Conditional quantile estimators:  
A small sample theory*

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November 9, 2020

Abstract

This paper studies small sample properties and bias of just-identified instrumental variable quantile regression (IVQR) estimators, nesting order statistics and classical quantile regression. We propose a theoretical framework for analyzing small sample properties based on a novel approximation of the discontinuous sample moments with a Hölder continuous process. Using this approximation, we derive remainder bounds for the asymptotic linear expansions of exact and k-step estimators of IVQR models. Furthermore, we derive a bias formula for exact IVQR estimators up to order $o \left( \frac{1}{n} \right)$. The bias contains components that cannot be consistently estimated and depend on the particular numerical estimation algorithm. To circumvent this problem, we propose a novel 1-step adjustment of the estimator, which admits a feasible bias correction. Monte Carlo evidence suggests that our formula removes a substantial portion of the bias for sample sizes as small as $n = 50$. We suggest using exact estimators, when possible, to achieve the smallest bias. Otherwise, applying 1-step corrections may improve the higher-order bias and MSE of any consistent estimator.

JEL Classification: C21, C26.

Keywords: non-smooth estimators, KMT coupling, Hungarian construction, higher-order stochastic expansion, order statistic, bias correction, mixed integer linear programming (MILP), exact estimators, k-step estimators, instrumental variable quantile regression.

*We thank seminar participants at UC Davis and USC for comments. All errors and omissions are our own.

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1 Introduction

Since the seminal contribution of Koenker and Bassett (1978), conditional quantile methods have become popular for modeling the impact of covariates on the entire distribution of the outcome variable of interest (for extensive reviews, see Koenker, 2005; Koenker et al., 2017). In this paper, we consider the instrumental variable quantile regression (IVQR) model (Chernozhukov and Hansen, 2005, 2006), which nests unconditional quantiles and the classical quantile regression (QR) model (see Chernozhukov and Hansen, 2013; Chernozhukov et al., 2017b, for recent reviews). Many interesting empirical applications of conditional quantile methods feature small sample sizes, either as a result of a limited number of observations (e.g., Adrian et al., 2019) or when estimating tail quantiles (e.g., Chernozhukov et al., 2017a). The goal and main contribution of this paper is to develop a small sample theory for conditional quantile estimators.

For most existing IVQR estimators, only the first order large-sample theory has been developed. This theory provides normal approximations that are accurate up to order \( o_p\left(\frac{1}{\sqrt{n}}\right) \). In small samples, the error of this approximation may affect the relative performance of first order equivalent estimators. This observation motivates the study of the higher order properties of IVQR estimators, such as their higher order bias and mean squared error (MSE).

The classical results on the higher order properties of moment estimators (e.g., Nagar, 1959; Rilstone et al., 1996; Newey and Smith, 2004) rely on smoothness of the sample moment conditions. These results do not apply to IVQR estimators because the IVQR sample moments are discontinuous, which gives rise to two main challenges. First, the sample moment conditions do not have an exact solution in most cases, so the error in the estimating equations cannot be made arbitrarily small. Second, one component of the higher-order remainder is the difference between the values of the sample moment computed at the true and the estimated parameter values. Typically, for models with smooth sample moment functions, the increments of the corresponding empirical process are approximately linear in the argument such that this difference is \( O_p\left(\frac{1}{n}\right) \) for \( \sqrt{n} \)-consistent estimators. This property does not hold for IVQR models. For example, for unconditional quantiles, the discontinuity makes this difference nearly \( O_p\left(\frac{1}{n^{3/4}}\right) \) instead of \( O_p\left(\frac{1}{n}\right) \) (e.g., Kiefer, 1967; Jureckova and Sen, 1987).

\(^1\)Our results do not cover instrumental variables approaches based on the local average treatment effects framework (e.g., Abadie et al., 2002; Frölich and Melly, 2013; Melly and Wüthrich, 2017; Wüthrich, 2020) and triangular models (e.g., Ma and Koenker, 2006; Lee, 2007; Imbens and Newey, 2009, among others). An alternative approach to conditional distribution modeling is based on distribution regression (Foresi and Peracchi, 1995; Chernozhukov et al., 2013); distribution regression can also be used for estimating IVQR models (Wüthrich, 2019). Distribution and quantile regression are not nested in general and, thus, the results in our paper do not directly apply to distribution regression models.
We make three main contributions to the literature on IVQR models. First, we establish a novel coupling of the weighted empirical cumulative distribution function (CDF) with a tight Hölder continuous process with a uniform remainder bound. Second, based on this coupling result, we derive remainder rates for asymptotic linear expansions of exact (moment) estimators, which exactly minimize a norm of the sample moments, as well as k-step estimators of just-identified models. Third, we derive an asymptotic bias formula for exact estimators of just-identified models that captures the bias up to order $o\left(\frac{1}{n}\right)$.

**Coupling of sample moments.** In Theorem 1, we develop a novel coupling of the stochastically weighted empirical CDF process by a tight Hölder continuous process with exponent $\frac{1}{2} - \gamma$ for any small $\gamma > 0$ and a uniformly bounded $O_p\left(\frac{\log n}{n}\right)$ error. The expectation of the absolute approximation error depends only on the support of the random weights and the sample size; the expected Hölder constant depends on both the bound of the conditional probability density function (PDF) of the outcome and the support of the weights. Our result is a generalization of the KMT coupling of the uniform empirical process (Komlós et al., 1975, 1976; Pollard, 2002), which additionally exploits the smoothness of the conditional CDF. Under some conditions on the regressors and instruments, this result translates into a coupling for the IVQR sample moments with the same rates.

**Bahadur-Kiefer representations.** IVQR estimators are non-linear. Nevertheless, it is well-known that such estimators can be approximately represented as sample averages of score functions. Such asymptotic linear representations are called *Bahadur-Kiefer representations* (Bahadur, 1966; Kiefer, 1967). It is important to have a bound on the non-linear remainder term to make statements on the precision of the normal approximation and computationally attractive multiplier bootstrap procedures. This motivated an extensive literature; see, for example, Chapter 4 in Jurečková et al. (2012) and the references therein.

Using our coupling result, we derive Bahadur-Kiefer expansions for $\sqrt{n}$-consistent IVQR estimators under some support restrictions on the regressors (see Theorem 2). Namely, one can use such an estimator as a starting point and subtract a product of the inverse Jacobian of the population moments multiplied by the vector of sample moments evaluated at the estimator. After such a 1-step *Newton correction*, any $\sqrt{n}$-consistent estimator has an asymptotic linear representation with nearly $O_p\left(\frac{1}{n^{3/4}}\right)$ remainder rate.

To our knowledge, this paper is the first to provide an explicit rate of the remainder in the Bahadur-Kiefer representation for general IVQR estimators. The available results were limited to order statistics (Bahadur, 1966; Kiefer, 1967) and classical QR with exogenous

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2In practice, the population Jacobian moments can be replaced by a $\sqrt{n}$-consistent estimator.
regressors (e.g., Zhou et al., 1996; Knight, 2002). The existing results used specific properties of order statistics or classical QR. Our approach is based on general IVQR moment conditions and, thus, is much more broadly applicable. We show that the availability of a Bahadur-Kiefer expansion with nearly $O_p\left(\frac{1}{n^{3/4}}\right)$ rate does not depend on the specific structure of the estimator, but rather on the Hölder continuity of the IVQR sample moments.

Our results also have important implications for the higher-order properties of k-step IVQR estimators (e.g., Zhu, 2019) and complement the findings in Robinson (1988) for estimators based on smooth sample moment conditions. Unlike with smooth extremum estimators, additional Newton steps may not result in convergence of k-step estimators to the exact minimizer of the GMM objective function. However, we show that $\sqrt{n}$-consistent estimators become equivalent up to nearly $O_p\left(\frac{1}{n^{3/4}}\right)$ after a single Newton correction (see Theorems 3 and 4). This is in contrast to Andrews (2002a,b) who shows that 1-step corrected smooth estimators are equivalent with higher precision, $O_p\left(\frac{1}{n^{3/2}}\right)$.

Finally, we consider exact estimators that exactly minimize some vector norm of the IVQR sample moments. Such estimators can be implemented using mixed integer programming (MIP) techniques (Chen and Lee, 2017; Zhu, 2019). We use the coupling result to prove that the norm of exact estimators attains nearly $O_p\left(\frac{1}{n}\right)$ asymptotic rate and establish a higher order expansion of exact estimators of just identified models up to $o_p\left(\frac{1}{n}\right)$ (see Theorem 5). It implies, in particular, that any two exact minimizers (possibly corresponding to different norms) are equivalent up to nearly $O_p\left(\frac{1}{n}\right)$. This equivalence applies to the exact estimators proposed by Chen and Lee (2017) and Zhu (2019), which minimize the $\ell_2$ and $\ell_\infty$ norm, respectively.

**Bias correction formula.** An important practical aspect of the small sample performance of IVQR estimators is their bias. To our knowledge, there is no complete characterization of the higher-order bias of IVQR estimators up to $o\left(\frac{1}{n}\right)$ even in the sample quantile case. We provide a formula for the bias of the exact moment estimators in just-identified models up to $o\left(\frac{1}{n}\right)$. The bias has four components: (1) the bias from non-zero sample moments at the optimum; (2) the bias from the covariance of the linear influence of a single observation.

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3In a related work, Portnoy et al. (2012) developed a non-linear Gaussian approximation of classical QR. His saddle path approximation approach is convenient for analyzing the approximate density of the estimator but is not well-suited for analyzing its bias beyond the order of magnitude or equivalence of various estimators.

4Pouliot (2019) proposed an alternative MIP approach based on the inverse quantile regression estimator of Chernozhukov and Hansen (2006). We further discuss this approach in Section 3.4.

5The interesting recent work by Lee et al. (2017, 2018) proposed formulas based on generalized function heuristic (e.g., Phillips, 1991) that partially corrects the $\frac{1}{n}$-terms in the bias of IVQR and classical QR. We discuss their work in more detail in Section 4, where we show that the bias formula has additional, even larger, terms.
and the sample moments; (3) the bias from the positive measure of IVQR residuals at 0; (4) the bias from the typical higher-order quadratic component of the population moment conditions (e.g., Rilstone et al., 1996).

Components (1) and (3) appear because of the discontinuity of the sample moments. These two components depend on the realization of the estimator and, to our knowledge, there exists no approach to consistently estimate them. Therefore, we suggest a symmetric 1-step correction of the exact estimators, which is the average of two Newton corrections based on IVQR sample moments corresponding to \((\tau, Y_i)\) and \((1 - \tau, -Y_i)\). This correction eliminates bias components (1) and (3). The remaining components of the asymptotic bias take the form of weighted-average derivatives, such that standard kernel-based procedures can be used (e.g., Powell, 1986; Powell et al., 1989).

Our general bias formula has implications for order statistics. Specifically, order statistics can be represented as exact moment estimators, so that our results also apply in that case. We show that our asymptotic bias formula matches the well-known exact bias formula corresponding to uniformly distributed data up to \(O(\frac{1}{n})\).

We validate our formula in a Monte Carlo study for multiple designs where analytical formulas for the bias components are available. The simulations show that the asymptotic bias correction removes a substantial portion of the finite sample bias for sample size as small as \(n = 50\).

**Practical implications.** There are two important take-aways from our analysis. First, exact moment estimators enjoy nearly-\(\frac{1}{n}\) bias, which can be further reduced using our bias correction formula. This property is particularly relevant for the small sample analysis. Second, for settings with larger samples, where exact estimators may be computationally prohibitive, our result suggest that a 1-step correction of existing estimators (e.g., Chernozhukov and Hong, 2003; Chernozhukov and Hansen, 2006; Kaido and Wüthrich, 2019) may improve higher order bias and MSE.\(^6\)

**Outline.** In Section 2, we present the setup and explain the complications arising from discontinuous sample moments. Section 3 presents the coupling result, provides Bahadur-Kiefer expansions of generic IVQR estimators, equivalence results, and expansions for exact estimators. In Section 4, we derive the bias correction formula, suggest a feasible implementation, and present a Monte Carlo validation of our formula. Finally, Section 5 discusses the implications of our theoretical analysis for empirical practice.

\(^6\)Under additional smoothness conditions on PDF of the data, the estimators based on smoothed estimation equations studied in Kaplan and Sun (2017) and de Castro et al. (2019) may have a smaller higher-order MSE than 1-step estimators.
2 Setup and background

In this section, we introduce the IVQR model, review existing approaches for deriving stochastic expansions of estimating equations estimators, and discuss the complications in deriving higher-order results arising from discontinuous sample moments.

2.1 The IVQR model

The IVQR model was introduced by Chernozhukov and Hansen (2005, 2006). This model nests univariate sample quantiles and classical QR (Koenker and Bassett, 1978).

Consider a setting with a continuous outcome variable $Y$, a $(k \times 1)$ vector of regressors $W$, and a $(k' \times 1)$ vector of instruments $Z$. Every observation $(Y_i, W_i, Z_i)$, $i = 1, \ldots, n$, is jointly drawn from a distribution $P$. Throughout the paper it is assumed that $(Y_i, W_i, Z_i)$ is i.i.d.

Consider the linear-in-parameters structural model

$$ Y = W'\theta_0(U), \quad U \mid Z \sim \text{Uniform}(0, 1), \quad (1) $$

where $\theta_0 : (0, 1) \rightarrow \Theta \subset \mathbb{R}^k$ is a measurable function and the mapping $\tau \mapsto W'\theta_0(\tau)$ is nondecreasing in $\tau$ for almost every realization of $W$. For any quantile level $\tau \in (0, 1)$, this model implies

$$ \mathbb{E}(1\{Y \leq W'\theta_0(\tau)\} \mid Z) = \mathbb{E}(1\{U \leq \tau\} \mid Z) = \tau. \quad (2) $$

When $W = Z = 1$, $\theta_0(\tau)$ is the unconditional $\tau$-quantile. When $W = Z$, this model reduces to the classical QR model of Koenker and Bassett (1978), and $W'\theta_0(\tau)$ is the conditional $\tau$-quantile of $Y$ given $W$. When $W \neq Z$ and $Z$ is a vector of instruments, the conditional moment restriction (2) is the main testable implication of the linear-in-parameters IVQR model of Chernozhukov and Hansen (2006), and $W'\theta_0(\tau)$ is the structural (conditional potential outcome) $\tau$-quantile.

The conditional moment restriction (2) implies the following unconditional moment restriction for estimation

$$ \mathbb{E}[(1\{Y \leq W'\theta_0(\tau)\} - \tau)Z] = 0 \quad (3) $$

Throughout the paper, we will often omit the dependence on the quantile $\tau$ unless it causes confusion. We will use the following notation for the unconditional moment restrictions as function of $\theta \in \Theta$,

$$ g(\theta) \triangleq \mathbb{E}(1\{Y \leq W'\theta\} - \tau). \quad (4) $$

To abstract from additional complications arising from the estimation of the optimal weight-
ing matrix, we focus on the just-identified case where $k = k'$.

The following example shows that model (1) nests the classical location and location-scale models with endogeneity.

**Example 1 (Location-scale model).** Consider the heteroscedastic model

$$Y = W'\beta + (W'\gamma)F^{-1}(U), \quad U \mid Z \sim \text{Uniform}(0, 1),$$

where $F$ is a continuous CDF and $(W'\gamma)$ is almost surely positive, $W' = (1, X')$. A special case of model (5) is the homoscedastic linear instrumental variable (IV) model,

$$Y = W'\beta + F^{-1}(U), \quad U \mid Z \sim \text{Uniform}(0, 1).$$

Unlike the conventional IV mean regression, the model (6) allows for heavy-tailed data, for example, when $F(\cdot)$ is the CDF of the Cauchy distribution. The structural model (1) nests model (5) with $\theta(U) = \beta + \gamma F^{-1}(U)$ and model (6) with $\theta(U) = \beta + e_1 F^{-1}(U)$, where $e_1 = (1, 0, \ldots, 0)^\prime \in \mathbb{R}^k$.

### 2.2 Complications due to discontinuous sample moments

Consider any estimator $\hat{\theta}$ of $\theta_0$ that approximately solves

$$\hat{g}(\theta) = 0,$$

where $\hat{g}(\theta)$ is the sample analog of moment condition (3), $\hat{g}(\theta) \triangleq \mathbb{E}_n(1\{Y \leq W'\theta\} - \tau)$ for $\theta \in \Theta$. Here we used notation $\mathbb{E}_n m$ as a shortcut for expectation with respect to the empirical measure, i.e., $\frac{1}{n} \sum_{i=1}^n m_i$. The standard way to get an approximate solution to (7) is to minimize $||\hat{g}(\theta)||_p$, where $|| \cdot ||_p$ is an $\ell^p$ norm on $\mathbb{R}^k$.

In the case where the sample moment conditions $\hat{g}(\theta)$ are a.s. differentiable, the higher order expansions of the exact and $\ell^2$-approximate solutions of (7) were studied correspondingly in Rilstone et al. (1996) and Newey and Smith (2004).\(^7\) One can understand the nature of the standard argument using the following tautology

$$\frac{\hat{g}(\hat{\theta}) - g(\theta_0)}{(i)} = \frac{1}{\sqrt{n}} \mathbb{G}_n(\theta_0) + \frac{1}{\sqrt{n}} (g(\hat{\theta}) - g(\theta_0)) + \frac{1}{\sqrt{n}} (\mathbb{G}_n(\hat{\theta}) - \mathbb{G}_n(\theta_0)),$$

\(^7\)We abstract from the additional higher-order bias terms arising from overidentification and estimation of the GMM weighting matrix.
where the empirical process $G_n(\cdot)$ is defined as

$$G_n(\theta) \triangleq \sqrt{n}(\hat{g}(\theta) - g(\theta)), \quad \theta \in \Theta.$$  \hfill (9)

Under the standard assumptions (e.g., Rilstone et al., 1996, Assumptions B and C), including the Lipschitz continuity of the sample moments and non-degeneracy of their Jacobian, the following results hold:\footnote{See also Assumptions 2(b) and 2(c) in Newey and Smith (2004).}

(i) can be made arbitrarily small since (7) admits exact solution;

(ii) is zero for correctly specified models;

(iii) is asymptotically normal by the CLT;

(iv) is approximately linear in $(\hat{\theta} - \theta_0)$ with an error that is $O_p(\frac{1}{n})$ by the Taylor theorem applied to $g(\theta)$ at $\theta_0$;

(v) is $O_p(\frac{1}{n})$ by a.s. Lipschitz continuity of $\hat{g}(\theta)$.

