Perturbation symmetries in viscoelastic pipe flows

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The fluctuations to the laminar FENE-P viscoelastic pipe flow are shown to exhibit leading-order power-law behaviours and the expected odd-even parities with respect to the radial coordinate that depend on the azimuthal wavenumber, \( n \). The analysis provides means for the derivation of the regularity conditions at the centreline, and allows for a complete stability analysis of three-dimensional perturbations for a general integer value of \( n \), which still remains a challenge for FENE-P models. It is shown here that the symmetry and analytic behaviours of the velocity and pressure fields of the Newtonian counterpart are both preserved in this flow, and the reason is elucidated. For \( |n| = 1 \), the perturbations of the correlations between the axial component and the radial or azimuthal components of the end-to-end polymer vector exhibit behaviour similar to that of the velocity fluctuations close to the centreline, and traced to the uniformity of axial traction with respect to azimuthal direction. For all values of \( n \), the fluctuation to the end-to-end length polymer molecules vanishes at the centreline. The ansatz for the conformations tensors derived here using heuristics have been proved in two different ways, namely, the method of Frobenius and by making use of the observations from Fourier analysis.

I. INTRODUCTION

Over the past decades, the well-known phenomenon of turbulent drag reduction associated with the addition of heavy-molecular long-chain polymers [1] has motivated research on the emergence of modal instabilities. Some groups have analyzed the disturbances in shear-thinning fluids using generalized Newtonian fluid models, such as the Carreau model [2, 3], or by considering a range of dumbbell constitutive models, such as the Oldroyd-B, or the more elaborate FENE-CR and FENE-P models. Given the theoretical backing that such dumbbell models receive from the kinetic theory [4], we focus our attention on one of these canonical models, namely the FENE-P (which stands for “Finitely Extensible Nonlinear Elastic model with Peterlin approximation”) model for this paper as it predicts shear thinning.

The vast majority of linear stability analyses that use dumbbell models were based on geometries with Cartesian symmetries [e.g. see, 5–11]. Among those performed in cylindrical

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coordinates, the analyses systematically excluded the point of singularity at the null radial co-
ordinate [e.g. see, 12]. Although the pipe geometry finds a commonplace in applications, it has
been largely excluded from non-Newtonian studies due to the difficulty of specifying the regularity
conditions at the pipe centre.

On the other hand, such regularity conditions are well known in the case of Newtonian pipe
flows. In one way of treating this singular behaviour, the conditions of uniformity of velocity fields
along the azimuthal direction in the limit of reaching the centre [13, 14] are used as boundary con-
ditions. Another approach consists of circumventing this singular point by using the symmetries
of the perturbations predicted by its analytic behaviour while the domain is artificially extended to
$-1 \leq r/R \leq 1$, where $R$ is the pipe radius [15–17]. A third way for treating this singularity, still
in the Newtonian case, consists of deriving the regularity conditions at the pipe centre ($r = 0$) by
making use of the perturbations analytic behaviour. [18].

In most stability analyses of non-Newtonian flows performed in circular geometry [19–23],
only axisymmetric disturbances with the azimuthal wavenumber $n = 0$ were studied under upper-
convected Maxwell (UCM) or Oldroyd-B models. Studying this particular mode does not pose
a problem, since the terms exhibiting an explicit dependence in $nr^{-1}$ in the linearized governing
equations for conformation tensor would vanish.

Miller and Rallison [24] considers the mode with $n = 1$ besides $n = 0$ for the UCM and
Oldroyd-B models. For $n = 1$, the azimuthal perturbation velocity, $w'$, and radial perturbation
velocity, $v'$, follow a relation $v' + iw' = 0$ at the centreline, a result observed by Khorrami et al.
[14] and established by Lewis and Bellan [25] as a property of any vector in all physical problems
in the plane perpendicular to the polar axis. This fact, together with the property that $v'(r)$ and
$u'(r)$ are even, sets $nr^{-1}(v' + iw') = 0$ in the governing equations for polymer stress components.

However, carrying out a generalized stability analysis with $|n| \geq 2$ will warrant more robust
regularity conditions. For modes with $|n| = 2$, this singularity could still be removable if the
leading order power-law behaviour with respect to radial coordinate is known. When the power-
law behaviour of the unknowns would remain unaccounted explicitly in the analysis, the terms
with the factors of $nr^{-1}u'$, $nr^{-1}v'$ or $nr^{-1}w'$ (where $u'$ is the axial perturbation velocity) in the
governing equations would pose difficulty even when the vanishing conditions, namely, $u' = v' =
w' = 0$ at $r = 0$ found by the analysis of Khorrami et al. [14] are used.

Further, neither the UCM and nor the Oldroyd-B models considered in Miller and Rallison [24]
account for shear-thinning, hence the effect of shear-rate dependent viscosity would be obscured
under such models.

More recently, Malik et al. [26] observed that the laminar profiles of the velocity field and con-
formation tensor of the FENE-P viscoelastic pipe flow exhibit odd or even symmetries. Moreover,
Malik et al. [26] postulated that these properties of the laminar profiles can be exploited to derive
the symmetries and leading-order power-law behaviours of the perturbations to this flow similarly
to the way it is classically achieved in the Newtonian case. These properties can eventually be
exploited to circumvent the singularity at the pipe centre as carried out by Priymak and Miyazaki [15] and Meseguer and Trefethen [17], or alternatively, to derive a set of more robust regularity conditions for all values of $n$ as in Malik and Skote [18].

In the present paper, we derive such leading-order power-law behaviour, and analyze the symmetries of perturbations under the FENE-P constitutive viscoelastic model. We find that these perturbations indeed exhibit odd/even parities depending on the value of $n$. In fact, the change in parities with $n$ is to be expected in any physical systems, as noted by the analysis of Fourier modes by Lewis and Bellan [25]. As noted in that paper, these leading order behaviour and parities of scalars and vectors in axial planes can be arrived without the need to analyze the governing equations. However, the complexity gets enhanced when these physical quantities of vectors in nature (velocity, for example) forces certain tensors. As we show here by Frobenius method, the leading order behaviour and parities of the tensor components could have signatures of the physical problem (i.e., governing equations) at hand unlike the vectors in axial planes and scalars.

We also note that the perturbations to the correlations between the axial component and the radial or azimuthal components of the polymer end-to-end vector exhibit the same behaviour as the perturbation velocities in these respective directions close to centreline, which become evident for $|n|=1$ as shown in Sec. III.

Since the Oldroyd-B model can be seen as a special case of the FENE-P model—with the Peterlin function set equal to unity—these derived properties can be more generally applied to various dumbbell models [e.g. see, 4].

II. FENE-P MODEL AND LAMINAR PROFILES

The unsteady flow of dilute polymers modeled as dumbbells is governed by

$$u_t + u \cdot \nabla u = -\nabla p + Re^{-1} \left[ \beta \nabla^2 u + (1 - \beta) \nabla \cdot \tau \right],$$

(1)

$$c_t + u \cdot \nabla c - c \cdot \nabla u - (\nabla u)^T \cdot c = -\tau,$$

(2)

where the subscript “$t$” refers to the time derivative, $\tau = (f c - I)/W$ is the elastic stress of polymer, $f = (L^2 - 3)/(L^2 - \text{tr}(c))$ is the Peterlin function under the FENE-P model, $\beta = \mu_\text{s}/\mu$, $W = \lambda U_c/R$, and $Re = \rho U_c R/\mu$ is the Reynolds number. The quantities $\mu_\text{s}$ and $\mu$ refer to the solvent viscosity and total viscosity, respectively, while $U_c$ is the centreline velocity when no polymer is added. In addition, $R$, $\lambda$ and $L$ refer to the pipe radius, relaxation time, and the mean length of extensibility of polymer molecules, respectively. In Eq. (2), the conformation tensor is $c_{ij} = \langle \tilde{R}_i \tilde{R}_j \rangle$, where $\tilde{R}_i$ classically represents the end-to-end vector of the polymer molecule, and $I$ is the identity matrix. Lastly, the velocity field $u = (u, v, w)^T$ is considered in cylindrical coordinates $(x, r, \theta)$ with axial ($\hat{e}_1$), radial ($\hat{e}_2$) and azimuthal ($\hat{e}_3$) directions, respectively (see Fig 1). With these notations, the divergence of the elastic stress in Eq. (1) is given by $\nabla \cdot \tau =$
was obtained by Oliveira [29].

\[ [\tau_{11x} + \tau_{21r} + (\tau_{21} + \tau_{31})/r] \hat{e}_1 + [\tau_{12x} + \tau_{22r} + (\tau_{22} - \tau_{33} + \tau_{32})/r] \hat{e}_2 + [\tau_{13x} + \tau_{23r} + (\tau_{23} + \tau_{32} + \tau_{33})/r] \hat{e}_3. \]

The suffixes \( x, r \) and \( \theta \) imply differentiation with respect to these variables.

