Split-then-Combine simplex combination and selection of forecasters.

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Abstract

This paper considers the Split-Then-Combine (STC) approach (Arroyo and de Juan, 2014) to combine forecasts inside the simplex space, the sample space of positive weights adding up to one. As it turns out, the simplicial statistic given by the center of the simplex compares favorably against the fixed-weight, average forecast. Besides, we also develop a Combine-After-Selection (CAS) method to get rid of redundant forecasters. We apply these two approaches to make out-of-sample one-step ahead combinations and subcombinations of forecasts for several economic variables. This methodology is particularly useful when the sample size is smaller than the number of forecasts, a case where other methods (e.g., Least Squares (LS) or Principal Component Analysis (PCA)) are not applicable.

JEL Codes: C63, C65, C823, C83

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1 Introduction

Forecasters have access to a wide variety of information and forecasting techniques, thus leading to a considerable degree of heterogeneity or redundancy among them. A weighted average forecast is expected to perform better than individual ones because this way we can diversify away idiosyncratic forecast misspecifications, thus reducing the variance of the forecast. The simplest example is the (fixed weight) arithmetic average. More sophisticated methods that make use of varying weights usually do not improve the average in empirical applications because of the instability of the estimated weights (a problem known as forecast combination puzzle, Stock and Watson (2004)); in particular, when an increasing number of forecasts requires us to estimate an increasing number of weights (a problem known as the curse of dimensionality). The forecast combination puzzle has been considered by Smith and Wallis (2009), who pointed out that the failure of more sophisticated combination methods is due to the estimation of the combining weights. On the other hand, Barrow and Kourentzes (2016) found that the average forecast does not perform well in the presence of irregular data, suggesting the use of the median forecast as the best combination method. Similar results are found by Genre et al. (2013).

With forecast (or model)-specific combinations, forecasting is often based on predicting the same variable independently by forecasters. However, analysts who are interested in forecasting a variable from a specific source should not ignore the forecasts from other competing sources. A forecast combination is in fact influenced by all the forecasts; hence, the relationship among individual forecasts is lost when forecasts are independently analyzed. Only a few methods have been suggested that incorporate dependence between forecasts. Multivariate models could incorporate dependence between forecasts if we knew such a dependence. Alternatively, we can engage straightaway with weight distributions based on given individual forecast errors, as dependence between weights can be incorporated directly, thus increasing forecast accuracy. We follow this idea and propose a forecast accuracy method for economic variables.

One important difference between modeling forecast-specific combinations and weight distributions is that weights are directly dependent on each other on an aggregated level. Constraints on weights to have non-negative values and sum up to one lead to spurious effects on their covariance structure. In particular, each row or column of the variance matrix of a vector of weights sums up to zero. Given that the variances are always positive, this implies that some covariances are forced towards negative values.

Independent modeling and forecasting with forecast-specific combinations is not only unattractive since it ignores dependence patterns among (relative) weights, but also because weights often fail to be coherent in the sense of the erratic way in which the covariance associated with two specific weights can fluctuate in sign as we move from a full combination to lower and lower dimensional subcombinations. In fact, there is no relationship between the variance matrix of a subcombination and that of the full combination. Besides, variances may display different rank orderings as we form subcombinations, which could
lead to implausible forecasts.

Also, avoided forecasts in a subcombination will result in an increase of weights for some other forecasts. In combinations, not only are there common elements in the denominator of two weights, but also elements common to numerator and denominator in each weight. Avoided forecasts in a subcombination thus affect both the numerator and the denominator, and the dependence between forecasts is therefore not as easy to predict.

All combinations are subcombinations of a larger one. Since the covariance between two weights depends on which other forecasts are reported in the dataset, there is no guarantee that a plot of a subcombination exhibits similar or even compatible patterns with the plot of the original dataset, even if the forecasts not included in the subcombination are irrelevant (redundant).

There is thus incoherence of the correlation between weights as a measure of dependence. Note, however, that the ratio of two weights remains unchanged when we move from a full combination to a subcombination. Therefore, as long as we work with scale invariant functions (i.e., ratios), we shall be subcombinationally coherent.

Since standard descriptive statistics (e.g., arithmetic mean and standard deviation) are not informative with combinations, in this paper we propose a time-varying method to combine, select, and recombine forecasts based on Aitchison (1982, 1986), who characterizes compositions as vectors having a relative scale and identifies its sample space with the simplex. More crucial than the constraining property of compositional data is the scale-invariant property of this kind of data. Indeed, when we are considering only few forecasts of a full combination we are not working with constrained data but our data are still compositional. This approach has been successfully applied to various fields; see, for instance, Billheimer et al (2001), Egozcue and Pawlowsky-Glahn (2005), and van den Boogaart, Tolosana, and Bren (2009), a software package available now to deal with compositional data. To our knowledge, it has not been applied to combinations of forecasts. Compositional Data Analysis (CODA) is a well-established set of statistical methods for the analyses of compositional data, defined in general as a data vector with positive elements summing to a constant value and thereby containing only relative information (Pawlowsky-Glahn and Buccianti, 2011). Thus, CODA enables a coherent and correct modeling of dependent weights by recognizing the sign and sum constraints.

Traditional decomposition techniques provide inconsistent results when applied to compositional data as they do not recognize the implicit constraints of summing to a constant (Aitchison, 1982, 1986): mathematically, compositional data lie in the bounded space of the simplex while traditional decomposition techniques are defined for data in the real space. Aitchison (1986) showed that by making log-ratio transformations it is possible to express compositional data in the real space where the data can be analyzed with conventional models and then transformed back into the simplex. We make use of the centered log-ratio (CLR) transformation to express the weights in the real space. The CLR transformation takes the logarithm of each weight divided by the geometric mean. This transformation maintains the initial constraint in the weights as its ele-
ments sum to 0 by construction but resulting values are real. The inverse CLR transformation takes the data back to the simplex with the closure operator \( C \) that divides the exponential of each entry by the sum of all entries. Aitchison (1986) also defined addition and subtraction operations obeying conventional rules of arithmetic, maintaining the result of the operation in the simplex. These two operations applied to vectors of weights are defined, respectively, as the closures of the ratio and product of weights. They will be used in our analysis and are essential when modeling weight distributions. The difference of two vectors of weights measures the distance between them in compositional data similarly to subtraction on the real axis. The sum of two vectors of weights is the opposite operation and can be compared with addition on the real axis.

The analysis that is presented in this paper uses the Split-Then-Combine (STC) approach of Arroyo and de Juan Fernández, 2014, to generate the weights of a combination. Because they are restricted to be positive and sum up to one, we propose the center of the simplex \( g \) as our basic simplicial combination vector. To get a subcombination of forecasts, we develop a Combination-After-Selection (CAS) procedure to recombine the best subset of forecasts, again with positive weights adding up to one. Finally, we compare both the full combination and the CAS subcombination with the benchmark average forecast, that has been shown hard to beat in the literature. It is important to note that our combination vector is just the gravity center of the simplex whose weights are not estimated, thus avoiding the combination puzzle.

