ON RELATIVE ESSENTIAL SPECTRA OF BLOCK OPERATOR MATRICES AND APPLICATION

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Abstract. In this paper, we investigate relative essential spectra of $2 \times 2$ block operator matrix using the Fredholm perturbation theory. Furthermore, an example for two-group transport equations is presented to illustrate the validity of the main results.

1. Introduction

Numerous mathematical and physical problems lead to operator pencils, $L - \lambda M$ (see for example [8, 18]). Recently, the spectral theory of operator pencils attracts the attention of many mathematicians. Moreover, the motivations for studying the $M$-essential spectra of block operator matrix are various and meaningful in transport theory.

In this paper, we are mainly concerned with the study of the spectral theory for pencils of the form

$$L_0 - \lambda M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \lambda \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

considered on the product Banach space $X \times Y$, where $M$ is a bounded operator. The operator $A$ (resp. $D$) acts on the Banach space $X$ (resp. $Y$) and has the domain $D(A)$ (resp. $D(D)$) and the intertwining operator $B$ (resp. $C$) is defined on the domain $D(B)$ (resp. $D(C)$) and acts from $Y$ into $X$ (resp. from $X$ into $Y$).

The block operator matrix of the form (1) is densely defined with domain given by

$$\mathcal{D}(L_0 - \lambda M) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)) \text{ such that } \Gamma_X x = \Gamma_Y y \right\},$$

Received April 4, 2015; Revised September 21, 2015.

2010 Mathematics Subject Classification. 47A10, 47A53, 34K08.

Key words and phrases. matrix operator, Fredholm perturbations, $M$-essential spectra, transport operator.
where $\Gamma_X$ (resp. $\Gamma_Y$) is a linear operator acting from $X$ (resp. $Y$) into a Banach space $Z$. The closure of the block operator matrix $L_0, L := T_0$, is discussed in details in the paper of [25] under some assumptions on the entries components. The study of the block operator matrix is the subject of many authors under different assumptions. In this direction some issues may be found in the literature, we can quote for example [1, 2, 10, 16, 24]. Recently, an account research and a wide panorama of methods to investigate the spectral theory of block operator matrices is given in [3, 4, 5, 11, 14, 25]. More precisely, the description of various essential spectra of a block operator matrix $L$ appears in [3, 5, 11, 14] to improve and generalize some results given by [1, 2, 24] for block operator matrices in Banach spaces under some compactness assumptions.

However, it should be noted that several results for the authors cited in the papers of [1, 2, 4, 14, 24] are aimed at providing methods for dealing with spectral theory for operator in the form $L_0 - \lambda M$ where $M = I$.

The purpose of this work consists principally in extending results given in [4] and we concern ourselves exclusively with the investigation of some $M$-essential spectra of unbounded $2 \times 2$ block operator matrices for pencils of the form $L_0 - \lambda M$, where $M$ is a bounded operator defined on the product of two Banach spaces $X \times Y$ under a coupling condition between the two components of its elements.

To do this, we firstly establish some results on right and left-Fredholm perturbations theory (see Theorems 2.2 and 2.3). Eventually, we dispose different conditions in terms of the Fredholm, right and left-Fredholm perturbations to prove the Fredholmness perturbations of block $2 \times 2$ operator matrix having the form

$$\begin{pmatrix} L - \lambda M & -1 \\ L\lambda_0 - \lambda M & \end{pmatrix}^{-1}$$

(see Section 3), which allows us to investigate the stability of their $M$-essential spectra in terms of Schur-complement whose $M_4$-essential spectra is easier to calculate. Moreover, the use of the $M$-resolvent, the Fredholm perturbations theory and the lower factorization allows us to formulate some supplements to many results presented in [4, 14] and to ameliorate the description of the $M$-essential spectra of two-group transport equations without knowing the totality of the relative essential spectra of the operator $A$, but only the relative essential spectra of its restriction which is more general than the one provided by [4, 14].

The present paper consists of four sections: In Section 2, we present some basic notations and auxiliary lemmas connected to the main body of the paper. We advise that Section 3 constitutes the main results. Section 4 is devoted to illustrate our abstract results to two group transport operator in Banach spaces.

## 2. Preliminary results

Let $X$ and $Y$ be two Banach spaces. Throughout this section, $T$ denotes a linear operator from $X$ into $Y$ with domain $D(T) \subset X$ and range $R(T) \subset Y$. By $C(X,Y)$ (resp. $\mathcal{L}(X,Y)$), we designate the set of all closed, densely
defined linear operators (resp. the set of all bounded linear operators) from $X$ into $Y$ and by $\mathcal{K}(X,Y)$ the subset of all compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, $\mathcal{N}(T)$ denotes the null space of $T$. The nullity $\alpha(T)$ of $T$ is defined as the dimension of $\mathcal{N}(T)$ and the deficient $\beta(T)$ of $T$ is defined as the codimension of $\mathcal{R}(T)$ in $Y$.

Let $S$ be a non null bounded operator from $X$ into $Y$. For $T \in \mathcal{C}(X,Y)$, we define the $S$-resolvent set of $T$ by:

$$\rho_S(T) := \{ \lambda \in \mathbb{C} : T - \lambda S \text{ has a bounded inverse} \}$$

and the $S$-spectrum of $T$ by:

$$\sigma_S(T) := \mathbb{C} \setminus \rho_S(T).$$

In what follows, we need to introduce some important classes of operators. The set of upper semi-Fredholm operators from $X$ into $Y$ is defined by:

$$\Phi_+(X,Y) = \{ T \in \mathcal{C}(X,Y) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y \},$$

and the set of lower semi-Fredholm operators from $X$ into $Y$ is defined by:

$$\Phi_-(X,Y) = \{ T \in \mathcal{C}(X,Y) : \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y \}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ (resp. $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$) denotes the set of Fredholm (resp. semi-Fredholm) operators from $X$ into $Y$. If $T \in \Phi(X,Y)$, the number $i(T) := \alpha(T) - \beta(T)$ is called the index of $T$.

The set $\Phi_{T,S}$ is defined as:

$$\Phi_{T,S} = \{ \lambda \in \mathbb{C} : T - \lambda S \in \Phi(X,Y) \}.$$

A complex number $\lambda$ is in $\Phi_{+T,S}$, $\Phi_{-T,S}$ or $\Phi_{T,S}$ if $T - \lambda S$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{C}(X,Y), \mathcal{K}(X,Y), \Phi(X,Y), \Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X), \mathcal{C}(X), \mathcal{K}(X), \Phi(X), \Phi_+(X)$ and $\Phi_-(X)$, respectively.

Recall the following results established in [20].

