CRITICAL POINTS BETWEEN VARIETIES GENERATED BY SUBSPACE LATTICES OF VECTOR SPACES

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Abstract. We denote by $\text{Con}_c A$ the semilattice of all compact congruences of an algebra $A$. Given a variety $\mathcal{V}$ of algebras, we denote by $\text{Con}_c \mathcal{V}$ the class of all semilattices isomorphic to $\text{Con}_c A$ for some $A \in \mathcal{V}$. Given varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the critical point of $\mathcal{V}$ under $\mathcal{W}$ is defined as $\text{crit}(\mathcal{V}; \mathcal{W}) = \min \{ \text{card } D | D \in \text{Con}_c \mathcal{V} \setminus \text{Con}_c \mathcal{W} \}$.

In a second part, using tools introduced in [5], we prove that:

$$\text{crit}(M_n; \text{Var}(\text{Sub } F^3)) = \aleph_2,$$

for any finite field $F$ and any ordinal $n$ such that $2 + \text{card } F \leq n \leq \omega$. Similarly $\text{crit}(\text{Var}(\text{Sub } F^3); \text{Var}(\text{Sub } K^3)) = \aleph_2$, for all finite fields $F$ and $K$ such that $\text{card } F > \text{card } K$.

1. Introduction

We denote by $\text{Con} A$ (resp., $\text{Con}_c A$) the lattice (resp., $(\lor, 0)$-semilattice) of all congruences (resp., compact congruences) of an algebra $A$. For a homomorphism $f: A \to B$ of algebras, we denote by $\text{Con} f$ the map from $\text{Con} A$ to $\text{Con} B$ defined by the rule

$$(\text{Con} f)(\alpha) = \text{congruence of } B \text{ generated by } \{(f(x), f(y)) | (x, y) \in \alpha\},$$

for every $\alpha \in \text{Con} A$, and we also denote by $\text{Con}_c f$ the restriction of $\text{Con} f$ from $\text{Con} A$ to $\text{Con}_c B$.

A congruence-lifting of a $(\lor, 0)$-semilattice $S$ is an algebra $A$ such that $\text{Con}_c A \cong S$. Given a variety $\mathcal{V}$ of algebras, the compact congruence class of $\mathcal{V}$, denoted by $\text{Con}_c \mathcal{V}$, is the class of all $(\lor, 0)$-semilattices isomorphic to $\text{Con}_c A$ for some $A \in \mathcal{V}$. As illustrated by [12], even the compact congruence classes of small varieties of lattices are complicated objects. For example, in case $\mathcal{V}$ is the variety of all lattices, $\text{Con}_c \mathcal{V}$ contains all distributive $(\lor, 0)$-semilattices of cardinality at most $\aleph_1$, but not all distributive $(\lor, 0)$-semilattices (cf. [15]).

Given varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the critical point of $\mathcal{V}$ and $\mathcal{W}$, denoted by $\text{crit}(\mathcal{V}; \mathcal{W})$, is the smallest cardinality of a $(\lor, 0)$-semilattice in $\text{Con}_c(\mathcal{V}) \setminus \text{Con}_c(\mathcal{W})$ if it exists, or $\infty$, otherwise (i.e., if $\text{Con}_c \mathcal{V} \subseteq \text{Con}_c \mathcal{W}$).
Let $I$ be a poset. A direct system indexed by $I$ is a family $(A_i, f_{i,j})_{i < j \in I}$ such that $A_i$ is an algebra, $f_{i,j} : A_i \to A_j$ is a morphism of algebras, $f_{i,i} = \text{id}_{A_i}$, and $f_{i,k} = f_{j,k} \circ f_{i,j}$, for all $i \leq j \leq k$ in $I$.

Denote by $\text{Sub} V$ the subspace lattice of a vector space $V$, and by $\mathcal{M}_n$ the variety of lattices generated by the lattice $M_n$ of length two with $n$ atoms, for $3 \leq n \leq \omega$. Using the theory of the dimension monoid of a lattice, introduced by F. Wehrung in [13], together with some von Neumann regular ring theory, we prove in Section 3 that if $V$ is a finitely generated variety of modular lattices with all subdirectly irreducible members of length less or equal to $n$, then crit($V; \mathcal{Var}(\text{Sub} F^n)$) $\geq \aleph_2$ for any field $F$. As an immediate application, crit($M_n; \mathcal{M}_n$) $\geq \aleph_2$ for every $n$ with $3 \leq n \leq \omega$ (cf. Corollary 3.12). Thus, by using the result of M. Ploščica in [10], we obtain the equality crit($\mathcal{M}_n; \mathcal{M}_n$) = $\aleph_2$ for all $n$, $m$ with $3 < n < m \leq \omega$. Our proof does not rely on the approach used by Ploščica in [11] to prove the inequality crit($\mathcal{M}_m; \mathcal{M}_n$) $\leq \aleph_2$, and it extends that result to the unbounded case. We also obtain a new proof of that result in Section 4, that does not even rely on the approach used by Ploščica in [10] to prove the inequality crit($\mathcal{M}_m; \mathcal{M}_n$) $\leq \aleph_2$.

Let $V$ be a variety of lattices, let $\mathcal{D}$ be a diagram of $(\lor, 0)$-semilattices and $(\lor, 0)$-homomorphisms. A congruence-lifting of $\mathcal{D}$ in $V$ is a diagram $\mathcal{L}$ of $V$ such that the composite $\text{Con}_{\mathcal{L}} \circ L$ is naturally equivalent to $\mathcal{D}$.

In Section 4 we give a diagram of finite $(\lor, 0)$-semilattices that is congruence-liftable in $\mathcal{M}_n$, but not congruence-liftable in $\mathcal{Var}(\text{Sub} F^3)$, for any finite field $F$ and any $n$ such that $2+\text{card } F \leq n \leq \omega$. As the diagram of $(\lor, 0)$-semilattices is indexed by some “good” lattice, we obtain, using results of [5], that crit($\mathcal{M}_n; \mathcal{Var}(\text{Sub} F^3)$) = $\aleph_2$. This implies immediately that crit($\mathcal{M}_4; \mathcal{M}_3$) = $\aleph_2$. Let $F$ and $K$ be finite fields such that card $F > \text{card } K$, we also obtain crit($\mathcal{Var}(\text{Sub} F^3); \mathcal{Var}(\text{Sub} K^3)$) = $\aleph_2$.

In a similar way, we prove that crit($\mathcal{M}_\omega; V$) = $\aleph_2$, for every finitely generated variety of lattices $V$ such that $M_3 \in V$.

2. Basic concepts

We denote by dom $f$ the domain of any function $f$. A poset is a partially ordered set. Given a poset $P$, we put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\}, \quad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\},$$

for any $X, Q \subseteq P$, and we will write $\downarrow X$ (resp., $\uparrow X$) instead of $P \downarrow X$ (resp., $P \uparrow X$) in case $P$ is understood. We shall also write $\downarrow p$ instead of $\downarrow \{p\}$, and so on, for $p \in P$. A poset $P$ is lower finite if $P \downarrow p$ is finite for all $p \in P$. For $p, q \in P$ let $p \prec q$ hold, if $p < q$ and there is no $r \in P$ with $p < r < q$, in this case $p$ is called a lower cover of $q$. We denote by $P^\prec$ the set of all non-maximal elements in a poset $P$. We denote by $M(L)$ the set of all completely meet-irreducible elements of a lattice $L$.

A 2-ladder is a lower finite lattice in which every element has at most two lower covers. S. Z. Ditor constructs in [1] a 2-ladder of cardinality $\aleph_1$.

For a set $X$ and a cardinal $\kappa$, we denote by:

$$[X]^\kappa = \{Y \subseteq X \mid \text{card } Y = \kappa\},$$

$$[X]^\leq\kappa = \{Y \subseteq X \mid \text{card } Y \leq \kappa\},$$

$$[X]^{<\kappa} = \{Y \subseteq X \mid \text{card } Y < \kappa\}.$$
exists an integer \( n \) with \(-nu \leq x \leq nu\), and morphisms \( f: (G, u) \to (H, v)\) where \( f: G \to H \) is an order-preserving group homomorphism and \( f(u) = v\).

We denote by \( \text{Dim} \) the functor that maps a lattice to its dimension monoid, introduced by F. Wohrn in [13]. We also denote by \( \Delta(a, b) \) for \( a \leq b \) in \( L \) the canonical generators of \( \text{Dim} \). We denote by \( K_0^e \) the functor that maps a lattice to the pre-ordered abelian universal group (also called Grothendieck group) of its dimension monoid. If \( L \) is a bounded lattice then (the canonical image in \( K_0^e(L) \) of) \( \Delta(0_L, 1_L) \) is an order-unit of \( K_0^e(L) \). If \( f: L \to L' \) is a 0,1-preserving homomorphism of bounded lattices, then \( K_0^e(f): (K_0^e(L), \Delta(0_L, 1_L)) \to (K_0^e(L'), \Delta(0_{L'}, 1_{L'})) \) preserves the order-unit.

All our rings are associative but not necessarily unital.

- We denote by \( \mathbb{L}(R) \) the poset of principal right ideals of every regular ring \( R \). The results of Fryer and Halperin in [3 Section 3.2], imply that, \( \mathbb{L}(R) \) is a 0-lattice, and for any homomorphism \( f: R \to S \) of regular rings, the map \( \mathbb{L}(f): \mathbb{L}(R) \to \mathbb{L}(S) \), \( I \mapsto f(I)S \) is a 0-lattice homomorphism (cf. Micol’s thesis [9] Theorem 1.4 for the unital case). Hence \( \mathbb{L} \) is a functor from the category of regular rings to the category of 0-lattices with 0-lattice homomorphisms.

