Pentad and triangular structures behind the Racah matrices

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ABSTRACT

Somewhat unexpectedly, the study of the family of twisted knots revealed a hidden structure behind exclusive Racah matrices $S$, which control non-associativity of the representation product in a peculiar channel $R \otimes R \otimes R \rightarrow R$. These $S$ are simultaneously symmetric and orthogonal, and therefore admit two decompositions: as quadratic forms, through the skew Schur and Macdonald functions — what makes Racah matrices calculable. Moreover, the second exclusive Racah matrix for $R$, which control non-associativity of the representation product in a peculiar channel, can lead to further insights about representation theory, knot invariants and Macdonald-Kerov functions.

Triangular structures are long believed to be intimately related to integrability. For example, triangular are the pseudo-differential "dressing operators" in the description of KdV/KP hierarchies [1]. In a complementary development, triangularity in the form of Gauss decomposition proved to be crucial for the free-field description of generic conformal theories [2]. Since then triangular transforms appeared in many places, in particular, in the description of Macdonald-Kerov functions, of their "generalized" multi-time deformations and, somewhat surprisingly, in the study of "evolution" along the family of twist knots — which had direct implication to the theory of Racah matrices (6j-symbols) [3], and allowed to explicitly calculate some of them, what remained an unsolvable problem for quite a long period of time. A purpose of this short note is a brief review of this newly emerging and promisingly diverse field.

1. Another example of the fundamental importance is description of Macdonald [4] and, more generally, Kerov functions [5] as triangular transformation of Schur functions, see [7] for a recent reminder. In this case one just applies a triangular orthogonalization procedure for the scalar product

$$\langle p^\Delta | p^{\Delta'} \rangle^{(g)} = z_{\Delta} \cdot \delta_{\Delta, \Delta'} \cdot \prod_{i=1}^{l_{\Delta}} g_{i}$$

(1)

on the space of time-variables $p_k$. Here $p^\Delta$ is the monomial basis, $p^\Delta = \prod_{i=1}^{l_{\Delta}} p_{i}$, labeled by Young diagrams $\Delta = [\delta_1 \geq \delta_2 \geq \ldots \geq \delta_{l_{\Delta}} > 0] = [\ldots, 2^{m_2}, 1^{m_1}]$, with $z_{\Delta} = \prod_k k^{m_k} \cdot m_k!$. It is well known, that Schur functions $\chi_R \{ p \}$ form another orthonormal basis when all $g_i = 1$,

$$\langle \chi_R | \chi_{R'} \rangle^{(f)} = \delta_{R,R'}$$

(2)

and Kerov functions are defined as their triangular transformation, which diagonalizes the product for arbitrary $g$:

$$\text{Ker}_{R}^{(g)} \{ p \} = \chi_R \{ p \} + \sum_{R' < R} K_{R|R'}^{(g)} \cdot \chi_{R'} \{ p \}$$

$$\langle \text{Ker}_{R}^{(g)} | \text{Ker}_{R'}^{(g)} \rangle^{(g)} = ||\text{Ker}_{R}^{(g)}||^2 \cdot \delta_{R,R'}$$

(3)

It is triangular w.r.t. lexicographical ordering of the Young diagrams of the same size, for $R = [r_1 \geq r_2 \geq \ldots \geq r_{l_{R}} > 0]$

$$R > R' \text{ if } r_1 > r'_1 \text{ or if } r_1 = r'_1, \text{ but } r_2 > r'_2, \text{ or if } r_1 = r'_1 \text{ and } r_2 = r'_2, \text{ but } r_3 > r'_3, \text{ and so on}$$

(4)
Alternative ordering, when lexicographically ordered are transposed Young diagrams, gives risen to a non-equivalent set of dual Kerov functions, and both sets nicely complement each other, for example in Cauchy formula and all its far-going implications. Actually parameters $g_k$ appear in many formulas through Schur functions, depending on them as on the new, additional set of time-variables, what can imply an essentially new twist in the theory of integrable systems. The best-known particular example of Kerov functions is provided by the set of Macdonald polynomials, associated with the strangely-looking choice

$$g_n^{Mac} = \frac{\{q^n\}}{\{p^n\}}$$  \hspace{1cm} (5)

Empirically this choice is distinguished by a tremendous simplification of many formulas, especially those for the Littlewood-Richardson coefficients, what allows to preserve close relation to conventional representation theory. However, an a priori reason for this simplification, at the level of triangular structures, remains an open puzzle.

