TRIALITY AND ALGEBRAIC GROUPS OF TYPE $3D_4$

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Abstract. We determine which simple algebraic groups of type $3D_4$ over arbitrary fields of characteristic different from 2 admit outer automorphisms of order 3, and classify these automorphisms up to conjugation. The criterion is formulated in terms of a representation of the group by automorphisms of a trialtarian algebra: outer automorphisms of order 3 exist if and only if the algebra is the endomorphism algebra of an induced cyclic composition; their conjugacy classes are in one-to-one correspondence with isomorphism classes of symmetric compositions from which the induced cyclic composition stems.

1. Introduction

Let $G_0$ be an adjoint Chevalley group of type $D_4$ over a field $F$. Since the automorphism group of the Dynkin diagram of type $D_4$ is isomorphic to the symmetric group $S_3$, there is a split exact sequence of algebraic groups

$1 \to G_0 \to \text{Aut}(G_0) \to S_3 \to 1.$

Thus, $\text{Aut}(G_0) \cong G_0 \rtimes S_3$; in particular $G_0$ admits outer automorphisms of order 3, which we call trialtarian automorphisms. Adjoint algebraic groups of type $D_4$ over $F$ are classified by the Galois cohomology set $H^1(F, G_0 \rtimes S_3)$ and the map induced by $\pi$ in cohomology

$\pi_* : H^1(F, G_0 \rtimes S_3) \to H^1(F, S_3)$

associates to any group $G$ of type $D_4$ the isomorphism class of a cubic étale $F$-algebra $L$. The group $G$ is said to be of type $1D_4$ if $L$ is split, of type $2D_4$ if $L \cong F \times \Delta$ for some quadratic separable field extension $\Delta/F$, of type $3D_4$ if $L$ is a cyclic field extension of $F$ and of type $6D_4$ if $L$ is a non-cyclic field extension. An easy argument given in Theorem 4.1 below shows that groups of type $2D_4$ and $6D_4$ do not admit trialtarian automorphisms defined over the base field. Trialtarian automorphisms of groups of type $1D_4$ were classified in [3], and by a different method in [2]: the adjoint groups of type $1D_4$ that admit trialtarian automorphisms are the groups of proper projective similitudes of 3-fold Pfister quadratic spaces; their trialtarian automorphisms are shown in [3, Th. 5.8] to be in one-to-one correspondence with the symmetric composition structures on the quadratic space. In the present paper,

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we determine the simple groups of type $^3D_4$ that admit trialitarian automorphisms, and we classify those automorphisms up to conjugation.

Our main tool is the notion of a trialitarian algebra, as introduced in [7, Ch. X]. Since these algebras are only defined in characteristic different from 2, we assume throughout (unless specifically mentioned) that the characteristic of the base field $\mathbb{F}$ is different from 2. In view of [7, Th. (44.8)], every adjoint simple group $G$ of type $D_4$ can be represented as the automorphism group of a trialitarian algebra $T = (E, L, \sigma, \alpha)$. In the datum defining $T$, $L$ is the cubic étale $\mathbb{F}$-algebra given by the map $\pi_*$ above, $E$ is a central simple $L$-algebra with orthogonal involution $\sigma$, known as the Allen invariant of $G$ (see [1]), and $\alpha$ is an isomorphism relating $(E, \sigma)$ with its Clifford algebra $C(E, \sigma)$ (we refer to [7, §43] for details). We show in Theorem 4.1 that if $G$ admits an outer automorphism of order 3 modulo inner automorphisms, then $L$ is either split (i.e., isomorphic to $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$), or it is a cyclic field extension of $\mathbb{F}$ (so $G$ is of type $^1D_4$ or $^3D_4$), and the Allen invariant $E$ of $G$ is a split central simple $L$-algebra. This implies that $T$ has the special form $T = \text{End} \Gamma$ for some cyclic composition $\Gamma$. We further show in Theorem 4.2 that if $G$ carries a trialitarian automorphism, then the cyclic composition $\Gamma$ is induced, which means that it is built from some symmetric composition over $\mathbb{F}$, and we establish a one-to-one correspondence between trialitarian automorphisms of $G$ up to conjugation and isomorphism classes of symmetric compositions over $\mathbb{F}$ from which $\Gamma$ is built.

The notions of symmetric and cyclic compositions are recalled in §2. Trialitarian algebras are discussed in §3 which contains the most substantial part of the argument: we determine the trialitarian algebras that have semilinear automorphisms of order 3 (Theorem 3.1) and we classify these automorphisms up to conjugation (Theorem 3.5). The group-theoretic results follow easily in §4 by using the correspondence between groups of type $D_4$ and trialitarian algebras.

Notation is generally as in the Book of Involutions [7], which is our main reference. For an algebraic structure $S$ defined over a field $F$, we let $\text{Aut}(S)$ denote the group of automorphisms of $S$, and write $\text{Aut}(S)$ for the corresponding group scheme over $F$.

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2. CYCLIC AND SYMMETRIC COMPOSITIONS

Cyclic compositions were introduced by Springer in his 1963 Göttingen lecture notes ([8], [9]) to get new descriptions of Albert algebras. We recall their definition from [9] and [7, §36.B], restricting to the case of dimension 8.

Let $F$ be an arbitrary field (of any characteristic). A cyclic composition (of dimension 8) over $F$ is a 5-tuple $\Gamma = (V, L, Q, \rho, *)$ consisting of

- a cubic étale $F$-algebra $L$;
- a free $L$-module $V$ of rank 8;
- a quadratic form $Q: V \rightarrow L$ with nondegenerate polar bilinear form $b_Q$;
- an $F$-automorphism $\rho$ of $L$ of order 3;

1 A cyclic composition is called a normal twisted composition in [8] and [9].
• an $F$-bilinear map $*: V \times V \to V$ with the following properties: for all $x, y, z \in V$ and $\lambda \in L$, 
\[
(\lambda x) * y = \rho(\lambda)(x * y), \quad x * (y\lambda) = (x * y)\rho^2(\lambda), \\
Q(x * y) = \rho(Q(x)) \cdot \rho^2(Q(y)), \\
b_Q(x * y, z) = \rho(b_Q(y * z, x)) = \rho^2(b_Q(z * x, y)).
\]

These properties imply the following (see [7, §36.B] or [9, Lemma 4.1.3]): for all $x, y \in V$, 
\[
(1) \quad (x * y) * x = \rho^2(Q(x))y \quad \text{and} \quad x * (y * x) = \rho(Q(x))y.
\]

Since the cubic étale $F$-algebra $L$ has an automorphism of order 3, $L$ is either a cyclic cubic field extension of $F$, and $\rho$ is a generator of the Galois group, or we may identify $L$ with $F \times F \times F$ and assume $\rho$ permutes the components cyclically. We will almost exclusively restrict to the case where $L$ is a field; see however Remark 2.3 below.

Let $\Gamma' = (V', L', Q', \rho', *)$ be also a cyclic composition over $F$. An isotopy $^2$ $\Gamma \to \Gamma'$ is defined to be a pair $(\nu, f)$ where $\nu: (L, \rho) \to (L', \rho')$ is an isomorphism of $F$-algebras with automorphisms (i.e., $\nu \circ \rho = \rho' \circ \nu$) and $f: V \to V'$ is a $\nu$-semilinear isomorphism for which there exists $\mu \in L^\times$ such that 
\[
Q'(f(x)) = \nu(\mu)\rho^2(\mu) \cdot Q(x) \quad \text{and} \quad f(x) * f(y) = \nu(\mu)f(x * y)
\]
for $x, y \in V$. The scalar $\mu$ is called the multiplier of the isotopy. Isotopies with multiplier 1 are isomorphisms. When the map $\nu$ is clear from the context, we write simply $f$ for the pair $(\nu, f)$, and refer to $f$ as a $\nu$-semilinear isotopy.

