Anisotropic Asymptotics
and
High Energy Scattering

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Abstract

Recently E.Verlinde and H.Verlinde have suggested an effective two-dimensional theory describing the high-energy scattering in QCD. In this report we attempt to clarify some issues of this suggestion. We consider anisotropic asymptotics of correlation functions for scalar and gauge theories in four dimensions. Anisotropic asymptotics describe behaviour of correlation functions when some components of coordinates are large as compare with others components. It is occurred that (2+2) anisotropic asymptotics for 4-points functions are related with the well known Regge regime of scattering amplitudes.

We study an expansion of correlation functions with respect to the rescaling parameter $\lambda$ over a part of variables (anisotropic $\lambda$-expansion). An effective theory describing the anisotropic limit of free scalar field contains two 2 dim conformal theories. One of them is a conformal theory in configuration space and another one is a conformal theory in momentum space. In some special cases, in particular for the Wilson line correlators in gauge theories, the leading term of the anisotropic expansion involves only one of the conformal theories and it can be described by an effective theory with an action being a dimensional reduction of the original action.
1 Introduction

Recently E.Verlinde and H.Verlinde [1] have suggested a new approach to high energy scattering in QCD. They performed a rescaling of the longitudinal coordinates inside the Yang-Mills action and reduced the full theory to a two-dimensional sigma-model in the transversal subspace. Moreover they have assumed a special ansatz for the truncated action which was engaged to reproduce the log s dependence of the high energy of amplitudes in QCD.

Several papers there appeared in which the approach of [1] have been discussed [2]-[6]. Quantum corrections to the longitudinal dynamics in the Verlindes’ approach have been considered in [2] A nonperturbative approach to the Regge regime in QCD based on an anisotropic lattice gauge theory has been suggested in [3].

The necessity of non-perturbative study of QCD in the Regge regime of large energies $\sqrt{s} \to \infty$ and fixed momentum transfers $q, |q| \sim 1 Gev$ has been emphasized by Nachtmann [7]. Let us recall that problems of nonperturbative investigations of QCD have been discussed for many years. Some hopes were laid on an analogy between four-dimension Yang-Mills theory and two-dimensional chiral field [8, 9]. There is a well known conjecture that the long-distance dynamics of gauge theories in the confining phase is described in terms of two-dimensional conformal field theory [10, 11]. In the recent years Lipatov [12] has made suggestions on a relationship between QCD at high energies and a two-dimensional field theory (see also [13] for further developments). This approach is based on the solution of the unitarity condition. Recently Faddeev and Korchemsky [14] have found that the Lipatov two dimensional effective theory is completely integrable and they have studied it by using of a generalized Bethe ansatz.

The Verlindes’ approach is based on elegant rescaling arguments but there are some questions which deserves a further study. In particular, it is not enough clear why one should use an ansatz [1] which fixes log s expansion and which makes contact with the usual perturbative answer and reggeization. Also there are questions about ultraviolet and infrared divergences. Some aspects of these problems have been discussed in [1]-[3]

In this talk we discuss some of these questions. We will show that the rescaling arguments can be understand by using the notion of anisotropic asymptotics. Let us explain what we mean by the anisotropic asymptotics. If $x^\mu$ are space-time coordinates, one denotes $x^\mu = (y^\alpha, z^i)$, where $\alpha = 0, 1, i = 2, 3$ for 2+2 decomposition and $\alpha = 0, i = 1, 2, 3$ for 1+3 decomposition. We are interested in the evaluation of asymptotics of correlation functions for a field $\Phi_j(\lambda y, z),$

$$<\Phi_{j_1}(\lambda y_1, z_1)\Phi_{j_2}(\lambda y_2, z_2)\cdots \Phi_{j_n}(\lambda y_n, z_n)> \tag{1.1}$$

when $\lambda \to 0$ (or $\lambda \to \infty$), i.e. when some of variables are much larger than others. If one rescales all the variables then one deals with the usual short or large distance behaviour of the theory. These isotropic asymptotics are given by the factors defined by ultraviolet or infrared dimensions of the corresponding Green function multiplied on corresponding anomalous dimensions defined by the standard renormalization group
The leading terms of the anisotropic asymptotics can be expected to be given by anisotropic dimension analysis and be related with anisotropic operator product expansion (AOPE)

\[ \Phi_{i_1}(\lambda y_1, z_1)\Phi_{i_2}(\lambda y_2, z_2) \sim \sum_{n_1,n_2} C_{n_1,n_2}^{i_1,i_2}(y_1, y_2, \lambda) \mathcal{O}_{n_1,n_2}(y_1, z_1, z_2) \]  

Matveev, Muradyan and Tavkhelidze have used a generalized dimensional analysis to derive an automodel behaviour of differential cross-sections for hadrons-hadrons reactions.

Our approach to anisotropic asymptotics was stimulated by recent consideration of stochastic limit in QFT where in fact the (1 + 3) anisotropic asymptotics has been considered, i.e. an asymptotic behaviour of quantum field \( \Phi(\lambda t, x) \) after time rescaling was evaluated.

There is an analogy between the anisotropic \((1, d - 1)\) \( \lambda \) rescaling and the high temperature limit of lattice gauge theory.

