CANONICAL EXTENSIONS OF LOCAL SYSTEMS

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Abstract. A local system on a complex manifold $M$ can be viewed in two
ways—either as a locally free sheaf $\mathcal{H}$ on $M$, or as a union of covering spaces
$T(\mathcal{H}) \to M$. When $M$ is an open set in a bigger manifold $\overline{M}$, the local
system will generally not extend to $\overline{M}$, because of local monodromy. This
paper proposes an extension of the local system as an analytic space over $\overline{M}$,
in the case when $\overline{M} \setminus M$ has normal crossing singularities, and the local system
is unipotent along $\overline{M} \setminus M$. The analytic space is obtained by taking the closure
of $T(\mathcal{H})$ inside the total space of Deligne’s canonical extension of the vector
bundle $\mathcal{O}_M \otimes \mathcal{H}$ to $\overline{M}$. It is not normal, but its normalization is locally toric.

1. Introduction

In his book about differential equations with regular singular points [1], Deligne
introduced the so-called “canonical extension” of a vector bundle with flat connec-
tion. The most important case of his construction is the following. Say $V$ is a
holomorphic vector bundle on a complex manifold $M$, equipped with a flat connec-
tion $\nabla$ (which means that $\nabla \circ \nabla = 0$). Now suppose that $M$ is an open subset in
a bigger complex manifold $\overline{M}$, in such a way that

(1) the complement $\overline{M} \setminus M$ is a divisor with normal crossing singularities, and
(2) the local monodromy of $\nabla$ near points of $\overline{M} \setminus M$ is unipotent.

In this situation, Deligne shows that $V$ extends in a canonical manner to a vector
bundle $\mathcal{V}$ on $\overline{M}$, whose characteristic property is that in any local frame for $\mathcal{V}$ near
points of $\overline{M} \setminus M$, the connection matrix for $\nabla$ has at worst logarithmic poles with
nilpotent residues.

Local systems with integer coefficients are one source for flat vector bundles; if $\mathcal{H}$
is a local system of (finitely generated, free) abelian groups, then $V = \mathcal{O}_M \otimes \mathcal{H}$,
together with the natural connection, is such a bundle. So if $\mathcal{H}$ is unipotent
along $\overline{M} \setminus M$, i.e., has unipotent monodromy near points of $\overline{M} \setminus M$, then Deligne’s
construction applies, and there is a canonical extension $\mathcal{V}$ for the vector bundle. It
is then natural to ask:

Question. Does the original local system also extend in some way?

The present paper proposes an answer to this question. It should be clear that,
because of monodromy, $\mathcal{H}$ cannot in general be extended to $\overline{M}$ as a local system.
We return therefore to the more old-fashioned view of a local system as a space,
instead of as a sheaf—the total space $T = T(\mathcal{H})$ is a covering space of $M$, typically
infinite-sheeted and with countably many components. Now $T$ is naturally embed-
ded into the total space of the vector bundle $\mathcal{V}$, and we shall answer the question.

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The construction of $\mathcal{V}$, in local coordinates, is reviewed in the appendix.
posed above by determining its closure $\mathcal{T}$ inside the total space of the canonical extension $\mathcal{V}$.

Perhaps surprisingly, this closure is still an analytic space with good properties. For instance, while $\mathcal{T}$ need not be normal, its normalization is locally toric, and thus has controlled singularities, in a sense. This, and other things, will follow from the explicit local equations for $\mathcal{T}$ that are given in Section 5.

As discussed in Section 7, the space $\mathcal{T}$ also has a sort of universal property that might justify calling $\mathcal{T}$—or perhaps its normalization—the “canonical extension” of the local system $\mathcal{H}$. It is in this sense that the title of the paper should be understood.

A precise description of $\mathcal{T}$ will be given below, so only two small issues need be addressed here. One is that points of $\mathcal{T}\setminus\mathcal{T}$ arise from monodromy-invariant points in $\mathcal{H}$. This is what one would expect; indeed, $\mathcal{T}$ is also “canonical” in the sense that it includes all points that are invariant under any part of the local monodromy near $M\setminus M$. The other, perhaps more surprising issue is that, even though $\mathcal{T}$ itself is discrete over $M$, the fibers of $\mathcal{T}$ over $M$ need not be discrete. In fact, they are unions of affine spaces, of dimension possibly as big as $\dim M - 1$.

Note. The construction of $\mathcal{T}$ arose in a project that Herb Clemens and myself are working on. Hoping that the result might be of independent interest, I am presenting it by itself, without any of the original context. I warmly thank Herb Clemens for many useful discussions and for his continuing help.

2. Statement of the result

This section gives a precise statement of the main result. Let $M$ be a complex manifold of dimension $n$, embedded as an open subset into a larger complex manifold $\overline{M}$, in such a way that $\overline{M}\setminus M$ is a divisor with only normal crossing singularities. Every point in $\overline{M}$ thus has a neighborhood isomorphic to $\Delta^n$, with holomorphic coordinates $t_1, \ldots, t_n$, in which the divisor $\overline{M}\setminus M$ is defined by an equation of the form $t_1 \cdots t_r = 0$.

On $M$, we assume that we are given a local system $\mathcal{H}$, with fiber $H \cong \mathbb{Z}^d$ a finitely generated free $\mathbb{Z}$-module. Up to isomorphism, it is determined by the corresponding monodromy representation

$$\rho: G \to \text{Aut}_\mathbb{Z}(H),$$

where $G$ is the fundamental group of $M$ (for some choice of basepoint).

We shall assume that the local system $\mathcal{H}$ is unipotent; that is to say, in a neighborhood $\Delta^n$ of each point, the fundamental group of $\Delta^n \cap M$ should act by unipotent transformations on the fiber of $\mathcal{H}$. It is then possible to extend the holomorphic vector bundle $\mathcal{V} = \mathcal{H} \otimes \mathcal{O}_M$ to a vector bundle $\overline{\mathcal{V}}$ on $\overline{M}$, using Deligne’s construction.

The total space $T = T(\mathcal{H})$ of the local system is naturally a subset of the total space $T(\overline{\mathcal{V}})$ of this canonical extension. We are going to extend $T$ in the maximal possible way, by taking its closure inside the total space of the vector bundle. The resulting space is surprisingly nice, as witnessed by the following theorem.

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2Here, and in the following, $\Delta \subseteq \mathbb{C}$ denotes the open unit disk in the complex plane, and $\Delta^* = \Delta \setminus \{0\}$ the punctured unit disk.
Theorem 1. Let $\overline{T}$ be the closure of the total space $T$ of the local system, taken inside the total space of the canonical extension of $H \otimes O_M$ over $\overline{M}$. Then $\overline{T}$ has the following three properties:

(i) $\overline{T}$ is a reduced analytic subset of $T(\overline{V})$.

(ii) The projection map $p: \overline{T} \rightarrow \overline{M}$ is holomorphic, and $p^{-1}(M) = T$.

(iii) The normalization of $\overline{T}$ is locally toric.

