Computer algebra in gravity:
Programs for (non-)Riemannian spacetimes. I

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Abstract

Computer algebra programs are presented for application in general relativity, in electrodynamics, and in gauge theories of gravity. The mathematical formalism used is the calculus of exterior differential forms, the computer algebra system applied Hearn’s Reduce with Schrüfer’s exterior form package Excalc. As a non-trivial example we discuss a metric of Plebański & Demiański (of Petrov type D) together with an electromagnetic potential and a triplet of post-Riemannian one-forms. This whole geometrical construct represents an exact solution of a metric-affine gauge theory of gravity. We describe a sample session and verify by computer that this exact solution fulfills the appropriate field equations.– Computer programs are described for the irreducible decomposition of (non-Riemannian) curvature, torsion, and nonmetricity. file cpc9.tex, 1998-03-18

1 Introduction

General relativity (GR), that is, Einstein’s gravitational theory, is one of those scientific subjects in which computer algebra methods were used as soon as they became available. In fact, in some instances, general relativity was taken as a guinea pig since this theory is notorious for its lengthy and messy calculations. Just remember that the Riemann curvature 2-form $R_{\alpha\beta} := g_{\beta\gamma} R_{\alpha}^{\gamma} = -R_{\beta\alpha}$, here $g_{\beta\gamma}$ are the components of the metric, characterizes the local geometry of four-dimensional spacetime. It has, in these dimensions, 20 independent components. And Einstein’s field equation has 10 independent components. These numbers alone make it plausible that it is desirable to give the corresponding calculations to a computer, see Brans
[3]. Since they are analytic, numerics doesn’t help, rather symbol manipulation programs or, as they are called today, computer algebra systems are necessary.

This is even more important for generalizations of Einstein’s theory, like the Einstein–Cartan theory, the Poincaré gauge theory, or the metric-affine theory of gravity. In these gauge theories of gravitation, see Gronwald & Hehl [11] for a review, spacetime carries additional post-Riemannian structures, namely torsion $T^\alpha$ (a vector valued 2-form) with $6 \times 4$ independent components, and/or nonmetricity $Q_{\alpha\beta} = Q_{\beta\alpha}$ (a symmetric tensor-valued 1-form) with $4 \times 10$ components. Moreover, in general, the curvature loses its antisymmetry. Thus, a metric-affine space, in four dimensions, is described by the $36 + 24 + 40$ ‘field strengths’ $R_{\alpha\beta}$, $T^\alpha$, and $Q_{\alpha\beta}$, respectively. For some physicist this is too much and considered to be a plethora of arbitrariness. This is like the argument against Einstein’s theory, which was taken quite seriously in some quarters, that the 10 gravitational potentials of Einstein’s theory are too much as compared to only 1 potential in Newton’s gravitational theory. But consistency and beauty cannot be measured in numbers of components! In [16], e.g., one can find the arguments which make us believe that the microstructure of spacetime requires post-Riemannian degrees of freedom as represented by torsion and nonmetricity.

Let us come back to general relativity. A readable survey of the computer algebra systems in use for general relativity has been given by Hartley [12]. In the same book [17], see also [14], there is a whole chapter on computer algebra methods as used in relativity. Suppose you would like to use such methods in general relativity or gauge theories of gravity. Considering the ample computer resources one has available today, it seems reasonable to us to turn to a general purpose system like Macsyma, Maple, Mathematica, and/or Reduce and, on top of one of these, to use specialized packages for general relativity/differential geometry, see [12]. For getting familiar with the corresponding general relativity packages, we can recommend the respective articles or books of McLenaghan [23] for Maple, of Parker & Christensen [26] and Soleng [34] for Mathematica, and of McCrea [22] for Reduce. Some specific relativistic applications have been discussed, e.g., in Maple, see [28], in Mathematica, see [19,39], and in Reduce, see [43].

The general relativity computer algebra packages can, as a rule, also be used for non-Riemannian spacetimes and the corresponding gravitational gauge theories. Usually the inclusion of torsion and nonmetricity is not difficult to program. However, there is not too much literature available on that subject. For torsion theories there exist, amongst others, the papers of Aman et al. [1,2], McCrea [22], Schrüfer et al. [33], Tertychniy and Obukhova [36], and Zhytnikov [44]. The packages $GRG$ [44] and $GRG_{EC}$ [36] (EC stands for Einstein-Cartan theory) are Reduce based systems.
There is even less literature available for metric-affine spaces and the corresponding gauge theories. Such a spacetime carries, besides the coframe (of 1-forms) \( \vartheta^\alpha = e_i^\alpha \, dx^i \), the metric \( g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = g_{ij} \, dx^i \otimes dx^j \) and the linear connection 1-form \( \Gamma^\alpha_{\beta\gamma} = \Gamma^i_{\alpha\beta} \, dx^i \) as totally independent gravitational ‘potentials’: \((g_{\alpha\beta}, \vartheta^\alpha, \Gamma^\alpha_{\beta\gamma})\). For the metric-affine gauge theory of gravity (MAG), which encompasses the Poincaré gauge theory, the Einstein-Cartan theory, and general relativity as specific subcases, we developed a couple of computer algebra programs which were and are used in our respective groups for the search of exact solutions, e.g. We would like to explain and demonstrate them. They were developed over the last eight years or so and different people were involved: Dermott McCrea†, Werner Esser, Frank Gronwald, Ralf Hecht, Yuri Obukhov, Roland Puntigam, Sergei Tertychniy, Romualdo Tresguerres, Eugen Vlachynsky, and, more recently, José Socorro.

We will proceed as follows: The general metric-affine framework, including all the conventions, will be taken from [16], see also [11]. Our computer algebra programs use Cartan’s calculus of exterior differential forms as implemented on the computer by means of Hearn’s computer algebra system Reduce [13] and Schrüfer’s corresponding Excalc package [32]. We will always have in mind the special case of general relativity, i.e., we can specialize to general relativity if desirable. This is important if one wants to extract the post-Riemannian structures from the computed expressions, i.e., the deviations from the corresponding Riemannian spacetime. We are trying to program our MAG procedures as some kind of master programs which can be immediately and efficiently specialized to Riemann-Cartan or Riemann spacetimes.

In this sense, we turn, in Sec.2, to the curvature 2-form of a metric-affine space with its \( 16 \times 16 \) independent components. We will irreducibly decompose it under the Lorentz group and discover thereby typical post-Riemannian pieces, besides the pieces which emerge already in a Riemannian space. This automatized irreducible decomposition of the curvature may also be useful for people who are not concerned with gravitational gauge theories but just with non-Riemannian structures in differential geometry, i.e., we believe that Sec.2 should be of more general interest than the rest of the paper.

