I. INTRODUCTION AND SUMMARY

In an earlier work ([1], hereafter referred to as Paper 0), the geometry of a slowly rotating black hole deformed by tidal forces was determined under the assumption that the tidal forces are weak and vary slowly with time. The construction relied on perturbative techniques, and the metric of the deformed black hole was expressed as a perturbation of a background Kerr metric with mass $M$ and dimensionless spin $\chi^a \ll 1$. The perturbation introduces new terms to the metric, and these are constructed from tidal moments $E_{ab}$ and $B_{ab}$ that provide, at leading order in the tidal deformation, a complete characterization of a generic tidal environment. In particular, the perturbation includes terms that arise from the coupling between $\chi^a$ and $E_{ab}$, and the coupling between $\chi^a$ and $B_{ab}$; these capture all consequences of the dragging of inertial frames on the black hole’s tidal deformation. The metric was cast in light-cone coordinates $(v, r, \theta, \phi)$ that possess a clear geometrical meaning: the advanced-time coordinate $v$ is constant on light cones that converge toward the black hole, the angular coordinates $(\theta, \phi)$ are constant on the null generators of each light cone, and the radial coordinate $r$ is an affine parameter on each generator.

In this paper we replace the black hole by a compact body of mass $M$, radius $R$, and dimensionless spin $\chi^a \ll 1$, and rely on the fact that to first order in $\chi^a$, the external geometries of the unperturbed bodies are identical (differences occur at second and higher orders). We generalize the perturbative solution of Paper 0 to account for the presence of matter inside $r = R$. The new solution is no longer required to be regular at $r = 2M$ (which marks the position of the black hole’s horizon), and as a consequence we find that it contains terms that were absent from the black-hole metric. These describe the body’s response to the tidal deformation, and they come with dimensionless multiplicative constants known as gravitational Love numbers. In the case of a nonrotating body ($\chi^a = 0$) [2–4], two types of Love numbers appear in the metric: a gravito-electric Love number $K_2^{el}$ that measures the body’s response to the tidal forces associated with $E_{ab}$, and a gravito-magnetic Love number $K_2^{mag}$ that measures the body’s response to those associated with $B_{ab}$. With rotation included to first order, we find here that the description of the external geometry of a tidally deformed body requires the introduction of four new numbers, which we call rotational-tidal Love numbers. The first two, denoted $\mathcal{E}^a$ and $\mathcal{S}^a$ below, are associated with terms that couple $\chi^a$ to $E_{ab}$ in the metric; the remaining two, denoted $\mathcal{B}^a$ and $\mathcal{R}^a$, are associated with terms that couple $\chi^a$ to $B_{ab}$. The label $\mathcal{E}$ indicates that $\mathcal{E}^a$ and $\mathcal{B}^a$ are associated with quadrupolar terms in the metric; the label $\mathcal{S}$ indicates that $\mathcal{S}^a$ and $\mathcal{R}^a$ are associated with octupolar terms. All six Love numbers are gauge-invariant in the usual sense of perturbation theory: While the form of the solution may change under a gauge transformation (understood as an infinitesimal transformation of the background coordinates), the value of the Love numbers are preserved. And our construction guarantees that $K_2^{el} = K_2^{mag} = \mathcal{E}^a = \mathcal{S}^a = \mathcal{B}^a = \mathcal{R}^a = 0$ for a slowly rotating black hole.

1 A different set of Love numbers, $k_2^{el}$ and $k_2^{mag}$, was introduced in Ref. [3]. It is related to the set used here by $k_2^{el} = (2M/R)^5 K_2^{el}$ and $k_2^{mag} = (2M/R)^5 K_2^{mag}$. The former notation for the gravito-magnetic Love number was unfortunate, because Favata [4] has shown that $K_2^{mag}$ scales as $(R/M)^4$ instead of $(R/M)^5$. 

Tidal deformation of a slowly rotating material body. I. External metric

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Our considerations in this paper are limited to the exterior geometry of a slowly rotating, tidally deformed body. In this context, the new Love numbers \((\mathbb{E}^q, \mathbb{S}^q, \mathbb{B}^q, \mathbb{R}^q)\), like the old Love numbers \((K_{2e}^q, K_{2e}^{\text{mag}}, K_{2q}^q)\), must be left undetermined. The determination of the Love numbers requires the construction of an internal geometry based on an assumed model for the body’s matter content, and a match of the internal and external geometries at the matter’s boundary; this procedure will be detailed in Paper I\(^\text{[7]}\). The numerical values adopted by the Love numbers reflect the details of internal structure, and an external measurement of the tidal properties of the body can therefore reveal otherwise inaccessible aspects of the interior.

This observation has motivated a recent surge of activity in the development of a relativistic theory of tidal deformation and dynamics, in the context of the measurement of tidal effects in gravitational waves emitted by neutron-star binaries \([6, 19]\) and during the capture of solar-mass compact binaries by supermassive black holes \([20–25]\). The Love number \(K_{2e}^q\) of a neutron star was implicated, along with its moment of inertial \(I\) and its rotational quadrupole moment \(Q\), in the remarkable \(I\)-Love-\(Q\) relations \([26–32]\), and tidal invariants have been inserted within point-particle actions to account for the tidal response of an extended body \([33–37]\). The geometry of a tidally deformed black hole was constructed beyond the small rotation limit \([38, 39]\) and beyond perturbation theory \([40]\).

We begin our work in Sec. III with the construction of tidal potentials obtained from \(\chi_a, \mathcal{E}_{ab}, \text{and } B_{ab}\). These form the fundamental building blocks of the perturbed metric, which is calculated in Sec. III in the light-cone coordinates \((v, r, \theta, \phi)\). To illustrate the features of this metric, it is helpful to introduce Cartesian coordinates \(x^a = (x, y, z)\) defined in the usual way from the spherical polar coordinates \((r, \theta, \phi)\). In this notation, the time-time component of the metric obtained in Sec. III takes the form of

\[
g_{vv} = -1 + \frac{2M}{r} \left[ 1 + \cdots + 2K_{2e}^q \left( \frac{2M}{r} \right)^5 (1 + \cdots) \right] \mathcal{E}_{ab} x^a x^b + \frac{1}{2} \left( \frac{2M}{r} \right)^2 \left[ 1 + \cdots + 8\mathbb{E}^q \left( \frac{2M}{r} \right)^3 (1 + \cdots) \right] \chi^c \epsilon_{cda} \mathbb{E}_b^d x^a x^b + 2M(1 + \cdots) \chi^b B_{ab} x^a - \frac{(2M)^2}{2r^3} \left[ 1 + \cdots + 4\mathbb{R}^q \left( \frac{2M}{r} \right)^4 (1 + \cdots) \right] \chi^{(a} B_{bc)} x^a x^b x^c, \tag{1.1}
\]

in which \(\epsilon_{abc}\) is the completely antisymmetric permutation symbol, all indices are raised and lowered with the Euclidean metric \(\delta_{ab}\), the notation \((abc)\) indicates symmetrization of all indices and removal of all traces, and ellipses designate relativistic corrections of order \(2M/r\) and higher.

The first two terms in \(g_{vv}\) originate from the background metric, and the following terms proportional to \(\mathcal{E}_{ab} x^a x^b\) describe a quadrupolar tidal deformation of the geometry; the decaying piece involving \(K_{2e}^q\) represents the body’s response to the applied tidal field. The next sequence of terms describes another quadrupolar deformation that arises from the coupling between \(\chi^a\) and \(\mathcal{E}_{ab}\); the decaying piece involving \(\mathbb{E}^q\) represents the body’s response to this coupling. Following this we find a dipolar deformation that results from the coupling between \(\chi^a\) and \(B_{ab}\). The presence of a dipole indicates that the body is accelerated in the tidal field, the acceleration being measured by \(a_a = -MB_{ab} \chi^b\); this is the Mathisson-Papapetrou spin force \([41–43]\) discussed in detail in Paper 0. And finally, the last set of terms describes an octupolar deformation that also arises from the coupling between \(\chi^a\) and \(B_{ab}\); the decaying piece involving \(\mathbb{R}^q\) represents the body’s response. The time-time component of the metric features only the Love numbers \(K_{2e}^q, \mathbb{E}^q, \text{and } \mathbb{R}^q\); the remaining Love numbers appear in the remaining components of the metric.

The light-cone conditions placed on the metric of Sec. III only partially determine the \((v, r, \theta, \phi)\) coordinates, and as a result, the metric features a number of arbitrary constants that serve to further specify the choice of gauge. While the coordinate system is still geometrically meaningful, the residual gauge freedom implies that the uninteresting gauge parameters must be determined alongside the physically interesting Love numbers, making the task more cumbersome than it has to be. (An instance of this additional burden can be observed in Ref. \([3]\).) To facilitate the determination of the Love numbers in Paper II, in Sec. IV we present another version of the external metric, this time adopting the standard Boyer-Lindquist \((t, r, \theta, \phi)\) coordinates for the background metric, and the familiar Regge-Wheeler gauge for the perturbation.

