Viscosity, wave damping and shock wave formation in cold hadronic matter

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Abstract

We study linear and nonlinear wave propagation in a dense and cold hadron gas and also in a cold quark gluon plasma, taking viscosity into account and using the Navier-Stokes equation. The equation of state of the hadronic phase is derived from the nonlinear Walecka model in the mean field approximation. The quark gluon plasma phase is described by the MIT equation of state. We show that in a hadron gas viscosity strongly damps wave propagation and also hinders shock wave formation. This marked difference between the two phases may have phenomenological consequences and lead to new QGP signatures.
I. INTRODUCTION

Before the appearance of the RHIC data, a tremendous effort had been dedicated to the search of quark gluon plasma (QGP) signatures and to determine ways of experimentally disentangling the QGP from a hadron gas (HG) [1]. Among the most promising signatures we could find $J/\psi$ suppression and enhancement of strangeness production. All the proposed signatures were based on medium properties which were very different in each one of the two phases. Charmonium suppression, for example, was based on the QGP properties of deconfinement and plasma screening [2]. Strangeness enhancement was based on the chiral symmetry restoration, which is fully realized in QGP but only partially in a hadron gas [3].

In the most recent years the search for QGP signatures had lower priority, mostly because we believe that QGP has been already produced at RHIC and at LHC and now the focus should be on the study of its properties [4]. We believe that it is still interesting to look for QGP signatures and in this work we take the first steps to propose a new one, based on viscosity.

Recently we have learned many things about the QGP and one of the most striking is that it is an almost perfect fluid with very small viscosity [4, 5]. We have also learned, albeit with much larger uncertainties, that the hadron gas has a very large viscosity [6, 7]. According to some estimates, the viscosity coefficients (both bulk and shear) in the two phases can differ by two orders of magnitude or more! This difference is so big that it motivates us to look for observables which are sensitive to it. So far the existing calculations have only addressed the effects of viscosity in the QGP phase and the shear and bulk viscosity coefficients were varied in a limited range. These changes produce some visible but not very large effects on global observables such as rapidity and transverse momentum distributions, flow coefficients and particle correlations [8].

We will investigate the viscosity effects on wave propagation in a quark gluon plasma and also in a hadron gas. These waves may be caused, for example, by fluctuations in baryon number or energy density. These fluctuations may be produced by inhomogeneous initial conditions, which, as pointed out in [9], are the result of quantum fluctuations in the densities of the two colliding nuclei and also in the energy deposition mechanism. These fluctuations and their phenomenological implications have been studied extensively [10–14] because they may be responsible for the angular correlations of particle emission observed in
heavy-ion experiments. There are also hydrodynamic fluctuations [9], which are the result of finite particle number effects in a given fluid cell. This generates local thermal fluctuations of the energy density (and flow velocity) which propagate throughout the fluid. Furthermore, there may be fluctuations induced by energetic partons, which have been scattered in the initial collision of the two nuclei and propagate through the medium, loosing energy and acting as a source term for the hydrodynamical equations. Finally, there may be also freeze-out fluctuations, which may be caused by finite particle number effects during and after the freeze-out of the hydrodynamically expanding fluid.

According to our current picture of high energy heavy ion collisions, a dense and hot hadron gas may exist at later stages of the collisions, when the fluid has cooled down and the quark-hadron phase transition has taken place. A dense hadron gas may also exist and play a more important role in lower energy collisions, such as those which will be performed at FAIR [15]. Finally, a dense and cold hadron gas is expected to exist in the core of compact stars. It is not clear which mechanism might be responsible for the production of waves in the hadronic phase, but it is nevertheless necessary to know how waves propagate in a HG. In particular, we need to know to what extent waves generated in the QGP survive the passage through a hadron gas.

At intermediate energy proton-nucleus reactions, where we do not expect to form QGP, the impinging proton may be absorbed by the nuclear medium creating a perturbation in baryon density, which then propagates through the target. This wave may have a relatively large amplitude and must then be treated beyond the linearization approximation. This was first done in [16], where it was found that, for a perfect fluid and for a particular equation of state, a Korteweg - de Vries (KdV) soliton is formed. In [17] a similar analysis was carried out with an equation of state derived from the nonlinear Walecka model with the same conclusion. The formalism was subsequently extended to relativistic hydrodynamics in [18], to the case of spherical waves in [19] and finally to a hadron gas at finite temperature in [20], where a more detailed numerical solution of the differential equations was performed and a wider class of relativistic mean field models was investigated. For some equations of state and/or some approximation schemes we find KdV solitons and for others we do not find them, but even when there is no soliton we always observe that localized pulses propagate for long distances (from 10 up to 20 fm !) without loosing strength. In [21] and [22] we completed the studies of perfect fluids including the quark gluon plasma at zero and finite
temperatures. Even for the QGP, a fluid with a very different equation of state, the main features of pulse propagation were still very similar to those found in the previous works. A really strong difference was observed in [23] where viscosity was introduced for the first time in our study of waves. In that work we studied the time evolution of cylindrical "hot spots" (or simply "tubes") in a hot and viscous quark gluon plasma. We could clearly see that a small viscosity strongly damps and broadens the tubes during their expansion and they are more easily mixed with the background fluid.

In the present work we perform a systematic comparison between waves (both linear and nonlinear) in a hadron gas and in a quark gluon plasma, focusing on the effects of viscosity. For simplicity we shall consider only the case where the temperature is zero and derive the wave equations for perturbations in a non-relativistic viscous fluid. We will then find the solutions of these equations and see how they change when we change the viscosity coefficients.

