Ideals of largest weight in constructions based on directed graphs

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**Recommended Citation**

Kelarev, A V.; Susilo, Willy; Miller, Mirka; and Ryan, Joe, "Ideals of largest weight in constructions based on directed graphs" (2016). *Faculty of Engineering and Information Sciences - Papers: Part A*. 5671.  
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Keywords
directed, graphs, constructions, ideals, weight, largest

Disciplines
Engineering | Science and Technology Studies

Publication Details
Kelarev, A. V., Susilo, W., Miller, M. & Ryan, J. (2016). Ideals of largest weight in constructions based on directed graphs. Bulletin of Mathematical Sciences and Applications, 15 8-16.

This journal article is available at Research Online: https://ro.uow.edu.au/eispapers/5671
Ideals of Largest Weight in Constructions Based on Directed Graphs

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Keywords: balanced graphs, incidence semirings, semiring constructions, ideals, maximum weights.

Abstract. We introduce a new construction based on directed graphs. It provides a common generalization of the incidence rings and Munn semirings. Our main theorem describes all ideals of the largest possible weight in this construction. Several previous results can be obtained as corollaries to our new main theorem.

Introduction

Many interesting results have been obtained recently in the literature devoted to properties of graphs (see, for example, [5, 6, 7, 8, 9, 12, 19, 23] as well as algebraic constructions (see, for example [13, 16, 18, 20, 22]). Incidence algebras of directed graphs, have been studied for a long time (cf. [15, 22]). On the other hand, Munn semirings have also been considered by several authors too (cf. [2]). They were defined by analogy with the Munn algebras, which are very well known (cf. [21]).

The present paper introduces Munn incidence semirings as a common generalization of the incidence algebras and Munn semirings. A complete definition of this new construction is given in Section . The main result of this paper is Theorem 1 in Section . It describes all ideals of the largest possible weight in Munn incidence semirings. These results belong to the large area studying the weights of ideals in various constructions, which are related to the applications of ideals in data mining, cryptography and information security (cf. [1, 4]). For instance, the ideals of the largest possible weight have been described for certain classes of the polynomial quotient rings in [17], structural matrix semirings in [10], Munn semirings in [2], and incidence semirings in [1, 3]. It is important to describe the weights of ideals in the new and more general construction of the Munn incidence semirings for the following reasons. First, the ideals in a more general construction may lead to a discovery of systems with better properties crucial for cryptographic applications. Second, it is nice to unify already known facts in such a way that several previous results can be derived as corollaries to a new and more general theorem. In particular, proofs of the main results of the previous papers [1], and [3] can now be derived from the main theorem of the present article.

Preliminaries

We use standard terminology and refer the readers to [6, 11, 13, 14, 22] for more detailed explanations of notions used in the present article. Throughout the word ‘digraph’ means a directed graph without multiple parallel edges but possibly with loops, and $D = (V, E)$ is a digraph with the set $V$ of vertices and the set $E$ of edges. The set of all positive integers is denoted by $\mathbb{N}$.

Here we use the standard definition of a semiring without assuming that every semiring contains an identity element, see [2] and [10].
For any subset $S$ of a semiring $Q$, the ideal generated by $S$ in $Q$ is denoted by $\text{id}(S)$ and is defined as the set of all sums of these elements and their multiples, i.e.,

$$\text{id}(S) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m_i} \ell_{ij} s_i r_{ij} \mid s_i \in S, \ell_{ij}, r_{ij} \in Q \cup \{1\} \right\}.$$  \hspace{1cm} (1)

It is easily seen that the set $\text{id}(S)$ is a subsemiring of $Q$ closed for the multiplication by the elements of $Q$, i.e., $\text{id}(S)Q + Q \text{id}(S) \subseteq \text{id}(S)$.

Recall that a semiring $Q$ is said to be idempotent if $x + x = x$ for all $x \in Q$. A semiring $Q$ is said to be zero-divisor-free, if $xy = 0$ implies $x = 0$ or $y = 0$, for any $x, y \in Q$.

Let $Q$ be a semiring, $I$ and $\Lambda$ finite nonempty sets, and let $P$ be a $\Lambda \times I$-matrix with entries in $Q$. A Munn semiring over $Q$ with sandwich-matrix $P$ is the set $M(Q; I, \Lambda; P)$, consisting of all $I \times \Lambda$ matrices over $Q$, equipped with the usual addition and multiplication $\cdot$ defined by $A \cdot B = APB$, for $A, B \in M(Q; I, \Lambda; P)$.

