REPRESENTATION OF $n$-ABELIAN CATEGORIES IN ABELIAN CATEGORIES

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Abstract. Let $\mathcal{M}$ be a small $n$-abelian category. We show that the category of absolutely pure group valued functors over $\mathcal{M}$, denote by $L_2(\mathcal{M}, G)$, is an abelian category and $\mathcal{M}$ is equivalent to a full subcategory of $L_2(\mathcal{M}, G)$ in such a way that $n$-kernels and $n$-cokernels are precisely exact sequences of $L_2(\mathcal{M}, G)$ with terms in $\mathcal{M}$. This gives a higher-dimensional version of the Freyd-Mitchell embedding theorem for $n$-abelian categories.

1. Introduction

For a positive integer $n$, $n$-cluster tilting subcategories of abelian categories was introduced by Iyama [4, 7] (see also [5, 6] and [9]) to develop the higher-dimensional analogs of Auslander-Reiten theory. In these subcategories kernels and cokernels don’t necessarily exist and replace with $n$-kernels and $n$-cokernels. Recently, Jasso [8] introduced $n$-abelian categories which are an axiomatization of $n$-cluster tilting subcategories. He proved that any $n$-cluster tilting subcategory of an abelian category is $n$-abelian.

Freyd in [2] asked the following general question: "Given a category how nicely can it be represented in an abelian category?" Note that in homological algebra it is very convenient to have a concrete abelian category, for that allows one to check the behavior of morphisms on actual elements of the sets underlying the objects. Freyd in [1] proved that for every small abelian category there exists an exact, covariant embedding into the category of abelian groups (see also [12]). This shows that if a statement concerning exactness and commutativity of a diagram is true in the category of abelian groups then it is true in a general abelian category. If an abelian category $\mathcal{A}$ has a projective generator $P$, then there is an exact fully faithful functor from $\mathcal{A}$ to the category of modules over $\text{End}_A(P)$ and if $\mathcal{A}$ has a generating set of projectives, then there is an exact fully faithful functor from $\mathcal{A}$ to the category of modules over a ring with several objects [1, 13]. Mitchell in [14] proved that "every small abelian category admits a full, exact and covariant embedding into a category of $R$-modules for some ring $R$". The fullness of the embedding functor implies that the statement concerning the existence of morphisms in a diagram in a general abelian category is true providing that it is true in the category of modules. Let $\mathcal{A}$ be an abelian category, $(\mathcal{A}, G)$ be the category of additive group valued functors and $L(\mathcal{A}, G)$ be the full subcategory of $(\mathcal{A}, G)$ consisting of all left exact functors. Gabriel proved that $L(\mathcal{A}, G)$ is an abelian category with an injective cogenerator [3] and Mitchell proved that the Yoneda functor $\mathcal{A} \to L(\mathcal{A}, G)$ is exact and fully faithful [14].

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Let $\mathcal{M}$ be a small $n$-abelian category with a generating set of projectives, $\mathcal{P}$ be the category of projective objects in $\mathcal{M}$ and $F : \mathcal{M} \to \text{mod}\mathcal{P}$ be the functor defined by $F(X) = \mathcal{M}(-, X)|_{\mathcal{P}}$. Jasso in [8] proved that if there exists an exact duality $D : \text{mod}\mathcal{P} \to \text{mod}\mathcal{P}^{op}$, then $F$ is fully faithful and the essential image of $F$ is an $n$-cluster tilting subcategory of $\text{mod}\mathcal{P}$. Kvamme in [10] proved that the existence of the exact duality is unnecessary.

In this paper we prove the higher-dimensional analogs of the Freyd-Mitchell embedding theorem for $n$-abelian categories. For a small $n$-abelian category $\mathcal{M}$ we consider the category of additive group valued functors $(\mathcal{M}, \mathcal{G})$ and the representation functor $H : \mathcal{M} \to (\mathcal{M}, \mathcal{G})$. This functor is only left $n$-exact but not $n$-exact (see definition 3.1). To fix this, we find a subcategory $L_2(\mathcal{M}, \mathcal{G})$ of $L(\mathcal{M}, \mathcal{G})$ which is an abelian category consist of the essential image of $H$. We show that the functor $H : \mathcal{M} \to L_2(\mathcal{M}, \mathcal{G})$ is $n$-exact and reflect $n$-exact sequences. Then we can describe each small $n$-abelian category $\mathcal{M}$ as a full subcategory of an abelian category $L_2(\mathcal{M}, \mathcal{G})$ such that left $n$-exact sequences, right $n$-exact sequences and $n$-exact sequences are precisely exact sequences of the abelian category $L_2(\mathcal{M}, \mathcal{G})$ with terms in $\mathcal{M}$. This shows that any statement concerning exactness of a finite diagram, commutativity of a finite diagram and the existence of morphisms in a finite diagram in a general $n$-abelian category is true providing that it is true in the abelian categories.

The paper is organized as follows. In section 2 we recall the definitions of $n$-abelian categories and $n$-cluster tilting subcategories and recall some results that we need in the rest of the paper. For further information and motivation of definitions the readers are referred to [7, 4, 8].

1.1. Notation. Throughout this paper, unless otherwise stated, $n$ always denotes a fixed positive integer, $\mathcal{M}$ is a fixed $n$-abelian category and $\mathcal{G}$ is the abelian category of all abelian groups.

2. $n$-ABELIAN CATEGORIES

In this section we recall the definition of $n$-abelian category and recall some results that we need in the rest of the paper. For further information and motivation of definitions the readers are referred to [7, 4, 8].

2.1. $n$-abelian categories. Let $\mathcal{M}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{M}$. A weak cokernel of $f$ is a morphism $g : B \to C$ such that for all $C' \in \mathcal{M}$ the sequence of abelian groups

$$\mathcal{M}(C, C') \xrightarrow{(g,C')} \mathcal{M}(B, C') \xrightarrow{(f,C')} \mathcal{M}(A, C')$$

is exact. The concept of weak kernel is defined dually.
Let $d^0 : X^0 \to X^1$ be a morphism in $\mathcal{M}$. An $n$-cokernel of $d^0$ is a sequence 

$$(d^1, \ldots, d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

of objects and morphisms in $\mathcal{M}$ such that for all $Y \in \mathcal{M}$ the induced sequence of abelian groups

$$0 \to \text{Hom}(X^{n+1}, Y) \to \text{Hom}(X^n, Y) \to \cdots \to \text{Hom}(X^1, Y) \to \text{Hom}(X^0, Y)$$

is exact [8]. The concept of $n$-kernel of a morphism is defined dually.