Our analysis improves forecast-specific combinations by using a CODA-based combination vector of weights. We find that just the simplicial mean \( g \) provides in general more accurate forecasts than the average forecast (which is just the neutral point in the simplex) in combinations as well as subcombinations. CODA enables coherent modeling of forecast-specific combinations where dependences between forecasts are explicitly modeled, so a relative improvement in the weight for one forecast leads to a decline in the relative weight for the remaining ones. Thus, CODA models provide a more satisfactory combination as relative dependence between forecasts is taken into account.

The paper is organized as follows: the next section describes the STC approach both in the Euclidean and simplex spaces. Then, we explain the CAS strategy. In the empirical application, in section 4, we pull out information provided by panels of quarterly periodicity from a pool of expert forecasters for the US macroeconomy over the period 1991–2018. Forecast accuracy of simplicial combinations are compared with the uniform benchmark arithmetic average. Finally, some concluding remarks complete the paper.

## 2 The Split-Then-Combine (STC) approach

Arroyo and de Juan (2014) proposed the Split-Then-Combine approach to generate combinations across \( J \) forecasts \( \left( \hat{y}_{t,j} \right) \), \( j = 1, 2, \ldots, J \), along \( t = 1, 2, \ldots, T \)
periods using the expression:

\[ \hat{Y}^{(m)}_{t} = \omega^{(m)}_{t,1} \hat{Y}^{(m)}_{t,1} + \omega^{(m)}_{t,2} \hat{Y}^{(m)}_{t,2} + \ldots + \omega^{(m)}_{t,J} \hat{Y}^{(m)}_{t,J} \]

where the weights \( \omega^{(m)}_{t,j} \) vary in two dimensions: (1) from one period to the next; and (2) from one panel to another. A panel is a division of the frequency data. For example, if we are working with monthly data, we will have 12 panels, one for each month; if we work with quarterly data, we will have four panels, one for each quarter. Panels take into account the different behavior of the time series among seasons, but STC can also be applied to time series with lower frequency than quarterly or monthly data.

The weights of the STC approach must satisfy two restrictions: be positive and sum up to one; the latter, to avoid biased combinations if individual forecasts are unbiased. Arroyo and de Juan (2014) developed the STC in the Euclidean Space. The analysis presented in this paper extends and improves the STC in the simplex space (Aitchison, 1986).

In order to see the differences between both methods, we first briefly review the STC approach in the Euclidean space; then, we expand the STC approach to the simplex space.

2.1 The STC approach in the Euclidean Space

Table 1 shows how the STC approach works in the Euclidean space. Columns 2 to 5 show the forecasts of the variable of interest for panel \( m \). Each element of this column represents the forecast of each forecaster for a given period. For instance, \( \hat{Y}^{(m)}_{2,1} \) is the forecast from forecaster 2 for period 1 in panel \( m \). The 6th column shows the cross average by period for the \( J \) forecasts; that is, \( \hat{Y}^{(m)}_{J,1} \) is the average of the \( J \) forecasts for the first forecasting period. The 6th row shows the time average by forecaster, that is, \( \hat{Y}^{(m)}_{1,T_1} \) is the average over time of all the forecasts from the first forecaster. Column 7 reports the real data of the variable and the 7th row shows the precision of each forecast average with respect to the overall average \( \hat{Y}^{(m)}_{J,T_1} \). This measure is used to construct the weights \( \omega \) that will be assigned to each forecast in the STC approach in the Euclidean space.

Insert Table 1 around here

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1 See Bujosa-Brun et al (2019) for an application of the STC approach to annual data with only one panel.
The STC weights $\omega$ are then computed using the information up to time $T_1$ for each panel using the expression:

$$
\omega^{(m)}_{j,T_1} = \left( \frac{\hat{Y}^{(m)}_{j,T_1} - \bar{Y}^{(m)}_{J,T_1}}{\sum_{j=1}^{J} \bar{Y}^{(m)}_{j,T_1} - \bar{Y}^{(m)}_{J,T_1}} \right)^{-2}
$$

and these weights are then used to form the STC combination in $T_1 + 1$ for panel $m$:

$$
\hat{Y}^{(m)}_{T_1+1} = \omega^{(m)}_{1,T_1} \hat{Y}^{(m)}_{1,T_1+1} + \omega^{(m)}_{2,T_1} \hat{Y}^{(m)}_{2,T_1+1} + \ldots + \omega^{(m)}_{J,T_1} \hat{Y}^{(m)}_{J,T_1+1}
$$

This expression must be computed for each panel, $m = 1, 2, ..., M$. These weights satisfy two restrictions: they are positive and add up to one. Once we get forecasts at $T_1 + 1$, we re-compute the weights by rolling over another one-step-ahead combination for $T_1 + 2$, and so on, always satisfying the same two restrictions.

The unit-sum and non-negativeness constraints on weights, however, give rise to a number of known issues that make inappropriate the Euclidean geometry. Thus, standard methods from multivariate statistics are inapplicable for computing weights in combinations. Among others: (i) weights cannot be normally distributed due to the bounded range of their values; (ii) due to the constant sum constraint, each row $i$ of the variance-covariance matrix of a vector of random weights adds up to zero, $\sum_{j=1}^{J} C[W_i, W_j] = C[W_i, \sum_{j=1}^{J} W_j] = C[W_i, 1] = 0$ giving rise to the singularity of these matrices; that is, the variance-covariance matrix is orthogonal to the $J \times 1$ vector of ones. A classical way to get rid of singularity is to erase one weight, but results will depend on which one is erased, not being an operation that is permutation invariant; (iii) Since $\sum_{j \neq i} C[W_i, W_j] = -V[W_i] < 0$, some covariances between weights are forced towards negative values, leading to negative bias. In particular, with only two weights, $V[W_1] = V[W_2]$ and their correlation is $-1$. Hence, correlations are not free to range over the usual interval $(-1, +1)$; (iv) With negative bias, what is the meaning of zero correlation between two weights in a combination? What correlations between restricted weights would have arisen if the corresponding unrestricted weights had been uncorrelated? Even when two restricted weights are correlated, it is by no means safe to conclude that the corresponding unrestricted weights quantities are correlated (many-to-one function from the Euclidean space to the simplex space); (v) by definition, not only is there a common forecast in the denominator of two weights, but also there is a common forecast in the numerator and denominator of each weight, resulting in spurious correlation between weights; (vi) most importantly, there is no relationship between the variance-covariance matrix of a subcombination of weights and the
subvariance-covariance matrix of the same weights in the full combination. Besides, variances may display different and unrelatable rank orderings as we form subcombinations. This is known as subcombinational incoherence. For instance, the covariance between weight $W_1$ and $W_2$ along time in a full combination of 4 weights may be completely different, even in sign, from the covariance of the corresponding weights $S_1 = W_1 / (W_1 + W_2 + W_3)$ and $S_2 = W_2 / (W_1 + W_2 + W_3)$ in a subcombination where the fourth weight is excluded. There is thus incoherence of the correlation between weights as a measure of dependence. Note, however, that the ratio of two components remains unchanged when we move from full combination to a subcombination: $S_1 / S_2 = W_1 / W_2$, so that as long as we work with scale invariant functions (i.e., ratios), we shall be subcomputationally coherent; (vii) the covariance between two weights depends on those others who have been included in the combination. Hence, there is no guarantee that the behavior of a pair of weights in a combination exhibits similar or even compatible patterns with that of the same pair of weights in alternative subcombinations (and all combinations are possible subcombinations of a larger one), even if the forecasts not included are irrelevant (i.e., nonsense of scatterplots for pairs of weights over time); (viii) finally, since the construction of subcombinations from combinations is similar to the construction of combinations from unrestricted weights, we may expect the same difficulties in relating variance-covariance matrices of vectors of weights in the simplex and those in the the Euclidean space.

