Twisted Quantum Deformations of Lorentz and Poincaré algebras

V.N. Tolstoy †

Institute of Nuclear Physics, Moscow State University, 119 992 Moscow, Russia; e-mail: tolstoy@nucl-th.sinp.msu.ru

Abstract

We discussed twisted quantum deformations of $D = 4$ Lorentz and Poincaré algebras. In the case of Poincaré algebra it is shown that almost all classical $r$-matrices of S. Zakrzewski classification can be presented as a sum of subordinated $r$-matrices of Abelian and Jordanian types. Corresponding twists describing quantum deformations are obtained in explicit form. This work is an extended version of the paper arXiv:0704.0081v1[math.QA].

1 Introduction

The quantum deformations of relativistic symmetries are described by Hopf-algebraic deformations of Lorentz and Poincaré algebras. Such quantum deformations are classified by Lorentz and Poincaré Poisson structures. These Poisson structures given by classical $r$-matrices were classified already some time ago by S. Zakrzewski in [1] for the Lorentz algebra and in [2] for the Poincaré algebra. In the case of the Lorentz algebra a complete list of classical $r$-matrices involves the four independent formulas and the corresponding quantum deformations in different forms were already discussed in literature (see [3, 4, 5, 6, 7]). In the case of Poincaré algebra the total list of the classical $r$-matrices, which satisfy the homogeneous classical Yang-Baxter equation, consists of 20 cases which have various numbers of free parameters. Analysis of these twenty solutions shows that eighteen of them can be presented as a sum of subordinated $r$-matrices which are of Abelian and Jordanian types. Corresponding twists describing quantum deformations are obtained in explicit form.

This work is extended version of the paper [8].
2 Preliminaries

Let $r$ be a classical $r$-matrix of a Lie algebra $\mathfrak{g}$, i.e. $r \in \wedge^2 \mathfrak{g}$ and $r$ satisfies to the classical Yang–Baxter equation (CYBE)

\[
[r^{12}, r^{13} + r^{23}] + [r^{13}, r^{23}] = \Omega ,
\]

(2.1)

where $\Omega$ is $\mathfrak{g}$-invariant element, $\Omega \in (\wedge^3 \mathfrak{g})^\mathfrak{g}$. We consider two types of the classical $r$-matrices and corresponding twists.

Let the classical $r$-matrix $r = r_A$ has the form

\[
r_A = \sum_{i=1}^{n} y_i \wedge x_i ,
\]

(2.2)

where all elements $x_i, y_i$ ($i = 1, \ldots, n$) commute among themselves. Such an $r$-matrix is called of Abelian type. The corresponding twist is given as follows

\[
F_{r_A} = \exp \frac{\tilde{r}_A}{2} = \exp \left( \frac{1}{2} \sum_{i=1}^{n} x_i \wedge y_i \right) .
\]

(2.3)

This twisting two-tensor $F := F_{r_A}$ satisfies the cocycle equation

\[
F^{12}(\Delta \otimes \text{id})(F) = F^{23}(\text{id} \otimes \Delta)(F) ,
\]

(2.4)

and the "unital" normalization condition

\[
(\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1 .
\]

(2.5)

The twisting element $F$ defines a deformation of the universal enveloping algebra $U(\mathfrak{g})$ considered as a Hopf algebra. The new deformed coproduct and antipode are given as follows

\[
\Delta^{(F)}(a) = F\Delta(a)F^{-1} , \quad S^{(F)}(a) = uS(a)u^{-1}
\]

(2.6)

for any $a \in U(\mathfrak{g})$, where $\Delta(a)$ is a co-product before twisting, and

\[
u = \sum_i f_i^{(1)} S(f_i^{(2)})
\]

(2.7)

if $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$.

Let the classical $r$-matrix $r = r_{J_n}(\xi)$ has the form

\[
r_{J_n}(\xi) = \xi \left( \sum_{\nu=0}^{n} y_\nu \wedge x_\nu \right) ,
\]

(2.8)

where the elements $x_\nu, y_\nu$ ($\nu = 0, 1, \ldots, n$) satisfy the relations

\[
[x_0, y_0] = y_0 , \quad [x_0, x_i] = (1 - t_i)x_i , \quad [x_0, y_i] = t_i y_i ,
\]

\[
[x_\nu, y_j] = \delta_{ij} y_0 , \quad [x_i, x_j] = [y_i, y_j] = 0 , \quad [y_0, x_j] = [y_0, y_j] = 0 ,
\]

(2.9)

1Here entering the parameter deformation $\xi$ is a matter of convenience.

2It is easy to verify that the two-tensor (2.8) indeed satisfies the homogenous classical Yang-Baxter equation (2.1) (with $\Omega = 0$), if the elements $x_\nu, y_\nu$ ($\nu = 0, 1, \ldots, n$) are subject to the relations (2.9).
(i, j = 1, . . . , n), (t_i ∈ C). Such an r-matrix is called of Jordanian type. The corresponding twist is given as follows [9, 10]

\[ F_{r_{J_n}} = \exp \left( \sum_{i=1}^{n} x_i \otimes y_i \ e^{-2t_i \sigma} \right) \exp(2x_0 \otimes \sigma) , \]  

(2.10)

where \( \sigma := \frac{1}{2} \ln(1 + \xi y_0) \).³

Remark. The zero component \( r_{J_0}(\xi) := \xi y_0 \wedge x_0 \) in (2.8) itself is the classical Jordanian r-matrix and the corresponding Jordanian twist is given by the formula (2.10) for \( n = 0 \), i.e. \( F_{r_{J_0}} = \exp(2x_0 \otimes \sigma) \).

Let \( r \) be an arbitrary r-matrix of \( \mathfrak{g} \). We denote a support of \( r \) by \( \text{Sup}(r) \). The following definition is useful.

Definition 2.1 Let \( r_1 \) and \( r_2 \) be two arbitrary classical r-matrices. We say that \( r_2 \) is subordinated to \( r_1 \), \( r_1 \succ r_2 \), if \( \delta_{r_1}(\text{Sup}(r_2)) = 0 \), i.e.

\[ \delta_{r_1}(x) := [x \otimes 1 + 1 \otimes x, r_1] = 0 , \ \forall x \in \text{Sup}(r_2) . \]  

(2.11)

If \( r_1 \succ r_2 \) then \( r = r_1 + r_2 \) is also a classical r-matrix (see [18]). The subordination enables us to construct a correct sequence of quantizations. For instance, if the r-matrix of Jordanian type (2.8) is subordinated to the r-matrix of Abelian type (2.2), \( r_A \succ r_J \), then the total twist corresponding to the resulting r-matrix \( r = r_A + r_J \) is given as follows

\[ F_r = F_{r_J} F_{r_A} . \]  

(2.12)

The further definition is also useful.

Definition 2.2 A twisting two-tensor \( F_r(\xi) \) of a Hopf algebra \( U(\mathfrak{g}) \), satisfying the conditions (2.4) and (2.5), is called locally r-symmetric if the expansion of \( F_r(\xi) \) in powers of the parameter deformation \( \xi \) has the form

\[ F_r(\xi) = 1 + cr + \mathcal{O}(\xi^2) . . . \]  

(2.13)

where \( r \) is a classical r-matrix, and \( c \) is a numerical coefficient, \( c \neq 0 \).

It is evident that the Abelian twist (2.3) is globally r-symmetric and the twist of Jordanian type (2.10) does not satisfy the relation (2.13), i.e. it is not locally r-symmetric. It is a matter of direct verification to prove the following theorem.

