Energy-Momentum Distribution: Some Examples

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Abstract

In this paper, we elaborate the problem of energy-momentum in
General Relativity with the help of some well-known solutions. In
this connection, we use the prescriptions of Einstein, Landau-Lifshitz,
Papapetrou and Möller to compute the energy-momentum densities
for four exact solutions of the Einstein field equations. We take the
gravitational waves, special class of Ferrari-Ibanez degenerate solution,
Senovilla-Vera dust solution and Wainwright-Marshman solution. It
turns out that these prescriptions do provide consistent results for
special class of Ferrari-Ibanez degenerate solution and Wainwright-
Marshman solution but inconsistent results for gravitational waves
and Senovilla-Vera dust solution.

Keyword: Energy-Momentum Distribution

1 Introduction

In the theory of General Relativity (GR), the energy-momentum conservation
laws are given by

\[ T_{a}^{b} = 0, \quad (a, b = 0, 1, 2, 3), \]  

where \( T_{a}^{b} \) denotes the energy-momentum tensor. In order to change the co-
variant divergence into an ordinary divergence so that global energy-momentum

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conservation, including the contribution from gravity, can be expressed in the usual manner as in electromagnetism, Einstein formulated [1] the conservation law in the following form

\[
\frac{\partial}{\partial x^b}(\sqrt{-g}(T^b_a + t^b_a)) = 0. \tag{2}
\]

Here \( t^b_a \) is not a tensor quantity and is called the gravitational field pseudo-tensor. Schrödinger showed that the pseudo-tensor can be made vanish outside the Schwarzschild radius using a suitable choice of coordinates. There have been many attempts in order to find a more suitable quantity for describing the distribution of energy and momentum due to matter, non-gravitational and gravitational fields. The proposed quantities which actually fulfill the conservation law of matter plus gravitational parts are called gravitational field pseudo-tensors. The choice of the gravitational field pseudo-tensor is not unique. Because of this, quite a few definitions of these pseudo-tensors have been proposed.

Misner at el. [2] showed that the energy can only be localized in spherical systems. But later on, Cooperstock and Sarracino [3] proved that if energy is localizable for spherical systems, then it can be localized in any system. Einstein was the first to construct a locally conserved energy-momentum complex [4]. After this attempt, many physicists including Tolman [5], Landau-Lifshitz [6], Papapetrou [7], Bergmann [8] and Weinberg [9] introduced different definitions for the energy-momentum complex. These definitions can only give meaningful results if the calculations are performed in Cartesian coordinates. In 1990, Bondi [10] argued that a non-localizable form of energy is not allowed in GR. After this, the idea of quasi-local energy was introduced by Penrose and other researchers [11-13]. In this method, one can use any coordinate system while finding the quasi-local masses to obtain the energy-momentum of a curved spacetime. Bergqvist [14] considered seven different definitions of quasi-local mass and showed that no two of these definitions give the same result. Chang at el. [15] showed that every energy-momentum complex can be associated with a particular Hamiltonian boundary term and hence the energy-momentum complexes may also be considered as quasi-local.

Möller [16,17] proposed an expression which is the best to make calculations in any coordinate system. He claimed that his expression would give the same results for the total energy and momentum as the Einstein’s energy-momentum complex for a closed system. Lessner [18] gave his opinion that
Möller’s definition is a powerful concept of energy and momentum in GR. However, Möller’s prescription was also criticized by some people [10,19-21]. Komar’s complex [21], though not restricted to the use of Cartesian coordinates, is not applicable to non-static spacetimes. Thus each of these energy-momentum complex has its own drawbacks. As a result, these ideas of the energy-momentum complexes could not lead to some unique definition of energy in GR.

Scheidegger [22] raised doubts whether gravitational radiation has well-defined existence. Rosen [23] investigated whether or not cylindrical gravitational waves have energy and momentum. He used the energy-momentum pseudo-tensors of Einstein and Landau-Lifshitz and carried out calculations in cylindrical coordinates. He found that the energy-momentum densities vanish. The results obtained fit in the conjecture of Scheidegger that a physical system cannot radiate gravitational energy. Two years later, Rosen [24] realized the mistake and carried out the calculations in Cartesian coordinates. He found that energy-momentum densities are non-vanishing and reasonable. After that Rosen and Virbhadra [25] calculated the energy-momentum densities of gravitational waves in Einstein complex and found to be finite and reasonable results. Numerous attempts have been made to resolve the problem of energy-momentum localization but still remains un-resolved. This problem first appeared in electromagnetism which turns out to be a serious matter in GR due to the non-tensorial quantities.

Virbhadra et al. [25-31] explored several spacetimes for which different energy-momentum complexes show a high degree of consistency in giving the same and acceptable energy-momentum distribution. Aguirregabiria et al. [32] showed that five different energy-momentum complexes gave the same result for any Kerr-Schild class (including the Schwarzschild, Reissner-Nordström, Kerr and Vaidya metrics). Xulu [33-35] extended this investigation and found same energy distribution in the Melvin magnetic and Bianchi type I universe. Chamorro and Virbhadra [36] and Xulu [37] studied the energy distribution of charged black holes with a dilaton field.

