Non-convex regularization of bilinear and quadratic inverse problems by tensorial lifting

Robert Beinert© and Kristian Bredies©

Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstraße 36, 8010 Graz, Austria

E-mail: robert.beinert@uni-graz.at and kristian.bredies@uni-graz.at

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Abstract
Considering the question: how non-linear may a non-linear operator be in order to extend the linear regularization theory, we introduce the class of dilinear mappings, which covers linear, bilinear, and quadratic operators between Banach spaces. The corresponding dilinear inverse problems cover blind deconvolution, deautoconvolution, parallel imaging in MRI, and the phase retrieval problem. Based on the universal property of the tensor product, the central idea is here to lift the non-linear mappings to linear representatives on a suitable topological tensor space. At the same time, we extend the class of usually convex regularization functionals to the class of diconvex functionals, which are likewise defined by a tensorial lifting. Generalizing the concepts of subgradients and Bregman distances from convex analysis to the new framework, we analyse the novel class of dilinear inverse problems with non-convex regularization terms and establish convergence rates with respect to a generalized Bregman distance under similar conditions than in the linear setting. Considering the deautoconvolution problem as specific application, we derive satisfiable source conditions and validate the theoretical convergence rates numerically.

Keywords: ill-posed inverse problems, bilinear and quadratic equations, non-convex Tikhonov regularization, convergence rates, deautoconvolution

(Some figures may appear in colour only in the online journal)
1. Introduction

Nowadays the theory of inverse problems has become one of the central mathematical approaches to solve recovery problems in medicine, engineering, and life sciences. Some of the main applications are computed tomography (CT), magnetic resonance imaging (MRI), and deconvolution problems in microscopy, see for instance [BB98, MS12, Ram05, SW13, Uhl03, Uhl13] being recent monographs as well as many other publications.

The beginnings of the modern regularization theory for ill-posed problems are tracing back to the pioneering works of Tikhonov [Tik63a, Tik63b]. Between then and now, the theory has been heavily extended and covers linear and non-linear formulations in the Hilbert space as well as in the Banach space setting. In order to name at least a few of the numerous monographs, we refer to [BG94, EHN96, Hof86, Lou89, LP13, Mor84, SKHK12, TA77, TLY98]. Due to the enormous relevance of the research topic, the published literature embraces many further monographs as well as a vast number of research articles.

Especially for linear problem formulations, the analytical framework is highly developed and allows the general treatment of inverse problems with continuous operators, see for instance [EHN96, MS12, SKHK12] and references therein. In addition to the sophisticated analysis, efficient numerical implementations of the solution schemes are available for practitioners [EHN96, Sch11]. The interaction between analysis and numerics are one reason for the great, interdisciplinary success of the linear regularization theory for ill-posed inverse problems.

If the linear operator in the problem formulation is replaced by a non-linear operator, the situation changes dramatically. Depending on the operator, there are several regularization approaches with different benefits and drawbacks [EHN96, Gra11, HKPS07, SGG+09]. One standard approach to regularize non-linear operators is to introduce suitable non-linearity conditions and to restrict the set of considered operators. These constraints are mostly based on properties of the remainder of the first-order Taylor expansion [BO04, EKN89, HKPS07, RS06]. In an abstract way, this approach allows the generalization of well-understood linear results by controlling the deviation from the linear setting. Unfortunately, the validation of the required assumptions for a specific non-linear operator is a non-trivial task.

In order to extend the linear theory to the non-linear domain further, our idea is to introduce a class of operators that covers many interesting applications for practitioners and, at the same time, allows a general treatment of the corresponding inverse problems. More precisely, we introduce the class of dilinear operators that embraces linear, bilinear, and quadratic mappings between Banach spaces. Consequently, our novel class of dilinear inverse problems covers formulations arising in imaging and physics [SGG+09] like blind deconvolution [BS01, JR06], deautoconvolution [GH94, GHB+14, ABHS16], parallel imaging in MRI [BBM04], or phase retrieval [DF87, Mil90, SSD+06].

The central idea behind the class of dilinear operators is the universal property of the topological tensor product, which enables us to lift a continuous but non-linear mapping to a linear operator. Owing to the lifting, we get immediate access to the linear regularization theory. On the downside, a simple lifting of the non-linear inverse problem causes an additional non-convex rank-one constraint, which is similarly challenging to handle than the original non-linear problem. For this reason, most results of the linear regularization theory are not applicable for the lifted problem and cannot be transferred to the original (unlifted) inverse problem. In order to overcome this issues, we use the tensorial lifting indirectly and generalize the required concepts from convex analysis to the new framework.

The recent literature already contains some further ideas to handle inverse problems arising from bilinear or quadratic operators. For instance, each quadratic mapping between separable...
Hilbert spaces may be factorized into a linear operator and a strong quadratic isometry so that the corresponding inverse problem can be decomposed into a possibly ill-posed linear and a well-posed quadratic part, see [Fle14, Fle18]. In order to determine a solution, one can now apply a two-step method. Firstly, the ill-posed linear part is solved by a linear regularization method. Secondly, the well-posed quadratic problem is solved by projection onto the manifold of symmetric rank-one tensors. The main drawback of this approach is that the solution of the linear part has not to lie in the range of the well-posed quadratic operator such that the second step may corrupt the obtained solution. This issue does not occur if the forward operator of the linear part is injective, which is generally not true for quadratic inverse problems.

Besides the forward operator, one can also generalize the used regularization from usually convex to non-convex functionals. An abstract analysis of non-convex regularization methods for bounded linear operators between Hilbert spaces has been introduced in [Gra10], where the definition of the subdifferential and the Bregman distance have been extended with respect to an arbitrary set of functions. On the basis of a variational source condition, one can further obtain convergence rates for these non-convex regularization methods. Similarly to [Gra10], we employ non-convex regularizations—however—with the tensorial lifting in mind.

As mentioned above, the deautoconvolution problem is one specific instance of a dilinear inverse problem [GHB'14, ABHS16]. Although the unregularized problem can have two different solutions at the most, the deautoconvolution problem is everywhere locally ill posed [GH94, FH96, Ger11]. Nevertheless, with an appropriate regularization, very accurate numerical solutions can be obtained [CL05, ABHS16]. Establishing theoretical convergence rates for the applied regularization is, unfortunately, very challenging since most conditions for the non-linear theory are not fulfilled [BH15, ABHS16]. For more specific classes of true solutions, for instance, the class of all trigonometric polynomials or some subset of Sobolev spaces, however, the regularized solutions converge to the true solution with a provable rate, see [BFH16] or [Jan00, DL08] respectively. Applying our novel regularization theory, we establish convergence rates with respect to the generalized Bregman distance under a source-wise representation of the subdifferential of the regularization functional. In other words, we generalize the classical range source condition in a specific manner fitting the necessities of dilinear inverse problems. The main advantage of this approach is that the derived convergence rates for non-convex penalty terms do not rely on abstract smoothness and non-linearity conditions, which usually are not fulfilled for the autoconvolution operator [BH15, ABHS16].

In this paper, we show that the essential results of the classical regularization theory with bounded linear operators and convex regularization functionals may be extended to bilinear and quadratic forward operators. At the same time, we allow the regularization functional to be non-convex in a manner being comparable with the non-linearity of the considered operator.

Since our analysis is mainly based on the properties of the topological tensor product [DF93, Rya02], we firstly give a brief survey of tensor spaces and the tensorial lifting in section 2. Our main foci are here the different interpretations of a specific tensor. Further, we introduce the set of dilinear operators and show that each dilinear operator may be uniquely lifted to a linear operator. Analogously, in section 3, the class of diconvex functionals is defined through a convex lifting. In order to study diconvex regularization methods, we generalize the usually convex subdifferential and Bregman distance with respect to dilinear mappings. Further, we derive sum and chain rules for the new dilinear subdifferential calculus.

Our main results about dilinear—bilinear and quadratic—inverse problems are presented in section 4. Under suitable assumptions being comparable to the assumptions in the classical
linear theory, dilinear inverse problems are well posed, stable, and consistent. Using a source-wise representation as in the classical range condition, we obtain convergence rates for the data fidelity term and the BREGMAN distance between the true solution of the undisrupted and the regularized problem.

In section 5, we apply our non-convex regularization theory for dilinear inverse problems to the deautoconvolution problem and study the required assumptions in more detail. Further, we reformulate the source-wise representation and obtain an equivalent source condition, which allows us to construct suitable true solutions. With the numerical experiments in section 6, we verify the derived rates for the deautoconvolution numerically and give some examples of possible signals fulfilling the source condition.

2. Tensor products and dilinear mappings

The calculus of tensors has been invented more than a century ago. Since then tensors have been extensively studied from an algebraic and topological point of view, see for instance [DF93, DFS08, Lic66, Rom08, Rya02] and references therein. One of the most remarkable results in tensor analysis, which is the starting point for our study of non-linear inverse problems, is the universal property of the tensor product that allows us to lift a bilinear mapping to a linear one. In order to get some intuition about tensor calculus on BANACH spaces and about the different interpretations of a specific tensor, we briefly survey the required central ideas about tensor products and adapt the lifting approach to the class of dilinear forward operators.

The tensor product of two real BANACH spaces \( X_1 \) and \( X_2 \) denoted by \( X_1 \otimes X_2 \) can be constructed in various ways. Following the presentations of Ryan [Rya02] and Diestel [DFS08], we initially define a tensor as a linear operator acting on bounded bilinear forms. More precisely, the tensor product \( X_1 \otimes X_2 \) is defined by

\[
\left( u \otimes v \right)(A) = A(u, v)
\]

On the basis of this construction, the tensor product \( X_1 \otimes X_2 \) of the real BANACH spaces \( X_1 \) and \( X_2 \) now consists of all finite linear combinations \( w = \sum_{n=1}^{N} \lambda_n u_n \otimes v_n \) with \( u_n \in V_1 \), \( v_n \in V_2 \), \( \lambda_n \in \mathbb{R} \), and \( N \in \mathbb{N} \).

There are several approaches to define an appropriate norm on the tensor product \( X_1 \otimes X_2 \). Here we employ the projective norm, which allows us to lift each bounded bilinear operator to a bounded linear operator on the tensor product. More precisely, the projective norm \( \| \cdot \|_\pi \) is defined by

\[
\| w \|_\pi := \inf \left\{ \sum_{n=1}^{N} \| u_n \| \| v_n \| : w = \sum_{n=1}^{N} u_n \otimes v_n \right\},
\]

where the infimum is taken over all finite representations of \( w \in X_1 \otimes X_2 \), see for instance [DF93, Rya02]. The projective norm on \( X_1 \otimes X_2 \) belongs to the reasonable crossnorms, which means that \( \| u \otimes v \|_\pi = \| u \| \| v \| \) for all \( u \in X_1 \) and \( v \in X_2 \), see [DFS08, Rya02].
The completion of the tensor product $X_1 \otimes X_2$ with respect to the projective norm now forms the projective tensor product $X_1 \hat{\otimes} X_2$.

If one of the Banach spaces $X_1$ or $X_2$ is a dual space, the projective tensor product can be embedded into the space of bounded linear operators. More precisely, we have $X_1^* \otimes_{\pi} X_2 \subset \mathcal{L}(X_1, X_2)$ and $X_1 \otimes_{\pi} X_2^* \subset \mathcal{L}(X_2, X_1)$, see for instance [Won79]. Depending on the situation, a tensor $w = \sum_{n=1}^{\infty} \lambda_n u_n \otimes v_n$ in $X_1^* \otimes_{\pi} X_2$ or $X_1 \otimes_{\pi} X_2^*$ generates the linear mapping

$$L_w(\phi) := \sum_{n=1}^{N} \lambda_n u_n(\phi) v_n \quad \text{or} \quad R_w(\psi) := \sum_{n=1}^{N} \lambda_n v_n(\psi) u_n$$

(1)

respectively. In the Hilbert space setting, the projective tensor product $H_1 \otimes_{\pi} H_2$ of the Hilbert spaces $H_1$ and $H_2$ corresponds to the trace class operators, and the projective norm is given by $\|w\|_{\pi} = \sum_{n=1}^{\infty} \sigma_n(w)$ for all $w \in H_1 \otimes_{\pi} H_2$, where $\sigma_n(w)$ denotes the $n$th singular value of $w$, see [Wer02].

The main benefit of equipping the tensor product with the projective norm is that each bounded bilinear mapping $A: X_1 \times X_2 \rightarrow W$ can be uniquely lifted to a linear operators $\tilde{A}: X_1 \otimes_{\pi} X_2 \rightarrow W$ by $\tilde{A}(\sum_{n=1}^{\infty} \lambda_n u_n \otimes v_n) = \sum_{n=1}^{\infty} \lambda_n A(u_n, v_n)$. More precisely, the lifting of bounded bilinear operators can be stated in the following form, see for instance [Rya02].

**Proposition 2.1 (Lifting of bounded bilinear mappings).** Let $A: X_1 \times X_2 \rightarrow Y$ be a bounded bilinear operator, where $X_1, X_2,$ and $Y$ denote real Banach spaces. Then there exists a unique bounded linear operator $\tilde{A}: X_1 \otimes_{\pi} X_2 \rightarrow Y$ so that $A(u, v) = \tilde{A}(u \otimes v)$ for all $(u, v)$ in $X_1 \times X_2$.

The lifting of a bilinear operator in proposition 2.1 is also called the universal property of the projective tensor product. Vice versa, each bounded linear mapping $\tilde{A}: X_1 \otimes_{\pi} X_2 \rightarrow Y$ uniquely defines a bounded bilinear mapping $A$ by $A(u, v) = \tilde{A}(u \otimes v)$, which gives the canonical identification

$$\mathcal{B}(X_1 \times X_2, Y) = \mathcal{L}(X_1 \otimes_{\pi} X_2, Y).$$

Consequently, the topological dual of the projective tensor product $X_1 \otimes_{\pi} X_2$ is the space $\mathcal{B}(X_1 \times X_2)$ of bounded bilinear forms, where a specific bounded bilinear mapping $A: X_1 \times X_2 \rightarrow \mathbb{R}$ acts on an arbitrary tensor $w = \sum_{n=1}^{\infty} \lambda_n u_n \otimes v_n$ by $\langle A, w \rangle = \sum_{n=1}^{\infty} \lambda_n A(u_n, v_n)$, see for instance [Rya02].

In order to define the novel class of dilinear operators, we restrict the projective tensor product to the subspace of symmetric tensors. Assuming that $X$ is a real Banach space, we call a tensor $w \in X \otimes_{\pi} X$ symmetric if and only if $w = w^t$, where the transpose of $w = \sum_{n=1}^{\infty} \lambda_n u_n \otimes v_n$ is given by $w^t := \sum_{n=1}^{\infty} \lambda_n v_n \otimes u_n$. In the following, the closed subspace of the symmetric tensors spanned by $u \otimes u$ with $u \in X$ is denoted by $X \otimes_{\pi, \text{sym}} X$. Considering that the annihilator

$$(X \otimes_{\pi, \text{sym}} X)^\perp := \{ A \in \mathcal{B}(X \times X) : A(w) = 0 \text{ for all } w \in X \otimes_{\pi, \text{sym}} X \}$$

consists of all antisymmetric bilinear forms, which means $A(u, v) = -A(v, u)$, the topological dual of $X \otimes_{\pi, \text{sym}} X$ is isometric isomorphic to the space $\mathcal{B}_{\text{sym}}(X \times X)$ of symmetric, bounded bilinear forms, see for instance [Meg98].

On the basis of these preliminary considerations, we are now ready to define the class of dilinear operators, which embraces linear, quadratic, and bilinear mappings as discussed below.
Definition 2.2 (Dilinear mappings). Let $X$ and $Y$ be real Banach spaces. A mapping $K: X \to Y$ is dilinear if there exists a linear mapping $\tilde{K}: X \times (X \otimes_{\pi, \text{sym}} X) \to Y$ such that $K(u) = \tilde{K}(u, u \otimes u)$ for all $u \in X$.

Hence, the dilinear mappings are the restrictions of the linear operators from $X \times (X \otimes_{\pi, \text{sym}} X)$ into $Y$ to the diagonal $\{(u, u \otimes u) : u \in X\}$. Since the representative $\tilde{K}$ acts on a Cartesian product, we can always find two linear mappings $\tilde{A}: X \to Y$ and $\tilde{B}: X \otimes_{\pi, \text{sym}} X \to Y$ so that $\tilde{K}(u, w) = \tilde{A}(u) + \tilde{B}(w)$. A dilinear mapping $K$ is bounded if the representative linear mapping $\tilde{K}$ is bounded. It is worth to mention that the representative $\tilde{K}$ of a bounded dilinear operator $K$ is uniquely defined. The breve character $\breve{}$ henceforth denotes this unique lifting of a bounded dilinear mapping.

Example 2.3 (Linear mappings). One of the easiest examples of a dilinear operator are the linear operators $A: X \to Y$ with the representative $\tilde{A}(u, u \otimes u) = A(u)$. Consequently, the dilinear operators can be seen as a generalization of linear mappings.

