Quantum Mechanical Probability of Internally Electrodynamic Particle(s)

J.X. Zheng-Johansson
Institute of Fundamental Physics Research, 611 93 Nyköping, Sweden
E-mail: jxzj@iofpr.org

Abstract.
A distribution of electromagnetic fields presents a statistical assembly of a particular type, which is at scale $h$ a quantum statistical assembly itself and has also been instrumental to a concrete demonstration of the basic probability assumption of quantum mechanics. Of specific concern in this discussion is an extensive wave train of radiation fields, described by a total wave function $\psi$, which are continuously (re)emitted and (re)absorbed by an oscillatory (point) charge of a zero rest mass and yet a finite dynamical mass, with the waves and charge together making up an extensive undulatory IED particle. The IED particle will as any real particle be subject to interactions with the environmental fields and particles, hence to excitations, and therefore will explore all possible states over time; at scale $h$ the states are discrete. On the basis of the principles of statistics and statistical mechanics combined with first principles solutions for the IED particle, we derive for the specific statistical system of the IED particle the probability functions in position space, of a form $|\psi|^2$, and in dynamical-variable space.

1. Introduction
It has been a long established fact in experiment and formal theory that matter particles manifest both the characteristics of corpuscle and wave, termed as matter waves ($\Psi$), and are at the scale of Planck constant $h$ dominated by quantum mechanics which formally is formulated on the basis of the wave equations of E Schrödinger, W Heisenberg and P Dirac[1]. The interpretation of the quantum mechanical wave function $\Psi$ is based on the probability assumption of M Born[2] that the state of a particle, assumed corpuscular, is completely specified by its $\Psi(x,t)$, with $\Psi(x,t)$ being generally complex, and $|\Psi(x,t)|^2$ represents the statistical probability of finding the particle at position $x$. This probability has a key feature distinct from that of classical systems: it is additionally constrained by the Heisenberg uncertainty relations for the conjugate dynamical variables [3]. In phase space, this probability is associated with a smallest volume, equal to the fundamental constant $h$ of M Planck raised to power $fN$ for a $N$ particle system of $f$ degrees of freedom, which is accessible to a microscopic state.

Quantum mechanics has proven successful in predicting a broad range of quantum mechanical properties of particles, especially properties involving the quantization of dynamical variables, for which the quantum theoretical predictions are directly comparable with the overall vast quantity of experimental data accumulated over a century or so to this date. However, it is up to the present not satisfactorily understood that what is waving with a matter wave, what is accordingly statistical with it in position as well as dynamical variable spaces, and what is the origin of the Planck constant which defines a smallest volume in phase space?

Their understandings come especially in question when our purpose is in order that the probability, and the fundamental constant $h$ here, can be inferred based on a common minimal set of first principles’ laws which also govern the remaining basic properties of particles. Further
to this, quantum mechanics as interpreted through a statistical corpuscular-particle picture meets difficulties in (coherently) accounting for certain basic experimental phenomena, including in particular the basic coherent wave phenomena such as diffraction and (self) interference[4]. Furthermore, the corpuscular particle with a given rest mass does not furnish a mechanical scheme for accounting for the origin of mass, the cause of gravity and the origin of relativistic effect, among others (see [4] for a recent fuller review).

On the whole, we appear to have in hand a comprehensive, rigorous mathematical structure for quantum mechanics and yet an incomplete conceptual grasp of the underlying mechanical content. Conceptually, we are confronted with various internal clashes. These concern for examples the dual corpuscle and wave aspects in a single object, the desired wave form of spreading and the corpuscular particle assumption at any one time, and the relationships between prediction, measurement and actuality of the state of a particle as a subject of debate ever since the foundation of quantum mechanics. The difficulties and clashes, as one would identify upon an examination, are the inevitable results of descriptions based on the corpuscular particle picture. Although, it cannot not be overemphasised that, the current description represents the best theory which is as accurately as can be made in the given framework.

From the standpoint that nature constructs and operates itself by mechanical means, indefinitely down to as small a scale as it takes to sum up to the lager-scale manifestations in question, one may anticipate, as it has already been demonstrated to certain extent in [5a-m], that the conceptual clashes will all of a sudden vanish once a realistic model of particle, one with a realistic internal structure and dynamical scheme, is constructed.

Based on overall experimental observations as input information, the experimentally established de Broglie wave and relations from the outset, the author developed in [5a-m] an internally electrodynamic (IED) particle model, which briefly states that a single-charged material particle, like the electron, proton, etc., is composed of (i) an oscillatory point-like charge $q$ (as source) with a characteristic frequency $\Omega$ and zero rest mass, and (ii) the electromagnetic waves $E, B$ generated by the oscillating charge. The IED particle thus defined is geometrically an extensive object, where the term "particle" stands no longer for a "geometrical point". Nevertheless, it maintains the other usual aspects pertinent of a "particle", such that it separates one distinct kind of material species from others.

The IED process is itself built on the experimental facts as reflected through a minimal set of established basic, or, first principles laws (see [4] for a recent review). Based on the first principles solutions for the IED process, it has become feasible to predict a range of basic properties of particle as demonstrated in [5a-m].

