MINIMAL MODELS OF SEMI-LOG-CANONICAL PAIRS

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Abstract. We compare the minimal model of a log canonical pair with the minimal model of its reduced boundary. These results are then used to study the existence of the minimal model of a semi-log-canonical pair using its normalization.

In birational geometry, it is frequently necessary to work not just with log canonical pairs \((X, \Delta)\), but with their non-normal variants, called semi-log-canonical pairs. Such pairs appear when one tries to compactify the moduli spaces of varieties and in inductive arguments.

Many properties of log canonical pairs have been generalized to the semi-log-canonical setting [Amb03, Amb11, Fuj14, Fuj17, Kol13], but it was observed in [Kol11] that log canonical rings of semi-log-canonical pairs are not always finitely generated and some flips of semi-log-canonical pairs do not exist. Note that, by contrast, abundance holds for a semi-log-canonical pair iff it holds for its normalization; this was proved in increasing generality in [Miy88, Kol92, KMMc94, Fuj00, Gon13, FG14, HX16].

The aim of this note is to describe some conditions that guarantee the existence of minimal models for certain semi-log-canonical pairs. Our assumptions are rather restrictive, but they may be close to being optimal. The key is to understand an even simpler question involving log canonical pairs: How does the boundary of a log canonical pair change under a flip?

This is a very natural problem, that first appeared explicitly in Tsuchida’s treatment of semi-stable flips [Miy87], later in Shokurov’s approach that reduces flips to special flips [Sho92, Kol92] and in [HMX14, Sec.4]; see also [BP12].

We are thus led to the following general questions.

Question 1. Let \((X, D + \Delta)\) be an lc pair that is projective over a base scheme \(S\) with relatively ample divisor \(H\), where all divisors in \(D\) appear with coefficient 1. Set \((X^0, D^0 + \Delta^0) := (X, D + \Delta)\) and for \(i = 1, \ldots, m\) let

\[
\phi^i : (X^{i-1}, D^{i-1} + \Delta^{i-1}) \to (X^i, D^i + \Delta^i)
\]

be the steps of the \((X, D + \Delta)\)-MMP with scaling of \(H\); see Definition [11]. Let \(\rho : \tilde{D} \to D\) be the normalization. Do the restrictions

\[
\phi^i_{\tilde{D}} := \phi^i |_{\tilde{D}^{i-1}} : (\tilde{D}^{i-1}, \Diff_{\tilde{D}} \Delta^{i-1}) \to (\tilde{D}^i, \Diff_{\tilde{D}} \Delta^i)
\]

form the steps of the MMP starting with \((\tilde{D}^0, \Diff_{\tilde{D}} \Delta^0) := (\tilde{D}, \Diff_{\tilde{D}} \Delta)\) and with scaling of \(\rho^*H\)?

Notation 2. We follow the terminology and notation of [KM98, Kol13].

From now on, whenever we write a divisor as \(D + \Delta\), we assume that all irreducible components of \(D\) appear with coefficient 1 (\(\Delta\) may also contain divisors with coefficient 1).
Let $\rho : \bar{D} \to D$ denote the normalization. The different of $\Delta$ on $\bar{D}$ is denoted by $\text{Diff}_\Delta \Delta$. It is a $\mathbb{Q}$-divisor on $\bar{D}$ that satisfies a natural $\mathbb{Q}$-linear equivalence

$$K_{\bar{D}} + \text{Diff}_\Delta \Delta \sim_\mathbb{Q} \rho^* (K_X + D + \Delta). \quad (2.1)$$

See [Kol13 4.2] for a precise definition and its main properties. In order to avoid secondary sub and superscripts, we usually write $\text{Diff}_\Delta \Delta^i$ instead of the more precise $\text{Diff}_\bar{D} \Delta^i$.

In the original definition, a step of the MMP corresponds to an extremal ray $[CKM88]$. By (2.1), any contraction of an extremal ray on $X$ induces the contraction of an extremal face on $D$, but the face may well have dimension $> 1$. In an MMP with scaling of an ample divisor, the steps correspond to certain contractions of extremal faces. The divisor $H$ plays a very minor role in the sequel, but it makes it possible for us to tell exactly which MMP steps we get.

It turns out that a positive answer to Question 1 gives a positive answer to the following problem on slc pairs.

