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INJECTION-SUCTION CONTROL FOR NAVIER-STOKES EQUATIONS WITH SLIPAGE

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Abstract. We consider a velocity tracking problem for the Navier-Stokes equations in a 2D-bounded domain. The control acts on the boundary through an injection-suction device and the flow is allowed to slip against the surface wall. We study the well-posedness of the state equations, linearized state equations and adjoint equations. In addition, we show the existence of an optimal solution and establish the first order optimality condition.

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1. Introduction

The goal of this article is to study an optimal boundary control problem for viscous incompressible fluids, filling a bounded domain \( \Omega \subset \mathbb{R}^2 \), and governed by the Navier-Stokes equations with non-homogeneous Navier slip boundary conditions

\[
\begin{aligned}
\partial_t y + \text{div}(y \otimes y) - \nabla p &= \Delta y, & \text{div } y &= 0, & \text{in } \Omega_T = (0, T) \times \Omega, \\
y \cdot n &= a, & [2D(y) n + \alpha y] \cdot \tau &= b & \text{on } \Gamma_T = (0, T) \times \Gamma, \\
y(0, x) &= y_0(x) & \text{in } \Omega,
\end{aligned}
\]

(1.1)

where \( y = y(t, x) \) is the velocity, \( p = p(t, x) \) is the pressure and the condition verifies

\[
\text{div } y_0 = 0 \quad \text{in } \Omega.
\]

(1.2)

Here \( D(y) = \frac{1}{2}[\nabla y + (\nabla y)^T] \) is the rate-of-strain tensor; \( n \) is the external unit normal to the boundary \( \Gamma \in C^2 \) of the domain \( \Omega \) and \( \tau \) is the tangent unit vector to \( \Gamma \), such that \( (n, \tau) \) forms a standard orientation in \( \mathbb{R}^2 \). The function \( \alpha = \alpha(t, x) \) is a so-called friction coefficient. The quantity \( a \) corresponds to inflow and outflow fluid through \( \Gamma \), satisfying the natural condition

\[
\int_{\Gamma} a(t, x) \, d\gamma = 0 \quad \text{for any } t \in [0, T].
\]

(1.3)

In the literature, the Navier-Stokes equations are usually studied with the Dirichlet boundary condition \( y = g \) on \( \Gamma_T \), however it is well known that for small values of the viscosity, the Dirichlet boundary conditions is a source of problems due to the adherence of fluid particles to the boundary and the creation of a strong boundary layer. The laminar flow is often disturbed by the boundary layer breaking away from the surface. This flow separation region results in increased overall drag. On the other hand, theoretical studies and practical experimental (see [7], [10]-[17], [20], [37], [38]) emphasize the importance of the surface roughness on the slip behavior of the fluid particles on the surface wall. Accordingly, slip type boundary conditions, which were firstly introduces by Navier in 1823, have renewed interest in order to describe the physical phenomena in an appropriate way.

In this work, we consider a tracking problem with an injection-suction control through the boundary, by allowing simultaneously the fluid to slip in a natural way.
along the boundary, and aim to solve the control problem and state the first order optimality condition.

Let us mention that boundary control is of main importance in several branches of the industry, for instance in the aviation industry extensive research has been carried out concerning the implementation of injection-suction devices to control the motion of the fluid (see [3], [5], [6], [33], [40]).

From the mathematical point of view, the boundary control in general is technically hard to deal with (see [22], [23]), in the case of the slip boundary condition, the tangent component of the velocity field being part of the solution is not given in advance, which requires a very careful management of the boundary terms, that appear in the state equation, linearized state equations as well as in the adjoint equations.

In this article we consider a quadratic cost functional, which depends on the boundary control variables and with a desired target velocity, and prove the existence of an optimal control, furthermore, we establish the first order optimality condition. We recall that the optimality condition is a very difficult issue when dealing with nonlinear systems, since it requires the well-posedness of the boundary values problems for the state equation linearized state equation and the adjoint equation. In addition, we should verify that the linearized state and the adjoint state are related by a suitable integration by parts formula.

The plan of the present paper is as follows. In Section 2 we present the general setting, by introducing the appropriate functional spaces and some necessary classical inequalities. The formulation of the problem and the main results are stated in Section 3. Section 4 deals with the well-posedness of the state equations. In Section 5 we show that the control-to-state mapping is Lipschitz continuous. Section 6 is devoted to the well-posedness of the linearized state equations. In Section 7 we verify that the Gâteaux derivative of the control-to-state mapping corresponds to the solution of the linearized state equation. Section 8 deals with the formulation of the adjoint equations and to the study of the existence and uniqueness of the solutions. In Section 9 we deduce the duality relation between the linearized state and the adjoint state. Finally, in Section 10 we prove the main result of the article, Theorems 3.1 and 3.2.

2. General setting

We define the spaces

\[ H = \{ v \in L^2(\Omega) : \text{div } v = 0 \text{ in } \mathcal{D}'(\Omega), \ v \cdot n = 0 \text{ in } H^{-1/2}(\Gamma) \}, \]

\[ V = \{ v \in H^1(\Omega) : \text{div } v = 0 \text{ a.e. in } \Omega, \ v \cdot n = 0 \text{ in } H^{1/2}(\Gamma) \}. \]

In what follows we will frequently use the standard inequality

\[ uv \leq \varepsilon u^2 + \frac{v^2}{4\varepsilon}, \quad \forall \varepsilon > 0, \quad (2.1) \]

Young’s inequality

\[ uv \leq \frac{u^p}{p} + \frac{v^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \forall p, q > 1, \quad (2.2) \]

and the equality

\[ -\int_{\Omega} \Delta v \cdot \psi \, dx = -\int_{\Gamma} (2D(v)n) \cdot \psi \, d\gamma + \int_{\Omega} 2D(v) : D(\psi) \, dx, \quad (2.3) \]

which is valid for any \( v \in H^2(\Omega) \cap V \) and \( \psi \in H^1(\Omega) \).
The following results are well-known, and can be found on the pages 62, 69 of [28], p. 125 of [35], Lemma 2 of [42] and [36].

**Lemma 2.1.** Let us denote by $\mathbf{v}_\Omega = \int_\Omega \mathbf{v} \, dx$. For any $\mathbf{v} \in H^1(\Omega)$ the Gagliardo–Nirenberg–Sobolev

$$
\|\mathbf{v} - \mathbf{v}_\Omega\|_{L_q(\Omega)} \leq C\|\mathbf{v}\|^{2/q}_{L_q(\Omega)}\|\nabla \mathbf{v}\|^{1-2/q}_{L_2(\Omega)}, \quad \forall q \geq 2,
$$

the trace interpolation inequality

$$
\|\mathbf{v} - \mathbf{v}_\Omega\|_{L_2(\Gamma)} \leq C\|\mathbf{v}\|^{1/2}_{L_2(\Omega)}\|\nabla \mathbf{v}\|^{1/2}_{L_2(\Omega)}
$$

are valid.

Moreover if $\mathbf{v} \in \mathcal{V}$ satisfies the Navier boundary condition $[2D(\mathbf{v}) \mathbf{n} + \alpha \mathbf{v}] \cdot \mathbf{t} = 0$ on the boundary $\Gamma$ with $\alpha \neq 0$, then Korn’s inequality

$$
\|\mathbf{v}\|_{H^1} \leq C\|D(\mathbf{v})\|_{L_2(\Omega)}
$$

is also valid. Here the constants $C$ depend only on the domain $\Omega$.

We notice that any vector $\mathbf{v} \in \mathcal{V}$ satisfies the condition $\mathbf{v}_\Omega = 0$, since

$$
\int_\Omega v_j \, dx = \int_\Omega \text{div}(\mathbf{v} x_j) \, dx = \int_\Gamma x_j (\mathbf{v} \cdot \mathbf{n}) \, d\gamma = 0 \quad \text{for} \quad j = 1, 2.
$$

We should mention that as in the previous Lemma as well as throughout the article, we will represent by $C$ a generic constant that can assume different values from line to line.

Let us define the space $C([0,T]; L_2(\Omega))$ of continuous functions on $[0,T]$ with values in $L_2(\Omega)$, endowed by the norm $\|\mathbf{v}\|_{C([0,T]; L_2(\Omega))} = \max_{t \in [0,T]} \|\mathbf{v}(t)\|_{L_2(\Omega)}$ and the space

$$
\mathcal{W}(0,T; \Omega) = L_2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)),
$$

provided with the norm

$$
\|\mathbf{v}\|_{\mathcal{W}(0,T; \Omega)} = \|\mathbf{v}\|_{L_2(0,T; H^1(\Omega))} + \|\mathbf{v}\|_{H^1(0,T; H^{-1}(\Omega))}.
$$

We remember the following interpolation result, given in [30] (see Proposition 3.1, p. 18 and Theorem 3.1, p. 125).

**Lemma 2.2.** The embedding

$$
\mathcal{W}(0,T; \Omega) \hookrightarrow C([0,T]; L_2(\Omega))
$$

is a continuous and linear mapping, that is there exists a constant $C$, depending only on $\Omega$, such that

$$
\|\mathbf{v}\|_{C([0,T]; L_2(\Omega))} \leq C\|\mathbf{v}\|_{\mathcal{W}(0,T; \Omega)} \quad \text{for any} \quad \mathbf{v} \in \mathcal{W}(0,T; \Omega).
$$

Finally, for $p \in (2, +\infty)$ let us set the space

$$
\mathcal{H}_p(0,T; \Gamma) = \left( H^1(0,T; H^{-2/(p-1)}(\Gamma)) \cap L_2(0,T; W_p^{1-\frac{2}{p}}(\Gamma)) \right) \times L_2(\Gamma_T),
$$

endowed with the norm

$$
\|(a,b)\|_{\mathcal{H}_p(0,T; \Gamma)} = \|a\|_{L_2(0,T; W_p^{1-\frac{2}{p}}(\Gamma))} + \|\partial_t a\|_{L_2(0,T; W_p^{1-\frac{2}{p}}(\Gamma))} + \|b\|_{L_2(\Gamma_T)}.
$$

In this work we consider the data $a, b, \alpha$ and $\mathbf{v}_0$ in the following Banach spaces

$$
(a,b) \in \mathcal{H}_p(0,T; \Gamma) \quad \text{for} \quad p \in (2, +\infty),
$$

$$
\alpha \in L_\infty(\Gamma_T) \cap H^1(0,T; L_\infty(\Gamma)), \quad \mathbf{v}_0 \in H.
$$
3. Formulation of the problem and main results

The main goal of this paper is to control the solution of the system (1.1) by a boundary control \((a, b)\), which belongs to the space of admissible controls \(A\) that is defined as a bounded and convex subset of \(H_p(0, T; \Gamma)\).