It has been one basic solution obtained earlier [5a,c,d] that the total energy $\varepsilon$ of an IED particle is in direct proportion to the quadratic of the complex total wave function $\psi(x,t)$, i.e. $\varepsilon \propto |\psi(x,t)|^2$, where $|\psi(x,t)|^2$ plays an apparent role of probability in position space. In this paper we shall deduce the $|\psi(x,t)|^2$ as a probability in position space more rigorously, and in turn its counterpart in dynamical-variable space, on the basis of the basic principles of statistics and statistical mechanics, the first-principles solutions for single and many IED particles, and the relevant experimental indications.

2. Wave mechanics of single IED particle. Probability density in position space

Consider that an IED particle travelling at a velocity $v$ along the $x$ axis between two non-absorbing massive walls at $x = 0$ and $L$ in the presence of an external potential field $V(x)$. Its constituent point-like charge $q$, as source, is endowed with a total mechanical energy (or Hamiltonian) $\varepsilon_q$ at initial time, and will over time oscillate continuously at a characteristic frequency $\Omega$, and also travelling at velocity $v$ in the $x$ direction. Two opposite travelling, Doppler-displaced electromagnetic waves of fields $E_j, B_j$, which together with the $q$ constitute the IED particle, are generated by the charge along each direction, where $j = \dag$ indicates the component wave travelling in the direction parallel with $v$, and $j = \|$ antiparallel with $v$. 
In regions excluding the charge, assuming that, except for $\mathcal{V}(x)$, no other charges and currents present, the $E^j, B^j$ fields are subject to the Maxwell’s equations

$$\nabla \cdot E^j = 0, \quad \nabla \cdot B^j = 0, \quad \nabla \times E^j = 0 + \frac{1}{c^2} \partial_t E^j, \quad \nabla \times B^j = -\partial_t B^j,$$

where $\partial_t = \frac{\partial}{\partial t}$, and $c^2 = c^2 + \mathcal{V}/m$, with $c = \sqrt{1/c_0^2}$ the velocity of light in a free vacuum and $m$ the particle’s dynamical mass expressed in (6) below. Setting $E^j = E_q \varphi^j$, with $E_q$ the amplitude and $\varphi$ the (real) dimensionless fields, or wave functions, their superposition gives the real part of a complex total wave function $\psi = E/E_q$, $\text{Re}[\psi] = \sum \psi^j = E_q = \varphi^j + \varphi^q$. Further restricting for illustration here the waves to be propagated along the $x$ axis only, we obtain from the Maxwell’s equations a total wave equation $\frac{d^2 \psi}{d t^2} = (\mathcal{V}^2 + \mathcal{V}(x)/m) \nabla^2 \psi$. The equation, after routine transformations under conditions of small $\mathcal{V}$ and an insured existence of stationary-state solutions [5c], reduces to

$$\ddot{\psi} = \hat{H}\psi, \quad \dot{\psi} = i\hbar \partial_t, \quad \hat{H} = \hbar^2/m + \mathcal{V}(x), \quad \dot{p} = (\hbar/\imath) \nabla.$$

If after $N_\varphi$ cycles of continuous oscillations the charge had emitted its entire initial energy $\varepsilon_q$ without reabsorption, nor with supply of external energy, $\varepsilon_q$ is then converted entirely to the energy of total wave $\psi$ generated by it, $\varepsilon = \sqrt{\varepsilon^2 E^2} = \varepsilon_q$. The total wave $\psi$ extends as a wave train of a mean total length $L_{\psi} = \sqrt{\varepsilon^2 \mathcal{V}^2} = N_\varphi \lambda$.

In the presence of reflection walls assumed earlier, the radiation waves $\varphi^q, \varphi^q$, and thus the total wave $\psi$, will be reflected back from the walls to the charge, be reabsorbed and re-emitted by it, repeatedly. Assume that the particle has reached an equilibrium state in $[0, L]$, with $L$ being relatively small. Its total energy ($\varepsilon_{\text{tot}}$) will ordinarily be carried in part by the charge and in part by the wave, by the fractions $a_1$ and $a_2$, with $a_1, a_2 \leq 1$, $a_1 + a_2 = 1$, and $\varepsilon_{\text{tot}} = a_1 \varepsilon_q + a_2 \varepsilon_q$. The extensiveness of the IED particle is determined by its wave components, the total $\psi$ in question here, and is of our primary concern below. $\psi$ will be based on in the discussions below.

Our primary concern here is "probability". We thus start by directly writing down the total electromagnetic energy density, in terms of its total field $E(x, t) = E_q \psi(x, t)$, in the usual form $\varepsilon_0(x, t) = \varepsilon_0 |E(x, t)|^2$. $\varepsilon(t) = \frac{1}{\varepsilon_0} \int_0^L \varepsilon_0(x, t) dx = \frac{L}{\varepsilon_0} \int_0^L \varepsilon_0(x, t) dx$ gives the total wave energy at time $t$. It is here that the complex form of $\psi$ finds its clear physical significance: The complex $\psi$ ensures that $\varepsilon$ is a mechanical energy, or, Hamiltonian $\propto |E|^2 = |\text{Re}[E]|^2 + |\text{Im}[E]|^2$, which includes both an inertial component $\propto (\text{Re}[E])^2$ carrying the kinetic energy of the field $E$, and an elastic component $\propto (\text{Im}[E])^2$ carrying the restoring elastic potential energy of the vacuum which acquires an induced shear elasticity in the presence of the charge $q$ [5a.g.h.].