This paper explores some more examples to investigate this problem. The paper is organized as follows. In section 2, we shall briefly mention different prescriptions to evaluate energy-momentum distribution. Sections 3-6 are devoted for the evaluation of energy-momentum densities for the four particular spacetimes. The last section contains summary of the results obtained.
2 Energy-Momentum Complexes

We shall use four different prescriptions, i.e., Einstein, Landau-Lifshitz, Papapetrou and Möller, to evaluate the energy-momentum density components of different spacetimes. For the sake of completeness, we briefly give these formulae as details are available elsewhere [38-41].

The energy-momentum complex of Einstein [42] is given by

\[ \Theta^b_a = \frac{1}{16\pi} H^b_{a,c}, \quad (a, b, \ldots = 0, 1, 2, 3), \]  

(3)

where

\[ H^b_{a,c} = \frac{g_{ad}}{\sqrt{-g}} \left[ -g(g^{bd}g^{ce} - g^{cd}g^{be}) \right]_{,e}. \]  

(4)

It is to be noted that \( H^b_{a,c} \) is anti-symmetric in indices \( b \) and \( c \). \( \Theta^0_0 \) is the energy density, \( \Theta^0_i \) \( (i = 1, 2, 3) \) are the components of momentum density and \( \Theta^0_i \) are the energy current density components.

The prescription of Landau-Lifshitz [6] is defined as

\[ L^{ab} = \frac{1}{16\pi} \ell^{abcd}, \]  

(5)

where

\[ \ell^{abcd} = -g(g^{ab}g^{cd} - g^{ad}g^{cb}). \]  

(6)

\( L^{00} \) represents the energy density of the whole system including gravitational and \( L^{0i} \) represent the components of the total momentum density. \( \ell^{abcd} \) has symmetries of the Riemann curvature tensor. It is clear from Eq. (7) that \( L^{ab} \) is symmetric with respect to its indices.

The symmetric energy-momentum complex of Papapetrou [7] is given as

\[ \Omega^{ab} = \frac{1}{16\pi} N^{abcd}, \]  

(7)

where

\[ N^{abcd} = \sqrt{-g}(g^{ab}\eta^{cd} - g^{ac}\eta^{bd} + g^{cd}\eta^{ab} - g^{bd}\eta^{ac}) \]  

(8)

and \( \eta^{ab} \) is the Minkowski spacetime. The quantities \( N^{abcd} \) are symmetric in its first two indices \( a \) and \( b \). The locally conserved quantities \( \Omega^{ab} \) contain contribution from the matter, non-gravitational and gravitational field. The quantity \( \Omega^{00} \) represents energy density and \( \Omega^{0i} \) are the momentum density components.
The coordinate independent prescription by Möller [17] is defined as

$$M^b_a = \frac{1}{8\pi} K^{bc}_{a,c},$$  \hspace{1cm} (9)

where

$$K^{bc}_{a} = \sqrt{-g}(g_{ad,e} - g_{ae,d})g^{be}g^{cd}. \hspace{1cm} (10)$$

Here $K^{bc}_{a}$ is antisymmetric, $M^0_0$ is the energy density, $M^i_0$ are the momentum density components and $M^0_i$ are the energy current density components. We shall apply these prescriptions to particular examples.

### 3 Gravitational Waves

The general line element of gravitational waves [46] is given by

$$ds^2 = e^{-M}(dt^2 - dx^2) - e^{-U}(e^{-V}dy^2 + e^{V}dz^2),$$  \hspace{1cm} (11)

where $U$, $V$ and $M$ are functions of $t$ and $x$ only. In the case of a stiff perfect fluid, the Einstein field equations imply that $U$ satisfies the wave equation

$$U_{tt} - U_{xx} = 0$$  \hspace{1cm} (12)

and $U$, $V$ satisfy the linear equation

$$V_{tt} - U_{t}V_{t} - V_{xx} + U_{x}V_{x} = 0.$$  \hspace{1cm} (13)

It may be noted that the solution describing the closed FRW stiff fluid model can be given by

$$e^{-U} = \sin 2t \sin 2x, \hspace{0.5cm} V = \ln \tan x,$$

$$M = -\ln \sin 2t - \ln \gamma, \hspace{0.5cm} \sigma = \sqrt{3} \ln \tan t,$$

where $0 < t < \frac{\pi}{2}$, $0 < x < \frac{\pi}{2}$ and $\gamma$ is constant. A stiff perfect fluid can be associated with a potential $\sigma(t, x)$ such that the density and 4-velocity of the fluid are given by

$$16\pi \rho = e^M (\sigma^2_t - \sigma^2_x), \hspace{0.5cm} u_a = \frac{\sigma_a}{(\sigma_b \sigma^b)^{\frac{1}{2}}}$$
and the fluid potential $\sigma$ satisfies

$$\sigma_{tt} - U_t \sigma_t - \sigma_{xx} + U_x \sigma_x = 0. \quad (14)$$

A gravitational wave with toroidal wavefront can be obtained by taking [47]

$$U = -\ln t - \ln \rho, \quad V = \ln t - \ln \rho + \tilde{V}(t, \rho),$$

where $\tilde{V}$ has the form

$$\tilde{V}(t, \rho) = \int_{\frac{1}{2}}^{\infty} \phi(k)(t\rho)^k H_k\left(\frac{t^2 + \rho^2 - a^2}{2t\rho}\right) dk$$

with an arbitrary function $\phi(k)$ and

$$M = \frac{1}{2k} a_k (t^2 - \rho^2)(t\rho)^{k-1} H_{k-1} - \frac{1}{2} (t\rho)^{2k} a_k^2 [k^2 H_k^2 - \frac{(t^2 - \rho^2)^2}{4t^2 \rho^2} H_{k-1}^2],$$

where the dimension of $a_k$ is $L^{-2k}$. Now we calculate energy-momentum distribution by using the four prescriptions.

