Stabilization of Deterministically Chaotic Systems by Interference and Quantum Measurements: The Ikeda Map Case

Mauro Fortunato\(^{(a)}\), Gershon Kurizki\(^{(b)}\), and Wolfgang P. Schleich\(^{(a)}\)

\(^{(a)}\) Abteilung für Quantenphysik, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany

\(^{(b)}\) Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

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We propose a method which can effectively stabilize fixed points in the classical and quantum dynamics of a phase-sensitive chaotic system with feedback. It is based on feeding back a selected quantum sub-ensemble whose phase and amplitude stabilize the otherwise chaotic dynamics. Although the method is rather general, we apply it to realizations of the inherently chaotic Ikeda map. One suggested realization involves the Mach-Zender interferometer with Kerr nonlinearity. Another realization involves a trapped ion interacting with laser fields.

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Deterministic (Hamiltonian) chaos in a classical system manifests itself by the extreme (exponential) sensitivity of the evolution to initial conditions, thus making long-time predictions on the dynamical observables of the system practically impossible\(^{(1)}\). Yet, it has been recently recognized\(^{(2)}\) that it is possible to control chaos: by applying specifically designed time-dependent perturbations, regular periodic orbits can be stabilized within chaotic phase-space domains\(^{(3)}\). The quantum dynamics underlying classical chaos has been extensively investigated for some time now\(^{(4)}\). Of particular relevance here is the finding\(^{(5)}\) that quantum spread can cause a quasiclassical state initially confined within regular phase-space regions to eventually access chaotic regions. It is therefore natural to ask: what is the quantum analog of classical chaos control and how can we deal with chaotic behavior induced by quantum spread?

It is our purpose here to propose the quantum control of deterministically chaotic two-dimensional systems, whose evolution is governed by their phase \((\text{mod} \ 2\pi)\), and involves phase-dependent feedback. The proposed control is based on two elements. First, creation of phase correlations between the stabilized system and a similar system acting as a “stabilizer”. If the stabilized and stabilizer systems are classically correlated and are not entangled, then their phase correlations allow us to enforce regular evolution (stabilization) of an inherently-chaotic stabilized system, by feeding it with the output of the stabilizer system. However, whenever these systems are appreciably entangled, this stabilization may not suffice. If we merely ignore (trace out) the stabilizer states, and feed back the resulting statistical mixture of stabilized-system states, then stabilization may eventually fail, because one of the states in the mixture may venture into a chaotic region of phase space (see below). This calls for the second element of proposed control: a quantum measurement of the entangled stabilized and stabilizer systems, which selects a particular state of the stabilized system and feeds it back into this system. The chaotic trend due to entanglement-induced spread is thus suppressed by suitable projections which localize the system in regular phase-space regions. Such manipulation of the evolution, i.e., selection of a certain state corresponding to the desired outcome of the measurement, is the essence of the conditional measurement (CM) approach to quantum state control\(^{(6)}\). The price one has to pay for guiding the evolution by CMs is that one must perform as many trials as implied by the success probability of the required CM until it is accomplished (see below). However, we stress that this price is unavoidable, since there is no other remedy for the chaotic effects of entanglement. Since the quantum version of our scheme is non-unitary, it does not correspond to traditional control theory, in which a force driving the system can be identified. This essential difference stems from our objective, namely, to combat the unwanted effects of entanglement. We finally note that all of the above considerations apply on timescales much shorter than dissipation (decoherence) time.

Although the outlined proposal is quite general, we shall apply it here to systems governed by the Ikeda map\(^{(7)}\). To this end, we shall analyze the Ikeda map quantum-mechanically, focusing on the quasiclassical limit, thereby extending its existing classical analysis. Two different realizations will be considered: (i) a Mach-Zender interferometer with Kerr nonlinearity; (ii) a trapped ion interacting with traveling- and standing-wave fields.

We first consider the following realization of the Ikeda map, which differs from its standard realization in a ring cavity\(^{(8)}\). The basic block of the proposed realization is a Mach-Zender interferometer (MZI) with a Kerr-nonlinear element in one arm, such that the output of one of the ports of this interferometer is fed back into the input port, along with a fixed input field (see block I in Fig.\(^{(9)}\)). The input-output operator transformation for this MZI is
now replaced by \( \cos(\theta) \) feedback. The input ports are interferometers with Kerr nonlinearities (indicated by \( K \)) and stabilizer (Block II) systems: two coupled Mach-Zehnder \( \kappa \) with a 50–50 beam splitter at port \( c \), while in the case of the \( a \) (via the thin-line beam splitter).

\[ \hat{\mathcal{O}}_{\text{total}}(\text{system + stabilizer}) \text{ setup the output} \]

\[ 50–50 \text{ beam splitters, while the 45° is fed back into port} \]

\[ a \] (via the dashed-line beam splitter), while in the case of the \( \hat{\mathcal{O}}_{\text{system}} \) that the feedback parameter \( R \) is 0 (vacuum input in port \( b \))... When the Kerr effect is significant. Upon inverting Eq. (4)

\[ \text{with} \quad \hat{a}_{\text{in}} = a_{\text{in}}/\sqrt{2}, \quad \text{in that the feedback parameter} \quad R \]

\[ 0 \quad \text{large enough} \quad \text{corresponding to excessive feedback, and} \]

\[ \varphi, \kappa \neq 0 \quad \text{see Fig. 3(a)} \).

