Cosmic ray threshold in an asymptotically dS spacetime

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Abstract

We discuss the threshold anomaly in ultra-high energy cosmic ray physics by assuming that the matter world just be perturbation of a de Sitter universe, which is consistent with the recent astronomical observations: about two thirds of the whole energy in the universe is contributed by a small positive cosmological constant. One-particle states are presented explicitly. It is noticed that the dispersion relation of free particles is dependent on the degrees of freedom of angular momentum. This fact can be regarded as the effects of the cosmological constant on kinematics of particles.

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1 Introduction

The origin of the ultra high energy cosmic rays (UHECRs) is one of the outstanding puzzles of modern astrophysics. Present understanding of the phenomena responsible

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for the production of UHECR is still limited. Currently, there are generally two scenarios to produce the UHECRs. One is the “bottom-up” acceleration scenario with some astrophysical objects as sources\cite{1, 2}. The other is called “top-down” scenario in which UHECR particles are from the decay of certain sufficiently massive particles originating in the early Universe\cite{3}.

Decades ago, Greisen, Zatsepin and Kuz’mín (GZK)\cite{4} discussed the propagation of UHECRs through the cosmic microwave background radiation (CMBR). Due to photopion production process by the CMBR, the UHECR particles will lose their energies drastically down to a theoretical threshold, which is about $5 \times 10^{19}\text{eV}$. The mean free path for this process is only a few Mpc\cite{5}. This is the so-called GZK cutoff. However, we have observed indeed hundreds of events with energies above $10^{19}\text{eV}$ and about 20 events above $10^{20}\text{eV}$\cite{6, 7, 8, 9, 10}. The puzzle can be considered to be some cosmic ray threshold anomaly: energy of an expected threshold is reached but the threshold has not been observed yet. Although some physicists argued that the GZK cutoff might actually have already been seen\cite{11}, it need more evidences and careful analysis to become the consensus in scientific circles. In this paper, we will take the existence of UHECR paradox as an acceptable working assumption, on the basis of which we try to unravel the puzzle.

The cosmic-ray paradox was suggested to be solved by the departures from Lorentz symmetry in the end of the last century\cite{12, 13}. The basic idea is to introduce tiny Lorentz-violating terms into the standard model Lagrangian and derive a parameter-dependent threshold. As long as the parameters are taken some appropriate values, the threshold will be above the energies of all observed events. A related but alternative scheme is proposed\cite{14}, in which Lorentz symmetry is broken by Planck-scale effects. Another approach is to assume deformed Lorentz symmetry, constructing the theory of doubly special relativity (DSR)\cite{15, 16, 17}. In the approach, a new dispersion relation is written down and then an enhanced threshold is obtained.

The recent astronomical observations on supernovae\cite{18, 19} and CMBR\cite{20} show that about two thirds of the whole energy in the Universe is contributed by dark energy. The simplest model for dark energy is that it is devoted by a small positive cosmological constant ($\Lambda$). Then, the Universe can be regarded as a de Sitter (dS) spacetime in the zeroth approximation. The physics in dS spacetime has been discussed extensively (see, for example,\cite{21, 22, 23}).

In our previous paper\cite{24}, we have discussed the possibility of the cosmological constant as the origin of threshold anomaly in a dS spacetime. In a kind of simplified case, we obtained a positive conclusion. However there are two imperfections in Ref.\cite{24}, which are also motivations for this paper. Firstly, although the dominated ingredient in the Universe is dark energy, the matter also contributes about 1/3 to the density of the Universe. The challenging task is to find a way of dealing with the universe dominated by the dark energy and matter. At least at the present stage of the evolution of the
universe, one has few reasons to omit effects of any of the two ingredients. Therefore, we need to investigate the Universe including matter in order to more approach to the Universe nowadays. Secondly, in [24], only some simplified cases are considered, which are not necessary the true ones in our Universe and will be improved on in current paper.

