Laplacian Spectra of Regular Graph Transformations *

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Abstract

Given a graph $G$ with vertex set $V(G) = V$ and edge set $E(G) = E$, let $G^l$ be the line graph and $G^c$ the complement of $G$. Let $G^0$ be the graph with $V(G^0) = V$ and with no edges, $G^1$ the complete graph with the vertex set $V$, $G^+ = G$ and $G^- = G^c$. Let $B(G)$ ($B^c(G)$) be the graph with the vertex set $V \cup E$ and such that $(v, e)$ is an edge in $B(G)$ (resp., in $B^c(G)$) if and only if $v \in V$, $e \in E$ and vertex $v$ is incident (resp., not incident) to edge $e$ in $G$. Given $x, y, z \in \{0, 1, +, -\}$, the $xyz$-transformation $G^xyz$ of $G$ is the graph with the vertex set $V(G^xyz) = V \cup E$ and the edge set $E(G^xyz) = E(G^x) \cup E((G^y)^l) \cup E(W)$, where $W = B(G)$ if $z = +$, $W = B^c(G)$ if $z = -$, $W$ is the graph with $V(W) = V \cup E$ and with no edges if $z = 0$, and $W$ is the complete bipartite graph with parts $V$ and $E$ if $z = 1$. In this paper we obtain the Laplacian characteristic polynomials and some other Laplacian parameters of every $xyz$-transformation of an $r$-regular graph $G$ in terms of $|V|$, $r$, and the Laplacian spectrum of $G$.

Key words: regular graph, $xyz$-transformation, Laplacian spectrum, Laplacian characteristic polynomial.

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1 Introduction

The graphs in this paper are simple and undirected. All notions on graphs and matrices that are used but not defined here can be found in [1, 5, 6, 8, 22].

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Let $G$ denote the set of simple undirected graphs. Various important results in graph theory have been obtained by considering some functions $F : G \to G$ or $F_s : G_1 \times \ldots \times G_s \to G$ called operations or transformations (here each $G_i = G$) and by establishing how these operations affect certain properties or parameters of graphs. The complement, the $k$-th power of a graph, and the line graph are well known examples of such operations. The Bondy-Chvátal and Ryzáček closers of graphs are very useful operations in graph Hamiltonicity theory \[1\]. (Strengthenings and extensions of the Ryzáček result are given in \[9\]). Some graph operations introduced by A. Kelmans (see, in particular, \[10\] \[13\]) turn out to be monotone with respect to various partial order relations on the set of graphs. For that reason these operations turned out to be very useful in obtaining non-trivial results on graphs of given size with various extreme properties (with the maximum number of spanning trees and some other Laplacian parameters of graphs, with the maximum reliability of graphs having randomly deleted edges, etc.), see, for example, \[11\] \[12\]. The operation of voltage lifting on a base graph introduced by Gross and Tucker can be generalized to digraphs \[4\] \[7\]. Using this operation one can obtain the derived covering (di)graph and deduce the relationship between the adjacency characteristic polynomials of the base (di)graph and its derived covering (di)graph \[4\] \[3\] \[19\] \[25\].

In this paper we consider certain graph operations depending on parameters $x, y, z \in \{0, 1, +, -\}$. These operations induce functions $T_{xyz} : G \to G$. We put $T_{xyz}(G) = G^{xyz}$ and call $G^{xyz}$ the $xyz$-transformation of $G$. We describe for all $x, y, z \in \{0, 1, +, -\}$ the Laplacian characteristic polynomials and some other Laplacian parameters of $xyz$-transformations of an $r$-regular graph $G$. This descriptions revealed the following fact interesting in itself: if $G$ is $r$-regular, then the Laplacian spectrum of $G^{xyz}$ is uniquely defined by $|V(G)|$, $r$, and the Laplacian spectrum of $G$; moreover, the Laplacian eigenvalues are the roots of a quadratic polynomial with the coefficients depending on $|V(G)|$, $r$, and the Laplacian spectrum of $G$. Furthermore, for $(xyz) \in \{(00+), (0+ +), (+0+)\}$ the number of spanning trees of $G^{xyz}$ are uniquely defined by $|V(G)|$, $r$, and the number of spanning trees of $G$ (see Theorem \[2.5\] and Corollaries \[3.5\] \[3.9\] and \[3.12\] below). The approach we have used to obtain all these formulas may also be useful in further research along this line. The results of this paper may be considered as a natural and useful extension of Section 2 “Operations on Graphs and the Resulting Spectra” in book \[2\].

The Reciprocity Theorem \[16\] (see also Theorem \[2.6\] below) provides for every graph $G$ the relation between the Laplacian characteristic polynomial of $G$ and its complement $G^c$. For that reason it is sufficient to describe the Laplacian characteristic polynomials of
graph $xyz$-transformations up to the graph operation of taking the complement.

In Section 2 we introduce main notions, notation, simple observations, and some preliminaries. In Section 3 we describe (up to the complementarity) the Laplacian characteristic polynomials of the transformations $G^{xyz}$ with $\{x,y,z\} \cap \{0,1\} \neq \emptyset$. In Section 4 we describe the Laplacian characteristic polynomials of transformations $G^{xyz}$ with $\{x,y,z\} \cap \{0,1\} = \emptyset$, i.e. with $x, y, z \in \{+,-\}$. In Sections 5 we consider some transformations of cycles and show that different transformations of the same graph may be isomorphic. Section 6 contains some additional remarks. In the Appendix we provide for all $x, y, z \in \{0,1,+,−\}$ the list of formulas for the Laplacian characteristic polynomials and the number of spanning trees of the ($xyz$)-transformations of an $r$-regular graph $G$ in terms of $r$, the number of vertices, the number of edges, and the Laplacian spectrum of $G$. This catalog may be pretty useful for further research in the spectrum graph theory.

2 Some notions, notation, and preliminaries

Let $G = (V,E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. The degree $d(v,G)$ of vertex $v \in V$ is the number of vertices in $G$ adjacent to $v$. Let $t(G)$ denote the number of spanning trees of $G$.

Given two graphs $G$ and $H$, an isomorphism from $G$ to $H$ is a bijection $\alpha$ from $V(G)$ to $V(H)$ such that $(u,v) \in E(G) \iff (\alpha(u),\alpha(v)) \in E(H)$.

The complement $G^c$ of a graph $G$ is the graph with vertex set $V(G^c) = V(G)$ and $(u,v) \in E(G^c) \iff (u,v) \notin E(G)$ for any $u,v \in V(G)$ and $u \neq v$.

The line graph $G^l$ of a graph $G$ is the graph with vertex set $E(G)$ and two vertices are adjacent in $G^l$ if and only if the corresponding edges in $G$ are adjacent. Graphs $G$ and $H$ are called isomorphic if there exists an isomorphism from $G$ to $H$.

For a graph $G = (V,E)$, let $G^0$ be the graph with $V(G^0) = V$ and with no edges, $G^1$ the complete graph with $V(G^1) = V$, $G^+ = G$, and $G^- = G^c$. Let $B(G) \ (B^c(G))$ be the graph with the vertex set $V \cup E$ and such that $(v,e)$ is an edge in $B(G)$ (resp., in $B^c(G)$) if and only if $v \in V$, $e \in E$, and vertex $v$ is incident (resp., not incident) to edge $e$ in $G$. For example, in Figure 1 $G^{00+} = B(G)$ and $B^c(G)$ is obtained from $G^{01-}$ by deleting the edge connecting two white vertices.

The graph transformations we are going to discuss are defined as follows.

**Definition 2.1.** Given a graph $G = (V,E)$ and three variables $x,y,z \in \{0,1,+,−\}$, the $xyz$-transformation $G^{xyz}$ of $G$ is the graph with the vertex set $V(G^{xyz}) = V \cup E$ and the
edge set $E(G^{xyz}) = E(G^x) \cup E((G^y)^z) \cup E(W)$, where $W = B(G)$ if $z = +$, $W = B^c(G)$ if $z = -$, $W$ is the graph with $V(W) = V \cup E$ and with no edges if $z = 0$, and $W$ is the complete bipartite graph with parts $V$ and $E$ if $z = 1$.

Examples of $xyz$-transformations of a 3-vertex path are given in Figure 1. Graphs $G^{++}$ and $G^{00+}$ are called in [2] the total graph and the subdivision graph of $G$, respectively.