In Sec.3, as an example for testing our methods, we turn to an exact (electrovac) solution of the Einstein-Maxwell equations of Petrov type D as formulated by Plebański and Demiański (P&D) [27], see also García Díaz [9] and Debever, Kamran, and McLenaghan [4,5]. After a short review of the Einstein-Maxwell equations in exterior calculus, we develop the computer code for the orthonormal coframe of the P&D solution and compute its curvature.

Subsequently, we will determine the electromagnetic energy-momentum distribution of the P&D solution. For this purpose we bring in our Maxwell program code which is suitable for electromagnetic fields on top of an arbitrary differ-
entiable manifold, whatever metric and whatever connection, see [30,29]. This
so-called metric-free formulation of Maxwell’s equations (the metric enters
only via the Hodge star into the constitutive assumption) is extremely concise
and well adapted to executing such computations within geometrical theories
of gravitation.

We combine the decomposition method of Sec.2 with this Maxwell program.
We can now easily verify the Einstein-Maxwell equations and find, in addition,
a very compact form for Weyl’s conformal curvature 2-form.

In Sec.4, we turn our attention again to non-Riemannian geometries. We de-
scribe programs for the irreducible decomposition of torsion and nonmetricity.

In Sec.5, the P&D metric of GR is extended to an exact solution in the frame-
work of metric-affine gravity. For that purpose a non-Riemannian covector
triplet has to be introduced, see [37,38,41,25,42,6,29,24,40,15]. Using our de-
composition programs as described in Secs.2 and 4, we are able to test this
whole geoemtro-physical construct provided we also use our Maxwell package.
It should be pointed out that this exact solution of metric-affine gravity was
only found by means of an extensive use of computer algebra. In Sec.6 we will
conclude with a discussion.

2 Curvature in non-Riemannian and Riemannian spaces, its irre-
duce decompostion by machine

Before we turn to the programs, let us report on the irreducible decomposition
of the curvature 2-form

\[ R_\alpha^\beta = d\Gamma_\alpha^\beta - \Gamma_\alpha^\gamma \wedge \Gamma_\gamma^\beta = \frac{1}{2} R_{\mu\alpha}^\beta \partial^\mu \wedge \partial^\nu = \frac{1}{2} R_{ij}^\alpha^\beta dx^i \wedge dx^j, \]  

(1)

see [16] for a discussion and the relevant literature. Besides the linear con-
nection \( \Gamma_\alpha^\beta \), we have a metric \( g \) available with signature \((-+++\)). We may
lower the second index on the curvature two-form. We consider the irreducible
decomposition of \( R_\alpha^\beta \) in 4 dimensions under the Lorentz group. The first step
is to separate it into its antisymmetric and symmetric parts,

\[ R_\alpha^\beta = W_\alpha^\beta + Z_\alpha^\beta \sim \text{curv2}(-a,-b) := w2(-a,-b) + z2(-a,-b), \]  

(2)

with

\[ W_\alpha^\beta := R_{[\alpha\beta]} \sim w2(-a,-b) := (\text{curv2}(-a,-b) - \text{curv2}(-b,-a))/2 \]  

(3)
and
\[ Z_{\alpha\beta} := R_{[\alpha\beta]} \sim z2(-a, -b) := (\text{curv}2(-a, -b) + \text{curv}2(-b, -a))/2. \quad (4) \]

We displayed at the same time the computer names (identifiers) of the quantities emerging and the formulae in Reduce-Excalc syntax. A tutorial on the use of Excalc is contained in [35], e.g. We recall that lower indices are denoted by a minus sign. For convenience, in Excalc, we use Latin indices for the frames instead of Greek ones as in the formulae. The rank \( p \) of a form (if \( p > 0 \)) is usually indicated by a corresponding Arabic number in the computer name of the quantity, that is, the curvature is a 2-form, thus we call it \( \text{curv}2 \). The Einstein form is a 3-form or \( \text{einstein}3(a) \), etc.

### 2.1 Rotational curvature

The antisymmetric piece \( W_{\alpha\beta} \), the ‘rotational curvature’, can be decomposed, similarly as in a Riemann-Cartan spacetime, into 6 pieces,

\[ W_{\alpha\beta} = (1)W_{\alpha\beta} + (2)W_{\alpha\beta} + (3)W_{\alpha\beta} + (4)W_{\alpha\beta} + (5)W_{\alpha\beta} + (6)W_{\alpha\beta} \sim \text{weyl} + \text{paircom} + \text{pscalar} + \text{ricsymf} + \text{ricanti} + \text{scalar}. \quad (5) \]

Here \( \text{weyl} \) corresponds to \( (1)W_{\alpha\beta} \) etc.

We can ‘contract’ the 2-form \( W_{\alpha\beta} \) in different ways by means of the frame \( e_\alpha \sim e(-a) \), the coframe \( \vartheta^\alpha \sim o(a) \), and the Hodge star \( * \sim # \). The corresponding truncated quantities read:

\[ W^\alpha := e_\beta [W^{\alpha\beta}] \sim \quad \text{wone}1(a) := e(-b) \! \wedge w2(a, b), \quad (6) \]
\[ W := e_\alpha [W^\alpha] \sim \quad \text{wzero} := e(-a) \! \wedge \text{wone}1(a), \quad (7) \]
\[ X^\alpha := * (W^{\beta\alpha} \! \wedge \vartheta_\beta) \sim \quad \text{xone}1(a) := # (w2(b, a) \! \wedge o(-b)), \quad (8) \]
\[ X := e_\alpha [X^\alpha] \sim \quad \text{xzero} := e(-a) \! \wedge \text{xone}1(a). \quad (9) \]

They can further be decomposed into

\[ \Psi_\alpha := X_\alpha - \frac{1}{4} \vartheta_\alpha \wedge X - \frac{1}{2} e_\alpha [(\vartheta^\beta \wedge X_\beta)], \quad (10) \]
\[ \Phi_\alpha := W_\alpha - \frac{1}{4} W \vartheta_\alpha - \frac{1}{2} e_\alpha [(\vartheta^\beta \wedge W_\beta)], \quad (11) \]

or

\[ \text{psi}1(-a) := \text{xone}1(-a) - (1/4) o(-a) \! \wedge \text{xzero} \]
\[ -(1/2) e^{-(a)} \lvert (o(b)^xone1(-b)) \]
\[ \phi1(-a) := wone1(-a) - (1/4) wzero \cdot o(-a) 
- (1/2) e^{-(a)} \lvert (o(b)^wone1(-b)) \]

and then 'blown up' to the 2-forms

\[ \text{paircom2}(-a, -b) \sim -{}^* (\vartheta[^{\alpha}] \wedge \Phi[^{\beta}]), \quad (12) \]
\[ \text{pscalar2}(-a, -b) \sim -\frac{1}{12} {}^* (X \wedge \vartheta[^{\alpha}] \wedge \vartheta[^{\beta}]), \quad (13) \]
\[ \text{ricsymf2}(-a, -b) \sim -\vartheta[^{\alpha}] \wedge \Phi[^{\beta}], \quad (14) \]
\[ \text{ricanti2}(-a, -b) \sim -\frac{1}{2} \vartheta[^{\alpha}] \wedge e[^{\beta}] [(\vartheta[^{\gamma}] \wedge W[^{\gamma}])]. \quad (15) \]
\[ \text{scalar2}(-a, -b) \sim -\frac{1}{12} W[^{\alpha}] \wedge \vartheta[^{\beta}], \quad (16) \]
\[ \text{weyl2}(-a, -b) \sim W[^{\alpha\beta}] - \sum_{I=2}^{6} (I) W[^{\alpha\beta}]. \quad (17) \]

To be concrete, let us display (12), e.g., as a Reduce-Excalc command:

\[ \text{paircom2}(-a, -b) := -(1/2) \# (o(-a)^\psi1(-b) - o(-b)^\psi1(-a)); \]

The hash sign \# represents the Hodge star operator.

