Width of Percolation Transition in Complex Networks

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Abstract

It is known that the critical probability for the percolation transition is not a sharp threshold, actually it is a region of non-zero width $\Delta p_c$ for systems of finite size. Here we present evidence that for complex networks $\Delta p_c \sim \frac{l}{N^{\nu_{\text{opt}}}}$, where $l \sim N^{\nu_{\text{opt}}}$ is the average length of the percolation cluster, and $N$ is the number of nodes in the network. For Erdős-Rényi (ER) graphs $\nu_{\text{opt}} = 1/3$, while for scale-free (SF) networks with a degree distribution $P(k) \sim k^{-\lambda}$ and $3 < \lambda < 4$, $\nu_{\text{opt}} = \frac{(\lambda - 3)}{(\lambda - 1)}$. We show analytically and numerically that the survivability $S(p, l)$, which is the probability of a cluster to survive $l$ chemical shells at probability $p$, behaves near criticality as $S(p, l) = S(p_c, l) \cdot \exp[(p - p_c)l/p_c]$. Thus for probabilities inside the region $|p - p_c| < p_c/l$ the behavior of the system is indistinguishable from that of the critical point.

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I. INTRODUCTION:

Recently the subject of networks has received much attention. It was realized that many systems in the real world, such as the Internet, can be successfully modeled as networks. Other examples include social networks such as the web of social contacts, and biological networks such as the protein interaction network and metabolic networks [1, 2, 3]. The problem of percolation on networks has also been studied extensively (e.g. [4]). Using percolation theory we can describe the resilience of the network to breakdown of sites or links [5, 6], epidemic spreading [7], and the properties of the optimal path in a highly network with highly fluctuating weights on the links [8].

A typical percolation system consists of a \( d \)-dimensional grid of length \( L \), in which the nodes or links are removed with some probability \( 1 - p \), or are considered “conducting” with probability \( p \) (e.g. [9, 10]). Below some critical probability \( p_c \) the system becomes disconnected into small clusters, i.e., it becomes impossible to cross from one side of the grid to the other by following the conducting links. Percolation is considered a geometrical phase transition exhibiting universality, critical exponents, upper critical dimension at \( d = 6 \) etc. It was noted by Coniglio [11] that for systems of finite size \( L \) the transition from connected to disconnected state has a width \( \Delta p_c \sim \frac{1}{L^{1/\nu}} \), where \( \nu \) is a critical exponent related to the correlation length.

Percolation on networks was studied also from a mathematical point of view [4, 12, 13]. It was found that in Erdős-Rényi (ER) graphs with an average degree \( \langle k \rangle \) the percolation threshold is: \( p_c = \frac{1}{\langle k \rangle} \). Below \( p_c \) the graph is composed of small clusters (most of them trees). As \( p \) approaches \( p_c \) trees of increasing order appear. At \( p = p_c \) a giant component emerges and loops of all orders abruptly appear. Nevertheless, for graphs of finite size \( N \) it was found that the percolation threshold has a finite width \( \Delta p_c \sim \frac{1}{N^{1/\nu}} \) [13], meaning that all attributes of criticality are present in the system in the range \([p_c - \Delta p_c, p_c + \Delta p_c]\). For example: The number of loops is negligible below \( p_c + \Delta p_c \) [20].

In this paper we study the *Survivability* of the network near the critical threshold. The survivability \( S(p, l) \) is defined to be the probability of a connected cluster to “survive” up to \( l \) chemical shells in a system with conductance probability \( p \) [14] (i.e the probability that there exists at least one node at chemical distance \( l \) from a randomly chosen node on the same cluster). At the critical point \( p_c \), the survivability decays as a power-law: \( S(p_c, l) \sim l^{-x} \),
where $x$ is a universal exponent. Below $p_c$ the survivability decays exponentially to zero, while above $p_c$ it decays (exponentially) to a constant. Here we will derive analytically and numerically the functional form of the survivability above and below the critical point. We will show that near the critical point $S(p, l) = S(p_c, l) \cdot \exp[(p - p_c)l/p_c]$. Thus, given a system with a maximal chemical length $l$, for probabilities inside the range $|p - p_c| < \frac{p_c}{l}$ the behavior of the system is indistinguishable from that of the critical point. Hence we get $\Delta p_c \sim \frac{p_c}{l}$.

The maximal chemical length $l$ at criticality is actually the length of the percolation cluster, which was found to be: $l \sim N^{\nu_{opt}}$ where $N$ is the number of nodes in the network. For Erdős-Rényi (ER) graphs $\nu_{opt} = 1/3$, while for scale-free (SF) networks with a degree distribution $P(k) \sim k^{-\lambda}$ and $3 < \lambda < 4$, $\nu_{opt} = (\lambda - 3)/(\lambda - 1)$ [8].

