Relativistic Motion with Viscosity: II Stokes’s Law of Resistance

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Abstract
The deduction of a relativistic and mildly relativistic equation of motion in the presence of a drag force proportional to the velocity is presented. The obtained results are used to model the trajectory of the supernova SN1993J and the light curves of gamma-ray bursts.

Keywords
Supernovae, General Supernovae, Individual (SN 1993J) ISM, Supernova Remnants GRB, Individual (GRB 130427A) GRB, Individual (GRB 120521C) GRB, Individual (GRB 130606A)

1. Introduction
A relativistic treatment of the equation of motion in the presence of a resistive force proportional to the velocity has been investigated in the following models: a model for the Newtonian scattering of photons [1], a motion through a uniform adiabatic medium on the steady-state accretion of matter onto a Schwarzschild black hole [2], an extreme mass-ratio inspirals around strongly accreting supermassive black holes [3], and ultra-relativistic detonations in the framework of the cosmological first-order phase transitions [4]. In Section 2, this paper explores the relativistic law of motion in the presence of viscosity proportional to the velocity. Section 3 is devoted to the astrophysical applications.

2. The Equation of Motion
2.1. The Classic Case
We assume a one-dimensional motion with a resistive force of Stokes type [5], $F_{\text{res}} = -Amv(t)$, where $A$ is a constant, $m$ is the considered mass and $v(t)$ is the velocity. The differential equation which governs the motion is
\[ v(t) = \frac{v_0 e^{-At}}{e^{-t_0t}}, \]  

which has an analytical solution in an explicit form

\[ v(t; A, v_0, t_0) = v(t) = \frac{v_0 e^{-At}}{e^{-t_0t}}, \]

where \( v_0 \) is the velocity at \( t = t_0 \). The equation of motion in the explicit form is

\[ r(t; A, v_0, t_0, r_0) = -\frac{v_0}{A} \left( e^{-At} - e^{-t_0t} \right) e^{t_0t} + t_0 \]

where \( r_0 \) is the distance at \( t = t_0 \). The numerical value of the constant \( A \) is

\[ A = \frac{\ln \left( \frac{v_1}{v_0} \right)}{t_0 - t_1}, \]

where \( v_1 \) is the velocity at \( t = t_1 \).

### 2.2. The Relativistic Case

We assume a one-dimensional motion with a resistive force of Stokes type, \( F_{\text{res}} = -Am v(t) \), where \( A \) is a constant, \( m_0 \) is the considered rest mass and \( v(t) \) is the velocity. Newton’s second law in special relativity is:

\[ F = \frac{d}{dt} \left( \frac{m_0 v(t)}{\sqrt{1 - \frac{(v(t))^2}{c^2}}} \right), \]

where \( F \) is the force, \( m_0 \) is the rest mass, \( c \) is the velocity of light and \( v(t) \) is the velocity; see Equation (7.16) in [6]. The first order differential equation in the velocity which governs the relativistic motion is

\[ \frac{d}{dt} v(t) = -A (v(t)). \]

An analytical solution to the above first order differential does not exist; however, a solution exists for \( v(t) \) in an implicit form for the time

\[ t = \frac{N}{D}, \]

\[ N = \left( -2c^3 + 2cv_0^2 \right) \sqrt{c^2 - v^2} - \left( -2t_0A + \ln \left( \sqrt{c^2 - v^2} - c \right) \right) \]

\[ - \ln \left( c + \sqrt{c^2 - v^2} \right) - \ln \left( \sqrt{c^2 - v_0^2 - c} + \ln \left( c + \sqrt{c^2 - v_0^2} \right) \right) c^2 \]

\[ - 2c \sqrt{c^2 - v_0^2 - v_0^2} \left( -2t_0A + \ln \left( \sqrt{c^2 - v^2} - c \right) - \ln \left( c + \sqrt{c^2 - v^2} \right) \right) \]

