A NOTE ON COHOMOLOGY OF CLIFFORD ALGEBRAS

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Abstract. In this article we construct a cochain complex of a complex Clifford algebra with coefficients in itself in a combinatorial fashion and we call the corresponding cohomology by Clifford cohomology. We show that Clifford cohomology controls the deformation of a complex Clifford algebra and can classify them up to Morita equivalence. We also study Hochschild cohomology groups and formal deformations of the algebra of smooth sections of a complex Clifford algebra bundle over an even dimensional orientable Riemannian manifold $M$ which admits a Spin$^c$ structure.

1. Introduction

Algebraic deformation theory of associative algebras was developed by M. Gerstenhaber [3] [4]. In algebraic deformation theory of associative algebras the Hochschild cohomology group $HH^n(A)$ of an associative algebra $A$ with coefficients in $A$ plays the key role. The second Hochschild cohomology group $HH^2(A)$ of $A$ has a one-to-one correspondence to the set of all equivalence classes of non isomorphic infinitesimal deformations of $A$ while the triviality of $HH^2(A)$ implies that any formal deformation of $A$ is equivalent to null deformation. Again $HH^3(A) = 0$ forces that any infinitesimal deformation of $A$ can be extended to a formal deformation. Thus primarily the second and third Hochschild cohomology groups of an associative algebra $A$ controls the deformations of $A$.

Now let $Ass$ denotes the category of associative algebras over some field $k$ of characteristic zero, $\mathcal{A}$ be a particular subcategory of $Ass$ over $k$ having some extra structure. It is natural to ask whether the algebras belonging to this subcategory allows deformations and for this it is necessary to look for a suitable deformation cohomology which should be constructed using the extra structure. Let $H^*_\mathcal{A}$ be such a cohomology theory for $\mathcal{A}$. It should be imminent that the cochain complex defining $H^*_\mathcal{A}$ would have to be different from the Hochschild cochain complex. By this we mean that if the $n$th cochain space of $A \in \mathcal{A}$ is $C^n(A, A) = \text{hom}_k(A^\otimes n, A)$, then $\delta_A$ is not a scalar multiple of $\delta_{Hoch}$, where $\delta_{Hoch}$ and $\delta_A$ are respective coboundary maps from $C^n(A, A)$ to $C^{n+1}(A, A)$ in Hochschild cochain complex and cochain complex associated to $H^*_\mathcal{A}$.

The main theme of this note is to illustrate the above question by considering the subcategory $\mathcal{A}$ of $Ass$ to be the subcategory of complex Clifford algebras.
Clifford algebras were invented by William K. Clifford who introduced a new multiplication rule into Grassmann’s exterior algebra $\bigwedge \mathbb{R}^n$. A Clifford algebra is a unital associative algebra and generalizes the real numbers, complex numbers and Hamilton’s quaternions and plays important roles in geometry and theoretical physics. Clifford algebras can be seen as deformations of exterior algebras. Now it is well known that complex Clifford algebras are rigid i.e. any deformation is equivalent to null deformation with respect to the Hochschild cohomology. The Hochschild cohomology groups of complex Clifford algebras are easy to compute due to the fact that a complex Clifford algebra is either isomorphic to a complex central simple algebra or isomorphic to a direct sum of two isomorphic complex central simple algebras.

In this article we construct a cochain complex of a complex Clifford algebra $A$ with coefficients in $A$, for every arbitrary choice of an ordered orthogonal basis $B$, in a combinatorial fashion which is not of Hochschild cochain complex type. We call the corresponding cohomology by Clifford cohomology and will be denoted by $H^*_C(B)(A)$. It turns out that deformations of $A \in A$, where $A$ is the subcategory of complex Clifford algebras of Ass, is controlled by $H^*_C(B)(A)$ and it also classifies $A$ upto Morita equivalence.

In section 2 and 3 we briefly recall some facts about Clifford algebras and deformations of associative algebras respectively. In section 4 we introduce Clifford cohomology. In the final section 5, we obtained some interesting observations about formal deformations of the algebra of smooth sections of a complex Clifford algebra bundle over an even dimensional orientable Riemannian manifold $M$ which admits a $\text{Spin}^c$ structure by computing its Hochschild cohomology groups.

2. Clifford algebra

We start by recalling some basic facts about Clifford algebras [10] [11]. There are lots of literatures on Clifford algebras to be mentioned. Let $V$ be an $n$ dimensional complex vector space equipped with a non degenerated quadratic form $q$ and $T(V) = \bigoplus_{k=0}^{\infty} \otimes^k V$ be the tensor algebra over $V$.

**Definition 2.1.** A Clifford algebra $C(n)$ over $V$ is defined to be the quotient algebra $T(V)/I_q$, where $I_q$ is the two-sided ideal generated by elements of the form $v \otimes v − q(v)$ for all vectors $v$ in $V$. The product in $C(n)$ is called the Clifford product.

