Analysis of solutions to a model parabolic equation with strongly singular diffusion

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Abstract
We consider a singular parabolic equation of form

\[ u_t = u_{xx} + \frac{\alpha^2}{2} (\text{sgn} u_x)_x. \]

Solutions to this kind of equations exhibit competition between smoothing due to one-dimensional Laplace operator and tendency to create flat facets due to operator \((\text{sgn} u_x)_x\). We present results concerning analysis of qualitative behaviour and regularity of the solutions. Our main result states that locally (between moments when facets merge), the evolution is described by a system of free boundary problems for \(u\) in intervals between facets coupled with equations of evolution of facets. In particular, we provide a proper law governing evolution of endpoints of facets. This leads to additional regularity of solutions.

1 Introduction

We consider a formally parabolic equation of form

\[ u_t = u_{xx} + \frac{\alpha^2}{2} (\text{sgn} u_x)_x \]  \(1\)

for a function \(u = u(t,x)\) defined on a time-space rectangle \([0,T] \times I\), with \(I = \]0,1[\). Here, \(\alpha\) is a positive constant coefficient. As we are anyway interested in analysing a model case, we supply (1) with uniform Dirichlet boundary conditions

\[ u|_{[0,T] \times \partial I} = 0. \]  \(2\)
The differential operator on the right hand side of (1) may be written in the divergence form

\[ L(u_x)_x = \left( 1 + \frac{\alpha}{2 \| u_x \|} \right) u_x \]

and thus we see that (1) may be seen as a one-dimensional heat flow with diffusion coefficient singular whenever \( u_x = 0 \). This particular kind of singularities in diffusion coefficient are exhibited in models of crystal growth – graphs of solutions display stable flat parts (facets) corresponding to boundary of crystal. For this reason, various problems associated with operators of form \((\text{sgn } u_x)_x\) were considered by many authors, see e.g. [2, 3]. However, these authors usually considered generic singular \( L \), motivated by modeling crystals whose optimal shapes are polygons.

On the other hand, in our case \( L \) is strictly monotone. From the applications viewpoint, this corresponds to modeling growth of crystals (lumps) of metal – in this case the optimal shape still exhibits facets, but also has smoothly rounded edges (see e.g. [7]). From the viewpoint of pure mathematics, (1) displays competition between standard diffusion operator \( u_{xx} \) which tends to smoothen solutions and strong directional diffusion operator \((\text{sgn } u_x)_x\) that tends to create facets. The equation (1) was investigated by Mucha and Rybka, who collected several basic observations concerning the behaviour of its solutions in [5]. In particular, they obtained some regularity results in the language of Sobolev spaces and noticed that in any moment of time \( t > 0 \) the solutions do not allow isolated extremal points (which are immediately turned into facets of finite length) nor facets embedded in monotone graph (which are immediately destroyed). Furthermore, they seeked to analyse fine behaviour of endpoints of facets. For this purpose, they considered solutions to (1) for a certain class of initial data and provided a condition deciding whether a facet will grow or shrink. However, as we will see, their initial data were not regular (in the sense appropriate to (1)), as the one-sided second derivative on the facet endpoint was not equal to the “crystalline curvature” of facet (proportional to inverse of its length).

In this paper, we recollect basic regularity properties of solutions and characterise regular evolutions given by (1), obtaining following results
there exists a unique solution \( u \in C([0, T]; L^2(I)) \cap L^2(0, T; H^1_0(I)) \) to (1,2),

the solution becomes instantly regularized so that for any \( \delta > 0 \), \( u_t \in L^\infty(\delta, T; L^2(I)) \cap L^2(\delta, T; H^1_0(I)) \) (but typically \( u_t \notin C(\delta, T; L^2(I)) \)),

in almost every moment of time \( t > 0 \) there exists a finite subdivision of \( I \) into a finite number of intervals of flatness of solution \( F_k(t) \) of length bounded away from 0 and intervals of monotonicity \( I_k(t) \) with \( u \in H^3(I_k(t)) \),

locally the solution can be described by a system of free boundary problems for evolution of \( u \) in \( I_k(t) \) and intervals \( I_k(t) \) themselves, the solution to this system has additional regularity; in particular we provide the law of evolution of endpoints of \( I_k \), \( F_k \) and obtain \( I_k, F_k \in H^1(0, T) \).