\[ - \ln \left( \sqrt{c^2 - v_0^2 - c} + \ln \left( c + \sqrt{c^2 - v_0^2} \right) \right) \left( c - v \right) \left( c + v \right). \]
and
\[ D = 2A \left( c^2 - v^2 \right) \left( c^2 - v_0^2 \right), \]  
where \( v_0 \) is the velocity at \( t = t_0 \). The constant \( A \) can be derived from the following formula
\[ A = \frac{NN}{DD}, \]  
where
\[ \begin{align*}
NN &= \left( -2c^2 + 2cv_0^2 \right) \sqrt{c^2 - v_0^2} + \left( c + v_0 \right) \left( c - v_0 \right) \left( \ln \left( \sqrt{c^2 - v_1^2} - c \right) 
- \ln \left( c + \sqrt{c^2 - v_0^2} \right) - \ln \left( c + \sqrt{c^2 - v_0^2} \right) \right) c^2 \\
&+ 2 \sqrt{c^2 - v_0^2} c - v_0^2 \left( \ln \left( \sqrt{c^2 - v_1^2} - c \right) - \ln \left( c + \sqrt{c^2 - v_0^2} \right) 
- \ln \left( c + \sqrt{c^2 - v_0^2} \right) \right),
\end{align*} \]  
and
\[ DD = 2 \left( t_0 - t_1 \right) \left( c^2 - v_1^2 \right) \left( c^2 - v_0^2 \right), \]
where \( v_1 \) is the velocity at \( t = t_1 \).

### 2.3. The Mildly-Relativistic Case

The first order differential equation for the mildly-relativistic motion is
\[ \frac{dv}{dt} + \frac{3v(t)^2 \left( \frac{d}{dt} v(t) \right)}{2c^2} = -Av(t), \]
which has solution
\[ v\left(t; t_0, v_0\right) = W\left( -e^{\frac{-At}{2}} \right), \]
where \( W \) is the Lambert W function \([7]\) and
\[ B = -\frac{4v_0 Ac^2 + 4 \ln \left( v_0 \right) c^2 + 3v_0^2}{2c^2}, \]
with \( v_0 \) being the velocity at \( t = t_0 \). The trajectory in the mildly relativistic case is
\[ r\left(t; t_0, r_0, v_0\right) = -\frac{e^{\frac{-At}{2}} c^2 v_0 \left( W\left( D \right) + 3 \left( \frac{4v_0^2}{c^2} \right)^{\frac{3}{2}} - 3e^{-At} \sqrt{W\left( D \right)} \left( Av_0 + v_0 \right) c^2 + \frac{v_0^3}{2} \right)}{3e^{\frac{-At}{2}} c^2 A}, \]
where
with \( r_0 \) being \( r \) at \( t = t_0 \). The constant \( A \) can be derived in the mildly relativistic case by the following formula
\[
A(t_0, t_1, v_0) = \frac{-v_0^2 \left(4 \ln \left(\frac{v_0}{v_1}\right) c^2 + 3v_0^2 - 3v_1^2\right)}{4v_0^2 c^3 (t_0 - t_1)},
\] (18)
where \( v_1 \) is the velocity at \( t = t_1 \).

2.4. Astrophysical Luminosity

The mechanical relativistic luminosity is
\[
L_{m,r} = 4\pi r(t)^2 \frac{1}{1 - \beta(t)^2} \rho_0 \left(\frac{r}{r_0}\right)^2 \frac{v(t)}{c} \beta(t),
\] (19)
where \( r(t) \) is the temporary radius of the expansion, \( r_0 \) is the radius at \( t = t_0 \), \( \rho_0 \) is the density at \( t = t_0 \), \( d \) is a shape parameter and \( \beta(t) = \frac{v(t)}{c} \).

The observed luminosity, \( L_{obs} \), is assumed to scale as
\[
L_{obs} = C_{obs} L_{m,r} \left(1 - e^{-\tau}\right),
\] (20)
where \( C_{obs} \) is a constant that allows the match between theory and observations, and \( -\tau \) is the optical thickness.