**Definition 2.1.** Let $X$ and $Y$ be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to have a left (resp. a right) Fredholm inverse if there exists an operator $T_l \in \mathcal{L}(Y,X)$ (resp. $T_r \in \mathcal{L}(Y,X)$) such that $T_l T - I \in \mathcal{K}(X)$ (resp. $TT_r - I \in \mathcal{K}(Y)$). The operators $T_l$ (resp. $T_r$) is called left (resp. right) Fredholm inverse of $T$.

We will denote by $\Phi_l(X,Y)$ (resp. $\Phi_r(X,Y)$) the set of operators which have left (resp. right) Fredholm inverse.

We denote the sets $\Phi_l^b(X,Y)$, $\Phi_h^b(X,Y)$, $\Phi^l(X,Y)$, $\Phi^h(X,Y)$ and $\Phi^b(X,Y)$ by $\Phi_l(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_r(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$ respectively.

Our concern in this paper is mainly the following $S$-essential spectra:

$$\sigma_{el,S}(T) := \{ \lambda \in \mathbb{C} : T - \lambda S \notin \Phi_l(X,Y) \} = \mathbb{C} \setminus \Phi_{lT,S},$$

$$\sigma_{er,S}(T) := \{ \lambda \in \mathbb{C} : T - \lambda S \notin \Phi_r(X,Y) \} = \mathbb{C} \setminus \Phi_{rT,S},$$
Let $S$ be said to be strictly singular if the restriction of $S$ to a subspace of $X$ is not an homeomorphism.

The family of weakly compact operators from $X$ to $Y$ is denoted by $W(X,Y)$. If $X = Y$ the family of weakly compact operators on $X$, $W(X) := W(X,X)$ is a closed two-sided ideal of $L(X)$ containing $K(X)$ (cf. [7]).

Definition 2.3. Let $X$ and $Y$ be two Banach spaces. An operator $S \in L(X,Y)$ is said to be strictly singular if the restriction of $S$ to any infinite-dimensional subspace of $X$ is not an homeomorphism.

Let $S(X,Y)$ denote the set of strictly singular operators from $X$ to $Y$.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [15] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators, we refer to [15]. Note that $S(X,Y)$ is a closed subspace of $L(X,Y)$. In general, if $X = Y$, $S(X) := S(X,X)$ is a closed two-sided ideal of $L(X)$ containing $K(X)$.

If $X$ is a Hilbert space, then $S(X) = K(X)$. The class of weakly compact operators in $L_1$-spaces (resp. $C(\Omega)$-spaces with $\Omega$ is a compact Hausdorff space) is nothing else than the family of strictly singular operators on $L_1$-spaces (resp. $C(\Omega)$-spaces) (see [22, Theorem 1]).

In the following, we introduce some definitions on Fredholm perturbations:
Definition 2.4. Let $X$ and $Y$ be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.

(i) $F$ is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$.

(ii) $F$ is called a left (resp. right) Fredholm perturbation if $U + F \in \Phi_l(X,Y)$ (resp. $U + F \in \Phi_r(X,Y)$) whenever $U \in \Phi_l(X,Y)$ (resp. $U + F \in \Phi_r(X,Y)$).

We denote by $\mathcal{F}(X,Y)$ the set of Fredholm perturbations and by $\mathcal{F}_l(X,Y)$ (resp. $\mathcal{F}_r(X,Y)$) the set of left (resp. right) Fredholm perturbations.

If $X = Y$ we write $\mathcal{F}(X), \mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$ for $\mathcal{F}(X,X), \mathcal{F}_l(X,X)$ and $\mathcal{F}_r(X,X)$ respectively.

Remark 2.1. Let $\Phi^b(X,Y)$, $\Phi^l_l(X,Y)$ and $\Phi^r_r(X,Y)$ denote respectively the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_l(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_r(X,Y) \cap \mathcal{L}(X,Y)$. If in Definition 2.4 we replace $\Phi(X,Y)$, $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^l_l(X,Y)$ and $\Phi^r_r(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^l_l(X,Y)$ and $\mathcal{F}^r_r(X,Y)$ respectively.

The set of Fredholm perturbations $\mathcal{F}^b(X,Y)$ was introduced and investigated in [9]. In particular, it is shown that $\mathcal{F}^b(X,Y)$ is a closed two-sided ideal of $\mathcal{L}(X,Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

In [13], it was proved that if $X = Y$, then $\mathcal{F}^l_l(X) := \mathcal{F}^l_l(X,X)$ and $\mathcal{F}^r_r(X) := \mathcal{F}^r_r(X,X)$ are two-sided ideals of $\mathcal{L}(X)$, satisfying:

$$K(X,Y) \subseteq \mathcal{F}^l_l(X,Y) \subseteq \mathcal{F}^b(X,Y)$$

and

$$K(X,Y) \subseteq \mathcal{F}^r_r(X,Y) \subseteq \mathcal{F}^b(X,Y).$$

Let us recall the following results on Fredholm perturbations theory of $2 \times 2$ block operator matrix introduced by [13].

Theorem 2.1 ([13, Theorems 3.1-3.2]). Let $X_1$ and $X_2$ be two Banach spaces and $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ where $F_{ij} \in \mathcal{L}(X_j,X_i), i,j = 1,2$. Then

(i) $F \in \mathcal{F}^b(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^b(X_j,X_i)$, $\forall i,j = 1,2$.

(ii) $F \in \mathcal{F}^l_l(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^l_l(X_j,X_i)$, $\forall i,j = 1,2$.

(iii) $F \in \mathcal{F}^r_r(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^r_r(X_j,X_i)$, $\forall i,j = 1,2$.

Theorem 2.2. Let $A \in \mathcal{C}(X,Y)$ and $S \in \mathcal{L}(X,Y)$. Then

(i) If $F \in \mathcal{F}^b(X,Y)$, then $\sigma_{er,S}(F + A) = \sigma_{er,S}(A)$.

(ii) If $F \in \mathcal{F}^l_r(X,Y)$, then $\sigma_{el,S}(F + A) = \sigma_{el,S}(A)$.

Proof. (i) Let $\lambda \in \mathbb{C}$ such that $\lambda S - A \in \Phi_r(X,Y)$. Since $F \in \mathcal{F}^b(X,Y)$, then $\lambda S - A \in \Phi^b(X,Y)$ if and only if $\lambda S - A - F \in \Phi^b(X,Y)$. Hence $\sigma_{er,S}(A) = \sigma_{er,S}(A + F)$.