Analogously, we can also translate the remaining definitions of the irreducible components into Reduce-Excalc. We remember that the rank of a form in Reduce-Excalc has to be declared explicitly by means of the \texttt{pform} declaration. In this way we arrive at the decomposition file:

\%
% Irred. decomposition of the rotational curvature w2(-a,-b) *
%***************************************************************
%***************************************************************
% file rotcurvdecomp.exi, 1998-03-15
% input: coframe o(a), frame e(a), metric g(a,b), curv2(a,b)

\texttt{pform \{wzero,xzero\}=0,}
\texttt{\{wone1(a),xone1(a),psi1(a),phi1(a)\}=1,}
\texttt{\{w2(a,b),paircom2(a,b),pscalar2(a,b),ricsymf2(a,b),}
\texttt{ricanti2(a,b),scalar2(a,b), weyl2(a,b)\}=2}$
\texttt{w2(-a,-b) := (1/2)*(curv2(-a,-b)-curv2(-b,-a))}$
\texttt{wone1(a) := e(-b)_\lvert w2(a,b)$}
\texttt{wzero := e(-b)_\lvert wone1(b)$}
\texttt{xone1(a) := #(w2(b,a)^o(-b))}$
\texttt{xzero := e(-a)_\lvert xone1(a)$}
\texttt{psi1(-a) := xone1(-a)-(1/4)o(-a)^xzero}$
\(-1/2)\cdot e(-a) \cdot (o(b)\cdot xone1(-b))$

\(\phi1(-a) := \text{wone1}(-a) - (1/4)\cdot o(-a)\cdot wzero - (1/2)\cdot e(-a) \cdot (o(b)\cdot xone1(-b))$

\(\text{paircom2}(-a,-b) := -(1/2)\cdot \#(o(-a)\cdot \text{psi1}(-b) - o(-b)\cdot \text{psi1}(-a));$

\(\text{pscalar2}(-a,-b) := -(1/12)\cdot \#(\text{xzero}\cdot o(-a)\cdot o(-b));$

\(\text{ricsymf2}(-a,-b) := -(1/2)\cdot (o(-a)\cdot \phi1(-b) - o(-b)\cdot \phi1(-a));$

\(\text{ricanti2}(-a,-b) := -(1/4)\cdot (o(-a)\cdot (e(-b) \cdot (o(c)\cdot xone1(-c))) - o(-b)\cdot (e(-a) \cdot (o(c)\cdot xone1(-c))));$

\(\text{scalar2}(-a,-b) := -(1/12)\cdot wzero\cdot o(-a)\cdot o(-b);$

\(\text{wyl2}(-a,-b) := \text{w2}(-a,-b)\cdot \text{paircom2}(-a,-b)\cdot \text{pscalar2}(-a,-b)\cdot \text{ricsymf2}(-a,-b)\cdot \text{ricanti2}(-a,-b)\cdot \text{scalar2}(-a,-b);$

\text{clear wzero, xzero, wone1(a), xone1(a), psi1(a), phi1(a)$}

$end$

%**********************************************************************

Before this program segment can be used, we have to specify a coframe \(o(a)\),
the frame \(e(-a)\) dual to it, a metric \(g\), and a connection \(\gamma(-a,b)\), and
to compute the curvature \(\text{curv2}(-a,b)\) beforehand. This will be done below.

Already in a Riemann-Cartan space (with metric compatible connection), i.e.,
in the case of vanishing nonmetricity, there emerge these 6 pieces of the cur-
vature tensor with their 10 + 9 + 1 + 9 + 6 + 1 independent components. In a
Riemannian space, we have \(\text{paircom} = \text{pscalar} = \text{ricanti} = 0\), or

\[
R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} , \quad R_{[\alpha\beta\gamma\delta]} = 0 , \quad Ric_{[\alpha\beta]} = 0 , \tag{18}
\]

with \(Ric_{\alpha\beta} := R_{\gamma\alpha\beta}^\gamma\). Then only \text{wyl}, \text{ricsymf} (ricci symmetric and tracefree),
and \text{scalar} survive with their, as is well known from general relativity, 10 +
9 + 1 independent components. In other words, the torsion \(T^\alpha\), via the first
Bianchi identity \(DT^\alpha = R_{\beta\alpha}^\beta \wedge \vartheta^\beta\), induces the non-
vanishing of \text{paircom}, \text{pscalar}, and \text{ricanti}. Obviously, (18) represents the 16 algebraic identities
of a Riemannian curvature tensor.

2.2 Strain curvature = segmental curvature \(\oplus\) shear curvature

The symmetric part \(Z_{\alpha\beta}\) of the curvature two-form, the 'strain curvature', is
more involved than the rotational curvature. First of all, we split it into a
tracefree (or shear) and a trace (or dilation) part:

\[
Z_{\alpha\beta} = Z'_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} Z , \quad Z := Z_{\gamma}^\gamma . \tag{19}
\]
The dilation piece $g_{\alpha\beta}Z_{\gamma}/4$ is Weyl’s *segmental curvature*, which we recognize as one irreducible piece of the strain curvature. The rest, the shear curvature $\mathcal{Z}_{\alpha\beta}$, will be ‘contracted’:

\[
\mathcal{Z}_\alpha := e^\beta \mathcal{Z}_{\alpha\beta}, \quad \Delta := \frac{1}{2} (\vartheta^\alpha \wedge \mathcal{Z}_\alpha), \quad Y_\alpha := * (\mathcal{Z}_{\alpha\beta} \wedge \vartheta^\beta). \quad (20)
\]

An even finer decomposition is possible:

\[
\Xi_\alpha := \mathcal{Z}_\alpha - \frac{1}{2} e_\alpha (\vartheta^\gamma \wedge Z_\gamma), \quad \Upsilon_\alpha := Y_\alpha - \frac{1}{2} e_\alpha (\vartheta^\gamma \wedge Y_\gamma). \quad (21)
\]

