Killing Horizons and Spinors

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We study the near horizon geometry of generic Killing horizons constructing suitable coordinates and taking the appropriate scaling limit. We are able to show that the geometry will always show an enhancement of symmetries, and, in the extremal case, will develop a causally disconnected “throat” as expected. We analyze the implications of this to the Kerr/CFT conjecture and the attractor mechanism. We are also able to construct a set of special (pure) spinors associated with the horizon structure using their interpretation as maximally isotropic planes. The structure generalizes the usual reduced holonomy manifold in an interesting way and may be fruitful to the search of new types of compactification backgrounds.

Keywords: Kerr-CFT, Extremal Black Hole, Attractor Mechanism, SUSY and Reduced Holonomy
I. INTRODUCTION

The emergence of the supergravity equations of motion as the requisite of conformal invariance of the worldsheet of the superstring \[1, 2\] may be one of the most powerful ideas proposed by string theory. It addresses at once the problem of background independence of the superstring \[3\], if partially, and provides a geometrical view of the renormalization group (RG) flow of a number of theories which have $\sigma$-models as an effective description at some point in coupling space \[4\]. Also, it provides a definition of the “off-shell” string as a $\sigma$-model away from the conformal fixed point \[5, 6\].

The RG flow of two-dimensional field theories is moderately well-known and their fixed points are known to follow from maximization principles \[7\]. The geometrization of the flow provided by string theory allows one to treat one of the directions of the target manifold as a “cut-off” (Liouville) direction, directly controlling the energy of the exchanges in the effective field theory description. In this sense, one can treat the RG flow as “displacement” in that Liouville direction, with the effective field theory living in the transverse dimensions.

One can then tackle, using geometrical tools, the RG flow of a variety of quantum field theories – at least those which arise from a background of string theory. Versions of the “c-theorem” in higher dimensions \[8, 9\] were the first application of this idea. Later, it was used to argue for an appearance of a mass gap in thermal backgrounds in supersymmetric QCD \[10\]. In this geometric context, the predictability of the RG flow stems in large part from certain algebraic structures of the target space. These structures can be used to “protect” sectors of the dual theory and talk about fixed points at the end of the flow process. In the geometrical description (here dubbed the “gravity side”), the RG flow process occurs in the “radial direction”, and has two natural endpoints: i) the fixed point of some rotational symmetry (“$r = 0$”), or ii) an event horizon of a Black Hole. Studying the near horizon geometry of the black hole then becomes necessary for a description of the theory at the IR fixed point.

The foremost example of such algebraic structures arise in compactifications that preserve $N = 1$ supersymmetry in four dimensions \[8, 9\]. This requisite imposes that the compactified space has a complex structure of the Calabi-Yau type \[11\]. One can then use arguments based on supersymmetry to predict that certain properties of the theory at the ultra-violet will not be spoiled at low energies. Two questions arise at this point: i) whether non-renormalization
theorems can be generalized to – possibly – non-supersymmetric backgrounds, and ii) which geometrical features of the target space are necessary in order to claim that a sector will be protected under the RG flow.

From the “gravity side”, such structures have been less well studied. Supersymmetric structures have deep ties with special holonomy manifolds [12], but the geometrical condition for no mass gap is weaker. For instance, the first version of the Kerr/CFT correspondence [13–15] hinted that a local conformal symmetry should occur for any extreme (zero-temperature) black hole. In fact, the global part of the symmetry was found in an earlier work by Bardeen and Horowitz [16]. The Kerr/CFT correspondence showed that the local version of the symmetry provides a suitable “gauge slice” of the diffeomorphism invariance, which can be assigned to physically independent, macroscopic distinguishable degrees of freedom. Much of the initial properties were model-dependent, but later work by the authors [17] showed that only basic facts, like the laws of black hole thermodynamics (BHT), were necessary to construct the conformal symmetry, again in the extremal case.

The present work explores further this result from the $\sigma$-model side (“gauge side”), locating the type of algebraic structures responsible for the protection of (part of) the spectrum under the RG flow. Some geometrical results are needed along the way. In particular, we will show that generically (i.e. independently of the Lagrangean of the theory), the set of conserved charges associated with remaining symmetries – including Kaluza-Klein charges – are set on the horizon independently of their value at infinity. We will dwell on these issues in sections 2, 3 and 4. Furthermore, their boundary values satisfy a maximum principle, stemming from the first law of BHT. This can been interpreted as a geometrical verification of the “attractor mechanism” [18–22].

The latter part of the paper (section 5 onwards) is dedicated to the structures responsible for the existence of the protected sector. In order to relate them with the known reduced holonomy case, we will need to present some elementary notions of algebraic geometry and spinors (section 5). We tie the discussion (section 6) with some prospects for the construction of relevant backgrounds.
II. KILLING HORIZONS

As stated above, the endpoint of the RG flow process in the geometrical picture may be, in general, an event horizon of a black hole. In this section we will give a geometrical description of this fact, and derive an analogue of the zeroth law of the black hole thermodynamics (BHT) in a Lagrangean-independent way.

The event horizon of a black hole is formally defined as the boundary of the causal past of an observer sitting at (null or spacelike) infinity \[23\]. Intuitively, this notion entails the idea that, far enough in the future, the geometry is sufficiently “settled” (strong asymptotic predictability) that one can speak about a particular region of space.

As always, in order to be easily workable, the notion of event horizon presupposes the existence of a symmetry of the spacetime. Thus, as it is usual, we consider for the sake of workability event horizons that are Killing horizons as well. Observe however that this strategy is not of itself without hurdles. In fact, even the existence of conserved charges like the mass and angular momentum is not automatic in general relativity. It also asks for the existence of suitable symmetries like time translation and rotation around an axis in order to be unambiguously defined.

Much of the ensuing discussion will follow an essay by Carter \[24\], with a few extra added results which are relevant for our construction. Newer treatments can be found in \[25\] and \[26\]. To the best of our knowledge, a derivation of the laws of black hole thermodynamics in generic dimensions based solely on the concept of Killing horizon and integrability like the one we present here is new. Of course, knowledge of some specific form of the dynamics will be necessary to study the near-horizon geometry, and the conclusion that the notion of algebraic speciality plays an important role there was anticipated by the work of R. Milson and collaborators, as for instance \[27\].

Let us begin by recalling the usual four-dimensional case.

We denote by \(k^a\) and \(m^a\) the time-translation and axial rotation Killing vector fields (KVFs), respectively. By default, these vectors are tangent to integral lines of functions \(t\) and \(\phi\) and are commuting \([k^a, m^b] = 0\). The event horizon will be defined as the region where a combination of these vectors becomes null. From the commuting condition, we have that

\[
\begin{align*}
k_a &= -V(dt)_a + W(d\phi)_a, \\
m_a &= W(dt)_a + X(d\phi)_a
\end{align*}
\] (1)
The event horizon is then the region where a combination of the inner products become null, that is, \( V = k^ak_a, X = m^am_a \) and \( W = k^am_a \) are such that

\[
\rho = VX + W^2 \to 0,
\]

where \( \rho \) can be thought of as the determinant of the restriction of the metric to the plane formed from \( k^a \) and \( m^a \). When \( \rho \) becomes zero, this plane becomes null and finite acceleration timelike curves cease to exist in it. The most important result that follows is that the combination of \( k^a \) and \( m^a \) that becomes null at this point,

\[
\chi^a = k^a + \Omega_H m^a,
\]

is actually also a Killing vector field. To see that, define \( \Omega_H = -W/X \) and consider

\[
X^2 d\Omega_H = -XdW + WdX.
\]

Note that

\[
X^2 d\Omega_H \wedge dt \wedge d\phi = X^2 d\Omega_H \wedge k \wedge m = \rho^2 dm \wedge m.
\]

Now, given that the far right hand side vanishes, then \( d\Omega_H \wedge k \wedge m = 0 \). This means that \( d\Omega_H \) is parallel to \( k^a \) and \( m^a \), but this is not possible since \( k^a \) and \( m^a \) are Killing vector fields and \( \Omega_H \) is a scalar function constructed out of the metric. The quantity \( \Omega_H = -W/X \) is thus constant throughout the horizon. It corresponds to its angular velocity.

