continuing to $-\infty$. Then for $a > 0, b > 0$ and $\alpha \in \mathbb{R}$,
\[
\frac{1}{2\pi i} \int_{\tilde{\Gamma}} \exp \left[ az + bz^{-1} - \alpha \log(z)/2 \right] \, dz \left( \frac{a}{b} \right)^{(1-\alpha)/2} = \frac{\sin(\alpha \pi)}{\pi} \int_0^{\infty} \exp \left[ -\lambda \cosh(u) + (1 - \alpha)u \right] \, du \\
+ \frac{1}{\pi} \int_0^{\pi} \exp \left[ \lambda \cos(\theta) \right] \cos((1 - \alpha)\theta) \, d\theta \\
= I_{\alpha-1}(\lambda)
\]
(7.58)
where $\lambda = 2\sqrt{ab}$ and $I_\beta(x)$ denotes the modified Bessel function of the first kind.

**Proof.** We first make the substitution $z = (b/a)^{1/2} u$ to find,
\[
\frac{1}{2\pi i} \int_{\tilde{\Gamma}} \exp \left[ az + bz^{-1} - \alpha \log(z)/2 \right] \, dz = \frac{1}{2\pi i} \int_G \exp \left[ \lambda (u + u^{-1}) - \frac{1}{2} \log(u) \right] \, du (b/a)^{(1-\alpha)/2}
\]
(7.59)
where $\lambda = 2(ab)^{1/2}$. Now we break up $\tilde{\Gamma}$ into a circle $|z| = 1$ and two horizontal segments at $\pm i0$ and $\text{Re}[z] \leq -1$. The line from $-\infty$ to $-1$ sitting in the lower half-plane at $-i0$ can be parameterized by $u = -e^t, t > 0$. This portion of the integral contributes (dropping the $b/a$ factor)
\[
\frac{1}{2\pi i} \int_0^{\infty} \exp \left[ -\lambda \cosh(t) + (1 - \alpha)t + \alpha i \pi \right] \, dt
\]
(7.60)
The contribution from the other line segment is identical except that the term $\alpha i \pi$ becomes $-\alpha i \pi$, due to the branch cut of the logarithm. For the circle we parameterize it as $z = e^{i\pi}$ for $-\pi \leq t \leq \pi$. The circle contributes,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[ \lambda \cos(\theta) + i(1 - \alpha)\theta \right] \, d\theta = \frac{1}{\pi} \int_0^{\pi} \exp \left[ \lambda \cos(\theta) \cos((1 - \alpha)\theta) \right] \, d\theta.
\]
(7.61)
We find the claim after adding up all of the contributions. The integral representation for the Bessel function is on page 181 of [19] (see also [23]).

Now, one can check that $I_\beta(x)$ satisfy,
\[
x^2 I''_\beta(x) + x I'_\beta(x) - (x^2 + \beta^2) I_\beta(x) = 0.
\]
(7.62)
Making the substitution $\varphi(x) := \sqrt{2} I_{1/2}(x)$ we see that $\varphi''(x) = \varphi(x)$ and so $\varphi(x)$ is a linear combination of $\sinh(x)$ and $\cosh(x)$. Since $I_{1/2}(0)$ is finite, we see that $\varphi(x)$ is a linear multiple of $\sinh(x)$. From the equations,
\[
I'_\beta(\lambda) = \frac{\beta}{\lambda} I_\beta(\lambda) + I_{\beta+1}(\lambda)
\]
\[
I'_\beta(\lambda) = -\frac{\beta}{\lambda} I_\beta(\lambda) + I_{\beta-1}(\lambda),
\]
(7.63)
we see that $\varphi'(x) = \sqrt{x}I_{-1/2}(x)$. Then,

$$\lim_{x \to 0} \sqrt{x}I_{-1/2}(x) = \lim_{x \to 0} \frac{1}{\pi} \int_0^\infty \exp[-x \cosh(t) + t/2] dt$$

$$= \lim_{x \to 0} \frac{1}{\pi} \int_0^\infty \exp[-x \cosh(t)] 2 \sinh(t/2) dt$$

$$= \lim_{x \to 0} \frac{1}{\pi} \int_0^\infty \exp[-2x \cosh(t/2)] 2 \sinh(t/2) dt$$

$$= \lim_{x \to 0} \frac{1}{\pi} \int_0^\infty 4 \exp[-2xt^2] dt = \frac{1}{\sqrt{2\pi}}. \quad (7.64)$$

Therefore,

$$I_{1/2}(x) = \frac{1}{\sqrt{2\pi}} \frac{\sinh(x)}{\sqrt{x}}, \quad I_{-1/2}(x) = \frac{1}{\sqrt{2\pi}} \frac{\cosh(x)}{\sqrt{x}}. \quad (7.65)$$

Note that once we have the formula for one Bessel function, we can find the others an integer value of $\beta$ apart due to the recursions,

$$I_{\beta-1}(x) = \frac{1}{x^\beta} \frac{d}{dx} x^\beta I_{\beta}(x), \quad I_{\beta+1}(x) = x^\beta \frac{d}{dx} x^{-\beta} I_{\beta}(x). \quad (7.66)$$

From the explicit formula for the Bessel function $I_{-1/2}$ we have,

**Lemma 7.8.** There are $c > 0$ and $C > 0$ such that for any $a > 0$ and $b > 0$, with $\lambda = 2\sqrt{ab}$,

$$c \left(1 + \frac{1}{\sqrt{\lambda}}\right) + \frac{e^\lambda}{\lambda^{1/2}} \leq I_{-1/2}(\lambda) \leq Ce^\lambda + \frac{C}{\lambda^{1/2}}. \quad (7.67)$$

We will use all of the above with

$$a := (\beta + \bar{m}_N(\gamma))c_\beta/2, \quad b := v_1^2\theta/(2c_\beta). \quad (7.68)$$

With this notation, the leading order contribution in the second estimate of Proposition 7.6 is,

$$\frac{1}{2\pi i} \int_F \exp[au - \frac{1}{2} \log(u) + b/u - a - b]c_\beta N^{-1} du = b^{1/4}a^{-1/4}I_{-1/2}(2\sqrt{ab}) \exp[-a - b]c_\beta N^{-1} \quad (7.69)$$

In the asymptotics for $I_{-1/2}(2\sqrt{ab})$ we have an exponential term $e^{2\sqrt{ab}}$ which could cause problems. However, it is balanced by the $e^{-a-b}$ term as follows.

$$2\sqrt{ab} - a - b = -(\sqrt{a} - \sqrt{b})^2 = \frac{(a-b)^2}{\sqrt{a} + \sqrt{b}}. \quad (7.70)$$

Now, on the event $\mathcal{F}_{\varepsilon_1}$,

$$2(a - b) = c_\beta(\beta - 1) - v_1^2\theta c_\beta^{-1} + O(N^{-1/3+\varepsilon_1/10}) = 1 + O(N^{-1/3+\varepsilon_1/10}), \quad (7.71)$$

where we used the definition of $c_\beta$ in the second equality. Since $a \geq c$ we therefore conclude that,

$$b^{1/4}a^{-1/4}I_{-1/2}(2\sqrt{ab}) \exp[-a - b] \geq \frac{1}{\sqrt{a}} \quad (7.72)$$

and $N^{-\varepsilon_1/10} \leq a^{-1/2} \leq C$. In summary, we have on $\mathcal{F}_{\varepsilon_1}$ that

$$\frac{1}{N} \log Z_{N,\beta,h} = \frac{1}{2} G(\gamma) + \frac{1}{N} \left( (1 - N/2) \log(\beta) + \frac{N}{2} \log(2\pi) + \frac{1}{2} \log(N) \right)$$

$$+ \frac{1}{N} \log \left( c_\beta N^{-1}(b/a)^{1/4}I_{-1/2}(2\sqrt{ab})e^{-a-b} \right) + O(N^{-1-1/3+\varepsilon_1}). \quad (7.73)$$