**Definition 2.1.** ([11, Definition 2.4]) Let $\mathcal{M}$ be an additive category. A left $n$-exact sequence in $\mathcal{M}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

such that $(d^0, \ldots, d^{n-1})$ is an $n$-kernel of $d^n$. The concept of right $n$-exact sequence is defined dually. An $n$-exact sequence is a sequence which is both a right $n$-exact sequence and a left $n$-exact sequence.

**Definition 2.2.** ([8, Definition 3.1]) An $n$-abelian category is an additive category $\mathcal{M}$ which satisfies the following axioms:

1. (A0) The category $\mathcal{M}$ is idempotent complete.
2. (A1) Every morphism in $\mathcal{M}$ has an $n$-kernel and an $n$-cokernel.
3. (A2) For every monomorphism $d^0 : X^0 \to X^1$ in $\mathcal{M}$ and for every $n$-cokernel $(d^1, \ldots, d^n)$ of $d^0$, the following sequence is $n$-exact:

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}.$$ 

4. (A3) For every epimorphism $d^n : X^n \to X^{n+1}$ in $\mathcal{M}$ and for every $n$-kernel $(d^0, \ldots, d^{n-1})$ of $d^n$, the following sequence is $n$-exact:

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}.$$ 

**Definition 2.3.** ([8, Definition 3.14]) Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ be a generating-cogenerating full subcategory of $\mathcal{A}$. $\mathcal{M}$ is called an $n$-cluster tilting subcategory of $\mathcal{A}$ if $\mathcal{M}$ is functorially finite in $\mathcal{A}$ and

$$\mathcal{M} = \{X \in \mathcal{A} \mid \forall i \in \{1, \ldots, n-1\}, \text{Ext}^i(X, \mathcal{M}) = 0\} = \{X \in \mathcal{A} \mid \forall i \in \{1, \ldots, n-1\}, \text{Ext}^i(\mathcal{M}, X) = 0\}.$$ 

Note that $\mathcal{A}$ itself is the unique 1-cluster tilting subcategory of $\mathcal{A}$.

The following result gives a rich source of $n$-abelian categories.

**Theorem 2.4.** ([8, Theorem 3.16]) Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\mathcal{A}$. Then $\mathcal{M}$ is an $n$-abelian category.
Recall that the mapping cone \( C(f) \) of \( f \) is the following complex

\[
\begin{align*}
X^0 &\xrightarrow{d^0_X} X^1 & \cdots & \xrightarrow{d^{n-2}_X} X^{n-1} & \xrightarrow{d^{n-1}_X} X^n \\
Y^0 &\xrightarrow{d^0_Y} Y^1 & \cdots & \xrightarrow{d^{n-2}_Y} Y^{n-1} & \xrightarrow{d^{n-1}_Y} Y^n \\
\end{align*}
\]

where

\[
d^k_C = \begin{pmatrix} -\frac{d^k_X}{f^k} & 0 \\ \frac{d^{k-1}_Y}{f^k} & 1 \end{pmatrix} : X^k \oplus Y^{k-1} \rightarrow X^{k+1} \oplus Y^k,
\]

for each \( 0 \leq k \leq n \).

**Definition 2.5.** ([8, Definition 2.11]) Let \( M \) be an additive category, \( X : X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \) a complex in \( M \) and \( f^0 : X^0 \rightarrow Y^0 \) a morphism in \( M \). An \( n \)-pushout diagram of \( X \) along \( f^0 \) is a morphism of complexes

\[
\begin{align*}
X^0 &\xrightarrow{d^0_X} X^1 \oplus Y^0 & \cdots & \xrightarrow{d^{n-1}_X} X^{n-1} \oplus Y^{n-1} & \xrightarrow{d^n_X} X^n \\
Y^0 &\xrightarrow{d^0_Y} Y^1 & \cdots & \xrightarrow{d^{n-1}_Y} Y^{n-1} & \xrightarrow{d^n_Y} Y^n \\
\end{align*}
\]

such that the mapping cone \( C = C(f) \) of \( f \) is a right \( n \)-exact sequence. The concept of \( n \)-pullback diagram is defined dually.

**Theorem 2.6.** ([8, Theorem 3.8]) Let \( M \) be an additive category which satisfies axioms (A0) and (A1) of \( n \)-abelian category,

\[
X : X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n
\]

a complex in \( M \) and \( f^0 : X^0 \rightarrow Y^0 \) a morphism in \( M \). Then the following statements hold:

(i) There exists an \( n \)-pushout diagram

\[
\begin{align*}
X^0 &\xrightarrow{d^0_X} X^1 & \cdots & \xrightarrow{d^{n-1}_X} X^{n-1} & \xrightarrow{d^n_X} X^n \\
Y^0 &\xrightarrow{d^0_Y} Y^1 & \cdots & \xrightarrow{d^{n-1}_Y} Y^{n-1} & \xrightarrow{d^n_Y} Y^n \\
\end{align*}
\]

(ii) Suppose, moreover, that \( M \) is an \( n \)-abelian category. If \( d^0_X \) is a monomorphism, then \( d^0_Y \) is also a monomorphism.
In the following crucial proposition we give a necessarily and sufficient conditions for when a complex in an \( n \)-abelian category is a right \( n \)-exact sequence. One direction of the proposition has been proved in [8, Proposition 3.13]. Note that in the proof we do not need to use a good \( n \)-pushout diagram (see [8]).

