THE WORLD SHEET REVISITED

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Abstract
We investigate the mathematical structure of the world sheet in two-dimensional conformal field theories.
1 On to the world sheet

One way physicists think about conformal field theory is in terms of a sigma model, defined as a “quantization” of a space of maps from a two-dimensional world sheet $\Sigma$ to a target space $M$. Much effort has been spent making the idea of quantization more precise. The notion of vertex (operator) algebras is one result of these efforts. It captures basic features of a conformal field theory associated to $M$ (e.g. a free boson model when $M$ is flat) and thereby some aspects of the target space $M$ itself that are relevant for the quantized theory. In this note we want to show that it is worthwhile and, for a deeper understanding of conformal field theory, even indispensable to think about the structure of the world sheet $\Sigma$ as well.

A natural starting point for the discussion of $\Sigma$ is the the structure of a real, two-dimensional manifold. Such a manifold can have various additional structure:

- it can be smooth
- it can be compact
- it can be orientable, and if so, it can be oriented
- it can have boundaries
- it can have a conformal structure
- if it is orientable, it can have a holomorphic structure
- it can have metric, with two possible choices of the signature
- it can have a (generalized) spin structure

In different parts of the physics and mathematics literature on conformal field theory different structures are needed. In this note we present an attempt to clarify their relationship. From the outset, the reader should be aware of one basic feature: There is nothing like “CFT©”. On the contrary, the term conformal field theory is used to refer to various different physical situations, and different purposes can and do require different axiomatizations.

Let us start with the question whether a metric, if present, has Euclidean, i.e. $(++)$, or Lorentzian, i.e. $(+-)$, signature. If we wish to think about conformal field theory as a (toy) model for four-dimensional, “realistic” quantum field theories, we are tempted to require a Lorentzian signature. In fact, a lot of work has been done in this setting (for a review see [17]).

But – in contrast to the situation in the Euclidean case – the requirement of the existence of a metric of Lorentzian signature severely restricts the topology of $\Sigma$. Indeed, a compact manifold admits a metric of Lorentzian signature if and only if it admits a line element. This is the case precisely if its Euler characteristic vanishes. In two dimensions, a metric of Lorentzian signature is therefore an interesting structure only when the manifold is either non-compact or when it is a torus, which indeed can arise as a compactification of two-dimensional Minkowski space $\mathbb{R}^2$. We will not study this type of CFT in the remainder of this note.

Most ‘modern’ applications of CFT indeed require a compact world sheet. This is more or less evident in the application to two-dimensional critical phenomena, since most samples have finite size. (It is worth noting that compactness persists in the scaling limit that should be used to investigate universal aspects of critical phenomena.)

With regard to the application to string theory, compactness requires a few more words of explanation. The following naive picture is frequently suggested. A string moves in a target space $M$ of Lorentzian signature. It sweeps out a world sheet, which is endowed with the

\[\text{This is not the conformal compactification, which is obtained by adding three points at future and past timelike as well as spacelike infinity, but a separate compactification of two light-like directions.}\]
induced metric and thus has Lorentzian signature. The scattering of two ingoing particles to two outgoing particles is then described (at the so-called tree level) by a picture of the following form:

At first sight this diagram suggests non-compactness of the world sheet. But this first impression is misleading. One way to see this is to understand the real meaning of the “boundaries” of the world sheet in this picture, which are to be thought of as parametrized circles. In fact, they are not physical boundaries, but rather their role is to indicate the presence of “asymptotic states”. Those, in turn, correspond to the insertion of suitable vertex operators on the world sheet. An appropriate way to interpret the situation is therefore to obtain local coordinates around each insertion of a vertex operator, which can be achieved by gluing a standard disk

\[ D = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \]

to each of the parametrized circles.

