Wide Morita contexts, relative injectivity and equivalence results

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Abstract

We extend Morita theory to abelian categories by using wide Morita contexts. Several equivalence results are given for wide Morita contexts between abelian categories, widely extending equivalence theorems for categories of modules and comodules due to Kato, Müller and Berbec. In the case of Grothendieck categories we derive equivalence results by using quotient categories. We apply the general equivalence results to rings with identity, rings with local units, graded rings, Doi-Hopf modules and coalgebras.

Mathematics Subject Classification: 16D90, 18E15, 16W30, 16W50.

Keywords: wide Morita context, quotient category, ring with local units, graded ring, Doi-Hopf module, Hopf algebra (co)action.

0 Introduction and preliminaries

Morita contexts appeared in the work of Morita on equivalence of categories of modules over rings with identity. A fundamental result of Morita says that the categories of modules over two rings with identity $R$ and $S$ are equivalent if and only if there exists a strict Morita context connecting $R$ and $S$. Morita contexts have been used to the study of group actions on rings and Galois theory for commutative rings (see [16]). A Morita theory for rings with local units was developed in [3]. Several Morita contexts were constructed in connection to Galois

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theory for Hopf algebra actions and coactions (see [11], [8], [4]), where Hopf-Galois extensions are characterized by the surjectivity of one of the Morita maps. As an application, the finite dimensional version of the duality theorem of Blattner-Montgomery ([6]) was deduced and explained in a nice way by using Morita contexts and Hopf-Galois theory in [29].

A Morita context gives rise to an equivalence of categories if and only if it is strict, i.e. if both Morita maps are surjective. A natural question that was posed was how far is an arbitrary Morita context from an equivalence of categories. An answer is given by the Kato-Müller Theorem (see [20], [24]), which briefly says (in the formulation of Müller) that if the rings $R$ and $S$ are connected by a Morita context, then certain quotient categories of $R - \text{mod}$ and $S - \text{mod}$ are equivalent.

A concept dual to Morita contexts was constructed for coalgebras by Takeuchi in [28], where he defines what is now known as a Morita-Takeuchi context connecting two coalgebras, and proves that the categories of comodules over two coalgebras $C$ and $D$ are equivalent if and only if $C$ and $D$ are connected by a strict Morita-Takeuchi context. A result dual to the Kato-Müller Theorem for Morita-Takeuchi contexts and the associated categories of comodules was proved by Berbec in [5].

Inspired by [26], where an equivalence result for the subcategories of $R - \text{mod}$ and $S - \text{mod}$ consisting of the trace-torsionfree trace-accessible modules was proved in the case where the rings $R$ and $S$ are connected by a Morita context, Castaño Iglesias and Gomez Torrecillas define in [9], [10] the concept of a wide Morita context for abelian categories. A datum $(F, G, \eta, \rho)$ is called a right wide Morita context between the abelian categories $\mathcal{A}$ and $\mathcal{B}$ if $F : \mathcal{B} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{B}$ are right exact functors, and $\eta : F \circ G \to 1_\mathcal{A}$ and $\rho : G \circ F \to 1_\mathcal{B}$ are natural transformations with the property that $G\eta = \rho G$ and $F\rho = \eta F$. A left wide Morita context is a datum which is a right wide Morita context when regarded between the dual categories. Both Morita contexts and Morita-Takeuchi contexts can be regarded as particular wide Morita contexts, so this approach can be seen as a way to unify (as much as possible) Morita theory for modules and comodules.

The aim of this paper is two-fold. On one hand, we study general properties of wide Morita contexts and we explain how they allow extending Morita theory to abelian categories. As a generalization of Morita’s result, we show that an equivalence of abelian categories is essentially a strict wide Morita context. On the other hand we give several equivalence results which widely extend the results of Kato, Müller and Berbec. We give a general equivalence result for wide Morita contexts between abelian categories, and we derive several equivalence theorems for the case where the categories are Grothendieck. The Müller type equivalence result for Grothendieck categories seems to be the most interesting since the quotient categories are also Grothendieck categories, and this general framework can be applied to a large number of examples. We apply the general results about wide Morita contexts to rings with identity, rings with local units, graded rings, Doi-Hopf modules and coalgebras. Some of the results obtained in this way are known, but we explain them from a general point of view, and some others are new.
The content of the paper is as follows. In Section 1 we recall the definition of right wide Morita contexts and left wide Morita contexts, and present several general properties of these. We also define a composition for wide Morita contexts and show that it is associative. We define the concepts of isomorphic right wide Morita contexts, and we show that the invertible right wide Morita contexts (with respect to the composition previously defined) are exactly the strict right wide Morita contexts, i.e. those ones defining an equivalence of categories. On the other hand, we show that any equivalence of abelian categories arises from a strict right wide Morita context. A fundamental fact that we prove is that to a right wide Morita context for which $F$ and $G$ are left adjoint functors, we can associate a left wide Morita context whose functors are the right adjoints of $F$ and $G$. In Section 2 we consider the concept of relative injective object with respect to a subcategory and another object, and study related properties. This allows us to define a concept of a closed object with respect to a subcategory. For any right wide Morita context $\Gamma$ between the abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that the functors $F$ and $G$ have right adjoints, we construct subcategories $\mathcal{C}_\Gamma$ of $\mathcal{A}$, and $\mathcal{D}_\Gamma$ of $\mathcal{B}$ such that the category of $\mathcal{C}_\Gamma$-closed objects of $\mathcal{A}$ and the category of $\mathcal{D}_\Gamma$-closed objects of $\mathcal{B}$ are equivalent. In Section 3 we consider the dual situation, by defining relative projective objects and the dual results for left wide Morita contexts. Since the dual of an abelian category is abelian, these results are dual to the ones in Section 2, so they follow directly by dualization, and we did not include direct proofs. Even if they follow directly by dualization, we include them since they are interesting for several applications. In Section 4 we consider that $\mathcal{A}$ and $\mathcal{B}$ are Grothendieck categories. Then it makes sense to consider the smallest localizing category $\overline{\mathcal{C}}_{\Gamma}$ that contains $\mathcal{C}_\Gamma$, and discuss the connection between $\overline{\mathcal{C}}_{\Gamma}$-closed objects and $\mathcal{C}_\Gamma$-closed objects. We prove a general equivalence result by using quotient categories and the results of Section 2. In Section 5 we apply the general equivalence result to some particular cases. If $\mathcal{A}$ and $\mathcal{B}$ are categories of modules over rings with identity, we obtain as a particular case Kato-Müller Theorem. We get new similar equivalence results for Morita contexts associated to graded rings and to rings with local units. The results of Section 3 are applied to left wide Morita contexts obtained by taking the Hom functors associated to a Morita context. We obtain results of Kato and Ohtake as particular cases. We also apply the general equivalence result to Doi-Hopf modules. As particular situations we obtain equivalence results for Hopf-Galois extensions and for the case where a total integral exists. We derive the Weak Structure Theorem for Hopf-Galois extension as a special case. In Section 6 we consider left wide Morita contexts between Grothendieck categories. In the case the functors $F$ and $G$ commute with direct limits we obtain a new equivalence result. In Section 7 we apply it to Morita-Takeuchi contexts and derive as a particular case the theorem of Berbec, and also to Hopf-Galois coextensions.

For notations and basic concepts we refer to [22] for general category theory issues, to [19], [18] and [27] for things related to abelian categories, Grothendieck categories and quotient categories, and to [14] and [23] for coalgebras, Hopf algebras and Hopf-Galois theory. If $\mathcal{A}$ is a category, by a subcategory of $\mathcal{A}$ we always mean a full subcategory. If $\mathcal{A}$ is an abelian category, then the subcategory $\mathcal{C}$ of $\mathcal{A}$ is closed if it is closed under subobjects, factor objects, and arbitrary direct sums. If moreover $\mathcal{C}$ is closed under extensions, it is called a localizing.
subcategory. By functor we always mean a covariant functor. If \( f : X \to Y \) and \( g : Y \to Z \) are two morphisms in a category, their composition is denoted by \( g \circ f \). The same notation is used for composition of functors. All the categories we work with are abelian, and all functors are additive. If \( T, S : \mathcal{A} \to \mathcal{B} \) are two functors, then a natural transformation \( \eta : T \to S \) is called an epimorphism (monomorphism) if \( \eta(X) \) is an epimorphism (monomorphism) for any \( X \in \mathcal{A} \).

1 Wide Morita contexts, equivalence of abelian categories and adjunctions

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two abelian categories. Following [9], [10] a datum \((F, G, \eta, \rho)\) is called a right wide Morita context between the categories \( \mathcal{A} \) and \( \mathcal{B} \) if \( F : \mathcal{B} \to \mathcal{A} \) and \( G : \mathcal{A} \to \mathcal{B} \) are right exact functors, and \( \eta : F \circ G \to 1_\mathcal{A} \) and \( \rho : G \circ F \to 1_\mathcal{B} \) are natural transformations with the property that \( G \eta = \rho G \) and \( F \rho = \eta F \). Note that in this case \((G, F, \rho, \eta)\) is a right wide Morita context between the categories \( \mathcal{B} \) and \( \mathcal{A} \), therefore any general result that we prove for \( F \), respectively \( \eta \), also holds for \( G \), respectively \( \rho \).

Dually, \((F, G, \eta, \rho)\) is called a left wide Morita context between \( \mathcal{A} \) and \( \mathcal{B} \) if \( F : \mathcal{B} \to \mathcal{A} \) and \( G : \mathcal{A} \to \mathcal{B} \) are left exact functors, \( \eta : 1_\mathcal{A} \to F \circ G \) and \( \rho : 1_\mathcal{B} \to G \circ F \) satisfy \( G \eta = \rho G \) and \( F \rho = \eta F \).

**Proposition 1.1** Let \( \Gamma = (F, G, \eta, \rho) \) be a right wide Morita context. If \( \eta \) is an epimorphism, then \( \eta \) is a natural equivalence. Thus if \( \eta \) and \( \rho \) are epimorphisms, the functors \( F \) and \( G \) give an equivalence between the categories \( \mathcal{A} \) and \( \mathcal{B} \).

**Proof:** Assume that \( \eta \) is an epimorphism. Let \( M \in \mathcal{A} \). We have the exact sequence

\[
0 \longrightarrow \text{Ker } \eta(M) \xrightarrow{i} (F \circ G)(M) \xrightarrow{\eta(M)} M \longrightarrow 0
\]

where \( i \) is the inclusion morphism. Since \( F \circ G \) is right exact, we obtain the commutative diagram

\[
\begin{array}{c}
(F \circ G)(\text{Ker } \eta(M)) \\
\eta(\text{Ker } \eta(M))
\end{array}
\xrightarrow{(F \circ G)(i)}
\begin{array}{c}
(F \circ G)((F \circ G)(M)) \\
\eta((F \circ G)(M))
\end{array}
\xrightarrow{(F \circ G)(\eta(M))}
\begin{array}{c}
(F \circ G)(M) \\
\eta(M)
\end{array}
\longrightarrow
0
\]

But

\[
\eta((F \circ G)(M)) = F(\rho(G(M))) = (F \circ G)(\eta(M))
\]
and

\[ \eta((F \circ G)(M)) \circ (F \circ G)(i) = (F \circ G)(\eta(M)) \circ (F \circ G)(i) = (F \circ G)(\eta(M) \circ i) = (F \circ G)(0) = 0 \]

so \( i \circ \eta(\text{Ker } \eta(M)) = 0 \). Since \( i \) is a monomorphism, we have \( \eta(\text{Ker } \eta(M)) = 0 \), and by the hypothesis we get \( \text{Ker } \eta(M) = 0 \). We conclude that \( \eta(M) \) is an isomorphism. Similarly (or by using the remark that general facts about \( \eta \) also hold for \( \rho \)) one proves that \( \rho \) is an isomorphism whenever it is an epimorphism, so the last part of the statement follows.

The terminology of the following definition is inspired by the one for classical Morita contexts. It will be clear in Section 5 that in fact wide Morita contexts generalize classical Morita contexts.

**Definition 1.2** A right wide Morita context \( \Gamma = (F,G,\eta,\rho) \) is called strict if \( \eta \) and \( \rho \) are epimorphisms (so then by Proposition 1.1, \( F \) and \( G \) define an equivalence between \( A \) and \( B \) with natural equivalences \( \eta : F \circ G \to 1_A \) and \( \rho : G \circ F \to 1_B \)).

**Remark 1.3** Let \( \Gamma = (F,G,\eta,\rho) \) be a right wide Morita context between the categories \( A \) and \( B \). Then we can regard \( F \) and \( G \) as functors between the dual categories \( A^0 \) and \( B^0 \). It is easy to see that in this way \( \Gamma \) becomes a left wide Morita context between the categories \( A^0 \) and \( B^0 \). Note that \( A^0 \) and \( B^0 \) are also abelian categories.

Similarly, any left wide Morita context can be regarded as a right wide Morita context between the dual categories. In this way we will be able to transfer results from right to left wide Morita contexts.

A first example of how the above remark can be applied is the following.

**Proposition 1.4** Let \( (F,G,\eta,\rho) \) be a left wide Morita context. If \( \eta \) (respectively \( \rho \)) is a monomorphism, then it is a natural equivalence.

Now we define a composition operation for right wide Morita contexts. Let \( \Gamma = (F,G,\eta,\rho) \) be a right wide Morita context between the categories \( A \) and \( B \), and let \( \Delta = (U,V,\epsilon,\delta) \) be a right wide Morita context between the categories \( B \) and \( C \). Define the natural transformations

\[ \gamma : F \circ U \circ V \circ G \to 1_A, \quad \gamma(X) = \eta(X) \circ F(\epsilon(G(X))) \text{ for } X \in A \]
\[ \pi : V \circ G \circ F \circ U \to 1_C, \quad \pi(Z) = \delta(Z) \circ V(\rho(U(Z))) \text{ for } Z \in C \]

Shortly we denote \( \gamma = \eta \circ F \epsilon G \) and \( \pi = \delta \circ V \rho U \). With these notations we have
Proposition 1.5 \((F \circ U, V \circ G, \gamma, \pi)\) is a right wide Morita context between the categories \(\mathcal{A}\) and \(\mathcal{C}\). We call this context the composition of \(\Gamma\) and \(\Delta\), and we denote it by \(\Gamma \circ \Delta\).