Moreover, the exact solution to (7) is unique by the assumed non-degeneracy of the Jacobian $\partial_\theta \hat{g}(\hat{\theta})$. However, the IVQR sample moment conditions $\hat{g}(\theta)$ are a.s. discontinuous and have a degenerate Jacobian, which gives rise to three main complications.

**Non-existence of exact solutions to the estimating equations.** Even in the just-identified setting ($k = k'$), equations (7) do not have an exact solution in most cases. As a result, we cannot assume that the error in the estimating equations is arbitrarily small as illustrated in the following example.

**Example 2.** Suppose we are interested in the $\tau$-quantile of a one-dimensional r.v. $Y_i$. The corresponding estimating equations take the form

$$\mathbb{E}_n(1\{Y \leq \theta\}) - \tau = 0.$$  \hfill (10)

Equation (10) can be satisfied exactly only if $\tau n$ is an integer. In general, the best approximate solution $\hat{\theta}$ can take any value in $[Y_{(\lfloor \tau n \rfloor)}, Y_{(\lfloor \tau n \rfloor + 1)})$.\footnote{We use the notation $\lfloor x \rfloor$ for the integer part of a real number $x$ and $Y_{(k)}$ for the $k$-th order statistic.} Then by definition,

$$\mathbb{E}_n(1\{Y \leq \hat{\theta}\}) = \frac{\lfloor \tau n \rfloor}{n} \text{ or } \frac{\lfloor \tau n \rfloor + 1}{n},$$  \hfill (11)

which implies $\mathbb{E}_n(1\{Y \leq \hat{\theta}\}) \neq \tau$ in either case. As a result, $\hat{g}_\tau(\hat{\theta}) = O_p(\frac{1}{n})$. \hfill \(\square\)
Example 2 illustrates that even in the simplest case where \( W_i = Z_i = 1 \) for \( i = 1, \ldots, n \), the estimating equations can have an error that is bounded away from zero by \( O_p(\frac{1}{n}) \) for most values of \( \tau \). In the general case with non-constant regressors \( W_i \) and instrumental variables \( Z_i \), the error in the estimating equations may be bounded away from 0 even if \( \tau n \) is an integer. However, in Theorem 5 in Section 3, we show that in the general case there exists an approximate solution to (7) achieving nearly \( O_p(\frac{1}{n}) \) value of the objective function, as in the univariate Example 2. This error is non-negligible and appears in the expansion up to order \( O_p(\frac{1}{n}) \).

**Non-uniqueness of the best approximate solution.** In the case of sample moments with non-degenerate Jacobian, the exact solution to the estimating equations is unique by the implicit function theorem. By contrast, due to the presence of the indicator function, the IVQR sample moment conditions are step functions that have a degenerate Jacobian. As a result, the set of solutions with the same value of the objective function has non-negligible diameter. This is true even if the estimating equations admit exact solutions. To illustrate, let us revisit Example 2.

*Example 3 (Example 2 cont.).* Suppose that \( \tau n \) is an integer. Suppose further that \( Y_i \) has a uniform distribution with support \([0, 1]\). Consider two approximate solutions of (10), \( \hat{\theta} = Y_{(\tau n)} \) and \( \hat{\theta}^* = Y_{(\tau n)} + (1 - \epsilon)(Y_{(\tau n + 1)} - Y_{(\tau n)}) \) for some small \( \epsilon > 0 \). Both solutions will have the norm of the error exactly equal to 0. The well known formula for the order statistic in the uniform distribution case is \( E(Y_{(j)}) = \frac{j}{n+1} \), so the difference in the means of the solutions is \( E(\hat{\theta}^* - \hat{\theta}) = \frac{(1-\epsilon)}{n+1} \). \(^{10}\) As a result, \( \hat{\theta}^* - \hat{\theta} \geq O_p(\frac{1}{n}) \) since \( \hat{\theta}^* \geq \hat{\theta} \) a.s. \( \square \)

This example shows that different solutions with equivalent error norm can have stochastic expansions that differ by \( O_p(\frac{1}{n}) \). Corollary 1 of Theorem 4 in Section 3 shows that equivalence of different exact solutions holds with nearly \( \frac{1}{n} \) rate even in for conditional quantile estimators.

**Non-Lipschitz sample moments.** Typically, for models with smooth sample moments the empirical process \( G_n(\theta) \) has a Lipschitz constant \( K_n \) that is bounded in probability in a neighborhood of \( \theta_0 \).\(^{11}\) By this property, for any \( \sqrt{n} \)-consistent estimator \( \hat{\theta} \),

\[
\|G_n(\hat{\theta}) - G_n(\theta_0)\| \leq K_n \|\hat{\theta} - \theta_0\| = O_p\left(\frac{1}{\sqrt{n}}\right).
\] \(^{12}\)

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\(^{10}\)See Ahsanullah et al. (2013, Example 8.1).

\(^{11}\)See for example Assumption C in Rilstone et al. (1996) with \( s \geq 2 \).
This result is instrumental in showing that the precision of the Gaussian approximation of
the estimating equations estimator \( \sqrt{n} \left( \hat{\theta} - \theta_0 \right) \) is \( O_p \left( \frac{1}{\sqrt{n}} \right) \).

Unfortunately, such a high-precision linear approximation is not available for the IVQR
model since property (12) does not hold (see Example 4 below). To study the precision of
this approximation, we exploit the special structure of the IVQR moment conditions that
resemble the empirical CDF. This structure is particularly evident in the special case of the
sample quantile estimator, where the sample moment conditions can be directly represented
using the empirical CDF, \( \hat{F}_Y(y) = \mathbb{E}_n 1 \{ Y \leq y \} \). When \( Y \) is uniformly distributed on \([0, 1]\),
the empirical CDF admits the following strong approximation, which was first derived by
Komlós et al. (1975, 1976).

**Theorem** (KMT coupling, Theorem 26 on p.252 in Pollard (2002)). There exists a
Brownian Bridge \( \{ B^o(y) : 0 \leq y \leq 1 \} \) with continuous sample paths, and a uniform empirical
process \( \mathbb{F}_n(y) \triangleq \sqrt{n}(\hat{F}_Y(y) - F_Y(y)) \), for which
\[
P \left\{ \sup_{0 \leq y \leq 1} |\mathbb{F}_n(y) - B^o(y)| \geq C_1 \frac{x + \log n}{\sqrt{n}} \right\} \leq C_0 \exp(-x) \text{ for all } x \geq 0, \tag{13}
\]
with constants \( C_1 \) and \( C_0 \) that depend on neither \( n \) nor \( x \).

The KMT theorem suggests that \( O_p \left( \frac{1}{\sqrt{n}} \right) \) is unattainable even in the case of the sample
quantile of a uniform r.v. as the following example shows.

**Example 4** (Example 2 cont.). For a uniform \( Y_i \), the CDF is \( F_Y(y) = y \) for \( y \in [0, 1] \). The
KMT theorem implies the following a.s. representation (cf. Pollard, 2002, p.255) for the
sample moment conditions:
\[
\mathbb{G}_n(\theta) = \mathbb{F}_n(\theta) = B(\theta) - \theta B(1) + R_n(\theta), \tag{14}
\]
where \( \sup_{0 \leq \theta \leq 1} R_n(\theta) = O_p \left( \frac{\log n}{\sqrt{n}} \right) \). \( B(1) \) is a standard Gaussian r.v. and
\( B(\theta) \triangleq B^o(\theta) + \theta B(1) \) is the standard Brownian motion. It follows from the Kolmogorov-Chentsov theorem
(see Lemma A.5 in Appendix A.2) that we can represent
\[
B(\theta) - B(\theta_0) = O_p \left( \|\theta - \theta_0\|^{\frac{1}{2} - \gamma} \right). \tag{15}
\]
Therefore, the increment of the empirical process \( \mathbb{G}_n \) evaluated at a random point \( \hat{\theta} \) has the

\(^{12}\text{Theorems 1.7 and 1.8 in Dudley (2014) provide a specific choice of constants. By Theorem 1.8, one can}
\text{choose } C_1 = 12 \text{ and } C_0 = 2 \text{ for } n \geq 2.\)
following representation for each $n$,

$$
\mathbb{G}_n(\hat{\theta}) - \mathbb{G}_n(\theta_0) = O_p\left(\frac{1}{n^{1/4 - \gamma}}\right) + R_n(\hat{\theta}) - R_n(\theta_0).
$$

The remainder here is uniformly bounded since $|R_n(\theta) - R_n(\theta_0)| \leq 2 \sup_{0 \leq t \leq 1} R_n(t)$. As a result, for any $\sqrt{n}$-consistent estimator $\hat{\theta}$ and any small $\gamma > 0$, we get

$$
\mathbb{G}_n(\hat{\theta}) - \mathbb{G}_n(\theta_0) = O_p\left(\frac{1}{n^{1/4 - \gamma}}\right).
$$

This results in a nearly $O_p \left(\frac{1}{n^{3/4}}\right)$ term in the stochastic expansion of $\hat{\theta}$, see Equation (8).

Theorem 1 of Section 3 provides an analog of the KMT theorem that accommodates the general IVQR model and shows that the sample moments can be approximated by a Hölder continuous process such that the remainder rate of nearly $O_p \left(\frac{1}{n^{3/4}}\right)$ still holds.

## 3 Higher order properties of IVQR estimators

In the previous section, we illustrated the key ideas behind the stochastic expansion of a simple quantile estimator. We now develop these ideas and obtain a stochastic (asymptotically linear) expansion of IVQR estimators in the general case that allows for multiple endogenous regressors. In addition, we provide some equivalence results.

### 3.1 Coupling of sample moments with a Hölder continuous process

The IVQR sample moment conditions are discontinuous functions of the parameter $\theta$, which invalidates standard arguments for deriving higher-order properties. As we have seen in the previous section in the case of the sample quantile estimator, the sample moment conditions can be approximated by a Brownian bridge process, which has continuous trajectories and admits a strong version of stochastic equicontinuity property that we call $\ell_1$ Hölder continuity. Namely, expectation of $\ell_1$ norm of the increments of the process indexed by $\theta$ is bounded by a polynomial function of increments of $\theta$.

Consider an empirical process

$$
Z_n(y, a) \triangleq \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( Z_i 1\{Y_i \leq y, A_i = a\} - E Z_i 1\{Y_i \leq y, A_i = a\} \right), \text{ for } a \in \{0, 1\}.
$$

**Theorem 1.** Suppose that $Z \in \mathbb{R}$ and $|Z| < m < \infty$ a.s.. Suppose further that $Y$ has density conditional on $Z$ and $A$ that is bounded by $\bar{f}$. Then, for any $a \in \{0, 1\}$, the process
$Z_n(y,a)$ can be a.s. represented as

$$Z_n(y,a) = Z(y,a) + R_n(y,a), \tag{19}$$

where $Z(y,a)$ is a zero mean process with a.s. Hölder continuous paths that has increments with bounds

$$\limsup_{r \to 0^+} \sup_{y_1,y_2 \in \mathbb{R}, |y_2 - y_1| < r} \frac{|Z(y_2,a) - Z(y_1,a)|}{\sqrt{\Psi(|y_2 - y_1|)}} \leq 4\sqrt{2}m, \ a.s. \tag{20}$$

$$\mathbb{E} \sup_{y_1 \neq y_2} \frac{|Z(y_2,a) - Z(y_1,a)|}{|y_2 - y_1|^\frac{1}{2} - \gamma} \leq m\tilde{c}_\gamma \bar{f}^{-\frac{1}{2} - \gamma}, \text{ for any } \gamma \in (0, 1/2), \tag{21}$$

where $\Psi(x) \triangleq x \log(1/x)$ and $R_n(y,a)$ is such that $\mathbb{E}(R_n(y,a)) = 0$ for all $y \in \mathbb{R}$ and

$$\mathbb{E} \sup_{y \in \mathbb{R}^+} |R_n(y,a)| \leq m\tilde{c}_1 \log n + \tilde{c}_0 \sqrt{n}, \tag{22}$$

with constants $\tilde{c}_1$, $\tilde{c}_0$, and $\tilde{c}_\gamma$ that do not depend on $n$, $x$, or the distribution of $(Y,Z,A)$.

**Proof.** See Appendix A. \qed

Equation (19) corresponds to the a.s. representation implied by the KMT theorem. The moment bounds on the remainder term could be replaced by the corresponding tail probability bounds as in the KMT theorem. We found it more convenient to work with moment bounds for deriving stochastic expansions.

Equation (21) generalizes Example 4 to the case of a stochastically weighted CDF, which is the key result for deriving stochastic expansions and equivalence results in the later subsections.\(^{13}\) This equation shows the relationship between the bound on the conditional density, $\bar{f}$, and the modulus of continuity of the IVQR sample moments. Whether the modulus of continuity $r^{1/2 - \gamma}$ in Equation (21) can be replaced with $r^{1/2}$ up to a log term is an open question.\(^{14}\)

In the proof, we treat the support of the instruments as fixed. However, we can allow the diameter of the support, $m$, to grow with sample size at a slow rate. In particular, if $Z$ is a sub-Gaussian or a sub-exponential random variable, the effective rate of growth of $m$ will be $\sqrt{\log n}$ or $\log n$ correspondingly. We leave this extension for future work.

\(^{13}\)There is literature on weighted empirical processes (e.g., Csorgo et al., 1986) that assumes that the weights are deterministic. These results could be used to study classical QR with fixed designs, but are not suitable for analyzing general IVQR estimators.

\(^{14}\)The recent work by Fischer and Nappo (2009) suggests that it may be possible to achieve $r^{1/2}$ up to a log term.
3.2 Bahadur-Kiefer representations of IVQR estimators

To derive the stochastic expansions, we impose the following condition on the regressors and instruments.

**Assumption 1 (Regressors and instruments).**

1. There exists $m < \infty$ such that $\|Z\| < m$ and $\|W\| < m$ a.s.
2. $\|W_i\| > 0$ a.s. and $\tilde{W}_i \triangleq W_i/\|W_i\| \in \{\tilde{w}_1, \ldots, \tilde{w}_s\} \subset \mathbb{R}^k$ where $s \ll n$.

Bounded support of the instruments is crucial for applying Theorem 1, see the discussion in Section 3.1. This assumption holds automatically when the instruments are discrete, which is common in empirical applications. Assumption 1.2, or sparsity restrictions on the support of the regressors $W_i$, allows extending Theorem 1 to the case of conditional moments. In particular, the sparsity is used to reduce the $k$-dimensional empirical process to $s$ univariate processes to which Theorem 1 can be applied.

The previous work on higher order properties of quantile regression estimators (e.g., Zhou et al., 1996; Knight, 2002) assumed that the regressors are exogenous. The sparsity Assumption 1.2 allows us to obtain the Bahadur-Kiefer expansion for (endogenous) IVQR models.

We impose the following additional restrictions on the conditional density.

**Assumption 2 (Conditional density).**

1. The conditional density of $Y_i$ given $(W_i, Z_i)$, $f_Y(y|W, Z)$, exists and is a.s. uniformly bounded in $y$ on $\text{supp}(Y)$ by $\bar{f}$.
2. $f_Y(y|W, Z)$ is a.s. twice continuously differentiable on $\text{supp}(Y)$.

The uniform bound in Assumption 2.1 is essential for applying the coupling in Theorem 1. This assumption is standard in the IVQR literature (e.g., Chernozhukov and Hansen, 2006, Assumption R3).

Under Assumption 2.2, the Jacobian of the population moment functions $G(\cdot) \triangleq \partial_\theta g(\cdot)$ is well-defined and twice continuously differentiable (see Lemma B.1). We use this property for second-order Taylor expansion of the population moments. Such smoothness assumptions are standard in the literature on higher-order properties of moment-based estimators (e.g., Rilstone et al., 1996; Newey and Smith, 2004).

Next, we impose identification assumptions.

---

15For example, Assumption 2.1 rules out the model $Y = W^T \theta_0(U)$, where $\theta_0(U)$ does not depend on $U$. 
Assumption 3 (Identification).

1. \( \theta_0 \) is the unique solution to \( g(\theta) = 0 \) over \( \Theta \), where \( \theta_0 \) is in the interior of the compact set \( \Theta \).

2. The Jacobian of the moment functions evaluated at \( \theta_0 \), \( G(\theta_0) \), is of full rank.

Assumption 3 is commonly imposed in GMM settings; see, for example, Chernozhukov and Hansen (2006) and Kaido and Wüthrich (2019) for primitive conditions for global identification in linear IVQR models. Note that Assumption 3.1 implies that the support of \( \tilde{W}_i \) has more than \( k \) points, \( s \geq k \). The full rank assumption is necessary for asymptotic normality of the estimator and underlies our higher order expansion.