In terms of these newly introduced quantities, the irreducible decomposition of $Z_{\alpha\beta}$ may be written as

\[
Z_{\alpha\beta} = (1) Z_{\alpha\beta} + (2) Z_{\alpha\beta} + (3) Z_{\alpha\beta} + (4) Z_{\alpha\beta} + (5) Z_{\alpha\beta}, \quad (22)
\]

\[
\sim z\text{curvone} + z\text{curvtwo} + z\text{curvthree} + \text{dilcurv} + z\text{curvfive},
\]

where

\[
z\text{curvtwo}2(-a,-b) \sim -\frac{1}{2} \left\{ \vartheta_\alpha \wedge \Upsilon_\beta \right\}, \quad (23)
\]

\[
z\text{curvthree}2(-a,-b) \sim \frac{1}{3} \left\{ 2 \vartheta_\alpha \wedge (e_\beta \wedge \Delta) - g_{\alpha\beta} \Delta \right\}, \quad (24)
\]

\[
\text{dilcurv}2(-a,-b) \sim \frac{1}{4} g_{\alpha\beta} Z, \quad (25)
\]

\[
z\text{curvfive}2(-a,-b) \sim \frac{1}{2} \vartheta_\alpha \wedge \Xi_\beta, \quad (26)
\]

\[
z\text{curvone}2(-a,-b) \sim Z_{\alpha\beta} - \sum_{I=2}^{5} (I) Z_{\alpha\beta}. \quad (27)
\]

The corresponding Reduce-Excalc code reads as follows:

```reduce
%**************************************************************
% Irred. decomposition of the strain curvature z2(-a,-b) *
%**************************************************************
% file straincurvdecomp.exi, 1998-03-15
% input: coframe o(a), frame e(a), metric g(a,b), curv2(a,b)

pform {ztracef1(a),yy1(a),xi1(a),upsilon1(a)}=1,
{z2(a,b),ztracef2(a,b),delta2,
zcurvone2(a,b), zcurvtwo2(a,b),zcurvthree2(a,b),
dilcurv2(a,b),zcurvfive2(a,b)}=2$
```

8
Let us now turn to general relativity (GR) in order to have a decent electrovac solution of GR as a starting point for our further considerations, i.e. an exact solution of the Einstein equation with a distribution of electromagnetic energy-momentum as source. The Einstein-Maxwell equations with cosmological constant $\lambda$ read:

$$G^\alpha_\beta + \lambda \eta^\alpha_\beta = 2 \mathcal{T}^\alpha_\beta, \quad dH = 0, \quad dF = 0, \quad H = *F. \quad (28)$$

The Einstein 3-form $G^\alpha_\beta \sim \text{einstein3}(-a)$ and the electromagnetic energy-momentum 3-form $\mathcal{T}^\alpha_\beta \sim \text{maxenergy3}(-a)$ are defined by

$$G^\alpha_\beta := \frac{1}{2} \eta^{\alpha_\beta_\gamma} R^\beta_\gamma$$

and

$$\mathcal{T}^\alpha_\beta = e^\alpha_\beta L_{\text{Max}} + (e^\alpha_\beta F) \wedge H, \quad \text{with} \quad L_{\text{Max}} := -\frac{1}{2} F \wedge H, \quad (30)$$
respectively. The excitation $H \sim \text{excit2}$ and the electromagnetic field strength $F \sim \text{farad2}$ are given by

$$H = -\mathcal{H} \wedge dt + \mathcal{D}, \quad F = E \wedge dt + B$$  \hfill (31)

($\mathcal{H}$ magnetic and $\mathcal{D}$ electric excitation, $E$ electric and $B$ magnetic field). The ubiquitous $\eta$-basis is dual to the $\vartheta$-basis, that is, see [16],

$$\eta_\alpha = \star \vartheta_\alpha, \quad \eta_{\alpha\beta} = \star (\vartheta_\alpha \wedge \vartheta_\beta), \quad \text{etc.}$$  \hfill (32)

In a Riemannian space, we can decompose the Einstein 3-form into 2 irreducible pieces\footnote{In a metric-affine space, the definition (29) kills the strain curvature and also 3 of the rotational curvature pieces. We find:}

$$G_\alpha = \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge \left[ (^{(4)} R^{\beta\gamma}) + (^{(6)} R^{\beta\gamma}) \right]$$

$$= \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge (^{(4)} R^{\beta\gamma} - \frac{1}{4} R \eta_\alpha), \quad \text{with} \quad R := e_\alpha \lbrack e_\beta \rbrack R^{\alpha\beta}. \hfill (34)$$

If we substitute this into the Einstein equation in (28), then it splits into two independent pieces, a tracefree one and the trace:

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge (^{(4)} R^{\beta\gamma}) = 2 \frac{\text{Max}}{T_{\alpha}}, \quad R = 4\lambda. \hfill (35)$$

Note that $T_{\alpha}$ is tracefree: $\vartheta^\alpha \wedge T_{\alpha} = 0$. It is now evident of how we can directly use our decomposition program of the last section in order to verify the field equations (35). But, of course, we could also compute the Einstein 3-form directly according to its definition (29).

### 3.1 P&D coframe $o(a)$ and electromagnetic potential $\text{pot1}$

The P&D solution is described in terms of coordinates $x^\mu = (\tau, q, p, \sigma) \sim (\tau, q, p, \sigma)$. We use Eq.(3.30) of Plebanski and Demianski [27], see also [9,4,5,10]. Then the orthonormal coframe reads,

$$G_\alpha = \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge \sum_{I=4}^{6} (^{(I)}W^{\beta\gamma}). \hfill (33)$$

Consequently, in a metric-affine space, besides $\text{ricsymf}$ and $\text{scalar}$, only $\text{ricanti}$ surfaces as an additional quantity in the Einstein 3-form.
\[\vartheta^0 = \frac{1}{H} \sqrt{\frac{Q}{\Delta}} (d\tau - p^2 d\sigma) ,\]
\[\vartheta^1 = \frac{1}{H} \sqrt{\frac{\Delta}{Q}} dq,\]
\[\vartheta^2 = \frac{1}{H} \sqrt{\frac{\Delta}{P}} dp,\]
\[\vartheta^3 = \frac{1}{H} \sqrt{\frac{P}{\Delta}} (d\tau + q^2 d\sigma) ,\] (36)

with the metric
\[g = -\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3 ,\] (37)

or
\[g = \frac{1}{H^2} \left\{ -\frac{Q}{\Delta} (d\tau - p^2 d\sigma)^2 + \frac{\Delta}{Q} dq^2 + \frac{\Delta}{P} dp^2 + \frac{P}{\Delta} (d\tau + q^2 d\sigma)^2 \right\} .\] (38)

The functions \(P, Q, \Delta,\) and \(H\) are polynomial in \(p\) and \(q\). With the parameters \(m, n, e_o, g_o, b, \epsilon, \mu,\) they read:

\[P := (b - g_o^2) + 2np - \epsilon p^2 + 2m\mu p^3 - \left[ \mu^2 \left( b + e_o^2 \right) + \frac{\lambda}{3} \right] p^4,\]
\[Q := (b + e_o^2) - 2mq + \epsilon q^2 - 2m\mu q^3 - \left[ \mu^2 \left( b - g_o^2 \right) + \frac{\lambda}{3} \right] q^4,\]
\[\Delta := p^2 + q^2,\]
\[H := 1 - \mu pq .\] (39)

We recover the original version of P&D [27], if we put \(b := \gamma - \lambda/6\) and \(\mu = 1.\) The parameter \(\mu,\) see [10], generalizes the P&D solution slightly.

