Width of the chaotic layer: maxima due to marginal resonances

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Abstract

Modern theoretical methods for estimating the width of the chaotic layer in presence of prominent marginal resonances are considered in the perturbed pendulum model of nonlinear resonance. The fields of applicability of these methods are explicitly and precisely formulated. The comparative accuracy is investigated in massive and long-run numerical experiments. It is shown that the methods are naturally subdivided in classes applicable for adiabatic and non-adiabatic cases of perturbation. It is explicitly shown that the pendulum approximation of marginal resonance works good in the non-adiabatic case. In this case, the role of marginal resonances in determining the total layer width is demonstrated to diminish with increasing the main parameter $\lambda$ (equal to the ratio of the perturbation frequency to the frequency of small-amplitude phase oscillations on the resonance). Solely the “bending effect” is important in determining the total amplitude of the energy deviations of the near-separatrix motion at $\lambda \gtrsim 7$. In the adiabatic case, it is demonstrated that the geometrical form of the separatrix cell can be described analytically quite easily by means of using a specific representation of the separatrix map. It is shown that the non-adiabatic (and, to some extent, intermediary) case is most actual, in comparison with the adiabatic one, for the physical or technical applications that concern the energy jumps in the near-separatrix chaotic motion.

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1 Introduction

Width of the chaotic layers near separatrices of Hamiltonian systems is subject to large jump-like perturbations (upon variation of parameters) due to emergence of marginal resonances at the borders of the layers [1, 2]. The width is maximal, or close to maximal, when a prominent marginal resonance is present at the border of the chaotic layer, and its separatrix chaotic layer is in heteroclinic connection with the main layer. The width is maximal when the two layers are on the brink of heteroclinic disconnection.

Formulas for the maximal width and for critical values of parameters were derived in [2] in the perturbed pendulum model of nonlinear resonance. This is the model that naturally arises in a wide range of problems of physics and celestial mechanics, when interactions of nonlinear resonances are considered [1]. The field of applicability of the theory [2] is the realm of non-adiabatic chaos [3, p. 813]. In fact, the model [2] covers the realm of non-adiabatic (and, to some extent, intermediary) chaos for any system described by the separatrix map. As found in [3], the border between slow (adiabatic) and fast (non-adiabatic) chaos is rather sharp and is located at \( \lambda \approx 1/2 \) for any system described by the separatrix map, given by Eqs. (2) in [3] (see Eqs. (11) below). The quantity \( \lambda \) is the “adiabaticity parameter” [1], equal to the ratio of the perturbation frequency to the frequency of small-amplitude phase oscillations on the resonance.

The non-adiabatic approximation was adopted in [2] on several reasons. The separatrix map theory, which forms the basis for the subsequent theories of the near-separatrix energy jumps, including [2, 3, and 4], was initially developed and believed to be applicable in the non-adiabatic case \( \lambda \gg 1 \), see [1, 4]. (Later on, in [6], the separatrix map theory was shown to be applicable in the adiabatic domain. An analytical description of marginal resonances in this domain was performed in [5] by the method of so-called resonance invariants.) The study [2] was inherently aimed at real applications (in celestial mechanics), where the adiabatic limit was not actual. Besides, the boundary in \( \lambda \) between slow and fast chaos had not been yet identified, neither it had been known to be sharp, until the boundary was revealed in [3]. Of course, in a moderately adiabatic case, say, at \( \lambda \) as low as 0.1, the
theory \cite{2} can be also used, as a rough extrapolation.

2 The model of nonlinear resonance and the separatrix map

For the model of nonlinear resonance we take that adopted in \cite{2, 3}. Under general conditions \cite{7, 11, 8}, a model of nonlinear resonance is provided by the Hamiltonian of nonlinear pendulum with periodic perturbations. A number of problems on non-linear resonances in mechanics and physics is described by the Hamiltonian

$$H = \frac{Gp^2}{2} - F\cos\varphi + a\cos(\varphi - \tau) + b\cos(\varphi + \tau)$$

(1)

(see e.g. \cite{6}). The first two terms in Eq. (1) represent the Hamiltonian $H_0$ of the unperturbed pendulum, and the last two terms represent the periodic perturbations; $\varphi$ is the pendulum angle (the resonance phase angle), $p$ is the momentum; $\tau$ is the phase angle of perturbation: $\tau = \Omega t + \tau_0$, where $\Omega$ is the perturbation frequency, and $\tau_0$ is the initial phase of the perturbation. The quantities $F, G, a, b$ are constants. We assume that $F > 0, G > 0$, and $a = b$.