Examples of cyclic compositions can be obtained by scalar extension from symmetric compositions over $F$, as we now show. Recall from [7, §34] that a symmetric composition (of dimension 8) over $F$ is a triple $\Sigma = (S, n, *)$ where $(S, n)$ is an 8-dimensional $F$-quadratic space (with nondegenerate polar bilinear form $b_n$) and $*: S \times S \to S$ is a bilinear map such that for all $x, y, z \in S$
\[
n(x * y) = n(x)n(y) \quad \text{and} \quad b_n(x * y, z) = b_n(x, y * z).
\]

If $\Sigma' = (S', n', *)$ is also a symmetric composition over $F$, an isotopy $\Sigma \to \Sigma'$ is a linear map $f: S \to S'$ for which there exists $\lambda \in F^\times$ (called the multiplier) such that 
\[
n'(f(x)) = \lambda^2 n(x) \quad \text{and} \quad f(x) * f(y) = \lambda f(x * y) \quad \text{for} \ x, y \in S.
\]

Note that if $f: \Sigma \to \Sigma'$ is an isotopy with multiplier $\lambda$, then $\lambda^{-1}f: \Sigma \to \Sigma'$ is an isomorphism. Thus, symmetric compositions are isotopic if and only if they are isomorphic. For an explicit example of a symmetric composition, take a Cayley (octonion) algebra $(C, \cdot)$ with norm $n$ and conjugation map $\overline{\cdot}$. Letting $x * y = \overline{xy}$ for $x, y \in C$ yields a symmetric composition $\overline{C} = (C, n, \cdot)$, which is called a para-Cayley composition (see [7, §34.A]).

Given a symmetric composition $\Sigma = (S, n, *)$ and a cubic étale $F$-algebra $L$ with an automorphism $\rho$ of order 3, we define a cyclic composition $\Sigma \otimes (L, \rho)$ as follows:
\[
\Sigma \otimes (L, \rho) = (S \otimes_F L, L, n_L, \rho, *)
\]
\[\text{The term used in [7, p. 490] is similarity.} \]
where \( n_L \) is the scalar extension of \( n \) to \( L \) and \(*\) is defined by extending \(*\) linearly to \( S \otimes_F L \) and then setting

\[
x \ast y = (\text{Id}_S \otimes \rho)(x) \ast (\text{Id}_S \otimes \rho^2)(y) \quad \text{for } x, y \in S \otimes_F L.
\]

(See [7] (36.11)). Clearly, every isotopy \( f : \Gamma \to \Gamma' \) of symmetric compositions extends to an isotopy of cyclic compositions \( (\text{Id}_L, f) : \Sigma \otimes (L, \rho) \to \Sigma' \otimes (L, \rho) \).

Observe for later use that the map \( \tilde{\rho} = \text{Id}_S \otimes \rho \in \text{End}_F(S \otimes_F L) \) defines a \( \rho \)-semilinear automorphism

\[
\tilde{\rho} : \Sigma \otimes (L, \rho) \xrightarrow{\sim} \Sigma \otimes (L, \rho)
\]

such that \( \tilde{\rho}^3 = \text{Id} \).

We call a cyclic composition that is isotopic to \( \Sigma \otimes (L, \rho) \) for some symmetric composition \( \Sigma \) induced. Cyclic compositions induced from para-Cayley symmetric compositions are called reduced in [7].

**Remark 2.1.** Induced cyclic compositions are not necessarily reduced. This can be shown by using the following cohomological argument. We assume for simplicity that the field \( F \) contains a primitive cube root of unity \( \omega \). There is a cohomological invariant \( g_3(\Gamma) \in H^3(F, \mathbb{Z}/3\mathbb{Z}) \) attached to any cyclic composition \( \Gamma \). The cyclic composition \( \Gamma \) is reduced if and only if \( g_3(\Gamma) = 0 \) (we refer to [9] [§8.3] or [7] [§40] for details). We construct an induced cyclic composition \( \Gamma \) with \( g_3(\Gamma) \neq 0 \). Let \( \alpha, \beta \in F^\times \) and let \( A(\alpha, \beta) \) be the \( F \)-algebra with generators \( \alpha, \beta \) and relations \( \alpha^3 = \alpha, \beta^3 = \beta, \beta\alpha = \omega\alpha\beta \). The algebra \( A(\alpha, \beta) \) is central simple of dimension 9 and the space \( A^0 \) of elements of \( A(\alpha, \beta) \) of reduced trace zero admits the structure of a symmetric composition \( \Sigma(\alpha, \beta) = (A^0, n, \ast) \) (see [7] (34.19)). Such symmetric compositions are called Okubo symmetric compositions. From the Elduque–Myung classification of symmetric compositions [4] p. 2487 (see also [7] (34.37)), it follows that symmetric compositions are either para-Cayley or Okubo. Let \( L = F(\gamma) \) with \( \gamma^3 = c \in F^\times \) be a cubic cyclic field extension of \( F \), and let \( \rho \) be the \( F \)-automorphism of \( L \) such that \( \gamma \mapsto \omega\gamma \). We may then consider the induced cyclic composition \( \Gamma(\alpha, \beta, c) = \Sigma(\alpha, \beta) \otimes (L, \rho) \). Its cohomological invariant \( g_3(\Gamma(\alpha, \beta, c)) \) can be computed by the construction in [9] [§8.3]: Using \( \omega \), we identify the group \( \mu_3 \) of cube roots of unity in \( F \) with \( \mathbb{Z}/3\mathbb{Z} \), and for any \( u \in F^\times \) we write \([u]\) for the cohomology class in \( H^1(F, \mathbb{Z}/3\mathbb{Z}) \) corresponding to the cube class \( uF^\times \) under the isomorphism \( F^\times / F^\times = H^1(F, \mathbb{Z}/3\mathbb{Z}) \) arising from the Kummer exact sequence (see [7] p. 413). Then \( g_3(\Gamma(\alpha, \beta, c)) = \) the cup-product \([a] \cup [b] \cup [c] \in H^3(F, \mathbb{Z}/3\mathbb{Z})\). Thus any cyclic composition \( \Gamma(\alpha, \beta, c) \) with \([a] \cup [b] \cup [c] \neq 0 \) is induced but not reduced.

Another cohomological argument can be used to show that there exist cyclic compositions that are not induced. We still assume that \( F \) contains a primitive cube root of unity \( \omega \). There is a further cohomological invariant of cyclic compositions \( f_3(\Gamma) \in H^3(F, \mathbb{Z}/2\mathbb{Z}) \) which is zero for any cyclic composition induced by an Okubo symmetric composition [3] and is given by the class in \( H^3(F, \mathbb{Z}/2\mathbb{Z}) \) of the 3-fold Pfister form which is the norm of \( C \) if \( \Gamma \) is induced from the para-Cayley \( C \) (see for example [7] [§40]). Thus a cyclic composition \( \Gamma \) with \( f_3(\Gamma) \neq 0 \) and \( g_3(\Gamma) \neq 0 \) is not induced. Such examples can be given with the help of the Tits process used for constructing Albert algebras (see [7] [§39 and §40]). However, for example, cyclic

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[3] The fact that \( F \) contains a primitive cubic root of unity is relevant for this claim.
compositions over finite fields, $p$-adic fields or algebraic number fields are reduced, see [3, p. 108].

**Examples 2.2.** (i) Let $F = \mathbb{F}_q$ be the field with $q$ elements, where $q$ is odd and $q \equiv 1 \mod 3$. Thus $F$ contains a primitive cube root of unity and we are in the situation of Remark 2.1. Let $L = \mathbb{F}_{q^3}$ be the (unique, cyclic) cubic field extension of $F$, and let $\rho$ be the Frobenius automorphism of $L/F$. Because $H^3(F, \mathbb{Z}/3\mathbb{Z}) = 0$, every cyclic composition over $F$ is reduced; moreover every 3-fold Pfister form is hyperbolic, hence every Cayley algebra is split. Therefore, up to isomorphism there is a unique cyclic composition over $F$.

(ii) Assume that $F$ contains a primitive cube root of unity and that $F$ carries an anisotropic 3-fold Pfister form $n$. Let $C$ be the non-split Cayley algebra with norm $n$ and let $\tilde{C}$ be the associated para-Cayley algebra. For any cubic field extension $(L, \rho)$ the norm $n_L$ of the cyclic composition $\tilde{C} \otimes (L, \rho)$ is anisotropic. Thus it follows from the Elduque–Myung classification that any symmetric composition $\Sigma$ such that $(L, \rho)$ is isotopic to $\tilde{C} \otimes (L, \rho)$ must be isomorphic to $\tilde{C}$.

