Effects of Changes in the Leading Coefficient of Fifth-Order Manabe’s Polynomial

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Abstract. How a change in the coefficient of the highest-degree term of a characteristic polynomial affects the locations of its poles is an important issue in, for instance, determining the reference model with different orders in control system designs. The root-locus technique is used to study the range of the leading coefficient for which a dynamic property of an (n+1)-th order polynomial remain close to that of the n-th order polynomial. As an example, it is shown that the fifth-order Manabe polynomial leads to the step response that is close to the fourth-order Manabe polynomial for a wide range of leading coefficient values; about 70 to 80% of the stability critical value, beyond which the polynomial becomes unstable.

1. Introduction
When building a system model, the number and values of its parameters need to be identified, which implies that the system order needs also be specified [1]. The analysis of a plant and the design of its controller are usually carried out assuming a specific order and there are a large number of methods available [2]-[4]. However, they have to be repeated when a different order is chosen. This is not only a cumbersome task but can also be inconclusive, since the results obtained for a particular order are often not easy to compare with those obtained for different orders. This is true for a general class of polynomials with different orders, different coefficients and, therefore, different characteristics. However, these polynomials can still have common features as well. For instance, reference polynomials with desired properties in magnitudes and phases have commonly been used in filter designs [5]-[6], such as Bessel, Butterworth, and Chebyshev filters, whose characteristic polynomials have coefficients that depend on their order. Polynomials of different order and coefficients also appear when they are obtained using order-reduction methods [7]-[8].

In the field of control, systems with desired properties are often specified in terms of a characteristic polynomial with an appropriate set of coefficients [9]-[11]. When the polynomials have different set of coefficients before and after order changes, it is not clear to see how these two polynomials may be related. For the purpose of comparing polynomials of different orders, it is ideal that the difference exists only in a single parameter, such as the leading coefficient; the coefficient of the highest order term. Under such a condition, it will be useful to know if there is a range of leading coefficient, for which the order could be reduced or otherwise; that is, some dynamic properties remain more or less the same when all the coefficients are the same except for the leading coefficients.

Manabe’s polynomial is used in [11] as a reference model for the design of control systems, where there is room for order changes. The source of this flexibility is the fact that n-th order Manabe...
polynomial can be reduced to the \((n-1)\)-th order, simply by removing the \(n\)-th-order term. While transient-response waveforms may be altered somewhat, this process at least preserves the stability. As an example, the present paper considers the fourth-order Manabe polynomial and fifth-order polynomials that is created by adding to it the fifth-order term. Besides Manabe’s, there are other polynomials that have similar properties, such as Kessler’s, Naslin’s, and Kitamori’s polynomials [9]-[11]. These are called order-canonical polynomials in the present paper, where a higher order polynomial is obtained simply by adding the higher order term without modifying other non-leading coefficients. Furthermore, a lower order polynomial can be obtained by simply ignoring the highest-order term. It is, then, informative to know how such a change in the order affects the associated dynamic properties.

The paper is organized as follows: After Introduction Section, Section 2 reviews characteristic polynomials and the root-locus technique with a view to order changes. Section 3 investigates the fourth to fifth order change in Manabe’s characteristic polynomials, with simulation results and observations. Section 4 draws some Conclusions.

2. Characteristic polynomials with different order

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2.1. Some notes on polynomials undergoing order change

A characteristic polynomial is often written as

\[ a_n s^n + \cdots + a_2 s + a_0 \]  

(1)

When \(a_n \neq 0\), this can be written as a monic polynomial with the leading coefficient normalized as unity, which is to say that the order has been declared to be \(n\). Therefore, monic polynomials are not convenient to consider order changes. For instance, a monic second-order polynomial is not obtained as a special case of a monic third order polynomial; in a pole-explicit monic form, they are written,

\[
(s + p_1)(s + p_2) = s^2 + (p_1 + p_2)s + p_1 p_2 \\
(s + p_1)(s + p_2)(s + p_3) = s^3 + (p_1 + p_2 + p_3)s^2 \\
+ (p_1 p_2 + p_2 p_3 + p_3 p_1)s + p_1 p_2 p_3,
\]

(2)

While in a non-monic form, they are

\[
(\tau_1 s + 1)(\tau_2 s + 1) = \tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1 \\
(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1) = \tau_1 \tau_2 \tau_3 s^3 \\
+ (\tau_1 \tau_2 + \tau_2 \tau_3 + \tau_3 \tau_1)s^2 + (\tau_1 + \tau_2 + \tau_3)s + 1.
\]

(3)

In the latter form, since the second-order case is obtained by letting \(\tau_3\) approach zero, a result obtained for the third order polynomial should apply to the second order case by letting \(\tau_1\) be zero. In this case, two of the three poles are fixed and only the third pole changes. While all the coefficients in equation will change by some amounts in general when the order changes, their values are small if \(\tau_3\) is small. This leads to the idea of changing only the leading coefficient, while keeping all other coefficients constant. With this concept, an \(n\)-th order system may be expressed as a higher-order system by adding the \((n+1)\)-th order term with the coefficient changed from zero to a small nonzero number.

