The propagation characteristics of Airy beams is investigated and fully described under the traveling waves approach analogous to that used for non-diffracting Bessel beams. This is possible when noticing that Airy functions are in fact Bessel functions of fractional order \( \frac{1}{3} \). We show how physical principles impose restrictions such that the non-diffracting Airy beams cannot be of infinite extent as has been argued, and introduce for the first time quantitative expressions for the maximum transverse and longitudinal extent of Airy beams. We show that under the appropriate physical conditions it is possible to obtain higher-order Airy beams.

Bessel beams are known to belong to a class of optical beams described by the (2 + 1)-dimensional Helmholtz equation, and they have the property of being non-diffracting [1, 2] and self-healing [3, 4] on propagation. Originally, Bessel beam properties have been studied using the diffraction Rayleigh-Sommerfeld or Fresnel-Kirchhoff integrals. Implicitly, this approach considers the Bessel beam to be of infinite transverse extent [1]. Alternatively, a more straightforward and physically sound description of these beams can be done in terms of traveling Hankel waves [2]. This description based on the differential Helmholtz wave equation provides a better understanding of the physical origin of these beams and their intriguing properties. Even further, this same approach allowed to demonstrate the existence of other families of non-diffracting beams described by fundamental solutions of the separable (2+1) Helmholtz wave equation. These families of beams show propagation characteristics analogous to the Bessel beams [5, 6].

In recent years, another kind of non-diffracting beam has been reported but its propagation is governed by the paraxial wave equation in (1 + 1) dimensions, the so-called Airy beams [7]. These beams do not belong to the same class as those of the Helmholtz equation and thus it might be expected that their physical properties are not actually described in the same terms. Nonetheless, an interesting fact that is hardly discussed in all the published literature on Airy beams is that the Airy functions are Bessel functions of fractional order equal to 1/3. This allows the application of the aforementioned travelling Hankel wave description. In this work we show that Airy beams have similar properties to those of Bessel beams due to the fact that the former are the result of the superposition of counter-propagating Hankel traveling waves of fractional order. We also demonstrate that the proposed wave approach imposes the condition under physical principles for these beams to be of finite extent contrary to the “ideal” infinite Airy beam.

The aim of this paper is to provide a better understanding of the fundamental nature of the travelling wave methodology applied to Airy beams. In particular we show that all the known propagation characteristics of Airy beams are straightforwardly and intuitively understood using the Hankel travelling wave approach and that the focusing features of (1 + 1)-dimensional Airy beams can only be described in clear and simple terms with this methodology, which is similar to that for (2 + 1)-dimensional Bessel beams. In Section I we provide a brief account of the travelling wave description for Bessel beams as a way to motivate its application to Airy beams. Section II explains the relationship between Airy and Bessel functions and in Section III we show the characteristic of the finite Airy beam in a direct manner. Finally in Section IV we show that there can exist higher order Airy beams by discussing the physical conditions that enable this possibility.

I. TRAVELLING HANKEL WAVES AND BESSEL BEAMS

Light propagation in linear media is described with the use of the scalar wave equation for the electric field
\( E(r, t) \) given by
\[
\nabla^2 E(r, t) = \frac{1}{v^2} \frac{\partial E(r, t)}{\partial t}.
\]
(1)

In cylindrical coordinates \( r = (r, z) \), and \( v \) corresponds to the speed of light in the medium in question. It is possible to solve Equation (1) by separation of variables ending up with ordinary differential equations for the variables \( r, z \) and \( t \). The separation constants can be chosen such that \( \omega^2/v^2 = k_r^2 + k_z^2 \equiv |\mathbf{k}|^2 \) so that we can think of \( \mathbf{k} \) as the wavevector. In this manner, the equation for the radial coordinate from Equation (1) is given by:
\[
\frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} + \left(k_r^2 - \frac{m^2}{r^2}\right) H = 0,
\]
(2)

