On a Duality in Calogero-Moser-Sutherland Systems

To Vladimir Igorevich Arnol’d on his 60th birthday

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We point out a map between the dynamics of a non-relativistic system of $N$ particles in one dimension interacting via the pair-wise potentials $U_I(q) = \frac{\nu^2}{4R^2\sin^2(q/R)}$ and the one of the particles with the pair potential $U_{II}(q) = \frac{\nu^2}{q^2}$ and the external potential $U_{ext} = \frac{\omega^2 q^2}{2}$. The natural relation between the frequency $\omega$ and the radius $R$ is: $\omega R = 1$.

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1. Formulation of the problem and the main statement

Consider a system of indistinguishable particles on a circle \( S^1_R \) of the radius \( R \), interacting with the pair-wise potential

\[
U_I(q) = \frac{\nu^2}{4R^2 \sin^2 \left( \frac{q}{2R} \right)}
\]  

From the Hamiltonian point of view the system has the phase space:

\[
M_I = \left[ T^*(S^1_R)^N \right]/S_N
\]

where \( S_N \) is the \( N \)-th symmetric group. The coordinates in the phase space will be denoted as \((p_i, q_i)\) where \( q_i \) is the angular coordinate on the circle \( S^1_R \) and \( p_i \) is the corresponding momentum. The Hamiltonian \( H_I \) which corresponds to (1.1) has the natural form:

\[
H_I = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i \neq j} U_I(q_i - q_j)
\]  

This is a well-known Sutherland model \([1]\). The second model of our interest is the Calogero model \([2]\), which describes the particles on a real line \( \mathbb{R} \) with the Hamiltonian

\[
H_{II} = \sum_i \frac{p_i^2}{2} + \frac{\omega^2 q_i^2}{2} + \sum_{i<j} U_{II}(q_i - q_j), \quad U_{II}(q) = \frac{\nu^2}{q^2}
\]  

The phase space is

\[
M_{II} = \left[ T^*\mathbb{R}^N \right]/S_N
\]

The systems \( I \) and \( II \) are integrable. Each Hamiltonian \( H_{I,II} \) is contained in the commuting family of Hamiltonians \( H_{I,II}^{(k)} \) with \( k = 0, \ldots, N - 1 \). We claim that these two integrable systems are equivalent in the following sense:

The main result. There exists an injective map

\[
\pi : M_{II} \to M_I
\]

such that

a. \( H_{II}^{(k)} = \pi^* H_I^{(k)} \)

b. \( H_I = H_{I}^{(1)}, \ H_{II} = H_{II}^{(0)} \)

c. Locally \( \pi \) is a symplectomorphism. In physical terms, the image of the map \( \pi \) consists of the configurations of particles on a circle, which all have roughly speaking positive momenta. More precisely, the action variables must be non-negative in appropriate normalization. Of course, the sign of the action variable is not uniquely defined. What we mean by positivity here is that under some choice of the signs all the action variables must be non-negative. This condition defines a chamber in the phase space \( M_I \) which is the image of \( \pi \).
2. Construction via Hamiltonian reduction

Recall the Hamiltonian reductions giving rise to the systems $I$ and $II$ respectively. The main idea behind the construction is to realize the (classical) motion due to the Hamiltonians $H_I, H_{II}$ as a projection of the simple motion on a somewhat larger phase space $E$.

Construction of the system $I$. In the $I$ case (we essentially follow [4] here) one starts with the symplectic manifold $X_I = T^* G \times \mathbb{C}^N$ where $G = U(N)$ with the canonical Liouville form

$$\Omega_I = i \text{Tr} \delta(p \wedge \delta gg^{-1}) + \frac{1}{2i} \delta v^+ \wedge \delta v$$

(2.1)

Here $p$ represents the cotangent vector to the group $G$. We think of it as of the Hermitian matrix. The manifold $X_I$ is acted on by $G$ (by conjugation on the $T^* G$ factor and in a standard way on $\mathbb{C}^N$). The action is Hamiltonian with the moment map:

$$\mu_I = p - g^{-1}pg - v \otimes v^+$$

(2.2)

