Quantification of the effects of uncertainties in turbulent flows through generalized Polynomial Chaos

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Abstract. Statistical methodologies based on surrogate-models have proved to be an efficient approach to quantify the physical properties of turbulent flows. The underlying idea is to parametrize the space of possible solutions via a computationally inexpensive approximation model, which is then used to generate samples for the statistical tool at hand.

In the following homogeneous isotropic turbulence (HIT) decay and Large-eddy simulation (LES) subgrid-scale modeling are considered as stochastic processes and their sensitivity to uncertainties in the energy spectrum shape is investigated by a surrogate-model approach based on the generalized Polynomial Chaos (gPC) approximation.

The initial spectrum shape at large scales drives the long-time evolution of the physical quantities in HIT: this sensitivity is recovered even at high Reynolds number. In particular, a universal asymptotic behavior in which kinetic energy decays as $t^{-1}$ is not observed.

The statistical average of the Smagorinsky subgrid model constant is close to the asymptotic Lilly-Smagorinsky value if the LES filter cut is applied in the inertial range at high Reynolds numbers, while a significant variance is recovered if the cut if performed in the dissipation range or if moderate Reynolds number are considered.

1. Introduction

Along with the rapid increase of computational resources availability in the last years, the use of numerical simulations to predict the evolution of turbulent flows is a common practice in the scientific community. Indeed, the numerical simulation can describe the behavior of turbulent flows in a wide range of applications spanning from direct numerical simulation of basic flows, which is necessary to understand the underlying physics of turbulent phenomena, up to industrial complex flows devoted to the recovery of statistical low order moments. The prediction of turbulent flows in which moving objects, massive flow separation or combustion are present it is a challenging field of application, due to the nature of the approximations that have to be made to simulate this kind of flow with realistic computational resources. Even though new insights about error control have been recovered when using promising mathematical approaches, like Large-eddy simulation, the validation of numerical results is still a open issue.

One possible way to assess the quality of the results of numerical simulations could be the comparison with experiments, even though this approach has always been a controversial task to perform. It is well known that turbulent flow experiments are extremely sensitive to
perturbations in boundary and inlet conditions, as well as in measurements: these epistemic uncertainties may rarely be reduced or canceled, leading to a difficult comparisons with single numerical simulations. Indeed, a numerical simulation is a deterministic process that cannot capture adequately uncertainties affecting an experiment. Simulation databases built over a significant parameter space may lead to a more consistent comparison with experimental data; anyway brute-force statistical methods, like Monte Carlo sampling, lead to a systematic exploration of the simulation parameter space that is not realistic in practical cases, requiring an enormous number of deterministic simulations. This obviously leads to an unaffordable computational costs. Therefore, it is important to reduce the number of simulations performed while using a method well suited for globally non-convex cost function. A key idea is to parameterize the space of possible solutions via a simple, computationally inexpensive model, and to use this model to generate samples for the optimization algorithm. Such a model is often referred to as the response surface of the system to be optimized, leading to the definition of a so-called surrogate-model based optimization methodology. Despite a huge amount of work has been devoted to the design of efficient optimization techniques, the definition of a robust, general-purpose reliable technique is still a research topic.

The difficulty to retrieve a robust approach rises principally from the underlying stochasticity related to turbulent phenomena: the efficacy of the mathematical model to represent the probabilistic distribution associated to the physical moments analyzed is crucial to recover accurate results. Mathematical tools developed for the probability theory look like natural efficient candidates to elaborate the model included in the response surface reconstruction. Indeed, the sensitivity of experiments to the boundary or initial condition may be numerically modeled considering a degree of uncertainty in the free parameters present in the set up of the numerical simulation. In the context of the response surface reconstruction methodology, the quantification of uncertainties may be exploited by the probabilistic surrogate-model itself, being the model based on statistical or non-statistical approaches. Recalling that a statistical approach may lead to prohibitive computational resources required, a non statistical probabilistic method looks as a perfect candidate to effectively reconstruct response surfaces of turbulent flow dynamics with reasonable computational resources.

