THE NON-COMMUTATIVE WEIL ALGEBRA

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Abstract. For any compact Lie group $G$, together with an invariant inner product on its Lie algebra $g$, we define the non-commutative Weil algebra $W_G$ as a tensor product of the universal enveloping algebra $U(g)$ and the Clifford algebra $Cl(g)$. Just as the usual Weil algebra $W_G = Sg^* \otimes \wedge g^*$, $W_G$ carries the structure of an acyclic, locally free $G$-differential algebra and can be used to define equivariant cohomology $H_G(B)$ for any $G$-differential algebra $B$. The main result of this paper is the construction of an isomorphism $Q : W_G \to W_G$ of the two Weil algebras as $G$-differential spaces. Furthermore, we prove that the corresponding vector space isomorphism from the usual equivariant cohomology $H_G(B)$ to the equivariant cohomology $H_G(B)$ is in fact a ring isomorphism. This generalizes the Duflo isomorphism $(Sg)^G \cong U(g)^G$ between the ring of invariant polynomials and the ring of Casimir elements. We extend our considerations to Weil algebras and equivariant cohomology with generalized coefficients, where the algebra $U(g)$ is replaced by the convolution algebra $E'(G)$ of distributions on $G$.

1. INTRODUCTION

Let $G$ be a connected Lie group with Lie algebra $g$. The Duflo map is a vector space isomorphism $Duf : S(g) \to U(g)$ between the symmetric algebra and the universal enveloping algebra which, as proved by Duflo [7], restricts to a ring isomorphism from the algebra of invariant polynomials $S(g)^G$ onto the center $U(g)^G$ of the universal enveloping algebra. For semi-simple $g$ the Duflo map coincides with the Harish-Chandra isomorphism. The Duflo map extends to a map of compactly supported distributions, $Duf : E'(g) \to E'(G)$, which is a ring homomorphism for the $G$-invariant parts. For generalizations of Duflo’s theorem see the papers of Kashiwara-Vergne [13] and Kontsevich [14].

In this paper we will obtain analogues of the Duflo map and of Duflo’s theorem in the context of equivariant cohomology of $G$-manifolds. Our result will involve a non-commutative version of the de Rham model of equivariant cohomology. Throughout we will assume that the group $G$ is compact. Let

$$W_G = Sg^* \otimes \wedge g^*$$

the Weil algebra. Given a basis $e_a$ of $g$ let $f_{ab}^c$ the structure constants of $g$, and denote by $v^a$, $y^a$ the generators of $Sg^*$ and $\wedge g^*$ corresponding to the dual basis. The coadjoint

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* Partially supported by NFR under contract FU 11672-302.
† Partially supported by NSERC grant # 72011899 and by a Connaught grant.
action of $G$ on $\mathfrak{g}^*$ induces an action on $W_G$, with generators $L_a$. Let derivations $\iota_a$ be given on generators by $\iota_a y^b = \delta^b_a$ and $\iota_a v^b = 0$. In [6], H. Cartan shows that the derivation

\begin{equation}
\begin{aligned}
d y^a &= v^a - \frac{1}{2} f_{ijk} y^j y^k, \\
d v^a &= -f_{ijk} y^j v^k
\end{aligned}
\end{equation}

(1)

gives $W_G$ the structure of a $G$-differential algebra. In particular, $d$ is a differential and $[\iota_a, d] = L_a$.

Suppose $g$ comes equipped with an invariant inner product, used to identify $g \cong g^*$. Let $\text{Cl}(g)$ be the corresponding Clifford algebra and let the non-commutative Weil algebra $W_G$ be its tensor product with the universal enveloping algebra:

$$W_G = U(g) \otimes \text{Cl}(g).$$

Take the basis $e_a$ to be orthonormal, and let $u_a, x_a$ be the corresponding generators of $U(g)$ and $\text{Cl}(g)$. Again $W_G$ carries a $G$-action induced by the adjoint action; let $L_a$ be its generators and let $\iota_a$ be the derivation extension of $\iota_a x_b = \delta_{ab}$ and $\iota_a u_b = 0$. We show that there exists a derivation $d$ on $W_G$ which on generators is given by formulas analogous to (1):

\begin{equation}
\begin{aligned}
dx_a &= u_a - \frac{1}{2} f_{ijk} x_j x_k, \\
du_a &= -f_{ijk} x_j u_k.
\end{aligned}
\end{equation}

(2)

Moreover $W_G$ is still a $G$-differential algebra, that is, $d$ squares to zero and Cartan’s formula continues to hold. A new feature is that the derivations $\iota_a, L_a, d$ can be written as commutators. In particular, $d = \text{ad}(\mathcal{D})$ where

$$\mathcal{D} = u_a x_a - \frac{1}{6} f_{abc} x_a x_b x_c$$

squares to the quadratic Casimir element

$$\mathcal{D}^2 = \frac{1}{2} u_a u_a - \frac{1}{48} f_{abc} f_{abc}.$$ 

The latter is naturally interpreted as a Laplace operator on $G$ and $\mathcal{D}$ as a Dirac operator.

The main result of this paper is the construction of a vector space isomorphism $Q : W_G \to W_G$ (called the quantization map) which intertwines the derivations $\iota_a, L_a, d$. Put differently, $Q$ induces a second, non-commutative algebra structure on the Weil algebra for which $\iota_a, L_a, d$ continue to be derivations. On the symmetric algebra $S_{g^*} \otimes 1$, $Q$ restricts to the Duflo map while on the exterior algebra $1 \otimes \wedge g$, it restricts to the inverse of the symbol map $\sigma : \text{Cl}(g) \to \wedge g$. However, $Q$ is not the direct product $\text{Duf} \otimes \sigma^{-1}$ but has the more complicated form

$$Q = (\text{Duf} \otimes \sigma^{-1}) \circ \exp(-\frac{1}{2} T_{ab} \iota_a \iota_b)$$

where $T_{ab}$ is a certain solution of the classical dynamical Yang-Baxter equation [8].

We also consider Weil algebras with generalized coefficients, $\hat{W}_G = E'(g^*) \otimes \wedge g^*$ and $\hat{W}_G = E'(G) \otimes \text{Cl}(g)$. Here $S_{g^*}$ is identified with the subalgebra of $E'(g^*)$ consisting of distributions with support at the origin and $U(g)$ with the subalgebra of $E'(G)$ consisting
of distributions with support at the group unit. The quantization map extends to a map $Q: \hat{W}_G \to \hat{W}_G$ which still intertwines the derivations $\iota_a, L_a, d$ but is no longer an isomorphism.

Given any $G$-differential algebra $B$, for example the de Rham complex $\Omega^*(M)$ of a $G$-manifold $M$, the equivariant cohomology $H_G(B)$ is defined as the cohomology algebra of the basic subcomplex of $W_G \otimes B$. It is a module over the ring $(Sg^*)^G$ of invariant polynomials. Replacing $W_G$ with $\hat{W}_G, \hat{W}_G, \hat{W}_G$ we also define equivariant cohomology algebras $\hat{H}_G(B), H_G(B), \hat{H}_G(B)$, which are modules over the rings $\mathcal{E}'(g^*)^G, U(g)^G, \mathcal{E}'(G)^G$, respectively.

The quantization map $Q$ induces a vector space isomorphism $Q: H_G(B) \to H_G(B)$ and a linear map $Q: \hat{H}_G(B) \to \hat{H}_G(B)$. Our second main result is that both of these maps are ring homomorphisms. For $B = \mathbb{R}$ this is the Duflo theorem for compact $G$.

In the case $B = \Omega(M)$ where $M$ is a compact oriented $G$-manifold, there is a push-forward map $\int: \hat{H}_G(\Omega(M)) \to \mathcal{E}'(G)^G$. In a sequel to this paper, we will explain how the localization formula in equivariant cohomology (see [4], [2]) carries over to give a formula for the Fourier coefficients of this map.

In another sequel [1] we use the equivariant cohomology groups $\hat{H}_G(\Omega(M))$ to construct Liouville forms for Lie group valued moment maps, with applications to moduli spaces of flat connections on surfaces.

**Acknowledgments.** We would like to thank N. Berline, R. Bott, P. Etingof, V. Guillemin, B. Kostant, A. Recknagel, S. Sternberg, M. Vergne and C. Woodward for many stimulating discussions. We are particularly indebted to Michele Vergne for the important suggestion of using the duality between the Weil algebras $\hat{W}_G, \hat{W}_G$ and spaces of differential forms $\Omega(g), \Omega(G)$. In particular, this idea improved the result about the ring structures in Section 8.

2. Review of the Weil algebra

In this section we review H. Cartan’s algebraic version of equivariant de Rham theory. Following more recent references (e.g. [18, 12, 10]) we will phrase his theory in “super” terminology. Throughout this paper we work over the field $\mathbb{R}$ of real numbers, and “algebra” will always mean algebra with a unit element.

2.1. Super-notation. A super-vector space is a vector space $B$ together with a linear map (parity operator) $\epsilon \in \text{End}(B)$ satisfying $\epsilon^2 = \text{Id}_B$. The $+1$ eigenspace of $\epsilon$ is denoted $B_{\text{even}}$ and the $-1$ eigenspace $B_{\text{odd}}$. A homomorphism of super-vector spaces is a linear map preserving $\mathbb{Z}_2$-gradings. Ungraded vector spaces $B$ will be viewed as “super” by putting $B = B_{\text{even}}$. The tensor product of two super-vector spaces $B_1, B_2$ is a super-vector space. If $B_j$ are super-algebras this tensor product denotes the super-($\mathbb{Z}_2$-graded) tensor product. Similarly commutators $[a, b] = \text{ad}(a)b$ in a super-algebra always mean super-commutators, derivations always mean super-derivations. The space
Der$(B)$ of derivations of a super-algebra $B$ is a super-Lie algebra, and there is a super-Lie homomorphism $\text{ad} : B \to \text{Der}(B)$. An odd derivation $d \in \text{Der}(B)$ is called a differential if $d \circ d = 0$.

For any element $b \in B$ we denote by $b^L$ the operator of left multiplication by $b$ and by $b^R$ the operator by graded right multiplication. Thus $\text{ad}(b) = b^L - b^R$. For commutative super-algebras we drop the superscripts.

By a graded super-vector space we mean a $\mathbb{Z}$-graded vector space $B^\star = \bigoplus_k B^k$, viewed as a super-vector space by putting $\epsilon = (-1)^k$ on $B^k$. A filtered super-vector space is defined to be a super-vector space $B = B(\star)$, together with an increasing $\mathbb{Z}$-filtration

$$\ldots \subseteq B^{(k)} \subseteq B^{(k+1)} \subseteq \ldots$$

such that $\epsilon(B^{(k)}) \subseteq B^{(k)}$ for all $k$ and the associated graded space $\text{Gr}^*(B)$, with $\mathbb{Z}_2$-grading induced from $\epsilon$, is a graded super-vector space.

### 2.2. G-differential algebras.

Let $G$ be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Choose a basis $e_a$ of $\mathfrak{g}$, and let $e^a \in \mathfrak{g}^*$ be the dual basis. Let $f_{ab}^c$ be the structure constants defined by

$$[e_a, e_b] = f_{ab}^c e_c.$$

Let $\mathfrak{g}^*$ be the graded super-Lie algebra defined as follows. As vector spaces $\mathfrak{g}^0 \simeq \mathfrak{g}^{-1} \simeq \mathfrak{g}$, while $\mathfrak{g}^1 \cong \mathbb{R}$ and $\mathfrak{g}^k = \{0\}$ for $k \neq -1, 0, 1$. Letting $L_a, t_a$ denote the basis elements in $\mathfrak{g}^0, \mathfrak{g}^{-1}$ corresponding to $e_a$ and $d$ the generator of $\mathfrak{g}^1$ the bracket relations are defined as

$$[L_a, t_b] = f_{ab}^c t_c,$$
$$[L_a, L_b] = f_{ab}^c L_c,$$
$$[t_a, d] = L_a.$$

For any $G$-manifold $M$ there is a natural representation of $\mathfrak{g}$ on the commutative super-algebra of differential forms $B = \Omega^\star(B)$, where $t_a$ are interpreted as contractions with generating vector fields

$$(e_a)_M := \frac{\partial}{\partial t} |_{t=0} \exp(-te_a)^*$$
on $M$, $L_a$ as Lie derivatives, and $d$ as the exterior differential. More generally, one defines:

**Definition 2.1.** A $G$-differential space is a super-space $B$, together with a super Lie algebra homomorphism $\rho : \mathfrak{g} \to \text{End}(B)$. The horizontal subspace $B_{\text{hor}}$ is the space fixed by $\mathfrak{g}^{-1}$, the invariant subspace $B^G$ is the space fixed by $\mathfrak{g}^0$, and the space $B_{\text{basic}}$ of basic elements is their intersection. If $B$ is graded/filtered and the map $\rho$ preserves degrees/ filtration degrees we call $B$ a graded/filtered $G$-differential space. A $G$-differential algebra is a super-algebra $B$, together with a structure of a $G$-differential space such that $\rho$ takes values in derivations of $B$. 
Definition 2.2. A homomorphism between (graded/filtered) $G$-differential spaces/algebras $(B_1, \rho_1)$ and $(B_2, \rho_2)$ is a homomorphism of (graded/filtered) super-spaces/algebras $\phi : B_1 \to B_2$ with $A \circ \rho_1 = \rho_2$.

The idea of a $G$-differential algebra was introduced by H. Cartan in [6], Section 6 and appears in the literature under various names. We follow the terminology from [17], although in this reference it is also required that $B$ carries a (Frechet) topology and the representation of $\hat{g}^0 = g$ exponentiates to a differentiable $G$-action on $B$. Obviously, the associated graded algebra of any filtered $G$-differential algebra is a graded $G$-differential algebra.

The basic examples for $G$-differential algebras are as follows. We will always write $\iota_a, L_a, d$ in place of $\rho(\iota_a), \rho(L_a)$ and $\rho(d)$.

Example 2.3 (Trivial $G$-differential algebra). The trivial $G$-differential algebra is $\mathbb{R}$ with the trivial representation $\rho = 0$. If $(B, \rho)$ is any (possibly graded or filtered) $G$-differential algebra, the injection

$$\mathbb{R} \to B, \ t \mapsto t I,$$

where $I \in B$ is the unit element, is a homomorphism of (graded/filtered) $G$-differential algebras. This follows since every derivation annihilates the unit element.