In the framework of non-relativistic fluid dynamics we can derive a theory of dissipative fluids which is remarkably successful: the Navier-Stokes (NS) theory [24, 26]. We can use this theory to investigate the evolution of density perturbations in a hadron gas and in a quark gluon plasma. Perturbations are usually studied with the linearization formalism [24, 27], which is the simplest way to study small deviations from equilibrium and to obtain a wave equations. The propagation of perturbations through a QGP has been investigated in several works with the help of a linearized version of the hydrodynamics of perfect fluids [10, 11] and also of viscous fluids [12]. We will later go beyond linearization and study the effects of viscosity on the propagation of nonlinear waves. To this end we employ the well established reductive perturbation method (RPM) [28, 33].

This work is organized as follows. In the next section we present the non-relativistic viscous fluid dynamics. In section III we briefly review the main expressions of the equation of state of both the hadron gas and the quark gluon plasma at zero temperature. In section IV we combine the hydrodynamical equations with the equations of state and derive the wave equations and present a numerical study. The last section is devoted to some concluding remarks.
II. NON-RELATIVISTIC VISCOUS FLUID DYNAMICS

In what follows we employ the natural units $c = 1$ and $\hbar = 1$. The metric is $g^{\mu\nu} = \text{diag}(+,-,-,-)$. The non-relativistic Navier-Stokes equation is given by [24–26]:

$$\frac{\partial v^i}{\partial t} + v^k \frac{\partial v^i}{\partial x^k} = -\frac{1}{\rho} \frac{\partial p}{\partial x^i} - \frac{1}{\rho} \frac{\partial \Pi^{ki}}{\partial x^k}$$  \hspace{1cm} (1)

with the viscous tensor given by:

$$\Pi^{ki} = -\eta \left( \frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i} - \frac{2}{3} \delta^{ki} \frac{\partial v^l}{\partial x^l} \right) - \zeta \delta^{ki} \frac{\partial v^l}{\partial x^l}$$  \hspace{1cm} (2)

The viscous coefficients are $\eta$ for the shear viscosity and $\zeta$ for the bulk viscosity. We shall consider $(i = k = l = x)$ for the one dimensional cartesian case, so the Navier-Stokes equation (1) with (2) becomes:

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 v_x}{\partial x^2}$$  \hspace{1cm} (3)

where $\rho$ is the mass density. The continuity equation for the mass density $\rho$ is given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$  \hspace{1cm} (4)

In the nuclear medium, the mass density is related to the baryon density through $\rho = M \rho_B$, where $M$ is the nucleon mass. The continuity equation for the baryon density in one cartesian dimension is:

$$\frac{\partial \rho_B}{\partial t} + v_x \frac{\partial \rho_B}{\partial x} + \rho_B \frac{\partial v_x}{\partial x} = 0$$  \hspace{1cm} (5)

III. EQUATION OF STATE

A. Hadron gas

The theory of nuclear matter has experienced a continuous progress. Recent developments include the use of effective field theory with chiral power counting (see for example [34]). For our purposes it is enough to work with a theory which reproduces the main features of dense nuclear matter. In this section we review the derivation of the equation of state from the nonlinear Walecka model [35–37] of cold nuclear matter. The Lagrangian density is given by:

$$\mathcal{L} = \bar{\psi} \left[ \gamma_\mu (i \partial^\mu - gV^\mu) - (M - g_S \phi) \right] \psi + \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m_S^2 \phi^2 \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
+ \frac{1}{2} m_{V}^{2} V_{\mu}V^{\mu} - \frac{b}{3} \phi^{3} - \frac{c}{4} \phi^{4}\)\(6\)

where \(F_{\mu\nu} = \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu}\). The degrees of freedom are the baryon field \(\psi\), the neutral scalar meson field \(\phi\) and the neutral vector meson field \(V_{\mu}\), with the respective couplings and masses. The equation of state is obtained through the usual mean field theory (MFT) approximation which consists in considering the meson fields as classical fields. This classical approach is based on the assumption that the baryonic sources are intense, their coupling to the meson fields is strong and the infinite nuclear matter is static, homogeneous in space and isotropic. From the calculations performed in [35–37] we have the energy density:

\[\varepsilon = \frac{g_{V}^{2}}{2m_{V}^{2}} \rho_{B}^{2} + \frac{m_{S}^{2}}{2g_{S}^{2}} (M - M^{*})^{2} + \frac{b}{3g_{S}^{3}} (M - M^{*})^{3} + \frac{c}{4g_{S}^{4}} (M - M^{*})^{4} + \frac{\gamma_{s}}{(2\pi)^{3}} \int_{0}^{k_{F}} d^{3}k \sqrt{k^{2} + M^{*2}}\] \(7\)

where \(\gamma_{s} = 4\) is the nucleon degeneracy factor and the effective mass of the nucleon is defined as \(M^{*} = M - g_{S} \phi_{0}\). In numerical calculations we will use \([35–37]\) \(M = 939\) \(MeV\), \(g_{V} = 9.182\) and \(m_{V} = 783\) \(MeV\). The baryon density \(\rho_{B}\) up to the Fermi level \((k_{F})\) is given by:

\[\psi^\dagger \psi \equiv \rho_{B} = \frac{\gamma_{s}}{6\pi^{2}} k_{F}^{3}\] \(8\)

The nucleon effective mass is determined by the self-consistency relation obtained from the minimization of \(\varepsilon(M^{*})\) with respect to \(M^{*}\):

\[M^{*} = M - \frac{g_{S}^{2}}{m_{S}^{2}} \frac{\gamma_{s}}{(2\pi)^{3}} \int d^{3}k \frac{M^{*}}{\sqrt{k^{2} + M^{*2}}} + \frac{g_{S}^{2}}{m_{S}^{2}} \left[ \frac{b}{g_{S}^{3}} (M - M^{*})^{2} + \frac{c}{g_{S}^{4}} (M - M^{*})^{3} \right]\] \(9\)