For the Einstein prescription, we need the following non-zero components of $H_{a}^{bc}$

$$H_0^{01} = -H_0^{10} = 2U'e^{-(2M+3U)}, \quad (15)$$
$$H_1^{01} = 2Ue^{-U}, \quad (16)$$

where dot and prime mean differentiating w.r.t. $t$ and $x$ respectively. Using Eqs.(15)-(16) in Eq.(3), we obtain the following components of energy, momentum and energy current densities

$$\Theta_0^0 = \frac{e^{-(2M+3U)}}{8\pi} [U'' - 2U'M' - 3U'''], \quad (17)$$
$$\Theta_0^1 = \frac{e^{-(2M+3U)}}{8\pi} [U'(2M + 3U) - U''], \quad (18)$$
$$\Theta_1^0 = \frac{e^{-U}}{8\pi}(\dot{U}' - \ddot{U}U'), \quad (19)$$
$$\Theta_2^0 = 0 = \Theta_3^0 = \Theta_3^2.$$

The non-zero components of $\ell_{abcd}$ are used in the Landau-Lifshitz complex

$$\ell^{0011} = -e^{-2U}, \quad (21)$$
$$\ell^{0101} = e^{-2U}. \quad (22)$$
Substituting these values in Eq.(5), it follows the components of energy and momentum (energy current) densities in Landau-Lifshitz prescription

\begin{align*}
L^{00} &= \frac{e^{-2U}}{8\pi}(U'' - 2U'^2), \\
L^{10} &= L^{01} = \frac{e^{-2U}}{8\pi}(2U'\dot{U} - \dot{U}''), \\
L^{20} &= L^{02} = 0 = L^{30} = L^{03}.
\end{align*}

(23)

(24)

(25)

For Papapetrou prescription, the non-zero components of \( N^{abcd} \) are the following

\begin{align*}
N^{0011} &= -2e^{-U}, \\
N^{1001} &= N^{1010} = e^{-U}.
\end{align*}

(26)

(27)

When we make use of these values in Eq.(7), it yields the following components of energy and momentum (energy current) densities

\begin{align*}
\Omega^{00} &= \frac{e^{-U}}{8\pi}(U'' - U'^2), \\
\Omega^{10} &= \Omega^{01} = \frac{e^{-U}}{8\pi}(\dot{U}'U'' - \dot{U}''), \\
\Omega^{20} &= \Omega^{02} = 0 = \Omega^{30} = \Omega^{03}.
\end{align*}

(28)

(29)

(30)

The following non-zero components of \( K^{bc}_a \) are required in Möller prescription

\begin{align*}
K^{01}_0 = -K^{10}_0 = -M'e^{-U}.
\end{align*}

(31)

Consequently, the components of energy, momentum and energy current densities become

\begin{align*}
M^0_0 &= \frac{e^{-U}}{8\pi}(M'U'' - M''), \\
M^1_0 &= \frac{e^{-U}}{8\pi}(\dot{M}' - \dot{U}M'), \\
M^0_1 &= 0 = M^2_0 = M^3_0 = M^3_0.
\end{align*}

(32)

(33)

(34)

We have obtained energy-momentum distribution for a general line element of the gravitational waves. The energy-momentum distribution for the colliding and toroidal gravitational waves can be found by substituting the corresponding values of \( U, V, M \).
4 Special Class of Ferrari-Ibanez Degenerate Solution

The special class of Ferrari-Ibanez degenerate solution [48] is given by

\[ ds^2 = (1 + \sigma \sin t)^2(dt^2 - dz^2) - \frac{(1 - \sigma \sin t)}{(1 + \sigma \sin t)} dx^2 - \cos^2 z(1 + \sigma \sin t)^2 dy^2, \quad (35) \]

where \( \sigma = \pm 1 \) is an arbitrary constant and \( t, z \) are timelike and spacelike coordinates respectively. If we take the coordinate transformation

\[ r = 1 + \sin t, \quad \tau = \sqrt{2}x, \quad \theta = \frac{\pi}{2} - z, \quad \phi = \sqrt{2}y, \quad m = 1, \]

then the line element (38) reduces to the Schwarzschild metric. It is to be noted that there is a curvature singularity for \( \sigma = -1 \) and \( t = \frac{\pi}{2} \). However, the spacetime appears to be regular for \( \sigma = 1 \) and \( 0 \leq t \leq \frac{\pi}{2} \).