**Question 3.** Let $(X, \Delta)$ be an slc pair that is projective over a base scheme $S$ with normalization $\pi : (\bar{X}, \bar{D} + \bar{\Delta}) \to (X, \Delta)$, conductor $\bar{D} \subset \bar{X}$ and $H$ an ample divisor on $X$. Set $(\bar{X}^0, \bar{D}^0 + \bar{\Delta}^0) := (\bar{X}, \bar{D} + \bar{\Delta})$ and for $i = 1, \ldots, m$ let

$$\bar{\phi}^i : (\bar{X}^{i-1}, \bar{D}^{i-1} + \bar{\Delta}^{i-1}) \to (\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)$$

be the steps of the $\bar{D}$-$\text{MMP}$ with scaling of $\pi^* H$. Do we get

$$\phi^i : (X^{i-1}, \Delta^{i-1}) \to (X^i, \Delta^i),$$

which form the steps of the $(X, \Delta)$-$\text{MMP}$ with scaling of $H$ and such that $(\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)$ is the normalization of $(X^i, \Delta^i)$?

**Example 4.** We give 2 types of examples showing that in Question 1 we usually do not get the steps of the $(D, \text{Diff}_\Delta \Delta)$-$\text{MMP}$.

(4.1) Start with a smooth variety $X'$, a smooth divisor $D' \subset X'$ and another smooth divisor $C' \subset D'$. Assume that $K_{X'} + D'$ is ample. Set $X := B_{C'}X'$ with exceptional divisor $E$ and let $D \subset X$ denote the birational transform of $D'$.

For any $1 \geq \epsilon > 0$, $(X, D + \epsilon E)$ is an lc pair whose canonical model is $(X', D')$ and $(D', 0)$ is its own canonical model.

However, $(D, \epsilon \text{Diff}_D E) \cong (D', \epsilon C')$ is different from $(D', 0)$.

Note further that $K_X + D$ is the pull-back of $K_{X'} + D'$, hence semiample and big. Thus the stable base locus of $K_X + D + \epsilon E$ is $E$. If $1 > \epsilon > 0$ then the only log canonical center of $(X, D + \epsilon E)$ is $D$ and the other log centers are $E$ and $E \cap D$; see Definition 3. Thus the stable base locus contains the log centers but not the log canonical center.

Here are some concrete examples.

(4.1.1) Let $X'$ be a smooth surface, $D' \subset X'$ a smooth rational curve and $C' \subset D'$ a set of 3 points. Then $(D, \text{Diff}_D E) \cong (D', C')$ has ample log canonical class but $(D', 0) \cong (\mathbb{P}^1, 0)$ has negative log canonical class.

(4.1.2) For dim $X' \geq 3$ it can also happen that the $(D, \epsilon \text{Diff}_D E)$-$\text{MMP}$ tells us to contract $C'$. Take $X' = \mathbb{P}^3$ and let $D' \subset X'$ be a smooth surface of degree 5 that contains a line $C'$. Then the self-intersection of $C'$ is $-3$, thus for $1 \geq \epsilon > \frac{1}{3}$ the first (an only) step of the $(D, \epsilon \text{Diff}_D E)$-$\text{MMP}$ is to contract $C'$. 


Let $B$ be a smooth curve and $f : X \to B$ be a flat family of surfaces with quotient singularities and such that $K_X$ is $Q$-Cartier. Let $g : X \to Z$ be a flipping contraction. (For concrete examples, see [KM98, 2.7] or the list in [KM02].) Thus there is a closed point $0 \in B$ such that $g$ is an isomorphism over $B \setminus \{0\}$. Set $D := X_0$ and let $C \subset D$ denote the flipping curve. Our example is the pair $(X, D)$. Here $\text{Diff}_D 0 = 0$, hence we need to compare the MMP for $(X, D)$ with the MMP for $(D, 0)$.

Over $0 \in B$ we have a birational contraction $g_0 : X_0 \to Z_0$ that contracts $C \subset X_0$ to a point. Moreover $(C \cdot K_{X_0}) = (C \cdot K_X) < 0$, thus $Z_0$ is again log terminal and the contraction $g_0 : X_0 \to Z_0$ is a step in the MMP for $X_0 = D$.

However, since $g : X \to Z$ a flipping contraction, the special fiber of the flip $g^+ : X^+ \to Z$ is another surface $X_0^+ \to Z_0$ with a new exceptional curve $C^+ \subset X_0^+$ such that $(C^+ \cdot K_{X_0^+}) = (C^+ \cdot K_{X^+}) > 0$. Thus $X_0^+$ is not the canonical model of $X_0$ and $X_0 \to X_0^+$ is not even a step of any minimal model program.

We can easily arrange that $K_{X^+}$ is ample. In this case the stable base locus of $K_X$ is the flipping curve $C \subset X_0 = D$. The only log canonical center of $(X, D)$ is $D$ which is not contained in the stable base locus of $K_X$.

