abstract. Let $X$ be a Banach space. We study the circumstances under which there exists an uncountable set $A \subset X$ of unit vectors such that $\|x - y\| > 1$ for distinct $x, y \in A$. We prove that such a set exists if $X$ is quasi-reflexive and non-separable; if $X$ is additionally super-reflexive then one can have $\|x - y\| \geq 1 + \varepsilon$ for some $\varepsilon > 0$ that depends only on $X$. If $K$ is a non-metrizable compact, Hausdorff space, then the unit sphere of $X = C(K)$ also contains such a subset; if moreover $K$ is perfectly normal, then one can find such a set with cardinality equal to the density of $X$; this solves a problem left open by S. K. Mercourakis and G. Vassiliadis.

1. Introduction

The study of distances between unit vectors in Banach spaces has a long history that originates perhaps with the classical Riesz lemma ([30]), whose centennial is to be celebrated soon. There are two closely related problems in the geometry of Banach spaces that are often considered—given a Banach space $X$, what are the possible cardinalities of equilateral sets in $X$ (a set in a metric space is equilateral if all of its members are equidistant from each other) and what are the possible cardinalities of sets of unit vectors in $X$ that are apart by a certain distance (typically greater than one)?

One of the early results concerning equilateral sets is due to Petty who studied distances between unit vectors in finite-dimensional Banach spaces ([28]). It is therefore a natural question of whether every infinite-dimensional Banach space contains an infinite equilateral set. This is the case for large spaces of cardinality at least $2^c$ as observed by Terenzi ([36]) and one can always re-norm the space in such a way that the new norm has arbitrarily small Banach–Mazur distance from the old one and there exists an infinite equilateral set with respect to the new norm ([24, 33]). Hypotheses such as containing a copy of $c_0$ or having a uniformly smooth norm are also sufficient for the existence of an infinite equilateral set ([8, 24]). In general, however, this is not the case ([11, 35]).

Kottman ([20]) provided a powerful extension of the Riesz lemma. According to his result, the unit sphere of every infinite-dimensional Banach space contains an infinite set

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Date: April 21, 2015.

2010 Mathematics Subject Classification. 46B20, 46B04 (primary), and 46E15, 46B26 (secondary).

Key words and phrases. Kottman’s theorem, the Elton–Odell theorem, unit sphere, equilateral set, quasi-reflexive space, super-reflexive space, cardinal function.

The research of the second-named author was supported by the Polish Ministry of Science and Higher Education in the years 2013-14, under Project No. IP2012011072.
of vectors that are in distance greater than one from each other. (We shall call such sets \((1+)\)-separated.) In [4, pp. 7-8], one may find a very elegant and surprisingly short proof of Kottman’s theorem whose authorship is explained by Diestel as follows: ‘We were shown this proof by Bob Huff who blames Tom Starbird for its simplicity.’ Elton and Odell ([6]) gave a beautiful, Ramsey-theoretic proof of the following theorem: for each infinite-dimensional Banach space \(X\) there exists \(\varepsilon > 0\) and an infinite \((1+\varepsilon)\)-separated subset of the unit sphere of \(X\). (Given \(\delta > 0\), a set is \(\delta\)-separated if all of its members are in the distance at least \(\delta\) from each other.) Kryczka and Prus ([21]) proved that if \(X\) is non-reflexive then the unit sphere of \(X\) contains an infinite, \(\sqrt{4}\)-separated subset.

All the above-mentioned constructions of separated subsets of the unit sphere yield actually sequences (countable sets). Of course, in the case of separable Banach spaces this is the best one may expect as all discrete subsets of separable metric spaces are countable. What happens beyond the separable case? The aim of this paper is to answer this question at least partially.

Our starting point is the observation by Elton and Odell ([6, Remark on p. 109]) who noticed that for all \(\varepsilon > 0\) each \((1+\varepsilon)\)-separated subset of the unit sphere of \(c_0(\omega_1)\) is countable. For the reader’s convenience, we present the proof in Section 2. Curiously enough, the unit sphere of \(c_0(\omega_1)\) contains an uncountable \((1+)\)-separated subset (it was mentioned without a proof in [10, p. 12]). The proof is so simple that we include it here.

Towards a contradiction, assume that each \((1+)\)-separated set of unit vectors in \(c_0(\omega_1)\) is countable and take one, \(\{f_n : n \in \mathbb{N}\}\) say, that is maximal with respect to inclusion and whose each member assumes the value 1. The union \(D\) of the supports of all \(f_n\)’s \((n \in \mathbb{N})\) is countable, thus \(D = \{\alpha_1, \alpha_2, \ldots\}\) for some \(\alpha_k < \omega_1\) \((k \in \mathbb{N})\). Pick \(\alpha_0 \notin D\) and set \(f(\alpha_0) = 1\), \(f(\alpha_k) = -\frac{1}{k}\) \((k \in \mathbb{N})\) and \(f(\alpha) = 0\) otherwise. Then \(f \in c_0(\omega_1)\), \(\|f\| = 1\) and \(\|f - f_n\| > 1\) for each \(n\) which contradicts the maximality of \(\{f_n : n \in \mathbb{N}\}\).

We prove that for a substantial class of non-separable Banach spaces one can find an uncountable \((1+)\)-separated subset of the unit sphere.

**Theorem A.** Let \(X\) be a non-separable Banach space. Then

(i) if \(X\) is quasi-reflexive, the unit sphere of \(X\) contains an uncountable \((1+)\)-separated subset;

(ii) if \(X\) is super-reflexive, for each regular cardinal number \(\kappa\) that does not exceed the density of \(X\) there exist \(\varepsilon > 0\) and a \((1+\varepsilon)\)-separated subset of the unit sphere of \(X\) that has cardinality \(\kappa\).

In particular, the unit sphere of a super-reflexive space of density \(\omega_1\) contains an uncountable \((1+\varepsilon)\)-separated subset for some \(\varepsilon > 0\).

(iii) if \(X\) is a dual space of a Banach space of density bigger than continuum that admits a projectional resolution of the identity, the unit sphere of \(X\) contains an uncountable \((1+)\)-separated subset.

A Banach space \(X\) is called **quasi-reflexive** if \(X^{**}/X\) is finite-dimensional; in particular, reflexive spaces are quasi-reflexive, *a fortiori*. A Banach space \(X\) is **super-reflexive** if each
ultrapower of $X$ is reflexive. The first part of this result is Theorem 3.1, the second clause is Theorem 3.2, whereas the third one is Theorem 3.8. We borrowed the very idea of the proof of Theorem 3.1 from Starbird’s proof of Kottman’s theorem ([4, p. 7–8]) which is very finitary in its nature—it deals with norm estimates of certain linear combinations. We employ the transfinite induction in order to extract an uncountable $(1+)$-separated subset, however a good number of obstacles must be circumvented as, for instance, our ‘generalised linear combinations’ are no longer finite.

In Section 4, we specialise to the class of Banach spaces of continuous functions on locally compact Hausdorff spaces. Propositions 4.2, 4.3, and 4.4 had been obtained independently of Mercourakis and Vassiliadis ([25]) at the beginning of 2014—we take this opportunity to acknowledge their priority as these results overlap with [25, Theorem 2]. The proofs we provide are in most cases different and it is for the reader to decide whether they are more elementary or not.

We also solve a problem left open by Mercourakis and Vassiliadis ([25, p. 5]) who asked whether the unit sphere of $C(K)$, where $K$ is non-metrisable, contains an uncountable $(1+)$-separated subset. We prove actually a stronger result relating the cardinality of such sets to the density of $C(K)$, that is the minimal cardinality of a dense subset of $C(K)$.

**Theorem B.** Let $K$ be a non-metrisable, compact Hausdorff space. Then, the unit sphere of $C(K)$ contains an uncountable, $(1+)$-separated subset. Furthermore,

(i) if $K$ is not perfectly normal, then the unit sphere of $C(K)$ contains an uncountable 2-equilateral set;

(ii) if $K$ is perfectly normal, then the unit sphere of $C(K)$ contains a $(1+)$-separated subset of cardinality equal to the density of $C(K)$.

The proof of Theorem B is a conjunction of Proposition 4.7 with Theorem 4.10.

2. Preliminaries and auxiliary results

We work with real Banach spaces however most of the results can be easily generalised to the case of complex scalars. We adapt the von Neumann definition of an ordinal number and we identify cardinal numbers with initial ordinal numbers. By $c$, we denote the cardinality of continuum, whereas $\omega_1, \omega_2, \ldots$ are the first and, respectively, the second etc. uncountable ordinal number. We follow the fashion that is very alive in certain circles to keep the symbols $\omega, \omega_1, \omega_2, \ldots$ etc. for infinite cardinal numbers (thus, $\omega = \aleph_0, \omega_1 = \aleph_1, \ldots$). Given a cardinal number $\lambda$, we denote by $\text{cf} \lambda$ the cofinality of $\lambda$, that is, the smallest cardinal number $\kappa$ such that $\lambda$ can be written as a union of $\kappa$ many sets each of cardinality less than $\lambda$. A cardinal number $\lambda$ is regular if $\lambda = \text{cf} \lambda$ and singular otherwise. For the brevity of notation, we introduce the following cardinal functions.