The scalar-product approach is technically very powerful, however it has a conceptual drawback: there is no explanation why original basis, which needs to be triangularly transformed, consists of Schur functions rather than any other set, obtained by some orthogonal transformation, which always preserves both the scalar product and Cauchy formula.

2. The next example is provided by generalized Macdonald functions (GMF) \cite{8}, which play the crucial role in description of AGT relations \cite{9}. In the simplest two-times case they are triangular transforms from the factorized basis $M_{R_1}\{p\} \cdot M_{R_2}\{\bar{p}\}$:

$$M_{R_1,R_2}\{p,\bar{p}\} = M_{R_1}\{p\} \cdot M_{R_2}\{\bar{p}\} + \sum_{(R_1',R_2')<(R_1,R_2)} C_{R_1,R_2,R_1',R_2'}(A,q,t) \cdot M_{R_1'}\{p\} \cdot M_{R_2'}\{\bar{p}\} =$$

$$= \chi_{R_1}\{p\} \cdot \chi_{R_2}\{\bar{p}\} + \sum_{(R_1',R_2')<(R_1,R_2)} C_{R_1,R_2,R_1',R_2'}(A,q,t) \chi_{R_1'}\{p\} \cdot \chi_{R_2'}\{\bar{p}\}$$  \hspace{1cm} (6)

with triangular $C$ and $C$ depending on a deformation parameter $Q = A^2$. Ordering for the pairs of equal size, $|R_1| + |R_2| = |R_1'| + |R_2'|$, is defined by the rule

$$(R_1,R_2) > (R_1',R_2') \text{ if } |R_2| > |R_2'| \text{ or } |R_2| = |R_2'| \text{ and } R_2 > R_2' \text{ or } R_2 = R_2' \text{ and } R_1 > R_1'$$  \hspace{1cm} (8)

GMF have no nice definition in terms of the scalar products (the standard approach exploits orthogonality to another basis – of the “dual” GMF), see \cite{10} for discussion and references. Instead they are specified in a different way – as eigenfunctions of the triangularly-deformed Hamiltonians \cite{11}, the simplest one being

$$\hat{H}(F)\{p,\bar{p}\} = \frac{1}{t^2 - 1} \left\{ -(1 + Q^{-1}) \cdot F\{p,\bar{p}\} + \text{res}_{z=0} \left( \exp \left( \sum_n \frac{(1 - t^{-2n})p_nz^n}{n} \right) F\left\{ p_k + \frac{q^{2k} - 1}{z^k}, \bar{p}_k \right\} + \right. \\ + Q^{-1} \cdot \exp \left( \sum_n \frac{(1 - t^{-2n})z^n}{n} \left( 1 - (t/q)^{2n} \right) p_n + \bar{p}_n \right) F\left\{ p_k, \bar{p}_k + \frac{q^{2k} - 1}{z^k} \right\} \right\} \right\}$$  \hspace{1cm} (9)

It is triangular in the sense that the mixing term (underlined) between the ordinary Ruijsenaars Hamiltonians lowers the level $|R_2|$ of $M_{R_2}\{\bar{p}\}$ and trades this for increase of the level $|R_1|$ of $M_{R_1}\{p\}$.

These are well known, still somewhat mysterious Hamiltonians. Hamiltonian approach has an advantage that it selects the basis, in particular, distinguishes Schur functions $\chi\{p\}$ from monomials $p^\lambda$, but instead it is considerably less efficient technically. And even conceptually it is not always revealing triangular structures in explicit way.