II. ERDŐS-RÉNYI GRAPHS:

Consider an ER graph with a mean degree $\langle k \rangle$, and each link having a probability $p$ to conduct. We define $N_l(x) = n_0 + n_1 x + n_2 x^2 + n_3 x^3 + \ldots$ to be the generating function of the number of sites that exists on layer (i.e. chemical shell) $l$ starting from a random node on the graph (for a conduction probability $p$).

The generating function for the degree distribution of a randomly chosen node in the network is $G_0(x) = \sum P(k) \cdot x^k$ and the generating function for the number of links emerging from a node reached by following a randomly chosen link is $G_1(x) = \sum \frac{1}{k} \langle k \rangle k P(k) \cdot x^{k-1}$ [15]. Taking into account the probability $p$ for conduction, we have:

$$G_1(x) = 1 - p + p \sum \frac{1}{k} \langle k \rangle k P(k) \cdot x^{k-1}. \quad (1)$$

We can now write the following recursive relation [16]:

$$N_{l+1}(x) = G_1(N_l(x)), \quad (2)$$

which means that the probability $n_i^{(l+1)}$ for having $i$ nodes at layer $l + 1$ is composed of the probability of reaching a vertex by following a link, and reaching $i$ nodes at layer $l$ by following all branches emerging from that vertex - see sketch in Fig. 1.

It can be seen that $N_l(0) = n_0$ is the probability that there are 0 nodes at layer $l$, i.e., the probability to die before layer $l$. Thus $\epsilon_l = 1 - N_l(0)$ is the probability to survive up to
layer $l$. From (2) we have:

$$N_{l+1}(0) = G_1(N_l(0))$$  \hfill (3)

$$1 - \epsilon_{l+1} = G_1(1 - \epsilon_l)$$  \hfill (4)

$$1 - \epsilon_{l+1} = 1 - p + p \sum \frac{1}{\langle k \rangle} k P(k)(1 - \epsilon_l)^{k-1}$$  \hfill (5)

For ER graphs $G_0(x) = G_1(x) = e^{(k)(x-1)}$ (for $p = 1$), Thus:

$$1 - \epsilon_{l+1} = 1 - p + pe^{(k)(1-\epsilon_l-1)} = 1 - p + pe^{-k}\epsilon_l$$  \hfill (6)

$$\epsilon_{l+1} = p - pe^{-k}\epsilon_l$$  \hfill (7)

Setting $\delta \equiv p - p_c$, where $p_c = \frac{1}{\langle k \rangle}$, and expanding by series we get:

$$\epsilon_{l+1} = p_c + \delta - (p_c + \delta) \left( 1 - \langle k \rangle \epsilon_l + \frac{1}{2} \langle k \rangle^2 \epsilon_l^2 - \ldots \right)$$  \hfill (8)

Leaving only expressions up to second order in $\delta$ and $\epsilon_l$ (we assume that $p < p_c$ and thus $\epsilon_l \ll 1$ for large $l$) we get:

$$\epsilon_{l+1} \approx \epsilon_l - \frac{1}{2} \langle k \rangle \epsilon_l^2 + \delta \langle k \rangle \epsilon_l$$  \hfill (9)

$$\frac{d\epsilon_l}{dl} \approx - \frac{1}{2} \langle k \rangle \epsilon_l^2 + \frac{\delta}{p_c} \cdot \epsilon_l$$  \hfill (10)

At criticality, $\delta = 0$ and the solution to this equation is: $\epsilon_l \sim l^{-1}$. The additional term suggests the following solution near criticality: $\epsilon_l \sim l^{-1} \cdot \exp \left( \frac{1}{p_c} \delta l \right)$.

In terms of survivability this can be written as:

$$S(p, l) = S(p_c, l) \cdot \exp \left( \frac{1}{p_c} (p - p_c) l \right).$$  \hfill (11)

In order to check this result we numerically solved the survivability $S(p, l)$ near $p_c$ according to the exact enumeration method presented in [17] [22]. Fig. 2 shows the survivability $S(p, l)$ for different values of $p$. For $p = p_c$ the survivability decays as a power law, while above and below there is an exponential decay, either to zero (for $p < p_c$) or to a constant (for $p > p_c$). Fig. 3 shows that all curves of the survivability $S(p, l)$ from Fig. 2 can be rescaled such that they all collapse. Moreover, scaled survivabilities from all different graphs with
different values of \( \langle k \rangle \) (i.e., different values of \( p_c \)) also collapse on the same curve. However, equation (11) is true only below the percolation threshold where there is no giant component. Above the percolation threshold there is an exponential decay to a non-zero constant, and the generalized expression is:

\[
S(p, l) = S(p_c, l) \cdot \exp\left(-\frac{1}{p_c} |p - p_c| l\right) + pP_{\infty},
\]

(12)

Where \( pP_{\infty} \) is the probability for a randomly chosen site to be inside the percolation cluster [23]. Indeed, setting \( \epsilon_{l+1} = \epsilon_l \) in equation (7) the resulting “steady state” solution is \( pP_{\infty} \) [13] [24].