The so-called separatrix (or “whisker”) map

$$w_{i+1} = w_i - W \sin \tau_i,$$

$$\tau_{i+1} = \tau_i + \lambda \ln \frac{32}{|w_{i+1}|} \quad (\text{mod } 2\pi),$$

(2)

written in the present form and explored in \cite{7, 9, 11} and first introduced (in implicit form) in \cite{11}, describes the motion in the vicinity of the separatrices of Hamiltonian \cite{11}. The quantity $w$ denotes the relative (with respect to the unperturbed separatrix value) pendulum energy $w \equiv \frac{H_0}{F} - 1$; $\tau$ is the phase angle of perturbation (the same as in Eq. (1)).

The constants $\lambda$ and $W$ are the two basic parameters. The parameter $\lambda$ is the ratio of $\Omega$, the perturbation frequency, to $\omega_{0} = (FG)^{1/2}$, the frequency of the small-amplitude pendulum oscillations. The parameter $W$ in the case of $a = b$ is given by

$$W = \varepsilon \lambda (A_2(\lambda) + A_2(-\lambda)) = \varepsilon \frac{4\pi \lambda^2}{\sinh \frac{\lambda}{2}}$$

(3)
Here $A_2(\lambda) = 4\pi \lambda \frac{\exp(\pi \lambda/2)}{\sinh(\pi \lambda)}$ is the value of the Melnikov–Arnold integral as defined in [1]. We use the notation $\varepsilon \equiv a/F = b/F$ for the relative amplitude of perturbation. Formula (3) differs from that given in [1, 8] by the term $A_2(-\lambda)$, which is small for $\lambda \gg 1$. However, its contribution is significant for $\lambda$ small [2], i.e., in the case of adiabatic chaos. Analytical expressions for $W$ at $a \neq b$ are given in [6].

One iteration of map (2) corresponds to one period of the pendulum rotation or a half-period of its libration. The motion of system (1) is mapped by Eqs. (2) asynchronously [2]: the relative energy variable $w$ is taken at $\phi = \pm \pi$, while the perturbation phase $\tau$ is taken at $\phi = 0$. The desynchronization can be removed by a special procedure [2, 6]. The synchronized separatrix map gives correct representation of the sections of the phase space of the near-separatrix motion both at high and low perturbation frequencies; this was found in [6] by direct comparison of phase portraits of the separatrix map to the corresponding sections obtained by numerical integration of the original systems. This testifies good performance of both the separatrix map theory and the Melnikov theory, describing the splitting of the separatrices.

3 Analytical estimating the width maxima

Let us present, for clarity, the resulting formulas of the method [2] for estimating the locations and heights of the peaks (Eqs. (8), (9), and (10) in [2]).

According to [2], the approximate condition for the tangency of the unperturbed separatrix of a marginal integer resonance to the border of the primary chaotic layer (equivalently, the approximate condition for the maximal energy jumps in the near-separatrix motion) is given by the equation

$$W(\varepsilon, \lambda) = W_t^{(m)}(\lambda),$$

where $W(\varepsilon, \lambda)$ and $W_t^{(m)}(\lambda)$ are expressed as follows. The expression for $W(\varepsilon, \lambda)$ is determined by the choice of the system. For the pendulum with the harmonically driven point of suspension it is given by Eq. (3):

$$W(\varepsilon, \lambda) = \varepsilon \frac{4\pi \lambda^2}{\sinh \frac{\pi \lambda}{2}}.$$
The expression for $W_t^{(m)}(\lambda)$ does not depend on the choice of the system. In the pendulum model of the marginal resonance it has the universal form:

$$W_t^{(m)}(\lambda) = \frac{32}{\lambda^3} \left[ (1 + \lambda^2)^{1/2} - 1 \right]^2 \exp \left( -\frac{2\pi m}{\lambda} \right),$$

where $m$ is the order of the marginal resonance (see details in [2]).

