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Restricting directions for Kakeya sets

Anthony Gauvan

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Abstract

We prove that the Kakeya maximal conjecture is equivalent to the $\Omega$-Kakeya maximal conjecture. This completes a recent result in [2] where Keleti and Mathé proved that the Kakeya conjecture is equivalent to the $\Omega$-Kakeya conjecture. Moreover, we improve concrete bound on the Hausdorff dimension of a $\Omega$-Kakeya set: for any Borel set $\Omega$ in $S^{n-1}$, we prove that if $X \subset \mathbb{R}^n$ contains for any $e \in \Omega$ a unit segment oriented along $e$ then we have

$$d_X \geq \frac{6}{11} d_\Omega + 1$$

where $d_E$ denotes the Hausdorff dimension of a set $E$.

1 Introduction

The Kakeya problem is a central question in harmonic analysis which can be formulated in different ways; it is also related to restriction theory and arithmetic. A measurable set $X$ in $\mathbb{R}^n$ is said to be a Kakeya set if for any direction $e \in S^{n-1}$ it contains a unit segment $T_e$ oriented along $e$. The Kakeya conjecture concerns the Hausdorff dimension of Kakeya set $X$.

Conjecture 1 (Kakeya conjecture). If $X$ is a Kakeya set in $\mathbb{R}^n$ then

$$d_X = d_{S^{n-1}} + 1 = n.$$ 

This conjecture has been proved by Davies in the plane in [1]. For $n \geq 3$, a vast amount of techniques have been developed in order to tackle this issue; we invite the reader to look at [3] or [6] to see the extent of the techniques that might be deployed. Here, we will simply say that, specialists are able to prove that if $X$ is a Kakeya set in $\mathbb{R}^n$ then

$$d_X \geq (\frac{1}{2} + \epsilon_n)d_{S^{n-1}} + 1$$

where $\epsilon_n > 0$ is a dimensional constant. There exists a more quantitative version of the Kakeya conjecture and we need to introduce the Kakeya maximal operator to state it. We define the Kakeya maximal function

$$K_\delta f : S^{n-1} \to \mathbb{R}_+$$

at scale $\delta > 0$ of a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ as

$$K_\delta f(e) := \sup_{a \in \mathbb{R}^n} \frac{1}{|T_{e,\delta}(a)|} \int_{T_{e,\delta}(a)} |f(x)| dx$$

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where \( e \in \mathbb{S}^{n-1} \) and \( T_{e, \delta}(a) \) stands for the tube in \( \mathbb{R}^n \) with center \( a \), oriented along the direction \( e \), of length 1 and radius \( \delta \). It appears that any quantitative information on \( \|K_\delta\|_{\sigma,p} \) provides lower bound on the dimension of any Kakeya set \( X \): for any \( 1 < p < \infty \) and \( \beta > 0 \) such that \( n - \beta p > 0 \), if we have
\[
\|K_\delta\|_{\sigma,p} \lesssim \delta^{\beta}
\]
then the Hausdorff dimension of any Kakeya set in \( \mathbb{R}^n \) is at least \( n - \beta p \). In regards of this fact, the following conjecture is called the Kakeya maximal conjecture, it is stronger than the Kakeya conjecture.

**Conjecture 2** (Kakeya maximal conjecture). For any \( \epsilon > 0 \) we have
\[
\|K_\delta\|_{\sigma,n} \lesssim \delta^{-\epsilon}.
\]

In this text, we are concerned with a natural generalization of the Kakeya problem. Given an arbitrary Borel set of directions \( \Omega \subset \mathbb{S}^{n-1} \), we say that a set \( X \) in \( \mathbb{R}^n \) is a \( \Omega \)-Kakeya set if for any \( e \in \Omega \) there exists a unit segment \( T_e \) oriented along \( e \) included in \( X \). What can be said about the dimension of a \( \Omega \)-Kakeya set? The following conjecture seems plausible.

**Conjecture 3** (\( \Omega \)-Kakeya conjecture). For any Borel set \( \Omega \) in \( \mathbb{S}^{n-1} \); if \( X \) is a \( \Omega \)-Kakeya set then
\[
d_X \geq d_\Omega + 1.
\]

At least three questions can be asked. First, if we know that the Kakeya conjecture is true, can we say something about the \( \Omega \)-Kakeya conjecture? Secondly, can we state a maximal version of the \( \Omega \)-Kakeya conjecture? Lastly, if there exists a \( \Omega \)-Kakeya maximal conjecture, what can we said about it given the Kakeya maximal conjecture? Very recently, Keleti and Mathé gave a positive answer to the first question in [2].