The vector \( \chi^a \) is orthogonal to the horizon: the axial rotation vector \( m^a \) is normal to it. Also, since it is orthogonal to a surface of constant \( \rho \), it is also hypersurface orthogonal there, satisfying Frobenius’ condition

\[
\chi \wedge d\chi = \chi_{[a} \nabla_b \chi_{c]} = 0 \quad \text{at the horizon.}
\]

We will see that this condition poses a powerful constraint on the geometry near the horizon in the general case.

The conditions above are suitably generalized to higher dimensional spacetimes.

Consider an even \( D \)-dimensional spacetime with \( D/2 \) commuting (dual) KVFs, denoted
by \{\Psi_i\}. This setup corresponds to the physical situation where one has a “vacuum” space-
time which allows for \(D/2\) independent conserved quantities. In more dimensions they will
add up to the total mass, the (Weyl subalgebra of) the angular momenta, as well as charges
arising from Kaluza-Klein compactifications, be they abelian or non-abelian.

In general, one considers the problem of finding \(D/2\) coordinates \(\{\phi^i\}\) covering the orbits
of each of those KVFs. In these coordinates, the vectors associated to the \(\Psi_i\) will be \(\partial/\partial\phi^i\).
The condition that such \(\{\phi^i\}\) exist is actually a zero curvature problem. Indeed, if there are
\(\lambda_{ij}\) functions such that
\[
\Psi_i = \lambda_{ij} d\phi^j,
\]
then
\[
d\Psi_i = \mu_{ik} \wedge \Psi_k. \tag{7}
\]
The “connection” \(\mu_{jk}^k\) is actually a flat one, since the above two equations imply that
\[
\mathcal{F}_j^k = d\mu_{jk}^k + \mu_{jl}^l \wedge \mu_{jk}^k = 0, \quad \mu_{jk}^k = (\lambda^{-1})^{kl} d\lambda_{lj}.
\]
The vanishing of \(\mathcal{F}_j^k\) can be interpreted as an existence condition for the \(\lambda_{ij}\) (also known
as flat parametrization). The equation (7) is called Frobenius’ theorem for dual (covariant)
 vectors. In contrast to the usual connections, the \(\lambda_{ij}\) are the components of the matrix of
inner products between the \(\Psi_i\) and are actually symmetric, and thus they do not form a
group. Specifically, the separation between the orbits of the KVFs \(\{\phi^i\}\) and the transverse
space form a sort of a bundle which is not a principal one.

### A. Compatible Killing Vector Fields

Defining the killing horizon as the region where one combination of the KVFs become
null allow us to probe the near horizon geometry. In essence, the situation is not much
different from the Euclidian case where any isometry near a fixed point can be seen as a
rotation.

A null eigenvector of \(\lambda_{ij}\) implies that one combination of the 1-forms \(\Psi_i\) becomes null.
As such the coordinates \(\{\phi^i\}\) are not suitable for describing the horizon itself. Since \(\lambda_{ij}\) is
symmetric, one can decompose it as

$$\lambda = zP + X,$$  \hspace{1cm} (8)

where $z$ is a selected eigenvalue, which the one going to zero at the horizon, $P$ is the (orthogonal) projector to the would-be ($z = 0$) null eigenspace, and $X = (1 - P)\lambda(1 - P)$ is the restriction of $\lambda$ to the orthogonal subspace.

Define a particular combination of KVFs

$$\chi = \Omega^i \Psi_i = \Omega_i (d\phi^i), \quad \text{with} \quad \Omega_i = \lambda_{ij} \Omega^j.$$  \hspace{1cm} (9)

If $\chi$ is to be a KVF, then the $\Omega^i$ have to be constant. Now, consider

$$\chi \wedge d\chi = -\frac{1}{2}(\Omega_i d\Omega_j - \Omega_j d\Omega_i) \wedge d\phi^i \wedge d\phi^j$$  

$$= -\frac{1}{2}(\Omega_i d\Omega_j - \Omega_j d\Omega_i)(\lambda^{-1})^{ik}(\lambda^{-1})^{jl} \wedge \Psi_k \wedge \Psi_l,$$  \hspace{1cm} (10)

the last step being necessary since the coordinates $\{\phi^i\}$ are not all valid at the horizon. The inverse of (8) is

$$\lambda^{-1} = \frac{1}{z} P + Y, \quad \text{where} \quad Q \equiv XY = YX = 1 - P.$$  \hspace{1cm} (11)

Substituting this back to the last line of (10) and using that both $\chi$ and $\Psi_i$ are well-defined at $z \to 0$, we find that

$$P^{ik}(\Omega_i d\Omega_j - \Omega_j d\Omega_i) P^{jl} = \mathcal{O}(z^2), \quad P^{ik}(\Omega_i d\Omega_j - \Omega_j d\Omega_i) Y^{jl} = \mathcal{O}(z).$$  \hspace{1cm} (11)

The first condition is trivial if the rank of $P$ is one. Let us consider this case first, and comment about what happens in general. Assuming that $Y^{ij}$ is non-degenerate on the orthogonal subspace, the second condition implies

$$P^{ik}(\Omega_i d\Omega_j - \Omega_j d\Omega_i) Q^{jl} = \mathcal{O}(z),$$  \hspace{1cm} (12)

which, specializing to the case where $\chi$ is actually the null KVF at the boundary – we can
take $P$ to project onto the first component, and $Q$ onto the others – results in

$$d \left( \frac{\Omega_i}{\Omega_1} \right) = \mathcal{O}(z)$$

(13)

which means that the $\omega_i = \Omega_i/\Omega_1$ are constant over the horizon. Apart from the temperature, which we discuss later, this implies the zeroth law of BHT, and does not depend on details of the dynamics, just on the presence of a Killing horizon.

If the rank of $P$ is greater than one, there will be families of conserved intrinsic quantities,

$$\Omega_\mu d\Omega_i - \Omega_i d\Omega_\mu = \Omega_\mu^2 d \left( \frac{\Omega_i}{\Omega_\mu} \right) = \mathcal{O}(z)$$

where $\mu$ ranges over the image of $P$ and $i$ over the orthogonal complement. This alters little the discussion that follows, but the differences, especially when defining coordinates, will be outlined in the appropriate sections.

Going back to the case where $P$ is of rank one, the conditions (11) can be solved by requiring that all $d\Omega_i$ are parallel, that is, there is a function $\beta$ such that $d\Omega_i = \Omega_i d\beta + \mathcal{O}(z)$. Thus,

$$\nabla_a \chi_b = d\Omega_i \wedge d\phi^i = d\beta \wedge \Omega_i (d\phi^i) + \mathcal{O}(z) = d\beta \wedge \chi + \mathcal{O}(z),$$

(14)

that is to say, $\chi$ is hypersurface orthogonal at the horizon ($z = 0$). By contracting the equation above with $\chi$ itself, one arrives at

$$\beta = \log |\chi^a \chi_a| + \mathcal{O}(z).$$

(15)

So far we have only shown that the “potentials” $\omega_i$ are constant over the horizon. We will postpone the discussion about the “temperature” $\kappa$ to the next subsection. With this provision, the discussion above shows that the existence of a Killing horizon gives enough constraints on the “intrinsic” quantities (like the temperature, angular velocities and others associated with Kaluza-Klein charges) that render them constant over the horizon. We have thus arrived at a Lagrangean independent formulation of the zeroth law of BHT, as anticipated by Carter.
B. Hypersurface Orthogonal Killing Vector Fields

Let us discuss (14) in more detail. Suppose that we have a spacetime with a Killing vector field (KVF) \( \chi^a \). Let us assume, for simplicity, that this KVF is hypersurface orthogonal, \( \chi_a \nabla_b \chi_c = 0 \). The preceding discussion shows that this holds “to order \( z \)” near the Killing horizon, so every equation in this section will be valid up to \( \mathcal{O}(z) \).

From the Killing equation \( \nabla_a \chi^a \chi^b = 0 \), we obtain the identity

\[
\nabla_a \nabla_b \chi_c = R_{cabd} \chi^d. \tag{16}
\]

Now, Frobenius’ theorem implies that

\[
\chi^a R_{a[bc]} = 0. \tag{17}
\]

Einstein manifolds (\( R_{ab} = \frac{R}{\mathcal{V}} g_{ab} \)) satisfy naturally this condition, but of course this condition is less restrictive.

