Mathematical Models in Isotropic Cosmology

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Abstract. An axiomatic approach to the mathematical models of the isotropic cosmology.
Preface

1. This work comes from my own need to understand the mathematical basics of modern Cosmology, and it is therefore the result of a research (at the present incomplete) aimed to collect the countless notions of this discipline in a logically ordered system, starting from a number of postulates.

2. The postulates proposed here are very simple and intuitive. They include the isotropy principle (the homogeneity principle comes out as a theorem). As a consequence, the mathematical models that follow cannot describe the complex physical phenomena occurring at the beginning of the universe, as well as other current phenomena regarding the dark matter and the background radiation. Nevertheless, despite the extreme simplification, one of the surprising facts is that the obtained estimates of the age of the universe as well as of other quantities are in full agreement with those advanced by astrophysicists. Anyway, as for any axiomatic theory, our approach is of course open to criticisms, adjustments and extensions.

3. The philosophy adopted in conducting this work is based on the following rules:

   (i) To distinguish the various subjects under investigation between those having a purely observational or geometrical character (they are incorporated into the chapter of the cosmic kinematics) and those that involve dynamical concepts and laws (as energy-momentum tensors, Einstein field equations, etc.).

   (ii) Accompany the mathematical analysis with the geometric vision, and perform ordinary differential calculus on differential manifolds avoiding the use of special kinds of coordinates. It will not be necessary to use, for example, the typical coordinates of the Friedman-Robertson-Walker metric.

   (iii) Pay attention to the logical order in which the definitions and theorems have to be placed.

   (iv) Pay a special attention to the dimensionality of the physical or geometrical objects. This is a way for testing the correctness of the calculations and of the numerical evaluations.

   (v) Last but not least, be free of preconceptions. For example, do not think ab initio that the space-time of the relativistic cosmology is a manifold with a Lorentzian metric (as everybody knows) because we have to find out this fact, at the appropriate time, as a theorem. For this reason we cannot start from the celebrated Weyl principle, which imposes from the very beginning the existence of a Lorentzian metric for which the world-lines of the galaxies are time-like curves. In my opinion this principle lies at a too advanced position. In mathematical terms, the Weyl principle ranks in the category of Riemannian (or semi-Riemannian) manifolds, while it seems more appropriate to begin from a more general category, as that of the differentiable manifolds. In doing so, we get two advantages: we can locate primary concepts and their relationships in the right order and at the right place and, second, we do not lose secondary but
noteworthy concepts, which otherwise would remain hidden.

4. That said, the schedule of this work is the following.

- **CHAPTER 1** (Cosmic kinematics). We compose the basic geometrical structures of the **cosmic space-time**, as the set of all the **cosmic events**. This set is the union of two **singular events** \( \alpha \) and \( \omega \), representing the beginning (or birth) and the end (or death) of the universe, and a four-dimensional manifold \( M \) of **non-singular events**. The manifold will be endowed with two fibrations (Fig. 1).

![Figure 1: The basic fibration of the cosmic space-time.](image)

The first fibration is made of three-dimensional manifolds \( S_t \), parametrized by a **cosmic time** \( t \). They represent the sets of simultaneous events and will be called **spatial sections**. Each \( S_t \) will be endowed with a Riemannian structure with metric tensor \( g_t \). The **isotropy principle** implies that each metric \( g_t \) has a constant curvature, varying with \( t \). The second fibration is made of the **world-lines** of the galaxies, which are transversal to the spatial sections. Since the world-lines do not intersect each other, this theory does not contemplate fragmentations or collisions of galaxies. We postulate that the set \( Q \) of the galactic world-lines (which is nothing but the set of all the galaxies) is endowed with the structure of a three-dimensional manifold: it will be called the **quotient manifold** and it will play a basic role in this theory. A theorem establishes that the quotient manifold \( Q \) is endowed with a metric \( \tilde{g} \), called **quotient metric**, in such a way that any spatial section \(( S_t, g_t)\) is isomorphic to \(( Q, \tilde{g})\), so that any spatial section can be taken as a representative of the quotient manifold.

In any cosmological theory the so-called **scale factor** plays a key role. It is a function of the cosmic time \( t \), commonly denoted as \( a(t) \), that appears as a conformal factor of the spatial metrics. In our theory, however, the notion of scale factor arises as a conformal factor linking two spatial metrics through the
Therefore it is a two-variable function in the cosmic time: \( a(t_1, t_2) \). This fact offers significant opportunities and advantages in the mathematical analysis of cosmological phenomena, which instead escape in using the one-variable version of the scale factor. For example, if we fix a value of the second variable \( t_2 \) (which we will call reference time) and leave \( t_1 = t \) free to run as the only independent variable, then a theorem shows that two scale factors obtained in this way, with different values of the reference time, say \( a(t, t_2) \) and \( a(t, t'_2) \), differ by a constant factor. As a consequence, we can always impose to the scale factor the normalization condition
\[
a(t_2, t_2) = 1
\]
to be satisfied under the free choice of a reference time \( t_2 \). This allows us to establish an effective test for the physical validity of equations involving the scale factor, which must be invariant under the choice of \( t_2 \). Once a reference time is fixed, the scale factor \( a(t, t_2) \) becomes the principal cosmic function from which, in principle, we should derive the evolution of several other observational quantities (like the Hubble parameter, the energy density, the matter density, etc.). For this reason the graph of a scale factor \( a(t, t_2) \) will be called profile of the universe.

- **Chapter 2** (Cosmic connections). At the end of Chapter 1, through the introduction of the notion of free particle, we are led to presume the existence in space-time of one (or more) linear symmetric connections that, somehow, are ‘adapted’ to the geometric structures introduced so far in \( M \). In this chapter we prove the existence of a family of cosmic connections which are the natural prelude to the formulation of a dynamics and depend on an indeterminate (but not arbitrary) function of the cosmic time. The assignment of such a function through a bridge-postulate will mark the passage from kinematics to dynamics. We will examine two possible bridge-postulates. The first one leads towards a generalization of the Newtonian space-time, where we could build-up a Newtonian cosmic dynamics. The second one consists in assuming the existence of special particles (read photons) wandering in the cosmos with a constant ‘peculiar velocity’. The surprising result (theorem) is that:
   
   (i) The cosmic connection is the Levi-Civita connection of a space-time metric. (ii) This metric has necessarily a Lorentzian signature. (iii) The galactic world-lines are time-like geodesics and the world-lines of the photons are null geodesics.

   This makes us to move towards a relativistic cosmic dynamics founded on the Einstein field equations (Chapter 4).

- **Chapter 3** is devoted to the preparation of the elements that we need for the formulation of the dynamics (Ricci tensor, Einstein tensor, etc.). It is pointed out a remarkable fact which greatly simplifies the calculations: due
to the isotropy principle any symmetric two-tensor $T^{\alpha\beta}$ having a geometrical or physical meaning is fully determined by two functions $\phi(t)$ and $\psi(t)$ only, which we call characteristic functions. In generic co-moving coordinates the components of such a tensor are

\[
\begin{cases}
T^{00} = \phi(t) = a \text{ function of } t \\
T^{0a} = 0 \\
T^{ab} = \psi(t) \tilde{g}^{ab}(\tilde{q}) = a \text{ function of } t \times \text{ the quotient metric } \tilde{g}^{ab}
\end{cases}
\]

Then it can be proved that the conservation law and the Einstein field equations result in ordinary differential equations involving the scale factor and the two characteristic functions of the momentum-energy tensor. Significant general results are obtained without specifying the form the characteristic functions.

**• Chapter 4.** The choice of the energy-momentum tensor is the first topic of this chapter, devoted to the relativistic cosmic dynamics. We will consider the standard energy-momentum tensor of a perfect fluid

\[
T^{\alpha\beta} = (\epsilon + p) U^\alpha U^\beta + p g^{\alpha\beta}
\]

where $\epsilon(t)$ is the energy density, $p(t)$ is the intergalactic pressure and $U^\alpha$ is the unitary four-velocity of the galactic fluid

\[
U^\alpha \text{ def } c^{-1} d\gamma^\alpha/dt; \quad \begin{cases}
U^0 = 1 \\
U^a = 0
\end{cases} \quad g_{\alpha\beta} U^\alpha U^\beta = -1.
\]

It follows that the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$ and the Einstein field equations reduce respectively to two dynamical equations only:

\[
\begin{align*}
a \dot{\epsilon} + 3(\epsilon + p) \dot{a} &= 0 \\
\frac{\dot{a}^2}{c^2} &= \frac{1}{3} a^2 (\Lambda + \chi \epsilon) - \tilde{K}
\end{align*}
\]

where $\tilde{K}$ is the curvature of the quotient manifold.

**• Chapter 5.** A further crucial step is the choice of an equation of state that ties the three unknown functions $\epsilon(t)$, $p(t)$ and $a(t)$. In this chapter we deal with the simplest possible case of a linear barotropic fluid, whose equation of state is

\[ p = w \epsilon, \]

$w$ being a constant parameter. Thus, we are faced with various models whose behaviour depends on the sign of the constant curvature $\tilde{K}$. However, the models with negative spatial curvature are ‘a priori’ excluded from this theory because, since it can be proved that the dynamical equations imply a radial speed expansion permanently greater than the light speed. In turn, also the positive spatial curvature is proved to be inadmissible by a mathematical argument based on a
reasonable (although rough) estimate of the present-day matter density. As a consequence, we confine ourselves to study the models with zero spatial curvature. These models are in fact very simple and do not take into account of all the complex quantum-physical phenomenology occurring in the evolution of the universe, especially in the vicinity of its creation (as said above). Nevertheless they reveal the typical features of the isotropic cosmology and could serve as a starting point for the creation of finer models. For example, in the gallery of the possible profiles of the universe we find that of Fig. 2, which is perfect agreement with that appearing in the Nobel Lecture by G. Riess [14] (Fig. 3).

Figure 2: One of the eligible profiles of the universe.

Figure 3: From Riess Nobel Lecture,

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5 The Nobel Prize in Physics 2011 was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess "for the discovery of the accelerating expansion of the Universe through observations of distant supernovae".
A crucial and fortunate circumstance is that the dynamical equations of a flat barotropic model are solvable in terms of elementary functions (exponential or hyperbolic functions). This allows us to get the exact expressions of all the observational variables related to the scale factor. The only approximation is then due to the numerical calculation and to the estimate of the observational data.

• Chapter 6. In order to avoid the common misconceptions on the various notions of horizons, pointed out for example in [3], we afford the so-called ‘horizon problems’ on the basis of what we have learned from the previous chapter and with a significant graphic method. When applied to the flat dust-matter model, this method gives numerical results in very good agreement with the current observational data.

S. B., May 2016.
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Chapter 1

Cosmic kinematics

1.1 The postulates of the cosmic space-time

In an axiomatic formulation of cosmology the concept of event is a primary (or 'primitive') concept as that of point in Euclidean geometry.

1st Postulate. The history of the universe is made of events, whose set will be called cosmic space-time and denoted by \( \mathcal{E} \). There are two critical events, denoted by \( \alpha \) and \( \omega \), representing the beginning and the end of the universe, respectively. All other events are called regular and form a four-dimensional manifold \( M \):

\[ \mathcal{E} = \alpha \cup M \cup \omega \]

The second primary concept is that of cosmic particle.

2nd Postulate. The life of a cosmic particle is a sequence of regular events forming a smooth curve in \( M \), called world-line of the particle.
In view of the large-scale approach to cosmology, the cosmic particles are identified with galaxies. The third primary concept is that of cosmic fluid.

\[3^{rd} \text{ Postulate.} \]

(i) The cosmic fluid is made of cosmic particles whose world-lines form a congruence of curves filling the whole manifold \( M \).

(ii) The set \( Q \) of all the galactic world-lines has the structure of a three-dimensional differentiable manifold such that the canonical projection \( \rho: M \to Q \) is a surjective submersion. We call \( Q \) the quotient manifold.

The third primary concept is that of cosmic time. A cosmic time is a regular mapping from \( M \) to an open interval of \( \mathbb{R} \),

\[ t: M \to (t_\alpha, t_\omega) \subseteq \mathbb{R}, \]

which associates to each regular event \( e \in M \) its date \( t(e) \). Two cosmic times are said to be equivalent if they are related by an affine transformation

\[ t \mapsto \tilde{t} = \lambda t + \mu, \quad \lambda, \mu \in \mathbb{R}, \quad \lambda > 0 \]

The regularity property of the function \( t \) implies that the inverse image of any real number \( t \in (t_\alpha, t_\omega) \) is a three-dimensional submanifold \( S_t \). Two such submanifolds \( S_t \) and \( S_{t'} \) do not intersect and form a foliation of the whole manifold \( M \). The submanifolds \( S_t \) which we call spatial sections. Two events of an \( S_t \) have, by definition, the same date \( t \); thus, they are simultaneous. Consequently a cosmic time provides a cosmic chronology: given two events \( e_1 \) and \( e_2 \) we say that

- \( e_1 \) and \( e_2 \) are simultaneous \( \text{if } t(e_1) = t(e_2) \)
- \( e_1 \) occurs before \( e_2 \) \( \text{if } t(e_1) < t(e_2) \)
- \( e_1 \) occurs after \( e_2 \) \( \text{if } t(e_1) > t(e_2) \)
It follows that two equivalent cosmic times provides the same cosmic chronology. The chronology can be extended to the critical events by setting

\[ t(\alpha) = t_\alpha, \quad t(\omega) = t_\omega. \]

So, the event \( \alpha \) (beginning of the universe) occurs before all other events and \( \omega \) (end of the universe) occurs after all other events.

It is crucial to highlight the following. By means of a reversible smooth transformation \( t \mapsto \bar{t} \) any smooth mapping over an open interval \( t: M \to (t_\alpha, t_\omega) \) can be transformed into a smooth mapping over the whole real line, \( t: M \to \mathbb{R} = (-\infty, +\infty) \), preserving the above chronology. Consequently, if we want to give meaning to a concept like the duration of the universe then we must exclude such a general transformation of the cosmic time. So, having assumed as permissible the affine transformations, the life-time-interval of the universe is of four types:

| Life-time interval | Beginning | End        |
|--------------------|-----------|------------|
| \((-\infty, +\infty)\) | infinite past | infinite future |
| \((-\infty, t_\omega]\) | infinite past | finite time \( t_\omega \) |
| \([t_\alpha, t_\omega] \) | finite time \( t_\alpha \) | finite time \( t_\omega \) |
| \([t_\alpha, +\infty) \) | finite time \( t_\alpha \) | infinite future |

Table 1.1: Life of the universe.

4th Postulate. There exist a cosmic time and a cosmic chronology, as defined above, such the foliation \( S_t \) of the spatial sections is transversal to the world-lines of the cosmic fluid.
The transversality condition means that a world-line is nowhere tangent to a spatial section. By virtue of well-known arguments of differential geometry, the $3^{rd}$ and the $4^{th}$ postulates imply that the restriction of the projection $\rho$ to any spatial section $S_t$ is a diffeomorphism, and consequently

**Theorem 1.1** – The spatial sections and the quotient manifold are diffeomorphic manifolds.

Another consequence of this postulate is the following.

**Theorem 1.2** – Any coordinate system $\bar{q} = (q^a) = (q^1, q^2, q^3)$ on an open domain $U \subseteq Q$ generates a coordinate system $(t, q^a)$ on the open subset of $M$ made of the world-lines determined by $U$ (Fig. 1.1).

![Figure 1.1: Co-moving coordinates and spatial metrics.](image)

Coordinates on $M$ of this type are called **co-moving coordinates**\(^1\). The coordinates $\bar{q} = (q^a)$ are Lagrangian coordinates of the galactic fluid: they have a constant value on each world-line of $U$. In this way the coordinates $\bar{q}$ can be interpreted as coordinates on each spatial section $S_t$. So, they will be called **spatial coordinates**\(^2\). At this point of our route the cosmic space-time is equipped with two **transversal trivial fibrations:**

(i) A fibration over the open real interval $(t_\alpha, t_\omega)$ (beginning and of the universe) with fibers diffeomorphic to the quotient manifold $Q$.

(ii) A fibration over the quotient manifold $Q$ with fibers the galactic world-lines which are diffeomorphic to the interval $(t_\alpha, t_\beta)$.

\(^1\) Since $t$ is constant on each spatial section, co-moving coordinates are also called **synchronous coordinates**.

\(^2\) Greek indices $\alpha, \beta, \ldots$ will run from 0 to 3. Latin indices $a, b, \ldots$ will run from 1 to 3.
1.2. Manifolds with constant curvature

**5th Postulate.** Each spatial section \( S_t \) is endowed with a positive-definite metric tensor \( g_t \) smoothly depending on \( t \).

In other words, we think of each \( S_t \) as a three-dimensional Riemannian manifold where the metric tensor components \( g_{ab}(t, \tilde{q}) \) in co-moving coordinates are smooth functions of \( t \).

The **Copernican principle** assumes that neither the Sun nor the Earth are in a central, specially favored position in the universe. This principle is extended to cosmology with the following **isotropy principle**.

**6th Postulate.** On each spatial section \( S_t \) there is no distinguished vector field having any epistemological meaning.

**Theorem 1.3** – Any scalar field on \( M \) having an epistemological meaning is a function of the cosmic time \( t \) only i.e., it is constant on each \( S_t \).

*Proof* – By means of the metric \( g_t \) we can define the gradient of such a scalar field which is then a distinguished vector field on \( S_t \). This is in contrast with the isotropy principle. \( \square \)

**Theorem 1.4** – Each spatial section \( (S_t, g_t) \) is a manifold with constant curvature \( K(t) \).

*Proof* – The Ricci tensor \( R_t \) of the metric \( g_t \) must be proportional to the metric tensor itself, \( R_t = \lambda_t g_t \), otherwise the existence of distinguished Ricci directions would be in contrast with the isotropy principle. In turn, the factor \( \lambda_t \) must be constant on \( S_t \) because of Theorem 1.3. Thus \( (S_t, g_t) \) is an Einstein manifold. It is known that an Einstein manifold of dimension 3 has constant curvature. \( \square \)

### 1.2 Manifolds with constant curvature

In this section we recall the main features of the manifolds with constant curvature. For the Riemann curvature tensor and the Ricci tensor of a linear symmetric connection \( \Gamma \) we will refer to the following definitions:

\[
R^\nu{}_{\alpha\beta\mu} \overset{\text{def}}{=} \partial_\mu \Gamma^\nu{}_{\alpha\beta} - \partial_\beta \Gamma^\nu{}_{\mu\alpha} + \Gamma^\ell{}_{\alpha\beta} \Gamma^\nu{}_{\mu\ell} - \Gamma^\ell{}_{\mu\alpha} \Gamma^\nu{}_{\beta\ell}
\]

\[
R_{\alpha\beta} \overset{\text{def}}{=} R^\mu{}_{\alpha\mu\beta} = \partial_\mu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\mu\alpha} + \Gamma^\sigma{}_{\alpha\beta} \Gamma^\mu{}_{\sigma\mu} - \Gamma^\sigma{}_{\alpha\mu} \Gamma^\mu{}_{\beta\sigma}
\]

3 A high level reference to this topic is [1]. A classical reference for a simpler approach, sufficient for our needs, is [17].

4 Our definition of the Riemann tensor is that of [10] and [1]. On the contrary, our Ricci tensor is the opposite of that of [1].
where $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$ are the symbols of $\Gamma$ in any coordinate system.

A Riemannian manifold with metric tensor $g_{\alpha\beta}$ is said to be of constant curvature $K$ when the totally covariant Riemann tensor

$$R^\lambda_{\alpha\mu\beta} \overset{\text{def}}{=} g^\lambda_\nu R^\nu_{\alpha\mu\beta}$$

of the Levi-Civita connection satisfies the equation

$$R_{\alpha\beta\gamma\delta} = K (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

equivalent to

$$R^\alpha_{\beta\gamma\delta} = K (\delta^\alpha_\gamma g_{\beta\delta} - \delta^\alpha_\delta g_{\beta\gamma}),$$

It follows that

$$R_{\alpha\beta} = (n - 1) K g_{\alpha\beta} \quad R = n(n - 1) K$$

where $n$ is the dimension of the manifold and $R$ is the Ricci scalar (or scalar curvature)

$$R \overset{\text{def}}{=} g^{\alpha\beta} R_{\alpha\beta}$$

For a conformal transformation $\bar{g}_{\alpha\beta} = \alpha g_{\alpha\beta}$ with a constant factor $\alpha$ the Riemann tensor components are invariant: $\bar{R}^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta}$. Hence, from (1.3),

$$\bar{K} (\delta^\alpha_\gamma \bar{g}_{\beta\delta} - \delta^\alpha_\delta \bar{g}_{\beta\gamma}) = K (\delta^\alpha_\gamma g_{\beta\delta} - \delta^\alpha_\delta g_{\beta\gamma}),$$

and consequently $\alpha \bar{K} = K$. This shows that

$$\bar{g}_{\alpha\beta} = \alpha g_{\alpha\beta} \quad (\alpha = \text{constant}) \implies \bar{K} = \frac{K}{\alpha}$$

With a similar argument one can show that: two metric tensors $\bar{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ on a same manifold, with the same signature and with constant curvatures of the same sign are conformal $\bar{g}_{\alpha\beta} = \alpha g_{\alpha\beta}$ with a positive constant conformal factor $\alpha$.

### 1.3 Dimensional analysis

To test the correctness of the formulas that we will write, it is important to consider the physical dimension of the involved objects. We will denote the
1.3. Dimensional analysis

The physical dimension of an object $X$ by the symbol $\text{Dim} \ (X)$ is the basic physical dimensions are

$$
\begin{align*}
\text{Dim (time)} &= T \\
\text{Dim (length)} &= L \\
\text{Dim (mass)} &= M \\
\text{Dim (dimensionless quantity)} &= 1
\end{align*}
$$

Then the dimension of any object $X$ will be expressed by the product of positive or negative integer powers of these symbols,

$$\text{Dim} \ (X) = T^a L^b M^c, \quad a, b, c \in \mathbb{Z}.$$ 

For instance:

| Object       | Dim     |
|--------------|---------|
| area         | $L^2$   |
| volume       | $L^3$   |
| velocity     | $LT^{-1}$ |
| acceleration | $LT^{-2}$ |
| angle        | 1       |
| angular velocity | $T^{-1}$ |

Table 1.2: Dimension of the basic geometric and kinematic quantities.

| Object         | Symbol | Dim         | Note                       |
|----------------|--------|-------------|----------------------------|
| force          | $F$    | $MLT^{-2}$  | mass $\times$ acceleration |
| pressure       | $P$    | $ML^{-1}T^{-2}$ | force/area               |
| energy (work)  | $E$    | $ML^2T^{-2}$ | work $= force \times length$ |
| energy density | $\epsilon$ | $ML^{-1}T^{-2}$ | energy/volume            |
| mass density   | $\rho$ | $ML^{-3}$   | mass/volume               |

Table 1.3: Dimension of the basic physical quantities.

The coordinates of a manifold can be dimensionless (e.g. angles) or with a physical dimension (time, length,...). About the co-moving coordinates introduced by Theorem 1.2, we assume that:

(i) The cosmic time $t$ is time-dimensional: $\text{Dim} \ (t) = T$.
(ii) The spatial coordinates $q^a$ are length-dimensional: $\text{Dim} \ (q^a) = L$.

\[\text{The symbol } [X] \text{ is more commonly used.}\]
Then it will be convenient to replace the time coordinate $t$ with a length-dimensional coordinate $q^0$ via the simple relationship

\[ q^0 = \kappa t \]

where $\kappa$ is an arbitrary constant with the dimension of a velocity: $\text{Dim} (\kappa) = L T^{-1}$. It is immaterial the numerical value of this constant. In this way we get **length-dimensional co-moving coordinates**.

Consequently:

\[ \text{Dim} (g_{\alpha\beta}) = \text{Dim} (g^{\alpha\beta}) = 1, \]
\[ \text{Dim} (\Gamma_{\alpha\beta,\gamma}) = \text{Dim} (\Gamma^\gamma_{\alpha\beta}) = L^{-1}, \]
\[ \text{Dim} (R^\nu_{\alpha\mu\beta}) = L^{-2}, \]
\[ \text{Dim} (R_{\alpha\beta}) = \text{Dim} (R) = L^{-2}. \]

**Remark 1.1** – Do not pay attention to the dimension of the coordinates is unfortunately a widespread harmful habit, which produces confusion and mistakes in the writing and interpreting of the tensorial-type formulas. For example, if you use angular coordinates (for instance on a sphere or hyper-sphere with fixed radius) which are dimensionless, then $\text{Dim} (g_{\alpha\beta}) = L^2$ and consequently,

\[
\begin{align*}
\text{Dim} (g^{\alpha\beta}) &= L^{-2}, \\
\text{Dim} (\Gamma_{\alpha\beta,\gamma}) &= L^2, \\
\text{Dim} (\Gamma^\gamma_{\alpha\beta}) &= 1,
\end{align*}
\]

\[
\begin{align*}
\text{Dim} (R^\nu_{\alpha\mu\beta}) &= 1, \\
\text{Dim} (R_{\alpha\beta}) &= 1, \\
\text{Dim} (R) &= L^{-2}.
\end{align*}
\]

Note that the Ricci scalar still maintains its dimension $L^{-2}$, as should be. But if you use coordinates of mixed type (dimensionless, length-dimensional, time-dimensional,...) then the dimension of the objects listed above depends on the indices. This may create a big confusion.