Theorem 2.3 Let \( F_r(\xi) \) be a twisting two-tensor of a Hopf algebra \( U(\mathfrak{g}) \), and \( \omega := \sqrt{u} \) where \( u \) is give by the formula (2.7), then the new twisting two-tensor \( F_r^{(\omega)}(\xi) \),

\[ F_r^{(\omega)}(\xi) := \omega^{-1} \otimes \omega^{-1} F_r(\xi) \Delta(\omega) , \]  

(2.14)

is locally r-symmetric.

³Similar twists for Lie algebras \( \mathfrak{sl}(n) \), \( \mathfrak{so}(n) \) and \( \mathfrak{sp}(2n) \) were firstly constructed in the papers [11, 12, 13, 14].

⁴The support \( \text{Sup}(r) \) is a subalgebra of \( \mathfrak{g} \) generated by the elements \( \{x_i, y_i\} \) if \( r = \sum_i y_i \wedge x_i \).
3 Quantum deformations of Lorentz algebra

The results of this section in different forms were already discussed in literature (see [3, 4, 5, 6]).

The classical canonical basis of the $D = 4$ Lorentz algebra, $\mathfrak{o}(3, 1)$, can be described by the anti-Hermitian six generators $(h, e_{\pm}, h', e'_{\pm})$ satisfying the following non-vanishing commutation relations\(^5\):

\[
[h, e_{\pm}] = \pm e_{\pm} , \quad [e_+, e_-] = 2h ,
\]

\[
[h, e'_\pm] = \pm e'_\pm , \quad [h', e_\pm] = \pm e'_\pm , \quad [e_\pm, e'_\mp] = \pm 2h' ,
\]

\[
[h', e'_\pm] = \mp e_\pm , \quad [e'_+, e'_-] = -2h ,
\]

and moreover

\[
x^* = -x \quad (\forall \ x \in \mathfrak{o}(3, 1)) .
\]

A complete list of classical $r$-matrices which describe all Poisson structures and generate quantum deformations for $\mathfrak{o}(3, 1)$ involve the four independent formulas [1]:

\[
r_1 = \alpha e_+ \wedge h ,
\]

\[
r_2 = \alpha (e_+ \wedge h - e'_+ \wedge h') + 2\beta e'_+ \wedge e_+ ,
\]

\[
r_3 = \alpha (e'_+ \wedge e_- + e_+ \wedge e'_-) + \beta (e_+ \wedge e_- - e'_+ \wedge e'_-) - 2\gamma h \wedge h' ,
\]

\[
r_4 = \alpha (e'_+ \wedge e_- + e_+ \wedge e'_-) - 2h \wedge h' \pm e_+ \wedge e'_+ .
\]

If the universal $R$-matrices of the quantum deformations corresponding to the classical $r$-matrices (3.5)–(3.8) are unitary then these $r$-matrices are anti-Hermitian, i.e.

\[
r_j^* = -r_j \quad (j = 1, 2, 3, 4) .
\]

Therefore the $*$-operation (3.4) should be lifted to the tensor product $\mathfrak{o}(3, 1) \otimes \mathfrak{o}(3, 1)$. There are two variants of this lifting: direct and flipped, namely,

\[
(x \otimes y)^* = x^* \otimes y^* \quad (* - \text{direct}) ,
\]

\[
(x \otimes y)^* = y^* \otimes x^* \quad (* - \text{flipped}) .
\]

We see that if the "direct" lifting of the $*$-operation (3.4) is used then all parameters in (3.5)–(3.8) are pure imaginary. In the case of the "flipped" lifting (3.11) all parameters in (3.5)–(3.8) are real.

The first two $r$-matrices (3.5) and (3.6) satisfy the homogeneous CYBE and they are of Jordanian type. If we assume (3.10), the corresponding quantum deformations were described detailed in the paper [6] and they are entire defined by the twist of Jordanian type:

\[
F_{r_1} = \exp(2h \otimes \sigma) , \quad \sigma = \frac{1}{2} \ln(1 + \alpha e_+) \quad (3.12)
\]

\(^5\)Since the real Lie algebra $\mathfrak{o}(3, 1)$ is standard realification of the complex Lie $\mathfrak{sl}(2, \mathbb{C})$ these relations are easy obtained from the defining relations for $\mathfrak{sl}(2, \mathbb{C})$, i.e. from (3.1).
for the \( r \)-matrix (3.5), and

\[
F_{r_2} = \exp \left( \frac{i \beta}{\alpha^2} \sigma \wedge \varphi \right) \exp \left( h \otimes \sigma - h' \otimes \varphi \right),
\]

\[
\sigma = \frac{1}{2} \ln \left[ \left( 1 + \alpha e_+ \right)^2 + \left( \alpha e'_+ \right)^2 \right], \quad \varphi = \arctan \frac{\alpha e'_+}{1 + \alpha e_+}
\]  

(3.13)

for the \( r \)-matrix (3.6). It should be recalled that the twists (3.12) and (3.13) are not locally \( r \)-symmetric. Using the representation of the Jordanian twist \( F_{r_1} \) in binomial series form (see \([15, 16]\))

\[
F_{r_1} = (1 \otimes 1 + \alpha e_+ \otimes 1 + 1 \otimes \alpha e_+) \frac{h \otimes 1 + 1 \otimes h}{2}.
\]

(3.15)

we can easy obtain the explicit form of the element (2.7):

\[
u = 1 + \sum_{k>0} \frac{(-\alpha)^k}{k!} h(h-1) \cdots (h-k+1) e_+^k = \exp (-\alpha h e_+)
\]

(3.16)

and therefore

\[
\omega := \sqrt{\nu} = \exp \left( -\frac{1}{2} \alpha h e_+ \right).
\]

(3.17)

By Theorem 2.3 the locally \( r \)-symmetric Jordanian twist is given as follows\(^6\)

\[
F_{r_1}^{(\omega)} = \exp \left( \frac{\alpha}{2} \left( h e_+ + h \otimes e_+ + e_+ \otimes h \right) \right) \exp \left( 2h \otimes \sigma \right) \times
\]

\[
\times \exp \left( -\frac{\alpha}{2} \left( h e_+ + h \otimes e_+ + e_+ \otimes h \right) \right).
\]

(3.18)

In a similar way one can obtain the locally \( r \)-symmetric expression for the twist \( F_{r_2} \), (3.13).