It is easy to see that $D$ must have at least 1 noncanonical singularity that is also contained in $C$. This gives a 0-dimensional log center of $(X, D)$ that is contained in the stable base locus.

**Example 5.** Every counter example to Question 1 where $D$ is normal, gives a counter example to Question 3 as follows.

Let $b \in B$ be a smooth, projective, pointed curve of genus $\geq 1$. We can glue $(X, D+\Delta)$ to $(B \times D, \{b\} \times D + B \times \text{Diff}_D \Delta)$ along $D$ to get an slc pair $(Y, \Delta_Y)$ whose normalization is the disjoint union of $(X, D+\Delta)$ and $(B \times D, \{b\} \times D + B \times \text{Diff}_D \Delta)$.

On $(X, D+\Delta)$ we get the steps of the $(X, D+\Delta)$-MMP

$$\phi^i : (X^{i-1}, D^{i-1} + \Delta^{i-1}) \to (X^i, D^i + \Delta^i)$$

and these restrict to

$$\phi^i_D : (D^{i-1}, \text{Diff}_D \Delta^{i-1}) \to (D^i, \text{Diff}_D \Delta^i).$$

Let us denote the steps of the $(D, \text{Diff}_D \Delta)$-MMP by

$$\psi_i : (D_{i-1}, \text{Diff}_D \Delta_{i-1}) \to (D_i, \text{Diff}_D \Delta_i).$$

Then the steps of the $(B \times D, \{b\} \times D + B \times \text{Diff}_D \Delta)$-MMP are given by

$$(B \times D_{i-1}, \{b\} \times D_{i-1} + B \times \text{Diff}_D \Delta_{i-1}) \to (B \times D_i, \{b\} \times D_i + B \times \text{Diff}_D \Delta_i).$$

If $(D^i, \text{Diff}_D \Delta^i) \not\cong (D_i, \text{Diff}_D \Delta_i)$, then we can not glue the resulting pairs

$$(X^i, D^i + \Delta^i) \text{ and } (B \times D_i, \{b\} \times D_i + B \times \text{Diff}_D \Delta_i).$$

Thus the $(Y, \Delta_Y)$-MMP does not exist.

We give positive answers to Questions 1 and 3 when the singularities of $(X, D+\Delta)$ (resp. of $(X, D+\Delta)$) are mild along the exceptional locus of $\phi$ (resp. of $\phi$). We use discrepancies to make this assertion precise.

**Definition 6.** Let $(X, \Theta)$ be an lc pair. An irreducible subvariety $W \subset X$ is called a log canonical center (resp. a log center) of $(X, \Theta)$ if there is a divisor $E$ over $X$ such that $\text{center}_X E = W$ and $a(E, X, \Theta) = -1$ (resp. $a(E, X, \Theta) < 0$).
Assume next that \( \Theta = D + \Delta \) and let \( \rho : \tilde{D} \to D \) denote the normalization. By adjunction \cite{Kol13} 4.9, \( W \subseteq \tilde{D} \) is a log center of \( (\tilde{D}, \text{Diff} \Delta) \) iff \( \rho(W) \) is a log center of \( (X, D + \Delta) \). See \cite{Kol13} Chap.7 for more on log centers.

From now on we assume that the base scheme \( S \) is essentially of finite type over a field of characteristic 0. Our main result is the following.

**Theorem 7.** Using the notation and assumptions of Question 3, assume in addition that the intersection of \( D \) with the exceptional locus of 
\[
\Phi^m := \phi^m \circ \cdots \circ \phi^1 : X \longrightarrow X^m
\]
does not contain any log center of \( (\bar{X}, D + \Delta) \). Then the maps
\[
\phi^i : (\bar{D}^{i-1}, \text{Diff}_D \Delta^{i-1}) \longrightarrow (\bar{D}^i, \text{Diff}_D \Delta^i)
\]
form the steps of the MMP starting with \( (\bar{D}^0, \text{Diff}_D \Delta^0) := (\bar{D}, \text{Diff}_D \Delta) \) and with scaling of \( \rho^*H \).

**Remark 8.** As the Examples (4.1.1–2) show, we need to avoid all log centers, not just the log canonical centers.

It can happen that \( \phi^i \) is an isomorphism along \( D^{i-1} \). Thus the precise claim is that each \( \phi^i_D \) is either an isomorphism or an MMP step. (The literature is somewhat inconsistent. Usual definitions of MMP steps allow isomorphisms, but in many statements they are tacitly excluded.)