The anisotropic rescaling gives generally a more complicated effective action as compare with isotopic rescaling since after making a change of variables in the functional integral one gets an anisotropic action with the small parameter \( \lambda \) in front of some terms with derivatives. However some simplifications take place for the case of massless gauge theory. An essential difference between scalar and vectors theories is due to that in the latter case the anisotropic rescaling acts different on different components and just in the zero order on rescaling parameter we get a nontrivial action. In this case one deals in some sense with a dimensional reduction. However the naive rescaling arguments do not always work because we are working with distributions. In particular, the naive derivation of the effective action cannot be used even for the free scalar field.

As we will see the anisotropic asymptotics corresponding to the \((2+2)\) decomposition of 4-point correlation function is related with behaviour of scattering amplitudes in the Regge regime (\( s \) large, \( t \) is fixed). In the standard study of the Regge regime in QFT one first expands the functional integral in the perturbation series with respect to the coupling constant and then calculates the limit \( s \to \infty \) for \( t \) fixed for each diagram and finally sum up the leading terms. It is rather remarkable that in the perturbation theory the leading terms exhibit the following structure

\[ A(s, t) \sim \sum_{n} g^{4n} (\ln s)^n I_n(t) \]  

where \( I_n(t) \) can be represented by two-dimensional Feynman diagrams in the transversal space (\( z \)-space). This representation takes place for all theories and it is interesting to understand an origin of such representation without doing an examination of individual diagrams. We would like to emphasize that we are going to perform an expansion with respect to \( \lambda \) in the functional integral, i.e. this approach is a non-perturbative one. After the evaluation the leading term with respect to \( \lambda \) one can try to take into account next terms in the \( \lambda \)-expansion. So it seems that the \( \lambda \)-expansion looks like a systematic method similar to the semiclassical expansion or to the renormalization
group approach. Apparently it should exist a renormalization group approach corresponding to the anisotropic rescaling and the corresponding Callan-Symanzik equations could give, in particular, the Regge form of the amplitude (see [20]).

2 Asymptotics of correlation functions for scalar theories

2.1 Isotropic asymptotics of correlation functions

It is well known that asymptotics of perturbative connected correlations functions in a local field theory for small coordinates are governed by the ultraviolet dimension $\text{dim}_{n,k}$ of the corresponding diagram

$$< \phi(\lambda x_1) \phi(\lambda x_2) \ldots \phi(\lambda x_n) >^{(k)} \sim_{\lambda \rightarrow 0} (\lambda)^{-\text{dim}_{n,k}} G^{(k)}(x_1, x_2, \ldots x_n).$$

(2.1)

Here $k$ means the k-order of the perturbative theory.

The asymptotics (2.1) follows from the dimensional analysis. Indeed, considering for definiteness the case of selfinteracting scalar field $\phi$ in $d$-dimensional space-time

$$< \phi(\lambda x_1) \phi(\lambda x_2) \ldots \phi(\lambda x_n) > =$$

$$\int \phi(\lambda x_1) \ldots \phi(\lambda x_n) \exp\{i \int d^d x \left[ \frac{1}{2} (\partial \phi)^2 - \frac{m^2}{2} \phi^2 - V(\phi) \right] \} d\phi$$

(2.2)

and performing the change of variables

$$\phi(\lambda x) = (\lambda) \frac{2-d}{2} \tilde{\phi}(x)$$

(2.3)

in the path integral and also the change of variables $x \rightarrow \lambda x$ in the action one gets

$$< \phi(\lambda x_1) \ldots \phi(\lambda x_n) > = (\lambda)^{n \frac{2-d}{2}} \int \tilde{\phi}(x_1) \ldots \tilde{\phi}(x_n) \cdot$$

$$\exp\{i \int d^d x \left[ \frac{1}{2} (\partial \tilde{\phi})^2 - \lambda^2 \frac{m^2}{2} \tilde{\phi}^2 - \lambda^d V(\lambda \frac{2-d}{2} \tilde{\phi}) \right] \} d\tilde{\phi}.$$  

(2.4)

To get the ultraviolet asymptotic, i.e. $\lambda \rightarrow 0$ one can neglect the mass term and we see that the ultraviolet behaviour is given by the UV index of the corresponding diagram.

In particular, for $d = 4$, $V(\phi) = g \phi^4$ (2.4) gives

$$< \phi(\lambda x_1) \ldots \phi(\lambda x_n) > \sim_{\lambda \rightarrow 0} (\lambda)^{-n} < \phi(x_1) \ldots \phi(x_n) >_{m=0}$$

(2.5)

Ultraviolet divergences do not destroy this formal estimation at least for renormalizable theories and (2.1) takes place after suitable renormalizations. In this case using the renormalization group equations one can get the logarithmic corrections to the
formula \((2.1)\). Let us assume that we are working in the BPHZ subtraction scheme with a subtraction point \(\mu\) then from the dimensional analysis it follows

\[
\tilde{G}_{\text{ren}}(\{\frac{p_i}{\lambda}\}, g, \mu) = \left(\frac{1}{\lambda}\right)^{\text{dim}_{n,k}} \tilde{G}_{\text{ren}}(\{p_i\}, g, \lambda \mu),
\]

(2.6)