First, it is worth highlighting that in the steady laminar case, with \( \mathbf{c} = \mathbf{C}(r), \mathbf{u} = \mathbf{U}(r) = [U(r), 0, 0] \), and \( F(r) = (L^2 - 3)/(L^2 - \text{tr}(\mathbf{C})) \), the solution of Eqs. (1)–(2) is given by:

\[
F(r) = 1 + (\zeta_1^{1/3} + \zeta_2^{1/3} - 2)/(3\beta), \quad (3)
\]

\[
U(r) = 2 \int_0^1 r' F(r') [1 - U(r - r')] \beta [F(r') - 1] + 1 dr', \quad (4)
\]

\[
C_{11}(r) = (2W^2U_r^2 + F^2)/F^3, \quad (5)
\]

\[
C_{22}(r) = C_{33} = 1/F, \quad (6)
\]

\[
C_{12}(r) = W U_r/F^2, \quad (7)
\]

where \( \zeta_1 = a + \sqrt{a^2 - 1} \), \( \zeta_2 = a - \sqrt{a^2 - 1} \) and \( a = 1 + 108\beta W^2 r^2/L^2 \) with positive square-root and real cubic-root implied [26, 27]. Note that \( U(r) \) denotes the Heaviside step function, and that all functions in Eqs. (3)–(7) hold even or odd parities with respect to \( r \). Specifically, \( C_{12} \) has odd parity while \( F, U, C_{11}, C_{22} \) and \( C_{33} \) have even parities. These properties will be used to obtain the symmetries and leading-order behaviours of the perturbations in what follows. For completeness, it is worth mentioning that an alternate formulation of the FENE-P model, its laminar mean profiles, and its relation to the Phan-Thien and Tanner model can be found in Cruz et al. [27] and Poole et al. [28]. The pioneering solution for the mean-flow for an inviscid solvent was obtained by Oliveira [29].

III. STRUCTURE OF SMALL PERTURBATIONS

Let the mean-state be perturbed with \( \mathbf{q}' \exp[i(\alpha x + n\theta - \omega t)] \) where \( \mathbf{q}' = (u', p', \mathbf{c}')^T, n \in \mathbb{Z}, \alpha \in \mathbb{R}, \) and \( \omega \in \mathbb{C} \). Let \( G = \partial F/\partial [\text{tr}(\mathbf{C})], \) i.e., \( G = (L^2 - 3)/(L^2 - \text{tr}(\mathbf{C}))^2 \) and \( \Gamma = (1 - \beta)/(ReW) \). The amplitude of the fluctuating part of \( f \) is given by \( f' = \text{tr}(\mathbf{c}')G \). The linearised equations for small perturbations read:

\[
\mathcal{L}u' = -U_r u' - i\alpha p' + \Gamma [(G_1 + i\alpha F)c_{11}' + G_1(c_{22}' + c_{33}') + C_{12}GD(c_{11}' + c_{22}' + c_{33}')]\]

FIG. 1. Schematic diagram of the flow set-up with associated notations in cylindrical coordinates.

\[ \tau_{11x} + \tau_{21r} + (\tau_{21} + \tau_{31})/r] \hat{e}_1 + [\tau_{12x} + \tau_{22r} + (\tau_{22} - \tau_{33} + \tau_{32})/r] \hat{e}_2 + [\tau_{13x} + \tau_{23r} + (\tau_{23} + \tau_{32} + \tau_{33})/r] \hat{e}_3. \]
\[(F_r + Fr^{-1} + FD)c_1 + iFr^{-1}c_1 + \text{in} Fr^{-1}c_1\]  
\[\mathcal{L}v' = -p'_r - \beta(r^2\text{Re})^{-1}(v' + i2nu') + \Gamma \left[ G_2c_1 + (G_2 + F_r + Fr^{-1})c_2 + i\alpha Fc_2 \right] 
+ (G_2 - Fr^{-1})c_3 + C_{22}GD(c_1' + c_3') + (F + C_{22}G)Dc_2' + i\text{in} Fr^{-1}c_3' \]  
\[\mathcal{L}w' = -inr^{-1}p'_r - \beta(r^2\text{Re})^{-1}(w' - i2nu') + \Gamma \left[ i\alpha Fc_1 + (F_r + 2Fr^{-1} + FD)c_2 \right] 
+ i\text{in} C_{33}Gr^{-1}(c_1' + c_2' + inr^{-1}(F + 33G)c_3') \]  
\[G_3c_1 = -C_{11}r^{-1}u' + (i\alpha C_1 + \chi C_2 - C_{11}GW^{-1}(c_1' + c_2' + c_3') \]  
\[G_3c_2 = (i\alpha C_1 - C_{12}r - C_{12}D)c_1' + (i\alpha C_2 + 2C_{22}D)u' + \left( U_r - C_{12}GW^{-1} \right)c_2' \]  
\[G_3c_3 = (i\alpha C_1 - 2C_{22} + 2C_{22}D)c_2' + (i\alpha C_2 - C_{33}r^{-1} + C_{22}D)u' + \left( C_{33}r^{-1}u' \right)c_3' \]  
\[G_3c_3' = (2C_{33}r^{-1} - C_{33}r)u' + 2\text{in} C_{33}r^{-1}w' - C_{33}GW^{-1}(c_1' + c_2' + c_3') \]

where \[\mathcal{L} = i(\alpha U - \omega) - \beta\text{Re}^{-1} [D^2 + r^{-1}D - (\alpha^2 + 2n^2r^{-2})] \]
\[D = \frac{d}{dr}, G_1(r) = (i\alpha C_1 + C_{12} + \chi C_2 - C_{11}GW^{-1}) \]
\[G_2(r) = (i\alpha C_1 + C_{22}r) \]
\[G_3(r) = i(\alpha U - \omega) + FW^{-1} \]

As a matter of convention, we use the suffix \(r \) to represent the derivative with respect to \( r \) of mean flow variables, and operator \( D \) to imply the same for the fluctuations.

### A. Analytic behaviour

As can be noted from the operator \( \mathcal{L} \), Eqs. (8)–(10) exhibit a singularity at \( r = 0 \). Hence, regularity conditions are required to preclude non-analytic solutions. If each term of Eqs. (8)–(16) goes like \( \sim r^j \), where \( j \in \mathbb{Z}^+ \) can be different for each term, then in the limit of \( r \to 0 \), these equations would become redundant as each of them would have factors of zeros. However, Eqs. (8)–(16) can convey useful information in the limit of \( r \to 0 \) if the greatest common factor, say \( r^i \), of all the terms is factored out, thus serving to enforce the boundary conditions at the pipe centre [18]. To get the actual value of this integer \( i \), the behaviour of the unknowns in this limit is indispensable and can be derived by means of a Taylor analysis around \( r = 0 \).

Let \( \chi_j(r) \) with \( j = 0, \ldots, 9 \) be analytic with Taylor expansions, \( \sum_{k=0}^{\infty} r^k \chi_{j,k} \) for all \( j \). We use these family of functions to represent parts of velocities, pressure and conformation tensors as shown below. A Taylor analysis of Eqs. (8)–(16) shows that the solutions take the form

\[(p', u', v', w') = \begin{cases} 
(\chi_0ystem
and that the functions $\chi_j(r)$ are even functions of $r$, and $\chi_2(r) = 0$ for $\alpha = 0$. The detailed derivation of Eqs. (17)–(19) is given in the appendices A–D by the method of Frobenius. However, these solution forms can also be arrived at via couple of other methods. A heuristic method is given in the next subsection. As an observation, it should be noted that the behaviours of $u', v'$, and $w'$ shown in Eq. (17) are same as those for the Newtonian case [15, 17, 18].

As a third way for arriving at the ansatz in Eqs. (17)-(19), one can follow the method of Fourier analysis [25]. This analysis predicts the leading order power law behaviour and the parities of the scalars such as $\beta$ according to Eqs. (11)–(16). In Appendix E, we show this for the case of $(n = 0; \alpha \neq 0)$ and $(n = 0; \alpha = 0)$. The other cases can be handled in the same way to arrive at the ansatz in Eqs. (17)-(19).

1. Heuristic derivation

Here we present a heuristic approach to derive the functional forms of solution set in Eqs. (17)–(19). Let us first consider the limit of $\beta \to 1$ where the contributions due to polymer stress in Eqs. (8)–(10) are identically null. The case of $\beta \neq 1$ is addressed at the end of this subsection.

