BOUNDS FOR SPECTRAL RADIUS OF NONNEGATIVE TENSORS USING MATRIX-DIGRAPH-BASED APPROACH

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Abstract. We obtain the improved results of the upper and lower bounds for the spectral radius of a nonnegative tensor by its majorization matrix’s digraph. Numerical examples are also given to show that our results are significantly superior to the results of related literature.

1. Introduction. Nonnegative tensors are an important extension of nonnegative matrices, and there are many results regarding their eigenvalues and eigenvectors, see [5, 11, 12, 13, 1, 9, 17, 4]. The eigenvalue problem of a tensor has wide applications; see [5, 11, 2, 3, 16, 14].

Let \( \mathbb{R} \) be the real field. Consider an \( m \)-th order \( n \)-dimensional tensor \( \mathbf{A} \) which consists of \( n^m \) entries in \( \mathbb{R} \):

\[
\mathbf{A} = (a_{i_1,i_2,...,i_m})_{i_1,i_2,...,i_m=1}^n \text{ with } a_{i_1,i_2,...,i_m} \in \mathbb{R}.
\]

A tensor \( \mathbf{A} \) is called nonnegative if \( a_{i_1,i_2,...,i_m} \geq 0 \). Denote the set of all \( m \)-th order \( n \)-dimensional tensor as \( \mathbb{R}^{[m,n]}_+ \).

An \( m \)-th order \( n \)-dimensional tensor \( \mathbf{I} = (\delta_{i_1,i_2,...,i_m}) \) is called the unit tensor, if

\[
\delta_{i_1,i_2,...,i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}
\]

Definition 1.1. ([10]) A nonnegative matrix \( \hat{\mathbf{A}} \) is called the majorization associated to nonnegative tensor \( \mathbf{A} \), if the \((i,j)\)-th element of \( \hat{\mathbf{A}} \) is defined to be \( a_{ij,...,j} \) for any \( i, j \in 1, \ldots, n \).

Denote the digraph [15] of majorization matrix \( \hat{\mathbf{A}} \) as \( \Gamma(\hat{\mathbf{A}}) \), and if \( a_{ij,...,j} \neq 0 \), \( i \neq j \), \( e_{ij} \) is the directed edge.

In 2005, Lim [5] and Qi [11] defined the eigenvalues of a tensor, respectively.

Definition 1.2. If there are a complex number \( \lambda \) and a nonzero complex vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \) such that

\[
\mathbf{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},
\]

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then \( \lambda \) is called an eigenvalue of \( A \) and \( x \) is termed as an eigenvector of \( A \) associated with \( \lambda \), \( Ax^{m-1} \) and \( x^{[m-1]} \) are vectors, whose \( i \)-th entries are

\[
(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^n a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \quad i \in \langle n \rangle = \{1, 2, \ldots, n\}
\]

and \( (x^{[m-1]})_i = x_i^{m-1} \), respectively.

Similarly to the nonnegative matrix theory, the spectral radius \( \rho(A) \) of a tensor \( A \) is defined as

\[
\rho(A) = \sup \{ |\lambda| : \lambda \in \text{spec}(A) \},
\]

where \( \text{spec}(A) \) is the set of eigenvalues of a tensor \( A \).

Moreover, \cite{1} generalized the concept of irreducible matrices to irreducible tensors.

**Definition 1.3.** \cite{1}, Definition 2.1) An \( m \)-th order \( n \)-dimensional tensor \( A \) is called reducible if there exists a nonempty proper index subset \( I \subset \langle n \rangle \) such that

\[
a_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.
\]

If \( A \) is not reducible, then \( A \) is irreducible.

**Remark 1.** From the Definition 1.3, it is easy to know that if \( \tilde{A} \) is irreducible, then \( A \) is irreducible.

In \cite{1}, the eigenvalues and eigenvectors of a nonnegative tensor gave the following results.

**Theorem 1.4.** \cite{1}, Theorem 1.3) If \( A \) is an \( m \)-th order \( n \)-dimensional nonnegative tensor, then there exist \( \lambda_0 \geq 0 \) and a nonnegative vector \( x_0 \) such that

\[
Ax_0^{m-1} = \lambda_0 x_0^{[m-1]}.
\]

**Theorem 1.5.** \cite{1}, Theorem 1.4) If \( A \) is an \( m \)-th order \( n \)-dimensional nonnegative tensor, then the pair \( (\lambda_0, x_0) \) in Equation (1) satisfy:

(i) \( \lambda_0 > 0 \) is an eigenvalue.