**Theorem 9.** Using the notation and assumptions of Question 3, assume in addition that the intersection of \( D \) with the exceptional locus of 
\[
\Phi^m_X := \tilde{\phi}^m \circ \cdots \circ \tilde{\phi}^1 : \bar{X} \longrightarrow \bar{X}^m
\]
does not contain any log center of \( (\bar{X}, \bar{D} + \bar{\Delta}) \).

Then the first \( m \) steps of the \( (X, \Delta) \)-MMP with scaling of \( H \) exist
\[
\phi^i : (X^{i-1}, \Delta^{i-1}) \longrightarrow (X^i, \Delta^i),
\]
and \( (\bar{X}^i, \bar{D}^i + \bar{\Delta}^i) \) is the normalization of \( (X^i, \Delta^i) \).

Proof. Let \( (X, \Delta) \) be an slc pair with normalization \( (\bar{X}, \bar{D} + \bar{\Delta}) \to (X, \Delta) \), where \( \bar{D} \subseteq \bar{X} \) is the conductor. Let \( \rho : \bar{D}^n \to \bar{D} \) denote its normalization.

The gluing theory of \cite{Kol13} Chap.5] says that there is a (regular) involution 
\[
\tau : (\bar{D}^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}) \to (\bar{D}^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}),
\]
and \( X \) is obtained from \( \bar{X} \) by identifying the equivalence classes of the relation generated by \( \tau \) on \( \bar{X} \).

Next let
\[
\tilde{\phi}^i : (\bar{X}^{i-1}, \bar{D}^{i-1} + \bar{\Delta}^{i-1}) \longrightarrow (\bar{X}^i, \bar{D}^i + \bar{\Delta}^i)
\]
be the steps of the \( (\bar{X}, \bar{D} + \bar{\Delta}) \)-MMP with scaling of \( \pi^*H \) and assume that Theorem 7 applies. Then
\[
\tilde{\phi}^i_D : ((\bar{D}^{i-1})^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}^{i-1}) \longrightarrow ((\bar{D}^i)^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}^i)
\]
are steps of the \( (\bar{D}^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}) \)-MMP with scaling of \( \rho^*\pi^*H \). Since both \( \text{Diff}_{\bar{D}^n} \bar{\Delta} \) and \( \rho^*\pi^*H \) are \( \tau \)-invariant, the \( \tau \)-action descends to give (regular) involutions
\[
\tau^i : ((\bar{D}^i)^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}^i) \to ((\bar{D}^i)^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}^i).
\]
Let \( Z^i \subseteq \bar{X}^i \) denote the intersection of \( D^i \) with the exceptional locus of
\[
(\phi^i \circ \cdots \circ \phi^1)^{-1} : \bar{X}^i \longrightarrow \bar{X}.
\]
By our assumption, \( Z^i \) does not contain any of the log centers of \( (\bar{X}^i, \bar{D}^i + \Delta^i) \). Thus \( \tau^i \) defines a finite equivalence relation on \( \bar{X}^i \) by [Kol13, 9.55]. Therefore the geometric quotient \( \pi^i : \bar{X}^i \to X^i \) of \( \bar{X}^i \) by the equivalence relation generated by \( \tau^i \) exists by [Kol13, 9.21]. Next [Kol13, 5.38] shows that \((X^i, \Delta^i)\) is slc. By Lemma 12, the resulting rational map

\[ \phi^i : (X^{i-1}, \Delta^{i-1}) \to (X^i, \Delta^i) \]

is an MMP step with scaling of \( H \).

Note that if \( X \) is a normal crossing variety [Kol13, 1.7] then the log centers of \((X, 0)\) are exactly the log canonical centers of \((X, 0)\), which are also the strata of \( X \), so the important distinction between log centers and log canonical centers is not visible in this case.

The normalization \( \pi : (\bar{X}, \bar{D}) \to X \) is a normal crossing pair. It is conjectured that \((\bar{X}, \bar{D})\) has a minimal model. This is currently known if \( K_{\bar{X}} + \bar{D} \) has non-negative Kodaira dimension (on every irreducible component) and the dimension is \( \leq 5 \) [Bir10].

If a minimal model \( \phi : X \to X^{\min} \) exists, then its normalization \((\bar{X}^{\min}, \bar{D}^{\min})\) is a dlt pair whose canonical class is nef and big. The abundance conjecture predicts that its canonical class is semi-ample, but this is known only if the dimension is \( \leq 4 \) [Has10]. However, if abundance holds for \((\bar{X}^{\min}, \bar{D}^{\min})\) then [HX16] implies that the canonical class of \( X^{\min} \) is also semi-ample. In particular, the canonical ring of \( X \) is finitely generated.