In this limit of $\beta \to 1$, the velocities in Eq. (17) acquire the behaviour of the Newtonian flow. The behaviours of $c'_{ij}$ are determined by $(u', v', w')$ according to Eqs. (11)–(16). First, let us consider the case $n = 0$. We take into consideration the parities of the mean flow variables shown at the end of Sec. II. These imply that $C_{11}, C_{22}, C_{33}, G, G_1$ and $G_3$ are even functions with constant-like behaviours as $r \to 0$, and that $U_r, C_{12}$ and $G_2$ are odd and linear to the leading order. From Eq. (17), we have $w' = r\chi_3$. Therefore, the terms $C_{22}Dw' - r^{-1}C_{33}w'$ and $i\alpha C_{12}w'$ in the RHS of Eq. (15) goes like $r^2$ times an even function in the limit $r \to 0$. Since, $G_3$ is even, Eq. (15) implies $c'_{23} = r^2\chi_9$ as given by Eq. (19). Similarly, Eq. (15) suggests that $c'_{13} \sim r$ multiplied by an even function of $r$ to the leading order, hence the definition $c'_{13} = r\chi_8$ in Eq. (19). Similar analyses establish the behaviours of $c'_{11}, c'_{22}, c'_{33}$ and $c'_{12}$ as in Eqs. (18)–(19) though their equations are coupled. It is worth adding that these coupled equations can be solved algebraically
for each of the variables and analyzed individually. Here, only three variables are independent, namely $c'_{11} + c'_{33}$, $c'_{12}$ and $c'_{13}$. In addition, the equation for $c'_{11} + c'_{33}$ can be formed by adding Eq. (11) and Eq. (16).

Now, let us consider the case $|n| = 1$ for $B \rightarrow 1$. A similar analysis can be carried out after noting the following element. The terms $2C'_{33}(v' + n\dot{w}')/r$ and $2C'_{33}(-w' + n\dot{v}')/r$ on the RHS of Eq. (16) and Eq. (15), respectively, are of $O(r)$, since $v' + n\dot{w}' \sim r^2$. This can be derived using the fact that Khorrami et al. [14] have shown that $v' + n\dot{w}' = 0$ at the centreline for $|n| = 1$. Since $v'$ and $w'$ are even functions from Eq. (17) for $|n| = 1$, $(v' + n\dot{w}')_{r=0} = 0$ implies $v' + n\dot{w}' \sim r^2$. While analyzing the terms for $n = \pm 1$, one should note that $(c'_{12} + n\dot{c}'_{13})_{r=0} = 0$ from Eq. (12) and Eq. (13).

Further for $|n| = 1$, the term $F(c'_{12} + n\dot{c}'_{13})/r$ in Eq. (8) deserves attention. One should note that $(c'_{12} + n\dot{c}'_{13})_{r=0} = 0$ from Eq. (12) and Eq. (13). Therefore the arguments in the above paragraph for $(v' + n\dot{w}')/r$ applies, hence $(c'_{12} + n\dot{c}'_{13})/r = 0$ as $r \rightarrow 0$.

For $n \geq 2$, all terms of Eqs. (11)–(16) remain, and the analysis is essentially the same as for the case $n = 0$. Finally, let us consider the case $B \neq 1$, whereby there is a non-trivial contribution from the polymer stress terms in Eqs. (8)–(10). It should be noted that the functional forms of the solution set given by Eqs. (17)–(19) will still remain valid, as long as the parity of each of these polymer terms in Eqs. (8)–(10) conforms with the parities of other terms of these equations for each $n$. If the parities are indeed the same, the polymer terms only amount to a modification in the multiplicative constants in the Taylor expansion of the other terms in Eqs. (8)–(10) allowing the velocity field to preserve the form as in Eq. (17). This, in turn, implies the preservation of Eqs. (18)–(19) by Eqs. (11)–(16). We find that that turns out to be the case. With the parities of the mean flow variables taken into consideration, every term of the polymer stress is in perfect ‘harmony’ with other terms.

2. Physical significance and further discussions

The physical significance is that $\lim_{r \rightarrow 0}[(c'_{1j} \hat{e}_1 \hat{e}_j)_{\theta}] = 0$, which is similar to the condition $\lim_{r \rightarrow 0}[(u')_{\theta}] = 0$ that results in $(v' + n\dot{w}')_{r=0} = 0$. The condition $\lim_{r \rightarrow 0}[(c'_{1j} \hat{e}_1 \hat{e}_j)_{\theta}] = 0$ implies that the traction vector along $\hat{e}_1$ is uniform along $\hat{e}_3$. As evident from Eq. (12) and Eq. (13), $c'_{12}$ and $c'_{13}$ at the centreline are due to $v'$ and $w'$, respectively, to the leading order. This implies a shearing of polymers against the restoring tendency by $v'$ and $w'$, but without stretching, given that the stretching terms are of higher orders. Since $(v' + n\dot{w}')_{r=0} = 0$, this results in $(c'_{12} + n\dot{c}'_{13})_{r=0} = 0$. In fact, even the mean $C_{12}$ arises due to the restoring tendency and mean fluid shear, $U_r$, and results in shear thinning. (See Malik et al. [26] for more details).

In another way of looking at this result for $|n| = 1$, one should note that the axial traction vector $c'_{1j} \hat{e}_1 \hat{e}_j$ (apart from the factor of $F$ that is neglected here), is predominantly in the plane perpendicular to axis, since $c'_{11}$ is of higher order than $c'_{12}$ and $c'_{13}$. Therefore, the the general result
for such vectors from Fourier analysis by Lewis and Bellan [25] apply, which would predict the current observation, i.e., \((c'_{12} + n\bar{c}_{13})_{r=0} = 0\).

Further, the components of the axial traction, \(c'_{ij}\hat{e}_i\hat{e}_j\) have the same behaviour as the velocity vector \((u', v', w')\) for all values of \(n\) as long as \(\alpha \neq 0\). This indicates the close relation between the axial traction and the velocities, where the components of the former inherits these properties from the latter via Eqs. 11-13.

The Eq. 18 suggests that for \(|n| \geq 2\) that the stretching of the polymer molecules in the plane perpendicular to the axial direction is dominant over the same along the axial direction close to the centreline. The particular value of \(|n| = 2\) deserves attention. Only for these values of \(n\), the components \(c'_{22}\) and \(c'_{33}\) do not vanish at the centre. However the sum, \(c'_{22} + c'_{33}\) indeed vanishes, which implies that, under these linear level dynamics, the molecules are in random rotational motion in the plane perpendicular to the polar axis around \(r = 0\). However, we note that the trace vanishes near centreline for all values of \(n\).

### IV. CONCLUSION AND OUTLOOK

In this paper, we have shown that the centreline behaviour of the components of \(c'\) exhibits odd or even parities depending on \(n\), similarly to that of the velocity field. In fact these properties of \(c'\) are forced upon by the the similar properties the velocity field exhibits. These leading-order behaviour of the solution set and the parities, which are reported for the first time, plays a crucial role in: (i) determining the centreline regularity conditions, and (ii) carrying out the stability analysis in a robust manner.

We found that the near-centre behaviour of the the pressure and velocity fluctuations are same as their Newtonian counterparts. The reason for this can be tracked as follows. The leading order power law behaviour and parities of \((v', w')\) — a vector perpendicular to axial direction, and \(p'\) — a scalar, are independent of physical problems [25]. However, such behaviours of the \(u'\) is determined by the governing equations of the problem at hand, i.e., the continuity equation in the present case. Since this equation is same both in Newtonian and non-Newtonian cases, the Newtonian behaviours of for these variables were preserved in the non-Newtonian case.

As observed in the Newtonian case, the use of these symmetries not only help avoid spurious modes, but also help improving the accuracy of the solutions and eigenvalues, especially for large azimuthal wavenumbers as demonstrated in [18].

The modes with \(|n| = 1\) are of important to flow instability in pipe geometry. The Newtonian limit of this flow exhibits least-decaying modes for \(|n| = 1\), after setting aside the case of Stokes modes with \((n - 0; \alpha = 0)\) which are immune to transient growth [18]. For \(|n| = 1\), we observed that \(c'_{12}\) and \(c'_{13}\) do not vanish at the centreline, similarly to \(v'\) and \(w'\), and \((c'_{12} + n\bar{c}_{13})_{r=0} = 0\) holds for the uniformity of the axial traction vector in the azimuthal direction close to \(r = 0\), again in accordance with a similar condition, \((v' + n\bar{w'})_{r=0} = 0\) for the velocity vector, where the
latter result is known in the Newtonian case. Nonetheless, with the leading-order behaviours for the other components obtained in this work, this condition is automatically satisfied by Eq. (12) and Eq. (13). Lastly, since the symmetries of the mean profiles of the FENE-P model are also applicable to the Oldroyd-B model, the findings are also applicable there.

Data availability: The data that supports the findings of this study are available within the article.

Appendix A: Derivation of analytic behaviour for \(|n| \geq 2\)

To address the regular singularity in Eqs. (8)–(10), let \(p' = r^s \chi_0(r)\) and \(u' = r^m \chi_1(r)\) where \(m \geq 0\) and \(s \geq 0\) are integers to be determined. The \(\chi_j(r)\)'s are general analytic functions, which are shown to be even with respect to \(r\) in what follows.