Proof: We first note that since \(\epsilon : U \circ V \to 1_B\) is a natural transformation, then for any morphism \(u : Y_1 \to Y_2\) in the category \(\mathcal{B}\), we have
\[
\epsilon(Y_2) \circ (U \circ V)(u) = u \circ \epsilon(Y_1) \quad (1)
\]
Let \(Z \in \mathcal{C}\). Then we have
\[
(F \circ U)(\pi(Z)) = (F \circ U)(\delta(Z)) \circ (F \circ U \circ V)(\rho(U(Z)))
\]
\[
= F(\epsilon(U(Z))) \circ F((U \circ V)(\rho(U(Z))))
\]
\[
= F(\epsilon(U(Z))) \circ (U \circ V)(\rho(U(Z)))
\]
\[
= F(\rho(U(Z))) \circ \epsilon((G \circ F \circ U)(Z)) \quad \text{(by 1)}
\]
\[
= \eta(F(U(Z))) \circ F(\epsilon((G \circ F \circ U)(Z)))
\]
\[
= \gamma((F \circ U)(Z))
\]
showing that \((F \circ U)\pi = \gamma(F \circ U)\). In a similar way we can prove that \((V \circ G)\gamma = \pi(V \circ G)\).

The composition of right wide Morita contexts is associative, as the following result shows.

Proposition 1.6 Let us consider three right wide Morita contexts: \(\Gamma\) from \(\mathcal{A}\) to \(\mathcal{B}\), \(\Delta\) from \(\mathcal{B}\) to \(\mathcal{C}\), and \(\Sigma\) from \(\mathcal{C}\) to \(\mathcal{D}\). Then \((\Gamma \circ \Delta) \circ \Sigma = \Gamma \circ (\Delta \circ \Sigma)\).

Proof: If \(\Gamma = (F, G, \eta, \rho)\), \(\Delta = (U, V, \epsilon, \delta)\), and \(\Sigma = (P, Q, \alpha, \beta)\), it follows directly from the definition that
\[
(\Gamma \circ \Delta) \circ \Sigma = \Gamma \circ (\Delta \circ \Sigma) = (F \circ U \circ P, Q \circ V \circ G, \eta \circ F \epsilon G \circ FU \alpha VG, \beta \circ Q \delta P \circ QV \rho UP)
\]

If \(\mathcal{A}\) is an abelian category, \(1_A : \mathcal{A} \to \mathcal{A}\) is the identity functor, and \(Id_{1_A} : 1_A \to 1_A\) is the identity natural transformation, then clearly we have a right (and also left) wide Morita context \(1_A = (1_A, 1_A, Id_{1_A}, Id_{1_A})\) from \(\mathcal{A}\) to \(\mathcal{A}\). We call \(1_A\) the identity wide Morita context. It is obvious that for any right wide Morita context \(\Gamma\) from \(\mathcal{A}\) to \(\mathcal{B}\) we have \(1_A \circ \Gamma = \Gamma\) and \(\Gamma \circ 1_B = \Gamma\). Now we define a concept of isomorphism between wide Morita contexts.
Definition 1.7 Let $\Gamma = (F, G, \eta, \rho)$ and $\Delta = (P, Q, \alpha, \beta)$ be two right wide Morita contexts between $\mathcal{A}$ and $\mathcal{B}$. We say that $\Gamma$ and $\Delta$ are isomorphic, and we write $\Gamma \simeq \Delta$, if there exist natural equivalences $u : F \to P$ and $v : G \to Q$ such that for any $X \in \mathcal{A}$ the diagram

\[
\begin{array}{ccc}
F(G(X)) & \xrightarrow{\eta(X)} & X \\
\downarrow{u(G(X))} & & \downarrow{1_X} \\
P(G(X)) & & P(v(X)) \\
\downarrow{P(v(X))} & & \downarrow{\alpha(X)} \\
P(Q(X)) & \xrightarrow{\eta(X)} & X \\
\end{array}
\]

is commutative, and for any $Y \in \mathcal{B}$ the diagram

\[
\begin{array}{ccc}
G(F(Y)) & \xrightarrow{\rho(Y)} & Y \\
\downarrow{v(F(Y))} & & \downarrow{1_Y} \\
Q(F(Y)) & & Q(u(Y)) \\
\downarrow{Q(u(Y))} & & \downarrow{\beta(Y)} \\
Q(P(Y)) & \xrightarrow{\beta(Y)} & Y \\
\end{array}
\]

is commutative. Shortly, we write these as $\alpha \circ P v \circ u G = \eta$ and $\beta \circ Q u \circ v F = \rho$.

Remark 1.8 With the notations from Definition 1.7 we have the following.

1. If $\Gamma \simeq \Delta$, then $\Delta \simeq \Gamma$, with the latter isomorphism defined by the natural equivalences $u^{-1}$ and $v^{-1}$.
2. Assume that $\Gamma \simeq \Delta$. If $\eta$ is an epimorphism, then so is $\alpha$. Also, if $\rho$ is an epimorphism, then $\beta$ is an epimorphism.

Definition 1.9 A right wide Morita context $\Gamma$ between $\mathcal{A}$ and $\mathcal{B}$ is called invertible if there exists a wide right Morita context $\Delta$ between $\mathcal{B}$ and $\mathcal{A}$ such that $\Gamma \circ \Delta \simeq 1_{\mathcal{A}}$ and $\Delta \circ \Gamma \simeq 1_{\mathcal{B}}$.

The following result shows that the invertible right wide Morita contexts are exactly the strict right wide Morita contexts.
**Proposition 1.10** Let $\Gamma = (F,G,\eta,\rho)$ be a right wide Morita context between the abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then the following assertions are equivalent.

1. $\Gamma$ is invertible.
2. $\Gamma$ is a strict right wide Morita context, i.e. $F$ and $G$ define an equivalence between the categories $\mathcal{A}$ and $\mathcal{B}$ with natural equivalences $\eta : F \circ G \to 1_\mathcal{A}$ and $\rho : G \circ F \to 1_\mathcal{B}$.

**Proof:** (1)$\Rightarrow$(2) Assume that there exists a right wide Morita context $\Sigma = (U,V,\epsilon,\delta)$ between $\mathcal{B}$ and $\mathcal{A}$ such that $\Gamma \circ \Sigma \simeq 1_\mathcal{A}$ and $\Sigma \circ \Gamma \simeq 1_\mathcal{B}$. Thus $(F \circ U, V \circ G, \eta \circ F \epsilon G, \delta \circ V \rho U) \simeq (1_\mathcal{A}, 1_\mathcal{A}, \text{Id}_{1_\mathcal{A}}, \text{Id}_{1_\mathcal{A}})$. By Remark 1.8(2) we have that $\eta$ and $\delta$ are epimorphisms. Similarly we get that $\rho$ and $\epsilon$ are epimorphisms, so $\Gamma$ and $\Sigma$ are strict.

(2)$\Rightarrow$(1) Let $\Delta = (G,F,\rho,\eta)$, a right wide Morita context. Then we have that

$$\Gamma \circ \Delta = (F \circ G, F \circ G, \eta \circ F \rho G, \eta \circ F \rho G) \simeq 1_\mathcal{A}$$

Indeed, we have the natural equivalences $\eta : F \circ G \to 1_\mathcal{A}$ on the role of $u$, and also $\eta$ on the role of $v$ from Definition 1.7. Moreover

$$\text{Id}_{1_\mathcal{A}} \circ 1_\mathcal{A} \eta \circ \eta(F \circ G) = \eta \circ \eta(F \circ G) = \eta \circ F \rho G$$

Similarly $\Delta \circ \Gamma \simeq 1_\mathcal{B}$, and this ends the proof.

We have seen that a strict right wide Morita context defines an equivalence of categories. The next result shows that any equivalence between abelian categories arises like this. These generalize the classical result of Morita which tells that two categories of modules are equivalent if and only if there exists a strict Morita context connecting them.

**Proposition 1.11** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{B} \to \mathcal{A}$, $G : \mathcal{A} \to \mathcal{B}$ two functors defining an equivalence of categories. Then there exists a wide Morita context $(F,G,\eta,\rho)$ between $\mathcal{A}$ and $\mathcal{B}$.

**Proof:** Since $F$ and $G$ define an equivalence of categories, there exist two natural equivalences $u : F \circ G \to 1_\mathcal{A}$ and $v : G \circ F \to 1_\mathcal{B}$. Let $Y \in \text{cal} \mathcal{B}$. Then $u(F(Y)) : (F \circ G \circ F)(Y) \to F(Y)$ is a morphism in $\mathcal{A}$. Since $F$ is fully faithful, there exists a unique morphism $w(Y) : (G \circ F)(Y) \to Y$ in $\mathcal{B}$ such that $F(w(Y)) = u(F(Y))$. We show that $w : G \circ F \to 1_\mathcal{B}$ is a natural transformation, i.e. for any morphism $f : Y_1 \to Y_2$ in $\mathcal{B}$ we have $w(Y_2) \circ (G \circ F)(f) = f \circ w(Y_1)$. Since $F$ is fully faithful, this is equivalent to $F(w(Y_2)) \circ (F \circ G \circ F)(f) = F(f) \circ F(w(Y_1))$. By the definition of $w$, this is the same to $u(F(Y_2)) \circ (F \circ G)(f) = F(f) \circ u(F(Y_1))$, and this is true since $u : F \circ G \to 1_\mathcal{A}$ is a natural transformation.

Since $uF = Fw$, $u$ is a natural equivalence and $F$ is fully faithful, we get that $w$ is also a natural equivalence.
It remains to show that \( Gu = wG \), and then we have that \((F,G,u,w)\) is a right wide Morita context. Denote \( H = F \circ G \). Since \( u : H \to 1_A \) is a natural transformation, we have that \( u(X) \circ H(u(X)) = u(X) \circ u(H(X)) \) for any \( X \in A \). Since \( u(X) \) is an isomorphism, this shows that \( uH = Hu \), i.e. \((F \circ G)(u(X)) = u((F \circ G)(X))\). Since \( uF = Fw \), this implies that \( F(G(u(X))) = F(w(G(X))) \). But \( F \) is fully faithful, so then \( G(u(X)) = w(G(X)) \), showing that \( Gu = wG \), and this ends the proof.

We end the section by discussing right wide Morita contexts for which the two functors are left adjoints. Let \( \Gamma = (F,G,\eta,\rho) \) be a right wide Morita context between the abelian categories \( A \) and \( B \). Assume that \((F,F',\alpha,\beta)\) is an adjunction, i.e. \( F' \) is a right adjoint of \( F \), and \( \alpha : 1_B \to F' \circ F \) and \( \beta : F \circ F' \to 1_A \) are the unit and the counit of the adjunction. This means that both compositions \( F' \circ G \circ G' \circ (F' \circ F) \circ F' \overset{\alpha}{\Rightarrow} F' \overset{\beta}{\Rightarrow} F' \circ F \) and \( F \overset{\alpha}{\Rightarrow} F \circ F' \overset{\beta}{\Rightarrow} F \) are identities. Also consider an adjunction \((G,G',\gamma,\delta)\), so \( \gamma : 1_A \to G' \circ G \), \( \delta : G \circ G' \to 1_B \), and both compositions \( G' \circ G \circ G' \overset{\gamma}{\Rightarrow} G' \overset{\delta}{\Rightarrow} G' \circ G \) and \( G \overset{\gamma}{\Rightarrow} G \circ G' \overset{\delta}{\Rightarrow} G \) are identities. It is known that the functors \( F' \), \( G' \) are left exact. By [22, page 101] we can define the compositions of these adjunctions

\[
(F,F',\alpha,\beta) \circ (G,G',\gamma,\delta) = (F \circ G, G' \circ F', G' \circ \alpha G \circ \gamma, \beta \circ \delta F')
\]

and

\[
(G,G',\gamma,\delta) \circ (F,F',\alpha,\beta) = (G \circ F, F' \circ G', F' \circ F \circ \alpha, \delta \circ G \circ \beta G')
\]

We recall that the composition of the adjunctions is associative ([22, page 102]). Now we can prove the following key result.

**Proposition 1.12** Let \( \Gamma = (F,G,\eta,\rho) \) be a right wide Morita context between the abelian categories \( A \) and \( B \) such that the functors \( F \) and \( G \) have right adjoints \( F' \) and \( G' \). Then there exist natural transformations \( \eta' : 1_A \to G' \circ F' \) and \( \rho' : 1_B \to F' \circ G' \) induced by \( \Gamma \) such that \((F',G',\eta',\rho')\) is a left wide Morita context.

**Proof:** We keep the notations above for the adjunctions. By [22, Theorem 2, page 98], there exists a unique natural transformation \( \eta' : 1_A \to G' \circ F' \) (called the conjugate of \( \eta \)) such that the diagram

\[
\begin{array}{ccc}
\text{Hom}(M,N) & \overset{=}\longrightarrow & \text{Hom}(M,N) \\
\downarrow \text{Hom}(\eta(M),1_N) & & \downarrow \text{Hom}(1_M,\eta'(N)) \\
\text{Hom}((F \circ G)(M),N) & \underset{\Phi}{\longrightarrow} & \text{Hom}(M,(G' \circ F')(N))
\end{array}
\]
is commutative for any \( M,N \in \mathcal{A} \), where \( \Phi \) is the isomorphism associated to the natural transformation \( G' \circ \alpha \circ \gamma \) or \( \beta \circ F \delta F' \).

Similarly the conjugate \( \rho' : 1_\mathcal{B} \to F' \circ G' \) of \( \rho \) is defined. By [22, Theorem 2, page 102], we have that the conjugate of \( G \eta = G \circ G \) is \( \eta' \circ 1/\mathcal{G}' = \eta' G' \), and the conjugate of \( \rho' \) is \( G' \circ \rho' \). Since \( G \eta = \rho G \) and the conjugate is unique, we have that \( \eta' G' = G' \circ \rho' \). Similarly we see that \( \rho F' = \rho G \), which shows that \( (F', G', \eta', \rho') \) is a left wide Morita context.

\[ \]

### 2 Relative injectivity and right wide Morita contexts

Let \( \mathcal{A} \) be an abelian category, and let \( \mathcal{C} \) be a subcategory of \( \mathcal{A} \). If \( X \) and \( M \) are two objects of \( \mathcal{A} \), we say that \( M \in \mathcal{C} \)-injective if for any monomorphism \( i : X' \to X \in \mathcal{A} \) such that \( X/X' \in \mathcal{C} \), and any morphism \( f : X' \to M \), there exists \( g : X \to M \) such that \( g \circ i = f \).