3. In particular, we can refer to sec.1 and wonder why the Ruijsenaars Hamiltonian,

$$\hat{H} = \oint dz \exp \left( \sum_n \frac{t^{n}}{n} p_n z^n \right) \exp \left( \sum_n \frac{(q/t)^{n}}{z^n} \frac{q^n}{\partial p_n} \right)$$  \hspace{1cm} (10)

defines the ordinary Macdonald polynomials, which are its eigenfunctions, as triangular transformations of Schurs. Here and below we use the standard notation $\{x\} = x - x^{-1}$. We give just an example of how it works, with the help of Cauchy formula,

$$\exp \left( \sum_n \frac{p_n\bar{p}_n}{n} \right) = \sum_Y \chi_Y\{p\} \chi_Y\{p'\}$$  \hspace{1cm} (11)
recently reviewed for these purposes in \[12\]. In our case the role of \(p'\) will be played by a peculiar set \(p'_k = \{t^k\}\), and we use the fact that Schur functions at this locus are non-vanishing only for the single-hook diagrams \(Y = [\alpha + 1,1^b]\):

\[
\chi_{\alpha+1,1^b} \{ p_k = \{t^k\} \} = (-)^b \cdot t^{a-b} \cdot \{t\}
\]

Therefore

\[
\exp \left( \sum_n \frac{t^n}{n} p_n z^n \right) = \sum_{a,b} (-)^b \cdot t^{a+b+1} \cdot t^{a-b} \cdot \{t\} \cdot \chi_{\alpha+1,1^b} \{ p \}
\]

\[
\exp \left( \sum_n \left( \frac{q}{t} \right)^n \frac{q^n}{z^n} \frac{\partial}{\partial p_n} \right) \chi_R \{ p \} = \sum_{a,b} (-)^b \cdot \left( \frac{q}{t} \right)^{a+b+1} \cdot q^{a-b} \cdot \{q\} \cdot \chi_{R/[\alpha+1,1^b]} \{ p \}
\]

where \(\chi_{R/Y}\) are the skew Schur functions, arising when an operator \(\chi_Y \{ \frac{k \partial}{\partial p} \} \) acts on \(\chi_R \{ p \}\). Thus

\[
\frac{\mathcal{H} - 1}{\{t\} \{q\}} \chi_R = \frac{q}{t} \cdot \chi_{[1]} \chi_{R/[1]} + \left( \frac{q}{t} \right)^2 \left( t \cdot \chi_2 - \frac{1}{t} \cdot \chi_{[1,1]} \right) \left( q \cdot \chi_{R/[2]} - \frac{1}{q} \cdot \chi_{R/[1,1]} \right) + \ldots
\]

For \(R = [1,1]\) we get:

\[
\frac{q}{t} (\chi_{[2]} + \chi_{[1,1]}) - \frac{1}{q} \left( \frac{q}{t} \right)^2 \left( t \chi_{[1]} - \frac{1}{t} \cdot \chi_{[1,1]} \right) = \frac{q(1 + t - 2)}{t} \chi_{[1,1]}
\]

For \(R = [2]\) the eigenfunction is a triangular transform \(\chi_{[2]} + \alpha \chi_{[1,1]}\). Then we get

\[
(1 + \alpha) \frac{q}{t} (\chi_{[2]} + \chi_{[1,1]}) + \left( \frac{q - \alpha}{q} \right) \left( \frac{q}{t} \right)^2 \left( t \chi_{[1]} - \frac{1}{t} \chi_{[1,1]} \right) = \frac{q}{t} \left( 1 + q^2 \right) \chi_{[2]} + \left( 1 - \frac{q^2}{t^2} + \alpha + \frac{\alpha}{t^2} \right) \chi_{[1,1]}
\]

and this is an eigenfunction, provided

\[
1 - \frac{q^2}{t^2} + \alpha + \frac{\alpha}{t^2} = \alpha \cdot (1 + q^2) \quad \Rightarrow \quad \alpha = \frac{\{q/t\}}{\{qt\}}
\]

what is exactly the coefficient, defining the Macdonald function \(M_{[2]} \{ p \} \sim \chi_{[2]} \{ p \} + \left( \frac{q/t}{qt} \right) \chi_{[1,1]} \{ p \}\).

Note that in the case of Schur functions, this \(\mathcal{H}\) is actually not a single operator, but an entire one-parametric family of commuting Hamiltonians, depending on a free parameter \(q = t\) and no higher Hamiltonians are actually needed. However, in Macdonald and generalized Macdonald cases they get more important. In the most interesting realization they involve skewing over diagrams with a larger number of hooks.

