Ground state representations of topological groups

Karl-Hermann Neeb, Francesco G. Russo

January 14, 2022

Abstract

Let $\alpha : \mathbb{R} \to \operatorname{Aut}(G)$ define a continuous $\mathbb{R}$-action on the topological group $G$. A unitary representation $(\pi^\alpha, \mathcal{H})$ of the extended group $G^\alpha := G \rtimes \mathbb{R}$ is called a ground state representation if the unitary one-parameter group $\pi^\alpha(e, t) = e^{itH}$ has a non-negative generator $H \geq 0$ and the subspace $\mathcal{H}^0 := \ker H$ of ground states generates $\mathcal{H}$ under $G$. In this paper we introduce the class of strict ground state representations, where $(\pi^\alpha, \mathcal{H})$ and the representation of the subgroup $G^\alpha_0 := \operatorname{Fix(\alpha)}$ on $\mathcal{H}^0$ have the same commutant. The advantage of this concept is that it permits us to classify strict ground state representations in terms of the corresponding representations of $G^\alpha_0$. This is particularly effective if the occurring representations of $G^\alpha_0$ can be characterized intrinsically in terms of concrete positivity conditions.

To find such conditions, it is natural to restrict to infinite dimensional Lie groups such as (1) Heisenberg groups (which exhibit examples of non-strict ground state representations); (2) Finite dimensional groups, where highest weight representations provide natural examples; (3) Compact groups, for which our approach provides a new perspective on the classification of unitary representations; (4) Direct limits of compact groups, as a class of examples for which strict ground state representations can be used to classify large classes of unitary representations.

Keywords: positive energy representation, ground state, holomorphic induction, Heisenberg group, compact group

MSC2010: Primary: 22E45, 22E66. Secondary: 43A75, 43A65

Contents

1 Introduction 2

2 Generalities on ground state representations 6
  2.1 Minimal representations .................................................. 6
  2.2 Ground state representations .............................................. 8
  2.3 Strict ground state representations ..................................... 10

3 Lie theoretic aspects 12
  3.1 Regularity conditions for actions on Lie groups ..................... 12
  3.2 Momentum sets and positive cones ..................................... 15

4 Heisenberg and oscillator groups 18
5 Holomorphic induction
5.1 The geometric setup for holomorphic induction ........................................ 24
5.2 Holomorphically induced representations .................................................. 26
5.3 The setting with $\alpha$ .............................................................. 27

6 Finite dimensional groups ................................................................. 29
6.1 Generalities ........................................................................... 29
6.2 Application to compact Lie groups ...................................................... 33

7 Compact non-Lie groups ....................................................................... 37

8 Ground state representations of direct limits .......................................... 38
8.1 Ground state representations ................................................................. 38
8.2 Some infinite dimensional unitary groups ........................................... 39

A Arveson spectral theory ......................................................................... 42
A.1 Arveson spectral subspaces ................................................................. 42
A.2 Applications to unitary representations ................................................ 45

B Positive definite kernels .......................................................................... 45

C Bosonic Fock space .................................................................................. 46

D Spaces with the finest locally convex topology ....................................... 47

1 Introduction

Let $G$ be a (Hausdorff) topological group and $\alpha : \mathbb{R} \to \text{Aut}(G)$ be a homomorphism defining a continuous action of $\mathbb{R}$ on $G$ and let

$$G^\alpha := \text{Fix}(\alpha) := \{ g \in G : (\forall t \in \mathbb{R}) \alpha_t(g) = g \}$$

be the closed subgroup of fixed points. The semidirect product $G^\alpha := G \rtimes_{\alpha} \mathbb{R}$ is a topological group with respect to the product topology, and a unitary representation $(\pi^\alpha, \mathcal{H})$ of $G^\alpha$ on a complex Hilbert space $\mathcal{H}$ always has the form

$$\pi^\alpha(g, t) = \pi(g)U_t, \quad (1)$$

where $(\pi, \mathcal{H})$ is a unitary representation of $G$ and $(U_t)_{t \in \mathbb{R}}$ is a unitary one-parameter group on $\mathcal{H}$ satisfying the covariance condition

$$U_t \pi(g)U_{-t} = \pi(\alpha_t(g)) \quad \text{for} \quad t \in \mathbb{R}, g \in G. \quad (2)$$

Writing $U_t = e^{itH}$ with a selfadjoint operator $H$ (Stone’s Theorem, [Ru73, Thm. 13.38]), we call $\pi^\alpha$, resp., the pair $(\pi, U)$ a positive energy representation of $(G, \alpha)$ if $H \geq 0$. If, in addition, for the minimal energy space $\mathcal{H}^0 := \ker H$, the subset $\pi(G)\mathcal{H}^0$ spans a dense subspace of $\mathcal{H}$, we call $(\pi, \mathcal{H})$ a ground state representation. 1 One expects that that ground state representations are determined by the representation $(\pi^0, \mathcal{H}^0)$ of $G^\alpha$ on the “minimal energy space” $\mathcal{H}^0$, but this is in

1In [JN21] the term “ground state representation” is used in a slightly more general context where the minimal eigenvalue of $H$ is not necessarily 0, but this is only a matter of terminology.
general not the case (Example 4.8). To make up for this defect, we introduce the concept of a strict ground state representation (Definition 2.12), where we require, in addition, that the (injective) restriction map from the commutant \( \pi(G)' \) of \( \pi(G) \) to the commutant \( \pi^0(G^0)' \) (a morphism of von Neumann algebras) is surjective. This class of representations of \( G \) is completely determined by the corresponding representations of \( G^0 \) and we want to use it to classify strict ground state representations in terms of the corresponding representations of \( G^0 \). This is particularly effective if the occurring representations of \( G^0 \) can be characterized intrinsically in terms of concrete positivity conditions.

To find such conditions, it is natural to restrict to the class of, possibly infinite dimensional, Lie groups. For these groups the method of holomorphic induction which has been developed in [Ne13] for Banach–Lie groups and extended in [Ne14b, App. C] to certain Fréchet–Lie groups, can be used to construct strict ground state representations.

We discuss four classes of groups:

1. Heisenberg groups \( \text{Heis}(V, \sigma) \), i.e., the canonical central extension of the additive groups of a symplectic vector space \((V, \sigma)\) by the circle group \( \mathbb{T} \). These groups provide in particular examples of non-strict ground state representations. As these group arise naturally in Quantum Field Theory in the context of the canonical commutation relations, we were able to use old results of M. Weinless [We69] to derive a structure theorem for ground state representations under some extra conditions on the \( \mathbb{R} \)-action. In this context our results complement the theory of semibounded representations of these group, developed in [NZ13] and [Ze14].

2. Finite dimensional Lie groups, as a class of groups whose well-developed structure theory permits to understand the intricacies on the conditions to be imposed on the \( \mathbb{R} \)-action on \( G \). Here highest weight representations, and the more general class of semibounded representations provide natural examples of strict ground state representations ([Ne00]).

3. Connected compact groups, as a class of topological groups which is rather close to Lie groups. They are projective limits of connected Lie groups and we show that this approximation can be aligned with the \( \mathbb{R} \)-action on \( G \). This permits us to show that all unitary representations are strict ground state representations, and this provides a novel perspective on the classification of unitary representations of connected compact Lie groups (Subsection 6.2). If \( \alpha \) is defined by \( \alpha_t(g) = \exp(td)g\exp(-td) \) for a regular element \( d \in \mathfrak{g} \) (the Lie algebra of \( G \)), then \( G^0 = T \) is a maximal torus and the approach to the classification in terms of ground state representations leads to the Cartan–Weyl Theorem on the classification in terms of their highest (lowest) weights (cf. [Wa72, p. 209]).

4. Direct limits of compact groups ([Gl05]); as a class of Lie groups for which strict ground state representations can be studied systematically with the methods developed in this paper. We only briefly discuss some concrete examples to give an impression of how this can be done in principle.

In the representation theory of Lie groups, ground states have classically been studied in the context of highest or lowest weight representations, which require a much finer structure theoretic context (cf. [KR87]). For this specific class of representations (and for the more general class of semibounded representations), methods similar to ours have also been used in the following contexts:

- double extensions of Hilbert–Lie groups ([MN16])
- twisted loop groups with values in Hilbert–Lie groups ([MN17])
• Hermitian Banach–Lie groups of compact type, i.e., automorphism groups of Banach hermitian symmetric spaces ([Ne12, §8])

• the particular Schatten class groups $U_1(\mathcal{H})$ and $U_2(\mathcal{H})$ ([Ne12, App. D])

• for groups of local gauge transformations, positive energy and ground state representations have been treated from a similar perspective in [JN21, §§3, 9].

The specific features appearing in these papers suggest that a general theory of ground state representations for general topological groups could be a useful tool to classify important classes of unitary representations of groups which are not locally compact.

We now describe the structure of this paper in some more detail. In Section 2 we develop the basic concepts. First we explain how the concept of a minimal implementing group from the theory of operator algebraic dynamical systems translates into our context (Subsection 2.1). It provides the language to define minimal positive energy representations and ground state representations. In this context, the Borchers–Arveson Theorem implies that, for a ground state representation $(\pi, U, \mathcal{H})$ of $(G, \alpha)$, the one-parameter group $(U_t)_{t \in \mathbb{R}}$ is redundant in the sense that it is completely determined by the representation $\pi$ of $G$ and the assumption that the generating subspace $\mathcal{H}^0$ is fixed pointwise by $U$ (see also [Ne14] for a formulation of the Borchers–Arveson Theorem in the context of topological groups). After discussing some elementary properties of ground state representations, we introduce the new concept of a strict ground state representation in Subsection 2.3. The purpose of this concept is to classify ground state representations of $G$ in terms of representations of $G^0$ arising on some $\mathcal{H}^0$.

In Section 3 we develop for Lie groups methods to identify these representations of $G^0$ in intrinsic terms. We formulate four conditions: (L1) $G$ is a Lie group (modelled over a locally convex space), (L2) $\alpha$ defines a smooth $\mathbb{R}$-action on $G$, (L3) the subgroup $G^0$ of fixed points is a Lie group, and (L4) the Lie algebra $\mathfrak{g}$ of $G$ is the direct sum of the Lie algebra $\mathfrak{g}^0$ of $G^0$ and $D(\mathfrak{g})$, where $D$ is the infinitesimal generator of the induced $\mathbb{R}$-action on $\mathfrak{g}$. Note that conditions (L1-3) are automatic if $G$ is a finite dimensional Lie group, but (L4) corresponds to $\ker(D^2) = \ker(D)$. If these conditions are satisfied and $p_0: \mathfrak{g} \to \mathfrak{g}^0$ is the projection with kernel $D(\mathfrak{g})$, then the closed convex cone

$$C_{\alpha} \subseteq \mathfrak{g}^0,$$

generated by elements of the form $p_0([Dx, x])$, $x \in \mathfrak{g}$, turns out to play an important role. The main result in Section 3 is Theorem 3.11 which asserts that, for every ground state representation $(\pi, U, \mathcal{H})$, we have

$$C_{\alpha} \subseteq C_{\pi^0} := \{x \in \mathfrak{g}^0: -i\partial\pi^0(x) \geq 0\},$$

where $\partial\pi^0(x)$ is the infinitesimal generator of the unitary one-parameter group $(\pi^0(\exp tx))_{t \in \mathbb{R}}$. This is a necessary condition for a representation $(\pi^0, \mathcal{H}^0)$ of $G^0$ to appear in a ground state representation of $G$. Unfortunately it is not sufficient in general (Remark 6.17).

As a consequence of the Borchers–Arveson Theorem, for abelian groups $G$ with a faithful ground state representation, the $\mathbb{R}$-action on $G$ is trivial. Therefore the simplest non-trivial examples arise from 2-step nilpotent groups. We therefore discuss Heisenberg groups $G = \text{Heis}(V, \sigma)$ in Section 4, where $\mathbb{R}$ acts on $G$ through a symplectic one-parameter group $\beta: \mathbb{R} \to \text{Sp}(V, \sigma)$. In this context condition (L4) turns into the weak splitting condition, which implies a tensor factorization of ground state representations that can be derived from old results of M. Weinless [We69]. Actually an example where (L4) is violated led us to an example of a non-strict ground state representation (Example 4.8).
Section 5 introduces the powerful technique of holomorphic induction as a means to construct ground state representations. It deals with unitary representations that can be realized in Hilbert spaces of holomorphic sections of complex vector bundles whose fibers are Hilbert spaces. Unfortunately, holomorphic induction requires rather fine geometric assumptions on the Lie groups. But it provides effective tools to determine if a given representation can be realized in this setup, and then one can typically conclude that it is a strict ground state representation. If $G$ is a Banach–Lie group and $D: \mathfrak{g} \to \mathfrak{g}$ is a bounded derivation for which $0$ is isolated in $\text{Spec}(D)$ and the norm on $\mathfrak{g}$ is $\alpha$-invariant ($D$ is elliptic), then Theorem 5.8, our main result in Section 5 is one of our key tools to identify strict ground state representations. In the following sections it is applied to finite dimensional Lie groups, where the necessary requirements are verified more easily.

This is why we first turn to finite dimensional Lie groups in Section 6. Again, an old result, C. Moore’s Eigenvector Theorem ([Mo80]), turns out to be quite helpful. We use it to see that, if an irreducible $G$-representation $(\pi, \mathcal{H})$ with discrete kernel has ground states for an $\mathbb{R}$-action corresponding to an inner derivation $D = \text{ad} \; \mathbf{d}$, then $\mathbf{d}$ must be an elliptic element, i.e., $\mathbb{R} \cdot \text{ad} \; \mathbf{d}$ is a torus group, or, equivalently, $\text{ad} \; \mathbf{d}$ is semisimple with purely imaginary spectrum. Our setup applies particularly well to compact connected Lie groups. In this context, for any $\alpha: \mathbb{R} \to \text{Aut}(G)$, we show in Theorem 6.15 that every unitary representation $(U, \mathcal{H})$ of $G$ is a strict ground state representation, and that these correspond precisely to the representations $(\pi^0, \mathcal{H}^0)$ of the connected subgroup $G^0$ satisfying $C_\alpha \subseteq C_{\pi^0}$. If the derivation $D = \text{ad} \; \mathbf{d}$ is such that $\mathbf{d}$ is a regular element, then $G^0$ is a maximal torus, and this result reproduces the Cartan–Weyl classification of irreducible representations, but it also works for any $\mathbf{d}$. In Section 7 this result is extended to general connected compact topological groups (Theorem 7.3), which provides a novel global perspective on the classification for these groups. We conclude this paper with Section 8 which is devoted to countable direct limits of finite dimensional Lie groups. These are always locally convex Lie groups by Glöckner’s Theorem ([Gl05]). For direct limits of compact Lie groups, Theorem 8.2 generalizes the characterization of representations $(\pi^0, \mathcal{H}^0)$ of $G^0$ which extend to ground state representations in terms of the positivity condition $C_\alpha \subseteq C_{\pi^0}$. It thus reduces the corresponding classification problem from $G$ to $G^0$. We discuss some examples where $G^0$ is abelian (a direct limit of tori), so that concrete classification results can be obtained with the Bochner Theorem for nuclear groups ([Ba91]).

We include four short appendices: In Appendix A we recall Arveson’s concept of spectral subspaces which is an important tool to formulate the splitting conditions in Section 3. Similarly Appendix B recalls the vector-valued version of the Gefland–Neimark–Segal (GNS) correspondence between operator-valued positive definite functions and unitary representations generated by Hilbert subspaces. Some facts on bosonic Fock spaces are collected in Appendix C because they are needed in our discussion of Heisenberg groups in Section 4. Finally, Appendix D contains a key observation that we use in Section 7 for the reduction from general compact groups to finite dimensional ones and also at some point in Section 8.

Notation

- $\mathcal{H}$ denotes a complex Hilbert space, the scalar product is linear in the second argument, $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on $\mathcal{H}$, and $\mathcal{U}(\mathcal{H})$ the unitary group. We call a subset $E \subseteq \mathcal{H}$ total if $\|E\| := \text{span}E = \mathcal{H}$.
- For a set $S \subseteq \mathcal{B}(\mathcal{H})$ of bounded operators, its commutant is denoted
  $$S' := \{ A \in \mathcal{B}(\mathcal{H}) : (\forall B \in S) \; AB = BA \}.$$
A $*$-subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is called a von Neumann algebra if it equals its own bicommutant: $\mathcal{M} = \mathcal{M}'' := (\mathcal{M}')'$ (cf. [BR02, §2.4]). Then its center is $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$. 

5
• For a selfadjoint operator $H = H^*$ on $\mathcal{H}$ we write $H \geq 0$ if its spectrum $\text{Spec}(H)$ is contained in $[0, \infty)$. We then say that the corresponding unitary one-parameter group $(e^{itH})_{t \in \mathbb{R}}$ has positive spectrum.

• For a (continuous) unitary representation $(\pi, \mathcal{H})$ of a Lie group $G$ with Lie algebra $\mathfrak{g}$ and exponential function $\exp: \mathfrak{g} \to G$, we write $\partial \pi(x)$ for the skew-adjoint infinitesimal generator of the unitary one-parameter group $\pi(e^{tx})$, so that we have $\pi(e^{tx}) = e^{i\partial \pi(x)}$ in the sense of measurable functional calculus for unbounded normal operators (cf. Definition 3.4). To $\pi$ we associate two convex cones in the Lie algebra:

$$W_\pi := \{ x \in \mathfrak{g} : \inf(\text{Spec}(-i\partial \pi(x))) > -\infty \} \supseteq C_\pi := \{ x \in \mathfrak{g} : -i\partial \pi(x) \geq 0 \}. \quad (3)$$

• $G^0 = G \rtimes_\alpha \mathbb{R}$ for a homomorphism $\alpha: \mathbb{R} \to \text{Aut}(G)$.

2 Generalities on ground state representations

In this section we discuss three aspects of ground state representations $(\pi, U, \mathcal{H})$ for a pair $(G, \alpha)$. In Subsection 2.1 we explain how the concept of a minimal implementing group from the theory of operator algebraic dynamical systems translates into our context. It provides the language to define minimal positive energy representations and ground state representations. In Subsection 2.2 we take a closer look at elementary properties of ground state representations. The main new concept we introduce is that of a strict ground state representation (Subsection 2.3). Roughly speaking, strictness means that the representation $(\pi, \mathcal{H})$ of $G$ decomposes in the same way as the representation $(\pi^0, \mathcal{H}^0)$ of the fixed point group $G^0$ on the minimal energy space $\mathcal{H}^0$. In particular, strictness implies that $(\pi^0, \mathcal{H}^0)$ determines $(\pi, \mathcal{H})$. The purpose of this concept is to classify ground state representations of $G$ in terms of the representations $(\pi^0, \mathcal{H}^0)$ of $G^0$ extending to ground state representations of $G$. A key problem is to identify these representations of $G^0$ in intrinsic terms. We shall see in Section 3 how this problem can be addressed for Lie groups.

2.1 Minimal representations

The following refinement of the Borchers–Arveson Theorem can be found in [JN21, Thm. 3.7]. Part (i) is in [BR02, Thm. 3.2.46]. We also refer to [BGN20, Thm. 4.14, Lemma 4.17] for a detailed discussion of this circle of ideas.

**Theorem 2.1.** (Borchers–Arveson Theorem) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Further, let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous unitary one-parameter group on $\mathcal{H}$ for which $\mathcal{M}$ is invariant under conjugation with the operators $U_t$, so that we obtain a one-parameter group $\alpha: \mathbb{R} \to \text{Aut}(\mathcal{M})$ by $\alpha_t(M) := \text{Ad}(U_t)M := U_tMU_t^*$ for $M \in \mathcal{M}$. If $U_t = e^{itH}$ with $H \geq 0$, then the following assertions hold:

(i) There exists a strongly continuous unitary one-parameter group $U^0_t = e^{iH_0}$ in $\mathcal{M}$ with $\text{Ad}(U^0_t) = \alpha_t$ and positive spectrum. It is uniquely determined by the requirement that it is minimal in the sense that, for any other unitary one-parameter group $V_t(\mathcal{H})$, such that $\text{Ad}(V_t) = \alpha_t$ for $t \in \mathbb{R}$, the unitary one-parameter group $(V_tU^{-1}_t)_{t \in \mathbb{R}}$ in the commutant $\mathcal{M}'$ has positive spectrum.

(ii) If $U^0_T = 1$ for some $T > 0$ and $\mathcal{F} \subseteq \mathcal{H}$ is an $\mathcal{M}$-invariant subspace, then the subspace $\mathcal{F}^0 := \mathcal{F} \cap \ker H_0$ satisfies $[\mathcal{M}\mathcal{F}_0] = \mathcal{F}$. 

6
(iii) If $T \in \mathbb{R}$ satisfies $\alpha_T = \text{id}_\mathcal{M}$, then $U_T^0 = 1$.

The Borchers–Arveson Theorem applies naturally to positive energy representations ([JN21, Cor. 3.9]), where it takes the following form:

**Corollary 2.2.** Let $(\pi, U, \mathcal{H})$ be a positive energy representation of $(G, \alpha)$ and $(U_t^0)_{t \in \mathbb{R}}$ be the minimal positive implementing unitary one-parameter group in $\pi(G)''$ from Theorem 2.1. Then $U$ factorizes as

$$U_t = U_t^0 W_t \quad \text{for} \quad t \in \mathbb{R},$$

where $(W_t)_{t \in \mathbb{R}}$ is a unitary one-parameter group with positive spectrum in the center $\mathcal{Z}(\pi^b(G)''')$ of the von Neumann algebra $\pi^b(G)'''$ generated by $\pi^b(G)$, where $\pi^b(g, t) = \pi(g)U_t$.

**Proof.** We apply Theorem 2.1 to the von Neumann algebra $\mathcal{M} := \pi(G)'$. As $W_t := U_t^{-1} U_t$ commutes with $\pi(G)$, it is contained in $\pi(G)'$. Further, $U_{-t}^0 t \in \pi(G)'''$ implies $W_t \in \pi^b(G)''$. Since $W_t$ commutes with $\pi(G)$ and $U_t$, it is central in $\pi^b(G)'''$.

**Corollary 2.3.** ([Ne14, Thm. 2.5]) If $(\pi, U, \mathcal{H})$ is an irreducible positive energy representation of $G^b = G \rtimes_{\alpha} \mathbb{R}$, then its restriction to $G$ is also irreducible.

**Proof.** By Schur’s Lemma, the irreducibility of $\pi^b$ implies that $\pi^b(G)' = \mathbb{C}1$. Hence $W_t \in \mathbb{T}1$ implies $U_t \in \pi(G)'''$. Therefore $\mathbb{C}1 = \pi^b(G)' = \pi(G)'$, so that the irreducibility of $\pi$ follows from Schur’s Lemma.

The concept of a minimal representation that we now define is inspired by the situation in Corollary 2.2. We shall see in Proposition 2.7 that the concept of a ground state representation is more restrictive, but it is this extra assumption of the existence of “sufficiently many” ground states that permits us to obtain more information to classify representations.

**Definition 2.4.** (Minimal and ground state representations) A positive energy representation $(\pi, U, \mathcal{H})$ of $G^b$ with $U_t = e^{i t H}$ is called

- **minimal** if the one-parameter group $U$ is minimal with respect to the von Neumann algebra $\pi(G)'''$, i.e., $U = U^0$ (see [BGN20, §5] for a discussion of this concept in the context of von Neumann algebras).

- **a ground state representation** if $[\pi(G)\mathcal{H}^0] = \mathcal{H}$ holds for the minimal energy subspace

$$\mathcal{H}^0 := \ker H.$$

In view of the factorization in Corollary 2.2, we adopt the point of view that we understand positive energy representations if we know the minimal ones. For those, the extension of the representation $\pi$ of $G$ to $G^b$ is uniquely determined by the Borchers–Arveson Theorem. As the extendability of a unitary representation $(\pi, \mathcal{H})$ of $G$ to a positive energy representation of $G^b$ is an intrinsic property, we also call $(\pi, \mathcal{H})$ a positive energy representation of $(G, \alpha)$ if it extends to a positive energy representation of $G^b$. Here we keep in mind that the minimal extension with $U = U^0$ provides a natural extension $\pi^b$ to $G^b$ with the same commutant, hence with the same closed invariant subspaces.

With this terminology, Theorem 2.1 has the following consequence:

**Corollary 2.5.** Let $(\pi, \mathcal{H})$ be a positive energy representation of $(G, \alpha)$. If $\alpha_T = \text{id}_\mathcal{G}$ for some $T > 0$, then $U_T^0 = 1$ and $(\pi, \mathcal{H})$ is a ground state representation.
We write \( \hat{G} \) for the set of equivalence classes of irreducible unitary representations of the topological group \( G \). Since the problem of parametrizing \( \hat{G} \) for an infinite dimensional Lie group \( G \) is rather intractable (cf. [Ne14c]), the positive energy condition with respect to some \( \alpha \) provides a regularity condition that selects a class of representations for which one expects concrete classification results ([JN19], [Ne12, Ne14, Ne14b]).

**Remark 2.6.** Every \( \mathbb{R} \)-action \( \alpha \) on \( G \) specifies a subset \( \hat{G}(\alpha) \subseteq \hat{G} \) of irreducible positive energy representations with respect to \( \alpha \) (cf. Corollary 2.3), and these subsets of \( \hat{G} \) are expected to be more tractable for specific \( \alpha \)’s than general irreducible representations whose classification is elusive.

### 2.2 Ground state representations

It is a natural problem to describe the positive energy representations \((\pi, \mathcal{H})\) of \((G, \alpha)\) in terms of simpler data. For ground state representations, the necessary information should be provided by the representation of \( G^0 \) on \( \mathcal{H}^0 \). Here the following observation is an important tool.