We denote by \( \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) = \{ X \in \mathcal{A} \mid M \in \mathcal{C} \text{ is } X \text{-injective} \} \). If \( \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) = \mathcal{A} \), i.e. \( M \) is \( \mathcal{C} \)-injective for any \( X \in \mathcal{A} \), we simply say that \( M \) is \( \mathcal{C} \)-injective.

**Proposition 2.1** With the above notations, the following assertions hold.

1. \( \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) \) is closed under factor objects.
2. If \( \mathcal{C} \) is closed under extensions, \( X \in \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) \), and \( X' \) is a subobject of \( X \), then \( X' \in \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) \).
3. If \( \mathcal{A} \) is a Grothendieck category and \( \mathcal{C} \) is closed under subobjects, then \( \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) \) is closed under direct sums.

**Proof:** (1) Assume that \( M \) is \( \mathcal{C} \)-injective, and let \( Y \) be a subobject of \( X \). We show that \( M \) is \( \mathcal{C} \)-injective. Let \( X'/Y \leq X/Y \) such that \( X/X' \in \mathcal{C} \). Denote by \( p : X \to X/Y \) and \( p' : X' \to X'/Y \) the projection morphisms, and by \( i : X' \to X \) and \( j : X'/Y \to X/Y \) the inclusion morphisms such that \( p \circ i = j \circ p' \). Since \( X \in \mathcal{I}^{-1}(\mathcal{M}, \mathcal{C}) \), we see that there exists \( h : X \to M \) such that \( h \circ i = f \circ p' \). But \( h(Y) = (h \circ i)(Y) = (f \circ p')(Y) = 0 \), so \( h \) factorizes through \( X/Y \), i.e. there exists \( g : X/Y \to M \) with \( g \circ p = h \). Then we have \( g \circ j \circ p' = g \circ p \circ i = h \circ i = f \circ p' \), and since \( p' \) is an epimorphism we have that \( g \circ j = f \), which shows that \( M \) is \( \mathcal{C} \)-injective.

(2) Let \( K \) be a subobject of \( X' \) such that \( X'/K \in \mathcal{C} \), and let \( f : K \to M \) be a morphism. Denote by \( j : K \to X' \) and \( i : X' \to X \) the inclusion morphisms. We have the exact sequence

\[
0 \to X'/K \to X/K \to X/X' \to 0
\]

and since \( \mathcal{C} \) is closed under extensions we obtain that \( X/K \in \mathcal{C} \). Since \( M \) is \( \mathcal{C} \)-injective, there exists \( g : X \to M \) such that \( g \circ i \circ j = f \). Then \( g \circ i : X' \to M \), and \( (g \circ i) \circ j = f \), so \( M \) is \( \mathcal{C} \)-injective.
Let \((X_i)_{i \in I}\) be a family in \(I^{-1}(M, \mathcal{C})\), and let \(X = \oplus_{i \in I} X_i\). Let \(K \leq X\) with \(X/K \in \mathcal{C}\), and let \(h : K \to M\) be a morphism. The family
\[
\mathcal{F} = \{ f : L \to M | K \leq L \leq X \text{ and } f|_K = h \}
\]
is inductive when ordered in the obvious way by inclusion, and then by Zorn’s Lemma it has a maximal element \(\overline{h} : N \to M\). Let \(N_i = N \cap X_i\). We have that \(X_i/N_i = X_i/N \cap X_i \simeq X_i + N/N \leq X/N\), and since \(\mathcal{C}\) is closed under subobjects, we also have \(X_i/N_i \in \mathcal{C}\).

Assume that \(N \neq X\). Then there is \(i \in I\) such that \(X_i\) is not a subobject of \(N\), and then \(N_i\) is not a subobject of \(X_i\). Let \(q : N_i \to N\) be the inclusion morphism. Since \(M\) is \(\mathcal{C}\)-\(X_i\)-injective, there exists \(u : X_i \to M\) extending \(\overline{h} \circ q\). But the restrictions of \(u\) and \(\overline{h}\) to \(N_i\) are equal, and then it is easy to see that there exists \(\overline{u} : N_i + X_i \to M\) extending both \(u\) and \(\overline{h}\) (the argument is exactly as in [1, Proposition 1.13]). Then \(\overline{u} \in \mathcal{F}\) and \(\overline{h} < \overline{u}\), a contradiction. We conclude that \(N\) must be the whole of \(X\), and then \(M\) is \(\mathcal{C}\)-\(X_i\)-injective.

Let \(\mathcal{A}\) and \(\mathcal{B}\) be abelian categories and let \(\Gamma = (F,G,\eta,\rho)\) be a right wide Morita context between \(\mathcal{A}\) and \(\mathcal{B}\). For any \(M \in \mathcal{A}\) we write \(I_M = \text{Im} \eta(M)\), and for any \(N \in \mathcal{B}\) we denote \(J_N = \text{Im} \rho(N)\). If \(f : M \to M'\) is a morphism in \(\mathcal{A}\), the commutative diagram

\[
\begin{array}{ccc}
FG(M) & \xrightarrow{FG(f)} & FG(M') \\
\downarrow \eta(M) & & \downarrow \eta(M') \\
M & \xrightarrow{f} & M'
\end{array}
\]

shows that \(f(I_M) \subseteq I_{M'}\). A similar result holds for the objects \(J_N, N \in \mathcal{B}\). Moreover, if \(f : M \to M'\) is an epimorphism, we have \(f(I_M) = I_{M'}\) since the functors \(F\) and \(G\) are right exact. In particular we have that \(I_{M/I_M} = 0\).

We consider the following class
\[
\mathcal{C}_\Gamma = \{ X \in \mathcal{A} | f(I_M) = 0 \text{ for any } M \text{ and } f : M \to X \} \\
= \{ X \in \mathcal{A} | f \circ \eta(M) = 0 \text{ for any } M \text{ and } f : M \to X \}
\]

**Proposition 2.2** With the above notations, the following assertions hold.
(1) For any \(M \in \mathcal{A}\) we have \(M/I_M \in \mathcal{C}_\Gamma\).
(2) \(M \in \mathcal{C}_\Gamma\) if and only if \(I_M = 0\). Moreover, if \(M \in \mathcal{A}\) and \(K \leq M\) is a subobject such that \(M/K \in \mathcal{C}_\Gamma\), then \(I_M \leq K\).
(3) \(\mathcal{C}_\Gamma\) is closed under subobjects and factor objects. If \(\mathcal{A}\) has direct products, then \(\mathcal{C}_\Gamma\) is closed under direct products (i.e. \(\mathcal{C}_\Gamma\) is a TTF-class). Moreover, if \(\mathcal{A}\) and \(\mathcal{B}\) have direct sums and \(F,G\) commute with the direct sums, then \(\mathcal{C}_\Gamma\) is closed under direct sums.
(4) If the category $\mathcal{A}$ has a family of generators $(U_i)_{i \in I}$, then $\{U_i/I_{U_i}|i \in I\}$ is a family of generators of the category $\mathcal{C}_\Gamma$.

(5) If $I_{M'} = I_M$ for any $M$, then $\mathcal{C}_\Gamma$ is closed under extensions.

**Proof:**

(1) Let $N \in \mathcal{A}$ and let $f : N \to M/I_M$ be a morphism. Since $f(I_N) \leq I_{M/I_M} = 0$, so $f(I_N) = 0$, showing that $M/I_M \in \mathcal{C}_\Gamma$.

(2) If $I_M = 0$, then we have $M \in \mathcal{C}_\Gamma$ by (1). Conversely, let $M \in \mathcal{C}_\Gamma$. Then for the morphism $I_M : M \to M$ we have $0 = 1_M(I_M) = I_M$. The last part follows by considering the natural projection $\pi : M \to M/K$. Then $\pi(I_M) = I_{M/K} = 0$, so $I_M \leq K$.

(3) Consider an exact sequence

$$0 \to X' \xrightarrow{i} X \xrightarrow{\pi} X'' \to 0$$

in $\mathcal{A}$, and assume that $X \in \mathcal{C}_\Gamma$. Let $M \in \mathcal{A}$ and $f : M \to X'$. Then $i \circ f : M \to X$, so $(i \circ f)(I_M) = 0$. Since $i$ is a monomorphism, we have that $f(I_M) = 0$, so then $X' \in \mathcal{C}_\Gamma$.

Since $\pi$ is an epimorphism, we have that $\pi(I_X) = I_{X''}$. But $X \in \mathcal{C}_\Gamma$ shows that $I_X = 0$, and then we also have $I_{X''} = 0$, i.e. $X'' \in \mathcal{C}_\Gamma$.

Let now $(X_i)_{i \in I}$ be a family of objects in $\mathcal{C}_\Gamma$, $M \in \mathcal{A}$ and $f : M \to \prod_{i \in I} X_i$ an arbitrary morphism. If $\pi_i : \prod_{j \in I} X_j \to X_i$ are the natural projections, we have that $(\pi_i \circ f)(I_M) = 0$ for any $i$, so $f(I_M) = 0$. We conclude that $\prod_{i \in I} X_i \in \mathcal{C}_\Gamma$.

Assume now that $\mathcal{A}$ and $\mathcal{B}$ have direct sums, and $F$ and $G$ commute with direct sums. Let $(M_i)_{i \in I}$ be a family of objects in $\mathcal{C}_\Gamma$, $\oplus_{i \in I} M_i$ their direct sum, and $q_j : M_j \to \oplus_{i \in I} M_i$, the natural embedding for any $j \in I$. Since $F$ and $G$ commute with the direct sums, so does $F \circ G$, therefore $(F \circ G)(\oplus_{i \in I} M_i) \simeq \oplus_{i \in I} (F \circ G)(M_i)$. Since $\eta$ is a natural transformation we have that

$$\eta(\oplus_{i \in I} M_i) \circ (F \circ G)(q_j) = q_j \circ \eta(M_j) = 0$$

for any $j \in I$, and we conclude that $\eta(\oplus_{i \in I} M_i) = 0$, which shows that $\oplus_{i \in I} M_i \in \mathcal{C}_\Gamma$.

(4) Let $X \in \mathcal{C}_\Gamma$ and $X' < X$ a strict subobject. Since $(U_i)_{i \in I}$ is a family of generators of $\mathcal{A}$, there exist $i \in I$ and a morphism $f : U_i \to X$ such that $\text{Im} f$ is not a subobject of $X'$. Since $f(I_{U_i}) \leq I_X = 0$ (by (3)), then $f$ factorizes through a morphism $g : U_i/I_{U_i} \to X$. Clearly $\text{Im} g = \text{Im} f$ is not a subobject of $X'$, and this ends the proof.

(5) Let

$$0 \to X' \xrightarrow{i} X \xrightarrow{\pi} X'' \to 0$$

be an exact sequence with $X', X'' \in \mathcal{C}_\Gamma$. If $f : M \to X$ is a morphism, then $(\pi \circ f)(I_M) = 0$, so there exists $g : I_M \to X'$ such that $i \circ g = f$. By hypothesis we have $g(I_{I_M}) = 0$, so then $g(I_M) = 0$. We conclude that $f(I_M) = 0$ and $X \in \mathcal{C}_\Gamma$.

**Remark 2.3** We note that part (1) of Proposition 2.2 shows that $\mathcal{C}_\Gamma = \{X \in \mathcal{A} | \eta(X) = 0\}$. 

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Proposition 2.4 Let $M \in \mathcal{A}$. Then $\text{Ker} \eta(M) \in \mathcal{C}_\Gamma$.

Proof: Denote by $K = \text{Ker} \eta(M)$ and by $i: K \to (F \circ G)(M)$ the inclusion morphism. Let $U \in \mathcal{A}$ be an object and $f: U \to K$ a morphism. We have the commutative diagram

$$
\begin{array}{cccccc}
(F \circ G)(U) & \xrightarrow{\eta(U)} & U \\
\downarrow (F \circ G)(f) & & & f \\
(F \circ G)(K) & \xrightarrow{\eta(K)} & K \\
\downarrow (F \circ G)(i) & & & i \\
(F \circ G)((F \circ G)(M)) & \xrightarrow{\eta((F \circ G)(M))} & (F \circ G)(M)
\end{array}
$$

Since $\eta F = F \rho$ and $\rho G = G \eta$, we have $\eta((F \circ G)(M)) = (F \circ G)(\eta(M))$, and then

$$
i \circ f \circ \eta(U) = i \circ \eta(K) \circ (F \circ G)(f)
= (F \circ G)(\eta(M)) \circ (F \circ G)(i) \circ (F \circ G)(f)
= (F \circ G)(\eta(M) \circ i \circ f)
= 0
$$

Since $i$ is a monomorphism we have $f \circ \eta(U) = 0$, showing that $K \in \mathcal{C}_\Gamma$.

Corollary 2.5 If $(U_i)_{i \in I}$ is a family of generators of the category $\mathcal{A}$ and $\eta(U_i)$ is an epimorphism for any $i \in I$, then $\eta$ is a natural equivalence.

Proof: By Proposition 2.2(4), we see that $\mathcal{C}_\Gamma = 0$. Then by Proposition 2.4 we have $\text{Ker} \eta(M) = 0$ for any $M \in \mathcal{A}$, so $\eta$ is a natural equivalence.

Definition 2.6 Let $M$ be an object of the abelian category $\mathcal{A}$, and $\mathcal{C}$ be an arbitrary subcategory of $\mathcal{A}$. We say that $M$ is $\mathcal{C}$-torsion free if for any $X \in \mathcal{C}$ and any monomorphism $i: X \to M$, we must have $X = 0$. If $M$ is $\mathcal{C}$-torsion free and $\mathcal{C}$-injective, then $M$ is called $\mathcal{C}$-closed.

Now we can characterize the $\mathcal{C}_\Gamma$-closed objects by a categorial property.
Proposition 2.7  An object $M \in A$ is $C_G$-closed if and only if for any $U \in A$ the natural map

$$
\phi = \text{Hom}(\eta(U), 1_M) : \text{Hom}(U, M) \to \text{Hom}((F \circ G)(U), M), \quad \phi(\beta) = \beta \circ \eta(U)
$$

is bijective.

