Entanglement at the Soft-Hair Horizon

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Abstract

We study entanglement between two modes of a free bosonic and fermionic field as seen by two relatively accelerating observers at the soft-hair horizon of a four-dimensional supertranslated Schwarzschild black hole. First, we associate the shock wave to the near horizon geometry by explicit construction of map between their metric as well as relation between wave form factor and soft hair function. We then compute mutual information and entanglement negativity as a measure of entanglement between these two observers and show that their angular dependence at the soft-hair horizon. We also show that for vanishing soft hair function our results are equal to that of an ordinary Schwarzschild black hole case.

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1 Introduction

Over the last few decades entanglement has evolved as a central attention in the study of diverse physical phenomena ranging from quantum many-body systems to the process of black hole formation and various other issues in quantum gravity. It is well known that for a bipartite \((A \cup B)\) system in a pure state \(\rho\), the entanglement entropy characterizes the amount of entanglement which is given by the von-Neumann entropy of the reduced density matrix \(\rho_A = \text{Tr}_B \rho\) of the subsystem \(A\) as

\[
S_A = -\text{Tr}(\rho_A \log \rho_A).
\]

Another important quantity that is constructed from the algebraic sum of entanglement entropies known as mutual information:

\[
I(A : B) = S_A + S_B - S_{AB},
\]

that is symmetric in subsystems \(A\) and \(B\). It is important to note that \(I(A : B)\) has not all the properties to be an entanglement measure, but it provides an upper bound on the total amount of correlations between the subsystems \(A\) and \(B\). In quantum field theory (QFT) entanglement entropy is obtained by a \textit{replica trick} developed by the authors in [1–3]. For holographic characterization of the entanglement entropy in dual conformal field theories (CFTs) see [4–20] (and references therein).

It is well known that entanglement entropy fails to characterize mixed state entanglement as it involves correlations irrelevant to the entanglement of the specific mixed state. To resolve this intricate issue in quantum information theory Vidal and Werner in [21] proposed a computable measure termed as \textit{entanglement negativity} which characterized the upper bound on the distillable entanglement whose non-convexity and monotonicity property was proved by Plenio in [22]. For a bipartite system \(A \cup B\) in a pure and mixed state \(\rho\), the entanglement negativity is defined as the logarithm of the trace norm of the partially transposed reduced density matrix as [21]

\[
\mathcal{E} = \log \text{Tr} |\rho^{T_B}| = \log(1 + 2 \sum_{\lambda_i < 0} |\lambda_i|),
\]

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for a bipartite system \(A \cup B\) in a pure and mixed state \(\rho\), the entanglement negativity is defined as the logarithm of the trace norm of the partially transposed reduced density matrix as [21]
where $\lambda_i$ are the negative eigenvalues of the matrix $\rho^{T_B}$. The partial transposed reduced density matrix $\rho^{T_B}$ of $\rho$ is obtained by exchanging the subsystem $B$’s qubit as $|m \; n\rangle\langle p \; q| \to |m \; q\rangle\langle n \; p|$. Similar to the case of entanglement entropy, the authors in [23–25] described the computation of the entanglement negativity for bipartite mixed states in QFTs through a replica technique. For holographic characterization of the entanglement negativity of bipartite pure and mixed states in dual CFTs see [26–47] (and references therein).

On the other hand, investigation of entanglement in the non-inertial frame has witnessed a surge in interest in the past few decades [48–58]. Inertial observers are described by the Minkowski coordinates while Rindler coordinates are best suited for uniformly accelerated observers. A uniformly accelerated observer cannot access the entire spacetime since it is constrained to move in one of the causally disconnected Rindler wedges. Therefore, the observer must trace over the inaccessible part thus losing information about the total state of the system. As a consequence, the observer detects a thermal state which is known as Unruh effect [59].

The authors in [50, 51] studied the entanglement between two free modes of bosonic and fermionic fields as seen by an inertial observer Alice detecting one of the mode and a uniformly accelerating observer Rob detecting the other. Since the latter follows a hyperbolic trajectory in the Rindler coordinates which emerges as the near-horizon geometry of a Schwarzschild black hole, the above-mentioned scenario can be seen as entanglement between qubits located at horizon and that being sent away. If the Hawking radiation would carry information, as a popular resolution to the information loss paradox, this entanglement could also contribute to the time correlation between early and late radiation. On the other hand, Hawking, Perry and Strominger noticed an infinite family of degenerate vacua associated with BMS supertranslation at null infinity. They suggested an infinite numbers of soft hairs of black hole are responsible for storage of were-claimed-lost information [60–62]. However, the black hole loses its spherical isometry after generic supertranslation and a global Rindler coordinates becomes unavailable. Nevertheless, it can be shown that locally the near-horizon geometry is no more than a shock wave background and we suggest to probe this anisotropy by placing the entangled qubit at different locations near horizon.

