NEW CLASS OF VOLterra INTEGRO-DIFFERENTIAL EQUATIONS WITH FRACTAL-FRACTIONAL OPERATORS: EXISTENCE, UNIQUENESS AND NUMERICAL SCHEME

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Abstract. In this paper, we introduce a new fractional integro-differential equation involving newly introduced differential and integral operators so-called fractal-fractional derivatives and integrals. We present a numerical scheme that is convenient for obtaining solution of such equations. We give the general conditions for the existence and uniqueness of the solution of the considered equation using Banach fixed-point theorem. Both the suggested new equation and new numerical scheme will considerably contribute for our readers in theory and applications.

1. Introduction and statement of problem. Fractional calculus which is a generalization of classical calculus has a wide range in numerous scientific fields [[18]-[24]]. It allows us to construct some new equations and one of these equations is integro-differential equations which will be handled in this study. Integro-differential equations are equations which involve both integral and differential operators. In this aspect, these equations have become more interesting in creating new ideas in the last years [[11]-[4]]. Volterra integro-differential equations which are an important class of these equations have arise widely in the mathematical modelling of many physical and biological processes, for example biological species coexisting together with increasing and decreasing rate of growth, electromagnetic theory, Wilson-Cowan model and many more [15]. Although they have considerably been studied in science and engineering, fractional integro-differential equations with mixed fractional operators have been newly introduced [[24]-[21]].

We shall mention some researches about fractional integro-differential equations in recent years. In [21], Mahdy give least squares method aid of Hermite polynomials for solving a linear system of fractional integro-differential equations with Caputo derivative. In [23], a numerical scheme based on these operational matrices and the typical Tau method is proposed for solving some nonlinear fractional differential equations. In [25], Ravichandran et. al. deal with neutral integro-differential systems with impulsive conditions where fractional derivative with Mittag-Leffler kernel is used. In [19], numerical scheme for a new Volterra integro-differential equation with fractal-fractional operators is presented. Ahmed et. al. study the existence of the mild solution and also discuss the sufficient conditions for approximate controllability and null controllability of for Sobolev-type impulsive fractional differential equations in [3]. In [13], Balachandran et. al. present sufficient conditions

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for boundary controllability of various classes of Sobolev-type nonlinear systems including integro-differential systems in Banach spaces. Tate et al. investigate the existence and other properties of solution of nonlinear fractional integro-differential equations with constant coefficient in [28]. Sakthivel et al. deal with controllability of nonlinear neutral evolution integro-differential systems with infinite delay in [26].

In this study, we introduce a new integro-differential equation which include mixed fractal-fractional operators. Briefly, this equation can be written such as;

\[ FFP_0 D_{\alpha,\tau}^t u(x,t) = f(x,t,u) + FFE_0 J_{\alpha,\tau}^t[K(x,t,u)] \]

where derivative is expressed by Caputo fractal-fractional derivative and integral is Caputo-Fabrizio fractal-fractional integral. Also, we can take as;

\[ FFE_0 D_{\alpha,\tau}^t u(x,t) = f(x,t,u) + FFP_0 J_{\alpha,\tau}^t[K(x,t,u)] \]

where derivative is Caputo-Fabrizio fractal-fractional derivative and integral is Caputo fractal-fractional integral.

This new mathematical equation is constructed with a new concept of differentiation introduced by Atangana, who presented the fractal-fractional derivative of a given function with power law, exponential decay law and the generalized Mittag-Leffler function in his study [12]. Also, we present numerical scheme for solution of our new equation [22]. In [11], a new partial integro-differential equation with mixed fractional operators has been suggested and numerical simulation for such equation has been presented. In this study, we modify classical derivative and integral operators by fractal-fractional differential and integral operators. Therefore, this study is highly important for opening new doors of investigation in modelling of physical and biological phenomena.

The organization of this study is as follows. In section 2, we give some concepts about fractal-fractional calculus. In section 3, we handle a new fractional integro-differential equation with mixed fractal-fractional operators. We present existence and uniqueness conditions for solution of the suggested equation. In section 4, we construct a new numerical scheme for the considered equation and this numerical method is applicable and convenient in solving such equations. In section 5, we give some examples for different values of fractional order \( \alpha \) and fractal order \( \tau \) and numerical simulation for the newly introduced equation. In section 6, we discuss the results about numerical solution of this equation.