In this paper, we present a scenario in which the dark energy is supposed to come from the cosmological constant and that the spacetime is a dS one, and the matter in the Universe is dealt with as a perturbation around the dS background. The perturbation of the spacetime geometry is derived firstly. Then we discuss the motion of a free particle in the asymptotically de Sitter Universe, and its kinetics is set up primarily. Meanwhile, we derive a general form of dispersion relation for free particles moving in the Universe. This formalism is used to investigate the UHECR propagating in the cosmic microwave background. We obtain explicitly the corrections of the GZK threshold for the UHECR particles interacting with soft photons, which are dependent on the cosmological constant as supposed in the beginning of the paper. We show how the threshold varies with a positive cosmological constant and additional degrees of freedom of the angular momentums of interacting particles. It should be noticed that, for a positive cosmological constant, the theoretic threshold tends to be above the energies of all the observed events. Thus, we may conclude that the tiny but nonzero cosmological constant is a possible origin of the threshold anomaly of the UHECR.

The paper is organized as follows. In Section 2, we discuss the Friedmann equation for a constant curvature spacetime with a homogeneous density perturbation. An explicit solution including metric and corresponding connection is presented. The section 3 is devoted to the investigation of kinematics. Approximate conservation laws of momentum and angular momentum are obtained along the geodesics. By solving equations of motion of a free particle, we obtain a remarkable dispersion relation, which includes degrees of freedom of angular momentum. In Section 4, by taking effects of a tiny but nonzero positive cosmological constant into account, we show that the theoretic threshold is above the energies of all the observed UHECR events. In the last section, we present conclusions and remarks.

2 Perturbation of the Friedmann equations

The Einstein equation with cosmological constant is

$$R_{ab} - \frac{1}{2}g_{ab}R - \Lambda g_{ab} = -8\pi G T_{ab}.$$  (1)

The stress-energy tensor in Eq. (1) can be of the general perfect fluid form

$$T_{ab} = (\rho + p)U_a U_b - p g_{ab},$$  (2)
where \( p, \rho \) are the proper pressure and energy density, respectively.

While observations of the distribution of galaxies in our Universe show clustering of galaxies on a wide range of distance scales, on the largest scales the galaxy distribution appears to be homogeneous and isotropic, namely, \( p \) and \( \rho \) are only functions of variable \( t \). The homogeneous and isotropic universe is described by the Robertson-Walker metric

\[
ds^2 = dt^2 - \hat{a}^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right],
\]

where \( k \) denotes spatial curvature of the spacetime.

Thus, the general evolution equations for homogenous, isotropic universe read

\[
3\ddot{\hat{a}} = -4\pi G(\rho + 3p)\dot{\hat{a}}
\]

\[
\dot{\hat{a}}\ddot{\hat{a}} + 2\dot{\hat{a}}^2 + 2k = 4\pi G(\rho - p)\dot{\hat{a}}^2,
\]

where \( \rho = \rho_\Lambda + \rho_m \), \( \rho_\Lambda \) and \( \rho_m \) denote the energy density contributed from cosmological constant and matter, respectively.

The stress-energy tensor should be covariantly conserved, i.e., \( T^a_{\ b} = 0 \), which reduces to

\[
\dot{\rho} = -3\frac{\dot{\hat{a}}}{\hat{a}}(\rho + p).
\]

It should be noticed that \( \rho_m/\rho_\Lambda \) is a small parameter for the evolution of the Universe (at least in the present stage). To discuss deviation of physical quantities in an asymptotically de Sitter spacetime from those in a dS one, we would introduce a small parameter \( \epsilon \) characterizing the effect of the matter on the spacetime geometry. Then, one can assume the scale factor \( a(t) \) has a perturbation by the matter density \( \rho_m \) around a dS background,

\[
\dot{a}(t) = a(t)[1 + \epsilon f(t)] = R \cosh(t/R)[1 + \epsilon f(t)],
\]

where \( R := \sqrt{3/\Lambda} \).