As emphasized previously, the complete collection of Love numbers \((K_{2e}^q, K_{2e}^{\text{mag}}, \mathbb{E}^q, \mathbb{S}^q, \mathbb{B}^q, \mathbb{R}^q)\) for a selected stellar model must be determined by matching an internal metric to the external metric provided in Sec. III or Sec. IV. For a given equation of state for cold matter — the functions \(p(\rho)\) and \(\epsilon(\rho)\), in which \(\rho\) is the rest-mass density, \(p\) the pressure, and \(\epsilon\) the density of internal (thermodynamic) energy — the Love numbers can be expressed as functions of

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\(^2\) Philippe Landry and Eric Poisson, in preparation.
$M/R$, the stellar compactness. While it is known that $K_2^3$ scales as $(R/M)^5$ and $K_2^{\text{mag}}$ scales as $(R/M)^4$ — modulo relativistic corrections of order $M/R$ and higher — our considerations in this paper give us no information regarding the expected dominant scaling of the remaining Love numbers with $R/M$. In order to gain some insight into this matter, in Sec. IV we exploit post-Newtonian methods and attempt to calculate the Love numbers of a rigidly rotating ball of incompressible fluid. Based on this calculation, we conclude that $\mathcal{E}^q \sim (R/M)^3$ and $\mathcal{E}^o \sim (R/M)^3$ for a generic body with an arbitrary equation of state. Our results for the remaining Love numbers are more tentative. Here we conclude that $\mathcal{B}^q \sim (R/M)^3$ and $\mathcal{B}^o \sim (R/M)^4$, with an admission that one of these relations might be off by one power of $R/M$: it is thus possible that $\mathcal{B}^q$ actually scales as $(R/M)^2$, or that $\mathcal{B}^o$ actually scales as $(R/M)^3$. These provisional assignments will be confirmed once a proper matching with an internal metric is carried out in Paper II.

As this work was reaching completion we were made aware of an independent effort\footnote{Paolo Pani, Leonardo Gualtieri, Andrea Maselli, and Valeria Ferrari, in preparation.} to describe the external geometry of a slowly rotating material body deformed by tidal forces. This work generalizes ours in the sense that it constructs the external metric to second order in the dimensionless spin $\chi^o$. It is also a restricted version of our own efforts, in the sense that it allows only for axisymmetric tidal environments with vanishing $\mathcal{B}_{ab}$.

II. TIDAL POTENTIALS

The construction of tidal potentials is presented in great detail in Sec. II of Paper 0 \cite{2014PhRvD..89d4009P}. Here we summarize the main results, and introduce the new moments $\hat{\mathcal{E}}_{ab}$ and $\hat{\mathcal{B}}_{ab}$ that were missed in the earlier work.

The potentials are obtained by combining $\chi_a$, $\mathcal{E}_{ab}$, and $\Omega^{q}$, and $\mathcal{E}^q := x^q/r = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$ in various irreducible ways, with each potential carrying a specific multipole order $\ell$ and a specific parity label (even or odd). The coupling of $\chi_a$ and $\mathcal{E}_{ab}$ produces the odd-parity tensors

$$\mathcal{F}_a := \mathcal{E}_{ab} \chi^b, \quad \mathcal{F}_{abc} := \mathcal{E}_{(ab} \chi_{c)}$$

(2.1)

in which the angular brackets designate the operation of symmetrization and trace removal, so that $\mathcal{F}_{abc}$ is a symmetric-tracefree (STF) tensor. It produces also the even-parity tensor

$$\hat{\mathcal{E}}_{ab} := 2\chi^c \epsilon_{c[d(\mathcal{E}^d)]}$$

(2.2)

in which $\epsilon_{abc}$ is the antisymmetric permutation symbol. The coupling of $\chi_a$ and $\mathcal{B}_{ab}$ produces the even-parity tensors

$$\mathcal{K}_{a} := \mathcal{B}_{ab} \chi^b, \quad \mathcal{K}_{abc} := \mathcal{B}_{(ab} \chi_{c)}$$

(2.3)

and the odd-parity tensor

$$\hat{\mathcal{B}}_{ab} := 2\chi^c \epsilon_{c[d(\mathcal{B}^d)]}$$

(2.4)

The independent components of the STF tensors $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, $\hat{\mathcal{E}}_{ab}$, $\mathcal{F}_a$, $\mathcal{F}_{abc}$, $\mathcal{K}_{a}$, $\hat{\mathcal{B}}_{ab}$, and $\mathcal{K}_{abc}$ can be packaged in spherical-harmonic coefficients $\mathcal{E}_m^a$, $\mathcal{B}_m^a$, $\hat{\mathcal{E}}_m^a$, $\mathcal{F}_m$, $\mathcal{F}_m^a$, $\mathcal{K}_m^a$, $\hat{\mathcal{B}}_m^a$, and $\mathcal{K}_m^a$, respectively. The definitions are summarized in Table III.

The tidal potentials are decomposed in scalar, vector, and tensor spherical harmonics, functions of the angular coordinates $\hat{\theta}^A = (\theta, \phi)$. The decomposition involves the scalar harmonics of Table III the even-parity harmonics

$$Y^\ell_m := D_A Y^\ell_m, \quad Y^\ell_{AB} := \left[ D_A D_B + \frac{1}{2} (\ell + 1) \Omega_{AB} \right] Y^\ell_m$$

(2.5)

and the odd-parity harmonics

$$X^\ell_m := -\epsilon^B_A D_B Y^\ell_m, \quad X^\ell_{AB} := -\frac{1}{2} (\epsilon^C_A D_B + \epsilon^C_B D_A) D_C Y^\ell_m.$$

(2.6)

Here $\Omega_{AB} = \text{diag}(1, \sin^2 \theta)$ is the metric on a unit two-sphere, and $D_A$ is the covariant-derivative operator compatible with this metric; $\epsilon_{AB}$ is the Levi-Civita tensor on the unit two-sphere ($\epsilon_{\theta \phi} = \sin \theta$), and its index is raised with $\Omega_{AB}$, the matrix inverse to $\Omega_{AB}$. It should be noted that the tensorial harmonics are tracefree, in the sense that $\Omega_{AB} Y^\ell_{AB} = \Omega_{AB} X^\ell_{AB} = 0$.\footnote{Paolo Pani, Leonardo Gualtieri, Andrea Maselli, and Valeria Ferrari, in preparation.}
The relations between $K^d_A$, $K_d$, and $\chi E_{m}^q$ are identical to those between $F_{m}^d$, $F_{a}$, and $\chi E_{m}^a$. The relations between $B_{m}^q$, $B_{ab}$, and $\chi E_{m}^q$ are identical to those between $E_{m}^q$, $E_{ab}$, and $\chi E_{m}^a$. Finally, the relations between $K^d_{m}$, $K_{abc}$, and $\chi E_{m}^q$ are identical to those between $F_{m}^d$, $F_{abc}$, and $\chi E_{m}$. 

The decomposition of the tidal potentials in spherical harmonics is described by

\begin{align}
\chi_{A}^d & = \sum_{m} \chi_{m}^{d} T_{A}^{1m}, \\
F_{A}^{d} & = \sum_{m} F_{m}^{d} T_{A}^{1m}, \\
K^{d} & = \sum_{m} K_{m}^{d} T_{1m}, \\
K_{A}^{d} & = \sum_{m} K_{m}^{d} T_{A}^{1m}, \\
E^{q} & = \sum_{m} E_{m}^{q} T_{2m}, \\
E_{A}^{q} & = \frac{1}{2} \sum_{m} E_{m}^{q} T_{A}^{2m}, \\
E_{AB}^{q} & = \sum_{m} E_{m}^{q} T_{AB}^{2m}, \\
\hat{E}^{q} & = \sum_{m} \hat{E}_{m}^{q} T_{2m}, \\
\hat{E}_{A}^{q} & = \frac{1}{2} \sum_{m} \hat{E}_{m}^{q} T_{A}^{2m}, \\
\hat{E}_{AB}^{q} & = \sum_{m} \hat{E}_{m}^{q} T_{AB}^{2m}, \\
B_{A}^{q} & = \frac{1}{2} \sum_{m} B_{m}^{q} T_{A}^{2m}, \\
B_{AB}^{q} & = \sum_{m} B_{m}^{q} T_{AB}^{2m}, \\
\hat{B}_{A}^{q} & = \frac{1}{2} \sum_{m} \hat{B}_{m}^{q} T_{A}^{2m}, \\
\hat{B}_{AB}^{q} & = \sum_{m} \hat{B}_{m}^{q} T_{AB}^{2m}, \\
F_{A}^{o} & = \frac{1}{3} \sum_{m} F_{m}^{o} T_{A}^{3m}, \\
F_{AB}^{o} & = \frac{1}{3} \sum_{m} F_{m}^{o} T_{AB}^{3m}
\end{align}
In Paper 0 the metric of a slowly rotating, tidally deformed black hole was presented in terms of the potentials, without recognizing that these are in fact associated with the moments of Eqs. (2.2) and (2.4). Similarly, we have that