Let $G$ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. The incidence matrix $Q(G)$ of $G$ is the $(V \times E)$-matrix $\{q_{ij}\}$, where $q_{ij} = 1$ if vertex $v_i$ is incident to edge $e_j$ and $q_{ij} = 0$, otherwise. Let $A(G)$ be the $(V \times V)$-matrix $\{a_{ij}\}$, where $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise. Let $D(G)$ be the (diagonal) $(V \times V)$-matrix $\{d_{ij}\}$, where $d_{ii} = d(v_i, G)$ and $d_{ij} = 0$ for $i \neq j$. The matrices $A(G)$, $D(G)$ and $L(G) = D(G) - A(G)$ are called the adjacency matrix, the degree matrix, and the Laplacian matrix of $G$, respectively. The adjacency polynomial, the adjacency spectrum and the adjacency eigenvalues of $G$ are the characteristic polynomial $A(\alpha, G) = \det(\alpha I - A(G))$, the spectrum, and the eigenvalues of $A(G)$, respectively. Similarly, the Laplacian polynomial, the Laplacian spectrum and the Laplacian eigenvalues of $G$ are the characteristic polynomial $L(\lambda, G) = \det(\lambda I - L(G))$, the spectrum, and the eigenvalues of $L(G)$, respectively. Let $I_n$ be the identity $(n \times n)$-matrix and $J_{mn}$ the all-ones $m \times n$-matrix.

Since $A(G)$ and $L(G)$ are symmetric matrices, their eigenvalues are real numbers. It is easy to see that each row sum of $L(G)$ is equal to zero and that $L(G)$ is a symmetric positive semi-definite matrix [2, 15]. Therefore all eigenvalues $\lambda_i(G)$ of $L(G)$ are real non-negative numbers and one of them is equal to zero; we order them in the descendant
order:
\[ \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0. \]
The set \( Sp(G) = \{ \lambda_1(G), \ldots, \lambda_n(G) \} \) is called the Laplacian spectrum of \( G \).

Some graph properties of the transformations \( G^{xyz} \) with \( x, y, z \in \{+,-\} \) have been discussed and obtained in [17, 23, 24].

For a regular graph \( G \), the adjacency characteristic polynomials and the adjacency spectrum of \( G^{00+} \), \( G^{+0+} \), \( G^{0++} \) and the total graph \( G^{+++} \) are given in [2] (pages 63 and 64). The adjacency characteristic polynomials and the adjacency spectrum of the other seven \( G^{xyz} \) with \( x, y, z \in \{+,-\} \) are obtained in [26]. The definition of \( xyz \)-transformation can be easily extended to digraphs, which is also a generalization of the digraph transformations defined by Liu and Meng [18]. Zhang, Lin, and Meng have described the adjacency characteristic polynomials of \( D^{00+} \), \( D^{+0+} \), \( D^{0++} \) and the total digraph \( D^{+++} \) for any digraph \( D \) [27]. The adjacency characteristic polynomials of other \( D^{xyz} \) of a regular digraph \( D \) with \( x, y, z \in \{+,-\} \) are obtained in [18].

Very few results are known for the Laplacian spectra of transformations. In 1967 A. Kelmans published the following results on the Laplacian polynomial of \( G^{0++} \), \( G^{0+0} \), \( G^{00+} \), and \( G^l \) for a regular graph \( G \). These results are included in the survey papers [20] (Theorem 3.8) and [21] (Theorem 1.4.2) with an error, namely, graph \( G^{0++} \) is mistakenly called the total graph of \( G \).

**Theorem 2.2.** [15] Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then
\[
L(\lambda, G^{0++}) = (\lambda - r - 1)^n(\lambda - 2r - 2)^{m-n}L\left(\frac{\lambda^2 - (r + 2)\lambda}{\lambda - r - 1}, G\right).
\]

**Theorem 2.3.** [15] Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then
\[
L(\lambda, G^l) = (\lambda - 2r)^{m-n}L(\lambda, G) \quad \text{and} \quad L(\lambda, G^{0+0}) = \lambda^n L(\lambda, G^l).
\]

**Theorem 2.4.** [15] Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then
\[
L(\lambda, G^{00+}) = (-1)^n(\lambda - 2)^{m-n}L(\lambda(r + 2 - \lambda), G).
\]

Since \( nt(G) = (-1)^{n-1}\lambda^{-1}L(\lambda, G)|_{\lambda=0} \), where \( n = v(G) \) [16], we have from Theorems 2.2, 2.3, and 2.4,

**Theorem 2.5.** [15] Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then
\[
t(G^{0++}) = \frac{n}{m+n}2^{m-n}(r+1)^{m-1})(r+2)t(G),
\]
\[ t(G^{00+}) = \frac{n}{m+n} 2^{m-n}(r+2)t(G), \]

and

\[ t(G^l) = \frac{n}{m} 2^{m-n} r^{m-n} t(G). \]

A set \( \mathcal{A} \) of graphs is closed under complementarity if for every graph in \( \mathcal{A} \) its complement is also in \( \mathcal{A} \). Given a graph \( G \) and \( S \subseteq \{0, 1, +, -\} \), let \( \mathcal{F}(G, S) \) denote the set of graphs \( G^{xyz} \) such that \( x, y, z \in S \). If \( \mathcal{F}(G, S) \) is closed under complementarity, then in order to find the Laplacian polynomial for the graphs in \( \mathcal{F}(G, S) \), it is sufficient to find the solutions for a “half” of graphs in \( \mathcal{F}(G, S) \) and to obtain the solutions for the graphs of the other “half” using the Reciprocity Theorem 2.6 [16]. It is easy to see that \( \mathcal{F}(G, \{+, -\}) \) and \( \mathcal{F}(G, \{0, 1, +, -\}) \) (and therefore \( \mathcal{F}(G, \{0, 1, +, -\} \setminus \{+, -\}) \)) are closed under complementarity. Hence the Reciprocity Theorem below can be used for these classes of transformations.

**Theorem 2.6.** [16] Let \( G \) be a simple graph with \( n \) vertices. Then

(a1) \( \lambda_i(G) + \lambda_{n-i}(G^c) = n \) for every \( i \in \{1, \ldots, n-1\} \) or, equivalently,

(a2) \( (n-\lambda)L(\lambda, G^c) = (-1)^{n-1} \lambda L(n-\lambda, G) \).

Moreover, the matrices \( L(G) \) and \( L(G^c) \) are simultaneously diagonalizable.

Similar results for weighted graphs are obtained in [14].

We will also use the following two classical and simple facts on matrices.

**Lemma 2.7.** [6, 8] Let \( A \) and \( D \) be square matrices. Then

\[
\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{cases} |A| |D - CA^{-1}B|, & \text{if } A \text{ is invertible}, \\ |D| |A - BD^{-1}C|, & \text{if } D \text{ is invertible}. \end{cases}
\]

The following two useful and very well known lemmas are obvious.

**Lemma 2.8.** Given a graph \( G \) with \( m \) edges, let \( G^l \) be the line graph of \( G \) and \( Q \) the incidence matrix of \( G \). Then

(a1) \( QQ^\top = D(G) + A(G) \) and

(a2) \( Q^\top Q = 2I_m + A(G^l) \).

**Lemma 2.9.** Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges and let \( A \) and \( Q \) be the adjacency and the incidence matrix of \( G \), respectively. Let \( k \) be a positive integer. Then
Lemma 2.10. Let $G$ be an $r$-regular graph with $n$ vertices, $A(G) = A$, and
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0 \] the list of the Laplacian eigenvalues of $G$. Let $P(x, y)$ be a polynomial with two variables and real coefficients. Then matrix $P(A, J_{nn})$ has the eigenvalues $\sigma_n = P(r, n)$ and $\sigma_i = P(r - \lambda_i, 0)$ for $i = 1, 2, \ldots, n - 1$.

Proof. Since the Laplacian matrix $L = L(G)$ is symmetric and real, there is a list $B = \{X_1, X_2, \cdots, X_n\}$ of mutually orthogonal eigenvectors of $L$, where $X_i$ corresponds to $\lambda_i$ and $X_n = J_{n1}$. Since $A = rI_n - L$, clearly $AX_i = (r - \lambda_i)X_i$ for each $i$. Since $B$ is an orthogonal basis, $J_{nn}X_i = 0$ for each $i \neq n$. Clearly, $J_{nn}J_{n1} = nJ_{n1}$ and $J_{nn}^2 = nJ_{nn}$.

By Lemma 2.9, $AJ_{nn} = J_{nn}A = rJ_{nn}$. Therefore, $P(A, J_{nn}) X_n = P(r, n) X_n$ and $P(A, J_{nn}) X_i = P(r - \lambda_i, 0) X_i$ for each $i \neq n$. □

The arguments in this proof are similar to those in [14].

