Rotating black holes with equal-magnitude angular momenta in $d = 5$ Einstein-Gauss-Bonnet theory

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ABSTRACT: We construct rotating black hole solutions in Einstein-Gauss-Bonnet theory in five spacetime dimensions. These black holes are asymptotically flat, and possess a regular horizon of spherical topology and two equal-magnitude angular momenta associated with two distinct planes of rotation. The action and global charges of the solutions are obtained by using the quasilocal formalism with boundary counterterms generalized for the case of Einstein-Gauss-Bonnet theory. We discuss the general properties of these black holes and study their dependence on the Gauss-Bonnet coupling constant $\alpha$. We argue that most of the properties of the configurations are not affected by the higher derivative terms. For fixed $\alpha$ the set of black hole solutions terminates at an extremal black hole with a regular horizon, where the Hawking temperature vanishes and the angular momenta attain their extremal values. The domain of existence of regular black hole solutions is studied. The near horizon geometry of the extremal solutions is determined by employing the entropy function formalism.

KEYWORDS: Einstein-Gauss-Bonnet theory, black holes, numerical solutions.
1. Introduction

In $d \geq 5$ dimensions, the gravity action may be modified to include higher order curvature terms while keeping the equations of motion to second order, provided the higher order terms appear in specific combinations \[1\]. In five dimensions, this leads to the so-called Einstein-Gauss-Bonnet (EGB) theory, which contains quadratic powers of the curvature. The Gauss-Bonnet (GB) term appears in the one-loop corrected effective action of heterotic string theory \[2, 3\], and also in the low-energy effective action for the compactification of M-theory on a Calabi-Yau threefold \[4\]. Inclusion of this term in the action leads to a variety of new features (see the recent reviews \[5, 6\]). In particular, the black holes of EGB theory do not in general obey the Bekenstein-Hawking area law, but the entropy formula includes a new contribution coming from the higher curvature terms in the action \[7, 8\]. Also, for $d = 5$, the GB term implies the existence of a branch of small static black holes which are thermodynamically stable \[9\].

However, the complexity of EGB theory makes the task of finding closed form solutions a highly non-trivial task. The only known analytic solutions correspond to the counterparts
of the Schwarzschild-Tangherlini black holes [10, 11], and a variety of physically interesting solutions were found only numerically, (e.g. the black strings [12, 13] and the black rings [14]). In particular, no rotating EGB solutions are known yet in closed form, and it was proven in [15] that the Kerr-Schild ansatz may not work in Lovelock theory. Nevertheless, a number of partial results support the idea that EGB generalizations of the Myers-Perry (MP) solutions [16] actually exist [17, 18].

The main purpose of this work is to present numerical evidence for the existence of a class of asymptotically flat, rotating solutions in $d = 4+1$ EGB theory. These solutions have a spherical horizon topology and two equal-magnitude angular momenta. This restriction leads to a system of coupled nonlinear ordinary differential equations, which are solved numerically within a nonperturbative approach. The same approach has been employed to construct Einstein-Maxwell [19, 20], Einstein-Maxwell-Chern-Simons [21], and Einstein-Yang-Mills [22] rotating black hole solutions in higher dimensions. Also, the existence of the $AdS$ counterparts of the solutions addressed in this work was established in [23].

Our results suggest that the EGB rotating black holes with two equal angular momenta share most of the features of the corresponding Einstein gravity MP solutions. In particular, for a fixed value of the GB coupling constant, the sets of black hole solutions terminate when they reach an extremal black hole, where the Hawking temperature vanishes and the equal-magnitude angular momenta assume their extremal values. These extremal black holes obey the attractor mechanism assuming their near horizon geometry involves an $AdS_2$ factor. As a new feature, a GB term in the action leads to the existence of a branch of rotating black holes with a positive specific heat at constant angular velocity at the horizon.

This paper is organized as follows: in the next Section we recall the EGB action and present the ansatz and boundary conditions. Section 3 contains a discussion of the counterterm approach for asymptotically flat solutions in EGB theory, together with the computation of the action and global charges for rotating black holes with equal magnitude angular momenta. We present our numerical results in Section 3, and exhibit the physical properties of these solutions and their domain of existence. The near horizon geometry of the extremal solutions is also studied there by using the entropy function formalism [24]. We conclude in Section 4 with some further remarks.

2. The model

2.1 The action

We consider the Einstein-Hilbert action supplemented by the GB term:

$$I = \frac{1}{16\pi G} \int_M d^5 x \sqrt{-g} \left( R + \frac{\alpha}{4} L_{GB} \right),$$

where $R$ is the Ricci scalar and

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}$$

(2.1)
is the GB term. Variation of the action (2.1) with respect to the metric tensor results in the equations of EGB gravity

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{\alpha}{4} H_{\mu\nu} = 0, \quad (2.3) \]

where

\[ H_{\mu\nu} = 2 (R_{\mu\sigma\kappa\tau} R_{\nu}^{\sigma\kappa\tau} - 2 R_{\mu\rho\sigma\kappa} R_{\nu}^{\rho\sigma} - 2 R_{\mu\sigma} R_{\nu}^{\sigma} + RR_{\mu\nu}) - \frac{1}{2} L_{GB} g_{\mu\nu}. \quad (2.4) \]

For a well-defined variational principle, one has to supplement the action (2.1) with the Gibbons-Hawking surface term

\[ I_b^{(E)} = -\frac{1}{8\pi G} \int_{\partial M} d^4 x \sqrt{-h} K, \quad (2.5) \]

and its counterpart for EGB gravity \[ I_b^{(GB)} = -\frac{\alpha}{16\pi G} \int_{\partial M} d^4 x \sqrt{-h} \left( J - 2G_{ab} K^{ab} \right), \quad (2.6) \]

where \( h_{ab} \) is the induced metric on the boundary, \( K \) is the trace of the extrinsic curvature of the boundary, \( G_{ab} \) is the Einstein tensor of the metric \( h_{ab} \) and \( J \) is the trace of the tensor

\[ J_{ab} = \frac{1}{3} (2K K_{ac} K_c^b + K_{cd} K^{cd} K_{ab} - 2K_{ac} K^{cd} K_{db} - K^2 K_{ab}). \quad (2.7) \]

An interesting feature of EGB gravity is the presence of two branches of static solutions, distinguished by their behaviour for \( \alpha \to 0 \) [10]. In this paper we shall restrict our analysis of rotating EGB black hole solutions to those, whose static limit corresponds to the branch of static solutions with a well defined Einstein gravity limit.

### 2.2 The ansatz and boundary conditions

While rotating EGB black holes will generically possess two independent angular momenta and a more general topology of the event horizon\(^1\) we restrict here to configurations with equal-magnitude angular momenta and a spherical horizon topology. A suitable metric ansatz reads [13]

\[ ds^2 = \frac{dr^2}{f(r)} + g(r) d\theta^2 + h(r) \sin^2 \theta (d\varphi_1 - w(r) dt)^2 + h(r) \cos^2 \theta (d\varphi_2 - w(r) dt)^2 \]
\[ + (g(r) - h(r)) \sin^2 \theta \cos^2 \theta (d\varphi_1 - d\varphi_2)^2 - b(r) dt^2, \]

where \( \theta \in [0, \pi/2], (\varphi_1, \varphi_2) \in [0, 2\pi] \), and \( r \) and \( t \) denote the radial and time coordinate, respectively. Also, this metric ansatz admits a simpler expression in terms of the left-invariant 1-forms \( \sigma_i \) on \( S^3 \), with

\[ ds^2 = \frac{dr^2}{f(r)} + \frac{1}{4} g(r) (\sigma_1^2 + \sigma_2^2) + \frac{1}{4} h(r) (\sigma_3 + 2w(r) dt)^2 - b(r) dt^2, \quad (2.9) \]

\(^1\)Recently, also static black rings were obtained in \( d = 5 \) EGB gravity by employing a numerical approach [14].
where $\sigma_1 = \cos \psi d\tilde{\theta} + \sin \psi \sin \theta d\phi$, $\sigma_2 = -\sin \psi d\tilde{\theta} + \cos \psi \sin \theta d\phi$, $\sigma_3 = d\psi + \cos \theta d\phi$ and $2\theta = \tilde{\theta}$, $\phi_1 - \phi_2 = \phi$, $\phi_1 + \phi_2 = \psi$.