The last two \( r \)-matrices (3.7) and (3.8) satisfy the non-homogeneous (modified) CYBE and they can be easily obtained from the solutions of the complex algebra \( \mathfrak{o}(4, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \) which describes the complexification of \( \mathfrak{o}(3,1) \). Indeed, let us introduce the complex basis of Lorentz algebra \( \mathfrak{o}(3,1) \simeq \mathfrak{sl}(2; \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \) described by two commuting sets of complex generators:

\[
H_1 = \frac{1}{2} \left( h + ih' \right), \quad E_{1 \pm} = \frac{1}{2} \left( e_{\pm} + ie'_{\pm} \right),
\]

\[
H_2 = \frac{1}{2} \left( h - ih' \right), \quad E_{2 \pm} = \frac{1}{2} \left( e_{\pm} - ie'_{\pm} \right),
\]

(3.19)

(3.20)

which satisfy the non-vanishing relations (compare with (3.1))

\[
[H_k, E_{k \pm}] = \pm E_{k \pm}, \quad [E_{k +}, E_{k -}] = 2H_k \quad (k = 1,2).
\]

(3.21)

The \( * \)-operation describing the real structure acts on the generators \( H_k \), and \( E_{k \pm} \) \( (k = 1,2) \) as follows

\[
H_1^* = -H_2, \quad E_{1 \pm}^* = -E_{2 \pm}, \quad H_2^* = -H_1, \quad E_{2 \pm}^* = -E_{1 \pm}.
\]

(3.22)

\(^6\)Another form of the locally \( r \)-symmetric Jordanian twist was presented in \([17]\) (see also \([8]\)).
The classical $r$-matrix $r_3$, (3.7), and $r_4$, (3.8), in terms of the complex basis (3.19), (3.20) take the form

$$r_3 = r_3' + r_3'',$$
$$r_3' := 2(\beta + i\alpha)E_{1+} \wedge E_{1-} + 2(\beta - i\alpha)E_{2+} \wedge E_{2-},$$
$$r_3'' := 4r\gamma H_2 \wedge H_1,$$

and

$$r_4 = r_4' + r_4'',$$
$$r_4' := 2i\alpha(E_{1+} \wedge E_{1-} - E_{2+} \wedge E_{2-} - 2H_1 \wedge H_2),$$
$$r_4'' := 2i\lambda E_{1+} \wedge E_{2+}.$$

For the sake of convenience we introduce parameter\(^7\lambda\) in $r_4''$. It should be noted that $r_3'$, $r_3''$ and $r_4'$, $r_4''$ are themselves classical $r$-matrices. We see that the $r$-matrix $r_3'$ is simply a sum of two standard $r$-matrices of $\mathfrak{sl}(2; \mathbb{C})$, satisfying the anti-Hermitian condition $r^* = -r$. Analogously, it is not hard to see that the $r$-matrix $r_4$ corresponds to a Belavin-Drinfeld triple [18] for the Lie algebra $\mathfrak{sl}(2; \mathbb{C}) \oplus \overline{\mathfrak{sl}}(2; \mathbb{C})$. Indeed, applying the Cartan automorphism $E_{2+} \rightarrow E_{2+}$, $H_2 \rightarrow -H_2$ we see that this is really correct (see also [19]).

We firstly describe quantum deformation corresponding to the classical $r$-matrix $r_3$, (3.23). Since the $r$-matrix $r_3''$ is Abelian and it is subordinated to $r_3'$ therefore the algebra $\mathfrak{o}(3, 1)$ is firstly quantized in the direction $r_3'$ and then an Abelian twist corresponding to the $r$-matrix $r_3''$ is applied. We introduce the complex notations $z_\pm := \beta \pm i\alpha$. It should be noted that $z_+ = z_1^+$ if the parameters $\alpha$ and $\beta$ are real, and $z_- = -z_1^-$ if the parameters $\alpha$ and $\beta$ are pure imaginary. From structure of the classical $r$-matrix $r_3'$ it follows that a quantum deformation $U_{r_3'}(\mathfrak{o}(3, 1))$ is a combination of two $q$-analogs of $U(\mathfrak{sl}(2; \mathbb{C}))$ with the parameter $q_{z_+}$ and $q_{z_-}$, where $q_{z_\pm} := \exp z_\pm$. Thus $U_{r_3'}(\mathfrak{o}(3, 1)) \cong U_{q_{z_+}}(\mathfrak{sl}(2; \mathbb{C})) \otimes U_{q_{z_-}}(\overline{\mathfrak{sl}}(2; \mathbb{C}))$ and the standard generators $q_{z_+}^{\pm H_1}$, $E_{1\pm}$ and $q_{z_-}^{\pm H_2}$, $E_{2\pm}$ satisfy the following non-vanishing defining relations

$$q_{z_+}^{H_1}E_{1\pm} = q_{z_+}^{\pm 1}E_{1\pm}q_{z_+}^{H_1}, \quad [E_{1+}, E_{1-}] = \frac{q_{z_+}^{2H_1} - q_{z_+}^{-2H_1}}{q_{z_+} - q_{z_+}^{-1}},$$
$$q_{z_-}^{H_2}E_{2\pm} = q_{z_-}^{\pm 1}E_{2\pm}q_{z_-}^{H_2}, \quad [E_{2+}, E_{2-}] = \frac{q_{z_-}^{2H_2} - q_{z_-}^{-2H_2}}{q_{z_-} - q_{z_-}^{-1}}.$$

In this case the co-product $\Delta_{r_3'}$ and antipode $S_{r_3'}$ for the generators $q_{z_+}^{\pm H_1}$, $E_{1\pm}$ and $q_{z_-}^{\pm H_2}$, $E_{2\pm}$ can be given by the formulas:

$$\Delta_{r_3'}(q_{z_+}^{\pm H_1}) = q_{z_+}^{\pm H_1} \otimes q_{z_+}^{\pm H_1},$$
$$\Delta_{r_3'}(E_{1\pm}) = E_{1\pm} \otimes q_{z_+}^{H_1} + q_{z_+}^{H_1} \otimes E_{1\pm},$$
$$\Delta_{r_3'}(q_{z_-}^{\pm H_2}) = q_{z_-}^{\pm H_2} \otimes q_{z_-}^{\pm H_2},$$
$$\Delta_{r_3'}(E_{2\pm}) = E_{2\pm} \otimes q_{z_-}^{H_2} + q_{z_-}^{H_2} \otimes E_{2\pm},$$

\(^7\)We can reduce this parameter $\lambda$ to $\pm 1$ by automorphism of $\mathfrak{o}(4, \mathbb{C})$.\n
The \(*\)-involution describing the real structure on the generators (3.22) can be adapted to the quantum generators as follows

\[
(q_{z_+}^{±H_1})^* = q_{z_+}^{±H_2}, \quad E_{1±}^* = -q_{z_±}^{±1} E_{1±}, \quad (3.31)
\]
\[
(q_{z_-}^{±H_2})^* = q_{z_-}^{±H_1}, \quad E_{2±}^* = -q_{z_-}^{±1} E_{2±}. \quad (3.32)
\]

and there exit two \(*\)-liftings: \textit{direct} and \textit{flipped}, namely,

\[
(a \otimes b)^* = a^* \otimes b^* \quad (\text{\textit{direct}}), \quad (3.34)
\]
\[
(a \otimes b)^* = b^* \otimes a^* \quad (\text{\textit{flipped}}). \quad (3.35)
\]

for any \(a \otimes b \in U_{r_3'}(\mathfrak{o}(3, 1)) \otimes U_{r_3'}(\mathfrak{o}(3, 1))\), where the \(*\)-direct involution corresponds to the case of the pure imaginary parameters \(\alpha, \beta\) and the \(*\)-flipped involution corresponds to the case of the real deformation parameters \(\alpha, \beta\). It should be stressed that the Hopf structure on \(U_{r_3'}(\mathfrak{o}(3, 1))\) satisfy the consistency conditions under the \(*\)-involution

\[
\Delta_{r_3'}(a^*) = (\Delta_{r_3'}(a))^*, \quad S_{r_3'}((S_{r_3'}(a^*))^*) = a \quad (\forall x \in U_{r_3'}(\mathfrak{o}(3, 1)). \quad (3.36)
\]