**Proposition 2.7.** (Ground state representations are minimal) For a ground state representation \((\pi, U, \mathcal{H})\), the following assertions hold:

1. \((\pi, U, \mathcal{H})\) is minimal, hence in particular \( U_\mathbb{R} \subseteq \pi(G)'' \).
2. Let \( P_0 \) denote the orthogonal projection onto \( \mathcal{H}^0 \). Then the restriction map
   
   \[ R: \pi(G)' \to (P_0\pi(G)''P_0)' \subseteq B(\mathcal{H}^0), \quad A \mapsto P_0AP_0 \]
   
   is an isomorphism of von Neumann algebras whose range is contained in \( \pi^0(G^0)' \).

**Proof.** (i) follows by applying [BGN20, Prop. 5.4] to the C*-algebra \( \pi(G)'' \).

(ii) By (i), \( U_t = U_t^0 \in \mathcal{M} := \pi(G)'' \), so that the orthogonal projection \( P_0 \) of \( \mathcal{H}^0 \) is contained in \( \mathcal{M} \). That \( \mathcal{H}^0 \) is generating under \( \pi(G) \) shows that the central support of \( P_0 \) in \( \mathcal{M} \), i.e., the minimal central projection \( Z \) with \( ZP_0 = P_0 \), is \( 1 \). Therefore [BGN20, Lemma 3.14(iv)] implies that the restriction map \( R \) is an isomorphism onto the commutant of the von Neumann algebra \( \mathcal{M}^0 := P_0\mathcal{M}P_0 \supseteq \pi^0(G^0)' \), hence contained in \( \pi^0(G^0)' \). \( \square \)

We recall that a unitary representation \((U, \mathcal{H})\) of a group \( G \) is called factorial if the von Neumann algebra \( U(G)'\) is a factor, i.e., its center is trivial: \( Z(U(G)') = \mathbb{C}1 \).

**Lemma 2.8.** Let \((\pi, U, \mathcal{H})\) be a factorial minimal positive energy representation and \( U_t = e^{it\mathcal{H}} \).

If \( \mathcal{H}^0 := \ker \mathcal{H} \) is non-zero, then \([\pi(G)\mathcal{H}^0] = \mathcal{H} \), so that \((\pi, \mathcal{H})\) is a ground state representation.

**Proof.** The subspace \( \mathcal{F} := [\pi(G)\mathcal{H}^0] \) is invariant under the von Neumann algebra \( \mathcal{M} := \pi(G)'' \) generated by \( \pi(G) \) because it is \( \pi(G) \)-invariant. Since \( \mathcal{H}^0 \) is also invariant under \( \mathcal{M}' \), the same holds for \( \mathcal{F} \). Therefore \( \mathcal{F} \) is invariant under \((\mathcal{M} \cup \mathcal{M}')'' = B(\mathcal{H}) \). Here we use that factoriality implies that \((\mathcal{M} \cup \mathcal{M}')' = \mathcal{M}' \cap \mathcal{M}'' = C1 \). Invariance under \( B(\mathcal{H}) \) clearly entails \( \mathcal{F} = \mathcal{H} \). \( \square \)

The following lemma is a useful elementary tool to verify the positive energy and the ground state condition for direct sums.

**Lemma 2.9.** Let \((\pi, \mathcal{H})\) be a unitary representation of \( G \) which is a direct sum of unitary subrepresentations \((\pi_j, \mathcal{H}_j)_{j \in J}\). Then \((\pi, \mathcal{H})\) is a positive energy (ground state) representation for \((G, \alpha)\) if and only if all the representations \((\pi_j, \mathcal{H}_j)_{j \in J}\) have this property.

In particular, every subrepresentation of a positive energy (ground state) representation inherits this property.
Proof. “⇒”: If $(\pi, \mathcal{H})$ is of positive energy, then $U_t^0 \in \pi(G)''$ for every $t \in \mathbb{R}$ (Theorem 2.1), so that the $U_t^0$ preserve all the subspaces $\mathcal{H}_j$. Therefore its restriction to $\mathcal{H}_j$ shows that all representations $(\pi_j, \mathcal{H}_j)$ are of positive energy since the one-parameter groups $(U_t^0|_{\mathcal{H}_j})_{t \in \mathbb{R}}$ have positive spectrum.

If, in addition, $(\pi, U, \mathcal{H})$ is a ground state representation, then Proposition 2.7(i) implies that $U = U_0^0$ is minimal, so that $U_R \subseteq \pi(G)''$. Therefore $\mathcal{H}_0^0 = \ker H$ is invariant under the commutant $\pi(G)'$, and this implies that

$$\mathcal{H}_0^0 = \sum_{j \in J} \mathcal{H}_j \cap \mathcal{H}_0^0 \cong \bigoplus_{j \in J} \mathcal{H}_j^0.$$ 

As $\pi(G)\mathcal{H}_0^0$ is total in $\mathcal{H}$, projection to $\mathcal{H}_j$ shows that $\pi_j(G)\mathcal{H}_j^0$ is total in $\mathcal{H}_j$, i.e., $(\pi_j, U_j, \mathcal{H}_j)$ is a ground state representation.

“⇐”: If all representations $(\pi_j, \mathcal{H}_j)$ are of positive energy, then $U_j|_{\mathcal{H}_j} := U_j^0$ defines a positive unitary one-parameter group with the correct commutation relations, so that $(\pi, \mathcal{H})$ is of positive energy.

If all representations $(\pi_j, \mathcal{H}_j)$ are ground state representations, then $\mathcal{H}_0^0 = \ker H_0 = \hat{\bigoplus}_{j \in J} \mathcal{H}_j^0$ satisfies $\pi(G)\mathcal{H}_0^0 = \mathcal{H}$, and therefore $(\pi, \mathcal{H})$ is a ground state representation. \hfill \square

In the following proposition we take a closer look at the special case where $\alpha$ is given by conjugation with a one-parameter subgroup $\gamma : \mathbb{R} \to G$, i.e., by inner automorphism. Then the spectrum of $\pi \circ \gamma$ determines the positive energy properties of $(\pi, \mathcal{H})$.

**Proposition 2.10.** (The inner case) Let $\gamma : \mathbb{R} \to G$ be a continuous one-parameter group and suppose that $\alpha$ is defined by

$$\alpha_t(g) = \gamma(t)g\gamma(t)^{-1} \quad \text{for} \quad t \in \mathbb{R}, g \in G.$$ 

We consider a unitary representation $(\pi, \mathcal{H})$ of $G$ and write $\pi(\gamma(t)) = e^{itH}$ for $t \in \mathbb{R}, H^* = H$. Then the following assertions hold:

(i) If $H$ is bounded from below, then there exists $c \in \mathbb{R}$ such that $U_t := e^{it(H+c\mathbf{1})}$ defines a positive energy representation $(\pi, U, \mathcal{H})$.

(ii) If $(\pi, \mathcal{H})$ is factorial, then it is of positive energy for $(G, \alpha)$ if and only if $H$ is bounded from below.

(iii) $(\pi, \mathcal{H})$ is a positive energy representation for $(G, \alpha)$ if and only if it is a direct sum of representations $(\pi_j, \mathcal{H}_j)_{j \in J}$ for which the corresponding generators $H_j$ of $\pi_j \circ \gamma$ are bounded from below.

**Proof.** (i) is clear.

(ii) If $(\pi, U, \mathcal{H})$ is a positive energy representation, we have

$$U_t^0 = \pi(\gamma(t))V_t \quad \text{with} \quad V_t \in Z(\pi(G)'') = \pi(G)'' \cap \pi(G)'.$$ 

(4)

If $\pi$ is factorial, i.e., $Z(\pi(G)'') = \mathbb{C}\mathbf{1}$, then $V_t = e^{it\mathbf{1}}$ for some $c \in \mathbb{R}$. For $U_t^0 = e^{itH}$, this implies that $\pi(\gamma(t)) = e^{it(H_0-c\mathbf{1})}$ has the generator $H_0 - c\mathbf{1}$ which is bounded from below by $-c\mathbf{1}$. The converse follows from (i).

(iii) Let $(\pi, \mathcal{H})$ be a positive energy representation and $U_t^0 = e^{itH}V_t$ be as in (4) with $V_t \in Z(\pi(G)'')$. If $V_t = e^{itZ}$, then all spectral subspaces $\mathcal{H}_n := \hat{P}(\mathbb{Z})([n, n+1])$, $n \in \mathbb{Z}$, of $Z$ are $\pi(G)$-invariant, hence lead to subrepresentations $(\pi_n, \mathcal{H}_n)_{n \in \mathbb{Z}}$. On $\mathcal{H}_n$ the restriction $Z$ of $Z$ has spectrum in $[n, n+1]$, hence is bounded. Therefore $H_n = H_0^0 - Z_n$ is bounded from below.
If, conversely, the representations \((\pi_n, \mathcal{H}_n)_{n \in \mathbb{N}}\) are such that the generators \(H_n\) are bounded from below, say \(H_n \geq c_n \mathbf{1}\), then \(U_t^\pi := e^{it(H_n - c_n \mathbf{1})}\) defines a covariant positive energy representation \((\pi_n, U^n, \mathcal{H}_n)\). Now Lemma 2.9 implies that the direct sum also is a positive energy representation. \(\square\)

**Example 2.11.** (a) If the \(\mathbb{R}\)-action on \(G\) is trivial, then every unitary representation \((\pi, \mathcal{H})\) of \(G\) is a ground state representation with \(\mathcal{H} = \mathcal{H}^0\) and \(U_t^0 = \mathbf{1}\) for \(t \in \mathbb{R}\).

(b) If \((\pi, \mathcal{H})\) is a positive energy representation, then \(\pi(Z(G))\) is pointwise fixed by \(\text{Ad}(U_t)\). In particular, for an abelian group \(G\) and a faithful positive energy representation \(\pi\), we have \(\alpha_t = \text{id}_G\) for \(t \in \mathbb{R}\) (cf. [Ne14]).

Therefore the simplest class of groups with non-trivial positive energy representations are two-step nilpotent groups \(G\), for which Heisenberg groups are the simplest examples. Then \(\pi((G, G)) \subseteq \mathfrak{T}_1\) for every irreducible representation, which leads to oscillator groups \(G^n\). We discuss this special case in Section 4. For these groups the existence of semibounded representations (cf. Definition 3.5) and the existence of non-trivial ground state representations has been studied in [NZ13] and [Zc14].

### 2.3 Strict ground state representations

We now identify a class of ground state representations for which the subgroup \(G^0\) is “large” in a suitable sense. This is the class of strict ground state representations, which are determined by the representation \((\pi^0, \mathcal{H}^0)\) of \(G^0\). In Example 4.8 we shall see a ground state representation which is not strict.

**Definition 2.12.** We call a ground state representation \((\pi, \mathcal{H})\) for \((G, \alpha)\) strict if every operator on \(\mathcal{H}^0\) commuting with \(\pi^0(G^0)\) extends to an operator on \(\mathcal{H}\) commuting with \(\pi(G)\). In view of Proposition 2.7, this is equivalent to the following identity of the von Neumann algebras in \(B(\mathcal{H}^0)\):

\[
\pi^0(G^0)'' = P_0 \pi(G)'' P_0.
\]

As \(\pi(G)\) spans a weakly dense subspace of \(\pi(G)''\), the von Neumann algebra \(P_0 \pi(G)'' P_0 \subseteq B(\mathcal{H}^0)\) is generated by \(P_0 \pi(G) P_0\) which always contains \(\pi^0(G^0)\), hence also \(\pi^0(G^0)''\).

**Problem 2.13.** Suppose that \(\alpha\) is periodic. Is every ground state representation of \((G, \alpha)\) strict?

**Remark 2.14.** If the ground state representation \((\pi, \mathcal{H})\) of \((G, \alpha)\) is irreducible, then Schur’s Lemma implies \(\pi(G)'' = B(\mathcal{H})\) which leads to \(P_0 \pi(G)'' P_0 = P_0 B(\mathcal{H}) P_0 = B(\mathcal{H}^0)\). Therefore \((\pi, \mathcal{H})\) is strict if and only if \((\pi^0, \mathcal{H}^0)\) is also irreducible, i.e., \(\pi^0(G^0)'' = B(\mathcal{H}^0)\).

**Remark 2.15.** Since von Neumann algebras are generated by their projections, a ground state representation is strict if and only if the map \(\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{H}^0\) defines a bijection between the closed \((\pi(G))\)-invariant subspaces of \(\mathcal{H}\) and the closed \(\pi^0(G^0)\)-invariant subspaces of \(\mathcal{H}^0\). As this map is injective (cf. Proposition 2.7(i)), the main point is its surjectivity.

**Example 2.16.** (An operator algebraic example of a strict ground state representation) Let \(\mathcal{M} \subseteq B(\mathcal{H})\) be a von Neumann algebra and \(G := U(\mathcal{M})\) be its unitary group. For a unitary one-parameter group \(U_t = e^{itH}\) in \(\mathcal{M}\) we obtain a continuous action on \(G\) by \(\alpha_t(g) = U_t g U_{-t}\) for \(t \in \mathbb{R}, g \in U(\mathcal{M})\). We assume that \(H \geq 0\) and \(U = U^0\) in the sense of the Borchers–Arveson Theorem (Theorem 2.1). Then the identical representation of \(G\) on \(\mathcal{H}\) is a positive energy representation.

It is a ground state representation if and only if \(\mathcal{H}^0 = \ker H\) satisfies \([U(\mathcal{M})\mathcal{H}^0] = \mathcal{H}\), which is equivalent to \([\mathcal{M}\mathcal{H}^0] = \mathcal{H}\). For the projection \(P_0\) onto \(\mathcal{H}^0\), which is contained in \(\mathcal{M}\), this is equivalent to its central support being equal to \(\mathbf{1}\) ([BGN20, Lemma 3.14]).
The group $G^0$ is the centralizer of $U_R$ in $G = U(M)$, hence contained in the subalgebra

$$P_0MP_0 \oplus (1 - P_0)M(1 - P_0)$$

and it contains the unitary group $U(P_0MP_0)$. We conclude that $\pi^0(G^0) = U(P_0MP_0)$, and the von Neumann algebra generated by this group is $P_0MP_0 = P_0G^\prime P_0$. Therefore the ground state representation of $(G, \alpha)$ on $H$ is strict.

**Proposition 2.17.** If $(\pi, H)$ is a strict ground state representation, then every subrepresentation is also a strict ground state representation.

*Proof.* Let $(\rho, F)$ be a subrepresentation of $(\pi, H)$ and $P_F$ denote the orthogonal projection onto $F$. Then $Q_0 := P_F P_0$ is the projection onto $F^0$, and Lemma 2.9 implies that $(\rho, F)$ is a ground state representation. Let $A \in B(F^0)$ commute with $\rho^0(G^0)$. Extending $A$ by 0 on the orthogonal complement of $F^0$ in $H^0$, we obtain an operator $A' \in \pi^0(G^0)'$. This operator commutes with $P_0\pi(G)P_0$ by strictness of $(\pi, H)$, and therefore $A$ commutes with $Q_0\rho(G)Q_0 = P_F P_0\pi(G)P_0P_F$. Hence $(\rho, F)$ is also strict. \[\square\]

**Definition 2.18.** We say that the pair $(G, \alpha)$ has the unique extension property if two ground state representations $(\pi_j, H_j)_{j=1,2}$ for which the $G^0$-representations $(\pi_j^0, H_j^0)$ and $(\pi_2^0, H_2^0)$ are equivalent, the representations $\pi_1$ and $\pi_2$ are unitarily equivalent, that is, the following diagram commutes:

$$
\begin{array}{ccc}
(\pi_1^0, H_1^0) & \rightarrow & (\pi_1, H_1) \\

\downarrow \phi_0 & & \downarrow \pi \\

(\pi_2^0, H_2^0) & \rightarrow & (\pi_2, H_2)
\end{array}
$$

The following lemma is a key to some of our main results below, in particular to Theorem 8.2.

**Proposition 2.19.** (Strictness and unique extension) A pair $(G, \alpha)$ has the unique extension property if and only if every ground state representation of $(G, \alpha)$ is strict. If this is the case, then, for ground state representations $(\pi_j, H_j)_{j=1,2}$ and any unitary $G^0$-equivalence $\Phi^0 : H_1^0 \rightarrow H_2^0$, there exists a unique $G$-equivalence $\Phi : H_1 \rightarrow H_2$ extending $\Phi^0$.

*Proof.* We first observe that the second statement on the uniqueness of $\Phi$ follows from the fact that we must have $\Phi(\pi_1(g)\xi) = \pi_2(g)\Phi^0(\xi)$ for $g \in G$, $\xi \in H_1^0$, and $\pi_1(G)H_1^0$ is total in $H_1$.

We now show the first statement. Suppose first that $(G, \alpha)$ has the unique extension property and that $(\pi, H)$ is a ground state representation. In view of Proposition 2.7(ii), it suffice to show that $R$ is surjective. Since the von Neumann algebra $\pi^0(G^0)'$ is generated by its unitary elements $V$, it suffices to observe that the unique extension property implies that any such $V$ extends uniquely to an element $\tilde{V} \in \pi(G)'$. Therefore every ground state representation is strict.

Now we assume that every ground state representation is strict. Let $(\pi_j, H_j)_{j=1,2}$ be two ground state representations and $V : H_1^0 \rightarrow H_2^0$ be a unitary $G^0$-equivalence. Then

$$W : H_1^0 \oplus H_2^0 \rightarrow H_2^0 \oplus H_2^0, \quad W(\xi, \eta) := (V^*\eta, V\xi)$$

is a unitary element in the commutant of $(\pi_1^0 \oplus \pi_2^0)(G^0)$. As $\rho := \pi_1 \oplus \pi_2$ is a ground state representation by Lemma 2.9, it is strict. Hence there exists an element $\tilde{W} \in \rho(G)'$ extending $W$. As the restriction map $R$ is an injective homomorphism of $*$-algebras, $\tilde{W}$ is unitary. Further, $W H_1^0 = H_2^0$ implies that $W H_1 = H_2$, so that $\tilde{V} := \tilde{W}|_{H_1} : H_1 \rightarrow H_2$ is a unitary $G$-equivalence extending $V$. \[\square\]
Remark 2.20. If \((\pi, H)\) is a ground state representation and \(P_0\) the orthogonal projection onto \(H^0\), then
\[
\varphi(g) := P_0 \pi(g) P_0 \in B(H^0)
\]
defines a positive definite \(B(H^0)\)-valued function on \(G\) with
\[
\varphi(h_1 gh_2) = \pi^0(h_1) \varphi(g) \pi^0(h_2) \quad \text{for} \quad g \in G, h_1, h_2 \in G^0
\]
and in particular \(\varphi|_{G^0} = \pi^0\). The requirement \([\pi(G)H^0] = H\) implies that the representation \((\pi, H)\) is equivalent to the GNS representation \((\pi_\varphi, H_\varphi)\) in the reproducing kernel Hilbert space \(H_\varphi \subseteq C(G, H^0)\). We refer to Proposition B.1 in Appendix B for a precise formulation of the vector-valued Gelfand–Naimark–Segal (GNS) construction. From this perspective, the unique extension property asserts that the representation \((\pi^0, H^0)\) of \(G^0\) determines the function \(\varphi\) if \(\pi_\varphi\) is a ground state representation with \(H^0_\varphi = H^0\).

3 Lie theoretic aspects

To formulate necessary conditions for a representation \((\pi^0, H^0)\) to extend to a ground state representation for \((G, \alpha)\), it is instructive to take a closer look at the context of, possibly infinite dimensional, Lie groups.

3.1 Regularity conditions for actions on Lie groups

We assume that

(L1) \(G\) is a Lie group with Lie algebra \(\mathfrak{g}\), modeled on a locally convex space (cf. [Ne06]).

(L2) \(\alpha\) is smooth, so that \(G^0\) is a Lie group. The \(\mathbb{R}\)-action on its Lie algebra \(\mathfrak{g} = L(G)\) is denoted by \(\alpha_\mathfrak{g}^\theta := L(\alpha_\theta) \in \text{Aut}(\mathfrak{g})\) and we write \(D := \frac{d}{dt}\big|_{t=0} \alpha_\mathfrak{g}^\theta \in \text{der}(\mathfrak{g})\) for the infinitesimal generator of this one-parameter group.

(L3) \(G^0\) is a Lie group with Lie algebra \(\mathfrak{g}^0 = \text{Fix}(\alpha^\theta)\).

(L4) The subspace \(\mathfrak{g}_+ := D(\mathfrak{g})\) complements \(\mathfrak{g}^0\) in the sense that \(\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}_+\) as topological vector spaces. Then \(p_0 : \mathfrak{g} \to \mathfrak{g}^0\) is an \(\alpha^\theta\)-invariant continuous projection onto the subspace of fixed points.

We define the \(\alpha\)-cone in \(\mathfrak{g}^0\) by
\[
C_\alpha := \text{conv}\{p_0([Dx, x]) : x \in \mathfrak{g}\} \subseteq \mathfrak{g}^0
\]
for the closed convex cone generated by all elements \(p_0([Dx, x])\).

Remark 3.1. (a) If \(\alpha^\theta\) is continuous and periodic, i.e., \(\alpha_\mathfrak{g}^\theta = \text{id}_\mathfrak{g}\) for some \(T > 0\), and \(\mathfrak{g}\) is complete, then
\[
p_0(x) := \frac{1}{T} \int_0^T \alpha_\mathfrak{g}^\theta(x) \, dt
\]
is a projection onto the subspace \(\mathfrak{g}^0\) of fixed points and (L4) is satisfied.

(b) If \(\mathfrak{g}\) is finite dimensional, (L4) means that the generalized 0-eigenspace of \(D\) coincides with the eigenspace. Clearly, this is the case if \(D\) is semisimple.
Remark 3.2. (The diagonalizable case) Suppose that the one-parameter group $\alpha^g$ of Lie algebra automorphisms can be written as $\alpha^g_t = e^{t \mathrm{ad} d}$ for some $d \in g$ for which $\mathrm{ad} d$ is diagonalizable on $g_C$ with purely imaginary eigenvalues:

$$g_C = \bigoplus_{\lambda \in \mathbb{R}} g_C^\lambda(-i d) \quad \text{with} \quad g_C^\lambda(-i d) = \{ z \in g_C : [d, z] = i \lambda z \}.$$ 

Defining $(x + iy)^* := -x + iy$ for $x, y \in g$, we have $g = \{ z \in g_C : z^* = -z \}$ and

$$[d, g] = [d, g] \cap g = \{ z = \sum_{\lambda \neq 0} z_\lambda : z^* = -z \}.$$ 

For $x = \sum x_\lambda \in g$ we then have $-x = x^* = \sum x_\lambda^*$ with $x_\lambda^* = -x_\lambda$. Write $p_0 : g_C \to g_C^0$ for the fixed point projection introduced by (L4). It is given by the 0-component $x_0 := p_0(x)$.

$$p_0([d, x], x]) = \sum_{\lambda, \mu} i p_0(\lambda[x_\lambda, x_\mu]) = \sum_{\lambda} i \lambda[x_\lambda, x_{-\lambda}] = \sum_{\lambda} -i \lambda[x_\lambda, x_\lambda]^* = 2i \sum_{\lambda > 0} \lambda[x_\lambda^*, x_\lambda], \quad (7)$$

where we have used that

$$(-\lambda)[x_{-\lambda}^*, x_{-\lambda}] = (-\lambda)[-x_\lambda, -x_\lambda] = \lambda[x_\lambda^*, x_\lambda].$$

We conclude that

$$C_\alpha = \overline{\text{conv}} \{ i[x_\lambda^*, x_\lambda] : \lambda > 0, x_\lambda \in g_C^\lambda(-i d) \} \quad (8)$$

(see (6)).

If $\alpha^g$ is continuous and periodic and $g$ is complete, then the sum of the eigenspaces is a dense subalgebra (Remark 3.1).