The required components of \( H^b_{ac} \) for Einstein complex are

\[ H^0_{03} = -H^3_{00} = 2 \sin z(1 - \sigma^2 \sin^2 t)^{\frac{1}{2}}, \quad (36) \]
\[ H^0_{33} = 2 \cos z \sin t \cos t \left(1 - \sigma^2 \sin^2 t\right)^{\frac{1}{2}}. \quad (37) \]

Thus the components of energy, momentum and energy current densities with \( \sigma = \pm 1 \) become

\[ \Theta^0_0 = \frac{1}{8\pi} \cos t \cos z, \quad (38) \]
\[ \Theta^3_0 = \frac{1}{8\pi} \sin t \sin z = -\Theta^0_3, \quad (39) \]
\[ \Theta^1_0 = \Theta^1_1 = 0 = \Theta^0_2 = \Theta^2_0. \quad (40) \]

For Landau-Lifshitz prescription, the non-zero components of \( \ell^{abcd} \) are as follows:

\[ \ell^{0033} = -\cos^2 z(1 - \sigma^2 \sin^2 t), \quad (41) \]
\[ \ell^{0303} = \cos^2 z(1 - \sigma^2 \sin^2 t). \quad (42) \]

The components of energy and momentum (energy current) densities are

\[ L^{00} = \frac{1}{8\pi} \cos t \cos z, \quad (43) \]
\[ L^{30} = L^{03} = \frac{1}{16\pi} \sin 2t \sin 2z, \quad (44) \]
\[ L^{10} = L^{01} = 0 = L^{20} = L^{32}. \quad (45) \]
The energy-momentum densities in Papapetrou complex can be found by using the components of \( N^{abcd} \) as

\[
N^{0033} = -2 \cos z (1 - \sigma^2 \sin^2 t)^{\frac{3}{2}}, \tag{46}
\]
\[
N^{3003} = N^{3030} = 2 \cos z (1 - \sigma^2 \sin t)^{\frac{1}{2}}. \tag{47}
\]

As a result, the components of energy and momentum (energy current) densities turn out to be

\[
\Omega^{00} = \frac{1}{8\pi} (\cos t \cos z), \tag{48}
\]
\[
\Omega^{30} = \Omega^{03} = \frac{1}{8\pi} (\sin t \sin z), \tag{49}
\]
\[
\Omega^{10} = \Omega^{01} = 0 = \Omega^{20} = \Omega^{02}. \tag{50}
\]

We see that energy density becomes the same in three prescriptions.

For Möller prescription, the required components of \( K^{bc}_a \) are

\[
K^{03}_3 = -2\sigma \cos t \sin z \frac{(1 - \sigma \sin t)^{\frac{1}{2}}}{(1 + \sigma \sin t)^{\frac{1}{2}}}, \tag{51}
\]
and the components of energy, momentum and energy current densities are

\[
M^{00}_0 = 0, \tag{52}
\]
\[
M^{00}_3 = -\frac{1}{4\pi} \cos t \sin z \frac{(1 - \sin t)^{\frac{1}{2}}}{(1 + \sin t)^{\frac{1}{2}}}, \tag{53}
\]
\[
M^{01}_1 = 0 = M^{02}_0 = M^{20}_0 = M^{30}_0. \tag{54}
\]

This shows that energy and momentum become constant.

## 5 Senovilla-Vera Dust Solution

The Senovilla-Vera dust solution [49] in the fluid co-moving coordinates is given by

\[
ds^2 = dt^2 - t^2 dx^2 - Y^2 dy^2 - (te^{-x})^{1-k} dz^2,
\]
where

\[
Y = c_-(te^{-x})^{b'_-} + m^2 (te^x)^b + c_+(te^{-x})^{b'_+},
\]
\[
b = \frac{1}{2} (1 + k), \quad b_\pm = \frac{1}{2} (1 \pm \sqrt{2-k^2}), \quad -1 < k < 1
\]
and $c_-, c_+$ are constants.

Einstein complex gives the components of $H^{bc}_a$ as

\[
H^{01}_0 = (te^{-x})^{1-k} \left[ 2c_- b_- t^b_+ e^{-xb} - m^2 b t^{b-1} e^{xb} + c_+ b_+ t^{b+1} e^{-xb+} \right] + Y(1-k)t^{-1},
\]
\[
H^{10}_1 = - (te^{-x})^{1-k} \left[ 2t c_- b_+ t^b_+ e^{-xb-} + m^2 b t^{b-1} e^{xb} + c_+ b_+ t^{b+1} e^{-xb+} \right] + (1-k)Y.
\] (56)

Substituting these values in Eq.(3), we obtain the components of energy, momentum and energy current densities as

\[
\Theta^0_0 = \frac{1}{16\pi} \left[ (1-k)(te^{-x})^{1-k} c_- b_- t^{b-1} e^{-xb} - m^2 b t^{b-1} e^{xb} \right.
+ c_+ b_+ t^{b+1} e^{-xb+} \right] + 2(t e^{-x})^{1-k} c_- b_- t^{b+1} e^{xb} + m^2 b^2 t^{b-1} e^{xb} \\
+ c_+ b_+ t^{b+1} e^{-xb+} \right] + (1-k)^2 t \frac{2}{t^{(1-k)/2}} e^{-x} \left[ \{c_-(te^{-x})^{-1/2} \}
+ m^2 (te^{-x})^b + c_+(te^{-x})^{b+} \right] + (1-k)t^{-1/2} e^{-x} \left[ \{c_- (te^{-x})^{b-} - m^2 b (te^{-x})^b + c_+ b_+ (te^{-x})^{b+} \} \right].
\] (58)