Now the corresponding bilinear form of $q$ is $\beta_q(u,v) = \frac{1}{2}(q(u + v) − q(u) − q(v))$. We note that in $C(n)$,

$q(u + v) = (u + v)^2 = u^2 + uv + vu + v^2 = q(u) + uv + vu + q(v)$

and so $uv + vu = 2\beta_q(u,v)$. Thus if $u$ and $v$ are orthogonal vectors then $uv = −vu$ in $C(n)$.

Now if we choose an orthogonal basis $B = \{v_1, \cdots, v_n\}$ of $V$ then the set $\mathcal{F}$ consisting of $2^n$ many elements given by $\mathcal{F} = \{1, E_{m_1m_2\cdots m_k}\}$, $1 \leq k \leq n$ and $1 \leq m_1 < m_2 < \cdots < m_k \leq n$ form a basis of the vector space $C(n)$ where $E_{m_1m_2\cdots m_k} = v_{m_1}v_{m_2}\cdots v_{m_k}$ (here right side is the Clifford product of $v_{m_1}, \cdots, v_{m_k}$) and thus $\dim C(n) = 2^n$. 

Periodicity of Clifford algebras: Let $M(k)$ denotes the matrix algebra of $k \times k$ complex matrices. It is known that there is an isomorphism between $C(n+2)$ and $C(n) \otimes_{\mathbb{C}} M(2)$. Now from this isomorphism and along with the fact that $C(0) \cong \mathbb{C}$ and $C(1) \cong \mathbb{C} \oplus \mathbb{C}$ it follows that:

\[
C(2n) \cong M(2^n) \\
C(2n+1) \cong M(2^n) \oplus M(2^n).
\]

In particular, $C(n+2)$ and $C(n)$ are Morita equivalent.

3. Deformation of associative algebras and Hochschild cohomology

Let us start with a short review of deformation theory of associative algebras [2] [3] [4]. Let $A$ be an associative algebra over a field $k$ of characteristic zero, $k[[t]]$ be the formal power series ring over $k$ and $A[[t]]$ is the formal power series over $A$ which is a $k[[t]]$ algebra.

**Definition 3.1.** A formal deformation of $A$ with base $k[[t]]$ is a $k[[t]]$-bilinear multiplication law $\mu_t : A[[t]] \otimes_{k[[t]]} A[[t]] \rightarrow A[[t]]$ on the spaces $A[[t]]$ of formal power series in a variable $t$ with coefficients in $A$, satisfying the following properties:

\[
\mu_t(a, b) = \mu_0(a, b) + \mu_1(a, b)t + \mu_2(a, b)t^2 + \cdots \quad \text{for all } a, b \in A,
\]

where $\mu_0(a, b) = ab$ is the original multiplication in $A$, and $\mu_t$ is associative, which is equivalent to the equation

\[
\mu_t(\mu_t(a, b), c) = \mu_t(a, \mu_t(b, c)) \quad \text{for } a, b, c, \in A,
\]

or, equivalently,

\[
\sum_{i+j=k, i, j \geq 0} (\mu_i(\mu_j(a, b)), c) - \mu_i(a, \mu_j(b, c)) = 0
\]

for all $a, b, c \in A$ and for each $k \geq 1$.

If one chooses $\mu_i = 0$ for all $i \geq 1$ then the deformation of $A$ is called null deformation.

Let $\mu_t = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots$ and $\mu'_t = \mu'_0 + \mu'_1 t + \mu'_2 t^2 + \cdots$ be two deformations of $A$. Now we say $\mu_t$ and $\mu'_t$ are are equivalent if there exists a $k[[t]]$ linear map $u : A[[t]] \rightarrow A[[t]]$ defined by $u = id_A + \phi_1 t + \phi_2 t^2 + \cdots$, $\phi_i \in \text{Hom}_k(A, A)$, $i \in \mathbb{N}$, such that

\[
u \circ \mu_t = \mu'_t \circ (u \otimes u).
\]

If every formal deformation of $A$ is equivalent to null deformation then $A$ is called rigid.

The main tool in studying deformation theory of an associative algebra $A$ is the Hochschild cochain complex $C^*(A, A) :$

\[
0 \rightarrow C^0(A, A) \xrightarrow{\delta_{Hoch}} \cdots \xrightarrow{\delta_{Hoch}} C^n(A, A) \xrightarrow{\delta_{Hoch}} C^{n+1}(A, A) \xrightarrow{\delta_{Hoch}} \cdots
\]
where $C^0(A, A) = A$ and $C^n(A, A) = \text{hom}_k(A^\otimes n, A)$ is the space of Hochschild $n$-cochains, i.e., the $n$-linear maps $f$ on $A$ with values in $A$. The differential $\delta_{\text{Hoch}} : C^n(A, A) \to C^{n+1}(A, A)$ is defined by:

$$(\delta_{\text{Hoch}} f)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 f(a_1 \otimes \cdots \otimes a_n)$$

$$+ \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n) + (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) a_n.$$

It turns out that $\delta_{\text{Hoch}}^2 = 0$ and Hochschild cohomology of $A$ with coefficients in $A$ is defined by $HH^*(A) = H^*(C^*(A, A); \delta_{\text{Hoch}})$. Hochschild cohomology groups are invariant under Morita equivalence [for details regarding Hochschild cohomology see [5] [9]].

