Abstract

The general formula of the $\pi^-p$ atom strong energy-level shift in the 1s state is derived in the next-to-leading order in the isospin breaking, and in all orders in chiral expansion. Isospin-breaking corrections to the level shift are explicitly evaluated at order $p^2$ in ChPT. The results clearly demonstrate the necessity to critically reaccess the values of the $\pi N$ scattering lengths, extracted from the energy-level shift measurement by means of the potential model-based theoretical analysis.

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In the experiment performed at PSI [1,2], isospin-symmetric S-wave $\pi N$ scattering lengths $a_{0^+}^+$ and $a_{0^+}^-$ are obtained from the measurements of the $\pi^-p$ atom strong energy-level shift $\varepsilon_{1s}(\pi^-p)$, its decay width into $\pi^0n$ state $\Gamma_{1s}(\pi^-p)$, and the strong energy-level shift of the $\pi^-d$ atom $\varepsilon_{1s}(\pi^-d)$. These three measurements are claimed to be fairly consistent within the error bars, and - by using the technique described in Ref. [1] - yield the strong $\pi N$ scattering lengths in an accuracy which is unique for hadron physics: $a_{0^+}^+ = (1.6 \pm 1.3) \times 10^{-3} M_{\pi^-}^{-1}$ and $a_{0^+}^- = (86.8 \pm 1.4) \times 10^{-3} M_{\pi^-}^{-1}$. The scattering length $a_{0^+}^-$ may be used as an input in the Goldberger-Miyazawa-Oehme [4] sum rule to determine the $\pi NN$ coupling constant. Recently, a new experiment has been approved on the pionic hydrogen [5] that will allow one to measure $\Gamma_{1s}(\pi^-p)$ with a much higher accuracy and thus, in principle, to determine

\footnote{In Refs. [1,2], the quantities $b_0 = a_{0^+}^+$ and $b_1 = -a_{0^+}^-$ are used. Below, we follow the conventions of [3]}
the $\pi N$ scattering lengths from the data on pionic hydrogen alone. This, in its turn, will enable one to vastly reduce the model-dependent uncertainties that come from the analysis of three-body problem [6]. Further, at CERN, the DIRAC collaboration conducts the experiment aimed at the measurement of the $\pi^+\pi^-$ atom lifetime within the 10% accuracy that will allow one to determine the difference of $\pi\pi$ scattering lengths $a_0 - a_2$ with a 5% precision [7]. The DIRAC measurement provides a critical test for large/small condensate scenario in QCD: should it turn out that the quantity $a_0 - a_2$ is different from the value predicted in standard Chiral Perturbation Theory (ChPT) [8], one has to conclude that the chiral symmetry breaking in QCD proceeds differently from the standard picture [9]. Finally, the DEAR collaboration [10] at the DAΦNE facility (Frascati) plans to measure the energy level shift and lifetime of the $1s$ state in $K^-p$ and $K^-d$ atoms - with considerably higher precision than in the previous experiment carried out at KEK [11] for $K^-p$ atoms. It is expected [10] that this will result in a precise determination of the $I = 0, 1$ $S$-wave scattering lengths - although, of course, one will again be faced with the three-body problem already mentioned. It will be a challenge for theorists to extract from this new information on the $K N$ amplitude at threshold a more precise value of e.g. the kaon-nucleon sigma term and of the strangeness content of the nucleon [12].

The relations between the characteristics of hadronic atoms and hadronic scattering lengths in the leading order in isospin breaking are given by the well-known formulae by Deser et al. [13] - consequently, the measurement of these characteristics provides one with the unique possibility to directly extract these scattering lengths from the data. In order to perform the above-mentioned elaborate tests of QCD at low energy, the precision of the leading-order formula is, however, not sufficient. The theoretical calculations of the hadronic atom observables should be necessarily carried out at the accuracy that matches the accuracy of the experimental measurements. To this end, one may invoke the potential model, assuming, e.g. that the strong interactions which are described by energy-independent local potentials, do not violate the isospin symmetry. This framework, applied to the case of the $\pi^+\pi^-$ atom decay [14], failed, however, to agree with the magnitude and even the sign of the correction term to the leading-order formula, evaluated independently in ChPT [15–18]. The reason for this discrepancy is now well understood: the simple potential model described above, does not include the full content of the isospin-breaking effects in QCD. For this reason, in our opinion, it is legitimate to question the applicability of the potential model to the calculation of the energy spectrum and decays of other hadronic atoms (e.g., the $\pi^-p$ atom), as well. These calculations should be also reexamined on the basis of low-energy effective theories where the isospin-breaking effects are completely taken into account.

The present paper is aimed at establishing the precise relation between the
strong energy-level shift of the $\pi^-p$ atom in the 1s state, and strong $\pi N$ scattering lengths. This relation, valid at next-to-leading order in isospin breaking, can be used to extract $\pi N$ scattering lengths from the measured energy spectrum of the $\pi^-p$ atom. So far, only potential model calculations have been utilized for this purpose [1]. In the present paper, however, we use the non-relativistic effective Lagrangian approach and incorporate all isospin-breaking effects. The decay of $\pi^-p$ atom will be considered in a separate publication. Note that the same approach has been already applied to study the $\pi^+\pi^-$ atom decay in ChPT [17,18].