\[
\begin{align*}
Y^{1,0} &= C \\
Y^{1,1c} &= S \cos \phi \\
Y^{1,1s} &= S \sin \phi \\
Y^{2,0} &= 1 - 3C^2 \\
Y^{2,1c} &= 2SC \cos \phi \\
Y^{2,1s} &= 2SC \sin \phi \\
Y^{2,2c} &= S^2 \cos 2\phi \\
Y^{2,2s} &= S^2 \sin 2\phi \\
Y^{3,0} &= C(3 - 5C^2) \\
Y^{3,1c} &= \frac{3}{2}S(1 - 5C^2) \cos \phi \\
Y^{3,1s} &= \frac{3}{2}S(1 - 5C^2) \sin \phi \\
Y^{3,2c} &= 3S^2C \cos 2\phi \\
Y^{3,2s} &= 3S^2C \sin 2\phi \\
Y^{3,3c} &= S^3 \cos 3\phi \\
Y^{3,3s} &= S^3 \sin 3\phi
\end{align*}
\]

TABLE II. Spherical-harmonic functions \(Y^{\ell m}\). The functions are real, and they are listed for the relevant modes \(\ell = 1\) (dipole), \(\ell = 2\) (quadrupole), and \(\ell = 3\) (octupole). The abstract index \(m\) describes the dependence of these functions on the angle \(\phi\); for example \(Y^{\ell,2}\) is proportional to \(\sin 2\phi\). To simplify the expressions we write \(C := \cos \theta\) and \(S := \sin \theta\).

\[
K^m = \sum_m K^m Y^{3m}, \quad K_A^0 = \frac{1}{3} \sum_m K^m Y^{3m}_A, \quad K_{AB}^0 = \frac{1}{3} \sum_m K^m Y^{3m}_{AB}.
\] (2.7i)

It should be noted that tensorial potentials are not defined when \(\ell = 1\). The relations between \(\hat{E}_m^q\) and \(\chi \hat{E}_m^q\) displayed in Table II imply that the tidal potentials associated with \(\hat{E}_{AB}\) can also be expressed as

\[
\begin{align*}
\hat{E}_m^q &= -\chi \partial_\phi E_m^q, \\
\hat{E}_A^q &= -\chi \partial_\phi \hat{E}_A^q, \\
\hat{E}_{AB}^q &= -\chi \partial_\phi \hat{E}_{AB}^q.
\end{align*}
\] (2.8)

Similarly, we have that

\[
\begin{align*}
\hat{B}_A^q &= -\chi \partial_\phi B_A^q, \\
\hat{B}_{AB}^q &= -\chi \partial_\phi \hat{B}_{AB}^q.
\end{align*}
\] (2.9)

In Paper 0 the metric of a slowly rotating, tidally deformed black hole was presented in terms of the \(\phi\)-differentiated potentials, without recognizing that these are in fact associated with the moments of Eqs. (2.2) and (2.4).

III. EXTERNAL METRIC OF A TIDALLY DEFORMED BODY — LIGHT-CONE GAUGE

The external metric of an isolated, slowly rotating body of mass \(M\) and dimensionless spin \(\chi := |\chi^a| \ll 1\) can be expressed as

\[
d^2s = -f dv^2 + 2 dvdr + r^2 d\Omega^2 - 2\frac{\chi M^2}{r} \sin^2 \theta dv d\phi,
\] (3.1)

where \(f := 1 - 2M/r\) and \(d\Omega^2 := \Omega_{AB} d\theta^A d\theta^B := d\theta^2 + \sin^2 \theta d\phi^2\). The metric is displayed in coordinates \((v, r, \theta, \phi)\) that are tied to the behavior of incoming null geodesics that are tangent to converging light cones. It is easy to show that each surface \(v = \text{constant}\) is a null hypersurface, and that its null generators move with constant values of \(\theta\) and \(\phi\); \(-r\) is an affine parameter on each null geodesic.

The metric of a slowly rotating body immersed in a tidal field produced by remote matter is obtained by perturbing the background metric of Eq. (3.1). The methods to construct the perturbation are described in detail in Paper 0 [1], for the specific case in which the body is a black hole. We adopt these methods here (with very few details provided), and make the required changes to account for the presence of matter and the absence of an event horizon.

In this section we continue to work in light-cone coordinates, so that the coordinates \((v, r, \theta, \phi)\) keep their geometrical meaning in the perturbed spacetime: \(v\) continues to be constant on each converging light cone, \(\theta\) and \(\phi\) continue to be
TABLE III. Radial functions appearing in the metric of Eq. (3.2), expressed in terms of \( x := r/(2M) \), \( f := 1 - 1/x \), and a number of integration constants. All functions within square brackets behave as \( 1 + O(1/x) \) when \( x \gg 1 \).

\[

e^\theta_1 = f^2 + \frac{\gamma^2}{\bar{\beta}} \left[ -30x^3(x - 1)^2 \ln f + \frac{\gamma}{\bar{\beta}} x(2x - 1)(1 + 6x - 6x^2) \right] K_2^2
\]

\[
e^\theta_2 = f^2 - \left( 20x^4(x - 1) \ln f + \frac{\gamma}{\bar{\beta}} x(1 + 2x + 6x^2 - 12x^3) \right) K_2^3
\]

\[
e^\theta_3 = 1 - \frac{\gamma}{\bar{\beta}} + \frac{\gamma}{\bar{\beta}} \left[ -15x^3(2x^2 - 1) \ln f + 5x^2(1 - 3x - 6x^2) \right] K_2^2
\]

\[
\beta^1_1 = f - \frac{1}{x} \left[ 20x^4(x - 1) \ln f - \frac{\gamma}{\bar{\beta}} x(1 + 2x + 6x^2 - 12x^3) \right] M_{\text{mag}}^{(1,1)}
\]

\[
\beta^1_2 = 1 - \frac{\gamma}{\bar{\beta}} + \frac{\gamma}{\bar{\beta}} \left[ -15x^3(2x^2 - 1) \ln f + 5x^2(1 - 3x - 6x^2) \right] K_2^{\text{mag}}
\]

\[
e^\theta_4 = \frac{1}{x^2} \left[ 60x^5(x - 1)^2 \ln f + 10x^3(x - 1)(1 - 9x + 12x^2) \ln f - \frac{\gamma}{\bar{\beta}} x(1 + 48x^2 + 108x^3 - 72x^4) \right] K_2^2
\]

\[
e^\theta_5 = \frac{1}{x^2} \left[ -30x^3(x - 1)^2 \ln f + \frac{\gamma}{\bar{\beta}} x(2x - 1)(1 + 6x - 6x^2) \right] K_2^3
\]

\[
e^\theta_6 = \frac{1}{x^2} \left[ 30x^3(x - 1)^2 \ln f + \frac{\gamma}{\bar{\beta}} x(2x - 1)(1 + 6x - 6x^2) \right] K_2^2
\]

\[
\beta^2_1 = \frac{1}{x} \left[ 20x^4(x - 1) \ln f - \frac{\gamma}{\bar{\beta}} x(1 + 2x + 6x^2 - 12x^3) \right] M_{\text{mag}}^{(2,1)}
\]

\[
\beta^2_2 = 1 - \frac{\gamma}{\bar{\beta}} + \frac{\gamma}{\bar{\beta}} \left[ -15x^3(2x^2 - 1) \ln f + 5x^2(1 - 3x - 6x^2) \right] K_2^{\text{mag}}
\]

\[
k^1_1 = \frac{1}{x^2} \left[ 15x^3(x - 1)(2x - 1)(5x - 1) \ln f - \frac{\gamma}{\bar{\beta}} x(1 + 2x - 34x^2 + 144x^3 - 120x^4) \right] K_2^{\text{mag}}
\]

\[
k^1_2 = 1 + \frac{\gamma}{\bar{\beta}} \left[ -30x^3(x - 1) \ln f - \frac{\gamma}{\bar{\beta}} x(2x - 1)(1 + 6x + 30x^2) \right] K_2^{\text{mag}}
\]

\[
k^1_3 = \frac{1}{x^2} \left[ -10x^3(x - 1)(1 + 3x + 140x^2 + 420x^3 + 280x^4) \ln f + \frac{\gamma}{\bar{\beta}} x(1 + 12x + 34x^2 + 244x^3 - 3640x^4 + 6720x^5 - 3360x^6) \right] K_2^{\text{mag}}
\]

\[
k^2_1 = \frac{1}{x^2} \left[ -20x^4(2x^2 - 1)(x - 1) \ln f + 7x^2(1 + 10x - 130x^2 + 240x^3 - 120x^4) \right] M_{\text{mag}}^{(2,1)}
\]