Let $X$ be a Banach space. Denote by $S_X$ the unit sphere of $X$. We define

$$k(X) = \sup\{|A| : A \subseteq S_X, \|x - y\| > 1 \text{ for all distinct } x, y \in A\}$$

$$eo(X) = \sup\{|A| : A \subseteq S_X, \|x - y\| \geq 1 + \epsilon \text{ for some } \epsilon > 0 \text{ and all distinct } x, y \in A\}.$$
The attentive reader will note that the names of the just-defined cardinal functions refer to the aforementioned theorems of Kottman and Elton–Odell, respectively.

Given a Banach space \( X \), denote by \( d(X) \) the minimal cardinality of a dense subset of \( X \). Thus, we have the following immediate inequality:

\[
\text{eo}(X) \leq k(X) \leq d(X).
\]

Let \( \Gamma \) be a set. Then the vector space \( c_0(\Gamma) \) that consists of all scalar-valued functions \( f \) on \( \Gamma \) such that the set \{ \( \gamma \in \Gamma : |f(\gamma)| \geq \varepsilon \) \} is finite for all \( \varepsilon > 0 \) is a Banach space when endowed with the supremum norm. We regard \( c_0(\omega_1) \) as a paradigm example of a Banach space \( X \) for which the numbers \( k(X) \) and \( \text{eo}(X) \) differ. We have proved already in the introduction that \( k(c_0(\omega_1)) = \omega_1 \) and we mentioned that it is a result of Elton and Odell ([6, p. 109]) that \( \text{eo}(c_0(\omega_1)) = \omega \). Let us record this formally here; we present also a detailed proof.

**Proposition 2.1.** \( \omega = \text{eo}(c_0(\omega_1)) < k(c_0(\omega_1)) = \omega_1 \).

**Proof.** It is enough to prove that \( \text{eo}(c_0(\omega_1)) \) is countable.

Assume that for some \( \varepsilon > 0 \) there exists an uncountable \((1 + \varepsilon)\)-separated set \( A \subset S_{c_0(\omega_1)} \).

For each \( x \in A \) define

\[
F_x = \left\{ \alpha \in [0, \omega_1) : |x(\alpha)| > \frac{\varepsilon}{2} \right\}
\]

which is, of course, a finite set. According to the \( \Delta \)-system lemma ([22, Theorem 1.5]), there exists an uncountable subfamily \( B \) of \( A \) and a finite set \( \Delta \subset \omega_1 \) such that

\[
F_x \cap F_y = \Delta \quad (x, y \in B, x \neq y).
\]

Consider any two distinct members \( x \) and \( y \) of \( B \). If \( \alpha \in \omega_1 \setminus \Delta \), then at least one of the coordinates \( x(\alpha), y(\alpha) \) lies inside the interval \( [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \), hence \( |x(\alpha) - y(\alpha)| \leq 1 + \frac{\varepsilon}{2} \).

Therefore, the validity of \( \|x - y\| \geq 1 + \varepsilon \) may only be witnessed by coordinates from \( \Delta \). Define a ‘pattern function’ \( p : B \to \{-1, 1\}^\Delta \) by

\[
p(x)(\alpha) = \begin{cases} +1 & \text{if } x(\alpha) > \frac{\varepsilon}{2}, \\ -1 & \text{if } x(\alpha) < -\frac{\varepsilon}{2} \end{cases} \quad (x \in B, \alpha \in \Delta).
\]

There must exist two distinct vectors \( x, y \in B \) with \( p(x) = p(y) \). Consequently,

\[
|x(\alpha) - y(\alpha)| \leq 1 - \frac{\varepsilon}{2} \quad (\alpha \in \Delta).
\]

Therefore, \( \|x - y\| \leq 1 + \frac{\varepsilon}{2} \) and we arrive at a contradiction. \( \square \)

We proceed now to general results concerning \( k(X) \) and \( \text{eo}(X) \). The following lemma is a well-known observation; we provide a proof for the sake of completeness.

**Lemma 2.2.** Let \( X \) be a normed space and let \( x, y \in X \) be non-zero vectors such that \( \|x\|, \|y\| \leq 1 \) and \( \|x - y\| > 1 \). Then

\[
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq \|x - y\|.
\]
Proof. Assume with no loss of generality that \(|x| > |y|\) and consider the function given by \(g(t) = |x - ty|\) \((t \in \mathbb{R})\); it is easy to verify that this is a convex function. We have
\[
\left\| \frac{x}{|x|} - \frac{y}{|y|} \right\| = \frac{1}{|x|} \left| x - \frac{|x|}{|y|} y \right| \geq g\left(\frac{|x|}{|y|}\right).
\]
Since \(g(0) = |x| \leq 1\) and \(g(1) = |x - y| > 1\), the convexity of \(g\) implies that \(g(|x|/|y|)\) must be at least equal to \(g(1)\).

\[\square\]

**Proposition 2.3.** Suppose that \(X\) and \(Y\) are Banach spaces and \(Y\) is isometric to a quotient of \(X\). If for some \(\varepsilon > 0\) there exists a \((1 + \varepsilon)\)-separated subset \(\{y_i: i \in I\}\) of unit vectors in \(Y\), then for all \(\delta \in (0, \varepsilon)\) there exists a \((1 + \delta)\)-separated subset of the unit sphere of \(X\) with cardinality \(|I|\). Consequently, \(eo(Y) \leq eo(X)\).

**Proof.** Let \(\pi: X \to Y\) be the quotient map and choose \(\eta \in (0, \frac{\delta}{1 + \delta})\). For each \(i \in I\) let \(u_i \in X\) so that \(\pi(u_i) = y_i\) and \(|u_i| \leq 1 + \eta\). Set \(x_i = u_i/|u_i|\) \((i \in I)\). We have
\[
|x_i - x_j| = \left| \frac{u_i}{|u_i|} - \frac{u_j}{|u_j|} \right| \geq \frac{\left| \pi(u_i - u_j) \right|}{1 + \eta} \geq \frac{|y_i - y_j|}{1 + \eta} > 1 + \delta.
\]

\[\square\]

**Example 2.4.** Let \(JL\) be the Johnson–Lindenstrauss space \((\|x\|_{\infty})\). Denote by \(M\) the canonical copy of \(c_0\) in \(JL\). The quotient space \(JL/M\) is isometric to \(\ell_2(c)\) hence \(eo(JL) \geq c\). On the other hand, the cardinality of \(JL\) itself is \(c\), so \(eo(JL) = c\).

**Proposition 2.5.** Let \(X\) be a Banach space and let \(M \subseteq X\) be a reflexive subspace. Set \(Y = X/M\). Then \(k(Y) \leq k(X)\).

**Proof.** Since \(M\) is reflexive, for each \(y \in Y\) there is \(x \in X\) such that \(\pi(x) = y\) and \(|x| = |y|\). Given a \((1+)-\text{separated}\) family of unit vectors in \(Y\), for each member of this family choose an appropriate lifting and use the fact that \(\pi\) does not increase the norm.

\[\square\]

### 3. Reflexive spaces

#### 3.1. Quasi-reflexive spaces

We start this section with giving a generalisation of Starbird’s argument, mentioned in the Introduction, to the non-separable setting.

**Theorem 3.1.** Let \(X\) be a non-separable quasi-reflexive Banach space. Then \(k(X)\) is uncountable.

Before going into action, we need some preparations.

Let \(y^* \in X^*\) and \((x^*_n)_{n=1}^{\infty}\) be a bounded sequence in \(X^*\). Denote by \(\text{span}_{\mathbb{Q}}\{x^*_n: n \in \mathbb{N}\}\) the set of all (finite) linear combinations of \(x^*_n\)'s with rational coefficients. We say that \(y^*\) is a generalised combination of \(x^*_n\)'s \((n \in \mathbb{N})\) if for some enumeration \(\{z^*_1, z^*_2, \ldots\}\) of the set \(\text{span}_{\mathbb{Q}}\{x^*_n: n \in \mathbb{N}\}\cap B_X\) there exists a measure \(\mu \in \mathfrak{ba}(\mathcal{P}\mathbb{N})\) (here \(\mathfrak{ba}(\mathcal{P}\mathbb{N})\) denotes the family of all scalar-valued, bounded, finitely additive measures on the \(\sigma\)-algebra \(\mathcal{P}\mathbb{N}\) of all subsets of \(\mathbb{N}\); we identify this space with \(\ell^*_\infty\)) such that
\[
\langle y^*, x \rangle = \int_{\mathbb{N}} \langle z^*_n, x \rangle \mu(\text{d}n) \quad (x \in X).
\]

(3.1)
It is a well-known result from linear algebra that given finitely many linear functionals $y^*, x_1^*, \ldots, x_N^*$, so that $y^*$ vanishes whenever all $x_i^*$’s vanish, the functional $y^*$ must be a linear combination of $x_i^*$’s (see, e.g., [31, Lemma 3.9]). The following lemma yields an infinite version of this statement. We shall use the fact that for a quasi-reflexive space $X$ every total subspace $M$ of $X^*$ is norming (see, e.g., [29, 32]), that is, for some positive constant $c$ we have

$$\sup\{|\langle x^*, x \rangle|: x^* \in M \cap B_{X^*}\} \geq c\|x\| \quad (x \in X).$$

(In fact, this property characterises quasi-reflexive spaces, as was shown by Davis and Lindenstrauss in [3].)