2 Basic properties of solutions

Formally, the equation (1) may be viewed as a parabolic inclusion

\[
u_t + Lu \geq 0
\]

in the sense of \( H^{-1}(I) \) with \( Lu = L(u_x)_x \) supplied with uniform boundary condition (2), where \( L \) is treated as a maximal monotone graph

\[
L(p) = \begin{cases} 
  p - \frac{\alpha}{2} & \text{if } p < 0, \\
  -\frac{\alpha}{2} & \text{if } p = 0, \\
  p + \frac{\alpha}{2} & \text{if } p > 0.
\end{cases}
\]

The multifunction \( L \) is a subderivative of \( J(p) = \frac{1}{2}(p^2 + \alpha|p|) \). Thus, the operator \( L \) may be defined as a subderivative of a functional \( J \) defined on \( L^2(I) \) by

\[
J(u) = \frac{1}{2} \int_1 u_x^2 + \alpha|u_x|
\]

whenever \( u \in H^1_0(I) \) and \( J(u) = +\infty \) otherwise. Clearly, \( D(J) = H^1_0(I) \) and \( J \) is an equivalent norm on \( H^1_0(I) \). Furthermore, \( J \) is convex and lower semicontinuous (in particular, if \( (u_n) \subset D(J) \) converges to \( u \in L^2(I) \setminus D(J) \), then \( J(u_n) \to \infty \)). Let us now calculate formally the subderivative \( \partial J \).
Proposition 1. We have

$$D(\partial J) = \left\{ u \in H^1_0(I) \text{ such that there exists} \sigma \in H^1(I) \text{ satisfying } \sigma \in L(u_x) \text{ in } I \right\}$$

and

$$\partial J = \{-\sigma_x: \sigma \in H^1(I), \sigma \in L(u_x) \text{ in } I\}.$$ 

Proof. Let $$u \in D(J) = H^1_0(I)$$. Whenever $$w \in \partial J$$, we have

$$J(u + \varphi) \geq J(u) + (w, \varphi)$$

for any $$\varphi \in L^2(I)$$, with $$(\cdot, \cdot)$$ denoting the standard scalar product in $$L^2(I)$$. Clearly, we may assume $$\varphi \in H^1_0(I)$$. Then (7) becomes

$$\frac{1}{2} \int_I |u_x + \varphi_x|^2 + \alpha |u_x + \varphi_x| - \frac{1}{2} \int_I |u_x|^2 + \alpha |u_x| \geq (w, \varphi)$$

which we transform into form

$$\frac{1}{2} \int_I |\varphi_x|^2 + \frac{\alpha}{2} \int_{\{u_x = 0\}} |\varphi_x| + \int_{\{u_x \neq 0\}} u_x \varphi_x + \frac{\alpha}{2} \text{sgn } u_x \varphi_x \geq (w, \varphi).$$

The first term on the left hand side is positive and higher-order, and therefore (applying transformation $$\varphi \mapsto \lambda \varphi$$ with $$\lambda \in \mathbb{R}_+$$ and passing to the limit $$\lambda \to 0^+)$$ we may omit it in (9). Now, from the form of (9) we deduce that $$w \in \partial J$$ whenever there exists a selection $$\sigma \in u_x + \frac{\alpha}{2} \text{sgn } u_x$$ such that

$$\int_I \sigma \varphi_x \geq (w, \varphi).$$

Applying transformation $$\varphi \mapsto -\varphi$$ to (10), we observe that inequality in (10) may be replaced by equality. Therefore $$\sigma \in H^1(I)$$ and asserted representation of $$w$$ holds.

