Nodal curves on K3 surfaces

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Abstract. In this paper, we study the Severi variety $V_{L,g}$ of genus $g$ curves in $|L|$ on a general polarized K3 surface $(X, L)$. We show that the closure of every component of $V_{L,g}$ contains a component of $V_{L,g-1}$. As a consequence, we see that the general members of every component of $V_{L,g}$ are nodal.

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1. Introduction

It was proved that every complete linear system on a very general polarized K3 surface $(X, L)$ contains a nodal rational curve [C1] and furthermore every rational curve in $|L|$ is nodal, i.e., has only nodes $xy = 0$ as singularities [C2]. The purpose of this note is to prove an analogous result on singular curves in $|L|$ of geometric genus $g > 0$.

For a line bundle $A$ on a projective surface $X$, we use the notation $V_{A,g}$ to denote the Severi varieties of integral curves of geometric genus $g$ in the complete linear series $|A| = \mathbb{P}H^0(A)$. For a K3 surface $X$, it is well known that every component of $V_{A,g}$ has the expected dimension $g$. Furthermore, using theory of deformation of maps, one can show that $\nu : \hat{C} \to X$ is an immersion for $\nu$ the normalization of a general member $[C] \in V_{A,g}$ if $g > 0$ [HM, Chap. 3, Sec. B].

It was claimed that a general member of $V_{A,g}$ is nodal on every projective K3 surface $X$ and every $A \in \text{Pic}(X)$ as long as $g > 0$ in [C1, Lemma 3.1]. However, as kindly pointed out to the author by Edoardo Sernesi [DS, Sec. 3.3], the proof there is wrong. So this note provides a partial fix for this
problem, albeit only for singular curves in the primitive class \(|L|\) on a general polarized K3 surface \((X, L)\). Our main theorem is

**Theorem 1.1.** For a general polarized K3 surface \((X, L)\), every (irreducible) component of \(\overline{V}_{L,g}\) contains a component of \(V_{L,g-1}\) for all \(1 \leq g \leq \alpha_p(L)\), where \(\overline{V}_{L,g}\) is the closure of \(V_{L,g}\) in \(|L|\) and \(\alpha_p(L) = L^2/2+1\) is the arithmetic genus of \(L\).

Clearly, the above theorem, combining with the fact that every rational curve in \(|L|\) is nodal \([\text{C2}]\), implies the following corollary by induction:

**Corollary 1.2.** For a general polarized K3 surface \((X, L)\), the general members of every component of \(V_{L,g}\) are nodal for all \(0 \leq g \leq \alpha_p(L)\).

It was proved in \([\text{KLM}, \text{Theorem } 1.3, 5.3 \text{ and Remark } 5.6]\) that the general members of every component of \(V_{L,g}\) are not trigonal for \(g \geq 5\). Combining with \([\text{DS, Theorem } B.4]\), it shows that the corollary holds for \(5 \leq g \leq \alpha_p(L)\). Of course, we have settled it for all genus \(g\) here. As an application, it shows that the genus \(g\) Gromov-Witten invariant computed in \([\text{BL}]\) is the same as the number of genus \(g\) curves in \(|L|\) passing through \(g\) general points.

A comprehensive treatment for \(\overline{V}_{mL,g}\) is planned in a future paper.

As another potential application of Theorem 1.1, we want to mention the conjecture of the irreducibility of universal Severi variety \(V_{L,g}\) on K3 surfaces:

**Conjecture 1.3.** Let \(\mathcal{K}_p\) be the moduli space of polarized K3 surfaces \((X, L)\) of genus \(p = \alpha_p(L)\) and let

\[
\mathcal{V}_{L,g} = \{(X, L, C) : (X, L) \in \mathcal{K}_p, C \in V_{L,g}\}
\]

be the universal Severi variety of genus \(g\) curves in \(|L|\) over \(\mathcal{K}_p\). Then \(\mathcal{V}_{L,g}\) is irreducible.