**Lemma 3.2.** Let $X$ be a quasi-reflexive Banach space. Suppose that $y^* \in X^*$ and let $(x_n^*)_{n=1}^\infty$ be a bounded sequence in $X^*$ such that

$$\bigcap_{n=1}^\infty \ker(x_n^*) \subseteq \ker(y^*).$$

Then $y^*$ is a generalised combination of $x_n^*$’s ($n \in \mathbb{N}$).

**Proof.** Set $N = \bigcap_{n=1}^\infty \ker(x_n^*)$ and let $z_1^*, z_2^*, \ldots$ be any enumeration of all vectors from $\text{span}_Q\{x_n^*\}_{n=1}^\infty \cap B_{X^*}$. Define a linear map $\Lambda: X \to \ell_\infty$ by

$$\Lambda(x) = (\langle z_n^*, x \rangle)_{n=1}^\infty \quad (x \in X)$$

and observe that, by our assumption, we have $\langle y^*, x \rangle = \langle y^*, x' \rangle$ whenever $\Lambda(x) = \Lambda(x')$. This makes it possible to define a linear map $f: \ell_\infty \supset \Lambda(X) \to \mathbb{R}$ by $\langle f, \Lambda(x) \rangle = \langle y^*, x \rangle$.

Since quotients of a quasi-reflexive space are quasi-reflexive too ([21 Corollary 4.2]), the space $X/N$ is quasi-reflexive. For any functional $x^* \in X^*$ that vanishes on $N$ define $\hat{x}^* \in (X/N)^*$ by $\langle \hat{x}^*, x + N \rangle = \langle x^*, x \rangle$; of course, $\|x^*\| = \|\hat{x}^*\|$ (cf. [31 Theorem 4.9(b)]). As the set $\{\hat{x}_n^*\}_{n=1}^\infty$ is total in $(X/N)^*$, the subspace $\text{span}_Q\{\hat{x}_n^*\}_{n=1}^\infty$ is norming, whence so is $\text{span}_Q\{\hat{x}_n^*\}_{n=1}^\infty$. So, let $c > 0$ be so that

$$\sup_{n \in \mathbb{N}}|\langle \hat{z}_n^*, x + N \rangle| \geq c\|x + N\|_{X/N} \quad (x \in X).$$

For any $x \in X$ pick $y_x \in N$ for which $\|x + y_x\| \leq 2\|x + N\|_{X/N}$ and note that

$$\|\Lambda(x)\|_{\ell_\infty} = \sup_{n \in \mathbb{N}}|\langle z_n^*, x \rangle|$$

$$\geq \sup_{n \in \mathbb{N}}|\langle \hat{z}_n^*, x + N \rangle|$$

$$\geq c\|x + N\|_{X/N}$$

$$\geq \frac{1}{2} \cdot c\|x + y_x\|.$$

Therefore,

$$|\langle f, \Lambda(x) \rangle| = |\langle y^*, x + y_x \rangle| \leq \|y^*\| \cdot \|x + y_x\| \leq \frac{2}{c}\|y^*\| \cdot \|\Lambda(x)\|_{\ell_\infty}.$$
which shows that $f$ is a continuous functional. So, let $F \in \ell^*_\infty \cong \text{ba}(\mathcal{P}\mathbb{N})$ be any norm-preserving extension of $f$ and let $\nu \in \text{ba}(\mathcal{P}\mathbb{N})$ be the measure corresponding to $F$. Then, for every $x \in X$ we have
\[
\langle y^*, x \rangle = \langle F, \Lambda(x) \rangle = \langle F, (z_n^*)_{n=1}^\infty \rangle = \int_N \langle z_n^*, x \rangle \nu(dn),
\]
which completes the proof. \hfill \Box

Remark 3.3. Note that the above proof gives the upper estimate
\[
(3.2) \quad \|\mu\| \leq \frac{2}{c} \|y^*\|
\]
for the variation norm of the measure $\mu$ representing $y^*$ via formula (3.1). Here, $c$ depends only on the given sequence $(x_n^*)_{n=1}^\infty$. Note also that Lemma 3.2, and especially inequality (3.2), is related to a result of Ostrovskii [27] saying that for any quasi-reflexive space $X$ and any absolutely convex set $A \subset X^*$ (in particular, any subspace) we have
\[
A_{w^*} = \bigcup_{n=1}^\infty A \cap nB_{X^*_{w^*}},
\]
the set on the right-hand side being called the weak* derived set of $A$. Inequality (3.2) gives a uniform bound on the norms of combinations of $z_n^*$'s that approximate any given functional being a generalised combination of $z_n^*$'s. (For our purposes it is more natural to use generalised combinations rather than the weak* closure.)

Proof of Theorem 3.1. According to a result by Civin and Yood [2, Theorem 4.6]), $X$ contains a non-separable reflexive subspace, whence we may suppose that $X$ itself is reflexive in which case we have $d(X) = d_{w^*}(X^*)$, where $d_{w^*}(X^*)$ is the minimal cardinality of a weak*-dense subset of $X^*$, thereby $X^*$ is non-separable in the weak*-topology.

We shall construct a sequence $(x_\alpha)_{\alpha<\omega_1} \subset X$ of unit vectors such that $\|x_\alpha - x_\beta\| > 1$ whenever $0 \leq \beta < \alpha < \omega_1$. Choose any unit vector $x_0$ in $X$ and pick $x_0^* \in S_{X^*}$ with $\langle x_0^*, x_0 \rangle = 1$. Now, fix any $\alpha < \omega_1$ and assume that we have already chosen vectors $(x_\beta)_{0 \leq \beta < \alpha} \subset S_X$ and functionals $(x_\beta^*)_{0 \leq \beta < \alpha} \subset S_{X^*}$ such that
\begin{enumerate}
  \item $\langle x_\beta^*, x_\beta \rangle = 1$ for every $\beta < \alpha$;
  \item $\|x_\beta - x_\gamma\| > 1$ for all $0 \leq \beta < \gamma < \alpha$;
  \item $\bigcap_{\gamma \neq \beta} \ker(x_\gamma^*) \not\subseteq \ker(x_\beta^*)$ for every $\beta < \alpha$.
\end{enumerate}

We claim that there exists a vector $y \in X$ satisfying $\langle x_\beta^*, y \rangle < 0$ for each $\beta < \alpha$. Indeed, by (iii), for every $\beta < \alpha$ we may find a unit vector
\[
z_\beta \in \bigcap_{\gamma \neq \beta} \ker(x_\gamma^*)
\]
such that $\langle x_\beta^*, z_\beta \rangle < 0$. Take any summable sequence $(c_\beta)_{\beta<\alpha}$ of positive numbers. Then, $y = \sum_{\beta<\alpha} c_\beta z_\beta$ has the desired property.
Set \( N = \bigcap_{\beta < \alpha} \ker(x^*_\beta) \). Since \( (\perp N)^\perp = N^w \) and \((X^*, w^*)\) is non-separable, there exists a non-zero vector \( x \in N \). Let \( c > 0 \) be the constant occurring in inequality (3.2) corresponding to (any enumeration of) the sequence \((x^*_\beta)_{\beta < \alpha}\). Choose \( K > 0 \) so large that

\[
\|y + Kx\| > \frac{3}{c}\|y\|
\]

and define

\[
x_\alpha = \frac{y + Kx}{\|y + Kx\|}.
\]

Let also \( x^*_\alpha \) be any norm-one functional satisfying \( \langle x^*_\alpha, x_\alpha \rangle = 1 \).

In order to verify condition (ii) for \( \alpha + 1 \) in the place of \( \alpha \), observe that for each \( \beta < \alpha \)

\[
\|x_\alpha - x_\beta\| \geq \|\langle x^*_\beta, x_\alpha - x_\beta \rangle\| = 1 - \frac{\langle x^*_\beta, y \rangle}{\|y + Kx\|} > 1,
\]

as \( \langle x^*_\beta, y \rangle \) is negative.