Remark. We have $$D(\partial J) \subset H^2(I) \subset C^1(I)$$. Indeed, the function $$u_{xx} = \sigma_x 1_{\{u_x \neq 0\}}$$ (defined independently of $$\sigma$$), is distributional second derivative of $$u$$ and belongs to $$L^2(I)$$. Furthermore, it is an easy observation that $$D(\partial J)$$ is dense in $$L^2(I)$$.

Equipped with the above observations concerning $$L$$, we may use semigroup theory to obtain basic existence and regularity result for the inclusion (4) (see [1], Theorem 2.1 in Chapter IV).
Proposition 2. Let $u_0 \in L^2(I)$. The problem with initial condition $u_0$ has a unique solution

$$u \in C([0,T];L^2(I)) \cap L^2(0,T;H^1_0(I))$$

which satisfies

$$u \in H^1(\delta,T;L^2(I)) \text{ for every } 0 < \delta < T,$$

$$u(t) \in D(\mathcal{L}) \text{ for a.e. } t \in [0,T].$$

Moreover, if $u_0 \in H^1(I)$ then

$$u \in H^1(0,T;L^2(I)) \cap L^\infty(0,T;H^1_0(I)).$$

The remark below Proposition 1 states that the regularity properties of $\mathcal{L} = \partial J$ are, in a sense, at least as good as those of the (one-dimensional) Laplace operator. However, the dissipation in $\mathcal{L}$ is essentially stronger than that of $\Delta$, so higher regularity could be expected. The following proposition (in a way, a corollary of Proposition 1) captures this additional regularity. Roughly, it states that $u \in D(\mathcal{L})$ if and only if $u \in H^2(I)$ and $I$ may be divided into a finite number of (non-degenerate) intervals where $u$ is constant and intervals where $u$ is monotone.

Proposition 3. Let $u \in D(\partial J)$. Then, there exists $n + 1$ open intervals $I_k = [a_k, b_k] \subset I$, $k = 0, \ldots, n$ and $n$ open intervals $F_k = [b_k - 1, a_k] \subset I$, $k = 1, \ldots, n$ with $0 = a_0 < b_0 < a_1 < b_1 < \ldots < a_k < b_k < \ldots < a_n < b_n = 1$ such that

(i) all of $I_k$ and $F_k$ are pairwise disjoint,

(ii) $I = \bigcup I_k \cup \bigcup F_k$,

(iii) $u_x = 0$ in each $F_k$, $F_k$ is a maximal open interval with this property and $u_x$ is convex or concave in some open interval $I$ compactly containing $F_k$.

(iv) $u$ is monotone in each $I_k$. 

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(v) $|F_k| \geq n \frac{a^2}{E(u)^2}$, where $E(u) = \inf_{\sigma_x \in \partial J(u)} \|\sigma_x\|_{L^2(I)}$.

On the other hand, if $u \in H^2(I)$ and a finite decomposition $\{I_k, F_k\}$ of $I$ satisfies conditions (i-iv), then $u \in D(\partial J)$ and (v) holds.

Proof. The existence of a (possibly infinite) decomposition of $I$ satisfying properties (i-iv) is an obvious consequence of continuity of $u_x$. Finiteness follows from property (v). To prove property (v), we observe that for any $F_k$ and any $\sigma_x \in \partial J(u)$ we have

$$\int_{F_k} \sigma_x^2 \geq \int_{F_k} \left( \frac{\sigma(a_k) - \sigma(b_{k-1})}{|F_k|} \right)^2 = \frac{\alpha^2}{|F_k|}. \quad (11)$$

Indeed, the inequality in (11) is a consequence of the fact that the affine function minimizes the functional $\int_a^b u_x^2$ in $H^1$ with prescribed boundary values. The equality follows from continuity of $\sigma$ and property (iii) of the decomposition, as we necessarily have

$$\lim_{x \to b_{k-1}} \sigma(x) = \lim_{p \to 0^\pm} L(p) = \pm \frac{\alpha}{2}, \quad \lim_{x \to a_k^+} \sigma(x) = \lim_{p \to 0^+} L(p) = \mp \frac{\alpha}{2}.$$}

Now, assume that $u \in H^2(I)$ and a finite decomposition $\{I_k, F_k\}$ of $I$ satisfying conditions (i-iv) exists. Then we define $\sigma(x)$ as $L(u_x(x))$ whenever $u_x(x) \neq 0$. Now, we consider the case that $u_x(x) = 0$. If $x \in F_k$ and $u$ is non-decreasing (resp. non-increasing) in $I_k$, we put $\sigma(x) = \frac{\alpha}{2}$ (resp. $\sigma(x) = -\frac{\alpha}{2}$).