Given the KVF \( \chi^a \), we define its norm \( z \) and its gradient \( N_a \), respectively, as

\[
\chi^a \chi_a = z, \quad N_a = \frac{1}{2} \nabla_a z = -\chi^c \nabla_c \chi_a. \tag{18}
\]

We then note that \( N_a \) is basically the acceleration of the integral curves of \( \chi^a \) with \( N^a \chi_a = 0 \). The norm of \( N^a \) is

\[
N_a N^a = -\kappa^2 z. \tag{19}
\]

Later we interpret \( \kappa^2 \) as the “surface gravity”. We note that \( \kappa \) is imaginary if both \( \chi^a \) and \( N^a \) are space-like.

As we assumed that the KVF is hypersurface orthogonal, its covariant derivative can be calculated readily \( [23] \) as

\[
\nabla_a \chi^b = -\frac{\chi^c \nabla_b (\chi^c \chi_a)}{\chi^d \chi_d} = -\frac{2}{\alpha} \chi^c \nabla_c N_b. \tag{20}
\]

This, in turn, means that

\[
(\nabla^a \chi^b) (\nabla_a \chi_b) = \frac{2}{z^2} (\chi^a \chi_a) (N_b N^b) = -2\kappa^2 \tag{21}
\]
The normalized bivector normal to surface levels of $z$,

$$\epsilon_{ab} = \frac{1}{\kappa} \nabla_a \chi_b = -\frac{2}{\kappa z} \chi_{[a} N_{b]}, \quad \text{with} \quad \epsilon^{ab} \epsilon_{ab} = -2, \quad (22)$$

is a purely geometrical quantity which is well-defined at the horizon $H$ ($z = 0$) even in the extremal limit $\kappa \to 0$. Using the identity (16), we can also compute the covariant derivative of $N_a$,

$$\nabla_a N_b = -\frac{\kappa^2}{\alpha} \left( \chi_a \chi_b - \frac{1}{\kappa^2} N_a N_b \right) + \chi^c R_{cabd} \chi^d = -\kappa^2 \epsilon_{ac} \epsilon^b_c + \chi^c R_{cabd} \chi^d. \quad (24)$$

From the above facts, we can compute with a little effort, the following formulas,

$$\nabla_a \kappa^2 = -\kappa \chi_b R_{abcd} \epsilon^{cd} = \Sigma N_a - \kappa \omega_{ab} \chi_b, \quad (25)$$

$$\nabla_{[a} \epsilon_{bc]} = -\frac{1}{2\kappa^2} \epsilon_{[ab} \nabla_c] \kappa^2 + \frac{1}{\kappa} R_{[cba]d} \chi^d = -\frac{1}{2\kappa^2} \epsilon_{[ab} \omega_{c]\rangle d} \chi^d, \quad (26)$$

where in the final step, Bianchi identity $R_{[abc]}d = 0$ was used, and the quantities $\Sigma$ and $\omega_{ab}$ are defined in terms of the decomposition

$$R_{abcd} \epsilon^{cd} = -\Sigma \epsilon_{ab} + \omega_{ab}; \quad \omega_{ab} = \omega_{[ab]}, \quad \text{and} \quad \omega_{ab} \epsilon^{ab} = 0. \quad (27)$$

We thus see that $\Sigma$ is associated with the area of the plane defined by $\epsilon_{ab}$ (it may be thought of as the generalization of the expansion parameter of congruences of curves) $\omega_{ab}$ is associated with the parallel transport perpendicular to this plane (it may be thought of as the generalization of the rotation or twist parameter of congruences of curves).

Movement normal to the surfaces with constant $z$ is generated by a vector parallel to $N^a$, properly normalized, so that

$$n^a \nabla_a z = 1 \longrightarrow n^a = -\frac{1}{2\kappa^2 z} N^a. \quad (28)$$

We think of $n^a$ as $\partial/\partial z$. Then, with the above formulas, we can compute the Lie derivative
of $\sigma_{ab}$ with respect to $n^a$ – recall that $\mathcal{L}_\xi \sigma = \xi \cdot d\sigma + d(\xi \cdot \sigma)$, arriving at

$$
\mathcal{L}_n \sigma_{ab} = -\frac{\Sigma}{4\kappa^2} \sigma_{ab} + \frac{1}{4\kappa^2} \left( \omega_{ab} + \frac{2}{z} \chi_{[a} \omega_{b]c} \chi^c \right) = -\frac{\Sigma}{4\kappa^2} \sigma_{ab}. \tag{29}
$$

Note that the vanishing of the term between brackets is a result of the Bianchi identity $R_{[abc]d} = 0$. From the derivative of $\kappa^2$, (25), we obtain

$$
\mathcal{L}_n \kappa^2 = \frac{1}{2} \Sigma. \tag{30}
$$

This, along with (29), means that the tensor $\kappa \sigma_{ab}$ is independent of $z$. We will use this fact in the next subsection to derive the approximate form of the metric near the horizon. For now, we consider the term inside the brackets in (29), rewritten as

$$
\chi_{[a} \omega_{b]c} \chi^c = -\frac{z}{2} \omega_{ab} \tag{31}
$$

which can be translated to

$$
\omega_{ab} \chi^b \propto \chi_a + \mathcal{O}(z). \tag{32}
$$

This has a deep implication to (25). The derivative of the surface gravity $\nabla_a \kappa$ points in the $\epsilon_{ab}$ plane, up to terms of order $z$. It is thus constant over the horizon. Later we will see that $N_a$ actually becomes proportional to $\chi_a$ at the horizon, but for now this fact suffices to finish the proof on the zeroth law of BHT.

Let us now introduce the null vectors

$$
\xi^a_{\pm} = \frac{1}{\kappa} N^a \pm \chi^a. \tag{33}
$$

After some algebra, we may show that

$$
\nabla_a (\xi_{\pm})_b = -\kappa \epsilon_{ac} \epsilon^e_b + \frac{1}{\kappa} \left( \delta^e_b + \frac{1}{\kappa^2 z} N^e N_b \right) \chi^c R_{caded} \chi^d \pm \kappa \epsilon_{ab}. \tag{34}
$$
and then

\[ \xi^b_+ \nabla_b \xi^a_+ = -2\kappa \xi^a_+ - zh^{ab} \nabla_b \kappa, \quad (35) \]

\[ \xi^b_- \nabla_b \xi^a_- = -zh^{ab} \nabla_b \kappa, \quad (36) \]

where \( h_{ab} = g_{ab} - \epsilon^c_a \epsilon_{bc} \) is the “angular part” of the metric. The last term in the above equation projects out the \( N^a \) component of the derivative of \( \kappa \).

Let us define the vectors

\[ l^a = \frac{1}{2} \xi^a_+ \quad \text{and} \quad n^a = \epsilon^a_b \xi^b_+, \quad (37) \]

so that both have well-defined limits as we take \( z \to 0 \). Such limit is easily computed for \( l^a \). Indeed, since \( N^a \) becomes null at the horizon, and it is always orthogonal to \( \chi^a \), it must be proportional to \( \chi^a \) itself. From the definition of \( \kappa \), we can easily see that

\[ l^a = \chi^a \quad \text{at the horizon.} \quad (38) \]

The limit of \( n^a \) is a little more subtle. From its definition \((34)\), it can be thought of to be proportional to \( \xi^a_- \), but in such a way that it does not vanish at the horizon. Indeed, recall that \( \epsilon_{ab} = 2n_{[a}l_{b]} \) everywhere. From the definition of \( \xi^a_\pm \), \((33)\), we obtain

\[ n^a = \frac{1}{\alpha} \left( \frac{1}{\kappa} N^a - \chi^a \right) \quad \text{at the horizon.} \quad (39) \]

In other words, it is a vector normal to the horizon in such a way that \( n^al_a = -1 \).

Since \( \kappa \) is constant at the horizon \( \mathcal{H} \), we find, by inspecting \((34)\) that \( l^a \) is tangent to null geodesics if \( \kappa \neq 0 \) at the horizon. The “angular” component of the derivative of \( \kappa \) will tell us the transverse acceleration of these null geodesics.