Thus Theorem 9 implies the following. Conjecturally, the dimension restrictions should not be necessary.

**Corollary 10.** Let \( X \) be a pure dimensional, projective, normal crossing variety.
Assume that \( K_X \) has non-negative Kodaira dimension on every irreducible component of \( X \) and its stable base locus does not contain any stratum of \( X \).

1. If \( \dim X \leq 5 \) then \( X \) has a minimal model \( \phi : X \to X^{\min} \), \( \phi \) is a local isomorphism at all log canonical centers and \( X^{\min} \) is semi-dlt [Kol13, 5.19].

2. If \( \dim X \leq 4 \) then the canonical ring of \( X \) is finitely generated. □

Before we start the proof of Theorem 8, we need to define what a step of an MMP is.

**Definition 11 (MMP steps).** An MMP step is a diagram of \( S \)-schemes

\[
\begin{array}{ccc}
(X, \Theta) & \xrightarrow{\phi} & (X', \Theta') \\
\downarrow{f} & \nearrow{Z} & \downarrow{f'}
\end{array}
\]

with the following properties.

1. \((X, \Theta)\) and \((X', \Theta')\) are pure dimensional lc pairs,
2. \( \phi \) is birational,
3. \( f, f' \) are projective and generically finite,
4. \(- (K_X + \Theta) \) is \( f \)-ample and \( K_X' + \Theta' \) is \( f' \)-ample,
5. \( f' \) has no exceptional divisors and
6. \( \Theta' = \phi_* \Theta \).

Note that (3) and (6) together imply that \( \phi \) is a rational contraction, that is, \( \phi^{-1} \) has no exceptional divisors.
For slc pairs, one needs to pay extra attention to the non-normal locus, and there are various possible definitions. However, if \( \phi \) is a local isomorphism at all codimension 1 singular points, then the above definition works without changes. This is the only case that we use in the sequel.

We frequently call \( \phi : (X, \Theta) \rightarrow (X', \Theta') \) an MMP step if it sits in a diagram as in \((11.1)\) for suitable \( Z \). Note that \( Z \) is not uniquely determined by \( \phi : (X, \Theta) \rightarrow (X', \Theta') \); if \( Z \rightarrow Z_1 \) is finite then we can replace \( Z \) by \( Z_1 \). The usual choice is to take the unique \( Z \) such that \( f_* \mathcal{O}_X = \mathcal{O}_Z \). However, the latter condition is not preserved when passing to the normalization of \( X \) or to a divisor in \( X \). Thus allowing different choices of \( Z \) is convenient for us.

If \( H \) is a \( \mathbb{Q} \)-Cartier divisor on \( X \) then \((11.1)\) is an MMP step with scaling of \( H \) if, in addition,

1. \( H \) is \( f' \)-ample, \(-H' := -\phi_* H \) is \( f' \)-ample,
2. \( K_X + \Theta + cH \) is numerically \( f \)-trivial for some \( c \in \mathbb{Q} \), (this implies that \( K_{X'} + \Theta' + cH' \) is numerically \( f' \)-trivial) and \( K_{X'} + \Theta' + cH' \) is numerically \( f' \)-trivial),
3. \( K_X + \Theta + cH \) has positive degree on every proper, irreducible curve \( C \subset X \) that is not contracted by \( f \) (and lies over a closed point of \( Z \)).

In practice we start with a pair \( (X, \Theta + cH) \) such that \( K_X + \Theta + cH \) is ample over \( S \). We then decrease the value of \( c' \) until we reach \( c \leq c' \) such that \( K_X + \Theta + cH \) is nef but not ample. If a multiple of \( K_X + \Theta + cH \) is semiample, it gives us \( f : X \rightarrow Z \); see \([BCHM10]\) for details.

The following comparison result is clear from the definition.

**Lemma 12.** Let \( (X, \Theta) \) and \( (X', \Theta') \) be pure dimensional slc pairs with normalizations \( \pi : (\bar{X}, \bar{D} + \bar{\Theta}) \rightarrow (X, \Theta) \) and \( \pi' : (\bar{X}', \bar{D}' + \bar{\Theta}') \rightarrow (X', \Theta') \). Then \((11.1)\) is an MMP step ifff

\[
(\bar{X}, \bar{D} + \bar{\Theta}) \overset{\phi}{\rightarrow} (\bar{X}', \bar{D}' + \bar{\Theta}') \quad (12.1)
\]

is an MMP step, where \( \bar{f} = f \circ \pi \) and \( \bar{f}' = f' \circ \pi' \).