If we approach the conjecture along the line of argument of J. Harris for the irreducibility of Severi variety of plane curves \([\text{H}]\), we need to establish two facts:

- Every component of \(\overline{V}_{L,g}\) contains a component of \(\mathcal{V}_{L,0}\).
- \(\mathcal{V}_{L,0}\) is irreducible and the monodromy action on the \(p\) nodes of a rational curve \(C \in V_{L,0}\) is the full symmetric group \(\Sigma_p\) as \((X, L, C)\) moves in \(\mathcal{V}_{L,0}\).

The second fact comes easily for plane curves, while the establishment of the first fact is the focus of Harris’ proof (see also \([\text{HM, Chap. 6, Sec. E}]\)). The situation for \(\mathcal{V}_{L,g}\) is somewhat reversed at the moment: the first fact follows from our main theorem, while the difficulty lies in the second fact:

**Conjecture 1.4.** Let \(\mathcal{V}_{L,0}\) be the universal Severi variety of rational curves in \(|L|\) over the moduli space \(\mathcal{K}_p\) of polarized K3 surfaces \((X, L)\) of genus \(p\)
and let

\[ W_{L,0} = \{(X, L, C, s_1, s_2, \ldots, s_p) : (X, L, C) \in V_{L,0}, C_{\text{sing}} = \{s_1, s_2, \ldots, s_p\}\}. \]  

(1.2)

Then \( W_{L,0} \) is irreducible.

Our above discussion shows that Conjecture 1.4 implies 1.3.

**Conventions.** We work exclusively over \( \mathbb{C} \). A K3 surface in this paper is always projective. A polarized K3 surface is a pair \((X, L)\), where \( X \) is a K3 surface and \( L \) is an indivisible ample line bundle on \( X \).

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2. Proof of Theorem 1.1

We start with the following observation:

**Proposition 2.1.** Let \( W \) be a component of \( V_{L,g} \) for a polarized K3 surface \((X, L)\) with \( \text{Pic}(X) = \mathbb{Z} \). The following are equivalent:

1. The closure \( \overline{W} \) of \( W \) in \(|L|\) contains a component of \( V_{L,g-1} \).
2. \( \dim(\overline{W} \setminus W) = g - 1 \).
3. For a set \( \sigma \) of \( g - 1 \) general points on \( X \), \( W \cap \Lambda_{\sigma} \) is not projective (i.e. complete), where \( \Lambda_{\sigma} \subset |L| \) is the locus of curves \( C \in |L| \) passing through \( \sigma \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious. Since every curve in \(|L|\) is integral, we have

\[ \overline{W} \setminus W \subset \bigcup_{i < g} V_{L,i}. \]  

(2.1)

And since \( \dim V_{L,i} \leq i \), we have (2) \( \Rightarrow \) (1).

Let \( \partial W = \overline{W} \setminus W \). Obviously, \( \dim(\partial W \cap \Lambda_{\sigma}) = \dim \partial W - (g - 1) \). Therefore, (2) \( \Rightarrow \) (3). On the other hand, if \( W \cap \Lambda_{\sigma} \) is not complete, then there exists \( C_{\sigma} \in \partial W \) passing through \( \sigma \). Then \( \dim \partial W \geq g - 1 \). So (3) \( \Rightarrow \) (2). \( \square \)

So it suffices to show that \( W \cap \Lambda_{\sigma} \) is not complete for every component \( W \) of \( V_{L,g} \). We prove this using a degeneration argument similar to the one in [C2]. A general K3 surface can be specialized to a Bryan-Leung (BL) K3 surface \( X_0 \), which is a K3 surface with Picard lattice

\[ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}. \]  

(2.2)

It can be polarized by the line bundle \( C + mF \), where \( C \) and \( F \) are the generators of \( \text{Pic}(X_0) \) satisfying \( C^2 = -2, CF = 1 \) and \( F^2 = 0 \). A general polarized K3 surface of genus \( m \) can be degenerated to \((X_0, C + mF)\). Such \( X_0 \) has an elliptic fibration \( X_0 \to \mathbb{P}^1 \) with fibers in \(|F|\). For a general BL
K3 surface $X_0$, there are exactly 24 nodal fibers in $|F|$. A key fact here is that every member of $|C + mF|$ is “completely” reducible in the sense that it is a union of $C$ and $m$ fibers in $|F|$ (counted with multiplicities).