**B. QGP**

The equation of state of the plasma phase is derived from the MIT bag model which describes a gas of noninteracting quarks and gluons and takes into account nonperturbative effects through the bag constant \(B\) \([32]\). This constant is interpreted as the energy needed to create a bubble (or bag) in the QCD physical vacuum. The baryon density is given by:

\[\rho_{B} = \frac{1}{3} \frac{\gamma_{Q}}{(2\pi)^{3}} \int d^{3}k \left[ n_{\vec{k}} - \bar{n}_{\vec{k}} \right]\] \(10\)

with

\[n_{\vec{k}} \equiv n_{\vec{k}}(T) = \frac{1}{1 + e^{(k - \frac{3}{2} \mu)/T}} \quad \text{and} \quad \bar{n}_{\vec{k}} \equiv \bar{n}_{\vec{k}}(T) = \frac{1}{1 + e^{(k + \frac{3}{2} \mu)/T}}\] \(11\)
where $\mu$ is the baryon chemical potential. At zero temperature the expression for the baryon density reduces to:

$$\rho_B = \frac{2}{3\pi^2} k_F^3$$  \hspace{1cm} (12)

and $k_F$ refers to highest occupied momentum level. The energy density and the pressure are given respectively by:

$$\varepsilon = \mathcal{B} + \frac{\gamma_G}{(2\pi)^3} \int d^3k \, k \, (e^{k/T} - 1)^{-1} + \frac{\gamma_Q}{(2\pi)^3} \int d^3k \, k \left[ n_\vec{k} + \bar{n}_\vec{k} \right]$$  \hspace{1cm} (13)

and

$$p = -\mathcal{B} + \frac{1}{3} \left\{ \frac{\gamma_G}{(2\pi)^3} \int d^3k \, k \, (e^{k/T} - 1)^{-1} + \frac{\gamma_Q}{(2\pi)^3} \int d^3k \, k \left[ n_\vec{k} + \bar{n}_\vec{k} \right] \right\}$$  \hspace{1cm} (14)

The statistical factors are $\gamma_G = 2$(polarizations) $\times 8$(colors) = 16 for gluons and $\gamma_Q = 2$(spins) $\times 2$(flavors) $\times 3$(colors) = 12 for quarks. From the above expressions we extract the speed of sound $c_s$:

$$c_s^2 = \frac{\partial p}{\partial \varepsilon} = \frac{1}{3}$$  \hspace{1cm} (15)

For the cold plasma at zero temperature and high baryon density the quark distribution functions become step functions. Inserting (12) into (13) and into (14) we find the energy density and pressure as functions of the baryon density, respectively:

$$\varepsilon(\rho_B) = \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} + \mathcal{B}$$  \hspace{1cm} (16)

and

$$p(\rho_B) = \frac{1}{3} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} - \mathcal{B}$$  \hspace{1cm} (17)

IV. WAVE EQUATIONS

A. Linear waves

Before addressing the nonlinear wave equations, we revisit the linear wave equation approach in viscous hydrodynamics. The non-relativistic Navier-Stokes equation may provide a simple wave equation for perturbations in the baryon density based on the linearization formalism [24, 27]. The use of this formalism can be justified when small perturbations are considered in a fluid at rest with background pressure $p_0$ and background baryon density $\rho_0$. For simplicity we consider perturbations which depend only on the $x$ space coordinate.
and on time. We first define the dimensionless variables for the baryon density, for the fluid velocity and for the pressure:

\[ \hat{\rho}(x,t) = \frac{\rho_B(x,t)}{\rho_0}, \quad \hat{v}_x(x,t) = \frac{v_x(x,t)}{c_s} \quad \text{and} \quad \hat{p}(x,t) = \frac{p(x,t)}{p_0} \]  

(18)

The expression for the speed of sound, \( c_s \), is:

\[ c_s^2 = \frac{\partial p}{\partial \varepsilon} \Rightarrow p \sim c_s^2 \varepsilon \]  

(19)

In the non-relativistic limit \( \varepsilon \approx \rho \), where \( \rho \) is the matter density, which is related to the baryon density through \( \rho = M \rho_B \). We can then write:

\[ p = c_s^2 M \rho_B \]  

(20)

Rewriting (11), (5) and (20) in one cartesian dimension and using (18) we find:

\[ M \rho_0 \hat{\rho} \left( c_s \frac{\partial \hat{v}_x}{\partial t} + c_s^2 \hat{v}_x \frac{\partial \hat{v}_x}{\partial x} \right) + p_0 \frac{\partial \hat{p}}{\partial x} = \left( \zeta + \frac{4}{3} \eta \right) c_s \frac{\partial^2 \hat{v}_x}{\partial x^2} \]  

(21)

\[ \frac{\partial \hat{\rho}}{\partial t} + c_s \hat{v}_x \frac{\partial \hat{\rho}}{\partial x} + c_s \rho \frac{\partial \hat{v}_x}{\partial x} = 0 \]  

(22)

and

\[ p_0 \hat{p} = M \rho_0 c_s^2 \hat{\rho} \]  

(23)

The perturbations are described by the following expansions of the dimensionless variables:

\[ \hat{\rho} = 1 + \delta \rho_B, \quad \hat{v}_x = \delta v_x \quad \text{and} \quad \hat{p} = 1 + \delta p \]  