which is the Bessel differential equation of order \( m \) and its solutions are given by the \( m \)-th order Bessel function, \( J_m(k_r r) \) and the \( m \)-th order Neumann function, \( N_m(k_r r) \). In many cases the Neumann functions mentioned above are discarded due to the singularity they present at the origin. However, it has been proved by one of the authors [2] that these functions do indeed carry physical meaning. Also, the solutions \( J_m \) and \( N_m \) cannot be used separately to describe the propagation of light, because they do not satisfy the Sommerfeld radiation condition independently. For cylindrical waves this condition reads,
\[
\lim_{r \to \infty} r^{1/2} \left(\frac{dH}{dr} - ik_r H\right) = 0,
\]
(3)
and it tells us that a wave equation cannot have waves coming from an infinite distance. Thus, the solution needed to describe propagating waves must be given by the complex superposition of the \( J_m \) and \( N_m \) functions. This leads to the so-called Hankel waves:
\[
H^{(1)}_m(k_r r) = J_m(k_r r) + iN_m(k_r r), \quad (4)
H^{(2)}_m(k_r r) = J_m(k_r r) - iN_m(k_r r), \quad (5)
\]

Once the azimuthal and longitudinal wave components are incorporated, \( \exp(ik_z z) \), the wavefronts of the solutions for the Helmholtz wave equation in cylindrical coordinates are conic helicoids. \( H^{(1)}_m \) and \( H^{(2)}_m \) are related to outgoing (convex) and incoming (concave) cone solutions respectively. The Hankel waves, in combination with the temporal part, describe cylindrical wavefronts that collapse and are generated at the longitudinal \( z \)-axis. This is the physical origin of the singularity of the Hankel functions, the longitudinal axis is simultaneously sink and source for the incoming and outgoing cylindrical wave components, respectively. In that sense, since incoming waves become outgoing, there is a region where both of them interfere leaving only the Bessel function:
\[
E_{in}(r, \varphi, z, t) + E_{out}(r, \varphi, z, t) = 2J_m(k_r r) \exp(\im \varphi + ik_z z - \im \omega t).
\]
(6)

Recalling the Sommerfeld radiation condition, Equation (3), the incoming wave must be generated at a finite distance implying that Bessel beams must have finite transverse extent and thus a finite propagation distance. The interference of both Hankel waves only occurs within a conic region, and it is only within this region that the Bessel beam can be formed. This region is therefore called the region of existence of the Bessel beam which can be exploited in applications such as the design of laser resonators whose output is related to a Bessel beam.

As we have mentioned before, Bessel beams are non-diffractive, i.e. they propagate without spreading or changing their shape within their region of existence. Similarly, they show the property of self-healing that occurs when the beam is partially blocked, i.e. it reconstructs itself after some distance [4]. These properties can straightforwardly be understood in terms of traveling Hankel waves that provides a clear frame for the physics behind of them and others like the evolution of focused Bessel beams [11].

II. TRAVELLING WAVES APPROACH TO AIRY BEAMS

Berry and Balazs introduced the idea of the “non-spreading Airy wave-packet” by solving a force-free Schrödinger equation [12]. The solution propagates without change along a parabolic trajectory and thus show acceleration. The relationship between the force-free Schrödinger equation and the paraxial wave equation indicates that non-diffractive Airy beams are therefore possible and indeed they have been observed [7]. Furthermore, it has been reported that, similar to Bessel beams, Airy beams also have the property of self-healing [13].

Airy beams can be obtained when considering the propagation of a plane-polarised beam in a linear medium described by the normalised paraxial wave equation
\[
-\im \frac{\partial U}{\partial \xi} + \frac{1}{2} \frac{\partial^2 U}{\partial s^2} = 0,
\]
(7)
where \( U(s, \xi) \) is the electric field envelope that depends on the normalised coordinates \( s = x/x_0 \) and \( \xi = z/kx_0^2 \), with \( x_0 \) being a given transverse scale and \( k \) is the wavenumber. Since we know that the Airy beam remains invariant while propagating along a curved trajectory we can define an accelerating variable \( \Gamma = s - \frac{a}{4} \xi^2 + v \xi \) with \( a \) and \( v \) being real constants. We can now write the electric field envelope as follows \( U(s, \xi) = w(\Gamma) \exp(\im \theta(\Gamma, \xi)) \), which leads to the following ordinary differential equation
for $w(\Gamma)$

$$w'' - \alpha \Gamma w = \frac{A^2}{w^3},$$

(8)

and $\theta(\Gamma, \xi)$ is given by

$$\theta(\Gamma, \xi) = A \int_{0}^{\Gamma} \frac{d\nu}{w^2} \left( \frac{a}{2} \xi - v \right) + \frac{a^2}{24} \xi^3 - \frac{av}{4} \xi^2 + \frac{v}{2} \xi,$$

