The Duality of the Volumes and the Numbers of Vertices of Random Polytopes

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Abstract
An identity due to Efron dating from 1965 relates the expected volume of the convex hull of \( n \) random points to the expected number of vertices of the convex hull of \( n + 1 \) random points. Forty years later this identity was extended from expected values to higher moments. The generalized identity has attracted considerable interest. Whereas the left-hand side of the generalized identity—concerning the volume—has an immediate geometric interpretation, this is not the case for the right-hand side—concerning the number of vertices. A transformation of the right-hand side applying an identity for elementary symmetric polynomials overcomes the blemish. The arising formula reveals a duality between the volumes and the numbers of vertices of random polytopes.

Keywords Random polytope · Convex hull · Moment · Duality · Elementary symmetric polynomial

Mathematics Subject Classification 60D05 · 52A22

1 Introduction
Write \( \mathcal{K}^d \) for the set of all convex bodies (convex compact sets with non-empty interiors) in \( \mathbb{R}^d \). Fix \( K \in \mathcal{K}^d \), and choose points \( x_1, \ldots, x_n \in K \) randomly, independently, and according to the uniform distribution on \( K \). The set \( K_n = \text{conv} \{ x_1, \ldots, x_n \} \) is a random polytope. Denote the volume of \( K_n \) by the random variable \( V_n \) and the number of vertices of \( K_n \) by the random variable \( N_n \).
Efron [10, p. 335, formulae (3.5) and (3.7)] proved the identity

\[
\frac{EV_n}{\text{vol } K} = 1 - \frac{EN_{n+1}}{n+1}.
\] (1.1)

The left-hand side has an immediate geometric interpretation: Assume that \(n\) points have been chosen at random. Then \(EV_n/\text{vol } K\) is the probability that a further point chosen at random falls into the convex hull of the \(n\) points, which means that it is not a vertex of the convex hull of the \(n + 1\) points. Likewise, the right-hand side can be interpreted geometrically: The ratio \(EN_{n+1}/(n + 1)\) is the probability that any of \(n + 1\) points chosen at random is a vertex of their convex hull and hence not contained in the convex hull of the remaining \(n\) points.

In [7, p. 127, formula (1.8)] Efron’s identity (1.1) was extended to higher moments:

\[
\frac{EV_k^n}{(\text{vol } K)^k} = E \prod_{i=1}^{k} \left( 1 - \frac{N_{n+k}}{n+i} \right).
\] (1.2)

This result has attracted considerable interest; see in particular Sect. 8.2.3 in the book by Schneider and Weil [20], the survey articles (chapters of books) by Reitzner [17], Hug [12], and Schneider [19], as well as the research articles by Cowan [9], Groeneboom [11], Beermann and Reitzner [5], and Kabluchko et al. [13]. The identity (1.2) is also mentioned in a survey paper (book chapter) by Calka [8] and in research papers by Reitzner [15], Kabluchko et al. [14], as well as Brunel [6]. A consequence of the identity is stated in a survey article (book chapter) by Bárány [1], in a further survey article by Bárány [2], and in a survey article—different from the one mentioned above—by Schneider [18], as well as in research articles by Reitzner [16] and Bárány and Reitzner [3, 4].

The extension of the geometric interpretation of the left-hand side of (1.1) to the left-hand side of (1.2) is straightforward: Assume again that \(n\) points have been chosen at random. Then \(EV_k^n/(\text{vol } K)^k\) is the probability that each of \(k\) further points chosen at random falls into the convex hull of the \(n\) points, which means that none of the specified \(k\) points among the \(n + k\) points chosen at random is a vertex of the convex hull of the \(n + k\) points.

The right-hand side of (1.2) has no obvious geometric meaning which explains that the right-hand side is equivalent to the left-hand side. Writing it in the form

\[
\frac{(-1)^k n!}{(n + k)!} E \prod_{i=1}^{k} (N_{n+k} - (n+i)),
\]

we identify \(\prod_{i=1}^{k} (N_{n+k} - (n+i))\) as the generating function of the elementary symmetric polynomials \(\sigma_j(n+1, \ldots, n+k), j = 0, \ldots, k\), in the variables \(n+1, \ldots, n+k\). It will turn out in the proof of Theorem 1 that decomposing these polynomials in a
suitable way gives rise to a transformation of (1.2) into

\[ \frac{\mathbb{E} V_n^k}{(\text{vol } K)^k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\mathbb{E}(N_n+k)^{(j)}}{(n+k)^{(j)}}, \tag{1.3} \]

where \( \mathbb{E}(N_n+k)^{(j)} = \mathbb{E} N_{n+k}(N_{n+k-1}) \cdots (N_{n+k-j+1}) \) is the \( j \)-th factorial moment of \( N_{n+k} \) and \( (n+k)^{(j)} = (n+k)(n+k-1) \cdots (n+k-j+1) = (n+k)!/(n+k-j)! \).