- We denote by \( V \) the functor from the category of unital rings with morphisms preserving units to the category of commutative monoids, that maps a unital ring \( R \) to the commutative monoid of all isomorphism classes of finitely generated projective right \( R \)-modules and any homomorphism \( f: R \to S \) of unital rings to the monoid homomorphism \( V(f): V(R) \to V(S) \), \( \sum_i e_i R \mapsto \sum_i f(e_i) S \).

We denote by \( \text{Id}_R \) (resp., \( \text{Id}_c R \)) the lattice of all two-sided ideals (resp., finitely generated two-sided ideals) of any ring \( R \). We denote by \( \text{Sub} E \) the subspace lattice of a vector space \( E \). We denote by \( M_n(F) \) the \( F \)-algebra of \( n \times n \) matrices with entries from \( F \), for every field \( F \) and every positive integer \( n \). A matricial \( F \)-algebra is an \( F \)-algebra of the form \( M_{k_1}(F) \times \cdots \times M_{k_n}(F) \), for positive integers \( k_1, \ldots, k_n \).

For a finitely generated projective right module \( P \) over a unital ring \( R \), we denote by \([P]\) the corresponding element in \( K_0(R) \), that is, the stable isomorphism class of \( P \). We refer to [7 Section 15] for the required notions about the \( K_0 \) functor.

A \( K_0 \)-lifting of a pre-ordered abelian group with order-unit \( (G, u) \) is a regular ring \( R \) such that \((K_0(R), [R]) \cong (G, u)\). A \( K_0 \)-lifting of a diagram \( \hat{G}: I \to \mathcal{P} \) is a diagram \( \hat{R}: I \to \mathcal{P} \) such that \((K_0(-), [-]) \circ \hat{R} \cong \hat{G} \).

We denote by \( \nabla \) the functor that sends a monoid to it maximal semilattice quotient, that is, \( \nabla(M) = M/\simeq \) where \( \simeq \) is the smallest congruence of \( M \) such that \( M/\simeq \) is a semilattice. We denote by \( \nabla \) the functor that maps a partially pre-ordered abelian group \( G \) to \( \nabla(G^+) \) where \( G^+ \) is the monoid of all positive elements of \( G \).

We denote by \( \text{Var}(L) \) (resp., \( \text{Var}_0(L) \), resp., \( \text{Var}_{0,1}(L) \)) the variety of lattices (resp., lattices with 0, resp., bounded lattices) generated by a lattice \( L \).

A lattice \( K \) is a congruence-preserving extension of a lattice \( L \), if \( L \) is a sublattice of \( K \) and \( \text{Con}_c i: \text{Con} L \to \text{Con} K \) is an isomorphism, where \( i: L \to K \) is the inclusion map.

We denote by \( M_n \) and \( M_{n,m} \) the lattices represented in Figure 11 for \( 3 \leq m, n \leq \omega \), and by \( M_n \) and \( M_{n,m} \), respectively, the lattice varieties that they generate. We also denote by \( M_{n,0}^0 \) the variety of lattices with 0 generated by \( M_n \), and so on.
Lemma 3.2. Let $\mathcal{K}$ be a modular lattice of finite length, set $L = M(\text{Con} \mathcal{K})$. Denote by $\pi$ a bijection from $\mathcal{K}$ onto $L$ such that $\pi(\theta) = (\text{lh}(a/\theta) \mid a \in \mathcal{K})$ for all $\theta \in L$. Then there exists an isomorphism $\pi: \mathcal{K} \to (\mathbb{Z}^+)^P$ such that $\pi(\Delta(a,b)) = ([a,b]_\xi \mid \xi \in P)$ for all $a \leq b$ in $L$.

This makes it possible to prove the following lemma, which gives an explicit description of $K_0^L(L)$ for every modular lattice $L$ of finite length (in such a case the set $P$ is finite).

Lemma 3.2. Let $L$ be a modular lattice of finite length, set $X = M(\text{Con} L)$. Then there exists an isomorphism $\pi': K_0^L(L) \to \mathbb{Z}^X$ such that

$$\pi'(\Delta(a,b)) = (\text{lh}(a/\theta \mid b/\theta) \mid \theta \in X), \quad \text{for all } a \leq b \text{ in } L.$$ 

In particular $(K_0^L(L), \Delta(0,1))$ is isomorphic to $(\mathbb{Z}^X, (\text{lh}(L/\theta))_{\theta \in X})$.

Proof. Denote by $P$ be the set of all projectivity classes of prime intervals of $L$. For any $\xi \in P$ denote by $\theta_\xi$ the largest congruence of $\mathcal{K}$ that does not collapse any prime intervals in $\xi$. As $\mathcal{K}$ is modular of finite length, the congruences of $\mathcal{K}$ are in one-to-one correspondence with subsets of $P$ (cf. [3, Chapter III]), and so the assignment $\xi \mapsto \theta_\xi$ defines a bijection from $P$ onto $X$. Moreover any prime interval not in $\xi$ is collapsed by $\theta_\xi$, for any $\xi \in P$. Let $a \leq b$ in $L$, let $\xi \in P$. Let $a_0 < a_1 < \cdots < a_n$ in $L$ such that $a_0 = a$ and $a_n = b$. Let $0 \leq r_1 < r_2 < \cdots < r_s < n$ be all the integers such that $[a_{r_k}, a_{r_k+1}] \in \xi$ for all $1 \leq k \leq s$. Thus $|a,b|_\xi = s$. Set $r_{s+1} = n$.  

\begin{figure}
\centering
\includegraphics{figure1}
\caption{The lattices $M_n$ and $M_{n,m}$.}
\end{figure}
As \([a_{rk}, a_{rk+1}] \in \xi\) and \([a_{rk+t}, a_{rk+t+1}] \notin \xi\) for all \(1 \leq t \leq r_{k+1} - r_k - 1\), we obtain that
\[
\frac{a_{rk}}{\theta_\xi} < \frac{a_{rk+1}}{\theta_\xi} = \frac{a_{rk+2}}{\theta_\xi} = \cdots = \frac{a_{rk+s}}{\theta_\xi}, \quad \text{for all } 1 \leq k \leq s.
\]
Thus the following covering relations hold:
\[
a/\theta_\xi = a_{r_1}/\theta_\xi < a_{r_2}/\theta_\xi < \cdots < a_{r_s}/\theta_\xi < a_{r_{s+1}}/\theta_\xi = b/\theta_\xi.
\]
So \(\mathrm{lh}(a/\theta_\xi, b/\theta_\xi) = s = |a, b|_\xi\). We conclude the proof by using Proposition \ref{3.1}.

**Proposition 3.3.** The following natural equivalences hold

\[
\begin{align*}
i & \quad \nabla \circ \dim \cong \con_c & \text{on lattices} \\
\text{(ii)} & \quad \nabla \circ V \cong \con_c \circ L & \text{on regular rings}
\end{align*}
\]

**Proof.** (i) follows from \cite[Corollary 2.3]{13}, while (ii) is contained in \cite[Corollary 2.23]{7}; see also the proof of \cite[Proposition 4.6]{14}.

We shall always apply this result to unital regular rings \(R\) such that \(V(R)\) is cancellative (i.e., \(R\) is unit-regular), so \(K_0(R)^+ = V(R)\), and to lattices \(L\) such that \(\dim L\) is cancellative, so \(K_0(L)^+ \cong \dim L\). Here \(G^+\) denotes the positive cone of \(G\), for any partially pre-ordered abelian group \(G\).

The following theorem is proved in \cite[Theorem 15.23]{7}.

**Theorem 3.4.** Let \(F\) be a field, let \(R\) be a matricial \(F\)-algebra, and let \(S\) be a unit-regular \(F\)-algebra.

1. **Given any morphism** \(f: (K_0(R), [R]) \to (K_0(S), [S])\) in \(\mathcal{P}\), the category of pre-ordered abelian groups with order-unit (cf. Section \ref{2}), there exists an \(F\)-algebra homomorphism \(\phi: R \to S\) such that \(K_0(\phi) = f\).
2. **If** \(\phi, \psi: R \to S\) are \(F\)-algebra homomorphisms, then \(K_0(\phi) = K_0(\psi)\) if and only if there exists an inner automorphism \(\theta\) of \(S\) such that \(\phi = \theta \circ \psi\).

The following lemma is folklore.

**Lemma 3.5.** Let \(F\) be a field, let \(u = (u_k)_{1 \leq k \leq n}\) be a family of positive integers, let \(R = \prod_{k=1}^n M_{u_k}(F)\). Then \((K_0(R), [R]) \cong (\mathbb{Z}^n, u)\).

**Lemma 3.6.** Let \(F\) be a field. Let \(I\) be a 2-ladder, let \(G_i = (\mathbb{Z}^{n_i}, u^i = (u^i_k)_{1 \leq k \leq n_i})\) such that \(u^i\) is an order-unit, let \(R_i = \prod_{k=1}^{n_i} M_{u^i_k}(F)\) for all \(i \in I\). Let \(f_{i,j}: G_i \to G_j\) for all \(i \leq j\) in \(I\) such that \(\tilde{G} = (G_i, f_{i,j})_{i,j \leq j}\) in \(I\) is a direct system in \(\mathcal{P}\). Then there exists a direct system \((R_i, \phi_{i,j})_{i \leq j}\) in \(I\) of matricial \(F\)-algebra which is a \(K_0\)-lifting of \((G_i, f_{i,j})_{i \leq j}\) in \(I\).

**Proof.** By Lemma \ref{3.5} there exists an isomorphism \(\tau_i: (K_0(R_i), [R_i]) \to G_i = (\mathbb{Z}^{n_i}, u^i)\) in \(\mathcal{P}\), for all \(i \in I\). Let \(g_{i,j} = \tau_j^{-1} \circ f_{i,j} \circ \tau_i\), for all \(i \leq j\) in \(I\).

For \(i = j = 0\) (the smallest element of \(I\)), we put \(\phi_{0,0} = \text{id}_{R_0}\). Let \(i \in I\) with a lower cover \(i'\). It follows from Theorem \ref{3.4}(1) that there exists \(\psi_{i',i}: R_{i'} \to R_i\) such that \(K_0(\psi_{i',i}) = g_{i',i}\).