3 Laplacian spectra of $G^{xyz}$ with $\{x, y, z\} \cap \{0, 1\} \neq \emptyset$

Given a graph $G$ with $n$ vertices and $m$ edges, we always denote by $A, D$ and $Q$, the adjacency matrix, the degree matrix and the incidence matrix of $G$, respectively, and so if $G$ is an $r$-regular graph, then $D = rI_n$ and $2m = rn$. We put $\lambda_i(G) = \lambda_i$, and so $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ is the list of the Laplacian eigenvalues of $G$.

3.1 Laplacian spectra of $G^{xyz}$ with $z = 0$

We start with the following simple observation.

Theorem 3.1. Let $G$ be a graph with $n$ vertices and $m$ edges and let $x, y \in \{0, 1, +, -\}$. Then $L(\lambda, G^{xyz}) = L(\lambda, G^x)L(\lambda, (G')^y)$.

Since $L(\lambda, G^0) = \lambda^n$, $L(\lambda, G^+) = L(\lambda, G)$ and $L(\lambda, G^-) = \lambda(\lambda - n)^{n-1}$, we can calculate $L(\lambda, G^{xyz})$ for $x, y \in \{0, 1, +, -\}$ from Theorems 2.3, 2.6 and 3.1.
Theorem 3.2. Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then

(a1) $L(\lambda, G^{x00}) = \lambda^m L(\lambda, G^x)$ and $L(\lambda, G^{x10}) = \lambda(\lambda - m)^{m-1} L(\lambda, G^x)$ for $x \in \{0, 1, +\}$,

(a2) $L(\lambda, G^{-00}) = (-1)^n (\lambda - n)^{-1} \lambda^{m+1} L(n - \lambda, G)$,

(a3) $L(\lambda, G^{-10}) = (-1)^n \lambda^2 (\lambda - n)^{-1} (\lambda - m)^{m-1} L(n - \lambda, G)$,

(a4) $L(\lambda, G^{0-0}) = (-1)^n (\lambda - m)^{-1} \lambda^{m+1} (\lambda - m + 2r)^{m-n} L(m - \lambda, G)$,

(a5) $L(\lambda, G^{0+0}) = \lambda (\lambda - 2r)^{m-n} L(\lambda, G)$,

(a6) $L(\lambda, G^{1+0}) = \lambda (\lambda - n)^{-1} (\lambda - 2r)^{m-n} L(\lambda, G)$,

(a7) $L(\lambda, G^{+-0}) = (-1)^n \lambda (\lambda - m)^{-1} (\lambda - m + 2r)^{m-n} L(m - \lambda, G) L(\lambda, G)$ and

$L(\lambda, G^{-0-}) = \lambda^2 (\lambda - m)^{-1} (\lambda - n)^{-1} (\lambda - m + 2r)^{m-n} L(m - \lambda, G) L(n - \lambda, G)$.

3.2 Laplacian spectra of $G^{xyz}$ for $z = +$ and $x, y \in \{0, 1\}$

Theorem 3.3. Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then

$L(\lambda, G^{00+}) = \lambda (\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n-1} \{\lambda^2 - \lambda(r + 2) + \lambda_i\}$

or equivalently,

$L(\lambda, G^{00+}) = (-1)^n (\lambda - 2)^{m-n} L(\lambda(r + 2) - \lambda), G)$.

Proof. It is easy to see that

$A(G^{00+}) = \begin{pmatrix} 0 & Q \\ Q^\top & 0 \end{pmatrix}$ and $D(G^{00+}) = \begin{pmatrix} rI_n & 0 \\ 0 & 2I_m \end{pmatrix}$.

We know that $L(G^{00+}) = D(G^{00+}) - A(G^{00+})$ and $L(\lambda, G^{00+}) = \det(\lambda I_n - L(G^{00+}))$. Therefore

$L(\lambda, G^{00+}) = \begin{vmatrix} (\lambda - r)I_n & Q \\ Q^\top & (\lambda - 2)I_m \end{vmatrix}$.

Clearly, it is sufficient to prove our claim for $\lambda \neq 2$. Now using Lemmas 2.7 we obtain:

$L(\lambda, G^{00+}) = |(\lambda - 2)I_m| \times |(\lambda - r)I_n - \frac{1}{\lambda - 2} Q I_m Q^\top|$. 

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By Lemma 2.8 (a1), \( QQ^T = rI_n + A \). Therefore

\[
L(\lambda, G^{00+}) = (\lambda - 2)^{m-n} \times |B|, \quad \text{where} \quad B = (\lambda - 2)(\lambda - r)I_n - (rI_n + A).
\]

Obviously, \(|B|\) equals the product of its eigenvalues. By Lemma 2.10, the eigenvalues of \( B \) are

\[
\sigma_n = (\lambda - 2)((\lambda - r)) - (r + r) = \lambda(\lambda - r - 2) \quad \text{and}
\]

\[
\sigma_i = (\lambda - 2)(\lambda - r) - (r + r - \lambda_i) = \lambda^2 - (r + 2)\lambda + \lambda_i \quad \text{for} \quad i = 1, 2, \ldots, n - 1.
\]

Since \(|B| = \prod_{i=1}^{n} \sigma_i\), we have:

\[
L(\lambda, G^{00+}) = \lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n} (\lambda^2 - (r + 2)\lambda + \lambda_i). \qquad \square
\]

**Corollary 3.4.** If \( G \) is an \( r \)-regular graph with \( n \) vertices and \( m \) edges, then \( G^{00+} \) has \( m - n \) Laplacian eigenvalues equal to 2 and the following \( 2n \) Laplacian eigenvalues

\[
\frac{1}{2}(r + 2 \pm \sqrt{(r+2)^2 - 4\lambda_i}), \quad i = 1, 2, \ldots, n.
\]

**Corollary 3.5.** Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

\[
t(G^{00+}) = \frac{n}{m + n}(r + 2)^{m-n}t(G).
\]

Similarly, we can prove the following theorem.

**Theorem 3.6.** Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

(a1) \( L(\lambda, G^{10+}) = \lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n} ((\lambda - 2)(\lambda - n - r) - 2r + \lambda_i) \),

(a2) \( L(\lambda, G^{01+}) = \lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n} \prod_{i=1}^{n} ((\lambda - r)(\lambda - m - 2) - 2r + \lambda_i) \) and

(a3) \( L(\lambda, G^{11+}) = \lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n} \prod_{i=1}^{n} ((\lambda - n - r)(\lambda - m - 2) - 2r + \lambda_i) \).

**3.3 Laplacian spectra of \( G^{xyz} \) for \( z = + \) and \(|\{x, y\} \cap \{0, 1\}| = 1\)**

**Theorem 3.7.** Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

\[
L(\lambda, G^{+0+}) = \lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n} ((\lambda - 2)(\lambda - n - \lambda_i) - 2r + \lambda_i)
\]

or equivalently,

\[
L(\lambda, G^{+0+}) = (\lambda - 2)^{m-n}(\lambda - 3)^n L\left(\frac{\lambda^2 - \lambda(r + 2)}{\lambda - 3}, G\right).
\]

**Proof.** The adjacency matrix and the degree matrix of \( G^{+0+} \) are

\[
A(G^{+0+}) = \begin{pmatrix} A & Q \\ Q^T & 0 \end{pmatrix} \quad \text{and} \quad D(G^{+0+}) = \begin{pmatrix} 2rI_n & 0 \\ 0 & 2I_m \end{pmatrix}.
\]
Therefore

\[ L(\lambda, G^{0+}) = \begin{vmatrix} \lambda - 2r & I_n + A & Q \\ Q^T & (\lambda - 2)I_m \end{vmatrix}. \]

Clearly, it is sufficient to prove our claim for \( \lambda \neq 2 \). Now using Lemmas 2.7 we obtain:

\[ L(\lambda, G^{0+}) = (\lambda - 2)^m |(\lambda - 2r)I_n + A - \frac{1}{\lambda - 2}QQ^T|. \]

By Lemma 2.8 (a1), \( QQ^T = rI_n + A \). Therefore

\[ L(\lambda, G^{0+}) = (\lambda - 2)^m B, \] where \( B = (\lambda - 2)(\lambda - 2r)I_n + (\lambda - 2)A - (rI_n + A). \)