Furthermore, if \( H \) is a \( \mathbb{Q} \)-Cartier divisor on \( X \) then \((11.1)\) is an MMP step with scaling of \( H \) ifff \((12.1)\) is an MMP step with scaling of \( \pi^* H \). \( \square \)

Next we consider a generalization of MMP steps.

**Definition 13.** A diagram as in \((11.1)\) is called a sub-MMP step if

1. the assumptions \((11.2–5)\) hold,
2. \( f' \) is allowed to have exceptional divisors and
3. \( \text{coeff}_{G'} \Theta' \leq \text{coeff}_{G'} \Theta \) for every divisor \( G' \subset X' \) that is not \( f' \)-exceptional.

(By Lemma \[14\] this inequality then holds for all divisors over \( X \).)

The following example is good to keep in mind. Let \( X \) be a smooth surface and \( C \subset X \) a smooth, rational curve with self-intersection \( \leq -3 \). Let \( X \rightarrow X' \) denote the contraction of \( C \).

Then \( (X, C) \rightarrow (X, 0) \) and \( (X', 0) \rightarrow (X, 0) \) are both sub-MMP step. Thus \( \phi \) can be an isomorphism on the underlying varieties yet a non-trivial sub-MMP step.

The main reason for this definition is Lemma \[16\] but first we prove that the usual discrepancy inequalities (cf. \([KM98\ 3.38]\) or \([Kol13\ 1.19 \text{ and } 1.22]\)) also hold for sub-MMP steps.
Lemma 14. Consider a sub-MMP step of lc pairs

\[(X, \Theta) \xrightarrow{\phi} (X', \Theta') \]

where \(f, f'\) are birational. Then \(a(E, X', \Theta') \geq a(E, X, \Theta)\) for every divisor \(E\) over \(X\). Furthermore, for every \(E\), the following are equivalent.

1. \(a(E, X', \Theta') > a(E, X, \Theta)\).
2. \(\phi\) is not a local isomorphism at the generic point of center \(X\).
3. \(\phi^{-1}\) is not a local isomorphism at the generic point of center \(X\).
4. Either \(f\) or \(f'\) has positive dimensional fiber over the generic point of center \(Z\).

Proof. Let \(Y\) be the normalization of the main component of the fiber product \(X \times_Z X'\) with projections \(X \xrightarrow{g'} Y \xrightarrow{g} X'\). Write

\[K_Y \sim_Q g^*(K_X + \Theta) - F\]

and

\[K_Y \sim_Q g'^*(K_{X'} + \Theta') - F'\]  \hspace{1cm} (14.5)

where \(g, F = \Theta\) and \(g', F' = \Theta'\). Thus

\[F' - F \sim_Q g'^*(K_{X'} + \Theta') - g^*(K_X + \Theta)\]  \hspace{1cm} (14.6)

Note that \((f' \circ g')_*(F - F') = f_* \Theta - f'_* \Theta'\) is effective by assumption (13.3). Therefore \(F - F'\) is effective by [KM98, 3.39], proving the required inequality.

It is clear that (1) \(\Rightarrow\) (2), (2) \(\Leftrightarrow\) (3) and (2) \(\Rightarrow\) (4). Thus assume (4).

By [KM98, 3.39] the support of \(F - F'\) contains \(\text{Ex}(f' \circ g')\). Arguing similarly we get that it also contains \(\text{Ex}(f \circ g)\). Thus \(a(E, X', \Theta') > a(E, X, \Theta)\) if either \(f\) or \(f'\) has positive dimensional fiber over the generic point of center \(Z\). \(\square\)

Corollary 15. A sub-MMP step \(\phi : (X, \Theta) \to (X', \Theta')\) is an MMP step iff \(a(G', X', \Theta') = a(G', X, \Theta)\) for every divisor \(G' \subset X'\).

Proof. If \(\phi\) is an MMP step then \(\Theta' = \phi_* \Theta\), hence \(a(G', X', \Theta') = a(G', X, \Theta)\) for every divisor \(G' \subset X'\).