Regarding condition (iii), we start by showing that \( x^*_\alpha \) is not a generalised combination of \( x^*_\beta \)'s \((\beta < \alpha)\). If it were, then for some enumeration \( z^*_1, z^*_2, \ldots \) of \( \text{span}_Q \{x^*_\beta\}_{\beta < \alpha} \cap B_{X^*} \) and some measure \( \mu \in ba(PN) \) we would have

\[
\langle x^*_\alpha, z \rangle = \int_N \langle z^*_n, z \rangle \mu(dn) \quad (z \in X).
\]

Moreover, in view of Remark 3.3 we may assume that \( \|\mu\| \leq 2/c \). Set \( z = y + Kx \). For each \( \varepsilon > 0 \) we may then find \( N \in \mathbb{N} \) and scalars \( a_1, \ldots, a_N \) so that

\[
\sum_{j=1}^N |a_j| \leq \|\mu\| \leq \frac{2}{c} \quad \text{and} \quad \left| \langle x^*_\alpha, y + Kx \rangle - \sum_{j=1}^N a_j \langle z^*_j, y + Kx \rangle \right| < \varepsilon.
\]

However, the left-hand side of the latter inequality equals to

\[
\left| \|y + Kx\| - \sum_{j=1}^N a_j \langle z^*_j, y \rangle \right| \geq \|y + Kx\| - \|y\| \sum_{j=1}^N |a_j| > \left( \frac{3}{c} - \|\mu\| \right) \|y\| \geq \frac{\|y\|}{c},
\]

hence we arrive at a contradiction in the case where \( \varepsilon < \|y\|/c \). Now, Lemma 3.2 implies that

\[
\bigcap_{\beta < \alpha} \ker(x^*_\beta) \nsubseteq \ker(x^*_\alpha).
\]

Finally, suppose that for some \( \beta_0 < \alpha \) we have

\[
\ker(x^*_\alpha) \cap \bigcap_{\beta < \alpha} \ker(x^*_\beta) \subseteq \ker(x^*_{\beta_0}).
\]
Appealing once again to Lemma 3.2 we infer that for some enumeration \( \{ w^*_n, w^*_2, \ldots \} \) of the set \( \text{span}_q \{ x^*_n \cup \{ x^*_\beta : \beta < \alpha, \beta \neq \beta_0 \} \} \) and for some measure \( \nu \in \text{ba}(\mathcal{P}\mathcal{N}) \) we have

\[
\langle x^*_n, z \rangle = \int \langle w^*_n, z \rangle \nu(\text{d}n) \quad (z \in X).
\]

For each \( n \in \mathbb{N} \) let \( \theta_n \) be the coefficient of \( x^*_n \) in the linear combination \( w^*_n \). Putting \( z = x \) in the above equality, and using the fact that \( x \) belongs to the kernels of all \( x^*_\beta \) (\( \beta < \alpha \)), we obtain \( \int \theta_n \nu(\text{d}n) = 0 \). On the other hand, for every \( z \in X \) we have

\[
\langle x^*_n, z \rangle = \langle x^*_n, z \rangle \int \theta_n \nu(\text{d}n) + \int \langle \tilde{w}^*_n, z \rangle \nu(\text{d}n),
\]

where \( \tilde{w}^*_n \) is the linear combination resulting from \( w^*_n \) by truncating \( \theta_n x^*_n \). Since the first integral vanishes, we see that the kernel of \( x^*_n \) contains \( \bigcap_{\beta < \alpha, \beta \neq \beta_0} \ker(x^*_\beta) \) which is false due to the induction hypothesis. Therefore, condition (iii) for \( \alpha + 1 \) instead of \( \alpha \) has been fully verified and the proof is complete.

\[ \square \]

3.2. Super-reflexive spaces. Now, we shall show that Theorem 3.1 may be strengthened for super-reflexive spaces. We require a piece of terminology before explaining the result.

Let \( \Lambda \) be a set and let \( (X_\alpha)_{\alpha \in \Lambda} \) be a family of Banach spaces. Given \( p \in [1, \infty) \), the \( \ell_p \)-sum of \( (X_\alpha)_{\alpha \in \Lambda} \) is the Banach space \( Z = (\bigoplus_{\alpha \in \Lambda} X_\alpha)_{\ell_p} \) that consists of all tuples \( x = (x_\alpha)_{\alpha \in \Lambda} \) such that \( x_\alpha \in X_\alpha \) (\( \alpha \in \Lambda \)) and \( \|x\| = (\sum_{\alpha < \lambda} \|x_\alpha\|^p)^{1/p} < \infty \). Given \( x = (x_\alpha)_{\alpha \in \Lambda} \in Z \), the support of \( x \) is the set \( \text{supp} \, x = \{ \alpha \in \Lambda : x_\alpha \neq 0 \} \).

Let \( X \) be a non-separable Banach space and set \( \lambda = d(X) \). A family \( (P_\alpha)_{\omega \leq \alpha < \lambda} \) of norm-one linear projections on \( X \) is called a projectional resolution of the identity in \( X \) (a PRI, for short) if

(i) \( P_\omega = 0 \) and \( P_\lambda = I_X \), the identity map on \( X \),
(ii) \( d(P_\alpha(X)) \leq |\alpha| \) for each \( \alpha < \lambda \),
(iii) \( P_\omega P_\beta = P_\beta P_\alpha = P_{\min(\alpha, \beta)} \) for all \( \omega \leq \alpha < \beta \leq \lambda \),
(iv) for every \( x \in X \) the map \( \alpha \mapsto P_\alpha x \) is continuous in the order-norm topology.

Condition (iii) implies that the ranges of \( P_\alpha \) (\( \omega \leq \alpha < \lambda \)) are well-ordered. When talking about any PRI, we shall always assume without loss of generality that for different \( \alpha, \beta \) the ranges of \( P_\alpha \) and \( P_\beta \) are different. We want also to emphasise that it follows from condition (iv) that if \( W \subset X \) is a subspace with \( d(W) < \lambda \) then for some \( \alpha < \lambda \) we have \( W \subseteq P_\alpha(X) \). We refer to [13, Chapter 13.2] for more information concerning projectional resolutions of the identity.

Let \( X \) be a non-separable, super-reflexive space. By a result of Lindenstrauss [23], there exists a PRI for \( X \), say \( (P_\alpha)_{\omega \leq \alpha < \lambda} \). Our idea relies on the observation due to Benyamini and Starbird [11, p. 139]) who, building on work of James [15], observed that for any \( \varepsilon > 0 \) there exists \( p \in (1, \infty) \) such that the operator
Moreover, if (3.4)\( e \prec (3.4) \)
c has norm at most \( 2 + \varepsilon \). This follows from James’ theorem (see [15] Theorem 4]) which says that for each super-reflexive Banach space \( X \) and any constants \( 0 < c < 1/(2K) \), \( C > 1 \) there are exponents \( 1 < q < p < \infty \) such that for every normalised basic sequence \( (e_n)_{n=1}^{\infty} \subset X \) with basis constant \( K \), and any scalars \( (a_n)_{n=1}^{N} \), we have

\[
(3.4) \quad c \cdot \left( \sum_{n=1}^{N} |a_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{N} a_n e_n \right\| \leq C \cdot \left( \sum_{n=1}^{N} |a_n|^q \right)^{1/q}.
\]

Moreover, if \( (e_n)_{n=1}^{\infty} \) is monotone, then \( c \) may be taken to be arbitrarily close to \( \frac{1}{2} \). Once \( c < \frac{1}{2} \) is chosen (together with the corresponding value of \( p \)), the norm of \( T \) equals at most \( \frac{1}{2} \); moreover, \( c \) may be chosen as close to \( \frac{1}{2} \) as we wish, since any sequence of vectors picked from different blocks of the form \( (P_{\beta+1} - P_\beta)(X) \) forms a monotone basic sequence.

**Lemma 3.4.** Let \( X \) be super-reflexive space such that \( d(X) \) has uncountable cofinality and let \( T \) be given by (3.3). Then there exists a positive constant \( \gamma_{\min} \) such that for each closed subspace \( Y \subseteq X \) with \( d(X/Y) < d(X) \) we have \( \left\| T|_Y \right\| \geq \gamma_{\min} \).

**Proof.** Assume, on the contrary, that there is a sequence of closed subspaces \( Y_n \subset X \) \( (n \in \mathbb{N}) \) so that \( d(X/Y_n) < d(X) \) and \( \left\| T|_{Y_n} \right\| \to 0 \) as \( n \to \infty \). Since the cofinality of \( d(X) \) is uncountable, the intersection of all \( Y_n \)'s is a closed subspace \( Y \) of \( X \) such that still we have \( d(X/Y) < d(X) \) (so \( Y \) is non-zero), and also \( T|_Y = 0 \). This is, of course, impossible as \( T \) is one-to-one. \( \square \)

**Theorem 3.5.** For each super-reflexive Banach space \( X \) we have \( eo(X) = d(X) \).

**Proof.** The separable case follows from the Elton–Odell theorem, so let us suppose that \( X \) is non-separable. Since every singular cardinal number is a limit of an increasing transfinite sequence of regular cardinals, without loss of generality we may suppose that \( \lambda = d(X) \) itself is regular. In particular, it has uncountable cofinality.

