Orbits of particles and black hole thermodynamics in a spacetime with torsion

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We derive the static spherically symmetric vacuum solution for a spacetime with non-vanishing torsion by solving the field equations analytically. The effects of torsion appear as a single parameter in the line element. For the positive values of this parameter, the resulting line element is found to be of the Reissner-Nordstrom type. This parameter is related to the spin of matter and acts as a torsion ‘charge’ much like the electric charge in conventional Reissner-Nordstrom geometry. We also analyze the existence and stability of the orbits for both massless and massive particles in this setup and compare the results to the corresponding case in general relativity. We also derive the first law of black hole thermodynamics for a black hole with torsion and define the black hole temperature and entropy in terms of its mass and torsion charge.

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I. INTRODUCTION

The first hint that black holes may behave as thermodynamical systems came from Bekenstein [1, 2, 3]. He suggested that a black hole may have an entropy proportional to its surface area, however he was not able to determine the exact relation. A little later, Bardeen, Carter, and Hawking discovered that there is analogy between the laws of black hole mechanics and four laws of classical thermodynamics, but with certain limitations, as in order for the laws of black hole mechanics to be true thermodynamical equations, black holes ought to have non-zero temperatures. This was thought to be impossible at the time because black holes could only absorb and not emit particles and radiation [5]. This conundrum was solved when by applying quantum field theory to black holes, Hawking realized that they indeed emit thermal radiation and as a result have a non-zero temperature related to their surface gravity \( \kappa \) by the relation \( T = \frac{\kappa}{2 \pi} \). [6, 7]. This discovery led to the famous Hawking - Bekenstein formula for the black hole Entropy \( \Sigma = \frac{A}{4} \), where \( A \) is the surface area of the event horizon (In the above and subsequent relations we use Planck units \( \hbar = c = G = 4\pi\epsilon_0 = 1 \)). This relation has profound theoretical implications, as it relates quantum mechanical effects through the entropy to gravity and can lead us to the ultimate quantum theory of gravity.

On the other hand, torsion appears naturally in many proposed theories of quantum gravity [8, 9, 10, 11]. More specifically the low energy effective Lagrangian of string theory has shown to be equivalent to a Brans-Dicke generalization of a metric theory of gravity with torsion [12, 13]. It is also well known that the corresponding field strength of the Kalb-Ramond field in string theory can act like a torsion field in the background geometry. [14]. The Kalb-Ramond field also appears in noncommutative field theory [15] where the presence of torsion is also established [16].

By the above arguments, studying the black hole thermodynamics in the presence of torsion seems to be a matter of interest when one wants to look for quantum effects in gravity. There are several important previous works in this subject. In [17] the role of torsion in three-dimensional quantum gravity is investigated by studying the partition function of the Euclidean theory in Riemann-Cartan spacetime. In [18], the formalism of spacetime thermodynamics was extended to Einstein-Cartan theory of gravity. In [19] it has been shown that the presence of spacetime torsion does not affect the entropy-area relation. In the present paper we study orbits of particles around a black hole and black hole thermodynamics in gauge theories of gravity were torsion is present. These theories are of a great importance from a field theoretical point of view as they involve localization of spacetime symmetries, much like the localization scheme of internal symmetries present in the standard model of particle physics. Gauge theories of gravity typically involve spin of particles in the gravitational interactions and as a result may provide a convenient way to study some quantum effect on gravitational phenomena, specially at high energies.

The most general gauge theories of gravity are equipped with a metric and a general linear connection and are usually called ‘metric-affine’ theories of gravity and are described by a \((L_g, g)\) space. If we introduce the metricity condition in these theories, we will get the Riemann-Cartan space \(U_4\) and the resulting theory is called the Poincaré gauge theory of gravity (PGT) which applies the localization scheme to the global Poincaré group of transformations [20, 21]. The dynamical variables in this theory are tetrad and spin connection fields and the associated field strengths are curvature and torsion tensors, which are coupled to energy-momentum and spin-density tensors respectively. There are several important special cases of PGT, namely General relativity (vanishing torsion), teleparallel theory (vanishing curva-
ture) and Einstein-Cartan theory where the Lagrangian is set to be equal to the Einstein-Hilbert Lagrangian of general relativity. Einstein-Cartan theory offers the simplest generalization of general relativity and has been studied extensively in literature \cite{22}. In this case the torsion is completely determined by the spin density tensor and can not propagate \cite{22}. However, propagating torsion modes can be present in Poincaré gauge theory of gravity with general quadratic Lagrangian \cite{24}.