From the first approximation of the covariant conservation equation

\[
\dot{\rho} + 3H(\rho + p) = 0,
\]

and

\[
\rho = \rho_\Lambda + \rho_m = \frac{3}{8\pi G} \left( \frac{1}{R^2} + \epsilon \rho(t) \right),
\]

one can obtain

\[
\epsilon f(t) = \frac{\rho_0 R^2}{2} \left( \tanh(t/R) \cot^{-1}[\sinh(t/R)] - \frac{1}{\cosh(t/R)} \right).
\]
In the derivation, the asymptotic condition \( \lim_{t \to \infty} f(t) \to 0 \) has been used. The asymptotically dS spacetime with the above form of matter can be realized as a four dimensional hypersurface embedded in a five dimensional flat space

\[
ds^2 = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 - (d\xi^4)^2,
\]

such that

\[
(\dot{\xi}^0)^2 - (\dot{\xi}^1)^2 - (\dot{\xi}^2)^2 - (\dot{\xi}^3)^2 - (\dot{\xi}^4)^2 = -\dot{R}^2(t) .
\]

\( \dot{\xi}^\mu (\mu = 0, 1, 2, 3, 4 ) \) is related to the coordinate \((t, r, \theta, \phi)\) in Eq. (3) as follows

\[
\dot{\xi}^0(t = 0) = \int_0^t \sqrt{1 + \dot{a}^2(\tau)} d\tau , \quad \dot{\xi}^A = \xi^A(1 + \epsilon f) , \quad (A = 1, 2, 3, 4) ,
\]

where

\[
\begin{align*}
\xi^1 &= r \cosh(t/R) \sin \theta \cos \phi, \\
\xi^2 &= r \cosh(t/R) \sin \theta \sin \phi, \\
\xi^3 &= r \cosh(t/R) \cos \theta, \\
\xi^4 &= \sqrt{R^2 - r^2 \cosh(t/R)}. 
\end{align*}
\]

Applying Eqs. (6) and (9) to the expression of \( \dot{\xi}^0 \) in (12), one obtains

\[
\dot{\xi}^0 = \xi^0 + \epsilon R \cosh^2(t/R) \sinh(t/R) f(t) + \epsilon \rho_0 R^3 (\coth(2t/R) - 1) ,
\]

where

\[
\xi^0 = R \sinh(t/R).
\]

Similarly, in the course of deriving Eq. (14), the condition \( \lim_{t \to \infty} \dot{\xi}^0 = \xi^0 \) is used. \( \dot{R}(t) \) in Eq. (11) reads

\[
\dot{R}(t) = R \left( 1 - \epsilon \rho_0 R^2 \sinh(t/R)(\coth(2t/R) - 1) \right) .
\]

It is obvious that

\[
\lim_{t \to \infty} \dot{R}(t) = R .
\]

For convenience, we write the metric in terms of the Beltrami coordinates \( x^a = R \frac{\xi^a}{\xi^4} \), \((\xi^4 \neq 0 , a = 0, 1, 2, 3)\) [26]-[29]. Due to the additional matter, the metric of the 4-dimensional spacetime in Beltrami coordinates has a perturbed form

\[
ds^2 = \hat{g}_{ab} dx^a dx^b = (g_{ab} + \epsilon h_{ab}) dx^a dx^b ,
\]

5
where \( g_{ab} \) is the metric in the empty dS spacetime with the form

\[
g_{ab} = \frac{\eta_{ab}}{\sigma} + \frac{\eta_{ac} \eta_{bd} x^c x^d}{\sigma^2 R^2}, \quad \eta_{ab} = \text{diag}(1, -1, -1, -1),
\]

\[
\sigma := \sigma(x, x) = 1 - \frac{\eta_{ab} x^a x^b}{R^2} > 0
\]

and \( h_{ab} \) is the perturbation from the additional matter.