(iii) Finally, we observe that the cyclic compositions of type $\Gamma(a, b, c)$, described in Remark 2.1, have invariant $g_3$ equal to zero if $c = a$. Since the $f_3$-invariant is also zero, they are all isotopic to the cyclic composition induced by the split para-Cayley algebra. Thus we can get (over suitable fields) examples of many mutually non-isomorphic symmetric compositions $\Sigma(a, b)$ that induce isomorphic cyclic compositions $\Gamma(a, b, c)$.

Of course, besides this construction of cyclic compositions by induction from symmetric compositions, we can also extend scalars of a cyclic composition: if $\Gamma = (V, L, Q, \rho, \ast)$ is a cyclic composition over $F$ and $K$ is any field extension of $F$, then $\Gamma_K = (V \otimes_F K, L \otimes_F K, Q_K, \rho \otimes \text{Id}_K, \ast_K)$ is a cyclic composition over $K$.

**Remark 2.3.** Let $\Gamma = (V, L, Q, \rho, \ast)$ be an arbitrary cyclic composition over $F$ with $L$ a field. Write $\theta$ for $\rho^2$. We have an isomorphism of $L$-algebras

$$\nu: L \otimes_F L \to L \times L \times L \quad \text{given by} \quad \ell_1 \otimes \ell_2 \mapsto (\ell_1 \ell_2, \rho(\ell_1)\ell_2, \theta(\ell_1)\ell_2).$$

Therefore, the extended cyclic composition $\Gamma_L$ over $L$ has a split cubic étale algebra. To give an explicit description of $\Gamma_L$, note first that under the isomorphism $\nu$ the automorphism $\rho \otimes \text{Id}_L$ is identified with the map $\tilde{\rho}$ defined by $\tilde{\rho}(\ell_1, \ell_2, \ell_3) = (\ell_2, \ell_3, \ell_1)$. Consider the twisted $L$-vector spaces $\rho V$, $\theta V$ defined by

$$\rho V = \{\rho x \mid x \in V\}, \quad \theta V = \{\theta x \mid x \in V\}$$

with the operations

$$\rho(x + y) = \rho x + \rho y, \quad \theta(x + y) = \theta x + \theta y, \quad \rho(x \lambda) = (\rho x)\rho(\lambda), \quad \theta(x \lambda) = (\theta x)\theta(\lambda)$$

for $x, y \in V$ and $\lambda \in L$. Define quadratic forms $\theta Q: \rho V \to L$ and $\rho Q: \theta V \to L$ by

$$\rho Q(\rho x) = \rho(Q(x)) \quad \text{and} \quad \theta Q(\theta x) = \theta(Q(x)) \quad \text{for} \quad x \in V,$
This homomorphism is injective because this is the main case for the purposes of this paper.

We may then consider the quadratic form

\[ Q \times \rho Q \times \rho Q : V \times \rho V \times \rho V \to L \times L \times L \]

and the product \( \circ : (V \times \rho V \times \rho V) \times (V \times \rho V \times \rho V) \to (V \times \rho V \times \rho V) \) defined by

\[ (x, \rho x, \rho y) \circ (y, \rho y, \rho y) = (\rho x \ast \rho y, \rho x \ast y, \rho x \ast \rho y). \]

Straightforward calculations show that the \( F \)-vector space isomorphism \( f : V \otimes_F L \to V \times \rho V \times \rho V \) given by

\[ f(x \otimes \xi) = (x \xi, (\rho x) \xi, (\rho^2 x) \xi) \quad \text{for } x \in V \text{ and } \xi \in L \]

defines with \( \nu \) an isomorphism of cyclic compositions

\[ \Gamma_L \isoto (V \times \rho V \times \rho V, L \times L \times L, Q \times \rho Q \times \rho Q, \bar{\nu}, \circ). \]

## 3. Trialitarian algebras

In this section, we assume that the characteristic of the base field \( F \) is different from 2. Trialitarian algebras are defined in [7, §43] as 4-tuples \( T = (E, L, \sigma, \alpha) \) where \( L \) is a cubic étale \( F \)-algebra, \((E, \sigma)\) is a central simple \( L \)-algebra of degree 8 with an orthogonal involution, and \( \alpha \) is an isomorphism from the Clifford algebra \( C(E, \sigma) \) to a certain twisted scalar extension of \( E \). We just recall in detail the special case of trialitarian algebras of the form \( \text{End} \Gamma \) for \( \Gamma \) a cyclic composition, because this is the main case for the purposes of this paper.

Let \( \Gamma = (V, L, Q, \rho, \ast) \) be a cyclic composition (of dimension 8) over \( F \), with \( L \) a field, and let \( \theta = \rho^2 \). Let also \( \sigma_Q \) denote the orthogonal involution on \( \text{End}_L V \) adjoint to \( Q \). We will use the product \( \ast \) to see that the Clifford algebra \( C(V, Q) \) is split and the even Clifford algebra \( C_0(V, Q) \) decomposes into a direct product of two split central simple \( L \)-algebras of degree 8. Using the notation of Remark 2.3 to any \( x \in V \) we associate \( L \)-linear maps

\[ \ell_x : \rho V \to \rho V \quad \text{and} \quad r_x : \rho V \to \rho V \]

defined by

\[ \ell_x(\rho y) = x \ast \rho y = \theta(x \ast y) \quad \text{and} \quad r_x(\rho y) = \theta \ast \rho y = \rho(z \ast x) \]

for \( y, z \in V \). From [2] it follows that for \( x \in V \) the \( L \)-linear map

\[ \alpha_*(x) = \left( \begin{array}{cc} 0 & r_x \\ \ell_x & 0 \end{array} \right) : \rho V \oplus \rho V \to \rho V \oplus \rho V \]

given by \((\rho y, \rho z) \mapsto (r_x(\rho z), \ell_x(\rho y))\) satisfies \( \alpha_*(x)^2 = Q(x) \text{Id} \). Therefore, there is an induced \( L \)-algebra homomorphism

\[ \alpha_* : C(V, Q) \to \text{End}_L(\rho V \oplus \rho V). \]

This homomorphism is injective because \( C(V, Q) \) is a simple algebra, hence it is an isomorphism by dimension count. It restricts to an \( L \)-algebra isomorphism

\[ \alpha_{*0} : C_0(V, Q) \isoto \text{End}_L(\rho V) \times \text{End}_L(\rho V), \]

see [7, (36.16)]. Note that we may identify \( \text{End}_L(\rho V) \) with the twisted algebra \( \rho(\text{End}_L V) \) (where multiplication is defined by \( \rho f_1 \cdot \rho f_2 = \rho(f_1 \circ f_2) \)) as follows: for
a cyclic composition, is defined to be an isomorphism of $F$-algebras with involution $\alpha_{\sigma_0} \colon C(\text{End}_L V, \sigma_Q) \to \rho(\text{End}_L V) \times \theta(\text{End}_L V)$.

Thus, $\alpha_{\sigma_0}$ depends only on $\text{End}_L V$ and $\sigma_Q$. The trialitarian algebra $\text{End}_\Gamma$ is the 4-tuple

$$\text{End}_\Gamma = (\text{End}_L V, L, \sigma_Q, \alpha_{\sigma_0}).$$

An isomorphism of trialitarian algebras $\text{End}_\Gamma \cong \text{End}_\Gamma'$, for $\Gamma' = (V', L', Q', \rho', \ast')$ a cyclic composition, is defined to be an isomorphism of $F$-algebras with involution $\varphi : (\text{End}_L V, \sigma_Q) \to (\text{End}_{L'} V', \sigma_{Q'})$ subject to the following conditions:

(i) the restriction of $\varphi$ to the center of $\text{End}_L V$ is an isomorphism $\varphi|_L : (L, \rho) \to (L', \rho')$, and

(ii) the following diagram (where $\theta' = \rho'^2$) commutes:

$$
\begin{array}{ccc}
C(\text{End}_L V, \sigma_Q) & \xrightarrow{\alpha_{\sigma_0}} & \rho(\text{End}_L V) \times \theta(\text{End}_L V) \\
\downarrow{\scriptstyle C(\varphi)} & & \downarrow{\scriptstyle \rho \times \theta \varphi} \\
C(\text{End}_{L'} V', \sigma_{Q'}) & \xrightarrow{\alpha_{\sigma_0}} & \rho'(\text{End}_{L'} V') \times \theta'(\text{End}_{L'} V')
\end{array}
$$

For example, it is straightforward to check that every isotopy $(\nu, f) : \Gamma \to \Gamma'$ induces an isomorphism $\text{End}_\Gamma \to \text{End}_{\Gamma'}$ mapping $g \in \text{End}_L V$ to $f \circ g \circ f^{-1} \in \text{End}_{L'} V'$. As part of the proof of the main theorem below, we show that every isomorphism $\text{End}_\Gamma \cong \text{End}_{\Gamma'}$ is induced by an isotopy; see Lemma 5.3 (A cohomological proof that the trialitarian algebras $\text{End}_\Gamma$, $\text{End}_{\Gamma'}$ are isomorphic if and only if the cyclic compositions $\Gamma$, $\Gamma'$ are isotopic is given in [7 (44.16)].)

