High-m kink/tearing modes in cylindrical geometry

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Abstract

The global ideal kink equation, for cylindrical geometry and zero beta, is simplified in the high poloidal mode number limit and used to determine the tearing stability parameter, $\Delta'$. In the presence of a steep monotonic current gradient, $\Delta'$ becomes a function of a parameter, $\sigma_0$, characterising the ratio of the maximum current gradient to magnetic shear and $x_s$, characterising the separation of the resonant surface from the maximum of the current gradient. In equilibria containing a current ‘spike’, so that there is a non-monotonic current profile, $\Delta'$ also depends on two parameters: $\kappa$, related to the ratio of the curvature of the current density at its maximum to the magnetic shear and $x_s$, which now represents the separation of the resonance from the point of maximum current density. The relation of our results to earlier studies of tearing modes and to recent gyrokinetic calculations of current driven instabilities, is discussed, together with potential implications for the stability of the tokamak pedestal.

Keywords: tearing mode, radially global, magnetohydrodynamic stability, gyrokinetic simulation

(Some figures may appear in colour only in the online journal)

1. Introduction

Most studies of micro-instabilities consider those driven by gradients of density or ion and electron temperatures. However the toroidal current gradient is also a potential source of instability. Indeed, this is the instability drive for tearing modes and it was even proposed [1] that it could drive high mode number ideal magnetohydrodynamic (MHD) kink instability. This analysis was based on the periodic, cylindrical, ideal MHD equation for the perturbed magnetic flux function, $\Psi = \psi(r) \exp[i(m\theta - nz/R)]$, in the tokamak limit, $B_\theta/B_z \sim r/R \ll 1$:

$$ \frac{d}{dr} \frac{d\psi}{dr} - \frac{m^2}{r} \psi - \frac{\mathcal{J}}{1/q - n/m} \psi = 0, $$

(1)

where $r (0 < r < a)$ is the radial coordinate with $'$ denoting a radial derivative, $\theta$ the azimuthal angle and $z$ the axial co-ordinate, $2\pi R$ being the periodicity length in the $z$-direction and we have introduced ‘poloidal’ and ‘toroidal’ mode numbers, $m$ and $n$, respectively. The magnetic field in the $z$-direction is $B_0$, the safety factor $q = rB_\theta/R B_0$ and the normalised current density, $\mathcal{J}$, is defined by

$$ \mathcal{J} = \frac{4\pi}{c} \frac{RJ(r)}{B_0}. $$

(2)

Expanding the denominator of (1), $(1/q - n/m)$, in the vicinity of the resonant position, $r_s$, where, $m - nq(r_s) = 0$, so that

$$ \frac{1}{q} - \frac{n}{m} \approx -\frac{sx}{mq}, $$

(3)

with

$$ x = m(r - r_s)/r_s. $$

(4)
denoting a new (dimensionless) local radial variable, this equation takes the local form:

\[
\frac{d^2\psi}{dx^2} - \left(1 + \frac{\sigma_0}{x}\right)\psi = 0, \tag{5}
\]

where

\[
\sigma_0 = -\frac{r_0^2 r'_{r'}}{ns}, \tag{6}
\]

with \(s = r q'/q\) being the magnetic shear at \(r = r_0\). Equation (5) has been the basis for a number of studies at high \(m\). Thus Kadomtsev and Pogutse [1] used it to claim that ideal MHD instability is possible, for steep current gradients or low shear, if \(\sigma_0 > 2\),

\[
\tag{7}
\]

while Wesson [2] and Strauss [3] investigated the dependence of the tearing mode index, \(\Delta'\) [4] on \(\sigma_0\). Hegna and Callen [5] used a generalisation of (5) that took account of the effect of toroidal and plasma shaping on metric coefficients, to estimate \(\Delta'\) in more general, toroidal, devices. Furthermore, the local gyrokinetic code GS2 has recently [6] been used to study current gradient driven instabilities in collisionless plasmas, again finding instability above a critical value of \(\sigma_0\). However this code is based on the ballooning transformation and this also relies on a linear expansion of \(q\) about the rational surface.