Our strategy is the follows. First, we write down the general non-relativistic Lagrangian that describes the interactions between nucleons, pions and photons at a very small momenta. This Lagrangian is then used - by means of Feshbach’s formalism [19] - to calculate the ground-state energy of the $\pi^-p$ atom. The energy is expressed in terms of the couplings of the non-relativistic Lagrangian which, in its turn, through the matching condition are related to the relativistic $\pi^-p$ elastic scattering amplitude at threshold, calculated in ChPT. At the last step, we analyze the isospin-symmetry breaking contributions to this amplitude at $O(p^2)$ in ChPT that, using the already established relation between the amplitude and the bound-state energy, provides us with the isospin-breaking correction to the 1s energy level of the $\pi^-p$ atom.

The preliminary remarks are in order. The isospin breaking in the $\pi N$ system is due to two physically distinct sources: electromagnetic effects being proportional to the fine structure constant $\alpha$, and up and down quark mass difference $m_d - m_u$. It is convenient to introduce the common counting for these two effects. In particular, one may define\footnote{Note that in Refs. [17,18], a slightly different convention $\alpha \sim (m_d - m_u)^2 \sim \delta$ was adopted. The counting rule is purely a matter of convention: if we assume the same rule as in the present paper, the $O((m_d - m_u)^2)$ contributions which emerge in Refs. [17,18], will be merely relegated to higher orders in the perturbation expansion.} the small parameter $\delta$ so that $\alpha \sim (m_d - m_u) \sim \delta$. With this definition, both $\Delta M_\pi^2 = O(\delta)$ and $m_p - m_n = O(\delta)$. Further, in the leading order in the fine-structure constant, the binding energy of the $\pi^-p$ atom ground state is given by the well-known non-relativistic Coulomb-Schrödinger value $E_1 = -\frac{1}{2} \mu_c \alpha^2$, where $\mu_c$ denotes the reduced mass of $\pi^-p$ system. The leading-order strong energy-level shift is given by the strong amplitude times the square of the Coulomb wave function at the origin and, consequently, is of order $O(\alpha^3)$. We are interested in the leading order isospin-breaking corrections to the strong shift - thus we have to evaluate the bound-state energy at $O(\delta^4)$ that stands either for $O(\alpha^4)$, or for $O(\alpha^3(m_d - m_u))$.\footnote{There is one exception from this counting rule. We include the effect due to the electron vacuum polarization as well. Formally this contribution is of order $\alpha^5$, but it is amplified by the large factor $(M_{\pi^+}/m_e)^2$.}
We emphasize that in this paper we deal with the shift of the ground-state level while the quantity that is experimentally measured is the transition energy between $3p - 1s$ levels. The strong shift in the $3p$ state is, however, very small (see [1] and references therein), and for the time being is neglected in our approach.

Now, we start with writing down the non-relativistic Lagrangian which will be used to calculate the energy of the $\pi^-p$ atom. This Lagrangian, in general, consists of an infinite tower of operators with an increasing mass dimension— it contains all possible terms that are allowed by discrete $P$, $C$, $T$ symmetries, rotational invariance and gauge invariance. The convenient building blocks for this Lagrangian, therefore, are: the non-relativistic pion and nucleon fields, their covariant derivatives, and electric ($E$) and magnetic ($B$) fields. For the general procedure of the derivation, we refer to Ref. [20] which deals with the same problem in detail, in the context of QED, and merely display the result

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \psi^\dagger \left\{ i\mathcal{D}_t - m_p + \frac{\mathcal{D}^2}{2m_p} + \frac{\mathcal{D}^4}{8m^3_p} + \cdots \right. \\
- c_p \frac{e\sigma B}{2m_p} - c_p \frac{e(\mathcal{D}E - E\mathcal{D})}{8m^2_p} - c_p \frac{i\sigma(\mathcal{D} \times E - E \times \mathcal{D})}{8m^2_p} + \cdots \left. \right\} \psi \\
+ \chi^\dagger \left\{ i\partial_t - m_n + \frac{\nabla^2}{2m_n} + \frac{\nabla^4}{8m^3_n} + \cdots \right. \\
- c_n \frac{e\sigma B}{2m_n} - c_n \frac{e(\nabla E - E\nabla)}{8m^2_n} - c_n \frac{i\sigma(\nabla \times E - E \times \nabla)}{8m^2_n} + \cdots \left. \right\} \chi \\
+ \sum_{\pm} \pi^\dagger_{\pm} \left\{ i\mathcal{D}_t - M_{\pi^\pm} + \frac{\mathcal{D}^2}{2M_{\pi^\pm}} + \frac{\mathcal{D}^4}{8M^3_{\pi^\pm}} + \cdots \right. \\
\mp c^R \frac{e(\mathcal{D}E - E\mathcal{D})}{6M^2_{\pi^\pm}} + \cdots \left. \right\} \pi_{\pm} \\
+ \pi^\dagger_0 \left\{ i\partial_t - M_{\pi^0} + \frac{\nabla^2}{2M_{\pi^0}} + \frac{\nabla^4}{8M^3_{\pi^0}} + \cdots \right. \left. \right\} \pi_0 \\
+ d_1 \psi^\dagger \psi \pi^\dagger_0 \pi_0 + d_2 (\psi^\dagger \chi \pi^\dagger_0 \pi_0 + h.c.) + d_3 \chi^\dagger \chi \pi^\dagger_0 \pi_0 + \cdots. \tag{1}
$$