Fix any positive number \( c < \frac{1}{2} \) and let \( p \in (1, \infty) \) be so that the first inequality in (3.4) holds true. Keeping the above notation, let us say that the operator \( T \) is **bounded by a pair** \( (\gamma, \delta) \) if \( \|T\| \leq \delta \) and \( \gamma \leq \|T|_Y \| \) for every subspace \( Y \subseteq X \) with \( d(X/Y) < \lambda \).

First, we claim that our assertion will follow whenever we show that there exists a pair \( (\gamma, \delta) \) with \( \gamma/\delta > 2^{-1/p} \) and such that \( T \) is bounded by \( (\gamma, \delta) \). To see this choose a unit vector \( x_0 \in X \) such that \( \|Tx_0\| \geq \gamma \). Let \( \beta < \lambda \) and suppose that we have already chosen unit vectors \( x_\alpha \in X \) \( (\alpha < \beta) \) such that \( \|Tx_\alpha\| \geq \gamma \) and the supports of \( Tx_\alpha \) \( (\alpha < \beta) \) are pairwise disjoint. We are now in a position to choose a unit vector \( x_\beta \in X \) such that \( \|Tx_\beta\| \geq \gamma \) and the support of \( Tx_\beta \) is disjoint from the supports of \( Tx_\alpha \) for all \( \alpha < \beta \).

Indeed, if this were impossible, we could build a transfinite, linearly independent sequence of unit vectors \( (w_\xi)_{\xi < \lambda} \) in \( X \) with the following properties:
(i) \(\|Tw_\xi\| \geq \gamma (\xi < \lambda)\);
(ii) for each \(\xi < \lambda\) there exists \(\eta < \beta\) such that \(\text{supp} Tw_\xi \cap \text{supp} Tx_\eta \neq \emptyset\).

The possibility of choosing a transfinite sequence \((w_\xi)_{\xi<\lambda}\) that satisfies (i) follows from the fact that \(T\) is bounded by \((\gamma, \delta)\).

For each \(\omega \leq \alpha < \lambda\) the space \((P_{\alpha+1} - P_{\alpha})(X)\), as a subspace of \(X\), is reflexive and has density at most equal to the cardinality of \(\alpha\), hence so does its dual space. Therefore, as \(\lambda \geq |\beta|\), there must exist:

(i) \(\alpha_0 \in \bigcup_{\xi<\beta} \text{supp} T x_\xi\),
(ii) a norm-one functional \(x^* \in (P_{\alpha+1} - P_{\alpha})(X)^*\),
(iii) a subsequence \((w_{\xi_n})_{n=1}^\infty\) of \((w_\xi)_{\xi<\lambda}\), and
(iv) a positive constant \(c\)

such that \(\langle x^*, Tw_{\xi_n} \rangle \geq c\) for all \(n \in \mathbb{N}\). Passing to a further subsequence and replacing \(x^*\) with its negative if necessary, we may suppose that for all \(n \in \mathbb{N}\) we have \(\langle x^*, Tw_{\xi_n} \rangle \geq c\).

Note that
\[
\|Tw_{\xi_1} + \ldots + Tw_{\xi_n}\| \geq \langle x^*, Tw_{\xi_1} + \ldots + Tw_{\xi_n} \rangle \geq nc \quad (n \in \mathbb{N}).
\]
By James’ inequality (3.14),
\[
\|w_{\xi_1} + \ldots + w_{\xi_n}\| \leq C \cdot n^{1/q} \quad (n \in \mathbb{N})
\]
for some \(C > 0\) and \(q \in (1, \infty)\) independent of \(n\). Setting
\[
y_n = \frac{w_{\xi_1} + \ldots + w_{\xi_n}}{\|w_{\xi_1} + \ldots + w_{\xi_n}\|} \quad (n \in \mathbb{N}),
\]
we conclude that \(\|Ty_n\| \geq \frac{cn}{Cn^{1/q}} \to \infty\) as \(n \to \infty\); a contradiction.

Now, for distinct \(x_\alpha, x_\beta\) \((\alpha, \beta < \lambda)\) we have
\[
\|x_\alpha - x_\beta\| \geq \frac{1}{\delta} \|Tx_\alpha - Tx_\beta\| = \frac{1}{\delta} (\|Tx_\alpha\|^p + \|Tx_\beta\|^p)^{1/p} \geq \frac{\gamma}{\delta} \cdot 2^{1/p} > 1.
\]
This shows that our assertion is true whenever \(T\) is bounded by a pair \((\gamma, \delta)\) satisfying \(\gamma/\delta > 2^{-1/p}\).

Set \(\delta_0 = \frac{1}{c}\). Since \(\|T\| \leq \delta_0\), \(T\) is bounded by a pair \((\gamma_0, \delta_0)\), where
\[
\gamma_0 = \sup \{ \gamma : \|T|Y\| \geq \gamma \text{ for every subspace } Y \subseteq X \text{ with } d(X/Y) < \lambda \}.
\]
If \(\gamma_0 > 2^{-1/p}\delta_0\), then we are done by the first part of the proof. Assume the opposite, that is, the supremum at the right-hand side is at most \(2^{-1/p}\delta_0\). Fix any sequence \((\eta_n)_{n=0}^{\infty}\) of real numbers strictly larger than 1 and such that \(\prod_{n=0}^{\infty} \eta_n\) converges. Since
\[
\gamma_0 < 2^{-1/p} \eta_0 \delta_0 =: \delta_1,
\]
there exists a subspace \(X_1 \subset X\) with \(d(X/X_1) < \lambda\) and such that \(\|T|X_1\| \leq \delta_1\). Now, define \(\gamma_1\) analogously as \(\gamma_0\) replacing \(X\) by \(X_1\). Applying again the first part of the proof.
to \( T|_{X_1} \), we know that the proof is accomplished whenever \( \gamma_1 > 2^{-1/p} \delta_1 \). If this is not the case, we continue our process. At the \( n \)-th step we have \( \delta_n = 2^{-1/p} \eta_{n-1} \delta_{n-1} \), whence

\[
\delta_n = 2^{-n/p} \delta_0 \cdot \prod_{j=0}^{n-1} \eta_j \xrightarrow{n \to \infty} 0.
\]

On the other hand, Lemma 3.4 gives \( \gamma_n \geq \gamma_{\text{min}} \) for each \( n \in \mathbb{N} \). Hence, if our process did not terminate, then at some point we would have arrived at the absurdity \( \gamma_n > \delta_n \), which completes the proof. \( \square \)

**Remark 3.6.** One cannot extend the above technique to the class of all reflexive spaces. Indeed, Hájek constructed a non-separable Tsirelson-like reflexive space \( X \) whose no non-separable subspace admits an injective, bounded linear map into \( \ell_p(\lambda) \) for some uncountable cardinal number \( \lambda \). Nonetheless, it is easily verifiable that \( d(X) = eo(X) = \omega_1 \).

**Remark 3.7.** The supremum appearing in the definition of \( eo(X) \) need not be attained, even in the case where \( X \) is reflexive. Indeed, let \( (p_n)_{n=1}^{\infty} \) be a sequence of real numbers with \( p_1 > 1 \) that increase to \( \infty \) as \( n \to \infty \). Consider the Banach space

\[
X = \left( \bigoplus_{n \in \mathbb{N}} \ell_{p_n}(\omega_n) \right)_{\ell_2}.
\]

Then \( X \) is reflexive, as an \( \ell_2 \)-sum of reflexive spaces, and \( d(X) = \omega_\omega \). For each \( p \in [1, \infty) \), infinite subsets of the unit sphere of \( \ell_p \) are separated by at most \( 2^{1/p} \) ([38, Theorem 16.9]) and since \( 2^{1/p} \to 1 \) as \( p \to \infty \), we conclude that the unit sphere of \( X \) does not contain \((1 + \varepsilon)\)-separated subsets of cardinality \( \omega_\omega (\varepsilon > 0) \).

The space \( X \) is not super-reflexive, though. It would interesting to find out what happens in the super-reflexive case.

### 3.3. Dual Banach spaces of large density that have a PRI.

The last result of this chapter is devoted to Banach spaces that are duals of spaces with density bigger than continuum that have a PRI. This class of spaces is quite rich as it contains all duals of weakly compactly generated Banach spaces of large density. The following result shows, in a sense, how far one can go with the original argument of Kottman.

**Theorem 3.8.** Let \( X \) be a Banach space which admits a PRI and satisfies \( d(X) > c \). Then \( k(X^*) \) is uncountable.

**Proof.** Let \( \lambda = d(X) \); we may assume that \( \lambda = c^+ \), the successor of the continuum. If \( (P_\alpha)_{\omega \leq \alpha < \lambda} \) is a PRI for \( X \), then [17, Lemma 3] yields that \( (P_\alpha^*)_{\omega \leq \alpha < \lambda} \) forms a PRI for \( X^* \) and, of course, all the projections \( (P_\alpha^*)_{\omega \leq \alpha < \lambda} \) are \( w^*-to-w^* \) continuous.