(9)

where $A = A(\xi)$ is assumed to be a constant. We can think of the parameters $a$ and $v$ above as the acceleration and velocity of the beam, respectively. Without loss of generality, let us consider the case where $a = 1$ and $v = 0$ and $A = 0$; Equation (8) becomes:

$$w'' - \left( s - \frac{1}{4} \xi^2 \right) w = 0,$$

(10)

which is the well-known Airy differential equation [14] and therefore the solution can be expressed as:

$$U(s, \xi) = \text{Ai}(s - \frac{\xi^2}{4}) \exp \left[ i \left( \frac{s \xi}{2} - \frac{\xi^3}{12} \right) \right].$$

(11)

where $\text{Ai}(\cdot)$ is the Airy function. The expression above implies that the intensity of the beam has the profile of the modulus squared of the Airy function, i.e.

$$I(s, \xi) = \left| \text{Ai}(s - \frac{\xi^2}{4}) \right|^2.$$

(12)

The argument of the Airy function in Equation (11) shows that the Airy beam follows a parabolic trajectory that can be interpreted as having a transverse acceleration [12]. It is also clear that it does not change neither its profile nor its amplitude on propagation when observed along this parabolic trajectory. In this sense the Airy beam can be regarded as being a non-diffracting beam, as it is the case for the Bessel beam discussed in Section [13]. It may seem that although both Airy and Bessel beams are non-diffracting, the physics are not described in a similar way given that they do not belong to the same class of beams. However, in what follows we will show that Airy beams can be better understood when described by the travelling waves approach used in the treatment of Bessel beams outlined in Section [14].

In order to elucidate the properties and behaviour of Airy beams, let us recall Airy’s differential equation, namely

$$\frac{d^2w}{ds'^2} \mp sw = 0,$$

(13)

whose solutions are given by the Airy functions $\text{Ai}(s)$ and $\text{Bi}(s)$ [14], where we are using normalised coordinates. The $\text{Bi}(s)$ function is defined as the solution with the same amplitude of oscillation as $\text{Ai}(s)$ as $s$ goes to minus infinity and differs in phase by $\pi/2$. We will therefore concentrate on the behaviour of the $\text{Ai}(s)$ in the rest of this discussion. The Airy differential equation is defined in the entire real space and thus, the second term in Equation (13) can have either a positive or a negative sign depending on whether the value of $s$ is positive or negative. Using either of the two signs yields the same Airy solution with the only difference that one will be the mirror reflection of the other with respect to the vertical axis, $s = 0$.

A simple calculation shows that by making the change of variable $w = \sqrt{s}Z_{1/3}(\frac{2}{3}s^{3/2})$, Equation (13) can be transformed into the Bessel differential equation of order 1/3, with $Z_{1/3}$ being the cylindrical Bessel function of order 1/3 [13,16]. In a similar way, the Airy functions can be expressed in terms of modified Bessel functions of order 1/3, $K_{1/3}$, as follows [13]:

$$w(s) = \frac{1}{\pi} \sqrt{\frac{s}{3}} K_{1/3} \left( \frac{2}{3}s^{3/2} \right),$$

(14)

$$w(s) = \frac{1}{\pi} \sqrt{\frac{s}{3}} 2^{2/3} i^{1/3} H^{(1)}_{1/3} \left( \frac{2}{3} i s^{3/2} \right).$$

(15)

We can distinguish two important cases; let us consider the argument of these Bessel functions, $K_{1/3}$ to be $\chi = \frac{2}{3}|s|^{3/2}$. On the one hand, when $s \geq 0$, the profile has a monotonic decreasing behaviour and is proportional to the modified Bessel function $K_{1/3}$:

$$\text{Ai}(s) = \frac{1}{\pi} \sqrt{\frac{s}{3}} K_{1/3}(\chi).$$

(16)