In this paper we study black hole thermodynamics and particle orbits in Poincaré gauge theory of gravity. The organization of the paper is as follows: In section II, we briefly review the gauge theories of gravity with torsion and present the main equations for Poincaré gauge theory of gravity. In section III, static spherically symmetric solutions to the Poincaré field equations was derived. In section IV, the effective potential and orbits of particles is set to be equal to the Einstein-Hilbert Lagrangian of general relativity. Einstein-Cartan theory offers the simplest generalization of general relativity and has been studied \cite{25}. In a static spherically symmetric spacetime with torsion \cite{25}, the effective potential and orbits of particles is studied for photons and massive particles. Section V devotes to black hole thermodynamics in the presence of torsion. Finally a brief review and discussion of the main results is given in the conclusion.

II. GAUGE THEORIES OF GRAVITY WITH TORSION

In PGT the gravitational field is described by both curvature and torsion tensors. These in turn can be expressed in terms of tetrad and spin connection as

\[ R_{\mu
u}^i j = 2 \left( \partial_{[\mu} T_{\nu]}^i j + \Gamma_{[\mu k} T_{\nu] k}^i j \right) \]

\[ T_{\mu
u}^i = 2 \left( \partial_{[\mu} e_{\nu]}^i + \Gamma_{[\mu j} e_{\nu]}^i \right), \quad T_\mu = T_{\mu\nu}^\nu \]  

where \( e_{\mu}^i \) is the tetrad field and \( g_{\mu\nu} = \eta_{i\jmath} e_{\mu}^i e_{\nu}^j \), (2)

is the spacetime metric. The spin connection is related to the usual holonomic connection by the relation

\[ \Gamma_{\mu
u}^{\nu} = e_{\mu}^j e_{\nu}^k \Gamma_{ij}^{k} + e_{\mu}^j \partial_{[\mu} e_{\nu]}^{\nu} \]  

(3)

Here the Greek indices refer to holonomic coordinate bases and the Latin indices refer to the Local Lorentz frame. The most general Lagrangian of the theory is a quadratic function built by irreducible decompositions of curvature and torsion. Here we assume a Lagrangian in the form

\[ L_g = -\frac{a_0}{2} R + \frac{b}{24} R^2 + \frac{a_1}{8} \left( T_{\nu\sigma\mu} T^{\nu\sigma\mu} + 2 T_{\nu\sigma\mu} T^{\nu\sigma\mu} - 4 T_{\nu} T^{\nu\mu} \right) \]  

(4)

Where \( a_0, a_1 \) and \( b \) are coupling constants and \( R \) is the Ricci scalar. The field equations then is given by the variation of the Lagrangian with respect to the tetrad and spin connection fields and have the general form \cite{25}

\[ \nabla_\nu H_{\mu\nu}^{\mu} - E_{\nu}^{\mu} = T_\nu^{\nu}, \]  

\[ \nabla_\nu H_{\mu\nu}^{\nu} - E_{\nu}^{\nu} = S_{\nu}^{\nu}, \]  

where

\[ H_{\mu\nu}^{\mu} := \frac{\partial L_G}{\partial g_{\nu\mu}} = 2 \frac{\partial L_G}{\partial T_{\nu\mu}^{\nu}}, \]  

\[ H_{\mu\nu}^{\nu} := \frac{\partial L_G}{\partial \Gamma_{\mu\nu}^{\nu}} = 2 \frac{\partial L_g}{\partial R_{\mu\nu}^{\nu}}, \]  

and

\[ E_{\nu}^{\mu} := e_{\nu\mu} e L_G - T_{\nu\rho} H_{\rho}^{\mu} - R_{\nu\rho} H_{\rho}^{\mu}, \]

\[ E_{\nu}^{\nu} := H_{\nu|\nu}^{\nu}. \]  

(9)

(10)

The source terms here are energy-momentum and spin density tensors respectively and are defined by

\[ T_{\mu}^{\nu} := \frac{\partial L_M}{\partial \Gamma_{\mu\nu}^{\nu}}, \]  

\[ S_{\nu}^{\mu} := \frac{\partial L_M}{\partial \Gamma_{\mu\nu}^{\nu}}, \]  

(11)

(12)

where \( L_M \) is the matter Lagrangian and \( e \) is the determinant of the tetrad.