With radial vorticity \(\eta' \equiv r^{-1}nu' - \alpha w'\), we have \(u' = \text{id}^{-1}(\alpha v' + \alpha r Dv' - \eta' r)\) and \(w' = \text{id}^{-1}(nu' + nr Dv' + \alpha^2 r \eta')\) [13]. An analysis, after substituting \(u' = r^m \chi_1(r)\) and requiring analyticity for \(\eta'\), suggests that \(v' = r^{m-1} \chi_2(r)\) and \(w' = r^{m-1} \chi_3(r)\). The addition of Eq. (11) and Eq. (16) yields,

\[
\begin{align*}
\overline{\alpha}_1 (c'_{11} + c'_{33}) + \overline{b}_1 c'_{22} - 2U_r c'_{12} &= (2C_{33}r^{-1} - C_{33r} - C_{11r})v' + 2(i\alpha C_{11} + C_{12} D)v' + 2inC_{33} w' r^{-1},
\end{align*}
\]

where \(\overline{\alpha}_1 = \text{G}_3 + (C_{11} + C_{33})G/W\), \(\overline{b}_1 = (C_{11} + C_{33})G/W\). The Eqs. (12), (14) and (A1), can be solved and yield

\[
\begin{align*}
c'_{11} + c'_{33} &= \frac{(\overline{d}_1 G_3 + 2U_r \overline{d}_3) \overline{b}_1}{(\overline{\alpha}_1 G_3 + 2U_r C_{12} G/W) \overline{b}_1} - \frac{(b_1 G_3 + 2U_r b_3) \overline{d}_2}{(b_1 G_3 + 2U_r b_3) C_{22} G/W},
\end{align*}
\]

where \(\overline{d}_2 = G_3 + C_{22} G/W\), \(\overline{b}_3 = C_{12} G/W - U_r\) and \(\overline{d}_1\) is the RHS of Eq. (A1). The functions \(\overline{d}_2\) and \(\overline{d}_3\) are the terms on the RHS of Eq. (14) and Eq. (12), respectively, that have explicit dependence on velocity fluctuations. Substituting the above expressions for \(u', v'\) and \(w'\) into Eq. (A2) and carrying out a Taylor analysis round \(r = 0\), we find that

\[
c'_{11} + c'_{33} \sim r^{m-2}.
\]

Substituting Eq. (A3) into Eq. (14), we find \(c'_{22} \sim r^{m-2}\). Therefore, let \(c'_{22} = r^{m-2} \chi_5(r)\). However, using Eq. (A3) and \(c'_{22} \sim r^{m-2}\) in Eq. (11) and Eq. (16) does not predict the leading-order behaviours of \(c'_{11}\) and \(c'_{33}\). The three possibilities arising from Eq. (A3) are

\[
(c'_{11}, c'_{33}) = \begin{cases} (r^{m-2} \chi_4, r^{m-2} \chi_6) & \text{(option 1)}, \\ (r^{m-2} \chi_4, r^{s1} \chi_6) & \text{(option 2)}, \\ (r^{s2} \chi_4, r^{m-2} \chi_6) & \text{(option 3)} \end{cases}
\]
with $s_1 > m - 2$ and $s_2 > m - 2$. For now, let us assume that option (1) is true. However, we will prove in Sec. A 2, that the option (3) is in fact the correct one, and that $s_2 = m$. Using option (1) of Eq. (A4), analyses of Eq. (12), Eq. (15) and Eq. (13) suggest $c'_{12} = r^{m-1} \chi_7(r)$, $c'_{23} = r^{m-2} \chi_9(r)$ and $c'_{13} = r^{m-1} \chi_8(r)$, respectively.

Let us make the following simplifying substitutions for the mean-flow. Since we are interested only in finding the leading-order behaviour and parities of the unknowns without any intention to know exactly the values of constants $\chi_{j,k}$ in the Taylor expansion, we can represent the mean-flow variables by the first term in their expansion in powers of $r$ since this term would contain the leading-order behaviour and parity. Without any loss of generality for the stated purpose to retain only the leading-order behaviour and parity, we carry out the following assignments, $C := C_1, H := H_4$ for which will further be refined later. Substituting the above into Eqs. (9)–(10), we get

In Eq. (A5), if $s < m - 2$, the equations formed by the coefficients of $r^{k+s}$ form a trivial solution for $\chi_{0,k}$ for each $k = 0, \ldots, (m-3-s)$. Therefore, without any loss of generality, we can choose $s = m - 2$. If in reality $s > m - 2$ while we had chosen $s = m - 2$, then this fact will show up by giving the solution for the coefficients $\chi_{0,k}$ for $k = 0, \ldots, s - m + 1$ upon which we will be able to correct this exponent $s$ such that $\chi_{0,0} \neq 0$. (In fact this happens in the following, and we will see that $s = m$.) To summarize, the solution forms of the unknowns currently stand as

$$
\begin{align*}
(p', u', v', w') &= \left(r^{m-2} \chi_0, r^{m-1} \chi_2, r^{m-1} \chi_3, r^{m-1} \chi_8, r^{m-2} \chi_9, r^{m-2} \chi_9\right), \\
(c'_{11}, c'_{22}, c'_{33}, c'_{12}, c'_{13}, c'_{23}) &= \left(r^{m-2} \chi_4, r^{m-2} \chi_5, r^{m-2} \chi_6, r^{m-1} \chi_7, r^{m-1} \chi_8, r^{m-1} \chi_9\right),
\end{align*}
$$

(A6)

(A7)

which will further be refined later. Substituting the above into Eqs. (9)–(10), we get

$$
\sum_{k=0}^\infty \{ H_{12} \chi_{1,k} r^{k+m} - \beta Re^{-1} [(k + m)^2 - n^2] \chi_{1,k} r^{k+m-2} \} = \sum_{k=0}^\infty \{ -H_{2} \chi_{2,k} r^{k+m} - i\alpha \chi_{0,k} r^{k+s} + \Gamma ([H_6 + i\alpha H_3] \chi_{4,k} + H_6 (\chi_{5,k} + \chi_{6,k}) + H_5 H_7 (k + m - 2) (\chi_{4,k} + \chi_{5,k} + \chi_{6,k}) + H_3 ((k + m) \chi_{7,k} + i n \chi_{8,k}) ] r^{k+m-2} + H_4 \chi_{7,k} r^{k+m} \} .
$$

(A5)
Similarly, upon substituting Eq. (A6) and Eq. (A7) into Eqs. (11)–(16), we get

\[
\sum_{k=0}^{\infty} \left\{ H_{12} \chi_{3,k} r^{k+m+1} - \beta R e^{-1} \left[ \{(k + m - 1)^2 - n^2 - 1\} \chi_{3,k} + 2i \chi_{2,k} \right] r^{k+m-1} \right\}
\]

\[
= \sum_{k=0}^{\infty} \left\{ -i n \chi_{0,k} r^{k+m-1} + \Gamma \left[ (H_4 \chi_{9,k} + i \alpha H_3 \chi_{8,k}) r^{k+m+1} + [H_3(k + m) \chi_{9,k} + i n (H_5 \chi_{9} + H_3 \chi_{6,k} + H_5 H_9 (\chi_{4,k} + \chi_{5,k}))] r^{k+m-1} \right\}. \quad (A9)
\]

Similarly, upon substituting Eq. (A6) and Eq. (A7) into Eqs. (11)–(16), we get

\[
A_1 \sum_{k=0}^{\infty} \chi_{4,k} r^{k+m-2} = \sum_{k=0}^{\infty} \left\{ [-H_{13} \chi_{2,k} + 2(i \alpha H_{10} + H_7 (k + m)) \chi_{1,k}] r^{k+m-2} - H_{10} H_3 W^{-1} (\chi_{5,k} + \chi_{6,k}) r^{k+m-2} \right\} \quad (A10)
\]

\[
A_2 \sum_{k=0}^{\infty} \chi_{5,k} r^{k+m-2} = \sum_{k=0}^{\infty} \left\{ (2i \alpha H_7 - H_{14}) \chi_{2,k} r^{k+m-2} + 2H_9 (k + m - 1) \chi_{2,k} r^{k+m-2} - H_9 H_5 W^{-1} (\chi_{4,k} + \chi_{6,k}) r^{k+m-2} \right\} \quad (A11)
\]

\[
A_2 \sum_{k=0}^{\infty} \chi_{6,k} r^{k+m-2} = \sum_{k=0}^{\infty} \left\{ 2H_9 [\chi_{2,k} + i \chi_{3,k}] r^{k+m-2} - H_{14} \chi_{2,k} r^{k+m-2} - H_9 H_5 W^{-1} (\chi_{4,k} + \chi_{6,k}) r^{k+m-2} \right\} \quad (A12)
\]

\[
H_{11} \sum_{k=0}^{\infty} \chi_{7,k} r^{k+m-1} = \sum_{k=0}^{\infty} \left\{ [H_9(k + m) \chi_{1,k} + (i \alpha H_{10} + (k + m - 2) H_7) \chi_{2,k} + (H_2 - H_7 H_5 W^{-1}) \chi_{4,k} - H_7 H_5 W^{-1} (\chi_{4,k} + \chi_{6,k})] r^{k+m-1} + i \alpha H_7 \chi_{1,k} r^{k+m+1} \right\} \quad (A13)
\]