Let $X$ be a family of K3 surfaces of genus $m$ over a smooth quasi-projective curve $T$ such that $X_0$ is a general BL K3 surface for a point $0 \in T$, $X_t$ are K3 surfaces of Pic $(X_t) = \mathbb{Z}$ for $t \neq 0$ and $L$ is a line bundle on $X$ with $L_0 = C + mF$. After a base change, there exists $W \subset V_{L,g}$ flat over $T$ such that $W_t$ is a component of $V_{L_t,g}$ for all $t \neq 0$. Let $\sigma$ be a set of $g - 1$ general sections of $X/T$. It suffices to prove that $W_t \cap \Lambda_\sigma$ is not projective for $t$ general.

By stable reduction, there exists a family $f : Y \to X$ of genus $g$ stable maps over a smooth surface $S$ with the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi} & T
\end{array}
$$

(2.3)

where $S$ is flat and projective over $T$, $f_sY_s \in \overline{W}_t \cap \Lambda_\sigma$ on $X_t$ for all $s \in S_t$ and $t \in T$ and $S$ dominates $\overline{W} \cap \Lambda_\sigma$ via the map sending $s \to (f_sY_s)$. In other words, $f : Y \to X$ is the stable reduction of the universal family over $\overline{W}$ such that $f : Y_s \to X$ is the normalization of a general member $G \in W_t$ passing through the $g - 1$ points $\sigma(t)$ for $s \in S_t$ general and $t \neq 0$.

Let us consider the moduli map $\rho : S \to \overline{M}_g \times T$ sending $s \to ([Y_s], \pi(s))$, where $\overline{M}_g$ is the moduli space of stable curves of genus $g$ with $M_g$ its open subset parameterizing smooth curves. To show that $W_t \cap \Lambda_\sigma$ is not complete, it suffices to show that

$$
\rho^{-1}(\Delta \times T) \cap S_t \neq \emptyset
$$

(2.4)

for $t \neq 0$, where $\Delta = \overline{M}_g \backslash M_g$ is the boundary divisor of $\overline{M}_g$.

Let $F_1, F_2, \ldots, F_{g-1} \subset X_0$ be $g - 1$ fibers in $|F|$ passing through the $g - 1$ points $\sigma(0)$, respectively. Since $\sigma(0)$ are in general position, $F_1, F_2, \ldots, F_{g-1}$ are $g - 1$ general fibers in $|F|$ and $\sigma(0) \cap C = \emptyset$.

For every $s \in S_0$, $f_sY_s \in |C + mF|$ passes through $\sigma(0)$. Therefore, we must have

$$
f_sY_s = C + m_1F_1 + m_2F_2 + \ldots + m_{g-1}F_{g-1} + M_s
$$

(2.5)

for some $m_1, m_2, \ldots, m_{g-1} \in \mathbb{Z}^+$. Since the curves in $W_t \cap \Lambda_\sigma$ cover $X_t$ for $t \neq 0$, $f$ is surjective. Hence $f_sY_s$ covers $X_0$ as $s$ moves in $S_0$. Therefore, $M_s$ contains a moving fiber in $|F|$. More precisely, there exists a component $\Gamma$ of $S_0$ such that $\cup_{s \in \Gamma} M_s = X_0$.