\[
k^2_2 = \frac{1}{x^2} \left[ 20x^4(2x^2 - 1)(x - 1) \ln f + \frac{\gamma}{\bar{\beta}} x(2x^2 - 1)(1 + 6x^2 + 36x^3) \right] K_2^{\text{mag}}
\]

\[
f^1_1 = \frac{1}{x^2} \left[ -30x^3(x - 4) \ln f - \frac{\gamma}{\bar{\beta}} x(2x - 1)(1 + 6x + 30x^2) \right] K_2^2
\]

\[
f^1_2 = \frac{1}{x^2} \left[ -10x^3(x - 4) \ln f - \frac{\gamma}{\bar{\beta}} x(2x - 1)(1 + 6x + 30x^2) \right] K_2^2
\]

\[
f^2_1 = \frac{1}{x} \left[ 20x^4(3x^2 - 2)(x - 1) \ln f + \frac{\gamma}{\bar{\beta}} x(1 + 5x + 30x^2 - 210x^3 + 180x^4) \right] M_{\text{mag}}^{(2,1)}
\]

\[
f^2_2 = \frac{1}{x} \left[ -20x^4(3x^2 - 2)(x - 1) \ln f + \frac{\gamma}{\bar{\beta}} x(1 + 5x + 30x^2 - 210x^3 + 180x^4) \right] K_2^{\text{mag}}
\]

\[
\text{constant on each generator, and } -r \text{ continues to be an affine parameter on each generator. These requirements imply that } g_{rr} = 1, g_{\theta r} = 0 \text{, so that } g_{uu}, g_{ov}, g_{oA}, \text{ and } g_{AB} \text{ are the only nonvanishing components of the metric.}
\]

The perturbed metric is written as

\[
g_{uv} = - f - \frac{\gamma^2}{\bar{\beta}} e^\theta_1 c^\theta + r^2 e^\theta_1 c^\theta + r^2 k^d_1 k^d - r^2 k^o_1 k^o, \quad (3.2a)
\]

\[
g_{ov} = 1, \quad (3.2b)
\]

\[
g_{AV} = 2Mf^2 \kappa_2 \kappa_2 - \frac{2}{3} r^2 (e^\theta_1 c^\theta - b^o_1 B^o_1) + r^2 (e^\theta_1 c^\theta - b^o_1 B^o_1) - r^2 (f^2_1 F^a_1 - k^d_1 k^d) + r^2 (f^2_1 F^a_1 + k^d_1 k^d), \quad (3.2c)
\]

\[
b_{AB} = r^2 \Omega_{AB} - \frac{1}{3} r^4 (e^\theta_1 c^\theta_{AB} - b^o_1 B^o_1_{AB}) + r^4 (e^\theta_1 c^\theta_{AB} - b^o_1 B^o_1_{AB}) - r^4 (f^2_1 F^a_1 F^a_1 - k^d_1 k^d), \quad (3.2d)
\]

in which \( e^n, b^n, c^n, b^n, k^n, f^n, \) and \( f^n_o \) are functions of \( r \) that are determined by solving the vacuum Einstein field equations. The radial functions are listed in Table III.
The strategy to solve the field equations is as follows. First, we switch perspectives and consider the metric of Eq. (3.2) to be a perturbation of the Schwarzschild solution expressed in \((v, r, \theta, \phi)\) coordinates — this is Eq. (3.1) with \(\chi\) set equal to zero.

Second, we manufacture a first-order perturbation to this new background metric by introducing the rotational potential \(\chi^4\) and the purely tidal potentials constructed from \(\mathcal{E}_{ab}\) and \(\mathcal{B}_{ab}\). With the spherical harmonic decompositions of Eqs. (2.7), the linearized field equations reduce to a system of coupled differential equations for the radial functions inserted in front of these potentials. Schematically, these take the form of

\[
\mathcal{L}_k^j w_k^j = 0, \quad (3.3)
\]

where \(\mathcal{L}_k^j\) is a second-order differential operator, and \(w_k^j(r)\) is the collection of radial functions that appear in the first-order metric perturbation. (In practice, the equations for the rotational, \(\ell = 1\) perturbation decouple from the equations for the tidal, \(\ell = 2\) perturbations, and the equations for \(\{e_1^3, e_4^3, e_7^3\}\) decouple from the equations for \(\{b_1^3, b_2^3\}\).) Equation (3.3) is then integrated, and the general solution is formed from two linearly independent modes, one decaying as \(r\) increases, the other growing. In the case of the rotational perturbation, the decaying mode produces the angular-momentum term in Eq. (3.1), and the growing mode is set to zero, because it corresponds to an uninteresting coordinate transformation to a rotating frame. In the case of the tidal perturbation created by \(\mathcal{E}_{ab}\) (respectively \(\mathcal{B}_{ab}\)), the growing mode represents the external tidal field, and the decaying mode represents the body’s response to this tidal field, measured by the gravitational Love number \(K_2^\text{mag}\) (respectively \(K_2^\text{mag}\)), which appears in the solution as an integration constant. The solution for \(b_2^3\) is also observed to involve \(c\), an additional integration constant that represents a residual gauge freedom that preserves the geometrical meaning of the \((v, r, \theta, \phi)\) coordinates.

Third, the first-order perturbation is used as a seed to construct a second-order perturbation that accounts for the coupling between \(\chi^4\) and \(\mathcal{E}_{ab}\), and the coupling between \(\chi^4\) and \(\mathcal{B}_{ab}\). This perturbation ignores terms quadratic in \(\chi^4\), and terms quadratic in \(\mathcal{E}_{ab}\) and \(\mathcal{B}_{ab}\). The composition of \(\ell = 1\) and \(\ell = 2\) spherical harmonics produces the dipole, quadrupole, and octupole potentials listed in Sec. III and insertion of the radial functions gives rise to the metric of Eq. (3.2). Substitution of this metric in the vacuum field equations yields differential equations of the schematic form

\[
\mathcal{L}_k^j w_k^j = S^j(w_1^j), \quad (3.4)
\]

where \(w_k^j\) is the collection of radial functions that appear in the second-order perturbation, \(S^j\) is a set of source terms constructed from the first-order radial functions, and \(\mathcal{L}_k^j\) is the same differential operator as in Eq. (3.3). The general solution to Eq. (3.4),

\[
w_k^j = w_k^j(\text{particular}) + w_k^j(\text{decaying}) + w_k^j(\text{growing}), \quad (3.5)
\]

is a linear superposition of a particular solution to the system of differential equations, a decaying solution to the homogeneous system \(\mathcal{L}_k^j w_k^j = 0\), and a growing solution to the same homogeneous system.³

In Table III, the decaying solutions to the homogeneous system are identified with the integration constants \(c^g\), \(\bar{\mathcal{S}}^g\), \(\bar{\mathcal{B}}^g\), and \(\bar{\mathcal{R}}^g\), and the growing solutions are identified with \(\mathcal{E}^g\), \(\bar{\mathcal{S}}^{d.o.}\), \(\bar{\mathcal{B}}^{d.o.}\), and \(\bar{\mathcal{R}}^{d.o.}\). In addition to these, the solutions depend on gauge constants \(\gamma^{d.q.e.}\) and \(c^{d.q.o.}\) that (like \(c\)) specify the residual freedom of the light-cone gauge. It is useful to note that with \(K_2^\text{mag} = K_2^\text{mag} = \mathcal{E}^g = \bar{\mathcal{S}}^g = \bar{\mathcal{B}}^g = \bar{\mathcal{R}}^g = 0\), the solutions of Table III reduce to those obtained in Paper I.³

Many of the integration constants introduced in Table III do not have a physical meaning, and can be set equal to zero without loss of generality. We have already put in this category the gauge constants \(c\), \(\gamma^{d.q.e.}\), and \(c^{d.q.o.}\), which merely serve to further specify the coordinate system. We can also single out as unphysical the constants \(\bar{\mathcal{E}}^g\), \(\bar{\mathcal{S}}^{d.o.}\), \(\bar{\mathcal{B}}^{d.o.}\), and \(\bar{\mathcal{R}}^{d.o.}\), which come with the growing modes of the radial functions.

Let us first examine the constant \(\bar{\mathcal{S}}^d\), which appears in \(f_4^d\). It is easy to see that this term can be removed from \(g_{vA}\) by making the replacement

\[
\chi^a - 8M^2 \bar{\mathcal{S}}^d \mathcal{F}^a \to \chi^a \quad (3.6)
\]

in the metric. This shift of the original spin vector is unobservable, and its effect on other terms in the metric scale as \(\chi^2\) and can be neglected. Thus, we see that \(\bar{\mathcal{S}}^d\) can be set equal to zero without producing a different physical situation.