III. SOLVING FIELD EQUATIONS

First we derive the static spherically symmetric vacuum solutions to the Poincaré field equations. This solution would describe the spacetime outside a static spherically symmetric black hole with torsion. The corresponding solution in general relativity is the Schwarzschild metric. For a Lagrangian in the form of \cite{14}, the explicit form of the field equations are \cite{25}

\[ \nabla_\mu R + \frac{2}{3} \left( \frac{R + 6\mu}{b} \right) T_\mu = 0 \]  

(13)

\[ a_0 \left( \bar{R}_\mu\nu - \frac{1}{2} g_{\mu\nu} \bar{R} \right) - \frac{b}{6} \bar{R} \left( R_{(\mu\nu)} - \frac{1}{4} g_{\mu\nu} R \right) \]

\[ - \frac{2b}{3} \left( \nabla_{(\mu} T_{\nu)} - g_{\mu\nu} \nabla_\rho T^{\rho} \right) - \frac{b^2}{9} \left( 2 T_{\mu} T^{\mu} + g_{\mu\nu} T_{\mu} T^{\nu} \right) = - \tau_{\mu\nu} \]  

(14)

Where \( \mu := a_1 - a_0 \), \( \bar{R}_\mu\nu \) and \( \bar{R} \) are the Riemannian Ricci tensor and scalar respectively and \( \nabla \) is the covariant
derivative with respect to the Levi-Civita connection of GR. For black holes, the most general static line element with desired symmetries is

\[ ds^2 = -e^{\nu(r)} dt^2 + e^{-\nu(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \]  

(15)

In PGT, torsion must also satisfy the Killing equation \( L_\xi T^\mu_{\nu \rho} = 0 \) where \( L_\xi \) is the Lie derivative in the direction of \( \xi \). By using this condition, the explicit form of the torsion tensor for a static spherically symmetric spacetime can be written as [26, 27, 28]

\[
T^t_{\ tr} = -T^{r \ t} = a(r), \ T^{r \ \theta} = -T^{\theta \ r} = k(r) \sin(\theta) e^{\nu(r)} \\
T^{r \ \phi} = -T^{\phi \ r} = -T^{t \ \phi} = k(r) \sin(\theta) \\
T^{\theta \ \phi} = -T^{\phi \ \theta} = -T^{t \ t} = h(r) \\
T^{\theta \ r} = -T^{r \ \theta} = T^{\phi \ r} = T^{r \ \phi} = g(r) e^{\nu(r)} \\
T^{\phi \ \theta} = T^{\phi \ t} = T^{\phi \ \phi} = g(r) e^{\nu(r)} \\
T^{\theta \ \phi} = T^{r \ \phi} = -T^{\phi \ \theta} = g(r)
\]

(16)

where \( a(r), k(r), h(r) \) and \( g(r) \) are four unknown functions. Using (15), (16) and the field equations (13) and (14), we get the following differential equations

\[
\nu'(r) \left[ (4g(r) - 2a(r) - \nu'(r) - \frac{4}{r}) \nu'(r) 
\right. \\
+ \frac{1}{r} \left[ \frac{2}{r} - 4a(r) + 8g(r) \right] - 4a'(r) + 8g'(r) - 3\nu''(r) \\
+ 2 \left( 2g(r) - a(r) - \frac{2}{r} \right) \nu''(r) \\
+ \frac{4}{r} \left[ 2g'(r) - a'(r) + \frac{1}{r} (a(r) - 2g(r)) \right] \\
+ \frac{4}{r} \left( 1 - e^{-\nu(r)} \right) - \nu'''(r) - 2a''(r) + 4g''(r) \]= 0 \]  

(17)

\[
+ r^2 \left[ 2a(r) + 4g'(r) - \nu''(r) + \frac{8}{r} g(r) - \frac{4}{r} a(r) \right] \\
-2 + 2e^{-\nu(r)} \]  

\[
+ r^4 \left( 4g'(r) + \nu''(r) - 2a'(r) - 8g(r)a(r) + \frac{8}{r} g(r) + \frac{4}{r} a(r) \right) \\
+ r^2 \left( 4b(r)k(r) - 2 + 2e^{-\nu(r)} \right) + 2k(r)^2 \right) \]  