Arguing as above we derive the item (ii). \(\square\)

To close this section, we state a straightforward, but useful result to provide the stability of the $S$-right and $S$-left spectra.
**Theorem 2.3.** Let $X$ be a Banach space, $T_1, T_2$ two closed densely defined linear operators on $X$ and $S$ an invertible operator on $X$.

(i) If for some $\lambda_0 \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator 
\[(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}^b(X),\]

then 
\[\sigma_{er,S}(T_1) = \sigma_{er,S}(T_2).\]

(ii) If for some $\lambda_0 \in \rho_S(T_1) \cap \rho_S(T_2)$, the operator 
\[(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}^b(X),\]

then 
\[\sigma_{el,S}(T_1) = \sigma_{el,S}(T_2).\]

**Proof.** Let $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$. The proof of this theorem is based on the following relation 
\[T_1 - \lambda S = (\lambda - \lambda_0)S \left[ (\lambda - \lambda_0)^{-1}S^{-1} - (T_1 - \lambda_0 S)^{-1} \right] (T_1 - \lambda_0 S).\]

Since $T_1 - \lambda_0 S$ is one to one and onto, then
\[
\begin{align*}
\alpha(T_1 - \lambda S) &= \alpha \left[ (\lambda - \lambda_0)^{-1}S^{-1} - (T_1 - \lambda_0 S)^{-1} \right], \\
\beta(T_1 - \lambda S) &= \beta \left[ (\lambda - \lambda_0)^{-1}S^{-1} - (T_1 - \lambda_0 S)^{-1} \right].
\end{align*}
\]

This shows that $\lambda \in \Phi_{T_1,S,r}$ (resp. $\lambda \in \Phi_{T_1,S,l}$) if and only if $(\lambda - \lambda_0)^{-1} \in \Phi_{(T_1 - \lambda_0 S)^{-1},S^{-1},r}$ (resp. $(\lambda - \lambda_0)^{-1} \in \Phi_{(T_1 - \lambda_0 S)^{-1},S^{-1},l}$).

Combining Theorem 2.2 and the fact that $(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}^b(X)$ (resp. $(\lambda_0 S - T_1)^{-1} - (\lambda_0 S - T_2)^{-1} \in \mathcal{F}^b(X)$), we get
\[
\lambda \in \sigma_{er,S}(T_1) \iff \lambda \in \sigma_{er,S}((T_1 - \lambda_0 S)^{-1}) \cap \sigma_{er,S}((T_2 - \lambda_0 S)^{-1})
\]
\[
\lambda \in \sigma_{el,S}(T_2)
\]
\[
\lambda \in \sigma_{el,S}(T_1) \iff \lambda \in \sigma_{el,S}((T_1 - \lambda_0 S)^{-1}) \cap \sigma_{el,S}((T_2 - \lambda_0 S)^{-1})
\]
\[
\lambda \in \sigma_{el,S}(T_2).
\]

This achieves the proof of the theorem. \(\square\)

**3. Fredholm perturbations of block operators matrices and stability of their $M$-essential spectra**

Let $X$, $Y$ and $Z$ be three Banach spaces and $\Gamma_X$ (resp. $\Gamma_Y$) be the linear operator from $X$ (resp. $Y$) into $Z$. In the Banach space $X \times Y$, we consider the linear operator $L_0 - \lambda M$ given by the block operator matrix 
\[
L_0 - \lambda M := \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} - \lambda \begin{pmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{pmatrix},
\]
where $M$ is a bounded operator, $A$ (resp. $D$) is a densely defined closable (resp. closed) linear operator in $X$ (resp. $Y$) and $B$ (resp. $C$) is a linear operator from $Y$ (resp. $X$) into $X$ (resp. $Y$). The domain of $L_0 - \lambda M$ is given by:

$$
D(L_0 - \lambda M) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (D(A) \cap D(C)) \times (D(B) \cap D(D)) : \Gamma_X x = \Gamma_Y y \right\}.
$$

The main purpose of this section is to discuss the $M$-essential spectra of the closure of the matrix operator $L_0$ denoted by $L$ and defined on the product of Banach spaces $X \times Y$. First of all, we shall make some hypotheses.

Let $X,Y$ and $Z$ three Banach spaces and assume that:

(H1) $A$ is a closable, densely defined linear operator.

It follows from this hypothesis that, $\mathcal{D}(A)$, equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ can be completed to a Banach space $X_A$ which coincides with $\mathcal{D}(\widehat{A})$ the domain of the closure of $A$.

(H2) $\mathcal{D}(A) \subset \mathcal{D}(\Gamma_X) \subset X_A$ and $\Gamma_X$ is bounded as a mapping from $X_A$ into $Z$.

(H3) The set $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$ is dense in $X$ with $\rho_{M_1}(A_1) \neq \emptyset$ for $A_1 := A\mid_{\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)}$.

Remark 3.1. From (H1)-(H3), one can easily check that $\Gamma_X(\mathcal{D}(A_1)) = \{0\}$ and that the operator $A_1$ is closed.

Now, let us recall the following lemma:

Lemma 3.1 ([25, Lemma 3.1]). Assume that the hypotheses (H1)-(H3) are satisfied. Then, for any $\lambda \in \rho_{M_1}(A_1)$, the following assertions hold:

(i) $\mathcal{D}(A) := \mathcal{D}(A_1) \oplus \mathcal{N}(A_{\lambda,M_1})$, where the operator $A_{\lambda,M_1}$ is defined on $\mathcal{D}(A)$ by $A_{\lambda,M_1} := (A - \lambda M_1)$.

(ii) The restriction $\Gamma_\lambda := \Gamma_X|_{\mathcal{N}(A_{\lambda,M_1})}$ is injective.

(iii) $\mathcal{R}(\Gamma_\lambda) = \Gamma_X(\mathcal{N}(A_{\lambda,M_1})) = \Gamma_X(\mathcal{D}(A))$ does not depend on $\lambda$.

As a direct consequence of the last lemma, we let, for $\lambda \in \rho_{M_1}(A_1)$, the following operator $K_\lambda$ defined by:

$$
K_\lambda := (\Gamma_\lambda)^{-1} := (\Gamma_X|_{\mathcal{N}(A_{\lambda,M_1})})^{-1} : \Gamma_X(\mathcal{D}(A)) \rightarrow \mathcal{N}(A_{\lambda,M_1}).
$$

In other words, $Kz = x$ means that $x \in \mathcal{D}(A), A_{\lambda,M_1}x = 0$ and $\Gamma_X x = z$.