All of these issues lead us to consider the simplex space as the best one to overcome these problems.

### 2.2 The STC approach in the Simplex Space

Consider a $T \times J$ panel $\hat{Y}$ of $T$ out-of-sample forecasts $\hat{Y}_{t,j}$ produced over time by $J$ forecasters on some variable of interest $Y_t$, and $A$ be its related panel of prediction accuracies $a_{t,j} \equiv (\hat{Y}_{t,j} - Y_t)^{-2} \in \mathbb{R}_+$. Then, the matrix $\mathbb{W} \equiv \begin{pmatrix} w_{1,1} & \ldots & w_{1,J} \\ \vdots & \ddots & \vdots \\ w_{T,1} & \ldots & w_{T,J} \end{pmatrix}$ $\equiv \begin{pmatrix} w^\prime_{1\bullet} \\ \vdots \\ w^\prime_{T\bullet} \end{pmatrix} \equiv \begin{pmatrix} w_{1\bullet} & \ldots & w_{J\bullet} \end{pmatrix}$

with weights $w_{t,j} \equiv a_{t,j} / \sum_{j=1}^{J} a_t$ represents $T$ combination vectors $w_{1\bullet}, \ldots, w_{T\bullet}$ such that $w_{t,j} \geq 0$ for all $t$ and $j$, and $\sum_{j=1}^{J} w_{t,j} = 1$ for all $t$. Thus, $w^\prime_{t\bullet}$ is just a $1 \times J$ point in a simplex space $S^{J-1}$ of positive weights adding up to one of dimension $J - 1$. The function $C : \mathbb{R}_+^J \mapsto S^{J-1}$ that transforms a vector of precisions $a_{t\bullet} \in \mathbb{R}_+^J$ into a vector of weights $w_{t\bullet} \in S^{J-1}$ is called a closure transformation $w_{t\bullet} = C (a_{t\bullet})$. Since this operator cancels out any constant, $C (ca_{t\bullet}) = C (a_{t\bullet})$, it is scale invariant. Hence, we just need to work with scale
invariant functions (e.g., ratios or logratios). Every statement about vectors in \( S^{J-1} \) will be fully expressed in terms of logratios in \( \mathbb{R}^{J-1}_+ \) with inferences transformed back from \( \mathbb{R}^{J-1}_+ \) into combinational statements in \( S^{J-1} \). In particular, we will use the center \( g \) of \( \mathbb{W} \) as our benchmark simplicial statistic, which is based on the centered logratio transformation \( \text{clr} : S^{J-1} \rightarrow \mathbb{R} \),

\[
  x_{t,j} = \text{clr}(w_{t,j}) := \ln w_{t,j} - \frac{1}{J} \sum_{j=1}^{J} \ln w_{t,j} = \ln \frac{w_{t,j}}{\prod_{j=1}^{J} w_{t,j}} = \ln \frac{w_{t,j}}{g(w)} \quad (1)
\]

where \( g(w) \), the geometric average of the \( J \) forecasts for the \( t \)th observation, is the gravity center of \( \mathbb{W} \). This function may be interpreted as a bijection \( S^{J-1} \leftrightarrow \mathbb{H}^{J-1} \) between \( S^{J-1} \) and a vector subspace \( \mathbb{H}^{J-1} := \{ x_{t,\bullet} \in \mathbb{R}^J_+ : \sum_{j=1}^{J} x_{t,j} = 0 \} \) of \( \mathbb{R}_+^J \) orthogonal to the vector of ones. The inverse \( \text{clr}^{-1} \) transformation is then defined by

\[
  \text{clr}^{-1}(x_{t,\bullet}) := C(\exp x_{t,\bullet}) = C(w_{t,\bullet}/g(w_{\bullet})) = C(w_{\bullet}) = w_{\bullet} \in S^{J-1} \quad (2)
\]

that is, \( \text{clr}^{-1} \) allows us to go from \( \mathbb{R}^{J-1}_+ \) back to \( S^{J-1} \). Finally, the center \( g \) of \( \mathbb{W} \) is given by

\[
  g \equiv \text{clr}^{-1}(\overline{w}) = C\left( \prod_{t=1}^{T} w_{t,1}^{1/T}, \ldots, \prod_{t=1}^{T} w_{t,J}^{1/T} \right) \equiv C\left( g(w_1), \ldots, g(w_J) \right) \quad (3)
\]

that is, the point in the simplex given by the closure of the geometric averages of weights over time will be the combination vector of the STC simplex.

While vectors of weights are subcombinationally incoherent, the ratio of two weights remains unchanged when we move from a full combination to a subcombination; that is, \( a_{t,i}/a_{t,j} = w_{t,i}/w_{t,j} = s_{t,i}/s_{t,j} \) for all \( t \). Hence, as long as we work with ratios or logratios, we shall be subcombinationally coherent. Therefore, we only consider relative precision among forecasts: each weight in a combination vector will have no meaning on itself isolated from the others. In particular, the variation matrix \( \Upsilon \) with elements given by the sample variance over time, \( \Upsilon_{i,j} = \text{var}\left( \ln \frac{w_{i,j}}{\prod_{j=1}^{J} w_{t,j}} \right) \), with diagonal elements all 0, will be used to define the total variation in \( \mathbb{W} \) as \( v^2 := \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \Upsilon_{i,j} \). Then, \( v \) will be a proper measure of distance among forecasts in cluster analysis, from perfect association (\( v = 0 \)) to perfect independence (\( v = +\infty \)).

### 3 Combination-after-Selection (CAS)

The combination after selection procedure looks for those forecasts that best or orthogonally contribute to improve the simplex STC full combination \( g \in S^{J-1} \). Thus, we preselect from \( g \in S^{J-1} \) those forecasts whose weights \( (w_1, \ldots, w_I) \) are greater than the benchmark average \( C(1,J) = 1/J \) weight, the neutral point.
in the simplex. Then, we convert this subvector into the CAS subcombination \( \mathcal{C}(w_1, ..., w_I) = (s_1, ..., s_I) \in S^{I-1} \) inside a simplex of a lower dimension \( I - 1 \) so that \( s_1 > 0, ..., s_I > 0 \) and \( s_1 + ... + s_I = 1 \). Sometimes, especially when \( J >> T \), we perform another subsequent selection by choosing those forecasts from the CAS preselection that are orthogonal to each other, avoiding this way redundant forecasts. To do this, we have also carried out cluster and biplot (Gabriel (1971)) analyses.