The cost functional is given by

\[
J(a, b, y) = \frac{1}{2} \int_{\Omega_T} |y - y_d|^2 \, dx \, dt + \int_{\Gamma_T} \left( \frac{\lambda_1}{2}|a|^2 + \frac{\lambda_2}{2}|b|^2 \right) \, d\gamma \, dt
\]

(3.1)

where \(y_d \in L_2(\Omega_T)\) is a desired target field and \(\lambda_1, \lambda_2 \geq 0\). We aim to control the solution \(y\) minimizing the cost functional (3.1) for an appropriate \((a, b)\) \(\in A\). More precisely, our goal is to solve the following problem

\[
(P) \quad \minimize_{(a, b)} \{ J(a, b, y) : (a, b) \in A \text{ and } y \text{ is the solution of the system (1.1) for the minimizing } (a, b) \in A \}.
\]

The first main result of this article establishes the existence of solution for the control problem \((P)\).

**Theorem 3.1.** Let \(A\) be a bounded convex subset of \(H_p(0, T; \Gamma)\). Then there exists at least one solution for the problem \((P)\).

Now we give the formulation of the second main result which deals with first order necessary optimality condition for the problem \((P)\).

**Theorem 3.2.** Assume that \((a^*, b^*, y^*)\) is a solution of the problem \((P)\). In addition assume that \(a^*\) belongs to \(H^1(0, T; L_\infty(\Gamma))\). Then there exists a unique solution

\[
p^* \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; H^2(\Omega)), \quad \pi^* \in L_2(0, T; H^1(\Omega))
\]

of the adjoint system

\[
\begin{aligned}
-\partial_t p^* - 2D(p^*) y^* + \nabla \pi^* &= \Delta p^* + (y^* - y_d), & & \text{in } \Omega_T, \\
\text{div } p^* &= 0, & & \text{in } \Omega_T, \\
p^* \cdot n &= 0, & & \text{on } \Gamma_T, \\
2D(p^*) n + (a^* + \alpha) p^* \cdot \tau &= 0 & & \text{on } \Gamma_T, \\
p^*(T) &= 0 & & \text{in } \Omega, \\
\end{aligned}
\]

(3.2)

verifying the optimality condition

\[
\int_{\Gamma_T} \left\{ (f - a^*)(\pi^* + (p^* \cdot y^* + (2D(p^*) n) \cdot n)) \\
+ (b^* - g)(p^* \cdot \tau) + \lambda_1 a^*(a^* - f) + \lambda_2 b^*(b^* - g) \right\} \, d\gamma \, dt \geq 0
\]

(3.3)

for all \((f, g) \in H_p(0, T; \Gamma)\).

4. State equation

In this section, we study the well-posedness of the state equation (1.1) and deduce estimates for the state in terms of the control variables. Such estimates will be fundamental to study the regularity (continuity, differentiability) of the control-to-state mapping. Our strategy relies on Galerkin’s approximation method, by taking into account some useful results on elliptic equations and compactness arguments.

Let us introduce the notion of solution to the system (1.1), which should be understood in the weak sense, according to the next definition.
**Definition 4.1.** The weak solution of the system (1.1) is a divergence free function $y \in L^2(0, T; H^1(\Omega))$, satisfying the boundary condition 

$$y \cdot n = a \quad \text{on} \quad \Gamma_T$$

and being the solution of the integral equality

$$\int_{\Omega_T} \{-y \cdot \partial_t \psi + ((y \nabla) y) \cdot \psi + 2 D(y) : D(\psi)\} \, dx \, dt = \int_{\Gamma_T} (b - \alpha(y \cdot \tau))(\psi \cdot \tau) \, d\gamma \, dt + \int_{\Omega} y_0 \cdot \psi(0) \, dx$$

(4.1)

for any $\psi \in H^1(0, T; V)$ with $\psi(T) = 0$.

The well-posedness of the system (1.1) will be presented at the end of this section. Before we establish crucial intermediate results.

Let us introduce the function $a = \nabla h_a$, where $h_a$ is the solution of the system

$$\begin{cases}
-\Delta h_a = 0 & \text{in} \ \Omega, \\
\frac{\partial h_a}{\partial n} = a & \text{on} \ \Gamma.
\end{cases}$$

(4.2)

The function $a$ satisfies Calderon-Zygmund’s estimates

$$||a||_{C(\overline{\Omega})} \leq C||a||_{W^2_\sigma(\Omega)} \leq C_p||a||_{W^1_p(\Omega)},$$

$$||\partial_t a||_{L^2(\Omega_T)} \leq C||\partial_t a||_{W^{-1}_2(\Gamma)} \quad \text{a.e. on} \ (0, T).$$

(4.3)

where the constants $C_p$ depend on $2 < p < \infty$ (see [34], Theorem 9.9, p. 230 in [20] and Theorem 1.8, p. 12 & Theorem 1.10, p. 15 in [21]). Accounting the regularity (2.7) and the embedding theorem $H^1(0, T) \hookrightarrow C([0, T])$ (also we refer to Lemma 2.2) we have that

$$a \in L^2(0, T; C(\overline{\Omega})), \quad \partial_t a \in L^2(\Omega_T),$$

$$a \in C([0, T]; L^2(\Omega)).$$

(4.4)

The existence of solution for the system (4.6) will be shown by Galerkin’s method. There exists a sequence $\{e_k\}_{k=1}^\infty \subset H^3(\Omega)$, being a basis for $V$ and an orthonormal basis for $H$, which satisfies the Navier slip boundary condition

$$2D(e_k) n + \alpha e_k \cdot \tau = 0$$

(4.5)

on $\Gamma_T$ by Lemma 2.2. of [10] (see also Theorem 1 of [32]).

For any fixed $n = 1, 2, \ldots$, let $V_n = \text{span} \{e_1, \ldots, e_n\}$ and set $y_n = u_n + a$ with

$$u_n(t) = \sum_{k=1}^n c^{(n)}_k(t) \ e_k$$

being the solution of the integral equation

$$\int_{\Omega} \partial_t y_n \cdot \psi \, dx + \int_{\Omega} \{(y_n \cdot \nabla) y_n \} \cdot \psi + 2 D(y_n) : D(\psi)\} \, dx = \int_{\Gamma} (b - \alpha(y_n \cdot \tau))(\psi \cdot \tau) \, d\gamma, \quad \forall \psi \in V_n,$$

$$u_n(0) = u_{n,0}.$$

(4.6)

Here $u_{n,0}$ is the orthogonal projection of $u_0(x) = y_0(x) - a(0, x) \in H$ onto the space $V_n$.

In the following Proposition we will show the solvability of the system (4.6).
Proposition 4.1. Under the assumptions (2.6) the system (4.6) has a solution \( y_n = u_n + a_n \), such that
\[
\|u_n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|D(u_n)\|_{L^2(\Omega_T)}^2 + \|\nabla u_n\|_{L^2(\Gamma)}^2 \\
\leq C(\|u_n(0)\|_{L^2(\Omega)}^2 + ||(a, b)||_{H^s(0, T; \Gamma)}^2 + 1)\exp(C(\|a, b\|_{H^s(0, T; \Gamma)}^2 + 1))
\]
(4.7)
and
\[
\|\partial_t y_n\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq C(\|y_0\|_{L^2(\Omega)}^2 + ||(a, b)||_{H^s(0, T; \Gamma)}^2 + 1).
\]
(4.8)

Proof. The equation (4.6) defines a system of ordinary differential equations in \( \mathbb{R}^2 \) with locally Lipschitz nonlinearities. Hence there exists a local-in-time solution \( u_n \) in the space \( C([0, T_n]; V_n) \). The global-in-time existence of \( u_n \) follows from a priori estimate (4.7), which is valid for any \( n = 1, 2, \ldots \). Therefore we focus our attention on the deduction of the estimate (4.7).

By firstly writing the equation (4.6) in terms of \( u_n \) and \( a_n \), taking \( \psi = e_k \), multiplying by \( c_k(n) \) and summing on \( k = 1, \ldots, n \), we derive
\[
\frac{1}{2} \frac{d}{dt} \int \Omega |u_n|^2 \, dx + \int \Gamma |D(u_n)|^2 \, d\gamma + \int \Gamma \alpha(u_n \cdot \tau)^2 \, d\gamma \\
= \int \Gamma \left\{ -\frac{\alpha}{2} (u_n \cdot \tau)^2 + (b - \alpha(a \cdot \tau))(u_n \cdot \tau) \right\} \, d\gamma \\
- \int \Omega \left[ \partial_t a + ((u_n + a) \cdot \nabla) a \right] \cdot u_n \, dx \\
- 2 \int \Omega (a : D(u_n)) \, dx = I_1 + I_2 + I_3.
\]
(4.9)

Considering the inequality (2.11) for an appropriate \( \varepsilon > 0 \) and the inequalities (2.4)–(2.6), the terms \( I_1, I_2 \) and \( I_3 \) are estimated as follows
\[
I_1 \leq (\|a\|_{L^\infty(\Gamma)} + 1)\|u_n\|_{L^2(\Gamma)}^2 + \|b - \alpha(a \cdot \tau)\|_{L^2(\Gamma)}^2 \\
\leq C(\|a\|_{W_{\frac{1}{2}, \Gamma}^1}^2 + 1)\|u_n\|_{L^2(\Omega)}^2 + \frac{1}{3} |D(u_n)|_{L^2(\Omega)}^2 \\
+ C(\|b\|_{L^2(\Gamma)}^2 + \|a\|_{L^\infty(\Gamma)}^2 |a|_{W_{\frac{1}{2}, \Gamma}^1}^2),
\]
\[
I_2 \leq \left( \|\partial_t a\|_{L^2(\Omega)} + \|a\|_{C(\overline{\Omega})} \|\nabla a\|_{L^2(\Omega)} \right) \|u_n\|_{L^2(\Omega)} + \|\nabla a\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)}^2 \\
\leq \left( \|\partial_t a\|_{W_{\frac{1}{2}, \Gamma}^1} + \|a\|_{W_{\frac{1}{2}, \Gamma}^1} \right) \|u_n\|_{L^2(\Omega)} + \|\nabla a\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)}^2 \\
+ C |a|_{W_{\frac{1}{2}, \Gamma}^1}^2 \|u_n\|_{L^2(\Omega)} + \frac{1}{3} |D(u_n)|_{L^2(\Omega)}^2
\]
and
\[
I_3 \leq C |D(a)|_{L^2(\Omega)} + \frac{1}{3} |D(u_n)|_{L^2(\Omega)}^2 \\
\leq C |a|_{W_{\frac{1}{2}, \Gamma}^1} + \frac{1}{3} |D(u_n)|_{L^2(\Omega)}^2.
\]

Combining the estimates of the terms \( I_1, I_2 \) and \( I_3 \) and (4.9), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + \int \Omega |D(u_n)|^2 \, dx + \int \Gamma \alpha(u_n \cdot \tau)^2 \, d\gamma \\
\leq h(t)(\|u_n\|_{L^2(\Omega)}^2 + \|u_n\|_{L^2(\Omega)} + 1)
\]
with
\[
h(t) = C \left[ 1 + (1 + \|u_n\|_{L^2(\Omega)}^2) \right] \|a|_{W_{\frac{1}{2}, \Gamma}^1}^2 + \|\partial_t a\|_{W_{\frac{1}{2}, \Gamma}^1} + \|b\|_{L^2(\Gamma)}^2
\]
which belongs to $L_1(0,T)$ due to (4.3) and (2.7). Applying Gronwall’s inequality, we deduce (4.7).

