On the Theory and Practice of Thin Walled Structures

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Abstract We consider the problem of satisfaction of boundary conditions when the generalized stress vector is given on the surfaces for elastic plates and shells. This problem was open also both for refined theories in the wide sense and hierarchical type models. This one for hierarchical models was formulated by Vekua. In nonlinear cases the bending and compression-extension processes did not split and for this aim we cited von Kármán type system without variety of ad hoc assumptions since, in the classical form of this system of DEs one of them represents the condition of compatibility but it is not an equilibrium equation. Thus, we created the mathematical theory of refined theories both in linear and nonlinear cases for anisotropic nonhomogeneous elastic plates and shells, approximately satisfying the corresponding system of partial differential equations and boundary conditions on the surfaces. The optimal and convenient refined theory might be chosen easily by selection of arbitrary parameters; preliminarily a few necessary experimental measurements have been made without using any simplifying hypotheses. The same problem is solved for hierarchical models too.

Elasticity cannot be linear!
Ph. Ciarlet, [1, p. 286]

1 Introduction

Let us consider the equilibrium equations of the elastic body in the form [1, 2]:

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\[ \partial_j (\sigma_{ij} + \sigma_{kj} u_{i,k}) = f_i, \quad x \in \Omega_h = D(x,y) \times [h^- (x,y), h^+ (x,y)]. \quad (1) \]

The boundary conditions:

\[ T_3 = \sigma_3 + \sigma_{3j} u_{i,j} = g^\pm_i, \quad x \in S^\pm = D \times \{h^\pm\}, \quad T_3 = (T_{13}, T_{23}, T_{33})^T, \quad (2) \]

\[ l [\partial_1, \partial_2, \partial_3] (x,u) = g, \quad x \in S = \partial D \times [h^-, h^+]. \quad (3) \]

The relation between the displacement vector \( u = (u_1, u_2, u_3) \), the symmetrical strain \( \varepsilon \) and stress \( \sigma \) tensors satisfy the Cauchy formulae and Hooke’s law:

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{i,k} u_{j,k}), \quad \varepsilon = A \sigma, \quad \sigma = B \varepsilon. \quad (4) \]

Above and below we used the basic notations according to [i.e. 2, pp. xiv-xv] which are same to usually notations from well-known books and articles. For example, the repetition of an the index denotes summation; small Latin and Greek indices assume the values of 1, 2, 3 and 1, 2 accordingly, unless otherwise stipulated. In a reference to a subsection of a section the first number denotes the number of the section, the second one denotes the subsection. \( \partial \partial = \partial_i = \partial_i \) is a derivative by \( x_i \), \( \partial \partial = \partial_t = \partial_t \) is the derivative with respect to time, \( \delta_{ij} \) is Kronecker’s symbol. \( A, B = A^{-1} \) named as the compliance and stiffness matrices, \( \Omega_h (x) = D(x,y) \times [h^-, h^+] \) is 3dim cylindrical domain, \( 2h = h^+ - h^- \) is a thickness, \( S^\kappa, S \) are face and lateral surfaces.

The paper is dedicated to the problem of the satisfaction of the boundary conditions on surfaces \( S^\pm \) of the elastic plates. Although the main part of this problem was solved in [2], but some statements needs improvements. It is well known, that the problem of satisfaction of boundary conditions on surfaces is important for all refined theories of von Kármán-Mindlin-Reissner (KMR) type except Reissner’s [3] and Ambartsumian’s [4] models, which has evident gaps. Wholly, this problem depends on the justification of the calculus of variations. As it is known, the violation of Riemann “Dirichlet Principle (DP)” was shown in the considerable examples of Weierstass and Hadamard. In the case of the Dirichlet boundary condition the justification of DP was shown by Hilbert [5] (for the bilinear functionals) and Razmadze [6, 7] (for 1dim problems in general cases). In case of Neumann (natural) conditions the principle step was made by Rektorys [8]. Here we also study this problem and we constructed an example for the elastic plates when the stress vector is given on \( S^\pm \) and found the exact solution. If we use the Legendre polynomials as a basis by means of Vekua method [9] we obtain the unstable process. This fact demonstrates, the existence of ”Vekua problem” with regard of the satisfaction of boundary conditions on \( S^\pm \) and which was studied by Vekua carefully, but incompletely [9, ch. I, 11, ch. II, 2]. We have studied this problem for the hierarchical models in case of the isotropic homogeneous elastic plate. Also we investigated and defined the functional spaces of admissible solutions. The results are given below using some main statements from [2, ch. II, 6.2-3].
2 Investigation of Stability Problems of Vekua Models for Elastic Prismatic Shells

Over 20 years Vekua studied the problem of constructing of 2D hierarchical models for an arbitrary integer \( N \), especially for \( N = 0, 1, 2 \) without using any physical and geometrical hypothesis. Different versions of the models correspond to the linear theory of isotropic elastic plates and shells with the variable thickness according to (1)-(4) in monograph [9]. Numerous of scientists has been worked on this problem (see i.e. references in [10]). Vekua used the following way: for the relations (1), (3) by means of (2) the Galerkin method was applied using as the basis the system of the Legendre polynomials \( \{ p_n(x), p_n(\pm 1) = (\pm 1)^n \} \). In addition, new expressions of the type (7.2 c), [9, ch. I, 7.2] were introduced. Those expressions were named as the normalized moments of the field of stresses which are coordinated with boundary conditions. By the expression (8.4 a,b), (8.9) [9, 8.1] which represent 2dim boundary value problems Vekua has constructed the approximate solutions of (1)-(4) in the form:

\[
(u, \sigma) = \sum_{s=0}^{\infty} \left( \hat{u}(x,y), \hat{\sigma}(x,y) \right) p_s(z),
\]

which "are not compatible with boundary data on face surfaces \( S^+, S^- \). Therefore these approximations may prove to be rather rough values near the face surfaces" [9, page 79]. We called it as the "Vekua problem". In [9] the solution corresponding to the hierarchical BVPs for any integer \( N \) by additional functions satisfying also the approximate system of DEs was corrected. This function depends on the sum of differences of Legendre polynomials with respect to indices in the form (11.7) [9, ch. 1]:

\[
U_0 = A_m(x,y)(p_{m+1}(\zeta) - p_{m-1}(\zeta)) + A_{m+1}(x,y)(p_{m+2}(\zeta) - p_m(\zeta)),
\]

\[
\zeta = \frac{z-\bar{h}}{2h}, \quad m > N + 2.
\]

When \( N, m \) tends to infinity, the problem is open.

You can look for another way of investigating this problem in [9, ch. II, 2]. Here for the displacement vector and stress tensor the Taylor series is used near the point \( z = 0 \) and the boundary conditions are approximately satisfied on the surfaces. Besides, the case is considered when the approximation has the second order.