For such solutions the isometry group is enhanced from $R \times U(1)^2$ to $R \times U(2)$, where $R$ denotes the time translation. This symmetry enhancement allows to factorize the angular dependence and thus leads to ordinary differential equations.

Without fixing a metric gauge, a straightforward computation leads to the following reduced action for the system

$$A_{\text{eff}} = \int \! dr dt \; L_{\text{eff}}, \quad \text{with} \quad L_{\text{eff}} = L_E + \frac{\alpha}{4} L_{\text{GB}}, \quad (2.10)$$

with

$$L_E = \sqrt{\frac{f h}{b}} \left( b' g' + \frac{g}{2 h} b' h' + \frac{b}{2 g} g'^2 + \frac{b}{h} g' h' + \frac{1}{2} g h w'^2 + \frac{2 b}{f} (4 - \frac{h}{g}) \right), \quad (2.11)$$

$$L_{\text{GB}} = \sqrt{\frac{f h}{b}} \left( \frac{4 h}{g} b' g' + 2(4 g - 3 h)(\frac{b' h'}{h} + h w'^2) - \frac{f}{2 h} b' h' g'^2 - \frac{1}{2} f h g'^2 w'^2 \right),$$

where a prime denotes a derivative with respect to $r$. The corresponding equations for the metric functions $b$, $f$, $h$ and $w$ are found by taking the variation of $A_{\text{eff}}$ with respect to $a$, $b$, $f$ and $g$ and by fixing afterwards the metric gauge. (This procedure is equivalent to solving the EGB equations directly, but it is technically simpler.) The equation for $w(r)$ admits the first integral

$$w' h \sqrt{\frac{f h}{b}} (g - \frac{\alpha}{4} \frac{f g'^2}{g} - 16 + \frac{12 h}{g}) = \text{const.}, \quad (2.12)$$

which is useful in the numerical calculations. The remaining equations are rather long, and we do not include them here. In the following we fix the metric gauge by taking $g(r) = r^2$.

Also, the EGB equations (2.13) imply the following relations which will be important in the later discussion

$$\frac{1}{\sin^2 \theta} (R_{\varphi}^t + \frac{\alpha}{4} H_{\varphi}^t) = \frac{1}{\cos^2 \theta} (R_{\psi}^t + \frac{\alpha}{4} H_{\psi}^t) = -\frac{1}{2 r^2} \sqrt{\frac{f}{b h}} \frac{d}{dr} \left[ \frac{\sqrt{f h}}{b} h w' \left( -r^2 + \frac{f - 4 + \frac{3 h}{r^2}}{f \frac{h}{r}} \right) \right],$$

$$R_t^t + \frac{\alpha}{4} (H_t^t + \frac{1}{2} L_{\text{GB}}) = \frac{1}{2 r^2} \sqrt{\frac{f}{b h}} \frac{d}{dr} \left[ \frac{\sqrt{f h}}{b} \left( r^2 h w' - b' \right) \right] + \alpha \left( \frac{f - 4 + \frac{h}{r^2} + \frac{r f h'}{h} b'}{r^2} \right) + (4 - \frac{3 h}{r^2} + f) h w' + r f h w'^2 \right] \right]. \quad (2.14)$$

The horizon of these black hole solutions is a squashed $S^3$ sphere. It resides at the constant value of the radial coordinate $r = r_H > 0$, and is characterized by $f(r_H) = 2$.

The numerical calculations are performed in two gauges. Besides the gauge $g(r) = r^2$, we also employ the gauge choice in (15, 19) (with $b(r) = f(r)$, $f = f(r)/m(r)$, $g(r) = r^2 m/f(r)$, $h(r) = m^2 f(r)$, and $w(r) = \bar{w}(r)/r$), corresponding to isotropic coordinates.
b(r_H) = 0. Restricting to nonextremal solutions, the following expansion holds near the event horizon:

\[
\begin{align*}
    f(r) &= f_1 (r - r_H) + O(r - r_H)^2, \\
    h(r) &= h_H + O(r - r_H), \\
    b(r) &= b_1 (r - r_H) + O(r - r_H)^2, \\
    w(r) &= \Omega_H + w_1 (r - r_H) + O(r - r_H)^2.
\end{align*}
\] (2.15)

For a given event horizon radius, the essential parameters characterizing the event horizon are \(f_1, \ b_1, \ h_H, \ \Omega_H\) and \(w_1\) (with \(f_1 > 0, \ b_1 > 0\), which fix all higher order coefficients in (2.15). (These constants are related in a complicated way to the global charges of the solutions.) The (constant) horizon angular velocity \(\Omega_H\) is defined in terms of the Killing vector \(\chi = \partial/\partial t + \Omega_1 \partial/\partial \varphi_1 + \Omega_2 \partial/\partial \varphi_2\) which is null at the horizon. For the solutions within the ansatz (2.8), the horizon angular velocities are equal, \(\Omega_1 = \Omega_2 = \Omega_H\).

A straightforward calculation gives the following asymptotic expansion for the metric functions, involving three parameters \(U, \ V\) and \(W\):

\[
\begin{align*}
    b(r) &= 1 + \frac{U}{r^2} + \left(2W^2 - UV + 3U^2 \alpha \right) \frac{1}{3r^6} + \left(U^2 V + U - 2W^2 \alpha\right) \frac{1}{3r^8} + O\left(\frac{1}{r^{10}}\right), \\
    f(r) &= 1 + \frac{U}{r^2} + \frac{V}{r^4} + \left(-W^2 - UV + U^2 \alpha \right) \frac{1}{r^6} + \left(\frac{23}{30} U(W^2 + UV) + \frac{14}{15} (W^2 - 3UV) \alpha\right) \frac{1}{r^8} + O\left(\frac{1}{r^{10}}\right), \\
    h(r) &= r^2 + \frac{V}{r^4} - \frac{W^2 + UV}{r^4} + \left(\frac{9U}{10} (W^2 + UV) + \frac{2}{5} (W^2 - 3UV) \alpha\right) \frac{1}{r^6} + \left(\frac{5U}{10} + 6U^2 (W^2 + UV) - 7U (W^2 - 3UV) \alpha\right) \frac{1}{r^8} + O\left(\frac{1}{r^{10}}\right), \\
    w(r) &= \frac{W}{r^4} - \frac{W(U - U \alpha)}{r^8} + O\left(\frac{1}{r^{10}}\right),
\end{align*}
\] (2.16)

which guarantees that the Minkowski spacetime background is approached at infinity.