Proof. To show that $\overline{T}$ is an analytic subset, we need to show that it is locally defined by analytic equations inside the complex manifold $T(\overline{V})$. For an arbitrary point of $M$, take a neighborhood isomorphic to $\Delta^n$ in which $M \setminus M$ is defined by the equation $t_1 \cdots t_r = 0$. Then Proposition 6 in Section 5 below states precisely that the closure of $\overline{T}$ over $\Delta^n$ is a reduced analytic subset, and this establishes (i).

The assertion in (ii) that $p^{-1}(M) = T$ follows from the corresponding statement over each coordinate neighborhood $\Delta^n$, also proved below (see the discussion at the end of Section 4). Finally, the statement about the normalization of $\overline{T}$ may be found in Section 6. □

3. The local situation

Determining the closure $\overline{T}$ inside the total space of the canonical extension is really a local problem, and we may restrict our attention to what happens in a small polydisk neighborhood of a point $P \in \overline{M}$. Let $\Delta^n \subseteq \overline{M}$ be such a neighborhood, with local holomorphic coordinates $t_1, \ldots, t_n$ centered at the point $P$ in question. We assume that, in these coordinates, $\overline{M} \setminus M$ is defined by the equation $t_1 \cdots t_r = 0$. When $P$ is a boundary point, we get $r > 0$, but the case of a point in $M$ is included by taking $r = 0$. In any case, we have $(\Delta^n)^n \subseteq M$.

To avoid having to treat various cases based on the value of $r$, we will restrict the local system $H$ to the set $(\Delta^n)^n$, and compute the closure of only this piece inside the total space of the canonical extension over $\Delta^n$. We shall argue later, at the end of Section 4, that the result is the same.

The fundamental group of $(\Delta^n)^n$ is isomorphic to $\mathbb{Z}^n$. Let $H \cong \mathbb{Z}^d$ be the fiber of the local system at some point in $(\Delta^n)^n$; by assumption, the monodromy action of $\mathbb{Z}^n$ on $\mathbb{Z}^d$ is by unipotent matrices. Let $T_j \in \text{Aut}_\mathbb{Z}(\mathbb{Z}^d)$ be the matrix corresponding to the $j$-th standard generator of $\mathbb{Z}^n$, and put

$$ N_j = -\log T_j = \sum_{n=1}^{\infty} \frac{1}{n}(\text{id} - T_j)^n. $$

This is well-defined because $(\text{id} - T_j)^n = 0$ for large values of $n$. The matrices $N_j$ are nilpotent, with rational entries, and commute with one another.

In the given system of coordinates, we now describe how the local system is embedded into the total space of the canonical extension. To begin with,

$$ \mathbb{H}^n \rightarrow (\Delta^n)^n, \quad (z_1, \ldots, z_n) \mapsto \left(e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}\right), $$

is the universal covering space of $(\Delta^n)^n$. The canonical extension of $H \otimes O_{(\Delta^n)^n}$ over $\Delta^n$ is isomorphic to the trivial vector bundle

$$ O_{\Delta^n} s_1 \oplus \cdots \oplus O_{\Delta^n} s_d; $$

here $s_i$ is the section of $H \otimes O_{(\Delta^n)^n}$ on $(\Delta^n)^n$ whose pullback to $\mathbb{H}^n$ is given by the map

$$ \tilde{s}_i: \mathbb{H}^n \rightarrow \mathbb{Z}^d, \quad \tilde{s}_i(z) = e^{z_1 N_1 + \cdots + z_n N_n} e_i, $$
being one of the standard basis elements of \( Z^d \) (see the appendix for details). The total space of the canonical extension is thus isomorphic to \( \Delta^n \times \mathbb{C}^d \), using this frame.

When the local system is pulled back to the universal covering space \( \mathbb{H}^n \), it becomes of course trivial. At any given point \( z = (z_1, \ldots, z_n) \) of \( \mathbb{H}^n \), a class \( h \in \mathbb{Z}^d \) in the fiber of the trivial local system has coordinates \( e^{-(z_1 N_1 + \cdots + z_n N_n)} h \) with respect to the given framing for the canonical extension. It follows that the point \((z, h) \in \mathbb{H}^n \times \mathbb{Z}^d \) has coordinates \((e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \cdots + z_n N_n)} h)\) in \( \Delta^n \times \mathbb{C}^d \). The total space of the local system, when embedded into that of the canonical extension, is thus the image of the holomorphic map

\[
f : \mathbb{H}^n \times \mathbb{Z}^d \to \Delta^n \times \mathbb{C}^d,
\]

defined by the rule

\[
(1) \quad (z_1, \ldots, z_n, h) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \cdots + z_n N_n)} h).
\]

The closure of this image will be computed in the following section.

4. Description of the closure

We now determine which points inside the total space of the canonical extension belong to the closure of the local system. As explained in the previous section, this is a local question; we chose a neighborhood \( \Delta^n \subseteq \overline{M} \) of an arbitrary point \( P \in \overline{M} \), and study the closure over that neighborhood.

According to the description above, the total space of the local system over \((\Delta^*)^n\) is the image of the holomorphic map \( f \) given in (1). As it stands, that map is not one-to-one; when the real parts \( x_j = \text{Re} z_j \) are restricted to \( 0 \leq x_1, \ldots, x_n < 1 \), however, every point in the image is parametrized only once.

The remainder of this section is devoted to proving the following proposition, which describes the points in the closure of the image of the map \( f \). As written, it only makes a statement about points that lie over the origin in \( \Delta^n \), which is to say over the point \( P \in \overline{M} \). But as we are free to place \( P \) wherever we please, we really get a description of all the points in the closure.

**Proposition 2.** A point in \( \Delta^n \times \mathbb{C}^d \) over \((0, \ldots, 0) \in \Delta^n \) is in the closure of the image of \( f \) if, and only if, it is of the form \((0, \ldots, 0, e^{-(w_1 N_1 + \cdots + w_n N_n)} h)\); here \( h \in \mathbb{Z}^d \) is such that \( a_1 N_1 h + \cdots + a_n N_n h = 0 \) for certain positive integers \( a_1, \ldots, a_n \), while \( w_1, \ldots, w_n \in \mathbb{C} \) can be arbitrary complex numbers.

