ON SMALL DUAL RINGS

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Abstract. A ring $R$ is called right (small) dual if every (small) right ideal of $R$ is a right annihilator. Left (small) dual rings can be defined similarly. And a ring $R$ is called (small) dual if $R$ is left and right (small) dual. It is proved that $R$ is a dual ring if and only if $R$ is a semilocal and small dual ring. Several known results are generalized and properties of small dual rings are explored. As applications, some characterizations of QF rings are obtained through small dualities of rings.

Keywords: Small dual rings; dual rings; semilocal rings, QF rings.

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1. Introduction

Throughout this paper rings are associative with identity. For a subset $X$ of a ring $R$, the left annihilator of $X$ in $R$ is $l(X) = \{r \in R: rx = 0 \text{ for all } x \in X\}$. Right annihilators are defined analogously. We write $J$, $Z_l$, $Z_r$, $S_l$ and $S_r$ for the Jacobson radical, the left singular ideal, the right singular ideal, the left socle and the right socle of $R$, respectively. Let $M$ be an $R$-module, $N \subseteq_{ess} M$ means that $N$ is an essential submodule of $M$. We denote by $f = c$-that $f$ is a left multiplication map by the element $c \in R$. Right multiplication map can be defined similarly. Let $R$ be a ring. We write $M_R$ for the category of right $R$-modules. And $M_n(R)$ denotes the ring of all $n \times n$ matrices over $R$.

Recall that a ring $R$ is called right dual if every right ideal $I$ of $R$ is a right annihilator. Left dual rings can be defined similarly. And $R$ is called a dual ring if it is left and right dual. Dual rings were firstly investigated by Baer [2], Hall [5], Kaplansky [6], and discussed in detail by Hajarnavis, Norton [4]. There are several generalizations of right dual rings. Recall that a ring $R$ is
called right Kasch if every maximal right ideal of $R$ is a right annihilator. Left Kasch rings are defined analogously. And $R$ is called a Kasch ring if it is both left and right Kasch. Kasch rings are named in honor of Friedrich Kasch. These rings were discussed in detail by Morita [10], Nicholson, Yousif [13] and so on. A ring $R$ is called right (left) quasi-dual if every essential right (left) ideal of $R$ is a right (left) annihilator. $R$ is called quasi-dual if it is two-sided quasi-dual. These rings were studied by Page and Zhou [14].

In this article, we discuss the condition that every small right ideal of a ring $R$ is a right annihilator. And these rings are called right small dual rings. Recall that a right ideal $I$ of $R$ is small if for any proper right ideal $K$ of $R$, $I + K \neq R$. Left small dual rings can be defined similarly. A ring $R$ is called small dual if it is both left and right small dual. By definition, a small right ideal is a “superfluous” part of $R$. We will give a short proof in Theorem 2.12 to show that a right small dual ring $R$ is also a right dual ring if it is semilocal. Hence it is proved in Theorem 3.1 that a ring $R$ is a dual ring if and only if it is a semilocal and small dual ring if and only if it is a semiperfect and small dual ring. Some other properties of small dual rings are explored and several known results are improved. As applications, we obtain some characterizations of QF rings given by small dualities. Recall that a ring $R$ is called a quasi-Frobenius (QF) ring if it is a right artinian and two-sided dual ring.

2. Results

**Definition 2.1.** A ring $R$ is called right small dual if every small right ideal $I$ of $R$ is a right annihilator, that is, $\text{rl}(I) = I$. Left small dual rings can be defined similarly. $R$ is called small dual if $R$ is left and right small dual.

Recall that a ring $R$ is called a semiprimitive ring if the Jacobson radical $J$ of $R$ is zero. Since each small one-sided ideal of $R$ is contained in $J$, it is clear that every semiprimitive ring is small dual.