The location $\lambda = \lambda_m$ of an $m$th peak at any value of $\varepsilon$ can be found by solving the functional equation (4) with respect to $\lambda$ at any $\varepsilon$ and $m \geq 1$ numerically. This can be easily done, e.g., in the Maple computer algebra system [11]. The extreme value of the relative energy during the energy jump is given by

$$w_m = \pm \left[ 64 \exp \left( -\frac{2\pi m}{\lambda_m} \right) - \lambda_m W_t^{(m)}(\lambda_m) \right].$$

Recently Soskin and Mannella [4] presented a theoretical method for calculation of maximal width of the separatrix chaotic layer, which is suitable for a wide class of the periodically perturbed one-degree of freedom Hamiltonian systems. Their theory describes the shape of the resulting peak in the “frequency of perturbation — relative energy” coordinates, its location in the frequency of perturbation and its height in the relative energy. Besides, it provides a general approach in the framework of Soskin and Mannella’s classification of the periodically perturbed one-degree of freedom Hamiltonian systems. The pendulum model was not used in [4] for description of the marginal resonance. Instead, a transition from a discrete mapping to continuous regular-like representation of the motion in some limit and an analysis of the marginal resonance Hamiltonian were used to describe the perturbed border of the layer. Note that the system parameters are designated differently in [4] as compared to our designations in [2]: (i) the relative frequency of perturbation is $\lambda$ in [2], but $\omega_f$ in [4], so that $\lambda = \omega_f$; (ii) the relative amplitude of perturbation is $\varepsilon$ in [2], but $h/2$ in [4], so that $\varepsilon = h/2$.

The formula for the location of the $m$th peak $\lambda_m$ derived in [4] in the adiabatic approximation $h \ll 1$ is

$$\lambda_m \approx -\frac{2\pi m}{\ln \frac{4}{\varepsilon}} = -\frac{2\pi m}{\ln \frac{\varepsilon}{2}},$$

(Eq. 63 in [4]), and the corresponding theoretical value of the maximal half-width of the chaotic layer is
\[ |w_m| \approx 2(4e + 1)h = 4(4e + 1)\epsilon \approx 23.75h \quad (9) \]

(Eq. 72 in [4]). The limit \( h \to 0 \) in the description of the peaks is adiabatic, because \( \lambda_m \to 0 \) if \( h \to 0 \).

4 Computing the width maxima

At each value of \( \lambda \) the half-width can be measured by two methods, as described in [3]. The first one was proposed in [9, 1] and developed and extensively used in [12]. It is based on calculation of the minimum period \( T_{\text{min}} \) of the motion in the chaotic layer. The half-width is determined by the formula [9, 1, 12]:

\[ w_b = 32 \exp(-\omega_0 T_{\text{min}}). \quad (10) \]

The minimum period corresponds to the maximum energy deviation from the unperturbed separatrix value. This formula directly follows from the second line of Eqs. (2).

The second method consists in the direct continuous measuring of the relative energy deviation from the unperturbed separatrix \( w = \frac{H}{p} - 1 \) in the course of integration, and fixing the extremum one.

In this work, I compute the width of the chaotic layer near the separatrices of Hamiltonian (11) directly, i.e., by the second method. The integration of the equations of motion, given by Hamiltonian (11), has been performed by the integrator by Hairer et al. [13]. It is an explicit 8th order Runge–Kutta method due to Dormand and Prince, with the step size control.

The integration time interval is chosen to be \( 10^4 \), in the units of periods of perturbation. Each unit is divided in \( 10^5 \) equal segments; the trajectories have been output at the end of each segment, to provide, with such time resolution, the calculation of the time period and the relative energy deviation of the motion. Further increasing the integration time interval or decreasing the length of the segments have been checked to leave the estimates of the width unchanged within 3–4 significant digits; i.e., the estimates are saturated enough.