It turns out that for a formal deformation of $A$ as defined above the coefficient $\mu_1$ is a Hochschild 2-cocycle, that is, $\delta_{\text{Hoch}}(\mu_1) = 0$, and is called the infinitesimal of the deformation.

**Definition 3.2.** An infinitesimal deformation of $A$ is a deformation of the form $A[[t]]/(t^2)$. More generally, a one parameter deformation of order $n$, is a deformation with base $k[[t]]/(t^{n+1})$, given by $\mu_t$ modulo $t^{n+1}$. In this case, the associativity condition in the definition above holds for the 2-cochains $\mu_i$ for $0 \leq i \leq n$.

Next comes the question of extending an infinitesimal deformation to a full-blown deformation. If we start with an arbitrary Hochschild 2-cocycle $\mu_1$, it need not be an infinitesimal of a formal deformation. If it be so, then we say $\mu_1$ is integrable. The integrability of $\mu_1$ implies an infinite sequence of relations which may be interpreted as the vanishing of the obstructions to the integration of $\mu_1$.

Suppose we have a deformation of $A$ of order $n \geq 1$ given by multiplication $\mu_t$ modulo $t^{n+1}$ and we would like to extend this to a deformation of order $n + 1$. Then, $\mu_t$ modulo $t^{n+2}$ must be associative. This gives rise to a 3-cochain

$$G(a, b, c) = \Sigma_{i+j=n+1} \mu_i(a, \mu_j(b, c)) - \mu_i(a, \mu_j(b, c)), \ i > 0 \ j > 0 \text{ and } a, b, c \in A.$$  

It turns out that $G$ is a 3-cocycle and is called the obstruction cocycle.

We wish to end this section by quoting the following well-known theorems:

**Theorem 3.3.** There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of $A$ and the second Hochschild cohomology $HH^2(A)$ of $A$ with coefficients in itself.

**Theorem 3.4.** Let $A$ be an associative algebra such that $HH^2(A) = 0$. Then all formal deformations of $A$ are equivalent to null deformation, in other words, $A$ is rigid.

**Theorem 3.5.** A deformation of $A$ of order $n$ extends to a deformation of order $n + 1$ if and only if the cohomology class of the associated obstruction cocycle $G$ vanishes. Thus, if $HH^3(A) = 0$ then, any Hochschild 2-cocycle is integrable.

4. **Clifford cohomology**

In this section we will construct a cochain complex of a Clifford algebra $C(n)$ over an $n$ dimensional complex vector space $V$ equipped with a non degenerated
quadratic form \( q \), which we call the *Clifford cochain complex*. Firstly for every choice of an ordered orthogonal basis \( B \) of \( V \) we will define a bilinear product \( \sigma_B \) on \( C(n) \) and then use it to define the *Clifford cochain complex*. It turns out that if \( B \) consists of orthonormal vectors then the *Clifford cochain complex* of \( C(n) \) coincides with the Hochschild cochain complex.

We start with a finite set \( X = \{1, 2, \cdots, n\} \). Let \( S \) be the set of all finite sequences in \( X \) and we define a “2-shuffle”, denoted by \( sh_i \), on \( \{m_1, \cdots, m_i, m_{i+1}, \cdots, m_k\} \in S \) by

\[
\{m_1, \cdots, m_i, m_{i+1}, \cdots, m_k\} \xrightarrow{sh_i} \{m_1, \cdots, m_{i+1}, m_i, \cdots, m_k\}
\]

for \( 1 \leq i \leq k - 1 \).

Now let \( P(X) \) be the set of all ordered subsets \( \{m_1, m_2, \cdots, m_k\} \) of \( X \) such that \( m_1 < m_2 < \cdots < m_k \), \( 1 \leq m_i \leq n \) for \( i = 1, 2, \cdots, k \). We consider the empty set, \( \{\} \), as a member of \( P(X) \) as follows:

\[
\{m_1, m_2, \cdots, m_k\} \triangleleft \{t_1, t_2, \cdots, t_s\}, \text{ where } \{t_1, t_2, \cdots, t_p\} \text{ is obtained by applying the minimum number of 2-shuffles } \tau \text{ on the sequence }
\]

\{m_1, m_2, \cdots, m_k, t_1, t_2, \cdots, t_s\} \text{ to get a monotonic increasing sequence and then deleting } m_i \text{ and } t_j \text{ if } m_i = t_j, 1 \leq i \leq k, 1 \leq j \leq s. \text{ We note that } p = \text{card}(\{m_1, m_2, \cdots, m_k\}) \triangle \{t_1, t_2, \cdots, t_s\} \text{ (here } \triangle \text{ means symmetric difference of two sets). Let us explain this by an example:}

We take \( X = \{1, 2, 3, 4\} \) and we compute \( \{1, 3, 4\} \triangle \{2, 3\} \). We see \( \{1, 3, 4\} \triangle \{2, 3\} = \{1, 2, 4\} \). This is obtained by successively applying 2-shuffles \( sh_3, sh_2 \) and \( sh_4 \) on the sequence \( \{1, 3, 4, 2, 3\} \) to get \( \{1, 2, 3, 3, 4\} \) and then finally deleting 3 and 3 to get \( \{1, 2, 4\} \). Here \( \tau = 3 \).