2. Preliminaries. In this section, we present some important definitions and mathematical concepts on the fractal-fractional calculus.

**Definition 1.** Suppose that \( f(t) \) be continuous and fractal differentiable on an open interval \((a,b)\) with order \( \tau \), then the fractal-fractional derivative of \( f(t) \) with order \( \alpha \) in the Riemann-Liouville sense having power law type kernel is given by:

\[ FFP_0 D_{\alpha,\tau}^t (f(t)) = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt^\tau} \int_0^t (t-s)^{m-\alpha-1} f(s) \, ds, \]

where \( m-1 < \alpha, \tau \leq m \in \mathbb{N} \) and

\[ \frac{df(s)}{dt^\tau} = \lim_{t \to s} \frac{f(t) - f(s)}{t^\tau - s^\tau}. \]

**Definition 2.** Suppose that \( f(t) \) be continuous and fractal differentiable on an open interval \((a,b)\) with order \( \tau \), then the fractal-fractional derivative of \( f(t) \) with
order α in the Riemann-Liouville sense having exponentially decaying type kernel is given by:

\[
\text{FFE} D_{0,t}^{\alpha,\tau} (f (t)) = \frac{M (\alpha)}{(1 - \alpha)} \frac{d}{dt} \int_0^t \exp \left[ -\frac{\alpha}{1 - \alpha} (t - s) \right] f (s) \, ds,
\]

where \( \alpha > 0, \tau \leq m \in \mathbb{N} \) and \( M (0) = M (1) = 1 \).

**Definition 3.** Suppose that \( f (t) \) be continuous on an open interval \((a, b)\), then the fractal-fractional integral of \( f (t) \) with order \( \alpha \) having power law type kernel is given by:

\[
\text{FFE} J_{0,t}^{\alpha,\tau} (f (t)) = \frac{\tau}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha - 1} s^{\tau - 1} f (s) \, ds.
\]

**Definition 4.** Suppose that \( f (t) \) be continuous on an open interval \((a, b)\), then the fractal-fractional integral of \( f (t) \) with order \( \alpha \) having exponentially decaying type kernel is given by:

\[
\text{FFE} J_{0,t}^{\alpha,\tau} (f (t)) = \frac{\alpha \tau}{M (\alpha)} \int_0^t s^{\tau - 1} f (s) \, ds + \frac{\tau (1 - \alpha) t^{\tau - 1} f (t)}{M (\alpha)}.
\]

**Definition 5.** Suppose that \( f (t) \) be continuous on opened interval \( I \), the fractal-Laplace transform of order \( \alpha \) is given by:

\[
\text{L} \left( \frac{\alpha}{f (t)} \right) = \int_0^\infty \exp \left( -pt \right) t^{\alpha - 1} f (t) \, dt, \quad \alpha > 0.
\]

3. **Volterra type with fractal-fractional operators.** In this section, we consider new fractal-fractional integro-differential equation

\[
\text{FFE} D_{0,t}^{\alpha,\tau} u (x, t) = f (x, t, u) + \text{FFE} J_t^{\alpha,\tau} [K (x, t, u)].
\]

In this equation, the differential is Caputo fractal-fractional derivative and integral is the Caputo-Fabrizio fractal fractional integral operator. We rewrite above equation such as

\[
\text{FFE} D_{0,t}^{\alpha,\tau} u (x, t) = f (x, t, u) + \frac{\tau t^{\tau - 1} (1 - \alpha)}{M (\alpha)} K (x, t, u)
\]

\[
+ \frac{\tau \alpha}{M (\alpha)} \int_0^t s^{\tau - 1} K (x, s, u) \, ds.
\]

Now by taking Caputo fractal-fractional integral on both side, we have

\[
u (x, t) - u (x, 0) = \frac{\tau}{\Gamma (\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} f (x, s, u) \, ds
\]

\[
+ \frac{\tau t^{\tau - 1} (1 - \alpha)}{M (\alpha)} J_0^\alpha [K (x, t, u)]
\]

\[
+ \frac{\tau \alpha}{M (\alpha)} \int_0^t s^{\tau - 1} C_0^\alpha J_0^\alpha K (x, l, u) \, dl
\]

\[
= \frac{\tau}{\Gamma (\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} f (x, s, u) \, ds
\]

\[
+ \frac{\tau^2 t^{\tau - 1} (1 - \alpha)}{M (\alpha) \Gamma (\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} K (x, s, u) \, ds
\]

\[
+ \frac{\tau^2 \alpha}{M (\alpha) \Gamma (\alpha)} \int_0^t s^{\tau - 1} (t - s)^{2\alpha - 1} K (x, s, u) \, ds.
\]
Now we shall define the mapping $T$ such as:

$$T u = u = u(x, 0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} f(x, s, u) \, ds$$

$$+ \frac{\tau^2 t^{\tau - 1} (1 - \alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} K(x, s, u) \, ds$$

$$+ \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{2\alpha - 1} K(x, s, u) \, ds. \quad (4)$$