\[
H_{11} \sum_{k=0}^{\infty} \chi_{8,k} r^{k+m-1} = \sum_{k=0}^{\infty} \left\{ [i n \chi_{9,k} + (i \alpha H_{10} + (k + m - 2) H_7) \chi_{3,k} + H_2 \chi_{9,k}] r^{k+m-1} \right\} \quad (A14)
\]

\[
H_{11} \sum_{k=0}^{\infty} \chi_{9,k} r^{k+m-2} = \sum_{k=0}^{\infty} \left\{ H_9 [(k + m - 2) \chi_{3,k} + i n \chi_{2,k}] r^{k+m-2} + i \alpha H_7 \chi_{3,k} r^{k+m} \right\}, \quad (A15)
\]

where \( A_1 = H_{11} + H_{10} H_5 W^{-1}, A_2 = H_{11} + H_9 H_5 W^{-1} \). The continuity equation reads

\[
\sum_{k=0}^{\infty} \left\{ i \alpha \chi_{1,k} r^{k+m} + [(m + k) \chi_{2,k} + i n \chi_{3,k}] r^{m-2+k} \right\} = 0. \quad (A16)
\]

1. Determination of \( m \)

The leading-order terms of Eq. (A16) satisfy \( m \chi_{2,0} + i n \chi_{3,0} = 0 \). With this information, the leading-order terms of Eq. (A5) and Eqs. (A8)–(A15) read

\[
-(\beta/Re) (m^2 - n^2) \chi_{1,k} = -i \alpha \chi_{0,k} + \Gamma [(H_6 + i \alpha H_3) \chi_{4,0} + H_6 (\chi_{5,0} + \chi_{6,0})] + H_5 H_7 (m - 2) (\chi_{4,0} + \chi_{5,0} + \chi_{6,0}) + H_3 (m \chi_{7,0} + i n \chi_{8,0}) \], \quad (A17)
\]

\[
-(\beta/Re) (m^2 - n^2) \chi_{2,0} = (2 - m) \chi_{0,0} + \Gamma [H_3 (\chi_{5,0} - \chi_{6,0} + i n \chi_{9,0})]
\]
\[ A_1 \chi_{4,0} = - H_{10} H_5 W^{-1}(\chi_{5,0} + \chi_{6,0}), \quad (A20) \]

\[ A_2 \chi_{5,0} = 2(m - 1) H_9 \chi_{2,0} - H_9 H_5 W^{-1}(\chi_{4,0} + \chi_{6,0}), \quad (A21) \]

\[ A_2 \chi_{6,0} = -2(m - 1) H_9 \chi_{2,0} - H_9 H_5 W^{-1}(\chi_{4,0} + \chi_{5,0}), \quad (A22) \]

\[ H_{11} \chi_{7,0} = m H_9 \chi_{1,0} + [i \alpha H_{10} + (m - 2) H_7] \chi_{2,0} + H_2 \chi_{5,0} \]

\[ - H_7 H_5 W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}), \quad (A23) \]

\[ H_{11} \chi_{8,0} = i n \chi_{2,0} + \imath mn^{-1}[i \alpha H_{10} + (m - 2) H_7] \chi_{2,0} + H_2 \chi_{9,0} \]

\[ H_{11} \chi_{9,0} = i n \chi_{2,0} + \imath mn^{-1}(m^2 + n^2 - 2m) \chi_{2,0}. \quad (A25) \]

This system of Eqs. (A17)–(A25) has nine unknowns: \( \{\chi_{0-2,0}, \chi_{4-9,0}\} \). Therefore, the condition that the determinant has to vanish gives the value for the parameter \( m \). However, this \( 9 \times 9 \) system can be simplified to a \( 3 \times 3 \) system as follows. Adding Eq. (A21) and Eq. (A22), we get

\[ (A_2 + H_9 H_5 W^{-1})(\chi_{5,0} + \chi_{6,0}) = -2H_9 H_5 W^{-1} \chi_{4,0}. \quad (A26) \]

Solving Eq. (A26) and Eq. (A20) gives \( \chi_{5,0} + \chi_{6,0} = 0 \) and

\[ \chi_{4,0} = 0. \quad (A27) \]

Further, \( m \times \) Eq. (A21) + \( \imath n \times \) Eq. (A25) results in \( m \chi_{5,0} + \imath n \chi_{9,0} = (m^2 - n^2) H_9 H_{11}^{-1} \chi_{2,0} \). Similarly, \( m \times \) Eq. (A23) + \( \imath n \times \) Eq. (A24) results in \( m \chi_{7,0} + \imath n \chi_{8,0} = (m^2 - n^2)[H_9 H_{11}^{-1} \chi_{1,0} + H_2 H_9 H_{11}^{-2} \chi_{2,0}] \). Similarly, \( m \times \) Eq. (A25) + \( \imath n \times \) Eq. (A22) results in \( m \chi_{9,0} + \imath n \chi_{6,0} = i(m^2 - n^2)(m - 2) H_9 (n H_{11})^{-1} \chi_{2,0} \). Substituting this information into Eqs. (A17)–(A19), we get

\[ -A_3 (m^2 - n^2) \chi_{1,0} - A_4 (m^2 - n^2) \chi_{2,0} = -i \alpha \chi_{0,0}, \quad (A28) \]

\[ -A_3 (m^2 - n^2) \chi_{2,0} = (2 - m) \chi_{0,0}, \quad (A29) \]

\[ A_5 (m^2 - n^2)(m - 2) \chi_{2,0} = \chi_{0,0}, \quad (A30) \]

respectively, where \( A_3 = \beta R e^{-1} + \Gamma H_3 H_9 H_{11}^{-1}, A_4 = \Gamma H_3 H_9 H_2 H_{11}^{-2} \) and \( A_5 = \beta (m R e^{-1} + \Gamma H_3 H_9 (n H_{11})^{-1} \). The Eqs. (A28)–(A30) are three homogeneous equations for three unknowns. Therefore, the condition that the determinant has to vanish for non-trivial solution for \( \{\chi_{0-2,0}\} \) gives rise to \( m = |n| \). However, the current choice of \( s = m - 2 \) needs correction, since Eqs. (A28)–(A30) state that \( \chi_{0,0} = 0 \) for this choice of \( m = |n| \). This will be addressed later after establishing the parity.
2. Parity and solution form

To determine the parity, if any, of \( \chi_j(r) \)'s with respect to \( r \), let us look at the equations governing \( \chi_{j,1} \). If these equations would result in trivial solutions, then there would be an even symmetry for \( \chi_j(r) \) as explained below. The next higher order terms of Eq. (A5) and Eqs. (A8)–(A16) forms a linear homogeneous system \( L_{ij} \chi_{j,1} = 0 \) where \( j = 0–9 \) and \( L_{ij} \)'s are components of a \( 10 \times 10 \) matrix. Since there are no free undetermined parameters in \( L \), its determinant may not vanish. This gives \( \chi_{j,1} = 0 \) for each \( j \). Since the 10 equations, Eq. (A5) and Eqs. (A8)–(A16) couples only alternate terms in the series expansion, we get the following recursion relations for the Taylor coefficients:

\[
\chi_{i,k+2} = L_{ij}^{(k)} \chi_{j,k} \quad \text{for} \quad k = 0, 1, \ldots , \tag{A31}
\]

where \( L_{ij}^{(k)} \) is the corresponding linear system that could be defined from the Eq. (A5) and Eqs. (A8)–(A16) for every higher order. Equation (A31) implies that \( \chi_{j,k} = 0 \) for every odd \( k \) since \( \chi_{j,1} = 0 \) and \( \chi_{j,k} \) are non-trivial for every even \( k \) since the solution set, \( \{ \chi_{j,0} \} \) is non-trivial. This proves the even parity of \( \chi_j(r) \).

The leading-order behaviour for \( c'_{11} \equiv r|n|-2 \chi_4(r) \) can be corrected as follows. By Eq. (A27) and the established parity, we have \( \chi_4,0 = \chi_{4,1} = 0 \). This suggests that \( \chi_4(r) \) itself is having a leading-order behaviour of \( r^2 \) which can used to redefine \( c'_{11} \) as \( c'_{11} \equiv r|n| \chi_4(r) \). Note that such redefinition does not affect the solution obtained for \( m = |n| \). Such a redefinition implies that the LHS of Eq. (A20) is zero at the lowest order, suggesting \( \chi_{5,0} + \chi_{6,0} = 0 \), which in turn is a result that is consistent with Eq. (A26).