Clearly \( \{m_1, m_2, \cdots, m_k\} \triangle \{m_1, m_2, \cdots, m_k\} = \{\} \), \( 1 \leq k \leq n \), is obtained by applying \( \frac{k(k-1)}{2} \) many 2-shuffles on the sequence \( \{m_1, \cdots, m_k, m_1, \cdots, m_k\} \) and then by deleting everything.

Let \( B = \{v_1, \cdots, v_n\} \) be an ordered orthogonal basis of an \( n \) dimensional complex vector space \( V \) and the set \( F = \{1, E_{m_1 m_2 \cdots m_k}\} \), where \( E_{m_1 m_2 \cdots m_k} = v_1 v_2 \cdots v_m \), \( 1 \leq k \leq n \), \( 1 \leq m_1 < m_2 < \cdots < m_k \leq n \), form a basis of \( C(n) \). Now we define a map \( \sigma_B : F \times F \rightarrow C(n) \) by

\[
E_{m_1 m_2 \cdots m_k} \circ_B E_{t_1 t_2 \cdots t_s} = (-1)^{p+1} E_{t_1 t_2 \cdots t_p},
\]

where \( \{t_1, t_2, \cdots, t_p\} = \{m_1, m_2, \cdots, m_k\} \circ \{t_1, t_2, \cdots, t_s\} \) if \( \{m_1, m_2, \cdots, m_k\} \neq \{t_1, t_2, \cdots, t_s\} \),

\[
E_{m_1 m_2 \cdots m_k} \circ_B E_{m_1 m_2 \cdots m_k} = (-1) \frac{k(k-1)}{2}
\]

and

\[
1 \circ_B E_{m_1 m_2 \cdots m_k} = E_{m_1 m_2 \cdots m_k} = E_{m_1 m_2 \cdots m_k} \circ_B 1.
\]

Finally we extend \( \circ_B \) bilinearly on \( C(n) \times C(n) \) and get a linear map \( \sigma_B : C(n) \otimes C(n) \rightarrow C(n) \).

**Lemma 4.1.** If \( B' = \{e_1, e_2, \cdots, e_n\} \) is an ordered orthonormal basis of \( V \) then \( \sigma_{B'} \) coincides with the Clifford product in \( C(n) \).
Proof. Let $\overline{E}_{m_1m_2\cdots m_k} = e_{m_1}e_{m_2}\cdots e_{m_k}$. Now the lemma follows from the fact that
\[
\overline{E}_{m_1m_2\cdots m_k} \circ_B \overline{E}_{t_1t_2\cdots t_p} = (-1)^p \overline{E}_{t_1t_2\cdots t_p} = e_{m_1}e_{m_2}\cdots e_{m_k}e_{t_1}e_{t_2}\cdots e_{t_p}.
\]
if $\{m_1, \ldots, m_k\} \neq \{t_1, \ldots, t_p\}$,
\[
\overline{E}_{m_1m_2\cdots m_k} \circ_B \overline{E}_{m_1m_2\cdots m_k} = (-1)^{\frac{k(k-1)}{2}} = e_{m_1}e_{m_2}\cdots e_{m_k}e_{m_1}e_{m_2}\cdots e_{m_k}
\]
and
\[
1 \circ_B \overline{E}_{m_1m_2\cdots m_k} = \overline{E}_{m_1m_2\cdots m_k} = e_{m_1}e_{m_2}\cdots e_{m_k} = \overline{E}_{m_1m_2\cdots m_k} \circ_B 1.
\]
\[\square\]

Now let $C^q(C(n)) = \text{Hom}_C(C(n)^{\otimes q}, C(n))$, for all $q \geq 0$, $C^0(C(n)) = C(n)$ and $\delta_{\text{Cl}(B)} : C^q(C(n)) \to C^{q+1}(C(n))$ be the coboundary map defined by
\[
(\delta_{\text{Cl}(B)} f)(a_0 \otimes a_1 \otimes \cdots \otimes a_g) = a_0 \mathcal{B}_{B} f(a_1 \otimes \cdots \otimes a_q)
\]
\[+ \sum_{i=1}^{q} (-1)^i f(a_0 \otimes \cdots \otimes \hat{a_i} \otimes a_{i-1} \mathcal{B}_{B} a_i \otimes \cdots \otimes a_q) + (-1)^{q+1} f(a_0 \otimes \cdots \otimes a_{q-1} \mathcal{B}_{B} a_q),
\]
where $f \in C^q(C(n))$. We note that if $B' = \{e_1, e_2, \ldots, e_n\}$ is an ordered orthonormal basis of $V$ then $\delta_{\text{Cl}(B')} = \delta_{\text{Hoch}}$ by Lemma 4.1.