To start with we consider a maximally entangled pure state (Bell state) in an inertial frame and study its entanglement at the hairy horizon of a four-dimensional supertranslated Schwarzschild black hole. We consider two observers Alice and Rob, as pictured in the Fig. 1, share an initially entangled state at the past timelike infinity $i^{-}$ before the formation of the black hole. Once the black hole is formed by the shock wave, they fall towards it and Rob eventually hovers outside the horizon whereas Alice continues its journey across the horizon. We investigate the entanglement between the two modes of a free bosonic and fermionic field as detected by Alice and Rob by computing mutual information and entanglement negativity. We show that mutual information and entanglement negativity for both bosonic and fermionic case become angular dependent by considering a specific of soft hair function. It has been discussed in [63] that the supertranslated Schwarzschild metric induces uneven surface gravity at the hairy horizon, and we show that this causes different degradation of entanglement between Alice and Rob. Furthermore, we show that for vanishing soft hair function, our results mimic the usual behavior as obtained in [70] for bosonic fields and in [51] for fermionic fields.

The article is organized as follows. In section 2 we derive the shock wave background as the near-horizon geometry of Schwarzschild black hole under supertranslation. In section 3, we study entanglement between two relatively accelerating observers of free modes of bosonic and fermionic fields at the soft hair horizon of a four-dimensional supertranslated Schwarzschild black hole by computing mutual information and entanglement negativity. In the final section
Figure 1: A Gedanken scenario that Alice and Rob share an entangled qubit at past timelike infinity $i^-$. Alice then free-falls towards the black hole while Rob hangs around at the horizon. The degradation of entanglement might depend on Rob’s angular location if the horizon were supertranslated by soft charges.

4, we present a brief summary and our conclusions.

2 Near hairy horizon geometry and shock wave background

We propose a 4-dimensional spherical shock wave in Minkowski spacetime by the following metric, after the construction in [64, 65]

$$ds^2 = -du \, dv + f(z^A) \delta(u - u_0) \, du^2 + r^2 \gamma_{AB} dz^A dz^B,$$

(4)

$$= -du \, \hat{v} - \Theta(u - u_0) \partial_B f(z^A) du \, dz^B + r^2 \gamma_{AB} dz^A dz^B.$$  

(5)

Here $u = t + r$, $v = t - r$ are the light-cone coordinates, and $z^A$ are transverse Fubuni-Study coordinates of 2-sphere. The function $f(z^A)$ is non-negative [66] which describes the shock wave form factor. The hat coordinate is obtained by the supertranslation shift

$$\hat{v} = v - \Theta(u - u_0) f(z^A),$$

(6)

where $\Theta(u - u_0)$ is the heaviside step function. $T_{uu} = \delta(u - u_0) T(z^A)$ is the only non-zero component of the stress-tensor, localized on the shock wave front $u = u_0$ which propagates on the negative $r$ direction. At late stage, a Schwarzschild black hole is formed by collapsing this shock wave which is depicted in Fig. 2. In the following, we will show this shock wave metric is related to the near-horizon metric of a 4-dimensional supertranslated Schwarzschild black hole.

Recall under supertranslation, the 4-dimensional hairy Schwarzschild black hole in isotropic coordinate $(t, \rho, z^A)$ becomes [67]:

$$ds^2 = -\frac{(1 - \frac{M}{2\rho})^2}{(1 + \frac{M}{2\rho})^2} dt^2 + (1 + \frac{M}{2\rho})^4 \left( d\rho^2 + \left( (\rho - E)^2 + U \right) \gamma_{AB} + (\rho - E) C_{AB} \right) dz^A dz^B,$$

(7)

where scalars $E, U$ and tensor $C_{AB}$ are functions of $C(z^A)$, defined as follows:
Figure 2: A black hole is formed by a shock wave where the soft hair function $C(z^A)$ is point-wisely mapped to the waveform factor $f(z^A)$ via the eq. (14).

\[
C_{AB} = -(2D_A D_B - \gamma_{AB} D^2)C, \\
E = \frac{1}{2} D^2 C + C - C_{0,0}, \\
U = \frac{1}{8} C_{AB} C^{AB}.
\] (8)