If we retake norm on both side equality (4), then we write the following

$$||Tu|| = \left\| u(x, 0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} f(x, s, u) \, ds \right\|$$

$$+ \frac{\tau^2 t^{\tau - 1} (1 - \alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} ||K(x, s, u)|| \, ds$$

$$+ \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{2\alpha - 1} ||K(x, s, u)|| \, ds \right\| \quad (5)$$

and then

$$||Tu|| \leq ||u(x, 0)|| + \frac{\tau^2 t^{\tau - 1} (1 - \alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} ||f(x, s, u)|| \, ds$$

$$+ \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{2\alpha - 1} ||K(x, s, u)|| \, ds. \quad (6)$$

Assume that $K$, $f$ and $u$ are continuous and bounded, we have

$$||u(x, 0)|| \leq M_1$$

$$||f(x, t, u)|| \leq M_2$$

$$||K(x, t, u)|| \leq M_3. \quad (7)$$

Let $s = ty$, then we write the following inequality

$$||Tu|| \leq M_1 + \frac{\tau^2 t^{\tau - 1} M_2 (1 - \alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} B(\tau, \alpha)$$

$$+ \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{2\alpha - 1} B(\tau, \alpha) \quad (8)$$

Thus we have $||Tu|| < M < \infty$. Also, we evaluate

$$||Tu - Tv|| = \left\| u(x, 0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} [f(x, s, u) - f(x, s, v)] \, ds \right\|$$

$$+ \frac{\tau^2 t^{\tau - 1} (1 - \alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{\alpha - 1} [K(x, s, u) - K(x, s, v)] \, ds$$

$$+ \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t - s)^{2\alpha - 1} [K(x, s, u) - K(x, s, v)] \, ds \right\| \quad (9)$$

We assume that the functions $f$ and $K$ are Lipschitz, then we have

$$||f(x, t, u) - f(x, t, v)|| \leq c_1 ||u - v||$$

$$||K(x, t, u) - K(x, t, v)|| \leq c_2 ||u - v||. \quad (10)$$
Thus we can write the following inequality
\[
\|Tu - Tv\| \leq \frac{\tau}{\Gamma(\alpha)} \tau^{\alpha - 3} B(\tau, \alpha) c_1 \|u - v\|
\]
\[
+ \frac{\tau^2 t^{\alpha - 1} (1 - \alpha) M(t) \tau^{\alpha - 3} B(\tau, \alpha) c_2}{M(\alpha) \Gamma(\alpha)} \|u - v\|
\]
\[
+ \frac{\tau}{M(\alpha) \Gamma(\alpha)} \tau^{\alpha - 3} B(\tau, \alpha) c_2 \|u - v\|
\]
and
\[
\|Tu - Tv\| \leq \left( \frac{\tau}{\Gamma(\alpha)} \tau^{\alpha - 3} B(\tau, \alpha) c_1 + \frac{\tau^2 t^{\alpha - 1} (1 - \alpha) M(t) \tau^{\alpha - 3} B(\tau, \alpha) c_2}{M(\alpha) \Gamma(\alpha)} \right) \|u - v\|.
\]
So
\[
\|Tu - Tv\| \leq L \|u - v\|.
\]
Now the contraction is obtained if \(L < 1\).