The simple example in this section demonstrates that triangularity is not always obvious at the Hamiltonian level. Even more difficult is the inverse problem – to construct the deformation of Hamiltonians from the triangular deformation of eigenfunctions. In particular the generalization of Hamiltonians from Macdonald to Kerov functions and thus the very definition of generalized Kerov functions remain open questions. A possible new insight can be provided by a very different appearance of triangular Hamiltonian, which comes supplemented by additional “pental” structure, which, if better understood, can serve as a tool to fixing numerous ambiguities in the triangular-transformation (dressing) approach.

4. Pentad structure was discovered in the study of double-twist knots \[13\]–\[18\], we refer to \[18\] for a summary, pictures and notation. These are the simplest arborescent knots, with reduced colored HOMFLY-PT polynomials \[19\] given by a very simple formula \[20\]

\[
H_R^{(m,n)} = d_R \cdot \langle \emptyset | \tilde{S} \tilde{T}^{2m} \tilde{S} \tilde{T}^{2n} \tilde{S} | \emptyset \rangle
\]

where \(\tilde{T}\) is the colored \(\mathcal{R}\)-matrix in representation \(R\), while \(\tilde{S}\) is the exclusive Racah matrix \((6j\)-symbol), relating the two maps of representation products \((R \otimes \hat{R}) \otimes R \rightarrow R\) and \(R \otimes (\hat{R} \otimes R) \rightarrow R\). Dependence of knot invariants on “evolution parameters” \(n\) and \(m\) is controlled by the eigenvalues of \(\tilde{T}\), and the conventional technique \[21\] is to work in the basis, where \(\tilde{T}\) is diagonal. Then the only non-trivial ingredient is the symmetric and orthogonal Racah matrix \(\tilde{S} = \tilde{S}^{tr}, \tilde{S}^2 = I\). It heavily depends on \(R\), but we suppress the label \(R\) in \(\tilde{S}_R = \tilde{S}\) to make the formulas readable. The problem is that it is not actually known from representation theory and somewhat
difficult to calculate from the first principles. Thus what happened is that instead of being used to calculate HOMFLY-PT polynomials, $\mathcal{S}$ was instead deduced from the intuition about the twist knots. The benefit for know theory is that once known, the same $\mathcal{S}$ can be used to calculate colored HOMFLY-PT polynomials for all other arborescent knots and links. Moreover, the insights about Racah matrices, revealed in this way, can probably be used for more general mixing matrices \[22\], where they can be also combined with the powerful eigenvalue hypothesis \[23\].

The main knot-theory implication is the differential expansion \[24\], implying the separation of representation $R$ and knot/link $\mathcal{K}$ dependencies

$$H^\mathcal{K}_R = \sum_X Z_X^R F^\mathcal{K}_X$$ \hspace{1cm} (19)

where $X$ are Young diagrams, somehow restricted (through vanishing of the Z-factors) for the given $R$. It appears that the interplay between \[18\] and \[19\] for the double twist family is highly non-trivial – and is actually enough to obtain explicit expression for Racah matrix $\mathcal{S}$. The procedure is fully described in \[18\] for all rectangular $R$, where there are no multiplicities in the product $R \otimes \mathcal{R}$, but it actually remains the same for arbitrary $R$, some technical details still remain to be worked out in this case. The main discovery \[17\] is that to rewrite \[18\], one needs to actually refrain from diagonalizing $\mathcal{T}$: instead of diagonal "Hamiltonian" one needs to use triangular – and this is what makes the story resembling that in s.2 above. However, this time we can see additional miracles and more structures.

To be more precise, we substitute evolution with the diagonal matrix $\mathcal{T}$ by that with the triangular KNTZ matrix $\mathcal{B}$ \[17\] \[18\]. At the first step we rewrite \[18\] by insertion of unity decompositions with the help of an auxiliary matrix $U$:

$$\langle 0 | \mathcal{S} T^{2m} S \mathcal{T}^2 S \mathcal{S} | 0 \rangle = \langle 0 | U^{tr} (U^{tr})^{-1} \mathcal{S} T^{2m+2} \mathcal{S} U^{tr} (U^{tr})^{-1} \mathcal{S} T^{-2} \mathcal{T}^{-2} \mathcal{S} U^{-1} (U^{tr})^{-1} \mathcal{S} T^{2n+2} \mathcal{S} U^{-1} | 0 \rangle$$ \hspace{1cm} (20)