As will be shown later, a stable \(n\)-th order polynomial becomes unstable as the \((n+1)\)-th order coefficient is increased from zero and the range of such leading coefficient values becomes smaller as the order increases, in general. In this sense, stability becomes more difficult to retain as the order increases, since the leading coefficients have to be chosen smaller and smaller, if the order have to be made higher. Furthermore, the stability of a polynomial can change when the sign of leading
coefficient changes. For example, one of the poles makes a drastic change in the following example as
the leading coefficient changes its signs, from 0.001 to 0 and then to -0.001, as
\[
0.001s^3 + s^2 + s + 1 = (s + 999)(s + 0.5 \pm j0.8666)
\]
\[
0s^3 + s^2 + s + 1 = (s + 0.5 \pm j0.86603)
\]
\[
-0.001s^3 + s^2 + s + 1 = (s - 1001)(s + 0.5 \pm j0.86545).
\]
In a similar vein, it is known that the higher the order of a stable system is, the more fragile its
stability becomes [12]. The situation is even worth in the discrete-time case using the popular shift
operator, although it can be expected that one using the so-called delta operator is as good, or worse,
as the continuous-time case [13].
In the rest of the paper, the polynomials will be normalized assuming \(a_0 \neq 0\) and all coefficients are
assumed to be positive as a necessary condition. Introducing the time-constant as
\[
\tau = \frac{a_i}{a_0}
\]
(5)
the characteristic polynomial can be written as
\[
F_s(s) = \alpha_\tau^1 (\tau s)^n + \cdots + (\tau s) + 1,
\]
(6)
In this case, time is scaled by the time-constant. Using this notation, the Routh stability criterion
gives stability conditions for the \((n+1)\)-th order system, which reduces to the \(n\)-th order conditions
by letting the highest order coefficient to zero. For instance, the necessary and sufficient conditions for
stability of the fifth-order polynomial are
\[
\alpha_5 > 0, \alpha_4 > 0, \alpha_3 \alpha_1 > \alpha_5 \alpha_2, \quad \alpha_4 > (\alpha_5^2 - \alpha_3) \alpha_2,
\]
(7)
\[
\alpha_5^2 - (\alpha_5 \alpha_3 - \alpha_2^2 \alpha_1 + 2\alpha_3 \alpha_6) \alpha_6
\]
\[
- \alpha_5 \alpha_3 \alpha_2 \alpha_6 - \alpha_6^2 \alpha_1 - \alpha_4 \alpha_6 < 0
\]
(8)
Which reduces to those for the fourth-order case by letting \(\alpha_5 \to 0\). Similarly, stability conditions
for the third-order case can be obtained when \(\alpha_4\) is set to zero, in addition to \(\alpha_5\). In the following, time
is assumed to have been scaled so that \(\tau s\) can be replaced with \(s\); i.e., \(a_i = a_i\).
Of all possible linear-time-invariant systems expressed in the transfer function form, those
considered in the present study are systems, whose denominator is a canonical polynomial with certain
desired properties and whose numerator is only a gain (no finite zeros). Such transfer functions are
often used, for instance, as target models in model-reference types of control systems [14]. The order
of a reference model is often the relative degree of the plant, which changes depending on how the
numerator order is chosen for the plan besides the denominator order. Also, the order increases in the
design of an input saturation filter, which spreads the input with a large magnitude over multiple
sampling instants with smaller magnitude [15]. Inclusions of noise removing filters can also be
considered as the order increase in a control system. Effects of such order changes are not easily
studied analytically.