For the solution of this equation can be written as
\[
\xi_n = u_n(x, t) - u_{n-1}(x, t)
\]
where
\[
u_n(x, t) = \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha - 1} (t - s)^{\alpha - 1} f(x, s, u_{n-1}) \, ds
\]
\[
+ \frac{\tau^2 t^{\alpha - 1} (1 - \alpha) M(t) \tau^{\alpha - 3} B(\tau, \alpha) c_2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha - 1} (t - s)^{\alpha - 1} K(x, s, u_{n-1}) \, ds
\]
\[
+ \frac{\tau}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha - 1} (t - s)^{\alpha - 1} B(\tau, \alpha) c_2 \, ds
\]
such that finally
\[
u_n(x, t) = \sum_{j=0}^n \xi_j.
\]

**Theorem 6.** The equation 1 has a unique solution \(u(x, t)\) within \(0 < \tau < t < T\) under condition that \(K(x, t, u)\) and \(f(x, t, u)\) are integrable.
\[
\|u(x, t)\| < M(x, 0),
\]
\[
\|f(x, t, u)\| < f_1,
\]
\[
\|K(x, t, u)\| < k_1.
\]
Here \(f\) and \(K\) are continuous. Then there exists a function \(U\) such that
\[
\|u(x, t)\| < U(x, t)
\]
where \(U\) is continuous and is the solution of the equation (2), that is
\[
\ddot{u}(x, t) = u(x, 0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha - 1} (t - s)^{\alpha - 1} f(x, s, U) \, ds
\]
\[
+ \frac{\tau^2 t^{\alpha - 1} (1 - \alpha) M(t) \tau^{\alpha - 3} B(\tau, \alpha) c_2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha - 1} (t - s)^{\alpha - 1} K(x, s, U) \, ds
\]
\[
+ \frac{\tau}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha - 1} (t - s)^{\alpha - 1} B(\tau, \alpha) c_2 \, ds.
\]
Proof. We shall rewrite the suggested equation as follows;

\begin{equation}
\begin{aligned}
    u(x, t) &= u(x, 0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} f(x,s,u) \, ds \\
    &+ \frac{\tau^2 t^{\alpha-1}(1-\alpha)}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} K(x,s,u) \, ds \\
    &+ \frac{\tau^2 \alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} K(x,s,u) \, ds.
\end{aligned}
\end{equation}

Thus

\begin{equation}
\begin{aligned}
    \|u(x, t)\| &\leq \|u(x, 0)\| + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} \|f(x,s,u)\| \, ds \\
    &+ \frac{\tau^2 t^{\alpha-1}(1-\alpha)}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} \|K(x,s,u)\| \, ds \\
    &+ \frac{\tau^2 \alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} \|K(x,s,u)\| \, ds.
\end{aligned}
\end{equation}

Now we can easily evaluate the following

\begin{equation}
\begin{aligned}
    u(x, t) - \|u(x, t)\| > \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} f_1(x,s,u-\|u\|) \, ds \\
    &+ \frac{\tau^2 t^{\alpha-1}(1-\alpha)}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} k_1(x,s,-\|u\|) \, ds
    + \frac{\tau^2 \alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} k_1(x,s,u-\|u\|) \, ds.
\end{aligned}
\end{equation}

Since \(f_1\) and \(k_1\) are continuous, then \(u(x, t) - \|u(x, t)\| > 0\). Thus we complete the proof. \(\square\)

**Theorem 7.** Let \(u(x, 0) = G(x, t)\), thus under the condition that \(G(x, t), K(x,t,u), f(x,t,u), \Delta G(x,t), \Delta K(x,t,u)\) and \(\Delta f(x,t,u)\) are continuous and bounded, if \(u_{exact}(x, t)\) is the exact solution of our equation

\begin{equation}
\begin{aligned}
    u_{exact}(x, t) &= G(x, t) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} f(x,s,u) \, ds \\
    &+ \frac{\tau^2 t^{\alpha-1}(1-\alpha)}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} K(x,s,u) \, ds \\
    &+ \frac{\tau^2 \alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} K(x,s,u) \, ds
\end{aligned}
\end{equation}

then

\begin{equation}
\|u_{exact}(x, t) - u(x, t)\| = o(\Delta g_1) + o(\Delta f_1) + o(\Delta k_1).
\end{equation}
where
\[ \|G(x,t)\| < g_1, \quad \|\Delta G(x,t)\| < \Delta g_1, \]
\[ \|K(x,t,u)\| < k_1, \quad \|\Delta K(x,t,u)\| < \Delta k_1, \quad (25) \]
\[ \|f(x,t,u)\| < f_1, \quad \|\Delta f(x,t,u)\| < \Delta f_1. \]