Now we consider deformation of the quantum algebra \(U_{r_3'}(\mathfrak{o}(3, 1))\) (secondary quantization of \(U(\mathfrak{o}(3, 1))\)) corresponding to the additional \(r\)-matrix \(r_3''\), (3.23). Since the generators \(H_1\) and \(H_2\) have the trivial coproduct

\[
\Delta_{r_3'}(H_k) = H_k \otimes 1 + 1 \otimes H_k \quad (k = 1, 2), \quad (3.37)
\]

therefore the unitary two-tensor

\[
F_{r_3''} = q_{r_3''}^{H_1 \wedge H_2} \quad (F_{r_3'}^* = F_{r_3''}^{-1}) \quad (3.38)
\]
satisfies the cocycle condition (2.4) and the "unitil" normalization condition (2.5). Thus the complete deformation corresponding to the \(r\)-matrix \(r_3''\) is the twisted deformation of \(U_{r_3'}(\mathfrak{o}(3, 1))\), i.e. the resulting coproduct \(\Delta_{r_3}\) is given as follows

\[
\Delta_{r_3}(x) = F_{r_1''} \Delta_{r_1'}(x) F_{r_3''}^{-1} \quad (\forall x \in U_{r_1'}(\mathfrak{o}(3, 1)) \quad (3.39)
\]

and in this case the resulting antipode \(S_{r_3}\) does not change, \(S_{r_3} = S_{r_3'}\). Applying the twisting two-tensor (3.38) to the formulas (3.27)-(3.30) we obtain

\[
\Delta_{r_3}(q_{z_+}^{±H_1}) = q_{z_+}^{±H_1} \otimes q_{z_+}^{±H_1}, \quad (3.40)
\]
\[
\Delta_{r_3}(E_{1+}) = E_{1+} \otimes q_{z_+}^{H_1} q_{r_1}^{H_2} + q_{z_+}^{-H_1} q_{r_1}^{-H_2} \otimes E_{1+}, \quad (3.41)
\]
\[
\Delta_{r_3}(E_{1-}) = E_{1-} \otimes q_{z_+}^{H_1} q_{r_1}^{-H_2} + q_{z_+}^{-H_1} q_{r_1}^{H_2} \otimes E_{1-}, \quad (3.42)
\]
\[
\Delta_{r_3}(q_{z_-}^{±H_2}) = q_{z_-}^{±H_2} \otimes q_{z_-}^{±H_2}, \quad (3.43)
\]
\[
\Delta_{r_3}(E_{2+}) = E_{2+} \otimes q_{z_-}^{H_2} q_{r_1}^{-H_1} + q_{z_-}^{-H_2} q_{r_1}^{H_1} \otimes E_{2+}, \quad (3.44)
\]
\[
\Delta_{r_3}(E_{2-}) = E_{2-} \otimes q_{z_-}^{H_2} q_{r_1}^{H_1} + q_{z_-}^{-H_2} q_{r_1}^{-H_1} \otimes E_{2-}. \quad (3.45)
\]
Next, we describe quantum deformation corresponding to the classical \( r\)-matrix \( r'_4 \) (3.24). Since the \( r\)-matrix \( r'_4(\alpha) := r'_4 \) is a particular case of \( r_3(\alpha, \beta, \gamma) := r_3 \), namely \( r'_4(\alpha) = r_3(\alpha, \beta = 0, \gamma = \alpha) \), therefore a quantum deformation corresponding to the \( r\)-matrix \( r'_4 \) is obtained from the previous case by setting \( \beta = 0, \gamma = \alpha \), and we have the following formulas for the coproducts \( \Delta_{r'_4} \):

\[
\begin{align*}
\Delta_{r'_4}(q^+H_k) &= q^+H_k \otimes q^+H_k \quad (k = 1, 2), \\
\Delta_{r'_4}(E_{1+}) &= E_{1+} \otimes q^+H_1 + q^+H_2 \otimes E_{1+}, \\
\Delta_{r'_4}(E_{1-}) &= E_{1-} \otimes q^+H_1 - q^+H_2 \otimes E_{1-}, \\
\Delta_{r'_4}(E_{2+}) &= E_{2+} \otimes q^-H_1 + q^-H_2 \otimes E_{2+}, \\
\Delta_{r'_4}(E_{2-}) &= E_{2-} \otimes q^-H_1 - q^-H_2 \otimes E_{2-},
\end{align*}
\]

where we set \( \xi := i\alpha \).

Consider the two-tensor

\[
F_{r'_4} := \exp_{q^\xi} \left( -2t\lambda E_{1+}q^+H_1 \otimes E_{2+}q^+H_2 \right).
\]

Using properties of \( q\)-exponentials (see [20]) is not hard to verify that \( F_{r'_4} \) satisfies the cocycle equation (2.4). Thus the quantization corresponding to the \( r\)-matrix \( r_4 \) is the twisted \( q\)-deformation \( U_{r'_4}(\mathfrak{o}(3, 1)) \). Explicit formulas of the co-products \( \Delta_{r'_4}(\cdot) = F_{r'_4} \Delta_{r'_4}(\cdot) F^{-1}_{r'_4} \) and antipodes \( S_{r'_4}(\cdot) \) will be presented in the outgoing paper [7].

### 4 Quantum deformations of Poincaré algebra

The Poincaré algebra \( \mathcal{P}(3, 1) \) of the 4-dimensional space-time is generated by 10 elements: the six-dimensional Lorentz algebra \( \mathfrak{o}(3, 1) \) with the generators \( M_i, N_i \ (i = 1, 2, 3) \):

\[
\begin{align*}
[M_i, M_j] &= \epsilon_{ijk} M_k, \\
[M_i, N_j] &= \epsilon_{ijk} N_k, \\
[N_i, N_j] &= -\epsilon_{ijk} M_k,
\end{align*}
\]

and the four-momenta \( P_0, P_j \ (j = 1, 2, 3) \) with the standard commutation relations:

\[
\begin{align*}
[M_j, P_k] &= \epsilon_{jkl} P_l, \\
[N_j, P_k] &= -\epsilon_{jkl} P_l, \\
[M_j, P_0] &= 0, \\
[N_j, P_0] &= -tP_j.
\end{align*}
\]

The physical generators of the Lorentz algebra, \( M_i, N_i \ (i = 1, 2, 3) \), are related with the canonical basis \( h, h', e_\pm, e'_\pm \) as follows

\[
\begin{align*}
h &= tN_3, & e_\pm &= t(N_1 \pm M_2), \\
h' &= tM_3, & e'_\pm &= t(M_1 \mp N_2).
\end{align*}
\]
The subalgebra generated by the four-momenta $P_0, P_j$ ($j = 1, 2, 3$) will be denoted by $\mathbf{P}$, and we also set $P_\pm := P_0 \pm P_3$.