III. SCALE-FREE GRAPHS:

Scale-free graphs can be taken to have a degree distribution of the form \( P(k) = c k^{-\lambda} \) where \( c \approx (\lambda - 1) m^{\lambda - 1} \) [6]. In order to solve equation (5) we have to evaluate:

\[
G_1(1 - \epsilon_l) = \frac{1}{\langle k \rangle} \sum k P(k) (1 - \epsilon_l)^{k-1}
\]

(13)

Expanding by powers of \( \epsilon_l \), and inserting \( P(k) = c k^{-\lambda} \) with \( 3 < \lambda < 4 \), we get [18]:

\[
\sum k P(k) (1 - \epsilon_l)^{k-1} \approx \langle k \rangle - (\langle k(k-1) \rangle \epsilon_l + \frac{c}{2} \Gamma(4 - \lambda) \epsilon_l^{\lambda - 2}
\]

(14)

Thus equation (5) becomes:

\[
1 - \epsilon_{l+1} \approx 1 - p + \frac{p}{\langle k \rangle} \left( \langle k \rangle - (\langle k(k-1) \rangle \epsilon_l + \frac{c}{2} \Gamma(4 - \lambda) \epsilon_l^{\lambda - 2}\right)
\]

(15)

Taking \( p = p_c + \delta \):

\[
1 - \epsilon_{l+1} \approx 1 - (p_c + \delta) + \frac{p_c + \delta}{\langle k \rangle} \left( \langle k \rangle - (\langle k(k-1) \rangle \epsilon_l + \frac{c}{2} \Gamma(4 - \lambda) \epsilon_l^{\lambda - 2}\right).
\]

(16)

Substituting \( p_c = \frac{\langle k \rangle}{\langle k(k-1) \rangle} \) we get:

\[
1 - \epsilon_{l+1} \approx 1 - \epsilon_l + p_c \frac{c}{2 \langle k \rangle} \Gamma(4 - \lambda) \cdot \epsilon_l^{\lambda - 2} - \frac{1}{p_c} \cdot \delta \epsilon_l + \frac{c}{2 \langle k \rangle} \Gamma(4 - \lambda) \cdot \delta \epsilon_l^{\lambda - 2}
\]

(17)

Setting \( A \equiv p_c \frac{c}{2 \langle k \rangle} \Gamma(4 - \lambda) \) we get:

\[
\epsilon_{l+1} - \epsilon_l \approx -A \cdot \epsilon_l^{\lambda - 2} + \frac{1}{p_c} \cdot \delta \epsilon_l - \frac{A}{p_c} \cdot \delta \epsilon_l^{\lambda - 2}
\]

(18)
\[ \epsilon_{l+1} - \epsilon_l \approx -A \cdot \epsilon_l^{\lambda - 2} + \frac{1}{p_c} \cdot \delta (\epsilon_l - A \cdot \epsilon_l^{\lambda - 2}) . \] (19)

For large \( l \), \( \epsilon_l \ll 1 \), and taking into account that \( \lambda - 2 > 1 \) we have \( \epsilon_l^{\lambda - 2} \ll \epsilon_l \). Therefore:

\[ \frac{d\epsilon_l}{dl} \approx -A \cdot \epsilon_l^{\lambda - 2} + \frac{1}{p_c} \cdot \delta \epsilon_l . \] (20)

For \( \delta = 0 \) the solution is \( \epsilon_l \sim l^{-x} \) with \( x = \frac{1}{\lambda - 3} \) [16]. The additional term suggests the following solution near criticality: \( \epsilon_l \sim l^{-x} \cdot \exp \left( \frac{\delta l}{p_c} \right) \) [25]. A similar form can be found for \( \lambda > 4 \) [26]. The scaling form for SF networks is also confirmed by numerical simulations as shown in Figures 4 and 5.

IV. SUMMARY AND CONCLUSIONS

The scaling form of the survivability near the critical probability obeys the following scaling relation (for \( p < p_c \)):

\[ S(p, l) = S(p_c, l) \cdot \exp \left( \frac{p - p_c}{\Delta p_c} \right) . \] (21)

Where \( \Delta p_c = \frac{p_c}{l} \). Given a system with a maximal chemical length \( l \), for all values of conductivity \( p \) inside the range \( [p_c - \Delta p_c, p_c + \Delta p_c] \) the survivability behaves similar to the power law \( S(p_c, l) \sim l^{-x} \) found at criticality. Thus, the width of the critical threshold is \( \Delta p_c = \frac{p_c}{l} \).