Proof: For $U \in A$ we denote by $\eta_1(U) : (F \circ G)(U) \to I_U$ the corestriction of $\eta(U) : (F \circ G)(U) \to U$ to $I_U$, and by $j : I_U \to U$ the inclusion morphism. Note that $\eta(U) = j \circ \eta_1(U)$.

Assume that $M$ is $C_G$-closed. Let $\alpha : (F \circ G)(U) \to M$ be a morphism. We consider the following diagram.

$$
\begin{array}{ccc}
\text{Ker } \eta(U) & \xrightarrow{i} & (F \circ G)(U) \\
& \alpha \downarrow & \downarrow \eta_1(U) \\
M & \xrightarrow{j} & I_U
\end{array}
$$

As in the proof of Proposition 2.4, $i$ denotes the inclusion morphism. Since $\text{Ker } \eta(U) \in C_G$ (by Proposition 2.4) and $M$ is $C_G$-torsion free, we have that $\alpha(\text{Ker } \eta(U)) = 0$. Thus there exists a unique morphism $\overline{\alpha} : I_U \to M$ such that $\overline{\alpha} \circ \eta_1(U) = \alpha$. Since $U/I_U \in C_G$ and $M$ is $C_G$-injective, we see that there exists a morphism $\beta : U \to M$ such that $\overline{\alpha} = \beta \circ j$. Then $\alpha = \beta \circ j \circ \eta_1(U) = \beta \circ \eta(U) = \phi(\beta)$, showing that $\phi$ is surjective.

Now if $\phi(\beta) = \beta \circ \eta(U) = 0$ for some $\beta$, we have that $\beta(I_U) = 0$, so then there exists $\overline{\beta} : U/I_U \to M$ such that $\overline{\beta} \circ p = \beta$, where $p : U \to U/I_U$ is the natural projection. But $U/I_U \in C_G$ and $M$ is $C_G$-torsion free, so we must have $\overline{\beta} = 0$. Hence $\beta = 0$, so $\phi$ is injective.

Therefore $\phi$ a bijection, and it is obviously natural.

Conversely, assume that $\phi$ is bijective for any $U$. Define $p : \text{Hom}(I_U, M) \to \text{Hom}((F \circ G)(U), M)$ by $p(f) = f \circ \eta_1(U)$, and $q : \text{Hom}(U, M) \to \text{Hom}(I_U, M)$ by $q(f) = f \circ j$. Since $\eta_1(U)$ is an epimorphism, $p$ is injective. We have the commutative diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{p} & \text{Hom}((F \circ G)(U), M) \\
\downarrow q & & \downarrow \phi \\
\text{Hom}(U, M)
\end{array}
$$
This shows that \( p \) is surjective, so then it is bijective. Hence \( q \) is also bijective.

If \( U \) is a subobject of \( M \) which is in \( C_T \), we have that \( I_U = 0 \), and this shows that \( \text{Hom}(U,M) = 0 \). Thus \( M \) is \( C_T \)-torsion free.

Now if we take an arbitrary \( U \), and \( K \) a subobject of \( U \) such that \( U/K \in C_T \), we have by Proposition 2.2 that \( I_U \leq K \). We thus have \( I_K \leq I_U \leq K \leq U \). If \( f : K \to M \) is a morphism, it restricts to a morphism \( f_1 : I_U \to M \). Since \( q : \text{Hom}(U,M) \to \text{Hom}(I_U,M) \) is an isomorphism, there exists \( h : U \to M \) extending \( f_1 \). Denote by \( f' \) the restriction of \( h \) to \( K \). Let \( f_2 \) be the restriction of \( f \) to \( I_K \) (which is also the restriction of \( f_1 \) to \( I_K \)). Then clearly the restriction of \( f' \) to \( I_K \) is \( f_2 \), and the natural isomorphism \( \text{Hom}(I_K,M) \cong \text{Hom}(K,M) \) shows that \( f' = f \). Thus \( h \) extends \( f \), and this proves that \( M \) is \( C_T \)-injective. 

We can prove now the main result of this section.

**Theorem 2.8** Let \( \Gamma = (F,G,\eta,\rho) \) be a right wide Morita context between the abelian categories \( \mathcal{A} \) and \( \mathcal{B} \), such that the functors \( F \) and \( G \) have right adjoints \( F' \) and \( G' \). Denote by \( C_T \) (respectively \( D_T \)) the subcategory of \( \mathcal{A} \) (respectively \( \mathcal{B} \)) defined by the natural morphism \( \eta \) (respectively \( \rho \)). Then the categories of \( C_T \)-closed objects of \( \mathcal{A} \) and \( D_T \)-closed objects of \( \mathcal{B} \) are equivalent via the functors \( F',G' \).

**Proof:** By Proposition 1.12, we can associate a left wide Morita context \( (F',G',\eta',\rho') \) to \( \Gamma \). Let \( M \in \mathcal{A} \) be \( C_T \)-closed. By Proposition 2.7 there is a natural bijection

\[
\phi = \text{Hom}(\eta(U),1_M) : \text{Hom}(U,M) \to \text{Hom}((F \circ G')(U),M), \quad \phi(\beta) = \beta \circ \eta(U)
\]

where \( U \in \mathcal{A} \). Since \( G' \circ F' \) is a right adjoint of \( F \circ G \), we have a natural bijection

\[
\psi : \text{Hom}((F \circ G')(U),M) \to \text{Hom}(U,(G' \circ F')(M))
\]

The construction of \( \eta' \) as the conjugate of \( \eta \) (see the proof of Proposition 1.12) shows that

\[
\psi \phi = \text{Hom}(1_U,\eta'(M)) \quad \text{Since} \quad \psi \quad \text{and} \quad \phi \quad \text{are bijections, then so is} \quad \text{Hom}(1_U,\eta'(M)).
\]

This implies that \( \eta'(M) \) is an isomorphism. Similarly, if \( N \in \mathcal{B} \) is \( D_T \)-closed, then the natural isomorphism \( \rho'(N) : N \to (F' \circ G')(N) \) is an isomorphism.

On the other hand, for any object \( V \in \mathcal{B} \) we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(V,F'(M)) & \xrightarrow{\text{Hom}(\rho(V),1_{F'(M)})} & \text{Hom}((G \circ F)(V),F'(M)) \\
\cong & & \cong \\
\text{Hom}(F(V),M) & \xrightarrow{\text{Hom}(F(\rho(V)),1_M)} & \text{Hom}((F \circ G \circ F)(V),M)
\end{array}
\]
where the vertical arrows are the natural bijections coming from the adjunction given by $F$ and $F'$. Since $F\rho = \eta F$ and $M \in A$ is $C_\Gamma$-closed, we have that $\text{Hom}(F(\rho(V)), 1_M) = \text{Hom}(\eta F(V), 1_M)$ is a bijection, and then also the top horizontal arrow $\text{Hom}(\rho(V), 1_{F'(M)})$ is a bijection. This shows that $F'(M)$ is $D_\Gamma$-closed, by Proposition 2.7. Similarly $G'(N)$ is $C_\Gamma$-closed for any $N \in B$ which is $D_\Gamma$-closed. These show that $F'$ and $G'$ induce an equivalence between the subcategory of $C_\Gamma$-closed objects of $A$ and the subcategory of $D_\Gamma$-closed objects of $B$.

Corollary 2.9 Let $\Gamma = (F, G, \eta, \rho)$ be a right wide Morita context between the abelian categories $A$ and $B$, such that the functors $F$ and $G$ have right adjoints $F'$ and $G'$, and $\eta$ is an epimorphism. Then $A$ and the category of $D_\Gamma$-closed objects of $B$ are equivalent via the functors $F'$ and the restriction of $F$.

Proof: Since $\eta$ is an isomorphism, we have $C_\Gamma = 0$, so the category of all $C_\Gamma$-closed objects of $A$ is the whole of $A$. Denote by $D$ the subcategory of $B$ consisting of all $D_\Gamma$-closed objects. We know from Theorem 2.8 that $F'$ is an equivalence between $A$ and $D$. Since $F$ is a left adjoint of $F'$, then the restriction $F : D \to A$ is still a left adjoint of $F'$, when regarded as a functor from $A$ to $D$. But then $F'$ is an equivalence, so $F : D \to A$ is also an equivalence.

3 The dual case: relative projectivity and left wide Morita contexts

In this section we consider the dual concepts of the ones presented in the Section 2. As we will see, this is useful for understanding several examples. Since the dual of an abelian category is also an abelian category, we can dualize the definitions and results directly.

So let $A$ be an abelian category and $C$ a subcategory of $A$. If $M$ and $X$ are two objects of $A$, we say that $M$ is $C$-$X$-projective if $M$ is $C^0$-$X$-injective as an object of the dual category $A^0$. It is clear that $M$ is $C$-$X$-projective if for any epimorphism $p : X \to X'$ in $A$, such that $\text{Ker}(p) \in C$, and any morphism $f : P \to X'$, there exists a morphism $g : M \to X$ such that $p \circ g = f$. We denote $P^{-1}(M, C) = \{ X \in A \mid M \text{ is } C-\text{X-projective} \}$. If $P^{-1}(M, C) = A$, we say that $M$ is $C$-projective. We have the following properties.

Proposition 3.1 With the above notations, the following assertions hold.
(1) $P^{-1}(M, C)$ is closed under subobjects.
(2) If $C$ is closed under extensions, $X \in P^{-1}(M, C)$, and $p : X \to X'$ is an epimorphism such that $\text{Ker}(p) \in C$, then $X' \in P^{-1}(M, C)$.
(3) If $A$ is a Grothendieck category and $C$ is closed under factor objects, then $P^{-1}(M, C)$ is
closed under direct products. In particular $\mathcal{P}^{-1}(M, C)$ is closed at finite direct sums.

(4) If $\mathcal{A}$ is a Grothendieck category, $\mathcal{C}$ is a closed subcategory, and $M$ is a finitely generated object, then $\mathcal{P}^{-1}(M, C)$ is closed under arbitrary direct sums.

**Proof:** Parts (1)-(3) follow directly by dualizing Proposition 2.1. Part (4) can be proved as [2, Proposition 16.12].

Let $M$ be an object of $\mathcal{A}$, and $\mathcal{C}$ be an arbitrary subcategory of $\mathcal{A}$. We say that $M$ is $\mathcal{C}$-cotorsion free if for any $X \in \mathcal{C}$ and any epimorphism $f : M \to X$, we have $f = 0$. This is of course equivalent to the fact that $M$ is $\mathcal{C}^0$-torsion free when regarded in the dual category $\mathcal{A}^0$.

Let now $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $\Gamma = (F, G, \eta, \rho)$ be a left wide Morita context between $\mathcal{A}$ and $\mathcal{B}$. For any $M \in \mathcal{A}$ we denote $K_M = \text{Ker} \eta(M)$, and for any $N \in \mathcal{B}$ we denote $L_N = \text{Ker} \rho(N)$. By looking at the definition of $I_M$ and $J_N$ in Section 2, it is natural to consider these objects $K_M$ and $L_N$, since when regarded in the dual category, the image of a morphism becomes a coimage. If $f : M \to M'$ is a morphism in $\mathcal{A}$, we have that $f(K_M) \subseteq K_{M'}$. Dually to the definition we made in Section 2, define now

$$
\mathcal{C}_\Gamma = \{ X \in \mathcal{A} | \eta(M) \circ f = 0 \text{ for any } M \text{ and } f : X \to M \}
$$
$$
= \{ X \in \mathcal{A} | \text{Im} f \leq K_M \text{ for any } f : M \to X \}
$$

If we regard $\Gamma$ as a right wide Morita context between the dual categories, then $\mathcal{C}_\Gamma$ is exactly the class associated to $\eta$. Similarly one can consider the subcategory $\mathcal{D}_\Gamma$ associated to $\rho$. The following results are dual to Propositions 2.2 and 2.4, Corollary 2.5, and Proposition 2.7.

**Proposition 3.2** With the above notations, the following assertions hold.

1. For any $M \in \mathcal{A}$ we have $K_M \in \mathcal{C}_\Gamma$.
2. $M \in \mathcal{C}_\Gamma$ if and only if $K_M = M$.
3. $\mathcal{C}_\Gamma$ is closed under subobjects and factor objects. If $\mathcal{A}$ has direct sums, then $\mathcal{C}_\Gamma$ is closed under direct sums (i.e. $\mathcal{C}_\Gamma$ is a closed subcategory). Moreover, if $\mathcal{A}$ and $\mathcal{B}$ have direct products and the functors $F$ and $G$ commute with direct products, then $\mathcal{C}_\Gamma$ is closed under direct products (i.e. it is a TTF-class).
4. If the category $\mathcal{A}$ has a family of cogenerators $(Q_i)_{i \in I}$, then $\{K_{Q_i} | i \in I\}$ is a family of cogenerators of the category $\mathcal{C}_\Gamma$.

**Proposition 3.3** Let $M \in \mathcal{A}$. Then $\text{Coker} \eta(M) \in \mathcal{C}_\Gamma$.

**Corollary 3.4** If $(Q_i)_{i \in I}$ is a family of cogenerators of the category $\mathcal{A}$ and $\eta(Q_i)$ is a monomorphism for any $i \in I$, then $\eta$ is a natural equivalence.
Proposition 3.5 An object $M \in A$ is $\mathcal{C}_\Gamma$-cotorsion free and $\mathcal{C}_\Gamma$-projective if and only if for any $U \in A$, the map

$$\phi : \text{Hom}(M, U) \to \text{Hom}(M, (F \circ G)(U)) \quad \phi(\beta) = \eta(U) \circ \beta$$

is bijective.

Denote by $\mathcal{A}_{\Gamma,\text{proj}}$ the subcategory of $A$ consisting of all objects that are $\mathcal{C}_\Gamma$-cotorsion free and $\mathcal{C}_\Gamma$-projective. Similarly we denote by $\mathcal{B}_{\Gamma,\text{proj}}$ the subcategory of $B$ consisting of all objects that are $\mathcal{D}_\Gamma$-cotorsion free and $\mathcal{D}_\Gamma$-projective. The dual of Theorem 2.8 and Corollary 2.9 are the following.