We will use such local coordinates instead of parametrized boundaries. We then arrive at a compact world sheet. This leads to a conflict with a possible Lorentzian signature of a metric on the world sheet; we avoid it by always considering Euclidean metrics on the world sheet. Thus we must give up the idea that the metric on the world sheet is related to the pull back of a metric on the target space. Note, however, that we do stick to a Lorentzian signature for target space\(^2\)

This point of view is consistent with the following, somewhat minimalistic, view of string theory: It constitutes a perturbation theory whose combinatorics is captured in terms of surfaces and not, as in the case of usual Lagrangian quantum field theory, in terms of graphs, the Feynman diagrams. A useful analogy is supplied by the manner in which Kontsevich’s prescription for deformation quantization \(^3\) can be visualized in terms of amplitudes of a topological string theory on a disk \(^4\).

It turns out to be crucial to distinguish the ‘fake’ boundaries encountered above from real physical boundaries of world sheets. A first, mathematical, distinction is provided by the fact that the physical boundaries are not parametrized. To understand the difference between physical boundaries and fake boundaries more clearly, it is instructive to consider some applications of physical boundaries:

- The first application is provided by theories of open strings, which appeared already in the early days of string theory. They arose in an attempt to describe strong interactions, interpreting mesons, i.e. particles composed of a quark and an antiquark, as open strings with the two charges that correspond to quarks and antiquarks attached to the end-points of the string. An open string is an interval; when it moves through space-time, it sweeps out a real two-dimensional surface \(\Sigma\) with a boundary that is swept out by the end-points of the interval.

\(^2\) This statement applies to the bosonic string and the \(N=1\) superstring. The \(N=2\) string requires a four-dimensional target space of signature \((+,+,−,−)\).
Thus in this picture each boundary component corresponds to the world line of a charged particle.

- In the application of conformal field theory to defects in solid state physics, points on the boundary of the world sheet carry defects while the interior models the medium in which the defect is situated. Thus, again the boundary is the world line of a particle-like defect with definite physical properties.

These examples indicate that physical boundaries have measurable physical properties and cannot be disposed of by gluing small disks to them. Moreover, these properties can change along a component of a boundary. As a consequence, physical boundaries can themselves support insertions, at which so-called boundary fields can change physical properties of the boundary.

We are thus led to study compact real surfaces, possibly with boundaries, with a Euclidean metric on it. Actually, a metric is a bit too much of a structure: We did not take into account the conformal symmetry so far. The presence of this symmetry implies that only the conformal class of the metric matters. That is, metrics \( g \) and \( g' \) that differ by a local rescaling,

\[
g'(p) = e^{\phi(p)} g(p)
\]

with \( \phi \) a sufficiently well behaved function on \( \Sigma \), should be identified. To be precise, we must be slightly more careful: If the Virasoro central charge does not vanish, the so-called Weyl anomaly forces us to fix even a projective structure on \( \Sigma \). (A projective structure on a Riemann surface is an equivalence class of coverings by holomorphic charts such that all transition functions are Möbius transformations, i.e. fractional linear transformations.)

Typically, many inequivalent conformal structures exist for a given topological manifold. They are parametrized by a moduli space. On can therefore consider “one and the same” conformal field theory on a family of different spaces for the world sheet. This important idea does not have any direct analogue in conventional quantum field theory.

In string theory, we must go even further: Since no conformal structure is preferred, the choice of a conformal structure must be regarded as an auxiliary datum and hence be eliminated. Accordingly, scattering amplitudes in string theory are defined as integrals of conformal field theory correlation functions over the moduli space of conformal structures. In this context, the following observation becomes relevant: The moduli space is typically not compact, but it can be compactified by including manifolds with singularities that are not too bad. It is therefore common to consider conformal field theories also on spaces that are more general than smooth manifolds and that in particular can possess ordinary double points. From a purely mathematical point of view, the idea to consider conformal field theories in families over moduli space and to extend them to singular manifolds has been successful, too; the corresponding factorization rules form the basis of most approaches to the Verlinde formula [3, 8, 23].