(ii) \( x_0 > 0 \), i.e. all entries of \( x_0 \) are positive.

(iii) If \( \lambda \) is an eigenvalue with a nonnegative eigenvector, then \( \lambda = \lambda_0 \). Moreover, the nonnegative eigenvector is unique up to multiplicative constant.

(iv) If \( \lambda \) is an eigenvalue of \( A \), then \( |\lambda| \leq \lambda_0 \).

From the results of (ii) and (iv) in Theorem 1.5, it is known that the spectral radius of a nonnegative tensor is also an eigenvalue.

Given a tensor \( A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{|m,n|}_+ \), denote \( r_i(A) \) is the \( i \)-th row sum of \( A \), that is,

\[
r_i(A) = \sum_{i_2, \ldots, i_m = 1}^n a_{i_2 \ldots i_m}, \quad i \in \langle n \rangle,
\]

\[
r_i(A) = r_i(A) - a_{i \ldots}, \quad i \in \langle n \rangle,
\]

\[
r'_i(A) = \sum_{\delta_{i_2 \ldots i_m} = 0}^n a_{i_2 \ldots i_m}, \quad i \in \langle n \rangle,
\]

\[
r_i(A) = r_i(A) - a_{i \ldots} - r'_i(A), \quad i \in \langle n \rangle,
\]

\[
r'_i(A) = r_i(A) - a_{i j \ldots}, \quad i \neq j, \quad i, j \in \langle n \rangle.
\]
Lemma 1.6. ([18], Lemma 3.3) If $0 \leq A < C$, then $\rho(A) \leq \rho(C)$.

Lemma 1.7. If $A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}_+$, then $\rho(A) \geq a_{i_1 i_2 \ldots i_m}$, $i \in \langle n \rangle$.

In [18], the upper and lower bounds for the spectral radius of a nonnegative tensor were given, which all depended only on the entries of $A$.

Theorem 1.8. ([18], Lemma 5.2) If $A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}_+$, then
\[
\min_{i \in \langle n \rangle} \tau_i(A) \leq \rho(A) \leq \max_{i \in \langle n \rangle} \tau_i(A).
\]

In 2016, [6] gave the improved results of the upper and lower bounds for the spectral radius of a nonnegative tensor.

Theorem 1.9. ([6], Theorem 3.3) If $A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}_+$ with $n \geq 2$, then
\[
\Omega_{\min} \leq \rho(A) \leq \Omega_{\max},
\]
where
\[
\Omega_{\min} = \min_{i \neq j} \Omega_{i,j}(A), \quad \Omega_{\max} = \max_{i \neq j} \Omega_{i,j}(A)
\]
and
\[
\Omega_{i,j}(A) = \frac{1}{2} \left\{ a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} + a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} + r_i^j(A) + \sqrt{\left( a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} - a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} + r_i^j(A) \right)^2 + 4a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m}} \right\}.
\]

In 2015, [7] gave the further improved results of the upper and lower bounds for the spectral radius of a nonnegative tensor.

$\Theta_i = \{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j \in \{2, \ldots, m\}, \text{ where } i, i_2, \ldots, i_m \in \langle n \rangle\}$,

$\Theta_i = \{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j \in \{2, \ldots, m\}, \text{ where } i, i_2, \ldots, i_m \in \langle n \rangle\}$,

\[
r_i^{\Theta_i}(A) = \sum_{(i_2, \ldots, i_m) \in \Theta_i} |a_{i_1 i_2 \ldots i_m}|, \quad r_i^{\overline{\Theta}_i}(A) = \sum_{(i_2, \ldots, i_m) \in \overline{\Theta}_i} |a_{i_1 i_2 \ldots i_m}|.
\]

Theorem 1.10. ([7], Theorem 5) If $A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}_+$ with $n \geq 2$, then
\[
\Delta_{\min} \leq \rho(A) \leq \Delta_{\max},
\]
where
\[
\Delta_{\min} = \min_{i \neq j} \Delta_{i,j}(A), \quad \Delta_{\max} = \max_{i \neq j} \Delta_{i,j}(A)
\]
and
\[
\Delta_{i,j}(A) = \frac{1}{2} \left\{ a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} + a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} + r_i^{\Theta_i}(A) + \sqrt{\left( a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} - a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} + r_i^{\Theta_i}(A) \right)^2 + 4a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m}} \right\}.
\]

In 2015, [8] also gave the results of the upper and lower bounds for the spectral radius of a nonnegative tensor.