### 1.4 Scale factor and quotient metric

**Theorem 1.5** – There exists a two-variable function $a(t_1, t_2)$ such that

\[ g(t_1) = a^2(t_1, t_2) g(t_2) \]

\[ ^7 \text{Later, when dealing with cosmic dynamics, we shall be led to consider } \kappa = c, \text{ the light speed}. \]
where \( g(t_1) \) and \( g(t_2) \) are the metric tensors on the spatial sections \( S_{t_1} \) and \( S_{t_2} \).

Proof – Each spatial section \( S_t \) has a constant curvature \( K(t) \). Since two spatial sections \( S_{t_1} \) and \( S_{t_2} \) are diffeomorphic, the constant curvatures \( K(t_1) \) and \( K(t_2) \) have the same sign (or are both equal to 0). In accordance with what has been said at the end of Section 1.2, the metrics \( g(t_1) \) and \( g(t_2) \) are conformal with a positive constant conformal factor \( a^2 : g(t_1) = a^2 g(t_2) \). □

Actually this constant is a dimensionless positive function \( a^2(t_1, t_2) \) of the two dates \( t_1 \) and \( t_2 \). We assume, without loss of generality, that this function is positive. From the definition (1.11) it follows that it obeys the following rules:

\[
\begin{align*}
    a(t, t) &= 1, \\
    a(t_1, t_2) a(t_2, t_3) &= a(t_1, t_3) \quad \text{(composition rule)}, \\
    a(t_1, t_2) &= \frac{1}{a(t_2, t_1)}.
\end{align*}
\]

(1.12)

If we fix a reference time \( t^\# \) then by virtue of Theorem 1.1 (page 4) we can take the spatial section \( S_{t^\#} \) as a representative of the quotient manifold \( Q \). Then \( Q \) is endowed with the quotient metric

\[
\tilde{g} \overset{\text{def}}{=} g(t^\#)
\]

and (1.11) provides the equation

\[
g(t) = a^2(t, t^\#) \tilde{g}
\]

(1.14)

**Notation.** The geometrical objects associated with the quotient metric – or which belong to the geometry of the quotient manifold – will be marked with a tilde \( \tilde{\cdot} \). An exception will be the arc element of the quotient metric \( \tilde{g} \), denoted by \( ds \) – see for instance equation (1.15) below.

This factorization equation will be widely applied in the following. An equivalent form is

\[
ds_t = a(t, t^\#) ds
\]

(1.15)

where \( ds_t \) is the arc element of \( g(t) \).

\textsuperscript{8} Note that it is not commutative in \((t_1, t_2)\). The use of the scale factor must be attentive to the location of the two variables \((t_1, t_2)\).
We call **scale factor** the positive function $a(t, t_z)$ \( \textsuperscript{9} \) It has to be regarded as a function of $t$, but determined by the preset value of $t_z$. When there is no danger of confusion, we can simply denote it by $a(t)$,

\[
(1.16) \quad a(t) \overset{\text{def}}{=} a(t, t_z)
\]

Note that $a(t_z) = a(t_z, t_z) = 1$. For this reason $t_z$ will be also called **normalization time**.

**Remark 1.2** - The composition rule in (1.12) implies that if $a(t_1, t_2)$ is constant then this constant is necessarily equal to 1. For $a(t) = \text{constant} = 1$ we have the so-called **static universe**: $g(t) = g^t$ for all $t$. 

**Remark 1.3** - The scale factor plays a central role in cosmology because, as a function of the cosmic time, it contains all the information concerning the **evolution of the universe**. It is determined by dynamical equations established in the next chapter. 

\( \textsuperscript{9} \) It is also called **scale parameter** or **expansion factor**. It is also denoted by the symbol $R(t)$, here used for the radius of the universe.
1.4. Scale factor and quotient metric

Remark 1.4 – In the current literature it is not sufficiently emphasized the fact that the scale factor $a(t)$ is defined up to the choice of a reference time. This oversight precludes the recognition of some important facts. It should be borne in mind that, in order to have a physical meaning, any formula involving $a(t)$ and its derivatives must be invariant under the change of a reference time.

Remark 1.5 – Let $a(t, t_♯)$ and $a(t, t♭)$ be the scale parameters associated with two different reference times $t_♯$ and $t♭$. By applying the composition rule (1.12) we get

\[ a(t, t_♯) = a(t, t♭) a(t♭, t_♯) \]

This shows that the scale factors associated with two different reference dates differ by a constant factor (depending on the reference dates).

Remark 1.6 – The metric tensor $g(t) = a^2(t, t_♯) g(t_♯)$ does not depend, by definition, on the choice of the reference time $t_♯$, so that

\[ a^2(t, t_♯) g(t_♯) = a^2(t, t♭) g(t♭) \]

For $t = t_♯$ we get the relation

\[ g(t_♯) = a^2(t_♯, t♭) g(t♭) \]

which is in agreement with (1.14).

Remark 1.7 – We can write the factorization equation (1.14) in terms of the coordinate $q^0 = \kappa t$ by setting

\[ a(t) = A(q^0) \]

\[ g_{ab}(q^0, \bar{q}) = A^2(q^0) \bar{g}_{ab}(\bar{q}) \]

As a rule, we will use a small letter to denote a scalar function of $t$ and the corresponding capital letter to denote the same scalar as a function of $q^0$: $f(t) = F(q^0)$.

Remark 1.8 – The quotient manifold is endowed with the Levi-Civita connection coming from the quotient metric, henceforth denoted by $\tilde{\Gamma}$, whose Christoffel symbols are

\[ \tilde{\Gamma}^c_{ab} \overset{\text{def}}{=} \frac{1}{2} \bar{g}^{cd} (\partial_a \bar{g}_{bd} + \partial_b \bar{g}_{da} - \partial_d \bar{g}_{ab}) \]

where $\bar{g}^{ab}$ are the contravariant components of the quotient metric. It is easy to check that these symbols are invariant under the change of the reference date. It follows that the $\Gamma$-geodesics, as unparametrized curves, are not affected by the change of the reference time.
Remark 1.9 – Since the co-moving coordinates \( q^a \) are \( L \)-dimensional (Section 1.3) the metric tensor components \( \tilde{g}_{ab} \) are dimensionless:

\[
\begin{align*}
\text{Dim} (\tilde{g}^{ab}) & = 1, \\
\text{Dim} (\tilde{\Gamma}^c_{ab}) & = L^{-1}.
\end{align*}
\]

Remark 1.10 – On the quotient manifold the \textbf{arc-element} \( ds \) of any curve \( q^a = \gamma^a(\xi) \) (with generic parameter \( \xi \)) is defined by

\[
(1.23) \quad ds \overset{\text{def}}{=} \sqrt{\tilde{g}_{ab}} dq^a dq^b = \sqrt{\tilde{g}_{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi}} d\xi.
\]

In turn, the \textbf{arc-length} \( s \) is defined by

\[
(1.24) \quad s(\xi_1) - s(\xi_0) \overset{\text{def}}{=} \int_{\xi_0}^{\xi_1} ds = \int_{\xi_0}^{\xi_1} \sqrt{\tilde{g}_{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi}} d\xi.
\]

It is a privileged parameter for any curve on \( Q \), since

\[
(1.25) \quad \sqrt{\tilde{g}_{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi}} = 1
\]

This follows from

\[
\frac{ds}{d\xi} = \sqrt{\frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi}}
\]

and

\[
\tilde{g}_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} = \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi} \left( \frac{d\xi}{ds} \right)^2 = 1.
\]

However, the arc-element and the arc-length are not invariant under the change of the reference time \( t^\# \). If we denote by \( ds^\# \) and \( ds^* \) the arc-elements defined in the quotient metrics \( g(t^\#) \) and \( g(t^\flat) \), then equation (1.19) is equivalent to

\[
(1.26) \quad ds^\# = \frac{1}{a(t^\#, t^\flat)} ds^*.
\]

1.5 The Hubble law

Any curve in the quotient manifold, of length \( \ell^\# \), is carried along the galactic world-lines and generates curves of length \( \ell(t) \) on each spatial section \( S_t \). By virtue of (1.15), this length is given by

\[
(1.27) \quad \ell(t) = a(t, t^\#) \ell^\#.
\]

It follows that

\[
(1.28) \quad \dot{\ell}(t) = \dot{a}(t, t^\#) \ell^\#.
\]
Theorem 1.6 – The derivative \( \dot{\ell}(t) \) obeys the rule

\[
(1.29) \quad \dot{\ell}(t) = h(t) \ell(t)
\]

where

\[
(1.30) \quad h(t) \overset{\text{def}}{=} \frac{\dot{a}(t, t_0)}{a(t, t_0)}
\]

does not depend on the reference time \( t_0 \).

Proof – From (1.28) and (1.27) we get

\[
(1.31) \quad \dot{\ell}(t) = \frac{\dot{a}(t, t_0)}{a(t, t_0)} \ell(t).
\]

The scale factors relative to different normalization times differ by a multiplicative constant. So, the ratio (1.30) remains unchanged.

---

A dot over a symbol will denote the derivative with respect to \( t \).

---

Figure 1.3: Synchronous and co-moving distance.
The above argument can be applied to the case of geodesic curves, where the ‘lengths’ are ‘distances’. In particular, we can consider two distances between two galaxies $A$ and $B$:

(i) The **co-moving distance** $\tilde{\ell}_{AB}$: this is the length of the geodesic joining $A$ to $B$ in the quotient metric $\tilde{g}$. This distance is a constant depending on $A$ and $B$ only (Fig. 1.3).

(ii) The **synchronous distance** $\ell_{AB}(t)$ at the time $t$: this is the length of the shortest geodesic joining $A$ to $B$ in the spatial section $S_t$ with metric $g(t)$. The derivative $\dot{\ell}_{AB}(t)$ with respect to $t$ of the synchronous distance $\ell_{AB}(t)$ is called the **recession velocity** of the galaxies $A$ and $B$.

By applying equations (1.27) and (1.29) to these distances we find the two relationships

\begin{align}
\ell_{AB}(t) &= a(t) \tilde{\ell}_{AB} \\
\dot{\ell}_{AB}(t) &= h(t) \ell_{AB}(t)
\end{align}

In the second of these we recognize the celebrated **Hubble law**, and in $h(t)$ the **Hubble parameter**.

**Remark 1.11** – The Hubble parameter $h(t)$ is not affected by the choice of the reference time $t_\sharp$. In the present theory the Hubble law is a matter of pure Kinematics: it is a consequence of the postulates so far stated for the structure of the cosmic space-time. In other words, *it does not depend on whatsoever dynamical setting.*

### 1.6 Isotropic vectors and tensors

**Definition 1.1** – A tensor on space-time is **isotropic** if it meets the isotropy principle: it does not generates distinguished vector fields tangent to the spatial sections.

Since the isotropy principle is one of the main postulates, only isotropic tensors are **admissible** in the present cosmological theory. The **isotropic scalar** have been considered in Theorem 1.3. Now we characterize the isotropic vectors and two-tensors.

**Theorem 1.7** – A vector field $V^\alpha$ is isotropic if and only if its components in a co-moving coordinate system $(q^0, q^a)$ are of the following type:

\[
\begin{align*}
V^0 &= \text{function of } q^0 \text{ only}, \\
V^a &= 0.
\end{align*}
\]
1.6. Isotropic vectors and tensors

Proof – If \( V^a \) is also a function of the coordinates \( \tilde{q} \) then its gradient defines a distinguished direction on each spatial section. If \( V^a \neq 0 \) then a distinguished direction field is defined on each spatial section. ■

Theorem 1.8 – A contravariant symmetric two-tensor \( T^{\alpha\beta} \) is isotropic if and only if its components are of the following type:

\[
\begin{align*}
T^{00} &= \Phi(q^0) = \text{a function of } q^0 \text{ only,} \\
T^{0a} &= 0, \\
T^{ab} &= \Psi(q^0) \tilde{g}^{ab}(\tilde{q}) = \text{a function of } q^0 \text{ times } \tilde{g}^{ab},
\end{align*}
\]

where \( \tilde{g}^{ab} \) are the contravariant components of the quotient metric \( \tilde{g}_{ab} \). A similar result holds for an admissible covariant symmetric tensor:

\[
\begin{align*}
T_{00} &= \Phi(q^0) = \text{a function of } q^0 \text{ only,} \\
T_{0a} &= 0, \\
T_{ab} &= \Psi(q^0) \tilde{g}_{ab}(\tilde{q}) = \text{a function of } q^0 \text{ only times } \tilde{g}_{ab}.
\end{align*}
\]

Proof – \( T^{\alpha\beta} = J_0^\alpha J_\beta^\beta T^{\alpha'\beta'} \), \( J_\alpha^\alpha \equiv \frac{\partial q^{\alpha'}}{\partial q^\alpha} \), \( J_\alpha^\beta \equiv \frac{\partial q^\alpha}{\partial q^{\beta'}} \).

For a transformation of co-moving coordinates leaving \( q^0 \) invariant we have \( J_0^{\alpha'} = 1 \), \( J_0^\alpha = 0 \), \( J_0^\beta = 0 \), \( J_\alpha^0 = 0 \). Thus

\[
\begin{align*}
T^{00} &= J_0^\alpha J_\beta^\beta T^{\alpha'\beta'} = (J_0^0)^2 T^{0'0'} = T^{0'0'}. \\
T^{0a} &= J_0^\alpha J_\beta^b T^{\alpha'\beta'} = J_0^\alpha J_\beta^b T^{0'b'} = J_\beta^b T^{0'b'}. \\
T^{ab} &= J_\alpha^a J_\beta^b T^{\alpha'\beta'} = J_\alpha^a J_\beta^b T^{a'b'}.
\end{align*}
\]

This shows that: \( T^{00} \) is a scalar, so it must be a function of \( q^0 \) only; \( T^{0a} \) is a vector, so it must vanish; \( T^{ab} \) is a symmetric tensor on each spatial section generating eigenfields, so it must be proportional to the spatial metric \( g_{ab}(q^0, \tilde{q}) = A^2(q^0) \tilde{g}_{ab}(\tilde{q}) \). ■

Remark 1.12 – According to this theorem any admissible symmetric two-tensor \( T^{\alpha\beta} \) is fully determined by two functions \( \Phi \) and \( \Psi \) of \( q^0 \) only. •

Theorem 1.9 – No skew-symmetric isotropic two-tensor is admissible in the cosmic space-time.

Proof – A skew-symmetric two-tensor \( A^{\alpha\beta} \) gives rise to a spatial vector field \( A^{\alpha a} \) and to a spatial skew-symmetric tensor field \( A^{ab} \). The isotropy principle

\[\text{In other words: only symmetric two-tensors are admissible in the isotropic models of the universe.}\]
implies $A^{\alpha} = 0$. Any antisymmetric tensor field $A^{ab}$ on a three-dimensional Riemannian space has a real eigenvector. This is in contrast with the isotropy principle. Thus $A^{ab} = 0$. ■

**Remark 1.13** – Theorem 1.9 shows that the existence in space-time of a single or a finite number of electro-magnetic fields is incompatible with the isotropic cosmology. However, a continuous distribution of electro-magnetic fields such as those emitted from galaxies, may not generate any particular direction. Such fields are then admissible. ●

**Remark 1.14** – Since the torsion $\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}$ of a linear connection is a skew-symmetric tensor in the lower indices, only symmetric connections are allowed in space-time. ●

### 1.7 The cosmic radius and the angular distance

For cosmological models with non-flat spatial sections – $K(t) \neq 0$ – we can define the **cosmic radius** $R(t) > 0$ through the equation

$$K(t) = \frac{\varepsilon}{R^2(t)}, \quad \varepsilon = \pm 1$$

with $\varepsilon = \pm 1$ according to the sign of the curvature. If we set

$$\tilde{K} \overset{\text{def}}{=} K(t)$$

$$\tilde{R} \overset{\text{def}}{=} R(t)$$

then we have\(^\dagger\)

$$\varepsilon \tilde{K} = \frac{1}{\tilde{R}^2}$$

and, due to the general formula \(1.27\), we have

$$R(t) = a(t, t) \tilde{R}$$

$$K(t) = \frac{\tilde{K}}{a^2(t, t)}, \quad \forall \ t$$

By putting in particular $t = t_s$ in these equations we get

$$R(t_s) = a(t_s, t) \tilde{R}$$

$$K(t_s) = \frac{\tilde{K}}{a^2(t_s, t)}$$

\(^\dagger\) Note that the product $\varepsilon \tilde{K}$ is always positive.
Furthermore, from $h = \dot{a}/a$ we get

\begin{equation}
(1.43) \quad h(t) = \frac{\dot{R}(t)}{R(t)} = \frac{d \log R(t)}{dt}
\end{equation}

This formula gives the Hubble parameter in terms of the cosmic radius. We call $\dot{R}$ the \textbf{radial velocity} of the universe. Finally, from (1.32) and (1.39) it follows that for any pair of galaxies

\begin{equation}
(1.44) \quad \psi_{AB} \overset{\text{def}}{=} \frac{\ell_{AB}}{R} = \frac{\ell_{AB}(t)}{\dot{R}(t)} = \text{constant}
\end{equation}

We call \textbf{angular distance} of two galaxies the dimensionless constant $\psi_{AB}$. Its geometrical meaning is explained in the next section. As a consequence, the Hubble law can be written as

\begin{equation}
(1.45) \quad \dot{\ell}_{AB}(t) = \psi_{AB} \dot{R}(t)
\end{equation}

Note that the angular distance $\psi_{AB}$ does not depend on the choice of the reference date $t\#$.

\section*{1.8 Topological types of the universe}

The isotropic cosmological models that can be constructed on the basis of the postulates so far stated have two characteristic elements: (i) the scale parameter $a(t)$, a function of the cosmic time $t$ defined on a real (bounded or unbounded) interval $(t_\alpha, t_\omega)$ (beginning and end of the universe) which contains all the information about the evolution of the cosmos (expansion, contraction, etc.) and (ii) the sign of the constant curvature $\tilde{K}$ of the quotient manifold. However, it must be observed that this description is incomplete because it lacks a third characteristic element: the topology of the quotient manifold (that is the topology of the spatial sections).

There are several topological types of three-dimensional manifolds with constant curvature. The types that are commonly considered in cosmology are the following three ones:

(i) $Q \simeq S_3$, the three-dimensional sphere, positive curvature,

(ii) $Q \simeq H_3$, the three-dimensional pseudo-sphere, negative curvature,

(iii) $Q \simeq E_3$, the three-dimensional Euclidean space, zero curvature.

The corresponding models are called (i) \textbf{closed universe}, (ii) \textbf{open universe} and (iii) \textbf{flat universe}. In this section we take a look at these three...
types of universes, but we have to point out right now that the non-flat universes are incompatible with the barotropic dynamics studied in Chapter 5.

1.8.1 Closed universe

In a closed universe the spatial sections $S_t$ are three-dimensional sphere of radius $R(t)$ immersed in the Euclidean affine space $\mathbb{R}^4 = (w, x, y, z)$ and centered at the origin $O$ of the coordinates; they represented by the equation

$$w^2 + x^2 + y^2 + z^2 = R^2(t).$$

Their curvature is $K(t) = R^{-2}(t) > 0$. This geometrical vision differs from that of the space-time where the submanifolds $S_t$ form a foliation. Here they contract and expand in time according to the function $R(t)$. What we get is a sort of movie which we call **radial diagram** (Fig. 1.4). One of the spheres, corresponding to a reference date $t_4$, can be identified with the quotient manifold $Q$. Any galaxy is represented by a point moving along a straight line crossing the origin $O$. At any date $t$ two galaxies $A$ and $B$ stay on the sphere of radius
$R(t)$ and are separated by a circular arc of maximal radius (i.e. by a geodesic arc) of length $\ell_{AB}(t)$. Then the straight lines joining $A$ and $B$ to the center $O$ form, in agreement with (1.44), an angle $\psi_{AB}$ such that $\ell_{AB}(t) = \psi_{AB} \, R(t)$, which remains constant in time. The maximal distance between two galaxies is $\pi \, R$, the half of the length $2 \, \pi \, R$ of a maximal (geodesic) circle. Then the maximal angular distance is $\psi_{\text{max}} = \pi$.

### 1.8.2 Open universe

In this model the spatial sections $S_t$ are three-dimensional hyperboloid $\mathbb{H}_3$ of radius $R_t$

$$w^2 - x^2 - y^2 - z^2 = R^2(t)$$

immersed in the Minkowski affine space $M_4 = (w, x, y, z)$ with signature $(- + + +)$. The hyperboloid $\mathbb{H}_3$ is the set of points $P$ such that the vector $OP$ is time-like and with positive component with respect the time-like coordinate $w$. It results to be a space-like three-dimensional surface whose curvature at the date $t$ is $K(t) = -R^{-2}(t) < 0$ (Fig. 1.5). Remarks similar to those concerning the closed universe hold for the open universe. The only difference is that in the open universe the angular distance is not bounded.

![Radial diagram of the open universe](image)
1.8.3 Flat universe

In this model the concept of radius of the universe does not make sense. So we must refer to the scale parameter \( a(t) \) only. The spatial sections are the three-dimensional planes \( w = \text{constant} \) immersed in the Euclidean affine space \( \mathbb{R}^4 = (w, x, y, z) \).

1.9 Co-moving volumes and conserved densities

Let \( U \) be a domain in the quotient manifold \( Q \) with a finite volume
\[
\tilde{V}(U) = \int_U \sqrt{\det[g_{ab}]} \ dq^1 \wedge dq^2 \wedge dq^3.
\]
Carried along the world-lines, \( U \) generates domains \( U_t \subset S_t \) with finite volumes
\[
\mathcal{V}(U, t) = \int_{U_t} \sqrt{\det[g_{ab}(t, \tilde{q})]} \ dq^1 \wedge dq^2 \wedge dq^3.
\]

**Theorem 1.10** – The ratio \( \mathcal{V}(U, t)/a^3(t) \) does not depend on \( t \):

\[
(1.46) \quad \frac{\mathcal{V}(U, t)}{a^3(t)} = \tilde{V}(U) = \text{constant} > 0
\]

**Proof** – The factorization \( g_{ab}(t, \tilde{q}) = a^2(t) \tilde{g}_{ab}(\tilde{q}) \) implies
\[
\mathcal{V}(U, t) = \int_{U_t} \sqrt{\det[g_{ab}(t, \tilde{q})]} \ dq^1 \wedge dq^2 \wedge dq^3
\]
\[
= a^3(t) \int_{U_t} \sqrt{\det[\tilde{g}_{ab}(\tilde{q})]} \ dq^1 \wedge dq^2 \wedge dq^3 = a^3(t) \tilde{V}(U). \quad \blacksquare
\]

**Theorem 1.11** – For any scalar function of the cosmic time \( \mu(t) \) the following equations are equivalent,

\[
\mu(t) \mathcal{V}^u(U, t) = \text{const.} \quad \forall U \iff \mu(t) a^{3u}(t) = \text{const.}
\]

\[
(1.47) \quad a \dot{\mu} + 3 u \mu \dot{a} = 0 \iff h = -\frac{\dot{\mu}}{3u \mu}
\]

When these equations are satisfied we say that \( \mu(t) \) is a **conserved density of order** \( u \).

**Proof** – Write equation \( (1.46) \) as \( \mathcal{V}(U, t) = \tilde{V}(U) a^3(t) \). Then the condition \( \mu(t) \mathcal{V}^u(U, t) = \text{const.} \) is equivalent to \( \mu(t) \tilde{V}^u(U) a^{3u}(t) = \text{const.} \), hence to \( \mu(t) a^{3u}(t) = \text{const.} \). By differentiation we get the second equivalence. The last equivalence is due to the definition of the Hubble factor: \( h = \dot{a}/a \).

The typical case is the mass (or matter) density for which \( u = 1 \). But there are other density of physical interest for which \( u \neq 1 \) (see Chapter 5).
1.10 Cosmic monitor and free particles

The quotient manifold will play a crucial role in the sequel. In order to make such an abstract concept more accessible we can think of it as a cosmic monitor whose pixels, which are bright fixed points, represent the galaxies. Of course we need an effort of imagination because such a monitor is neither flat nor two-dimensional: it is a (possibly curved) three-dimensional screen. On the cosmic monitor there is also a clock showing the cosmic time $t$.

Imagine a very special person, the cosmic watcher, sitting in front of (or better, sitting inside) the monitor. Since the cosmic monitor is endowed with a metric (the quotient metric $\tilde{g}$) the cosmic watcher is able to recognize distinguished curves, called geodesics, connecting any pair of galaxies with a minimal (or stationary) distance. Then the cosmic watcher is able to measure the co-moving distance $\ell_{AB}$ of two galaxies (Fig. 1.6).

![Figure 1.6: The cosmic space-time and the cosmic monitor.](image)

**6th Postulate.** There are bodies, other than galaxies, running in the universe. We call them particles. The life of a particle is represented by a world-line in space-time transversal to the spatial sections.

The cosmic watcher cannot see the world-line $\gamma(t)$ of a particle. What he
can see on the monitor is only the projection \( \tilde{\gamma}(t) \) of \( \gamma(t) \). He looks at \( \tilde{\gamma}(t) \) as the **motion of a point**. Then he can measure the **monitor speed** \( v(t) \)

\[
(1.48) \quad v(t) = \frac{ds}{dt}
\]

where \( s \) is the arc-length along \( \tilde{\gamma}(t) \).