A selected CAS subcombination \( \mathcal{CS} : S^{J-1} \mapsto S^{I-1} \) will be viewed as taking place in two stages: a selection of \( I < J \) forecasts by a selecting \( I \times J \) matrix \( \mathcal{S} \), followed by its closure,

\[
\mathcal{CS}(g) = \mathcal{C}(\mathcal{S}g) := \left( \frac{w_1, ..., w_I}{w_1 + ... + w_I} \right)' = (s_1, ..., s_I)'
\]  

(4)

For \( I = 3 \), the CAS subcombination can be represented in a ternary diagram by barycentric coordinates (height of the point over the side of the triangle opposite to it). Similarly, for \( I = 4 \), it can be represented by a tetrahedron where each possible 3-forecast subcombination vector is found by projecting every 4-forecast vector onto the side opposite to the vertex corresponding to the removed forecast.

### 3.1 CAS from clusters of forecasts

Forecasts are seldom homogeneous. Often there are several subgroups, corresponding to a different, unknown subpopulation, with a distinct behavior. In order to find possible subgroups of forecasts, we apply a hierarchical algorithm of agglomeration with two steps: first, we consider all forecasts as isolated groups; then, we proceed upwards using two clustering criteria:

1. The ward criteria: Harmonic weighted average distance between forecasts in a cluster.
2. The complete criteria: Maximum distance between forecasts in a cluster.

We define redundant forecasts those who belong to the same cluster. Our CAS subcombination of clusters is formed by first combining the forecasts in each cluster, then by finding the simplicial center of all clusters.

The dendrogram contains information on the marginal distribution of each coordinate (although, it does not contain information on the relationship between coordinates). Each coordinate is represented in the horizontal axis. The vertical bar going up from each one of these coordinate axes represents the variance of that specific coordinate, and the contact point is the coordinate mean.
3.2 CAS from a biplot of forecasts

To better visualize the structure of $\mathbb{W}$, we just need to center it through a translation by the inverse of its center:

$$
\mathbb{W}_c \equiv \left( \begin{array}{c}
\left( \frac{w_{t,j} / g(w_{\bullet,j})}{\sum_{h=1}^J w_{t,h} / g(w_{\bullet,h})} \right)
\end{array} \right)_{t=1,...,T}
$$

This has also the effect of moving the center $g$ of $\mathbb{W}$ to the benchmark average $C(1,J)$. Moreover, if we scale $\mathbb{W}_c$ by powering to $\nu^{-1}$, we obtain a combination matrix with unit total variance, but with the same relative contribution of each logratio in the variation matrix.

The biplot represents simultaneously the rows and columns of a centered $T \times J$ matrix by means of a rank-2 approximation which, in the least squares sense, is provided by the singular value decomposition of $\mathbb{W}_c$. Observations are represented as dots and forecasts as arrows from the center of the plot. Redundant forecasts, lying on a common line, will show a one-dimensional pattern.

3.3 Selection strategy

The CAS approach that selects forecasts from the center $g$ of $\mathbb{W}$ can be summarized in the following steps:

1. Given a $T \times J$ table $\hat{Y}$ of $J$ forecasters over $T$ time periods in a given season (month or quarter in our cases) compute the related $T \times J$ table $\mathbb{A}$ of $1 \times J$ vectors $a'_{\bullet,t}$ of prediction accuracies for each time period $t \in [1,T]$.

2. Convert $\mathbb{A}$ into a $T \times J$ table $\mathbb{W}$ of combination vectors $w'_{\bullet,t}$ of weights inside the simplex; that is, weights in each row of $\mathbb{W}$ are positive and add up to one.

3. For each $t \in [1,T]$, compute a $1 \times J$ vector of logweights in differences with respect to its average over $J$ using the clr transformation.

4. Calculate the gravity center $g$ of $\mathbb{W}$ as a combination vector of forecasts.

5. Select the CAS subcombination of those forecasts with simplicial weights larger than $1/J$. Where appropriate, when $J >> T$, make another sub-selection from the previous CAS by applying cluster and biplot analyses to find non-redundant forecasts with orthonormal coordinates.

6. Go back to the simplex with the $\text{clrInv}$ transformation.

7. Repeat steps 1-6 for all panels.

8. Generate rolling, out-of-sample, one-step-ahead combination vectors to forecast next year’s corresponding seasons and compare them to their realized value.

Once again, this procedure can also be applied to data with lower periodicity than monthly or quarterly data. See for example, Bujosa et al (2019) for an application of the simplicial methods proposed in this paper using annual data.
4 Empirical application

We apply the STC in the Simplex and CAS combination procedures to the variables defined in table 2, where we include their definition and the samples used to form the combinations of forecasts. Here, we deal with forecasts obtained from the Survey of Professional Forecasters (SPF) from the Federal Reserve Bank of Philadelphia. Blanks in the Survey due to the entry and exit of forecasters are fulfilled following the same strategy as in Poncela et al (2011), that is, we only consider one-step-ahead forecasts and select only those forecasters without missing data. When there is a missing datum, we use the two-steps-ahead forecast to fill it. Forecasters with more than four consecutive missing data are excluded. For each sample, we only take into account balanced panels. This strategy is also used in Lahiri, Peng and Zhao (2015). Because of the entry and exit of forecasters in the survey, we also analyze different sample sizes, depending on the number of included forecasters. In table 3, we show, for each variable, the number of forecasters chosen in each subsample. The combinations of forecasts are computed for the periods 2015 to 2018. Note that, in some samples, the number of forecasts is larger than the number of observations, a fact that cannot be treated with other methods (e.g., regression and PCA).

To analyze the prediction accuracy of combinations, we look at four well-known measures: Mean Error (ME), Root Mean Squared Error (RMSE), Mean Absolute Percentage Error (MAPE), and Median Absolute Percentage Error (MdAPE). Although in general these measures produce similar results, there are some differences depending on the type of the combination considered.

We compute three kinds of combinations: two with varying weights (STC in the simplex, S_STC and CAS) and the fixed-weight arithmetic average combination (AVE).

4.1 General results

We have analyzed 1266 values of accuracy measures. General results are shown in table 4. According to the type of weights, they favored fixed weights in 469 cases (37%) and varying weights in 797 (63%). With respect to the latter, 194 (15.3%) favored S_STC and 603 (47.6%) CAS. In general, a time-varying simplicial combination outperforms the fixed weights; in particular, the S_STC gives is best in 15.3% of cases while CAS is in 47.6%. Based on this overall

\footnote{We consider as benchmark the average because our objective functions are symmetric. As suggested by a referee, we also examine the Median as our objective function. The results, however, are very similar and are available upon request.}
result, we may conclude that selecting forecasts improve the combination for the period 2015 – 2018.

Although in general simplicial combinations generate better results than the simple average of forecasts, the figures vary depending on the number of forecasts in relation to the observations (years) used to construct the weights. In table 4, we include the summary for each type of combination. The SIMPLEX column compares varying-weight combinations (S\textit{STC} plus CAS) with the fixed-weight one (AVE). When \( J < T \), CAS is 6.89 points higher than AVE, but when \( J > T \), this figure goes to 14.41 (more than twice larger); that is, selection of forecasts works very well when its number is greater than that of the years used to compute the weights, precisely when some other methods can’t say anything about it.