Now we show (4.3). The integration by parts gives

$$
\int_{\Omega} ((y_n \cdot \nabla) y_n) \cdot \psi \, dx = \int_{\Gamma} a(y_n \cdot \psi) \, d\gamma - \int_{\Omega} ((y_n \cdot \nabla) \psi) \cdot y_n \, dx.
$$

Therefore, the identity (4.6) permit to deduce

$$(\partial_t y_n, \psi)_{L^2(\Omega)} \leq C \left( \|a\|_{L^\infty(\Gamma)} \|y_n\|_{H^1(\Omega)} + \|y_n\|_{L^4(\Omega)}^2 \right) \|\psi\|_{H^1(\Omega)}
$$

$$
+ \|D(y_n)\|_{L^2(\Omega)} \|D(\psi)\|_{L^2(\Omega)}
$$

$$
+ \|b\|_{L^2(\Gamma)} + \|\sqrt{\sigma} y_n\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)}.
$$

gives

$$
\|\partial_t y_n\|_{H^{-1}(\Omega)} = \sup_{\psi \in H^1_0(\Omega)} \{ (\partial_t y_n, \psi)_{L^2(\Omega)} : \|\psi\|_{H^1(\Omega)} = 1 \}
$$

$$
\leq C \left( \|a\|_{L^\infty(\Gamma)} \|y_n\|_{H^1(\Omega)} + \|y_n\|_{L^4(\Omega)}^2 \right) + \|D(y_n)\|_{L^2(\Omega)} + \|b\|_{L^2(\Gamma)} + \|\sqrt{\sigma} y_n\|_{L^2(\Gamma)}.
$$

taking into account (4.4) we have

$$
\int_0^T \left( \|y_n\|_{L^2(\Omega)}^2 \right)^2 \, dt \leq \int_0^T \left( \|y_n\|_{L^2(\Omega)}^{1/2} \right)^4 \, dt
$$

$$
\leq C \|y_n\|_{L^\infty(0,T;L^2(\Omega))}^2 \|y_n\|_{L^2(0,T;H^1(\Omega))} + \|y_n\|_{L^\infty(0,T;L^2(\Omega))}^2
$$

that yields (4.8) by (4.3) and (4.4).

\[ \square \]

**Theorem 4.1.** Assume that the hypothesis (2.7) hold, then the system (1.1) has a unique weak solution $y$, such that

$$
y \in C ([0,T]; L^2(\Omega)) \cap L^2 (0,T; H^1(\Omega)), \quad \partial_t y \in L^2 (0,T; H^{-1}(\Omega)). (4.10)
$$

Moreover, the following estimates hold

$$
\|y\|_{C([0,T];L^2(\Omega))} + \|y\|_{L^2(0,T;H^1(\Omega))} + \|\sqrt{\sigma} y\|_{L^2(\Gamma)}
$$

$$
\leq C \left( \|y_0\|_{L^2(\Omega)}^2 + \|(a,b)\|_{H_p(0,T;\Gamma)}^2 \right) + 1 + \exp \left( C \|(a,b)\|_{H_p(0,T;\Gamma)}^2 \right), (4.11)
$$

$$
\|\partial_t y\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \|y_0\|_{L^2(\Omega)}^2 + \|(a,b)\|_{H_p(0,T;\Gamma)}^2 + 1. (4.12)
$$

**Proof.** The estimates (4.3), (4.4), (4.7) and (4.8) imply that the sequence of the functions

$$
u_n \in L^2 (0,T; V), \quad \partial_t u_n \in L^2 (0,T; H^{-1}(\Omega)),
$$

are uniformly bounded, for $n = 1, 2, \ldots, \infty$, so we can apply the compactness argument of (II) and take a suitable subsequence of $\{u_n\}$, such that

$$
y_n = u_n + \alpha \rightharpoonup y = u + a \quad \text{weakly in} \quad L^\infty (0,T; L^2(\Omega)) \cap L^2 (0,T; H^1(\Omega)),
$$

$$
\partial_t y_n \rightharpoonup \partial_t y \quad \text{weakly in} \quad L^2 (0,T; H^{-1}(\Omega)),
$$

$$
y_n \rightarrow y \quad \text{strongly in} \quad L^2 (\Omega T).
$$

Hence integrating over the time interval $(0,T)$ and passing to the limit as $n \rightarrow \infty$ in (4.10), we deduce that the function $y = u + a$ is a weak solution of (1.1) in the sense of the definition (4.1).

The properties $y \in L^2 (0,T; H^1(\Omega))$, $\partial_t y \in L^2 (0,T; H^{-1}(\Omega))$ and Lemma 2.2 yield

$$
y \in C ([0,T]; L^2(\Omega)),
$$
which gives a meaning for the initial condition for $y$ in (1.1). Finally, accounting (1.3)-(1.4), we derive (4.11)-(4.12).

The uniqueness result is a direct consequence of Proposition 5.1 that we will show in the following section. □

5. Lipschitz Continuity of the Control-to-State Mapping

This section is devoted to the study of the Lipschitz continuity to the state $y$ as a function of the control variables $a, b$. This regularity result will be necessary in Section 7 in order to analyse the Gâteaux differentiability of this function.

**Proposition 5.1.** Let $(y_1, p_1)$ and $(y_2, p_2)$ be two weak solutions for the system (1.1) with two corresponding boundary conditions $a_1, b_1$ and $a_2, b_2$, but with the same initial condition $y_0$. Denoting by $\hat{y} = y_1 - y_2$, we have

$$
\|\hat{y}\|_{C([0,T], L^2(\Omega))} + \|D(\hat{y})\|_{L^2(\Omega)} + \|\sqrt{\alpha} \hat{y}\|_{L^2(\Omega_T)} \leq C\|\hat{\alpha}\|_{H^1(0,T; \Gamma)}^2
$$

with $\hat{\alpha} = a_1 - a_2$.

**Proof.** Let us denote $\hat{a} = \nabla h_{\hat{a}}$, where $h_{\hat{a}}$ is the solution of the system (1.2) with $a = \hat{a}$.

We easily verify that the functions $w = \hat{y} - \hat{a}, \hat{\rho} = p_1 - p_2$ satisfy the system

$$
\begin{aligned}
\partial_t w + (\nabla w) \cdot \nabla w - \nabla \hat{\rho} = \Delta w + F, \quad \text{div } w = 0 \quad \text{in } \Omega_T, \\
w \cdot n = 0, \quad [2D(w) n + \alpha w] \cdot \tau = \hat{b} \quad \text{on } \Gamma_T, \\
w(0, x) = \hat{a} \cdot n = 0 \quad \text{in } \Omega
\end{aligned}
$$

with $F = -\partial_t \hat{a} + \Delta \hat{a} - ((\nabla \hat{a} + \hat{a}) \cdot \nabla) y_1 - (\nabla \hat{a} + \hat{a}) \cdot \nabla \hat{a}$ and $\hat{b} = \hat{b} - [2D(\hat{a}) n + \alpha \hat{a} \cdot \tau]$.

Therefore multiplying the first equation in (5.2) by $w$ and integrating over $\Omega$, we obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w|^2 \, dx + 2 \int_{\Omega} |D(w)|^2 \, dx + \int_{\Gamma} \alpha (w \cdot \tau)^2 \, d\gamma \\
= \int_{\Omega} \left\{ - \frac{\alpha_2}{2}(w \cdot \tau)^2 + (\hat{b} - \alpha (\hat{a} \cdot \tau))(w \cdot \tau) \right\} \, dx \\
- \int_{\Omega} [\partial_t \hat{a} + ((\nabla \hat{a} + \hat{a}) \cdot \nabla) y_1] \cdot w \, dx \\
- \int_{\Omega} (\nabla \hat{a} \cdot \nabla) w \, dx - \int_{\Omega} 2D(\hat{a}) : D(w) \, dx \\
= J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

Let us estimate the term $J_1$. By (2.5), (2.7), (4.4) and the embedding $W_{p-\frac{2}{p}}(\Gamma) \hookrightarrow L^\infty(\Gamma)$, we deduce

$$
J_1 \leq (\alpha_2 \|L^\infty(\Gamma) + 1)\|w\|^2_{L^2(\Gamma)} + C \left( \|\hat{\alpha}\|^2_{L^2(\Gamma)} + \|\alpha\|^2_{L^\infty(\Gamma)} (\hat{a} \cdot \tau)^2_{L^2(\Gamma)} \right)
\leq f_1(t)\|w\|^2_{L^2(\Omega)} + \frac{1}{4} \|D(w)|^2_{L^2(\Omega)} + C \left( \|\hat{b}\|^2_{L^2(\Gamma)} + \|\hat{a}\|^2_{W^{-1}_{p-\frac{2}{p}}(\Gamma)} \right)
$$
with \( f_1(t) = C(\|a_2\|_{W^1_\infty(\Gamma)} + 1)^2 \in L_1(0, T) \) by (2.7). The term \( J_2 \) is estimated as follows

\[
J_2 \leq \left( \|\partial t \tilde{a}\|_{L^2(\Omega)} + \|\tilde{a}\|_{C^{1+\alpha}(\Omega)} \|\nabla y_1\|_{L^2(\Omega)} \right) \|w\|_{L^2(\Omega)} + \|\nabla y_1\|_{L^2(\Omega)} \|w\|_{L^4(\Omega)}^2
\]

\[
\leq \left( \|\partial t \tilde{a}\|_{L^2(\Omega)} + \|\tilde{a}\|_{C^{1+\alpha}(\Omega)} \sqrt{J_2(\Gamma)} \|w\|_{L^2(\Omega)}
+ \|\nabla y_1\|_{L^4(\Omega)} \|w\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \right) \leq f_2(t)\|w\|_{L^2(\Omega)}^2 + C \left( \|\partial t \tilde{a}\|^2_{W^1_2(\Gamma)} + \|\tilde{a}\|^2_{W^1_2(\Gamma)} + \frac{1}{4} \|D(w)\|^2_{L^2(\Omega)} \right)
\]

with \( f_2(t) = C(1 + \|\nabla y_1\|_{L^2(\Omega)})^2 \in L_1(0, T) \) by (4.10). Using (2.3) for \( v = y_2 \) and (2.4) for \( v = w \), we have

\[
J_3 \leq \|\nabla y_2\|_{L^4(\Omega)} \|\nabla \tilde{a}\|_{L^2(\Omega)} \|w\|_{L^4(\Omega)} \leq C \|\nabla \tilde{a}\|_{L^2(\Omega)} \|\nabla y_2\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}^{1/2}
\]

\[
\leq C \|\nabla \tilde{a}\|^2_{L^2(\Omega)} + \|\nabla y_2\|_{L^4(\Omega)} \|w\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \leq f_3(t)\|w\|_{L^2(\Omega)}^2 + C \|\tilde{a}\|^2_{W^1_2(\Gamma)} + \frac{1}{4} \|D(w)\|^2_{L^2(\Omega)}
\]

with \( f_3(t) = C\|\nabla y_2\|_{L^2(\Omega)} \|\nabla \tilde{a}\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \), we have

\[
J_4 \leq C \|D(\tilde{a})\|_{L^2(\Omega)}^2 + \frac{1}{4} \|D(w)\|_{L^2(\Omega)}^2 \leq C \|\tilde{a}\|^2_{W^1_2(\Gamma)} + \frac{1}{4} \|D(w)\|_{L^2(\Omega)}^2.
\]