Consider the infeasible linear and unbiased estimator

\[
\hat{\theta}_1 \triangleq \theta_0 - G^{-1}(\theta_0)\hat{g}(\theta_0) = \theta_0 - G^{-1}(\theta_0)\frac{1}{n} \sum_{i=1}^{n} Z_i (1\{Y_i \leq W_i^\prime \theta_0\} - \tau).  \tag{23}
\]

The linear structure of this estimator makes it easy to study precision of the asymptotic normal approximation using the corresponding result for multivariate sample means. Under some regularity conditions, the estimator \( \hat{\theta}_1 \) is first order equivalent to most IVQR estimators (e.g., Chernozhukov and Hansen, 2006; Kaplan and Sun, 2017; Kaido and Wüthrich, 2019). However, it is important to have a bound on the higher order terms to make statements on the precision of the normal approximation and computationally attractive multiplier bootstrap procedures. Rilstone et al. (1996) and Newey and Smith (2004) provide results for smooth objective functions suggesting the higher-order properties of first order equivalent estimators may be very different. Kaplan and Sun (2017) derive higher-order properties of smoothed IVQR estimators. GMM and classical QR estimators of conditional quantiles are based on non-smooth sample objective functions such that these results are not applicable in our context (see Section 2.2).

For conditional quantile models, remainder bounds in asymptotic linear representation are only available for classical QR (e.g., Zhou et al., 1996; Knight, 2002). Theorem 1 allows us to obtain such bounds for a much broader class of estimators. Consider an infeasible single Newton step correction corresponding to the minimization of \( \|\hat{g}(\theta)\|_2 \), \( T(\theta) \triangleq \theta - G^{-1}\hat{g}(\theta) \). We establish asymptotic linearity for a consistent estimator of \( \theta_0 \) after applying a single Newton step with an explicit bound on the remainder term.
To state the result, it is useful to introduce the following notation. For $\theta \in \Theta$, we define

$$g^c(\theta) \triangleq \mathbb{E}1\{Y \leq W'\theta\} Z,$$

$$(24)$$

$$B_n^\circ(\theta) \triangleq \sqrt{n}(\mathbb{E}_{n} 1\{Y \leq W'\theta\} Z - g^c(\theta)),$$

$$(25)$$

$$B_n(\theta) \triangleq B_n^\circ(\theta) - B_n^\circ(\theta_0).$$

$$(26)$$

**Theorem 2.** Suppose that assumptions 1-3 hold. Suppose further that $\hat{\theta} = \theta_0 + R_n$. Then

$$\hat{\theta}_{1\text{-step}} \triangleq T(\hat{\theta}) = \hat{\theta} + O_p\left(\frac{sm^2 R_n^{1-\gamma}}{n^{1/2}}\right) + O_p(||R_n||^2).$$

$$(27)$$

**Proof.** By Lemma B.2, $\hat{\theta}$ satisfies

$$G(\theta_0)(\hat{\theta} - \theta_0) + \frac{1}{2}(\hat{\theta} - \theta_0)' \partial_\theta G(\theta_0)(\hat{\theta} - \theta_0)$$

$$= \hat{g}(\hat{\theta}) - \frac{1}{\sqrt{n}} B_n^\circ(\theta_0) - \tau(\mathbb{E}Z - \mathbb{E}_n Z) - \frac{1}{\sqrt{n}} B_n(\hat{\theta}) + O_p\left(||R_n||^3\right).$$

$$(28)$$

By Assumption 2, we have

$$\hat{\theta} - G^{-1}(\theta_0)\hat{g}(\hat{\theta}) = \hat{\theta} - G^{-1}(\theta_0) \frac{1}{\sqrt{n}} B_n(\hat{\theta}) + O_p\left(||R_n||^2\right).$$

By Lemma B.3 which uses Theorem 1, $\frac{1}{\sqrt{n}} B_n(\hat{\theta}) = O_p\left(\frac{sm^2 R_n^{1-\gamma}}{n^{1/2}}\right)$. This result completes the proof.

In particular, Theorem 2 implies that $\hat{\theta}_{1\text{-step}}$ has an asymptotically linear representation with nearly $O_p\left(\frac{1}{n^{1/4}}\right)$ rate if $R_n = o_p\left(\frac{1}{n^{1/4}}\right)$. Moreover, if $R_n = O_p\left(\frac{1}{\sqrt{n}}\right)$, then for the two step estimator $\hat{\theta}_{2\text{-step}} \triangleq T(T(\hat{\theta}))$, we get the nearly conventional order of the remainder, $O_p\left(\frac{1}{n^{1/2}}\right)$, as in the sample quantile case (see p.122 in Jurečková et al., 2012). This implication for the 2-step estimator is analogous to the results in Robinson (1988) for smooth estimators.\textsuperscript{16} Theorem 2 also complements Zhu (2019), who established first order equivalence of $k$-step estimators and GMM estimators for the IVQR model, by providing an explicit bound on the remainder term.

\textsuperscript{16}The same implication would hold for $k$-step estimators with $G^{-1}$ replaced by its $\sqrt{n}$-consistent estimator. See discussion of such estimators in Section 4.3.
3.3 Some higher order equivalence results for IVQR estimators

In stark contrast with the smooth extremum estimators (e.g., Andrews, 2002a,b), additional Newton steps may not result in convergence of the $k$-step estimator to the exact minimizer of the GMM objective function. The reason is that the Newton steps are not guaranteed to reduce the objective function $\|\hat{g}(\hat{\theta})\|$ below $O_p\left(\frac{sm2}{n^{3/4}}\right)$ since this objective is non-smooth. The following theorem gives a generic result about the norm of the objective for a generic estimator $\hat{\theta}$.

**Theorem 3.** Suppose that Assumptions 1–3 hold. Any estimator of form $\hat{\theta} = \hat{\theta}_1 + R_n$ with $\|R_n\| = o_p\left(\frac{1}{\sqrt{n}}\right)$ satisfies

$$\hat{g}(\hat{\theta}) = G(\theta_0)R_n + O_p\left(\frac{sm2}{n^{3/4}}\right).$$

for any small $\gamma > 0$.

**Proof.** Since $\|R_n\| = o_p\left(\frac{1}{\sqrt{n}}\right)$, $\hat{\theta} = \theta_0 + O_p\left(\frac{1}{\sqrt{n}}\right)$. As in the proof of Theorem 2, we have

$$R_n = \hat{\theta} - \hat{\theta}_1 = G^{-1}(\theta_0)\hat{g}(\hat{\theta}) - G^{-1}(\theta_0)\frac{1}{\sqrt{n}}B_n(\hat{\theta}) + O_p\left(\frac{1}{n}\right).$$

By Lemma B.3, which uses Theorem 1, $\frac{1}{\sqrt{n}}B_n(\hat{\theta}) = O_p\left(\frac{sm2\|\hat{\theta} - \theta_0\|^2}{n^{1/2}}\right) = O_p\left(\frac{sm2}{n^{3/4}}\right)$ for any small $\gamma > 0$. So equation (30) becomes

$$\hat{g}(\hat{\theta}) = O_p\left(\frac{sm2}{n^{3/4}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + G(\theta_0)R_n.$$ (31)

The term $O_p\left(\frac{1}{\sqrt{n}}\right)$ is negligible when compared to the first term, which completes the proof. \[\square\]

The previous theorem suggests that the order of magnitude of the sample moments evaluated at the estimated value $\hat{\theta}$ additively depends on the remainder of the Bahadur-Kiefer expansion. The following statement provides the converse result, i.e. estimators with close value of the $\hat{g}(\hat{\theta})$ are equivalent up the order of magnitude of the difference in the sample moments evaluated at each of them.

**Theorem 4.** Suppose that Assumptions 1–3 hold. Consider any pair of asymptotically linear estimators $\hat{\theta} = \hat{\theta}_1 + o_p\left(\frac{1}{\sqrt{n}}\right)$ and $\hat{\theta}^* = \hat{\theta}_1 + o_p\left(\frac{1}{\sqrt{n}}\right)$. If $\hat{g}(\hat{\theta}) - \hat{g}(\hat{\theta}^*) = O_p\left(\frac{|R_n|}{\sqrt{n}}\right)$ for some bounded sequence $R_n$, then the following is true

$$\hat{\theta} - \hat{\theta}^* = O_p\left(\frac{|R_n|}{\sqrt{n}}\right) + O_p\left(\frac{m^3s^2}{n^{1/2}}\right).$$ (32)
Proof. By Lemma B.2, both estimators satisfy representation (28). Then, for any small \( \gamma > 0 \),

\[
\hat{\theta} - \hat{\theta}^* = -G^{-1}(\theta_0) B_n(\hat{\theta}) - B_n(\hat{\theta}^*) \sqrt{n} + G^{-1}(\theta_0) (\hat{g}(\hat{\theta}) - \hat{g}(\hat{\theta}^*)) + O_p \left( \frac{s^2 m^3}{n^{5/4 - \gamma}} \right)
\]

(33)

The result follows immediately from Lemma B.6.

One direct implication of Theorems 3 and 4 is that any \( \sqrt{n} \)-consistent estimators of \( \theta_0 \) become equivalent up to nearly \( O_p \left( \frac{1}{n^{5/4}} \right) \) after a single Newton step correction. This result is reminiscent of the results obtained by Andrews (2002a) for smooth extremum estimators equations, although the non-differentiability of the objective function results in equivalence of order nearly \( \frac{1}{n^{5/4}} \) instead of \( \frac{1}{n^{3/2}} \).

### 3.4 Stochastic expansion of exact estimators

The previous results hold for generic estimators. Theorem 3 shows the connection between the remainder in the Bahadur-Kiefer expansion and the sample moments evaluated at the estimator. It is therefore useful to study estimators that precisely minimize a finite dimensional \( \ell_p \) norm of the sample moments,

\[
\hat{\theta}_{\ell_p} = \arg\min_{\theta \in \Theta} ||g(\theta)||_p.
\]

(34)

This class of exact estimators includes GMM, which corresponds to \( || \cdot ||_2 \) norm as in Chen and Lee (2017) for just-identified models, and the estimator proposed by Zhu (2019), which corresponds to \( || \cdot ||_{\infty} \). In the Monte Carlo section of this paper we consider \( || \cdot ||_1 \) for computational convenience.\(^{17}\)

The following theorem provides a bound on the minimal norm of the sample moments for the exact estimator and an explicit stochastic expansion up to order nearly \( \frac{1}{n^{5/4}} \).

We use the following notation,

\[
G \triangleq G(\theta_0),
\]

(35)

\[
\partial_\theta G_j \triangleq \frac{\partial^2 g_j(\theta_0)}{\partial \theta \partial \theta^r}.
\]

(36)

For any \( x \in \mathbb{R}^k \), we will use \( x' \partial_\theta G x \) to denote a vector with components \( x' \partial_\theta G_j x \) for \( j = 1, \ldots, k \).

\(^{17}\)The cases \( || \cdot ||_1 \) and \( || \cdot ||_{\infty} \) have computationally convenient MILP representations, while the MILP formulation for \( || \cdot ||_2 \) proposed in Chen and Lee (2017) has many more decision variables.
Theorem 5. Suppose that Assumptions 1–3 hold. Consider $\hat{\theta}_p$ obtained from program (34) for some $p \in [1, \infty]$. Then
\[
\hat{\theta}_p - G^{-1}\hat{g}(\hat{\theta}_p) = \hat{\theta}_1 - G^{-1}\left[\frac{B_n(\hat{\theta}_p)}{\sqrt{n}} + \frac{1}{2}(\hat{\theta}_1 - \theta_0)'\partial_0 G(\hat{\theta}_1 - \theta_0)\right] + R_n, \tag{37}
\]
where $\hat{g}(\hat{\theta}_p) = O_p\left(\frac{m^3s^2}{n^{1-\gamma}}\right)$, $B_n(\hat{\theta}_p) = O_p\left(\frac{sm^{3/2}}{n^{1/4-\gamma}}\right)$, and $R_n = O_p\left(\frac{sm^{3/2}}{n^{5/4-\gamma}}\right)$, for any small $\gamma > 0$.

Proof. By Lemma B.2 applied to $\hat{\theta}_p$,
\[
\hat{\theta}_p - G^{-1}\hat{g}(\hat{\theta}_p) = \hat{\theta}_1 - G^{-1}\left[\frac{B_n(\hat{\theta}_p)}{\sqrt{n}} + \frac{1}{2}(\hat{\theta}_1 - \theta_0)'\partial_0 G(\hat{\theta}_1 - \theta_0)\right] + O_p(||\hat{\theta}_p - \theta_0||^3). \tag{38}
\]
By Lemma B.5, $\hat{\theta}_p - \theta_0 = O_p\left(\frac{1}{\sqrt{n}}\right)$. The result $B_n(\hat{\theta}_p) = O_p\left(\frac{sm^{3/2}}{n^{1/4-\gamma}}\right)$ follows from Lemma B.3. Under Assumptions 1–3, Lemma B.7, which is based on the Hölder continuity of $B_n$, yields
\[
\hat{g}(\hat{\theta}_p) = O_p\left(\frac{m^3s^2}{n^{1-\gamma}}\right). \tag{39}
\]
To complete the proof, notice that
\[
(\hat{\theta}_1 - \theta_0)'\partial_0 G(\hat{\theta}_1 - \theta_0) = (\hat{\theta}_1 - \theta_0)'\partial_0 G(\hat{\theta}_1 - \theta_0) + O_p\left(\frac{sm^{3/2}}{n^{5/4-\gamma}}\right). \tag{40}
\]

Theorem 5 implies that any two estimators $\hat{\theta}_p$ and $\hat{\theta}_{L_2}$ yield sample moments that differ by at most $O_p\left(\frac{m^3s^2}{n^{1-\gamma}}\right)$. Then Theorem 4 implies the following corollary.

Corollary 1. Under Assumptions 1–3, difference between any two solutions to (34), possibly corresponding to different norms, is at most $O_p\left(\frac{m^3s^2}{n^{1-\gamma}}\right)$ for any small $\gamma > 0$.

This corollary generalizes Example 3 to conditional quantile models. In particular, Corollary 1 implies the equivalence of the exact MIP estimators proposed by Chen and Lee (2017) and Zhu (2019). Pouliot (2019) also proposed an a dual MIP approach based on the inverse quantile regression estimator of Chernozhukov and Hansen (2006). His MIP formulation has sample moment equations constraints, which generally cannot be satisfied exactly. Therefore, the asymptotic rate of the tolerance level on the sample moment constraints would determine the order of equivalence of the dual estimator and $\hat{\theta}_p$. 
4 Higher-order bias of the exact estimator

4.1 Bias formula

Here we provide the bias formula for \( \hat{\theta}_{\ell_p} \) that is a corner solution to program (34). We say that the solution is a corner solution if it is a corner of the polygon that corresponds to the argmin set of (34). We focus on corner solutions because such solutions are the default output of exact MILP solvers. Importantly, the proof of the bias formula only relies on stochastic equicontinuity of the sample moment empirical process and does not require the stronger Hölder property in Theorem 1. As a result, the formula holds even if the sparsity Assumption 1.2 is violated.

Note that the true parameter \( \theta_0 \) has an equivalent alternative definition as a solution to

\[
g^*(-\theta_0) \triangleq \mathbb{E}[1\{Y \leq W'(-\theta_0)\} - (1 - \tau)Z] = 0. \quad (41)
\]

The sample analog of (41), \( \hat{g}^*(-\theta) \) is different from \( \hat{g}(\theta) \), and thus would typically deliver a different corner solution, which we refer to as \( \hat{\theta}_{\ell_p}^* \). To resolve this apparent arbitrary choice of the corner solutions, Theorem 6 below is written in a symmetric form. Namely, the choice of the sample moment conditions, \( \hat{g}(\theta) \) or \( \hat{g}^*(-\theta) \), does not affect the resulting bias formula (49), which is symmetric with respect to permutation of \((Y, \theta, \tau, \hat{\theta}_{\ell_p})\) and \((-Y, -\theta, 1 - \tau, -\hat{\theta}_{\ell_p}^*)\).

We use the following notation,

\[
\varepsilon_i \triangleq Y_i - W_i'\theta_0, \\
\kappa_1(\tau) \triangleq \left( \tau - \frac{1}{2} \right) \mathbb{E}f_\varepsilon(0|W, Z)ZW'G^{-1}Z, \\
\kappa_2(\tau) \triangleq \frac{\mathbb{E}\hat{g}(\hat{\theta}_{\ell_p}) + \hat{g}^*(-\hat{\theta}_{\ell_p})}{2} = \frac{1}{2} \mathbb{E}_n Z_i \{ \hat{\varepsilon}_i = 0 \}, \\
\hat{\varepsilon}_i \triangleq Y_i - W_i'\hat{\theta}_{\ell_p}, \\
\Omega \triangleq \text{Var}(m(Y, W, Z, \theta_0)), \quad (47)
\]

where the operator \( \text{vec}(\cdot) \) denotes the standard matrix vectorization.