**Proposition 2.7.** Let \( \mathcal{M} \) be an \( n \)-abelian category and

\[
X : X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}
\]
a complex in \( \mathcal{M} \). Then, for every \( k \in \{0, 1, \ldots, n\} \) and every \( l \in \{1, 2, \ldots, n\} \) there exist morphisms \( g_k^l : Y^l_k \rightarrow Y^{l-1}_k \) (with \( Y^0_k = X^k \)) and \( p_k^{l-1} : Y^{l-1}_k \rightarrow Y^l_{k+1} \) satisfying the following properties:

(i) For every \( k \in \{0, 1, \ldots, n\} \) the diagram

\[
\begin{array}{ccccccccc}
Y^n_k & \xrightarrow{g^n_k} & Y^{n-1}_k & \xrightarrow{g^{n-1}_k} & \cdots & \xrightarrow{g^1_k} & Y^1_k & \xrightarrow{g^0_k} & X^k \\
\downarrow{p^{n-1}_k} & & \downarrow{p^1_k} & & & & \downarrow{p^0_k} & & \\
0 & \xrightarrow{g^n_{k+1}} & Y^n_{k+1} & \xrightarrow{g^{n-1}_{k+1}} & \cdots & \xrightarrow{g^1_{k+1}} & Y^2_{k+1} & \xrightarrow{g^0_{k+1}} & Y^1_{k+1}
\end{array}
\]

commutes.

(ii) The sequence \((g^n_k, \ldots, g^1_k)\) is an \( n \)-kernel of \( f^k \).

(iii) The diagram

\[
\begin{array}{ccccccccc}
Y^n_k & \xrightarrow{g^n_k} & Y^{n-1}_k & \xrightarrow{g^{n-1}_k} & \cdots & \xrightarrow{g^1_k} & Y^1_k & \xrightarrow{g^0_k} & X^k \\
\downarrow{p^{n-1}_k} & & \downarrow{p^1_k} & & & & \downarrow{p^0_k} & & \\
0 & \xrightarrow{g^n_{k+1}} & Y^n_{k+1} & \xrightarrow{g^{n-1}_{k+1}} & \cdots & \xrightarrow{g^1_{k+1}} & Y^2_{k+1} & \xrightarrow{g^0_{k+1}} & Y^1_{k+1}
\end{array}
\]

is an \( n \)-pullback diagram.

Moreover the morphism \([p_k^0, g^2_{k+1}] : X^k \oplus Y^2_{k+1} \rightarrow Y^1_{k+1}\) is an epimorphism for every \( k \) if and only if the complex \( X \) is a right \( n \)-exact sequence. In this case we can choose the objects \( Y^l_k, 1 \leq l \leq n \) and morphisms \( g^l_k, 1 \leq l \leq n \) in such a way that the diagram 2.1 is both \( n \)-pullback and \( n \)-pushout diagram.

(iv) If the complex \( X \) is a right \( n \)-exact sequence and \( k \neq 0 \), then the sequence \(((g^{k-1}_k, \ldots, g^1_k, f^k, \ldots, f^n))\) is an \( n \)-cokernel of the morphism \( g^k_k \).

**Proof.** The first part proved in Proposition 3.13 of [8]. Now assume that for every \( k \)

\[
[p_k^0, g^2_{k+1}] : X^k \oplus Y^2_{k+1} \rightarrow Y^1_{k+1}
\]
is an epimorphism. We show that the complex \( X \) is a right \( n \)-exact sequence. It is clear that \( f^n \) is an epimorphism. Let \( u : X^{k+1} \rightarrow M \) be a morphism such that \( uf^k = 0 \). Since \( X^k \oplus Y^1_{k+1} \rightarrow Y^1_{k+1} \) is an epimorphism, \( ug^1_{k+1} = 0 \). Since by assumption

\[
0 \rightarrow (Y^1_{k+2}, M) \rightarrow (X^{k+1} \oplus Y^2_{k+2}, M) \rightarrow (Y^1_{k+1} \oplus Y^3_{k+2}, M)
\]
is exact, there is a morphism $v : Y_{k+2}^1 \to M$ such that $u = vp_{k+1}^1$ and $vg_{k+2}^2 = 0$. Since $g_{k+2}^1$ is a weak cokernel of $g_{k+2}^2$ there is a morphism $w : X^{k+2} \to M$ such that $v = wg_{k+2}^1$. Now it is easy to see that $u = wf^{k+1}$ and hence $f^{k+1}$ is a weak cokernel of $f^k$. Then the complex $X$ is a right $n$-exact sequence.

**Remark 2.8.** We remind that in the proof of Proposition 3.13 of [8] we can construct the following commutative diagram inductively.

First we construct the right-hand column and then by taking $n$-pullback we construct another columns.

**Lemma 2.9.** Consider the following commutative diagram in an abelian category.

\[
\begin{array}{cccccc}
X^0 & \rightarrow & X^1 & \rightarrow & \cdots & \rightarrow & X^{n-1} & \rightarrow & X^n & \rightarrow & X^{n+1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y_0^1 & \rightarrow & Y_1^1 & \rightarrow & \cdots & \rightarrow & Y_{n-1}^1 & \rightarrow & Y_n^1 & \rightarrow & Y_{n+1}^1 = X^{n+1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y_0^2 & \rightarrow & Y_1^2 & \rightarrow & \cdots & \rightarrow & Y_{n-1}^2 & \rightarrow & Y_n^2 & \rightarrow & Y_{n+1}^2 = 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y_0^{n-1} & \rightarrow & Y_1^{n-1} & \rightarrow & \cdots & \rightarrow & Y_{n-1}^{n-1} & \rightarrow & Y_n^{n-1} & \rightarrow & Y_{n+1}^{n-1} = 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y_0^n & \rightarrow & Y_1^n & \rightarrow & \cdots & \rightarrow & Y_{n-1}^n & \rightarrow & Y_n^n & \rightarrow & Y_{n+1}^n = 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

Diagram 2.2
Diagram 2.3
Assume that the right-hand column is exact and for every $k$ the mapping cone

$$0 \to B_k^n \to B_k^{n-1} \oplus B_{k+1}^n \to \cdots \to A_k \oplus B_{k+1}^2 \to B_{k+1}^1$$

is exact. Then the top row is exact if and only if for every $k$ the morphism $A_k \oplus B_{k+1}^2 \to B_{k+1}^1$ is an epimorphism.

