The relation between symmetries and coincidence and collinearity of polygon centers and centers of multisets of points in the plane

Luis Felipe Prieto-Martínez

Abstract There are several remarkable points, defined for polygons and multisets of points in the plane, called centers (such as the centroid). To make possible their study, there exists a formal definition for the concept of center in both cases. In this paper, the relation between symmetries of polygons and multisets of points in the plane and the coincidence and collinearity of their centers is studied. First, a precise statement for the problem is given. Then, it is proved that, given a polygon or a multiset of points in the plane, a given point in the plane is a center for this object if and only if it belongs to the set of points fixed by its group of symmetries.

Keywords Polygon · Triangle Center · Polygon center · Finite multiset of points in the plane · Group of Symmetries

Mathematics Subject Classification (2010) Primary 51M04 · Secondary 51M15

1 Introduction

Associated to the every triangle $P$ there are four famous points known as the classical centers. Those are the incenter, the centroid, the circumcenter and the orthocenter. But there are many other remarkable points, associated to $P$, which are also called “centers”. Thousands of them are known at this moment (see the web site of the Encyclopedia of Triangle Centers [9]).

But, what must satisfy a point to be called “center”? In a series of articles published in the 1990’s (see, for example, [7, 8]) C. Kimberling indicated the importance of giving a formal definition of triangle center as a function and not as a “remarkable” point obtained with a geometric construction. Recently,
the ideas by Kimberling has been generalized for polygons with four or more sides in [4,12].

In this paper we deal, not only with centers of \( n \)-gons but with \( n \)-multisets of points in the plane (multiset with \( n \) elements). Polygons are multisets of points in the plane endowed with a notion of adjacency between their elements, called vertices. Polygons (and their centers) are basic objects in Elementary Geometry and multisets of points in \( \mathbb{R}^N \) (and their centers) are very important in problems in Applied Mathematics (see for instance [11] to see an application of the centroid in the study of tumor growth).

Concerning this theory of centers, in the bibliography we can find (not explicitly stated) the following:

**Principle:** As more coincidence and collinearity of centers occur for a given \( n \)-gon (resp. \( n \)-multiset of points in the plane), the more regular it is.

See for example [3] for the case of quadrilaterals, [4,12] for general \( n \)-gons and [3,10] for \( n \)-multisets of points in \( \mathbb{R}^N \).

Let us consider, as inspiring examples for our study, the following results:

- For a triangle with not all its vertices collinear, the incenter and the orthocenter coincide if and only if the triangle is equilateral (see [6]).
- For a triangle with not all its vertices collinear, the incenter, the centroid and the orthocenter are collinear if and only if the triangle is isosceles (see [5]).
- For a set of four different points \( \{V_1, V_2, V_3, V_4\} \) in the plane such that any of its elements is in the convex hull of the other three, the centroid and the Fermat-Torricelli point coincide if and only \( V_1, V_2, V_3, V_4 \) correspond to the vertices of a parallelogram (see [1], where the results is stated for quadrilaterals).
- For a convex quadrilateral, the centroid and the centroid of the boundary coincide if and only if it is a parallelogram (see [2]).

The main target of this paper is, in relation to the **Principle** above, to solve the following:

**Main Problem:** For a given polygon \( P \) (resp. multiset \( \tilde{P} \)), determine the set of points in the plane that can be viewed as centers, according to the formal definition of center as a function (see Definitions 1 and 2).

This problem was already explored (but not solved) in [4,12] for \( n \)-gons and in [3,10] for \( n \)-multisets of points in \( \mathbb{R}^N \). We prove that the answer is (exactly) the set of fixed points by the group of symmetries of \( P \) (resp. \( \tilde{P} \)) and is given in Theorem 6 (for \( n \)-multisets of points in the plane) and Theorem 9 (for \( n \)-gons).

The fact that the set of centers must be contained in this set of fixed points is easy to verify and the main ideas already appear in the bibliography cited above (anyway, a proof fitting our approach is included here). But to solve the Main Problem there are two difficulties: (a) find two \( n \)-gon centers such that they coincide if and only if the corresponding \( n \)-gon has rotational symmetry (resp. for \( n \)-multisets) and (b) find three \( n \)-gon centers such that they are
collinear if and only if the corresponding \( n \)-gon has an axis of symmetry (resp. for \( n \)-multisets).

The point is that, for general values of \( n \), it is not easy (it may be impossible) to find simple examples of centers with these properties. It will be necessary to construct artificially two \( n \)-multiset centers and two \( n \)-gon centers with this purpose of detecting asymmetry. This centers also apply for non-covex and non-simple \( n \)-gons, which is one of the strengths of the results herein.

These processes of detecting asymmetry can be reduced, in turn, to solving a nice problem (and its weighted version) concerning cyclic convex \( n \)-gons, which may be of its own interest.

**First Step Problem:** Let \( c \) be a circle with center \( O \). Find a function \( \overline{B}_n \) that assign to each set of points \( \{V_1, \ldots, V_n\} \subset c \) a point in \( c \cup \{O\} \) such that (1) \( \overline{B}_n(\{V_1, \ldots, V_n\}) = O \) if and only if there is a rotation fixing the set \( \{V_1, \ldots, V_n\} \) and (2) it commutes with similarities, that is, for every similarity \( T \), \( \overline{B}_n \) satisfies
\[
\overline{B}_n(\{T(V_1), \ldots, T(V_n)\}) = T(\overline{B}_n(\{V_1, \ldots, V_n\})).
\]

The notation and formal definitions (including the one of \( n \)-multiset and \( n \)-gon center) required for the rest of the paper are introduced in Section 2. The First Step Problem is solved in Section 3. Finally, in Sections 4 and 5 we include the proof of Theorems 6 and 9 respectively.

### 2 Notation and definition of center

The most basic objects through the rest of this paper are \( n \)-multiset (multisets of \( n \) elements for \( n \geq 1 \)) of points in the plane (just referred as \( n \)-multiset in the following). The elements will be referred as vertices. If all the elements are different we call \( n \)-set to the corresponding object. Let \( \overline{P}_n \) be the set consisting in all \( n \)-multisets of points in \( \mathbb{R}^2 \).