Combining the above deduced estimates of the terms \( J_1, J_2, J_3, J_4 \) and (5.3), we obtain

\[
\frac{d}{dt}\|w\|_{L^2(\Omega)}^2 + \int_{\Omega} |D(w)|^2 dx + \int_{\Gamma} \alpha(\omega + \tau)^2 d\gamma \leq f(t)\|w\|_{L^2(\Omega)}^2
\]

\[
+ C \left\{ \|\tilde{a}\|_{L^2(\Gamma)} + \|\partial t \tilde{a}\|_{W^1_2(\Gamma)} + \|\tilde{a}\|_{W^1_2(\Gamma)} \right\}
\]

with \( f(t) = f_1(t) + f_2(t) + f_3(t) \) and \( f_4(t) = L_1(0, T) \). Applying Gronwall’s inequality, we deduce

\[
\|w\|_{L^2(0,T; L^2(\Omega))}^2 + \|D(w)\|_{L^2(\Omega^2)}^2 \leq C \left\{ \|\tilde{a}\|_{L^2(\Omega)}^2 + \int_{0}^{T} \left( \|\tilde{a}\|_{L^2(\Gamma)}^2 + \|\partial t \tilde{a}\|_{W^1_2(\Gamma)} + \|\tilde{a}\|_{W^1_2(\Gamma)} \right) dt \right\}, (5.4)
\]

Therefore, taking into account that \( \tilde{w} = w + \tilde{a} \) and (1.3) to (1.5), we derive (5.1). □

6. Linearized state equation

This section deals with the well-posedness of the linearized state equation. Let us mention that the existence and uniqueness of the linearized state is of main importance to analyse the Gateaux derivative of the control-to-state mapping. Moreover, its regularity plays a key role in the deduction of the duality property, relating the linearized state with the adjoint state. We recall that such duality relation allows to write the first order derivative of the cost functional in terms of the adjoint state, yielding the so-called first order optimality condition.
Let us consider the solution \( y \) of the state system (1.1), then the corresponding linearized system reads as follows

\[
\begin{aligned}
&\partial_t z + (z \cdot \nabla) y + (y \cdot \nabla) z + \nabla \pi = \Delta z, \quad \text{div } z = 0, \quad \text{in } \Omega_T, \\
z \cdot n = f, \quad [2D(z) n + \alpha z] \cdot \tau = g \quad \text{on } \Gamma_T, \\
z(0) = 0 \quad \text{in } \Omega
\end{aligned}
\]

(6.1)

with the boundary data

\[
(f, g) \in H_p(0, T; \Gamma) \quad \text{with } p \in (2, +\infty) \quad \text{as in (2.7). (6.2)}
\]

Let us define \( f = \nabla h_f \) with \( h_f \) being the solution of the system (4.2). Then the function \( f \) satisfies the estimates

\[
||f||_{C(\Omega)} \leq C ||f||_{W_2^1(\Omega)} ,
\]

\[
||\partial_t f||_{L_2(\Omega_T)} \leq C ||\partial_t f||_{W_2^{-1}(\Gamma)} \quad \text{a.e. on } (0, T),
\]

(6.3)

and

\[
f \in L_2(0, T; C(\Omega)), \quad \partial_t f \in L_2(\Omega_T),
\]

\[
f \in C([0, T]; L_2(\Omega)).
\]

(6.4)

**Definition 6.1.** The weak solution of the system (6.1) is the divergence free function \( z \in L_2(0, T; H^1(\Omega)) \) satisfying the boundary condition

\[
z \cdot n = f \quad \text{on } \Gamma_T
\]

and being the solution of the integral equality

\[
\int_{\Omega_T} \left\{ -z \cdot \partial_t \psi + \left[ (z \cdot \nabla) y + (y \cdot \nabla) z \right] \cdot \psi + 2 D(z) : D(\psi) \right\} \, dx \, dt
\]

\[
= \int_{\Gamma_T} (g - \alpha(z, \tau))(\psi \cdot \tau) \, d\gamma \, dt,
\]

which is valid for all \( \psi \in H^1(0, T; V) : \psi(T) = 0. \)

In what follows we will establish the solvability of the system (6.1)

**Proposition 6.1.** Under the assumptions (6.2) there exists a unique weak solution \( z \) for the system (6.1), such that

\[
z \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; H^1(\Omega)),
\]

\[
\partial_t z \in L_2(0, T; H^{-1}(\Omega))
\]

and

\[
||z||_{C([0, T]; L_2(\Omega))}^2 + ||D(z)||_{L_2(\Omega_T)}^2 + ||\sqrt{\alpha z}||_{L_2(\Gamma_T)}^2 \leq C ||(f, g)||_{H_p(0, T; \Gamma)}^2.
\]

(6.5)

**Proof.** Let us consider as in the Section 4 the subspace \( V_n = \text{span} \{ e_1, \ldots, e_n \} \) of \( V \) and the sequence \( \{ e_k \}_{k=1}^\infty \subset H^3(\Omega) \) being the orthogonal basis for \( V \) and the orthonormal basis for \( H \), satisfying the Navier slip boundary condition (4.5).

For any fixed \( n = 1, 2, \ldots \), we define \( z_n = z_n + f \), where

\[
z_n(t) = \sum_{k=1}^n r_k^{(n)}(t) \, e_k
\]
is the solution for the differential equation

\[
\int_{\Omega} \partial_t \mathbf{z}_n \cdot \psi \, dx + \int_{\Omega} \{ [(\mathbf{z}_n \cdot \nabla) \mathbf{y} + (\mathbf{y} \cdot \nabla) \mathbf{z}_n] \cdot \psi + 2 D(\mathbf{z}_n) : D(\psi) \} \, dx = \int_{\Gamma} (g - \alpha (\mathbf{z}_n \cdot \tau)) (\psi \cdot \tau) \, d\gamma, \quad \forall \psi \in V_\alpha,
\]

\[
z_n(0) = \mathbf{z}_{n,0}.
\]

(6.6)

Here \(\mathbf{z}_{n,0}\) is the orthogonal projections in \(H\) of \(\bar{z}_0(\mathbf{x}) = \mathbf{z}_0(\mathbf{x}) - \mathbf{f}(0, \mathbf{x})\) onto the space \(V_\alpha\). Since the equation (6.6) is a system of linear ordinary differential equations in \(\mathbb{R}^2\), there exists a global-in-time solution \(\tilde{\mathbf{z}}_n\) in the space \(C([0, T]; V_\alpha)\).

Let us show the validity of (6.6) for \(\mathbf{z} = \mathbf{z}_n\). If we write the equation (6.6) in terms of \(\tilde{\mathbf{z}}_n\) and choose the test function \(\psi = \tilde{\mathbf{z}}_n\), we deduce

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\tilde{\mathbf{z}}_n|^2 \, dx + 2 \int_{\Omega} |D(\tilde{\mathbf{z}}_n)|^2 \, dx + \int_{\Gamma} \alpha (\tilde{\mathbf{z}}_n \cdot \tau)^2 \, d\gamma
\]

\[
= \int_{\Gamma} \left\{ -\frac{\alpha}{2} (\tilde{\mathbf{z}}_n \cdot \tau)^2 + (g - \alpha (\mathbf{f} \cdot \tau)) (\tilde{\mathbf{z}}_n \cdot \tau) \right\} \, d\gamma
\]

\[
- \int_{\Omega} \partial_t \mathbf{f} + ((\tilde{\mathbf{z}}_n + \mathbf{f}) \cdot \nabla) \mathbf{y} \cdot \tilde{\mathbf{z}}_n \, dx
\]

\[
- \int_{\Omega} [(\mathbf{y} \cdot \nabla) \mathbf{f}] \cdot \tilde{\mathbf{z}}_n \, dx - \int_{\Omega} 2 D(\mathbf{f}) : D(\tilde{\mathbf{z}}_n) \, dx
\]

\[
= J_1 + J_2 + J_3 + J_4.
\]

(6.7)

Let us estimate the terms \(J_1, J_2\) and \(J_3\). We have

\[
J_1 \leq (\|f\|_{L^\infty(\Gamma)} + 1) \|\tilde{\mathbf{z}}_n\|^2_{L^2(\Gamma)} + C \left( \|g\|^2_{L^2(\Gamma)} + \|\alpha\|^2_{L^\infty(\Gamma)} (\|\mathbf{f} \cdot \tau\|^2_{L^2(\Gamma)}) \right)
\]

\[
\leq C \left( \|f\|_{W^{-1/2, 1}_{p, \Gamma}} + 1 \right) \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)} \|\nabla \tilde{\mathbf{z}}_n\|_{L^2(\Omega)} + C \left( \|g\|^2_{L^2(\Gamma)} + \|\mathbf{f}\|^2_{W^{-1/2, 1}_{p, \Gamma}} \right)
\]

\[
\leq h_1(t) \|\tilde{\mathbf{z}}_n\|^2_{L^2(\Omega)} + \frac{1}{4} \|D(\tilde{\mathbf{z}}_n)|^2_{L^2(\Omega)} + C \left( \|g\|^2_{L^2(\Gamma)} + \|\mathbf{f}\|^2_{W^{-1/2, 1}_{p, \Gamma}} \right)
\]

with \(h_1(t) = C (\|a\|_{W^{-1/2, 1}_{p, \Gamma}} + 1)^2 \in L_1(0, T)\) by (2.7).