Unfortunately such treatments are not entirely consistent because, for \(\sigma_0 \sim O(1)\), the truncated expansion of \((q - mn)\) employed is inadequate. To see this is the case recall \(\tilde{J}(r)\) and \(q(r)\) are related through Ampère’s equation,

\[
\tilde{J} = \frac{1}{r} \frac{d}{dr} \left( \frac{r^2}{q} \right), \tag{8}
\]

which can be written \(\tilde{J} = (2 - s)/q\), so that

\[
r r^2 \approx (2s^2 - 3s - w)/q, \tag{9}
\]

with \(w\) denoting the quantity \(r q'/q\). Now if we restrict attention to positive values of the current density, \(\tilde{J}(r)\), then \(s < 2\) and \([2s^2 - 3s] - O(1)\). The condition \(s \tilde{J}(r) \gg 1\) then implies \(|w| \gg 1\) so that terms in \(q'/q\) must be retained in the expansion of \(q\) around \(r_0\), giving \(\sigma_0 = w/(ms)\). Consequently, if the parameter \(\sigma_0\) is of order unity, as in [1–3], the tearing equation must now take the form:

\[
\frac{d^2\psi}{dx^2} - \left(1 + \frac{\sigma_0}{x (1 + \sigma_0 x/2)}\right)\psi = 0. \tag{10}
\]

Equation (10), however, of necessity describes a scenario in which there are two resonant surfaces; a situation which cannot be the case for a monotonic \(q(r)\) even when \(\sigma_0 \sim O(1)\). We conclude that both (5) and (10) must give a flawed description of stability of monotonic \(q(r)\) profiles. Clearly, higher order terms in the expansion of \(q - mn\) are required. Thus, remarkably, there is no truly local theory for such high-\(m\) instabilities. We note that a comparison of analytic resistive tearing mode growth rates with those obtained from a numerical code at low-\(m\) also required the inclusion of more derivatives of \(q(r)\) to obtain agreement when the resistivity and \(\Delta'\) were relatively high [7].

In this paper we reconsider the solution of (1) at high \(m\), taking full account of the structure of \(\tilde{J}(r)\) and \(q(r)\). Two scenarios are considered: (i) a monotonic \(\tilde{J}(r)\) with a steep gradient, modelled by a ‘tanh function’; and (ii) the effect of a positive current ‘spike’, possibly arising in the pedestal region of a tokamak in H-mode due to bootstrap currents. This latter case can lead to a region of greatly reduced shear, or even non-monotonic \(q(r)\) and multiple resonances. The outcomes of our calculations are self-consistent forms for \(\Delta'\) which can be used to investigate high-\(m\) tearing mode instability. A value of \(\Delta'\) that accounts for key nonlocal features of the global structure of the self-consistent \(q\) and current profiles is required to properly interpret the GS2 results alluded to above. The quantity \(\Delta'\) also plays a role in determining the saturated amplitude of magnetic islands arising from tearing mode instability. However other parameters may play a role in determining this amplitude [8].

2. Reduction of the tearing equation for high-\(m\): monotonic current profiles

In this section we consider a specific example for the current density \(\tilde{J}(r)\) which has a steep gradient and exploit the property \(m \gg 1\) to expand (1) close to the resonant surface at \(r_0\). The current density employed is

\[
\tilde{J}(r) = \frac{J_0}{2} \left\{ \frac{1 - \tanh(\lambda (2r^2 - r_0^2)/(2r_0^2))}{\tanh(\lambda (a^2 - r_0^2)/(2r_0^2))} \right\}, \tag{11}
\]

where \(r_0\) is the point of steepest current gradient and we will be interested in large values of the parameter \(\lambda \sim O(m)\). The current profile of (11) is shown in figure 1 for \(J_0 = 1, \lambda = 4\) and \(r_0/a = 0.5\). Integrating the Ampère equation, (8), yields an expression for the resonant denominator in the tearing equation, (1), viz.:

\[
\frac{1}{q(r)} - \frac{n}{m} = \frac{J_0 r_0^2}{2 \lambda t} \left\{ \frac{1}{r^2} \log \left[ \cosh \frac{\lambda (r^2 - r_0^2)}{2r_0^2} \right] \right\} - \frac{1}{r^2} \log \left[ \cosh \frac{\lambda (a^2 - r_0^2)}{2r_0^2} \right] + \log \left[ \cosh \frac{\lambda}{2} \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) \right],
\]

\[t \equiv \tanh \left[ \frac{\lambda}{2r_0^2} (a^2 - r_0^2) \right]. \tag{12}
\]
where \( r_s \) denotes the location of the resonance, i.e.
\[
q(r_s) = \frac{m}{n}, \tag{13}
\]
The numerator of the resonant term in (1) is readily evaluated to give
\[
r\hat{J}^* = -\frac{J_0 \lambda r^2}{2r_0^2} \sech^2 \left( \frac{\lambda}{2r_0^2} (r^2 - r_0^2) \right). \tag{14}
\]
We next introduce the local radial variable \( x = m(r - r_0)/r_0 \), where we have now chosen \( r_0 \) rather than \( r_s \) as origin and corresponding resonance position, \( x_s = m(r_s - r_0)/r_0 \) and express \( \hat{J}(x) \) and \( q(x) \) as functions of \( x \) around \( r = r_0 \), noting that these expressions are not confined to small values of \( x \), since we only require \(|x|/m \ll 1\). Therefore, using \( t = 1 \) and ignoring \( O(1/m^2) \) corrections in \((r/r_0)^2 = 1 + (2x/m)\), we find
\[
\hat{J}(x) = \frac{J_0}{2} [1 - \tanh(px)], \tag{15}
\]
\[
\frac{1}{q(x)} - \frac{1}{q(x_s)} = -\frac{J_0}{2 \lambda} \left[ p(x - x_s) + \log \left[ \cosh(px) \right] \right. 
- \log \left[ \cosh(px_s) \right], \tag{16}
\]
\[
r\hat{J}^* = -\frac{J_0}{2} \sech^2(px), \tag{17}
\]
\[
p \equiv \frac{\lambda}{m} \tag{18}
\]
where integrating (8) from \( r = 0 \) to \( r_0 \) gives \( J_0 q(r_0) = 2 + O(1/\lambda) \) for \( \lambda \ll m \gg 1 \). When \( x_s = 0 \), we may write the kink/tearing equation in the form of (5), with
\[
\sigma_0 \rightarrow \sigma(x) \equiv \frac{p^2 x \sech^2(px)}{px + \log \left[ \cosh(px) \right]}. \tag{19}
\]
Since we can identify \( p = \sigma(0) \), there is again a single parameter, \( \sigma(0) \), determining stability, with
\[
\sigma(x) = \sigma(0) f(x), f(y) = \frac{y \sech^2(y)}{y + \log \left[ \cosh(y) \right]} \tag{20}
\]
Thus we can consider (19) as a semi-localised generalisation of (6) in which \( \sigma_0 \) acquires a form factor, \( f \). The function \( \sigma(x) \) is shown in figure 2 for \( \sigma(0) = 1 \), from which it is clear that \( \sigma(x) \) is very far from being constant, as assumed in the local treatments of [1] and [2].

Furthermore, if we assume that the mode numbers, \( m \) and \( n \) are such that the resonant surface lies at \( x = x_s \), i.e. that \( m/n = q(x_s) \) rather than \( m/n = q(0) \), the tearing equation can be written in what we will refer to as the semi-local form:
\[
\frac{d^2\psi}{dx^2} - \psi \left( 1 + \frac{p^2 \sech^2(px)}{p(x - x_s) + \log \left[ \cosh(px) \right] - \log \left[ \cosh(px_s) \right] \} \right) = 0, \tag{21}
\]
with dependence on two parameters, \( p \equiv \sigma(0) \) and \( x_s \).