50
Furthermore, since on \( \mathcal{F}_{\varepsilon_1} \),
\[
\bar{m}(\gamma) + 1 = \mathcal{O}(N^{-1/3+\varepsilon_1/10}), \tag{7.74}
\]
we find the estimate,
\[
\left| \log \left( (b/\bar{a})^{1/4} I_{-1/2}(2\sqrt{ab})e^{-a-b} \right) - \log \left( (b/\bar{a})^{1/4} I_{-1/2}(2\sqrt{ab})e^{-\bar{a}-b} \right) \right| \leq N^{-1/3+\varepsilon_1} \tag{7.75}
\]
where
\[
\bar{a} := \frac{c_\beta (\beta - 1)}{2}. \tag{7.76}
\]
For \( G(\gamma) \), we have on \( \mathcal{F}_{\varepsilon_1} \)
\[
G(\gamma) = (\beta - 1)(\lambda_1 - 2) + 2\beta - \int \log(2 - x) \rho_{\text{nc}}(x) dx + \mathcal{O}(N^{-1+\varepsilon_1}). \tag{7.77}
\]
These estimates are summarized in the following

**Proposition 7.9.** For sufficiently small \( \varepsilon_1 > 0 \) we have on \( \mathcal{F}_{\varepsilon_1} \) that,
\[
\frac{1}{N} \log Z_{N,\beta,h} = \frac{1}{2} G(\gamma) + \frac{1}{N} \left( (1 - N/2) \log(\beta) + \frac{N}{2} \log(2\pi) + \frac{1}{2} \log(N) \right) + \mathcal{O}(N^{-1/3+\varepsilon_1}), \tag{7.78}
\]
as well as
\[
\frac{1}{N} \log Z_{N,\beta,h} = \frac{\beta - 1}{2} (\lambda_1 - 2) + 2\beta - \int \log(2 - x) \rho_{\text{nc}}(x) dx + \mathcal{O}(N^{-1+\varepsilon_1}). \tag{7.79}
\]

From the above, we conclude Theorem 2.13.

### 7.2 Overlap between two replicas

In this section we will calculate a few moments of the overlap between two replicas \( N^{-1} \sigma^{(1)} \cdot \sigma^{(2)} \). We will build on the estimates established in the previous subsection and prove the remaining results of Section 2.4.

We start with the following representation from Proposition 4.2.
\[
\frac{1}{N \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle} = \frac{\int_{\Gamma^2} \frac{\theta}{N^{\beta\theta} T^{-\frac{1}{2}}} (-w)[M]^{-w} \exp \left[ \frac{N}{2} (G(z) + G(w)) \right] dz dw}{\int_{\Gamma^2} \exp \left[ \frac{N}{2} (G(z) + G(w)) \right] dz dw}, \tag{7.82}
\]
where \( \Gamma \) denotes the vertical line in the complex plane \( \gamma + it \). We will again use the approximate steepest descent contour \( \Gamma_\kappa \) as defined in the previous subsection. It will be convenient to introduce
\[
f(z) := \frac{N}{2} (\beta + \bar{m}(\gamma)) z - \frac{1}{2} \log \left( 1 + \frac{Nz}{c_\beta} \right) + \frac{v^2 \theta}{2c_\beta} \left( \frac{1}{1 + \frac{Nz}{c_\beta}} - 1 \right). \tag{7.83}
\]
For \( |z| \leq N^{-1+\kappa} \) we have that on \( \mathcal{F}_{\varepsilon_1} \),
\[
f(z) = f(Nz c_\beta^{-1}) + \mathcal{O}(N^{-1/3+\kappa+\varepsilon_1/10}), \tag{7.84}
\]
for \( \varepsilon_1 \) sufficiently small. From this observation as well as Lemma 7.5 we have the first estimate of the following lemma. The second estimate easily follows from the fact that the first term in the definition of \( \tilde{f} \) dominates the others for large \( z \).
Lemma 7.10. Let $z \in \Gamma_\kappa$. Then for any $c > 0$ there is a $c' > 0$ such that
\[
\text{Re}[\tilde{f}(z)] \leq -Nc_\beta^{-1}c'(\text{Re}[z] - cc_\beta N^{-1}) \mathbf{1}_{\{\text{Re}[z] \leq -cc_\beta N^{-1}\}} + N^{-1/3+\varepsilon}. \quad (7.85)
\]
Also,
\[
\text{Re}[\tilde{f}(z)] \leq -cN|E| \quad (7.86)
\]
if $E \leq -N^\kappa/N$.

The following proposition contains our first estimates for the quantities in the numerator and denominator.

Proposition 7.11. For all sufficiently small $\varepsilon_1 > 0$, the following hold on $\mathcal{F}_{\varepsilon_1}$; we have
\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \exp \left[ \frac{N}{2} (G(z) - G(\gamma)) \right] dz = \frac{c_\beta}{N} \int_{\Gamma} \exp \left[ au + bu^{-1} - \frac{1}{2} \log(u) - a - b \right] du + \mathcal{O}(N^{-1+3\kappa-2/3+\varepsilon_1}), \quad (7.87)
\]
where
\[
a = \frac{c_\beta}{2} \beta + \tilde{m}(\gamma), \quad b = \frac{v_1^2 \theta}{2c_\beta}. \quad (7.88)
\]
Moreover,
\[
\int_{\Gamma_2} \frac{\theta}{N^2 \beta} v^T \frac{1}{(M - w)(M - z)} v \exp \left[ \frac{N}{2} (G(z) + G(w)) \right] dzdw = \frac{c_\beta^2}{N^2} \int_{\Gamma_2} \exp \left[ a(u + w) + b(u^{-1} + w^{-1}) - \frac{1}{2} (\log(u) + \log(w)) - 2a - 2b \right] \frac{2b}{c_\beta \beta v} dudw + \mathcal{O}(N^{-2+4\kappa+\varepsilon_1-2/3}). \quad (7.89)
\]
where $\Gamma$ is a keyhole contour encircling $u = 0$ and continuing along above and below the negative real axis.