Eventually (in the next section) we will also denote by \( \overline{P}^* \) to the set of \( n \)-multisets with their vertices labelled with a natural number, that is, the sets of pairs \( \{(V_1, l_1), \ldots, (V_n, l_n)\} \) where \( (V_i, l_i) \subset \mathbb{R}^2 \times \mathbb{N} \).

For \( n \geq 3 \), a \( n \)-gon is a \( n \)-multiset \( \overline{P} \) which elements are called vertices endowed with some notion of adjacency \( \mathcal{A} \) between these points (symmetric and irreflexive) that additionally satisfies (1) each vertex is adjacent to (exactly) two vertices and (2) for every pair of vertices \( V, V' \) there exist two (and only two) sequences of vertices \( V_1, \ldots, V_k \), that only intersect in their endpoints, such that \( V = V_1, V' = V_k \) and such that, for \( i = 1, \ldots, k-1 \), \( V_i \) is adjacent to \( V_{i+1} \). The set of all \( n \)-gons will be denoted by \( \mathcal{P}_n \).

The sides of a \( n \)-gon are the segments joining two adjacent vertices. The rest of segments joining vertices of the \( n \)-gon are called diagonals. Note that, for the case \( n = 3 \), every 3-multiset has a unique triangle (3-gon) structure.
In the notation above, any sequence of vertices \((V_1,\ldots,V_n)\) in \(\tilde{P}\) such that, for \(i = 1,\ldots,k-1, V_{i+1}\) is adjacent to \(V_i\) is called a **polygonal chain** starting at \(V_1\) (sometimes, this concept refers to the set of sides joining these vertices). This polygonal chain is **closed** if \(k = n + 1\) and so \(V_1 = V_k\). Let \(r_1,\ldots,r_{k-1}\) where \(r_i\) denotes the side with endpoints \(V_i,V_{i+1}\). We say that the polygonal chain is **simple** if these segments only intersect in their endpoints and only with the following and the previous one. For the case \(k = n + 1\) we consider \(r_1\) to be “the following one” of \(r_n\) and, if the chain is simple, we say that the polygon \(P\) is simple.

**Comment on the definition of polygon:** We may assume that the \(n\)-gon has a labelling for its vertices \((V_1,\ldots,V_n)\) in such a way that \(V_i, V_j\) are adjacent if and only if \(i,j\) are consecutive (modulo \(n\)). This labelling is not unique. Let us denote by \(S_n\) to the set of permutations of \(\{1,\ldots,n\}\). Consider \(D_n < S_n\) to be the dihedral subgroup, that is, the one generated by the permutations \(\rho, \sigma\) given by
\[
\rho(i) = i + 1 \mod n, \quad \sigma(i) = n - i + 2 \mod n.
\]
Then two labellings correspond to the same \(n\)-multiset if and only if there exists some \(\alpha \in D_n\) or \(\alpha \in S_n\) in each case, by
\[
(V_1,\ldots,V_n) \mapsto (V_{\alpha(1)},\ldots,V_{\alpha(n)})
\]
This second approach is more similar to the one in [2][4][12].

To avoid confusion, in general, the objects related to \(n\)-multisets will be denoted with tildes and the ones related to \(n\)-gons without them.

Denote by \(E(2)\), \(S(2)\), with \(E(2) < S(2)\), to the groups of plane rigid motions and plane similarities, respectively. \(E(2)\), \(S(2)\) act on \(\tilde{P}_n\), \(\mathcal{P}_n\) and \(\mathcal{P}_n^*\). The action of any \(T \in S(2)\) is given by
\[
\begin{align*}
\{V_1,\ldots,V_n\} &\mapsto \{T(V_1),\ldots,T(V_n)\} \\
\{\{(V_1,\ldots,V_n),\mathcal{A}\} &\mapsto \{\{T(V_1),\ldots,T(V_n)\}, T(\mathcal{A})\} \\
where \((W_1,W_2) \in T(\mathcal{A}) \iff (W_1,W_2) \in \mathcal{A}\} \\
\{(V_1,l_1),\ldots,(V_n,l_n)\} &\mapsto \{\{(T(V_1),l_1),\ldots,(T(V_n),l_n)\}
\end{align*}
\]

The **group of symmetries** of a given \(n\)-multiset \(\tilde{P}\) (resp. of a \(n\)-gon \(P\) or of a \(n\)-multiset with its vertices labelled \(\tilde{P}_n\)) is the subgroup of plane rigid motions \(T \in E(2)\) such that \(T(\tilde{P}) = \tilde{P}\) (resp. \(T(P) = P\), \(T(\tilde{P}^*) = \tilde{P}^*\)).

We say that some \(\tilde{P} \in \tilde{P}_n\) (resp. \(P \in \mathcal{P}_n\) or \(\mathcal{P}_n^*\)) is **rotationally symmetric** or order \(k\) if there is a rotation of order \(k\) in its group of symmetries. We say that some \(\tilde{P} \in \tilde{P}_n\) (resp. \(P \in \mathcal{P}_n\) or \(\mathcal{P}_n^*\)) is **axially symmetric** if there is some reflection with respect to a line in its group of symmetries.

**Remark:** Let \(P \in \mathcal{P}\) and \(\tilde{P}\) be the \(n\)-multiset associated to \(P\) (containing its vertices). Note that the group of symmetries of \(P\) is a subgroup of the group of symmetries of \(\tilde{P}\) and both groups may not be equal. As a consequence, the set of fixed points of the group of symmetries of \(P\) contains the set of fixed points by the group of symmetries of \(\tilde{P}\) and may be larger.

With this notation, we have the following:
**Definition 1** For \( n \geq 1 \), let \( \tilde{F}_n \) be a non-empty subset of \( \tilde{P}_n \) closed with respect to similarities. A **\( n \)-gon center** is a function \( \tilde{X}_n : \tilde{F}_n \to \mathbb{R}^2 \) that commutes with respect to similarities.

**Definition 2** For \( n \geq 3 \), let \( F_n \) be a non-empty subset of \( P_n \) closed with respect to similarities. A **\( n \)-gon center** is a function \( X_n : F_n \to \mathbb{R}^2 \) that commutes with respect to similarities.