Here \(\tilde{G}_{\text{ren}}\) is the Fourier transformation of the renormalized Green function \(G_{\text{ren}}\) and asymptotics for small \(x\) are related with asymptotics of \(\tilde{G}_{\text{ren}}\) for large momenta,

\[
< \phi(\lambda x_1)\phi(\lambda x_2) \cdots \phi(\lambda x_n) >_{\text{ren,}\mu} = \left(\lambda\right)^{-d(n-1)} \int e^{i \sum p_i x_i} \delta^d(\sum p_i) \tilde{G}_{\text{ren}}(\{\frac{p_i}{\lambda}\}, g, \mu) \prod dp_i.
\]

(2.7)

In the l.h.s. of (2.6) one has now the renormalized Green functions but with the subtraction point different from the initial one. To restore the subtraction point one can use the renormalization group invariance and compensate the shift of the subtraction point by the change of the coupling constant and the renormalization of the wave function

\[
\tilde{G}_{\text{ren}}(\{p_i\}, g, \lambda \mu) = \xi(\frac{\mu}{\lambda})^{-k/2} \tilde{G}_{\text{ren}}(\{p_i\}, g(\frac{\mu}{\lambda}), \mu)
\]

(2.8)

here \(\xi\) is anomalous dimension. One finally gets the following well-known formula for the Fourier transformation of the correlation functions

\[
\tilde{G}_{\text{ren}}(\{\frac{p_i}{\lambda}\}, g, \mu) = \xi(\frac{\mu}{\lambda})^{-k/2} \left(\frac{1}{\lambda}\right)^{\text{dim}_{n,k}} \tilde{G}_{\text{ren}}(\{p_i\}, g(\frac{\mu}{\lambda}), \mu)
\]

(2.9)

### 2.2 \((2+2)\)-anisotropic asymptotics for scalar theories

Let us study the behaviour of correlation functions when only some components of coordinates in a preferable frame are supposed to be large or small. One of the most important examples corresponds to the \((2+2)\)-decomposition and it describes the case when all longitudinal components in the central mass frame are assumed much smaller than transversal ones. Let \(x^\mu\) be coordinates in the 4-dimensional Minkowski space-time and denote \(x^\mu = (y^\alpha, z^i), \alpha = 0, 1, \ i = 2, 3\). The \(2+2\) anisotropic asymptotics of Green functions of scalar selfinteracting theory describes the behaviour of the following correlation functions

\[
G_n(\{\lambda y_i, z_i\}) = < \phi(\lambda y_1, z_1)\phi(\lambda y_2, z_2)\cdots\phi(\lambda y_n, z_n) >
\]

(2.10)

for \(\lambda \to 0\).

For the scalar self interacting theory the correlation function (2.10) in the Euclidean regime is given by

\[
< \phi(\lambda y_1, z_1)\phi(\lambda y_2, z_2)\cdots\phi(\lambda y_n, z_n) > = \int \phi(\lambda y_1, z_1)\phi(\lambda y_2, z_2)\cdots\phi(\lambda y_n, z_n) \cdot \exp\{-\int d^4x \left[\frac{1}{2}(\partial_\alpha \phi)^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi)\right]\} d\phi.
\]

(2.11)
Performing in (2.11) the rescaling
\[ \phi(\lambda y, z) = \tilde{\phi}(y, z) \]  
and the change of variables in the action \( y \to \lambda y \) one gets
\[
G_n(\{\lambda y_i, z_i\}) = \int \tilde{\phi}(y_1, z_1)\tilde{\phi}(y_2, z_2)\cdots\tilde{\phi}(y_n, z_n) \exp\{-\int d^4x[\frac{1}{2}(\partial_\alpha \tilde{\phi})^2 + \frac{1}{2}\lambda^2(\partial_i \tilde{\phi})^2 + \lambda^2V(\tilde{\phi})]\} d\tilde{\phi}.
\]  
(2.13)

The anisotropic Green functions \( G_4(\{\lambda y_A, z_A\}) \), \( A = 1, 2, \ldots n \) with small longitudinal coordinates are related with the Green function with large longitudinal components of momenta \( G_n(\{\frac{p_A}{\lambda}, p_Ai\}) \)
\[
G_n(\{\frac{p_A}{\lambda}, p_Ai\}) = \int \prod_{A=1}^n dx_A e^{i \sum_A \frac{p_A}{\lambda} y_A + p_Ai z_A} G_n(\{y_A, z_A\}) = \\
(\lambda)^n \int \prod_{A=1}^n dy_A' dz_A' e^{i \sum_A \frac{p_A}{\lambda} y_A + p_Ai z_A} G_n(\{\lambda y_A', z_A\}) 
\]  
(2.14)

Therefore 1PI Green functions with large longitudinal components in momentum space can be expressed in terms of Green functions with rescaled longitudinal components
\[
G^{1PI}(\{\frac{p_A}{\lambda}, p_Ai\}) = \frac{1}{\lambda^2} G^{1PI}_{\lambda}(\{p_A, p_Ai\})
\]  
(2.15)

calculated in the theory with the Lagrangian
\[
L_\lambda = \frac{1}{2}(\partial_\alpha \tilde{\phi})^2 + \frac{\lambda^2}{2}(\partial_i \tilde{\phi})^2 + \lambda^2V(\tilde{\phi})
\]  
(2.16)

The extra term \( 1/\lambda^2 \) in (2.15) comes from \( \delta \)-function describing the momentum conservation.

Now we consider the asymptotics of the Green functions for the theory with the action (2.16) when \( \lambda \to 0 \). Let us start with the free action.