Let us consider the case when the boundary value problem of the theory of elasticity is a 1dim problem and thus we have: \( u_1 = u_2 = \varepsilon_{ai} = \sigma_{ai} = f_a = 0, h = 1, \sigma_{33} = (\lambda + 2\mu)u_{33} \). Then we get the following boundary value problem:

\[
-u''(x) = f(x), \quad u'(-1) = \alpha, \quad u'(1) = \beta.
\]

As \( z(x) = u(x) - \frac{\alpha + \beta}{2} x - \frac{\beta - \alpha}{4} x^2 + u_0 \), problem (5) is equivalent to the following one:
\[-z''(x) = f(x) + \frac{\beta - \alpha}{2}, \quad z'(1) = z'(1) = 0. \tag{6}\]

For simplicity we assume that \( f(x) - \frac{\beta - \alpha}{2} = p_1(x) \) and consider the following coordinate system:

\[
q_k(x) = -(2k+1) \int_{-1}^{x} (x-t)p_k(t)\,dt = \frac{1}{2k+3}(p_{k+2} - p_k) - \frac{1}{2k-1}(p_k - p_{k-2}), \quad k = 0, 1, 2, \ldots,
\]

\[-q_0 = \frac{1}{3}(p_3 - p_0), \quad -q_1 = \frac{1}{5}(p_3 - p_1), \quad q'(\pm 1) = 0.\]

We will find the solution of (6) as the set: \( z(x) = \sum_{k=0}^{n} z_k q_k(x) \). Then by the projective method

\[
(-z'', -q_0(x)) = z'(x)q_0(x)|_{x=-1} = 0,
\]

\[
(-z'', -q_1) = \int_{-1}^{x} z'(p'_3 - p'_1)\,dx = (p_1, p_3 - p_1) \Rightarrow
\]

\[-z_1 + \frac{3}{7}z_3 = -1,
\]

\[
(-z'', -q_2) = z'(x)q_2|_{x=-1} + \int_{-1}^{x} \sum_{k=0}^{n} z_k q_k p_1\,dx \Rightarrow
\]

\[-\frac{1}{3}z_0 + \frac{2}{3}z_2 + \frac{7}{3}z_4 = 0,
\]

\[-\frac{1}{4n-1}z_{2n-2} + \frac{2}{4n-1}(4n+1)z_{2n} - \frac{1}{4n+3}z_{2n+2} = 0, \quad (n = 2, 4, \ldots),
\]

\[z_1 - z_3 = \frac{1}{3}, \quad -\frac{1}{5}z_1 + \frac{1}{5}z_3 = \frac{1}{15},
\]

\[-\frac{1}{4n-1}z_{2n-1} + \frac{2}{4n-1}(4n+3)z_{2n} - \frac{1}{4n+3}z_{2n+1} = 0, \quad (n = 1, 3, 5, \ldots) \Rightarrow
\]

\[z_1 = -\frac{1}{3}, \quad z_0 = z_2 = 0, \quad (n = 2, 3, \ldots),
\]

as matrices of both systems are irresoluble and by the theorem of Olga Taussky-Todd are nonsingular ones. Thus the solution of problem (6) has the following form:

\[z(x) = \frac{1}{3}q_1(x), \quad i.e. \quad -z''(x) = p_1, \quad z'(1) = 0.
\]

Now if in (5) we put \( f(x) = p_1(x), \alpha = \beta \) and \( u(x) = \sum_{k=0}^{n} u_k p_k(x) \) by using methodology of [9] we obtain \( u(x) = \left(\frac{2}{5} + \alpha\right)p_1(x) - \frac{1}{15}p_3(x) \). The first summand presence here demonstrates unstable process same to [8, ch. 21, example 21.2].

In [2, ch. II, 6.3] we investigated the problem of construction and justification of Vekua type systems using methodology of [8] in case of natural conditions.
By using the Galerkin method for DEs (1) we obtain that the components of the stress vector \( \sigma_3 \) for systems of DEs considered in [9] and [2] are different. For models from [9] the condition (2) is not satisfied as underlined in [9, 11]. Let us return to the initial problem (1)-(4) and consider the linear case. In the above-mentioned works was considered the case when the components of the exterior tension vector \( \sigma_3 \) is given on \( S^\pm \). The problems of satisfying these boundary conditions for any approximations were different among proposed systems. For some models they are natural, while for others they appear to be the main ones in the sense of variational methods (see Rektorys [8]). We construct a class of operator equations actually coinciding with systems (7.9 a,b), (7.18 h,i) or (8.16) [9]. For the sake of brevity, we shall denote it by \((V)\).

Let us use this expansion into Fourier-Legendre for incomplete series components of stress tensor. By virtue of boundary conditions on \( S^\pm \) we have:

\[
\sigma_{\alpha\beta} = \sum_{k=0}^{\infty} \sigma_{\alpha\beta} p_k \left( \frac{z}{h} \right),
\]

\[
\sigma_3 = \frac{(h+z)g^+ + (h-z)g^-}{2h} + \sum_{s=1}^{\infty} \sigma_{3j} \left[ p_{s+1} \left( \frac{z}{h} \right) - p_{s-1} \left( \frac{z}{h} \right) \right],
\]

At first we construct the basic Vekua type hierarchical 2-dim model which approximates the linear boundary value problem for homogeneous isotropic plates (for details see [2, Ch. II, part 6.3]). Then equilibrium equations in terms of components of the stress tensor will be equivalent to the following infinite system

\[
c_m h \sigma_{\alpha\beta,\beta} + (2m+1)c_m \sigma_{\alpha\beta} = f_{\alpha} - h c_0 \delta_{m0} \frac{g_{\alpha}^+ - g_{\alpha}^-}{2},
\]

\[
c_m h \left( \sigma_{\alpha3,3} - \sigma_{\alpha3,\alpha} \right) + (2m+1)c_m \sigma_{33} = f_3 - h c_0 \delta_{m0} \frac{g_{\alpha,\alpha}^+ + g_{\alpha,\alpha}^-}{2}
\]

\[-h c_1 \delta_{m1} \frac{g_{\alpha,\alpha}^+ - g_{\alpha,\alpha}^-}{2} - h c_0 \delta_{m0} \frac{g_3^+ + g_3^-}{2}\]

where

\[
m = \int_{-h}^{h} f(x_1,x_2,t) p_m \left( \frac{t}{h} \right) dt, \quad c_m = \frac{2}{2m+1}, \quad m = 0,1,2,\ldots.
\]

Hooke’s law takes the following form:
\[c_m h \sigma_{11}^m = (\lambda + 2\mu) c_m h u_{1,1}^m + \lambda c_m h u_{2,2}^m + \lambda (2m + 1)c_m \sum_{k \geq m(2)}^{k+1} u_3,\]

\[c_m h \sigma_{12}^m = \mu h c_m \left( u_{1,2}^m + u_{2,1}^m \right),\]

\[c_m h \sigma_{22}^m = \lambda c_m h u_{1,1}^m + (\lambda + 2\mu) c_m h u_{2,2}^m + \lambda (2m + 1)c_m \sum_{k \geq m(2)}^{k+1} u_3,\]

\[c_m h \left( \sigma_{3\alpha}^{m-1} - \sigma_{3\alpha}^{m+1} \right) = \mu h c_m u_{3,\alpha} + \mu (2m+1)c_m \sum_{k \geq m(2)}^{k+1} u_{\alpha},\]

\[-hc_0 \delta_{m0} \frac{g_+^\alpha + g_-^\alpha}{2} - hc_1 \delta_{m1} \frac{g_+^\alpha - g_-^\alpha}{2},\]

\[c_m h \left( \sigma_{33}^{m-1} - \sigma_{33}^{m+1} \right) = \lambda h c_m u_{3,\alpha} + (\lambda + 2\mu) (2m+1)c_m \sum_{k \geq m(2)}^{k+1} u_{3,\alpha},\]

\[-hc_0 \delta_{m0} \frac{g_+^3 + g_-^3}{2} - hc_1 \delta_{m1} \frac{g_+^3 - g_-^3}{2}.\]

Here and (often) below the following note is used:

\[\sum_{k \geq (s)}^k u = u + i + u + i + 2u + \ldots, \quad \sum_{k \leq (s)}^k u = u + i - u + i - 2u + \ldots.\]

Formulae (9) and (10) make it possible to obtain an explicit form of Vekua type system in displacement components. For this purpose we use Hooke’s law for values \(\sigma_{3\alpha}\) and condition (2). We shall have:

\[g_+^\alpha = \mu \sum_{k=0}^{m} \left( u_{3,\alpha} + \frac{k(k+1)}{2h} u_{\alpha} \right),\]

\[g_-^\alpha = \mu \sum_{k=0}^{m} (-1)^k \left( u_{3,\alpha} - \frac{k(k+1)}{2h} u_{\alpha} \right),\]

and

\[g_+^3 = \sum_{k=0}^{\infty} \left( \lambda u_{3,\alpha} + (\lambda + 2\mu) \frac{k(k+1)}{2h} u_3 \right),\]

\[g_-^3 = \sum_{k=0}^{\infty} (-1)^k \left( \lambda u_{3,\alpha} - (\lambda + 2\mu) \frac{k(k+1)}{2h} u_3 \right).\]