For stationary state, for which $\varepsilon_0(x, t)$ by definition does not change with time,

$$\varepsilon_0(x) = \varepsilon |\psi(x, t)|^2 = \varepsilon |\psi(x)|^2, \quad \varepsilon = \int_0^L \varepsilon_0(x) dx = \int_0^L \varepsilon \int_0^L |\psi(x, t)|^2 dx,$$

where $\varepsilon = L_{\psi}\varepsilon_0 E_q^2$ for $\psi$ being normalized in $[0, L]$,

$$\int_0^L |\psi(x, t)|^2 dx = 1 \quad \text{and thus } \varepsilon = \varepsilon_0.$$  

The corresponding complex wave function is $\psi(x, t) = \psi(x) e^{-i(\omega t + \alpha_0)}$, where $\omega = \gamma \Omega$, with $\gamma = 1/\sqrt{1 - \mathcal{V}^2/c^2}$ [5a.e,d]; $\alpha_0$ is an initial phase which affects the field configurations of the individual component waves (see further after eqn. 9) but not the total particle state (since $|e^{i\alpha_0}|^2 = 1$) and can be here simply set to $\alpha_0 = 0$. The total linear momentum is $p = \int_0^L p_0(x) dx$, with $p_0(x) = \frac{\varepsilon |\psi(x)|^2}{\varepsilon}$; in the case of $\mathcal{V}(x) = \varepsilon_0$ being constant, $p = (\varepsilon - \varepsilon_0)/c$. 


The \( \varepsilon \), in terms of the total wave \( \psi \) of a frequency \( \omega \), is simultaneously subject to the Planck energy equation,

\[
\varepsilon_{q,nq} = n_q (b_0/2\pi) \omega, \quad n_q = 1, 2, \ldots, \quad b_0 = h; \quad \text{(for } n_q = 1) : \varepsilon_q = \hbar \omega, \quad \varepsilon = \varepsilon_q \quad \text{(5)}
\]

(5) can be obtained as a solution to the energy equation (internal work), of a form analogous to the Schrödinger equation, for the harmonically oscillating charge \( q \) being point like and yet extensive at scale \( \sim 10^{-18} \) m, and of a dynamical mass \( M_q \)[5c,j]. Furthermore, for the wave train travelling rectilinearly at a finite velocity \( c \) which resembles a rigid object[5a,c,e] of a finite inertial mass \( (m) \), we may employ Newton’s law of inertia and obtain \( p_0(x) \) in relation to density of mass, \( m_0(x) \), (setting \( \mathcal{V} = 0 \) for defining mass here) and in turn the integration as

\[
m_0(x) = \frac{p_0(x)}{c} = \frac{\varepsilon_0(x)}{c^2} = \frac{\varepsilon}{c^2}|\psi(x,t)|^2, \quad m = \frac{\varepsilon}{c^2} \int_{0}^{L} |\psi(x,t)|^2 dx = \frac{\varepsilon}{c^2} = \frac{p}{c}. \quad \text{(6)}
\]

If \( \psi(x,t) \) were an arbitrary function of \( t \), the proportionality relation \( \varepsilon_0(x) \propto |\psi(x,t)|^2 \) of (3) would not generally hold, and \( \varepsilon \) would be generally dependent on \( t \). However under the condition of stationary state we assumed, (3) holds generally, as may be readily verified by substituting \( \psi(x,t) = \psi(x)e^{-i(\varepsilon/c^2)\hbar t} \) in the wave equation \( \hat{\psi} = \hat{H}\psi \) of (2), and multiplying it by \( \psi^* \) from the left. Its left side gives (3a), or rewritten as

\[
\varepsilon_0(x) = \psi^* \hat{\psi} = \int_{0}^{L} \psi^* \hat{\psi} dx = \int_{0}^{L} \psi^* \hat{H} \psi dx \quad \text{(7)}
\]

where (7b) is given by integrating (7a) on both sides.

For the specific case of \( \mathcal{V}(x) = 0 \) and \( \nu = 0 \), (2) has the solution \( \psi = \frac{1}{\sqrt{L}} e^{i(Kx-\Omega t)} \), where \( K = \lim_{\nu_1/\nu_2-0} k(= \omega/c) = Mc^2/h \), \( \Omega = \lim_{\nu_1/\nu_2-0} \omega = Mc^2/h \) and \( M = \lim_{\nu_1/\nu_2-0} m[5a,c,e] \); with this \( \psi \) in (2) we obtain \( \hbar \Omega \psi = \frac{b^2K^2}{2} \psi \). Subtracting the resulting equation from (2) of a generally finite \( \mathcal{V} \) and \( \nu \), taking limit at \( \nu^2/c^2 \to 0 \) and at scale of the de Broglie wavevector \( k_d = (\frac{2}{\hbar})k (<< k) \), \( \psi = \lim_{\nu_1/\nu_2-0,k_d/K<1} \psi = \psi(x)e^{-i\Omega_d t} \), with \( \Omega_d = \frac{1}{b\nu_2} \Omega \), we obtain a Schrödinger form of wave equation governing the particle’s kinetic motion \([5b,c]\), \( \hat{\psi}_v \psi = \hat{H}_v \psi \)

where \( \hat{\psi}_v = \hat{\psi} - \hbar \Omega, \hat{H}_v = \hat{H} - \frac{b^2K^2}{2M} = \frac{\hbar^2}{2M} + \mathcal{V} \), and \( \hat{P}_v = \frac{\hbar}{b} \nabla \). The corresponding energy density and integration follow to be

\[
\varepsilon_0(x) = \Psi^* \hat{\psi}_v \Psi = \int_{0}^{L} |\Psi(x,t)|^2 dx, \quad \varepsilon_v = \int_{0}^{L} \Psi^* \hat{\psi}_v \Psi dx = \int_{0}^{L} \Psi^* \hat{H}_v \Psi dx, \quad \text{(8)}
\]

where \( \hat{\psi}_v = \hat{\psi} \), for \( \int_{0}^{L} |\Psi|^2 dx = 1 \); for the case of \( \mathcal{V}(x) = \mathcal{V}_0 = \text{constant} \), \( \varepsilon_v = \frac{1}{2} M \nu^2 + \mathcal{V}_0 \).