(24)

where \( \delta \rho_B, \delta v_x \) and \( \delta p \), denote small deviation of the baryon density, velocity and pressure from their equilibrium values respectively. When the expansions (24) are inserted into the relevant equations in consideration, the “linearization approximation” is performed by neglecting the \( O(\delta^2) \) terms [24, 27]. Inserting (24) into (21), (22), (23) and performing the linearization, we find respectively:

\[ M \rho_0 c_s \frac{\partial}{\partial t} \delta v_x + p_0 \frac{\partial}{\partial x} \delta p = \left( \zeta + \frac{4}{3} \eta \right) c_s \frac{\partial^2 \delta v_x}{\partial x^2} \]  

(25)

\[ \frac{\partial}{\partial t} \delta \rho_B + c_s \frac{\partial}{\partial x} \delta v_x = 0 \]  

(26)

and

\[ p_0(1 + \delta p) = M \rho_0 c_s^2 (1 + \delta \rho_B) \]  

(27)
Differentiating (26) with respect to $x$ and inserting it into (25) we find:

$$M\rho_0 c_s \frac{\partial}{\partial t} \delta v_x + p_0 \frac{\partial}{\partial x} \delta p = - \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial}{\partial t} \frac{\partial}{\partial x} \delta \rho_B$$

(28)

Performing the derivative of (28) with respect to $x$ and using (27), we find:

$$M\rho_0 c_s \frac{\partial}{\partial t} \frac{\partial}{\partial x} \delta v_x + M\rho_0 c_s^2 \frac{\partial^2}{\partial x^2} \delta \rho_B = - \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} \delta \rho_B$$

(29)

Inserting the time derivative of (26) into (29) we obtain the following viscous wave equation for the baryonic density perturbation $\delta \rho_B$:

$$\frac{\partial^2}{\partial x^2} \delta \rho_B - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \delta \rho_B = -\nu \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} \delta \rho_B$$

(30)

where

$$\nu = \frac{1}{M\rho_0 c_s^2} \left( \zeta + \frac{4}{3} \eta \right)$$

(31)

and making use of the Ansatz:

$$\delta \rho_B(x, t) = Ae^{ikx-\omega t}$$

(32)

in (30) we find the following dispersion relation $\omega(k)$:

$$\omega^2 = c_s^2 k^2 - \nu \omega k^2$$

(33)

where $\nu > 0$ and $k \in \mathbb{R}$. The angular frequency is decomposed in two components:

$$\omega = \omega_R + i\omega_I$$

(34)

where $\omega_R \in \mathbb{R}$ and $\omega_I \in \mathbb{R}$. Inserting (34) in (33) we find:

$$\omega_R^2 = c_s^2 k^2 - \frac{\nu^2 k^4}{4} \quad \text{and} \quad \omega_I = -\frac{\nu^2 k^2}{2} < 0$$

(35)

From the above we draw two important conclusions concerning the velocities:

i) The phase velocity $v_p$ is given by:

$$v_p^2 = \left( \frac{\omega_R}{k} \right)^2 = c_s^2 - \left( \frac{\nu k}{2} \right)^2$$

(36)

We can observe that one of the effects of viscosity is to reduce $v_p$.

ii) The group velocity $v_g$ is calculated using $|\omega| = c_s k$:

$$v_g = \frac{d|\omega|}{dk} = c_s$$

(37)
Inserting \((35)\) into \((34)\) and the resulting value of \(\omega\) into the Ansatz \((32)\), we find the solution of \((30)\):

\[
\delta \rho_B(x,t) = A e^{-\nu k^2 t/2} e^{i(kx - \nu k^2 t/2)} \quad (38)
\]

with

\[
\Re\{\delta \rho_B(x,t)\} = A e^{-\nu k^2 t/2} \cos \left( kx - \sqrt{c_s^2 k^2 - \nu^2 k^4/4} \right) \quad (39)
\]

Since in the derivation of the above equation we have only used the general expression \((19)\), it is valid both for the QGP and HG phases. The differences appear because the constants \(\nu\) and \(c_s\) are different. The former varies over a much wider range of values whereas the latter is never very different from \(1/3\). For the sake of simplicity we will choose a constant speed of sound \(c_s^2 = 1/3\) for the two phases and all the differences will come from \(\nu\). From \((31)\) we see that the changes in density and viscosity are correlated in such a way that in a hadron gas \(\eta\) and \(\zeta\) are large and \(\rho_0\) is relatively small with \(\nu\) being large, whereas in the plasma phase the opposite happens and \(\nu\) is small. As an example we study the effects of viscosity on the waves propagating in a cold and dense system. In this case the perturbation \(\delta \rho_B\) is a deviation of the baryon density from the background density \(\rho_0\).

In numerical calculations the viscosity coefficients \(\zeta\) and \(\eta\) may be taken from \([6, 7]\) and used in \((31)\). Using the exact values of these coefficients is not really crucial for our purposes. It is enough to know that \(\eta\) is dramatically different in different phases. The phase velocity must be real and therefore in \((39)\) we must have:

\[
k < \frac{2c_s}{\nu} \quad (40)
\]

From this inequality we see that, since \(\nu_{\text{QGP}} << \nu_{\text{HG}}\) the maximum value of \(k\) is much smaller in the HG than in the QGP and consequently in a hadron gas there are much less modes than in a quark gluon plasma. From \((39)\) we see that for a hadron gas with high viscosity there is a strong damping of the wave due to the exponential factor, whereas in the plasma the oscillations persist for a very long time.

We close this section emphasizing that waves produced by small perturbations, those which can be treated with the linearization approximation, behave quite differently in a hadron gas and in a quark gluon plasma. This pronounced difference comes essentially from the very different viscosity coefficients in these two phases. In a hadron gas waves are damped and stalled, whereas in a quark gluon plasma they can propagate nearly undisturbed.
Since there are many possible sources of perturbations and since they do not have to be necessarily very small, this study has to be extended to stronger perturbations and the resulting nonlinear waves. This is the subject of the next section.