Proof. On the event $\mathcal{F}_{\varepsilon_1}$, we have the estimate for $z \in \Gamma_\kappa$:
\[
\left| \frac{1}{N^2} \sum_i \frac{1}{|\lambda_i - z|^2} \right| \leq N^{\varepsilon_1/10}. \quad (7.90)
\]
This estimate also holds if $\eta \geq N^{-1}$. Hence, from Lemma 7.4 we can replace the contours in the integrals in the numerator and denominator by $\Gamma_\kappa$ with only an error of size $\mathcal{O}(e^{-N^\eta})$ some $c > 0$. For $z \in \Gamma_\kappa$ we have the estimate, which holds on the event $\mathcal{F}_{\varepsilon_1}$,
\[
\frac{N}{2} (G(z + \gamma) - G(\gamma)) = \tilde{f}(z) + \mathcal{O}(N^{2\kappa+\varepsilon_1/10-2/3}). \quad (7.91)
\]
From Lemma 7.10 we then conclude that on $\mathcal{F}_{\varepsilon_1}$,
\[
\int_{\Gamma_\kappa} \exp \left[ \frac{N}{2} (G(z + \gamma) - G(\gamma)) \right] dz = \int_{\Gamma_\kappa} \exp \left[ \tilde{f}(z) \right] dz + \mathcal{O}(N^{-1+3\kappa-2/3+\varepsilon_1}). \quad (7.92)
\]
For the quantity in the denominator we have on the event $\mathcal{F}_{\varepsilon_1}$,
\[
\frac{1}{N^2} v^T \frac{1}{(M - \gamma - z)(M - \gamma - w)} v = \frac{v_1^2}{c_\beta^2 (1 + \frac{Nz}{c_\beta})(1 + \frac{Nw}{c_\beta})} + \mathcal{O}(N^{-2/3+\varepsilon_1/10}) \quad (7.93)
\]
and so,
\[
\int_{\Gamma_2} \frac{1}{N^2} v^T \frac{1}{(M - w)(M - z)} v \exp \left[ \frac{N}{2} (G(z) + G(w) - 2G(\gamma)) \right] dzdw = \int_{\Gamma_2} \exp \left[ \tilde{f}(z) + \tilde{f}(w) \right] \frac{v_1^2}{c_\beta^2 (1 + \frac{Nz}{c_\beta})(1 + \frac{Nw}{c_\beta})} dzdw + \mathcal{O}(N^{-2/3+4\kappa+\varepsilon_1}). \quad (7.94)
\]
Due to the second estimate of Lemma 7.10 we may then turn $\Gamma_\kappa$ into a keyhole contour by adding in the portion above and below negative real axis. The claim then follows from the substitution $Nz/c_\beta + 1 = u$.

Now from (7.72) and Lemma 7.7 we have that,

$$N^{-1-\varepsilon_1/10} \leq \frac{c^2}{N} \int_\Gamma \exp \left[ au + bu^{-1} - \frac{1}{2} \log(u) - a - b \right] du \leq N^{\varepsilon_1/10-1} \quad (7.95)$$

for any sufficiently small $\varepsilon_1 > 0$ on $F_{\varepsilon_1}$. Hence, on the event $F_{\varepsilon_1}$, we have by applying Lemma 7.7 and the previous proposition,

$$\frac{1}{N} \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle = \frac{I_{1/2} (\lambda)^2 (c_\beta \beta)^{-1}}{(b/a)^{1/2} I_{-1/2} (\lambda)^2} + O(N^{-2/3+2\varepsilon_1+10\varepsilon}) \quad (7.96)$$

by taking $\kappa > 0$ and $\varepsilon_1 > 0$ sufficiently small. By direct calculation:

$$\frac{I_{1/2} (\lambda)^2 (c_\beta \beta)^{-1}}{(b/a)^{1/2} I_{-1/2} (\lambda)^2} = \frac{\beta + \tilde{m}(\gamma)}{\beta} \left( \frac{\tanh(\sqrt{v^2 \theta (\beta + \tilde{m}(\gamma))})}{\beta} \right)^2 \quad (7.97)$$

From this and the estimate

$$|\tilde{m}(\gamma) - \tilde{m}(\lambda_1(M))| \leq N^{-2/3+\varepsilon_1/10} \quad (7.98)$$

which holds on $F_{\varepsilon_1}$, we conclude Theorem 2.15. Theorem 2.16 follows from a Taylor expansion developing $\tilde{m}(\lambda_1(M))$ around $-1$ the the calculation,

$$\frac{d}{dm} \frac{\beta + m}{\beta} \left( \frac{\tanh(\sqrt{v^2 \theta (\beta + m)})}{\beta} \right)^2 \bigg|_{m=-1} = \frac{1}{\beta} \tanh^2(\tilde{\lambda}) + \sqrt{\lambda} \frac{\tanh(\tilde{\lambda})}{\cos^2(\tilde{\lambda})} = \frac{\tanh(\lambda)}{\beta} \frac{\sinh(\lambda) \cosh(\lambda)}{\beta \cos^2(\lambda)} \quad (7.99)$$

as well as Lemma C.1, where we defined $\tilde{\lambda} := \sqrt{2v^2 \theta (\beta - 1)}$.

We now turn to calculating the second moment of the overlap between two replicas. From Proposition 4.2 we have

$$\frac{1}{N^2} \langle (\sigma^{(1)} \cdot \sigma^{(2)})^2 \rangle = \int_{\mathbb{R}^2} \left\{ \frac{1}{N^2} \sum_i \frac{1}{(\lambda_i - z)(\lambda_i - w)} - \frac{\theta}{N^3} v^T v \frac{1}{(M - z)^2(M - w)} + \frac{1}{(M - w)^2(M - z)} \right\} dv dw \quad (7.100)$$

$$+ \frac{\theta^2}{N^4} \left[ \frac{1}{v^T (M - z)(M - w)} \right]^2 \exp \left[ \frac{N}{2} (G(z) + G(w) - 2G(\gamma)) \right] \frac{d^2}{d \gamma^2} = AB^{-2} \quad (7.101)$$

For the denominator we use the same analysis as above. Arguing along similar lines to Proposition 7.11 we have the chain of estimates that hold on $F_{\varepsilon_1}$,

$$A = \frac{\beta^2}{\beta} \int_{\mathbb{R}^2} \left\{ \frac{c^2}{c_\beta} (1 + \frac{Nz}{c_\beta})(1 + \frac{NW}{c_\beta}) + \frac{2\theta v^2}{c_\beta} (1 + \frac{Nz}{c_\beta})(1 + \frac{NW}{c_\beta}) \right\} \exp \left[ \frac{N}{2} (G(z) + G(w) - 2G(\gamma)) \right] d\gamma$$

$$= \frac{\beta^2}{N^2} \int_{\mathbb{R}^2} \left\{ \frac{1}{c_\beta u w} + \frac{2\theta v^2}{c_\beta} \frac{1}{u^2 w} + \frac{\theta^2 v^4}{c_\beta^2 u^2 w^2} \right\} \exp \left[ a(u + w) + b(u^{-1} + w^{-1}) - \frac{1}{2} (\log(u) + \log(w)) - 2a - 2b \right] d\gamma$$

$$\quad \times \exp \left[ \int_{\mathbb{R}^2} (a(u + w) + b(u^{-1} + w^{-1}) - \frac{1}{2} (\log(u) + \log(w)) - 2a - 2b) \right] d\gamma = \frac{N^{2-4\kappa - 2/3+\varepsilon_1}}{\beta^2} \quad (7.102)$$
Hence, using Lemma 7.7 we see that on $\mathcal{F}_{\varepsilon_1}$,

$$
\frac{1}{N^2} \langle (\sigma^{(1)}, \sigma^{(2)})^2 \rangle = \frac{1}{c_\beta^2 b^2 I_{-1/2}^2} \left( I_{1/2}(\lambda)^2 + 2\lambda I_{3/2}(\lambda)I_{1/2}(\lambda) + \lambda^2 I_{3/2}^2(\lambda) \right) + \mathcal{O}(N^{-2/3+10\varepsilon+2\varepsilon_1}).
$$