Theorem 1.11. ([8], Theorem 2.17) If $A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}_+$ is irreducible, then
\[
\min_{i,j} \left\{ a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} \left( \frac{1}{\tau_i(A)^2} - 1 \right) + \tau_i(A) \right\} \leq \rho(A) \leq \max_{i,j} \left\{ \tau_i(A) - a_{i_1 i_2 \ldots i_{j-1} i_{j+1} \ldots i_m} (1 - \tau_i(A)^2) \right\},
\]
where
\[
\tau(A) = \left( \frac{\min_i r_i(A) - \min_{i,j} a_{ij} \cdots j}{\max_i r_i(A) - \max_{i,j} a_{ij} \cdots j} \right)^{\frac{1}{2m-1}}.
\]

If exist \( \bar{i}_0 \in \arg \min_{i \in \langle n \rangle} r_i(A) \) and \( \bar{j}_0 \in \langle n \rangle \), such that \( a_{\bar{i}_0 \bar{j}_0 \cdots \bar{j}_0} = 0 \) and \( a_{\bar{i}_0 \bar{j}_0 \cdots \bar{j}_0} = 0 \), respectively, then Theorem 1.11 is the same as Theorem 1.8.

In this paper, we give the new upper and lower bounds for the spectral radius of a nonnegative tensor, which further enriches the corresponding conclusions about the estimation of the upper and lower bounds for the spectral radius of a nonnegative tensor.

2. Main result. In this section, the new upper and lower bounds for the spectral radius of a nonnegative tensor are obtained by using its majorization matrix’s digraph.

**Theorem 2.1.** If \( A = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_{+} \), then
\[
\min_{e \in \Gamma(\hat{A})} \tilde{\kappa}_{i,j}(A) \leq \rho(A) \leq \max_{e \in \Gamma(\hat{A})} \kappa_{i,j}(A),
\]
where
\[
\kappa_{i,j}(A) = \frac{1}{2} \left\{ a_{i \cdots i} + a_{j \cdots j} + r_i'(A) + \sqrt{(a_{i \cdots i} - a_{j \cdots j} + r_i'(A))^2 + 4r_i'(A)r_j'(A)} \right\}.
\]

**Proof.** For any \( A = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_{+} \), there are two cases:

(i) If \( \hat{A} \) is irreducible, then \( A \) is irreducible. From Theorem 1.4 and Theorem 1.5, \( \rho(A) \) is the eigenvalue of \( A \) and its corresponding eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \) is a positive vector, then
\[
Ax^{m-1} = \rho(A)x^{m-1}.
\]
Without loss of generality, suppose that
\[
x_{t_1} \geq x_{t_2} \geq \cdots \geq x_{t_{n-1}} \geq x_{t_n} > 0.
\]
Since \( \hat{A} \) is irreducible, there exists \( j \neq t_1 \), such that \( a_{t_1 \cdots j} \neq 0 \). Assume
\[
a_{t_1 t_2 \cdots t_l} = 0, \ l = 2, 3, \ldots, s - 1, \ a_{t_1 t_s \cdots t_s} \neq 0 (2 \leq s \leq n),
\]
then \( e_{t_1 t_s} \in \Gamma(\hat{A}) \).
Firstly, we prove
\[
\rho(A) \leq \kappa_{t_1 t_s}(A) \leq \max_{e_{ij} \in \Gamma(\hat{A})} \kappa_{i,j}(A).
\]
From (2), we have
\[
\sum_{i_2, \ldots, i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \rho(A)x_{t_1}^{m-1},
\]
Thus, we have

$$\rho(A) = a_{t_1 \cdots t_1} x_{t_1}^{m-1}$$

and then

$$= \sum_{i_2, \ldots, i_m = 1}^{n} a_{t_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

$$\delta_{t_1 i_2 \cdots i_m} = 0$$

$$\leq r'_{t_1}(A) x_{t_1}^{m-1} + \tilde{r}_{t_1}(A) x_{t_1}^{m-1},$$

i.e.,

$$(\rho(A) - a_{t_1 \cdots t_1} - r'_{t_1}(A)) x_{t_1}^{m-1} \leq \tilde{r}_{t_1}(A) x_{t_1}^{m-1}.$$  \hfill (3)

Similarly, from (2), we have

$$\sum_{i_2, \ldots, i_m = 1}^{n} a_{t_2 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \rho(A) x_{t_2}^{m-1},$$

and then

$$(\rho(A) - a_{t_2 \cdots t_2}) x_{t_2}^{m-1} = \sum_{i_2, \ldots, i_m = 1}^{n} a_{t_2 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \leq r_{t_2}(A) x_{t_2}^{m-1}.$$ \hfill (4)