**Theorem 6.** Consider any corner solution \( \hat{\theta}_{\ell_p} \) of program (34). Suppose that Assump-


\[ \mathbb{E}\hat{\theta}_p - \theta_0 = \frac{1}{n} G^{-1} \left[ \mathbb{E}\hat{g}(\hat{\theta}_p) - \kappa_1(\tau) - \kappa_2(\tau) - \frac{1}{2} Q' \text{vec}(\Omega) \right] + o\left(\frac{1}{n}\right). \] (49)

**Proof.** Following the proof of Theorem 5, Lemma B.2 implies

\[ \hat{\theta}_p - G^{-1}\hat{g}(\hat{\theta}_p) = \hat{\theta}_1 - G^{-1} \left[ \frac{B_n(\hat{\theta}_p)}{\sqrt{n}} + \frac{1}{n}(\hat{\theta}_1 - \theta_0)' \partial_\theta G(\hat{\theta}_1 - \theta_0) \right] + R_n, \] (50)

where \( R_n = o_p\left(\frac{1}{n}\right) \) is implied by Lemma B.5. Lemma C.1 implies

\[ \frac{1}{\sqrt{n}} \mathbb{E}B_n(\hat{\theta}_p) = \frac{1}{n} \kappa_1(\tau) + \frac{1}{n} \kappa_2(\tau) + o\left(\frac{1}{n}\right), \] (51)

For a correctly specified models \( \mathbb{E}\hat{\theta}_1 = \theta_0 \) and

\[ \mathbb{E}(\hat{\theta}_1 - \theta_0)' \partial_\theta G_j(\hat{\theta}_1 - \theta_0) = \mathbb{E}\hat{g}'(\theta_0) (G^{-1})_j \partial_\theta G_j G^{-1} \hat{g}(\theta_0) \] (52)
\[ = \text{vec}((G^{-1})_j \partial_\theta G_j G^{-1})' \text{vec}(\text{Var}(m(Y_i, W_i, Z_i, \theta_0))) \] (53)
\[ = \frac{1}{n} Q_j' \text{vec}(\Omega). \] (54)

The statement of the theorem follows immediately. \( \square \)

The formula (49) has four components:

1. \( \mathbb{E}G^{-1}\hat{g}(\hat{\theta}_p) \), the bias from non-zero sample moments;

2. \( G^{-1}\frac{\kappa_1(\tau)}{n} \), the bias from covariance of linear influence of a single observation on \( \hat{\theta}_p \) and the sample moments;

3. \( G^{-1}\frac{\kappa_2(\tau)}{n} \), the bias from non-zero probability of \( \hat{\varepsilon} = 0 \);

4. \( G^{-1}\frac{Q' \text{vec}(\Omega)}{2n} \), the bias from non-uniform conditional distribution of \( Y \) given \((W, Z)\).

Component (1) is equal to zero when it is possible to find exact zeros to the sample moments. That is generally only possible in the univariate case when \( \tau n \) is an integer. Component (2) is generally equal to zero only at the median \( (\tau = \frac{1}{2}) \) and is linear in \( \tau \). Component (3) is always present for corner solutions. Component (4) is typically present in most non-linear estimators with non-zero Hessian of the score function (see, for example, Rilstone et al., 1996). In particular, in the location-scale model (Example 1) this component captures deviations from the uniform distribution in the error term.
It is instructive to study those four components in the uniform sample quantile case, where exact formulas are available.

**Example 5 (Example 3 cont.)** Suppose we are interested in estimating the \( \tau \)-quantile of a uniformly distributed outcome variable \( Y_i \). This is a special case of the general IVQR model with \( W = Z = 1 \), \( f_Y(y) = 1\{0 \leq y \leq 1\} \). Then sample moments take form

\[
\hat{g}(\theta) = \mathbb{E}_n(1\{Y \leq \theta\} - \tau),
\]
\[
\hat{g}^*(-\theta) = \mathbb{E}_n1\{-Y \leq -\theta\} - (1 - \tau),
\]
\[
\kappa_2(\tau) = \frac{1}{n}.
\]

The derivatives of the moment conditions \( g(\theta) = g^*(-\theta) = 0 \) are \( G = 1, \frac{\partial \theta}{\partial G} = 0 \). In the univariate case all norms \( \| \cdot \|_p, \ p \in [1, \infty] \), are equivalent to the absolute value function. The order statistic \( Y(k) \) with \( k = \lfloor \tau n \rfloor \) is the corner solution to minimization of \( |\hat{g}(\theta)| \), while \( Y(k+1) \) corresponds to \( |\hat{g}^*(-\theta)| \). Then

\[
G^{-1}\frac{\hat{g}(Y(k)) - \hat{g}^*(-Y(k))}{2} = \frac{1}{2n} \sum_{i=1}^{n} 1\{Y_i \leq Y(k)\} = \frac{1}{2n},
\]
\[
G^{-1}\frac{\hat{g}(Y(k+1)) - \hat{g}^*(-Y(k+1))}{2} = -\frac{1}{2n} \sum_{i=1}^{n} 1\{Y_i \leq Y(k+1)\} = -\frac{1}{2n}.
\]

First, consider the asymptotic bias formula (49) in the case where \( k = \lfloor \tau n \rfloor = \tau n \), so that \( \hat{g}(Y(k)) = \hat{g}^*(Y(k+1)) = 0 \) and

\[
\mathbb{E} Y(k) - \tau = -\frac{1}{2n} + \frac{1}{n} \left( \frac{1}{2} - \tau \right) + o\left(\frac{1}{n}\right) = -\frac{\tau}{n} + o\left(\frac{1}{n}\right),
\]
\[
\mathbb{E} Y(k+1) - \tau = \frac{1}{2n} + \frac{1}{n} \left( \frac{1}{2} - \tau \right) + o\left(\frac{1}{n}\right) = \frac{1 - \tau}{n} + o\left(\frac{1}{n}\right).
\]

The corresponding exact formulas indeed coincide up to a higher order term,

\[
\mathbb{E} Y(k) - \tau = \frac{\tau n}{n + 1} - \tau = \frac{-\tau}{n + 1},
\]
\[
\mathbb{E} Y(k+1) - \tau = \frac{\tau n + 1}{n + 1} - \tau = \frac{1 - \tau}{n + 1}.
\]

In the general case, there is an additional bias term coming from the non-zero sample moments, for example \( \hat{g}(Y(k)) = \frac{\lfloor \tau n \rfloor - \tau n}{n} \). This additional term induces saw-like shape of the bias as a function of \( \tau \). Figure 1 illustrates the exact and the asymptotic bias formula (scaled by \( n \)) for \( n = 10 \).
Figure 1: Exact and asymptotic bias (scaled by $n$) for $\hat{\theta}_\ell = Y(\lfloor \tau n \rfloor)$ and $\hat{\theta}^*_\ell = Y(\lfloor \tau n + 1 \rfloor)$, where $Y \sim \text{Uniform}(0, 1)$, $n = 10$.

### 4.2 Comparison with the literature

To our knowledge, without sparsity, the near-$\frac{1}{n}$ order of the bias is only available for sample quantiles based on specific distributions (Ahsanullah et al., 2013) as well as classical QR (Portnoy et al., 2012).\(^\text{18}\) With the help of the sparsity assumption, however, Theorem 2 and Theorem 5 suggest that the biases of k-step and exact estimators are nearly $O\left(\frac{1}{n^{3/4}}\right)$ and $O\left(\frac{1}{n}\right)$, respectively. Bias correction of the exact estimator can reduce the bias to $o\left(\frac{1}{n}\right)$ (and possibly nearly $O\left(\frac{1}{n^{5/4}}\right)$ under Assumption 1.2).

An important implication of our result is that the exact estimators have a smaller bias than estimators based on smoothed estimating equations and generic estimators. The bias of such estimators can be bounded by their higher-order MSE. Theorems 2–3 suggest that the higher-order MSE and the corresponding bias can be as large as $O\left(\frac{1}{n^{3/4}}\right)$. Kaplan and Sun (2017) show that MSE of IVQR estimators can be reduced using smoothed estimating equations approach. In particular, they can guarantee that the bias of the smoothed estimator is as small as MSE, which is $O(n^{-\alpha})$ for some $\frac{1}{2} < \alpha < 1$, where $\alpha$ depends on the smoothness of the conditional density of $Y$ given $(W, Z)$. Theorem 6 shows that the 1-step corrected exact

\(^\text{18}\)Portnoy et al. (2012) derived the near-$\frac{1}{n}$ error of the Gaussian of the QR process, which implies the order of the bias. However, this analysis does not yield an explicit bias formula, and thus is not suitable for deriving a feasible bias correction procedure.
estimator has bias of order $1/n$, which is substantially lower that the bias of both the generic first-order equivalent estimators and smoothed estimators even under favorable smoothness assumptions.

Finally, we would like to mention an interesting approach to higher order bias analysis of non-smooth estimators that is based on the generalized functions heuristic (e.g., Phillips, 1991). The recent work of Lee et al. (2017, 2018) derived an asymptotic bias formula for classical QR and IVQR under the assumption that the sample moments are equal to zero. Example 2 shows that this assumption is violated even in simplest cases. Moreover, the generalized function approach neglects the multiplicity of solutions, see Example 3. As a result, the bias formulas in Lee et al. (2017, 2018) neglect Components (1) and (3) of the bias formula (49) in Theorem 6. Both the analytical Example 5 as well as the Monte Carlo evidence presented in Section 4.4 suggest that these two components contribute substantially to the higher-order bias.

### 4.3 Feasible bias correction

The theoretical bias formula (49) has a feasible counterpart that can be used for bias correction of an exact estimator $\hat{\theta}_{\ell_p}$.

Bias Components (1) and (3) depend on the realization of $\hat{\theta}_{\ell_p}$ and, thus, cannot be consistently estimated. Therefore, we consider the following symmetric 1-step correction of the $\hat{\theta}_{\ell_p}$,

$$\hat{\theta}_{sym} = \hat{\theta}_{\ell_p} - G^{-1} \left( \frac{\hat{g}(\hat{\theta}_{\ell_p}) - \hat{g}^*(-\hat{\theta}_{\ell_p})}{2} \right).$$

(64)

Such 1-step estimator is first order equivalent to $\hat{\theta}_{\ell_p}$ as implied by Lemma B.5 in the Appendix.

Our key insight is to use the following symmetric 1-step estimator $\hat{\theta}_{sym}$, which admits a concise bias formula,

$$E\hat{\theta}_{sym} - \theta_0 = -\frac{1}{n} G^{-1} \left[ \kappa_1(\tau) + \frac{1}{2} Q' vec(\Omega) \right] + o \left( \frac{1}{n} \right),$$

(65)

that can be consistently estimated using sample analogs. The corresponding bias-corrected estimator is then defined as follows,

$$\hat{\theta}_{bc} = \hat{\theta}_{sym} + \frac{1}{n} G^{-1} \left[ \kappa_1(\tau) + \frac{1}{2} Q' vec(\Omega) \right],$$

(66)

\[\text{We emphasize that the symmetric one 1-step estimator is different from standard 1-step estimators as considered, for instance, in Zhu (2019).}\]
Estimators of the Jacobian $G$ and the variance of the sample moments $\Omega$ are readily available (e.g., Powell, 1986; Chernozhukov and Hansen, 2006; Angrist et al., 2006). The parameter $\kappa_1(\tau)$ has the same structure as $G$ and can be estimated accordingly. The remaining parameter $Q$ takes the form of a weighted-average derivative of the density of $\varepsilon$ conditional on $(Z, W)$. Consistent kernel estimators of parameters with such structure were considered, for example, in Powell et al. (1989). In sum, feasible bias correction can proceed based on well-established plug-in estimators. In particular, this approach does not rely on any resampling or simulation-based methods. This is important in practice as exact estimators can be computationally quite expensive.

4.4 Monte Carlo validation of the bias formula

To validate our bias formula, we perform Monte Carlo simulations based on the following simple location model:

$$ Y = W'\beta + F^{-1}(U), \quad U \mid W \sim \text{Uniform}(0, 1), \quad (67) $$

where $W = (1, X)'$, $\beta = (0, 1)'$, and $X \sim \text{Uniform}(0, 1)$. In this model, the conditional quantile of $Y$ given $W$ is

$$ Q_Y(\tau|W) = W'\theta_0(\tau), \quad (68) $$

where $\theta_0(\tau) = (F^{-1}(\tau), \beta)$. We use the location model (67) because there are explicit formulas for all the bias components, so that we avoid any approximation errors. We consider three different choices for $F$:

| DGP1 (Uniform) | $F(y) = \int_{-\infty}^{y} 1\{t \in [0, 1]\} dt$ |
|----------------|-----------------------------------------------|
| DGP2 (Triangular) | $F(y) = \int_{-\infty}^{y} 1\{t \in [0, 1]\} t dt$ |
| DGP3 (Cauchy) | $F(y) = \int_{-\infty}^{y} \frac{1}{\pi(1+(4t^2))} dt$ |

The asymptotic bias formula in the case of DGP1 is expected to be precise even for small $n$ since all the higher-order derivatives of $f_Y(y|W, Z)$ are zero. By construction, $Q = 0$ such that Component (4) of the bias formula (49) is zero. DGP2 has non-zero first derivative of $f_Y(y|W, Z)$. Unlike DGP1, this DGP has $Q \neq 0$, which allows assess the contribution of Component (4). At the same time, second and higher-order derivatives of $f_Y(y|W, Z)$ are zeros, such that the population moments are quadratic in $\theta$. As a result, the asymptotic formula then fully captures the shape of the population moments. In addition, $f_Y(y|W, Z)$ is asymmetric around the median. This feature is useful to illustrate the effects of a low $f_Y(y|W, Z)$ and thus large $G^{-1}$ on the bias. Finally, DGP3 is more complex and features all the four components of the bias. The bias correction under DGP3 is less precise than under
DGP1 and DGP2 since the influence of the neglected higher-order terms of the population moments is \( o\left(\frac{1}{n}\right) \). The remaining bias can potentially be quite substantial for tail quantities and for small \( n \).

We use the MILP formulations defined in Appendix D to compute \( \hat{\theta}_{\ell_1} \). We compare bias of \( \hat{\theta}_{\ell_1} = (\hat{\theta}_{\ell_1,1}, \hat{\theta}_{\ell_1,2})' \) and \( \hat{\theta}_{bc} = (\hat{\theta}_{bc,1}, \hat{\theta}_{bc,2})' \) computed using formula (66). We use sample size \( n = 50 \) and perform 20000 Monte Carlo simulations for a grid of values for \( \tau \). The result of the experiments are summarized in Tables 1 and 2.

### Table 1: Rescaled bias of the exact estimator \( \hat{\theta}_{L_1} \)

| \( \tau \) | 0.10 | 0.15 | 0.20 | 0.25 | 0.50 | 0.75 | 0.80 | 0.85 | 0.90 |
|------------|------|------|------|------|------|------|------|------|------|
| \( n\mathbb{E}(\hat{\theta}_{\ell_1,1} - \theta_{0,1}) \) | 0.83 | 0.62 | 0.59 | 0.51 | 0.27 | -0.09 | -0.15 | -0.25 | -0.33 |
| \( n\mathbb{E}(\hat{\theta}_{\ell_1,2} - \theta_{0,2}) \) | -0.30 | -0.05 | -0.12 | -0.11 | -0.50 | -0.75 | -0.75 | -0.75 | -0.72 |

DGP1 (Uniform)

| \( \tau \) | 0.10 | 0.15 | 0.20 | 0.25 | 0.50 | 0.75 | 0.80 | 0.85 | 0.90 |
|------------|------|------|------|------|------|------|------|------|------|
| \( n\mathbb{E}(\hat{\theta}_{\ell_1,1} - \theta_{0,1}) \) | 0.50 | 0.17 | 0.16 | 0.01 | -0.02 | -0.20 | -0.19 | -0.25 | -0.25 |
| \( n\mathbb{E}(\hat{\theta}_{\ell_1,2} - \theta_{0,1}) \) | -0.25 | 0.09 | -0.01 | 0.02 | -0.35 | -0.37 | -0.41 | -0.33 | -0.37 |

DGP2 (Triangular)

| \( \tau \) | 0.10 | 0.15 | 0.20 | 0.25 | 0.50 | 0.75 | 0.80 | 0.85 | 0.90 |
|------------|------|------|------|------|------|------|------|------|------|
| \( n\mathbb{E}(\hat{\theta}_{\ell_1,1} - \theta_{0,1}) \) | -13.61 | -6.62 | -2.43 | -1.57 | 0.16 | 2.14 | 2.91 | 7.77 | 16.17 |
| \( n\mathbb{E}(\hat{\theta}_{\ell_1,2} - \theta_{0,2}) \) | 2.41 | 2.55 | 0.23 | 0.24 | -0.33 | -1.43 | -1.19 | -5.02 | -8.05 |

DGP2 (Cauchy)

To evaluate precision of the Monte Carlo integration, we report MCSE, standard errors based on CLT with sample size 20000. We report 3 \( \times \) MCSE to account for the joint testing of 18 hypotheses. Despite its approximate nature, the asymptotic bias correction formula systematically reduces the bias across all the designs and quantile levels. In the case of DGP1, the bias correction leaves a remainder bias that is statistically indistinguishable from zero for a given number of simulations. This result is consistent with the exact comparison presented in Figure 1. Bias correction for DGP2 is very precise at the right tail of the distribution, since the asymptotic formula precisely captures the population moments. The remaining bias in the left tail is still significant. This is expected since \( f_Y(y|W,Z) \) is close to zero on the left.