2.2. Order-canonical polynomials
Although an infinite number of stable characteristic-polynomials with desirable properties exist, most
of them have coefficients that are specific to the order of the polynomial chosen have vary as the order
changes. While this is inconvenient when comparing systems of different orders, there are
polynomials whose coefficients are independent of the system’s order. An example is the Kitamori
canonical polynomial [9] given by
\[
\alpha_0 = \alpha_1 = 1, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{15}{100}, \alpha_4 = \frac{3}{100}, \alpha_5 = \frac{3}{1000}, \alpha_i = \ldots
\]  

The one used in the present paper is that proposed by Manabe in [10]-[11]. In terms of coefficients that appear in equation, it is one given by

\[
\alpha_0 = \alpha_1 = 1, \alpha_2 = \frac{2}{5}, \alpha_3 = \frac{2}{25}, \alpha_4 = \frac{1}{125}, \alpha_5 = \frac{1}{2500},
\]

\[
\ldots, \alpha_i = \left(\frac{1}{5}\right)^{i-1} \left(\frac{1}{2}\right)^{i-1}, \quad (i \geq 4).
\]  

It should be emphasized that these parameters are constants that decrease in magnitude as the order of the term increases. Thus, Manabe’s polynomials with different orders are given by the same set of coefficients \(\alpha_i\), with the difference only in the time constant \(\tau\). A set of polynomials whose coefficients are given by the same set of \(\alpha_i\) will be called the order-canonical polynomial in this study. In view of equation, it is apparent that all the poles move as the leading coefficient changes.

The stability indices are defined [10], for \(i = 1, 2, \ldots, n-1\), as

\[
\gamma_i = \frac{a_i^2}{a_{i+1}a_{i+1}} = \left(\frac{a_i}{a_{i+1}}\right) \left(\frac{a_{i+1}}{a_i}\right)^{-1}
\]  

(11)

Which are useful for assessing relative stability of polynomials and used extensively for the design of control systems based on the coefficient-diagram [11]. Equation also gives

\[
a_i = \frac{\tau^i}{\gamma_{i+1}^2 \gamma_{i-2}^2 \cdots \gamma_1^2}
\]  

(12)

for \(i = 1, 2, \ldots, n-1\). In terms of stability indices, Manabe’s polynomial is one with coefficients that satisfy

\[
\gamma_1 = 2.5, \quad \gamma_2 = 2.0 \quad (i \geq 2)
\]  

(13)

A system whose transfer function has Manabe’s polynomial as its characteristic polynomial are often used as reference models for controller designs. It has such characteristics as fast speed of response, no overshoot, short settling time, and robustness [11]. It is stated there that the choice of \(\gamma_1 = 2.7\) makes the response of third-order system non overshooting, while the choice of \(\gamma_1 = 2.0\), called Kessler’s polynomial, yields 8% overshoot. Smaller values of 1.7 ~ 2.0 (Naslin’s) make the overshoot larger.

2.3. Root-Locus Analysis

Consider the n-th order characteristic equation denoted by

\[
D_n(s) = a_n s^n + \cdots + a_1 s + 1 = 0
\]  

(14)

The root-locus method can be used to see how the roots move as one of the coefficients, \(a_\alpha\), is changed; i.e. by modifying equation (14) as

\[
1 + \frac{a_\alpha}{a_n s^n + \cdots + a_1 s + 1} s^i = 0
\]  

(15)

If this coefficient is chosen to be the leading coefficient, the coefficient of the highest-degree term, equation is written as

\[
1 + a_\alpha s^n D_{n-1}(s) = 0
\]  

(16)
where
\[ D_{n,1}(s) = a_n s^{n-1} + \cdots + a_1 s + 1 \]  
(17)
so that equation is written as
\[ D_n(s) = a_n s^n + D_{n,1}(s) = 0 \]  
(18)
Those polynomials that satisfy equations and are order-canonical polynomials. The root-locus plots show how the poles move as the order of the canonical polynomial increases from \((n-1)\) to \(n\). In fact, this shows how poles of \(D_{n,1}\), and one at infinity, move as \(a_n\) changes from zero to positive infinity, towards the \(n\) multiple poles at the origin. Since some of the \(n\) poles that approach the origin do so from the unstable region for \(n\geq2\), there must be an upper limit for \(a_n\) for stability. This will be called the stability-critical value of the leading coefficient. For \(n=1\) and 2, the stability critical value is infinite.

In the rest of the present paper, \(D_{n,1}(s)\) is assumed to be stable with desirable characteristics, such as fast speed of response and adequate damping, for instance. Since the main interests of the present study is about the change in pole location \(s\) when and after the order changes, equation is used in the simulation studied with \(n=5\) as an example.