Example 2.4 (Quadratic mappings). A further example of dilinear mappings are the quadratic mappings. For this, we recall that a mapping $Q: X \to Y$ is quadratic if there exists a bounded symmetric bilinear mapping $A: X \times X \to Y$ such that $Q(u) = A(u, u)$ for all $u \in X$.

Since each bilinear operator is uniquely liftable to the tensor product $X \otimes_{\pi} X$ by proposition 2.1, the representative of $Q$ is just given by $\tilde{Q}(u, u \otimes u) = \tilde{A}(u, u)$, where $\tilde{A}$ is the restriction of the lifting of $A$ to the subspace $X \otimes_{\pi, \text{sym}} X$.

Example 2.5 (Bilinear mappings). Finally, the dilinear mappings also cover the class of bounded bilinear operators. For this, we replace the Banach space $X$ by the Cartesian product $X_1 \times X_2$, where $X_1$ and $X_2$ are arbitrary real Banach spaces. Given a bounded bilinear operator $A: X_1 \times X_2 \to Y$, we define the symmetric bilinear mapping $B: (X_1 \times X_2) \times (X_1 \times X_2) \to Y$ by $B((u_1, v_1), (u_2, v_2)) = \frac{1}{2} A(u_1, v_2) + \frac{1}{2} A(u_2, v_1)$. Using the lifting $\tilde{B}$ of $B$, we obtain the representative $\tilde{A}((u, v), (u, v) \otimes (u, v)) = \tilde{B}((u, v) \otimes (u, v))$ for all $(u, v)$ in $X_1 \times X_2$.

3. Generalized subgradient

Exploiting the tensorial lifting in section 2, we can immediately apply the linear regularization theory. Since the solutions of the lifted problem do not have to be of rank one, the success of this approach is limited for general operators. In order to incorporate the rank-one constraint, we follow a different approach by generalizing the linear theory suitably.

One of the drawbacks of the dilinear operators in definition 2.2 is that dilinear mappings $K: X \to \mathbb{R}$ do not have to be convex. Hence, the application of the usual subgradient to dilinear operators is limited. To surmount this issue, we generalize the concept of convexity and of the usual subgradient, see for instance [BC11, ET76, Roc70], to our setting. In the following, we denote the real numbers extended by $+\infty$ and $-\infty$ by $\overline{\mathbb{R}}$. For a mapping $F$ between a real Banach space and the extended real numbers $\overline{\mathbb{R}}$, the effective domain is the section

$$\text{dom}(F) := \{ u : F(u) < +\infty \}.$$ 

The mapping $F: X \to \overline{\mathbb{R}}$ is proper if it is never $-\infty$ and not everywhere $+\infty$.

Definition 3.1 (Diconvex mappings). Let $X$ be real Banach space. A mapping $F: X \to \overline{\mathbb{R}}$ is diconvex if there exists a proper, convex mapping $\tilde{F}: X \times (X \otimes_{\pi, \text{sym}} X) \to \overline{\mathbb{R}}$ such that $F(u) = \tilde{F}(u, u \otimes u)$ for all $u \in X$.
Since each proper, convex mapping \( F : X \to \mathbb{R} \) may be represented by the convex mapping \( \tilde{F}(u, u \otimes u) = F(u) \) on \( X \times (X \otimes \pi, \text{sym} X) \), we can view the diconvex mappings as a generalization of the set of proper, convex mappings. The central notion behind this definition is that each dilinear functional is by definition also diconvex. However, differently from the dilinear operators, the representative \( \tilde{F} \) of a diconvex mapping \( F \) does not have to be unique. One can show that one sufficient condition for diconvexity is the existence of a continuous, diaffine minorant \( A \), see appendix A, where diaffine means that there exists a continuous, affine mapping \( \tilde{A} : X \times (X \otimes \pi, \text{sym} X) \to Y \) such that \( A(u) = \tilde{A}(u, u \otimes u) \) for all \( u \) in \( X \). We now generalize the classical subgradient and subdifferential to the class of (proper) diconvex mappings.

**Definition 3.2 (Dilinear subgradient).** Let \( F : X \to \mathbb{R} \) be a diconvex mapping on the real Banach space \( X \). The dual element \((\xi, \Xi) \in X^* \times (X \otimes \pi, \text{sym} X)^*\) is a dilinear subgradient of \( F \) at \( u \) if \( F(u) \) is finite and if
\[
F(v) \geq F(u) + \langle \xi, v - u \rangle + \langle \Xi, v \otimes v - u \otimes u \rangle
\]
for all \( v \) in \( X \). The union of all dilinear subgradients of \( F \) at \( u \) is the dilinear subdifferential \( \partial \beta F(u) \). If no dilinear subgradient exists, the dilinear subdifferential is empty.

If the mapping \( F \) is convex, then the dilinear subdifferential obviously contains the usual subdifferential of \( F \). More precisely, we have \( \partial \beta F(u) \supseteq \partial F(u) \times \{0\} \). Where the usual subgradient consists of all linear functionals entirely lying below the mapping \( F \), the dilinear subgradient consists of all dilinear mappings below \( F \). In this context, the dilinear subdifferential can be interpreted as the W-subdifferential introduced in [Gra10] with respect to the family of dilinear functionals whereas the bilinear part \( \Xi \) here does not have to be negative semi-definite. If we think at the one-dimensional case \( F : \mathbb{R} \to \mathbb{R} \), the dilinear subdifferential embraces all parabola beneath \( F \) at a certain point \( u \). Since each diconvex mapping \( F \) has at least one convex representative \( \tilde{F} \), we next investigate how the representative \( \tilde{F} \) may be used to compute the dilinear subdifferential \( \partial \beta F(u) \).

**Definition 3.3 (Representative subgradient).** Let \( F : X \to \mathbb{R} \) be a diconvex mapping on the real Banach space \( X \) with representative \( \tilde{F} : X \times (X \otimes \pi, \text{sym} X) \to \mathbb{R} \). The dual element \((\xi, \Xi) \in X^* \times (X \otimes \pi, \text{sym} X)^*\) is a representative subgradient of \( F \) at \( u \) with respect to \( \tilde{F} \) if \((\xi, \Xi)\) is a subgradient of the representative \( \tilde{F} \) at \((u, u \otimes u)\). The union of all representative subgradients of \( F \) at \( u \) is the representative subdifferential \( \partial \tilde{F}(u) \) with respect to \( \tilde{F} \).

Since the representative \( \tilde{F} \) of a diconvex mapping \( F \) may not be unique, the representative subgradient usually depends on the choice of the mapping \( F \). Comparing definitions 3.2 and 3.3, one can easily verify that a representative subgradient is as well a dilinear subgradient, which simply means \( \partial \tilde{F}(u) \subseteq \partial \beta F(u) \) for each representative \( \tilde{F} \). Here the question arises whether there exists a certain representative \( \tilde{F} \) such that the representative subdifferential actually coincides with the dilinear subdifferential.

Indeed, we can always construct an appropriate representative by considering the convexification of \( F \) on \( X \times (X \otimes \pi, \text{sym} X) \). In this context, the convexification \( \text{conv} \ G \) of an arbitrary functional \( G : X \to \mathbb{R} \) on the Banach space \( X \) is the greatest convex function majorized by \( G \) and can be determined by
\[
\text{conv} \ G(u) = \inf \left\{ \sum_{n=1}^{N} \alpha_n G(u_n) : u = \sum_{n=1}^{N} \alpha_n u_n \right\},
\]
(2)
where the infimum is taken over all convex representations \( u = \sum_{n=1}^{N} \alpha_n u_n \) with \( n \in \mathbb{N} \), \( u_n \in X \), and \( \alpha_n \in [0, 1] \) so that \( \sum_{n=1}^{N} \alpha_n = 1 \), see for instance \( [Roc70] \). For a diconvex functional \( F: X \to \overline{\mathbb{R}} \), we now consider the convexification of

\[
F_\otimes (u, w) := \begin{cases} 
F(u) & w = u \otimes u, \\
+\infty & \text{else.}
\end{cases}
\]

Obviously, the mapping \( \text{conv} \ F_\otimes \) as supremum of all convex functionals majorized by \( F_\otimes \) is a valid representative of \( F \) since there exists at least one convex representative \( \tilde{F} \) with \( F(u) = \tilde{F}(u, u \otimes u) \) for all \( u \) in \( X \). Further, the convexification \( \text{conv} \ F_\otimes \) is also proper since the representative \( \tilde{F} \) has to be proper.

**Theorem 3.4 (Equality of subdifferentials).** Let \( F: X \to \mathbb{R} \) be a diconvex mapping on the real Banach space \( X \). Then the representative subdifferential with respect to \( \text{conv} \ F_\otimes \) and the dilinear subdifferential coincide, i.e.

\[
\dot{\partial} F(u) = \partial F(u).
\]

**Proof.** For any dilinear subgradient \((\xi, \Xi) \in \partial F(u)\), and for any possible, finite convex combination \( \sum_{n=1}^{N} \alpha_n (u_n, u_n \otimes u_n) \) with \( u_n \in X \), \( \alpha_n \in [0, 1] \), and \( \sum_{n=1}^{N} \alpha_n = 1 \), definition 3.2 implies

\[
\sum_{n=1}^{N} \alpha_n F_\otimes (u_n, u_n \otimes u_n) \geq \sum_{n=1}^{N} \alpha_n \left[ F_\otimes (u, u \otimes u) + (\xi, u_n - u) + (\Xi, u_n \otimes u_n - u \otimes u) \right] = F_\otimes (u, u \otimes u) + \langle \xi, \sum_{n=1}^{N} \alpha_n u_n - u \rangle + \langle \Xi, \sum_{n=1}^{N} \alpha_n u_n \otimes u_n - u \otimes u \rangle.
\]

Taking the infimum over all convex combinations \((v, w) = \sum_{n=1}^{N} \alpha_n (u_n, u_n \otimes u_n)\) for an arbitrary point \((v, w)\) in \( X \times (X \otimes_{\pi, \text{sym}} X) \), we obtain

\[
\text{conv} F_\otimes (v, w) \geq \text{conv} F_\otimes (u, u \otimes u) + \langle \xi, v - u \rangle + \langle \Xi, w - u \otimes u \rangle.
\]

Consequently, the dilinear subgradient \((\xi, \Xi)\) is contained in the representative subdifferential \( \dot{\partial} F(u) \) with respect to \( \text{conv} F_\otimes \). \qed

Similarly to the classical (linear) subdifferential, the dilinear subdifferential of a sum \( F + G \) contains the sum of the single dilinear subdifferentials of \( F \) and \( G \), which is an immediate consequence of definition 3.2.

**Proposition 3.5 (Sum rule).** Let \( F: X \to \mathbb{R} \) and \( G: X \to \mathbb{R} \) be diconvex mappings on the real Banach space \( X \). Then the dilinear subdifferential of \( F + G \) and the dilinear subdifferentials of \( F \) and \( G \) are related by

\[
\partial F(u) + \partial G(u) \subset \partial F + \partial G(u).
\]

Differently from the classical (linear) subdifferential, we cannot transfer the chain rule to the dilinear/diconvex setting. The main reason is that the composition of a diconvex mapping \( F \) and a dilinear mapping \( K \) has not to be diconvex, since a representative of \( F \circ K \) cannot simply be constructed by composing the representatives of \( F \) and \( K \). Therefore, the chain rule can only be transferred partly. For an arbitrary bounded linear operator \( K: X \to Y \), we recall that there exists a unique bounded linear operator \( K \otimes \pi: X \otimes \pi \to Y \otimes \pi \) such that
\[(K \otimes K)(u \otimes v) = K(u) \otimes K(v),\] see for instance [Rya02]. In the following, the restriction of the lifted operator \(K \otimes \pi K\) to the symmetric subspace is denoted by \(K \otimes_{\pi, \text{sym}} K\). Moreover, we recall that the Cartesian product \(K \otimes \pi K\) of the mappings \(K\) and \(K \otimes \pi K\) is defined by

\[\[K \otimes \pi K\](u, w) := (K(u), K \otimes \pi K(w)).\]

**Proposition 3.6 (Chain rule for linear operators).** Let \(K : X \to Y\) be a bounded linear mapping and \(F : Y \to \mathbb{R}\) be a diconvex mapping on the real Banach spaces \(X\) and \(Y\). Then the dilinear subdifferential of \(F \circ K\) is related to the dilinear subdifferential of \(F\) by

\[\left( K \otimes \pi, \text{sym} K \right)^* \partial F(K(u)) \subset \partial F(K(u)).\]

**Proposition 3.7 (Chain rule for convex functionals).** Let \(K : X \to Y\) be a bounded linear mapping and \(F : Y \to \mathbb{R}\) be a convex mapping on the real Banach spaces \(X\) and \(Y\). Then the dilinear subdifferential of \(F \circ K\) is related to the linear subdifferential of \(F\) by

\[\tilde{K}^* \partial F(K(u)) \subset \partial F(K(u)).\]

Both chain rules can be proven straightforwardly; so we omit the simple proofs at this position. The details can be found in appendix B.

Since the representative subdifferential is based on the classical subdifferential on the lifted space, the stronger classical sum and chain rules obviously remain valid whenever the representatives fulfil the necessary conditions. For instance, one functional of the sum or the outer functional of the composition has to be continuous at some point. At least in the finite-dimensional setting, this condition is satisfied if the diconvex functional is finite on some open set. The central idea behind the following proposition is that one can construct an appropriate simplex in the effective domain of the lifted functional. The technical construction of this simplex together with the proofs of the resulting representative sum and chain rules can be found in appendix B.

**Proposition 3.8 (Continuity of the representative).** Let \(\tilde{F} : X \times (X \otimes \pi, \text{sym} X) \to \mathbb{R}\) be a representative of the diconvex mapping \(F : X \to \mathbb{R}\) on the real Banach space \(X\). If \(F\) is finite on a non-empty, open set, then the representative \(\tilde{F}\) is continuous in the interior of the effective domain \(\text{dom}(\tilde{F})\).

On the basis of this observation, we obtain the following computation rules for the representative subdifferential on finite-dimensional Banach spaces, which follow immediately from proposition 3.8 and the classical sum and chain rules, see for instance [BC11, ET76, Roc70, Sho97].

**Proposition 3.9 (Representative sum rule).** Let \(F : X \to \mathbb{R}\) and \(G : X \to \mathbb{R}\) be diconvex functionals on the real finite-dimensional Banach space \(X\) with representatives \(\tilde{F}\) and \(\tilde{G}\). If there exists a non-empty, open set where \(F\) and \(G\) are finite, then

\[\tilde{\partial}[F + G](u) = \tilde{\partial}F(u) + \tilde{\partial}G(u)\]

for all \(u\) in \(X\) with respect to the representative \(\tilde{F} + \tilde{G}\) of \(F + G\).
Proposition 3.10 (Representative chain rule for linear operators). Let $K : X \to Y$ be a bounded, surjective linear mapping and $F : Y \to \mathbb{R}$ be a diconvex functional with representative $\tilde{F}$ on the real Banach spaces $X$ and $Y$. If $Y$ is finite-dimensional, and if there exists a non-empty, open set where $F$ is finite, then
\[
\tilde{\partial}(F \circ K)(u) = (K \times (K \otimes_{\pi, \text{sym}} K))^* \tilde{\partial}F(K(u))
\]
for all $u$ in $X$ with respect to the representative $\tilde{F} \circ (K \times (K \otimes_{\pi, \text{sym}} K))$ of $F \circ K$.

Proposition 3.11 (Representative chain rule for convex functionals). Let $K : X \to Y$ be a bounded bilinear mapping and $F : Y \to \mathbb{R}$ be a proper, convex functional on the real Banach spaces $X$ and $Y$. If there exists a non-empty, open set where $F$ is bounded from above, and if the interior of the effective domain $\text{dom}(F)$ and the range $\text{ran}(K)$ are not disjoint, then
\[
\tilde{\partial}(F \circ K)(u) = \tilde{K}^* \partial F(K(u))
\]
for all $u$ in $X$ with respect to the representative $F \circ \tilde{K}$ of $F \circ K$.

Although the bilinear and the representative subdifferential of a diconvex functional $F$ coincide with respect to the representative $\text{conv} F$, the established computation rules for the representative subdifferential cannot be transferred to the bilinear subdifferential in general. A counterexample for the sum rule is given below. The main reason for this shortcoming is that the convexification of $(F + G) \otimes$ does not have to be the sum of the convexifications of $F \otimes$ and $G \otimes$. An analogous problem occurs for the composition.

Counterexample 3.12. One of the simplest counterexamples, where the sum rule is failing for the bilinear subdifferential, is the sum of the absolute value function $| \cdot | : \mathbb{R} \to \mathbb{R}$ and the indicator function $\chi : \mathbb{R} \to \mathbb{R}$ of the interval $[-1, 1]$ given by
\[
\chi(t) := \begin{cases} 
0 & t \in [-1, 1] \\
+\infty & \text{else.}
\end{cases}
\]
As mentioned above, for a function from $\mathbb{R}$ into $\mathbb{R}$, the bilinear subdifferential consists of all parabolae beneath that function at a certain point. Looking at the point zero, we have the bilinear subdifferentials
\[
\partial | \cdot |(0) = \{ t \mapsto c_1 t^2 + c_2 t : c_2 \leq 0, c_1 \in [-1, 1] \} \quad \text{and} \quad \partial \chi(0) = \{ t \mapsto c_2 t^2 : c_2 \leq 0 \}.
\]
The sum of the bilinear subdifferentials thus consists of all parabolae with leading coefficient $c_2 \leq 0$ and linear coefficient $c_1 \in [-1, 1]$. However, the bilinear subdifferential of the sum also contains parabolae with positive leading coefficient, see the schematic illustrations in figure 1.