(18)

\[
\left[ (4g(r) - 2a(r) - \nu'(r) - \frac{4}{r}) \nu'(r) - 2a'(r) + 4g'(r) \right] \\
- \nu''(r) + \frac{8}{r} g(r) - \frac{4}{r} a(r) - \frac{2}{r^2} (1 - e^{-\nu(r)}) \right] \\
\times \left[ 2g'(r) - 4g(r)a(r) + \frac{2}{r}(a(r) + g(r)) \\
+ \frac{k(r)}{r^2} \left( \frac{k(r)}{r^2} + 2h(r) \right) \right] = 0 \]  

(19)

\[
a_0 \left( r\nu'(r) + 1 - e^{-\nu(r)} \right) \\
+ \frac{1}{24r^4} \left\{ b e^{\nu(r)} \left[ (4g(r) - 2a(r) - \nu'(r) - \frac{4}{r}) r^2 \nu'(r) \right] \\
+ r^2 \left( -2a'(r) + 4g'(r) - \nu''(r) + \frac{8}{r} g(r) - \frac{4}{r} a(r) \right) \\
-2 + 2e^{-\nu(r)} \]  

\[
+ r^4 \left( -4g'(r) + \nu''(r) + 2a'(r) + 8g(r)a(r) - \frac{4}{r} a(r) \right) \\
+ r^2 \left( -4h(r)k(r) - 2 + 2e^{-\nu(r)} \right) - 2k(r)^2 \right) \]  

(20)

\[
a_0 \left[ \left( \nu'(r) + \frac{2}{r} \right) \nu'(r) + \nu''(r) \right] \\
- \frac{b}{3} e^{\nu(r)} \left[ (4g(r) - 2a(r) - \nu'(r) - \frac{4}{r}) \nu'(r) - 2a'(r) \right] \\
+ 4g'(r) - \nu''(r) + \frac{8}{r} g(r) - \frac{4}{r} a(r) - \frac{2}{r^2} (1 - e^{-\nu(r)}) \]


\[
x \left[ \frac{g(r)}{r} - \frac{1}{2r^2} \left( 1 - e^{-\nu(r)} \right) + \frac{1}{2} \nu'(r) \left( a(r) + \frac{1}{2} \nu'(r) \right) \right] + \frac{1}{2} \nu'(r) + \frac{1}{4} \nu''(r) = 0
\]

(21)

Where a prime denotes differentiation with respect to \( r \). After solving the differential equations analytically, we can obtain the solution for 5 unknown functions \( \nu(r), a(r), g(r), h(r) \) and \( k(r) \). However equations (18) and (20) are not independent of each other. This leads to a dependent solution for torsion functions \( h(r) \) and \( k(r) \). Torsion solutions of the system are presented in the appendix. The solution for the metric function \( \nu(r) \) is

\[
\nu(r) = \ln \left( 1 - \frac{c_1}{r} + \frac{c_2}{r^2} \right)
\]

(22)

Substituting this relation in (15), we get the equivalent to the Schwarzschild metric in PGT. For positive values of constant \( c_2 \), this is similar to the Reissner-Nordstrom solution in general relativity. This result is consistent with the results of reference 26. It should be noted that our solutions are obtained for general values of coupling constants \( a_0, a_1 \) and \( b \). The solutions have 3 constants of integration \( c_1, c_2 \) and \( c_3 \) (The constant \( c_3 \) appears in the solutions for torsion functions given in the appendix). For interpreting these constants, we consider a limiting case of the differential equations (17-21) when all components of the torsion tensor are set to zero. If we consider this particular case, \( a(r) = h(r) = g(r) = k(r) = 0 \), then equation (19) becomes trivial and we have a set of differential equations just in \( \nu(r) \). By solving equation (17) for \( \nu(r) \) in this case, we obtain the following form

\[
\nu(r) = \ln \left( 1 - \frac{c_4}{r} + \frac{c_5}{r^2} - \frac{6b^2}{12} \right)
\]

(23)

But this solution must satisfy all other equations. Substituting (20) in equations (18), (20) and (21), and solving for the constants \( c_4, c_5 \) and \( c_6 \), gives \( c_5 = c_6 = 0 \). In this case, as one can see, the Schwarzschild metric of GR will be recovered from (23). Combining these results, we find that the constant \( c_1 \) is related to the mass of the black hole. Also constants \( c_2 \) and \( c_3 \) should be related to the spin effects which induce torsion in the spacetime. From now on we write the solution (22) for \( \nu(r) \) as

\[
\nu(r) = \ln \left( 1 - \frac{2m}{r} + \frac{S}{r^2} \right)
\]

(24)