³ The growing and decaying solutions can be unambiguously identified. The decaying solution decreases with increasing \(r\), faster than any term in the growing solution, and it is singular in the limit \(r \to 2M\). The growing solution increases with \(r\), and is regular in this limit.
Turning next to the terms involving the constants \( \mathcal{E}^q \) and \( \mathcal{B}^q \), we see they also can be removed by performing the shifts

\[
\mathcal{E}_{ab} - \mathcal{E}^q \hat{\mathcal{E}}_{ab} \to \mathcal{E}_{ab}, \quad \mathcal{E}^q - \mathcal{E}^q K^c_2 \to \mathcal{E}^q
\]

and

\[
\mathcal{B}_{ab} - \mathcal{B}^q \hat{\mathcal{B}}_{ab} \to \mathcal{B}_{ab}, \quad \mathcal{B}^q - \mathcal{B}^q K^\text{mag}_2 \to \mathcal{B}^q.
\]

These shifts also are unobservable, and their effects on other terms in the metric can also be neglected. We therefore conclude that \( \mathcal{E}^q \) and \( \mathcal{B}^q \) can be set equal to zero without loss of generality.

The case of the constants \( \mathcal{R}^o \) and \( \mathcal{S}^o \) is more subtle, because the metric of Eq. (3.2) does not include a tidal perturbation generated by octupole moments \( \mathcal{E}_{abc} \) and \( \mathcal{B}_{abc} \) that could be shifted by rotational corrections. But this situation merely reflects an incompleteness of our description, which can easily be supplemented with the missing octupole terms (see Refs. [14, 15]). With the octupole contribution included, we find that the terms involving \( \mathcal{R}^o \) and \( \mathcal{S}^o \) can be eliminated by performing the shifts

\[
M \mathcal{E}_{abc} + 3 \mathcal{R}^o \mathcal{F}_{abc} \to M \mathcal{E}_{abc}, \quad \mathcal{R}^o - 2 \hat{\mathcal{R}}^o K^c_3 \to \hat{\mathcal{R}}^o
\]

and

\[
\frac{4}{3} MB_{abc} + 3 \mathcal{S}^o \mathcal{F}_{abc} \to \frac{4}{3} MB_{abc}, \quad \mathcal{S}^o + 2 \hat{\mathcal{S}}^o K^\text{mag}_3 \to \hat{\mathcal{S}}^o,
\]

where \( K^c_3 \) and \( K^\text{mag}_3 \) are the \( \ell = 3 \) gravitational Love numbers. The factor of \( \frac{4}{3} \) in front of \( B_{abc} \) reflects the definition of the \( \ell = 3 \) tidal potentials adopted by Poisson and Vlasov [15]: its original source is the choice of normalization for the tidal moments made in Ref. [46]. Once again the conclusion is that \( \mathcal{R}^o \) and \( \mathcal{S}^o \) can be set equal to zero without altering the physical situation.

With the constants \( c, \gamma, e^3, e^4, \mathcal{E}^q, \mathcal{S}^q, \mathcal{B}^q \), and \( \hat{\mathcal{R}}^o \) dismissed as physically uninteresting, we are left with a metric that depends on the two original Love numbers \( K^c_2 \) and \( K^\text{mag}_2 \), as well as the remaining four integration constants \( \mathcal{E}^q, \mathcal{S}^q, \mathcal{B}^q \), and \( \hat{\mathcal{R}}^o \). These are attached to the decaying solutions of the homogeneous perturbation equations, and we interpret them as a new class of Love numbers associated with the coupling between the body’s rotation and the external tidal field. Fittingly, we may refer to \( \mathcal{E}^q, \mathcal{S}^q, \mathcal{B}^q \), and \( \hat{\mathcal{R}}^o \) as rotational-tidal Love numbers.

The new Love numbers are thus identified as integration constants in front of the decaying solutions to the homogeneous perturbation equations. They are gauge-invariant: While the precise expression of the decaying solution will be altered by a change of gauge, its identity as a decaying solution will be preserved, and the numerical value of \( \mathcal{E}^q, \mathcal{S}^q, \mathcal{B}^q \), and \( \hat{\mathcal{R}}^o \) will also be preserved by the gauge transformation. The gauge-invariant nature of the Love numbers can also be recognized from a computation of gauge-invariant quantities. For example, the metric of Eq. (3.2) can be used to calculate the Newman-Penrose quantity \( \psi_0 \) in a null tetrad adapted to the principal null congruence of the slowly rotating background spacetime; as is well known, \( \psi_0 \) vanishes in the background spacetime and is therefore gauge-invariant in the perturbed spacetime. We carried out this computation but choose not to disclose the results here to avoid displaying the long expressions. We state nevertheless that \( \psi_0 \) is indeed observed to depend on \( \mathcal{E}^q, \mathcal{S}^q, \mathcal{B}^q \), and \( \hat{\mathcal{R}}^o \).

IV. EXTERNAL METRIC OF A TIDALLY DEFORMED BODY — REGGE-WHEELER GAUGE

In this section we repeat the calculations carried out in the preceding section, but express the metric perturbation in the Regge-Wheeler gauge instead of the light-cone gauge. And instead of the light-cone coordinates \((v, r, \theta, \phi)\), we cast the background metric in the standard Boyer-Lindquist coordinates \((t, r, \theta, \phi)\), so that the line element is now expressed as

\[
ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 - 2 \frac{\chi M^2}{r} \sin^2 \theta \, d\theta d\phi,
\]

where \( f := 1 - 2M/r \) and \( d\Omega^2 := \Omega_{AB} d\theta^A d\theta^B := d\theta^2 + \sin^2 \theta d\phi^2 \). It is important to note that the coordinate \( \phi \) adopted here is distinct from the \( \phi_{LC} \) featured in Sec. III (and denoted \( \phi \) there); we have the relations \( d\nu = dt + f^{-1} dr \) and \( d\phi_{LC} = d\phi + (2\chi M^2/r^3 f) dr \) between the coordinate systems.

The Regge-Wheeler gauge conditions are formulated at the level of each \( \ell m \) mode of the metric perturbation. For the even-parity sector generated by \( \mathcal{E}_{ab}, \hat{\mathcal{E}}_{ab}, \mathcal{K}_a, \) and \( \mathcal{K}_{ab} \), we decompose the metric perturbation \( p_{\alpha\beta} \) as

\[
p_{ab} = \sum_{\ell m} p_{ab} Y_{\ell m}, \quad p_{aB} = \sum_{\ell m} p_{aB} Y_{\ell m} Y^B_{A}, \quad p_{AB} = r^2 \sum_{\ell m} (K^{\ell m} \Omega_{AB} Y_{\ell m} + G^{\ell m} Y_{AB}),
\]

where \( K^{\ell m} \) and \( G^{\ell m} \) are the spherical harmonic tensors associated with the tidal moments made in Ref. [46]. Once again the conclusion is that \( \mathcal{E}^q, \mathcal{S}^q, \mathcal{B}^q \), and \( \hat{\mathcal{R}}^o \) can be set equal to zero without altering the physical situation.
TABLE IV. Radial functions appearing in the metric of Eq. (4.4), expressed in terms of \( x \).

where \( x^a = (t, r) \) and \( \theta^A = (\theta, \phi) \), and we impose \( J_{LM}^{\ell m} = 0 = G^{\ell m} \) for \( \ell = 2 \) and \( \ell = 3 \). For \( \ell = 1 \), the tensorial harmonics \( Y_{LM}^a \) and the associated perturbation variables \( G^{\ell m} \) are not defined, and we adopt instead \( K^{1 m} = 0 \) as a gauge condition, along with \( J_{LM}^{1 m} = 0 \). For the odd-parity sector generated by \( B_{ab}, \tilde{B}_{ab}, F_a, \) and \( F_{abc} \), we decompose the metric perturbation as

\[
p_{ab} = 0, \quad p_{AB} = \sum_{LM} \tilde{h}_{LM}^{\ell m} \chi_{LM}^{\ell m}, \quad p_{AB} = \sum_{LM} h_{LM}^{\ell m} \chi_{LM}^{\ell m}
\]

and impose \( \tilde{h}_{LM}^{\ell m} = 0 \) for \( \ell = 2 \) and \( \ell = 3 \). For \( \ell = 1 \), \( X_{LM}^{1 m} \) and \( \tilde{h}_{LM}^{\ell m} \) are not defined, and we adopt instead \( h_{LM}^{1 m} = 0 \) as a gauge condition; this choice is motivated by the fact that \( h_{LM}^{1 m} \) is found to vanish by virtue of the field equations.