From Lemma 1.7, we have $\rho(A) \geq a_{t_2 \cdots t_s}$. So multiplying (4) with (7) gives

$$(\rho(A) - a_{t_1 \cdots t_1} - r'_{t_1}(A))(\rho(A) - a_{t_2 \cdots t_2}) x_{t_1}^{m-1} x_{t_2}^{m-1} \leq \tilde{r}_{t_1}(A) r_{t_2}(A) x_{t_1}^{m-1} x_{t_2}^{m-1}.$$  

Note that $x_{t_1} \geq x_{t_2} > 0$, we get

$$(\rho(A) - a_{t_1 \cdots t_1} - r'_{t_1}(A))(\rho(A) - a_{t_2 \cdots t_2}) \leq \tilde{r}_{t_1}(A) r_{t_2}(A).$$

Furthermore,

$$\rho(A) \leq \frac{1}{2} \left\{ a_{t_1 \cdots t_1} + a_{t_2 \cdots t_2} + r'_{t_1}(A) + \sqrt{(a_{t_1 \cdots t_1} - a_{t_2 \cdots t_2} + r'_{t_1}(A))^2 + 4\tilde{r}_{t_1}(A) r_{t_2}(A)} \right\}.$$  

Thus, we have

$$\rho(A) \leq \kappa_{t_1, t_s}(A) \leq \max_{e_{ij} \in \Gamma(\tilde{A})} \kappa_{i,j}(A).$$

Since $\tilde{A}$ is irreducible, there exists $j \neq t_n$, such that $a_{t_n \cdots j} \neq 0$. Assume

$$a_{t_n t_1 \cdots t_l} = 0, \quad l = 2, 3, \ldots, r - 1, \quad a_{t_n t_r \cdots t_r} \neq 0(2 \leq r \leq n),$$

then $e_{t_n t_r} \in \Gamma(\tilde{A})$.

Now, we prove

$$\rho(A) \geq \kappa_{t_n, t_r}(A) \geq \min_{e_{ij} \in \Gamma(\tilde{A})} \kappa_{i,j}(A).$$

From (2), we have

$$\sum_{i_2, \ldots, i_m = 1}^{n} a_{t_2 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \rho(A) x_{t_n}^{m-1},$$

and

$$\sum_{i_2, \ldots, i_m = 1}^{n} a_{t_2 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \rho(A) x_{t_r}^{m-1}.$$
Similarly to the above proof, it is easy to obtain
\[
\rho(A) \geq \kappa_{t_n, t_r}(A) \geq \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A).
\]

Therefore, when \( \tilde{A} \) is irreducible, we have
\[
\min_{e_{ij} \in \Gamma'(\tilde{A})} \kappa_{i,j}(A) \leq \rho(A) \leq \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A).
\]

(ii) If \( \tilde{A} \) is reducible, for any \( \varepsilon > 0 \), denote
\[
\tilde{A}(\varepsilon) = A + \Theta(\varepsilon), \quad A(\varepsilon) = A + \Theta(\varepsilon), \quad \Theta(\varepsilon) = (\theta_{ij} \cdots (\varepsilon))_{n \times n},
\]
then \( \tilde{A}(\varepsilon) \) is a reducible matrix. From the proof of (i) in Theorem 2.1, we have
\[
\min_{e_{ij} \in \Gamma'(A(\varepsilon))} \kappa_{i,j}(A(\varepsilon)) \leq \rho(A(\varepsilon)) \leq \max_{e_{ij} \in \Gamma(A(\varepsilon))} \kappa_{i,j}(A(\varepsilon)),
\]
and \( \rho(A(\varepsilon)) \) is a continuous function of \( \varepsilon \), letting \( \varepsilon \to 0 \), we obtain
\[
\min_{e_{ij} \in \Gamma'(A)} \kappa_{i,j}(A) \leq \rho(A) \leq \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A).
\]

Corollary 1. If \( A = (a_{i_1i_2 \cdots i_m}) \in \mathbb{R}_{+}^{m\times n} \), then
\[
\min_{i \neq j} \kappa_{i,j}(A) \leq \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \leq \min_{j-i=1,1-n} \kappa_{i,j}(A) \leq \max_{i \neq j} \kappa_{i,j}(A).
\]

From the proof of the Theorem 2.1, we also obtain

Corollary 2. If \( A = (a_{i_1i_2 \cdots i_m}) \in \mathbb{R}_{+}^{m\times n} \), then
\[
\min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \leq \rho(A) \leq \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A).
\]