**Lemma 4.2.** For any choice of an ordered orthogonal basis $B$ of $V$, $\delta^2_{\text{Cl}(B)} = 0$.

Proof. Let $B = \{v_1, \ldots, v_n\}$ be an ordered orthogonal basis and $B' = \{e_1, e_2, \ldots, e_n\}$ is an ordered orthonormal basis of $V$, $E_{m_1m_2\cdots m_k} = v_{m_1}v_{m_2}\cdots v_{m_k}$ and $\overline{E}_{m_1m_2\cdots m_k} = e_{m_1}e_{m_2}\cdots e_{m_k}$. We take the vector space isomorphism $\psi : C(n) \to C(n)$ defined by
\[
\psi(E_{m_1m_2\cdots m_k}) = E_{m_1m_2\cdots m_k}
\]
and consider the isomorphism $\psi^* : C^q(C(n)) \to C^q(C(n))$ defined by
\[
f \mapsto \psi^* f,
\]
where
\[
\psi^* f(a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1}) = \psi^{-1}[f(\psi^{\otimes q}(a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1}))].
\]
It is noted earlier that $\delta_{\text{Cl}(B')} = \delta_{\text{Hoch}}$ and by Lemma 4.1, $\mathcal{B}_{B'}$ coincides with the Clifford product in $C(n)$. Now it is an easy check that the following diagram
\[
\begin{array}{c}
C^q(C(n)) \xrightarrow{\psi^*} C^q(C(n)) \\
\downarrow \quad \downarrow \psi^* \\
C^q(C(n)) \xrightarrow{\delta_{\text{Hoch}}} C^{q+1}(C(n)) \\
\end{array}
\]
commutes for all $q \geq 0$ and therefore $\psi^*(\delta^2_{\text{Cl}(B')} f) = \delta^2_{\text{Hoch}}(\psi^* f) = 0$, $f \in C^q(C(n))$. Finally as $\psi^*$ is an isomorphism so $\delta^2_{\text{Cl}(B')} f = 0$. This completes the proof. \[\square\]

We define the **Clifford cochain complex** $(C^*(C(n)); \delta_{\text{Cl}(B)})$ of the Clifford algebra $C(n)$ associated to an ordered orthogonal base $B$ of $V$ by
\[
0 \to C(n) \xrightarrow{\delta_{\text{Cl}(B)}} \cdots \xrightarrow{\delta_{\text{Cl}(B)}} C^q(C(n)) \xrightarrow{\delta_{\text{Cl}(B)}} C^{q+1}(C(n)) \xrightarrow{\delta_{\text{Cl}(B)}} \cdots
\]
and the Clifford cohomology of $C(n)$ associated to $B$ by
\[ H^*_\text{Cl}(B)(C(n)) = H^*(\text{C}(C(n)); \delta_{\text{Cl}(B)}). \]

**Theorem 4.3.** For any ordered orthogonal base $B$ of $V$, $H^*_\text{Cl}(B)(C(n)) \cong HH^*(C(n))$.

*Proof.* The proof at once follows from the commutativity of the diagram in Lemma 4.2 along with the fact that $\psi^*$ is an isomorphism. \(\square\)

**Corollary 4.4.** Up to isomorphism Clifford cohomology groups of $C(n)$ are independent of the choice of an ordered orthogonal base of $V$.

*Proof.* It readily follows from Theorem 4.3. \(\square\)

**Remark 4.5.** It follows from the construction of Clifford cochain complex that in general the coboundary maps $\delta_{\text{Cl}(B)}$ are not scalar multiples of $\delta_{\text{Hoch}}$ for arbitrary choices of an ordered orthogonal basis $B$ and consequently it is not of Hochschild cochain complex type while Clifford cohomology being isomorphic to Hochschild cohomology, controls the deformations of complex Clifford algebras.

Let $B = \{v_1, \cdots, v_n\}$ is an ordered orthogonal basis of an $n$ dimensional complex vector space $V$.

**Proposition 4.6.** If $n$ is odd, then $v_1v_2\cdots v_n$ is a $0$ cocycle in the Clifford cochain complex associated to $B$.