Proof. Since the right-hand column and all mapping cones are exact, all other columns are also exact. Assume that for every $k$ the morphism $A_k \oplus B_{k+1}^2 \to B_{k+1}^1$ is an epimorphism. Consider the following commutative diagram:
We want to show that \( \text{Ker}(f^k) \subseteq \text{Im}(f^{k-1}) \). Let \( a^k \in \text{Ker}(f^k) \), then \( g^k_{k+1}p^k(a^k) = 0 \). Since all columns are exact, \( p^k(a^k) = g^2_{k+1}(b^2_{k+1}) \) for some \( b^2_{k+1} \in B^2_{k+1} \) and so

\[
[p^k, g^2_{k+1}](a^k, -b^2_{k+1}) = 0,
\]

where \([p^k, g^2_{k+1}] : A^k \oplus B^2_{k+1} \rightarrow B^1_{k+1} \) is the induced morphism. Thus there exist \( b^1_k \in B^1_k \) and \( b^3_{k+1} \in B^3_{k+1} \) such that

\[
(a^k, -b^2_{k+1}) = \begin{pmatrix}
-g^1_k \\ 0 \\ g^2_{k+1}
\end{pmatrix}
\begin{pmatrix}
 b^1_k \\ b^3_{k+1}
\end{pmatrix}.
\]

This means that \( a^k = g^1_k(b^1_k) \). Since the mapping cones are exact, \( b^1_k = p^{k-1}(a^{k-1}) + g^2_k(b^2_k) \) for some \( a^{k-1} \in A^{k-1} \) and \( b^2_k \in B^2_k \). Therefore

\[
a^k = g^1_k(b^1_k) = g^1_kp^{k-1}(a^{k-1}) = f^{k-1}(a^{k-1})
\]

and so \( a^k \in \text{Im}(f^{k-1}) \). An easy calculation shows that \( \text{Im}(f^{k-1}) \subseteq \text{Ker}(f^k) \) and the result follows. By the similar argument we can prove the other direction. \( \square \)

3. THE CATEGORY OF GROUP VALUED FUNCTORS

In this section we first provide some preliminaries on the functor category \((\mathcal{M}, \mathcal{G})\) and then we construct the subcategory \( \mathcal{L}_2(\mathcal{M}, \mathcal{G}) \) of absolutely pure group valued functors of \((\mathcal{M}, \mathcal{G})\) which is an abelian category with injective cogenerator.

3.1. \( n \)-exact functors. In this subsection we recall the definitions of left \( n \)-exact, right \( n \)-exact and \( n \)-exact functors from an \( n \)-abelian category to an abelian category and show that a functor is \( n \)-exact if and only if it is both left and right \( n \)-exact.

**Definition 3.1.** [13] Let \( \mathcal{M} \) be an \( n \)-abelian category, \( \mathcal{A} \) an abelian category and \( F : \mathcal{M} \rightarrow \mathcal{A} \) a covariant functor.

(i) \( F \) is called left \( n \)-exact if for any left \( n \)-exact sequence \( X^0 \xrightarrow{f_0^0} X^1 \xrightarrow{f_1^1} \cdots \xrightarrow{f^{n-1}_n} X^n \xrightarrow{f^n_n} X^{n+1} \) in \( \mathcal{M} \), \( 0 \rightarrow F(X^0) \rightarrow F(X^1) \rightarrow \cdots \rightarrow F(X^n) \rightarrow F(X^{n+1}) \) is an exact sequence of \( \mathcal{A} \).

(ii) \( F \) is called right \( n \)-exact if for any right \( n \)-exact sequence \( X^0 \xrightarrow{f^n_0} X^1 \xrightarrow{f^1_1} \cdots \xrightarrow{f^{n-1}_n} X^n \xrightarrow{f^n_n} X^{n+1} \) in \( \mathcal{M} \), \( F(X^0) \rightarrow F(X^1) \rightarrow \cdots \rightarrow F(X^n) \rightarrow 0 \) is an exact sequence of \( \mathcal{A} \).

(iii) \( F \) is called \( n \)-exact if for any \( n \)-exact sequence \( X^0 \xrightarrow{f_0^0} X^1 \xrightarrow{f_1^1} \cdots \xrightarrow{f^{n-1}_n} X^n \xrightarrow{f^n_n} X^{n+1} \) in \( \mathcal{M} \), \( 0 \rightarrow F(X^0) \rightarrow F(X^1) \rightarrow \cdots \rightarrow F(X^n) \rightarrow F(X^{n+1}) \rightarrow 0 \) is an exact sequence of \( \mathcal{A} \).

The contravariant left \( n \)-exact (resp., right \( n \)-exact, \( n \)-exact) functors are defined similarly.

**Proposition 3.2.** Let \( \mathcal{M} \) be an \( n \)-abelian category and \( \mathcal{A} \) an abelian category. A covariant functor \( F : \mathcal{M} \rightarrow \mathcal{A} \) is an \( n \)-exact functor if and only if it is both left and right \( n \)-exact functor.
Proof. If $F$ is left $n$-exact and right $n$-exact, then it is obvious that $F$ is $n$-exact. Now assume that $F$ is $n$-exact. We show that it is right $n$-exact. Let

$$X : X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

be a right $n$-exact sequence in $\mathcal{M}$. First we construct the diagram 2.2 for $X$. In the diagram 2.2 of $X$ the complex

$$Y^n \xrightarrow{g^1_{n+1}} Y^n_{n+1} \xrightarrow{g^1_{n+1}} \cdots \xrightarrow{g^1_{n+1}} Y^1_{n+1} \xrightarrow{g^1_{n+1}} X^{n+1}$$

is an $n$-exact sequence and the mapping cones of $n$-pullback $n$-pushout diagrams are also $n$-exact sequences. Thus applying $F$ they send to exact sequences in $\mathcal{A}$. After applying $F$ we have a commutative diagram 2.3, where $F(X^i) = A^i$ and $F(Y^j_{l}) = B^j_l$. Then by Lemma 2.9 $F(X^0) \rightarrow F(X^1) \rightarrow \cdots \rightarrow F(X^n) \rightarrow F(X^{n+1}) \rightarrow 0$ is an exact sequence. Similarly we can see that $F$ is a left $n$-exact functor. \hfill \Box

3.2. Reminder on basic properties. Let $\mathcal{M}$ be an $n$-abelian category, we denote by $(\mathcal{M}, \mathcal{G})$ the category of all additive functors from $\mathcal{M}$ to the category of all abelian groups. In this subsection we recall some basic properties of the category $(\mathcal{M}, \mathcal{G})$. The reader can find proofs in [I].