3. Calculation of the tearing mode index, \( \Delta' \)

In this section we consider reconnecting instabilities which might be driven by the energy source associated with a steep monotonically increasing current gradient, i.e. instabilities which are driven by positive values of the tearing index, \( \Delta' \), defined as the jump in \( \psi'/\psi \) as in (4). However we first discuss the ideal MHD stability properties of (21), noting that these can be determined by computing the stability properties of the two Newcomb [9] sub-intervals, \([0, r_s] \) and \([r_s, a] \), or, in terms of the ‘local’ variables, \( x_s [-\infty, x_0] \) and \([x_0, +\infty] \). This analysis has been carried out numerically and no instability found at any finite values of the parameters \( p \) and \( x_s \) (i.e. in Newcomb terms, in shooting a solution which is regular at one end-point of a sub-interval, no zero of \( \psi(r) \) is encountered before the other end of the sub-interval is reached). In particular, the value \( p = \sigma(0) = 2 \) is NOT a marginal point for ideal MHD instability, contrary to [1].

Returning to the issue of tearing mode stability, we calculate \( \Delta(p, x_s) \), defined as the jump in \( \psi'(x)/\psi(x) \) at \( x_s \) for the solution of (21) which vanishes at \( x = -\infty \) and at \( x = +\infty \). The global value of \( \Delta' \) is then related to \( \Delta \) by:
\[
r_s \Delta' = m \Delta. \tag{22}
\]

Figure 3 shows \( \Delta \) as a function of \( p \) when \( x_s = 0 \) and figure 4 shows \( \Delta \) as a function of \( x_s \) for \( p = 1 \) (solid line) and \( p = 0.67 \) (dashed line). The somewhat surprising content of figure 4, namely that the most unstable location for the resonant surface is at large negative values of \( x_s \), is an artefact of our semi-local approximation, as discussed below.

We have also performed global calculations of \( \Delta = [r_s \Delta'(r_s)]/m \) with \( \hat{J}(r) \) given by (11). The results of such global calculations are shown in figure 5 where, for the purpose of comparing with figure 4, (22) has been used to transform ‘global’ data into ‘local’ variables, as in figure 4. In figure 5 the parameters are \( r_0/a = 0.5 \) and \( \lambda = m = 8 \) (solid curve) and \( \lambda = m = 12 \) (dashed curve). Figure 5 shows that the most unstable value of \( \Delta \) occurs at a finite negative value of \( x_s \). A more careful inspection of the derivation of (21) reveals that, at large negative values of \( x_s \) the shear, \( \psi(x_s) \), becomes exponentially small, \( \sim \exp(-p x_s) \), so that terms of order \( 1/m \), which have been neglected, can compete, leading to the behaviour...

![Figure 2](image-url)
seen in figure 5, for large, but finite, m. Not surprisingly, the Δ' values obtained and therefore stability, are sensitive to the global structure of the ideal MHD region since equation (21) contains less global information than equation (11), thereby resulting in the differences in Δ shown in figures 4 and 5.

4. Comparison with previous studies

For comparison with the previous results of Kadomtsev and Pogutse [1] and Wesson [2], we have also calculated the tearing stability index Δ0(p) obtained by computing solutions of (5) with σ0 = p.

Figure 6 shows (dashed curve) the resulting Δ0(p) as a function of p from (5). For comparison the solid line shows Δ(p) of figure 3, computed with σ = σ(x) of (19). Ideal MHD marginality, where the value of Δ0(p) becomes infinite, apparently occurs at p = 2 which corresponds to the prediction [1] of ideal kink instability beyond this value. At the value p = 2, an exact solution of (5) in the inner Newcomb sub-interval is ψ = x e^x, which vanishes at both end-points, again demonstrating ideal marginality. However, as noted in the foregoing discussion, this is an incorrect prediction and is not found in full cylindrical solutions, or in our semi-localised version (19) with the correct σ(x) dependence, which yields the solid curve in figure 6. For small values of p, Δ0 is negative and changes sign at p ≈ 1, as reported in [2].