We shall proceed the remainder of the discussions in terms of the total wave \( \psi \) and energy \( \varepsilon \), etc., i.e., the “relativistic” forms, and in one dimension. The conclusions can all be carried in a straightforward way over for \( \psi \), \( \hat{\varepsilon}_v \), etc. by taking the limit at \( \frac{\nu^2}{c^2} \to 0 \) and scale \( k_d \), and to three dimensions[5c].

Unambiguously, the total energy or mass of the IED particle is as any real particle an absolute measure of the existence of the particle. The IED particle of a spatially extensive energy and mass as of (3), or (7), and (6) therefore is extensive, here apparently across \([0, L]\) (though practically across \( L_{\psi} \)). And its presence at position \( x \) in \([0, L]\) is measured by the energy density \( \varepsilon_0(x) \), or alternatively mass density \( m_0(x) \), which for stationary state contains each the common factor \( |\psi|^2 \), denoting by \( \rho_1(x) \) as

\[
\rho_1(x) = |\psi(x,t)|^2, \quad \hat{\mathcal{K}} \rho_1 = \psi^*(x,t) \hat{\mathcal{K}} \psi(x,t) = |\psi(x,t)|^2 \hat{\mathcal{K}}. \quad \text{(9)}
\]

where in (9b) \( \rho_1(x) \) is written in the more general form of being acted on by a dynamical variable operator \( \hat{K} \).
The foregoing descriptions relevant to \( \rho_i(x) \) may be summarized as: (i) \( \rho_i(x) \) is a distributed function in \([0, L]\) describing electromagnetic fields which are implicitly random because the initial phase \( \alpha_0 \) of \( \psi \) (see after eqn. 4) is in general the result of random interactions (Sec. 3).

(ii) \( \rho_i(x) \) is a measure of the fraction of the total energy or mass of the particle presenting at \( x \). And \( \varepsilon \) and \( m \) are the \( \rho_i(x) \)-weighted integrations across \([0, L]\), and therefore the expectations of total energy and mass. (iii) \( 0 \leq \rho_i(x) \leq 1 \) according to (3) and (4); in particular, \( \rho_i(x) = 0 \) if no portion of the particle is found at \( x \), and \( \rho_i(x) = 1 \) if the entire particle is found at \( x \), hence a corpuscle.

Points (i)-(iii) furnish \( \rho_i(x) = |\psi(x, t)|^2 \), or \( \rho_i(x) = |\Psi(x, t)|^2 \) at scale \( k \), with all of the essential qualifications for \( \rho_i(x) \) to be defined as the probability, or probability density here, for finding a corresponding portion \( \rho_i(x) \) of the IED particle at position \( x \) in a given stationary state. This conclusion is in complete accord with the (mathematical) probability assumption in the formal theory of quantum mechanics.

3. Time evolution of single particle state. Probability density in dynamical variable space

Let now the \( N = 1 \) IED particle in the one-dimensional box of volume \( V = L \) be subject to interactions with its environment, the wall particles and/or environmental radiation fields, maintained at a constant temperature \( T \). The particle is here described in the quantum-mechanical terms by the wave equation (2) for all regimes, and the description will reduce to the classical mechanics limit when the particle’s energy levels \( \varepsilon_r \)s are on the scale \( \hbar \) densely nested with one another. Assume that the interactions are weak, infrequent, and elapse over a brief time \( \delta t \), only on each occurrence. So the IED particle will maintain a given stationary state \( \nu \) (specified by e.g. a total energy, linear momentum, and spin, etc.) for a finite time interval \( t_{\nu} \gg \delta t_{\nu} \), but, subjected to interactions with environment, will explore all possible states on a time scale \( t \gg t_{\nu} \).

This may for example describe the realistic case of an electron moving in a fixed orbit about the nucleus of an isolated atom until an emission/absorption of radiation, or a conduction electron moving at a constant velocity in a small metal specimen until next collision, or a spin-half atom traversing a magnetic field with a fixed spin orientation as in the Stern-Gerlach experiment (Sec. 3) until externally perturbed. Under the condition of \( t_{\nu} \gg \delta t_{\nu} \), contributions from transient states may be neglected, and the thermodynamic properties are essentially determined by the stationary states.

Instead of making observations over time, we now "compact" a large \( N \) number of the states evolved over time (\( t' \)) into \( N \) replicas which each describe the one particle system here under condition of constant \( T, V, N(=1) \), and are uniformly distributed over a total \( N \) number of possible quantum states of the particle. We thus obtain a canonical ensemble. Each \( \nu \)th (eigen) state is associated with an (eigen) wave function \( \psi_{\nu}(x, t) = \psi_{\nu}(x)e^{-i(\varepsilon_{\nu}/\hbar)t} \), energy \( \varepsilon_{\nu} \), etc., given as the solution of (2), with \( \nu = 1, 2, \ldots, N \). Each \( \nu \)th state occupies a volume element in the phase space spanned here by \( x, p_x \); the trajectory of state in this space corresponds to the time evolution of state of the actual particle. With (2) being a linear equation, the linear sum of \( \psi_{\nu}(x) \)s,

\[
\psi_{\text{ens}}(x, t) = \sum_{\nu=1}^{N} c_{\nu}(t)\psi_{\nu}(x)
\]  

must again be a solution of (2), where \( c_{\nu}(t) \) is a "weight coefficient" or "amplitude" of state \( \nu \) and is in general complex.