We define values \(g^+ \pm g^-\), entering (9). We shall have:
In these expressions

\[ g^+_{\alpha} + g^-_{\alpha} = 2\mu \sum_{k=0}^{m} \left( \frac{2k}{u_{3,\alpha} + \frac{(k+1)(2k+1)}{2k+1} u_{\alpha}} \right), \]

\[ g^+_\alpha - g^-_\alpha = 2\mu \sum_{k=0}^{m} \left( \frac{2k+1}{u_{3,\alpha} + \frac{k(2k+1)}{2k+1} u_{\alpha}} \right), \]

\[ g^+_3 + g^-_3 = 2\sum_{k=0}^{m} \left( \lambda \frac{2k}{u_{3,\alpha} + \frac{(k+1)(2k+1)}{2k+1} u_{\alpha}} \right), \]

\[ g^+_3 - g^-_3 = 2\sum_{k=0}^{m} \left( \lambda \frac{2k+1}{u_{3,\alpha} + \frac{k(2k+1)}{2k+1} u_{\alpha}} \right), \]

From equations (10), summing up the three last formulae, for values \( m \) of \( \sigma_{3\alpha} \) we obtain:

\[
\sum_{s \leq m(2)} \left( \frac{s-2}{\sigma_{3\alpha} - \sigma_{3\alpha}} \right) = -\sigma_{3\alpha} = \mu \sum_{s \leq m(2)} \left( \frac{s-1}{\mu \alpha} + \frac{k}{h} \sum_{s \leq m(2)} (2s-1) \sum_{k \geq i(2)} k \right)
\]

\[-\frac{1}{2} (g^+_\alpha + g^-_\alpha) \sum_{s \leq m(2)} \delta_{s-1,0} - \frac{1}{2} (g^+_\alpha - g^-_\alpha) \sum_{s \leq m(2)} \delta_{s-1,1} \]

Similarly

\[-\sigma_{33} = \mu \sum_{s \leq m(2)} \left( \frac{s-1}{\mu \alpha + \frac{\lambda + 2\mu}{h}} \frac{2s-1}{k} \right) \sum_{k \geq i(2)} k \]

\[-\frac{1}{2} (g^+_3 + g^-_3) \sum_{s \leq m(2)} \delta_{s-1,0} - \frac{1}{2} (g^+_3 - g^-_3) \sum_{s \leq m(2)} \delta_{s-1,1} \]

In these expressions \( \sigma_{3\alpha} = 0 \) is assumed.

Now, by using formulae (11) from the latter representations after some computations, we get

\[
\sigma_{3\alpha} = \mu \sum_{s \geq (m+1)(2)} \left[ \frac{s}{u_{3,\alpha} + \frac{1}{2h} ((s+1)(s+2) - m(m+1)) u_{\alpha}} \right],
\]

\[
\sigma_{33} = \sum_{s \geq (m+1)(2)} \left[ \lambda \frac{s}{u_{3,\alpha} + \frac{1}{2h} ((s+1)(s+2) - m(m+1)) u_{\alpha}} \right].
\]

Taking into account the last formulae, as well as (10), after obvious simplifications with respect to components of the displacement vector we obtain the following infinite system of Vekua's differential equations:
Here

\[ u_+ = (u_1, u_2)^T, \quad f_+ = (f_1, f_2)^T, \quad g_+ = (g_1, g_2)^T, \]

\((l_2 u_+, u_+) = \mu (\Delta u_\alpha, u_\alpha) + (\lambda + \mu) (\text{grad}u_+, u_+).\]

From system (10), evidently, for values \(\sigma_{\alpha 3}\) we have:

\[
m^{-1} \sigma_{\alpha 3} = \sigma_{\alpha 3} + \mu u_{3, \alpha} + \mu \frac{2m + 1}{h} \sum_{k \geq m(2)} \frac{m}{2} k^{m + 1} u_\alpha - \frac{1}{2} \sum_{k \geq m(2)} \frac{m}{2} k^{m + 1} u_\alpha \left( g_\alpha^+ - g_\alpha^- \right) \delta_{\mu 0} - \frac{1}{2} \sum_{k \geq m(2)} \frac{m}{2} k^{m + 1} u_\alpha \left( g_\alpha^+ - g_\alpha^- \right) \delta_{\mu 1},
\]

\[
m^2 \sigma_{\alpha 3} = \sum_{k \geq m(2)} \left( g_\alpha^+ + g_\alpha^- \right) \delta_{\mu 0} + \left( g_\alpha^+ - g_\alpha^- \right) \delta_{\mu 1}, \quad m = 1, 2, \ldots
\]

\[
m^3 \sigma_{\alpha 3} = \mu \sum_{k \geq (m + 1)(2)} \left( \lambda + \mu \right) \frac{h}{2} \left( (k + 1)(k + 2) - m(m + 1) \right) u_\alpha + \frac{1}{2} \left( (k + 1)(k + 2) - m(m + 1) \right) u_\alpha.
\]

Analogously,

\[
m \sigma_{33} = \sum_{k \geq (m + 1)(2)} \left( \lambda \frac{h}{2} \left( (k + 1)(k + 2) - m(m + 1) \right) u_\alpha + \frac{1}{2} \right)
\]

Taking into account these formulae we obtain

\[
\frac{1}{2} \left( g_\alpha^+ + g_\alpha^- \right) \sum_{m = 1}^{\infty} \delta_{m - 1, 0} + \frac{1}{2} \left( g_\alpha^+ - g_\alpha^- \right) \sum_{m = 1}^{\infty} \delta_{m - 1, 1} - m \sigma_{33} u_\alpha
\]

\[
= \mu \sum_{k < (m + 1)(2)} \left( \lambda \frac{h}{2} \left( (k + 1)(k + 2) - m(m + 1) \right) u_\alpha + \frac{1}{2} \right).
\]

Hence for values \(\sigma_{\alpha 3}\) we have:
\[ m_{\alpha 3} = \frac{1}{2} (g^+_{3\alpha} + g^-_{3\alpha}) \sum_{m \geq 1(1)}^{\infty} \delta_{m-1,0} + \frac{1}{2} (g^+_{3\alpha} - g^-_{3\alpha}) \sum_{m \geq 1(1)}^{\infty} \delta_{m-1,1} - \mu \sum_{k \leq (m-1)(2)}^{\infty} \left( \frac{k}{2h} + \frac{1}{(k+1)(k+2)} \right) u_{3,\alpha} \]  

Similarly for \[ m_{33} \] we shall have:

\[ m_{33} = \frac{1}{2} (g^+_{33} + g^-_{33}) \sum_{m \geq 1(1)}^{\infty} \delta_{m-1,0} + \frac{1}{2} (g^+_{33} - g^-_{33}) \sum_{m \geq 1(1)}^{\infty} \delta_{m-1,1} - \sum_{k \leq (m-1)(2)}^{\infty} \left[ \frac{\lambda + 2\mu}{2h} + \frac{k}{(k+1)(k+2)} \right] u_3. \]