From Eq. (18), the following results can be obtained

\[
h_{0a} = 0, \quad h_{ij} = -2 f(t) \frac{\sigma R^2 + (x^0)^2}{\sigma(x) R^2} \, 3 \, g^B_{ij} = -2 f(t) R^{-2} a^2(t) \, 3 \, g^B_{ij}, \quad i, j = 1, 2, 3;
\]

where \( 3 \, g^B_{ij} \) is the Beltrami metric on 3-sphere

\[
3 \, g^B_{ij} = -\frac{\delta_{ij}}{\sigma_3(x)} - \frac{\delta_{ik} \delta_{jl} x^k x^l}{\sigma_3^2(x) R^2},
\]

\[
\sigma_3(x) = 1 + R^{-2} \delta_{ij} x^i x^j.
\]

In order to investigate the motion of free particles, Christoffel coefficients are listed as follows

\[
\hat{\Gamma}^a_{bc} = \Gamma^a_{bc} + \varepsilon \tilde{\Gamma}^a_{bc}, \quad \Gamma^a_{bc} = (\eta_{cd} \delta^a_c + \eta_{cd} \delta^b_c) \frac{x^d}{R^2 \sigma(x)};
\]

\[
\tilde{\Gamma}^0_{0i} = \frac{\rho_0}{2} \tan(t/R) \cot^{-1}[\sinh(t/R)] x^i, \quad \tilde{\Gamma}^0_{ij} = \frac{\rho_0 R}{2} \{ \cosh(t/R)[1 + \tan^2(t/R)] \cot^{-1}[\sinh(t/R)] - 2 \tan(t/R) \} \sigma^{1/2} 3 \, g^B_{ij} \]

\[
= -\frac{\rho_0}{R} \tan(t/R) \cot^{-1}[\sinh(t/R)] \frac{x^i x^j}{\sigma_3^{3/2}},
\]

\[
\tilde{\Gamma}^0_{ij} = \frac{\rho_0 R}{2 \sqrt{\sigma_3}} \cot^{-1}[\sinh(t/R)] \delta^i_j, \quad \tilde{\Gamma}^k_{ij} = -\frac{\rho_0}{2} \tan(t/R) \cot^{-1}[\sinh(t/R)] \frac{x^i \delta^k_j + x^j \delta^k_i}{\sigma_3}.
\]

It is not difficult to calculate the deviation of the geodesics in an asymptotically dS spacetime from those in the empty dS one

\[
0 = \frac{d^2 x^a}{ds^2} + \hat{\Gamma}^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = \sigma \frac{d}{ds} \left( \frac{1}{\sigma} \frac{dx^a}{ds} \right) + \varepsilon \tilde{\Gamma}^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds}.
\]
In the asymptotically dS spacetime with homogeneous matter, one can define the 4-momentum formally as

$$\hat{P}^a := \frac{1}{\sigma} \frac{dx^a}{ds}, \quad \text{and} \quad \hat{E} := \hat{P}^0.$$  

From Eq. (23), one has

$$\frac{d\hat{P}^a}{dx^0} = -\epsilon \tilde{\Gamma}^a_{bc} \hat{P}^b dx^c \Rightarrow \frac{d\hat{P}^a}{dx^0} = -\epsilon \tilde{\Gamma}^a_{bc} \hat{P}^b v^c,$$

where $v^c$ is a constant defined as

$$v^a := \frac{dx^a}{dx^0}.$$  

For the particle moving along a one-dimensional curve, one can set $x^2 = x^3 = 0$ without loss of generality. That is, one can discuss the motion of a free particle in $x^0 - x^1$ plane. In this case, the deviations of energy and momentum in the perturbed dS universe from those in the empty dS universe can be calculated numerically. In Fig. 1, we present a plot for the relative errors of energy and momentum vs. the time coordinate $x^0$ (we use the convention $c = 1$ throughout the paper).