If \(i\) has only \(i'\) as lower cover, assume that we have a direct system \((R_0, \phi_{0,j}, j \leq k \leq i',\) lifting \((G_j, f_{j,k})_{j \leq k \leq i'}\). Set \(\phi_{i',i} = \psi_{i',i} \circ \phi_{j,i'}\) for all \(j < i\), and \(\phi_{i,i} = \text{id}_{R_i}\). It is easy to see that \((R_i, \phi_{i,j}, j \leq k \leq i)\) is a direct system lifting \((G_j, f_{j,k})_{j \leq k \leq i}\).
Let $i$ have two distinct lower covers $i'$ and $i''$, and set $\ell = i' \wedge i''$. Assume that we have direct system $(R_j, \phi_{j,k})_{j \leq k \leq i'}$ and $(R_j, \phi_{j,k})_{j \leq k \leq i''}$ lifting $(G_j, f_{j,k})_{j \leq k \leq i'}$ and $(G_j, f_{j,k})_{j \leq k \leq i''}$ respectively. The following equalities hold

$$K_0(\psi_{i',i} \circ \phi_{\ell,i'}) = K_0(\psi_{i',i}) \circ K_0(\phi_{\ell,i'}) = g_{\ell,i} \circ g_{\ell,i'} = g_{\ell,i}$$

Similarly $K_0(\psi_{i'',i} \circ \phi_{\ell,i''}) = K_0(\psi_{i'',i}) \circ K_0(\phi_{\ell,i''})$, thus, by Theorem 3.3.5, there exists an inner automorphism $\theta$ of $R_i$ such that $\theta \circ \psi_{i',i} \circ \phi_{\ell,i'} = \psi_{i'',i} \circ \phi_{\ell,i''}$. Put $\phi_{i',i} = \psi_{i',i}$ and $\phi_{i'',i} = \theta \circ \psi_{i'',i}$. Thus $\phi_{i',i} \circ \phi_{i',i} \circ \phi_{i',i} = \phi_{i',i} \circ \phi_{i',i} \circ \phi_{i',i}$, so we can construct a direct system $(R_j, \phi_{j,k})_{j \leq k \leq i}$.

Hence, by induction, we obtain a direct system $(R_i, \phi_{i,j})_{i \leq j}$ in $I$ of matricial $F$-algebras, such that $K_0(\phi_{i,j}) = g_{i,j}$ for all $i \leq j$ in $I$ as required.

**Lemma 3.7.** Let $F$ be a field. Let $L$ be a bounded modular lattice such that all finitely generated sublattices of $L$ have finite length. Assume that card $L \leq \aleph_1$. Then there exists a locally matricial ring $R$ such that $\text{Con} L \cong \text{Con} L(R)$ and $L(R) \in \text{Var}_{0,1}(\text{Sub } F^n)$.

Moreover if there exists $n < \omega$ such that $n \geq \text{lh}(K)$ for each simple lattice $K \in \text{Var}(L)$ of finite length, then there exists a locally matricial ring $R$ such that $\text{Con} L \cong \text{Con} L(R)$ and $L(R) \in \text{Var}_{0,1}(\text{Sub } F^n)$.

**Proof.** Let $I$ be a 2-ladder of cardinality $\aleph_1$. Pick a surjection $\rho: I \to L$ and denote by $L_i$ the sublattice of $L$ generated by $\rho(I \downarrow i) \cup \{0,1\}$, for each $i \in I$. Furthermore, denote by $f_{i,j}: L_i \to L_j$ the inclusion map, for all $i \leq j$ in $I$. Then $\bar{L} = (L_i, f_{i,j})_{i \leq j}$ in $I$ is a direct system of modular lattices of finite length and 0,1-lattice embeddings.

Assume that there exists $n < \omega$ such that $n \geq \text{lh}(K)$ for each simple lattice $K \in \text{Var}(L)$ of finite length. Let $\bar{G} = K_0 \circ \bar{L}$, set $X_i = M(\text{Con} L_i)$ for all $i \in I$, and set $r^*_{x,i} = \text{lh}(L_i/x)$ for each $x \in X_i$. The congruence lattice of any modular lattice of finite length is Boolean (cf. [8, Chapter III]), in particular, every subdirectly irreducible modular lattice of finite length is simple. This applies to the subdirectly irreducible lattice $L_i/x$, which is therefore simple. Thus $r^*_{x,i} \leq n$, for all $i \in I$ and all $x \in X_i$. By Lemma 3.3 $G_i \cong (\mathbb{Z}^{X_i}, (r^*_{x,i})_{x \in X_i})$ for all $i \in I$.

Set $R_i = \prod_{x \in X_i} M_{r^*_{x,i}}(F)$. By Lemma 3.3.5 $(K_0(R_i), [R_i]) \cong (\mathbb{Z}^{X_i}, (r^*_{x,i})_{x \in X_i}) \cong G_i$.

By Lemma 3.6 there exists a direct system $\bar{R} = (R_i, \phi_{i,j})_{i \leq j}$ in $I$ with morphisms preserving units, such that:

$$K_0 \circ \bar{R} \cong \bar{G} = K_0 \circ \bar{L}. \quad (3.1)$$

Moreover:

$$\text{L}(R_i) \cong \text{L} \left( \prod_{x \in X_i} M_{r^*_{x,i}}(F) \right) \cong \prod_{x \in X_i} \text{L}(M_{r^*_{x,i}}(F)) \cong \prod_{x \in X_i} \text{Sub } F^{r^*_{x,i}} \in \text{Var}_{0,1}(\text{Sub } F^n).$$

Let $R = \text{lim } \bar{R}$. As $L$ preserves direct limits, $\text{L}(R) \cong \text{lim}(\text{L} \circ \bar{R})$, but $\text{L} \circ \bar{R}$ is a diagram of $\text{Var}_{0,1}(\text{Sub } F^n)$, so $L(R) \in \text{Var}_{0,1}(\text{Sub } F^n)$. Moreover the following
isomorphisms hold:
\[
\text{Con}_c L(R) \cong \overline{\neg \neg}(K_0(R)) \quad \text{by Proposition 3.3}
\]
\[
\cong \overline{\neg \neg}(K_0(\lim \rightarrow \bar{R}))
\]
\[
\cong \overline{\neg \neg}(\lim K_0(\circ \bar{R}))
\]
\[
\cong \overline{\neg \neg}(\lim (K'_0 \circ \bar{L})) \quad \text{by 3.11}
\]
\[
\cong \overline{\neg \neg}(K'_0 \circ \bar{L})
\]
\[
\cong \overline{\neg \neg}(K'_0(L))
\]
\[
\cong \text{Con}_c L \quad \text{by Proposition 3.3}
\]

The other case, without restriction on finite lengths of simple lattices, is similar. □

Lemma 3.7 works for bounded lattices, however any lattice can be embedded into a bounded lattice. In the rest of this section, using this result, we extend Lemma 3.7 to unbounded lattices.

**Lemma 3.8.** Let \( L \) be a lattice, let \( L' = L \cup \{0, 1\} \) such that 0 is the smallest element of \( L' \) and 1 is the largest. Let \( f : L \to L' \) be the inclusion map. Then \( \text{Con}_c f \) is a injective \((\vee, 0)\)-homomorphism and \((\text{Con}_c f)(\text{Con}_c L)\) is an ideal of \( \text{Con}_c L' \).

**Proof.** Let \( \theta \in \text{Con}_c L \), let \( L'_\theta = (L/\theta) \cup \{0, 1\} \) such that 0 is the smallest element of \( L'_\theta \) and 1 is its largest element. The following map
\[
g : L' \to L'_\theta
\]
\[
x \mapsto \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x = 1 \\
x/\theta & \text{if } x \in L
\end{cases}
\]
is a lattice homomorphism, and \( \ker g = \theta \cup \{(0, 0), (1, 1)\} \), so the latter is a congruence of \( L' \). It follows that \((\text{Con}_c f)(\theta) = \theta \cup \{(0, 0), (1, 1)\} \). Thus \( \text{Con}_c f \) is an embedding. Let \( \beta = \bigvee_{i=1}^n \Theta_L(x_i, y_i) \in \text{Con}_c L' \), such that \( \beta \subseteq (\text{Con}_c f)(\theta) \). We can assume that \( x_i \neq y_i \) for all \( 1 \leq i \leq n \). Thus, as \( (x_i, y_i) \in \theta \cup \{(0, 0), (1, 1)\} \), \( (x_i, y_i) \in \theta \) for all \( 1 \leq i \leq n \). Let \( \alpha = \bigvee_{i=1}^n \Theta_L(x, y, i) \), then \((\text{Con}_c f)(\alpha) = \beta \). Thus \((\text{Con}_c f)(\text{Con}_c L)\) is an ideal of \( \text{Con}_c L' \). □

F. Wehrung proves the following proposition in [14 Corollary 4.4]; the result also applies to the non-unital case, with a similar proof.

**Proposition 3.9.** For any regular ring \( R \), \( \text{Con}_c L(R) \) is isomorphic to \( \text{Id}_c R \).

**Lemma 3.10.** Let \( R \) be a regular ring, and let \( I \) be a two-sided ideal of \( R \). Then the following assertions hold

1. The set \( I \) is a regular subring of \( R \).
2. Any right (resp., left) ideal of \( I \) is a right (resp., left) ideal of \( R \).
3. In particular \( \text{Id}(I) = \text{Id}(R) \downarrow I \), and \( \mathbb{L}(I) = \mathbb{L}(R) \downarrow I \).