*Proof.* If $m \in C(n)$ then $\delta_{\text{Cl}(B)}(m)(a) = n\overline{a} - a\overline{m}$ for all $a \in C(n)$. Now if $n$ is odd then we note that $v_1v_2\cdots v_n\overline{a} = a\overline{v_1v_2\cdots v_n}$ for all $a \in C(n)$ and consequently $v_1v_2\cdots v_n$ is a $0$ cocycle in the Clifford cochain complex associated to $B$. \(\square\)

**Proposition 4.7.** If $n$ is odd then
\[ H^i_{\text{Cl}(B)}(C(n)) = \begin{cases} \mathbb{C} \oplus \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \]
and if $n$ is even then
\[ H^i_{\text{Cl}(B)}(C(n)) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \]

*Proof.* First we note that $H^*(C(n)) \cong HH^*(C(n))$ [Theorem 4.3]. Let $n$ be odd. Then $C(n)$ is Morita equivalent to $C(1) \cong \mathbb{C} \oplus \mathbb{C}$. As Hochschild cohomology is invariant under Morita equivalence therefore $HH^*(C(n)) \cong HH^*(\mathbb{C} \oplus \mathbb{C}) \cong HH^*(\mathbb{C}) \oplus HH^*(\mathbb{C})$. Now if $n$ is even then $C(n)$ is Morita equivalent to $C(0) \cong \mathbb{C}$ and $HH^*(C(n)) \cong HH^*(\mathbb{C})$. Again it is known that for any field $k$, $HH^0(k) = k$ and $HH^i(k) = 0$ for $i > 0$ ([9], 1.5.5) and this completes the proof. \(\square\)

We end this section by showing that in the category of complex Clifford algebras, Clifford cohomology associated to any ordered orthogonal basis can classify algebras up to Morita equivalence.

**Theorem 4.8.** Let $W, V$ are complex vector spaces of dimension $m$ and $n$ with $B, B'$ are any two ordered orthogonal basis of them respectively. Then the Clifford algebras
$C(m)$ and $C(n)$ over $W$ and $V$ are Morita equivalent if and only if $H^*_C(B)(C(m)) \cong H^*_C(B)(C(n))$.

**Proof.** If $C(m)$ and $C(n)$ are Morita equivalent then $HH^*(C(m)) \cong HH^*(C(n))$ and it follows from Theorem 4.3 that $H^*_C(B)(C(m)) \cong H^*_C(B)(C(n))$. Conversely, let $H^*_C(B)(C(m)) \cong H^*_C(B)(C(n))$. Now it follows from Proposition 4.7 that $m$ and $n$ must be both even or both odd and consequently $C(m)$ and $C(n)$ are Morita equivalent. □

5. **Formal deformations of smooth sections of complex Clifford algebra bundle**

The aim of this last section is to study Hochschild cohomology groups and formal deformations of the algebra of smooth sections of a complex Clifford algebra bundle over an even dimensional orientable Riemannian manifold $M$ which admits a $\text{Spin}^c$ structure. It turns out that if $M$ is 2-dimensional then the algebra of smooth sections of the complex Clifford algebra bundle over $M$ is highly non-rigid in the sense that it admits infinitely many inequivalent formal deformations.

We start by recalling very briefly some facts about $\text{Spin}$ and $\text{Spin}^c$ manifolds (for details see [8]).

**Definition 5.1.** Let $M$ be an orientable Riemannian manifold of dimension $n$, $P_{\text{SO}}(M)$ be the oriented orthonormal frame bundle over $M$ and $\text{Spin}(n)$ is the double covering group of $\text{So}(n)$. The manifold $M$ is said to have a $\text{Spin}$ structure if there exists a $\text{Spin}(n)$ bundle $P_{\text{Spin}}(M)$ over $M$ and an equivariant bundle map: $P_{\text{Spin}}(M) \to P_{\text{SO}}(M)$.

The complex analogue of $\text{Spin}(n)$ group is $\text{Spin}^c(n) = \text{Spin}(n) \times_{\mathbb{Z}_2} U(1) \subset \text{R}(n) \otimes \mathbb{C}$, where $\text{R}(n)$ denotes the real Clifford algebra over $\mathbb{R}^n$ equipped with a positive definite form.

**Definition 5.2.** We say $M$ admits a $\text{Spin}^c$ structure if there exists a $\text{Spin}^c$ bundle $P_{\text{Spin}^c}(M)$ over $M$, a $U(1)$ bundle $P_U(M)$ over $M$ and an equivariant bundle map: $P_{\text{Spin}^c}(M) \to P_{\text{SO}}(M) \times P_U(M)$.

**Definition 5.3.** Let $k = \mathbb{R}$ or $\mathbb{C}$, and $E \to M$ be a smooth $k$-vector bundle over a manifold $M$. Then $E$ is called a bundle of $k$-algebras, if each fibre $E_x$ is a $k$-algebra for any $x \in M$, such that the algebra operations are smooth.

**Definition 5.4.** Suppose $E \to M$ be a smooth $k$-vector bundle over a manifold $M$. A $k$-vector bundle $S$ over $M$ is said to be bundle of $E$-module if there is a smooth bundle map: $E \otimes S \to S$ that makes $S_x$ a $E_x$-module for each $x \in M$.