Looking at \((25)\), we learn that the condition that these null vector fields are geodesic is equivalent to the vanishing of \( \omega_{ab} \). When this tensor vanishes, the canonical bivector \( \epsilon^{ab} \propto \xi^a_+ \xi^b_- \) satisfies

\[ R_{abcd} \epsilon^{ab} = -\Sigma \epsilon_{cd}. \quad (40) \]

In other words, \( \epsilon_{ab} \) is an eigen-bivector of the Riemann tensor. It can be easily checked that
for Einstein spaces, $R_{ab} = \frac{R}{2} g_{ab}$, $\epsilon_{ab}$ is also an eigen-bivector of the Weyl tensor. In fact, using the decomposition of the Riemann tensor, we have for Einstein spaces that

$$
\Sigma = \frac{1}{2} \epsilon^{ab} \epsilon^{cd} R_{abcd} = -\frac{2}{n(n-1)} R + \frac{1}{2} \epsilon^{ab} \epsilon^{cd} C_{abcd}. \tag{41}
$$

Further, by inspecting the form of the derivative of $l^a$ and $n^a$, we can see that

$$
R_{abcd} l^b l^d = \frac{\Sigma}{4} l_c, \quad R_{abcd} n^b n^d = \frac{\Sigma}{4} n_c, \quad \text{at the horizon.} \tag{42}
$$

Therefore both null directions are repeated principal null directions. Principal null directions arise naturally in the algebraic classification of the Weyl tensor, which in turn was important for the hunt of solutions of General Relativity in four dimensions. Incidentally, the conditions in (42) in four dimensions are known as the Bel criteria [28] for type D solutions, and happen for all known Kerr-Newman solutions. In higher dimensions, the notion of algebraic speciality is a bit more involved [29], and vectors satisfying (42) are sometimes called Weyl-aligned null directions [27]. The detailed structure of Weyl-aligned null directions in higher dimensional Killing horizons can be found in [30] and [31].

C. Null Vector on the Near Horizon: Extremal Case

We now turn to the problem of defining suitable null vectors near the horizon of an extremal black hole.

Consider the derivative of $\xi^a_{\pm}$, from which we obtain

$$
l^a \nabla_a l^b = -\kappa l^b - \frac{1}{2} z h^{bc} \nabla_c \kappa. \tag{43}
$$

We first consider second term on the right hand side which is proportional to the Riemann tensor $\nabla_a \kappa = \frac{1}{2} \chi^b R_{abcd} \epsilon^{cd}$. Near the horizon we have

$$
\chi^a R_{abcd} \chi^d = -\frac{\Sigma}{2 \kappa^2} N_c N_c + \ldots, \tag{44}
$$

where we left out terms at higher order in $z$. At the horizon, $N^a = \kappa \chi^a$ are parallel to each
other, so that
\[ \chi^a R^d_{abcd} \chi^d = -\frac{\Sigma}{4} \chi_b \chi_c + \mathcal{O}(z). \] (45)

Therefore \( \chi^a \) is a doubly-repeated principal null vector. From this we conclude that the second term in the right hand side of (43), that is, \( h^{bc} \nabla_c \kappa \) is of order \( \mathcal{O}(z^{1/2}) \), and then (43)
\[ l^a \nabla_a l^b = -\kappa l^b + \mathcal{O}(z^{3/2}). \] (46)

In the extremal (\( \kappa \to 0 \)) limit, only terms of highest order are kept. This means that \( l^b \), after the limit is taken, can be seen as a geodesic vector field. Moreover, the curvature-dependent term of (34) can also be seen to vanish in the limit, being itself of order \( z^{3/2} \). Then, after the limit is taken,
\[ \nabla_{l_b} = -\Gamma_{l}^{mn} l_m. \] (47)

This means that the null vector \( l_b \) is hypersurface orthogonal, that is, \( l_{[a} \nabla_b l_{c]} = 0 \) even away from the horizon. The null vector is a principal null direction, which vanishing expansion, shear and twist.

The usefulness of defining such a null vector in the near horizon of an extremal black hole stems from the following property. Let \( m^a_i \) and \( \bar{m}^a_i \) be null (complex) vectors spanning the space orthogonal to \( l^a \) and \( n^a \), constructed in such a way that the commutators between \( m^a_i \) and \( l^a \) vanish. Parametrize a generic vector in the orthogonal space by
\[ \eta^a = z^i m^a_i + \bar{z}^i \bar{m}^a_i. \] (48)

Upon parallel transport under \( l^a \), we obtain
\[ l^a \nabla_a z^i = -\rho_{ij} z^j + \sigma_{ij} \bar{z}^j, \] (49)
where \( \rho_{ij} = m^a_i m^b_j \nabla_a l_b \) and \( \sigma_{ij} = m^a_i \bar{m}^b_j \nabla_a l_b \) both vanish due the expression of the derivative of \( l^a \) in (47). This means that one can define a complex structure in the subspace generated by the \( m^a_i \) and \( \bar{m}^a_i \) and (47) will guarantee that such structure can be parallel-transported throughout the variable.

The above parallel-transported structure is related, but weaker, than that of SUSY, which asks for the existence of a covariantly constant spinor. We will digress in the following section.
about the similarities and differences of both SUSY and the above.

In the following sections we will discuss more about the geometrical interpretation of these conditions. We just note that in four dimensions, the null vectors $l^a$ and $n^a$ satisfying (42) are related to integrability properties.

The laws of BHT state that $\kappa$ is constant over the horizon $H$, which in turn means that the tensor $\omega_{ab}$ vanishes, and $\epsilon^{ab}$ is a principal bivector. However, we will see that the two null vectors $\xi^a_{\pm}$ degenerate on $H$ to $\pm \chi^a$ and thus fail to define distinct principal null vectors. If $\omega_{ab} = 0$ hold throughout, then the structure outlined in this section will carry on to the whole manifold.

III. THE NEAR HORIZON GEOMETRY

In this section we will state several conditions on the relations obtained above in the case where the KVF becomes null, $z \to 0$. This condition holds for stationary black holes, and in several cases may be used as the telltale signal of an event horizon.

A. Coordinates

At first one can think of the construction above as happening throughout the spacetime. We will show, however, that a significant simplification happens on the structure near a Killing horizon $H$, where $z = 0$.

We have assumed that the KVF $\chi^a$ is hypersurface orthogonal, its covariant derivative being given by (20). From this, it is readily verified that the 1-form $\frac{1}{z} \chi_a$ is closed. Then, locally, there is a function $u$ such that

$$\chi_a = -z(du)_a. \tag{50}$$

We can think of $u$ and $z$ as local coordinates on the spacetime, parameterizing the “non-angular” directions, that is, they span the analogue of the “$r-t$” plane. In these coordinates, the bivector $\epsilon_{ab}$ has the form

$$\epsilon_{ab} = \frac{1}{\kappa} (dz)_a (du)_b. \tag{51}$$

This form is suitable to study the behavior near the horizon $z = 0$. Now, (30) and (29)
allow us to expand both $\kappa$ and $\epsilon_{ab}$ near $z = 0$. Thus,

$$\kappa^2(z) = \kappa_0^2 + \frac{\Sigma_0}{2}z + \mathcal{O}(z^2), \quad (52)$$

where the subscript 0 means evaluation at the horizon. These expressions can be used to expand the metric $g_{ab}$ near the region $z = 0$. We split the metric as the semi-direct sum of the "$r-t$" plane and the angular coordinates, that is,

$$g_{ab} = \epsilon_{ac} \epsilon^{c}_{b} + h_{ab}. \quad (53)$$

The "$r-t$" plane has line element

$$ds^2 = dr^2 - f(r)^2 du^2, \quad (54)$$

with $r(z)$ a function chosen so that $dr$ is the normalized vector parallel to $N_a$. Also, we have $z = f(r)^2$, and then,

$$N_a = -\frac{1}{2}dz = -f(r)f'(r)dr, \quad N^aN_a = -f^2(r)[f'(r)]^2 = -\kappa^2 z. \quad (55)$$

The function $f(r)$ is then determined from the knowledge of $\kappa$ as

$$\kappa = |f'(r)|, \quad (56)$$

which is known near $z = 0$. We can then distinguish two cases.

1. **Case $\kappa_0 > 0$**

   This is the best known case. Since the laws of BHT tell that $\kappa_0$ is constant at the horizon, we have $\lim_{r\to 0}|f(r)| = \kappa_0$ and then $f(r) = \kappa_0 r$ for small $r$. The line element is then

   $$ds^2 = -dr^2 + \kappa_0^2 r^2 du^2, \quad (57)$$

   which shows the well known property that an analytic continuation for imaginary $u$ will display a conical singularity unless $u$ is identified with period $2\pi/\kappa_0$. We will borrow some
terminology – however appropriate – and dub this case “elliptic”.