**Example 2.2.** The ring of integers $\mathbb{Z}$ is a right small dual ring which is not right dual.
Proof. Since \( Z \) is a semiprimitive ring, \( Z \) is right small dual. It is easy to see that \( Z \) is not right dual. \( \square \)

Example 2.3. (Björk Example) Let \( F \) be a field and assume that \( a \mapsto \bar{a} \) is an isomorphism \( F \to \bar{F} \subseteq F \), where the subfield \( \bar{F} \neq F \). Let \( R \) denote the left vector space on basis \( \{1, t\} \), and make \( R \) into an \( F \)-algebra by defining \( t^2 = 0 \) and \( ta = \bar{a}t \) for all \( a \in F \). Then

1. \( R \) is left small dual but not right small dual.
2. \( M_2(R) \) is not left small dual.

Proof. (1). By [13, Example 2.5, Example 7.3], \( R \) is a left CF ring (every left ideal is a left annihilator of finite elements in \( R \)) but not left mininjective. Recall that a ring \( R \) is called left mininjective if every homomorphism from a minimal left ideal of \( R \) to \( _R R \) can be extended to one from \( _R R \) to \( _R R \). Since \( R \) is left CF, \( R \) is left dual. Thus \( R \) is left small dual. But \( R \) is not right small dual, if \( R \) is right small dual, by following Proposition 2.6 (4), \( R \) is left mininjective. This is a contradiction.

(2). By [13, Example 2.5], \( R \) is a left artinian ring which is not QF. Thus \( M_2(R) \) is a semilocal ring. If \( M_2(R) \) is left small dual, by following Theorem 2.12, \( M_2(R) \) is a left dual ring. Thus \( M_2(R) \) is right \( P \)-injective by [13, Lemma 5.1]. Recall that a ring \( R \) is called right \( P \)-injective (right \( n \)-injective) if every homomorphism from a principal (\( n \)-generated) right ideal of \( R \) to \( _R R \) can be extended to one from \( _R R \) to \( _R R \). Since \( R \) is left artinian and right \( 2 \)-injective, \( R \) is a right \( 2 \)-injective ring with ACC on right annihilators. Thus, by [13, Corollary 3], \( R \) is a QF ring. This is a contradiction. \( \square \)

Recall that a right \( R \)-module \( M \) is called torsionless if it can be cogenerated by \( R_R \). That is, \( M \) can be embedded into a direct product of \( R_R \).

Lemma 2.4. [13, Lemma 1.40] If \( T \) is a right ideal of \( R \), then \( rl(T) = T \) if and only if \( R/T \) is torsionless.

Proposition 2.5. The following are equivalent for a ring \( R \).
(1) \( R \) is right small dual.

(2) \( r(l(T) \cap Rb) = T + r(b) \) for all small right ideals \( T \) and all \( b \in R \).

(3) \( R/T \) is torsionless for every small right ideal \( T \) of \( R \).

Proof. (1)\( \Leftrightarrow \) (3) is clear by Lemma 2.4. (1)\( \Leftrightarrow \) (2) can be obtained from \[12\] Lemma 2.1. For the sake of completeness, we provide the proof. (2)\( \Rightarrow \) (1) is obvious if we set \( b = 1 \). For (1)\( \Rightarrow \) (2), it is clear that \( T + r(b) \subseteq r(l(T) \cap Rb) \).

Now we prove the converse. Assume \( x \in r(l(T) \cap Rb) \) and \( y \in l(bT) \). It is easy to see \( yb \in l(T) \cap Rb \). Thus \( ybx = 0 \). This informs that \( l(bT) \subseteq l(bx) \).

So \( bx \in rl(bx) \subseteq rl(bT) \). Since \( T \) is a small right ideal of \( R \), \( bT \) is also small. As \( R \) is right small dual, \( rl(bT) = bT \). This implies that \( bx \in bT \).

So there exists \( t \in T \) such that \( bx = bt \). Thus \( x - t \in r(b) \). Therefore, \( x = t + x - t \in T + r(b) \).

Proposition 2.6. Let \( R \) be a right small dual ring. Then

(1) \( l(J) \subseteq \text{ess} \, R \).

(2) \( J \subseteq Z \).

(3) \( rl(T) = T \) for every finitely generated semisimple right ideal \( T \) of \( R \).