Lemma 3.2 ([25, Lemma 3.2]). For every $\lambda, \mu \in \rho_{M_1}(A_1)$ and under the assumptions (H1)-(H3), we have:

$$
(3) \quad K_\lambda - K_\mu = (\lambda - \mu)(A_1 - \lambda M_1)^{-1}M_1K_\mu.
$$

If $K_\lambda$ is closable for some $\lambda \in \rho_{M_1}(A_1)$, then it is closable for all $\lambda$, with closure satisfying

$$
\overline{K_\lambda - K_\mu} = (\lambda - \mu)(A_1 - \lambda M_1)^{-1}M_1\overline{K_\mu}.
$$

Now, we suppose that:
Theorem 3.1. Under the assumptions with closure, we infer that, for
\( \lambda \in \rho_{M_1}(A_1) \), the operator \( F(\lambda) := (C - \lambda M_3)(A_1 - \lambda M_1)^{-1} \) is bounded from \( X \) into \( Y \).

Remark 3.2. (i) Combining the closed graph theorem with the above assumption, we infer that, for \( \lambda \in \rho_{M_1}(A_1) \), the operator \( F(\lambda) := (C - \lambda M_3)(A_1 - \lambda M_1)^{-1} \) is bounded from \( X \) into \( Y \).

(ii) If the assumptions (H1)-(H3) are satisfied, then for \( \lambda \in \rho_{M_1}(A_1) \) and \( x \in D(A) \), we have
\[
(A - \lambda M_1)x = (A_1 - \lambda M_1)(I - K_1 \Gamma_X)x.
\]
In addition, we will assume that:

(H5) For some (hence for all) \( \lambda \in \rho_{M_1}(A_1) \), \( K_\lambda \) is a bounded operator from \( \Gamma_X(D(A)) \) into \( X \), its extension by continuity to \( \overline{\Gamma_X(D(A))} \) is denoted by \( \overline{\Gamma_X} \).

(H6) \( D \in C(Y) \) with \( \rho_{M_1}(D) \neq \emptyset \).

(H7) \( D(B) \cap D(D) \subset D(\Gamma_Y) \), the set
\[
Y_1 := \{ y \in D(B) \cap D(D) \text{ such that } \Gamma_Y y \in \Gamma_X(D(A)) \}
\]
is dense in \( Y \). We denote by \( \overline{\Gamma_Y} \) the continuous extension of \( \Gamma_Y \) on the all space \( Y \).

(H8) The operator \( B \) is densely defined and for some (hence for all) \( \lambda \in \rho_{M_1}(A_1) \), the operator \( (A_1 - \lambda M_1)^{-1}B \) is bounded on its domain.

For \( \lambda \in \rho_{M_1}(A_1) \), the operator \( S_\lambda := D + (C - \lambda M_3)[K_\lambda \Gamma_Y - (A_1 - \lambda M_1)^{-1}(B - \lambda M_2)] \) is defined on the set \( Y_1 \), which is dense in \( Y \) according to (H7).

We also introduce the following assumptions:

(H9) For some (hence for all) \( \lambda \in \rho_{M_1}(A_1) \), the operator \( C[-K_\lambda \Gamma_Y + (A_1 - \lambda M_1)^{-1}B] \) is bounded on \( Y_2 \) where
\[
Y_2 := \{ y \in D(B) \cap D(\Gamma_Y) \text{ such that } \Gamma_Y y \in \Gamma_X(D(A)) \}
\]
is dense in \( Y \) such that the restriction of \( \Gamma_Y \) to this set is bounded as an operator from \( Y \) into \( Z \).

(H10) The set \( Y_1 \) is a core of \( D \).

Having formulate the above assumptions, the following theorem holds:

Theorem 3.1. Under the assumptions (H1)-(H10), the operator \( L_0 \) is closable with closure
\[
L = \lambda M + \begin{pmatrix} I & 0 \\ F(\lambda) & I \end{pmatrix} \begin{pmatrix} A_1 - \lambda M_1 & 0 \\ 0 & D + \overline{R_\lambda} - \lambda M_4 \end{pmatrix} \begin{pmatrix} I & G(\lambda) \\ 0 & I \end{pmatrix},
\]
where
\[
G(\lambda) = -\overline{K_\lambda} \Gamma_Y + (A_1 - \lambda M_1)^{-1}(B - \lambda M_2),
\]
and
\[
R_\lambda = -(C - \lambda M_3)[-K_\lambda \Gamma_Y + (A_1 - \lambda M_1)^{-1}(B - \lambda M_2)].
\]
Proof. Taking account of assumptions (H3) and (H6), we can easily check that $A_1$ and $D$ are closed operators.

Obviously, combining the fact that $C [-K_1 \Gamma_Y + (A_1 - \lambda M_1)^{-1} B]$ is bounded and densely defined on $Y_2$ with the boundedness of the operators $M_2$ and $M_3$, we deduce that, for $\lambda \in \rho_{M_4}(A_1)$, the operator
\[
\left( C - \lambda M_3 \right) [-K_1 \Gamma_Y + (A_1 - \lambda M_1)^{-1} B - \lambda M_2]
\]
is bounded, everywhere defined and hence it is bounded on the dense set $Y_2$.

This together with the fact that $Y_1$ is a core of $D$, we conclude that $\mathcal{S}_\lambda$ is closable for every $\lambda \in \rho_{M_4}(A_1)$ with closure
\[
\mathcal{S}_\lambda = D \left( C - \lambda M_3 \right) [-K_1 \Gamma_Y + (A_1 - \lambda M_1)^{-1} B - \lambda M_2]
= D + \mathcal{R}_\lambda.
\] (5)

Now, applying [25, Theorem 3.1], we deduce that the operator $L_0$ is closable and its closure $L := \mathcal{R}_0$ is given by (4).

\[ \square \]

Remark 3.3. Under the assumptions (H1)-(H10), Eq. (5) allows us to write, for $\lambda \in \rho_{M_4}(A_1) \cap \rho_{M_4}(D) \cap \rho_{M_4}(\mathcal{S}_\lambda)$, the $M_4$-resolvent of the operator $\mathcal{S}_\lambda$ as:
\[
(\mathcal{S}_\lambda - \lambda M_4)^{-1} = (D - \lambda M_4)^{-1} + (\mathcal{S}_\lambda - \lambda M_4)^{-1} - (D - \lambda M_4)^{-1}
= (D - \lambda M_4)^{-1} + (\mathcal{S}_\lambda - \lambda M_4)^{-1} (D - \mathcal{S}_\lambda) (D - \lambda M_4)^{-1}
= (D - \lambda M_4)^{-1} - (\mathcal{S}_\lambda - \lambda M_4)^{-1} \mathcal{R}_\lambda (D - \lambda M_4)^{-1}
\]
or\[
(\mathcal{S}_\lambda - \lambda M_4)^{-1} = (D - \lambda M_4)^{-1} - (D - \lambda M_4)^{-1} (\mathcal{S}_\lambda - D) (\mathcal{S}_\lambda - \lambda M_4)^{-1}
= (D - \lambda M_4)^{-1} - (D - \lambda M_4)^{-1} \mathcal{R}_\lambda (\mathcal{S}_\lambda - \lambda M_4)^{-1}
\]