The concept of an incidence semiring was defined in [1] as an exact analogue of an incidence algebra. Let $Q$ be a semiring. The incidence algebra of $D = (V, E)$ over $Q$ is denoted by $I_D(Q)$ and is defined as the set consisting of zero 0 and all finite sums $\sum_{i=1}^{n} q_i(u_i, v_i)$, where $n \geq 1$, $0 \neq q_i \in Q$, $(u_i, v_i) \in E$, endowed with the usual addition and with multiplication defined by the distributive law and the rule

$$(u_1, v_1) \cdot (u_2, v_2) = \begin{cases} (u_1, v_2) & \text{if } v_1 = u_2 \text{ and } (u_1, v_2) \in E, \\ 0 & \text{otherwise}, \end{cases}$$

for all $(u_1, v_1), (u_2, v_2) \in E$, see [13, 22].

**Main Results**

Let $D = (V, E)$ be a digraph with the (possibly infinite) set $V$ of vertices and set $E$ of edges, and let $I$ and $\Lambda$ be nonempty sets such that $V = I \cup \Lambda$. Let $Q$ be a semiring, and let $P = [p_{\lambda i}]$ be a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in Q$, for all $\lambda \in \Lambda$, $i \in I$. For $i \in I, \lambda \in \Lambda$, denote by $e_{i\lambda}$ the standard elementary $I \times \Lambda$ matrix with 1 in the intersection of $i$-th row and $\lambda$-th column and zeros in all other entries. Denote by $M_D(Q; I, \Lambda; P)$ the set of all $I \times \Lambda$ matrices $x$ of the form

$$x = \sum_{j=1}^{n} x_{ij\lambda_j} e_{ij\lambda_j},$$  \hspace{1cm} (3)

where $(i_j, \lambda_j) \in E \cap (I \times \Lambda)$ and $0 \neq x_{ij\lambda_j} \in Q$, for $j = 1, \ldots, n$. Endow this set with the usual componentwise addition of matrices and with a multiplication $\cdot$ defined by the distributive laws and the rule

$$x_{i_1\lambda_1} e_{i_1\lambda_1} \cdot x_{i_2\lambda_2} e_{i_2\lambda_2} = \begin{cases} (x_{i_1\lambda_1} p_{\lambda_1 i_2} x_{i_2\lambda_2}) e_{i_1\lambda_2} & \text{if } (i_1, \lambda_2) \in E, \\ 0 & \text{otherwise}, \end{cases}$$

for any $(i_1, \lambda_1), (i_2, \lambda_2) \in E \cap (I \times \Lambda)$ and any $x_{i_1\lambda_1}, x_{i_2\lambda_2} \in Q$. If the multiplication defined by (4) is associative, then we say that $M_D(Q; I, \Lambda; P)$ is a Munn incidence semiring of the digraph $D$. Lemmas 2 and 3 in Section show that Munn incidence semirings of digraphs are a common generalisation of the standard Munn semirings and incidence algebras, including the incidence algebras of infinite digraphs.

Let $M_D(Q; I, \Lambda; P)$ be a Munn incidence semiring of the digraph $D = (V, E)$. For each element $x$ in $M_D(Q; I, \Lambda; P)$, the weight $\text{wt}(x)$ of $x$ is equal to the number of nonzero coefficients $x_{ij\lambda_j}$ in the finite sum (3) for $x$ (cf. [14, §2.13]). If $S$ is a subset of $M_D(Q; I, \Lambda; P)$, then the weight of $S$ is denoted by $\text{wt}(S)$ and is defined as the minimum weight of a nonzero element in $S$. 


For any subset $S$ of $F = M_D(Q; I, \Lambda; P)$ and any $i \in I$, $\lambda \in \Lambda$, $A \subseteq E$, $X \subseteq I$, $Y \subseteq \Lambda$, we use the following notation: $S_{iA} = \{ q \in S \mid q \in Q \}$, $S_A = S \cap \sum_{(i, \lambda) \in A} F_{i\lambda}$, $S_X = S \cap \sum_{i \in X, \lambda \in Y} F_{i\lambda}$, $S_{X*} = S \cap \sum_{i \in X, \lambda \in \Lambda} F_{i\lambda}$, $S_{Y*} = S \cap \sum_{i \in I, \lambda \in Y} F_{i\lambda}$. Besides, we put $S_{X\lambda} = S_{X(\lambda)}$, $S_{iY} = S_{(i)Y}$, $S_{i*} = S_{(i)*}$, $S_{\lambda} = S_{(\lambda)}$ and assume that $S_{0Y} = S_{X0} = S_{0*} = S_0 = S_{00} = 0$.