The electromagnetic potential \(A\) appropriate for this solution can be expressed as follows, see [27] (\(e_o = \) electric and \(g_o = \) magnetic charge):

\[A = \frac{1}{\Delta} [(e_o q + g_o p) d\tau + (g_o q - e_o p) p q d\sigma] \]
\[= \frac{H}{\sqrt{\Delta}} \left( \frac{e_o q}{\sqrt{Q}} \vartheta^0 + \frac{g_o p}{\sqrt{P}} \vartheta^3 \right) .\] (40)

In the following we will show of how one can put this metric into Excalc. In an interactive Reduce 3.6 session, the Excalc package can be loaded by
the command `load_package excalc$`. In the next step, Excalc has to be acquainted with the rank of the differential forms to be used in the code. This is achieved by the `pform` declaration. Subsequently it has to be specified by means of `fdomain` on which variables these forms do depend. Afterwards one defines the coframe:

```
%***************************************************************
% Specifying the coframe o(a)                     *
%***************************************************************
% file coframe.exi, 1998-03-15
% no prior input

load_package excalc$

pform {hh,sqrtqq,sqrtpp,delta,polynomq,polynomp}=0$
fdomain hh=hh(p,q),sqrtqq=sqrtqq(q),sqrtpp=sqrtpp(p),delta=
delta(p,q),polynomq=polynomq(q),polynomp=polynomp(p)$

coframe o(0) = ((1/hh)*sqrtqq/sqrt(delta))*(d tau-p**2*d sigma),
o(1) = ((1/hh)*sqrt(delta)/sqrtqq)* d q,
o(2) = ((1/hh)*sqrt(delta)/sqrtpp)* d p,
o(3) = ((1/hh)*sqrtpp/sqrt(delta))*(d tau+q**2*d sigma)

with
metric g = -o(0)*o(0) + o(1)*o(1) + o(2)*o(2) + o(3)*o(3)$
frame e$
$end$
%***************************************************************
```

Here the frame \( e_{\alpha} \sim e(-a) \) is dual to the coframe.

In accordance with (39), we introduced the functions:

\[
\begin{align*}
hh & \sim 1 - \mu p q, \\
delta & \sim p^2 + q^2, \\
sqrtqq & \sim \sqrt{Q}, \\
sqrtpp & \sim \sqrt{P}.
\end{align*}
\]

(41)

Here \( Q \) and \( P \), see (39), are polynomials of quartic order in \( q \) and \( p \), respectively. For the time being, we will only read in \( \Delta \) and \( H \), since they are not very complicated:

\[
\begin{align*}
delta & := p**2 + q**2; \\
hh & := 1 - \mu u * p * q;
\end{align*}
\]

However, the explicit form of \( \sqrt{Q} \) and \( \sqrt{P} \) will be left open.
It is an old trick to treat square roots of functions in the way we did it. Bringing in square roots explicitly, will usually slow down a program appreciably or even may blow it up beyond the capacity of the computer. We can express the derivatives of $\sqrt{Q}$ and $\sqrt{P}$ by means of the chain rule, also their squares get a new name:

\[
\begin{align*}
\sqrt{pp} & := \sqrt{pp} \\
\sqrt{qq} & := \sqrt{qq} \\
\sqrt{pp}^2 & := \text{polynomp} \\
\sqrt{qq}^2 & := \text{polynomq}
\end{align*}
\]

3.2 The curvature $\text{riem2}(a,b)$ of the P&D solution

Let us first bring in the metric volume element and the $\eta$-basis:

This is simple geometry. Now we compute connection and curvature:

\[
\begin{align*}
\text{eta0}(a,b,c,d) & := 0, \text{eta1}(a,b,c) = 1, \text{eta2}(a,b) = 2, \text{eta3}(a) = 3, \text{eta4} = 4 \\
\text{eta4} & := 1 \\
\text{eta3}(a) & := \text{e}(a) \_\_ \text{eta4} \\
\text{eta2}(a,b) & := \text{e}(b) \_\_ \text{eta3}(a) \\
\text{eta1}(a,b,c) & := \text{e}(c) \_\_ \text{eta2}(a,b) \\
\text{eta0}(a,b,c,d) & := \text{e}(d) \_\_ \text{eta1}(a,b,c)
\end{align*}
\]

$\text{end}$

This is simple geometry. Now we compute connection and curvature:

\[
\begin{align*}
\text{chris1}(a,b) & := \text{Riemannian connection} \\
\text{riem2}(a,b) & := \text{curvature}
\end{align*}
\]

$\text{factor}\ ^,\text{o}(0),\text{o}(1),\text{o}(2),\text{o}(3)$

\[
\begin{align*}
\text{Riemannconx chris1} \\
\text{because of our conventions, we have to take the transp. of it}
\end{align*}
\]
chris1(-a,b):=chris1(b,-a)$

pform {riem2(a,b),curv2(a,b)}=2, {einstein3(a),cosmolog3(a)}=3, grscalar=0$
% antisymmetric riem2; since we want to turn non-Riem. later

riem2(-a,b) := d chris1(-a,b) - chris1(-a,c) ^ chris1(-c,b)$
grscalar := e(-a) _| (e(-b) _| riem2(a,b))$

einstein3(a) := (1/2) * eta1(a,b,c) ^ riem2(-b,-c);
cosmolog3(a) := lam * eta3(a)$

curv2(a,b) := riem2(a,b)$
$end$
%******************************************************************************

3.3 Starting an interactive Reduce session

Let us start such a session by calling Reduce and reading in the coframe and the $\eta$ files. Then we specify $\Delta$, $H$, etc. and call the Riemann file:

reduce
in "coframe.exi"$
in "eta.exi"$
delta := p**2 + q**2;
hh := 1 - mu*p*q;

@sqrtpp,p) := @(polynomp,p)/(2*sqrtpp);
@sqrtqq,q) := @(polynomq,q)/(2*sqrtqq);

sqrtpp**2 := polynomp;
sqrtqq**2 := polynomq;

in "riemann.exi"$

We hasten to point out that we copied the commands listed above from the original tex-file of this paper directly into a separate Reduce window. Thereby our Reduce-Excalc code is on line computer checked and, thus, absolutely trustworthy.