Here $F_{\mu\nu}$ stands for the electromagnetic field strength tensor, $\psi, \chi, \pi_{\pm}$ and $\pi_0$ denote the non-relativistic field operators for proton, neutron, charged and neutral pion fields, and $\mathcal{D}_t \psi = \partial_t \psi - ieA_0 \psi, \mathcal{D} \psi = \nabla \psi + ieA \psi, \partial_t \pi_{\pm} = \partial_t \pi_{\pm} \mp ieA_0 \pi_{\pm}, \mathcal{D} \pi_{\pm} = \nabla \pi_{\pm} \pm ieA \pi_{\pm}$ are the covariant derivatives acting on the proton and charged pion fields, respectively. The ellipses stand for the higher-dimensional operators and the counterterms that are needed to render the off-shell Green functions finite in the perturbation theory. The
couplings $c_F^p$, $c_D^p$, $c_S^p$, $c_F^n$, $c_D^n$, $c_S^n$, $c^R$, $d_1$, $d_2$, $d_3$, $\cdots$ should be determined from matching of non-relativistic and relativistic amplitudes (see below). In general, the matching procedure determines the same combinations of the non-relativistic couplings that are needed in the bound-state calculations [21].

The Lagrangian displayed in Eq. (1), contains an infinite tower of operators with the increasing mass dimension. This Lagrangian is used in the perturbation theory to calculate the scattering amplitudes in the one-nucleon, one-pion and one-nucleon-one-pion sectors, with any number of photons, to a given accuracy in the coupling constant $e$ and in the expansion in external 3-momenta.

At any order in the perturbative expansion, these amplitudes can be renormalized by adding the appropriate counterterms to the Lagrangian (1). Moreover, these amplitudes coincide with the same amplitudes calculated in the relativistic theory, provided the couplings in the non-relativistic Lagrangian are matched in a sufficient accuracy in $e$. The perturbative calculations in the non-relativistic theory are carried out using the “threshold expansion” [22] in all Feynman integrals - this ensures that the counting rules are obeyed so that the higher-dimensional operators do not contribute to the momentum expansion of the amplitudes at lower orders. In particular, this ensures that the mass parameters in the Lagrangian (1) coincide with the physical masses of the particles - there is no mass renormalization. Further, the same non-relativistic Lagrangian is used to calculate the bound-state characteristics. Here, the powers of external momenta “translate” into the powers of $e$ and the threshold expansion should be again used to ensure the validity of counting rules. For the details, we refer the reader to Ref. [21] and references therein. In actual calculations, one always deals with the truncated Lagrangian. In particular, in the present paper we aim at the evaluation of the ground-state energy of the $\pi^-p$ atom at the accuracy $O(\delta^4)$. It can be demonstrated that then it is necessary to construct the non-relativistic Lagrangian that describes correctly all contributions to the $\pi^-p$ elastic scattering amplitude at $O(\delta)$, which do not vanish at physical threshold. The Lagrangian displayed in Eq. (1), is sufficient for this purpose. It can be shown that all operators that can be constructed (including the counterterms), except the ones given in Eq. (1), will not contribute to the $\pi^-p$ scattering amplitude at $O(\delta)$ at threshold neither at tree level, nor through loops.

Below, we consider briefly the non-relativistic couplings $c_F^p$, $c_D^p$, $c_S^p$, $c_F^n$, $c_D^n$, $c_S^n$, $c^R$. At the accuracy we are working, it is sufficient to match these constants at $O(e^0)$. To this end, one may consider the nucleon and pion electromagnetic form-factors in the external field $A_\mu$. The necessary details, including different normalization of states in the relativistic and non-relativistic theories, are provided in [20], where the same problem is treated in the context of QED. Below, we merely display the result of the matching
\[ c_p^F = 1 + \mu_p \quad \text{and} \quad c_n^F = \mu_n \]
\[ c_p^D = 1 + 2\mu_p + \frac{4}{3} m_p^2 < r_p^2 > \quad \text{and} \quad c_n^D = 2\mu_n + \frac{4}{3} m_n^2 < r_n^2 > \]
\[ c_p^S = 1 + 2\mu_p \quad \text{and} \quad c_n^S = 2\mu_n \]
\[ c_p^R = M_{\pi^+}^2 < r_{\pi^+}^2 > \quad \text{and} \quad c_n^R = 2\mu_n + 4 \frac{3}{m_n^2} < r_{\pi^-}^2 >, \quad (2) \]

where \( \mu_p \) and \( \mu_n \) denote the anomalous magnetic momenta of proton and neutron, respectively, and \( < r_p^2 >, \quad < r_n^2 >, \quad < r_{\pi^+}^2 > \) stand for the charge radii of proton, neutron and charged pion. The remaining constants \( d_i, \quad i = 1 \cdots 3 \) are determined from matching the \( \pi N \) scattering amplitudes in different channels.