On the other hand, when $s < 0$, the profile can be considered as the superposition of two waves which are essentially Hankel functions of order 1/3, i.e.

$$\text{Ai}(-s) = \frac{1}{2} \sqrt{\frac{s}{3}} \left[ e^{\frac{i\pi}{6}} H^{(1)}_{1/3}(\chi) + e^{-\frac{i\pi}{6}} H^{(2)}_{1/3}(\chi) \right].$$

(17)

We can now define the following functions:

$$\text{Ai}H^{(1)}(s) = \frac{1}{2} \sqrt{\frac{s}{3}} e^{\frac{i\pi}{6}} H^{(1)}_{1/3}(\chi),$$

(18)

$$\text{Ai}H^{(2)}(s) = \frac{1}{2} \sqrt{\frac{s}{3}} e^{-\frac{i\pi}{6}} H^{(2)}_{1/3}(\chi).$$

(19)

With these definitions, we can now write the Airy function in a more compact form as follows:

$$\text{Ai}(-s) = \text{Ai}H^{(1)}(s) + \text{Ai}H^{(2)}(s).$$

(20)

We identify these functions as the travelling Hankel components of the Airy beam, in analogy to the way it was done for Bessel beams in Section [13]. We note that it is possible to obtain an analytical form of the phase of these Hankel components by considering the asymptotic expansion of Hankel functions, which approximate very well to the original function from the first maximum [14].
Using this expansion we have that

\[
AiH^{(1)}(s) \simeq \sqrt{\frac{s}{6\pi\chi}} e^{i\phi},
\]

(21)

\[
AiH^{(2)}(s) \simeq \sqrt{\frac{s}{6\pi\chi}} e^{-i\phi},
\]

(22)

with \( \phi = \chi - \frac{\pi}{4} \), and their wavefront being the opposite of each other. In Figure 1, we can see the wavefronts of each of the components \( AiH^{(1)}(s) \) and \( AiH^{(2)}(s) \). Each phase determines a geometric wavefront, and thus according to geometrical optics, the rays that make up each beam will propagate perpendicularly to this front [17]. In fact, when the Airy beam is cut, it will propagate along trajectories determined by the rays of the \( AiH^{(2)}(s) \) beam. We note that these waves must satisfy the Sommerfeld radiation condition that imposes the restriction of Airy beams being of finite extent and, similar to Bessel beams, they can only exist within a finite region of space.

In order to show that it is in fact the \( AiH^{(2)} \) component the one that bears the property of the parabolic caustics we carried out the propagation of each component independently. In Figure 2, we see the propagation of the \( AiH^{(1)} \) component that simply travels away from the propagation axis diminishing its amplitude. In Figure 3, the parabolic caustic associated to the normal rays to its concave wavefront can indeed be appreciated. We note that the caustic determined by the rays defines the parabolic motion of the main maximum of the beam under consideration [12]. This behaviour is similar to that produced by a third order aberration (coma). The numerical simulations shown in this paper use a computational window such that the dimensionless parameters are \( s \in [-30, 30] \) and \( \xi \in [0, 5] \), and we have used a super-Gaussian window \( t(x) = \exp\left(-x^2/t_0^2\right)^{50} \), \( t_0 \) the width of the window, to reduce diffraction effects introduced if a hard aperture was used instead and emphasize the propagation features of the AiH beams.

![FIG. 1. (Color online) Schematic wavefronts of the \( AiH^{(1)}(s) \) and \( AiH^{(2)}(s) \) functions (arbitrary units).](image1)

![FIG. 2. (Color online) Behaviour of the independent propagation of the \( AiH^{(1)}, a) \), and of the \( AiH^{(2)}, b) \) component of the Airy beam in arbitrary dimensionless units.](image2)

![FIG. 3. (Color online) Propagation of an obstructed Airy beam (in arbitrary dimensionless units) showing self-healing. Notice the presence of two shadows. The red dot at the edge of the figure indicates the position where the obstruction has been located.](image3)

In regards to focusing, we note that the Airy beam shows the non-common behaviour of presenting two different focusing regions, one for \( AiH^{(2)}(s) \) in which focusing is observed and a second for \( AiH^{(1)}(s) \), farther away,
where a defocusing beam appears. This behaviour can be seen in Figure 4, where we have marked the two regions as I and II. This behaviour is easily explained by noting that the composing Hankel waves, besides having opposite travelling directions, have opposite wave front curvatures: one is positive focusing and the other is negative defocusing. When passing through the positive lens these add or subtract accordingly.