3. Astrophysical Applications

The astrophysical units are chosen to be pc for the length and years for the time: the constant \( A \) is therefore expressed in \( \frac{1}{yr} \). A test for the quality of the fits is represented by the merit function
\[
\chi^2 = \sum_j \left(\frac{r_{th,j} - r_{obs,j}}{\sigma_{obs,j}}\right)^2,
\]
where \( r_{th,j}, r_{obs,j} \) and \( \sigma_{obs,j} \) are the theoretical radius, the observed radius and the observed uncertainty, respectively.

3.1. Application to SN 1993J

Figure 1 reports the numerical trajectory, of SN 1993J for which observational parameters are available \[8\] [9] with data as in Table 1.

3.2. Application to GRBs

A first example is applied to the light curve (LC) of GRB 130427A, which was the most luminous gamma-ray burst in the last 30 years; see Figure 1 in \[10\]. Figure 2 reports the X-flux as a function of the time and the relative theoretical data, with data as in Table 2.
Figure 1. Numerical radius (full line) and astronomical data of SN 1993J with vertical error bars.

Figure 2. Flux in the X-ray as a function of time in seconds for GRB 130427A (empty stars) and theoretical curve as given by Equation (20) (full line) when $r_e = \infty$ with data are as in Table 2.

Table 1. Numerical values for the parameters of Stokes's theoretical model applied to SN 1993J.

| model     | values                        | $\chi^2$ |
|-----------|-------------------------------|----------|
| Stokes's  | $r_0 = 3.0 \times 10^{-3}$ pc; $v_0 = 13800$ km/s; $A = 0.07 \frac{1}{\text{years}}$ | 85.7     |

A second example is applied to the LC in X-ray of GRB 120521C 2, see Figure 2 in [11], which is reported in Figure 3, with temporal behavior of the optical depth as in Figure 4.

A third example is given by the LC in X-ray of GRB 130606A, see Figure 2 in [11], which is reported in Figure 5, with the temporal behavior of the optical depth as in Figure 6.
Figure 3. Flux in the X-ray as function of time in seconds for GRB 120521C (empty stars) and theoretical curve as given by Equation (20) (full line), with $\nu \tau$ as in Figure 4 and with data as in Table 2.

Figure 4. The time dependence of $\nu \tau$ (empty stars) for GRB 120521C and a logarithmic polynomial approximation of degree 5 (full line). Parameters as in Table 2.

Table 2. Numerical values of the parameters for the theoretical model.

| GRB name | theoretical parameters |
|----------|------------------------|
| GRB 130427A | $r_g = 9.9 \times 10^{-3}$ pc; $t_u = 1.0 \times 10^{-3}$ year; $\beta_g = 0.9$; $A = 1 \frac{1}{\text{pc}}$; $d = 3.1$ |
| GRB 120521C | $r_g = 1.0 \times 10^{-4}$ pc; $t_u = 1.0 \times 10^{-4}$ year; $\beta_g = 0.9$; $A = 10000 \frac{1}{\text{pc}}$; $d = 3$ |
| GRB 130606A | $r_g = 1.0 \times 10^{-5}$ pc; $t_u = 1.0 \times 10^{-5}$ year; $\beta_g = 0.9$; $A = 1000 \frac{1}{\text{pc}}$; $d = 2$ |
4. Conclusions

We analyzed the one-dimensional relativistic motion in the presence of a resistive force proportional to the velocity. An analytical solution for the velocity was derived in an implicit form, see Equation (7). In the mildly relativistic case, we derived an analytical solution for both the velocity, see Equation (14), and the distance, see Equation (16), in terms of the Lambert W function.

A first test to evaluate the constant $A$ in an astrophysical environment is on SN 1993J. A full relativistic treatment of the LC for GRBs was done for GRB 130427A, GRB 120521C and GRB 130606A in the framework of the optical thickness with a time dependence.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.
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