That \( \psi_{\text{ens}}(x, t) \) is a linear sum of \( \psi_{\nu} \)s implies only the linearity of the wave equation. \( \psi_{\text{ens}}(x, t) \) does not describe the total displacement at \( x \) due to the superposition of \( \psi_{\nu} \)s, since the \( \psi_{\nu} \)s occur in reality at different times and never meet.
Multiplying $\psi_\nu^*(x)$ on both sides of (10) from the left and integrating over $(0, L)$, we obtain, for the $\psi_\nu$s being orthonormal ($\int_0^L \psi_\nu^*(x) \psi_\nu(x) dx = \delta_{\nu', \nu}$, $\delta_{\nu', \nu}$ being the Kronecker delta),

$$\int_0^L \psi_\nu^*(x) \psi_{\text{ens}}(x, t) dx = \int_0^L \psi_\nu^*(x) \sum_{\nu'} c_{\nu'}(t) \psi_{\nu'}(x) dx = c_{\nu}(t).$$

(11)

For the canonical ensemble of a dimension $\mathbb{N}$ assumed large, its average energy, proportional to the integral below, is evidently a constant. And the integral itself must als0 be a constant, and is normalised to one as

$$\int_0^L \psi_{\text{ens}}^*(x, t) \psi_{\text{ens}}(x, t) dx = 1,$$

or

$$\sum_{\nu=1}^{N} c_{\nu}^* c_{\nu} \int_0^L |\psi_\nu(x)|^2 dx = \sum_{\nu=1}^{N} |c_{\nu}(t)|^2 = 1,$$

(12)

where $|c_{\nu}(t)|^2 = c_{\nu}^* c_{\nu}(t)$. With $\psi_\nu$ for $\psi$ in (2), multiplying on both sides of the $\nu$th equation by $c_{\nu}$ from the left for $\nu = 1, \ldots, \mathbb{N}$, summing the equations over all $\nu$, we obtain a corresponding wave equation,

$$\xi \psi_{\text{ens}} = \hat{H} \psi_{\text{ens}}, \quad \xi = i\hbar \partial_t, \quad \hat{H} = -(\hbar^2 / m) \nabla^2 + V(x).$$

(13)

With $\psi_\nu$ for $\psi$, making similar algebraic operations as in leading to $\varepsilon$ of (7), we obtain the energy of $\nu$th state, $\int_0^L \psi_\nu^*(x, t) i\hbar \partial_t \psi_\nu(x, t) dx = \varepsilon_\nu$. From the significance of $c_{\nu}$ as the amplitude of state $\nu$ in $\psi_{\text{ens}}$ defined in (10), the energy expectation over the states of the ensemble follows to be

$$\langle \varepsilon_\nu \rangle = \int_0^L \psi_{\text{ens}}^*(x, t) i\hbar \partial_t \psi_{\text{ens}}(x, t) dx,$$

and

$$\langle \varepsilon_\nu \rangle = \sum_{\nu'=1}^{N} c_{\nu'}^* c_{\nu} \int_0^L \varepsilon_\nu c_{\nu}(t) \psi_\nu(x) dx = \sum_{\nu=1}^{N} \varepsilon_\nu \rho_\nu, \quad \rho_\nu = |c_{\nu}(t)|^2,$$

(14)

where for the final expression for $\langle \varepsilon_\nu \rangle$, $\int_0^L |\psi_\nu|^2 dx = 1$ is used. The ensemble average $\langle \varepsilon_\nu \rangle$ of (14) is equivalent to the time average provided that the IED particle here is ergodic, an assumption usually made for real particles. By a similar reasoning as for $\rho_\nu(x)$ earlier (as $\rho_\nu(x) = |\psi_\nu(x, t)|^2$ here), the $\rho_\nu = |c_{\nu}(t)|^2$ given in (14) has all of the qualifications to be defined as the probability of finding the particle at state $\nu$. The function $\rho_\nu$ is related to $\varepsilon_\nu$ through a Boltzmann factor, $\exp(-c_{\nu}/k_B T)$ which is a proper problem of statistical mechanics.

Given an arbitrary dynamical variable $K_\nu(x)$ of an operator $\hat{K}_\nu(x)$ (which may also be an independent variable like the $x$, or the $\sigma$ later) of the single particle for $\nu$th state at position $x$, from the preceding discussions its expectation in both the position and dynamical-variable spaces follow to be

$$\langle K_\nu \rangle = \int_0^L \sum_{\nu'=0}^{\infty} c_{\nu'}^* K_\nu(x) \sum_{\nu'=0}^{\infty} c_{\nu'} \psi_\nu(x) dx = \sum_{\nu=0}^{\infty} |c_{\nu}|^2 \langle K_\nu \rangle, \quad \langle K_\nu \rangle = \int_0^L \psi_\nu^*(x) \hat{K}_\nu \psi_\nu(x) dx.$$

(15)

The $\psi_\nu$s have been introduced in the above as describing events taking place in reality in a random fashion one after another in time, only one at a time regardless of being probed by an observer or not. This feature is in direct accord with the eigen-state solutions of Schrödinger equation for a corresponding system and, as can be expounded case by case, with overall experiments.