We intend to show how the chaotic dynamics of this device can be made regular (stabilized) by connecting it to a similar block (block II in Fig. 3) via an interface. The second nonlinear MZI (block II) will then be shown to act as a “stabilizer”. The corresponding output-input operator transformations for the second block are

\[ \left( \begin{array}{c} \hat{r} \\ \hat{q} \end{array} \right) = i e^{i \hat{\theta}_{\text{II}}/2} \left( \begin{array}{cc} \cos \hat{\theta}_{\text{II}}/2 & -\sin \hat{\theta}_{\text{II}}/2 \\ \sin \hat{\theta}_{\text{II}}/2 & \cos \hat{\theta}_{\text{II}}/2 \end{array} \right) \left( \begin{array}{c} \hat{f} \\ \hat{\varepsilon} \end{array} \right), \quad (4) \]

where \( \hat{\theta}_{\text{II}} = \varphi_{\text{II}} + \kappa_{\text{II}}(\hat{f} + i\hat{\varepsilon}) \hat{f}(\hat{f} + i\hat{\varepsilon}) \).

In order to achieve stabilization we impose the conditions

\[ \frac{1}{2} (\varphi_{\text{II}} - \varphi_{\text{I}}) = \frac{1}{2} (\chi_{\text{II}}^0 l_{\text{II}} l_{\text{II}}^0 - \chi_{\text{I}}^0 l_{\text{I}} l_{\text{I}}^0) = \pi/2 - \delta, \quad (5a) \]

\[ \frac{1}{2} (\kappa_{\text{II}} - \kappa_{\text{I}}) = \frac{1}{4} (\chi_{\text{II}}^{(3)} l_{\text{II}} - \chi_{\text{I}}^{(3)} l_{\text{I}}) = 0. \quad (5b) \]

At the level of classical wave optics, when the operators are replaced by \( c \)-numbers, these conditions ensure a fixed, \( j \)-independent (intensity-independent) feedback factor for the \( r \)-mode, namely, \( \hat{R} = \cos[(\hat{\theta}_{\text{II}} - \hat{\theta}_{\text{II}})/2]/\sqrt{2} = \sin \delta/\sqrt{2} \). By choosing \( R \leq 0.1 \) we cause the dynamics to be stable [Fig. 2(b)] at the classical level. This result implies that, classically, stabilization is achievable by additional rotation and phase shift of the two-mode basis, forcing the system from the unstable into the stable domain. These additional rotation and phase shift are caused by an external “device” (the stabilizer) which, although similar to the controlled system, is \textit{not part of its dynamics}, but rather the analog of the Ramsey apparatus which rotates the states of a two-level system \[. \]

If the controlled system is near the edge of the stability domain \( \cos(\theta)/2 \) just slightly larger than 0.14, then the required additional rotation and phase shift \( \hat{\theta}_{\text{II}}/2 \) can be small and amount to a \textit{weak perturbation} of the parameters, as in certain existing control methods \[. \] What singles out our classical stabilization method is that it exploits the interference of two correlated (coherent) evolutions.

The more intricate part of our proposal is concerned with correcting for the \textit{unavoidable} effects of quantum spread and entanglement, which cause the breakdown of the quantum-classical correspondence in phase space, when the Kerr effect is significant. Upon inverting Eq. (4) and using Eqs. (5), it can be readily verified that the output states corresponding to the modes \( r \) and \( q \) will in general be entangled, because of the cross-terms \[a_{\text{in}} b + \text{h.c.}\] and their equivalent \[f_{\text{in}} \hat{\varepsilon} + \text{h.c.}\] in the phase-shift operators \( \hat{\theta}_{\text{II}} \).

We are primarily interested in quasiclassical coherent-state inputs, which can be readily compared to their classical counterparts.
the chaotic-domain and the regular-domain amplitudes. Their counterparts in the other mode, with mean amplitude \( \alpha \) and \( \kappa \) and \( R \), are

\[
|\text{in}\rangle = |\alpha\rangle_a |0\rangle_b = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \alpha^n (\hat{a}^\dagger)^n |0\rangle_a |0\rangle_b .
\]  

We then characterize our input density matrix by using the Q-function quasiprobability distribution, which is diagonal in the coherent-state basis, that is,

\[
\hat{\rho}_F^{(\text{in})} = \int d^2 \alpha d^2 \beta P(\alpha, \beta) |\alpha, \beta\rangle_a, b \langle \alpha, \beta| ,
\]

with \( P(\alpha, \beta) \approx g(\alpha - \alpha_{\text{in}}, \beta - 0) \), \( g \) being a narrow Gaussian. The resulting output, in general, will then be the entangled state

\[
\hat{\rho}_F^{(\text{out})} = \int d^2 \alpha d^2 \beta P(\alpha, \beta) |\text{out}\rangle r, q \langle \text{out}| ,
\]

where the state \( |\text{out}\rangle r, q \) is in general an entangled state of field modes \( r \) and \( q \).