### 4.2 Results by method of combination, variable and accuracy criteria

\textit{Insert table 5 around here}

Table 5 shows the percentage of beats by variable and accuracy criteria for each combination procedure. The following comments are worth mentioning:

1. Fixed weights works better with RMSE and MAPE, although it never reaches 50% of cases and, on average, never beats CAS.
2. CAS is the best with Mean Error and MdAPE, reaching, on average, 50% of cases.
3. S\textit{STC} full combination is the worst except for PGDP with MAPE and MdAPE, matching CAS with Mean Error. This is also true for RLSGOV with MdAPE.
4. AVE is the best for NGDP, EMP, TBILL, RRESIN and RFEDGOV while CAS is the best for INDPORD, HOUSING, BOND, RGDP, RCONSUM and RNRESIN. Being the best is never achieved by the full S\textit{STC} combination.

### 4.3 Results by number of forecasts and accuracy criteria

Table 6 shows the results of each combination by the number of forecasts and accuracy criteria.

\textit{Insert table 6, panel a) and b) around here}
In panel a), we present the number of beats of each combination which we then added by fixed (AVE) and varying (SIMPLEX) weights in panel b). There is a clear advantage of SIMPLEX by around 30 points with Mean Error and MdAPE. The difference is even greater when J > T. As we can see from panel a), this is due to CAS. The difference further increases when J > T. With MdAPE, for example, the difference goes from 7.19 percentage points when J < T to 25.53 when J > T with respect to AVE.

4.4 Results according to the variability of the forecasts

The basic idea under this section is the following: a fixed-weight combination assigns the same weight to forecasts, so if variability among them is small, then the average will work well in the same direction, however wrong it may be ('precisely' wrong) unless they are unbiased. On the other hand, when variability is high, it is better to assign different weights. This is in line with the results obtained by Jose and Winkler (2008) when comparing the accuracy of the average with trimmed and Winsorized averages and the results by Genre et al (2016) using the ECB survey of professional forecasters. In this later paper, they find that some combination methods outperform the simple average of forecasts in variables with heterogeneity of forecasters and apparent bias.

In order to verify this hypothesis, we compute the variation coefficient (VC) of each variable for each combination and forecast period from 2015 to 2018. We also plotted the forecasts for each period. In fact, this issue forms part of the selection procedure presented in this paper. The method is based on the orthogonality of forecasts; that is, it looks for selecting those forecasts that do not share common information. In this empirical application, the forecasts come from the Survey of Professional Forecasters (SPF) and may have common information in forming their forecasts. This is the reason why we expect some forecasts to be highly correlated (even redundant) and others with low correlation. Then, the CAS procedure takes advantage of this situation and usually generates better results.

The main comments that can be pointed out are the following:

1. When all the forecasts included in the sample are highly correlated and their plots show a similar behavior, AVE is usually the best combination. A clear example of this situation is shown in figure 4 where we plot the forecasts for the variable NGDP for all the samples.

2. When some of the forecasts are correlated but their plots differ somewhat, S_STC is better because of its varying-weight allocation. Figure 5 shows this situation for the variable RLSGOV.

3. In a mixed situation with some forecasts highly correlated and some others not so, CAS is the best because it only selects non-redundant forecasts. In figure 6 we show this situation for the variable UNEMP.

*In order to save space, these results are available upon request.*
4. In general, with low correlated forecasts, varying-weight combinations generate better results: the \textit{S\_STC} procedure, when the forecasts show a similar behavior, and the \textit{CAS} procedure, when they don’t. Figure 7 shows a clear example of this situation for the variable \textit{HOUSING}.

Table 7 shows the VC and results for the aforementioned variables\textsuperscript{[5]}. The analysis of the VC will be done jointly with figures 1 to 4.

\textit{Insert table 7 and figures 1 to 4 around here}

1. **Variable NGDP**: All the graphs plotted in figure 1 show very little variation between forecasts. The VC in each sample is very low, suggesting that \textit{AVE} should be used. Looking at the combination results, \textit{AVE} is the winner in all the samples with the exception of sample 4. In this case, \textit{CAS} generates the best forecasts for all the forecasting periods. Notice that in the graph for sample 4, although the forecasts follow a similar behavior, there are some of them with different patterns that can be used to improve the forecast combination through the \textit{CAS} method.

2. **Variable RLSGOV**: The behavior of the forecasts for this variable is different from the one observed before. In this case, the forecasts seem to have a similar behavior, but the correlation between them is not too high. Then, assigning different weights generates better combinations. Looking at figure 2, we can see that \textit{S\_STC} obtains very good results in 2017 and perhaps in 2016. Our perception from the graph is confirmed in table 7: varying-weight combinations outperform the fixed-weight one. This situation is also supported by the VC, which shows higher values than the observed for the \textit{NGDP}. So, in this case, the fact that not all the forecasts show the same pattern leads to better forecasting results with varying-weight methods.

3. **Variable UNEMP**: The VC of this variable in table 7 clearly shows higher values than the observed for the previous variables. This fact can indicate that the average forecast may not be the best combination in this case. Looking at figure 3, not all the forecasts have the same pattern. This favors the varying-weight combinations, \textit{S\_STC} and \textit{CAS}, the latter being the one that beats more times. Therefore, in this case, selection is better than a full combination either fixed \textit{AVE} or varying \textit{S\_STC}.

4. **Variable HOUSING**: Figure 4 shows the behavior of the forecasts for the variable \textit{HOUSING}. This is a clear example for \textit{CAS} to form a combination. Different behavior of some forecasts and high VC are the clues to select forecasts to obtain better forecasting results. Although there is a common behavior of some forecasts, the selection of orthogonalized forecasts improves the results.

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\textsuperscript{[5]}The VC, figures and results for the other variables are available upon request. They have been omitted to save space.
Similar results are confirmed for the other variables analyzed in the empirical application. As a matter of fact, high VC and different behavior might be the clues to consider CAS as the best subcombination to forecast a variable.

4.5 Results according to the forecast ability

When the Diebold and Mariano (1995) or Giacomini and White (2006) tests are not appropriate, it might be interesting to break down the MSFE into three components (bias, variance, and covariance) to assess which of them holds sway over a given MSFE:

$$MSFE = \frac{1}{H} \sum_{h=1}^{H} (\bar{Y}_{T+h} - Y_{T+h})^2 \equiv \left( \bar{Y}_H - \bar{Y}_H \right)^2 + \left( sd(\hat{Y}_f) - sd(Y_f) \right)^2 + 2 (sd(Y_f)) (sd(Y_f)) (1 - corr[\hat{Y}_f, Y_f])$$

where $\bar{Y}_H$ is an $H$-period average forecast, $\bar{Y}_H$ is the corresponding average for the realized values, $sd(\hat{Y}_f)$ is the standard deviation of the forecasts, $sd(Y_f)$ is the standard deviation of the realized values, and $corr[\hat{Y}_f, Y_f]$ is the correlation between forecasts and realized values. Then, proportions are defined as follow:

Bias proportion: $\frac{(\bar{Y}_H - \bar{Y}_H)^2}{MSFE}$

Variance proportion: $\frac{(sd(\hat{Y}_f) - sd(Y_f))^2}{MSFE}$

Covariance proportion: $\frac{2 (sd(Y_f)) (sd(Y_f)) (1 - corr[\hat{Y}_f, Y_f])}{MSFE}$

Finally, we study which one contributes more to the MSFE. A ranking of preferences may be given by the following four situations:

1. The best will be when there are little bias and variance (hence, high covariance proportion).
2. The next one will be when there is little bias, but high variance (hence, low covariance proportion).
3. Bad situations happen when the bias is high: either with high variance,
4. Or the worst, with low variance (‘precisely’ wrong).