The 2-form $\epsilon_{ab}$ has the canonical form from (51),

$$\epsilon_{ab} = \frac{1}{\kappa_0} (dz)[a(du)]_b,$$  \hspace{1cm} (58)

which can be easily used to define a canonical volume form on a topological sphere that intersects $\mathcal{H}$, as in the first law of BHT.

2. Case $\kappa_0 = 0$

This is the case of foremost interest to us. It is dubbed “hyperbolic”. If $\kappa_0 = 0$, then we have to go to next order in $z$ in (52).

$$\kappa^2 = \frac{\Sigma_0}{2} z = \frac{\Sigma_0}{2} f(r)^2.$$  \hspace{1cm} (59)

This gives us the following differential equation for $f(r)$:

$$f'(r) = \Gamma f(r) \quad \Rightarrow \quad f(r) = \exp(\Gamma r),$$  \hspace{1cm} (60)

with $\Gamma^2 = \Sigma_0/2$. The metric is then

$$ds^2 = -dr^2 + \exp(2\Gamma r)du^2,$$  \hspace{1cm} (61)

which has constant negative (sectional) curvature.

The volume form $\epsilon_{ab}$ can be cast as

$$\epsilon_{ab} = \frac{1}{\Gamma} \frac{(dz)[a(du)]_b}{z^{1/2}},$$  \hspace{1cm} (62)

so that the canonical variable is now $\rho = z^{1/2}$.

Note that in both cases the isometry of the “$r-t$” plane has been enlarged (or enhanced). Instead of just a line of symmetries coming from the KVF, they are now symmetries of the Minkowski plane $\text{SO}(1,1)$ in the case of positive $\kappa_0$ and the ubiquitous $\text{SL}(2,\mathbb{R})$ in the hyperbolic case. Whether the action of those new generators keep the full metric is not
known.

Along with the above two cases, there is another where \( \kappa_0 = \Sigma_0 = 0 \), which we call “parabolic”. It has also a canonical form of the metric, but no enlarged symmetry or constant curvature.

### IV. THE BOTTOMLESS PIT

In this section we study the near-horizon geometry in the extremal case. We show that the enhancement of symmetries discussed in the previous section can be carried over to the near horizon region. Also, we show that the near horizon region becomes causally disconnected with the asymptotic region.

Very close to the horizon \( z = 0 \), the integral curves of \( \xi_{\pm}^a \) come closer and closer to being geodesics. From the equation (34), one sees that the affine parameters \( w^\pm \) of such geodesics are related to the affine parameters \( x^\pm \) of \( \xi_{\pm}^a \) by

\[
w^\pm = \exp \left[ \int x^\pm \ dx' \kappa \right]. \tag{63}
\]

From this we can again distinguish two cases. If \( \kappa \to \kappa_0 \neq 0 \) at the horizon, then \( w^\pm \) has a minimum value, and then the horizon is incomplete in the past. Geodesics which asymptote the horizon as we take \( x^\pm \to -\infty \) will be incomplete in the past as well, because for geodesics sufficiently close to the horizon the affine parameter can be approximated by \( w \approx e^{\kappa_0 x} \). At the horizon, the point on which \( w^\pm = 0 \) is actually a fixed point of the Killing vector field \( l^a \), and it is called bifurcation point \[32\].

The situation changes somewhat when \( \kappa \to 0 \) at the horizon. Now we can take \( w^\pm = x^\pm \) and the geodesics can be indefinitely extended. Introducing coordinates such that \( \chi^a = \partial / \partial u \) and \( \rho^a = -2\kappa^2 z \partial / \partial z \), then \( \xi_{\pm}^a \) is given by

\[
\xi_{\pm}^a = -2\kappa z \frac{\partial}{\partial z} \pm \frac{\partial}{\partial u}. \tag{64}
\]

Given that \( \kappa \to 0 \) at the horizon, the affine parameter of \( \xi_{\pm}^a \) will depend crucially on the
behavior of $\kappa$ close to the horizon. By (30), we have the expansion

$$\kappa(z)^2 = \kappa_0^2 + \frac{1}{2} \Sigma_0 z + O(z^2)$$

(65)

where the subscript 0 refers to quantities being computed at the horizon. Integrating (52) with $\kappa_0 = 0$ for the affine parameters $x^\pm$, we have

$$x^\pm = \pm u + A z^{1/2} = \pm u + A \rho.$$  

(66)

These have two interesting features. First and foremost, unlike the case where $\kappa_0 \neq 0$, the curves can be continued indefinitely, meaning that the region near the black hole has a causal infinity – the “throat is bottomless” or “the bottomless pit” \cite{24} – which can be thought of as a decoupling limit of the induced theory near the boundary and the asymptotic region far from the horizon. We will have more to say about the holographic interpretation below.

The second feature can be seen as a consequence of the first. One notes that, in order to focus in the near-horizon $z \approx 0$ region, one can make a scale transformation, which for $\kappa_0 \neq 0$, involves the logarithm of the coordinate $z$,

$$\xi^a_\pm \rightarrow \lambda^{-1} \xi^a_\pm \Rightarrow u \rightarrow \lambda u \quad \text{and} \quad z \rightarrow z + A^{-1} \log \lambda.$$  

(67)

For $\kappa_0 = 0$, the transformation is different,

$$\xi^a_\pm \rightarrow \lambda^{-1} \xi^a_\pm \Rightarrow u \rightarrow \lambda u \quad \text{and} \quad \rho \rightarrow \lambda \rho.$$  

(68)

This last transformation allows for a holographic interpretation. The coordinate $\rho$ actually changes scales without any dimensionful parameter (like $\kappa_0$). We will argue below that the absence of dimensions in the $\rho$ coordinate can be thought of as the signal of a fixed point on the renormalization group flow.

We can now make a simple argument supporting the so-called “attractor mechanism” conjectured to hold for extremal black holes \cite{18, 19}. Indeed, from (13) we can argue the following regarding the values of the angular velocities $\Omega^i$ at the horizon. It was shown that those are constant on the horizon. Now, by the argument given above, these values at the horizon are causally disconnected from the asymptotic region in the extremal case.
Given this, one can set values for the angular velocities – the intrinsic quantities in BHT – independently from the values of these quantities at infinity. As argued above, the same reasoning can be applied to the values of any field that can be obtained from Kaluza-Klein reduction of gravity.

V. SUCH SPINORS AT THE HORIZON

In this section we discuss the previous results in the “gauge picture”, using integrability of null planes and their relation to spinors to rewrite the geometrical results obtained for the near horizon geometry in a setting suitable to discuss implications in the field theory side of the gauge/gravity duality. The construction we have outlined in the preceding sections made use of the notion of Killing horizon to single out preferred null directions on the manifold. We will now explore the relationship of generic null (isotropic) subspaces and spinors. The goal is to derive a condition on these subspaces compatible with extremality.

Consider a Lorentzian, even \( D = 2n \)-dimensional manifold with real Vielbeine \( e^i \). From those we can construct a null basis \( \{ l^i, n^i \} \) satisfying

\[
\begin{aligned}
l^i \cdot l^j &= n^i \cdot n^j = 0, \\
l^i \cdot n^j &= \delta^{ij}.
\end{aligned}
\]

Now, some of the elements of this basis will be complex, so we will consider the complexified tangent space arising from generic complex combinations of \( \{ l^i, n^i \} \). This construction can be thought of as splitting the tangent space at each point into an isotropic vector space \( V \) and its dual \( V^\ast \): \( T_p M \cong V \oplus V^\ast \). Observe that \( \{ l^i \} \) forms a basis of \( V \) and \( \{ n^i \} \) forms a basis of \( V^\ast \). The “natural” pairing between \( V^\ast \) and \( V \) is given by the metric, with the direct sum structure stemming from the linearity of the inner product and the isotropy of \( V \) and \( V^\ast \), that is,

\[
\langle (v_1 + w_1), (v_2 + w_2) \rangle = v_1 \cdot w_2 + w_1 \cdot v_2, \quad \text{with } v_i \in V, \ w_i \in V^\ast.
\]

Anyway, the particular choice of null basis is not relevant for the following discussion, since any two of them are related by an element of \( \text{SO}(2n) \). As algebras, the splitting is

\[
\text{so}(V \oplus V^\ast) = \text{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^\ast,
\]
involving particular two forms (exterior products) of the \( \{l^i\} \) and the \( \{n^i\} \).