**Theorem 1** (Keleti-Mathé). If the Kakeya conjecture is true then the \( \Omega \)-Kakeya conjecture is also true.

The proof of this Theorem relies on fine notions concerning Hausdorff and packing dimension and we invite the reader to look at [2] for more details.

### 2 Notations

We will work in the euclidean space \( \mathbb{R}^n \) with \( n \geq 3 \) endowed with the Lebesgue measure and the euclidean distance; if \( U \) is a measure set in \( \mathbb{R}^n \) we denote by \( |U| \) its \( n \)-dimensional Hausdorff measure and by \( |U|_k \) its \( k \)-dimensional Hausdorff measure for \( k < n \). Also we denote by \( d_U \) its Hausdorff dimension and by \( \text{diam} (U) \) its diameter. We denote by \( \sigma \) the spherical surface measure on \( \mathbb{S}^{n-1} \) and \( \mu \) will stand for a probability measure on \( \mathbb{S}^{n-1} \); we will denote by \( S_\mu \) its support. The surface measure \( \sigma \) will usually not charge the support of \( \mu \) i.e. we will have \( \sigma(S_\mu) = 0 \). We will see that we need to focus on the study of \( (\mu, p) \)-norm of
\[
K_\delta : L^p(\mathbb{R}^n) \to L^p(\mathbb{S}^{n-1}, \mu)
\]
where \( \mu \) is an arbitrary measure on \( \mathbb{S}^{n-1} \) i.e. we will be interested in estimating the following quantity
\[
\|K_\delta\|_{\mu,p} := \sup_{\|f\|_p \leq 1} \left( \int_{\mathbb{S}^{n-1}} (K_\delta f(e))^p \, d\mu \right)^{\frac{1}{p}}.
\]
Hence the notation $\|K_\delta\|_{\mu,p}$ emphasizes the dependence on the probability measure $\mu$ set on the target space.

3 Results

We are going to formulate the appropriate maximal version of the Ω-Kakeya conjecture; then we will prove that the Kakeya maximal conjecture implies the Ω-Kakeya maximal conjecture. In other words, we prove the maximal analog to Theorem 1. Our approach is close to the approach initiated by Mitsis in [5]; here we work in higher dimension. We will start by proving the following Proposition.

**Proposition 1.** Let $\mu$ be an arbitrary probability measure on $S^{n-1}$ and suppose we have $1 < p < \infty$ and $\beta > 0$ such that $n - \beta p > 0$. Suppose that we have

$$\|K_\delta\|_{\mu,p} \lesssim n,p,\beta \delta^{-\beta}.$$  

In this case, for any Borel set of direction $\Omega$ containing the support $S_\mu$ of $\mu$, the Hausdorff dimension of any $\Omega$-Kakeya set $X$ is at least $n - \beta p$.

In regards of this Proposition, we will call the following Conjecture the Ω-Kakeya maximal conjecture.

**Conjecture 4 (Ω-Kakeya maximal conjecture).** Fix any probability $\mu$ defined on $S^{n-1}$ satisfying

$$|\mu|_d := \sup_{e \in S^{n-1}, r > 0} \mu(B_{e,r}) r^{-d} \leq 1.$$  

Then for any $\epsilon > 0$ we have

$$\|K_\delta\|_{\mu,n} \lesssim n,d,\epsilon \delta^{-\frac{d+1}{n}(1+\epsilon)}.$$  

Using Frostman’s Lemma, one can easily checked that the Ω-Kakeya maximal conjecture implies the Ω-Kakeya conjecture. One of our main result is the following.

**Theorem 2.** If the Kakeya maximal conjecture is true then the Ω-Kakeya maximal conjecture.

In particular, since in the plane $\mathbb{R}^2$ we do have $\|K_\delta\|_{\sigma,2} \lesssim \delta^{-\epsilon}$ for any $\epsilon > 0$, this gives another proof of the Ω-Kakeya conjecture in the plane; recall that this Theorem has also been established by Mitsis in [5].