![Figure 1: The solid line describes the relative error of one-dimensional momentum for a free particle moving in the perturbed dS universe, while the dashed line describes its relative error of energy.](image)

These results show clearly that, as a feasible approximation, we can substitute energy and momentum in the empty dS universe for the counterparts in perturbed one provided that the distance between source of UHECRs and the earth is less than 100Mpc.
3 Kinematics around dS spacetime

In the asymptotically dS spacetime with homogeneous matter one can still define the five dimensional angular momentum $M^{\mu\nu}$ of a free particle with mass $m_0$ as the form

$$\hat{M}^{\mu\nu} = m_0 \left( \hat{\xi}^\mu \frac{d\hat{\xi}^\nu}{ds} - \hat{\xi}^\nu \frac{d\hat{\xi}^\mu}{ds} \right),$$  \hspace{1cm} (25)

where $s$ is a parameter along the geodesic. In the Universe, there is no translation invariance and so that one can not introduce a momentum vector. However, it should be noticed that, at least somehow, one may define formally a counterpart of the 4–momentum $P$ for a free particle in the Universe as

$$\hat{P}^a := R^{-1} \hat{M}^{4a} = m_0 \sigma^{-1} \frac{dx^a}{ds}.$$  \hspace{1cm} (26)

In the same manner, the counterparts of the four dimensional angular momentum $J^{ab}$ can be assigned as $\hat{J}^{ab} := \hat{M}^{ab}$.

It is easy to show that $\xi^0 := \sigma(x, x)^{-1/2} x^0$ is invariant under the spatial transformations. Thus, we can say two spacelike events are simultaneous if they satisfy

$$\sigma(x, x)^{-1} x^0 = \xi^0 = \text{const.}.$$  \hspace{1cm} (27)

Therefore, it is convenient to discuss physics of the asymptotically dS spacetime in the coordinate ($\xi^0, x^a$). In this coordinate, the metric can be rewritten into the form

$$ds^2 = \frac{d\xi^0 d\xi^0}{1 + \lambda \xi^0 \xi^0} - (1 + \lambda \xi^0 \xi^0)(1 + 2 \epsilon f) \left[ \frac{d\rho^2}{(1 + \rho^2)^2} + \frac{\rho^2}{1 + \lambda \rho^2} d\Omega^2 \right],$$  \hspace{1cm} (28)

where $\rho^2 := \sum_{a=1}^3 x^a x^a$ and $d\Omega^2$ denotes the metric on 2-dimensional sphere $S^2$.

The Klein-Gordon equation describes motions of a scalar field $\phi(x)$,

$$\left( \Box - m_0^2 \right) \phi(x) = 0,$$  \hspace{1cm} (29)

where $\Box$ is the d’Alembertian operator defined as

$$\Box := -\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b).$$  \hspace{1cm} (30)

The Dirac equation for the spinor field $\Psi(x)$ has the form

$$[-i \gamma^a (\partial_a - \Gamma_a) + m_0] \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = 0,$$  \hspace{1cm} (31)

where $\Gamma_a$ is the Ricci rotational coefficient. For free spin 1/2 particles, it is easy to check that the components $\psi_\alpha(x)$ of a spinor satisfy the relation

$$\left( \Box - m_0^2 \right) \psi_\alpha(x) = 0, \quad \alpha = 1, 2.$$  \hspace{1cm} (32)
In Lorentz gauge, one can simplify the Maxwell equation without source as
\[ \Box A^a = 0. \] (33)

From Eqs. \([29], [32]\) and \([33]\), one knows that scalar fields and components of spinor and vector fields can be described uniformly as \((m_0 = 0 \text{ for vector field})\)
\[ \left( \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b) - m_0^2 \right) \Phi(\xi^0, x^i) = 0. \] (34)

In the coordinates \((\xi^0, x^i)\), one can rewrite the d’Alembertian operator as the following form
\[ \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b) = -(1 + \lambda \xi^0 \xi^0) \partial_{\xi^0}^2 - \left( 4 \lambda \xi^0 (1 + \lambda \xi^0 \xi^0) \frac{df}{d\xi^0} \right) \partial_{\xi^0} \]
\[ + (1 - 2 \epsilon f) \left( 1 + \lambda \xi^0 \xi^0 \right)^{-1} (1 + \lambda \rho^2) \left[ \partial^2 \rho + 2 \rho^{-1} \partial \rho \right] \]
\[ + (1 - 2 \epsilon f) \left( 1 + \lambda \xi^0 \xi^0 \right)^{-1} (1 + \lambda \rho^2) \rho^{-2} \partial^2 u, \] (35)

where \(\lambda = R^{-2}, uu' = 1\) and \(\partial^2 u\) denotes the Laplacian operator on \(S^2\).