We show that the trialitarian algebra $\text{End}_\Gamma$ admits a $\rho$-semilinear automorphism of order 3 if and only if $\Gamma$ is reduced. More precisely:

**Theorem 3.1.** Let $\Gamma = (V, L, Q, \rho, \ast)$ be a cyclic composition over $F$, with $L$ a field.

(i) If $\Sigma$ is a symmetric composition over $F$ and $f : \Sigma \otimes (L, \rho) \to \Gamma$ is an $L$-linear isotopy, then the automorphism $\tau_{(\Sigma, f)} = \text{Int}(f \circ \hat{\rho} \circ f^{-1})|_{\text{End}_L V}$ of $\text{End}_\Gamma$, where $\hat{\rho}$ is defined in [8], is such that $\tau_{(\Sigma, f)}|_L = \text{Id}$ and $\tau_{(\Sigma, f)}|_{\Gamma} = \rho$. The automorphism $\tau_{(\Sigma, f)}$ only depends, up to conjugation in $\text{Aut}_F(\text{End}_{\Gamma})$, on the isomorphism class of $\Sigma$.

(ii) If $\text{End}_\Gamma$ carries an $F$-automorphism $\tau$ such that $\tau|_L = \rho$ and $\tau^3 = \text{Id}$, then $\Gamma$ is reduced. More precisely, there exists a symmetric composition $\Sigma$ over $F$ and an $L$-linear isotopy $f : \Sigma \otimes (L, \rho) \to \Gamma$ such that $\tau = \tau_{(\Sigma, f)}$.

**Proof.** (i) It is clear that $\tau_{(\Sigma, f)}^3 = \text{Id}$ and $\tau_{(\Sigma, f)}|_L = \rho$. For the last claim, note that if $g : \Sigma \otimes (L, \rho) \to \Gamma$ is another $L$-linear isotopy, then $f \circ g^{-1}$ is an isotopy of $\Gamma$, hence $\text{Int}(f \circ g^{-1})$ is an automorphism of $\text{End}_{\Gamma}$, and

$$\tau_{(\Sigma, f)} = \text{Int}(f \circ g^{-1}) \circ \tau_{(\Sigma, g)} \circ \text{Int}(f \circ g^{-1})^{-1}.$$ 

The proof of claim (ii) relies on three lemmas. Until the end of this section, we fix a cyclic composition $\Gamma = (V, L, Q, \rho, \ast)$, with $L$ a field. We start with some general
observations on $\rho$-semilinear automorphisms of $\text{End}_L V$. For this, we consider the inclusions

$$L \hookrightarrow \text{End}_L V \hookrightarrow \text{End}_F V.$$ 

The field $L$ is the center of $\text{End}_L V$, hence every automorphism of $\text{End}_L V$ restricts to an automorphism of $L$.

**Lemma 3.2.** Let $\nu \in \{\text{Id}_L, \rho, \theta\}$ be an arbitrary element in the Galois group $\text{Gal}(L/F)$. For every $F$-linear automorphism $\varphi$ of $\text{End}_L V$ such that $\varphi|_L = \nu$, there exists an invertible transformation $u \in \text{End}_F V$ such that $\varphi(f) = u \circ f \circ u^{-1}$ for all $f \in \text{End}_L V$. The map $u$ is uniquely determined up to a factor in $L^\times$; it is $\nu$-semilinear, i.e., $u(x\lambda) = u(x)\nu(\lambda)$ for all $x \in V$ and $\lambda \in L$. Moreover, if $\varphi \circ \sigma_Q = \sigma_Q \circ \varphi$, then there exists $\mu \in L^\times$ such that

$$Q(u(x)) = \nu(\mu \cdot Q(x)) \quad \text{for all } x \in V.$$

**Proof.** The existence of $u$ is a consequence of the Skolem–Noether theorem, since $\text{End}_L V$ is a simple subalgebra of the simple algebra $\text{End}_F V$: the automorphism $\varphi$ extends to an inner automorphism $\text{Int}(u)$ of $\text{End}_F V$ for some invertible $u \in \text{End}_F V$. Uniqueness of $u$ up to a factor in $L^\times$ is clear because $L$ is the centralizer of $\text{End}_L V$ in $\text{End}_F V$, and the $\nu$-semilinearity of $u$ follows from the equation $\varphi(f) = u \circ f \circ u^{-1}$ applied with $f$ the scalar multiplication by an element in $L$.

Now, suppose $\varphi$ commutes with $\sigma_Q$, hence for all $f \in \text{End}_L V$

$$u \circ \sigma_Q(f) \circ u^{-1} = \sigma_Q(u \circ f \circ u^{-1}).$$

Let $\text{Tr}_\nu(Q)$ denote the transfer of $Q$ along the trace map $\text{Tr}_{L/F}$, so $\text{Tr}_\nu(Q): V \to F$ is the quadratic form defined by $\text{Tr}_\nu(Q)(x) = \text{Tr}_{L/F}(Q(x))$. The adjoint involution $\sigma_{\text{Tr}_\nu(Q)}$ coincides on $\text{End}_L V$ with $\sigma_Q$, hence from (5) it follows that $\sigma_{\text{Tr}_\nu(Q)}(u)u$ centralizes $\text{End}_L V$. Therefore, $\sigma_{\text{Tr}_\nu(Q)}(u)u = \mu$ for some $\mu \in L^\times$. We then have $b_{\text{Tr}_\nu(Q)}(u(x), u(y)) = b_{\text{Tr}_\nu(Q)}(x, y\mu)$ for all $x, y \in V$, which means that

$$\text{Tr}_{L/F}(b_Q(u(x), u(y))) = \text{Tr}_{L/F}(\mu b_Q(x, y)).$$

Now, observe that since $u$ is $\nu$-semilinear, the map $c: V \times V \to L$ defined by $c(x, y) = \nu^{-1}(b_Q(u(x), u(y)))$ is $L$-bilinear. From (4), it follows that $c - \mu b_Q$ is a bilinear map on $V$ that takes its values in the kernel of the trace map. But the value domain of an $L$-bilinear form is either $L$ or $\{0\}$, and the trace map is not the zero map. Therefore, $c - \mu b_Q = 0$, which means that

$$\nu^{-1}(b_Q(u(x), u(y))) = \mu b_Q(x, y) \quad \text{for all } x, y \in V,$$

hence $Q(u(x)) = \nu(\mu \cdot Q(x))$ for all $x \in V$. \qed

Note that the arguments in the preceding proof apply to any quadratic space $(V, Q)$ over $L$. By contrast, the next lemma uses the full cyclic composition structure: Let again $\nu \in \{\text{Id}_L, \rho, \theta\}$. Given an invertible element $u \in \text{End}_F V$ and $\mu \in L^\times$ such that for all $x \in V$ and $\lambda \in L$

$$u(x\lambda) = u(x)\nu(\lambda) \quad \text{and} \quad Q(u(x)) = \nu(\mu \cdot Q(x)),$$

we define an $L$-linear map $\beta_u: {}^\nu V \to \text{End}_L (\rho V \oplus {}^\theta V)$ by