For each limit point, there is an arc (for suitably small \( \varepsilon > 0 \))

\[
\Delta(\varepsilon) \to \Delta^n \times \mathbb{C}^d,
\]

of the form

\[
t \mapsto (t^{a_1} e^{2\pi i w_1}, \ldots, t^{a_n} e^{2\pi i w_n}, e^{-(w_1 N_1 + \cdots + w_n N_n)} h),
\]

contained in the image of \( f \) for \( t \neq 0 \), and passing through the limit point at \( t = 0 \).
One half of this is easy to prove—if \( h \in \mathbb{Z}^d \) satisfies \( a_1 N_1 h + \cdots + a_n N_n h = 0 \) for positive integers \( a_1, \ldots, a_n \), then every point of the form
\[
(0, \ldots, 0, e^{-(w_1 N_1 + \cdots + w_n N_n)} h)
\]
is in the closure of the image of \( f \). Indeed, taking the imaginary part of \( z \in \mathbb{H} \) sufficiently large to have \( \text{Im}(a_j z + w_j) > 0 \) for all \( j \), we get
\[
f(a_1 z + w_1, \ldots, a_n z + w_n, h) = (e^{2 \pi i a_1 z} e^{2 \pi i w_1}, \ldots, e^{2 \pi i a_n z} e^{2 \pi i w_n}, e^{-\sum (a_j z + w_j) N_j} h)
\]
\[
= (t_1 e^{2 \pi i w_1}, \ldots, t^n e^{2 \pi i w_n}, e^{-\sum w_j N_j} e^{-z \sum a_j N_j} h)
\]
having set \( t = \exp(2 \pi i z) \). For \( t \neq 0 \), these points are all in the image of \( f \); as \( t \to 0 \), in other words, as \( \text{Im} z \to \infty \), they approach the point \( (0, \ldots, 0, e^{-(w_1 N_1 + \cdots + w_n N_n)} h) \), which is consequently in the closure.

To prove the converse, we take a sequence of points in the image that converges to some point of \( \{(0, \ldots, 0)\} \times \mathbb{C}^d \), and show that its limit is of the stated form. So let
\[
(z(m), h(m)) = (z_1(m), \ldots, z_n(m), h(m)) \in \mathbb{H}^n \times \mathbb{Z}^d
\]
be a sequence of points such that \( f(z(m), h(m)) \) converges to a point over \( (0, \ldots, 0) \). This means that each \( y_j(m) = \text{Im} z_j(m) \) is going to infinity, and that the sequence of vectors
\[
e^{-\sum z_j(m) N_j} h(m) \in \mathbb{C}^d
\]
is convergent as \( m \to \infty \). Changing the values of \( h(m) \), if necessary, we may in addition assume that the real parts \( x_j(m) = \text{Re} z_j(m) \) satisfy \( 0 \leq x_j(m) \leq 1 \).

In the course of the argument, we shall frequently have to pass to a subsequence of \( (z(m), h(m)) \). To avoid clutter, this will not be indicated in the notation—in each case, the subsequence will be denoted by the same letters \( (z(m), h(m)) \) as the original sequence. Since it should not lead to any confusion, we shall avail ourselves of this convenient device.

Keeping this convention in mind, we now proceed in several steps.

**Step 1.** The sequence of real parts \( x_j(m) \) is bounded, for each \( j = 1, \ldots, n \), and we can thus pass to a subsequence where each \( x_j(m) \) converges. The vectors
\[
e^{\sum x_j(m) N_j} e^{-\sum z_j(m) N_j} h(m) = e^{-i \sum y_j(m) N_j} h(m)
\]
still form a convergent sequence in this case, and so the \( x_j(m) \) really play no role for the remainder of the argument.

**Step 2.** While all imaginary parts \( y_j(m) \) are going to infinity, this may happen at greatly different rates. To make their behavior more tractable, we use the following technique, borrowed from the paper by Cattani, Deligne, and Kaplan [1, p. 494]. Let \( y(m) = (y_1(m), \ldots, y_n(m)) \). By taking a further subsequence, we can arrange that
\[
y(m) = \tau_1(m) \theta^1 + \cdots + \tau_r(m) \theta^r + \eta(m),
\]
where \( \theta^1, \ldots, \theta^r \in \mathbb{R}^n \) are constant vectors with nonnegative components, and where the ratios
\[
(2) \quad \frac{\tau_1(m)}{\tau_2(m)}, \ldots, \frac{\tau_{r-1}(m)}{\tau_r(m)}, \frac{\tau_r(m)}{}
\]
are all going to infinity. The remainder term $\eta(m)$, on the other hand, is convergent. We can even assume that

$$0 \leq \theta^1_j \leq \theta^2_j \leq \cdots \leq \theta^r_j$$

for all $j$, because $y_j(m) \to \infty$, all components of the last vector $\theta^r$ have to be positive real numbers.

Now define

$$N(m) = \sum_{j=1}^n (y_j(m) - \eta_j(m)) N_j.$$

As in Step 1, the convergence of the expression $e^{-i\sum \eta_j(m) N_j}$ makes the $\eta_j$ essentially irrelevant to the rest of the argument—the sequence $e^{-iN(m)} h(m)$ is still a convergent sequence.

**Step 3.** For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$N^\alpha = \prod_{j=1}^n N_{\alpha_j}^j.$$

Since the $N_j$ are commuting nilpotent operators, $N^\alpha = 0$ whenever $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is sufficiently large.

We can thus let $p \geq 0$ be the smallest integer for which there is a subsequence of $(z(m), h(m))$ with

$$N^\alpha h(m) = 0 \text{ for all multi-indices } \alpha \text{ with } |\alpha| \geq p + 1.$$

Passing to this subsequence, we find that when $|\alpha| = p$, the sequence

$$N^\alpha e^{-i\sum y_j(m) N_j} h(m) = N^\alpha h(m)$$

is convergent. However, it takes its values in a discrete set (in fact, there is an integer $M > 0$ such that each coordinate of $N^\alpha h(m)$ is in $\mathbb{Z}[1/M]$, and $M$ depends only on $\alpha$ and the $N_j$), and so it has to be eventually constant. If we remove finitely many terms from the sequence, we can therefore achieve that

$$h^\alpha = \text{def } N^\alpha h(m)$$

is constant whenever $|\alpha| = p$. Moreover, we have $N(m) h^\alpha = 0$ by the choice of $p$.

**Step 4.** At this point, we can use an inductive argument to get the conclusion of Step 3 for all multi-indices $\alpha$ with $|\alpha| \leq p$. Thus let us assume that we already have a subsequence $(z(m), h(m))$ for which $h^\alpha = N^\alpha h(m)$ is constant and $N(m) h^\alpha = 0$, whenever $\alpha$ is a multi-index with $p' \leq |\alpha| \leq p$. If $p' > 0$, we now show how to get the same statement with $p'$ replaced by $p' - 1$.

Consider a multi-index $\alpha$ with $|\alpha| = p' - 1$. Then

$$N^\alpha e^{-iN(m)} h(m) = N^\alpha h(m) - iN(m) N^\alpha h(m) + \sum_{s=1}^{p-p'} (-i)^{s+1} N(m)^s \cdot N(m) N^\alpha h(m)$$

(3)
is again convergent. Since $\alpha + e_j$ has length $p'$, we see that

$$N(m)N^\alpha h(m) = \sum_{j=1}^{n} (y_j(m) - \eta_j(m))N^{\alpha + e_j}h(m) = \sum_{j=1}^{n} (y_j(m) - \eta_j(m))h^{\alpha + e_j};$$

by the inductive hypothesis, the last term in (3) is actually zero.