To solve the equation of motion, one writes the field \(\Phi(\xi^0, x^i)\) into the form
\[ \Phi(\xi^0, \rho, u) = T(\xi^0) U(\rho) Y_{lm}(u). \]

Thus, one transforms the equation of motion into \([23][29]\),
\[ \left[ (1 + \lambda \xi^0 \xi^0)^2 (1 + 2 \epsilon f) \partial_{\xi^0}^2 + \left( 4 \lambda \xi^0 (1 + \lambda \xi^0 \xi^0) (1 + 2 \epsilon f) + 3 \epsilon \frac{df}{d\xi^0} (1 + \lambda \xi^0 \xi^0)^2 \right) \partial_{\xi^0} \right. \]
\[ \left. + m_0^2 (1 + 2 \epsilon f) (1 + \lambda \xi^0 \xi^0) + (\epsilon^2 - m_0^2) \right] T(\xi^0) = 0, \]
\[ \left[ \partial^2 + \frac{2}{\rho} \partial_{\rho} - \left( \frac{m_0^2 - \epsilon^2}{(1 + \lambda \rho^2)^2} + \frac{l(l + 1)}{\rho^2 (1 + \lambda \rho^2)} \right) \right] U(\rho) = 0, \]
\[ \left[ \partial^2_u + l(l + 1) \right] Y_{lm}(u) = 0, \] (36)

where \(Y_{lm}(u)\) is the spherical harmonic function and \(\epsilon\) is the constant from separating variables.

For the irrelevance of the expression of \(T(\xi^0)\) to our discussion, we can focus our attention on the last two equations. The solutions for the radial equation of the field is
\[ U(\rho) \sim \rho^l (1 + \lambda \rho^2)^{k/2} F \left( \frac{1}{2} (l + k + 1), \frac{1}{2} (l + k), l + \frac{3}{2}; - \lambda \rho^2 \right), \] (37)

where \(k\) denotes the radial quantum number
\[ k^2 - 2k - \lambda^{-1}(\epsilon^2 - m_0^2) = 0 . \]
To be normalizable, the hypergeometric function in the radial part of the wavefunction has to break off, leading to the quantum condition
\[ l + k = -2n, \quad (2n \in \mathbb{N}). \] (38)

Then, one obtains the dispersion relation for a free particle moving in the Universe
\[ E^2 = m_0^2 + \varepsilon^2 + \lambda(2n + l)(2n + l + 2), \] (39)
where the term \( \varepsilon^2 \) denotes what is independent of the parameters \( n \) and \( l \).

### 4 UHECR threshold

In this section, we investigate the UHECR threshold in the covariant framework of kinematics in an asymptotically dS spacetime set up in the preceding sections.

One considers the head-on collision between a soft photon of energy \( E_\gamma \), momentum \( \mathbf{q} \) and a high energy particle \( m_1 \) of energy \( E_1 \), momentum \( \mathbf{p}_1 \), which leads to the production of two particles \( m_2, m_3 \) with energies \( E_2, E_3 \) and momentums \( \mathbf{p}_2, \mathbf{p}_3 \), respectively. From the energy and momentum conservation laws
\[ E_1 + E_\gamma = E_2 + E_3, \]
\[ p_1 - q = p_2 + p_3. \] (40)

In the C. M. frame, \( m_2 \) and \( m_3 \) are at rest when the threshold is reached, so they have the same velocity in the lab frame and there exists the following relation
\[ \frac{p_2}{p_3} = \frac{m_2}{m_3}. \] (41)