In Table 1, I present the numerical-experimental estimates of \( \lambda_1 \) and \( |w_1|/h \), obtained in this way, i.e., by the direct integrations of the continuous system (11). (It should be stressed that iterating the separatrix map is not used here.) The numerical-experimental estimates are given alongside with
the theoretical estimates obtained by the methods [4] and [2]. The necessary formulas are given above. As follows from the data in Table 1, a direct comparison of the numerical-experimental and theoretical results for a set of 5 values of \( h \) (namely, \( h = 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, \) and \( 10^{-2} \)), spanning the whole range of the perturbation strengths considered in [4], shows that the theory [2] performs worse than [4] when it is out of its range of applicability (i.e., when \( \lambda_1 \lesssim \frac{1}{2} \)), but it is definitely more accurate than [4] when it is in its range of applicability (when \( \lambda_1 \gtrsim \frac{1}{2} \)). Judging by the absolute deviation of the theoretical values of \( \lambda_1 \) from its numerically measured values, the theory [2] is 4.9 times better than the theory [4] in the case of \( h = 10^{-2} \), 1.5 times better in the case of \( h = 10^{-3} \), and 1.1 times better in the case of \( h = 10^{-4} \). (The corresponding deviations are less in the given proportions.) Judging by the absolute deviation of the theoretical values of \( |w_1|/h \) from its numerically measured values, the theory [2] is 3.2 times better than the theory [4] in the case of \( h = 10^{-2} \), 2.0 times better in the case of \( h = 10^{-3} \), and 1.4 times better in the case of \( h = 10^{-4} \). Therefore, the theory [2] is definitely more accurate than the theory [4] in the realm of non-adiabatic chaos.

| \( h \) | \( 10^{-6} \) | \( 10^{-5} \) | \( 10^{-4} \) | \( 10^{-3} \) | \( 10^{-2} \) |
|--------|--------|--------|--------|--------|--------|
| \( \lambda_1 \) num., this paper | 0.420 | 0.499 | 0.615 | 0.800 | 1.156 |
| \( \lambda_1 \) theor. [4] | 0.413 | 0.487 | 0.593 | 0.758 | 1.049 |
| \( \lambda_1 \) theor. [2] | 0.433 | 0.515 | 0.635 | 0.828 | 1.178 |
| \( |w_1|/h \) num., this paper | 26.44 | 27.27 | 28.06 | 28.09 | 26.69 |
| \( |w_1|/h \) theor. [4] | 23.75 | 23.75 | 23.75 | 23.75 | 23.75 |
| \( |w_1|/h \) theor. [2] | 31.68* | 31.51 | 31.14 | 30.24 | 27.60 |

* This value is conditional, because \( \lambda_1 < 1/2 \).

In fact, the comparative results of the performance of the theories [4] and [2], presented in Table 1, are expectable and natural, because the theory [4] was developed for the case of adiabatic chaos, whereas the theory [2] was developed for the cases of intermediary and non-adiabatic chaos. It is evident that the relatively strong perturbations \( h \gtrsim 10^{-3} \) (corresponding to \( \lambda_1 \gtrsim 0.8 \)) are most common in physical and technical applications, because the applications usually concern interactions in multiplets of resonances of comparable
strengths; see e.g. [14]. Therefore, in the range of perturbation amplitudes most common in applications, the theory [2] performs much better than [4] in predicting the locations and heights of the peaks.

From Table 1 it is apparent that, in the given range of $h$, the predictions of the theory [4] are smaller than the numerical results and the predictions of the theory [2] are larger than the numerical results, for both $\lambda_1$ and $|w_1|/h$. First of all, note that the deviations of the theoretical $\lambda_1$ and $|w_1|/h$ values from the actual ones are coherent (i.e., they are both negative or both positive), because the chaotic layer width, when it is close to a maximal one, increases with increasing $\lambda$, until the peak value is achieved and the width sharply drops at the moment of heteroclinic disconnection of the primary and secondary chaotic layers.

The “generalized separatrix split”, used in the theory [4] as an addend in the sum representing the total maximal width of the chaotic layer, is smaller than the width of the primary chaotic layer when the perturbation is non-adiabatic; therefore the values of $\lambda_1$ predicted by the theory [4] in the non-adiabatic and intermediary cases are expected to be smaller than the actual ones. What is more, the “generalized separatrix split” is not taken into account at all in deriving the approximate theoretical estimates in [4]. Thus, on increasing $\lambda$, the moment of heteroclinic disconnection is somewhat underestimated.

