A REFINEMENT OF SOME PREVIOUS RESULTS OF BERNARDARA-MARCOLLI-TABUADA AND ORNAGHI-PERTUSI

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ABSTRACT. The Voevodsky nilpotence conjecture was proved by Bernardara-Marcolli-Tabuada for certain quadric fibrations, intersections of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, and Moishezon manifolds, and later by Ornaghi-Pertusi for certain cubic fourfolds and Gushel-Mukai fourfolds. In this paper we refine these results by using the algebraic equivalence relation instead of the nilpotence equivalence relation. Along the way, we address also certain cases of Küchle fourfolds, families of Sextic del Pezzo surfaces and families of Fano fourfolds of K3 type.

1. INTRODUCTION

Let $k$ be a base field. Given a smooth proper $k$-scheme $X$, let us denote by $Z^\ast(X)_\mathbb{Q}$ the (graded) $\mathbb{Q}$-vector space of algebraic cycles on $X$. It is well-known that one can define many different equivalence relations on $Z^\ast(X)_\mathbb{Q}$, such as the rational equivalence relation $\sim_{\text{rat}}$, the algebraic equivalence relation $\sim_{\text{alg}}$, the nilpotence equivalence relation $\sim_{\text{nil}}$, the numerical equivalence relation $\sim_{\text{num}}$, etc; consult [12]. Voevodsky conjectured in [44] that the nilpotence and numerical equivalence relations agree. This conjecture was proved by Bernardara-Marcolli-Tabuada [8] for certain quadric fibrations, intersections of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, and Moishezon manifolds, and later by Ornaghi-Pertusi [38] for certain cubic fourfolds and Gushel-Mukai fourfolds. As shown by Voevodsky in [44], every algebraic cycle which is algebraically trivial is also nilpotently trivial. Consequently, it is natural to ask if the aforementioned results also hold when the nilpotence equivalence relation is replaced by the algebraic equivalence relation? Our main result is an affirmative answer to this question:

**Theorem 1.1 (Refinement).** The algebraic and numerical equivalence relations agree for certain quadric fibrations, intersections of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, Moishezon manifolds, cubic fourfolds and Gushel-Mukai fourfolds. In addition, they also agree for certain Küchle fourfolds, families of Sextic del Pezzo surfaces and families of Fano fourfolds of K3 type.

Theorem 1.1 follows from the combination of Corollaries 5.4, 5.6, 5.17, 5.22, 5.26 with Theorems 5.7, 5.8, 5.11, 5.12, 5.14 and 5.28; consult §5.2–§5.9 and §5.11 below.

**Remark 1.2.** Theorem 1.1 does not hold for every smooth proper $k$-scheme! For example, it follows from the pioneering work of Griffiths [14] that if $X$ is a general quintic 3-fold, then the algebraic and numerical equivalence relations on $Z^\ast(X)_\mathbb{Q}$ do not agree; consult also the later works of Clements [10] and Voisin [46] for further examples.

In addition to Theorem 1.1, we also prove that the quotient between any two equivalence relations is invariant under Homological Projective Duality in the sense of Kuznetsov [22]; consult Theorem 5.19 below. This allows us to explicitly compute the quotient between the rational and the algebraic equivalence relations for certain 5-folds; consult Corollaries 5.23, 5.24 below.

2. PRELIMINARIES

Throughout this paper, $k$ denotes a base field.

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1 Thanks to the work of Kahn-Sebastian, Matsusaka, Voevodsky, and Voisin (see [15, 39, 44, 45]), Voevodsky’s conjecture is known in the case of curves, surfaces, and abelian 3-folds (when $k$ is of characteristic zero).
2.1. Dg categories. We recall here some basic concepts on differential graded categories; for a survey, consult [16].

Let \((C(k), \otimes, k)\) be the category of (cochain) complexes of \(k\)-vector spaces. A differential graded (dg) category \(A\) is a category enriched over \(C(k)\) and a dg functor \(F : A \to B\) is a functor enriched over \(C(k)\). Note that every \(k\)-algebra \(A\) gives rise to a dg category with a single object. Recall also that, given any quasi-compact quasi-separated \(k\)-scheme \(X\), the category of perfect complexes \(\text{perf}(X)\) admits a canonical dg enhancement \(\text{perf}_d(X)\); see [16] §4.6.

Let \(A\) be a dg category. The opposite dg category \(A^{op}\) has the same objects of \(A\) and \(A^{op}(x, y) := A(y, x)\) and the category \(\text{H}^0(A)\) has the same objects of \(A\) and \(\text{H}^0(A)(x, y) = \text{H}^0(A(x, y))\); consult [16] §2.2. A right dg \(A\)-module is a dg functor \(M : A^{op} \to C_{dg}(k)\) with values in the dg category of complexes of \(k\)-vector spaces. The category of dg modules \(\text{C}(A)\) has the right dg \(A\)-modules as objects and the morphisms of dg functors as morphisms. The localization of \(\text{C}(A)\) with respect to the class of quasi-isomorphism is called the derived category \(\mathcal{D}(A)\) of \(A\). We will write \(\mathcal{D}_c(A)\) for the full subcategory of compact objects of \(\mathcal{D}(A)\); consult [16] §3.2.

A dg functor \(F : A \to B\) is a Morita equivalence if it induces an equivalence on the derived categories \(\mathcal{D}(A) \simeq \mathcal{D}(B)\); see [16] §4.6]. Given dg categories \(A\) and \(B\), its tensor product \(A \otimes B\) is the dg category whose set of objects is \(\text{obj}(A) \times \text{obj}(B)\) and \((A \otimes B)((x, w), (y, z)) := A(x, y) \otimes B(w, z)\). Following [16] §2.3], this construction gives a symmetric monoidal structure on the dg category \(\text{dgcat}(k)\) (of essentially small) dg categories and dg functors over the base field \(k\). A dg \(A\)-\(B\)-bimodule is a dg functor \(B : A \otimes B^{op} \to C_{dg}(k)\), i.e., a right dg \(A^{op} \otimes B\)-module. Given a dg functor \(F : A \to B\) there exists an induced dg \(A\)-\(B\)-bimodule associated to \(F\) defined as \(\_F : A \otimes B^{op} \to C_{dg}(k), (x, z) \mapsto B(z, F(x))\).

Following Kontsevich [17,18,19], a dg category \(A\) is called smooth if the dg \(A\)-\(A\)-bimodule \(i_A\) belongs to the subcategory \(\mathcal{D}_c(A^{op} \otimes A)\) and it is called proper if \(\sum_i \dim \text{H}^i(A(x, y)) < \infty\) for every pair of objects \(x, y \in A\). The dg category of the perfect complexes \(\text{perf}_d(X)\) associated to a smooth proper \(k\)-scheme \(X\) is an example of a smooth proper dg category; see [39] Example 1.42(ii)]. We shall write \(\text{dgcat}_{sp}(k)\) for the full subcategory of smooth proper dg categories.

Given dg categories \(A\) and \(B\) and a \(B\)-\(A\)-bimodule \(B\), consider the dg category \(T(A; B)\) whose set of objects is the disjoint union of the sets of objects of \(A\) and \(B\), and whose complexes of morphisms are defined as follows:

\[
T(A; B)(x, y) := \begin{cases} 
A(x, y) & \text{if } x, y \in A \\
B(x, y) & \text{if } x \in A, y \in B \\
B(y, x) & \text{if } x \in A, y \in B \\
0 & \text{if } y \in A, x \in B.
\end{cases}
\]

By construction, we have two canonical inclusions \(i_A : A \to T(A; B; B)\) and \(i_B : B \to T(A; B; B)\).

**Lemma 2.1.** Let \(A\) and \(B\) be two dg categories and \(B\) a dg \(B\)-\(A\)-bimodule. Given a dg category \(C\) we have a canonical identification between the dg categories \(T(A; B; B) \otimes C\) and \(T(A \otimes C, B \otimes C; B \otimes C)\).

**Proof.** The proof is simple and we leave it to the reader. \(\square\)

2.2. Noncommutative motives. We recall here some basic concepts on noncommutative motives. For a book, resp. survey, consult [39], resp. [40]. Recall from [39] §4.1] the construction of the category of noncommutative Chow motives with \(\mathbb{Q}\)-coefficients \(\text{NChow}(k)\). This category is \(\mathbb{Q}\)-linear, additive, idempotent complete, rigid symmetric monoidal (i.e., all its objects are dualizable) and is equipped with a symmetric monoidal functor \(U(-) : \text{dgcat}_{sp}(k) \to \text{NChow}(k)\). Moreover, given any two smooth proper dg categories \(A\) and \(B\), we have the following computation:

\[(2.1)\]

\[
\text{Hom}_{\text{NChow}(k)}(U(A)_{\mathbb{Q}}, U(B)_{\mathbb{Q}}) := K_0(\mathcal{D}_c(A^{op} \otimes B))_{\mathbb{Q}} = K_0(A^{op} \otimes B)_{\mathbb{Q}}.
\]

Furthermore, the composition law in \(\text{NChow}(k)\) is induced by the (derived) tensor product of bimodules.

Finally, recall from [39] §4.4 & §4.6] the construction of the categories of noncommutative Voevodsky motives \(\text{NVoev}(k)\) and noncommutative numerical motives \(\text{NNum}(k)\). These categories are also \(\mathbb{Q}\)-linear, additive, idempotent complete, and rigid symmetric monoidal.

3. Equivalence relations

Let \(A\) be a smooth proper dg category. In this section, we introduce three different equivalence relations on the rational Grothendieck group \(K_0(A)_{\mathbb{Q}} := K_0(\mathcal{D}_c(A))_{\mathbb{Q}}\) and compare them with their classical commutative counterparts.

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2When \(X\) is quasi-projective, this dg enhancement is moreover unique, see [32] Theorem 2.12]
3.1. Algebraic equivalence relation. Given a smooth connected \( k \)-scheme \( T \) and a \( k \)-rational point \( p : \text{Spec}(k) \to T \), consider the associated pull-back dg functor \( p^* : \text{perf}_{dg}(T) \to \text{perf}_{dg}(\text{Spec}(k)) \). Note that \( \text{perf}_{dg}(\text{Spec}(k)) \) and \( k \) are Morita equivalent.

**Definition 3.1.** An element \( \alpha \in K_0(\mathcal{A})_{/\sim_{\text{alg}}} \) is called algebraically trivial if it belongs to the image of the homomorphism
\[
\bigoplus_{(T,p,q)} K_0(\mathcal{A} \otimes \text{perf}_{dg}(T))_{Q} \xrightarrow{\oplus (K_0(id \otimes p^*)_{Q} - K_0(id \otimes q^*)_{Q})} K_0(\mathcal{A})_{Q},
\]
where the direct sum ranges over all smooth connected \( k \)-schemes \( T \) equipped with two \( k \)-rational points \( p \) and \( q \).

**Remark 3.2.** In the particular case where \( k \) is algebraically closed, any two \( k \)-rational points of \( T \) can be joined by a smooth projective \( k \)-curve. Hence, in this particular case, it suffices to consider the triples \( (C,p,q) \) where \( C \) is a smooth projective \( k \)-curve equipped with two \( k \)-rational points \( p \) and \( q \).