We are left with the task of defining $\sigma$ in the intervals $F_k$. As we have already defined $\sigma$ in each $a_k$ and $b_k$, we extend it continuously to $F_k$ by appropriate affine functions. We observe that the function $\sigma$ we obtained belongs to $H^1(I)$.

Formally, we may write

$$u_{tt} = L(u_x)_{xx} = (L'(u_x)u_{tx})_x. \quad (12)$$

As $L' > 0$ in $D'(\mathbb{R})$, we could expect (12) to yield additional regularity of solutions to (4), but due to lack of proper definition of the term $L'(u_x)$ in (12) we need to proceed by approximation. Hence, let us denote by $J_\varepsilon$ smoothened versions of $J$ given by

$$J_\varepsilon(p) = \frac{1}{2}p^2 + \alpha(\varepsilon + p^2)^{\frac{1}{2}}$$
and by $L_\varepsilon$ its derivative

$$L_\varepsilon(p) = J'_\varepsilon(p) = p + \alpha \frac{p}{(\varepsilon + p^2)^\frac{3}{2}}.$$ 

In particular we have

$$1 \leq L'_\varepsilon(p) = 1 + \frac{\alpha \varepsilon}{(\varepsilon + p^2)^\frac{3}{2}} \leq 1 + \frac{\alpha}{\varepsilon \frac{3}{2}}.$$ 

Analysing the approximate problem

$$u^\varepsilon_t = L_\varepsilon(u^\varepsilon_x)u^\varepsilon_x \text{ in } I, \quad (13)$$

$$u^\varepsilon = 0 \text{ on } \partial I \quad (14)$$

we obtain the following result.

**Proposition 4.** Let $u$ be the unique solution to (4) with $u_0 \in L^2(I)$. Then, for any $\delta > 0$ we have

$$u_t \in L^2(\delta, T; H^1_0(I)) \cap L^\infty(\delta, T; L^2(I)).$$

*Proof.* Using either the semigroup theory [1] or fixed point methods [4] we obtain the existence of weak solutions to (13, 14) in $C([0, T]; L^2(I)) \cap L^2(0, T; H^1_0(I)) \cap H^1(\delta, T; L^2(I)) \cap L^2(\delta, T; H^2(I))$ for any $\delta > 0$. The time derivative of approximation $w^\varepsilon = u^\varepsilon_t$ satisfies formally

$$w^\varepsilon_t = (L'_\varepsilon(u^\varepsilon_x)u^\varepsilon_x)_x \text{ in } I, \quad (15)$$

$$w^\varepsilon = 0 \text{ on } \partial I. \quad (16)$$

Thus, as $L'_\varepsilon(u^\varepsilon_x)$ is essentially bounded in $[0, T] \times I$ for any given $\varepsilon > 0$, we may solve the problem (15, 16) in $[\delta, T]$ with (well-defined for almost all $\delta$) initial condition $w^\varepsilon_0 = u_t(\delta, \cdot) \in L^2(I)$. Using e.g. Proposition 4.1 in Chapter III of [6], we get unique solution $w^\varepsilon$ in the class $C([\delta, T]; L^2(I)) \cap L^2(\delta, T; H^2_0(I))$ which clearly coincides with $u^\varepsilon_t$. Testing the problem (15, 16) with the solution $w^\varepsilon$ we obtain following estimate independent of $\varepsilon$

$$\frac{1}{2} \text{ess sup}_{t \in [\delta, T]} \|w^\varepsilon(t, \cdot)\|_{L^2(I)}^2 + \|w^\varepsilon_x\|_{L^2(0,T;L^2(I))}^2 \leq \frac{1}{2} \|u_t(\delta, \cdot)\|_{L^2(I)}^2.$$ 

As $w^\varepsilon \to u_t$ in $\mathcal{D}'(I)$, we arrive at the assertion. \qed
Corollary 5. Propositions 2 and 4 imply that \( u_x \in C^1([\delta, T] \times \bar{I}) \).