Multivectors of \( V^* \) are generic linear combinations of exterior products of the \( \{n^i\} \),

\[
\varphi = a + a_in^i + a_{ij}n^i \wedge n^j + \ldots. \tag{72}
\]
The elements of the full space \( V \oplus V^* \) acts on those multivectors via the geometric product,

\[
(v + w)\varphi = v \cdot \varphi + w \wedge \varphi, \tag{73}
\]
where the contraction is always done with the first index. Repeating the action, one gets

\[
(v + w)(v + w)\varphi = (v \cdot w)\varphi = \langle (v + w), (v + w) \rangle \varphi. \tag{74}
\]
The above geometric action is then implemented as the Clifford product on the space of multivectors. In other words, isotropic multivectors can be seen as spinors. Simple counting shows that the space of (simple) isotropic multivectors is identified with Dirac spinors, whereas restriction to multivectors of either odd or even degree results in chiral (Weyl) spinors.

Given a (non-vanishing) spinor \( \varphi \), one defines its *annihilating space* \( L_\varphi \) as the elements of \( V \oplus V^* \), such that

\[
(v + w)\varphi = 0. \tag{75}
\]
As a direct consequence, one notes that any element of \( V \) is a null (isotropic) vector, since for any (non-vanishing) \( \varphi \in V^* \),

\[
(v + w)(v + w)\varphi = 0 \implies \langle (v + w), (v + w) \rangle = 0. \tag{76}
\]
Then, in the complexified vector space, \( L_\varphi \) can have at most (complex) dimension \( n \). If this is the case, \( L_\varphi \) is called *maximally isotropic* and \( \varphi \) is called a *pure spinor* \( \text{[33]} \). One such spinor is \( \varphi = 1 \), which is annihilated by all elements of \( V \) (via \( \text{(73)} \)). Therefore \( L_1 \) has maximal dimension.

A known result \( \text{[33]} \) states that all pure spinors are related by the action of an element of \( \text{SO}(2n) \). So, to any given maximally isotropic space \( L \), one can find a single (up to scaling)
pure spinor $\varphi_L$ such that $L$ is its annihilating space. The idea is that $L$ can be “rotated” to $V$ by a suitable element of $\text{SO}(2n)$, and then this element will bring $\varphi_L$ to 1. Given the decomposition (71), this element is unique up to endomorphisms of $L$, which leave $\varphi_L$ invariant up to scaling.

We may now consider the introduction of a connection on the manifold, so that we can parallel transport the above structures over the manifold. For that we should deal with the geometric product using Leibniz rule

$$\nabla (v \varphi) = (\nabla v) \varphi + v \nabla \varphi,$$

which has to vanish if $v$ is an element of $L_\varphi$. Requiring that the right hand side vanishes imposes that $\nabla v$ has only components along $L_\varphi$ and $\nabla \varphi$ is proportional to $\varphi$.

The condition that the splitting $T_p M \to V \oplus V^*$ holds throughout the manifold is now easily seen as either of the equivalent notions:

1. Parallel transport of any vector in $V$ stays into $V$, that is, the holonomy is reduced from $\text{SO}(2n)$ to $\text{SU}(n)$.

2. There is a parallel, covariantly constant, (pure) spinor, associated via (72) to a maximally isotropic plane.

Either condition is related to remaining supersymmetry. The construction makes sense in the complexified tangent space, so the choice of “real form” (whether the signature is Euclidian or Lorentzian) has to be compatible with an integrability structure. In practice this means that the SUSY charges satisfy the correct reality condition leading to the correct algebra, so that they anticommute to the Hamiltonian.

As we have stressed throughout, all the discussion applies when the integrable null plane is maximal.

In the preceding sections we built the argument that one has (necessarily) a principal null direction on a Killing horizon. This will be necessarily real, and can be extended to the near-horizon in the extremal case. Extra commuting Killing vector fields related to “transverse” symmetries will also be able to generate similar structures. The word “transverse” merely states that the conserved quantities arising from the symmetries are constructed uniquely from the symplectic structure (cf. discussion in [35]). In the maximal case, we can define
$D/2$ null, orthogonal but not necessarily real lines bundled to a maximally isotropic plane. From the preceding discussion, these are associated with a parallel spinor. One can then see that the construction resembles very closely the relation between reduced holonomy and SUSY, although the remark about signature can prevent the charges thus defined to play a role in organizing the spectrum of the theory.

In four dimensions, the conditions arising from the existence of integrable maximal isotropic planes result in familiar structures. The Goldberg-Sachs theorem relates the existence of integrable isotropic planes to repeated principal spinors. One assigns null vectors belonging to a null plane by

$$\ell^a = \sigma^a_{\alpha\dot{\alpha}} \pi^{\alpha} \bar{\pi}^{\dot{\alpha}}, \quad (78)$$

with $\sigma^a_{\alpha\dot{\alpha}}$ the van der Waerden symbols (also chiral Pauli matrices or soldering form). To relate with the discussion above, we may think of a spinor basis $\{\iota^\alpha, o^\alpha\}$ generating the two-dimensional $V$. We will omit the identification between spinors and vectors and write $\ell^{\alpha\dot{\alpha}}$ for a vector from now on. The spinor $\bar{\pi}^{\dot{\alpha}}$ is generic, but the plane contains the real vector $\iota^\alpha \bar{\iota}^{\dot{\alpha}}$. The condition of integrability is

$$\iota^\beta \iota^\alpha \nabla_{\alpha\dot{\alpha}} \iota_{\beta} = 0. \quad (79)$$

This merely states that any commutator of vectors formed from $\iota^\alpha \bar{\pi}^{\dot{\alpha}}$ belongs to the same plane. The condition can be trimmed slightly since one can freely rescale the spinor, so that

$$\iota^\alpha \nabla_{\alpha\dot{\alpha}} \iota_{\beta} = 0. \quad (80)$$

Now, using the definition of the curvature spinor,

$$\nabla_{\alpha(\dot{\alpha}} \nabla^\alpha_{\beta)} \kappa_{\gamma} = \Psi_{\alpha\beta\gamma}\delta^\kappa - 2\Lambda \kappa_{(\alpha} \epsilon_{\beta)\gamma}, \quad (81)$$

one arrives from (80) at an algebraic condition involving the spinorial version of the Weyl tensor $\Psi_{\alpha\beta\gamma}\delta$, that is,

$$\Psi_{\alpha\beta\gamma}\delta \iota^\alpha \iota^\beta \iota^\gamma = 0. \quad (82)$$

The spinor space is two-dimensional, hence $\pi_{\alpha} \ell^\alpha = 0$ implies that $\pi^\alpha$ and $\iota^\alpha$ are proportional. Furthermore, $\Psi_{\alpha\beta\gamma}\delta$ can be factorized into the principal spinors: $\Psi_{\alpha\beta\gamma}\delta = \kappa_{(\alpha}^1 \kappa_{\beta}^2 \kappa_{\gamma}^3 \kappa_{\delta)}^4$. The
condition (82) means that $\iota^a$ will appear twice in this decomposition. So the spinor associated with the null plane is necessarily repeated. We can then work out two cases, whether there are one or two integrable null planes, and the Weyl tensor is either one of the following two forms,

$$\Psi_{\alpha\beta\gamma\delta} = A\iota_{\alpha\iota\beta\iota\gamma\iota\delta}, \quad \text{or} \quad \Psi_{\alpha\beta\gamma\delta} = A\iota_{\alpha\iota\beta\iota\gamma\iota\delta}. \quad (83)$$

These correspond to the Petrov type N and D Weyl tensor, respectively. Associated with the spinor(s) one can assign

$$\Phi_{\alpha\beta} = A\iota_{\alpha\iota\beta} \quad \text{or} \quad \Phi_{\alpha\beta} = A\iota_{\alpha\iota\beta} + A'\iota_{\alpha\iota\beta}. \quad (84)$$

One can then construct the Killing-Yano antisymmetric tensor $G_{ab} = G_{[ab]} = \Phi_{\alpha\beta}\epsilon_{\alpha\beta} + \text{c.c.}$. At this point the coefficients $A$ and $A'$ are generic functions. The Killing-Yano tensor satisfies $\nabla (aG_{b)c} = 0$ and its existence is also tied to integrability properties of the space-time. In fact, the integrability structure is exactly the same as outlined above. Killing-Yano tensors have been important in the integration of the geodesic equation of type D solutions [37, 38] and in the separability of the wave equations in those backgrounds [39, 40].