We introduce the following sets of edges:

\[
E_L = \{(u, v) \in E \mid \forall w, x \in V : (w, x), (w, v) \in E \Rightarrow p_{xu} = 0 \}, \tag{5}
\]

\[
E_R = \{(u, v) \in E \mid \forall w, x \in V : (w, x), (u, x) \in E \Rightarrow p_{uv} = 0 \}. \tag{6}
\]

For any vertex $v \in V$, define the following sets of vertices:

\[
\text{In}(v) = \{ i \in I \mid (i, v) \in E \}, \tag{7}
\]

\[
\text{Out}(v) = \{ \lambda \in \Lambda \mid (v, \lambda) \in E \}, \tag{9}
\]

\[
\text{Out}_{P}(v) = \{ \lambda \in V \mid \exists x : p_{vx} \neq 0, (x, \lambda) \in E \}. \tag{10}
\]

For any $m \in \mathbb{N}$, denote by $K_L(m)$ the set of all pairs $(Y, \lambda)$, where $\lambda \in \Lambda$ and $Y \subseteq \text{In}(\lambda)$ are such that $|Y| = m$, $(i, \lambda) \in E_L$ for all $i \in I$, and the intersection $\text{Out}(i) \cap \text{Out}_{P}(\lambda)$ is equal to the same set for all $i \in Y$, i.e., $\text{Out}(i) \cap \text{Out}_{P}(\lambda) = \text{Out}(i_2) \cap \text{Out}_{P}(\lambda)$, for all $i_1, i_2 \in Y$.

Likewise, denote by $K_F(m)$ the set of all pairs $(i, Y)$, where $i \in I$ and $Y \subseteq \text{Out}(i)$ are such that $|Y| = m$, $(i, \lambda) \in E_R$ for all $\lambda \in Y$, and the intersection $\text{In}_{P}(i) \cap \text{In}(\lambda)$ is equal to the same set for all $\lambda \in Y$, i.e., $\text{In}_{P}(i) \cap \text{In}(\lambda_1) = \text{In}_{P}(i) \cap \text{In}(\lambda_2)$, for all $\lambda_1, \lambda_2 \in Y$.

Further, we assume that $D = (V, G)$ is a finite digraph. Then the cardinality $N_Z = |E_R \cap E_L|$ is an integer. Besides, denote by $N_L$ the largest positive integer such that the set $K_L(N_L)$ is not empty, and put $N_L = 0$ if such integers do not exist. Likewise, denote by $N_R$ the largest positive integer such that the set $K_F(N_R)$ is not empty, and put $N_R = 0$ if such integers do not exist.

Let us define the following subsets of the Munn incidence semiring $M_D(Q; I, \Lambda; P)$:

\[
A_L = \left\{ \sum_{i \in Y} q_{i\lambda} e_{i\lambda} \mid (Y, \lambda) \in K_L(N_L), 0 \neq q_{i\lambda} \in Q \text{ for all } i \right\}, \tag{11}
\]

\[
A_R = \left\{ \sum_{\lambda \in Y} q_{i\lambda} e_{i\lambda} \mid (i, Y) \in K_F(N_R), 0 \neq q_{i\lambda} \in Q \text{ for all } \lambda \right\}, \tag{12}
\]

\[
A_Z = \left\{ \sum_{(i, \lambda) \in E_R \cap E_L} q_{i\lambda} e_{i\lambda} \mid 0 \neq q_{i\lambda} \in Q \text{ for all } (i, \lambda) \in E_R \cap E_L \right\}. \tag{13}
\]

Our main theorem describes ideals that have the largest possible weight among the weights of all ideals in Munn incidence semirings.

**Theorem 1.** Let $Q$ be a zero-divisor-free idempotent semiring with identity element, let $F$ be a Munn incidence semiring $M_D(Q; I, \Lambda; P)$ of the finite digraph $D = (V, E)$, and let $T$ be an ideal of $F$ such that $T$ has the largest possible weight. Then the following conditions are satisfied:

(a) if $\text{wt}(T) > 1$, then there exists a generator $x$ in $A_L \cup A_R \cup A_Z$ such that $\text{wt}(x) = \text{wt}(T)$ and $x \in T$;

(b) $\text{wt}(T) = \max\{1, N_L, N_R, N_Z\}$.
Technical Lemmas and Proofs

Let us begin with a few technical properties of the Munn incidence semirings. The following two lemmas show that Munn incidence semirings of digraphs are a common generalization of the incidence algebras and Munn semirings.