$$\beta_u(x) = \begin{pmatrix} 0 & \nu(\mu)^{-1}r_u(x) \\ \ell_u(x) & 0 \end{pmatrix} \in \text{End}_L (\rho V \oplus {}^\theta V) \quad \text{for } x \in V.$$
Then from (2) we get $\beta_u(x)^2 = \nu(Q(x)) = \nu Q(\nu x)$. Therefore, the map $\beta_u$ extends to an $L$-algebra homomorphism

$$\beta_u : C(\nu V, \nu Q) \rightarrow \text{End}_L(\nu V \oplus \theta V).$$

Just like $\alpha_*$ in [4], the homomorphism $\beta_u$ is an isomorphism. We also have an isomorphism of $F$-algebras $C(\nu \cdot) : C(V, Q) \rightarrow C(\nu V, \nu Q)$ induced by the $F$-linear map $x \mapsto \nu x$ for $x \in V$, so we may consider the $F$-automorphism $\psi_u$ of $\text{End}_L(\nu V \oplus \theta V)$ that makes the following diagram commute:

$$\begin{array}{ccc}
C(V, Q) & \xrightarrow{\alpha_*} & \text{End}_L(\nu V \oplus \theta V) \\
\downarrow & & \downarrow \psi_u \\
C(\nu \cdot) & \xrightarrow{\beta_u} & \text{End}_L(\nu V \oplus \theta V)
\end{array}$$

(7)

**Lemma 3.3.** The $F$-algebra automorphism $\psi_u$ restricts to an $F$-algebra automorphism $\psi_{u_0}$ of $\text{End}_L(\nu V) \times \text{End}_L(\theta V)$. The restriction of $\psi_{u_0}$ to the center $L \times L$ is either $\nu \times \nu$ or $(\nu \times \nu) \circ \varepsilon$ where $\varepsilon$ is the switch map $(l_1, l_2) \mapsto (l_2, l_1)$. Moreover, if $\psi_{u_0}|_{L \times L} = \nu \times \nu$, then there exist invertible $\nu$-semilinear transformations $u_1, u_2 \in \text{End}_F V$ such that

$$\psi_u(f) = \begin{pmatrix} \nu u_1 & 0 \\ 0 & \nu u_2 \end{pmatrix} \circ f \circ \begin{pmatrix} \nu u_1^{-1} & 0 \\ 0 & \nu u_2^{-1} \end{pmatrix} \quad \text{for all } f \in \text{End}_L(\nu V \oplus \theta V).$$

For any pair $(u_1, u_2)$ satisfying this condition, we have

$$u_2(x \ast y) = u(x) \ast u_1(y) \quad \text{and} \quad u_1(x \ast y) = \theta \nu(\mu)^{-1}(u_2(x) \ast u(y)) \quad \text{for all } x, y \in V.$$

**Proof.** The maps $\alpha_*$ and $\beta_u$ are isomorphisms of graded $L$-algebras for the usual $(\mathbb{Z}/2\mathbb{Z})$-gradings of $C(V, Q)$ and $C(\nu V, \nu Q)$, and for the "checker-board" grading of $\text{End}_L(\nu V \oplus \theta V)$ defined by

$$\text{End}_L(\nu V \oplus \theta V)_0 = \text{End}_L(\nu V) \times \text{End}_L(\theta V)$$

and

$$\text{End}_L(\nu V \oplus \theta V)_1 = \begin{pmatrix} 0 & \text{Hom}_L(\theta V, \nu V) \\ \text{Hom}_L(\nu V, \theta V) & 0 \end{pmatrix}.$$}

Therefore, $\psi_u$ also preserves the grading, and it restricts to an automorphism $\psi_{u_0}$ of the degree 0 component. Because the map $C(\nu \cdot)$ is $\nu$-semilinear, the map $\psi_u$ also is $\nu$-semilinear, hence its restriction to the center of the degree 0 component is either $\nu \times \nu$ or $(\nu \times \nu) \circ \varepsilon$.

Suppose $\psi_{u_0}|_{L \times L} = \nu \times \nu$. By Lemma 3.2 (applied with $\nu V \oplus \theta V$ instead of $V$), there exists an invertible $\nu$-semilinear transformation $v \in \text{End}_F(\nu V \oplus \theta V)$ such that $\psi_u(f) = \nu v f \circ \nu^{-1}$ for all $f \in \text{End}_F(\nu V \oplus \theta V)$. Since $\psi_{u_0}$ fixes $\begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}$, the element $v$ centralizes $\begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}$, hence $v = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}^{-1}$ for some invertible $u_1, u_2 \in \text{End}_F V$.

The transformations $u_1$ and $u_2$ are $\nu$-semilinear because $v$ is $\nu$-semilinear. From the commutativity of (7) we have $v \circ \alpha_*(x) = \beta_u(\nu x) \circ v = \alpha_*(u(x)) \circ v$ for all $x \in V$. By the definition of $\alpha_*$, it follows that

$$u_1(z \ast x) = \theta \nu^{-1}(\mu)(u_2(z) \ast u(x)) \quad \text{and} \quad u_2(x \ast y) = u(x) \ast u_1(y) \quad \text{for all } y, z \in V.$$
Lemma 3.4. Let $\nu \in \{\text{Id}_L, \rho, \theta\}$. For every $F$-linear automorphism $\varphi$ of $\text{End} \Gamma$ such that $\varphi|_L = \nu$, there exists an invertible transformation $u \in \text{End}_F V$, uniquely determined up to a factor in $L^\times$, such that $\varphi(f) = u \circ f \circ u^{-1}$ for all $f \in \text{End}_L V$.

Every such $u$ is a $\nu$-semilinear isotopy $\Gamma \to \Gamma$.

Proof. The existence of $u$, its uniqueness up to a factor in $L^\times$, and its $\nu$-semilinearity, were established in Lemma 3.2. It only remains to show that $u$ is an isotopy.

Since $\varphi$ is an automorphism of $\text{End} \Gamma$, it commutes with $\sigma_Q$, hence Lemma 3.2 yields $\mu \in L^\times$ such that $Q(u(x)) = \nu(\mu \cdot Q(x))$ for all $x \in V$. We may therefore consider the maps $\beta_u$ and $\psi_u$ of Lemma 3.3. Now, recall from [7, (8.8)] that $C_0(V, Q) = C(\text{End}_L V, \sigma_Q)$ by identifying $x \cdot y$ for $x, y \in V$ with the image in $C(\text{End}_L V, \sigma_Q)$ of the linear transformation $x \otimes y$ defined by $z \mapsto x \cdot y$ for $z \in V$. We have

$$\varphi(x \otimes y) = u \circ (x \otimes y) \circ u^{-1}: z \mapsto u(x \cdot b_Q(y, u^{-1}(z))) \quad \text{for } x, y, z \in V.$$ 

Since $u$ is $\nu$-semilinear and $Q(u(x)) = \nu(\mu \cdot Q(x))$ for all $x \in V$, it follows that

$$u(x \cdot b_Q(y, u^{-1}(z))) = u(x) \cdot \nu(b_Q(y, u^{-1}(z))) = u(x) \cdot \nu(\mu^{-1} b_Q(u(y), z)).$$

Therefore, $\varphi(x \otimes y) = \nu(\mu)^{-1} u(x) \otimes u(y)$ for $x, y \in V$, hence the following diagram (where $\beta_u$ and $C(\nu \cdot \cdot)$ are as in (4)) is commutative:

$$
\begin{array}{ccc}
C_0(V, Q) & \xrightarrow{C(\nu \cdot \cdot)} & C_0(\nu^V, \nu Q) \\
\downarrow \beta_u|_{C_0(\nu^V, \nu Q)} & & \downarrow \beta_u|_{C_0(\nu^V, \nu Q)} \\
C_0(V, Q) & \xrightarrow{\alpha \cdot \cdot} & \text{End}_L(\nu^V) \times \text{End}_L(\theta^V)
\end{array}
$$

On the other hand, the following diagram is commutative because $\varphi$ is an automorphism of $\text{End} \Gamma$:

$$
\begin{array}{ccc}
C_0(V, Q) & \xrightarrow{\alpha \cdot \cdot} & \text{End}_L(\nu^V) \times \text{End}_L(\theta^V) \\
\downarrow \beta_u|_{C_0(\nu^V, \nu Q)} & & \downarrow \beta_u|_{C_0(\nu^V, \nu Q)} \\
C_0(V, Q) & \xrightarrow{\alpha \cdot \cdot} & \text{End}_L(\nu^V) \times \text{End}_L(\theta^V)
\end{array}
$$