(4) Every homomorphism from a small principal left ideal to \( R \) can be extended to one from \( R \) to \( R \). In particularly, \( R \) is left mininjective.

Proof. (1) and (2) are obtained by \[20\] Proposition 3.2.

We will prove (3) by induction on \( \text{G.dim} \ (T) \) which is the Goldie (or called uniform) dimension of \( T \). If \( \text{G.dim} \ (T) = 1 \), then \( T \) is a minimal right ideal of \( R \). By \[3\] Lemma 10.22, \( T \) is either nilpotent or a direct summand of \( R \). If \( T \) is nilpotent then it is contained in \( J \), which is the biggest small right (or left) ideal of \( R \). This informs that \( T \) is small. Thus \( T \) is a right annihilator.

If \( T \) is a direct summand of \( R \), it is clear that \( T \) is a right annihilator.

Now we assume that \( rl(T) = T \) for every semisimple right ideal \( T \) of \( R \) with \( \text{G.dim} \ (T) \leq n \), where \( n \geq 1 \). We are going to show that \( rl(K) = K \) for every semisimple right ideal \( K \) of \( R \) with \( \text{G.dim} \ (K) = n + 1 \). If \( K \) is small then \( K \) is already a right annihilator. If \( K \) is not small, it is easy to see that \( K = T \oplus eR \) where \( \text{G.dim} \ (T) = n \), \( e^2 = e \in R \), and \( eR \) is a minimal right ideal of \( R \). Thus
\( r_l(K) = r_l(T \oplus eR) = r_l(I(T) \cap I(e)) = r_l(I(T) \cap R(1-e)) \). Since \( T \) and \( (1-e)T \) are semisimple right ideals of \( R \) with \( G.l.dim ((1-e)T) \leq G.l.dim (T) = n \). By assumption, \( r_l(T) = T \) and \( r_l((1-e)T) = (1-e)T \). Applying a similar proof in Proposition 2.5 “(1) \Rightarrow (2)”, we have \( r_l(I(T) \cap R(1-e)) = T + r_l(1-e) = T + eR = K \).

(4). If \( Rt \) is a small principal left ideal of \( R \), then \( tR \) is a small right ideal of \( R \). By assumption, \( r_l(tR) = tR \). Let \( f \) be an \( R \)-linear map from \( Rt \) to \( R e R \). It is clear that \( I(t) \subseteq I(f(t)) \). Thus \( f(t) \subseteq r_l(f(t)) \subseteq r_l(t) = tR \). Hence \( f = \cdot c \) is a right multiplication map by some element \( c \in R \). Since a minimal left ideal is either small or a direct summand of \( R e R \), it is clear that a right small dual ring is left mininjective. □

**Question 2.7.** If \( R \) is a right small dual ring, is \( r_l(S_r) = S_r \)?

By (3) in the above proposition, the answer is “Yes” if \( S_r \) is finitely generated as a right \( R \)-module.

**Theorem 2.8.** Let \( R \) be a right small dual ring and \( e \) an idempotent of \( R \) such that \( ReR = R \). Then \( eRe \) is also a right small dual ring.

**Proof.** Let \( S = eRe \). Since \( ReR = R \), it is clear that \( R \) and \( S \) are equivalent rings with equivalence functors \( F = (- \otimes_R Re) : M_R \to M_S \) and \( G = (- \otimes_S eR) : M_S \to M_R \). According to Proposition 2.5, we will show that \( S/K \) is torsionless for every small right ideal \( K \) of \( S \). It is clear that \( G(S/K) \cong G(S)/G(K) \) and \( G(K) \) is small in \( G(S) \cong eR \). Thus, by [1, Proposition 21.6 (3)], \( S/K \) is cogenerated by \( S \) if and only if \( eR/T \) is cogenerated by \( eR \), where \( T \) is a small submodule of \( eR \). Now we only need to prove that \( eR/T \) can be cogenerated by \( eR \). It is clear that \( T \) is also a small right ideal of \( R \). As \( R \) is right small dual, by Proposition 2.5, \( eR/T \hookrightarrow R/T \) which is a torsionless right \( R \)-module. This informs that \( eR/T \) can be cogenerated by \( R e R \). Since \( ReR = R \), \( eR \) is a progenerator of \( M_R \). It is easy to see that \( R \) can be cogenerated by \( eR \). Hence \( eR/T \) can also be cogenerated by \( eR \). Thus \( S \) is a right small dual ring. □
The following example shows that if $R$ is small dual then $eRe$ may not be small dual, where $e^2 = e \in R$.