### 2.3 Known solutions and slowly rotating black holes

For the metric ansatz (2.8), the EGB field equations (2.3) possess two well known exact solutions. First, the MP black holes [16] with equal-magnitude angular momenta are found for \(\alpha = 0\) (i.e., without GB term). Expressed in terms of the event horizon radius and the horizon angular velocity\(^3\) (which are the control parameters in our numerical approach), this solution reads

\[
\begin{align*}
    f(r) &= 1 - \frac{1}{1 - r_H^2 \Omega_H^2} \left(\frac{r_H}{r}\right)^2 + \frac{r_H^2 \Omega_H^2}{1 - r_H^2 \Omega_H^2} \left(\frac{r_H}{r}\right)^4, \\
    h(r) &= r^2 \left(1 + \left(\frac{r_H}{r}\right)^4 \frac{r_H^2 \Omega_H^2}{1 - r_H^2 \Omega_H^2}\right), \\
    b(r) &= 1 - \left(\frac{r_H}{r}\right)^2 \frac{1}{1 - (U \alpha r^4)^4} r_H^2 \Omega_H^2, \\
    w(r) &= \left(\frac{r_H}{r}\right)^4 \frac{\Omega_H}{1 - (U \alpha r^4)^4} r_H^2 \Omega_H^2. 
\end{align*}
\] (2.17)

Therefore, for a MP black hole, the relevant parameters in the event horizon expansion (2.15) are

\[
\begin{align*}
    f_1 &= \frac{2(1 - 2r_H^2 \Omega_H^2)}{r_H (1 - r_H^2 \Omega_H^2)}, \\
    b_1 &= \frac{2}{r_H} \frac{r_H^2 \Omega_H^2}{1 - 2r_H^2 \Omega_H^2}, \\
    w_1 &= \frac{4\Omega_H}{r_H} (r_H^2 \Omega_H^2 - 1),
\end{align*}
\] (2.18)

\(^3\)This metric is usually expressed in terms of the mass parameters \(M\) and the angular momentum parameter \(a\), with \(M = r_H^2/2(1 - r_H^2 \Omega_H^2)\) and \(a = r_H^2 \Omega_H\).
while the constants $U$, $V$ and $W$ in the far field expansion (2.16) have the following expression\(^4\)

\[
V = \frac{r_H^6 \Omega_H^2}{1 - r_H^2 \Omega_H^2}, \quad U = -\frac{r_H^2}{1 - r_H^2 \Omega_H^2}, \quad W = \frac{r_H^4 \Omega_H}{1 - r_H^2 \Omega_H^2}.
\] (2.19)

The $d = 5$ MP black holes with equal-magnitude angular momenta emerge smoothly from the static Schwarzschild-Tangherlini black hole when the event horizon velocity $\Omega_H$ is increased from zero. For a given event horizon radius, the solutions exist up to a maximal value of the horizon angular velocity $\Omega_H^{(c)} = \frac{1}{\sqrt{2}r_H}$. Expressed in terms of the mass-energy $E$ and the equal-magnitude angular momenta $|J_1| = |J_2| = J$, this bound reads $27\pi J^2 / 8G < E^2$. The extremal solution saturating this bound has a regular but degenerate horizon. Further details on the properties of MP black holes with equal-magnitude angular momenta are found in [25, 26].

The second exact solution corresponds to the generalization [10] of the static Schwarzschild solution with a GB term and reads (note our restriction to the branch with Einstein gravity limit)

\[
w(r) = 0, \quad h(r) = r^2 \quad \text{and} \quad f(r) = b(r) = 1 + \frac{r^2}{\alpha} \left(1 - \sqrt{1 + \frac{\alpha}{r^4}(\alpha + 2r_H^2)}\right).
\] (2.20)

This solution exists for all $r_H > 0$ and $\alpha > -r_H^2$ (with $r_H$ the event horizon radius). Without entering into details, let us mention the existence of some substantial differences between the thermodynamics of the static $d = 5$ EGB black hole solutions\(^5\) and their Einstein gravity counterparts (restricting to the physical case $\alpha > 0$). If the black holes are large enough, $r_H^2 > \alpha$, then they behave like their Einstein gravity counterparts since the specific heat is negative, $C_p = T_H(\partial S/\partial T_H) < 0$. A different behaviour is found for $r_H^2 / \alpha < 1$, since in that case $T_H \sim r_H$ and thus $C_p > 0$. This implies the existence of a branch of small five-dimensional EGB black holes that is thermodynamically stable [9] (see e.g. [5] for a review of these aspects).

To the best of our knowledge, there is no exact solution of the EGB equations describing non-static configurations\(^6\). However, slowly rotating black holes can be found by considering perturbation theory around the static solution (2.20) in terms of the rotation parameter $a$ (see e.g. [17]). For our case the slowly rotating solution then contains in its non-diagonal metric elements the function $w(r)$

\[
w(r) = \frac{a(2r_H^2 + \alpha)}{r^4 \left(1 + \sqrt{1 + \frac{\alpha(2r_H^2 + \alpha)}{r^4}}\right)},
\] (2.21)

that is linear in the perturbative parameter $a$, while the other metric functions remain unchanged to this order in $a$.

\(^4\)Note that since $V = -W^2 / \mathcal{U}$, there are only two free parameters in the far field expansion, which fix the mass and equal-magnitude angular momenta of the solutions.

\(^5\)For $d > 5$, the thermodynamical properties of the solutions are similar to the Einstein gravity case.

\(^6\)See however, the results in [15]. However, the solution there is very special (e.g. the value of the GB coupling constant is fixed by the cosmological constant) and does not describe a black object.
3. The boundary counterterm approach in $d = 5$ Einstein-Gauss-Bonnet theory and the global charges

3.1 A counterterm for the $d = 5$ asymptotically flat space

It is well known that the gravitational action contains divergences even at the tree-level – that arise from integrating over the infinite volume of spacetime. A common approach - background subtraction - uses a second, reference spacetime to identify divergences which should be subtracted from the action. After subtracting the (divergent) action of the reference background, the resulting action will be finite.

At a conceptual level, the background subtraction method is not entirely satisfactory, since it relies on the introduction of a spacetime which is auxiliary to the problem. In some cases the choice of reference spacetime is ambiguous - for example NUT-charged solutions (see e.g. the discussion in [27]). For asymptotically $AdS$ spacetimes, this problem is solved by adding additional surface terms to the action [28]. These counterterms are built up with curvature invariants of a boundary $\partial M$ (which is sent to infinity after the integration), and thus they do not alter the bulk equations of motion. This yields a finite action and mass of the system.

The generalization of this procedure to the asymptotically flat case was considered in [29, 30, 31]. Moreover, as discussed in [32], a renormalized stress-tensor can be defined by varying the total action (including the counterterms) with respect to the boundary metric. The conserved quantities can be constructed from this stress tensor via the algorithm of Brown and York [33].

However, all studies in the literature of asymptotically flat configurations have considered the case of counterterms in Einstein gravity only. We have found that for five-dimensional asymptotically flat EGB solutions with a boundary topology $S^3 \times R$, the action can be regularized by the following counterterm:

$$ I_{ct} = -\frac{1}{8\pi G} \int_{\partial M} d^4x \sqrt{-h} \Psi(\mathcal{R}), \quad (3.1) $$

where $\mathcal{R}$ is the Ricci scalar of the induced metric on the boundary $h_{ij}$ and

$$ \Psi(\mathcal{R}) = \sqrt{\frac{3}{2}} \mathcal{R} \left(1 + \frac{\alpha}{9} \mathcal{R}\right). \quad (3.2) $$

For $\alpha = 0$, this reduces to the counterterm expression proposed in [31] for Einstein gravity with the same asymptotic structure.

Varying the total action (which contains the Gibbons-Hawking boundary term) with respect to the boundary metric $h_{ij}$, we obtain the divergence-free boundary stress-tensor

$$ T_{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h_{ab}} = \frac{1}{8\pi G} \left( K_{ab} - h_{ab}K + \frac{\alpha}{2} (Q_{ab} - \frac{1}{3} Q h_{ab}) - 2 \frac{d\Psi}{d\mathcal{R}} \mathcal{R}_{ab} + 2 \Psi h_{ab} - 2 h_{ab} \Box \Psi + \Psi_{ab} \right), \quad (3.3) $$

where $K_{ab}$ is the extrinsic curvature of the boundary and

$$ Q_{ab} = 2K K_{ac} K^c_b - 2K_{ac} K^{cd} K_{db} + K_{ab}(K_{cd} K^{cd} - K^2) + 2K \mathcal{R}_{ab} + \mathcal{R} K_{ab} - 2K^{cd} \mathcal{R}_{cadb} - 4 \mathcal{R}_{ac} K^c_b, $$

The general expression of the counterterms and the boundary stress tensor in EGB theory with a cosmological constant is presented in [35, 36].
while $R_{abcd}$, $R_{ab}$ denote the Riemann and Ricci tensors of the boundary metric.