Then the fact is that $U$ can be adjusted in such a way that the five facts are simultaneously true:

- $\mathcal{B} := U \mathcal{S} T^2 \mathcal{S} U^{-1}$ is triangular: $\mathcal{B}_{XY} \neq 0$ iff $Y \subset X$
- $U_{X\emptyset} = 1$ or 0
- $\langle X | \mathcal{B} U | \emptyset \rangle = \delta_{X,Y}$
- $Z_X^R := d_R \cdot \langle 0 | \mathcal{S} T^{2m} \mathcal{T}^{-2} \mathcal{S} U^{-1} | X \rangle$
- $d_R \cdot \langle X | (U^{tr})^{-1} \mathcal{S} T^{-2} \mathcal{T}^{-2} \mathcal{S} U^{-1} | Y \rangle = \frac{\mathcal{B}_{X,Y}}{X^2} \cdot \delta_{X,Y}$ is diagonal matrix, with the same $Z_R^X$

Taken together, these facts mean that \[18\] is indeed rewritten in the form \[19\] with the original mysterious factorization conjecture \[13\]

$$F^{(m,n)}_X = \frac{F^{(m)}_X F^{(n)}_X}{\Lambda_X}$$ \hspace{1cm} (21)

with $\Lambda'_X = F^{(1)}_X = \mathcal{B}_{XY} - \text{a generically known monomial } \Lambda'_X = q^{\alpha_X} \cdot A^{\beta_X}$, and the triangular evolution formula \[17\] for the twist-family coefficient

$$F^{(m)}_X = \langle X | \mathcal{B}^{m+1} U | \emptyset \rangle = \sum_Y \langle X | \mathcal{B}^{m+1} | Y \rangle$$ \hspace{1cm} (22)

5. Explicit example of the double twist family actually allows to understand what is the set $X$ in \[19\] – then by universality of the differential expansion this remains true for arbitrary knots. Thus $X$ are actually the elements of the representations product $R \otimes \mathcal{R}$ (remarkably, they are in one-to-one correspondence with those of $R \otimes \mathcal{R}$, what allows the whole story about the differential expansion to be self-consistent). In the case of $R \otimes \mathcal{R}$ these are actually composite diagrams of the type $X = (\lambda, \lambda')$ with $\lambda, \lambda' \subset R$ and $|\lambda| = |\lambda'|$. Moreover, for rectangular $R = [r^s]$ only diagonal composites with $\lambda' = \lambda$ contribute, what allows to label $X$ in \[19\] in this case by subdiagrams $\lambda$ of $R$. In non-rectangular case there are also non-diagonal composites, moreover, some diagrams appear with multiplicities – and these two facts appear to be intimately related, especially in the formulas for $\mathcal{S}$ and $\mathcal{B}$. Still the entire construction remains the same.
6. The next crucial fact about it is that the matrix $B$ is explicitly known, at least for arbitrary rectangular representations $R = \rho^{\nu}$, where diagrams $X$ can actually be identified with the sub-diagrams of $R$:

$$B_{\lambda\mu} := (-)^{|\lambda|-|\mu|} \Lambda_{\lambda} \cdot \frac{\chi_{\lambda}^{0} \cdot \chi_{\mu}^{0}}{\chi_{\lambda}^{0} \cdot \chi_{\mu}^{0}}$$

$$\text{(23)}$$

It is expressed through $\Lambda := \bar{T}^2$ and through the skew Schur functions and thus is automatically triangular. Label "0" denotes restriction of Schur functions to

$$p_k = p_k^0 := \{q\}^{k} \{k\}$$

with $t = q$. In the second (boxed) version and with Schur substituted by Macdonald functions the same formula provides a positive rectangularly-colored super-polynomial $[17][18]$.