Generalizing the classical Fermat rule, see for instance [BC11], we obtain a necessary and sufficient optimality criterion for the minimizer of a diconvex functional based on the bilinear subdifferential calculus.

Theorem 3.13 (Fermat’s rule). Let $F : X \to \overline{\mathbb{R}}$ be a proper functional on the real Banach space $X$. Then $u^* \in \text{argmin} \{ F(u) : u \in X \}$ is a minimizer of $F$ if and only if $0 \in \partial F(u^*)$.

Proof. By definition, zero is contained in the bilinear subgradient of $F$ at $u^*$ if and only if $F(v) \geq F(u^*)$ for all $v$ in $X$, which is equivalent to $u^*$ being a minimizer of $F$. \qed
Using the dilinear subgradient, we now generalize the classical Bregman distance, see for instance [BC11, IJ15], to the dilinear/diconvex setting.

**Definition 3.14 (Dilinear Bregman distance).** Let $F: X \to \mathbb{R}$ be a diconvex mapping on the real Banach space $X$. The dilinear Bregman domain $\Delta_{\beta, \text{dom}}(F)$ of the mapping $F$ is the union of all points in $X$ with non-empty dilinear subdifferential, i.e.

$$\Delta_{\beta, \text{dom}}(F) := \{ u \in X : \partial_{\beta} F(u) \neq \emptyset \}.$$ 

For every $u \in \Delta_{\beta, \text{dom}}(F)$ and $(\xi, \Xi) \in \partial_{\beta} F(u)$, the dilinear Bregman distance of $v$ and $u$ with respect to $F$ and $(\xi, \Xi)$ is given by

$$\Delta_{\beta, (\xi, \Xi)}(v, u) := F(v) - F(u) - \langle \xi, v - u \rangle - \langle \Xi, v \otimes v - u \otimes u \rangle.$$ 

In the same way as the dilinear subdifferential, the above dilinear Bregman distance is a generalization of the classical Bregman distance. Indeed, for a ‘classical’ subgradient of the form $(\xi, 0)$, the dilinear Bregman distance coincides with the usual definition. Hence, if we replace the Banach space $X$ by a Hilbert space $H$ and choose the derivative as dilinear subgradient, the Bregman distance of the squared norm coincides with the squared norm itself. Since the dilinear subdifferential of a Hilbert norm is, however, no singleton, we get a whole family of Bregman distances, which gives us more flexibility to derive convergence rates under satisfiable source conditions.

### 4. Dilinear inverse problems

During the last decades, the theory of inverse problems has become one of the central mathematical tools for data recovery problems in medicine, engineering, and physics. Many inverse problems are ill posed such that finding numerical solutions is challenging. Although the regularization theory of linear inverse problems is well established, especially with respect to convergence and corresponding rates, solving non-linear inverse problems remains problematic, and many approaches depend on assumptions and source conditions that are difficult to verify or to validate. Based on the tensorial lifting, we will show that the linear regularization theory on Banach spaces can be extended to the non-linear class of dilinear inverse problems.

To be more precise, we consider the Tikhonov regularization for the dilinear inverse problem

$$K(u) = g^\dagger,$$
where \( K : X \to Y \) is a bounded dilinear operator between the real \textsc{Banach} spaces \( X \) and \( Y \), and where \( g^\dagger \) denotes the given data without noise. In order to solve this type of inverse problems, we study the \textsc{Tikhonov} functional

\[
J_\alpha(u) := \| K(u) - g^\dagger \|_p^p + \alpha R(u),
\]

where \( g^\dagger \) represent a noisy version of the exact data \( g^1 \), where \( p \geq 1 \), and where \( R \) is some appropriate diconvex regularization term. To verify the well-posedness and the regularization properties of the \textsc{Tikhonov} functional \( J_\alpha \), we rely on the well-established non-linear theory, see for instance [HKPS07]. For this, we henceforth make the following assumptions, which are based on the usual requirements for the linear case, see [IJ15, assumption 3.1].

**Assumption 4.1.** Let \( X \) and \( Y \) be real \textsc{Banach} spaces with predual \( X_* \) and \( Y_* \), where \( X_* \) is separable or reflexive. Assume that the data fidelity functional see for instance [Meg98, theorem 2.6.23] and [Lax02, theorem 10.7] respectively.

We require that every bounded sequence \( g^\dagger \) contains a weakly convergent subsequence, where \( g^\dagger \) is some appropriate diconvex regularization term. To verify the well-posedness and the regularization properties of the \textsc{Tikhonov} functional \( J_\alpha \), we rely on the well-established non-linear theory, see for instance [HKPS07] and [Lax02, theorem 10.7] respectively.

**Remark 4.2.** Since \( X_* \) is a separable or reflexive \textsc{Banach} space, we can henceforth conclude that every bounded sequence \( (u_n)_{n \in \mathbb{N}} \) in \( X \) contains a weakly convergent subsequence, see for instance [Meg98, theorem 2.6.23] and [Lax02, theorem 10.7] respectively.

For the non-linear regularization theory in [HKPS07], which covers a much more general setting, the needed requirements are much more sophisticated and comprehensive, but one can easily verify that assumption 4.1 is compatible with these requirements.

Since the dilinear operator \( K \) is the composition of \( \tilde{K} \) and the lifted operator \( \hat{K} \), the weak* continuity in assumption 4.1. (iii) may be transferred to the representative \( K \).

The main problem with this approach is that the predual of the tensor product \( X \otimes_{\pi, \text{sym}} X \) does not have to exist, even if the predual \( X_* \) of the real \textsc{Banach} space \( X \) is known. Therefore, we equip the symmetric tensor product \( X \otimes_{\pi, \text{sym}} X \) with an appropriate topology.

The key observation to introduce a suitable weaker topology is here that each finite-rank tensor \( \omega := \sum_{n=1}^N \lambda_n \phi_n \otimes \psi_n \) in \( X_* \otimes X \) uniquely defines a bilinear form \( B_\omega : X \times X \to \mathbb{R} \) by \( B_\omega(u, v) := \sum_{n=1}^N \lambda_n u(\phi_n) v(\psi_n) \). If we consider the closure of \( X_* \otimes X \) with respect to the norm

\[
\| \omega \|_c := \| B_\omega \| = \sup \left\{ \frac{B_\omega(u, v)}{\| u \| \| v \|} : u, v \in X \setminus \{0\} \right\}
\]

of \( \mathcal{B}(X \times X) \), we obtain the injective tensor product \( X_* \otimes_e X_* \), see [DF93, Rya02]. Since the space of bilinear forms \( \mathcal{B}(X \times X) \) is the topological dual of \( X \otimes_{\pi} X \), the injective product \( X_* \otimes_e X_* \) of the predual \( X_* \) is a family of linear functionals on \( X \otimes_{\pi} X \).

If the \textsc{Banach} space \( X \) has the approximation property, i.e. the identity operator can be approximated by finite-rank operators on compact sets, then the canonical mapping from \( X \otimes_{\pi} X \) into \( (X_* \otimes_e X_*)^* \) becomes an isometric embedding, see [Rya02, theorem 4.14]. In this case, the injective tensor product \( X_* \otimes_e X_* \) even defines a separating family, and the projective tensor product \( X \otimes_{\pi} X \) together with the topology induced by the injective tensor...
Lemma 4.3 (Weak* continuity of the tensor mapping). Let \( X \) be a real Banach space with predual \( X_\ast \). The mapping \( \otimes : X \to X \otimes_s X, u \mapsto u \otimes u \) is sequentially weakly* continuous with respect to the topology induced by the injective tensor product \( X_\ast \otimes_s X_\ast \).

**Proof.** Let \( \phi \otimes \phi \) be a rank-one tensor in \( X_\ast \otimes_s X_\ast \), and let \((u_n)_{n \in \mathbb{N}}\) be a weakly* convergent sequence in the Banach space \( X \). Without loss of generality, we postulate that the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded by \( \|u_n\| \leq 1 \). Under the assumption that \( u \) is the weak* limit of \((u_n)_{n \in \mathbb{N}}\), we observe

\[
\lim_{n \to \infty} (\phi \otimes \phi)(u_n \otimes u_n) = \lim_{n \to \infty} \phi(u_n) \phi(u_n) = \phi(u) \phi(u) = (\phi \otimes \phi)(u \otimes u).
\]

Obviously, this observation remains valid for all finite-rank tensors in \( X_\ast \otimes_s X_\ast \). Now, let \( \omega \) be an arbitrary tensor in \( X_\ast \otimes_s X_\ast \). For every \( \varepsilon > 0 \), we find a finite-rank approximation \( \tilde{\omega} \) of the tensor \( \omega \) such that \( \|	ilde{\omega} - \omega\|_\varepsilon \leq \frac{\varepsilon}{2} \). Hence, for suitable large \( n \), we have

\[
|\omega(u_n \otimes u_n) - \omega(u \otimes u)| \\
\leq |\omega(u_n \otimes u_n) - \tilde{\omega}(u_n \otimes u_n)| + |\tilde{\omega}(u_n \otimes u_n) - \tilde{\omega}(u \otimes u)| + |\tilde{\omega}(u \otimes u) - \omega(u \otimes u)| \\
\leq \varepsilon
\]

and thus \( \lim_{n \to \infty} \omega(u_n \otimes u_n) = \omega(u \otimes u) \) for all linear functionals \( \omega \) in \( X_\ast \otimes_s X_\ast \). \( \square \)

Being the composition of a linear mapping \( K : X \times (X \otimes_s X) \to Y \) and the sequential weakly* continuous tensor mapping \( u \mapsto (u, u \otimes u) \), the dilinear mapping \( K \) may inherit the required continuity in assumption 4.1 from its representative. Similarly, the continuity of the
A diconvex functional can be inherited. Since the following statements immediately follow from lemma 4.3, we omit the proofs.

**Proposition 4.4 (Sequential weak* continuity of a dilinear operator).** Let \( X \) and \( Y \) be real Banach spaces with preduals \( X_* \) and \( Y_* \). If the representative \( \bar{K} : X \times (X \otimes X_*) \to Y \) is sequentially weak* continuous with respect to the weak* topology on \( X \) and the weak* topology on \( X \otimes X_* \) induced by \( X_* \otimes X_* \), then the related dilinear operator \( K : X \to Y \) is sequentially weak* continuous.

**Proposition 4.5 (Sequential weak* lower semi-continuity of a diconvex mapping).** Let \( X \) be a real Banach space with predual \( X_* \). If the representative \( F : X \times (X \otimes X_*) \to \mathbb{R} \) is sequentially weak* lower semi-continuous with respect to the weak* topology on \( X \) and the weak* topology on \( X \otimes X_* \) induced by \( X_* \otimes X_* \), then the related diconvex mapping \( F : X \to \mathbb{R} \) is sequentially weak* lower semi-continuous.

### 4.1. Well-posedness and regularization properties

We now return to the well-posedness, stability, and consistency of the Tikhonov regularization of the dinlinear inverse problem \( K(u) = g^\dagger \). In other words, we study the regularization properties of the variational regularization

\[
\min_{u \in X} J_\alpha(u) \quad \text{with} \quad J_\alpha(u) = \|K(u) - g^\dagger\|^p + \alpha R(u).
\]

The main difference of the introduced dilinear regularization and the non-linear Tikhonov regularization in Banach spaces, see for instance [HKPS07], is the non-convex regularization functional \( R \). Since the usually required convexity of \( R \) is not needed to verify well-posedness, stability, and consistency, these properties immediately follow from the well-established non-linear regularization theory. For the sake of completeness and convenience, we briefly summarize the central results with respect to our setting. The detailed proofs can be found in [HKPS07].

Firstly, the Tikhonov functional \( J_\alpha \) in (4) is well posed in the sense that the minimum of the regularized problem \( \min \{ J_\alpha(u) : u \in X \} \) is attained, and that the related minimizer is thus well defined.

**Theorem 4.6 (Well-posedness).** Under assumption 4.1, for every \( \alpha > 0 \), there exists at least one minimizer \( u^\alpha \) to the functional \( J_\alpha \) in (4).

Stability of a variational regularization method means that the minimizers \( u^\alpha \) of the Tikhonov functional \( J_\alpha \) weakly* depend on the noisy data \( g^\dagger \). If the regularization functional \( R \) satisfies the so-called H-property, which means that every weakly* convergent sequence \( (u_n)_{n \in \mathbb{N}} \) with limit \( u \) and with \( R(u_n) \to R(u) \) strongly converges to \( u \), see for instance [IJ15, Wer02], the dependence between the solution of the regularized problem and the corrupted data is actually strong.

**Theorem 4.7 (Stability).** Let the sequence \( (g_n)_{n \in \mathbb{N}} \) in \( Y \) be convergent with limit \( g^\dagger \in Y \). Under assumption 4.1, the sequence \( (u_n)_{n \in \mathbb{N}} \) of minimizers of the Tikhonov functional \( J_\alpha \) in (4) with \( g_n \) in place of \( g^\dagger \) contains a weakly* convergent subsequence to a minimizer \( u^\alpha \) of \( J_\alpha \). If the minimizer of \( J_\alpha \) is unique, then the complete sequence \( (u_n)_{n \in \mathbb{N}} \) converges weakly*. If the functional \( R \) possesses the H-property, then the sequence \( (u_n)_{n \in \mathbb{N}} \) converges in norm topology.
Finally, the Tikhonov regularization of a dilinear inverse problem is consistent; so the minimizer $u_\delta$ weakly* converges to a solution $u^\dagger$ of the unperturbed problem $K(u) = g^\dagger$ if the noise level $\delta$ goes to zero. More precisely, the solutions $u_\delta$ converges to an $R$-minimizing solution, which simply means that $R(u^\dagger) \leq R(u)$ for all further solutions $u$ of the dilinear equation $K(u) = g^\dagger$. Although the $R$-minimizing solution is not unique, the existence of at least one $R$-minimizing solution is ensured by assumption 4.1, see for instance [HKPS07, IJ15].

**Theorem 4.8 (Consistency).** Let $(g^\delta_n)_{n \in \mathbb{N}}$ be a sequence of noisy data with $\delta_n := \|g^\delta_n - g^\dagger\| \to 0$. Under assumption 4.1, the sequence of minimizers $(u_\delta_n)_{n \in \mathbb{N}}$ contains a weakly* convergent subsequence whose limit is an $R$-minimizing solution $u^\dagger$ if the sequence of regularization parameters $(\alpha_n)_{n \in \mathbb{N}} = (\alpha(\delta_n))_{n \in \mathbb{N}}$ satisfies

$$\lim_{n \to \infty} \frac{\delta_n}{\alpha_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \alpha_n = 0.$$  

If the $R$-minimizing solution $u^\dagger$ is unique, then the entire sequence $(u_\delta_n)_{n \in \mathbb{N}}$ converges weakly*. If the functional $R$ possesses the $H$-property, then the sequence $(u_\delta_n)_{n \in \mathbb{N}}$ converges in norm topology.

### 4.2. Convergence analysis under source-wise representations

Based on the dilinear subdifferential calculus, we now analyse the convergence behaviour of the variational regularization method for dilinear inverse problems $K(u) = g^\dagger$ if the noise level $\delta$ goes to zero. In the following, we assume that the dilinear operator $K : X \to Y$ maps from a real Banach space into a real Hilbert space $Y$. Further, we restrict ourselves to the squared Hilbert norm $\| \cdot \|^2$ as data fidelity functional $S$ in (4), which means that we consider the Tikhonov functional

$$J_\alpha(u) := \|K(u) - g^\dagger\|^2 + \alpha R(u). \quad (5)$$

On the basis of assumption 4.1, we recall that the dilinear inverse problem $K(u) = g^\dagger$ always possesses an $R$-minimizing solution $u^\dagger$ with respect to the regularization functional $R$. Consequently, Fermat’s rule (theorem 3.13) implies that zero must be contained in the dilinear subdifferential $\partial_2(R + \chi_{\{K(u)=g^\dagger\}})(u^\dagger)$, where the indicator function $\chi_{\{K(u)=g^\dagger\}}(u)$ is 0 if $u$ is a solution of the inverse problem $K(u) = g^\dagger$ and $+\infty$ else. Applying the sum and chain rule in propositions 3.5 and 3.7, we have

$$\text{ran}K^* + \partial_2 R(u^\dagger) \subset \partial_2 (R + \chi_{\{K(u)=g^\dagger\}})(u^\dagger).$$

Without further assumptions, the sum and chain rule here only yield an inclusion. Although we always have $0 \in \partial_2 (R + \chi_{\{K(u)=g^\dagger\}})(u^\dagger)$, we hence cannot guarantee $0 \in \text{ran}K^* + \partial_2 R(u^\dagger)$ or, equivalently, $\text{ran}K^* \cap \partial_2 R(u^\dagger) \neq \emptyset$. Against this background, we postulate the regularity assumption that the range of the adjoint operator and the dilinear subdifferential are not disjoint. In other words, we assume the existence of a source-wise representation

$$K^* \omega = (\xi^\dagger, \Xi^\dagger) \in \partial_2 R(u^\dagger)$$  

for some $\omega$ in $Y$.