5. Reduction of the tearing equation for high-m: non-monotonic current profiles

In this section we investigate a different, but possibly important, scenario in which a fairly localised positive current spike occurs relatively near the plasma edge [10, 11], where resistivity is high and the inductive current density is small. The non-linear theory of external kink modes in the presence of such a current spike has been studied by Eriksson and Wahlberg [12]. Bootstrap currents in the region of a steep pedestal, or localized current drive, may produce just such a situation. Figure 7 shows an example for the current density given by,

\[ \tilde{J}(r) = \tilde{J}_b(r) + \tilde{J}_c(r), \]

\[ = J_0[1 - (r/a)^2] + J_1e^{-\mu(r/c - 1)^2}. \]
with \(J_0 = 2.5, J_1 = 1, \mu = 64\) and the peak of the current spike at \(r_f/a = 0.8\). In figure 7 the dashed curve represents the resulting safety factor, \(q(r)\), indicating that the magnetic shear becomes small in the region of the current spike.

It might be thought that the most unstable location for a tearing mode resonance would be at the minimum of the current density \((r/a = 0.65\) in figure 7\), since the gradient of the current density is destabilising on both sides of the resonance in this case, that is, \(J/[(1/q) - 1/q(r_s)] < 0\) so its sign is the opposite of the stabilizing \(m^2\) line bending term in (1). Surprisingly, however, this is not the case, as can be seen from figure 8 which displays the value of \(r_s \Delta(r_s)\) for an \(m = 4\) mode, calculated from (1), as \(r_s\) is moved across the \(\hat{J}(r)\) profile, regarding \(n\) as a continuous variable.

Figure 8 does show that when \(r_s\) is located at the maximum of the current spike, \(r_s/a \approx 0.79\) in figure 7, the tearing index, \(\Delta\) is strongly stabilising (i.e. negative) as one would expect since the current gradients on both sides of \(r_s\) are stabilising in this case, that is, \(J/[(1/q) - 1/q(r_s)] > 0\), enhancing the stabilizing \(m^2\) line bending term in (1). However, values of \(r_s\) quite close to the maximum of the current density are very unstable.

To model this in a high-\(m\), localised analysis, we expand \(\hat{J}(r)\) and \(q(r)\) locally around the maximum of the current density, approximately at \(r_1\), the maximum of the current spike. Around this point, \(\hat{J}(r)\) is dominated by the current spike, \(\hat{J}_1(x)\), and is a local function, \(\hat{J}(x)\), if we order the parameter \(\mu \sim O(m)\). However, the \(q\) profile in this region is determined by both the extended ‘inductive’ current profile \(\hat{J}_0(r)\) and by the current spike. It therefore contains both a slowly varying part, due to \(\hat{J}_0(r)\) and a rapidly varying part, due to \(\hat{J}_1(r)\) and the result can be a greatly reduced magnetic shear, as seen in figure 7. The resulting local tearing equation has been derived in the appendix A, for the current profile of (23). However a simpler derivation expands \(q(r)\) locally around the point, \(r_1\), at the maximum of \(\hat{J}(r)\) and orders the weakened shear, \(s \sim 1/m\) and \(r_1^2 q^\prime / q \sim m \gg 1\). Thus:

\[
q(x) = q(r_1) + r_1 q^\prime x + \frac{1}{6} r_1^2 q^\prime q^\prime x^3.
\]  

(24)

Then, constructing \(\hat{J}'(x)\) from (23), the high-\(m\) tearing mode equation (1), can be written in the form:

\[
\frac{d^2 \psi}{dx^2} = \psi \left[ 1 + \frac{\kappa x}{x(1 + \frac{1}{\pi^2} + x^2) - x(1 + \frac{1}{\pi^2} + x^2)} \right],
\]

\[
\kappa = \frac{r_1^2 q^\prime}{m^2 r_1 q^\prime}.
\]  

(25)

where \(x_s\) is again the location of the resonance. We note from (24) that a monotonic \(q\) profile requires \(\kappa > 0\).

As in section 3, stability depends on two parameters, \(\kappa\) and \(x_s\). In order to reduce the parameter space in (25), we have focused on three cases:

\((a)\) \(\kappa = 8\) monotonically increasing \(q\),

\((b)\) \(\kappa = 64\) with weaker shear at \(x = x_s\),

\((c)\) \(\kappa = -8\) non-monotonic \(q(r)\).