Example 2.4 (Exterior algebra). Take $B = \wedge g^*$, equipped with the coadjoint $G$-action. Let $y^a \in \wedge^1 g^*$ be the basis elements corresponding to $e^a$, let $\iota_a = \iota(e_a)$ be the contraction, $L_a$ the Lie derivative $L_a = -f_{ab}^c y^b \iota_c$, and let $d$ be given by Koszul’s formula

$$(3) \quad d = -\frac{1}{2} f_{bc}^a y^b y^c \iota_a = \frac{1}{2} y^a L_a.$$

Then $d \circ d = 0$ and $[\iota_a, d] = L_a$ so that $\wedge g^*$ is a $G$-differential algebra. Since $G$ is compact, every cocycle has a representative in the invariant part $(\wedge^* g^*)^G$. Since $d$ vanishes on this subspace it follows that $H^*(\wedge g^*, d) \cong (\wedge^* g^*)^G$.

Example 2.5 (Weil algebra). Let $S g^*$ be the symmetric algebra, equipped with the $G$-action induced by the coadjoint action of $G$ on $g^*$. If we denote by $v^a \in S^1 g^*$ the generators corresponding to the basis $e^a \in g^*$, the Lie derivative is given by the formula

$$L_a v^c = -f_{ab}^c v^b.$$ 

The Weil algebra is the graded commutative super-algebra $W_G = \oplus_{l=0}^{\infty} W^l_G$, where

$$W^l_G = \bigoplus_{j+2k=l} S^k g^* \otimes \wedge^j g^*.$$ 

We will write $v^a$ in place of $v^a \otimes 1 \in W^2_G$ and $y^a$ in place of $1 \otimes y^a \in W^1_G$. Let $W^*_G$ be equipped with the diagonal $G$-action and corresponding Lie derivative $L_a = 1 \otimes L_a +$
Let \( L_a \otimes 1 \in \text{Der}^0(W_G) \). Let \( \iota_a = 1 \otimes \iota_a \in \text{Der}^{-1}(W_G) \), and let the differential on \( W_G \) be given by the formula

\[
d = y^a(L_a \otimes 1) + (v^a - \frac{1}{2} f^a_{bc} y^b y^c)\iota_a.
\]

On generators,

\[
dy^a = v^a - \frac{1}{2} f^a_{jk} y^j y^k,
\]

\[
dv^a = -f^a_{jk} y^j v^k.
\]

It is easily verified that \([\iota_a, d] = L_a\) and \(d \circ d = 0\) so that \(W_G^*\) has the structure of a \(G\)-differential algebra. Cartan shows in [5] that \((W_G, d)\) is acyclic, that is, \(H^k(W_G) = 0\) for \(k \neq 0\), \(H^0(W_G) = \mathbb{R}\). The horizontal algebra is \((W_G)_{\text{hor}} \cong Sg^*\) and the basic subalgebra is the algebra of invariant polynomials \((Sg^*)^G\).

2.3. Algebraic connections. A \(G\)-differential algebra \(B\) is called locally free if there exists an element \(\theta = \theta^a e_a \in (B^{\text{odd}} \otimes g)^G\), called (algebraic) connection form with \(\iota_a \theta = e_a\).

If \(B\) is graded/filtered we require in addition that \(\theta \in (B^1 \otimes g)^G\) resp. \(\theta \in (B^{(1)} \otimes g)^G\). The invariance condition means that \(L_a \theta^c = -f^c_{ab} \theta^b\). The \(G\)-differential algebra \(B = \Omega^*(M)\) is locally free if and only if the \(G\)-action on \(M\) is locally free. In this case \(\theta^a\) are connection forms in the usual sense. Note that the basic subcomplex \(B_{\text{basic}}\) is naturally isomorphic to the differential forms on the quotient space:

\[
B_{\text{basic}}^* = \Omega^*(M / G).
\]

The exterior algebra \(B = \wedge g^*\) is locally free, with connection forms \(\theta^a = y^a\). The Weil algebra \(W_G\) is locally free, with connection forms \(\theta^a = y^a\). Together with acyclicity this motivates the interpretation of \(W_G\) as the algebraic model for the de Rham complex of the classifying bundle \(EG\).

2.4. Equivariant cohomology. Given any two \(G\)-differential algebras \(B_1\) and \(B_2\), the tensor product \(B_1 \otimes B_2\) is naturally a \(G\)-differential algebra. If \(B_1\) is locally free then the tensor product \(B_1 \otimes B_2\) is locally free. In particular, \(W_G \otimes B\) is locally free for any \(G\)-differential algebra \(B\). This motivates the following definition.

Definition 2.6. Let \(B\) be any \(G\)-differential algebra. The equivariant cohomology of \(B\) is super-algebra defined as the cohomology of the basic subcomplex,

\[
H_G(B) := H((W_G \otimes B)_{\text{basic}}).
\]

If \(B\) is a graded/filtered super-algebra then \(H_G(B)\) inherits a grading/filtration.

In the special case \(B^* = \Omega^*(M)\), and if \(G\) is compact, it can be shown that \(H^*_G(\Omega^*(M))\) equals the topological equivariant cohomology \(H^*_G(M) := H^*(EG \times_G M)\). For non-compact \(G\) this statement is false in general.

Remarks 2.7. a. If \(B\) is locally free, \(H_G(B) = H(B_{\text{basic}})\).
The equivariant cohomology of the trivial $G$-differential algebra is $H_G(\mathbb{R}) = (S^*g)^G$.

Any homomorphism of $G$-differential spaces/algebras $\phi : B_1 \to B_2$ induces a super-space/algebra homomorphism $H_G(B_1) \to H_G(B_2)$. It follows that for any $G$-differential algebra $B$, the natural homomorphism $\mathbb{R} \to B$ induces an algebra homomorphism $H_G(\mathbb{R}) \to H_G(B)$ making $H_G(B)$ into a module over the ring of invariant polynomials $H_G(\mathbb{R}) = (S^*g)^G$. If $M$ is an oriented compact manifold, the integration map $\int_M : \Omega^*(M) \to \mathbb{R}$ is a chain map, and therefore defines an equivariant linear map $\int_M : H_G(M) \to S^*(g)^G$.

**Definition 2.8.** Two homomorphisms $\phi_1, \phi_2 : B_1 \to B_2$ between $G$-differential spaces are called $G$-chain homotopic if there an odd linear map $h : B_1 \to B_2$ which commutes with contractions $\iota_a$ and Lie derivatives $L_a$ and which satisfies $[d, h] = \phi_1 - \phi_2$.

If $\phi_1, \phi_2 : B_1 \to B_2$ are chain homotopic homomorphisms of $G$-differential spaces, then the induced maps in cohomology coincide.

### 3. The non-commutative Weil algebra

All of the examples of $G$-differential algebras discussed in the previous section are super-commutative. In this section we will consider two non-commutative examples: The Clifford algebra $\text{Cl}(g)$ and its tensor product with the universal enveloping algebra, $U(g) \otimes \text{Cl}(g)$.

#### 3.1. The Clifford algebra.

Suppose that $g$ comes equipped with an invariant inner product, used to identify $g \cong g^*$. Let $\text{Cl}(g)$ be the Clifford algebra of $g$, i.e. the quotient of the tensor algebra by the ideal generated by all $(\mu, \mu) - 2\mu \otimes \mu$ with $\mu \in g$. It inherits from the tensor algebra a natural $\mathbb{Z}_2$-grading and filtration,

$$\mathbb{R} = \text{Cl}^{(0)}(g) \subset \text{Cl}^{(1)}(g) \subset \ldots.$$ 

The filtration and grading are compatible, so that $\text{Cl}(g)$ is a filtered super-algebra. There is a canonical isomorphism

$$\text{Gr}^*(\text{Cl}(g)) \cong \wedge^* g.$$ 

Let us choose the basis $e_a$ of $g$ to be orthonormal so that $e_a = e^a$. Using the metric to pull down indices we write $f_{abc} = f_{abc}^e$. If we denote the generators of $\text{Cl}(g)$ corresponding to $e_a$ by $x_a$ the defining relations are $[x_a, x_b] = \delta_{ab}$.

The symbol map $\sigma : \text{Cl}(g) \to \wedge g$ is the vector space isomorphism given on generators by

$$\sigma(x_{j_1} \ldots x_{j_k}) = y_{j_1} \wedge \ldots \wedge y_{j_k}$$

for $j_1 < \ldots < j_k$. It does not depend on the choice of basis.

Henceforth we will sometimes drop $\sigma$ from the notation and just think of $\text{Cl}(g)$ as $\wedge g$ with a different product structure $\odot$, defined by

$$y \odot y' = \sigma(\sigma^{-1}(y)\sigma^{-1}(y')).$$
The operators of Clifford multiplication by $x_a$ from the left or right become
\[ x_a^L = y_a + \frac{1}{2} t_a, \quad x_a^R = y_a - \frac{1}{2} t_a. \]

The relation between Clifford multiplication and exterior multiplication is described in [13], Theorem 16: Let $\iota_a^1$ and $\iota_a^2$ be the contraction operators for the first resp. second factor in $\wedge g \otimes \wedge g$.

**Lemma 3.1.** The algebra structure on $\wedge g$ induced by the symbol map is the map $\circ : \wedge g \otimes \wedge g \to \wedge g$ given by composition of the operator $\exp(-\frac{1}{2} \iota_a^1 \iota_a^2)$ on $\wedge g \otimes \wedge g$ with exterior multiplication on $\wedge g$.

**Proof.** Since the product $\circ$ is associative, it suffices to check $\sigma(xx') = \sigma(x) \circ \sigma(x')$ for the case $x = x_j$ for some $j$, and $x' = x_{j_1} \ldots x_{j_k}$ with $j_1 < \ldots < j_k$. In both of the sub-cases $j \in \{j_1, \ldots, j_k\}$ and $j \notin \{j_1, \ldots, j_k\}$ this is easily verified. \hfill \Box

Let us now consider the $G$-action on $\text{Cl}(g)$ induced by the adjoint action on $g$. The corresponding Lie derivative is a commutator
\[ L_a = \text{ad}(g_a), \]
with
\[ g_a = -\frac{1}{2} f_{ars} x_r x_s \in \text{Cl}^{(2)}(g). \]

Let
\[ \gamma = \frac{1}{3} x_a g_a = -\frac{1}{6} f_{abc} x_a x_b x_c \in \text{Cl}^{(3)}(g)^G. \]

**Proposition 3.2.** The Clifford algebra $\text{Cl}(g)$ with derivations
\[ \iota_a = \text{ad}(x_a), \quad L_a = \text{ad}(g_a), \quad d = \text{ad}(\gamma) \]
is a filtered $G$-differential algebra, having $\wedge g$ as its associated graded $G$-differential algebra. The cohomology is trivial in all filtration degrees (except if $g$ is abelian, in which case $d = 0$).

**Proof.** The required commutation relations for the operators $L_a, \iota_a$ and $d$ follow from those for the elements $g_a, x_a$ and $\gamma$. For example $[g_a, \gamma] = L_a \gamma = 0$ shows $[L_a, d] = 0$, and $[x_a, \gamma] = \iota_a \gamma = g_a$ shows $[\iota_a, d] = 0$. The only non-trivial commutation relation to check is that $[d, d] = 0$, or equivalently that $[\gamma, \gamma] = 2 \gamma^2$ is in the center of $\text{Cl}(g)$. In fact $\gamma^2$ is a scalar (cf. Kostant [13]):
\[ \gamma^2 = -\frac{1}{48} f_{abc} f_{abc}. \]

To see this we compute the symbol $\sigma(\gamma^2) = \sigma(\gamma) \circ \sigma(\gamma)$ using Lemma 3.1. Here $\sigma(\gamma) = -\frac{1}{6} f_{abc} y_a y_b y_c$. The terms $\sigma(\gamma)$ and $\iota_a \iota_b \sigma(\gamma)$ square to zero since they have odd degree, and the term $\iota_a \sigma(\gamma) = \sigma(g_a)$ squares to zero by the Jacobi identity. Since
\[ \frac{1}{3!} (-\frac{1}{2} \iota_a^1 \iota_a^2)^3 \sigma(\gamma) \otimes \sigma(\gamma) = -\frac{1}{48} (\iota_a^1 \iota_a^2)^3 \sigma(\gamma) \otimes \sigma(\gamma) = -\frac{1}{48} f_{abc} f_{abc}, \]
Equation (7) follows. This shows that the operators $d$, $L_a$ and $\iota_a$ define a representation of $\hat{g}$ on $\text{Cl}(g)$. To verify that these operators have the required filtration degrees we check on generators:

$$L_a x_b = f_{abc} x_c, \quad \iota_a x_b = \delta_{ab}, \quad dx_a = -\frac{1}{2} f_{a r s} x_r x_s.$$  

These equations show also that the associated graded $G$-differential algebra on $\wedge^\star g = \text{Gr}^\star (\text{Cl}(g))$ is the standard one. Finally, to see that the cohomology of $(\text{Cl}(g), d)$ is trivial (if $g$ is non-abelian) we note that

$$[d, \gamma] = [\gamma, \gamma] = 2\gamma^2 = -\frac{1}{24} f_{abc} f_{abc},$$

so that

$$H := -\frac{24}{f_{abc} f_{abc}} \gamma$$

is a homotopy operator.

Below we will need the following description of the differential $d = \text{ad}(\gamma)$ in terms of the identification $\sigma : \text{Cl}(g) \cong \wedge g$. It shows that $\text{ad}(\gamma)$ is the Koszul differential plus an extra cubic term:

**Proposition 3.3.** Under the identification $\text{Cl}(g) \cong \wedge g$ by the symbol map,

$$\text{ad}(\gamma) = -\frac{1}{2} f_{abc} y_b y_c \iota_a - \frac{1}{24} f_{abc} \iota_a \iota_b \iota_c.$$

**Proof.** Using (5) we compute

$$\gamma^L = -\frac{1}{6} f_{abc} \left( (y_a + \frac{1}{2} \iota_a) (y_b + \frac{1}{2} \iota_b) (y_c + \frac{1}{2} \iota_c) \right),$$

$$\gamma^R = -\frac{1}{6} f_{abc} \left( (y_a - \frac{1}{2} \iota_a) (y_b - \frac{1}{2} \iota_b) (y_c - \frac{1}{2} \iota_c) \right).$$

Taking the difference $\text{ad}(\gamma) = \gamma^L - \gamma^R$, the terms cubic and linear in $y_a$’s cancel, and the remaining terms yield the Proposition.  