The most important multiset center is the **centroid** \( \tilde{C}_n : \tilde{P}_n \to \mathbb{R}^2 \), which is given by

\[
\tilde{C}_n(\{V_1, \ldots, V_n\}) = \frac{1}{n}V_1 + \ldots + \frac{1}{n}V_n.
\]

One more illustrative example (the circumcenter) will be explained at the beginning of the following section.

Associated to every \( n \)-multiset center \( \tilde{Z}_n : \tilde{F}_n \to \mathbb{R}^2 \) there is a **\( n \)-gon center** \( Z_n : F_n \to \mathbb{R}^2 \), defined for the family \( F_n \) of \( n \)-gons \( P \) which \( n \)-multiset of vertices \( \tilde{P} \) is in \( \tilde{F}_n \) and given by \( Z_n(P) = \tilde{Z}(\tilde{P}) \). So we also have a \( n \)-gon version of the centroid \( C_n : P_n \to \mathbb{R}^2 \) that maps each \( n \)-gon to the centroid of its vertices.

The converse is not true: not for very \( n \)-gon center there is associated some \( n \)-multiset center. For example, for \( n = 4 \), the function \( D_4 : F_4 \to \mathbb{R}^2 \) that maps, when defined, each tetragon \( P \) to its diagonal crosspoint is a 4-gon center but there is no finite multiset center associated to it, since its definition depends on the adjacency relation \( A \) of the vertices of the polygon.

**Comment:** In this paper, we commit an abuse of notation which is very extended in this context. We call “center” to both, the function \( \tilde{Z}_n \) (resp. \( Z_n \)) and the point \( \tilde{Z}_n(\tilde{P}) \) (resp. the point \( Z_n(P) \)) corresponding to a given multiset \( \tilde{P} \) (resp. the point \( Z_n(P) \)) in its domain.

### 3 Some centers for families of cyclic \( n \)-multisets

We say that a \( n \)-set \( \tilde{P} \) is **cyclic** if all its elements belong to a circle. The center of this circle is called the **circumcenter** of \( \tilde{P} \). The function \( \mathfrak{M}_n : F_n \to \mathbb{R}^2 \), for \( F_n \) being the family of cyclic \( n \)-sets, that maps every cyclic \( n \)-set to its circumcircle is a \( n \)-multiset center.

Any cyclic \( n \)-set \( \tilde{P} \) has a natural notion of adjacency between its elements (so cyclic \( n \)-multisets endowed with this adjacency relation can be viewed as \( n \)-gons). For \( n \geq 3 \), two points \( V, W \) in \( \tilde{P} \) are adjacent if there is no other point of \( P \) for some of the two circular arcs with endpoints \( V, W \).

Let us start with the following lemma. It is required to guarantee that some of the algorithms below produce an output.

**Lemma 3** Let \( \tilde{P}, \tilde{Q} \) be two sets of points inscribed in the same circle \( c \) with center \( O \) and consisting, respectively, in the vertices of a regular \( n \)-gon and a regular \( m \)-gon.
(i) The set $\tilde{P} \cup \tilde{Q}$ is not rotationally symmetric if and only if $m, n$ are coprimes.

Provided that $m, n$ are coprime, then:

(ii) $\tilde{P}, \tilde{Q}$ intersect in at most one point.

(iii) If there are two pairs of points $(V_1, W_1), (V_2, W_2) \in \tilde{P} \times \tilde{Q}$ such that they are adjacent in $\tilde{P} \cup \tilde{Q}$ and $\angle(V_1, O, W_1) = \angle(V_2, O, W_2) = \alpha < \pi$, then $\tilde{P} \cup \tilde{Q}$ is axially symmetric with respect to the segment bisector of $V_1, V_2$ (or, equivalently, of $W_1, W_2$). Moreover, the midpoints $M_1, M_2$ of the circular arcs corresponding to these angles cannot be antipodal. See the right hand side of Figure 1.

(iv) There are not three pairs of adjacent points $(V_1, W_1), (V_2, W_2), (V_3, W_3) \in \tilde{P} \times \tilde{Q}$ such that $\angle(V_1, O, W_1) = \angle(V_2, O, W_2) = \angle(V_3, O, W_3)$.

![Fig. 1 Illustrations for Statement (iii). The configuration on the left hand side is not possible.](image)

Proof: Statements (i) and (ii) are immediate.

(iii) Suppose that there are two pairs of points $(V_1, W_1), (V_2, W_2) \in \tilde{P} \times \tilde{Q}$ such that $\angle(V_1, O, W_1) = \angle(V_2, O, W_2) = \alpha$.

Note that $W_1, W_2$ cannot be “at the same side” of $V_1, V_2$, respectively (the circular arcs from $V_1$ to $W_1$ and from $V_2$ to $W_2$ cannot have the same orientation, this would correspond to the picture at the left hand side of Figure 1). To see this, let $\tilde{P}'$ be the set obtained from $\tilde{P}$ by a rotation of angle $\alpha$ in the sense of the circular arc from $V_1$ to $W_1$. Then $\tilde{P'}, \tilde{Q}$ intersect at two points, which is a contradiction with Statement (ii).

So the segment bisector of the segments $V_1, V_2$ and $W_1, W_2$ coincide. To see that $\tilde{P} \cup \tilde{Q}$ is symmetric with respect to this line, we can use that any regular $n$-gon is symmetric with respect to any of the segment bisectors of its sides and diagonals.

Finally, suppose that the midpoints of these arcs are antipodal. Then, we are (modulo congruences) in the situation in Figure 2. The angle $\pi - \alpha$ and $\pi + \alpha$ are multiples of $\frac{2\pi}{m}$ and of $\frac{2\pi}{n}$. So $\alpha$ is a multiple of $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

The case $\alpha = 0$ is not possible (contradiction with Statement (i)). The
case \( \alpha = \frac{\pi}{m} = \frac{\pi}{n} \) is a contradiction with the fact that \( m, n \) are coprime. Finally, the case \( \alpha = k \frac{\pi}{m} \) (resp. \( \alpha = k \frac{\pi}{n} \)) for \( k \geq 2 \) contradicts that \( V_1, V_2 \) are adjacent.