Obviously, \( |B| \) equals the product of its eigenvalues. By Lemma 2.10 the eigenvalues of \( B \) are

\[ \sigma_n = (\lambda - 2)(\lambda - 2r) + (\lambda - 2)r - (r + r) = \lambda(\lambda - r - 2) \] and

\[ \sigma_i = (\lambda - 2)(\lambda - r - \lambda_i) - 2r + \lambda_i \text{ for } i = 1, 2, \ldots, n - 1. \]

Since \( |B| = \prod_{i=1}^{n} \sigma_i \), we have:

\[ L(\lambda, G^{0+}) = \lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - 2)(\lambda - r - \lambda_i) - 2r + \lambda_i\}. \]

\[ \square \]

Corollary 3.8. If \( G \) is an \( r \)-regular graph with \( n \) vertices and \( m \) edges, then \( G^{0+} \) has \( m - n \) Laplacian eigenvalues equal to \( 2 \) and the following \( 2n \) Laplacian eigenvalues

\[ \frac{1}{2}(r + 2 + \lambda_i \pm \sqrt{(r + 2 + \lambda_i)^2 - 12\lambda_i}), \text{ i = 1, 2, \ldots, n.} \]

Corollary 3.9. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

\[ t(G^{0+}) = \frac{n}{m + n}(r + 2)2^{m-n}3^{n-1}t(G), \]

and so

\[ t(G^{0+}) = 3^{n-1}t(G^{00+}). \]

Theorem 3.10. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

\[ L(\lambda, G^{0++}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\} \]

or equivalently,

\[ L(\lambda, G^{0++}) = (\lambda - r - 1)^n(\lambda - 2r - 2)^{m-n}L\left(\frac{\lambda^2 - (r + 2)\lambda}{\lambda - r - 1}, G\right). \]
Proof. The adjacency matrix and the degree matrix of $G^{0++}$ are

$$A(G^{0++}) = \begin{pmatrix} 0 & Q \\ Q^T & A(G^l) \end{pmatrix} \quad \text{and} \quad D(G^{0++}) = \begin{pmatrix} rI_n & 0 \\ 0 & 2rI_m \end{pmatrix}.$$ 

Since by Lemma 2.8 (a2), $A(G^l) = Q^T Q - 2I_m$, we have:

$$L(\lambda, G^{0++}) = \begin{vmatrix} (\lambda - r)I_n & Q \\ Q^T & (\lambda - 2r - 2)I_m + Q^T Q \end{vmatrix}.$$ 

Clearly, it is sufficient to prove our claim for $\lambda \neq 2r + 2$. Using Lemmas 2.7 we obtain:

$$L(\lambda, G^{0++}) = (\lambda - 2r - 2)^m |(\lambda - r)I_n - \frac{r + 1 - \lambda}{\lambda - 2r - 2} QQ^T|. $$

By Lemma 2.8 (a1), $QQ^T = rI_n + A$. Therefore

$$L(\lambda, G^{0++}) = (\lambda - 2r - 2)^{m-n} |B|, \quad \text{where} \quad B = (\lambda - r)(\lambda - 2r - 2)I_n - (r + 1 - \lambda)(rI_n + A).$$

Obviously, $|B|$ equals the product of its eigenvalues. By Lemma 2.10 the eigenvalues of $B$ are

$$\sigma_n = (\lambda - r)(\lambda - 2r - 2) - (r + 1 - \lambda)(r + r) = \lambda(\lambda - r - 2)$$

$$\sigma_i = (\lambda - r)(\lambda - 2 - \lambda_i) - 2r + \lambda_i \quad \text{for} \quad i = 1, 2, \ldots, n - 1.$$ 

Since $|B| = \prod_{i=1}^n \sigma_i$, we have:

$$L(\lambda, G^{0++}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}. \quad \square$$

Corollary 3.11. If $G$ is an $r$-regular graph with $n$ vertices and $m$ edges, then $G^{0++}$ has $m - n$ Laplacian eigenvalues equal to $2r + 2$ and the following $2n$ Laplacian eigenvalues

$$\frac{1}{2} \left( r + 2 + \lambda_i \pm \sqrt{\lambda_i^2 - 2r\lambda_i + r^2 + 4r + 4} \right), \quad i = 1, 2, \ldots, n.$$
Theorem 3.13. Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then
\[
L(\lambda, G^{0-}) = \lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - m - 2 - \lambda_i) - 2r + \lambda_i\},
\]
\[
L(\lambda, G^{-0}) = \lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - 2) - 2r + \lambda_i\},
\]
\[
L(\lambda, G^{-1 +}) = \lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - m - 2) - 2r + \lambda_i\},
\]
\[
L(\lambda, G^{-1 -}) = \lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - m - 2) - 2r + \lambda_i\},
\]
\[
L(\lambda, G^{1-}) = \lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r - \lambda_i)(\lambda - m - 2) - 2r + \lambda_i\}
\]
and
\[
L(\lambda, G^{1+}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}.
\]

3.4 Laplacian spectra of $G^{xyz}$ for $z \in \{1, -\}$

Using the Reciprocity Theorem 2.6 it is easy to find from Theorems 3.2, 3.3, 3.6, 3.7, 3.10 and 3.13 the Laplacian characteristic polynomial of every $G^{xyz}$ with $\{x, y\} \cap \{0, 1\} \neq \emptyset$ and $z \in \{1, -\}$ (see Appendix).

4 Laplacian spectra of $G^{xyz}$ with $x, y, z \in \{+, -\}$

In this section we will describe the Laplacian characteristic polynomials and the Laplacian spectra of transformations $G^{xyz}$ of an $r$-regular graph $G$ for $x, y, z \in \{+, -\}$ in terms of the Laplacian spectrum of $G$, $r$, $v(G) = n$, $r$ (and $e(G) = m = \frac{1}{2}rn$).

Theorem 4.1. Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then
\[
L(\lambda, G^{+++}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r - \lambda_i)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}.
\]

Proof. The adjacency matrix and the degree matrix of $G^{+++}$ are
\[
A(G^{+++}) = \begin{pmatrix} A & Q \\ Q^\top & A(G^t) \end{pmatrix} \quad \text{and} \quad D(G^{+++}) = \begin{pmatrix} 2rI_n & 0 \\ 0 & 2rI_m \end{pmatrix}.
\]
Since $L(G^{+++}) = D(G^{+++}) - A(G^{+++})$, $L(\lambda, G^{+++}) = \det(\lambda I_n - L(G^{+++}))$, and by
Lemma 2.8 (a2), \( A(G^d) = Q^T Q - 2I_m \), we have:

\[
L(\lambda, G^{+++}) = \begin{vmatrix}
(\lambda - 2r)I_n + A & Q \\
Q^T & (\lambda - 2r - 2)I_m + Q^T Q
\end{vmatrix}
\]

\[
= \begin{vmatrix}
(\lambda - 2r)I_n + A & Q \\
-Q^T((\lambda - 2r - 1)I_n + A) & (\lambda - 2r - 2)I_m
\end{vmatrix}.
\]

Clearly, it is sufficient to prove our claim for \( \lambda \neq 2r + 2 \). Using Lemmas 2.7 we obtain:

\[
L(\lambda, G^{+++}) = |(\lambda - 2r - 2)I_m| \times |(\lambda - 2r)I_n + A + (\lambda - 2r - 2)^{-1}QQ^T((\lambda - 2r - 1)I_n + A)|.
\]

By Lemma 2.8 (a1), \( QQ^T = rI_n + A \). Therefore \( L(\lambda, G^{+++}) = (\lambda - 2r - 2)^{m-n} \times |B| \), where

\[
B = |(\lambda - 2r - 2)((\lambda - 2r)I_n + A) + (rI_n + A)((\lambda - 2r - 1)I_n + A)|.
\]

Obviously, \( |B| \) equals the product of its eigenvalues. By Lemma 2.10 the eigenvalues of \( B \) are

\[
\sigma_i = (\lambda - 2r - 2)((\lambda - 2r) + r - \lambda_i) + (r + r - \lambda_i)((\lambda - 2r - 1) + r - \lambda_i)
\]

\[
= (\lambda - r - \lambda_i)(\lambda - 2 - \lambda_i) - 2r + \lambda_i \text{ for } i = 1, 2, \ldots, n.
\]

Since \( |B| = \prod_{i=1}^{n} \sigma_i \) and \( \lambda_n = 0 \), we have:

\[
L(\lambda, G^{+++}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r - \lambda_i)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}.
\]

Theorem 4.2. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

\[
L(\lambda, G^{++}) = \lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r - \lambda_i)(\lambda - m - 2 + \lambda_i) - 2r + \lambda_i\}.
\]