### 3.2. The non-commutative Weil algebra.

Let $U(g)$ be the universal enveloping algebra, with its natural filtration $\mathbb{R} = U^{(0)}(g) \subset U^{(1)}(g) \subset \ldots$. The associated graded algebra is $\text{Gr}^\star U(g) = S^\star g$. We denote the generators of $U(g)$ corresponding to $e_a \in g$ by $u_a$. The extension to $U(g)$ of the adjoint action has Lie-derivative $L_a = \text{ad}(u_a)$. Define a filtered super-algebra $W^{(l)}_G$, where

$$W^{(l)}_G = \bigoplus_{j + 2k = l} U^{(k)}(g) \otimes \text{Cl}^{(j)}(g),$$

equipped with the diagonal $G$-action. Then $\text{Gr}^\star (W_G) = W^\star_G$ is just the usual Weil algebra, using the identification $g^\star \cong g$. We will show that $W_G$ has the structure of a filtered $G$-differential algebra.
The generators \( u_a = u_a \otimes 1 \) and \( x_a = 1 \otimes x_a \) have degree 2 and 1, respectively. The Lie derivatives are \( L_a = \text{ad}(u_a + g_a) \), and the derivations \( \iota_a \) on \( \text{Cl}(\mathfrak{g}) \) extend to derivations \( \iota_a \) on \( \mathcal{W}_G \). The basic subspace of \( \mathcal{W}_G \) for the derivations \( \iota_a \) and the \( G \)-action is \( (\mathcal{W}_G)_{\text{basic}} = U(\mathfrak{g})^G \otimes 1 \). In particular, \( (\mathcal{W}_G)_{\text{basic}} \) is a subspace of the center \( Z(\mathcal{W}_G) \).

Introduce an element \( D \in (\mathcal{W}_G^{(3)})^G \) by
\[
D := x_a u_a + \gamma.
\]

**Proposition 3.4.** The square \( D^2 \) is given by
\[
D^2 = \frac{1}{2} u_a u_a + \gamma^2 = \frac{1}{2} u_a u_a - \frac{1}{48} f_{abc} f_{abc}.
\]

**Proof.** We calculate:
\[
D^2 = \frac{1}{2} [D, D] = \frac{1}{2} [u_a x_a, u_b x_b] + \gamma^2 + [u_a x_a, \gamma]
= \frac{1}{2} u_a u_a + \frac{1}{2} [u_a, u_b] x_a x_b + \gamma^2 + u_a [x_a, \gamma]
= \frac{1}{2} u_a u_a + \frac{1}{2} f_{abc} u_c x_a x_b + \gamma^2 + u_a g_a
= \frac{1}{2} u_a u_a + \gamma^2.
\]

**Remark 3.5.** As suggested by the formula for \( D^2 \), the element \( D \) may be viewed as a Dirac operator. Indeed let \( \text{Cl}(T^* G) \) be the Clifford algebra bundle for the left-invariant Riemannian metric on \( G \) which coincides with the given inner product on \( \mathfrak{g} = T_e G \). Identify \( x_a \) with left-invariant sections of \( \text{Cl}(T^* G) \) and \( e_a \) with left-invariant (co-)vector fields. Let \( \nabla^L_a = \nabla^L_{e_a} \) denote covariant derivatives with respect to left-trivialization. The operators \( \nabla^L_a \) satisfy the commutation relations \( [\nabla^L_a, \nabla^L_b] = f_{abc} \nabla^L_c \) and \( [\nabla^L_{e_a}, x_b] = 0 \). Hence setting \( u_a = \nabla^L_a \) we identify \( U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{g}) \) with an algebra of left-invariant differential operators acting on the bundle \( \wedge T^* G \). Furthermore \( \nabla_a = \nabla^L_a - \frac{1}{6} f_{abc} x_b x_c \) becomes a left-invariant connection and \( D = x_a \nabla_a \) the associated Dirac operator. (It is not the de-Rham Dirac operator, however.) The above Dirac operator appears in the Physics literature, see Fröhlich, Grandjean and Recknagel [3] (Section 3).

A generalization of this Dirac operator, with fascinating applications to representation theory was considered by Kostant [10].

Since \( D^2 \in (\mathcal{W}_G)_{\text{basic}} \) is a central element, \( d := \text{ad}(D) \) is a differential. On generators,
\[
d x_a &= u_a - \frac{1}{2} f_{ajk} x_j x_k, \\
d u_a &= -f_{ajk} x_j u_k,
\]
which shows that \( d \) has filtration degree 1.
Theorem 3.6. The non-commutative Weil algebra $\mathcal{W}_G$, equipped with derivations
\[
\begin{align*}
\iota_a &= \text{ad}(x_a), \\
L_a &= \text{ad}(u_a + g_a), \\
d &= \text{ad}(\mathcal{D})
\end{align*}
\]
is a filtered $G$-differential algebra, with associated graded $G$-differential algebra equal to
the standard Weil algebra $\mathcal{W}_G$ (identifying $g^* \cong g$).

Proof. We have already shown that $[d, d] = 0$. Since $\mathcal{D}$ is $G$-invariant, $[g_a, \mathcal{D}] = L_a \mathcal{D} = 0$
so that $[L_a, d] = 0$. Furthermore $[x_a, \mathcal{D}] = u_a - \frac{1}{2} f_{ars} x_r x_s = u_a + g_a$
shows $[\iota_a, d] = L_a$. All other commutators are obvious. The formulas for $d, \iota_a, L_a$ on
generators show that the associated graded $G$-differential algebra is just the standard
one on $\mathcal{W}_G$.

To compare the differential on $\mathcal{W}_G$ with the differential on the commutative Weil algebra
$\mathcal{W}_G$, use the symbol map to identify $\mathcal{W}_G = U(g) \otimes \text{Cl}(g) \cong U(g) \otimes \wedge g$.

Proposition 3.7. Under the identification $\sigma : \mathcal{W}_G \cong U(g) \otimes \wedge g$, the non-commutative
Weil differential is given by the formula
\[
d^{\mathcal{W}} = y_a(L_a \otimes 1) + \left(\frac{u_a^L + u_a^R}{2} - \frac{1}{2} f_{abc} y_b y_c\right) \iota_a - \frac{1}{24} f_{abc} \iota_a \iota_b \iota_c.
\]

Proof. Using (5) we compute:
\[
\text{ad}(x_a u_a) = (y_a + \frac{1}{2} \iota_a) u_a^L - (y_a - \frac{1}{2} \iota_a) u_a^R
\]
\[
= (u_a^L - u_a^R) y_a + \frac{1}{2} (u_a^L + u_a^R) \iota_a.
\]
Together with Proposition 3.3 this proves Proposition 3.7.

Definition 3.8. If $B$ is any $G$-differential algebra define the cohomology $\mathcal{H}_G(B)$ as the
super-algebra
\[
\mathcal{H}_G(B) := H((\mathcal{W}_G \otimes B)_{\text{basic}}, d).
\]
If $B$ carries a filtration $B = B^{(*)}$ then $\mathcal{H}_G(B)$ inherits a filtration $\mathcal{H}^{(*)}(B)$.

Any homomorphism $B_1 \to B_2$ of (filtered) $G$-differential algebras induces an homomorphism of (filtered) super-algebras $\mathcal{H}_G(B_1) \to \mathcal{H}_G(B_2)$. In particular, for any $G$-differential algebra $B$ the natural embedding $\mathbb{R} \to B^{(0)}$, $t \mapsto t I$ makes $\mathcal{H}_G(B)$ into an algebra over the ring of Casimir operators $\mathcal{H}_G(\mathbb{R}) = U(g)^G$. If $M$ is an oriented compact manifold, the integration map $\int_M : \Omega(M) \to \mathbb{R}$ induces a linear map in cohomology, $\int_M : \mathcal{H}_G(\Omega(M)) \to U(g)^G$. 
4. The Cartan model

In \cite{5}, Section 6 Cartan shows that the equivariant cohomology of a $G$-differential algebra can be computed in an equivalent, simpler “model” $C_G(B) := (Sg^* \otimes B)^G$ with differential $d_G = 1 \otimes d - v^a \otimes \iota_a$. The goal of this section is to develop a similar Cartan model for $H_G(B)$. We first review the standard Cartan model, following the exposition in Guillemin-Sternberg \cite{10}.

4.1. Cartan model for $H^G_B$. A simple and transparent way of obtaining the Cartan model from the Weil model was found by Kalkman \cite{12}, following earlier work by Mathai-Quillen \cite{18}. Let the Kalkman operator $\phi$ be defined by

$$\phi = \exp(y^a \otimes \iota_a) : W_G \otimes B \to W_G \otimes B.$$ 

One checks that $y^a \otimes \iota_a \in \text{Der}(W_G \otimes B)$ is an even derivation so that $\phi$ defines an algebra isomorphism (preserving degrees/filtration degrees if $B$ is graded/filtered). The key property of the Kalkman map $\phi$ is as follows.

**Proposition 4.1** (\cite{12}, Eq. (1.23)). The conjugate of the contraction operator $\iota_a = \iota_a \otimes 1 + 1 \otimes \iota_a$ under the map $\phi$ is

$$\text{Ad}_{\phi} \iota_a = \iota_a \otimes 1.$$ 

**Proof.** This follows by writing $\text{Ad}_{\phi} = \sum_j \frac{1}{j!} \text{ad}^j(y^r \otimes \iota_r)$ and

$$\text{ad}(y^r \otimes \iota_r)(\iota_a \otimes 1 + 1 \otimes \iota_a) = -1 \otimes \iota_a,$$

$$\frac{1}{2!} \text{ad}(y^r \otimes \iota_r)(\iota_a \otimes 1 + 1 \otimes \iota_a) = 0.$$ 

Since the kernel of the operators $\iota_a \otimes 1$ is just $Sg^* \otimes B$, it follows that $\phi$ restricts to an algebra isomorphism $(W_G \otimes B)_{\text{hor}} \to Sg^* \otimes B$, and by equivariance it restricts further to an algebra isomorphism $(W_G \otimes B)_{\text{basic}} \to C_G(B) = (Sg^* \otimes B)^G$.

The algebra isomorphism $(W_G \otimes B)_{\text{basic}} \to C_G(B)$ is due to Cartan \cite{6} and Mathai-Quillen \cite{18} while the extension to an algebra automorphism of $W_G \otimes \Omega(M)$ is due to Kalkman \cite{12}. It can also be described as follows. Let

$$P_{\text{hor}} = \prod_\beta \iota_\beta y^\beta : W_G \to S(g) \hookrightarrow W_G$$

be the horizontal projection. On the invariant subspace $(W_G \otimes B)^G$, the Kalkman map agrees with the operator $P_{\text{hor}} \otimes 1$ since

$$\exp(y^a \otimes \iota_a) = \prod_\beta (1 + y^\beta \otimes \iota_\beta) = \prod_\beta (1 - y^\beta \iota_\beta \otimes 1) = \prod_\beta (\iota_\beta y^\beta \otimes 1) = P_{\text{hor}} \otimes 1.$$ 

(We use the convention that we sum over roman indices but not over greek indices.) The Cartan differential $d_G$ is defined by the condition

$$d_G \circ (P_{\text{hor}} \otimes 1) = (P_{\text{hor}} \otimes 1) \circ (d \otimes 1 + 1 \otimes d)$$
on \((W_G \otimes B)_{\text{basic}}\). Recall formula (4) for \(d \otimes 1\). Application of \(P_{\text{hor}} \otimes 1\) annihilates all terms involving \(y_a\)'s, and on the subspace \((W_G \otimes B)_{\text{basic}}\), the operator \(v^a(\iota_a \otimes 1)\) may be replaced by \(-v^a(1 \otimes \iota_a)\) which then commutes with \((P_{\text{hor}} \otimes 1)\). Hence
\[
(P_{\text{hor}} \otimes 1) \circ (d \otimes 1 + 1 \otimes d) = (1 \otimes d - v^a \otimes \iota_a) \circ (P_{\text{hor}} \otimes 1)
\]
on \((W_G \otimes B)_{\text{basic}}\). Therefore
\[
d_G = 1 \otimes d - v^a \otimes \iota_a : C_G(M) \to C_G(M).
\]
Thus, the equivariant cohomology \(H_G(B)\) can be computed as the cohomology of the complex \((C_G(B), d_G)\):
\[
H_G(B) = H(C_G(B), d_G).
\]

4.2. Cartan model for \(H_G(B)\). We now describe the Cartan model for the equivariant cohomology \(H_G(B)\). Let \(B\) be any \(G\)-differential algebra. The non-commutative analogue of the Kalkman isomorphism is the \(G\)-equivariant operator
\[
\phi := \exp(x_a \otimes \iota_a) : W_G \otimes B \to W_G \otimes B.
\]
Note that if \(B\) is filtered, the map \(\phi\) preserves filtration degrees. By a calculation parallel to that for the usual Weil algebra, \(\phi\) coincides on \((W_G \otimes B)_{\text{hor}}\) with the operator \(P_{\text{hor}} \otimes 1\) where \(P_{\text{hor}} : W_G \to U(\mathfrak{g})\) is the horizontal projection
\[
P_{\text{hor}} = \prod_{\beta} \iota_{\beta} x^{L}_{\beta} = \prod_{\beta} \iota_{\beta} y_{\beta}.
\]
Likewise, the proof of Proposition 4.1 carries over to show that
\[
\text{Ad}_{\phi} \iota_a = \iota_a \otimes 1.
\]
Thus \(\phi\) defines a vector space isomorphism \((W_G \otimes B)_{\text{hor}} \to U(\mathfrak{g}) \otimes B\), and by equivariance
\[
(W_G \otimes B)_{\text{basic}} \cong (U(\mathfrak{g}) \otimes B)^{G}.
\]
We call \(C_G(B) := (U(\mathfrak{g}) \otimes B)^{G}\) the non-commutative Cartan model. The new differential \(d_G\) on \(C_G(B)\) is once again obtained from the condition
\[
d_G \circ (P_{\text{hor}} \otimes 1) = (P_{\text{hor}} \otimes 1) \circ (d \otimes 1 + 1 \otimes d)
\]
on \((W_G \otimes B)_{\text{basic}}\). One can read off an expression for \(d_G\) from Proposition 3.7. Application of \((P_{\text{hor}} \otimes 1)\) annihilates all terms involving \(y_a\)'s, and on the subspace \((W_G \otimes B)_{\text{basic}}\) we may replace \(\iota_a \otimes 1\) by \(-1 \otimes \iota_a\) which then commutes with \((P_{\text{hor}} \otimes 1)\).