**Lemma 2.** Every incidence algebra $I_D(Q)$ of a digraph $D = (V, E)$ is isomorphic to the Munn incidence semiring $M_D(Q; V, V; P)$ of the same digraph $D$, where $P$ is the identity $|V| \times |V|$-matrix.

*Proof* follows immediately from the definitions of the Munn incidence semiring and incidence algebra. □

**Lemma 3.** Every $M(Q; I, \Lambda; P)$ is isomorphic to the Munn incidence semiring $M_D(Q; I, \Lambda; P)$ of the digraph $D$, where $D$ is the complete digraph on the set $I \cup \Lambda$ of vertices.

*Proof* follows immediately from the definitions of the Munn incidence semiring and classical Munn semiring. □

We say that the digraph $D$ is $P$-balanced if, for all $i_1, \lambda_1, i_2, \lambda_2, i_3, \lambda_3 \in V$ such that $(i_1, \lambda_1), (i_2, \lambda_2), (i_3, \lambda_3) \in E \cap (I \times \Lambda)$ and $p_{\lambda_1i_2}, p_{\lambda_2i_3} \neq 0$, the following equivalence holds:

\[(i_1, \lambda_2) \in E \iff (i_2, \lambda_3) \in E. \quad (14)\]

**Lemma 4.** The set $M_D(Q; I, \Lambda; P)$ is a Munn incidence semiring of the digraph $D = (V, E)$ if and only if $D$ is $P$-balanced.

*Proof.* Suppose that the digraph $D$ is $P$-balanced. Then we are going to verify that the associative law $x(yz) = (xy)z$ is satisfied, for all $x, y, z \in M_D(Q; I, \Lambda; P)$. The distributive law and equality (3) imply that it is enough to consider the case where $x = q_{i_2}e_{i_2}e_{\lambda_2}$, $y = q_{i_3}e_{i_3}e_{\lambda_3}$, $z = q_{i_4}e_{i_4}e_{\lambda_4}$, for some $q_{i_2}, q_{i_3}, q_{i_4} \in Q$ and $(i_1, \lambda_1), (i_2, \lambda_2), (i_3, \lambda_3), (i_4, \lambda_4) \in E \cap (I \times \Lambda)$.

If $p_{\lambda_1i_2} = 0$ or $p_{\lambda_2i_3} = 0$, then we get $x(yz) = 0 = (xy)z$, as required. Likewise, if $(i_1, \lambda_1) \notin E$, then it follows that $x(yz) = 0 = (xy)z$, too. Further, we assume that $p_{\lambda_1i_2}, p_{\lambda_2i_3} \neq 0$ and $(i_1, \lambda_1) \in E$.

If $(i_1, \lambda_1), (i_2, \lambda_2) \notin E$, then (4) tells us that $xy = 0$, and so $(xy)z = 0$. Since $D$ is $P$-balanced, we get $(i_2, \lambda_3) \notin E$. Therefore $yz = 0$ and $x(yz) = 0$, and so the associative law holds.

It remains to consider the case where $(i_1, \lambda_1) \in E$. Then $(i_1, \lambda_1) \in E$, because $D$ is $P$-balanced. Therefore we get $x(yz) = (q_{i_3}p_{\lambda_1i_2}q_{i_3})e_{i_3}e_{\lambda_3}z = (q_{i_2}p_{\lambda_1i_2}q_{i_2}p_{\lambda_2i_3}q_{i_2})e_{i_2}e_{\lambda_2}e_{i_4}e_{\lambda_4} = x \cdot (q_{i_3}p_{\lambda_2i_3}q_{i_3})e_{i_3}e_{\lambda_3} = x(yz).$ Thus, the associative law holds true, as required.

Conversely, let us assume that $M_D(Q; I, \Lambda; P)$ is a semiring. Then we have to verify that the equivalence (14) holds. By way of contradiction, suppose that there exist $i_1, \lambda_1, i_2, \lambda_2, i_3, \lambda_3, i_4, \lambda_4$ such that $(i_1, \lambda_1), (i_2, \lambda_2), (i_3, \lambda_3), (i_4, \lambda_4) \in E \cap (I \times \Lambda)$ and $p_{\lambda_1i_2}, p_{\lambda_2i_3} \neq 0$.

If $(i_1, \lambda_1) \notin E$ and $(i_2, \lambda_2) \notin E$, then (4) yields that $xy = 0$, and so $(xy)z = 0$. By (4), we get $x(yz) \neq 0$, a contradiction with the associative law.