For DGP3, the bias reduction is even more noticeable than in the previous two designs. However, for \( \theta_{0,1} \), the remaining bias is still significant for most \( \tau \neq 0.5 \); for \( \theta_{0,2} \), the remaining bias is statistically indistinguishable from zero at all but two quantiles. The comparison with DGP2 suggests that the large size of the remainder bias is partially due to the fact the asymptotic formula only captures the quadratic terms of the population moments, and partially due to the low density \( f_Y(y|W,Z) \) away from the median.
Table 2: Rescaled bias of the bias corrected estimator $\hat{\theta}_{bc}$

| $\tau$ | 0.10 | 0.15 | 0.20 | 0.25 | 0.50 | 0.75 | 0.80 | 0.85 | 0.90 |
|-------|------|------|------|------|------|------|------|------|------|
| $nE(\hat{\theta}_{bc,1} - \theta_{0,1})$ | -0.10 | -0.07 | 0.00 | 0.03 | -0.03 | 0.02 | -0.01 | 0.09 | 0.09 |
| $3\times$MCSE | 0.10 | 0.11 | 0.12 | 0.13 | 0.14 | 0.13 | 0.12 | 0.11 | 0.09 |
| $nE(\hat{\theta}_{bc,2} - \theta_{0,2})$ | 0.02 | -0.00 | -0.09 | -0.10 | 0.07 | 0.02 | 0.09 | -0.05 | -0.02 |
| $3\times$MCSE | 0.17 | 0.19 | 0.20 | 0.22 | 0.25 | 0.22 | 0.20 | 0.19 | 0.16 |

| $\tau$ | 0.10 | 0.15 | 0.20 | 0.25 | 0.50 | 0.75 | 0.80 | 0.85 | 0.90 |
|-------|------|------|------|------|------|------|------|------|------|
| $nE(\hat{\theta}_{bc,1} - \theta_{0,1})$ | -0.18 | -0.19 | -0.04 | -0.12 | -0.06 | -0.03 | -0.02 | -0.00 | 0.01 |
| $3\times$MCSE | 0.14 | 0.14 | 0.13 | 0.13 | 0.10 | 0.07 | 0.07 | 0.06 | 0.05 |
| $nE(\hat{\theta}_{bc,2} - \theta_{0,2})$ | 0.11 | 0.20 | 0.01 | 0.13 | 0.09 | 0.02 | 0.01 | 0.00 | -0.00 |
| $3\times$MCSE | 0.24 | 0.24 | 0.23 | 0.22 | 0.18 | 0.13 | 0.12 | 0.10 | 0.09 |

| $\tau$ | 0.10 | 0.15 | 0.20 | 0.25 | 0.50 | 0.75 | 0.80 | 0.85 | 0.90 |
|-------|------|------|------|------|------|------|------|------|------|
| $nE(\hat{\theta}_{bc,1} - \theta_{0,1})$ | -3.88 | -2.70 | -0.37 | -0.40 | 0.07 | 0.72 | 0.39 | 3.26 | 4.96 |
| $3\times$MCSE | 1.18 | 0.60 | 0.34 | 0.25 | 0.13 | 0.25 | 0.34 | 0.71 | 1.27 |
| $nE(\hat{\theta}_{bc,2} - \theta_{0,2})$ | -0.41 | 1.79 | -0.13 | 0.18 | -0.16 | -0.78 | 0.12 | -2.85 | -1.85 |
| $3\times$MCSE | 1.96 | 0.97 | 0.59 | 0.42 | 0.23 | 0.42 | 0.58 | 1.02 | 1.93 |

5 Implications for empirical practice

Here we summarize the implications of our paper for the empirical practitioner.

Many empirical applications of IVQR models feature small sample problems, either due to a limited number of overall observations or when estimating tail quantiles. In practice, researchers are often interested in size control when testing hypotheses such that it is preferable to use estimators with low bias.

Our analysis suggests that exact estimators that minimize a norm of the sample moments have provable near $\frac{1}{n}$ bias (Theorem 6). The bias correction formula that we developed for exact estimators can reduce this bias even further. Corresponding results are not available for other estimators.\(^{20}\)

One limitation of the exact estimators is that they are computationally expensive. However, recent evidence (Zhu, 2019) shows that state-of-art MILP solvers can efficiently tackle sample sizes up to 1000 observations using consumer-grade desktop computers.\(^{21}\) Since higher order bias correction is particularly important for small and medium samples, we recommend using exact estimators when computationally feasible. For larger samples,

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\(^{20}\)Portnoy et al. (2012) established near $\frac{1}{n}$ bias for classical QR, but did not provide a bias correction approach.

\(^{21}\)MILP problems for tail quantile estimators remain tractable even with larger sample sizes.
we recommend a 1-step Newton correction of any $\sqrt{n}$-consistent approximate estimator like inverse quantile regression (Chernozhukov and Hansen, 2006)$^{22}$, Quasi-Bayesian MCMC (Chernozhukov and Hong, 2003), or fixed-point estimators (Kaido and Wüthrich, 2019). Under conditions of Theorem 2, such approximate estimators would be equivalent up to nearly $O_p(\frac{1}{n^{3/4}})$. So the choice should be based on computational speed considerations.

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$^{22}$Inverse quantile regression is computationally tractable only with very few endogenous regressors.
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Appendix

A Coupling of IVQR sample moments, proofs

A.1 Proof of Theorem 1

The proof consists of four steps below which depend on auxiliary lemmas in Section A.2. Throughout the proof, we denote the linear functional $G_n \triangleq \sqrt{n}(E_n - E).

Step 1 (bound on the approximation error). First, let us split $Z$ into a sum of a positive and negative parts $Z = Z^+ - Z^- \triangleq \max(Z, 0) - \max(-Z, 0)$. We will consider the positive part $Z^+$ only, since the argument for the negative part is analogous.

Denote $Z_0 \triangleq [0, m], \quad Z_{j,\ell} \triangleq \left( \frac{jm}{2^\ell}, \frac{(j+1)m}{2^\ell} \right)$, $j \in \{0, \ldots, 2^\ell - 1\}, \ \ell \geq 0$. (70)

Hence $Z_{j,\ell}$ is a subinterval of $[0, m]$ with length $m2^{-\ell}$ and midpoint $m(j+\frac{1}{2})/2^\ell$. The intervals are nested such that the union of $Z_{2j,\ell+1}$ and $Z_{2j+1,\ell+1}$ constitutes $Z_{j,\ell}$.

Denote the union of even numbered subintervals at level $\ell > 0$ as $Z_{\ell}$, i.e.

$$Z_{\ell} \triangleq \bigcup_{j=0}^{2^\ell-1} Z_{2j,\ell}. \quad (71)$$

Using these intervals, we can define an approximating sequence of simple random variables

$$Z^{+,*} \triangleq \left( 1 - \sum_{\ell=1}^{\bar{\ell}} \frac{1}{2^\ell} 1\{Z \in Z_{\ell}\} \right) \frac{m}{2^{\bar{\ell}}} \quad (72)$$

By construction, $|Z^+ - Z^{+,*}| \leq m/2^{\bar{\ell}}$.

Hence, the empirical process $Z^+_n(y, a) \triangleq G_n1\{Y \leq y, A = a\}Z^+$ can be written as

$$m^{-1}Z^+_n(y, a) = G_n1\{Y \leq y, A = a, Z \in Z_0\} - \sum_{\ell=1}^{\infty} 2^{-\ell} G_n1\{Y \leq y, A = a, Z \in Z_{\ell}\} \quad (73)$$

$$\triangleq Z^{+,0}_n(y, a) - \sum_{\ell=1}^{\infty} 2^{-\ell} Z^{+,\ell}_n(y, a), \quad (74)$$

\footnote{This is the Hungarian construction (see, for example, p.252 in Pollard, 2002).}
where the last line defines $Z^{+\ell, n}$, for $\ell \geq 0$.

By Lemmas A.2 and A.3, for each $\ell \geq 0$, there exists an approximating Brownian bridge $Z^{+, \ell}(y, a)$, such that the remainder

$$R^{+, \ell}_n(y, a) \triangleq Z^{+, \ell}_n(y, a) - Z^{+, \ell}(y, a)$$

(75)

satisfies the bound $\mathbb{E} \sup_{y \in \mathbb{R}} |R^{+, \ell}_n(y, a)| \leq \tilde{c}_1 \frac{\log n + \tilde{c}_0}{\sqrt{n}}$.

Now, with a convenient abuse of notation, define

$$Z^{+, \bar{l}}_n(y, a) \triangleq Z^{+, 0}_n(y, a) - \sum_{\ell=0}^{\bar{l}} 2^{-\ell} Z^{+, \ell}_n(y, a),$$

(76)

$$Z^{+, \bar{l}}(y, a) \triangleq Z^{+, 0}(y, a) - \sum_{\ell=0}^{\bar{l}} 2^{-\ell} Z^{+, \ell}(y, a).$$

(77)

The total remainder of the approximation of $Z^{+, \bar{l}}_n$ by $Z^{+, \bar{l}}$ is then

$$R^{+, \bar{l}}_n(y, a) \triangleq Z^{+, \bar{l}}_n(y, a) - Z^{+, \bar{l}}(y, a) = R^{+, \ell}_n(y, a) - \sum_{\ell=1}^{\bar{l}} 2^{-\ell} R^{+, \ell}_n(y, a).$$

(78)

By the triangle inequality,

$$\mathbb{E} \sup_{y \in \mathbb{R}} |R^{+, \bar{l}}_n(y, a)| \leq \sum_{\ell=0}^{\bar{l}} 2^{-\ell} \mathbb{E} \sup_{y \in \mathbb{R}} |R^{+, \ell}_n(y, a)| \leq \left( \sum_{\ell=0}^{\bar{l}} 2^{-\ell} \right) \tilde{c}_1 \frac{\log n + \tilde{c}_0}{\sqrt{n}}.$$  

(79)

Note that the bound $\sum_{\ell=0}^{\bar{l}} 2^{-\ell} \leq 2$ does not depend on $\bar{l}$. Then we can define an approximating process

$$Z^+(y, a) \triangleq m Z^{+, \bar{l}}(y, a),$$

(80)

so that

$$Z^{+}_n(y, a) = Z^+(y, a) + R^{+}_n(y, a),$$

(81)

where

$$\mathbb{E} \sup_{y \in \mathbb{R}} |R^{+}_n(y, a)| \leq \left( \sum_{\ell=0}^{\bar{l}} 2^{-\ell} \right) \tilde{c}_1 \frac{\log n + \tilde{c}_0}{\sqrt{n}} m + m \mathbb{E} \sum_{\ell=\bar{l}+1}^{\infty} 2^{-\ell} \mathbb{E} \sup_{y \in \mathbb{R}} |Z^{+, \ell}_n(y, a)|$$

(82)

$$\leq \left( \sum_{\ell=0}^{\bar{l}} 2^{-\ell} \right) \tilde{c}_1 \frac{\log n + \tilde{c}_0}{\sqrt{n}} m + m \sum_{\ell=\bar{l}+1}^{\infty} 2^{-\ell} \mathbb{E} \sup_{y \in \mathbb{R}} |Z^{+, \ell}_n(y, a)|.$$  

(83)
By the KMT Theorem,

\[ \mathbb{E} \sup_{y \in \mathbb{R}} |Z_n^{+, \ell}(y, a)| \leq \mathbb{E} \sup_{y \in \mathbb{R}} |Z_n^{+, \ell}(y, a)| + \mathbb{E} \sup_{y \in \mathbb{R}} |R_n^{+, \ell}(y, a)|. \]  

(84)

The first term on the right is bounded by the expectation of a supremum of a standard Brownian bridge over [0, 1], denoted as constant \( c_B \). The constant \( c_B \) is finite by the definition of a Brownian bridge and the tail bounds for Brownian motions (Shorack and Wellner, 2009, p.34). The second term is bounded by \( \tilde{c}_1 \frac{\log n + \tilde{c}_0}{\sqrt{n}} \). Note that both bounds do not depend on \( \ell \), and hence

\[ \mathbb{E} \sup_{y \in \mathbb{R}} |R_n^{+, \ell}(y, a)| \leq 2\tilde{c}_1 \frac{\log n + \tilde{c}_0 + c_B/(2\tilde{c}_1)}{\sqrt{n}} m + m 2^{-(\tilde{\ell} - 1)} c_B. \]  

(85)

We can take \( \tilde{\ell} = 1 + \log_2(\sqrt{n}) \) so that

\[ \mathbb{E} \sup_{y \in \mathbb{R}} |R_n^{+, \ell}(y, a)| \leq 2\tilde{c}_1 \frac{\log n + \tilde{c}_0 + c_B/(2\tilde{c}_1)}{\sqrt{n}} m + c_1 \frac{\log n + c_0}{\sqrt{n}} m, \]  

(86)

for appropriately defined constants \( c_0, c_1 \).

The bound on the remainder of the analogous approximation for \( Z^- \) is the same. To complete the argument, take \( Z(y, a) \triangleq Z^+(y, a) - Z^-(y, a) \) and apply the triangle inequality to bound the total remainder.

**Step 2** (\( Z^{+, \ell}(\cdot, a) \) is a.s. 1/2-Hölder, up to a log term). Denote \( \psi(y) = P\{Y \leq y, Z \in Z^+, A = a\} \). Since \( Z^{+, \ell}(y, a) = B^{(\ell)}(y) - y B^{(\ell)}(1) \) for some Brownian motion \( B^{(\ell)} \), we have

\[ Z^{+, \ell}(y_2, a) - Z^{+, \ell}(y_1, a) = B^{(\ell)}(\psi(y_2)) - B^{(\ell)}(\psi(y_1)) - (\psi(y_2) - \psi(y_1)) B^{(\ell)}(1). \]  

(87)

Therefore,

\[ \frac{|Z^{+, \ell}(y_2, a) - Z^{+, \ell}(y_1, a)|}{\sqrt{\Psi(f|y_2 - y_1|)}} \leq \frac{|B^{(\ell)}(\psi(y_2)) - B^{(\ell)}(\psi(y_1))|}{\sqrt{\Psi(f|y_2 - y_1|)}} + \frac{|\psi(y_2) - \psi(y_1)| \cdot |B^{(\ell)}(1)|}{\sqrt{\Psi(f|y_2 - y_1|)}} \]  

\[ = \frac{|B^{(\ell)}(\psi(y_2)) - B^{(\ell)}(\psi(y_1))|}{\sqrt{2\Psi(|\psi(y_2) - \psi(y_1)|)}} + \sqrt{2 \frac{\Psi(|\psi(y_2) - \psi(y_1)|)}{\Psi(f|y_2 - y_1|)}} \]  

\[ + \sqrt{|\psi(y_2) - \psi(y_1)|} \cdot \frac{|\psi(y_2) - \psi(y_1)|}{f|y_2 - y_1| \log(1/f|y_2 - y_1|)} \cdot |B^{(\ell)}(1)|. \]  

(88)

(89)

(90)

The goal of this step is to derive an a.s. bound of \( \lim sup \sup_{r \to 0^+} \sup_{0 < |y_2 - y_1| \leq r} \) of the left-hand side.

By Lévy’s modulus of continuity theorem (Theorem 10.6 in Schilling and Partzsch, 2014),
we have\[
\limsup_{r \to 0^+} \sup_{x_1, x_2 \in [0,1], |x_2 - x_1| \leq r} \frac{|B^\ell(x_2) - B^\ell(x_1)|}{\sqrt{2\Psi(r)}} = 1 \quad \text{a.s.}, \tag{91}
\]
so, since $\psi(\cdot) \in [0,1]$, the lim sup of term (i) is a.s. bounded by 1 too.

In the second term (ii), we have, for $|y_2 - y_1|$ small enough,

\[
0 < \frac{\Psi(|\psi(y_2) - \psi(y_1)|)}{\Psi(\bar{f}|y_2 - y_1|)} \leq 1, \tag{92}
\]

since $\Psi(x)$ is positive and strictly increasing in $(0, e^{-1})$ (see Lemma 10.4 in Schilling and Partzsch, 2014). Hence lim sup sup of the term (ii) is bounded by $\sqrt{2}$ a.s..