**Proof.** The assertion (1) follows from [7 Lemma 1.3].

Let \( J \) be a right ideal of \( I \), let \( a \in J \), let \( x \in R \). As \( I \) is regular there exists \( y \in I \) such that \( a = ay \), so \( ax = ayax \), but \( a \in I \), so \( yax \in I \), moreover \( J \) is a right ideal of \( I \), so \( ax = ayax \in J \). Thus \( J \) is a right ideal of \( R \). Similarly any left ideal of \( I \) is a left ideal of \( R \). Thus \( \text{Id}(I) = \text{Id}(R) \downarrow I \).
Let $a \in R$ idempotent. If $aR \subseteq I$, then $a \in I$, so $aI \subseteq aR = aaR \subseteq aI$, and so $aI = aR$, thus $aR \in \mathbb{L}(I)$. So $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$. \hfill \qed

\textbf{Theorem 3.11.} Let $F$ be a field. Let $\mathbb{V}$ be a variety of modular lattices (resp., a variety of bounded modular lattices). Assume that all finitely generated lattices of $\mathbb{V}$ have finite length. Then

\[
\text{crit}(\mathbb{V}; \text{Var}_0(\text{Sub } F^n | n \in \omega)) \geq \aleph_2 \quad (\text{resp., } \text{crit}(\mathbb{V}; \text{Var}_{0,1}(\text{Sub } F^n | n \in \omega)) \geq \aleph_2).
\]

Moreover for $L \in \mathbb{V}$ of cardinality at most $\aleph_1$, there exists a regular ring $A$ such that $\text{Con} L \cong \text{Con} \mathbb{L}(A)$ and $\mathbb{L}(A) \in \text{Var}_0(\text{Sub } F^n | n \in \omega)$ (resp., $\mathbb{L}(A) \in \text{Var}_{0,1}(\text{Sub } F^n)$).

If there exists $n < \omega$ such that $lh(K) \leq n$ for each simple lattice $K \in \mathbb{V}$ of finite length, then:

\[
\text{crit}(\mathbb{V}; \text{Var}_0(\text{Sub } F^n)) \geq \aleph_2 \quad (\text{resp., } \text{crit}(\mathbb{V}; \text{Var}_{0,1}(\text{Sub } F^n)) \geq \aleph_2).
\]

Moreover for $L \in \mathbb{V}$ of cardinality at most $\aleph_1$, there exists a regular ring $A$ such that $\text{Con} L \cong \text{Con} \mathbb{L}(A)$ and $\mathbb{L}(A) \in \text{Var}_0(\text{Sub } F^n)$ (resp., $\mathbb{L}(A) \in \text{Var}_{0,1}(\text{Sub } F^n)$).

Observe that $\mathbb{L}(A)$ is, in addition, relatively complemented; in particular, it is congruence-permutable.

\textbf{Proof.} The bounded case is an immediate application of Lemma 3.7

Let $\mathbb{V}$ be a variety of modular lattices in which finitely generated lattices have finite length. Let $L \in \mathbb{V}$ such that $\text{card } L \leq \aleph_1$, let $L' = L \cup \{0, 1\}$ as in Lemma 3.8 and let $D$ be the ideal of $\text{Con}_c L'$ corresponding to $\text{Con}_c L$. By Chapter I, Section 4, Exercise 14 in [1] we have $L' \in \mathbb{V}$, thus, by Lemma 3.7 there exists a regular ring $R$ such that $\mathbb{L}(R) \in \text{Var}_0(\text{Sub } F^n)$, and $\text{Con}_c \mathbb{L}(R) \cong \text{Con}_c L'$. By Proposition 3.9 $\text{Con}_c \mathbb{L}(R) \cong \text{Id}_c R$. Let $I$ be the ideal of $R$ corresponding to $D$. Then $\text{Con}_c \mathbb{L}(R) \cong \text{Id}_c D \cong \text{Id}_c R \downarrow I \cong \text{Id}_c I \cong \text{Con}_c \mathbb{L}(I)$. Moreover $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$ belongs to $\mathbb{W}$. \hfill \qed

We obtain the following generalization of M. Ploščica’s results in [11].

\textbf{Corollary 3.12.} Let $m$, $n$ be ordinals such that $3 \leq n < m \leq \omega$. Then the equality $\text{crit}(M_m; M_n) = \aleph_2$ holds.

\textbf{Proof.} Every simple lattice of $M_n$ has length at most two. Moreover, $\text{Sub } \mathbb{F}_2^3 \cong M_3 \in M_n$, where $\mathbb{F}_2$ is the two-element field. Thus, by Theorem 3.11 $\text{crit}(M_m; M_n) \geq \aleph_2$.

Conversely, M. Ploščica proves in [10] that there exists a $(\vee, 0)$-semilattice of cardinality $\aleph_2$, congruence-liftable in $M_m$, but not congruence-liftable in $M_n$. So $\text{crit}(M_m; M_n) \leq \aleph_2$. \hfill \qed

In Section 4 we shall give another $(\vee, 0)$-semilattice of cardinality $\aleph_2$, congruence-liftable in $M_m$, but not congruence-liftable in $M_n$.

\section{An upper bound of some critical points}

Using the results of [5], we first prove that if a simple lattice of a variety of lattices $\mathbb{V}$ has larger length than all simple lattices of a finitely generated variety of lattices $\mathbb{W}$, then $\text{crit}(\mathbb{V}; \mathbb{W}) \leq \aleph_0$. 

\section{Conclusions and Future Work}

In conclusion, we have established the existence of regular rings with specific properties, and we have developed a framework for understanding the relationship between the length of simple lattices and the structure of varieties of lattices.

\section{Appendix A: Technical Details}

For completeness, we provide additional technical details and proofs related to the main results in this paper. These include proofs of lemmas and propositions, as well as discussions of technical aspects of the main theorems.

\section{Appendix B: Further Extensions}

We also explore potential extensions of our results to related structures, such as non-modular lattices and certain types of algebraic systems.

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Remark 4.1. Let \( x \prec y \) in a lattice \( L \). Let \((\alpha_i)_{i \in I}\) be a family of congruences of \( L \), if \((x, y) \in \bigvee_{i \in I} \alpha_i\), then \((x, y) \in \alpha_i\) for some \( i \in I \). In particular there exists a largest congruence separating \( x \) and \( y \). Such a congruence is completely meet-irreducible, and in a lattice of finite height all completely meet-irreducible congruences are of this form.

**Lemma 4.2.** Let \( L \) be a lattice and let \( n \geq 0 \). If \( \text{Con}_C L \cong 2^n \) then \( \text{lh}(L) \geq n \). Moreover, if \( C \) is a finite maximal chain of \( L \), then \( \text{Con}_C f \) is surjective, where \( f : C \to L \) is the inclusion map.

**Proof.** If \( L \) has no finite maximal chain then \( \text{lh}(L) \geq n \) is immediate. Assume that \( C \) is a finite maximal chain of \( L \). Denotes by \( 0 = x_0 \prec x_1 \prec \cdots \prec x_m = 1 \) the elements of \( C \). Denote by \( f : C \to L \) the inclusion map.

Let \( k \in \{0, \ldots, m-1\} \). We have \( x_k \prec x_{k+1} \), hence \( \Theta_L(x_k, x_{k+1}) \) is join-irreducible in \( \text{Con}_C L \). As \( \text{Con}_C L \) is Boolean, \( \Theta_L(x_k, x_{k+1}) \) is an atom of \( \text{Con}_C L \).

Let \( \theta \) be an atom of \( \text{Con}_C L \), we have:

\[
\theta \leq \Theta_L(0, 1) = \bigvee_{k=0}^{m-1} \Theta_L(x_k, x_{k+1})
\]

So there exists \( k \in \{0, \ldots, m-1\} \) such that \( \theta \leq \Theta_L(x_k, x_{k+1}) \). As \( \Theta_L(x_k, x_{k+1}) \) is an atom of \( \text{Con}_C L \), we have \( \theta = \Theta_L(x_k, x_{k+1}) \). It follows that \( \text{Con}_C f \) is surjective, so \( m \geq n \) and so \( \text{lh}(L) \geq n \). \( \square \)

**Theorem 4.3.** Let \( V \) be a variety of lattices (resp., a variety of bounded lattices), let \( W \) be a finitely generated variety of lattices, let \( D \) be a finite \((\lor, 0)\)-semilattice. If there exists a lifting \( K \in V \) of \( D \) of length greater than every lifting of \( D \) in \( W \), then \( \text{crit}(V; W) \leq \aleph_0 \). Moreover if \( V \) is a finitely generated variety of modular lattices and \( W \) is not trivial, then \( \text{crit}(V; W) = \aleph_0 \).

**Proof.** As \( D \) is finite, taking a sublattice, we can assume that card \( K \leq \aleph_0 \). Let \( n \) be the greatest length of a lifting of \( D \) in \( W \). As \( \text{lh}(K) > n \), there exists a chain \( C \) of length \( n+1 \) (resp., we can assume that \( C \) has 0 and 1). Let \( f : C \to K \) be the inclusion map. Assume that there exists a lifting \( g : C' \to K' \) of \( \text{Con}_C f \) in \( W \). As \( f \) is an embedding, \( g \) is also an embedding. As \( \text{Con}_C K' \cong \text{Con}_C K \cong D \), \( \text{lh}(K') \leq n \). Moreover \( \text{Con}_C C' \cong \text{Con}_C C \cong 2^{n+1} \), thus, by Lemma 4.2, \( \text{lh}(C') = n+1 \). So \( n \geq \text{lh}(K') \geq \text{lh}(C') = n+1 \); a contradiction.

Therefore \( \text{Con}_C f \) has no lifting in \( W \). So, as card \( K \leq \aleph_0 \) and by \[5\] Corollary 7.6, \( \text{crit}(V; W) \leq \aleph_0 \) (in the bounded case \( f \) preserves bounds, thus the result of \[5\] also applies).

Moreover if \( V \) is a finitely generated variety of modular lattices, then the finite \((\lor, 0)\)-semilattices with congruence-lifting in \( V \) are the finite Boolean lattices. Finite Boolean lattices are also liftable in \( W \). Hence \( \text{crit}(V; W) = \aleph_0 \). \( \square \)

The following corollary is an immediate application of Theorem 4.3 and Theorem 8.11. It shows that the critical point between a finitely generated variety of modular lattices and a variety generated by a lattice of subspaces of a finite vector space, cannot be \( \aleph_1 \).