The above homomorphism \( (3.1) \) gives rise to the algebraic equivalence relation \( \sim_{\text{alg}} \) on \( K_0(\mathcal{A})_{Q} \). In what follows, we will write \( K_0(\mathcal{A})_{Q}/\sim_{\text{alg}} \) for the associated quotient, i.e., for the cokernel of \( (3.1) \).

**Lemma 3.3.** The functor \( K_0(-)_{Q}/\sim_{\text{alg}} : \text{dgcat}_{sp}(k) \to \text{Vect}(Q) \) sends Morita equivalences to isomorphisms.

*Proof.* Let \( F : \mathcal{A} \to \mathcal{B} \) be a Morita equivalence in \( \text{dgcat}_{sp}(k) \). Since the functor \( K_0(-)_{Q} \) is an additive invariant (consult \cite[§2]{39}), \( K_0(F)_{Q} \) is an isomorphism. Moreover, since Morita equivalences are preserved by tensor products over a field (consult \cite[§1.6.4]{39}), we have a Morita equivalence \( \text{F} \otimes \text{id}_{\text{perf}_{dg}(T)} \) and, consequently, an isomorphism \( K_0(F)_{Q}/\sim_{\text{alg}} \). We then obtain the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{(T,p,q)} K_0(\mathcal{A} \otimes \text{perf}_{dg}(T))_{Q} & \xrightarrow{3.1} & K_0(\mathcal{A})_{Q} \\
\oplus_{(T,p,q)} K_0(\mathcal{B} \otimes \text{perf}_{dg}(T))_{Q} & \xrightarrow{3.1} & K_0(\mathcal{B})_{Q}
\end{array}
\]

Thanks to Definition 3.1, we hence conclude from (3.2) that \( K_0(F)_{Q}/\sim_{\text{alg}} \) is an isomorphism. \( \square \)

**Lemma 3.4.** Let \( \mathcal{A} \) be a smooth proper dg category such that \( H^0(\mathcal{A}) = \langle b, c \rangle \) admits a semi-orthogonal decomposition in the sense of Bondal-Orlov \cite{39}. Let \( b_{dg} \) and \( c_{dg} \) denote, respectively, the dg enhancement of \( b \) and \( c \) induced from \( \mathcal{A} \). Under these assumptions, the inclusions of \( b_{dg} \) and \( c_{dg} \) into \( \mathcal{A} \) induce an isomorphism:
\[
K_0(b_{dg})_{Q}/\sim_{\text{alg}} \oplus K_0(c_{dg})_{Q}/\sim_{\text{alg}} \xrightarrow{\sim} K_0(\mathcal{A})_{Q}/\sim_{\text{alg}}.
\]

*Proof.* Consider the dg category \( T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}) \), which is known to be smooth and proper; see \cite[Proposition 4.9]{30}. Following the proof of \cite[Proposition 2.2]{39}, the inclusion of \( T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}) \) into \( \mathcal{A} \) is a Morita equivalence. Therefore, since the functor \( K_0(-)_{Q} \) is an additive invariant, we obtain an induced isomorphism between \( K_0(T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}))_{Q} \) and \( K_0(\mathcal{A})_{Q} \). Consequently, in order to finish the proof, it suffices to show that the inclusions of \( b_{dg} \) and \( c_{dg} \) into \( T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}) \) induce an isomorphism
\[
K_0(b_{dg})_{Q}/\sim_{\text{alg}} \oplus K_0(c_{dg})_{Q}/\sim_{\text{alg}} \xrightarrow{\sim} K_0(T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}))_{Q}/\sim_{\text{alg}}.
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
(\bigoplus_{(T,p,q)} K_0(b_{dg} \otimes \text{perf}_{dg}(T))_{Q}) & \oplus & (\bigoplus_{(T,p,q)} K_0(c_{dg} \otimes \text{perf}_{dg}(T))_{Q}) \\
\bigoplus_{(T,p,q)} K_0(T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}) \otimes \text{perf}_{dg}(T))_{Q} & \xrightarrow{3.1} & K_0(T(b_{dg}, c_{dg}; \text{id}_{\mathcal{A}}))_{Q}
\end{array}
\]

(3.3)
with exact rows and whose vertical arrows are induced by the inclusions of $b^{dg}$ and $c^{dg}$ into $T(b^{dg}, c^{dg}; idA)$. We need to prove that (3.3) is an isomorphism. By diagram chasing, we observe that (3.3) is surjective. Since the middle vertical map of the above diagram is an isomorphism, Lemma 2.1 implies that, for a fixed $(T, p, q)$, the corresponding map

$$K_0(b^{dg} \otimes \text{perf}_{dg} T)_Q \oplus K_0(c^{dg} \otimes \text{perf}_{dg} T)_Q \rightarrow K_0(T(b^{dg}, c^{dg}; idA) \otimes \text{perf}_{dg} T)_Q$$

is an isomorphism as well. This implies that the left-hand-side vertical map of the above diagram is surjective. Consequently, we conclude (once again by diagram chasing) that (3.3) is moreover injective and hence an isomorphism.

\[ \square \]

Lemma 3.5. Let $X$ be a smooth proper $k$-scheme and $\mathbb{B}_0$ an Azumaya algebra over $X$. The canonical dg functor $i_X, \mathbb{B}_0 : \text{perf}_{dg}(X) \rightarrow \text{perf}_{dg}(X, \mathbb{B}_0)$ induces an isomorphism:

$$K_0(i_X, \mathbb{B}_0)_Q : K_0(\text{perf}_{dg}(X))_Q \rightarrow K_0(\text{perf}_{dg}(X, \mathbb{B}_0))_Q.$$

Proof. Thanks to [42, Propositions 8.3 and 8.17], the canonical dg functor $i_X, \mathbb{B}_0$ induces an isomorphism $U(i_X, \mathbb{B}_0)_Q$ in the category of noncommutative Chow motive $\text{NChow}(k)_Q$. As a consequence, since the functor $U(\cdot)_Q$ is symmetric monoidal, the morphism $U(i_X, \mathbb{B}_0 \otimes id_{\text{perf}_{dg}(T)})_Q$ is also an isomorphism. Thanks to the computation (2.1), this implies that the following two maps are invertible:

$$K_0(i_X, \mathbb{B}_0)_Q : K_0(\text{perf}_{dg}(X))_Q \rightarrow K_0(\text{perf}_{dg}(X, \mathbb{B}_0))_Q;$$

$$K_0(i_X, \mathbb{B}_0 \otimes id_{\text{perf}_{dg}(T)})_Q : K_0(\text{perf}_{dg}(X) \otimes \text{perf}_{dg}(T))_Q \rightarrow K_0(\text{perf}_{dg}(X, \mathbb{B}_0) \otimes \text{perf}_{dg}(T))_Q.$$

Therefore, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{(T, p, q)} K_0(\text{perf}_{dg}(X) \otimes \text{perf}_{dg}(T))_Q & \xrightarrow{3.1} & K_0(\text{perf}_{dg}(X))_Q \\
\bigoplus_{(T, p, q)} K_0(i_X, \mathbb{B}_0 \otimes id_{\text{perf}_{dg}(T)})_Q & \xrightarrow{3.1} & K_0(\text{perf}_{dg}(X, \mathbb{B}_0))_Q
\end{array}
\]

Thanks to Definition 5.1, we hence conclude from (5.1) that $K_0(i_X, \mathbb{B}_0)_Q/\sim_{\text{alg}}$ is an isomorphism. \[ \square \]

3.2. Nilpotence equivalence relation. Given an integer $m \geq 1$ consider the following functor:

$$\prod_{i=1}^{m} D_c(A) \longrightarrow D_c(A^{\otimes m}) \quad \{M_i\}_{1 \leq i \leq m} \longrightarrow \otimes_{i=1}^{m} M_i.$$

The functor (3.5) is triangulated in each one of its variables. Hence, it gives rise to the following multilinear pairing:

$$\prod_{i=1}^{m} K_0(A)_Q \longrightarrow K_0(A^{\otimes m})_Q \quad \{\otimes_{i=1}^{m} M_i\}.$$

Definition 3.6. An element $\alpha \in K_0(A)_Q$ is called nilpotently trivial if there is an integer $m \geq 1$ such that the image of $(\alpha, \ldots, \alpha)$ under the multilinear pairing (3.6) is equal to 0.\[ \square \]

The above multilinear pairing (3.6) gives rise to the nilpotence equivalence relation $\sim_{\text{nil}}$ on $K_0(A)_Q$. In what follows, we will write $K_0(A)_Q/\sim_{\text{nil}}$ for the associated quotient.

Remark 3.7. Recall from (2.2) the following computation in the category of noncommutative Chow motives:

$$\text{Hom}_{\text{NChow}(k)_Q}(U(k)_Q, U(A)_Q) = K_0(D_c(k^{op} \otimes A))_Q = K_0(A)_Q.$$

This enables the following re-phrasing of the nilpotence equivalence relation: an element $\alpha \in K_0(A)_Q$ is nilpotently trivial if and only if the associated morphism $\alpha : U(k)_Q \rightarrow U(A)_Q$ in $\text{NChow}(k)_Q$ is $\otimes$-nilpotent, or, equivalently, if and only if the associated morphism $\alpha : U(A^{op})_Q \rightarrow U(k)_Q$ in $\text{NChow}(k)_Q$ is $\otimes$-nilpotent.
3.3. Numerical equivalence relation. Consider the following Euler bilinear pairing:

\[ \chi : K_0(A)_\mathbb{Q} \times K_0(A)_\mathbb{Q} \rightarrow \mathbb{Q}, \quad ([M], [N]) \mapsto \sum_n (-1)^n \dim_k \text{Hom}_D(A)(M, N[n]). \]

Although the bilinear pairing (3.7) is not symmetric nor skew-symmetric, its left and right kernels agree; consult [39 §4]. Let us then write \( \ker(\chi) \) for (unique) kernel.

**Definition 3.8.** An element \( \alpha \in K_0(A)_\mathbb{Q} \) is called numerically trivial if it belongs to \( \ker(\chi) \). In other words, we have \( \chi(\alpha, \beta) = 0 \) (or, equivalently, \( \chi(\beta, \alpha) = 0 \)) for every \( \beta \in K_0(A)_\mathbb{Q} \).

Definition 3.8 gives rise to the numerical equivalence relation \( \sim_{num} \) on \( K_0(A)_\mathbb{Q} \). In what follows we will write \( K_0(A)_\mathbb{Q}/\sim_{num} \) for the associated quotient.

3.4. Comparison between the different equivalence relations. In this subsection we compare the different equivalence relations introduced in §3.1 and §3.3.

**Theorem 3.9.** Given an element \( \alpha \in K_0(A)_\mathbb{Q} \), we have the implications \( \alpha \sim_{\text{alg}} 0 \Rightarrow \alpha \sim_{\text{nil}} 0 \Rightarrow \alpha \sim_{\text{num}} 0 \).