Finally, in the term (iii),

\[
\sqrt{|\psi(y_2) - \psi(y_1)|} \leq \sqrt{\bar{f}|y_2 - y_1|}, \tag{93}
\]

and therefore

\[
\limsup_{r \to 0^+} \sup_{0 < |y_2 - y_1| \leq r} \sqrt{|\psi(y_2) - \psi(y_1)|} = 0 \quad \text{a.s.} \tag{94}
\]

Since

\[
\frac{|\psi(y_2) - \psi(y_1)|}{\bar{f}|y_2 - y_1| \log(1/\bar{f}|y_2 - y_1|)} \leq \frac{|\psi(y_2) - \psi(y_1)|}{\bar{f}|y_2 - y_1|} \leq 1, \tag{95}
\]

and\[
\limsup_{r \to 0^+} \sup_{0 < |y_2 - y_1| \leq r} |B^\ell(1)| = |B^\ell(1)| \quad \text{a.s.}, \tag{96}
\]

Combining the bounds for lim sup sup of terms (i), (ii) and (iii), we obtain

\[
\limsup_{r \to 0^+} \sup_{0 < |y_2 - y_1| \leq r} \frac{|Z^{+,\ell}(y_2, a) - Z^{+,\ell}(y_1, a)|}{\sqrt{\Psi(|y_2 - y_1|)}} \leq \sqrt{2} \quad \text{a.s.} \tag{97}
\]

**Step 3 (Z(\cdot, a) is a.s. 1/2-Hölder, up to a log term).** By the definition of $Z^{+,\ell}(y, a)$ in Equation (80),

\[
m^{-1}(Z^{+,\ell}(y_2, a) - Z^{+,\ell}(y_1, a)) = Z^{+,0}(y_2, a) - Z^{+,0}(y_1, a) - \sum_{\ell=1}^{\frac{\ell}{2\ell}} \bar{f} \cdot (Z^{+,\ell}(y_2, a) - Z^{+,\ell}(y_1, a)). \tag{98}
\]

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Dividing by \( \sqrt{\Psi(f|y_2 - y_1|)} \) and using the triangular inequality, we obtain

\[
m^{-1} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{\sqrt{\Psi(f|y_2 - y_1|)}} \leq \frac{|Z^{+,0}(y_2, a) - Z^{+,0}(y_1, a)|}{\sqrt{\Psi(f|y_2 - y_1|)}} + \sum_{\ell=1}^{\bar{\ell}} \frac{1}{2^{\ell}} \frac{|Z^{+,\ell}(y_2, a) - Z^{+,\ell}(y_1, a)|}{\sqrt{\Psi(f|y_2 - y_1|)}}. \tag{99}
\]

Taking \( \limsup \sup \) of both sides and multiplying by \( m \) yields

\[
\limsup_{r \to 0^+} \sup_{0 < |y_2 - y_1| \leq r} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{\sqrt{\Psi(f|y_2 - y_1|)}} \leq m(\sqrt{2} + \sqrt{2}) = 2m\sqrt{2}. \tag{100}
\]

The case of \( Z^- \) is analogous and yields the overall bound

\[
\limsup_{r \to 0^+} \sup_{0 < |y_2 - y_1| \leq r} \frac{|Z(y_2, a) - Z(y_1, a)|}{\sqrt{\Psi(f|y_2 - y_1|)}} \leq 4m\sqrt{2}. \tag{101}
\]

**Step 4 (Z is \((1/2 - \gamma)\)-Hölder in expectation).** First, note that, for any fixed \( \gamma \in (0, \frac{1}{2}) \),

\[
\mathbb{E} \sup_{y_1 \neq y_2} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|y_2 - y_1|^{\frac{1}{2}-\gamma}} \\
\leq \mathbb{E} \sup_{y_1 \neq y_2, \psi(y_1) \neq \psi(y_2)} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma}} \left( \frac{|\psi(y_2) - \psi(y_1)|}{|y_2 - y_1|} \right)^{\frac{1}{2}-\gamma} + \mathbb{E} \sup_{y_1 \neq y_2, \psi(y_2) = \psi(y_2)} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|y_2 - y_1|^{\frac{1}{2}-\gamma}} \\
\leq \mathbb{E} \sup_{\psi(y_1) \neq \psi(y_2)} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma}} \cdot \bar{f}^{\frac{1}{2}-\gamma}. \tag{102}
\]

By the definition of \( Z^+(y, a) \) in Equation (80),

\[
m^{-1}(Z^+(y_2, a) - Z^+(y_1, a)) = Z^{+,0}(y_2, a) - Z^{+,0}(y_1, a) - \sum_{\ell=1}^{\bar{\ell}} \frac{1}{2^{\ell}} \cdot (Z^{+,\ell}(y_2, a) - Z^{+,\ell}(y_1, a)).
\]

Dividing by \( |\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma} \) and using the triangular inequality, we obtain

\[
m^{-1} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma}} \leq \frac{|Z^{+,0}(y_2, a) - Z^{+,0}(y_1, a)|}{|\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma}} + \sum_{\ell=1}^{\bar{\ell}} \frac{1}{2^{\ell}} \cdot \frac{|Z^{+,\ell}(y_2, a) - Z^{+,\ell}(y_1, a)|}{|\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma}}. \tag{103}
\]

Taking \( \mathbb{E} \sup_{\psi(y_1) \neq \psi(y_2)} \) of both sides and multiplying by \( m \) yields

\[
\mathbb{E} \sup_{\psi(y_1) \neq \psi(y_2)} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|\psi(y_2) - \psi(y_1)|^{\frac{1}{2}-\gamma}} \leq 2C_m, m < \infty, \tag{104}
\]

where the bound holds by Lemma A.5, since \( Z^{+,\ell}, \ell \geq 0, \) are Brownian bridges.
Combining with \((102)\) yields

\[
E\sup_{y_1 \neq y_2} \frac{|Z^+(y_2, a) - Z^+(y_1, a)|}{|y_2 - y_1|^\frac{1}{2} - \gamma} \leq 2C_\gamma m f^\frac{1}{2} - \gamma,
\]

which is the definition of \((1/2 - \gamma)\)-Hölder continuity of \(Z^+\) in expectation.

The case of \(Z^-\) is analogous and yields the overall bound

\[
E\sup_{y_1 \neq y_2} \frac{|Z(y_2, a) - Z(y_1, a)|}{|y_2 - y_1|^\frac{1}{2} - \gamma} \leq 4C_\gamma m f^\frac{3}{2} - \gamma.
\]

\(\square\)

### A.2 Auxiliary Lemmas

**Lemma A.1.** Let \(Y\) be a r.v. with CDF \(F(y)\). Then there exist a uniformly distributed r.v. \(V\) such that \(F^{-1}(V) = Y\) a.s.

**Proof.** This result follows immediately from Proposition 3.2 in Shorack (2017). \(\square\)

Consider a simple r.v. \(A \in \{0, 1\}\) defined on the same probability space with \(Y\). Let

\[
G_n(y, a) \triangleq \frac{1}{\sqrt{n}} \sum_{i=1}^n (1\{Y_i \leq y, A_i = a\} - P\{Y_i \leq y, A_i = a\}) \quad \text{for} \quad a \in \{0, 1\}.
\]

**Lemma A.2.** For all \(a \in \{0, 1\}\), there exist a tight Brownian Bridge \(\{B^\circ(t) : 0 \leq t \leq 1\}\) such that, for all \(x \geq 0\),

\[
P \left( \sup_{y \in \mathbb{R}} |G_n(y, a) - B^\circ(P\{Y \leq y, A = a\})| \geq c_1 \frac{x + \log n}{\sqrt{n}} \right) \leq c_0 \exp(-x),
\]

where the constants \(c_1\) and \(c_0\) do not depend on \(n, a,\) and \(x\).

**Proof.** First, consider the case \(a = 1\). Note that for any CDF \(F\) and its left-continuous inverse \(F^{-1}\) the following holds: for any \(p \in [0, 1]\) and \(x \in \mathbb{R}\),

\[
F^{-1}(p) \leq x \text{ if and only if } p \leq F(x).
\]

Let \(V\) be the uniform r.v. such that \(F^{-1}(V) = Y\) a.s. as in Lemma A.1. Then

\[
\{Y \leq y, A = 1\} = \{F^{-1}(V) \leq y, A = 1\} = \{V \leq F^{-1}(y), A = 1\} = \{V + 2(1 - A) \leq F(y)\}.
\]

Now let \(\tilde{F}\) be the CDF of \(V + 2(1 - A)\) and let \(U\) be the uniform r.v. such that \(\tilde{F}^{-1}(U) =

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\( V + 2(1 - A) \) a.s. as in Lemma A.1. Then
\[
\{ V + 2(1 - A) \leq F(y) \} = \{ \tilde{F}^{-1}(U) \leq F(y) \} = \{ U \leq \tilde{F}(F(y)) \}
\]

By the KMT theorem, there exist a tight Brownian bridge \( B^o(u) \), such that
\[
R_n \triangleq \sup_{0 \leq u \leq 1} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1\{ U_i \leq u \} - P\{ U_i \leq u \}) - B^o(u) \right| \tag{108}
\]
is uniformly tight and satisfies, for any \( x > 0 \),
\[
P \left\{ R_n \geq c_1 \frac{x + \log n}{\sqrt{n}} \right\} \leq c_0 \exp(-x), \tag{109}
\]
where constants \( c_1 \) and \( c_0 \) depend neither on \( x \) nor \( n \).

Therefore,
\[
R_Y^n \triangleq \sup_{y \in \mathbb{R}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1\{ Y_i \leq y, A_i = 1 \} - P\{ Y_i \leq y, A_i = 1 \}) - B^o(P\{ Y_i \leq y, A_i = 1 \}) \right|
= \sup_{y \in \mathbb{R}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1\{ U_i \leq \tilde{F}(F(y)) \} - P\{ U_i \leq \tilde{F}(F(y)) \}) - B^o(P\{ U_i \leq \tilde{F}(F(y)) \}) \right|
\leq \sup_{0 \leq u \leq 1} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1\{ U_i \leq u \} - P\{ U_i \leq u \}) - B^o(u) \right| = R_n
\]

This establishes that \( R^n_y \leq R_n \) a.s., which means that \( R^n_y \) also satisfies the tail bound (109).

The case \( a = 0 \) is analogous since \( A = 0 \) is equivalent to \( A^c = 1 \), where \( A^c = 1 - A \). This completes the proof.

\[\square\]

**Lemma A.3.** Let a sequence of r.v. \( R_n \geq 0 \) satisfy
\[
P \left\{ R_n \geq c_1 \frac{x + \log n}{\sqrt{n}} \right\} \leq c_0 \exp(-x) \text{ for all } x \geq 0. \tag{110}
\]

Then
\[
\mathbb{E} R_n \leq \tilde{c}_1 \frac{\log n + \tilde{c}_0}{\sqrt{n}}. \tag{111}
\]
Proof. Define \( \xi_n \equiv \frac{\sqrt{n}}{c_1} R_n \) and consider the following decomposition
\[
\mathbb{E} \xi_n = \mathbb{E} \xi_n 1\{\xi_n \leq \log n\} + \mathbb{E} \xi_n 1\{\xi_n > \log n\}.
\]
The first term is bounded by \( \log n \). The second term can be bounded as follows:
\[
\mathbb{E} \xi_n 1\{\xi_n > \log n\} = \mathbb{E} (\xi_n - \log n) 1\{\xi_n - \log n > 0\} + \log n \cdot P(\xi_n - \log n > 0) \\
\leq \mathbb{E} \xi_n + c_0 \log n,
\]
where \( \xi_n = (\xi_n - \log n) 1\{\xi_n - \log n > 0\} \).

By assumption of the lemma, for \( x \geq 0 \),
\[
P\{\xi_n \geq x\} = P\left\{ R_n \geq c_1 \frac{x + \log n}{\sqrt{n}} \right\} \leq c_0 \exp(-x) \quad \text{for all} \quad x \geq 0,
\]
Therefore, by the Fubini theorem (e.g., Kallenberg, 2006, Lemma 3.4), \( \mathbb{E} \xi_n \leq c_0 \). Overall, we obtain
\[
\mathbb{E} \xi_n \leq \log n + c_0 + c_0 \log n = c_0 + (1 + c_0) \log n \quad (112)
\]
or, substituting the expression for \( \xi_n \),
\[
\mathbb{E} R_n \leq \frac{c_1(1 + c_0) \log n + c_1 c_0}{\sqrt{n}}. \quad (113)
\]
Taking \( \tilde{c}_1 = c_1(1 + c_0) \) and \( \tilde{c}_0 = c_0/(1 + c_0) \) completes the proof.

Lemma A.4. If a sequence of stochastic processes \( \theta \mapsto X_n(\theta) \) defined on \( \Theta \subset \mathbb{R}^k \) satisfies the bound
\[
\mathbb{E} \sup_{\theta' \neq \theta} \frac{|X_n(\theta') - X_n(\theta)|}{||\theta' - \theta||^{\frac{1}{2} - \gamma}} \leq C_\gamma < \infty \quad (114)
\]
for some constant \( C_\gamma \) that does not depend on \( n \), then for any random sequences \( \theta'_n, \theta_n \) we have
\[
X_n(\theta'_n) - X_n(\theta_n) = O_p \left( ||\theta'_n - \theta_n||^{\frac{1}{2} - \gamma} \right). \quad (115)
\]

Proof. For \( M > 0 \), Markov’s inequality implies
\[
P \left( |X_n(\theta'_n) - X_n(\theta_n)| > M ||\theta'_n - \theta_n||^{\frac{1}{2} - \gamma} \right) \leq P \left( \sup_{\theta' \neq \theta} \frac{|X_n(\theta') - X_n(\theta)|}{||\theta' - \theta||^{\frac{1}{2} - \gamma}} > M \right) \leq M^{-1} \mathbb{E} \sup_{\theta' \neq \theta} \frac{|X_n(\theta') - X_n(\theta)|}{||\theta' - \theta||^{\frac{1}{2} - \gamma}} \leq M^{-1} C_\gamma \quad (116) \]
Therefore, the left-hand side can be made arbitrarily small (for all \( n \)) by choosing large enough \( M \). The statement of the theorem follows. \( \square \)
Lemma A.5. Any sequence of Brownian bridges \( t \mapsto BB_n(t) \) defined on \( \Theta = [0, 1] \) satisfies the bound (114).

Proof. Since \( BB_n(t) = BM_n(t) - tBM_n(1) \) for some Brownian motion \( BM_n(\cdot) \), we have

\[
BB_n(t') - BB_n(t) = BM_n(t') - BM_n(t) - (t' - t)BM_n(1). \tag{118}
\]

For the first term in (118), we have

\[
E \sup_{t' \neq t} \frac{|BM_n(t') - BM_n(t)|}{|t' - t|^{\frac{1}{2} - \gamma}} \leq E \sup_{0 < |t' - t| < 1} \frac{|BM_n(t') - BM_n(t)|}{|t' - t|^{\frac{1}{2} - \gamma}} + E|BM_n(1) - BM_n(0)| < \infty, \tag{119}
\]

where the last inequality holds since the first term on the right is finite by Theorem 10.1 and Corollary 10.2 in Schilling and Partzsch (2014).

For the second term in (118), we have

\[
E \sup_{t' \neq t} \frac{|(t' - t)BM_n(1)|}{|t' - t|^{\frac{1}{2} - \gamma}} = E \sup_{t' \neq t} |t' - t|^{\frac{1}{2} - \gamma}|BM_n(1)| \leq E|BM_n(1)| < \infty. \tag{120}
\]

Combining (119) and (120) yields

\[
E \sup_{t' \neq t} \frac{|BB_n(t') - BB_n(t)|}{|t' - t|^{\frac{1}{2} - \gamma}} \leq E \sup_{t' \neq t} \frac{|BM_n(t') - BM_n(t)|}{|t' - t|^{\frac{1}{2} - \gamma}} + E \sup_{t' \neq t} \frac{|(t' - t)BM_n(1)|}{|t' - t|^{\frac{1}{2} - \gamma}} < \infty. \tag{121}
\]

The conclusion follows since no bounds in this proof depend on \( n \). \qed
B Bahadur-Kiefer representation, proofs

**Lemma B.1.** Under Assumptions 1.1 and 2.2, \( g(\theta) \) is three times continuously differentiable.

**Proof.** By definition, \( g(\theta) = \mathbb{E}(1\{Y \leq W'\theta\} - \tau)Z = \mathbb{E}(F_Y(W'\theta|W, Z) - \tau)Z \). The result then follows from the dominated convergence theorem. \( \square \)

**Lemma B.2.** Suppose Assumptions 1.1 and 2.2 hold. Then for any estimator \( \hat{\theta} \), we have a representation

\[
\hat{g}(\hat{\theta}) = \frac{1}{\sqrt{n}}B_n^c(\theta_0) + \tau(\mathbb{E}Z - \mathbb{E}_nZ) + \frac{1}{\sqrt{n}}B_n(\hat{\theta}) + g(\theta_0) + G(\theta_0)(\hat{\theta} - \theta_0) + \frac{1}{2}(\hat{\theta} - \theta_0)'\partial_\theta G(\theta_0)(\hat{\theta} - \theta_0) + O_p\left(\|\hat{\theta} - \theta_0\|^2\right).
\]  

**(122)**

**Proof.** By definition

\[
\hat{g}(\hat{\theta}) = \mathbb{E}_n1\{Y \leq W'\hat{\theta}\}Z - \tau\mathbb{E}_nZ,
\]

\[
= \frac{1}{\sqrt{n}}B_n^c(\hat{\theta}) + g^c(\hat{\theta}) - \tau\mathbb{E}_nZ
\]

\[
= \frac{1}{\sqrt{n}}B_n^c(\theta_0) + \frac{1}{\sqrt{n}}B_n(\hat{\theta}) + \tau(\mathbb{E}Z - \mathbb{E}_nZ) + g(\hat{\theta})
\]

**(125)**

By Lemma B.1, \( g(\cdot) \) is three times continuously differentiable. The Taylor theorem implies that there exist a neighborhood of \( \theta_0 \) such that for any \( \theta \) in the neighborhood,

\[
g(\theta) = g(\theta_0) + G(\theta_0)(\theta - \theta_0) + \frac{1}{2}(\theta - \theta_0)'\partial_\theta G(\theta_0)(\theta - \theta_0) + R(\theta),
\]

**(126)**

where \( R(\theta) = O\left(\|\theta - \theta_0\|^2\right) \). Then (122) follows immediately. \( \square \)

**Lemma B.3.** Suppose Assumptions 1, 2.1 and 3.1 hold. For any pair of estimators \( \hat{\theta} \) and \( \hat{\theta}^* \in \mathbb{R}^d \), the following property holds

\[
B_n(\hat{\theta}) - B_n(\hat{\theta}^*) = O_p\left(sm(mf\|\hat{\theta} - \hat{\theta}^*\|)^{\frac{3}{2} - \gamma}\right) + O_p\left(\frac{sm\log n}{\sqrt{n}}\right).
\]

**(127)**

**Proof.** Assumption 3.1 implies that \( \theta_0 \) is well-defined. Define the random variables \( \varepsilon_i \triangleq Y_i - W_i'\theta_0 \) and \( \bar{\varepsilon}_i \triangleq \frac{\varepsilon_i}{\|W_i\|} \), as well as binary variables \( A^q \triangleq 1\{W = \bar{w}_q\} \) for \( q = 1, \ldots, s \). Consider the corresponding empirical process

\[
B_n^2(e_1, e_2) \triangleq \mathbb{G}_n(1\{\bar{\varepsilon} \leq e_2 - \bar{w}'_q\theta_0, A^q = 1\}Z)
\]

\[ - \mathbb{G}_n(1\{\bar{\varepsilon} \leq e_1 - \bar{w}'_q\theta_0, A^q = 1\}Z),
\]

**(128)**

\(41\)
where \( e_1, e_2 \in \mathbb{R} \) and the empirical process operator is defined as \( G_n \equiv \sqrt{n}(E_n - E) \). By Assumption 1.2, events \( \{\bar{W} = \bar{w}_q\} \), \( q = 1, \ldots, s \), form a partition of the probability space, so

\[
B_n(\hat{\theta}) - B_n(\hat{\theta}^*) = G_n([1(\bar{\varepsilon} \leq \bar{W}'(\hat{\theta} - \theta_0)) - 1(\bar{\varepsilon} \leq \bar{W}'(\hat{\theta}^* - \theta_0))]Z)
\]

(129)

\[
= \sum_{q=1}^s B_n^2(\bar{w}'_q \hat{\theta}, \bar{w}'_q \hat{\theta}^*).
\]

(130)

By Assumption 2.1, \( f_\varepsilon(e|W, Z) \equiv f_Y(e + W'\theta_0|W, Z) \leq \bar{f} \). By Assumption 1.1, it follows that \( f_\varepsilon(e|W, Z) \leq m \bar{f} \). Now apply (21) from Theorem 1 and Lemma A.4 to \( B_n^q(\cdot, \cdot) \) for each \( q = 1, \ldots, s \) to get

\[
B_n^q(\bar{w}'_q \hat{\theta}, \bar{w}'_q \hat{\theta}^*) = O_p\left(m(m\bar{f}|\hat{\theta} - \hat{\theta}^*|)^{\frac{1}{2}}\right) + O_p\left(\frac{m\log n}{\sqrt{n}}\right),
\]

(131)

\[
\leq O_p\left(m(m\bar{f}|\hat{\theta} - \hat{\theta}^*|)^{\frac{1}{2}}\right) + O_p\left(\frac{m\log n}{\sqrt{n}}\right),
\]

(132)

where the last inequality follows by the Cauchy–Schwarz inequality.