**Theorem 4.9 (Convergence rate).** Let $u^\dagger$ be an $R$-minimizing solution of the dilinear inverse problem $K(u) = g^\dagger$. Under the source condition $K^* \omega = (\xi^\dagger, \Xi^\dagger) \in \partial_2 R(u^\dagger)$ for some
\(\omega\) in \(Y\) and under assumption 4.1, the minimizers \(u^\delta\) of the Tikhonov regularization \(J_\alpha\) in (5) converge to \(u^1\) in the sense that the dilinear Bregman distance between \(u^\delta\) and \(u^1\) with respect to the regularization functional \(R\) is bounded by
\[
\Delta_{\beta,\langle \xi, \Xi \rangle}(u^\delta, u^1) \leq \left( \frac{\delta}{\sqrt{\alpha}} + \frac{\alpha}{2} \|\omega\| \right)^2
\]
and the data fidelity term by
\[
\|K(u^\delta) - g^\delta\| \leq \delta + \alpha \|\omega\|.
\]

**Proof.** Inspired by the proof for the usual subdifferential in [IJ15], the desired convergence rate for the dilinear subdifferential can be established in the following manner. Since \(u^\delta\) is a minimizer of the Tikhonov functional \(J_\alpha\) in (5), \(u^\delta\) and \(u^1\) satisfy
\[
\|K(u^\delta) - g^\delta\|^2 + \alpha R(u^\delta) \leq \|K(u^1) - g^\delta\|^2 + \alpha R(u^1).
\]
Remembering \(K(u^1) = g^1\), we can bound the norm on the right-hand side by \(\|g^1 - g^\delta\|^2 \leq \delta^2\).

Rearranging the last inequality and exploiting the source condition, we get
\[
\|K(u^\delta) - g^\delta\|^2 + \alpha \Delta_{\beta,\langle \xi, \Xi \rangle}(u^\delta, u^1)
\]
\[
\leq \delta^2 - \alpha \langle \xi, u^\delta - u^1 \rangle - \alpha \langle \Xi, u^\delta \odot u^\delta - u^1 \odot u^1 \rangle
\]
\[
= \delta^2 - \alpha \langle \omega, K(u^\delta) - u^1 \rangle
\]
\[
= \delta^2 - \alpha \langle \omega, K(u^\delta) - g^1 \rangle
\]
\[
= \delta^2 - \alpha \langle \omega, u^\delta - g^\delta \rangle - \alpha \langle \omega, g^1 - g^\delta \rangle.
\]

Rearranging the terms, completing the square, and applying Cauchy–Schwarz’s inequality, we obtain
\[
\left\|K(u^\delta) - g^\delta + \frac{\alpha}{2} \omega \right\|^2 + \alpha \Delta_{\beta,\langle \xi, \Xi \rangle}(u^\delta, u^1) \leq \delta^2 + \alpha \delta \|\omega\| + \frac{\alpha^2}{4} \|\omega\|^2 = \left( \delta + \frac{\alpha}{2} \|\omega\| \right)^2,
\]
which proves the convergence rate for the Bregman distance. The second convergence rate follows immediately by applying the reverse triangle inequality.

**Remark 4.10.** If we apply the a priori parameter choice rule \(\alpha \sim \delta\), then the Bregman distance \(\Delta_{\beta,\langle \xi, \Xi \rangle}(u^\delta, u^1)\) as well as the data fidelity term \(\|K(u^\delta) - g^\delta\|\) converges to zero with a rate of \(\Theta(\delta)\).

### 4.3. Convergence analysis under variational source conditions

The convergence rate established in the last subsection is mainly based on the source-wise representation \(K^*\omega \in \partial_2 R(u^1)\). A second approach to derive convergence rates is to employ variational source conditions. Especially for continuous operators and convex regularization functionals, this strategy has been extensively studied in the past, see for instance [HKPS07, Gra10] and references therein. In the following, we generalize the convergence results in [HKPS07] to dilinear inverse problems with diconvex regularization. In so doing, we consider
again the Tikhonov functional $J_\alpha$ in (5) and assume that every $u \in X$ fulfills the variational source condition

$$\langle \xi^\dagger, u^\dagger - u \rangle + \langle \Xi^\dagger, u^\dagger \otimes u^\dagger - u \otimes u \rangle \leq \beta_1 \Delta_{\beta, (\xi^\dagger, \Xi^\dagger)}(u, u^\dagger) + \beta_2 \| K(u) - K(u^\dagger) \|,$$

(8)

where $(\xi^\dagger, \Xi^\dagger)$ denotes a specific dilinear subgradient in $\partial_\delta R(u^\dagger)$, and where $\beta_1$ and $\beta_2$ are two constants with $\beta_1 \in [0, 1)$ and $\beta_2 \in [0, \infty)$.

**Theorem 4.11 (Convergence rate).** Let $u^\dagger$ be an R-minimizing solution of the dilinear inverse problem $K(u) = g^\dagger$. Under the variational source condition (8) and under assumption 4.1, the minimizers $u^\alpha_\delta$ of the Tikhonov regularization $J_\alpha$ in (5) converge to $u^\dagger$ in the sense that the dilinear Bregman distance between $u^\alpha_\delta$ and $u^\dagger$ with respect to the regularization functional $R$ is bounded by

$$\Delta_{\beta, (\xi^\dagger, \Xi^\dagger)}(u^\alpha_\delta, u^\dagger) \leq \frac{\delta^2 + \alpha \delta \beta_2 + \frac{1}{2} (\alpha \beta_2)^2}{\alpha (1 - \beta_1)}$$

and the data fidelity term by

$$\| K(u^\alpha_\delta) - g^\dagger \| \leq \sqrt{2} \left( \delta^2 + \alpha \delta \beta_2 + \frac{1}{2} (\alpha \beta_2)^2 \right).$$

**Proof.** Following the first lines in the proof of theorem 4.9, and using the variational source condition (8) instead of the source-wise representation (6), we obtain

$$\| K(u^\alpha_\delta) - g^\dagger \|^2 + \alpha \Delta_{\beta, (\xi^\dagger, \Xi^\dagger)}(u^\alpha_\delta, u^\dagger) \leq \delta^2 + \alpha \beta_1 \Delta_{\beta, (\xi^\dagger, \Xi^\dagger)}(u^\alpha_\delta, u^\dagger) + \alpha \beta_2 \| K(u^\alpha_\delta) - g^\dagger \|^2 + \delta,$$

which can be rearranged, by combining the norms and Bregman distances, to

$$\| K(u^\alpha_\delta) - g^\dagger \|^2 + \alpha \beta_1 \Delta_{\beta, (\xi^\dagger, \Xi^\dagger)}(u^\alpha_\delta, u^\dagger) \leq \delta^2 + \alpha \beta_2 \| K(u^\alpha_\delta) - g^\dagger \|^2,$$

(9)

Applying Young’s inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$ to $\alpha \beta_2 \| K(u^\alpha_\delta) - g^\dagger \|^2$, we can verify the inequality

$$-\frac{1}{2} \| K(u^\alpha_\delta) - g^\dagger \|^2 \leq -\alpha \beta_2 \| K(u^\alpha_\delta) - g^\dagger \|^2 + \frac{1}{2} (\alpha \beta_2)^2.$$

(10)

Adding $\frac{1}{2} (\alpha \beta_2)^2$ to both sides of (9) and employing (10), we obtain the conjectured convergence rates.

Applying the parameter choice rule $\alpha \sim \delta$ again, both error bounds in theorem 4.11 converge to zero with a rate of $\Theta(\delta)$. Since the source-wise representation (6) implies

$$\langle \xi^\dagger, u^\dagger - u \rangle + \langle \Xi^\dagger, u^\dagger \otimes u^\dagger - u \otimes u \rangle = \langle \omega, K(u^\dagger, u^\dagger \otimes u^\dagger) - K(u, u \otimes u) \rangle \leq \| \omega \| \| K(u^\dagger) - K(u) \|,$$

we can interpret the variational source condition (8) as a generalization of the source-wise representation (6). Usually, the verification of a variational source condition is very challenging for a specific inverse problem. On the other side, if the forward operator, the regularization term, and the true solution are sufficiently smooth, one can prove that, for certain convex regularizers, a specially constructed variational source condition holds true [WSH18, HW15].
5. Deautoconvolution problem

In order to give a non-trivial example for the practical relevance of the developed dilinear regularization theory, we consider the deautoconvolution problem, where one wishes to recover an unknown signal \( u : \mathbb{R} \to \mathbb{C} \) with compact support from its kernel-based autoconvolution

\[
\mathcal{A}_k[u](t) := \int_{-\infty}^{\infty} k(s, t) u(s) u(t-s) ds \quad (t \in \mathbb{R}),
\]

(11)

where \( k : \mathbb{R}^2 \to \mathbb{C} \) is an appropriate kernel function. Problems of this kind occur in spectroscopy, optics, and stochastics for instance, see [BH15].

Following the model in [BH15], we assume that \( u \) is a square-integrable complex-valued signal on the interval \([0, 1]\), or, in other words, \( u \in L^2_\mathbb{C}([0, 1]) \). To ensure that the integral in (11) is well defined, we extend the signal \( u \) outside the interval \([0, 1]\) with zero and restrict ourselves to bounded kernels \( k \in L^2_\mathbb{C}([0, 1] \times [0, 2]) \). Considering the support of \( u \), we may moreover assume

\[
\text{supp} \ k \subset \{(s, t) : 0 \leq s \leq 1 \text{ and } s \leq t \leq s + 1\}
\]

(12)

and

\[
k(s, t) = k(t-s, t) \quad (0 \leq s \leq 1, s \leq t \leq s + 1).
\]

(13)

The symmetry property (13) can be demanded in general because of the identity

\[
\mathcal{A}_k[u](t) = \int_{-\infty}^{\infty} k(s, t) + k(t-s, t) \frac{1}{2} u(s) u(t-s) ds.
\]

After these preliminary considerations, we next verify that the kernel-based autoconvolution \( \mathcal{A}_k \) is a bounded dilinear operator. For this, we exploit that the autoconvolution fulfils the required assumptions and to analyse the source-wise

\[
\text{supp} \ k \subset \{(s, t) : 0 \leq s \leq 1 \text{ and } s \leq t \leq s + 1\}
\]

(12)

and

\[
k(s, t) = k(t-s, t) \quad (0 \leq s \leq 1, s \leq t \leq s + 1).
\]

(13)

The symmetry property (13) can be demanded in general because of the identity

\[
\mathcal{A}_k[u](t) = \int_{-\infty}^{\infty} k(s, t) + k(t-s, t) \frac{1}{2} u(s) u(t-s) ds.
\]

Although the deautoconvolution problem is ill posed in general, the unperturbed inverse problem can have at most two different solutions, see [GHB’14]. The original proof is based on Titchmarsh’s convolution theorem.

**Proposition 5.1 (Ambiguities deautoconvolution).** Let \( g^\dagger \) be a function in \( L^2_\mathbb{C}([0, 2]) \), and let the kernel of the autoconvolution be \( k \equiv 1 \). If \( u \in L^2_\mathbb{C}([0, 1]) \) is a solution of the deautoconvolution problem \( \mathcal{A}_k[u] = g^\dagger \), then \( u \) is uniquely determined up to the global sign.

We proceed by considering the perturbed deautoconvolution problem

\[
\mathcal{A}_k[u] = g^\delta \quad \text{with} \quad \|g^\delta - g^\dagger\| \leq \delta.
\]

Since the autoconvolution is a bounded dilinear mapping, we can apply the developed dilinear regularization theory to the Tikhonov functional

\[
J_\alpha(u) := \|\mathcal{A}_k[u] - g^\delta\|^2 + \alpha\|u\|^2,
\]

(14)

where we choose the squared norm of \( L^2_\mathbb{C}([0, 1]) \) as regularization term. In order to verify that the autoconvolution fulfils the required assumptions and to analyse the source-wise
representation (6), firstly, we need a suitable representation of the kernel-based autoconvolution and of the dual and predual spaces of $L^2_{c,\text{sym}}([0,1]) \otimes_{\pi,\text{sym}} L^2_{c,\text{sym}}([0,1])$.

For this, we recall that each tensor $w$ in the tensor product $H \otimes_{\pi} H$, where $H$ is an arbitrary Hilbert space, may be interpreted as a trace class or a nuclear operator in sense of the mapping $L_\infty$ or $R_\infty$ in (1). More precisely, the projective tensor product $H \otimes_{\pi} H$ is isometrically isomorphic to the space of compact operators $\mathcal{K}(H)$, see for instance [Wer02, section VI.5]. Further, the injective tensor product $H \otimes_{\epsilon} H$ is here isometrically isomorphic to the space of compact operators $\mathcal{K}(H)$, see [Wer02, Satz VI.6.4]. Finally, the dual space $(H \otimes_{\epsilon} H)^*$ of the projective tensor product coincides with the space of linear operators $\mathcal{L}(H)$, see [Wer02, Satz VI.6.4]. The action of an operator $T$ in $\mathcal{K}(H)$ or $\mathcal{L}(H)$ on an arbitrary tensor $w$ in $H \otimes_{\pi} H$ is here given by the tensorial lifting of the bilinear mapping

$$(u,v) \mapsto \langle Tu,v \rangle$$

or by the Hilbert–Schmidt inner product if $w$ is interpreted as nuclear operator. The corresponding spaces for the symmetric projective tensor product $H \otimes_{\pi,\text{sym}} H$ coincide with the related self-adjoint operators. In particular, these identifications are valid for the real Hilbert space $L^2_c([0,1])$.

To handle the autoconvolution $\mathcal{A}_k$, we split this mapping into a linear integral part $g_k : L^2_{c,\text{sym}}([0,1]^2) \to L^2_{c,\text{sym}}([0,2])$ and into a quadratic part $\circ : L^2_{c,\text{sym}}([0,1]) \to L^2_{c,\text{sym}}([0,1]^2)$, where $L^2_{c,\text{sym}}([0,1]^2)$ denotes the subspace of the symmetric square-integrable functions on the square $[0,1] \times [0,1]$ defined by

$L^2_{c,\text{sym}}([0,1]^2) := \{ u \in L^2_{c,\text{sym}}([0,1]^2) : u(s,t) = u(t,s) \text{ for almost every } 0 \leq s,t \leq 1 \}$. More precisely, we employ the factorization $\mathcal{A}_k = g_k \circ \circ$ where the operators $g_k$ and $\circ$ are given by

$$g_k[w](t) := \int_{-\infty}^{\infty} k(s,t) w(s,t-s) ds \quad (t \in \mathbb{R}) \quad (15)$$

and

$$\circ[u](s,t) := u(s) u(t) \quad (s,t \in \mathbb{R}). \quad (16)$$

On the basis of these mappings, we can determine the lifting of the autoconvolution $\mathcal{A}_k$.

**Lemma 5.2 (Lifting of the autoconvolution).** The unique (quadratic) lifting of the kernel-based autoconvolution $\mathcal{A}_k = g_k \circ \circ$ is given by

$$\tilde{\mathcal{A}}_k = g_k \circ \tilde{\circ},$$

the unique dilinear lifting by

$$\tilde{(0,\mathcal{A})} = (0,g_k \circ \tilde{\circ}).$$

For the dilinear lifting, we here use a matrix-vector-like representation. More precisely, the mapping $(0,\tilde{\mathcal{A}}_k)$ is defined by

$$\tilde{(0,\tilde{\mathcal{A}}_k)} \left( \begin{array}{c} u \\ w \end{array} \right) := 0[u] + \tilde{\mathcal{A}}_k[w]$$

with $u \in L^2_{c,\text{sym}}([0,1])$ and $w \in L^2_{c,\text{sym}}([0,1])$. 

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Proof of lemma 5.2. Firstly, we notice that the norms of the integral operator \( \mathcal{J}_k \) and the quadratic operator \( \circ \) are bounded by \( \| k \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \) and 1 respectively. Hence, the complete mapping \( \mathcal{F}_k = \mathcal{J}_k \circ \circ \) is bounded. The quadratic operator now possesses a unique lifting \( \tilde{\circ} \), see proposition 2.1, which leads us to

\[
\mathcal{F}_k[u \otimes u] = (\mathcal{J}_k \circ \tilde{\circ})[u \otimes u] = (\mathcal{J}_k \circ \circ)[u] = \mathcal{F}_k[u];
\]

so the defined mapping \( \mathcal{F}_k \) is the unique quadratic lifting of the autoconvolution \( \mathcal{A}_k \). Since the dilinear representative of a quadratic mapping is completely independent of the first component space, see example 2.4, the first component is always mapped to zero, which completes the proof. \( \square \)

With the factorization of the kernel-based autoconvolution in mind, we next determine the related adjoint operator of the lifting \( \tilde{\mathcal{A}}_k \).

