Local vortex nucleation and the surface mode spectrum of large condensates

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A combination of analytical and numerical approaches obtains the complete dispersion curve for surface excitations in a condensate held in a plane linear potential. This improvement on previous approximate results yields an accurate formula for the local Landau critical velocity for vortex nucleation at the surface of a sufficiently large condensate, which agrees very well with recent experiments. The dispersion curve for surface modes at a hard wall potential is also presented, for contrast.

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Critical velocities for vortex nucleation in dilute trapped Bose-Einstein condensates have recently been investigated theoretically [1] and experimentally [2]. It is clear topologically and energetically that unstable surface excitations are the seeds from which vortices grow, and this fixes the characteristic scales of the problem. Quantitative comparison between theory and experiment, however, requires more than order-of-magnitude estimates. A precise but general theory may be based on the idea that the Thomas-Fermi (TF) surface of a large condensate enters a 'bulk regime' in which the physics of vortex nucleation is local. A merit of this local theory is that it can be applied to generic condensate flows, which cannot naturally be described in terms of global surface modes of definite angular momentum.

When vortices are produced 'gently' enough that one can consider them to grow from a quasi-steady state, the local theory of vortex nucleation proceeds in three steps. The first is determining the condensate density and velocity field in the vortex-free steady state, as a function of experimental control parameters such as atom number and stirring frequency. The second is determining the local critical velocity over the TF surface (which in the presence of stirring beams may be multiply connected), and so assessing when and where the surface velocity field may be locally supercritical. The third is computing the rate at which supercritical surface excitations grow into nonlinear vortices. The first step is in general a difficult problem in three-dimensional hydrodynamics, though in some cases it can be performed analytically. The second step is the subject of this paper. The third step again depends on experimental details, such as kinetics of the thermal cloud, strength of stirring potential, and timescale over which control parameters are changed.

Whatever issues arise in steps one and three, the Landau criterion [3] is universal in step two: an energetically unstable surface mode must appear for vortices to enter the cloud. For sufficiently large condensates, such as some in recent experiments, the local Landau critical velocity may be determined by numerically solving one universal equation.

Near the Thomas-Fermi (TF) surface of a large trapped condensate, the trapping potential is approximately linear. If the condensate is large enough, and does not have too extreme an aspect ratio, local physics should also be insensitive to the curvature of the TF surface. So, following Al Khawaja, Pethick and Smith [4] (hereafter AKPS), one can approximate a generic TF surface by replacing the actual trapping potential with a linear ramp \( V = Fx \) for constant \( F \). The equations simplify by rescaling from physical to dimensionless variables, defining \( \bar{x} = \delta^{-1}x_{ph}, t = \tau^{-1}t_{ph}, \psi = \delta \sqrt{8\pi\hbar a_{ph}} \), where \( \delta \) and \( \tau \) are the characteristic surface length and time scales

\[
\delta = \left( \frac{\hbar^2}{2MF} \right)^{1/3}, \quad \tau = \left( \frac{2M\hbar}{F^2} \right)^{1/3},
\]

\( M \) is the particle mass, \( a \) is the s-wave scattering length, and \( \psi_{ph} \) is the macroscopic wavefunction. (The chemical potential factor \( e^{-i\mu_{ph}} \) has been extracted from \( \psi_{ph} \), which is normalized so that \( |\psi_{ph}|^2 \) is the density of particles in the condensate.) In the dimensionless variables the Gross-Pitaevskii equation then reads

\[
\imath \partial_t \psi = -\nabla^2 \psi + x\psi + |\psi|^2 \psi, \tag{2}
\]

where the origin of \( x \) is at the TF surface. Setting \( \partial_t \psi = 0 \) determines the stationary background solution \( \psi = f(x) \), where \( f \) is taken to be real. It may be found to good accuracy numerically [5].