For non-rectangular representations the story is a little more involved: $\bar{S}$ splits into independent blocks, with only one block contributing to arborescent calculus. The same is true for $B$, which also acquires explicit $N$-dependence ($A = q^N$) in additional rows and columns, and thus is not fully described by (23). For example, for $R = [2,1]$ the triangular matrix $B$ is

\[
\begin{array}{|c|c|c|c|}
\hline
1 & -A^2 & 0 & 0 \\
\hline
-A^2 & A^2 & 0 & 0 \\
\hline
\frac{A^4}{q^2} - \frac{A^4}{[2][q]} & 0 & \frac{A^4}{q^2} \sqrt{\frac{[N+2]}{[N-2]}} & A^2 \\
\frac{q^2 A^4}{[2]} & -\frac{q^3 A^4}{[N]} \sqrt{\frac{[N-2]}{[N+2]}} & -\frac{q^3}{2} A^4 & 0 \\
\frac{[3] A^6}{[2]} - \frac{[3] A^6}{[2][q]} & \frac{[3] A^6}{[2]} & -\frac{[3] A^6}{[q][2]} & A^6 \\
\frac{[3] A^6}{[2][q]} - \frac{[3] A^6}{[2]} & \frac{[3] A^6}{[2][q]} \sqrt{\frac{[N+2][N-2]}{[N][N-2]}} + A^6 & \frac{[3] A^6}{[2][q]} + \frac{[3] A^6}{[2][q]} \sqrt{\frac{[N+2][N-2]}{[N][N-2]}} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

The $2 \times 2$ block in the right low corner decouples, and also vanishing is the matrix element $U_{X;Y}$ with $X$ from the third line, thus vanishing are sums of all the matrix elements along each line, with omission of the ones in the third column. All contributing $Z$-factors are nicely factorized, just as they are for rectangular case, the non-factorized expression in [24].

$$Z^{[1]}_{[2,1]} = D_3 D_{-3} + D_2 D_0 + D_0 D_{-2} = \frac{[3]}{[2]} D_0^0 + \frac{[3]^2}{[2]} D_2 D_{-2}$$

is actually a sum of two factorized $Z$-factors from the second and the forth lines.

7. As we saw, the matrix $B$ is very simple. Moreover, it is universal, it does not actually depend on $R$, specification of $R$ just cuts out a piece, restricted to relevant lines/columns $X$, for which the $R$-dependent $Z$-factors are non-vanishing.

Matrices $\bar{S}$ and $U$ are instead heavily $R$-dependent and rather complicated. However, they are easily restored, if one knows another universal triangular matrix $E$, made from the eigenfunctions of $B$:

$$B E = E \Lambda \iff B = E \Lambda E^{-1}$$

Then $E = U \bar{S}$ and

$$d_R \cdot T^{-2} S T^{-2} = E^{tr} \frac{Z}{N} E$$

(27)
Thus $\mathcal{E}$ diagonalizes the triangular evolution operator ("Hamiltonian") $\mathcal{B}$, and at the same time it diagonalizes the Racah matrix $\mathcal{S}$, but this time – not as an operator, but as a quadratic form. Instead, $\bar{\mathcal{S}}$ can be diagonalized as an operator, and then the diagonalizing matrix is the second exclusive Racah matrix $S$ [20]:

$$\bar{T}\bar{S}\bar{T} = ST^{-1}S^{-1}$$

Thus we get a peculiar ordered pentad of matrices:

$$
\begin{align*}
&\bar{T} \\
&\downarrow \\
&\mathcal{B} \leftrightarrow \mathcal{E} \\
&\downarrow \\
&\bar{\mathcal{S}} \leftrightarrow S
\end{align*}
$$

with one diagonal matrix in the first line, two universal triangular matrices in the second line and two $R$-dependent (non-universal) and non-triangular in the third line. One can also add $U$ and $T$ to make a septet.

8. If $\mathcal{E}$ in the first line was just an eigenfunction matrix for $\mathcal{B}$, it would be ambiguous: one could multiply it by any diagonal matrix from the right. Moreover, when $\bar{T}^2$ has coincident entries, the corresponding block can even be non-diagonal. Such modification, however, affects orthogonality of $\bar{\mathcal{S}}$, the normalization condition $U_{\chi\emptyset} = 1$ and the expressions for $Z$. Any of them can be used to fix the ambiguity – and this is why the pentad structure seems to be more rigid than just the triangular one. In fact, one can rewrite the knot polynomial in a form which is free from the ambiguity,

$$F_X^{(m)} = \sum_{Y:U_{\chi\emptyset} = 1} \langle X | \mathcal{E} \bar{T}^{2m} \mathcal{E}^{-1} | Y \rangle$$

but there is no such non-ambiguous formula for $\bar{\mathcal{S}}$. Also, ambiguity in $\mathcal{E}$ could be used to get rid of the diagonal matrix $Z/N'$ at the r.h.s. of (27), but this is also not so simple the factor could be easily absorbed if the freedom to multiply $\mathcal{E}$ from the left – but instead we can do this only from the right. Thus reconstruction of $\bar{\mathcal{S}}$ from the known $\mathcal{B}$ is in fact a more delicate procedure than that of building the HOMFLY-PT polynomial. We illustrate it by the very simplest example of the fundamental representation $R = [1]$. In this case:

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ -A^2 & A^2 \end{pmatrix}, \quad \bar{T}^2 = \begin{pmatrix} 1 & 0 \\ 0 & A^2 \end{pmatrix} \quad \Rightarrow \quad \mathcal{E} = \begin{pmatrix} 1 & 0 \\ \frac{A^2}{c} & c \end{pmatrix}$$

where $c$ is the ambiguous parameter. Also yet-unknown in this approach is $Z' = Z_{[1]}/N_{[1]}$ in (27):

$$\bar{\mathcal{S}} = d_{[1]} \cdot \bar{T}^2 \mathcal{E} \frac{Z}{N} \mathcal{E} \bar{T}^2 = d_{[1]} \cdot \begin{pmatrix} 1 & 0 \\ 0 & A^2 \end{pmatrix} \begin{pmatrix} 1 & \frac{A^2}{c} \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Z' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{A^2}{c} & c \end{pmatrix} = d_{[1]} \cdot \begin{pmatrix} 1 + \frac{cA^2Z}{A^2-1} \\ \frac{cA^2Z}{A^2-1} \end{pmatrix}$$

If we require this matrix to be orthogonal, we fix $c^{-1} = d_{[1]} \cdot A\sqrt{Aq^2 \{Aq\} \{A/q\}}$ and $Z' = -\{Aq\} \{A/q\}$.

Deriving a general formula for $\mathcal{E}$ is an open and challenging problem. Even in the simplest case of the single-line (symmetric) $R$ explicit solution

$$\mathcal{E}_{nm} = \frac{[n]!}{[m]![n-m]!} \prod_{j=n+m}^{2m-1} \{Aq^j\} \cdot c_m$$

satisfies $\mathcal{B}\mathcal{E} = \mathcal{E}\bar{T}^2$ due to rather exotic combinatorial identity:

$$q^{2Nn} \sum_{i=0}^{n} \left( -\{q\} \right)^i \cdot \frac{[n]!}{[n-i]!\{q\}!} \cdot \frac{[N]!}{[N-i]!} \cdot q^{i(i+1)/(N+n)i} = 1$$

satisfies $\mathcal{B}\mathcal{E} = \mathcal{E}\bar{T}^2$ due to rather exotic combinatorial identity:
which is a $q$-deformation of
\[
A^n \cdot \sum_{i=0}^{n} (-)^{n-i} \frac{n!}{(n-i)!i!} (A - A^{-1})^{n-i} A^i = A^n \cdot (A - (A - A^{-1}))^n = 1 \tag{35}
\]

Since Racah matrix $\bar{S}$ is quadratic in $\mathcal{E}$, explicit knowledge of this matrix is important to generalization of hypergeometric series [26] for $\bar{S}$ and $S$ from symmetric to arbitrary representations $R$.

9. If we treat like $\mathcal{B}$ the Hamiltonians $\mathcal{H}$ from sec.2 and 3, then $\mathcal{E}$ is the matrix of their eigenfunction, i.e. for Macdonald-Kerov theory from secs.1 and 2 these would be Macdonald, Kerov or generalized Macdonald functions (more precisely, $\mathcal{E}$ would be inverse of triangular Kostka matrices $K$). The question is what are the other three matrices $\bar{S}, S$ and $U$. The first of them, $\bar{S}$ looks like the matrix of scalar products for Schurs, provided Macdonald/Kerov functions are orthogonal. The rigidity of pentad structure can help to explain what distinguishes particular choices of bases for symmetric functions and especially for GMF. Hopefully this can also help to resolve the long-standing problems of Kerov Hamiltonians and higher Hamiltonians for GMF, and all the way further, to generalized Kerov functions.

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