![Diagram](image)

**Fig. 2** \( M_1, M_2 \) cannot be antipodal.

(iv) Suppose that there are such three pairs of points. Then two of the points \( W_{i_1}, W_{i_2} \) for \( i_1, i_2 \in \{1, 2, 3\} \) are “at the same side” of the corresponding points \( V_{i_1}, V_{i_2} \in \tilde{P} \) in the sense precised before. So we can repeat the argument in the first paragraph of the proof of Statement (iii).

Now we describe our algorithms. Each algorithm describes one of the functions \( \Phi, \tilde{A}_n, \tilde{B}_n, \tilde{B}^*_n \) which domain is in \( \tilde{P}_n \) or \( \tilde{P}^*_n \) and which image is in \( \tilde{P}_n \) or \( \mathbb{R}^2 \). For us the following fact of immediate proof for each case will be very important:

**Remark:** The functions \( \Phi, \tilde{A}_n, \tilde{B}_n, \tilde{B}^*_n \), described in Algorithms 1, 2, 3, 4, commute with respect to similarities (in fact, \( \tilde{A}_n \) and \( \tilde{B}_n \) are \( n \)-multiset centers). In particular, \( \tilde{B}_n \) is the answer for the First Step Problem.

| Algorithm 1 (definition of \( \Phi \)) |
|-------------------------------------|
| **INPUT:**  | a cyclic \( n \)-set \( P \in \tilde{P}_n \) contained in a circle \( c \) with circumcenter \( O \) which is rotationally symmetric of order \( k \geq 2 \). |
| **OUTPUT:** | a \( k \)-set \( \Phi(\tilde{P}) \) consisting in the vertices of some regular \( k \)-gon inscribed in \( c \) (or the endpoints of a diameter, for \( k = 2 \)). |
| (1)         | If \( P \) already is the \( k \)-set consisting in the vertices of some regular \( k \)-gon inscribed in \( c \), then \( \Phi(\tilde{P}) = \tilde{P} \). |
| (2)         | In other case, for each element \( V \) in \( \tilde{P} \), consider the sequence \( V_1, \ldots, V_n \) where \( V_1 = V \), and, for \( i = 1, \ldots, n - 1 \), \( V_{i+1} \) is the element in \( \tilde{P} \) adjacent to \( V_i \) and such that the circular arc from \( V_i \) to \( V_{i+1} \) is positively oriented. |
For each element $V$ in $\tilde{P}$, and for the notation described above, define the sequence $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i$ is the angle $\angle(V_i, O, V_{i+1})$ (corresponding to the circular arc from $V_i$ to $V_{i+1}$).

There are exactly $k$ elements in $\tilde{P}$ such that the corresponding sequence $(\alpha_1, \ldots, \alpha_n)$ is minimal with respect to the lexicographic order. Let $\tilde{Q}_+$ be this $k$-set.

For each element $V$ in $\tilde{P}$, consider the sequence $W_1, \ldots, W_n$ where $W_1 = V$, and, for $i = 1, \ldots, n - 1$, $W_{i+1}$ is the element in $\tilde{P}$ adjacent to $W_i$ and such that the circular arc from $W_i$ to $W_{i+1}$ is negatively oriented.

For each element $V$ in $\tilde{P}$, and for the notation described above, define the sequence $(\beta_1, \ldots, \beta_n)$ such that $\beta_i$ is the angle $\angle(W_i, O, W_{i+1})$ (corresponding to the circular arc from $W_i$ to $W_{i+1}$).

There are exactly $k$ elements in $\tilde{P}$ such that the corresponding sequence $(\beta_1, \ldots, \beta_n)$ is minimal with respect to the lexicographic order. Let $\tilde{Q}_-$ be this $k$-set.

The case $\tilde{Q}_+ = \tilde{Q}_-$ is not possible. So, there are $k$ circular arcs oriented in the positive order starting in an element $V$ in $\tilde{Q}_+$ and ending in an element $W$ in $\tilde{Q}_-$ adjacent (in $\tilde{Q}_+ \cup \tilde{Q}_-$) to $V$. $\Phi(\tilde{P})$ is the set of midpoints of these circular arcs.

**Algorithm 2** (definition of $\tilde{A}_n$)

| INPUT: | a cyclic $n$-set $P \in \mathcal{P}_n$ contained in a circle $c$ with circumcenter $O$, not containing any rotationally symmetric subset. |
| OUTPUT: | a point $\tilde{A}_n(P)$ in $c$. |

If $P$ consists in a single point $V$, then $\tilde{A}_n(P) = V$. 

Fig. 3 Illustration of Algorithm 1. The set of black dots (●) corresponds to the input and the set of white dots (○) to the output.
(2) If \( \tilde{P} \) consist in two points \( V,W \), then \( \tilde{A}_n(\tilde{P}) \) is the midpoint of the smallest circular arc among the two of them with endpoints \( V,W \).

(3) In other case, consider the pairs \( \{V_1,W_1\},\ldots,\{V_r,W_r\} \subset \tilde{P} \), consisting in two adjacent elements in \( \tilde{P} \) that maximize \( \angle(V,O,W) \) (we take the angle corresponding to the circular arc not containing any other element in \( \tilde{P} \)).

(4) If \( \tilde{P} = \bigcup_{i=1}^{r} \{V_i,W_i\} \), then define a new \( \tilde{P} \) consisting in the midpoints of the circular arcs corresponding to the maximal angles described above and go back to Step (1).

(5) In other case, define a new \( \tilde{P} \) obtained from the old one subtracting \( V_1,\ldots,V_r,W_1,\ldots,W_r \) and go back to Step (1).

---

**Algorithm 3 (definition of \( \tilde{B}_n \))**

**INPUT:** a cyclic \( n \)-set \( \tilde{P} \in \mathcal{P}_n \) with circumcenter \( O \) and contained in the circle \( c \).

**OUTPUT:** a point \( \tilde{B}_n(\tilde{P}) \) in \( c \cup \{O\} \) such that \( \tilde{B}_n(\tilde{P}) = O \) if and only if \( \tilde{P} \) is rotationally symmetric.

1. If \( \tilde{P} \) is rotationally symmetric, then define \( \tilde{B}_n(\tilde{P}) = O \).

2. In other case, let us consider the sets \( \tilde{Q}_2,\ldots,\tilde{Q}_r \) such that for every \( i \in \mathbb{Z} \), the rotation of angle \( \frac{i}{r}2\pi \) maps \( V \in \tilde{P} \) such that \( \tilde{Q}_i \) contains the set of elements \( \tilde{Q}_i \) contains the set of elements \( V \in \tilde{P} \) such that \( \tilde{Q}_i \) contains the set of elements.