Now, we are in the position to express the main result of this section based on Fredholm perturbations theory to describe the $M$-essential spectra of a block operator matrix $L$. To do this, we introduce for an arbitrary fixed $\lambda_0 \in \rho_{M_4}(A_1)$, the following block diagonal matrix $L_{\lambda_0}$:
\[
L_{\lambda_0} := \begin{pmatrix} A_1 & 0 \\ 0 & D + \mathcal{R}_\lambda \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & \mathcal{S}_{\lambda_0} \end{pmatrix}.
\]

Theorem 3.2. Assume that the assumptions (H1)-(H10) are fulfilled. Then, if for some (hence for all) $\lambda_0 \in \rho_{M_4}(A_1) \cap \rho_{M_4}(D)$, we have:

(i) $M_2$, $M_3$, $(D - \lambda_0 M_4)^{-1} C (A_1 - \lambda_0 M_1)^{-1}$, and $[-K_1 \Gamma_Y + (A_1 - \lambda_0 M_1)^{-1} B] (D - \lambda_0 M_4)^{-1}$ are Fredholm perturbations, then, for $\lambda \in \rho_{M}(L) \cap \rho_{M}(L_{\lambda_0})$,
\[
(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} \in \mathcal{F}_d(X \times Y),
\]
in particular,
\[
\sigma_{e_4,M}(L) = \sigma_{e_4,M}(A_1) \cup \sigma_{e_4,M}(D + \mathcal{R}_\lambda),
\]
\[
\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M}(A_1) \cup \sigma_{e_5,M}(D + \mathcal{R}_\lambda).
\]
If $C_{σ_{e,M_1}}(A_1)$ is connected, then

$$\sigma_{e,M}(L) = \sigma_{e,M_1}(A_1) \cup \sigma_{e,M_4}(D + \overline{R}_{\lambda_0}).$$

Moreover, if $C_{σ_{e,M_1}}(A_1)$ and $C_{σ_{e,M_4}}(D + \overline{R}_{\lambda_0})$ are connected, then

$$\sigma_{e,M}(L) = \sigma_{e,M_1}(A_1) \cup \sigma_{e,M_4}(D + \overline{R}_{\lambda_0}).$$

(iii) $M_2, M_3, (D - λ_0M_4)^{-1}C(A_1 - λ_0M_1)^{-1}$ and $[-K_{\lambda_0}Y + (A_1 - λ_0M_1)^{-1}B](D - λ_0M_4)^{-1}$ are right-Fredholm perturbations, then, for $λ ∈ \rho_M(L) \cap \rho_M(L_{λ_0})$,

$$(L - λM)^{-1} - (L_{λ_0} - λM)^{-1} \in F^b(X × Y),$$

in particular,

$$σ_{er,M}(L) = σ_{er,M_1}(A_1) \cup σ_{er,M_4}(D + \overline{R}_{\lambda_0}).$$

(iii) $M_2, M_3, (D - λ_0M_4)^{-1}C(A_1 - λ_0M_1)^{-1}$ and $[-K_{\lambda_0}Y + (A_1 - λ_0M_1)^{-1}B](D - λ_0M_4)^{-1}$ are left-Fredholm perturbations, then, for $λ ∈ \rho_M(L) \cap \rho_M(L_{λ_0})$,

$$(L - λM)^{-1} - (L_{λ_0} - λM)^{-1} \in F^b(X × Y),$$

in particular,

$$σ_{el,M}(L) = σ_{el,M_1}(A_1) \cup σ_{el,M_4}(D + \overline{R}_{\lambda_0}).$$

Proof. Let $λ_0 ∈ \rho_M(A_1)$ and $λ ∈ \mathbb{C}$ such that $λ ∈ \rho_M(L) \cap \rho_M(L_{λ_0})$. According to Eq. (4) and Remark 3.3, the representation of $(L - λM)^{-1} - (L_{λ_0} - λM)^{-1}$ can be written as:

$$(L - λM)^{-1} - (L_{λ_0} - λM)^{-1}$$

$$= \begin{pmatrix}
G(λ)(D - λM_4)^{-1}F(λ) & -G(λ)(D - λM_4)^{-1} \\
-σ(λ_0 - λM_4)^{-1}σ_0(D - λM_4)^{-1}F(λ) & +G(λ)(D - λM_4)^{-1}σ_0(σ_0 - λM_4)^{-1}
\end{pmatrix}
-\begin{pmatrix}
-(D - λM_4)^{-1}F(λ) & (σ_0 - λM_4)^{-1} - (σ_0 - λM_4)^{-1}
\end{pmatrix}.
\]

Based on Theorems 2.1 and 2.3, we will prove the Fredholmness perturbation of $(L - λM)^{-1} - (L_{λ_0} - λM)^{-1}$, hence it remains to show that all entries of this block operator matrix are Fredholm perturbations.

(i) For details the proof of this assertion, we infer from the assumptions that:

$$(D - λM_4)^{-1}F(λ) = (D - λM_4)^{-1}C(A_1 - λM_1)^{-1} - λ(D - λM_4)^{-1}M_3(A_1 - λM_1)^{-1}$$

and

$$G(λ)(D - λM_4)^{-1}$$

$$= [-K_{\lambda_0}Y + (A_1 - λM_1)^{-1}(B - λM_2)](D - λM_4)^{-1}$$

$$= [-K_{\lambda_0}Y + (A_1 - λM_1)^{-1}B - λ(A_1 - λM_1)^{-1}M_3](D - λM_4)^{-1}$$
are Fredholm perturbations. According to the boundedness property of the operators \( G(\lambda), (\mathcal{S}_\lambda - \lambda M_4)^{-1} \), we get the Fredholm
p perturbations of each operators \( G(\lambda)(D - \lambda M_4)^{-1}F(\lambda), G(\lambda)(\mathcal{S}_\lambda - \lambda M_4)^{-1}\mathcal{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda), G(\lambda)(D - \lambda M_4)^{-1}\mathcal{R}_\lambda(\mathcal{S}_\lambda - \lambda M_4)^{-1} \) and \((\mathcal{S}_\lambda - \lambda M_4)^{-1}\mathcal{R}_\lambda(D - \lambda M_4)^{-1}F(\lambda)\).