S. Zakrzewski has shown in [2] that each classical $r$-matrix, $r \in \mathcal{P}(3, 1) \wedge \mathcal{P}(3, 1)$, has a decomposition

$$ r = a + b + c , $$

where $a \in \mathbf{P} \wedge \mathbf{P}$, $b \in \mathbf{P} \wedge \mathfrak{o}(3, 1)$, $c \in \mathfrak{o}(3, 1) \wedge \mathfrak{o}(3, 1)$ satisfy the following relations

$$ [[c, c]] = 0 , $$

$$ [[b, c]] = 0 , $$

$$ 2[[a, c]] + [[b, b]] = t\Omega \quad (t \in \mathbb{R}, \Omega \neq 0) , $$

$$ [[a, b]] = 0 . $$

Here $[[\cdot, \cdot]]$ means the Schouten bracket. Moreover a total list of the classical $r$-matrices for the case $c \neq 0$ and also for the case $c = 0, t = 0$ was found.\(^8\) The results are presented in the following table taken from [2]:

| $c$ | $b$ | $a$ | $\#$ | $N$ |
|-----|-----|-----|-----|-----|
| $\gamma h' \wedge h$ | 0 | $\alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2$ | 2 | 1 |
| $\gamma e_+ \wedge e_+$ | $\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$ | 0 | 1 | 2 |
| $\beta_1 b_{P_+}$ | $\alpha P_+ \wedge P_1$ | 1 | 3 |
| $\gamma_1 (P_1 \wedge e_+ + P_2 \wedge e_+')$ | $P_+ \wedge (\alpha P_1 + \alpha_2 P_2) - \gamma_1^2 P_1 \wedge P_2$ | 2 | 4 |
| $\gamma (h \wedge e_+)$ | 0 | 0 | 1 | 5 |
| $-h' \wedge e_+'$ | $\beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+$ | 0 | 1 | 6 |
| $+\gamma_1 e_+' \wedge e_+$ | 0 | 1 | 7 |
| $\beta_1 (P_1 \wedge e_+' + P_2 \wedge e_+')$ | $\alpha P_+ \wedge P_2$ | 1 | 8 |
| $\beta_1 P_+ \wedge (h + \chi e_+)$, $\chi = 0, \pm 1$ | $\beta_1 P_+ \wedge e_+$ | $\alpha_1 P_+ \wedge P_1 + \alpha_2 P_+ \wedge P_2$ | 2 | 10 |
| $\beta_1 P_+ \wedge e_+$ | $\alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2$ | 1 | 11 |
| $\beta_1 P_3 \wedge h'$ | $\alpha P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ | 2 | 13 |
| $\beta_1 P_2 \wedge h'$ | $\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ | 2 | 14 |
| $\beta_1 P_+ \wedge h$ | $\alpha P_1 \wedge P_+ + \alpha_1 P_1 \wedge P_2$ | 1 | 15 |
| $P_+ \wedge (\beta_1 h + \beta_2 h')$ | $\alpha_1 P_1 \wedge P_2$ | 1 | 17 |
| $\beta_1 b_{P_+}$ | $\alpha_1 P_1 \wedge P_4$ | 0 | 19 |
| $\beta_1 b_{P_+}$ | $\alpha_1 P_1 \wedge P_2$ | 0 | 20 |
| $\beta_1 P_0 \wedge P_+ + \alpha_2 P_1 \wedge P_2$ | $\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ | 1 | 21 |

Table 1. Normal forms of $r$ for $c \neq 0$ or $t = 0$.

\(^8\)Classification of the $r$-matrices for the case $c = 0, t \neq 0$ is an open problem up to now.
Where \( b_{P_+} \) and \( b_{P_2} \) are given as follows:

\[
\begin{align*}
  b_{P_+} & = P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h, \\
  b_{P_2} & = 2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-.
\end{align*}
\] (4.11) (4.12)

The table lists 21 cases labelled by the number \( N \) in the last column. In the forth column (labelled by \#) the number of essential parameters (more precisely - the maximal number of such parameter) involved in deformation are indicated. This number is in any cases less than the number of parameters actually using in the table. Moreover we introduced an additional parameter \( \gamma \) in the component \( a \) (in the cases 2,3,4,5,6) and a parameter \( \beta_1 \) in the component \( b \) (in the cases 7–18) and also a parameter \( \alpha_1 \) in the component \( a \) (in the cases 19–21)\(^9\). The final reduction of the number of the actual parameters can be achieved using of automorphisms of the Poincaré algebra \( P(3,1) \) (see details in [2]).

Now we would like to show that each \( r \)-matrices of the given table, \( r_N \ (1 \leq N \leq 21) \), can be presented as a sum of subordinated \( r \)-matrices which almost all are of Abelian and Jordanian types and we write down possible twisting two-tensors. Firstly we analyze classical \( r \)-matrices for the case \( c \neq 0 \).

1). The first \( r \)-matrix \( r_1 \),

\[
r_1 = \gamma h' \wedge h + \alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2,
\] (4.13)

is a sum of two subordinated Abelian \( r \)-matrices

\[
r_1 = r'_1 + r''_1, \quad r'_1 \succ r''_1,
\]

\[
r'_1 = \alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2,
\]

\[
r''_1 = \gamma h' \wedge h.
\] (4.14)

Therefore the total twist defining quantization in the direction to this \( r \)-matrix is the ordered product of two the Abelian twits

\[
F_{r_1} = F_{r'_1} F_{r''_1} = \exp(\gamma h \wedge h') \exp(\alpha P_- \wedge P_+ + \tilde{\alpha} P_2 \wedge P_1).
\] (4.15)

2). The second \( r \)-matrix \( r_2 \),

\[
r_2 = \gamma e'_+ \wedge e_+ + \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \beta_2 P_+ \wedge h',
\] (4.16)

is a sum of three subordinated \( r \)-matrices where one of them is of Jordanian type and two are of Abelian type

\[
r_2 = r'_2 + r'''_2 + r''''_2, \quad r'_2 \succ r'''_2, \quad r'_2 + r'''_2 \succ r''''_2,
\]

\[
r'_2 = \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h),
\]

\[
r'''_2 = \gamma e'_+ \wedge e_+,
\]

\[
r''''_2 = \beta_2 P_+ \wedge h'.
\] (4.17)

\(^9\)In the original paper by S. Zakrzewski [2] all these additional parameters are equal to 1 and the numbers in the forth column correspond to this situation.
The corresponding twisting two-tensor is given by the following formula

\[ F_{r_2} = F_{r_2''} F_{r_2'} F_{r_2} \]  

(4.18)

where

\[ F_{r_2'} = \exp(\beta_1(e_+ \otimes P_1 - e'_+ \otimes P_2)) \exp(2h \otimes \sigma_+) \]

\[ F_{r_2''} = \exp(\gamma e_+ \wedge e'_+) \]  

(4.19)

\[ F_{r_2'''} = \exp(\frac{\beta_2}{\beta_1} h' \wedge \sigma_+) \]

Here and below we set \( \sigma_+ := \frac{1}{2} \ln(1 + \beta_1 P_+) \).

3). The third \( r \)-matrix \( r_3 \),

\[ r_3 = \gamma e_+ \wedge e_+ + \beta_1(P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \alpha P_+ \wedge P_1 \]  

(4.20)

can be presented as a sum of three subordinated \( r \)-matrices where one of them is of Jordanian type and two are of Abelian type

\[ r_3 = r'_3 + r''_3 + r'''_3 , \quad r'_3 \succ r''_3 , \quad r'_3 + r''_3 \succ r'''_3 , \]

\[ r'_3 = P_1 \wedge (\beta_1 e_+ - \alpha P_+) - \beta_1 P_2 \wedge e'_+ + \beta_1 P_+ \wedge h \]

\[ r''_3 = \gamma e_+ \wedge (e_+ - \frac{\alpha}{\beta_1} P_+) \]

\[ r'''_3 = \frac{\gamma \alpha}{\beta_1} e'_+ \wedge P_+ \]  

(4.21)

The corresponding twist is given by the following formula

\[ F_{r_3} = F_{r_3'} F_{r_3''} \]  