On the other hand, if $(i_1, \lambda_1) \in E$ and $(i_2, \lambda_2) \notin E$, then (4) implies that $yz = 0$; whence $x(yz) = 0$. By (4), we get $(xy)z \neq 0$, a contradiction again. This completes the proof.

The following lemma is easy and well known.

**Lemma 5.** ([11]) Every idempotent semiring $Q$ is zero-sum-free, i.e., for all $q_1, \ldots, q_n \in Q$, the equality $q_1 + \cdots + q_n = 0$ implies that $q_1 = \cdots = q_n = 0$.

For any semiring $Q$, the left annihilator of $Q$ is the set $\text{Ann}_L(Q) = \{ x \in Q \mid xQ = 0 \}$, and the right annihilator of $Q$ is the set $\text{Ann}_R(Q) = \{ x \in Q \mid Qx = 0 \}$. It follows immediately that $\text{Ann}_L(Q)$ and $\text{Ann}_R(Q)$ are ideals of the semiring $Q$. 

Lemma 6. Let $Q$ be a zero-divisor-free idempotent semiring with identity element, and let $F$ be a Munn incidence semiring $M_D(Q; I, \Lambda; P)$ of the digraph $D = (V, E)$. Then $\text{Ann}_R(F) = F_{E_L}$ and $\text{Ann}_L(F) = F_{E_R}$.

Proof. Put $T_R = \text{Ann}_R(F)$. First, let us prove the inclusion $T_R \supseteq F_{E_L}$. Suppose to the contrary that there exists a nonzero element $x$ in $F_{E_L}$ such that $x \notin T_R$. Then we have $Fx \neq 0$, and so there exists $qe_{ij\lambda} \in F$ such that $qe_{ij\lambda}x \neq 0$. The definition of $F_{E_L}$ and equality (3) show that $x = \sum_{j=1}^{n} q_j e_{ij\lambda}$, for some $n > 0$, $0 \neq q_j \in Q$, $q_j \in Q$, $(i_j, \lambda_j) \in E_L$. It follows that $qe_{ij\lambda}(q_j e_{ij\lambda}) \neq 0$, for some $j$. Hence (4) yields that $p_{\mu_j} \neq 0$ and $(i, \lambda_j) \in E$. Therefore (5) implies that $(i_j, \lambda_j) \notin E_L$. This contradicts the choice of $x$ in $F_{E_L}$ and shows that $x$ belongs to $T_R$. Thus, $T_R \supseteq F_{E_L}$.

Second, let us prove the reversed inclusion. Choose any element $x$ in $T_R$. Equality (3) implies that $x = \sum_{j=1}^{n} q_j e_{ij\lambda}$, for some $n > 0$, $0 \neq q_j \in Q$, $q_j \in Q$, $(i_j, \lambda_j) \in E$. Suppose to the contrary that $x \notin F_{E_L}$.

Then the definition of $F_{E_L}$ shows that there exists $j$ such that $(i_j, \lambda_j)$ belongs to $E \setminus E_L$. It follows from (5) that there exists $(w, \mu) \in E$ such that $(w, \lambda_j) \in E$ and $p_{\mu_j} \neq 0$. Therefore (4) yields that $e_{wp_j q_j e_{i\lambda}} = (p_{\mu_j} q_j) e_{wp_j} \neq 0$ in $F$, where $p_{\mu_j} q_j \neq 0$ because $Q$ is zero-divisor-free. However, $e_{wp_j q_j e_{i\lambda}}$ is a summand of $e_{wp_j}x$ with coefficient $q_j$. Since $Q$ is an idempotent semiring, Lemma 5 shows that this summand does not cancel with other summands of $e_{wp_j}x$. Therefore $e_{wp_j}x \neq 0$. This contradicts the choice of $x$ in $T_R$, and shows that $x$ belongs to $F_{E_L}$.

Thus, $T_R = F_{E_L}$, as required. The proof of the second equality $\text{Ann}_L(F) = F_{E_R}$ is dual and we omit it.

Lemma 7. Let $Q$ be a zero-divisor-free idempotent semiring with identity element, and let $F$ be a Munn incidence semiring $M_D(Q; I, \Lambda; P)$ of the digraph $D = (V, E)$. Then $\text{wt}(\text{id}_R(x)) = \text{wt}(x) = N_L$, for all $x \in A_L$.

Proof. Choose any element $x$ in $A_L$. In view of (11), we can write it down as $x = \sum_{i \in Y} q_{i\lambda} e_{i\lambda}$, for some $(Y, \lambda) \in K_L(N_L)$ and $0 \neq q_{i\lambda} \in Q$. The definition of $K_L(N_L)$ yields that $|Y| = N_L$. Thus, we have $\text{wt}(x) = |Y| = N_L$.