Hence we have shown:

**Proposition 4.2.** The differential \(d_G\) on \(C_G(B) = (U(\mathfrak{g}) \otimes B)^{G}\) is given by
\[
d_G = 1 \otimes d - \frac{1}{2}(u^L_a + u^R_a) \otimes \iota_a + \frac{1}{24} f_{abc} (1 \otimes \iota_a \iota_b \iota_c).
\]
In particular, \(d_G \circ d_G = 0\) on \(C_G(B)\).

It follows that the equivariant cohomology \(H_G(B)\) of any \(G\)-differential algebras is equal to the cohomology of the complex \(C_G(B)\) with differential \(d_G\).
4.3. Ring structure of the non-commutative Cartan model. In contrast to the commutative Weil model the non-commutative Kalkman map is not an algebra isomorphism (since $x_a \otimes \iota_a$ is not a derivation). In this section we compute the new algebra structure on the Cartan model $\mathcal{C}_G(B)$ induced by the Kalkman map.

Suppose $B$ is a $G \times G$-differential algebra. Passing to the diagonal action it becomes a $G$-differential algebra. The ring structure for $W_G$ defines a natural map $(W_G \otimes B)_{\text{basic}} \rightarrow (W_G \otimes B)_{\text{basic}}$, where on the left hand side mean basic elements for the $G \times G$-action and on the right hand side for the $G$-action. Correspondingly, we obtain a chain map between Cartan models, $\gamma : \mathcal{C}_{G \times G}(B) \rightarrow \mathcal{C}_G(B)$.

**Proposition 4.3.** For any $G \times G$-differential algebra $B$, the map between Cartan models induced by the ring structure of $W_G$ reads

$$(\text{Mult}_{U(\mathfrak{g})} \otimes 1) \circ \exp(-\frac{1}{2}(1 \otimes \iota_1^a \iota_2^a)) : \mathcal{C}_{G \times G}(B) \rightarrow \mathcal{C}_G(B)$$

where $(\text{Mult}_{U(\mathfrak{g})}$ is the multiplication map for the universal enveloping algebra $U(\mathfrak{g})$.

**Proof.** Let $\text{Mult}_{W_G} = \text{Mult}_{U(\mathfrak{g})} \otimes \text{Mult}_{\mathfrak{cl}(\mathfrak{g})}$ be the multiplication map for $W_G$. The map $\gamma$ is defined by the condition $\gamma \circ (P_{\text{hor}} \otimes 1) = (P_{\text{hor}} \otimes 1) \circ (\text{Mult}_{W_G} \otimes 1)$ on basic elements $(W_{G \times G} \otimes B)_{\text{basic}}$. According to Lemma 3.1, under the identification $\sigma : \mathfrak{cl}(\mathfrak{g}) \cong \wedge \mathfrak{g}$, Clifford multiplication is given by composition of the operator $\exp(-\frac{1}{2} \iota_a^1 \iota_a^2)$ followed by wedge product. On basic elements, we can replace $\iota_a^1 \otimes 1$ by $-(1 \otimes \iota_a^1)$, and therefore $\exp(-\frac{1}{2}(1 \otimes \iota_a^1 \iota_a^2))$ with $\exp(-\frac{1}{2}(1 \otimes \iota_a^1 \iota_a^2))$, which then commutes with $P_{\text{hor}} \otimes 1$. \hfill \Box

For any $G$-differential algebra $B$ define an associative ring structure on $\mathcal{C}_G(B)$ as follows. Let $\iota_1^a, \iota_2^a$ be the contraction operators on $B \otimes B$ with respect to the first and second $G$-factor. Define a new ring structure $\odot$ on $B$ as a composition of the operator $\exp(-\frac{1}{2} \iota_a^1 \iota_a^2)$ on $B \otimes B$ with multiplication map $B \otimes B \rightarrow B$. The ring structure extends to a ring structure $\odot$ on the Cartan model $\mathcal{C}_G(B)$. It follows from Proposition 4.3 (applied to $B \otimes B$) that the ring structure $\odot$ on $\mathcal{C}_G(B)$ is a chain map for $d_G$, or equivalently that $d_G$ is a derivation for $\odot$.

5. Generalized coefficients

We will often find it useful to embed commutative and non-commutative Weil algebras into “Weil algebras with generalized coefficients”. This point of view will make many of the subsequent constructions much more natural, and also it is dictated by the applications to moment map theory developed in [1].

We point out that already for the commutative Weil model, we introduce generalized coefficients in a way different from Kumar-Vergne [17] since we are working in the Fourier transformed picture.
5.1. **The Weil algebra with generalized coefficients.** Let \( \mathcal{E}'(\mathfrak{g}^*) \) be the convolution algebra of compactly supported distributions on \( \mathfrak{g}^* \). The subalgebra of distributions with support at the origin is canonically isomorphic to \( S\mathfrak{g}^* \), by identifying the generators \( v^a \) with
\[
v^a = \frac{d}{dt} \bigg|_{t=0} \delta_{te^a} = -\frac{\partial}{\partial \mu_a} \delta_0.
\]
We view the coordinate functions \( \mu_a \) as multiplication operators \( \langle \mu_a u, \phi \rangle = \langle u, \mu_a \phi \rangle \).

Then \([\mu_a, v^b] = \delta^b_a\) and the Lie derivative \( L_a \) for the conjugation action on \( \mathcal{E}'(\mathfrak{g}^*) \) can be written
\[
L_a = f^c_{ab} v^b \mu_c.
\]

The **Weil algebra with generalized coefficients** is the \( G \)-differential algebra
\[
\widehat{\mathcal{W}}_G = \mathcal{E}'(\mathfrak{g}^*) \otimes \wedge \mathfrak{g}^*,
\]
where the differential is given by formula (4). The multiplication map on \( \widehat{\mathcal{W}}_G \) extends to a map
\[
\text{Mult}_W : \widehat{\mathcal{W}}_{G \times G} \to \widehat{\mathcal{W}}_G
\]
from the completion \( \widehat{\mathcal{W}}_{G \times G} \) of \( \widehat{\mathcal{W}}_G \otimes \widehat{\mathcal{W}}_G \). This map is the direct product of the push-forward under the addition map \( \text{Add}_\ast : \mathcal{E}'(\mathfrak{g}^* \times \mathfrak{g}^*) \to \mathcal{E}'(\mathfrak{g}^*) \) on the distribution part and the multiplication map on the exterior algebra part.

Given a \( G \)-differential algebra \( B \) we define the cohomology with generalized coefficients as the super-algebra
\[
\widehat{H}_G(B) := H((\widehat{\mathcal{W}}_G \otimes B)_{\text{basic}}).
\]
It is naturally a module over the ring \( \widehat{H}_G(\mathbb{R}) = \mathcal{E}'(\mathfrak{g}^*)^G \) of invariant compactly supported distributions. The embedding \( S(\mathfrak{g}^*) \hookrightarrow \mathcal{E}'(\mathfrak{g}^*) \) induces a natural embedding \( W_G \hookrightarrow \widehat{\mathcal{W}}_G \) and for any \( G \)-differential algebra \( B \) a map in cohomology \( H_G(B) \to \widehat{H}_G(B) \).

The Weil differential simplifies if we conjugate it by the function \( \tau_0 \in C^\infty(\mathfrak{g}^*, \wedge \mathfrak{g}^*) \) given as
\[
\tau_0(\mu) = \exp(-\frac{1}{2} f^a_{bc} \mu_a y^b y^c)
\]
where \( \mu = \mu_a e^a \).

**Lemma 5.1.** The conjugate of the differential \( d \) and the contraction \( \iota_a \) by the function \( \tau_0 \) are given by
\[
\text{Ad}(\tau_0^{-1}) \ d = v^a \iota_a \\
\text{Ad}(\tau_0^{-1}) \ i_a = i_a - f^c_{ab} \mu_c y^b
\]
Proof. We compute \( \text{Ad}(\tau_0)(v^r \iota_r) = \sum_j \frac{1}{j!} \text{ad}^j(-\frac{1}{2} f_{bc}^a \mu_a v^b y^c)(v^r \iota_r): \)

\[
\text{ad}( -\frac{1}{2} f_{bc}^a \mu_a y^b y^c)(v^r \iota_r) = -\frac{1}{2} f_{bc}^a y^b y^c \iota_a + f_{bc}^a \mu_a v^b y^c = -\frac{1}{2} f_{bc}^a y^b y^c \iota_a + (L_c \otimes 1) y^c,
\]

where the last term vanishes by the Jacobi identity for \( g \). Similarly

\[
\frac{1}{2}! \text{ad}^2(-\frac{1}{2} f_{bc}^a \mu_a y^b y^c)(v^r \iota_r) = \frac{1}{2} f_{bc}^a f_{rs}^c \mu_a y^b y^s y^r = 0
\]

Lemma 5.1 has the following interpretation. Consider the natural pairing of \( \hat{W}_G \) with the space \( \Omega^*(g^*) = C^\infty(g^*) \otimes \wedge g \) of smooth differential forms on \( g^* \). All the operators on \( \hat{W}_G \) that we are interested in are dual to certain operators on \( \Omega^*(g^*) \). In particular,

\[
v^a = \left( \frac{\partial}{\partial \mu_a} \right)^*, \quad y^a = -\iota \left( \frac{\partial}{\partial \mu_a} \right)^*, \quad \iota_a = -(d \mu_a)^*, \quad \mu_a = (\mu_a)^*.
\]

and by Lemma 5.1,

\[
\text{Ad}(\tau_0^{-1})d = v^a \iota_a = -d^* \quad \text{(10)}
\]

\[
\text{Ad}(\tau_0^{-1}) \iota_a = \iota_a - f_{ab}^c \mu_c y^b = -(d \mu_a + \iota_a)^* \quad \text{(11)}
\]

Note that \( \Omega^*(g) \) with derivations \( \iota_a = \iota_a + d \mu_a, L_a, d \) is a \( G \)-differential space, so that \( \hat{W}_G \) is (in an obvious sense) the dual \( G \)-differential space. Letting \( \text{Mult}_W : \hat{W}_G \otimes \hat{W}_G \to \hat{W}_G \) denote the multiplication in the Weil algebra, the composition \( \tau_0^{-1} \circ \text{Mult}_W \circ (\tau_0 \otimes \tau_0) \) is dual to the co-product \( \Omega^*(g^*) \to \Omega(g^*) \otimes \Omega(g^*) \) given by pull-back under the addition map \( \text{Add} : g^* \times g^* \to g^* \).

As applications we have:

**Proposition 5.2.** For all fixed \( \mu \), the element \( \tau_0(\mu) \delta_{\mu} \in \hat{W}_G \) is closed.

**Proof.** We verify:

\[
d \tau_0(\mu) \delta_{\mu}(\nu) = d \tau_0(\nu) \delta_{\mu}(\nu) = \tau_0(\nu)(v^a \iota_a) \delta_{\mu}(\nu) = 0.
\]

\( \square \)

**Proposition 5.3.** The element

\[
\Lambda_0 = \exp(-y_a d \mu_a) \tau_0(\mu) \delta_{\mu} \in \hat{W}_G \otimes \Omega(g^*)
\]

has the properties

\[
d \Lambda_0 = 0, \quad \iota_a \Lambda_0 = -d \mu_a \Lambda_0.
\]
Proof. Since \( \exp(-y_a d \mu_a) \delta_{\mu} \in \widehat{W}_G \otimes \Omega(g^*) \) can be viewed as the kernel of the identity map which obviously commutes with contractions and with the differential, this follows from Equations (10) and (11).

As another application of Lemma 5.1, let us show that \( \widehat{W}_G \) is acyclic. The space of compactly supported distributions is a direct sum

\[
\mathcal{E}'(g^*) = \mathbb{R} \oplus \mathcal{E}'(g^*)_+\]

where \( \mathbb{R} \) is embedded as multiples of \( \delta_0 \) and \( \mathcal{E}'(g^*)_+ = \{ u \mid \langle u, 1 \rangle = 0 \} \) is the space of distributions of integral 0. Similarly, \( \wedge g^* = \mathbb{R} \oplus (\wedge g^*)_+ \) where \( (\wedge g^*)_+ \) consists of elements of positive degree, and

\[
\widehat{W}_G = \mathbb{R} \oplus (\widehat{W}_G)_+.
\]

Let \( \Pi : \widehat{W}_G \to \mathbb{R} \hookrightarrow \widehat{W}_G \) be projection defined by this splitting. The differential \( v^a t_a \) preserves the decomposition and vanishes on \( \mathbb{R} \), hence \( \Pi \) is a chain map.

**Proposition 5.4.** There exists an odd operator \( h : \widehat{W}_G \to \widehat{W}_G \) satisfying

\[
[h, v^a t_a] = \text{Id} - \Pi.
\]

Hence \( h \) provides a chain homotopy and \( \widehat{W}_G \) is acyclic: \( H(\widehat{W}_G) = \mathbb{R} \).

Proof. Under the pairing of \( \widehat{W}_G \) with \( \Omega(g^*) \), the projection \( \Pi \) becomes dual to the projection map \( \pi : \Omega(g^*) \to \mathbb{R} \hookrightarrow \Omega(g^*) \) induced by the inclusion of the origin. Let \( h^{Rh} : \Omega^*(g^*) \to \Omega^{*-1}(g^*) \) be the standard de-Rham homotopy operator, so that

\[
[d^{Rh}, h^{Rh}] = \text{Id} - \pi.
\]  

Let \( h : \widehat{W}_G \to \widehat{W}_G \) be minus the dual operator to \( h^{Rh} \). The Proposition follows by taking the dual of (12).

An alternative proof of Proposition 5.4 can be given along the lines of Kumar-Vergne [17], Proposition 18.