FIG. 4. (Color online) Behaviour of a focussed Airy beam (in arbitrary dimensionless units), observe the two regions due to the focusing of the two Hankel travelling waves. In the edge of the figure we have indicated the profile of the Airy beam as well as the two wavefronts of the beam.

A simple geometrical analysis under this consideration easily explains the two observed regions. In Figure 5 we show a schematic of the optical arrangements including a lens; the wavefronts have been marked in two different colours. Notice how each region is generated by the focusing of the rays coming from each of the two wavefronts, i.e. from each of the Hankel waves that compose the Airy beam. We can think of each region to be formed due to the simultaneous incidence of each Hankel component of the Airy beam. We would like to emphasise that this is the case regardless of any interference pattern caused by the functions AiH\(^{(1)}\) and AiH\(^{(2)}\). However, we can indeed go further and assume that it is possible to make use of a packet that is either only the real or only the imaginary part of one of these two components, say AiH\(^{(1)}\) for example.

Finally, we can now take a more general view of the behaviour of an Airy-Gauss beam when it is off-Axis. In Figure 6 we show the propagation of the Airy-Gauss beam, where it is clear that each of the Hankel components provides the two contrasting behaviours of focusing and defocusing mentioned before.

FIG. 5. Schematic ray tracing of the double focusing of an Airy beam related to that shown in Figure 4.

FIG. 6. (Color online) Propagation of off-axis Airy-Gauss beam using arbitrary dimensionless units

As we have seen, the application of the travelling Hankel wave approach has provided us with a straightforward description of the propagation characteristics of the Airy beam under different circumstances. We will now show how to determine the propagation distance of finite energy Airy beams.

III. FINITE AIRY BEAMS

It is usually argued that non-diffracting beams, such as the Airy beam, require an infinite amount of energy for their generation. This might be to give an explanation for the non-diffraction feature or because the range of the mathematical function that describes the profile is infinite. However, in this Section we show that from a physical perspective this cannot be the case as the beam must be constrained by physical principles. One of them was mentioned above for the Bessel beams that also apply for the Airy beams and this is the Sommerfeld radiation condition. Within the travelling wave approach, to have an infinite Airy beam would require having sources at infinity.

For Airy beams, whose profile exhibits reduction of the separation distance between intensity peaks, there exists another more basic physical constraint that we discuss now. The distance between two consecutive peaks
in the transverse Airy intensity profile should not become smaller than the wavelength of the light used. If that were the case we would end up with an unphysical situation. This situation is analogous to the treatment, for instance, of wave excitations with a fractal boundary [18, 19]; although mathematically the fractal structure continues to infinitely small scales, in reality there are physical constraints that avoid this situation.

Next, we proceed to find a critical value of the beam extent after which its profile must dampen, giving rise to a finite energy Airy beam. In order to tackle the issue, let us take the asymptotic expansion for large arguments of the Airy function given by the corresponding Bessel functions of order 1/3:

$$\text{Ai}(\alpha s) \propto \cos \left( \frac{2}{3} (\alpha s)^{\frac{3}{2}} - \frac{\pi}{4} \right),$$

where we have taken the normalised coordinates and $\alpha$ is a parameter that allows us to change the frequency in the Airy function. We need to find the distance between two consecutive extreme points of the equation above, see Figure 7. These points occur when the following conditions are met: For minima we have

$$s_{\text{min}} = \frac{1}{\alpha} \left( \frac{3\pi}{2} \right)^{\frac{4}{3}} \left( 2l - \frac{3}{4} \right)^{\frac{2}{3}},$$

and for maxima:

$$s_{\text{max}} = \frac{1}{\alpha} \left( \frac{3\pi}{2} \right)^{\frac{4}{3}} \left( 2l + \frac{1}{4} \right)^{\frac{2}{3}},$$

where $l$ is a non-negative integer that provides us with information about the number of cycles that have occurred for a particular value of $l$.