For a general point $s \in \Gamma$, $M_s$ contains a general fiber $F_s$ in $|F|$. Therefore, $Y_s$ has components $\widehat{F}_{1,s}, \widehat{F}_{2,s}, \ldots, \widehat{F}_{g-1,s}, \widehat{F}_s$ dominating $F_1, F_2, \ldots, F_{g-1}, F_s$, respectively. And since $p_\sigma(Y_s) = g$, $\widehat{F}_{1,s}, \widehat{F}_{2,s}, \ldots, \widehat{F}_{g-1,s}, \widehat{F}_s$ are all elliptic.
curves. Indeed, it is very easy to see that its moduli $[Y_s]$ in $\overline{M}_g$

$$[Y_s] = [\tilde{C}_s \cup \tilde{F}_{1,s} \cup \tilde{F}_{2,s} \cup ... \cup \tilde{F}_{g-1,s} \cup \tilde{F}_s]$$ (2.6)

is a smooth rational curve $\tilde{C}_s$ with $g$ elliptic “tails” $\tilde{F}_{1,s}, \tilde{F}_{2,s}, ..., \tilde{F}_{g-1,s}, \tilde{F}_s$ attached to it, where $\tilde{C}_s$ is the component of $Y_s$ dominating $C$. Of course, when $g \leq 2$, $\tilde{C}_s$ is contracted under the moduli map.

Note that $\tilde{F}_{1,s}, \tilde{F}_{2,s}, ..., \tilde{F}_{g-1,s}, \tilde{F}_s$ are isogenous to $F_1, F_2, ..., F_{g-1}, F_s$, respectively. As $s$ moves on $\Gamma$, $F_s$ moves in $|F|$. So $\hat{F}_s$ has varying moduli. This shows that $\rho$ maps $S$ generically finitely onto its image. That is,

$$\dim \rho(S) = 2.$$ (2.7)

Furthermore, when $F_s$ becomes one of 24 nodal fibers in $|F|$, $\hat{F}_s$ becomes a union of rational curves. Therefore, there exists $b \in \Gamma$ such that $\hat{F}_b$ is a connected union of rational curves with normal crossings and $p_\rho(\hat{F}_b) = 1$. The moduli $[Y_b]$ of $Y_b$ is thus a smooth rational curve with $g - 1$ elliptic tails and one nodal rational curve attached to it. Consequently,

$$\rho(b) \in \Delta_0 \times T$$ (2.8)

where $\Delta_0$ is the component of $\Delta$ whose general points parameterize curves of genus $g - 1$ with one node. Combining (2.7), (2.8) and the fact that $\Delta_0$ is $\mathbb{Q}$-Cartier, we conclude that

$$\rho(S) \cap (\Delta_0 \times T) \neq \emptyset$$ has pure dimension 1. (2.9)

Therefore, for every connected component $G$ of $\rho^{-1}(\Delta_0 \times T)$, we have

$$\dim \rho(G) = 1.$$ (2.10)

If $\rho^{-1}(\Delta_0 \times T) \cap S_t \neq \emptyset$ for $t \neq 0$, then (2.4) follows and we are done. Otherwise,

$$\rho^{-1}(\Delta_0 \times T) \subset S_0.$$ (2.11)

Let $G$ be the connected component of $\rho^{-1}(\Delta_0 \times T)$ containing the point $b$. Then $G \subset S_0$ and $\dim \rho(G) = 1$.

Let $B$ be an irreducible component of $G$ passing through $b$. For $Y_b$, we have

$$f_s Y_b = C + m_1 F_1 + m_2 F_2 + ... + m_{g-1} F_{g-1} + M_b$$ (2.12)

with $M_b$ supported on the union $F_{g}^\sim$ of 24 nodal rational curves in $|F|$. Therefore, for $s \in B$ general, $M_s$ must also be supported on $F_{g}^\sim$; otherwise, $M_s$ contains a general member $F_s$ of $|F|$, the moduli $[Y_s]$ of $Y_s$ is given by (2.6) and $[Y_s] \not\in \Delta_0$. Consequently, $M_s \equiv M_b$ for all $s \in B$ and $\rho$ is constant on $B$.

For a component $Q$ of $G$ with $q \in B \cap Q \neq \emptyset$, the same argument shows that $M_s \equiv M_q$ is supported on $F_{g}^\sim$ for all $s \in Q$ and $\rho$ is constant on $Q$. And since $G$ is connected, we can use this argument to show that $\rho$ is constant on every component of $G$, i.e., constant on $G$. This contradicts (2.10).
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