On the other hand, the values of $\lambda_1$ predicted by [2] are somewhat overestimated in the given range of $h$. This is due to the following reason. When the value of $\lambda_m$ corresponds to a slow or intermediary perturbation, the separatrix cell of the marginal resonance is deformed in comparison with the lenticular form in the pendulum model (this deformation is evident in Fig. 2 discussed below). The deformation increases with decreasing $\lambda_m$. In particular, the “lower half-width” (the minimal distance between the resonance center and the lower branch of the separatrix) of the cell becomes smaller and the “upper half-width” greater than the single half-width value given by the pendulum model, while the total width of the cell remains rather close to that in the pendulum model. Thus the theory [2] uses a somewhat overestimated value of the lower half-width, and on increasing $\lambda$ in the vicinity of the peak the moment of heteroclinic disconnection is overestimated.

All the peaks with $m \geq 2$ are located at $\lambda_m > \lambda_1$, see e.g. Eq. [8]. Therefore, in the range of $h$ covered in Table 1, the peaks of order $m$ higher than 1 are all situated more deeply in the domain of non-adiabatic chaos than those with $m = 1$, and thus the theory [2] is expected to dominate over
in accuracy even to a greater degree. As my numerical experiments show, this is indeed the case; see Table 2 (constructed in the same way as Table 1, except that the case of $m = 2$ is considered).

Table 2: The locations and heights of the peaks ($m = 2$)

| $h$          | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-3}$ | $10^{-2}$ |
|--------------|-----------|-----------|-----------|-----------|-----------|
| $\lambda_2$ num., this paper | 0.853     | 1.015     | 1.245     | 1.603     | 2.181     |
| $\lambda_2$ theor. [4]      | 0.827     | 0.974     | 1.186     | 1.515     | 2.097     |
| $\lambda_2$ theor. [2]      | 0.867     | 1.029     | 1.262     | 1.618     | 2.188     |
| $|w_2|/h$ num., this paper    | 28.11     | 28.01     | 25.97     | 23.00     | 16.27     |
| $|w_2|/h$ theor. [4]         | 23.75     | 23.75     | 23.75     | 23.75     | 23.75     |
| $|w_2|/h$ theor. [2]         | 30.01     | 28.87     | 26.79     | 22.92     | 16.29     |

For example, let us take $m = 2$ and $h = 10^{-2}$. Then the numerical-experimental values of $\lambda_2$ and $|w_2|/h$ turn out to be 2.1808 and 16.27, respectively. The corresponding values predicted by the theory [4] are 2.0974 and 23.75, and the values predicted by the theory [2] are 2.1885 and 16.29. Thus, at $h = 10^{-2}$, the theory [2] in comparison with [4] is $\approx 11$ times more accurate in predicting $\lambda_2$ and is $\approx 370$ times more accurate in predicting $|w_2|/h$.

5 Dynamical validity of the marginal resonance model

Note that no adiabatic limits were obtained or analyzed in [2], because the theory [2] was constructed for another (non-adiabatic) domain of application. The approximation of marginal resonance in the pendulum model works good in the range of applicability of the theory [2], i.e., in the non-adiabatic domain (contrary to an opinion expressed in [4]). Indeed, if this model were invalid, than the good accord between the theory [2] and the numerical-experimental data, as demonstrated above, would be merely a coincidence. But this cannot be the case on the following reason. In the theory [2], the maximal half-width of the layer is the sum of the half-width of the primary (unperturbed) chaotic layer and the width of the marginal resonance. The expression for the width of the primary layer is derived independently from that for the marginal
resonance, and therefore, if the pendulum model for the marginal resonance were invalid, the resulting error in the sum could not be compensated by chance. This means that the good performance of the formulas in estimating the maximal half-width directly testifies the good performance in estimating the width of the marginal resonance and, consequently, its dynamical model.