**Proof.** We start by proving the implication \( \alpha \sim_{\text{alg}} 0 \Rightarrow \alpha \sim_{\text{nil}} 0 \). Thanks to Proposition 3.10 below, we can assume without loss of generality that \( k \) is algebraically closed. Take \( \alpha \in K_0(A)_\mathbb{Q} \) an algebraically trivial element. Recall from Remark 3.7 that it is enough to show that the associated morphism \( \alpha : U(A^\text{op})_\mathbb{Q} \rightarrow U(k)_\mathbb{Q} \) in the category \( \text{NChow}(k)_\mathbb{Q} \) is \( \otimes \)-nilpotent. Since \( k \) is algebraically closed and the nilpotently trivial elements form a \( \mathbb{Q} \)-linear subspace of \( K_0(A)_\mathbb{Q} \), we can assume that there exists a single smooth projective curve \( C \), equipped with two \( k \)-rational points \( p \) and \( q \), and an element \( \beta \in K_0(A \otimes \text{perf}_{\text{dg}}(C))_\mathbb{Q} \) such that \( \alpha = K_0(\text{id}_A \otimes p^*): \beta - K_0(\text{id}_A \otimes q^*) \beta \); see Remark 3.2. Making use of the following computations

\[
\text{Hom}_{\text{NChow}(k)_\mathbb{Q}}(U(A^\text{op})_\mathbb{Q}, U(\text{perf}_{\text{dg}}(C))_\mathbb{Q}) \simeq K_0(A \otimes_k \text{perf}_{\text{dg}}(C))_\mathbb{Q},
\]

we observe that the associated morphism \( \alpha : U(A^\text{op})_\mathbb{Q} \rightarrow U(k)_\mathbb{Q} \) can be written as the following composition

\[
\alpha : U(A^\text{op})_\mathbb{Q} \xrightarrow{\beta} U(\text{perf}_{\text{dg}}(C))_\mathbb{Q} \xrightarrow{U(p^*)_\mathbb{Q} - U(q^*)_\mathbb{Q}} U(k)_\mathbb{Q},
\]

where \( \beta \) is the morphism associated to the element \( \beta \in K_0(A \otimes \text{perf}_{\text{dg}}(C))_\mathbb{Q} \). Consequently, once we show that \( U(p^*)_\mathbb{Q} - U(q^*)_\mathbb{Q} \) is \( \otimes \)-nilpotent, we conclude that \( \alpha \) is \( \otimes \)-nilpotent too. Let \( \text{Chow}(k)_\mathbb{Q} \) be the classical category of Chow motives; see Manin [33]. This category is \( \mathbb{Q} \)-linear, additive, idempotent complete, and rigid symmetric monoidal. In addition, it has a symmetric monoidal functor \( h(-)_\mathbb{Q} : \text{SmProp}(k)^{\text{op}} \rightarrow \text{Chow}(k)_\mathbb{Q} \) defined on smooth proper \( k \)-schemes. Following [39, Theorem 4.3], there exists a \( \mathbb{Q} \)-linear, fully-faithful, symmetric monoidal functor \( \Phi \) making the following diagram commutative:

\[ \begin{array}{ccc}
\text{SmProp}(k)^{\text{op}} & \xrightarrow{X \mapsto \text{perf}_{\text{dg}}(X)} & \text{dgcat}_{\text{sp}}(k) \\
\downarrow h(-)_\mathbb{Q} & & \downarrow \tau \\
\text{Chow}(k)_\mathbb{Q} & \phi & \text{NChow}(k)_\mathbb{Q}, \\
\end{array} \]

where we denote by \( \text{Chow}(k)_\mathbb{Q}/(- \otimes \mathbb{Q}(1)) \) the orbit category with respect to the Tate motive \( \mathbb{Q}(1) \); consult [39 §4.2] for the definition of the orbit category. In [44 Proposition 3.1] it is proven that the morphism \( h(p)_\mathbb{Q} - h(q)_\mathbb{Q} : h(\text{Spec}(k))_\mathbb{Q} \), corresponding to the degree zero cycle \( p - q \) on \( C \), is \( \otimes \)-nilpotent. Since both functors \( \tau \) and \( \Phi \) are symmetric monoidal, the commutativity of diagram (3.8) hence implies that \( U(p^*)_\mathbb{Q} - U(q^*)_\mathbb{Q} \) is \( \otimes \)-nilpotent. This concludes the proof.

We now prove the implication \( \alpha \sim_{\text{nil}} 0 \Rightarrow \alpha \sim_{\text{num}} 0 \). Recall first from [34 §5 and Proposition 6.2] that an element \( \alpha \in K_0(A)_\mathbb{Q} \) is numerically trivial if and only if for every \( \beta \in K_0(A^\text{op})_\mathbb{Q} \) the associated composition

\[
U(k)_\mathbb{Q} \xrightarrow{\alpha} U(A)_\mathbb{Q} \xrightarrow{\beta} U(k)_\mathbb{Q}
\]

is equal to zero. Let \( \alpha \in K_0(A)_\mathbb{Q} \) be a nilpotently trivial element. Bearing in mind Remark 3.7 the associated morphism \( \alpha : U(k)_\mathbb{Q} \rightarrow U(A)_\mathbb{Q} \) is \( \otimes \)-nilpotent. Let us assume by absurd that \( \alpha \) is not numerically trivial. In this case, there would
exist an element $\beta \in K_0(A^{\text{op}})_Q$ such that the associated morphism $\beta : U(A)_Q \to U(k)_Q$ composed with $\alpha$ is different from zero. Using the fact that $\beta \circ \alpha \in \text{Hom}_{\text{Chow}(k)}(U(k)_Q, U(k)_Q) \simeq \mathbb{Q}$, we can further assume without loss of generality that $\beta \circ \alpha$ is the identity. But, this would then imply that $\beta \circ \alpha$ is not $\otimes$-nilpotent, which contradicts the assumption that $\alpha$ is $\otimes$-nilpotent. In conclusion, $\alpha$ is also numerically trivial. \hfill \Box

**Proposition 3.10.** Let $\overline{k}/k$ be a fixed algebraic closure of $k$. Given an element $\alpha \in K_0(A)_Q$, the following holds:

$$\alpha \sim_{\text{alg}} 0 \Rightarrow (\alpha \otimes_k \overline{k}) \sim_{\text{alg}} 0 \quad \text{and} \quad \alpha \sim_{\text{nil}} 0 \Leftrightarrow (\alpha \otimes_k \overline{k}) \sim_{\text{nil}} 0.$$  

**Proof.** It follows from [35, Proposition 7.2] that if $A \in \text{dgcat}_{sp}(k)$, then $A \otimes_k \overline{k} \in \text{dgcat}_{sp}(\overline{k})$. Note also that given a smooth connected $k$-scheme $T$, $\text{perf}_{dg}(T) \otimes_k \overline{k}$ is a smooth $\overline{k}$-linear dg category which is canonically Morita equivalent to $\text{perf}_{dg}(\overline{T})$, where $\overline{T} := T \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$. Hence, the implication $\alpha \sim_{\text{alg}} 0 \Rightarrow (\alpha \otimes_k \overline{k}) \sim_{\text{alg}} 0$ follows from diagram (3.9) below (where $T'$ is a smooth connected $\overline{k}$-scheme equipped with two $\overline{k}$-rational points $p'$ and $q'$):

$$
\begin{align*}
\bigoplus_{(T', p', q')} K_0((A \otimes_k \overline{k}) \otimes_{\overline{k}} \text{perf}_{dg}(T'))_Q & \xrightarrow{3.1} K_0(A \otimes_k \overline{k})_Q \\
\bigoplus_{(T, p, q)} K_0(A \otimes \text{perf}_{dg}(T))_Q & \xrightarrow{3.1} K_0(A)_Q.
\end{align*}

(3.9)

We now prove the equivalence $\alpha \sim_{\text{nil}} 0 \Leftrightarrow (\alpha \otimes_k \overline{k}) \sim_{\text{nil}} 0$. For any integer $m \geq 1$, we have the commutative diagram:

$$
\begin{align*}
\prod_{i=1}^m K_0((A \otimes_k \overline{k})_Q \otimes_{\overline{k}} \text{perf}_{dg}(T^m))_Q & \xrightarrow{3.1} K_0((A \otimes_k \overline{k})^{\otimes m})_Q \\
\prod_{i=1}^m K_0((A \otimes_k \overline{k})_Q) & \xrightarrow{3.1} K_0((A \otimes_k \overline{k})^{\otimes m})_Q.
\end{align*}

(3.10)

Moreover, we have a canonical identification $A \otimes^{\otimes m} \otimes_k \overline{k} \simeq (A \otimes_k \overline{k})^{\otimes m}$. Therefore, by applying Lemma 3.11 below to the dg category $A^{\otimes m}$, we conclude that the homomorphism $K_0(- \otimes_k \overline{k})_Q$ is injective. Consequently, the proof follows now from (3.10). \hfill \Box

**Lemma 3.11.** Given an algebraic closure $\overline{k}/k$, the induced homomorphism $K_0(- \otimes_k \overline{k})_Q : K_0(A)_Q \to K_0(A \otimes_k \overline{k})_Q$ is injective.

**Proof.** Assume that $\overline{k}/k$ is a finite field extension of degree $d$. In this case, the restriction along the field extension $k \to \overline{k}$ yields an homomorphism $\text{Res}_{\overline{k}/k} : K_0(A \otimes_k \overline{k})_Q \to K_0(A)_Q$ such that $\text{Res}_{\overline{k}/k} \circ K_0(- \otimes_k \overline{k})_Q$ is equal to multiplication by $d$. Therefore, since $K_0(A)_Q$ is a $\mathbb{Q}$-vector space, the induced homomorphism $K_0(- \otimes_k \overline{k})_Q$ is injective. In the case where the field extension $\overline{k}/k$ is infinite, $\overline{k}$ identifies with the colimit of the filtrant diagram $\{k_i\}_{i \in I}$ of all the intermediate field extensions $\overline{k}/k_i/k$ which are finite over $k$. Since both functors $- \otimes_k \overline{k}$ and $K_0(-)_Q$ preserve filtrant (homotopy) colimits, we have that $K_0(A \otimes_k \overline{k})_Q \simeq \text{colim}_{i \in I} K_0(A \otimes_k k_i)_Q$ and injectivity of $K_0(- \otimes_k \overline{k})_Q$ follows now from the finite field extension case. \hfill \Box

3.5. Classical equivalence relations on algebraic cycles. Given a smooth proper $k$-scheme $X$, recall from [12, Corollary 18.3.2] that the following map is invertible:

$$K_0(\text{perf}_{dg}(X))_Q \xrightarrow{3.11} Z^*(X)_Q/\sim_{\text{rat}} \quad [F] \mapsto \text{ch}(F) \cdot \sqrt{\text{td}_X},$$

where $\text{ch}(F)$ stands for the Chern character of $F$ and $\text{td}_X$ for the Todd class of $X$. 