Taking into account these expression in (10) we obtain the infinite system of differential equations according to Vekua’s system \((V)\) in the following form:

\[
\begin{align*}
& l_2 u_+ + h^{-1}(2n + 1) \text{grad} \left( \frac{\lambda \sum_{i \geq m(2)}^{i+1} u_3 - \mu \sum_{i \leq n(2)}^{i+1} u_3}{i(i+1)} \right) \\
& - \frac{\mu h^{-2} 2n + 1}{2} \sum_{i \leq n(2)}^{i+1} i(i+1) u_+ = \frac{n}{\Delta}\sum_{k \leq (m-1)(2)}^{\infty} \frac{k}{(k+1)(k+2)} u_3 \\
& - \left( \frac{g^+_{33} + g^-_{33}}{2} \sum_{i \geq 1(1)}^{\infty} \delta_{i,0} + (g^+_{33} + g^-_{33}) \sum_{i \leq 1(1)}^{i+1} \delta_{i-1,0} + (g^+_{33} - g^-_{33}) \sum_{i \geq 1(1)}^{\infty} \delta_{i-1,1} \right),
\end{align*}
\]

(12)

The comparison of these equations (12) with those of \((V)\) proves their identity for \(N = 0, 1, 2\). When \(N \geq 3\), the main parts (containing only second order partial derivatives) of systems (7.18 h, i) [9] and (12) are different. Then [9, page 52] we read: the (7.18 h, i) is a strong elliptic system of PDEs for \(N \geq 3\), "but we do not rewrite this one in a more expanded form and shall not deal with the investigation of problems of existence and uniqueness in the general form". Evidently, in order to obtain effective values a priori in the form of energy inequalities for Vekua’s operator with fixed \(N\) together with highest derivatives, we should pay attention to the explicit form of summands with derivatives of zero and first order from unknown moments \(u_{h,1}, u_{h,3}\) \((n = 0, 1, 2, \ldots)\) appearing in system (12). Thus, we constructed (12) corresponding to the equations (1). Reduced boundary conditions, originated by
the data on the lateral surfaces \( S \) and the construction of which is not difficult, should be added to these systems. For this purpose we should multiply equalities (3) by Legendre polynomials \( p_i \left( \frac{z}{h} \right) \) and integrate them between \(-h \) and \( h \). If Hooke’s law and other representations from (4) are used, then we come up to the finite reduced boundary conditions, defined on \( \partial D \).

Now let us return to the linear case and use the variational principle, when the system (1) with reduced conditions (3) is the identity to the Euler-Lagrange equation for the initial problem with the vector components of \( \sigma_i \) given on surfaces \( S^\pm \).

Let us consider the reduced systems, generated by the basic system (12). If we bound ourselves \( N + 1 \) vector equations, the following results will be true [2, ch. II, 6.3, pp. 72-77].

**Theorem 1.** Let the boundary conditions on the lateral boundary \( S \) corresponding to the linear problem (1)-(3), be homogeneous and such that the equalities hold:

\[
\left( \begin{array}{c} a \\ u_{\alpha, \beta} \\ m \\ \end{array} \right) = \int_D \left( \begin{array}{c} a \\ u_{\alpha, \beta} \\ m \\ \end{array} \right) u_i dx_1 dx_2 = - \left( \begin{array}{c} a \\ u_{\alpha, \beta} \\ m \\ \end{array} \right),
\]

where \( u_i \) are desired coefficients of the expansion function \( u_i \). Then the operator of the theory of plates, corresponding to the reduced systems (12), satisfies an inequality of Korn’s type with a constant, independent of \( N \),

\[
-(L_N U_N, U_N) \geq \left( \frac{\| \text{grad} \hat{U}_N \|_1^2 + \| \text{div} \hat{U}_N \|_1^2 + 2 \| \hat{U}_N \|_2^2}{\kappa^2} \right),
\]

where

\[
\begin{align*}
\left( \begin{array}{c} m \\ \alpha \\ \beta \\ n \end{array} \right)_1 &= \frac{1}{(2m + 1)(2n + 1)} \left( \begin{array}{c} m \\ \alpha \\ \beta \\ n \end{array} \right), \\
\left( \begin{array}{c} m \\ \alpha \\ \beta \\ n \end{array} \right)_2 &= h^{-2} \frac{(2m + 1)(2n + 1)}{\sqrt{(2m + 1)(2n + 1)}} \left( \sum_{i \geq m(2)} \frac{i+1}{u_i}, \sum_{i \geq n(2)} \frac{i+1}{v_i} \right), \\
U_N &= \left( \begin{array}{c} 0 \\ u_1, \ldots, u_N \end{array} \right)^T = (u_1, u_2, u_3)^T = (u_+, u_3)^T, \quad \hat{u} = 0, \ n > N.
\end{align*}
\]

**Theorem 2.** It is required to find the solution for the reduced system (12) in the domain \( D(x_1, x_2) \), satisfying homogeneous Dirichlet boundary conditions at \( \partial D \) and being a trace of functions, defined by inclusion \( u_i(x, y, z) \in W_p^{2+\alpha}(\Omega_h), \ \alpha \geq 0, \ p \geq 1 \). Then is true the positive definiteness of Vekua’s type operator for problem (1)-(3) and will follow in case of Dirichlet’s boundary conditions on \( S \)

\[
-(L_N U_N, U_N) \geq \mu \left( \kappa^2 \| \hat{U}_N \|_1^2 + 2 \| \hat{U}_N \|_2^2 \right),
\]

where \( \kappa^2 \) is a constant in Friedrich’s inequality.
Theorem 3. It is required to find the solution for the reduced system (12) in \( D \) when the boundary of \( \partial D \) domain is free and above same inclusion for \( u_i \) is fulfilled. Then we have

\[
-\frac{1}{2h}(L_{v_1}U, U) \geq 2\mu \left( \kappa_3^2 \|U\|_1^2 + \kappa_2^2 \|U\|_2^2 \right),
\]

where \( c_1, c_2 \) are constants depending on \( D \) and \( c_3, c_4 \) - the constants of Poincaré inequality. We have used above the following notations: \( U = \begin{pmatrix} U_1 & U_2 & U_3 \end{pmatrix}^T \), \( L_N, L_{v_1} \) are operators, corresponding to 2dim approximate systems (12), and 3dim linear problems (1), (3) for any \( N \leq \infty \) when in (2): \( \sigma_3^\pm = g_3^\pm = 0 \).

Thus theorem 3 represents also a different proof of Korn’s inequality.

In addition, for 1-dim models according (5) these system (8.16 b) ([9.8]) if \( f = 0 \) have the following form:

\[
-(\lambda + 2\mu)(2k + 1) \left( \dot{U}^{k+1} + U_3^{k+1} + \cdots \right) + (k + \frac{1}{2}) [\delta_{ik} - (g_3^+ + (-1)^i g_3^-)] = 0,
\]

where \( U' = (2k + 1) \left( \dot{U}^{k+1} + U_3^{k+1} + \cdots \right) \). While systems (6.13) from [2] are true for \( \forall N \).

Let us consider the problem of satisfaction of boundary conditions on \( S^\pm \) for the class of refined theories in the wide sense [2, Ch. I, 3]. Here is necessary to note that, among the refined theories we found that the models of Reissner and Ambartsumian satisfy these conditions.

By [3] we have

\[
\begin{align*}
\sigma_{a3} &= \frac{3Q_\alpha}{h} \left[ 1 - \frac{z^2}{h^2} \right], & \sigma_{33} &= -\frac{3q}{4} \left[ \frac{2}{3} - \frac{z}{h} + \frac{1}{3} \left( \frac{z}{h} \right)^3 \right], \\
\sigma_{333,|z=\pm h|} &= -\frac{3q}{4} \left[ -\frac{1}{h} + \frac{z^2}{h^3} \right]_{z=\pm h} = 0.
\end{align*}
\]

Here for the linear case according to (2) \( g_3^+ = 0, g_3^- = -q \). We stress the fact that the boundary condition (18): \( \sigma_{333,|z=\pm h|} = 0 \) is an artificial and odd condition. Then the third equation of (1) for the linear and isotropic case when \( f_3 \equiv 0, -h \leq z \leq h \) is satisfied on \( S^\pm \), i.e. \( \sigma_{333,|z=\pm h|} = 0 \).