Therefore, $\beta_u|_{C_0(\nu^V, \nu Q)} \circ C(\nu \cdot \cdot)|_{C_0(V, Q)} = (\nu \cdot \cdot \theta^\nu \varphi) \circ \alpha \cdot \cdot$. By comparing with [7], we see that $\psi_{u_0} = \nu \cdot \cdot \theta^\nu \varphi$, hence $\psi_{u_0}|_{L \times L} = \nu \times \nu$. Lemma 3.3 then yields $\nu$-semilinear transformations $u_1, u_2 \in \text{End}_F V$ such that

$$\psi_u(f) = (\nu_{u_1} \circ 0 \circ \nu_{u_2}) \circ f \circ (\nu_{u_1^{-1}} \circ 0 \circ \nu_{u_2^{-1}}) \quad \text{for all } f \in \text{End}_L(\nu^V \oplus \theta^V),$$

hence $\psi_{u_0} = \text{Int}(\nu_{u_1}) \times \text{Int}(\theta_{u_2})$. But we have $\psi_{u_0} = \nu \cdot \cdot \theta^\nu \varphi = \text{Int}(\nu_{u_1}) \times \text{Int}(\theta_{u_1})$. Therefore, multiplying $(u_1, u_2)$ by a scalar in $L^\times$, we may assume $u = u_1$ and $u_2 = \zeta u$ for some $\zeta \in L^\times$. Lemma 3.3 then gives

$$\zeta u(x \ast y) = u(x) \ast u(y) \quad \text{and} \quad u(x \ast y) = \theta \nu(\mu)^{-1} ((\zeta u(x)) \ast u(y)) \quad \text{for all } x, y \in V.$$ 

The second equation implies that $u(x \ast y) = \rho(\zeta) \theta \nu(\mu)^{-1} (u(x) \ast u(y))$. By comparing with the first equation, we get $\rho(\zeta) \theta \nu(\mu)^{-1} = \zeta^{-1}$, hence $\nu(\mu) = \rho(\zeta) \theta(\zeta)$. Therefore, $(\nu, u)$ is an isotopy $\Gamma \to \Gamma$ with multiplier $\nu^{-1}(\zeta)$. \qed
We start with the proof of claim (ii) of Theorem 3.4. Suppose $\tau$ is an $F$-automorphism of $\text{End}\Gamma$ such that $\tau|_L = \rho$ and $\tau^3 = \text{Id}$. By Lemma 3.3 we may find an invertible $\rho$-semilinear transformation $t \in \text{End}_F V$ such that $\tau(f) = t \circ f \circ t^{-1}$ for all $f \in \text{End}_L V$, and every such $t$ is an isotopy of $\Gamma$. Since $\tau^3 = \text{Id}$, it follows that $t^3$ lies in the centralizer of $\text{End}_L V$ in $\text{End}_F V$, which is $L$. Let $t^3 = \xi \in L^\times$. We already know by Theorem 3.1 that the map induced by $\Sigma$\footnote{The canonical map $f : \Sigma \rightarrow \text{End}_F V \otimes_F (\text{End}_L V)^\ast$ yields an isomorphism of cyclic compositions $\Sigma \otimes (L, \rho, \xi) \rightarrow \Gamma'$ with multiplier $\xi$, and \textit{[5]} implies that $Q'(t(x)) = \rho(Q'(x))$ and $t(x) \ast' t(y) = t(x \ast y)$ for all $x, y \in V$.} is isotopy $f$ such that $\tau$ may find an isotopy $(\nu, u \circ t \circ u^{-1} = \xi \ast' t'$. Hence, $\xi \in F^\times$. The $F$-subalgebra of $\text{End}_F V$ generated by $L$ and $t$ is a crossed product $(L, \rho, \xi)$; its centralizer is the $F$-subalgebra $(\text{End}_L V)^\ast$ fixed under $\tau$, and we have

$$\text{End}_F V \cong (L, \rho, \xi) \otimes_F (\text{End}_L V)^\ast.$$ 

Now, $\text{deg}(L, \rho, \xi) = 3$ and $\text{deg}(\text{End}_L V)^\ast = 8$, hence $(L, \rho, \xi)$ is split. Therefore $\xi = N_{L/F}(\eta)$ for some $\eta \in L^\times$. Substituting $\eta^{-1}t$ for $t$, we get $t^3 = \text{Id}_V$, and $t$ is still a $\rho$-linear isotopy of $\Gamma$. Let $\mu \in L^\times$ be the corresponding multiplier, so that for all $x, y \in V$

$$(8) \quad Q'(t(x)) = \rho(\mu)Q'(x) \quad \text{and} \quad t(x) \ast' t(y) = t(x \ast y).$$

From the second equation we deduce that $t^3(x) \ast t^3(y) = N_{L/F}(\mu)t^3(x \ast y)$ for all $x, y \in V$, hence $N_{L/F}(\mu) = 1$ because $t^3 = \text{Id}_V$. By Hilbert’s Theorem 90, we may find $\zeta \in L^\times$ such that $\mu = \zeta \theta(\zeta)^{-1}$. Define $Q' = \rho(\mu)Q$ and let $x \ast' y = \zeta \ast(x \ast y)$ for $x, y \in V$. Then $\text{Id}_V$ is an isotopy $\Gamma \rightarrow \Gamma' = (V, L, Q', \rho, \ast')$ with multiplier $\zeta$, and \textit{[5]} implies that

$$Q'(t(x)) = \rho(Q'(x)) \quad \text{and} \quad t(x) \ast' t(y) = t(x \ast y) \quad \text{for all} \quad x, y \in V.$$ 

Now, observe that because $t$ is $\rho$-semilinear and $t^3 = \text{Id}_V$, the Galois group of $L/F$ acts by semilinear automorphisms on $V$, hence we have a Galois descent (see \textit{[4]} (18.1))): the fixed point set $S = \{x \in V \mid t(x) = x\}$ is an $F$-vector space such that $V = S \otimes_F L$. Moreover, since $Q'(t(x)) = \rho(Q'(x))$ for all $x \in V$, the restriction of $Q'$ to $S$ is a quadratic form $n : S \rightarrow F$, and we have $Q' = n_L$. Also, because $t(x \ast' y) = t(x) \ast' t(y)$ for all $x, y \in V$, the product $\ast'$ restricts to a product $\ast$ on $S$, and $\Sigma = (S, n, \ast)$ is a symmetric composition because $\Gamma'$ is a cyclic composition. The canonical map $f : S \otimes_F L \rightarrow V$ yields an isomorphism of cyclic compositions $f : \Sigma \otimes (L, \rho) \rightarrow \Gamma'$, hence also an isotopy $f : \Sigma \otimes (L, \rho) \rightarrow \Gamma$. We have $t = f \circ \rho \circ f^{-1}$, hence $\tau$ is conjugation by $f \circ \rho \circ f^{-1}$.

\textbf{Theorem 3.5.} The assignment $\Sigma \mapsto \tau_{(\Sigma, f)}$ induces a bijection between the isomorphism classes of symmetric compositions $\Sigma$ for which there exists an $L$-linear isotopy $f : \Sigma \otimes (L, \rho) \rightarrow \Gamma$ and conjugacy classes in $\text{Aut}_F(\text{End}\Gamma)$ of automorphisms $\tau$ of $\text{End}\Gamma$ such that $\tau^3 = \text{Id}$ and $\tau|_L = \rho$.