3. Define \( \tilde{Q}_\infty = \tilde{P} \setminus \left( \bigcup_{i=2}^{r} \tilde{Q}_i \right) \).

4. If \( \tilde{Q}_\infty \neq \emptyset \), then \( \tilde{B}_n(\tilde{P}) = \tilde{A}_n(\tilde{P}) \).

5. In other case consider, among all the pairs of integers \( (i,j) \) such that \( \tilde{Q}_i,\tilde{Q}_j \neq \emptyset \) and \( gcd(i,j) = 1 \), the maximal one with respect to the lexicographic order.

6. Let \( \tilde{Q} = \Phi(\tilde{Q}_i) \cup \Phi(\tilde{Q}_j) \).
There are at most two pairs \( \{V, W\} \) of adjacent points in \( \tilde{Q} \) minimizing the angle \( \angle(V, O, W) \) (Statement (iii) in Lemma 3). Suppose that this minimum is \( \alpha \).

If there is only one such pair, then \( \tilde{B}_n(\tilde{P}) \) is the midpoint of the circular arc with endpoints \( V, W \) (the smallest one among the two of them).

If there are two of them \( \{V_1, W_1\} \) and \( \{V_2, W_2\} \), then consider the two midpoints \( M_1, M_2 \) of the circular arcs with endpoints \( V_1, W_1 \) and \( V_2, W_2 \) (the one not containing any element in \( \tilde{Q} \)). \( M_1, M_2 \) are not antipodal (Statement (iii) in Lemma 3). So \( \tilde{B}_n(\tilde{P}) \) is the midpoint of the circular arc with endpoints \( M_1, M_2 \) (the smallest one among the two of them).

**Algorithm 4 (definition of \( \tilde{B}^*_n \))**

**INPUT:** a cyclic \( n \)-set with its vertices labelled \( P^* \in P^*_n \) with circumcenter \( O \) and contained in the circle \( c \).

**OUTPUT:** a point \( \tilde{B}^*_n(\tilde{P}^*) \) in \( c \cup \{O\} \) such that \( \tilde{B}^*_n(\tilde{P}^*) = O \) if and only if \( \tilde{P}^* \) is rotationally symmetric.

1. Suppose that all the elements in \( P^* \) have the same label. Then \( \tilde{B}^*_n(\tilde{P}^*) = \tilde{B}_n(\tilde{P}) \), where \( \tilde{P} \) is the set obtained from \( \tilde{P}^* \) removing the labels.
2. Suppose that there are \( k \) different labels, \( k \geq 2 \). Let us assume that they are 1, \ldots, \( k \) (in other case we do a relabelling preserving the order). Let \( \tilde{Q}^*_1, \ldots, \tilde{Q}^*_k \) be the sets of points in \( \tilde{P}^* \) corresponding to the same label.
3. Suppose that at least one of the sets \( \tilde{Q}^*_1, \ldots, \tilde{Q}^*_k \) is not rotationally symmetric, and suppose that \( i \) is the maximal subindex with this property. Then \( \tilde{B}^*_n(\tilde{P}^*) = \tilde{B}_n(\tilde{Q}^*_i) \), where \( \tilde{Q}^*_i \) is the set obtained from \( \tilde{Q}^*_i \) removing the labels.
4. In other case, for \( i = 1, \ldots, k \), define \( d_i \) to be the order of the rotational symmetry.
5. Consider the pair of sets such that \( gcd(d_i, d_j) = 1 \) and, corresponding to them, the maximal pair of subindices \( (i, j) \) (with respect to the lexicographic order).
6. Let \( \tilde{Q} = \Phi(\tilde{Q}_i) \cup \Phi(\tilde{Q}_j) \), where \( \tilde{Q}_i, \tilde{Q}_j \) are the sets obtained removing the labels.

Follow Steps (7), (8), (9), (10) and (11) in Algorithm 3 (replacing \( \tilde{B}_n(\tilde{P}) \) by \( \tilde{B}^*_n(\tilde{P}^*) \)).
4 Main result for \( n \)-multisets

In the following, let us denote by \( \tilde{A}_n, \tilde{B}_n \) and \( \tilde{C}_n \) to the elements in \( \tilde{P}_n \) which group of symmetries has one fixed point, a line of fixed points and a plane of fixed points, respectively.

**Lemma 4** (definition of \( \tilde{X}_n \)) There exists a \( n \)-multiset center \( \tilde{X}_n : \tilde{P}_n \to \mathbb{R}^2 \), that we will call the center of rotational asymmetry, such that \( \tilde{C}_n(\tilde{P}) = \tilde{X}_n(\tilde{P}) \) if and only if \( \tilde{P} \in \tilde{A}_n \).

**Proof:** For every \( \tilde{P} \in \tilde{A}_n \), we define \( \tilde{X}_n(\tilde{P}) = \tilde{C}_n(\tilde{P}) \). For each fixed \( P = \{V_1, \ldots, V_n\} \in \tilde{B}_n \cup \tilde{C}_n \) we define \( \tilde{X}_n(P) \) as follows.

There is a finite set of circles \( c_1, \ldots, c_k \), labelled in decreasing order or radius (that can be 0) centered in \( \tilde{C}_n(\tilde{P}) \) and intersecting \( \tilde{P} \).

Define \( X_1, \ldots, X_m \) where \( X_i = c_i \cap r_i \). For each of these points \( X_i \), consider the sequence \( (a_{i1}, \ldots, a_{ik}) \) where \( a_{ij} \) is the number of points in \( r_i \cap c_j \).

Define any set \( \tilde{Q}^* = \{(X_1, l_1), \ldots, (X_m, l_m)\} \) where \( l_i \) is some natural number such that \( l_i \leq l_j \) if and only if \( (a_{i1}, \ldots, a_{ik}) \leq (a_{j1}, \ldots, a_{jk}) \) (in the lexicographic order). Then \( \tilde{X}_n(P) = \tilde{B}_n^*(\tilde{Q}^*) \).