**5.2. The non-commutative Weil algebra with generalized coefficients.** Just as for the usual Weil algebra it is important for applications to introduce generalized coefficients in the non-commutative Weil algebra. For this we identify \( u_a \) with the distribution on \( G \),

\[
u_a = \frac{d}{dt} \bigg|_{t=0} \delta_{\exp(tec)};
\]

and extend to an algebra homomorphism \( U(g) \to \mathcal{E}'(G) \). In this way \( U(g) \) is identified with the space of distributions on \( G \) with support at \( e \). The embedding \( G \to \mathcal{E}'(G) \), \( g \mapsto \delta_g \) satisfies \( \delta_g * \delta_{g_2} = \delta_{g_1 g_2} \) where \( u_1 * u_2 \) is the convolution of compactly supported distributions on \( G \), defined as push-forward under group multiplication. For all \( u \in \mathcal{E}'(G) \), \( \delta_g * u = (g \cdot u) * \delta_g \) which implies that the Lie derivative is given by a commutator.
\( L_a u = [u_a, u] \). We define the Weil algebra with generalized coefficients as the super-algebra
\[
\hat{W}_G := \mathcal{E}'(G) \otimes \text{Cl}(g).
\]
The operators \( \iota_a, L_a \) and \( d \) extend naturally to \( \hat{W}_G \) and make it into a filtered \( G \)-differential algebra. As for the commutative Weil model, the multiplication map extends to a map
\[
\text{Mult}_W : \hat{W}_{G \times G} \to \hat{W}_G
\]
given as push-forward under group multiplication on the \( \mathcal{E}'(G \times G) \) factor and by Clifford multiplication on the \( \text{Cl}(g \times g) \) factor. Given a \( G \)-differential algebra \( B \) define the super-algebra \( \hat{H}_G(B) \) as the cohomology of the complex \( (\hat{W}_G \otimes B)_\text{basic} \),
\[
\hat{H}_G(B) = H((\hat{W}_G \otimes B)_\text{basic}).
\]
It is a module over the convolution algebra \( \hat{H}_G(\mathbb{R}) = \mathcal{E}'(G)^G \) of invariant distributions. As in the commutative case, the embedding \( U(g) \hookrightarrow \mathcal{E}'(G) \) induces an embedding \( \hat{W}_G \to \hat{W}_G \) and a map \( \mathcal{H}_G(B) \to \hat{H}_G(B) \) for any \( G \)-differential algebra \( B \). For any compact oriented \( G \)-manifold \( M \) the integration map descends to a map
\[
\int_M : \hat{H}_G(\Omega(M)) \to \mathcal{E}'(G)^G.
\]
The Cartan model with generalized coefficients is
\[
\hat{C}_G(B) = (\mathcal{E}'(G) \otimes B)^G,
\]
with differential and product structure given by the same formulas as in Section 4. Integration over \( G \) defines a map
\[
\hat{C}_G(B) \to B^G
\]
which becomes a chain map if \( B^G \) is equipped with the new differential
\[
\text{(13)} \quad d + \frac{1}{24} f_{abc} t_a t_b t_c.
\]
This map can also be described in the Weil model: Let \( \Pi : \hat{W}_G \to \mathbb{R} \) the composition of the horizontal projection \( P_{\text{hor}} : \hat{W}_G \to \mathcal{E}'(G) \) followed by integration over \( G \). Then \( \Pi \) defines a map
\[
\Pi \otimes 1 : \hat{W}_G \otimes B \to B
\]
which on basic elements agrees, under the isomorphism \( P_{\text{hor}} \otimes 1 : (\hat{W}_G \otimes B)^G \cong C_G(B) \) with the map defined above. In particular, the image of any basic element is closed under the differential \( \text{(13)} \).
5.3. The pairing between $\hat{W}_G$ and $\Omega(G)$. Identify $\Omega^*(G) = C^\infty(G) \otimes \wedge g^*$ by means of left-invariant Maurer-Cartan forms $\theta_a$, and identify $\hat{W}_G \cong \mathcal{E}'(G) \otimes \wedge g$ using the symbol map. In this section we study the pairing between $\hat{W}_G$ and $\Omega(G)$ given by these identifications.

We first describe the group analogue of the function $\tau_0$. We will assume that the group $G$ is a direct product of a connected, simply connected group and a torus.

Let $\text{Spin}(g)$ be the spin group, defined as the image of $\text{so}(g) \subset \text{Cl}(2)$ even under the (Clifford) exponential map. The $G$-action on $\text{Cl}(g)$ defines a homomorphism

$$g \rightarrow \text{so}(g) \subset \text{Cl}(g), \mu \mapsto \mu a$$

where $g_a$ was defined in (6). Using the assumption on $G$ it exponentiates to a map $\tau : G \rightarrow \text{Spin}(g) \hookrightarrow \text{Cl}(g)$. Thus

$$\tau(\exp \mu) = \exp(-\frac{1}{2} f_{abc} \mu_a x_b x_c)$$

which is similar to the definition of $\tau_0$ (3). By definition, $\tau(g_1)\tau(g_2) = \tau(g_1g_2)$ and $g \cdot x = \text{Ad}(\tau(g))x$ for all $x \in \text{Cl}(g)$. The two maps $G \rightarrow \text{Cl}(g)$, $g \mapsto \tau(g)$ and $G \rightarrow \mathcal{E}'(G)$, $g \mapsto \delta_g$ combine into a map $G \rightarrow \hat{W}_G$, $g \mapsto \tau(g)\delta_g$. It has the property

$$\text{Ad}(\tau(g)\delta_g)w = g \cdot w$$

for all $g \in G$ and all $w \in \hat{W}_G$.

**Proposition 5.5.** For all fixed $g \in G$ the element $\tau(g)\delta_g \in \hat{W}_G$ is closed.

**Proof.** Using the fact (15) that conjugation by $\tau(g)\delta_g$ is the $G$-action on the Weil algebra, and that the element $\delta$ is invariant, we find

$$d\tau(g)\delta_g = [\delta, \tau(g)\delta_g] = \tau(g)\delta_g (g^{-1} \cdot \delta - \delta) = 0.$$ 

The function $\tau \in C^\infty(G) \otimes \text{Cl}(g)$ acts on $W_G$ by multiplication. This action commutes with $L_a$ since $\tau$ is equivariant. Let us conjugate the differential and contractions by the action of $\tau$. Let $\theta_a$ and $\overline{\theta}_a$ be the left/right invariant Maurer-Cartan forms on $G$ and let

$$\eta = \frac{1}{12} f_{abc} \theta_a \theta_b \theta_c$$

be the canonical 3-form. It has the property $i_a \eta = -\frac{1}{2} d(\theta_a + \overline{\theta}_a)$, from which one deduces that setting

$$\tilde{i}_a = i_a + \frac{1}{2} (\theta_a + \overline{\theta}_a), \quad \tilde{L}_a = L_a, \quad \tilde{d} = d + \eta$$

gives $\Omega(G)$ the structure of a $G$-differential space.
Proposition 5.6. The conjugates by \( \tau \) of the Weil differential and contractions are dual to the differential \( d \) and contractions \( \iota_a \) on \( \Omega(G) \):

\[
\text{Ad}(\tau^{-1})(d) = -(d + \eta)^*, \quad \text{Ad}(\tau^{-1})(\iota_a) = -(\iota_a + \frac{1}{2}(\theta_a + \overline{\theta}_a))^*.
\]

Proof. The Weil differential is

\[
d = \text{ad}(\mathcal{D}) = \mathcal{D}^L - \mathcal{D}^R = (x_a^L u_a^L - x_a^R u_a^R) + \text{ad}(\gamma)
\]

Since \( \gamma \) is \( G \)-invariant, \( \text{Ad}(\tau^{-1}) \text{ad}(\gamma) = \text{ad}(\gamma) \). We have

\[
\text{Ad}(\tau^{-1})x_a^L = (\text{Ad}_{\gamma^{-1}}) a_b x_b^L, \quad \text{Ad}(\tau^{-1})x_a^R = x_a^R
\]

and

\[
\text{Ad}(\tau^{-1})u_a^L = u_a^L - (\text{Ad}_{\gamma^{-1}}) a_b g_b^L, \quad \text{Ad}(\tau^{-1})u_a^R = u_a^R - g_a^L.
\]

Hence

\[
\text{Ad}(\tau^{-1})(u_a^L x_a^L - u_a^R x_a^R)
= (u_a^L - (\text{Ad}_{\gamma^{-1}}) a_b g_b^L)(\text{Ad}_{\gamma^{-1}}) a_x x_x^L - (u_a^R - g_a^L)x_a^R
= (u_a^R x_a^L + \frac{1}{2} f_{abc} x_a^L x_b^L x_c^L) - (u_a^R x_a^R + \frac{1}{2} f_{abc} x_a^L x_b^L x_c^R)
= (u_a^R x_a + \frac{1}{2} f_{abc} x_a^L x_b^L x_c - \frac{1}{2} f_{abc} x_a^L x_b^L x_c) + \frac{1}{2} f_{abc} x_a^L x_b^L x_c
= (u_a^R x_a + \frac{1}{2} f_{abc} x_a^L x_b^L x_c - \frac{1}{2} f_{abc} x_a^L x_b^L x_c)
\]

This result combines with the expression for \( \text{ad}(\gamma) \) from Proposition 3.3 to

\[
\text{Ad}(\tau^{-1})(d) = u_a^R x_a + \frac{1}{2} f_{abc} x_a^L x_b^L x_c + \frac{1}{12} f_{abc} x_a^L x_b^L x_c
\]

which is indeed dual to \(-(d + \eta)^*\). The calculation for the contractions is simpler:

\[
\text{Ad}(\tau^{-1})\iota_a = (\text{Ad}_{\gamma^{-1}}) a_b x_b^R - x_a^R = (\text{Ad}_{\gamma^{-1}} - I) a_b y_b + \frac{1}{2} (\text{Ad}_{\gamma^{-1}} + I) a_b t_b
\]

which is dual to \(-\iota_a + \frac{1}{2}(\theta_a + \overline{\theta}_a)) \) \].

\[ \square \]

Proposition 5.7. The element

\[ \Lambda = \exp(-x_a \overline{\theta}_a) \tau(g) \delta_g = \tau(g) \exp(-x_a \theta_a) \delta_g \in \hat{\mathcal{W}}_G \otimes \Omega(G) \]

has the properties,

\[ d\Lambda = -\eta \Lambda, \quad \iota_a \Lambda = -\frac{1}{2}(\theta_a + \overline{\theta}_a) \Lambda. \]

Proof. This follows from Proposition 5.6 since \( \exp(-x_a \theta_a) \delta_g \in \hat{\mathcal{W}}_G \otimes \Omega(G) \) is the kernel of the identity map. \[ \square \]

Let us also describe the co-multiplication on \( \Omega(G) \) which by duality gives rise to the multiplication on \( \hat{\mathcal{W}}_G \).
Proposition 5.8. Let \( \text{Mult}_W : \widehat{W}_G \otimes \widehat{W}_G \to \widehat{W}_G \) be the multiplication map for the Weil algebra. The composition \( \tau^{-1} \circ \text{Mult}_W \circ (\tau \otimes \tau) : \widehat{W}_G \otimes \widehat{W}_G \to \widehat{W}_G \) is dual to the map
\[
\exp(-\frac{1}{2}\theta_a^2) \circ \text{Mult}^*_G : \Omega(G) \to \Omega(G \times G)
\]
where \( \text{Mult}_G : G \times G \to G \) is the group multiplication and the superscripts denote pull-backs to the respective \( G \)-factor.

Proof. Since \( \tau^{-1} \Lambda^1 \Lambda^2 \) is the kernel of the multiplication map on \( \widehat{W}_G \), it suffices to show that
\[
\Lambda^1 \Lambda^2 = \exp(-\frac{1}{2}\theta_a^2)(1 \otimes \text{Mult}^*_G)\Lambda.
\]
Write \( \Lambda^1 = \exp(-x_a\overline{\theta}_a^1)\tau(g_1)\delta_g \) and similarly for \( \Lambda^2 \). We calculate:
\[
\begin{align*}
\Lambda^1 \Lambda^2 &= \exp(-x_a\overline{\theta}_a^1)\tau(g_1)\delta_g \exp(-x_b\overline{\theta}_b^2)\tau(g_2)\delta_g \\
&= \exp(-x_a\overline{\theta}_a^1)\exp(-x_b(\text{Ad}_{g_1})_{cb}\overline{\theta}_c^2)\tau(g_1)\delta_g \tau(g_2)\delta_g \\
&= \exp(-\frac{1}{2}\theta_a^1(\text{Ad}_{g_1})_{ba}\overline{\theta}_b^2) \exp(-x_a(\overline{\theta}_a^1 + (\text{Ad}_{g_1})_{ba}\overline{\theta}_b^2))\tau(g_1 g_2)\delta_{g_1 g_2} \\
&= \exp(-\frac{1}{2}\theta_a^2)(1 \otimes \text{Mult}^*_G)\Lambda
\end{align*}
\]
For the last equality we used that
\[
\text{Mult}^*_G \overline{\theta}_a = \overline{\theta}_a^1 + (\text{Ad}_{g_1})_{ba}\overline{\theta}_b^2,
\]
and for the third equality we used that
\[
\exp(\kappa_a^1 x_a) \exp(\kappa_a^2 x_a) = \exp(-\frac{1}{2}\kappa_a^1 \kappa_a^2) \exp((\kappa_a^1 + \kappa_a^2) x_a)
\]
if \( \kappa_a^1, \kappa_a^2 \) are elements of a commutative super-algebra. \(\square\)

6. The quantization map \( \mathcal{Q} : W_G \to W_G \)

In this Section we construct an explicit isomorphism of \( W_G \) and \( W_G \) as \( G \)-differential spaces. Notice that by contrast, the exterior algebra \( \wedge g^* \) and the Clifford algebra \( \text{Cl}(g) \) are not isomorphic as differential spaces (unless \( g \) is abelian): As we have seen the cohomology of \( \text{Cl}(g) \) is trivial in all degrees while the cohomology \( \wedge g, d \) for the Lie algebra differential is not.

6.1. The Duflo map. The Birkhoff-Witt symmetrization map \( S(g) \to U(g) \) is the unique \( G \)-module isomorphism sending \( (\mu_a v_a)\) to \((\mu_a v_a)^k \), for all \( k \geq 0 \) and all \( \mu \in g \). Under the identification of \( U(g) \) with distributions on \( G \) supported at the identity, and \( S(g) \) with distributions on \( g \) supported at 0, this isomorphism is induced by push-forward under the exponential map: \( \exp_* : \mathcal{E}'(g) \to \mathcal{E}'(G) \).

Duflo \( \text{[7]} \) introduced a different isomorphism \( S(g) \cong U(g) \) given by composition of \( \exp_* \) with multiplication by the square root of the Jacobian of the exponential map,
$J^{\frac{1}{2}}: \mathfrak{g} \to \mathbb{R}$. (Recall that the Jacobian $J$ has a globally defined smooth square root $J^{\frac{1}{2}}$ with $J^{\frac{1}{2}}(0) = 1$.) The Duflo map

$$\text{Duf} := \exp_* \circ J^{\frac{1}{2}}$$

has the important property that it induces an algebra isomorphism $S(\mathfrak{g})^G \to U(\mathfrak{g})^G$. We will also refer to the map $\text{Duf} := \exp_* \circ J^{\frac{1}{2}}: \mathcal{E}'(\mathfrak{g}) \to \mathcal{E}'(G)$ as the Duflo map; it restricts to a ring homomorphism for the convolution algebras, $\mathcal{E}'(\mathfrak{g})^G \to \mathcal{E}'(G)^G$.