(7.103)

From (7.63), we have $I_{3/2}(\lambda) = I_{-1/2}(\lambda) - I_{1/2}(\lambda)/\lambda$ and so,

$$
I_{1/2}(\lambda)^2 + 2\lambda I_{3/2}(\lambda)I_{1/2}(\lambda) + \lambda^2 I_{3/2}^2 = \lambda^2 I_{-1/2}(\lambda).
$$

(7.104)

Therefore,

$$
\frac{1}{N^2} \langle (\sigma^{(1)}, \sigma^{(2)})^2 \rangle = \frac{1}{c_\beta^2 b^2 I_{-1/2}^2} \lambda^2 I_{-1/2}^2(\lambda) + \mathcal{O}(N^{-2/3+10\varepsilon+2\varepsilon_1})
$$

$$
= \frac{1}{\beta^2} (\beta + \tilde{m}(\gamma))^2 + \mathcal{O}(N^{-2/3+10\varepsilon+2\varepsilon_1}).
$$

(7.105)

We conclude Theorem 2.17 from this and our previous estimate of $N^{-1}\langle \sigma^{(1)}, \sigma^{(2)} \rangle$ from Theorem 2.15.

Finally, we turn to the proof of Theorem 2.14. It will suffice to show that the squared overlap $N^{-2}(\sigma^{(1)}, \sigma^{(2)})^2$ concentrates with respect to the Gibbs measure, with high probability with respect to the disorder variables. For this, we will show that its variance is $o(1)$ with high probability. We must therefore calculate the fourth moment of the overlap. We will prove the following.

**Proposition 7.12.** For all sufficiently small $\varepsilon > 0$ there is an $\varepsilon_2 > 0$ so that we have with probability at least $1 - N^{-\varepsilon_2}$ that,

$$
\frac{1}{N^4} \mathbb{E} \left( (\sigma^{(1)}, \sigma^{(2)})^4 \right) = \left( \frac{\beta + \tilde{m}(\gamma)}{\beta} \right)^4 + \mathcal{O}(N^{-2/3+\varepsilon}).
$$

(7.106)

**Proof.** We apply Proposition 4.3 to find the representation,

$$
\frac{1}{N^4} \mathbb{E} \left( (\sigma^{(1)}, \sigma^{(2)})^4 \right) = DB^{-2}
$$

(7.107)

where,

$$
D := \frac{1}{\beta^4} \int_{\mathbb{T}_2} \frac{dzdw}{\beta^4} \exp \left[ \frac{N}{2} \left( G(z) + G(w) - 2G(\gamma) \right) \right] \left\{ 6N^{-2}F_{1zzw} + 3(F_1)^2 
+ 3\beta^{-2}(F_{2zz}^2 + F_{2w}^2) + F_2^4 + 4N^{-2}F_2F_{2zz} + 6N^{-2}F_{2zz}^2 + 6F_2^2(-N^{-1}F_{2zz} - N^{-1}F_{2w} + F_1)
-6N^{-3}(F_{2zzw} + F_{2wzz}) - 6N^{-1}F_1(F_{2zz} + F_{2w}) \right\}
$$

(7.108)

where,

$$
F_1(z, w) = \frac{1}{N^2} \sum_i \frac{1}{(\lambda_i - z)(\lambda_i - w)}, \quad F_2(z, w) = \frac{\theta}{N^2} \sum_i \frac{v_i^2}{(\lambda_i - z)(\lambda_i - w)}.
$$

(7.109)

In the definition of $D$ above we have suppressed the arguments of $F_1$ and $F_2$ as they are all just the integration variables $(z, w)$. It is no problem to argue as in our calculations of the first and second moments of the overlap to move the contour $\Gamma$ to $\Gamma_{\varepsilon}$, and then expand the function $G(z) - G(\gamma)$ that appears in the exponential around $f(z)$ along $\Gamma_{\varepsilon}$ on the event $\mathcal{F}_{\varepsilon_1}$. Similarly, for $z, w \in \Gamma_{\varepsilon}$ we use the
following estimates which hold on the event $F_{\varepsilon_1}$

\[
F_1(\gamma + z, \gamma + w) = \frac{1}{c_3^2} \left( 1 + \frac{Nz}{c_3} \right) \left( 1 + \frac{Nw}{c_3} \right) + O(N^{-2/3+\varepsilon_1/10})
\]

\[
N^{-2}F_{1zw}(\gamma + z, \gamma + w) = \frac{1}{c_3^2} \left( 1 + \left( \frac{Nz}{c_3} \right)^2 \left( 1 + \frac{Nw}{c_3} \right)^2 \right) + O(N^{-4/3+\varepsilon_1/10})
\]

\[
F_2(\gamma + z, \gamma + w) = \frac{\theta v_1^2}{c_3^2} \left( 1 + \frac{Nz}{c_3} \right) \left( 1 + \frac{Nw}{c_3} \right) + O(N^{-2/3+\varepsilon_1/10})
\]

\[
N^{-1}F_{2z}(\gamma + z, \gamma + w) = -\frac{\theta v_1^2}{c_3^2} \left( 1 + \left( \frac{Nz}{c_3} \right)^2 \left( 1 + \frac{Nw}{c_3} \right)^2 \right) + O(N^{-1+\varepsilon_1/10})
\]

\[
N^{-2}F_{2zw}(\gamma + z, \gamma + w) = \frac{\theta v_1^2}{c_3^2} \left( 1 + \left( \frac{Nz}{c_3} \right)^2 \left( 1 + \frac{Nw}{c_3} \right)^2 \right) + O(N^{-4/3+\varepsilon_1/10})
\]

\[
N^{-3}F_{2zw}(\gamma + z, \gamma + w) = -2 \frac{\theta v_1^2}{c_3^2} \left( 1 + \left( \frac{Nz}{c_3} \right)^3 \left( 1 + \frac{Nw}{c_3} \right)^2 \right) + O(N^{-5/3+\varepsilon_1/10}). \tag{7.110}
\]

Changing the contour from $\Gamma_\kappa$ to the keyhole $\tilde{\Gamma}$ after making the same change of variables $1 + \frac{Nz}{c_3} = u$ incurs the same exponential error as before. Calculating all of the contributions from $F_1$, $F_2$ and their derivatives yields the following, where we drop the error $O(N^{-2/3-2+4\varepsilon_1})$ for brevity:

\[
\frac{1}{(2\pi)^2} D = e^{-2a-2b} \left\{ 9I_{3/2}(\lambda)^2(a/b)^3/2 + (2b)^4 I_{7/2}(\lambda)^2(a/b)^7/2 + (4! + 6 + 6)(2b)^2 I_5\gamma_2(\lambda)^2(a/b)^5/2 \\
+ 12(2b)^3 I_5\gamma_2(\lambda)I_{7/2}(\lambda)(a/b)^3 + 12(2b)(2 + 1)I_{3/2}(\lambda)I_5\gamma_2(\lambda)(a/b)^2 + 6I_{7/2}(\lambda)I_{3/2}(\lambda)(2b)^2(a/b)^5/2 \right\}
\]

\[
= \frac{a^{3/2}e^{-2a-2b}}{\beta^4 N^2 c_3^2 b^{5/2}} \times \left\{ 9I_{3/2}(\lambda)^2 + I_{7/2}(\lambda)^2 3^4 + 36I_5\gamma_2(\lambda)^2 \lambda^2 \\
+ 12I_5\gamma_2(\lambda)I_{7/2}(\lambda)\lambda^3 + 36I_{3/2}(\lambda)I_5\gamma_2(\lambda)\lambda + 6I_{7/2}(\lambda)I_{3/2}(\lambda)\lambda^2 \right\}. \tag{7.111}
\]