Remark. If (12) was a regular parabolic equation, one would be able to obtain \( u_t \) at least in \( C([\delta, T]; L^2(I)) \). The reasoning in the proof of Proposition 4 does not lead to such regularity, as the required estimate on \( L^2(\delta, T; H^{-1}(I)) \) norm of \( w^\varepsilon \) does not hold.

3 Characterisation of regular evolutions

Proposition 4 implies that \( u_t(t, \cdot) \) is continuous on \( I \) in almost all moments of time \( t > \delta \). As it is known in some cases that facets persist and their speed is equal to the quotient of jump in \( L \) (in our case \( \alpha \)) and length of the facet, one may try to construct regular evolutions for (11) locally as solutions to a system of free boundary problems

\[
\begin{align*}
\text{(17)} &\quad u_t = u_{xx} \quad \text{in } I_k(t) \quad \text{for all } t \in ]0, T[ , k = 0, \ldots , n, \\
\text{(18)} &\quad u_x(t, a_k(t)) = u_x(t, b_{k-1}(t)) = 0 \quad \text{for all } t \in ]0, T[ , k = 1, \ldots , n, \\
\text{(19)} &\quad u(t, a_k(t)) = u(t, b_{k-1}(t)) = u(F_k(t)) \quad \text{for all } t \in ]0, T[ , k = 1, \ldots , n, \\
\text{(20)} &\quad u(t,0) = u(t,1) = 0 \quad \text{for all } t \in ]0, T[ , \\
\text{(21)} &\quad \frac{d}{dt}u(F_k(t)) = (-1)^k \frac{\alpha}{|F_k(t)|} \quad \text{for all } t \in ]0, T[ , \\
\text{(22)} &\quad u(0, \cdot) = u_0 \quad \text{in } \{0\} \times I_{0,k} \quad \text{for all } k = 0, \ldots , n.
\end{align*}
\]

Here, we assume that \( u_0 \in D(L) \) and therefore I can be decomposed into families \( I_{0,k} = ]a_k^0, b_k^0[ , k = 0, \ldots , n \) and \( F_k(0) = F_{0,k} = ]b_{k-1}^0, a_k^0[ \) of subintervals as in Proposition 3. We denote by \( I_k(t) = ]a_k(t), b_k(t)[ , F_k(t) = ]b_{k-1}(t), a_k(t)[ \) postulated evolutions of these intervals in time. We also introduce notation \( (a, b) = (a_0, \ldots , a_n, b_0, \ldots , b_n) \).

Proposition 6. Let \( u_0 \in D(L) \cap H^3 \left( \bigcup_{k=0, \ldots , n} I_{0,k} \right) \). Assuming that \( T \) is small enough, there exists a unique solution \((u, (a, b))\) to (17-22) satisfying

\[
\begin{align*}
\|u_{xxxx}(t, \cdot)\|_{L^2(I_k)} &\in L^2(0, T), \quad \|u_{xxx}(t, \cdot)\|_{L^2(I_k)} \in L^\infty(0, T), \\
(a, b) &\in H^1(0, T)^{2n+2}.
\end{align*}
\]
Proof. In order to solve (17-22), we consider differentiated equation for \( u \) for all \((t, x)\)
\[
  w_t = w_{xx}. 
\]
(23)