A category is called complete if all small limits exist in it. Dually, a category is called cocomplete if all small colimits exist in it. A cocomplete abelian category with a generator in which direct limit of exact sequences is exact, is called a Grothendieck category. The category $(\mathcal{M}, \mathcal{G})$ is a complete abelian category that all limits compute pointwise. Thus it is not hard to see that it is a Grothendieck category. If for every $X \in \mathcal{M}$ we denote the functor $\text{Hom}(X, -) \in (\mathcal{M}, \mathcal{G})$ by $H^X$, then the Yoneda lemma state that $H^X$ is a projective object, and $\Sigma_{X \in \mathcal{M}} H^X$ is a generator for $(\mathcal{M}, \mathcal{G})$.

Theorem 3.3. ([I] Theorem 6.25) In a Grothendieck category with a generator, every object has an injective envelope.

Proposition 3.4. ([I] Proposition 3.37) Let $\mathcal{A}$ be a complete abelian category with a generator. Every object in $\mathcal{A}$ may be embedded in an injective object if and only if $\mathcal{A}$ has an injective cogenerator.

Proposition 3.5. (See [I] Proposition 7.11) If an object $E \in (\mathcal{M}, \mathcal{G})$ is injective, then it is a right $n$-exact functor.

Proof. Let $X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^n \rightarrow X^{n+1}$ be a right $n$-exact sequence. Then we have the following exact sequence in $(\mathcal{M}, \mathcal{G})$

$$0 \rightarrow H^{X^{n+1}} \rightarrow H^{X^n} \rightarrow \cdots \rightarrow H^{X^1} \rightarrow H^{X^0}$$

Since $\text{Hom}(-, E) : (\mathcal{M}, \mathcal{G}) \rightarrow \mathcal{G}$ is an exact functor, applying this functor to the above sequence and using the Yoneda lemma the result follows. \hfill \Box
3.3. The subcategory of mono functors. A right $n$-exact functor is $n$-exact if and only if it carries monomorphisms to monomorphisms. A functor $F \in (\mathcal{M}, \mathcal{G})$ is called a mono functor if it preserves monomorphisms. Thus by proposition 3.5 an injective mono functor is an $n$-exact functor. We denote by $\mathcal{M}(\mathcal{M}, \mathcal{G})$, the full subcategory of $(\mathcal{M}, \mathcal{G})$ consist of all mono functors.

Lemma 3.6. (See [1] Lemma 7.12] Let $M \in \mathcal{M}(\mathcal{M}, \mathcal{G})$, and $M \hookrightarrow E$ be an essential extension of $M$ in $(\mathcal{M}, \mathcal{G})$. Then $E \in \mathcal{M}(\mathcal{M}, \mathcal{G})$.

Proof. Assume that $E$ is not mono. Then there is a monomorphism $f : X^0 \to X^1$ and $0 \neq x \in E(X^0)$ such that $E(f)(x) = 0$. We produce a subfunctor $F$ of $E$ generated by $x$. For every $Y \in \mathcal{M}$, define

$$F(Y) := \{ y \in E(Y) | \text{there is a } h : X^0 \to Y \text{ such that } E(h)(x) = y \}.$$ 

Since $M \subseteq E$ is essential, there is an object $Y^0$ such that $F(Y^0) \cap M(Y^0) \neq 0$. Let $0 \neq y \in F(Y^0) \cap M(Y^0)$. By the construction of $F$ there is a morphism $h : X^0 \to Y^0$ such that $E(h)(x) = y$. Let

$$
\begin{array}{cccccc}
X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n \\
\downarrow h & & \downarrow t & & & & \downarrow & & \downarrow \\
Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n \\
\end{array}
$$

be an $n$-pushout diagram where the top row is right $n$-exact. By Theorem 2.6, $g$ is a monomorphism. Since $M$ is a mono functor $M(g)(y) \neq 0$ and hence $E(g)(y) \neq 0$. On the other hand, $E(g)(y) = E(g)E(h)(x) = E(t)E(f)(x) = 0$, which is a contradiction. $\square$

By the above lemma, $\mathcal{M}(\mathcal{M}, \mathcal{G}) \subseteq (\mathcal{M}, \mathcal{G})$ is a full subcategory closed under subobjects, products and essential extensions. Therefore all results of section 7.2 of [1] are valid. We summarize basic results of section 7.2 of [1] in the following proposition.

An object $M$ of $\mathcal{M}(\mathcal{M}, \mathcal{G})$ is called torsion if for any object $N$ of $\mathcal{M}(\mathcal{M}, \mathcal{G})$, $\text{Hom}(M, N) = 0$.

Proposition 3.7. The inclusion functor $I : \mathcal{M}(\mathcal{M}, \mathcal{G}) \hookrightarrow (\mathcal{M}, \mathcal{G})$ has a left adjoint $\mathcal{M} : (\mathcal{M}, \mathcal{G}) \to \mathcal{M}(\mathcal{M}, \mathcal{G})$ such that for each $F \in (\mathcal{M}, \mathcal{G})$ the morphism $F \to \mathcal{M}(F)$ is an epimorphism. The kernel of this morphism is the maximal torsion subobject of $F$.

$\mathcal{M}(\mathcal{M}, \mathcal{G})$ is not in general an abelian category. There may be a monomorphism in $\mathcal{M}(\mathcal{M}, \mathcal{G})$ which is not a kernel of a morphism in $\mathcal{M}(\mathcal{M}, \mathcal{G})$. To fix this we introduce the subcategory of absolutely pure functors.