Corollary 3. If \( A = (a_{i_1i_2 \cdots i_m}) \in \mathbb{R}_{+}^{m\times n} \), then
\[
\min \left\{ \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A), \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \right\} \leq \rho(A)
\]
\[
\leq \max \left\{ \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A), \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \right\}.
\]

Theorem 2.2. If \( A = (a_{i_1i_2 \cdots i_m}) \in \mathbb{R}_{+}^{m\times n} \) is irreducible, \( \overline{N} = \{ i \in \langle n \rangle | \Gamma_{A}^+(i) = \emptyset \} \neq \emptyset \), then
\[
\min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A), \min_{i \in \overline{N}} \max_{j \neq i \in \overline{N}} \kappa_{i,j}(A) \leq \rho(A)
\]
\[
\leq \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A), \max_{i \in \overline{N}} \min_{j \neq i \in \overline{N}} \kappa_{i,j}(A).
\]
where
\[ \kappa_{i,j}(A) = \frac{1}{2} \left\{ a_{i\cdots i} + a_{j\cdots j} + r'(A) + \sqrt{(a_{i\cdots i} - a_{j\cdots j} + r'(A))^2 + 4r_t(A)r_j(A)} \right\}. \]

Proof. If \( A = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{m,n} \) is irreducible. From Theorem 1.4 and Theorem 1.5, \( \rho(A) \) is the eigenvalue of \( A \) and its corresponding eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \) is a positive vector, then
\[ Ax^{m-1} = \rho(A)x^{m-1}. \]  
(5)

(i) Let \( \Gamma^+_A(i) \neq \emptyset, \ i \in \langle n \rangle \). Without loss of generality, suppose that \( x_{t_1} \geq x_{t_2} \geq \cdots \geq x_{t_{n-1}} \geq x_{t_n} > 0 \).

Since \( \Gamma^+_A(i) \neq \emptyset, \ i \in \langle n \rangle \), there exists \( j \neq t_1 \), such that \( a_{t_1j} \neq 0 \). Assume \( a_{t_1t_1\cdots t_l} = 0, \ l = 2, 3, \ldots, s - 1, \ a_{t_1t_s\cdots t_s} \neq 0 (2 \leq s \leq n) \),
then \( \epsilon_{t_1t_s} \in \Gamma(A) \).

Firstly, we prove
\[ \rho(A) \leq \kappa_{t_1t_s}(A) \leq \max_{\epsilon_{ij} \in \Gamma(A)} \kappa_{ij}(A). \]

From (5), we have
\[ \sum_{i_2,\ldots,i_m=1}^n a_{i_2\cdots i_m}x_{i_2}\cdots x_{i_m} = \rho(A)x^{m-1}_{t_1}, \]
and then
\[ (\rho(A) - a_{t_1\cdots t_1})x^{m-1}_{t_1} = \sum_{i_2,\ldots,i_m=1}^n a_{i_1i_2\cdots i_m}x_{i_2}\cdots x_{i_m} = \sum_{i_2,\ldots,i_m=1}^n a_{i_1i_2\cdots i_m}x_{i_2}\cdots x_{i_m} \]
\[ = \sum_{i_2,\ldots,i_m=1}^n a_{i_1i_2\cdots i_m}x_{i_2}\cdots x_{i_m} + \sum_{i_1,\ldots,i_m=1}^n a_{i_1i_2\cdots i_m}x_{i_2}\cdots x_{i_m} \]
\[ \leq r'(A)x^{m-1}_{t_1} + \tilde{r}_A \rho(A)x^{m-1}_{t_1}, \]
i.e.,
\[ (\rho(A) - a_{t_1\cdots t_1} - r'(A))x^{m-1}_{t_1} \leq \tilde{r}_A \rho(A)x^{m-1}_{t_1}. \]  
(6)

Similarly, from (5), we have
\[ \sum_{i_2,\ldots,i_m=1}^n a_{i_2\cdots i_m}x_{i_2}\cdots x_{i_m} = \rho(A)x^{m-1}_{t_s}, \]
and then
\[ (\rho(A) - a_{t_s\cdots t_s})x^{m-1}_{t_s} = \sum_{i_2,\ldots,i_m=1}^n a_{i_2\cdots i_m}x_{i_2}\cdots x_{i_m} \leq r_t(A)x^{m-1}_{t_1}. \]  
(7)