**Example 2.9.** Let $R$ be the algebra of matrices over a field $K$ of the form

$$
R = \begin{bmatrix}
  a & x & 0 & 0 & 0 \\
  0 & b & 0 & 0 & 0 \\
  0 & 0 & c & y & 0 \\
  0 & 0 & 0 & a & 0 \\
  0 & 0 & 0 & b & z \\
  0 & 0 & 0 & 0 & c
\end{bmatrix}, \ a, b, c, x, y, z \in K.
$$

Set $e = e_{11} + e_{22} + e_{44} + e_{55}$, which is a sum of canonical matrix units. Then $e$ is an idempotent of $R$ such that $ReR \neq R$. $R$ is a small dual ring. But $eRe$ is not small dual.

**Proof.** [7, Example 9] informs that $R$ is a QF ring and $eRe$ is not a dual ring. Since $R$ is QF, $R$ is small dual and $eRe$ is a semilocal ring. If $eRe$ is small dual, by following Theorem 2.12, $eRe$ is a dual ring. This contradiction implies that $eRe$ is not small dual. \hfill \Box

For an $R$-module $N$, we write $N^{m \times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of $N$. To be convenient, write $N^n = N^{1 \times n}$ and $N_n = N^{n \times 1}$. Let $M_R$, $R_N$ be $R$-modules. If $X \subseteq M^{l \times m}$, $S \subseteq R^{m \times n}$ and $Y \subseteq N^{n \times k}$. Define

$$
r_{R^{m \times n}}(X) = \{s \in R^{m \times n} : xs = O_{l \times n}, \forall x \in X\},$$

$$l_{R^{m \times n}}(Y) = \{s \in R^{m \times n} : sy = O_{m \times k}, \forall y \in Y\}.$$

Example 2.3 shows that a matrix ring over a left small dual ring may not be left small. Next we have

**Theorem 2.10.** The following are equivalent for a ring $R$.

1. $M_n(R)$ is a right small dual ring.
2. $\forall K_R \leq J_n$, every $n$-generated right $R$-module $R_n/K$ is torsionless.
3. $\forall K_R \leq J_n$ and $b \in R_n$, $l_{R^n}(K) \subseteq l_{R^n}(b)$ implies $b \in K$.
4. $\forall K_R \leq J_n$, $r_{R_n}l_{R^n}(K) = K$. 


Proof. (1)$\iff$(2). Set $S = M_n(R)$ and $P = R_n$. Then $P$ is a left $S$-right $R$-bimodule. By [1] Theorem 22.2, Corollary 22.4, $R$ and $S$ are equivalent rings with equivalence functors $F = \text{Hom}_R(SP_R, -) : M_R \to M_S$ and $G = (- \otimes_S P) : M_S \to M_R$. By Proposition 2.5, $S$ is right small dual if and only if $S/K$ is cogenerated by $S_S$ for every small right ideal $K$ of $S$. Since $G(S/K) \cong G(S)/G(K)$, by [1] Proposition 21.6 (3)], $S/K$ is cogenerated by $S$ if and only if $G(S)/G(K)$ is cogenerated by $G(S) \cong P$. According to [1] Proposition 21.6 (5], $K$ is a small right ideal of $S$ if and only if $G(K)$ is a small submodule of $G(S)$. Now replace $G(S)$ by $P$. Since $J_n$ is the radical of $R_n$, it is clear that (1)$\iff$(2).