The strategy to integrate the vacuum Einstein field equations is readily adapted from Sec. [III] and we obtain a perturbed metric of the form

\[
g_{tt} = -f - v^2_c \chi_{tt} \hat{g}^q + v^2_c \chi_{tt} \hat{g}^q + r^2 k^d \hat{K}^d - r^2 k^0 \hat{K}^0,
\]

\[
g_{tr} = r^2 v^2_c \theta^r \hat{g}^q,
\]

\[
g_{rr} = -f - v^2_c \chi_{rr} \hat{g}^q + v^2_c \chi_{rr} \hat{g}^q + r^2 k^d \hat{K}^d - r^2 k^0 \hat{K}^0,
\]

\[
g_{tA} = \frac{2 M^2}{r} (\chi^a + \frac{3 r^2 k^d f^d}{3 r^2 k^d f^d} f^d - r^3 f^d F^d_A + r^3 f^d F^d_A),
\]

\[
g_{rA} = -r^2 k^0 \hat{g}^q,
\]

\[
g_{AB} = \frac{2 M^2}{r} \chi^a \Omega_{AB}^a - r^4 \chi^q \Omega_{AB}^q + r^4 k^0 \Omega_{AB}^0 \hat{g}^p,
\]

with the radial functions listed in Table IV.
The perturbation of the internal potential must have a pure quadrupolar form proportional to \( \delta U \). We find that the boundary must be an equipotential surface; we obtain the constant of proportionality is obtained by continuity of \( \delta U \) and \( \delta R \) to these quantities, as well as a perturbation \( \delta R \) to the position of the boundary. With a tidal environment characterized by a quadrupole moment \( \mathcal{E}_{ab} \), we have that \( \delta U \) can be expressed quite generally as

\[
\delta U_{\text{out}} = -\frac{1}{2} \left[ 1 + 2k_2 (R/r)^5 \right] \mathcal{E}_{ab} x^a x^b, \tag{5.1}
\]

in terms of a gravitational Love number \( k_2 \); the first term (which grows as \( r^2 \)) represents the external tidal field, while the second term (decaying as \( r^{-3} \)) represents the body’s response to the tidal forces. With this information, \( \delta R \) is deduced from the requirement that the boundary must be an equipotential surface; we obtain

\[
\delta R = -\frac{R^4}{2GM} (1 + 2k_2) \mathcal{E}_{ab} \Omega^a \Omega^b. \tag{5.2}
\]

The perturbation of the internal potential must have a pure quadrupolar form proportional to \( \mathcal{E}_{ab} x^a x^b \), and the constant of proportionality is obtained by continuity of \( \delta U_{\text{in}} \) and \( \delta U_{\text{out}} \) at \( r = R \); we find

\[
\delta U_{\text{in}} = -\frac{1}{2} (1 + 2k_2) \mathcal{E}_{ab} x^a x^b. \tag{5.3}
\]

The density perturbation is obtained from the new assignment \( \rho + \delta \rho = \rho_0 \Theta(R + \delta R - r) \). Expansion of the step function to first order in \( \delta R \) and use of Eq. (5.2) produces

\[
\delta \rho = -\frac{3R}{8\pi G} (1 + 2k_2) \mathcal{E}_{ab} \Omega^a \Omega^b \delta(r - R). \tag{5.4}
\]

To obtain the pressure perturbation we appeal to the statement of hydrostatic equilibrium, \( \partial_a \delta \rho = \rho \partial_a \delta U_{\text{in}} + \delta \rho \partial_a U_{\text{in}} \). The angular components of this equation integrate to \( \delta p = \rho_0 \partial_a U_{\text{in}} \), and we get

\[
\delta p = -\frac{3M}{8\pi R^3} (1 + 2k_2) \mathcal{E}_{ab} x^a x^b. \tag{5.5}
\]
It can be verified that $p + \delta p$ properly vanishes when $r = R + \delta R$. The results for $\delta R$, $\delta p$, and $\delta \rho$ can alternatively be derived on the basis of the Lagrangian displacement vector

$$\xi_a = - \frac{R^3}{2GM} (1 + 2k_2) \xi_{ab} x^b. \quad (5.6)$$

Finally, $k_2$ can be determined from the discontinuity in $\partial_r \delta U$ at $r = R$ implied by Poisson’s equation $\nabla^2 \delta U = -4\pi G \delta \rho$ and the $\delta$-function of Eq. (5.4). A short calculation reveals that the discontinuity must be given by $[\partial_r \delta U] = \frac{2}{5} (1 + 2k_2) R \xi_{ab} \omega^a \Omega^b$, where $[\partial_r \delta U] := \partial_r U_{\text{out}} - \partial_r U_{\text{in}}$ evaluated at $r = R$. Substituting our previous results for $\delta U_{\text{in}}$ and $\delta U_{\text{out}}$, we arrive at

$$k_2 = \frac{3}{4}, \quad 1 + 2k_2 = \frac{5}{2}. \quad (5.7)$$

This is the well-known value of the gravitational Love number for an incompressible fluid.

The body is taken to be rotating rigidly with a uniform angular velocity $\omega^a$. The rotation is assumed to be slow, and we work consistently to first order in $\omega^a$. The unperturbed velocity field inside the body is $v_a = \epsilon_{abc} \omega^b x^c$, the spin angular momentum of the unperturbed body is given by $S^a = \frac{2}{5} M R^2 \omega^a$, and its dimensionless version is $\chi^a = (c/GM^2) S^a$, or

$$\chi^a = \frac{2}{5} \frac{cR^2}{GM} \omega^a. \quad (5.8)$$

In terms of this the velocity field becomes

$$v_a = \frac{5GM}{2cR^2} \xi_{abc} \chi^b x^c. \quad (5.9)$$

The tidal deformation creates a perturbation $\delta v_a = v^b \partial_b \xi_a - \xi^b \partial_b v_a$ of the velocity field, which can be calculated from the Lagrangian displacement vector of Eq. (5.6). We obtain

$$\delta v_a = \frac{25 R}{8c} \xi_{ab} x^b. \quad (5.10)$$

The tidal perturbation changes the body’s angular momentum by $\delta S_a = \epsilon_{abc} \int x^b \delta j^c d^3x$, where $\delta j_a = \delta \rho v_a + \rho \delta v_a$. A short calculation reveals that the shift in the dimensionless spin is given by

$$\delta \chi^a = \frac{5}{4} \frac{R^3}{GM} \xi^b \chi^b = \frac{5}{4} \frac{R^3}{GM} F^a. \quad (5.11)$$

**B. Vector potential**

Our post-Newtonian calculation is based on the metric

$$g_{00} = -1 + \frac{2}{c^2} U + O(c^{-4}), \quad g_{0a} = -\frac{4}{c^3} U_a + O(c^{-5}), \quad g_{ab} = \delta_{ab} \left(1 + \frac{2}{c^2} U\right) + O(c^{-4}), \quad (5.12)$$

which involves a vector potential $U_a$ in addition to the Newtonian potential $U$ encountered previously. The coupling between $\chi^a$ and $\xi_{ab}$ is captured by the vector potential, which satisfies the field equation

$$\nabla^2 U_a = -4\pi G \rho v_a. \quad (5.13)$$

An examination of the post-Newtonian equations (see, for example, Sec. 8.1 of Ref. [47]) reveals that the coupling does not appear in $g_{00}$ at first post-Newtonian (1PN) order, and we therefore omit the 1PN terms at order $c^{-4}$. Similarly, consideration of $g_{ab}$ at 2PN order reveals that the coupling does not appear within the omitted $c^{-4}$ terms. The unperturbed vector potential is

$$U_a^{\text{out}} = \frac{(GM)^2}{2cR^3} \epsilon_{abc} \chi^b x^c \quad (5.14)$$

outside the body, and

$$U_a^{\text{in}} = \frac{(GM)^2}{4cR^3} \left(5 - 3r^2/R^2\right) \epsilon_{abc} \chi^b x^c \quad (5.15)$$
inside the body. The additional factor of $c^{-1}$ results from expressing the angular-momentum vector in terms of its dimensionless version of Eq. (5.8); this promotes $g_{oa}$ to a quantity of $1.5$PN order.