Theorem 3.12. The above map (3.14) induces the following isomorphisms:

\[ K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}/_{\sim_{\text{alg}}} \xrightarrow{\sim} Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{alg}}} K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}/_{\sim_{\text{nil}}} K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}/_{\sim_{\text{num}}} \xrightarrow{\sim} Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{num}}}. \]

Proof. The left-hand-side and middle isomorphisms in (3.12) follow from Lemmas 3.13, 3.14 below. In order to prove the right-hand-side in (3.12), note that, given any two perfect complexes \( F, G \in \text{perf}(X) \), we have the following equalities

\[ \chi([F],[G]) = K_0(\pi_*([\text{RHom}(F,G)])_{\mathbb{Q}} = K_0(\pi_*([F^* \otimes_X G]))_{\mathbb{Q}}, \]

where \( \pi : X \to \text{Spec}(k) \) stands for the structure map of \( X \) and \( \text{RHom}(-,-) \), resp. \((-)^*\), stands for the internal Hom, resp. (contravariant) duality, of the category \( \text{perf}_{\text{dg}}(X) \). Hence, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \times K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} & \xrightarrow{\chi} & \mathbb{Q} \\
K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \times K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} & \xrightarrow{K_0((-)^* \otimes X^* \sim)} & K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \\
K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \times K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} & \xrightarrow{\cdots} & K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \\
\text{ch}(-) \cdot \sqrt{\text{td}_X} \times \text{ch}(-) \cdot \sqrt{\text{td}_X} & \xrightarrow{\sim} & \text{ch}(-) \cdot \text{td}_X \\
Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{rat}}} \times Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{rat}}} & \xrightarrow{\sim} & Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{rat}}} \end{array}
\]

The square (1) is commutative, thanks to (3.13). The commutativity of the squares (2) and (3) is obvious; note that \( \mathbb{Q} = K_0(\text{Spec}(k))_{\mathbb{Q}} \). The square (4) is commutative since \( \text{ch}(-) \) is multiplicative. Finally, the square (5) is commutative because of the Grothendieck-Riemann-Roch formula; see [14, Theorem 5.26]. Note that the right vertical map of the square (5), i.e., \( \text{ch}(-) \cdot \text{td}_{\text{Spec}(k)} \), is the identity map. Indeed, since \( K_0(\text{Spec}(k))_{\mathbb{Q}} = \mathbb{Q} \) is a graded ring concentrated in degree zero, we have \( \text{td}_{\text{Spec}(k)} = 1 \) and the ring map \( \text{ch}(-) \) from \( K_0(\text{Spec}(k))_{\mathbb{Q}} \) to \( Z^*(\text{Spec}(k))_{\mathbb{Q}}/_{\sim_{\text{rat}}} = \mathbb{Q} \) is the identity. We already know that \( \text{ch}(-) \cdot \sqrt{\text{td}_X} \) is an isomorphism and that \( K_0((-)^*)_{\mathbb{Q}} \times \text{id} \) is an isomorphism because both \((-)^*\) and \( \text{id} \) are isomorphisms. This explains diagram (3.14). Finally, recall that an algebraic cycle \( \mu \in Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{rat}}} \) is called numerically trivial if \( \pi_*(\nu \cdot \mu) = 0 \) for all \( \nu \in Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{rat}}} \). Therefore, given an element \( F \in K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \), we conclude from the outer commutative square of diagram (3.14) that \( F \) is numerically trivial if \( \text{ch}(F) \cdot \sqrt{\text{td}_X} \) is numerically trivial. This concludes the proof of Theorem 3.12. \( \square \)

Lemma 3.13. The Chern character \( \text{ch}(-) : K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}} \xrightarrow{\sim} Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{rat}}} \) induces the following isomorphisms:

\[ K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}/_{\sim_{\text{alg}}} \xrightarrow{\sim} Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{alg}}} \quad \quad K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}}/_{\sim_{\text{nil}}} \xrightarrow{\sim} Z^*(X)_{\mathbb{Q}}/_{\sim_{\text{nil}}}. \]

Proof. From [13, Lemma 4.26] we have that, for \( T \) a smooth connected \( k \)-scheme, the following dg functor is a Morita equivalence:

\[ - \boxtimes : \text{perf}_{\text{dg}}(X) \otimes \text{perf}_{\text{dg}}(T) \longrightarrow \text{perf}_{\text{dg}}(X \times T). \]

Moreover, by the properties of Chern character (see [12, page 282, (iii)]), one obtains the commutative diagram:
where \( p \) and \( q \) stand for \( k \)-rational points of a smooth connected \( k \)-scheme \( T \). Recall that an algebraic cycle \( \mu \in \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{rat}} \) is called algebraically trivial if it belongs to the image of \( \bigoplus_{(T,p,q)}((id \otimes p)^* - (id \otimes q)^*) \). Therefore, the left-hand-side isomorphism in (3.15) follows from the above commutative diagram (3.16).

In order to prove the right-hand-side isomorphism in (3.15), note that a repeated use of [13] Lemma 4.26 shows that the following dg functor

\[ - \boxdot - \cdots - \boxdot - \colon \text{perf}_{dg}(X)^{\otimes m} \rightarrow \text{perf}_{dg}(X^{\otimes m}) \]

is a Morita equivalence. Therefore, for any integer \( m \geq 1 \) we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\prod_{i=1}^{m} K_0(\text{perf}_{dg}(X))_\mathbb{Q} & \longrightarrow & K_0(\text{perf}_{dg}(X)^{\otimes m})_\mathbb{Q} \\
\bigoplus_{(T,p,q)} & \approx & K_0(\mathbb{Q}) \\
\bigoplus_{(T,p,q)} Z^*(X \times T)_\mathbb{Q}/\sim_{\text{rat}} & \longrightarrow & Z^*(X)^{\otimes m}_\mathbb{Q}/\sim_{\text{rat}}
\end{array}
\]

(3.17)

Recall that an algebraic cycle \( \mu \in \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{rat}} \) is called nilpotently trivial if there is a positive integer \( m \) such that \( \mu^m = 0 \). Therefore, the right-hand-side isomorphism in (3.15) follows from the above commutative diagram (3.17).

**Lemma 3.14.** The homomorphism \(- \sqrt{\text{td} X} : \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{rat}} \rightarrow \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{rat}} \) induces the following isomorphisms:

\[
\mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{alg}} \xrightarrow{\sim} \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{alg}} \quad \quad \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{nil}} \xrightarrow{\sim} \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{nil}}.
\]

**Proof.** Note that in order to prove the left-hand-side of (3.18) it suffices to show that the following square is commutative:

\[
\begin{array}{ccc}
\bigoplus_{(T,p,q)} & \approx & \bigoplus_{(T,p,q)} \\
\bigoplus_{(T,p,q)} Z^*(X \times T)_\mathbb{Q}/\sim_{\text{rat}} & \longrightarrow & Z^*(X)_\mathbb{Q}/\sim_{\text{rat}}
\end{array}
\]

where \( p \) and \( q \) stand for \( k \)-rational points of a smooth connected \( k \)-scheme \( T \). Let \( \pi_X \) and \( \pi_T \) be, respectively, the projections from \( X \times T \) to \( X \) and \( T \). Consider the pull-backs \( \pi_X^* : \mathbb{Z}^*(X)_\mathbb{Q}/\sim_{\text{rat}} \rightarrow \mathbb{Z}^*(X \times T)_\mathbb{Q}/\sim_{\text{rat}} \) and \( \pi_T^* : \mathbb{Z}^*(T)_\mathbb{Q}/\sim_{\text{rat}} \rightarrow \mathbb{Z}^*(X \times T)_\mathbb{Q}/\sim_{\text{rat}} \).

It is known, by the proof of [13] Lemma 10.6, that \( \sqrt{\text{td} X \times T} = \pi_X^*(\sqrt{\text{td} X}) \cdot \pi_T^*(\sqrt{\text{td} T}) \). Consequently, we conclude that

\[
(id \times p)^* (\sqrt{\text{td} X \times T}) = (id \times p)^* (\sqrt{\text{td} X}) \cdot (id \times p)^* (\sqrt{\text{td} T}).
\]
\[
\begin{align*}
&= (id \times p)^* (\pi_X^* (\sqrt{td_X}) \cdot (id \times p)^* (\pi_T^* (\sqrt{td_T})) \\
&= (\pi_X \circ (id \times p))^* (\sqrt{td_X}) \cdot (\pi_T \circ (id \times p))^* (\sqrt{td_T}) \\
&= id^* (\sqrt{td_X}) \cdot (\pi_T \circ (id \times p))^* (\sqrt{td_T}) \\
&= \sqrt{td_X} \cdot (\pi_T \circ (id \times p))^* (\sqrt{td_T}).
\end{align*}
\]

Recall from [14, §5.2] that the degree zero term of \( td_T \), in \( H^0(T; \mathbb{Q}) \), is 1. Therefore, choose for \( \sqrt{td_T} \) the cohomology class whose degree zero term is 1. Since \( Z^*(\text{Spec}(k))_{\mathbb{Q}/\sim_{rat}} = \mathbb{Q} \) and \( p^* \) is a graded ring morphism, the following equalities hold for every \( \nu \in Z^*(X \times T)_{\mathbb{Q}/\sim_{rat}} \).

\[
(id \times p)^* (\nu \cdot \sqrt{td_{X \times T}}) = (id \times p)^* (\nu) \cdot (id \times p)^* (\sqrt{td_{X \times T}})
\]

This implies that the above diagram (3.19) is commutative.

Now, in order to prove the right-hand-side of (3.18), note that the homomorphism \( \{\mu_i\}_{1 \leq i \leq m} \mapsto \mu_1 \times \ldots \times \mu_m \) can be written as the following composition:

\[
\prod_{i=1}^{m} Z^*(X)_{\mathbb{Q}/\sim_{rat}} \xrightarrow{\prod \pi_i^*} \prod_{i=1}^{m} Z^*(X^{\times m})_{\mathbb{Q}/\sim_{rat}} \xrightarrow{\text{mult}} Z^*(X^{\times m})_{\mathbb{Q}/\sim_{rat}},
\]

where \( \pi_i \) stands for the \( i \)th projection map from \( X^{\times m} \) to \( X \). Consequently, since the pull-back homomorphism \( \pi_i^* \) is multiplicative, if an algebraic cycle \( \mu \in Z^*(X)_{\mathbb{Q}/\sim_{rat}} \) is nilpotently trivial, then so is \( \mu \cdot \sqrt{td_X} \). Conversely, since the homomorphism \( \pi_i^* \) is multiplicative and \( \sqrt{td_X} \) is invertible, if \( \mu \cdot \sqrt{td_X} \) is nilpotently trivial, then \( \mu \) is also necessarily nilpotently trivial. \( \square \)

4. Kernels

In this section, we consider the quotients between the different equivalence relations introduced in \( \text{3} \) We start by fixing some useful notations that will be used in the remainder of the paper.

**Notation 4.1.** Let \( X \) be a smooth proper \( k \)-scheme and \( \mathcal{A} \) a smooth proper dg category (over \( k \)).

(i) Given equivalence relations \( \sim_1 \) and \( \sim_2 \) on a set \( S \), we will write \( \sim_1 \triangleright \sim_2 \) if, for all \( a, b \in S \), \( a \sim_1 b \) implies \( a \sim_2 b \).