We remark that the supertranslated radius $\rho_s = \sqrt{(\rho - C - C_{0,0})^2 + \|D C\|^2}$. Here $C_{0,0}$ refers to the zero mode in spherical harmonic expansion, which generates a time translation, and the square of norm $\|D C\|^2 \equiv \gamma_{AB} D^A C D^B C$. The metric reduces to the no-hair Schwarzschild solution for vanishing soft hair function $C$. We now take the near-horizon limit of hair metric eq. (7), say $\rho_s \simeq M^2 + M x$ for $x \ll 1$ [63]:

\[
ds^2 \simeq -x^2 dt^2 + \kappa^{-2} dx^2 + \cdots,
\] (9)

where we identify the part in Rindler coordinates and obtain the angle-dependent surface gravity as

\[
\kappa = \sqrt{M^2 - 4 \|D C\|^2}.
\] (10)

For small hair function $C$, the surface gravity in eq. (10) maybe further expanded as

\[
\kappa \simeq (\kappa_0 - \epsilon), \\
\kappa_0 = \frac{1}{4 M}, \quad \epsilon = \frac{\|D C\|^2}{2 M^3}.
\] (11)

We can identify $\kappa_0$ as the surface gravity without any soft hair, and regard $\epsilon$ as a correction to the
surface gravity $\kappa$. As an explicit example, we will adopt the following soft hair function [68,69]

$$C = MeY_{2,0}(\theta, \phi) = Me\sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1), \quad (12)$$

for sufficiently small $\epsilon'$ and $(\theta, \phi)$ in $Y_{2,0}(\theta, \phi)$. Then $\epsilon$ maybe computed from eq. (11), which is given as

$$\epsilon = \frac{45 \sin^2 2\theta \epsilon' \pi}{32 M}. \quad (13)$$

Note that $\epsilon' \leq .42$ in eq. (13) to ensure the positivity of the surface gravity $\kappa$ in eq. (11). Thus, the angular dependence of the surface gravity in eq. (10) is manifest through the correction eq. (13).

Recall that we are interested in investigating the entanglement between the free falling observer, Alice and the accelerating observer, Rob at the soft hair horizon. For simplicity, we assume that Rob approaches the hairy black hole at a constant angle, therefore the hair function remains constant. At each angular patch $U_i(z_A)$ or equivalently $U_i(\theta_i, \phi_i)$, the Rindler part of metric eq. (9) can be brought into the shock wave form eq. (4) under the transformation$^1$:

$$u = -\kappa^{-1}_i x e^{-\kappa_i t},$$
$$\hat{v} = \kappa^{-1}_i x e^{\kappa_i t}, \quad (14)$$

where the constant $\kappa_i = \kappa|_{z_A}$. By studying the transition between two neighboring patches, one can further find out the differential relation between hair function $C(z^A)$ and wave form factor $f(z^A)$:

$$\partial_{z^A}||D^A(z^A)|| \propto \partial_{z^A} f(z^A) \quad (15)$$

We leave the construction detail in the appendix. In summary, Rob near the hairy horizon can be regarded as a Rindler observer in a shock wave background. In the next section, we will take advantage of this equivalence and compute his entanglement with free-falling observer Alice.

3 Entanglement at the soft-hair horizon of a supertranslated Schwarzschild black hole

We now proceed to study the entanglement between two observers Alice and Rob of a 4-dimensional supertranslated Schwarzschild black hole background by computing mutual information and entanglement negativity. As mentioned before, we consider Alice and Rob share an entangled state (Bell state) at the same initial point in flat Minkowski spacetime before the formation of the black hole. Once the black hole is formed by the shock wave, Alice free falls into the black hole, while Rob accelerates following a hyperbolic trajectory near the soft hair horizon of a supertranslated Schwarzschild black hole. The maximally entangled Bell state is given as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_R + |1\rangle_A |1\rangle_R), \quad (16)$$

Note that we assume Alice has a detector which only detects $|n\rangle_A$ mode while Rob detects only mode $|n\rangle_R$. The states corresponding to Rob $|n\rangle_R$ must be expressed in terms of the black hole coordinates [70] in order to express what Rob observes in the curved spacetime near the soft

$^1$For small enough patch, one can also assume the wave form $f(z^A)$ remains constant.
3.1 Entanglement for bosonic field