\[
\Theta^1_0 = \frac{(te^{-x})^{1-k}}{16\pi} \left[ (1-k)^2 \left[ c_- (te^{-x})^{b-} + m^2 (te^{-x})^b + c_+ (te^{-x})^{b+} \right] \\
+ (1-k)c_- b_- (te^{-x})^{b-} - m^2 b (te^{-x})^b + c_+ b_+ (te^{-x})^{b+} \right] \\
- t(1-k) \left[ c_- b_- t^{b-1} e^{-xb} + m^2 b t^{b-1} e^{xb} + c_+ b_+ t^{b+1} e^{-xb+} \right] \\
- 2t c_- b_- t^{b-1} e^{-xb} - m^2 b^2 t^{b-1} e^{xb} + c_+ b_+ t^{b+1} e^{-xb+} \right] \right],
\] (59)

\[
\Theta^1_0 = \frac{1}{16\pi} \left[ (1-k)^2 t \left[ \frac{2}{t^{(1-k)/2}} e^{-x} \right] \left[ c_- (te^{-x})^{b-} + m^2 (te^{-x})^b \right] \\
+ c_+ (te^{-x})^{b+} \right] + (1-k) t \left[ \frac{2}{t^{(1-k)/2}} e^{-x} \right] \left[ c_- b_- (te^{-x})^{b-} - m^2 b (te^{-x})^b \right] \\
+ c_+ b_+ (te^{-x})^{b+} \right] - (1-k) t \left[ \frac{2}{t^{(1-k)/2}} e^{-x} \right] \left[ c_- b_- t^{b-1} e^{-xb} \right] \\
+ m^2 b t^{b-1} e^{xb} + c_+ b_+ t^{b+1} e^{-xb+} \right] - 2t \left[ \frac{2}{t^{(1-k)/2}} e^{-x} \right] \left[ c_- b_- t^{b-1} e^{-xb} - m^2 b t^{b-1} e^{xb} + c_+ b_+ t^{b+1} e^{-xb+} \right] \right],
\] (60)

\[
\Theta^2_0 = 0 = \Theta^0_3 = \Theta^3_3.
\] (61)

In the Landau-Lifshitz prescription, we use the component of $\ell^{abcd}$

\[
\ell^{1001} = - \ell^{0011} = (te^{-x})^{1-k} \left[ c_- (te^{-x})^{b-} \\
+ m^2 (te^{-x})^b + c_+ (te^{-x})^{b+} \right].
\] (62)
The components of energy and momentum (energy current) densities are

\[
L^{00} = \frac{1}{16\pi}[(1-k)^2(te^{-x})^b\{c_-(te^{-x})^b - m^2(te^x)^b + c_+(te^{-x})^b} \] \right)^2
+ 4(1-k)(te^{-x})^{1-k}\{c_-(te^{-x})^b - m^2(te^x)^b + c_+(te^{-x})^b}\}
\times \{c_-b_-(te^{-x})^b - m^2b(te^x)^b + c_+b_+(te^{-x})^b\}
- 2(te^{-x})^{1-k}\{c_-b_-(te^{-x})^b - m^2b(te^x)^b + c_+b_+(te^{-x})^b\}^2
+ 2(te^{-x})^{1-k}\{c_-b_-(te^{-x})^b - m^2b(te^x)^b + c_+b_+(te^{-x})^b\}
\times \{c_-b_-(te^{-x})^b + m^2b(te^x)^b + c_+(te^{-x})^b\}]. \tag{63}
\]

\[
L^{10} = \frac{1}{16\pi} L^{01} = \frac{1}{16\pi} \left[2\{c_-(b_- + 1-k)(te^{-x})^{b_- +1-k} - m^2b(te^x)^b(te^{-x})^{1-k}\}
+ m^2(1-k)(te^x)^b(te^{-x})^{1-k} + c_+b_1 + 1-k)(te^{-x})^{b_1 +1-k}\}
\times \{c_-b_2 t^{b_-1}e^{-xb^2} - m^2b^2 t^{b_-1}e^{xb^2} + c_+b_2 t^{b_1 +1-k}\}
- (1-k)\{c_-b_-(te^{-x})^b - k^b e^{-x(b_+ +1-k)}
+ m^2b^b e^{-x(1-k-b)}(1-k-b)
+ c_+(b_1 + 1-k)t^{b_+ +1-k}e^{-x(b_+ +1-k)}\}]. \tag{64}
\]

\[
L^{20} = L^{02} = 0 = L^{30} = L^{03}. \tag{65}
\]

For Papapetrou complex, the following components of \(N^{abcd}\) are used

\[
N^{0011} = -(t^2 + 1)(t^{-\frac{1+k}{2}}e^{-x(\frac{1-k}{2})})^2 \left[c_-(te^{-x})^{b_-} + m^2(te^x)^b + c_+(te^{-x})^b\right], \tag{66}
\]

\[
N^{0101} = e^{-x(\frac{1-k}{2})} (t^{-1+k} + t^{-\frac{1+k}{2}}) \left[c_-(te^{-x})^{b_-} + m^2(te^x)^b + c_+(te^{-x})^b\right]. \tag{67}
\]