(2)$\implies$(3). Let $K_R \leq J_n$ and $1_{R^n}(K) \subseteq 1_{R^n}(b)$ for some $b \in R_n$. If $b \notin K$ then $0 \neq \overline{b} = b + K \in R_n/K$. Since $R_n/K$ is torsionless, there exists an $R$-linear map $f : R_n/K \to R_R$ such that $f(\overline{b}) \neq 0$. Now set $g : R_n \to R$ with $g(x) = f(\overline{x}), \forall x \in R_n$. Then $g$ is a right $R$-linear map from $R_n$ to $R_R$ with $g(K) = 0$ and $g(b) \neq 0$. It is not difficult to see that there exists $c \in R^n$ such that $g(x) = cx, \forall x \in R_n$. Thus $cK = g(K) = 0$ implies that $c \in 1_{R^n}(K) \subseteq 1_{R^n}(b)$. So $g(b) = cb = 0$. This is a contradiction.

(3)$\implies$(2). Let $K_R \leq J_n$. By [1] Proposition 8.10 (2], it is equivalent to prove that for each $0 \neq \overline{b} \in R_n/K$, there exists a right $R$-homomorphism from $R_n/K$ to $R_R$ such that $g(\overline{b}) \neq 0$. If $0 \neq \overline{b}$ then $b \notin K$. (3) informs that $1_{R^n}(K) \not\subseteq 1_{R^n}(b)$. Thus there exists $c \in R_n$ such that $cK = 0$ and $cb \neq 0$. Now define a map $g : R_n/K \to R_R$ with $g(\overline{x}) = cx$. It is clear that $g$ is a well-defined right $R$-linear map from $R_n/K$ to $R_R$ with $g(\overline{b}) = cb \neq 0$.

(3)$\implies$(4). Let $K_R \leq J_n$. It is obvious that $K \subseteq r_{R_n}1_{R^n}(K)$. If $b \in r_{R_n}1_{R^n}(K)$ then $1_{R^n}(K) \subseteq 1_{R^n}(b)$. Applying (3), $b \in K$.

(4)$\implies$(3). Let $K_R \leq J_n$. If $1_{R^n}(K) \subseteq 1_{R^n}(b)$ then $b \in r_{R_n}1_{R^n}(b) \subseteq r_{R_n}1_{R^n}(K) = K$.

\[\square\]

If $R$ is a local ring, then every proper right ideal of $R$ is small. It is clear that a local and right small dual ring is right dual. In Theorem 2.12, we will
show that a semilocal and right small dual ring is right dual. Recall that a
right ideal $I$ of a ring $R$ has a weak supplement in $R$ if there exists a right
ideal $K$ of $R$ such that $I + K = R$ and $I \cap K$ is a small right ideal of $R$.

Lemma 2.11. The following are equivalent for a ring $R$.

(1) $R$ is right semilocal.
(2) Every right ideal $I$ of $R$ has a weak supplement in $R$.
(3) Every left ideal $I$ of $R$ has a weak supplement in $R$.

Proof. See [9, Corollary 3.2].\hfill $\Box$

Theorem 2.12. If $R$ is a semilocal ring, then $R$ is right small dual if and only
if $R$ is right dual.

Proof. We only need to prove the necessity. Let $I$ be any right ideal of $R$. We
will show that $I$ is a right annihilator. According to Lemma 2.4, it is equivalent
to prove that $R/I$ is torsionless. Since $R$ is semilocal, by Lemma 2.11, $I$ has a
weak supplement $K$ in $R$. That is, $I + K = R$ and $I \cap K$ is small. As $R$ is right
small dual, by Proposition 2.5, $\frac{R}{I + K} \cong \frac{R^A}{I + K}$ for some set $A$. Thus we have

$$\frac{R}{I} = \frac{I + K}{I} \cong \frac{K}{I \cap K} \hookrightarrow \frac{R}{I \cap K} \hookrightarrow \frac{R^A}{I \cap K}.$$  

This shows that $R/I$ is torsionless. Hence $I$ is a right annihilator.\hfill $\Box$

Since semiperfect rings and left (or right) perfect rings are semilocal, we
have

Corollary 2.13. [19, Lemma 1.5] Suppose $R$ is semiperfect and $rl(K) = K$ for
every small right ideal $K$ of $R$. Then $rl(T) = T$ for every right ideal $T$ of $R$.