Combining the Duflo map with the inverse of the symbol map, and using the inner product to identify $\mathfrak{g} \cong \mathfrak{g}^*$, we obtain a map between Weil algebras

$$(\text{Duf} \times \sigma^{-1}) : \hat{W}_G \to \hat{W}_G.$$ 

In the following Section 6.2 we show that one can do much better: There exists a map $Q: \hat{W}_G \to \hat{W}_G$ which we call the quantization map such that $Q$ is a homomorphism of $G$-differential spaces.

### 6.2. Quantization of Weil algebra

The definition of the quantization map $Q: \hat{W}_G \to \hat{W}_G$ involves a certain skew-symmetric tensor field $T_{ab}$, given as follows. Let $\mathfrak{g}_\# \subset \mathfrak{g}$ be the open subset on which the exponential map is a local diffeomorphism, that is $\mathfrak{g}_\# = \mathfrak{g} \setminus J^{-1}(0)$. Let $e^L_a$ be the left-invariant vector field on $G$ which equals $e_a$ at the group unit, and $e^R_a$ the corresponding right-invariant vector field. It turns out (see Appendix A) that at any point $\mu \in \mathfrak{g}_\#$, the pull-back of the half-sum $e^L_a + e^R_a$ under the exponential map differs from the constant vector field $\frac{\partial}{\partial \mu_a}$ only by a vector tangent to the $G$-orbit through $\mu$. It follows that there is a unique tensor field $T: \mathfrak{g}_\# \to \mathfrak{g} \otimes \mathfrak{g}$ such that $T(\mu)$ takes values in $\mathfrak{g}^\perp_{\mu} \otimes \mathfrak{g}^\perp_{\mu}$ and

$$(16) \quad \exp^* \left( \frac{e^L_a + e^R_a}{2} \right) - \frac{\partial}{\partial \mu_a} = T_{ab}(e_b)_{\mathfrak{g}}.$$ 

We verify in Appendix A that $T$ is explicitly given as

$$T \in C^\infty(\mathfrak{g}_\# , \mathfrak{g} \otimes \mathfrak{g}), \quad T_{ab}(\mu) := f(\text{ad}_{\mu})_{ab},$$

where $f(s)$ is the function

$$f(s) = \frac{1}{s} - \frac{1}{2} \cotanh \left( \frac{s}{2} \right).$$

In particular $T_{ab} = -T_{ba}$. The product $J^{\frac{1}{2}} \exp \left( -\frac{1}{2} T_{ab} t^a t^b \right)$ is a smooth function on $\mathfrak{g}$ with values in operators on $\hat{W}_G$, since the zeroes of $J^{\frac{1}{2}}$ compensate the singularities of $T$. The main result of this paper is presented in the following Theorem.

**Theorem 6.1.** The quantization map

$$Q := (\text{Duf} \times \sigma^{-1}) \circ \exp \left( -\frac{1}{2} T_{ab} t^a t^b \right) : \hat{W}_G \to \hat{W}_G$$
is a homomorphism of $G$-differential spaces. That is, $\mathcal{Q}$ satisfies

$$ \mathcal{Q} \circ L_a = L_a \circ \mathcal{Q}, \quad \mathcal{Q} \circ t_a = t_a \circ \mathcal{Q}, \quad \mathcal{Q} \circ d = d \circ \mathcal{Q}. $$

**Proof.** The properties $\mathcal{Q} \circ L_a = L_a \circ \mathcal{Q}$ and $\mathcal{Q} \circ t_a = t_a \circ \mathcal{Q}$ is obvious. Let us show that $\mathcal{Q}$ is a chain map. For clarity we denote the differential on $\hat{\mathcal{W}}_G$ by $d^W$ and the differential on $\hat{\mathcal{W}}_G$ by $d^W$.

We want to compare the expression for $d^W$ obtained in Proposition 3.7 to

$$ \text{Ad} \left( J^2 \exp(-\frac{1}{2}T_{ab}t_a t_b) \right) d^W. $$

Using the invariance property $[L_j \otimes 1, T_{ab}] + f_{jar} T_{rb} + f_{jbs} T_{as} = 0$ we compute:

$$ \text{ad}\left(-\frac{1}{2} T_{ab} t_a t_b \right) d^W $$

$$ = - \frac{1}{2} \frac{\partial T_{rs}}{\partial \mu_a} t_r t_s t_a + \frac{1}{4} T_{rs} f_{abc} [t_r t_s, y_b y_c] t_a - T_{rs t_r t_s} [t_s, y_j] L_j \otimes 1 - \frac{1}{2} [T_{rs}, L_j \otimes 1] y_j t_r t_s $$

$$ = - \frac{1}{2} \frac{\partial T_{rs}}{\partial \mu_a} t_r t_s t_a - f_{ajk} T_{jr} y_k t_a - \frac{1}{2} T_{bc} f_{abc} t_a + f_{jrk} T_{ks} y_j t_r t_s - T_{jr t_r} (L_j \otimes 1) $$

$$ = - \frac{1}{2} T_{bc} f_{abc} t_a - \frac{1}{2} \frac{\partial T_{rs}}{\partial \mu_a} t_r t_s t_a - T_{jr t_r} (L_j \otimes 1), $$

$$ = - \frac{1}{2} \frac{\partial T_{rs}}{\partial \mu_a} t_r t_s t_a - T_{jr t_r} (L_j \otimes 1), $$

$$ = - \frac{1}{2} T_{jk} T_{kr} f_{juk t_a t_v} v. $$

It follows that $\text{Ad}(\exp(-\frac{1}{2}T_{ab} t_a t_b)) d^W$ is given by the formula

$$ d^W - T_{ab} (L_b \otimes 1) t_a - \frac{1}{2} \left( \frac{\partial T_{rs}}{\partial \mu_a} + T_{ar} f_{rbs} T_{sc} \right) t_a t_b t_c - \frac{1}{2} f_{abc} T_{bc} t_a. $$

In Appendix A we show that the tensor field $T_{ab}$ is a solution of the classical dynamical Yang-Baxter equation

$$ \text{Cycl}_{abc} \left( \frac{\partial T_{ab}}{\partial \mu_c} + T_{ar} f_{rbs} T_{sc} \right) = \frac{1}{4} f_{abc}, $$

where $\text{Cycl}_{abc}$ means the sum over cyclic permutations of the indices $a, b, c$. Therefore,

$$ \text{Ad} \left( e^{-\frac{1}{2} T_{ab} t_a t_b} \right) (d^W) = d^W - T_{ab} (L_b \otimes 1) t_a - \frac{1}{24} f_{abc} t_a t_b t_c - \frac{1}{2} f_{abc} T_{bc} t_a. $$

Conjugating further by $J^2$ adds $\frac{1}{2} \frac{\partial \ln J}{\partial \mu_a}$ to this expression. As shown in Appendix A, the derivatives of $J$ can be expressed in terms of the tensor field $T_{ab}$ as follows:

$$ f_{abc} T_{bc} - \frac{\partial \ln J}{\partial \mu_a} = 0. $$
Hence,
\[
\text{Ad} \left( J^\frac{1}{2} e^{-\frac{1}{2} T_{ab} a_t b_t } \right) d^W = d^W - T_{ab}(L_b \otimes 1) t_a - \frac{1}{24} f_{abc} a_t b_t c_t.
\]

The definition of \( T \) implies that
\[
\frac{u_a^L + u_a^R}{2} \circ \exp = \exp \circ (v_a - T_{ab}(L_b \otimes 1))
\]
Together with Proposition 3.7, this shows that
\[
\exp \circ \text{Ad} \left( J^\frac{1}{2} e^{-\frac{1}{2} T_{ab} a_t b_t } \right) d^W = d^W \circ \exp,
\]
which means that \( Q \) is a chain map. An alternative proof of this fact will be given in Section 7.

\[\text{Remark 6.2.}\]
Notice that the quantization map restricts to the Duflo map \( \text{Duf} : \mathcal{E}'(g) \to \mathcal{E}'(G) \) on the subalgebra \( \mathcal{E}'(g) \otimes 1 \subseteq \hat{W}_G \), and to the inverse of the symbol map \( \sigma \) on \( 1 \otimes \wedge g \). Furthermore, \( Q \) restricts to an isomorphism of \( G \)-differential spaces \( Q : W_G \to \hat{W}_G \).

As a direct consequence to Theorem 6.1, we have:

\[\text{Corollary 6.3.}\]
For any \( G \)-differential algebra \( B \), the quantization map \( Q \) induces a linear isomorphism \( H^G(B) \cong \hat{H}_G(B) \) and a linear map \( \hat{H}_G(B) \to \hat{H}_G(B) \).

6.3. Quantization map in Cartan model. Suppose that \( B \) is a \( G \)-differential algebra. The following proposition describes the chain map
\[(\hat{W}_G \otimes B)_{\text{basic}} \to (\hat{W}_G \otimes B)_{\text{basic}}, \]
induced by \( Q \) in terms of the Cartan model:

\[\text{Proposition 6.4.}\]
The chain map between the Cartan models
\[(\mathcal{E}'(g) \otimes B)^G \to (\mathcal{E}'(G) \otimes B)^G \]
induced by \( Q \) is given by
\[\text{Duf} \circ \exp \left( - \frac{1}{2} T_{ab} (1 \otimes t_a t_b) \right).\]

\[\text{Proof.}\]
We have to show that on the subspace of basic elements \( (\hat{W}_G \otimes B)_{\text{basic}} \),
\[(P_{\text{hor}} \otimes 1) \circ Q = \text{Duf} \circ \exp \left( - \frac{1}{2} T_{ab} (1 \otimes t_a t_b) \right) \circ (P_{\text{hor}} \otimes 1)\]
Since basic elements are in particular horizontal, the operator \( \exp \left( - \frac{1}{2} T_{ab} (t_a t_b \otimes 1) \right) \) appearing in the definition of \( Q \) can be replaced with \( \exp \left( - \frac{1}{2} T_{ab} (1 \otimes t_a t_b) \right) \), which then commutes with \( P_{\text{hor}} \otimes 1 \).
7. The transpose of the quantization map

Let \( \varpi \in \Omega^2(\mathfrak{g}) \) be the image of the closed form \( \exp^* \eta \in \Omega^3(\mathfrak{g}) \) under the de Rham homotopy operator \( \Omega^*(\mathfrak{g}) \to \Omega^{*-1}(\mathfrak{g}) \). Explicitly,

\[
\varpi_\mu = -\frac{1}{2} g(\text{ad}_\mu)_{ab} d\mu_a d\mu_b
\]

where \( g(s) \) is the function

\[
g(s) = \frac{\sinh(s) - s}{s^2}.
\]

**Theorem 7.1.** Under the pairings of \( \hat{\mathcal{W}}_G \) with \( \Omega(\mathfrak{g}) \) and of \( \hat{\mathcal{W}}_G \) with \( \Omega(G) \), the composition \( \tau^{-1} \circ Q \circ \tau_0 \) is dual to the map \( e^\varpi \circ \exp^* : \Omega(G) \to \Omega(\mathfrak{g}) \).

Since this map is a chain map for the differentials \( d \) on \( \Omega(\mathfrak{g}^*) \) and \( d + \eta \) on \( \Omega(G) \), this gives an alternative proof of the fact that \( Q \) is a chain map. To prove this result note that \( \tau^{-1} Q(\Lambda_0) \in \hat{\mathcal{W}}_G \otimes \Omega(\mathfrak{g}^*) \) is the integral kernel of the quantization map, while \( \tau^{-1} \exp^* \Lambda \) is the kernel of the map \( \exp^* : \hat{\mathcal{W}}_G \to \hat{\mathcal{W}}_G \). Hence, the theorem will follow once we show that \( Q(\Lambda_0) = e^\varpi \exp^* \Lambda \).

We begin by computing the quantization of the form \( \tau_0(\mu) \delta_\mu \). This will require the following Lemma.

**Lemma 7.2.** Let \( V \) be an oriented Euclidean vector space of even dimension \( \dim V = 2n \), and suppose \( S \in \text{so}(V) \) is invertible. Let \( e_1 \ldots e_{2n} \) be an oriented orthonormal basis of \( V \). Then

\[
\frac{1}{\text{Pf}(S)} \exp\left(\frac{1}{2}(S^{-1})_{ab} t_a t_b\right) \exp\left(\frac{1}{2} S_{ab} e_a \wedge e_b\right) = e_1 \wedge \ldots \wedge e_{2n}
\]

where \( \text{Pf}(S) \) is the Pfaffian of \( S \).

**Proof.** Block-diagonalizing \( S \) one reduces the Lemma to the case \( \dim V = 2 \) and \( S_{11} = S_{22} = 0 \), \( S_{12} = -S_{21} = s \). In this case the equation becomes

\[
\frac{1}{s} \exp\left(\frac{1}{s} t_1 t_2\right) \exp(se_1 \wedge e_2) = e_1 \wedge e_2
\]

which follows immediately by writing \( \exp(se_1 \wedge e_2) = 1 + se_1 \wedge e_2 \) and \( \exp\left(\frac{1}{s} t_1 t_2\right) = 1 + \frac{1}{s} t_1 t_2 \).

**Proposition 7.3.** For all \( \mu \in \mathfrak{g} \),

\[
Q \tau_0(\mu) \delta_\mu = \tau(\exp \mu) \delta_{\exp \mu}.
\]
Proof. Recall that the set of regular elements $G_{\text{reg}}$ is the set of all elements whose stabilizer is a maximal torus. Since both sides depend continuously on $\mu$ we may assume $\mu \in \exp^{-1}(G_{\text{reg}})$.