Consequently, the components of energy and momentum (energy current) densities turn out to be

\[
\Omega^{00} = \frac{-(t^2 + 1)(t^{-\frac{1+k}{2}}e^{-x(\frac{1-k}{2})})^2}{16\pi} \left[(1-k)^2\{c_-(te^{-x})^{b_-}\}
+ m^2(te^x)^b + c_+(te^{-x})^b\} + (1-k)\{c_-b_-(te^{-x})^b\}
- m^2b(te^x)^b + c_+b_+(te^{-x})^b\} + \{c_-b_2^2 (te^{-x})^b\}
- m^2b^2 (te^x)^b + c_+b_2^2 (te^{-x})^b\}]. \tag{68}
\]
\[ \Omega^{10} = \Omega^{01} = \frac{1}{16\pi} \left( 1 - \frac{k}{2} \right) e^{-\frac{\sqrt{1+k}}{2} \left( t^{-(2+k)(1+k)} \right)} \{ (t^{-(2+k)}(1+k) \]
\[ + t^{-(\frac{3+k}{2})(1+k)} \times (c_- (te^{-x})b_- + m^2(te^x)b) \]
\[ + c_+ (te^{-x})b_+) + (t^{-(1+k)} + t^{-(\frac{1+k}{2})}) \times (c_- b_- t^{b_-e^{-x}}b_- \]
\[ + m^2 bt^{b-1}e^{bx} + c_+ b_+ t^{b_+e^{-x}}b_+] \}
\[ + e^{-x \frac{1-k}{2}} \{ (t^{-(2+k)}(1+k) + t^{-(\frac{3+k}{2})(1+k)} \]
\[ \times (c_- b_- t^{b_-e^{-x}}b_- - m^2 bt^{b-1}e^{bx} + c_+ b_+ t^{b_+e^{-x}}b_+) \]
\[ - (t^{-(1+k)} + t^{-(\frac{1+k}{2})}) \times (c_- b_- t^{b_-e^{-x}}b_- \]
\[ - m^2 b^2 t^{b-1}e^{bx} + c_+ b_+ t^{b_+e^{-x}}b_+) \} \]
\[ \Omega^{20} = \Omega^{02} = 0 = \Omega^{30} = \Omega^{03}. \]

(69)

The required component of \( K^a_b \) in Möller prescription is
\[ K_1^{01} = 2(te^{-x})^{\frac{1-k}{2}} [c_- (te^{-x})b_- + m^2(te^x)b + c_+ (te^{-x})b+] \]  
and the components of energy, momentum and energy current densities become
\[ M_0^0 = 0 = M_0^1 = 0, \]
\[ M_1^0 = -\frac{1}{4\pi} [c_- (\frac{1-k}{2} + b_-)(te^{-x})^{\frac{1-k}{2}b_-} \]
\[ + (\frac{1-k}{2})m^2 (te^x)b(te^{-x})^{\frac{1-k}{2}b_+} - m^2 b(te^x)b(te^{-x})^{\frac{1-k}{2}} \]
\[ + c_+ (\frac{1-k}{2} + b_+)(te^{-x})^{\frac{1-k}{2}b_+}], \]
\[ M_2^0 = 0 = M_0^2 = M_0^3 = M_0^3. \]  

(70)

(71)

This gives constant energy and momentum.

6 Wainwright-Marshman Solution

The line element of Wainwright-Marshman solution [50] is given as follows
\[ ds^2 = t^{2m}e^n(dt^2 - dx^2) - t^{\frac{4}{3}}dy^2 - 2\omega t^{\frac{2}{3}}dydz - (t^{\frac{4}{3}}\omega^2 + t^{\frac{2}{3}})dz^2, \]  

(75)
where \( \omega = \omega(t - x) \) is an arbitrary function, \( n = n(t - x) \) is determined according to \( n' = (\omega')^2 \) and \( m \) is constant. For this metric, energy-momentum turns out to be constant in the Einstein, Landau-Lifshitz and Papapetrou prescriptions. In the Möller’s prescription, the components of energy, momentum and energy current densities are

\[
M_0^0 = \frac{tn''}{8\pi}, \quad (76)
\]

\[
M_0^1 = -\frac{n'}{8\pi}, \quad (77)
\]

\[
M_1^0 = 0 = M_2^0 = M_3^0 = M_0^3. \quad (78)
\]

This shows that energy and momentum can be constant only if \( n \) is constant.

7 Summary and Discussion

This paper continues the investigation of comparing various distributions presented in the literature. It is devoted to discuss the burning problem of energy-momentum in the framework of GR and four different energy-momentum complexes have been used to find the energy-momentum distribution. We have applied the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller to investigate energy-momentum distribution for various spacetimes. The summary of the results (only non-zero quantities) can be given in the form of tables in the following:

**Table 1(a) Gravitational Waves: Einstein Complex**

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| \( \Theta_0^0 \)          | \( \frac{e^{-2U'M'}}{8\pi}(U'' - 2U'M' - 3U'^2) \) |
| \( \Theta_0^1 \)          | \( \frac{e^{-2U'M'}}{8\pi}(2U'M' + 3U - U') \) |
| \( \Theta_1^0 \)          | \( \frac{e^{-2U'M'}}{8\pi}(U' - UU') \) |