In this way, the distance between two consecutive extremes is thus given by:

$$\Delta s = \frac{1}{\alpha} \left( \frac{3\pi}{2} \right)^{\frac{4}{3}} \left( 2l - \frac{3}{4} \right)^{\frac{2}{3}} \left( 2l - \frac{3}{4} \right)^{\frac{2}{3}},$$

We know that the distance between two consecutive peaks in the beam intensity cannot be smaller than a transverse wavelength $\lambda_c$, i.e. $\Delta s \geq \lambda_c/2$. This argument provides a physical criterion for the cut-off point of a finite Airy beam after which it must dampen.

To define the critical transverse wavelength $\lambda_c$ we must also take into consideration that Airy beams propagate within the paraxial regime as governed by Equation (7). For this purpose we require to provide the maximum angle allowed for a ray in the wavefront to be considered paraxial. Curiously enough, to date in the literature there is not an established quantitative criterion for such angle and, as noticed by Beck several decades ago: “The term paraxial rays is a relative one and to some extent a matter of arbitrary choice.” It will depend on the tolerance error that is planned to be accepted [20].

We will introduce the quantitative criterion of paraxiality put forward by Agrawal, Siegman and others [21, 22] where paraxial optical beams can be focused or diverge at angles up to a maximum critical value of $\theta_c = \pi/6$. With this in mind, we consider a ray perpendicular to the wavefront of $\text{Ai}H^{(2)}$ at the cut-off position. When the wavevector $\vec{k}$ to this ray reaches the maximum paraxial angle $\theta_c = \pi/6$, recalling that $|\vec{k}| = 2\pi/\lambda$, a simple calculation gives $\lambda_c = 2\lambda$. Thus, substituting $\Delta s = 2\lambda$ in Equation (26) and solving for $l_c$ we can get the position $s_{\text{max}}(l_c)$ where the oscillations of the Airy profile must dampen.

A. Maximum propagation distance of finite Airy beams

We now propose a method for determining the propagation distance of the finite Airy beam based on the observations made above. Consider the parabolic trajectory followed by the main maximum of the Airy beam and take a ray coming from the wavefront at the opposite extreme of the window and find the point of intersection. Figure 8 shows this situation; in order to guide the eye we have marked the region of existence of the Airy beam given by the ray as well as the parabolic trajectory followed by the beam (white dotted lines).

We know that for an Airy beam propagating in a homogeneous medium, the normalised solution is given by Equation (11) and thus it is clear that the trajectory of
the main maximum satisfies the following equation:

\[ s - \left( \frac{\xi}{2} \right)^2 = a_1 \]  

where \( a_1 = -1.0187297 \ldots \) is the first zero of the derivative of the Airy function, that is to say, the position of the first intensity maximum at the onset of propagation. Furthermore, we know that light rays are normal to the wavefront and therefore in this case, the equation for the ray we are interested in can be expressed as

\[ \xi = s_0^{-1/2}(s - s_0), \]

where \( s_0 \) is the point where the beam is truncated, in other words, where the edge of the window is located. Using Equation (27) and Equation (28) for the ray we have that

\[ \xi = 2\left(\sqrt{|s_0|} \pm \sqrt{2|s_0| - |a_1|}\right), \]

\[ s = 3s_0 \pm 2\sqrt{|s_0|}\sqrt{2|s_0| - |a_1|}. \]

The solution we seek is the one with the negative sign because it is the first intersection between the parabola and the ray, i.e. the point where the propagation ends. In this way, the propagation distance is given by:

\[ \xi_{\text{max}} = 2\left(\sqrt{|s_0|} - \sqrt{2|s_0| - |a_1|}\right). \]

This expression defines the maximum propagation distance for a finite Airy beam. This is, to our knowledge, the first time that the propagation distance for Airy beams is formally defined.