### 2.2.1 Free Propagator

Consider the action
\[
I_\lambda = \int d^4x\left[\frac{1}{2}(\partial_\alpha \phi)^2 + \frac{1}{2}\lambda^2(\partial_i \phi)^2\right]
\]  
(2.17)

One could think that an effective action for \( \lambda = 0 \) is just
\[
I = \int d^4x \frac{1}{2}(\partial_\alpha \phi)^2
\]
However we will easily see that in fact the effective action contains another term which can be interpreted as a conformal theory in momentum space. The free propagator for the action \((2.17)\) has the form

\[
G_{\lambda}(y, z) \equiv \frac{1}{(2\pi)^4} \int \frac{e^{ikx}}{k_{\alpha}^2 + \lambda^2 k_i^2} dk = \frac{1}{(2\pi)^2} \frac{1}{\lambda^2 y^2 + z^2}
\]

(2.18)

and is related with the standard propagator as

\[
G_{\lambda}(y, z) = G(\lambda y, z)
\]

(2.19)

Let us examine the limit of \((2.18)\) for \(\lambda \to 0\) in the sense of theory of distributions, i.e. consider the asymptotic behaviour of the integral

\[
(G_{\lambda}, f) = \frac{1}{(2\pi)^2} \int d^2 y d^2 z \frac{f(y, z)}{\lambda^2 y^2 + z^2}
\]

(2.20)

when \(\lambda \to 0\). Here \(f(y, z)\) is a test function. We cannot simply remove \(\lambda^2\) from denominator since the integral over \(y\)-variables diverges at \(z = 0\). One subtracts this divergence. One has

\[
(2\pi)^2 (G_{\lambda}, f) = \int d^2 y \int_{|z| \leq 1} d^2 z \frac{f(y, z) - f(y, 0)}{\lambda^2 y^2 + z^2} + \left( \int d^2 y \int_{|z| > 1} d^2 z \frac{f(y, 0)}{\lambda^2 y^2 + z^2} \right) \frac{1}{(2\pi)^2} \frac{1}{\lambda^2 y^2 + z^2}
\]

(2.21)

One takes \(\lambda = 0\) in the two first integrals and the answer can be regarded as a regularized version of \(1/z^2\). The third integral can be calculated explicitly. One gets

\[
G(\lambda y, z) = \frac{1}{4\pi} \delta^{(2)}(z) \ln \frac{1}{\lambda^2 y^2} + \frac{1}{4\pi^2} \frac{1}{z^2} + o(1)
\]

(2.22)

Here

\[
\frac{1}{z^2} \equiv \text{Reg} \frac{1}{z^2}
\]

\[
(\frac{1}{z^2}, f) = \int d^2 y \int_{|z| \leq 1} d^2 z \frac{f(y, z) - f(y, 0)}{z^2} + \int d^2 y \int_{|z| > 1} d^2 z \frac{f(y, z)}{z^2}
\]

(2.23)

One can interpret the formula \((2.22)\) by saying that one has two 2 dim conformal field theories here. The first one corresponds to the first term in \((2.22)\) and it is the standard conformal theory but only living in 4 dim space that gives the factor \(\delta^{(2)}(z)\). And one can interpret the second term in \((2.22)\) as the propagator for another conformal theory but living now in momentum space, i.e. we interpret \(z\)-coordinates as momenta.

For the Fourier transformation one has

\[
\tilde{G}_{\lambda}(k_{\alpha}, k_i) = \frac{1}{k_{\alpha}^2 + \lambda^2 k_i^2} = \pi \delta^{(2)}(k_{\alpha}) \ln \frac{1}{\lambda^2 k_i^2} + \frac{1}{(k_{\alpha})^2} + o(1),
\]

(2.24)
\(\alpha = 1, 2, \ i = 3, 4\). Here \(\frac{1}{(k_\alpha)^2} = \text{Reg} \frac{1}{(k_\alpha)^2}\) is understood in the sense of equation \((2.23)\). Integrating the R.H.S. of \((2.24)\) over \(k_\alpha\) and \(k_i\) by using the relations

\[
\int e^{ipy} \frac{d^2 p}{p^2} = \pi \ln \frac{1}{y^2} - 2\pi C_0, \quad C_0 = \int_0^1 \frac{1 - J_0(u)}{u} du - \int_1^\infty \frac{J_0(u) du}{u},
\]

\[
\pi \int e^{ikz} \ln \frac{1}{k^2} \frac{d^2 k}{z^2} = (2\pi)^2 + (2\pi)^3 C_0 \delta(z)
\]

where \(J_0\) is the Bessel function, we get the R.H.S. of \((2.22)\).