The result (127) follows immediately from (130) and (131).

\[\square\]

**Lemma B.4.** Under Assumptions 1.1, 2.1, and 3.1, any estimator \( \hat{\theta}_{\ell_p} \) that minimizes \( \theta \mapsto \|\hat{g}(\theta)\|_p \) satisfies \( \hat{\theta}_{\ell_p} \overset{p}{\to} \theta_0 \).

**Proof.** By Assumption 3.1,

\[
\arg\min_{\theta \in \Theta} \|g(\theta)\|_p = \theta_0.
\]

(133)

Assumptions 2.1 and 1 imply that the function class \( \theta \mapsto m(Y, W, Z, \theta) \) is Donsker and thus Glivenko-Cantelli, and hence\(^{24}\)

\[
\sup_{\theta \in \Theta} |\hat{g}(\theta) - g(\theta)| = \sup_{\theta \in \Theta} |(E - E_n)m(Y, W, Z, \theta)| \overset{a.s.}{\to} 0.
\]

(134)

By the argmin theorem (Theorem 2.1 in Newey and McFadden, 1994), applied to \( Q_n(\theta) = \|\hat{g}(\theta)\|_p \), we get \( \hat{\theta}_{\ell_p} \overset{p}{\to} \theta_0 \).

\[\square\]

**Lemma B.5.** Under Assumptions 1.1, 2, 3, for any estimator \( \hat{\theta}_{\ell_p} \) that minimizes \( \|\hat{g}(\theta)\|_p \),\(^{24}\)

---

\(^{24}\)See Chernozhukov and Hansen (2006, Lemma B.2) for a more detailed discussion.
we have

\[ \hat{\theta}_p = \hat{\theta}_1 + o_p \left( \frac{1}{\sqrt{n}} \right), \]  

\[ \| \hat{g}(\hat{\theta}_p) \|_p = o_p \left( \frac{1}{\sqrt{n}} \right), \]  

(135)

(136)

where \( \hat{\theta}_1 \) is introduced in equation (23).

**Proof.** The proof proceeds in four steps.

**Step 1.** Notice that under the assumptions of the lemma, the empirical process \( B_n(\theta) \) is Donsker and stochastically equicontinuous (see Chernozhukov and Hansen, 2006, Lemma B.2).

**Step 2.** By definition, \( \hat{\theta}_1 \) can be written as

\[ \hat{\theta}_1 = \theta_0 - G^{-1} \left[ \tau(\mathbb{E}Z - \mathbb{E}_nZ) + \frac{1}{\sqrt{n}} B_n^\circ(\theta_0) \right] = \theta_0 + O_p \left( \frac{1}{\sqrt{n}} \right), \]  

(137)

where \( \theta_0 \) and \( G^{-1} \triangleq \partial g(\theta_0) \) are well-defined by Assumption 3 and CLT holds as an implication of 1.1.

By Lemma B.2 and Step 1,

\[ \hat{g}(\hat{\theta}_1) = \frac{1}{\sqrt{n}} B_n^\circ(\theta_0) + \tau(\mathbb{E}Z - \mathbb{E}_nZ) + \frac{1}{\sqrt{n}} B_n(\hat{\theta}_1) 
+ g(\theta_0) + G(\theta_0)(\hat{\theta}_1 - \theta_0) + \frac{1}{2}(\hat{\theta}_1 - \theta_0)' \partial_\theta G(\theta_0)(\hat{\theta}_1 - \theta_0) + O_p \left( \frac{n^{-\frac{3}{2}}}{\sqrt{n}} \right) \] 

(138)

\[ = \frac{1}{\sqrt{n}} B_n(\hat{\theta}_1) + O_p \left( \frac{1}{n} \right) = O_p \left( n^{-\frac{1}{2}} \right). \]  

(139)

Since \( \hat{\theta}_p \) is defined as the estimator with the minimal norm,

\[ \| \hat{g}(\hat{\theta}_p) \|_p \leq \| \hat{g}(\hat{\theta}_1) \|_p = O_p \left( n^{-\frac{1}{2}} \right). \]  

(140)
Step 3. By Lemma B.2 and Steps 1 and 2, $\hat{\theta}_p$ satisfies

$$G(\theta_0)(\hat{\theta}_p - \theta_0) + \frac{1}{2}(\hat{\theta}_p - \theta_0)' \partial_{\theta} G(\theta_0)(\hat{\theta}_p - \theta_0)$$

$$= \hat{g}(\hat{\theta}_p) - \frac{1}{\sqrt{n}} B_n(\theta_0) - \tau(\mathbb{E} Z - \mathbb{E}_n Z) + \frac{1}{\sqrt{n}} B_n(\hat{\theta}_p) + O_p \left( \|\hat{\theta}_p - \theta_0\|^3 \right)$$

$$= O_p \left( \frac{1}{\sqrt{n}} \right) + O_p \left( \|\hat{\theta}_p - \theta_0\|^3 \right).$$

(141)

By Lemma B.4, $\|\hat{\theta}_p - \theta_0\| \xrightarrow{p} 0$. By Assumption 3.2,

$$\hat{\theta}_p - \theta_0 + O_p \left( \|\hat{\theta}_p - \theta_0\|^2 \right) = O_p \left( \frac{1}{\sqrt{n}} \right),$$

(143)

$$O_p \left( \|\hat{\theta}_p - \theta_0\| \right) = O_p \left( \frac{1}{\sqrt{n}} \right),$$

(144)

which implies $\hat{\theta}_p = \theta_0 + O_p \left( \frac{1}{\sqrt{n}} \right)$. So consistency of $\hat{\theta}_p$ implies by Step 1 that

$$B_n(\hat{\theta}_p) = B_n^o(\hat{\theta}_p) - B_n^o(\theta_0) = o_p(1).$$

(145)

Step 4. Consider $\hat{\theta}_2 \triangleq \hat{\theta}_1 - G^{-1} \frac{B_n(\hat{\theta}_1)}{\sqrt{n}} = \hat{\theta}_1 + o_p \left( \frac{1}{\sqrt{n}} \right)$. Then by the stochastic equicontinuity of $B_n$ and Lemma B.2,

$$\hat{g}(\hat{\theta}_2) = \frac{B_n(\hat{\theta}_2) - B_n(\hat{\theta}_1)}{\sqrt{n}} + o_p \left( \frac{1}{\sqrt{n}} \right) = o_p \left( \frac{1}{\sqrt{n}} \right).$$

(146)

This implies

$$\|\hat{g}(\hat{\theta}_p)\|_p \leq \|\hat{g}(\hat{\theta}_2)\|_p = o_p \left( n^{-\frac{1}{2}} \right).$$

(147)

The remaining result then follows from (142).

\[\Box\]

Lemma B.6. Suppose Assumptions 1, 2.1 and 3.2 hold. Then for any estimators $\hat{\theta}$ and $\hat{\theta}^*$ that satisfy

$$\hat{\theta} - \hat{\theta}^* = -G^{-1}(\theta_0) \frac{B_n(\hat{\theta}) - B_n(\hat{\theta}^*)}{\sqrt{n}} + \frac{R_n}{\sqrt{n}},$$

(148)

for some sequence of random vectors $R_n = O_p(1)$, we have, for any small positive $\gamma$,

$$\hat{\theta} - \hat{\theta}^* = O_p \left( \frac{m^3 s^2 \bar{f}}{n^{1-\gamma}} \right) + O_p \left( \frac{\|R_n\|}{\sqrt{n}} \right).$$

(149)
Proof. By definition
\[ B_n(\hat{\theta}) - B_n(\hat{\theta}^*) = B_n^c(\hat{\theta}) - B_n^c(\hat{\theta}^*). \quad (150) \]

By Lemma B.3, for any \( \gamma \in (0, \frac{1}{2}) \),
\[ B_n(\hat{\theta}) - B_n(\hat{\theta}^*) = O_p \left( sm \frac{\log n}{\sqrt{n}} \right). \quad (151) \]

By equation (148), we get
\[ \hat{\theta} - \hat{\theta}^* = O_p \left( \frac{sm \|\hat{\theta} - \hat{\theta}^*\|^\frac{1}{2} - \gamma}{\sqrt{n}} \right) + \frac{\zeta_n}{\sqrt{n}}. \quad (152) \]

where \( \zeta_n = O_p \left( \frac{sm \log n}{\sqrt{n}} \right) + R_n \). There are two possibilities. If \( \zeta_n \) converges to zero slower than \( O_p \left( sm \|\hat{\theta} - \hat{\theta}^*\|^\frac{1}{2} - \gamma \right) \), then, according to the big-O notation,
\[ \hat{\theta} - \hat{\theta}^* = O_p \left( \frac{\zeta_n}{\sqrt{n}} \right), \quad (153) \]

which implies the result of the lemma since \( \frac{\log n}{n} = o(n^{-1-\gamma}) \). The other possibility is \( \zeta_n = o_p \left( sm \sqrt{\bar{f}} \|\hat{\theta} - \hat{\theta}^*\| \right) \). Then we can ignore \( \zeta_n \), i.e.
\[ \|\hat{\theta} - \hat{\theta}^*\| = \frac{sm \|\hat{\theta} - \hat{\theta}^*\|^\frac{1}{2} - \gamma}{\sqrt{n}} O_p(1), \quad (154) \]

which implies
\[ \|\hat{\theta} - \hat{\theta}^*\| = \left( O_p \left( \frac{m^{\frac{3}{1+\gamma}} s \bar{f}^{\frac{1}{2} - \gamma}}{\sqrt{n}} \right) \right)^{1/2+\gamma} = O_p \left( m^{\frac{3}{1+2\gamma}} s^{\frac{2}{1+2\gamma}} \frac{\bar{f}^{\frac{1}{2} - \gamma}}{n^{\frac{1}{1+2\gamma}}} \right). \quad (155) \]

This again implies equation (153) and completes the proof. \( \square \)

Lemma B.7. Under Assumptions 1–3, for any estimator \( \hat{\theta} \) that minimizes \( \|\hat{g}(\theta)\|_p \), we have, for all small enough \( \gamma > 0 \),
\[ \hat{g}(\hat{\theta}_p) = O_p \left( \frac{m^3 s^2}{n^{1-\gamma}} \right). \quad (156) \]

Proof. The proof proceeds in three steps. Throughout the proof we will treat \( \bar{f} \) as a constant to simplify algebra.
Step 1. By Lemmas B.3 and B.5,

\[ B_n(\hat{\theta}_{\ell_p}) = \mathcal{O}_p \left( \frac{sm \log n}{\sqrt{n}} \right) = \mathcal{O}_p \left( \frac{sm^\frac{3}{2}}{n^{1-\gamma}} \right). \]  

(157)

Since \( \bar{f} \) is a constant, we can ignore the second term.

Step 2. Consider the following estimator

\[ \hat{\theta}_2 \triangleq \hat{\theta}_1 - \frac{G^{-1}B_n(\hat{\theta}_{\ell_p})}{\sqrt{n}} \]  

(158)

By Lemma B.2, we get

\[ \hat{g}(\hat{\theta}_2) = \frac{1}{\sqrt{n}}B_n^o(\theta_0) + (\tau\mathbb{E}Z - \tau\mathbb{E}_nZ) + \frac{1}{\sqrt{n}}B_n(\hat{\theta}_2) \]

\[ + g(\theta_0) + G(\theta_0)(\hat{\theta}_2 - \theta_0) + (\hat{g}_2 - \theta_0) \frac{\partial G(\theta_0)}{\partial \theta}(\hat{\theta}_2 - \theta_0) + \mathcal{O}_p \left( \frac{1}{n^{3/2}} \right), \]  

(159)

(160)

Then, by definition of \( \hat{\theta}_2 \),

\[ \hat{g}(\hat{\theta}_2) = \frac{B_n(\hat{\theta}_2) - B_n(\hat{\theta}_{\ell_p})}{\sqrt{n}} + (\hat{\theta}_2 - \theta_0) \frac{\partial G(\theta_0)}{\partial \theta}(\hat{\theta}_2 - \theta_0) + \mathcal{O}_p \left( \frac{1}{n^{3/2}} \right). \]  

(161)

Also, by definition,

\[ B_n(\hat{\theta}_2) - B_n(\hat{\theta}_{\ell_p}) = B_n^o(\hat{\theta}_2) - B_n^o(\hat{\theta}_{\ell_p}). \]  

(162)

By Lemma B.3,

\[ B_n(\hat{\theta}_2) - B_n(\hat{\theta}_{\ell_p}) = \mathcal{O}_p \left( sm^\frac{3}{2}\frac{\|\hat{\theta}_2 - \hat{\theta}_{\ell_p}\|^{1-\gamma}}{\sqrt{n}} + \mathcal{O}_p \left( \frac{sm \log n}{\sqrt{n}} \right) \right). \]  

(163)

Then (161) becomes

\[ \hat{g}(\hat{\theta}_2) = \mathcal{O}_p \left( \frac{sm^\frac{3}{2}\|\hat{\theta}_2 - \hat{\theta}_{\ell_p}\|^{1-\gamma}}{\sqrt{n}} + \mathcal{O}_p \left( \frac{sm \log n}{\sqrt{n}} \right) \right). \]  

(164)

By Lemma B.2 applied to \( \hat{\theta}_{\ell_p} \) and the definition of \( \hat{\theta}_1 \),

\[ \hat{\theta}_{\ell_p} = \hat{\theta}_1 + G^{-1}\hat{g}(\hat{\theta}_{\ell_p}) - \frac{G^{-1}B_n(\hat{\theta}_{\ell_p})}{\sqrt{n}} - G^{-1}(\hat{\theta}_{\ell_p} - \theta_0) \frac{\partial G(\theta_0)}{\partial \theta}(\hat{\theta}_{\ell_p} - \theta_0) + \mathcal{O}_p \left( \frac{1}{n^{3/2}} \right). \]  

(165)
So by (160) and the definition of \( \hat{\theta}_2 \), we get

\[
\hat{\theta}_{\ell p} - \hat{\theta}_2 = G^{-1} \hat{g}(\hat{\theta}_{\ell p}) + O_p(n^{-1}). \tag{166}
\]

So (164) becomes

\[
\hat{g}(\hat{\theta}_2) = O_p \left( \frac{sm^{\frac{3}{2}}}{\sqrt{n}} \left\| \hat{g}(\hat{\theta}_{\ell p}) \right\|^{\frac{3}{2} - \gamma} \right) + O_p \left( \frac{sm \log n}{n} \right). \tag{167}
\]

**Step 3.** From (167), we obtain

\[
\| \hat{g}(\hat{\theta}_{\ell p}) \|_p \leq \| \hat{g}(\hat{\theta}_2) \|_p = O_p \left( \frac{sm^{\frac{3}{2}}}{\sqrt{n}} \left\| \hat{g}(\hat{\theta}_{\ell p}) \right\|^{\frac{3}{2} - \gamma} \right) + O_p \left( \frac{sm \log n}{n} \right). \tag{168}
\]

On the right hand side of this inequality, suppose that the first term dominates the second term. Then \( \| \hat{g}(\hat{\theta}_{\ell p}) \| = O_p \left( \frac{sm \log n}{n} \right) \), which has order no larger than \( O \left( \frac{m^3 s^2}{n^{1 - \gamma}} \right) \), from which the statement of the lemma follows.

Otherwise, if the second term dominates the first term, we have

\[
\| \hat{g}(\hat{\theta}_{\ell p}) \|^{\frac{3}{2} + \gamma} \leq O_p \left( \frac{sm^{\frac{3}{2}}}{\sqrt{n}} \right)
\]

or, after exponentiation,

\[
\| \hat{g}(\hat{\theta}_{\ell p}) \| \leq O_p \left( \frac{sm^{\frac{3}{2}} m^{\frac{3}{2}}}{n^{1 - \frac{1}{2} \gamma}} \right) \leq O_p \left( \frac{s^2 m^3}{n^{1 - \gamma}} \right),
\]

which implies the statement of the lemma.

\[\square\]

**C Higher order bias, proofs**

**Lemma C.1.** Consider a corner solution \( \hat{\theta} = \hat{\theta}_{\ell p} \). Under Assumptions 1.1, 2, and 3 we have

\[
\mathbb{E} \frac{1}{\sqrt{n}} B_n(\hat{\theta}) = \frac{1}{n} \kappa(\tau) + o \left( \frac{1}{n} \right)
\]

where

\[
\kappa(\tau) \triangleq \mathbb{E} \left( \tau - \frac{1}{2} \right) f_{\varepsilon_1}(0|W_1, Z_1) \mathbb{E} \left( \frac{\hat{g}(\hat{\theta}) + \hat{g}^*(-\hat{\theta})}{2} \right).
\]

[47]
Proof. The proof proceeds in five steps.