To summarize, we have shown analytically and numerically the the survivability in ER and SF graphs scales according to equations (11) and (12) near the critical point. This implies that the width of the critical region in networks of finite size scales as \( \Delta p_c = \frac{p_c}{l} \), where \( l \) is the chemical length of the percolation cluster. For ER graphs, \( l \sim N^{1/3} \), while for SF networks with \( 3 < \lambda < 4 \), \( l \sim N^{(\lambda - 3)/(\lambda - 1)} \).

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Given a certain conduction probability $p$, the probability to reach $i$ nodes at layer $l+1$ is found by summing up the probabilities to follow a link to a node (whose outgoing degree is described by $G_1(x)$) and reaching $i$ nodes at layer $l$ from that node.

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[21] Indeed, solving equation (10) taking $\delta \ll 1$ and the initial condition $\epsilon_{l=0} = 1$ we get: $\epsilon_l = \frac{2}{\langle k \rangle} l^{-1} \exp\left(\frac{1}{pc}\delta l\right)$.

[22] This method assumes that near the critical point there is a negligible number of loops and thus the network behaves similar to a cayley tree with the same degree distribution $P(k)$ as the ER network.

[23] $S(p, l \to \infty)$ is the probability that is we start from a randomly chosen conducting site, we will survive an infinite chemical distance. This equals the probability $p$ that the randomly chosen site is conducting, multiplied by the probability $P_\infty$ that it resides in the giant component.

[24] $P_\infty$ obeys the transcendental equation: $P_\infty = 1 - e^{-(k)pP_\infty}$.

[25] Solving equation (20) with $\delta \ll 1$ and the initial condition $\epsilon_{l=0} = 1$ we get: $\epsilon_l = \frac{1}{((\lambda-3)A)^{1/(\lambda-3)}} l^{-1/(\lambda-3)} \exp\left(\frac{1}{pc}\delta l\right)$.

[26] In this range is the behavior is similar to ER graphs [19].
FIG. 2: (Color online) The survivability $S(p, l)$ for an ER graphs with $\langle k \rangle = 5$, numerically calculated for different values of $p$: $p_c, p_c \pm 5 \cdot 10^{-4}$, $p_c \pm 3 \cdot 10^{-4}$, $p_c \pm 1 \cdot 10^{-4}$, $p_c \pm 6.66 \cdot 10^{-5}$, and $p_c \pm 3.33 \cdot 10^{-5}$. For $p = p_c$ the survivability decays to zero according to a power law: $S(p_c, l) \sim l^{-1}$. For $p < p_c$, $S(p, l) \rightarrow 0$, while for $p > p_c$, $S(p, l) \rightarrow Const$. The decay is exponential (to zero or to a constant) according to equations (11) and (12)
FIG. 3: (Color online) Scaling of the survivability for different values of $p$, $l$, and $\langle k \rangle$. Shown is 
\[
\frac{S(p,l) - S(p,\infty)}{S(p_c,l)} \text{ vs. } \frac{1}{p_c |p - p_c| l}
\] for ER graphs with $\langle k \rangle = 5$ (unfilled symbols) and $\langle k \rangle = 10$ (filled symbols). The collapse of all curves on an exponential function (for large $l$) shows that indeed the scaling relations (11) and (12) are correct.
FIG. 4: (Color online) The survivability $S(p,l)$ for a SF network with $\lambda = 3.5$, numerically calculated for different values of $p$: $p_c$, $p_c \pm 6 \cdot 10^{-2}$, $p_c \pm 4 \cdot 10^{-2}$, $p_c \pm 2 \cdot 10^{-2}$, $1.33 \cdot 10^{-2}$, and $p_c \pm 6.66 \cdot 10^{-3}$. For $p = p_c$ the survivability decays to zero according to a power law: $S(p_c,l) \sim l^{-2}$. For $p < p_c$, $S(p,l) \rightarrow 0$, while for $p > p_c$, $S(p,l) \rightarrow Const$. The decay is exponential (to zero or to a constant) according to equations $11$ and $12$. 

\[ S(l,p) \]
FIG. 5: (Color online) Scaling of the survivability for different values of $p$, $l$, and $\lambda$. Shown is $\frac{S(p,l) - S(p,\infty)}{S(p_c,l)}$ vs. $\frac{1}{p_c} |p - p_c| l$ for SF graphs with $\lambda = 3.5$ (filled symbols) and $\lambda = 5$ (unfilled symbols). Due to numerical difficulties only curves with $p < p_c$ are shown. All curves collapse on an exponential function according to relation (11).

The exact enumeration method is limited here to small chemical distances $l$ due to the upper cutoff of the scale-free distribution.