**Lemma 5.3 (Adjoint of the autoconvolution).** The adjoint operator \( \tilde{\mathcal{A}}_k^* : L^2_{\text{C}}([0, 2]) \to (L^2_{\text{C}}([0, 1]) \otimes_{\text{sym}} L^2_{\text{C}}([0, 1]))^* \) of the quadratic lifting of the kernel-based autoconvolution \( \mathcal{A}_k \) is given by \( \tilde{\mathcal{A}}_k^* = \tilde{\circ}^* \circ \mathcal{J}_k^* \) with

\[
\mathcal{J}_k^*[\phi](s, t) = k(s, s + t) \phi(s + t) \quad (s, t \in [0, 1]),
\]

where \( \mathcal{J}_k^* \) is a mapping between \( L^2_{\text{C}}([0, 2]) \) and \( L^2_{\text{C}}([0, 1]) \), and the adjoint operator of the dilinear lifting by

\[
(0, \tilde{\mathcal{A}}_k^*)^T = (0, \tilde{\circ}^* \circ \mathcal{J}_k^*)^T.
\]

For the adjoint dilinear lifting, we again use a matrix-vector-like representation. In more detail, the mapping \( (0, \tilde{\mathcal{A}}_k^*)^T \) is defined by

\[
\begin{pmatrix}
0 \\
\tilde{\mathcal{A}}_k^* \phi
\end{pmatrix} := \begin{pmatrix}
0[\phi] \\
\tilde{\mathcal{A}}_k^*[\phi]
\end{pmatrix}
\]

with \( \phi \in L^2_{\text{C}}([0, 2]). \)

**Proof of lemma 5.3.** The representation of the adjoint integral operator \( \mathcal{J}_k^* \) can be directly verified by

\[
\langle \phi, \mathcal{J}_k[w] \rangle_R = \Re \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(s, t - s) k(s, t) \phi(t) ds \, dt \right] = \Re \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(s, t) k(s, t + s) \phi(t + s) ds \, dt \right] = \langle \mathcal{J}_k^*[\phi], w \rangle_R
\]

for all \( w \in L^2_{\text{C}}([0, 1]^2) \) and \( \phi \in L^2_{\text{C}}([0, 2]). \) The remaining assertion immediately follows from the properties of the adjoint. \( \square \)

The next ingredients for the dilinear regularization theory in section 4 are the required conditions in assumption 4.1. The central idea to verify the sequentially weakly* continuity of the autoconvolution operator is to prove the corresponding continuity of the quadratic operator \( \circ \) by exploiting proposition 4.4. Since the explicit argumentation is more than technical, the proof can be found in appendix D.

**Lemma 5.4 (Sequentially weak* continuity).** The quadratic operator \( \circ \) defined in (16) is sequentially weakly* continuous.
Using lemma 5.4, we can now verify that the Tikhonov regularization of the deautoconvolution problem satisfies the required assumptions for the developed regularization theory in section 4.

**Lemma 5.5 (Verification of required assumptions).** The Tikhonov functional \( J_\alpha \) in (14) related to the kernel-based deautoconvolution \( A_k \) fulfills the requirements in assumption 4.1.

**Proof.** We briefly verify the needed requirements step by step.

(i) Obviously, the Tikhonov functional \( J_\alpha \) in (14) is coercive since the regularization term coincides with the squared Hilbert space norm.

(ii) For the same reason, the regularization term is sequentially weakly* lower semi-continuous, see for instance [Meg98, theorem 2.6.14].

(iii) In lemma 5.4, we have already proven that the quadratic operator \( \circ \) of the factorization \( A_k = g_k \circ \circ \) is sequentially weakly* continuous. Further, the obvious norm-to-norm continuity of the integral operator \( g_k \) implies the weak-to-weak continuity of \( g_k \), see for instance [Meg98, proposition 2.5.10]. Since \( L_2^2([0,1]^2) \) and \( L_2^2([0,2]) \) are Hilbert spaces, the mapping \( g_k \) is weakly* to weakly* continuous as well. Consequently, the composition \( A_k = g_k \circ \circ \) is sequentially weakly* continuous as required. \( \square \)

Since the Tikhonov functional \( J_\alpha \) for the deautoconvolution problem fulfills all constraints in assumption 4.1, we can employ the developed dilinear/diconvex regularization theory in section 4. Besides the well-known well-posedness, stability, and consistency of the regularized deautoconvolution problem, the convergence rate introduced by theorem 4.9 is far more interesting. In order to study the employed source condition, we have to compare the range of the adjoint autoconvolution operator \( (0, \bar{A}_k^*)^T \), see lemma 5.3, and the dilinear subdifferential of the squared Hilbert norm on \( L_2^2([0,1]) \).

More generally, we initially determine the dilinear subdifferential for the norm on an arbitrary Hilbert space \( H \). For this purpose, we exploit that the dual space \( (H \otimes_{\mathbb{R},\text{sym}} H)^* \) is isometrically isomorphic to the space of self-adjoint, bounded linear operators. As mentioned above, the action of a specific self-adjoint, bounded linear operator \( \Phi \) on an arbitrary symmetric tensor \( w \) in \( H \otimes_{\mathbb{R},\text{sym}} H \) is given by the lifting of the quadratic map \( u \mapsto \langle \Phi[u], u \rangle_H \) with \( u \in H \). In the following, we use the dual pairing notation \( \langle \Phi, w \rangle_{H \otimes_{\mathbb{R},\text{sym}} H} \) to refer to this action. With this preliminary considerations, the dilinear subdifferential of an arbitrary Hilbert norm is given in the following manner, where the operator \( \text{Id} \) denotes the identity and \( S_- (H) \) the set of all self-adjoint and negative semi-definite operators on \( H \).

**Theorem 5.6 (Dilinear subdifferential of Hilbert norms).** Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle_H \) and norm \( \| \cdot \|_H \). The dilinear subdifferential of the squared Hilbert norm is given by

\[
\partial_\beta \| u \|_H^2 = \left\{ \langle T u, \text{Id} + T \rangle : T \in S_- (H) \right\}.
\]

**Proof.** Firstly, we notice that the \( p \)th power of the Hilbert norm is differentiable with Gâteaux derivative

\[
D \| u \|_H^p = p \| u \|_H^{p-2} u \quad (u \in H, p > 1).
\]

see for instance [BL11]. Further, the squared Hilbert norm is itself a dilinear mapping with representative \( (u, w) \mapsto \langle \text{Id}, w \rangle_{H \otimes_{\mathbb{R},\text{sym}} H} \) and \( \| u \|^2 = \langle \text{Id}, u \otimes u \rangle_{H \otimes_{\mathbb{R},\text{sym}} H} \). The Taylor expansion at some point \( u \) in \( H \) is given by
\[ \|v\|^2_H = \|u\|^2_H + (2u, v - u)_H + \langle \text{Id}, (v - u) \otimes (v - u) \rangle_{H \otimes \pi_{\text{sym}} H}. \]

Since all bilinear mappings are differentiable in the Hilbert space setting, the linear part of the Taylor expansion is fixed, and the only possibility to construct a bilinear mapping beneath the squared Hilbert norm is to add a negative semi-definite quadratic part \( T \). In this manner, we obtain

\[ \|v\|^2_H \geq \|u\|^2_H + (2u, v - u)_H + \langle \text{Id} + T, (v - u) \otimes (v - u) \rangle_{H \otimes \pi_{\text{sym}} H} \]

\[ = \|u\|^2_H + \langle -2Tu, v - u \rangle_H + \langle \text{Id} + T, v \otimes v - u \otimes u \rangle_{H \otimes \pi_{\text{sym}} H}, \]

which yields the claimed bilinear subgradients. 

Similarly to the bilinear subdifferential of the squared Hilbert norm, we use the identification of the dual space \((L^2_\text{sym}([0, 1]))^*\) with the space of self-adjoint, bounded linear operators to describe the range of the adjoint \((0, A_k^*)^T\) of the bilinearly lifted autocorrelation. More precisely, using the unique identification (D.3) and (D.4), and incorporating the factorization \( A_k^* = \bigcirc^* \circ g_k^* \), we may write the range of the adjoint lifted operator as

\[ \text{ran}(0, A_k^*)^T = \left\{(0, \Phi_{A_k^*}[\phi]) : \phi \in L^2_\text{sym}([0, 2])\right\}, \quad (17) \]

where the self-adjoint, bounded integral operator \( \Phi_{A_k^*} \) is given by

\[ \Phi_{A_k^*}[\phi][t] := \int_0^1 k(s, s + t) \phi(s + t) u(s) ds \quad (t \in [0, 1]). \quad (18) \]

In our specific setting, the operator \( \Phi_{A_k^*}[\phi] \) is additionally compact since the range of \( \bigcirc^* \) is contained in the injective tensor product as discussed in the proof of lemma 5.4.

Comparing the range of \((0, A_k^*)^T\) and the subdifferential of the squared Hilbert norm in theorem 5.6, we notice that an element \((-2Tu^t, \text{Id} + T) \circ \partial \| \cdot \|^2(\mu^t)\) is contained in the range of the adjoint \((0, A_k^*)^T\) if and only if

\[ Tu^t = 0 \quad \text{and} \quad \text{Id} + T = \Phi_{A_k^*}[\phi] \]

for some \( \phi \) in \( L^2_\text{sym}([0, 2]) \). Since \( T \) is a self-adjoint, negative semi-definite operator, the spectrum \( \sigma(\Phi_{A_k^*}[\phi]) = \sigma(\text{Id} + T) \) of eigenvalues of \( \Phi_{A_k^*}[\phi] \) is bounded from above by one. Considering that

\[ \Phi_{A_k^*}[\phi][v] = \lambda v \quad \text{implies} \quad \Phi_{A_k^*}[\phi][iv] = -\lambda iv \quad (19) \]

for every eigenfunction \( v \) related to the eigenvalue \( \lambda \), we see that the spectrum of \( \Phi_{A_k^*}[\phi] \) is moreover symmetric, which means

\[ \sigma(\Phi_{A_k^*}[\phi]) \subset [-1, 1]. \]

Further, the equation

\[ Tu^t = (\Phi_{A_k^*}[\phi] - \text{Id})u^t = 0 \]

yields that \( u^t \) has to be an eigenfunction of \( \Phi_{A_k^*}[\phi] \) with respect to the eigenvalue one. In summary, the source-wise representation (6) may thus be rewritten in the following form.

**Theorem 5.7 (Source condition—deautoconvolution).** For a norm-minimizing solution \( u^t \) of the kernel-based deautoconvolution problem \( A_k[u] = g^t \), the source condition (6) is fulfilled if and only if there exists a \( \phi \) in \( L^2_\text{sym}([0, 2]) \) such that
\[ \Phi_{\delta^*}[u] = 1 \quad \text{and} \quad \Phi_{\delta^*}[u^\dagger] = u^\dagger \] (20)

for the integral operator
\[ \Phi_{\delta^*}[u](t) := \int_0^1 k(s, s + t) \phi(s + t) u(s) \, ds \quad (t \in [0, 1]). \] (21)

Starting from an arbitrary integral operator \( \Phi_{\delta^*}[\phi] \) or, more precisely, from a function \( \phi \in L^2_\delta([0, 1]) \), we can easily construct solutions to the deautoconvolution problem that fulfil the source condition (6) or, equivalently, (20) and (21) by rescaling the spectrum \( \sigma(\Phi_{\delta^*}[\phi]) \) appropriately and by choosing \( u^\dagger \) from the eigenspace of the eigenvalue one. For the trivial kernel \( k \equiv 1 \), the eigenfunction condition simply means that
\[ (\phi \ast u^\dagger(-\cdot)) |_{[0,1]} = u^\dagger. \] (22)

Hence, the source-wise representation (6) can only be fulfilled for continuous functions. More generally, if the integral operator \( \Phi_{\delta^*}[\phi] \) possesses smoothing properties, the norm-minimizing solution \( u^\dagger \) has to be arbitrary smooth. Our observations also cover the case of incomplete measurements, which can be modelled by setting the kernel \( k(s, t) = 0 \) for the corresponding \( t \in [0, 1] \).

Applying theorem 4.9 to the deautoconvolution problem, we obtain the following convergence rate.

**Corollary 5.8 (Convergence rate—deautoconvolution).** Let \( u^\dagger \) be a norm-minimizing solution of the kernel-based deautoconvolution problem \( \mathcal{A}_k[u] = g^\dagger \). If there exist an integral operator \( \Phi_{\delta^*}[\phi] \) fulfilling the conditions (D.3), the minimizer \( u^\dagger_{\alpha} \) of the Tikhonov regularization \( J_\alpha \) in (14) converges to \( u^\dagger \) in the sense that the Bregman distance between \( u^\dagger_\alpha \) and \( u^\dagger \) with respect to the norm of \( L^2_\delta([0, 1]) \) is bounded by
\[ \Delta_{\beta,0}\Phi_{\delta^*}[\phi](u^\dagger_\alpha, u^\dagger) \leq \left( \frac{\delta}{\sqrt{\alpha}} + \frac{\sqrt{\alpha}}{2} \| \phi \| \right)^2 \]
and the data fidelity term by
\[ \| \mathcal{A}_k[u^\dagger_\alpha] - g^\dagger \| \leq \delta + \alpha \| \phi \|. \]

Although corollary 5.8 gives a convergence rate for the deautoconvolution problem, the employed Bregman distance is usually very weak. Knowing that, besides the true solution \( u^\dagger \), the function —\( u^\dagger \) also solves the deautoconvolution problem, we notice that the Bregman distance \( \Delta_{\beta,0}\Phi_{\delta^*}[\phi] \) cannot measure distances in direction of \( u^\dagger \). Since the squared Hilbert norm is itself a dilinear functional, in the worst case, it can happen that the Bregman distance is constantly zero, and thus the derived convergence rate is completely useless. Depending on the integral operator \( \Phi_{\delta^*}[\phi] \) or, more precisely, on the eigenspace \( E_1 \) of the eigenvalue one, we can estimate the Bregman distance from below in the following manner.

**Theorem 5.9 (Bregman distance—deautoconvolution).** Let \( \Phi_{\delta^*}[\phi] \) be an integral operator of the form (20) and (21) with distinct positive eigenvalues \( 1 = \lambda_1 > \lambda_2 > \ldots \) and related finite-dimensional eigenspaces \( E_1, E_2, \ldots \). The corresponding dilinear Bregman distance \( \Delta_{\beta,0}\Phi_{\delta^*}[\phi] \) is bounded from below by
\[ \Delta_{\beta,0}\Phi_{\delta^*}[\phi](v, u^\dagger) \geq (1 - \lambda_2) \left\| P_{E_v^\perp} (v - u^\dagger) \right\|^2, \]
where \( P_{E_1} \) denotes the orthogonal projection onto the orthogonal complement of the eigenspace \( E_1 \).

**Proof.** The self-adjoint, compact integral operator \( \Phi_{q^*} : L^2_0([0,1]) \to L^2_0([0,1]) \) possesses a symmetric spectrum, where the eigenspace of the eigenvalue \(-\lambda_n\) is given by \( iE_n \), see (19); hence, the spectral theorem implies that the action of \( \Phi_{q^*} \) is given by

\[
\Phi_{q^*}[v] = \sum_{n=1}^{\infty} \lambda_n P_{E_n}(v) - \lambda_n P_{iE_n}(v) \quad (v \in L^2_0([0,1])),
\]

where \( P_{E_n} \) and \( P_{iE_n} \) denote the projections onto the eigenspace \( E_n \) and \( iE_n \) respectively. Considering that \( u^1 \) is an eigenfunction in \( E_1 \), we may write the Bregman distance for the squared Hilbert norm as

\[
\Delta_{\beta,(0,\Phi_{q^*})}(v, u^1) = \| v \|^2 - \| u^1 \|^2 - \left\langle \Phi_{q^*}[v], v \otimes v - u^1 \otimes u^1 \right\rangle
\]

Using the spectral representation of \( \Phi_{q^*} \), and denoting the kernel of \( \Phi_{q^*} \) by \( E_0 \), we have the estimation

\[
\Delta_{\beta,(0,\Phi_{q^*})}(v, u^1) = \| P_{E_0}(v) \|^2 + \sum_{n=1}^{\infty} (1 - \lambda_n) \| P_{E_n}(v) \|^2 + (1 + \lambda_n) \| P_{iE_n}(v) \|^2
\]

\[
\geq (1 - \lambda_2) \left[ \| P_{E_0}(v) \|^2 + \sum_{n=2}^{\infty} \| P_{E_n}(v) \|^2 + \sum_{n=1}^{\infty} \| P_{iE_n}(v) \|^2 \right]
\]

\[
= (1 - \lambda_2) \| P_{E_1}(v - u^1) \|^2,
\]

which yields the assertion.