Proof. From the definition of \( G^{++} \), we have:

\[
A(G^{++}) = \begin{pmatrix}
A & Q \\
Q^T & J_{nm} - I_m - A(G^d)
\end{pmatrix}.
\]

For every \( z \in V(G^{++}) \), \( d(z, G^{++}) = 2r \) if \( z \in V(G) \) and \( d(z, G^{++}) = 2 + m - 1 - (2r - 2) = m - 2r + 3 \) if \( z \in E(G) \). Therefore,

\[
D(G^{++}) = \begin{pmatrix}
2rI_n & 0 \\
0 & (m - 2r + 3)I_m
\end{pmatrix}.
\]
Then $L(\lambda, G^{++}) = |M|$, where

$$M = \begin{pmatrix}
(\lambda - 2r)I_n + A & Q \\
Q^\top & (\lambda - m + 2r - 2)I_m + J_{mn} - Q^\top Q
\end{pmatrix}.$$ 

By Lemma 2.8 (a), $J_{mn}Q = 2J_{mn}$. Hence multiplying the first row of the block matrix $M$ by $Q^\top - \frac{1}{2}J_{mn}$ and adding the result to the second row of $M$, we obtain a new matrix $M' = \begin{pmatrix}
(\lambda - 2r)I_n + A & Q \\
Q^\top((\lambda - 2r + 1)I_n + A) - \frac{1}{2}J_{mn}((\lambda - 2r)I_n + A) & (\lambda - m + 2r - 2)I_m
\end{pmatrix}$.

Clearly, $L(\lambda, G^{++}) = |M| = |M'|$. Obviously, it is sufficient to prove our claim for $\lambda \neq m - 2r + 2$. Now using Lemmas 2.7 we obtain:

$$L(\lambda, G^{++}) = (\lambda - m + 2r - 2)^{m-n} \times |B|,$$

where

$$B = (\lambda - m + 2r - 2)((\lambda - 2r)I_n + A) - QQ^\top((\lambda - 2r + 1)I_n + A) + \frac{1}{2}QJ_{mn}((\lambda - 2r)I_n + A).$$

By Lemma 2.8 (a1), $QQ^\top = rI_n + A$ and by Lemma 2.9 (a2), $QJ_{mn} = rJ_{mn}$. Therefore

$$B = (\lambda - m + 2r - 2)((\lambda - 2r)I_n + A) - (rI_n + A)((\lambda - 2r + 1)I_n + A) + \frac{1}{2}J_{mn}((\lambda - 2r) + A).$$

Obviously, $|B|$ equals the product of its eigenvalues and $nr = 2m$. By Lemma 2.10 the eigenvalues of $B$ are

$$\sigma_n = (\lambda - m + 2r - 2)(\lambda - 2r + r) - 2r(\lambda - 2r + 1 + r) + \frac{nr}{2}(\lambda - 2r + r) = \lambda^2 - \lambda(r + 2)$$

and

$$\sigma_i = (\lambda - m + 2r - 2)(\lambda - 2r + r - \lambda_i) - (2r - \lambda_i)(\lambda - 2r + r + 1 - \lambda_i)$$

$$= (\lambda - r - \lambda_i)(\lambda - m - 2 + \lambda_i) - 2r + \lambda_i$$

for $i = 1, 2, \cdots, n - 1$.

Since $|B| = \prod_{i=1}^{n} \sigma_i$, we have:

$$L(\lambda, G^{++}) = \lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - r - \lambda_i)(\lambda - m - 2 + \lambda_i) - 2r + \lambda_i\}.\quad \square$$

**Theorem 4.3.** Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then

$$L(\lambda, G^{++}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}.$$ 

**Proof.** From the definition of $G^{++}$ we have:

$$A(G^{++}) = \begin{pmatrix}
J_{nn} - I_n - A & Q \\
Q^\top & A(G^t)
\end{pmatrix}.$$
For every vertex \( z \) in \( G^{-+} \), we have: 
\[
d(z, G^{-+}) = n - 1 \text{ if } z \in V(G) \text{ and } d(z, G^{-+}) = 2r \text{ if } z \in E(G),
\]
and so 
\[
D(G^{-+}) = \begin{pmatrix}
(n - 1)I_n & 0 \\
0 & 2rI_m
\end{pmatrix}.
\]

By Lemma 2.8 (a2), \( A(G^l) = 2I_m - Q^TQ \). Therefore 
\[
L(\lambda, G^{-+}) = \begin{vmatrix}
(\lambda - n + 1)I_n + J_{nn} - I_n - A & Q \\
Q^T & (\lambda - 2r)I_m + Q^TQ - 2I_m
\end{vmatrix}.
\]

Multiplying the first row of the above block matrix by \(-Q^T\) and adding the result to the second row we obtain:
\[
L(\lambda, G^{-+}) = \begin{vmatrix}
(\lambda - n)I_n + J_{nn} - A & Q \\
Q^T - Q^T((\lambda - n)I_n + J_{nn} - A) & (\lambda - 2r - 2)I_m
\end{vmatrix}.
\]

Clearly, it is sufficient to prove our claim for \( \lambda \neq 2r + 2 \). Using Lemmas 2.7 we obtain:
\[
L(\lambda, G^{-+}) = (\lambda - 2r - 2)^{m-n} \times |B|,
\]
where 
\[
B = (\lambda - 2r - 2)((\lambda - n)I_n + J_{nn} - A) - QQ^T((1 - \lambda + n)I_n - J_{nn} + A).
\]

Since by Lemma 2.8 (a1), \( QQ^T = D(G) + A(G) = rI_n + A \), we have:
\[
B = (\lambda - 2r - 2)((\lambda - n)I_n + J_{nn} - A) - (rI_n + A)((1 - \lambda + n)I_n - J_{nn} + A).
\]

Obviously, \( |B| \) is the product of the eigenvalues of \( B \). By Lemma 2.10 the eigenvalues of \( B \) are
\[
\sigma_n = (\lambda - 2r - 2)(\lambda - n + n - r) - (r + r)(1 - \lambda + n - n + r) = \lambda(\lambda - r - 2) \text{ and }
\]
\[
\sigma_i = (\lambda - 2r - 2)(\lambda - n - r + \lambda_i) - (r + r - \lambda_i)((1 - \lambda + n) + r - \lambda_i)
\]
\[
= (\lambda - 2 - \lambda_i)(\lambda - n - r + \lambda_i) - 2r + \lambda_i \text{ for } i = 1, 2, \ldots, n - 1.
\]

Since \( |B| = \prod_{i=1}^{n} \sigma_i \), we have:
\[
L(\lambda, G^{-+}) = \lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}.
\]

\(\square\)

Similarly we can prove the following theorem for \( G^{--} \):
Theorem 4.4. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Then

\[
L(\lambda, G^{--}) = \lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - m - 2 + \lambda_i) - 2r + \lambda_i\}.
\]

Now we can use Reciprocity Theorem 2.6 to obtain from Theorems 4.1 - 4.4 the Laplacian characteristic polynomials of the corresponding complement graphs \( G^{xyz} \).

Theorem 4.5. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges and let \( s = n+m \). Then

(a1) \( L(\lambda, G^{--}) = \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \times \prod_{i=1}^{n-1} \{(\lambda - s + r + \lambda_i)(\lambda - s + 2 + \lambda_i) - 2r + \lambda_i\}, \)

(a2) \( L(\lambda, G^{--}) = \lambda(\lambda - s + r + 2)(\lambda - n - 2r + 2)^{m-n} \times \prod_{i=1}^{n-1} \{(\lambda - s + r + \lambda_i)(\lambda - n - 2 + \lambda_i) - 2r + \lambda_i\}, \)

(a3) \( L(\lambda, G^{++}) = \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \times \prod_{i=1}^{n-1} \{(\lambda - m + r - \lambda_i)(\lambda - s + 2 + \lambda_i) - 2r + \lambda_i\}, \) and

(a4) \( L(\lambda, G^{++}) = \lambda(\lambda - s + r + 2)(\lambda - n - 2r + 2)^{m-n} \times \prod_{i=1}^{n-1} \{(\lambda - m + r - \lambda_i)(\lambda - n + 2 - \lambda_i) - 2r + \lambda_i\}.
\]

Proof. As we have mentioned above, the claims (a1) - (a4) can be easily proven from Theorems 4.1 - 4.4 respectively, using Reciprocity Theorem 2.6. We give below the proof of claim (a1). The proofs of the remaining claims (a2) - (a4) are similar.