One performs the reduction at the central level of the moment map, i.e. takes the manifold $\mu_I^{-1}(-\nu \cdot \text{Id})$ and takes its quotient by $G$ (as it is invariant). Explicitly, one solves the equation

$$p - g^{-1}pg - v \otimes v^+ = -\nu \text{Id}$$

(2.3)

up to the $G$-action. The way to do it is to fix a gauge

$$g = \exp \left( \frac{i}{R} \text{diag}(q_1, \ldots, q_N) \right)$$

and then solve for $p$ and $v$. One has:

$$v_i = \sqrt{\nu}, \quad p_{ij} = Rp_i \delta_{ij} + \nu \frac{1 - \delta_{ij}}{e^{\frac{(q_i - q_j)}{R}} - 1}$$

As a result one gets the reduced phase space:

$$M_I = \left[ T^* (S^1)^N \right] / S_N$$

(2.4)

with the canonical symplectic structure

$$\Omega_I^{\text{red}} = \sum_i \delta p_i \wedge \delta q_i$$
The functions on $X_I$ invariant under the action of $G$ descend down to $M_I$. Moreover the Poisson-commuting functions descend to Poisson-commuting ones in the reduced Poisson structure. The Hamiltonian $H_I$ comes from the quadratic casimir $\text{Tr}p^2$. The complete set of functionally independent integrals is given by:

$$H_I^{(k)} = \frac{1}{R^{k+1}} \text{Tr}p^{k+1}, \quad k = 0, \ldots, N - 1$$  \hspace{1cm} (2.5)$$

Construction of the system $II$. In the $II$ case one starts (again, essentially following [3] [4]) with the symplectic manifold $X_{II} = T^*g \times \mathbb{C}^N$ where $g = \text{Lie}U(N)$ with the canonical Liouville form

$$\Omega_{II} = \text{Tr}\delta(P \wedge \delta Q) + \frac{1}{2i} \delta v^+ \wedge \delta v$$  \hspace{1cm} (2.6)$$

The manifold $X_{II}$ is acted on by $G$ (by conjugation on the $T^*g$ factor and in a standard way on $\mathbb{C}^N$). The action is Hamiltonian with the moment map:

$$\mu_{II} = [P, Q] - v \otimes v^+$$  \hspace{1cm} (2.7)$$

One performs the reduction at the central level of the moment map, i.e. takes the manifold $\mu_{II}^{-1}(-\nu \cdot \text{Id})$ and takes its quotient by $G$ (as it is invariant). Explicitly, one solves the equation

$$[P, Q] - v \otimes v^+ = -\nu \text{Id}$$  \hspace{1cm} (2.8)$$

up to the $G$-action. The way to do it is to fix a gauge

$$Q = \text{diag}(q_1, \ldots, q_N)$$

and then solve for $P$ and $v$. As a result one gets the reduced phase space:

$$M_I = \left[ T^*\mathbb{R}^N \right] / S_N$$  \hspace{1cm} (2.9)$$

The functions on $X_{II}$ invariant under the action of $G$ descend down to $M_{II}$. Moreover the Poisson-commuting functions descend to Poisson-commuting ones in the reduced Poisson structure. The Hamiltonian $H_{II}$ comes from the quadratic casimir $\text{Tr}(P^2 + \omega^2 Q^2)$. A convenient set of functionally independent integrals is given by:

$$H_{II}^{(k+1)} = \text{Tr}(ZZ^+)^{k+1}, \quad k = 0, \ldots, N - 1$$  \hspace{1cm} (2.10)$$
with the matrix:

\[ Z = P + i\omega Q \in \mathfrak{g}_C \]  

(2.11)

The explicit form of \( Z \), solving (2.7) is

\[ Z = \text{diag}(p_i + i\omega q_i) + \|i\nu(1 - \delta_{ij})\| \frac{q_i - q_j}{q_i - q_j} \]