Example 3.3. (Twisted loop groups) Important examples where $\alpha$ is periodic arise as follows. Let $K$ be a Lie group with a complete Lie algebra $k$, and $\Phi \in \text{Aut}(K)$ be of finite order $\Phi^n = \text{id}_K$. We consider the twisted loop group

$$\mathcal{L}_\Phi(K) := \{ \xi \in C^\infty(\mathbb{R}, K) : (\forall x \in \mathbb{R}) \xi(t + 1) = \Phi^{-1}(\xi(t)) \}.$$ 

This is a Lie group with algebra

$$\mathcal{L}_\Phi(k) := \{ \xi \in C^\infty(\mathbb{R}, k) : (\forall x \in \mathbb{R}) \xi(t + 1) = \varphi^{-1}(\xi(t)) \}, \quad \text{where} \quad \varphi = L(\Phi) \in \text{Aut}(k)$$

is the induced automorphism of the Lie algebra $k$ of $K$. Then

$$(\alpha_t \xi)(x) := \xi(x + t) \quad (9)$$

defines a smooth action of $\mathbb{R}$ on $G := \mathcal{L}_\Phi(K)$ with $\alpha_N = \text{id}_G$. Therefore (L4) follows from Remark 3.1(a). The infinitesimal generator of the automorphism group $\alpha^g = L(\alpha_t)$ (acting also by (9)) is given by $D\xi = \xi'$. The subgroup of $\alpha$-fixed points is the subgroup

$$G^0 \cong K^\Phi$$

of constant elements with values in the subgroup $K^\Phi$ of $\Phi$-fixed points in $K$. If

$$k_C^\Phi = \{ x \in k_C : \varphi^{-1}(x) = e^{\frac{2\pi i n}{N}} x \} \quad \text{for} \quad n \in \mathbb{Z},$$

then

$$G = \mathcal{L}_\Phi(K)^0,$$
denotes the $\varphi$-eigenspaces in $\mathfrak{k}$, then $\mathfrak{k}^\alpha = \mathfrak{k}^\alpha_{-\varphi}$. Now
\[
\hat{\mathcal{L}}(\mathfrak{k}) = \mathfrak{k}^\alpha \otimes \mathfrak{e}_n, \quad \text{where} \quad \mathfrak{e}_n(t) = e^{\frac{2\pi it}{N}},
\]
are the $D$-eigenspaces in $\mathcal{L}(\mathfrak{k}) \cong \hat{\mathcal{L}}(\mathfrak{t})$ corresponding to the eigenvalue $\frac{2\pi in}{N}$. The expansion as Fourier series $x = \sum_{n \in \mathbb{Z}} x_n$ with $x_n \in \mathcal{L}(\mathfrak{k})$ converges in $\mathcal{L}(\mathfrak{k})$ by Harish–Chandra’s Theorem ([Wa72, Thm. 4.4.21]) and
\[
\text{im}(D) = \left\{ x = \sum_{n \in \mathbb{Z}} x_n : x_n \in \mathcal{L}(\mathfrak{k}) : x_0 = 0 \right\}.
\]
From Remark 3.2 we know that the cone $C_\alpha \subseteq \mathfrak{t}^\alpha$ is generated by the brackets
\[
[(y_n \otimes e_n)^*, y_n \otimes e_n] = [y_n^* \otimes e_{-n}, y_n \otimes e_n] = [y_n^*, y_n] \otimes 1, \quad y_n \in \mathfrak{k}^\alpha, n > 0.
\]
Therefore
\[
C_\alpha = \text{cone}\{i[y_n^*, y_n] : n > 0, y_n \in \mathfrak{k}^\alpha \}.
\]
From $\mathfrak{k}^\alpha_{-\varphi} = \mathfrak{k}^\alpha$ it follows that, for $0 < n \leq N$, $z_{2N-n} := y_n^* \in \mathfrak{k}^\alpha_{-\varphi}$ with $2N - n > 0$, and
\[
[y_n^*, y_n] = [z_{2N-n}, z_{2N-n}^*] = -[z_{2N-n}, z_{2N-n}^*].
\]
Hence the cone $C_\alpha$ is a linear space which coincides with $\mathfrak{g}^0 \cap [\mathfrak{g}, \mathfrak{g}]$, which is an ideal in $\mathfrak{g}^0$.

To create a situation with a non-trivial cone $C_\alpha$, which by Theorem 3.11 below is necessary for the existence of ground state representations with trivial kernel, one has to pass to a central extension of the loop algebras:
\[
\hat{\mathcal{L}}(\mathfrak{k}) := \mathbb{R} \oplus_\sigma \mathcal{L}(\mathfrak{k}) \quad \text{where} \quad \sigma(\xi, \eta) := \frac{1}{2\pi} \int_0^1 \kappa(\xi(t), \eta'(t)) \, dt,
\]
and the bracket is given by
\[
[(t, \xi), (s, \eta)] = (\sigma(\xi, \eta), [\xi, \eta]).
\]
Here $\kappa$ is a positive definite invariant bilinear form on $\mathfrak{k}$ which is also $\varphi$-invariant. Then the elements $i[y_n^*, y_n]$ generating $C_\alpha$ have a non-trivial central component:
\[
i\sigma(y_n^* \otimes e_{-n}, y_n \otimes e_n) = i\kappa(y_n^*, y_n) \frac{1}{2\pi} \int_0^1 e^{-n(t)}e_n'(t) \, dt = i\kappa(y_n^*, y_n) \frac{1}{2\pi} \frac{2\pi in}{N} = -\kappa(y_n^*, y_n) \frac{n}{N}.
\]
For $a, b \in \mathfrak{k}$, the complex bilinear extension of $\kappa$ satisfies
\[
-\kappa((a + ib)^*, a + ib) = -\kappa(-a + ib, a + ib) = \kappa(a, a) + \kappa(b, b).
\]
Therefore all elements in $C_\alpha$ have a non-negative central component and $C_\alpha$ is non-trivial.

If $\mathfrak{k}$ is abelian, this construction simply leads to a Heisenberg algebra, a class of examples discussed in Section 4 below; see in particular [SeG81] for the case where $\alpha$ is periodic. For more details on (twisted) loop groups with values in compact Lie groups, we refer to [PS86], [Ne14b], [MN16, MN17], [JN21].
3.2 Momentum sets and positive cones

Definition 3.4. Let $G$ be a Lie group and $(\pi, \mathcal{H})$ be a unitary representation of $G$. An element $\xi \in \mathcal{H}$ is called a smooth vector if its orbit map

$$\pi^\xi : G \to \mathcal{H}, \quad g \mapsto \pi(g)\xi$$

is smooth. The smooth vectors form a $\pi(G)$-invariant subspace $\mathcal{H}^\infty \subseteq \mathcal{H}$, and the representation $(\pi, \mathcal{H})$ is said to be smooth if $\mathcal{H}^\infty$ is dense in $\mathcal{H}$. This is always the case if $G$ is finite dimensional, but not in general ([BN08]).

On $\mathcal{H}^\infty$ the derived representation $d\pi$ of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ is defined by

$$d\pi(x)v := \frac{d}{dt}\big|_{t=0} \pi(\exp tx)v.$$

For a smooth representation the invariance of $\mathcal{H}^\infty$ under $\pi(G)$ implies that, for $x \in \mathfrak{g}$, the operator $i \cdot d\pi(x)$ on $\mathcal{H}^\infty$ is essentially selfadjoint (cf. [RS75, Thm. VIII.10]) and that its closure, coincides with the selfadjoint generator $i\partial\pi(x)$ of the unitary one-parameter group $\pi_x(t) := \pi(\exp tx)$, i.e., $\pi_x(t) = e^{t\partial\pi(x)}$ for $t \in \mathbb{R}$.

Definition 3.5. (a) Let $\mathbb{P}(\mathcal{H}^\infty) = \{[v] := \mathbb{C}v : 0 \neq v \in \mathcal{H}^\infty\}$ denote the projective space of the subspace $\mathcal{H}^\infty$ of smooth vectors. The map

$$\Phi_\pi : \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}' \quad \text{with} \quad \Phi_\pi([v])(x) = \frac{\langle v, -i \cdot d\pi(x)v \rangle}{\langle v, v \rangle}$$

is called the momentum map of the unitary representation $\pi$. The operator $i \cdot d\pi(x)$ is symmetric so that the right hand side is real, and since $v$ is a smooth vector, it defines a continuous linear functional on $\mathfrak{g}$. We also observe that we have a natural action of $G$ on $\mathbb{P}(\mathcal{H}^\infty)$ by $g.[v] := [\pi(g)v]$, and the relation

$$\pi(g)d\pi(x)\pi(g)^{-1} = d\pi(\text{Ad}(g)x)$$

immediately implies that $\Phi_\pi$ is equivariant with respect to the coadjoint action of $G$ on the topological dual space $\mathfrak{g}'$.

(b) The weak-$*$-closed convex hull $I_\pi \subseteq \mathfrak{g}'$ of the image of $\Phi_\pi$ is called the (convex) momentum set of $\pi$. In view of the equivariance of $\Phi_\pi$, it is an $\text{Ad}^*(G)$-invariant weak-$*$-closed convex subset of $\mathfrak{g}'$.

(c) The momentum set $I_\pi$ provides complete information on the extreme spectral values of the selfadjoint operators $i \cdot \partial\pi(x)$:

$$\sup(\text{Spec}(i\partial\pi(x))) = s_\pi(x) := \sup\{I_\pi, -x\} \quad \text{for} \quad x \in \mathfrak{g} \quad \text{(10)}$$

(cf. [Ne08, Lemma 5.6]). This relation shows that $s_\pi$ is the support functional of the convex subset $I_\pi \subseteq \mathfrak{g}'$, which implies that it is lower semicontinuous and convex. It is obviously positively homogeneous. The semibounded unitary representations are those for which the set $I_\pi$ is semi-equicontinuous in the sense that its support function $s_\pi$ is bounded in a neighborhood of some $x_0 \in \mathfrak{g}$.

The closed convex cone

$$C_\pi := \{x \in \mathfrak{g} : -i\partial\pi(x) \geq 0\} \quad \text{(11)}$$

$$I_\pi^* := \{x \in \mathfrak{g} : (\forall\alpha \in I_\pi) \alpha(x) \geq 0\} \quad \text{(10)}$$

is called the positive cone of $\pi$. 

15
Definition 3.6. We call a ground state representation \((\pi, \mathcal{H})\) of the Lie group \(G\) smooth if the subspace
\[
\mathcal{H}^{0,\infty} := \mathcal{H}^0 \cap \mathcal{H}^\infty
\]
is dense in \(\mathcal{H}^0\). This implies in particular that the representation \((\pi^0, \mathcal{H}^0)\) of \(G^0\) is also smooth.

Lemma 3.7. If \((L1/2)\) hold and \((\pi, \mathcal{H})\) is a smooth ground state representation of \(G\), then the extended representation \((\pi^\alpha, \mathcal{H})\) of \(G^\alpha\) is smooth.

Proof. The assumptions imply that \(\mathcal{H}^{0,\infty}\) is contained in the space \(\mathcal{H}^\infty(G^\alpha)\) of smooth vectors for \(G^\alpha\). As \(\pi(G)\mathcal{H}^{0,\infty}\) is total in \(\mathcal{H}\) by the ground state condition and \(\mathcal{H}^\infty(G^\alpha)\) is \(\pi(G)\)-invariant, this subspace is dense.

Definition 3.8. Let \(E\) be a locally convex space and let \(\alpha : \mathbb{R} \to \text{GL}(E), \ t \mapsto \alpha_t\) be a group homomorphism. Then \(\alpha\) is called

(a) equicontinuous, if the subset \(\{\alpha_t : t \in \mathbb{R}\} \subset \text{End}(E)\) is equicontinuous (cf. Definition A.1). (b) polynomially bounded, if for every continuous seminorm \(p\) on \(E\), there exists a 0-neighborhood \(U \subseteq E\) and \(N \in \mathbb{N}\) such that
\[
\sup_{x \in U} \sup_{t \in \mathbb{R}} \frac{p(\alpha_t(x))}{1 + |t|^N} < \infty.
\]

Remark 3.9. If \(E\) is finite dimensional, then \(\alpha\) is polynomially bounded if and only if the spectrum of its infinitesimal generator \(A\) is purely imaginary. However, for an infinite dimensional Hilbert space \(\mathcal{H}\), there exists a one-parameter group \(\alpha : \mathbb{R} \to \text{GL}(\mathcal{H})\) with \(\|\alpha_t\| = e^{\|t\|}\) whose generator has purely imaginary spectrum, cf. [vN96, Example 1.2.4].

From [NSZ15, Prop. 3.2] we quote the following sufficient condition for the density of \(\mathcal{H}^{0,\infty}\) in \(\mathcal{H}^0\).

Proposition 3.10. Suppose that

- the one-parameter group \((\alpha_t^\mathfrak{g})_{t \in \mathbb{R}}\) of Lie algebra automorphisms is polynomially bounded,
- \((\pi, \mathcal{H})\) is a smooth positive energy representation of \((G, \alpha)\), and
- there exists an \(\varepsilon > 0\) such that \(\text{Spec}(-i\partial \pi(\mathfrak{d})) \cap [0, \varepsilon] = \{0\}\) (spectral gap condition),

then \(\mathcal{H}^{0,\infty}\) is dense in \(\mathcal{H}^0\).

Let \(\mathfrak{d} := (0, 1) \in \mathfrak{g}^\mathbb{R} = \mathfrak{g} \rtimes_{\text{ad}} \mathbb{R}\) be the element implementing \(D\), so that \(Dx = [\mathfrak{d}, x]\) for \(x \in \mathfrak{g}\). In the setting specified above, we formulate in the following theorem a necessary positivity condition that a representation \((\pi^0, \mathcal{H}^0)\) arising in a ground state representation of \((G, \alpha)\) has to satisfy.

Theorem 3.11. (\(C_\alpha\)-positivity Theorem) Suppose that \((L1-4)\) are satisfied. If \((\pi, \mathcal{H})\) is a smooth ground state representation of the Lie group \(G\), then the cone \(C_\alpha\) introduced in (6) satisfies
\[
C_\alpha \subseteq C_{\pi^0} = \{x \in \mathfrak{g}^0 : -i\partial \pi^0(x) \geq 0\}. \tag{12}
\]

Proof. Let \(\xi \in \mathcal{H}^0 \cap \mathcal{H}^\infty, \ t \in \mathbb{R}\) and \(x \in \mathfrak{g}\). Then \(\pi(\exp tx)\xi\) is a smooth vector for \(G^\alpha\), hence contained in the domain of the infinitesimal generator \(H\) of \(U\), and we have
\[
f(t) := \langle \pi(\exp tx)\xi, H\pi(\exp tx)\xi \rangle \geq 0 \quad \text{for} \quad t \in \mathbb{R}
\]
because $H \geq 0$. As $f(0) = \langle \xi, H\xi \rangle = 0$, we also have
\[ f'(0) = \langle \partial\pi(x)\xi, H\xi \rangle + \langle H\xi, \partial\pi(x)\xi \rangle = 0 \]
and $f''(0) \geq 0$, and this is what we shall exploit. To this end, we rewrite $f$ as
\[ f(t) = \langle \xi, \pi(\exp-tx)H\pi(\exp tx)\xi \rangle = \langle \xi, \pi(\exp-tx)(-i\partial\pi^a(d))\pi(\exp tx)\xi \rangle \]
\[ = -i\langle \xi, \partial\pi^a(e^{-t\text{ad}_x}d)\xi \rangle. \]
This immediately leads to
\[ 0 \leq f''(0) = -i\langle \xi, \partial\pi([x, [x, d]])\xi \rangle = -i\langle \xi, \partial\pi([Dx, x])\xi \rangle. \quad (13) \]
Next we observe that, for $y \in \mathfrak{g}$, we have
\[ \langle \xi, \partial\pi(Dy)\xi \rangle = \frac{d}{dt} \bigg|_{t=0} \langle \xi, \partial\pi(e^{tD}y)\xi \rangle = \frac{d}{dt} \bigg|_{t=0} \langle \xi, U_it\partial\pi(y)U_{-t}\xi \rangle \]
\[ = \frac{d}{dt} \bigg|_{t=0} \langle U_{-t}\xi, \partial\pi(y)U_{-t}\xi \rangle = \frac{d}{dt} \bigg|_{t=0} \langle \xi, \partial\pi(y)\xi \rangle = 0 \]
because $U_it = \xi$ for all $t \in \mathbb{R}$. Therefore $\langle \xi, \partial\pi(z)\xi \rangle = 0$ for $z \in \mathfrak{g}_+$, and thus (L4) entails
\[ \langle \xi, \partial\pi([Dx,x])\xi \rangle = \langle \xi, \partial\pi(p_0([Dx,x]))\xi \rangle = \langle \xi, \partial\pi^0(p_0([Dx,x]))\xi \rangle. \]
Since $\mathcal{H}^{0,\infty}$ is dense in $\mathcal{H}^0$ and invariant under $\pi^0(G^0)$, it follows that the operators $i \cdot \partial\pi^0(x_0)|_{\mathcal{H}^{0,\infty}}$, $x_0 \in \mathfrak{g}^0$, are essentially self-adjoint with closure equal to $i \cdot \partial\pi^0(x_0)$. We therefore obtain
\[ -i\partial\pi^0(p_0([Dx,x])) \geq 0 \quad \text{for every} \quad x \in \mathfrak{g} \]
and thus $C_\alpha \subseteq C_{\pi^a}$. \hfill \Box

**Example 3.12.** In the context of Remark 3.2, for the representation of $G^0$ on $\mathcal{H}^0$, the condition $C_\alpha \subseteq C_{\pi_0}$ in Theorem 3.11 is equivalent to
\[ \partial\pi^0([x^\lambda, x_\lambda]) \geq 0 \quad \text{for} \quad \lambda > 0, \quad \text{and} \quad [d, x_\lambda] = i\lambda x_\lambda, x_\lambda \in \mathfrak{g}_\mathbb{C}. \quad (14) \]

In the following we shall encounter various circumstances, where (12) is also sufficient for a representation $(\pi^0, \mathcal{H}^0)$ of $G^0$ to extend to a ground state representation of $G$. In some cases we can derive a positivity property similar to (12) from the positive energy condition. The idea for the following proposition is taken from [JN21].

**Proposition 3.13.** Let $(\pi, U, \mathcal{H})$ be a positive energy representation of $(G, \alpha)$ for which the extension to $G^0$ is smooth. If $x \in \mathfrak{g}$ satisfies $(\text{ad } x)^2Dx = 0$, then
\[ -i\partial\pi([Dx,x]) \geq 0. \quad (15) \]

**Proof.** Let $\pi^a(g,t) := \pi(g)e^{itH}$ be the extension of $\pi$ to $G^0$. We proceed as in the proof of Theorem 3.11 for any $\xi \in \mathcal{H}^{\infty}$. Our assumption implies that the smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ is non-negative. Now $[[Dx,x],x] = 0$ leads to
\[ f(t) = -i\langle \xi, \partial\pi(e^{-t\text{ad}_x}d)\xi \rangle = -i\langle \xi, Hz\xi \rangle - it\langle \xi, \partial\pi^a(Dx)\xi \rangle - \frac{it^2}{2}\langle \xi, \partial\pi^a([Dx,x])\xi \rangle \geq 0 \]
for $t \in \mathbb{R}$. This implies that $-\frac{t^2}{2}\langle \xi, \partial\pi([Dx,x])\xi \rangle \geq 0$, so that (15) follows. \hfill \Box
In this section we discuss ground state representations of Heisenberg groups. Here an old result by M. Weinless [We69] can be used to obtain crucial information on the structure of ground state representations.

We consider a locally convex space $V$, endowed with a continuous alternating form $\sigma: V \times V \to \mathbb{R}$. We further assume that $\sigma$ is weakly symplectic, i.e., non-degenerate. The corresponding Heisenberg group is the central extension

$$G = \text{Heis}(V, \sigma) := \mathbb{T} \times V \quad \text{with} \quad (z, v)(z', v') = (zz'e^{i\sigma(v,v')}, v + v').$$

(16)

It is a Lie group with Lie algebra $\mathfrak{g} = \mathbb{R} \oplus V$, on which the bracket is

$$[[z, v], [z', v']] = (\sigma(v, v'), 0).$$

Any smooth action $\alpha: \mathbb{R} \to \text{Aut}(G)$ fixing all elements in the central circle $\mathbb{T} \times \{0\}$ corresponds to a one-parameter group $\beta: \mathbb{R} \to \text{Sp}(V, \sigma)$ whose infinitesimal generator is denoted $D_V \in \mathfrak{sp}(V, \sigma)$.

Then the infinitesimal generator $D$ of $\alpha$ has the form

$$D(z, v) = (0, D_V v) \quad \text{for} \quad z \in \mathbb{R}, v \in V.$$

(17)

This implies in particular that

$$G^0 = \mathbb{T} \times \ker(D_V) = \mathbb{T} \times V^\beta,$$

which also is a Heisenberg group. The group

$$G^\beta := G \rtimes_{\alpha} \mathbb{R}$$

is called the associated oscillator group (cf. [NZ13]).

We define the effective subspace of $V$ as

$$V_{\text{eff}} := \overline{V_\beta}, \quad V_\beta := \text{span}\{\beta_t(v) - v: t \in \mathbb{R}, v \in V\}.$$

From the invariance of $\sigma$ under $\beta$, we immediately obtain

$$V^\beta = \{v \in V: (\forall t \in \mathbb{R}) \beta_t(x) = x\} = V_{\beta}^\perp = V_{\text{eff}}^\perp.$$

(18)

Then $G_{\text{eff}} := \mathbb{T} \times V_{\text{eff}}$ is a closed $\alpha$-invariant subgroup of $G$ and

$$G_{\text{eff}}^\beta = G_{\text{eff}} \rtimes_{\alpha} \mathbb{R} \leq G^\beta = G \rtimes_{\alpha} \mathbb{R}$$

is a normal subgroup for which the quotient $G^\beta/G_{\text{eff}}^\beta$ is abelian, and $G^0$ commutes with $G_{\text{eff}}^\beta$.

We consider unitary representations $(\pi, \mathcal{H})$ of $G$ with $\pi(z, 0) = z1$ for $z \in \mathbb{T}$. These representations can also be viewed as projective representations of the abelian group $(V, +)$ (see [JN19] for generalities on projective Lie group representations).

For any element $x = (z, v) \in \mathfrak{g}$, we have

$$[Dx, x] = [(0, D_V v), (z, v)] = (\sigma(D_V v, v), 0) \in \mathfrak{g}(\mathfrak{g}).$$

Hence Proposition 3.13 shows that the positive energy condition for $\pi$ implies the following positivity condition on $D_V \in \mathfrak{sp}(V, \sigma)$:

$$\sigma(D_V v, v) = -i\partial \pi((\sigma(D_V v, v), 0)) \geq 0 \quad \text{for} \quad v \in V$$

(19)

(cf. [BGN20, Ex. 4.26, Prop. 4.27]).
Definition 4.1. (Weinless conditions) The triple \((V, \sigma, \beta)\) defines a \textit{boson single particle space} in the sense of [We69]. Weinless defines a \textit{positive energy Bose–Einstein field} over \((V, \sigma, \beta)\) as a quadruple \((H, W, \Omega, U)\), consisting of a complex Hilbert space \(H\), a unit vector \(\Omega \in H\), a continuous unitary one-parameter group \((U_t)_{t \in \mathbb{R}}\) on \(H\), and a map \(W : V \to U(H)\), with the following properties:

(W1) \(W(x)W(y) = e^{i\frac{\sigma}{2}x \cdot y}W(x + y)\) for \(x, y \in V\) (Weyl relations).

(W2) \(W(\beta_t(x)) = U_tW(x)U_{-t}\) for \(x \in V, t \in \mathbb{R}\) (\(\beta\)-equivariance).

(W3) \(U_t\Omega = \Omega\) for \(t \in \mathbb{R}\).

(W4) \(U_t = e^{itH}\) with \(H \geq 0\) (\(U\) has positive spectrum).

(W5) The unitary one-parameter groups \(W^x_t := W(tx), x \in V\), are strongly continuous (regularity).

(W6) \(W(V)\Omega\) is total in \(H\).

These requirements translate naturally into our context. Relation (W1) means that \(\pi(z, v) := zW(v)\) defines a unitary representation of \(\text{Heis}(V, \sigma)\) and (W2,3,4,6) imply that \((\pi, U, H)\) defines a ground state representation with cyclic vector \(\Omega\). Condition (W5) is a rather weak continuity requirement which is in particular satisfied if \(W\) is strongly continuous. Therefore every continuous positive energy representation \((\pi, U, H)\) with a cyclic ground state vector \(\Omega\) defines by \(W(x) := \pi(1, x)\) a positive energy Bose–Einstein field over \((V, \sigma, \beta)\).

The following uniqueness result is a variant of the Stone–von Neumann Uniqueness Theorem, which fails in general for infinite dimensional spaces \(V\), but the existence of \(\beta\) implements additional structure that can be used to obtain a similar uniqueness for ground state representations if \(V_\beta\) is large ([BR02],[Ka79,Ka85]).

Theorem 4.2. (Weinless’ Uniqueness Theorem; [We69, Thm. 4.1]) Let \((H, W, \Omega, U)\) be a positive energy Bose–Einstein field over \((V, \sigma, \beta)\), i.e., (W1-6) are satisfied, and \(V_\beta = \text{span}\{\beta_t(x) - x : x \in V, t \in \mathbb{R}\}\).

Then there exists an, up to unitary equivalence unique, complex Hilbert space \(H\) and a symplectic linear map \(j : (V_\beta, \sigma) \to (H, 2\text{Im}\langle\cdot, \cdot\rangle)\) such that

(j1) \(j(V_\beta)\) is dense in \(H\), and

(j2) there exists a unitary one-parameter group \(U^H_t = e^{itB}\) on \(H\) with \(j \circ \beta_t = U^H_t \circ j\) for \(t \in \mathbb{R}\) and \(B > 0\).

We further have

(j3) \(\langle \Omega, W(x)\Omega \rangle = e^{-\frac{1}{4}\|j(x)\|^2}\) for \(x \in V_\beta\).

(j4) If \(\xi \in H^U_t\) is \(U\)-fixed and orthogonal to \(\Omega\), then \(W(V_\beta)\Omega \perp \xi\).

The Weinless Theorem has interesting consequences for the structure of strongly continuous ground state representations.

Proposition 4.3. Let \((\pi, U, H)\) be a continuous ground state representation of \((\text{Heis}(V, \sigma), \alpha)\) with \(\pi(z, 0) = z1\) for \(z \in \mathbb{T}\) and cyclic ground state vector. Then the following assertions hold:
(a) The linear map \( j : V_\beta \to \mathcal{H} \) extends to a continuous linear map \( j : V_{\text{eff}} = \overline{V_\beta} \to \mathcal{H} \).

(b) On the subspace \( \mathcal{K}_0 := \overline{W(V_{\text{eff}})\Omega} \) we have an irreducible representation \( \rho_0 \) of \( G_{\text{eff}}^0 \) with a smooth ground state vector \( \Omega \) and \( (\mathcal{K}_0)^0 = \mathbb{C}\Omega \).

(c) The representation \( \rho_1 \) of \( G_{\text{eff}}^0 \) on the \( G_{\text{eff}}^0 \)-invariant subspace \( \mathcal{K}_1 \subseteq \mathcal{H} \) generated by \( \mathcal{H}^0 \) is equivalent to the representation

\[
\mathcal{K}_1 \cong \mathcal{H}^0 \hat{\otimes} \mathcal{K}_0, \quad \rho_1(g) = 1 \otimes \rho_0(g) \quad \text{for} \quad g \in G_{\text{eff}}^0. \tag{20}
\]

In particular, the commutant of \( \rho_1(G_{\text{eff}}^0) \) is \( \rho_1(G_{\text{eff}}^0)' \cong B(\mathcal{H}^0) \otimes 1 \).