Spanos & Ghanem (1989) pioneered the computational use of the Polynomial Chaos (PC) expansion method in solid mechanics: this approach proved well-suited to solve stochastic differential equations. The Polynomial Chaos representation is actually a projection of the considered random process over an orthogonal basis, in order to achieve a spectral decomposition based on a second-order norm. The basis coefficients, which are recovered during the projection operation, provide quantitative estimates of the modeled random process solution. In the present work we will consider the generalized Polynomial Chaos (gPC) method (Lucor et al. (2007)) which has the advantage that Gaussian and non-Gaussian random processes can be investigated. Even if this mathematical method does not furnish a rigorous probabilistic framework it has been successfully applied to solve stochastic differential equations (Lucor & Karniadakis (2004)) and it has recently been applied to sensitivity analysis of turbulent flows with significant results (Lucor et al. (2007); Ko et al. (2008)).

In the following two generalized polynomial chaos (gPC) based stochastic analysis dealing with different turbulence topics will be presented and investigated. The stochasticity will be propagated in the problem considering the free parameters of the energy spectrum functional form as random variables over a bounded space. Indeed, many of the characteristics of the energy spectrum are elusive to experimental validation and it is not possible to recover univoque results even if direct numerical simulations are performed due to boundary condition saturation effects combined with numerical error.

The paper is structured as follows: in Section 2 the energy spectrum is introduced and commented, while practical details of the gPC stochastic approach are reported in Section 3.
The results from the two stochastic analysis are presented in Section 4 while concluding remarks and observations are reported in Section 5.

2. Energy spectrum functional forms

The sensitivity analysis reported in the present paper will be performed propagating uncertainties in the free parameters of the energy spectrum functional form. In particular, the recent kinetic energy spectrum functional forms proposed by Meyers & Menevau (2008) has been chosen: this model actually accounts for all known features of the kinetic energy spectrum and its precision has been assessed through the comparison with real energy spectra at moderate Reynolds numbers. The formulation of the model is:

\[ E(k) = C_k \varepsilon^{2/3} k^{-5/3} f_L(kL) f_\eta(k\eta) \]  

where \( \beta \) is the intermittency correction applied in correspondence of the inertial range of the spectrum. It is taken equal to \( \beta = \mu/9 \), with \( \mu \) being one of the random variables selected for the analysis. \( C_k \) is the Kolmogorov constant, \( L = E^{3/2}/\varepsilon \) is the integral length scale, \( \eta \) is the Kolmogorov length scale and \( \varepsilon \) is the rate of energy dissipation. The functions \( f_L \) and \( f_\eta \) shape the spectrum at very large and very small scales, respectively, and are expressed as:

\[ f_L(kL) = \left( \frac{kL}{(kL)^p + \alpha_3} \right)^{5/3 + \beta + \sigma} \]  
\[ f_\eta(k\eta) = e^{-\alpha_1 k\eta} B(k\eta) \]  

The bottleneck correction term \( B(k\eta) \) is:

\[ B(k\eta) = 1 + \frac{\alpha_2 (k\eta/\alpha_4)^{\alpha_3}}{1 + (k\eta/\alpha_4)^{\alpha_3}} \]  

The two shape functions \( f_L \) and \( f_\eta \) introduce several arbitrary parameters: \( \sigma \) is related to the slope of the energy spectrum at very large scales, while \( p \) drives the position and the smoothness of the energy peak. Parameters \( \alpha_1 - \alpha_4 \) govern the shape of the spectrum at high wave numbers while \( \alpha_5 \) controls it at low wavenumbers. Meyers & Menevau (2008) propose to compute the \( \alpha_i, i = 1, 5 \) parameters by solving a system of five constraint equations: the constraints are formulated with the aim to recover target values for the turbulent kinetic energy, the turbulent dissipation (or equivalently the enstrophy), the palinstrophy (or equivalently the longitudinal velocity derivative skewness) or to recover some universal feature of the compensated dissipation spectrum. Considering the compensated formulation of the energy spectrum,

\[ F(k\eta) = \frac{k^{5/3} E(k)}{\varepsilon^{2/3}} \]  

the conservation law constraint equations may be reformulated as:

\[ \int_0^{+\infty} x^{-5/3} F(x Re^{-3/4}) dx = 1 \]  
\[ \int_0^{+\infty} x^{1/3} Re^{-3/4} F(x) dx = \frac{1}{2} \]  
\[ \int_0^{+\infty} x^{7/3} Re^{-3/4} F(x) dx = \frac{-7S_3(\mu, Re)}{12\sqrt{15}} \]  

where \( S_3 \) is the skewness of the longitudinal velocity derivative. This quantity, which is dependent on the Reynolds number, is is recovered by the numerical interpolation \( S_3 = \)
$$\left( \frac{S_{ref}^{\lambda}}{Re_{\lambda}^{\xi/2}} \right) \left( \frac{Re_{\lambda}^{\xi/2}}{S_{ref}^{\lambda}} \right)$$, where $\xi = 9\mu/16$ and $S_{ref}^{\lambda}$ and $Re_{\lambda}^{\xi/2}$ are two reference points taken from experimental measurements. The two last constraint used are recovered by observed results of direct numerical simulations: the considered data show an approximate invariance of the position and the intensity of the peak of the compensated dissipation spectrum. This set of five equation define the parameter set $\alpha_i$, known the other free parameters as well as the Reynolds number.