Thus the sequence $N^\alpha h(m) - iN(m)N^\alpha h(m)$ is itself convergent, implying convergence of its real and imaginary parts separately. As before, the sequence of real parts $N^\alpha h(m)$ has to be eventually constant, and after omitting finitely many terms, we can assume that it is constant. Let

$$h^\alpha \overset{\text{def}}{=} N^\alpha h(m)$$

be that constant value. Then the convergence of the imaginary part

$$N(m)h^\alpha = N(m)N^\alpha h(m) = \sum_{i=1}^{r} \tau_i(m) \sum_{j=1}^{n} \theta_j^i N^{\alpha + \epsilon_j}h(m) = \sum_{i=1}^{r} \tau_i(m) \sum_{j=1}^{n} \theta_j^i h^{\alpha + \epsilon_j},$$

together with the behavior of the $\tau_i(m)$ described in (2), shows that

$$\sum_{j=1}^{n} \theta_j^i h^{\alpha + \epsilon_j} = 0$$

for all $i$. But this says that, in fact, $N(m)h^\alpha = 0$. The statement is thus proved for all multi-indices $\alpha$ of length $|\alpha| = p' - 1$ as well.

**Step 5.** From Step 4, we conclude that, on a suitable subsequence, $h^\alpha = N^\alpha h(m)$ is constant for all $\alpha$, and satisfies $N(m)h^\alpha = 0$. In particular, $h(m)$ is itself constant, equal to a certain element $h = h(0,...,0) \in \mathbb{Z}^d$. Moreover, we have $N(m)h = 0$ for all $m$.

On the one hand, we now find that, along the subsequence we have chosen in the previous steps, our original convergent sequence simplifies to

$$e^{-\sum z_j(m)N_j h(m)} = e^{-\sum (x_j(m) + i\eta_j(m))N_j h} = e^{-\sum (x_j(m) + i\eta_j(m))N_j h}.$$

If we set $w_j = \lim_{m \to -\infty} (x_j(m) + i\eta_j(m))$, then the limit of the sequence is of the form $e^{-\sum w_j N_j h}$, which was part of the assertion in Proposition 2.

On the other hand, we conclude from

$$N(m)h = \sum_{i=1}^{r} \tau_i(m) \sum_{j=1}^{n} \theta_j^i N_j h = 0$$

that

$$\sum_{j=1}^{n} \theta_j^i N_j h = 0$$

for all $i = 1, \ldots, r$. 
Step 6. By Step 5, we know that the $n$ vectors $N_j h$ are linearly dependent; the coefficients $\theta_r^j$ in the relation (for $i = r$) are positive real numbers. But as the vectors themselves are in fact in $\mathbb{Q}^d$, we can also find a relation with positive rational coefficients. Taking a suitable multiple, we then obtain positive integers $a_1, \ldots, a_n$ satisfying
\[ \sum_{j=1}^{n} a_j N_j h = 0. \]

The remaining assertion of the proposition is thereby established, and this finishes the proof.

For later use, we record the result of the six steps in the following proposition.

Proposition 3. Let $(z(m), h(m)) \in \mathbb{H}^n \times \mathbb{Z}^d$ be a sequence of points with $x_j(m) = \text{Re} z_j(m) \in [0, 1]$, and assume that $f(z(m), h(m))$ converges to a point in $\Delta^n \times \mathbb{C}^d$ over $(0, \ldots, 0) \in \Delta^n$. Then there is a subsequence, still denoted $(z(m), h(m))$, for which $h(m)$ is constant.

A technical point. For the sake of convenience, we had restricted the local system from the open set $U = \Delta^n \cap M$—whose complement in $\Delta^n$ is the normal crossing divisor with equation $t_1 \cdots t_r = 0$—to $(\Delta^*)^n$, and computed the closure of only this smaller piece. We now have to argue that this makes no difference.

So let $T_U \subseteq U \times \mathbb{C}^d$ be the total space of the local system over $U$; each connected component of $T_U$ is a covering space of $U$, and therefore the subset $T_U \cap ((\Delta^*)^n \times \mathbb{C}^d)$ is itself dense in $T_U$. But from this circumstance, it follows immediately that it has the same closure in $\Delta^n \times \mathbb{C}^d$ as $T_U$. Since $T_U$ is already closed inside of $U \times \mathbb{C}^d$, we see in particular that all points of $\overline{T_U} \setminus T_U$ have to lie over the boundary divisor $t_1 \cdots t_r = 0$.

(This last fact can also be inferred from Proposition 2. For if we take $P$ to be a point in $M$, the local system is already defined at $P$, and trivial on a neighborhood $\Delta^n$ of that point. Consequently, the local monodromy over $(\Delta^*)^n$ is also trivial, and Proposition 2 shows that taking the closure is only adding back a copy of $\mathbb{Z}^d$ over $P$. This means that we get back the original fiber $\mathcal{F}_P$ of the local system.)

5. Local equations

Now that we know which points are in the closure, we need to show that $\overline{T}$ is an analytic space. We shall do this by finding explicit local equations, over the same coordinate neighborhoods that were used in the previous section. Consequently, $\Delta^n \subseteq \overline{T}$ will continue to denote a neighborhood of an arbitrary point in $\overline{T}$, with local holomorphic coordinates $t_1, \ldots, t_n$.

It has already been pointed out that the total space of the local system over $M$ has countably many connected components. Locally, over the much smaller open set $\Delta^n$, those components may break up even further. The map
\[ f: \mathbb{H}^n \times \mathbb{Z}^d \to \Delta^n \times \mathbb{C}^d, \]
parametrizing the total space of the local system over $\Delta^n$, was defined above by the rule
\[ (z_1, \ldots, z_n, h) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}, e^{-(z_1 N_1 + \cdots + z_n N_n)} h). \]

If we take any element $h \in H = \mathbb{Z}^d$, the image of
\[ f(h): \mathbb{H}^n \to \Delta^n \times \mathbb{C}^d, \]
denoted by \( C(h) \), is one of the local connected components of the total space. Obviously, two such components \( C(h_0) \) and \( C(h_1) \) are either the same (which is the case if \( h_0 \) and \( h_1 \) are in the same \( \mathbb{Z}^n \)-orbit), or disjoint; this means that, typically, each component is equal to \( C(h) \) for infinitely many \( h \in H \).

In the previous section, we have described the closure of \( \bigcup_{h \in H} C(h) \) inside of \( \Delta^n \times \mathbb{C}^d \), but only as a set. To show that this closure is actually an analytic space, we need to give holomorphic equations that define it inside \( \Delta^n \times \mathbb{C}^d \). We first observe that, as a matter of fact,

\[
\bigcup_{h \in H} C(h) = \bigcup_{h \in H} \overline{C(h)}.
\]

This is because of the description of the closure given in Proposition 2—any point in the closure is already in the closure of one of the components \( C(h) \).

Next, as would be expected if the closure is an analytic space, only finitely many of the \( C(h) \) can come together at the boundary. This is expressed in the following lemma.

**Lemma 4.** At most finitely many distinct \( \overline{C(h)} \) can meet at any given point in \( \Delta^n \times \mathbb{C}^d \).