A most clear-cut experimental indication is provided by the Stern-Gerlach experiment (Z Phys 8, 110, 1921). Here, a silver atom, of a net spin $\frac{1}{2} \hbar$ due to its outermost-shell electron and being ordinarily random oriented, is let travel at a constant velocity ($v$) in the $x$ direction across a region $d$ applied with an inhomogeneous transverse ($z$-direction) magnetic field $B_z$ of a finite gradient $\frac{dB_z}{dz}$ ($> 0$). In the $B_z$ field its spin magnetic moment $\mu_z = -\frac{\hbar}{2} g_s \mu_B$ is quantized in orientation (Zeeman effect), $\sigma = +1$ or $-1$. If the atom deterministically maintains a fixed
spin orientation, say \( \sigma = +1 \), provided also that no other perturbation presents, the eigen state function being thus \( \psi = \psi_+ \), then across the entire \( d \) region the atom will constantly be acted by a force \( F_z = \frac{1}{2} g_j \mu_0 \frac{d}{m} \) in the fixed \( +z \) direction for a duration \( t = d/\nu \), and at the exit of \( d \) be displaced by a finite distance \( z_1 = \frac{1}{2} \frac{(d^2)}{m} \) in the \( +z \) direction (in the experiment the atom also traverses an additional free flight distance \( D \) which is not of direct concern here). Or if \( \sigma = -1 \), the state function being thus \( \psi = \psi_- \), then \( F_z \) and \( z_1 \) change each to the \(-z\) direction. The above is as observed in the experiment.

On the contrary, if during each one trip the atom were in a mixture of both spin up and down states, so \( \psi = \frac{1}{2} \psi_+ + \frac{1}{2} \psi_- \), then across \( d \) the atom would be acted by forces of alternating directions. The so produced net displacement at the exit must on average be zero, \( z_1 = 0 \), which is in contradiction with the experimental observation.

### 4. The state vector space

What is mainly new in the foregoing is the concretisation of the physics underlying the probabilities in both the position and dynamical variable spaces instrumented by the IED model and solutions. The involved mathematical structure and operations are of no essential difference from the usual quantum mechanics, since the governing equations at scale \( k_d \) are the common Schrödinger equation. This feature holds through the remainder of discussions.

As in the unified mathematical formulation of quantum mechanics, the description in the (position) coordinate space can be transformed to one in an \( \mathbb{R} \)-dimensional linear state vector space (the Hilbert space for \( \mathbb{R} \) being infinite), which for the single IED particle here is spanned by the \( \psi \)'s as coordinate axes, or basic vectors, of a total number \( \mathbb{R} \), assuming the \( \psi \)'s orthonormal in \((0, L)\). For many (IED) particles, we have the Fock space. The linear combination of the \( \mathbb{R} \) basic vectors \( \psi_i \)'s, \( \psi_{\text{ens}} = \sum c_i \psi_i \) of (10), represents in this space a "(general) state" vector which describes based on Sec. 3 the state of the (canonical) ensemble; \( c_i \) is a component of the vector \( \psi_{\text{ens}} \). And \( |c_i|^2 \) is the probability of "finding" the physical system, the single particle in \((0, L)\) here, in \( n \)th (stationary) state at a time point \( t' \), on the time evolution axis \( t' \).

With the \( \psi_i \)'s being discrete, the linear mathematical operations in the space spanned by the \( \psi_i \)'s may be naturally carried out in terms of matrix analysis. If for example \( \mathcal{X}_{ai}, \mathcal{X}_{bj} \) \((i, j = 1, 2, \ldots, \mathbb{R})\) are given as two dynamical variables, their two general states are given by

\[
\psi_{\text{ens}, a}(x, t) = \sum_i a_i(t) \psi_i(x), \quad \psi_{\text{ens}, b}(x, t) = \sum_j b_j(t) \psi_j(x).
\]

Their scalar product for example is defined by

\[
(\psi_a, \psi_b) = \int_0^L \sum_{\nu} a_\nu^* \psi_\nu (x) \sum_{\nu'} b_{\nu'} \psi_{\nu'}(x) dx = \sum_{\nu} a_\nu^* b_{\nu} = (a_1^*, a_2^*, \ldots, a_\mathbb{R}^*),
\]

\[
\begin{pmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_\mathbb{R}
\end{pmatrix}
= (a|b),
\]

where \( \int_0^L \psi_\nu^* \psi_{\nu'} dx = \delta_{\nu\nu'} \) is used; \( \psi_a = \langle a | \) corresponds in the final expression to a \( \mathbb{R} \times 1 \) matrix with its \( \mathbb{R} \) elements being imaginary and arranged in one raw, and \( \psi_b = | b \rangle \) a \( 1 \times \mathbb{R} \) matrix with its \( \mathbb{R} \) elements being real and in one column. If \( a = b = c \), (16) returns the result of (12). The operator \( \hat{H} \) or \( \mathcal{X} \) in general is an \( \mathbb{R} \times \mathbb{R} \) matrix.