To prove that $\text{wt}(\text{id}_R(x)) = \text{wt}(x)$, choose a nonzero element $y$ in $\text{id}_R(x)$. The inequality $\text{wt}(\text{id}_R(x)) \leq \text{wt}(x)$ is obvious, and so it remains to verify that $\text{wt}(y) \geq \text{wt}(x)$.

It follows from (1) that $y$ can be written down in the form $y = \sum_{j=1}^{k} \ell_j x r_j$, for some $\ell_j, r_j \in F \cup \{1\}$. We may assume that only nonzero summands $\ell_j x r_j$ have been included in representation for $y$ given above. The definition of $K_L(N_L)$ tells us that $(i, \lambda) \in E_L$, for all $i \in Y$. Lemma 6 shows that $x$ belongs to $\text{Ann}_R(F)$. This forces all products $\ell_j x$ to be equal to zero whenever $\ell_j$ is in $F$. It follows that $\ell_1 = \ldots = \ell_k = 1$.

In view of (3) each element $r_j \in F$, that occurs in the expression for $y$ given above, can be represented in the form $r_j = \sum_{a=1}^{b} h_j(a) e_{i_j(a)\lambda_j(a)}$, for some $b \in \mathbb{N}, h_j(a) \in Q, i_j(a) \in I, \lambda_j(a) \in \Lambda$. Substituting these representations in the sum for $y$ given above and applying the distributive law, to simplify notation we may assume that from the very beginning each element $r_j \neq 1$ itself has the form $r_j = h_j e_{i_j\lambda_j}$, for some $0 \neq h_j \in Q, (i_j, \lambda_j) \in E$.

Given that $x \in A_L$ and $(Y, \lambda) \in K_L(N_L)$, the last condition in the definition of $K_L(N_L)$ tells us that the intersection $\text{Out}(i) \cap \text{Out}_p(\lambda)$ is equal to one and the same set for all $i \in Y$, i.e., $\text{Out}(i) \cap \text{Out}_p(\lambda) = \text{Out}(i) \cap \text{Out}_p(\lambda)$, for all $i_1, i_2 \in Y$. Since $x r_j \neq 0$, it follows from (4) that $p_{\lambda_j} \neq 0$ and that there exists $i_j$ in $Y$ such that $\lambda_j$ belongs to $\text{Out}(i_j) \cap \text{Out}_p(\lambda)$. Consequently, $\lambda_j$ is in $\text{Out}(i) \cap \text{Out}_p(\lambda)$, for all $i \in Y$.

Fix any $i \in Y$ and consider the summand $q_i e_{i\lambda}$ of $x$. Since $Q$ is zero-divisor-free and $p_{\lambda_j} \neq 0$, we get $q_i p_{\lambda_j} h_j \neq 0$. Hence (4), (9) and (10) imply that $q_i e_{i\lambda} \cdot r_j = q_i e_{i\lambda} \cdot h_j e_{i_j\lambda_j} = (q_i p_{\lambda_j}) h_j e_{i_j\lambda_j} \neq 0$.

It follows that $\text{wt}(x r_j) = |Y| = \text{wt}(x)$, for each $r_j \neq 1$. The equality $\text{wt}(x r_j) = \text{wt}(x)$ also holds trivially for each $r_j = 1$ in the sum for $y$. Since $Q$ is an idempotent semiring, it follows from Lemma 5 that $\text{wt}(y) \geq \text{wt}(x)$. By the choice of $x$, this means that $\text{wt}(\text{id}_R(x)) = \text{wt}(x)$. This completes the proof.
Lemma 8. Let $Q$ be a zero-divisor-free idempotent semiring with identity element, and let $F$ be a Munn incidence semiring $M_D(Q; \Lambda; P)$ of the digraph $D = (V, E)$. Then $\text{wt}(\text{id}(x)) = \text{wt}(x) = N_R$, for all $x \in A_R$.

Proof. The proof of Lemma 8 is dual to the proof of Lemma 7 and we omit it.

Lemma 9. Let $Q$ be a zero-divisor-free idempotent semiring with identity element, and let $F$ be a Munn incidence semiring $M_D(Q; \Lambda; P)$ of the digraph $D = (V, E)$. Then $\text{wt}(x) = \text{wt}(\text{id}(x)) = N_Z$, for all $x \in A_Z$.