Choose any orientation of $\mathfrak{g}_\mu^\perp$. We will apply the Lemma with $V = \mathfrak{g}_\mu^\perp$. Let $\text{Pf}_{\mathfrak{g}_\mu^\perp}$ denote the Pfaffian of an operator on the orthogonal complement of $\mathfrak{g}_\mu = \ker(\text{ad}_\mu)$ (using the metric induced from $\mathfrak{g}$ and any choice of orientation). The operator $\text{ad}_\mu$ has kernel $\ker(\text{ad}_\mu) = \mathfrak{g}_\mu$ and is invertible on $\mathfrak{g}_\mu^\perp$. The square root of the Jacobian of the exponential map is a quotient of two Pfaffians (see e.g. [11], p. 105):

$$J^\frac{1}{2}(\mu) = \frac{\det \frac{1}{2} \text{Pf}_{\mathfrak{g}_\mu^\perp}(2 \sinh(\text{ad}_\mu/2))}{\det \frac{1}{2} \text{Pf}_{\mathfrak{g}_\mu^\perp}(\text{ad}_\mu)}.$$ 

Consider the skew-symmetric tensor fields $r_0 \in \mathcal{C}^\infty(\mathfrak{g}_{\text{reg}}, \mathfrak{g} \wedge \mathfrak{g})$ and $r \in \mathcal{C}^\infty(G_{\text{reg}}, \mathfrak{g} \wedge \mathfrak{g})$ on the set of regular elements $\mathfrak{g}_{\text{reg}}$ resp. $G_{\text{reg}}$ introduced in the Appendix.

We can re-write the quantization map as

$$Q = (\exp \times \sigma^{-1}) \det \frac{1}{2}(\text{cosh}(\text{ad}_\mu/2)) \mathcal{T}^{-1} \mathcal{T}_0$$

where

$$\mathcal{T}_0 = \frac{1}{\det \frac{1}{2} \text{Pf}_{\mathfrak{g}_\mu^\perp}(\text{ad}_\mu)} \exp(-\frac{1}{2}(r_0)_{ab} t_a t_b)$$

and

$$\mathcal{T} = \frac{1}{\det \frac{1}{2} \text{Pf}_{\mathfrak{g}_\mu^\perp}(\tanh(\text{ad}_\mu/2))} \exp(-\frac{1}{2}(r_{ab}) t_a t_b).$$

As a special case of [3], Proposition 3.13, the symbol of $\tau$ is given by

$$\sigma(\tau(g)) = \det \frac{1}{2}(\text{cosh}(\text{ad}_\mu/2)) \exp \left(- \frac{\text{Ad}_g - 1}{\text{Ad}_g + 1} (r_{ab}) y_a y_b \right)$$

where $g = \exp(\mu)$. Together with Lemma 7.2 this shows $\det \frac{1}{2}(\text{cosh}(\text{ad}_\mu/2)) \mathcal{T}_0 \tau_0 = \mathcal{T} \sigma(\tau)$. 

Proposition 7.4. The image of $\Lambda_0$ under the quantization map is given by

$$(\mathcal{Q} \otimes 1)\Lambda_0 = \exp(\varpi) (1 \otimes \exp^*) \Lambda.$$ 

Proof. By continuity it suffices to verify the equation on the open dense subset $\mathfrak{g}_{\text{reg}}$ of regular elements of $\mathfrak{g}$. We define 1-forms $\kappa_a \in \Omega^1(\mathfrak{g}_{\text{reg}})$ by

$$\kappa_a := (r_0)_{ab} \mu_b.$$ 

At any point $\mu \in \mathfrak{g}_{\text{reg}}$ we have a decomposition of the tangent space into the spherical part, i.e. the tangent space to the orbit $(T_{\mu}\mathfrak{g})^{sp} = \text{im}(\text{ad}_\mu)$, and its orthogonal complement, the radial part $(T_{\mu}\mathfrak{g})^{rad} = \ker(\text{ad}_\mu)$ (spanned by the Cartan subalgebra containing
µ). Correspondingly every vector field on \( g_{\text{reg}} \), and dually every 1-form, decomposes into radial and spherical part. The spherical part is spanned by the forms \( \kappa_a \). We have

\[(d\mu_a)^{sp} = (\text{ad}_\mu)_{ab}\kappa_b, \quad (\exp^*\theta_a)^{sp} = (1 - e^{-\text{ad}_{\mu}})_{ab}\kappa_b\]

while

\[(d\mu_a)^{rad} = (\exp^*\theta_a)^{rad}.\]

Then the definition of \( \varpi \) can be re-written \( \varpi = \varpi_2 - \varpi_1 \) where

\[\varpi_2 = -\frac{1}{2}(\sinh(\text{ad}_\mu))_{ab}\kappa_a\kappa_b,\]
\[\varpi_1 = -\frac{1}{2}(\text{ad}_\mu)_{ab}\kappa_a\kappa_b.\]

The form \( \varpi_1 \in \Omega^2(\mathfrak{g}_{\text{reg}}) \) combines nicely with \( \Lambda_0 \):

\[\exp(\varpi_1)\Lambda_0 = \exp(\varpi_1 - y_a d\mu_a - \frac{1}{2}f_{abc}\mu_ay_by_c)\delta_\mu\]
\[= \exp(-y_a(d\mu_a)^{rad})\exp(-\frac{1}{2}f_{abc}\mu_a(y_b + \kappa_b)(y_c + \kappa_c))\delta_\mu\]
\[(19)\]
\[= \exp(-y_a(d\mu_a)^{rad})\exp(\kappa_a\ell_a)\tau_0(\mu)\delta_\mu.\]

We will see that there exists a similar expression for \( \exp(\varpi_2)\exp^*\Lambda \). We will need the following

**Lemma 7.5.** Let \( V \) be an oriented Euclidean vector space, and \( S \in \text{so}(V) \). Let \( x_a \in \text{Cl}(V) \) be the generators corresponding to some choice of oriented, orthonormal basis of \( V \), and let \( \kappa_a \) be odd elements in some commutative super-algebra \( A \). Then the following identity in \( \text{Cl}(V) \otimes A \) holds:

\[(20)\]
\[\exp(-\ell_a\kappa_a)\exp\left(\frac{1}{2}S_{ab}x_ax_b\right) = \exp(\varpi_2)\exp(-x_r\gamma_r)\exp\left(\frac{1}{2}S_{ab}x_ax_b\right)\]

where \( \gamma_r = (1 - e^S)_{rs}\kappa_s \) and \( \varpi_2 = \frac{1}{2}(\sinh(S))_{ab}\kappa_a\kappa_b. \)

**Proof.** Block-diagonalizing \( S \) we can assume \( \dim V = 2 \) and that \( S_{12} = -S_{21} = s \) and \( S_{11} = S_{22} = 0 \). Then

\[\exp\left(\frac{1}{2}S_{ab}x_ax_b\right) = \exp(sx_1x_2) = \cos\left(\frac{s}{2}\right) + 2\sin\left(\frac{s}{2}\right)x_1x_2.\]

Furthermore, \( \varpi_2 = \sin(s)\kappa_1\kappa_2 \) and

\[\gamma_1 = 2\sin(s/2)(\sin(s/2)\kappa_1 - \cos(s/2)\kappa_2),\]
\[\gamma_2 = 2\sin(s/2)(\sin(s/2)\kappa_2 + \cos(s/2)\kappa_1).\]

Equation (20) becomes

\[(1 + \kappa_1\ell_1 + \kappa_2\ell_2 - \kappa_1\kappa_2\ell_1\ell_2)(\cos(s/2) + 2\sin(s/2)x_1x_2)\]
\[= (1 + \sin(s)\kappa_1\kappa_2)(1 - x_1\gamma_1 - x_2\gamma_2 - x_1x_2\gamma_1\gamma_2)(\cos(s/2) + 2\sin(s/2)x_1x_2)\]

which is verified by an elementary calculation.
Using the Lemma, we can write
\[ \exp(\omega_2) \exp^* \Lambda = \exp(-x_a \exp^* \tau^a) \exp(\kappa_a \iota_a) \tau(\exp \mu) \delta_{exp \mu}. \]

Since the quantization map is equivariant, and since it intertwines the contractions on \( \hat{W}_G \) and \( \hat{W}_G \), Equations (19), (21) show that
\[ Q(e^{\omega_1} \Lambda_0) = e^{\omega_2} \exp^* \Lambda. \]

8. Ring structure

In the Section 6 we established that the quantization map induces maps in cohomology \( Q : \hat{H}_G(B) \to \hat{H}_G(B) \) and \( Q : H_G(B) \to H_G(B) \). Are these maps algebra homomorphisms?

To answer this question let us introduce two homomorphisms of \( G \)-differential algebras.
\[ \phi_1, \phi_2 : \hat{W}_G \otimes \hat{W}_G \to \hat{W}_G \]
by
\[ \phi_1 = \text{Mult}_W \circ (Q \otimes Q), \]
\[ \phi_2 = Q \circ \text{Mult}_W. \]

**Theorem 8.1.** The maps \( \phi_1, \phi_2 : \hat{W}_G \otimes \hat{W}_G \to \hat{W}_G \) are \( G \)-chain homotopic (cf. Definition 2.8). They restrict to \( G \)-chain homotopic maps \( W_G \otimes W_G \to W_G \). It follows that for any \( G \)-differential algebra \( B \), the quantization map induces a ring isomorphism \( H_G(B) \to H_G(B) \) and a ring homomorphism \( \hat{H}_G(B) \to \hat{H}_G(B) \).

Taking \( B \) to be the trivial \( G \)-differential algebra \( B = \mathbb{R} \) we recover the fact that the Duflo map induces a ring homomorphism \( \mathcal{E}'(g)^G \to \mathcal{E}'(G)^G \) and a ring isomorphism \( (Sg)^G \cong U(g)^G \).

**Proof.** For \( j = 1, 2 \) let \( \mathcal{K}_j \in \hat{W}_G \otimes \Omega(g \times g) \) be defined as
\[ \mathcal{K}_j = \phi_j(\Lambda_0 \otimes \Lambda_0). \]
then \( \tau^{-1} \mathcal{K}_j \) are integral kernels for the maps \( \phi_j \). From the fact that these are chain maps intertwining contractions, one obtains
\[ d \mathcal{K}_j = 0, \quad \iota_a \mathcal{K}_j = -\text{Add}_g^*(d \mu_a) \mathcal{K}_j. \]
Since \( \text{Mult}_W(\Lambda_0 \otimes \Lambda_0) \) is just \( (1 \otimes \text{Add}_g^*)\Lambda_0 \) (pull-back under the addition map \( \text{Add}_g : g \times g \to g \)),
\[ \mathcal{K}_2 = \text{Add}_g^* Q(\Lambda_0) = \text{Add}_g^* (e^{\omega} \exp^* \Lambda). \]
Notice that \( e^{\omega} \exp^* \Lambda \) is an invertible element of the algebra \( \hat{W}_G \otimes \Omega(g) \), and the same is true for its pull-back under \( \text{Add}_g \). Hence \( \mathcal{K}_2^{-1} \) is defined, and the product
\[ N = \mathcal{K}_2^{-1} \mathcal{K}_1 \in \hat{W}_G \otimes \Omega(g \times g) \]
is $G$-basic. Since $\phi_j(1 \otimes 1) = 1$ for both $j = 1, 2$, the pull-back of both $K_j$, hence also of $N$ to the origin in $g \times g$ is the identity element $1 \in \widehat{W}_G$. Since $g \times g$ retracts equivariantly to the origin, it follows that there exists an odd element $\Gamma \in (\widehat{W}_G \otimes \Omega(g \times g))_{\text{basic}}$ with $N = 1 + d\Gamma$. Multiplying this identity by $K_2$ we find

\[ K_2 = K_1 + d(K_2 \Gamma). \]

The element $K_2 \Gamma$ has the property

\[ \iota_a(K_2 \Gamma) = -d\mu_a(K_2 \Gamma). \]

Consequently, $\tau^{-1}(K_2 \Gamma)$ is the kernel for an odd linear map

\[ h : \widehat{W}_G \otimes \widehat{W}_G \to \widehat{W}_G \]

that provides a $G$-chain homotopy between $\phi_1$ and $\phi_2$. We notice that $h$ restricts to a $G$-chain homotopy $W_G \otimes W_G \to W_G$ between the restrictions of $\phi_1$ and $\phi_2$. \hfill \Box

**Remark 8.2.** The first part of Theorem 8.1 shows more generally that if $B$ is a $G \times G$-differential space, the natural diagram

\[ \begin{CD} \widehat{H}_{G \times G}(B) @>>> \widehat{H}_G(B) \\
@VVV @VVV \\
\widehat{H}_{G \times G}(B) @>>> \widehat{H}_G(B) \end{CD} \]

where the vertical arrows are quantization maps and the horizontal arrows are induced by multiplication in the Weil algebras, commutes.

The simplest non-trivial example illustrating the ring isomorphism $H_G(B) \cong \mathcal{H}_G(B)$ is the rotation action of $G = SU(2)$ on the 2-sphere $S^2 \subset \mathbb{R}^3$. As is well-known, the equivariant cohomology $H_G(\Omega(S^2))$ as an algebra over $\mathbb{R}$ is a polynomial ring in one generator $[\omega]$ of degree 2. We can normalize $[\omega]$ by the condition $[\omega]^2 = [\lambda]$ where $\lambda = v_a v_a \in S(g)$ is the generator of the ring of invariant polynomials. Let us verify explicitly that $Q([\omega]^2) = Q([\omega])^2$.

To describe a representative for $[\omega]$ let us choose the basis of $g$ according to the standard identification $su(2) \cong \mathbb{R}^3$, so that the structure constants are given by the totally anti-symmetric tensor $f_{abc} = \varepsilon_{abc}$. Also, let us view $S^2$ as the unit sphere in $\mathbb{R}^3$, with coordinates $n_a$. In the Cartan model, a representative for $[\omega]$ is given by

\[ \omega = \frac{1}{2} f_{abc} n^a d n^b d n^c + n^a v^a. \]

The representatives satisfy

\[ \omega^2 = \lambda + d_G(f_{abc} v^a n^b d n^c). \]
The image of \( \lambda \) under the quantization map, which on \((S\mathfrak{g})^G\) restricts to the Duflo map \( \text{Duf} \), is the quadratic Casimir,

\[
Q(\lambda) = u^a u^a + \frac{1}{4} \in U(\mathfrak{g})^G.
\]

Applying the quantization map to \( \omega \), one obtains

\[
Q(\omega) = \frac{1}{2} f_{abc} n^a n^b n^c + n^a u^a.
\]

Using

\[
\frac{1}{8} \sum_{\alpha, \beta} (\alpha \beta Q(\omega))^2 = \frac{1}{4} n^a n^a = \frac{1}{4},
\]

we find that the square of \( Q(\omega) \) with respect to the ring structure \( \odot \) is

\[
Q(\omega) \odot Q(\omega) = Q(\omega)^2 + \frac{1}{2} \sum_{\alpha} (\alpha Q(\omega))^2 + \frac{1}{8} \sum_{\alpha, \beta} (\alpha \beta Q(\omega))^2
\]

\[
= u^a u^a + d_G(f_{abc} u^a n^b n^c) + \frac{1}{4}
\]

\[
= Q(\lambda) + d_G(f_{abc} u^a n^b n^c) = Q(\omega^2).
\]

**Appendix A. Properties of the tensor \( T_{ab} \)**

In this Section we prove the properties of the tensor field \( T \) which were used in this paper. Let us first verify that the definition \( T(\mu) = f(\text{ad}_\mu) \), where

\[
f(s) = \frac{1}{s} - \frac{1}{2} \coth \left( \frac{s}{2} \right),
\]

coincides with the definition given in the text.