Since \( G^{++} \) and \( G^{--} \) are complement, we can apply Reciprocity Theorem to obtain from Theorem 4.1

\[
L(\lambda, G^{--}) = (-1)^{s-1} \frac{\lambda}{s-\lambda} L(s, \lambda, G^{++})
\]

\[
= (-1)^{s-1} \frac{\lambda}{s-\lambda} (s-\lambda)(s-\lambda - r - 2)(s-\lambda - 2r - 2)^{m-n} \prod_{i=1}^{n-1} \{((s-\lambda - r - \lambda_i)(s-\lambda - 2\lambda_i) - 2r + \lambda_i\}
\]

\[
= \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \prod_{i=1}^{n-1} \{(\lambda - s + r + \lambda_i)(\lambda - s + 2 + \lambda_i) - 2r + \lambda_i\}.
\]

From the above results it follows that the transformations \( G^{xyz} \) have the following common Laplacian spectrum properties.

Theorem 4.6. Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges and \( F = G^{xyz} \), where \( z \in \{+, -\} \). Then \( F \) and \( F^c \) have, respectively, the Laplacian eigenvalue
(a1) \( r + 2 \) and \( m + n - r - 2 \) of multiplicity one if \( z = + \),
(a2) \( 2r + 2 \) and \( m + n - 2r - 2 \) of multiplicity \( m - n \) if \( (y, z) = (+, +) \),
(a3) \( m - 2r + 2 \) and \( n + 2r - 2 \) of multiplicity \( m - n \) if \( (y, z) = (-, +) \),
(a4) \( 2 \) and \( m + n - 2 \) of multiplicity \( m - n \) if \( (y, z) = (0, +) \), and
(a5) \( m + 2 \) and \( n - 2 \) of multiplicity \( m - n \) if \( (y, z) = (1, +) \).

The proofs of Theorems 3.6(a1), 3.6(a2), and 4.4 can be found in [?].

5 Transformation graphs of cycles

In this section we first describe some \( xyz \)-transformations of the 4-cycle and the 5-cycle and the Laplacian spectra of these transformations. After that we consider \( xyz \)-transformations of any cycle and show that some different \( xyz \)-transformations of the same cycle may be isomorphic.

Figure 2: The 4-cycle \( C_4 \) and its transformations \( C_4^{+++} \) (middle) and \( C_4^{++-} \) (right)

Let \( C_n \) be the cycle with \( n \) vertices. It is known (see, for example, [2]) that
\[
Sp(C_n) = \{ 2 - 2 \cos \frac{2\pi i}{n} : i = 1, \ldots, n \}.
\]

Let \( G \) be a 4-cycle \( C_4 \). It is easy to see (and it follows from the above formula for \( C_n \)) that \( Sp(C_4) = \{4, 2^{(2)}, 0\} \). Then

(c1) \( C_4^{+++} \) and \( C_4^{++-} \) are isomorphic and by Theorem 4.1 or 4.5

\[
Sp(C_4^{+++}) = Sp(C_4^{++-}) = \{6^{(2)}, 4 + \sqrt{2}^{(2)}, 4, [4 - \sqrt{2}]^{(2)}, 0\},
\]

(c2) \( C_4^{-++} \) and \( C_4^{---} \) are isomorphic and by Theorem 4.4 or 4.5

\[
Sp(C_4^{-++}) = Sp(C_4^{---}) = \{[4 + \sqrt{2}]^{(2)}, 4, 2^{(2)}, 0\}, \text{ and}
\]
(c3) $C_4^{-++}$ and $C_4^{++-}$ are isomorphic and by Theorem 4.2 or 4.3

$$Sp(C_4^{-++}) = Sp(C_4^{++-}) = \{6, [4 + \sqrt{2}]^{(2)}, 4, [4 - \sqrt{2}]^{(2)}, 2, 0\}.$$  

It is easy to prove that $B(C_4)$ and $B^c(C_4)$ are isomorphic. Therefore $C_4^{xy+}$ and $C_4^{xy-}$ are isomorphic for any $x, y \in \{0, 1, +, -\}$.

It is also interesting to consider some transformations of the 5-cycle $C_5$ (the pentagon) because $C_5$ is isomorphic to its complement and $C_5$ is also isomorphic to its line graph. By the above formula for $C_n$, we have:

$$Sp(C_5) = \{[\frac{1}{2}(5 + \sqrt{5})]^{(2)}, [\frac{1}{2}(5 - \sqrt{5})]^{(2)}, 0\}.$$  

Then

(p1) $C_5^{+++}$ is a 4-regular graph (see Fig. 5) and by Theorem 4.1

$$Sp(C_5^{+++}) = \{[\frac{1}{2}(9 + \sqrt{5} + \sqrt{6 - 2\sqrt{5}})]^{(2)}, [\frac{1}{2}(9 + \sqrt{5} - \sqrt{6 - 2\sqrt{5}})]^{(2)}, 4,$$
Figure 5: The 5-cycle $C_5$ and its transformation $C_5^{++}$

Figure 6: $C_5^{--}$

$[\frac{1}{2}(9 - \sqrt{5} + \sqrt{6 - 2\sqrt{5}})^{(2)}, \frac{1}{2}(9 - \sqrt{5} - \sqrt{6 - 2\sqrt{5}})^{(2)}, 0],$

(p2) $C_5^{--}$ is a 4-regular graph (see Fig. 6) and by Theorem 4.4,

$Sp(C_5^{--}) = \{[\frac{1}{2}(13 + \sqrt{5} + \sqrt{6 + 2\sqrt{5}})]^{(2)}, [\frac{1}{2}(13 + \sqrt{5} - \sqrt{6 - 2\sqrt{5}})]^{(2)},$

$[\frac{1}{2}(13 - \sqrt{5} + \sqrt{6 + 2\sqrt{5}})]^{(2)}, [\frac{1}{2}(13 - \sqrt{5} - \sqrt{6 - 2\sqrt{5}})]^{(2)}, 4, 0\},$

(p3) $C_5^{---}$ and $C_5^{++}$ are isomorphic 4-regular graphs (see Fig. 7) and by Theorem 4.2 or 4.3,

$Sp(C_5^{---}) = Sp(C_5^{++}) = \{[\frac{1}{2}(9 + \sqrt{11 + 2\sqrt{5}})]^{(2)}, [\frac{1}{2}(9 + \sqrt{11 - 2\sqrt{5}})]^{(2)},$

$4, [\frac{1}{2}(9 + \sqrt{11 - 2\sqrt{5}})]^{(2)}, [\frac{1}{2}(9 - \sqrt{11 + 2\sqrt{5}})]^{(2)}, 0\}.$
By (c3) and (p3), $G^{-+}$ and $G^{+-}$ are isomorphic if $G$ is either $C_4$ or $C_5$. As we will see below, a more general claim is true not only for $C_4$ and $C_5$ but for any cycle $C$.

**Theorem 5.1.** Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. If $m = n$, then $G^{xyz}$ and $G^{yxz}$ are isomorphic for all $x, y, z \in \{0, 1, +, -\}$.

**Proof.** By the Reciprocity Theorem 2.6, it is sufficient to prove our claim for $x, y \in \{0, 1, +, -\}$ and $z \in \{0, +\}$. Since $m = n$ and $nr = 2m$, we have $r = 2$, and so $G$ is 2-regular. Then $G$ is a disjoint union of cycles. If $A$ and $B$ are disjoint graphs, then $(A \cup B)^{xy0} = A^{xy0} \cup A^{xy0}$ and $(A \cup B)^{xy+} = A^{xy+} \cup A^{xy+}$. Therefore it is sufficient to prove our claim for a connected graph $G$. In this case $G$ is a cycle on $n$ vertices and we can assume that $V(G) = V = \{v_1, \ldots, v_n\}$ and $E(G) = E = \{e_i : i = 1, \ldots, n\}$, where $e_i = v_iv_{i+1}$ for $i = 1, \ldots, n$ and $i + 1$ is considered mod $n$, and so $e_n = v_nv_1$. Let for each $i$, $\alpha(v_i) = \sigma(v_{i+1}) = e_i$. Then both $\alpha$ and $\sigma$ are isomorphisms from $G$ to $G^l$. Recall that $G^+ = G$, $G^- = G^c$, $G^0$ is the the graph with $V(G^0) = V$ and with no edges, and $G^1$ the a complete graph with $V(G^1) = V$. Hence for every $x \in \{0, 1, +, -\}$, both $\alpha$ and $\sigma$ are isomorphisms from $G^x$ to $(G^l)^x$. Put $\pi|_V = \alpha$ and $\pi|_E = \sigma^{-1}$. Since $G^{xy0}$ is a disjoint union of $G^x$ and $(G^l)^y$, we have: $\pi$ is an isomorphism from $G^{xy0}$ to $G^{yx0}$.