Corollary 2.14. [20, Theorem 3.11] If $R$ is a left (or right) perfect, then $R$ is
small dual if and only if $R$ is dual.

Next we will get a similar result as that in Theorem 2.12 when the small
right ideals of $R$ are restricted to be $n$-generated. We define a ring $R$ to be
$J$-regular if $R/J$ is a von Neumann regular ring.
Lemma 2.15. [17, Proposition 2.3] The following are equivalent for a ring $R$.
1. $R$ is $J$-regular.
2. Every principal right (or left) ideal of $R$ has a weak supplement in $R$.
3. Every finitely generated right (or left) ideal of $R$ has a weak supplement in $R$.

Lemma 2.16. [17, Lemma 2.9] Let $R$ be a ring, $b, r_i, a_i \in R$, $i = 1, 2, \ldots, n$, such that $b + \sum_{i=1}^{n} a_i r_i = 1$. Then $bR \cap \sum_{i=1}^{n} a_i R = \sum_{i=1}^{n} ba_i R$.

Using a similar proof in Theorem 2.12, we have

Theorem 2.17. Let $R$ be a $J$-regular ring. For a given integer $n \geq 1$, every $n$-generated small right ideal of $R$ is a right annihilator if and only if every $n$-generated right ideal of $R$ is a right annihilator.

Proof. We only need to prove the necessity. Let $I = a_1 R + \cdots + a_n R$ be an $n$-generated right ideal of $R$, by Lemma 2.4, we need to show that $R/I$ is torsionless as a right $R$-module. Since $R$ is right $J$-regular, by Lemma 2.15, $I$ has a weak supplement in $R$. Thus, there exists a right ideal $K$ of $R$ such that $I + K = R$ and $I \cap K \subseteq J$. It is easy to see there are $r_1, \ldots, r_n \in R$, $b \in K$ such that $b + \sum_{i=1}^{n} a_i r_i = 1$ and $I \cap bR \subseteq I \cap K \subseteq J$. Therefore, $I \cap bR$ is a small right ideal of $R$. By Lemma 2.16, $I \cap bR = \sum_{i=1}^{n} ba_i R$ is $n$-generated. By the assumption, $I \cap bR$ is a right annihilator. According to Lemma 2.4, $R/(I \cap bR)$ is torsionless. Thus

$$\frac{R}{I} = \frac{I + bR}{I} \cong \frac{bR}{I \cap bR} \hookrightarrow \frac{R}{I \cap bR} \cong \frac{R_A}{R} \text{ for some set } A.$$  

□

Recall that a ring $R$ is called semiregular if $R$ is $J$-regular and idempotents can lift modulo $J$. It is clear that a semiregular or semilocal ring is $J$-regular, so we have

Corollary 2.18. Let $R$ be a semiregular or semilocal ring. For a given integer $n \geq 1$, every $n$-generated small right ideal of $R$ is a right annihilator if and only if every $n$-generated right ideal of $R$ is a right annihilator.
Corollary 2.19. [20, Proposition 3.6] Suppose that $R$ is a semiregular and right small dual ring. Then

1. $R$ is a left $C2$ ring.
2. $J = Z_l$.

Proof. By the above corollary, every principle right ideal of $R$ is a right annihilator. Thus $R$ is left $P$-injective by [13, Lemma 5.1]. Then the left is clear by [13, Proposition 5.10, Theorem 5.14]. □

3. Applications

Recall that a ring $R$ is called right simple injective if every homomorphism $f$ from a right ideal $I$ of $R$ to $R_R$ with simple image can be extended to one from $R_R$ to $R_R$. Left simple injective rings can be defined analogously.

Theorem 3.1. The following are equivalent for a ring $R$.