In more than four dimensions, however, one can encounter a lower dimensional integrable null plane which cannot be seen as the real or imaginary part of a maximal one. A spinor associated to such plane is called impure. In fact, a lower dimensional null plane is not associated to a single spinor, but with a number of them. One can write schematically

$$\varphi = \sum_i a_i\psi_i, \quad (85)$$

where $\psi_i$ are pure spinors, but in general $\varphi$ is not. The coefficients $a_i$ are arbitrary. Geometrically, the $\varphi$ is associated to a null plane of annihilators $L_\varphi$ which is the intersection of the annihilators of each of the pure spinors $L_{\psi_i}$. One can now state the condition that $L_\varphi$ is integrable by demanding that the set of pure spinors $\{\psi_i\}$ satisfy the involutive property

$$\nabla \psi_i = \sum_j a_{ij}\psi_j. \quad (86)$$

This condition is milder than SUSY at six or more dimensions.

The conclusion of the above discussion is that Killing horizons have associated with them
a set of pure spinors \( \{\psi_i\} \) satisfying the involutive property (86). When the horizon has zero temperature, the property can be extended to the near-horizon geometry. The number of pure spinors in the set reflects the dimension of the integrable null plane: in the one-dimensional case one has \( d/2 - 1 \) pure spinors whose intersection gives a particular null line. When the set of pure spinors consists of just one element, the geometry of the spacetime is of reduced holonomy and one can use the standard arguments to show that SUSY charges can be defined. In the general case, however, the condition seems milder and may be of relevance in the quest for backgrounds for AdS/CFT, among other applications.

We will close the section by discussing examples in 6 dimensions. The notation chosen when writing (86) tries to make clear that \( \{\psi_i\} \) should be thought of as Dirac spinors. Let us work explicitly with the basis for spinors in 3 complex dimensions,

\[
\{1, n_1, n_2, n_3, n_1 \wedge n_2, n_1 \wedge n_3, n_2 \wedge n_3, n_1 \wedge n_2 \wedge n_3\}.
\]  

(87)

Each element of the basis is a pure spinor. For example, \( L_1 = \{\ell_1, \ell_2, \ell_3\} \). Any sum of same chirality – Weyl – spinors either has dimension 3 or 1, like \( n_1 + n_1 \wedge n_2 \wedge n_3 \), whose annihilator is \( n_1 \). In order to obtain a two-dimensional plane, one has to consider the sum of terms with opposite chirality: the annihilator of \( 1 + n_1 \) is, for instance, \( \{\ell_2, \ell_3\} \). The condition (86) may then involve the sum of spinors of different chiralities. It would be interesting to understand its relation to usual constructions of supersymmetry and supergravity.

VI. DISCUSSION

Let us now apply the geometrical construction outlined to the case where the dual field theory has a sigma-model description. From the geometrical side, the horizon is, by definition, a surface of last contact. It is usually the limit of coordinates considered to be “natural” by an asymptotic observer. This surface is necessarily null, and by the usual results repeated in the preceding sections it is also integrable.

Given an integrable null surface in spacetime, we can either associate it with a (perhaps partial) splitting of spacetime into isotropic spaces \( V \otimes V^* \), or associate it with a number of null spinors via the involutive property (86). The first approach was crucial to the development of special solutions in general relativity, including the Kerr-Newman family,
and ultimately to the twistor programme. The second has deep ties to supersymmetry, which played a role akin to integrability – perhaps “amenability” would be a more suitable word – in field theory. We will argue that they should be thought of as equivalent physically as well as mathematically.

Let us begin with the case where there is a maximally isotropic null integrable plane. By definition, this means that the splitting \( T_p M \to V \oplus V^* \) can be done consistently throughout the manifold. For Euclidian signature, such spaces with special holonomy are called Kähler manifolds. As a rule, the existence of this special holonomy combined with some restriction on the geometry – like, for instance, the manifold being Einstein (\( R_{ab} = \frac{R}{2} g_{ab} \)) is enough to preclude non-trivial quantum corrections to spring up in the higher-order effective theory. In fact, if the scalar curvature is zero (and hence the second Chern class also vanishes), the manifold has a well-defined spinor charge and is in fact a supersymmetric background. The geometric spinors defined above can be integrated to generate a spinorial charge. The usual non-renormalizability theorems \([41–43]\) can be invoked to protect the potential from generating a mass gap, at least perturbatively.

To sum up, the constancy of spinors can then be related to the constancy, or integrability, of the splitting between a maximally isotropic space \( V \) and its dual. Heuristically, the integrability of the splitting is in turn related to the absence of quantum corrections to the geometry.

This property can be really appreciated from the gauge/gravity perspective. Suppose we have a two-dimensional sigma-model with target manifold \( M \). Usually the geometry of \( M \) is set in the ultraviolet scale of the sigma-model and the geometry changes as the scale goes down. Upon certain conditions, the scale itself can be seen as an extra dimension \([44–46]\). In some special cases, the renormalization group (RG) flow equations can be seen as the truncation of the Einstein equations for the extended space.

The nature of such truncation seems a bit mysterious at first. The RG flow (Callan-Symanzik) equations are of first order in the scale parameter, whereas the Einstein Equations are of second order if we pick as the scale parameter the coordinate spanning the extra dimension. In fact, in the usual example of this correspondence, the extra dimension is the radial coordinate \( r \) of AdS\(_5\), so that the metric \( h_{ab} \) induced in the space transverse to \( \nabla_a r \) satisfies, rather trivially,

\[
\frac{\partial h_{ab}}{\partial r} = \frac{1}{R} h_{ab}.
\]
The extrinsic curvature $K_{ab}$ is then proportional to the metric. From the “other half” of Einstein’s equations, one learns that

$$\frac{\partial K_{ab}}{\partial r} = R[h]_{ab}. \quad (89)$$

Therefore only for Einstein spaces the constraint that the extrinsic curvature is proportional to the metric will hold throughout the evolution in the parameter $r$.

Our understanding is that the flow along the $r$ direction should be understood as geodesic flow. It is generated by the $r^a = (\partial/\partial r)^a$ vector field, conjugate to the gradient of $r$, $r^a \nabla_a r = 1$. We will choose coordinates so that the normalized geodesic vector field $n^a$ is parallel to $r^a$, $r^a = N n^a$, $n^a n_a = 1$. The treatment parallels that of congruences of timelike geodesics. Note that the geodesics we are talking about are spacelike. We define the induced metric on the leaves of constant $r$,

$$h_{ab} = g_{ab} - n_a n_b. \quad (90)$$

The derivative of $n^a$, related to the extrinsic curvature $K_{ab}$, is defined by

$$B_{ab} = \nabla_b n_a, \quad \text{so that} \quad K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab} = B_{(ab)}. \quad (91)$$

The interpretation of the operator $B_{ab}$ is similar to its timelike counterpart. If one defines a basis of vectors tangent to the surface of constant $r$ by $m_i^a$, obeying the compatibility condition

$$\mathcal{L}_n m_i^a = n^b \nabla_b m_i^a - m_i^b \nabla_b n^a = 0, \quad (92)$$

then the parallel transport by $n^a$ will be

$$\delta m_i^a = \epsilon n^b \nabla_b m_i^a = \epsilon m_i^b \nabla_b n^a = \epsilon B^a_b m_i^b. \quad (93)$$

So, as the flow through $r$ progresses, the basis $m_i^a$ will be twisted and turned by the exponential of the operator $B^a_b$. If we take the usual action form for the $\sigma$-model,

$$S_\sigma = \int d^2 x \sum_i \sqrt{g} g^{\alpha \beta} h_{ab} \partial_\alpha m_i^a \partial_\beta m_i^a + \ldots, \quad (94)$$
then the (worldsheet) derivatives of the (spacetime) vectors $m^a_i$ play the role of the primaries $\mathcal{O}_i$. The action above is of course schematic: we do not consider fermions, and the $r$ direction arises only at the effective action level. In the holographic RG spirit \[44\], movement in the $r$ direction corresponds to changing of the energy scale of the $\sigma$-model.