For each \( \alpha \in [\omega, \lambda) \) we define recursively \( x_\alpha \in X \) and \( x_\alpha^* \in X^* \) as follows. First, take a unit vector \( x_\omega \in X \) and a norm-one functional \( x_\omega^* \in (P_{\omega+1}^* - P_\omega^*)(X^*) \) such that \( \langle x_\omega, x_\omega^* \rangle = 1 \). Now, given any \( \alpha \in (\omega, \lambda) \) assume that we have already chosen unit vectors \( (x_\beta)_{\beta < \alpha} \) and norm-one functionals \( (x_\beta^*)_{\beta < \alpha} \) such that \( x_\beta \in (P_{\xi_{\beta+1}}^* - P_{\xi_\beta})(X^*) \) and \( \langle x_\beta, x_\beta^* \rangle = 1 \) for each \( \omega \leq \beta < \alpha \),
where \((\xi_\beta)_{\beta < \alpha}\) is a strictly increasing sequence in \([\omega, \lambda)\). Notice that \(P_\alpha^*(X^*)\) is a weak*-closed subspace of \(X^*\) that does not separate points of \(X\) (in fact, \(\perp P_\alpha^*(X^*) = \ker(P_\alpha)\)), which easily implies that there exist:

(i) an ordinal \(\xi_\alpha\) with \(\sup\{\xi_\beta : \beta < \alpha\} < \xi_\alpha < \lambda\),

(ii) a norm-one functional \(x^*_\alpha \in (P_{\xi_\alpha+1}^* - P_{\xi_\alpha}^*)(X^*)\), and

(iii) a unit vector \(x_\alpha \in \bigcap_{\beta < \alpha} \ker(x^*_\beta)\) such that \(\langle x_\alpha, x^*_\alpha \rangle = 1\).

This completes the recursive construction. For simplicity, we may assume that \(\xi_\alpha = \alpha\) for every \(\alpha \in [\omega, \lambda)\).

For each \(\alpha \in [\omega, \lambda)\) we define a functional \(\tilde{x}_\alpha \in X^{**}\) as the composition of the evaluation functional at \(x_\alpha\) with the projection \(P_{\alpha+1}^*\), that is,

\[
\langle x^*, \tilde{x}_\alpha \rangle = \langle x_\alpha, P_{\alpha+1}^* x^* \rangle \quad (x^* \in X^*).
\]

Since \(x^*_\beta \in (P_{\beta+1}^* - P_{\beta}^*)(X^*)\) for every \(\beta \in [\omega, \lambda)\), we have:

\[
\langle x^*_\beta, \tilde{x}_\alpha \rangle = \langle x_\alpha, P_{\alpha+1}^* x^*_\beta \rangle = \begin{cases} 
\langle x_\alpha, 0 \rangle = 0 & \text{if } \alpha < \beta, \\
\langle x_\alpha, x^*_\beta \rangle = 1 & \text{if } \alpha = \beta, \\
\langle x_\alpha, x^*_\beta \rangle = 0 & \text{if } \alpha > \beta.
\end{cases}
\]

Therefore, we have obtained a system \(\{ (x^*_\omega, \tilde{x}_\omega) \}_{\omega < \alpha < \lambda} \subset X^* \times X^{**}\) satisfying the following conditions:

(i) \(\|x^*_\omega\| = \|\tilde{x}_\omega\| = 1\) for each \(\omega \leq \alpha < \lambda\);

(ii) \(\langle x^*_\omega, \tilde{x}_\omega \rangle = \delta_{\alpha \beta}\) for all \(\omega \leq \alpha, \beta < \lambda\);

(iii) \(\tilde{x}_\alpha\) is weak*-continuous for each \(\omega \leq \alpha < \lambda\).

Now, define

\[
\mathcal{A} = \{ x = (x^*_\omega, \tilde{x}_\omega)_{\omega < \alpha < \lambda} \in \mathbb{R}^{[\omega, \lambda)} : \|x^*\| \leq 1 \text{ and } \langle x^*, \tilde{x}_\omega \rangle \neq 0 \text{ for countably many } \alpha \text{'s} \}.
\]

Plainly, \(\mathcal{A}\) enjoys the following three properties:

(a1) \(e_\alpha \in \mathcal{A}\) for every \(\omega \leq \alpha < \lambda\), where \(e_\alpha\) is the \(\alpha\)'th vector from the canonical basis of the linear space \(\mathbb{R}^{[\omega, \lambda)}\);

(a2) if \(x \in \mathcal{A}\), then \(-x \in \mathcal{A}\);

(a3) \(\text{supp}(x) := \{ \omega \leq \alpha < \lambda : x(\alpha) \neq 0 \}\) is countable for every \(x \in \mathcal{A}\). (We write \(x(\alpha)\) for \(\langle x^*, \tilde{x}_\alpha \rangle\), the \(\alpha\)'th coordinate of \(x\).)

Assume, in search of a contradiction, that every uncountable subset of the unit ball of \(X^*\) contains two distinct elements at distance larger than 1. Then, the set \(\mathcal{A}\) satisfies also the additional condition:

(a4) for every uncountable set \(\mathcal{B} \subseteq \mathcal{A}\) there exist \(x, y \in \mathcal{B}\) with \(x \neq y\) such that \(x - y \in \mathcal{B}\).

For any two sequences \(x, y \in \mathcal{A}\) we shall say that \(y\) extends \(x\) if and only if there exists a third sequence \(z \in \mathcal{A}\) such that the following conditions are satisfied:

(e1) \(|y(\alpha)| \geq |x(\alpha)|\) for each \(\alpha \in \text{supp}(x)\);

(e2) \(y(\alpha) = z(\alpha)\) for each \(\alpha \in \text{supp}(x)\);
(e3) there is an ordinal number $\beta$ with $\sup(\text{supp}(x)) < \beta < \lambda$ such that $y(\beta) > 0$ and $z(\beta) = -1$.

In such a case we say that $z$ is a witness of $y$ extending $x$ at the $\beta^\text{th}$ coordinate. By a chain starting with an element $x \in A$ we mean any sequence $(x_\alpha)_{0 \leq \alpha < \xi}$, where $\xi \geq 0$ is an ordinal number, for which $x_0 = x$ and $x_{\alpha + 1}$ extends $x_\alpha$ for every $\alpha \geq 0$ with $\alpha + 1 < \xi$.

Let $(x_\alpha)_{0 \leq \alpha < \xi}$ be a chain and, for any $\alpha \geq 0$ with $\alpha + 1 < \xi$, let $z_\alpha$ be a witness of $x_{\alpha + 1}$ extending $x_\alpha$ at the $\beta^\alpha$ coordinate. Then, according to (e3), we have $z_\alpha(\beta_\alpha) = -1$ and $x_{\alpha + 1}(\beta_\alpha) > 0$ for all $\alpha$'s as above. Moreover, by (e1) and (e2) we obtain $z_\gamma(\beta_\alpha) > 0$ whenever $\gamma > \alpha$. Consequently, for any two ordinals $\alpha, \beta$ with $1 \leq \alpha + 1 < \beta + 1 < \xi$ at least one of the coordinates of $z_\beta - z_\alpha$ is larger than one, hence $z_\beta - z_\alpha \notin A$. Therefore, condition (a4) implies that every chain in $A$ is at most countable. On the other hand, in view of the Kuratowski–Zorn lemma, for every $x \in A$ there exists a maximal chain starting with $x$.

Now, we claim that every such maximal chain contains a maximal element, i.e., a sequence $y \in A$ which extends $x$ and which has no further extension.

Indeed, let $C$ be a maximal chain starting with some $x \in A$ for which there is no maximal extension. Then, there is a countable sequence $(y_n)_{n=1}^\infty \subset C$ so that $y_{n+1}$ extends $y_n$ for each $n \in \mathbb{N}$, but there is no $y \in C$ extending all $y_n$'s. Let $y^*_n \in X^*$ be given so that $y_n = (y^*_n, \tilde{x}_\alpha)_{\omega \leq \alpha < \lambda}$ ($n \in \mathbb{N}$).

Take $z^*$ to be any weak*-cluster point of $\{y^*_n: n \in \mathbb{N}\}$. Then, obviously $z^*$ lies in the unit ball of $X^*$ and $\langle z^*, \tilde{x}_\alpha \rangle = 0$ for all but countably many $\alpha$'s, since the same property is shared by each $y^*_n$. Thus, $z^*$ gives rise to the element $z = (\langle z^*, \tilde{x}_\alpha \rangle)_{\omega \leq \alpha < \lambda}$ of $A$ which extends each $y_n$ by the very definition. (Here, we employed the weak*-continuity of $P_\alpha$'s.)