3. Generalized polynomial chaos

The main characteristics of the generalized polynomial chaos approach are here introduced and discussed; we address the reader to the works of Ghanem & Spanos (1991) and of Le Maître & Knio (2011) for an exhaustive description of the numerical approach adopted. Generalized polynomial chaos is a non-statistical method used to solve stochastic differential (SDE) and stochastic partial differential equations (SPDE).

Let us consider a probability space $(\Omega, A, P)$ where $\Omega$ is the event space, $A \subset 2^\Omega$ its $\sigma$-algebra and $P$ its probability measure. Being $\omega$ an element of the event space, we define a random field $X(\omega)$ such that it maps the probability space into a function space $V$, $X : \Omega \to V$. In the following we will consider second-order random fields, i.e. those satisfying the relation:

$$E(X, X) < +\infty$$

where $E$ denotes the expectation of a random variable. Starting from these assumptions, gPC is an orthogonal decomposition allowing second-order random fields to be represented through a set of random variables $\xi(\omega)$. The random field $X(\omega)$ can be expressed by its Galerkin projection over a polynomial orthogonal basis taking the following form:

$$X(\omega) = a_0 B_0 + \sum_{i_1=1}^{\infty} a_{i_1} B_1(\xi_{i_1}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} a_{i_1 i_2} B_2(\xi_{i_1}, \xi_{i_2}) + ...$$

where $\xi = (\xi_1, ..., \xi_N)^T$ is a $N$-dimensional random vector and $B_i$ is a polynomial of order $i$ depending on the $\sigma$-algebra of $\xi$. A generalized polynomial basis can be generated by selecting the univariate basis that is most optimal for each random input and then applying the products as defined by the multi-index to define a mixed set of multivariate polynomials. Similarly, the multivariate weighting functions involve are recovered by a product of the one-dimensional weighting functions. Equation 9 can be reformulated using a term-based indexing instead of a order-based indexing. Let $\Phi_k(\xi(\omega))$ be a single polynomial, the pseudospectral approximation of the random process solution can be written as follows:

$$X(\omega) = \sum_{k=0}^{\infty} a_k \Phi_k(\xi(\omega))$$

in which there is a one-to-one correspondence between $a_{i_1, i_2, ..., i_n}$ and $a_k$ and between $B_n(\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_n})$ and $\Phi_k(\xi)$. The polynomial expansion is truncated to a finite limit considering the elements of the basis up to the maximum polynomial order $p$ selected by the user. The total number of terms $N$ in a complete polynomial chaos expansion of arbitrary order $p$ for a response function involving $n$ uncertain input variables is given by:

$$N = 1 + \sum_{s=1}^{p} \frac{1}{s!} \prod_{r=0}^{s-1} (n + r) = \frac{(n + p)!}{n!p!}$$
The orthogonality of the polynomials is set through the relation

\[ \langle \Phi_i \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij} \]  \hspace{1cm} (12)

where \( \langle \cdot \rangle \) denotes an ensemble average. This inner product is defined over the measure \( W(\xi) \) of the random variables as follows:

\[ \langle f(\xi)g(\xi) \rangle = \int_{\omega \in \Omega} f(\xi)g(\xi)dP(\omega) = \int_{\omega \in \Omega} f(\xi)g(\xi)W(\xi)d\xi \]  \hspace{1cm} (13)

Thanks to the orthogonality of the polynomial basis, each coefficient of the Galerkin projection (10) can be recovered through the following definition:

\[ a_k = \frac{\langle X, \Phi_k \rangle}{\langle \Phi_k^2 \rangle} = \frac{1}{\langle \Phi_k^2 \rangle} \int_{\omega \in \Omega} X\Phi_k \rho \xi d\xi(\xi) \]  \hspace{1cm} (14)

where each inner product involves a multidimensional integral over the support range of the weighting function.