(d) The subspace \( \mathcal{K}_1 \) is also invariant under \( G^0 \), which acts on it by \( \rho_1(g) = \pi^0_0(g) \otimes 1 \) for \( g \in G^0 \).

**Proof.** (a) From (j3) we derive that \( j \) is bounded on a 0-neighborhood in \( V_\beta \), hence continuous and therefore extends to a continuous linear map on \( V_{\text{eff}} \).

(b) Clearly, \( \mathcal{K}_0 \) is invariant under \( G_{\text{eff}}^0 \) with cyclic vector \( \Omega \). Since the corresponding positive definite function

\[
\varphi(z, x, t) = \langle \Omega, zW(x)U_0\Omega \rangle = z e^{-\frac{i}{2}||j(x)||^2}, \quad z \in \mathbb{T}, x \in V_{\text{eff}}, t \in \mathbb{R},
\]

is smooth, \( \Omega \) is a smooth vector ([Ne10, Thm. 7.2]).

Further, (j4) implies that \( (\mathcal{K}_0)^0 = \mathcal{K} \cap \mathcal{H}^0 = \mathbb{C}\Omega \) is one-dimensional. As \( \Omega \) is cyclic, the representation of the commutant \( \rho_0(G_{\text{eff}}^0)' \) on \( (\mathcal{K}_0)^0 = \mathbb{C}\Omega \) is faithful, hence the commutant is one-dimensional and thus \( (\rho_0, \mathcal{K}_0) \) is irreducible.

(c) Let \( \xi \in \mathcal{H}^0 \) be a unit vector, and consider the representation of \( G^0 \) on the subspace \( \mathcal{H}_\xi \) generated by \( W(V)\xi \). Then Weinless’ Theorem applies to the representation of \( G^0 \) on \( \mathcal{H}_\xi \), and it follows that, for \( x \in V_{\text{eff}} \), we also have \( \langle \xi, W(x)\xi \rangle = e^{-\frac{i}{2}||j(x)||^2} \). If \( (\xi_j)_{j \in J} \) in an orthonormal bases of \( \mathcal{H}^0 \), it follows that the subspaces \( \mathcal{K}_{\xi_j} \) generated by \( W(V_{\text{eff}})\xi_j \) are mutually orthogonal, and the GNS Theorem implies that the representation on this cyclic subspace is equivalent to \( (\rho_0, \mathcal{K}_0) \). This proves (c).

(b) The subgroup \( G^0 \) commutes with \( G_{\text{eff}}^0 \) by (18) and acts on the subspace \( \mathcal{H}^0 \) of ground state vectors by the representation \( \pi^0 \). Hence it also preserves the subspace \( \mathcal{K}_1 \). By (c), it acts on the left tensor factor, which is a multiplicity space for the action of \( G_{\text{eff}}^0 \).

**Theorem 4.4.** (Factorization Theorem for ground state representations) Suppose that \((V, \sigma, \beta)\) satisfies the weak splitting condition \(^2\)

\[
V = V^0 + V_{\text{eff}}. \tag{WSC}
\]

Then every ground state representation \((\pi, U, \mathcal{H})\) of \( G^0 \) factorizes as a tensor product \( \mathcal{H} = \mathcal{H}^0 \hat{\otimes} \mathcal{K}_0 \) with

\[
\pi(g) = \pi^0(g) \otimes 1 \quad \text{for} \quad g \in G^0 \quad \text{and} \quad \pi(g) = 1 \otimes \rho_0(g) \quad \text{for} \quad g \in G_{\text{eff}}^0,
\]

where the representation \((\rho_0, \mathcal{K}_0)\) of \( G_{\text{eff}}^0 \) is irreducible and equivalent to the pullback of the canonical Fock representation of \( \text{Heis}(V_{\text{eff}}, \sigma) \) on \( \mathcal{F}_+(\mathcal{H}) \), defined by the positive definite function

\[
\varphi(z, x) = ze^{-\frac{i}{2}||j(x)||^2}
\]

(see Appendix C and Remark 4.6 below). We further have \( \pi^0(G^0)' = \pi(G)' \cong \pi^0(G^0)' \). In particular, \((\pi, U, \mathcal{H})\) is strict.

\(^2\)Condition (WSC) is a weaker version of the splitting condition (SC) that plays a crucial role in the construction of ground state representations by holomorphic induction in Section 5.
Proof. Our assumption implies that $W(V^β + V_β)Ω ⊆ CW(V_β)W(V^β)Ω ⊆ W(V_β)H^0$ is total in $H$. Therefore $H = K_1$, and the assertion follows from Proposition 4.3. □

Remark 4.5. (a) If $β$ is periodic of period $T > 0$ and $V$ is complete, then the fixed point projection

$$p_0 : V → V^β, \quad p_0(v) = \frac{1}{T} \int_0^T β_t(v) \, dt$$

satisfies $p_0(V) = V^β$ and $\ker(p_0) = V_{eff}$ by the Peter–Weyl Theorem ([HM06]). We therefore have $V = V^β ⊕ V_{eff}$ in this case.

(b) Suppose that $V^β + V_{eff}$ is dense in $V$, i.e., that the weak splitting condition (WSC) is satisfied. As $V^β$ and $V_{eff}$ are $σ$-orthogonal, it follows that $(V^β, σ)$ is also symplectic.

For every $t \neq 0$, the range of the map $V → V_{eff}, v → β_t(v) − v$ generates the same closed subspace as its restriction to $V_{eff}$. We therefore have $(V_{eff})_{eff} = V_{eff}$.

The direct sum $V' := V^β ⊕ V_{eff}$ carries a natural symplectic form and the addition map $i : V → V$ is symplectic and $R$-equivariant with dense range. Therefore the adjoint

$$j : V → V' ≃ (V^β)' ⊕ V_{eff}', \quad j(v) = (σ(v, ·), σ(·, v))$$

is injective.

Remark 4.6. (a) If $V = H^R$ (the real subspace underlying a complex Hilbert space $H$) and $σ(v, w) = 2 \text{Im}(v, w)$ and $β_t = e^{itD}$ with $D ≥ 0$, then second quantization leads to a positive energy representation $(π, U, H)$ on the symmetric Fock space

$$H := \overset{∞}{\bigoplus}_{n ∈ N_0} S^n(H)$$

with cyclic unit vector $Ω ∈ S^0(H)$ for which we have

$$⟨Ω, π(x)Ω⟩ = e^{-\frac{1}{2}∥x∥^2} \quad \text{for} \quad x ∈ V = H$$

(cf. Appendix C). We conclude that the irreducible representation $(ρ_0, K_0)$ in Theorem 4.4 is equivalent to the canonical representation of $G_{eff}$ on the Fock space $F_+(H)$. Note that

$$H^0 = F_+(\ker D)$$

is one-dimensional if and only if $D > 0$.

(b) Suppose that $V = V_0 ⊕ H^R$ is a direct sum of two symplectic spaces, where $H$ is a complex Hilbert space and $β_t(v_0 + v_1) = v_0 + e^{itD}v_1$ for $t ∈ R$, $v_0 ∈ V_0, v_1 ∈ H$ and $D = D^* ≥ 0$ on $H$. For any unitary representation $(ρ, K)$ of Heis($V_0, σ_0$) with $ρ(z, 0) = z1$ for $z ∈ T$, we obtain on

$$H := K ⊗ F_+(H) \quad \text{by} \quad π(z, v_0 + v_1) := π(z, v_0) ⊗ πF_+(v_1)$$

a ground state representation of Heis($V, σ$), where $H^0 = K ⊗ F_+(\ker D)$. Therefore one cannot expect to draw any finer conclusion on the representation of $G_{eff}$ on $H^0$.

(c) As the representation of $G_{eff}$ on $K_0$ is smooth, Proposition 3.13 implies that, for $x ∈ V_{eff}$, we have

$$σ(D_V(x), x) = ⟨Ω, -i∂x(∥Dx, x∥)Ω⟩ ≥ 0. \quad (21)$$

As $j : V_{eff} → H$ is symplectic, we obtain with $U^H_t = e^{-itD_H}$ for $t ∈ R$ the relation

$$σ(D_V(x), x) = 2 \text{Im}(j(D_V(x)), j(x)) = 2 \text{Im}(-iD_Hj(x), j(x)) = 2(D_Hj(x), j(x)). \quad (22)$$

This implies that $D_H ≥ 0$. 

21
Lemma 4.7. If there exists a ground state representation \((\pi, U, \mathcal{H})\) of \((\text{Heis}(V, \sigma), \alpha)\) with \(\pi(z, 0) = z1\) for \(z \in \mathbb{T}\), then the following assertions hold:

(a) The subspace \(V_{\text{eff}} \subseteq V\) is symplectic, i.e., the restriction of \(\sigma\) to \(V_{\text{eff}}\) is non-degenerate and

\[
V_{\text{eff}} \cap V^\beta = \{0\}.
\]  

(b) The map \(j: V_{\text{eff}} \rightarrow (\mathcal{H}, 2\text{Im}(\cdot, \cdot))\) is injective.

(c) If \(\Omega\) is an eigenvector for some \(x \in V_{\text{eff}}\), then \(x = 0\).

(d) If \(x \in V\) with \(W(x)\Omega \in \mathcal{H}^0\), then \(x \in V^\beta\).

Proof. (a) In view of (18), we have to verify (23). Let \(x \in V_{\text{eff}} \cap V^\beta\). Then

\[
W(x) = \rho_1(x) = 1 \otimes \rho(x) = \pi^0(x) \otimes 1
\]

in the terminology of Proposition 4.3, and thus \(W(x) \in \mathbb{T}1 = \pi(\mathbb{T} \times \{0\})\). This contradicts the injectivity of \(\pi\) (Remark C.1).

(b) follows immediately from (a) and the fact that \(j\) is symplectic.

(c) If \(W(x)\Omega = \lambda \Omega\) for some \(\lambda \in \mathbb{T}\), then (j3) implies \(\lambda = e^{-\frac{1}{2}||j(x)||^2}\), so that \(j(x) = 0\). This entails \(x = 0\) because \(j\) is injective by (b).

(d) For every \(t \in \mathbb{R}\), we have

\[
W(\beta_t(x))\Omega = U_t W(x)\Omega = W(x)\Omega
\]

because \(W(x)\Omega \in \mathcal{H}^0\). Therefore \(\Omega\) is an eigenvector of \(W(x)^{-1}W(\beta_t(x)) \in W(\beta_t(x) - x)\). Now (c) shows that \(\beta_t(x) = x\), hence that \(x \in V^\beta\). \(\square\)

Example 4.8. We present an example where \(\sigma\) is non-degenerate on \(V_{\text{eff}}\) but \(V_{\text{eff}} + V^\beta\) is not dense in \(V\). We consider the Banach space

\[
V := C([0, 1], \mathbb{C}) \quad \text{with} \quad \sigma(f, g) := 2\text{Im} \int_0^1 \overline{f(x)}g(x) \, dx = -i \int_0^1 \overline{f(x)}g(x) - g(x) f(x) \, dx,
\]

edowed with the symplectic \(\mathbb{R}\)-action, defined by

\[
(\beta_s f)(x) = e^{-ist} f(x).
\]

Then \(V^\beta = \{0\}\), and since \(\sigma\) is the imaginary part of a hermitian scalar product, it is non-degenerate. By (18), the vanishing of \(V^\beta\) implies that \(\sigma\) is non-degenerate on

\[
V_{\text{eff}} \subseteq \{f \in V: f(0) = 0\}.
\]

As all functions in \(V_{\text{eff}}\) vanish in 0, this subspace is not dense in \(V\).

The uniqueness of \(j\) implies that \(\mathcal{H} = L^2([0, 1])\), where \(j: V_{\text{eff}} \hookrightarrow \mathcal{H}\) is the canonical inclusion. Although this inclusion is injective, there exist positive energy representations of Heis\((V, \sigma)\) that are not multiples of Fock space representations (cf. Remark 4.6(a)).

For any unitary representation \(\kappa\) of the additive group \((\mathbb{C}, +)\), we obtain by \(f \mapsto \kappa(f(0))\) a unitary representation of \(V\). Let \((\rho_0, K_0)\) be the cyclic Fock representation of Heis\((V, \sigma)\), specified by the positive definite function \(\varphi(z, f) = ze^{-\frac{1}{4}||f||^2}\) and let \((\kappa, \mathcal{E})\) be a cyclic representation of \(\mathbb{C}\). 

22
with cyclic vector $\Omega^0$. Recall that this implies that $\mathcal{E} \cong L^2(\mathbb{R}^2, \mu)$ with a finite positive measure $\mu$, $\Omega^0 = 1$ and

$$(\kappa(x + iy)F)(a, b) = e^{i(axy + by)}F(a, b).$$

We define on the tensor product $\mathcal{H} := \mathcal{E} \otimes K_0$ a unitary positive energy representation of $\text{Heis}(V, \sigma)^0$ by

$$\pi(z, f, t) := \kappa(f(0)) \otimes \rho_0(z, f, t).$$

Its minimal energy subspace is $\mathcal{H}^0 = \mathcal{E} \otimes \kappa \cong \mathcal{E}$, which is clearly cyclic. The commutant $\pi(\text{Heis}(V, \sigma)^0)'$ is isomorphic to the commutant $\kappa(V)' = \kappa(V)^n$; where we use that $\kappa$ is cyclic and thus $\kappa(V)^n$ maximal abelian. As the cyclic vector $\Omega^0 \in \mathcal{E}$ separates $\kappa(V)' \cong \pi(\text{Heis}(V, \sigma)^0)'$, it is also cyclic for $\text{Heis}(V, \sigma)$. Therefore $(\pi, \mathcal{H})$ defines a positive energy Bose–Einstein field on $\mathcal{H}$.

From the isomorphism of the commutants $\kappa(V)' \cong \pi(V) = \pi(\text{Heis}(V, \sigma)^0)'$ and Schur's Lemma, it follows that the representation $\pi$ is irreducible if and only if $\mu$ is a point measure, i.e., $\kappa$ is simply a character of the group $(\mathbb{C}, +)$. Then $\mathcal{H} = K_0$ and $\pi(z, f, t) = \kappa(f(0))\rho_0(z, f, t)$.

We also note that $\pi^0(G^0) = T1$, so that the inclusion $\kappa(\mathbb{C})' \subseteq \pi^0(G^0)' = B(\mathcal{H}^0)$ is proper if and only if $\kappa$ is not irreducible. We conclude that $\pi$ is strict if and only if $\dim \mathcal{E} = 1$, i.e., $\kappa$ is irreducible.

As a consequence of the preceding discussion, we record:

**Proposition 4.9.** There exist pairs $(G, \alpha)$, where $G$ is a Banach–Lie group and $\alpha$ is a smooth action, such that not all smooth ground state representations are strict.

**Remark 4.10.** One cannot replace the density assumption (WSC) in Theorem 4.4 by the assumption that $V^\beta + V_\beta$ is $\sigma$-weakly dense, i.e., that $\sigma$ is non-degenerate on $V^\beta$. If $V^\beta = \{0\}$, then $V_\beta$ is $\sigma$-weakly dense, and we have this situation in Example 4.8. The construction of the non-strict ground state representations in this example show that the conclusion of Theorem 4.4 is invalid in this example.

**Remark 4.11.** (a) If $\dim V < \infty$, then $V^\beta = V^\perp_{\text{eff}}$ intersects $V_{\text{eff}}$ trivial, hence is a linear complement.

(b) Without the assumption of $V_{\text{eff}}$ being symplectic, the sum $V_{\text{eff}} + V^\beta$ need not be dense in $V$, as the following example shows. On $V = \mathbb{R}^2$ with $\mathfrak{sp}(V, \sigma) \cong \mathfrak{sl}_2(\mathbb{R})$, we consider the nilpotent element

$$D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sp}(V, \sigma) \quad \text{and} \quad \beta_t = e^{tD} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Then $V^\beta = \mathbb{R}e_2 = V_\beta$ are both one-dimensional, hence not symplectic. It follows in particular, that any map $j : V_\beta \to H$ into a complex Hilbert space $H$ with dense range is trivial. Therefore Weinless’ Theorem implies that $\mathbb{R}e_2$ annihilates all ground state vectors (see also Theorem 6.6(ii) below).

(c) From (18) it follows that

$$(V^\beta + V_{\text{eff}})^{\perp_{\sigma}} = (V^\beta + V_\beta)^{\perp_{\sigma}} = V^\beta \cap (V^\beta)^{\perp_{\sigma}}$$

is the radical of the restriction of $\sigma$ to $V^\beta$. Therefore the non-degeneracy of $\sigma$ on $V^\beta$ is equivalent to $V^\beta + V_\beta$ being $\sigma$-weakly dense in $V$.

**Remark 4.12.** (Kay’s Uniqueness Theorem)

(a) A single particle structure for $(V, \sigma, \beta)$ is a triple $(j, H, U^H)$, consisting of a complex Hilbert space $H$, a real linear map $j : V \to H$ and a unitary one-parameter group $U^H_t = e^{itB}$ with $B > 0$ such that:

$$j(V) = H, \quad \sigma(v, w) = 2 \text{Im}(j(v), j(w)) \quad \text{for} \quad v, w \in V, \quad \text{and} \quad j \circ \beta_t = U^H_t \circ j \quad \text{for} \quad t \in \mathbb{R}. \quad \text{(ii)}$$
Conditions (j1/2) in Theorem 4.2 imply that \((j, H, \mathcal{U}^H)\) defines a single particle structure for \((V, \sigma, \beta)\). According to Kay's Uniqueness Theorem ([Ka79, Ka85]), any two single particle structures for \((V, \sigma, \beta)\) are unitarily equivalent. Note that single particle structures on \(V\) can only exist if \(V^\beta = \{ 0 \}\) because \(j\) is supposed to be injective and \(U^H\) has no non-zero fixed points on \(H\).

(b) If we start with a complex Hilbert space \(H\) and \(V = H^\mathbb{R}\) (the underlying real space) with \(\sigma = 2 \text{Im}(\langle \cdot, \cdot \rangle)\), then any unitary one-parameter group \(\beta_t = e^{itA}\) on \(H\) defines a boson single particle space \((V, \sigma, \beta)\) and \(V = \ker A \oplus V_\beta\).

Suppose that \(\ker A = 0\) and write \(A = R|A|\) for the polar decomposition of \(A\). Then \(R\) is a unitary involution and \(I := iR\) defines a new complex structure on \(H\); we write \(H\) for the resulting complex Hilbert space. Then the identity map \(j : H \to \mathcal{U}\) is symplectic and \(\beta_t = e^{it|A|}\), so that \((j, H, \beta)\) is the unique single particle structure for \((V, \sigma, \beta)\).

(c) An important special case arises from the translation action \((\beta_t f)(z) = f(e^{it}z)\) of \(\mathbb{R}\) on \(H = L^2(\mathbb{T})\). Then

\[
\sigma(\xi, \eta) = 2 \int_0^1 \text{Im}(\xi(t)\eta(t)) \, dt = \frac{1}{i} \int_0^1 \overline{\xi(t)}\eta(t) - \overline{\eta(t)}\xi(t) \, dt
\]

In this case \((Af)(z) = zf'(z)\) with \(\ker A = \mathbb{C}1\) (the constant functions). Therefore \(R\) corresponds to the Hilbert transform on \(1^1 \subseteq L^2(\mathbb{S}^1)\).

5 Holomorphic induction

One can use the technique of holomorphic induction to show that certain ground state representations are strict and to obtain further regularity properties. In [Ne14b, App. C], it is shown that this technique, developed in [Ne13] for Banach–Lie groups, also applies to Fréchet–BCH–Lie groups satisfying the conditions (SC) and (A1-3) below.

In this section we first describe the geometric environment on the Lie group and Lie algebra level needed for holomorphic induction (Subsection 5.1) and define this concept in Subsection 5.2. All this is combined with a homomorphism \(\alpha : \mathbb{R} \to \text{Aut}(G)\) in Subsection 5.3. If \(G\) is a Banach–Lie group and \(D \in \text{der}(\mathfrak{g})\) is a bounded operator for which \(0\) is isolated in \(\text{Spec}(D)\) and the norm on \(\mathfrak{g}\) is \(\alpha\)-invariant \((D\) is elliptic), then this leads to Theorem 5.8, the main result of this section.

5.1 The geometric setup for holomorphic induction

Let \(G\) be a Lie group with the smooth exponential function \(\exp : \mathfrak{g} \to G\). If, in addition, \(G\) is analytic and the exponential function is an analytic local diffeomorphism in 0, then \(G\) is called a \(BCH\)-Lie group (for Baker–Campbell–Hausdorff). Then the Lie algebra \(\mathfrak{g}\) is a \(BCH\)-Lie algebra, i.e., there exists an open 0-neighborhood \(U \subseteq \mathfrak{g}\) such that for \(x, y \in U\) the Hausdorff series

\[
x * y = x + y + \frac{1}{2}[x, y] + \cdots
\]

converges and defines an analytic function \(U \times U \to \mathfrak{g}, (x, y) \mapsto x * y\). The class of \(BCH\)-Lie groups contains in particular all Banach–Lie groups ([Ne06, Prop. IV.1.2], [GN]).

Let \(G\) be a connected Fréchet–BCH–Lie group \(G\) with Lie algebra \(\mathfrak{g}\). We further assume that there exists a complex \(BCH\)-Lie group \(G_C\) with Lie algebra \(\mathfrak{g}_C\) and a natural map \(\eta : G \to G_C\) for which \(U(\eta)\) is the inclusion \(\mathfrak{g} \to \mathfrak{g}_C\). Assume that \(G^0 \subseteq G\) is a Lie subgroup for which \(M := G/G^0\) carries the structure of a smooth manifold with a smooth \(G\)-action. We also assume the existence of closed \(\text{Ad}(G^0)\)-invariant complex subalgebras \(\mathfrak{g}_C^+ \subseteq \mathfrak{g}_C\) for which the complex conjugation
\[ \sigma_g(x + iy) = x - iy \text{ on } g \mathbb{C} \text{ satisfies } \sigma_g(g^\pm_{\mathbb{C}}) = g^\pm_{\mathbb{C}} \] and we have a topological direct sum decomposition
\[ g_{\mathbb{C}} = g^+_\mathbb{C} \oplus g^0_{\mathbb{C}} \oplus g^-_{\mathbb{C}} \] (SC)
(cf. (L3/4) in Section 3). We put
\[ q := g^+_\mathbb{C} \times g^0_{\mathbb{C}} \text{ and } p := g \cap (g^+_\mathbb{C} \oplus g^-_{\mathbb{C}}), \]
so that \( g = g^0 \oplus p \) is a topological direct sum. We assume that there exist open symmetric convex 0-neighborhoods
\[ U_{g_{\mathbb{C}}} \subseteq g_{\mathbb{C}}, \quad U_p \subseteq p \cap U_{g_{\mathbb{C}}}, \quad U_{g^0} \subseteq g^0 \cap U_{g_{\mathbb{C}}}, \quad U_{g^+_\mathbb{C}} \subseteq g^+_\mathbb{C} \cap U_{g_{\mathbb{C}}}, \quad \text{ and } \ U_q \subseteq q \cap U_{g_{\mathbb{C}}}, \]
such that the BCH-product is defined and holomorphic on \( U_{g_{\mathbb{C}}} \times U_{g_{\mathbb{C}}}, \) and the following maps are analytic diffeomorphisms onto an open subset:
(A1) \( U_p \times U_{g^0} \rightarrow g, (x, y) \mapsto x \ast y. \)
(A2) \( U_p \times U_q \rightarrow g_{\mathbb{C}}, (x, y) \mapsto x \ast y. \)
(A3) \( U_{g^+_\mathbb{C}} \times U_q \rightarrow g_{\mathbb{C}}, (x, y) \mapsto x \ast y. \)

**Examples 5.1.** (a) If \( \dim G < \infty \) and \( D \) is elliptic, then all these assumptions are satisfied for the positive/negative spectral subspaces \( g^\pm_{\mathbb{C}} \) of the derivation \( i \cdot D \) of \( g_{\mathbb{C}}. \)
(b) Let \( G \) be a simply connected Banach–Lie group for which \( g_{\mathbb{C}} \) also is the Lie algebra of a Banach–Lie group and \( M = G/G^0 \) is a Banach homogeneous space. If the subalgebras \( g^\pm_{\mathbb{C}} \subseteq g_{\mathbb{C}} \) satisfy the splitting condition (SC), then (A1-3) follow directly from the Inverse Function Theorem. This is the context of [Ne13].