Among the various curves \( \tilde{\gamma}(t) \) that the cosmic watcher can see there are also geodesics. He argues that they represent distinguished particles whose world lines in space-time have some special features. But he cannot claim that these world-lines are geodesics because he does not know if in space-time there is any special metric or a connection. Anyway, he proposes the following formal definition:

**Definition 1.2**  – A **free particle** (or **free-falling particle**) is a particle whose monitor motion \( \tilde{\gamma}(t) \) is a geodesic. No matter if the cosmic time \( t \) is an affine parameter or not.

---

\[14\] In fact this distinction will be matter of a postulate (page 37).

Thus, we are encouraged to investigate on the existence of linear connections on space-time that are, in a sense, *adapted* to the geometric structures so far
1.11. Local reference frames and peculiar velocity

In order to locate an event in space-time we need to know where it occurs and when. To do this we need to assign a reference frame made of the congruence of world-lines of particles of an ideal ‘body’ and of a transversal foliation $S_t$ of simultaneous events parametrized by a ‘time’ $t$. Then we can say that a certain event occurs in a point of a certain ‘body’ and at a certain ‘date’ $t$.

In cosmology we have a privileged reference frame: the galactic world-
lines and the cosmic time $t$. Note that this is similar to what happens in Newtonian space-time (which is an affine space): there is an absolute time $t$, a foliation $\mathcal{S}_t$ made of three-dimensional affine subspaces, and a class of equivalence of reference frames, whose world-lines are parallel straight lines, called inertial frames. Instead, in Special Relativity i.e., in the Minkowski space-time (which is still an affine space) there is not a privileged time but a class of equivalence of reference frames, whose world-lines are parallel straight lines, still called inertial frames. Each one of them determines an orthogonal foliation of three-dimensional affine subspaces, hence a time $t$ which is ‘relative’ to the frame.

Consider now the intergalactic journey of a particle $P$ from a galaxy $A$ to a galaxy $C$. Let $t_d$ and $t_a$ be the dates of departure and arrival (see Fig. 1.7, page 22). At a date $t \in (t_d, t_a)$ the particle $P$ crosses a galaxy $B_t$. Then we are faced with the following data:

\[
\begin{align*}
\ell_{AB}(t) &= \text{the isochronous distance from } A \text{ to } B_t \text{ measured at the actual time } t. \\
\tilde{\ell}_{AB}(t) &= \text{the co-moving distance from } A \text{ to } B_t \text{ measured by the cosmic watcher.} \\
a(t) &= \text{the scale parameter, unknown to the cosmic watcher.}
\end{align*}
\]

They are linked by equation (1.32)

\[\ell_{AB}(t) = a(t) \tilde{\ell}_{AB}(t).\]

It follows that \[\dot{\ell}_{AB}(t) = \dot{a}(t) \tilde{\ell}_{AB}(t) + a(t) \dot{\tilde{\ell}}_{AB}(t) \implies (1.49) \dot{h}(t) \ell_{AB}(t) + a(t) \dot{\tilde{\ell}}_{AB}(t), \quad h(t) \overset{\text{def}}{=} \frac{\dot{a}(t)}{a(t)}.
\]

These formulas are quite similar to the composition law of velocities in classical mechanics. The galaxy $B_t$ can be interpreted as a local reference frame that moves with respect to the main (fixed) reference frame $A$. Then the first term $\dot{\ell}_{AB}$ is the absolute velocity of the particle $P$ (the velocity with respect to the main frame $A$). The second term $\dot{a} \tilde{\ell}_{AB} = h \ell_{AB}$ plays the role of dragging velocity. The third term $a \dot{\tilde{\ell}}_{AB}$ is the relative velocity, that is the velocity of the particle with respect to the moving frame $B_t$. Hence equation (1.49) can be read as

\[
\begin{align*}
\{ \text{absolute velocity of } P \text{ w. r. to the frame } A \} &= \text{dragging velocity of the galaxy } B_t \iff \dot{\ell}_{AB}. \\
+ \text{relative velocity of } P \text{ w. r. to the frame } B_t. \\
\end{align*}
\]

In cosmology these three velocities are called total velocity, recession velo-
ity and peculiar velocity, respectively:

\[
\begin{align*}
\text{total velocity } v_{\text{tot}}(P/A) &= v_{\text{rec}}(B_t/A) + v_{\text{pec}}(P/B_t) \\
\quad &= \ell_{AB_t} = \dot{a} \ell_{AB_t} + a \dot{\ell}_{AB_t}
\end{align*}
\]

Here the slash symbol / stands for with respect to. The total velocity pertains the galaxy A and the particle P when it crosses the galaxy B_t at the date \( t \). The recession velocity pertains the two galaxies A and B_t only. The peculiar velocity pertains the particle P and the galaxy B_t only. The cosmic watcher is able to measure \( \ell_{AB_t} = s(t) - s(t_d) \) and \( \dot{\ell}_{AB_t} = \dot{s}(t) \) only.

\[
v_{\text{rec}}(B_t/A) = h(t) \ell_{AB_t} = \dot{a}(t) \left[ s(t) - s(t_d) \right]
\]

\( \dot{a}(t) \) unknown to the cosmic watcher

\( s(t) \) known to the cosmic watcher

\[
v_{\text{pec}}(P/B_t) = a(t) \dot{\ell}_{AB_t} = a(t) \frac{ds}{dt}
\]

\( a(t) \) unknown to the cosmic watcher

\( \dot{s}(t) \) known to the cosmic watcher

This last formula provides the definition of peculiar velocity of a particle.
2.1 Preamble

A connection $\Gamma$ on a manifold $M$ is a mathematical device which determines a transport of vectors along curves. A connection is linear if the transport commute with linear combinations of vectors (with constant coefficients). In the domain of any given coordinate system a linear connection is determined by symbols or coefficients $\Gamma^\gamma_{\alpha\beta}$ which are functions of the coordinates. Then the transport equations of a vector $v^\alpha(\xi)$ along any curve $q^\alpha = \gamma^\alpha(\xi)$ are

$$\frac{d\gamma^\alpha}{d\xi} + \Gamma^\gamma_{\alpha\beta} v^\alpha \frac{d\gamma^\beta}{d\xi} = 0$$

By means of a linear connection we can also define the acceleration vector along any curve $q^\alpha = \gamma^\alpha(\xi)$:

$$a^\gamma(\xi) \overset{\text{def}}{=} \frac{d^2\gamma^\alpha}{d\xi^2} + \Gamma^\gamma_{\alpha\beta} \frac{d\gamma^\alpha}{d\xi} \frac{d\gamma^\beta}{d\xi}$$

When the acceleration is parallel to the velocity, namely when

$$a^\gamma(\xi) = \lambda(\xi) \frac{d\gamma^\gamma}{d\xi}$$

then the curve is said to be a geodesic of the connection $\Gamma$ or, briefly, a $\Gamma$-geodesic. Thus the geodesics are characterized by the geodesic equations

$$\frac{d^2\gamma^\gamma}{d\xi^2} + \Gamma^\gamma_{\alpha\beta} \frac{d\gamma^\alpha}{d\xi} \frac{d\gamma^\beta}{d\xi} = \lambda(\xi) \frac{d\gamma^\gamma}{d\xi}$$

1 Sometimes called parallel transport.
2.1. Preamble

The acceleration (2.2) of a curve (as well as the velocity) strictly depends on the choice of the parameter $\xi$ of the curve. As shown by the following theorem, it is always possible to determine a parameter $\xi$ for which the acceleration vanishes, so that the geodesic equations read

$$
\frac{d^2\gamma^\gamma}{d\xi^2} + \Gamma^\gamma_{\alpha\beta} \frac{d\gamma^\alpha}{d\xi} \frac{d\gamma^\beta}{d\xi} = 0
$$

Such a parameter is said to be an affine parameter, since it is determined up to an affine transformation (with constant coefficients). The comparison with the transport equations (2.1) shows that a $\Gamma$-geodesic is a curve whose velocity vector, with respect to an affine parameter, is invariant under the parallel transport.

**Theorem 2.1** – The transformations $\bar{\xi} = \bar{\xi}(\xi)$ satisfying the differential equation

$$
\frac{d\bar{\xi}}{d\xi} = \exp \int \lambda(\xi) \, d\xi.
$$

determine all the affine parameters preserving the orientation of the geodesic.

**Proof** – Under a change of parameter equations (2.4) transform to

$$
\frac{d\bar{\xi}}{d\xi} \frac{d\gamma^\gamma}{d\xi} \frac{d\bar{\xi}}{d\xi} + \Gamma^\gamma_{\alpha\beta} \frac{d\gamma^\alpha}{d\xi} \frac{d\gamma^\beta}{d\xi} \left( \frac{d\bar{\xi}}{d\xi} \right)^2 = \lambda(\xi) \frac{d\bar{\xi}}{d\xi} \frac{d\gamma^\gamma}{d\xi}
$$

$$
\Rightarrow \left( \frac{d\bar{\xi}}{d\xi} \right)^2 \frac{d\gamma^\gamma}{d\xi} + \frac{d\bar{\xi}}{d\xi} \frac{d\gamma^\gamma}{d\xi} = \lambda(\xi) \frac{d\bar{\xi}}{d\xi} \frac{d\gamma^\gamma}{d\xi}
$$

$$
\Rightarrow \left[ \frac{d\gamma^\gamma}{d\xi} + \Gamma^\gamma_{\alpha\beta} \frac{d\gamma^\alpha}{d\xi} \frac{d\gamma^\beta}{d\xi} \right] \left( \frac{d\bar{\xi}}{d\xi} \right)^2 = \left[ \lambda(\xi) \frac{d\bar{\xi}}{d\xi} - \frac{d\bar{\xi}}{d\xi} \frac{d\gamma^\gamma}{d\xi} \right] \frac{d\gamma^\gamma}{d\xi}.
$$

The parameter $\bar{\xi}$ is affine if and only if

$$
\lambda(\xi) \frac{d\bar{\xi}}{d\xi} - \frac{d\bar{\xi}}{d\xi} \frac{d\gamma^\gamma}{d\xi} \left( \frac{d\bar{\xi}}{d\xi} \right) = 0 \iff \lambda(\xi) = \frac{d}{d\xi} \left( \frac{d\bar{\xi}}{d\xi} \right) \iff \lambda(\xi) = \frac{d}{d\xi} \left( \frac{d\bar{\xi}}{d\xi} \right)
$$

$$
\iff \lambda(\xi) = \frac{d}{d\xi} \frac{d}{d\xi} \left( \frac{d\bar{\xi}}{d\xi} \right) \iff \lambda(\xi) = \frac{d}{d\xi} \left( \log \frac{d\bar{\xi}}{d\xi} \right)
$$

under the assumption that $d\bar{\xi}/d\xi > 0$. ■
2.2 Connections in the cosmic space-time

Due to the isotropy principle only symmetric connections, for which $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$, are admissible. In homogeneous co-moving coordinates $(q^a, q^0)$ we classify the symbols according to the number of the 0–indices:

\begin{equation}
\begin{cases}
\Gamma^a_{00}, \\
\Gamma^0_{a0}, \quad \Gamma^0_{0a} = \Gamma^0_{0a}, \\
\Gamma^c_{a0}, \quad \Gamma^0_{ab} = \Gamma^0_{ba}, \\
\Gamma^c_{ab} = \Gamma^0_{ba}.
\end{cases}
\end{equation}

Then the transport equations (2.1) and the geodesic equations (2.4) read, respectively,

\begin{equation}
\begin{aligned}
\frac{dv^0}{d\xi} + \Gamma^0_{a0} v^a \frac{d\gamma^0}{d\xi} + \Gamma^0_{00} v^0 \frac{d\gamma^0}{d\xi} + \Gamma^0_{ab} v^a \frac{d\gamma^b}{d\xi} = 0, \\
\frac{dv^c}{d\xi} + \Gamma^c_{a0} v^a \frac{d\gamma^0}{d\xi} + \Gamma^c_{00} v^0 \frac{d\gamma^0}{d\xi} + \Gamma^c_{ab} v^a \frac{d\gamma^b}{d\xi} = 0.
\end{aligned}
\end{equation}

Remark 2.1 – If a curve is transversal to the spatial sections, then the coordinate $q^a(t)$ can be taken as a parameter. So, $\gamma^0(q^0) = q^0$ and consequently $d\gamma^0/dq^0 = 1$. Then the transport equations (2.8) and the geodesic equations (2.9) reduce respectively to

\begin{equation}
\begin{aligned}
\frac{dv^0}{dq^0} + \Gamma^0_{a0} v^a + \Gamma^0_{00} v^0 \frac{d\gamma^a}{dq^0} + \Gamma^0_{ab} v^a \frac{d\gamma^b}{dq^0} = 0, \\
\frac{dv^c}{dq^0} + \Gamma^c_{a0} v^a + \Gamma^c_{00} v^0 \frac{d\gamma^a}{dq^0} + \Gamma^c_{ab} v^a \frac{d\gamma^b}{dq^0} = 0.
\end{aligned}
\end{equation}

2.3 The basic requirements

We look for a linear connection $\Gamma = (\Gamma^\gamma_{\alpha\beta})$ that meets the postulates of the cosmic kinematics or, in other words, that is adapted to the geometrical

\footnote{See Remark 1.14, page 16.}
2.3. The basic requirements

structures introduced on the space-time: the congruence of the galactic world-lines and the spatial foliation. We will translate these features into precise requirements.

**1\(^{st}\) Requirement.** The galactic world-lines are geodesics of \(\Gamma\) with affine parameter \(q^0\).

**2\(^{nd}\) Requirement.** The property of a vector to be tangent to the spatial foliation is preserved by the \(\Gamma\)-transport along the galactic world-lines.

Henceforth we will call spatial every vector tangent to a spatial section. In co-moving coordinates a spatial vector is characterized by the condition \(v^a = 0\).

**Theorem 2.2** – The 1\(^{st}\) requirement is satisfied if and only if \(\Gamma_{0a} = 0\) and \(\Gamma_{c0} = 0\).

**Proof** – A galactic world-line is parametrized by \(q^0\) and characterized by the condition \(q^a = \text{constant}\). In this case the geodesic equations (2.11) give

\[
\Gamma_{0a} = \lambda, \quad \Gamma_{c0} = 0.
\]

The parameter \(q^0\) is affine if and only if \(\lambda = 0\). ■

**Theorem 2.3** – The 2\(^{nd}\) requirement is satisfied if and only if (i) \(\Gamma_{0a} = 0\) and (ii) the transport of a spatial vector along a galactic world-line is represented by the equations

\[
\frac{dv^c}{dq^0} + \Gamma^c_{a0} v^a = 0.
\]

**Proof** – For a spatial vector \(v^a = 0\) and the transport equations (2.10) with \(\gamma^a = \text{constant}\) reduce to

\[
\left\{
\begin{array}{l}
\Gamma_{a0}^0 v^a = 0, \\
\frac{dv^c}{dq^0} + \Gamma_{a0}^c v^a = 0.
\end{array}
\right.
\]

By virtue of the 3\(^{rd}\) postulate each spatial section \(S_t\) is endowed with a positive-definite metric tensor \(g_{\ell}\). Hence, a further ‘natural’ requirement is the following.

**3\(^{rd}\) Requirement.** The norm of the spatial vectors is preserved by the \(\Gamma\)-transport along the galactic world-lines.
The norm of a spatial vector $v^a(\xi)$ along a curve $q^a(\xi)$ is defined by:

\[
(2.13) \quad ||v(\xi)|| \overset{\text{def}}{=} g_{ab}(q^a, \tilde{q}) v^a(\xi) v^b(\xi) = A^2(q^a) \tilde{g}_{ab}(\tilde{q}) v^a(\xi) v^b(\xi).
\]

**Theorem 2.4** — If the 2nd requirement is satisfied then the connection meets the 3rd requirement if and only if $\Gamma^b_{ab} = H \delta^b_a$, where $H(q^a)$ is the Hubble parameter.

**Proof** — A galactic world-line can be parametrized by $q^a$. From the definition (2.13) it follows that ($'=d/dq^a$)

\[
\left\{ \begin{align*}
\frac{d||v||}{dq^a} &= 2 AA' \tilde{g}_{ab} v^a v^b + 2 A^2 \tilde{g}_{ab} v^a \frac{dv^b}{dq^a} \\
\text{use the transport equations (2.12)} \frac{dv^b}{dq^a} &= -\Gamma^b_{co} v^c,
\end{align*} \right.
\]

\[
= 2 AA' \tilde{g}_{ab} v^a v^b - 2 A^2 \tilde{g}_{ab} v^a \Gamma^b_{co} v^c = 2 AA \tilde{g}_{ab} v^a (A' v^b - H^{-1} \Gamma^b_{co} v^c).
\]

The 3rd requirement is equivalent to

\[
\frac{d||v||}{dq^a} = 0 \iff \tilde{g}_{ab} v^a (v^b - H^{-1} \Gamma^b_{co} v^c) = 0.
\]

According to this last equations the spatial vectors $v^a$ and $v^a + x^a$, with $x^a \overset{\text{def}}{=} -H^{-1} \Gamma^b_{co} v^c$, must be orthogonal. This is absurd unless $v^a + x^a = 0$ i.e., $v^b = H^{-1} \Gamma^b_{co} v^c \iff \Gamma^b_{ab} = H \delta^b_a$.

---

3 Recall (2.20), $a(t) = A(q^a)$. 

---

[Figure 2.1: The parallel transport requirement.]
At this point, the table of the symbols of the connection $\Gamma$ is the following:

\begin{align}
\Gamma^0_{00} &= 0, \quad \Gamma^0_{a0} = 0, \quad \Gamma^c_{00} = 0 \\
\Gamma^b_{0a} &= H(q^a) \delta^b_a \\
\Gamma^0_{ab} &= F(q^a) \bar{g}_{ab}
\end{align}

(2.14)

The geodesic equations (2.9) in a generic parameter $\xi$ reduce to

\begin{align}
\frac{d^2 \gamma^a}{d\xi^2} + \Gamma^a_{ab} \frac{d\gamma^b}{d\xi} = \lambda \frac{d\gamma^a}{d\xi}, \\
\frac{d^2 \gamma^c}{d\xi^2} + \Gamma^c_{ab} \frac{d\gamma^a}{d\xi} = \lambda \frac{d\gamma^c}{d\xi} + 2H \frac{d\gamma^c}{d\xi} = \lambda \frac{d\gamma^c}{d\xi}.
\end{align}

(2.15)

For $\xi = q^a$,

\begin{align}
\Gamma^a_{ab} \frac{d\gamma^a}{dq^b} = \lambda \\
\frac{d}{dq^b} \frac{d\gamma^b}{dq^a} + \Gamma^c_{ab} \frac{d\gamma^c}{dq^a} = \lambda \frac{d\gamma^c}{dq^b} + 2H \frac{d\gamma^c}{dq^b} = \lambda \frac{d\gamma^c}{dq^b}.
\end{align}

(2.16)

Assuming that the above three requirements are met, now we ask the connection $\Gamma$ to meet the 4th postulate: the isotropy principle.

**4th Requirement.** The connection $\Gamma$ is isotropic in the sense that it does not give rise to distinguished vector fields tangent to the spatial sections.

**Theorem 2.5** – If the connection is isotropic then

\begin{align}
\Gamma^a_{ab} = F(q^a) \bar{g}_{ab}(\bar{q})
\end{align}

(2.17)

where $F(q^a)$ is a function of $q^a$ only.

**Proof** – Going back to the definition (2.3) of geodesic, $a^a = \lambda \frac{d\gamma^a}{d\xi}$, we observe that the multiplier $\lambda$ is a scalar since $a^a$ and $d\gamma^a/d\xi$ are vectors. Then the left hand side of the first equation (2.16) is a scalar. It follows that $\Gamma^a_{ab}(q^a, \bar{q})$ are the components of a spatial covariant symmetric tensor for each fixed $q^a$. It generates distinguished eigenvectors unless it is of the form (2.17) (see also Section 1.6 below). □

Now the table of the $\Gamma$-symbols is

\begin{align}
\Gamma^0_{00} &= 0, \quad \Gamma^0_{a0} = 0, \quad \Gamma^c_{00} = 0 \\
\Gamma^b_{0a} &= H(q^a) \delta^b_a, \quad \Gamma^0_{ab} = F(q^a) \bar{g}_{ab} \\
F(q^a) \text{ and } \Gamma^c_{ab} \text{ to be determined}
\end{align}

(2.18)
The \( \Gamma \)-geodesic equations (2.15) in a generic parameter \( \xi \) reduce to

\[
\begin{align*}
\frac{d^2 \gamma^a}{d\xi^2} + F \tilde{g}^{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi} &= \lambda \frac{d\gamma^a}{d\xi}, \\
\frac{d^2 \gamma^c}{d\xi^2} + \Gamma^c_{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi} + 2H \frac{d\gamma^c}{d\xi} &= \lambda \frac{d\gamma^c}{d\xi}.
\end{align*}
\]

and for \( \xi = q^0 \) to

\[
\begin{align*}
F \tilde{g}^{ab} \frac{dq^a}{dq^0} \frac{dq^b}{dq^0} &= \lambda, \\
\frac{d}{dq^0} \frac{d\gamma^c}{dq^0} + \Gamma^c_{ab} \frac{d\gamma^a}{dq^0} \frac{d\gamma^b}{dq^0} + 2H \frac{d\gamma^c}{dq^0} &= \lambda \frac{d\gamma^c}{dq^0}.
\end{align*}
\]

### 2.4 Projection of geodesics

Let us return to the end of Section 1.10 where the cosmic watcher noted the presence of geodesics on his monitor. Arguing that these \( \Gamma \)-geodesics may result from the projection of geodesics of a connection in space-time we are led to the following

**5th Requirement.** \( \Gamma \)-geodesics project onto \( \tilde{\Gamma} \)-geodesics.

This means that if \( \gamma(\xi) \) is a geodesic for the connection \( \Gamma \) then the projected curve \( \tilde{\gamma}(\xi) \) must be a geodesic for the Levi-Civita connection \( \tilde{\Gamma} \) of the quotient metric \( \tilde{g} \). Note that this requirement is already satisfied by the galactic world-lines. In fact they are \( \Gamma \)-geodesics which projects onto single points of the quotient manifold which may be interpreted as **singular geodesics**.

Impose this 5th requirement to those geodesics \( \gamma \) which can be parametrized by \( q^0 \) and by the arc-length \( s \) of their projected curves \( \tilde{\gamma} \) (see Fig. 2.2). We will call them **regular \( \Gamma \)-geodesics**. The arc-length \( s \) is a distinguished parameter for any curve on \( Q \) since

\[
\tilde{g}^{ab} \frac{dq^a}{ds} \frac{dq^b}{ds} = 1
\]

and is an affine parameter for all \( \tilde{\Gamma} \)-geodesics,

\[
q^0 = \gamma^0(s) \text{ is a } \tilde{\Gamma} \text{-geodesic } \iff \frac{d^2 \gamma^c}{ds^2} + \tilde{\Gamma}^c_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} = 0.
\]

---

\[A \text{ parametrized curve is interpreted as a motion. So, a point represents the motion of a point at rest.}\]
The reversible relationship between the parameters \( q^0 \) and \( s \) is represented by a function\(^5\)

\[
V(q^0) \overset{\text{def}}{=} \frac{ds}{dq^0} > 0
\]

**Theorem 2.6** – The 5\(^{th}\) requirement implies that the symbols \( \Gamma^c_{ab} \) coincide with the Christoffel symbols \( \Gamma^c_{ab} \) of the quotient metric

\[
\Gamma^c_{ab}(q^0, \bar{q}) = \Gamma^c_{ab}(\bar{q})
\]

and that the function \( V(q^0) > 0 \) (2.23) satisfies the equation

\[
\frac{d \log V}{dq^0} - F V^2 + 2H = 0
\]

\(^5\) Actually, the reversibility condition is \( \frac{dq^0}{ds} \neq 0 \). The condition \( \frac{dq^0}{ds} > 0 \) is not restrictive: it simply means that the two parameters \( q^0 \) and \( s \) are assumed to be with the same orientation.
Proof – Taking into account equation (2.21) the geodesic equations (2.20) are equivalent to

\[
\begin{aligned}
V^2 F &= \lambda, \\
V^2 \left( \frac{d}{ds} \frac{d\gamma^c}{ds} + \Gamma^c_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} \right) &= \left( V^3 F - V' - 2 H V \right) \frac{d\gamma^c}{ds}.
\end{aligned}
\]  

By replacing the expression of \( \lambda \) given by the first equation in the second set we get the characteristic equations of the regular \( \Gamma \)-geodesics in the parameter \( s \):

\[
\begin{aligned}
\frac{d}{ds} \frac{d\gamma^c}{ds} + \Gamma^c_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} &= V^{-1} \left( V^2 F - (\log V) - 2 H \right) \frac{d\gamma^c}{ds}.
\end{aligned}
\]

These equations involve only the parametric equations \( q^a = \gamma^a(q^0) \) and their first and second derivatives. Consequently, they are satisfied by the projected curve \( \gamma(s) \). But, in accordance with the 5th requirement, this projected curve must be a \( \Gamma \)-geodesic. As a consequence, by the comparison with the \( \Gamma \)-geodesic equations (2.22) we get

\[
\begin{aligned}
\left( \Gamma^c_{ab} - \tilde{\Gamma}^c_{ab} \right) \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} &= V^{-1} \left( V^2 F - (\log V) - 2 H \right) \frac{d\gamma^c}{ds}.
\end{aligned}
\]

These equations must hold for all geodesics. Since the left sides are quadratic in the \( s \)-velocities and the right sides are linear, both sides must vanish. ■

The following is in a sense the inverse of the previous theorem.