**Theorem 3.** For any Borel set $\Omega$ in $S^1$, if $X$ is a $\Omega$-Kakeya set then

$$d_X \geq d_\Omega + 1.$$  

At this point, it is interesting to note Theorems 1 and 2 cannot provide partial result to the Ω-Kakeya conjecture. For example, say we can prove that if $X$ is a Kakeya set then we have

$$d_X \geq \frac{3}{4}(n - 1) + 1.$$  

In this situation, we cannot use Theorems 1 and 2 - neither their methods of proof - to show that for any $\Omega$-Kakeya set $Y$, we have

$$d_Y \geq \frac{3}{4}d_\Omega + 1.$$  

Hence, in order to obtain further partial result on the Ω-Kakeya conjecture, we are going to employ Bourgain’s arithmetic argument in order to prove the following Theorem.
Theorem 4. For any Borel set $\Omega$ in $\mathbb{S}^{n-1}$ and any $\Omega$-Kakeya set $X$ in $\mathbb{R}^n$ we have
\[ d_X \geq \frac{6}{11} d_\Omega + 1. \]

In [7], Venieri proved that if $\Omega$ is $d$-Ahlfors regular then a $\Omega$-Kakeya set $X$ has Hausdorff dimension greater than $\frac{d+2}{2} + \frac{1}{2}$. Theorem 4 strengthens this result since it gives a better estimate for large $n$ and also since we do not make assumptions concerning the set of directions $\Omega$.

4 Proof of Proposition 1

We let $\mu$ be an arbitrary probability measure on $\mathbb{S}^{n-1}$ and suppose we have $1 < p < \infty$ and $\beta > 0$ such that $n - \beta p > 0$. We also suppose that we have
\[ \|K_\delta\|_{\mu,p} \lesssim n,p,\beta \delta^{-\beta}. \]

We fix then an arbitrary Borel set of directions $\Omega$ which contains $S_\mu$ and we let $X$ included in $\mathbb{R}^n$ be a $\Omega$-Kakeya set; we are going to prove that we have
\[ d_X \geq n - \beta p. \]

Fix an arbitrary $\alpha \in (0, n - \beta p)$. Consider a covering of $X$ by balls $B_i = B(x_i, r_i)$ such that $r_i < 1$ for any $i \in I$. We will show that we have
\[ \sum_{i \in I} r_i^\alpha \gtrsim \alpha 1 \]
which gives $d_X \geq \alpha$. For $e \in \Omega$, let $T_e \subset X$ be a unit segment oriented along the direction $e$; for $k \geq 1$ we order the balls $B_i$ by their radii defining
\[ I_k = \left\{ i \in I : r_i \approx \frac{1}{2^k} \right\}. \]

We also define
\[ \Omega_k = \left\{ e \in \Omega : \left| T_e \cap \bigcup_{i \in I_k} B_i \right| \geq \frac{1}{2k^2} \right\}. \]

It is not difficult to show that we have $\Omega = \bigcup_{k \geq 1} \Omega_k$. We are going to fatten a little bit every segment $T_e$ in order to deal with tubes. We define for $k \geq 1$ the set
\[ Y_k' = \bigcup_{i \in I_k} B(x_i, 2r_i). \]

For $e \in \Omega$, by simple geometry we have the following inequality
\[ |T_{e,2^{-k}} \cap Y_k'| \gtrsim \frac{1}{k^2} \left| T_{e,2^{-k}} \right|. \]

Hence for any $e \in \Omega_k$ we have $K_{2^{-k}} 1_{Y_k'}(e) \gtrsim \frac{1}{k^2}$. Using our hypothesis on $K_\delta$, we obtain
\[ \mu(\Omega_k) \lesssim \mu \left( \left\{ K_{2^{-k}} 1_{Y_k'} \geq \frac{1}{2k^2} \right\} \right) \lesssim_{n,p,\beta} k^{2p}2^{k\beta p} \left| Y_k' \right|. \]
Since we have $|Y_k| \lesssim_{n} 2^{-k n \# I_k}$ it follows that $\mu(\Omega_k) \leq k^{2p} 2^{-k(n-\beta p) \# I_k}$. We have selected $\alpha < n - \beta p$ and so we have by polynomial comparison $k^{2p} 2^{-k(n-\beta p)} \lesssim_{n} 2^{-k \alpha}$. Hence we have

$$\sum_{i \in I} v_i^\alpha \geq \sum_{k} 2^{-\alpha k} \# I_k \geq_{\alpha} \sum_{k} \mu(\Omega_k) \geq \mu(\Omega) = 1$$

since $\Omega$ contains the support $\mathcal{S}_\mu$ of $\mu$.

## 5 Proof of Theorem 2

We are going to prove Theorem 2 proving the following estimate.