\textbf{Proof.} We already know by Theorem 3.4 that the map induced by $\Sigma \mapsto \tau_{(\Sigma, f)}$ is onto. Therefore, it suffices to show that if the automorphisms $\tau_{(\Sigma, f)}$ and $\tau_{(\Sigma', f')}$ associated to symmetric compositions $\Sigma$ and $\Sigma'$ are conjugate, then $\Sigma$ and $\Sigma'$ are isomorphic. Assume $\tau_{(\Sigma, f')} = \varphi \circ \tau_{(\Sigma, f)} \circ \varphi^{-1}$ for some $\varphi \in \text{Aut}_F(\text{End}\Gamma)$, and let $t = f \circ \rho \circ f^{-1}$, $t' = f' \circ \rho \circ f'^{-1}$ in $\text{End}\Gamma$ be the $\rho$-semilinear transformations such that $\tau_{(\Sigma, f)} = \text{Int}(t)|_{\text{End}_L V}$ and $\tau_{(\Sigma', f')} = \text{Int}(t')|_{\text{End}_L V}$. By Lemma 3.4 we may find an isotopy $(\nu, u) : \Gamma \rightarrow \Gamma$ such that $\varphi = \text{Int}(u)|_{\text{End}_L V}$. The equation $\tau_{(\Sigma', f')} = \varphi \circ \tau_{(\Sigma, f)} \circ \varphi^{-1}$ then yields $\text{Int}(t')|_{\text{End}_L V} = \text{Int}(u \circ t \circ u^{-1})|_{\text{End}_L V}$, hence there exists $\xi \in L^\times$ such that $u \circ t \circ u^{-1} = \xi \ast' t'$. Because $t^3 = \text{Id}_V$, we have $N_{L/F}(\xi) = 1$, hence Hilbert’s Theorem 90 yields $\eta \in L^\times$ such that $\xi = \rho(\eta)\eta^{-1}$.
Then \( \eta^{-1}u : \Gamma \to \Gamma \) is a \( \nu \)-semilinear isotopy such that \( (\eta^{-1}u) \circ t \circ (\eta^{-1}u)^{-1} = \xi t' \),
and we have a commutative diagram
\[
\begin{array}{c}
\Sigma \otimes (L, \rho) \xrightarrow{\hat{G}^{-1} \circ (\eta^{-1}u) \circ f} \Sigma' \otimes (L, \rho) \\
\downarrow \hat{\rho} \quad \quad \quad \downarrow \hat{\rho} \\
\Sigma \otimes (L, \rho) \xrightarrow{\hat{G}^{-1} \circ (\eta^{-1}u) \circ f} \Sigma' \otimes (L, \rho)
\end{array}
\]

The restriction of \( G^{-1} \circ (\eta^{-1}u) \circ f \) to \( \Sigma \) is an isotopy of symmetric compositions \( \Sigma \to \Sigma' \); a scalar multiple of this map is an isomorphism \( \Sigma \to \Sigma' \).

\[ \square \]

4. Trialitarian automorphisms of groups of type \( D_4 \)

Let \( F \) be a field of characteristic different from 2. By [7, (44.8)], for every adjoint simple group \( G \) of type \( D_4 \) over \( F \) there is a trialitarian algebra \( T = (E, L, \sigma, \alpha) \) such that \( G \) is isomorphic to \( \text{Aut}_L(T) \). Since the correspondence between trialitarian algebras and adjoint simple groups of type \( D_4 \) is actually shown in [7, (44.8)] to be an equivalence of groupoids, we have \( \text{Aut}(G) \cong \text{Aut}_F(T) \) if \( G = \text{Aut}_L(T) \). We then have a commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \text{Aut}_L(T) & \longrightarrow & \text{Aut}_F(T) & \longrightarrow & \text{Aut}_F(L) & \longrightarrow & 1 \\
& \Phi & \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\
1 & \longrightarrow & G & \longrightarrow & \text{Aut}(G) & \longrightarrow & (\mathfrak{S}_3)_L & \longrightarrow & 1
\end{array}
\]

where \( \Phi \) maps every \( F \)-automorphism \( \tau \) of \( T \) to conjugation by \( \tau \), and \( (\mathfrak{S}_3)_L \) is a (non-constant) twisted form of the symmetric group \( \mathfrak{S}_3 \). Here \( \text{Aut}_F(L) \) is the group scheme given by \( \text{Aut}_F(L)(R) = \text{Aut}_{R, \text{alg}}(L \otimes F R) \) for any commutative \( F \)-algebra \( R \). Thus, the type of the group \( G \) is related as follows to the type of \( L \) and to \( \text{Aut}_F(L) \):

(i) type \( 1D_4 \): \( L \cong F \times F \times F \) and \( \text{Aut}_F(L)(F) \cong \mathfrak{S}_3 \);
(ii) type \( 2D_4 \): \( L \cong F \times \Delta \) (with \( \Delta \) a quadratic field extension of \( F \)) and \( \text{Aut}_F(L)(F) \cong \mathfrak{S}_2 \);
(iii) type \( 3D_4 \): \( L \) a cyclic cubic field extension of \( F \) and \( \text{Aut}_F(L)(F) \cong \mathbb{Z}/3\mathbb{Z} \);
(iv) type \( 4D_4 \): \( L \) a non-cyclic cubic field extension of \( F \) and \( \text{Aut}_F(L)(F) = 1 \).

**Theorem 4.1.** Let \( G \) be an adjoint simple group of type \( D_4 \) over \( F \). If \( \text{Aut}(G)(F) \) contains an outer automorphism \( \varphi \) such that \( \varphi^3 \) is inner, then \( G \) is of type \( 1D_4 \) or \( 3D_4 \), and in the trialitarian algebra \( T = (E, L, \sigma, \alpha) \) such that \( G \cong \text{Aut}_L(T) \), the central simple \( L \)-algebra \( E \) is split.

**Proof.** Since the image \( \pi(\varphi) \in (\mathfrak{S}_3)_L(F) \) has order 3, it is clear from the characterization of the various types above that \( G \) cannot be of type \( 2D_4 \). If \( G \) is of type \( 6D_4 \), then after extending scalars from \( F \) to \( L \) we get as new cubic algebra \( L \otimes_F L \cong L \times (\Delta \otimes_F L) \), where \( \Delta \), the discriminant of \( L \), is a quadratic field extension. Thus, the group \( G_L \) has type \( 2D_4 \); but the outer automorphism \( \varphi \) extends to an outer automorphism of \( G_L \) such that \( \varphi^3 \) is inner, in contradiction to the preceding case. Therefore, the type of \( G \) is \( 1D_4 \) or \( 3D_4 \). If \( G \) is of type \( 1D_4 \), then the algebra \( E \) is split by [5, Example 15] or by [2, Theorem 13.1]. If \( G \) is of type \( 3D_4 \), then after scalar extension to \( L \) the group \( G_L \) has type \( 1D_4 \), so \( E \otimes_F L \) is split. Therefore, the Brauer class of \( E \) has 3-torsion since it is split by a cubic
extension. But it also has 2-torsion since $E$ carries an orthogonal involution, hence $E$ is split.

For the rest of this section, we focus on trialitarian automorphisms (i.e., outer automorphisms of order 3) of groups of type $3D_4$. Let $G$ be an adjoint simple group of type $3D_4$ over $F$, and let $L$ be its associated cyclic cubic field extension of $F$. Thus,

$$\mathcal{G}_3(L)(F) = \text{Gal}(L/F) \cong \mathbb{Z}/3\mathbb{Z}.$$

If $G$ carries a trialitarian automorphism $\varphi$ defined over $F$, then $\pi: \text{Aut}(G)(F) \to \text{Gal}(L/F)$ is a split surjection, hence $\text{Aut}(G)(F) \cong G(F) \rtimes (\mathbb{Z}/3\mathbb{Z})$. Therefore, it is easy to see that for any other trialitarian automorphism $\varphi'$ of $G$ defined over $F$, the elements $\varphi$ and $\varphi'$ are conjugate in $\text{Aut}(G)(F)$ if and only if there exists $g \in G(F)$ such that $\varphi' = \text{Int}(g) \circ \varphi \circ \text{Int}(g)^{-1}$. When this occurs, we have $\pi(\varphi) = \pi(\varphi')$.

**Theorem 4.2.**

(i) Let $G$ be an adjoint simple group of type $3D_4$ over $F$. The group $G$ carries a trialitarian automorphism defined over $F$ if and only if the trialitarian algebra $T = (E, L, \sigma, \alpha)$ (unique up to isomorphism) such that $G \cong \text{Aut}_L(T)$ has the form $T \cong \text{End} \Gamma$ for some reduced cyclic composition $\Gamma$.