The dispersion relation for surface excitations may be found by solving the Bogoliubov equations for small perturbations \( \psi \rightarrow f + \eta \delta \psi \), with infinitesimal \( \eta \) and

\[
\delta \psi = u(x) e^{i(k \cdot \bar{x}_{ph} - \omega t_{ph})} + v^*(x) e^{-i(k \cdot \bar{x}_{ph} - \omega t_{ph})} = u(x) e^{i(qy - \Omega t)} + v^*(x) e^{-i(qy - \Omega t)}. \tag{3}
\]

Here the excitation’s wave vector is taken to lie along the \( y \) axis, and the dimensionless frequency \( \Omega = \omega \tau \) and wave number \( q = k\delta \) are defined. Note that far outside the TF surface, for \( x \rightarrow \infty, f(x) \propto e^{-\frac{\delta}{2}x^{3/2}} \rightarrow 0 \), and hence
Since it turns out that \( \Omega > q^2 \), this means that the growth of \( q \) does not just lead to vortex formation, but actually is the approach of vortices to the TF surface from infinity.

It is convenient to solve for the dispersion relation \( \Omega(q) \) by iterating the Bogoliubov equations to obtain the fourth order eigenvalue problem for \( s(x) = u - v \):

\[
\Omega^2 s = (\hat{H}_3 + q^2)(\hat{H}_1 + q^2)s
\]

where \( \hat{H}_n \equiv -\partial_x^2 + x + nf^2(x) \).

Discretizing \( x \) and representing the second derivatives with finite differences converts this fourth order ODE directly into a matrix diagonalization problem, which can be solved numerically. Taking \( \Omega(q) \) to be the square root of the lowest eigenvalue \( \Omega^2 \) for given \( q \), one finds the function shown in Figure 1. (For any \( q \) there is obviously a discrete spectrum of \( \Omega^2 \), but the critical velocity is set by the lowest branch.) Numerical solution becomes difficult at small \( q \), where even the lowest lying eigenstate \( s(x) \) extends far enough into the condensate that a large grid is needed to contain it; but fortunately the problem remains easily tractable until the numerical curve has already converged onto the \( \sqrt{q} \) asymptotic behaviour as \( q \to 0 \), found analytically by AKPS [11]. At large \( q \), simple perturbation theory implies that \( \Omega(q) = q^2 + E_{g2} + O(q^{-2}) \), where \( E_{g2} = 1.17 \) is the lowest eigenvalue of \( \hat{H}_2 \). (This is the same as the ‘Popov approximation’ of neglecting the anomalous average in the Bogoliubov equations for \( u \) and \( v \).) Figure 1 shows that the exact dispersion curve approaches these asymptotic forms closely for \( q \lesssim 0.3 \) and for \( q \gtrsim 1.5 \).

For comparison, Figure 1 also shows the approximation of AKPS [11], which in our dimensionless units is

\[
\Omega_{\text{AKPS}}^2 = 2q + 4q^4 \left[ 0.15 - \ln(q) \right],
\]

and was derived to give the first corrections at small \( q \) beyond the leading term \( \Omega^2 \sim 2q \). According to Figure 1, it clearly does very well, tracking the exact dispersion curve closely for \( q \lesssim 0.6 \). This approximation was never intended to be accurate at larger \( q \), however, and it obviously becomes pathological for \( q \gtrsim 1 \); its application to vortex nucleation in [1] is therefore only qualitatively accurate. The recent [11], on the other hand, is based on the high \( q \) regime, in the sense that it fits numerical data to the functional form \( q^2 + \text{const.} \) and so underestimates \( v_c \) by about 15%, because this form is not really valid near \( q_c \). For comparison with [11], in the range \( 0 \leq q \leq 2 \) the numerical result shown in Figure 1 is indistinguishable, on the scale of the plot, from

\[
\Omega_{0-2}^2 = 2q + 1.35 q^3 + 0.711 q^4.
\]

This formula becomes inaccurate at \( q \gg 2 \), where it is better to use the asymptotic Popov result.