Using this classification, we show in Table 8 the bias, variance, and covariance proportions for the combination procedures with lowest MSFE.\(^\text{6}\)

\(\text{6The specific values for the bias, variance, and covariance proportions for each variable, each sample, and each combination procedure are available upon request.}\)
Looking at the results shown in Table 8, we can conclude that the CAS combination with lowest MSFE is classified in the best situation more than 50% of the time, whereas the AVE does in the third situation almost 50% of the time. The \textit{S\textsubscript{STC}} combination is also classified in the third situation 60% of the time. So, as a general result, we can conclude that CAS shows the lowest MSFE where it is ‘precisely’ right.

5 Conclusions

In this paper, we have used the Split-Then-Combine (\textit{STC}) approach to build positive weights that sum up to one. Because of these two restrictions (positiveness and adding up to 1), most methods from multivariate statistics are inapplicable for combinational datasets, giving rise to a number of issues that make inappropriate the Euclidean geometry. Instead, the Aitchison geometry considers combinations of forecasts inside the simplex, the sampling space of positive weights adding up to one. Basic transformations from the simplex space to the real space and back to the simplex space allow us to define different simplicial combinations with time-varying weights. In addition, we develop new strategies to construct Combinations after Selection (CAS) simplicial sub-combinations by selecting those forecasts in a full simplicial combination that assign higher weights than the one allocated by the benchmark average or, where appropriate, a simplicial subcombination based on orthogonal clusters of redundant forecasts.

The methodology can be summarized in these steps: first, we split experts’ forecasts by seasons to assess their relative forecast performance that periodically evolve over time. Second, we choose as a combination vector the gravity center of the simplex by means of an isometric, centered logratio transformation. Then, we select forecasts inside a simplex of lower dimension. Finally, we make rolling, truly out-of-sample, one-step-ahead combinations of forecasts, even in cases where the sample size is smaller than the number of forecasts. Once a new observation is known, we recalculate the weights that we then keep one-step-ahead to form a new out-of-sample combination.

We present experimental results with a pool of expert forecasters of the US macroeconomy over the period 1991–2018. In most cases, the Combination after Selection strategy improves the average (neutral combination in the simplex space) with different criteria of forecasting accuracy, and works very well even when the number of forecasts is greater than the number of observations.

As a general rule, we can conclude that when there are a high number of heterogeneous forecasts to be combined, the best way to form a combination is by selecting a CAS simplicial subcombination formed by the most weighted, non-redundant forecasts.

Compliance with ethical standards:
Ethical approval: This article does not contain any studies with human or animal participants performed by any of the authors.

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### Table 1: STC approach in the Euclidean Space

| Panel $m$ | 1                  | 2                  | ... | $J$ | $\overline{Y}_{J,t}^{(m)}$ | Real data |
|-----------|-------------------|-------------------|-----|----|---------------------|-----------|
| 1         | $\hat{Y}_{1,1}^{(m)}$ | $\hat{Y}_{2,1}^{(m)}$ | ... | $\hat{Y}_{J,1}^{(m)}$ | $\overline{Y}_{J,1}^{(m)}$ | $Y_{1}^{(m)}$ |
| 2         | $\hat{Y}_{1,2}^{(m)}$ | $\hat{Y}_{2,2}^{(m)}$ | ... | $\hat{Y}_{J,2}^{(m)}$ | $\overline{Y}_{J,2}^{(m)}$ | $Y_{2}^{(m)}$ |
| ...       | ...               | ...               | ... | ... | ...                 | ...       |
| $T_1$     | $\hat{Y}_{1,T_1}^{(m)}$ | $\hat{Y}_{2,T_1}^{(m)}$ | ... | $\hat{Y}_{J,T_1}^{(m)}$ | $\overline{Y}_{J,T_1}^{(m)}$ | $Y_{T_1}^{(m)}$ |
| $\overline{Y}_{J,T_1}$ | $\overline{Y}_{1,T_1}$ | $\overline{Y}_{2,T_1}$ | ... | $\overline{Y}_{J,T_1}$ | $\overline{Y}_{J,T_1}$ |                       |
| Fixed     | $\overline{Y}_{1,T_1}^{(m)} - \overline{Y}_{J,T_1}^{(m)}$ | $\overline{Y}_{2,T_1}^{(m)} - \overline{Y}_{J,T_1}^{(m)}$ | ... | $\overline{Y}_{J,T_1}^{(m)} - \overline{Y}_{J,T_1}^{(m)}$ |                       |
Table 2: Definition of the main variables used in the application.

| Variable  | Definition                                                                 | Sample          |
|-----------|---------------------------------------------------------------------------|-----------------|
| NGDP      | Forecasts for the quarterly level of nominal GDP. SA. billions $          | 1991 Q1 - 2018 Q4 |
| PGDP      | Forecasts for the quarterly level of the chain-weighted GDP price index. SA. Index. Base year 1992 | 1991 Q1 - 2018 Q4 |
| UNEMP     | Forecasts for the quarterly average unemployment rate. SA. % points        | 1991 Q1 - 2018 Q4 |
| EMP       | Forecasts for the quarterly average level of nonfarm payroll employment. SA. Thousands of jobs. | 2004 Q1 - 2018 Q4 |
| INDPROD   | Forecasts for the quarterly average level of the index of industrial prod. SA. Index. | 1991 Q1 - 2018 Q4 |
| HOUSING   | Forecasts for the quarterly average level of housing starts. SA. millions. | 1991 Q1 - 2018 Q4 |
| TBILL     | Forecasts for the quarterly average 3-months Treasury Bill rates. % points | 1991 Q1 - 2018 Q4 |
| BOND      | Forecasts for the quarterly average level of Moody's Aaa corporate Bond yield. % points | 1991 Q1 - 2018 Q4 |
| RGDP      | Forecasts for the quarterly chain-weighted real GDP. SA. annual rate. Base years 1992 - 1995, fixed weighted real GDP | 1991 Q1 - 2018 Q4 |
| RCONSUM   | Forecasts for the quarterly chain-weighted real personal consumption expenditures. SA, annual rate, base years 1992 - 1995. | 1991 Q1 - 2018 Q4 |
| RNRESIN   | Forecasts for the quarterly chain-weighted real nonresidential fixed investment. SA, annual rate, base years 1992 - 1995. | 1991 Q1 - 2018 Q4 |
| RRESINV   | Forecasts for the quarterly chain-weighted real residential fixed investment. SA, annual rate, base years 1992 - 1995 | 1991 Q1 - 2018 Q4 |
| RFEDGOV   | Forecasts for the quarterly chain-weighted real federal government consumption and gross investment. SA, annual rate, base years 1992-95 | 1991 Q1 - 2018 Q4 |
| RLSGOV    | Forecasts for the quarterly level of chain-weighted real state and local government consumption and gross investment. SA. annual rate. base years 1992 - 1995 | 1991 Q1 - 2018 Q4 |
| CPI       | Forecasts for the headline CPI inflation rate. SA, annual rate, % points. Quarterly forecasts are annualized quarter-overquarter percent changes of the quarterly average price index level | 1991 Q1 - 2018 Q4 |