It is convenient to use the approximate formulae of dispersion relations \( E_\gamma^2 = q^2 + \lambda_\gamma^*, \) for the soft photons and the ultra high energy particles
\[ E_i = \sqrt{m_i^2 + \frac{p_i^2}{2} + \lambda_i^*} \approx p_i + \frac{m_i^2}{2p_i} + \frac{\lambda_i^*}{2p_i} , \quad (i = 1, 2, 3). \] (43)

where \( \lambda_\gamma^* := \lambda(l_\gamma + 2n_\gamma)(l_\gamma + 2n_\gamma + 2) \approx \lambda l_\gamma(l_\gamma + 2) \) and \( \lambda_i^* := \lambda(l_i + 2n_i)(l_i + 2n_i + 2) \approx \lambda l_i(l_i + 2) \) with the conjecture that \( l \gg n \).

The obtained threshold can be expressed as the form
\[ E_{\text{th}, \lambda} \approx \frac{(m_2 + m_3)^2 - m_2^2 + \lambda_2^* \left(1 + \frac{m_3}{m_2}\right) + \lambda_3^* \left(1 + \frac{m_2}{m_3}\right) - \lambda_1^*}{2 \left(E_\gamma + \sqrt{E_\gamma^2 - \lambda_\gamma^*}\right)}. \] (44)
The usual GZK threshold could be recovered when the parameter $\lambda^*$, which is dependent on the cosmological constant, runs to zero.

The conservation law of the angular momentum imposes a constraint on the parameters $\lambda^*$,

$$\lambda_1^* + \lambda_2^* + 2\lambda L_1 \cdot L_\gamma = \lambda_2^* + \lambda_3^* + 2\lambda L_2 \cdot L_3.$$  

Making use of the relation, one can rewrite the $\lambda^*$ dependent terms of the threshold as the following

$$\frac{\lambda_2^* m_2 + \lambda_3^* m_3 + \lambda_2^* + 2\lambda L_1 \cdot L_\gamma - 2\lambda L_2 \cdot L_3}{2 (E_\gamma + \sqrt{E^2_\gamma - \lambda_\gamma^*})}. \quad (45)$$

If $\lambda_2^*$ and $\lambda_3^*$ take value of the same order with $\lambda_\gamma^*$ (less than the square of energy of a soft photon), the $\lambda^*$ dependent terms can be omitted \cite{24}. We will investigate the case of $\lambda_2^* + \lambda_3^* \gg \lambda_\gamma^*$, and the threshold (44) is of the form

$$E_{\text{th}, \lambda} \simeq \frac{(m_2 + m_3)^2 - m_2^2 + \lambda_2^* m_3 + \lambda_3^* m_3 - 2\lambda L_2 \cdot L_3}{2 (E_\gamma + \sqrt{E^2_\gamma - \lambda_\gamma^*})}. \quad (46)$$

Now, we can study the photopion production processes of the UHECR interaction with the CMBR

$$p + \gamma \rightarrow p + \pi.$$  

The corresponding threshold for this process is given by

$$E_{\text{th}, \lambda, \pi} \simeq \frac{(m_N + m_\pi)^2 - m_N^2 + \lambda_N^* m_\pi + \lambda_\pi^* m_N - 2\lambda L_N \cdot L_\pi}{2 (E_\gamma + \sqrt{E^2_\gamma - \lambda_\gamma^*})}. \quad (47)$$

To show the behavior of the threshold in the $\lambda^*$-parameter space clearly, we should discuss some limit cases in detail.

In the case that the out-going nucleon has zero angular momentum, the threshold (47) reduces as

$$E_{\text{th}, \lambda, \pi} \simeq \frac{(m_N + m_\pi)^2 - m_N^2 + \lambda_\pi^* m_N}{2 (E_\gamma + \sqrt{E^2_\gamma - \lambda_\gamma^*})}. \quad (48)$$

we provide a plot for the dependence of the threshold $E_{\text{th}, \lambda, \pi}$ on the cosmological constant and angular momentums (the in-coming photon and out-going pion) in Fig. 2.