The ratio \( \mathbb{E}(N_n+k)^{(j)}/(n+k)^{(j)} \) has a simple geometric interpretation: It is just the probability that \( j \) specified points in a set of \( n+k \) random points are vertices of the convex hull of the \( n+k \) points. Consequently, by the inclusion–exclusion principle,

\[ \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \frac{\mathbb{E}(N_n+k)^{(j)}}{(n+k)^{(j)}} \]

is the probability that at least one of \( k \) specified points among \( n+k \) points chosen at random is a vertex of the convex hull of the \( n+k \) points. The complementary probability, i.e., the probability that none of the \( k \) specified points is a vertex of the convex hull of the \( n+k \) points, is just given by the right-hand side of (1.3).

Trivially, (1.1) is equivalent to

\[ \frac{\mathbb{E} N_{n+1}}{n+1} = 1 - \frac{\mathbb{E} V_n}{\text{vol } K}. \]

We prove that, more generally, the identity

\[ \frac{\mathbb{E}(N_{n+k})^{(k)}}{(n+k)^{(k)}} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\mathbb{E} V_{n+k-j}^j}{(\text{vol } K)^j} \tag{1.4} \]

is dual to the identity (1.3) in the sense that (1.3) implies (1.4), and (1.4) implies (1.3).

2 Transformation Based on the Decomposition of Certain Elementary Symmetric Polynomials

Write \( \sigma_j(x_1, \ldots, x_k) \) for the \( j \)-th elementary symmetric polynomial in the variables \( x_1, \ldots, x_k \), i.e.,

\[ \sigma_j(x_1, \ldots, x_k) = \sum_{1 \leq i_1 < \ldots < i_j \leq k} x_{i_1} \cdots x_{i_j}. \]

The elementary symmetric polynomials generalize the binomial coefficients: \( \sigma_j(x_1, \ldots, x_k) = \binom{k}{j} \) if \( x_1 = \ldots = x_k = 1 \). Correspondingly, the generating function
of the binomial coefficients

\[(t - 1)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} t^{k-j}\]

extends to

\[\prod_{i=1}^{k} (t - x_i) = \sum_{j=0}^{k} (-1)^j \sigma_j(x_1, \ldots, x_k) t^{k-j},\]

and the recurrence relation \(\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}\) extends to

\[\sigma_j(x_1, \ldots, x_k) = x_k \sigma_{j-1}(x_1, \ldots, x_{k-1}) + \sigma_j(x_1, \ldots, x_{k-1}).\] (2.1)

The generating function and the recurrence relation for the binomial coefficients suggest to define \(\binom{k}{j} = 1\) if \(j = 0\) and \(\binom{k}{j} = 0\) if \(j \neq 0\) and \(k < j\). Likewise, the generating function and the recurrence relation for the elementary symmetric polynomials suggest to define \(\sigma_j(x_1, \ldots, x_k) = 1\) if \(j = 0\) and \(\sigma_j(x_1, \ldots, x_k) = 0\) if \(j \neq 0\) and \(k < j\). Here \(j\) and \(k\) are also allowed to be negative.

To transform the product on the right-hand side of (1.2), each of the elementary symmetric polynomials \(\sigma_j(n+1, \ldots, n+k), j = 0, \ldots, k\), in the variables \(n+1, \ldots, n+k\), which occur there implicitly, is now decomposed into a sum of products of elementary symmetric polynomials such that the first factor is an elementary symmetric polynomial in just the integers 1, 2, \ldots, whereas the second factor is the elementary symmetric polynomial of maximal degree, i.e., just the product of all variables.

**Proposition** Denote by \(\sigma_j(x_1, \ldots, x_k)\) the \(j\)-th elementary symmetric polynomial in the \(k\) variables \(x_1, \ldots, x_k\). Then

\[\sigma_j(x + 1, \ldots, x + k) = \sum_{i=0}^{j} \binom{k}{i} \sigma_{j-i}(1, \ldots, k-i-1) \sigma_i(x + 1, \ldots, x + i).\]