The Lagrangian (1) still contains the terms that do not contribute to the ground-state energy level shift of the \( \pi^- p \) atom in the accuracy we are working. In order to simplify this Lagrangian, in the Coulomb gauge we exclude the \( A_0 \) field by using the equations of motion, and neglect high-dimensional operators that arise in a result of this operation. Further, we neglect the operators that contribute to the \( \pi N \) scattering sectors with non-zero total charge, and to the spin-flip part of the \( \pi^- p \) elastic scattering amplitude (since the ground state of the \( \pi^- p \) atom is an \( S \)-wave state, the spin-flip part does not contribute due to the rotational invariance). In a result, we arrive at the simplified Lagrangian which is better suited for the bound-state calculations

\[ \mathcal{L}' = \frac{1}{2} A^2 - \frac{1}{2} B^2 - e^2 (\psi^\dagger \psi) \Delta^{-1} (\pi^+ \pi^-) \]
\[ + \psi^\dagger \left\{ i\partial_t - m_p + \frac{\nabla^2}{2m_p} + \frac{\nabla^4}{8m_p^3} + \frac{ie}{2m_p} (\nabla A + A \nabla) \right\} \psi \]
\[ + \chi^\dagger \left\{ i\partial_t - m_n + \frac{\nabla^2}{2m_n} + \frac{\nabla^4}{8m_n^3} \right\} \chi \]
\[ + \sum_{\pi^\pm} \pi^\dagger_{\pi^\pm} \left\{ i\partial_t - M_{\pi^+} + \frac{\nabla^2}{2M_{\pi^+}^2} + \frac{\nabla^4}{8M_{\pi^+}^3} \right\} \pi_{\pi^\pm} \]
\[ + \pi_0^\dagger \left\{ i\partial_t - M_{\pi^0} + \frac{\nabla^2}{2M_{\pi^0}^2} + \frac{\nabla^4}{8M_{\pi^0}^3} \right\} \pi_{\pi_0} \]
\[ + d'_1 \psi^\dagger \psi \pi^+ \pi^- + d_2 (\psi^\dagger \chi \pi^+ \pi_0 + \text{h.c.}) + d_3 \chi^\dagger \chi \pi^+_0 \pi^-_0, \quad (3) \]

where

\[ d'_1 = d_1 - e^2 \lambda \quad \text{and} \quad \lambda = \left( \frac{c_p^D}{8m_p^2} + \frac{c_n^D}{6M_{\pi^+}^2} \right) = 1 + 2\mu_p + \frac{1}{6} (\langle r_p^2 \rangle + \langle r_{\pi^+}^2 \rangle) \quad (4) \]
For the calculation of the energy-level shift of $\pi^- p$ atom ground state we again, as in Ref. [17] use the Feshbach’s formalism [19]. To this end, we first obtain the total Hamiltonian of the system from the Lagrangian (3) by using the canonical formalism

$$H = H_0 + H_C + H_R + eH_\gamma + H_S + e^2\lambda H_\lambda = H_0 + H_C + V,$$

(5)

where $H_0$ is the free Hamiltonian describing photons and non-relativistic pions and nucleons. Further, $H_\Gamma = \int d^3x H_\Gamma$, $\Gamma = C, R, \gamma, S, \lambda$, and

$$H_C = e^2 (\psi^\dagger \psi) \Delta^{-1} (\pi_+^\dagger \pi_-),$$

$$H_R = -\psi^\dagger \frac{\nabla^4}{8m_\pi^3} \psi - \chi^\dagger \frac{\nabla^4}{8m_n^3} \chi - \sum_{\pm} \frac{\nabla_+^4}{8M_{\pi^+}^3} \pi_+^\dagger \pi_+ - \frac{\nabla_0^4}{8M_{\pi^-}^3} \pi_0,$$

$$H_\gamma = -\frac{i}{2m_p} \psi^\dagger (\nabla A + A\nabla) \psi - \sum_{\pm} \frac{\pm i}{2M_{\pi^+}} \pi_+^\dagger (\nabla A + A\nabla) \pi_+,$$

$$H_S = -d_1 \psi^\dagger \psi \pi_+^\dagger \pi_- - d_2 (\psi^\dagger \chi \pi_+^\dagger \pi_0 + h.c.) - d_3 \chi^\dagger \chi \pi_0^\dagger \pi_0,$$

$$H_\lambda = \psi^\dagger \psi \pi_+^\dagger \pi_.
$$

(6)

We treat the problem in the perturbation theory: $H_0 + H_C$ is the unperturbed Hamiltonian, whereas $V$ is considered as a perturbation. The ground-state solution of the unperturbed Schrödinger equation in the CM frame $(\tilde{E}_1 - H_0 - H_C)|\Psi_0(0, s)\rangle = 0$, with $\tilde{E}_1 = m_p + M_{\pi^+} + E_1$, is given by

$$|\Psi_0(0, s)\rangle = \int \frac{d^3p}{(2\pi)^3} \Psi_0(p) b^\dagger_+(p, s) a^\dagger_-(-p) |0\rangle,$$

(7)

where $a_\dagger^\pm(p)$ and $b^\dagger_+(p, s)$ denote creation operators for non-relativistic $\pi^-$ and proton acting on the Fock space vacuum, $s$ is the projection of the proton (atom) spin, and $\Psi_0(p)$ stands for the non-relativistic Coulomb wave function in the momentum space.