**B. Nonlinear waves in a hadron gas**

We start recalling the relation between the mass density and baryon density, \( \rho = M \rho_B \), which used in equation (3) yields:

\[
\rho_B \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = -\frac{1}{M} \frac{\partial p}{\partial x} + \frac{1}{M} \left( \frac{\partial \varepsilon}{\partial \rho_B} \right) \frac{\partial^2 v_x}{\partial x^2}
\]  

(41)

From the first law of thermodynamics at zero temperature we have:

\[
d\varepsilon = \mu_B d\rho_B
\]

where

\[
\mu_B = \frac{\partial \varepsilon}{\partial \rho_B}
\]

(42)

Inserting (42) into the Gibbs relation at zero temperature:

\[
d\varepsilon + dp = \rho_B d\mu_B + \mu_B d\rho_B
\]

we find

\[
dp = \rho_B d\mu_B
\]

and then finally:

\[
dp = \rho_B d\left( \frac{\partial \varepsilon}{\partial \rho_B} \right) \quad \text{and consequently} \quad \frac{\partial p}{\partial x} = \rho_B \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon}{\partial \rho_B} \right)
\]

(43)

Inserting (43) into (41) we find:

\[
\rho_B \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = -\frac{1}{M} \rho_B \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon}{\partial \rho_B} \right) + \frac{1}{M} \left( \frac{\partial \varepsilon}{\partial \rho_B} \right) \frac{\partial^2 v_x}{\partial x^2}
\]

(44)

where the relevant quantity which carries information about the equation of state is \( \frac{\partial \varepsilon}{\partial \rho_B} \), which can be computed from (7). In the present calculation we will neglect the weak dependence of \( M^* \) and also of the integral term in (7) on \( \rho_B \). We obtain:

\[
\frac{\partial \varepsilon}{\partial \rho_B} = \frac{g v^2}{m v^2 \rho_B}
\]

(45)

Inserting (45) into (44) we find:

\[
\rho_B \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = \frac{1}{M} \left[ -\frac{g v^2}{m v^2 \rho_B} \frac{\partial \rho_B}{\partial x} + \left( \frac{4}{3} \eta \right) \frac{\partial^2 v_x}{\partial x^2} \right]
\]

(46)
which is the Navier-Stokes equation for the hadron phase.

We next combine (46) and (5) to obtain a nonlinear wave equation. In dealing with these equations we use the reductive perturbation method (RPM) [28–33], which is a technique which allows us to go beyond linearization and to preserve nonlinearities, dissipative and dispersive terms in the wave equations. We have already used the RPM in our previous works to study nonlinear waves in relativistic and non-relativistic ideal hydrodynamics and also to study the evolution of flux in relativistic viscous hydrodynamics [23]. A comprehensive review of the technique and its applications in hydrodynamics of strongly interacting fluids can be found in [32].

The background baryon density is \( \rho_0 \), upon which the perturbation propagates. We now rewrite (5) and (46) using (18) to obtain:

\[
\frac{\partial \hat{\rho}}{\partial t} + c_s \hat{v}_x \frac{\partial \hat{\rho}}{\partial x} + c_s \hat{\rho} \frac{\partial \hat{v}_x}{\partial x} = 0 \quad (47)
\]

and

\[
\hat{\rho} \left( \frac{\partial \hat{v}_x}{\partial t} + c_s \hat{v}_x \frac{\partial \hat{v}_x}{\partial x} \right) = -\rho_0 \frac{g_{V^2}}{m_V^2} \hat{\rho} + \frac{1}{M \rho_0} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 \hat{v}_x}{\partial x^2} \quad (48)
\]

Following [28–31], we define the “stretched coordinates”:

\[
\xi = \sigma^{1/2} \left( x - c_s t \right) / L \quad \text{and} \quad \tau = \sigma^{3/2} c_s t / L \quad (49)
\]

where \( L \) is a characteristic length scale of the problem (typically the radius of a heavy ion) and \( \sigma \) is a small \((0 < \sigma < 1)\) expansion parameter. We also perform the following transformation of the viscosity coefficients [39, 40]:

\[
\zeta = \sigma^{1/2} \tilde{\zeta} \quad \text{and also} \quad \eta = \sigma^{1/2} \tilde{\eta} \quad (50)
\]

From the stretched coordinates (49) we have the operators:

\[
\frac{\partial}{\partial x} = \frac{\sigma^{1/2}}{L} \frac{\partial}{\partial \xi} \quad , \quad \frac{\partial^2}{\partial x^2} = \frac{\sigma}{L^2} \frac{\partial^2}{\partial \xi^2} \quad \text{and} \quad \frac{\partial}{\partial t} = -c_s \frac{\sigma^{1/2}}{L} \frac{\partial}{\partial \xi} + c_s \frac{\sigma^{3/2}}{L} \frac{\partial}{\partial \tau} \quad (51)
\]

Using (50) and (51) we transport (47) and (48) from the \((x, t)\) space to the \((\xi, \tau)\) space, obtaining:

\[
- \frac{\partial \hat{\rho}}{\partial \xi} + \sigma \frac{\partial \hat{\rho}}{\partial \tau} + \hat{v}_x \frac{\partial \hat{\rho}}{\partial \xi} + \hat{\rho} \frac{\partial \hat{v}_x}{\partial \xi} = 0 \quad (52)
\]

and

\[
\hat{\rho} \left( - \frac{\partial \hat{v}_x}{\partial \xi} + \sigma \frac{\partial \hat{v}_x}{\partial \tau} + \hat{v}_x \frac{\partial \hat{v}_x}{\partial \xi} \right) = -\left( \frac{g_{V^2} \rho_0}{M c_s m_V^2} \right) \hat{\rho} + \frac{\sigma}{M \rho_0 c_s L} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 \hat{v}_x}{\partial \xi^2} \quad (53)
\]
We now perform the expansion of the dimensionless baryon density and the dimensionless fluid velocity given by \((18)\) around their equilibrium values:

\[
\hat{\rho} = \frac{\rho_B}{\rho_0} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \sigma^3 \rho_3 + \ldots \tag{54}
\]

\[
\hat{v}_x = \frac{v_x}{c_s} = \sigma v_{x1} + \sigma^2 v_{x2} + \sigma^3 v_{x3} + \ldots \tag{55}
\]