In which \( m \) and \( S \) are some charges related to field strengths of curvature and torsion, respectively. The black hole metric (15) then will be

\[
f(r) = 1 - \frac{2m}{r} + \frac{S}{r^2}
\]

(26)

The roots of \( f(r) \) are black hole horizons

\[
R_\pm = m \pm \sqrt{m^2 - S}
\]

(27)

with the following condition

\[
m^2 \geq S
\]

(28)

\( R_+ \) is outer horizon and can be interpreted as the Schwarzschild radius of the black hole. From now on we rename \( R_+ \) as \( R \). For the positive values of \( S \), we have \( R_{PGT} < R_{GR} \) while for negative values of \( S \), the opposite is true i.e. \( R_{PGT} > R_{GR} \). As expected, in the limiting case of \( S = 0 \), the Schwarzschild radius is equal in Poincaré gauge theory and general relativity.

IV. EFFECTIVE POTENTIAL AND ORBITS FOR A BLACK HOLE WITH TORSION

For the line element (24), there exist two killing vectors \( \xi \) associated with energy \( E \) and the angular momentum \( L \) per unit mass which are conserved quantities of motion along the geodesics

\[
\xi_\mu = \left( - (1 - \frac{2m}{r} + \frac{S}{r^2}), 0, 0, 0 \right)
\]

(29)

\[
\xi_\mu = \left( 0, 0, 0, r^2 \sin^2(\theta) \right)
\]

(30)

In equatorial plane \( \theta = \pi/2 \) the line element is simplified as follows

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\theta^2 + r^2 \sin^2(\theta)d\phi^2
\]

(31)

We also have for \( E \) and \( L \)

\[
E = -P_t = -g_{tt}P^t = f(r) \frac{dt}{d\lambda} \rightarrow \frac{dt}{d\lambda} = \frac{E}{f(r)}
\]

(32)

\[
L = P_\phi = g_{\phi\phi}P^\phi = r^2 \frac{d\phi}{d\lambda} \rightarrow \frac{d\phi}{d\lambda} = \frac{L}{r^2}
\]

(33)

Where \( P \) is energy-momentum 4-vector and \( \lambda \) is an affine parameter.

With substituting relations (32) and (33) in line element (31) and some straightforward calculations we obtain relation as follows

\[
k = \left( \frac{ds}{d\lambda} \right)^2 = -\frac{E^2}{f(r)} + \frac{1}{f(r)} \left( \frac{dr}{d\lambda} \right)^2 + \frac{L^2}{r^2}
\]

(34)
where $k$ is equal to $-1$, $+1$ and 0 for timelike, spacelike and null geodesics, respectively. Equation \(8\) can be written as

$$
\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2m}{r} + \frac{S}{r^2}\right)\left(-k + \frac{L^2}{r^2}\right) \quad (35)
$$

From the above relation, we define the effective potential as follows

$$
V_{\text{eff}}(r) = \left(1 - \frac{2m}{r} + \frac{S}{r^2}\right)\left(-k + \frac{L^2}{r^2}\right) \quad (36)
$$

The position of the orbits for the effective potential is given by the condition

$$
\left.\frac{dV_{\text{eff}}}{dr}\right|_{r_c} = 0 \quad (37)
$$

The orbits are stable if

$$
\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r_c} > 0 \quad (38)
$$

The quantities $m$ and $L^2$ in \(8\) are positive, however $S$ could be positive, negative or zero. We will examine orbits of particles for different values of $S$ and compare the result to the orbits in general relativity ($S = 0$). For photons ($k = 0$), we have

$$
\left.\frac{dV_{\text{eff}}}{dr}\right|_{r_c} = 0 \quad \Rightarrow \quad r_c = \frac{3m}{2} \pm \frac{1}{2}\sqrt{9m^2 - 8S} \quad (40)
$$

This result is independent of $L$. We now look for the stability of these two orbits. For the $r_{c+}$ we have

$$
\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r_{c+}} = -64 \frac{L^2\left(3m\sqrt{9m^2 - 8S} + 9m^2 - 8S\right)}{(3m + \sqrt{9m^2 - 8S})^6} \quad (41)
$$

The condition \(28\) ensures that $9m^2 - 8S$ in the above equation will always be positive or zero; as $m$ and $L^2$ are also positive or zero, \(\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r_{c+}}\) will always be negative or zero in this case and as a result we conclude that the orbit given by $r_{c+}$ is not stable. For the $r_{c-}$ orbit we have

$$
\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r_{c-}} = 64 \frac{L^2\left(3m\sqrt{9m^2 - 8S} - 9m^2 + 8S\right)}{(-3m + \sqrt{9m^2 - 8S})^6} \quad (42)
$$