Provided the boundary geometry has an isometry generated by a Killing vector $\xi^i$, a conserved charge

$$\Omega_\xi = \oint_\Sigma d^3S^i \xi^j T_{ij}$$

(3.4)

can be associated with a closed surface $\Sigma$. Physically this means that a collection of observers on the hypersurface with metric $h_{ij}$ all observe the same value of $\Omega_\xi$, provided this surface has an isometry generated by $\xi$.

To test the counterterm (3.1), we have verified that the known expressions for the mass and action are recovered for the case of the Schwarzschild-Tangherlini solution in EGB. Neglecting the second term in (3.2) (i.e. considering only the Einstein gravity counter-term) results in the occurrence of an unphysical constant term proportional with $\alpha$ in the expressions of $I_d$ and $E$.

However, one should remark that the proposal (3.1) inherits all ambiguities already present in the Einstein gravity case. In particular, there is no rigorous justification for considering that expression, and, in fact the results in [31] show that the counterterm choice in Einstein gravity is not unique. This suggests that other more complicated expressions are possible for the EGB case as well. Moreover, unlike the AdS case, the counterterm depends on the boundary topology (for example, we have verified that the coefficient in front of (3.1) is different for black string solutions).

These problems are avoided in the proposal put forward by Mann and Marolf in [37] for asymptotically flat spacetimes and further generalized in [38] to extended objects, like black strings and $p$–brane spacetimes. In this approach, the conserved quantities are constructed essentially using the electric part of the Weyl conformal tensor. The generalization of the Mann and Marolf prescription to Lovelock gravity solutions is an interesting open problem.

### 3.2 The action and global charges of solutions

The computation of the boundary stress-tensor $T_{ab}$ for the solutions of interest in this work is straightforward, and, for the asymptotic expression (2.16), we find the following expressions for the relevant components

$$T_{t\varphi_1} = \frac{1}{8\pi G} \frac{2W \sin^2 \theta}{r^3} + O(1/r^5), \quad T_{t\varphi_2} = \frac{1}{8\pi G} \frac{2W \cos^2 \theta}{r^3} + O(1/r^5), \quad T_t = \frac{1}{8\pi G} \frac{3U}{2r^3} + O(1/r^5).$$

(3.5)

The energy $E$ and the equal-magnitude angular momenta $J$ are the charges associated with the Killing vectors $\partial/\partial t$, $\partial/\partial \varphi_1$, and $\partial/\partial \varphi_2$, respectively. Computed according to (3.4) these quantities are

$$E = -\frac{3V_3}{16\pi G} U, \quad J = \frac{V_3}{8\pi G} W,$$

(3.6)

where $V_3 = 2\pi^2$ denotes the area of the unit three-dimensional sphere.

\[8\] However, the expressions for the regularized action and the conserved quantities will be the same for any counterterm choice.
Other quantities of interest are the Hawking temperature \( T_H = 1/\beta \) and the area \( A_H \) of the black hole horizon

\[
T_H = \frac{\sqrt{b_1 f_1}}{4\pi}, \quad A_H = \sqrt{h_H r_H^2} V_3. \tag{3.7}
\]

The thermodynamics of the EGB black holes can be formulated via the path integral approach \[33, \, 41\]. Following the standard prescription, one computes the classical bulk action evaluated on the equations of motion, by replacing the \( R + \frac{\alpha}{4} L_{GB} \) volume term with

\[
2(R_t^t + \frac{\alpha}{4}(H_t^t + \frac{1}{2} L_{GB})). \tag{2.14}
\]

Then one makes use of (2.14) to express the volume integral of this quantity as the difference of two boundary integrals. The boundary integral on the event horizon is simplified by using the identity (2.13) which provides the following relation between the asymptotic parameter \( W \) (which fixes the equal-magnitude angular momenta) and the horizon data which enter (2.15):

\[
W = \frac{1}{4} h_H \sqrt{\frac{f_1 h_H}{b_1}} w_1(-r_H^2 + \alpha(\frac{3h_H}{r_H^2} - 4)). \tag{3.8}
\]

A straightforward calculation using the asymptotic expressions (2.16) shows that the divergences of the boundary integral at infinity, together with the contributions from \( I_b^{(E)} \) and \( I_b^{(GB)} \), are regularized by \( I_{cl} \). As a result, one finds the following finite expression for the classical action in terms of the event horizon data and the asymptotic parameters \( U \) and \( W \)

\[
I_{cl} = \frac{V_3}{4G} \left( - \left( \frac{\sqrt{h_H r_H^2}}{b_1} + \frac{3U}{\sqrt{b_1 f_1}} + \frac{4\Omega_H W}{\sqrt{b_1 f_1}} \right) + \frac{1}{\sqrt{r_H^2}} (h_H - 4r_H^2) \right). \tag{3.9}
\]

We have verified that a similar result for \( I_{cl} \) is obtained when using instead the standard background subtraction regularization procedure (with a Minkowski spacetime background).

Upon application of the Gibbs-Duhem relation to the partition function, one finds the entropy

\[
S = \beta(E - 2\Omega_H J) - I_{cl}, \tag{3.10}
\]

which is the sum of one quarter of the event horizon area (the Einstein gravity term) plus a GB correction

\[
S = S_E + S_{GB}, \quad \text{with} \quad S_E = \frac{V_3}{4G} r_H^2 \sqrt{h_H}, \quad S_{GB} = \alpha \frac{V_3}{4G} \sqrt{h_H} (4 - \frac{h_H}{r_H^2}). \tag{3.10}
\]

It is interesting to note that the above expression for the entropy can also be written in Wald’s form \[8\] as an integral over the event horizon

\[
S = \frac{1}{4G} \int_{\Sigma_h} d^3x \sqrt{\hat{h}} (1 + \frac{\alpha}{2} \hat{R}), \tag{3.11}
\]

(\( \hat{h} \) is the determinant of the induced metric on the horizon and \( \hat{R} \) is the event horizon curvature).
4. The rotating EGB solutions

4.1 Non-extremal black holes

4.1.1 General features

In the absence of closed form solutions, we relied on numerical methods to solve the EGB equations. In this work\(^9\), we integrated the system of coupled non-linear ordinary differential equations with appropriate boundary conditions which follow from (2.15), (2.16), by using a standard solver \(^{11}\). This solver involves a Newton-Raphson method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure. Typical mesh sizes include \(10^2 - 10^3\) points. The solutions in this work have a typical relative accuracy of \(10^{-6}\).

In our approach, the input parameters\(^{10}\) are the GB coupling constant \(\alpha\), the event horizon radius \(r_H\) and the horizon angular velocity \(\Omega_H\) (or equivalently, the equal-magnitude angular momenta \(J\) through the parameter \(W\)). Physical quantities characterizing the solutions can then be extracted from the numerical solutions.

For most of the analysis, we set \(r_H = 1\), which does not spoil the generality of the results, since it corresponds to fixing the scale of the problem. Also, to simplify the problem, we restrict our integration to the region outside the event horizon\(^{11}\).