In this section we compute mutual information and entanglement negativity for two free bosonic modes between Alice and Rob. The vacuum $|0\rangle_R$ and the first-excited state $|1\rangle_R$ of Rob may be expressed in terms of two-mode squeezed state as [70]

$$
|0\rangle_R = \sqrt{1 - e^{-2\pi\omega/\kappa}} \sum_{n=0}^{\infty} e^{-n\pi\omega/\kappa} |n\rangle_{in} \otimes |n\rangle_{out},
$$

$$
|1\rangle_R = \left(1 - e^{-2\pi\omega/\kappa}\right) \sum_{n=0}^{\infty} \sqrt{n+1} e^{-n\pi\omega/\kappa} |n\rangle_{in} \otimes |n+1\rangle_{out},
$$

where $\{|n\rangle_{in}\}$ and $\{|n\rangle_{out}\}$ are the orthonormal bases for the inside and outside region of the event horizon respectively. Employing eq. (11), the above eq. (17) maybe expressed as follows

$$
|0\rangle_R = \sqrt{1 - (1 - 2\alpha)e^{-2x}} \sum_{n=0}^{\infty} e^{-nx(1 - n\alpha)} |n\rangle_{in} \otimes |n\rangle_{out},
$$

$$
|1\rangle_R = \sqrt{1 - (1 - 2\alpha)e^{-2x}} \sum_{n=0}^{\infty} \sqrt{n+1} e^{-nx(1 - n\alpha)} |n\rangle_{in} \otimes |n+1\rangle_{out},
$$

where

$$
x = 4\pi\omega M, \quad \alpha = \frac{45\pi\omega M e^2 \sin^2 2\theta}{2}.
$$

Note that henceforth we will denote $|n\rangle_{in} \otimes |n\rangle_{out}$ simply as $|nm\rangle$. Furthermore, using eq. (18), eq. (16) maybe expressed in terms of Minkowski modes for Alice and black hole modes in the inside and outside regions of the soft hair horizon for Rob. Since Rob is causally disconnected from the inside region of the black hole, it is required to trace over the states from this part, and consequently we obtained the joint mixed density matrix of Alice and Rob which is given as [70]

$$
\rho_{AR} = (1 - (1 - 2\alpha)e^{-2x}) \sum_{n=0}^{\infty} (1 - 2\alpha n) e^{-2nx} \rho_n,
$$

where

$$
\rho_n = \frac{1}{2} \left( |0n\rangle \langle 0n| + \sqrt{(n+1)(1 - (1 - 2\alpha)e^{-2x})} (|0n\rangle \langle 1n+1| + |1n+1\rangle \langle 0n|) \\
+ (n+1)(1 - (1 - 2\alpha)e^{-2x}) |1n+1\rangle \langle 1n+1| \right).
$$

It is important to notice that the density matrix $\rho_{AR}$ has $2 \times 2$ block structure in the basis $\{|0 n\rangle, |1 n+1\rangle\}_{n=0}^{\infty}$, i.e the elements of the $(n, n+1)$ block of the matrix $\rho_{AR}$ is

$$
\frac{1}{2}(1 - 2\alpha n) e^{-2nx}(1 - (1 - 2\alpha)e^{-2x}) \left( \frac{1}{\sqrt{(n+1)(1 - (1 - 2\alpha)e^{-2x})}} \sqrt{(n+1)(1 - (1 - 2\alpha)e^{-2x})} \right).
$$

The entanglement entropy of this joint state $\rho_{AR}$ maybe computed by finding the eigenvalues of the above matrix, which is given as

$$
-7-
$$
Figure 3: (a) Mutual information $I(A : R)$ is plotted as a function of $\theta$ with a fixed field mode frequency $\omega = 1/4\pi$ and $\epsilon' = 0.3$ for different black hole mass $M = 0.03$ (green curve), $M = 0.05$ (red curve) and $M = 0.07$ (blue curve). (b) Mutual information $I(A : R)$ is plotted as a function of black hole mass $M$ for vanishing soft hair function $C = 0$.

\[ S(\rho_{AR}) = -\frac{1}{2} \sum_{n=0}^{\infty} (1 - 2\alpha n) e^{-2nx}(1 - (1 - 2\alpha)e^{-2x}) \left[ 1 + (n + 1)(1 - (1 - 2\alpha)e^{-2x}) \right] \]
\[ \times \log_2 \frac{1}{2} (1 - 2\alpha n) e^{-2nx}(1 - (1 - 2\alpha)e^{-2x}) \left[ 1 + (n + 1)(1 - (1 - 2\alpha)e^{-2x}) \right]. \] (23)