(ii) Given an equivalence relation \( \sim \) on \( Z^*(X)_{\mathbb{Q}} \), resp. on \( K_0(\mathcal{A})_{\mathbb{Q}} \), let us write

\[
Z^*_{\sim}(X)_{\mathbb{Q}} := \{ \alpha \in Z^*(X)_{\mathbb{Q}} | \alpha \sim 0 \} \quad \text{resp.} \quad K_{0, \sim}(\mathcal{A})_{\mathbb{Q}} := \{ \alpha \in K_0(\mathcal{A})_{\mathbb{Q}} | \alpha \sim 0 \}
\]

for the \( \mathbb{Q} \)-subspace of \( \sim \)-trivial elements.

(iii) Given equivalence relations \( \sim_1 \triangleright \sim_2 \) on \( Z^*(X)_{\mathbb{Q}} \), resp. on \( K_0(\mathcal{A})_{\mathbb{Q}} \), let us write

\[
Z^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}} := \frac{Z^*_{\sim_2}(X)_{\mathbb{Q}}}{Z^*_{\sim_1}(X)_{\mathbb{Q}}} \quad \text{resp.} \quad K_{0, \sim_2/\sim_1}(\mathcal{A})_{\mathbb{Q}} := \frac{K_{0, \sim_2}(\mathcal{A})_{\mathbb{Q}}}{K_{0, \sim_1}(\mathcal{A})_{\mathbb{Q}}}
\]

for the associated quotient.

(iv) Given equivalence relations \( \sim_1 \triangleright \sim_2 \) on \( Z^*(X)_{\mathbb{Q}} \), resp. on \( K_0(\mathcal{A})_{\mathbb{Q}} \), let us write

\[
q_{\sim_2/\sim_1}^X : Z^*(X)_{\mathbb{Q}/\sim_1} \to Z^*(X)_{\mathbb{Q}/\sim_2} \quad \text{resp.} \quad q_{\sim_2/\sim_1}^{\mathcal{A}, nc} : K_0(\mathcal{A})_{\mathbb{Q}/\sim_1} \to K_0(\mathcal{A})_{\mathbb{Q}/\sim_2}
\]

for the quotient map.

(v) Given equivalence relations \( \sim_1 \triangleright \sim_2 \) on \( K_0(\text{perf}_{dg}(X))_{\mathbb{Q}} \), let \( q_{\sim_2/\sim_1}^{X, nc} : K_0(\text{perf}_{dg}(X))_{\mathbb{Q}/\sim_1} \to K_0(\text{perf}_{dg}(X))_{\mathbb{Q}/\sim_2} \) be the quotient map.
Remark 4.2. (i) The equivalence relations $\sim_{alg}$, $\sim_{nil}$ and $\sim_{num}$ agree for smooth proper $k$-schemes $X$ of dim $\leq 2$; see [11 §3.2.7] and [12 §19.3.5]. Moreover, $\sim_{rat}$, $\sim_{alg}$, $\sim_{nil}$ and $\sim_{num}$ agree for 0-dimensional $k$-schemes.

(ii) In general, we have $\ker(q^X_{\sim_{rat}}) \neq 0$; consult Remark 1.2.

(iii) For every smooth projective complex curve $X$ of positive genus $g$, we have that $Z^*_{\sim_{alg}}/\sim_{rat}(X)_Q \simeq (\mathbb{Q}/\mathbb{Z})^{g\cdot 2g}$. Indeed, as $X$ is a curve, we have $Z^*_{\sim_{alg}}/\sim_{rat}(X)_Q = Z^1_{\sim_{alg}}/\sim_{rat}(X)_Q$ which, thanks to [12 §19.3.5], is isomorphic to the $\mathbb{Q}$-vector space of torsion points of the Albanese variety $\text{Alb}(X)$. Since $X$ is a curve, we have $\text{Alb}(X) \simeq \text{Jac}(X)$ (the Jacobian variety of $X$) and it is well-known that the $\mathbb{Q}$-vector space of torsion points of $\text{Jac}(X)$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{g\cdot 2g}$.

(iv) Voevodsky’s nilpotence conjecture asserts that $\ker(q^X_{\sim_{num}}/\sim_{nil}) = 0$.

Remark 4.3. Let $\sim_{1}, \sim_{2} \in \{\sim_{rat}, \sim_{alg}, \sim_{nil}, \sim_{num}\}$.

(i) Given a smooth proper $k$-scheme $X$, we have $Z^*_{\sim_{2}/\sim_{1}}(X)_Q \simeq \ker(q^X_{\sim_{2}/\sim_{1}})$.

(ii) Given a smooth proper dg category $\mathcal{A}$ (over $k$), we have $K_{0,\sim_{2}/\sim_{1}}(\mathcal{A})_Q \simeq \ker(q^{\mathcal{A},nc}_{\sim_{2}/\sim_{1}})$.

Corollary 4.4 (of Theorem 3.12). Given a smooth proper $k$-scheme $X$ and $\sim_{1}, \sim_{2} \in \{\sim_{rat}, \sim_{alg}, \sim_{nil}, \sim_{num}\}$, we have an isomorphism $\ker(q^{X}_{\sim_{2}/\sim_{1}}) \simeq \ker(q^{\mathcal{A},nc}_{\sim_{2}/\sim_{1}})$. Equivalently, we have an isomorphism $Z^*_{\sim_{2}/\sim_{1}}(X)_Q \simeq K_{0,\sim_{2}/\sim_{1}}(\text{perf}_{dg}(X))_Q$.

Proof. Recall from Theorem 3.12 that the map (3.11) induces an isomorphism $K_{0}(\text{perf}_{dg}(X))_Q/\sim \simeq Z^*(X)_Q/\sim$, where $\sim \in \{\sim_{rat}, \sim_{alg}, \sim_{nil}, \sim_{num}\}$. Consequently, we have the following commutative diagram:

$$
\begin{array}{ccc}
\ker(q^{X}_{\sim_{2}/\sim_{1}}) & \xrightarrow{\simeq} & K_{0}(\text{perf}_{dg}(X))_Q/\sim_{1} \\
\downarrow & & \downarrow \simeq \text{(3.11)} \\
\ker(q^{\mathcal{A},nc}_{\sim_{2}/\sim_{1}}) & \xrightarrow{\simeq} & K_{0,\sim_{2}/\sim_{1}}(\mathcal{A})_Q/\sim_{2} \end{array}
$$

(4.1)

Since both rows in (4.1) are exact, we hence conclude that the left-hand-side vertical homomorphism in (4.1) is also invertible.

□

Proposition 4.5. Let $\mathcal{A}$ be a smooth proper dg category such that $\text{H}^0(\mathcal{A}) = \langle b, c \rangle$ admits a semi-orthogonal decomposition in the sense of Bondal-Orlov [9]. Let $b^{dg}$ and $c^{dg}$ denote, respectively, the dg enhancement of $b$ and $c$ induced from $\mathcal{A}$. Under these assumptions, the inclusions of $b^{dg}$ and $c^{dg}$ into $\mathcal{A}$ induce an isomorphism

$$
\ker(q^{b^{dg},nc}_{\sim_{2}/\sim_{1}}) \oplus \ker(q^{c^{dg},nc}_{\sim_{2}/\sim_{1}}) \xrightarrow{\sim} \ker(q^{A,nc}_{\sim_{2}/\sim_{1}}) \text{ with } \sim_{1}, \sim_{2} \in \{\sim_{rat}, \sim_{alg}, \sim_{nil}, \sim_{num}\}.
$$

Equivalently, we have an induced isomorphism $K_{0,\sim_{2}/\sim_{1}}(b^{dg})_Q \oplus K_{0,\sim_{2}/\sim_{1}}(c^{dg})_Q \xrightarrow{\sim} K_{0,\sim_{2}/\sim_{1}}(\mathcal{A})_Q$.

Proof. Since the dg category $\mathcal{A}$ is smooth and proper, it follows from the proof of [8] Lemma 2.1 that the dg categories $b^{dg}$ and $c^{dg}$ are smooth and proper. Moreover, as explained in [40 §3.2], we have the following computations:

$$
\text{Hom}_{\text{N-Voev}(k)_Q}(U(k)_Q, U(A)_Q) \simeq K_{0}(\mathcal{A})_Q/\sim_{nil} \quad \text{Hom}_{\text{N-Num}(k)_Q}(U(k)_Q, U(A)_Q) \simeq K_{0}(\mathcal{A})_Q/\sim_{num}.
$$

As $\text{H}^0(\mathcal{A}) = \langle b, c \rangle$, we deduce from [8] Proposition 2.2 and Theorem 2.9 that the induced morphism $U(b^{dg})_Q \oplus U(c^{dg})_Q \to U(A)_Q$ is invertible. Therefore, making use of the computations (4.2), we obtain induced isomorphisms:

$$
K_{0}(b^{dg})_Q/\sim_{nil} \oplus K_{0}(c^{dg})_Q/\sim_{nil} \xrightarrow{\sim} K_{0}(\mathcal{A})_Q/\sim_{nil} \quad K_{0}(b^{dg})_Q/\sim_{num} \oplus K_{0}(c^{dg})_Q/\sim_{num} \xrightarrow{\sim} K_{0}(\mathcal{A})_Q/\sim_{num}.
$$

Thanks to the computation (2.1), we have an induced isomorphism $K_{0}(b^{dg})_Q \oplus K_{0}(c^{dg})_Q \xrightarrow{\sim} K_{0}(\mathcal{A})_Q$. Note that thanks to Lemma 3.4, the inclusions of $b^{dg}$ and $c^{dg}$ into $\mathcal{A}$ also induce an isomorphism $K_{0}(b^{dg})_Q/\sim_{alg} \oplus K_{0}(c^{dg})_Q/\sim_{alg} \xrightarrow{\sim} K_{0}(\mathcal{A})_Q/\sim_{alg}$. Consequently, we have the following commutative diagram:
The proofs are, in most cases, adaptations of the ones given in [8] and [38].

Let $\mathcal{A}$ and $\mathcal{B}$ be two smooth proper dg categories.

(i) Given $F: \mathcal{A} \rightarrow \mathcal{B}$ a Morita equivalence, the homomorphism $K_0(F)\sim$ is invertible when $\sim \in \{\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}\}$.

(ii) For $\sim_1, \sim_2 \in \{\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}\}$ and $\mathcal{A}$ Morita equivalent to $\mathcal{B}$, we have $\ker(q_{\sim_2/\sim_1}) \simeq \ker(q_{\sim_2/\sim_1})$, or, equivalently, $K_0(\sim_{\sim_2/\sim_1}(\mathcal{A}) \simeq K_0(\sim_{\sim_2/\sim_1}(\mathcal{B})\sim$.

Proof. We start by proving item (i). Since $K_0(\sim)\sim$ is an additive invariant, we have that $K_0(F)\sim$ is an isomorphism. Thanks to Lemma 3.5, we have that $K_0(\sim_{\sim_2/\sim_1}(\mathcal{A}) \simeq K_0(\sim_{\sim_2/\sim_1}(\mathcal{B})\sim$.