Now, we are going to evaluate the energy-level shift of the ground state due to the perturbation Hamiltonian $V$. To this end, we again, as in Ref. [17], look for the poles of the scattering operator $T(z) = (H_C + V)(H_C + V)G_0(z)T(z)$ on the second Riemann sheet in the complex $z$-plane: the real and imaginary parts of the pole position give, by definition, the energy and the decay width of the metastable bound state. The free and Coulomb Green operators are defined as $G_0(z) = (z - H_0)^{-1}$ and $G(z) = (z - H_0 - H_C)^{-1}$, respectively. Further, we define the “Coulomb-pole removed” Green function as $G(z) = G(z)(1 - \Pi)$, where $\Pi$ denotes the projector onto the Coulomb ground
state $\Psi_0$ (7). The $\pi N$ scattering states in the sector with the total charge 0 are defined as $|P, p; s\rangle_+ = b_{\eta}^\dagger (p_1, s) a_{\eta}^\dagger (p_2) |0\rangle$ and $|P, p; s\rangle_0 = b_{\eta}^\dagger (p_1, s) a_{\eta}^0 (p_2) |0\rangle$ ($a_{\eta}^\dagger (p)$ and $b_{\eta}^\dagger (p, s)$ denote the creation operators for the $\pi^0$ meson and neutron, respectively). The center-of-mass mass and relative momenta are defined by $P = p_1 + p_2$, $p = \eta_2 (A) p_1 - \eta_1 (A) p_2$, $A = +, 0$, and $\eta_1^{(+)} = m_p/(m_p + M_{\pi^+})$, $\eta_2^{(+)} = M_{\pi^+}/(m_p + M_{\pi^+})$, $\eta_1^{(0)} = m_n/(m_n + M_{\pi^0})$, $\eta_2^{(0)} = M_{\pi^0}/(m_n + M_{\pi^0})$. We remove the CM momenta from the matrix elements of any operator $R(z)$ by introducing the notation

$$A(P, q; s| R(z)|0, p; s')_B = (2\pi)^3 \delta^3(P) (q, s| r_{AB}(z)| p; s'). \quad (8)$$

In general, the expression for the matrix elements of $r_{AB}(z)$ contains the spin-nonflip ad the spin-flip parts that are defined in the following manner

$$(q, s| r_{AB}(z)| p; s') = \delta_{ss'} (q| r_{AB}^\eta (z)| p) + i (\sigma_{ss'} \cdot [p \times q]) (q| r_{AB}^f (z)| p). \quad (9)$$

Below, we define the “Coulomb-pole removed” transition operator that satisfies the equation

$$M(z) = V + V \hat{G}(z) M(z). \quad (10)$$

With the use of the Feshbach’s formalism and the definitions given above, it is straightforward to demonstrate that the scattering operator $T(z)$ develops the pole at $z = \bar{z}$ where $\bar{z}$ is the solution of the following equation

$$\bar{z} - \bar{E}_1 - (\Psi_0| m_{++}^n (z) |\Psi_0) = 0, \quad (11)$$

where $(p|\Psi_0) = \Psi_0(p)$ and $m_{++}^n (z)$ is related to $M(z)$ through the definitions (8) and (9).

In order to get the shift of the ground-state energy, the quantity $m_{++}^n (z)$ is determined perturbatively from Eq. (10). The iteration series that emerge from Eq. (10), can be truncated since the higher-order terms do not contribute to the energy at $O(\delta^4)$. Using the explicit expression of $V$ given by Eq. (5), replacing $\hat{G}(z)$ by $\hat{G}_0(z)$ whenever possible, and retaining only those terms that contribute at the accuracy we are working, the operator $M(z)$ can be written in the form $M(z) = U(z) + W(z)$, where

$$U(z) = H_R + e^2 H_{\gamma} G_0(z) H_{\gamma} + e^2 \lambda H_{\lambda},$$

$$W(z) = H_S + H_S G(z) H_S + H_S G_0(z) H_S G_0(z) H_S, \quad (12)$$

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At the accuracy $O(\delta^4)$, the energy of the bound state is equal to $\tilde{E} = \tilde{E}_1 + \Delta E_{1e}^\text{em} + \epsilon_{1s}$, where

$$
\Delta E_{1e}^\text{em} = \Re (\Psi_0|u_{++}^n(\tilde{E}_1)|\Psi_0) + E^\text{vac}, \quad \epsilon_{1s} = \Re (\Psi_0|w_{++}^n(\tilde{E}_1)|\Psi_0). \quad (13)
$$

Here $u_{++}^n(z)$, $w_{++}^n(z)$ are related to $U(z)$, $W(z)$ through the definitions (8) and (9), and $E^\text{vac}$ stands for the contribution due to the electron vacuum polarization (the corresponding term is added “by hand”, see below).