From Lemma 1.7, we have \( \rho(A) \geq a_{t_1\cdots t_s} \). So multiplying (6) with (7) gives
\[ (\rho(A) - a_{t_1\cdots t_1} - r'(A)) \rho(A) - a_{t_s\cdots t_s})x^{m-1}_{t_1}x^{m-1}_{t_s} \leq \tilde{r}_A \rho(A)x^{m-1}_{t_1}x^{m-1}_{t_s}. \]
Note that \( x_{t_1} \geq x_{t_2} > 0 \), we get
\[
(\rho(A) - a_{t_1, \ldots, t_1} - r_{t_1}'(A)) (\rho(A) - a_{t_2, \ldots, t_2}) \leq \bar{r}_{t_1}(A) r_{t_2}(A).
\]
Furthermore,
\[
\rho(A) \leq \frac{1}{2} \left\{ a_{t_1, \ldots, t_1} + a_{t_2, \ldots, t_2} + r_{t_1}'(A) + \sqrt{(a_{t_1, \ldots, t_1} - a_{t_2, \ldots, t_2} + r_{t_1}'(A))^2 + 4\bar{r}_{t_1}(A) r_{t_2}(A)} \right\}.
\]
Thus, we have
\[
\rho(A) \leq \kappa_{t_1, t_2}(A) \leq \max_{\epsilon_{i,j} \in \Gamma(A)} \kappa_{i,j}(A).
\]
Since \( \Gamma^+_A(i) \neq \emptyset \), \( i \in \langle n \rangle \), there exists \( j \neq t_n \), such that \( a_{t_n, \ldots, j} \neq 0 \). Assume
\[
a_{t_n, t_l, \ldots, t_l} = 0, \ l = 2, 3, \ldots, r - 1, \ a_{t_n, t_r, \ldots, t_r} \neq 0 (2 \leq r \leq n),
\]
then \( \epsilon_{t_n, t_r} \in \Gamma(A) \).
Now, we prove
\[
\rho(A) \geq \kappa_{t_n, t_r}(A) \geq \min_{\epsilon_{i,j} \in \Gamma(A)} \kappa_{i,j}(A).
\]
From (5), we have
\[
\sum_{i_2, \ldots, i_m = 1}^n a_{t_n, i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} = \rho(A) x_{t_n}^{m-1},
\]
and
\[
\sum_{i_2, \ldots, i_m = 1}^n a_{t_r, i_2 \ldots i_m} x_{i_2} \cdots x_{i_m} = \rho(A) x_{t_r}^{m-1}.
\]
Similarly to the above proof, it is easy to obtain
\[
\rho(A) \geq \kappa_{t_n, t_r}(A) \geq \min_{\epsilon_{i,j} \in \Gamma(A)} \kappa_{i,j}(A).
\]
(ii) If \( \Gamma^+_A(i) = \emptyset \), \( i \in \bar{N} \), we denote \( A(\varepsilon) = (a_{ij-j}(\varepsilon))_{n \times n} \), take any one \( j \in \bar{N} \setminus \{i\} \), such that \( a_{ij-j}(\varepsilon) = \varepsilon \), otherwise \( a_{ij-j}(\varepsilon) = a_{ij-j} \), then \( \Gamma^+_{A(\varepsilon)}(i) \neq \emptyset \).
From Theorem 1.4 and Theorem 1.5, \( \rho(A(\varepsilon)) \) is the eigenvalue of \( A(\varepsilon) \) and its corresponding eigenvector \( x(\varepsilon) = (x_1(\varepsilon), x_2(\varepsilon), \ldots, x_n(\varepsilon))^T \) is a positive vector, then
\[
A(\varepsilon)x(\varepsilon)^{m-1} = \rho(A(\varepsilon))x(\varepsilon)^{m-1}.
\]
Similarly to the proof in (i), it is easy to obtain
\[
\min_{\epsilon_{i,j} \in \Gamma(A(\varepsilon))} \kappa_{i,j}(A(\varepsilon)) \leq \rho(A(\varepsilon)) \leq \max_{\epsilon_{i,j} \in \Gamma(A(\varepsilon))} \kappa_{i,j}(A(\varepsilon)).
\]
When \( \varepsilon \to 0 \), we have \( \lim_{\varepsilon \to 0} \rho(A(\varepsilon)) = \rho(A) \), then
\[
\min \left\{ \min_{\epsilon_{i,j} \in \Gamma(A)} \kappa_{i,j}(A), \min_{i \in N, j \neq i \in N} \kappa_{i,j}(A) \right\} \leq \rho(A)
\]
\[
\leq \max \left\{ \max_{\epsilon_{i,j} \in \Gamma(A)} \kappa_{i,j}(A), \min_{i \in N, j \neq i \in N} \kappa_{i,j}(A) \right\}.
\]
Corollary 4. If $A = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}_+$ is irreducible, $\Gamma^{\frac{1}{4}}(i) \neq \emptyset$, then
\[
\min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \leq \rho(A) \leq \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A),
\]
where
\[
\kappa_{i,j}(A) = \frac{1}{2} \left\{ a_{i\ldots i} + a_{j\ldots j} + r_i'(A) + \sqrt{(a_{i\ldots i} - a_{j\ldots j} + r_i'(A))^2 + 4r_i(A)r_j(A)} \right\}.
\]