Proposition 4.7. Let $X$ be a smooth proper $k$-scheme and $\mathcal{B}_0$ an Azumaya algebra over $X$.

(i) The canonical dg functor $i_{X,\mathcal{B}_0}: \text{perf}_{dg}(X) \rightarrow \text{perf}_{dg}(X,\mathcal{B}_0)$ induces an isomorphism

$$K_0(i_{X,\mathcal{B}_0})\sim: \text{perf}_{dg}(X)\sim \rightarrow \text{perf}_{dg}(X,\mathcal{B}_0)\sim \text{with } \sim \in \{\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}\}.$$

(ii) For $\sim_1, \sim_2 \in \{\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}\}$, we have $\ker(q_{\sim_2/\sim_1}) \simeq \ker(q_{\sim_2/\sim_1})$, or, equivalently, an isomorphism

$$K_0(\sim_{\sim_2/\sim_1}(\text{perf}_{dg}(X))\sim \simeq K_0(\sim_{\sim_2/\sim_1}(\text{perf}_{dg}(X,\mathcal{B}_0))\sim.$$

Proof. To prove item (i), we start by noting that, following [22] Corollary 3.1, we have $K_0(\text{perf}_{dg}(X))\sim \simeq K_0(\text{perf}_{dg}(X,\mathcal{B}_0))\sim$. Moreover, thanks to Lemma 3.5, we have that $K_0(\text{perf}_{dg}(X))\sim \simeq K_0(\text{perf}_{dg}(X,\mathcal{B}_0))\sim$. Following [22] Propositions 8.3 and 8.7, the canonical dg functor $i_{X,\mathcal{B}_0}$ induces an isomorphism $U(\text{perf}_{dg}(X))\sim \simeq U(\text{perf}_{dg}(X,\mathcal{B}_0))\sim$. So, the same reasoning in Proposition 4.6 allows us to conclude that

$$K_0(\text{perf}_{dg}(X))\sim \simeq K_0(\text{perf}_{dg}(X,\mathcal{B}_0))\sim.$$

This finishes the proof of item (i). To prove item (ii), note that item (i) yields the following commutative diagram:

Since both rows in (4.4) are exact, the left-hand-side vertical homomorphism in (4.4) is also invertible. □

5. Applications

In this section, making use of the mathematical tools developed in §3-§4, we prove Theorem 5.1 (as well as Theorem 5.19). The proofs are, in most cases, adaptations of the ones given in [8] and [38].
5.1. Full exceptional collection.

Corollary 5.1 (of Proposition 4.5). Let \( X \) be a smooth proper \( k \)-scheme such that \( \text{perf}(X) \) admits a full exceptional collection, cf. [23] §1.1. The equivalence relations \( \sim_{\text{alg}} \) and \( \sim_{\text{num}} \) agree on \( Z^*(X)_Q \).

Proof. Just consider Remark 4.2(i), Corollary 4.4 and Proposition 4.5.

5.2. Severi-Brauer varieties.

Theorem 5.2. Let \( X \) be a Severi-Brauer variety. The equivalence relations \( \sim_{\text{alg}} \) and \( \sim_{\text{num}} \) agree on \( Z^*(X)_Q \).

Proof. Let \( A \) be the unique central simple \( k \)-algebra such that \( X = \text{SB}(A) \). Following [5] Theorem 4.1, we have the semi-orthogonal decomposition:

\[
\text{perf}(X) = \langle \text{perf}(k), \text{perf}(A), \ldots, \text{perf}(A^\otimes n) \rangle,
\]

where \( n \) stands for the dimension of \( X \). Since the tensor product of central simple \( k \)-algebras is still a central simple \( k \)-algebra, Remark 4.2(i), Corollary 4.4 and Propositions 4.5, 4.7(ii) together imply that the equivalence relations \( \sim_{\text{alg}} \) and \( \sim_{\text{num}} \) agree on \( Z^*(X)_Q \).

5.3. Quadric fibrations. Take \( S \) a smooth projective \( k \)-scheme and \( q : Q \to S \) a flat quadric fibration of relative dimension \( n \) with \( Q \) smooth. Let \( C_0 \) be the sheaf of even parts of the Clifford algebra associated to \( q \); see [1] §1 [23] §3. Recall from [23] §3.5-§3.6 that when the discriminant divisor of \( q \) is smooth and \( n \) is even (resp., odd) we have a discriminant double cover \( \tilde{S} \to S \) (resp., a square root stack \( \tilde{S} \) equipped with an Azumaya algebra \( B_0 \).

Theorem 5.3. Under the above assumptions, with \( \sim_1, \sim_2 \in \{ \sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}} \} \), the following holds:

\begin{itemize}
  \item[(i)] We have \( Z^*_{\sim_2/\sim_1}(Q)_Q \cong K_{0, \sim_2/\sim_1}(\text{perf}_{\text{dg}}(S, C_0))_Q \oplus Z^*_{\sim_1}(S)_Q^{\otimes n} \).
  \item[(ii)] If the discriminant divisor of \( q \) is smooth and \( n \) is even, then \( Z^*_{\sim_2/\sim_1}(Q)_Q \cong Z^*_{\sim_2/\sim_1}(\tilde{S})_Q \oplus Z^*_{\sim_1}(S)_Q^{\otimes n} \).
  \item[(iii)] If the discriminant divisor of \( q \) is smooth and \( n \) is odd, then \( Z^*_{\sim_2/\sim_1}(Q)_Q \cong K_{0, \sim_2/\sim_1}(\text{perf}_{\text{dg}}(\tilde{S}, B_0))_Q \oplus Z^*_{\sim_2/\sim_1}(S)_Q^{\otimes n} \).
\end{itemize}

Proof. Follow the reasoning of the proof of [8] Theorem 1.2 with the following changes: to prove item (i), use Corollary 4.4 instead of [8] Theorem 1.1 and Proposition 4.5 instead of [8] equation (5.2) (in order to conclude that \( \ker(q_{\text{nc}}) \) is isomorphic to \( \ker(q_{\text{nc}}) \oplus \ker(q_{\text{nc}}^{\otimes n}) \); to prove item (ii), consider Proposition 4.7 and use Corollary 4.4 instead of [8] Theorem 1.1; finally, in order to prove (iii), it is enough to replace the reference [8] Theorem 1.1 by Corollary 4.5.

Corollary 5.4. When \( \dim(S) \leq 2 \), the following holds:

\begin{itemize}
  \item[(i)] If the discriminant divisor of \( q \) is smooth and \( n \) is even, then the equivalence relations \( \sim_{\text{alg}} \) and \( \sim_{\text{num}} \) on \( Z^*(Q)_Q \) agree.
  \item[(ii)] If the discriminant divisor of \( q \) is smooth and \( n \) is odd, then \( Z^*_{\sim_2/\sim_1}(Q)_Q \cong K_{0, \sim_2/\sim_1}(\text{perf}_{\text{dg}}(\tilde{S}, B_0))_Q \).
\end{itemize}

Proof. Since \( \dim(\tilde{S}) = \dim(S) \), the proof follows from Remark 4.2(i).

5.4. Intersection of quadrics. Let \( X \) be a smooth complete intersection of \( r \) quadric hypersurfaces in \( \mathbb{P}^m \). The linear span of these \( r \) quadrics gives rise to a hypersurface \( Q \subseteq \mathbb{P}^{r-1} \times \mathbb{P}^m \), and the projection into the first factor to a flat quadric fibration \( q : Q \to \mathbb{P}^{r-1} \) of relative dimension \( m - 1 \).

Theorem 5.5. Under the above assumptions, with \( \sim_1, \sim_2 \in \{ \sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}} \} \), the following holds:

\begin{itemize}
  \item[(i)] We have \( Z^*_{\sim_2/\sim_1}(X)_Q \cong K_{0, \sim_2/\sim_1}(\text{perf}_{\text{dg}}(\mathbb{P}^{r-1}, C_0))_Q \).
  \item[(ii)] If the discriminant divisor of \( q \) is smooth and \( m \) is odd, then \( Z^*_{\sim_2/\sim_1}(X)_Q \cong Z^*_{\sim_2/\sim_1}(\mathbb{P}^{r-1})_Q \).
  \item[(iii)] If the discriminant divisor of \( q \) is smooth and \( m \) is even, then \( Z^*_{\sim_2/\sim_1}(X)_Q \cong K_{0, \sim_2/\sim_1}(\text{perf}_{\text{dg}}(\mathbb{P}^{r-1}, B_0))_Q \).
\end{itemize}

Proof. The proof is similar to the one given in [8] §7 with the following adaptations: to prove item (i), replace the reference to the proof of [8] Theorem 1.2(i) by the actual proof of Theorem 5.3(i) and consider Corollary 4.4 instead of [8] Theorem 1.1; the proofs of items (ii)-(iii) follow a similar reasoning to the proofs of Theorem 5.3(ii)-(iii).

Corollary 5.6. When \( r \leq 3 \), the following holds:
5.5. Moishezon manifolds. A Moishezon manifold $X$ is a compact complex manifold such that the field of meromorphic functions on each component of $X$ has transcendence degree equal to the dimension of the component. As proved in [37], $X$ is a smooth projective $\mathbb{C}$-scheme if and only if it admits a Kähler metric. In the remaining cases, it is shown in [2] that $X$ is a proper algebraic space over $\mathbb{C}$. Let $Y \to \mathbb{P}^2$ be one of the non-rational conic bundles described by Artin and Mumford in [31], and $X \to Y$ a small resolution. In this case, $X$ is a smooth (not necessarily projective) Moishezon manifold.

**Theorem 5.7.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $\mathbb{Z}^*(X)_{\mathbb{Q}}$ agree.

**Proof.** The proof of [8, Theorem 1.14] can be adapted as follows: replace the reference to the proof of [8, Theorem 1.2(i)] by Theorem 5.7. □

5.6. Cubic fourfolds and Gushel-Mukai fourfolds. Recall that a cubic fourfold is a smooth complex hypersurface of degree 3 in $\mathbb{P}^5$ and consult [35, §2.2] for the definition of an (ordinary/ special) Gushel-Mukai fourfold.

In what follows, we adapt [38, Theorems (A)-(D)] to our context.

**Theorem 5.8.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $\mathbb{Z}^*(X)_{\mathbb{Q}}$ agree when $X$ is a cubic fourfold or an ordinary generic Gushel-Mukai fourfold.

**Proof.** Apply the same reasoning of the proof of [38, Theorem (A)], use Remark 4.2(i), and replace the equivalence relation $\sim_{\text{nil}}$ by the equivalence relation $\sim_{\text{alg}}$. □

**Remark 5.9.** (i) Let $X$ be a cubic fourfold. Recall from [27] that the Kuznetsov category $\mathcal{A}_X$ of $X$ is defined as a certain semi-orthogonal component of $\text{perf}(X)$. Therefore, thanks to Theorem 5.8 Corollary 4.4 and Proposition 4.5, we have $\ker(q^{\text{alg}, \text{nc}}_{X, \text{num}}/\sim_{\text{alg}}) = 0$, i.e., the equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $\mathcal{A}^\text{dg}_X$ agree, where $\mathcal{A}^\text{dg}_X$ denotes the dg enhancement of $\mathcal{A}_X$ induced from $\text{perf}_{\text{dg}}(X)$.