Using explicit expressions (6), we obtain

$$
\Re (\Psi_0|u_{++}^n(\tilde{E}_1)|\Psi_0) = -\frac{5}{8} \alpha^2 \mu_c^4 \left\{ \frac{1}{d_1 - 3} - \Gamma'(1) - \ln 4\pi \right\} + \ln \left( \frac{2\mu_c^2}{\mu^2} \right) - 1, \quad (14)
$$

$$
\Re (\Psi_0|w_{++}^n(\tilde{E}_1)|\Psi_0) = \frac{\alpha^3 \mu_c^3}{\pi} \left( -d_1 + d_2 \left( \xi + \frac{\alpha \mu^2}{\pi} (\ln \alpha - 1) \right) + \frac{\mu_0 q_0}{4\pi^2} d_2 d_3 \right),
$$

where $\mu_0$ is the reduced mass of the $\pi^0 n$ system, and

$$
\xi = \frac{\alpha \mu_c^2}{2\pi} \left\{ \left( \mu^2 \right)^{d-3} \left( \frac{1}{d - 3} - \Gamma'(1) - \ln 4\pi \right) + \ln \left( \frac{2\mu_c^2}{\mu^2} \right) - 1 \right\},
$$

$$
q_0 = \left[ 2\mu_0 (m_p + M_{\pi^+} - m_n - M_{\pi^0}) \right]^{1/2}. \quad (15)
$$

Here $d$ and $\mu$ denote the dimension of space and the scale of the dimensional regularization, respectively (we use the dimensional regularization scheme throughout the paper).

The vacuum polarization contribution, which is calculated separately, should be added to the electromagnetic energy shift. The modification of Coulomb potential due to electron vacuum polarization is given by the well-known expression [16]

$$
\Delta V^\text{vac}(r) = -\frac{4}{3} \alpha^2 \int \frac{d^3k}{(2\pi)^3} \ e^{ikr} \int_0^\infty \frac{ds}{s+k^2} \frac{1}{s} \left( 1 + \frac{2m_e^2}{s} \right) \sqrt{1 - \frac{4m_e^2}{s}}, \quad (16)
$$

where $m_e$ denotes the electron mass. Applying the first-order perturbation theory, we find

$$
E^\text{vac} = \int d^3r \, \Psi_0^2(\mathbf{r}) \Delta V^\text{vac}(r) = -\frac{\alpha^3 \mu_c^3}{3\pi} \eta^2 \Phi(\eta), \quad \eta = \frac{\alpha \mu_c}{m_e}, \quad (17)
$$

$$
\eta^2 \Phi(\eta) = \frac{1}{\eta^3} \left( 2\pi - 4\eta + \frac{3\pi}{2} \eta^2 - \frac{11}{3} \eta^3 \right) + \frac{2\eta^4 - \eta^2 - 4}{\eta^3 \sqrt{\eta^2 - 1}} \ln(\eta + \sqrt{\eta^2 - 1}).
$$
Table 1. Contributions to the electromagnetic binding energy of the $\pi^-p$ atom (eV). Higher-order corrections: vacuum polarization correction (order $\alpha^3$) and vertex correction have not been calculated.

| Corrections                                    | Notation | Ref. [1]     | Our         |
|-----------------------------------------------|----------|--------------|-------------|
| Point Coulomb, KG equation                    | $E^{\text{KG}}$ | $-3235.156$ | $-3235.156$ |
| Finite size effect (proton, pion)             | $E^{\text{fin}}$ | $0.102$   | $0.100$     |
| Vacuum polarization, order $\alpha^2$        | $E^{\text{vac}}$ | $-3.246$   | $-3.241$    |
| Relativistic recoil, proton spin and           |          |              |             |
| anomalous magnetic moment                     | $E^{\text{rel}}$ | $0.047$   | $0.047$     |
| Vacuum polarization, order $\alpha^3$        |          | $-0.018$   |             |
| Vertex correction                              |          | $0.007$     |             |

The calculation of the electromagnetic energy-level shift is now complete. In order to compare our results with those given in Ref. [1], it is convenient to define

$$\tilde{E}_1 + \Delta E_1^{\text{em}} = E^{\text{KG}} + E^{\text{fin}} + E^{\text{vac}} + E^{\text{rel}},$$

(18)

where

$$E^{\text{KG}} = -\frac{1}{2} \mu_c \alpha^2 \left( 1 + \frac{5\alpha^2}{4} \right),$$

$$E^{\text{fin}} = \frac{2}{3} \alpha^4 \mu_c^3 \left( \langle r_p^2 \rangle + \langle r_\pi^2 \rangle \right),$$

(19)

$$E^{\text{rel}} = \frac{1}{2} \mu_c \alpha^4 \left( \frac{\mu_c}{4(m_p + M_\pi)} + \frac{m_p^2}{(m_p + M_\pi)^2} \right) - 1 - \frac{2\mu_p M_{\pi^+}^2}{(m_p + M_\pi)^2},$$

and $E^{\text{vac}}$ is given by Eq. (17). In Table 1 we collect various contributions to the electromagnetic energy-level shift calculated using the same input parameters, as in Ref. [1]. As can be seen from Table 1, the calculated values of the electromagnetic shift almost coincide.