![Figure 1. Dimensionless surface mode dispersion relation \( \Omega(q) \) (solid curve). This curve is numerical for \( q > 0.1 \); for \( q < 0.1 \) it is simply \( \sqrt{q} \). The tangent to \( \Omega(q) \) through the origin has slope 2. The approximation of Al Khawaja, Pethick and Smith is shown for comparison. Dotted curves show the asymptotic behaviours \( \sqrt{2q} \) and \( q^2 + E_{g2} \) (see text).](image-url)

The main result which we can derive from our numerical knowledge of \( \Omega(q) \) is an accurate value for the Landau critical velocity due to surface excitations, which in units of \( \delta/\tau \) is simply the slope of the tangent to \( \Omega(q) \) through the origin. Numerically this slope is indistinguishable from 2. This leads to the expression in physical units

\[
v_c = \left( \frac{2\hbar F}{M^2} \right)^{1/3} = \frac{\hbar}{M} \delta^{-1}.
\]

This formula can immediately be obtained as an order of magnitude estimate, of course; but the fact that it is precisely accurate is nontrivial. Despite its simple appearance, I have been unable to derive this result analytically, and it may not actually be exact; but it is accurate at least to within a few tenths of one percent. Another result which we obtain is the wavenumber \( q_c \) at which \( \Omega(q)/q \) is minimized, which is approximately \( q_c \approx 0.89 \). This number is close to \( 2^{-1/6} \); but since \( \Omega(q) \) has rather small curvature near \( q = q_c \), the numerical uncertainty in \( q_c \) is too large to make this identification more than a conjecture.

The applicability of the local surface critical velocity [6] to recent experiments may be assessed as follows. Since \( \delta \) is the characteristic depth to which surface modes of \( q \sim 1 \) penetrate the condensate, the bulk regime requires \( \delta/R = \epsilon \ll 1 \) (where \( R \) is the TF radius of the condensate at its rotational equator) so that surface modes are indeed confined to within the range over which the
harmonic potential is accurately linear. Since $\pi \delta$ is also the characteristic surface wavelength of the critical excitations, neglecting the curvature of the TF surface requires that $\delta$ is much smaller than either radius of curvature. In prolate traps rotated about the long axis, the shortest radius of curvature is just $R$; but in oblate traps the curvature in the axial ($z$) direction is greater, and we need the more stringent condition $\varepsilon = (\omega_R^2 \delta)/(\omega_R^2 R) \ll 1$.

For a rotationally symmetric harmonic trap, the quantity $(q_a R/\delta - 1)$ may be compared to the multipolarity $l_c$ of the rotationally symmetric surface mode, and the surface critical velocity can be converted into the dimensionless critical rotation frequency as a fraction of radial trapping frequency, given in the Thomas-Fermi limit of large particle number $N$ by

$$\chi = \frac{\omega_c}{\omega_p} = 2^{1/3} \left( \frac{\hbar \omega_p}{M \omega_0^2 a^2} \right)^{1/3} (15 N)^{-1/5}. \quad (9)$$

For experiments reported to date, computing $F$ using the TF density profile for the reported numbers of atoms yields

| Experiment  | $N$   | $\varepsilon$ | $q_a R/\delta$ | $\chi$ | $\chi_{\text{obs}}$ |
|-------------|-------|---------------|-----------------|--------|------------------|
| MIT stiff   | $1.5 \times 10^4$ | 0.045         | 19.6            | 0.30   | 0.29             |
| MIT weak    | $5 \times 10^5$   | 0.030         | 29.8            | 0.24   | 0.27             |
| ENS         | $3 \times 10^3$   | 0.11          | 8.4             | 0.46   | 0.65             |
| JILA        | $6 \times 10^6$   | 0.12          | 30.7            | 0.24   | 0.3(5 ± 3)       |
| Oxford      | $2 \times 10^4$   | 0.72          | 9.9             | 0.42   | 0.56             |

The values of $\varepsilon$ (which include oblateness factors of 4 and 8 for the JILA and Oxford traps respectively) indicate that the local surface theory should describe the MIT experiments very well, and should apply to the JILA and ENS experiments with fair accuracy, but is not expected to fit the measurements at Oxford (which were made at various eccentricities, not reflected in the table, and to which the global hydrodynamic analysis of [37] is relevant instead).