A natural candidate for the RG-flow parameter in the generic case is the scale factor $\theta$, defined as the trace of $B_{ab}$, as found in the decomposition

$$B_{ab} = \theta h_{ab} + \omega_{[ab]} + \sigma_{(ab)},$$

where the “shear” $\sigma_{(ab)}$ is the traceless part of the extrinsic curvature. Since the role of $\theta$ is to dilate the vectors, $m^b_i$, the above assignment seems plausible. If $\omega_{[ab]}$ or $\sigma_{(ab)}$ are not equal to zero, then the transverse metric $h_{ab}$, seen as the couplings of the sigma model is changing not only the scale, but also the relative couplings between the sigma-model observables (in this case, the primaries $\partial X^a$ and $\bar{\partial} X^a$). The classical geometric equations dictating the change of $B_{ab}$ are known as the Raychaudhuri equations \[23\]. Computing the second derivative of $B_{ab}$, we have

$$n^c \nabla_c B_{ab} = -B_{ac} B^c_b + n^c R_{abcd} n^d,$$ or $$\mathcal{L}_n B_{ab} = n^c B^c_a B_{cb} + n^c R_{abcd} n^d,$$

where one can use the decomposition of the Riemann tensor into the pure trace, traceless transverse 2-tensor $S_{ab} = R_{ab} - \frac{1}{n-1} R g_{ab}$ and the Weyl tensor $C_{abcd}$:

$$R_{abcd} = \frac{2}{(n-1)(n-2)} R g_{[cd]} g_{ab} + \frac{2}{n-2} (g_{a[c} S_{d]b} - g_{b[c} S_{d]a}) + C_{abcd}.$$  

One can then split \[96\] into equations for each of the terms in \[95\]:

$$\mathcal{L}_n \theta = -\theta^2 - \frac{1}{n-1} (\sigma_{ab} \sigma^{ab} - \omega_{ab} \omega^{ab}) - \frac{1}{n-2} R_{ab} n^a n^b;$$

$$\mathcal{L}_n \omega_{ab} = 0;$$

$$\mathcal{L}_n \sigma_{ab} = \theta \sigma_{ab} + \sigma_{ac} \sigma^c_b - \omega_{ac} \omega^c_b - \frac{1}{n-1} h_{ab} (\sigma_{cd} \sigma^{cd} - \omega_{cd} \omega^{cd})$$

$$+ h_{a}^c h_{b}^d S_{cd} + C_{cbad} n^c n^d.$$ 

If $B_{ab}$ turns into a pure scale transformation, we can say that we arrived at a fixed point of
the RG flow process, in which the primaries no longer change. One sees from the equations above that near the horizon of a black hole which is an Einstein manifold, the conditions for the endpoint are met because $n^a$ effectively becomes a repeated principal null vector. Therefore the source terms on the right hand side vanish. One should also note that \textit{a priori} the choice of initial condition $\omega_{ab} = 0$ is maintained as one follows through the flow. In physical terms, the tensor $\sigma_{ab}$ encodes the non-trivial renormalization process. One could in principle have a non-trivial $\omega_{ab}$ by introducing a “lapse vector” $N^a$, and then the Lie derivatives of a generic tensor $T_{ab}$ would be modified into

$$\mathcal{L}_n T_{ab} = \frac{1}{N} \left( \frac{\partial}{\partial r} T_{ab} - N^c D_c T_{ab} - T_{cb} D_a N^c - T_{ac} D_b N^c \right), \quad (101)$$

where $D_a$ is the covariant derivative associated with $h_{ab}$, the projection of $\nabla_a$ to the surfaces of constant $r$.

Coming back to the Callan-Symanzik equation, the fact that the Einstein equations are second order entails the fact that the observables entering the (gauge independent) correlation functions are themselves changing with the energy scale, that is,

$$\delta \mathcal{O}_i = \sum_j B_{ij} \mathcal{O}_j. \quad (102)$$

Such mixing can only happen for theories for which there are a large number of operators sharing the same scaling dimension. One notes that this variation mixes different correlation functions in the Callan-Symanzik equation and can be thought of as a higher order correction. Of course, near a conformal fixed point, the only change up to first order in the beta function is that of the scaling dimension, corresponding to a diagonal $B_{ab}$.

We have two examples illustrating the above discussion. For asymptotically anti-de Sitter (aAdS) space-times, the metric can be written as

$$ds^2 = \frac{dr^2}{r^2} + h_{\mu\nu}(z, x^\rho) dx^\mu dx^\nu \quad (103)$$

with $h_{\mu\nu}$ approaching a conformally flat metric as we take $r \to 0$. In this regime, the condition $K_{ab} = \theta h_{ab}$ is satisfied and the usual lore says we are arriving at a (UV) conformal fixed point. The addition of marginal or relevant perturbations – like the addition of a
subleading mass term – will mix the primaries as the flow goes down. Perhaps the most famous example of this mixing happens when we have several U(1) global symmetries. The RG flow mixes the generators of each of the U(1) factors. If spontaneous symmetry breaking occurs along the way, then the generators of the remnant U(1) symmetries at the infrared fixed point may be radically different from those proposed at the ultraviolet. In some important applications, such generators can be found from a variational principle – “anomaly maximization”, presumably for the same reasons the algebraic structure made explicit above is relevant for extremal black holes.

The second example comes in the guise of the many versions of the “c-theorem”, a generalization of the famous work in two dimensions [7]. General relativity coupled to matter satisfying the strong energy condition will always have $\theta < 0$ as $r$ increases, moving $h_{ab}$ away from the UV point. This is in tune with the irreversibility of the RG-flow, in which the number of degrees of freedom decreases under the RG-flow. Geometrically, $\theta$ counts the change of small elements of area with $r$. The assignment of some degrees of freedom with the area is then natural from BHT.

VII. CONCLUSIONS & PERSPECTIVES

In this article we showed that the “geometric definition” of a black-hole in terms of local existence of a null Killing vector field entails a great deal of information about the local geometry. We were able to show that an enhancement of the geometry should happen, but whether this enhancement can be extended to the near horizon limit seems to be feasible only in the extremal case, where it is causally disconnected from the asymptotic region. Given the absence of a mass scale as well as the disconnection with asymptotic observers, it is less of a surprise that one can elect a class of gauge-inequivalent metrics that counts the Hawking-Bekenstein entropy, in the spirit of Kerr-CFT.

Also, one of the major consequences of the analysis is that the disconnection of the region near the horizon and the asymptotic is independent of the dynamics and the dimension of the theory, and is a direct consequence of extremality. In particular, one can then dictate values for the global charges in the near horizon regime are independent from those in the asymptotic region. These global charges can be thought of basically any charge that can be obtained from a suitable Kaluza-Klein compactification of pure gravity, so it encompasses
not only the charges associated with the Killing vector fields, but also scalar charges, flavor charges, abelian and non-abelian charges as well as monopoles. These are fixed from the Second Law, which now can be stated purely from the geometry, not depending on the details of the dynamics or supersymmetry. Hence, the construction outlined here provides a geometrical verification of the attractor mechanism which relies solely on the integrable structure of the near-horizon region.

The second part of this work dealt with the algebraic structures behind the extremality. It is known that the near horizon limit displays an enhancement of (super)symmetries for supergravity backgrounds. We claimed that even in the non-supersymmetric case one deals with the involutive property of null planes, which are naturally associated with spinors. Thus the requirement of supersymmetry is not mandatory in the generic case, and might as well be replaced with the integrability of (lower-dimensional) null planes. In terms of spinors, the latter translates into the involutive property described in (86). The same classical integrable structure which is phenomenologically interesting for SUGRA backgrounds also allows for the solutions of Einstein equations in pure gravity. Quantum mechanically, these spaces should have a geometric description of their gravitational degrees of freedom, just like in Kerr/CFT. If in four dimensions one encounters essentially the known cases (either a Calabi-Yau or complex charges), the situation changes in 6 dimensions, where one can have interesting cases of non-maximal null planes [47]. Also, it would be interesting to rewrite the SUSY algebra in terms of the splitting proposed here, which we believe should help in the search of backgrounds of 10d SUGRA.

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