**Theorem 2.7** – Let \( \gamma \) be a regular curve on space-time, that is a curve which can be parametrized by \( q^0 \) as well as by the arc-length \( s \) of the projected curve \( \tilde{\gamma} \). Assume that (i) equations (2.24) and (2.25) are satisfied and that (ii) the projected curve \( \tilde{\gamma} \) is a \( \tilde{\Gamma} \)-geodesic. Then \( \gamma \) is a \( \Gamma \)-geodesic.

Proof – Under the assumptions (i) and (ii)

\[
\begin{aligned}
\Gamma^c_{ab} &= \tilde{\Gamma}^c_{ab} \\
\frac{d}{ds} \frac{d\gamma^c}{ds} + \tilde{\Gamma}^c_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} &= 0
\end{aligned}
\]

and equations (2.26) reduce to

\[
\begin{aligned}
V^2 F &= \lambda, \\
V^2 F - (\log V) - 2 H V &= 0
\end{aligned}
\]

The second equation is satisfied because of the assumption (i). The first equation gives the expression of \( \lambda \) and in this context is irrelevant. Hence, equations (2.20) are satisfied. On the other hand, as we have seen in the previous proof, equations (2.20) are equivalent to equations (2.20) which in turn are the \( \Gamma \)-geodesic equations in the parameter \( q^0 \). ■
2.5 Cosmic connections

The arguments of this chapter can be summarized in the following definition and theorem.

**Definition 2.1** – A **cosmic connection** is a linear symmetric connection on the cosmic space-time satisfying the five requirements listed above, which are compatible with the postulates of the cosmic kinematics stated in the first chapter.

**Theorem 2.8** – In any co-moving coordinate system the symbols of a cosmic connection are

\[
\begin{align*}
\Gamma^a_{00} &= 0, \quad \Gamma^a_{ao} = 0, \quad \Gamma^c_{00} = 0 \\
\Gamma^b_{ao} &= H(q^o) \delta^b_a, \quad \Gamma^a_{ab} = F(q^o) \tilde{g}_{ab} \\
\Gamma^c_{ab} &= \tilde{\Gamma}^c_{ab}
\end{align*}
\]

where \(H(q^o)\) is the Hubble parameter, \(\tilde{\Gamma}^c_{ab}\) are the Christoffel symbols of the quotient metric \(\tilde{g}\), and \(F(q^o)\) is a function satisfying the equation

\[
\frac{d \log V}{dq^o} + 2H = FV^2
\]

\[
V(q^o) \overset{\text{def}}{=} \frac{ds(t_0)}{dq^o} = \frac{1}{a(t_0, t_\ast)} ds(t_\ast)
\]

along any regular \(\Gamma\)-geodesic.

Despite the complexity of the above discussion, the result (2.28) is very simple.

**Remark 2.2** – Having in mind Remark [1.4] we observe that the five requirements of a cosmic connection are expressed in a geometrical way which is manifestly invariant with respect to the choice of a reference date (take also into account Remark [1.8]). Nevertheless, we can observe this invariance directly from the expressions (2.28) of the symbols. (i) The symbols \(\Gamma^b_{ao} = H(q^o) \delta^b_a\) are invariant since the Hubble factor is invariant (Theorem [1.6]). (ii) The coefficients \(\Gamma^c_{ab} = \tilde{\Gamma}^c_{ab}\) are invariant (Remark [1.8]). (iii) About the undetermined function \(F\) entering the symbols \(\Gamma^a_{ab} = F(q^o) \tilde{g}_{ab}\), we observe that due to equation (1.26)

\[
ds(t_o) = \frac{1}{a(t_o, t_\ast)} ds(t_\ast)
\]

from the definition (2.30) it follows that

\[
V(t_o, q^o) = \frac{ds(t_o)}{dq^o} = \frac{1}{a(t_o, t_\ast)} \frac{ds(t_\ast)}{dq^o} = \frac{1}{a(t_o, t_\ast)} V(t_\ast, q^o),
\]
thus
\[ d \log V(t_0, q^0) = d \log V(t_*, q^0). \]
Hence, the left-hand side of equation (2.29) is invariant, so that \( F V^2 \) must be invariant:
\[ F(t_0, q^0) V^2(t_0, q^0) = F(t_*, q^0) V^2(t_*, q^0). \]
Due to (2.29), this equation is equivalent to
\[ (2.32) \]
\[ F(t_0, q^0) = a^2(t_0, t_*) F(t_*, q^0) \]
In turn, due to (1.19), this last equation is equivalent to the invariance of \( \Gamma^{\mu}_{\alpha \beta} = F(q^0) \tilde{g}_{\alpha \beta} \). 

**Remark 2.3** – The role of the undetermined function \( F(q^0) \) raises a subtle argument. In fact equation (2.29) involve not only the functions \( F \) and \( H \), which participate in the definition of the connection, but also the function \( V \) which instead, by definition, is linked to the structure of the regular geodesics. This is a paradox that may cast doubt on the correctness of this equation which in a sense is ‘hybrid’. This paradox will be clarified in the following.
The assignment of a function \( F(q^0) \) characterizing a cosmic connection must be the consequence of a postulate. In this regard we observe that a connection provides the basis of a dynamics, so that the postulate we are arguing will form a ‘bridge’ between the Cosmic Kinematics and the Cosmic Dynamics. We will consider two of these bridge-postulates that open the way to two different dynamics (Fig. 2.3).

2.6 The Newtonian cosmic connection

1st Bridge–postulate. The cosmic time \( t \) is an affine parameter for the world-lines of the free particles.

Remind that the world-line of a particle is (by definition) transversal to the spatial foliation (paragraph 2 of Section 1.10). Note that, due the first requirement of a cosmic connection (page 29), the cosmic time is an affine parameter of the galactic geodesics. Thus, this bridge-postulate strictly concerns with free-particles which are not galaxies. Consequently, in the following the parametric functions \( \gamma^a(q^0) \) are assumed to be not all constant.

**Theorem 2.9** – The parameter \( q^a = \kappa t \) is affine for all transversal geodesics of a cosmic connection if and only if \( F(q^0) = 0 \).

**Proof** – Write the first geodesic equation (2.19) for \( \xi = q^0 \) and put \( \lambda = 0 \):

\[
F \tilde{g}_{ab} \frac{d\gamma^a}{dq^0} \frac{d\gamma^b}{dq^0} = 0. \quad [*]
\]

\[
\tilde{g}_{ab} \frac{d\gamma^a}{dq^0} \frac{d\gamma^b}{dq^0} = 0 \iff \frac{d\gamma^a}{dq^0} = 0 \implies \text{absurd}
\]

because of the transversality assumption. Then \( [*] \implies F = 0 \). ■

We conclude that there is a unique cosmic connection meeting the above postulate. We call it **Newtonian cosmic connection** since a cosmic space-time equipped with this connection is a generalization of the Newtonian space-time of classical mechanics (Fig. 2.4).

In the Newtonian space-time:

1. The manifold \( M \) is an affine four-dimensional space.
2. The spatial sections are Euclidean three-dimensional affine spaces.
3. The galactic world-lines are parallel straight lines and represents the motion of the so-called fixed stars. The congruence of these lines is an **inertial reference frame**, as well as any other congruence of parallel lines transversal to the foliation \( S \).
4. The world-lines of the free-falling particles are transversal straight lines (law of inertia).

5. The cosmic time \( t \) is the \textbf{absolute time}.

6. The expansion factor is constant and equal to 1, and the Hubble parameter is \( h = 0 \); thus \( H = 0 \). Consequently if the coordinates \( \bar{q} \) are Cartesian, then all symbols \( \Gamma^\gamma_{\alpha \beta} \) vanishes. The cosmic connection is flat and coincides with the canonical connection of an affine space.

\[ (2.33) \quad a(t) \frac{ds}{dt} = c = \text{constant} \]

Since the peculiar velocity is in fact the velocity with respect to a local frame of reference, this postulate clearly falls within the relativistic vision. Thus, the connection we are going to define will be called \textbf{relativistic cosmic connection}.

\textbf{Theorem 2.10} – \textit{There is a unique cosmic connection compatible with the ex-}
istence of special particles. The function $F(q^o)$ is given by

\begin{equation}
F = \frac{\kappa^2}{c^2} A^2 H
\end{equation}

\[\text{Figure 2.5: Special particle.}\]

**Proof** – Due to (2.33) the monitor speed (1.48) of a special particle is

\begin{equation}
v(t) \overset{\text{def}}{=} \frac{ds}{dt} = \frac{c}{a(t)}
\end{equation}

Then in the parameter $q^o$ the monitor speed is expressed by the function

\begin{equation}
V(q^o) \overset{\text{def}}{=} \frac{ds}{dq^o} = \frac{c}{\kappa A(q^o)}.
\end{equation}

Note that this ‘speed’ coincides with the function $V(q^o)$ defined in (2.23) for which equation (2.24) holds,

\[\frac{d\log V}{dq^o} - FV^2 + 2H = 0.\]

Due to (2.36) this equation is equivalent to

\[\frac{d\log A^{-1}}{dq^o} + 2H - F \frac{c^2}{\kappa^2} A^{-2} = 0.\]
As $H = (\log A)'$, we have $H - F \frac{c^2}{\kappa^2} A^{-2} = 0$ and we find equation (2.34). ■

**Remark 2.4** – The definition (2.34) of $F$ shows that the cosmic connection depends explicitly on $c$. (the functions $A$ and $H$ do not depend on $c$). Then $c$ plays the role of **universal constant**.

However, as an alternative to the second bridge-postulate, one could consider a third one:

**3rd Bridge–postulate.** There exist two (or more) **special particles** whose peculiar velocities (1.52) are universal constants:

$$a(t) \frac{ds_1}{dt} = c_1, \quad a(t) \frac{ds_2}{dt} = c_2, \ldots$$

Then we will have two (or more) superposed cosmic connections and two (or more) light-particles, with which we can build up something that we could call **multi-relativistic cosmic dynamics** (science fiction?).

**Remark 2.5** – Remind that the constant $\kappa$ has been introduced at the beginning of our discussion for a dimensional consistency in the correlation between the cosmic time $t$ and the length-dimensional coordinate $q^0$: $q^0 = \kappa t$. Its numerical value has been left arbitrary, but fixed. This constant has been present throughout our discussion, and it is also present in the definition (2.34) of $F$. Then there is no loss of generality in considering

$$\kappa = c$$

It follows from (2.36) that for a special particle

$$\frac{ds}{dq^0} = V = A^{-1}$$

The symbols of the relativistic cosmic connection are

$$\begin{align*}
\Gamma_{00} &= 0 \\
\Gamma_{0c} &= 0 \\
\Gamma_{0a} &= 0
\end{align*}$$

$$\begin{align*}
\Gamma_{ab} &= H \delta_a^b \\
\Gamma_{ab} &= A^2 H \tilde{g}_{ab}(\tilde{q}) \\
\Gamma_{ab} &= \tilde{\Gamma}_{ab}
\end{align*}$$

**Theorem 2.11** – The world-line of a special particle is a geodesic of the relativistic cosmic connection.\(^6\)

---

\(^6\) In other words: a special particle is a free particle of the relativistic cosmic connection.
2.7. The relativistic cosmic connection

Proof – The world-line of a special particle is a curve transversal to the spatial sections. The characteristic equations \( 2.27 \) of the \( \Gamma \)-geodesics in the parameter \( s \) read

\[
\frac{d}{ds} d\gamma^c + \Gamma^c_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} = V^{-1} \left( V^2 F - (\log V)' - 2H \right) \frac{d\gamma^c}{ds}.
\]

Due to \( 2.38 \), \( V = A^{-1} \), the coefficient at the right hand side vanishes:

\[
V^2 F - (\log V)' - 2H = H + (\log A)' - 2H = 0.
\]

It follows that \( \frac{d^2\gamma^c}{ds^2} + \Gamma^c_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} = 0 \) and we can apply Theorem \( 2.7 \). 

If we accept the existence of 'special particles' then at the bifurcation point of Fig. 2.6, we turn right. On this way we are going towards a relativistic formulation of the isotropic cosmology. In fact at the first station we will find a surprise: as a consequence of our postulates the space-time admits in a canonical way a metric with Lorentzian signature.

![Figure 2.6: Towards the relativistic cosmology.](image-url)
2.8 The canonical cosmic metric

**Theorem 2.12** – The relativistic cosmic connection is the Levi-Civita connection of the Lorentzian metric

\[
(2.40) \quad g_{\alpha\beta} \begin{cases} 
 g_{00} = \alpha = \text{const.} \\
 g_{ao} = 0 \\
 g_{ab} = -\alpha A^2 \bar{g}_{ab}
\end{cases} 
\]

**Proof** – According to Theorem 1.8 the components of any metric have the form

\[
g_{\alpha\beta} = \begin{cases} 
 g_{00} = \alpha(q^0), \\
 g_{ao} = 0, \\
 g_{ab} = \beta(q^i) \bar{g}_{ab}
\end{cases} \iff g^{\alpha\beta} = \begin{cases} 
 g^{00} = \alpha^{-1}, \\
 g^{0a} = 0, \\
 g^{ab} = \beta^{-1} \bar{g}_{ab}.
\end{cases}
\]

Computation of the first-kind Christoffel \( \Gamma_{\alpha\beta,\gamma} = \frac{1}{2} \left( \partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta} \right) \):

\[
2 \Gamma_{00,\gamma} = \partial_{0}g_{0\gamma} + \partial_{\gamma}g_{00} - \partial_{\gamma}g_{00} = \begin{cases} 
 2 \Gamma_{00,a} = \partial_{a}g_{0\gamma} + \partial_{\gamma}g_{0a} - \partial_{a}g_{00} = \alpha'. \\
 2 \Gamma_{00,a} = \partial_{a}g_{0\gamma} + \partial_{\gamma}g_{0a} - \partial_{a}g_{00} = 0.
\end{cases}
\]

\[
2 \Gamma_{ab,\gamma} = \partial_{a}g_{b\gamma} + \partial_{b}g_{a\gamma} - \partial_{\gamma}g_{ab} = \begin{cases} 
 2 \Gamma_{ab,0} = \partial_{0}g_{b\gamma} + \partial_{\gamma}g_{0b} - \partial_{b}g_{00} = 0, \\
 2 \Gamma_{ab,0} = \partial_{0}g_{b\gamma} + \partial_{\gamma}g_{0b} - \partial_{b}g_{00} = \beta'.
\end{cases}
\]

Computation of the second-kind Christoffel \( \Gamma_{\alpha\beta}^\gamma = g^{\gamma\delta} \Gamma_{\alpha\beta,\delta} \):

\[
\Gamma_{00}^\gamma = g^{\gamma0} \Gamma_{00,0} = \Gamma_{00} = g^{0a} \Gamma_{00,a} = \frac{1}{2} \alpha^{-1} \alpha' = \frac{1}{2} (\log \alpha)',
\]

\[
\Gamma_{00}^c = g^{00} \Gamma_{00,0} = g^{cd} \Gamma_{00,d} = 0.
\]

\[
\Gamma_{aa}^0 = g^{0a} \Gamma_{aa,0} = g^{0a} \Gamma_{aa,a} = 0.
\]

\[
\Gamma_{aa}^c = g^{ac} \Gamma_{aa,a} = g^{cd} \Gamma_{aa,d} = \frac{1}{2} \beta^{-1} \bar{g}_{cd} \beta' \bar{g}_{dc}.
\]

\[
\Gamma_{ab}^\gamma = g^{\gamma\delta} \Gamma_{ab,\delta} = \begin{cases} 
 \Gamma_{ab} = g^{0\delta} \Gamma_{ab,0} = g^{0\delta} \Gamma_{ab,0} = -\frac{1}{2} \alpha^{-1} \beta' \bar{g}_{ab}, \\
 = g^{cd} \Gamma_{ab,d} = g^{cd} \Gamma_{ab,d} = \beta^{-1} \bar{g}_{cd} \beta \Gamma_{ab,d} = \bar{g}_{ab}.
\end{cases}
\]

Summary:

\[
\begin{align*}
\Gamma_{00}^0 &= \frac{1}{2} (\log \alpha)', \\
\Gamma_{00}^c &= \frac{1}{2} (\log \beta)' \delta_a^c, \\
\Gamma_{aa}^0 &= 0, \\
\Gamma_{ab} &= -\frac{1}{2} \alpha^{-1} \beta' \bar{g}_{ab}, \\
\Gamma_{ab}^c &= \bar{g}_{ab}.
\end{align*}
\]

\footnote{Note that the Newtonian connection, for which \( F = 0 \), cannot admit a cosmic metric.}
2.8. The canonical cosmic metric

These symbols coincide with those of the relativistic cosmic connection (2.39)

\[
\begin{align*}
\Gamma^0_{00} &= 0, \\
\Gamma^c_{00} &= 0, \\
\Gamma^0_{a0} &= 0, \\
\Gamma^c_{ab} &= \bar{\Gamma}^c_{ab},
\end{align*}
\]

\[
\begin{align*}
\Gamma^c_{a0} &= H \delta^c_a, \\
\Gamma^0_{ab} &= K \bar{g}_{ab}, \\
\Gamma^0_{c0} &= 0, \\
\Gamma^c_{ab} &= \bar{\Gamma}^c_{ab},
\end{align*}
\]

\[
H = A^{-1} A' = (\log A)',
\]

\[
K = A A' = A^2 H
\]

if and only if

\[
\begin{align*}
\alpha &= \text{constant}, \\
\frac{1}{2} (\log \beta)' &= (\log A)', \quad \iff \quad A^{-2} \beta = \text{constant} = \gamma, \\
-\frac{1}{2} \alpha^{-1} \beta' &= A A'.
\end{align*}
\]

\[
\begin{align*}
\alpha &= \text{constant}, \\
\beta &= \gamma A^2, \\
\beta' &= -2 \alpha A A'.
\end{align*}
\]

\[
\begin{align*}
\alpha &= \text{constant}, \\
\gamma &= -\alpha, \\
\beta &= -\alpha A^2.
\end{align*}
\]

Without loss of generality we can take \( \alpha = -1 \) and consider the metric

\[
(2.41) \quad g_{\alpha\beta} : \begin{cases} 
\begin{align*}
g_{00} &= -1 \\
g_{0a} &= 0 \\
g_{ab} &= A^2 (q^a) \bar{g}_{ab}(\bar{q})
\end{align*}
\end{cases}
\]

with signature \(-+++\) as the canonical cosmic metric. The contravariant components are

\[
(2.42) \quad g^{\alpha\beta} : \begin{cases} 
\begin{align*}
g^{00} &= -1 \\
g^{0a} &= 0 \\
g^{ab} &= A^{-2} (q^a) \bar{g}^{ab}(\bar{q})
\end{align*}
\end{cases}
\]

**Theorem 2.13** – The galactic world-lines are time-like geodesics of the canonical cosmic metric orthogonal to the spatial sections.

**Proof** – By virtue of the first requirement of a cosmic connection (p. 29) the galactic world-lines are geodesics of the relativistic connection, thus of the cosmic metric. Since in co-moving coordinates \( g_{0a} = 0 \), these world lines are orthogonal to the spatial sections, thus they are time-like. \( \blacksquare \)
2.9 Light particles

By the second bridge-postulate (Section 2.7) we have introduced the notion of special particle.

**Theorem 2.14** – The world-lines of the special particles are light-like geodesics of the canonical cosmic metric.

**Proof** – For any world-line we have

\[
g_{\alpha\beta} \frac{d\gamma^\alpha}{dq^\alpha} \frac{d\gamma^\alpha}{dq^\alpha} = - \left( \frac{d\gamma^0}{dq^0} \right)^2 + g_{ab} \frac{d\gamma^a}{dq^a} \frac{d\gamma^b}{dq^b} = -1 + A^2 g_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} \left( \frac{ds}{dq^0} \right)^2.
\]

Since

\[
g_{ab} \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} = 1
\]

we get

\[
g_{\alpha\beta} \frac{d\gamma^\alpha}{dq^\alpha} \frac{d\gamma^\alpha}{dq^\alpha} = A^2 \left( \frac{ds}{dq^0} \right)^2 - 1.
\]

For a photon equation (2.38) \[\frac{ds}{dq^0} = A^{-1}\] holds. Then equation (2.43) gives

\[
g_{\alpha\beta} \frac{d\gamma^\alpha}{dq^0} \frac{d\gamma^\alpha}{dq^0} = A^2 A^{-2} - 1 = 0.
\]

This proves that the world-line of a special particle is a light-like curve. We already know that the world-line of a special particle is a geodesic (Theorem 2.11, page 40).

Special particles are very strange particles: they are never at rest and have the same peculiar velocity in whatever local reference frame. This is in accordance with the theory of propagation of the electro-magnetic waves. Then by virtue of Theorem 2.14 the concept of special particle can be identified with that of electro-magnetic signal or that of photon in a broadest sense i.e., as a particle associated with light (visible or non-visible).

2.10 Sub-luminal particles

**Definition 2.2** – A sub-luminal particle is a particle (2nd postulate of Kinematics, page 21) whose world-line \(\gamma^\alpha(t)\) is a time-like curve admitting a parameter \(\tau\), called proper time, such that

\[
g_{\alpha\beta} \frac{d\gamma^\alpha}{d\tau} \frac{d\gamma^\beta}{d\tau} = -c^2 \frac{d\tau}{dt} > 0
\]

\[\text{The proper time is oriented towards the future, as the cosmic time } t.\]
Theorem 2.15 – Sub-luminal particles have peculiar velocity less than $c$ and along their world-lines $d\tau < dt$.

Proof –

\[
\text{(2.43)} \iff g_{\alpha\beta} \frac{d\gamma^\alpha}{d\tau} \frac{d\gamma^\alpha}{dq^\alpha} (\frac{d\tau}{dq^\alpha})^2 = A^2 \left( \frac{ds}{dq^0} \right)^2 - 1.
\]

\[
\text{(2.44)} \implies -c^2 \left( \frac{d\tau}{dq^0} \right)^2 = A^2 \left( \frac{ds}{dq^0} \right)^2 - 1 \implies 1 - \left( \frac{d\tau}{dt} \right)^2 = c^{-2} a^2 \left( \frac{ds}{dt} \right)^2.
\]

\[
\text{(1.52): } v_{\text{pec}} \overset{\text{def}}{=} a(t) \frac{ds}{dt} \implies 1 - \left( \frac{d\tau}{dt} \right)^2 = c^{-2} v_{\text{pec}}^2 \implies \begin{cases}
\text{if } v_{\text{pec}} \neq 0: & 1 - \left( \frac{d\tau}{dt} \right)^2 > 0 \text{ i.e. } \left( \frac{d\tau}{dt} \right)^2 < 1.
\text{if } v_{\text{pec}} = 0: & 1 - \left( \frac{d\tau}{dt} \right)^2 = 0 \text{ i.e. } \frac{d\tau}{dt} = 1.
\end{cases}
\]

Remark 2.6 – Galaxies have zero peculiar velocity. Thus galaxies are sub-luminal particles with $\tau = t$. ●

2.11 Geodesics of the cosmic metric

For later use we summarize here some fundamental equations concerning the geodesics of the relativistic cosmic metric. The symbols of the relativistic cosmic connection are given in (2.39). The non-identically-vanishing symbols are

\[
\begin{align*}
\Gamma^b_{a0} &= \frac{A'}{A} \delta^b_a = H \delta^b_a, \\
\Gamma^0_{ab} &= A A' \tilde{g}_{ab} = A^2 H \tilde{g}_{ab} \\
\Gamma^c_{ab} &= \tilde{\Gamma}^c_{ab}
\end{align*}
\]

As a consequence, the geodesic equations (2.44) for a curve $\gamma^0(\xi)$ with a generic parameter $\xi$ read

\[
\begin{align*}
\frac{d^2\gamma^0}{d\xi^2} + A^2 H \tilde{g}_{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi} &= \lambda \frac{d\gamma^0}{d\xi}, \\
\frac{d^2\gamma^c}{d\xi^2} + \tilde{\Gamma}^c_{ab} \frac{d\gamma^a}{d\xi} \frac{d\gamma^b}{d\xi} + 2 H \frac{d\gamma^c}{d\xi} \frac{d\gamma^0}{d\xi} &= \lambda \frac{d\gamma^c}{d\xi}.
\end{align*}
\]

\textsuperscript{9} $d\tau < dt$: the proper time is runs slower than the cosmic time: twins paradox.
In the parameter $q^0$
\[
\gamma^0 = q^0 \implies \begin{cases} 
A^2 H \tilde{g}_{ab} \frac{d^a \gamma}{dq^a} \frac{d^b \gamma}{dq^b} = \lambda, \\
\frac{d^a \gamma^c}{dq^a} + \Gamma^c_{ab} \frac{d^a \gamma}{dq^a} \frac{d^b \gamma}{dq^b} = (\lambda - 2 H) \frac{d^c \gamma}{dq^c}.
\end{cases}
\]
In the parameter $t$: 
\[
\gamma^0 = ct \implies \begin{cases} 
A^2 H \tilde{g}_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} = \lambda c, \\
\frac{d^a \gamma^c}{dt^2} + \Gamma^c_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} = (\lambda - 2 c H) \frac{d^c \gamma}{dt}.
\end{cases}
\]
Since $A' = \frac{da}{dt} = \frac{\dot{a}}{c}$, so that $H = A'/A = \frac{\dot{a}}{c a} = \frac{h}{c}$, these last equations become
\[
\implies \begin{cases} 
a^2 h \tilde{g}_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} = \lambda c^2, \\
\frac{d^2 \gamma^c}{dt^2} + \Gamma^c_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} = (\lambda - 2 h) \frac{d^c \gamma}{dt}.
\end{cases}
\]
\[
\implies \begin{cases} 
\gamma^0 = ct, \\
\frac{d^2 \gamma^c}{dt^2} + \Gamma^c_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} = \frac{h}{c^2} \left( a^2 \tilde{g}_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} - 2 c^2 \right) \frac{d^c \gamma}{dt}.
\end{cases}
\]
These are the general geodesic equations in the cosmic time parameter $t$. Observe that for any world-line (transversal to the spatial sections)
\[
g_{\alpha\beta} \frac{d\gamma^\alpha}{dt} \frac{d\gamma^\alpha}{dt} = - \left( \frac{d\gamma^0}{dt} \right)^2 + a^2 \tilde{g}_{ab} \frac{d\gamma^a}{dt} \frac{d\gamma^b}{dt} = - c^2 + a^2 \tilde{g}_{ab} \frac{d\gamma^a}{dt} \frac{d\gamma^b}{dt}.
\]
For the world-line of a photon 
\[
g_{\alpha\beta} \frac{d\gamma^\alpha}{dt} \frac{d\gamma^\alpha}{dt} = 0 \implies a^2 \tilde{g}_{ab} \frac{d\gamma^a}{dt} \frac{d\gamma^b}{dt} = c^2,
\]
and equations (2.48) reduce to
\[
\begin{cases} 
\gamma^0 = ct, \\
\frac{d^2 \gamma^c}{dt^2} + \Gamma^c_{ab} \frac{d^a \gamma}{dt} \frac{d^b \gamma}{dt} = -h \frac{d^c \gamma}{dt}.
\end{cases}
\]
These are the light-like geodesic equations in the parameter $t$. Note that $t$ is not an affine parameter (because $h \neq 0$).
2.12 Comments on the Weyl principle

The standard texts of cosmology refer to the postulate of Weyl and the cosmological principle as basic statements on which to build up models of the evolution of the universe. They are essentially formulated as follows:

**Weyl’s principle**: In cosmic space-time the world-lines of the galaxies form a bundle of non-intersecting time-like geodesics orthogonal to a series of space-like hyper-surfaces.