**Proof** The formula is correct if \(k = 1\). Assume that it has been verified for elementary symmetric polynomials in less than \(k\) variables. Applying the recurrence relation (2.1) we obtain

\[\sigma_j(x + 1, \ldots, x + k) = (x + k) \sum_{i=0}^{j-1} \binom{k-1}{i} \sigma_{j-i-1}(1, \ldots, k-i-2) \sigma_i(x + 1, \ldots, x + i)\]

\[+ \sum_{i=0}^{j} \binom{k-1}{i} \sigma_{j-i}(1, \ldots, k-i-2) \sigma_i(x + 1, \ldots, x + i).\]
Splitting the factor \( x + k \) by which the first sum is multiplied into \( x + i + 1 \) and \( k - i - 1 \), observing that \((x + i + 1) \sigma_i(x + 1, \ldots, x + i) = \sigma_{i+1}(x + 1, \ldots, x + i + 1)\), and adapting the summation index, we get

\[
\sigma_j(x + 1, \ldots, x + k) = \sum_{i=1}^{j} \binom{k - 1}{i - 1} \sigma_{j-i}(1, \ldots, k - i - 1) \sigma_i(x + 1, \ldots, x + i) + \sum_{i=0}^{j-1} \binom{k - 1}{i} (k - i - 1) \sigma_{j-i-1}(1, \ldots, k - i - 2) \sigma_i(x + 1, \ldots, x + i) + \sum_{i=0}^{j} \binom{k - 1}{i} \sigma_{j-i}(1, \ldots, k - i - 2) \sigma_i(x + 1, \ldots, x + i).
\]

If the index is taken from 0 to \( j \) in all three sums, only terms are added which have the value zero. According to (2.1),

\[
(k - i - 1) \sigma_{j-i-1}(1, \ldots, k - i - 2) + \sigma_{j-i}(1, \ldots, k - i - 2) = \sigma_{j-i}(1, \ldots, k - i - 1),
\]

and as \( \binom{k-1}{i-1} + \binom{k-1}{i} = \binom{k}{i} \), the claimed formula arises.

\[\square\]

**Theorem 1** Let \( K \in K^d, n \in \mathbb{N}, \) and \( k \in \mathbb{N}. \) Then

\[
\frac{\text{EV}^k_n}{(\text{vol } K)^k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} E(N_{n+k})_{(j)},
\]

where \( E(N_{n+k})_{(j)} = E N_{n+k}(N_{n+k} - 1) \cdots (N_{n+k} - j + 1) \) is the \( j \)-th factorial moment of \( N_{n+k} \) and \( (n + k)_{(j)} = (n + k)(n + k - 1) \cdots (n + k - j + 1) \).

**Proof** We start from [7, Thm. 1], which states that

\[
\frac{\text{EV}^k_n}{(\text{vol } K)^k} = E \prod_{i=1}^{k} \left(1 - \frac{N_{n+k}}{n + i}\right).
\]

The right-hand side can equivalently be written in the form

\[
\frac{n!}{(n + k)!} \sum_{j=0}^{k} (-1)^{k-j} \sigma_j(n + 1, \ldots, n + k) E N_{n+k}^{k-j}.
\]
The proposition transforms this into
\[
\frac{n!}{(n+k)!} \sum_{j=0}^{k} (-1)^{k-j} \sum_{i=0}^{j} \binom{k}{i} \sigma_{j-i}(1, \ldots, k-i-1) \sigma_i(n+1, \ldots, n+i) \, EN_{n+k}^{k-j}.
\]

Interchanging the order of summation yields
\[
\frac{n!}{(n+k)!} \sum_{i=0}^{k} \binom{k}{i} \sigma_i(n+1, \ldots, n+i) \sum_{j=i}^{k} (-1)^{k-j} \sigma_{j-i}(1, \ldots, k-i-1) \, EN_{n+k}^{k-j}.
\]

Since
\[
\sum_{j=i}^{k} (-1)^{k-j} \sigma_{j-i}(1, \ldots, k-i-1) \, EN_{n+k}^{k-j} = (-1)^{k-i} E(N_{n+k}(k-i))
\]
and
\[
\frac{n!}{(n+k)!} \sigma_i(n+1, \ldots, n+i) = \frac{1}{(n+k)(k-i)},
\]
we obtain
\[
\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \frac{E(N_{n+k}(k-i))}{(n+k)(k-i)},
\]
which is identical with the claimed expression. \(\Box\)

The subsequent theorem provides a geometric interpretation of the \(j\)-th factorial moment of \(N_{n+k}\), \(j = 0, \ldots, k\), occurring in Theorem 1. We write \(m\) instead of \(n + k - j\) hoping to facilitate understanding.

**Theorem 2** Let \(K \in K^d\), \(m \in \mathbb{N}\), and \(j \in \mathbb{N}\). Then the probability that \(j\) points distributed independently and uniformly in \(K\) are vertices of the convex hull of these \(j\) and \(m\) further points distributed independently and uniformly in \(K\) is given by
\[
\frac{E(N_{m+j})(j)}{(m+j)(j)},
\]
where \(E(N_{m+j})(j)\) denotes the \(j\)-th factorial moment of \(N_{m+j}\) and \((m+j)(j) = (m+j)(m+j-1) \cdots (m+1)\).