(4.22)

where

\[ F_{r_3'} = \exp((\beta_1 e_+ - \alpha P_+) \otimes P_1 - \beta_1 e'_+ \otimes P_2) \exp(2h \otimes \sigma_+) \]

\[ F_{r_3''} = \exp(\gamma (e_+ - \frac{\alpha}{\beta_1} P_+) \wedge e'_+) \]  

(4.23)

\[ F_{r_3'''} = \exp(\frac{\gamma \alpha}{\beta_1^2} \sigma_+ \wedge e'_+) \]

4). The fourth \( r \)-matrix \( r_4 \),

\[ r_4 = \gamma (e'_+ \wedge e_+ + \beta_1 P_1 \wedge e_+ + \beta_1 P_2 \wedge e'_+ - \beta^2 P_1 \wedge P_2) + P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) \]  

(4.24)

is a sum of two subordinated \( r \)-matrices of Abelian type

\[ r_4 = r'_4 + r''_4 , \quad r'_4 \succ r''_4 , \]

\[ r'_4 = P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) \]  

(4.25)

\[ r''_4 = \gamma (e'_+ + \beta_1 P_1) \wedge (e_+ - \beta_1 P_2) \]
The corresponding twist is given by the following formula

\[ F_{r_4} = F_{r_4}^r F_{r_4}^s , \]  

(4.26)

where

\[ F_{r_4}^r = \exp\left((\alpha_1 P_1 + \alpha_2 P_2) \wedge P_\gamma \right) , \]
\[ F_{r_4}^s = \exp\left(\gamma (e_+ - \beta_1 P_1) \wedge (e'_+ + \beta_1 P_2) \right) . \]  

(4.27)

Remark. The parameter \( \beta_1 \) can be removed by the rescaling automorphism \( \beta_1 P_\nu \rightarrow P_\nu \)
\((\nu = 0, 1, 2, 3)\).

5). The fifth \( r \)-matrix \( r_5 \) is the \( r \)-matrix of the Lorentz algebra, (3.6) of the Section 3, and
the corresponding twist is given by the formulas (3.13) and (3.14) of the previous Section 3.

6). The sixth \( r \)-matrix \( r_6 \),

\[ r_6 = \gamma h \wedge e_+ + \beta_1 (2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-) + \]
\[ + \beta_2 P_2 \wedge e_+ , \]  

(4.28)

is a sum of three subordinated \( r \)-matrices

\[ r'_6 = r'_6 + r''_6 , \quad r'_6 \succ r''_6 , \quad r'_6 + r''_6 \succ r'''_6 , \]
\[ r''_6 = \beta_1 (2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-) , \]
\[ r'''_6 = \gamma h \wedge e_+ , \]
\[ r'''_6 = \beta_2 P_2 \wedge e_+ . \]  

(4.29)

The \( r \)-matrix \( r''_6 \) is of Jordanian type and the \( r \)-matrix \( r'''_6 \) is of Abelian type while the first \( r \)-matrix \( r'_6 \) is not Abelian and Jordanian type because the total \( r \)-matrix (4.28) satisfy the
non-homogeneous classical Yang-Baxter equation (4.9) with \( t \neq 0 \). In terms of the generators
\( M_i, N_i \, (i = 1, 2, 3) \) the \( r \)-matrix \( r'_6 \) has the form

\[ r'_6 = 2t \beta_1 (P_1 \wedge M_3 - P_3 \wedge M_1 - P_0 \wedge N_2) . \]  

(4.30)

Unfortunately a quantum deformation corresponding to this \( r \)-matrix is unknown to us, and
it is very likely that it can be obtain by contraction procedure from the \( q \)-analog \( U_q(\mathfrak{so}(5)) \) in
the same way as for the \( \kappa \)-Poincar‘e deformation (see [21]).

Now we will analyze classical \( r \)-matrices when \( c = 0, \ t = 0 \). In this case there are twelve
solutions with \( b \neq 0, \ r_N \, (7 \leq N \leq 18) \), and three solutions with \( b = 0, \ r_N \, (19 \leq N \leq 21) \).

7). The seventh \( r \)-matrix \( r_7 \),

\[ r_7 = \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \beta_2 P_+ \wedge h' , \]  

(4.31)

is a partial case of the \( r \)-matrix \( r_2(\gamma, \beta_1, \beta_2) := r_2 \), namely \( r_7 = r_2(\gamma = 0, \beta_1, \beta_2) \), therefore
the corresponding twist is defined by the formulas (4.18) and (4.19) where \( \gamma = 0 \).

8). The eighth \( r \)-matrix \( r_8 \),

\[ r_8 = \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \beta_2 P_+ \wedge e_+ , \]  

(4.32)
is a sum of two subordinated $r$-matrices of Jordanian and Abelian types

$$ \begin{align*}
r_8 &= r'_8 + r''_8, \\
r'_8 &= \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h), \\
r''_8 &= \beta_2 P_+ \wedge e_+.
\end{align*} $$

(4.33)

The corresponding twisting two-tensor is given by the following formula

$$ F_{r_8} = F_{r'_8} F_{r''_8}, $$

(4.34)

where

$$ \begin{align*}
F_{r'_8} &= \exp(\beta_1 (e_+ \otimes P_1 - e'_+ \otimes P_2)) \exp(2h \otimes \sigma_+), \\
F_{r''_8} &= \exp(\frac{\beta_2}{\beta_1} e_+ \wedge \sigma_+).
\end{align*} $$

(4.35)

9). The two-tensor $\tilde{r}_9$, which corresponds to the ninth row of Table 1, has the form:

$$ \begin{align*}
\tilde{r}_9 &= P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+ + \beta_1 P_+ \wedge (h + \chi e_+)) \\
&\quad + \alpha P_+ \wedge P_2,
\end{align*} $$

(4.36)

Unfortunately we could not prove that $\tilde{r}_9$ for $\chi \neq 0$ satisfies the equation (4.9) and thus we consider the modified variant $r_9$:

$$ \begin{align*}
r_9 &= P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) + P_+ \wedge (\beta_1 h + \chi (\beta_1 e_+ + \beta_2 e'_+)) \\
&\quad + \alpha P_+ \wedge P_2,
\end{align*} $$

(4.37)

This two-tensor satisfies the equations (4.9) and (4.10) and therefore it is a classical $r$-matrix. We can present $r_9$ as a sum of two subordinated $r$-matrices of Jordanian and Abelian types

$$ r_9 = r'_9 + r''_9, \quad r'_9 \succ r''_9, $$

(4.38)

where

$$ \begin{align*}
r'_9 &= P_+ \wedge \left( \beta_1 h + \alpha P_2 + \frac{\alpha \beta_2}{\beta_1} P_1 \right) + \\
&\quad + P_1 \wedge \left( \beta_1 e_+ + \beta_2 e'_+ + \frac{\alpha \beta_2}{\beta_1} P_+ \right), \\
r''_9 &= \chi P_+ \wedge \left( \beta_1 e_+ + \beta_2 e'_+ + \frac{\alpha \beta_2}{\beta_1} P_+ \right).
\end{align*} $$

(4.39)

The corresponding twisting two-tensor is given by the following formula

$$ F_{r_9} = F_{r'_9} F_{r''_9}, $$

(4.40)

where

$$ \begin{align*}
F_{r'_9} &= \exp\left( (\beta_1 e_+ + \beta_2 e'_+ + \frac{\alpha \beta_2}{\beta_1} P_+) \otimes P_1 \right) \times \\
&\quad \times \exp\left( 2(h + \frac{\alpha}{\beta_1} P_2 + \frac{\alpha \beta_2}{\beta_1^2} P_1) \otimes \sigma_+ \right),
\end{align*} $$