In the paper [5] the matrix \( Z \) was called \( L^+ \) and together with \( L^- = Z^\dagger \) they formed a pair of "Lax-like" matrices. It was shown that \( L^\pm \) evolve according to the following generalization of Lax equations:

\[ i\dot{L}^\pm = [M, L^\pm] \pm \omega L^\pm \]  

(2.12)

Clearly, this is just the simple flow in the \( Z, Z^\dagger \) space:

\[ Z(t) = e^{-i\omega t} Z(0) \]

projected onto the space of eigenvalues of \( Q = \frac{1}{2i\omega}(Z - Z^\dagger) \). See [6][7] for the study of the relations between the trajectories of the system \( II \) and the \( R \to \infty \) limit of the system \( I \).

3. The Map \( \pi \).

3.1. The Polar Decomposition

The map \( \pi \) is constructed first on the level of unreduced systems, and then by \( G \)-equivariance descends to the map of the reduced phase spaces. The map \( \pi \) sends \( v \) identically to \( v \). Let us discuss its \( Z \mapsto (p, g) \) part (we also denote it by \( \pi \)). The map \( \pi \) sends \( Z \) to its polar decomposition:

\[ \pi(Z) = (p, g) \quad Z = \omega^\frac{1}{2} p^\frac{1}{2} g, \]

\[ g \in G, g^\dagger g = 1, p^\dagger = p, \quad p = \frac{1}{\omega} ZZ^\dagger \quad (3.1) \]

The image of \( \pi \) is clearly the space \( U \) of all pairs \( (p, g) \) where \( p \) is a Hermitian matrix with non-negative eigenvalues, and \( g \) is a unitary matrix. It is a subset of \( T^*G \), which we denote as \( T^*_+ G \). Now it is a straightforward computation to check that

\[ \pi^* \Omega_{II} = \Omega_I \]  

(3.2)
Clearly, $\pi$ commutes with the action of $G$:

$$\pi(UZU^{-1}; Uv) = U \cdot (p, g; v) \equiv (UpU^{-1}, UgU^{-1}; Uv) \quad (3.3)$$

Finally, the Hamiltonians $H_{II}^{(k+1)} = \mathrm{Tr}(ZZ^\dagger)^{k+1}$ map to $(\omega R)^{k+1} H_I^{(k)} = \mathrm{Tr}(\omega p)^{k+1}$. So, in order to achieve (a.) of (1.6) one must choose

$$\omega = \frac{1}{R} \quad (3.4)$$

Under the map $\pi$ the quadratic Hamiltonian $H_I$ of the Calogero model is mapped to the linear (total momentum) Hamiltonian $H_{II}^{(0)} = \sum_i p_i$ of the Sutherland model. The quadratic Hamiltonian of the Sutherland system is mapped to the quartic hamiltonian of Calogero system.

### 3.2. Generalizations

One can obviously generalize the construction to cover the spin generalizations of the Calogero [8] as well as of the Sutherland models (trigonometric case of [9]). In fact, the map $\pi$ remains the same, the only change is that one starts with a bigger phase space: $T^*g \times S$, where the symplectic $G$-manifold $S$ produces spin degrees of freedom (for example, one take as $S$ a general coadjoint orbit of $G$, see [10] for a discussion of a general spin system). The map $\pi$ is again an identity on $S$ and the same (3.1) on $T^*g$ mapping it to $T^*_+G$.