Source: Survey of Professional Forecasters documentation. SA = Seasonal Adjusted.
Table 3: Variables, samples and number of forecasters

| Variable   | Sample (1) | Sample (2) | Sample (3) | Sample (4) | Sample (5) |
|------------|------------|------------|------------|------------|------------|
|            | T  | J  | T  | J  | T  | J  | T  | J  | T  | J  |
| NGDP       | 24 | 3  | 20$^a$ | 6  | 15$^d$ | 10 | 9$^g$ | 18 | 5  | 22 |
| PGDP       | 24 | 3  | 20$^a$ | 6  | 15$^d$ | 10 | 9$^g$ | 20 | 5  | 25 |
| UNEMP      | 24 | 4  | 20$^a$ | 6  | 15$^d$ | 12 | 9$^g$ | 22 | 5  | 27 |
| EMP        |    |    |       |    |       |    |       |    |    |    |
| INDPROD    | 24 | 4  | 19$^b$ | 8  | 15$^d$ | 12 | 9$^g$ | 21 | 5  | 26 |
| HOUSING    | 24 | 4  | 19$^b$ | 10 | 15$^d$ | 15 | 10$^f$ | 19 | 5  | 26 |
| TBILL      | 24 | 5  | 19$^b$ | 8  | 15$^d$ | 11 | 9$^g$ | 19 | 5  | 24 |
| BOND       | 24 | 3  | 19$^b$ | 5  | 15$^d$ | 7  | 9$^g$ | 13 | 5  | 17 |
| RRESINV    | 24 | 5  | 20$^a$ | 9  | 15$^d$ | 13 | 9$^g$ | 19 | 5  | 28 |
| RGDP       | 24 | 5  | 20$^a$ | 9  | 15$^d$ | 14 | 9$^g$ | 25 | 5  | 31 |
| RCONSUM    | 24 | 5  | 20$^a$ | 9  | 16$^c$ | 13 | 10$^f$ | 20 | 5  | 29 |
| RNREIN     | 24 | 5  | 20$^a$ | 9  | 16$^c$ | 13 | 10$^f$ | 20 | 5  | 29 |
| RFEDGOV    | 24 | 5  | 20$^a$ | 9  | 16$^c$ | 13 | 10$^f$ | 19 | 5  | 28 |
| RLSGOV     | 24 | 5  | 20$^a$ | 9  | 16$^c$ | 13 | 10$^f$ | 19 | 5  | 28 |
| CPI        | 24 | 5  | 20$^a$ | 8  | 16$^c$ | 12 | 10$^f$ | 19 | 5  | 29 |

$T =$ number of periods, $J =$ number of forecasters, Sample (1): 1991 - 2014; Sample (2) a) 1995 - 2014; b) 1996 - 2014; Sample (3) c) 1999 - 2014; d) 2000 - 2014; e) 2001 - 2014; Sample (4) f) 2005 - 2014; g) 2006 - 2014; Sample (5) 2010 - 2014; For the EMP variable the samples are: (1) 2004-2014; (2) 2005-2014; (3) 2007-2014 and (4) 2010-2014

Table 4: Summary between combinations depending on $J$ and $T$

| J < T     | AVERAGE | S_STC | CAS  | SIMPLEX | TOTAL |
|-----------|---------|-------|------|---------|-------|
| (%        | (282)   | (111) | (332) | (443)   | (725) |
| (38.90)   | (15.31) | (45.79)| (61.10)| (57.27) |
| J > T     | 189     | 85    | 267  | 352     | 541   |
| (%        | (34.94) | (15.71)| (49.35)| (65.06)| (42.73) |
| TOTAL     | 469     | 194   | 603  | 797     | 1266  |
| (%)       | (37.05) | (15.32)| (47.63)| (62.95)|       |

Number of times that an accuracy measure favored a combination procedure.
Table 5: Results for each combination procedure by variable and accuracy criteria. Percentages of beats

| Variable | Mean Error | RMSE | MAPE | MdAPE |
|----------|------------|------|------|-------|
|          | AVE | S_STC | CAS | AVE | S_STC | CAS | AVE | S_STC | CAS | AVE | S_STC | CAS |
| NGDP     | 47.62 | 14.29 | 38.10 | 47.83 | 13.04 | 39.13 | 56.52 | 13.04 | 30.43 | 66.67 | 9.52 | 23.81 |
| PGDP     | 21.74 | 39.13 | 39.13 | 41.67 | 33.33 | 25.00 | 30.77 | 42.31 | 26.92 | 32.00 | 40.00 | 28.00 |
| UNEMP    | 28.57 | 9.52  | 61.9 | 42.86 | 14.29 | 42.86 | 38.10 | 9.52  | 52.38 | 38.10 | 9.52  | 52.38 |
| EMP      | 72.22 | 11.11 | 16.67 | 82.35 | 5.88  | 11.76 | 78.95 | 5.26  | 15.79 | 78.95 | 5.26  | 15.79 |
| INDPROM  | 20.00 | 25.00 | 55.00 | 25.00 | 5.00  | 70.00 | 23.81 | 9.52  | 66.67 | 9.09  | 13.64 | 77.27 |
| HOUSING  | 0.00  | 15.00 | 85.00 | 4.76  | 19.05 | 76.19 | 4.76  | 9.52  | 85.71 | 0.00  | 9.09  | 90.91 |
| TBILL    | 75.00 | 0.00  | 25.00 | 75.00 | 5.00  | 20.00 | 70.00 | 10.00 | 20.00 | 50.00 | 9.09  | 40.91 |
| BOND     | 10.00 | 5.00  | 85.00 | 20.00 | 10.00 | 70.00 | 15.00 | 15.00 | 70.00 | 10.00 | 15.00 | 75.00 |
| RRESIN   | 57.14 | 4.76  | 38.10 | 50.00 | 22.73 | 27.27 | 57.14 | 9.52  | 33.33 | 54.55 | 9.09  | 36.36 |
| RGDP     | 4.76  | 4.76  | 90.48 | 8.70  | 17.39 | 73.91 | 9.09  | 13.64 | 77.27 | 9.09  | 9.09  | 81.82 |
| RCONSUM  | 13.64 | 18.18 | 68.18 | 28.57 | 9.52  | 61.90 | 31.82 | 9.09  | 59.09 | 25.00 | 5.00  | 70.00 |
| RNRESIN  | 20.00 | 25.00 | 55.00 | 31.82 | 22.73 | 45.45 | 30.00 | 20.00 | 50.00 | 25.00 | 8.33  | 66.67 |
| RFEDGOV  | 42.86 | 19.05 | 38.10 | 65.00 | 25.00 | 10.00 | 71.43 | 14.29 | 14.29 | 55.00 | 20.00 | 25.00 |
| RLGOV    | 45.00 | 30.00 | 25.00 | 55.00 | 25.00 | 20.00 | 40.00 | 25.00 | 35.00 | 31.82 | 40.91 | 27.27 |
| CPI      | 52.63 | 10.53 | 36.84 | 45.00 | 5.00  | 50.00 | 50.00 | 15.00 | 35.00 | 50.00 | 10.00 | 40.00 |
| MEAN     | 34.08 | 15.42 | 50.50 | 41.57 | 15.53 | 42.90 | 40.49 | 14.71 | 44.79 | 35.68 | 14.24 | 50.08 |
|          | Mean Error | RMSE  | MAPE | MdAPE |
|----------|------------|-------|------|-------|
| AVE      | S_MT       | CAS   |      |       |
| J < T    | 58         | 26    | 92   |       |
| (%)      | (32.95)    | (14.77) | (52.27) |       |
| J > T    | 45         | 22    | 64   |       |
| (%)      | (34.35)    | (16.79) | (48.85) |       |
| TOTAL    | 103        | 48    | 156  |       |
| (%)      | (33.55)    | (15.64) | (50.81) |       |