**The cosmological principle**: On large scales the universe is spatially homogeneous and spatially isotropic.

1st comment: As mentioned in the Introduction, the Weyl postulate put cosmology in the framework of general relativity from the very beginning. In our approach the second part of this postulate is a theorem (Theorem 2.13) while the first part is contained in our third postulate. One might argue why not accept from the very beginning the Weyl postulate instead to spend so a long time starting from several postulates. The answer is that in this longer way we do not lose the knowledge of important facts which are not strictly pertinent to the theory of general relativity. For instance, the Hubble law as well as several other results are the subject of theorems valid regardless of any dynamical assumptions on the evolution of the galactic fluid.

2nd comment about the cosmological principle: in our approach isotropy implies homogeneity (Theorem 1.3). In fact this follows at once from the fourth postulate concerning the existence of a metric on each spatial section.

At the end of these first two chapters it is worth emphasizing that so far, as well as in the following discussion, we did not use any special coordinate system like, for instance, one of those which are commonly used on manifolds with constant curvature.

---

10 See e.g. [12].
11 Concerning, for instance, the scale parameter and the quotient metric (Section 1.4), the cosmic connections (this chapter) and the symmetric tensors (Chapter 3).
Chapter 3

Fundamental symmetric tensors

3.1 Symmetric tensors and conservation equations

Theorem 3.1 – Let \( \nabla_\alpha \) be the covariant derivative with respect to a general cosmic connection \( \Gamma \) \(^{(2.28)}\) and \( T^{\alpha\beta} \) the components \(^{(1.34)}\) of an isotropic symmetric tensor. The four conservation equations \( \nabla_\alpha T^{\alpha\beta} = 0 \) are equivalent to the single equation

\[
\Phi' + 3 (H \Phi + F \Psi) = 0
\]

Proof – It is sufficient to prove that

\[
\nabla_\alpha T^{\alpha\beta} = \Phi' + 3 (H \Phi + F \Psi)
\]

\[
\nabla_\alpha T^{\alpha0} = 0
\]

\[
\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^\gamma_{\alpha\gamma} T^{\gamma\beta} + \Gamma^\gamma_{\beta\gamma} T^{\alpha\gamma} \implies
\]

\[
\nabla_\alpha T^{\alpha0} = \partial_\alpha T^{\alpha0} + \Gamma^\gamma_{\alpha\gamma} T^{\gamma0} + \Gamma^\gamma_{\beta\gamma} T^{\alpha\gamma} = \partial_\alpha T^{\alpha0} + \Gamma^\alpha_{\alpha0} T^{\gamma0} + \Gamma^\alpha_{\beta0} T^{\alpha\gamma} + \Gamma^\alpha_{\gamma0} T^{\alpha\gamma} = \Phi' + (\Gamma^\alpha_{\alpha0} + \Gamma^\gamma_{\beta0}) \Phi + \Gamma^\alpha_{\beta0} \Psi \tilde{g}^{ab} = \nabla_\alpha T^{\alpha0} = \partial_\alpha T^{\alpha0} + \Gamma^\alpha_{\alpha0} T^{\gamma0} + \Gamma^\alpha_{\beta0} T^{\alpha\gamma} + \Gamma^\alpha_{\gamma0} T^{\alpha\gamma}
\]

\[
\nabla_\alpha T^{ab} = \partial_\alpha T^{ab} + \Gamma^\alpha_{\alphaa} T^{ab} + \Gamma^\alpha_{\betab} T^{ab} + \Gamma^\alpha_{\gammag} T^{ab} = \partial_\alpha T^{ab} + \Gamma^\alpha_{\alphaa} T^{ab} + \Gamma^\alpha_{\betab} T^{ab} + \Gamma^\alpha_{\gammag} T^{ab} + \Gamma^\alpha_{\gammag} T^{ao} + \Gamma^\alpha_{\gammag} T^{ao}
\]

\[
\Psi (\partial_\alpha \tilde{g}^{ab} + \Gamma^\alpha_{\alphaa} \tilde{g}^{ab} + \Gamma^\alpha_{\betab} \tilde{g}^{ab} + \Gamma^\alpha_{\gammag} \tilde{g}^{ab}) + \Gamma^\alpha_{\gammag} \Phi.
\]

Use \( (2.28) \),

\[
\left\{
\begin{array}{l}
\nabla_\alpha T^{\alpha0} = \Phi' + 3 (H \Phi + F \Psi),
\n\nabla_\alpha T^{\alpha0} = \Psi (\partial_\alpha \tilde{g}^{ab} + \tilde{g}^{ab} + \tilde{g}^{ab}) = \Psi \tilde{g}^{ab} = 0.
\end{array}
\right.
\]

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3.2 Ricci tensor of a cosmic connection

Theorem 3.2 – The components of the Ricci tensor of a general cosmic connection \((\mathbb{T}_{\mathbb{C}})\) are

\[
\begin{align*}
R_{00} &= -3(H' + H^2) \\
R_{0a} &= 0 \\
R_{ab} &= (F' + HF + 2\bar{K})\bar{g}_{ab}
\end{align*}
\]

where \(\bar{K}\) is the curvature constant of the quotient metric \(\bar{g}\).

Proof – The Ricci tensor components are defined by (see (1.2))

\[
R_{\alpha\beta} = \partial_{\mu}\Gamma^\mu_{\alpha\beta} - \partial_{\beta}\Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\sigma\mu} \Gamma^\sigma_{\alpha\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\alpha\mu}.
\]

1. Computation of \(R_{00}\):

\[
R_{00} = \partial_{0}\Gamma^{0}_{00} - \partial_{0}\Gamma^{0}_{0\alpha} + \Gamma^{0}_{\sigma\mu} \Gamma^{\sigma}_{00} - \Gamma^{0}_{\sigma\beta} \Gamma^{\sigma}_{0\mu} = -\partial_{0}\Gamma^{0}_{00} - \Gamma^{0}_{0\sigma} \Gamma^{\sigma}_{00} = -\partial_{0}\Gamma^{0}_{00} - \Gamma^{0}_{0\sigma} \Gamma^{\sigma}_{00} = -3(H' + H^2).
\]

2. Computation of \(R_{ab}\):

\[
R_{ab} = \partial_{a}\Gamma^{c}_{bc} + \partial_{b}\Gamma^{c}_{ac} + \Gamma^{c}_{b\beta} \Gamma^{\beta}_{ac} - \Gamma^{c}_{b\gamma} \Gamma^{\gamma}_{ac} = \partial_{a}\Gamma^{c}_{bc} + \partial_{b}\Gamma^{c}_{ac} + \Gamma^{c}_{b\beta} \Gamma^{\beta}_{ac} - \Gamma^{c}_{b\gamma} \Gamma^{\gamma}_{ac}.
\]

Reordering:

\[
R_{ab} = \partial_{b}\Gamma^{c}_{ac} + \partial_{a}\Gamma^{c}_{bc} + \Gamma^{c}_{b\beta} \Gamma^{\beta}_{ac} - \Gamma^{c}_{b\gamma} \Gamma^{\gamma}_{ac} = \partial_{b}\Gamma^{c}_{ac} + \partial_{a}\Gamma^{c}_{bc} + \Gamma^{c}_{b\beta} \Gamma^{\beta}_{ac} - \Gamma^{c}_{b\gamma} \Gamma^{\gamma}_{ac}.
\]

The first four terms give the Ricci-tensor components \(\bar{R}_{ab}\) of \(\bar{g}\)

\[
\bar{R}_{ab} = F' \bar{g}_{ab} + 3HF\bar{g}_{ab} = H(\partial_{a}\bar{g}_{bc} + \partial_{b}\bar{g}_{ac} + \partial_{c}\bar{g}_{ab} - \partial_{a}\bar{g}_{bc} - \partial_{b}\bar{g}_{ac} - \partial_{c}\bar{g}_{ab})
\]

\[
\bar{R}_{ab} = F' \bar{g}_{ab} + 3HF\bar{g}_{ab} = H(\partial_{a}\bar{g}_{bc} + \partial_{b}\bar{g}_{ac} + \partial_{c}\bar{g}_{ab} - \partial_{a}\bar{g}_{bc} - \partial_{b}\bar{g}_{ac} - \partial_{c}\bar{g}_{ab})
\]

\[
\bar{R}_{ab} = F' \bar{g}_{ab} + 3HF\bar{g}_{ab} = H(\partial_{a}\bar{g}_{bc} + \partial_{b}\bar{g}_{ac} + \partial_{c}\bar{g}_{ab} - \partial_{a}\bar{g}_{bc} - \partial_{b}\bar{g}_{ac} - \partial_{c}\bar{g}_{ab})
\]

Theorem 3.3 – The covariant components of the Ricci tensors of the cosmic connections take the form

\[
\begin{align*}
\text{Newtonian:} & \quad \begin{cases} 
R_{00} = -3A^{-1}A'' \\
R_{0a} = 2\bar{K}\bar{g}_{ab} 
\end{cases} \\
\text{Relativistic:} & \quad \begin{cases} 
R_{00} = -3A^{-1}A'' \\
R_{ab} = (2A^2 + AA' + 2\bar{K})\bar{g}_{ab}
\end{cases}
\end{align*}
\]

Proof – (i) For \(\Gamma = 0\) (Newtonian connection) equations (3.3) reduce to

\[
\begin{align*}
R_{00} &= -3(H' + H^2) \\
R_{0a} &= 2\bar{K}\bar{g}_{ab}.
\end{align*}
\]
Since $H = (\log A)' = A^{-1} A'$, we find
\[
H^2 + H' = A^{-2} (A')^2 - A^{-2} (A')^2 + A^{-1} A'' = A^{-1} A'' ,
\]
and (3.4) are proved. (ii) For $F = A^2 H$ (relativistic connection)
\[
F + H F = (A^2 H)' + A^2 H^2 = 2 A A' H + A^2 H' + A^2 H^2
\]
(use again $H = A^{-1} A'$)
\[
= 2 (A')^2 + A^2 [A^{-1} A'' - A^{-2} (A')^2] + (A')^2 = 2 (A')^2 + A A'' .
\]
Enter this result in equations (3.3) and (3.5) are proved. ■

3.3 Einstein tensor of the relativistic cosmic connection

For the relativistic cosmic connection we can compute the mixed and the contravariant components of the Ricci tensor by raising the indices of the components (3.5) by means of the contravariant components (2.42) of the metric. We find that
\[
\begin{align*}
R^0_0 &= 3 A^{-1} A'' \\
R^a_b &= A^{-2} \left( 2 (A')^2 + A A'' + 2 \tilde{K} \right) \delta^a_b
\end{align*}
\]
(3.6)
\[
\begin{align*}
R^{00} &= -3 A^{-1} A'' \\
R^{ab} &= A^{-4} \left( 2 (A')^2 + A A'' + 2 \tilde{K} \right) \tilde{g}^{ab}
\end{align*}
\]
(3.7)
From (3.6) we derive the Ricci scalar curvature
\[
R \overset{\text{def}}{=} R^\alpha_\alpha = 3 \left[ A^{-1} A'' + A^{-2} \left( 2 (A')^2 + A A'' + 2 \tilde{K} \right) \right] \implies \\
R = 6 A^{-2} \left( A'^2 + A A'' + \tilde{K} \right)
\]
(3.8)
As a consequence we can compute the contravariant components of the Einstein tensor $G^{\alpha\beta} \overset{\text{def}}{=} R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}$:
\[
\begin{align*}
G^{00} &= R^{00} - \frac{1}{2} R g^{00} = -3 A^{-1} A'' + 3 A^{-2} \left( A'^2 + A A'' + \tilde{K} \right) \\
&= 3 A^{-2} \left( A'^2 + \tilde{K} \right).
\end{align*}
\]
\(^1\)As we will see, we are more interested in these components, rather than in the covariant or mixed ones.
3.4 The intervention of the cosmic time

So far we have used the coordinate \( q^0 \) as a parameter. However, in the perspective of applying in a dynamical context the formulas so far found, we should rewrite them using as a parameter the cosmic time \( t \). For this purpose we observe that

\[
\begin{align*}
q^0 &= c t \\
A(q^0) &= a(t) \\
A'(t) &= \frac{\dot{a}}{c}, \\
A''(t) &= \frac{\ddot{a}}{c^2}.
\end{align*}
\]

The Ricci curvature (3.8) and the contravariant components (3.9) of the Einstein tensor become

\[
\begin{align*}
R_{\text{Ricci}} &= 6 c^{-2} a^{-2} \left( \dot{a}^2 + a \ddot{a} + c^2 \bar{K} \right) \\
G^{00} &= 3 c^2 \dot{a}^2 \left( \dot{a}^2 + c^2 \bar{K} \right) \\
G^{ab} &= -\frac{1}{c^2} a^2 \left( 2 a \ddot{a} + \dot{a}^2 + c^2 \bar{K} \right) \bar{g}^{ab}.
\end{align*}
\]

The contravariant components (3.13) of an isotropic symmetric two-tensor take the form

\[
\begin{align*}
T^{00} &= \phi(t) = \text{a function of } t \text{ only}, \\
T^{0a} &= 0, \\
T^{ab} &= \psi(t) \bar{g}^{ab}(\bar{q}) = \text{a function of } t \text{ times } \bar{g}^{ab};
\end{align*}
\]

We call \( \phi(t) \) and \( \psi(t) \) the \textbf{characteristic functions} of the symmetric tensor \( T^{\alpha \beta} \).
Theorem 3.4 – For an isotropic symmetric tensor \((3.13)\) the conservation equations \(\nabla_\alpha T^{\alpha\beta} = 0\) are equivalent to the single equation

\[
(3.14) \quad a \dot{\phi} + 3 \dot{a} (\phi + a^2 \psi) = 0
\]

In turn, this equation is equivalent to

\[
(3.15) \quad (\phi a^3)' + 3 a^4 \dot{a} \psi = 0
\]

Proof – Apply the rules \((3.10)\) to equation \((3.1)\):

\[
\Phi' + 3 (H \Phi + F \Psi) = 0 \iff \frac{\dot{\phi}}{c} + 3 \left( \frac{\dot{a}}{ca} \phi + \frac{a \dot{a}}{c} \psi \right) = 0 \iff (3.14). \quad \blacksquare
\]
Chapter 4

Relativistic cosmic dynamics

4.1 The principles of the relativistic cosmic dynamics

In the previous chapters we have constructed the geometrical background we need for passing to the foundation of the cosmology. We have seen that the evolution of the universe can be described by a single function of the cosmic time, the scale parameter $a(t)$. Our goal is now to state physical laws governing the evolution of $a(t)$.

1st Postulate. We found the cosmic dynamics on the principles of the cosmic kinematics (first chapter) and on the existence of photons (2nd bridge-postulate).

The cosmic space-time is then equipped with the cosmic metric (2.41)

\[
\begin{align*}
g_{00} & = -1 \\
g_{a0} & = 0 \\
g_{ab} & = a^2(t) \tilde{g}_{ab}(\tilde{q})
\end{align*}
\]

The contravariant components are

\[
\begin{align*}
g^{00} & = -1 \\
g^{a0} & = 0 \\
g^{ab} & = a^{-2}(t) \tilde{g}^{ab}(\tilde{q})
\end{align*}
\]
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2nd Postulate. The evolution of the scale parameter $a(t)$ is governed by the Einstein field equations

\[
R^\alpha{}_{\beta} + \left(\Lambda - \frac{1}{2} R\right) g^\alpha{}_{\beta} = \chi T^\alpha{}_{\beta}
\]

equivalent to

\[
G^\alpha{}_{\beta} = \chi T^\alpha{}_{\beta} - \Lambda g^\alpha{}_{\beta}
\]

where $T^\alpha{}_{\beta}$ is the energy tensor, $\Lambda \geq 0$ is the cosmological constant and the constant $\chi$ is given by

\[
\chi = \frac{8\pi G_N}{c^4},
\]

where $G_N$ is the Newtonian gravitational constant.

Dimensional analysis of the Einstein equations:

| Object       | Dim            | Note |
|--------------|----------------|------|
| $\Lambda$    | $L^{-2}$       | [a]  |
| $\chi T^\alpha{}_{\beta}$ | $L^{-2}$       | [a]  |
| $T^\alpha{}_{\beta}$ | $M L^{-1} T^{-2}$ (energy density) | [b]  |
| $\chi$      | $M^{-1} L^{-1} T^2$ | [c]  |
| $G_N$        | $M L^3 T^{-2}$  | [d]  |

[a] According to our conventions (Section 1.3) the coordinates $q^\alpha$ are length-dimensional, thus the metric tensor components $g_{\alpha\beta}, g^\alpha{}_{\beta}$ are dimensionless and $\text{Dim} (R^\alpha{}_{\beta}) = \text{Dim} (R) = L^{-2}$. Then from the Einstein equations (4.3) it follows that: $\text{Dim} (\Lambda) = \text{Dim} (\chi T^\alpha{}_{\beta}) = L^{-2}$.

[b] Equations (4.12) below shows that $\text{Dim} (T^{00}) = \text{Dim} (e)$ and $\text{Dim} (T^{ab}) = \text{Dim} (p)$. From the entries of Table 1.3 (page 7) $\text{Dim} (e) = \text{Dim} (p) = M L^{-1} T^{-2}$.

[c] $\text{Dim} (\chi) = \text{Dim} (\chi T^\alpha{}_{\beta})/\text{Dim} (T^\alpha{}_{\beta}) = L^{-2}/(M L^{-1} T^{-2})$.

[d] $\text{Dim} (\chi) = L^{-4} T^4 \cdot \text{Dim} (G_N) \implies \text{Dim} (G_N) = \text{Dim} (\chi) L^4 T^{-4}$

$= M^{-1} L^{-1} T^2 L^4 T^{-4} = M^{-1} L^3 T^{-2}$. 
4.2 The energy-momentum tensor

According the above postulates, the core of a cosmological model is the choice of an energy-momentum tensor $T^{\alpha \beta}$ for the galactic fluid. As for any isotropic symmetric tensor, it must have the form (4.5):

\[(4.5) \quad T^{\alpha \beta} = \phi(t) \partial_t \otimes \partial_t + \psi(t) g^{ab} \partial_a \otimes \partial_b \]

The two characteristic functions $\phi(t)$ and $\psi(t)$ must satisfy the conservation equation

\[(4.6) \quad a \dot{\phi} + 3 a \dot{a} (\phi + a^2 \psi) = 0 \]

which is equivalent to the four conservation equations $\nabla_{\alpha} T^{\alpha \beta} = 0$ (Theorem 4.4). An equivalent form of this equation is

\[(4.7) \quad (\phi a^3) = -3 a^4 \dot{a} \psi \]

Nothing can be changed in definition (4.5), in equation (4.6) and in equation (4.7) without violating our postulates. The only degrees of freedom we will have in the following will be the choice of the energy-momentum tensor (Section 4.3), which reduces to the choice of two characteristic functions (of time) and of an equation of state binding these functions (Section 5.1).

**Theorem 4.1** – Let $T^{\alpha \beta}$ be an isotropic energy-momentum tensor with characteristic functions $\phi(t)$ and $\psi(t)$. Then:

(i) The ten Einstein equations are equivalent to the two differential equations

\[(4.8) \quad \begin{cases} \frac{\ddot{a}}{a} = \frac{1}{3} a^2 (\Lambda + \chi \phi) - \bar{K} \\ 2 \frac{\ddot{\chi}}{a^2} = a \left[ \frac{3}{2} \Lambda - \chi (\psi a^2 + \frac{1}{3} \psi) \right] \end{cases} \]

(ii) Due to the conservation law (4.6) the second equation (4.8) is a differential consequence of the first one.

**Proof** – (i) The contravariant components of the metric tensor and of the Einstein tensor are given in (3.12). Then the Einstein field equations

\[\begin{align*} G^{00} &= \chi T^{00} - \Lambda g^{00}, \\ G^{ab} &= \chi T^{ab} - \Lambda g^{ab}. \end{align*} \]
are equivalent to
\[
\begin{align*}
\frac{3}{c^2 a^2} (\dot{a}^2 + c^2 \tilde{K}) &= \chi \phi + \Lambda \\
-\frac{1}{c^2 a^4} (2 a \ddot{a} + \dot{a}^2 + c^2 \tilde{K}) \tilde{g}^{ab} &= \chi \psi \tilde{g}^{ab} - \Lambda a^{-2} \tilde{g}^{ab} \\
\dot{a}^2 &= \frac{1}{3} c^2 a^2 \left( \chi \phi + \Lambda \right) - c^2 \tilde{K} \\
\left( 2 a \ddot{a} + \dot{a}^2 + c^2 \tilde{K} \right) \tilde{g}^{ab} &= c^2 a^4 \left( \Lambda a^{-2} - \chi \psi \right) \tilde{g}^{ab} \\
\dot{a}^2 &= c^2 \left[ \frac{1}{3} a^2 \left( \Lambda + \chi \phi \right) - \tilde{K} \right] \\
2 a \ddot{a} + \dot{a}^2 + c^2 \tilde{K} &= c^2 a^2 \left( \Lambda - \chi \psi a^2 \right)
\end{align*}
\]

Substitute the first equation into the second one:
\[
\begin{align*}
\dot{a}^2 &= c^2 \left[ \frac{1}{3} a^2 \left( \Lambda + \chi \phi \right) - \tilde{K} \right] \\
2 a \ddot{a} + \dot{a}^2 + c^2 \tilde{K} &= c^2 a^2 \left( \Lambda - \chi \psi a^2 \right)
\end{align*}
\]

(ii) Differentiate the first equation (4.8):
\[
(4.9) \quad \frac{2 \dot{a} \ddot{a}}{c^2} = \frac{2}{3} \left( \chi \phi + \Lambda \right) a \dot{a} + \frac{1}{3} \chi a^2 \dot{\phi}.
\]

Substitute into this equation the conservation law (4.10) in the form \( a \dot{\phi} = -3 \dot{a} \left( \phi + a^2 \psi \right) \). Then
\[
\begin{align*}
\text{[4.10]} \quad \frac{2 \dot{a} \ddot{a}}{c^2} &= \frac{2}{3} \left( \chi \phi + \Lambda \right) a \dot{a} - \chi \left( \phi + a^2 \psi \right) a \dot{a} \\
\Rightarrow \quad \frac{2 \ddot{a}}{c^2 a} &= \frac{2}{3} \left( \chi \phi + \Lambda \right) - \chi \left( \phi + a^2 \psi \right) \\
\Rightarrow \quad \frac{2 \ddot{a}}{c^2 a} &= \frac{4}{3} \Lambda - \chi \left( \frac{3}{4} \phi + a^2 \psi \right). \quad \leftrightarrow \quad \text{second equation [4.8]}. \quad \blacksquare
\end{align*}
\]

4.3 The energy-momentum tensor of the galactic fluid

Theorem 4.2 – Let \( V(t) \) be the volume of any arbitrary co-moving portion of the galactic fluid, \( \epsilon(t) \) and \( p(t) \) the energy density and the pressure in that
4.3. The energy-momentum tensor of the galactic fluid

portion. Then the energy conservation law

\[ \frac{d}{dt} (\epsilon V) = -p \frac{dV}{dt} \tag{4.10} \]

holds if and only if the energy-momentum tensor of the galactic fluid is that of a perfect fluid\(^1\)

\[ T^{\alpha\beta} = (e + p) U^\alpha U^\beta + pg^{\alpha\beta} \tag{4.11} \]

where \( U^\alpha \) is the unitary four-velocity of the galactic fluid

\[ U^\alpha \overset{\text{def}}{=} c^{-1} \frac{\gamma^\alpha}{dt}; \left\{ \begin{array}{l} U^0 = 1, \\ U^a = 0, \\ g_{\alpha\beta} U^\alpha U^\beta = -1. \end{array} \right. \]

Note that

\[ \left\{ \begin{array}{l} T^{00} = \epsilon + p - p = \epsilon, \\ T^{ab} = 0, \\ T^{ab} = pg^{ab}. \end{array} \right. \tag{4.12} \]

\textbf{Lemma 4.1} – The conservation law \((4.10)\) is equivalent to the equation

\[ a \dot{\epsilon} + 3 (\epsilon + p) \dot{a} = 0 \tag{4.13} \]

\textbf{Proof} – Due to \((4.10)\),

\[ \frac{V(t)}{a^3(t)} = \text{const.} = \frac{V(t_*)}{a^3(t_*)}. \]

\[ V = \frac{V(t_*)}{a^3(t_*)} a^3(t), \quad \dot{V} = 3 \frac{V(t_*)}{a^3(t_*)} a^2(t) \dot{a}(t) = 3 \frac{\dot{a}}{a} V. \]

\[ \leftrightarrow \dot{\epsilon} V + \epsilon \dot{V} = -p \dot{V} \leftrightarrow \dot{\epsilon} V + (\epsilon + p) \dot{V} = 0 \]

\[ \leftrightarrow \dot{\epsilon} + 3 (\epsilon + p) h = 0. \quad \blacksquare \]

\textbf{Proof of Theorem 4.2} – Equation \((4.13)\) is in perfect agreement with equation \((4.14)\)\( a \phi + 3 \dot{a} (\phi + a^2 \psi) = 0 \) with

\[ \phi = \epsilon(t), \quad \psi = a^{-2} p \tag{4.14} \]

Then from \((4.5)\) we get \((4.12)\). \(\blacksquare\)

\(^1\) See [16] p. 127, [10] p. 132, [6] p. 14, [7] p. 23. Due to the different conventions, there are changes of sign.
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Remark 4.1 – Since $U^\beta U_\beta = -1$, the four-velocity $U^\alpha$ is an eigenvector of $T^\alpha_\beta$ with eigenvalue $-\epsilon$:

$$T^\alpha_\beta U_\beta = -\epsilon U^\alpha.$$  
Moreover, any vector $X^\alpha$ orthogonal to $U^\alpha$ is an eigenvector of $T^\alpha_\beta$ with eigenvalue $p$:

$$T^\alpha_\beta X_\beta = p X^\alpha.$$  

Remark 4.2 – With the substitution (4.14) $\phi = \epsilon(t), \quad \psi = a^{-2}p$ the dynamical equations (4.8) read respectively

$$\frac{\dot{a}^2}{c^2} = \frac{1}{3} a^2 (\Lambda + \chi \epsilon) - \tilde{K}$$ (4.15)

$$\frac{\ddot{a}}{c^2} = \frac{1}{2} a \left[ \frac{2}{3} \Lambda - \chi (p + \frac{1}{3} \epsilon) \right]$$ (4.16)

Equation (4.16) is a differential consequence of (4.15) and of the conservation law (4.13) (Theorem 4.1, item (iii))2.