Step 1. Note that
\[
\frac{1}{\sqrt{n}} \mathbb{E} B_n(\hat{\theta}) = \frac{1}{\sqrt{n}} \mathbb{E} \left( B_n'(\hat{\theta}) - B_n'(\theta_0) \right)
\]
\[= \mathbb{E} \left( 1\{Y \leq W'\hat{\theta}\} Z \right) - \mathbb{E} g(\hat{\theta}). \tag{173} \]

Step 2. Define \( \tilde{\epsilon}_i \triangleq Y_i - W'_i\hat{\theta} \) and split the first term in the equation above as follows:
\[
\mathbb{E} 1\{Y_i \leq W'_i\hat{\theta}\} Z_i = \mathbb{E} 1\{\tilde{\epsilon}_i = 0\} Z_i + \mathbb{E} 1\{\tilde{\epsilon}_i < 0\} Z_i. \tag{175} \]

Lemma B.5 implies
\[
\hat{\theta} = \theta_0 - \frac{1}{n} G^{-1} \sum_{i=1}^{n} (1\{Y_i \leq W'_i\theta_0\} - \tau) Z_i + R_n, \tag{176} \]
where \( R_n = o_p(n^{-1/2}) \). We can use this structure to isolate an influence of observation \( i \),
\[
\frac{1}{n} \lambda_i \triangleq -\frac{1}{n} W'_i G^{-1} Z_i (1\{Y_i \leq W'_i\theta_0\} - \tau). \]
The indicator \( 1\{\tilde{\epsilon}_i < 0\} \) can be rewritten as
\[
1 \left\{ Y_i < W'_i\hat{\theta}_{-i} + \frac{1}{n} \lambda_i \right\}, \tag{177} \]
where
\[
\hat{\theta}_{-i} \triangleq \theta_0 - \frac{1}{n} G^{-1} \sum_{j=1, j \neq i}^{n} (1\{Y_j \leq W'_j\theta_0\} - \tau) Z_i \tag{178} \]
is equal to \( \hat{\theta} \) without the linear influence of the observation \( i \). Then, using Taylor’s theorem, the term
\[
\mathbb{E} \mathbb{Z}_i P(Y_i < W'_i\hat{\theta}_{-i} + \frac{1}{n} \lambda_i |1\{Y_i \leq W'_i\theta_0\}, Z_i, W_i) \]
can be represented as
\[
\mathbb{E} \mathbb{Z}_i P(Y_i < W'_i\hat{\theta}_{-i} | Z_i, W_i) + \mathbb{E} \frac{1}{n} Z_i \lambda_i f_Y(W'_i\hat{\theta}_{-i} | \hat{\theta}_{-i}, \lambda_i, Z_i, W_i) + O \left( \frac{1}{n^2} \right). \tag{179} \]

By Assumption 2, \( f_Y(y|W_i, Z_i) \) is uniformly bounded and
\[
\mathbb{E} \phi(W_i, Z_i) f_Y(y|\hat{\theta}_{-i}, \lambda_i, W_i, Z_i) = \mathbb{E} \phi(W_i, Z_i) f_Y(y|W_i, Z_i) \leq \mathbb{E} \phi(W_i, Z_i) \bar{f} \tag{180} \]
for any non-negative measurable function \( \phi(W_i, Z_i) \). The same is true for the derivative of the density. So \( P(f_Y(y|\hat{\theta}_{-i}, \lambda_i, W_i, Z_i) = \infty) = 0 \) and \( P(\partial f_Y(y|\hat{\theta}_{-i}, \lambda_i, W_i, Z_i) = \infty) = 0 \), which justifies the Taylor expansion above. By a.s. smoothness of \( f_Y(y|\hat{\theta}_{-i}, \lambda_i, Z_i, W_i) \) and
equation (176),

\[ \mathbb{E} Z_i f_Y(W_i' \hat{\theta}_{-i} | \hat{\theta}_{-i}, W_i, \lambda_i, Z_i) = \mathbb{E} Z_i f_Y(W_i' \theta_0 | W_i, Z_i, 1\{Y_i \leq W_i' \theta_0\}) + O \left( \frac{1}{\sqrt{n}} \right). \quad (181) \]

Notice that the side of the density in the Taylor expansion depends on the direction of the deviation \( \lambda_i^* \triangleq -\frac{1}{n} W_i' G^{-1} Z_i 1\{Y_i \leq W_i' \theta_0\}, \)

\[ \mathbb{E} Z_i \lambda_i^* f_Y(W_i' \theta_0 | \lambda_i^*, Z_i, W_i) = \]

\[ = \mathbb{E} \left( -Z_i W_i' G^{-1} Z_i 1\{Y_i \leq W_i' \theta_0\} \lim_{t \to 0} \frac{1\{W_i' \theta_0 - t1\{\lambda_i^* \leq 0\} < Y_i \leq W_i' \theta_0 + t1\{\lambda_i^* > 0\}\}}{t} \right) \quad (182) \]

\[ = -\mathbb{E} Z_i W_i' G^{-1} Z_i 1\{-W_i' G^{-1} Z_i < 0\} \lim_{t \to 0} 1\{\varepsilon_i \leq 0\} \frac{1\{-t < \varepsilon_i \leq 0\}}{t}, \quad (183) \]

So (175) becomes

\[ \mathbb{E} Z_i P(Y_i < W_i' \hat{\theta}_{-i} | Z_i, W_i) + \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} Z_i (\tau - 1\{W_i' G^{-1} Z_i > 0\}) f_\varepsilon(0 | W_i, Z_i) \]

\[ + \mathbb{E} 1\{\hat{\varepsilon}_i = 0\} Z_i + O \left( \frac{1}{n^{3/2}} \right). \quad (184) \]

**Step 3.** Now consider the second term \( \mathbb{E} g^\circ(\hat{\theta}) \). Let \( (Y_{n+1}, W_{n+1}, Z_{n+1}) \) be a copy of \( (Y, W, Z) \), which is independent of the sample \( \{Y_i, W_i, Z_i\}_{i=1}^n \). Also define \( \lambda_{n+1,i} = -\frac{1}{n} W_{n+1}' G^{-1} Z_i (1\{Y_i \leq W_i' \theta_0\} - \tau) \), which satisfies \( \mathbb{E} \lambda_{n+1,i} = 0 \). Then

\[ \mathbb{E} g^\circ(\hat{\theta}) = \mathbb{E} 1\{Y_{n+1} \leq W_{n+1}' \hat{\theta}\} Z_{n+1} \]

\[ = \mathbb{E} P\{Y_{n+1} \leq W_{n+1}' \hat{\theta}_{-i} - \frac{1}{n} \lambda_{n+1,i} | W_{n+1}, Z_{n+1}\} Z_{n+1} \]

\[ = \mathbb{E} P\{Y_{n+1} < W_{n+1}' \hat{\theta}_{-i} | W_{n+1}, Z_{n+1}\} Z_{n+1} + O \left( \frac{1}{n^{3/2}} \right). \quad (185) \]

Combining this equality with (188) yields

\[ \mathbb{E} \left( 1\{Y \leq W' \hat{\theta}\} Z - \mathbb{E} g(\hat{\theta}) \right) \]

\[ = \mathbb{E} Z_i P\{Y_i < W_i' \hat{\theta}_{-i} | W_i, Z_i\} - \mathbb{E} Z_{n+1} P\{Y_{n+1} < W_{n+1}' \hat{\theta}_{-i} | W_{n+1}, Z_{n+1}\} \]

\[ + \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} Z_i (\tau - 1\{W_i' G^{-1} Z_i > 0\}) f_\varepsilon(0 | W_i, Z_i) + \mathbb{E} 1\{\hat{\varepsilon}_i = 0\} Z_i + O \left( \frac{1}{n^{3/2}} \right). \quad (186) \]
\textbf{Step 4.} By expansion (176)

\[ \hat{\theta}_i = \theta_0 - \frac{1}{n} G^{-1} \sum_{j=1, j \neq i}^n (1\{\varepsilon_j \leq 0\} - \tau) Z_j + R_n. \]  

(195)

Define

\[ \zeta_{i,n} \triangleq \frac{n - 1}{n} \frac{1}{n - 1} G^{-1} \sum_{j=1, j \neq i}^n (1\{\varepsilon_j \leq 0\} - \tau) Z_j, \]  

(196)

so that \( \zeta_{i,n} \) is a zero mean r.v. that is independent of \( Y_i \). Therefore,

\[ \mathbb{E}Z_i P\{Y_i < W_i' \hat{\theta}_i|W_i, Z_i\} = \mathbb{E}Z_i P(Y_i - W_i' \zeta_{i,n} < W_i' W_i' R_n|W_i, Z_i) \]  

(197)

\[ = \mathbb{E}Z_i P(\xi_i < W_i' R_n|W_i, Z_i), \]  

(198)

where \( \xi_i \triangleq Y_i - W_i' \zeta_{i,n} \) is a r.v. with PDF conditional on \( (W_i, Z_i) \) by Assumption 2.1 and \( \zeta_{i,n} \) is independent of \( Y_i \).

Apply the Taylor theorem to obtain

\[ \mathbb{E}Z_i P\{Y_i < W_i' \hat{\theta}_i|W_i, Z_i, W_i' R_n\} \]  

(199)

\[ = \mathbb{E}Z_i P(\xi_i < W_i' \theta_0|W_i, Z_i) + \mathbb{E}W_i' R_n f_{\xi_i}(W_i' \theta_0|W_i, Z_i, R_n) \]  

(200)

\[ + \frac{1}{2} \mathbb{E}R_n W_i \partial f_{\xi_i}(W_i' \theta_0|W_i, Z_i, R_n) W_i' R_n + \mathbb{E}O_p \left( \|R_n\|^3 \right). \]  

(201)

By the Bahadur expansion (176), \( R_n = O_p\left(\frac{1}{\sqrt{n}}\right) \). Hence, (201) becomes

\[ \mathbb{E}Z_i P(Y_i - W_i' \zeta_{i,n} < W_i' \theta_0|W_i, Z_i) + \mathbb{E}W_i' R_n f_{\xi_i}(0|W_i, Z_i, R_n) + o\left(\frac{1}{n}\right). \]  

(202)

Similarly,

\[ \mathbb{E}Z_{n+1} P\{Y_{n+1} < W_{n+1}' \hat{\theta}_{i,n}|W_{n+1}, Z_{n+1}\} \]  

(203)

\[ = \mathbb{E}Z_{n+1} P(Y_{n+1} - W_{n+1}' \zeta_{i,n} < W_{n+1}' \theta_0|W_{n+1}, Z_{n+1}) \]  

(204)

\[ + \mathbb{E}W_{n+1}' R_n f_{\xi_{n+1}}(0|W_{n+1}, Z_{n+1}, R_n) + o\left(\frac{1}{n}\right) \]  

(205)

\[ = \mathbb{E}Z_i P(Y_i - W_i' \zeta_{i,n} < W_i' \theta_0|W_i, Z_i) + \mathbb{E}Z_{i,n} W_i' (\mathbb{E}R_n) f_{\xi_i}(0|W_i, Z_i) + o\left(\frac{1}{n}\right). \]  

(206)
Note that
\[
\lim_{t \to 0} \mathbb{E} Z_i W_i'(R_n - \mathbb{E} R_n)(f_{\varepsilon_i}(0|W_i, Z_i, R_n) - f_{\varepsilon_i}(0|W_i, Z_i)) = 0.
\] (207)

\[
\begin{aligned}
&= \mathbb{E} Z_i W_i'(R_n - \mathbb{E} R_n) \lim_{t \to 0} \frac{1}{t} (\mathbb{E}(1\{0 < \varepsilon_i \leq t\}|W_i, Z_i, R_n) - \mathbb{E}(1\{0 < \varepsilon_i \leq t\}|W_i, Z_i)) \\
&= \lim_{t \to 0} \mathbb{E} Z_i W_i'(R_n - \mathbb{E} R_n) \frac{1}{t} (1\{0 < \varepsilon_i \leq t\} - 1\{0 < \varepsilon_i \leq t\}) = 0.
\end{aligned}
\] (208)

To summarize, (194) becomes
\[
\begin{aligned}
\mathbb{E} \left(1\{Y \leq W' \hat{\theta}\} Z\right) &= \mathbb{E} g^\circ(\hat{\theta}) \\
&= \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} (\tau - 1\{W_i' G^{-1} Z_i > 0\}) f_{\varepsilon_i}(0|W_i, Z_i) + \mathbb{E} 1\{\varepsilon_i = 0\} Z_i + o \left(\frac{1}{n}\right).
\end{aligned}
\] (210)

**Step 5.** Formula (211) can be rewritten as
\[
\begin{aligned}
\mathbb{E} \left(1\{Y \leq W' \hat{\theta}\} Z\right) &= \mathbb{E} g^\circ(\hat{\theta}) \\
&= \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} (\tau - 1\{W_i' G^{-1} Z_i > 0\}) f_{\varepsilon_i}(0|W_i, Z_i) + \mathbb{E} 1\{\varepsilon_i = 0\} Z_i + o \left(\frac{1}{n}\right) \\
&= \mathbb{E} 1\{\varepsilon_i = 0\} Z_i - \mathbb{E} \left(1\{Y_i \geq W' \hat{\theta}\} Z_i\right) + \mathbb{E} \left(1\{Y_n+1 \geq W'_{n+1} \hat{\theta}\} Z_{n+1}\right) \\
&= \mathbb{E} 1\{\varepsilon_i = 0\} Z_i - \mathbb{E} \left(1\{-Y_i \leq W'(\varepsilon)\}\right) + \mathbb{E} \left(1\{-Y_n+1 \leq W'_{n+1}(-\hat{\theta})\}\right) Z_{n+1} \\
&= \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} (\tau - 1\{W_i' G^{-1} Z_i > 0\}) f_{\varepsilon_i}(0|W_i, Z_i) + \mathbb{E} 1\{\varepsilon_i = 0\} Z_i + o \left(\frac{1}{n}\right).
\end{aligned}
\] (213)

This implies
\[
\begin{aligned}
\frac{1}{2} \mathbb{E} 1\{\varepsilon_i = 0\} Z_i &= \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} (\tau - 1\{W_i' G^{-1} Z_i > 0\}) - 1/2 f_{\varepsilon_i}(0|W_i, Z_i) + o \left(\frac{1}{n}\right)
\end{aligned}
\] (217)

and
\[
\mathbb{E} 1\{\varepsilon_i = 0\} Z_i = \mathbb{E} \left(\frac{\hat{g}(\hat{\theta}) + \hat{g}^\ast(-\hat{\theta})}{2}\right).
\] (218)

Hence, equation (211) can be rewritten as
\[
\begin{aligned}
\mathbb{E} \left(1\{Y \leq W' \hat{\theta}\} Z\right) &= \mathbb{E} g^\circ(\hat{\theta}) \\
&= \frac{1}{n} \mathbb{E} Z_i W_i' G^{-1} (\tau - 1/2) f_{\varepsilon_i}(0|W_i, Z_i) + \mathbb{E} \left(\frac{\hat{g}(\hat{\theta}) + \hat{g}^\ast(-\hat{\theta})}{2}\right) + o \left(\frac{1}{n}\right).
\end{aligned}
\] (220)
D MILP implementation

Consider the following exact estimator,

$$\hat{\theta}_{\ell_1} = \arg\min_{\theta \in \Theta} \| \hat{g}(\theta) \|_1.$$  \hfill (221)

The underlying optimization problem can be equivalently reformulated as a mixed integer linear program (MILP) with special ordered set (SOS) constraints,

$$\begin{align*}
\min_{e, \theta, r, s, t} & \quad \ell' t \\
\text{s.t.} & \\
\varepsilon_i &= r_i - s_i = Y_i - W_i' \theta, \quad i = 1, \ldots, n, \\
(r_i, e_i) &\in SOS_1, \quad i = 1, \ldots, n, \\
(s_i, 1 - e_i) &\in SOS_1, \quad i = 1, \ldots, n, \\
r_i &\geq 0, \quad s_i \geq 0, \quad i = 1, \ldots, n, \\
e_i &\in \{0, 1\}, \\
- t_l &\leq Z_l'(e - \tau \ell) \leq t_l, \quad l = 1, \ldots, d.
\end{align*}$$

where $Z_l$ is an $n \times 1$ vector of realizations of instrument $l$. All constraints except the last one coincide with the ones derived by Chen and Lee (2017) in Appendix C.1. The last constraint ensures that the objective function is the $\ell_1$ norm of the just identifying moment conditions.

We also considered the big-M formulation while performing the Monte Carlo analyses. The big-M formulation has certain computational advantages, although the arbitrary choice of tuning parameters may result in sub-optimal solutions. This problem is more prominent for tail quantiles. Consistent with our theory, the choice of tuning parameters in the big-M formulation may affect the asymptotic bias. We prefer the above SOS formulation because it does not depend on tuning parameters as the big-M MILP/MIQP formulations considered in Chen and Lee (2017); Zhu (2019).\textsuperscript{25}

\textsuperscript{25}These papers pick the value of the tuning parameter $M$ as a solution to a linear program that in turn depends on the choice of an arbitrary box around a linear IV estimate. This is problematic if there is a lot of heterogeneity in the coefficients across quantiles. Moreover, in the linear model with heavy tailed residuals, the linear IV estimator is not consistent.