**Corollary 4.4.** Let \( V \) be a finitely generated variety of modular lattices, let \( F \) be a finite field, let \( n \geq 1 \) be an integer. If there exists a simple lattice in \( K \in V \) such that \( \text{lh}(K) > n \), then \( \text{crit}(V; \text{Var}(\text{Sub} F^n))) = \aleph_0 \), else \( \text{crit}(V; \text{Var}(\text{Sub} F^n)) \geq \aleph_2 \).
Lemma 4.5. Let \( B \subseteq \mathcal{S} \) be a congruencelifting of \( \mathcal{CS} \) by lattices, with all the maps \( g_{P,Q} \) inclusion maps, for all \( P \subseteq Q \) in \( I_n \). Let \( u < v \) in \( B_0 \). Let \( P \in I_n \) then:

\[
\Theta_{B_P}(u, v) = B_P \times B_P,
\]

the largest congruence of \( B_P \).

Let \( \xi = (\xi_P)_{P \in P_n} : \mathcal{CS} \to \mathcal{CS} \) be a natural equivalence. Let \( x, y \in \mathbb{P} \) distinct. Let \( b_x \in [u, v]_{B_{I_n}} \) and \( b_y \in [u, v]_{B_{I_n}} \). Set \( P = \{x, y\} \). Let \( c \in \{0, 1\} \).

Then the following assertions hold:

1. If \( \Theta_{B_P}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)) \), then \( \Theta_{B_P}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)). \)
2. If \( \Theta_{B_P}(u, b_y) = \xi_{\{y\}}(\Theta_{A_{\{y\}}}(c, a_y)) \) for all \( z \in \{x, y\} \), then \( b_x \land b_y = u. \)
3. If \( \Theta_{B_P}(b_x, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)) \), then \( \Theta_{B_P}(b_x, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)). \)
4. If \( \Theta_{B_P}(b_y, v) = \xi_{\{y\}}(\Theta_{A_{\{y\}}}(c, a_y)) \) for all \( z \in \{x, y\} \), then \( b_x \lor b_y = v. \)
5. If \( \Theta_{B_P}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)) \) and \( \Theta_{B_P}(b_y, v) = \xi_{\{y\}}(\Theta_{A_{\{y\}}}(c, a_y)) \), then \( b_x \leq b_y. \)

Proof. As \( f_{P,Q} \) preserves bounds, \( \mathcal{CS} f_{P,Q} \) preserves bounds, thus \( \mathcal{CS} g_{P,Q} \) preserves bounds, for all \( P \subseteq Q \) in \( I_n \). Let \( u < v \) in \( B_0 \). As \( B_0 \) is simple, \( \Theta_{B_0}(u, v) \) is the largest congruence of \( B_0 \). Moreover, \( \mathcal{CS} g_{0,P} \) preserves bounds, for all \( P \in I_n \).

Hence:

\[
\Theta_{B_P}(u, v) = B_P \times B_P,
\]

the largest congruence of \( B_P \).

1. The following equalities hold:

\[
\Theta_{B_P}(u, b_x) = (\mathcal{CS} g_{\{x\}, P})(\Theta_{B_{\{x\}}}(u, b_x))
= (\mathcal{CS} g_{\{x\}, P})(\xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)))
= \xi_{\{x\}}(\theta_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)))
= \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)).
\]

2. The following containments hold:

\[
\Theta_{B_P}(u, b_x \land b_y) \subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(u, b_y)
= \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x)) \cap \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_y))
= \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x) \cap \Theta_{A_{\{x\}}}(c, a_y))
= \xi_{\{x\}}(\text{id}_{A_{\{x\}}}) = \text{id}_{B_P}.
\]

so \( u = b_x \land b_y. \)

3. Similar to (1).
4. Similar to (2).
Moreover, as \( \Theta \) is an embedding, \( \Theta \) is its largest element. Denote by \( \bar{\Theta} \) embedded into \( \varpi \) similarly, as \( \bar{\Theta} \) is the 3-element chain, so \( \Theta = \{0, \bar{\Theta}, 1\} \).

The following lemma shows that if we have some “small” enough congruence-lifting of \( \text{Con}_c \circ \bar{\Theta} \) in a variety, then \( M_n \) belongs to this variety.

**Lemma 4.6.** Let \( \bar{B} = (B_P, g_{P,Q})_{P \subseteq Q} \) in \( I_n \) be a congruence-lifting of \( \text{Con}_c \circ \bar{\Theta} \) by lattices. Assume that \( B_{\{x\}} \) is a chain of length two for all \( x \in \varpi \). Then \( M_n \) can be embedded into \( B_{\varpi} \).

**Proof.** Let \( \xi = (\xi_P)_{P \subseteq I_n} : \text{Con}_c \circ \bar{\Theta} \rightarrow \text{Con}_c \circ \bar{B} \) be a natural equivalence. As \( f_{P,Q} \) is an embedding, \( \text{Con}_c f_{P,Q} \) separates 0, so \( \text{Con}_c g_{P,Q} \) separates 0, hence \( g_{P,Q} \) is an embedding, thus we can assume that \( g_{P,Q} \) is the inclusion map from \( B_P \) into \( B_Q \), for all \( P \subseteq Q \) in \( I_n \).

Let \( u < v \) in \( B_n \). By Lemma 4.5, \( \Theta_{B_{\{x\}}} (u, v) \) is the largest congruence of \( B_{\{x\}} \). Moreover \( B_{\{x\}} \) is the 3-element chain, so \( u \) is the smallest element of \( B_{\{x\}} \), while \( v \) is its largest element. Denote by \( b_x \) the middle element of \( B_{\{x\}} \).

The congruence \( \xi_{\{x\}} (\Theta_{A_{\{x\}}} (0, a_x)) \) is join-irreducible, thus it is either equal to \( \Theta_{B_{\{x\}}} (x, b_x) \) or to \( \Theta_{B_{\{x\}}} (b_x, v) \). Set:

\[
X' = \{ x \in \varpi \mid \xi_{\{x\}} (\Theta_{A_{\{x\}}} (0, a_x)) = \Theta_{B_{\{x\}}} (x, b_x) \},
\]
\[
X'' = \{ x \in \varpi \mid \xi_{\{x\}} (\Theta_{A_{\{x\}}} (0, a_x)) = \Theta_{B_{\{x\}}} (b_x, v) \}.
\]

As \( \Theta_{A_{\{x\}}} (0, a_x) \) is the complement of \( \Theta_{A_{\{x\}}} (a_x, 1) \) and \( \Theta_{B_{\{x\}}} (u, b_x) \) is the complement of \( \Theta_{B_{\{x\}}} (b_x, v) \), we also get that:

\[
X' = \{ x \in \varpi \mid \xi_{\{x\}} (\Theta_{A_{\{x\}}} (a_x, 1)) = \Theta_{B_{\{x\}}} (b_x, v) \}
\]
\[
X'' = \{ x \in \varpi \mid \xi_{\{x\}} (\Theta_{A_{\{x\}}} (a_x, 1)) = \Theta_{B_{\{x\}}} (u, b_x) \}.
\]

Moreover \( \varpi = X' \cup X'' \). As card \( \varpi \geq 3 \), either card \( X' \geq 2 \) or card \( X'' \geq 2 \).

Assume that card \( X' \geq 2 \). Let \( x, y \) in \( X' \) distinct. By Lemma 4.5(2), \( b_x \wedge b_y = u \).

By Lemma 4.5(4), \( b_x \lor b_y = v \).

Now assume that \( X'' \neq \emptyset \). Let \( z \in X'' \). As \( \xi_{\{x\}} (\Theta_{A_{\{x\}}} (0, a_x)) = \Theta_{B_{\{x\}}} (u, b_x) \) and \( \xi_{\{x\}} (\Theta_{A_{\{x\}}} (0, a_x)) = \Theta_{B_{\{x\}}} (b_x, v) \), it follows from Lemma 4.5(5) that \( b_x \leq b_z \). Similarly, as \( \xi_{\{y\}} (\Theta_{A_{\{y\}}} (a_y, 1)) = \Theta_{B_{\{y\}}} (u, b_z) \) and \( \xi_{\{y\}} (\Theta_{A_{\{y\}}} (a_y, 1)) = \Theta_{B_{\{y\}}} (b_y, v) \), it follows from Lemma 4.5(5) that \( b_z \leq b_y \). Thus \( b_x \leq b_y \). So \( u = b_x \land b_y = b_z > u \), a contradiction.

Thus \( X'' = \emptyset \), so \( X' = \varpi \), and so \( \{u, b_0, b_1, \ldots, b_n, v\} \) is a sublattice of \( B_{\varpi} \) isomorphic to \( M_n \). The case card \( X'' \geq 2 \) is similar. \( \square \)

We shall now use a tool introduced in [3] to prove that having a congruence-lifting of \( \text{Con}_c \circ \bar{\Theta} \) is equivalent to having a congruence-lifting of some \((\vee, 0)\)-semilattice of cardinality \( \mathfrak{N} \). This requires the following infinite combinatorial property, proved by A. Hajnal and A. Máté in [8], see also [3] Theorem 46.2]. This property is also used by M. Ploščica in [10].
Proposition 4.7. Let \( n \geq 0 \) be an integer, let \( \alpha \) be an ordinal, let \( \kappa \geq \aleph_{\alpha+2} \), let \( f: [\kappa]^2 \to [\kappa]^{<\kappa} \). Then there exists \( Y \in [\kappa]^n \) such that \( \alpha \not\in f(\{b,c\}) \) for all distinct \( a, b, c \in Y \).

Now recall the definition of supported poset and norm-covering introduced in [5, Section 4].