After a few seconds (with a Sun sparcstation) the curvature will be in the output in terms of the up to now unspecified functions sqrtqq, sqrtpp,
Nevertheless, if we read in the rotational and the strain curvature decomposition programs of above,

in "rotcurvdecomp.exi"

in "straincurvdecomp.exi"

we immediately find \texttt{weyl2}, \texttt{ricsymf2}, \texttt{scalar2} as non-vanishing, whereas the whole rest vanishes, as it must be in a Riemannian spacetime. Before we look into more details of the surviving curvature pieces, we discuss the coupling to the electromagnetic field.

### 3.4 The electromagnetic field as source of gravity: \texttt{maxenergy3(a)}

Our Maxwell program has been published earlier, see [30]. It has to be preceded by the coframe statement and the specification of the frame \(e(a)\). This has been already executed in the last subsection. Therefore we can take the electromagnetic potential from (40) and type it in directly into the Maxwell program:

```plaintext
%***************************************************************
% Maxwell equations metric free *
%***************************************************************
% file maxwell.exi, 1998-03-15
% prior input: coframe o(a), frame e(a), metric g(a,b)

% potential is prescribed
% field strength farad2, excitation excit2, and the left hand
% sides of the Maxwell equations are defined

pform pot1=1, {farad2,excit2}=2, {maxhom3,maxinh3}=3$

pot1 := (1/delta)*( (ee*q+gg*p) * d tau 
+ (gg*q-ee*p)*p*q * d sigma);  
farad2 := d pot1;  
maxhom3 := d farad2;  
excit2 := # farad2;  
maxinh3 := d excit2;  

% Maxwell Lagrangian and energy-momentum current are assigned

pform lmax4=4, maxenergy3(a)=3$

lmax4 := -(1/2) * farad2 \wedge excit2;
```
maxenergy3(-a) := e(-a) _| lmax4 + (e(-a) _| farad2) ^ excit2;
$end$

%*******************************************************************************

In continuing our interactive Reduce session, we read in this file:

in "maxwell.exi"

Both Maxwell equations are fulfilled. The electromagnetic energy-momentum is determined. If we type in

pot1;

the frame version of $A$ will be displayed, cf. (40). The Einstein equation with cosmological constant and electromagenetic field source can be written down:

cform einsteinmax3(a)=3$
einsteinmax3(a):=einstein3(a) + cosmolog3(a) - 2*maxenergy3(a);

The left hand side, namely $einsteinmax3(a)$ if evaluated, has already a very compact form. Then, substituting the polynomials $\mathcal{P}$ and $\mathcal{Q}$ of (39),

$% \text{ substitution}$

polynomp := bb - gg**2 + 2*n*p - epsi*p**2 + 2*mu*m*p**3
- (lam/3 + mu**2*(bb + ee**2))*p**4;

polynomq := bb + ee**2 - 2*m*q + epsi*q**2 - 2*n*mu*q**3
- (lam/3 + mu**2*(bb - gg**2))*q**4;

einsteinmax3(a) := einsteinmax3(a);
showtime;

the Einstein equation turns out to be fulfilled, too.

3.5 \textit{Conformal curvature two-form} \textit{weyl2(a,b) of the P\&D solution}

Consequently the P\&D metric is a solution of the Einstein-Maxwell equations, indeed. The P\&D gravitational field is characterized by the conformal curvature 2-form

$$C_{\alpha\beta} := (1)R_{\alpha\beta} = \frac{1}{2} C_{\mu\nu\alpha\beta} \psi^\mu \wedge \psi^\nu . \quad (42)$$

Therefore it is desirable to put this in a compact form. Our decomposition program ran already. Thus we only have to collect the result by turning the
expansion off and calling for the Weyl 2-form:

\[
\text{off exp$\$}
\]
\[
\text{weyl2}(a,b):=\text{weyl2}(a,b);
\]

We find thereby (Heinicke [18]):

\[
C^{\alpha\beta} = f(\alpha, \beta) \left\{ -A \partial^\alpha \wedge \partial^\beta + B^* \left( \partial^\alpha \wedge \partial^\beta \right) \right\}, \quad (43)
\]

with

\[
A := \frac{(\mu pq - 1)^3}{(p^2 + q^2)^3} \left[ m \left( 3q^2 - p^2 \right) q + n \left( p^2 - 3q^2 \right) p - \left( e_0^2 + g_0^2 \right) (\mu pq + 1) \left( p^2 - q^2 \right) \right]
\]
\[
B := \frac{(\mu pq - 1)^3}{(p^2 + q^2)^3} \left[ m \left( 3q^2 - p^2 \right) p - n \left( q^2 - 3p^2 \right) q - \left( e_0^2 + g_0^2 \right) (\mu pq + 1)2pq \right]
\]

and

\[
f(\alpha, \beta) = \begin{cases} 2 & \text{for } (\alpha, \beta) \in \{(0,1), (1,0), (2,3), (3,2)\} \\ -1 & \text{else} \end{cases} \quad (44)
\]

Of course, the Weyl 2–form is a real quantity. Accordingly, we can also represent it by its self–dual part, see [20] \((i= \text{imaginary unit})\),

\[
{+_C}_{\alpha\beta} := \frac{1}{2} \left( _C \alpha_{\beta} + i{^*}_C \alpha_{\beta} \right) = \frac{1}{2} {+_C}_{\mu\nu\alpha\beta} \partial^\mu \wedge \partial^\nu, \quad (45)
\]
or, in computer code, by

\[
\text{on complex$\$}
\]
\[
\text{pform selfdualweyl2(a,b)=2$}
\]
\[
\text{selfdualweyl2(a,b) := (1/2) * (weyl2(a,b) + i * #weyl2(a,b));}
\]

It can be very compactly displayed [18],

\[
+_C^{\alpha\beta} = f(\alpha, \beta) \mathcal{C} \left\{ \partial^\alpha \wedge \partial^\beta + i^* \left( \partial^\alpha \wedge \partial^\beta \right) \right\}, \quad (46)
\]

where

\[
\mathcal{C} := -\frac{1}{2}(A + iB) = -\frac{(\mu pq - 1)^3}{2(p - iq)^3} \left( n - im - \left( e_0^2 + g_0^2 \right) \frac{\mu pq + 1}{p + iq} \right). \quad (47)
\]
Often it is thought that such a compact representation is only possible in a null frame (i.e., by means of a Newman-Penrose tetrad). Obviously, however, this is not true. It is an effect of the power of the moving frame aspect of exterior calculus, not of that of a null coframe.

By the same token, we computed also the two quadratic curvature invariants \( *(W_{\alpha \beta} \wedge * W^{\alpha \beta}) \) and \( *(W_{\alpha \beta} \wedge W^{\alpha \beta}) \), but their discussion is beyond the scope of this paper.