Differentiating boundary condition (18) we obtain
\[
  0 = u_{tt}(\cdot, a_k) + u_{xx}(\cdot, a_k)\dot{a}_k = w_t(\cdot, a_k) + w(\cdot, a_k)\dot{a}_k, \\
  0 = u_{tt}(\cdot, b_{k-1}) + u_{xx}(\cdot, b_{k-1})\dot{b}_{k-1} = w_t(\cdot, b_{k-1}) + w(\cdot, b_{k-1})\dot{b}_{k-1}
\]
(24)
in \([0, T]\) for each \( k = 1, \ldots, n \). On the other hand, (19), (21) and (18) imply
\[
  (-1)^k \frac{\alpha}{|x|} = u_t(\cdot, a_k) + u_x(\cdot, a_k)\dot{a}_k = w(\cdot, a_k), \\
  (-1)^k \frac{\alpha}{|x|} = u_t(\cdot, b_{k-1}) + u_x(\cdot, b_{k-1})\dot{b}_{k-1} = w(\cdot, b_{k-1})
\]
(25)
in \([0, T]\) for each \( k = 1, \ldots, n \). Differentiating (20) leads to
\[
  w(t, 0) = w(t, 1) = 0 \quad \text{for all } t \in [0, T]. 
\]
(26)

Finally, from (22) we get
\[
  w(0, \cdot) = u_{0,xx} \quad \text{in } \{0\} \times I_k(0) \quad \text{for all } k = 0, \ldots, n. 
\]
(27)

We prove the existence of solutions to (23-27) by means of Banach fixed point theorem. First, we rescale each part of the system to fixed interval \( I \), namely we introduce \( \tilde{w}^k \) defined by
\[
  \tilde{w}^k(t, x) = \begin{cases} 
    w(t, b_0(t)x + x \frac{\alpha}{\alpha_1(t) - b_0(t)}), & \text{if } k = 0 \\
    w(t, a_k(t) + (b_k(t) - a_k(t))x) - (-1)^k \frac{\alpha}{a_k(t) - b_{k-1}(t)} \\
    -x(-1)^{k+1} \frac{\alpha}{a_{k+1}(t) - b_k(t)}, & \text{if } k = 1, \ldots, n-1 \\
    w(t, a_n(t) + (b_n(t) - a_n(t))x) - (-1)^n \frac{\alpha}{a_n(t) - b_{n-1}(t)}, & \text{if } k = n
  \end{cases}
\]
(28)
for all \((t, x) \in [0, T] \times I, k = 0, \ldots, n \). Functions \( \tilde{w}^k \) are expected to satisfy equations
\[
  \tilde{w}_t^k = \frac{1}{(b_k - a_k)^2} \tilde{w}_{xx}^k + \frac{\dot{a}_k + x(b_k - a_k)}{b_k - a_k} \tilde{w}_x^k + f_{a,b}, 
\]
(29)
\[
  \tilde{w}^k(\cdot, 0) = \tilde{w}^k(\cdot, 1) = 0, 
\]
(30)
\[
  \tilde{w}_x^k(\cdot, 0) = \frac{\dot{a}_k}{b_k - a_k} \tilde{w}^k(\cdot, 0) - (-1)^{k+1} \frac{\alpha}{a_{k+1} - b_k}, 
\]
(31)
\[
  \tilde{w}_x^{k-1}(\cdot, 1) = \frac{b_{k-1}}{b_{k-1} - a_{k-1}} \tilde{w}_x^k(\cdot, 1) - (-1)^{k+1} \frac{\alpha}{a_{k+1} - b_k}, 
\]
(32)
Here, the number $m$ is chosen so that $b_{0,k} - a_{0,k} \geq 2m$ for $k = 0, \ldots, n$, $a_{0,k} - b_{0,k-1} \geq 2m$ for $k = 1, \ldots, n$. We also introduced notation

$$\|\tilde{w}\|_X = \max_{k=0,\ldots,n} \left( \int_0^T \left( \int_0^1 (\tilde{w}_x^k)^2 + \sup_{t \in [0,T]} \left( \int_0^1 (\tilde{w}_x^k)^2 \right) \right)^{\frac{1}{2}} \right),$$