### 4. The Quantum Case

The wave-functions of the Sutherland system can be described with the help of a quantum version of the reduction we sketched in the previous section [11][12][13]. In short, one fixes a representation $R$ of $G$. The simplest case is to take $R = S^{N\nu} \mathbb{C}^N$. Then one represents a wavefunction:

$$\psi(q_1, \ldots, q_N)$$

as a $R$-valued function $\Psi$ on $G$ with the following equivariance properties:

$$\Psi(UgU^{-1}) = T_R(U)\Psi(g) \quad (4.1)$$

Using the known behavior of $\Psi$ under the adjoint $G$ action one restricts $\Psi$ onto the maximal torus $T$, where (4.1) implies that $\Psi$ takes values in the $T$ - invariant subspace of $R$, which is one-dimensional for our choice of $R$. Therefore one gets a number-valued function. In
order to specify the wavefunction uniquely (up to a phase factor) one uses all quantum Hamiltonians: \( \hat{H}^{(k)}_I \) and normalization conditions as well:

\[
\hat{H}^{(k)}_I \Psi_\lambda = E^{(k)}_I(\lambda) \Psi_\lambda
\]

where \( \lambda \) is a spectral parameter.

Now the quantum version of a map \( \pi \) looks as follows. Given a function \( \Psi(g) \) obeying (4.1) one constructs a unique holomorphic function \( \Psi(Z) \), which coincides with \( \Psi(g) \) when \( Z = g \). Then one passes to the real polarization, using the relations like: \( Z = \frac{1}{i}(\frac{\delta}{\delta Q} - \omega Q) \). One arrives at a function \( \chi(Q) \), which enjoys certain \( G \)-equivariance properties. Therefore one can restrict \( \chi \) to the diagonal matrices \( Q = \text{diag}(q_1, \ldots, q_N) \) giving rise to the wavefunction of the Calogero system, which is the common eigenfunction of the family of Calogero quantum integrals \( \hat{H}^{(k)}_{II} \). The subtlety in this transformation is that the quantum version of the operators:

\[
\text{Tr}(ZZ^\dagger)^{k+1}
\]

is defined up to the normal ordering ambiguity, which allows one to shift \( \hat{H}^{(k)}_{II} \) by an integral linear combination of \( \hat{H}^{(k')}_{II} \) for \( k' < k \). In particular, the eigenvalues \( E^{(1)}_{II}(\lambda) \) may differ from \( E^{(0)}_I(\lambda) \) by a \( \lambda \) independent constant.

As an illustration of this, compare the formula for the spectrum of the Schrödinger operator of the Calogero model (borrowed essentially from [2]) and the trivial formula for the spectrum of the total momentum operator of the Sutherland model. The formula of [2] gives a spectrum of the operator

\[
\hat{H}_0 = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2} + \sum_{i<j} \left( \frac{g}{q_{ij}} + \frac{\omega_0^2}{4} q_{ij}^2 \right)
\]

(4.2)

where \( q_{ij} = q_i - q_j \) in the center-of-mass frame, i. e. on the set of functions, annihilated by \( \hat{P} = \frac{1}{i} \sum_i \frac{\partial}{\partial q_i} \). In order to map it to the problem considered in this paper we use that

\[
\hat{H} = \hat{H}_0 + N\omega^2 \frac{q_*^2}{2}, \quad q_* = \frac{1}{N} \sum_{i=1}^{N} q_i
\]

(4.3)

is the quantum counterpart of (L.4) for \( \omega = \omega_0 \sqrt{N/2} \) (this model also has been considered in [1]). Notice, that the center-of-mass motion can be separated (one can choose \( q_* \) as one
of coordinates, the rest of the coordinates being, say, the Jacobi coordinates). We only need to know that (2):

\[- \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2} = -\frac{1}{2N} \frac{\partial^2}{\partial q_i^2} + \ldots\]

to conclude that the spectrum of (4.3) is the sum of the one of (4.2) and \( \omega \left( \frac{1}{2} + n \right) \), \( n \in \mathbb{Z}, n \geq 0 \). Hence,

\[ E_{\vec{n}}^{Calogero} = \omega \left( \frac{N}{2} + \frac{N(N-1)}{2} \nu + \sum_{l=1}^{N} \ln_{l} \right), \ n_l \geq 0 \]  

(4.4)

where \( g = \nu(\nu - 1) \). This spectrum can be represented as follows:

\[ E_{\vec{n}}^{Calogero} = \omega \sum_{k=1}^{N} (n_k + (k-1) \nu + \frac{1}{2}), \ 0 \leq n_1 \leq \ldots \leq n_N \]  

(4.5)

The integers \( \vec{n} \) are the spectral parameters \( \lambda \) of the earlier discussion.