We now insert the expansions \((54)\) and \((55)\) into \((52)\) and into \((53)\), neglect the terms proportional to \(\sigma^n\) for \(n > 2\) and organize the equations as series in powers of \(\sigma\) and \(\sigma^2\), finding the following equations:

\[
\sigma \left\{ - \frac{\partial \rho_1}{\partial \xi} + \frac{\partial v_{x1}}{\partial \xi} \right\} + \sigma^2 \left\{ - \frac{\partial \rho_2}{\partial \xi} + \frac{\partial \rho_1}{\partial \tau} + v_{x1} \frac{\partial \rho_1}{\partial \xi} + \frac{\partial v_{x2}}{\partial \xi} + \rho_1 \frac{\partial \rho_1}{\partial \xi} \right\} = 0 \tag{56}
\]

and

\[
\sigma \left\{ - \frac{\partial v_{x1}}{\partial \xi} + \left( \frac{g v^2 \rho_0}{M c_s^2 m v^2} \right) \frac{\partial \rho_1}{\partial \xi} \right\} + \sigma^2 \left\{ - \frac{\partial v_{x2}}{\partial \xi} - \rho_1 \frac{\partial v_{x1}}{\partial \xi} + \frac{\partial v_{x1}}{\partial \tau} + v_{x1} \frac{\partial v_{x1}}{\partial \xi} \right\} = 0 \tag{57}
\]

Each bracket in the last two equations must vanish: \(\sigma \{ \ldots \} = 0\) and \(\sigma^2 \{ \ldots \} = 0\). The final step of RPM is collect the equations from each \(\sigma\) order to obtain the nonlinear wave equation. From the terms proportional to \(\sigma\) we have, in a simple way, after the integration with respect to \(\xi\) and setting the integration constant equals to zero, the following results:

\[
v_{x1} = \rho_1 \tag{58}
\]

and the algebraic relation:

\[
\frac{g v^2 \rho_0}{M c_s^2 m v^2} = 1 \tag{59}
\]

which gives the sound speed \((c_s)\):

\[
c_s^2 = \frac{g v^2 \rho_0}{M m v^2} \tag{60}
\]

From the terms proportional to \(\sigma^2\) we find:

\[
\frac{\partial \rho_2}{\partial \xi} - \frac{\partial v_{x2}}{\partial \xi} = \frac{\partial \rho_1}{\partial \tau} + v_{x1} \frac{\partial \rho_1}{\partial \xi} + \rho_1 \frac{\partial v_{x1}}{\partial \xi} \tag{61}
\]

and

\[
\left( \frac{g v^2 \rho_0}{M c_s^2 m v^2} \right) \frac{\partial \rho_2}{\partial \xi} - \frac{\partial v_{x2}}{\partial \xi} = - \frac{\partial v_{x1}}{\partial \tau} - v_{x1} \frac{\partial v_{x1}}{\partial \xi} + \rho_1 \frac{\partial v_{x1}}{\partial \xi}
\]
\(- \left( \frac{g v^2 \rho_0}{M c_s^2 m_v} \right) \rho_1 \frac{\partial \rho_1}{\partial \xi} + \frac{1}{M \rho_0 c_s L} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 v_{x1}}{\partial \xi^2} \) \tag{62}

Using (58) in (61) we find:
\[
\frac{\partial p_2}{\partial \xi} - \frac{\partial v_{x2}}{\partial \xi} = \frac{\partial p_1}{\partial \tau} + 2 \rho_1 \frac{\partial \rho_1}{\partial \xi} + \frac{1}{M \rho_0 c_s L} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 \rho_1}{\partial \xi^2} \tag{63}
\]

and using (58) and (59) in (62) we find:
\[
\frac{\partial p_2}{\partial \xi} - \frac{\partial v_{x2}}{\partial \xi} = \frac{\partial p_1}{\partial \tau} - \rho_1 \frac{\partial \rho_1}{\partial \xi} + \frac{1}{M \rho_0 c_s L} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 \rho_1}{\partial \xi^2} \tag{64}
\]

Combining (63) and (64) we find the Burgers equation in the \((\xi, \tau)\) space:
\[
\frac{\partial \rho_1}{\partial \tau} + \frac{3}{2} \rho_1 \frac{\partial \rho_1}{\partial \xi} = \frac{1}{2M \rho_0 c_s L} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 \rho_1}{\partial \xi^2} \tag{65}
\]

Returning (63) to the \((x, t)\) space (with the help of (50) and (51)) we obtain the following Burgers equation:
\[
\frac{\partial \delta \rho_B}{\partial t} + c_s \frac{\partial \delta \rho_B}{\partial x} + \frac{3}{2} c_s \frac{\partial \delta \rho_B}{\partial x} \frac{\partial \delta \rho_B}{\partial x} = \frac{1}{2M \rho_0} \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 \delta \rho_B}{\partial x^2} \tag{66}
\]

where \((54): \delta \rho_B \equiv \sigma \rho_1\) is the first deviation from the background baryon density \(\rho_0\).