The local theory actually agrees even better with the MIT experiments than the table indicates, because with the stirring beam applied the MIT traps were significantly asymmetric, and for the weak configuration even anharmonic. Hydrodynamics in the weaker trap is therefore not analytically tractable, but the stiff MIT trap remains harmonic with the stirring beam applied, and so the hydrodynamic solution for a rotating harmonically trapped condensate [3] yields the non-perturbative density and velocity fields, in the co-rotating frame. One can use this solution straightforwardly, for a given trap eccentricity and rotation frequency, to compute the local condensate velocity $v(\theta, \phi)$ over the whole TF surface, as well as the local surface force $F(\theta, \phi)$ (taking centrifugal effects into account), and hence the local critical velocity $v_c(\theta, \phi)$. The results show that the local flow velocity first reaches the local critical velocity at the poles of the cloud’s shortest axis, so that vortices will first form in these surface regions. One also finds that local criticality is reached in the stiff MIT trap at a slightly lower $\chi$ of 0.285. (Although angular momentum is not conserved in the asymmetric trap, the angle subtended by $2\pi \delta/q_a$ at the vortex nucleation radius (smallest semi-axis) yields the effective $l = 18$ cited in [3].)

The excellent agreement with the MIT experiments is likely due not only to the lower values of $\varepsilon$, but also to the facts that the MIT experiments were most sensitive in detecting vortices (due to condensate size and a ‘slicing’ technique to improve visibility), and applied strong stirring fields (so that the timescales for vortex nucleation were probably shorter for a given degree of supercriticality). As suggested in [6], the nucleation time in the weakly anisotropic ENS trap may be too long for visible vortices to be created, except in the large (and unstable [3]) velocity fields generated by rotations near the quadrupole resonance. Similarly, for the JILA system the $\chi = 0.24$ given by [3] is significantly lower than the lowest value 0.32 at which vortices may have been detected [14]. This discrepancy is enough to motivate further development of the local critical velocity theory, to determine whether $O(\varepsilon)$ corrections due to surface curvature may have large constant factors. It is also worth noting, however, the significant experimental uncertainty due to the indirect detection of small numbers of vortices through changes in the condensate aspect ratio [3].

And since vortex nucleation in the JILA experiments is through equilibration with the rotating thermal cloud, one can estimate the rate of surface instability growth using quantum kinetic theory, and show that vortex nucleation is slow until significantly supercritical velocities are reached. (When the thermal cloud’s mean velocity past the TF surface just exceeds $v_c$, [13] the highest gain over loss is for the mode with $q = q_c$, and is proportional to $(1 - \exp[-2\pi \omega_c (v - v_c)])$, where $\beta$ is the thermal cloud’s inverse temperature in our dimensionless surface units. Since experimental temperatures of $T \sim 100$ nK are much higher than the temperature corresponding to the surface critical velocity, this again indicates that the timescale for vortices to form may be quite long until the critical rotation frequency has been significantly exceeded.) Further study of such timescales is needed, but it appears that the JILA data do not demand any vortex nucleation mechanism other than surface mode excitation by the ‘wind’ of the rotating thermal cloud.

More recent observations at MIT [14] using a narrow, stick-like stirring beam have obtained $\chi$ as low as 0.08, where Eqn. [9] predicts 0.24. As in the rotating stiff trap, however, the local critical velocity must be compared to the actual local fluid velocity created over the entire TF surface, including the surface at the stirring beam. Detailed analysis of the 3D flow induced by the Gaussian beam will require substantial numerical effort, and is beyond the scope of this paper; but simplified sim-
ulations show that fluid velocities near Gaussian beams can easily be many times the velocities of the beams themselves. (The well-known hydrodynamic speed-up factor of 2 for a hard cylindrical stirrer is enhanced because the fluid density drops as it penetrates the beam.) Of course if the beams are too narrow in relation to their strength, the ratio of surface depth to curvature radius on the TF surface may not be sufficiently small for the zeroth order theory presented here to be accurate. On the other hand, if the \( \delta \) on the beam surface is actually smaller than the healing length \( \xi = (4\pi\hbar|\psi_{ph}|^2)^{-1/2} \) near the beam, the linearized theory will also fail, as the beam potential approaches the limit of a hard wall.

The surface mode spectrum of a hard wall can also be computed, if we can again neglect curvature. In this case we have an analytic solution for the stationary back-flow equations.