Proof. Take any element $x$ in $A_Z$. By (13), it can be recorded in the form $x = \sum_{(i, \lambda) \in E_R \cap E_L} q_{i\lambda} e_{i\lambda}$, for some $0 \neq q_{i\lambda} \in Q$. Clearly, $\text{wt}(x) = |E_R \cap E_L| = N_Z$. Choose a nonzero element $y$ in $\text{id}(x)$. It follows from (1) that $y$ can be written down as $y = \sum_{j=1}^{k} \ell_j x r_j$, for some $\ell_j, r_j \in F \cup \{1\}$.

Since $Q$ is an idempotent semiring with identity element, Lemma 6 tells us that $\text{Ann}_R(F) = F_{E_R}$ and $\text{Ann}_L(F) = F_{E_L}$. Hence we get $x \in F_{E_R} \cap F_{E_L} = F_{E_R} \cap F_{E_L} = \text{Ann}_R(F) \cap \text{Ann}_L(F)$. It follows that if $\ell_j \in F$ or $r_j \in F$, then $\ell_j x r_j = 0$ in the representation for $y$. Therefore we may assume that $\ell_j = r_j = 1$, for all summands of the representation of $y$ given above. This means that $\text{id}(x) = \mathbb{N}_x$.

Now, take any element $kx \neq 0$ in $\mathbb{N}_x$, where $k \in \mathbb{N}$. Suppose that $k q_{i\lambda} = 0$ for some summand $q_{i\lambda} e_{i\lambda}$ of $x$. Since $Q$ has an identity element $1_Q$, we infer that $k 1_Q = kx \neq 0$ in $Q$. The equalities $(k 1_Q) q_{i\lambda} = k q_{i\lambda} = 0$ mean that $k 1_Q$ is a zero divisor in $Q$. This contradicts the hypothesis that $Q$ is zero-divisor-free. It follows that, for every $k \in \mathbb{N}$ such that $kx \neq 0$, the weight of $kx$ is equal to $\text{wt}(x)$. We conclude that $\text{wt}(\text{id}(x)) = \text{wt}(x)$, which completes the proof.

Proof of Theorem 1. To prove condition (a) we suppose that $\text{wt}(T) > 1$. Choose a nonzero element $x$ of minimal weight in $T$. Then we have $\text{wt}(x) = \text{wt}(T) > 1$. Lemma 6 says that $\text{Ann}_L(F) = F_{E_R}$ and $\text{Ann}_R(F) = F_{E_L}$.

If $x \notin \text{Ann}_R(F) \cup \text{Ann}_L(F)$, then there exist $q_{1e_{i_1\lambda_1}}$ and $q_{2e_{i_2\lambda_2}}$ in $F$ such that the products $q_{1e_{i_1\lambda_1}} x$ and $q_{2e_{i_2\lambda_2}} x$ are nonzero. Equality (4) yields us that the product $y = q_{1e_{i_1\lambda_1}} x q_{2e_{i_2\lambda_2}}$ is nonzero too. Since $Q$ is zero-divisor-free, it also follows from (4) that $y \in F_{i_1\lambda_2}$; whence $\text{wt}(y) = 1$. However, $y$ belongs to $T$. This contradicts the hypothesis that $\text{wt}(T) = 1$, and shows that $x$ always belongs to the union $\text{Ann}_R(F) \cup \text{Ann}_L(F)$. Therefore the following three cases may occur.

Case 1. $x \in \text{Ann}_R(F) \cap \text{Ann}_L(F)$. Then $x \in F_{E_R} \cap F_{E_L} = F_{E_R} \cap F_{E_L}$, and so $\text{wt}(T) = \text{wt}(x) \leq N_Z$. On the other hand, it follows from the maximality of $\text{wt}(T)$ and Lemma 9 that $\text{wt}(T) \geq N_Z$. Therefore $\text{wt}(x) = N_Z$. Hence (13) shows that $x \in A_Z$, which means that condition (a) holds.

Case 2. $x \in \text{Ann}_R(F) \setminus \text{Ann}_L(F)$. Then $x \in F_{E_L}$. In view of (3), we infer

$$x = \sum_{j=1}^{n} q_j e_{i_j \lambda_j}$$

(15)

where $n \geq 0$, $0 \neq q_j \in Q$, and $(i_j, \lambda_j) \in E_L$. Since $x$ does not belong to $\text{Ann}_L(F)$, it follows that there exists $q e_{i\lambda} \in F$ such that $x q e_{i\lambda} \neq 0$.

Consider the set $Y = \{i_1, \ldots, i_n\}$. We are going to verify that the pair $(Y, \lambda)$ satisfies all conditions in the definition of $K_L(n)$.