### 4 Torsion and nonmetricity and their irreducible decompositions

In a metric-affine spacetime, the field strengths, besides the curvature two-form \( \text{curv2}(a, b) \), are given by the components of the torsion two-form

\[
\text{torsion2}(a) \sim T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma^\alpha_\beta \vartheta^\beta = \frac{1}{2} T^\alpha_{\mu \nu} \vartheta^\mu \wedge \vartheta^\nu, \quad (48)
\]

and the nonmetricity one-form

\[
\text{nonmet1}(-a, -b) \sim Q_{\alpha \beta} = -Dg_{\alpha \beta} = -dg_{\alpha \beta} + \Gamma^\gamma_{\alpha \beta} g_{\gamma \gamma} + \Gamma^\gamma_{\beta \gamma} g_{\alpha \gamma} = Q_{\alpha \beta} dx^i. \quad (49)
\]

Although torsion and nonmetricity are genuine field strengths, they can be reinterpreted as parts of the connection. The linear connection can be expressed in term of metric, coframe, torsion, and nonmetricity.

#### 4.1 Torsion torsion2(a) and its three irreducible pieces

For the torsion, we have the following irreducible decomposition

\[
T^\alpha = (1) T^\alpha + (2) T^\alpha + (3) T^\alpha \\
\sim \text{tentor} + \text{trator} + \text{axitor}, \quad (50)
\]

where

\[
\text{trator2}(a) \sim \frac{1}{3} \vartheta^\alpha \wedge (e_\beta [T^\beta]), \quad (51)
\]

\[
\text{axitor2}(a) \sim \frac{1}{3} e^\alpha [\vartheta^\beta \wedge T^\beta], \quad (52)
\]
tentor2(a) \sim T^\alpha - (2)T^\alpha - (3)T^\alpha, \quad (53)

or, in other ‘words’,

%**************************************************************%
% Irreducible decomposition of the torsion torsion2(a) *
%**************************************************************%

\text{% file torsiondecomp.exi, 1998-03-15}
\text{% input: o(a), e(a), g(a,b), torsion2(a)}

pform \{tentor2(a), trator2(a), axitor2(a)\}=2$

\text{trator2(a):= (1/3)*o(a)^ (e(-b) \_\_\_\_ torsion2(b));}
\text{axitor2(a):= (1/3)*e(a) \_\_\_\_ (o(-b) ^ torsion2(b));}
\text{tentor2(a):= torsion2(a)-trator2(a)-axitor2(a);}
$end$
\text{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%}

4.2 Nonmetricity \text{nonmet1(a,b) and its four irreducible pieces}

The Weyl one-form \( Q := Q_\alpha^\alpha/4 = -g^{\alpha\beta}Dg_{\alpha\beta}/4 \) is one \text{irreducible} piece of the nonmetricity. Thus we find the traceless part of the nonmetricity as

\[ Q_{\alpha\beta} := Q_{\alpha\beta} - Q g_{\alpha\beta}. \quad (54) \]

The traceless nonmetricity (its ‘deviator’, in the language of continuum mechanics) can be further ‘contracted’:

\[ \Lambda_{\alpha} := e^\beta \_\_\_\_ Q_{\alpha\beta}, \quad \Lambda := \Lambda_{\alpha} \vartheta^\alpha, \quad (55) \]
\[ \Theta_{\alpha} := *(Q_{\alpha\beta} \wedge \vartheta^\beta), \quad \Theta := \vartheta^\alpha \wedge \Theta_{\alpha}, \quad \Omega_{\alpha} := \Theta_{\alpha} - \frac{1}{3} e_{\alpha} \_\_\_\_ \Theta. \quad (56) \]

From these quantities we can build up the four irreducible pieces:

\begin{align*}
\text{binom1(-a, -b) & \sim \frac{2}{3} * \left( \vartheta_{(\alpha} \wedge \Omega_{\beta)} \right),} \\
\text{vecnom1(-a, -b) & \sim \frac{4}{9} \left( \vartheta_{(\beta} \Lambda_{\alpha)} - \frac{1}{4} g_{\alpha\beta} \Lambda \right),} \\
\text{conom1(-a, -b) & \sim g_{\alpha\beta} Q,} \\
\text{trinom1(-a, -b) & \sim Q_{\alpha\beta} - (2)Q_{\alpha\beta} - (3)Q_{\alpha\beta} - (4)Q_{\alpha\beta}.} 
\end{align*}
trinom represents a tensor of 3rd rank and binom one of 2nd rank, with vecnom and conom we denote the covector and the vector pieces, respectively. Clearly conom is equivalent to the Weyl covector. They add up according to

\[
Q_{\alpha\beta} = (1)Q_{\alpha\beta} + (2)Q_{\alpha\beta} + (3)Q_{\alpha\beta} + (4)Q_{\alpha\beta}
\sim \text{trinom} + \text{binom} + \text{vecnom} + \text{conom}.
\]

4.3 The post-Riemannian covector triplet

With propagating nonmetricity \( Q_{\alpha\beta} \) two types of charge are expected to arise: *One dilation charge* related via the Noether procedure to the trace of the nonmetricity, the Weyl covector \( Q = Q_i \, dx^i \). It represents the connection associated with gauging the scale transformations (instead of the \( U(1) \)-connection...
in the case of Maxwell’s field). Furthermore, *nine shear charges* are expected that are related to the remaining traceless piece \( \mathcal{Q}_{\alpha\beta} := Q_{\alpha\beta} - Q g_{\alpha\beta} \) of the nonmetricity with \( 4 \times (4 + 4 + 1) \) components.

For the torsion and nonmetricity field configurations, we concentrate on the simplest non–trivial case *with* shear. According to its irreducible decomposition (61), the nonmetricity contains two covector type pieces, namely \( {}^{(4)}Q_{\alpha\beta} = Q g_{\alpha\beta} \), the dilation piece, and

\[
{}^{(3)}Q_{\alpha\beta} = \frac{4}{9} \left( \partial_{(\alpha} e_{\beta)} \Lambda - \frac{1}{4} g_{\alpha\beta} \Lambda \right), \quad \text{with} \quad \Lambda = \partial^\alpha e^\beta \mathcal{Q}_{\alpha\beta}, \tag{62}
\]

a proper shear piece. Accordingly, our ansatz for the nonmetricity reads

\[
Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}. \tag{63}
\]

The torsion, in addition to its tensor piece, encompasses a covector and an axial covector piece. Let us choose only the covector piece as non–vanishing:

\[
T^\alpha = {}^{(2)}T^\alpha = \frac{1}{3} \partial^\alpha \wedge T, \quad \text{with} \quad T := e_\alpha \mathcal{Q}_\alpha. \tag{64}
\]

Thus we are left with the three non–trivial one–forms \( Q, \Lambda, \) and \( T \). We shall assume that this *triplet of one–forms* shares the spacetime symmetries, that is, its members are proportional to each other.