(4.41)
\[ F_{r'''} = \exp \left( \chi (e_+ + \frac{\beta_2}{\beta_1} e'_+ + \frac{\alpha \beta_2}{\beta_1} P_+) \land \sigma_+ \right). \] (4.42)

10). The tenth \( r \)-matrix \( r_{10} \),

\[ r_{10} = \beta (P_1 \land e'_+ + P_+ \land e_+) + \alpha_1 P_- \land P_1 + \alpha_2 P_+ \land P_2, \] (4.43)

can be presented as follows

\[ r_{10} = r_{10}' + r_{10}'' + r_{10}''', \] (4.44)

where

\[ r_{10}' = \beta_1 P_+ \land e_+, \]
\[ r_{10}'' = P_2 \land (2\alpha_1 P_1 - \alpha_2 P_+) , \]
\[ r_{10}''' = P_1 \land (\beta_1 e'_+ - \alpha_1 P_- + 2\alpha_1 P_2) . \]

All three \( r \)-matrices \( r_{10}', r_{10}'', r_{10}''' \) are Abelian and they have the following subordination

\[ r_{10}' \succ r_{10}'' ; \quad r_{10}' + r_{10}'' \succ r_{10}''' . \] (4.46)

The corresponding twisting two-tensor is given by the following formula

\[ F_{r_{10}} = F_{r_{10}'''} F_{r_{10}''} F_{r_{10}'} , \] (4.47)

where

\[ F_{r_{10}'} = \exp (\beta_1 e_+ \land P_+) , \]
\[ F_{r_{10}''} = \exp ((2\alpha_1 P_1 - \alpha_2 P_+) \land P_2) , \]
\[ F_{r_{10}'''} = \exp ((\beta_1 e'_+ - \alpha_1 P_- + 2\alpha_1 P_2) \land P_1) . \]

11). The eleventh \( r \)-matrix \( r_{11} \),

\[ r_{11} = \beta_1 P_2 \land e_+ + \alpha_1 P_+ \land P_1 + \alpha_2 P_- \land P_2 , \] (4.49)

is a sum of two subordinated \( r \)-matrices of Abelian type

\[ r_{11} = r_{11}' + r_{11}'' , \quad r_{11}' \succ r_{11}'' , \]
\[ r_{11}' = \alpha_1 P_+ \land P_1 , \]
\[ r_{11}'' = P_2 \land (\beta_1 e_+ - \alpha_2 P_-) . \]

The corresponding twisting two-tensor is given by the following formula

\[ F_{r_{11}} = F_{r_{11}'''} F_{r_{11}''} , \] (4.51)

where

\[ F_{r_{11}'} = \exp (\alpha_1 P_1 \land P_+) , \]
\[ F_{r_{11}''} = \exp ((\beta_1 e_+ - \alpha_2 P_-) \land P_2) . \] (4.52)
12). The two-tensor $\tilde{r}_{12}$, which corresponds to the twelfth row of Table 1, has the form:

$$\tilde{r}_{12} = \beta_1 p_+ \wedge e_+ + P_- \wedge (\alpha P_+ + \alpha_1 P_1 + \alpha_2 P_2) + \tilde{\alpha} P_+ \wedge P_2.$$  \hspace{1cm} (4.53)

We can show that $\tilde{r}_{12}$ for $\alpha_2 \neq 0$ does not satisfy the equation (4.10) that is the two-tensor $\tilde{r}_{12}$ is not any classical $r$-matrix provided that $\alpha_2 \neq 0$. In case of $\alpha_2 = 0$ the two-tensor is a classical $r$-matrix and unfortunately in this case we had not success to construct the corresponding twist.

13). The thirteenth $r$-matrix $r_{13}$,

$$r_{13} = \beta_1 p_0 \wedge h' + \alpha_1 p_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2,$$  \hspace{1cm} (4.54)

is a sum of two subordinated $r$-matrices of Abelian type

$$r_{13} = r_{13}' + r_{13}'' ; \quad r_{13}' \succ r_{13}'' ;$$  \hspace{1cm} (4.55)

$$r_{13}' = \alpha_1 p_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2,$$

$$r_{13}'' = \beta_1 p_0 \wedge h'.$$

The corresponding twisting two-tensor is given by the following formula

$$F_{r_{13}} = F_{r_{13}'} F_{r_{13}''} ;$$  \hspace{1cm} (4.56)

where

$$F_{r_{13}'} = \exp (\alpha_1 p_3 \wedge P_0 + \alpha_2 P_2 \wedge P_1) ,$$

$$F_{r_{13}''} = \exp (\beta_1 h' \wedge P_0) .$$  \hspace{1cm} (4.57)

14). The fourteenth $r$-matrix $r_{14}$,

$$r_{14} = \beta_1 p_3 \wedge h' + \alpha_1 p_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 ,$$  \hspace{1cm} (4.58)

is a sum of two subordinated $r$-matrices of Abelian type

$$r_{14} = r_{14}' + r_{14}'' ; \quad r_{14}' \succ r_{14}'' ;$$  \hspace{1cm} (4.59)

$$r_{14}' = \alpha_1 p_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 ,$$

$$r_{14}'' = \beta_1 p_3 \wedge h'.$$

The corresponding twisting two-tensor is given by the following formula

$$F_{r_{14}} = F_{r_{14}'} F_{r_{14}''} ;$$  \hspace{1cm} (4.60)

where

$$F_{r_{14}'} = \exp (\alpha_1 p_3 \wedge P_0 + \alpha_2 P_2 \wedge P_1) ,$$

$$F_{r_{14}''} = \exp (\beta_1 h' \wedge P_3) .$$  \hspace{1cm} (4.61)

15). The fifteenth $r$-matrix $r_{15}$,

$$r_{15} = \beta_1 p_+ \wedge h' + \alpha_1 p_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 .$$  \hspace{1cm} (4.62)
is a sum of two subordinated $r$-matrices of Abelian type
\[ r_{15} = r'_{15} + r''_{15} , \quad r'_{15} \succ r''_{15} , \]
\[ r'_{15} = \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 , \]
\[ r''_{15} = \beta_1 P_+ \wedge h . \]  
\hspace{1cm} (4.63)

The corresponding twisting two-tensor is given by the following formula
\[ F_{r_{15}} = F'_{r_{15}} \cdot F''_{r_{15}} , \]  
\hspace{1cm} (4.64)

where
\[ F'_{r_{15}} = \exp (\alpha_1 P_3 \wedge P_0 + \alpha_2 P_2 \wedge P_1) , \]
\[ F''_{r_{15}} = \exp (\beta_1 h \wedge P_+) . \]  
\hspace{1cm} (4.65)

16). The sixteenth $r$-matrix $r_{16}$,
\[ r_{16} = \beta_1 P_1 \wedge h + \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 , \]  
\hspace{1cm} (4.66)

is a sum of two subordinated $r$-matrices of Abelian type
\[ r_{16} = r'_{16} + r''_{16} , \quad r'_{16} \succ r''_{16} , \]
\[ r'_{16} = \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 , \]
\[ r''_{16} = \beta_1 P_1 \wedge h . \]  
\hspace{1cm} (4.67)