3. Numerical examples.

Example 1. Let $A \in \mathbb{R}^{[3,4]}_+$, where
\[
\begin{align*}
A(1,,:) &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, & A(2, :) &= \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}, \\
A(3,:) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}, & A(4, :) &= \begin{pmatrix} 3 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.
\end{align*}
\]
We know that $\tilde{A} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 1 \end{pmatrix}$ is an irreducible matrix.

From calculating $\rho(A) = 9.6363$. From Theorem 1.8, we have
\[
8 \leq \rho(A) \leq 11.
\]
From Theorem 1.9, we obtain
\[
\begin{align*}
\Omega_{1,2} &= 9.1692, & \Omega_{2,1} &= 10.0000, & \Omega_{1,3} &= 9.0000, & \Omega_{3,1} &= 8.0000, \\
\Omega_{1,4} &= 9.0000, & \Omega_{4,1} &= 10.5208, & \Omega_{2,3} &= 9.7958, & \Omega_{3,2} &= 8.0000, \\
\Omega_{2,4} &= 10.0000, & \Omega_{4,2} &= 11.0000, & \Omega_{3,4} &= 8.6235, & \Omega_{4,3} &= 11.0000.
\end{align*}
\]
Therefore,
\[
\Omega_{\min} = \min_{i \neq j} \Omega_{i,j}(A) = 8, \quad \Omega_{\max} = \max_{i \neq j} \Omega_{i,j}(A) = 11.
\]
i.e.,
\[
8 \leq \rho(A) \leq 11.
\]
From Theorem 1.11, we also obtain
\[
8 \leq \rho(A) \leq 11.
\]
From Theorem 1.10, we obtain
\[
\begin{align*}
\Delta_{1,2} &= 9.4162, & \Delta_{2,1} &= 9.5887, & \Delta_{1,3} &= 8.5574, & \Delta_{3,1} &= 8.3523, \\
\Delta_{1,4} &= 9.8102, & \Delta_{4,1} &= 10.2111, & \Delta_{2,3} &= 9.1521, & \Delta_{3,2} &= 8.6847, \\
\Delta_{2,4} &= 10.3899, & \Delta_{4,2} &= 10.6158, & \Delta_{3,4} &= 9.0000, & \Delta_{4,3} &= 9.7823.
\end{align*}
\]
Therefore,
\[
\Delta_{\min} = \min_{i \neq j} \Delta_{i,j}(A) = 8.3523, \quad \Delta_{\max} = \max_{i \neq j} \Delta_{i,j}(A) = 10.6158.
\]
i.e.,
\[
8.3523 \leq \rho(A) \leq 10.6158.
\]
From Theorem 2.1, we have
\[ \kappa_{1,2} = 9.1692, \quad \kappa_{2,3} = 9.7958, \quad \kappa_{3,4} = 8.6235, \quad \kappa_{4,1} = 10.5208. \]
Therefore,
\[ \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 8.6235, \quad \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 10.5208. \]
i.e.,
\[ 8.6235 \leq \rho(A) \leq 10.5208. \]

**Example 2.** Let \( A \in \mathbb{R}_{[3,4]}^{+} \), where
\[
A(1,\ldots) = \begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 2 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0
\end{pmatrix},
A(2,\ldots) = \begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 2 & 0
\end{pmatrix},
A(3,\ldots) = \begin{pmatrix}
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 0 \\
2 & 1 & 1 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix},
A(4,\ldots) = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 \\
1 & 1 & 0 & 1
\end{pmatrix}.
\]
We also know that \( \hat{A} = \begin{pmatrix}
1 & 2 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 3 & 1
\end{pmatrix} \) is a reducible matrix.