Rob’s density matrix is obtained by tracing over Alice’s state as $\rho_R = \text{Tr}_A(\rho_{AR})$, which reads
\[ \rho_R = \frac{1}{2} (1 - (1 - 2\alpha)e^{-2x}) \sum_{n=0}^{\infty} (1 - 2\alpha n) e^{-2nx} |n\rangle\langle n| + (n + 1)(1 - (1 - 2\alpha)e^{-2x})|n + 1\rangle\langle n + 1|, \] (24)

and Rob’s entropy is
\[ S(\rho_R) = -\frac{1}{2} \sum_{n=0}^{\infty} (1 - 2\alpha n) e^{-2nx}(1 - (1 - 2\alpha)e^{-2x}) \left[ 1 + n((1 + 2\alpha)e^{2x} - 1) \right] \]
\[ \times \log_2 \frac{1}{2} (1 - 2\alpha n) e^{-2nx}(1 - (1 - 2\alpha)e^{-2x}) \left[ 1 + n((1 + 2\alpha)e^{2x} - 1) \right]. \] (25)

Similarly, Alice’s density matrix is obtained by tracing over Rob’s state as $\rho_A = \text{Tr}_R(\rho_{AR})$, which is given as
\[ \rho_A = \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|), \] (26)

and the associated entanglement entropy is $S(\rho_R) = 1$. The mutual information $I(A : R)$ maybe computed between Alice and Rob by using eq. (2) which is plotted as function of $\theta$ as shown in Fig. 3a. Note that in Fig. 3b we plot mutual information $I(A : R)$ as a function of black hole mass (or inverse Hawking temperature) for vanishing soft hair function (i.e $C = 0$), which reproduces the mutual information for Alice and Rob as obtained for the case of ordinary Schwarzschild black hole in [70].

To compute entanglement negativity between Alice and Rob, we need to partially transpose the bipartite density matrix $\rho_{AR}$ in eq. (20). The partial transpose density matrix $\rho_{AR}^T_A$ is obtained by interchanging the Alice’s qubit (i.e $|m n\rangle\langle p q| \rightarrow |p n\rangle\langle m q|$) which has a $2 \times 2$ block.
structure in the basis \{|0 \ n+1\}, \{|1 \ n\}\}^\infty_{n=0}$, which is given as
\[
\frac{1}{2} \left( 1 - 2\alpha n \right) e^{-2nx} \left( 1 - (1 - 2\alpha) e^{-2x} \right) \left( n \left( (2\alpha + 1)e^{2x} - 1 \right) \right) \left( \sqrt{n+1} \right) \left( (1 - 1 - 2\alpha)e^{-2x} \right) \left( (1 - 2\alpha)e^{-2x} \right),
\]

The eigenvalues of the above matrix are as follows
\[
\lambda_1^n = -\frac{1}{4} (2\alpha n - 1) e^{-2(n+2)x} \left( 2\alpha + e^{2x} - 1 \right) \left( \xi_n + \sqrt{4e^{2x} \left( 2\alpha + e^{2x} (4\alpha^2 n + 1) - 1 \right) + \xi_n^2} \right),
\]
\[
\lambda_2^n = -\frac{1}{4} (2\alpha n - 1) e^{-2(n+2)x} \left( 2\alpha + e^{2x} - 1 \right) \left( \xi_n - \sqrt{4e^{2x} \left( 2\alpha + e^{2x} (4\alpha^2 n + 1) - 1 \right) + \xi_n^2} \right),
\]

where
\[
\xi_n = 1 - 2\alpha + (2\alpha + 1)ne^{4x} - ne^{2x}.
\]

The eigenvalues \(\lambda_2^n\) are negative for all possible values of \(x\) and \(\alpha\), and the associated entanglement negativity is obtained from eq. (3) which is given as follows
\[
E = \log(1 + 2 \sum_{n=0}^\infty |\lambda_2^n|).
\]

The entanglement negativity depends on the angular coordinate \(\theta\) through the function \(\alpha\) which we plot in Fig. 4a. For vanishing soft hair function \(C = 0\), the eq. (30) reproduces the entanglement negativity between Alice and Rob as that obtained for the case of an ordinary Schwarzschild black hole in [70], which we depict in Fig. 4b as a function of black hole mass \(M\).