Using

\[
I_{3/2}(\lambda) = I_{-1/2}(\lambda) - \frac{I_{1/2}(\lambda)}{\lambda} \\
I_{5/2}(\lambda) = I_{1/2}(\lambda)(1 + \frac{3}{\lambda^2}) - \frac{3}{\lambda} I_{-1/2}(\lambda) \\
I_{7/2}(\lambda) = I_{-1/2}(\lambda)(1 + \frac{15}{\lambda^2}) - I_{1/2}(\lambda)(\frac{6}{\lambda} + \frac{15}{\lambda^2}) \tag{7.112}
\]

we find for the term in braces in the last line of (7.111),

\[
9I_{3/2}(\lambda)^2 + I_{7/2}(\lambda)^2 3^4 + 36I_5\gamma_2(\lambda)^2 \lambda^2 + 12I_5\gamma_2(\lambda)I_{7/2}(\lambda)\lambda^3 + 36I_{3/2}(\lambda)I_5\gamma_2(\lambda)\lambda + 6I_{7/2}(\lambda)I_{3/2}(\lambda)\lambda^2 = \lambda^2 I_{-1/2}(\lambda)^2. \tag{7.113}
\]

Hence, on $F_{\varepsilon_1}$ we have for sufficiently small $\kappa$ and $\varepsilon_1 > 0$,

\[
\frac{1}{N^4} \langle (\sigma^{(1)} \cdot \sigma^{(2)})^4 \rangle = \left( \frac{\beta + \bar{m}(\gamma)}{\beta} \right)^4 + O(N^{-2/3+10\kappa+2\varepsilon_1}). \tag{7.114}
\]

This yields the claim. \qed

**Proof of Theorem 2.14.** Recalling our notation $R_{12} = N^{-1} \sigma^{(1)} \cdot \sigma^{(2)}$ we see that, for all $\varepsilon > 0$ there is an event on which the following holds,

\[
\langle 1_{|R_{12} - (1-\beta^{-1})^2| > \varepsilon} \rangle \leq N^{-2/3+\varepsilon\|} - 2. \tag{7.115}
\]
Let \( N^{-\delta} \geq t \geq N^{-1/3+\delta} \), and let \( p \) be
\[
 p_\pm := \langle 1_{|R_{12}|^2 (1-\beta^{-1})>t} \rangle. \tag{7.116}
\]
Then,
\[
 1 = p_+ + p_- + \langle 1_{|R_{12}^2 -(1-\beta^{-1})|>t(|R_{12}|+(1-\beta^{-1}))} \rangle \tag{7.117}
\]
For the latter term,
\[
 \langle 1_{|R_{12}^2 -(1-\beta^{-1})|>t(|R_{12}|+(1-\beta^{-1}))} \rangle \leq \langle 1_{|R_{12}^2 -(1-\beta^{-1})|>t(1-\beta^{-1})} \rangle \leq CN^{-2/3+\varepsilon}t^2 \tag{7.118}
\]
and so
\[
 1 = p_+ + p_- + \mathcal{O}(N^{-2/3+\varepsilon}t^2). \tag{7.119}
\]
From,
\[
 (1 - \beta^{-1})\tanh^2(\hat{\lambda}) = \langle R_{12} \rangle + \mathcal{O}(N^{-1/3+\varepsilon}) = (1 - \beta^{-1})(p_+ - p_-) + \langle |R_{12}|1_{|R_{12}|-(1-\beta^{-1})|>t} \rangle + \mathcal{O}(t + N^{-1/3+\varepsilon}) \tag{7.120}
\]
and from \(|R_{12}| \leq 1\),
\[
 \langle |R_{12}|1_{|R_{12}|-(1-\beta^{-1})|>t} \rangle \leq \langle 1_{|R_{12}^2 -(1-\beta^{-1})|>t} \rangle \leq CN^{-2/3+\varepsilon}t^{-2}, \tag{7.121}
\]
we conclude
\[
 p_\pm = \frac{1}{2} \pm \tanh(\hat{\lambda}) + \mathcal{O}(N^{-1/3+\varepsilon} + t + N^{-2/3+\varepsilon}t^{-2}). \tag{7.122}
\]
This yields the claim. \( \square \)

**A Proofs of Theorem 3.5 and Lemma 3.6**

We begin by proving Lemma (3.6). We start with the resolvent identity,
\[
 \frac{1}{M-z} = \frac{1}{H-z} = \frac{1}{M-z} \sum_{k=1}^{m} (V(M-z)^{-1})^k + \frac{1}{H-z} (V(M-z))^{-(m+1)}. \tag{A.1}
\]
Denote,
\[
 A_k := \frac{1}{M-z} \left( V \frac{1}{M-z} \right)^k, \quad R(z) := \frac{1}{M-z}. \tag{A.2}
\]
We first prove the following lemma.

**Lemma A.1.** Let \( C > 0 \) be a constant. On the event,
\[
 \max_{i,j} |R_{ij}| \leq C, \tag{A.3}
\]
we have for \( k \geq 2 \) and even \( p \),
\[
 \mathbb{E}_V |v^T A_k v|^p \leq C(k, p) \left( \frac{1}{N^p} + \sup_i |(Rv)_i - m_{\text{sc}}(z)v_i|^{2p} \right) \tag{A.4}
\]
for any unit vector \( v \).

**Proof.** We write,
\[
 v^T A_k v = \sum_{j_1 \ldots j_k} (Rv)_{j_1} R_{j_1,j_2} \ldots R_{j_{k-1},j_k} (Rv)_{j_k}. \tag{A.5}
\]
For even \( p \),
\[
 \mathbb{E}_V |v^T A_k v|^p = \sum_{j_1} (R^\# v)_{j_1} (R^\# v)_{j_2} (R^\# v)_{j_{k-1}} \ldots (R^\# v)_{j_k} M(j) \mathbb{E}_V [V_{j_1} V_{j_2} \ldots V_{j_k}], \tag{A.6}
\]

56
The term on the RHS can be taken to be less than $N$ by taking $\lambda$ of [SSK] by taking $\lambda$. The claim follows, again using Theorem 2.4.1 of [SSK].