For the massive case one has

\[
\tilde{G}_{\lambda,m}(k_\alpha, k_i) = \frac{1}{k_\alpha^2 + \lambda^2 k_i^2 + \lambda^2 m^2} = \pi \delta^{(2)}(k_\alpha) \ln \frac{1}{\lambda^2(k_i^2 + m^2)} + \frac{1}{k_\alpha^2} + o(1),
\]

### 2.2.2 Free generating functional

Let us consider a generating functional for free Green functions for large longitudinal momenta,

\[
Z(J_\lambda) = \int \exp\{\int d^4x \left[\frac{1}{2} (\partial_\alpha \phi)^2 + J_\lambda(x) \phi(x)\right]\} d\phi.
\]

We suppose that the source \(J_\lambda(x)\) has the Fourier transformation \(J_\lambda(p_\alpha, p_i) = J_1(\lambda k_\alpha, k_i)\) which nonvanish only for \(|p_\alpha| \leq |p_i|, \ \alpha = 0, 1\) and \(i = 2, 3\). One has

\[
Z(J_\lambda) = \exp\left\{\frac{i}{2} \int d^4k J_\lambda(k) \frac{1}{k^2} J_\lambda(-k)\right\} = \exp\left\{\frac{i}{2} \int d^4k J_1(\lambda k_\alpha, k_i) \frac{1}{k^2} J_1(\lambda k_\alpha, k_i)\right\}
\]

Performing a change of variables \(\lambda k_\alpha = k'_\alpha\) one can represent the generating functional as

\[
Z(J_\lambda) = \exp\left\{\frac{i}{2} \int d^4k J_1(k) \frac{1}{k_\alpha^2 + \lambda^2 k_i^2} J_1(-k)\right\}
\]

The behaviour of \(Z(J_\lambda)\) for \(\lambda \to 0\) is given by

\[
\exp\left\{\frac{i}{2} \int d^4k J_1(k) \left[\frac{1}{(k_\alpha)^2} + \pi \delta^{(2)}(k_\alpha) \ln \frac{1}{\lambda^2 k_i^2} + o(1)\right] J_1(-k)\right\}
\]

The contribution of the term with \(\delta^{(2)}\)-function dissipates due to the assumption about the support of the current \(J_1(k)\) and we can say that the effective action producing this generating functional is simply given by

\[
I = \int d^4x \frac{1}{2} (\partial_\alpha \phi)^2
\]

Let us stress that one gets the effective action \((2.31)\) only under the above assumptions on the support of the source \(J_1(k)\).
2.2.3 Long lines free generating functional

Let us consider a non-local 2-lines correlation function,

\[ G_{\Gamma_1 \Gamma_2} = \int_{\Gamma_1} \int_{\Gamma_2} G(x) d\Gamma_1 d\Gamma_2, \quad (2.32) \]

which is obtained by an integration of the usual 2-point Green function along some lines \( \Gamma_1 \) and \( \Gamma_2 \). For example, for the lines \( \Gamma_0 \) and \( \Gamma_1 \) which correspond to \( y_1 = 0, z_i = 0 \) and \( y_0 = y_{00}, z_i = z_{i0} \) (see fig.1) one has

\[ G_{\Gamma_0 \Gamma_1}(z_{i0}) = \int dy_0 \int dy_1 G(y_0, y_1, z_{i0}) \quad (2.33) \]

We assume an infrared regularization so that the integration in (2.33) is performed from \(-L\) to \(L\). We are interested in infrared asymptotics of this correlator when \(L \to \infty\). Performing the following change of variables

\[ y_\alpha = Ly'_\alpha \quad (2.34) \]

we get

\[ G_{\Gamma_0 \Gamma_1}(z_{i0}) = L^2 \int_{-1}^{1} dy'_0 \int_{-1}^{1} dy'_1 G(Ly'_0, Ly'_1, z_{i0}) = \int_{-1}^{1} dy'_0 \int_{-1}^{1} dy'_1 G(y'_0, y'_1, \frac{z_{i0}}{L}) \quad (2.35) \]

Therefore the infrared asymptotics of (2.33) is given by the following formula

\[ G_{\Gamma_0 \Gamma_1}(z_{i0}) = \int_{-1}^{1} dy'_0 \int_{-1}^{1} dy'_1 \left[ \frac{1}{4\pi^2} \frac{1}{y'^2} + \frac{1}{4\pi} \delta^{(2)}(y') \ln \frac{CL^2}{z^2} + o(1) \right] = \frac{1}{4\pi} \ln \frac{CL^2}{z^2} + \text{const.} \quad (2.36) \]
The Fourier transformation of the leading term of this asymptotics is given according to (2.25) and (2.26) by

\[ \tilde{G}_{\Gamma_0 \Gamma_1}(k_i) = \int d^2 z e^{i k_i z_i} G_{\Gamma_0 \Gamma_1}(z_i) = (2.37) \]

\[ \left( \frac{2\pi}{4\pi} \ln C L^2 + 8\pi^2 C_0 \right) \delta^2(k_i) + \frac{4\pi}{k_i^2} \]

This correlator for \( k_i \neq 0 \) can be regarded as correlator of 2d effective theory with a simple effective action

\[ S = \int d^2 z (\partial_i \phi(z))^2 \quad (2.38) \]

### 3 Anisotropic Asymptotics and Regge Regime

Let us show that (2+2) anisotropic asymptotics for 4-points functions are related with the Regge regime \((s >> t)\) of the scattering amplitudes.

Consider the 4-point 1PI Green function \( \tilde{G}(k_1, k_2, k_3, k_4) \) on the mass-shell, \( k_i^2 = m^2 \). One can choose a coordinate frame so that momenta of particles are \( k_1 = p_1 + q/2, k_2 = p_2 - q/2, k_3 = p_2 + q/2 \) and \( k_4 = p_1 - q/2 \) (see fig.2) with \( p_1, p_2 \) and \( q \) given by

\[ p_1 = \left( \frac{1}{2} \sqrt{s}, \frac{1}{2} \sqrt{-u}, 0, 0 \right), \quad (3.1) \]

\[ p_2 = \left( \frac{1}{2} \sqrt{s}, -\frac{1}{2} \sqrt{-u}, 0, 0 \right) \quad (3.2) \]

\[ q = (0, 0, q_i), \quad q_i = (q_2, q_3), \quad (3.3) \]

\( s, t \) and \( u \) are the Mandelstam variables and \( s + u + t = 4m^2 \).