The perturbation of the vector potential is sourced by the sum of $\delta_1 j_a = \delta \rho v_a$ and $\delta_2 j_a = \rho \delta v_a$, and correspondingly it is expressed as the sum of

$$
\delta_1 U_a(x) = G \int \frac{\delta_1 j_a(x')}{|x - x'|} d^3x', \quad \delta_2 U_a(x) = G \int \frac{\delta_2 j_a(x')}{|x - x'|} d^3x'.
$$

To evaluate this we rely on the addition theorem

$$
\frac{1}{|x - x'|} = \sum_{\ell m} \frac{4\pi}{2\ell + 1} r^{\ell+1} Y_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi),
$$

the identity [Eq. (1.171) of Ref. [47]; $L := a_1 a_2 \cdots a_\ell$ is a multi-index that includes a number $\ell$ of individual indices, $\Omega^L := \Omega^{a_1} \Omega^{a_2} \cdots \Omega^{a_\ell}$, and the angular brackets indicate that all traces are to be removed from $\Omega^L$]

$$
\sum_m Y_{\ell m}(\theta, \phi) \int Y_{\ell m}(\theta', \phi') \Omega^{(L)}(\theta) d\Omega' = \delta_{\ell, \ell'} \Omega^{(L)},
$$

and a decomposition of

$$
Z_a := \epsilon_{abc} \lambda^b \epsilon_{de} \Omega^c \Omega^d \Omega^e
$$

into irreducible components. This is accomplished with the identity $\Omega^c \Omega^d \Omega^e = \Omega^{(cde)} + \frac{1}{5} (\delta^{cd} \Omega^e + \delta^{ce} \Omega^d + \delta^{de} \Omega^c)$, which gives rise to $Z_a = \frac{2}{5} Z_a^1 + Z_a^3$ where

$$
Z_a^1 := \epsilon_{abc} \lambda^b \epsilon_{de} \Omega^c, \quad Z_a^3 := \epsilon_{abc} \lambda^b \epsilon_{de} \Omega^{(cde)}.
$$

These quantities can be expressed in terms of the tidal potentials [1]

$$
\mathcal{F}^a := \frac{1}{c^2} \epsilon_{abc} \lambda^b \mathcal{F}^c, \quad \mathcal{F}^a_o := \frac{1}{c^2} \epsilon_{abc} \lambda^b \mathcal{F}^c_o \Omega^d \Omega^e, \quad \mathcal{E}^a := \frac{1}{c^2} \mathcal{E}^a \Omega^d \Omega^e, \quad \mathcal{E}^a_o := \frac{1}{c^2} (\delta^a_b - \Omega_a \Omega^b) \mathcal{E}_b \Omega^e,
$$

which are Cartesian versions of the angular potentials introduced in Sec. III. the relation is given, for example, by $\mathcal{F}^a_A = \mathcal{F}^a \lambda^A$, where $\Omega^A := \nabla / \delta \mathcal{F}^a \Omega^A$. We have

$$
c^{-2} Z_a^1 = \frac{1}{2} (\mathcal{F}^a + \mathcal{E}^a + \Omega_a \mathcal{E}^a), \quad c^{-2} Z_a^3 = \frac{1}{15} (2 \mathcal{E}^a - 3 \Omega_a \mathcal{E}^a) - \mathcal{F}^a_o
$$

The computation of $\delta_1 U_a$ and $\delta_2 U_a$ proceeds by inserting

$$
\delta_1 j_a = -\frac{75}{32\pi} \frac{M}{c} Z_a \delta(r - R), \quad \delta_2 j_a = \frac{75}{32\pi} \frac{M}{c R^2} \mathcal{E}_{ab} b^b \Theta(R - r),
$$

as well as the addition theorem of Eq. (5.17), within Eq. (5.19). The angular integrals are evaluated with the help of Eq. (5.18), after involving the decomposition of $Z_a$ into $Z_a^1$ and $Z_a^3$. The final expressions,

$$
\delta_1 U_{a}^{\text{out}} = -\frac{75}{8} GMc \left[ \frac{1}{15 R^3} \left( \mathcal{F}^a + \mathcal{E}^a + \Omega_a \mathcal{E}^a \right) + \frac{1}{R^5} \left( \frac{2}{15} \mathcal{E}^a - \frac{1}{5} \Omega_a \mathcal{E}^a - \mathcal{F}^a_o \right) \right]
$$

and

$$
\delta_2 U_{a}^{\text{out}} = \frac{75}{8} GMc \left[ \frac{1}{15 R^3} \left( \mathcal{E}^a + \Omega_a \mathcal{E}^a \right) \right],
$$

are obtained after making use of Eq. (5.22). The complete perturbation is

$$
\delta U_{a}^{\text{out}} = -\frac{75}{8} GMc \left[ \frac{1}{15 R^3} \mathcal{F}^a + \frac{1}{105 R^5} (2 \mathcal{E}^a - 3 \Omega_a \mathcal{E}^a) - \frac{1}{R^5} \mathcal{F}^a_o \right],
$$

and we note that the terms proportional to $(R^3/r^2) \mathcal{E}_{ab}$ that appear in both $\delta_1 U_{a}^{\text{out}}$ and $\delta_2 U_{a}^{\text{out}}$ cancel out after taking the sum. We note also that the term proportional to $\mathcal{F}^a$ can be fully accounted for by the shift in spin vector described by Eq. (5.11). With the factors of $c^{-2}$ contained in the tidal potentials, we find that $\delta U_{a}^{\text{out}}$ scales as $GM/c$ and contributes to $g_{oa}$ at $1.5$PN order.
C. Light-cone coordinates

The post-Newtonian metric of Eq. (5.12) cannot be compared directly with the metrics obtained in Secs. III and IV because they are expressed in different coordinate systems. (The potentials $U$ and $U_a$ are now the full, perturbed potentials; they were previously denoted $U + \delta U$ and $U_a + \delta U_a$, respectively.) Here we transform the post-Newtonian metric to the light-cone coordinates of Sec. III, and compare the results with the metric of Eq. (3.2). For this purpose it is useful to look at the post-Newtonian metric as a perturbation of the Minkowski metric, write $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, and work consistently to first order in the perturbation $h_{\alpha\beta}$.

The first step is to perform the coordinate transformation

$$ct = cv - r, \quad x^a = r\Omega^a(\theta^A),$$  \hspace{1cm} (5.27)

which implies that $v$ is a null coordinate in flat spacetime. The transformation brings the Minkowski metric to the form

$$\eta_{00} = -1, \quad \eta_{0r} = 1, \quad \eta_{AB} = r^2\Omega_{AB},$$  \hspace{1cm} (5.28)

and the perturbation to the form

$$h'_{00} = \frac{2}{c^2}U, \quad (5.29a)$$
$$h'_{0r} = -2U - \frac{4}{c^3}U_a\Omega^a, \quad (5.29b)$$
$$h'_{0A} = \frac{4}{c^3}rU_a\Omega^a_A, \quad (5.29c)$$
$$h'_{rr} = \frac{4}{c^2}U + \frac{8}{c^3}U_a\Omega^a, \quad (5.29d)$$
$$h'_{rA} = \frac{4}{c^3}rU_a\Omega^a_A, \quad (5.29e)$$
$$h'_{AB} = \frac{2}{c^2}r^2U\Omega_{AB}. \quad (5.29f)$$

The second step is to perform a gauge transformation,

$$h_{\alpha\beta} \rightarrow h'_{\alpha\beta} = h_{\alpha\beta} - \nabla_{\alpha}\xi_{\beta} - \nabla_{\beta}\xi_{\alpha},$$  \hspace{1cm} (5.30)

to ensure that $v$ remains null in the perturbed spacetime. This requires $h'_{0r} = h'_{rr} = h'_{rA} = 0$, and a simple computation reveals that the gauge vector $\xi_{\alpha}$ is determined by

$$\partial_r\xi_0 = -\frac{2}{c^2}U - \frac{4}{c^3}U_a\Omega^a, \quad (5.31a)$$
$$\partial_r\xi_r = \frac{2}{c^2}U + \frac{4}{c^3}U_a\Omega^a, \quad (5.31b)$$
$$r^2\partial_r(r^{-2}\xi_A) = \frac{4}{c^3}rU_a\Omega^a_A - \partial_A\xi_r. \quad (5.31c)$$

With the null conditions satisfied, the remaining components of the metric perturbation become

$$h'_{00} = \frac{2}{c^2}U, \quad (5.32a)$$
$$h'_{0A} = -\frac{4}{c^3}rU_a\Omega^a_A - \partial_A\xi_0, \quad (5.32b)$$
$$h'_{AB} = \frac{2}{c^2}r^2U\Omega_{AB} - D_A\xi_B - D_B\xi_A - 2r(\xi_0 + \xi_r)\Omega_{AB}. \quad (5.32c)$$