C. Nonlinear waves in a QGP

From (17) we have:
\[
\frac{\partial p}{\partial x} = \frac{4}{9} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{1/3} \frac{\partial \rho_B}{\partial x} \tag{67}
\]

In the non-relativistic limit \(\varepsilon + p \approx \rho\) and from (16) and (17) we find:
\[
\rho = \frac{4}{3} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} \tag{68}
\]

Inserting (67) and (68) into (3) we obtain the following Navier-Stokes equation for the plasma phase:
\[
\frac{4}{3} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{4/3} \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = - \frac{4}{9} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_B^{1/3} \frac{\partial \rho_B}{\partial x} + \left( \zeta + \frac{4}{3} \eta \right) \frac{\partial^2 v_x}{\partial x^2} \tag{69}
\]

Repeating all the steps shown in the last section, we obtain from the terms of order \(\sigma\):
\[
v_{x1} = \rho_1 \quad \text{and} \quad c_s^2 = 1/3 \tag{70}
\]

and from terms of order \(\sigma^2\):
\[
\frac{4}{9 c_s^2} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_0^{4/3} \frac{\partial p_2}{\partial \xi} - \frac{4}{3} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_0^{4/3} \frac{\partial v_{x2}}{\partial \xi} = \frac{16}{9} \left( \frac{3}{2} \right)^{7/3} \pi^{2/3} \rho_0^{4/3} \rho_1 \frac{\partial v_{x1}}{\partial \xi} \]

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\[-\frac{4}{27c_s^2}\left(\frac{3}{2}\right)^{7/3}\pi^{2/3}\rho_0^{4/3}\rho_1\frac{\partial \rho_1}{\partial \xi} + \frac{4}{3}\left(\frac{3}{2}\right)^{7/3}\pi^{2/3}\rho_0^{4/3}\frac{\partial v_{x1}}{\partial \tau} + \frac{4}{3}\left(\frac{3}{2}\right)^{7/3}\pi^{2/3}\rho_0^{4/3}v_{x1}\frac{\partial v_{x1}}{\partial \xi} + \frac{1}{Lc_s}\left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^2 v_{x1}}{\partial \xi^2}\]\(^{(71)}\)

Inserting \((70)\) into \((71)\) and combining this result with \((63)\) we find the following Burgers equation in the \((x,t)\) space:

\[
\frac{\partial}{\partial t}\delta \rho_B + c_s\frac{\partial}{\partial x}\delta \rho_B + c_s\delta \rho_B\frac{\partial}{\partial x}\delta \rho_B = \left[\frac{8}{3}\left(\frac{3}{2}\right)^{7/3}\pi^{2/3}\rho_0^{4/3}\right]^{-1} \times \left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^2 \delta \rho_B}{\partial x^2}\]

\(^{(72)}\)

where, again, \(\delta \rho_B \equiv \sigma \rho_1\).

Comparing the above expression with \((66)\) we see that, apart from small numerical differences in the coefficients, nonlinear waves obey the same Burgers equation in both QGP and HG phases. We can thus anticipate that, as it was the case of linear waves, the most important differences of nonlinear wave propagation will come from the very different numerical values of the viscosity coefficients \(\eta\) and \(\zeta\).

### D. Shock waves

The Burgers equations \((66)\) and \((72)\) may be written in the general form:

\[
\frac{\partial}{\partial t}\delta \rho_B + c_s\frac{\partial}{\partial x}\delta \rho_B + \alpha\delta \rho_B\frac{\partial}{\partial x}\delta \rho_B = \mu\frac{\partial^2}{\partial x^2}\delta \rho_B\]

\(^{(73)}\)

where \(\alpha\) and \(\mu\) are the respective nonlinear and dissipative coefficients for the plasma and hadron phase given respectively by:

\[
\alpha = \begin{cases} 
\alpha_{QGP} = c_s \\
\alpha_{HG} = \frac{3}{2}c_s
\end{cases} \quad \text{and} \quad \mu = \begin{cases} 
\mu_{HG} = \frac{1}{2M\rho_0}\Delta_{HG} \\
\mu_{QGP} = \left[\frac{8}{3}\left(\frac{3}{2}\right)^{7/3}\pi^{2/3}\rho_0^{4/3}\right]^{-1} \times \Delta_{QGP}
\end{cases}
\]

\(^{(74)}\)

where for simplicity we have defined the “effective viscosity coefficient”:

\[
\Delta = \begin{cases} 
\Delta_{HG} \equiv \left(\zeta_{HG} + \frac{4}{3}\eta_{HG}\right) \\
\Delta_{QGP} \equiv \left(\zeta_{QGP} + \frac{4}{3}\eta_{QGP}\right)
\end{cases}
\]

\(^{(75)}\)

which assumes distinct values for QGP and HG phases.

Applying the hyperbolic tangent function method as described in \([41,43]\) we obtain the exact traveling wave solution of \((73)\) with the free parameter \(\lambda\):

\[
\delta \rho_B(x, t; \mu) = -\frac{2\mu\lambda}{\alpha} - \frac{2\mu\lambda}{\alpha} \tanh \left\{\lambda \left[ x - \left( c_s + \frac{\alpha}{3} \right) t \right] \right\}
\]

\(^{(76)}\)
For the particular choice $\lambda = -\alpha/6\mu$, to ensure $\delta \rho_B < 1$, the following wave:

$$\delta \rho_B(x, t; \mu) = \frac{1}{3} + \frac{1}{3} \tanh \left\{ -\frac{\alpha}{6\mu} \left[ x - \left( c_s + \frac{\alpha}{3} \right)t \right] \right\}$$

(77)

with the supersonic speed $v_s$:

$$v_s = c_s + \frac{\alpha}{3}$$

(78)

We follow the analysis performed in [41, 43] of the hyperbolic tangent profile (77). We fix $t$ for the QGP and HG phases and obtain the wave profiles shown in Fig. 1 with the parameters of Table I.