Definition 3.8. ([1] Page 144]) A subfunctor $F' \subseteq F$ in $\mathcal{M}(\mathcal{M}, \mathcal{G})$ is said to be a pure subfunctor if the quotient functor $\frac{F}{F'} \in \mathcal{M}(\mathcal{M}, \mathcal{G})$. A mono functor is called absolutely pure if and only if whenever it appears as a subfunctor of a mono functor it is a pure subfunctor. We denote by $\mathcal{L}_2(\mathcal{M}, \mathcal{G}) \subseteq \mathcal{M}(\mathcal{M}, \mathcal{G})$ the full subcategory of absolutely pure functors.
In the following proposition we summarize the basic properties of the category \( L_2(\mathcal{M}, \mathcal{G}) \).

**Proposition 3.9.** The inclusion functor \( I : L_2(\mathcal{M}, \mathcal{G}) \hookrightarrow \mathbb{M}(\mathcal{M}, \mathcal{G}) \) has a left adjoint \( R : \mathbb{M}(\mathcal{M}, \mathcal{G}) \to L_2(\mathcal{M}, \mathcal{G}) \) such that for each \( M \in \mathbb{M}(\mathcal{M}, \mathcal{G}) \) the morphism \( M \to R(M) \) is a monomorphism.

**Proof.** See section 7.2 of [1]. \( \square \)

**Theorem 3.10.** \( L_2(\mathcal{M}, \mathcal{G}) \) is an abelian category and every object of \( L_2(\mathcal{M}, \mathcal{G}) \) has an injective envelope.

**Proof.** See section 7.3 of [1]. \( \square \)

In the following remark we recall that how kernels and cokernels in \( L_2(\mathcal{M}, \mathcal{G}) \) are constructed.

**Remark 3.11.** Let \( L_1 \to L_2 \) be any morphism in \( L_2(\mathcal{M}, \mathcal{G}) \). If \( K \to L_1 \) is a kernel of \( L_1 \to L_2 \) in \((\mathcal{M}, \mathcal{G})\) it is also kernel in \( L_2(\mathcal{M}, \mathcal{G}) \). If \( L_2 \to F \) is a cokernel of \( L_1 \to L_2 \) in \((\mathcal{M}, \mathcal{G})\) then the cokernel in \( L_2(\mathcal{M}, \mathcal{G}) \) is the composition \( L_2 \to F \to \mathbb{M}(F) \to R(\mathbb{M}(F)) \). Thus a morphism in \( L_2(\mathcal{M}, \mathcal{G}) \) is epimorphism if and only if its cokernel in \((\mathcal{M}, \mathcal{G})\) is torsion.

4. REPRESENTATION OF \( n \)-ABELIAN CATEGORIES

In this section first we characterize the category of absolutely pu re functors. Then we give a representation of \( \mathcal{M} \) in \( L_2(\mathcal{M}, \mathcal{G}) \).

**Lemma 4.1.** ([1, Lemma 2.64]) Consider the following commutative diagram in an abelian category with exact columns and exact middle row.

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & B_{11} & B_{12} & B_{13} \\
\downarrow & \downarrow & \downarrow \\
0 & B_{21} & B_{22} & B_{23} \\
\downarrow & \downarrow & \\
0 & B_{31} & B_{32} \\
\downarrow & \\
0 &
\end{array}
\]

Then the top row is exact if and only if the bottom row is exact.

**Theorem 4.2.** (See [1, Theorem 7.27]) A mono functor \( M \in (\mathcal{M}, \mathcal{G}) \) is absolutely pure if and only if whenever \( X^0 \to X^1 \to \ldots \to X^n \to X^{n+1} \) is a left \( n \)-exact sequence in \( \mathcal{M} \), then \( 0 \to M(X^0) \to M(X^1) \to M(X^2) \) is an exact sequence of abelian groups.
Proof. Consider the exact sequence $0 \to M \to E \to F \to 0$, where $E$ is an injective envelope of $M$. First assume that $M$ is absolutely pure and consider an arbitrary left $n$-exact sequence $X^0 \to X^1 \to \ldots \to X^n \to X^{n+1}$. We have the following commutative diagram which satisfies the assumptions of Lemma 4.1.

$$
\begin{array}{ccc}
0 & \to & M(X^0) \\
\downarrow & & \downarrow \\
0 & \to & M(X^1) \\
\downarrow & & \downarrow \\
0 & \to & M(X^2) \\
\downarrow & & \downarrow \\
0 & \to & E(X^0) \\
\downarrow & & \downarrow \\
0 & \to & E(X^1) \\
\downarrow & & \downarrow \\
0 & \to & E(X^2) \\
\downarrow & & \downarrow \\
0 & \to & F(X^0) \\
\downarrow & & \downarrow \\
0 & \to & F(X^1) \\
\end{array}
$$

Since $M$ is absolutely pure, the bottom row is exact. Therefore by Lemma 4.1 the top row is exact and the result follows. Now assume that $M$ has desired property. Thus by Lemma 4.1, for any exact sequence $0 \to M \to E \to F \to 0$, $F$ is a mono functor. Now let $0 \to M \to N \to P \to 0$ be an exact sequence in $(\mathcal{M}, \mathcal{G})$ such that $N$ is a mono functor. We must show that $P$ is also a mono functor. Let $M \to E$ be an injective envelope of $M$ and construct the following commutative diagram.

$$
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & N \\
\downarrow & & \downarrow \\
0 & \to & P \\
\end{array}
$$

The right hand square is pullback and pushout diagram and hence the induced sequence $0 \to N \to P \oplus E \to F \to 0$ is exact. Because $\mathcal{M}((\mathcal{M}, \mathcal{G})$ is extension closed, $P$ is a mono functor and the result follows. □

The representation functor $H : \mathcal{M} \to (\mathcal{M}, \mathcal{G})$ send an object $X \in \mathcal{M}$ to $H^X$ that is a left $n$-exact functor. Thus by the above theorem $H$ factor through the subcategory $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$ and we have the following commutative diagram:
Theorem 4.3. The functor $\tilde{H} : \mathcal{M} \to \mathcal{L}_2(\mathcal{M}, \mathcal{G})$ is an $n$-exact functor.