(ii) Let $X$ be a Gushel-Mukai $n$-fold. Recall from [31] that the Gushel-Mukai category $\mathcal{A}_X$ of $X$ is defined as a certain semi-orthogonal component of $\text{perf}(X)$. Therefore, for $X$ a Gushel-Mukai fourfold, Theorem 5.8 Corollary 4.4 and Proposition 4.5 imply that $\ker(q^{\text{alg}, \text{nc}}_{X, \text{num}}/\sim_{\text{alg}}) = 0$, i.e., the equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $\mathcal{A}^\text{dg}_X$ agree.

Under the above notations, we obtain the analogous of [38, Theorem (B)]:

**Theorem 5.10.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $K_0(\mathcal{A}^\text{dg}_X)_{\mathbb{Q}}$ agree when $X$ is a cubic fourfold or an ordinary generic Gushel-Mukai fourfold.

We now adapt [38, Theorems (C) and (D)] to our context:

**Theorem 5.11.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $\mathbb{Z}^*(X)_{\mathbb{Q}}$ agree when $X$ is a generic Gushel-Mukai fourfold containing a plane $P$ of type $\text{Gr}(2,3)$.

**Proof.** The proof is similar to the one given in [38, §5]. Just bear in mind that one needs to use Corollary 4.4, Propositions 4.5 and 4.6, and also to consider Theorem 5.10 instead of [38, Theorem (B)]. □

**Theorem 5.12.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $\mathbb{Z}^*(X)_{\mathbb{Q}}$ agree when $X$ is an ordinary Gushel-Mukai fourfold containing a quintic del Pezzo surface.

**Proof.** Note first that the semi-orthogonal decomposition of $\text{perf}(X)$ consists only of $\mathcal{A}_X$ and of exceptional objects, [31, Proposition 2.3]. Therefore, Corollary 4.4 and a repeated use of Proposition 4.5 imply that $\ker(q^X_{\sim_{\text{num}}/\sim_{\text{alg}}})$ and $\ker(q^{\text{alg}, \text{nc}}_{X, \text{num}}/\sim_{\text{alg}})$ are isomorphic. To finish the proof, just consider Corollary 4.4, Proposition 4.6, and follow the proof of [38, Theorem (D)]. □

We are in conditions to generalize Theorem 5.10.

**Theorem 5.13.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $K_0(\mathcal{A}^\text{dg}_X)_{\mathbb{Q}}$ agree when $X$ is a cubic fourfold, an ordinary generic Gushel-Mukai fourfold, an ordinary Gushel-Mukai fourfold containing a plane of type $\text{Gr}(2,3)$ or an ordinary Gushel-Mukai fourfold containing a quintic del Pezzo surface.
5.7. Küchle fourfolds. A Küchle fourfold is the zero locus of a global section of a certain vector bundle on a specific Grassmannian. In particular, a Küchle fourfold of type $c_2$, denoted by $X_{c_2}$, is the zero locus of a global section of the vector bundle $\Lambda^2U^\perp(1) \oplus \mathcal{O}(1) \!\!\!\!\mathcal{O}$ on $\Gr(3, 8)$, where $U^\perp$ is the tautological vector subbundle of rank 5 on the Grassmannian $\Gr(3, 8)$ and $\mathcal{O}(1)$ stands for the ample generator of its Picard group; consult [20] [26] for further details.

**Theorem 5.14.** The equivalence relations $\sim_{\text{alg}}$ and $\sim_{\text{num}}$ on $Z^*(X_{c_2})\mathbb{Q}$ agree.

**Proof.** Thanks to [26, Corollary 4.12], the category $\perf(X_{c_2})$ admits a semi-orthogonal decomposition with 6 exceptional line bundles and a noncommutative K3 category $\mathcal{A}_X$ which is (Fourier-Mukai) equivalent to the non-trivial part of the derived category of a cubic fourfold $Z$. Consequently, Proposition [35, Remark 4.2], and Theorem 5.13 imply that $\ker(q_{\sim_{\text{num}}}) \simeq \ker(q_{\sim_{\text{alg}}})$ is trivial. Thanks to Corollary 4.3, this implies that $Z^*_{\sim_{\text{num}}/\sim_{\text{alg}}}(X_{c_2})\mathbb{Q}$ is trivial. \(\square\)

**Remark 5.15.** A consequence of Theorem 5.14 is that Voevodsky's nilpotence conjecture holds for $X_{c_2}$. To the best of the author's knowledge, this proves Voevodsky's nilpotence conjecture in new cases. Note that $X_{c_2}$ is not a Gushel-Mukai fourfold because it has Picard number greater than 1; see [11] [1].

5.8. Family of sextic del Pezzo surfaces. A sextic du Val del Pezzo surface is a normal integral projective surface $X$ with at worst du Val singularities and ample anticanonical class such that $K_X^2 = 6$. Take $S$ and $T$ smooth projective $k$-schemes and $f : T \to S$ a du Val family of sextic del Pezzo surfaces, i.e., $f$ is a flat morphism such that for every geometric point $s \in S$ the fiber $T_s$ of $T$ over $S$ is a sextic du Val del Pezzo surface. Following [25] [5], with $d = 2, 3$, let $\mathcal{M}_d$ denote the relative moduli stack of semi-stable sheaves on fibers of $T$ over $S$ with Hilbert polynomial $h_d(t) := (3t + d)(t + 1)$ and $Z_d$ the coarse moduli space of $\mathcal{M}_d$. Consequently, there are flat morphisms $Z_2 \to S$ and $Z_3 \to S$ with degree 2 and 3, respectively.

**Theorem 5.16.** Let $f : T \to S$ be a du Val family of sextic del Pezzo surfaces, and assume that the characteristic of $k$ is not 2 neither 3. Under these conditions, with $\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}$, we have an isomorphism:

$$Z^*_{\sim_{\text{rat}}/\sim_{\text{rat}}}(T)\mathbb{Q} \simeq Z^*_{\sim_{\text{alg}}/\sim_{\text{alg}}}(S)\mathbb{Q} \oplus Z^*_{\sim_{\text{nil}}/\sim_{\text{nil}}}(Z_2)\mathbb{Q} \oplus Z^*_{\sim_{\text{num}}/\sim_{\text{num}}}(Z_3)\mathbb{Q}.$$

**Proof.** Following [28, Theorem 5.2 and Proposition 5.11], the category $\perf(T)$ admits a semi-orthogonal decomposition

$$\perf(T) = \langle \perf(S), \perf(Z_2, \mathbb{B}_2), \perf(Z_3, \mathbb{B}_3) \rangle,$$

where $\mathcal{F}_2$ and $\mathcal{F}_3$ are, resp., certain sheaves of Azumaya algebras over $Z_2$, and $Z_3$, of order 2, and 3, resp. By considering Propositions 4.3 and 4.7 we hence conclude that

$$K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}((\perf_{\text{dg}}(T))\mathbb{Q}) \cong K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}((\perf_{\text{dg}}(S))\mathbb{Q} \oplus K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(\perf_{\text{dg}}(Z_2))\mathbb{Q} \oplus K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(\perf_{\text{dg}}(Z_3))\mathbb{Q}.$$

Now, an application of Corollary 4.3 finishes the proof. \(\square\)

**Corollary 5.17.** Let $f : T \to S$ be a du Val family of sextic del Pezzo surface and assume that the characteristic of $k$ is not 2 neither 3. If $\dim(S) \leq 2$, then the equivalence relations $\sim_{\text{num}}$ and $\sim_{\text{alg}}$ on $Z^*(T)\mathbb{Q}$ agree.

**Remark 5.18.** Note that, in the conditions of Corollary 5.17, Voevodsky's conjecture holds for $T$.

5.9. Homological Projective Duality. Let $X$ be a smooth projective $k$-scheme equipped with a line bundle $\mathcal{L}_X(1)$ and let us write $X \to \mathbb{P}(\mathcal{V})$ for the associated morphism where $V := H^0(X, \mathcal{L}_X(1))^*$. Assume that the triangulated category $\perf(X)$ admits a Lefschetz decomposition $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{k-1} \rangle$ with respect to $\mathcal{L}_X(1)$, where $\mathcal{A}_r(r) := \mathcal{A}_r \otimes \mathcal{L}_X(r)$, see [22, Definition 4.1]. Note that $\mathcal{A}_r(r) \simeq \mathcal{A}_r$. Bearing in mind [22, Definition 6.1], let $Y$ be the Homological Projective (HP)-dual of $X$, $\mathcal{L}_Y(1)$ the HP-dual line bundle, and $Y \to \mathbb{P}(\mathcal{V}^*)$ the morphism associated to $\mathcal{L}_Y(1)$. Given a linear subspace $L \subseteq V^*$, we consider the linear sections $X_L := X \times_{\mathbb{P}(\mathcal{V})} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(\mathcal{V})} \mathbb{P}(L)$. For a survey on HP Duality we invite the reader to consult [25].

**Theorem 5.19** (HP duality). Let $X$ and $Y$ be as above and assume that $X_L$ and $Y_L$ are smooth and that $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$. Consider the equivalence relations $\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}$ and assume moreover that $\ker(q_{\sim_{\text{num}}}) = 0$ (or, equivalently, that $K_{0, \sim_{\text{num}}}(\mathcal{A}_0^d)\mathbb{Q} = 0$), where $\mathcal{A}_0^d$ stands for the dg enhancement of $\mathcal{A}_0$ induced from $\perf_{\text{dg}}(X)$. Under these assumptions, we have an isomorphism $Z^*_{\sim_{\text{rat}}/\sim_{\text{rat}}}(X_L)\mathbb{Q} \simeq Z^*_{\sim_{\text{num}}/\sim_{\text{num}}}(Y_L)\mathbb{Q}$. 


Remark 5.20. Given a generic subspace $L \subseteq V^*$, the sections $X_L$ and $Y_L$ are smooth, and we have $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$. Moreover, by inductive use of Proposition 4.5, we have $\ker(A_{\sim_{\leq 5}/\sim_{\leq 1}}^nc) = 0$ whenever $A$ admits a full exceptional collection; see [25, §1.1] and Remark 4.2(i). This shows that the assumptions of Theorem 5.19 are quite mild.

Proof. The proof is very similar to the proof given in [8, §9]. Follow the same reasoning with the next three differences: firstly, where they write that a certain conjecture holds we write that the respective quotient $K_{0,\sim_{\leq 5}/\sim_{\leq 1}}L$ is trivial; secondly, apply Proposition 1.3 to conclude that $K_{0,\sim_{\leq 5}/\sim_{\leq 1}}(\mathbb{A}_j^{\leq 5})\mathbb{Q}$ is trivial; thirdly, apply Corollary 4.4 instead of [8, Theorem 1.1].