In constructing rotating EGB black holes, we made use of the existence of the closed form solutions (2.17) and (2.20), and employed them as starting configurations, when increasing gradually \(\alpha\) or \(\Omega_H\), respectively. Our results clearly show that any such MP solution admits generalizations in EGB theory. When starting instead from the EGB static black holes (2.20), a corresponding branch of rotating EGB solutions emerges smoothly for any value of the event horizon radius \(r_H\).

The profiles of the metric functions of typical EGB black hole solutions are presented in Figure 1a for a static \(\Omega_H = 0\) and a rotating solution with \(\Omega_H = 0.68\) (where both solutions have the same value of the GB coupling constant). The effects of the GB coupling constant are shown in Figure 1b, where rotating solutions for two values of \(\alpha\) and the same horizon angular velocity are shown (here the solution with \(\alpha = 0.01\) is very close to the MP configuration). One can see that, apart from a nonzero \(w(r)\), the rotation leads to non-constant values for \(h(r)/r^2\), while a nonzero \(\alpha\) leads to nontrivial deformations of all metric functions.

These rotating EGB solutions possess also an ergoregion inside of which the observers cannot remain stationary, and will move in the direction of the rotation. The ergoregion is bounded by the event horizon, located at \(r = r_H\) and the stationary limit surface, or ergosurface, located at \(r = r_e\), where the Killing vector \(\partial/\partial t\) becomes null, i.e. \(g_{tt} = -b(r_e) + h(r_e)w(r_e)^2 = 0\). One can see that the ergosurface does not intersect the horizon.

\(^{9}\)The numerical methods here are similar to those used in literature to find numerically black hole solutions with equal-magnitude angular momenta in Einstein-Maxwell theory \([14,21,22]\).

\(^{10}\)We have assumed that \(\Omega_H > 0\), which can always be achieved by \(t \to -t\) if necessary. Also, in string theory, the GB coefficient \(\alpha\) is positive, which is the only case considered here.

\(^{11}\)However, similar to the static case, the GB term is expected to drastically affect the geometry in the region inside the horizon of the rotating black holes (see e.g. \([21]\)).
The effect of the GB term is to increase the size of the ergoregion. For a fixed event horizon radius $r_H = 1$ and $\Omega_H = 0.33$ one finds $e.g.$ $r_c(\alpha = 0) \simeq 1.059$, $r_c(\alpha = 1) \simeq 1.08$ and $r_c(\alpha = 2) \simeq 1.104$.

Another interesting feature concerns the sign of the quantity $\rho_{\text{eff}} = \frac{4}{\alpha^4} H_{tt}$ which, via the modified Einstein equations $G_{\mu\nu} = -\frac{4}{\alpha^4} H_{\mu\nu} = T^\text{eff}_{\mu\nu}$, corresponds to a local ‘effective energy density’. This effective stress tensor, thought of as a kind of matter distribution, in principle may violate the weak energy condition. Indeed, this is the case for both black string [13] and black ring [14] solutions in EGB theory. However, we have found that similar to the case of the static spherically symmetric black holes (2.20), the ‘effective energy density’ is a strictly positive quantity for all rotating solutions investigated. This result together with those in [13, 14] leads to the conjecture that in order to have $\rho_{\text{eff}} > 0$ the solutions should possess a spherical topology of the event horizon and be cohomogeneity-1.

4.1.2 Domain of existence

Because EGB black holes are found starting both with the MP configurations and with the Schwarzschild-GB static solutions, we conclude that, for fixed $r_H$, rotating EGB black holes should exist in a given domain of the $(\alpha, \Omega_H)$ plane. To map out this domain, we fixed certain values of $\alpha > 0$ and increased $\Omega_H$ gradually. We then obtained numerical solutions up to a maximal value of the horizon angular velocity $\Omega_H^{(c)}$, that depends on $\alpha$.

The solutions are numerically robust but the integration becomes difficult, when the maximal value $\Omega_H^{(c)}$ is approached. However, for any $\alpha > 0$, no critical phenomenon (like $e.g.$ a bifurcation or the approach to a singular point) seems to arise there.

In order to clarify the issue of the limiting solutions for $\Omega_H \rightarrow \Omega_H^{(c)}$ for fixed $\alpha$, the study of the event horizon values $b'(r_H) = b_1$ and $f'(r_H) = f_1$ as functions of $r_H$ turned

\[ \text{Figure 1: The metric functions } f(r), b(r), w(r) \text{ and } h(r)/r^2 \text{ of typical EGB black hole solutions are shown for (a) two values of the horizon angular velocity } \Omega_H \text{ and (b) two values of the Gauss-Bonnet coupling constant } \alpha. \]
Figure 2: The ‘reduced’ dimensionless physical quantities (a) horizon area, (b) temperature, (c) horizon angular velocity, and (d) ratio between the Gauss-Bonnet and the Einstein gravity contributions to the total entropy are shown versus the dimensionless squared (equal-magnitude) angular momenta for several values of the Gauss-Bonnet coupling constant. The dots represent the data points. The spline-interpolated curves have been extrapolated to the extremal endpoints.

out to be crucial.

For $\Omega_H = 0$, we know that $b'(r_H) = 2r_H/(r_H^2 + \alpha)$, as can be seen from the explicit solution (2.20). Then, increasing $\Omega_H$, our numerical results show that $b'(r_H)$ and $f'(r_H)$ both decrease monotonically. This strongly suggests that they reach the value zero in the limit $\Omega_H \to \Omega_H^{(c)}$.

From these observations we conclude, that the families of rotating EGB black holes terminate at extremal configurations. All relevant quantities remain finite as $\Omega_H \to \Omega_H^{(c)}$, while the Hawking temperature vanishes in the limit. Moreover, we evaluated a number of curvature invariants (e.g. the scalar curvature $R$ and the Kretschmann scalar), and our extrapolations showed that these stay finite everywhere in that limit.

This behaviour is thus analogous to that of MP black holes in the extremal limit, where $\Omega_H^{(c)} = 1/\sqrt{2r_H}$. We have found this picture for any value of $\alpha$ that we considered.
Therefore we conjecture that rotating black holes with equal-magnitude angular momenta exist for any value of the GB coupling constant.

To illustrate these aspects, we show in Figure 2 the behaviour of the ‘reduced’ area of the horizon \( a_H \), the ‘reduced’ temperature \( t_H \), the ‘reduced’ horizon angular velocity \( \omega_H \), and the ratio \( S_{\text{GB}}/S_E \) as functions of the squared ‘reduced’ (equal-magnitude) angular momenta \( j^2 \) for several values of the GB coupling constant \( \alpha \). These ‘reduced’ dimensionless quantities are defined as follows

\[
\begin{align*}
    a_H &= \frac{3}{32} \sqrt{\frac{3}{2\pi G^3}} \frac{A_H}{E^{3/2}}, \\
    t_H &= 4 \sqrt{\frac{2\pi G}{3}} T_H \sqrt{E}, \\
    \omega^2_H &= \frac{8G}{3\pi} \Omega^2_H E, \\
    j^2 &= \frac{27\pi}{8G} J^2 E^{3/2}.
\end{align*}
\]

One can see that the pattern in Einstein gravity\(^\text{13}\) is recovered for all values of \( \alpha \) considered. A nonvanishing GB term decreases, however, the maximal values of the dimensionless quantities \( j^2 \), \( a_H \), and \( t_H \), while it decreases the maximal value of \( \omega_H \).

To determine the domain of existence of the black holes with respect to the GB coupling constant \( \alpha \), we exhibit in Figure 3 the values of \( \Omega_H \), \( J \) and \( E \) corresponding to critical solutions as functions of \( \alpha \). (Note, that these quantities are normalized there with respect to the corresponding ones for the MP solution). The effect of the GB interaction is clearly perceptible on the diagram. One should note that all these quantities have a nontrivial dependence on \( \alpha \).