\( \square \)

**Remark 5.10.** The quality of the convergence rate in corollary 5.8 thus crucially depends on the integral operator \( \Phi_{q^*} \) of the source condition. If the true solution \( u^1 \) is the only eigenfunction to the eigenvalue one, the minimizers \( u_n^* \) of the regularized perturbed problem converges nearly strongly to the true solution \( u^1 \) or \(-u^1 \) or, more precisely, converges to the \( \mathbb{R} \)-linear subspace spanned by \( u^1 \) in norm topology. If we choose \( \alpha \sim \delta \) and assume \( \dim E_1 = 1 \), then \( u_n^* \) converges in \( L^2_0([0,1]) / \text{span}\{u^1\} \) with a rate of \( O(\delta^\frac{1}{2}) \) strongly. If the true solution is a trigonometric polynomial or is contained in a certain subset of a Sobolev space, and if an appropriate variational source condition is fulfilled, one can moreover establish convergence rates based on the norm instead of the Bregman distance [BFH16, Jan00, DL08].

\( \square \)

### 6. Numerical simulations

Besides the theoretical regularization analysis in the previous sections, we now perform numerical experiments to verify the established convergence rates and to examine the error that is not covered by the Bregman distance. As before, we restrict ourselves to the deautoconvolution problem and, moreover, to the ‘kernelless’ setup with the trivial kernel \( k \equiv 1 \).
6.1. Construction of valid source elements

In a first step, we numerically construct signals that satisfy the source condition in theorem 5.7. For this, we approximate the integral operator $\Phi_{g^*_e} \colon L_2^1([0,1]) \to L_2^1([0,1])$ by applying the midpoint rule. More precisely, we assume

\[
\left( \frac{n}{N} \right) \approx \frac{1}{N} \sum_{k=0}^{N} \phi \left( \frac{n + k}{N} \right) u \left( \frac{k}{N} \right) \quad (n = 0, \ldots, N)
\]

for an appropriate large positive integer $N$. Starting from the source element $\phi$, we now determine the eigenfunction $u_1$ to the major eigenvalue $\lambda_1$. Exploiting the convolution representation of $\Phi_{g^*_e}$ in (22), we may compute the action of the integral operator $\Phi_{g^*_e}$ efficiently by applying the Fourier transform. The eigenfunction $u_1$ itself can be determined approximately by using the power iteration. To overcome the issue that the spectrum of $\Phi_{g^*_e}$ is symmetric, which means that $-\lambda_1$ is also an eigenvalue, see (19), we apply the power iteration to the operator $\Phi_{g^*_e} \circ \Phi_{g^*_e}$. In so doing, we obtain a vector $v_1$ in the span of $E_1 \cup iE_1$, where $E_1$ is the eigenspace with respect to the eigenvalue $\lambda_1$, and $iE_1$ with respect to $-\lambda_1$, see (19).

The projections to $E_1$ and $iE_1$ can, however, simply be computed by

\[
P_{E_1}(v_1) = \frac{1}{2} \left( v_1 + \Phi_{g^*_e}(v_1) \right)
\]

and

\[
P_{iE_1}(v_1) = \frac{1}{2} \left( v_1 - \Phi_{g^*_e}(v_1) \right).
\]

For the numerical experiments, we choose the signals $\phi^{(j)} := |\phi^{(j)}| e^{\arg(\phi^{(j)})}$ with

\[
|\phi^{(1)}(t)| := 2.7 e^{-((t - 2.1)^2 / 0.023)} + 4 e^{-((t - 1.5)^2 / 0.023)}
\]

\[
+ 4 e^{-((t - 0.143)^2 / 0.023)} + 2 e^{-((t - 1.4)^2 / 0.023)}
\]

\[
\arg(\phi^{(1)}(t)) := 5 \cos(7.867 t) - 2.3 \sin(25.786 t),
\]

\[
|\phi^{(2)}(t)| := 2.95 \mathbb{1}_{[0.415,0.458]}(t) + 6 \mathbb{1}_{[0.92,0.95]}(t)
\]

\[
\arg(\phi^{(2)}(t)) := 2 \cos(0.867 t) - 1.3 \sin(25.786 t),
\]

and

\[
|\phi^{(3)}(t)| := 0.645 \text{sinc}(9.855 (t - 1.2)) + 0.434 \text{sinc}(15.243 (t - 0.42))
\]

\[
+ 6.234 \text{sinc}(0.143 (t - 0.85))
\]

\[
\arg(\phi^{(3)}(t)) := 1.2 \cos(2.867 t) - 2.3 \sin(4.786 (t - 0.78)) + e^{0.643 t}.
\]

Here the indicator function $\mathbb{1}_{[t_1,t_2]}(t)$ is 1 if $t$ is contained in the interval $[t_1,t_2]$ and 0 else. Rescaling the source element with respect to the major eigenvalue $\lambda_1^{(j)}$ by $\frac{1}{\lambda_1^{(j)}} \phi^{(j)}$, we easily obtain norm-minimizing solutions $u_1^{(j)} := u_1^{(j)}(t)$ which satisfy the required source condition (20) and (21). The source elements $\phi^{(j)}$ and the related eigenfunctions $u_1^{(j)}$ are presented in figure 2. The results of the simulations here look quite promising in the sense that the class of functions $u_1^{(j)}$ satisfying (20) and (21) is rich on naturally occurring signals.

6.2. Validation of the theoretical convergence rate

To verify the convergence rate numerically, we have to solve the deautoconvolution problem for different noise levels. Referring to [Ger11] and [GHB+14], we here apply the GAUSS-NEWTON method to the discretized problem with forward operator
In order to solve the occurring equation systems iteratively, we use the conjugate gradient method with a suitable preconditioner and exploit the Toeplitz structure of the related system matrix, see for instance [RZR12]. For the exact signal $u^\ddagger$ arising from the source element $\phi(1)$, the numerical approximations $u_j^\ddagger$ corresponding to the minimizer $u^\delta_1$ of the regularized and non-regularized deautoconvolution problem are shown in figure 3. Besides the numerical ill-posedness of the discretized deautoconvolution problem for minor levels $\delta := ||g^\delta - g^\ddagger||$ of Gaussian noise, we can here see the smoothing effect of the applied Tikhonov regularization. Even for considerably higher noise levels like $\delta = 100||u^\ddagger||$, the reconstruction covers the main features of the unknown norm-minimizing solution $u^\ddagger$. The considered noise level $\delta$ here depends on the norm $||u^\ddagger||$ of the true signal and not on the norm $||g^\ddagger||$ of the exact data. Depending on our underlying numerical implementation, for the considered signal, the noise level $\delta = 100||u^\ddagger||$ corresponds to a Gaussian noise, whose norm approximately equals $||g^\ddagger||$. 

Figure 2. Numerical construction of norm-minimizing solutions $u^\ddagger := u_j^{(1)}$ satisfying the source condition in proposition 5.7 on the basis of an explicitly known source element $\phi(1)$. For the approximation of the integral operator $\Phi_{g^\ddagger}[\phi]$ in (23), a discretization with $N = 10^9$ samples has been used. (a) Magnitude of the source element $\phi(1)$. (b) Phase of the source element $\phi(1)$. (c) Magnitude of the eigenfunction $u_j^{(1)}$. (d) Phase of the eigenfunction $u_j^{(1)}$. 

$$\mathcal{A}_k[u] \left( \frac{n}{N} \right) \approx \frac{1}{N} \sum_{k=0}^{N} u \left( \frac{k}{N} \right) u \left( \frac{n-k}{N} \right).$$
Figure 3. Comparison between the norm-minimizing solution $u^\dagger$ and the numerically reconstructed signals $u_{\delta}^\dagger$ with and without regularization. The noise level for $u_1^\dagger$ and $u_3^\dagger$ amounts to $\delta = 0.05 \|u^\dagger\|$, and for $u_2^\dagger$ to $\delta = 100 \|u^\dagger\|$. The regularization for $u_1^\dagger$ and $u_2^\dagger$ corresponds to $\alpha = 100\delta$. The reconstruction $u_3^\dagger$ is computed without regularization or with $\alpha = 0$. (a) Magnitude of the approximation $u_{\delta}^\dagger$. (b) Phase of the approximation $u_{\delta}^\dagger$.

Figure 4. Comparison of the theoretical convergence rates in corollary 5.8 and theorem 5.9 with the numerical convergence rates for the source element $\phi^{(5)}$. (a) Bregman distance between $u_{\delta}^\dagger$ and $u^\dagger$. (b) Distance of $u_{\delta}^\dagger$ to ray $E_1$ spanned by $u^\dagger$. (c) Discrepancy between $A_k[u_{\delta}^\dagger]$ and $g^\dagger$. (d) Uncovered error part of $u_{\delta}^\dagger$ within the ray $E_1$ spanned by $u^\dagger$. 
Unfortunately, the result of the GAUSS–NEWTON method usually strongly depends on the start value, which have to be a very accurate approximation of the norm-minimizing solution $u^\dagger$ for very low noise levels $\delta$. For this reason, we extend the simulations for the convergence rate analysis by choosing 25 randomly created start values around the true solution $u^\dagger$ per noise level. Since we have constructed the norm-minimizing solution $u^\dagger$ and the exact data $g^\dagger$ from a specific source element $\phi$, besides the convergence rates in corollary 5.8, we have an explicit upper bound for the BREGMAN distance between the regularized solution $u^\alpha_{\delta}$ and the perturbed data $g^\delta$. The convergence rate analysis for the source element $\phi^{(3)}$ is presented in figure 4. Additionally, theorem 5.9 yields an upper bound for the distance $\|P_{E_1}(u^\alpha_{\delta})\|^2 = \|u^\alpha_{\delta} - P_{E_1}(u^\delta_{\alpha})\|^2$ between $u^\delta_{\alpha}$ and the ray $E_1$ spanned by $u^\dagger$. Here all three theoretical convergence rates and upper bounds match with numerical results.

If we consider the discrepancy $\|T_{\delta}[u^\alpha_{\delta}] - g^\delta\|$ in figure 4(c) more closely, we notice that the numerical and the theoretical rates coincides except for a multiplicative constant. Consequently, we cannot hope to improve the theoretical rate of $O(\delta)$. The BREGMAN distance and the distance to the spanned by $u^\dagger$ have a completely different behaviour. More precisely, we here have regions where the convergence rate is faster and regions where the convergence rate is slower than the theoretical rate of $O(\delta)$. Especially for the distance to the ray, it seems that the overall convergence rate is much faster than the theoretical rate. In some circumstances, our theoretical rate could thus be too pessimistic.

In this instance of the deautoconvolution problem, we can observe that the error $\|P_{E_1}(u^\alpha_{\delta} - u^\dagger)\|^2 = \|u^\alpha_{\delta} - P_{E_1}(u^\delta_{\alpha})\|^2$ within the ray $E_1$, which is not covered by corollary 5.8, numerically converges to zero superlinearly with a rate of $O(\delta^2)$. The shown numerical rate here strongly depends on the starting values of the GAUSS–NEWTON method, which have been chosen in a small neighbourhood around $u^\dagger$. Choosing starting values around $-u^\dagger$, we would observe the same convergence rate to $-u^\dagger$. In fact, the sequence $u^\delta_{\alpha}$ could be composed of two subsequences—one converging to $u^\dagger$ and the other to $-u^\dagger$.

7. Conclusion

Starting from the question: how non-linear may a non-linear forward operator be in order to extend the linear regularization theory, we have introduced the classes of dilinear and diconvex mappings, which corresponds to linear, bilinear, and quadratic inverse problems. Exploiting the tensorial structure behind these mappings, we have introduced two different concepts of generalized subgradients and subdifferentials—the dilinear and the representative generalization. We have shown that the classical subdifferential calculus can be partly transferred to both new settings. Although the representative generalization yields stronger computation rules, the related subdifferential unfortunately strongly depends on the non-unique convex representative of the considered diconvex mapping. Besides all differences, there exists several connections between the dilinear and representative subdifferential.

On the basis of these preliminary considerations, we have examined the class of dilinear inverse problems. Analogously to linear inverse problems, the (non-convex) regularizations in the dilinear setting are well posed, stable, and consistent. Using the injective tensor product, which is nearly a predual space of the projective tensor product, as topologizing family, we have seen that the required sequential weak* (semi-)continuity of the forward operator and regularization term may be inherited from the lifted versions. Moreover, we have derived a convergence rate analysis very similar to the linear setting under similar assumptions and requirements. This enables us to give explicit upper bounds for the discrepancy and the
BREGMAN distance between the solution of the regularized problem and the $R$-minimizing solution. One limitation of the presented approach is that the error is only measured with respect to a BREGMAN distance, which, in the worst case, can be nearly meaningless. Here the questions arise: how can we modify the range and variational source conditions to guarantee convergence with respect to an appropriate norm, and how can we obtain slower or optimal convergence rates.

In a last step, we have applied the developed theory to the deautoconvolution problem that appears in spectroscopy, optics, and stochastics. Although the requirements of the non-linear regularization theory are not fulfilled, our novel approach yields convergence rates based on a suitable range source condition and the dilinear BREGMAN distance. Depending on the source element, the BREGMAN distance is here surprisingly strong. In the best case, the solutions of the regularized problems converge strongly to the ray spanned by the true signal, which is the best possible rate with respect to the ambiguities of the problem. Using numerical experiments, we have considered different source elements and the corresponding norm-minimizing solutions, which shows that there exists signals satisfying the required source-wise representation. Finally, we have observed the established error bounds in the numerical simulations. Although the considered Tikhonov regularization may be solved by a GAUSS–NEWTON method, the questions remain how one can exploit the dilinear and diconvex structure of the problem to design more suitable algorithms.

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Appendix A. Sufficient conditions for diconvexity

As defined in section 3 the diconvexity of a functional $F: X \rightarrow \mathbb{R}$ is based on the existence of an appropriate convex lifting $\tilde{F}$. Here the question arises if we can find an much easier sufficient condition to guarantee the diconvexity. In this appendix, we prove that the existence of an diaffine minorant $A$, which means that the lifting $\tilde{A}$ is continuous and affine, is such a sufficient condition. In order to prove this assertion, we will exploit that each vector $(u, u \otimes u)$ is an extreme point of the convex hull of the diagonal $\{(u, u \otimes u) : u \in X\}$, which means that $(u, u \otimes u)$ cannot be written as a non-trivial convex combination of other points, see [Roc70].

Lemma A.1 (Extreme points of the convex hull of the diagonal). Let $X$ be a real BANACH space. Each point $(u, u \otimes u)$ with $u \in X$ is an extreme point of $\text{conv} \{(u, u \otimes u) : u \in X\}$.

Proof. For an element $(u, u \otimes u)$ with $u \in X$, we consider an arbitrary convex combination

$$(u, u \otimes u) = \sum_{n=1}^{N} \alpha_n (u_n, u_n \otimes u_n)$$

with $u_n \in X$, $\alpha_n \in [0, 1]$, and $\sum_{n=1}^{N} \alpha_n = 1$. Applying the linear functionals $(\phi, 0 \otimes 0)$ and $(0, \phi \otimes \phi)$ with $\phi \in X^*$ of the dual space $X^* \times (X \otimes_{\pi, \text{sym}} X)^*$, we get the identity
Due to the strict convexity of the square, this equation can only hold if \( \langle \phi, u_n \rangle = \langle \phi, u \rangle \) for every \( n \) between 1 and \( N \), and for every \( \phi \in X^* \). Consequently, all \( u_n \) coincide with \( u \), which shows that the considered convex combination is trivial, and that \( (u, u \otimes u) \) is an extreme point.

With the knowledge that the diagonal \( \{(u, u \otimes u) : u \in X\} \) contains only extreme points of its convex hull, we may now give an sufficient condition for a mapping being diconvex.

**Proposition A.2.** Let \( X \) be a real Banach space. If the mapping \( F : X \to \overline{R} \) has a continuous, diaffine minorant, then \( F \) is diconvex.

**Proof.** If \( F \) has a continuous, diaffine minorant \( \tilde{G} \) with representative \( \tilde{G} \), we can construct a representative \( \tilde{F} \) by

\[
\tilde{F}(u, w) := \begin{cases} 
F(u) & w = u \otimes u, \\
\tilde{G}(u, w) & w \neq u \otimes u, \text{ but } (u, w) \in \text{conv } \{(u, u \otimes u) : u \in X\}, \\
+\infty & \text{else}.
\end{cases}
\]

Since we firstly restrict the convex mapping \( \tilde{G} \) to the convex set \( \{(u, u \otimes u) : u \in X\} \) and secondly increase the function values of the extreme points on the diagonal, the constructed mapping \( \tilde{F} \) is convex. Obviously, the functional \( \tilde{F} \) is also proper and thus a valid representative.