Let us demonstrate the dynamical validity of the pendulum model for marginal resonance in the non-adiabatic domain graphically. In Fig. 1, the phase portrait of the separatrix map

\[
\begin{align*}
y_{i+1} &= y_i + \sin x_i, \\
x_{i+1} &= x_i - \lambda \ln |y_{i+1}| + c \pmod{2\pi},
\end{align*}
\]  

(11)

is presented at \( \lambda = 3 \) and \( c = 5.55 \mod 2\pi \). Form (11) (adopted in [2]) of the separatrix map is equivalent to the classical one (2). The variables \( x_i \equiv \tau_i + \pi \) and \( y_i \equiv w_i/W \) (where \( W \) is given by Eq. (5)) are the normalized time and energy, respectively. The parameter \( c \) is given by the following formula [2]:

\[
c = \lambda \ln \frac{32}{|W|}. 
\]  

(12)

Solely the chaotic component of the phase space at \( y \geq 0 \) is shown in Fig. 1. The phase portrait is synchronized [2]: the pairs \( x_{i-1}, (y_i + y_{i-1})/2 \) are drawn instead of \( x_i, y_i \), so that the portrait corresponds to a unified surface of section of phase space. The chosen values of \( \lambda \) and \( c \) correspond to the brink of heteroclinic disconnection between the primary chaotic layer (shown in black) and the secondary chaotic layer (the chaotic layer of the marginal resonance; shown in grey): a slightest increase in \( c \) separates the layers, and the width momentarily drops to that of the primary layer.

From Fig. 1, it is graphically evident that no “regular” approximation for the motion, like that in the theory [4], can describe precisely the conditions (the critical values of the parameters) for the maximal width in the non-adiabatic domain, because both the primary and secondary layers have substantial widths. These widths must be calculated and taken into account in any high-precision theory for estimating the conditions for the critical heteroclinic connection. Thus it is natural to develop any theory for estimating the layer width separately for adiabatic and non-adiabatic cases of perturbation.

It is also evident from Fig. 1 that the separatrix cell of the marginal resonance is qualitatively described by the theoretical pendulum cell. The
borders of the theoretical cell are depicted by the continuous curves. They are given by the formula

\[ y = y^{(n)} \pm 2 \left( \frac{y^{(n)}}{\lambda} \right)^{1/2} \cos \frac{x}{2}, \tag{13} \]

where \( y^{(n)} \) is the location of the center of an integer resonance of order \( n \):

\[ y^{(n)} = \exp \left( \frac{c - 2\pi n}{\lambda} \right), \tag{14} \]

as can be straightforwardly derived from Eqs. (11).

Of course, on decreasing \( \lambda \), the marginal resonance separatrix cell deforms more and more, and in the adiabatic realm the pendulum model is hardly applicable for its description. This is evident from the phase portrait in Fig. 2 (where \( \lambda = 0.001 \) and \( c = 0.0076008 \mod 2\pi \)): here the form of the separatrix cell is far from the well-known lenticular one characteristic for the pendulum case.

### 6 The bending effect

As already mentioned above, two methods for computation of the layer width were used in [3]. The first one, proposed in [1], is based on calculation of the minimal period of the motion in the chaotic layer. This minimal period can be converted to the layer half-width by means of Eq. (10). The second method consists in a direct continuous measuring of the relative energy deviation from the unperturbed separatrix in the course of numerical integration, and fixing the extreme deviation. In the case of the first method, the bending effect is averaged out [3]. The theoretical value of the maximal half-width of the chaotic layer is then given by

\[ |w_m| \approx 8e\hbar = 16e\varepsilon \approx 21.75\hbar, \tag{15} \]

instead of formula (9). One can see that the relative difference between (9) and (15) is rather small: about \( 10\% \approx 1/(4e) \). Compare Figs. 3 and 4 in [3]: the observed height of the first peak in Fig. 3 is less by \( \approx 10\% \approx 1/(4e) \) than that in Fig. 4. The reason is that the bending of the layer in the first case is averaged out, while in the second case it is present, because different methods (those described above) were used for measuring the width.
The bending effect is particularly important in the strongly non-adiabatic case, i.e., at \( \lambda \gg 1 \). The bending amplitude in the units of the relative energy \( w \), according to [2, 6], is equal to the product of the bending factor \( \delta(\lambda) \) (defined in [2]) and \( W(\lambda, \varepsilon) \) (given by Eq. (5)):

\[
W\delta = \frac{W}{\pi} \left\{ \text{Re} \left[ \psi \left( \frac{i\lambda}{2} \right) - \psi \left( \frac{i\lambda}{4} \right) \right] + \frac{1}{\lambda^2} - \ln 2 \right\} \sinh \frac{\pi \lambda}{2} \xrightarrow{\lambda \to \infty} \frac{8\varepsilon}{\lambda^2}, \tag{16}
\]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function, \( i \) is the imaginary unit. The primary layer half-width is given by

\[
w_b = \lambda W \xrightarrow{\lambda \to \infty} 8\pi \varepsilon \lambda^3 \exp \left( -\frac{\pi \lambda}{2} \right). \tag{17}
\]