\[
J_2 \leq \left( \|\partial_t \mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{C(\bar{\Omega})} \|\nabla \mathbf{y}\|_{L^2(\Omega)} \right) \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)}
\]

\[
\quad + \|\nabla \mathbf{y}\|_{L^2(\Omega)} \|\tilde{\mathbf{z}}_n\|^2_{L^2(\Omega)} \leq \left( \|\partial_t \mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{C(\bar{\Omega})} \right) \sqrt{h_2(t)} \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)}
\]

\[
\quad + \|\nabla \mathbf{y}\|_{L^2(\Omega)} \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)} \|\nabla \tilde{\mathbf{z}}_n\|_{L^2(\Omega)}
\]

\[
\leq h_2(t) \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)} + \frac{1}{4} \|D(\tilde{\mathbf{z}}_n)|^2_{L^2(\Omega)} + \left( \|\partial_t \mathbf{f}\|^2_{W^{-1/2, 1}_{p, \Gamma}} + \|\mathbf{f}\|^2_{W^{-1/2, 1}_{p, \Gamma}} \right)
\]

with \(h_2(t) = C (1 + \|\nabla \mathbf{y}\|_{L^2(\Omega)})^2 \in L_1(0, T)\) by (4.11). Reasoning as in Proposition 5.1 we derive

\[
J_3 \leq \|\mathbf{y}\|_{L^4(\Omega)} \|\nabla \mathbf{f}\|_{L^4(\Omega)} \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)} \|\nabla \tilde{\mathbf{z}}_n\|_{L^2(\Omega)} \leq \|\mathbf{y}\|_{L^4(\Omega)} \|\mathbf{f}\|_{L^4(\Omega)} \|\tilde{\mathbf{z}}_n\|^{1/2}_{L^2(\Omega)} \|\nabla \tilde{\mathbf{z}}_n\|^{1/2}_{L^2(\Omega)}
\]

\[
\leq \|\mathbf{y}\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{L^4(\Omega)} \|\tilde{\mathbf{z}}_n\|_{L^2(\Omega)} \|\nabla \tilde{\mathbf{z}}_n\|_{L^2(\Omega)}
\]

\[
\leq \|\mathbf{f}\|^2_{W^{-1/2, 1}_{p, \Gamma}} + h_3(t) \|\tilde{\mathbf{z}}_n\|^2_{L^2(\Omega)} + \frac{1}{4} \|D(\tilde{\mathbf{z}}_n)|^2_{L^2(\Omega)}
\]
with \( h_3(t) = C\|y\|^4_{L^4(\Omega)} \leq C \left( \|y\|_{L^2(\Omega)}^{1/2} \|\nabla y\|_{L^2(\Omega)}^{1/2} + \|y\|_{L^2(\Omega)} \right)^4 \in L^1(0,T) \) by (2.4) and (4.11). The last term \( J_4 \) is estimated as
\[
J_4 \leq C \|D(f)\|^2_{L^2(\Omega)} + \frac{1}{4} \|D(\tilde{z}_n)\|^2_{L^2(\Omega)}
\leq C \|f\|^2_{W^{1,1}_{p,\frac{1}{p}}(\Gamma)} + \frac{1}{4} \|D(\tilde{z}_n)\|^2_{L^2(\Omega)}.
\]

Therefore the above deduced estimates of the terms \( J_1, J_2, J_3, J_3 \) and (6.7) imply the inequality
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{z}_n\|^2_{L^2(\Omega)} + \int_\Omega |D(\tilde{z}_n)|^2 \, dx + \int_\Gamma \alpha (\tilde{z}_n \cdot \tau)^2 \, d\gamma \leq h(t) \|\tilde{z}_n\|^2_{L^2(\Omega)} + \int_\Gamma \bigg\{ \|g\|^2_{L^2(\tau)} + \|D(\tilde{z}_n)\|^2_{L^2(\Omega)} + \|f\|^2_{W^{1,1}_{p,\frac{1}{p}}(\Omega)} \bigg\}
\]
with \( h(t) = h_1(t) + h_2(t) + h_3(t) \in L^1(0,T) \). Hence Gronwall’s inequality gives
\[
\|\tilde{z}_n\|^2_{L^2(0,T;L^2(\Omega))} + \|D(\tilde{z}_n)\|^2_{L^2(\Omega)} + \|\sqrt{\alpha} \tilde{z}_n\|^2_{L^2(\Omega)} \leq C \left\{ \|f(0,x)\|^2_{L^2(\Omega)}
\right.
\]
\[
+ \int_0^T \left\{ \|g\|^2_{L^2(\Omega)} + \|\partial_t f\|^2_{W^{1,1}_{p,\frac{1}{p}}(\Omega)} + \|f\|^2_{W^{1,1}_{p,\frac{1}{p}}(\Omega)} \right\} \, dt \bigg\}. \tag{6.8}
\]
This estimate and (6.6) permit to obtain that the sequence
\[
\tilde{z}_n \to \tilde{z} \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)),
\]
\[
\partial_t \tilde{z}_n \to \partial_t \tilde{z} \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)),
\]
\[
\tilde{z}_n \to \tilde{z} \quad \text{strongly in} \quad L^2(\Omega_T). \tag{6.9}
\]
Passing on \( n \to \infty \) in (6.5), we deduce that
\[
\tilde{z} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \partial_t \tilde{z} \in L^2(0,T;H^{-1}(\Omega)).
\]
Hence \( z = \tilde{z} + f \) is the weak solution of (6.1), which satisfies (6.5) by Lemma 2.2 (6.8) and (6.2)-(6.4). The uniqueness result follows from the linearity of the system by taking into account the estimates (6.5). \( \square \)

7. Gâteaux differentiability of the control-to-state mapping

To deduce the necessary first-order optimality conditions, we should study the the Gâteaux differentiability of the cost functional \( J \), which requires the determination of the Gâteaux derivative of the control-to-state mapping. The goal of this section is to show that the Gâteaux derivative of the control-to-state mapping \( (a, b) \to y \), at a point \( (a, b) \), in any direction \( (f, g) \), exists and is given by the solution of the linearized system (6.1).

**Proposition 7.1.** For given \( (a, b) \) and \( y_0 \) satisfying (2.7) and \( (f, g) \in H_p(0,T;\Gamma) \), let us consider
\[
a_{\varepsilon} = a + \varepsilon f, \quad b_{\varepsilon} = b + \varepsilon g \quad \forall \varepsilon \in (0,1).
\]
If \((y, \pi)\) and \((y_\varepsilon, \pi_\varepsilon)\) are the solutions of (1.1) corresponding to \((a, b, y_0)\) and \((ae, b_\varepsilon, y_0)\), respectively, then the following representation holds

\[
y_\varepsilon = y + \varepsilon z + \varepsilon \delta_\varepsilon \quad \text{with} \quad \lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \|\delta_\varepsilon\|_{L^2(\Omega)}^2 = 0, \tag{7.1}
\]

where \(z \in C([0, T]; H) \cap L_2(0, T; V)\) is the solution of (6.1) satisfying the estimates (6.3).

Proof. It is straightforward to verify that \(z_\varepsilon = \frac{y_\varepsilon - y}{\varepsilon}\) and \(\pi_\varepsilon = \frac{\pi - \pi_0}{\varepsilon}\) satisfy the system

\[
\begin{aligned}
\partial_t z_\varepsilon + (y \cdot \nabla) z_\varepsilon + (z_\varepsilon \cdot \nabla) y_\varepsilon - \nabla \pi_\varepsilon &= \Delta z_\varepsilon, \quad \text{div} z_\varepsilon = 0 \quad \text{in} \quad \Omega_T, \\
z_\varepsilon \cdot n &= f, \quad [2D(z_\varepsilon) \cdot n + \alpha z_\varepsilon] \cdot \tau = g \quad \text{on} \quad \Gamma_T, \\
z_\varepsilon(0, x) &= 0 \quad \text{in} \quad \Omega
\end{aligned}
\tag{7.2}
\]

and \(\delta_\varepsilon = z_\varepsilon - z\) fulfills the system

\[
\begin{aligned}
\partial_t \delta_\varepsilon + (y \cdot \nabla) \delta_\varepsilon + (\delta_\varepsilon \cdot \nabla) y_\varepsilon + (z \cdot \nabla)(y_\varepsilon - y) - \nabla (\pi_\varepsilon - \pi) &= \Delta \delta_\varepsilon, \quad \text{div} \delta_\varepsilon = 0 \quad \text{in} \quad \Omega_T, \\
\delta_\varepsilon \cdot n &= 0, \quad [2D(\delta_\varepsilon) \cdot n + \alpha \delta_\varepsilon] \cdot \tau = 0 \quad \text{on} \quad \Gamma_T, \\
\delta_\varepsilon(0, x) &= 0 \quad \text{in} \quad \Omega. 
\end{aligned}
\tag{7.3}
\]

Multiplying the first equation of the last system by \(\delta_\varepsilon\) and integrating over \(\Omega\), we deduce

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\delta_\varepsilon|^2 \, dx + 2 \int_{\Omega} |D(\delta_\varepsilon)|^2 \, dx + \int_{\Gamma} \alpha(|\delta_\varepsilon| \cdot \tau)^2 \, d\gamma = - \int_{\Gamma} \frac{\alpha}{2} (\delta_\varepsilon \cdot \tau)^2 \, d\gamma
\leq I_1 + I_2 + I_3. \tag{7.4}
\]

Applying the inequalities (2.1), (2.4)-(2.6) and (4.4), the following estimates hold

\[
I_1 \leq C \|a\|_{L^\infty(\Gamma)} \|\delta_\varepsilon\|_{L^2(\Gamma)}^2 \leq C \|a\|^2_{W^{-1,2}_{p \varepsilon}(\Gamma)} \|\delta_\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{3} \|D(\delta_\varepsilon)\|_{L^2(\Omega)}^2,
\]

\[
I_2 \leq C \|y_\varepsilon\|_{H^1(\Omega)} \|\delta_\varepsilon\|_{L^2(\Omega)}^2 \leq C \|y_\varepsilon\|^2_{H^1(\Omega)} \|\delta_\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{3} \|D(\delta_\varepsilon)\|_{L^2(\Omega)}^2
\]

and

\[
I_3 \leq C \|y_\varepsilon - y\|_{H^1(\Omega)} \|z\|_{L^4(\Omega)} \|\delta_\varepsilon\|_{L^4(\Omega)} \\
\leq C \|y_\varepsilon - y\|^2_{H^1(\Omega)} + C \|z\|^4_{L^4(\Omega)} \|\delta_\varepsilon\|_{L^2(\Omega)} + \frac{1}{3} \|D(\delta_\varepsilon)\|^2_{L^2(\Omega)}.
\]

Then we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\delta_\varepsilon|^2 \, dx + \int_{\Omega} |D(\delta_\varepsilon)|^2 \, dx + \int_{\Gamma} \alpha(|\delta_\varepsilon| \cdot \tau)^2 \, d\gamma \\
\leq C \|f(t)\|_{L^2(\Omega)} \|\delta_\varepsilon\|_{L^2(\Omega)}^2 + C \|y_\varepsilon - y\|_{H^1(\Omega)}^2
\]

with \(f(t) = (\|a\|^2_{W^{-1,2}_{p \varepsilon}(\Gamma)} + \|y_\varepsilon\|^2_{H^1(\Omega)} + \|z\|^4_{L^4(\Omega)}) \in L_1(0, T)\) by (2.7), (4.11) and

\[
\|z\|_{L^2(\Omega)}^2 \leq \left( \|z\|_{L^2(\Omega)}^{1/2} \|\nabla z\|_{L^2(\Omega)}^{1/2} + \|z\|_{L^2(\Omega)} \right)^4 \in L_1(0, T) \tag{7.5}
\]

by (2.4), (6.3).
Applying Gronwall’s inequality and using (2.7), we deduce
\[ \| \delta \varepsilon \|_{L^2_\infty(0,T;L^2(\Omega))} + \| D(\delta \varepsilon) \|_{L^2(\Omega_T)} + \| \sqrt{\varepsilon} \delta \varepsilon \|_{L^2(\Omega_T)} \leq C \varepsilon^2 \| y - \tilde{y} \|_{L^2_0(0,T;H^1(\Omega))} \]
\[ \leq C \varepsilon^2 \| (f,g) \|_{L^2(0,T;L^2(\Omega))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \tag{7.6} \]
according to (5.1) and (1.3). On the other hand, using the same reasoning as for the state and linearized equation and the above estimates, we can also deduce that
\[ \| (\partial_t \delta, \psi) \|_{L^2(\Omega_T)} \leq \beta(t) \| \psi \|_{H^1(\Omega)}, \]
with \( \beta(t) \in L^2(0,T), \) which gives
\[ \| (\partial_t \delta) \|_{L^2(0,T;H^{-1}(\Omega))} < \infty. \tag{7.7} \]
Finally, (7.6) and (7.7) yield (7.1).