Proof of Theorem 3.5. Consider the first term in the product on the RHS. If $a_j = 1$, then it is bounded by 1 because $s_j \geq 2$. If $a_j = 1$, then it is bounded by

$$\frac{1}{N^{s_j/2}} \sum_i |(Rv)_i| \leq \frac{1}{N} \sum_i |(Rv)_i| \leq C \frac{\|v\|_1}{N} + \max_i |(Rv)_i - m_{sc}v_i| \leq C \frac{\|v\|_1}{N^{1/2}} + \max_i |(Rv)_i - m_{sc}v_i|.$$  

Note that $s_j \geq a_j$ (here is where we use that the $k$ in $A_k$ is $k \geq 2$). Then for $a_j \geq 2$, we have

$$\frac{1}{N^{s_j/2}} \sum_i |(Rv)_i|^a \leq \frac{1}{N^{a_j/2}}(m_{sc}v_i + (Rv)_i - m_{sc}v_i)^a \leq C \frac{\|v\|_2^a}{N^{a_j/2}} + C \max_i |(Rv)_i - m_{sc}v_i|^a.$$  

The claim follows after noting that $\sum_j a_j = 2p$. □

Proof of Lemma 3.6. The use the resolvent identity (A.1),

$$v^T \frac{1}{M - z}v - v^T \frac{1}{H - z}v = v^T A_1 v + \sum_{k=2}^m v^T A_k v + v^T \frac{1}{H - z}(V(M - z)^{-1})^{m+1}v.$$  

The term on the RHS can be taken to be less than $N^{-100}$ with overwhelming probability using Lemma A.4 of [SSK] by taking $m$ sufficiently large. The terms involving $A_k$ for $2 \leq k \leq m$ are handled using Lemma A.1 and the estimates of Theorem 3.3. Conditionally on $M$, the term $v^T A_1 v$ is a Gaussian with variance less than

$$\frac{1}{N} \sum_i |(Rv)_i|^4 \leq C \frac{\sup_i |(Rv)_i - m_{sc}v_i|^4}.$$  

The claim follows, again using Theorem 3.3. □

Proof of Theorem 3.5. Equation (3.15) was proven in [SSK], so it remains to prove (3.16). Fix a small $0 < \delta_0 < 0.1$ and small $\varepsilon > 0$, with $\varepsilon < 0.1$. We may assume that $|\lambda_i(M) - \lambda_i(H)| \leq N^{-1+\varepsilon}$ for $i = 1, 2$. Assume that the level repulsion events of Theorem 3.4 hold with $s = N^{-\delta_0}$. Let $\delta_1 > 0$ with $\delta_1 < 1$. Let $\eta_1 = N^{-2/3-\delta_1}$ and $\eta_2 = N^{-2/3-\delta_1}/2$. We denote by $\Gamma$ the contour in $\mathbb{C}$ that is a rectangle with sides parallel to the real and imaginary axes, symmetric across the real axis, centered at the point $\lambda_1(M)$ and horizontal side length $2\eta_2$ and vertical side length $\eta_1$.

Due to our assumptions, we have that only $\lambda_1(M)$ and $\lambda_2(H)$ are inside the contour $\Gamma$, and that $\lambda_2(M)$ and $\lambda_2(H)$ are at distance $N^{-2/3-\delta_1}/2$ at least from $\Gamma$. It follows that

$$(v^T u_1(M))^2 - (v^T u_1(H))^2 = \frac{1}{2\pi i} \int_{\Gamma} v^T (M - z)^{-1}v - v^T (H - z)^{-1}v dz.$$  

(A.14)
Fix a small $\delta_\varepsilon > 0$ with $\delta_\varepsilon < 0.1$. We estimate in the above integral the contribution from $|\text{Im}[z]| \leq N^{\delta_\varepsilon - 1}$. Due to the orientation of the integral, we may estimate the contribution as

$$
\int_{|y| \leq N^{\delta_\varepsilon - 1}} |v^T (A - (\lambda_1(M) + \eta_2 + iy))^{-1} v - v^T (A - (\lambda_1(M) - \eta_2 + iy))^{-1} v| \, dy
$$

(A.15)

for $A = M, H$. We bound first the contribution coming from $\lambda_1(A)$; recalling that $|\lambda_1(H) - \lambda_1(M)| \ll \eta_2$, we have

$$
\int_{|y| \leq N^{\delta_\varepsilon - 1}} \left| \frac{(v^T u_1(A))^2}{|\lambda_1(A) - (\lambda_1(M) + \eta_2 + iy)|} \right| \, dy \leq N^\varepsilon \int_{|y| \leq N^{\delta_\varepsilon - 1}} \frac{C}{\eta_2} \, dy \leq \frac{N^{2\varepsilon} N^{\delta_\varepsilon}}{N^2 \eta_2}.
$$

(A.16)

We denote $\eta_{lr} = N^{-2/3 - \delta_\varepsilon}$. For the contribution from the remaining eigenvalues, we have

$$
\int_{|y| \leq N^{\delta_\varepsilon - 1}} \sum_{i=2}^{N} \left| \frac{1}{\lambda_i(A) - (\lambda_i(M) - \eta_2 + iy)} - \frac{1}{\lambda_i(A) - (\lambda_i(M) + \eta_2 + iy)} \right| \, dy
$$

(A.17)

$$
\leq \frac{N^{2\varepsilon} \eta_2}{N} \int_{|y| \leq N^{\delta_\varepsilon - 1}} \sum_{i=2}^{N} \frac{1}{(\lambda_i(A) - \lambda_i(A))^2 + (\eta_{lr})^2} \, dy
$$

(A.18)

$$
\leq N^{2\varepsilon} \eta_2 N^{\delta_\varepsilon - 1} \sup_{|E - 2| \leq N^{\delta_\varepsilon - 2/3}} (\eta_{lr})^{-1} \text{Im}[N^{-1} \text{tr}(A - (E + i\eta_{lr}))^{-1}]
$$

(A.19)

$$
\leq \frac{N^{3\varepsilon} N^{\delta_\varepsilon} \eta_2}{N^2 \eta_{lr}^2} + N^{-1/3 + 3\varepsilon} \frac{N^{\delta_\varepsilon} \eta_2}{N \eta_{lr}}
$$

(A.20)

For the remaining portion of the vertical segments of $\Gamma$, we instead use (3.17). The error we get is

$$
\int_{\eta > |y| > N^{\delta_\varepsilon - 1}} N^\varepsilon \left( \frac{1}{\sqrt{N}} + \frac{1}{N^2 y^2} + \frac{y^{1/2} + N^{-1/3 + \varepsilon}}{Ny} + \frac{1}{N \sqrt{y} + N^{-2/3 + \varepsilon}} \right) \, dy
$$

(A.21)

$$
\leq N^{4\varepsilon} \left( \frac{\eta_1}{N^{1/2}} + \frac{\eta_2}{N^2 \eta_2^2} + \frac{\eta_1}{N \eta_1} + N^{-1/3} \right)
$$

(A.22)

The contribution of the horizontal segments are bounded by

$$
\int_{|x - \lambda_1(M)| \leq \eta_2} N^\varepsilon \left( \frac{1}{\sqrt{N}} + \frac{1}{N^2 \eta_1^2} + \frac{\eta_1^{1/2} + N^{-1/3 + \varepsilon}}{N \eta_1} + \frac{1}{N \sqrt{\eta_1} + N^{-2/3 + \varepsilon}} \right)
$$

(A.23)

$$
\leq N^{2\varepsilon} \left( \frac{\eta_2}{\sqrt{N}} + \frac{\eta_2}{N^2 \eta_2^2} + \frac{\eta_2}{N \sqrt{\eta_1} + N^{-1/3} \eta_1} \right)
$$

(A.24)

If we choose, e.g., $\eta_{lr} = N^{-2/3 - 0.1}$, $\eta_1 = \eta_2 = \eta_{lr}/2$, $\varepsilon = 10^{-10}$, and say $\delta_\varepsilon = 0.01$, then all of the errors (A.16), (A.20), (A.22) and (A.24) are seen to be $O(N^{-1 - \varepsilon})$ for some $c > 0$.