Theorem 3.6 Let $\Gamma = (F, G, \eta, \rho)$ be a left wide Morita context between the abelian categories $A$ and $B$, such that the functors $F$ and $G$ have left adjoints $F'$ and $G'$. Then the categories $\mathcal{A}_{\Gamma,\text{proj}}$ and $\mathcal{B}_{\Gamma,\text{proj}}$ are equivalent via the functors $F'$, $G'$.

Corollary 3.7 Let $\Gamma = (F, G, \eta, \rho)$ be a left wide Morita context between the abelian categories $A$ and $B$, such that the functors $F$ and $G$ have left adjoints $F'$ and $G'$. If $\eta$ is a monomorphism, then the categories $\mathcal{A}$ and $\mathcal{B}_{\Gamma,\text{proj}}$ are equivalent via the functors $F'$ and the restriction of $F$.

4 Wide Morita contexts over Grothendieck categories and equivalence results

Throughout this section we assume that $A$ is a Grothendieck category and $\mathcal{C}$ is a closed subcategory of $A$. Since $\mathcal{C}$ is closed under factor objects, an object $M \in A$ is $\mathcal{C}$-torsion free if for any object $X \in \mathcal{C}$ and any morphism $f : X \to M$, we have $f = 0$. We denote by $t(M)$ the sum of all subobjects of $M$ belonging to $\mathcal{C}$. Clearly $t(M)$ exists since $\mathcal{C}$ is closed under arbitrary direct sums and factor objects. In this way a left exact functor $t : A \to A$ is defined; it is called the preradical associated to $\mathcal{C}$. Clearly $M$ is $\mathcal{C}$-torsion free if and only if $t(M) = 0$. If $\mathcal{C}$ is a localizing subcategory, we have that $t(M/t(M)) = 0$.

For a closed subcategory $\mathcal{C}$, we denote by $\overline{\mathcal{C}}$ the smallest localizing subcategory containing $\mathcal{C}$. This is given by

$$\overline{\mathcal{C}} = \{ X \in A \mid \text{For any } X' < X, X/X' \text{ contains a non-zero object of } \mathcal{C} \}$$

If $t$ is the preradical associated to $\mathcal{C}$, we denote by $\overline{t}$ the preradical (which is in fact a radical) associated to $\overline{\mathcal{C}}$. We have the following characterization.
Theorem 4.1 Let $C$ be a closed subcategory of $A$, and let $M$ be an object of $A$. Then the following assertions are equivalent.

1. $M$ is $C$-closed.
2. $M$ is $C$-closed.
3. If $(U_i)_{i \in I}$ is a family of generators of the category $A$, then $M$ is $C$-$U_i$-injective for any $i \in I$, and $t(M) = 0$.

Proof: (1)⇒(2) is clear since $C \subseteq \overline{C}$.

(2)⇒(1) If $t(M) \neq 0$, then by the construction of $\overline{C}$, the object $t(M)$ contains a non-zero object belonging to $C$, and this would imply $t(M) \neq 0$, a contradiction. Therefore $t(M) = 0$.

We prove now that $M$ is $\overline{C}$-injective. Let $X \in A$ and $X' \leq X$ such that $X/X' \in \overline{C}$, and take $f : X' \to M$ be a morphism. A standard application of Zorn’s Lemma shows that there exists a maximal subobject $Y$ of $X$, with $X' \leq Y$ and there exists a morphism $g : Y \to M$ extending $f$. Hence $X/Y$ is a factor object of $X/X'$, so $X/Y \in \overline{C}$, and then there exists a subobject $Y < Z \leq X$ such that $Z/Y \in C$. Since $M$ is $C$-injective, there exists $h : Z \to M$ extending $g$. This is a contradiction to the maximality of $Y$. We conclude that $Y = X$ and $M$ is $\overline{C}$-injective.

(2)⇒(3) is clear.

(3)⇒(2) Let $X \in A$. Since $X$ is a factor object of a direct sum of $U_i$’s, and $M$ is $C$-$U_i$-injective for any $i$, we see by Proposition 2.1 that $M$ is $C$-$X$-injective.

Example 4.2 (i) Let $R$ be a ring with identity and $I$ be a two-sided ideal of $R$. We define the class $P_I = \{M \in R - \text{mod} | IM = 0\}$. It is easy to see that $P_I$ is a closed subcategory of $R - \text{mod}$. The smallest localizing subcategory of $R - \text{mod}$ containing $P_I$ is

$$C_I = \{M \in R - \text{mod} | \text{For any } M' < M, M/M' \text{ contains some } S \in P_I, S \neq 0\}$$

$$= \{M \in R - \text{mod} | \text{For any } M' < M, \text{ there is } m \in M - M' \text{ such that } Im \subseteq M'\}$$

Let us note that if the ideal $I$ is idempotent, i.e. $I^2 = I$, then $C_I = P_I$. An object $M \in R - \text{mod}$ is $C_I$-torsion free if and only if it is $P_I$-torsion free, and this means that

$$\text{Ann}_M(I) = \{x \in M | Ix = 0\} = 0$$

Now by Theorem 4.1, part (3), we have that $M$ is $C_I$-closed if and only if $\text{Ann}_M(I) = 0$ and any morphism $f : I \to M$ of $R$-modules can be (uniquely) extended to a morphism $g : R \to M$. We conclude that $M$ is $C_I$-closed if and only if the natural morphism

$$\alpha : M \to \text{Hom}_R(I, M), \quad \alpha(m)(a) = am, \quad a \in I, m \in M$$

is an isomorphism.

(ii) Let $R = \oplus_{\sigma \in G} R_{\sigma}$ be a $G$-graded ring, and let $R - \text{gr}$ be the category of left graded $R$-modules. If $M = \oplus_{\lambda \in G} M_{\lambda}$ is an object of this category, we can consider for any $\sigma \in G$ the
graded module $M(\sigma)$ such that $M(\sigma) = M$ as an $R$-module, and the homogeneous components are given by $M(\sigma)_\lambda = M_{\lambda\sigma}$ for any $\lambda \in G$. The object $M(\sigma)$ is called the $\sigma$-suspension of $M$. It is known that $R - \text{gr}$ is a Grothendieck category with a family of projective generators $\{R(\sigma)|\sigma \in G\}$ (see [25] for details).

Let $I$ be a graded ideal of $R$, and denote $\mathcal{P}_I = \{M \in R - \text{gr}| IM = 0\}$. Then $\mathcal{P}_I$ is a closed subcategory of $R - \text{gr}$. Moreover, $\mathcal{P}_I$ is rigid, i.e. if $M \in \mathcal{P}_I$, then $M(\sigma) \in \mathcal{P}_I$ for any $\sigma \in G$. The smallest localizing subcategory of $R - \text{gr}$ containing $\mathcal{P}_I$ is

$$\mathcal{C}_I = \{M \in R - \text{gr}| \text{For any } M' <_{R-\text{gr}} M, M/M' \text{ contains some } S \in \mathcal{P}_I, S \neq 0\}$$

Clearly $\mathcal{C}_I$ is also a rigid subcategory. As in (i) we obtain that if $M \in R - \text{gr}$, then $M$ is $\mathcal{C}_I$-closed if and only if the natural morphism

$$\alpha : M \rightarrow \text{HOM}_R(I, M), \quad \alpha(m)(a) = am, \quad a \in I, m \in M$$

is an isomorphism of graded modules. Recall that $\text{HOM}_R(I, M) = \oplus_{\sigma \in G}\text{HOM}_R(I, M)_\sigma$, where $\text{HOM}_R(I, M)_\sigma$ is the set of all linear maps of degree $\sigma$ (see [25] for details). It is clear from this that if $M$ is $\mathcal{C}_I$-closed, then $M(\sigma)$ is $\mathcal{C}_I$-closed for any $\sigma \in G$.

(iii) Let $R$ be a ring with local units, i.e. for any finite subset $X$ of $R$, there exists an idempotent element $e \in R$ such that $ex = xe = x$ for any $x \in X$ (or equivalently $X \subseteq eRe$); see [3] for details. For such an $R$ we have the Grothendieck category $R - \text{MOD}$, of all unital left $R$-modules. An $R$-module $M$ is unital if $RM = M$.

For any two-sided ideal $I$ of $R$, we can define a localizing category $\mathcal{C}_I$ as in (i). Also an object $M \in R - \text{MOD}$ is $\mathcal{C}_I$-closed if and only if the natural morphism

$$\alpha : M \rightarrow R\text{Hom}_R(I, M), \quad \alpha(m)(a) = am, \quad a \in I, m \in M$$

is an isomorphism.

The following is a wide generalization of the classical result of Kato and Müller (see [20], [24]), which is given for categories of modules.

**Theorem 4.3** Let $\Gamma = (F, G, \eta, \rho)$ be a right wide Morita context between the Grothendieck categories $\mathcal{A}$ and $\mathcal{B}$, such that the functors $F$ and $G$ have right adjoints $F'$ and $G'$. Denote by $\mathcal{C}_\Gamma$ (respectively $\mathcal{D}_\Gamma$) the subcategory of $\mathcal{A}$ (respectively $\mathcal{B}$) defined by the natural morphism $\eta$ (respectively $\rho$). Then the quotient categories $\mathcal{A}/\mathcal{C}_\Gamma$ and $\mathcal{B}/\mathcal{D}_\Gamma$ are equivalent via the functors $F'$, $G'$.

**Proof:** Proposition 2.2 shows that $\mathcal{C}_\Gamma$ is a closed subcategory of $\mathcal{A}$, and $\mathcal{D}_\Gamma$ is a closed subcategory of $\mathcal{B}$. By Theorem 4.1, an object $M \in \mathcal{A}$ is $\mathcal{C}_\Gamma$-closed if and only if it is $\mathcal{C}_\Gamma$-closed (and similarly for $\mathcal{D}_\Gamma$-closed objects). By [27, pages 195 and 213], the category of $\mathcal{C}_\Gamma$-closed objects is equivalent to the quotient category $\mathcal{A}/\mathcal{C}_\Gamma$, and this ends the proof.

The following result is a direct consequence of Corollary 2.9.
Corollary 4.4 If $\Gamma = (F,G,\eta,\rho)$ is a right wide Morita context between the Grothendieck categories $\mathcal{A}$ and $\mathcal{B}$, such that the functors $F$ and $G$ have right adjoints $F'$ and $G'$, and $\eta$ is an epimorphism, then the categories $\mathcal{A} \text{ and } \mathcal{B}/D_\Gamma$ are equivalent via the functors $F'$ and the functor induced by $F$.

Remark 4.5 Let $\Gamma = (F,G,\eta,\rho)$ is a right wide Morita context between the Grothendieck categories $\mathcal{A}$ and $\mathcal{B}$. Proposition 2.2(1) shows that $D_\Gamma = \{ Y \in \mathcal{B} \mid \rho(Y) = 0 \}$. If we take $Y \in \text{Ker } F$, i.e. $F(Y) = 0$, then $(G \circ F)(Y) = 0$, so $\rho(Y) = 0$, showing that $Y \in D_\Gamma$. Thus for any right wide Morita context we have $\text{Ker } F \subseteq D_\Gamma$. Assume now that $\eta$ is an epimorphism, and let $Y \in D_\Gamma$. Then $\eta(F(Y)) = F(\rho(Y)) = 0$. Since $\eta$ is in fact a natural equivalence, we must have $F(Y) = 0$, so $Y \in \text{Ker } F$. Therefore $\text{Ker } F = D_\Gamma$. Thus Corollary 4.4 can be reformulated by saying that $F$ induces an equivalence between the categories $\mathcal{B}/\text{Ker } F$ and $\mathcal{A}$.

5 Applications

In this section we apply the general equivalence results that we proved to several particular cases.

5.1 Morita contexts for rings with identity and the Kato-Müller Theorem

Let $R$ and $S$ be two rings with identity. A Morita context connecting $R$ and $S$ is a datum $(R,S,M_{S,R} M_{S,S} N_{R},\phi,\psi)$, where $M$ is an $R$-$S$-bimodule, $N$ is a $S$-$R$-bimodule, $\phi : M \otimes_S N \to R$ is a morphism of $R$-$R$-bimodules and $\psi : N \otimes_R M \to S$ is a morphism of $S$-$S$-bimodules such that

\begin{align*}
\phi(m \otimes n)m' &= m\psi(n \otimes m') \\
\psi(n \otimes m)n' &= n\phi(m \otimes n')
\end{align*}

for any $m, m' \in M$, $n, n' \in N$.

To such a context we associate two trace ideals: $I = \text{Im } \phi$, which is an ideal of $R$, and $J = \text{Im } \psi$, which is an ideal of $S$. Consider the categories $\mathcal{A} = R-\text{mod}$ and $\mathcal{B} = S-\text{mod}$, and define the functors $F : \mathcal{B} \to \mathcal{A}$, $F(Y) = M \otimes_S Y$, and $G : \mathcal{A} \to \mathcal{B}$, $G(X) = N \otimes_R X$.

We have a natural morphism $\eta : F \circ G \to 1_{R-\text{mod}}$, defined by

$$
\eta(X) : M \otimes_S N \otimes_R X \to X, \quad \eta(X)(m \otimes n \otimes x) = \phi(m \otimes n)x
$$

for any $X \in R-\text{mod}$. 

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We also have a natural morphism $\rho : G \circ F \to 1_{S\text{-mod}}$ defined by

$$\rho(Y) : N \otimes_R M \otimes_S Y \to Y, \quad \rho(Y)(n \otimes m \otimes y) = \psi(n \otimes m)y$$

It is straightforward to check that $\Gamma = (F, G, \eta, \rho)$ is a right wide Morita context. With the notation of Section 2 we have that $I_M = \text{Im} \eta(M) = (\text{Im} \phi)M = IM$ for any $M \in R - \text{mod}$. By using the definition, the closed subcategory $C_\Gamma$ associated to $\Gamma$, is exactly the subcategory $PI = \{ M \in R - \text{mod} | IM = 0 \}$, so the smallest localizing subcategory which contains $C_\Gamma$ is $C_I$ (see Example 4.2 (i)). Since the functors $F$ and $G$ have right adjoints

$$F' : R - \text{mod} \to S - \text{mod}, \quad F'(X) = \text{Hom}_R(M, X)$$

$$G' : S - \text{mod} \to R - \text{mod}, \quad G'(Y) = \text{Hom}_S(N, Y)$$

we see then by Theorem 4.3 that the quotient categories $R - \text{mod}/C_I$ and $S - \text{mod}/C_J$ are equivalent via the functors $F'$ and $G'$. This is exactly the Kato-Müller Theorem.