As a direct consequence of Proposition 7.1, we easily derive the following result on the variation for the cost functional (3.1).

**Proposition 7.2.** Assume that \((a, b), (f, g), y_0, z\) and
\[ a_\varepsilon = a + \varepsilon f, \quad b_\varepsilon = b + \varepsilon g, \quad \forall \varepsilon \in (0,1) \]
satisfy the assumptions of Proposition 7.1. Then we have
\[ J(a_\varepsilon, b_\varepsilon, y_\varepsilon) = J(a, b, y) + \varepsilon \left\{ \int_{\Omega_T} (y - y_d) \cdot z \, dx \, dt + \int_{\Gamma_T} (\lambda_1 af + \lambda_2 bg) \, d\gamma \, dt \right\} + o(\varepsilon), \]
where \( y, y_\varepsilon \) are the solutions of (1.1), corresponding to \((a, b, y_0), (a_\varepsilon, b_\varepsilon, y_0)\) and \( z \) is the solution of (7.1).

8. Adjoint equation

This section is devoted to the study of the adjoint system. The existence and uniqueness of the solution is shown by the same approach that we have considered to study the state and linearized state equations. Namely, we will use Galerkin’s approximations and compactness arguments.

Let \( y \) be the solution of the state equation (1.1) corresponding to the given data \((a, b, y_0)\). The adjoint system is given by
\[
\begin{cases}
-\partial_t p - 2D(p)y + \nabla \pi = \Delta p + U, & \text{div } p = 0 \quad \text{in } \Omega_T, \\
p \cdot n = 0 & \text{on } \Gamma_T, \\
p(T) = 0 & \text{in } \Omega.
\end{cases}
\tag{8.1}
\]

**Definition 8.1.** A function \( p \in L^2(0,T;V) \) is a weak solution of (8.1) if the integral equality
\[
\int_{\Omega_T} \{ p \cdot \partial_t \phi - (2D(p)y) \cdot \phi + 2D(p) : D(\phi) - U \cdot \phi \} \, dx \, dt
\]
\[
= -\int_{\Gamma_T} (\alpha + a)(p \cdot \tau) \phi \, d\gamma \, dt \tag{8.2}
\]
is valid for all \( \phi \in H^1(0,T;V) : \phi(0) = 0. \)

**Proposition 8.1.** Assume that \( U \in L^2(\Omega_T). \) Under the assumptions (2.7) there exists a unique weak solution \((p, \pi)\) for the system (8.1), such that
\[ p \in C([0,T];H) \cap L^2(0,T;V), \quad \pi \in H^{-1}(0,T;L^2(\Omega)). \]
Moreover, the following estimate holds
\[ \| p \|_{C([0,T];L^2(\Omega))} + \| D(p) \|_{L^2(\Omega_T)} + \| \sqrt{\alpha} p \|_{L^2(\Gamma_T)} \leq C \| U \|_{L^2(\Omega_T)}. \tag{8.3} \]
Proof. First, let us notice that according to p. 49-50 of [25] there exists a sequence \( \{ \tilde{e}_k \}_{k=1}^{\infty} \subset H^3(\Omega) \), being a basis for \( V \) and an orthonormal basis for \( H \), of eigenfunctions of the Stokes problem

\[
\begin{cases}
-\Delta \tilde{e}_k + \nabla \pi_k = \lambda_k \tilde{e}_k, & \text{in } \Omega, \\
\text{div } \tilde{e}_k = 0, & \text{in } \Omega, \\
\tilde{e}_k \cdot n = 0, & \text{on } \Gamma.
\end{cases}
\] (8.4)

For a more detailed description, we refer to a similar situation described in [19], p. 297-307: Theorem 2, p. 300 and Theorem 5, p. 305 (see also Definition 1-4 and Theorem 1-16, p. 63 of [4]).

The existence of solution for the system (8.1) will be shown by Galerkin’s method. For any fixed \( n = 1, 2, \ldots \), as in Proposition 4.1, we consider the subspace \( \tilde{V}_n = \text{span} \{ \tilde{e}_1, \ldots, \tilde{e}_n \} \) of \( V \) and define

\[
p_n(t) = \sum_{j=1}^{n} s_j^{(n)}(t) \tilde{e}_j
\] (8.5)
as the solution of the equation

\[
\int_{\Omega} \{-\partial_t p_n \cdot \psi - (2D(p_n)y) \cdot \psi + 2D(p_n) : D(\psi) - U \cdot \psi\} \, dx dt
\]

\[
\begin{aligned}
= \ - \int_{\Gamma} \{(a + \alpha)(p_n \cdot \tau)(\psi \cdot \tau)\} \, d\gamma, & \quad \forall \psi \in \tilde{V}_n, \\
p_n(T) = 0.
\end{aligned}
\] (8.6)

Since the equation (8.6) is a system of linear ordinary differential equations in \( \mathbb{R}^n \), there exists a global-in-time solution \( p_n \) in the space \( C([0, T]; \tilde{V}_n) \).

Now, we show the estimate (8.3) for \( \psi = p_n \). Taking \( \psi = \tilde{e}_j \) in (8.6), multiplying it by \( s_j^{(n)} \) and summing on \( j = 1, \ldots, n \), we verify that (8.6) holds for \( \psi = p_n \) yielding

\[
- \frac{1}{2} \frac{d}{dt} \int_{\Omega} |p_n|^2 \, dx + 2 \int_{\Omega} |D(p_n)|^2 \, dx + \int_{\Gamma} \alpha(p_n \cdot \tau)^2 \, d\gamma
\]

\[
= - \int_{\Gamma} \frac{\alpha}{2} |p_n|^2 \, d\gamma + \int_{\Omega} \left[ (\nabla p_n) \cdot U \right] \cdot p_n \, dx = J_1 + J_2.
\] (8.7)

Let us estimate the terms \( J_1 \) and \( J_2 \). We have

\[
J_1 \leq C \| \alpha \|_{L^\infty(\Gamma)} \| p_n \|^2_{L^2(\Gamma)} \leq C \| \alpha \|_{W^{1,\frac{4}{3}}(\Gamma)} \| p_n \|_{L^2(\Omega)} \| \nabla p_n \|_{L^2(\Omega)}
\]

\[
\leq h_1(t) \| p_n \|^2_{L^2(\Omega)} + \frac{1}{2} \| D(p_n) \|^2_{L^2(\Omega)}
\]

with \( h_1(t) = C \| \alpha \|^2_{W^{1,\frac{4}{3}}(\Gamma)} \in L^1(0, T) \) by (2.7). Applying the Gagliardo-Nirenberg-Sobolev inequality (2.4) with \( q = 4 \) and Young’s inequality (2.2), we obtain

\[
J_2 \leq \| \nabla p_n \|_{L^2(\Omega)} \| p_n \|_{L^4(\Omega)} \| y \|_{L^4(\Omega)} + \| U \|_{L^2(\Omega)} \| p_n \|_{L^2(\Omega)}
\]

\[
\leq h_2(t) \| p_n \|^2_{L^2(\Omega)} + \frac{1}{2} \| D(p_n) \|^2_{L^2(\Omega)} + \| U \|^2_{L^2(\Omega)}
\]

with \( h_2(t) = C(1 + \| y \|^2_{L^2(\Omega)} \| \nabla y \|^2_{L^2(\Omega)}) \in L^1(0, T) \) by (4.11). Therefore the above deduced estimates of the terms \( J_1 \), \( J_2 \) and (8.7) imply

\[
- \frac{1}{2} \frac{d}{dt} \| p_n \|^2_{L^2(\Omega)} + 2 \int_{\Omega} |D(p_n)|^2 \, dx + \int_{\Gamma} \alpha(p_n \cdot \tau)^2 \, d\gamma \leq h(t) \| p_n \|^2_{L^2(\Omega)} + \| U \|^2_{L^2(\Omega)}
\]
with \( h(t) = h_1(t) + h_2(t) \in L_1(0, T) \) depending only on the data of our problem \([11]\). Hence integrating the obtained inequality over the time interval \((t, T)\), we derive Gronwall’s inequality, which gives

\[
\|p_n\|^2_{L_\infty(0, T; L_2(\Omega))} + \|D(p_n)\|^2_{L_2(\Omega)} + \|\sqrt{\sigma} p_n\|^2_{L_2(\Omega)} \leq C \int_0^T \|U\|^2_{L_2(\Omega)} \, dt. \tag{8.8}
\]

This estimate and \([8.6]\) permit to conclude that the sequence

\[
\partial_t p_n \in L_2(0, T; H^{-1}(\Omega))
\]

is uniformly bounded on \( n = 1, 2, \ldots \), which allows to use the compactness argument of \([11]\). Therefore for a suitable subsequence of \( \{p_n\} \), we have that

\[
p_n \rightarrow p \quad \text{weakly in} \quad L_2(0, T; H^1(\Omega)), \quad \partial_t p_n \rightarrow \partial_t p \quad \text{weakly in} \quad L_2(0, T; H^{-1}(\Omega)),
\]

\[
p_n \rightarrow p \quad \text{strongly in} \quad L_2(\Omega_T). \tag{8.9}
\]

Taking the limit on \( n \to \infty \) in \([8.3]\), we derive that

\[
p \in L_2(0, T; H^1(\Omega)), \quad \partial_t p \in L_2(0, T; H^{-1}(\Omega))
\]

is the weak solution of \([8.1]\), satisfying \([8.3]\). By the result given on the page 208 of \([33]\), we deduce the existence of the pressure \( \pi \in H^{-1}(0, T; L_2(\Omega)) \).

The uniqueness follows from the linearity of the system and the estimates \([8.3]\). \( \square \)

In the next section, we will prove that the adjoint state \( p \) and the linearized state \( z \) are related through a suitable integration by parts formula. In order to give a meaning to certain boundary terms that will appear in that duality relation, it is necessary to improve the regularity properties of the adjoint state.