Further, \( \chi_{0,0} = \chi_{0,1} = 0 \), since the indicial solution \( m = |n| \) renders \( \chi_{0,0} = 0 \) by any of the Eqs. (A28)–(A30). This implies that we can adjust the exponents of \( r \) in the definition given be Eqs. (A6)–(A7) by increasing them by 2 for \( p' \) without any loss of generality. Such an increase in the exponent for \( r \) from \( m - 2 \) to \( m \) would not affect the very system given by Eqs. (A28)–(A30) that determined the value of \( m \) as \( m = |n| \), except that the RHS of this system would now be zeros at the lowest order, which still determines \( m = |n| \). The final solution form is

\[
(p', u', v', w')=(r^{|n|}X_0, r^{|n|}X_1, r^{|n|-1}X_2, r^{|n|-1}X_3) \quad \text{and} \quad (c'_{11}, c'_{22}, c'_{33}, c'_{12}, c'_{13}, c'_{23})=(r^{|n|}X_4, r^{|n|-2}X_5, r^{|n|-2}X_6, r^{|n|-1}X_7, r^{|n|-1}X_8, r^{|n|-2}X_9), \tag{A32}
\]

with Taylor expansions for \( \chi_j(r) \) as \( \chi_j(r) = \sum_{k=0}^{\infty} \chi_{j,2k} r^{2k} \).

Appendix B: The case of \( n = \pm 1 \)

For the ease of proceeding further, note that \( H_2 = -2, H_7 = -2W \) and \( H_3 = H_5 = H_9 = H_{10} = 1 \).

Since the governing equations for this case are not different from those we saw for the case of \( |n| \geq 2 \), the analysis we carried out for that case still applies here, except that a modification is
needed for the exponents of the leading order behaviour of $c_{22}^{'}, c_{23}^{'},$ and $c_{33}^{'}$. These components of $c'$ acquires new set of exponents for this case of $|n| = 1$, which are in fact an increment by two to the exponents predicted by substituting $|n| = 1$ into Eq. A33. This substitution give for these variables the following leading order behaviour.

\[(c_{22}^{'}, c_{33}^{'}, c_{23}^{'}) = (r^{-1}x_5, r^{-1}x_6, r^{-1}x_9).\]  \hspace{1cm} (B1)

The above Eq. (B1) apparently predicts that $c_{22}^{'}, c_{33}^{'}$ and $c_{23}^{'}$ are singular at the origin. However as we will show below the constant term in the Taylor expansion of $x_5, x_6$ and $x_9$ around $r = 0$ are zero.

Substituting $|n| = 1$ into Eqs. A32-A33, the variables in $(u', v', w', c_{11}^{'})$ take the form,

\[(u', v', w', c_{11}^{'}) = (rX_1, X_2, X_3, rX_4).\]  \hspace{1cm} (B2)

These variables are regular.

Since we found in Appendix A, that $m = |n|$, let us substitute $m = 1$ in Eq. (A25) we get $x_{9,0} = 0$. By the even parity nature, we get $x_{9,1} = 0$. This suggest that we can increase the exponent for leading order behaviour of $c_{23}^{'}$ by 2 in Eq. (B1). Therefore, the correct behaviour is $c_{23}^{'} = rX_9$. Now we focus our attention on $c_{22}^{'}$ and $c_{33}^{'}$. Adding Eq. (A21) and Eq. (A22), and substituting for $x_4,0$ from Eq. (A20) we get $x_{5,0} + x_{6,0} = 0$. By substituting this result into Eq. (A21) and Eq. (A22), and noting that $m = 1$, we get $x_{5,0} = x_{6,1} = 0$. Again, by the even parity of $x_j(r), x_{5,1} = x_{6,1} = 0.$ this allow us to increase the leading order exponents of $r$ in $c_{22}^{'}$ and $c_{33}^{'}$ by two. Therefore we arrive at the relations $c_{22}^{'} = rX_5$ and $c_{33}^{'} = rX_6$.

**Appendix C: Case $n = 0$ and $\alpha \neq 0$**

The governing equations after substituting the mean flow variables by their respective leading order terms become

\[i\alpha u' = -Dv' - r^{-1}v'\]  \hspace{1cm} (C1)

\[\mathcal{L}_1 u' = -H_2 rv' - i\alpha p' + \Gamma [(H_6 + i\alpha H_3)c_{11} + H_6 (c_{22}^{'} + c_{33}^{'}) + H_7 H_5 r D(c_{11} + c_{22}^{'} + c_{33}^{'}) + (H_4 r + H_3 r^{-1} + H_3 D)c_{12}^{'},\]  \hspace{1cm} (C2)

\[\mathcal{L}_1 v' = -p' - \beta (r^2 \text{Re})^{-1}v' + \Gamma [H_8 r c_{11} + (H_8 r + H_4 r + H_3 r^{-1}) c_{22}^{'} + i\alpha H_3 c_{12}^{'},\]  \hspace{1cm} (C3)

\[\mathcal{L}_1 w' = -\beta (r^2 \text{Re})^{-1}w' + \Gamma [i\alpha H_3 c_{13}^{'} + (H_4 r + 2H_3 r^{-1} + H_3 D)c_{23}^{'},\]  \hspace{1cm} (C4)

\[H_{11} c_{11}^{'} = -H_{13} rv' + 2(i\alpha H_{10} u' + H_7 rv' + H_2 r c_{12}^{'}) - H_{10} H_5 W^{-1} (c_{11}^{'} + c_{22}^{'} + c_{33}^{'})\]  \hspace{1cm} (C5)

\[H_{11} c_{12}^{'} = (i\alpha H_{10} - H_7 + H_7 r D) v' + (i\alpha H_7 r + H_9 D) u' + (H_2 - H_7 H_5 W^{-1}) r c_{22}^{'} - H_7 H_5 W^{-1} r (c_{11}^{'} + c_{33}^{'})\]  \hspace{1cm} (C6)
\[ H_{11}c_{13} = (\alpha H_{10} - H_7 + H_7r D)w' + H_2r c_{23} \quad \text{(C7)} \]
\[ H_{11}c_{22} = (2\alpha H_7r - H_{14}r + 2H_9D)v' - H_9H_5W^{-1}(c_{11} + c_{22} + c_{33}) \quad \text{(C8)} \]
\[ H_{11}c_{23} = (\alpha H_7r - H_9r^{-1} + H_9D)w' \quad \text{(C9)} \]
\[ H_{11}c_{33} = (2H_9r^{-1} - H_{14}r)v' - H_9H_5W^{-1}(c_{11} + c_{22} + c_{33}) \quad \text{(C10)} \]

where, the differential operator, \( \mathcal{L}_1 = i(\alpha H_1 - \omega) - \beta \text{Re}^{-1} [D^2 + r^{-1} D - \alpha^2] \). The Eqs. (C1)-(16) is a system for 10 unknowns namely, \( p', u', v', w', c_{11}', c_{22}', c_{33}', c_{12}', c_{13}', c_{23}' \). It should be noted that this system forms two different decoupled system for two sets of solutions, namely, \( \chi^{(1)} \equiv (p', u', v', c_{11}', c_{22}', c_{33}', c_{12}') \) and \( \chi^{(2)} \equiv (w', c_{13}', c_{23}') \). The \( \chi^{(1)} \) is governed by Eqs. (C1)-(C3), Eqs. (C5)-(C6), Eq. (C8) and Eq. (C10), while the set \( \chi^{(2)} \) is governed by Eq. (C4), Eq. (C7) and Eq. (C9).

1. The parity and behaviour of \( \chi^{(1)} \)

Let \( p' = r^s \chi_0(r) \) and \( u' = r^m \chi_1(r) \), where the exponents, \( m \) and \( s \) are to be determined. From Eq. (C1), we get \( v' = r^{m+1} \chi_2(r) \), by demanding analyticity of \( v' \) at \( r = 0 \). Now we proceed by using the values of some of the constants \( H_j \)'s listed in Appendix B.

Adding Eq. (C5) and Eq. (C10) we get,
\[
(H_{11} + 2W^{-1})(c_{11}' + c_{33}') + 2W^{-1}c_{22}' + 4rc_{12}' = 2r^m(i\alpha - 2Wm - 2WrD)\chi_1 \\
+ r^m[2 - r^2(H_{13} + H_{14})]\chi_2. \quad \text{(C11)}
\]

The Eq. (C6) and Eq. (C8) can be written as
\[
-2r(c_{11}' + c_{33}') + H_{11}c_{12}' = r^{m+1}(i\alpha - 2Wm - 2WrD)\chi_2 \\
+ r^{m-1}(-2Wio^2r^2 + m + rD)\chi_1, \quad \text{(C12)}
\]

\[
W^{-1}(c_{11}' + c_{33}') + (H_{11} + W^{-1})c_{22}' = r^m[-(4ioW + H_{14})r^2 + 2(m + 1) + 2rD]\chi_2. \quad \text{(C13)}
\]

The three Eqs. (C11)-(C13) can be solved for three unknown \( c_{11}', c_{33}', c_{22}' \), and \( c_{12}' \), and can be found to the leading order that \( c_{11}' + c_{33}' \sim r^m \), \( c_{22}' \sim r^s \), and \( c_{12}' \sim r^{-m-1} \) by analysis. Substituting these behaviours in Eq. (C5) and Eq. (C10), we find that \( c_{11}' \sim r^m \) and \( c_{33}' \sim r^m \), respectively. Therefore, let \( c_{11}' = r^m\chi_4(r) \), \( c_{22}' = r^m\chi_5(r) \), \( c_{33}' = r^m\chi_6(r) \), \( c_{12}' = r^{-m-1}\chi_7(r) \).