\[ \nabla^2 E + k^2 E = 0, \]

and let the medium have a dispersion relation such that the wave number is given by \( k^2(x) = k_0^2 + \beta \left[-k_0 - x^n + \frac{\beta}{\frac{n+2}{2}}\right] \), where \( \beta \) is a parameter that depends on the medium. We can now propose an Ansatz such that the electric field is given by \( E(x, z) = U(x, z) \exp\left[i\left(k_0 - \frac{\beta}{\frac{n+2}{2}}\right)z\right] \). In the paraxial approximation we have that

\[ 2i\left(k_0 - \frac{\beta}{\frac{n+2}{2}}\right) \frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial x^2} - \beta x^n U = 0. \]

If we require that \( \beta = 2k_0 \) the wave number is given by \( k^2(x) = -2k_0 x^n \) and Equation (33) is simplified into

\[ \frac{\partial^2 U}{\partial x^2} - \beta x^n U = 0, \]

which corresponds to a generalised form of the Airy differential equation. Making the transformation

\[ U(x) = \sqrt{Z} x^{\frac{n+2}{2}} \left( \frac{2\sqrt{\pi}}{n+2} \right), \]

substituting into Equation (34) and after some algebra, yields to

\[ Z''(\zeta) + \frac{1}{\zeta} Z'(\zeta) + \left(1 - \frac{1}{(n+2)^2 \zeta^2}\right) Z(\zeta) = 0 \]

where \( \zeta = \frac{2\sqrt{\pi}}{n+2} x^{\frac{n+2}{2}} \). Equation (36) is the Bessel differ-
potential equation whose solutions are the family of Bessel functions of order \( \frac{n}{2} \).

Equation (34) has been studied in detail by Swanson and Headley [34] who defined its solutions as \( A_n(x) \) and \( B_n(x) \). It is clear that when \( n = 1 \) the functions \( A_n(x) \) and \( B_n(x) \) become the standard Airy functions \( \text{Ai}(x) \) and \( \text{Bi}(x) \), respectively. An important remark is that they found that the general behaviour of the solutions is different depending on the parity of \( n \). This is something to be expected since the branches of the power law parabolae can have different signs depending on which side of the origin they extend to.

We can now refer to Equation (34) as the Airy differential equation of order \( n \). In other words, its solutions can be seen as higher order Airy functions, and similarly to the Airy functions, they can also be cast in terms of Bessel functions of fractional order. It should therefore be clear that given the adequate conditions in a power law gradient index medium it is then possible to obtain higher-order Airy beams.

We want to remark that the study presented in this section can also be applied to investigate the homologous problem in Quantum Mechanics for the Schrödinger equation of a particle confined within an infinite wall and a power law potential \( V(x) = \beta x^n \), with \( \beta \) constant [31–35]. For the particular case of the linear potential [36], the standing Airy wave packet arises from considering a particle subject to a constant force, e.g., gravitational force, that when it takes the value of zero the particle wave packet is still described by the Airy function, but accelerating away from the infinite wall as a consequence of removing the stabilising force [26].

V. CONCLUSIONS

In this paper we introduced the physical principles that govern the existence and propagation of Airy beams. We have shown that the non-diffracting characteristics of Airy beams can be explained under the formalism of travelling Hankel waves originally introduced to describe Bessel beams. This is possible due to the fact that Bessel and Airy functions are intimately related to each other, with the latter being the Bessel functions of fractional order equal to \( \frac{1}{2} \). We introduced the two Hankel components of the Airy beam, namely \( \text{AiH}^{(1)}(\cdot) \) and \( \text{AiH}^{(2)}(\cdot) \), and showed that the later bears the parabolic caustic property of the beam. It was shown that the superposition of these Hankel components fully explain in simple and straightforward terms propagation characteristics of the Airy beam, such as self-healing and double-focusing. Also, this approach allowed to establish for the first time the needed expression to compute the maximum propagation distance of finite energy Airy beams. We addressed physical constraints of why an “ideal” Airy beam of infinite extent cannot exist and provided a quantitative method to obtain the maximum extent of an Airy beam of a given wavelength. And finally, by studying the solution of the paraxial wave equation in power law GRIN media we demonstrated the possibility of creating higher-order Airy beams.

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