First, note that $\lambda \in \Lambda$, as required in the definition of $K_L(n)$.

Second, observe that the inequality $\text{wt}(x q e_{i\lambda}) \leq \text{wt}(x)$ is always true in $F$. The reversed inequality follows from the minimality of $\text{wt}(x)$ in $T$, because $x q e_{i\lambda} \in T$. Therefore $\text{wt}(x q e_{i\lambda}) = \text{wt}(x)$. This and (15) yield us that $q_j e_{i_j \lambda_j} \cdot q_j e_{i\lambda} \neq 0$, for all $j = 1, \ldots, n$. By (4), $p_{i,j} \neq 0$ and $(i_j, \lambda) \in E$, for all $j = 1, \ldots, n$. Therefore $Y \subseteq \text{In}(\lambda)$, as required in the definition of $K_L(n)$.

Third, given that $x$ belongs to the ideal $\text{Ann}_R(F)$ of $F$, it is clear that $x q e_{i\lambda}$ is in $\text{Ann}_R(F)$, too. Hence $(i_j, \lambda) \in E_L$, for all $i_j \in Y$, by Lemma 6.

Fourth, it is clear that $|Y| = n$, as required in the definition of $K_L(n)$, too.
Fifth, to prove the last property required in the definition of $K_L(n)$, suppose to the contrary that the intersections $\text{Out}(i_1) \cap \text{Out}_P(\lambda)$ and $\text{Out}(i_2) \cap \text{Out}_P(\lambda)$ are different for some $i_1, i_2$ in $Y$. Without loss of generality we may assume that there exists $\mu \in \Lambda$ that belongs to $\text{Out}(i_1) \cap \text{Out}_P(\lambda)$ but does not belong to $\text{Out}(i_2)$. Then $(i_1, \mu) \in E$ and $(i_2, \mu) \notin E$. Since $\mu \in \text{Out}_P(\lambda)$, by (10) there exists $j \in I$ such that $p_{i_1j} \neq 0$ and $(j, \mu) \in E$. Since $Q$ is a semiring with identity element, we have $e_{j\mu} \in F$. Therefore (4) implies that $q_1 e_{i_1\lambda} e_{j\mu} \neq 0$, but $q_1 e_{i_1\lambda} e_{j\mu} = 0$. It follows that $\text{wt}(x e_{j\mu}) < \text{wt}(x)$. This contradicts the minimality of $\text{wt}(x)$ in $T$, because $xe_{j\mu} \in T$. The contradiction shows that all intersections $\text{Out}(i_j) \cap \text{Out}_P(\lambda)$ are equal to one and the same set, for all $i_j \in Y$. Thus, we have proved that $(Y, \lambda)$ belongs to $K_L(n)$.

Further, the maximality of $N_L$ implies that $|Y| \leq N_L$. It follows from (15) that $xq e_{i\lambda} \in F_{Y\lambda}$. Hence we get $\text{wt}(x) = \text{wt}(x q e_{i\lambda}) \leq |Y| \leq N_L$. On the other hand, the maximality of $\text{wt}(T)$ and Lemma 7 show that $\text{wt}(T) \geq N_L$. Hence $\text{wt}(x) \geq N_L$. Therefore $\text{wt}(x q e_{i\lambda}) = \text{wt}(x) = N_L$, and so $|Y| \leq N_L$. Thus, we conclude that $n = \text{wt}(x) = N_L$ and $(Y, v) \in K_L(N_L)$. By (11), we get $x(g; i, \lambda) \in A_L$. Therefore condition (a) holds in this case too.

Case 3. $x \in \text{Ann}_L(F) \setminus \text{Ann}_R(F)$. Then the proof is dual to the proof of Case 2, and so condition (a) holds again. This completes the proof of condition (a).

To verify condition (b), first note that the inequality $\text{wt}(T) \geq \max\{N_L, N_R, N_Z\}$ follows immediately from Lemmas 7, 8 and 9 and the maximality of $\text{wt}(T)$. If $\text{wt}(T) = 1$, then we get $N_L, N_R, N_Z \leq 1$; whence condition (b) is met. On the other hand, if $\text{wt}(T) > 1$, then condition (a) implies that $\text{wt}(T) \leq \max\{N_L, N_R, N_Z\}$, and so (b) holds again. This completes the proof. \hfill $\square$

Acknowledgements

The first author was supported by Discovery grant DP0449469 from Australian Research Council. The second author was supported by ARC Discovery grant DP130101383. The third author was supported by Discovery grant DP0450294.

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