In [4] is considered a geometrically classical nonlinear case when in (4):

\[
\begin{align*}
&u_{k,i}u_{k,j} = u_{3,i}u_{3,j} (i, j = 1, 2), & u_{k,3}u_{k,3-i} = 0, (i = 0, 1, 2),
\end{align*}
\]

for a homogenous anisotropic plate with no more than 13 independent constants in Hooke’s law (4) of elastic plates with constant thickness. Ambartsumian transferred methodology of [3] satisfying the boundary conditions on the surfaces: \( \sigma_{33}|_{S^\pm} = g_{3i}^\pm \).
at $\sigma_{33,3}^{\pm} = -\sigma_{13,1}^{\pm} - \sigma_{23,1}^{\pm}$. If $f_3 = g_{\alpha}^{\pm} = 0$, $\sigma_{33,3}^{\pm} = 0$ and he studied the general case, $g_{\alpha}^{\pm} \neq 0$, and expressed the tangential components of the stress vector as

$$\sigma_{\alpha 3} = \frac{h + z}{2h} g_{\alpha}^{+} + \frac{h - z}{2h} g_{\alpha}^{-} + \frac{4}{3h^3} (h^2 - z^2) \phi_{\alpha}(x, y),$$

In this simple case from (1) follows that:

$$\phi_{\alpha,a} = h (g_{\alpha}^{+} + g_{\alpha}^{-}) + g_{3}^{+} - g_{3}^{-}.$$  

Then the nonlinear system of five DEs with respect to $u_{\alpha}(x, y), w(x, y)$ (correspondingly of averaged values of horizontal and normal components of the displacement vector), $\phi_{\alpha}$ is constructed. The linear part of leading two DEs of this system are second order relatively to $u_{\alpha}$, third one-the first order with respect to $\phi_{\alpha}$, the last two DEs have third order partial derivatives; nonlinear part contain correspondingly two, fourth and zero degrees of product of the first and second order derivatives under $w(x, y)$ The questions, connected with these types of systems are open. Like in the linear case by $\sigma_{33}^{\pm}|_{z=\pm h} = g_{3}^{\pm}$, according [3]

$$\sigma_{33,3}^{\pm}|_{z=\pm h} = \sigma_{\alpha 3,a}|_{z=\pm h} = g_{\alpha}^{\pm},$$

the same process was considered in the nonlinear case [4, ch. 2, 8], p.67: if $f_3 = 0$, $g_{\alpha}^{\pm} = 0$, then

$$\sigma_{33} = \frac{g_{1}^{+} + g_{1}^{-}}{2} + \frac{3z}{4h^3} (g_{3}^{+} - g_{3}^{-}) \left( h^2 - \frac{z^2}{3} \right), \quad \Rightarrow$$

$$\sigma_{33}|_{z=\pm h} = g_{3}^{\pm}, \quad \sigma_{33,3}|_{z=\pm h} = 0, \quad \sigma_{3,3,j}(x,y,\pm h) = 0.$$  

We remark that the complexity and obvious errors of the methodology in [4] are the result of using of the expression of the first order of $\phi_{\alpha,a}$ in the systems of DEs almost everywhere as well as the corresponding equation is the basis for constructing essential DE with respect to an averaging deflection. We underline that the methodology according to [3-4] are popular and this approach were used by Timoshenko, Donnel, Lukasiewicz, Morozov,...

Below we try to use a careful and correct approach to resolve this problem of satisfaction of (2) for sufficient general cases. With this aim, we use a methodology from [2, point 6.3] and consider the following relation for the nonlinear case too:

$$T_{3}(x, y, z) = \frac{(h + z) g_{3}^{+}}{2h} + \frac{(h - z) g_{3}^{-}}{2h} + \sum_{s=1}^{3} T_{3}(x, y) \left[ p_{s+1} \left( \frac{z}{h} \right) - p_{s-1} \left( \frac{z}{h} \right) \right], \quad (19)$$

For the simplicity and clearness let us consider the case when Lame coefficients $\lambda, \mu$ are constants. In this case the uniform of refined theories corresponding to the bending for deflection and generalized shearing forces is of the form
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\((DA^2 + 2h\rho \partial_t - 2DE^{-1}(1 + \nu)\rho \partial_t \Delta)w = \left(1 - \frac{h^2(1 + 2\gamma)(2 - \nu)}{3(1 - \nu)\Delta}\right)\)

\[\times (g_3^+ - g_3^-) + 2h(1 - \frac{2h^2(1 + 2\gamma)}{3(1 - \nu)}\Delta)|w, \partial| + h(g_{\alpha,\alpha} - g_{\alpha,\alpha})\]

\[- \int_{-h}^h \left( tf_{\alpha,\alpha} - \left(1 - \frac{1}{1 - \nu}(h^2 - t^2)\right)f_3\right) dt,\]

\(\left(\Delta^2 - \frac{1 - \nu^2}{E\rho \Delta \partial_t}\right)\varphi = -\frac{E}{2}[w, w]\)

\[+ \frac{\nu}{2}\left(\Delta - \frac{2\rho}{E} \partial_t\right)(g_3^+ + g_3^-) + \frac{1 + \nu}{2h}f_{\alpha,\alpha} + \]

Below we for simplicity consider only static cases. Then for an isotropic case we have:

\[D\Delta^2 u_3^* = - \left(1 - \frac{h^2(1 + 2\gamma)(2 - \nu)}{3(1 - \nu)}\right)(g_3^+ - g_3^-)\]

\[+ 2h \left(1 - \frac{2h^2(1 + 2\gamma)}{3(1 - \nu)}\right)\int_{-h}^h L[u_3^*, F_s] + (g_{\alpha,\alpha} + g_{\alpha,\alpha})\]

\[- \int_{-h}^h \left( tf_{\alpha,\alpha} - \left(1 - \frac{1}{1 - \nu}(h^2 - t^2)\right)f_3\right) dt + R_8 [u_3^*, \gamma],\]

\[Q_{\alpha 3} = \frac{1 + 2\gamma}{3}h^2\Delta Q_{\alpha 3} = -D\Delta u_3^*\]

\[+ \frac{h^2(1 + 2\gamma)}{3(1 - \nu)} \partial_t (g_3^+ - g_3^- + 2h(1 + \nu)L[u_3^*, F_s])\]

\[+ (g_{\alpha}^+ + g_{\alpha}^-) - \int_{-h}^h \left( tf_{\alpha} - \left(1 - \frac{1}{1 - \nu}(h^2 - t^2)f_3\right)\right) dt + R_{3 + \alpha} [Q_{\alpha 3}; \gamma],\]

We underline that \(L[u, v] = [u, v] = \partial_{11}u\partial_{22}v - 2\partial_{12}u\partial_{12}v + \partial_{11}v\partial_{22}u\) is the well-known Monge-Ampère operator. Now we investigate the influences of conditions (19) on the systems of differential equations (22)-(23), construction of which essentially depends on \(Q_{\alpha}, \psi_{\alpha}, I = (T_{33}, t),\)

\[\psi_{\alpha} = \frac{1}{2}(h^2 - t^2, \sigma_{\alpha 3}),\]

\[I = (T_{33}, t), \quad (u, v) = \int_{-h}^h u(x, y, t)v(x, y, t)dt.\]

Evidently, we have:

\[Q_{\alpha 3} = -2hT_{\alpha 3} + h(g_{\alpha}^+ + g_{\alpha}^-),\]

\[\psi_{\alpha} = \frac{2h^3}{3} \left( g_{\alpha}^+ + g_{\alpha}^- - \frac{17}{20} \int_{-h}^h T_{\alpha 3}(x, y, t)p_2 \left( \frac{L}{h} \right) dt.\]
fined theories depend on parameter $\gamma$ that all DEs systems of von KMR type follow from (1).

one of the equations of the corresponding system of DEs. In [11] we have proved solved problem. The point is that von Kármán, Love, Timoshenko, L. Landau, Lukasiewicz, Washizu considered Saint-Venant-Beltrami compatibility condition as isotropic elastic plate of constant thickness the subject of justification was an unsolved problem. The point is that von Kármán, Love, Timoshenko, L. Landau, Lukasiewicz, Washizu considered Saint-Venant-Beltrami compatibility condition as one of the equations of the corresponding system of DEs. In [11] we have proved that all DEs systems of von KMR type follow from (1).