$$\|(a,b)\|_Y = \max_{k=0,\ldots,n} \left( \|\tilde{u}_k\|_{L^2(0,T)} + \|\tilde{b}_k\|_{L^2(0,T)} \right)^{\frac{1}{2}}$$

for seminorms that introduce metrics on $X$ and $Y$. Further, we introduce operators $R : Y \to X$ solving the system (29,30,32) for $\tilde{w}^k$ given $a_k$ and $b_k$ and $S : X \to Y$ that solves the ODE system for $a_k, b_k$

$$\dot{a}_k = (-1)^k \frac{1}{\alpha} |a_k - b_{k-1}| |b_k - a_k| \tilde{w}_x^k (\cdot, 0),$$

$$\dot{b}_{k-1} = (-1)^k \frac{1}{\alpha} |a_k - b_{k-1}| |b_k - a_{k-1}| \tilde{w}_x^{k-1} (\cdot, 1)$$

(38)
that is a formal consequence of (30,31), given \( \tilde{w}^k \). We will now show that these operators are well defined and that the composed operator

\[
(\mathcal{R} \circ S, S \circ \mathcal{R}) : X \times Y \rightarrow X \times Y
\]

(39)
satisfies assumptions of Banach fixed point theorem provided that \( T \) is small enough.

First we consider well-posedness of the operator \( \mathcal{R} \). As \( a_k, b_k \in H^1(0, T) \), the problem of solving (29,30,32) is indeed locally well-posed in

\[
(C([0, T]; H^1_0(I)) \cap L^2(0, T; H^2(I)))^{n+1}
\]

and we have the following estimate on the solution \( \tilde{w}^k \)

\[
\frac{1}{2} \int_0^T \int_1 (\tilde{w}^k_{x,x})^2 + \text{ess sup}_{t \in [0,T]} \int_1 (\tilde{w}^k_x)^2 \\
\leq \int_0^T \int_1 (\dot{a}_k + x(\dot{b}_k - \dot{a}_k))^2 (\tilde{w}^k)^2 + \int_0^T \int_1 (b_k - a_k)^2 (f_{a,b})^2 + \int_1 \tilde{w}^k_x(0, \cdot)^2.
\]

(40)

Using inequalities

\[
\int_0^T \int_1 (\dot{a}_k + x(\dot{b}_k - \dot{a}_k))^2 (w^k_x)^2 \leq \|(a, b)\|^2_Y \text{ ess sup}_{t \in [0,T]} \int_1 (w^k_x)^2,
\]

\[
\int_0^T \int_1 (b_k - a_k)^2 (f^k)^2 \leq \int_0^T \int_1 (f_{a,b})^2 \leq \frac{11|a|^2}{m^2} \|(a, b)\|^2_Y,
\]

\[
\int_1 \tilde{w}^k_x(0, \cdot)^2 \leq 2\|u_{0,xxx}\|^2_{L^2(I_k)} + 2\frac{a^2}{m^2}
\]

and the definition of \( Y \) we obtain \( \tilde{w} \in X \).

Now, let \( \tilde{w} \in X \). Due to parabolic trace embedding

\[
C([0, T]; L^2(I)) \cap L^2(0, T; H^1(I)) \hookrightarrow L^4(0, T; L^2(\partial I))
\]

(41)

the problem of solving (38) is locally well-posed and we have inequalities

\[
\|(a, b)\|^2_Y \leq \frac{1}{\alpha^2} \max_{k=0,\ldots,n} \|\tilde{w}^k_x\|^2_{L^2(0,T;L^2(\partial I))} \\
\leq \frac{2}{\alpha^2} \max_{k=0,\ldots,n} \|\tilde{w}^k_x\|^2_{L^4(0,T;L^2(\partial I))} \leq \gamma \frac{2}{\alpha^2} \|\tilde{w}\|_X,
\]