In order to achieve the same stabilization as in the classical limit, our goal is to project out at each iteration a coherent state, and feed it back from port \( r \). It may be difficult to devise such a measurement always, since the entangled state \( |\rangle \) may have a complicated phase-space distribution. We therefore choose to impose an additional condition on the Kerr shift, corresponding to a fractional revival condition on the Kerr shifted Fock states. The condition for a fractional revival is \( \kappa = \kappa_1 + \kappa_{11} = \pi (m/l) \), where \( m \) and \( l \) are any integers. Then, for sufficiently large \( |\alpha| \) (\( |\alpha| \gg 1 \)), state \( |\rangle \) corresponds to an entanglement of a finite number of distinct coherent state components ("Schrödinger kittens") in modes \( r \) and \( q \) with different mean phases and amplitudes. A phase-plane plot of such a state reveals the origin of entanglement-induced chaotic behavior [Fig. 2(b)-inset]: certain components of the field mixture may be in the chaotic domain. Such a state allows us to project out the desired coherent-state component in \( r \) (localized in the regular domain), by a homodyne measurement of the \( q \)-component to which it is correlated. We illustrate this process by imposing the simplest (fractional) revival condition

\[
\kappa = \kappa_1 + \kappa_{11} = \frac{\pi}{2} \quad \Rightarrow \quad e^{i\kappa n^2} = \begin{cases} 1 & n \text{ even} \\ i & n \text{ odd} \end{cases} .
\]

This condition is realizable for Kerr-shifts of photons interacting with atoms in high-Q cavities, and, in a more straightforward fashion, for trapped ions interacting with lasers (see below). Under this condition, we can show by a series of manipulations that Eq. \( (6) \) reduces to

\[
|\text{out}\rangle r, q = \frac{1}{\sqrt{2}} \left[ e^{-i\pi/4} |\alpha e^{i\Phi} \cos \Phi\rangle_q - i\alpha e^{i\Phi} \sin \Phi\rangle_r \\ + e^{i\pi/4} - i\alpha e^{i\Phi} \sin \Phi\rangle_q |\alpha e^{i\Phi} \cos \Phi\rangle_r \right] ,
\]

where \( \Phi = (\varphi_1 + \varphi_{11})/2 \). This entanglement of two-mode Schrödinger-cat states implies that we have 50% probability of detecting the coherent state \( | - i\alpha e^{i\Phi} \sin \Phi\rangle_q \).
Such detection is known to be possible by balanced homodyning, which mixes $| -\alpha e^{i\phi}\sin \Phi \rangle_d$ with the appropriate local oscillator $|i\rangle$. After a successful CM, we perform a dyning, which mixes the resulting state [Eq. (11)] back into the $\alpha$-port. This result reduces [upon using the phase-space distribution Eq. (3)] to its classical counterpart Eq. (2) for $|\alpha| \gg 1$. The above analysis demonstrates the possibility of achieving complete classical-like stabilization by an appropriate CM, thus overcoming the effects of strong quantum entanglement. As stressed above, the price we have to pay is that we have to perform as many trials as implied by the success probability of the CM.

The outlined scheme can be fully implemented in an ion trap, where the ion interacts with both traveling and standing-wave fields. A two-mode classical traveling-wave pulse of area $\theta$ can perform an adiabatic rotation of the basis of two motional modes of the ion by the angle $\theta$: this is achievable via coupling to two orthogonally polarized internal states, or by alternative schemes, based on measurements of the internal states. The rotation of the basis of two motional modes then mimics the effect of beam splitters in the MZIs in Fig. 1. Kerr shift operators of the motional states $\hat{a}^\dagger \hat{a}$ are emulated by an off-resonant standing-wave field, whose $\kappa = \Omega^2 \tau / \Delta$, $\Omega$ being its Rabi frequency, $\tau$ its duration, and $\Delta$ the detuning. Schrödinger-cat motional states have already been prepared in ion traps.

To conclude, we have presented a scheme which enforces regular dynamical evolution towards a stable fixed point in an inherently chaotic system by interference (phase correlations) with a “stabilizer” at the level of classical (wave) optics. At the level of quantum description, the present scheme is the first attempt to remedy the unwanted chaotic effects of quantum spread induced by the system evolution and its entanglement with a “stabilizer” device. This is achieved by a conditional (homodyne) measurement of the stabilizer output, which suppresses entanglement-induced chaos by projecting out a quasiclassical coherent state localized in the regular phase-space region. Conditions for such stabilization have been presented for the quasiclassical analog of the Ikeda map [Eqs. (3), (4)–(11)]. Although the present scheme is quite distinct from existing schemes, it is in the vein of those classical schemes of chaos control in which the phase space is drastically changed, either by an external (driving) force (the Pyragas method and its extensions) or by spectral filtering, for the purpose of stabilization.

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* E-mail: for@physik.uni-ulm.de. Present address: Dipartimento di Matematica e Fisica, Università di Camerino, Via Madonna delle Carceri, I-62032 Camerino (MC), Italy.

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