(26), (27), (28)

Equation (25) has then been solved to obtain values of \(\Delta(x_s)\) as the resonant location, \(x_s\), is moved across the local \(q(x_s)\) structure. Results for cases (a) and (b), equations (26) and (27) respectively, are shown in figure 9. The solid curve corresponds to case (a) and the dashed one to case (b). In case (c), (28), we exclude consideration of the region of triple resonance, i.e. \(-1 < x_s < +1\) and figure 10 shows the value of \(\Delta\) when \(x_s\) falls outside this range. Consideration of the triple resonances in case (c) raises issues involving the different characteristic frequencies associated with tearing at each of the three resonant surfaces, these frequencies being determined by diamagnetic terms and by sheared equilibrium rotation. In addition, case (c) is likely to arise only after an equilibrium current profile has evolved through the very unstable weak shear scenario, case (b). It is therefore sufficient to note that as \(x_s\) approaches the location of \(q_{\text{min}}\) or \(q_{\text{max}}\), the value of \(\Delta\) becomes very large.

Positive values of \(\Delta\), with energy available to drive reconnection, are predicted for modes which are resonant close to, but not at, the local maximum of the current density. Comparison of figures 8 and 9 demonstrates the validity of the high-\(m\) equation (25).
6. GS2 Results

GS2 is a radially local gyrokinetic code modeling small scale instabilities \((k_a \gg 1)\) in a periodic flux-tube domain in toroidal geometry. Here, \(k_i = n q_i / r_s\) is the binormal wave number, where \(n\) is the toroidal mode number and the subscript \(r\) of a quantity refers to its value at the reference radius. The radial variation of the metric is not retained and all plasma and magnetic geometry parameters are linearized around their value at \(r_s\). In particular, \(q_i \approx q_i [1 + s (r - r_s)/r_s]\) is used: an approximation equivalent to (3). For brevity, henceforth we will drop the \(r\) subscripts. We use the low-flow version of GS2, which changes the current gradient to zero, thus there are no diamagnetic corrections to the mode structure. We choose to set the temperature and density gradients in the model break down close to the ideal MHD instability limit, \(\gamma \approx \Delta_k / k_b\), we find that \(\gamma / a \approx \Delta_k (k_a)^2 \beta_b^{-1}(sa) / (q R) [(m/T_e)/(m/T_i)]^{1/2}\). We may use the following rational approximation to describe the \(p\)-dependence of \(\Delta\) found from local ideal MHD calculations: \(\Delta \approx 4(p - 1)/(2 - p)\), shown as the dotted curve in figure 6. The dotted line in figure 11(a) represents the growth rate as estimated by the above expressions for \(\gamma\) and \(\Delta(p)\). Approximations in the model break down close to the ideal MHD instability limit, \(p = 2\), where the predicted values of \(\gamma\) and \(\Delta_k\) diverge.

Typical radial mode structures are shown in figure 11(b); the \(p\) values shown here are 1.5625 and 1.9375. Since GS2 solves the problem in ballooning angle \(\theta\) and not in \(x = (r - r_s)/r_s\), the plotted functions are obtained from the appropriate Fourier transform of the parallel component of the perturbed vector potential. The eigenfunctions \(\psi\) are normalized so that their value is 1 at the maximum of \(|\psi|\) appearing close to \(x = -1;\)

![Figure 9](image_url) Figure 9. Plot of \(\Delta(x_s)\), from (22) with \(\kappa = +8\) (solid curve) and \(\kappa = 64\) (dashed curve), corresponding to monotonically increasing \(q\) profiles.

![Figure 10](image_url) Figure 10. Equivalent plot to figure 9 but with \(\kappa = -8\) corresponding to reverse shear at \(x = 0\), i.e. to a locally non-monotonic \(q\) profile. \(\Delta\) values are only calculated when there is a single resonant surface, i.e. in the range \(|x| > 1\).
then the blue (red) curves represent the real (imaginary) part of $\psi$. The eigenfunctions do not change appreciably as $p$ is varied. In fact they very much resemble the well known marginally stable ideal MHD result which is of the form $xe^x$ for $x < 0$ and $0$ for $x > 0$ (indicated with black dotted curve in the figure).