**B Isotropic CLT for zero-diagonal GOE**

In this section we recall that $M$ is a matrix from the GOE with a zero diagonal. Let $v$ be a unit vector, and $\gamma > 2$. Define,

$$
\kappa := \gamma - 2.
$$

(B.1)

We prove the following theorem.

**Theorem B.1.** Fix $\varepsilon > 0$, and let $C \geq \kappa \geq N^{\varepsilon - 2/3}$. Then,

$$
V_N^{-1/2} N^{1/2} \kappa^{1/4} \left( v^T R(\gamma) v - m_{sc}(\gamma) \right)
$$

(B.2)

converges to a standard normal random variable. Here,

$$
V_N := \frac{\gamma + \sqrt{\gamma^2 - 4}}{\sqrt{\gamma + 2}} m_{sc}^2 \left( m_{sc}^2 + (1 - \|v\|_2^2)(1 - m_{sc}^2) \right)
$$

(B.3)

satisfies $c \leq V_N \leq C$. If $\kappa \to 0$ then $V_N \to 1$. 

58
Remark. In the following proof we work with expectations of matrix elements of the resolvent on the real line, \((M - E)^{-1}\). Due to integrability concerns, one should instead work with the regularization \(\text{Re}[(M - (E + i\eta))^{-1}]\) for, e.g., \(\eta = N^{-100}\). With overwhelming probability, the difference for any matrix element between these two quantities is \(\mathcal{O}(N^{-100})\). For notational convenience we omit this regularization in the proof below, but it is elementary to restore it and check that the proof goes through.

Proof. Define the characteristic function,

\[
\psi(\lambda) = \mathbb{E}[e(\lambda)], \quad e(\lambda) := \exp \left[ i\lambda N^{1/2} \kappa^{1/4} v^T R^\circ(\gamma) v \right],
\]

where we introduced the notation \(X^\circ := X - \mathbb{E}[X]\) for any random variable. We apply Stein’s method and calculate,

\[
\psi'(\lambda) = iN^{1/2} \kappa^{1/4} \sum_{i,j} v_i v_j \mathbb{E}[e(\lambda) R^\circ_{ij}]. \tag{B.5}
\]

From the matrix identity \(R(M - \gamma) = I\) and Gaussian integration by parts,

\[
\gamma \mathbb{E}[e(\lambda) R^\circ_{ij}] = \sum_{a \neq j} \mathbb{E}[e(\lambda)(R_{ia} M_{aj} - \mathbb{E}[R_{ia} M_{aj}])] \tag{B.6}
\]

\[
= \frac{1}{N} \sum_{a \neq j} \mathbb{E}[(\partial_{aj} e(\lambda)) R_{ia}] \tag{B.7}
\]

\[
- \frac{1}{N} \sum_{a \neq j} \mathbb{E}[e(\lambda)(R_{ia} R_{ja} + R_{ij} R_{aa})^\circ]. \tag{B.8}
\]

We begin with,

\[
\sum_{i,j} v_i v_j N^{1/2} \kappa^{1/4} \mathbb{E}[e(\lambda)(R_{ia} R_{ja})^\circ] = \kappa^{1/4} N^{-1/2} \mathbb{E}[e(\lambda)(v^T R v)^\circ] - \kappa^{1/4} N^{-1/2} \sum_{i,j} \mathbb{E}[e(\lambda) v_i (R_{ij} R_{jj})^\circ v_j]. \tag{B.9}
\]

The first term on the RHS is \(N^\varepsilon \mathcal{O}(N^{-1/2} \kappa^{-1/4})\) for any \(\varepsilon > 0\). For the second term,

\[
\kappa^{1/4} N^{-1/2} \sum_{i,j} \mathbb{E}[e(\lambda) v_i (R_{ij} R_{jj})^\circ v_j] = \kappa^{1/4} N^{-1/2} m_{sc} \mathbb{E}[e(\lambda)(v^T R v)^\circ] \tag{B.10}
\]

\[
+ \kappa^{1/4} N^{-1/2} \sum_{i,j} \mathbb{E}[e(\lambda) v_i (R_{ij} (R_{jj} - m_{sc}))^\circ v_j] \tag{B.11}
\]

\[
= \mathcal{O}(N^{-\varepsilon - 1}) + \mathcal{O}(N^\varepsilon N^{-1/2} \kappa^{-1/4}). \tag{B.12}
\]

where for the second line we used \(|R_{ij} (R_{jj} - m_{sc})| \leq N^\varepsilon (N \sqrt{\kappa})^{-1/2} + \delta_{ij} N^{\varepsilon} N^{-1/2} \kappa^{-1/4}\) with overwhelming probability, as well as the fact that \(|v|_1 \leq N^{1/2}\) for the \(i \neq j\) terms and \(|v|_2 \leq 1\) for the \(i = j\) terms. The next term we handle is,

\[
\kappa^{1/4} N^{-1/2} \sum_{i,j} v_i v_j \sum_{a \neq j} \mathbb{E}[e(\lambda)(R_{ia} R_{aa})^\circ] = m_{sc} \kappa^{1/4} N^{-1/2} \sum_{i,j} v_i v_j \sum_{a \neq j} \mathbb{E}[e(\lambda)(R_{ij})^\circ] \tag{B.13}
\]

\[
+ \kappa^{1/4} N^{-1/2} \sum_{a} \mathbb{E}[e(\lambda)(v^T R v(G_{aa} - m_{sc}))^\circ] \tag{B.14}
\]

\[
- \kappa^{1/4} N^{-1/2} \sum_{i,j} v_i v_j \mathbb{E}[e(\lambda)(R_{ij} (R_{jj} - m_{sc}))^\circ]. \tag{B.15}
\]

The term on the last line appeared above and is \(\mathcal{O}(N^\varepsilon N^{-1/2} \kappa^{-1/4})\). The term on the second line equals

\[
\kappa^{1/4} N^{-1/2} \sum_{a} \mathbb{E}[e(\lambda)(v^T R v(G_{aa} - m_{sc}))^\circ] = \kappa^{1/4} N^{1/2} \mathbb{E}[e(\lambda)(v^T R v(m_N - m_{sc}))^\circ] = \mathcal{O}(N^\varepsilon N^{-1/2} \kappa^{-3/4}). \tag{B.16}
\]

59
The term on the first line equals,
\[ m_{sc}N^{-1/2} \kappa^{1/4} \sum_{i,j} v_i v_j \sum_{a \neq j} \mathbb{E}[c(\lambda) (R_{ia})^2] = m_{sc} \kappa^{1/4} N^{1/2} \mathbb{E}[c(\lambda) (v^T R v)^2] + \mathcal{O}(N^{-1/2}). \tag{B.17} \]

We observe,
\[ \tilde{c}_{ia} e(\lambda) = -2\kappa^{1/4} N^{1/2} i \lambda e(\lambda) (Rv)_a (Rv)_j, \tag{B.18} \]
and so
\begin{align*}
- N^{-1/2} \kappa^{1/4} \sum_{i,j} v_i v_j \sum_{a \neq j} \mathbb{E}[\tilde{c}_{ia} e(\lambda) R_{ia}] &= 2\kappa^{1/2} i \lambda \sum_{a \neq j} \mathbb{E}[e(\lambda) (Rv)^2_v v_j (Rv)_j] \tag{B.19} \\
&= 2i \lambda \kappa^{1/2} \mathbb{E}[e(\lambda) v^T R^2 v v^T R v] - 2i \lambda \kappa^{1/2} \sum_j \mathbb{E}[e(\lambda) v_j (Rv)_j^3].
\end{align*}