**Proof.** We recall that the Volterra version of our equation is given as
\[
U(x,t) = g(\tau) + \int_0^t s^{\alpha-1} K(x,s,u) ds
\]
Thus after some manipulations, we obtain such as;
\[ \text{Applying fractal-Laplace transform on both side, we have the following} \]
\[ \text{So that} \]
\[ \text{The equation (26) can be rewritten as} \]
\[ \text{Applying fractal-Laplace transform on both side, we have the following} \]
\[ \text{Thus after some manipulations, we obtain such as;} \]
\[
\hat{G}(x,p) = \left(1 - \frac{\tau}{\Gamma(\alpha)p^\alpha} f_1 - \frac{\tau^2 t^{\alpha-1}(1-\alpha)}{M(\alpha)\Gamma(\alpha)p^\alpha k_1} \right) \hat{U}(x,p).
\]
\[
\hat{U}(x,p) = \frac{\hat{G}(x,p)}{p^\alpha + B} \quad (30)
\]
where
\[
A = 1 - \frac{\tau}{\Gamma(\alpha)} f_1 - \frac{\tau^2 t^{\alpha-1}(1-\alpha)}{M(\alpha)\Gamma(\alpha) k_1} \quad (31)
\]
\[ B = -\frac{\tau^2}{M(\alpha)\Gamma(\alpha)} k_1. \]

The function can be obtained as follows
\[ \hat{U}(x,p) = \frac{p^{2\alpha} \hat{G}(x,p)}{Ap^\alpha + B}. \]
and we have
\[ U(x,t) = \int_0^t G(x,\tau) M(x,t-\tau) \, d\tau \]  \hspace{1cm} (33)

where
\[ M(x,t) = L^{-1} \left( \frac{p^{2\alpha}}{Ap^\alpha + B} \right). \]  \hspace{1cm} (34)

Since \( M \) is in exponential form, thus \( \|U(x,t)\| \) is bounded exponentially. Now since \( u_{\text{exact}}(x,t) \) is the exact solution, then
\[ \|U(x,t)\| \leq g \exp \left[ \mu^{-1} t \right]. \]  \hspace{1cm} (35)

The inequality (35) leads us to the following statement
\[ \|u_{\text{exact}}(x,t)\| \leq (g + \Delta g) \exp \left[ \mu^{-1} t \right]. \]  \hspace{1cm} (36)

The function \( u_{\text{exact}}(x,t) \) is approximate solution of our equation, namely we have
\[ u_{\text{exact}}(x,t) = G(x,t) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} f(x,s,u_{\text{exact}}) \, ds \]
\[ + \frac{\tau^2 t^{\alpha-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} K(x,s,u_{\text{exact}}) \, ds \]
\[ + \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} K(x,s,u_{\text{exact}}) \, ds. \]  \hspace{1cm} (37)

We can write the error as follows
\[ R(x,t) = \Delta G(x,t) + \frac{\tau^2 t^{\alpha-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} \Delta f(x,s,u_{\text{exact}}) \, ds \]
\[ + \frac{\tau^2 t^{\alpha-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} \Delta K(x,s,u_{\text{exact}}) \, ds \]
\[ + \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} \Delta K(x,s,u_{\text{exact}}) \, ds. \]  \hspace{1cm} (38)

such that
\[ u - u_{\text{exact}} = R(x,t) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} f(x,\tau,u - u_{\text{exact}}) \, ds \]
\[ + \frac{\tau^2 t^{\alpha-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} K(x,\tau,u - u_{\text{exact}}) \, ds \]
\[ + \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\alpha-1} (t-s)^{2\alpha-1} K(x,\tau,u - u_{\text{exact}}) \, ds. \]  \hspace{1cm} (39)

Now taking the norm on both side, we write the following
\[ \|u(x,t) - u_{\text{exact}}(x,t)\| < \Delta g_1 + \Delta k_1 + \Delta f_1 \]  \hspace{1cm} (40)

with
\[ \max_{t \in [0,T]} \|u(x,t)\| = o(\Delta g_1) + o(\Delta k_1) + o(\Delta f_1). \]  \hspace{1cm} (41)
Now we shall write the following iterative formula

\[
  u_{n+1} (x, t) = G(x, t) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{\alpha-1} f(x, s, u_{n-1}(x, s)) \, ds \\
  + \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{\alpha-1} K(x, s, u_{n-1}(x, s)) \, ds \\
  + \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{2\alpha-1} K(x, s, u_{n-1}(x, s)) \, ds
\]