**Proof.** Suppose, to the contrary, that infinitely many distinct \( \overline{C(h)} \) met at a certain point \( Q \) of the closure. Such a point \( Q \) necessarily lies in the closure of infinitely many distinct sheets \( C(h) \). Moving the center \( P \) of the coordinate system, if necessary, we may assume that \( Q \) is a point over \((0, \ldots, 0) \) \( \in \Delta^n \). We can then find a sequence of points \((z(m), h(m)) \in \mathbb{H}^n \times \mathbb{Z}^d \), with \( 0 \leq \text{Re} z_j(m) \leq 1 \) for all \( j = 1, \ldots, n \), such that \( f(z(m), h(m)) \) converges to \( Q \), but all \( h(m) \) are distinct. But such a sequence cannot exist by Proposition 3. This contradiction proves that the number of components meeting at \( Q \) is indeed finite. \( \square \)

We are now ready to determine equations for each of the closed subsets \( \overline{C(h)} \) of \( \Delta^n \times \mathbb{C}^d \). Let \( h \in H \) be an arbitrary element; we will give finitely many holomorphic equations defining \( \overline{C(h)} \). As before, we break the argument down into several steps.

**Step 1.** According to Proposition 2 we get additional points in the closure \( \overline{C(h)} \) only when \( h \) is invariant under some part of the monodromy action. Thus we let \( S \subseteq \mathbb{Z}^n \) be the subgroup of elements that leave \( h \) invariant. As a subgroup of a free group, \( S \) is itself free, say of rank \( n - k \). If \( k = n \), then \( C(h) \) is already closed; only when \( k < n \) is the closure \( \overline{C(h)} \) strictly bigger than \( C(h) \). Since the first case is essentially trivial, we shall assume from now on that \( k < n \).

**Step 2.** The quotient \( \mathbb{Z}^n / S \) is a free abelian group.

**Proof.** Since \( \mathbb{Z}^n \) acts on \( H = \mathbb{Z}^d \) by unipotent transformations, we have that

\[
T_1^{a_1} \cdots T_n^{a_n} h = h \quad \text{if, and only if,} \quad a_1 N_1 h + \cdots + a_n N_n h = 0.
\]

This means that \( S \) is the kernel of the homomorphism

\[
\mathbb{Z}^n \to \mathbb{Q}^d, \quad (a_1, \ldots, a_n) \mapsto a_1 N_1 h + \cdots + a_n N_n h,
\]
and so the quotient $\mathbb{Z}^n/S$ embeds into $\mathbb{Q}^d$. Since $\mathbb{Q}^d$ is torsion-free, the same has to be true for $\mathbb{Z}^n/S$; being finitely generated, $\mathbb{Z}^n/S$ is then actually free. \[ \square \]

**Step 3.** Because of Step 2, we can now find an $n \times n$ matrix $A$, with integer entries and $\det A = 1$, whose last $n - k$ columns give a basis for the subgroup $S$. We then introduce new coordinates $(w_1, \ldots, w_n) \in \mathbb{C}^n$ by the rule

(4) \[ z_i = \sum_{j=1}^{n} a_{i,j} w_j. \]

Rewriting $z_1 N_1 + \cdots + z_n N_n$ in the form $w_1 M_1 + \cdots + w_n M_n$, where each

\[ M_j = \sum_{i=1}^{n} a_{i,j} N_i, \]

is still nilpotent, we now have $M_{k+1} h = \cdots = M_k h = 0$, while the remaining $k$ vectors $M_1 h, \ldots, M_k h$ are linearly independent. Instead of $f$, we can then use the parametrization

(5) \[ g: V \to \Delta^n \times \mathbb{C}^d, \quad (w_1, \ldots, w_n) \mapsto (t_1, \ldots, t_n, e^{-(w_1 M_1 + \cdots + w_k M_k)} h), \]

of the sheet $C(h)$ under consideration; here

\[ t_j = \prod_{s=1}^{n} e^{2\pi i a_{j,s} w_s}, \]

and the map $g$ is defined on the open subset $V \subseteq \mathbb{C}^n$ where all $|t_j| < 1$.

**Step 4.** We now analyze the term $e^{-(w_1 M_1 + \cdots + w_k M_k)} h$ in the parametrization $g$. As a matter of fact, the map

\[ \mathbb{C}^k \to \mathbb{C}^d, \quad (w_1, \ldots, w_k) \mapsto e^{-(w_1 M_1 + \cdots + w_k M_k)} h, \]

is a closed embedding, because the vectors $M_1 h, \ldots, M_k h$ are linearly independent. We will prove this by constructing an inverse—we show that there are polynomials $p_1(v), \ldots, p_k(v)$ in $v = (v_1, \ldots, v_d)$, such that whenever $v$ is in the image, one has

\[ (w_1, \ldots, w_k) = (p_1(v), \ldots, p_k(v)). \]

**Proof.** We construct suitable polynomials by induction on the number $k$ of variables. If $k = 0$, there is nothing to do. So let us assume that the existence of such polynomials is known for $k - 1 \geq 0$ variables, and let us establish it for $k$.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we write

\[ M^\alpha = M_1^{\alpha_1} \cdots M_k^{\alpha_k}; \]

these matrices are zero whenever $|\alpha|$ is sufficiently large. Among all multi-indices $\alpha$ with $M^\alpha h \neq 0$, select one of maximal length $|\alpha|$. Then $|\alpha| \geq 1$, because the vectors $M_j h$ are in particular nonzero, and without loss of generality we may assume that $\alpha_k \geq 1$. We have

\[ M^{\alpha - \epsilon_k v} = (\text{id} - w_1 M_1 - \cdots - w_{k-1} M_{k-1}) M^{\alpha - \epsilon_k} h - w_k M^\alpha h. \]

Because at least one of the components of $M^\alpha h$ is non-zero, we can now solve for $w_k$ in the form

\[ w_k = c_1 w_1 + \cdots + c_{k-1} w_{k-1} + l(v), \]
with \( c_1, \ldots, c_{k-1} \in \mathbb{Q} \), and \( l(v) \) a degree-one polynomial in \( v \). Substituting back, we obtain
\[ e^{l(v)M_k} v = e^{-w_1(M_1+c_1M_k)-\cdots-w_{k-1}(M_{k-1}+c_{k-1}M_k)} h, \]
and, by the inductive hypothesis, \( w_1, \ldots, w_{k-1} \) can now be expressed as polynomials in the coordinates of the vector \( e^{l(v)M_k} v \), since the vectors \( (M_i+c_iM_k)h \) are of course still linearly independent. It is thus possible to find polynomials in \( v \) such that
\[ (w_1, \ldots, w_{k-1}) = (p_1(v), \ldots, p_{k-1}(v)). \]
Then \( w_k = c_1 p_1(v) + \cdots + c_{k-1} p_{k-1}(v) + l(v) \) is also a polynomial in \( v \), and the assertion is proved. \( \square \)

**Step 5.** The result of Step 4 now gives us half of the equations for the closed subset \( C(h) \). Indeed, we have seen that if \( (t, v) \in \Delta^n \times \mathbb{C}^d \) is a point of \( C(h) \), then it is in the image of \( g \), and so its \( v \)-coordinates satisfy the relation
\[ v = e^{-(p_1(v)M_1+\cdots+p_k(v)M_k)} h. \]
In components, these are \( d \) polynomial equations for \( v = (v_1, \ldots, v_d) \). The same equations obviously have to hold for every point in the closure \( \overline{C(h)} \).