This orbit will be stable for $S > 0$ and unstable for $S < 0$. In general, there are no stable orbits of photons for negative $S$. For the limiting case of $S = 0$, we get

$$
\lim_{S \to 0} r_{c-} = 0, \quad \lim_{S \to 0} r_{c+} = 3m \quad (43)
$$

which is consistent with the GR case where there exist a single unstable orbit at $r = 3m$ for photons. Figure (1) shows the effective potential of equation \(36\). One
In the case of L \text{orbits given by} \ S_r \text{orbit is always stable while one stable orbit at } r = r_{c-}\text{. There are no stable orbits for negative } S \text{ as can be seen from the right figure.}

For massive particles, \( k = -1 \), we have

\[
\frac{dV_{\text{eff}}}{dr} = \frac{2m}{r^2} - \frac{2(S + L^2)}{r^3} + \frac{6mL^2}{r^4} - \frac{4SL^2}{r^5} \tag{44}
\]

In this case, equation (37) can be written as

\[
Mr^3 - (L^2 + S)r^2 + 3mL^2r - 2SL^2 = 0 \tag{45}
\]

In the limiting case of \( S = 0 \), we have

\[
r_c\left(Mr_c^2 - L^2r_c + 3mL^2\right) = 0 \tag{46}
\]

which gives the solutions as \( r_c = 0 \) and

\[
r_{c\pm} = \frac{L^2}{2m}\left(1 \pm \sqrt{1 - \frac{12m^2}{L^2}}\right) \tag{47}
\]

Using the stability condition (38), we find that the \( r_{c+} \) orbit is always stable while \( r_{c-} \) orbit is always unstable. In the case of \( L^2 = 12m^2 \) these two orbits will coincide at \( r_c = 6m \).

For general values of \( S \), equation (45) has three solutions given by

\[
r_{c1} = \frac{1}{3m}\left(L^2 + S + \frac{\alpha}{2} + \frac{2\beta}{\alpha}\right) \tag{48}
\]

\[
r_{c2,3} = \frac{1}{3m}\left(L^2 + S - \frac{\alpha}{4} - \frac{\beta}{\alpha}\right) \pm \frac{i}{6m}\left(\frac{\alpha}{2} - \frac{2\beta}{\alpha}\right) \tag{49}
\]

where

\[
\alpha = \left[4L^2\left(2L^4 - 27m^2(L^2 - S) + 6S(L^2 + S)\right) + 8S^3
\right.
\]

\[
+12\sqrt{3}mL\left(L^4(108m^4 - 126m^2S + 24S^2)
\right.
\]

\[
+L^6(8S - 9m^2) + L^2S^2(24S - 9m^2^2) + 8S^4\right]\frac{1}{4}
\]

\[
\beta = L^2(L^2 - 9m^2 + 2S) + S^2
\]

In order to determine the sign of the above solutions, we employ the Cardano method for solving cubic equations. Using the following change of parameter

\[
r_c = x + \frac{L^2 + S}{3m} \tag{50}
\]

equation (45) can be rewritten as

\[
x^3 + px + q = 0 \tag{51}
\]

where

\[
p = \frac{L^2(9m^2 - L^2 - 2S) - S^2}{3m^2} \tag{52}
\]

\[
q = \frac{L^2(27m^2L^2 - 2L^4 - 6L^2S - 27m^2S - 6S^2) + 2S^3}{27m^3} \tag{53}
\]

To determine whether the solutions are real or complex and also the sign of the solutions, we define

\[
\Delta = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \tag{54}
\]

Then, if \( \Delta > 0 \), there exist a single real solution to equation (51). In the case of \( \Delta < 0 \) there are three real solutions. Finally if \( \Delta = 0 \), there are three real solutions, two of which coincide.

Substituting (52) and (53) in (54), we get for the case \( S = 0 \)

\[
\Delta = -\frac{L^6}{12m^2}\left[L^2 - 12m^2\right] \tag{55}
\]

The sign of \( \Delta \) in (55) depends on the sign of the term inside the brackets. Interestingly this term is what determines the existence and stability of solutions in general relativity, as obvious from equation (47).