Conversely, if \(G' \subset X'\) is an \(f'\)-exceptional divisor then \(a(G', X', \Theta') > a(G', X, \Theta)\) by Lemma 14.2. Thus there are no \(f'\)-exceptional divisors and so \(\Theta' = \phi_* \Theta\). \(\square\)

Lemma 16. Let \(\phi : (X, \Theta) \to (X', \Theta')\) be an MMP step sitting in a diagram (17.1). Assume that \((X, \Theta)\) is lc, \(\Theta = D + \Delta\) where \(D\) is reduced with normalization \(\rho : \tilde{D} \to D\) and none of the irreducible components of \(D\) is contracted by \(\phi\). Then the diagram

\[\begin{array}{c}
(D, \text{Diff}_D \Delta) \xrightarrow{\phi_D} (\tilde{D}', \text{Diff}_{\tilde{D}'} \Delta') \\
\tilde{D} \xrightarrow{\phi_D} \tilde{D}'
\end{array}\]  \hspace{1cm} (16.1)

is a sub-MMP step.

Proof. Assumptions (17.2–4) are clear and (17.5) holds since

\[K_{\tilde{D}} + \text{Diff}_{\tilde{D}} \Delta \sim_Q \rho^*(K_X + D + \Delta)\]

It remains to show that (17.3) holds. More generally, we show that

\[a(E, \tilde{D}, \text{Diff}_{\tilde{D}} \Delta) \leq a(E, \tilde{D}', \text{Diff}_{\tilde{D}'} \Delta')\]  \hspace{1cm} (17.2)

for every divisor \(E\) over \(\tilde{D}\).
We may assume that $f, f'$ are birational. Let $Y$ be the normalization of the main component of the fiber product $X \times_Z X'$ with projections $X \xrightarrow{g} Y \xrightarrow{g'} X'$. As in (14.5) write
\[ g^*(K_X + D + \Delta) \sim_Q g'^*(K_{X'} + D' + \Delta') + F - F', \quad (16.3) \]
where $F - F'$ is effective by [KM98, 3.38] or by Lemma 14.

Let $D_Y$ denote the normalization of the birational transform of $D$ on $Y$. Restricting (16.3) to $D_Y$ we get
\[ (g|_{D_Y})^*(K_D + \text{Diff}_D \Delta) \sim_Q (g'|_{D_Y})^*(K_{D'} + \text{Diff}_{D'} \Delta') + F|_{D_Y} \quad (16.4) \]
and $F|_{D_Y}$ is also effective. \qed

**Corollary 17.** Using the notation and assumptions of Lemma 16 let $p \in \bar{D}$ be a point. Then $\phi_D : (\bar{D}, \text{Diff}_D \Delta) \rightarrow (D', \text{Diff}_{D'} \Delta')$ is a local isomorphism at $p$ iff $\phi : X \rightarrow X'$ is a local isomorphism at $\pi(p)$.

Note that the claims about $X$ and $D$ are different. As in Example 4.1, it can happen that $\phi_D : \bar{D} \rightarrow \bar{D}'$ is an isomorphism but $\text{Diff}_{D'} \Delta' \neq (\phi_D)_* \text{Diff}_D \Delta$.

**Proof.** If $\phi$ is a local isomorphism at $\pi(p)$ then clearly $\phi_D$ is a local isomorphism at $p$. Conversely, if $\phi_D : \bar{D} \rightarrow \bar{D}'$ is a local isomorphism at $p$ then the maps $g_D : D_Y \rightarrow \bar{D}$ and $g_{D'} : D_Y \rightarrow \bar{D}'$ are isomorphic to each other near $p$. By (16.4)
\[ g_{D'}^* \text{Diff}_{D'} \Delta' - g_D^* \text{Diff}_D \Delta = (g|_{D_Y})^*(F - F'). \]
If $\phi$ is not a local isomorphism at $\pi(p)$ then $\text{Supp}(F - F')$ contains $p$ by [KM98, 3.38] or by Lemma 14 thus $\text{Diff}_D \Delta \neq \text{Diff}_{D'} \Delta'$ in every neighborhood of $p$. \qed

**Proposition 18.** Using the notation of Lemma 16 assume in addition that $D \cap \text{Ex}(\phi)$ does not contain any log center of $(X, D + \Delta)$. Then (17) is an MMP step.

**Proof.** Assume to the contrary that (16.1) is not an an MMP step. Then, by Corollary 15 there is a divisor $G' \subset \bar{D}'$ such that
\[ a(G', \bar{D}, \text{Diff}_D \Delta) < a(G', \bar{D}', \text{Diff}_{D'} \Delta'). \quad (18.1) \]
Since $a(G', \bar{D}', \text{Diff}_{D'} \Delta') = -\text{coeff}_{D'} \text{Diff}_{D'} \Delta' \leq 0$, this implies that center $G'$ is a log center of $(\bar{D}, \text{Diff}_D \Delta)$. By adjunction [Kol13, 4.8], center $X G'$ is also a log center of $(X, D + \Delta)$.

