Alfvén QPOs in Magnetars

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Abstract. We investigate torsional Alfvén oscillations of relativistic stars with a global dipole magnetic field, via 2D numerical simulations. We find that a) there exist two families of quasi-periodic oscillations (QPOs) with harmonics at integer multiples of the fundamental frequency, b) the lower-frequency QPO is related to the region of closed field lines, near the equator, while the higher-frequency QPO is generated near the magnetic axis, c) the QPOs are long-lived, d) for the chosen form of dipolar magnetic field, the frequency ratio of the lower to upper fundamental QPOs is about 0.6, independent of the equilibrium model or of the strength of the magnetic field, and e) within a representative sample of EOSs and of various magnetar masses, the Alfvén QPO frequencies are given by accurate empirical relations that depend only on the compactness of the star and on the magnetic field strength. Compared to the observational frequencies, we also obtain an upper limit on the strength of magnetic field of SGR 1806-20 (if it is dominated by a dipolar component) between $\sim 3$ and $7 \times 10^{15}$ Gauss.

1. Introduction

The phenomenon of Soft Gamma Repeaters (SGRs) may allow us in the near future to determine fundamental properties of strongly magnetized, compact stars. SGRs produce giant flares with peak luminosities of $10^{44} - 10^{46}$ erg/s, which display a decaying tail for several hundred seconds. Up to now, three giant flares have been detected, SGR 0526-66 in 1979, SGR 1900+14 in 1998, and SGR 1806-20 in 2004. The timing analysis of the latter two events revealed several quasi-periodic oscillations (QPOs) in the decaying tail, whose frequencies are approximately 18, 26, 30, 92, 150, 625, and 1840 Hz for SGR 1806-20, and 28, 53, 84, and 155 Hz for SGR 1900+14 \cite{1}. Some of these oscillation frequencies are similar to those of torsional modes of the solid crust of a compact star. This observation is in support of the proposal that SGRs are magnetars (compact objects with very strong magnetic fields) \cite{2}. During an SGR event, torsional oscillations in the solid crust of the star could be excited \cite{3}, leading to the observed frequencies in the X-ray tail. Spherical stars have generally two types of oscillations, \textit{spheroidal} with polar parity and \textit{toroidal} with axial parity. The observed QPOs in SGR X-ray tails may originate from toroidal oscillations, since these could be excited more easily than poloidal oscillations, because they do not involve density variations. In Newtonian theory, there have been several investigations of torsional oscillations in the crust region of neutron stars (see e.g., \cite{4} for references), while only few studies have taken general relativity into account \cite{5, 6, 7, 8, 9, 10}. On the other hand, as an alternative possibility for explaining the QPO frequencies, one can also consider global Alfvén oscillations. Levin stressed the importance of crust-core coupling by a global magnetic field and of the existence of an Alfvén continuum \cite{11}, while Glampedakis et al. considered a model with simplified geometry, in which Alfvén oscillations form a discrete spectrum of normal modes.
that could be associated with the observed low-frequency QPOs [12]. In [13], the existence of a continuum was stressed further and it was shown that the edges or turning points of the continuum can yield long-lived QPOs. In addition, numerical simulations showed that drifting QPOs within the continuum become amplified near the frequencies of the crustal normal modes. Within this model, Levin suggested a likely identification of the 18Hz QPO in SGR 1806-20 with the lowest frequency of the MHD continuum or its first overtone. The above results were obtained in toy models with simplified geometry and Newtonian gravity, i.e., still we do not know how the Alfvén oscillations behave in a realistic magnetar models. Thus in this article, we perform two-dimensional numerical simulations of linearized Alfvén oscillations in magnetars. Our model improves on the previously considered toy models in various ways: relativistic gravity is assumed, various realistic equations of state (EOSs) are considered and a dipolar magnetic field is constructed. We do not consider the presence of a solid crust, but only examine the response of the ideal magnetofluid to a chosen initial perturbation. Unless otherwise noted, we adopt units of $c = G = 1$, where $c$ and $G$ denote the speed of light and the gravitational constant, respectively, while the metric signature is $(-, +, +, +)$. Extensive details of our work were presented in [14].

2. Magnetar Models
We adopt the ideal MHD approximation, for which the electric fields are zero for a comoving observer. And the stellar deformation due to the magnetic field is also neglected, because the magnetic energy is much smaller than the gravitational binding energy even for the magnetar models considered here. Therefore, the equilibrium configuration is static and spherically symmetric. To a non-magnetized TOV model, we superimpose an axisymmetric, dipolar magnetic field.

The perturbation equations can be derived by linearizing the equations of motion and Maxwell's equations. We focus only on the axisymmetric axial perturbations, i.e., $m = 0$, which are independent from the polar perturbations, under our assumptions. Additionally, we adopt the Cowling approximation by neglecting the metric perturbations, i.e. we assume $\delta g_{\mu\nu} = 0$.