(ii) Let $G = \text{Aut}_L(\text{End} \Gamma)$ for some reduced cyclic composition $\Gamma$. Every trialitarian automorphism $\varphi$ of $G$ has the form $\varphi = \text{Int}(\tau)$ for some uniquely determined $F$-automorphism $\tau$ of $\text{End} \Gamma$ such that $\tau^3 = \text{Id}$ and $\tau|_L = \pi(\varphi)$. For a given nontrivial $\rho \in \text{Gal}(L/F)$, the assignment $\Sigma \mapsto \text{Int}(\tau(\Sigma, f))$ defines a bijection between the isomorphism classes of symmetric compositions for which there exists an $L$-linear isotopy $f: \Sigma \otimes (L, \rho) \to \Gamma$ and conjugacy classes in $\text{Aut}(G)(F)$ of trialitarian automorphisms $\varphi$ of $G$ such that $\pi(\varphi) = \rho$.

**Proof.** Suppose first that $\varphi$ is a trialitarian automorphism of $G$, and let $G = \text{Aut}_L(T)$ for some trialitarian algebra $T = (E, L, \sigma, \alpha)$. Theorem 4.1 shows that the central simple $L$-algebra $E$ is split, hence $T = \text{End} \Gamma$ for some cyclic composition $\Gamma = (V, L, Q, \rho, \star)$ over $F$. Substituting $\varphi^{2^r}$ for $\varphi$ if necessary, we may assume $\pi(\varphi) = \rho$. The preimage of $\varphi$ under the isomorphism $\Phi_F: \text{Aut}_F(T)(F) \sim \text{Aut}(G)(F)$ (from 3.4) is an $F$-automorphism $\tau$ of $T$ such that $\varphi = \text{Int}(\tau)$, $\tau^3 = \text{Id}$, and $\tau|_L = \rho$. Since $\Phi_F$ is a bijection, $\tau$ is uniquely determined by $\varphi$. By Theorem 3.1(ii), the existence of $\tau$ implies that the cyclic composition $\Gamma$ is reduced.

Conversely, if $\Gamma$ is reduced, then by Theorem 3.1(i), the trialitarian algebra $\text{End} \Gamma$ carries automorphisms $\tau$ such that $\tau^3 = \text{Id}$ and $\tau|_L \neq \text{Id}_L$. For any such $\tau$, conjugation by $\tau$ is a trialitarian automorphism of $G$.

The last statement in (ii) readily follows from Theorem 3.4 because trialitarian automorphisms $\text{Int}(\tau)$, $\text{Int}(\tau')$ are conjugate in $\text{Aut}(G)(F)$ if and only if $\tau$, $\tau'$ are conjugate in $\text{Aut}_F(\text{End} \Gamma)$.

The following proposition shows that the algebraic subgroup of fixed points under a trialitarian automorphism of the form $\text{Int}(\tau(\Sigma, f))$ is isomorphic to $\text{Aut}(\Sigma)$, hence in characteristic different from 2 and 3 it is a simple adjoint group of type $G_2$ or $A_2$, in view of the classification of symmetric compositions (see §3).

**Proposition 4.3.** Let $G = \text{Aut}_L(\text{End}(\Sigma \otimes (L, \rho)))$ for some symmetric composition $\Sigma = (S, n, \star)$ over $F$ and some cyclic cubic field extension $L/F$ with nontrivial
automorphism $\rho$. The subgroup of $G$ fixed under the trialitarian automorphism $\text{Int}(\tilde{\rho})$ is canonically isomorphic to $\text{Aut}(\Sigma)$.

Proof (Sketch). Mimicking the construction of the map $\alpha_*$ in (4), we may use the product $\star$ to construct an $F$-algebra isomorphism $\alpha_*: C(S,n) \xrightarrow{\sim} \text{End}_F(S \otimes S)$ such that $\alpha_*(x)(y,z) = (z \star x, x \star y)$ for $x, y, z \in S$. This isomorphism restricts to an isomorphism $\alpha_{*0}: C_0(S,n) \xrightarrow{\sim} (\text{End}_F S) \times (\text{End}_F S)$.

Let $\text{Aut}(\text{End} \Sigma)$ be the group scheme whose rational points are the $F$-algebra automorphisms $\varphi$ of $(\text{End}_F S, \sigma_n)$ that make the following diagram commute:

$$
\begin{array}{ccc}
C(\text{End}_F S, \sigma_n) & \xrightarrow{\alpha_{*0}} & (\text{End}_F S) \times (\text{End}_F S) \\
\alpha_{*\circ} & & \varphi \times \varphi \\
C(\text{End}_F S, \sigma_n) & \xrightarrow{\alpha_{*0}} & (\text{End}_F S) \times (\text{End}_F S)
\end{array}
$$

Arguing as in Lemma 3.4, one proves that every such automorphism has the form $\text{Int}(u)$ for some isotopy $u$ of $\Sigma$. But if $u$ is an isotopy of $\Sigma$ with multiplier $\mu$, then $\mu^{-1}u$ is an automorphism of $\Sigma$. Therefore, mapping every automorphism $u$ of $\Sigma$ to $\text{Int}(u)$ yields an isomorphism $\text{Aut}(\Sigma) \xrightarrow{\sim} \text{Aut}(\text{End} \Sigma)$. The extension of scalars from $F$ to $L$ yields an isomorphism

$$
\text{PGL}(S) \xrightarrow{\sim} R_{L/F}(\text{PGL}(S \otimes_F L)^{\text{Int}(\tilde{\rho})}),
$$

which carries the subgroup $\text{Aut}(\text{End} \Sigma)$ to $G^{\text{Int}(\tilde{\rho})}$.

To conclude, we briefly mention without proof the analogue of Theorem 4.2 for simply connected groups, which we could have considered instead of adjoint groups. (Among simple algebraic groups of type $\text{D}_4$, only adjoint and simply connected groups may admit trialitarian automorphisms.)

Theorem 4.4. (i) For any cyclic composition $\Gamma = (V,L,Q,\rho,\star)$ over $F$, the group $\text{Aut}_L(\Gamma)$ is simple simply connected of type $\text{3D}_4$, and there is an exact sequence of algebraic groups

$$
1 \longrightarrow \mu_2 \longrightarrow \text{Aut}_L(\Gamma) \xrightarrow{\text{Int}} \text{Aut}_L(\text{End} \Gamma) \longrightarrow 1.
$$

(ii) A simple simply connected group of type $\text{3D}_4$ admits trialitarian automorphisms defined over $F$ if and only if it is isomorphic to the automorphism group of a reduced symmetric composition $\Gamma = (V,L,Q,\rho,\star)$. Conjugacy classes of trialitarian automorphisms of $\text{Aut}_L(\Gamma)$ defined over $F$ are in bijection with isomorphism classes of symmetric compositions $\Sigma$ for which there is an isotopy $\Sigma \otimes (L, \rho) \rightarrow \Gamma$.

Corollary 4.5. Every simple adjoint or simply connected group of type $\text{3D}_4$ over a finite field admits trialitarian automorphisms.

Proof. The Allen invariant is trivial, and cyclic compositions are reduced, see [9, §4.8].
Examples 4.6. (i) Let $F = \mathbb{F}_q$ be the field with $q$ elements, where $q$ is odd and $q \equiv 1 \mod 3$. As observed in Example 2.2(i), every symmetric composition over $F$ is isomorphic either to the Okubo composition $\Sigma$ or to the split para-Cayley composition $\tilde{C}$, and (up to isomorphism) there is a unique cyclic composition $\Gamma \cong \tilde{C} \otimes (L, \rho) \cong \Sigma \otimes (L, \rho)$ with cubic algebra $(L, \rho)$. Therefore, the simply connected group $\text{Aut}_L(\Gamma)$ and the adjoint group $\text{Aut}_L(\text{End} \Gamma)$ have exactly two conjugacy classes of trialitarian automorphisms defined over $F$. See also [6 (9.1)].

(ii) Example 2.2(ii) describes a cyclic composition induced by a unique (up to isomorphism) symmetric composition. Its automorphism group is a group of type $3^3D_4$ admitting a unique conjugacy class of trialitarian automorphisms.

(iii) In contrast to (i) and (ii) we get from Example 2.2(iii) examples of groups of type $3^3D_4$ with many conjugacy classes of trialitarian automorphisms.

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