**Step 6.** Next, we turn our attention to the remaining \( n \) coordinates \( (t_1, \ldots, t_n) \) of the parametrization \( g \). Each is of the form
\[ t_j = \prod_{s=1}^n e^{2\pi i a_{j,s} w_s}. \]
Letting \( u_j = \exp(2\pi i w_j) \), for \( j = k+1, \ldots, n \), we have
\[ t_j = u_{k+1}^{a_{j,k+1}} \cdots u_n^{a_{j,n}} \cdot e^{2\pi i (a_{j,1} w_1+\cdots+a_{j,k} w_k)}. \]
The shape of these products leads us to consider the algebraic map
\[ (\mathbb{C}^n)^{n-k} \to \mathbb{C}^n, \quad (u_{k+1}, \ldots, u_n) \mapsto (x_1, \ldots, x_n), \]
whose coordinates are given by
\[ x_j = \prod_{i=k+1}^n u_i^{a_{j,i}}. \]
Because the map is given by polynomials, the (topological) closure of its image is actually a closed algebraic subvariety of \( \mathbb{C}^n \), and as such defined by finitely many polynomial equations
\[ f_1(x_1, \ldots, x_n) = \cdots = f_e(x_1, \ldots, x_n) = 0. \]
In fact, because the original map is monomial, each \( f_b(x) \) can be taken as a binomial in the variables \( x_1, \ldots, x_n \). (We shall have to say more later about the toric structure of the image.)
Step 7. From Step 6, we can now deduce the remaining equations for \( \overline{C(h)} \). Indeed, a point \((t, v)\) in the image of \( g \) has to satisfy the equations

\[
 f_b\left(t_1 e^{-2\pi i \sum_{s \leq k} a_1, s w_s}, \ldots, t_n e^{-2\pi i \sum_{s \leq k} a_n, s w_s}\right) = 0
\]

for \( b = 1, \ldots, e \). From Step 4 we know, moreover, that \( w_s = p_s(v) \); therefore

\[
 \text{(B)} \quad f_b\left(t_1 e^{-2\pi i \sum_{s \leq k} a_1, s p_s(v)}, \ldots, t_n e^{-2\pi i \sum_{s \leq k} a_n, s p_s(v)}\right) = 0
\]

is another set of \( e \) holomorphic equations satisfied by the closure \( \overline{C(h)} \).

Step 8. It remains to see that the \( d + e \) equations in (A) and (B) really define \( \overline{C(h)} \), and not a bigger set. The trivial case is when \( h \) is not invariant under any part of the monodromy; here \( k = n \), and as pointed out in Step 1, \( C(h) \) is then already a closed set, and there is nothing to show. In the remaining case, when \( k < n \), we are free to place the point \( P \) (the center of our coordinate system \( t_1, \ldots, t_n \)) anywhere we like. A moment’s thought shows that it therefore suffices to consider solutions of the equations over \((0, \ldots, 0) \in \Delta^a\), and to prove that those have to lie in the closure of \( C(h) \).

So consider a point \((0, v) \in \Delta^a \times \mathbb{C}^d\) that satisfies the equations. On the one hand, the equations in (A) define the image of a closed embedding, as explained in Step 4; therefore, \( v = e^{-(w_1, M_1 + \cdots + w_k, M_k)} h \) for a unique point \((w_1, \ldots, w_k) \in \mathbb{C}^k\). Letting \( w = (w_1, \ldots, w_k, 0, \ldots, 0) \), and going back to the original coordinates \( z \) in (A), we get a point \((z_1, \ldots, z_n) \in \mathbb{C}^n\) such that

\[
v = e^{-(z_1, N_1 + \cdots + z_n, N_n)} h.
\]

On the other hand, the equations in (B) arose from the map defined in Step 6. Now (7) shows that the point \((0, \ldots, 0)\) can only be in the closure of the image when some linear combination of the exponent vectors \((a_{1,i}, \ldots, a_{n,i})\), for \( i = k+1, \ldots, n \), has positive coordinates. Since these vectors generate the subgroup \( S \), we thus get positive integers \( a_1, \ldots, a_n \) with

\[
a_1 N_1 h + \cdots + a_n N_n h = 0
\]

But by the description in Proposition 2, this says exactly that the point \((0, v)\) belongs to \( \overline{C(h)} \).

In summary, we have established the following two results. First, we can give local equations for each of the components \( \overline{C(h)} \).

Proposition 5. The closure \( \overline{C(h)} \) of the sheet \( C(h) \) in \( \Delta^a \times \mathbb{C}^d \) is an analytic subset, defined by \( d + e \) holomorphic equations in the coordinates \((t_1, \ldots, t_n, v_1, \ldots, v_d)\). These equations are, firstly,

\[
v = e^{-(p_1(v)M_1 + \cdots + p_k(v)M_k)} h,
\]

and, secondly,

\[
f_b\left(t_1 e^{-2\pi i \sum_{s \leq k} a_1, s p_s(v)}, \ldots, t_n e^{-2\pi i \sum_{s \leq k} a_n, s p_s(v)}\right) = 0
\]

for \( b = 1, \ldots, e \). In particular, as the closure of the complex submanifold \( C(h) \), the subset \( \overline{C(h)} \) is itself a reduced and irreducible analytic space.

Moreover, because only finitely many components can meet at any given point (by Lemma 4), we can conclude that the closure of the image of \( f \) is an analytic subset of \( \Delta^a \times \mathbb{C}^d \) as well.
Proposition 6. Let $T_U$ be the total space of the local system over the open set $U = M \cap \Delta^n$. The closure of $T_U$ inside of $\Delta^n \times \mathbb{C}^d$ is a reduced analytic subset with countably many irreducible components, each of the form $\overline{C(h)}$ for some $h \in \mathbb{Z}^d$.

6. Singularities

The analytic space $\overline{T}$, described in Theorem 1, will in general be singular at points not in $T$. This is apparent from the discussion in the previous section—on the one hand, several of the local components $C(h)$ may be coming together at the boundary (see Lemma 4); on the other hand, the local equations of the closure are such that singularities have to be expected even for each $C(h)$ itself (see Steps 6 and 7 in the previous section). There are two possible approaches to this problem—normalization, and resolution of singularities.

Normalization. For various applications, it is desirable to have at least a normal space; mostly because it is then possible to extend holomorphic maps that are naturally defined on $T$ to all of $\overline{T}$, by showing that they extend in codimension one. In addition, the normalization of $\overline{T}$ is an unexpectedly nice space.