Motivating by this, we analyze the solutions of equation (45) for general \( S \) in three different cases:

- **Case 1:** For \( L^2 < 12m^2 \) there exist a single stable orbit for \( S > 0 \) (Figure (2)). The radius of this orbit increases with increasing \( S \). There can be no orbits for \( S < 0 \), as one can see in the right figure.

- **Case 2:** For \( L^2 > 12m^2 \), there exist two stable orbits and one unstable orbit for \( S > 0 \) (Figure (3)). For \( S < 0 \) there exist one stable and one unstable orbit. The radius of stable orbits increase with increasing \( S \) for all values of \( S \). The opposite is true for unstable orbits. These stable orbits and their behaviors can also be seen in figure (4) for different values of \( S \). This figure shows that the radius of second stable orbit increases
FIG. 2: Behavior of effective potential of equation (36) for massive particles \( k = -1 \), case 1, \( L^2 < 12m^2 \): The left figure shows the effective potential for different values of orbital angular momentum per unit mass \( L \) with constant \( S = 0.7 \) and \( m = 1 \). The right figure shows the same effective potential for different values of \( S \) with constant \( L = 1 \) and \( m = 1 \).

FIG. 3: Behavior of effective potential of equation (36) for massive particles \( k = -1 \), case 2, \( L^2 > 12m^2 \): The left figure shows the effective potential for different values of orbital angular momentum per unit mass \( L \) with constant \( S = 0.7 \) and \( m = 1 \). The right figure shows different part of the same potential for \( S = 0.7 \) and \( S = -0.7 \).

with increasing \( L \).

- Case 3: For \( L^2 = 12m^2 \), there exist two stable orbits and one unstable orbit for \( S > 0 \). There are no orbits for \( S < 0 \). The behavior of orbits is the same as the case 2.

It is also interesting to examine the case of particles with zero orbital angular momentum per unit mass, \( L = 0 \). In this case we have

\[
\lim_{L \to 0} r_{c1} = \frac{S}{m} \quad (56)
\]

and the other solutions vanish in this limit. The stability condition takes the form

\[
\lim_{L \to 0} \left( \frac{d^2 V_{eff}}{dr^2} \right)_{r_{c1}} = \frac{2m^4}{S^3} \quad (57)
\]

which is always greater than zero for \( S > 0 \). This is an interesting result, as in GR there are no orbits for \( L = 0 \). As a consequence, the stable orbit for \( S > 0 \) given by (49) for spacetime with torsion, is a result of the interaction between the spin of particle and the background spacetime.
FIG. 4: Behavior of effective potential of equation (36) for massive particles \( (k = -1) \), case 2, \( L^2 > 12m^2 \). The left figure shows the effective potential for different values of \( S \) with constant \( L = 10 \) and \( m = 1 \). The right figure shows different part of the same potential.

V. BLACK HOLE THERMODYNAMICS WITH TORSION

We now turn our attention to black hole thermodynamics in the presence of torsion. We begin with the definition of the surface gravity

\[ \kappa = \frac{1}{2} f'(R) \]  
(58)

where \( f(r) \) and the horizon radius \( R \) now are given by equations (26) and (27) respectively. Using (26) and (58), we get

\[ \kappa = \frac{\sqrt{m^2 - S}}{\left( m + \sqrt{m^2 - S} \right)^2} \]  
(59)

In the GR limit, \( S = 0 \), this equation takes the form

\[ \lim_{S \to 0} \frac{\sqrt{m^2 - S}}{\left( m + \sqrt{m^2 - S} \right)^2} = \frac{1}{4m} = \kappa_{GR} \]  
(60)

One can see that in the appropriate limit, the surface gravity approaches its GR value. In order to obtain the first law of black hole thermodynamics in the presence of torsion, we consider a black hole with parameters \( m \) and \( S \) and assume that the black hole undergoes a change in the parameters by a quasi-static process to new parameter values \( m + \delta m \) and \( S + \delta S \). The surface area of the horizon, \( A = 4\pi R^2 \) is a function of the parameters \( m \) and \( S \) by the virtue of equation (27).

\[ A = 4\pi \left( m + \sqrt{m^2 - S} \right)^2 \]  
(61)

A change in this parameters gives

\[ \delta A = \frac{\partial A}{\partial m} \delta m + \frac{\partial A}{\partial S} \delta S \]  
(62)

Combining the last two equations we get

\[ \delta A = \frac{8\pi}{\sqrt{m^2 - S}} \delta m - \frac{4\pi}{\sqrt{m^2 - S}} \delta S \]  
(63)