**Remark 5.2.** (From Banach to Fréchet)
Let \( G_B \) be a Banach–Lie group with Lie algebra \( g_B, G^0_B \subseteq G_B \) and \( M_B = G_B/G^0_B \) be as above (cf. Example 5.1(b)). We assume that the splitting condition (SC) is satisfied. In addition, let \( \beta: \mathbb{R} \rightarrow \text{Aut}(G_B) \) be a one-parameter group of automorphisms defining a continuous \( \mathbb{R} \)-action on \( G_B \) and assume that the subalgebras \( g^\pm_{B, \mathbb{C}}, q_B \) and \( g^0 \) are \( \beta \)-invariant. Then the subgroup
\[ G := \{ g \in G_B : \beta^g: \mathbb{R} \rightarrow G_B, t \mapsto \beta_t(g) \text{ is smooth} \} \]
of \( G_B \) carries the structure of a Fréchet–BCH–Lie group with Lie algebra
\[ g := \{ x \in g_B : \mathbb{R} \rightarrow g_B, t \mapsto \text{L}(\beta_t)x \text{ is smooth} \}, \]
the Fréchet space of smooth vectors for the continuous \( \mathbb{R} \)-action on the Banach–Lie algebra \( g_B. \)
Likewise \( G^0 := G \cap G^0_B \) is a Lie subgroup of \( G \) for which \( M := G/G^0 \) is a smooth manifold consisting of the elements of \( M_B = G_B/G^0_B \) with smooth orbit maps with respect to the one-parameter group of diffeomorphisms induced by \( \beta \) via \( \beta^M_t(gG^0_B) = \beta_t(g)G^0_B. \)

Since the automorphisms \( \text{L}(\beta_t) \) of \( g \) resp., \( g_{\mathbb{C}} \) are compatible with the BCH multiplication, it is easy to see with [Ne14b, Lemma C.5] that conditions (A1-3) are inherited by the closed Fréchet subalgebras
\[ g^0 = g^0_B \cap g, \quad g^\pm_{\mathbb{C}} = (g^\pm_{\mathbb{C}})_B \cap g_{\mathbb{C}} \quad \text{ and } \quad q = q_B \cap g_{\mathbb{C}}. \]
5.2 Holomorphically induced representations

Condition (A1) implies the existence of a smooth manifold structure on $M = G/G^0$ for which $G$ acts analytically. Condition (A2) implies the existence of a complex manifold structure on $M$ which is $G$-invariant and for which the complex structure on the tangent space $T_{eG^0}(M) \cong g/g^0$ of $M$ in the base point $eG^0$ is determined by the identification with $g_C/q$. Finally, (A3) makes the proof of [Ne13, Thm. 2.6] work, so that we can associate to every bounded unitary representation $(\rho, V)$ of $G^0$ a holomorphic Hilbert bundle $V := G \times_{G^0} V$ over the complex $G$-manifold $M$ by defining $\beta : q \to gl(V)$ by $\beta(g^+_C) = \{0\}$ and $\beta|_{g^0} = d\rho$.

Definition 5.3. We write $\Gamma(V)$ for the space of holomorphic sections of the holomorphic Hilbert bundle $V \to M = G/G^0$ on which the group $G$ acts by holomorphic bundle automorphisms. A unitary representation $(\pi, H)$ of $G$ is said to be holomorphically induced from $(\rho, V)$ if there exists a $G$-equivariant linear injection $\Psi : H \to \Gamma(V)$ such that the adjoint of the evaluation map $ev_{eG^0} : H \to V = V_{eG^0}$ defines an isometric embedding $ev^*_{eG^0} : V \hookrightarrow H$. If a unitary representation $(\pi, H)$ holomorphically induced from $(\rho, V)$ exists, then it is uniquely determined ([Ne13, Def. 3.10]) and we call $(\rho, V)$ (holomorphically) inducible.

The concept of holomorphic inducibility involves a choice of sign. Replacing $g^+_C$ by $g^-_C$ changes the complex structure on $G/G^0$ and thus leads to a different class of holomorphically inducible representations of $G^0$.

The following two theorems contain key information on holomorphically induced representations. The first one describes properties of holomorphically induced representations and the second one provides a sufficient criterion for a representation to be holomorphically induced.

Theorem 5.4. ([Ne14b, Thm. C.2]) Assume (A1-3). If the unitary representation $(\pi, H)$ of $G$ is holomorphically induced from the bounded $G^0$-representation $(\rho, V)$, then the following assertions hold:

(i) $V \subseteq H^\omega$ consists of analytic vectors, i.e., their orbit maps $G \to H$ are real-analytic.

(ii) $R : \pi(G)' \to \rho(G^0)'$, $A \mapsto A|_V$ is an isomorphism of von Neumann algebras.

(iii) $d\pi(g^-_C)V = \{0\}$.

Proof. (i) follows from [Ne13, Lemma 3.5] and (ii) from [Ne13, Thm. 3.12]. Further (iii) follows from Equation (1) in the discussion preceding Theorem 3.12 in [Ne13].

Theorem 5.5. ([Ne13, Thm. 3.17]) Suppose that $(U, H)$ is a unitary representation of $G$ and $V \subseteq H$ is a $G^0$-invariant closed subspace such that

(HI1) The representation $(\rho, V)$ of $G^0$ on $V$ is bounded.

(HI2) $V \cap (H^\infty)^{g^-_C}$ is dense in $V$.

(HI3) $[\pi(G)V] = H$.

Then $(\pi, H)$ is holomorphically induced from $(\rho, V)$.
5.3 The setting with \( \alpha \)

Now let \( \alpha: \mathbb{R} \to \text{Aut}(G) \) define a smooth \( \mathbb{R} \)-action on \( G \) for which the action on the Lie algebra \( \mathfrak{g} \) is polynomially bounded and write \( D \in \text{der}(\mathfrak{g}) \) for the infinitesimal generator of \( \alpha \) (see (L1/2) in Section 3). We assume, in addition to (A1-3) in Subsection 5.1, that \( \alpha \) and the subspace \( \mathfrak{g}_+ \) are linked by the requirement that

\[
\mathfrak{g}_C^+ = \bigcup_{\delta > 0} \mathfrak{g}_C([\delta, \infty)),
\]

where \( \mathfrak{g}_C([\delta, \infty)) \) is the Arveson spectral subspace for the one-parameter group \( (\alpha_t^\delta)_{t \in \mathbb{R}} \) on \( \mathfrak{g}_C \) (cf. Appendix A). Applying Proposition A.8 to the Lie bracket \( \mathfrak{g}_C \times \mathfrak{g}_C \to \mathfrak{g}_C \), we see that \( \mathfrak{g}_C^+ \) is a closed complex subalgebra. For \( f \in S(\mathbb{R}) \), \( \alpha^\delta(f) := \int_\mathbb{R} f(t) \alpha_t^\delta \, dt \) and \( z \in \mathfrak{g}_C \), the relations \( \alpha^\delta(f)z = \alpha^\delta(f^\ast z) \) and the relation \( \widehat{f}(\xi) = f(-\xi) \) for the Fourier transform \( \widehat{f}(\xi) = \int_\mathbb{R} e^{i\xi x} f(x) \, dx \) imply that

\[
\mathfrak{g}_C := \sigma_\delta(\mathfrak{g}_C^+) = \bigcup_{\delta > 0} \mathfrak{g}_C((-\infty, -\delta]),
\]

where \( \sigma_\delta(x + iy) = x - iy \) is complex conjugation on \( \mathfrak{g}_C \).

**Example 5.6.** (a) Suppose that \( G \) is a Banach–Lie group and consider an element \( d \in \mathfrak{g} \) for which the one-parameter group \( e^{t \mathfrak{ad} d} \subseteq \text{Aut}(\mathfrak{g}) \) is bounded, i.e., preserves an equivalent norm. We call such elements, resp., the corresponding derivation \( D = \text{ad} d \) elliptic. Then

\[
G^0 = Z_G(\exp \mathbb{R}d) = Z_G(d) = \{ g \in G : \text{Ad}(g)d = d \}
\]

is a closed subgroup of \( G \), not necessarily connected, with Lie algebra \( \mathfrak{g}^0 = \mathfrak{j}_0(d) = \ker(\text{ad} d) \). Since \( \mathfrak{g} \) contains arbitrarily small \( e^{t \mathfrak{ad} d} \)-invariant 0-neighborhoods \( U \), there exists such an open 0-neighborhood with \( \exp_G(U) \cap G^0 = \exp_G(U \cap \mathfrak{g}^0) \). Therefore \( G^0 \) is a Lie subgroup of \( G \), i.e., a Banach–Lie group for which the inclusion \( G^0 \hookrightarrow G \) is a topological embedding.

Our assumption implies that \( \alpha_t^\delta := e^{t \mathfrak{ad} d} \) defines an equicontinuous one-parameter group of automorphisms of the complex Banach–Lie algebra \( \mathfrak{g}_C \). For \( \delta > 0 \), we consider the Arveson spectral subspace

\[
\mathfrak{g}_C^\delta := \mathfrak{g}_C([\delta, \infty]).
\]

By Lemma A.4, the splitting condition

\[
\mathfrak{g}_C = \mathfrak{g}_C^\delta \oplus \mathfrak{g}_C^0 \oplus \sigma_\delta(\mathfrak{g}_C^\delta)
\]

is satisfied for some \( \delta > 0 \) if and only if 0 is isolated in \( \text{Spec}(\text{ad} d) \).

Since \( \text{Ad}(G^0) \) commutes with \( e^{t \mathfrak{ad} d} \), the closed subalgebras \( \mathfrak{g}_C^\delta \subseteq \mathfrak{g}_C \) are invariant under \( \text{Ad}(G^0) \) and \( e^{t \mathfrak{ad} d} \). Now \( \mathfrak{p} := \mathfrak{g} \cap (\mathfrak{g}_C^\delta \oplus \mathfrak{g}_C^0) \) is a closed complement for \( \mathfrak{g}_C^\delta \) in \( \mathfrak{g} \), so that \( M := G/G^0 \) carries the structure of a Banach homogeneous space and \( \mathfrak{q} := \mathfrak{g}_C^0 + \mathfrak{g}_C^\delta \cong \mathfrak{g}_C^\delta \times \mathfrak{g}_C^0 \) defines a \( G \)-invariant complex manifold structure on \( M \).

(b) Let \( \mathfrak{g} \) be a real Hilbert–Lie algebra, i.e., \( \mathfrak{g} \) is a Lie algebra and a real Hilbert space, the Lie bracket is continuous and the operators \( \text{ad} x, x \in \mathfrak{g} \), are skew-symmetric. Then one can use spectral measures to see that \( \mathfrak{g}_C^\delta \) is the spectral subspace corresponding to the open interval \((0, \infty) \) (cf. Lemma A.10), so that the splitting condition

\[
\mathfrak{g}_C = \mathfrak{g}_C^\delta \oplus \mathfrak{g}_C^0 \oplus \mathfrak{g}_C^-
\]

is satisfied. In particular, 0 need not be isolated in the spectrum of \( \text{ad} d \) ([BRT07, Prop. 5.4]).
The following results are of key importance for the following. It contains the main consequences of Arveson’s spectral theory for the \( \mathbb{R} \)-actions on \( \mathfrak{g}_C \) and \( \mathcal{H}^\infty \).

**Proposition 5.7.** (A strictness criterion) Suppose that (27) holds, and that \((\pi, \mathcal{H})\) is a smooth ground state representation of \((G, \alpha)\), i.e., \(\mathcal{H}^0, \infty\) is dense in \(\mathcal{H}^0\), and that the representation \((\pi^0, \mathcal{H}^0)\) of \(G^0\) is bounded. Then \((\pi, \mathcal{H})\) is holomorphically induced from \((\pi^0, \mathcal{H}^0)\) and in particular strict.

**Proof.** First, Theorem A.12 implies that \(d\pi(\mathfrak{g}_C)\mathcal{H}^0, \infty = \{0\}\). Applying Theorem 5.5 to \(\mathcal{V} := \mathcal{H}^0\), we see that \((\pi, \mathcal{H})\) is holomorphically induced from \((\pi^0, \mathcal{H}^0)\), and Theorem 5.4(ii) implies strictness. \(\square\)

In the Banach case we can formulate more concrete sufficient conditions for strictness:

**Theorem 5.8.** Suppose that \(G\) is Banach and \(d \in \mathfrak{g}\) is elliptic with 0 isolated in Spec(\(\operatorname{ad} d\)) and \(\mathfrak{g}_C = \mathfrak{g}_C([-\infty, -\delta])\) for some \(\delta > 0\). Then the following assertions hold for any smooth representation \((\pi, \mathcal{H})\) for which \(-i\partial\pi(d)\) is bounded from below.

(a) The \(G^0\)-invariant subspace \(\mathcal{V} := (\mathcal{H}^\infty)^{\mathfrak{g}_C} \) satisfies \(\mathcal{H} = [\pi(G)\mathcal{V}]\).

(b) If the \(G^0\)-representation \(\rho(h) := \pi(h)|\mathcal{V}\) on \(\mathcal{V}\) is bounded, then \((\pi, \mathcal{H})\) is holomorphically induced from the representation \(\rho\) of \(G^0\) on \(\mathcal{V}\), \(\pi\) is semibounded, and \(d \in W^0\) (see (3)). In particular, \(\mathcal{V}\) consists of analytic vectors.

(c) In addition to (b), suppose that \(-i\partial\rho(d) \geq m1\) for \(m \in \mathbb{R}\). Then \(-i\partial\pi(d) \geq m1\) and the associated minimal positive energy representation of \((G, \alpha)\) for \(\alpha_t(g) = \exp(td)g\exp(-td)\) is strict with \(\mathcal{H}^0 = \mathcal{V}\).

**Proof.** (a) and (b) follow from [Ne13, Thms. 4.7, 4.14]. It remains to prove (c). Let \(P\) denote the spectral measure of \(-i\partial\pi(d)\). Our assumption implies that \(\mathcal{V} \subseteq P([m, \infty))\). As \(\mathcal{V}\) consists of analytic vectors, \(d\pi(U(g))\mathcal{V}\) is dense in \(\mathcal{H}\). Since \(\mathcal{V}\) is annihilated by \(\mathfrak{g}_C\), we have \(d\pi(U(g))\mathcal{V} = d\pi(U(g)^\perp)\mathcal{V}\) by the Poincaré–Birkhoff–Witt Theorem. Finally, we observe that the Spectral Translation Formula (Theorem A.12) implies that

\[
d\pi(U(g)^\perp)\mathcal{V} \subseteq \mathcal{V} + \mathcal{H}^\infty([\delta, \infty)) \subseteq \mathcal{H}^\infty([m, \infty)).
\]

(28)

We conclude that \(\mathcal{H}^\infty([m, \infty))\) is dense in \(\mathcal{H}\), i.e., that \(-i\partial\pi(d) \geq m1\).

Since \(d\) is central in \(\mathfrak{g}^0\) and \(\pi\) is holomorphically induced from \(\pi^0\), there exists a unitary one-parameter group \((U_t)_{t \in \mathbb{R}}\) in the commutant \(\pi(G)^t\) with \(U_t|\mathcal{V} = \pi^0(\exp td)\) (Theorem 5.4). As the restriction to \(\mathcal{V}\) is an isomorphism of von Neumann algebras, \(U_t = e^{itB}\), where \(B \geq 0\) is bounded. Now \(W_t := \pi(\exp td)U_t^{-1}\) defines a unitary one-parameter group implementing the same automorphisms as \(\exp(td)\) for which \(\mathcal{V}\) consists of fixed points. Now the same argument as in (a), applied to the extended Lie algebra \(\mathfrak{g} \times \mathbb{R}d\) and the representation of \(G \rtimes \mathbb{R}\) by \(\tilde{\pi}(g, t) = \pi(g)W_t\) implies that \((W_t)_{t \in \mathbb{R}}\) has positive spectrum and fixed point space \(\mathcal{H}^W = \mathcal{V}\) (see (28)). As \(\pi(G)\mathcal{V}\) is total in \(\mathcal{H}\), it follows that \((\pi, W, \mathcal{H})\) is a ground state representation, hence in particular minimal by Proposition 2.7. Theorem 5.4 further implies that it is strict. \(\square\)

The preceding theorem does not assert that \((\pi, \mathcal{H})\) is a ground state representation, but we have the following corollary. It provides a sufficient condition for a bounded representation to be ground state. It applies in particular to finite dimensional unitary representations of compact Lie groups. By the strong boundedness assumptions, it follows immediately from Theorem 5.8.

**Corollary 5.9.** If \(G\) is Banach and \(d \in \mathfrak{g}\) is elliptic with 0 isolated in Spec(\(\operatorname{ad} d\)), then every bounded representation of \(G\) is a strict ground state representation of \((G, \alpha)\) for \(\alpha_t(g) = \exp(td)g\exp(-td)\), where \(\mathcal{H}^0 = \mathcal{H}^\infty\).
The following theorem shows that, assuming that \((\pi, \mathcal{H})\) is semibounded with \(d \in W^0\) permits us to get rid of the quite implicit assumption that the \(G^0\)-representation on \(V\) is bounded. It is an important generalization of Corollary 5.9 to semibounded representations.

**Theorem 5.10.** ([Ne13, Thm. 4.12]) Let \((\pi, \mathcal{H})\) be a semibounded unitary representation of the Banach–Lie group \(G\) and let \(d \in W^0\) be an elliptic element for which 0 is isolated in \(\text{Spec}(\text{ad}d)\) and \(g_C = g_C([-\infty, -\delta])\) for some \(\delta > 0\). We write \(P : \mathcal{B}(\mathbb{R}) \to B(\mathcal{H})\) for the spectral measure of the unitary one-parameter group \(\pi_d(t) := \pi(\exp(td))\). Then the following assertions hold:

(i) The representation \(\pi|_{G^0}\) of \(G^0\) is semibounded and, for each bounded measurable subset \(B \subseteq \mathbb{R}\), the \(G^0\)-representation on \(P(B)\mathcal{H}\) is bounded.

(ii) The representation \((\pi, \mathcal{H})\) is a direct sum of representations \((\pi_j, \mathcal{H}_j)_{j \in J}\) for which there exist \(G^0\)-invariant subspaces \(D_j \subseteq (\mathcal{H}_j^\infty)^{\mathcal{H}_j}\) for which the \(G^0\)-representation \(\rho_j\) on \(\mathcal{V}_j := \overline{D_j}\) is bounded and \(\|\pi_j(G)\mathcal{V}_j\| = \mathcal{H}_j\). Then the representations \((\pi_j, \mathcal{H}_j)\) are holomorphically induced from \((\rho_j, \mathcal{V}_j)\).

(iii) If \((\pi, \mathcal{H})\) is irreducible and \(m := \inf \text{Spec}(-i\partial\pi(d))\), then \(P(\{m\})\mathcal{H} = (\mathcal{H}_\infty)^{\mathcal{H}_\infty}\) and \((\pi, \mathcal{H})\) is holomorphically induced from the bounded \(G^0\)-representation \(\rho\) on this space.

**Corollary 5.11.** In the context of Theorem 5.10, \((\pi, \mathcal{H})\) is a ground state representation and the direct summands \((\pi_j, \mathcal{H}_j)_{j \in J}\) are strict ground state representations.

**Proof.** First we use Theorem 5.8(c) to see that all representations \((\pi_j, \mathcal{H}_j)_{j \in J}\) are strict ground state representations. Now Lemma 2.9 shows that their direct sum \((\pi, \mathcal{H})\) is also a ground state representation. \(\square\)

**Problem 5.12.** Are direct sums of strict ground state representations always strict?

### 6 Finite dimensional groups

In this section we assume that \(G\) is finite dimensional, so that \(G^0\) is finite dimensional as well. Replacing \(G\) by \(G^0\), it suffices to consider the inner case, i.e., for some fixed \(d \in g\), we are interested in unitary representations \((\pi, \mathcal{H})\) for which \(-i\partial\pi(d)\) is bounded from below (cf. Proposition 2.10). Let \(I_\pi \subseteq g^*\) denote the momentum set of \(\pi\) (Definition 3.5).

#### 6.1 Generalities

From (3) we recall the that
\[
W_\pi = \{x \in g : -i\partial\pi(x) \text{ bounded below }\} = \{x \in g : \inf I_\pi(x) > -\infty\}
\]
(29)

is a convex cone in \(g\), invariant under \(\text{Ad}(G)\). It contains the positive cone of \(\pi\):
\[
C_\pi = \{x \in g : -i\partial\pi(x) \geq 0\}.
\]
(30)

We assume that \(\pi\) has discrete kernel, i.e., that the positive cone \(C_\pi\) of \(\pi\) is pointed. The linear subspace
\[
g_\pi := W_\pi - W_\pi \leq g
\]
is an ideal because \(W_\pi\) is an \(\text{Ad}(G)\)-invariant convex cone, and the restriction of \(\pi\) to the corresponding normal subgroup \(G_\pi \leq G\) is semibounded. Semibounded representations of finite dimensional
groups have been studied in detail and classified in [Ne00]. As \( d \in W_\pi \subseteq g_\pi \), we may further assume that \( g = g_\pi \) and restrict our discussion to semibounded representations. With [Ne00, Thm. XI.6.14] on the existence of a direct integral decompositions, many assertions can be reduced to the case of irreducible representations. This leaves us with the situation where:

(F1) \( \ker(\pi) \) is discrete, so that the cone \( C_\pi \) is pointed.

(F2) \( d \in W_\pi \) and \( \pi \) is semibounded.

(F3) \( \pi \) is irreducible.

**Remark 6.1.** From \( d \in W_\pi \) it follows that \( \text{Spec}(\text{ad} \: d) \subseteq i\mathbb{R} \) by [Ne00, Prop. VII.3.4] because discreteness of the kernel is equivalent to \( I_\pi \) spanning \( g^\ast \). As a consequence of the Jordan decomposition, this implies that the one-parameter group \( e^{R \: \text{ad} \: d} \subseteq \text{Aut}(g) \) is polynomially bounded because its semisimple component is bounded and its unipotent component is polynomially bounded (Remark 3.9).

**Definition 6.2.** A maximal abelian subspace \( t \subseteq g \) is called a \textit{compactly embedded Cartan subalgebra} if the closure of \( e^{\text{ad} \: t} \subseteq \text{Aut}(g) \) is compact. Let \( t \subseteq g \) be a compactly embedded Cartan subalgebra and \( g_C = t_C \oplus \bigoplus_{\alpha \in \Delta} g_C^\alpha \) the corresponding root decomposition, where

\[
g_C^\alpha = \{ y \in g_C : (\forall x \in t) \; [x, y] = \alpha(x)y \} \quad \text{and} \quad \Delta = \{ \alpha \in i t^\ast : g_C^\alpha \neq \{0\} \}.
\]

The elements of \( \Delta \) are called roots. We call a root \( \alpha \in \Delta \)

- \textit{compact} if there exists an \( x_\alpha \in g_C^\alpha \) with \( \alpha([x_\alpha, x_\alpha^\ast]) > 0 \) and write \( \Delta_k \subseteq \Delta \) for the set of compact roots.

- \textit{non-compact} if there exists a non-zero \( x_\alpha \in g_C^\alpha \) with \( \alpha([x_\alpha, x_\alpha^\ast]) \leq 0 \) and write \( \Delta_p \subseteq \Delta \) for the set of non-compact roots.

Then \( \dim g_C^\alpha = 1 \) for \( \alpha \in \Delta_k \) and there exists a unique element \( \alpha^\vee \in [g_C^\alpha, g_C^{-\alpha}] \) with \( \alpha(\alpha^\vee) = 2 \). The reflections \( r_\alpha : t \to t, r_\alpha(x) = x - \alpha(x)\alpha^\vee \) for \( \alpha \in \Delta_k \) generate the Weyl group \( W \).

**Lemma 6.3.** Let \( x \) be an element of the semisimple real Lie algebra \( g \) and \( x = x_s + x_n \) be its Jordan decomposition, where \( x_s \) is semisimple and \( x_n \) is nilpotent. Then the adjoint orbit of \( x \) contains all elements of the form \( x_s + tx_n, \; t > 0 \).

**Proof.** Since the Jordan decomposition and the adjoint orbit of \( x \) adapts to the decomposition of \( g \) into simple ideals, we may w.l.o.g. assume that \( g \) is simple.

Let \( q = l \times u \subseteq g \) denote the Jacobson–Morozov parabolic associated to the nilpotent element \( x_n \) ([HNO94]). Then \( x_s \in \ker(\text{ad} \: x_n) \subseteq q \) implies that \( x_s \in q \). As \( x_s \) is semisimple, it is conjugate under the group of inner automorphisms of \( q \) to an element of \( l \). By the Jacobson–Morozov Theorem, \( l \) contains a semisimple element \( h \) with \( [h, x_n] = 2x_n \) and \( h \in [x_n, g] \). In terms of this element, we have \( q = \sum_{n \geq 0} g_n(\text{ad} \: h) \) and \( l = \ker(\text{ad} \: h) \). We conclude that \( [h, x_s] = 0 \), so that \( e^{t \: \text{ad} \: h} \cdot x = x_s + e^{2t}x_n \) for \( t \in \mathbb{R} \).

We now come to the main result of this section. For its proof we shall use the following theorem ([Mo80, Thm. 1.1]), which is a formidable tool to exclude that certain Lie algebra elements have ground states.

---

3Every algebraic subgroup \( G \subseteq \text{GL}(V) \), \( V \) a finite dimensional real vector space, is a semidirect product \( G \cong U \rtimes L \), where \( U \) is unipotent and \( L \) is reductive. Moreover, for every reductive subgroup \( L_1 \subseteq G \) there exists an element \( g \in G \) with \( gL_1 g^{-1} \subseteq L \) ([Ho81, Thm. VIII.4.3]).
Theorem 6.4. (Moore’s Eigenvector Theorem) Let $G$ be a connected finite dimensional Lie group with Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$. Further, let $\mathfrak{n}_x \leq \mathfrak{g}$ be the smallest ideal of $\mathfrak{g}$ such that the induced operator $\text{ad}_{\mathfrak{g}/\mathfrak{n}_x} x$ on the quotient Lie algebra $\mathfrak{g}/\mathfrak{n}_x$ is elliptic, i.e., semisimple with purely imaginary spectrum. Suppose that $(\pi, \mathcal{H})$ is a continuous unitary representation of $G$ and $v \in \mathcal{H}$ an eigenvector for the one-parameter group $\pi(\exp \mathbb{R} x)$. Then

(a) $v$ is fixed by the normal subgroup $N_x := \langle \exp \mathfrak{n}_x \rangle \trianglelefteq G$, and

(b) the restriction of $i \partial \pi(x)$ to the orthogonal complement of the space $\mathcal{H}^{N_x}$ of $N_x$-fixed vectors has absolutely continuous spectrum.