Substituting these expressions for \( p', u', v', c_{11}', c_{22}', c_{33}' \) and \( c_{12}' \) into Eqs. (C1)-(C3), Eqs. (C5)-(C6), Eq. (C8) and Eq. (C10), and representing \( \chi_j(r) = \sum_{k=0}^{\infty} r^k \chi_{j,k} \), we get,
\[
i\alpha \chi_{1,k} = -(m + k + 2)\chi_{2,k} \quad \text{for each } k, \quad \text{(C14)}
\]

\[
\sum_{k=0}^{\infty} r^{k+m} \left\{ H_{12} - \beta \text{Re}^{-1}(m + k)^2r^{-2} \right\} \chi_{1,k} = \sum_{k=0}^{\infty} \left\{ 2r^{m+k+2}\chi_{2,k} - i\alpha r^{s+k}\chi_{0,k} \right\}
\]
\[ + \Gamma r^{m+k} [(H_6 + i\alpha)\chi_{4,k} + H_6(\chi_{5,k} + \chi_{6,k}) + (m + k)(\chi_{4,k} + \chi_{5,k} + \chi_{6,k}) + (H_4 + [m + k]r^{-2})\chi_{7,k}] \}, \quad (C15) \]

\[
\sum_{k=0}^{\infty} r^{k+1} \{ H_{12} - \beta \text{Re}^{-1} \left[(m+k+1)^2-1\right] \} \chi_{2,k} = \sum_{k=0}^{\infty} \{ -(s+k)r^{s+k-1}\chi_{0,k} + H_8 \chi_{4,k} + (H_8 + H_4 + r^{-2})\chi_{5,k} + i\alpha r^{-2}\chi_{7,k} + (H_8 - r^{-2})\chi_{6,k} + (m + k)r^{-2}(\chi_{4,k} + \chi_{6,k} + 2\chi_{5,k}) \}, \quad (C16) \]

\[
H_{11} \sum_{k=0}^{\infty} r^{k+m} \chi_{4,k} = \sum_{k=0}^{\infty} r^{k+m} \left\{ -H_{13} r^2 \chi_{2,k} + 2[i\alpha + H_7(m + k)]\chi_{1,k} - 4\chi_{7,k} - W^{-1}(\chi_{4,k} + \chi_{5,k} + \chi_{6,k}) \right\} ,
\]

\[
H_{11} \sum_{k=0}^{\infty} r^{k+m-1} \chi_{7,k} = \sum_{k=0}^{\infty} r^{k+m-1} \left\{ [i\alpha + 2W - 2W(m + k + 1)]r^2\chi_{2,k} + (-2i\alpha W r^2 + m + k)\chi_{1,k} + 2r^2(\chi_{4,k} + \chi_{6,k}) \right\} , \quad (C18) \]

\[
H_{11} \sum_{k=0}^{\infty} r^{k+m} \chi_{5,k} = \sum_{k=0}^{\infty} r^{k+m} \left\{ [-4i\alpha W + H_{14}]r^2 + 2(m + k + 1)]\chi_{2,k} - W^{-1}(\chi_{4,k} + \chi_{5,k} + \chi_{6,k}) \right\} , \quad (C19) \]

\[
H_{11} \sum_{k=0}^{\infty} r^{k+m} \chi_{6,k} = \sum_{k=0}^{\infty} r^{k+m} \left\{ (2 - H_{14} r^2)\chi_{2,k} - W^{-1}(\chi_{4,k} + \chi_{5,k} + \chi_{6,k}) \right\} \quad (C20) \]

In Eq. (C16), the lowest order non-pressure terms are of \( O(r^{m-1}) \) that occurs for \( k = 0 \). This suggests that the exponent \( s \lesssim m \), since that would result in \( \chi_{0,k} = 0 \) for \( k \leq m - s - 1 \). This shows that \( s \geq m \). Let us assume that \( s = m \) for now. (If the actual value of \( s \) is such that \( s > m \), this will show up with the result \( \chi_{0,k} = 0 \) for \( k \leq m - 1 \), which can be used to correct the value of \( s \).) After substituting, \( \chi_{2,0} = -i\alpha (m + 2)^{-1}\chi_{1,0} \) from Eq. (C14) into Eqs. (C15)-(C20), the leading order terms form the system,

\[
- \beta \text{Re}^{-1} m^2 \chi_{1,0} = \Gamma m \chi_{7,0}, \quad (C21) \]

\[
i\alpha \beta [(m + 2)\text{Re}]^{-1} [(m+1)^2-1] \chi_{1,0} = -m \chi_{0,0} + \Gamma \chi_{5,0} + i\alpha \chi_{7,0} - \chi_{6,0} + m(\chi_{4,0} + \chi_{5,0} + 2\chi_{5,0}), \quad (C22) \]

\[
H_{11} \chi_{4,0} = 2(i\alpha \chi_{1,0} - 2W m \chi_{1,0} - 2\chi_{7,0}) - W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}), \quad (C23) \]

\[
H_{11} \chi_{7,0} = m \chi_{1,0}, \quad (C24) \]

\[
H_{11} \chi_{5,0} = -2i\alpha (m + 2)^{-1}(m + 1)\chi_{1,0} - W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}), \quad (C25) \]

\[
H_{11} \chi_{6,0} = -2i\alpha (m + 2)^{-1}\chi_{1,0} - W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}) \quad (C26) \]

A set of non-trivial solution for the system given by Eqs. (C21)-(C26) would exist when \( m \) takes value such that the determinant of the coefficient matrix would vanish, which turns out to be \( m = 0 \). Note that for this value of \( m \), the Eq. (C22) allow any value for \( \chi_{0,0} \), that is \( \chi_{0,0} \) is
non-trivial. This confirms that the choice \( s = m(= 0) \) is the exponent for correct leading order behaviour for pressure around the pipe centre.

Since the set of Eqs. (C15)-(C20) couples every alternate higher order equations in powers of \( r \), the next higher order equations for \( \chi_{j,1} \) do not depend on \( \chi_{j,0} \). Since the free parameter \( m \) has been already fixed to obtain non-trivial solution for \( \chi_{j,0} \), the linear system \( \chi_{j,1}, L_{ij} \chi_{j,1} = 0 \) that governs \( \chi_{j,1} \) would allow only trivial values, i.e., \( \chi_{j,1} = 0 \) for each \( j \in \{0, 1, 2, 4-7\} \).

Let’s pay a particular attention to \( \chi_7(r) \). Since \( \chi_{7,0} = 0 \) by Eq. (C24) and \( \chi_{7,1} = 0 \) by parity, we redefine the earlier assumed exponent for the leading order behaviour for \( c'_{12} \). Earlier we had chosen it as \( c'_{12} = r^{m-1}\chi_7(r) \), which can be modified as \( c'_{12} = r\chi_7(r) \). Note that such definition does not affect the value determined for \( m \). Such redefinition would imply that the LHS of Eq. (C24) would be zero in the lowest order, which gives \( m = 0 \), a result that is consistent with earlier finding. Therefore, we arrive at

\[
(p', u', v', c_{11}', c_{12}', c_{22}', c_{33}', c_{12}) = (\chi_0, \chi_1, r\chi_2, \chi_4, \chi_5, \chi_6, r\chi_7). \tag{C27}
\]

2. The leading order behaviour and parities of \( \chi^{(2)} \)

Let \( w' = r^q\chi_3(r) \) where \( q \) is a positive integer required by analyticity that will be determined later. The Eq. (C9) and Eq. (C7) suggest the leading order behaviours for \( c'_{23} \) and \( c'_{13} \) as \( c'_{23} = r^{q-1}\chi_9(r) \) and \( c'_{13} = r^q\chi_8(r) \), respectively. Upon substituting these expressions in Eq. (C4), Eq. (C7) and Eq. (C9), and representing \( \chi_j(r) \)'s by their respective Taylor expansion around \( r = 0 \), we get,

\[
\sum_{k=0}^{\infty} r^{k+q} \left\{ H_{12} \chi_{3,k} - \beta \text{Re}^{-1} r^{-2} \left[ (q+k)^2 - 1 \right] \chi_{3,k} \right\} = \Gamma \sum_{k=0}^{\infty} r^{k+q} \left\{ i\alpha \chi_{8,k} + H_4 \chi_{9,k} + r^{-2} (k+q+1) \chi_{9,k} \right\}, \tag{C28}
\]

\[
H_{11} \sum_{k=0}^{\infty} r^{k+q} \chi_{8,k} = \sum_{k=0}^{\infty} r^{k+q} \left\{ [i\alpha - 2W(k+q-1)] \chi_{3,k} - 2\chi_{9,k} \right\}, \tag{C29}
\]

\[
H_{11} \sum_{k=0}^{\infty} r^{k+q-1} \chi_{9,k} = \sum_{k=0}^{\infty} r^{k+q-1} \left\{ [-2i\alpha W r^2 - (k+q-1)] \chi_{3,k} \right\}. \tag{C30}
\]