### 5. \( N \) weakly interacting particles in a mean interaction field

Let now \( N \) weakly interacting IED particles of the dynamical masses \( m_a, m_b, \ldots \), wave functions \( \psi_a(x_1, t), \psi_b(x_2, t), \ldots \), and attached with the coordinates \( x_1, x_2, \ldots \), be enclosed in the box of size \( L \) in an effective mean potential field \( \mathcal{V}(x_1, x_2, \ldots, x_N) = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j=1}^{N-1} \mathcal{V}_{ij}, \) with \( \mathcal{V}_{ij} \) a pairwise interaction potential. Each \( \psi_a(x_i, t) \) is governed by a wave equation given
after (2),
\[ \dot{\psi}_\alpha(x_i,t) = \hat{H}_\alpha \psi_\alpha(x_i,t), \quad \dot{\psi}_\alpha = i\hbar \partial_t, \quad \hat{H}_\alpha = -(\hbar^2/m_\alpha) \nabla_i^2 + \mathcal{V}(x_1,\ldots,x_N), \]
where \( \alpha = a, b, \ldots, \alpha_N, \ i = 1, 2, \ldots, N \). The \( N \psi_\alpha \)-waves are each distributed across the entire \( L \), thereby overlapping with one another. The total field at position \( x_i \) is \( \psi_\Sigma(x_i,t) = \sum_\alpha \psi_\alpha(x_i,t) \), which generally contains a finite portion of each of the \( N \) waves, or, particles. From a measurement of \( \psi_\Sigma(x_i,t) \) at \( x_i \), we are thus not able to distinguish which particle this is; that is, the \( N \) particles are indistinguishable, and in E. Schrödinger’s terminology (Proc Cam Phil Soc 31, 555, 1935), they are entangled with one another.

All of the expressions up to (16) can accordingly be written down for each \( \alpha \)-th particle. Of specific interest here is the probability density of finding particle \( \alpha \) at \( x_i \), \( \rho_\alpha(x_i,t) = |\psi_\alpha(x_i,t)|^2 \) given after (9), with \( \alpha = a, \ldots, \alpha_N, \ i = 1, \ldots, N \). The notion of \( |\psi_\alpha(x_i,t)|^2 \) as a probability may be advanced for the many-particle system through excising it in a few representative cases below, as we shall see, each returning the expected result of the formal quantum mechanics and (or) experiment.

(i) The probability of the simultaneous occurrence of \( N \) independent events that particle \( a \) is at \( x_1 \), \( b \) is at \( x_2 \), \ldots and \( \alpha_N \) is at \( x_N \) with the probabilities \( \rho_a(x_1), \rho_b(x_2), \ldots \) and \( \rho_N(x_{\alpha_N}) \), with the particles being in the mean field effectively non-interacting, is according to the usual multiplication rule for probabilities given by

\[ \prod_{\{a,i\}={a,1}}^{(\alpha_N,N)} \rho_\alpha(x_i) = |\psi_a(x_1)\psi_b(x_2)\ldots\psi_{\alpha_N}(x_N)|^2. \]

Because of indistinguishability, making simultaneous pair-wise coordinate permutations say from \( x_1 \) to \( x_2 \) for particle \( a \) and \( x_2 \) to \( x_1 \) for \( b \), leaves the product \( |\psi_a(x_2)\psi_b(x_1)| \) unchanged except for a change in sign if the permutations are antisymmetric; there are a total \( N! \) number of possible permutations. To ensure an always positive probability we should introduce as in usual practice a "Sign" into the product of \( N \) wave functions as \( \text{Sign}\psi_a(x_1)\psi_b(x_2)\ldots\psi_{\alpha_N}(x_N) \). Further dividing \( \sqrt{N!} \) for normalisation, the complete equation for the probability is therefore

\[ \rho_N(x_1,x_2,\ldots,x_N) = \sum_{i,i'} \psi_N(x_1,x_2,\ldots,x_N; t) = \sum_{i,i'} \prod_{\{a,i\}={a,1}}^{(\alpha_N,N)} \rho_\alpha(x_i), \]

where

\[ \psi_N(x_1,x_2,\ldots,x_N; t) = \text{Sign} \prod_{\{a,i\}={a,1}}^{(\alpha_{N,N})} \psi_a(x_1,t)\psi_b(x_2,t)\ldots\psi_{\alpha_N}(x_N,t), \]

with \text{Sign}= +1 for \( \psi_N \) symmetric and -1 antisymmetric in respect to simultaneous permutations between a pair of coordinates \( x_i \) and \( x_{i'} \).

One common application of the function form \( \psi_N \), a complex \( N \)-particle wave function, is the description of the simultaneous states of many electrons in an atom, or in the atoms of a condensed matter. Multiplying (17) by \( \frac{1}{\psi_a(x_1,t)} \) from the left, summing the equations (17) over all \( \alpha \), reorganising, we obtain a wave equation for \( \psi_N \) as

\[ \dot{\psi}_N + \mathcal{H}_N \psi_N = \mathcal{H}_N \psi_N, \quad \dot{\psi}_N = i\hbar \partial_t, \quad \mathcal{H}_N = -\sum_{(i,\alpha)=(1,)}^{(N,\alpha_N)} \frac{\hbar^2}{m_\alpha} \nabla_i^2 + NY. \]
arrived at $P$ as two events, which are here mutually exclusive and have the probabilities $\rho_{A}(P) = |\psi_{A}(P,t)|^2$ and $\rho_{B}(P) = |\psi_{B}(P,t)|^2$. The probability of the occurrence of at least one of the $n = 2$ two mutually exclusive events is according to the usual addition rule for probabilities given by