According to the local description of $\overline{T}$ given above, the process of normalizing $\overline{T}$ has two effects. Firstly, it separates all the local components $\overline{C(h)}$ at points where they meet, making them disjoint. Secondly, it normalizes each $\overline{C(h)}$ itself. From Step 6 in the previous section, we see that $\overline{C(h)}$ is locally isomorphic to a (typically non-normal) toric variety. Indeed, the map $g$ in (5), whose image is the sheet $C(h)$, is locally the product of a closed immersion and a map defined by monomials. As explained in the article by David Cox [3, p. 402], the closure of the image of a monomial map as in (6) is a non-normal toric variety, and after taking the normalization, one gets a toric variety in the usual sense. It follows that the normalization of each $\overline{C(h)}$ is locally toric; in particular, this means that it has only rational singularities. The same is therefore true for the normalization of $\overline{T}$ itself.

Resolution of singularities. A second possibility is to resolve all the singularities of $\overline{T}$. By construction, the total space $T$ of the local system is a nonsingular dense open subset of $\overline{T}$. Its complement $\overline{T} \setminus T$, being the preimage of the divisor $\overline{M} \setminus M$ under the holomorphic projection map from $\overline{T}$ to $\overline{M}$, is a closed analytic subspace. According to the results of Bierstone and Milman [4, p. 298], it is possible to resolve the singularities of $\overline{T}$ by blowing up, at the same time making the preimage of $\overline{T} \setminus T$ into a divisor with only normal crossing singularities. Since the centers of the blowups can be chosen to lie outside of $T$, the resulting complex manifold will still have $T$ as a dense open subset. Of course, the space one gets is as “canonical” as the resolution process.

Since the normalization of $\overline{T}$ is locally toric, one can also normalize first, and then use the older results on desingularizing toroidal embeddings [5, p. 94] to create a non-singular space from $\overline{T}$.

7. A universal property of the extension

In this section, we shall give some justification for calling the space $\overline{T}$ the “canonical extension” of the local system $\mathcal{H}$. The following proposition is the main result in this direction.
Proposition 7. Let $g : \overline{X} \to \overline{M}$ be an arbitrary holomorphic map from a reduced and normal analytic space $\overline{X}$ to $\overline{M}$. Assume that the open set $X = g^{-1}(M)$ is dense in $\overline{X}$, and that there is a factorization

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{g} & M \\
\downarrow & & \downarrow \\
X & \xrightarrow{s} & T
\end{array}
$$

as in the diagram. Then $s$ extends uniquely to a holomorphic map $s : \overline{X} \to \overline{T}$, making

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{g} & \overline{M} \\
\downarrow & & \downarrow \\
\overline{X} & \xrightarrow{s} & \overline{T}
\end{array}
$$

commute.

In order to prove this, we shall first reformulate the statement. Let $\mathcal{V}$ be the canonical extension of the flat vector bundle $\mathcal{V} = \mathcal{O}_M \otimes \mathcal{H}$ to $\overline{M}$; then $T$ is the closure of $T$ inside the total space of $\overline{\mathcal{V}}$. The map $s : X \to T$ gives a section of the pullback of the local system $g^{-1}\mathcal{H}$—as well as of the bundle $g^*\overline{\mathcal{V}}$—over $X$, that we continue to denote by $s$. Now the statement of the proposition is equivalent to saying that $s$ extends to a section of $g^*\overline{\mathcal{V}}$ over $X$. Indeed, such an extension (clearly unique if it exists) gives a map from $X$ to the total space of $\overline{\mathcal{V}}$, and since $X$ is mapped into $T$, the image has to be contained in the closure $\overline{T}$. Thus extending the map $s$ to $X$ is equivalent to extending the corresponding section $s$ of $g^*\overline{\mathcal{V}}$.

We now begin the proof by establishing the following special case.

Lemma 8. The conclusion of Proposition holds whenever $\overline{X} = \Delta$ is the unit disk, and $X = \Delta^\circ$.

Proof. Let $g : \Delta \to \overline{M}$ be the given morphism. Since the original local system $\mathcal{H}$ is unipotent along $\overline{M} \setminus M$, its pullback $g^{-1}\mathcal{H}$ to $\Delta^\circ$ has unipotent monodromy around $0 \in \Delta$. Thus the vector bundle $g^*\overline{\mathcal{V}}$ is the canonical extension of $\mathcal{O}_{\Delta^\circ} \otimes g^{-1}\mathcal{H}$ to $\Delta$.

As we said, it suffices to show that the section $s \in H^0(\Delta^\circ, g^*\overline{\mathcal{V}})$ extends to $\Delta$; but this follows very easily from the construction of the canonical extension. Indeed, if we let $N$ be the logarithm of the monodromy of $g^{-1}\mathcal{H}$ on $\Delta^\circ$, then the description on p. shows that the total space of $g^{-1}\mathcal{H}$, inside that of the bundle $g^*\overline{\mathcal{V}}$, is given by the image of the map

$$
\mathbb{H} \times \mathbb{Z}^d \to \Delta^\circ \times \mathbb{C}^d, \quad (z, h) \mapsto (e^{2\pi i z}, e^{-zN} h),
$$

in a suitable frame of $g^*\overline{\mathcal{V}}$. The section $s$ corresponds to a monodromy-invariant element $h \in \mathbb{Z}^d$, satisfying $Nh = 0$, and so the whole sheet $\Delta \times \{h\}$ lies in the closure of the image. This shows that $s$ extends over $0$, proving the lemma.

We can now turn to proving Proposition in general. Let $g : \overline{X} \to \overline{M}$ be the given map, and set $Z = \overline{X} \setminus X$. As before, $s \in H^0(X, g^*\overline{\mathcal{V}})$ denotes the section of the pullback bundle corresponding to the given factorization. In order to show that $s$ extends holomorphically to all of $\overline{X}$, it suffices to show that it extends in
codimension one, since $\overline{X}$ is normal (see [6, p. 118], for example). The singular locus of $\overline{X}$ has codimension at least two; it is therefore enough to prove that $s$ extends across those points $P \in Z$ where both $Z$ and $\overline{X}$ are nonsingular, and $\text{codim}_p(Z,X) = 1$.

This is a local question, and after choosing suitable local coordinates $z_1, \ldots, z_m$ on a small neighborhood $U$ of $P$ in $\overline{X}$, we can assume that $Z \cap U$ is defined by the equation $z_m = 0$. Applying the lemma to maps of the form

$$\Delta \to U, \quad t \mapsto (z_1, \ldots, z_{m-1}, t),$$

we see that the section $s$ extends across $P$. This completes the proof of the proposition.

**Note.** Since $\overline{T}$ itself is usually not a normal space, Proposition 7 is not quite as strong as one might like it to be. It is, however, easy to make examples of maps from non-normal spaces $X$ to $M$ (for instance, taking $X$ to be a nodal curve) where the statement is false. This suggests that the normalization of $\overline{T}$ is the space that deserves to be called the “canonical extension” of the local system $\mathcal{H}$.