### 3.2 Entanglement for fermionic field

In this section we proceed to compute mutual information and entanglement negativity between two free fermionic modes one is shared by Alice and the other one by Rob near soft hair horizon of a four dimensional supertranslated Schwarzschild black hole. The Kruskal vacuum state \(|0\rangle_R\) and the first-excited state \(|1\rangle_R\) of Rob maybe expressed in terms of two-mode squeezed state

![Figure 4](image-url)

Figure 4: (a) Entanglement negativity \(E\) is plotted as a function of \(\theta\) with a fixed field mode frequency \(\omega = 1/4\pi\) and \(e' = 0.3\) for different black hole mass \(M = 0.03\) (green curve), \(M = 0.05\) (red curve) and \(M = 0.07\) (blue curve). (b) Entanglement negativity \(E\) is plotted as a function of black hole mass \(M\) for vanishing soft hair function \(C = 0\).
The total density matrix of the system is

\[\rho = |0\rangle_R \cos \frac{\pi \omega}{\kappa} |0\rangle_{\text{out}} + \sin \frac{\pi \omega}{\kappa} |1\rangle_{\text{in}} |1\rangle_{\text{out}},\]  

(31)

where \(\kappa\) is the surface gravity at the soft hair horizon. Employing eq. (11), \(\cos \frac{\pi \omega}{\kappa}\) and \(\sin \frac{\pi \omega}{\kappa}\) in the above equation may be expanded as follows

\[\cos \frac{\pi \omega}{\kappa} = \frac{1}{\sqrt{1 + (1 - 2\alpha)e^{-2x}}},\]
\[\sin \frac{\pi \omega}{\kappa} = \frac{(1 - 2\alpha)e^{-2x}}{1 + (1 - 2\alpha)e^{-2x}},\]  

(32)

where \(\alpha\) and \(x\) are given in eq. (19). The Bell state in eq. (16) may then be expressed by using eqs. (31) and (32) as

\[|\psi\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{1}{1 + (1 - 2\alpha)e^{-2x}}} |00\rangle + \sqrt{\frac{(1 - 2\alpha)e^{-2x}}{1 + (1 - 2\alpha)e^{-2x}}} |01\rangle + |10\rangle \right).\]  

(33)

The total density matrix of the system is \(\rho = |\psi\rangle \langle \psi|\). The reduced density matrix for Alice-Rob \(\rho_{AR}\) is given by

\[\rho_{AR} = \frac{1}{2} \left( \begin{array}{cccc}
\frac{1}{1 + (1 - 2\alpha)e^{-2x}} & 0 & 0 & \frac{1}{\sqrt{1 + (1 - 2\alpha)e^{-2x}}} \\
0 & \frac{(1 - 2\alpha)e^{-2x}}{1 + (1 - 2\alpha)e^{-2x}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{1 + (1 - 2\alpha)e^{-2x}}} & 0 & 0 & 1
\end{array} \right),\]  

(35)

in the basis \{\(00, 01, 10, 11\}\} where \(\langle ab| = |a\rangle_A \langle b|_R\). The associated entanglement entropy of Alice and Rob is

\[S(\rho_{AR}) = -\frac{2 + (1 - 2\alpha)e^{-2x}}{2(1 + (1 - 2\alpha)e^{-2x})} \log \left( \frac{2 + (1 - 2\alpha)e^{-2x}}{2(1 + (1 - 2\alpha)e^{-2x})} \right) - \frac{(1 - 2\alpha)e^{-2x}}{2(1 + (1 - 2\alpha)e^{-2x})} \log \left( \frac{(1 - 2\alpha)e^{-2x}}{2(1 + (1 - 2\alpha)e^{-2x})} \right).\]  

(36)

The reduced density matrices of Rob \(\rho_R\) is

\[\rho_R = \frac{1}{2} \left( \begin{array}{cccc}
\frac{1}{1 + (1 - 2\alpha)e^{-2x}} & 0 & 0 & \frac{1}{\sqrt{1 + (1 - 2\alpha)e^{-2x}}} \\
0 & \frac{(1 - 2\alpha)e^{-2x}}{1 + (1 - 2\alpha)e^{-2x}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{1 + (1 - 2\alpha)e^{-2x}}} & 0 & 0 & 1
\end{array} \right),\]  

(37)
Figure 5: (a) Mutual information $I(A : R)$ is plotted as a function of $\theta$ with a fixed field mode frequency $\omega = 1/4\pi$ and $\epsilon' = 0.3$ for different black hole mass $M = .03$ (green curve), $M = .05$ (red curve) and $M = .07$ (blue curve). (b) Mutual information $I(A : R)$ is plotted as a function of black hole mass $M$ for vanishing soft hair function $C = 0$.

which has the following matrix representation in the basis $\{|00\rangle, |11\rangle\}$

$$
\rho_R = \frac{1}{2} \begin{pmatrix}
\frac{1+(1-2\alpha)e^{-2x}}{2(1+(1-2\alpha)e^{-2x})} & 0 \\
0 & \frac{1}{2(1+(1-2\alpha)e^{-2x})}
\end{pmatrix}.
$$