3. Torsional Oscillations of the Crust
Before showing the results for the Alfvén oscillations, we briefly review the properties of torsional oscillations of the crust, ignoring the coupling with the core region, for which details can be found in [7]. The effect of magnetic fields on crustal oscillations becomes important for $B > B_\mu$, where $B_\mu = 4 \times 10^{15}$ Gauss. The frequencies for fundamental modes depend strongly on the stellar properties, differing by about 30 ~ 50 %. On the other hand, frequencies of overtones are nearly independent of the harmonics index $\ell$, but depend strongly on both the EOS for crust and for the core region. Furthermore, using our numerical results for a large number of stellar models with different EOSs and different stellar masses, we derive the following empirical formula for the frequencies of torsional oscillations of the crust

$$\ell t_n \simeq \left[ 1 + \ell \alpha_n \left( \frac{B}{B_\mu} \right)^2 \right]^{1/2} \ell t_n^{(0)} ,$$

where $\ell t_n^{(0)}$ is the corresponding frequency in the limit of no magnetic field and $\ell \alpha_n$ and is a coefficient that depends on the properties of the stellar model. From these calculations we find that the attempt to explain the observed frequencies of QPOs is partially successful, but also notice that it may not be possible to explain all data by using only torsional oscillations of the crust. Especially the observed frequencies of 18, 26 and 30 Hz in SGR 1806-20 are difficult to be made compatible with only pure crustal frequencies, because the differences between these
observed frequencies are smaller than the frequency spacing of the crustal torsional oscillations with different harmonics index \( \ell \). This is one of the motivations to examine global Alfvén oscillation.

4. Alfvén Oscillations

In order to examine the global Alfvén oscillations, we consider the magnetar models with axisymmetric poloidal magnetic field and without crust regions. For out numerical calculations we add a Kreiss-Oliger numerical dissipation [15]. Then, defining our main perturbation variable \( Y \) through \( \delta u^\phi \sim \partial_t Y(t, r, \theta) \), (where \( u^\phi \) is the \( \phi \)-component of the 4-velocity) we are led to a single, two-dimensional, wave-like evolutionary equation

\[
A_{tt} \frac{\partial^2 Y}{\partial t^2} = A_{20} \frac{\partial^2 Y}{\partial r^2} + A_{11} \frac{\partial^2 Y}{\partial r \partial \theta} + A_{02} \frac{\partial^2 Y}{\partial \theta^2} + A_{10} \frac{\partial Y}{\partial r} + A_{01} \frac{\partial Y}{\partial \theta} + \epsilon_D D_4 Y, \tag{2}
\]

where the coefficients \( A_{tt}, A_{20}, A_{11}, A_{02}, A_{10}, \) and \( A_{01} \) are functions of the equilibrium properties and \( \epsilon_D D_4 Y \) corresponds to the 4th-order Kreiss-Oliger dissipation (see [14] for the detail of this equation). With appropriate boundary conditions we can thus obtain the time-evolution of oscillations throughout the magnetar interior. Fourier Transforms of this time-evolution at various points inside the star are shown in figure 1. At a first glance we can see the same peaks on each figure, whose fundamental frequency \( (n = 0) \) is around 41 Hz, with additional peaks being integer multiples of the fundamental. But for the panel at \( \theta \sim \pi/2 \), i.e., near the equatorial plane, we can see the additional peaks, for which the fundamental frequency is around 25 Hz. Using Levin’s toy model for the Alfvén continuum in magnetars [13], these results can be interpreted as the frequency peaks of two families of QPOs generated by the edges or turning points of an Alfvén continuum with the addition of overtones. Hereafter, we call these two families lower and upper QPOs and label \( n \)-th overtones as \( L_n \) and \( U_n \) correspondingly, with \( L_0 \) and \( U_0 \) being the fundamental QPOs.

In the left panel of Figure 2 we plot the phase of these Alfvén oscillations, for a representative case. From this figure one can see that the oscillations near the axis \( (\theta = 0) \) are practically at the same phase, while at other points inside the star the phase differs. This means that the Alfvén oscillations are not discrete normal modes, but form a continuum. The continuum can be seen more strongly near the equatorial plane. In the right panel of Figure 2 we show the evolution of \( \partial_t Y \) at the location inside the star where the effective amplitude of the fundamental upper QPO, \( U_0 \), attains its maximum value (see [14] for our definition of the effective amplitude). It is evident that, even though the oscillations are not discrete, the fundamental upper QPO is long-lived, since the amplitude of the oscillations barely diminishes with time.

5. Realistic EOS and Empirical Formula

We have obtained the lower and upper Alfvén QPO frequencies for a representative sample of magnetar models with realistic EOSs and various masses. These results are summarized in Table 1. In this Table we also show the ratio of the lower to upper QPO frequencies and we can see that these ratios are almost 0.6, independently of the stellar models. Additionally, we can observe that the overtones are nearly integer multiples of the fundamental frequencies such that

\[
f_{L_n} \simeq (n + 1)f_{L_0} \quad \text{and} \quad f_{U_n} \simeq (n + 1)f_{U_0}. \tag{3}\]
Figure 1. Fourier transform obtained from the 2D simulation with $B = B_\mu$, whose evolution time is 2 seconds. In the figures the different lines correspond to the different radial observers of inside star, at $r \sim 0$, $r \sim R/2$, and $r \sim R$ and the different panels correspond to different angular observers inside the star, at $\theta \sim 0$, $\theta \sim \pi/4$, and $\theta \sim \pi/2$. 
Figure 2. The left panel shows the phase of the Alfvén oscillations at different angular positions, while the right panel shows the time evolution of $\partial_t Y$ at a point where at the location inside the star where the effective amplitude of the fundamental upper QPO, $U_0$, attains its maximum value.