**Remark A.3.** If the Banach space \( X \) in proposition A.2 is finite-dimensional, then the reverse implication is also true. To validate this assertion, we restrict a given proper, convex representative \( \tilde{F} \) to the convex hull of \( \{(u, u \otimes u) : u \in X\} \), which here means that we set \( \tilde{F}(v, w) = +\infty \) for \( (v, w) \) outside the convex hull. Due to the fact that the relative interior of a convex set is non-empty in finite dimensions, see [Roc70, theorem 6.2], there exists a point \( (v, w) \) where the classical subdifferential \( \partial \tilde{F}(v, w) \) of the proper, convex representative \( \tilde{F} \) is non-empty, see for instance [Roc70, theorem 23.4]. In other words, we find a dual element \( (\xi, \Xi) \in X^* \times (X \otimes_{\tau, \text{sym}} X)^* \) such that

\[
\tilde{F}(v', w') \geq F(v, w) + \langle \xi, v' - v \rangle + \langle \Xi, w' - w \rangle
\]

for all \( v' \in X \) and \( w' \in X \otimes_{\tau, \text{sym}} X \). Obviously, the functional \( \tilde{A} \) given by

\[
\tilde{A}(v', w') := F(v, w) + \langle \xi, v' - v \rangle + \langle \Xi, w' - w \rangle
\]

defines a continuous, affine minorant of \( \tilde{F} \), and the restriction \( A(u) := \tilde{A}(u, u \otimes u) \) thus a diaffine minorant \( A \) of \( F \).

**Appendix B. Computation rules for the dilinear and representative subdifferential**

In this appendix, we give the detailed proofs for the dilinear and representative subdifferential calculus in section 3, which have been omitted. We start with the dilinear chain rule for the
composition of a linear operator $K: X \rightarrow Y$ and a diconvex functional $F: Y \rightarrow \mathcal{R}$. More precisely, we want to show

$$(K \times (K \otimes_{\pi, \text{sym}} K))^* \partial_{\beta} F(K(u)) \subset \partial_{\beta}(F \circ K)(u).$$

**Proof of proposition 3.6.** Firstly, we notice that the functional $F \circ K$ is diconvex with the representative $\tilde{F} \circ (K \times (K \otimes_{\pi, \text{sym}} K))$. Next, let us assume $(\xi, \Xi) \in \partial_{\beta} F(K(u))$, which is equivalent to

$$F(v) \geq F(K(u)) + \langle \xi, v - K(u) \rangle + \langle \Xi, v \otimes v - K(u) \otimes K(u) \rangle$$

for all $v$ in $Y$. Replacing $v \in Y$ by $K(v)$ with $v \in X$, we obtain

$$(F \circ K)(v) \geq (F \circ K)(u) + \langle \xi, K(v - u) \rangle + \langle \Xi, (K \otimes_{\pi, \text{sym}} K)(v \otimes v - u \otimes u) \rangle$$

for all $v$ in $X$. Thus, $(K \times (K \otimes_{\pi, \text{sym}} K))^*(\xi, \Xi)$ is contained in the subdifferential $\partial_{\beta} F \circ K(u)$. \hfill \Box

Next, we verify the chain rule for the composition of a dilinear mapping $K: X \rightarrow Y$ and a convex functional $F: Y \rightarrow \mathcal{R}$, which is given by

$$K^* \partial F(K(u)) \subset \partial_{\beta}(F \circ K)(u).$$

**Proof of proposition 3.7.** The functional $F \circ K$ is diconvex with convex representative $F \circ K$, since the lifted operator $\tilde{K}$ is linear. Next, we consider a linear subgradient $\xi$ of $F$ at $K(u)$, which means

$$F(v) \geq F(K(u)) + \langle \xi, v - K(u) \rangle$$

for all $v$ in $Y$. Replacing $v \in Y$ by $K(v)$ with $v \in X$, we obtain

$$(F \circ K)(v) \geq (F \circ K)(u) + \langle \xi, \tilde{K}(v - u, v \otimes v - u \otimes u) \rangle$$

for all $v$ in $X$. Consequently, $\tilde{K}^*(\xi)$ is contained in the subdifferential $\partial_{\beta}(F \circ K)(u)$. \hfill \Box

In order to show the computation rules for the representative subdifferential, the central idea is to show that, in the finite-dimensional setting, the representative $\tilde{F}$ of a diconvex mapping $F$ is continuous on the interior of its effective domain and to use the classical sum and chain rule, which can be found in [BC11, ET76, Roc70, Sho97]. For the continuity, we firstly show that the manifold of rank-one tensors is curved in a way such that the convex hull of each open set of the manifold contains an inner point with respect to the surrounding tensor product. In the following, we denote by $\mathcal{B}_{\epsilon}$ the closed $\epsilon$-ball around zero.

**Lemma B.1 (Local convex hull of rank-one tensors).** Let $u$ be a point of the real finite-dimensional Banach space $X$. The interior of the convex hull of the set

$$\{(u + v, (u + v) \otimes (u + v)) : v \in \mathcal{B}_{\epsilon} \} \subset X \times (X \otimes_{\pi, \text{sym}} X)$$

is not empty for every $\epsilon > 0$.

**Proof.** To determine a point in the interior, we construct a suitable simplex by using a normalized basis $(e_n)_{n=1}^N$ of the Banach space $X$. Obviously, the convex hull contains the points $(u + ce_n, (u + ce_n) \otimes (u + ce_n))$. Viewing the point $(u, u \otimes u)$ as new origin of $X \times (X \otimes_{\pi, \text{sym}} X)$, we obtain the vectors
(u + ee_n, (u + ee_n) ⊗ (u + ee_n)) - (u, u ⊗ u)
= (e_n, e (u ⊗ u) + ε (e_n ⊗ u) + ε^2 (e_n ⊗ e_n)),
(B.2)

where the first components again form a basis of the first component space X. Next, we consider the convex combination of \((u + ee_n, (u + ee_n) ⊗ (u + ee_n))\) and \((u - ee_n, (u - ee_n) ⊗ (u - ee_n))\) with weights \(\frac{1}{2}\). In this manner, we obtain

\[
\frac{1}{2} (u + ee_n, (u + ee_n) ⊗ (u + ee_n))
+ \frac{1}{2} (u - ee_n, (u - ee_n) ⊗ (u - ee_n)) - (u, u ⊗ u)
= (0, ε^2 (e_n ⊗ e_n)),
(B.3)
\]

Similarly, by considering the corresponding convex combination of \((u + \frac{ε}{2} (e_n + e_m), (u + \frac{ε}{2} (e_n + e_m)) ⊗ (u + \frac{ε}{2} (e_n + e_m)))\) and \((u - \frac{ε}{2} (e_n + e_m), (u - \frac{ε}{2} (e_n + e_m)) ⊗ (u - \frac{ε}{2} (e_n + e_m)))\) with \(n \neq m\), we get the vectors

\[
(0, \frac{ε^2}{4} ((e_n + e_m) ⊗ (e_n + e_m)))
= (0, \frac{ε^2}{4} (e_n ⊗ e_n)) + (0, \frac{ε^2}{4} (e_m ⊗ e_m)) + (0, \frac{ε^2}{4} (e_m ⊗ e_m)),
(B.4)
\]

Since the vectors \((e_n ⊗ e_m)_{n,m=1}^N\) form a basis of the tensor product \(X ⊗_π X\), see [Rya02], the vectors in (B.3) and (B.4) span the second component space \(X ⊗_π sym X\). Thus, the vectors (B.2)-(B.4) form a maximal set of independent vectors, and the convex hull of them cannot be contained in a true subspace of \(X × (X ⊗_π sym X)\). Consequently, the simplex spanned by the vectors (B.2)-(B.4) and zero contains an inner point. Since the constructed simplex shifted by \((u, u ⊗ u)\) is contained in the convex hull of (B.1), the assertion is established.

**Remark B.2.** Unfortunately, lemma B.1 does not remain valid for infinite-dimensional Banach spaces. For example, if the Banach space \(X\) has a normalized Schauder basis \((e_n)_{n∈N}\), we can explicitly construct a vector not contained in the convex hull of (B.1) but arbitrarily near to a given element of the convex hull. For this, we notice that the tensor product \(X ⊗_π X\) possesses the normalized Schauder basis \((e_n ⊗ e_m)_{n,m∈N}\) with respect to the square ordering, see [Rya02], and that the coordinates of an arbitrary rank-one tensor \(u ⊗ u = \sum_{(n,m)∈N^2} a_{nm} (e_n ⊗ e_m)\) with \(u = \sum_{n∈N} u_n e_n\) are given by \(a_{nm} = u_n u_m\). Consequently, the coordinates \(a_{nm}\) on the diagonal have to be non-negative. This implies that the convex hull of (B.1) only contains tensors with non-negative diagonal.

Now, let \((v, w)\) be an arbitrary element of the convex hull of (B.1). Since the representation \(\sum_{(n,m)∈N^2} b_{nm} (e_n ⊗ e_m)\) of the tensor \(w\) converges with respect to the square ordering, the coordinates \(b_{nm}\) form a zero sequence. Therefore, for each given \(δ > 0\), we find an \(N ∈ N\) such that \(b_{nm} < δ\) whenever \(n ≥ N\). Subtracting the vector \((0, δ (e_N ⊗ e_N))\) from \((v, w)\), we obtain an arbitrarily near vector to \((v, w)\) that is not contained in the convex hull, since one coordinate on the diagonal is strictly negative. Thus, the convex hull of (B.1) has an empty interior.

By means of lemma B.1, we can immediately establish proposition 3.8, which states the continuity of the representative in finite-dimension.
Proof of proposition 3.8. By assumption, there is a point \( u \in X \) such that \( F \) is finite on an \( \epsilon \)-ball \( B_\epsilon(u) \) around \( u \) for some \( \epsilon > 0 \). Consequently, the representative \( \tilde{F} \) is finite on the set
\[
\left\{ (u + v, (u + v) \otimes (u + v)) : v \in B_\epsilon \right\}.
\] (B.5)

Using the construction in the proof of lemma B.1, we find a simplex with vertices in (B.5) that contains an inner point \( (v, w) \) of the convex hull of (B.5) and hence of the effective domain \( \text{dom}(\tilde{F}) \). Since \( \tilde{F} \) is convex and finite on the vertices of the constructed simplex, the representative \( \tilde{F} \) is bounded from above on a non-empty, open neighbourhood around \( (v, w) \), which is equivalent to the continuity of \( \tilde{F} \) on the interior of the effective domain \( \text{dom}(\tilde{F}) \), see for instance [ET76, Sho97].

Based on the classical sum and chain rule in convex analysis and the continuity of the representative, we can now establish propositions 3.9–3.11. We begin with the proof of the sum rule
\[
\partial[F + G](u) = \partial F(u) + \partial G(u).
\]

Proof of proposition 3.9. In the proof of proposition 3.8, we have constructed an appropriate simplex to prove the existence of a point \( (v, w) \) in \( X \times (X \otimes_{\pi, \text{sym}} X) \) where the representative \( \tilde{F} \) is finite and continuous. Using the same simplex again, we see that the representative \( \tilde{G} \) is finite and continuous in the same point \( (v, w) \). Applying the classical sum rule—see for instance [Sho97, proposition II.7.7]—to the representative \( \tilde{F} + \tilde{G} \), we obtain
\[
\partial(\tilde{F} + \tilde{G})(u, u \otimes u) = \partial \tilde{F}(u, u \otimes u) + \partial \tilde{G}(u, u \otimes u)
\]
for all \( (u, u \otimes u) \) in \( X \times (X \otimes_{\pi, \text{sym}} X) \) and thus the assertion.

Remark B.3. In order to apply the classical sum rule in the proof of proposition 3.9, it would be sufficient if only one of the functionals \( \tilde{F} \) and \( \tilde{G} \) is continuous in \( (v, w) \). The assumption that \( F \) and \( G \) are finite at some non-empty, open set is thus stronger than absolutely necessary. On the other side, this assumption is needed to ensure that the effective domain of \( \tilde{G} \) and the interior of the effective domain of \( \tilde{F} \) have some point in common.

Next, we verify the representative chain rule
\[
\partial(F \circ K)(u) = (K \times (K \otimes_{\pi, \text{sym}} K))^* \partial F(K(u))
\]
for a surjective linear operator \( K : X \to Y \) and the diconvex Functional \( F : Y \to \mathbb{R} \).

Proof of proposition 3.10. Like in the proof of proposition 3.6, the mapping \( \tilde{F} \circ (K \times (K \otimes_{\pi, \text{sym}} K)) \) is a proper, convex representative of \( F \circ K \). Since the linear operator \( K \) is surjective, the mapping \( K \otimes_{\pi, \text{sym}} K \) is surjective too. In more detail, there exist finitely many vectors \( e_1, \ldots, e_N \) such that the images \( f_a = K(e_a) \) form a basis of the finite-dimensional Banach space \( Y \). Since the symmetric tensors
\[
f_a \otimes f_m + f_m \otimes f_a = (K \otimes_{\pi, \text{sym}} K)(e_a \otimes e_m + e_m \otimes e_a)
\]
form a basis of \( Y \otimes_{\pi, \text{sym}} Y \), see [Rya02, proposition 1.1], the bounded linear mapping \( K \otimes_{\pi, \text{sym}} K \) is also surjective. Using proposition 3.8, we thus always find a point \( (K \times (K \otimes_{\pi, \text{sym}} K))(v, w) \) where \( \tilde{F} \) is continuous. Now, the classical chain rule—see for instance [Sho97, proposition II.7.8]—implies
for all \( u \) in \( X \), which establishes the assertion. \( \square \)

**Remark B.4.** In the proof of proposition 3.10, the surjectivity of \( K \) implies the non-emptiness of the intersection between the interior of \( \text{dom}(\tilde{F}) \) and the range of \( K \times (K \otimes_{\pi, \text{sym}} K) \). So long as this intersection is not empty, proposition 3.10 remains valid even for non-surjective operators \( K \). The non-emptiness of the intersection then depends on the representative \( \tilde{F} \) and the mapping \( K \times (K \otimes_{\pi, \text{sym}} K) \). The intention behind proposition 3.10 has been to give a chain rule that only depends on properties of the given \( F \) and \( K \).

Finally, we give the proof of the second representative chain rule

\[
\partial(\tilde{F} \circ (K \times (K \otimes_{\pi, \text{sym}} K)))(u, u \otimes u) = (K \times (K \otimes_{\pi, \text{sym}} K)^* \partial \tilde{F}((K \times (K \otimes_{\pi, \text{sym}} K))(u, u \otimes u)) = (K \times (K \otimes_{\pi, \text{sym}} K)^* \partial \tilde{F}(K(u), K(u) \otimes K(u))
\]

for all \( u \) in \( X \).

**Proof of proposition 3.11.** Since the proper and convex mapping \( F \) is bounded from above on some non-empty, open set, the function \( \tilde{F} \) is continuous on the interior its effective domain \( \text{dom}\tilde{F} \), see for instance [ET76, proposition 2.5]. Consequently, we always find a point \( \tilde{K}(v, v \otimes v) \) where \( F \) is continuous, which allows us to apply the classical chain rule—see for instance [Sho97, proposition II.7.8]—to the representative \( F \circ \tilde{K} \). In this manner, we obtain

\[
\partial(F \circ \tilde{K})(u, u \otimes u) = \tilde{K}^* \partial \tilde{F}(\tilde{K}(u, u \otimes u))
\]

for all \( u \) in \( X \). \( \square \)

**Appendix C. Lifting of the semi-continuity in finite dimensions**

In section 4, we have seen that the sequential weak* lower semi-continuity of a diconvex mapping \( F \) can be inherited from the weak* lower semi-continuity of the convex lifting \( \tilde{F} \). At this point, one may ask oneself whether each sequentially weakly* lower semi-continuous, diconvex mapping possesses a sequentially weakly* lower semi-continuous, convex representative. At least for finite-dimensional spaces, where the weak* convergence coincides with the strong convergence, this is always the case. Remembering that all real \( d \)-dimensional BANACH spaces are isometrically isomorphic to \( \mathbb{R}^d \), we can restrict our argumentation to \( X = \mathbb{R}^d \) equipped with the EUKLIIDIAN inner product and norm. Further, the sequential weak* lower semi-continuity here coincides with the lower semi-continuity. The projective tensor product \( \mathbb{R}^d \otimes_{\pi, \text{sym}} \mathbb{R}^d \) becomes the space of symmetric matrices \( \mathbb{R}^{d \times d}_{\text{sym}} \) equipped with the HILBERT–SCHMIDT inner product and the FROBENIUS norm. Moreover, the space \( \mathbb{R}^{d \times d}_{\text{sym}} \) is spanned by the rank-one tensors \( u \otimes u = uu^T \) with \( u \in \mathbb{R}^d \). The dual space \( (\mathbb{R}^d \otimes_{\pi, \text{sym}} \mathbb{R}^d)^* \) may also be identified with the space of symmetric matrices \( \mathbb{R}^{d \times d}_{\text{sym}} \).

**Theorem C.1 (Lower semi-continuity in finite dimensions).** A diconvex mapping \( F: \mathbb{R}^d \to \mathbb{R} \) is lower semi-continuous if and only if there exists a lower semi-continuous representative \( \tilde{F}: \mathbb{R}^d \times \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R} \).
Proof. In proposition 4.5, we have in particular shown that the lower semi-continuity of $\tilde{F}$ implies the lower semi-continuity of $F$. Thus, it only remains to prove that each lower semi-continuous diconvex functional possesses a lower semi-continuous representative $\tilde{F}$.