From calculating \( \rho(A) = 12.4991 \). From Theorem 1.8, we have
\[ 10 \leq \rho(A) \leq 14. \]

From Theorem 1.9, we obtain
\[
\Omega_{1,2} = 12.8556, \quad \Omega_{2,1} = 12.1521, \quad \Omega_{1,3} = 13.0000, \quad \Omega_{3,1} = 14.0000, \\
\Omega_{1,4} = 13.0000, \quad \Omega_{4,1} = 10.0000, \quad \Omega_{2,3} = 13.0000, \quad \Omega_{3,2} = 14.0000, \\
\Omega_{2,4} = 13.0000, \quad \Omega_{4,2} = 10.0000, \quad \Omega_{3,4} = 13.4462, \quad \Omega_{4,3} = 10.9282.
\]
Therefore,
\[ \Omega_{\min} = \min_{i \neq j} \Omega_{i,j}(A) = 10, \quad \Omega_{\max} = \max_{i \neq j} \Omega_{i,j}(A) = 14. \]
i.e.,
\[ 10 \leq \rho(A) \leq 14. \]

From Theorem 1.11, we also obtain
\[ 10 \leq \rho(A) \leq 14. \]

From Theorem 1.10, we obtain
\[
\Delta_{1,2} = 12.5917, \quad \Delta_{2,1} = 12.4121, \quad \Delta_{1,3} = 13.3923, \quad \Delta_{3,1} = 13.7614, \\
\Delta_{1,4} = 11.7178, \quad \Delta_{4,1} = 11.0000, \quad \Delta_{2,3} = 12.8077, \quad \Delta_{3,2} = 13.5156, \\
\Delta_{2,4} = 11.1168, \quad \Delta_{4,2} = 10.6811, \quad \Delta_{3,4} = 13.0000, \quad \Delta_{4,3} = 11.3066.
\]
Therefore,
\[ \Delta_{\min} = \min_{i \neq j} \Delta_{i,j}(A) = 10.6811, \quad \Delta_{\max} = \max_{i \neq j} \Delta_{i,j}(A) = 13.7614. \]
i.e.,
\[ 10.6811 \leq \rho(A) \leq 13.7614. \]
From Theorem 2.1, we have
\[ \kappa_{1,2} = 12.8556, \kappa_{2,1} = 12.1521, \kappa_{2,3} = 12.3007, \]
\[ \kappa_{3,4} = 13.4462, \kappa_{4,1} = 10.7082, \kappa_{4,3} = 10.9282. \]
Therefore,
\[ \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 10.7082, \quad \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 13.4462. \]
i.e.,
\[ 10.7082 \leq \rho(A) \leq 13.4462. \]

From Corollary 2, we have
\[ \kappa_{1,2} = 12.8556, \kappa_{1,4} = 12.5574, \kappa_{2,1} = 12.1521, \]
\[ \kappa_{3,2} = 13.7284, \kappa_{3,4} = 13.4462, \kappa_{4,3} = 10.9282. \]
Therefore,
\[ \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 10.9282, \quad \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 13.7284. \]
i.e.,
\[ 10.9282 \leq \rho(A) \leq 13.7284. \]

From Corollary 3, we have
\[ \min \left\{ \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A), \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \right\} = 10.9282, \]
\[ \max \left\{ \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A), \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) \right\} = 13.4462. \]
i.e.,
\[ 10.9282 \leq \rho(A) \leq 13.4462. \]

From Corollary 4, we have
\[ \kappa_{1,2} = 12.8556, \kappa_{2,1} = 12.1521, \kappa_{3,4} = 13.4462, \kappa_{4,3} = 10.9282. \]
Therefore,
\[ \min_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 10.9282, \quad \max_{e_{ij} \in \Gamma(A)} \kappa_{i,j}(A) = 13.4462. \]
i.e.,
\[ 10.9282 \leq \rho(A) \leq 13.4462. \]

From Example 1 and Example 2, it is easy to see that the bounds in Corollary 3 are sharper than those in Theorem 1.8, Theorem 1.9, Theorem 1.10 and Theorem 1.11, and when $A$ is irreducible and $\Gamma_+(A)(i) \neq \emptyset$, compared with Corollary 3, Corollary 4 can reduce the number of calculations. Combining Corollary 3 with Theorem 1.10 and Theorem 1.11, we will get better the upper and lower bounds.

4. Conclusion. In this paper, we obtain the improved results of the upper and lower bounds for the spectral radius of a nonnegative tensor by using its majorization matrix’s digraph $\tilde{A}$. When $\tilde{A}$ is a sparse matrix, the number of calculations is greatly reduced, and in this case that conclusions about the upper and lower bounds for the spectral radius of a nonnegative tensor are improved.
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