Comparing Eqs. (16) and (17), one finds that the bending amplitude starts to dominate at \( \lambda \approx 7 \). This domination is exponential with \( \lambda \).

In the theory [2], the maximal half-width of the layer is the sum of the half-width of the primary (unperturbed) chaotic layer and the width of the marginal resonance. If \( \lambda \gg 1 \), the half-width of the primary layer, expressed in the units of the normalized relative energy \( y = w/W \) (where \( W \) is given by Eq. (5)), is given by \( y_b \approx \lambda \), whereas the width of the marginal resonance at the layer border is given by \( \Delta y \approx 2\pi \) [15]. Therefore, in the non-adiabatic limit \( (\lambda \to \infty) \) the relative (with respect to the primary layer half-width) jump in the energy due to the marginal resonance is just \( \approx 2\pi/\lambda \). Hence the role of marginal resonances in determining the total layer width diminishes with increasing \( \lambda \), and it becomes less important than that of the primary layer at \( \lambda \approx 2\pi \).

On the other hand, as we have seen, almost at the same value of \( \lambda \) (namely, at \( \lambda \approx 7 \)) the bending effect starts to dominate in determining the total amplitude of the energy deviations of the near-separatrix motion. Therefore, solely the bending effect is important in determining this amplitude at \( \lambda \gtrsim 7 \).

### 7 The adiabatic case

Returning to the adiabatic case, let us demonstrate that the geometrical form of the separatrix cell in that case (illustrated in Fig. 2) can be described analytically quite easily if one uses the separatrix map representation [11]. Formulas (8) and (15) can be derived as well, using this representation. This
is made as follows. Let us assume that the increments of $|x|$ and $|y|$ per iteration in Eqs. (11) are small compared to the total magnitudes of variation of the corresponding quantities. This is in spirit of an approximation proposed in [12] for the separatrix map in classical form for another problem. For description of the motion near the separatrix of a marginal integer resonance this assumption is valid, because the amplitude of variation of $y$ is much greater than 1, while the increment of $|y|$ per iteration is less than 1. (Note that in a general situation, when there are no marginal resonances, such an approximation is invalid; see discussion in [3].) Hence map (11) is reduced to the following differential equation:

$$\frac{dx}{dy} = -\lambda \ln |y| + c - 2\pi m \sin x,$$

where $m$ is the order of the marginal resonance. The $c$ parameter is determined by the parameters of the original Hamiltonian system, see Eq. (12).

Integrating Eq. (18), one has for the guiding curve at $y \geq 0$:

$$(\lambda - \lambda \ln y + c - 2\pi m)y = -\cos x + 1,$$

where the integration constant is set equal to 1, so that the curve is tangent to the axis $y = 0$. This corresponds to the critical situation: at $y = 0$ the motion is stochastized, but a slightest change of the map parameters can disconnect the curve from the axis $y = 0$, and then the motion is no more stochastized.

One can see that the geometrical form of the separatrix cell of the marginal resonance in Fig. 2 is described by Eq. (19) in a highly accurate way: the analytical curve visually coincides with the borders of the cell in the phase portrait given by the separatrix map.