Table 6 b) Number of beats of AVERAGE and SIMPLEX combinations by accuracy criteria and number of forecasts

|          | Mean Error | RMSE  | MAPE | MdAPE |
|----------|------------|-------|------|-------|
| AVE      | S_MT       | SIMPLEX |      |       |
| J < T    | 58         | 118   |      |       |
| (%)      | (32.95)    | (67.05) |       |       |
| J > T    | 45         | 86    |      |       |
| (%)      | (34.35)    | (65.65) |       |       |
| TOTAL    | 103        | 204   |      |       |
| (%)      | (33.55)    | (66.45) |       |       |
Table 7: Coefficients of variation for selected variables and results of the combination procedures by samples

| Sample (1) | NGDP CV | AVE  | S_STC | CAS | UNEMP CV | AVE  | S_STC | CAS | RLSGOV CV | AVE  | S_STC | CAS | HOUSING CV | AVE  | S_STC | CAS |
|------------|--------|------|-------|-----|----------|------|-------|-----|-----------|------|-------|-----|-----------|------|-------|-----|
| 2015       | 0.667  | 4    | 0     | 0   | 0.603    | 4    | 0     | 0   | 0.588     | 3    | 1     | 0   | 7.970     | 0    | 0     | 4   |
| 2016       | 0.405  | 0    | 4     | 0   | 2.150    | 0    | 3     | 1   | 0.724     | 2    | 0     | 2   | 11.287    | 0    | 0     | 4   |
| 2017       | 0.313  | 4    | 0     | 0   | 4.927    | 0    | 0     | 4   | 0.695     | 0    | 4     | 0   | 9.604     | 0    | 0     | 4   |
| 2018       | 0.409  | 4    | 0     | 0   | 1.575    | 2    | 0     | 2   | 0.324     | 4    | 1     | 0   | 9.850     | 0    | 4     | 4   |
| Sample (2) |        |      |       |     |          |      |       |     |           |      |       |     |            |      |       |     |
| 2015       | 0.830  | 4    | 0     | 0   | 3.447    | 1    | 0     | 3   | 0.834     | 0    | 0     | 4   | 6.813     | 0    | 1     | 4   |
| 2016       | 0.365  | 2    | 1     | 2   | 2.011    | 3    | 1     | 0   | 0.627     | 3    | 1     | 0   | 9.079     | 0    | 2     | 3   |
| 2017       | 0.245  | 0    | 1     | 3   | 4.080    | 0    | 0     | 4   | 0.566     | 1    | 3     | 0   | 8.518     | 0    | 1     | 2   |
| 2018       | 0.465  | 4    | 0     | 0   | 2.434    | 3    | 0     | 1   | 1.666     | 0    | 0     | 4   | 7.486     | 2    | 0     | 1   |
| Sample (3) |        |      |       |     |          |      |       |     |           |      |       |     |            |      |       |     |
| 2015       | 0.740  | 4    | 0     | 0   | 2.743    | 1    | 1     | 3   | 0.807     | 2    | 0     | 2   | 6.575     | 0    | 0     | 4   |
| 2016       | 0.342  | 1    | 2     | 2   | 1.829    | 4    | 0     | 0   | 0.550     | 3    | 1     | 0   | 8.528     | 0    | 0     | 4   |
| 2017       | 0.217  | 2    | 3     | 2   | 3.804    | 0    | 0     | 4   | 0.504     | 0    | 0     | 4   | 7.485     | 0    | 0     | 4   |
| 2018       | 0.392  | 4    | 0     | 0   | 2.951    | 2    | 1     | 1   | 1.418     | 1    | 3     | 0   | 6.476     | 0    | 0     | 4   |
| Sample (4) |        |      |       |     |          |      |       |     |           |      |       |     |            |      |       |     |
| 2015       | 0.590  | 1    | 0     | 3   | 2.253    | 0    | 0     | 4   | 0.725     | 3    | 1     | 0   | 6.188     | 0    | 0     | 4   |
| 2016       | 0.306  | 1    | 0     | 3   | 1.766    | 3    | 3     | 1   | 0.499     | 1    | 1     | 2   | 7.665     | 0    | 1     | 3   |
| 2017       | 0.248  | 0    | 0     | 4   | 3.104    | 1    | 0     | 3   | 0.446     | 0    | 4     | 0   | 6.814     | 0    | 0     | 5   |
| 2018       | 0.502  | 0    | 0     | 4   | 3.132    | 4    | 0     | 0   | 1.751     | 4    | 1     | 0   | 6.247     | 0    | 1     | 3   |
| Sample (5) |        |      |       |     |          |      |       |     |           |      |       |     |            |      |       |     |
| 2015       | 0.546  | 3    | 0     | 1   | 2.384    | 0    | 0     | 4   | 0.803     | 4    | 0     | 0   | 5.869     | 0    | 1     | 6   |
| 2016       | 0.360  | 4    | 0     | 2   | 1.358    | 0    | 0     | 4   | 1.920     | 1    | 3     | 0   | 6.403     | 0    | 0     | 4   |
| 2017       | 0.233  | 1    | 0     | 3   | 3.043    | 0    | 0     | 4   | 0.429     | 0    | 0     | 4   | 6.320     | 0    | 0     | 4   |
| 2018       | 0.459  | 4    | 0     | 0   | 3.463    | 3    | 0     | 1   | 1.708     | 3    | 1     | 0   | 5.825     | 0    | 4     | 0   |
Table 8: Classification of the combination procedures according to their forecast ability

|       | AVERAGE | S_STC  | CAS   |
|-------|---------|--------|-------|
|       | #       | %      | #     | %      | #     | %      |
| Case 1| 40      | 32.8   | 8     | 17.8   | 63    | 52.1   |
| Case 2| 7       | 5.7    | 6     | 13.3   | 16    | 13.2   |
| Case 3| 60      | 49.2   | 27    | 60.0   | 24    | 19.8   |
| Case 4| 15      | 12.3   | 4     | 8.9    | 18    | 14.9   |
| Total | 122     | 45     | 121   |

#: Proportions (bias, variance and covariance) of the best MSFE procedure included in specific cases.