Proof. Let $X^0 \to X^1 \to \ldots \to X^n \to X^{n+1}$ be an $n$-exact sequence in $\mathcal{M}$. We must show that $0 \to H^{X^{n+1}} \to H^{X^n} \to \ldots \to H^{X^1} \to H^{X^0} \to 0$ is an exact sequence in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$. This sequence is exact if and only if the induced sequence

$$0 \to \text{Hom}(H^{X^0}, E) \to \text{Hom}(H^{X^1}, E) \to \ldots \to \text{Hom}(H^{X^n}, E) \to \text{Hom}(H^{X^{n+1}}, E) \to 0$$

is exact, where $E$ is an injective cogenerator of $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$. By the Yoneda lemma the above sequence is isomorphic to the sequence

$$0 \to E(X^0) \to E(X^1) \to \ldots \to E(X^n) \to E(X^{n+1}) \to 0$$

which is exact, because $E$ is an $n$-exact functor. □

Lemma 4.4. Let $f : X \to Y$ be a morphism in $\mathcal{M}$.

(i) If $H^f : H^Y \to H^X$ is a monomorphism in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$, then $f$ is an epimorphism.

(ii) If $H^f : H^Y \to H^X$ is an epimorphism in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$, then $f$ is a monomorphism.

Proof. (i) Let $X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X$ be an $n$-kernel of $f$. Then applying $\tilde{H}$ we have that $0 \to H^Y \to H^X \to H^{X^{n+1}}$ is an exact sequence in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$. Since the kernel in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$ is coincide with the kernel in $(\mathcal{M}, \mathcal{G})$, it is obvious that $f$ is a cokernel of $X^{n-1} \to X$ and so is epimorphism.

(ii) By Remark 3.11, $H^f : H^Y \to H^X$ is an epimorphism. Then the functor $\frac{\text{Hom}(X, -)}{\text{Hom}(Y, -)}$ is a torsion functor. If $f$ is not a monomorphism, then there is an object $Z$ and a morphism $0 \neq g : Z \to X$ such that $fog = 0$. Then there exists a nonzero natural transformation $\frac{\text{Hom}(X, -)}{\text{Hom}(Y, -)} \to \text{Hom}(Z, -)$. Therefore $\frac{\text{Hom}(X, -)}{\text{Hom}(Y, -)}$ is not torsion which is a contradiction. □

In the following theorem we show that the functor $\tilde{H} : \mathcal{M} \to \mathcal{L}_2(\mathcal{M}, \mathcal{G})$ reflects $n$-exactness.

Theorem 4.5. The functor $\tilde{H} : \mathcal{M} \to \mathcal{L}_2(\mathcal{M}, \mathcal{G})$ satisfies the following statements:

(i) Let $X : X^0 \to X^1 \to \ldots \to X^n \to X^{n+1}$ be a sequence of objects and morphisms in $\mathcal{M}$. If $0 \to H^{X^{n+1}} \to H^{X^n} \to \ldots \to H^{X^1} \to H^{X^0}$ is an exact sequence in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$ then $X$ is a right $n$-exact sequence.
(ii) Let $X : X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ be a sequence of objects and morphisms in $\mathcal{M}$. If $H^{X^{n+1}} \to H^{X^n} \to \cdots \to H^{X^3} \to H^{X^0} \to 0$ is an exact sequence in $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$ then $X$ is a left $n$-exact sequence.

Proof. We only prove the statement (i) and the proof of the statement (ii) is similar.

Since $\tilde{H}$ is faithful, $X$ is a complex in $\mathcal{M}$. Let $X : X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ be a complex in $\mathcal{M}$ and consider the diagram 2.2 for it. Applying the functor $\tilde{H}$ we have the following diagram in the abelian category $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{H}(X^{n+1}) & \longrightarrow & \tilde{H}(X^n) & \longrightarrow & \tilde{H}(X^{n-1}) & \longrightarrow & \cdots & \longrightarrow & \tilde{H}(X^1) & \longrightarrow & \tilde{H}(X^0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{H}(X^{n+1}) & \longrightarrow & \tilde{H}(Y^1_n) & \longrightarrow & \tilde{H}(Y^n_{n-1}) & \longrightarrow & \tilde{H}(Y^1_{n-1}) & \longrightarrow & \tilde{H}(Y^1_1) & \longrightarrow & \tilde{H}(Y^1_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{H}(Y^2_n) & \longrightarrow & \tilde{H}(Y^n_{n-1}) & \longrightarrow & \tilde{H}(Y^2_{n-1}) & \longrightarrow & \tilde{H}(Y^2_1) & \longrightarrow & \tilde{H}(Y^2_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{H}(Y^{n-1}) & \longrightarrow & \tilde{H}(Y^{n-1}_{n-1}) & \longrightarrow & \tilde{H}(Y^{n-1}_{n-1}) & \longrightarrow & \tilde{H}(Y^{n-1}_1) & \longrightarrow & \tilde{H}(Y^{n-1}_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{H}(Y^n_n) & \longrightarrow & \tilde{H}(Y^n_{n-1}) & \longrightarrow & \tilde{H}(Y^n_{n-1}) & \longrightarrow & \tilde{H}(Y^n_1) & \longrightarrow & \tilde{H}(Y^n_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Diagram 4.1

Since the notion of abelian category is self-dual, the dual of Lemma 2.9 is also correct. By assumption, the top row of the diagram is exact and hence by the dual of Lemma 2.9, the morphism $d^k_C = \left[ \begin{array}{c} \tilde{H}(p^0_k) \\ \tilde{H}(g^2_{k+1}) \end{array} \right] : \tilde{H}(Y^1_{k+1}) \to \tilde{H}(X^k) \oplus \tilde{H}(Y^2_{k+1})$ is a monomorphism. By Lemma 4.4 $[p^0_k, g^2_{k+1}] : X^k \oplus Y^2_{k+1} \to Y^1_{k+1}$ is an epimorphism. Therefore by Proposition 2.7 the complex $X$ is a right $n$-exact sequence. \qed

We end this section with the following interesting question which is a goal of our next project.

**Question 4.6.** By the above notations let $\mathcal{L}(\mathcal{M}, \mathcal{G}) \subseteq \mathcal{L}_2(\mathcal{M}, \mathcal{G})$ be the full subcategory of left $n$-exact functors.