Example 5.21 (Linear sections of Grassmannians). Let us apply Theorem 5.19 to the case of linear sections of Grassmannians:

(i) For $W = k^{\leq 6}$, let $X_L$ be a generic linear section of codimension $r$ of the Grassmannian $\text{Gr}(2, W)$ under the Plücker embedding, and $Y_L$ the corresponding dual linear section of the cubic Pfaffian $Pf(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$. Note that $X_L$ and $Y_L$ are smooth and that $\dim(X_L) = 8 - r$ and $\dim(Y_L) = r - 2$ when $r \leq 6$.

(ii) For $W = k^{\leq 7}$, let $X_L$ be a generic linear section of codimension $r$ of the Grassmannian $\text{Gr}(2, W)$ under the Plücker embedding, and $Y_L$ the corresponding dual linear section of the cubic Pfaffian $Pf(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$. Note that $X_L$ and $Y_L$ are smooth and that $\dim(X_L) = 10 - r$ and $\dim(Y_L) = r - 4$ when $r \leq 10$.

Note also that [21, (11) and (12)], Proposition 1.3 and Remark 4.2 imply that for both classes (i)-(ii) there is a Lefschetz decomposition of $\text{perf}(\text{Gr}(2, W))$ and that $\ker(\sim_{\leq 5}/\sim_{\leq 1}) = 0$, where $\mathbb{A}_0$ is the first component of the Lefschetz decomposition of $\text{perf}(\text{Gr}(2, W))$.

Corollary 5.22. Let $X_L$ and $Y_L$ be as in the above classes (i)-(ii) and $\sim_1, \sim_2 \in \{\sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}}\}$. Under the assumption that $X_L$ and $Y_L$ are smooth, we have $Z^*_{\sim_1/\sim_2}(X_L)\mathbb{Q} \simeq Z^*_{\sim_1/\sim_2}(Y_L)\mathbb{Q}$. Moreover, the equivalence relations $\sim_{\text{num}}$ and $\sim_{\text{alg}}$ on $Z^*(X_L)$ agree when $r \leq 6$ (class (i)), and when $r \leq 6$ and $8 \leq r \leq 10$ (class (ii)).

Proof. The first statement is immediate from Theorem 5.19. The second statement follows from Remark 4.2(iii), except the case where $X_L$ and $Y_L$ are as in class (i) and $r = 5$. In this latter case, just follow the proof in [8, §8] and consider Remark 4.2(i).

Corollary 5.23. For $k = \mathbb{C}$, let $X_L$ and $Y_L$ be as above in class (i) and let $\dim L = r = 3$. Then $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(X_L)\mathbb{Q} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$.

Proof. In this case, $X_L$ is a 5-fold and $Y_L$ is an elliptic curve; see [21, §10]. Therefore, since $Y_L$ is of genus 1, $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(Y_L)\mathbb{Q}$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2 \times 1}$; see Remark 4.2(iii). Consequently, Theorem 5.19 implies $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(X_L)\mathbb{Q}$ isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$.

Corollary 5.24. For $k = \mathbb{C}$, let $X_L$ and $Y_L$ be as in the above class (ii) and let $\dim L = r$.

(i) If $r = 5$, then $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(X_L)\mathbb{Q} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 86}$.

(ii) If $r = 9$, then $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(Y_L)\mathbb{Q} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 30}$.

Proof. When $r = 5$, $X_L$ is a Fano 5-fold of index 2 and $Y_L$ is a curve of genus 43; see [21, §11]. Therefore, we deduce that $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(Y_L)\mathbb{Q}$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2 \times 43}$; see Remark 4.2(iii). Consequently, Theorem 5.19 implies that $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(X_L)\mathbb{Q}$ and $(\mathbb{Q}/\mathbb{Z})^{\oplus 86}$ are isomorphic.

When $r = 9$, $X_L$ is a curve of genus 15 and $Y_L$ is a Fano 5-fold of index 2; see [21, §11]. Hence, by combining Theorem 5.19 with Remark 4.2(iii), we conclude that $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(Y_L)\mathbb{Q}$, $Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(X_L)\mathbb{Q}$ and $(\mathbb{Q}/\mathbb{Z})^{\oplus 2 \times 15}$ are all isomorphic to each other.

Example 5.25 (Linear sections of determinantal varieties). Let $U$ and $V$ be two $k$-vector spaces of dimensions $m$ and $n$, respectively, with $m \leq n$ and $r$ an integer such that $0 < r < m$. As in [6], consider the determinantal variety $\mathcal{X}_{m,n} \subseteq \mathbb{P}(U \otimes V)$ defined as the locus of those matrices $M : V \to U^*$ with rank $\leq r$. It is known that $\mathcal{X}_{m,n}$ admits a canonical Springer resolution of singularities $\lambda_{m,n}^r : \mathbb{P}(Q \otimes U) \to \text{Gr}(r, U)$, where $Q$ stands for the tautological quotient on $\text{Gr}(r, U)$. Under these notations, let $X_L$ be a generic linear section of codimension $c$ of $\mathcal{X}_{m,n}$ under the map $\mathcal{X}_{m,n}^r \to \mathbb{P}(U \otimes V)$, and $Y_L$ the corresponding dual linear section of $\mathcal{X}_{m,n}^{r\perp}$ under the map $\mathcal{X}_{m,n}^{r\perp} \to \mathbb{P}(U^* \otimes V^*)$. From [6, §3] we have that $X_L$ and $Y_L$ are both...
smooth and, from \cite{[11]} §3, we have \( \dim(X_L) = r(m+n-r) - 1 - \dim(L) \) and \( \dim(Y_L) = r(m-n-r) - 1 + \dim(L) \). Moreover, by \cite{[6]} §3, Proposition 4.5 and Remark 4.2(i), there is a Lefschetz decomposition of \( \text{perf}(X_{\text{rat}}) \) and \( \ker(q_{X_{\text{rat}}/\text{rat}}) = 0 \).

Corollary 5.26. Let \( X_L \) and \( Y_L \) be as in Example 5.25 and \( \sim_1, \sim_2 \in \{ \sim_{\text{rat}}, \sim_{\text{alg}}, \sim_{\text{nil}}, \sim_{\text{num}} \} \). Under these assumptions, we have \( Z_{\sim_1/\sim_2}^*(X_L)_{\mathbb{Q}} \cong Z_{\sim_1/\sim_2}^*(Y_L)_{\mathbb{Q}} \). Moreover, the equivalence relations \( \sim_{\text{num}} \) and \( \sim_{\text{alg}} \) on \( Z^*(X_L) \) (and on \( Z^*(Y_L) \)) agree whenever \( \dim(L) \geq r(m+n-r) - 3 \) or \( \dim(L) \leq 3 + r(n-m+r) \).

5.10. Prime Fano threefolds and del Pezzo threefolds. A Fano variety is a smooth proper connected algebraic variety whose anticanonical class is ample. Following \cite{[29]} §5.4, we have that a prime Fano threefold is a Fano threefold \( X \) with Pic\((X) = \mathbb{Z}K_X \) whose genus \( g(X) \) is defined from \( (K_X)^3 = 2g(X) - 2 \) and it is known that \( 1 \leq g(X) \leq 12 \) and \( g(X) \neq 11 \). Moreover, for prime Fano threefolds of even genus there is a semi-orthogonal decomposition of \( \text{perf}(X) \) with a nontrivial component \( A_X \).

In the same way, following \cite{[29]} §5.4, a del Pezzo threefold is a Fano threefold \( Y \) with \( -K_Y = 2H \), for a primitive Cartier divisor class \( H \) and its degree is defined as \( d(Y) = H^3 \). It is known that \( 1 \leq d(Y) \leq 5 \) for del Pezzo threefolds of Picard rank 1. In addition, we have that a del Pezzo threefold admits a semi-orthogonal decomposition with a non-trivial component \( B_Y \); consult \cite{[24]} §3 for further details.

Let \( X \) be a prime Fano threefold with \( g(X) \in \{ 8, 10, 12 \} \). Following \cite{[24]} Theorem 3.8, there exists a unique del Pezzo threefold \( Y \) with degree \( d(Y) = \frac{g(X)}{2} - 1 \in \{ 3, 4, 5 \} \) such that \( A_X \cong B_Y \).

Proposition 5.27. Let \( X \) and \( Y \) be as above. Then, we have an isomorphism \( Z^*_{\sim_{\text{rat}}/\sim_{\text{rat}}}(X)_{\mathbb{Q}} \cong Z^*_{\sim_{\text{rat}}/\sim_{\text{rat}}}(Y)_{\mathbb{Q}} \).

Proof. Proposition 4.5 and Remark 4.2(i) imply that \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(X)_{\mathbb{Q}} \) is isomorphic to \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(A_X^{\text{alg}})_{\mathbb{Q}} \). Similarly, \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(Y)_{\mathbb{Q}} \) is isomorphic to \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(B_Y^{\text{alg}})_{\mathbb{Q}} \). Since the categories \( A_X \) and \( B_Y \) are (Fourier-Mukai) equivalent, we have, moreover, an isomorphism between \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(A_X^{\text{alg}})_{\mathbb{Q}} \) and \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(B_Y^{\text{alg}})_{\mathbb{Q}} \). This implies that \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(X)_{\mathbb{Q}} \) and \( K_{0, \sim_{\text{rat}}/\sim_{\text{rat}}}(Y)_{\mathbb{Q}} \) are isomorphic. Consequently, Corollary 4.4 allows us to conclude that \( Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(X)_{\mathbb{Q}} \) is isomorphic to \( Z^*_{\sim_{\text{alg}}/\sim_{\text{rat}}}(Y)_{\mathbb{Q}} \). \( \square \)

5.11. Fano fourfolds of K3 type. In \cite{[7]} a list of 64 Fano fourfolds of K3 type is given. By combining Remark 4.2(i), Corollary 4.4, Propositions 4.5, 4.7, and Theorem 5.13, one obtains the following theorem, where we use the notation in \cite{[7]}.

Theorem 5.28. The equivalence relations \( \sim_{\text{alg}} \) and \( \sim_{\text{num}} \) agree when \( X \) belongs to one of the following 59 families Fano fourfolds of type K3:

(i) 16 families of Fano fourfolds of K3 type obtained inside products of flag manifolds, where at least one projection is the blow up of a cubic fourfold: from C-2 to C-17.
(ii) 4 families of Fano fourfolds of K3 type obtained inside products of flag manifolds, where at least one projection is a blow up of a Gushel–Mukai fourfold: from GM-18 to GM-20 and GM-22.
(iii) 36 families of Fano fourfolds of K3 type that we obtained inside products of flag manifolds, where at least one projection is a blow up with center birational to a K3 surface, and no cubic or Gushel–Mukai fourfold is involved: from K3-24 to K3-35 and from K3-37 to K3-60.
(iv) 3 other families of Fano fourfolds of K3 type: from R-61 to R-63.

Remark 5.29. Theorem 5.28 is still valid for three more families of Fano fourfolds that are not of K3 type. Namely: A-65, A-67 and A-68; consult \cite{[7]} Appendix A).

Remark 5.30. Note that Theorem 5.28 and Remark 5.29 imply that Voevodsky’s nilpotence conjecture holds for all those families of Fano fourfolds.

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