The regime \( s >> t \) corresponds to

\[ s = \frac{\tilde{s}}{\lambda^2} \quad \text{with} \quad \tilde{s}, t \text{ fixed and} \quad \lambda \to 0. \]
Since in this regime \( u \) also can be represent as \( u = \frac{\bar{u}}{\lambda^2} \) with \( \bar{u} \) fixed, the longitudinal vectors \( p_1 \) and \( p_2 \) have the asymptotic form

\[
p_1 = \frac{\bar{p}_1}{\lambda}, \quad p_2 = \frac{\bar{p}_2}{\lambda}, \quad \text{with } \bar{p}_1, \bar{p}_2 \text{ fixed and } \lambda \to 0
\]

Therefore in the chosen coordinate frame the regime with \( s = \frac{\bar{s}}{\lambda^2}, \lambda \to 0 \) and \( \bar{s} \) and \( t \) fixed is described by the 1PI 4-point Green function \( G_4((\bar{p}_1/\lambda, \frac{q}{2}), (\bar{p}_2/\lambda, -\frac{q}{2}), (\bar{p}_1/\lambda, \frac{q}{2}), (\bar{p}_2/\lambda, -\frac{q}{2})) \) with large longitudinal coordinates in the momentum space.

### 3.1 4-point tree diagramm

Let us consider the diagram in fig.3 Assume that the c.m. coordinate frame is chosen as in fig.2 with momenta (3.1)-(3.2). In this frame the 1PI 4-point Green function \( G_4((\bar{p}_1/\lambda, \frac{q}{2}), (\bar{p}_2/\lambda, -\frac{q}{2}), (\bar{p}_1/\lambda, \frac{q}{2}), (\bar{p}_2/\lambda, -\frac{q}{2})) \) corresponding to the tree diagram (fig.2) according to (2.15) is equal \( \frac{q}{2i} \), i.e. we get the answer which can be described by the effective action (2.38) which is the effective action for the special non-local correlators (2.33). Assuming that the lines \( \Gamma_1 \) and \( \Gamma_2 \) go along two light-cone lines \( \Gamma_+ \) and \( \Gamma_- \) we get an interpretation of \( G_{\Gamma_+\Gamma_-} \) as a scattering amplitude of two ultrarelativistic particles.

### 3.2 4-point one-loop

Now we are going to consider loop corrections. The high-energy asymptotics of 1-loop diagram is well known \([15]\) \[3.1\]

\[
G_{box}^{1PI}(s, t) \underset{s \to \infty, \ t \ fixed}{\sim} g^4 \frac{1}{2s} \left[ \ln s - i\pi \right] J(q)
\]

where

\[
J(q) = \frac{1}{2} \int d^2k_i \frac{1}{k_i^2 + m^2} \cdot \frac{1}{(k_i - q_i)^2 + m^2}
\]

One can see immediately that the same answer describes the asymptotic behaviour for \( \lambda \to 0 \) of the 1-box diagramm corresponding to the rescaled action (2.16). Indeed
the asymptotics of the integral
\[
G_{1\text{box}}^{\lambda P I}(\tilde{p}_1, \tilde{p}_2, q) = \frac{\lambda^6 g^4}{2} \int dk_+ dk_- d^2k_i [k_+ k_- - \lambda^2 k_i^2 - \lambda^2 m^2 + i\epsilon]^{-1} [k_+ (k_- + \sqrt{2s}) - \lambda^2 (k_i - q_i) + i\epsilon]^{-1} [k_+ (k_- + \sqrt{2s}) - \lambda^2 (k_i - q_i) + i\epsilon]^{-1}
\]

(3.6)

\[\tilde{s} = (\tilde{p}_1 + \tilde{p}_2)^2,\]

Note that (3.7) takes place only for massive theory. The asymptotics of box diagram with massless internal lines and massive external line is

\[
G_{1\text{box}}^{\lambda P I}(\tilde{p}_1, \tilde{p}_2, q) \sim \lambda \to 0 \frac{i\pi}{2} g^4 \lambda^2 (\ln \frac{m^4}{\lambda^2 \tilde{s}})^2
\]

(3.8)

that is in the agreement with exact answer for the massless box diagram [24].