The polynomial family to be used must be a priori specified. The choice of the polynomials affects the speed of the convergence of the series so an accurate choice of the basis function may lead to a very accurate description of the random process with a moderate value of the maximum polynomial order \( p \). A polynomial family able to efficiently represent the input has a weight coefficient similar to the function \( W \) in equation 13 when using the Gaussian quadrature approach to compute the integrals, like we have done in the present work.

| Spectrum parameter | Range            | Reported variations |
|--------------------|------------------|---------------------|
| \( \sigma \)       | \([1, 4]\)        | \([1, 4] \cup +\infty\) |
| \( p \)            | \([1, 2]\)        | 1.5, 2              |
| \( C_k \)          | \([1.87, 2.13]\)  | \([1.3, 1.9]\)      |
| \( \mu \)          | \([0.2, 0.3]\)    | \([0.2, 0.3]\)      |

The gPC application is herein used in its non-intrusive approach, i.e. the considered random process is directly projected over the orthogonal basis spanning the random space, without any modification of the solver used to compute it. This means that the gPC is used as a postprocessing tool to reconstruct a continuous description of the solution over the uncertainty space, starting from a limited number of discrete samples (i.e. single deterministic simulations). In particular, the distribution of the samples over the uncertainty space has been selected in order to compute the coefficient of the polynomial expansion by Gaussian quadrature. We recall that the analysis have been performed by considering as random variables the free parameters present in the energy spectrum functional form: the range of the random variables related to uncertainties in the large scales of the energy spectrum is reported in table 1.

The probability density function \( pdf \) of the random variables has been considered as \textit{uniform} leading to the choice of the Legendre polynomial family: this is a well suited choice since the inner product weighting function is directly proportional with a factor 0.5 to the set probability density function.
4. Results

In this Section two sensitivity analysis are presented. In Section 4.1 the decay of homogeneous isotropic turbulence is considered as a stochastic process: the sensitivity of the power law decay exponents of different physical quantities is investigated considering uncertainties in the large scales of the energy spectrum formulation. The sensitivity of the Smagorinsky model constant to the characteristics of the energy spectrum is discussed in Section 4.2.

4.1. Homogeneous isotropic turbulence decay

The results reported in this section are a summary of the analysis presented in the paper by Meldi et al. (2011a). One of the oldest topic investigated in the field of turbulence theory is indeed the decay of isotropic turbulence, which is characterized by the existence and uniqueness of self-similar regimes. That means that global turbulent quantities like turbulence kinetic energy $q$ can be described as:

$$q(x) = A \left( x/M_u - x_0/M_u \right)^n \quad \text{or} \quad q(t) = A' (t - t_0)^n$$

Taylor’s frozen turbulence hypothesis relate the space and time-dependent evolution of the flow. Influential works by Kolmogorov (1941) and Batchelor & Proudman (1956) emphasized that the decay exponent of the global physical quantity (usually referred as $n$ for the turbulence kinetic energy) is significantly sensitive to the imposed initial conditions. More specifically, the shape of the spectrum at very large scales is observed to be of primary importance. Several spectrum shapes at very large scale are physically realizable in isotropic turbulence. This point, which is related to the existence of invariants in high-Reynolds decaying turbulence, has been discussed and investigated by many authors, being two the principal hypothesis reported in literature. Saffman (1967) showed that linear momentum conservation is linked to the invariance of the Birkhoff-Saffman invariant $L = \int \langle u \cdot u' \rangle d\mathbf{r}$. The associated kinetic energy spectrum behaves like $E(k \to 0) = Lk^2/4\pi^2$ and with these starting conditions the turbulent kinetic energy decays as $q(t) \sim t^{-6/5}$. If $L = 0$, the large scales behaves like $E(k \to 0) = Ik^4/24\pi^2$, where $I = \int \mathbf{r}^2 \langle u \cdot u' \rangle d\mathbf{r}$ is the Loitsyansky’s integral. In this case, the turbulent kinetic energy scales as $q(t) \sim t^{-10/7}$ and this resulting condition is referred to as Batchelor turbulence.

The theoretical value of the power law exponents may be recovered by theoretical analysis but comparison with experiments is not easily applicable for many reasons. In fact it is not possible to enforce the spectrum shape at very large scales in laboratory experiments and, in most cases, the large-scale spectrum shape is not directly measured but deduced from the measured decay law and theoretical relations which bridge between them.