**Definition 5.5.** The Clifford algebra bundle $\text{Cl}(M)$ over a smooth manifold $M$ is obtained from the tangent bundle $TM$ by replacing each fibre $T_xM$ by the Clifford algebra over $T_xM$. More precisely, the $\text{Cl}(M) = \bigcup_{x \in M} \text{Cl}(T_xM)$. The complex Clifford algebra bundle $\text{Cl}(M)$ is obtained from $\text{Cl}(M)$ by complexifying each fibre, that is, $\text{Cl}(M) = \text{Cl}(M) \otimes \mathbb{C}$.  


Remark 5.6. It is well-known that the real Clifford algebra bundle $Cl(M)$ over $M$ is $P_{SO}(M) \times_{SO(n)} R(n)$ where $SO(n) \to Aut(R(n))$ is the natural action. Moreover, if $M$ has a Spin structure then $Cl(M)$ can also be expressed as $P_{Spin}(M) \times_{Spin(n)} R(n)$ where the action of $Spin(n)$ on $R(n)$ is the adjoint action. If $M$ is $Spin^c$ then $Cl(M)$ can also be obtained as $P_{Spin}(M) \times_{Spin^c(n)} R(n) \otimes \mathbb{C}$, where the action of $Spin^c(n)$ on $R(n) \otimes \mathbb{C}$ is the adjoint action.

From now on we denote the algebra of smooth complex functions on $M$ by $C^\infty(M)$ and the algebra of smooth complex sections of a complex vector bundle $E$ over $M$ by $\Gamma^\infty(E)$. Clearly $\Gamma^\infty(Cl(M))$ is an associative unital algebra over $\mathbb{C}$.

Theorem 5.7. If $M$ is an orientable Riemannian manifold of dimension $2m$, $m \geq 1$, which admits a $Spin^c$ structure then the Hochschild cohomology group $HH^k(\Gamma^\infty(Cl(M)))$ is non trivial for $k \leq 2m$ and trivial for $k > 2m$.

Proof. First we note that as $M$ is of dimension $2m$, therefore the complex Clifford algebra bundle $Cl(M)$ is a bundle of complex matrix algebra $M(2^m)$ over $M$. Moreover the existence of a $Spin^c$ structure on $M$ ensures that there is a complex vector bundle $E$ of fiber dimension $2^m$ over $M$ which is a $Cl(M)$ module i.e. there is a continuous bundle map: $Cl(M) \otimes E \to E$ (see [8], Proposition II.3.8 for real version).

Now by the smooth version of Serre-Swan’s theorem ([12], Theorem 11.32), $\Gamma^\infty(E)$ is a finitely generated and projective module over $C^\infty(M)$. Also by Morita’s theorem (see e.g. [7], Sec.18; [1], Theorem 4.1.) it follows that $\Gamma^\infty(End(E)) \to \Gamma^\infty(E)$ is the endomorphism bundle) is Morita equivalent to $C^\infty(M)$ where $\Gamma^\infty(E)$ is an invertible $\Gamma^\infty(End(E)), C^\infty(M))$- bimodule (see [1], Example 4.2.). Again $E$ being a $Cl(M)$ module and as $Cl(M)$ is a bundle of complex matrix algebra $M(2^m)$, therefore $End(E) \cong Cl(M)$ and consequently $\Gamma^\infty(Cl(M))$ is Morita equivalent to $C^\infty(M)$.

It is known that Morita equivalent algebras have isomorphic Hochschild cohomology groups. Let $\wedge^k TM \otimes \mathbb{C}$ denotes the complexified $k$-th exterior bundle over $M$ and let $\chi^*(M) = \bigoplus_{k=0}^\infty \Gamma^\infty(\wedge^k TM \otimes \mathbb{C})$. Now we consider the Hochschild-Kostant-Rosenberg map $U: \chi^k(M) \to HH^k(C^\infty(M))$ defined by: $X \mapsto U(X), X \in \chi^k(M) = \Gamma^\infty(\wedge^k TM \otimes \mathbb{C})$ and $U(X)(f_1, \ldots, f_k) = \frac{1}{k!}X(df_1, \ldots, df_k)$ [X can be viewed as a multilinear alternating map: $X: \Omega^k(M) \times \cdots \times \Omega^1(M) \to C^\infty(M)$, where $\Omega^1(M)$ is the space of complexified 1 forms i.e. $\Omega^1(M) = \Gamma^\infty(T^*M \otimes \mathbb{C})$, $T^*(M)$ is the cotangent bundle of $M$]. We note that if $X = X_1 \wedge \cdots \wedge X_i \wedge \cdots \wedge X_k$, where $X_i \in \chi^1(M)$, then $U(X)(f_1, \ldots, f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{Sign } \sigma)X_{\sigma(1)}(f_1) \cdots X_{\sigma(k)}(f_k)$.