We conjecture that the domain of existence of rotating EGB black hole is the region below the (spline-interpolated) curve for \( \Omega_H^{(c)} \) (or equivalently below the (spline-interpolated) line for the equal-magnitude angular momenta \( J^{(c)} \).

\(^{13}\)The following relations hold for the solutions of Einstein gravity

\[
\begin{align*}
    a_H(j^2) &= \frac{1}{2} \left( 1 + \sqrt{1 - j^2} \right), \\
    t_H(j^2) &= 2 \left( 1 - \frac{1 - \sqrt{1 - j^2}}{j^2} \right), \\
    \omega^2_H(j^2) &= \frac{2(1 - \sqrt{1 - j^2})}{j^2} - 1.
\end{align*}
\]

Figure 3: The critical values of \( \Omega_H \), \( J \) and \( E \) corresponding to extremal limiting solutions are shown as a function of the Gauss-Bonnet coupling constant. The dots represent the data points, the curves are obtained by spline-interpolation.
To control the quality of these results, we have performed (a number of) these numerical calculations in two different gauges, finding excellent agreement for the physical parameters of the rotating EGB black holes. Our systematic analysis was limited to $\alpha \in [0, 3]$, because the GB term is supposed to emerge as a correction to the Einstein lagrangian; however, we also found families of solutions exhibiting the same pattern for larger values of $\alpha$.

4.1.3 Thermodynamical properties

Considering the thermodynamics of these solutions, the EGB black holes should satisfy the first law of thermodynamics

$$dE = T_H dS + 2\Omega_H dJ.$$  \hspace{1cm} (4.3)

One may regard the parameters $S, J$ as a complete set of extensive parameters for the mass-energy $E(S, J)$ and define the intensive parameters conjugate to them. These quantities are the temperature and the angular velocities. Also, for $\alpha \neq 0$, these solutions do not appear to satisfy any simple Smarr relation\textsuperscript{14}.

In the absence of an exact solution, we attempt here to analyze the thermodynamic stability of the rotating EGB solutions based on the available numerical data. Since we did not yet explore the full parameter space of solutions, the results below are only partial, and one cannot exclude the existence of new features outside the explored domain.

It is known that different thermodynamic ensembles are not exactly equivalent and may not lead to the same conclusions since they correspond to different physical situations

\textsuperscript{14}The Smarr relation in Einstein gravity is $E = 3(T_H S + 2\Omega_H J)/2$. The recent work \cite{43} proposes a Smarr relation for static black holes in Lovelock gravity. It would be interesting to extend the formalism in \cite{43} to the case of spinning solutions.
The entropy $S$ and mass-energy $E$ are shown vs. the Hawking temperature $T_H$ for fixed horizon angular velocity $\Omega_H$ and several values of the Gauss-Bonnet coupling constant.

In the canonical ensemble, we study black holes holding the temperature $T_H$ and the angular momenta $J$ fixed, the associated thermodynamic potential being the Helmholtz free energy $F[T_H, J] = E - T_H S$. In this case, the numerical analysis for several values of $\alpha$ indicates that the qualitative thermodynamical features of the MP solutions are also shared by their EGB generalizations. The configurations near extremality (i.e. with a small enough $T_H$) are thermally stable in a canonical ensemble since $C_J = T_H (\partial S / \partial T_H)|_J > 0$. However, there is also a branch of large black holes whose entropy is a decreasing function of $T_H$.

At the critical point, the specific heat goes through an infinite discontinuity, and a phase transition takes place. This is the picture one finds for the MP solution in Einstein gravity, in which case the specific heat changes the sign for $T_H^c \simeq 0.087396/(GJ)^{1/3}$. Interestingly, for a given $J$, the effect of the GB term is to decrease the critical value of the Hawking temperature.

In the grand canonical ensemble, on the other hand, we keep the temperature and the horizon angular velocity fixed. In this case the thermodynamics is obtained from the Gibbs potential $G[T_H, \Omega_H] = E - T_H S - 2 \Omega_H J = I_{cl}/\beta$. The first quantity of interest here is the specific heat at constant horizon angular velocity $C_{\Omega} = T_H (\partial S / \partial T_H)|_{\Omega_H > 0}$. A straightforward computation shows that for the Myers-Perry black hole this is a negative quantity

$$C_{\Omega} = -\frac{1}{4\pi GT_H^4} \frac{\sqrt{2 + x^2 - 2\sqrt{1 + x^2}(2 + 2x^2 + \sqrt{1 + x^2})}}{x^2(1 + x^2)^{3/2}(1 + x^2 + \sqrt{1 + x^2})} < 0, \quad \text{with} \quad x = \frac{\Omega_H}{\pi T_H}. \quad (4.4)$$
However, not completely unexpected\(^{15}\), our numerical results show the existence a branch of rotating black holes with \(C_\Omega > 0\) for any value of \(\alpha > 0\) considered.

To illustrate this feature, we show in Figure 5 the entropy and the mass-energy of solutions as functions of the Hawking temperature for several values of the GB coupling constant and an arbitrary fixed value of the horizon angular velocity (for better visualization, we used a logarithmic scale for \(S\)). We observe that the critical temperature where \(C_\Omega\) changes sign decreases with increasing \(\alpha\), while the Einstein gravity picture is recovered for large black holes, where \(C_\Omega < 0\).

Another response function of interest is the isothermal permittivity \(\epsilon_{T_H} = (\partial J/\partial \Omega_H)|_{T_H}\). The Einstein gravity result is
\[
\epsilon_{T_H} = \frac{V_3}{16\pi^5 G T_H} \left( \frac{5 + 6x^2 - \sqrt{1 + x^2}(5 + 3x^2)}{x^2(1 + x^2)^{3/2}(1 + \sqrt{1 + x^2})^2} \right),
\]
where \(x = \Omega_H/\pi T_H\), (4.5)
and thus \(\epsilon_{T_H} > 0\) for \(\Omega_H/\pi T_H < 0.77272\). Our results for \(\alpha = 0.1\) and \(\alpha = 1\) show that \(\epsilon_{T_H}\) changes sign also for EGB solutions and that, for a given \(T_H\), the solutions with a small \(\Omega_H\) have a positive isothermal permittivity (see Figure 6). Thermodynamic stability in the grand-canonical ensemble requires that the specific heat at constant angular momentum \(C_J\), the isothermal permittivity \(\epsilon_{T_H}\) as well as the specific heat at constant horizon angular velocity \(C_\Omega\) are positive. Although a systematic study of the domain of thermodynamic stability of the EGB solutions is beyond the purposes of this paper, our results show the existence of solutions which fulfill these conditions. The stability conditions hold for a set of rotating solutions emerging from small static black holes (that have positive specific heat).

We expect that this result is relevant for the properties of the corresponding \(d = 6\) rotating black string solutions in EGB theory (note that such solutions cannot be found

\[15\] We recall the existence of a branch of small static EGB black holes that is thermodynamically stable.
by simply uplifting the black holes in this paper). For example, the results in [17] indicate that the Gregory-Laflamme instability [18] persists up to extremality for all $d = 6$ Einstein gravity black strings with equal magnitude angular momenta. Based on the Gubser-Mitra conjecture [19] that correlates the dynamical and the thermodynamical stability, we conjecture that in a certain region of parameter space there are spinning EGB solutions which do not possess a Gregory-Laflamme instability. Also, not completely unrelated, it would be interesting to study the thermodynamic stability of these solutions from the perturbative corrections to the gravitational partition function (see [20] for a recent similar study of the Kerr-AdS black holes).

We conclude that the presence of a GB term in the lagrangian affects the thermodynamical properties of the solutions and allows for thermodynamically stable solutions, both in the macrocanonical and the canonical ensemble.