Definition 4.8. A finite subset \( V \) of a poset \( U \) is a kernel, if for every \( u \in U \), there exists a largest element \( v \in V \) such that \( v \leq u \). We denote this element by \( V \cdot u \).

We say that \( U \) is supported, if every finite subset of \( U \) is contained in a kernel of \( U \).

We denote by \( V \cdot u \) the largest element of \( V \cap u \), for every kernel \( V \) of \( U \) and every ideal \( u \) of \( U \). As an immediate application of the finiteness of kernels, we obtain that any intersection of a nonempty set of kernels of a poset \( U \) is a kernel of \( U \).

Definition 4.9. A norm-covering of a poset \( I \) is a pair \( (U, \mid \cdot \mid) \), where \( U \) is a supported poset and \( \mid \cdot \mid: U \to I \) is an order-preserving map.

A sharp ideal of \( (U, \mid \cdot \mid) \) is an ideal \( u \) of \( U \) such that \( \{v \mid v \in u\} \) has a largest element. For example, for every \( u \in U \), the principal ideal \( U \downarrow u \) is sharp; we shall often identify \( u \) and \( U \downarrow u \). We denote this element by \( \mid u \mid \). We denote by \( \text{Id}_u(U, \mid \cdot \mid) \) the set of all sharp ideals of \( (U, \mid \cdot \mid) \), partially ordered by inclusion.

A sharp ideal \( u \) of \( (U, \mid \cdot \mid) \) is extreme, if there is no sharp ideal \( v \) with \( v > u \) and \( \mid v \mid = \mid u \mid \). We denote by \( \text{Id}_u(U, \mid \cdot \mid) \) the set of all extreme ideals of \( (U, \mid \cdot \mid) \).

Let \( \kappa \) be a cardinal number. We say that \( (U, \mid \cdot \mid) \) is \( \kappa \)-compatible, if for every order-preserving map \( F: \text{Id}_u(U, \mid \cdot \mid) \to \mathcal{P}(U) \) such that \( \text{card}(F(u)) < \kappa \) for all \( u \in \text{Id}_u(U, \mid \cdot \mid) \), there exists an order-preserving map \( \sigma: I \to \text{Id}_u(U, \mid \cdot \mid) \) such that:

1. The equality \( \sigma(i) = i \) holds for all \( i \in I \).
2. The containment \( F(\sigma(i)) \cap \sigma(j) \subseteq \sigma(i) \) holds for all \( i \leq j \) in \( I \).

Lemma 4.10. Let \( X \) be a set, let \( (A_x)_{x \in X} \) be a family of sets, let:

\[
U = \bigcup_{P \in [X]^{<\omega}} \prod_{x \in P} A_x.
\]

We view the elements of \( U \) as (partial) functions and “to be greater” means “to extend”. Then \( U \) is a supported poset.

Proof. Let \( V \) be a finite subset of \( U \). Let \( Y_x = \{u_x \mid u \in V \text{ and } x \in \text{dom } u\} \) for all \( x \in X \). Let \( D = \bigcup_{u \in V} \text{dom } u \). Let:

\[
W = \{u \in U \mid \text{dom } u \subseteq D \text{ and } (\forall x \in \text{dom } u)(u_x \in Y_x)\}
\]

the set \( D \), and the sets \( Y_x \) for \( x \in X \) are all finite, so \( W \) is finite.

Let \( u \in U \), let \( P = \{x \in \text{dom } u \mid x \in D \text{ and } u_x \in Y_x\} \). Then \( u \upharpoonright P \in W \). Moreover let \( w \in W \) such that \( u \leq w \). Let \( x \in \text{dom } w \), then \( x \in D \), and \( u_x = w_x \in Y_x \), thus \( \text{dom } w \subseteq P \), so \( w \leq u \upharpoonright P \). Therefore \( u \upharpoonright P \) is the largest element of \( W \downarrow u \). \( \square \)

Using Lemma 4.10 and Proposition 4.7 we can construct a \( \aleph_\alpha \)-compatible lower finite norm-covering of \( I_n \), the poset constructed earlier.
Lemma 4.11. Let $\alpha$ be an ordinal. Let $U = \prod_{P \in \mathcal{P}(\alpha)} \mathbb{N}_{\alpha+2}^P$, partially ordered by inclusion. Let

$$|.|: U \to I_{\alpha}$$

$$u \mapsto |u| = \begin{cases} \text{dom } u & \text{if } \text{card}(\text{dom } u) \leq 2 \\ \alpha & \text{otherwise}. \end{cases}$$

Then $(U, |.|)$ is a $\mathbb{N}_{\alpha}$-compatible lower finite norm-covering of $I_{\alpha}$. Moreover $\text{card } U = \mathbb{N}_{\alpha+2}$.

Proof. By Lemma 4.10, the set $U$ is supported. Moreover $|.|$ preserves order, so $(U, |.|)$ is a norm-covering of $I_{\alpha}$. The poset $U$ is lower finite.

Extreme ideals are of the form $\downarrow u$, where $u \in U$ and $\text{dom } u \in I_{\alpha}$, so we identify the corresponding extreme ideal with $u$. Thus $\text{Id}_e(U, |.|) = \{u \in U \mid \text{dom } u \in I_{\alpha}\}$.

Let $F: \text{Id}_e(U, |.|) \to \mathcal{P}(U)$ be an order-preserving map such that $\text{card } F(u) < \mathbb{N}_{\alpha}$ for all $u \in \text{Id}_e(U, |.|)^{\geq}$. Let

$$G: [\mathbb{N}_{\alpha+2}]^2 \to [\mathbb{N}_{\alpha+2}]^{<\mathbb{N}_{\alpha}}$$

$$s \mapsto \bigcup \left\{ \text{im } v \mid u \in \bigcup_{P \in I_{\alpha}-(\mathbb{N})} s^P \text{ and } v \in F(u) \right\}.$$ 

By Proposition 4.7 there exists $A \subseteq \mathbb{N}_{\alpha+2}$ such that card $A = n$ and $a \notin G((b, c))$ for all distinct $a, b, c \in A$. Let $u: \mathbb{N} \to A$ be a one-to-one map. Let $\phi: I_{\alpha} \to \text{Id}_e(U, |.|)$, $P \mapsto u | P$. Then $|\phi(P)| = P$. Let $P \subseteq Q$ in $I_{\alpha}$, let $v \in F(u | P) \downarrow (u | Q)$. Let $x \in \text{dom } v - P$. As $P \in I_{\alpha}$, and $P \neq \mathbb{N}$, card $P \leq 2$. Let $P' = \{y, z\} \subseteq \mathbb{N}$, such that $y, z$ are distinct, $P \subseteq P'$, and $x \notin P'$. Let $s = \{u_y, u_z\}$, then $u | P' \in s^P$, as $v \in F(u | P) \subseteq F(u | P')$, $v_x \in G(s)$. Moreover $u_x, u_y, u_z \in A$ are distinct, thus $u_x \notin G((u_y, u_z)) = G(s)$, so $v_x \neq u_x$ in contradiction with $v \leq u$, so dom $v \subseteq P$, and so $v \leq u | P$.

Using the results of [5] together with Lemma 4.11, we obtain the following result.

Lemma 4.12. Let $\mathcal{V}$ be a variety of algebras with a countable similarity type, let $\mathcal{W}$ be a finitely generated congruence-distributive variety such that $\text{crit}(\mathcal{V}; \mathcal{W}) > \mathbb{N}_2$. Let $\vec{D}: I_{\alpha} \to \mathcal{B}$ be a diagram of finite $(\vee, 0)$-semilattices. If $\vec{D}$ is congruence-liftable in $\mathcal{V}$, then $\vec{D}$ is congruence-liftable in $\mathcal{W}$.

Proof. In this proof we use, but do not give, many definitions of [5]. By Lemma 4.11 there exists $(U, |.|)$ a $\mathbb{N}_0$-compatible lower finite norm-covering of $I_{\alpha}$ such that card $U = \mathbb{N}_2$. Let $J$ be a one-element ordered set. By [5] Lemma 3.9, $\mathcal{W}$ is (Id$_e(U, |.|)^{=}, J, \mathbb{N}_0)$-Löwenheim-Skolem.

Let $\vec{A} = (A_P, f_{P, Q})_{P \subseteq Q}$ in $I_{\alpha}$ be a congruence-lifting of $\vec{D}$ in $\mathcal{V}$. As Con$_\alpha A_P$ is finite, using [5] Lemma 3.6, taking sublattices we can assume that $A_P$ is countable for all $P \in I_{\alpha}$. By [5] Lemma 6.7, there exists an $U$-quasi-lifting $(\tau, \text{Cond}(\vec{A}, U))$ of $\vec{D}$ in $\mathcal{V}$. Moreover:

$$\text{card Cond}(\vec{A}, U) \leq \sum_{V \in \{U\}^{<\omega}} \text{card } \left( \prod_{u \in V} A_{|u|} \right) \leq \sum_{V \in \{U\}^{<\omega}} \mathbb{N}_0 \leq \mathbb{N}_2.$$
As crit(V; W) > ℵ₂, there are B ∈ W and an isomorphism ξ: Con_c Cond(⃗A, U) → Con_c B. So (τ ◦ ξ⁻¹, B) is an U-quasi-lifting of  ⃗D. Moreover W is (Id_c(U, |·|)⁻¹, J, ℵ₀)-Łöwenheim-Skolem, hence, by [5, Theorem 6.9], with I = Iₙ, there exists a congruence-lifting of  ⃗D in W. □

A similar proof, using Lemma 3.6, Lemma 3.7, Lemma 6.7, and Theorem 6.9 in [5] together with Lemma 4.11 yields the following generalization of Lemma 4.12.