1. $R$ is a dual ring.
2. $R$ is a semilocal and small dual ring.
3. $R$ is a semiperfect and small dual ring.
4. $R$ is a two-sided Kasch and two-sided simple injective ring.

Proof. (1)$\iff$(2)$\iff$(3). By [4, Theorem 3.9], A dual ring is a semiperfect ring. Since a semiperfect ring is semilocal, by Theorem 2.12, $R$ is dual if and only if $R$ is semilocal and small dual if and only if $R$ is semiperfect and small dual.

(1)$\iff$(4). See [13, Theorem 6.18]. □

Proposition 3.2. Let $R$ be a right small dual ring. Then $J$ is nilpotent if $R$ satisfies any one of the following chain conditions:

1. $R$ satisfies ACC on left annihilators.
2. $R$ satisfies ACC on essential left ideals.
3. $R$ satisfies ACC on essential right ideals.

Proof. (1) and (2) are obtained by [20, Corollary 3.3, Theorem 3.4].

(3). If $R$ satisfies ACC on essential right ideals, by [3, Lemma 2], $R/S_r$ is
right noetherian. Given chain $J \supseteq J^2 \supseteq J^3 \supseteq \cdots$, we have chain of right ideals $S_r \subseteq I(J) \subseteq I(J^2) \subseteq I(J^3) \subseteq \cdots$. Since $R/S_r$ is right noetherian, there exists integer $n \geq 1$ such that $I(J^n) = I(J^{n+1})$. Thus $rl(J^n) = rl(J^{n+1})$. As $R$ is right small dual, $J^n = rl(J^n) = rl(J^{n+1}) = J^{n+1}$. For each right ideal $I$ of $R$, set $\overline{I} = (I + S_r)/S_r$. Then $\overline{R}$ is noetherian. Thus $\overline{J^n}$ is finitely generated and $\overline{J^n} = \overline{J^{n+1}} = \overline{J^n}J$. By Nakayama’s Lemma ([1, Corollary 15.13]), $\overline{J^n} = \overline{0}$. Hence $J^n \subseteq S_r$. This informs that $J^{n+1} \subseteq S_rJ = 0$.

**Question 3.3.** Let $R$ be a right small dual ring. If $R$ satisfies ACC on right annihilators, is $J$ nilpotent?

**Proposition 3.4.** Let $R$ be a small dual ring. Then

1. $R$ is left and right mininjective. In particular, $S_r = S_l$.
2. $I(J) \subseteq_{ess} R$ and $r(J) \subseteq_{ess} R$.
3. $J \subseteq Z_l$ and $J \subseteq Z_r$.
4. $rl(T) = T$ and $lr(K) = K$ for every finitely generated semisimple right ideal $T$ and left ideal $K$ of $R$.

**Proof.** By Proposition 2.6 (4), $R$ is left and right mininjective. Thus, by [13, Theorem 2.21], $S_r = S_l$. (2), (3) and (4) are directly obtained by Proposition 2.6.

At last we show some characterizations of QF rings given by small dualities of ring.

**Lemma 3.5.** ([11, Proposition 3] Suppose $R$ is a left perfect, left and right simple injective ring. Then $R$ is a quasi-Frobenius ring.

**Lemma 3.6.** ([18, Theorem 2.5] If $R$ is a left and right mininjective ring with ACC on right annihilators in which $S_r \subseteq_{ess} R$, then $R$ is QF.

**Theorem 3.7.** The following are equivalent for a ring $R$.

1. $R$ is QF.
2. $R$ is a small dual and left perfect ring.
(3) $R$ is a small dual ring with ACC on right annihilators in which $S_r \subseteq_{ess} R_R$.

(4) $R$ is a small dual and semilocal ring with ACC on right annihilators.

(5) $R$ is a small dual and semilocal ring with ACC on essential right ideals.

Proof. It is obvious that (1)⇒(2), (3), (4), (5).

(2)⇒(1). If $R$ is a small dual and left perfect ring, by Theorem 3.1, $R$ is left and right simple injective. According to Lemma 3.5, $R$ is QF.