Is $\mathcal{L}(\mathcal{M}, \mathcal{G})$ an $n$-cluster tilting subcategory of $\mathcal{L}_2(\mathcal{M}, \mathcal{G})$?

The positive answer of this question shows that any small $n$-abelian category is an exact full subcategory of an $n$-cluster tilting subcategory and "who want more? (S. MacLane)".
5. Applications

Our main result in section 4 implies that any statement about commutativity, exactness and existence of a morphism in a diagram in an \( n \)-abelian category is true if and only if it is true in any abelian category. In this section, by using this fact, we prove some homological lemmas for \( n \)-abelian categories.

Lemma 5.1. Let \( \mathcal{M} \) be an \( n \)-abelian category, \( \mathcal{A} \) an abelian category and \( F : \mathcal{M} \to \mathcal{A} \) a functor.

(i) If \( F \) is a left \( n \)-exact functor and \( f : X \to Y \) is a weak kernel of \( g : Y \to Z \), then \( F(X) \to F(Y) \to F(Z) \) is exact.

(ii) If \( F \) is a right \( n \)-exact functor and \( g : Y \to Z \) is a weak kernel of \( f : X \to Y \), then \( F(X) \to F(Y) \to F(Z) \) is exact.

Proof. We only prove (i), the proof of (ii) is similar. If \( n = 1 \) the proof is obvious. Assume that \( n \geq 2 \). By [8, Proposition 3.7] we have the following left \( n \)-exact sequence

\[
X^0 \to \cdots \to X^3 \to X^2 \to X \to Y \to Z.
\]

Now by applying \( F \), the statement follows. \( \square \)

The following two theorems have been proved in [13] for small projectively generated \( n \)-abelian categories. We prove these theorems for general \( n \)-abelian categories.

Theorem 5.2. (5-Lemma). Let \( \mathcal{M} \) be an \( n \)-abelian category, and

\[
\begin{array}{cccccc}
X^0 & d^0_X & \to & X^1 & d^1_X & \to & X^2 & d^2_X & \to & X^3 & d^3_X & \to & X^4 \\
\downarrow f^0 & \downarrow f^1 & & \downarrow f^2 & \downarrow f^3 & & \downarrow f^4 \\
Y^0 & d^0_Y & \to & Y^1 & d^1_Y & \to & Y^2 & d^2_Y & \to & Y^3 & d^3_Y & \to & Y^4 \\
\end{array}
\]

be a commuting diagram in \( \mathcal{M} \). If \( f^1 \) and \( f^3 \) are isomorphisms, \( f^0 \) is an epimorphism, \( f^4 \) is a monomorphism and one of the following conditions holds, then \( f^2 \) is also an isomorphism.

(i) \( d^i_X \) and \( d^i_Y \) are weak cokernels of \( d^{i-1}_X \) and \( d^{i-1}_Y \) respectively, for \( i = 1, 2, 3, 4 \).

(ii) \( d^i_X \) and \( d^i_Y \) are weak kernels of \( d^{i+1}_X \) and \( d^{i+1}_Y \) respectively, for \( i = 0, 1, 2, 3 \).

Proof. We can construct a small \( n \)-abelian category consist of all objects in the diagram by adding \( n \)-kernel and \( n \)-cokernels. By Lemma 5.1, both rows of the diagram are exact in the corresponding abelian category. Since the five lemma is valid in abelian categories, the result follows by Theorems [13] and [13]. \( \square \)

Theorem 5.3. ((\( n+2 \)) \( \times \) (\( n+2 \))-Lemma). Let \( \mathcal{M} \) be an \( n \)-abelian category and
be a commuting diagram in $\mathcal{M}$ such that all columns are $n$-exact. Then $n + 1$ of the $n + 2$ rows are $n$-exact sequences implies the remainder.

**Proof.** The proof is similar to the proof of Theorem 5.2. □

The following proposition was proved in [8]. As an application of our main result in section 4, we give a much easier proof.

**Proposition 5.4.** Let $\mathcal{M}$ be an $n$-abelian category. Consider the following commutative diagram

\[
\begin{array}{cccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n & \downarrow{f^n} & \\
\end{array}
\]

Diagram 5.1

where the top row is an $n$-exact sequence and $d_Y^0$ is a monomorphism. Then the following statements are equivalent:

(i) The diagram is an $n$-pushout diagram.

(ii) The mapping cone of the diagram is an $n$-exact sequence.

(iii) The diagram is both an $n$-pushout and an $n$-pullback diagram.

(iv) There exists a commutative diagram

\[
\begin{array}{cccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{id} & X^{n+1} \\
\end{array}
\]

whose rows are $n$-exact sequences.
Proof. The equivalence of (i), (ii) and (iii) is obvious.

(i) \Rightarrow (iv). Since \( \tilde{H} : \mathcal{M} \to \mathcal{L}_2(\mathcal{M}, \mathcal{G}) \) is an embedding functor, for any \( X \in \mathcal{M} \) we denote \( \tilde{H}(X) \) by \( X \). Consider the diagram 5.1. By (ii)

\[
X^0 \xrightarrow{d^0_C} X^1 \oplus Y^0 \xrightarrow{d_1^1} \ldots \xrightarrow{d^{n-1}_C} X^n \oplus Y^{n-1} \xrightarrow{d^n_C} Y^n
\]

is an \( n \)-exact sequence. Then

\[
0 \to Y^n \xrightarrow{\tilde{H}(d^0_C)} X^n \oplus Y^{n-1} \xrightarrow{\tilde{H}(d^{n-1}_C)} \ldots \xrightarrow{\tilde{H}(d^1_C)} X^1 \oplus Y^0 \xrightarrow{\tilde{H}(d^n_C)} Y^0 \to 0
\]

is an exact sequence in \( \mathcal{L}_2(\mathcal{M}, \mathcal{G}) \). Since the top row is exact in \( \mathcal{L}_2(\mathcal{M}, \mathcal{G}) \) and the mapping cone is exact, the bottom row is also exact. Therefore, by Theorem 4.5, the bottom row is an \( n \)-exact sequence in \( \mathcal{M} \).

(iv) \Rightarrow (ii) is similar. \( \square \)

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