We can see that the function \( \tilde{X}_n : \tilde{P}_n \to \mathbb{R}^2 \) defined according to the previous rules is a \( n \)-multiset center and \( \tilde{C}_n(P) = \tilde{X}_n(P) \) if and only if \( P \in \tilde{A}_n \).

**Lemma 5** (definition of \( \tilde{Y}_n \)) There exists a \( n \)-multiset center \( \tilde{Y}_n : \tilde{P}_n \to \mathbb{R}^2 \), that will be called center of axial asymmetry, such that \( \tilde{C}_n(\tilde{P}), \tilde{X}_n(\tilde{P}), \tilde{Y}_n(P) \) are pairwise distinct and not collinear if and only if \( \tilde{P} \in \tilde{C}_n \).

**Proof:** For every \( \tilde{P} \notin \tilde{C}_n \), we define \( \tilde{Y}_n(\tilde{P}) = \tilde{C}_n(\tilde{P}) \). For each fixed \( \tilde{P} = \{V_1, \ldots, V_n\} \in \tilde{P}_n \), we define \( \tilde{Y}_n(P) \) as follows.

Let \( r \) be the line passing through \( \tilde{C}_n(P), \tilde{X}_n(P) \). The line \( r \) divides the plane in two half-planes, that we will denote as \( \mathcal{H}_1, \mathcal{H}_2 \).
There is a finite set of concentric circles $c_1, \ldots, c_k$ labelled in decreasing order of radius (that can be 0) centered in $\mathcal{C}_n(\hat{P})$ and intersecting $P$. There is a finite set of concentric circles $r_1, \ldots, r_m$ labelled in decreasing order of radius (that can be 0) centered in $\mathcal{H}_n(\hat{P})$ and intersecting $P$.

Let us assign to each point $X \in \mathcal{P}$ the label $(i, j)$ if $X \in c_i \cap r_j$ and consider the two multisets of labels $L_1, L_2$, each of them corresponding to the labels of the points in $\mathcal{H}_1, \mathcal{H}_2$, respectively.

Let the total order relation $\leq$, defined as

$$L_1 \leq L_2 \iff \min_{\text{lexic. order}}(L_1 \Delta L_2) \in L_1.$$  

Define $v$ to be the free unit vector perpendicular to $r$ and pointing to the first half-plane if $L_1 \leq L_2$ and the opposite one if $L_1 \geq L_2$. Let $\lambda = \sum_{V \in \mathcal{P}} \|V - \mathcal{C}_n(\hat{P})\|$. Then, we define $\mathcal{F}_n(\hat{P}) = \mathcal{C}_n(\hat{P}) + \lambda v$.

The function $\mathcal{P}_n : \mathcal{P}_n \to \mathbb{R}^2$ defined according to the previous method is a $n$-multiset center and $\mathcal{C}_n(\hat{P}), \mathcal{H}_n(\hat{P}), \mathcal{P}_n(\hat{P})$ are pairwise distinct and not collinear if and only if $\hat{P} \in \mathcal{C}_n$.

\[ \square \]

**Theorem 6 (Main Problem for $n$-multisets)**

(i) For every $n$-multiset center $\mathcal{F}_n : \mathcal{F}_n \to \mathbb{R}^2$ and every $\hat{P} \in \mathcal{F}_n$, $\mathcal{F}_n(\hat{P})$ always belongs to the set of points fixed by the group of symmetries of $\hat{P}$.

(ii) Conversely, for every point $X$ in the set of points fixed by the group of symmetries of a given $\hat{P} \in \mathcal{P}_n$, there exists a $n$-multiset center $\mathcal{F}_n : \mathcal{P}_n \to \mathbb{R}^2$ such that $\mathcal{F}_n(\hat{P}) = X$.

**Proof:** To prove the first statement, let $\hat{P} = \{V_1, \ldots, V_n\} \in \mathcal{F}_n$, suppose that $T$ is in the symmetry group of $\hat{P}$. See that, by definition of center,

$$T(\mathcal{F}_n(\hat{P})) = \mathcal{F}_n(T(V_1), \ldots, T(V_n)) = \mathcal{F}_n(\{V_1, \ldots, V_n\}) = \mathcal{F}_n(\hat{P}).$$

For the second statement we need the fact that, for any set of centers $\mathcal{X}_n^1, \ldots, \mathcal{X}_n^k$ the affine combination

$$\lambda_1 \mathcal{X}_n^1 + \ldots + \lambda_k \mathcal{X}_n^k$$

is also a center. So, for each $\hat{P} \in \mathcal{P}_n$, for every $X$ in the set of fixed points in its group of symmetries and for $\mathcal{C}_n, \mathcal{X}_n, \mathcal{P}_n$, being the centroid, the center of rotational asymmetry and the center of axial asymmetry, there is at least one center

$$\mathcal{F}_n = \lambda_1 \mathcal{C}_n + \lambda_2 \mathcal{X}_n + \lambda_3 \mathcal{P}_n$$

for some $\lambda_1 + \lambda_2 + \lambda_3 = 1$ such that $\mathcal{F}_n(\hat{P}) = X$.

\[ \square \]
5 Main result for \( n \)-gons

The main difficulty in this section is that we need to detect asymmetry in the adjacency relation, not in the vertices. In the following, let us denote by \( \mathcal{A}_n \), \( \mathcal{B}_n \) and \( \mathcal{C}_n \) to the elements in \( \mathcal{P}_n \) which group of symmetries has one fixed point, a line of fixed points and a plane of fixed points, respectively.

**Lemma 7 (definition of \( \mathcal{X}_n \))** There exists a \( n \)-gon center \( \mathcal{X}_n : \mathcal{P}_n \to \mathbb{R}^2 \), that we will call the center of rotational asymmetry, such that \( \mathcal{C}_n(P) = \mathcal{X}_n(P) \) if and only if \( P \in \mathcal{A}_n \).

*Proof:* Let every \( P \in \mathcal{P}_n \). Let us denote \( \tilde{P} \) to be the \( n \)-multiset of vertices of \( P \). If \( P \in \mathcal{A}_n \) we define \( \mathcal{X}_n(P) = \mathcal{C}_n(P) \). If \( P \notin \mathcal{A}_n \) and \( \tilde{P} \notin \mathcal{A}_n \), then we define \( \mathcal{X}_n(P) = \tilde{X}_n(\tilde{P}) \). In other case, that is, \( P \notin \mathcal{A}_n \) and \( \tilde{P} \in \mathcal{A}_n \), we define \( \mathcal{X}_n(P) \) as follows.