3. Simulation for fourth and fifth order polynomials

3.1. Step response
Consider the fifth-order Manabe polynomial given by
\[ M_s(s) = \frac{1}{2500} s^5 + \frac{1}{125} s^4 + \frac{2}{25} s^3 + \frac{2}{5} s^2 + s + 1 \]  
(19)
where \(\tau = 1\) in equations and, and whose stability indices satisfy conditions. The lower order polynomials are obtained simply by ignoring the leading coefficient in equation, successively. The step responses of the systems with these characteristic polynomials of order one to five are shown in Figure 1. It can be seen that above the third-order, the responses are more or less the same.

For comparisons, the step responses of systems with Kitamori characteristic polynomials, equation are shown in Figure 2, where
\[ K_s(s) = \frac{3}{1000} s^5 + \frac{3}{100} s^4 + \frac{15}{100} s^3 + \frac{1}{2} s^2 + s + 1 \]  
(20)

Lower order cases are obtained by simple truncation as in Manabe’s. It is seen that the step responses are more oscillatory than Manabe’s, as hinted by smaller stability indices of \(\gamma_1 = 2.0 < 2.5\) and \(\gamma_2 = 5/3 < 2.0\).

As an example of none order-canonical polynomials, Figure 3 shows the step responses of the fifth-order polynomial and the lower-order polynomials that are derived from it using the Routh table. The step responses look closer to those of Manabe’s than Kitamori’s in waveforms, but with a slightly larger overshoots. However, this polynomial is not canonical and its lower order polynomials have different set of coefficients, as
This suggests that the effects of order changes in a general polynomial may well be studied using simpler order-canonical polynomials. In the rest of the paper, Manabe’s fourth-order canonical polynomial is used with added fifth-order term.

3.2. Step response Stability critical value of leading coefficient
An order-canonical polynomial becomes less stable in general as the leading coefficient is increased closer to the stability critical value, at which the polynomial has roots on the imaginary axis, and becomes unstable for larger leading coefficients. For the case of a fifth-order polynomial, this value can be found based on, for instance, the Routh stability criterion. To find the critical value for $a_5$ such that the marginal stability is obtained, the Routh table can be used to obtain the condition as

\[ (-a_1 a_6 a_3^2) a_5^2 + (2a_1^2 a_1 a_6 a_5 + a_1 a_6 a_3 a_0 - a_1 a_3 a_0) a_5 + (-a_1 a_6 a_1^2 - a_3^2 a_0 a_6 + a_3^2 a_0 a_3) = 0 \]

(22)
which is a quadratic equation in $a_5$. Knowing $a_1$ to $a_4$, solve this equation for $a_5$ so that the critical value $a_5^*$ is obtained, which is the stability critical value of the leading coefficient.

### 3.3. Order changes

Using order-canonical polynomials, the order can be increased easily by adding a higher-order term with a sufficiently small coefficient. If this leading coefficient is chosen to be larger than the stability critical value, the system will be unstable. To make the order higher, more terms need to be added, but the coefficients must be chosen smaller and smaller as the order increases. Therefore, there will be a moment when the order cannot be increased further without abandoning stability in the digital implementation; when the leading coefficient must be smaller than the machine epsilon for stability, the order cannot be increased further.

Order-canonical forms can also be used in the context of order-reduction. Certain properties are expected to be preserved after order-reduction. The method of controller designs based on the so-called coefficient-diagram [11] of characteristic polynomials takes such properties as stability and step response performances. The order may be considered as reducible when the stability is preserved, has a similar amount of overshoot in the step response, and the difference in settling times remains small, after the order-reduction.

### 3.4. Root-locus plot

When $n=5$ and $D_{n-1}$ is chosen to be Manabe’s polynomial, equation gives

$$1 + a_5 \frac{s^5}{M_4(s)} = 0$$

(23)

When $M_4(s)$ is chosen to be Manabe’s polynomial with $\tau=1$, so that stability indices satisfy conditions, it follows that

$$M_4(s) = \frac{1}{125} s^4 + \frac{2}{25} s^3 + \frac{2}{5} s^2 + s + 1$$

(24)

The stability critical value for the leading coefficient $a_5$ is found to be

$$a_5^* = \frac{16 - 7 \sqrt{3}}{125 \sqrt{3}} = 1.24 \times 10^{-3}$$

(25)

The fifth-order polynomial to be considered can be written, then, as

$$D_5(s) = a_5 s^5 + M_4(s) = ka_5 s^5 + M_4(s)$$

(26)

As $a_5$ changes from zero to positive infinity, the four poles of $M_4(s)$ and one pole at infinity approach the multiple fifth-order poles at the origin, as shown in Figure 4. To show the sensitivity of pole movements to $a_5$, the locations of the poles are also indicated on the plot from $k=0.0$ to $1.1$ with the increment of $0.1$. It can be seen that the four poles move quickly near the break in/out point at $s=-5$ and gradually slow down as $k$ increases. When $a_5$ is $1/2500$, i.e., $k=0.32$, $D_5(s)$ equals $M_4(s)$.