**Lemma A.1.** The half-sum of the left- and the right-invariant vector fields satisfies

\[
\exp^* \frac{e^L_a + e^R_a}{2} = \frac{\partial}{\partial \mu_a} + T_{ab}(e_b)_g.
\]

**Proof.** Consider the function \( g(s) = \frac{1-e^{-s}}{s} \). The right-invariant Maurer-Cartan form \( \theta \) has the property \( \exp^* \theta_a = (g(\text{ad}_\mu))_{ab} d\mu_b \). Hence the right-invariant vector field satisfies

\[
\exp^* e^R_a = (g(\text{ad}_\mu))_{ba} \frac{\partial}{\partial \mu_b} = (\tilde{g}^{-1}(\text{ad}_\mu))_{ab} \frac{\partial}{\partial \mu_b}.
\]

where \( \tilde{g}(s) = g(-s) \). Similarly

\[
\exp^* e^L_a = (g^{-1}(\text{ad}_\mu))_{ab} \frac{\partial}{\partial \mu_b}.
\]

From this the Lemma follows since \( 1 + sf = (g^{-1} + \tilde{g}^{-1})/2 \). \( \square \)
Lemma A.2. The derivative of the Jacobian of the exponential map is given by the formula
\[ \frac{\partial \ln J}{\partial \mu_a} = f_{abc} T_{bc}. \]

Proof. It is well-known (see e.g. [11], p.105) that \( J(\mu) = \det(g(\text{ad}_\mu)) \) with \( g \) as in the proof of the previous Lemma. Therefore
\[
\frac{\partial \ln J}{\partial \mu_a} = \frac{\partial}{\partial \mu_a} \ln(\det(g(\text{ad}_\mu))) \\
= \frac{\partial}{\partial \mu_a} \text{tr}(\ln(g(\text{ad}_\mu))) \\
= \text{tr} \left( (\ln g)'(\text{ad}_\mu) \frac{\partial}{\partial \mu_a} (\text{ad}_\mu) \right) \\
= f_{abc} (\ln g)'(\text{ad}_\mu)_{bc}
\]
Since the anti-symmetric part of \( (\ln g)'(s) = (e^s - 1)^{-1} - s^{-1} \) is equal to \( f \) the final result is \( f_{abc} T_{bc} \).

The most interesting and most complicated property of \( T \) is the following result:

Lemma A.3 (Etingof-Varchenko [8]). The tensor \( T \) is a solution of the classical dynamical Yang-Baxter equation with coupling constant \( 1/4 \):
\[
\text{Cycl}_{abc} \left( \frac{\partial T_{bc}}{\partial \mu_a} + T_{ar} f_{rbs} T_{sc} \right) = \frac{1}{4} f_{abc},
\]
where \( \text{Cycl}_{abc} \) denotes the sum over cyclic permutations of \( a, b, c \).

This solution of the classical dynamical Yang-Baxter equation was obtained by Etingof-Varchenko as part of their general classification scheme, see [8], Theorems 3.1 and 3.14. The remainder of this section is devoted to a direct proof of (23), based on the orthogonal decompositions of vector \( X \) fields on \( g^* \) reg resp. \( G_{\text{reg}} \) into spherical and radial parts, \( X = X^{\text{sp}} + X^{\text{rad}} \). Here the spherical part \( X^{\text{sp}} \) is by definition tangent to orbits and the radial part \( X^{\text{rad}} \) orthogonal to orbits. Both radial and spherical vector fields are Lie-subalgebras of the Lie algebra of vector fields. It is convenient to introduce certain canonical skew-symmetric tensor fields \( v_0 \in C^\infty(g_{\text{reg}}^*, g \wedge g) \) and \( \mathbf{v} \in C^\infty(G_{\text{reg}}, g \wedge g) \) on the set of regular elements \( g_{\text{reg}}^* \) resp. \( G_{\text{reg}} \). Recall that \( g_{\text{reg}}^* \) is the set of all \( \mu \in g_{\text{reg}}^* \) such that the stabilizer \( G \mu \) under the coadjoint action has minimal dimension (so that it is a maximal torus of \( G \)). For all \( \mu \in g^* \) the operator \( \text{ad}_\mu \) (where we use the invariant inner product to identify \( g^* \cong g \)) is invertible on the subspace \( g_{\mu}^\perp \). Let \( \text{ad}_\mu^{-1} : g \to g \) be its extension to an operator on \( g \), defined to be 0 on \( g_{\mu} \). We set
\[
(24) \quad v_0(\mu)_{ab} = \text{ad}_\mu^{-1}(e_a) \cdot e_b.
\]
We remark that the definition of \( v_0 \) is in fact independent of the inner product on \( g \). Viewing \( v_0 \) as a 2-form on \( g_{\text{reg}}^* \), it restricts to the Kirillov-Kostant-Souriau form...
Lemma A.4. The tensor field $r$ (26) since $g$ commute we have ($g$ operator on all of $\mathfrak{g}$). The subset $\mathfrak{g}^1 = T_g(G \cdot g) \cong T_\mu(G \cdot \mu) \cong \mathfrak{g}_\mu^1 \subset \mathfrak{g}$. The skew-symmetric operator $\tanh(\frac{ad_\mu}{2})$ is invertible on $\mathfrak{g}_\mu^1 = \mathfrak{g}^1_g$. Let $\left(\tanh(\frac{ad_\mu}{2})\right)^{-1}$ be the extension of this inverse by 0 to an operator on all of $\mathfrak{g}$, and put
\[
(25) \quad r(g)_{ab} = \frac{1}{2}\left(\tanh(\frac{ad_\mu}{2})\right)^{-1} e_a \cdot e_b.
\]
Then
\[
(26) \quad T_{ab} = (r_0)_{ab} - \exp^* r_{ab}
\]
on coadjoint orbits. The subset $G_{\text{reg}}$ consist of all $g \in G$ such that the stabilizer $G_g$ is a maximal torus. Writing $g = \exp \mu$ the exponential map gives an isomorphism $\mathfrak{g}_\mu^1 = T_g(G \cdot g) \cong T_\mu(G \cdot \mu) \cong \mathfrak{g}_\mu^1 \subset \mathfrak{g}$. The skew-symmetric operator $\tanh(\frac{ad_\mu}{2})$ is invertible on $\mathfrak{g}_\mu^1 = \mathfrak{g}^1_g$. Let $\left(\tanh(\frac{ad_\mu}{2})\right)^{-1}$ be the extension of this inverse by 0 to an operator on all of $\mathfrak{g}$, and put
\[
(25) \quad r(g)_{ab} = \frac{1}{2}\left(\tanh(\frac{ad_\mu}{2})\right)^{-1} e_a \cdot e_b.
\]
Then
\[
(26) \quad T_{ab} = (r_0)_{ab} - \exp^* r_{ab}
\]
over $\exp^{-1}(G_{\text{reg}}) \subset \mathfrak{g}_{\text{reg}}$. Next, we examine the properties of the tensor fields $r_0$ and $r$.

Lemma A.4. The tensor field $r_0 \in C^\infty(\mathfrak{g}_{\text{reg}}, \Lambda^2 \mathfrak{g})$ satisfies the equation
\[
\text{Cycl}_{abc} \left( (\frac{\partial}{\partial \mu_a})^{rad}(r_0)_{bc} + (r_0)_{ak} f_{kld}(r_0)_{lc} \right) = 0.
\]

Proof. We begin by observing that at any point $\mu \in \mathfrak{g}_{\text{reg}}$, the tensor $\text{Cycl}_{abc}(\ldots)$ on the left hand side of this equation takes values in $\Lambda^3 \mathfrak{g}_\mu^1$. This follows because $r_0(\mu)$ takes values in $\Lambda^2 \mathfrak{g}_\mu^1$, and the same is true for radial derivatives $(\frac{\partial}{\partial \mu_a})^{rad}(r_0)(\mu)$.

We calculate the spherical part of $[\frac{\partial}{\partial \mu_a}, \frac{\partial}{\partial \mu_b}]$ in two ways. First, since partial derivatives commute we have $(\frac{\partial}{\partial \mu_a}, \frac{\partial}{\partial \mu_b})^{sp} = 0$. On the other hand, decomposing into radial and spherical parts we have
\[
[(\frac{\partial}{\partial \mu_a}, \frac{\partial}{\partial \mu_b})^{sp} = [(\frac{\partial}{\partial \mu_a})^{sp}, (\frac{\partial}{\partial \mu_b})^{sp}] + [(\frac{\partial}{\partial \mu_a})^{rad}, (\frac{\partial}{\partial \mu_b})^{sp}] + [(\frac{\partial}{\partial \mu_a})^{sp}, (\frac{\partial}{\partial \mu_b})^{rad}].
\]
By definition of $r_0$, the spherical part of $\frac{\partial}{\partial \mu_a}$ is $(r_0)_{ab}(e_b)g$. Since $r_0$ is equivariant,
\[
(e_a)g(r_0)_{bc} + f_{abk}(r_0)_{kc} + f_{abc}(r_0)_{bl} = 0.
\]
Hence the first term becomes
\[
[(\frac{\partial}{\partial \mu_a})^{sp}, (\frac{\partial}{\partial \mu_b})^{sp}] = [(r_0)_{ar}(e_r)g], (r_0)_{bs}(e_s)g]
\]
\[
= \text{Cycl}_{abc} \left( (r_0)_{ar} f_{rbs}(r_0)_{sc} \right) (e_c)g.
\]
The second term and third term add up to
\[
[(\frac{\partial}{\partial \mu_a})^{rad}, (\frac{\partial}{\partial \mu_b})^{sp}] + [(\frac{\partial}{\partial \mu_a})^{sp}, (\frac{\partial}{\partial \mu_b})^{rad}] = \left( (\frac{\partial}{\partial \mu_a})^{rad}(r_0)_{bc} + (\frac{\partial}{\partial \mu_b})^{rad}(r_0)_{ca} \right) (e_c)g.
\]
Since $(\frac{\partial}{\partial \mu_c})^{rad}(r_0)_{ab}(e_c)g = 0$ by orthogonality of radial and spherical vector fields, this can also be written
\[
\text{Cycl}_{abc} \left( (\frac{\partial}{\partial \mu_a})^{rad}(r_0)_{bc} \right) (e_c)g.
\]
We have thus shown
\[ \text{Cycl}_{abc} \left( \left( \frac{\partial}{\partial \mu^a} \right)^{rad} (\mathfrak{t}_0)_{bc} + (\mathfrak{t}_0)_{ak} f_{kbl} (\mathfrak{t}_0)_{lc} \right) (e_c)_\mathfrak{g} = 0. \]
which proves the Lemma since the vector fields $(e_c)_\mathfrak{g}$ span $q^\perp_\mu$ for all $\mu \in \mathfrak{g}_{\text{reg}}$. \hfill \Box

**Lemma A.5.** The tensor field $\mathfrak{t} \in C^\infty(G_{\text{reg}}, \wedge^2 \mathfrak{g})$ satisfies the equation
\[ \text{Cycl}_{abc} \left( \left( \frac{e^L_a + e^R_a}{2} \right)^{rad} \mathfrak{t}_{bc} + \mathfrak{t}_{ak} f_{kbl} \mathfrak{t}_{lc} \right) = \frac{1}{4} f_{abc}. \]

**Proof.** Since $T_{ab} = (\mathfrak{t}_0)_{ab} - \exp^* \mathfrak{t}_{ab}$, Equation (22) shows that the spherical part of $\frac{e^L_a + e^R_a}{2}$ is given by
\[ \left( \frac{e^L_a + e^R_a}{2} \right)^{sp} = \mathfrak{t}_{ab} (e_b)^G. \]
Using this Equation the proof of Lemma A.5 becomes parallel to that of Lemma A.4—the only difference being that
\[ \left[ \frac{e^L_a + e^R_a}{2}, \frac{e^L_b + e^R_b}{2} \right] = \frac{1}{4} f_{abc} (e_c)^G, \]
which accounts for the term $\frac{1}{4} f_{abc}$ on the right hand side. \hfill \Box

**Proof of Equation (23).** As a consequence of Equation (22), the pull-back to $\Phi^{-1}(G_{\text{reg}})$ of the radial part of $\frac{e^L_a + e^R_a}{2}$ is equal to the radial part of $\frac{\partial}{\partial \mu^a}$. Hence, combining Lemmas A.4 and A.5 we find that
\[
(27) \quad \text{Cycl}_{abc} \left( \left( \frac{\partial}{\partial \mu^a} \right)^{rad} T_{bc} + (\mathfrak{t}_0)_{ak} f_{kbl} (\mathfrak{t}_0)_{lc} - \exp^* \mathfrak{t}_{ak} f_{kbl} \exp^* \mathfrak{t}_{lc} \right) = \frac{1}{4} f_{abc}.
\]
The equivariance property of $T$ shows that
\[
\left( \frac{\partial}{\partial \mu^a} \right)^{sp} T_{bc} = (\mathfrak{t}_0)_{ar} (e_r)_\mathfrak{g} T_{bc} = - (\mathfrak{t}_0)_{ar} (f_{rbs} T_{sc} + f_{rcs} T_{bs}) = -2 (\mathfrak{t}_0)_{ar} f_{rbs} (\mathfrak{t}_0)_{sc} + (\mathfrak{t}_0)_{ar} f_{rbc} \exp^* \mathfrak{t}_{sc} + \exp^* \mathfrak{t}_{br} f_{rcs} (\mathfrak{t}_0)_{sa}.
\]
Taking the sum over cyclic permutations of $a, b, c$, adding to (27), and using $T_{ab} = (\mathfrak{t}_0)_{ab} - \exp^* \mathfrak{t}_{ab}$ we obtain the identity (23). \hfill \Box

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