The case in which \( \tilde{P} \) is collinear, requires a different approach. Let \( \{W_1, \ldots, W_m\} \) be the multiset of elements at maximal distance from \( \mathcal{C}_n(P) \). Consider the set of of sequences of integers (2 for each element)

\[
(a_{12}, \ldots, a_{1,n+1}), (b_{12}, \ldots, b_{1,n+1}), (a_{m2}, \ldots, a_{m,n+1}), (b_{m2}, \ldots, b_{m,n+1}),
\]

(1)
each pair corresponding to the two polygonal sequences \((A_1, \ldots, A_{i,n+1}), (B_1, \ldots, B_{i,n+1})\) starting at each \( W_i \) and such that \( a_{ij} \) (resp. \( b_{ij} \)) denote the distance from \( A_{i,j-1} \) to \( A_{ij} \) (resp. from \( B_{i,j-1} \) to \( B_{ij} \)) with positive sign if \( A_{ij} \) is closer to \( \mathcal{C}_n(P) \) than \( A_{i,j-1} \). These sequences in Equation (1) are all of them different. So only one of these sequences is minimal with respect to the lexicographic order. The point \( W_i \) corresponding to these sequence will be \( \mathcal{X}_n(P) \).

Suppose that not all the points in \( \tilde{P} \) are collinear. Define \( c_1, \ldots, c_k \) and \( r_1, \ldots, r_m \) as in the proof of Lemma 8.

Let \( \{W_1, \ldots, W_m\} \) be the multiset which elements are in \( P \cap C_1 \). For each \( W_i \), consider the two polygonal sequences starting at \( W_i \) and denote them by

\[
(A_{i1}, \ldots, A_{i,n+1}), \quad (B_{i1}, \ldots, B_{i,n+1}).
\]

If the polygon \( P \) is not simple, we do not have a natural notion of orientation. But in this setting, we will be able to establish a criteria to say that one of these two sequences is positively oriented and the other one negatively oriented. By hypothesis \( \tilde{P} \) is not collinear. Consider \( r \) to be the ray from \( \mathcal{C}_n(P) \) to \( W_i \). Let \( r_+ \) (resp. \( r_- \)) to be the ray from \( \mathcal{C}_n(P) \) to some of the points in \( P \setminus r \) such that the angle from the ray \( r \) to \( r_+ \) goes in the positive (resp. negative) sense and is minimal. From all the points of \( \tilde{P} \) in \( r_+ \) (resp. \( r_- \)), consider the furthest one from \( \mathcal{C}_n(P) \) and denote it by \( V_+ \) (resp. \( V_- \)). In the following, we suppose that \((A_{i1}, \ldots, A_{i,n+1})\) reaches \( V_+ \) before \( V_- \) and we will call it the positively oriented sequence. \((B_{i1}, \ldots, B_{i,n+1})\) will reach \( V_- \) before \( V_+ \) and will be called the negative oriented sequence.

Associated to this positively oriented sequence, we are going to define another sequence \((a_i, 2, \ldots, a_i, n+1)\). For \( j = 2, \ldots, n \), the element \( a_{ij} \) is a pair \((\rho, \alpha)\)
where $A_{ij} \in c_\rho$ and $\alpha$ is the angle $\angle(A_{ih}, c_\rho(P), A_{ij})$ where $h < j$ is the last index such that $A_{ih} \neq c_\rho(P)$ and we consider that $\angle(A_{ih}, c_\rho(P), A_{ij}) = 0$ if $A_{ij} = c_\rho(P)$. We define a sequence, $(b_{i2}, \ldots, b_{i,n+1})$ for the negatively oriented sequence in a similar way with the corresponding modifications. See Figure 6.

Note that a $n$-gon is rotationally symmetric if and only if, for any $i$, the corresponding $(a_{i2}, \ldots, a_{i,n+1})$ is periodic. As a consequence and since $P$ is not rotationally symmetric, then all the sequences must be different. Pick the point $Z_+ \in \{W_1, \ldots, W_m\}$ which sequence is minimal with respect to the lexicographic order (using, in turn, the lexicographic order to compare each position). Define $Z_-$ in a similar way.

We choose the one of the two possible such that from $Z_+$ to $Z_-$ is positively oriented (and from $Z_-$ to $Z_+$ is negatively oriented).

\[\square\]

**Lemma 8 (definition of $Y_n$)** There exists a $n$-gon center $Y_n : P_n \to \mathbb{R}^2$, that we will call the **center of rotational assymetry**, such that $c_\rho(P), X_n(P), Y_n(P)$ are different and not collinear if and only if $P \in c_\rho$. 

\[\text{Proof:}\] Let $P \in P_n$ and let $\tilde{P}$ denote the $n$-multiset of its vertices.

If $P \notin c_\rho$, we define $Y_n(P) = c_\rho(P)$. If $P \in c_\rho$ and $\tilde{P} \in \tilde{c}_\rho$, then $Y_n(P) = Y_n(\tilde{P})$. In other case, that is, $P \in c_\rho$ and $P \notin \tilde{c}_\rho$, then we define $Y_n(P)$ as follows.

Let us define $r, \mathcal{H}_1, \mathcal{H}_2, c_1, \ldots, c_k, r_1, \ldots, r_m$ as in the proof of Lemma 5 using $c_\rho(P), X_n(P)$ (instead of $\tilde{c}_\rho(P), \tilde{X}_n(P)$).