**Proposition 8.2.** Under the assumptions of Proposition \([8.1]\) and the additional regularity for the data

\[
a, \alpha \in H^1(0, T; L_\infty(\Gamma)),
\]

the pair

\[
p \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; H^2(\Omega)), \quad \pi \in L_2(0, T; H^1(\Omega)) \tag{8.10}
\]

satisfies the system \([8.7]\) in the usual sense.

**Proof.** Let us consider Galerkin’s approximations \( p_n \) defined in \([8.5]-[8.6]\). Since the function \( p_n(t, \cdot) \in H^3(\Omega) \cap V \) fulfills Navier’s boundary condition (see \([8.1]\)), then integrating by parts the equality \([8.6]\), we obtain

\[
\int_{\Omega} (\partial_t p_n + 2D(p_n)y + \Delta p_n + U) \cdot \psi \, dx = 0, \quad \forall \psi \in V_n,
\]

\[
p_n(T) = 0. \tag{8.11}
\]

Let us introduce the Helmholtz projector \( P_n : L_2(\Omega) \rightarrow \tilde{V}_n \) of \( V \) and define the function \( Ap_n = P_n(-\Delta p_n) = -\Delta p_n + \nabla \tilde{p}_n \in \tilde{V}_n \) for some \( \tilde{p}_n \in H^1(\Omega) \).

Taking \( \psi = \tilde{e}_j \) in \([8.11]\), multiplying it by \( \lambda_j s_j(n) \) and summing on \( j = 1, \ldots, n \), we verify that \([8.11]\) is valid for the test function \( \psi = Ap_n \), that implies the following equality

\[
-\int_{\Omega} \partial_t p_n \cdot Ap_n \, dx + \int_{\Omega} |Ap_n|^2 \, dx = \int_{\Omega} U \cdot Ap_n \, dx + \int_{\Omega} (2D(p_n)y) \cdot Ap_n \, dx.
\]
Applying (2.3) and accounting \( p_n \cdot n = 0 \) on \( \Gamma \), we have

\[
\int_\Omega \partial_t p_n \cdot A p_n \, dx = - \int_\Omega (2D(p_n)\n) \cdot \partial_t p_n \, d\gamma + \int_\Omega 2D(p_n) : D(\partial_t p_n) \, dx
\]

\[
= - \int_\Gamma (2D(p_n)\n) \cdot \tau | \partial_t(p_n \cdot \tau) \, d\gamma + \frac{d}{dt} \left( \int_\Omega |D(p_n)|^2 \, dx \right)
\]

\[
= \int_\Gamma (\alpha + a) (p_n \cdot \tau) | \partial_t(p_n \cdot \tau) \, d\gamma + \frac{d}{dt} \left( \int_\Omega |D(p_n)|^2 \, dx \right)
\]

\[
= \frac{1}{2} \int_\Gamma (\alpha + a) |\partial_t|(p_n \cdot \tau)^2 \, d\gamma + \frac{d}{dt} \left( \int_\Omega |D(p_n)|^2 \, dx \right)
\]

\[
= \frac{d}{dt} \left[ \int_\Gamma (\alpha + a) \frac{|p_n|^2}{2} \, d\gamma + \int_\Omega |D(p_n)|^2 \, dx \right]
- \int_\Gamma (\partial_t \alpha + \partial_t a) \frac{|p_n|^2}{2} \, d\gamma.
\]

Therefore

\[
- \frac{d}{dt} \left[ \int_\Gamma (\alpha + a) \frac{|p_n|^2}{2} \, d\gamma + \int_\Omega |D(p_n)|^2 \, dx \right] + \int_\Omega |Ap_n|^2 \, dx
\]

\[
= - \int_\Gamma (\partial_t \alpha + \partial_t a) \frac{|p_n|^2}{2} \, d\gamma + \int_\Omega U \cdot Ap_n \, dx + \int_\Omega (2D(p_n)\n) \cdot Ap_n \, dx
\]

\[
= I_1 + I_2 + I_3. \quad (8.12)
\]

Let us estimate the terms \( I_1, I_2 \) and \( I_3 \). We have

\[
I_1 = C ||\partial_t \alpha + \partial_t a||_{L_\infty(\Gamma)} ||p_n||_{L_2(\Gamma)}^2
\]

\[
\leq C ||\partial_t \alpha + \partial_t a||_{L_\infty(\Gamma)} ||p_n||_{L_2(\Omega)} ||\nabla p_n||_{L_2(\Omega)} \in L_1(0,T),
\]

uniformly bounded on \( n = 1, 2, \ldots \) by the hypothesis and (8.3). We also have

\[
I_2 = \int_\Omega U \cdot Ap_n \, dx \leq ||U||_{L_2(\Omega)} ||Ap_n||_{L_2(\Omega)} \leq C ||U||_{L_2(\Omega)}^2 + \frac{1}{4} ||Ap_n||_{L_2(\Omega)}^2.
\]

and

\[
I_3 \leq C ||y||_{L_6(\Omega)} ||D(p_n)||_{L_3(\Omega)} ||Ap_n||_{L_2(\Omega)}.
\]

Using Gagliardo-Nirenberg-Sobolev’s inequality (2.4) with \( q = 6 \) and with \( q = 3 \), respectively,

\[
||y||_{L_6(\Omega)} \leq C (||y||_{L_2(\Omega)}^{1/3} ||\nabla y||_{L_2(\Omega)}^{2/3}) + ||y||_{L_2(\Omega)} = f(t) \in L_3(0,T) \quad \text{by (4.4)},
\]

\[
||u||_{L_6(\Omega)} \leq C \left( ||u||_{L_2(\Omega)}^{2/3} ||\nabla u||_{L_2(\Omega)}^{1/3} + ||u||_{L_2(\Omega)} \right) \quad \text{for } u = D(p_n),
\]

we get

\[
I_3 \leq Cf(t) \left( ||\nabla p_n||_{L_2(\Omega)}^{2/3} ||Ap_n||_{L_2(\Omega)}^{1/3} + ||\nabla p_n||_{L_2(\Omega)} \right) ||Ap_n||_{L_2(\Omega)}.
\]

where we have used the inequality

\[
||p_n||_{H^2(\Omega)} \leq C ||Ap_n||_{L_2(\Omega)} \quad (8.13)
\]

which holds by the regular properties of the Stokes operator \( A \) (see Theorem 9 of [2] and Theorem 2 of [12]). Therefore applying Young’s inequality (2.2) and Korn’s inequality (2.6) we derive

\[
I_3 \leq Ch_1(t) ||D(p_n)||_{L_2(\Omega)}^2 + \frac{1}{4} ||Ap_n||_{L_2(\Omega)}^2
\]

with \( h_1(t) = f^3(t) \in L_1(0,T) \).
Therefore, the above deduced estimates for the terms $I_1$, $I_2$, $I_3$ and (8.12) imply

$$-\frac{d}{dt} \left[ \int_\Gamma (\alpha + a) \frac{|p_n|^2}{2} \ d\gamma + \int_\Omega |D(p_n)|^2 \ d\mathbf{x} \right] + \frac{1}{2} \|A p_n\|_{L^2(\Omega)}^2 \leq C \|\mathbf{U}\|_{L^2(\Omega)}^2 + C h_2(t) \left( \|D(p_n)\|_{L^2(\Omega)}^2 + 1 \right),$$

with some $h_2(t) \in L_1(0, T)$ depending only on the data. Hence (8.13) implies (8.14).

Integrating this inequality over the time interval $(t, T)$, we obtain

$$\|D(p_n(t))\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_t^T \|A p_n(s)\|_{L^2(0, T; L^2(\Omega))}^2 \ ds \leq C \|U\|_{L^2(0, T; L^2(\Omega))}^2 + C \int_t^T h_2(s) \left( \|D(p_n)\|_{L^2(\Omega)}^2 + 1 \right) \ ds.$$

Finally, with the help of the Korn inequality, we deduce

$$I = \left[ \int_\Gamma (\alpha + a) \frac{|p_n|^2}{2} \ d\gamma \right] \leq C |\alpha| + a \|\nabla p_n\|_{L^2(\Gamma)} \|p_n\|_{L^2(\Omega)} + \frac{1}{2} \|D(p_n)\|_{L^2(\Omega)}^2,$$

where $h_3(t) = C |\alpha + a|_{L^\infty(\Omega)}^2 \in L_\infty(0, T)$ by the hypothesis (2.7). Then we have the Gronwall inequality

$$\|D(p_n(t))\|_{L^2(0, T; L^2(\Omega))}^2 + \int_t^T \|A p_n(s)\|_{L^2(0, T; L^2(\Omega))}^2 \ ds \leq C \|U\|_{L^2(0, T; L^2(\Omega))}^2 + C \int_t^T h_3(s) \left( \|D(p_n)\|_{L^2(\Omega)}^2 + 1 \right) \ ds.$$

which gives

$$\|D(p_n)\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|A p_n\|_{L^2(\Omega_T)}^2 \leq C \|U\|_{L^2(\Omega_T)}^2 + 1.$$

Moreover we can take $\varphi = \tilde{c}_j$ in (8.13), multiply it by $\frac{d(x^{(n)}(t))}{dt}$ and summing on $j = 1, \ldots, n$, then we deduce that

$$\|\partial_t p_n\|_{L^2(\Omega)}^2 = \int_\Omega \|\partial_t p_n\|^2 \ dx = -\int_\Omega (2D(p_n)\nabla U + \Delta p_n + U) \cdot \partial_t p_n \ dx \leq C \|\partial_t p_n\|_{L^2(\Omega)} \left( \|D(p_n)\|_{L^4(\Omega)} \|\nabla U\|_{L^4(\Omega)} + \|\Delta p_n\|_{L^4(\Omega)} + \|U\|_{L^2(\Omega)} \right).$$

Since

$$\|\nabla y\|_{L^4(\Omega)} \leq \left( \|y\|_{L^2(\Omega)}^{1/2} \|\nabla y\|_{L^2(\Omega)}^{1/2} + \|y\|_{L^2(\Omega)} \right),$$

$$\|D(p_n)\|_{L^4(\Omega)} \leq \left( \|D(p_n)\|_{L^2(\Omega)}^{1/2} \|\nabla (D(p_n))\|_{L^2(\Omega)}^{1/2} + \|D(p_n)\|_{L^2(\Omega)} \right),$$

by (8.14), we obtain

$$\|\partial_t p_n\|_{L^2(\Omega_T)}^2 \leq C \left( \int_0^T \|D(p_n)\|_{L^2(\Omega_T)}^2 \|\nabla (D(p_n))\|_{L^2(\Omega_T)}^2 \ dt + \|U\|_{L^2(\Omega_T)}^2 \right) \leq C$$

for the constant $C$ being independent of $n$ by (1.1), (8.14) and (8.15).
Therefore (8.14), (8.15) and (8.16) imply that there exists a suitable subsequence of \{p_n\}, such that
\[
p_n \to p \quad \text{weakly in } L_2(0, T; H^2(\Omega)) \cap L_\infty(0, T; V),
\]
\[
\partial_t p_n \to \partial_t p \quad \text{weakly in } L_2(\Omega_T),
\]
\[
p_n \to p \quad \text{strongly in } L_2(\Omega T).
\]
Taking the limit on \(n \to \infty\) in (8.11), we derive that
\[
p \in L_2(0, T; H^2(\Omega)) \cap L_\infty(0, T; V), \quad \partial_t p \in L_2(\Omega T),
\]
satisfies the equality
\[
\int_{\Omega T} (\partial_t p + 2D(p)y + \Delta p + U) \cdot \psi \, dx = 0, \quad \forall \psi \in V, \text{ a.e. in } (0, T),
\]
\[
p(T) = 0.
\]
and has the regularity (8.10). Hence \(p\) fulfills the system (8.1) in the usual sense.