(42)

where \( u_0(x, t) = G(x, t) \). Also we evaluate

\[
  u_n(x, t) - u_{n-1}(x, t) = \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{\alpha-1} [f(x, s, u_{n-1}) - f(x, s, u_{n-2})] \, ds \\
  + \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{\alpha-1} [K(x, s, u_{n-1}) - K(x, s, u_{n-2})] \, ds \\
  + \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{2\alpha-1} [K(x, s, u_{n-1}) - K(x, s, u_{n-2})] \, ds
\]

(43)

and we write

\[
  \beta_n = u_n - u_{n-1} \implies \sum_{j=0}^n \beta_n = u_n. \tag{44}
\]

Then

\[
  \|u_n - u_{n-1}\| = \|\beta_n\| \leq \frac{\tau}{\Gamma(\alpha)} t^{\tau+\alpha-3} B(\tau, \alpha) \|f(x, t, u_{n-1}) - f(x, t, u_{n-2})\| \\
  + \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} t^{\tau+\alpha-3} B(\tau, \alpha) \|K(x, t, u_{n-1}) - K(x, t, u_{n-2})\| \\
  + \frac{\tau^2 \alpha}{M(\alpha) \Gamma(\alpha)} t^{\tau+2\alpha-3} B(\tau, \alpha) \|K(x, t, u_{n-1}) - K(x, t, u_{n-2})\|. \tag{45}
\]

Using the Lipschitz property of \( f \) and \( K \), then

\[
  \|u_n - u_{n-1}\| \leq \left( \frac{\tau}{\Gamma(\alpha)} t^{\tau+\alpha-3} B(\tau, \alpha) f_1 + \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} t^{\tau+\alpha-3} B(\tau, \alpha) k_1 \right) \|u_{n-1} - u_{n-2}\|. \tag{46}
\]

If we repeat same operations, we have the following If we repeat same operations, we have the following

\[
  \|u_n - u_{n-1}\| \leq \left( \frac{\tau}{\Gamma(\alpha)} t^{\tau+\alpha-3} B(\tau, \alpha) f_1 + \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} t^{\tau+\alpha-3} B(\tau, \alpha) k_1 \right)^n \|u_1 - u_0\|. \tag{47}
\]

When \( f \) and \( K \) are contraction and \( n \to \infty \), we write

\[
  \|u_n - u_{n-1}\| \to 0. \tag{48}
\]
We now prove uniqueness of the solution. We shall take two different solutions such that \(u_1(x, t)\) and \(u_2(x, t)\), then we evaluate

\[
u_1(x, t) - u_2(x, t) = \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau - 1} (t-s)^{\alpha-1} \left[f(x,s,u_1) - f(x,s,u_2)\right] ds
\]

\[+ \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t-s)^{\alpha-1} \left[K(x,s,u_1) - K(x,s,u_2)\right] ds
\]

\[+ \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau - 1} (t-s)^{2\alpha-1} \left[K(x,s,u_1) - K(x,s,u_2)\right] ds
\] (49)

Utilizing the Lipschitz property of \(K\) and \(f\), we have

\[
\|u_1(x,t) - u_2(x,t)\| \leq \frac{\tau}{\Gamma(\alpha)} t^{\tau+\alpha-3} \left|B(\tau,\alpha)\right| f_1 \int_0^t \|u_1 - u_2\| (t-s)^{\alpha-1} ds
\]

\[+ \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} t^{t+\alpha-3} \left|B(\tau,\alpha)\right| k_1 \int_0^t \|u_1 - u_2\| (t-s)^{\alpha-1} ds
\]

\[+ \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} t^{t+2\alpha-3} \left|B(\tau,\alpha)\right| k_1 \int_0^t \|u_1 - u_2\| (t-s)^{2\alpha-1} ds.
\] (50)

Nevertheless \(\|u_1 - u_2\| < \delta\), then

\[
\|u_1(x,t) - u_2(x,t)\| \leq \delta \left( \frac{\tau}{\Gamma(\alpha)} \left(\frac{t^{\tau-\alpha}}{\Gamma(\alpha) \Gamma(n+1)} + t^{\tau+\alpha-3} \left|B(\tau,\alpha)\right| f_1 \int_0^t \|u_1 - u_2\| (t-s)^{\alpha-1} ds
\]

\[+ \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \left(\frac{\tau^2}{\Gamma(\alpha) \Gamma(n+1)} + \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \left|B(\tau,\alpha)\right| k_1 \int_0^t \|u_1 - u_2\| (t-s)^{2\alpha-1} ds \right)
\] (51)

then for \(n \to \infty\), we obtain

\[
u_1(x,t) = u_2(x,t)
\] (52)

which shows that the solution of our equation is unique.