In order to complete the calculation of the strong energy-level shift, one has to match at the accuracy $O(\delta)$ the particular combination of the non-relativistic couplings $d_i$, $i = 1 \cdots 3$ that appears in Eq. (14). To this end, we consider the scattering operator

$$T_R(z) = V_R + V_R G_R(z) T_R(z),$$

10
\[ V_R = H_C + eH_\gamma + H_S + e^2\lambda H_\lambda, \quad G_R(z) = (z - H_0 - H_R)^{-1}. \] (20)

In the scattering operator \( T_R(z) \), all kinematical insertions contained in \( H_R \) are summed up in the external lines (see [21] for the details). We calculate the matrix element of the scattering operator \( T_R(z) \) between the \( \pi^-p \) states at \( O(\delta) \). After removing the CM momentum, the spin-nonflip part of this matrix element on energy shell is equal to

\[
(q|t^n_R(z)|p) = -\frac{4\pi\alpha}{|q - p|^2} - \frac{4\pi \alpha}{4m_p M^+} \frac{(q + p)^2}{|q - p|^2} + e^{2i\alpha\theta_C(|p|)} (q|\bar{t}^n_R(z)|p),
\]

(21)

where the (divergent) Coulomb phase in the dimensional regularization scheme is given by

\[
\theta_C(|p|) = \frac{\mu_c}{|p|} \mu^{d-3} \left( \frac{1}{d - 3} - \frac{1}{2} (\Gamma'(1) + \ln 4\pi) \right) + \frac{\mu_c}{|p|} \ln \frac{2|p|}{\mu},
\]

(22)

and

\[
\text{Re} (q|\bar{t}^n_R(z)|p) = -\frac{\pi \alpha \mu_c d_1}{|p|} + \frac{\alpha \mu_c^2 d_1^2}{\pi} \ln \frac{|p|}{\mu_c} + e^2\lambda
\]

\[ -d_1 + \frac{\mu_c d_0^2}{4\pi^2} d_2 d_3 + d_1^2 \xi + \cdots, \]

(23)

where ellipses stand for the terms that vanish at threshold.

In order to carry out the matching, we consider the elastic \( \pi^-p \) scattering amplitude \( T_{\pi^-p\to\pi^-p} \), calculated in the relativistic theory at \( O(\delta^4) \). This amplitude can be uniquely decomposed into the piece containing all diagrams that are made disconnected by cutting one photon line, and the remainder: \( T_{\pi^-p\to\pi^-p} = T_{ex} + T_1 \), with

\[
T_{ex} = \frac{e^2}{t} \bar{u}(p_1 s) \left\{ \gamma_\mu F_1(t) + i\sigma_{\mu\nu}(p_1 - q_1)^\nu \frac{F_2(t)}{2m_p} \right\} u(q_1 s') (p_2 + q_2)^\mu f(t),
\]

(24)

where \( (q_1, s), \ q_2 \) and \( (p_1, s') \), \( p_2 \) are the four-momenta and spin of outgoing/incoming particles, \( t = (q_1 - p_1)^2 \), and \( F_1(t) \) and \( f(t) \) denote, respectively, the nucleon and pion electromagnetic form factors. Then, from the matching

\[ \text{Note that, in order to have the same sign as in the non-relativistic definition, the sign of the relativistic amplitude is defined by } S = 1 - iT. \]
condition \( T_{\pi^-p\to\pi^-p} = (2E_qE_p\omega_q\omega_p)^{1/2}(q; s|t_{R,+}(z)|p; s') \), where \( E_q, E_p \) and \( \omega_q, \omega_p \) denote the relativistic energies of the outgoing/incoming nucleons and pions, respectively, one may conclude, that the spin-nonflip part \( T_1^n \) of the remainder amplitude \( T_1 \) should have the following threshold behavior

\[
\text{Re} \left\{ e^{-2\alpha\beta\epsilon(|p|)} T_1^n \right\} = \frac{B_1}{|p|} + B_2 \ln \frac{|p|}{\mu_c} - 8\pi(m_p + M_{\pi^+})A + \cdots,
\]

(25)

where \( p \) denotes the relative momentum of \( \pi^-p \) pair in the CM frame, \( B_1, B_2 \) do not depend on \( p \), and

\[
-\frac{2\pi}{\mu_c}A = -d_1 + \frac{\mu_c^2 d_0^2}{4\pi^2} d_2^2 d_3 + d_2^2 \xi
\]

(26)

where \( A \) is the regular part of \( \pi^-p \) scattering amplitude at threshold.

Substituting Eq. (26) into Eq. (14), we finally arrive at

\[
\epsilon_{1s} = -2\alpha^3 \mu_c^2 A (1 - 2\alpha\mu_c (\ln \alpha - 1) A) + \cdots,
\]

(27)

where the UV divergence contained in the quantity \( \xi \), has been cancelled.

The equation (27) is the main result of the present paper - it gives the relation between the measured strong energy-level shift of the \( \pi^-p \) atom and the \( \pi^-p \) scattering amplitude at threshold, defined by Eqs. (24) and (25). This relation is valid at \( O(\delta) \), and in all orders in the chiral expansion. The threshold amplitude still contains the isospin-breaking effects. Consequently, in order to extract from the experimental data the \( \pi N \) scattering lengths which are defined for the case with no isospin symmetry violation, one has to single out the isospin-breaking effects in the amplitude. To this end, one may invoke ChPT - below, we present the results of the calculations at the tree level.