Remark 4.3 – The acceleration equation (4.16) highlights a salient feature of the relativistic cosmic dynamics: it can be interpreted as a Newtonian dynamical equation of a point subjected to three forces competing with each other, namely

$$\left\{ \begin{array}{l}
F_\Lambda(a) \overset{\text{def}}{=} \frac{1}{3} c^2 \Lambda a, \\
F_p(a) \overset{\text{def}}{=} -\frac{1}{2} c^2 \chi p a, \\
F_\epsilon(a) \overset{\text{def}}{=} -\frac{1}{6} c^2 \chi \epsilon a.
\end{array} \right.$$  
Since $\Lambda$ and $\chi$ are positive constants, $F_\Lambda$ acts as a centrifugal force (with center $a = 0$), as well as $F_p$ and $F_\epsilon$ if $p(t)$ and $\epsilon(t)$ are negative. On the contrary, $F_p$ and $F_\epsilon$ are attractive towards $a = 0$ when $p(t)$ and $\epsilon(t)$ are positive.

4.4 Comments on the Friedman equations

The equations of Friedman are definitely the most cited equations in the texts of cosmology, where they appear written in various different forms. Actually, by Friedman equations one should understand the dynamical equations appearing in the original work Über die Krümmung des Raumes by A. Friedman (1922) which are written exactly as follows:

$$\left\{ \begin{array}{l}
\frac{dR}{dx}^2 + 2 \frac{R R''}{R^2} + \frac{c^2}{R^2} - \lambda = 0 \\
3 \frac{R^2}{R''} + \frac{3 c^2}{R^2} - \lambda = \kappa c^2 \vartheta
\end{array} \right.$$ (4.17)

Equations (4.15) and (4.16) are called Friedman equations, see Section 4.4.

2 Equations (4.15) and (4.16) are called Friedman equations, see Section 4.3.
where $\rho$ is declared to be the density of mass and $\kappa$ eine Konstante. The coordinate $x_4$ is time-dimensional and the signature of the metric is $(- - - +)$. These equations come from the Einstein field equations

$$ R_{ik} - \frac{1}{2} g_{ik} \bar{R} + \lambda g_{ik} = -\kappa T_{ik}, \quad \begin{cases} \bar{R} = g^{ik} R_{ik} \\ T_{ik} = 0, \quad i, k \neq 4 \\ T_{44} = c^2 \rho g_{44} \end{cases} $$

for $i = k = 1, 2, 3$ and for $i = k = 4$, respectively. Looking at the energy tensor components we observe that (i) Friedman takes into account the cosmological constant, (ii) the kinetic pressure $p$ is not present, so Friedman deals with a dust galactic fluid. Furthermore, as proved below, (iii) Friedman deals with a positive spatial curvature.

In our theory we have seen that the Einstein equations determined by the energy-momentum tensor (4.11) reduce to the differential equations (4.15) and (4.16), namely

$$ \begin{align*}
\dot{a}^2 &= \frac{1}{3} a^2 (\Lambda + \chi e) - \ddot{K}, \\
\dot{a}^2 &= \frac{1}{2} a \left[ \frac{2}{3} \Lambda - \chi (p + \frac{1}{3} e) \right].
\end{align*} $$

Let us compare these equations with the Friedman equations (4.17). To do this we rewrite them as

$$ \begin{align*}
\{ 4 \} & \quad 2 R R'' + R'^2 + c^2 - \lambda R^2 = 0, \\
\{ 5 \} & \quad R'^2 + c^2 - \frac{1}{3} (\lambda + \kappa c^2 \rho) R^2 = 0,
\end{align*} $$

Subtract side by side $\{ 4 \} - \{ 5 \}$:

$$ 2 R R'' - \lambda R^2 + \frac{1}{3} (\lambda + \kappa c^2 \rho) R^2 = 0. $$

Since $R \neq 0$, we get $2 R'' - \frac{2}{3} \lambda R + \frac{1}{3} \kappa c^2 \rho R = 0$, i.e.

$$ R'' = \frac{1}{6} R \left( 2 \lambda - \kappa c^2 \rho \right). $$

If in (4.20) we put

$$ \begin{align*}
\frac{dx_4}{dt} &= \dot{a} \\
R &= a \\
R' &= \dot{a} \\
R'' &= \ddot{a}
\end{align*} $$

then we get the equation $\ddot{a} = \frac{1}{6} a \left( 2 \lambda - \kappa c^2 \rho \right)$ which coincides with the second equation (4.18) with $p = 0$,

$$ \frac{\ddot{a}}{a^2} = \frac{1}{6} a \left( 2 \Lambda - \chi e \right). $$

---

$^3$ The comparison with our Einstein equations (4.3) $R^\alpha{}_{\beta} + (\Lambda - \frac{1}{2} R) g^{\alpha\beta} = \chi T^\alpha{}_{\beta}$ shows a difference of sign in the right side. This is due to the different signature of the metric.
provided that $2\lambda - \kappa c^2\varrho = c^2(2\Lambda - \chi\epsilon)$ i.e.

(4.22) \hspace{1cm} \lambda = c^2\Lambda, \quad \kappa \varrho = \chi\epsilon.

In turn, due to the substitutions (4.21), the second equation [5] in (4.19) reads

$$\ddot{a}^2 = \frac{1}{3}(\lambda + \kappa c^2\varrho) a^2 - c^2.$$

Due to (4.22),

$$\ddot{a}^2 = \frac{1}{3} c^2 a^2 (\Lambda + \chi\epsilon) - c^2.$$

This equation coincides with our first equation (4.18)

$$\frac{\dot{a}^2}{c^2} = \frac{1}{3} a^2 (\Lambda + \chi\epsilon) - \tilde{K}$$

provided that $\tilde{K} = 1$. This proves item (iii) above.
Chapter 5

Barotropic dynamics of a single-component universe

5.1 Preamble

The dynamics of the scale factor $a(t)$ is so far governed by the two first-order differential equations (4.15) and (4.13). These two equations involve three unknown functions $a(t)$, $\epsilon(t)$ and $p(t)$. We need a further equation. This equation should express the physical characteristics of the galactic fluid and should be dictated by physical arguments. The galactic fluid may have different components (typically mass, radiation, etc.). Assuming that the energy densities $\epsilon_i$ of these components are additive, we can write the total energy density $\epsilon$ as the sum

$$\epsilon = \sum_i \epsilon_i.$$ 

Besides the densities we have consider the pressures $p_i$ of each component. Then we must assume that all of these variables are bound each other by means of certain equations of state. In the case in which there is no interaction between the components, these equations are separated between them, that is to say of the type

$$p_i = f_i(\epsilon_i).$$

In the simplest case, we can consider linear equations

$$p_i = w_i \epsilon_i$$

where $w_i$ are dimensionless constants called barotropic parameters. In this chapter we confine our analysis to a single-component universe with the equation of state

$$p = w \epsilon$$

(5.1)
As a consequence, the **barotropic dynamics** will involve two functions in the cosmic time $t$, $a(t) > 0$ and $\epsilon(t)$, together with a constant parameter $w$. This dynamics is governed by the differential equations

\[
\begin{align*}
\ddot{a} + 3(w + 1)\epsilon \dot{a} &= 0 \tag{5.2} \\
\frac{\dot{a}^2}{c^2} &= \frac{1}{3} a^2 (\Lambda + \chi \epsilon) - \tilde{K} \tag{5.3} \\
\frac{\ddot{a}}{c^2} &= \frac{1}{2} a \left[ \frac{2}{3} \Lambda - \chi (w + \frac{1}{3}) \epsilon \right] \tag{5.4}
\end{align*}
\]

which are respectively called **fluid equation**, **velocity equation** and **acceleration equation**. Equations (5.2) and (5.4) come from equations (4.13) and (4.16) with the substitution $p = w \epsilon$. Equation (5.3) is nothing but equation (4.15). The acceleration equation (5.4) is a differential consequence of the fluid and the velocity equations (Remark 4.2). The velocity equation (5.3) can be written in the form

\[
\dot{h}^2 = \frac{1}{3} c^2 (\Lambda + \chi \epsilon) - \frac{c^2 \tilde{K}}{a^2} \tag{5.5}
\]

Our purpose is to find and classify all possible solutions $a(t)$ of these dynamical equations. We call them **profiles of the universe**. Such a classification should depend on the value of the parameter $w$ and on the value of the spatial curvature $\tilde{K}$, specially on its sign. Moreover, two profiles differing by a translation along the $t$-axis have to be considered equivalent.

### 5.2 Basic theorems

In the analysis of a barotropic dynamics it turns out to be convenient to replace the parameter $w$ with the new parameter

\[
u = w + 1 \tag{5.6}
\]

and introduce the new constants

\[
\begin{align*}
\lambda &\defeq \frac{1}{3} c^2 \Lambda > 0 \\
\mu &\defeq \frac{1}{3} \chi c^2 \epsilon \geq 0
\end{align*}
\]

Table of conversion:

\[
\begin{array}{|c|c|c|c|c|}
\hline
u & w + 1 & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{3} \\
\hline
w & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} & -\frac{3}{3} & 0 & \frac{1}{3} \\
\hline
\end{array}
\]

The parameter $u$ is used by other authors, and it is denoted also by $\gamma$ or $\Gamma$. 
5.2. Basic theorems

$\epsilon_\sharp$ being the value of $\epsilon(t)$ at the normalization time $t_\sharp$.

**Theorem 5.1** – The evolution $a(t,t_\sharp)$ of the scale factor is governed by the single first-order differential equation

\[
\dot{a}^2 = \lambda a^2 + \mu_\sharp a^{2-3u} - c^2 \tilde{K}
\]

which can also be written as

\[
h^2 = \frac{\dot{a}^2}{a^2} = \frac{\lambda}{a^{3u}} - \frac{c^2}{a^2} \tilde{K}.
\]

**Proof** – \([5.2] \iff \frac{\dot{e}}{e} + 3u \frac{\dot{a}}{a} = 0 \iff d \log e + 3u d \log a = 0 \iff \]

\[
\epsilon(t) a^{3u}(t,t_\sharp) = \text{constant in } t
\]

As $a(t_\sharp,t_\sharp) = 1$, equation \([5.10]\) is equivalent to

\[
\epsilon(t) a^{3u}(t,t_\sharp) = \epsilon_\sharp
\]

Substituting the expression $\epsilon(t) = \epsilon_\sharp a^{-3u}(t,t_\sharp)$ coming from this last equation into equation \([5.5]\) we get \([5.8]\) and \([5.7]\).

**Remark 5.1** – Remind that $\tilde{K} = K(t_\sharp)$ and $\epsilon_\sharp = \epsilon(t_\sharp)$. The dynamical equation \([5.9]\) must be invariant under any change $t_\sharp \mapsto t_\flat$ of the normalization time of the scale parameter $a(t,t_\sharp)$, as explained in Remark 1.4, page 11. We know that the definition of the Hubble parameter $h$ is invariant. Since also the constant $\lambda$ is invariant, the invariant condition of \([5.9]\) reduces to equation

\[
\frac{1}{3} \chi \epsilon(t) \frac{a^{3u}(t_\sharp)}{a^2(t,t_\sharp)} - \frac{K(t_\sharp)}{a^2(t,t_\sharp)} = \frac{1}{3} \chi \frac{\epsilon(t_\bullet)}{a^{3u}(t_\bullet,t_\sharp)} - \frac{K(t_\bullet)}{a^2(t_\bullet,t_\sharp)}.
\]

to be satisfied for all $t, t_\sharp, t_\bullet$. For $t = t_\bullet$ we get

\[
\frac{1}{3} \chi \epsilon(t_\sharp) \frac{a^{3u}(t_\bullet,t_\sharp)}{a^2(t_\bullet,t_\sharp)} - \frac{K(t_\sharp)}{a^2(t_\bullet,t_\sharp)} = \frac{1}{3} \chi \epsilon(t_\bullet) - K(t_\bullet).
\]

Due to \([1.34]\), $K(t_\bullet) = \frac{K(t_\sharp)}{a^2(t_\bullet,t_\sharp)}$, so that this last equation reduces to

\[
\epsilon(t_\sharp) = \epsilon(t_\bullet) a^{3u}(t_\bullet,t_\sharp).
\]

This equation holds for all values of $(t_\bullet, t_\sharp)$, and by putting $t_\bullet = t$ we get equation \([5.11]\) that, as we have seen above, is a consequence of the fluid equation \([5.2]\).

This proves that the dynamical equation \([5.9]\) satisfies the required invariance condition. ■
Remark 5.2 – By virtue of Theorem (1.11) (page 20) the following equations are equivalent:

\[(5.13) \quad \epsilon(t) V^U(U, t) = \text{const.} \quad \forall \ U = \text{co-moving domain},\]

\[(5.14) \quad \epsilon(t) a^3u(t, t) = \text{constant in} \ t\]

\[(5.15) \quad a \dot{e} + 3 u \epsilon \dot{a} = 0 \quad \text{(fluid equation)},\]

\[(5.16) \quad h = -\frac{1}{3u} \frac{\dot{t}}{t}.\]

This shows that the energy density \(\epsilon(t)\) is a conserved density of order equal to the barotropic parameter \(u\).

Theorem 5.2 – If the spatial curvature is negative then any physical length has a permanent superluminal expansion or contraction.

Note that this theorem holds whatever \(u\).

Proof – Let us go back to Section 1.3, equation (1.28), \(\hat{\ell}(t) = \dot{a}(t, t) \hat{\ell}\), where \(\hat{\ell}\) is the length of a curve (not necessarily a geodesic) on the quotient manifold and \(\ell(t)\) is the corresponding length on the spatial section \(S_t\). Due to equation (5.8) we have

\[(5.17) \quad \dot{\ell}^2(t) = \dot{a}^2(t, t) \ell^2 = \left[\lambda a^2 + \mu \dot{a}^{-3(w+1)} - c^2 |\hat{K}| \right] \ell^2.\]

If \(\hat{K} < 0\) then \(\dot{\ell}^2 = \left[\lambda a^2 + \mu \dot{a}^{-3(w+1)} + c^2 |\hat{K}| \right] \ell^2\). This shows that \(\dot{\ell}^2 > c^2\).

Remark 5.3 – The superluminal condition \(\ell > c^2\) of any physical length during all the whole duration of the universe has no physical sense. Thus, our theory leads to consider inadmissible the barotropic models with negative spatial curvature. Later on (Section 5.7) it will be shown that also the models with positive spatial curvature are inadmissible.

5.3 The profiles of the barotropic flat models

In the vastness of the cosmological mathematical models, the flat barotropic models have the rare property that the evolution in time of the scale factor admits an analytical expression in terms of elementary functions (exponentials, or hyperbolic functions), whatever the value of the parameter \(u\).

We will denote by \(a(u; t, t)\) the scale factor of a barotropic universe with barotropic parameter \(u = w + 1\) and reference time \(t\).
5.3. The profiles of the barotropic flat models

**Theorem 5.3** – The profiles $a(u; t, t_\sharp)$ of a barotropic flat model admit the following two equivalent representations

\[
\begin{align*}
(a(u; t, t_\sharp) &= \left[ \frac{1}{4} \frac{\chi}{\Lambda} \varepsilon_2 (e^{\beta t} - 1)^2 \right] \frac{1}{3u} \quad \text{exponential form} \\
(a(u; t, t_\sharp) &= \left[ \frac{1}{2} \frac{\chi}{\Lambda} \varepsilon_2 (\cosh(u\beta t) - 1) \right] \frac{1}{3u} \quad \text{hyperbolic form}
\end{align*}
\]

where

\[
\beta \overset{\text{def}}{=} \sqrt{3} \Lambda c \quad \text{Dim} (\beta) = T^{-1}
\]

and $\varepsilon_2$ is the value of the energy density $\varepsilon(t)$ at the reference time $t_\sharp$.

**Proof** – With $K_\sharp = 0$ equation (5.9) reads

\[
\dot{a}^2 \frac{a^2}{a} = \lambda + \frac{\mu_\sharp}{a^{3u}}
\]

and is equivalent to

\[
\frac{da}{a \sqrt{(1 + b a^{-3u})}} = \sqrt{\lambda} dt, \quad b \overset{\text{def}}{=} \frac{\mu_\sharp}{\lambda}.
\]

The left-hand side is integrable in terms of elementary functions:

\[
\int \frac{da}{a \sqrt{(1 + b a^{-3u})}} = \frac{1}{3u} \log \frac{\sqrt{a^{3u} + b + \sqrt{a^{3u}}}}{\sqrt{a^{3u} + b - \sqrt{a^{3u}}}} + \text{constant}.
\]

Thus from (5.22) we get

\[
\log \frac{\sqrt{a^{3u} + b + \sqrt{a^{3u}}}}{\sqrt{a^{3u} + b - \sqrt{a^{3u}}}} = 3u \sqrt{\lambda} (t - t_*)
\]

with an arbitrary $t_*$. However, there is no loss of generality in assuming $t_* = 0$. In this case $a(0) = 0$\(^2\) By setting $\beta = 3 \sqrt{\lambda} = \sqrt{3} \Lambda c$ we can write

\[
\frac{\sqrt{a^{3u} + b + \sqrt{a^{3u}}}}{\sqrt{a^{3u} + b - \sqrt{a^{3u}}}} = e^{u \beta t}.
\]

In order to solve this equation with respect to $a^{3u}$ we put

\[
X \overset{\text{def}}{=} e^{u \beta t}
\]

\(^2\) The physical meaning of a scale factor is invariant under translations along the $t$-axis.
and
\[ A \overset{\text{def}}{=} \sqrt{a^{3u} + b}, \quad B \overset{\text{def}}{=} \sqrt{a^{3u}}. \]

Note that \( A^2 - B^2 = b \). Then we have the following sequence of implications:
\[
\frac{A + B}{A - B} = X \implies A + B = X (A - B) \implies b = X (A - B)^2 \]
\[
\implies b = X (A^2 + B^2 - 2AB) \implies b = X (2A^2 + b - 2AB)
\]
\[
\implies \frac{1}{2} \frac{X - 1}{X} b = B (A - B). \quad \text{Due to [\ref{eq:5.7}],} \quad A - B = \sqrt{\frac{b}{X}}. \quad \text{Then}
\]
\[
\implies \frac{1}{2} \frac{X - 1}{X} b = B \sqrt{\frac{b}{X}} \implies \frac{1}{2} \frac{X - 1}{X} \sqrt{b} = B \implies B^2 = \frac{1}{4} \left( X - 1 \right)^2 \frac{b}{X}
\]
As \( B^2 = a^{3u}, \ X = e^{u \beta t} \) and \( b = \frac{\mu_2}{\lambda} \), we finally get
\[
a^{3u} = \frac{1}{4} \frac{\mu_2}{\lambda} \left( e^{u \beta t} - 1 \right)^2 e^{u \beta t}
\]

Due to the definitions \([5.7]\) we have
\[\frac{\mu_2}{\lambda} = \frac{X}{\Lambda} \epsilon_1,\]
\( \epsilon_1 \) being the value of \( \epsilon(t) \) at the normalization time \( t_2 \), and \([5.18]\) is proved. The profile \([5.19]\) follows from \([5.18]\) by observing that
\[2 \left( \cosh(z) - 1 \right) = e^z + e^{-z} - 2 = e^{-z} \left[ e^{2z} + 1 - 2 e^z \right] = \left( e^z - 1 \right)^2 e^z. \]

\textbf{Remark 5.4} – The profiles \([5.18]\) and \([5.18]\) show the convenience of introducing the \textit{dimensionless time}

\[x \overset{\text{def}}{=} \beta t\]

so that they assume the form
\[
a(u; x, x_2) = \left[ \frac{1}{4} \frac{X}{\Lambda} \epsilon_2 \left( e^{ux} - 1 \right)^2 \right] \frac{1}{3} e^{ux} \quad \text{\{exponential form\}}
\]
\[
a(u; x, x_2) = \left[ \frac{1}{2} \frac{X}{\Lambda} \epsilon_2 \left( \cosh(ux) - 1 \right) \right] \frac{1}{3} \quad \text{\{hyperbolic form\}}
\]

These profiles will be plotted later on (Section \[5.8\]) since we need more information about the magnitude of the constants that are involved. \[\blacksquare\]
5.4 The profiles of the Hubble parameter

Theorem 5.4 – The profiles $h(u; x)$ of the Hubble parameter of a barotropic flat model admit the following two equivalent representations

\begin{align*}
(5.28) \quad h(u; x) & \overset{\text{def}}{=} \frac{1}{a} \frac{da}{dx} = \frac{1}{3} \frac{e^{ux} + 1}{e^{ux} - 1} \\
(5.29) \quad h(u; t) & \overset{\text{def}}{=} \frac{1}{a} \frac{da}{dt} = \frac{1}{3} \beta \frac{e^{u \beta t} + 1}{e^{u \beta t} - 1}
\end{align*}

Proof – By setting, as above, $X \overset{\text{def}}{=} e^{ux}$ and observing that $X' = ux$ from (5.26) we get

\[ 3u \frac{d \log a}{dX} = 2 \frac{d \log(X - 1)}{dX} - \frac{d \log X}{dX} = \frac{2}{X - 1} - \frac{1}{X} = \frac{X + 1}{X(X - 1)} \implies \]

\[ \frac{da}{dX} = \frac{a}{3u} \frac{X + 1}{X(X - 1)} \]

\[ \implies \frac{da}{dx} = \frac{da}{dX} u X = \frac{a}{3u} \frac{X + 1}{X(X - 1)} u X \implies \]

\[ \frac{da}{dx} = \frac{1}{3} \frac{a}{X - 1} \frac{X + 1}{X - 1} = \frac{1}{3} \frac{a}{X - 1} e^{ux} + 1 \]

\[ \implies (5.28). \] As $\frac{da}{dt} = \beta \frac{da}{dx}$ we get (5.29). □

Figure 5.1: Graphs of $h(u; x)$. 

Remark 5.5 – The evolution of the Hubble parameter does not depend on the normalization time \( x_3 \), in accordance with Theorem 1.6 page 13.