(38)

The entanglement entropy of Rob is

$$
S(\rho_R) = -\frac{1}{2(1+(1-2\alpha)e^{-2x})} \log \left( \frac{1}{2(1+(1-2\alpha)e^{-2x})} \right) - \left(1 - \frac{1}{2(1+(1-2\alpha)e^{-2x})} \right) \log \left(1 - \frac{1}{2(1+(1-2\alpha)e^{-2x})} \right).
$$

(39)

Similarly, the density matrix of Alice $\rho_A$ is

$$
\rho_A = \frac{1}{2} \begin{pmatrix}
|0\rangle\langle 0| + |1\rangle\langle 1|
\end{pmatrix},
$$

(40)

which maybe written in the matrix form in the basis $\{|00\rangle, |11\rangle\}$ as

$$
\rho_A = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
$$

(41)

with entanglement entropy $S(\rho_A) = 1$. The mutual information $I(A : R)$ eq. (2) between Alice and Rob maybe obtained by employing eqs. (36) and (39), which is plotted as a function of the angular coordinate $\theta$ for different black hole mass $M$ in Fig. 5a. Note that for $\alpha = 0$ (i.e. vanishing soft hair function $C = 0$), the mutual information becomes equal to that of a no hair Schwarzschild black hole case as obtained in [51] which we plot as a function mass $M$ in Fig. 5b.

To compute entanglement negativity, it is required to obtain the partial transpose density matrix of $\rho_{AR}$ by interchanging the Alice’s qubit which is given as
Figure 6: (a) Entanglement negativity $E$ is plotted as a function of $\theta$ with a fixed field mode frequency $\omega = 1/4\pi$ and $\epsilon' = 0.3$ for different black hole mass $M = 0.03$ (green curve), $M = 0.05$ (red curve) and $M = 0.07$ (blue curve). (b) Entanglement negativity $E$ is plotted as a function of black hole mass $M$ for vanishing soft hair function $C = 0$.

The eigenvalues of the above matrix are as follows

$$
\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{2}, \quad \lambda_3 = \frac{1}{2} \frac{1}{1 + (1 - 2\alpha) e^{-2x}}, \quad \lambda_4 = -\frac{1}{2} \frac{1}{1 + (1 - 2\alpha) e^{-2x}}.
$$

The eigenvalue $\lambda_4$ is negative and the entanglement negativity is obtained by using eq. (3), which is given as

$$
E = \log(1 + 2|\lambda_4|),
= \log \left( \frac{2 + (1 - 2\alpha) e^{-2x}}{1 + (1 - 2\alpha) e^{-2x}} \right).
$$

The entanglement negativity in the above equation depends on the hair function through $\alpha$. In Fig. 6a, we plot entanglement negativity as a function of angle $\theta$ for various black hole mass. Vanishing value of $\alpha$ (i.e $C = 0$) represents the no hair black hole scenario and the corresponding entanglement negativity becomes equal to that of an ordinary Schwarzschild black hole case as computed in [51] which we plot as a function of black hole mass $M$ in Fig. 6b.

4 Summary and Conclusion

In this article we have studied entanglement between the two modes of a free bosonic and fermionic field as detected by two relatively accelerating observers Alice who is free falling and Rob who is hanging near the soft hair horizon of a four-dimensional supertranslated Schwarzschild black hole. The entanglement between these two observers was measured by investigating mutual information and entanglement negativity. In the bosonic case, the mutual information has the value 1 in the vacuum and gradually approaches 2 for massive black hole, while the negativity varies from zero to one. In the fermionic case, the mutual information shows similar behavior as the bosonic case, while the entanglement negativity has some finite value even for vacuum state and saturates to 1 for massive black hole. Both mutual information and
entanglement negativity for bosonic and fermionic field modes depend on the angular coordinates which is a consequence of the angle-dependent surface gravity at the soft hair horizon. The variation of surface gravity is justified by mapping its near horizon geometry to a shock wave Rindler background, where the soft hair function is closely related to the wave form factor. It was observed that for vanishing soft hair function mutual information and entanglement negativity for both bosonic and fermionic field modes become equal to that of ordinary Schwarzschild black hole case. We have following comments: At first, the degradation of entanglement can be explained by the thermal decoherence seen by an accelerating observer, whose acceleration can be identified as the Unruh effect. Nevertheless the Schwarzschild black hole is in fact thermally unstable due to its negative specific heat. Author in the [63] proposed the hair temperature