**Table 1(b) Gravitational Waves: Landau-Lifshitz Complex**

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| \( L_{00} \)             | \( \frac{e^{-2U'}}{8\pi}(U'' - 2U'^2) \) |
| \( L_{10} = L_{01} \)    | \( \frac{e^{-2U'}}{8\pi}(2U'U' - U') \) |
Table 1(c) Gravitational Waves: Papapetrou Complex

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| $\Omega^{00}$             | $\frac{e^{-\frac{U}{8\pi}}}{8\pi} (U'' - U'^2)$ |
| $\Omega^{10} = \Omega^{01}$ | $\frac{e^{-\frac{U}{8\pi}}}{8\pi} (UU' - U')$ |

Table 1(d) Gravitational Waves: Möller Complex

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| $M^{00}_0$                | $\frac{e^{-\frac{U}{8\pi}}}{8\pi} (M'U' - M'')$ |
| $M^{01}_0$                | $\frac{e^{-\frac{U}{8\pi}}}{8\pi} (M' - UM')$ |

Table 2(a) Ferrari-Ibanez Degenerate Solution: Einstein Complex

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| $\Theta^{00}_0$           | $\frac{1}{8\pi} (\cos t \cos z)$ |
| $\Theta^{30}_0$           | $\frac{1}{8\pi} (\sin t \sin z)$ |
| $\Theta^{03}_0$           | $\frac{1}{8\pi} (\sin t \sin z)$ |

Table 2(b) Ferrari-Ibanez Degenerate Solution: Landau-Lifshitz Complex

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| $L^{00}_1$                | $\frac{1}{8\pi} (\cos t \cos z)$ |
| $L^{01}_1 = L^{03}_1$     | $\frac{1}{16\pi} (\sin 2t \sin 2z)$ |

Table 2(c) Ferrari-Ibanez Degenerate Solution: Pappetrou Complex

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| $\Omega^{00}_1$           | $\frac{1}{8\pi} (\cos t \cos z)$ |
| $\Omega^{30}_1 = \Omega^{03}_1$ | $\frac{1}{8\pi} (\sin t \sin z)$ |
Table 2(d) Ferrari-Ibanez Degenerate Solution: Möller Complex

| Energy-Momentum Densities | Expressions |
|---------------------------|-------------|
| $M_3^0$                   | $-\frac{1}{4\pi} \cos t \sin 2 \left( \frac{(1-\sin t)^2}{(1+\sin t)^2} \right)$ |

Table 3(a) Senovilla-Vera Dust Solution: Einstein Complex

| E-M Densities | Expressions |
|---------------|-------------|
| $\Theta_0^0$ | $-\frac{1}{16\pi} \left[(1-k)(te^{-x})\right] \left\{ c_- b_- t^{b-1} e^{-xb_+} + m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} + 2(1-k) \right\} \right.$ $\left[ e^{-x} \right]^{(1-k)/2}$ $\left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} + (1-k)^2 t^{-\frac{(1+k)}{2}} e^{-x} \left[ e^{-x} \right]^{(1-k)/2} \left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} + (1-k) t^{-\frac{(1+k)}{2}} e^{-x} \left[ e^{-x} \right]^{(1-k)/2} \left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} - m^2 b(te^x)^b + c_+ b_+ (te^{-x})^{b_+}$ $\left. + m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\}$ $-2 t^{-\frac{(1+k)}{2}} e^{-x} \left[ e^{-x} \right]^{(1-k)/2} \left\{ c_- b_- t^{b-1} e^{-xb_+} - m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\}$ |
| $\Theta_0^1$ | $\left[ (1-k)^2 \right] \left[ e^{-x} \right]^{(1-k)/2} \left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} + (1-k) t^{-\frac{(1+k)}{2}} e^{-x} \left(1+k\right) \left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} - m^2 b(te^x)^b + c_+ b_+ (te^{-x})^{b_+}$ $\left. - t(1-k) \left\{ c_- b_- t^{b-1} e^{-xb_+} + m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\} - 2 t \left\{ c_- b_- t^{b-1} e^{-xb_+} + m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\}$ $- m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\}$ |
| $\Theta_1^0$ | $\left[ \frac{(1-k)}{2} \right] \left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} + (1-k) \left\{ c_- (te^{-x})^{b_-} + m^2 (te^x)^b + c_+ (te^{-x})^{b_+} \right\} - m^2 b(te^x)^b + c_+ b_+ (te^{-x})^{b_+}$ $\left. - t(1-k) \left\{ c_- b_- t^{b-1} e^{-xb_+} + m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\} - 2 t \left\{ c_- b_- t^{b-1} e^{-xb_+} + m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\}$ $- m^2 b^{b-1} e^{xb_+} + c_+ b_+ t^{b-1} e^{-xb_+} \right\}$ |
### Table 3(b) Senovilla-Vera Dust Solution: Landau-Lifshitz Complex