Analogously, one can see that the following right lower corner
\[
\begin{align*}
&= \frac{(\mathcal{S}_\lambda - \lambda M_4)^{-1} - (\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}}{(\mathcal{S}_\lambda - \lambda M_4)^{-1}(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}}
\end{align*}
\]
\[
= (\mathcal{S}_\lambda - \lambda M_4)^{-1}(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}(\mathcal{S}_\lambda - \lambda M_4)^{-1}
\]
\[
= (\mathcal{S}_\lambda - \lambda M_4)^{-1}(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}(\mathcal{S}_\lambda - \lambda M_4)^{-1}
\]
\[
= (\lambda - \lambda_0)(\mathcal{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}
\]
\[
= (\lambda - \lambda_0)(\mathcal{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}
\]
\[
= (\lambda - \lambda_0)(\mathcal{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}
\]
\[
= (\lambda - \lambda_0)(\mathcal{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}
\]
\[
= (\lambda - \lambda_0)(\mathcal{S}_\lambda - \lambda M_4)^{-1}F(\lambda)M_1G(\lambda_0)(\mathcal{S}_{\lambda_0} - \lambda M_4)^{-1}
\]
is a Fredholm perturbation since it is the product of bounded operators and the Fredholm perturbation operators \((D - \lambda M_4)^{-1}F(\lambda)\) and \(G(\lambda_0)(D - \lambda_0 M_4)^{-1}\).

Hence, according to [12, Theorem 2.2], we get
\[
\sigma_{e_{4,M}}(L) = \sigma_{e_{4,M}}(L_{\lambda_0}) = \sigma_{e_{4,M_1}}(A_1) \cup \sigma_{e_{4,M_4}}(D + \mathcal{R}_{\lambda_0}),
\]
with
\[
i(L - \lambda M) = i(A_1 - \lambda M_1) + i(D + \mathcal{R}_{\lambda_0} - \lambda M_4) = 0.
\]

Hence, from these two equalities, we have
\[
\sigma_{e_{5,M}}(L) \subseteq \sigma_{e_{5,M_1}}(A_1) \cup \sigma_{e_{5,M_4}}(D + \mathcal{R}_{\lambda_0}),
\]

According to [12, Lemma 2.1], we get
\[
\sigma_{e_{5,M}}(L) = \sigma_{e_{5,M}}(L_{\lambda_0}) = \sigma_{e_{5,M_1}}(A_1) \cup \sigma_{e_{5,M_4}}(D + \mathcal{R}_{\lambda_0}),
\]
and
\[
\sigma_{e_{6,M}}(L) = \sigma_{e_{6,M}}(L_{\lambda_0}) = \sigma_{e_{6,M_1}}(A_1) \cup \sigma_{e_{6,M_4}}(D + \mathcal{R}_{\lambda_0}).
\]
The use of Theorems 2.1, 2.2 and 2.3 allows us to reach the results of assertions (ii) and (iii) in a similar ways as in (i). \(\square\)
Remark 3.4. It is noted that, in the paper [14] the authors supposed that the operators $-K\lambda \Gamma_0 + (A_1 - \lambda)^{-1}B$ and $C(A_1 - \lambda)^{-1}$ are Fredholm perturbations. But in our case, we consider a weaker condition and we suppose only that $[-K\lambda \Gamma_0 + (A_1 - \lambda M_1)^{-1}(B - \lambda M_2)(D - \lambda M_4)^{-1}$ and $(D - \lambda M_4)^{-1}C(A_1 - \lambda M_1)^{-1}$ are Fredholm perturbations in order to investigate the $M$-essential spectra of the operator $L$ in term of its Schur-complement whose $M_4$-essential spectrum is easier to calculate. So, Theorem 3.2 may be regarded as an extension of [14, Theorem 3.3] to a larger class of operators.

The notion of Fredholm perturbations theory plays a crucial role in spectral theory. This notion is tested for two-group transport equations and is applicable to propose an abstract framework for the computation of the $M$-essential spectra of a one-dimensional problem of transport operator.

4. Application to two-group transport equations

In this section, we will apply our main results to study the $M$-essential spectra of a problem of transport equations acting in the space

$$X \times X := L_1([-a,a] \times [-1,1]; dx dv) \times L_1([-a,a] \times [-1,1]; dx dv), \quad a > 0,$$

and given by the following matrix of two-group transport operators:

$$L - \lambda M := \begin{pmatrix} T_1 - \lambda M_1 & K_{12} - \lambda M_2 \\ K_{21} - \lambda M_3 & T_2^H + K_{22} - \lambda M_4 \end{pmatrix}.$$

The operator $T_1$ is the closed linear operator defined by:

$$T_1 : \mathcal{D}(T_1) \subseteq X \rightarrow X$$

$$\psi \rightarrow T_1\psi(x,v) = -v \frac{\partial \psi}{\partial x}(x,v) - \sigma_1(v)\psi(x,v)$$

$$\mathcal{D}(T_1) := \mathcal{W} := \{ \psi \in X : v \frac{d\psi}{dx} \in X \}$$

and $T_2^H$ is the steaming operator:

$$T_2^H : \mathcal{D}(T_2^H) \subseteq X \rightarrow X$$

$$\psi \rightarrow T_2^H\psi(x,v) = -v \frac{\partial \psi}{\partial x}(x,v) - \sigma_2(v)\psi(x,v)$$

$$\mathcal{D}(T_2^H) = \{ \psi \in \mathcal{W} : \psi = H\psi^o \}.$$

The collision frequency $\sigma_j(\cdot) \in \mathcal{C}^\infty(-1,1)$, $\psi^o$ and $\psi^i$ represent respectively the outgoing and the incoming fluxes related by the boundary operator $H$. $\psi^o$ and $\psi^i$ belong respectively to the spaces

$$X^o := L_1([-a] \times [-1,0], |v| dv) \times L_1([a] \times [0,1], |v| dv) = X^o_1 \times X^o_2$$

and

$$X^i := L_1([-a] \times [0,1], |v| dv) \times L_1([a] \times [-1,0], |v| dv) = X^i_1 \times X^i_2.$$
where $\eta$ (see [6] for more details). The bounded operators $K_{ij}, (i, j) \in \{(1, 2), (2, 1), (2, 2)\}$ are defined on $X$ by:

$$K_{ij} : X \rightarrow X \quad \psi \rightarrow K_{ij} \psi(x, v) = \int_{-1}^{1} \kappa_{ij}(x, v, v') \psi(x, v') \ dv', \tag{7}$$

with kernels $\kappa_{ij}$ assumed to be measurable and the coefficients $M_i$ are defined by:

$$M_i : X \rightarrow X \quad \psi \rightarrow M_i \psi(x, v) = \eta_i(v) \psi(x, v), \quad i = 1, 4$$

where $\eta_i(\cdot) \in L^\infty(-1, 1)$ and $M_2, M_3$ are bounded operators on $X$.