4.2 Extremal solutions: near horizon geometry and the entropy function

Returning to the issue of the extremal solutions, we recall that the numerical integration in the neighbourhood of extremal black holes is very difficult, as noted also in other cases. The near horizon expansion of extremal configurations is still given by (2.13), with $f_1 = b_1 = 0$, and thus they satisfy a different set of boundary conditions at $r = r_H$ than the nonextremal configurations obtained in this work. Therefore finding such solutions explicitly is beyond the scope of this paper\textsuperscript{16}. Most likely, this should require a different parametrization for the metric, better suited to the extremal case.

However, we argue that the existence of the EGB generalizations of the extremal MP black holes for any $\alpha > 0$ is strongly suggested by the existence of an exact solution describing a rotating squashed $AdS_2 \times S^3$ spacetime. This solution would describe the neighbourhood of the event horizon of an extremal black hole. (The far field expression of the extremal solution is still given by (2.16), with a single essential parameter in the expansion there.)

Therefore we consider the following metric form (see the generic ansatz (2.9)) in corotating coordinates

$$ds^2 = v_1\left(\frac{dr^2}{r^2} - r^2 dt^2\right) + \frac{v_2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{v_2v_3}{4}(\sigma_3 + 2krdt)^2$$  

(4.6)

(i.e. for $b(r) = v_1r^2$, $f(r) = r^2/v_1$, $g(r) = v_2$, $h(r) = v_2v_3$, $w(r) = kr$ within the parametrization in this paper), such that the horizon is located\textsuperscript{17} at $r = 0$. This geometry describes a fibration of $AdS_2$ over the homogeneously squashed $S^3$ with symmetry group $SO(2,1) \times SU(2) \times U(1)$ [23].

The parameters $v_i, k$ satisfy a set of algebraic relations which result from the EGB equations. In what follows we choose to determine them by using the formalism proposed in [24], thus by extremizing an entropy function. This allows us also to compute the entropy of these black holes and to show that the solutions exhibit attractor behaviour.

\textsuperscript{16}Note, however, that we could construct with relatively good accuracy near-extremal black holes.

\textsuperscript{17}This position of the horizon can always be obtained by taking $r \rightarrow r - r_H$. 
Therefore let us denote by \( f(k, \vec{v}) \) the lagrangian density \( \sqrt{-g}L \) evaluated for the near horizon geometry (1.6) and integrated over the angular coordinates,

\[
f(k, \vec{v}) = \int d\bar{\theta}d\phi d\psi \sqrt{-g}L = \frac{1}{16\pi G} \int d\bar{\theta}d\phi d\psi \sqrt{-g}(R + \frac{\alpha}{4}L_{GB}). \tag{4.7}
\]

The metric field equations in the near horizon geometry (1.6) now correspond to \( \frac{\partial f}{\partial k} = J, \quad \frac{\partial f}{\partial v_i} = 0 \), with \( J \) the angular momenta of the solutions.

Then, following [24], we define the entropy function by taking the Legendre transform of the above integral with respect to the parameter \( k \),

\[
E(J, k, \vec{v}) = 2\pi \left( Jk - f(k, \vec{v}) \right). \tag{4.8}
\]

It follows as a consequence of the equations of motion that the constants \( k, \vec{v} \) are solutions of the equations

\[
\frac{\partial E}{\partial k} = 0, \quad \frac{\partial E}{\partial v_i} = 0. \tag{4.9}
\]

Then, the entropy associated with the black hole is given by \( S_{\text{extremal}} = E(J, k, \vec{v}) \) evaluated at the extremum (4.9). Further details on this formalism and explicit examples are given e.g. in [51], [52], [54].

For the metric ansatz (4.6), a straightforward calculation gives

\[
E(J, k, \vec{v}) = 2\pi \left[ Jk - \frac{\pi \sqrt{v_2 v_3}}{16Gv_1} \left( k^2 v_2^2 v_3 - 4v_1^2 (v_3 - 4) - 4v_1 v_2 - \alpha (k^2 v_2 v_3 (3v_3 - 4) - 4v_1 (v_3 - 4)) \right) \right], \tag{4.10}
\]

such that the explicit form of the equations (4.9) is

\[
\begin{align*}
\frac{\partial E}{\partial v_1} &= 0 \Rightarrow -16v_1^2 + 4v_1^3 + k^2 v_2^2 v_3 + \alpha k^2 v_2 v_3 (4 - 3v_3) = 0, \tag{4.11} \\
\frac{\partial E}{\partial v_2} &= 0 \Rightarrow -16v_2^2 + 12v_1 v_2 + 4v_1^2 v_3 - 5k^2 v_2^2 v_3 + \alpha (-4v_1 (v_3 - 4) + 3k^2 v_2 v_3 (3v_3 - 4)) = 0, \\
\frac{\partial E}{\partial v_3} &= 0 \Rightarrow -16v_1^2 + 4v_1 v_2 + 12v_1^2 v_3 - 3k^2 v_2^2 v_3 + \alpha (-4v_1 (3v_3 - 4) + 3k^2 v_2 v_3 (5v_3 - 4)) = 0,
\end{align*}
\]

and

\[
\frac{\partial E}{\partial k} = 0 \Rightarrow J = \frac{\pi k (v_2 v_3)^{3/2}}{8Gv_1} (v_2 + \alpha (4 - 3v_3)). \tag{4.12}
\]

In Einstein gravity the solution has a simple form in terms of \( J \)

\[
v_1 = \frac{1}{2} \left( \frac{GJ}{2\pi} \right)^{2/3}, \quad v_2 = \left( \frac{\sqrt{2}GJ}{\pi} \right)^{2/3}, \quad v_3 = 2, \quad k = \frac{1}{2}, \tag{4.13}
\]

and thus the known result \( E = S_{\text{extremal}} = \pi J \) is recovered.
In the limit of small $\alpha$, one can treat the GB term as a perturbation. In second order in $\alpha$, one finds the following solution of the system (4.11), (4.12) in terms of the global charge $J$

$$v_1 = \frac{1}{2} \left( \frac{GJ}{2\pi} \right)^{2/3} - \frac{\alpha}{6} + \left( \frac{\pi}{\sqrt{2GJ}} \right)^{3/2} \frac{\alpha^2}{9}, \quad v_2 = \left( \frac{\sqrt{2GJ}}{\pi} \right)^{2/3} + \frac{4}{3} \alpha + \left( \frac{2\pi}{GJ} \right)^{3/2} \frac{2\alpha^2}{9},$$

$$v_3 = 2 - 2 \left( \frac{2\pi}{GJ} \right)^{2/3} - \left( \frac{\pi}{GJ} \right)^{4/3} \frac{2^{1/3} 16\alpha^2}{3}, \quad k = \frac{1}{2} + \left( \frac{\pi}{\sqrt{2GJ}} \right)^{2/3} \frac{\alpha}{2} + \left( \frac{\pi}{GJ} \right)^{4/3} \frac{\alpha^2}{2^{2/3} 12},$$

while

$$\mathcal{E} = \pi J + 3 \left( \frac{J \pi^5}{2G^2} \right)^{1/3} \alpha - \left( \frac{\pi^7}{4G^4 J} \right)^{1/3} \frac{\alpha^2}{2}. \quad (4.15)$$

Therefore the approximate solutions exhibit a complicated behaviour in terms of $\alpha$ and $J$ and a perturbative approach may be misleading.