Here $\bar{T}_{\alpha 3}$. We can see that the boundary condition on the surfaces satisfying all refined theories depend on parameter $\gamma$. The system of DEs (22-23) contains Monge-Ampère operator for the nonlinear case. As it is known, even in the case of an isotropic elastic plate of constant thickness the subject of justification was an unsolved problem. The point is that von Kármán, Love, Timoshenko, L. Landau, Lukasiewicz, Washizu considered Saint-Venant-Beltrami compatibility condition as one of the equations of the corresponding system of DEs. In [11] we have proved that all DEs systems of von KMR type follow from (1).

We have the following relation (decomposition of Monge-Ampère operator):

$$\frac{1}{2} \int_{-h}^{h} (h^2 + t^2) \sigma_3 u_{\alpha,j} dt = \frac{2h^3}{3} \left( g_{\alpha} + g_{\alpha} - \frac{17}{20} \bar{T}_{\alpha 3} \right) + R[\psi_\alpha],$$

$$|R[\psi_\alpha]| \leq h^{5/2} (c_1 ||\sigma_3|| + c_2 ||\sigma_3 u_{\alpha,j}||),$$

$$I = \int_{-h}^{h} t \bar{T}_{\alpha 3} dt = \frac{(1 + 2\gamma)h^2}{3} (g^3_3 - g^3_3) - \int_{-h}^{h} t \sigma_3 u_{\alpha,j} dt,$$

$$\int_{-h}^{h} t \sigma_3 u_{\alpha,j} dt = O(h^2).$$

Now in (22)-(23) we change the components of normal rotations by functions $\bar{T}_{\alpha 3}$. We can see that the boundary condition on the surfaces satisfying all refined theories depend on parameter $\gamma$. The system of DEs (22-23) contains Monge-Ampère operator for the nonlinear case. As it is known, even in the case of an isotropic elastic plate of constant thickness the subject of justification was an unsolved problem. The point is that von Kármán, Love, Timoshenko, L. Landau, Lukasiewicz, Washizu considered Saint-Venant-Beltrami compatibility condition as one of the equations of the corresponding system of DEs. In [11] we have proved that all DEs systems of von KMR type follow from (1).

We have the following relation (decomposition of Monge-Ampère operator):

$$\partial_1 \partial_1\left[ \partial_2 \partial_2 (\partial_1 (\partial_1 u_{\alpha}) - \partial_2 (\partial_2 u_{\alpha})) - \partial_2 (\partial_1 u_{\alpha}) - \partial_2 (\partial_2 u_{\alpha}) \right]$$

$$= - (\partial_1 (\partial_1 u_{\alpha}) - \partial_2 (\partial_2 u_{\alpha}) + \partial_2 (\partial_1 u_{\alpha}) - \partial_1 (\partial_2 u_{\alpha})). \quad (M - A)$$

It is necessary that to system (20)-(21) we must add, for evidence, part of von Kármán type system (an isotropic case, see [11, formula (17)]:

$$(\lambda^s + 2\mu) \partial_1 \tau + \mu \partial_2 \omega = \frac{1}{2h} \bar{f}_1 + \mu (\partial_1 (\bar{u}_{3,2})^2 - \partial_2 (\bar{u}_{3,1} \bar{u}_{3,2})) + \lambda_1 (\sigma_{33,1}, 1), \quad (24)$$

$$(\lambda^s + 2\mu) \partial_2 \tau - \mu \partial_1 \omega = \frac{1}{2h} \bar{f}_2 + \mu (\partial_2 (\bar{u}_{3,1})^2 - \partial_1 (\bar{u}_{3,1} \bar{u}_{3,2})) + \lambda_1 (\sigma_{33,2}, 1). \quad (25)$$

Here $\tau = \bar{e}_{\alpha\alpha}, \omega = \bar{u}_{1,1} - \bar{u}_{2,1}$ are plane expansion and rotation, $\lambda_1 = \lambda / 2h(\lambda + 2\mu)$, nonlinear terms represent a decomposition of Monge-Ampère operator if in $M - A, u = v = u_3$.

We must remark that in the general case one can find for the anisotropic case the expressions (16) from [11]. The general transversality case if $c_{11} = c_{22} = c_{12} + c_{66}, b_{13} = b_{23} = b$, (see (22), [11]), might be interesting for applications in the following form:
the Reissner type form for bending process has the following face:

\[ c_{11} \partial_1 \tau + \frac{1}{2} c_{66} \partial_2 \omega = \frac{1}{2h} f_1 \]

\[-b b_{33} \frac{1}{2h} \int_{-h}^{h} \sigma_{33,1} dt - h c_{66} \left[ \partial_2 (\bar{u}_{3,1} \bar{u}_{3,2}) - \partial_1 (\bar{u}_{3,2}) \right]^2 + R_1^t, \] \quad (26)

\[ c_{11} \partial_2 \tau - \frac{1}{2} c_{66} \partial_1 \omega = \frac{1}{2h} f_2 \]

\[-b b_{33} \frac{1}{2h} \int_{-h}^{h} \sigma_{33,2} dt - h c_{66} \left[ \partial_1 (\bar{u}_{3,1} \bar{u}_{3,2}) - \partial_2 (\bar{u}_{3,1}) \right]^2 + R_2^t. \] \quad (27)

Below we give some extensions:

1. Let for anisotropic case the number of independent elastic modulus are taken according to [2, 2.2]. If now we used the corresponding expressions (2.17), (2.30) from [2]:

\[ u^*_\alpha = -u^\alpha, \frac{h^3}{3D(1 - \nu)} Q_{\alpha 3} + R\alpha \left[ u^\alpha, \gamma \right], \]

\[ Q_{\alpha 3} = -\frac{2h^3}{3} L_{\alpha 3} (\partial_1, \partial_2) u^3 + \frac{2h^3}{3} L_{\alpha \beta} (\partial_1, \partial_2) \psi^\beta_g + h (g^\alpha_\gamma + g^\gamma_\alpha) \]

\[ + \int_{-h}^{h} t [\alpha (b_1 + b_2) \sigma_{33} + c f_\alpha] dt + R_{2 + \alpha} [Q_{\alpha 3}; \gamma], \]

where

\[ \psi^\alpha_g = 3 (\delta h^3)^{-1} \int_{-h}^{h} t dt \int_{0}^{t} (b_{\alpha + 3, \alpha + 3} \sigma_{\alpha 3} - b_{45} \sigma_{33 - \alpha}) dt, \]

\[ \delta = b_{44} b_{55} - b_{45}^2, \]

\[ L_{\alpha 3} (\partial_1, \partial_2; c) = c_{aa} \partial_3^3 + 3 c_{aa} \partial_3^2 \partial_3 - \alpha + (c_{12} + 2 c_{66}) \partial_3 \partial_3 - \alpha + c_{3 - 6} \partial_3^3 - \alpha, \]

\[ L_{\alpha \beta} (\partial_1, \partial_2; c) = c_{aa} \partial_3^2 + 2 c_{aa} \partial_3 - \alpha + c_{66} \partial_3^2 - \alpha, \]

\[ L_{12} = L_{21} = c_{16} \partial_1^2 + (c_{12} + 2 c_{66}) \partial_1 + c_{26} \partial_2^2. \]

As well as the equations (2.36-37) [2] with respect to \( u^*_\alpha \) \( Q_{\alpha 3} \) are the same to (22), (23), it would be evident that all analogical conclusions are true also for the anisotropic case.