A simple algebra gives

\[ \delta m = \frac{\sqrt{m^2 - S}}{2\pi(m + \sqrt{m^2 - S})^2} \frac{\delta A}{4} + \frac{\delta S}{2(m + \sqrt{m^2 - S})} \]  
(64)

We define the temperature \( T \) and entropy \( \Sigma \) of the black hole in the presence of torsion by

\[ T = \frac{\sqrt{m^2 - S}}{2\pi(m + \sqrt{m^2 - S})^2}, \quad \Sigma = \pi \left( m + \sqrt{m^2 - S} \right)^2 \]  
(65)

If we compare equation (64) with the general law of black hole thermodynamics

\[ \delta m = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J + \Phi \delta Q \]  
(66)

we can see that the first term in the right-hand-side of equation (64) corresponds to the \( Td\Sigma \) term in the first law of classical thermodynamics and also to the \( \frac{1}{8\pi} \kappa \delta A \) term in equation (60). For a static black hole, the second term in (60) vanishes. Also if we define torsion 'charge' and 'potential' as
we can interpret the second term in the right-hand-side of equation (63) as the $\Phi \delta Q$ term in (66). This definition of the torsion potential is also consistent with (27).

VI. CONCLUSION

In this paper, we study static spherically symmetric solutions to the field equations in the Poincaré gauge theory of gravity. The effects of torsion appear as a sort of torsion ‘charge’ related to the spin of matter. We also study particle orbits around a black hole in this geometry for both massless and massive particles. For massless particles, there exist one stable orbit at $r = r_{c+}$ and one unstable one at $r = r_{c-}$, where $r = r_{c \pm}$ are given by equation (40). There are no stable orbits for negative $S$. For massive particles, there exist one stable orbit for $S > 0$ and no orbit for $S < 0$ in the case $L^2 < 12m^2$. For the case $L^2 > 12m^2$, there exist two stable orbits and one unstable orbit for $S > 0$. Also, there is one stable and one unstable orbit for $S < 0$. The case $L^2 = 12m^2$ is the same as $S > 0$ for the case $L^2 > 12m^2$. There is no orbit for $S < 0$. Remarkably for massive particles, there also exist a stable orbit even when orbital angular momentum per unit mass $L$ is set to zero. This suggests that this orbit is a result of the interaction between spin of particles with the torsion of the background geometry. Finally, we derive the first law of black hole thermodynamics by defining the temperature and entropy of a black hole with torsion in terms of its mass and torsion charge.

VII. APPENDIX

In this appendix we present the full solutions to the torsion function in the system of equations given by (17-21).

$$a(r) = \frac{(1 - \frac{c_1}{r} + \frac{c_2}{r^2})^{-1}}{2r^2} \left(c_1(r^2 - 1) - \frac{2a_0r}{c_1b}(1 - \frac{c_2}{r^2}) + 2r(1 + 2c_3) - 2c_2(1 - \frac{1}{r})\right)$$  (68)

$$g(r) = \frac{(1 - \frac{c_1}{r} + \frac{c_2}{r^2})^{-1}}{4r^2} \left(-c_1(r^3 + 2) - \frac{2a_0r}{c_1b}(1 - \frac{c_2}{r^2})\right)$$

$$Q_{\text{Torsion}} = S, \quad \Phi_{\text{Torsion}} = \frac{1}{2(m + \sqrt{m^2 - S})}$$  (67)

$$h(r) \left[r^3 + c_1r(c_1 - 2r) + c_2(2r(c_2 - c_1) + c_2)\right] = \left\{2k(r)^2(c_1br)^2 \left[r^2 + c_1(c_1 - 2r) + \left(\frac{c_2}{r}\right)^2 + 2c_2(1 - \frac{c_1}{r})\right]\right.$$

$$+2r(c_1br)^2 \left[c_1c_2r(r^3 - r^2 - 2) + c_1^2r^4(1 + \frac{r^3}{2})\right]$$

$$+2c_2(c_2(1 - 2r) + r^2(1 + 2c_3)) - 4c_3r^3(1 + 2c_3)\right\]$$

$$+4c_1a_0br^3 \left[c_2(c_1 + c_2(3 - \frac{1}{r}) - 2r(1 + 2c_3) - 3r^2) + r^3\right]$$

$$+4a_0r^2 \left(c_2(2r^2 - c_2) - r^4\right)\right\} \times \left(-\frac{1}{4rk(r)(c_1br)^2}\right)$$  (70)
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