Moreover, reasoning as in Proposition 1.2, p. 182 of [39], we derive that \(\pi \in L_2(0, T; H^1(\Omega))\).

\[\square\]

9. Duality property

In the next proposition we demonstrate the \textit{duality} property for the solution \(z\) of the linearized equation (6.1) and the adjoint pair \((p, \pi)\), being the solution of (8.1).

**Proposition 9.1.** The solution \(z\) of the system (6.1) and the solution \((p, \pi)\) of the adjoint system (8.1) verify the following duality relation
\[
\int_{\Omega T} z \cdot U \, dx dt = \int_{\Gamma_T} \{g(p \cdot \tau) + f [\pi - (p \cdot y) - 2 (D(p)n) \cdot n] \} \, d\gamma dt \quad (9.1)
\]

**Proof.** If we multiply (8.1) by \(z\), we have
\[
\int_{\Omega T} z \cdot U \, dx dt = \int_{\Omega T} z \cdot \{ -\partial_t p - 2D(p)y + \nabla \pi - \Delta p \} \, dx dt. \quad (9.2)
\]
The integration by parts gives the following three relations
\[
\int_{\Omega} z \cdot \nabla \pi \, dx = \int_{\Gamma} (z \cdot n) \, \pi \, d\gamma,
\]
\[
- \int_{\Omega} z \cdot (2D(p)y) \, dx = \int_{\Omega} [(y \cdot \nabla) z + (z \cdot \nabla) y] \cdot p \, dx - \int_{\Gamma} [(y \cdot n) (p \cdot z) + (z \cdot n) (p \cdot y)] \, d\gamma
\]
and
\[
- \int_{\Omega} z \cdot \Delta p \, dx = - \int_{\Gamma} 2 (D(p)n) \cdot z \, d\gamma + \int_{\Omega} 2D(p) : D(z) \, dx.
\]
by (2.3). Substituting these three relations in (9.2), we obtain
\[
\int_{\Omega T} z \cdot U \, dx dt = \int_{\Omega T} [-z \cdot \partial_t p + [(y \cdot \nabla) z + (z \cdot \nabla) y] \cdot p + 2D(p) : D(z)] \, dx dt
\]
\[
+ \int_{\Gamma_T} [(z \cdot n) \pi - (y \cdot n) (p \cdot z) + (z \cdot n) (p \cdot y) + (2D(p)n) \cdot z]] \, d\gamma dt.
\]
By another hand if we take $\psi = p \in L_2(0, T; V)$ in (9.2), we have
\[
\int_{\Omega_T} \left\{ -z \cdot \partial_t p + [(z \cdot \nabla) y + (y \cdot \nabla) z] \cdot p + 2D(z) : D(p) \right\} \, dx \, dt = \int_{\Gamma_T} (g - \alpha(z \cdot \tau))(p \cdot \tau) \, d\gamma \, dt,
\]
that implies
\[
\int_{\Omega_T} z \cdot U \, dx \, dt = \int_{\Omega_T} \left[ g(p \cdot \tau) - \alpha(z \cdot \tau)(p \cdot \tau) \right.
\]
\[
+ (z \cdot n) \pi - \left\{ (y \cdot n)(p \cdot z) + (z \cdot n)(p \cdot y) + (2D(p)n) \cdot z \right\} \, d\gamma \, dt.
\]
Accounting the boundary conditions for $y$, $z$ and $p$
\[
(y \cdot n) = a, \quad (z \cdot n) = f, \quad (p \cdot n) = 0,
\]
\[
(2D(p)n) \cdot \tau = -(a + \alpha)(p \cdot \tau),
\]
we obtain
\[
\int_{\Omega_T} z \cdot U \, dx \, dt = \int_{\Omega_T} \left[ g(p \cdot \tau) - \alpha(z \cdot \tau)(z \cdot \tau) + f \pi
\right.
\]
\[
- (a(p \cdot \tau)(z \cdot \tau) + f(p \cdot y)
\]
\[
+ (2D(p)n) \cdot n f - (a + \alpha)(p \cdot \tau)(z \cdot \tau) \right\} \, d\gamma \, dt
\]
\[
= \int_{\Omega_T} g(p \cdot \tau) + f \pi - \left\{ f(p \cdot y) + ((2D(p)n) \cdot n) f \right\} \, d\gamma \, dt
\]
that is
\[
\int_{\Omega_T} z \cdot U \, dx \, dt = \int_{\Omega_T} \left\{ g(p \cdot \tau) + f \pi - (p \cdot y) - 2(D(p)n) \cdot n \right\} \, d\gamma \, dt.
\]
which is the duality property (9.1). \qed

10. Proof of the main results

10.1. Proof of Theorem 5.1 Let us consider a minimizing sequence
\[
(a_n, b_n, y_{a_n, b_n}) \in A \times \left[ L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H^1(\Omega)) \right]
\]
of the cost functional $J$, namely
\[
\lim_n J(a_n, b_n, y_{a_n, b_n}) = \inf(\mathcal{P}).
\]
Since the sequence $(a_n, b_n)$ is bounded in $\mathcal{H}_p(0, T; \Gamma)$ there exists a subsequence, still indexed by $n$, such that
\[
(a_n, b_n) \to (a^*, b^*) \quad \text{weakly in} \quad \mathcal{H}_p(0, T; \Gamma).
\]
In addition, taking into account the estimate (11.1), we know that the sequence $(y_{a_n, b_n})$ is uniformly bounded on the index $n$ in the space $L_\infty(0, T; H) \cap L_2(0, T; H^1(\Omega))$, and $(\partial_t y_{a_n, b_n})$ is bounded in $L_2(0, T; H^{-1}(\Omega))$, then there exists a subsequence, still indexed by $n$, such that
\[
y_{a_n, b_n} \to y^* \quad \text{weakly in} \quad L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H^1(\Omega)),
\]
\[
\partial_t y_{a_n, b_n} \to \partial_t y^* \quad \text{weakly in} \quad L_2(0, T; H^{-1}(\Omega)),
\]
\[
y_{a_n, b_n} \to y^* \quad \text{strongly in} \quad L_2(\Omega_T).
\]
These convergence results allow to pass on the limit $n \to \infty$ in the variational formulation \((1.1)\) for \(y_{a_n,b_n}\) and in the equality \((4.4)\), showing that \(y^*\) satisfies the integral equality

\[
\int_{\Omega_T} \{-y^* \cdot \psi_t + ((y^* \cdot \nabla) y^*) \cdot \psi + 2D(y^*) \cdot D(\psi)\} dx dt
\]

\[
= \int_{\Gamma_T} (b^* - \alpha(y^* \cdot \tau))(\psi \cdot \tau) d\gamma dt + \int_{\Omega} y_0 \cdot \psi(0) dx, \tag{10.1}
\]

which holds for any \(\psi \in H^1(0, T; V)\) with \(\psi(T) = 0\). Therefore \((a^*, b^*, y^*)\) is a solution for the problem \((P)\).

**10.2. Proof of Theorem 3.2** Let \((a^*, b^*, y^*)\) be a solution of the problem \((P)\). According to Theorem 4.1 and Proposition 7.1 for any \((a, b) \in \mathcal{H}_p(0, T; \Gamma)\) the corresponding state equation \((1.1)\) has a unique solution \(y\) and the control-to-state mapping

\[(a, b) \to y\]

is the Gâteaux differentiable at \((a^*, b^*)\). For \(\varepsilon \in (0, 1)\) and \((f, g) \in \mathcal{H}_p(0, T; \Gamma)\), let us set \(a_\varepsilon = a^* + \varepsilon(f - a^*), b_\varepsilon = b^* + \varepsilon(g - b^*)\) and \(y_\varepsilon\) the corresponding state, being the solution of \((1.1)\).

Since \((a^*, b^*, y_{a^*,b^*}^*)\) is a optimal solution and \((a_\varepsilon, b_\varepsilon, y_\varepsilon)\) is admissible, we have

\[
\lim_{\varepsilon \to 0} \frac{J(a_\varepsilon, b_\varepsilon, y_\varepsilon) - J(a^*, b^*, y_{a^*,b^*}^*)}{\varepsilon} \geq 0.
\]

By taking into account Proposition 7.1 we deduce that

\[
\int_{\Omega_T} z_{f,g}^* \cdot (y^* - y_d) dx dt + \int_{\Gamma_T} (\lambda_1 a^* f + \lambda_2 b^* g) d\gamma dt \geq 0, \tag{10.2}
\]

where

\[z_{f,g}^* = \lim_{\varepsilon \to 0} \frac{y_\varepsilon^* - y^*}{\varepsilon}\]

is the unique solution of the linearized equation

\[
\begin{cases}
\partial_t z + (z \cdot \nabla) y^* + (y^* \cdot \nabla) z + \nabla \pi = \Delta z, & \text{in } \Omega_T, \\
z \cdot n = f - a^*, & |2D(z) n + \alpha z| \cdot \tau = g - b^*, & \text{on } \Gamma_T, \\
z(0) = 0, & \text{in } \Omega.
\end{cases}
\]

On the other hand, taking \(U = y - y_d\) and \(y = y^*\) in Proposition 8.2 we shows the existence of the adjoint state pair \((p^*, \pi^*)\) such that

\[p^* \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; H^2(\Omega)), \quad \pi^* \in L_2(0, T; H^1(\Omega))\]

that verifies the equation \((3.2)\). Moreover, considering \(z = z_{f,g}^*\) and \(U = y^* - y_d\) in the duality property \((3.1)\), we have

\[
\int_{\Omega_T} z_{f,g}^* \cdot (y^* - y_d) dx dt
\]

\[
= \int_{\Gamma_T} \{(f - a^*)[\pi + (p^* \cdot y^*) + 2(n \cdot D(p^*)) \cdot n] + (g - b^*) (p^* \cdot \tau)\} d\gamma dt.
\]

As a direct consequence of this equality and \((10.2)\), we obtain the necessary optimality condition \((3.3)\).

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