$$T_{\text{hair}} = \frac{\kappa}{2\pi},$$

which can be understood as the non-thermal temperature rather than usual Hawking temperature, driving subtle heat flow along angular direction by its gradient of soft hair function. In this paper, we investigate an example of hair function $C \sim \cos^2 \theta$. Its gradient $\partial_\theta C \sim \sin 2\theta$, which has a period $\pi/2$, consistent with the fluctuation of mutual information and negativity as shown in the Fig. 3a to Fig. 6a. Therefore according to our study, it is possible to indirectly observe the soft hair by measurement of fluctuated entanglement at the Schwarzschild horizon. Secondly, the soft hair Schwarzschild metric in consideration is more or less a classical background. If one believes in the entanglement nature of quantum gravity [71], we might instead consider an evaporating Schwarzschild black hole as an entangled state of the soft hair metric and radiation, where the metric is supertranslated after each coded emission. We will leave this for future study.

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Appendix

Here we attempt to construct the map between hair function $C(z^A)$ and wave form factor $f(z^A)$ by studying the shock wave geometry. We regard it as a product space of two-dimensional plane wave fiber and a two-sphere base. Each patch $U_i$ on the base is an infinitesimal solid-angular region centered at angular coordinate $z_i^A$. The plane wave fiber at each patch is coordinated by $(u_i, \hat{v}_i)$, which are related to the Schwarzschild near-horizon coordinate $(t, x)$ by eq. (14). Now consider in the region overlapped by two neighboring patches, say $U_1 \cap U_2$, we may adopt the coordinate $u_1$ in the patch $U_1$ and $\hat{v}_2$ in the patch $U_2$ and write the shock wave fiber metric eq. (4) (neglect the sphere part) as

$$ds^2 = -du_1 d\hat{v}_2 - \partial_{z^A} f(z_i^A) du_1 dz^A$$

$$= e^{-(\kappa_1 - \kappa_2)t} \left\{ -x^2 dt^2 + (\kappa_1 \kappa_2)^{-1} dx^2 - (\kappa_2^{-1} - \kappa_1^{-1}) x dx dt \right\} - \partial_{z^A} f(z_i^A) du_1 dz^A$$

$$\simeq \left( 1 - (\kappa_1 - \kappa_2)t + \cdots \right) \left\{ -x^2 dt^2 + (\kappa_1 \kappa_2)^{-1} dx^2 - (\kappa_2^{-1} - \kappa_1^{-1}) x dx dt \right\}$$

$$- \partial_{z^A} f(z_i^A) du_1 dz^A$$

(46)

\[ - 13 - \]
For infinitesimally close patches, $|\kappa_1 - \kappa_2| \ll 1$, this metric describes a near horizon geometry with surface gravity $(\kappa_1 \kappa_2)^{1/2}$ if the last two cross terms were canceled. To see it happen, we recall that by definition

$$d\rho_s \simeq d(M + Mx) = M dx, \quad d\rho_s \bigg|_{\rho_s = \frac{M}{2}} = \frac{\partial_A ||DC||^2}{\rho_s} dz^A = \frac{2}{M} \partial_A ||DC||^2 dz^A$$

(47)

and from (44),

$$x = \sqrt{-\kappa_1 \kappa_2 u_1 \hat{v}_2 e^{(\kappa_1 - \kappa_2) t/2}} \simeq \sqrt{-\kappa_1 \kappa_2 u_1 \hat{v}_2} (1 + (\kappa_1 - \kappa_2)t/2 + \cdots)$$

(48)

$$dt = d\left(\frac{-1}{\kappa_1 + \kappa_2} \ln \left(\frac{-\kappa_1 u_1}{\kappa_2 \hat{v}_2}\right)\right) = -\frac{1}{\kappa_1 + \kappa_2} \frac{du_1}{u_1} + \cdots$$

(49)

then the last two cross terms can be canceled if $^2$

$$\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \sqrt{-\frac{\hat{v}_2^*}{u_1^* \kappa_1 \kappa_2}} \frac{2}{M^2} \partial_A ||DC||^2 = \partial_A f.$$ 

(50)

Note we have assigned $u_1 = u_1^*, \hat{v}_2 = \hat{v}_2^*$ at which the shock wave forms the black hole. In this way we have explicitly constructed the differential relation between soft hair function and wave form factor.

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__Footnote 2__ One may instead use coordinate $\hat{v}_1$ in the patch $U_1$ and $u_2$ in the patch $U_2$. The result is expected to be similar but with indices 1, 2 swapped. Although there might be other construction of metric in the overlapped region, the result should be only different by order $O(\kappa_1 - \kappa_2)$. 

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