(3)⇒(1). By Proposition 3.4, $R$ is left and right mininjective. Then by Lemma 3.6, $R$ is a QF ring.

(4)⇒(1) and (5)⇒(1). If $R$ satisfies (4) or (5), by Proposition 3.2, $J$ is nilpotent. Thus $R$ is a semiprimary ring which is left and right perfect. According to (2)⇒(1), $R$ is a QF ring. □

Remark 3.8. From the above theorem, we have

(a). “left perfect” in (2) of the above theorem can not be replaced by “semiperfect”. By [11, Example 3], there is a commutative semiperfect, Kasch and simple injective ring $R$ which is not self-injective. Thus, by Theorem 3.1, $R$ is a small dual and semiperfect ring. But $R$ is not QF.

(b). “$S_r \subseteq_{ess} R_R$” and “semilocal” in (3), (4), (5) of the above theorem can not be removed. Because $\mathbb{Z}$ is a small dual and noetherian ring which is not QF.

(c). “Small dual” in (3) of the above theorem can not be replaced by “right small dual”. By [13, Example 8.16], there is a right Johns ring which is not right artinian. Recall that a ring $R$ is called right Johns if $R$ is right dual and right noetherian. And by [13, Lemma 8.7 (4)], if $R$ is a right Johns ring then $S_r \subseteq_{ess} R_R$.

(d). (5) gives partial affirmative answers to [16, Question 2.20].

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Graduate Texts in Math. 13 (Springer-Verlag, Berlin, New York, Heidelberg, 1992).

[2] R. Baer, Rings with duals, Amer. J. Math. 65 (1943) 569–584.
[3] N. V. Dung, D. V. Huynh and R. Wisbauer, Quasi-injective modules with acc or dcc on essential submodules, Arch. Math. 53 (1989) 252–255.

[4] C. R. Hajarnavis and N.C. Norton, On dual rings and their modules, J. Algebra 93 (1985) 253–266.

[5] M. Hall, A type of algebraic closure, Ann. of Math. 40 (1939) 360–369.

[6] I. Kaplansky, Dual rings, Ann. of Math. 49 (1948) 689–701.

[7] K. Koike, Dual rings and cogenerator rings, Math. J. Okayama Univ. 37 (1995) 99–103.

[8] T. Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Math. 131 (Springer-Verlag, Berlin, New York, Heidelberg, 1991).

[9] C. Lomp, On semilocal modules and rings, Comm. Algebra 27 (4) (1999) 1921–1935.

[10] K. Morita, On S-rings in the sense of F. Kasch, Nagoya Math. J. 27 (1966) 687–695.

[11] W. K. Nicholson and M. F. Yousif, On perfect simple-injective rings, Proc. Amer. Math. Soc. 125 (4) (1997) 979–985.

[12] W. K. Nicholson and M. F. Yousif, Annihilators and the CS-condition, Glasgow Math. J. 40 (1998) 213–222.

[13] W. K. Nicholson and M. F. Yousif, Quasi-Frobenius Rings, Cambridge Tracts in Math. 158 (Cambridge University Press, 2003).

[14] S. S. Page and Y. Q. Zhou, Quasi-dual rings, Comm. Algebra 28 (1) (2000) 489-504.

[15] E. A. Rutter, Rings with the principle extension property, Comm. Algebra 3 (3) (1975) 203-212.

[16] L. Shen, A note on quasi-Johns rings, Contemporary Ring Theory 2011, Proceedings of the Sixth China-Japan-Korea International Conference on Ring Theory, (World Scientific, 2012) 89–96.

[17] L. Shen, J-regular rings with injectivities, Algebra Colloq. 20 (2) (2013) 343-347.

[18] L. Shen and J.L. Chen, New characterizations of quasi-Frobenius rings, Comm. Algebra 34 (1997) 2157–2165.

[19] M. F. Yousif, On continuous rings, J. Algebra 191 (1997) 495–509.

[20] D. Zhou, D. Li and L. Guo, Annihilator conditions relative to a class of modules, Thai J. Math. 8(3) (2010) 419-428.

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