## 2 Doubling the world sheet

As we have seen, world sheets that are surfaces with boundaries should also be studied. Furthermore, string theories of type I suggest that we should include unorientable surfaces like the Klein bottle or the Möbius strip in our discussion as well. CFT on the Klein bottle may seem somewhat exotic, and you might decide to restrict your attention to orientable surfaces. But
even when you do so, this last remark should at least draw your attention to the fact that for the surfaces considered in the previous section we did not choose an orientation.

Physicists often talk about left moving and right moving modes, which suggests that both possible orientations should be considered simultaneously. If the world sheet $\Sigma$ has empty boundary (we will assume this in the next few paragraphs), this motivates us to consider along with $\Sigma$ also the total space $\hat{\Sigma}$ of its orientation bundle. Recall that the orientation bundle is a $\mathbb{Z}_2$ principal bundle, where the two points in the fiber correspond to the two possible choices of a local orientation. For example, the total space of the orientation bundle of a Klein bottle is a torus $\mathbb{C}/\mathbb{Z} + it\mathbb{Z}$ with $t$ real that is a twofold covering of the Klein bottle; the Klein bottle is obtained by identifying the points $z$ and $1 - \bar{z} + \frac{it}{2}$. When $\Sigma$ is orientable, the total space of the orientation bundle consists of two copies of $\Sigma$ endowed with opposite orientation.

Let us describe the geometry of this space in more detail. $\hat{\Sigma}$ forms a twofold cover over $\Sigma$ \[1, 5\]. This cover has two disjoint sheets if and only if $\Sigma$ is orientable. Interchanging the two sheets defines an anti-conformal involution $\sigma$ on $\hat{\Sigma}$, i.e. a map that reverses the orientation and preserves (the modulus of) angles. $\Sigma$ can be obtained from $\hat{\Sigma}$ as a quotient under this action:

$$\Sigma = \hat{\Sigma}/\sigma.$$ \quad (1)

In the physics literature this is referred to as a world sheet orbifold, parameter space orbifold or (un-)orientifold \[4, 8, 13\]. The total space $\hat{\Sigma}$ is not only orientable, it even possesses a canonical orientation. On the other hand, it also inherits a conformal structure from $\Sigma$. In two dimensions, together these data are equivalent to a complex structure on $\hat{\Sigma}$. This aspect of two-dimensional geometry supplies us with particularly powerful tools for the study of two-dimensional conformal field theories, as opposed to conformal field theories in higher dimensions. In fact, we believe that this feature is far more important than another peculiarity of two dimensions that is often emphasized: the fact that the conformal algebra is infinite-dimensional.

The construction can be easily extended to surfaces with boundary. We illustrate it in the case when $\Sigma$ is a disk: We only double the points in the interior of $\Sigma$, so that $\hat{\Sigma}$ is a sphere obtained by gluing two disks to each other along their boundary. The statements about $\hat{\Sigma}$ made above then remain true, but now the anti-conformal involution does not act freely any longer; its fixed points on $\hat{\Sigma}$ are in one-to-one correspondence with the boundary points of $\Sigma$.

It is worth to pause at this point and to note that we have now arrived at two different mathematical structures. First, a conformal real two-dimensional manifold $\Sigma$; second, its oriented cover $\hat{\Sigma}$, which is even a complex curve. (Both manifolds can in fact have singularities, a fact that, as already mentioned, we ignore for the moment.)

Correspondingly, there are actually two different types of theories that are referred to as conformal field theory: Chiral conformal field theory, which lives naturally on a complex curve such as $\hat{\Sigma}$, and full conformal field theory, which is defined on a real unoriented conformal manifold, possibly with boundary. The structure of a complex curve is frequently assumed in mathematical discussions of conformal field theory – like in the definition of conformal blocks, see e.g. \[14\] – for the world sheet itself. Most of the mathematical literature about conformal field theory therefore deals with chiral conformal field theory. (Still, chiral CFT also has direct physical applications. In particular it describes universality classes of the edge system of quantum Hall fluids (for a review, see \[10\]).)
3 Schemes and ringed spaces