Unfortunately, for a nonzero GB term in the action, it seems that the only possibility is to express the nonperturbative solution of the system (4.11), (4.12) in terms of the relative squashing parameter $v_3$, with

$$v_1 = - \frac{(v_2 + (4 - 3v_3)\alpha)(3v_2 - (v_3 - 4)\alpha)}{2(v_3 - 4)(3v_2 + (8 - 6v_3)\alpha)}, \quad (4.16)$$

$$J = \frac{\pi v_2 v_3}{G} \sqrt{(4 - v_3)(v_2 + \alpha(4 - 3v_3))}, \quad (4.17)$$

and

$$k = \frac{8GJv_1}{\pi(v_2 v_3)^{3/2}(v_2 + (4 - 3v_3)\alpha)}, \quad (4.18)$$

while the radius of the round $S^2$ sphere in the line element (4.4) is given by

$$v_2 = \frac{\alpha}{v_3 - 2} \left( 2v_3^2 - 7v_3 + 4 - \sqrt{5v_3^4 - 34v_3^3 + 73v_3^2 - 56v_3 + 16} \right). \quad (4.19)$$

Also, we notice that the relations (4.16)-(4.19) are invariant under the scaling

$$v_1 \to \lambda v_1, \quad v_2 \to \lambda v_2, \quad \mathcal{E} \to \lambda^{3/2} \mathcal{E}, \quad J \to \lambda^{3/2} J, \quad k \to k, \quad \text{and} \quad \alpha \to \lambda \alpha,$$

which shows that the solutions exist for any $\alpha \geq 0$.

Inserting these expressions into Eq. (4.10) we obtain for the entropy function of the extremal black hole:

$$\mathcal{E} = S_{\text{extremal}} = \frac{\pi^2}{2G} \sqrt{v_2 v_3 (v_2 - (v_3 - 4)\alpha)}, \quad (4.20)$$

(with $v_2(v_3)$ as implied by (4.19)).

---

18Here we restrict to the physical solution which recovers the general relativity limit as $\alpha \to 0$. 

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The dimensionless quantities $E/\alpha^{3/2}$, $k$, $v_1/\alpha$, $v_2/\alpha$ and $v_3$ are shown as functions of $J/\alpha^{3/2}$ (for a better visualisation, we employ here logarithmic scales).

As a check, we note that the result (4.20) agrees with Wald’s form (3.11) evaluated for the near horizon geometry (4.17). We also note that, in principle, $E$ can be expressed in terms of the conserved charge $J$ by inverting relation (4.17).

We conclude that a nonzero $\alpha$ may substantially affect the near horizon geometry of an extremal black hole. For example, the allowed range\footnote{This results from the physical condition that $v_i, J^2$ are strictly positive quantities.} for the relative squashing parameter $v_3$ is $0 < v_3 \leq 2$ (i.e. the GB term reduces the relative squashing of the solutions), with

$$v_1 = \frac{\alpha}{4} + \ldots, \quad v_2 = \frac{\alpha v_3^2}{2} + \ldots, \quad J = \frac{\pi \alpha^{3/2} v_3^3}{2G} + \ldots, \quad k = \frac{1}{\sqrt{2v_3^{3/2}}} + \ldots, \quad E = \frac{\sqrt{\pi} G}{G} (\alpha v_3)^{3/2} + \ldots,$$

as $v_3 \to 0$, and

$$v_1 = \frac{\alpha}{2 - v_3} + \ldots, \quad v_2 = \frac{4\alpha}{2 - v_3} + \ldots, \quad J = \frac{4\sqrt{2 \pi}}{G} \left( \frac{\alpha}{2 - v_3} \right)^{3/2} + \ldots, \quad k = \frac{1}{2} + \ldots, \quad E = \frac{4\sqrt{2 \pi} G}{G} \left( \frac{\alpha}{2 - v_3} \right)^{3/2} + \ldots,$$

as $v_3 \to 2$. In Figure 7 we exhibit a number of relevant dimensionless quantities as functions of the scaled angular momenta $J/\alpha^{3/2}$ (where for better visualisation, we use logarithmic scales). We observe that the ratio $E/J$ is no longer constant in EGB gravity.

However, finding local solutions in the vicinity of the horizon does not guarantee the existence of global asymptotically flat solutions. Further progress in this direction seems to require an explicit construction of the bulk extremal black hole solutions. For example, this would also allow to construct the $E(J)$ diagram for such configurations.

We close this Section by remarking that the study of these $AdS_2 \times S^3$ solutions in EGB theory is interesting in yet another context. In ref. \cite{ref55} it has been proposed that the near horizon geometry of an extremal Kerr black hole is holographically dual to a 2-dimensional chiral conformal field theory (CFT). This correspondence has been extended to various other examples of extremal spinning black holes in $d \geq 4$ dimensions, including
configurations with matter fields. These studies are based on the universality character of the near horizon geometry of extremal black holes. It would be interesting to consider also the case of such solutions with GB corrections, the configurations in this work being perhaps the simplest relevant example. In a Kerr/CFT context, the entropy formula (4.20) should be recovered by computing the central charge of a certain two-dimensional conformal algebra\textsuperscript{20}.

We hope to return to the study of extremal black holes in EGB theory in future work.

5. Further remarks

The main purpose of this paper was to present evidence for the existence of rotating black holes in $d = 4 + 1$ EGB theory. Representing generalizations of a particular class of MP black holes, the considered configurations possess a regular horizon of spherical topology and two equal-magnitude angular momenta. Our results indicate that the inclusion of a GB term in the action does not affect most of the qualitative features of the solutions. However, the presence of a GB term in the action has a tendency to stabilize the rotating black holes, leading to a branch of solutions with a positive specific heat at constant angular velocity at the horizon which does not exist for the MP solutions.

Also, analogous to the case of Einstein gravity, when the horizon angular velocity is increased, the black hole solutions reach a limiting extremal black hole with a regular horizon. Although we did not attempt to construct these extremal black holes, we gave further support for their existence in Section 4.2 by finding an exact $AdS_2 \times S^3$ rotating solution in EGB theory. This solution would describe the neighbourhood of the event horizon of an extremal EGB black hole.

A natural question that arose during our study was how to determine the various conserved quantities and the total action of the solutions. One particularly powerful approach to this problem is given by the counterterms method. In Section 3 we showed how to generalize the Einstein gravity counterterms in [29, 30, 31] by including the effects of a GB term in the bulk action. Although the counterterm method gives results that are equivalent to those obtained using the background subtraction method, we employed it here because it appears to be a more general technique than background subtraction. Moreover, it is interesting to explore the range of problems to which it applies, in particular for configurations with higher order curvature corrections in the gravity action.

The solutions obtained in this paper may provide a fertile ground for the further study of rotating configurations in EGB theory. For example, their generalization to include the effects of an electromagnetic field is straightforward. Also, in principle, by using the same techniques, there should be no difficulty to construct similar solutions in $d = 2N + 1$ dimensions with $N > 2$ equal-magnitude angular momenta.

Another interesting direction to consider in future work consists in finding the EGB generalizations of the $d = 5$ MP black holes with nonequal angular momenta, in particular

\textsuperscript{20}Note that one can associate a temperature $T = 1/2\pi k$ to the near horizon geometry (4.6). As can be seen from Figure 7, in EGB theory this temperature is no longer constant $T = 1/\pi$, presenting a nontrivial dependence on the dimensionless ratio $GJ/\alpha^{3/2}$. 

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the case with rotation in a single plane. Concerning the latter case, it would be interesting to see how the GB term would affect the properties of the solutions close to the extremal limit. Different from the solution (2.17), in this limit the Einstein gravity solution corresponds to a naked singularity. One might speculate that the higher derivative terms in the action might smoothen this singularity and lead to a physically reasonable solution. These aspects together with the investigation of the effects of the GB term on balanced black rings and thus the phase diagram are presently under investigation.

Acknowledgements
YB is grateful to the Belgian FNRS for financial support. BK gratefully acknowledges support by the DFG. The work of ER is carried out in the framework of Science Foundation Ireland (SFI) project RFP07-330PHY.

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