\[\Box\]

4. **Numerical scheme for the suggested equation.** We shall rewrite the suggested equation which is given by;

\[
\frac{F^\alpha}{F_0} D_t^\alpha u(x,t) = f(x,t,u) + \frac{\tau t^{\tau-1} (1-\alpha)}{M(\alpha)} K(x,t,u) + \frac{\tau}{M(\alpha)} \int_0^t s^{\tau-1} K(x,s,u) ds
\] (53)

This equation can be converted as follows;

\[
u(x,t) = u(x,0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{\alpha-1} f(x,s,u) ds
\]

\[+ \frac{\tau^2 t^{\tau-1} (1-\alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{\alpha-1} K(x,s,u) ds
\]

\[+ \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \int_0^t s^{\tau-1} (t-s)^{2\alpha-1} K(x,s,u) ds.
\] (54)
At the point $t_{n+1}$, then we write the equation (54) as follows

\[ u_{n+1}(x,t) = u_0(x,0) + \frac{\tau}{\Gamma(\alpha)} \int_0^{t_{n+1}} s^{\alpha-1} (t_{n+1} - s)^{\alpha-1} f(x,s,u) \, ds \]

\[ + \frac{\tau^2 t^{\alpha-1} (1 - \alpha)}{M(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} s^{\alpha-1} (t_{n+1} - s)^{\alpha-1} K(x,s,u) \, ds \]

\[ + \frac{\tau^2}{M(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} s^{\alpha-1} (t_{n+1} - s)^{2\alpha-1} K(x,s,u) \, ds. \]  

Then we have

\[ u_{n+1} = u_0 + \frac{\tau}{\Gamma(\alpha) + 2} \sum_{j=0}^{n} \left[ t_j^{\alpha-1} f(x_j, t_j, u_j) \right] \]

\[ + \frac{\tau^2 t^{\alpha-1} (1 - \alpha)}{M(\alpha) \Gamma(\alpha) + 2} \sum_{j=0}^{n} \left[ t_j^{\alpha-1} K(x_j, t_j, u_j) \right] \]

\[ + \frac{\tau^2}{M(\alpha) \Gamma(\alpha) + 2} \sum_{j=0}^{n} \left[ t_j^{\alpha-1} K(x_j, t_j, u_j) \right]. \]  

By using Lagrangian piecewise interpolation, we obtain the following numerical scheme

\[ u_{n+1} = u_0 + \frac{\tau (\Delta t)^{\alpha}}{\Gamma(\alpha) + 2} \sum_{j=0}^{n} \left[ t_j^{\alpha-1} f(x_j, t_j, u_j) \right] \]

\[ + \frac{\tau^2 t^{\alpha-1} (1 - \alpha) (\Delta t)^{\alpha}}{M(\alpha) \Gamma(\alpha) + 2} \sum_{j=0}^{n} \left[ t_j^{\alpha-1} K(x_j, t_j, u_j) \right] \]

\[ + \frac{\tau^2}{M(\alpha) \Gamma(\alpha) + 2} \sum_{j=0}^{n} \left[ t_j^{\alpha-1} K(x_j, t_j, u_j) \right]. \]  

5. Conclusion. Integro-differential equations involving both differential and integral operators plays an important role in modelling real world problems. Many studies related with integro-differential equations can be found in the literature, even recently several studies about fractional integro-differential equations have been done. Many theory about such equations have been established and also they have been solved numerically with different methods. Therefore, we need to have newly created equations in order to lead new studies. For this reason, we offered new class of integro-differential equations with mixed operators in this study.

In this paper, we deal with newly introduced fractional integro-differential equation which contains Caputo fractal-fractional derivative and Caputo Fabrizio fractal-fractional integral. Also we present necessary conditions about existence and uniqueness for solution of this mathematical equation. We finalize using the newly introduced numerical scheme for the suggested equation for different values of fractional order $\alpha$ and fractal order $\tau$. We can see that this numerical scheme is useful and practicable for solving such integro-differential equations. In this paper, we can
conclude that fractal-fractional differential operators are suitable and applicable mathematical tools to model real world problems.

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