**Theorem 5.** Fix $1 < p < \infty$ and let $\mu$ be a probability on $\mathbb{S}^{n-1}$ satisfying $|\mu|_{d} \leq 1$ for some $0 \leq d \leq n - 1$. In this case we have for any $\delta > 0$

$$\|K_{\delta} f\|_{\mu, p} \lesssim_{n, d, p} \delta \frac{n^{-d+1}}{p} \|K_{\delta}\|_{\sigma, p}.$$ 

This estimate comes from the fact that a function $K_{\delta} f$ is almost $\delta$-discrete. Observe that this estimate is not possible in general since the surface measure $\sigma$ typically does not charge the support $\mathcal{S}_\mu$ of the measure $\mu$ i.e. $\sigma(\mathcal{S}_\mu) = 0$. The following Lemma is a manifestation of the idea that we should not define the orientation of an object more precisely than its eccentricity.

**Lemma 1.** For any $\delta > 0$ and any directions $e_1, e_2 \in \mathbb{S}^{n-1}$ satisfying $|e_1 - e_2| \leq \delta$ we have

$$K_{\delta} f(e_1) \simeq_n K_{\delta} f(e_2)$$

for any locally integrable function $f$.

**Proof.** This comes from the fact that there is a dimensional constant $a_n > 1$ such that if we have two tubes $T_{e_1, \delta}, T_{e_2, \delta}$ with $|e_1 - e_2| < \delta$ then one can find $\vec{t} \in \mathbb{R}^n$ such that

$$\vec{t} + \frac{1}{a_n} T_{e_1, \delta} \subset T_{e_2, \delta} \subset \vec{t} + a_n T_{e_1, \delta}.$$ 

We can then relate the $(\sigma, p)$-norm of $K_{\delta} f$ with a discrete sum over a family $e_{\delta, \mathbb{S}^{n-1}} \subset \mathbb{S}^{n-1}$ which is $\delta$-separated and maximal for this property.

**Lemma 2.** For $f$ locally integrable and any family $e_{\delta, \mathbb{S}^{n-1}}$ of $\mathbb{S}^{n-1}$ which is maximal and $\delta$-separated, we have

$$\|K_{\delta} f\|_{\sigma, p} \simeq_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_{\delta} f(e)^p \delta^{n-1}.$$ 

**Proof.** On one hand we have

$$\int_{\mathbb{S}^{n-1}} K_{\delta} f(e)^p d\sigma(e) \lesssim_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_{\delta} f(e)^p \sigma(B(e, \delta)) \simeq_n \sum_{e \in e_{\delta, \mathbb{S}^{n-1}}} K_{\delta} f(e)^p \delta^{n-1}.$$ 

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On the other hand we have
\[
\int_{S^{n-1}} K_\delta f(e) p d\sigma(e) \gtrsim_n \sum_{e \in e_{\delta, S^{n-1}}} K_\delta f(e) p \sigma(B(e, \frac{\delta}{2})) \approx_n \sum_{e \in e_{\delta, S^{n-1}}} K_\delta f(e) p \delta^{n-1}
\]
which concludes. □

We can now prove Theorem 5.

Proof. Fix \( \delta > 0 \) and consider a family \((e_k)_{k \leq m} \subset S_\mu\) which is \( \delta \)-separated and whose cardinal is maximal; this implies that we have
\[
S_\mu \subset \bigcup_{k \leq m} B_{e_k, 2\delta}.
\]
For \( f \) in \( L^p(\mathbb{R}^n) \) we have then
\[
\int_{S_\mu} K_\delta f(e) d\mu(e) \lesssim_n \sum_{k \leq m} \int_{B_{e_k, 2\delta}} K_\delta f(e_k) \mu(B_{e_k, 2\delta}) \lesssim_n d \sum_{k \leq m} K_\delta f(e_k) \delta^d
\]
using Lemma 1 and the fact that \([\mu_d] \leq 1\). Now we complete the family \((e_k)_{k \leq m}\) into a \( \delta \)-separated family \( e_{\delta, S^{n-1}} \) which is maximal in \( S^{n-1} \). We have then
\[
\sum_{k \leq m} K_\delta f(e_k) \delta^d \leq \sum_{e \in e_{\delta, S^{n-1}}} K_\delta f(e) \delta^d \delta^{d+1-n} \sum_{e \in e_{\delta, S^{n-1}}} K_\delta f(e) p \delta^{n-1} \approx_n \delta^{d+1-n} \|K_\delta f\|_{p, \sigma}
\]
using the previous lemma. □

We can now prove Theorem 2 i.e. we can prove that the Kakeya maximal conjecture implies to the \( \Omega \)-Kakeya maximal conjecture. This simply comes from the fact that the \( \epsilon \)-loss can be easily transferred thanks to Theorem 5.