The relation between $\Sigma$ and $\hat{\Sigma}$ calls for a more conceptual explanation. As it turns out, it is convenient to regard the world sheet $\Sigma$ as a real scheme. The double is then just the complexification of this scheme:

$$\hat{\Sigma} = \Sigma \times_{\text{Spec}(R)} \text{Spec}(\mathbb{C}). \tag{2}$$

Let us consider a simple example to appreciate the situation – the affine scheme, given by the ring of polynomials in one variable over $\mathbb{R}$ respectively $\mathbb{C}$. Obviously, for the rings we have

$$\mathbb{C}[X] = \mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{C}. \tag{3}$$

The spectrum of (closed points of) $\mathbb{C}[X]$ is well known: All non-trivial prime ideals are of the form $X - \alpha$ with $\alpha \in \mathbb{C}$, hence $\text{Spec}(\mathbb{C}[X])$ is just the complex plane. The situation over $\mathbb{R}$ is quite a bit more subtle. Indeed, zeros of real polynomials are either real or come in complex conjugate pairs. Thus the prime ideals are either of the form $X - a$ with $a \in \mathbb{R}$, or $(X - \alpha)(X - \bar{\alpha})$ with $\alpha \in \mathbb{C} \setminus \mathbb{R}$. $\text{Spec}(\mathbb{R}[X])$ thus corresponds to the complex plane with complex conjugate numbers identified, which is topologically a half-plane.

Quite generally, a real scheme has complex points as well as real points. The real points correspond to the boundary of $\Sigma$, the complex points to the interior of $\Sigma$. Let us again consider the example of the ring of polynomials: A complex point is a ring homomorphism to $\mathbb{C}$, which for complex polynomials amounts to the evaluation of the polynomial at some value of $X$. For real polynomials, we can first of all consider real-valued points, i.e. ring homomorphisms to $\mathbb{R}$. They correspond to the evaluation of the polynomial at a real number. Evaluation at a complex number gives a homomorphism to the $\mathbb{R}$-algebra $\mathbb{C}$, and complex conjugate numbers give homomorphisms that are isomorphic over $\mathbb{R}$. The complex points in this example therefore correspond to points in the interior of $\Sigma$, which are called “bulk points” in the physics literature. Finally, on the complexification we have an action of the Galois group $\text{Gal}(\mathbb{C}, \mathbb{R}) \cong \mathbb{Z}_2$, whose non-trivial element is just the anti-conformal involution $\sigma$.

We can therefore summarize the situation concisely as follows. Chiral CFT lives on a complex curve, full CFT lives on a real curve. Accordingly, we wish to understand how full CFT on a real curve is connected to some chiral CFT on its complexification. It turns out that this relation can be formulated in a model independent manner. The mere fact that such a relation exists is good news indeed. It tells us that all the mathematical results about chiral conformal field theory, in particular the theory of vertex operator algebras, are relevant for conformal field theory on surfaces with boundaries as well.

Actually, this is not yet the end of the story. Certain classes of conformal field theories require to endow the world sheet with the structure of a scheme that is not a variety and thereby explore the full power of schemes.\footnote{The idea to regard the world sheet as a ringed space, i.e. as a topological space with a sheaf of rings on it, has already appeared in a different context: In \cite{2}, ringed spaces with nilpotents have been used to describe normal ordering in specific classes of chiral CFTs, so-called $b$-$c$ systems.} The best known example of such chiral conformal theories are superconformal theories. These are models of conformal field theory whose chiral algebra is a super vertex algebra \cite{14} (with a super-Virasoro vector, but this vector does not enter our discussion). In the physics literature it is standard lore that the introduction of
fermionic fields in the chiral algebra requires the choice of a spin structure on the world sheet. String theory correlators are then constructed by summing over all spin structures.