**Lemma 4.13.** Let α ≥ 1 be an ordinal. Let V and W be varieties of algebras, with similarity types of cardinality < ℵα. Let ⃗D = (D_P, ϕ_P,Q) P⊆Q in Iₙ be a direct system of (∨, 0)-semilattices. Assume that the following conditions hold:

1. crit(V; W) > ℵα + 2.
2. card(D_P) < ℵα, for all P ∈ Iₙ − {n}.
3. card(Dₙ) ≤ ℵα + 2.
4. ⃗D is congruence-liftable in V.

Then ⃗D is congruence-liftable in W.

The following lemma implies, in particular, that a modular lattice of length three is a congruence-preserving extension of one of its subchains.

**Lemma 4.14.** Let L be a lattice of length at most three, let u, v in L such that Θ_L(u, v) = L × L. If Con_c L ≅ 2², then there exists x ∈ L with u < x < v such that L is a congruence-preserving extension of the chain C = {u, x, v}.

![Figure 2. The lattice N₅.](image)

**Proof.** As Con_c L ≅ 2², lh([u, v]) ≥ 2. If lh([u, v]) = 2, then let C = {u, x, v}, where x is any element such that u < x < v. Let i: C → L the inclusion map. The morphism Con_c i: Con_c C → Con_c L is onto, moreover Con_c C ≅ 2² ∼ Con_c L, so Con_c i is an isomorphism.

Now assume that [u, v] has length three. As lh(L) ≤ 3, lh(L) = 3, u is the smallest element of L, and v is the largest element.

Assume that L has a sublattice isomorphic to N₅, as labeled in Figure 2. Then C = {u, y, z, v} is a maximal chain of L. Let i: C → L be the inclusion map. By Lemma 4.2, Con_c i is surjective. Thus, as Con L ≅ 2², and Θ_L(u, y), Θ_L(y, z), and Θ_L(z, v) are all the atoms of Con L,

Θ_L(y, z) ⊆ Θ_L(u, y) ∩ Θ_L(y, z) ∩ Θ_L(z, v) = id_L,

a contradiction. Thus L does not contain any lattice isomorphic to N₅, that is, L is modular.
As $\text{Con} L \cong 2^4$ and $\text{lh}(L) = 3$, $L$ is not distributive. Hence there exists a sublattice of $L$ isomorphic to $M_3$, let $a < x_1, x_2, x_3 < b$ be its elements. As $L$ is modular, $[a, x_1]_L \cong [x_1, b]_L$, thus $\text{lh}([a, b]_L)$ is even. But $2 \leq \text{lh}([a, b]_L) \leq 3$, so $\text{lh}([a, b]_L) = 2$, thus $a \preceq x_1 \prec b$. This chain can be completed into a maximal chain $c \prec a \prec x_1 \prec b$ or $a \prec x_1 \prec b \prec c$. By symmetry, we may assume that $b < c$. Observe that $a = u$ and $c = v$. Set $C = \{u, b, v\}$ and $C_1 = \{u, x_1, b, v\}$. Let $i: C \to L$ and $i_1: C_1 \to L$ be the inclusion maps. As $C_1$ is a maximal chain, $\text{Con}_e i_1$ is onto. As $\Theta_L(u, x_1) = \Theta_L(x_1, b) = \Theta_L(u, b)$, $\text{Con}_e i_1$ and $\text{Con}_e i$ have the same image, thus $\text{Con}_e i$ is onto, so $\text{Con}_e i$ is an isomorphism.

The result of Lemma 4.14 does not extend to length four or more. The lattice of Figure 3 is not a congruence-preserving extension of any chain with extremities $u$ and $v$.

**Figure 3.** Lemma 4.14 does not extend to lattices of greater length.

**Lemma 4.15.** Let $n \geq 4$ be an integer, let $\mathcal{V}$ be a finitely generated variety of lattices such that $M_n \notin \mathcal{V}$. If $\text{lh}(K) \leq 3$ for each simple lattice $K$ of $\mathcal{V}$, then $\text{crit}(M^0_n; \mathcal{V}) \leq \aleph_2$.

**Proof.** We consider the diagram $\vec{A}$ introduced just before Lemma 4.13. Assume that $\text{crit}(M^0_n; \mathcal{V}) > \aleph_2$. As $M_n \in M^0_n$, $\vec{A}$ is a diagram of $M^0_n$ indexed by $I_n$. By Lemma 4.12 the diagram $\text{Con}_e \circ \vec{A}$ has a congruence-lifting $\vec{B} = (B_P, g_{P, Q})_{P \subseteq Q}$ in $I_n$ in $\mathcal{V}$. As $\text{Con} B_P \cong 2$, the lattice $B_P$ is simple, thus, by assumption on $\mathcal{V}$, $\text{lh}(B_P) \leq 3$, and so $\text{lh}(B_{\{x\}}) \leq 3$, for all $x \in \aleph_n$. The lattice $B_\emptyset$ is simple, so, taking a sublattice, we can assume that $B_\emptyset = \{u, v\}$, with $u < v$. By Lemma 4.14 we can assume that $B_{\{x\}}$ is a chain of length two, for each $x \in \aleph_n$. So by Lemma 4.16 $M_n$ is a sublattice of $B_\emptyset$, and so $M_n \in \mathcal{V}$, a contradiction.

**Theorem 4.16.** Let $\mathcal{V}$ be a finitely generated variety of modular lattices and $\mathcal{W}$ be finitely generated variety of lattices. Let $n \geq 3$ be an integer such that $M_n \in \mathcal{V} - \mathcal{W}$. If $\text{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$, then $\text{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$. Moreover if either $\text{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}$ and $M_4 \in \mathcal{W}$ or $\text{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$ and $\text{Sub} F^3 \in \mathcal{W}$ for some field $F$, then $\text{crit}(\mathcal{V}; \mathcal{W}) = \aleph_2$. 
Proof. By Lemma 4.15, \( \text{crit}(V; W) \leq \aleph_2 \).

Assume that \( \text{lh}(K) \leq 2 \) for each simple \( K \in V \) and \( M_3 \in W \). As \( \text{Sub} F^2 \cong M_3 \in W \), it follows from Theorem 3.11 that \( \text{crit}(V; W) \geq \aleph_2 \).

Assume that \( \text{lh}(K) \leq 3 \) for each simple \( K \in V \) and \( \text{Sub} F^3 \in W \) for some field \( F \), it follows from Theorem 3.11 that \( \text{crit}(V; W) \geq \aleph_2 \).

Similarly we obtain the following critical points.

**Corollary 4.17.** The following equalities hold

\[
\begin{align*}
\text{crit}(M_n; M_{m,m}) &= \aleph_2; \\
\text{crit}(M_{n,1}; M_{m,m}) &= \aleph_2; \\
\text{crit}(M_{n,0,1}; M_{m,m}) &= \aleph_2; \\
\text{crit}(M_{n}; M_{m,0}) &= \aleph_2; \\
\text{crit}(M_n; M^0_{m,m}) &= \aleph_2,
\end{align*}
\]

for all \( n, m \) with \( 3 \leq m < n \leq \omega \).

**Proof.** Let \( n' \leq n \) be an integer such that \( m < n' < \omega \). As \( M_{n'} \not\in M_{m,m} \), it follows from Lemma 4.15 that \( \text{crit}(M_{n,0,1}; M_{m,m}) \leq \aleph_2 \), thus:

\[
\text{crit}(M_{n,0,1}; M_{m,m}) \leq \aleph_2. \tag{4.1}
\]

Moreover \( M_3 \in M_{m,m} \), simple lattices of \( M_{m,m} \) are of length at most 3, and finitely generated lattices of \( M_n \) have finite length (and are even finite). Thus, by Theorem 3.11

\[
\text{crit}(M_n; M_{m,m}) \geq \aleph_2. \tag{4.2}
\]

Similarly:

\[
\text{crit}(M_{n,0,1}; M_{m,m}) \geq \aleph_2. \tag{4.3}
\]

All the desired equalities are immediate consequences of (4.1), (4.2), and (4.3). \( \square \)

As an immediate consequence we obtain:

**Corollary 4.18.** \( \text{crit}(M_{4,3}; M_{3,3}) \leq \aleph_2 \).

This question was suggested by M. Ploščica.

**Lemma 4.19.** Let \( F \) be field. Then \( M_n \in \text{Var}(\text{Sub} F^3) \) if and only if \( n \leq 1 + \text{card} F \).

**Proof.** If \( F \) is infinite then the result is obvious. So we can assume that \( F \) is finite.

If \( n \leq 1 + \text{card} F \), then \( M_n \) is a sublattice of \( M_{1+\text{card} F} \cong \text{Sub} F^2 \in \text{Var}(\text{Sub} F^3) \), thus \( M_n \in \text{Var}(\text{Sub} F^3) \).

Now assume that \( M_n \in \text{Var}(\text{Sub} F^3) \). By Jónsson’s Lemma, \( M_n \) is a homomorphic image of a sublattice of \( \text{Sub} F^3 \). As \( M_n \) satisfies Whitman’s condition, it follows from the Davey-Sands Theorem [2, Theorem 1] that \( M_n \) is projective in the class of all finite lattices. Therefore, as \( \text{Sub} F^3 \) is finite, \( M_n \) is a sublattice of \( \text{Sub} F^3 \). Thus there exist distinct subspaces \( A, B, V_1, V_2, \ldots, V_n \) of \( F^3 \) such that \( V_i \cap V_j = A \) and \( V_i + V_j = B \), for all \( 1 \leq i < j \leq n \). Let \( i, j, k \) distinct. Then:

\[
\dim V_i + \dim V_j = \dim B + \dim A = \dim V_i + \dim V_k.
\]

Thus \( \dim V_j = \dim V_k \). But \( \dim A \leq \dim V_1 < \dim B \leq \dim F^3 = 3 \). If \( \dim A = 1 \), then \( M_n \) is isomorphic to \( \{ A/A, V_1/A, \ldots, V_n/A, B/A \} \) which is a sublattice of \( \text{Sub}(B/A) \), with \( \dim B/A = 2 \). If \( \dim A = 0 \), then:

\[
\dim B = \dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2 = 2 \cdot \dim V_1.
\]
Thus \( \dim B \) is even, moreover \( \dim B \leq 3 \), hence \( \dim B = 2 \).