The linearized theory will also fail, as the linear theory breaks down and the local surface mode dispersion curve lies between \( 2k\xi \) and the bulk dispersion curve \( k\xi \). As shown in Fig. 2, the result is that the surface mode dispersion curve \( \omega(k\xi) \) for a hard wall lies below the bulk dispersion curve, but not by enough to lower the critical velocity below the local speed of bulk sound.

![Figure 2](image)

**Figure 2.** Hard wall surface mode dispersion relation \( \omega(k\xi) \) (solid curve). Results are numerical only for \( q > 0.1 \); below this point the curve is simply \( 2k\xi \). For all \( k \), the surface mode dispersion curve lies between \( 2k\xi \) and the bulk dispersion curve \( k\xi \sqrt{1 + k^2\xi^2} \).

To summarize, in condensates of at least moderate size vortex nucleation at TF surfaces is governed by the local Landau critical velocity obtained from the dispersion relation for Bogoliubov surface modes. This velocity is \( \hbar/(M\delta) \) to high accuracy (not just of that order). Experiments at MIT support this theory well. Failure to see vortices at such low velocities elsewhere is likely due to long nucleation times. For very abrupt potentials, the linear theory breaks down and the local surface mode critical velocity approaches the local speed of bulk sound.

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[1] D.L. Feder, C.W. Clark, and B.I. Schneider, Phys. Rev. A **61**, 011601 (2000).
[2] P.O. Fedichev and G.V. Shlyapnikov, Phys. Rev. A **63**, 045601 (2001).
[3] S. Sinha and Y. Castin, cond-mat/0101292.
[4] A.E. Muryashev and P.O. Fedichev, cond-mat/0106462.
[5] M. Kraemer, L. Pitaevskii, S. Stringari, F. Zambelli, cond-mat/0106524.
[6] K.W. Madison, F. Chevy, W. Wohlleben and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000); K.W. Madison, F. Chevy, V. Breten and J. Dalibard, *ibid.* **86**, 4443 (2001).
[7] J.R. Abo-Shaeer, C. Raman, J.M. Vogels, and W. Ketterle, Science **292**, 476 (2001).
[8] P.C. Haljan, I. Coddington, P. Engels and E.A. Cornell, cond-mat/0106362.
[9] E. Hodby et al., cond-mat/0106262.
[10] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, Oxford, 1980). Ref. [2] discusses the application to condensate surface modes, in a class of pipe-like traps.
[11] U. Al Khawaja, C.J. Pethick, and H. Smith, Phys. Rev. A **60**, 1507 (1999).
[12] F. Dalfovo, L. Pitaevskii and S. Stringari, Phys. Rev. A **54**, 4213 (1996); E. Lundh, C.J. Pethick, and H. Smith, *ibid.* **55**, 2126 (1996). See also D. Margetis, *ibid.* **61**, 055601 (2000).
[13] A. Recati, F. Zambelli, and S. Stringari, cond-mat/0007152; J.J. García-Ripoll and V. Pérez-Garcia, cond-mat/0004351.
[14] The surface mode \( \chi = 0.40 \) cited in [3] is apparently based on the sum rule upper bound of F. Dalfovo and S. Stringari, Phys. Rev. A **63**, 011601(R) (2001), which is not expected to be accurate at the high \( l = 30 \) corresponding to \( q = q_c \) for this system.
[15] Imposing conservation of energy and momentum in collisions that generate surface excitations, one obtains that a surface mode of dimensionless wavenumber \( q \) can only be excited by incident thermal atoms whose momentum component in the direction of the surface mode is \( q_I = [\Omega(q) + q^2]/(2q) \). The same surface mode can only be de-excited by incident thermal atoms of \( q_I = [\Omega(q) - q^2]/(2q) \). For a Boltzmann distribution of thermal atoms \( P \propto \exp[-\beta(q_I - v/2)^2] \), therefore, the ratio between rates of gain and loss can only exceed unity for any surface mode \( q \) if \( v > v_c \). So even though the threshold velocity for an incident atom to excite surface modes at \( q_c \) is higher than \( v_c \), due to atomic recoil, the critical velocity for a thermal cloud of non-interacting atoms is still exactly \( v_c \).
[16] C. Raman et al., cond-mat/0106235.