The first term equals
\[ 2i \lambda \kappa^{1/2} \mathbb{E}[e(\lambda) v^T R^2 v v^T R v] = 2i \lambda \kappa^{1/2} m_{sc}^\prime m_{sc} \psi(\lambda) + \mathcal{O}(\|\lambda\| N^{1/2} \kappa^{-3/4}). \tag{B.21} \]

For the other term we note that,
\[ v_j (Rv)_j^3 = v_j ((Rv)_j - v_j m_{sc} + v_j m_{sc})^3 = m_{sc}^3 v_j^4 + m_{sc}^2 v_j^3 \mathcal{O}(N^2 N^{-1/2} \kappa^{-1/4}) + \mathcal{O}(v_j^2 N^{-1/2} \kappa^{-1/2} + v_j N^{-1/2} \kappa^{-3/4}), \]
and so
\[ 2i \lambda \kappa^{1/2} \sum_j \mathbb{E}[e(\lambda) v_j (Rv)_j^3] = 2i \lambda \kappa^{1/2} m_{sc}^3 \|v\|_4^4 \psi(\lambda) + \mathcal{O}(\|\lambda\| N^{-1/2}). \tag{B.22} \]

From the above calculations we see that,
\[ \gamma \psi'(\lambda) = -m_{sc} \psi'(\lambda) + (2\kappa^{1/2} m_{sc}^\prime m_{sc} - 2 \|v\|_4^4 m_{sc}^3 \kappa^{1/2}) \lambda \psi(\lambda) + \mathcal{O}(N^2 N^{-1/2} \kappa^{-3/4}). \tag{B.23} \]

We calculate the quantity which will be seen to be the variance,
\[ \frac{-2\kappa^{1/2} m_{sc}}{\gamma + m_{sc}} (m_{sc}^\prime - \|v\|_4^2 m_{sc}^2) = \left( \frac{\kappa^{1/2}}{1 - m_{sc}^2} \right) 2m_{sc}^4 \left( m_{sc}^2 + (1 - \|v\|_4^2)(1 - m_{sc}^2) \right). \tag{B.24} \]

A calculation gives
\[ \left( \frac{\kappa^{1/2}}{1 - m_{sc}^2} \right) = \frac{1}{2} \sqrt{\frac{\gamma^2 - 4}{\gamma + 2}} \approx 1. \tag{B.25} \]

Noting that \( 1 - m_{sc}^2 \geq 0 \) we see that
\[ V_N = \frac{\gamma + \sqrt{\gamma^2 - 4}}{\sqrt{\gamma + 2}} m_{sc}^4 \left( m_{sc}^2 + (1 - \|v\|_4^2)(1 - m_{sc}^2) \right) \tag{B.26} \]
is bounded below and above, \( c \leq V_N \leq C \), uniformly in \( N \). By Lévy’s continuity theorem for characteristic functions, this proves that the centered random variable \( v^T R v \) converges to a standard normal random variable after the stated normalization.

Next, we calculate the expectation. We have,
\[ \gamma \kappa^{1/4} N^{1/2} \mathbb{E}[v^T R v] = -\kappa^{1/4} N^{1/2} \sum_{i,j} v_i v_j N^{1/2} \kappa^{1/4} \sum_{a \neq j} \mathbb{E}[R_{ia} M_{aj}]. \tag{B.27} \]

The second term equals,
\begin{align*}
\sum_{i,j} v_i v_j N^{1/2} \kappa^{1/4} \sum_{a \neq j} \mathbb{E}[R_{ia} M_{aj}] &= - \sum_{i,j} v_i v_j N^{-1/2} \kappa^{1/4} \sum_{a \neq j} \mathbb{E}[R_{ia} R_{ja} + R_{ij} R_{aa}] \tag{B.28} \\
&= - N^{-1/2} \kappa^{1/4} \mathbb{E}[(v^T R^2 v)] - N^{1/2} \kappa^{1/4} \mathbb{E}[(v^T R v)m_N] + 2 \sum_{i,j} v_i v_j N^{-1/2} \kappa^{1/4} \mathbb{E}[R_{ij} R_{jj}]. \tag{B.29}
\end{align*}
Finally, this yields the claim.

These terms are $O(N^\varepsilon N^{-1/2}\kappa^{-1/4})$. Next,

$$N^{1/2}\kappa^{1/4}m_{sc}E[v^TRv] = m_{sc}N^{1/2}\kappa^{1/4}m_{sc}E[v^TRv] + O(N^{-1/2}\kappa^{-3/4}).$$

(B.33)

Finally,

$$N^{-1/2}\kappa^{1/4}m_{sc}E[v^TR^2v] = O(N^{-1/2}\kappa^{-1/4}).$$

(B.34)

Hence,

$$\kappa^{1/4}N^{1/2}m_{sc}E[v^TRv] = \kappa^{1/4}N^{1/2}m_{sc} + O(N^{-1/2}\kappa^{-3/4}).$$

(B.35)

This yields the claim. \qed

C Conditional probability statement

We require the following elementary lemma.

Lemma C.1. Let $X$ and $Z$ be random variables and $\mathcal{G}$ a sigma-algebra. Let $F$ be a bounded Lipschitz function with Lipschitz constant $\|F\|_L$ and let $\varepsilon_1, \varepsilon_2 > 0$ be constants so that,

$$\mathcal{P}(|X - Z| > \varepsilon_1) \leq \varepsilon_2. \quad (C.1)$$

Then,

$$E[|E[F(X) | \mathcal{G}] - E[F(Z) | \mathcal{G}]|] \leq 2\|F\|_L(\varepsilon_1 + \varepsilon_2), \quad (C.2)$$

and

$$\mathbb{P} [E[F(X) | \mathcal{G}] - E[F(Z) | \mathcal{G}] > \eta] \leq \frac{2\|F\|_L(\varepsilon_1 + \varepsilon_2)}{\eta}. \quad (C.3)$$

Proof. We have,

$$E[F(X) | \mathcal{G}] - E[F(Z) | \mathcal{G}] = E[(F(X) - F(Z))\mathbf{1}_{\{|X - Z| > \varepsilon_1\}} | \mathcal{G}] + E[(F(X) - F(Z))\mathbf{1}_{\{|X - Z| < \varepsilon_1\}} | \mathcal{G}]. \quad (C.4)$$

Then by conditional Jensen’s inequality,

$$\mathbb{E}[|E[(F(X) - F(Z))\mathbf{1}_{\{|X - Z| > \varepsilon_1\}} | \mathcal{G}] + E[(F(X) - F(Z))\mathbf{1}_{\{|X - Z| < \varepsilon_1\}} | \mathcal{G}]|$$

$$\leq 2\|F\|_L \mathbb{P}(|X - Z| > \varepsilon_1) + \|F\|_L \varepsilon_1, \quad (C.5)$$

which yields the first claim. The second is of course Markov’s inequality. \qed

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