TABLE I: Parameters of QGP and HG for Fig. 1.

|           | $\rho_0$ ($fm^{-3}$) | $c_s^2$ | $\Delta$ (GeV/fm$^2$) | $t$ (fm) |
|-----------|-----------------------|---------|------------------------|----------|
| **QGP**   | $10\rho_n$            | 1/3     | 0.05                   | 0.05     |
| **HG**    | $1.5\rho_n$           | 1/3     | 2.05                   | 0.05     |

FIG. 1: Burgers shock wave solution with different values of the effective viscosity coefficient $\Delta$ for QGP and HG.

In Fig. 1 we observe that in QGP phase the limit of small $\mu$, in (77) is given by the classical shock wave:

$$\lim_{\mu_{QGP} \to 0} \delta \rho_B(x, t; \mu_{QGP}) = \begin{cases} 2/3 & \text{for } x/t < c_s + \alpha/3 \\ 0 & \text{for } x/t > c_s + \alpha/3 \end{cases}$$

(79)
As in [41] we rewrite (73) as the following inhomogeneous continuity equation:

$$\frac{\partial}{\partial t} \delta \rho_B + \frac{\partial}{\partial x} \left( c_s \delta \rho_B + \frac{\alpha}{2} \delta \rho_B^2 \right) = \mu \frac{\partial^2}{\partial x^2} \delta \rho_B$$

(80)

where the limiting process $\mu \to 0$ given by (79) is a dissipative regularization of the conservation law:

$$\frac{\partial}{\partial t} \delta \rho_B + \frac{\partial}{\partial x} \left( c_s \delta \rho_B + \frac{\alpha}{2} \delta \rho_B^2 \right) = 0$$

(81)

For the HG phase in Fig. 1 the baryon perturbation (77) is quite different from QGP phase and we have:

$$\delta \rho_B(x,t; \mu_{HG} >> \mu_{QGP}) = \frac{1}{3}$$

(82)

due to higher background baryon density and higher viscous coefficients. From these figures we can conclude that, due to the large viscosity, shock waves can not be formed in HG. In contrast, they can be easily formed in a QGP. This difference may have phenomenological consequences.

### E. Wave packets

The most important feature of the nonlinear equation is that there is a competition between the nonlinear and dissipative coefficients. For some particular parameter choices there will be a balance between the two terms, in which case localized soliton-like configurations may propagate for long distances. The numbers employed here suggest that these localized configurations can only survive in the plasma. In order to illustrate this we will perform the numerical study of the time evolution of a soliton-like pulse. In this case the initial condition is given by:

$$\delta \rho_B(x,t_0) = A \text{sech}^2 \left[ \frac{x}{B} \right]$$

(83)

for both (66) and (72), with the amplitude $A$ and width $B$ as free parameters. The viscosity coefficients are extrapolated from [6, 7] and represented in the effective viscosity coefficient (75). The speed of sound in the HG is calculated by (60) and is also $c_s^2 = 1/3$.

In the analysis of the numerical solutions we must carry out a comprehensive study varying each parameter independently, keeping all others fixed. In particular, we study how the solution changes when we change the viscosity. Such a study was performed in [23] for the numerical solutions of a similar wave equation, derived from the relativistic Navier-Stokes equation applied to the QGP at high temperatures. The conclusion was that the
most relevant parameter was the viscosity. As for the others, we find that they have here the same relative importance as in [23]. In Fig. 2 we compare the hadron gas (on the two right panels) with the quark gluon plasma (on the two left panels) for the same scales. Comparing left and right we can observe the strong damping which happens in the hadron gas, which comes ultimately from the larger values of the viscosity coefficients. Comparing
the upper with the lower panels we see that increasing the initial amplitude of the pulse in both phases does not change much its evolution, damping and slowing down. In the plasma phase we can also observe the formation of a "wall" as can be seen in Fig. 2(a) and Fig. 2(c).

V. CONCLUSIONS

In this work we have studied wave propagation in cold and dense matter, both in a hadron gas and in a quark gluon plasma phase. In deriving wave equations from the equations of hydrodynamics, we have considered both smaller and larger amplitude waves. The former were treated with the linearization approximation while the latter were treated with the reductive perturbation method.

Linear waves were obtained by solving an inhomogeneous viscous wave equation and they have the familiar form of sinusoidal travelling waves multiplied by an exponential damping factor, which depends on the viscosity coefficients. Since these coefficients differ by two orders of magnitude, even without any numerical calculation we can conclude that, apart from extremely special parameter choices, in contrast to the quark gluon plasma there will be no linear wave propagation in a hadron gas.

Nonlinear waves were obtained by solving the Burgers equation, which was derived from the equations of hydrodynamics with the RPM. We could find an analytical solution of the Burgers equation which describes a shock wave. Varying the viscosity parameters in the appropriate range we could conclude that, with the exception of extremely special parameter choices, there is no shock formation in a hadron gas. If some external agent would try to create a sharp density discontinuity in a hadronic medium, viscosity would immediately smooth it and wash it out. In a quark gluon plasma, on the other hand, shocks may be formed. Exactly the same features were observed in the wave packet numerical solutions of the Burgers equation. In the hadron gas, the wave packet moves very slowly and its amplitude is very quickly reduced. In the quark gluon plasma there is a balance between nonlinearity and dissipation that prevents the wave from breaking and from dispersion. Density perturbations may “survive” longer in QGP than in HG.

We believe that our work may help in discriminating the quark gluon plasma from the hadron gas. The next and most difficult step is to connect the waves studied here with
observables and plug our analytical studies in realistic numerical simulations of low energy heavy ion collisions or compact stars. Work in this direction is in progress.

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