In the case where one of the two maps in the Morita context is surjective, we obtain the following result (see [12, Proposition 3.8]).

**Corollary 5.1** If $(R, S, R M_S, S N_R, \phi, \psi)$ is a Morita context such that $\phi$ is surjective, then $R - \text{mod}$ is equivalent to a quotient category of $S - \text{mod}$. More precisely, the categories $R - \text{mod}$ and $S - \text{mod}/C_J$ are equivalent via the functor induced by $F$.

**Proof:** Since $\phi$ is surjective we have that $\eta$ is an epimorphism. Now we apply Corollary 4.4.

The next result shows that in a special case any right wide Morita context between two categories of modules arises from a Morita context as we explained above in this subsection.

**Proposition 5.2** Let $\Delta = (P, Q, \alpha, \beta)$ be a right wide Morita context between the categories $A = R - \text{mod}$ and $B = S - \text{mod}$, where $R$ and $S$ are rings with identity, and assume that the functors $P$ and $Q$ commute with direct sums. Then there exists a Morita context $(R, S, M, N, \phi, \psi)$ connecting $R$ and $S$ such that $\Delta$ is isomorphic to the right wide Morita context $\Gamma$ defined by this Morita context.

**Proof:** Since $P$ is right exact and commutes with direct sums, there exists an $R, S$-bimodule $M$ such that $P \simeq F = M \otimes_S -$. Similarly there is an $S, R$-bimodule $N$ such that $Q \simeq G = N \otimes_R -$. Let $u : P \to F$ and $v : Q \to G$ natural equivalences. Then $Pv \circ uG : P \circ Q \to F \circ G$ and $Gu \circ vP : Q \circ P \to G \circ F$ are natural equivalences, so there exist natural transformations $\eta : F \circ G \to 1_A$ and $\rho : G \circ F \to 1_B$ such that

$$\eta \circ Fv \circ uQ = \alpha$$

(4)
and
\[ \rho \circ G u \circ v P = \beta \]  

The natural transformation \( \eta : F \circ G \to 1_A \) must be of the form \( \eta(X)(m \otimes n \otimes x) = \phi(m \otimes n)x \) for any \( X \in A, m \in M, n \in N \) and \( x \in X \), and similarly \( \rho(Y)(n \otimes m \otimes y) = \psi(n \otimes m)y \) for any \( Y \in B, n \in N, m \in M, y \in Y \). Moreover, the conditions \( P\beta = \alpha P \) and \( Q\alpha = \beta Q \) imply that \((R, S, M, N, \phi, \psi)\) is a Morita context connecting \( R \) and \( S \). Then \( \Gamma = (F, G, \eta, \rho) \) is the right wide Morita context associated to the Morita context \((R, S, M, N, \phi, \psi)\). Moreover, equations (4) and (5) show that \( u \) and \( v \) give an isomorphism between \( \Delta \) and \( \Gamma \).

### 5.2 Morita contexts for graded rings

Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) and \( S = \bigoplus_{\sigma \in G} S_{\sigma} \) be two \( G \)-graded rings, where \( G \) is a group. A graded Morita context is a datum \((R, S, M, N, \phi, \psi)\), where \( M \) is a graded \( R \)-\( S \)-bimodule, \( N \) is a graded \( S \)-\( R \)-bimodule, \( \phi : M \otimes_S N \to R \) is a morphism of graded \( R \)-\( R \)-bimodules and \( \psi : N \otimes_R M \to S \) is a morphism of graded \( S \)-\( S \)-bimodules such that equations (2) and (3) are satisfied (see [25]). In this case, the trace ideals of the context, \( I = \text{Im} \phi \) and \( J = \text{Im} \psi \) are graded two-sided ideals. To this graded Morita context we associate the right wide Morita context \( \Gamma = (F, G, \eta, \rho) \), where

\[ F : S - gr \to R - gr, \quad F(Y) = M \otimes_S Y \]
\[ G : R - gr \to S - gr, \quad G(X) = N \otimes_R X \]

and the morphisms \( \eta \) and \( \rho \) are given by the same formulas as in Subsection 5.1. The localizing subcategories associated to \( \Gamma \) are \( \mathcal{C}_I \subseteq R - gr \) and \( \mathcal{C}_J \subseteq S - gr \) (see Example 4.2 (ii)).

Since the right adjoint functor of \( F \) is \( F' : R - gr \to S - gr, \) \( F'(X) = \text{HOM}_R(M, X) \), and the right adjoint of \( G \) is \( G' : S - gr \to R - gr, \) \( G'(Y) = \text{HOM}_S(N, Y) \), then we obtain by Theorem 4.3 that the quotient categories \( R - gr/\mathcal{C}_I \) and \( S - gr/\mathcal{C}_J \) are equivalent via the functors \( F' \) and \( G' \).

### 5.3 Morita contexts for rings with local units

Let \( R \) and \( S \) be two rings with local units, and \( R - \text{MOD} \) and \( S - \text{MOD} \) the associated categories of unital modules. A Morita context for \( R \) and \( S \) is a datum \((R, S, R M_S S N_R, \phi, \psi)\) as in Subsection 5.1, with the condition that \( M \) and \( N \) are unital modules to the left and to the right. The tensor product is defined exactly as for rings with identity (see [3] for details). As for rings with identity we obtain by Theorem 4.3 an equivalence between the quotient categories \( R - \text{MOD}/\mathcal{C}_I \) and \( S - \text{MOD}/\mathcal{C}_J \) via the functors \( \text{SHom}_R(M, -) \) and \( \text{RHom}_S(N, -) \), where \( I = \text{Im} \phi \) and \( J = \text{Im} \psi \).
Also one can obtain a version of Corollary 5.1 exactly in the same way. Note that for instance this explains from a general point of view [4, Proposition 3.7].

5.4 Morita contexts and $I$-projective modules

Let us consider a Morita context $(R, S, M, N, \phi, \psi)$ connecting the rings with identity $R$ and $S$, and the right wide Morita context $\Gamma = (F, G, \eta, \rho)$ associated as in Subsection 5.1. Let $F'$ and $G'$ be the right adjoints of $F$ and $G$ described in Subsection 5.1, and let $\eta' : 1_{R\mathrm{-mod}} \to G' \circ F'$ be defined as follows

$$\eta'(X) : X \to (G' \circ F')(X) = \text{Hom}_S(N, \text{Hom}_R(M, X)),$$

for any $X \in R\mathrm{-mod}$, $x \in X$, $m \in M$ and $n \in N$. Similarly one defines $\rho' : 1_{S\mathrm{-mod}} \to F' \circ G'$.

In this way we obtain a left wide Morita context $\Gamma^\text{op} = (F', G', \eta', \rho')$ between the categories $R\mathrm{-mod}$ and $S\mathrm{-mod}$. We call $\Gamma^\text{op}$ the opposite of $\Gamma$. If $I = \text{Im} \phi$ and $J = \text{Im} \psi$, then $C_{\Gamma} = C_{\Gamma^\text{op}} = C_I$ and $D_{\Gamma} = D_{\Gamma^\text{op}} = C_J$.

If $P \in R\mathrm{-mod}$, then $P$ is $C_{\Gamma^\text{op}}$-projective if and only if $P$ is $I$-projective, i.e. for any epimorphism $u : M \to M'$ with $I\text{Ker} u = 0$, and any morphism $f : P \to M'$, there exists $g : P \to M$ such that $u \circ g = f$.

On the other hand, $P$ is $C_{\Gamma}$-cotorsion free if and only if $\text{Hom}(P, M) = 0$ whenever $IM = 0$, and it is also equivalent to the fact that $P = IP$. If we denote by

$$C_{I,\text{proj}} = \{ M \in R\mathrm{-mod} | M \text{ is } I \text{-projective and } IM = M \}$$

$$C_{J,\text{proj}} = \{ N \in S\mathrm{-mod} | N \text{ is } J \text{-projective and } JN = N \}$$

then we see by Theorem 3.6 that the categories $C_{I,\text{proj}}$ and $C_{J,\text{proj}}$ are equivalent via the functors $F$ and $G$.

Similar results can be obtained for the graded case and the local units case by using the opposite of the right wide Morita contexts defined in Subsections 5.2 and 5.3.

Remark 5.3 If $I$ is a two-sided ideal of a ring $R$, the concept of an $I$-flat module is defined in [21] as follows. The left $R$-module $M$ is called $I$-flat if for any exact sequence

$$0 \to N' \xrightarrow{u} N \to \text{Coker } u \to 0$$

of right $R$-modules such that $(\text{Coker } u)I = 0$, we have that the sequence of abelian groups

$$0 \to N' \otimes_R M \xrightarrow{u \otimes 1_M} N \otimes_R M \to \text{Coker } u \otimes_R M \to 0$$

is exact. Then one defines a category

$$C_{I,\text{flat}} = \{ M \in R\mathrm{-mod} | M \text{ is } I \text{-flat and } IM = M \}$$

It is proved in [21] that if the $R$-module $M$ is $I$-projective, then it is $I$-flat, and also that $C_{I,\text{proj}} = C_{I,\text{flat}}$. An equivalence result concerning the category $C_{I,\text{flat}}$, which is a particular case of our Theorem 3.6, is proved in [21].

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5.5 Applications to Doi-Hopf modules

We first recall some facts about coactions of Hopf algebras on algebras. Let $H$ be a Hopf algebra over a field $k$. Let $A$ be a right $H$-comodule algebra. This means that $A$ is an algebra, a right $H$-comodule with $H$-coaction given by $a \mapsto \sum a_0 \otimes a_1$ for any $a \in A$, and the comodule structure map from $A$ to $A \otimes H$ is an algebra morphism. The subspace of coinvariants with respect to this coaction is $A^{coH} = \{a \in A | \sum a_0 \otimes a_1 = a \otimes 1\}$, and it is a subalgebra of $A$.

We say that $M$ is an $(A, H)$-Doi-Hopf module (or simply a Doi-Hopf module) if $M$ is a left $A$-module and a right $H$-comodule (with $m \mapsto \sum m_0 \otimes m_1$), such that

$$\sum (am)_0 \otimes (am)_1 = \sum a_0 m_0 \otimes a_1 m_1$$

for any $a \in A$ and $m \in M$. We denote by $\mathcal{AM}^H$ the category whose objects are the Doi-Hopf modules, and in which the morphisms are the maps which are $A$-linear and $H$-colinear.

Assume that moreover the Hopf algebra $H$ is co-Frobenius, i.e. there exists a non-zero left integral $t \in H^*$. In this case the rational part $H^{rat}$ of the dual of $H$ is a subring without identity of the algebra $H^*$, but $H^{rat}$ has local units. We can form the smash product $A\#H^{rat}$, which is $A \otimes H^{rat}$ as a vector space (and the element $a \otimes h^*$ is denoted by $a\#h^*$), and has the multiplication given by

$$(a\#h^*)(b\#g^*) = \sum ab_0\#(h^* \leftarrow b_1)g^*$$

where $\leftarrow$ is the usual right action of $H$ on $H^*$. It is known that the category $\mathcal{AM}^H$ is isomorphic to the category $A\#H^{rat} - MOD$ of left unital $A\#H^{rat}$-modules. The $A\#H^{rat}$-module structure of a Doi-Hopf module $M$ is given by $(a\#h^*) \cdot m = \sum h^*(m_1)am_0$. We identify the categories $\mathcal{AM}^H$ and $A\#H^{rat} - MOD$, i.e. we freely regard a Doi-Hopf module as a unital module over the smash product, and also the other way around.

A Morita context $(A\#H^{rat}, A^{coH}, P, Q, [-, -], (-, -))$ connecting the smash product and the subalgebra of coinvariants was constructed in [4] (see also [14, Section 6.3]). We describe briefly this context. We note that in the case where $H$ is finite dimensional, this context is precisely the one of [11].

The first bimodule is $P = A_{\#H^{rat}} A^{coH}$ with the left module structure coming from the fact that $A$ itself is a Doi-Hopf module, and the right module structure obtained by restriction of scalars. The second bimodule is $Q = A^{coH} A_{\#H^{rat}}$ where the left module structure is the restriction of scalars, and the right $A\#H^{rat}$-module structure is defined by $b \leftarrow (a\#h^*) = \sum b_0a_0h^*(S^{-1}(b_1a_1)g)$, where $S$ is the antipode of $H$ (which is known to be bijective since $H$ is co-Frobenius), and $g$ is the distinguished group-like element of $H$ (i.e. $g$ is that group-like element for which the left integrals on $H$ are exactly the right $g$-integrals on $H$, see [14, Section 5.5] for details).

The bimodule maps $[-, -]$ and $(-, -)$ are defined by

$$[-, -]: P \otimes Q = A \otimes A^{coH} A \to A\#H^{rat},$$
\[ [-,-](a \otimes b) = [a,b] = \sum ab_0 # t \leftarrow b_1, \]

and

\[ (-,-) : Q \otimes P = A \otimes_{A^\#H^{\text{rat}}} A \to A^{coH}, \]

\[ (-,-)(a \otimes b) = (a,b) = t \cdot (ab) = \sum t(a_1 b_1)a_0 b_0. \]

The associated trace ideals are \( I = \text{Im} [-,-], \) an ideal of \( A^\#H^{\text{rat}}, \) and \( J = \text{Im} (-,-) = t \cdot A, \) an ideal of \( A^{coH}. \) The map \([-,-]\) is surjective if and only if the extension \( A/A^{coH} \) is \( H-\text{Galois} \) (see [4, Section 3]), and the map \((-,-)\) is surjective if and only if there exists a total integral for \( A, \) i.e. an \( H\)-comodule map from \( H \) to \( A \) that maps 1 to 1 (see [4, Proposition 3.6]). We apply the results of Subsection 5.3 to this particular Morita context.

We first note the following.