Now, we define by transfinite recursion a sequence $(x_\alpha)_{\omega \leq \alpha < \lambda}$ of maximal extensions as follows. First, let $x_0$ be any maximal extension of $e_\omega$. Now, if $\beta < \lambda$ and all the terms $x_\alpha$, for $\omega \leq \alpha < \beta$, have been already defined, we pick any ordinal $\gamma$ with

$$\sup_{0 \leq \alpha < \beta} \text{supp}(x_\alpha) < \gamma < \lambda$$

(recall that all maximal chains are countable so the entire support of all $x_\alpha$'s is not cofinal in $\lambda$ because $\lambda$, being a successor cardinal, is regular) and take $x_\beta$ to be any maximal extension of $e_{\gamma}$. The rest is the same as in Kottman’s proof; just instead of Ramsey’s theorem we need its variation for larger cardinals, that is, the Erdős–Rado theorem:

$$\mathfrak{c}^+ \to (\omega_1)^2_2$$

(cf. [16] Theorem 9.6). So, define a colouring $c: [\omega, \lambda]^2 \to \{0, 1\}$ of all 2-element subsets of $[\omega, \lambda]$ as

$$c(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } x_\alpha - x_\beta \in A, \\ 1 & \text{if } x_\alpha - x_\beta \notin A. \end{cases}$$

Note that by property (a2) this definition is well-posed. By the Erdős–Rado theorem, there exists an uncountable set $B \subset A$ such that

either $|B|^2 \subset c^{-1}(\{0\})$ or $|B|^2 \subset c^{-1}(\{1\})$. 

However, according to (a4), only the former possibility may occur, and by relabeling we may assume that $B = A$. Property (a4) implies that there are two ordinal numbers $\alpha$ and $\beta$ with $1 \leq \alpha + 1 < \beta < \beta + 1 < \lambda$ and such that $(x_\alpha - x_{\alpha + 1}) - (x_\beta - x_{\beta + 1}) \in A$. But also $x_\alpha - x_{\beta + 1} \in A$ which shows that $x_\alpha - x_{\beta + 1}$ is a witness of $x_\alpha - x_{\alpha + 1} - x_\beta + x_{\beta + 1}$ extending $x_\alpha$; a contradiction with the maximality of $x_\alpha$. \hfill \Box

Remark 3.9. As it was shown by Fabian and Godefroy ([7]), the dual of any Asplund space always admits a PRI. However, it may happen that the corresponding projections $P_\alpha$’s fail to be continuous in the weak∗ topology (for a specific example, see [7, Remark 5]). This shows that our hypothesis that $X$ itself has a PRI was crucial for our considerations.

4. Spaces of continuous functions

4.1. Compact Hausdorff spaces. Throughout this section $K$ stands for an infinite, compact, Hausdorff space unless otherwise stated. By $C(K)$ we denote the familiar Banach space of scalar-valued continuous functions on $K$ furnished with the supremum norm.

With the aid of the Urysohn lemma, it is easy to build a 2-separated (hence, by the triangle inequality, equilateral) sequence in the unit sphere of $C(K)$. Thus it is natural to ask whether $\text{eo}(C(K)) > \omega$ in the case where $K$ is non-metrizable. Very recently, Koszmider ([19]) established independence from the axioms of set theory of the answer to the above question. We offer therefore a number of sufficient conditions implying uncountability of $\text{eo}(C(K))$.

Proposition 4.1. Suppose that $\text{eo}(C(K))$ is countable. Then, given a closed subset $L \subseteq K$, $\text{eo}(C(L))$ is countable too. Moreover, if $C(L)$ contains a 2-equilateral set of cardinality $\kappa$, then so does $C(K)$.

Proof. Arguing by contraposition, let $L \subseteq K$ be a closed subspace and suppose that there exists an uncountable family $\{f_i : i \in I\}$ of unit vectors in the sphere of $C(L)$. Let $\hat{f}_i$ be a norm-preserving Tietze–Urysohn extension of $f_i$ to $K$ ($i \in I$). Then $\{\hat{f}_i : i \in I\}$ witnesses uncountability of $\text{eo}(C(K))$. \hfill \Box

Certainly (hereditary) separability of $K$ is a necessary condition for countability of $\text{eo}(C(K))$. To see this, note that if $K$ is non-separable, there exist families

\[ \{x_\alpha : \alpha < \omega_1\} \quad \text{and} \quad \{U_\alpha : \alpha < \omega_1\} \]

consisting of distinct points in $K$ and open subsets of $K$, respectively, with the property that for each $\alpha < \omega_1$ we have $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for all $\beta < \alpha$. Indeed, the closure of each countable set in $K$ is a proper subset of $K$ and by complete regularity proper closed subsets can be separated by open sets from elements in the complement. Using Urysohn’s lemma we may thus build continuous functions $f_\alpha : K \to [-1, 1]$ such that $f(x_\alpha) = 1$ and $f(x) = -1$ for all $x \in K \setminus U_\alpha$. It then follows that $\{f_\alpha : \alpha < \omega_1\}$ is a 2-separated subset of the unit sphere of $C(K)$. Taking into account Proposition 4.1 and the Tietze–Urysohn Extension Theorem, we arrive at the following conclusion.
Proposition 4.2. Suppose that $K$ is not hereditarily separable. Then $\text{eo}(C(K))$ is uncountable.

The case where $K$ is totally disconnected is even easier. Indeed, take an uncountable collection of clopen subets sets $U_i$ ($i \in I$), in which case 
$$\{1_{U_i} - 1_{K \setminus U_i} : i \in I\}$$
forms an uncountable 2-separated (hence equilateral) subset of the unit sphere in $C(K)$. Let us then record formally the following observation.

Proposition 4.3. Suppose that $K$ contains a closed, non-metrisable totally disconnected subspace. Then $\text{eo}(C(K))$ is uncountable.

One may wonder whether each non-metrisable compact, Hausdorff space $K$ contains a non-metrisable, totally disconnected closed subspace—this in the light of Proposition 4.3 would be sufficient to conclude uncountability of $\text{eo}(C(K))$. Nyikos (Example 6.17) constructed, assuming Jensen’s Diamond Principle $\lozenge$, a non-metrisable, compact manifold $K$ whose each non-metrisable, closed subspace contains a copy of the unit interval (thus, it is not totally disconnected). Therefore, one cannot hope to prove such a theorem about totally disconnected subspaces in ZFC only (this also follows from the main result of [19]). However Nyikos’ manifold is non-separable, hence by Proposition 4.2 the number $\text{eo}(C(K))$ is uncountable.

We conjecture that, assuming the Proper Forcing Axiom, every non-metrisable, perfectly normal compact space $K$ contains a non-metrisable totally disconnected closed subspace—according to Proposition 4.3, this would be sufficient to derive uncountability of $\text{eo}(C(K))$.

The existence of a non-separable Radon measure on $K$ is also sufficient for uncountability of $\text{eo}(C(K))$.

Proposition 4.4. Suppose that there exists a positive measure $\mu \in C(K)^*$ such that $L_1(\mu)$ is non-separable. Then
$$\text{eo}(C(K)) \geq d(L_1(\mu)) > \omega.$$  

Proof. For a Radon measure $\mu$ on a compact space, let $\lambda = d(L_1(\mu))$. The Hilbert space $\ell_2(\lambda)$ embeds isometrically into $L_1(\mu)$. Indeed, by Maharam’s theorem (Theorem 331L), $L_1(\mu)$ is isometric to $L_1(\{0, 1\}^\lambda)$. (Here, $L_1(\{0, 1\}^\lambda)$ denotes the $L_1$-space with respect to the Haar measure on the Cantor group $\{0, 1\}^\lambda$.) Consequently, we can find a collection of $\lambda$ many independent Gaussian random variables with mean zero and the same variance; their linear span is isometric to $\ell_2(\lambda)$.

Thus, by simple duality, $\ell_2(\lambda)$ is a quotient of $C(K)$. Since $\text{eo}(\ell_2(\lambda)) = \lambda$ (any orthonormal basis is a witness of this fact), by Proposition 2.3 the conclusion follows. $\square$

Remark 4.5. The conclusion of Proposition 4.4 extends to the class of all non-reflexive Grothendieck spaces. (A Banach space $X$ is Grothendieck if each bounded, linear operator $T : X \to c_0$ is weakly compact.) Indeed, Haydon (Theorem 1) proved that the dual space of every non-reflexive Grothendieck space contains an isometric copy of $L_1(\{0, 1\}^{\omega_1})$. 


so in this case we argue exactly as in the proof of Proposition 4.4 in order to produce an uncountable \((1 + \varepsilon)\)-separated subset of the unit sphere of \(X\); here \(\varepsilon\) can be taken to be arbitrarily close to \(\sqrt{2} - 1\).

If \(K\) is a Rosenthal compact space (that is, \(K\) is homeomorphic to a compact subset of the space of first-class Baire functions on a Polish space endowed with the topology of point-wise convergence), then for each Radon measure \(\mu\) on \(K\), the space \(L_1(\mu)\) is separable ([37, Theorem 2]). One may then wonder whether a potential counter-example should fall into this class. This is however not the case. Indeed, by Proposition 4.2 any counter-example \(K\) must be hereditarily separable, so if \(K\) is also Rosenthal compact, by [37, Theorem 4], it must contain a copy of the split interval. An appeal to Proposition 4.3 yields the following conclusion.