Corollary 6.5. Let $G$ be a connected finite dimensional Lie group. Suppose that $(\pi, \mathcal{H})$ is an irreducible unitary representation of $G$ with discrete kernel and that $d \in \mathfrak{g}$ is such that $\partial \pi(d)$ has an eigenvector in $\mathcal{H}$. Then $\text{ad}(d)$ is elliptic.

**Proof.** Suppose that $v$ is an eigenvector of $\partial \pi(d)$. Then Moore’s Eigenvector Theorem implies that the normal subgroup $N_d \trianglelefteq G$ fixes $v$. As $N_d$ is normal, the subspace $\mathcal{H}^{N_d}$ of $N_d$-fixed vectors is a $G$-subrepresentation, hence coincides with $\mathcal{H}$ by irreducibility. As $\ker(\pi)$ is discrete, $\mathfrak{n}_d = \{0\}$, and this means that $\text{ad}(d)$ is elliptic. □

Theorem 6.6. Let $(\pi, \mathcal{H})$ be an irreducible semibounded representation with discrete kernel of the finite dimensional connected Lie group $G$ and let $d \in W_\pi$. Then the following assertions hold:

(i) There exists a compactly embedded Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and a $\mathcal{W}$-invariant positive system $\Delta^+_p$ of non-compact roots such that

$$W_\pi \cap \mathfrak{t} = C_{\text{max}} := (i \Delta^+_p)^* := \{x \in \mathfrak{t} : (\forall \alpha \in \Delta^+_p) i\alpha(x) \geq 0\}$$

and $W_{\text{max}} := \overline{\text{Ad}(G)C_{\text{max}}}$ coincides with $W_\pi$.

(ii) $d$ has smooth ground states if and only if $\text{ad}(d)$ is elliptic.

(iii) If $\mathfrak{g}$ is simple hermitian, then $W_\pi = W_{\text{max}}$, and in particular $W_\pi$ is closed.

**Proof.** (i) From [Ne00, Thm X.4.1] and its proof we know that the set of extreme points $\text{Ext}(I_\pi)$ is a coadjoint orbit $O_\lambda$ for some $\lambda \in \mathfrak{t}^*$, that $I_\pi = \text{conv}(O_\lambda)$ and that there exists a $\mathcal{W}$-invariant positive system $\Delta^+_p$ such that

$$W_\pi \cap \mathfrak{t} = C_{\text{max}}.$$ 

As $W_\pi^0$ is an elliptic cone, i.e., its interior consist of elliptic elements ([Ne00, Prop. VII.3.4(c)]), it follows that

$$W_\pi = \overline{\text{Ad}(G)C_{\text{max}}} = W_{\text{max}}. \quad (31)$$

(ii) As in (i), we derive from [Ne00, Prop. VII.3.4(c)] that $W_\pi^0 = \text{Ad}(G)(W_\pi^0 \cap \mathfrak{t})$ because $W_\pi^0$ is elliptic. A ground state vector for $d$ exists if and only if the minimal spectral value

$$m := \inf \text{Spec}(-i \partial \pi(d))$$

is an eigenvalue. If this is the case, then Corollary 6.5 implies that $\text{ad}(d)$ is elliptic.

Now we assume, conversely, that $d \in W_\pi$ is elliptic. Then $\text{Ad}(G)d$ intersects the Cartan subalgebra $\mathfrak{t}$ ([Ne00, Thm. VII.1.4(v)])]. As $d \in W_\pi$ and $W_\pi \cap \mathfrak{t} = C_{\text{max}}$, it follows that $d$ is $\text{Ad}(G)$-conjugate to an element $d' \in C_{\text{max}}$. Therefore the t-weight decomposition of $\mathcal{H}_\lambda$ implies
that the \( m \)-eigenspace of \(-i\partial_t(d')\) contains a lowest weight vector (which is smooth), so that the minimal eigenspace \( H^\lambda \) contains a non-zero smooth vector.

(iii) Suppose that \( g \) is simple hermitian. To show that \( W_{\text{max}} \subseteq W_\pi \), let \( x \in W_{\text{max}} \) and write \( x = x_s + x_n \) for its Jordan decomposition. Then the adjoint orbit of \( x \) contains all elements \( x_s + tx_n, t > 0 \) (Lemma 6.3), so that \( x_s, x_n \in W_{\text{max}} \). Since \( x_s \) is nilpotent, we even have \( x_n \in W_{\text{min}} \) by [HNO94, Thm. III.9], and since \( W_{\text{min}} \subseteq C_\pi \) ([Ne00, Thm. X.4.1]), it follows that \( x_n \in C_\pi \). As \( x_s \) is elliptic, its adjoint orbit intersects \( t \cap W_{\text{max}} = C_{\text{max}} \subseteq W_\pi \), as we have seen in (i). This implies that every elliptic element in \( W_{\text{max}} \) is contained in \( W_\pi \). We thus obtain that

\[ x = x_s + x_n \in W_\pi + C_\pi \subseteq W_\pi, \]

and hence that \( W_{\text{max}} \subseteq W_\pi \), which implies equality by (31) above.

\[ \Box \]

Example 6.7. In the context of the preceding theorem, we note that, in general \( W_\pi \neq W_{\text{max}} \) because \( W_\pi \) need not be closed. Let

\[ \sigma(x, y) := \sum_{j=1}^n x_j y_{n+j} - y_j x_{n+j} \]

be the canonical symplectic form on \( \mathbb{R}^{2n} \) and \( d \in \mathfrak{sp}_{2n}(\mathbb{R}) \) an element for which the corresponding Hamiltonian function \( H_d(v) = \frac{1}{2}\sigma(dv, v) \) is positive definite. For the oscillator representation \((\pi, L^2(\mathbb{R}^n))\) of \( g = \mathfrak{heis}(\mathbb{R}^{2n}, \sigma) \times \mathbb{R}^d \), we have

\[ \overline{W_\pi} = \mathfrak{heis}(\mathbb{R}^{2n}, \sigma) \oplus \mathbb{R}^d, \]

a closed half space with boundary \( \mathfrak{heis}(\mathbb{R}^{2n}, \sigma) \). An element of \( \mathfrak{heis}(\mathbb{R}^{2n}, \sigma) \) corresponds to a semibounded operator if and only if it is central. Therefore \( W_\pi \) is not closed, hence different from \( W_{\text{max}} \).

Example 6.8. (The case of simple Lie algebras) (a) Assume that \( g \) is simple and that \( \alpha \) is non-trivial. As all derivations of \( g \) are inner, we have \( \alpha_t(g) = \exp(td)g\exp(-td) \) for some non-zero \( d \in g \). If \( G \) has a non-trivial positive energy representation, then \( g \) must be compact or hermitian ([Ne00, §§VII.2/3]).

- In the compact case all irreducible representations \((\pi, \mathcal{H})\) are finite dimensional, so that all operators \( \partial_t(x) \) for \( x \in g \) are bounded. Hence the positive energy condition is satisfied for every \( d \in g \) and ground states exist (cf. Corollary 5.9).

- In the hermitian case, there is a closed convex cone \( W_{\text{max}} \subseteq g \), such that there exists a positive energy representation if and only if \( d \in W_{\text{max}} \cup -W_{\text{max}} \) (Theorem 6.6(iii)). If \( d \in W_{\text{max}} \), then every irreducible representation \((\pi, \mathcal{H})\) for which \(-i\partial_t(d)\) is bounded from below is semibounded because the cone \( W_\pi = W_{\text{max}} \) has interior points. By Theorem 6.6(ii), the existence of ground states is equivalent to \( d \) being elliptic.

(b) In a hermitian Lie algebra, there exist two closed convex invariant cones \( W_{\text{min}} \subseteq W_{\text{max}} \) such that, for every non-trivial closed convex invariant cone \( W \subseteq g \) we have

\[ W_{\text{min}} \subseteq W \subseteq W_{\text{max}} \quad \text{or} \quad W_{\text{min}} \subseteq -W \subseteq W_{\text{max}}. \]

If \( W_{\text{min}} = W_{\text{max}} \), which is the case for \( g = \mathfrak{sp}_{2n}(\mathbb{R}) \), this means that \( W \) is unique up to sign. We therefore have

\[ W_{\text{min}} \subseteq C_\pi \subseteq W_\pi \subseteq W_{\text{max}} \]

for every positive energy representation, and thus \( C_\pi = W_\pi \).

In general the two cones \( C_\pi \) and \( W_\pi \) are different. Concrete examples are easily found for \( g = \mathfrak{su}_{1,2}(\mathbb{C}) \).
Remark 6.9. For finite dimensional Lie groups $G$ the classification of irreducible semibounded unitary representations easily boils down to a situation where one can apply Theorem 5.10. If the semibounded unitary representations of $G$ separate the points, then the set $\text{comp}(\mathfrak{g})$ of elliptic elements in $\mathfrak{g}$ has interior points ([Ne00, Prop. VII.3.4(c)]) and we may choose $\mathfrak{d}$ as an interior point which, in addition, is a regular element of $\mathfrak{g}$. Then $t := \mathfrak{g}^0 := \ker(\text{ad} \mathfrak{d})$ is a compactly embedded Cartan subalgebra and the corresponding subgroup $T := \exp(t) = G^0$ is abelian. For a semibounded representation $(\pi, \mathcal{H})$ of $G$ with discrete kernel we now choose a $\mathcal{W}$-invariant positive system $\Delta^+_p$ of non-compact roots such that $W_p \subseteq W_{\text{max}}$ and obtain $\mathfrak{d} \in C^0_{\text{max}}$. As $\mathfrak{d}$ is elliptic, Theorem 6.6 now implies that $\pi$ is holomorphically induced from $(\pi^0, \mathcal{H}^0)$. Further, this representation is irreducible, and since $G^0 = T$ is abelian, Schur’s Lemma leads to $\dim \mathcal{H}^0 = 1$. This means that $\pi^0(\exp x) = e^{i\lambda(x)}$ for some $\lambda \in \mathfrak{t}^*$, the lowest weight of the representation $\pi$ with respect to the positive system $\Delta^+ := \{\alpha \in \Delta : \alpha(-i\mathfrak{d}) > 0\}$. In this case $C_{\alpha} \subseteq C_{\alpha^0}$ is equivalent to

$$\lambda(\alpha^0) \leq 0 \quad \text{for} \quad \alpha \in \Delta^+_k \quad \text{and} \quad \lambda([x^*_\alpha, x_\alpha]) \geq 0 \quad \text{for} \quad \alpha \in \Delta^+_p, x_\alpha \in \mathfrak{g}^0_{\mathbb{C}}.$$  

As Remark 6.17 below shows, these conditions are in general not sufficient.

From [HN12, Cor. 14.3.10] we recall the following fact on the connectedness of the group $G^0$.

**Lemma 6.10.** If $G$ is connected and $\mathfrak{d}$ is elliptic, then the subgroup $G^0 = Z_G(\mathfrak{d})$ is connected.

### 6.2 Application to compact Lie groups

Theorem 6.15 below is the main result of the present subsection. It shows that all unitary representations of compact connected Lie groups are ground state representations for any continuous homomorphism $\alpha : \mathbb{R} \to \text{Aut}(G)$. The following lemma prepares the crucial information for its proof.

We first recall the root decomposition of the compact Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ (Definition 6.2) containing a fixed element $\mathfrak{d}$ that we may pick in $[\mathfrak{g}, \mathfrak{g}]$ as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. We have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha},$$

and obtain with

$$\mathfrak{g}^+_\mathbb{C} := \sum_{\pm \alpha(-i\mathfrak{d}) > 0} \mathfrak{g}_{\alpha}^0 \quad \text{and} \quad \mathfrak{g}^0_{\mathbb{C}} = \mathfrak{z}_{\mathbb{C}}(\mathfrak{d})_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha(-i\mathfrak{d}) = 0} \mathfrak{g}_{\alpha}^0 = \mathfrak{z}_{\mathbb{C}}(\mathfrak{d})$$

the triangular decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^+_\mathbb{C} \oplus \mathfrak{g}^0_{\mathbb{C}} \oplus \mathfrak{g}^-_{\mathbb{C}}$$

of $\mathfrak{g}_{\mathbb{C}}$ with respect to $-i \text{ad} \mathfrak{d}$ (cf. [Ne00, Ch. IX.5]). We choose a positive system $\Delta^+ \subseteq \Delta$ in such a way that

$$\Delta^+_p \supseteq \Delta^+_p^+ := \{\alpha \in \Delta : \alpha(-i\mathfrak{d}) > 0\}.$$  

Then $\Delta^0 := \Delta^0 \cap \Delta^+$ is a positive system in the root system

$$\Delta^0 := \{\alpha \in \Delta : \alpha(-i\mathfrak{d}) = 0\} \quad \text{of} \quad (\mathfrak{g}^0, \mathfrak{t}), \quad \text{and} \quad \Delta^+ = \Delta^0 \cup \Delta^+.$$  

**Proposition 6.11.** Let $G$ be a connected Lie group with compact Lie algebra $\mathfrak{g}$ and derivation $D = \text{ad} \mathfrak{d}$ for some $\mathfrak{d} \in [\mathfrak{g}, \mathfrak{g}]$. For an irreducible unitary representation $(\pi^0, \mathcal{H}^0)$ of $G^0 := Z_G(\mathfrak{d})$, the following conditions are equivalent:

33
Remark 6.12. The set \( \rho \in \hat{T} \) of chose characters satisfying Condition (iv) is invariant under the Weyl group \( \mathcal{W} := \mathcal{W}(g^0, t) \). Identifying the unitary dual \( \hat{G} \) of \( G \) with the orbit space \( \hat{T} / \mathcal{W} \) (Cartan–Weyl Theorem, [Wa72, p. 209]), it follows that the ground state representations for \( (G, \alpha) \) correspond to the subset \( \hat{T}_+ / \mathcal{W} \).

The above lemma characterizes the holomorphically inducible representations of \( G^0 \). The following proposition switches the perspective from \( G^0 \) to \( G \):

Proposition 6.13. Let \( G \) be a connected Lie group with compact Lie algebra \( g \) and derivation \( D = \text{ad}_d \) for some \( d \in g \). Then every irreducible unitary representations \( (\pi, \mathcal{H}) \) of \( G \) is holomorphically induced from the irreducible representations \( (\pi^0, \mathcal{H}^0) \) of \( G^0 \) on the minimal eigenspace of \( -i\partial\pi(d) \) and it is a strict ground state representation.

Proof. Since \( \mathcal{H} \) is finite dimensional, \( -i\partial\pi(d) \) has an eigenspace \( \mathcal{H}^0 \) for a minimal eigenvalue. This space carries an irreducible representation of the connected group \( G^0 \) (Lemma 6.10). This follows easily from \( U(g^0) = U(g^0_{\mathbb{C}})U(g^0_{\mathbb{C}})U(g^0_{\mathbb{C}}) \) which shows that every \( G^0 \)-invariant subspace of \( \mathcal{H}^0 \) generates an invariant subspace of \( \mathcal{H} \) under \( U(g^0_{\mathbb{C}}) \) (alternatively, we can use Theorem 5.4(ii)).

As \( \pi \) is irreducible, \( \mathcal{H}^0 \) generates \( \mathcal{H} \) under the action of \( G \). That \( (\pi, \mathcal{H}) \) is holomorphically induced from \( (\pi^0, \mathcal{H}^0) \) follows from Theorem 5.5 and that it is a strict ground state representation from Corollary 5.9. \( \square \)
Remark 6.14. The representation $\pi^0$ need not be one-dimensional if $t \neq g^0$, i.e., if $g^0$ is not abelian. A simple example is obtained for $G = U_3(\mathbb{C})$, the identical representation $(\pi, H)$ on $H = \mathbb{C}^3$, and

$$d = i \text{ diag}(1, 0, 0).$$

Then $H^0 = \mathbb{C}e_2 + \mathbb{C}e_3$ is 2-dimensional and

$$G^0 \cong U(\mathbb{C}e_1) \times U(\mathbb{C}e_2 + \mathbb{C}e_3) \cong \mathbb{T} \times U_2(\mathbb{C}).$$

Theorem 6.15. If $G$ is a compact connected Lie group and $\alpha : \mathbb{R} \to \text{Aut}(G)$ is a continuous homomorphism, then the following assertions hold:

(i) Every unitary representation of $G$ is a ground state representation.

(ii) A unitary representation $(\pi^0, H^0)$ of $G^0 = \text{Fix}(\alpha)$ extends to a ground state representation of $G$ if and only if it satisfies the positivity condition $C_\alpha \subseteq C_{\pi^0}$.

(iii) $(G, \alpha)$ has the unique extension property, i.e., every ground state representation of $(G, \alpha)$ is strict.

Proof. (i) As every unitary representation of $G$ is a direct sum of irreducible ones, Lemma 2.9 shows that it suffices to assume that $(\pi, H)$ is irreducible. Then the assertion follows from Proposition 6.13. (ii) The necessity of $C_\alpha \subseteq C_{\pi^0}$ follows from Theorem 3.11. To see that it is also sufficient, write $(\pi^0, H^0)$ as a direct sum of irreducible representations $(\pi^0_j, H^0_j)_{j \in J}$. By Proposition 6.11, the representations $(\pi^0_j, H^0_j)$ are holomorphically inducible to unitary representations $(\pi_j, H_j)$ and we can form their direct sum $(\pi, H)$, which naturally contains $(\pi^0, H^0)$ as a subrepresentation. Now the subspace $F \subseteq H$ generated by $\pi(G)H^0$ carries a ground state representation extending $\pi^0$.

(iii) We have to show that all ground state representations $(\pi, H)$ of $G$ are strict (cf. Proposition 2.19). As $H^0$ is $\pi(G)$-invariant, it decomposes according to the decomposition $\pi \cong \bigoplus_{[\rho] \in \hat{G}} \pi_{[\rho]}$ into isotypic $G$-representations. We have already seen that the passage from $\pi$ to $\pi^0$ defines for irreducible representations an injection $\hat{G} \hookrightarrow \hat{G}^0$, whose image has been characterized in Proposition 6.11. Hence there are no non-zero $G^0$-intertwining operators between different representations $\pi_{[\rho]}^0$. This reduces the problem to the case where $\pi$ is isotypic, where it follows from the fact that $\rho$ is holomorphically induced from $\rho^0$ (Proposition 6.13), and now strictness follows from Theorem 5.4(ii).

Remark 6.16. (Classification schemes)

(a) We think of the preceding theorem as a classification scheme for the irreducible representations of $G$. To recover the classical approach, let $d \in \mathfrak{g}$ be a regular element, i.e., $t := g^0 := \ker(\text{ad } d)$ is abelian. Then $T := G^0$ is a maximal torus of $G$, and the preceding theorem asserts that the irreducible unitary representations of $G$ can be parametrized in terms of those irreducible unitary representations of $T$ arising as ground state representations for $\alpha_t(y) = \exp(td)g \exp(-td)$. Since $T$ is abelian, its irreducible representations are characters. So Theorem 6.15 yields an injection

$$\hat{G} \hookrightarrow \hat{T}$$

whose range is the subset

$$\hat{T}_d := \{\chi \in \hat{T} : -i \cdot d\chi \in C^*_\alpha\}.$$

As $d$ is regular,

$$\Delta^+ := \{\alpha \in \Delta(\mathfrak{g}_C, t_C) : -i\alpha(d) > 0\}.$$
is a positive system of roots. Proposition 6.11 then implies that
\[ \hat{T}_d = \{ \chi \in \hat{T} : (\forall \alpha \in \Delta^+) \lambda(\alpha^\vee) \leq 0 \} \]
consists of all antidominant weights. We thus recover the Cartan–Weyl Classification of the irreducible \( G \)-representations in terms of lowest weights.

(b) The key point of the preceding theorem is that it does not require \( d \) to be regular. In any case we obtain an injection
\[ \hat{G} \hookrightarrow \hat{G^0} \]
and an irreducible representation \( \pi^0 \) of \( G^0 \) is contained in the range of this map if and only if \( C_\alpha \subseteq C_\pi^0 \). Note that \( C_\alpha \subseteq g^0 \) is a closed convex invariant cone in \( g^0 \), hence determined by the intersection with any Cartan subalgebra \( t \subseteq g^0 \) ([Ne00, Thm. VII.3.29]). Let \( \lambda^0 \in i t^* \) be an extremal weight of an irreducible representation \( \pi^0 \) of \( G^0 \). Then all weights of \( \pi^0 \) are contained in \( \text{conv}(W^0\lambda) \). Therefore \( C_\alpha \subseteq C_\pi^0 \) is equivalent to
\[ -i \cdot \lambda \in (C_\alpha \cap t)^* . \]
Hence the image of \( \hat{G} \) in \( \hat{G^0} \subseteq \hat{T} \) consists of all characters \( \chi \) which are lowest weights of \( G^0 \)-representation and, in addition, satisfy
\[ -i \cdot d\chi \in (C_\alpha \cap t)^* . \]

Remark 6.17. The assumptions of compactness and finite dimension in Theorem 6.15(i),(ii) are fundamental and cannot be removed. This is demonstrated by the following examples. We examine one case, disproving (i), and another one, disproving (ii).

(a) The group \( G = \text{SL}_2(\mathbb{R}) \) is a finite dimensional Lie group, hence locally compact, but not compact. It has irreducible unitary representations \((\pi, \mathcal{H})\) (the principal series) for which all non-zero elements \( d \in g \) correspond to unbounded hermitian operators \( i\partial\pi(d) \) which are neither bounded from below or above. Therefore (i) of Theorem 6.15 fails for \( \alpha_i(g) := \exp(td)g\exp(-td) \) and every non-zero \( d \in g \). The group \( G^0 = Z_G(d) \) is compact if and only if \( d \) is elliptic. Therefore, even if we require only \( G^0 \) to be compact, instead of the whole group \( G \), Theorem 6.15(i) fails.

(b) We illustrate another example, where Theorem 6.15(ii) fails. We consider the group \( G := \text{SU}_1,2(\mathbb{C}) \) and \( \alpha_i(g) = \exp(td)g\exp(-td) \) for \( d := i \text{diag}(1, -1, -1) \). Then \( \alpha \) has a compact group of fixed points
\[ G^0 \cong S(U_1(\mathbb{C}) \times U_2(\mathbb{C})) = \{ (\det(g)^{-1}, g) \in U_1(\mathbb{C}) \times U_2(\mathbb{C}) : g \in U_2(\mathbb{C}) \} \cong U_2(\mathbb{C}) . \]
The subspace \( t \) of diagonal matrices in \( g \) is a compactly embedded Cartan subalgebra. For a linear functional on \( t \), represented by
\[ \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 \]
and satisfying \( \lambda_1 > \lambda_3 \), the condition for the existence of a corresponding unitary highest weight representation is
\[ \lambda_3 - \lambda_1 \geq 1 \]
(cf. [NO98, Lemma I.6]).

However, the condition \( C_\alpha \subseteq C_\pi^0 \), only implies that \( \lambda_3 \geq \lambda_1 \). Therefore \( \lambda = (0, 1, 0) \) defines an irreducible representation of \( G^0 \) on \( \mathcal{H}^0 = \mathbb{C}^2 \) by \( \pi^0(g_1, g_2) = g_2 \), for which \( C_\alpha \subseteq C_\pi^0 \), but \( \pi^0 \) does not extend to a ground state representation. Therefore the conclusion of Theorem 6.15(ii) fails. Similar arguments apply to all hermitian Lie algebras of real rank \( r \geq 2 \).
7 Compact non-Lie groups

In this short subsection we show that the results on compact Lie groups in Section 6.2 can be extended to general compact groups.

Let $G$ be a compact group. Then the group $\text{Aut}(G)$ is a topological group with respect to the initial topology defined by the map $\text{Aut}(G) \to C(G, G)^{\mathbb{Z}_2}, \varphi \mapsto (\varphi, \varphi^{-1})$, where $C(G, G)_{co}$ denotes the set $C(G, G)$, endowed with the compact open topology. The continuity of the evaluation map $ev: C(G, G) \times G \to G$ implies that the action of $\text{Aut}(G)$ on $G$ is continuous.

A homomorphism $\alpha: \mathbb{R} \to \text{Aut}(G)$ is continuous if and only if it is continuous as a map into $C(G, G)_{co}$, which in turn is equivalent to the continuity if the corresponding action map

$$\alpha^\wedge: \mathbb{R} \times G \to G, \quad (t, g) \mapsto \alpha_t(g).$$

Here we use the Exponential Law for locally compact spaces, asserting that the map

$$C(X, C(Y, Z)_{co}) \to C(X \times Y, Z)_{co}, \quad f \mapsto f^\wedge, \quad f^\wedge(x, y) := f(x)(y) \quad (34)$$

is a homeomorphism ([GN, Prop. A.5.17]).

The Lie algebra $\mathfrak{g}$ of $G$ can be identified as a topological space with

$$\mathfrak{L}(G) := \text{Hom}(\mathbb{R}, G) \subseteq C(\mathbb{R}, G)_{co}.$$

Lemma 7.1. For a compact connected group $G$, the following assertions hold:

(i) The adjoint action $\text{Ad}: \text{Aut}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G), \text{Ad}(\varphi, \gamma) := \text{Ad}_\varphi \gamma := \varphi \circ \gamma$ is continuous.

(ii) Every continuous $\mathbb{R}$-action $\alpha$ on $G$ defines a continuous action on $\mathfrak{L}(G)$ by the automorphisms $\text{Ad}^\alpha_t \gamma := \alpha_t \circ \gamma$. Moreover, there exists a filter basis $(Q_j)_{j \in J}$ of $\alpha$-invariant closed subgroups such that $G/Q_j$ is a compact Lie group and $G \cong \varprojlim G/Q_j$.