In the case that the out-going pion has zero angular momentum, the UHECR threshold takes the form

$$E_{\text{th}, \lambda, N} \simeq \frac{(m_N + m_\pi)^2 - m_N^2 + \lambda_\pi^* m_N}{2 (E_\gamma + \sqrt{E^2_\gamma - \lambda_\gamma^*})}. \quad (49)$$

We present a plot for the dependence of the threshold $E_{\text{th}, \lambda, N}$ on the cosmological constant and angular momentums (the in-coming soft photon and out-going nucleon) in Fig. 3.
Figure 2: The cosmological constant and angular momentums (of in-coming soft photon and out-going pion) dependence of the threshold $E_{th,\lambda,\pi}^{UHECR}$ in the interaction between the UHECR protons and the CMBR photons ($\lambda^*_\pi$ in units of $m^2_\pi/10$).

Figure 3: The cosmological constant and angular momentums (of the in-coming soft photon and out-going nucleon) dependence of the threshold $E_{th,\lambda,N}^{UHECR}$ in the interaction between the UHECR protons and the CMBR photons ($\lambda^*_N$ in units of $m^2_N/10$).
Finally, if the out-going pion and nucleon have the same angular momentum, the UHECR threshold can be expressed as the following form

$$E_{\text{UHECR}}^{\text{th,} \lambda_N\pi} \simeq \frac{(m_N + m_\pi)^2 - m_N^2 + \lambda_N^* \left( \frac{m_N}{m_\pi} + \frac{m_\pi}{m_N} - 2 \right)}{2 \left( E_\gamma + \sqrt{E_\gamma^2 - \lambda_\gamma^*} \right)}. \quad (50)$$

We provide a plot for the dependence of the threshold $E_{\text{UHECR}}^{\text{th,} \lambda_N\pi}$ on the cosmological constant and angular momentums (the in-coming soft photon and out-going nucleon and pion) in Fig. 4.

From the above discussion, one can conclude that a tiny but nonzero cosmological constant may provide indeed sufficient corrections to the primary predicted threshold[4]. For the observed cosmological constant (which is around the level of $10^{-85}\text{GeV}^2$), if the CMBR possesses a quantum number $l_\gamma$ of the order of $10^{30}$, the threshold will be above the energies of all those observed UHECR particles. The predicted threshold should be upgraded to a more reasonable level. Now we can say that a possible origin of the cosmic ray threshold anomaly has been achieved. It is the cosmological constant that increases the GZK cut-off to a level above the observed UHECR events.
5 Conclusions and remarks

In this paper, we have showed a cosmological scenario of constant curvature spacetime with a homogeneous density perturbation. This is in agreement with the astronomical observations on supernovae and CMBR that about two thirds of the whole energy in the Universe is from dark energy and the lowest order description of the Universe may be a de Sitter spacetime. The matter world was dealt with as a perturbation around the de Sitter background within the framework of standard cosmology.

Kinematics in an asymptotically dS spacetime and, in particular, the perturbation around the dS background was presented. The exact conservation law of momentum and angular momentum in dS is violated by virtue of the additional homogeneous density perturbation. Numerical simulations show that, within the range that we are interested, the violation of the conservation law is small. A general form of dispersion relation for free particles moving in the Universe was obtained.

The perturbation around dS spacetime has been used to discuss the UHECR threshold anomaly. We obtained explicitly the corrections of the GZK threshold for the UHECR particles interacting with soft photons, which are dependent on the cosmological constant. We showed how the threshold varies with a positive cosmological constant and additional degrees of freedom of the angular momentums of interacting particles. It should be noticed that, for a positive cosmological constant, the theoretic threshold tends to be above the energies of all the observed events. Thus, we may conclude that the tiny but nonzero cosmological constant is a possible origin of the threshold anomaly of the UHECR.

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