Taking these $\psi(x)$ eigenfunctions, we can estimate $\Delta$ from the GS2 simulations. As also seen in figure 11(b), the eigenfunctions are affected by numerical error. In the calculation of $\Delta$, a division by $\psi(0)$ needs to be made, which amplifies small errors as $\psi(0)$ approaches 0, which happens close to the spurious ideal stability limit $p = 2$. Therefore, we are unable to determine $\Delta$ quantitatively.

Figure 11 shows the estimated values of $\Delta$ from GS2 simulations for a range of $p$ values (circle symbols and solid curve). The confidence intervals of the results are indicated with the shaded area, obtained by perturbing the eigenfunction within numerical uncertainties. As $p$ approaches 2, the uncertainties diverge; accordingly, we do not show values of $\Delta$ above $p = 1.875$. As a reference, we show the rational approximation of the ideal MHD result by the dotted line (this is the same as the dotted curve of figure 6).

7. Summary and conclusions

This investigation was stimulated, in part, by simulation results from GS2 [6], with a possible interpretation of an observed instability as an ideal current driven kink. Such a ‘ballooning space’ calculation assumes the neglect of $q''$ and all higher derivatives: i.e. it is equivalent to the approximations which lead to (5) in configuration space, an equation which, incorrectly, predicts ideal instability for values of $\sigma_0 > 2$, where $\sigma_0$ is related to the ratio of current gradient to magnetic shear. It therefore appears that the ‘ideal’ instability seen in GS2 is spurious. However, the calculations presented here show that the consequence of correctly retaining the full functional dependence of $\sigma(x)$ in, for example, (21), is to exclude the possibility of ideal kink instability, while still permitting unstable values of the tearing index, $\Delta'$, when $\sigma(0)$ exceeds a value around unity. Hence the mode identification in terms of a ‘tearing/kink’ drive looks entirely justified, but its identification as an ‘ideal kink’ as proposed by Kadomtsev and Pogutse [1], should be modified. The mode should be regarded as a high-$m$ tearing mode, driven unstable by a large value of $\Delta'$, with collisionless reconnection provided by electron physics in the resonant layer around $r_s$. As a consequence, it will be inappropriate to run GS2 with values of $\sigma_0 > 2$. It is also of interest to note that Hegna and Callen [5] used a modification of (5) to describe general geometry and as a simple way to derive a convenient formula for $\Delta'$ in toroidal systems. Although this may give a good approximation for values of $\sigma_0 < 1$, where $\Delta'$ is negative, it overestimates the instability drive for $\sigma_0 > 1$ and predicts ideal instabilities for $\sigma_0 > 2$, where none exist.

We have also investigated a situation with a non-monotonic profile of the current density, $\tilde{J}(r)$, as might result from bootstrap currents near a tokamak pedestal. Stability of high-$m$ modes is again governed by a local equation (25), depending on two parameters, $\kappa$ and $x_s$, where $\kappa$ is a measure of the ratio $r_s^2q''/m^2q$ and $x_s$ is the location of the resonance relative to the point of maximum $J$. As for the previous case, ideal instability does not occur for any values of the $\kappa$ and $x_s$ parameters, but positive values of the tearing index, $\Delta'$, can be found for sufficiently large values of $\kappa$. Such values of $\kappa$ can
arise from the low shear resulting from the near cancellation of contributions from the background current and the current spike. These situations can, typically, lead to $\Delta \sim O(1)$, implying $r_\Delta \sim O(m) \gg 1$. Since such a current spike can result from the bootstrap current occurring naturally in the pedestal region of a tokamak H-mode plasma, these observations have possible relevance for the interpretation of ELMs in terms of surface ‘peeling’ modes associated with tearing modes resonant within the pedestal. They also suggest the possibility of influencing ELM behaviour by driving reverse currents within the pedestal region. We note that the potentially large values of $\Delta'$ could overcome stabilising effects, such as the Glasser effect in a torus [15]. To ameliorate the deleterious effects of large ELMs on divertor target plates, resonant magnetic perturbations (RMPs) have been applied to produce magnetic islands with the intention of driving pedestal gradients below the MHD stability limit. The tearing stability of the non-symmetric equilibria is beyond the scope of this work but it is worth noting that the amplitude of such a RMP driven island depends on the value of $\Delta'$ calculated in this work [18].