Let $Q = \{W_1, \ldots, W_l\}$ to be the elements in $c_1 \cap r_1$ ($Q$ is a multiset with at most two different elements, both of them out of $r$). For each $W_i \in Q$, consider the polygonal chains with root at $W_i$

\[
(A_{i1}, \ldots, A_{i,n+1}), \quad (B_{i1}, \ldots, B_{i,n+1})
\]
Now define, associated to each of these elements $W_i$ a couple of sequences $(a_{i2}, \ldots, a_{i,n+1}), (b_{i2}, \ldots, b_{i,n+1})$. For $j = 1, \ldots, n$, the element $a_{ij}$ (similar for $b_{ij}$) is a triple $(x, y, z)$ such that $A_{ij} \in c_x \cap r_y$ and $z$ is defined as follows.

$$z = \begin{cases} 
-1 & \text{if the last predecessor of } A_{ij} \text{ out of } r \text{ is not in the same halfplane} \\
+1 & \text{if the last predecessor of } A_{ij} \text{ out of } r \text{ is in the same halfplane} \\
as A_{ij} \text{ or the earliest successor of } A_{ij} \text{ out of } r & 
\end{cases}$$

Note that $P$ is axially symmetric if for any (for all) $i = 1, \ldots, k$ we have that $(a_{i2}, \ldots, a_{i,n+1}) = (b_{i2}, \ldots, b_{i,n+1})$. Let us assume that, for every $i = 1, \ldots, k$, $(a_{i2}, \ldots, a_{i,n+1})$ is smaller than $(b_{i2}, \ldots, b_{i,n+1})$ in the lexicographic order where each position $a_{ij}$ or $b_{ij}$ is compared using, in turn, the lexicographic order for the triples $(x, y, z)$.

Moreover, for all $i = 1, \ldots, k$ the sequences $(a_{i2}, \ldots, a_{i,n+1})$ are all different. Let us choose $W_i$ to be the point in $Q$ corresponding to the smallest sequence with respect to the order described above. Then, we define $\tilde{\mathcal{Y}}_n(P) = W_i$.

The function $\mathcal{Y}_n : \mathcal{P}_n \to \mathbb{R}^2$ defined according to the previous method is a $n$-gon center and $X_n(P), Y_n(P)$ are pairwise distinct and not collinear if and only if $\tilde{P} \in C_n$.

\[ \square \]

**Theorem 9 (Main Problem for $n$-gons)**

(i) For every $n$-gon center $\mathfrak{Z}_n : \mathcal{F}_n \to \mathbb{R}^2$ and every $P \in \mathcal{F}_n$, $\mathfrak{Z}_n(P)$ always belongs to the set of points fixed by the group of symmetries of $P$.

(ii) Conversely, for every point $X$ in the set of points fixed by the group of symmetries of a given $P \in \mathcal{P}_n$, there exists a $n$-gon center $\mathfrak{Z}_n : \mathcal{P}_n \to \mathbb{R}^2$ such that $\mathfrak{Z}_n(P) = X$.

\[ \text{Proof:} \quad \text{The proof is exactly the same as in Theorem 6, but removing the tildes.} \quad \square \]

\[ \text{6 Final comments} \]

The intention of this notes is to provide a theoretical result relating the group of symmetries of a given multiset or polygon and the location of the centers of this object.

The centers $\tilde{\mathfrak{X}}_n, \tilde{\mathfrak{Y}}_n, \mathfrak{X}_n, \mathfrak{Y}_n$ are defined in terms of the center $\tilde{\mathfrak{B}}_n$. This last multiset center is not continuous, with respect to the most natural candidates for a topology for $\tilde{\mathcal{P}}_n$, $\mathcal{P}_n$, (such as the one induced by the Hausdorff distance for $\tilde{\mathcal{P}}_n$ and the corresponding quotient topology for $\mathcal{P}_n$). This make that the centers $\tilde{\mathfrak{X}}_n, \tilde{\mathfrak{Y}}_n, \mathfrak{X}_n, \mathfrak{Y}_n$ look rather artificial.
Now that the main results in this paper are proved and we know the answer for our Main Problem, we could look for sets of centers \( \tilde{X}_1, \ldots, \tilde{X}_n \) and \( X_1, \ldots, X_n \), among the ones with geometric meaning, such that if all of them coincide (resp. are collinear) the corresponding object has rotational symmetry (resp. axial symmetry), in a more similar fashion to the “inspiring results” listed in the introduction. To do so, the difficulty lies in solving equations of the type

\[
\tilde{Z}_1(P) = \tilde{Z}_2(P) \quad \text{or} \quad Z_1(P) = Z_2(P)
\]

in the indeterminate \( \tilde{P}, P \), respectively. This equation require some algebraic study and will be a matter for future work.

References

[1] Al-Sharif, A., Hajja, M., & Krasopoulos, P. T. (2009). Coincidences of centers of plane quadrilaterals. Results in Mathematics, 55(3), 231-247.
[2] Bachmann, F., & Schmidt, E. (2016). N-gons. In n-gons. University of Toronto Press.
[3] Edmonds, A. L. (2009). The center conjecture for equifacetal simplices. Advances in Geometry, 9, 563–570.
[4] Farré Puiggalí, M., & Prieto-Martínez, L. F. (2022). n-gon centers and central lines. arXiv e-prints, arXiv-2204.
[5] Franssen, W. N (2011). The distance from the incenter to the Euler line. Forum Geometricorum 11, 231–236.
[6] Isaacs, I. M. (2009). Geometry for college students (Vol. 8). American Mathematical Society.
[7] C. Kimberling (1993). Functional equations associated with triangle geometry. Aequationes Math. 45, 127–152.
[8] C. Kimberling (1994). Central Points and Central Lines in the Plane of a Triangle. Mathematics Magazine 67.3, 163–187.
[9] C. Kimberling, Encyclopedia of Triangle centers, https://faculty.evansville.edu/ck6/encyclopedia/etc.html
[10] Krantz, S. G., McCarthy, J. E., & Parks, H. R. (2006). Geometric characterizations of centroids of simplices. Journal of mathematical analysis and applications, 316(1), 87-109.
[11] Jiménez-Sánchez, J., Bosque, J. J., Londoño, G. A. J., Molina-García, D., Martínez, A., Pérez-Beteta, J., ... & Pérez-García, V. M. (2021). Evolutionary dynamics at the tumor edge reveal metabolic imaging biomarkers. Proceedings of the National Academy of Sciences, 118(6).
[12] Prieto-Martínez, L. F., & Sánchez-Cauce, R. (2021). Generalization of Kimberling’s concept of triangle center for other polygons. Results in Mathematics, 76(2), 1-18.
[13] Chris Van Tienhoven, Encyclopedia of Quadri-Figures, https://chrisvantienhoven.nl/