Table 1. Frequencies of lower and upper Alfvén QPOs and their ratios, for a representative sample of equilibrium models, constructed with various EOSs and masses and for $B = B_\mu$.

| Model      | M/R | $f_{L_0}$ (Hz) | $f_{U_0}$ (Hz) | ratio | $f_{L_1}$ (Hz) | $f_{U_1}$ (Hz) | ratio | $f_{U_2}$ (Hz) |
|------------|-----|----------------|----------------|-------|----------------|----------------|-------|----------------|
| A+DH_14   | 0.218 | 15.4 | 25.0 | 0.616 | 30.7 | 49.4 | 0.621 | 74.4 |
| A+DH_16   | 0.264 | 11.7 | 18.3 | 0.639 | 23.5 | 35.7 | 0.658 | 54.0 |
| WFF3+DH_14 | 0.191 | 17.9 | 29.8 | 0.601 | 36.2 | 59.2 | 0.611 | 89.8 |
| WFF3+DH_18 | 0.265 | 11.7 | 18.0 | 0.650 | 23.5 | 35.5 | 0.662 | 53.3 |
| APR+DH_14 | 0.171 | 20.4 | 34.1 | 0.598 | 41.3 | 68.6 | 0.602 | 104.6 |
| APR+DH_20 | 0.248 | 12.8 | 20.6 | 0.621 | 26.0 | 40.3 | 0.645 | 61.0 |
| L+DH_14   | 0.141 | 23.7 | 40.8 | 0.581 | 47.5 | 81.6 | 0.582 | 123.8 |
| L+DH_20   | 0.199 | 16.4 | 27.8 | 0.590 | 33.1 | 54.7 | 0.605 | 82.6 |

Updated empirical relations for improved boundary conditions can be found in [16, 17].

6. Constraints on Magnetic Field Strength

Using three empirical formulas, i.e., one for the frequencies of crustal torsional oscillations (1) and two for the Alfvén QPO frequencies, (4) and (5) we can attempt to identify several of the low-frequency QPOs observed in the X-ray tail of SGR 1806-20. In particular, one could identify the 18Hz and 30Hz observed frequencies with the fundamental lower and upper QPOs, correspondingly, while the observed frequencies of 92Hz and 150Hz would then be integer
multiples of the fundamental upper QPO frequency (three times and five times, correspondingly). With such an identification, our empirical relations (4) and (5) constrain the magnetic field strength of SGR 1806-20 (if it is dominated by a dipolar component) to be between $3 \times 10^{15}$ Gauss and $7 \times 10^{15}$ Gauss. Furthermore, an identification of the observed frequency of 26 Hz with the frequency of the fundamental torsional $\ell = 2$ oscillation of the magnetar’s crust implies a very stiff equation of state and a mass of about 1.4 to 1.6 $M_\odot$. For example, for the 1.4$M_\odot$ model constructed with EOS L+DH, one obtains the following frequencies: $f_{00} = 25.8$ Hz, $f_{L0} = 17.5$ Hz, $f_{U0} = 30.0$ Hz, $f_{U3} = 90.1$ Hz and $f_{U5} = 150.2$ Hz, for $B = 2.94 \times 10^{15}$ Gauss. Alternatively, one could also identify the 18 Hz and 30 Hz observed frequencies with overtones (which are also at a near 0.6 ratio). In this case, the strength of the magnetic field derived above is only an upper limit and the actual magnetic field may be weaker. Then, if one assumes that the observed frequency of 26 Hz is due to the fundamental $\ell = 2$ crust mode for a weak magnetic field, our numerical data agree best with a 1.4$M_\odot$ model constructed with an EOS of moderate stiffness. For example, for the 1.4$M_\odot$ model constructed with the APR+DH EOS one obtains $f_{00} = 25.9$ Hz, $f_{L1} = 17.7$ Hz, $f_{U1} = 30.0$ Hz, $f_{U5} = 90.1$ Hz and $f_{U9} = 150.1$ Hz, for $B = 1.77 \times 10^{15}$ Gauss.

7. Discussion

Using torsional oscillations of the magnetar models, we try to explain observed frequencies of QPOs in SGRs. This attempt is partially successful, when one considers only crustal torsional oscillations. To arrive at a more consistent model, we also consider global Alfvén oscillations, finding a continuum that allows for two families of QPOs. The QPOs produced near the magnetic axis are long-lived. Furthermore, we examined the possibility of explaining the observed data, using empirical formulas for the frequencies of crustal torsional oscillations and for global the Alfvén QPOs. Such a fit can constrain the strength of the magnetic field. In [16, 17], this investigation is continued, using improved boundary conditions and arriving at a simplified description of Alfvén oscillations along individual magnetic field lines.

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