The corresponding twisting two-tensor is given by the following formula
\[ F_{r_{16}} = F'_{r_{16}} \cdot F''_{r_{16}} , \]  
\hspace{1cm} (4.68)

where
\[ F'_{r_{16}} = \exp (\alpha_1 P_3 \wedge P_0 + \alpha_2 P_2 \wedge P_1) , \]
\[ F''_{r_{16}} = \exp (\beta_1 h \wedge P_1) . \]  
\hspace{1cm} (4.69)

17). The seventeenth $r$-matrix $r_{17}$,
\[ r_{17} = \beta_1 P_+ \wedge h + \alpha P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1 , \]  
\hspace{1cm} (4.70)

is a sum of two subordinated $r$-matrices of Jordanian and Abelian types
\[ r_{17} = r'_{17} + r''_{17} , \quad r'_{17} \succ r''_{17} \preceq r'_{17} , \]
\[ r'_{17} = P_+ \wedge (\beta_1 h + \alpha_2 P_1) , \]
\[ r''_{17} = \alpha_1 P_1 \wedge P_2 . \]  
\hspace{1cm} (4.71)

The corresponding twisting two-tensor is given by the following formula
\[ F_{r_{17}} = F'_{r_{17}} \cdot F''_{r_{17}} = F'_{r_{17}} \cdot F''_{r_{17}} , \]  
\hspace{1cm} (4.72)
where

\[ F_{r_{17}} = \exp \left( \left( h + \frac{\alpha_2}{\beta_1} P_1 \right) \otimes \sigma_+ \right), \]

\[ F_{r_{17}'} = \exp (\alpha_1 P_2 \wedge P_1). \] (4.73)

18). The eighteenth \( r \)-matrix \( r_{18} \),

\[ r_{18} = P_+ \wedge (\beta_1 h + \beta_2 h') + \alpha P_1 \wedge P_2, \] (4.74)

is a sum of two subordinated \( r \)-matrices of Abelian and Jordanian types

\[ r_{18} = r_{18}'' + r_{18}', \quad r_{18}' \succ r_{18}'', \]

\[ r_{18}' = \alpha P_1 \wedge P_2, \]

\[ r_{18}'' = P_+ \wedge (\beta_1 h + \beta_2 h'), \]

The corresponding twisting two-tensor is given by the following formula

\[ F_{r_{18}} = F_{r_{18}''} F_{r_{18}'}, \] (4.76)

where

\[ F_{r_{18}'} = \exp (\alpha P_2 \wedge P_1), \] (4.77)

\[ F_{r_{18}''} = \exp \left( \left( h + \frac{\beta_2}{\beta_1} h' \right) \otimes \sigma_+ \right). \]

19). – 21). The \( r \)-matrices:

\[ r_{19} = \alpha P_1 \wedge P_+, \] (4.78)

\[ r_{20} = \alpha P_1 \wedge P_2, \] (4.79)

\[ r_{21} = \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 \] (4.80)

are Abelian and their corresponding twists are given by the simple exponential formulas

\[ F_{r_{19}} = \exp (\alpha P_+ \wedge P_1), \] (4.81)

\[ F_{r_{20}} = \exp (\alpha P_2 \wedge P_1), \] (4.82)

\[ F_{r_{21}} = \exp (\alpha_1 P_3 \wedge P_0 + \alpha_2 P_2 \wedge P_1). \] (4.83)

**Conclusion**

In this paper twists which describe multiparameter quantum deformations of the Poincaré algebra were obtained in explicit form. These twists correspond to eighteen out of the twenty classical \( r \)-matrices of the Zakrzewski's classification, which satisfy the homogeneous classical Yang-Baxter equation. For this aid we used the notation of subordination for the classical \( r \)-matrices. These results can be extend on the Poincaré superalgebra. It should be noted that up to now there is not any classification of the classical \( r \)-matrices in spirit of the classification by S. Zakrzewski. However it turns out that it is possible to extend the Zakrzewski's classification to the Poincaré superalgebra by an addition of supercharges terms to the classical \( r \)-matrices \( r_i \) \((i = 1, 2, \ldots, 21)\). Theses results and corresponding twisting functions will be presented in a future paper.
References

[1] S. Zakrzewski, *Lett. Math. Phys.*, **32**, 11 (1994).

[2] S. Zakrzewski, *Commun. Math. Phys.*, **187**, 285 (1997); arXiv:q-alg/9602001v1.

[3] M. Chaichian and A. Demichev, *Phys. Lett.*, **B34**, 220 (1994).

[4] A. Mudrov, * Yadernaya Fizika, **60**, No.5, 946 (1997).

[5] A. Borowiec, J. Lukierski, V.N. Tolstoy, *Czech. J. Phys.*, **55**, 11 (2005); arXiv:hep-th/0510154v1.

[6] A. Borowiec, J. Lukierski, V.N. Tolstoy, *Eur. Phys. J.*, **C48**, 636 (2006); arXiv:hep-th/0604146v1.

[7] A. Borowiec, J. Lukierski, V.N. Tolstoy, in preparation.

[8] V.N. Tolstoy, arXiv:0704.0081v1[math.QA].

[9] V.N. Tolstoy, *Proc. of International Workshop "Supersymmetries and Quantum Symmetries (SQS’03)"*, Russia, Dubna, July, 2003, eds: E. Ivanov and A. Pashnev, publ. JINR, Dubna, p. 242 (2004); arXiv:math/0402433v1.

[10] V.N. Tolstoy, *Nankai Tracts in Mathematics "Differential Geometry and Physics". Proceedings of the 23-th International Conference of Differential Geometric Methods in Theoretical Physics (Tianjin, China, 20-26 August, 2005). Editors: Mo-Lin Ge and Weiping Zhang. World Scientific*, 2006, Vol. 10, 443-452; arXiv:math/0701079v1[math.QA].

[11] P.P. Kulish, V.D. Lyakhovsky and A.I. Mudrov, *Journ. Math. Phys.*, **40**, 4569 (1999).

[12] P.P. Kulish, V.D. Lyakhovsky and M.A. del Olmo, *Journ. Phys. A: Math. Gen.*, **32**, 8671 (1999).

[13] V.D. Lyakhovsky, A.A. Stolin and P.P. Kulish, *J. Math. Phys. Gen.*, **42**, 5006 (2000).

[14] D.N. Ananikyan, P.P. Kulish and V.D. Lyakhovsky, *St. Petersburg Math. J.*, **14**, 385 (2003).

[15] S.M. Khoroshkin, A.A. Stolin, V.N. Tolstoy, *Commun. Alg.* **26**, 1041 (1998).

[16] S.M. Khoroshkin, A.A. Stolin, V.N. Tolstoy, *Phys. Atomic Nuclei*, **64**, 2173 (2001).

[17] Ch. Ohn, *Lett. Math. Phys.*, **25**, 85 (1992).

[18] A.A. Belavin and V.G. Drinfeld, *Functional Anal. Appl.*, **16**, 159 (1983); translated from *Funktsional. Anal. i Prilozhen*, **16**, 1 (1982) (Russian).

[19] A.P. Isaev and O.V. Ogievetsky, *Phys. Atomic Nuclei*, **64**, 2126 (2001); arXiv:math/0010190v1[math.QA].

[20] S.M. Khoroshkin and V.N. Tolstoy, *Comm. Math. Phys.*, **141**, 599 (1991).

[21] J. Lukierski, A. Nowicki, H. Ruegg, and V.N. Tolstoy, *Phys. Lett.*, **B264**, 331 (1991).