Proof. (i) By the Exponential Law, the continuity of this map is equivalent to the continuity of the map

$$\text{Ad}^\wedge: \text{Aut}(G) \times \mathfrak{L}(G) \times \mathbb{R} \to G, \quad \text{Ad}^\wedge(\varphi, \gamma, t) := \varphi(\gamma(t)).$$

The continuity of this map follows from the continuity of the action of $\text{Aut}(G)$ on $G$ and the continuity of the evaluation map $\mathfrak{L}(G) \times \mathbb{R} \to G$.

(ii) follows immediately from (i). As a topological Lie algebra, $\mathfrak{g}$ is a projective limit of finite dimensional Lie algebras $(\mathfrak{g}_j)_{j \in J}$ ([HM07, Thm. 2.20, Lemma 3.20]) and, accordingly, its topological dual $\mathfrak{g}' \cong \varprojlim \mathfrak{g}'_j$ is a directed union of the finite dimensional subspaces $\mathfrak{g}'_j$.

We have seen above that any continuous one-parameter group $\alpha: \mathbb{R} \to \text{Aut}(G)$ defines a continuous action on $\mathfrak{g} \cong \mathfrak{L}(G)$. On its topological dual, we thus obtain a one-parameter group $\beta: \mathbb{R} \to \text{GL}(\mathfrak{g}')$ with continuous orbit maps with respect to the weak-* topology. As the weak-* topology on $\mathfrak{g}'$ is the finest locally convex topology ([HM07, Thm. A2.8]), Proposition D.1 shows that every $\mathfrak{g}'_j \subseteq \mathfrak{g}'$ is contained in a finite dimensional $\beta$-invariant subspace $W_j = \text{span}\{\beta_t(\mathfrak{g}'_j): t \in \mathbb{R}\}$.

Let $q_j: \mathfrak{g} \to \mathfrak{g}_j$ denote the quotient maps. Then $\mathfrak{g}'_j \cong \ker(q_j) \subseteq \mathfrak{g}'$ and thus

$$W_j^\perp = \bigcap_{t \in \mathbb{R}} \beta_t(\mathfrak{g}'_j)^\perp = \bigcap_{t \in \mathbb{R}} \text{Ad}_{\alpha_t}(\ker q_j) \subseteq \ker q_j$$

is an ideal of finite codimension. We further have $\bigcap_j W_j^\perp = \{0\}$. For the closed normal subgroups $Q_j := \overline{\exp(W_j^\perp)} \subseteq G$, the quotient $G/Q_j$ is a finite dimensional compact Lie group and $G \cong \varprojlim G/Q_j$. □
By the preceding lemma, we may write a compact connected Lie group $G$ as $\lim G/Q_j$, where the normal subgroups are $\alpha$-invariant and each $G_j = G/Q_j$ is a compact connected Lie group that inherits an $\mathbb{R}$-action $\alpha_j$ from $\alpha$.

Alternatively, this can be derived from the structure theory for the topological group $\text{Aut}(G)$ developed in [HM06, p. 264]. For a compact group $G$, the group $\text{Aut}(G)_0 \cong G'/Z(G')$ is a compact group with Lie algebra $g'$. Hence every continuous one-parameter group $\alpha : \mathbb{R} \to \text{Aut}(G)$ is obtained by the conjugation action of a one-parameter group $\gamma : \mathbb{R} \to G'$. In particular, all normal subgroups are $\alpha$-invariant. All this follows from the following structure theorem:

**Theorem 7.2.** ([HM06, Cor. 9.87]) Let $G$ be a compact connected group with maximal pro-torus $T$ and write $\text{Inn}(G) \subseteq \text{Aut}(G)$ for the subgroup of inner automorphisms. Then there is a totally disconnected closed subgroup $D$ of $\text{Aut}(G)$ contained in the normalizer

$$N_{\text{Aut}(G)}(T) = \{ \alpha \in \text{Aut}(G) : \alpha(T) = T \}$$

of $T$ in $\text{Aut}(G)$ such that

$$\text{Aut}(G) = \text{Inn}(G) \cdot D, \quad \text{Inn}(G) \cap D = \{ \text{id}_G \}, \quad \text{Aut}(G) = \text{Inn}(G) \rtimes \text{Out}(G), \quad \text{Out}(G) \simeq D.$$

In particular, $\text{Out}(G)$ is totally disconnected, $\text{Inn}(G) \simeq G/Z(G)$ is compact connected semisimple center-free and isomorphic to $G'/Z(G')$.

**Theorem 7.3.** Let $G$ be a connected compact group and $\alpha : \mathbb{R} \to \text{Aut}(G)$ be a continuous one-parameter group. Then every continuous unitary representation of $G$ is a strict ground state representation for $(G, \alpha)$.

**Proof.** We have $G \cong \lim G_j$ for compact connected Lie groups $G_j$ and quotient maps $q_j : G \to G_j$. By Lemma 7.1, we may assume that the kernels of the quotient maps $q_j$ are $\alpha$-invariant, so that we obtain $\mathbb{R}$-actions $\alpha^j$ on the connected compact Lie groups $G_j$.

By [Ne10, Thm. 12.2], every continuous unitary representation $(\pi, \mathcal{H})$ of $G$ is a direct sum of subrepresentations $(\pi_k, \mathcal{H}_k)$ with $\ker q_{j_k} \subseteq \ker \pi_k$ for some $j_k$. Lemma 2.9 now shows that a unitary representation $(\pi, \mathcal{H})$ of $G$ is a ground state representation if and only if this holds for all representations of Lie quotient groups $G_j$, and for these the assertion follows from Theorem 6.15.

Since any finite dimensional representation of $G$ factors through a representation of a Lie quotient group, the same argument as in the proof of Theorem 6.15 shows that the injectivity of the map $\tilde{G} \to \tilde{G}^0, [\pi] \mapsto [\pi^0]$ for compact connected Lie groups implies the same for $G$. Now we can argue as in the proof of Theorem 6.15(iii) to see that $(\pi, \mathcal{H})$ is a strict ground state representation. \[\square\]

## 8 Ground state representations of direct limits

After some general observation about direct limits, we discuss in this section some examples that show how the results on strict ground state representations on compact connected Lie groups (Theorem 6.15) can be extended to direct limits of such groups.

### 8.1 Ground state representations

Here we use the unique extension property of compact connected Lie groups $G$ to extend Theorem 6.15 to direct limit groups $G = \lim G_n$. By Glöckner’s Theorem [G05], countable direct limits of directed systems of connected finite dimensional Lie groups always exist in the category.
of locally convex Lie groups, and the so obtained Lie group is also the direct limit in the category of topological groups. Under additional assumptions on the \( G_n \), this provides in particular a classification of ground state representations in term of the corresponding representations \((\pi^0, \mathcal{H}^0)\) of the subgroup \( G^0 \). For abelian \( G^0 \) this leads in particular to direct integrals of lowest weight representations, but many other classes of representations appear naturally.

**Assumption:** In the following we assume that each subgroup \( G_n \subseteq G \) is \( \alpha \)-invariant and put

\[
\alpha_n(t) := \alpha(t)|_{G_n} \quad \text{for} \quad t \in \mathbb{R}.
\]

By Proposition D.1, the groups \( G_n \) can always be chosen such that this is the case. We also assume that the infinitesimal generators \( D_n \in \text{der}(\mathfrak{g}_n) \) of the corresponding one-parameter subgroups of \( \text{Aut}(\mathfrak{g}_n) \) are semisimple, so that \( \mathfrak{g}_n = D_n(\mathfrak{g}_n) \oplus \ker(D_n) \) and (L1)-(L4) in Section 3 are satisfied. In particular, the cone \( C_{\alpha_n} \) in (6) is defined.

**Lemma 8.1.** If all pairs \((G_n, \alpha_n)_{n \in \mathbb{N}}\) have the unique extension property for ground state representations, then so does \((G, \alpha)\).

**Proof.** Let \((\pi_j, \mathcal{H}_j)_{j=1,2}\) be two ground state representations of \( G \) and \( \varphi : \mathcal{H}_1^0 \to \mathcal{H}_2^0 \) be a unitary \( G^0 \)-equivalence. We write \( \mathcal{H}_j^n := [[\pi_j(G_n)\mathcal{H}_j^0]] \) for \( j = 1,2 \). As each \((G_n, \alpha_n)\) has the unique extension property and the representation of \( G_n \) on \( \mathcal{H}_j^n \) is ground state, there exist uniquely determined unitary \( G_n \)-equivaleces \( \Phi_n : \mathcal{H}_1^n \to \mathcal{H}_2^n \) extending \( \varphi \). For \( n < m \) uniqueness implies that \( \Phi_m|_{\mathcal{H}_1^n} = \Phi_n \). Therefore the \( \Phi_n \) combine to a unitary \( G \)-equivalence \( \Phi : \mathcal{H}_1 \to \mathcal{H}_2 \).

**Theorem 8.2.** Assume that each pair \((G_n, \alpha_n)\) has the unique extension property and that the condition \( C_{\alpha_n} \subseteq C_{\pi^0} \) (see (6)) is sufficient for the extendability of a unitary representation \((\pi^0, \mathcal{H}^0)\) of \( G^0 \) to \( G_n \). Then every unitary representation \((\pi^0, \mathcal{H}^0)\) of \( G^0 = \lim_{\to} G_n \) satisfying \( C_{\alpha} \subseteq C_{\pi^0} \) extends uniquely to a strict ground state representation of \( G \).

**Proof.** Each restriction \((\pi_0^0, \mathcal{H}^0)\) is a unitary representation of \( G_n \) satisfying \( C_{\alpha_n} \subseteq C_{\pi_0^n} \), hence extends uniquely to a unitary representation \((\pi_n, \mathcal{H}_n)\) of \( G_n \).

If \( n < m \), then \( \pi_0^n = \pi_0^m|_{\mathcal{H}_n^0} \), so that the unique extension property implies that the \( G_n \)-representation on the subspace \([\pi_0^m(G_n)\mathcal{H}^0]\) \( \subseteq \mathcal{H}_m \) is \( G_n \)-equivalent to \((\pi_n, \mathcal{H}_n)\). We thus obtain a natural isometric \( G_n \)-equivariant inclusion

\[
\varphi_{mn} : \mathcal{H}_n \to \mathcal{H}_m \quad \text{with} \quad \varphi_{mn}|_{\mathcal{H}_0^n} = \text{id}_{\mathcal{H}_0^n}.
\]

Here we identify \( \mathcal{H}^0 \) with a subspace of every space \( \mathcal{H}_n \). These embeddings also intertwine the minimal unitary one-parameter groups \( U_{n,t}^0 := e^{itH_n} \) implementing \( \alpha \) on \( \mathcal{H}_n \) and satisfying \( \mathcal{H}_n^0 = \mathcal{H}^0 = \ker H_n \).

For \( n < m < k \), we then have \( \varphi_{km} \circ \varphi_{mn} = \varphi_{kn} \), so that we obtain a unitary direct limit representation \((\pi, \mathcal{H}) = \lim_{\to} (\pi_n, \mathcal{H}_n)\) of the direct limit group \( G = \lim_{\to} G_n \). Since the restriction of this presentation is continuous on every \( G_n \) and \( G \) carries the direct limit topology, \( \pi \) is continuous. By construction, it is a ground state representation of \((G, \alpha)\). Lemma 8.1 implies that \((G, \alpha)\) has the unique extension property, so that the strictness of \((\pi, \mathcal{H})\) follows from Proposition 2.19.

### 8.2 Some infinite dimensional unitary groups

We consider the Lie group \( G := U_\infty(\mathbb{C}) = \lim_{\to} U_n(\mathbb{C}) \) and \( \alpha_t \in \text{Aut}(G) \) determined by

\[
\alpha_t(g) = e^{i\mathbf{d} g} e^{-i\mathbf{d}}, \quad \mathbf{d} = \text{diag}(i \cdot d_n)_{n \in \mathbb{N}}.
\]
Then \( G^0 \subseteq G \) is the subgroup preserving all eigenspaces of the diagonal operator \( \mathbf{d} \) on \( \mathbb{C}^{(n)} \).

We assume that the \( (d_n)_{n \in \mathbb{N}} \) are mutually different, so that

\[
T := G^0 \cong \mathbb{T}^{(n)}
\]

is the subgroup of diagonal matrices in \( G \). So \( G \) is a direct limit of the compact subgroups \( G_n \cong U_n(\mathbb{C}) \) and the abelian group \( G^0 \) is the direct limit of the tori \( T_n := T \cap U_n(\mathbb{C}) \cong \mathbb{T}^n \). Accordingly, the character group

\[
\hat{G}^0 \cong \lim \hat{T}^n \cong \lim \mathbb{Z}^n \cong \mathbb{Z}^N
\]

carries a natural totally disconnected, metrizable group topology. Taking the differential in \( \hat{T} \) identify the character group \( \hat{G} \) is the subgroup of diagonal matrices in \( U \) are eigenvectors of \( D \).

For \( d_n > d_m \) we thus obtain the generator

\[
-i[E_{nm}, E_{nm}] = -i[E_{nn}, E_{nm}] = -i(E_{mm} - E_{nn}),
\]

so that

\[
C_{\alpha} = i \text{ cone}\{ E_{nn} - E_{mm} : d_n > d_m \}.
\]

Hence \( C_{\alpha} \subseteq C_{\pi^0} \) is equivalent to

\[
\partial \pi^0(E_{nn} - E_{mm}) \geq 0 \quad \text{for} \quad d_n > d_m.
\]  

Since \( G^0 \) is abelian, all irreducible representations are one-dimensional. A character \( \chi_\lambda \in \hat{G}^0 \) with \( \chi_\lambda(\exp x) = e^{2\pi i \lambda(x)} \), \( \lambda \in \mathbb{R}^N \), satisfies the positivity condition (36) if and only if

\[
\lambda_n - \lambda_m \geq 0 \quad \text{for} \quad d_n > d_m.
\]  

We write \( \hat{T}(\mathbf{d}) \subseteq \hat{T} \cong \mathbb{Z}^N \) for the closed subgroup of all characters satisfying this condition.

By Theorems 8.2 and 3.11, a representations \((\pi^0, \mathcal{H}^0)\) of \( G^0 \) is extendable to a ground state representations of \( G \) if and only if \( C_{\alpha} \subseteq C_{\pi^0} \). In view of [Ba91, Prop. 7.9], the abelian group \( T = G^0 \) is nuclear because it is Hausdorff and a countable direct limit of compact abelian groups (which are nuclear). Therefore the Bochner Theorem for nuclear groups ([Ba91, Ch. 4] implies that its unitary representations are given by Borel spectral measures on the character group \( \hat{T} \), endowed with the topology of pointwise convergence, which is the product Borel structure on \( \hat{T} \cong \mathbb{Z}^N \). Therefore \( \pi^0 \) extends to a ground state representation if and only if its spectral measure is supported by the closed subset \( \hat{T}(\mathbf{d}) \). In particular, general ground state representations for \((G, \alpha)\) are direct integrals of unitary highest weight representations \((\pi_\lambda, \mathcal{H}_\lambda)\) with \( \lambda \) satisfying (37).

Remark 8.3. The classification results for unitary highest weight representations in [MN16] also fit into this context. In that paper one finds a description of all pairs \((\lambda, \mathbf{d})\), where \( \lambda \in \mathbb{R}^n \) and \( D = \mathbf{ad} \mathbf{d} \), for which the unitary highest weight representation \( L(\lambda) \) of \( \mathfrak{gl}_\infty(\mathbb{C}) \) with highest weight \( \lambda \) carries a positive energy representations for \((G, \alpha)\). This amounts to the condition that

\[
\inf\langle W\lambda - \lambda, -\mathbf{id} \rangle > -\infty
\]

holds for the corresponding Weyl group \( W \) (the group of all finite permutations in the present example). This condition is equivalent to the unitary highest weight representation \((\pi_\lambda, \mathcal{H}_\lambda)\) to be a ground state representation for \((G, \alpha)\), where the minimal eigenvalue of \(-\mathbf{id}\) is \( \lambda(\mathbf{id}) \).
Remark 8.4. The are also weight representations of $G = U\infty(\mathbb{C})$ which have no extremal weight, but which are ground state representations. We refer to [DP99] and [DMP00] for classification results for weight modules $V$ of $\mathfrak{su}_\infty(\mathbb{C}) \cong \mathfrak{sl}_\infty(\mathbb{C})$. As we have seen in [Ne04, Ex. V.9], some weight representations define bounded representations with no extremal weights. In particular, there are weight modules $V$ whose weight set $\mathcal{P}_V$ consists of all functionals of the form

$$-i\alpha = \chi_N - \chi_M, \quad \text{with} \quad N, M \subseteq \mathbb{N}, \quad N \cap M = \emptyset, \quad |N| = |M| < \infty.$$ 

As $\mathcal{P}_V = -\mathcal{P}_V$, the operator defined by $-i\mathbf{d} \in \mathbb{R}^N$ on $V$ is semibounded if and only if it is bounded, and this happens if and only if (up to an additive constant) $\mathbf{d}$ has finite support. Write $\mathbf{d} = \mathbf{d}_+ - \mathbf{d}_-$, where $\mathbf{d}_\pm$ are non-negative with finite support. Then

$$-i\alpha(\mathbf{d}) = \chi_N(\mathbf{d}) - \chi_M(\mathbf{d}) = \chi_N(\mathbf{d}_+) - \chi_N(\mathbf{d}_-) - \chi_M(\mathbf{d}_+) + \chi_M(\mathbf{d}_-)$$

is minimal if $\text{supp}(\mathbf{d}_+) \subseteq M$ and $\text{supp}(\mathbf{d}_-) \subseteq N$, which leads to the minimal value

$$-i\alpha(\mathbf{d}) = -\chi_N(\mathbf{d}_-) - \chi_M(\mathbf{d}_+) = -\sum_{n \in \mathbb{N}} |d_n|.$$ 

In particular, the representation on $V$ is a ground state representation for $G^0 = T$.

Remark 8.5. The situation does not change significantly if we replace the group $U\infty(\mathbb{C})$ by some Banach completion, such as $U_1(\mathcal{H})$ (completion in the trace norm), or the group $U_2(\mathcal{H})$ (completion in the Hilbert–Schmidt norm). The continuous unitary representations of these groups are simply those continuous unitary representations of $G$ which extend to these completions, so that we are dealing with less representations for the larger groups.

Example 8.6. Similar techniques apply to the direct limit $G = \lim_{\rightarrow} G_n$ of the groups $G_n := U_{2^n}(\mathbb{C})$ with their natural embedding, given by the connecting maps $g \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$. Then the abelian group $T := \lim_{\rightarrow} T_n$ can be identified with a subgroup of the mapping group $C(\{0, 1\}^\mathbb{N}, T)$ of $T$-valued functions on the Cantor set $\{0, 1\}^\mathbb{N}$. Here $T_n$ corresponds to the subgroup $C(\{0, 1\}^n, T) \cong T^{2^n}$. Let $f: \{0, 1\}^\mathbb{N} \rightarrow \mathbb{R}$ be an injective function, such as

$$f((a_n)) := \sum_{n=1}^{\infty} a_n 3^{-n}.$$ 

This function can be used to define an automorphism group $(\alpha_t)_{t \in \mathbb{R}}$ of $G$ with $G^0 = T$.

The embedding of $G$ into the $C^*$-algebra $\otimes_{n \in \mathbb{N}} M_2(\mathbb{C})$ leads to bounded representations of $G$, and all these are positive energy representations. Many of them have no ground states.

Example 8.7. The restricted direct product groups

$$G = U_2(\mathbb{C})^\mathbb{N} = \{(g_n)_{n \in \mathbb{N}} \in U_2(\mathbb{C})^\mathbb{N}; \ |\{n \in \mathbb{N}; g_n \neq 1\}| < \infty\}$$

are direct limits of the compact groups $G_n = U_2(\mathbb{C})^n$ are also natural examples. Up to equivalence, any one-parameter automorphism group $\alpha$ of $G$ is acting by $\alpha_t((g_n)) = (\exp(\mathbf{i}d_n g_n, \exp(-\mathbf{i}d_n)))$, where $\mathbf{d}_n = \mathbf{i} \text{diag}(x_n, -x_n) \in \mathfrak{su}_2(\mathbb{C})$ are diagonal matrices ($x_n \in \mathbb{R}$). If all $\mathbf{d}_n$ are non-zero, then

$$T := G^0 \cong (T^2)^\mathbb{N}.$$
is a nuclear abelian group (the subgroup of diagonal matrices). Its irreducible unitary representations are given by its character group
\[ \hat{T} \cong (\mathbb{Z}_2)^N \]
of sequences of pairs of integers \(((k_n, m_n))_{n \in \mathbb{N}}\). If all \(x_n\) are positive, we find that
\[ \hat{T}(\alpha) = \{((k_n, m_n))_{n \in \mathbb{N}} : (\forall n \in \mathbb{N}) \ k_n \geq m_n \} . \]

Let \((\pi_n, \mathcal{H}_n)\) denote the irreducible representation of \(U_2(\mathbb{C})\) with highest weight \((k_n, m_n)\) and \(v_n \in \mathcal{H}_n\) be a unit vector of lowest weight, which is a ground state vector for \(d_n\). Then the infinite tensor product
\[ \bigotimes_{n \in \mathbb{N}} (\mathcal{H}_n, v_n) \]
carries an irreducible unitary representation of \(G\) which a corresponding ground state representation. In the same way as for the group \(U_\infty(\mathbb{C})\), we see that the ground state representations of \(G\) correspond to those representations of \(T\) whose spectral measure is supported on the closed subset \(\hat{T}(\alpha)\).

### A Arveson spectral theory

In this appendix we collect the concepts relating to spectral subspaces for the action of a one-parameter group on a complete locally convex space. We follow \cite{Ar74} with some generalizations. We use these concepts for one-paramter groups of automorphisms of infinite dimensional Lie algebras.

#### A.1 Arveson spectral subspaces

**Definition A.1.** Let \(E\) and \(F\) be a locally convex spaces. We denote by \(\text{Hom}(E, F)\) the space of continuous linear maps from \(E\) to \(F\) and write \(\text{End}(E) := \text{Hom}(E, E)\). A subset \(Y \subset \text{Hom}(E, F)\) is called equicontinuous if for every open 0-neighborhood \(U\) in \(F\) there exists a 0-neighborhood \(W\) in \(E\) such that \(T(W) \subset U\) holds for every \(T \in Y\).

**Definition A.2.** (cf. \cite{NSZ15}, \cite{Ne13}) Let \(V\) be a complete complex locally convex space and let \(\alpha : \mathbb{R} \to \text{GL}(V)\), \(t \mapsto \alpha_t\) be a strongly continuous representation.

(a) Assume that \(\alpha\) is polynomially bounded (Definition 3.8), i.e., for every continuous seminorm \(p\), there exists a 0-neighborhood \(U \subset V\) and \(N \in \mathbb{N}\) such that
\[ \sup_{x \in U} \sup_{t \in \mathbb{R}} \frac{p(\alpha_t(x))}{1 + |t|^N} < \infty . \]

We define
\[ \alpha_f(v) := \int_{\mathbb{R}} f(t) \alpha_t(v) \, dt \quad \text{for} \quad v \in V, f \in \mathcal{S}(\mathbb{R}) . \]

Then \(\alpha_f \in \text{End}(V)\) and this yields a representation of the convolution algebra \((\mathcal{S}(\mathbb{R}), \ast)\) on \(V\). We define the spectrum of an element \(v \in V\) by
\[ \text{Spec}_\alpha(v; \mathcal{S}) := \{ y \in \mathbb{R} : (\forall f \in \mathcal{S}(\mathbb{R})) \ \alpha_f v = 0 \Rightarrow \hat{f}(y) = 0 \} . \]
which is the hull of the annihilator ideal of \( v \). Here we use the following version of the Fourier transform:

\[
\hat{f}(y) := \int_{\mathbb{R}} e^{ixy} f(x) \, dx.
\]

(39)

For a **closed** subset \( E \subseteq \mathbb{R} \), we now define the corresponding *Arveson spectral subspace*

\[ V(E; S) := \{ v \in V : \text{Spec}_\alpha(v; S) \subseteq E \}. \]

We define the spectrum of \((\alpha, V)\) by

\[
\text{Spec}_\alpha(V) := \{ y \in \mathbb{R} : (\forall f \in S(\mathbb{R})) \alpha_f = 0 \Rightarrow \hat{f}(y) = 0 \}. \]

We also put

\[
V^+ = \bigcup_{\delta > 0} V([\delta, \infty); S) \quad \text{and} \quad V^- = \bigcup_{\delta > 0} V((-\infty, -\delta]; S). \tag{40}
\]

We say that the **splitting condition** is satisfied if these subspace and the subspace \( V^0 := V(\{0\}) \) of fixed points (cf. Lemma A.6 below) satisfy

\[ V = V^+ \oplus V^0 \oplus V^- \]  

(\text{SC})

(b) If \( \alpha \) is equicontinuous, then (38) exists for all \( f \in L^1(\mathbb{R}) \) and we can define \( \text{Spec}_\alpha(v) \) and \( V(E) \) as above with by \( S(\mathbb{R}) \) replaced by \( L^1(\mathbb{R}) \), see [Ne13, Def. A.5(b)]. This was Arveson’s original context.