We define

$$\lambda_j^* := \inf_{v \in (-1, 1)} \sigma_j(v), \quad j = 1, 2$$

and

$$\mu_j^* := \inf_{v \in (-1, 1)} \eta_j(v), \quad j = 1, 4$$

and we assume that $\mu_j^* > 0, \ j = 1, 4$.

To verify the hypotheses of Theorem 3.2, we shall define the operator $L - \lambda M$ on the domain:

$$D(L - \lambda M) := \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in W \times D(T_2^H) : \psi_i^2 = \psi_2^2 \right\}$$

and we introduce the boundary operators $\Gamma_X$ and $\Gamma_Y$ as follows:

$$\Gamma_X : W \rightarrow X^i \quad \psi_1 \rightarrow \psi_1^i, \quad \Gamma_Y : W \rightarrow X^i \quad \psi_2 \rightarrow \psi_2^i = H \psi_2^i.$$ 

Let $A_1$ be the closed, densely defined linear operator with a non empty $M_1$-resolvent set defined as:

$$\left\{ \begin{array}{l} A_1 := T_1, \\
D(A_1) = \{ \psi_1 \in D(T_1) : \psi_1^i = 0 \}. \end{array} \right.$$ 

In order to verify assumption (H5), we will determine the solution of the equation:

$$(T_1 - \lambda M_1)\psi_1 = 0 \quad \text{for} \quad \psi_1 \in W.$$ 

A short computation shows that the operator $K_\lambda$ is bounded by $(\mu_1^* \text{Re} \lambda)^{-1}$ and is defined on $X^i$ by:

$$K_\lambda : X^i \rightarrow X, K_\lambda u := \chi_{(0, 1)}(v) K_\lambda^+ u + \chi_{(-1, 0)}(v) K_\lambda^- u \quad \text{with}$$

$$(K_\lambda^+ u)(x, v) := u(a, v) e^{-\left(\frac{\sigma_j(v)}{(1 + \lambda)(1 + \lambda)}\right) \sigma_j(v)}, \quad v \in (-1, 0)$$

$$(K_\lambda^+ u)(x, v) := u(-a, v) e^{-\left(\frac{\sigma_j(v)}{(1 + \lambda)(1 + \lambda)}\right) \sigma_j(v)}, \quad v \in (0, 1).$$

Consider the Schur-complement of the matrix $L - \lambda M$, which is formally given by the following expression:

$$S_\lambda := T_2^H + K_{22} - (K_{21} - \lambda M_3)[-K_\lambda \Gamma_Y + (T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2)]$$

for $\lambda \in \rho_{M_1}(T_1)$. 

Remark 4.1. It is easy to see that $D(S_\lambda)$ is a core for $T^H_2 + K_{22}$ since $T^H_2 + K_{22}$ is a closed, densely defined operator with a nonempty $M_4$-resolvent set.

In view of the previous remark, it is not difficult to see that $S_\lambda$ can be written for $\lambda \in \rho_{M_4}(T_1) \cap \rho_{M_4}(T_{H_2} + K_{22}) \cap \rho_{M_4}(S_\lambda)$, in the two ways:

(8) $(S_\lambda - \lambda M_4)^{-1} = (T_{H_2} + K_{22} - \lambda M_4)^{-1} - (S_\lambda - \lambda M_4)^{-1} R_\lambda (T_{H_2} + K_{22} - \lambda M_4)^{-1}$

or

(9) $(S_\lambda - \lambda M_4)^{-1} = (T_{H_2} + K_{22} - \lambda M_4)^{-1} - (T_{H_2} + K_{22} - \lambda M_4)^{-1} R_\lambda (S_\lambda - \lambda M_4)^{-1},$

where $R_\lambda := - (K_{21} - \lambda M_3) [ - K_3 F_\nu + (T_1 - \lambda M_1)^{-1} (K_{12} - \lambda M_2) ].$

Notice that the defined collision operators $K_{12}, K_{21}$ and $K_{22}$ act only on the velocity $v'$, so $x$ may be seen, simply, as a parameter in $[-a, a]$. Then, we will consider each of these operators as a function

$K_{ij}(\cdot) : x \in [-a, a] \rightarrow K(x) \in \mathcal{L}(L_1([-1, 1], dv)).$

**Definition 4.1** ([19]). A collision operator $K_{ij}$ in the form (7), is said to be regular if it satisfies the following conditions:

- the function $K_{ij}(\cdot)$ is measurable,
- there exists a compact subset $C \subset \mathcal{L}(L_1([-1, 1], dv))$ such that:
  - $K_{ij}(x) \in C$ a.e. on $[-a, a]$,
  - $K_{ij}(x) \in \mathcal{K}(L_1([-1, 1], dv))$ a.e. on $[-a, a]$.

where $\mathcal{K}(L_1([-1, 1], dv))$ is the set of compact operators on $L_1([-1, 1], dv)$.

We recall the following lemma established in [12].

**Lemma 4.1.** Let $\lambda \in \rho_{M_4}(T_1)$.

(i) If $\frac{K_{ij}(x,v,v')}{|v'|}$ defines a regular operator, then the operator

$$K_{21}(T_1 - \lambda M_1)^{-1}$$

is a weakly compact operator on $X$.

(ii) If $K_{12}(x,v,v')$ defines a regular operator, then the operator

$$(T_1 - \lambda M_1)^{-1} K_{12}$$

is weakly compact on $X$.

As a consequence for the previous lemma, the following result holds:

**Lemma 4.2.** Let $\lambda \in \rho_{M_4}(T_1)$.

(i) If $M_3$ is a Fredholm perturbation on $X$ with the kernel $\frac{K_{21}(x,v,v')}{|v'|}$ defines a regular operator, then $(T_{H_2} + K_{22} - \lambda M_4)^{-1} F(\lambda)$ is a Fredholm perturbation on $X$. 


(ii) If $M_2$ is a Fredholm perturbation on $X$ and $K_{12}$ is a regular operator, then the operator $(T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2)(T_{H_2} + K_{22} - \lambda M_4)^{-1}$ is a Fredholm perturbation on $X$.

Remark 4.2. It follows from Theorem 3.1 in [22] that $\mathcal{W}(X) = \mathcal{S}(X)$.