4 Anisotropic asymptotics for gauge theories

Let us now consider the anisotropic asymptotics of correlation functions for the gauge theory

\[
G_\alpha(\{\lambda y_j, z_j\}) = \langle A_{\mu_1}(\lambda y_1, z_1) \ldots A_{\mu_n}(\lambda y_n, z_n) \rangle \quad \text{for } \lambda \to 0.
\]

(4.1)

\(< ... >\) means the average

\[
\langle A_{\mu_1}(\lambda y_1, z_1) \ldots A_{\mu_n}(\lambda y_n, z_n) \rangle =
\]

\[
\int A_{\mu_1}(\lambda y_1, z_1) \ldots A_{\mu_n}(\lambda y_n, z_n) \exp i \int d^4x \left\{ \frac{1}{2\beta} (\partial_\lambda A_\mu) + \bar{c}Mc \right\} dAd\bar{c}dc,
\]

where \(F_{\mu\nu}\) is the field strength, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\), \(A_\mu = A_\mu^a \tau^a\), \(g\) is the coupling constant and \(\tau^a\) are the generators of the Lie algebra of the gauge group \(G = SU(N)\), \(M = \partial_\mu D_\mu\), \(D_\mu = \partial_\mu + A_\mu\). Let us note that the r.h.s. of (4.2) has a rather formal meaning. One has to assume some regularization procedure and moreover the functional integral being understood in the perturbation theory can be used to calculate correlation functions only for short distances. Let us ignore for a moment these subtleties and perform an estimation of the asymptotics of r.h.s. of (4.2) by using the anisotropic dimensional analysis. Performing the change of variables

\[
A_\alpha(\lambda y, z) = \frac{1}{\lambda} \tilde{A}_\alpha(y, z), \quad A_i(\lambda y, z) = \tilde{A}_i(y, z)
\]

(4.3)
in the path integral (4.2) and also the change of variables in the integral over $y$-space we see that the correlation functions (4.1) can be computed with the rescaled Yang-Mills action, i.e.

$$< A_{\mu_1}(\lambda y_1, z_1) ... A_{\mu_n}(\lambda y_n, z_n) > = \left( \frac{1}{\lambda} \right)^n < A_{\mu_1}(y_1, z_1) ... A_{\mu_n}(y_n, z_n) >_{S_{\lambda}}, \tag{4.4}$$

$$< A_{\mu_1}(y_1, z_1) ... A_{\mu_n}(y_n, z_n) >_{S_{\lambda}} = \int A_{\mu_1}(y_1, z_1) ... A_{\mu_n}(y_n, z_n) \exp \{ i S_{\lambda} \} \, dA, \tag{4.5}$$

where $\mu_{kj} = 0, 1$ for $j = 1, ... m$ and $\mu_{kj} = 2, 3$ for $j = m + 1, ... n$ and

$$S_{\lambda} = \frac{1}{4\lambda^2 g^2} \int d^4x \, \text{tr} \, (F_{\alpha\beta} F^{\alpha\beta}) + \frac{1}{2g^2} \int d^4x \, \text{tr} \, (F_{\alpha j} F^{\alpha j}) + \frac{\lambda^2}{4g^2} \int d^4x \, \text{tr} \, (F_{ij} F^{ij}) \tag{4.6}$$

+ gauge fixing terms

The action (4.6) has the form of Verlinde’s [1] rescaled action. Let us discuss the limit $\lambda \to 0$. Note that the rescaling (4.3) for the Wilson loop operator $W(\Gamma ) = P \exp \int_{\Gamma} A_\alpha dx^\alpha$ for loops belonging to the longitudinal plane gives the relation

$$< W(\Gamma_1^i) W(\Gamma_2^i) > = < W(\Gamma_1^i) W(\Gamma_2^i) >_{S'}, \tag{4.7}$$

where $\Gamma_1^i$ is the rescaled loop. In particular for the infinite long lines $\Gamma_+, \Gamma_-$, one gets at the formal level the relation

$$< W(\Gamma_+) W(\Gamma_-) > = < W(\Gamma_+) W(\Gamma_-) >_{S'}. \tag{4.8}$$

Now one can try to consider the parameter $\lambda$ like a parameter in the quasiclassical expansion. To make sense to such an expansion one has to use a regularization. A more suitable regularization is the lattice regularization. Since we intend to study the theory in the region of small longitudinal coordinates and large transversal coordinates and we want to take into account fluctuations in different direction with an approximately equal precision it is relevant to assume that the lattice spacing in the longitudinal and transversal directions are different so that there are equal number of points in the different directions (see fig 4) [3]. If the number of points in different directions is the same, i.e. $n_0 = n_1 = ... n_{D - 1}$ then we get a theory in an asymmetric space-time volume $L_0 L_1 ... L_{d - 1}$, $L_i$ is a typical size in the $i$-direction, with lattice spacing $a_0, a_1, ... a_{D - 1}$, such that $L_0 / L_1 = a_0 / a_1, ..., L_0 / L_D = a_0 / a_D$.

A general form of the lattice action on an asymmetric lattice with lattice spacing $a_0, a_1, ..., a_{D - 1}$ in $0, 1, ..., D$-1-directions has the form

$$S = \frac{1}{4g^2} a_0 a_1 ... a_{D - 1} \sum_{x, \mu, \nu} \frac{1}{(a_\mu a_\nu)^2} \text{tr} \, (U(\square_{\mu, \nu}) - 1), \tag{4.9}$$

Here $x$ are points of the 4-dimensional lattice, $\square_{\mu, \nu}$ is a single plaquette attached to the links $(x, x + \mu)$ and $(x, x + \nu)$, $U(\square_{\mu, \nu}) = U_{x, \mu} U_{x + \mu, \nu} U_{x + \nu, \mu} U_{x, \nu}$ and link variables
\(U_{x,\mu}\) are associated with the link between the lattice sites \(x\) and \(x + \hat{\mu}\). \(U_{x,\mu}\) belongs to a representation of the gauge group \(SU(N)\).