Finally as the Hochschild-Kostant-Rosenberg map $U: \chi^k(M) \to HH^k(C^\infty(M))$ is injective ([13], Cor: 6.2.47) and there are infinitely elements in $\chi^k(M)$ for $k \leq 2m$, therefore the non triviality of $HH^k(\Gamma^\infty(Cl(M)))$, $k \leq 2m$ follows from this. The triviality of $HH^k(\Gamma^\infty(Cl(M)))$ while $k > 2m$ follows from the fact that $\wedge^k T_p M \otimes \mathbb{C}$ is the zero vector space for $k > 2m$ and for all $p \in M$.

A star product on $M$ is a formal deformation of $C^\infty(M)$, i.e. an associative product $\star$ on the $\mathbb{C}[[t]]$ module $C^\infty(M)[[t]]$ given by: for $f, g \in C^\infty(M)$,

$$f \star g = fg + \sum_{i=1}^\infty \mu_i(f, g)t^i$$
where $\mu_i : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $i = 1, 2, \cdots$ are bi-differential operators. We denote the equivalence classes of star products by $Def(M)$.

Any Lie bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ which is compatible with the pointwise product on $C^\infty(M)$ via the Liebniz rule is called a Poisson structure on $M$. Given any star product $\star$ on $M$, it is known that $\{f,g\} = \frac{1}{i}(\mu_1(f,g) - \mu_1(g,f))$; $f, g \in C^\infty(M)$, is a Poisson structure on $M$ (see [1] Sec. 3.2.).

**Theorem 5.8.** If $M$ is a 2-dimensional $\text{Spin}^c$ manifold then $\Gamma^\infty(\text{Cl}(M))$ admits infinitely many inequivalent formal deformations.

**Proof.** As $M$ is $\text{Spin}^c$ therefore $\Gamma^\infty(\text{Cl}(M))$ is Morita equivalent to $C^\infty(M)$ (follows from the proof of Theorem 5.7.). Again as the set of equivalence classes of formal deformations is Morita invariant ([3], section 16) so it suffices to explore $Def(M)$. As $M$ is of dimension 2, therefore each complex bi-vector field $\pi \in \chi^2(M) = \Gamma^\infty(\bigwedge^2 TM \otimes \mathbb{C})$ induces a Poisson structure $\{\cdot, \cdot\}_\pi$ on $M$ defined by: $\{f,g\}_\pi = \pi(df,dg)$. Now by Kontsevich’s classification result ([6]; [1] Theorem 3.3.) distinct Poisson structures on $M$ corresponds to distinct elements in $Def(M)$. Finally as $\chi^2(M)$ is clearly an infinite set therefore $Def(M)$ is also infinite. This completes the proof. □

**Corollary 5.9.** $\Gamma^\infty(\text{Cl}(S^1 \times S^1))$ and $\Gamma^\infty(\text{Cl}(S^2))$ have infinitely many inequivalent formal deformations.

**Proof.** As $S^1 \times S^1$ is parallelizable so it is $\text{Spin}^c$ and tangent bundle of $S^2$ being stably trivial, it is a $\text{Spin}$ and therefore $\text{Spin}^c$. Now the proof readily follows from Theorem 5.8. □

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References

[1] Bursztyn H., Waldmann S., Classifying Morita Equivalent Star Products, Clay Mathematics Proceedings, Volume 16, 2012.

[2] Doubek M., Markl M., Zima P., Deformation Theory (Lecture Notes), arXiv: 0705.3719

[3] Gerstenhaber M., Schack D.S., Algebraic cohomology and deformation theory, in Deformation Theory of Algebras and Structures and Applications, Editors M. Hazewinkel and M. Gerstenhaber, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, Vol. 247, Kluwer Academic Publishers Group, Dordrecht, 1988.

[4] Gerstenhaber M., On the deformation of rings and algebras, Ann. of Math. (2) 79 (1964), 59-103.

[5] Kassel Christian., Homology and cohomology of associative algebras- A concise introduction to cyclic homology. Théorème-École thématique. Août 2004 à ICTP, Trieste (Italie), 2006. cel-00119891

[6] Kontsevich M., Deformation quantization of Poisson manifolds, Lett. Math. Phys. 56 (2003), 271–294.

[7] Lam T.Y. Lectures on Modules and Rings, Graduate Texts in Mathematics, Springer-Verlag, 1999.

[8] Lawson H.B., Michelshon Marie-Louise, Spin Geometry, Princeton University Press, 1990

[9] Loday J-L., Cyclic homology, Grundlehren der Mathematischen Wissenschaften, Vol. 301, Springer-Verlag, Berlin, 1992.

[10] Lounesto P., Clifford Algebras and Spinors, Cambridge University Press, Cambridge, 1997.

[11] Lundholm D., Svensson L., Clifford algebra, Geometric algebra and applications, arXiv: 0907.5356

[12] Nestruev J., Smooth manifolds and observables, vol. 220 in Graduate Texts in Mathematics, Springer-Verlag, 2003.

[13] Waldmann S., Poisson-Geometric und Deformationsquantisierung: Eine Einführung, Springer-Lehrbuch Masterclass, Springer, Heidelberg (2007).

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