Now we show that $\pi$ is also an isomorphism from $G^{xy+}$ to $G^{yx+}$. By definition of $G^{xy+}$, $E(G^{xy+}) = E(G^x) \cup E((G^l)^y) \cup E(W)$, where $E(W) = \{v_ie_i, v_{i+1}e_i : i = 1, \ldots, n\}$. Recall that each $e_i = v_iv_{i+1}$. Since $\pi(v_i) = \alpha(v_i) = e_i$ and $\pi(e_j) = \sigma^{-1}(e_j) = v_{j+1}$, vertices $\pi(v_i)$ and $\pi(e_j)$ are adjacent in $G^{yx+}$ if and only if $j + 1 = i$ or $j + 1 = i + 1$ which is equivalent to $i = j + 1$ or $i = j$. Therefore $v_i$ and $e_j$ are adjacent in $G^{xy+}$ if and only if $\pi(v_i)$ and $\pi(e_j)$ are adjacent in $G^{yx+}$.

$\square$
6 Some remarks and questions

(R1) Each factor of the Laplacian polynomials of $G^{xyz}$ ($x, y, z \in \{0, 1, +, -\}$) is a polynomial in $\lambda$ of degree one or two. Therefore the explicit formula for the Laplacian spectrum and the number of spanning trees of $G^{xyz}$ can be given in terms of those of $G$, respectively, as in Corollaries 3.4, 3.8, and 3.11.

(R2) Let $\mathcal{R}$ be the set of regular graphs. Obviously, if $G \in \mathcal{R}$, then $G^{c} \in \mathcal{R}$ and $G^{l} \in \mathcal{R}$. If $G$ is an $r$-regular graph, then $G^{+++}$ is $2r$-regular and $G^{---}$ is $(v(G) + e(G) - 2r - 1)$-regular, and so if $G \in \mathcal{R}$, then $G^{+++} \in \mathcal{R}$. In other words, the set $\mathcal{R}$ of regular graphs is closed under $c$-operation, $l$-operation, $(+++)$-operation, and $(---)$-operation. Therefore using the corresponding results described above, one can give an algorithm (and the computer program) that for any series $Z$ of $c$-, $l$-, $(++)$-, and $(--)$-operations and the Laplacian spectrum $Sp(G)$ of any $r$-regular graph $G$ provides the formula of the Laplacian spectrum of graph $F$ obtained from $G$ by the operation series $Z$ in terms of $r$, $v(G)$, and $Sp(G)$.

(R3) Examples and results in Section 5 show that there exists a regular graph $G$ such that $G^{xyz}$ and $G^{x'y'z'}$ are isomorphic although $(x, y, z) \neq (x', y', z')$, where $x, y, z \in \{0, 1, +, -\}$. It is also easy to see that if $K$ is a complete graph, then $K^{0yz} = K^{-yz}$ and $K^{x0z} = K^{x-z}$ as well as $K^{1yz} = K^{+yz}$ and $K^{x1z} = K^{x+z}$.

(R4) Suppose that a regular graph $G$ is uniquely defined by its Laplacian spectrum. Does it necessarily mean that $G^{xyz}$ is also uniquely defined by its Laplacian spectrum for every (or for some) $x, y, z \in \{+, -\}$?

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Appendix

Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges and let $s = n + m$, and so $2m = rn$ and $s$ is the number of vertices of $G^{xyz}$. The tables below provide the formulas for $L(\lambda, G^{xyz})$ and $t(G^{xyz})$ for all $x, y, z \in \{0, 1, +, -\}$ in terms of $n$, $m$, $r$, and the Laplacian eigenvalues of $G$. Obviously, $t(G^{xyz}) = 0$ if and only if $G^{xyz}$ is not connected. Note that for $z = 0$, graph $G^{xyz}$ is not connected, and so $t(G^{xyz}) = 0$. For $z = 1$, graph $G^{xyz}$ is connected, and so $t(G^{xyz}) > 0$.

The list of $L(\lambda, G^{xyz})$ and $t(G^{xyz})$ with $z = 0$.

| $xyz$ | $L(\lambda, G^{xyz})$ | $t(G^{xyz})$ |
|-------|-----------------------|--------------|
| 1 0 0 | $\lambda^{m+n}$       | 0            |
| 2 1 0 | $\lambda^{m+1}(\lambda - n)^{n-1}$ | 0 |
| 3 +0 0 | $\lambda^m L(\lambda, G)$ | 0 |
| 4 −0 0 | $(-1)^n(\lambda - n)^{-1}\lambda^{m+1}L(n - \lambda, G)$ | 0 |
| 5 0 1 0 | $\lambda^{n+1}(\lambda - m)^{m-1}$ | 0 |
| 6 1 1 0 | $\lambda^{2}(\lambda - m)^{m-1}(\lambda - n)^{n-1}$ | 0 |
| 7 +1 0 | $\lambda(\lambda - m)^{m-1}L(\lambda, G)$ | 0 |
| 8 −1 0 | $(-1)^n\lambda^{2}(\lambda - n)^{-1}(\lambda - m)^{m-1}L(n - \lambda, G)$ | 0 |
| 9 0 + 0 | $\lambda^n(\lambda - 2r)^{m-n}L(\lambda, G)$ | 0 |
| 10 1 + 0 | $\lambda(\lambda - n)^{n-1}(\lambda - 2r)^{m-n}L(\lambda, G)$ | 0 |
| 11 + + 0 | $(\lambda - 2r)^{m-n}L(\lambda, G)^2$ | 0 |
| 12 − + 0 | $(-1)^n(\lambda - 2r)^{m-n}\lambda(\lambda - n)^{-1}L(n - \lambda, G)L(\lambda, G)$ | 0 |
| 13 0 − 0 | $(-1)^n(\lambda - m)^{-1}\lambda^{n+1}(\lambda - m + 2r)^{m-n}L(m - \lambda, G)$ | 0 |
| 14 1 − 0 | $(-1)^n(\lambda - m)^{-1}\lambda^2(\lambda - n)^{n-1}(\lambda - m + 2r)^{m-n}L(m - \lambda, G)$ | 0 |
| 15 + − 0 | $(\lambda - m)^{-1}(\lambda - m + 2r)^{m-n}L(m - \lambda, G)L(\lambda, G)$ | 0 |
| 16 − − 0 | $\lambda^2(\lambda - m)^{-1}(\lambda - n)^{-1}(\lambda - m + 2r)^{m-n}L(m - \lambda, G)L(n - \lambda, G)$ | 0 |
The list of $L(\lambda, G^{xyz})$ and $t(G^{xyz})$ with $z = 1$.

| $xyz$ | $L(\lambda, G^{xyz})$ | $t(G^{xyz})$ |
|-------|------------------------|--------------|
| 1 0 0 1 | $\lambda(\lambda - s)(\lambda - n)^{m-1}(\lambda - m)^{n-1}$ | $n^{n-1}m^{n-1}$ |
| 2 1 0 1 | $\lambda(\lambda - s)^n(\lambda - n)^{m-1}$ | $n^{n-1}s^{n-1}$ |
| 3 +0 1 | $(\lambda - m)^{-1}(\lambda - n)^{m-1}(\lambda - m, G)$ | $n^{n-1}\prod_{i=1}^{m-1}(m + \lambda_i)$ |
| 4 −0 1 | $(-1)^n\lambda(\lambda - n)^{m-1}L(\lambda - s, G)$ | $n^{n-1}\prod_{i=1}^{n-1}(s - \lambda_i)$ |
| 5 0 1 1 | $\lambda(\lambda - s)^m(\lambda - m)^{n-1}$ | $s^{n-1}m^{n-1}$ |
| 6 1 1 1 | $\lambda(\lambda - s)^s^{-1}$ | $s^{s-2}$ |
| 7 +1 1 | $\lambda(\lambda - m)^{-1}(\lambda - s)^mL(\lambda - m, G)$ | $s^{m-1}\prod_{i=1}^{n-1}(m + \lambda_i)$ |
| 8 −1 1 | $(-1)^n\lambda(\lambda - s)^mL(\lambda - s, G)$ | $s^{m-1}\prod_{i=1}^{n-1}(s - \lambda_i)$ |
| 9 0 + 1 | $\lambda(\lambda - s)(\lambda - n)^{-1}(\lambda - m)^{n-1}$ | $m^{n-1}(n + 2r)^{m-n}\prod_{i=1}^{n-1}(n + \lambda_i)$ |
| 10 1 + 1 | $\lambda(\lambda - n)^{-1}(\lambda - s)^n(\lambda - n - 2r)^{m-n}$ | $s^{n-1}(n + 2r)^{m-n}$ |
| 11 + + 1 | $\lambda(\lambda - s)(\lambda - m)^{-1}(\lambda - n)^{-1}$ | $(n + 2r)^{m-n}$ |
| 12 − + 1 | $\lambda(\lambda - n)^{-1}(\lambda - n - 2r)^{m-n}$ | $(n + 2r)^{m-n}$ |
| 13 0 − 1 | $(-1)^n\lambda(\lambda - m)^{n-1}(\lambda - s + 2r)^mL(\lambda - s, G)$ | $m^{n-1}(s - 2r)^{m-n}\prod_{i=1}^{n-1}(s - \lambda_i)$ |
| 14 1 − 1 | $(-1)^n\lambda(\lambda - s)^{n-1}(\lambda - s + 2r)^mL(\lambda - s, G)$ | $s^{n-1}(s - 2r)^{m-n}\prod_{i=1}^{n-1}(s - \lambda_i)$ |
| 15 + − 1 | $(-1)^n\lambda(\lambda - m)^{-1}(\lambda - s + 2r)^{m-n}$ | $(s - 2r)^{m-n}$ |
| 16 − − 1 | $\lambda(\lambda - s)^{-1}(\lambda - s + 2r)^{m-n}L(\lambda - s, G)^2$ | $(s - 2r)^{m-n}\prod_{i=1}^{n-1}(s - \lambda_i)^2$ |
The list of $L(\lambda, G^{xyz})$ and $t(G^{xyz})$ with $z = +$.