### 5.5 Cosmological data

The dimensionless time \( x = \beta t \) has no practical significance as long as we do not know the value of the constant \( \beta \). The estimates of \( \beta \) and other constants introduced in this theory are listed in Table 5.2 and are inferred from Table 5.1 containing basic cosmological data taken from [2].

| name                      | symbol | estimate                              | note               |
|----------------------------|--------|---------------------------------------|--------------------|
| tropical year (2011)       |         | \( 31.5569522 \cdot 10^6 \text{s} \) |                    |
| speed of light             | \( c \) | \( 299.792458 \cdot 10^3 \text{km s}^{-1} \) |                    |
| age of the universe        | \( t_0 \) | \( \approx 13.81 \pm 0.05 \text{Gyr} \) |                    |
| Hubble parameter today     | \( H_0 \) | \( \approx 0.6882972691 \cdot 10^{-10} \text{yr}^{-1} \) |                    |
| gravitational constant     | \( G_N \) | \( \approx 6.67408 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \) |                    |
| dark energy density        | \( \Omega_\Lambda \) | \( 0.685_{+0.017}^{-0.016} \) |                    |

Table 5.1: Basic cosmological data.

| constant | dimension | estimate                                      | note             |
|----------|-----------|-----------------------------------------------|------------------|
| \( \Lambda \) | \( L^{-2} \) | \( \approx 1.087769524444 \cdot 10^{-52} \text{m}^{-2} \) | [1]              |
| \( \beta \) | \( T^{-1} \) | \( \approx \begin{Bmatrix} 5.41563983302 \cdot 10^{-18} \text{s}^{-1} \\ 0.170901087343 \text{Gyr}^{-1} \end{Bmatrix} \) | [2]              |
| \( \frac{1}{\beta} \) | \( T \) | \( \approx \begin{Bmatrix} 0.18465038865819 \cdot 10^{18} \text{s} \\ 5.8513378442864 \text{Gyr} \end{Bmatrix} \) |                  |
| \( \chi \) | \( L^{-1}M^{-1}T^2 \) | \( \approx 2.07657899185574 \cdot 10^{-43} \text{m}^{-1} \text{kg}^{-1} \text{s}^2 \) | [3]              |
| \( \frac{\Lambda}{\chi} \) | \( L^{-1}M T^{-2} \) | \( \approx 5.238276649771 \cdot 10^{-10} \text{m}^{-1} \text{kg} \text{s}^{-2} \) |                  |

Table 5.2: Supplementary data.
5.5. Cosmological data

Notes.

[1] Estimate of $\Lambda \equiv \frac{3H_0^2}{c^2} \Omega_\Lambda$:

$H_0 \simeq 0.6882972691 \cdot 10^{-10} \text{ yr}^{-1}$

$H_0^2 \simeq 0.47375313065051 \cdot 10^{-20} \text{ yr}^{-2}$

$3 H_0^2 \simeq 1.42125939195 \cdot 10^{-20} \text{ yr}^{-2}$

$\frac{3 H_0^2}{c} \simeq 0.004740811031176774967434 \cdot 10^{-20} \text{ yr}^{-2} \cdot 10^{-6} \text{ m}^{-1} \text{s}$

$\frac{3 H_0^2}{c^2} \simeq 1.581364342119899149509 \cdot 10^{-5} \cdot 10^{-20} \text{ yr}^{-2} \cdot 10^{-12} \text{ m}^{-2} \text{s}^2 [\ast]$

$\Lambda = 3 \frac{H_0^2}{c^2} \cdot \Omega_\Lambda \simeq 1.581364342119899149509 \cdot 10^{-37} \text{ yr}^{-2} \text{ m}^{-2} \text{s}^2 \cdot 0.685$

$\Lambda \simeq 1.0832457435213091744 \cdot 10^{-37} \text{ yr}^{-2} \text{ m}^{-2} \text{s}^2$

$yr/s = 3.15569522 \cdot 10^7$, $yr^2/s^2 = 9.9584123215308484 \cdot 10^{14}$

$\Lambda \simeq 0.108776952444422833994 \cdot 10^{-51} \text{ m}^{-2}$

[2] Estimate of $\beta \equiv \sqrt{3 \Lambda c}$:

$3 \Lambda \simeq 3.263308573333268501982 \cdot 10^{-52} \text{ m}^{-2}$

$\sqrt{3 \Lambda} \simeq 1.80646300820462002629 \cdot 10^{-26} \text{ m}^{-1}$

$c = 299.792458 \cdot 10^6 \text{ m} \text{s}^{-1}$

$\beta = \sqrt{3 \Lambda c} \simeq 541.56398333320223204638 \cdot 10^{-26} \text{ m}^{-1} \cdot 10^6 \text{ m} \text{s}^{-1}$

$\simeq 5.415639833320223204638 \cdot 10^{-18} \text{ s}^{-1}$. $1 \text{ yr} = 31.5569522 \cdot 10^6 \text{ s}$.

$\beta \simeq 1.709010873430351653021 \cdot 10^{-10} \text{ yr}^{-1}$.

[3] Estimate of $\chi \equiv \frac{8 \pi G_N}{c^4}$:

$c^{-4} \simeq 1.237990147236120239125 \cdot 10^{-34} \text{ m}^{-4} \text{s}^4$

$\pi \simeq 3.1415926535897932384626433$

$8 \pi c^{-4} \simeq 31.11408601418833454066 \cdot 10^{-34} \text{ m}^{-4} \text{s}^4$

$G_N = 6.67408(31) \cdot 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$

$\chi \simeq 207.657899185574 \cdot 10^{-34} \text{ m}^{-4} \text{s}^4 \cdot 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$

$\chi \simeq 2.07657899185574 \cdot 10^{-43} \text{ m}^{-1} \text{s}^2 \text{kg}^{-1}$
Chapter 5. Barotropic dynamics of a single-component universe

5.6 The age of the universe

**Theorem 5.5** – If $H_0$ is the present-day value of the Hubble parameter, then the age of the universe is

\[ x_0 = \beta t_0 = \frac{1}{u} \log \frac{3H_0 + \beta}{3H_0 - \beta} \]

**Proof** – Equation (5.28) is solvable with respect to $X = e^{ux}$:

\[ h(u, x) = \frac{1}{3} \left( \frac{X + 1}{X - 1} \right) \Rightarrow 3 (X - 1) h(u, x) = X + 1 \]

\[ \Rightarrow [3 h(u, x) - 1] X = 3 h(u, x) + 1 \Rightarrow X = \frac{3 h(u, x) + 1}{3 h(u, x) - 1} \]

Because of (5.29), $h(u, x) = \beta^{-1} h(u, t) \Rightarrow X = \frac{3 h(u, t) + \beta}{3 h(u, t) - \beta}$.

As $X = e^{ux} \Rightarrow$

\[ x = \beta t = \frac{1}{u} \log \frac{3h(u, t) + \beta}{3h(u, t) - \beta} \]

The profile (5.18) satisfies the initial condition $a(0) = 0$ with $t_0 = 0$. Then the beginning of the universe corresponds to $t = x = 0$, so that equation (5.33) applied to the present epoch provides the age of the universe.

According to the formula (5.32), for computing the age of the universe we only need the values of $H_0$ and $\beta = \sqrt{3/\Lambda}$:

\[
\begin{align*}
H_0 &\approx 0.6882972691 \cdot 10^{-10} \text{ yr}^{-1} \\
\beta &\approx 1.7090108734 \cdot 10^{-10} \text{ yr}^{-1}
\end{align*}
\]

\[ \Rightarrow u x_0 \approx \log 10.6043969244822 \quad \approx 2.361268719306985270849 \]

\[ \Rightarrow u x_0 \approx 2.3612687193 \]

\[ \Rightarrow u t_0 = \frac{u x_0}{\beta} \approx 13.81658101781 \cdot 10^9 \text{ yr} \]

**Remark 5.6** – For $u = 1$ this estimate is very close to that supplied by the astronomers [2] (2015) $t_0 \approx 13.81 \pm 0.05$ Gyr. This means that the primordial phase of radiation dominance has an irrelevant influence on the evaluation of the present-day age of the universe.
5.7 Inadmissibility of the positive curvature

The evaluation of equation (5.5) at the present day \( t_0 \) gives
\[
\frac{3 H_0^2}{c^2} = \Lambda + \chi \epsilon_0 - \frac{3 \tilde{K}}{a^2(t_0, t)}.
\]

This equation is invariant under the choice of the reference time (Remark 5.1). Then for \( t_\sharp = t_0 \) we get
\[
\frac{3 H_0^2}{c^2} - \Lambda - \chi \epsilon_0 = -3 \tilde{K}.
\]

If we assume \( \tilde{K} > 0 \) then the following inequalities hold:
\[
\frac{3 H_0^2}{c^2} - \Lambda - \chi \epsilon_0 < 0 \iff \frac{3 H_0^2}{c^2} - \Lambda - \chi \epsilon_0 < 0 \iff \frac{3 H_0^2}{c^2} - \Lambda < \chi \epsilon_0 \iff
\]
\[
(5.36)
\]
\[
\epsilon_0 > \frac{1}{\chi} \left( \frac{3 H_0^2}{c^2} - \Lambda \right) \overset{\text{def}}{=} \epsilon_*
\]

Go back to [*]:
\[
3 \frac{H_0^2}{c^2} \simeq 1.58136434211 \times 10^{-37} \cdot m^{-2} \cdot s^2 \cdot yr^{-2}, \quad 1 \text{yr} = 31.5569522 \cdot 10^6 \text{s}
\]
\[
\implies 3 \frac{H_0^2}{c^2} \simeq \frac{1.581364342119}{(31.5569522)^2} \cdot 10^{-43} \cdot m^{-2} \simeq 1.58796833 \cdot 10^{-46} \cdot m^{-2}.
\]

Due to the estimate of \( \Lambda \) and \( \chi \),
\[
\Lambda \simeq 0.10877695 \cdot 10^{-51} \cdot m^{-2} \quad \chi \simeq 2.07657899 \cdot 10^{-43} \cdot m^{-1} \cdot s^2 \cdot kg^{-1}
\]
we get
\[
\epsilon_* \simeq \frac{1}{2.07657899} \cdot 10^{43} \left( 1.58796833 \cdot 10^{-46} - 0.10877695 \cdot 10^{-51} \right) m^{-1} \cdot kg \cdot s^{-2}
\]
\[
\approx \frac{1}{2.07657899} \cdot 10^{43} \cdot 10^{-51} \left( 1.58796833 \cdot 10^5 - 0.10877695 \right) m^{-1} \cdot kg \cdot s^{-2}
\]
\[
\approx \frac{1.58796833}{2.07657899} \cdot 10^{43} \cdot 10^{-51} \cdot 10^5 \implies
\]
\[
\epsilon_* \simeq 0.764704 \cdot 10^{-3} \cdot m^{-1} \cdot kg \cdot s^{-2}
\]

This energy density corresponds to a mass density \( \rho_* = \epsilon_* c^{-2} \). Since \( c = 299.792458 \cdot 10^6 \text{m} \cdot \text{s}^{-1} \) and \( 0.764704/(299.792458)^2 \approx 8.50847 \cdot 10^{-6} \), we get
\[
\rho_* \simeq \frac{0.764704 \cdot 10^{-3} \cdot m^{-1} \cdot kg \cdot s^{-2}}{(299.792458)^2 \cdot 10^{12} m^2 \cdot s^{-2}} \simeq 8.50847 \cdot 10^{-6} \cdot 10^{-3} \cdot kg \cdot 10^{12} m^3
\]
\[
\rho_* \simeq 8.50847 \cdot 10^{-21} \cdot kg \cdot m^{-3}
\]
A rough present-day estimate of the mass density is

$$\rho_0 \approx (9.2 \pm 1.8) \cdot 10^{-27} \frac{kg}{m^3} \approx 10^{-26} \frac{kg}{m^3}.$$  

Then we see that the inequality (5.36) $\epsilon_0 > \epsilon_*$, which is equivalent to $\rho_0 > \rho_*$, is far to be satisfied even if the estimate of $\rho_0$ is not accurate. This proves that the assumption $K > 0$ is in strong contrast with the astronomical observations.

### 5.8 The ‘exact’ profiles of the flat barotropic universes

If we consider sufficiently reliable the estimate of the age of the universe found above, then we should consider equally reliable the choice of the present-day time $t_0$ as reference time for the scale factor. In doing so we get a ‘sufficiently reliable’ (or ‘exact’) numerical evaluation of the universe profile, for any value of the state parameter $u$.

**Theorem 5.6** – With the present-day reference time $x_0$ the profiles of the universe are

$$a(u; x, x_0) = \left[ c_0 \left( \cosh(ux) - 1 \right) \right]^\frac{1}{3u} = \left[ \frac{1}{2} c_0 \left( e^{ux} - 1 \right) \right]^\frac{1}{3u}.$$  

(5.37)  

$$c_0 \overset{\text{def}}{=} \frac{1}{\cosh(ux_0) - 1} \approx 0.2299194811$$  

(5.38)

**Proof** – With $x_0 = x_0$ the profiles (5.27) read

$$a(u; x, x_0) = \left[ \frac{1}{2} \frac{\chi}{\Lambda} \epsilon_0 \left( \cosh(ux) - 1 \right) \right]^\frac{1}{3u}.$$  

By imposing the normalization condition $a(u; x_0, x_0) = 1$ we get

$$\frac{1}{2} \frac{\chi}{\Lambda} \epsilon_0 \left( \cosh(ux_0) - 1 \right) = 1$$  

(5.39)

i.e.

$$\frac{1}{2} \frac{\chi}{\Lambda} \epsilon_0 = c_0$$  

(5.40)

with $c_0$ defined as in (5.38).

The graphs of $a(u; x, x_0)$ are plotted in Fig. 5.2 with respect to the variable $ux$ for some relevant values of $u$. Whatever $u$, they all pass through the point $(x_0, 1)$, as expected. In Fig. 5.3 the profiles are plotted with respect to the
variable $x$. In both representation we observe (i) a different way of approaching the origin $x = 0$ and (ii) the presence of inflection points $x_{ip}$ for certain values of $u$.

![Graph of $a(u; x, x_0)$ in the variable $ux$.](image)

**Figure 5.2:** Graphs of $a(u; x, x_0)$ in the variable $ux$.

**Theorem 5.7** – (i) The profiles approach the beginning of the universe $x = 0$ in different ways:

\[
\begin{align*}
    u > \frac{2}{3} & \implies \lim_{x \to 0} \frac{da}{dx} = +\infty. \\
    u = \frac{2}{3} & \implies \lim_{x \to 0} \frac{da}{dx} = \frac{1}{3}\sqrt{2c_0}. \\
    u < \frac{2}{3} & \implies \lim_{x \to 0} \frac{da}{dx} = 0.
\end{align*}
\]

(ii) For $u > \frac{2}{3}$ there is an inflection point at the time

\[
x_{ip}(u) = \frac{1}{u} \log \left[ 3u - 1 + \sqrt{(3u - 1)^2 - 1} \right] = \frac{1}{u} \text{arccosh}(3u - 1)
\]

**Remark 5.7** – This theorem indicates the value $u = \frac{2}{3}$ (corresponding to $w = -\frac{1}{3}$) as a **threshold parameter**: for any small variation of this value the profile $a(u, x)$ changes radically. Due to this sort of ‘instability’ we should consider **inadmissible** the case $u = \frac{2}{3}$.
Note that \( \frac{1}{2} \rightleftharpoons (5.31) = (5.37) = (i) (5.28) \).
5.9. The profiles of the energy density

\[ X = 3u - 1 \pm \sqrt{(3u - 1)^2 - 1}. \] With the - sign \( X < 1 \), rejected.

\[ X = e^{ux} \implies (5.42). \]

**Remark 5.8** – The inflection point marks the transition from decelerated to accelerated expansion.

| Component | \( w \) | \( u \) | \( x_{ip}(u) \) | \( t_{ip}(u) = \beta^{-1} x_{ip}(u) \) |
|-----------|--------|--------|----------------|-------------------------------|
| Matter    | 0      | 1      | 1.316958       | 7.70601 Gyr                   |
| Radiation | \( \frac{1}{3} \) | \( \frac{1}{3} \) | 1.322067         | 7.73591 Gyr                   |

**Table 5.3:** Estimate of the inflection time.

The two times \( x_{ip}(1) \) and \( x_{ip}(\frac{1}{3}) \) are very close. The phase of accelerated expansion starts \( \simeq 6.08 \div 6.11 \) billion years ago. Note that \( x_{ip} \) does not depend on the choice of the normalization time.

**5.9 The profiles of the energy density**

**Theorem 5.8** – In the barotropic flat models the profiles \( \epsilon(u; x) \) of the energy density do not depend on the reference time \( t_0 \) and admit the following two equivalent representations

\[
\epsilon(u; x) = \frac{\Lambda}{\chi} \frac{4 e^{ux}}{(1 - e^{ux})^2} \quad \text{or} \quad \epsilon(u; x) = \frac{\Lambda}{\chi} \frac{2}{\cosh(ux) - 1}.
\]

**Proof** –

\[
\begin{align*}
(5.26) & \implies a^{3u}(u; x; x) = \frac{1}{4} \epsilon_x \frac{\Lambda}{\chi} \left( e^{ux} - 1 \right)^2. \\
(5.27) & \implies a^{3u}(u; x, x) = \frac{1}{2} \epsilon_x \frac{\Lambda}{\chi} \left( \cosh(ux) - 1 \right). \\
(5.44) & \implies \epsilon(u; x, x) = \frac{\epsilon_x}{a^{3u}(u; x; x)} \implies (5.43). \]

**Remark 5.9** – The evolution of the energy density does not depend on the choice of the normalization time but only on the parameter \( u \) and on the ratio \( \Lambda/\chi \). As a consequence, since we know a ‘reliable’ numerical value of \( \Lambda/\chi \) (see Table 5.2)

\[ \frac{\Lambda}{\chi} \simeq 5.238276649771 \cdot 10^{-10} \text{ m}^{-1} \text{ kg} \text{s}^{-2} \]

then we can get a ‘reliable’ numerical estimate of the evolution of the energy density for any value of the parameter \( u \) (Fig. 5.4). The formula to be used for
plotting $\epsilon(u; x)$ is

$$\epsilon(u; x) = 5.238276649771 \times 2 \frac{1}{\cosh(u \ast x) - 1}.$$ 

Remark 5.10 – The present-day value $\epsilon_0$ of the energy density does not depend on the barotropic parameter $u$. Due to (5.40), (5.38) and (5.44) we have

$$\epsilon_0 = 2c_4 \frac{\Lambda}{\chi} \approx 2 \times 0.2299194811 \ast 5.238276649771 \cdot 10^{-10} m^{-1} kg s^{-2},$$

(5.45)

$$\epsilon_0 \approx 2.408763697 \cdot 10^{-10} m^{-1} kg s^{-2}.$$ 

5.10 The vanishing of the cosmological constant

With $\lambda = 0$ the dynamical equation (5.21) reads

$$\frac{\dot{a}^2}{a^2} = \frac{\mu_2}{a^{3u}}$$

and is equivalent to

$$\frac{da}{\sqrt{a^2 - 3u}} = \sqrt{\mu_2} dt.$$
Let us consider the case $u = 1$ (dust matter):

$$\sqrt{\dot{a}} da = \sqrt{\mu_u} dt \implies \frac{2}{3} a^2 = \sqrt{\mu_u} t \implies a^2 = \frac{3}{2} \sqrt{\mu_u} t \implies$$

$$a(1; t, t^\#) = \sqrt{\frac{9}{\pi} \mu_u} t^\frac{2}{3}$$

The Hubble factor evolution is

$$h = \frac{\dot{a}}{a} = \frac{2}{3} t^{-1}.$$

The evaluation of this formula for $t = t_0$ (today) gives the age of the universe:

$$t_0 = \frac{2}{3} H_0^{-1} = \frac{2}{3} t_0^H \quad (t_0^H \overset{\text{def}}{=} H_0^{-1} \text{ is the Hubble time})$$

Since $H_0 \simeq 0.6882972691 \cdot 10^{-10} \text{ yr}^{-1}$ we get

$$t_0 \simeq 9.68573744475 \cdot 10^9 \text{ yr}$$

which is far from the current estimate of the age of the universe. This is one of the many reasons that make it unacceptable to consider $\Lambda = 0$.

### 5.11 Superluminal recession speed and the Hubble radius

Let $\ell^\#$ be the distance of two galaxies at the reference time $t^\#$. This distance evolves with time according to the law (5.46)

$$\ell(u; t, t^\#) = a(u; t, t^\#) \ell^\#,$$
with an expansion speed \( \dot{\ell}(u; t, t_\sharp) = \dot{a}(u; t, t_\sharp) \ell_\sharp \) that may become greater than the light speed \( c \).

**Theorem 5.9** – The recession speed \( \dot{\ell}(u; t, t_\sharp) \) is superluminal

\[
\dot{\ell}(u; t, t_\sharp) \geq c
\]

in the time interval defined by the inequality

\[
(X + 1)^{3u} (X - 1)^{2 - 3u} \geq CX
\]

where the dimensionless constant \( C \) is defined by

\[
C \overset{\text{def}}{=} 4 \cdot 3^{\frac{3u}{2}} \frac{\Lambda^{1 - \frac{3u}{2}}}{\chi \epsilon_\sharp \ell_\sharp^{3u}}.
\]

This constant does not depend on the choice of the reference time \( t_\sharp \).

**Proof** – (i) \( \dot{\ell}(u; t, t_\sharp) = \beta \ell'(u; x, x_\sharp) \) and \( \ell' = a' \ell_\sharp \implies \)

\[
\dot{\ell} \geq c \iff \ell' \geq \frac{c}{\beta} \iff a' \ell_\sharp \geq \frac{1}{\sqrt{3 \Lambda}} \iff
\]

\[
a' \geq \frac{1}{\sqrt{3 \Lambda} \ell_\sharp}
\]

(ii) \( a' = \frac{da}{dx} = \frac{1}{3} a \frac{X + 1}{X - 1} \), (5.26) \( a(u; x, x_\sharp) = \left[ \frac{1}{4} \frac{X}{\Lambda} \frac{(X - 1)^2}{X} \right]^{3u} \)

\[
\implies a' = \frac{1}{3} \left[ \frac{1}{4} \frac{X}{\Lambda} \frac{(X - 1)^2}{X} \right]^{3u} \frac{X + 1}{X - 1}
\]

\[
\implies [a']^{3u} = \left[ \frac{1}{3} \right]^{3u} \frac{1}{4} \frac{X}{\Lambda} \frac{\epsilon_\sharp (X - 1)^2 (X + 1)^{3u}}{X (X - 1)^{3u}}.
\]

(iii) (5.49) \( \iff [a']^{3u} \geq \frac{1}{\sqrt{3 \Lambda}^{3u} \epsilon_\sharp^{3u}} \)

\[
\iff \left[ \frac{1}{3} \right]^{3u} \frac{1}{4} \frac{X}{\Lambda} \frac{\epsilon_\sharp (X - 1)^2 (X + 1)^{3u}}{X (X - 1)^{3u}} \geq \frac{1}{\sqrt{3 \Lambda}^{3u} \epsilon_\sharp^{3u}}
\]

\[
\iff \frac{(X - 1)^{2 - 3u} (X + 1)^{3u}}{X} \leq \frac{1}{\sqrt{3 \Lambda}^{3u} \epsilon_\sharp^{3u}} \iff \frac{4 \cdot 3^{3u} \Lambda}{\chi \sqrt{3 \Lambda}^{3u} \epsilon_\sharp^{3u}} \iff \frac{4 \cdot 3^{3u} \Lambda}{\chi \sqrt{3 \Lambda}^{3u} \epsilon_\sharp^{3u}} \iff
\]

\[
\dot{\ell} \geq c \iff (X - 1)^{2 - 3u} (X + 1)^{3u} \geq C X,
\]
5.11. Superluminal recession speed and the Hubble radius

where $C$ is defined as in (5.48). (iv) By virtue of (5.11) $\epsilon(t) a^{3u}(t, t') \ell_t^{3u} = \epsilon_\sharp \ell_t^{3u}$ = constant in $t$, i.e.

$$\epsilon(t) \ell_t^{3u}(t) = \epsilon_\sharp \ell_t^{3u}, \quad \forall \ t.$$  

This shows that the product $\epsilon_\sharp \ell_t^{3u}$ does not depend on the choice of the reference time $t_\sharp$. Hence, also $C$ is independent. ■

**Theorem 5.10** – The constant $C$ has the form

$$C = \left( \frac{L_\dashv \ell_\dashv}{\ell_\dashv} \right)^{3u}$$

Proof – 5.48: $C = \frac{4 \cdot 3^{3/2}}{\chi \epsilon_\sharp \ell_t^{3u}}$.

5.43: $\epsilon(u; x) = \frac{2}{\chi} \frac{\Lambda (1 - e^{ux})^2}{e^{ux}} = \frac{\Lambda}{\chi} \frac{2}{\cosh(ux) - 1}$

$$C = 4 \cdot 3^{3u} \frac{\Lambda^{1 - 3u}}{\chi \epsilon_\sharp \ell_t^{3u}} \cdot \left( \frac{1}{4} \frac{\Lambda - \frac{3u}{2}}{\ell_t^{3u}} \right) \cdot \frac{(1 - e^{ux})^2}{e^{ux}}

C = \left( \frac{3}{\Lambda} \right)^{3u} \left( \frac{1}{e^{ux}} \right)^{3u} \cdot \frac{1}{\ell_t^{3u}} = \left[ \frac{3}{\Lambda} \frac{1}{2} \left( \frac{(1 - e^{ux})^2}{e^{ux}} \right)^{3u} \cdot \frac{1}{\ell_t^{3u}} \right]^{3u} \Rightarrow

5.54.