(42)
\[ |a_k(T) - a_k(0)| \leq \int_0^T |\dot{a}_k| \leq \frac{1}{\alpha} \int_0^T |\dot{w}_x^k(\cdot, 0)| \leq \frac{2^{\frac{1}{\alpha}}}{a} \|\dot{w}_x^k(\cdot, 0)\|_{L^1(0, T)} \leq \gamma \frac{2^{\frac{1}{\alpha}}}{a} |\dot{w}|_X \] (43)

for each \( k = 0, \ldots, n \) and similarly with \( b_k \), where \( \gamma \) is the constant in the inequality

\[ \|f\|_{L^1(0,T;L^2(\partial I))} \leq \gamma \left( \int_0^T \int_1^T (|\dot{w}_x^k|^2 + \text{ess sup}_{t \in [0, T]} \int_1^T (|\dot{w}_x^k|^2)^{\frac{1}{2}} \right) \]

connected to the embedding \([H1]\). Thus, \( S \) is well defined provided that \( T \) is so small that

\[ \gamma^{\frac{T}{\alpha}} \left( 2 \max_{k=0,\ldots,n} \|u_{0,xx}\|_{L^2(I_k)} + \frac{15a^2}{m^2} \right) \leq \min(\frac{1}{2}, m). \]

We also see that under this assumption \((R \circ S, S \circ R)\) maps \( X \times Y \) into itself. We need yet to prove that this map is a contraction.

First we notice that if \( R\tilde{w} = (a, b) \) and \( R\tilde{v} = (c, d) \) then we have estimate

\[ \frac{1}{2} \int_0^T \int_1^T \left( \frac{\ddot{w}_x^k - \ddot{v}_x^k}{\dddot{w}_x^k - \dddot{v}_x^k} \right)^2 + \text{ess sup}_{t \in [0, T]} \int_1^T (|\dot{w}_x^k - \dot{v}_x^k|^2)^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} \int_0^T \int_1^T \left( \frac{1}{(b_k-a_k)^2} - \frac{1}{(d_k-c_k)^2} \right)^2 (\dddot{v}_x^k)^2 \]

\[ + \int_0^T \int_1^T \left( (\dddot{a}_k + x(\dddot{b}_k - \dddot{a}_k))\dot{w}_x^k - (\dddot{c}_k + x(\dddot{d}_k - \dddot{c}_k))\dot{v}_x^k \right)^2 \]

\[ + \int_0^T \int_1^T \left( (b_k - a_k)f_{a,b}^k - (d_k - c_k)f_{c,d}^k \right)^2 \] (44)

After a technical calculation involving several applications of triangle inequality and continuity of embedding \( H^1(0, T) \hookrightarrow C([0, T]) \) to (44) we obtain that \( R \) is Lipschitz continuous on \( Y \). Now, if \( S(a, b) = \tilde{w} \) and \( S(c, d) = \tilde{v} \), we can derive

\[ \max_{k=0,\ldots,n} \left( (\dddot{a}_k - \dddot{c}_k)^2 + (\dddot{b}_k - \dddot{d}_k)^2 \right) \]

\[ \leq \frac{8}{\alpha^2} \max_{k=0,\ldots,n} \left( (a_k - c_k)^2 + (b_k - d_k)^2 \right) \max_{k=0,\ldots,n} \|w_x^k\|_{L^2(\partial I)} \]

\[ + \frac{2}{\alpha^2} \|w_x^k - v_x^k\|_{L^2(\partial I)}^2 \] (45)

from (48). Invoking Gronwall’s inequality and

\[ \|f\|_{L^2(0,T;L^2(\partial I))} \leq T^\frac{1}{2} \|f\|_{L^1(0,T;L^2(\partial I))} \leq T^\frac{1}{2} |f|_X \] (46)
we obtain that $S$ is Lipschitz continuous with Lipschitz constant arbitrarily small for small $T$. Thus, choosing small enough $T$, we obtain existence of unique fixed point of (39) which clearly solves (29-32). Inverting the rescaling (28) yields a solution to the system of free boundary problems [23, 27]. Finally, we wish to return to the problem (17-22). We do it by solving the problem

$$u_{xx} = w \quad \text{in} \bigcup_{k=0,\ldots,n} I_k(t)$$

with boundary conditions (19,20) in each moment of time $t \in]0,T]$. □

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