If $1 < p < \infty$, $X_p$ is reflexive and then $\mathcal{L}(X_p) = \mathcal{W}(X_p)$. On the other hand, it follows from [9, Theorem 5.2] that $\mathcal{K}(X_p) \subseteq \mathcal{S}(X_p) \subseteq \mathcal{W}(X_p) \subseteq \mathcal{F}(X_p)$ with $p \neq 2$.

Now, let us denote by:

$$L_{\lambda_0} := \begin{pmatrix} T_1 & 0 \\ 0 & T_{H_2} + K_{22} + R_{\lambda_0} \end{pmatrix}.$$ 

The Fredholm perturbation theory is an important tool to describe the $M$-essential spectra and especially the $M$-essential spectra of an transport operator matrix $L$. In order to describe these subsets, for $\lambda \in \rho_M(L)$ and $\lambda \in \rho_M(L_{\lambda_0})$, we let:

\[
(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} = \begin{pmatrix} G(\lambda)[S_{\lambda} - \lambda M_4]^{-1}F(\lambda) & -G(\lambda)[S_{\lambda} - \lambda M_4]^{-1} \\
-[S_{\lambda} - \lambda M_4]^{-1}F(\lambda) & [S_{\lambda} - \lambda M_4]^{-1} - [S_{\lambda_0} - \lambda M_4]^{-1} \end{pmatrix}
\]

where

\[
\begin{align*}
G(\lambda) &= -K_{12} \Gamma_Y + (T_1 - \lambda M_1)^{-1}(K_{12} - \lambda M_2) \\
F(\lambda) &= (K_{21} - \lambda M_3)(T_1 - \lambda M_1)^{-1}.
\end{align*}
\]

The $M$-essential spectra of two-group transport operators can be described in the next theorem under additive Fredholm perturbations.

**Theorem 4.1.** If the operators $H, M_2, M_3$ are Fredholm perturbations, $K_{12}, K_{21}, K_{22}$ are regular operators and $L_{\lambda_0} - \lambda M$ is regular, then

\[
(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} \in \mathcal{F}^b(X \times X)
\]

in particular,

\[
\sigma_{ek,M}(L) = \{ \lambda \in \mathbb{C} : \Re \lambda \leq -\min(\frac{\lambda_1^2}{\mu_1^2}, \frac{\lambda_2^2}{\mu_1^2}) \}, \quad 4 \leq k \leq 6, r, l.
\]

**Proof.** According to Theorem 2.1, to characterize the Fredholm perturbations of the block operator matrix $(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1}$, it remains to provide the same property for all entries of this block operator matrix. To do this, for $\lambda \in \rho_M(T_1) \cap \rho_M(T_2^H + K_{22}) \cap \rho_M(S_{\lambda})$, we consider the operator $(S_{\lambda} - \lambda M_4)^{-1}F(\lambda)$ which can be expressed from Eq. (8) as:

\[
(S_{\lambda} - \lambda M_4)^{-1}F(\lambda) := (T_2^H + K_{22} - \lambda M_4)^{-1}F(\lambda)
\]

\[
- (S_{\lambda} - \lambda M_4)^{-1}R_{\lambda}(T_2^H + K_{22} - \lambda M_4)^{-1}F(\lambda).
\]

The use of Lemma 4.2 and Proposition 2 in [9] implies that

\[
(S_{\lambda} - \lambda M_4)^{-1}R_{\lambda}(T_2^H + K_{22} - \lambda M_4)^{-1}F(\lambda)
\]
is a Fredholm perturbation on $X$. Now, the fact that $F^b(X)$ is a closed two-sided ideal of $L(X)$ allows us to deduce from Eq. (10) that $(S_\lambda - \lambda M_k)^{-1} F(\lambda)$ is also a Fredholm perturbation.

Since the operator $H$ is a Fredholm perturbation on $X$, then $\Gamma_Y$ has also this property. This together with Lemma 4.2-(ii), Proposition 2 in [9] and Eq. (9), make us conclude that

$$G(\lambda)(S_\lambda - \lambda M_k)^{-1} := G(\lambda)(T_{2}^H + K_{22} - \lambda M_k)^{-1}$$

(11)

$$- G(\lambda)(T_{2}^H + K_{22} - \lambda M_k)^{-1} R(\lambda)(S_\lambda - \lambda M_k)^{-1}$$

is a Fredholm perturbation on $X$, for $\lambda \in \rho_M(T_1) \cap \rho_M(T_{2}^H + K_{22}) \cap \rho_M(S_\lambda)$.

In what follows, it easy to show from Eqs. (10) and (11) with Proposition 2 in [9], that the operators $G(\lambda)(S_\lambda - \lambda M_k)^{-1} F(\lambda)$ and $(S_\lambda - \lambda M_k)^{-1} - (S_{\lambda_0} - \lambda M_k)^{-1}$ are Fredholm perturbations on $X$.

For all claims cited above and from Theorem 2.1, we get

$$(L - \lambda M)^{-1} - (L_{\lambda_0} - \lambda M)^{-1} \in F^b(X \times X).$$

Therefore, by combining Remark 4.2, Theorems 2.2 in [12] and 2.3, we have

$$\sigma_{ek,M}(L) = \sigma_{ek,M}(L_{\lambda_0}) = \sigma_{ek,M}(T_1) \cup \sigma_{ek,M}(S_\lambda), \quad 4 \leq k \leq 6, r, l.$$  

If we combine the information about $\sigma_{ei,M}(T_1)$ and $\sigma_{ei,M}(S_\lambda)$, for $i = 1, \ldots, 6$ (see Section 4 in [25] for more details) with Eq. (2), we obtain the following result for the $M_j$-essential right and left spectra for $j = 1, 4$ of the operators $T_1$ and $S_\lambda$ as:

$$\sigma_{er,M}(T_1) = \sigma_{el,M}(T_1) = \sigma_{ek,M}(T_1) = \{ \lambda \in \mathbb{C} : \text{Re}\lambda \leq -\frac{\lambda_j}{\mu_i}, \} , \quad 4 \leq k \leq 6.$$  

$$\sigma_{er,M}(S_\lambda) = \sigma_{el,M}(S_\lambda) = \sigma_{ek,M}(S_\lambda) = \{ \lambda \in \mathbb{C} : \text{Re}\lambda \leq -\frac{\lambda_\rho}{\mu_i}, \} , \quad 4 \leq k \leq 6$$

which ends this proof. \□

**Conclusion:** In this paper, we provide some general results on right and left Fredholm perturbations. More specific perturbations results are stated until the paper where they are used to describe the Fredholm, right and left Fredholm perturbations of the difference between the resolvents of two block operator matrices which ensure the stability on their $M$-essential spectra under weaker conditions than proved in the papers of [4, 14, 25]. All the results are new and are not yet investigate considerably.

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