In order to evaluate the \( \pi N \) scattering amplitude, we use the effective chiral pion-nucleon Lagrangian at \( O(p^2) \) [23–25]. The amplitude is given by the following expression

\[
A = a_{0^+}^+ + a_{0^+}^- + \epsilon = \frac{1}{8\pi(m_p + M_{\pi^+})F_\pi^2} \left\{ m_p M_{\pi^+} - \frac{g_A^2 m_p M_{\pi^+}^2}{m_n + m_p + M_{\pi^+}} \right\} + m_p (-8c_1 M_{\pi^0}^2 + 4(c_2 + c_3) M_{\pi^+}^2 - 4e^2 f_1 - e^2 f_2),
\]

(28)

where \( c_i, f_i \) denote the low-energy constants in the \( O(p^2) \) chiral pion-nucleon Lagrangian, \( g_A \) is the nucleon axial-vector coupling constant [23–25], and in
our calculations we use the value $F_\pi = 92.4$ MeV. The scattering lengths $a_{0^+}^+, a_{0^-}^-$ are calculated in the isospin symmetric theory with $e = 0$, $m_u = m_d$ and where, by convention, the masses of the pions and nucleons are taken equal to the physical masses of the charged pion and proton, respectively. With this convention, the isospin-breaking correction to the $\pi^-p$ elastic scattering amplitude at $O(p^2)$ is equal to

$$\epsilon = \frac{m_p}{8\pi(m_p + M_{\pi^+})F_\pi^2} \left( 8c_1(M_{\pi^+}^2 - M_{\pi^0}^2) - 4e^2f_1 - e^2f_2 \right). \quad (29)$$

Below, we present the results of the numerical analysis for the isospin-breaking correction to the $\pi^-p$ atom energy-level shift. In order to evaluate the second term in the brackets in Eq. (27), one may safely approximate $A = a_{0^+}^+ + a_{0^-}^-$ and use the scattering lengths given in Ref. [2]: the total correction coming from this term amounts up to $+0.66 \times 10^{-2}$. Further, in order to evaluate the isospin-breaking correction to the $\pi^-p$ scattering amplitude at threshold given by Eq. (29), one has to specify the values of the $O(p^2)$ low-energy constants $c_1$, $f_1$ and $f_2$. We use the value of $c_1$ determined from the fit of the elastic $\pi N$ scattering amplitude at threshold to KA86 data [26]: $c_1 = -0.925$ GeV$^{-1}$. The value of the constant $f_2$ can be extracted from the proton-neutron electromagnetic mass difference [27]: $e^2f_2 = (-0.76 \pm 0.3)$ MeV. The determination of the constant $f_1$ from data is however, problematic. For this reason, in our analysis we have used order-of-magnitude estimate for this constant: $-|f_2| \leq f_1 \leq |f_2|$. The prediction for the $\pi^-p$ ground-state energy-level shift based on $O(p^2)$ calculations in ChPT, strongly deviate from the predictions of the potential model. The prediction for the total isospin-breaking correction in the strong energy-level shift, defined according to [1] $\epsilon_{1s} = \epsilon_{1s}^{LO}(1 + \delta_\epsilon)$, where $\epsilon_{1s}^{LO}$ stands for the leading-order shift given by Deser formula [13], is: $\delta_\epsilon = (-4.8 \pm 2.0) \cdot 10^{-2}$ at $O(p^2)$ in ChPT. This result differs more than twice from the prediction based on the potential model [1]: $\delta_\epsilon = (-2.1 \pm 0.5) \cdot 10^{-2}$, and the uncertainty caused mainly by a poor knowledge of the constant $f_1$, is four times larger than the estimate of the systematic error in the potential model. In demonstration of the above discussion, in Fig. 1 we confront the results of the present analysis with those of the potential model, using the experimental data on hadronic atoms taken from Ref. [2].

We would like to emphasize that the physical effects encoded in the low-energy constants $c_1$, $f_1$ and $f_2$, are absent in the potential model. Namely, $f_1$ and $f_2$ contain the effect of the direct quark-photon interaction, and the correction proportional to $c_1$ contains the effect coming from the dependence of the scattering amplitude on the quark mass - similar effect in the $\pi^+\pi^-$ case constitutes the tree-level “mass-shift” correction (see, e.g. [15,16,18]). For this reason, the discrepancy between the potential model predictions and the
Fig. 1. Determination of $\pi N$ scattering lengths from the pionic hydrogen and pionic deuterium measurements. Solid line corresponds to the energy-level shift calculations at $O(p^2)$ in ChPT, and the dashed lines - to the potential model results[1].

results of the present analysis based on ChPT, does not come to our surprise. It remains to be seen, how the $O(p^2)$ results in ChPT are altered by the loop corrections. A reliable estimate of the constant $f_1$ is also desirable.

To conclude, we have derived the general formula that relates the energy-level shift of the $\pi^- p$ atom to the $\pi N$ scattering amplitude at threshold. Numerical analysis carried out on the basis of this formula at $O(p^2)$ in ChPT, already indicates at the necessity to critically reaccess the values of the $\pi N$ scattering lengths, extracted from the energy-level shift measurement by means of the potential model-based theoretical studies.

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