**Proof** On the one hand, for any \(j\) out of \(m+j\) points distributed independently and uniformly in \(K\), the probability of being vertices of the convex hull of the \(m+j\) points is the same. Hence, as there are \(\binom{m+j}{j}\) possibilities to choose \(j\) points out of
$m + j$ points, $\binom{m+j}{j}$ times this probability gives the expected number of possibilities to choose $j$ points out of the $m + j$ points such that the chosen $j$ points are vertices of the convex hull of the $m + j$ points. On the other hand, the number of possibilities to choose $j$ points out of $m + j$ points such that the chosen $j$ points are vertices of the convex hull of the $m + j$ points is just the number of possibilities to choose $j$ points out of the vertices of the convex hull, i.e., $\binom{N_{m+j}}{j}$. Thus the expected number of possibilities is $E\binom{N_{m+j}}{j}$. Combining these two observations, we find that

$$
E\binom{N_{m+j}}{j} \frac{(m + j)}{(m + j)} = \frac{E(N_{m+j})}{(m + j)(j)}
$$

is just the probability in question. □

3 Duality

**Theorem 3** (dual version of Theorem 1) Let $K \in K^d$, $n \in \mathbb{N}$, and $k \in \mathbb{N}$. Then

$$
\frac{E(N_{n+k}(k))}{(n + k)(k)} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} EV_{n+k-j}^j (\text{vol } K)^j,
$$

where $E(N_{n+k}) = E N_{n+k}(N_{n+k} - 1) \cdots (N_{n+k} - k + 1)$ is the $k$-th factorial moment of $N_{n+k}$ and $(n + k)(k) = (n + k)(n + k - 1) \cdots (n + 1).

**Proof** Theorem 1, with $n$ and $k$ replaced by $n + k - i$ and $i, i = 0, \ldots, k$, can be written in the form $v_n^{(k)} = A_k n_n^{(k)}$, where

$$
v_n^{(k)} = \left( 1, \frac{EV_{n+k-1}}{\text{vol } K}, \frac{EV_{n+k-2}}{\text{vol } K^2}, \ldots, \frac{EV_n^k}{(\text{vol } K)^k} \right)^T
$$

and

$$
n_n^{(k)} = \left( 1, \frac{EN_{n+k}}{n + k}, \frac{E(N_{n+k})}{(n + k)(2)}, \ldots, \frac{E(N_{n+k})}{(n + k)(k)} \right)^T
$$

are $(k + 1)$-vectors and

$$
A_k = \left( (-1)^j \binom{i}{j} \right)_{i=0}^{k} \left( j \right)_{j=0}^{k}
$$

is a $(k + 1) \times (k + 1)$ matrix. The inverse of $A_k$ is $A_k$ itself. To see this, consider

$$
A_k^2 = \left( (-1)^j \binom{i}{j} \right)_{i=0}^{k} \left( j \right)_{j=0}^{k} \left( (-1)^l \binom{j}{l} \right)_{l=0}^{k}.
$$
The entries in the $i$-th row of the first matrix are zero if $j > i$, and the entries in the $l$-th column of the second matrix are zero if $j < l$. Consequently, if $i < l$, the inner product of the $i$-th row of the first matrix with the $l$-th column of the second matrix is zero. If $i \geq l$, the inner product of the $i$-th row of the first matrix with the $l$-th column of the second matrix is given by

$$\sum_{j=l}^{i} (-1)^{j+l} \binom{i}{j} \binom{j}{l} = \binom{i}{l} \sum_{j=0}^{i-l} (-1)^{j} \binom{i-l}{j},$$

which is one if $i = l$ and zero if $i > l$. Hence $v_n^{(k)} = A_k n_n^{(k)}$ implies $n_n^{(k)} = A_k v_n^{(k)}$, proving Theorem 3.

Finally we point out that, likewise, $n_n^{(k)} = A_k v_n^{(k)}$ implies $v_n^{(k)} = A_k n_n^{(k)}$, proving the duality of Theorems 1 and 3.

### 4 Concluding Remark

Alternatively, one could start with Theorem 2, the proof of which is based on the same approach as the proof of [7, Thm. 1]. As described in the introduction, the inclusion–exclusion principle then implies the right-hand side of (1.3), and the proposition verifies the equivalence of the right-hand side of (1.2) and the right-hand side of (1.3).

**Funding** Open access funding provided by the Paris Lodron University of Salzburg.

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