Now let us look at the Sutherland operator:

\[ \hat{H} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2} + \nu \frac{\nu - 1}{4R^2} \sum_{i>j} \frac{1}{\sin^2 \left( \frac{q_{ij}}{2R} \right)} \]  

(4.6)

One can easily deduce from (1) that the spectrum of this operator is given by:

\[ E_{\vec{n}}^{Sutherland} = \frac{1}{2R^2} \sum_{i} \left( n_i - \nu \left( \frac{N}{2} + \frac{N+1}{2} \right) \right)^2, \ 0 \leq n_1 \leq \ldots \leq n_N \]  

(4.7)

and the eigenvector with the eigenvalue \( E_{\vec{n}}^{Sutherland} \) has the following form (1):

\[ \Psi_{\vec{n}}(\vec{q}) = |\Delta(\vec{q})|^\nu \sum_{\vec{n}'} c_{\vec{n}',\vec{n}} \Phi_{\vec{n}'} \]  

(4.8)

where

\[ \Phi_{\vec{n}} = \sum_{\sigma \in S_N} e^{\frac{i\pi}{4} \sum_{k} q_k n_{\sigma(k)}}, \ \Delta(\vec{q}) = \prod_{i<j} \sin \left( \frac{q_{ij}}{2R} \right) \]  

(4.9)

The only important for us information about the coefficients \( c_{\vec{n}',\vec{n}} \) is that for all \( \vec{n}' \) such that \( c_{\vec{n}',\vec{n}} \neq 0 \) the total momentum is conserved: \( \sum_{k} n'_k = \sum_{k} n_k \). The total momentum eigenvalue is equal to:

\[ P_{\vec{n}} = \frac{1}{R} \sum_{k} \left( (n_k - \nu \left( \frac{N}{2} + \frac{N+1}{2} \right)) + \nu(k-1) + \frac{1}{2} \right) - \frac{N}{2R} \]  

(4.10)
As a periodic function in $q_k$'s, $\Psi$ can be expanded in Fourier series. The natural quantum analogue of the condition that the matrix $p \sim ZZ^\dagger$ has only positive eigenvalues is the condition that all Fourier modes of $\Psi$ are positive. A little inspection of the structure of $\Delta(\vec{q})$ implies that all $n_k$ must obey the condition

$$n_k \geq \frac{\nu}{2}(N - 1)$$

Now it is easy to see that up to the overall $\vec{n}$-independent shift (4.10) and (4.5) actually do coincide, provided that $\omega R = 1$. Note in conclusion that in the quantum problem the coupling $g$ is written as $g = \nu(\nu - 1)$ while in the classical as $\nu^2$. As far as the relation between the two systems is concerned the actual couplings $g$ are the same

$$g_I = g_{II}$$

5. Discussion

The phenomenon which we observed here is similar to a duality in integrable systems, recently discussed in [13]. It is very interesting to see whether this duality has anything to do with the relation between the $D$-brane matrix theories and gauge theories on $D$-probes. Among more direct extensions of this work one may try to find the analogue of the relation we pointed out for other root systems, as well as for elliptic systems. For the latter purpose the existence of Hamiltonian reduction [15] may be of some use.

The Sutherland model was called a system of Type II in Olshanetsky-Perelomov classification of integrable pair potentials [16], while the Calogero model is of the Type V. The systems I - III are the degenerations of the elliptic system (type IV) with the potential $U(q) = \wp(q)$. In that respect the system of the Type V was always a special case of such a classification. We have shown that it is actually a particular subsystem of the Type II model.

Notice that one of the corollaries of our result is that the action-angle variables for the two systems coincide. Also, it implies that the system, dual to the Calogero model in the sense of action-coordinate duality of [14] is nothing but the rational Ruijsenaars-Schneider model [17][18].

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