The aim of the present work is to further investigate the sensitivity of the power-law exponents with respect to details of the initial kinetic energy spectrum, the emphasis being put on large scales uncertainties. The free parameters $\sigma$, $p$ and $\mu$ present in the starting energy spectrum functional form are considered as epistemic uncertainties and modeled as random variables. The starting Reynolds number of the energy spectrum has been fixed at $Re_\lambda = 10^3$ and the self-decay of the physical quantities considered has been observed for $Re_\lambda > 400$.

The samples used to reconstruct the response surface are generated using a spectral model whose accuracy has been assessed for isotropic turbulence decay, namely the EDQNM model (Orszag (1970); Sagaut & Cambon (2008)). Statistical results are displayed in table 2 while the probability density functions (pdf) of the power-law exponent for several global quantities are reported in figure 1. The results show how a large variability of the coefficients is recovered and in particular a large number of occurrences tend to cluster close to the theoretical value recoverable with Comte-Bellot-Corrsin formula for $\sigma = 3$. The analysis of the partial variances reported in table 2 show that the power law coefficients are almost exclusively sensitive to $\sigma$ variations, a result that is in line with the theoretical analysis reported in literature. The
exponent $p$, which governs the smoothness of the energy spectrum peak, is observed to be the second leading parameter.

Table 2. Statistical properties of the recovered power-law exponents compared theoretical analysis formula. $n_d$, $\bar{n}$, $c_v$, $n_p$ and $\varsigma$ refer to the: deterministic value associated with the mean value of the random variables, statistical mean value, coefficient of variation (standard deviation divided by $\bar{n}$), most probable value and normalized partial variance associated to the subscript variable.

| Exponent | $n_d$ | $\bar{n}$ | $c_v$ | $n_p$ | $\varsigma$ | $\varsigma_p$ | $\varsigma_\mu$ | CBC formula |
|----------|-------|-----------|-------|-------|-------------|--------------|--------------|-------------|
| $q$      | -1.242| -1.227    | 8.6%  | -1.330| 9.85e-1     | 1.40e-2      | 4.53e-6      | $-\frac{2\sigma + 1}{\sigma + 3}$ |
| $L$      | 0.363 | 0.374     | 13.4% | 0.314 | 9.99e-1     | 9.46e-4      | 1.34e-6      | $\frac{\sigma + 3}{\sigma}$           |
| $Re_\lambda$ | -0.128| -0.118    | 45.6% | -0.173| 9.92e-1     | 7.05e-3      | 2.85e-6      | $\frac{1 - \sigma}{2(\sigma + 3)}$     |
| $\eta$  | 0.557 | 0.554     | 4.7%  | 0.579 | 9.76e-1     | 2.36e-2      | 5.91e-6      | $\frac{3\sigma + 5}{4(\sigma + 3)}$    |

The variability of the power law exponents is extremely dependent on the physical quantity considered and it is higher for Reynolds number exponents: this is due to coupling effects between variations of the kinetic energy and the characteristic lengthscale under consideration. Moreover, an asymptotic theoretical behavior for which the turbulent kinetic energy $q$ should decay as $t^{-1}$ at very high Reynolds number has not been recovered in this analysis, where high but finite Reynolds numbers have been considered.
4.2. Model coefficient for the Smagorinsky subgrid scale model

The results herein reported are investigated in detail in the work submitted to Physics of Fluids by Meldi et al. (2011a). The Smagorinsky subgrid scale model, presented by Smagorinsky (1963), is the oldest and probably most documented subgrid scale model used in large eddy simulations. The model, developed for high-Reynolds number applications, is build under the assumption that turbulence beyond the filter width is in local equilibrium. It is well known that, if the scale separation hypothesis is verified, i.e. large scales dynamics are not affected by filtered scales behavior, the small unresolved scales present a universal isotropic behavior and they are responsible for energy dissipation only.

As all the eddy viscosity models, the Smagorinsky model mimics the effect of the filtered scales on the flow by a statistical approach, introducing in the equations an amount of energy dissipation that is tuned through a model constant. Lilly (1967) showed that the asymptotic value of this constant $C_{S,\infty}$ could be assessed through a theoretical study based on dimensional analysis; anyway, if that value for the model constant is use in practical numerical simulations, the model usually introduces an unintended dissipation over the large scales.