At first sight, it is perhaps surprising that such high-$m$ calculations cannot always be reduced to a purely local calculation involving only the current gradient and magnetic shear at the rational surface, but instead requires that the complete structure of the $q$ profile be taken into account, albeit in a narrow region for a sharply localised gradient in the current profile. Consideration of this problem is beyond the scope of local gyrokinetic codes. Consequently the extension to a toroidal calculation must inevitably become two-dimensional, unlike problems amenable to the ballooning transformation as in, e.g., the local gyrokinetic code GS2.

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Appendix A. Approximate tearing equation for non-monotonic current profiles

In this appendix A we demonstrate that, using the current distribution of (23) with the parameter $\mu \sim O(m)$, the global kink/tearing equation (1), can be reduced, in the limit of high-$m$, to the form of (25) and that, for the case of the $m = 4$ mode investigated in figure 8, the equivalent $\kappa$ value is 51.1.

We first introduce the notation,

$$p_1 = \frac{\mu}{m}.$$  \hspace{1cm} (A.1)

and treat $p_1$ as $O(1)$ parameter. Then, in leading order of an expansion in $1/m$,

$$r^2 = -2p_1 J_{x},$$  \hspace{1cm} (A.2)

now with $x = m(r - r_1)/r_1$. We next construct an expression for $1/q(r) - 1/q(r_1)$ appearing in the denominator of the current drive term of (1), noting that the contributions of the inductive current, $\tilde{J}_0(r)$ and the localised current spike, $\tilde{J}_1(r)$, are simply additive, so that:

$$\frac{1}{q(r)} - \frac{1}{q(r_1)} = \frac{1}{r^2} \int_{r_1}^{r} s \textrm{d}s [\tilde{J}_0(s) + \tilde{J}_1(s)]$$

$$= \frac{J_0 J_1^2}{m} \left[ -\frac{3}{4} + 2r_1^2 - \frac{3}{4} r_1^4 \right] + \frac{J_{x}}{m} \left[ 1 - \frac{p_1 x^2}{3m} \right].$$  \hspace{1cm} (A.3)

where $p_1 \equiv r_1 / a$. In equation (A.3) we retained an $O(1/m)$ correction because the leading order term is small at low shear due to near cancellation of the contributions from $J_0$ and $J_1$ to $O(1/m)$. Finally, transforming the radial variable in (1) to $x$, the kink/tearing equation takes the form,

$$\frac{\textrm{d}^2 \psi}{\textrm{d}x^2} - \psi \left[ 1 - \frac{2p_1 J_{x}}{m[J_0 + J_0 J_1^2(-3/2 + 2r_1^2 - (3/4) r_1^4) - J_0 p_1 x^2/3m]} \right] = 0,$$  \hspace{1cm} (A.4)

which is precisely of the same form as (25) when $x_\ast = 0$, with the parameter, $\kappa$, given by:

$$\kappa = -\frac{2p_1 J_1}{m[J_0 + J_0 J_1^2(-3/2 + 2r_1^2 - (3/4) r_1^4)]}.$$  \hspace{1cm} (A.5)

the above is easily generalised for non zero values of $x_\ast$. Note that the above mentioned cancellation of terms in $J_0$ and $J_1$ means that $\kappa$ is formally $O(1)$ but can become very large and even change sign if the shear at $r_1$ reverses. For the parameters of figure 7, $J_0 = 2.5$, $J_1 = 1$, $\mu = 64$, $\tilde{r}_1 = 0.8$ and for $m = 4$, the equivalent value of $\kappa$ is 51.1.

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