There are several ways of looking at a spin structure. The naive idea of choosing a square root $S$ of the canonical bundle $K$, i.e. $(S)^{\otimes 2} = K$, is not necessarily the best way. Namely, in the theory of simple currents, one would like to generalize the situation to orders $N$ higher than two. However, on curves of general genus $g$ the equation $(S)^{\otimes N} = K$ does not possess any solution, due to the integrality of the degree of a line bundle. It is therefore better to recall that the category of spin curves is equivalent to the category of supersymmetric curves. A supersymmetric curve, on the other hand, should be seen as a ringed space with a sheaf of graded-commutative rings that contain nilpotent elements. Such ringed spaces, in turn, allow for generalizations.

A second class of models in which the structure of the world sheet must be extended are orbifold theories based on some orbifold group $G$. It has been advocated already long ago [13] that conformal blocks in such CFT models can be computed using covering surfaces. Instead of $\hat{\Sigma}$, one works with a (possibly branched) covering surface $\tilde{\Sigma}$ such that 

$$\tilde{\Sigma}/G = \hat{\Sigma}.$$ 

The similarity with (1) is striking, and again we would like to take it into account by extending the structure sheaf, as in (2). It therefore seems that the natural structure for the world sheet is the one of a ringed space, where the structure of the ring has to be adapted to the vertex algebra that underlies the theory.

This structure may sound complicated. But fortunately the following observation simplifies life: As shown in [4] in the case of complex curves, the Riemann-Hilbert correspondence implies that a complex modular functor can be described by an equivalent topological modular functor. For many purposes, in particular for the construction of a full CFT from a chiral CFT, one can therefore work entirely in a topological setting [9]. We expect that this pattern generalizes in such a way that the conformal field theory of orbifolds should find a natural topological counterpart in so-called $\pi$-manifolds [22]. This story, however, has yet to be unraveled.

4 Chiral and full CFT

Based on purely geometric considerations, we have been led to two distinct types of conformal field theories:

- Chiral conformal field theory, defined on closed oriented surfaces.
- Full conformal field theory, defined on unoriented (and possibly unorientable) surfaces which can have a boundary.

We conclude with a few comments on both classes of theories. The protagonists of chiral conformal field theory are the conformal blocks. In the algebro-geometric setting a conformal block corresponds to a certain vector bundle over the moduli space of curves; there is also a topological description as a vector space associated to an “extended surface” [21]. It is known [4] that to every vertex algebra one can associate a system of conformal blocks.

The following are, in our opinion, among the most pressing mathematical questions about chiral conformal field theory:

- Under which conditions on the vertex algebra do the conformal blocks possess good factorization properties?
Under which conditions is the tensor category of representations of a rational vertex algebra modular?

What are the good notions that allow to understand non-rational theories, including so-called non-compact conformal field theories?

(Note that one of the big virtues of the vertex algebras associated to certain infinite-dimensional Lie algebras is their ability to select a “good” subcategory of the representations of the Lie algebra, much like Lie groups do. This feature could become even more important in the study of non-compact conformal field theories that are based on non-compact forms of Lie algebras. The representation theory of those Lie algebras is a rich subject in itself, and it will be interesting to see what subcategories of representations are chosen by the associated vertex algebras.)

The protagonists of full conformal field theory are the correlation functions. In the applications of conformal field theory to statistical mechanics they encode physical quantities, such as scaling dimensions or critical exponents, and in string perturbation theory their integrals over moduli space provide scattering amplitudes.

One of the important insights in conformal field theory is that full CFT on a conformal surface Σ is closely connected to chiral CFT on the complex curve ̂Σ. The situation is in fact as beautiful as one could have hoped for: One can construct all correlation functions in terms of conformal blocks in an entirely model-independent manner (see e.g. [9, 11]). There is, however, still more to be uncovered; some of the category-theoretic tools that are needed for a complete understanding are developed in the contribution [12] to this volume.

Owing to these constructions, chiral conformal field theory in general, and notably vertex algebras and their representation theory, are in particular of direct relevance to the CFT description of certain solitonic sectors of string theory, so-called D-branes. These play an important role for various recent developments in string theory such as duality symmetries, dynamics of supersymmetric gauge theories, and black hole entropy. It can therefore be expected that vertex algebras and related mathematical structures will continue to play an important role in string theory.

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