5 \textbf{Applying the decomposition programs in metric-affine gravity:}

The P&D metric together with a non-Riemannian covector triplet as exact solution of a dilation-shear Lagrangian

In order to explore the potentialities of metric–affine gravity, see [11], and of the corresponding computer algebra programs, we will choose the simple non-trivial dilation-shear gravitational Lagrangian,

\[
V_{\text{dil–sh}} = -\frac{1}{2l^2} \left( R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge \star Q + \gamma T \wedge \star T \right) - \frac{1}{8} \alpha R^\alpha_\alpha \wedge \star R_\beta^\beta, \tag{65}
\]

with the dimensionless coupling constants \( \alpha, \beta, \) and \( \gamma \). We will choose units such that \( l^2 = 1 \). We add the Maxwell Lagrangian, see (30):

\[
L = V_{\text{dil–sh}} + L_{\text{Max}}. \tag{66}
\]
Explicitly we have:

\[
L = -\frac{1}{2l^2} \left( R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge \ast Q + \gamma T \wedge \ast T \right) - \frac{1}{8} R_\alpha^\alpha \wedge \ast R_\beta^\beta - \frac{1}{2} F \wedge \ast F .
\] (67)

Note the formal similarities of the curvature square Lagrangian with the segmental curvature $\frac{1}{2} R_\alpha^\alpha = d Q$ and the Maxwell Lagrangian with the electromagnetic field strength $F = d A$.

We will search for exact solutions of the field equations belonging to this Lagrangian. Since the matter part $L_{\text{Max}}$ does not depend on the connection $\Gamma_\alpha^\beta$, the hypermomentum $\Delta_\alpha^\beta := \delta L_{\text{Max}}/\delta \Gamma_\alpha^\beta$ vanishes, $\Delta_\alpha^\beta = 0$, and the only external current is the electromagnetic energy-momentum current $\Sigma_\alpha = T_\alpha$. We will only be able to find non-trivial solutions, if the coupling constants fulfill the constraint

\[
\gamma = -\frac{8}{3} \frac{\beta}{\beta + 6} .
\] (68)

This is the best we can do so far.

It has been shown by Obukhov et al. [24] — for the related work of Tucker et al., see [40] — how the general solution belonging to Lagrangians of the type (67) can be constructed: One starts with an electrovac metric of general relativity and the corresponding electromagnetic potential. We will take the P&D metric (38) and its potential (40). Then the post-Riemannian triplet of (63,64) is constructed which is patterned after the electromagnetic potential,

\[
Q = k_0 \omega , \quad \Lambda = k_1 \omega , \quad T = k_2 \omega ,
\] (69)

with the 1-form

\[
\omega = \frac{1}{\Delta} \left[ (N_e q + N_g p) d\tau + (N_g q - N_e p) p q d\sigma \right] = \frac{H}{\sqrt{\Delta}} \left( \frac{N_e q}{\sqrt{Q}} \delta^0 + \frac{N_g p}{\sqrt{P}} \delta^3 \right) .
\] (70)

In Excalc this input reads:

\[
%**************************************************************
% Triplet ansatz for nonmetricity and torsion
%**************************************************************
\]
% file triplet.exi, 1998-03-15
% prior input coframe o(a), frame e(a), metric g(a,b)

pform \{omega1, qq1, llam1, tt1\} = 1$

\omega1 := (1/\delta)*( (Ne*q+Ng*p) * d \tau + (Ng*q-Ne*p)*p*q * d \sigma);

qq1 := k0 * \omega1$
llam1 := k1 * \omega1$
tt1 := k2 * \omega1$
clear \omega1$
$end$
%******************************************************************************

These are the three one-forms which are fundamental for the exact solution. Now these one-forms have to be assigned to the corresponding pieces of the torsion $T^\alpha \sim \text{torsion2}(a)$, the contortion $K_{\alpha\beta} \sim \text{contor1}(-a,-b)$, and the nonmetricity $Q^\alpha_{\beta\gamma} \sim \text{nonmet1}(a,b)$:

%******************************************************************************
% Post-Riemannian assignments
%******************************************************************************
% file postriem.exi, 1998-03-15
% prior input o(a), e(a), g(a,b,), qq1, llam1, tt1

pform torsion2(a) = 2, \{contor1(a,b), nonmet1(a,b)\} = 1$
torsion2(a) := (1/3) * o(a) ^ tt1$
nonmet1(a,b) := (2/9)* ( o(b) * (e(a) _| llam1) +o(a) * (e(b) _| llam1) - (1/2)*g(a,b)*llam1 ) + qq1*g(a,b)$

contor1(-a,-b) := (1/2)* (e(-a) _| torsion2(-b) - e(-b) _| torsion2(-a)) -(1/2)* (e(-a) _| (e(-b) _| torsion2(-c)))^o(c)$
$end$
%******************************************************************************

The constants $k_0, k_1, k_2$ in front of the triplet can be expressed in terms of the coupling constants of the dilation-shear Lagrangian:

$$k_0 = -\frac{24}{\beta + 6}, \quad k_1 = -\frac{36\beta}{\beta + 6}, \quad k_2 = 6.$$ (71)
$N_e$ and $N_g$ are the quasi-electric and the quasi-magnetic dilation–shear–spin charges, respectively. Now, in metric-affine gravity, the polynomials depend also on these quasi-charges in a fairly trivial, i.e. in an additive way,

\[
P := (b - g_o^2 - G_o^2) + 2np - ep^2 + 2m\mu p^3 - \left[\mu^2 (b + e_o^2 + \varepsilon_o^2)\right]p^4,
\]

\[
Q := (b + e_o^2 + \varepsilon_o^2) - 2mq + eq^2 - 2n\mu q^3 - \left[\mu^2 (b - g_o^2 - G_o^2)\right]q^4,
\]

\[
\Delta := p^2 + q^2,
\]

\[
H := 1 - \mu pq.
\]

(72)

The post-Riemannian charges $N_e$ and $N_g$ are related to the post-Riemannian pieces $E_o$ and $G_o$, entering the polynomials, according to

\[
\varepsilon_o = k_0 \sqrt{\frac{\alpha}{2}} N_e, \quad \varepsilon_o = k_0 \sqrt{\frac{\alpha}{2}} N_g, \quad \text{with} \quad \alpha > 0.
\]

(73)

The field equations of metric-affine gravity can be expressed in terms of the irreducible components of curvature, torsion, and nonmetricity – and these quantities we have already computed. With a corresponding computer program we verified the correctness of the exact solution discussed in this section. The presentation of these programs, however, we will leave to another occasion.

6 Discussion

Effective work in gravity can nowadays only be done if one has the standard computer algebra methods and packages available. We hope to have shown that the Reduce-Excalc tool is very efficient in the context of the Einstein–Maxwell equations as well as in gauge theories of gravity, like metric-affine gravity, where the spacetime is a non-Riemannian manifold.

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