The assertion is obviously true for the constant functional $F \equiv +\infty$ with representative $\tilde{F}(u, w) := 0$ for one point $(u, w)$ with $w \neq u \otimes u$ and $\tilde{F}(..., \cdot) = +\infty$ otherwise. For the remaining functionals, the central idea of the proof is to show that the lower semi-continuous convexification $\overline{\text{conv}} F_\otimes$—the closure of the convex hull $\text{conv} F_\otimes$—of the mapping $F_\otimes$ in (3) is a valid representative of a lower semi-continuous mapping $F$, which means that $\overline{\text{conv}} F_\otimes(u, u \otimes u) = F(u)$ for all $u \in \mathbb{R}^d$. For the sake of simplicity, we assume that $F(u) > 0$ and thus $F_\otimes(u, u \otimes u) > 0$ for all $u$ in $\mathbb{R}^d$, which can always be achieved by subtracting a (continuous) dilinear minorant, see remark A.3.

For a fixed point $(u, u \otimes u)$ on the diagonal, we distinguish the following three cases:

(i) The point $(u, u \otimes u)$ is not contained in the relative closure of $\text{dom}(\overline{\text{conv}} F_\otimes)$, which implies $F(u) = \overline{\text{conv}} F_\otimes(u, u \otimes u) = +\infty$.

(ii) The point $(u, u \otimes u)$ lies in the relative interior of $\text{dom}(\overline{\text{conv}} F_\otimes)$. Since the effective domain $\text{dom}(\overline{\text{conv}} F_\otimes)$ is a subset of $\text{conv} \{(u, u \otimes u) : u \in \mathbb{R}^d\}$, and since $(u, u \otimes u)$ is an extreme point of the latter set, see lemma B.1, the point $(u, u \otimes u)$ thus has to be extreme with respect to the effective domain. Since the extreme points of a convex set are, however, contained in the relative boundary except for the zero-dimensional case, see [Roc70, corollary 18.1.3], the effective domains of $\overline{\text{conv}} F_\otimes$ and $F_\otimes$ have to consist exactly of the considered point $(u, u \otimes u)$. In this instance, the closed convex hull $\overline{\text{conv}} F_\otimes$ equals $F_\otimes$ and is trivially a sequentially weakly* continuous representative.

(iii) The point $(u, u \otimes u)$ is contained in the relative boundary of $\text{dom}(\overline{\text{conv}} F_\otimes)$.

To finish the proof, we have to show $F(u) = \overline{\text{conv}} F_\otimes(u, u \otimes u)$ for the third case.

In order to compute $\overline{\text{conv}} F_\otimes(u, u \otimes u)$, we apply [Roc70, theorem 7.5], which implies

$$\overline{\text{conv}} F_\otimes(u, u \otimes u) = \lim_{\lambda \not\to 1} \text{conv}(\lambda(u, u \otimes u) + (1 - \lambda)(v, w)),$$

where $(v, w)$ is some point in the non-empty relative interior of $\text{dom}(\text{conv} F_\otimes)$. Next, we take a sequence $(v_k, w_k) := \lambda_k(u, u \otimes u) + (1 - \lambda_k)(v, w)$ with $\lambda_k \in (0, 1)$ so that $\lim_{k \to \infty} \lambda_k = 1$ and consider the limit of the function values $\text{conv} F_\otimes(v_k, w_k)$. Since the complete sequence $(v_k, w_k)_{k=1}^\infty$ is contained in the relative interior of $\text{dom}(\text{conv} F_\otimes)$, see [Roc70, theorem 6.1], all functions values $\text{conv} F_\otimes(v_k, w_k)$ are finite and can be approximated by CARATHÉODORY’s theorem. More precisely, for every $\rho > 0$ and for every $k \in \mathbb{N}$, we can always find convex combinations $(v_k, w_k) = \sum_{n=1}^{N+1} \alpha_n^{(k)} (u_n^{(k)} \otimes u_n^{(k)})$, where $\alpha_n^{(k)}$ is in $[0, 1]$ so that $\sum_{n=1}^{N+1} \alpha_n^{(k)} = 1$, and where $N$ is an integer not greater than the dimension of $\mathbb{R}^d \times \mathbb{R}_{\text{sym}}^{d \times d}$, such that

$$\text{conv} F_\otimes(v_k, w_k) = \sum_{n=1}^{N+1} \alpha_n^{(k)} F_\otimes(u_n^{(k)} \otimes u_n^{(k)}) \leq \rho,$$

see [Roc70, corollary 17.1.5].

In the next step, we examine the occurring sequence of convex combinations in more detail. For this purpose, we define the half spaces

$$H^+_{\epsilon} := \{(v, w) : G(u, v) \geq \epsilon\} \quad \text{and} \quad H^-_{\epsilon} := \{(v, w) : G(u, v) \leq \epsilon\},$$

where $\epsilon > 0$ is a given precision.
with $u \in \mathbb{R}^d$, $\epsilon > 0$, and the functional $G_u : \mathbb{R}^d \times \mathbb{R}_{sym}^{d \times d} \to \mathbb{R}$ given by

$$G_u(v, w) := -\langle 2u, v \rangle_{\mathbb{R}^d} + \langle I, w \rangle_{\mathbb{R}_{sym}^{d \times d}},$$

where $I$ denotes the identity matrix. Obviously, a vector $(v, v \otimes v)$ is contained in the shifted half space $H_{\epsilon u}^+ + (u, u \otimes u)$ if and only if

$$G_u(v - u, v \otimes v - u \otimes u) = -\langle 2u, v - u \rangle_{\mathbb{R}^d} + \langle I, v \otimes v \rangle_{\mathbb{R}_{sym}^{d \times d}} - \langle I, u \otimes u \rangle_{\mathbb{R}_{sym}^{d \times d}}$$

$$= -\langle 2u, v - u \rangle_{\mathbb{R}^d} + \langle v, v \rangle_{\mathbb{R}^d} - \langle u, u \rangle_{\mathbb{R}^d}$$

$$= \|v - u\|^2 \geq \epsilon. \quad \text{(C.1)}$$

Similarly, the vector $(v, v \otimes v)$ is contained in $H_{\epsilon u}^- + (u, u \otimes u)$ if and only if $\|v - u\|^2 \leq \epsilon$.

As mentioned above, we now consider a sequence of convex combinations

$$\sum_{n=1}^{N+1} \alpha_n^{(k)}_u (u_n^{(k)}, u_n^{(k)} \otimes u_n^{(k)}) \to (u, u \otimes u),$$

where $\alpha_n^{(k)} \in [0, 1]$ so that $\sum_{n=1}^{N+1} \alpha_n^{(k)} = 1$, and where $N$ is some fixed integer independent of $k$. For $n = 1$, either the sequence $(u_n^{(k)})_{k \in \mathbb{N}}$ has a subsequence converging to $u$ or there exists an $\epsilon_1 > 0$ such that $\|u_n^{(k)} - u\|^2 \geq \epsilon_1$ for all $k \in \mathbb{N}$. In the second case, the complete sequence $(u_n^{(k)})_{k \in \mathbb{N}}$ is contained in the shifted half space $H_{\epsilon_1 u}^- + (u, u \otimes u)$. Thinning out the sequence of convex combinations by repeating this construction for the remaining indices $n = 2, \ldots, N$ iteratively, we obtain a subsequence of the form

$$\sum_{n=1}^{L} \alpha_n^{(k)} (u_n^{(k)}, u_n^{(k)} \otimes u_n^{(k)}) \to (u, u \otimes u),$$

where we assume that the first $L$ sequences $(u_n^{(k)})_{k \in \mathbb{N}}$ possess the accumulation point $u$ without loss of generality – if necessary, we rearrange the indices $n = 1, \ldots, N + 1$ accordingly. Thinning out the subsequence even further, we can also ensure that the coefficients $(\alpha_n^{(k)})_{k \in \mathbb{N}}$ converge for every index $n$.

In the following, we have to take special attention of the subsequence $(u_n^{(k)})_{k \in \mathbb{N}}$ not converging to $u$. Therefore, we consider the case where $\beta^{(k)} := \sum_{n=L+1}^{N+1} \alpha_n^{(k)} \otimes \beta^{(k)}$ does not become constantly zero after some index $\ell_0$ more precisely. Taking a subsequence with $\beta^{(k)} \neq 0$, and re-weighting the sequence of convex combinations, we now obtain

$$\left(1 - \beta^{(k)}\right) \sum_{n=1}^{L} \frac{\alpha_n^{(k)}}{1 - \beta^{(k)}} (u_n^{(k)}, u_n^{(k)} \otimes u_n^{(k)}) + \frac{\beta^{(k)}}{1 - \beta^{(k)}} \sum_{n=L+1}^{N+1} \frac{\alpha_n^{(k)}}{1 - \beta^{(k)}} (u_n^{(k)}, u_n^{(k)} \otimes u_n^{(k)}) \to (u, u \otimes u),$$

where $\epsilon > 0$ is chosen smaller than $\epsilon_{L+1}, \ldots, \epsilon_{N+1}$. Since the second sum is a convex combination of points in $H_{\epsilon u}^+ + (u, u \otimes u)$, the value of the sum is also contained in $H_{\epsilon u}^+ + (u, u \otimes u)$. Since $(u, u \otimes u)$ is not contained in the closed set $H_{\epsilon u}^+ + (u, u \otimes u)$ for $\epsilon > 0$ by construction, the coefficients $\beta^{(k)}$ could neither become constantly one. For an appropriate subsequence, the re-weighted convex combinations are thus well defined. Obviously, the first sum converges to $(u, u \otimes u)$. If we now assume that the sequence $(\beta^{(k)})_{k \in \mathbb{N}}$ does not converge to zero, the se-
ond sum has to converge to some point in \( H_{+}^{n} + (u, u \otimes u) \). Consequently, \((u, u \otimes u)\) is a non-trivial convex combination of itself with a further point in \( H_{+}^{n} + (u, u \otimes u) \). Since \((u, u \otimes u)\) is not contained in \( H_{+}^{n} + (u, u \otimes u) \), see (C.1), this is not possible. Hence, the coefficient \( \beta^{(f)} \) converges to zero, which is our main observation in this step.

Applying the subsequence construction above to the sequence of function values
\[
\left( \sum_{n=1}^{N+1} \alpha_n^{(k)} F_\otimes(u_n^{(k)}, u_n^{(k)} \otimes u_n^{(k)}) \right)_{k=1}^\infty,
\]
and exploiting the lower semi-continuity and non-negativity of \( F \), we can finally estimate the limit of the function values \( \text{conv} F_\otimes(v_k, w_k) \) by
\[
\lim_{k \to \infty} \text{conv} F_\otimes(v_k, w_k) + \rho \geq \liminf_{k \to \infty} \sum_{n=1}^{N+1} \alpha_n^{(t)} F_\otimes(u_n^{(t)}, u_n^{(t)} \otimes u_n^{(t)})
\geq \liminf_{k \to \infty} \frac{\alpha_n^{(t)}}{1 - \beta^{(t)}} \sum_{n=1}^L \alpha_n^{(t)} F_\otimes(u_n^{(t)}, u_n^{(t)} \otimes u_n^{(t)})
\geq \frac{L}{\rho} \liminf_{k \to \infty} \left[ \frac{\alpha_n^{(t)}}{1 - \beta^{(t)}} F(u_n^{(t)}) \right] \geq F(u)
\]
because \( 1 - \beta^{(t)} = \sum_{n=1}^L \alpha_n^{(t)} \) converges to one as discussed above. Since the accuracy \( \rho \) of the approximation can be chosen arbitrarily small, we thus have
\[
\text{conv} F_\otimes(u, u \otimes u) = \lim_{k \to \infty} \text{conv} F_\otimes(v_k, w_k) \geq F(u).
\]
Hence, \( \text{conv} F_\otimes(u, u \otimes u) \) equals \( F(u) \) for all \( u \in \mathbb{R}^d \), and the lower semi-continuous convex hull \( \text{conv} F_\otimes \) is a valid representative of \( F \).

**Appendix D. Sequential weak* continuity of the autoconvolution**

In order to apply the developed dilinear regularization theory to the deautoconvolution problem, the sequentially weak* continuity of the autoconvolution operator is required. In this appendix, we show the corresponding continuity for the quadratic part \( \otimes \) defined in (16). In the proof, we exploit that the adjoint lifted mapping \( \tilde{\otimes} \) allows us to interpret each function \( \omega \) in \( L^2_{\text{sym}}([0, 1]^2) \) as a bounded linear functional on the projective tensor product \( L^2_{\text{sym}}([0, 1]) \otimes \pi_{\text{sym}} L^2_{\text{sym}}([0, 1]) \). More precisely, the action of the function \( \omega \) on an arbitrary tensor \( w \) is simply given by
\[
w \mapsto \langle \omega, \tilde{\otimes}[w] \rangle_R = \Re \langle \omega, \tilde{\otimes}[w] \rangle_C.
\]

**Proof of lemma 5.4.** The central idea of the proof is to exploit proposition 4.4, which means that we have to show the sequential weak* continuity of the dilinear lifting \( \tilde{\otimes} \) with respect to the topology induced by \( L^2([0, 1]) \otimes_{\text{sym}} L^2([0, 1]) \). Since the dilinear lifting of a quadratic operator is independent of the first component space, see example 2.3, it is enough to show the sequential weak* continuity of the quadratic lifting \( \tilde{\otimes} \) in lemma 5.2. For this, we have to show
\[
\langle \omega, \tilde{\otimes}[w_n] \rangle_R \to \langle \omega, \tilde{\otimes}[w] \rangle_R
\]

(D.2)
for every \( \omega \in L^2_{\text{sym}}([0, 1]^2) \), and for every weakly* convergent sequence \( w_n \rightharpoonup w \).

As mentioned above, the symmetric injective tensor product \( L^2_{\text{sym}}([0, 1]) \otimes L^2_{\text{sym}}([0, 1]) \) is here isometrically isomorphic to the self-adjoint compact operators, where the action of a self-adjoint compact operator \( \Phi \) is given by the lifting of the quadratic form \( (u, u) \mapsto \langle \Phi[u], u \rangle \). This observation is the key component to establish the assertion. More precisely, if we can show that the action of an arbitrary symmetric function \( \omega \), which is equivalent to the testing in (D.2), corresponds to the lifting of a self-adjoint compact operator, then the assertion is trivially true.

For this purpose, we consider the action of \( \omega \) to a symmetric rank-one tensor \( u \otimes u \), which may be written as

\[
\langle \omega, \tilde{\omega} [u \otimes u] \rangle_R = \Re \left[ \int_0^1 \int_0^1 \omega(s, t) \overline{u(s)} u(t) ds \, dt \right] = \langle \Phi_\omega[u], u \rangle_R ,
\]

where the operator \( \Phi_\omega : L^2_{\text{sym}}([0, 1]) \rightarrow L^2_{\text{sym}}([0, 1]) \) is defined by

\[
\Phi_\omega[u](t) := \int_0^1 \omega(s, t) \overline{u(s)} ds \quad (t \in [0, 1]).
\]

The central observation is that the \( R \)-linear operator \( \Phi_\omega \) resembles a Fredholm integral operator. Due to the occurring conjugation, for a fixed point \((s, t)\), the multiplication with \( w(s, t) \) here only acts as an \( R \)-linear mapping instead of an usually \( C \)-linear mapping on \( u(s) \).

Similarly to the classical theory, see for instance [Wer02], we can approximate the kernel function \( \omega \) by a sequence of appropriate step functions \( \omega_n \) on a rectangular partition of \([0, 1]^2\) such that \( \omega_n \rightharpoonup \omega \) in \( L^2_{\text{sym}}([0, 1]^2) \). The ranges of the related operators \( \Phi_{\omega_n} \) are finite-dimensional, and because of the convergence of \( \Phi_{\omega_n} \) to \( \Phi_\omega \), the compactness of \( \Phi_\omega \) follows. Since \( \omega \) is symmetric, the operator \( \Phi_\omega \) is moreover self-adjoint. Consequently, the quadratic lifting \( w \mapsto \langle \omega, \tilde{\omega} [w] \rangle_R \) with \( w \in L^2_{\text{sym}}([0, 1]) \otimes_{\text{sym}} L^2_{\text{sym}}([0, 1]) \) of the quadratic form \( u \mapsto \langle \Phi_\omega[u], u \rangle \) is contained in the symmetric injective tensor product \( L^2_{\text{sym}}([0, 1]) \otimes_{\text{sym}} L^2_{\text{sym}}([0, 1]) \).

For a weakly* convergent sequence \( w_n \rightharpoonup w \), we thus have (D.2) for all \( \omega \) in \( L^2_{\text{sym}}([0, 1]^2) \), which shows the sequential weak* continuity of the quadratic lifting \( \tilde{\omega} \) and hence of the dilinear lifting \((0, \tilde{\omega})\). The sequential weak* continuity of the dilinear operator \( \tilde{\omega} \) now immediately follows from proposition 4.4.

\[\Box\]

**ORCID iDs**

Robert Beinert \( \odot \) https://orcid.org/0000-0002-7813-2762

Kristian Bredies \( \odot \) https://orcid.org/0000-0001-7140-043X

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