The fourth-order polynomial may be considered as the fifth, by treating the fifth pole located at negative infinity on the real axis. Table 1 shows how the poles moves along the real axis as $k$ is increased.

| Table 1. A The real pole locations as the leading coefficient increase. |
|---------------------------------------------------------------|
| $k$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| pole | -53.3631 | -19.2807 | -2.9447 | -2.6474 | -2.5147 |
| $k$ | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| pole | -2.4285 | -2.3646 | -2.3139 | -2.2719 | -2.2361 |
The unit step input is applied to the system with the transfer function given by $1/D_5$, and its response is simulated. Fig. 5 shows the step responses for $k$'s ranging from 0.0 to 0.7, which are almost identical; i.e., the fifth-order polynomials with $k$'s below 0.7 are almost the same as the fourth-order polynomial.
with $k=0$. There is no overshoot and the settling time is about 3 seconds, as hinted by $\tau=1$. The three stable poles with relatively slow frequencies are insensitive for this range of $k$’s, while the pair of complex poles are relatively far away from these three, implying their residues are relatively small. Beyond about $k=0.8$ and $0.9$, the step response starts to deviate from that of the fourth-order polynomial with slight oscillations (Figure 6). For $k=1.0$, the polynomial becomes marginally stable and its response goes into continuous oscillation. However, the amplitude of oscillation is relatively small compared with the steady-state value of unity.

For larger values of $k$, the step response become divergent, as can be seen in Fig. 7. While the rate of divergence increases as $k$ increase, which is largest when $k=10$, the natural frequency of oscillation decreases, as predicted in Fig. 4, where the system is now dominated by the unstable pair of poles, rather than three stable ones.

Figs. 8 and 9 shows the root-locus for the systems with the characteristic polynomial having the stability indices of 1.5 and 2.0 respectively. It has been observed by simulations that, in general, for polynomials with smaller stability indices (less stable polynomials), the dominant poles moves faster as the leading coefficient changes, affecting the characteristics behavior more than for polynomials with larger stability indices (more stable polynomials).

4. Conclusion

The order-canonical characteristic polynomials have been considered for studying the effects of increasing the order by changing the leading coefficient from zero to infinity. How the pole-locations and the accompanying step responses change are studied based on the root-locus plot and numerical simulations. The fourth-order Manabe polynomial, one of the order-canonical characteristic polynomials, has been modified to the fifth-order polynomial by adding the fifth-order term, whose coefficient changes from zero to positive infinity, while keeping other coefficients unchanged. It was found that the range of the leading coefficient such that the step responses have almost the same waveform as that of the fourth-order case was about 70 to 80% of the stability critical value, at which the polynomial with the increased order turns from a stable polynomial to an unstable polynomial. Considering this fact the other way around, if there is a fifth-order polynomial, which has the coefficients that are identical to Manabe’s up to the fourth-order term, but with the fifth-order coefficient being smaller than about 80% of the stability critical value, the fifth-order term can be ignored and treated as a fourth-order polynomial.

It was also found that the characteristic polynomial with lower degree of stability, i.e., those with smaller stability indices, the fifth-order term coefficient affects more sensitively the location of the dominant poles so that dynamic characteristics may change more noticeably. In contrast one with higher degree of stability, i.e., larger stability indices, locations of the dominant poles are less sensitive to the fifth-order term coefficient and often easier to reduce its order to four.
It should be noted that arguments made so far for continuous-time systems may be applied to
discrete-time systems expressed in delta form [13], due to the similarity of characteristic polynomials
in these two domains. The results should approach those of the continuous-time case at least as the
sampling period approaches zero. However, this does not apply to those commonly expressed in shift
form, where the coefficients have no similarity with the continuous-time values. In this case, not only
the leading coefficient but others have to be changed, making the analysis more cumbersome and
probably intractable.

Increasing the order by adding the higher-order term requires a smaller coefficient as the order
increases. At some point, this number becomes so small that it becomes smaller than the machine
epsilon. At this point, the order cannot be increased numerically further since the higher order
polynomial will have to be unstable. Therefore, a question arises as to if a real infinite-order system
can really be stable.

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