**Lemma 5.4** Let \( M \in A^\#H^{\text{rat}} - \text{MOD}. \) Then \( I \cdot M = 0 \) if and only if \( t \cdot M = 0. \) In particular, in the case where \( A/A^{coH} \) is \( H-\text{Galois}, t \cdot M = 0 \) implies that \( M = 0. \)

**Proof:** Let \( a, b \in A \) and \( m \in M. \) We have

\[
\sum (ab_0 # t \leftarrow b_1) \cdot m = \sum (t \leftarrow b_1)(m_1)ab_0 m_0 \\
= \sum t((bm)_1)a(bm)_0 \\
= a(t \cdot (bm))
\]

showing that \( I \cdot M = 0 \) if and only if \( t \cdot M = 0. \)

Thus the closed category \( \mathcal{P}_I \) (we keep the notation of Subsection 5.3) is given by

\[
\mathcal{P}_I = \{ M \in A^\#H^{\text{rat}} - \text{MOD} | I \cdot M = 0 \} \\
= \{ M \in A^\#H^{\text{rat}} - \text{MOD} | t \cdot M = 0 \}
\]

The smallest localizing subcategory containing \( \mathcal{P}_I \) is

\[
\mathcal{C}_I = \{ M \in A^\#H^{\text{rat}} - \text{MOD} | \text{For any } M' < M, M/M' \text{ contains some } S \neq 0 \text{ with } I \cdot S = 0 \} \\
= \{ M \in A^\#H^{\text{rat}} - \text{MOD} | \text{For any } M' < M, M/M' \text{ contains some } S \neq 0 \text{ with } t \cdot S = 0 \}
\]

Also we define \( \mathcal{P}_J = \{ N \in A^{coH} - \text{mod} | JN = 0 \}, \) and the smallest localizing category \( \mathcal{C}_J \) containing \( \mathcal{P}_J. \) Now we have the following result, which is a particular case of the results described in Subsection 5.3.

**Proposition 5.5** For a co-Frobenius Hopf algebra \( H \) and a right Hopf comodule algebra \( A, \) there is an equivalence between the quotient categories \( A^{M/H}/\mathcal{C}_I \) and \( A^{coH} - \text{mod}/\mathcal{C}_J \) induced by the above Morita context.
By the remarks above we have that $C_I = 0$ if and only if $A/A^{\text{co}H}$ is $H$-Galois, and $C_J = 0$ if and only if there exists a total integral for $A$. Using these facts, we obtain the following two particular cases of Proposition 5.5.

**Corollary 5.6** If $A$ is a right $H$-comodule algebra such that $A/A^{\text{co}H}$ is $H$-Galois, then the category of Doi-Hopf modules $A\mathcal{M}^H$ is equivalent to a quotient category of $A^{\text{co}H} - \text{mod}$.

**Corollary 5.7** ([4, Corollary 3.8]) If $A$ is a right $H$-comodule algebra such that there exists a total integral, then the category $A^{\text{co}H}$ is equivalent to a quotient category of $A\mathcal{M}^H$.

We can explain the equivalence from Corollary 5.6 in a more precise way. Assume that $A/A^{\text{co}H}$ is $H$-Galois, so $C_I = 0$. Let

$$F' : A\mathcal{M}^H \to A^{\text{co}H} - \text{mod}, \quad F'(X) = \text{Hom}_{A^{\#H}}(A, X)$$

be the right adjoint of the functor

$$F = A \otimes A^{\text{co}H} - : A^{\text{co}H} - \text{mod} \to A\mathcal{M}^H$$

It is easy to see that for any Doi-Hopf module $X$ we have a natural isomorphism

$$\text{Hom}_{A^{\#H}}(A, X) = \text{Hom}_{A\mathcal{M}^H}(A, X) \simeq X^{\text{co}H}$$

Therefore $F' \simeq (-)^{\text{co}H}$, the functor that takes the coinvariants of a Doi-Hopf module.

Let $\mathcal{D}$ be the subcategory of $A^{\text{co}H} - \text{mod}$ consisting of all $C_J$-closed objects. Then by Theorem 2.8, $F'$ is an equivalence between the categories $A\mathcal{M}^H$ and $\mathcal{D}$ (the inverse of this equivalence is $G'$, the right adjoint of the functor $G = A \otimes A^{\#H} -$). If we restrict $F$ to $\mathcal{D}$, we see that it is still a left adjoint of the equivalence functor $F'$ (regarded from $A\mathcal{M}^H$ to $\mathcal{D}$). We conclude that when restricted to $\mathcal{D}$, the functor $F$ is itself an equivalence, so $F \circ F' \simeq 1_{A\mathcal{M}^H}$. This means that for any Doi-Hopf module $M$, the natural morphism

$$\phi_M : A \otimes A^{\text{co}H} M^{\text{co}H} \to M, \quad \phi_M(a \otimes m) = am$$

is an isomorphism of Doi-Hopf modules. This is exactly the Weak Structure Theorem for Galois extensions. This result is not new. It was obtained in [4, Theorem 3.1], and it also follows from [17, 2.11] in presence of the fact that for an $H$-Galois extension $A/A^{\text{co}H}$ with $H$-co-Frobenius, the left $A^{\text{co}H}$-module $A$ is flat (see [4, Corollary 3.5]). However our approach gives a more categorial idea about how the Weak Structure Theorem arises.
6 Left wide Morita contexts and an equivalence theorem

Let \( \Gamma = (F, G, \eta, \rho) \) be a left wide Morita context between the Grothendieck categories \( \mathcal{A} \) and \( \mathcal{B} \). As in Section 3 we denote by

\[
\mathcal{C}_\Gamma = \{ M \in \mathcal{A} | \eta(M) = 0 \}
\]

the closed subcategory associated to \( \Gamma \). Let

\[
\overline{\mathcal{C}_\Gamma} = \{ X \in \mathcal{A} | \text{For any } X' < X, X/X' \text{ contains a non-zero object of } \mathcal{C}_\Gamma \}
\]

be the smallest localizing subcategory that contains \( \mathcal{C}_\Gamma \). Similarly we define the subcategory \( \mathcal{D}_\Gamma \) of \( \mathcal{B} \), and the smallest localizing subcategory \( \overline{\mathcal{D}_\Gamma} \) containing \( \mathcal{D}_\Gamma \). We see by the proof of Theorem 4.1 that an object \( M \in \mathcal{A} \) is \( \mathcal{C}_\Gamma \)-torsion free if and only if \( M \) is \( \mathcal{C}_\Gamma \)-torsion free.

**Lemma 6.1** If \( M \in \mathcal{A} \) is \( \mathcal{C}_\Gamma \)-torsion free, then \( \eta(M) : M \to (F\circ G)(M) \) is a monomorphism, and \( G(M) \) is \( \mathcal{D}_\Gamma \)-torsion free.

**Proof:** By Proposition 3.2 we have \( \text{Ker } \eta(M) \in \mathcal{C}_\Gamma \). Since \( M \) is \( \mathcal{C}_\Gamma \)-torsion free, we get that \( \text{Ker } \eta(M) = 0 \), so \( \eta(M) \) is a monomorphism.

Assume that \( G(M) \) were not \( \mathcal{D}_\Gamma \)-torsion free. Then there exists \( Y \in \mathcal{D}_\Gamma \), \( Y \neq 0 \) such that \( Y \leq G(M) \). Let \( i : Y \to G(M) \) be the inclusion morphism. Since \( Y \in \mathcal{D}_\Gamma \), we have \( \rho(Y) = 0 \). Then \( \rho(G(M)) \circ i = (G \circ F)(i) \circ \rho(Y) = 0 \). Since \( \rho G = G \eta \) we obtain that \( G\eta(M) \circ i = 0 \). But \( \eta(M) \) is a monomorphism and \( G \) is left exact, so \( G\eta(M) \) is also a monomorphism. This shows that \( i = 0 \), so \( Y = 0 \), a contradiction. We conclude that \( G(M) \) must be \( \mathcal{D}_\Gamma \)-torsion free.

**Lemma 6.2** (i) If \( M \in \mathcal{C}_\Gamma \), then \( G(M) \in \mathcal{D}_\Gamma \).

(ii) If the functors \( F \) and \( G \) commute with direct limits, then for any \( M \in \overline{\mathcal{C}_\Gamma} \) we have that \( G(M) \in \overline{\mathcal{D}_\Gamma} \).

**Proof:** (i) Since \( M \in \mathcal{C}_\Gamma \) we have \( \eta(M) = 0 \). Then \( \rho(G(M)) = G(\eta(M)) = 0 \), so \( G(M) \in \mathcal{M} \in \mathcal{D}_\Gamma \).

(ii) Let \( t \) be the preradical associated to the closed subcategory \( \mathcal{C}_\Gamma \). We define by transfinite recurrence the objects \( M_\alpha \) for any ordinal \( \alpha \). We first set \( M_1 = t(M) \in \mathcal{C}_\Gamma \). If \( \alpha \) is an ordinal such that \( M_\alpha \) is defined, we define \( M_{\alpha+1} \) such that \( M_{\alpha+1}/M_\alpha = t(M/M_\alpha) \). Finally, if \( \alpha \) is a limit ordinal, we put \( M_\alpha = \cup_{\beta<\alpha} M_\beta \).

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Since $M \in \mathcal{C}_T$, there exists an ordinal $\alpha_0$ such that $M = M_{\alpha_0}$. We prove by transfinite induction that $G(M_\alpha) \in \overline{\mathcal{D}}_T$ for any ordinal $\alpha \leq \alpha_0$. Indeed, for $\alpha = 1$ it follows by the assertion (i). If $\alpha$ is a limit ordinal, we have that

$$G(M_\alpha) = G(\cup_{\beta < \alpha} M_\beta) = \lim_{\beta < \alpha} G(M_\beta) \in \overline{\mathcal{D}}_T$$

since $G$ commutes with direct limits and $\overline{\mathcal{D}}_T$ is a localizing subcategory.

If $\alpha$ is an arbitrary ordinal, we have the exact sequence

$$0 \longrightarrow M_\alpha \longrightarrow M_{\alpha+1}/M_\alpha \longrightarrow 0$$

where $M_\alpha \in \mathcal{C}_T$ and $M_{\alpha+1}/M_\alpha \in \mathcal{C}_T \subset \mathcal{C}_T$. Since $G$ is left exact, we have the exact sequence

$$0 \longrightarrow G(M_\alpha) \longrightarrow G(M_{\alpha+1}) \longrightarrow G(M_{\alpha+1}/M_\alpha)$$

Since $G(M_{\alpha+1}/M_\alpha) \in \mathcal{D}_T$, we have that $\text{Im} G(\pi_{\alpha+1}) \in \mathcal{D}_T$. This shows that $G(M_{\alpha+1}) \in \overline{\mathcal{D}}_T$, since $\overline{\mathcal{D}}_T$ is closed under extensions.

**Proposition 6.3** If $M$ is $\overline{\mathcal{C}}_T$-closed, then $\eta(M) : M \rightarrow (F \circ G)(M)$ is an isomorphism. If $N \in \mathcal{B}$ is $\overline{\mathcal{D}}_T$-closed, then $\rho(N) : N \rightarrow (G \circ F)(N)$ is an isomorphism.

**Proof:** Since $M$ is $\overline{\mathcal{C}}_T$-torsion free, it is also $\mathcal{C}_T$-torsion free, so by Lemma 6.1 we have that $\eta(M)$ is a monomorphism and $(F \circ G)(M)$ is $\mathcal{C}_T$-torsion free. On the other hand Coker $\eta(M) \in \mathcal{C}_T$ by Proposition 3.3, so $\eta(M)$ is an essential monomorphism. Since $M$ is $\overline{\mathcal{C}}_T$-injective, there exists a morphism $g : (F \circ G)(M) \rightarrow M$ such that $g \circ \eta(M) = 1_M$. This shows that $(F \circ G)(M) \simeq \text{Im} \eta(M) \oplus \text{Coker} \eta(M)$. But $(F \circ G)(M)$ is $\overline{\mathcal{C}}_T$-torsion free, so Coker $\eta(M)$ must be 0, and then $\eta(M)$ is an isomorphism. Similarly we see that if $N \in \mathcal{B}$ is $\overline{\mathcal{D}}_T$-closed, then $\rho(N) : N \rightarrow (G \circ F)(N)$ is an isomorphism.

**Proposition 6.4** If $M \in \mathcal{A}$ is $\overline{\mathcal{C}}_T$-closed, then $G(M)$ is $\overline{\mathcal{D}}_T$-closed. Similarly, if $N \in \mathcal{B}$ is $\overline{\mathcal{D}}_T$-closed, then $F(M)$ is $\overline{\mathcal{C}}_T$-closed.

**Proof:** Assume that $M \in \mathcal{A}$ is $\overline{\mathcal{C}}_T$-closed. By Lemma 6.1 we have that $G(M)$ is $\mathcal{D}_T$-torsion free, so it is $\overline{\mathcal{D}}_T$-torsion free, too. Let $N \in \mathcal{B}$ be the closure of $G(M)$ with respect to the localizing subcategory $\overline{\mathcal{D}}_T$. This means that if $T' : \mathcal{B} \rightarrow \mathcal{B}/\overline{\mathcal{D}}_T$ is the natural functor associated to the quotient category $\mathcal{B}/\overline{\mathcal{D}}_T$, and $S'$ is a right adjoint of $T'$, then $N = (S' \circ T')(G(M))$. We have the exact sequence

$$0 \longrightarrow G(M) \xrightarrow{i} N \longrightarrow \text{Coker} i \longrightarrow 0$$
with \( \text{Coker } i \in \overline{D}_\Gamma \). Since \( F \) is left exact, we obtain the exact sequence

\[
0 \rightarrow (F \circ G)(M) \xrightarrow{F(i)} F(N) \xrightarrow{\theta} F(\text{Coker } i)
\]

with \( F(\text{Coker } i) \in \overline{C}_\Gamma \), by Lemma 6.2(ii). We have that \( \text{Im } \theta \in \overline{D}_\Gamma \). By Proposition 6.3 we have \( (F \circ G)(M) \simeq M \), so \( (F \circ G)(M) \) is \( \overline{D}_\Gamma \)-closed. Now the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{i} & (F \circ G)(M) \xrightarrow{F(i)} F(N) \xrightarrow{j} \text{Im } \theta \xrightarrow{} 0 \\
\downarrow & & \downarrow 1_{(F \circ G)(M)} \\
(F \circ G)(M) & & \\
\end{array}
\]

shows that there exists a morphism \( h : F(N) \rightarrow (F \circ G)(M) \) such that \( h \circ F(i) = 1_{(F \circ G)(M)} \). Hence \( F(N) \simeq \text{Im } F(i) \oplus \text{Im } \theta \). Since \( F(N) \) is \( \overline{C}_\Gamma \)-closed, then it is \( \overline{C}_\Gamma \)-torsion free, so \( \text{Im } \theta = 0 \). This shows that \( F(i) \) is an isomorphism. We have the morphisms

\[
\begin{array}{ccc}
M & \xrightarrow{G(\eta(M))} & (G \circ F \circ G)(M) \xrightarrow{(G \circ F)(i)} (G \circ F)(N) \\
\downarrow & & \downarrow \rho(N) \\
N & & \\
\end{array}
\]

where \( \eta(M) \) is an isomorphism and \( \rho(N) \) is also an isomorphism by Proposition 6.3. We conclude that \( G(M) \simeq N \), so \( G(M) \) is \( \overline{D}_\Gamma \)-closed.