The leading order terms form the system,

\[
-\beta \text{Re}^{-1} (q^2 - 1) \chi_{3,0} = \Gamma (q+1) \chi_{9,0}, \tag{C31}
\]

\[
H_{11} \chi_{8,0} = [i\alpha - 2W(q-1)] \chi_{3,0} - 2\chi_{9,0}, \tag{C32}
\]

\[
H_{11} \chi_{9,0} = -(q-1) \chi_{3,0}. \tag{C33}
\]

The determinant of coefficient matrix of the system given by Eq. (C31) and Eq. (C32) reads that \( q = 1 \). Similar to earlier cases, we have \( \chi_{3,1} = \chi_{8,1} = \chi_{9,1} = 0 \).
Now, since \(\chi_{9,0} = 0\) Eq. (C33), and \(\chi_{9,1} = 0\) by parity, we can redefine the exponent of the leading order behaviour for \(c_{23}'\) by increasing it by two. Therefore, we have,

\[
(w', c_{13}', c_{23}') = (r\chi_3, r\chi_8, r^2\chi_9).
\]

Note that such redefinition of \(c_{23}'\) as \(c_{23}' = r^2\chi_9\) would not affect the determined value for \(q = 1\). Such redefinition would mean the LHS of Eq. (C33) would be zero at the lowest order, which again gives \(q = 1\).

The even-party of each of \(\chi_j(r)\) is also immediate by the same arguments used for the case of \(|n| \geq 2\) and mentioned in the main paper.

**Appendix D: Case \(n = 0\) and \(\alpha = 0\)**

For this case, the continuity reads, \(rDv' + v' = 0\) which has \(v' = 0\) as the solution that is analytic at \(r = 0\). The other variables, follow the same leading order behaviour as in the previous case of \(n = 0\) and \(\alpha \neq 0\) except for \(c_{11}', c_{22}', c_{33}'\) and \(c_{13}'\). In the previous case, we found that \(c_{11}' = r^m\chi_4(r), c_{22}' = r^m\chi_4(r), c_{33}' = r^m\chi_4(r)\) with \(m = 0\). With this value for \(m\) for the present case of \(n = 0\) and \(\alpha = 0\), the Eq. (C23), Eq. (C25) and Eq. (C26) become

\[
\begin{align*}
H_{11}\chi_{4,0} &= -W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}), \\
H_{11}\chi_{5,0} &= -W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}), \\
H_{11}\chi_{6,0} &= -W^{-1}(\chi_{4,0} + \chi_{5,0} + \chi_{6,0}),
\end{align*}
\]

which has the solution, \(\chi_{4,0} = \chi_{5,0} = \chi_{6,0} = 0\). Due to the established even parity, \(\chi_{4,1} = \chi_{5,1} = \chi_{6,1} = 0\). Therefore the correct leading order behaviour for this case is \(c_{11}' = r^2\chi_4(r), c_{22}' = r^2\chi_4(r)\) and \(c_{33}' = r^2\chi_4(r)\). Similarly, the leading order exponent for \(c_{13}'\) needs to incremented by two since the Eq. (C32) reads that \(\chi_{8,0} = 0\) for \(\alpha = 0\). Therefore, \(c_{13}' = r^3\chi_8(r)\).

**Appendix E: Proof using the results of Fourier analysis for \(n = 0\)**

For \(n = 0\), the Fourier analysis by Lewis and Bellan [25] predicts \(v' = r\chi_2\) and \(u' = r\chi_3\) where \(\chi_2\) and \(\chi_3\) are even functions of \(r\). Since these results are independent of physical problems, these are applicable in the current non-Newtonian flow as well.

Substituting these results into the continuity equation, \(i\alpha u' - Dv' - r^{-1}v'\), we get \(u' = \chi_1\), which is also an even function of \(r\). Let the Taylor expansions for \(\chi_j(r)\) \((j = 1, 2, 3)\) be \(\chi_j(r) = \sum_{k=0}^{\infty} \chi_{2k} r^{2k}\).

Substituting these expressions for \(u', v'\) and \(w'\) into the Eqs. (C5)-(C10), and noting that \(H_2 = -2, H_7 = -2W\) and \(H_3 = H_5 = H_9 = H_{10} = 1\), and retaining only the leading order terms, we get
\[ H_{11}c'_{11} = -K_{1}r^{2} + \alpha K_{2} - 4rc'_{12} - W^{-1}(c'_{11} + c'_{22} + c'_{33}), \quad \text{(E1)} \]
\[ H_{11}c'_{12} = K_{3}r + 2r(c'_{11} + c'_{33}), \quad \text{(E2)} \]
\[ H_{11}c'_{13} = \alpha rK_{4} - 2rc'_{23}, \quad \text{(E3)} \]
\[ H_{11}c'_{22} = K_{5}r^{2} + K_{7} - W^{-1}(c'_{11} + c'_{22} + c'_{33}), \quad \text{(E4)} \]
\[ H_{11}c'_{23} = K_{6}r^{2}, \quad \text{(E5)} \]
\[ H_{11}c'_{33} = K_{7} + K_{8}r^{2} - W^{-1}(c'_{11} + c'_{22} + c'_{33}), \quad \text{(E6)} \]

where the constants, \( K_{i} \)'s are given by \( K_{1} = (H_{13}\chi_{2,0} + 2H_{7}\chi_{1,2}), K_{2} = 2i\chi_{1,0}, K_{3} = i\alpha\chi_{2,0} + i\alpha H_{7}\chi_{1,0} + 2\chi_{1,2}, K_{4} = i\chi_{3,0}, K_{5} = 2i\alpha H_{7}\chi_{2,0} - H_{14}\chi_{2,0}, k_{6} = (i\alpha H_{7}\chi_{3,0} + 2\chi_{3,2}), K_{7} = 2\chi_{2,0}, K_{8} = -H_{14}\chi_{2,0} \). The linear system given by Eqs. (E1)-(E6) for \( c'_{ij} \) can be solved to show that

\[ (c'_{11}, c'_{22}, c'_{33}, c'_{12}, c'_{13}, c'_{23}) = (\chi_{4}, \chi_{5}, \chi_{6}, r\chi_{7}, r\chi_{8}, r^{2}\chi_{9}), \quad \text{(E7)} \]

when \( \alpha \neq 0 \). Though \( c'_{11}, c'_{22} \) and \( c'_{33} \) are non-zero, the trace \( c'_{ii} \) vanishes at the centreline. The sum of Eq. E1, Eq. E4 and Eq. E6 shows that \( c'_{ii} \sim i\alpha\chi_{1,0} + 2\chi_{2,0} \) to the leading order. However, \( i\alpha\chi_{1,0} + 2\chi_{2,0} = 0 \) by the virtue for the continuity equation to the leading order.

When \( \alpha = 0 \), the Eq. (E1)-(E6) become,

\[ H_{11}c'_{11} = -K_{1}r^{2} - 4rc'_{12} - W^{-1}(c'_{11} + c'_{22} + c'_{33}), \quad \text{(E8)} \]
\[ H_{11}c'_{12} = K_{3}r + 2r(c'_{11} + c'_{33}), \quad \text{(E9)} \]
\[ H_{11}c'_{13} = -2rc'_{23}, \quad \text{(E10)} \]
\[ H_{11}c'_{22} = -W^{-1}(c'_{11} + c'_{22} + c'_{33}), \quad \text{(E11)} \]
\[ H_{11}c'_{23} = K_{6}r^{2}, \quad \text{(E12)} \]
\[ H_{11}c'_{33} = -W^{-1}(c'_{11} + c'_{22} + c'_{33}), \quad \text{(E13)} \]

where the information \( \nu' = 0 \), i.e., \( \chi_{2,j} = 0 \) from continuity equation is employed. The similar analysis of Eq. (E8)-(E13) reveals

\[ (c'_{11}, c'_{22}, c'_{33}, c'_{12}, c'_{13}, c'_{23}) = (r^{2}\chi_{4}, r^{2}\chi_{5}, r^{2}\chi_{6}, r\chi_{7}, r^{3}\chi_{8}, r^{2}\chi_{9}). \quad \text{(E14)} \]

Similar analysis can be performed for the cases, \( |n| = 1 \) and \( |n| \geq 2 \) using the predictions for \( \nu' \) and \( \nu' \) from Fourier analysis of Lewis and Bellan [25].

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