The alternative expression of $L_\dashv$ follows from $2 (\cosh(z) - 1) = (e^z - 1)^2$. ■

We can continue the analysis of the superluminal recession speed only with the specification of the barotropic parameter $u$. We will consider the case $u = 1$: dust-matter universe.

**Theorem 5.11** – For $u = 1$ the superluminal expansion condition 5.47 is equivalent to

$$\ell(1; t, t') \geq c \quad \iff f(C; X) = X^3 + (3 - C) X^2 + (3 + C) X + 1 \geq 0$$

with the constant $C$ given by

$$C = \left( \frac{L_\dashv}{\ell_\dashv} \right)^3$$

$$L_\dashv = \sqrt{3} \frac{(X_\dashv - 1)^2}{X_\dashv} \quad X_\dashv \equiv e^x$$
are plotted in Fig. 5.6 for various values of $C$ highlighted.

According to this theorem the analysis of the occurrence of the superluminal phenomenon is reduced to the analysis of the roots of the cubic polynomial $f(C; X)$, whose coefficients depends on the constant $C$. The graphs of $f(C; X)$ are plotted in Fig. 5.6 for various values of $C$. Some relevant facts must be highlighted:

1. As shown by (5.53) $C$ is the cube of the ratio of two lengths, $L_4$ and $\ell_2$. $L_4$ has the property of being computable regardless of the given value of galactic distance $\ell_2$ (see item 7 below).

2. The analysis makes sense only in the interval $X = e^t \geq 1$, corresponding to $t \geq 0$, since $t = 0$ is the date of birth of the universe.

3. Whatever $C$, $f(C, 0) = 1$ and $f(C, 1) = 8$: all graphs pass through the points (0,1) and (1,8). Furthermore, $f(C, X)$ has a real negative root close to $X = 0$ and the other real roots (if any) are located in the unbounded interval $X > 1$. 

---

Figure 5.6: Graphs of $f(C; X)$.
4. There exists a discriminant value

$$C_\Delta \simeq 10.3923$$

for which $f(C_\Delta, X)$ is tangent to the $X$-axis at a point $X_\Delta \simeq 3.7324$. The point $X_\Delta$ corresponds to the value

$$t_\Delta \simeq 7.706513 \text{ Gyr}$$

of the cosmic time.

5. For $C < C_\Delta$, there are no real roots $X > 1$ of $f(C, X)$ and we have a permanent superluminal recession speed. For $C > C_\Delta$ the polynomial $f(C, X)$ has two (positive) simple roots $X_1 < X_2$. The recession speed is subluminal in the interval $(X_1, X_2)$ delimited by these roots and containing $X_\Delta$.

6. There exists a special value $C_H > C_\Delta$ given by

$$C_H = \frac{(e^{x_0} + 1)^3}{e^{x_0}(e^{x_0} - 1)} \simeq 15.34304824$$

such that the polynomial $f(C_H, X)$ has a root

$$X_0 \simeq 10.604396924 > X_\Delta$$

corresponding to the present-day time

$$t_0 \simeq 13.816581 \text{ Gyr} \quad (x_0 \simeq 2.3612687193)$$

and a root

$$X_* \simeq 1.791299 < X_\Delta$$

corresponding to the cosmic time

$$t_* \simeq 3.41098505 \text{ Gyr} \quad (x_* \simeq 0.582941054).$$

---

3 $X_\Delta = e^{-\beta t_\Delta} \implies \beta t_\Delta = \log X_\Delta \simeq 1.317051 \implies t_\Delta = \beta^{-1} \log X_\Delta \simeq 7.706513.$

4 PROOF $f(C, X_0) \overset{\text{def}}{=} X_0^3 + (3 - C) X_0^2 + (3 + C) X_0 + 1$ with $X \overset{\text{def}}{=} e^{x_0}$, $x_0 \overset{\text{def}}{=} \beta t_0.$

$f(C, X_0) = 0 \iff X_0^3 + (3 - C) X_0^2 + (3 + C) X_0 + 1 = 0$

$\iff X_0^3 + 3 X_0^2 - C X_0^2 + 3 X_0 + C X_0 + 1 = 0$

$\iff X_0^3 + 3 X_0^2 + 3 X_0 + 1 = C X_0 (X_0 - 1) \iff (X_0 + 1)^3 = C X_0 (X_0 - 1)$

$\iff C_H = \frac{(X_0 + 1)^3}{X_0 (X_0 - 1)} \simeq 15.34304824$.

The estimate of the first root $X_1 < X_\Delta$ is a matter of numerical analysis. It follows that

$$t_* = \beta^{-1} \log X_* \simeq \frac{0.582941054}{0.170901087343} \text{ Gyr} \simeq 3.41098505 \text{ Gyr}.$$
Since the definition of $C$ does not depend on the choice of the reference time $t_2$ (Theorem 5.9) we can write the definition (5.53) for $C = C_H$ by taking the reference times $t_\star$ and $t_0$ of item 6 above:

\begin{equation}
C_H = \left( \frac{L(t_\star)}{\ell(t_\star)} \right)^3, \quad L(t_\star) = \sqrt{\frac{3}{\Lambda}} \sqrt{\frac{(X_\star - 1)^2}{X_\star}}.
\end{equation}

Since $L(t_\star)$ and $L(t_0)$ are computable (see below), and $C_H$ is known (item 4), from (5.59) we can derive the lengths $\ell(t_\star)$ and $\ell(t_0)$:

\begin{equation}
\begin{aligned}
\ell(t_\star) &= \sqrt[3]{\sqrt{3}} \Lambda, \\
\ell(t_0) &= \sqrt[3]{\sqrt{3}} \Lambda.
\end{aligned}
\end{equation}

For the way in which it has been defined, $\ell(t_0)$ is the present-day distance of two galaxies crossing the boundary beyond which the recession velocity exceeds the speed of light. This boundary is called the Hubble radius (of the Hubble sphere). In turn, $\ell(t_\star)$ is the distance at the time $t_\star$ at the early universe when the recession speed of these two galaxies crossed this boundary in the opposite sense: from superluminal to subluminal recession speed.

Remark 5.11 – Computation of $\ell(t_\star)$ and $\ell(t_0)$.

1. $\Lambda \simeq 1.08776952444422834 \cdot 10^{-52} m^{-2}$ (table 5.2) $\implies$

$$\frac{3}{\Lambda} \simeq 2.7579371664528356 \cdot 10^{52} m^2$$

$$\implies \sqrt{\frac{3}{\Lambda}} \simeq 1.66070381659489 \cdot 10^{26} m = 1.66070381659489 \cdot 10^{23} km.$$  

Conversion to light-years: $10^{23} km \simeq 10.570234105227 Gly$ $\implies$

$$\sqrt[3]{\sqrt{3}} \Lambda \simeq 17.554028120851 Gly$$

2. $C_H \simeq 15.34304824 \implies \sqrt[3]{C_H} \simeq 2.4848712052$

3. $X_\star \simeq 1.791299 \implies \frac{(X_\star - 1)^2}{X_\star} \simeq 0.349553 \implies \sqrt{\frac{(X_\star - 1)^2}{X_\star}} \simeq 0.7044297$

$$\implies L(t_\star) \simeq 17.55402812 \cdot 0.7044297 Gly \implies L(t_\star) \simeq 12.36557 Gly$$
4. \( \ell(t_*) = \frac{L(t_*)}{\sqrt{C_H}} \approx \frac{12.36557}{2.4848712052} \text{Gly} \implies \ell(t_*) \approx 4.97632 \text{Gly} \)

5. \( X_0 \approx 10.604396924 \implies \frac{(X_0 - 1)^2}{X_0} \approx 8.69869743 \equiv \frac{3}{X_0} \approx 2.056607463 \implies L(t_o) \approx 17.55402812 \cdot 2.056607463 \text{Gly} \implies \)

\[
L(t_o) \approx 36.10174523 \text{Gly}
\]

6. \( \ell(t_o) = \frac{L(t_o)}{\sqrt{C_H}} \approx \frac{36.10174523}{2.4848712052} \text{Gly} \implies \ell(t_o) \approx 14.52861828 \text{Gly} \bullet \)

We obtain the evolution of the Hubble radius in the cosmic time by suppressing the subscript 0 in (5.59) and (5.60):

\[
L(t) = \sqrt{\frac{3}{\Lambda}} \sqrt{\frac{e^x - 1}{e^x}} \quad C_H = \frac{(e^x + 1)^3}{e^x (e^x - 1)} \quad \implies \ell_H(t) = \frac{L(t)}{\sqrt{C_H}} = \sqrt{\frac{3}{\Lambda}} \sqrt[3]{\frac{e^x - 1}{e^x + 1}} \implies \\
\ell_H(t) = \sqrt{\frac{3}{\Lambda}} \frac{e^{\beta t} - 1}{e^{\beta t} + 1}
\]

Figure 5.7: Graph of the Hubble radius \( \ell_H(t) \).
Chapter 6

Transmission of photons

A human lives in a celestial body $O$ and observes in its sky another celestial body $B$ by means of optical or electro-magnetic devices. This requires that $B$ is capable of emitting electro-magnetic signals in form of ‘photons’ in a broad sense, i.e., in form of particles whose world-lines in space-time are light-like geodesics. We also assume that the world-lines of $B$ and $O$ are time-like geodesics belonging to the galactic fluid. The observer asks a number of questions:

How far is $B$? On what date the photons that I am receiving now have been issued by $B$? When $B$ has appeared in my sky? As long as it will be visible?

The focus of this chapter is to give an answer to these (and related) questions.

6.1 Emission-reception of photons

A photon is emitted by $B$ at the time $t_e$ and received by $O$ at the time $t_r$. Its world-line is depicted in Fig. 6.1. At any intermediate time $t$ (in the figure two of them are marked: $t_1 < t_2$) a length $\ell(t_e, t)$ is defined: it is the distance traveled by the photon till the time $t$ measured on the spatial section $S_t$.

If we look at the photon progression through the cosmic monitor, i.e. on the quotient manifold (Fig. 6.2) then we observe that it moves along a geodesic joining $B$ to $O$ with a speed given by equation (2.35):

$$v(t) = \frac{ds}{dt} = \frac{c}{a(t, t_r)}.$$  

(6.1)

On the other hand, the synchronous traveled distance in space-time (depicted

\[1\] In other words, we assume that $B$ and $O$ have to be considered as particles of the galactic fluid.

\[2\] It is a synchronous distance at the time $t$, as defined in Section 1.5.
in Fig. 6.1 is given by

\[(6.2) \quad \ell(t_e, t) = a(t, t_2) \ell(t_e, t; t_e),\]

Figure 6.1: Photon world-line from B to O.

\[
\ell(t_e, t; t_2) = c \int_{t_e}^{t} \frac{dt'}{a(t', t_2)}.
\]

Consequently, the photon will reach O at the time \(t_e\) if and only if

\[(6.4) \quad \ell(t_e, t_r; t_2) = c \int_{t_e}^{t_r} \frac{dt'}{a(t', t_2)} = \ell_{BO}^2 \quad \text{distance from B to O}\]

\(\therefore\) The symbol \(t_2\) has been inserted in this notation to emphasize the dependence on the reference time of the scale factor.
provided that the integral exists as a finite number. In this case we say that $B$ is visible to $O$ at the time $t_r$. This property is independent from the choice of $t_2$ and has a significant geometrical interpretation illustrated in Fig. 6.3: the shaded area delimited by the graph of $c/a(t, t_2)$ upon the emission-reception interval $[t_e, t_r]$ represents the integral of $0.4$ hence the co-moving distance $\ell_{BO}^t$. Note that, from the dimensional viewpoint, this ‘area’ is in fact a time times a velocity quantity, i.e. a length-dimensional quantity.

Figure 6.3: Geometrical interpretation of equation (6.4).

If a second photon is emitted at $\bar{t}_e > t_e$ then we have a different reception time $\bar{t}_r > t_r$. The two emission-reception intervals have (in general) different magnitudes: $\bar{t}_r - \bar{t}_e \neq t_r - t_e$. However, the two shaded areas over these intervals remain unchanged since both of them are equal to $\ell_{BO}^t$, according to equation (6.3) (Fig. 6.4). In other words, the shaded area behaves as a planar incompressible fluid constrained to stay under the graph of $c/a(t, t_2)$ and upon the emission-reception interval $[t_e, t_r]$. 

Figure 6.4: Shift of the emission-reception intervals preserving the shaded area.
6.2 Event horizon

The topic of the previous section is based on the existence of a finite reception time \( t_r \). However, one could object that the photon might not have enough time to reach \( O \) before the end of the universe (at the time \( t_\infty \)). In fact, this happens when

\[
\ell(t_e, t_\infty; t_4) \overset{\text{def}}{=} c \int_{t_e}^{t_\infty} \frac{dt'}{a(t', t_4)} < \ell^4_{BO}
\]

as illustrated in Fig. 6.5.

![Figure 6.5: Graphic representation of (6.5).](image)

Alternatively, one can consider the limit case where the reception time coincides with the finish time of the universe. In this case, denoting by \( t_* \) the emission time, we have

\[
\ell(t_*, t_\infty; t_4) \overset{\text{def}}{=} c \int_{t_*}^{t_\infty} \frac{dt'}{a(t', t_4)} = \ell^4_{BO}
\]

![Figure 6.6: Graphic representation of (6.6).](image)

If \( t_e > t_* \) is a second emission time, then \( \ell(t_e, t_\infty; t_4) < \ell(t_*, t_\infty; t_4) \) i.e.

\[
\ell(t_e, t_\infty; t_4) < \ell^4_{BO}
\]

and we fall in the previous case: any photon emitted after \( t_* \) never reaches \( O \). So, \( t_* \) is the boundary after which no event occurring on \( B \) will be observed from \( O \): it is the event boundary of \( B \).
This general argument finds a concrete and fruitful application in the case of a dust-matter flat universe. Indeed, in the case we have an estimate of the age of the universe \( t_0 \), so that we can take \( t_0 \) as reference time. Hence, by virtue of (5.37) and (5.38), the boundary \( t_* \) for which (6.6) holds,

\[
\ell(t_*, +\infty; t_0) \overset{\text{def}}{=} c \int_{t_*}^{+\infty} \frac{dt'}{a(t', t_0)} = \ell^0_{BO}
\]

is implicitly defined by equation

\[
\frac{c}{\sqrt{c_0}} \int_{t_*}^{+\infty} \frac{dt'}{\sqrt{\cosh(\beta t') - 1}} = \ell^0_{BO},
\]

where \( c_0 \approx 0.2299194811 \) (dimensionless) and \( \beta \approx 0.170901087343 \, \text{Gyr}^{-1} \).

For \( t_* = t_0 \) (today) we find what is called the (present-day) radius of the event horizon:

\[
R_{eh}(t_0) \overset{\text{def}}{=} \ell(t_0, +\infty; t_0) \approx 16.702920561 \, \text{Gly}
\]

The event horizon encloses the set of bodies \( B \) from which the photons emitted by \( t_0 \) onwards will never be received by an observer \( O \) in the future.

### 6.3 Particle horizon

According to the physics of the early universe there is a date \( t_* > 0 \) in which the photons began to spread freely in the universe, as a consequence of a phenomenon called recombination, whose current estimate is \( t_* \approx 378,000 \, \text{yr} \).

Thus, according to (6.3), the distance traveled by a photon from \( t_* \) to a time \( t \) is given by

\[
\ell(t_*, t; t') = c \int_{t_*}^{t} \frac{dz}{a(z, t')},
\]

If at the time \( t \) the co-moving distance \( \ell^0_{BO} \) is greater than \( \ell(t_*, t; t') \),

\[
\ell(t_*, t; t') < \ell^0_{BO},
\]

\(^4\) If the time \( t \) is expressed in Gyr units and we want \( \ell_{eh}(t_0) \) expressed in light-years units, then we have to put \( c = 1 \) in (6.12) (the speed of light is equal to one light-year per year).

Since

\[
\frac{1}{\sqrt{c_0}} \approx \frac{1}{\sqrt{0.2299194811}} \, \text{Gly} \approx 1.6323304282 \, \text{Gly},
\]

\( \beta \approx 0.170901087343 \, \text{Gyr}^{-1} \), and \( t_0 \approx 13.81658101781 \, \text{Gyr} \), the formula to be used for the estimation of the radius is

\[
\int_{13.81658101781}^{\text{very large}} \frac{dx}{\sqrt{\cosh(0.170901087343 \times x) - 1}}.
\]

Our estimate is in agreement with that provided in [9]: \( \approx 16.4 \, \text{Gly} \).
then the body $B$ is not yet visible to the observer $O$. In this regard it must be noted that, due to (1.17),

$$\ell(t_\star, t; t_2) = c \int_{t_2}^{t} \frac{dz}{a(z, t_2)} = \frac{c}{a(t_\star, t_2)} \int_{t_2}^{t} \frac{dz}{a(z, t_0)} = a(t_\star, t_0) \ell(t_\star, t_0; t_2)$$

and that, due to (1.19), $\ell^*_{BO} = a(t_\star, t_0) \ell^*_{BO}$. Hence, in changing the reference time, the inequality (6.11) remains invariant. More precisely: both members are multiplied by the same factor $a(t_\star, t_0)$.

In the case of a dust-matter flat model, for which we have an estimate of the age of the universe $t_0$, we can take $t_0$ as reference time. Hence, by virtue of (5.37) and (5.38), from (6.10) we get

(6.12) $$\ell(t_\star, t; t_0) = c \int_{t_2}^{t} \frac{dt'}{a(t', t_0)} = \frac{c}{\sqrt{c_0}} \int_{t_2}^{t} \frac{dt'}{\sqrt{\cosh(\beta t') - 1}}$$

where $c_0 \simeq 0.2299194811$. For $t = t_0$ we find

(6.13) $$R_{ph}(t_0) \overset{\text{def}}{=} \ell(t_\star, t_0; t_0) \simeq 45.627196784 \text{ Gly}$$

The meaning of this length is the following: a radiating body $B$ is currently not visible by an observer $O$ if

(6.14) $$R_{ph}(t_0) < \ell^*_{BO},$$

where $\ell^*_{BO}$ is the present-day proper distance of $B$ and $O$. $R_{ph}(t_0)$ is called the (present-day) radius of the visible universe or radius of the particle horizon\(^5\). The result (6.13) is in good agreement with the current estimate of $\simeq 46$ billion light years.

\(^5\) For this evaluation we follow the same procedure as for $R_{eh}(t_0)$, footnote of page 88. Since $t_\star \simeq 0.000378 \text{ Gyr}$ the formula to be used for this computation is

$$1.63233 * \int_{0.000378}^{13.816581} \frac{dx}{\sqrt{\cosh(0.1709 \times x) - 1}}.$$

\(^6\) See for instance [11] (Section 2.2) and [15].
6.4 Red-shift

Let us go back to the end of Section 6.1, p. 86. Assume that the emission time \( t_e \) of the second photon is very close to \( t_e \) (Fig. 6.7). From Fig. 6.8 we infer that the areas over the intervals \( I_{er} = [t_e, t_r] \) and \( I_{er} = [t_e, t_r] \) are both equal to the co-moving distance \( \ell_{BO} \). Since the central blank area is a common part of these two areas, these two shaded areas are equal. If the magnitude of the intervals \( I_e = [t_e, t_e] \) and \( I_r = [t_r, t_r] \) are ‘extremely smaller’ than the intervals \( I_{er} \) and \( I_{er} \), then the shaded areas can be considered equal to width \( \times \) height of the rectangles where they are contained. So, we can write with ‘great precision’ \(^7\)

\[
\frac{I_e}{a(t_e, t_f)} = \frac{I_r}{a(t_r, t_f)}
\]

i.e.,

\[
\frac{a(t_r, t_f)}{a(t_e, t_f)} = \frac{I_r}{I_e}
\]

\[(6.15)\]

\(^7\) This argument is taken, with minor modifications, from [8], pp.126-127.
This argument can be correctly applied when the two events of emission of the two photons correspond to successive crests of a monochromatic light-wave emitted by $B$ with wavelength

$$\lambda_e = c I_e.$$ 

In this case the two reception-events correspond to successive crests of the same light-wave received by $O$ with wavelength

$$\lambda_r = c I_r.$$ 

Then (6.15) is translated into equation

$$\frac{a(t_r, t_2)}{a(t_e, t_2)} = \frac{\lambda_r}{\lambda_e}$$

This formula describes the well-known spectral shift phenomenon:

\[
\begin{align*}
\{ \ a(t_r, t_2) > a(t_e, t_2) & \iff \lambda_r > \lambda_e \iff \\
\ a(t_r, t_2) < a(t_e, t_2) & \iff \lambda_r < \lambda_e \iff \\
\} \iff \text{shift of the original wavelength towards the red} \\
\iff \text{shift of the original wavelength towards the blue}
\end{align*}
\]

**Remark 6.1** - The term ‘shift’ sounds like ‘translational displacement’, and this may cause a misunderstanding. In fact, if we write (6.16) in the form

$$\lambda_r = \frac{a(t_r, t_2)}{a(t_e, t_2)} \lambda_e$$

then we observe that the spectrum of a galaxy is multiplied by $a(t_r, t_2)/a(t_e, t_2)$ and not translated as a whole.  •
6.5 Red-shift versus the emission-time

If we introduce the so-called red-shift parameter

\[ z \overset{\text{def}}{=} \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} - 1 \]

then equation (6.16) can be written as

\[ \frac{a(t_r, t_r)}{a(t_e, t_r)} = 1 + z \]

This formula can be used for determining the time of emission \( t_e \) of a photon from a galaxy \( B \) knowing the red-shift \( z \) observed from another galaxy \( O \). Indeed, since the reception time \( t_r \) is equal to the today time \( t_0 \), then by taking the reference time \( t_r \) of the profile \( a(t_r, t_r) \) equal to \( t_0 \) equation (6.18) reduces to

\[ \frac{1}{a(t_e, t_0)} = 1 + z \]

because \( a(t_o, t_o) = 1 \). As a consequence, if we know the analytic expression of the profile \( a(t, t_o) \) then from (6.19) we can extract the emission time \( t_e \) as a function of \( z \).

We apply this result to the barotropic flat model with \( u = 1 \) (pure matter).

**Theorem 6.1** - If we know the present-day red-shift \( z \) of a galaxy \( B \) observed from a galaxy \( O \), then the dimensionless emission time \( x_e \) is given by

\[ x_e = \arccosh(y) = \log \left( y + \sqrt{y^2 - 1} \right) \]

\[ y \overset{\text{def}}{=} \frac{1}{c_0(1 + z)^3} + 1 = \frac{1}{c_0} \left( \frac{\lambda_e}{\lambda_r} \right)^3 + 1, \quad c_0 \approx 0.2299194811 \]

**Proof** - From (5.37) we derive \( a^3(1; x, x_o) = c_0 \left( \cosh(x) - 1 \right) \)

\[ \Rightarrow c_0^{-1} a^3(1; x, x_o) = \cosh(x) - 1 \Rightarrow \cosh(x) = c_0^{-1} a^3(1; x, x_o) + 1 \]

\[ \Rightarrow x = \arccosh[c_0^{-1} a^3(1; x, x_o) + 1] \Rightarrow x_e = \arccosh[c_0^{-1} a^3(1; x_e, x_o) + 1]. \]

By virtue of (6.19) we obtain the emission dimensionless time \( x_e \). ■

The emission time \( t_e \) in Gyr is given by \( t_e = x_e/\beta \) with \( \beta^{-1} \approx 5.8513378 \) Gyr (table 5.2, page 68). Thus, the formula to be used for computing the emission cosmic time \( t_e \) is

\[ t_e(z) = 5.8513378 \times \arccosh \left( \frac{1}{0.22991948 \times (1 + z)^3} + 1 \right) \]
### 6.5. Red-shift versus the emission-time

#### Figure 6.9: Red-shift $z$ versus the emission time $t_e$.

#### Table 6.1: Red-shift $z$ versus the emission time $t_e$.

| red-shift $z$ | emission time $t_e$ (Gyr) |
|---------------|---------------------------|
| 0.0           | 13.8166                   |
| 0.0005        | ...                       |
| 0.001         | ...                       |
| 0.002         | ...                       |
| 0.003         | ...                       |
| 0.004         | ...                       |
| 0.005         | ...                       |
| 0.006         | ...                       |
| 0.1           | 12.4646                   |
| 0.2           | 11.2925                   |
| 0.3           | 10.2722                   |
| 0.4           | 9.3808                    |
| 0.5           | 8.5990                    |
| 0.6           | 7.9107                    |
| 0.7           | 7.3027                    |
| 0.8           | 6.7633                    |
| 0.9           | 6.2832                    |

| red-shift $z$ | emission time $t_e$ (Gyr) |
|---------------|---------------------------|
| 1.0           | 5.8543                    |
| 1.1           | 5.4696                    |
| 1.2           | 5.1234                    |
| 1.3           | 4.8109                    |
| 1.4           | 4.5278                    |
| 1.5           | 4.2705                    |
| 1.6           | 4.0360                    |
| 1.7           | 3.8216                    |
| 1.8           | 3.6251                    |
| 1.9           | 3.4445                    |
| 2.0           | 3.2782                    |
| 2.1           | 3.1246                    |
| 2.2           | 2.9824                    |
| 2.3           | 2.8505                    |
| 2.4           | 2.7280                    |
| 2.5           | 2.6138                    |
| 2.6           | 2.5073                    |
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