**Proposition 4.6.** Suppose that \(K\) is a non-metrizable, Rosenthal compact space. Then \(\text{eo}(C(K))\) is uncountable.

We say that \(K\) is **perfectly normal**, if each closed subset of \(K\) is \(\Sigma_2^1\). Thus, if \(U\) is an open subset of a perfectly normal space \(K\), then there exists a norm-one function \(f \in C(K)\) such that \(f(t) > 0\) for \(t \in U\) and \(f(t) = 0\) otherwise. The following fact is a part of [23, Theorem 2].

**Proposition 4.7.** Suppose that \(K\) is not perfectly normal. Then the unit sphere of \(C(K)\) contains an uncountable \(2\)-separated subset. In particular, \(\text{eo}(C(K))\) is uncountable.

**Definition 4.8.** We say that a continuous function \(f: K \to \mathbb{R}\) is **locally sign-changing** if for each non-isolated point \(z \in K\) with \(f(z) = 0\) the function \(f\) takes both positive and negative values in every neighbourhood of \(z\).

For instance, if \(K = [-1, 1]\), then \(f(x) = \sin x\) is a locally sign-changing function whereas \(g(x) = |x|\) is not.

**Lemma 4.9.** Suppose that \(K\) is perfectly normal. Then for each pair of distinct points \(x, y \in K\) there exists a norm-one, locally sign-changing function \(f \in C(K)\) such that \(f(t) = 1\) in some neighbourhood of \(x\) and \(f(t) = -1\) in some neighbourhood of \(y\).

**Proof.** Let \(g \in C(K)\) be a norm-one function such that \(g(t) = 1\) in some neighbourhood of \(x\) and \(g(t) = -1\) in some neighbourhood of \(y\). If \(g^{-1}(\{0\})\) has non-empty interior, by perfect normality of \(K\), let us choose a norm-one function \(h\) that is strictly positive on \(\text{int} g^{-1}(\{0\})\) and 0 otherwise. If the interior is already empty, take \(h = 0\).

Let \(w = g + h\); then \(w^{-1}(\{0\})\) has empty interior (in particular, it contains no isolated points). Consider all points \(z \in w^{-1}(\{0\})\) such that \(0 \leq w(t) < 1/4\) for \(t \in U^+_z\), where \(U^+_z\) is some open neighbourhood of \(z\), and let \(U^+\) be the union of all such sets \(U^+_z\). Take a function \(w^+\) of norm at most \(3/4\) that is strictly positive in \(U^+\) and zero otherwise.

Similarly, consider all points \(z \in w^{-1}(\{0\})\) such that \(-1/4 < w(t) \leq 0\) for \(t \in U^-_z\), where \(U^-_z\) is some open neighbourhood of \(z\), and let \(U^-\) be the union of all these sets \(U^-_z\). Take a function \(w^-\) of norm at most \(3/4\) that is strictly negative in \(U^-\) and zero otherwise.
Finally, observe that \( f = w + w^+ + w^- \) is a norm-one, locally sign-changing function with the desired properties. \( \square \)

It is a standard fact from the point-set topology that \( d(C(K)) \) is equal to \( w(K) \), the minimal cardinality of a basis for \( K \) (the weight). We prove that for \( C(K) \)-spaces with \( K \) perfectly normal, the number \( k(C(K)) \) is as large as possible.

**Theorem 4.10.** Suppose that \( K \) is perfectly normal. Then the unit sphere of \( C(K) \) contains a \((1+)\)-separated subset of cardinality \( w(K) \) that consists of locally sign-changing functions. In particular, \( k(C(K)) = w(K) \).

**Proof.** Assume, in search of contradiction, that each \((1+)\)-separated family consisting of norm-one, locally sign-changing functions has cardinality strictly less than \( w(C(K)) \) and take one, \( \{f_i : i \in I\} \) say, that is maximal (with respect to inclusion) subject to these conditions.

We claim that the family \( \{f_i : i \in I\} \) separates points in \( K \). Once it is proved, the Stone-Weierstrass theorem will immediately yield that the algebra \( \mathcal{A} \) generated by \( \{f_i : i \in I\} \) is dense in \( C(K) \). However, the density of \( \mathcal{A} \) is at most \(|I| < w(K)\), which will ultimately lead to a contradiction.

Assume that the functions \( \{f_i : i \in I\} \) do not separate points in \( K \). Thus, for some pair of distinct points \( x, y \in K \) and every \( i \in I \) we have \( f_i(x) = f_i(y) \). By Lemma 4.9, we may find a norm-one, locally sign-changing function \( f \in C(K) \) such that \( f(t) = 1 \) in some neighbourhood of \( x \) and \( f(t) = -1 \) in some neighbourhood of \( y \). Fix any \( i \in I \). If \( f_i(x) \neq 0 \), then we have

\[
\|f - f_i\| \geq \max\{|1 - f_i(x)|, |1 - f_i(x)|\} > 1.
\]

Otherwise (that is, if \( f_i(x) = 0 \)), we use the fact that \( f_i \) is locally sign-changing to pick a point \( t \) in the neighbourhood of \( x \) where \( f \) takes value 1 such that \( f_i(t) < 0 \). Thus

\[
\|f - f_i\| \geq |1 - f_i(t)| > 1.
\]

This is a contradiction with the maximality of \( \{f_i : i \in I\} \). \( \square \)

It is a tantalising problem whether the hypothesis of perfect normality in Theorem 4.10 may be dropped.

**4.2. Locally compact Hausdorff spaces.** We will employ Theorem B to obtain another class of spaces for which the cardinal function \( k \) assumes uncountable values.

**Proposition 4.11.** Let \( K \) be a non-separable, locally compact, Hausdorff space. Then \( k(C_0(K)) \) is uncountable.

**Proof.** Consider first the case where \( K \) contains a non-metrisable, compact subset \( L \). Then by Theorem B, \( k(C(L)) > \omega \). Using the Tietze–Urysohn extension theorem for locally compact spaces, we may extend each function \( f \in C(L) \) to a function from \( C_0(K) \) preserving the norm. Thus, \( k(C_0(K)) \) is uncountable.
Having eliminated the case where there exists a compact, non-metrizable subset of $K$, we arrive at the case where each compact subset of $K$ is second-countable, hence separable. Assume that each $(1+)$-separated subset of the unit sphere of $C_0(K)$ is countable. Similarly as in the Introduction, take a set $\{f_n: n \in \mathbb{N}\}$ that is maximal among all $(1+)$-separated subsets of the unit sphere of $C_0(K)$ consisting of functions assuming the value 1. Define

$$F = \{x \in K: f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\};$$

this is clearly a $\sigma$-compact subset of $K$. As $K$ is non-separable and its compact subsets are second-countable, $F$ is separable and hence, $F$ is a proper subset of $K$. Consider the function

$$g(x) = \sum_{k=1}^{\infty} \frac{|f_k(x)|}{2^k} \quad (x \in K).$$

We have $g(x) = 0$ for each $x \notin F$. Pick $x_0 \in K \setminus F$ and choose a norm-one function $h \in C_0(K)$ such that $h(x_0) = 1$ and $h(x) = 0$ for all $x \in F$. Let $f = h - g$. Then $\|f\| = 1$ and $\|f - f_n\| > 1$ for all $n \in \mathbb{N}$; a contradiction with the maximality of $\{f_n: n \in \mathbb{N}\}$. □

Unlike the compact case, if $K$ is a large enough discrete space then $k(C_0(K)) < d(C_0(K))$ (of course, $C_0(K) = c_0(|K|)$). Indeed, note that $d(c_0(c^+)) = c^+$, however, the following inequality holds true (which is due to P. Koszmider):

**Proposition 4.12.** $k(c_0(c^+)) \leq c$.

**Proof.** Assume that there exists a $(1+)$-separated family of unit vectors $A \subset c_0(c^+)$ with $|A| = c^+$. Consider the family of supports of elements of $A$, that is,

$$\mathcal{A} = \{\alpha < c^+: f(\alpha) \neq 0: f \in B\}. $$

Clearly, $|\mathcal{A}| = c^+$. By the generalised $\Delta$-system lemma ([22, Theorem 1.6]), there exist a subfamily $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = c^+$ and a countable set $\Delta$ such that $A \cap B = \Delta$ for all distinct $A, B \in \mathcal{B}$. Given $f, g \in A$ with supports in $\mathcal{B}$, $|f(\alpha) - g(\alpha)| > 1$ implies that $\alpha \in \Delta$. This is a contradiction because there are only continuum many real-valued functions on $\Delta$ yet we have $c^+$ different functions in $A$ with supports in $\mathcal{B}$. □

**Acknowledgements.** We are indebted to Marek Čuth and Yanqi Qiu for careful reading and detecting certain slips in the preliminary version of the manuscript.

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