For map (11), the unstable fixed point of a marginal integer resonance of order $m$ is situated at $x = \pi$, $y = \exp[(c - 2\pi m)/\lambda]$, and the stable fixed point (center) of the same resonance is situated at $x = 0 \mod 2\pi$, $y = \exp[(c - 2\pi m)/\lambda]$, see Eq. (14). Substituting the coordinates of the unstable fixed point in Eq. (19) and solving the resulting equation with respect to $c$, one has for the critical value of $c$:

$$c_m = 2\pi m - \lambda \ln \frac{\lambda}{2}.$$
Eq. (19): \( y = 0 \) and \( y = 2e/\lambda \). It is easy to check analytically that they correspond to two extrema of the \( y(x) \) function. (At \( x = \pi \) there is only one solution: \( y = 2/\lambda \).) Therefore, the maximal value of \( y \) is given by

\[
y_m = \frac{2e}{\lambda}.
\]

To connect the obtained values of \( c_m \) and \( y_m \) with the values of the parameters of the original Hamiltonian, recall that \( y = w/W \) and \( c = \lambda \ln(32/|W|) \), where, for the considered Hamiltonian model, \( W \approx 8\varepsilon \lambda \) if \( \lambda \ll 1 \) (this expression follows from Eq. (3); note that a good correspondence of this expression to the actual amplitude of the separatrix map derived numerically by integration of the original system was found in [12].) Hence \( w_m \approx 16\varepsilon \) and \( \lambda_m \approx -2\pi m/\ln(\varepsilon/2) \), in accord with Eqs. (15) and (8), respectively.

It is interesting that, as follows from formula (21) for \( y_m \) and the fact the \( y \) coordinate of the unstable fixed point is \( 2/\lambda \), the relative amplitude of the motion at the outermost border of the chaotic layer (i.e., the ratio of the maximal and minimal energies of the motion at the layer border) in the adiabatic limit is equal to \( e \approx 2.718 \).

8 Conclusions

In this paper, modern theoretical methods for estimating the width of the chaotic layer in presence of prominent marginal resonances have been considered in the perturbed pendulum model of nonlinear resonance. The fields of applicability of these methods have been explicitly and precisely formulated. The comparative accuracy has been investigated in massive and long-run numerical experiments.

It has been demonstrated that it is natural to develop any theory for estimating the layer width separately for adiabatic and non-adiabatic cases of perturbation. The comparative results of the numerical performance of the theories [4] and [2], given in Tables 1 and 2, unambiguously verify that the theory [4] is suitable in the adiabatic case, whereas the theory [2] is suitable in the intermediary and non-adiabatic cases.

It has been explicitly shown that the pendulum approximation of marginal resonance works good in the range of applicability of the theory [2], i.e., in the non-adiabatic domain. The role of marginal resonances in determining the total layer width has been shown to diminish with increasing the adiabaticity.
parameter $\lambda$, and to become less important than that of the primary layer at $\lambda \approx 2\pi$. On the other hand, almost at the same value of $\lambda$ (namely, at $\lambda \approx 7$) the bending effect starts to dominate in determining the total amplitude of the energy deviations of the near-separatrix motion. Therefore, solely the bending effect is important in determining this amplitude at $\lambda \geq 7$.

In the adiabatic case, it has been demonstrated that the geometrical form of the separatrix cell can be described analytically quite easily by means of using a specific representation of the separatrix map (namely, representation (11)). It has been found that the relative amplitude of the motion at the outermost border of the chaotic layer (i.e., the ratio of the maximal and minimal energies of the motion at the layer border) in the adiabatic limit is equal to $e \approx 2.718$.

The non-adiabatic (and, to some extent, intermediary) case has been shown to be most actual, in comparison with the adiabatic one, for the physical or technical applications that concern the energy jumps in the near-separatrix chaotic motion. The reason is that even the first peaks (with $m = 1$) appear in the realm of adiabatic chaos only if the relative strength of perturbation $\varepsilon$ attains microscopic values: Eq. (8) implies that for $\lambda_1$ to be less than 1/2 the value of $\varepsilon$ should be less than $2e^{-4\pi} \approx 10^{-5}$. In typical applications the strengths of perturbation are much greater usually. For the second and higher order peaks ($m \geq 2$) to appear in the realm of adiabatic chaos, $\varepsilon$ should be supermicroscopic.

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Figure 1: The phase portrait of the separatrix map at $\lambda = 3$ and $c = 5.55 \mod 2\pi$. The theoretical pendulum cell for the marginal resonance is shown by the continuous curves.

Figure 2: The phase portrait of the separatrix map at $\lambda = 0.001$ and $c = 0.0076008 \mod 2\pi$. 