$$\rho_{AB}(P) = |\psi_{A}(P,t)|^2 + |\psi_{B}(P,t)|^2. \quad (21)$$

(iii) If the incident particle beam of (ii) consists of ($N$) coherent waves, then the $n = 2$ events are no longer mutually exclusive, but are in a situation as stated by the usual conditional probability ($\rho_{AB} = \frac{1}{2}(\rho_{A}P_{A/B} + \rho_{B}P_{B/A})$); the expressions for $\rho_{AB}$ etc. are subject to how the events specifically depend on one another. Concretely, for the present case we can readily write down the total wave displacement at $P$ as $\psi_{AB}(P,t) = \psi_{A}(P,t) + \psi_{B}(P,t)$. But $|\psi_{AB}(P,t)|^2$ is according to Sec. 2 just the probability $\rho_{AB}(P)$, so

$$\rho_{AB}(P) = |\psi_{AB}(P,t)|^2 = |\psi_{A}(P,t) + \psi_{B}(P,t)|^2. \quad (22)$$

This will predict the interference fringes observed in double slit experiments. For $N$ identical particles as in (ii)-(iii) above, $m_{\alpha} = m$ for all $\alpha$; directly summing the (17)s over all $\alpha$s gives a total wave equation for $\psi_{\Sigma}$ as $i\hbar \frac{\partial \psi_{\Sigma}(x_{1}, t)}{\partial t} = \hat{H}_{1}\psi_{\Sigma}(x_{1}, t), \hat{H}_{1} = -\frac{\hbar^{2}}{m^{2}}\nabla_{x_{1}}^{2} + V'$.

The author expresses thanks to Scientist P.-I. Johansson for his continued moral and private funding support for the research, and to a community of national and international distinguished physicists for giving their invaluable moral support for the research.

References

[1] Dirac P.A.M. 1958 The Principles of Quantum Mechanics, 4th ed, (Oxford: Clarendon Press); Weyl H 1931 The theory of groups and quantum mechanics, English trans. H P Robertson (Mathuen Comp Ltd); Merzbacher E 1970 Quantum Mechanics, 2nd ed, (John Wiley & Sons, Inc.).

[2] Born M 1926 Zur Quantenmechanik der stövorgänge. Zeit fur Physik 37, 863-867; 1954 The statistical interpretation of quantum mechanics Nobel Lecture.

[3] Heisenberg W 1927 Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik, Z. Phys. 43:172; 1930 The Physical Principles of the Quantum Theory, C. Eckart and F. C. Hoyt German to English transl. (Univ. Chicago Press).

[4] Zheng-Johansson J X 2010 Internally electrodynamic particle model: its experimental basis and its predictions Phys. Atom. Nucl. 73 571-581 (Preprint arxiv:0812.3951).

[5] Zheng-Johansson J X and Johansson P-I (a) 2006 Unification of Classical, Quantum and Relativistic Mechanics and of the Four Forces (New York: Nova Science); (b) 2010 Inference of Basic Laws of Classical, Quantum and Relativistic Mechanics from First-Principles Classical-Mechanics Solutions (New York: Nova Sci.); (c) 2006 Inference of Schrödinger equation from classical mechanics solution Quantum Theory and Symmetries IV vol 2 Suppl. Bulg. J. Phys. 33, ed Dobrev VK (Sofia: Heron Press) pp 763-770 (Preprint arxiv:physics/0411345v5); (d) 2006 Developing de Broglie wave Prog. Phys. 4 32-35 (Preprint arxiv:physics/0608265); (e) 2006 Mass and mass–energy equation from classical-mechanics solution Phys. Ess. 19 544 (Preprint arxiv:physics/0510377); (f) (Zheng-Johansson J X) 2006 Spectral emission of moving atom Prog. Phys. 3 78-81 (Preprint arxiv:physics/0606161); (g) (Zheng-Johansson J X) Vacuum structure and potential Preprint arxiv:physics/0704.0131; (h) (Zheng-Johansson J X) Dielectric theory of the vacuum Preprint arxiv:physics/0612096; (i) (Zheng-Johansson J X, Johansson P-I, et al.) 2006 Depolarisation radiation force in a dielectric medium, its analogy with gravity Quantum Theory and Symmetries IV vol 2 Suppl. Bulg. J. Phys. 33 ed Dobrev VK (Sofia: Heron Press), pp 771-779; (Zheng-Johansson J X, Johansson P-I) arxiv:physics/0411245v4; (j) (Zheng-Johansson J X) Doebner-Goldin Equation for electrodynamic model particle, the implied applications Preprint arxiv:0801.4279 (talk at 7th Int. Conf. Symm. in Nonl. Math. Phys., Kyiv, 2007); (k) Zheng-Johansson J X 2008 Dirac equation for electrodynamic model particles J. Phys. Conf. Series 128, 012019, Proc. 5th Int. Symp. Quantum Theory and Symmetries, ed M Olmo (Valladolid, 2007); (l) (Zheng-Johansson J X) 2010 Self interference of single electrodynamic particle in double slit Proc. 6th Int. Symp. Quantum Theory and Symmetries ed A. Shapiro and Das (Lexington, 2009) (Preprint arxiv:1004.5000); (m) (J.X. Zheng-Johansson) 2003 Unification of classical and quantum mechanics & the theory of relative motion, Bull. Amer Phys Soc. G35.001-Gen. Phys. (Austin).