Since we are interested in the case when \(L_0 = L_1, L_2 = L_3\) and \(L_1/L_3 \to 0\) we put in (4.9) \(a_0 = a_1, a_2 = a_3\) and \(a_0 = \lambda a_3\), so we get

\[
S = \frac{1}{4\lambda^2 g^2} \sum x \sum_{\alpha,\beta} \text{tr} \left( U(\square_{\alpha,\beta}) - 1 \right) + \frac{1}{4 g^2} \sum x \sum_{\alpha,i} \text{tr} \left( U(\square_{\alpha,i}) - 1 \right) \tag{4.10}
\]

\[
+ \frac{\lambda^2}{4 g^2} \sum x \sum_{i,j} \text{tr} \left( U(\square_{i,j}) - 1 \right).
\]

Here \(\alpha\) and \(\beta\) are unit vectors in the longitudinal direction and \(i, j\) are unit vectors in the transversal direction. We also denote the points of 4-dimensional lattice as \(x = (y, z)\), where \(y\) and \(z\) are the points of two two-dimensional lattices, say, \(y\)-lattice (longitudinal) and \(z\)-lattice (transversal). Performing the \(\lambda \to 0\) limit in the lattice action (4.9) we get

\[
S_{\text{tr}} = \frac{1}{4 g^2} \sum x \sum_{\alpha,i} \text{tr} \left( U(\square_{\alpha,i}) - 1 \right), \tag{4.11}
\]

with \(U_{x,\alpha}\) being a subject of the relation

\[
U(\square_{\alpha,\beta}) = 1. \tag{4.12}
\]

Therefore \(U_{x,\alpha}\) is a zero-curvature lattice gauge field

\[
U_{x,\alpha} = V_x V_{x+\alpha}^+. \tag{4.13}
\]

Substituting (4.13) in (4.11) we get

\[
S_{\text{tr}} = \frac{1}{4 g^2} \sum x \sum_{\alpha,i} \text{tr} \left( U(\square_{\alpha,i}) - 1 \right), \tag{4.14}
\]

or

\[
S_{\text{tr}} = \frac{1}{4 g^2} \sum x \sum_{\alpha,i} \text{tr} \left( \bar{U}_{x+\alpha,i} \bar{U}_{x,i} - 1 \right) \tag{4.15}
\]

where

\[
\bar{U}_x = V_{x+1}^+ V_{x}^+. \tag{4.16}
\]

The action (4.13) is ultra-local in the transverse \(z\)-direction and it is the lattice chiral field action in the longitudinal \(y\)-direction. In the formal continuum limit \(a_0 \to 0\) we get

\[
S_{\text{c,l}} = \frac{1}{4 g^2} \int dy^0 dy^1 \sum_{z,\alpha} \text{tr} \left[ \partial_{\alpha} \bar{U}_{z,i}(y) \partial_{\alpha} \bar{U}_{z,i}(y) \right] \tag{4.17}
\]

with summation over the repeating indices \(\alpha = 0, 1\) and \(\bar{U}_{z,i}(y) = V_{z+1}^+(y) U_{z,i}^+(y) V_{z}(y)\).

To get its pure continuous version one has also to make in (4.17) and (4.16) a formal limit \(a_t \to 0\) under the assumption \(U_{x,i} = \exp(a_t A_i)\)

\[
S_{\text{c,c}} = \frac{1}{4 g^2} \int dy^0 dy^1 d^2 z_i \sum_{\alpha,i} \text{tr} \left[ \partial_{\alpha} \bar{A}_{i} \partial_{\alpha} \bar{A}_{i} \right] \tag{4.18}
\]
where
\[ \tilde{A}_i = V^+ D_i V, \quad D_i = \partial_i + A_i, \] (4.19)

Therefore in the formal limit \( a_i \to 0 \) the action (4.17) reproduces the Verlinde and Verlinde truncated action. However non-perturbatively there is an essential difference in the behaviour of two theories (4.18) and (4.17).

The lattice version of (4.8) for the zero curvature longitudinal gauge fields is simply reduced to the boundary values of the field \( V_z(y) \)
\[ V_+ (L, 0, z) = V_z (L, 0), \quad V_- (0, L, z) = V_z (0, -L). \] (4.20)
and the expectation value of these string operators is given by
\[ \langle V_+ (L, 0, 0) V_- (0, L, z) \rangle = \] (4.21)
\[ \int V_0 (-L, 0) V_0^+ (L, 0) V_z (0, -L) V_z^+ (0, L) \exp S^{\text{tr}} (V, U) dV dU \]
with \( S^{\text{tr}} (V, U) \) as in (4.13). In the continuum limit in the longitudinal direction we use \( S^{\text{tr}} \) given by (4.17). Our goal consists in calculation the functional integral (4.21) over gauge fields \( U \) to get an effective action describing an interaction of fields \( V_z (0, \pm L) \). The action (4.15) is the action for two-dimensional lattice non-linear \( \sigma \)-model in finite volume. Transversal dynamics arises from boundary effects for two-dimensional (longitudinal) non-linear \( \sigma \)-model. Therefore to get an effective action describing the transversal dynamics we have to calculate the Schr"{o}dinger functional for two-dimensional non-linear \( \sigma \)-model [10]. An exact solution of this problem is unknown. An approximate effective action is obtained in [3] under the assumption that the \( \sigma \)-model has massive excitations.

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