2. We consider also the case when the Lamé coefficients are variable. In this case, the Reissner type form for bending process has the following face:

\[
\left[ \left( (1 - \nu) D^{\frac{1}{2}} \mathbf{t}_{\alpha,1} \right)_{,1} + \left( (1 - \nu) D^{\frac{1}{2}} \mathbf{t}_{\alpha,2} \right)_{,2} \right]
\]

\[ + \left[ \left( (1 + \nu) D^{\frac{1}{2}} \mathbf{t}_{\alpha,a} \right)_{,a} + \left( 2 \nu D^{\frac{1}{2}} \mathbf{t}_{-\alpha,3 - \alpha} \right)_{,a} + \left( (1 - \nu) D^{\frac{1}{2}} \mathbf{t}_{-\alpha,a} \right)_{,3 - \alpha} \right] \]
\[-\frac{3(1 - \nu)}{h^2(1 + 2\gamma)} \left(\frac{1}{\tau_\alpha} + u^*_{3,\alpha}\right) = 2(f^*_{\alpha} + R_{\alpha+2}),\]
\[\frac{3}{1 + 2\gamma} \left[\frac{1 - \nu}{h^2} D \left(u^*_{3,\alpha} + \tau_\alpha\right)\right] = 2(f^*_{3} + R_3).\]

3. Let \( \Omega_h = D(x,y) \times (h_1(x,y), h_2(x,y)) \), \( 2h = h_2 - h_1, \hat{h} = \frac{1}{2}(h_1 + h_2) \). For this case the function \( v_\alpha(x,y,z) = u_{3,\alpha} - \frac{1}{\mu} \sigma_{33} \) has important weight. Instead of (22)-(23) we have:

\[
\frac{1}{h^3} D\partial_\alpha \left(h^3 \Delta u_{3,\alpha}(x,y,\hat{h})\right) = \sigma^+_{33} - \sigma^-_{33}
\]
\[+ \int_{\hat{h}}^h f_3 dt + \frac{h^2(2 - \nu)(1 + 2\gamma)}{3(1 - \nu)} (\Delta \sigma^+_{33} - \Delta \sigma^-_{33}) + \Phi_1 + R_1,\]
\[\frac{4}{3} \sigma_{33}(x,y,\hat{h}) - 2h^3 \Delta \sigma_{33}(x,y,\hat{h})\]
\[= -D\Delta u_{3,\alpha}(x,y,\hat{h}) - \frac{1 + 2\gamma}{3(1 - \nu)} (\sigma^+_{33,\alpha} - \sigma^-_{33,\alpha}) + \Phi_{1+\alpha} + R_{1+\beta}.
\]

We can see that for a variable thickness it is possible to use the same methodology, that was used for the normal stress vector in such expressions:

\[
\sigma_3 = \frac{h_2 - z}{2h} g^- + \frac{z - h_1}{2h} g^+ + \sum_{s=1}^{\infty} s \sigma_3(x,y) \left( p_{s+1} \left(\frac{2z - \hat{h}}{2h}\right) - p_{s-1} \left(\frac{2z - \hat{h}}{2h}\right)\right)
\]
\[= \left(a - \frac{l}{\tau}\right) p_0 + \left(b - \frac{\tau}{\sigma}\right) p_1 + \sum_{s=2}^{\infty} \left(\frac{s-1}{\tau} - \frac{s+1}{\sigma}\right) p_s \left(\frac{2z - \hat{h}}{2h}\right), \quad a, b = a, b(g^\pm, h),
\]
\[\sigma_3 = \sum_{s=0}^{\infty} \sigma(x,y) p_s \left(\frac{2z - \hat{h}}{2h}\right).
\]

4. New edge effect

If \( T_{33}|_{s= \pm} = g^\pm \), then by the value

\[
\varphi = \int_{-h}^{h} tT_{33} dt = \frac{(1 + 2\gamma)h^2}{3} (g^3_3 - g^-_3) + R, \quad (x,y \in \partial D).
\]

a new edge effect will be formed. This member is given in all boundary conditions corresponding to (3) (for details see [2, ch. I, 3.3.1].

5. On applications of the complex variable function theory.

The representations (20)-(23) allow to apply complex analysis. Let us preliminarily consider the equation (22) and underline the main members:

\[
D' \Delta [u, \varphi] = D'([\Delta u, \varphi] + [u, \Delta \varphi] + 2[\partial_\alpha u, \partial_\alpha \gamma]),
\]
\[(D' = 4h^3(1 + 2\gamma)/3(1 - \nu)), D\Delta^2 u. \quad (28)
\]
If \( \{ f \} \) denotes the physical dimension of value \( f \) then its evident that \( \{ \Delta^2 u \} = \{ \Delta [u, \varphi/E] \} \), \( E \) is modulus of elasticity. We have:

\[
\begin{align*}
[\Delta u, \varphi] &= \partial_{22} \varphi \partial_{11} \Delta u - 2 \partial_{12} \varphi \partial_{12} \Delta u + \partial_{11} \varphi \partial_{22} \Delta u, \\
[\partial_\alpha u, \partial_\alpha \varphi] &= \partial_{11\alpha} u \partial_{22\alpha} \varphi - 2 \partial_{12\alpha} u \partial_{12\alpha} \varphi - \partial_{22\alpha} u \partial_{11\alpha} \varphi.
\end{align*}
\]

Thus, the first summand type (29) of (28) may be define also the nonlinear wave processes in the static cases whereas the third order derivatives containing a sum- 
mand (30) with respect to function \( u = u(x, y) \) corresponds to 1 and 2-dim soliton solutions of Corteveg-de Vries or Kadomtsev-Petviashvili kind. As the second order derivatives of the function \( \varphi = \varphi(x, y) \) describe the stress tensors horizontal components, the summands \( [u, \Delta \varphi] \) correspond to the nonlinear part for the systems of the type (22-23).

The calculation and analysis of a symbolical determinant of these expressions show that the characteristic forms of the systems of type (22-23) may be positive, negative or zero as they represent arbitrary functions of \( x, y \). Let us consider the following operators and notations:

\[
\begin{align*}
z &= x + iy, \quad \bar{z} = x - iy, \quad x_1 = x, \quad x_2 = y, \quad u(x, y) = U(z, \bar{z}), \\
\partial_z &= \frac{1}{2} (\partial_1 + i \partial_2), \quad \partial_{\bar{z}} = \frac{1}{2} (\partial_1 - i \partial_2), \\
4 \partial_z \partial_{\bar{z}} &= \Delta, \quad 16 \partial_z \partial_z \partial_{\bar{z}} = \Delta^2, \quad 16 \partial_z \bar{z} \partial_{\bar{z}} |U(z, \bar{z}), V(z)| = E [u(x, y), V(x, y)].
\end{align*}
\]

Now we form the following iterative-direct (hybrid) method for finding the solution of rewriting in complex variables systems of PDEs (23)-(26) so:

Let \( [U(z, \bar{z})]^{(m)} \) denotes \( m \)-th approach for deflection \( u(x, y) \) which is calculated by known right-hand terms without \( R \) and \( m - 1 \)-th order approach of summand

\[
\begin{align*}
\frac{2 Eh}{16D} &\left( 1 - \frac{h^2(1 + 2\nu)}{3(1 - \nu)} \partial_z \partial_{\bar{z}} \right) \int_0^z \int_0^{\bar{z}} (z - \zeta) (\bar{z} - \bar{\zeta}) |U, V|^{(m-1)} d\zeta d\bar{\zeta}, \\
EV &= \Phi \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2} \right), \quad (31)
\end{align*}
\]