Meyers & Sagaut (2006) recently reformulated the theoretical value of the constant as follows:

$$C_S = \frac{C_{S,\infty}}{\gamma} \Phi^{-3/4} \sqrt{1 - \left(\frac{\gamma L}{C_{S,\infty} \Delta} \right)^{4/3}} \frac{1}{Re_L} \Phi$$

(16)

where $\Delta$ is the filtering width and $\Phi, \gamma$ are integral quantities dependent over the resolved part of the energy spectrum. This algebraic relation, which is recovered from the law of dissipation of energy, demonstrates that the optimum value of the Smagorinsky constant is dependent on the shape of the energy spectrum: a sensitivity analysis based on the gPC approach is investigated.

The analysis is performed at two different Reynolds numbers, namely $Re_\lambda = 250$ and $Re_\lambda = 1200$, considering uncertainties in the large and in the small scales of the spectrum. The random variables chosen to represent the effects of uncertainties in the large scales are respectively $\sigma$, $C_k$ and $\mu$. The considered bounded range of analysis, which is reported in table 1 with the values reported in literature, has been chosen to comply specific indications reported by Meyers & Meneau (2008). Small scales uncertainty has been simulated propagating a white noise in the parameters $\alpha_2, \alpha_3$ and $\alpha_4$ which shape the bottleneck correction term (see equation 3). The maximum intensity of the white noise is the 15% of the value recovered resolving the five equation constraint sistem reported in section 2.

Pdfs describing the distribution of the Smagorinsky constant value as a function of $\Delta/\eta$ are reported in figure 2. The pdfs are normalized at each $\Delta/\eta$ value by the peak magnitude of the pdf, so that the values are in the range [0, 1]. The integration over a chosen $\Delta/\eta$ value does not give back a unitary value, but the distribution and the position of the extreme values at each $\Delta/\eta$ section are conserved. Moreover, the visualization of the data is more clear even if the variance of the model constant is very sensitive to $\Delta/\eta$, as in the present case. Let us first consider the pdfs related to large scales uncertainties. In figure 2(b), i.e. the highest simulated Reynolds number, the pdfs of the model constant show a quasi-deterministic behavior if the LES filter cut is applied in the inertial range. The value at which the occurrences cluster is $\bar{C}_S \approx 0.182$, which is almost 5% larger than the asymptotic Lilly-Smagorinsky value. The presence of a bottleneck correction term in the functional form influences the results: this term actually lowers the slope of the energy spectrum in the inertial range, leading to an extra dissipation to be introduced by the subgrid-scale model. It also appears that, moving at lower $\Delta/\eta$ values, the extreme events tend to separate significantly and a clear most probable value is not anymore observable. If the same analysis is performed for $Re_\lambda = 250$ a completely different pdf is recovered, as in figure 2(a): an asymptotic behavior for high $\Delta/\eta$ values is not recovered and in particular a significant difference over the extreme values is observed. These results clearly point out the difficulties arising when trying to predict the suitable model constant in the absence of apriori informations.
Figure 2. Pdf of the Smagorinsky model constant at different $x$ values considering the energy spectrum at $Re_{\lambda} = 250$ (left column) and at $Re_{\lambda} = 1200$ (right column). Pfds are computed considering uncertainties in the large scales (upper row) and in the small scales (lower row).

If uncertainties in the small scales are considered, a more significant variance of the solution compared to the initial perturbation of the parameters can be appreciated. Both the pdfs, reported in figure 2(c) and 2(d), show that the model constant prediction becomes a deterministic process with increasing values of $\Delta/\eta$: in particular, an asymptotic behavior toward a value 5% larger than the Lilly-Smagorinsky value is observable for $Re_{\lambda} = 1200$. It is important to stress that, even if a limited degree of uncertainty has been introduced in the free parameters shaping the bottleneck correction term of the functional form, the extreme events span a quite large range of possible $C_S$ optimum values. Furthermore, the pdf behaves like the sum of two different uniform distributions if the LES filter cuts the energy spectrum close to the dissipation zone. This means that, fixed a $\Delta/\eta$ value, a most probable value of the model constant $C_S$ can be observed; however, the probability to recover significant differences is not negligible, depending of the characteristics of the energy spectrum.
5. Concluding remarks

The application of a stochastic approach based on generalized Polynomial chaos to turbulent flows has been reviewed. The results recovered from analysis dealing with different problems related to turbulence show a significant accuracy if compared to the computational resources required. The ease of implementation of this numerical tool, combined with the wide range of applications for which it can be naturally used, shows that the response surface methodology combined with probabilistic numerical methods promise to become a significant tool to predict the statistical moment of turbulent flows at very high Reynolds number in the next years.

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