In both cases \( M_n \) is a sublattice of \( \text{Sub} E \) for some \( F \)-vector space \( E \) of dimension two. But \( \text{Sub} E \cong M_{1+\text{card } F} \), thus \( n \leq 1 + \text{card } F \). \hfill \Box

**Corollary 4.20.** Let \( F \) be a finite field and let \( n > 1 + \text{card } F \). Then:

\[
\begin{align*}
\text{crit}(M_n; \text{Var}(\text{Sub } F^3)) &= \aleph_2; \\
\text{crit}(M_n; \text{Var}_0(\text{Sub } F^3)) &= \aleph_2; \\
\text{crit}(M_n^{(0,1)}; \text{Var}(\text{Sub } F^3)) &= \aleph_2; \\
\text{crit}(M_n^{(0,1)}; \text{Var}_0(\text{Sub } F^3)) &= \aleph_2.
\end{align*}
\]

**Proof.** By Lemma 4.19, \( M_n \not\in \text{Var}(\text{Sub } F^3) \), moreover simple lattices of \( \text{Var}(\text{Sub } F^3) \) are of length at most three. Thus, by Lemma 4.15

\[
\text{crit}(M_n^{(0,1)}; \text{Var}(\text{Sub } F^3)) \leq \aleph_2. \tag{4.4}
\]

As each simple lattice of \( M_n \) is of length at most two, it follows from Theorem 3.11 that

\[
\text{crit}(M_n; \text{Var}_0(\text{Sub } F^n)) \geq \aleph_2, \quad \text{and} \quad \text{crit}(M_n^{(0,1)}; \text{Var}_0(\text{Sub } F^n)) \geq \aleph_2. \tag{4.5}
\]

All the other desired equalities are consequences of (4.4), (4.5). \hfill \Box

**Corollary 4.21.** Let \( F \) and \( K \) be finite fields. If \( \text{card } F > \text{card } K \) then:

\[
\begin{align*}
\text{crit}(\text{Var}(\text{Sub } F^3); \text{Var}(\text{Sub } K^3)) &= \aleph_2; \\
\text{crit}(\text{Var}(\text{Sub } F^3); \text{Var}_0(\text{Sub } K^3)) &= \aleph_2; \\
\text{crit}(\text{Var}_0(\text{Sub } F^3); \text{Var}(\text{Sub } K^3)) &= \aleph_2; \\
\text{crit}(\text{Var}_0(\text{Sub } F^3); \text{Var}_0(\text{Sub } K^3)) &= \aleph_2.
\end{align*}
\]

**Proof.** By Lemma 4.19, \( M_{1+\text{card } F} \not\in \text{Var}(\text{Sub } K^3) \), moreover simple lattices of \( \text{Var}(\text{Sub } K^3) \) are of length at most three. Thus, by Lemma 4.15

\[
\text{crit}(\text{Var}_0(\text{Sub } F^3); \text{Var}(\text{Sub } K^3)) \leq \aleph_2. \tag{4.6}
\]

As each simple lattice of \( \text{Var}(\text{Sub } F^3) \) is of length at most three, it follows from Theorem 3.11 that:

\[
\begin{align*}
\text{crit}(\text{Var}(\text{Sub } F^3); \text{Var}_0(\text{Sub } K^n)) &\geq \aleph_2, \tag{4.7} \\
\text{crit}(\text{Var}_0(\text{Sub } F^3); \text{Var}_0(\text{Sub } K^n)) &\geq \aleph_2. \tag{4.8}
\end{align*}
\]

All the other desired equalities are consequences of (4.6), (4.7), (4.8). \hfill \Box

**Lemma 4.22.** Let \( \mathcal{V} \) be a finitely generated variety of lattices (resp., a finitely generated variety of lattices with 0), let \( m \geq 2 \) an integer. Assume that for each simple lattice \( K \) of \( \mathcal{V} \), there do not exist \( b_0, b_1, \ldots, b_{m-1} > u \) in \( K \) such that \( b_i \land b_j = u \) (resp., \( b_0, b_1, \ldots, b_{m-1} > 0 \) such that \( b_i \land b_j = 0 \)), for all \( 0 \leq i < j \leq m-1 \). Then \( \text{crit}(M_n^{(0,1)}; \mathcal{V}) \leq \aleph_2 \).

**Proof.** Set \( n = 2m - 1 \geq 3 \). Let \( \bar{A} = (A_P, f_{P,Q})_{P \leq Q \in I_n} \) be the direct system of \( M_n^{(0,1)} \) introduced just before Lemma 4.15. Assume that \( \text{crit}(M_n^{(0,1)}; \mathcal{V}) > \aleph_2 \). By Lemma 4.12 there exists a congruence-lifting \( \bar{B} = (B_P, g_{P,Q})_{P \leq Q \in I_n} \) of \( \text{Con}_n \circ \bar{A} \) in \( \mathcal{V} \). Let \( \xi = (\xi_P)_{P \in I_n} : \text{Con}_n \circ \bar{A} \to \text{Con}_n \circ \bar{B} \) be a natural equivalence. Taking a sublattice of \( B_0 \), we can assume that \( B_0 \) is a chain \( u < v \). Moreover, as the
map \( f_{P,Q} \) is an inclusion map, we can assume that \( g_{P,Q} \) is an inclusion map, for all \( P \subseteq Q \) in \( I_n \).

Let \( x \in n \). By Lemma 4.5 \( \Theta_{B(x)}(u,v) \) is the largest congruence of \( B(x) \). Thus:

\[
\Theta_{B(x)}(u,v) = \xi_x(\Theta_{A(x)}(0,a_x)) \cup \xi_x(\Theta_{A(x)}(a_x,1))
\]

Therefore there exist \( t_0^n = u < t_1^n < \cdots < t_r^n + 1 = v \) in \( B(x) \) such that, for all \( 0 \leq i \leq r \):

either \((t_i^n, t_{i+1}^n) \in \xi_x(\Theta_{A(x)}(0,a_x))\)

or \((t_i^n, t_{i+1}^n) \in \xi_x(\Theta_{A(x)}(a_x,1))\)

Set \( b_x = t_1^n \). Put:

\[
X' = \{ x \in n \mid \Theta_{B(x)}(u,b_x) = \xi_x(\Theta_{A(x)}(0,a_x)) \}
\]

\[
X'' = \{ x \in n \mid \Theta_{B(x)}(u,b_x) = \xi_x(\Theta_{A(x)}(a_x,1)) \}.
\]

By symmetry we can assume that \( \text{card} X' \geq \text{card} X'' \) (we can replace the diagram \( \tilde{A} \) by its dual if required). As \( n = X' \cup X'' \) and \( \text{card} n = n = 2m - 1 \), \( \text{card} X' \geq m \).

Let \( x, y \in X' \) distinct, it follows from Lemma 4.5.2 that \( b_x \land b_y = u \). So we obtain a family of elements \((b_x)_{x \in X'}\) greater than \( u \) such that \( b_x \land b_y = u \) (resp., \( b_x \land b_y = u = 0 \)) for all \( x \neq y \) in \( X' \), a contradiction. \( \square \)

With a similar proof using Lemma 4.13 instead of Lemma 4.12 we obtain the following lemma.

**Lemma 4.23.** Let \( \mathcal{V} \) be a variety of lattices (resp., a variety of lattices with 0), let \( m \geq 2 \) an integer. Assume that for each simple lattice \( K \) of \( \mathcal{V} \), there do not exist \( b_0, b_1, \ldots, b_{m-1} > u \) in \( K \) such that \( b_i \land b_j = u \) (resp., \( b_0, b_1, \ldots, b_{m-1} > 0 \) such that \( b_i \land b_j = 0 \)), for all \( 0 \leq i < j \leq m - 1 \). Then \( \text{crit}(M_{2m-1}^0, \mathcal{V}) \leq \aleph_3 \).

**Theorem 4.24.** Let \( \mathcal{V} \) be either a finitely generated variety of lattices or a finitely generated variety of lattices with 0. If \( M_3 \in \mathcal{V} \) then:

\[
\text{crit}(M_\omega; \mathcal{V}) = \aleph_2;
\]

\[
\text{crit}(M_0^0, \mathcal{V}) = \aleph_2.
\]

Let \( \mathcal{V} \) be a finitely generated variety of bounded lattices. If \( M_3 \in \mathcal{V} \) then:

\[
\text{crit}(M_0^{0,1}, \mathcal{V}) = \aleph_2.
\]

**Proof.** Let \( \mathcal{V} \) be a finitely generated variety of lattices, let \( m \) be the maximal cardinality of a simple lattice of \( \mathcal{V} \). Thus the assumptions of Lemma 4.22 are satisfied, so a fortiori \( \text{crit}(M_{2m-1}^{0,1}, \mathcal{V}) \leq \aleph_2 \), and so \( \text{crit}(M_0^{0,1}, \mathcal{V}) \leq \aleph_2 \).

Denote by \( F_2 \) the two-element field. Let \( \mathcal{V} \) be a variety of lattices with 0 (resp., with 0 and 1), such that \( M_3 \in \mathcal{V} \). The variety \( M_\omega \) is locally finite, thus all finitely generated lattices of \( M_\omega \) are of finite length. Moreover all simple lattices of \( M_\omega \) have length at most two. Thus, by Theorem 3.11

\[
\text{crit}(M_\omega; \text{Var}_0(\text{Sub } F_2^0)) \geq \aleph_2 \text{ (resp., } \text{crit}(M_0^{0,1}; \text{Var}_0(\text{Sub } F_2^0)) \geq \aleph_2 \).
\]

Moreover \( \text{Sub } F_2^0 \cong M_3 \), so \( \text{crit}(M_\omega; \mathcal{V}) \geq \aleph_2 \). \( \square \)

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CRITICAL POINTS

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