We do some operations for DEs (23) for shearing forces and for system (24-25). This system is equivalent to the following equation (see [11]):

\[
\Delta (\sigma_{11} + \sigma_{22}) = -\frac{E}{2} [w, w] + \frac{\nu}{2h} \int_{-h}^{h} \Delta \sigma_{33} dt + \frac{1 + \nu}{2h} \tilde{f}_{a, \alpha} \quad (K - R) 2.
\]

For

\[
V^{(m)} = V^{(m)}(z, \bar{z}) = -\frac{\mu}{\lambda + 2\mu} \int_0^z \int_0^{\bar{z}} (z - \zeta) (\bar{z} - \bar{\zeta}) [U^{(m-1)}, U^{(m-1)}] d\zeta d\bar{\zeta} + F(z, \bar{z}).
\]

We remark:
i. The correction of \((K - R)^2\) equation by summand depending from \(\Delta \sigma_{33}\) was considered by Lukasiewicz considering only effects of local loads [12].

ii. The invariant form of Monge- Ampère operator:

\[
[u(x,y), v(x,y)] = -4[\partial_{\bar{z}}U \partial_{\xi}V - 2\partial_{\bar{z}}U \partial_{\xi}V + \partial_{\bar{z}}U \partial_{\bar{z}}V] \\
= -4[U(z, \bar{z}), V(z, \bar{z})], \quad (M - A)_{ef},
\]

and when \(u = v\) we have

\[
[u, u] = 2\left(\partial_{11}u \partial_{22}u - (\partial_{12}u)^2\right) = -4[U(z, \bar{z}), V(z, \bar{z})].
\]

Thus, by means of complex analysis we reduced the systems of PDEs of KMR type to the pseudo-integral operator of second type. An iterative scheme, described by (31) corresponds to the solution of Volterra second type nonlinear integral equation. Whereas the processes by schemes generating from (22) contain both Volterra and Fredholm type operators with an arbitrary parameter \(\gamma\). The convergence for only pure Volterra type process (where \(\gamma = -0.5\)) depends also on the convenient selection of the initial functions \(U^{[0]}, V^{[0]}\). It is possible to apply some results of [13, Ch. XXIV, 476, example 4] to the equation \([u,u] + a^2 = 0\) for arbitrary function \(a = a(x,y)\). When \(\gamma \neq 0.5\) the convergence depends on the Fredholm operator:

\[
F_r(U, V) = \partial_{\bar{z}}\partial_{\xi} \lambda \int_0^\bar{z} \int_0^\xi (\bar{z} - \bar{\zeta})(z - \zeta) [U(\zeta, \bar{\zeta})V(\zeta, \bar{\zeta})d\bar{\zeta}d\zeta
\]

with an arbitrary parameter denoted for simplicity by \(\lambda\). The operator \(\lambda^{-1}F(U, V)\) depends on the behavior of expression which may generate different kinds of waves (shok, soliton) functions too and in the cases when they are uniformly bounded functions the process corresponding to applications of the Fredholm operator will be convergent as the corresponding operator will be a contracted one. More convenient may be Seidel’s type iterative scheme: let the initial value is \(U^{[0]} = \frac{1}{4}z^2\bar{z}^2\). Then in expressions of type (31) we used \(V^{[1]}\) defining from (32) and so on. The following theorem is true.

**Theorem 4.** Let us consider the following iterative process:

\[
V^{[m]}(z, \bar{z}) = \alpha \int_0^\bar{z} \int_0^\xi (z - \zeta)(\bar{z} - \bar{\zeta}) \left[U^{[m-1]}, U^{[m-1]}\right] d\zeta d\bar{\zeta}, \quad m = 1, 2, ..., \\
U^{[m]}(z, \bar{z}) = \beta \int_0^\bar{z} \int_0^\xi (z - \zeta)(\bar{z} - \bar{\zeta}) \left[U^{[m-1]}, V^{[m-1]}\right] d\zeta d\bar{\zeta} \\
+ c \int_0^\bar{z} \int_0^\xi \left[U^{[m-1]}, V^{[m]}\right] d\zeta d\bar{\zeta}, \quad m = 1, 2, ...
\]

then it is convergence for all finite \(a, b, c |\frac{4}{3} a| < \frac{1}{3} \). \(U^{[0]} = z^p\bar{z}^p\) and an integer \(n \geq 2\).

**Proof.** The essential moment is to estimation of the transition effect from \(m\) step to \(m + 1\) step. Let \(U^{[m]} = z^p\bar{z}^p\). The transition process contains two stages: the calculation of expressions of the type \([u,v]\) and corresponding integrals. It is evident
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that

\[ [U^m, U^{m+1}] = 2(p(p-1))^{2}z^{2p-2}z^{2p-2} - 2p^4z^{2p-2}z^{2p-2} = -2p^2(2p-1)z^{2p-2}z^{2p-2}, \]

then we have also:

\[ V^{m+1} = -2ap^2(2p-1)z^{2p-2}z^{2p-2} = -\frac{a}{2(2p-1)}z^{2p-2}z^{2p-2}, \]

and

\[ [U^m, V^{m+1}] = -\frac{2ap^2(3p-1)}{2p-1}z^{2p-2}z^{2p-2}, \quad c_p = \frac{2p^2(3p-1)}{2p-1}, \]

\[ I_1 = abc_p \int_0^z \int_0^\bar{z} z^{2p-2}z^{2p-2} \, dz \, d\bar{z} = \frac{abc_p}{4p^2(2p-1)^2}z^2z^2, \]

\[ I_2 = acc_p \int_0^z \int_0^\bar{z} z^{2p-2}z^{2p-2} \, dz \, d\bar{z} = \frac{acc_p}{(2p-1)^2}z^2z^2, \]

\[ c_p(2p-1)^{-2} < \frac{3}{4} + \frac{7}{8(p-1,5)}. \]

This relation show that if \(|c = \gamma| < \frac{4}{3}\) for all bounded functions \(a, b\) the above iterative process is convergence.

Remark. We calculated the systems (22), (23) approximately by the Euler-McLaurin quadrature formulae the summands of members contain \((\Delta \sigma_{33}, 1)\) by \(\Delta \sigma_{33}(x,y,\pm h)\) and \(\lim \Delta \sigma_{33}(x,y,\pm h - 0)\). Then we use the explicit representation of the Cauchy-Riemann nonhomogeneous system of DEs with respect to

\[ w(z,\bar{z}) = (\lambda^* + 2\mu) [z + \bar{z}, z - \bar{z}] + i\mu [z + \bar{z}, z - \bar{z}], \]

for

\[ \partial w(z,\bar{z}) = F(z,\bar{z}), \]

by Pompeiu formula (see i.e. (4.11) or (4.13) [14, ch. I. 4]).

The same processes are true for the anisotropic cases.

6. It is possible to use for an approximate solution by numerical methods the systems of type (24-25), (26-27) with boundary conditions generating by (3).

3 Conclusion

Thus, we created the mathematical theory for refined theories both in linear and nonlinear cases for anisotropic nonhomogeneous elastic plates and shells, approximately satisfying the corresponding system of partial differential equations and
boundary conditions on the surfaces. Now, the optimal and convenient refined theory might be chosen easily by selection of the parameter $\gamma$ after making a few necessary experimental measurements without using any simplifying hypotheses. We justified and give the right form to the von Kármán-Reissner-Mindlin type systems of refined theories. We demonstrated that the Monge-Ampère operator is a linear differential form of the first order of two nonlinear operators having applications in the nonlinear elasticity theory and has invariant form within to sign transform from the real to the complex variables.

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