**Conventions**

This short section lists various conventions that are used throughout the paper. The construction of Deligne’s canonical extension is also reviewed, in a form suitable for our proof.

**Fundamental group.** If $X$ is a topological space, with basepoint $x \in X$, we write $\pi_1(X,x)$ for its fundamental group. Given two closed paths $\gamma, \delta: [0,1] \to X$, with $\gamma(0) = \gamma(1) = \delta(0) = \delta(1) = x$, representing two elements of $\pi_1(X,x)$, their product is defined as

$$\gamma \delta: [0,1] \to X, \quad t \mapsto \begin{cases} 
\gamma(2t) & \text{if } 0 \leq t \leq 1/2, \\
\delta(2t-1) & \text{if } 1/2 \leq t \leq 1.
\end{cases}$$

Writing $\Delta$ for the unit disk in $\mathbb{C}$, and $\Delta^* = \Delta \setminus \{0\}$ for the punctured disk, we shall take the generator of $\pi_1(\Delta^*) \simeq \mathbb{Z}$ to be a loop that goes around the origin once, counter-clockwise.

**Action on the fiber.** For any covering space $p: Y \to X$ of $X$, the group $\pi_1(X,x)$ acts on the fiber $p^{-1}(x)$ by a left action; given $\gamma \in \pi_1(X,x)$ and a point $y \in Y$ with $p(y) = x$, one has $\gamma \cdot y = \tilde{\gamma}(0)$, where $\gamma$ has been lifted to a path $\tilde{\gamma}: [0,1] \to Y$ with $\tilde{\gamma}(1) = y$. Put more succinctly, $\gamma$ acts by parallel translation along the path $\gamma^{-1}$.

**Local systems.** A special case of this is the correspondence between local systems on $X$ and representations of the fundamental group. Given a local system $\mathcal{H}$ on $X$, say with fiber $H = \mathcal{H}_x$, each connected component of the total space of $\mathcal{H}$ is a covering space of $X$. One obtains a representation

$$\rho: \pi_1(X,x) \to \text{Aut}(H)$$

by letting the fundamental group act on the fiber. From the representation $\rho$, on the other hand, one can recover $\mathcal{H}$. Indeed, if $p: \tilde{X} \to X$ is the universal covering space of $X$ (assuming its existence), the quotient of $\tilde{X} \times H$ by the group action

$$\gamma \cdot (\tilde{x}, h) = (\gamma \cdot \tilde{x}, \rho(\gamma) h)$$
is isomorphic to the total space of \( \mathcal{H} \). From this description, it follows that the space of sections of \( \mathcal{H} \) over an open set \( U \subseteq X \) is given by
\[
\{ \tilde{s}: p^{-1}(U) \rightarrow H \mid \tilde{s}(\gamma \cdot y) = \rho(\gamma)\tilde{s}(y) \text{ for all } \gamma \in \pi_1(X, x), y \in p^{-1}(U) \}.
\]

**Deligne’s canonical extension.** Let \((\mathcal{V}, \nabla)\) be a flat holomorphic vector bundle on a complex manifold \( M \). We assume that \( M \) is an open subset of a bigger complex manifold \( \overline{M} \), in such a way that
1. \( \overline{M} \setminus M \) has normal crossing singularities, and
2. \( \nabla \) is unipotent along \( \overline{M} \setminus M \).

The second condition means that, near points of \( \overline{M} \setminus M \), the local monodromy for the local system of \( \nabla \)-flat sections should be unipotent. Deligne proves (see [H] pp. 91–5 for the precise statement) that \( (\mathcal{V}, \nabla) \) admits a unique extension to a vector bundle \( \overline{\mathcal{V}} \) on \( \overline{M} \), whose defining property is that, in any local frame for \( \overline{\mathcal{V}} \), the connection \( \nabla \) has only logarithmic poles along \( \overline{M} \setminus M \) with nilpotent residues.

For the purposes of this paper, we need a description of \( \overline{\mathcal{V}} \) in local coordinates, on a polydisk \( \Delta^n \). Thus let \( t_1, \ldots, t_n \) be local holomorphic coordinates near a point of \( \overline{M} \), and assume that \( \overline{M} \setminus M \) is defined by the equation \( t_1 \cdots t_r = 0 \). Restricting further, if necessary, it suffices to treat the case when \( M = (\Delta^n)^n \).

Let \( d \) be the rank of the bundle \( \mathcal{V} \), and \( V \) its fiber at some basepoint in \((\Delta^n)^n\). The fundamental group \( \mathbb{Z}^n \) of \((\Delta^n)^n\) acts on \( V \), by parallel translation, and we let \( T_j \) be the operator corresponding to the \( j \)-th standard generator of \( \mathbb{Z}^n \). By assumption, each \( T_j \) is a unipotent operator, and we can therefore define the nilpotent operators
\[
N_j = -\log T_j = \sum_{n=1}^{\infty} \frac{1}{n} (id - T_j)^n
\]
as their logarithms.\(^3\)

The vector bundle \( \overline{\mathcal{V}} \) has distinguished trivializations of the form
\[
\overline{\mathcal{V}} \simeq \mathcal{O}_{\Delta^n} s_1 \oplus \cdots \oplus \mathcal{O}_{\Delta^n} s_d,
\]
for certain special sections \( s_1, \ldots, s_d \) of \( \mathcal{V} \) over \((\Delta^n)^n\). To obtain the sections in question, pull \((\mathcal{V}, \nabla)\) back to the universal covering space
\[
p: \mathbb{H}^n \rightarrow (\Delta^n)^n,
p(z_1, \ldots, z_n) = (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}),
\]
where it becomes trivial (by virtue of being flat). By our conventions, the fundamental group \( \mathbb{Z}^n \) acts on \( \mathbb{H}^n \) by the rule
\[
(a_1, \ldots, a_n) \cdot (z_1, \ldots, z_n) = (z_1 - a_1, \ldots, z_n - a_n),
\]
and so sections of \( \mathcal{V} \) over \((\Delta^n)^n\) correspond to holomorphic maps \( \tilde{s}: \mathbb{H}^n \rightarrow V \) with the property that
\[
\tilde{s}(z - e_j) = T_j \tilde{s}(z)
\]
for all \( z \in \mathbb{H}^n \) and all \( j = 1, \ldots, n \).

Now let \( v_1, \ldots, v_d \in V \) be an arbitrary basis for \( V \). The maps
\[
\tilde{s}_i: \mathbb{H}^n \rightarrow V, \quad \tilde{s}_i(z) = e^{\sum z_i N_j} v_i,
\]
have the required invariance property, because
\[
\tilde{s}_i(z - e_j) = e^{-N_j} \tilde{s}_i(z) = T_j \tilde{s}_i(z),
\]
\(^3\)The minus sign is there to stay with the conventions of other authors, for example [2].
and thus define a frame of sections $s_1, \ldots, s_d$ for $\mathcal{V}$ on $(\Delta^r)^n$. These sections give the special trivialization of $\mathcal{V}$ in $\mathbf{S}$.

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