**Example A.3.** Suppose that \((\pi, \mathcal{H})\) is a continuous unitary representation of a finite dimensional Lie group \( G \) and \( d \in \mathfrak{g} \) is such that \( \text{Spec}(\text{ad} d) \subseteq i\mathbb{R} \). We claim that, on the Fréchet space \( \mathcal{H}^\infty \) of smooth vectors, the representation of \( \mathbb{R} \), defined by the unitary one-parameter group \( \pi_d(t) := \pi(\exp t d) \) is polynomially bounded (cf. [NSZ15]). The topology on \( \mathcal{H}^\infty \) is defined by the seminorms

\[
p_D(v) := \|d\pi(D)v\|, \quad D \in \mathcal{U}(\mathfrak{g}).
\]

Therefore

\[
p_D(\pi_d(t)v) = \|d\pi(D)\pi(\exp t d)v\| = \|d\pi(e^{-t \text{ad} d} D)v\|,
\]

and this expression has polynomial estimates because \( D \in \mathcal{U}(\mathfrak{g}) \) is contained in a finite dimensional \( \text{ad} d \)-invariant subspace \( F \) on which the one-parameter group \( e^{t \text{ad} d} \) is of polynomial growth.

As a consequence, the operators \( \pi_d(f) = \int_{\mathbb{R}} f(t) \pi_d(t) dt, f \in S(\mathbb{R}) \), leave the subspace \( \mathcal{H}^\infty \) invariant and the spectral subspaces \( \mathcal{H}^\infty(E; S) \) are defined for every closed subset \( E \subseteq \mathbb{R} \) in the sense of Definition A.2.

**Lemma A.4.** ([Ne13, Lemma A.16]) If \( V \) is a Banach space and \( D := \alpha'(0) \) is a bounded operator, i.e., \( \alpha : \mathbb{R} \to \text{Aut}(V) \) is norm continuous, then there exists a \( \delta > 0 \) such that the splitting condition (SC) is satisfied with

\[
V^+ = V([\delta, \infty)) \quad \text{and} \quad V^- = V((-\infty, -\delta])
\]

if and only if 0 is isolated in the spectrum of \( D \).

**Example A.5.** It is easy to see examples where the splitting condition (SC) fails. A very typical one is the Banach space

\[
V := C([-1, 1], \mathbb{C}) \quad \text{with} \quad (\alpha_t h)(x) = e^{itx} h(x).
\]
Then $\alpha_f(h)(x) = \hat{f}(x)h(x)$ shows that

$$\text{Spec}_\alpha(h) = \text{supp}(h) \quad \text{for} \quad h \in V.$$ 

This leads to

$$V^+ = \{ h \in V : h|_{[-1,0]} = 0 \}, \quad V^- = \{ h \in V : h|_{[0,1]} = 0 \} \quad \text{and} \quad V^0 = \{ 0 \}.$$ 

In particular all functions in $V^+ + V^0 + V^-$ vanish in 0, so that this is a proper subspace of $V$.

**Lemma A.6.** ([Ar74, p. 226]) For $\lambda \in \mathbb{R}$, we have

$$V(\{\lambda\}) = V_\lambda(\alpha) := \{ v \in V : (\forall t \in \mathbb{R}) \alpha_t(v) = e^{it\lambda}v \}.$$ 

**Remark A.7.**

(a) In [Ar74, p. 225] it is shown that (in the equicontinuous case)

$$V(E) = \{ v \in V : (\forall f \in L^1(G)) \text{ supp}(\hat{f}) \cap E = \emptyset \Rightarrow \alpha_f(v) = 0 \},$$

which implies in particular that $V(E)$ is a closed subspace, which is clearly $\alpha$-invariant. Note that the condition $\text{supp}(\hat{f}) \cap E = \emptyset$ means that $\hat{f}$ vanishes on a neighborhood of $E$.

(b) If $(E_j)_{j \in J}$ is a family of closed subsets of $\mathbb{R}$, then $V(\bigcap_{j \in J} E_j) = \bigcap_{j \in J} V(E_j)$ follows immediately from the definition.

(c) Lemma A.6 implies in particular that

$$V(\emptyset) = \{ 0 \}$$

because this subspace is contained in every eigenspace $V(\{\lambda\})$, $\lambda \in \mathbb{R}$.

The following proposition is an important technical tool.

**Proposition A.8.** ([Ne13, Prop. A.14]) Assume that $(\alpha_j, V_j)$, $j = 1, 2, 3$ are continuous equicontinuous representations of $\mathbb{R}$ on the complete locally convex complex spaces $V_j$ and that $\beta : V_1 \times V_2 \to V_3$ is a continuous equivariant bilinear map. Then we have for closed subsets $E_1, E_2 \subseteq \mathbb{R}$ the relation

$$\beta(V_1(E_1) \times V_2(E_2)) \subseteq V_3(\overline{E_1 + E_2}).$$

**Example A.9.** Let $\mathfrak{g}$ be a complete locally convex Lie algebra and let $x \in \mathfrak{g}$ be such that $\text{ad} x$ generates a continuous equicontinuous one-parameter group $\alpha : \mathbb{R} \to \text{Aut}(\mathfrak{g})$ of automorphisms, i.e., $\alpha$ is strongly differentiable with $\alpha'(0) = \text{ad} x$. Applying Definition A.2 to the $\mathbb{R}$-representation on $\mathfrak{g}$ defined by $\alpha$, we obtain for each closed subset $E \subseteq \mathbb{R}$ a spectral subspace $\mathfrak{g}C(E)$.

**Lemma A.10.** ([Ne13, Lemma 4.3]) Let $U_t := e^{itA}$ be a strongly continuous unitary one-parameter group with infinitesimal generator $A = A^*$. Then the following assertions hold:

(i) For each $f \in L^1(\mathbb{R})$, we have $U(f) = \hat{f}(A)$.

(ii) Let $P : \mathcal{B}(\mathbb{R}) \to B(H)$ be the unique spectral measure with $A = P(\text{id}_\mathbb{R})$. Then, for every closed subset $E \subseteq \mathbb{R}$, the range space $P(E)H$ coincides with the Arveson spectral subspace $\mathcal{H}(E)$. 

44
A.2 Applications to unitary representations

Proposition A.11. ([Ne13, Prop. 4.4]) Let \((\pi, \mathcal{H})\) be a smooth unitary representation of the Banach–Lie group \(G\), \(d \in \mathfrak{g}\) be elliptic, and \(P: \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})\) be the spectral measure of the unitary one-parameter group \(\pi_d(t) := \exp(\mathfrak{g} t d)\). Then the following assertions hold:

(i) \(\mathcal{H}^\infty\) carries a Fréchet structure for which \(\pi_d\) defines a continuous equicontinuous action of \(\mathbb{R}\) on \(\mathcal{H}^\infty\). In particular, \(\mathcal{H}^\infty\) is invariant under \(\pi_d(f)\) for every \(f \in L^1(\mathbb{R})\).

(ii) For every closed subset \(E \subseteq \mathbb{R}\), we have \(\mathcal{H}^\infty(E) = (P(E)\mathcal{H}) \cap \mathcal{H}^\infty\) for the corresponding spectral subspace.

(iii) For every open subset \(E \subseteq \mathbb{R}\), \((P(E)\mathcal{H}) \cap \mathcal{H}^\infty\) is dense in \(P(E)\mathcal{H}^\infty\). More precisely, there exists a sequence \((f_n)_{n \in \mathbb{N}}\) in \(L^1(\mathbb{R})\) for which \(\pi_d(f_n) \to P(E)\) in the strong operator topology and \(\text{supp}(f_n) \subseteq E\), so that \(\pi_d(f_n)v \in \mathcal{H}^\infty \cap P(E)\mathcal{H}^\infty\) for every \(v \in \mathcal{H}^\infty\).

Theorem A.12. (Spectral translation formula; [NSZ15, Thm. 3.1]) Assume that \(\mathfrak{g}\) is a complete locally convex Lie algebra, \(\alpha: \mathbb{R} \to \text{Aut}(G)\) defines a continuous action of \(\mathbb{R}\) on \(G\), and that the induced action on \(\mathfrak{g}\) is also continuous. Let \(\pi^\alpha(g, t) = \pi(g) U_t\) be a continuous unitary representation of \(G \times_{\alpha} \mathbb{R}\) on \(\mathcal{H}\) and let \(\mathcal{H}^\infty\) be the space of smooth vectors with respect to \(\pi\).

(i) If \(\alpha\) is equicontinuous, then
\[
d\pi(\mathfrak{g}(E))\mathcal{H}^\infty(F) \subseteq \mathcal{H}^\infty(E + F) \quad \text{for} \quad E, F \subseteq \mathbb{R} \quad \text{closed.}
\]

(ii) If \(\alpha\) is polynomially bounded, then
\[
d\pi(\mathfrak{g}(E; S))\mathcal{H}^\infty(F) \subseteq \mathcal{H}^\infty(E + F) \quad \text{for} \quad E, F \subseteq \mathbb{R} \quad \text{closed.}
\]

B Positive definite kernels

Let \(X\) be a set and \(\mathcal{E}\) a Hilbert space. A Hilbert subspace \(\mathcal{H} \subseteq \mathcal{E}^X\) of the linear space of \(\mathcal{E}\)-valued functions on \(X\) is said to have continuous point evaluations if the linear maps
\[
K_x: \mathcal{H} \to \mathcal{E}, \quad f \mapsto f(x)
\]
are continuous. Then the function
\[
K: X \times X \to B(\mathcal{E}), \quad K(x, y) := K_x K_y^*
\]
is called its reproducing kernel. As the kernel \(K\) determines the subspace \(\mathcal{H} \subseteq \mathcal{E}^X\) and its scalar product uniquely, we write \(\mathcal{H}_K \subseteq \mathcal{E}^X\) for the Hilbert space determined by \(K\) and \(\mathcal{H}_K^1 \subseteq \mathcal{H}_K\) for the subspace spanned by the functions \(K_y v, y \in X, v \in \mathcal{E}\). A kernel function \(K: X \times X \to B(\mathcal{E})\) is the reproducing kernel of some Hilbert space if and only if it is positive definite in the sense that, for any finite collection \((x_1, v_1), \ldots, (x_n, v_n) \in X \times \mathcal{E}\), the matrix \((K(x_j, x_k)v_k)_{1 \leq j, k \leq n}\) is positive semidefinite (cf. [Ne00, Ch. 1]).

If \(X = G\) is a group and the kernel \(K: G \times G \to \mathcal{E}\) is invariant under right translations, then it is of the form \(K(g, h) = \varphi(gh^{-1})\) for a function \(\varphi: G \to B(\mathcal{E})\). Such functions are called positive definite if the kernel \(K\) is positive definite.

The following proposition generalizes the well-known GNS construction to operator-valued functions on groups (cf. [NO18]).
Proposition B.1. (GNS-construction for groups) Let $E$ be a Hilbert space.

(a) Let $\varphi: G \to B(E)$ be a positive definite function with $\varphi(e) = 1_E$. Then $(U_\varphi^g f)(h) := f(hg)$ defines a unitary representation of $G$ on the reproducing kernel Hilbert space $H_\varphi := \mathcal{H}_K \subseteq E^G$ with kernel $K(g, h) = \varphi(gh^{-1})$ and the range of the isometric inclusion $K^*: E \to \mathcal{H}$ is a $G$-cyclic subspace, i.e., $U_\varphi^g K^* e$ spans a dense subspace of $\mathcal{H}$. We then have

$$\varphi(g) = K Ke U_\varphi g K_*^* e \quad \text{for} \quad g \in G. \quad (41)$$

(b) If, conversely, $(U, \mathcal{H})$ is a unitary representation of $G$ and $j: E \to \mathcal{H}$ is an isometric inclusion, then

$$\varphi: G \to B(E), \quad \varphi(g) := j^* U_g j$$

is a $B(E)$-valued positive definite function. If, in addition, $j(E)$ is cyclic, then $(U, \mathcal{H})$ is unitarily equivalent to $(U_\varphi^e, \mathcal{H}_\varphi)$.

C Bosonic Fock space

We start with the construction of the von Neumann algebras on the bosonic Fock space. For $v_1, \ldots, v_n \in \mathcal{H}$, we define

$$v_1 \cdots v_n := v_1 \vee \cdots \vee v_n := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

and $v^n := \vee^n v$, so that

$$\langle v_1 \vee \cdots \vee v_n, w_1 \vee \cdots \vee w_m \rangle = \sum_{\sigma \in S_n} \langle v_{\sigma(1)}, w_1 \rangle \cdots \langle v_{\sigma(n)}, w_m \rangle. \quad (42)$$

For every $v \in \mathcal{H}$, the series $\text{Exp}(v) := \sum_{n=0}^{\infty} \frac{1}{n!} v^n$ defines an element in $\mathcal{F}_+(\mathcal{H})$ and the scalar product of two such elements is given by

$$\langle \text{Exp}(v), \text{Exp}(w) \rangle = \sum_{n=0}^{\infty} \frac{n!}{(n!)^2} \langle v, w \rangle^n = e^{\langle v, w \rangle}.$$

These elements span a dense subspace of $\mathcal{F}_+(\mathcal{H})$, and therefore we have for each $x \in \mathcal{H}$ a unitary operator on $\mathcal{F}_+(\mathcal{H})$ determined by the relation

$$U_x \text{Exp}(v) = e^{-\langle x, v \rangle - \frac{|x|^2}{2}} \text{Exp}(v + x) \quad \text{for} \quad x, v \in \mathcal{H}. \quad (43)$$

A direct calculation then shows that

$$U_x U_y = e^{-i \text{Im}(x, y)} U_{x+y} \quad \text{for} \quad x, y \in \mathcal{H}. \quad (44)$$

To obtain a unitary representation, we have to replace the additive group of $\mathcal{H}$ by the Heisenberg group

$$\text{Heis}(\mathcal{H}) := T \times \mathcal{H} \quad \text{with} \quad (z, v)(z', v') := (zz'e^{-i \text{Im}(v, v')}, v + v').$$

For this group, we obtain with (44) a unitary representation

$$U: \text{Heis}(\mathcal{H}) \to U(\mathcal{F}_+(\mathcal{H})) \quad \text{by} \quad U_{(z, v)} := z U_v.$$
In this physics literature, all this is expressed in terms of the so-called Weyl operators

\[ W(v) := U_{iv/\sqrt{2}}, \quad v \in \mathcal{H} \]

satisfying the Weyl relations

\[ W(v)W(w) = e^{-i\text{Im}\langle v,w \rangle/2}W(v+w), \quad v, w \in \mathcal{H}. \]  

(45)

We also note that the vacuum vector \( \Omega := \text{Exp}(0) \in \mathcal{F}_+^c(\mathcal{H}) \) satisfies

\[ \langle \Omega, U_x\Omega \rangle = \langle \Omega, e^{-\|x\|^2/2}\text{Exp}(x) \rangle = e^{-\|x\|^2/2}. \]

Remark C.1. If \((V, \sigma)\) is a symplectic vector space, then the corresponding Weyl algebra \(C^*(V, \sigma)\) is the universal \(C^*\)-algebra with unitary generators \((W(v))_{v \in V}\), and the relations

\[ W(v_1)W(v_2) = e^{i\sigma(v_1,v_2)}W(v_1 + v_2) \]

([BR96, Thm. 5.2.8]).

The representations of this \(C^*\)-algebra are in one-to-one correspondence with the unitary representations \((\pi, \mathcal{H})\) of Heis\((V, \sigma)\) satisfying \(\pi(z,0) = z1\) for \(z \in \mathbb{T}\). An injective representation of Heis\((V, \sigma)\) is obtained on \(\ell^2(V)\) by

\[ W(x)\delta_y = e^{-i\sigma(x,y)}\delta_y, \quad x, y \in V, \quad \text{where} \quad \delta_y(z) = \delta_{yz} \]

is the canonical orthonormal basis in \(\ell^2(V)\). As the \(C^*\)-algebra \(C^*(V, \sigma)\) is simple by [BR96, Thm. 5.2.8], all its representations are injective, and therefore the corresponding representations \(\pi\) of Heis\((V, \sigma)\) are injective as well.

D Spaces with the finest locally convex topology

Let \(V\) be a countably dimensional real vector space, carrying the finest locally convex topology. This is the locally convex topology for which all seminorms \(p: V \to \mathbb{R}_+\) are continuous. So a net \((x_j)_{j \in J}\) converges in \(V\) to \(x\) if and only if, for every seminorm \(p\) on \(V\), we have \(p(x_j - x) \to 0\). From any basis of \(V\), we obtain an increasing sequence \((V_n)_{n \in \mathbb{N}}\) of finite dimensional linear subspaces for which \(V = \bigcup_n V_n\), and the topology on \(V\) is the direct limit topology with respect to the subspaces \(V_n\) ([GN, Ex. B.13.3]), i.e., a subset \(O \subseteq V\) is open if and only if \(O \cap V_n\) is an open subset of \(V_n\) for every \(n \in \mathbb{N}\). We refer to the survey paper [GGH10] for a discussion of more general final topologies on topological groups.

Proposition D.1. Let \(V\) be a real vector space, endowed with the finest locally convex topology, i.e., all seminorms on \(V\) are continuous. If \(\alpha: \mathbb{R} \to \text{GL}(V)\) is a homomorphism defining an action of \(\mathbb{R}\) on \(V\) with continuous orbit maps, then

(i) all \(\alpha\)-orbits lie in finite dimensional subspaces, and

(ii) there exists a locally finite endomorphism \(D\) such that \(\alpha_t = e^{tD}\) for all \(t \in \mathbb{R}\).

\(^4\)An endomorphism \(D \in \text{End}(V)\) is called locally finite if each \(v \in V\) is contained in a finite dimensional \(D\)-invariant subspace. Then \(e^{Dv} = \sum_{k=0}^{\infty} \frac{1}{k!}D^kv\) is defined and \((e^{tD})_{t \in \mathbb{R}}\) defines a one-parameter group of GL\((V)\).
Proof. Let \( v \in V \). Then \( \alpha_{[-1,1]}v \subseteq V \) is a compact subset, hence contained in a finite dimensional subspace \( W \) ([HM06, Prop. 7.25(iv)]). For \( f \in C_c^\infty(\mathbb{R}) \) with \( \text{supp}(f) \subseteq [-1,1] \) this implies that

\[
\alpha(f)v := \int_\mathbb{R} f(t)\alpha_t(v) \, dt = \int_{-1}^1 f(t)\alpha_t(v) \, dt \in W.
\]

For \( 0 < \varepsilon \leq 1 \), let \( W_\varepsilon \subseteq W \) denote the subspace generated by \( \alpha(f)v \) for \( \text{supp}(f) \subseteq [-\varepsilon,\varepsilon] \). Then \( W_\varepsilon \subseteq W_{\varepsilon'} \) for \( \varepsilon < \varepsilon' \), and by the finiteness of \( \dim W \), there exists an \( \varepsilon_0 \in (0,1) \) for which \( W_{\varepsilon_0} \) is minimal. Then \( W_\varepsilon = W_{\varepsilon_0} \) for \( 0 < \varepsilon \leq \varepsilon_0 \). As \( \alpha(\delta_n)v \to v \) for any sequence \( (\delta_n)_{n \in \mathbb{N}} \) in \( C_c^\infty(\mathbb{R}, [0, \infty)) \) with \( \text{supp}(\delta_n) \subseteq [-\frac{1}{n}, \frac{1}{n}] \) and \( \int_{\mathbb{R}} \delta_n = 1 \), it follows that \( v \in W_{\varepsilon_0} \). Since \( W_{\varepsilon_0} \) consists of smooth vectors for \( \alpha \), this implies in particular that \( v \in V^\infty \).

We conclude that all orbit maps in \( V \) are smooth. Therefore \( Dw := \left. \frac{d}{dt} \right|_{t=0} \alpha_t w \) defines an element of \( \text{End}(V) \). From \( \alpha_t(v) \subseteq W \) for \( |t| \leq 1 \), it follows that \( D^k v \in W \) for \( k \in \mathbb{N} \). Therefore \( U := \text{span}\{D^k v : k \in \mathbb{N}_0\} \) is a finite dimensional \( D \)-invariant subspace of \( W \). Let \( D_U := D|_U \). Then

\[
\frac{d}{dt}(\alpha_t e^{tD_U}v) = \alpha_t(-D + D_U)e^{tD_U}v = 0,
\]

so that \( \alpha_t v = e^{tD_U}v \). This shows that \( U \) is \( \alpha \)-invariant and that \( \alpha_t = e^{tD} \) holds in the sense of exponentials of locally finite operators. \( \square \)

References

[Ar74] Arveson, W., *On groups of automorphisms of operator algebras*, J. Funct. Anal. **15** (1974), 217–243

[Ba91] Banaszczyk, W., *“Nuclear Groups,”* Lecture Notes in Mathematics 1466, Springer, Berlin, 1991

[BGN20] Beltită, D., Grundling, H., and K.-H. Neeb, *State space geometry and covariant representations for singular actions on C*-algebras*, Diss. Math. **549** (2020), 1–94; arXiv:math.OA:1708.01028

[BN08] Beltită, D., and K.-H. Neeb, *A non-smooth continuous unitary representation of a Banach–Lie group*, J. Lie Theory **18** (2008), 933–936

[BRT07] Beltită, D., T. S. Ratiu, and A. B. Tumpach, *The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits*, J. Funct. Anal. **247** (1) (2007), 138–168

[BR02] Bratteli, O., and D.W. Robinson, *“Operator Algebras and Quantum Statistical Mechanics I,”* 2nd ed., Texts and Monographs in Physics, Springer, Berlin, 2002

[BR96] Bratteli, O., and D. W. Robinson, *“Operator Algebras and Quantum Statistical Mechanics II,”* 2nd ed., Texts and Monographs in Physics, Springer, Berlin, 1996

[DP99] Dimitrov, I., and I. Penkov, *Weight modules of direct limit Lie algebras*, Int. Math. Res. Not. **5** (1999), 223–249

[DMP00] Dimitrov, I., O. Mathieu and I. Penkov, *On the structure of weight modules*, Trans. Amer. Math. Soc. **352** (6) (2000), 2857–2869, Erratum in Trans. Amer. Math. Soc. **356** (8) (2004), 3403–3404
[Gl05] Glöckner, H., *Fundamentals of direct limit Lie theory*, Compos. Math. 141 (2005), 1551–1577 3, 5, 38

[GGH10] Glöckner, H., R. Gramlich, and T. Hartnick, *Final Group Topologies, Kac-Moody Groups and Pontryagin Duality*, Israel J. Math. 177 (2010), 49–101 47

[GN] Glöckner, H., and K.-H. Neeb, “Infinite dimensional Lie groups, Vol. I, Basic Theory and Main Examples,” book in preparation, 2021 24, 37, 47

[HN12] Hilgert, J., and K.-H. Neeb, “Structure and Geometry of Lie Groups,” Springer, Berlin, 2012 33

[HNO94] Hilgert, J., K.-H. Neeb, and B. Ørsted, *The geometry of nilpotent orbits of convex type in hermitian Lie algebras*, J. Lie Theory 4 (2) (1994), 185–235 30, 32

[Ho81] Hochschild, G. P., “Basic Theory of Algebraic Groups and Lie Algebras,” Graduate Texts in Mathematics 75, Springer, Heidelberg, 1981 30

[HM06] Hofmann, K.H., and S.A. Morris, “The Structure of Compact Groups,” de Gruyter, Berlin, 2013. 21, 38, 48

[HM07] Hofmann, K.H., and S.A. Morris, “The Lie Theory of Connected Pro-Lie Groups,” Tracts in Mathematics 2, European Mathematical Society, Zürich, 2007 37

[JN19] Janssens, B., and K.-H. Neeb, *Projective unitary representations of infinite-dimensional Lie groups*, Kyoto J. Math. 59 (2) (2019), 293–341; arXiv:math.RT.1501.00939 8, 18

[JN21] Janssens, B., and K.-H. Neeb, *Positive energy representations of gauge groups I: Localization*, in preparation, 2021 2, 4, 6, 7, 14, 17

[KR87] Kac, V. G., and A. K. Raina, “Highest weight representations of infinite dimensional Lie algebras,” Advanced Series in Math. Physics, World Scientific, Singapore, 1987 3

[Ka79] Kay, B. S., *A uniqueness result in the Segal–Weinless approach to linear Bose fields*, J. Math. Physics 20 (1979), 1712–1713 19, 24

[Ka85] Kay, B. S., *A uniqueness result for quasi-free KMS states*, Helv. Phys. Acta 58 (1985), 1017–1029 19, 24

[MN16] Marquis, T., and K.-H. Neeb, *Positive energy representations for locally finite split Lie algebras*, Int. Math. Res. Not. 21 (2016), 6689–6712; arXiv:math.RT.1507.06077 3, 14, 40

[MN17] Marquis, T., and K.-H. Neeb, *Positive energy representations of double extensions of Hilbert loop algebras*, J. Math. Soc. Jpn 69 (4) (2017), 1485–1518 3, 14

[Mo80] Moore, C. C., *The Mautner phenomenon for general unitary representations*, Pac. J. Math. 86 (1) (1980), 155–169 5, 30

[Ne00] Neeb, K.-H., “Holomorphy and Convexity in Lie Theory,” Expositions in Mathematics 28, de Gruyter, Berlin, 2000 3, 30, 31, 32, 33, 34, 36, 45

[Ne01] Neeb, K.-H., *Locally finite Lie algebras with unitary highest weight representations*, Manuscr. Math. 104 (3) (2001), 343–358
[Ru73] Rudin, W., “Functional Analysis,” McGraw Hill, New York, 1973 2

[SeG81] Segal, G., *Unitary representations of some infinite-dimensional groups*, Comm. Math. Phys. 80 (1981), 301–342 14

[Wa72] Warner, G., “Harmonic analysis on semisimple Lie groups I,” Springer, Berlin, 1972 3, 14, 34

[We69] Weinless, M., *Existence and uniqueness of the vacuum for linear quantized fields*, J. Funct. Anal. 4 (1969), 350–379 3, 4, 18, 19

[Ze14] Zellner, C., *Semibounded unitary representations of oscillator groups*, PhD Thesis, Friedrich–Alexander University Erlangen–Nuremberg, 2014 3, 10