**Theorem 6.5** Let \( \Gamma = (F, G, \eta, \rho) \) be a left wide Morita context between the Grothendieck categories \( \mathcal{A} \) and \( \mathcal{B} \) such that the functors \( F \) and \( G \) commute with direct limits. Then the quotient categories \( \mathcal{A}/\overline{C}_\Gamma \) and \( \mathcal{B}/\overline{D}_\Gamma \) are equivalent via the restriction of the functors \( F \) and \( G \).

**Proof:** Since \( \mathcal{A}/\overline{C}_\Gamma \) is the subcategory of all \( \overline{C}_\Gamma \)-closed objects of \( \mathcal{A} \), and \( \mathcal{B}/\overline{D}_\Gamma \) is the subcategory of all \( \overline{D}_\Gamma \)-closed objects of \( \mathcal{B} \), the result follows from Propositions 6.3 and 6.4.
7 Applications to Morita-Takeuchi contexts

In this section we apply the results of Section 6 to left wide Morita contexts arising from Morita-Takeuchi contexts.

7.1 Morita-Takeuchi contexts and a theorem of Berbec

Let $C$ and $D$ be two coalgebras over a field. A Morita-Takeuchi context connecting $C$ and $D$ is a datum $(C, D, M, N, \phi, \psi)$, where $M$ is a $C, D$-bicomodule, $N$ is a $D, C$-bicomodule, $\phi : C \to M \boxtimes_D N$ is a morphism of $C, C$-bicomodules, and $\psi : D \to N \boxtimes_C M$ is a morphism of $D, D$-bicomodules such that $(1_M \boxtimes \psi) \circ \gamma_{M,D} = (\phi \boxtimes 1_M) \circ \gamma_{C,M}$ and $(1_N \boxtimes \phi) \circ \gamma_{N,C} = (\psi \boxtimes 1_N) \circ \gamma_{D,N}$, where $\boxtimes$ is the cotensor product, $\gamma_{M,D} : M \to M \boxtimes_D D$ and $\gamma_{D,N} : N \to N \boxtimes_C C$ are the natural isomorphisms.

To such a Morita-Takeuchi context we associate a left wide Morita context $\Gamma = (F, G, \eta, \rho)$ between the categories of right comodules $M^D$ and $M^C$ (see [10]), where the functors $F$ and $G$ are defined by

\[ F : M^C \to M^D, \quad F(X) = X \boxtimes_C M \]
\[ G : M^D \to M^C, \quad G(Y) = Y \boxtimes_D N \]

and the natural morphisms $\eta : 1_{M^D} \to F \circ G$ and $\rho : 1_{M^C} \to G \circ F$ are defined by

\[ \eta(Y) = (1_Y \boxtimes_D \psi) \circ \gamma_{Y,D} \quad \text{for any } Y \in M^D \]
\[ \rho(X) = (1_X \boxtimes_C \phi) \circ \gamma_{X,C} \quad \text{for any } X \in M^C \]

The functors $F$ and $G$ are left exact and commute with direct limits. Denote $A = \ker \phi$, which is a subcoalgebra of $C$, and $B = \ker \psi$, which is a subcoalgebra of $D$. Let $\mathcal{C}_\Gamma$ be the closed subcategory of $M^C$ defined by $\Gamma$ as in Section 2. We have that

\[ \mathcal{C}_\Gamma = \{ M \in M^C | M \boxtimes_C A \simeq M \} \]
\[ = \{ M \in M^C | \rho_M(M) \subseteq M \otimes A \} \]
\[ = \{ M \in M^C | A^\perp M = 0 \} \]

where $A^\perp$ is the subspace of $C^*$ consisting of all maps that annihilates $A$ (see [14, Sections 1.2 and 2.5]). The smallest localizing subcategory containing $\mathcal{C}_\Gamma$ is

\[ \overline{\mathcal{C}_\Gamma} = \{ M \in M^C | A^\perp_{\infty} M = 0 \} \]

where $A_{\infty} = \cup_{n \geq 1} \land^n A$ (here $\land$ is the usual wedge, see [14], [23]).

Now we can derive in a natural way as a particular case of Theorem 4.3 the following result of Berbec (see [5]).
Corollary 7.1 Let \((C, D, M, N, f, g)\) be a Morita-Takeuchi context connecting the coalgebras \(C\) and \(D\), and let \(\Gamma\) be the associated left wide Morita context as above. Then the quotient categories \(\mathcal{M}_C/C\Gamma\) and \(\mathcal{M}_D/D\Gamma\) are equivalent.

Proof: The cotensor product functors are left exact and commute with direct limits, so the result follows directly from Theorem 6.5.

In the particular case where one of the maps of the Morita-Takeuchi context is injective, we obtain the following result (see [13, Proposition 2.2]).

Corollary 7.2 Let \((C, D, M, N, f, g)\) be a Morita-Takeuchi context such that \(f\) is injective. Then the category \(\mathcal{M}_C\) is equivalent to a quotient category of \(\mathcal{M}_D\).

7.2 Applications to Hopf-Galois coextensions

Let \(H\) be a finite dimensional Hopf algebra over a field \(k\), and let \(C\) be a left \(H\)-comodule coalgebra. This means that \(C\) is a coalgebra (with comultiplication \(c \mapsto \sum c_1 \otimes c_2\)) and a left \(H\)-comodule (with coaction \(c \mapsto \sum c_{(-1)} \otimes c_{(0)}\)) such that

\[
\sum c_{(-1)} \otimes c_{(0)1} \otimes c_{(0)2} = \sum c_{(1)} c_{2(-1)} \otimes c_{1(0)} \otimes c_{2(0)}
\]

and

\[
\sum \varepsilon(c_{(0)}) c_{(-1)} = \varepsilon(c) 1_H
\]

for any \(c \in C\). We can form the smash coproduct \(C \bowtie H\), which is \(C \otimes H\) as a \(k\)-vector space, with the element \(c \otimes h\) denoted by \(c \bowtie h\), and has a coalgebra structure with counit \(\varepsilon_C \bowtie \varepsilon_H\) and comultiplication given by

\[
\Delta(c \bowtie h) = \sum (c_1 \bowtie c_{2(-1)} h_1) \otimes (c_{2(0)} \bowtie h_2)
\]

We also consider the factor coalgebra \(\overline{C} = C/CH^{*+}\), where \(C\) is regarded as a right \(H^{*}\)-module, and \(H^{*+} = \text{Ker } \varepsilon_{H^{*}}\). Let \(t \in H^{*}\) be a left integral on \(H\), and \(a \in H\) the distinguished grouplike element. Then we have a Morita-Takeuchi context \((C \bowtie H, \overline{C}, C, f, g)\) as follows (see [15, Theorem 1.1] for details). The left and right \(\overline{C}\)-comodule structures on \(C\) come via the natural projection \(C \rightarrow \overline{C}\). The left coaction of \(C \bowtie H\) on \(C\) is given by

\[
c \mapsto \sum c_{1(0)} \otimes (c_{2(0)} \bowtie S^{-1}(c_{1(-1)}c_{2(-1)}) a)
\]

The maps \(f\) and \(g\) are defined by

\[
f : C \bowtie H \rightarrow C \square \overline{C}, \quad f(c \bowtie h) = \sum c_1 \square t(c_{2(-1)} h) c_{2(0)}
\]
\[ g : \overline{C} \rightarrow C \square_{C \triangleright \triangleleft H} C, \quad g(\tau) = \sum t(c_1(-1)c_2(-1)c_1(0)\square c_2(0) \]

where \( \tau \) denotes the class of \( c \in C \) modulo the coideal \( CH^+ \).

We can apply Corollary 7.1 to this Morita-Takeuchi context, and we find that certain quotient categories of \( \mathcal{M}^{C\triangleright \triangleleft H} \) and \( \mathcal{M}^\overline{C} \) are equivalent. This may be reformulated if we take into account that the category \( \mathcal{M}^{C\triangleright \triangleleft H} \) is isomorphic to the category of right \( C, H \)-comodules, consisting of all objects that are right \( C \)-comodules and right \( H \)-comodules, and the two comodule structures satisfy a compatibility condition (see [7]).

If moreover \( C/\overline{C} \) is an \( H^* \)-Galois coextension, which is equivalent to the map \( f \) in the Morita-Takeuchi context being injective (see [15, Theorem 1.2]), then we obtain by Corollary 7.2 that the category \( \mathcal{M}^{C\triangleright \triangleleft H} \) is equivalent to a quotient category of \( \mathcal{M}^\overline{C} \).

On the other hand, in the case where \( H \) is cosemisimple (or equivalently \( H^* \) is semisimple), we have by [7, Proposition 3.7 and the comments before it] that \( \overline{C} \simeq C^{coH} \), the associated coalgebra of coinvariants. In this case it is easy to see that the map \( g \) is injective. Indeed, if \( g(\tau) = 0 \), then by applying \( I \otimes \varepsilon_C \), we get that \( c \cdot t = 0 \). Since \( H \) is cosemisimple we can choose \( t \) such that \( t(1) = 1 \). Then \( (\varepsilon - t)(1) = 0 \), so \( \varepsilon - t \in H^+ \). Hence \( c = c \cdot \varepsilon = c \cdot (\varepsilon - t) \in CH^+ \), so \( \tau = 0 \). Thus for cosemisimple \( H \) we obtain by Corollary 7.2 that the category \( \mathcal{M}^\overline{C} \) is equivalent to a quotient category of \( \mathcal{M}^{C\triangleright \triangleleft H} \).

References

[1] T. Albu and C. Nastasescu, Relative finiteness in module theory, Marcel Dekker, 1984.
[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, GTM 13, 2-nd edition, Springer-Verlag, 1992.
[3] P.N. Anh and L. Márki, Morita equivalences for rings without identity, Tsukuba J. Math. 11 (1987), 1-16.
[4] M. Beattie, S. Dăscălescu, and Ş. Raianu, Galois Extensions for Co-Frobenius Hopf Algebras, J. Algebra 198 (1997), 164-183.
[5] I. Berbec, The Morita-Takeuchi theory for quotient categories, Comm. Algebra 31 (2003), No.2, 843-858.
[6] R. Blattner, S. Montgomery, A duality theorem for Hopf module algebras, J. Algebra 95 (1985), 153-172.
[7] S. Caenepeel, S. Dăscălescu and Ş. Raianu, Cosemisimple Hopf algebras coacting on coalgebras, Comm. Algebra 24 (1996), 1649-1677.
[8] C. Cai and H. Chen, Coactions, smash products, and Hopf modules, J. Algebra 167 (1994), 85-99.

[9] F. Castaño Iglesias and J. Gomez Torrecillas, Wide Morita contexts, Comm. Algebra 23 (1995), 601-622.

[10] F. Castaño Iglesias and J. Gomez Torrecillas, Wide Morita contexts and equivalences of comodule categories, J. Pure Appl. Algebra 131 (1998), 213-225.

[11] M. Cohen, D. Fischman and S. Montgomery, Hopf Galois extensions, smash products and Morita equivalence, J. Algebra 133 (1990), 351-372.

[12] M. Cohen, Ş. Raianu and S. Westreich, Semiinvariants for Hopf algebra actions, Israel J. Math. 88 (1994), 279-306.

[13] S. Dăscălescu, C. Năstăescu, Ş. Raianu and F. Van Oystaeyen, Graded coalgebras and Morita-Takeuchi contexts, Tsukuba J. Math. 19 (1995), 395-407.

[14] S. Dăscălescu, C. Năstăescu and Ş. Raianu, Hopf algebras: an introduction, Pure and Applied Math. 235 (2000), Marcel Dekker.

[15] S. Dăscălescu, Ş. Raianu, Y. H. Zhang, Finite Hopf-Galois Coextensions, Crossed Co-products and Duality, J. Algebra 178 (1995), 400-413.

[16] F. De Meyer and E. Ingraham, Separable algebras over commutative rings, Springer Verlag, 1971.

[17] Y. Doi, M. Takeuchi, Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action and Azumaya algebras, J. Algebra 121 (1989), 488-516.

[18] C. Faith, Algebra: Rings, Modules and Categories, I, Springer, Berlin, 1973.

[19] P. Gabriel, Des categories abeliennes, Bull. Soc. Math. France 90 (1962), 323-448.

[20] T. Kato, U-distinguished modules, J. Algebra 25 (1973), 15-24.

[21] T. Kato and K. Ohtake, Morita context and equivalence, J. Algebra 61 (1979), 360-366.

[22] S. Mac Lane, Categories for the working mathematician, Springer Verlag, 1994.

[23] S. Montgomery, Hopf algebras and their actions on rings, CBMS Reg. Conf. Series 82, Providence, R.I., 1993.

[24] B. J. Müller, The quotient category of a Morita context, J. Algebra 28 (1974), 389-407.

[25] C. Năstăescu and F. Van Oystaeyen, Graded ring theory, North Holland, 1982.
[26] W. K. Nicholson and J. F. Watters, Morita context functors, Math. Proc. Cambridge Phil. Soc. 103 (1988), 399-408.

[27] B. Stenström, Rings of quotients, Springer Verlag, 1975.

[28] M. Takeuchi, Morita Theorems for Categories of Comodules, J. Fac. of Sci. Univ. Tokyo, 24 (1977), 629-644.

[29] M. Van den Bergh, A duality theorem for Hopf algebras, in Methods in Ring Theory, NATO ASI Series vol. 129, Reidel, Dordrecht, 1984, 517-522.

[30] R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, 1991.