Finally (18.1) also shows that $\phi$ is not a local isomorphism at the generic point of center $X G'$. \qed

**19 (Proof of Theorem 7).** By assumption none of the irreducible components of $D$ is contained in $\text{Ex}(\Phi^m)$, thus the maps $\phi_D^m$ are birational. They sit in diagrams
\[ (\bar{D}^{i-1}, \text{Diff}_{\bar{D}} \Delta^{i-1}) \xrightarrow{\phi_D^m} (\bar{D}^i, \text{Diff}_{\bar{D}} \Delta^i) \quad (19.1) \]
that are sub-MMP steps by Lemma 16.

If $\phi_D^m$ is not an MMP step then, by Corollary 15 there is a divisor $G^m \subset \bar{D}^m$ such that
\[ a(G^m, \bar{D}^{m-1}, \text{Diff}_{\bar{D}} \Delta^{m-1}) < a(G^m, \bar{D}^m, \text{Diff}_{\bar{D}} \Delta^{m-1}) \leq 0. \]
Combining with the inequalities
\[ a(G^m, \bar{D}^{i-1}, \text{Diff}_D \Delta^{i-1}) \leq a(G^m, \bar{D}, \text{Diff}_D \Delta^i) \]
of Lemma 14.1, we get that
\[ a(G^m, \bar{D}, \text{Diff}_D \Delta) < a(G^m, X^m, \text{Diff}_D \Delta^m) - 1 \leq 0. \]
Thus center \( \bar{D} \) of \( G^m \) is log center of \( (\bar{D}, \text{Diff}_D \Delta) \). By adjunction [Kol13 4.8], its image in \( X \) is a log center of \( (X, D + \Delta) \) that is contained in \( \text{Ex}(\Phi^m) \).

Note that Proposition 18 almost implies Theorem 7 except that it is not quite clear how to compare \( \text{Ex}(\Phi^m) \subset X \) with the \( \text{Ex}(\phi^i) \subset X^{i-1} \) that are needed to directly apply Proposition 18. The following variant of the concept of exceptional set gives a clearer picture and a slightly different way of deriving Theorem 7.

**Definition 20 (Divisorial exceptional set).** Let \( \phi : X \to X' \) be a birational map of schemes that are proper over \( S \). The divisorial exceptional set of \( \phi \), denoted by \( \text{DEx}(\phi) \), is the set of all divisors \( E \) over \( X \) such that \( \phi \) is not a local isomorphism at the generic point of center of \( E \).

Thus, the usual exceptional set \( \text{Ex}(\phi) \subset X \) is the union of the centers of the divisors in \( \text{DEx}(\phi) \). The advantage of divisorial exceptional sets is that we can compare them for different birational models.

**Lemma 21.** Let \( \phi_i : (X^{i-1}, \Delta^{i-1}) \to (X^i, \Delta^i) \) be a sequence of MMP steps. Then
1. \( \text{DEx}(\phi^m \circ \cdots \circ \phi^1) = \{ E : a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m) \} \) and
2. \( \text{DEx}(\phi^m \circ \cdots \circ \phi^1) = \text{DEx}(\phi^1) \cup \cdots \cup \text{DEx}(\phi^m) \).

Proof. The containments
\[ \text{DEx}(\phi^m \circ \cdots \circ \phi^1) \supset \{ E : a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m) \} \]
\[ \text{DEx}(\phi^m \circ \cdots \circ \phi^1) \subset \text{DEx}(\phi^1) \cup \cdots \cup \text{DEx}(\phi^m) \]
are clear. For a single MMP step \( \phi : (X, \Delta) \to (X', \Delta') \), [KM98 3.38] shows that
\[ \text{DEx}(\phi) = \{ E : a(E, X, \Delta) < a(E, X', \Delta') \}. \]
Combining with the inequalities \( a(E, X^{i-1}, \Delta^{i-1}) \leq a(E, X^i, \Delta^i) \) we obtain that \( a(E, X^0, \Delta^0) \leq a(E, X^m, \Delta^m) \)
and
\[ a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m) \iff E \in \text{DEx}(\phi^1) \cup \cdots \cup \text{DEx}(\phi^m). \]
This shows that
\[ \{ E : a(E, X^0, \Delta^0) < a(E, X^m, \Delta^m) \} = \text{DEx}(\phi^1) \cup \cdots \cup \text{DEx}(\phi^m), \]
which completes the proof. \( \square \)

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