| xyz | $L(\lambda, G^{xyz})$ | $t(G^{xyz})$ |
|-----|------------------------|--------------|
| 1 0 0+ | $\lambda(\lambda - r - 2)(\lambda - 2)^{m-n} \prod_{i=1}^{n-1} \{\lambda^2 - \lambda(r + 2) + \lambda_i\}$ | $ns^{-1}(r + 2)2^{m-n}t(G)$ |
| 2 1 0+ | $\lambda(\lambda - r - 2)(\lambda - 2)^{m-n}$ | $s^{-1}(r + 2)2^{m-n} \prod_{i=1}^{n-1} (2n + \lambda_i)$ |
| 3 +0+ | $\lambda(\lambda - r - 2)(\lambda - 2)^{m-n}$ | $ns^{-1}(r + 2)2^{m-n}3^{n-1}t(G)$ |
| 4 -0+ | $\lambda(\lambda - r - 2)(\lambda - 2)^{m-n}$ | $s^{-1}(r + 2)2^{m-n} \prod_{i=1}^{n-1} (2n - \lambda_i)$ |
| 5 0 1+ | $\lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 6 1 1+ | $\lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 7 +1+ | $\lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 8 -1+ | $\lambda(\lambda - r - 2)(\lambda - m - 2)^{m-n}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 9 0 ++ | $\lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n}$ | $ns^{-1}(r + 2)2^{m-n}(r + 1)^{m-1}t(G)$ |
| 10 1 ++ | $\lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n}$ | $s^{-1}(r + 2)(2r + 2)^{m-n}$ |
| 11 ++ ++ | $\lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n}$ | $ns^{-1}(r + 2)(2r + 2)^{m-n}t(G)$ |
| 12 - + ++ | $\lambda(\lambda - r - 2)(\lambda - 2r - 2)^{m-n}$ | $s^{-1}(r + 2)(2r + 2)^{m-n}$ |
| 13 0 - + | $\lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
| 14 1 - + | $\lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
| 15 + - ++ | $\lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
| 16 - - ++ | $\lambda(\lambda - r - 2)(\lambda - m + 2r - 2)^{m-n}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |

| xyz | $L(\lambda, G^{xyz})$ | $t(G^{xyz})$ |
|-----|------------------------|--------------|
| 1 0 0+ | $\prod_{i=1}^{n-1} \{(\lambda - 2)(\lambda - n - r) - 2r + \lambda_i\}$ | $ns^{-1}(r + 2)2^{m-n}t(G)$ |
| 2 1 0+ | $\prod_{i=1}^{n-1} \{(\lambda - 2)(\lambda - n - r) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)2^{m-n} \prod_{i=1}^{n-1} (2n + \lambda_i)$ |
| 3 +0+ | $\prod_{i=1}^{n-1} \{(\lambda - 2)(\lambda - n - r) - 2r + \lambda_i\}$ | $ns^{-1}(r + 2)2^{m-n}3^{n-1}t(G)$ |
| 4 -0+ | $\prod_{i=1}^{n-1} \{(\lambda - 2)(\lambda - n - r) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)2^{m-n} \prod_{i=1}^{n-1} (2n - \lambda_i)$ |
| 5 0 1+ | $\prod_{i=1}^{n-1} \{(\lambda - n - r)(\lambda - m - 2) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 6 1 1+ | $\prod_{i=1}^{n-1} \{(\lambda - n - r)(\lambda - m - 2) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 7 +1+ | $\prod_{i=1}^{n-1} \{(\lambda - r - \lambda_i)(\lambda - m - 2) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 8 -1+ | $\prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - m - 2) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m + 2)^{m-n}$ |
| 9 0 ++ | $\prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - 2r - 2) - 2r + \lambda_i\}$ | $ns^{-1}(r + 2)2^{m-n}(r + 1)^{m-1}t(G)$ |
| 10 1 ++ | $\prod_{i=1}^{n-1} \{(\lambda - n - r)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(2r + 2)^{m-n}$ |
| 11 ++ ++ | $\prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}$ | $ns^{-1}(r + 2)(2r + 2)^{m-n}t(G)$ |
| 12 - + ++ | $\prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - 2 - \lambda_i) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(2r + 2)^{m-n}$ |
| 13 0 - + | $\prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - m - 2 - \lambda_i) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
| 14 1 - + | $\prod_{i=1}^{n-1} \{(\lambda - n - r)(\lambda - m - 2 - \lambda_i) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
| 15 + - ++ | $\prod_{i=1}^{n-1} \{(\lambda - r)(\lambda - m + 2r - 2 - \lambda_i) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
| 16 - - ++ | $\prod_{i=1}^{n-1} \{(\lambda - n - r + \lambda_i)(\lambda - m - 2 - \lambda_i) - 2r + \lambda_i\}$ | $s^{-1}(r + 2)(m - 2r + 2)^{m-n}$ |
The list of \( L(\lambda, G_{xyz}) \) and \( t(G_{xyz}) \) with \( z = -n \):

| \( xyz \) | \( L(\lambda, G_{xyz}) \) | \( t(G_{xyz}) \) |
|------|--------------------------|--------------------------|
| 1 0 0 | \( \lambda(\lambda - s + r + 2)(\lambda - n + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n - 2)^{m-n} \) |
| 2 1 0 | \( \lambda(\lambda - s + r + 2)(\lambda - n + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n - 2)^{m-n} \) |
| 3 +0 | \( \lambda(\lambda - s + r + 2)(\lambda - n + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n - 2)^{m-n} \) |
| 4 -0 | \( \lambda(\lambda - s + r + 2)(\lambda - n + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n - 2)^{m-n} \) |
| 5 0 1 | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2)^{m-n} \) |
| 6 1 1 | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2)^{m-n} \) |
| 7 +1 | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2)^{m-n} \) |
| 8 -1 | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2)^{m-n} \) |
| 9 0 + | \( \lambda(\lambda - s + r + 2)(\lambda - n - 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n + 2r - 2)^{m-n} \) |
| 10 1 + | \( \lambda(\lambda - s + r + 2)(\lambda - n - 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n + 2r - 2)^{m-n} \) |
| 11 ++ | \( \lambda(\lambda - s + r + 2)(\lambda - n - 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n + 2r - 2)^{m-n} \) |
| 12 -- | \( \lambda(\lambda - s + r + 2)(\lambda - n - 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(n + 2r - 2)^{m-n} \) |
| 13 0 -- | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2r - 2)^{m-n} \) |
| 14 1 -- | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2r - 2)^{m-n} \) |
| 15 ++ | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2r - 2)^{m-n} \) |
| 16 -- | \( \lambda(\lambda - s + r + 2)(\lambda - s + 2r + 2)^{m-n} \) | \( s^{-1}(s - r - 2)(s - 2r - 2)^{m-n} \) |