We focus our attention on the piece of the metric associated with $\delta U_a$, which captures the coupling between $\chi^a$ and $E_{ab}$. We insert Eq. (5.26) within Eq. (5.31) and obtain

$$\delta\xi_0 = \frac{75}{2}\frac{GM}{c^2} \left( \frac{1}{105} \frac{R_5}{r^5} \hat{\xi}^a + \beta_0 \right), \quad (5.33a)$$
\[\delta \xi_r = \frac{75 \, GM}{2 \, c^2} \left( -\frac{1}{105} \frac{R^5}{r^3} \hat{\mathbf{e}}_q + \beta_r \right), \quad (5.33b)\]
\[\delta \xi_A = \frac{75 \, GM}{2 \, c^2} \left( \frac{1}{30} R^3 F_A^q - \frac{1}{28} \frac{R^5}{r^2} \sum_{q=1}^{4} q \beta_A + r \partial_A \beta_r + r^2 \beta_A \right), \quad (5.33c)\]

where \(\beta_0, \beta_r,\) and \(\beta_A\) are arbitrary functions of \(\theta^A\). For our purposes here it is sufficient to restrict the residual gauge freedom to functions of the form
\[\beta_0 = p_0^q R^2 \hat{\mathbf{e}}_q, \quad \beta_r = p_r^q R^2 \hat{\mathbf{e}}_q, \quad \beta_A = p^q R F_A^q + p^q R F_A^q, \quad (5.34)\]
in which \(p_0^q, p_r^q, p^q,\) and \(p^q\) are arbitrary dimensionless coefficients. Inserting the gauge vector within Eq. (5.32) produces
\[\delta h'_{0A} = \frac{75 \, GM}{2 \, c^2} \left[ \frac{1}{15} \frac{R^3}{r} F_A^q - 2p_0^q R^2 \hat{\mathbf{e}}_q - \frac{1}{7} \frac{R^5}{r^3} F_A^q \right], \quad (5.35a)\]
\[\delta h'_{AB} = \frac{75 \, GM}{2 \, c^2} \left[ 2(2p_r^q - p_0^q) R^2 r \Omega_{AB} \hat{\mathbf{e}}_q - 2p_r^q R^2 r \hat{\mathbf{e}}_q + \left( \frac{1}{14} \frac{R^5}{r^2} - 2p^q R r^2 \right) F_{AB}^q \right]. \quad (5.35b)\]

We notice that terms proportional to \((R^5/r^3) \hat{\mathbf{e}}_q^0\), which used to appear in \(\delta h_{0A}\), no longer appear in \(\delta h'_{0A}\). We can further eliminate the term proportional to \(\Omega_{AB} \hat{\mathbf{e}}_q\) in \(\delta h'_{AB}\) by setting \(2p_r^q = p_0^q\); this constitutes a refinement of the light-cone gauge.

By comparing \(\delta h'_{0A}\) and \(\delta h'_{AB}\) with the metric of Eq. (5.22), we can read off the post-Newtonian expressions for the relevant radial functions. We have
\[\hat{e}_q^3 = -\frac{75 \, p_0^q \, GM \, R^2}{c^2 \, r^3}, \quad (5.36a)\]
\[\hat{e}_q^4 = \frac{75 \, GM \, R^3}{2 \, c^2 \, r^3}, \quad (5.36b)\]
\[f_4^d = \frac{5 \, GM \, R^3}{2 \, c^2 \, r^3}, \quad (5.36c)\]
\[f_4^p = -\frac{75 \, GM \, R^5}{14 \, c^2 \, r^5}, \quad (5.36d)\]
\[f_4^q = -\frac{75 \, GM \, (R^5 - 28 p^q R)}{28 \, c^2 \, r^6}. \quad (5.36e)\]

Comparing these with the expressions listed in Table III, we find a precise match at this leading, 1.5PN order. Our expression for \(\hat{e}_q^3\) matches the \(\frac{1}{5} x^{-3.7} \hat{\mathbf{e}}_q\) term in Table III and we see that the constant \(p_0^q\) can be related to \(\gamma^q\). Similarly, our \(\hat{e}_q^4\) corresponds to the \(\frac{1}{5} x^{-3.7} \hat{\mathbf{e}}_q\) term in Table III. Our expression for \(f_4^d\), which reflects the shift of the body’s angular-momentum vector created by the tidal perturbation, matches the expected \(x^{-4} \hat{\mathbf{e}}_d\) term from Table III and we recall that the parameter \(\delta h_{0A}^q\) was indeed related to a shift of \(\hat{\mathbf{e}}^q\) in Sec. III. Our expression for \(f_4^q\) matches the expected \(2 x^{-6} \hat{\mathbf{e}}_q\) term from Table III and in this case we can assign the precise value
\[\left( \frac{GM}{c^2} \right)^5 \hat{\mathbf{e}}^0 = -\frac{75}{1792} R^5. \quad (5.37)\]

to the associated rotational-tidal Love number. This assignment is confirmed by comparing \(f_4^d\) with the expression of Table III and in addition, we see that the gauge constant \(p^q\) can be related to \(\gamma^q\).

Our calculation cannot produce an expression for \(\hat{e}_q^4\), which occurs at a higher post-Newtonian order, and it also cannot produce the terms that couple \(\chi^q\) to \(B_{ab}\) in the metric of Eq. (5.22). In addition, our calculation did not produce the expected \(-2 x^{-5} \hat{\mathbf{e}}_q\) term in \(\hat{e}_q^4\), nor the expected \(\frac{4}{3} x^{-5} \hat{\mathbf{e}}_q\) term in \(\hat{e}_q^4\). The reason for this must be that these terms occur at a higher post-Newtonian order. A match at 1.5PN order would have implied a scaling of the form \((GM/c^2)^5 \hat{\mathbf{e}}_q \propto (GM/c^2)^4 R^4 \hat{\mathbf{e}}_q\). In the absence of such a match, we may expect that the term appears instead at 2.5PN order, with an expected scaling of \((GM/c^2)^5 \hat{\mathbf{e}}_q \propto (GM/c^2)^2 R^3 \hat{\mathbf{e}}_q\).

As a final note, we mention that our post-Newtonian calculation did not produce the expected \(\frac{1}{4} x^{-1} \hat{\mathbf{e}}_d\) term in \(f_4^d\). The fault here does not lie with a failure to go to a sufficiently high post-Newtonian order. It lies instead with the omission of a dipolar solution to \(\nabla^2 U_a = 0\) as an additional piece of the vector potential of Eq. (5.20). It is easy to show that the homogeneous term \(\delta \chi U_a = \frac{1}{2} \chi R F_A^q\) is indeed the required missing piece.
D. Conclusion: Scaling of rotational-tidal Love numbers

The consistency between the post-Newtonian metric obtained in this section and the exact metric obtained in Sec. III is pleasing and reassuring, but our main purpose was not to demonstrate this consistency. Our goal was instead to seek guidance in the scaling of \( \mathcal{E}^3, \mathcal{B}^8, \mathcal{E}^9, \mathcal{R}^8 \), the new class of rotational-tidal Love numbers, with the body’s radius \( R \). The calculations presented here allow us to anticipate that for strongly self-gravitating bodies with arbitrary internal structure, \( \mathcal{E}^9 \) will definitely scale as \( R^5 \), and \( \mathcal{E}^8 \) will likely scale as \( R^5 \).

Our calculations give us no direct guidance regarding \( \mathcal{B}^8 \) and \( \mathcal{B}^9 \), but inspection of the equations of post-Newtonian theory (including the statement of hydrostatic equilibrium) indicates that at leading order, the coupling between \( \chi^a \) and \( B_{ab} \) produces terms of order \( c^{-7} \) in both \( g_{00} \) and \( \ddot{B}_{ab} \). Matching this observation with the \( x^{-5} \mathcal{B}^8 \) terms in Table III produces the expectation that \( \mathcal{B}^8 \) should scale as \((GM/c^2)^5 \mathcal{B}^8 \propto (GM/c^2)^2 R^4 \). Similarly, matching the \( c^{-7} \) scaling with the \( x^{-5} \mathcal{B}^9 \) terms in Table III returns an expected scaling of \((GM/c^2)^5 \mathcal{B}^9 \propto (GM/c^2)^2 R^3 \) for \( \mathcal{B}^9 \). Either one (but not both) of these estimates could be off if the suppression of post-Newtonian order observed in the case of \( \mathcal{E}^9 \) also occurred here. Thus, for example, if the \( \mathcal{B}^8 \) terms appeared only at order \( c^{-9} \) in the post-Newtonian metric, then \( \mathcal{B}^8 \) would scale instead as \((GM/c^2)^5 \mathcal{B}^8 \propto (GM/c^2)^3 R^2 \).

Based on this combination of definite results and educated guesswork, we obtain the expected scalings of the rotational-tidal Love numbers with the body’s radius \( R \). We express this as

\[
\mathcal{E}^3 = \mathcal{E}^3 \left( \frac{R}{GM/c^2} \right)^3, \quad \mathcal{B}^8 = \mathcal{B}^8 \left( \frac{R}{GM/c^2} \right)^5, \quad \mathcal{B}^9 = \mathcal{B}^9 \left( \frac{R}{GM/c^2} \right)^3, \quad \mathcal{R}^8 = \mathcal{R}^8 \left( \frac{R}{GM/c^2} \right)^4,
\]

where \( \mathcal{E}^3, \mathcal{B}^8, \mathcal{B}^9, \) and \( \mathcal{R}^8 \) are scalefree versions of the rotational-tidal Love numbers. These are expected to be approximately independent of \((GM/c^2R)\) when the body is weakly self-gravitating, but to acquire a dependence upon this quantity when the internal gravity becomes strong. We recall the caveat made in the previous paragraph: our considerations cannot guarantee that the expected scalings of \( \mathcal{B}^8 \) and \( \mathcal{R}^8 \) (and even \( \mathcal{E}^3 \)) are not altered by a suppression of post-Newtonian order.

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