| E-M Densities | Expressions |
|---------------|-------------|
| \( L^{00} \) | \[
\frac{1}{16\pi}[(1-k)^2(te^{-x})^b c_-(te^{-x})^b_+ + m^2(te^x)^b \\
+ c_+(te^{-x})^b_+]^2 + 4(1-k)(te^{-x})^{1-k} c_-(te^{-x})^b_+ \\
+ m^2(te^x)^b + c_+(te^{-x})^b_+ \{c_-(te^{-x})^b_+ \\
- m^2b(te^x)^b + c_+(te^{-x})^b_+ \} - 2(te^{-x})^{1-k} \]
\[
c_-(te^{-x})^b_+ - m^2b(te^x)^b + c_+(te^{-x})^b_+ \}^2 \\
+ 2(te^{-x})^{1-k} c_-(te^{-x})^b_+ - m^2b(te^x)^b \\
+ c_+(te^{-x})^b_+ \{c_-(te^{-x})^b_+ + m^2(te^x)^b \\
+ c_+(te^{-x})^b_+ \} \]
| \( L^{10} = L^{01} \) | \[
\frac{1}{16\pi} \{2[c_-(b_- + 1 - k)(te^{-x})^b_+ - 1-k \\
- m^2b(te^x)^b(te^{-x})^{1-k} + m^2(1-k)(te^x)^b(te^{-x})^{1-k} \\
+ c_+(b_+ + 1 - k)(te^{-x})_+ - k + 1-k] \{c_-(te^{-x})^b_+ \\
+ m^2b(te^x)^b + c_+(te^{-x})^b_+ \{c_-(te^{-x})^b_+ \\
- m^2b(te^x)^b + c_+(te^{-x})^b_+ \}] \}
\]

### Table 3(c) Senovilla-Vera Dust Solution: Papapetrou Complex

| E-M Densities | Expressions |
|---------------|-------------|
| \( \Omega^{00} \) | \[
- (t^2+1)(t+1)(e^{-1-t}k)^2 \{c_-(te^{-x})^b_+ \\
+ m^2(te^x)^b + c_+(te^{-x})^b_+ \} + (1-k) \{c_-(te^{-x})^b_+ \\
- m^2b(te^x)^b + c_+(te^{-x})^b_+ \} + \{c_-(te^{-x})^b_+ \\
- m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \} \}
| \( \Omega^{10} = \Omega^{01} \) | \[
\frac{1}{16\pi} \{1-k \} e^{-2(1-t)^2} \{(t-(2+k)(1+k) \\
+ t^{-1}(1+k) e^{-1-t}k \}
\]
\[
+ c_+(te^{-x})^b_+ - t^-1(1+k) e^{-1-t}k \}
\]
\[
+ m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \{c_-(te^{-x})^b_+ \\
- m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \} \}
\]
\[
+ e^{-2(1-t)^2} \{(t-(2+k)(1+k) + \frac{1}{2} t^{(3+k)} \\
(1-t)^2 \}
\]
\[
+ (t-(1+k) + t^-1(1+k) e^{-1-t}k \}
\]
\[
+ m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \{c_-(te^{-x})^b_+ \\
- m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \} \}
\]
\[
- (t-(1+k) + t^-1(1+k) e^{-1-t}k \}
\]
\[
+ m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \{c_-(te^{-x})^b_+ \\
- m^2b^2(te^x)^b + c_+(te^{-x})^b_+ \} \}
\]

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Table 3(d) Senovilla-Vera Dust Solution: Möller Complex

| E-M Densities | Expressions |
|---------------|-------------|
| $M^0_1$       | $-\frac{1}{4\pi} c_-(\frac{1-k}{2} + b_-)(te^{-x})^{(\frac{1-k}{2})} + b_- + (\frac{1-k}{2})m^2(te^{x})^b(te^{-x})^{(\frac{1-k}{2})} + b_+ - m^2b(te^x)^b(te^{-x})^{(\frac{1-k}{2})} + c_+(\frac{1-k}{2} + b_+)(te^{-x})^{(\frac{1-k}{2})} + b_+ | |

From the above tables, it is concluded that the energy-momentum densities turn out to be finite and well-defined in all the prescriptions for the spacetimes under consideration. We find that the three prescriptions, i.e., Einstein, Landau-Lifshitz and Papapetrou complexes provide the same energy distribution for the special class of Ferrari-Ibanez degenerate solution while Möller prescription gives constant energy and momentum (Table 2). If we take $t$ or $z = \frac{\pi}{2}$, energy becomes constant and for $t = 0 = z$, momentum turns out to be constant in the remaining prescriptions as well. For the Wainwright-Marshman solution, energy and momentum is constant in all the prescriptions except Möller where energy and momentum densities are constant for $n$ to be constant. The metric exhibiting asymptotic silence-breaking singularities, Senovilla-Vera dust solution yields constant energy-momentum only in Möller prescription while it has different non-vanishing energy-momentum densities in the remaining prescriptions. The energy-momentum densities turn out to be different for gravitational waves in all the prescriptions (Table 1).

It is worth mentioning that the results of energy-momentum distribution for the two examples turn out to be same in all the prescriptions. These results justify the viewpoint of Virbhadra and his collaborators [26-37] that different energy-momentum complexes may provide some basis to define a unique quantity. However, the remaining two examples give different energy-momentum densities in different prescriptions. This difference is due to the fact that different energy-momentum complexes, which are pseudo-tensors, are not covariant objects. This is in accordance with the equivalence principle [2] which implies that the gravitational field cannot be detected at a point. This also supports the viewpoint of Cooperstock [3] that energy can not be localized. Notice that each expression may have a geometrically and physically clear significance associated with the boundary conditions.
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