Proof. Fix any probability \( \mu \) defined on \( S^{n-1} \) satisfying for some \( d \in [0, n-1], [\mu_d] \leq 1 \). Thanks to Theorem 5, if the Kakeya maximal conjecture is true then we have
\[
\|K_\delta\|_{\mu, n} \lesssim_{n, d} \delta^{(d+1)/n - 1} \|K_\delta\|_{\sigma, n} \lesssim_{n, d, \epsilon} \delta^{(d+1)/n - (1+\epsilon)}
\]
i.e. the \( \Omega \)-Kakeya maximal conjecture is true. □

6 Proof of Theorem 4

The proof of Theorem 4 follows Bourgain’s arithmetic argument for the classic Kakeya problem; this method relies on the following two results. The first one allow us to give an upper bound on the difference set \( A - B \).

Theorem 6 (Sum-difference Theorem). Fix any \( \delta > 0 \) and suppose that \( A, B \) are finite subset of \( \delta \mathbb{Z}^n \) such that \( \#A, \#B \leq N \). If \( G \subset A \times B \) satisfies
\[
\#\{a + b : (a, b) \in G\} \lesssim N
\]
then we have \( \#\{a - b : (a, b) \in G\} \leq N^{2 - \frac{1}{d}} \).
The second Theorem needed is due to Heath-Brown [8]: it states that if $S$ is a large subset of \{0, \ldots , M\} for $M$ large enough then $S$ contains an arithmetic progression of length 3.

**Theorem 7 (Heath-Brown).** There exists an integer $M_0$ such that if $M > M_0$ is a integer and if $S$ is a subset of \{0, \ldots , M\} such that 

$$\#S \geq \frac{M}{\log(M)^c}$$

then $S$ contains a subset of the form \{m, m+m', m+2m'\} $\subset S$. Here $c > 0$ is an absolute constant.

For the sake of clarity, we have decomposed the proof of Theorem 4 in two steps. We will denote by $C(A, \delta)$ the smallest number of balls of radius $\delta$ needed to cover the set $A$.

**Decomposition of the $\Omega$-Kakeya set**

To begin with, we may suppose that $\Omega$ is contained in a small spherical cap; concretely we suppose that for any $e = (e_1, \ldots, e_n) \in \Omega$ we have $e_n > \frac{1}{2}$. We fix then $d < d_\Omega$ arbitrarily close and we use Frostman's Lemma to obtain a probability $\mu$ such that $S_\mu \subset \Omega$ and also $|\mu|_d \leq 1$. We consider then a $\Omega$-Kakeya set $X$ and we suppose that $X$ is contained in [0, 1]$^n$. For any $e \in \Omega$ we will denote by $T_e$ a unit segment oriented along $e$ contained in $X$. Finally we fix $s > d_X$ and we will prove that we have

$$s \geq \frac{6}{11}d + 1.$$

We fix $\epsilon \in (0, 1)$ arbitrarily small and we let $\eta \in (0, 1)$ such that defining $\delta_k = 2^{-2^n}k$ we have for any $k \geq 1$,

$$\delta_k^{s+\epsilon} \leq \delta_k^{\eta b}. $$

Since $s > d_X$, for arbitrary large $k_0$, we can cover $X$ by a countable collection of balls $\{B_i\}_{i \in I}$ and such that for any $i \in I$, we have $\text{diam}(B_i) < \delta_k$ and $\sum_{i \in I} \text{diam}(B_i)^s < 1$. In addition, we take $k_0$ so large that we have

$$k_0 \max(\delta_k, \delta_k^\eta b) < 1. $$

We denote by $Y$ the union of the balls $\{B_i\}_{i \in I}$, i.e.

$$Y := \bigcup_{i \in I} B_i$$

and for $k \geq k_0$ we will denote by $I_k := \{i \in I_k : \delta < \text{diam}(B_i) \leq \delta\}$ and also $Y_k := \bigcup_{i \in I_k} B_i$. We can control the size of $\#I_k$.

**Claim 1.** We have $\#I_k \delta_k^{s+\epsilon} < 1$.

**Proof.** By definition of $I_k$ and since we have $\delta_k^{s+\epsilon} \leq \delta_k^{s}$ and $\sum_{i \in I} \text{diam}(B_i)^s < 1$, we obtain

$$\#I_k \delta_k^{s+\epsilon} \leq \#I_k \delta_k^{s} \leq \sum_{i \in I_k} \text{diam}(B_i)^s < 1.$$
step 1: refinement to a single scale

We wish to work at a single scale with respect to this covering. Hence we are going to exhibit a $k \geq k_0$ such that there is a specific subset $\Omega_k \subset \Omega$ adapted to the covering $\{B_i\}_{i \in I_k}$: on one hand $\Omega_k$ is large and on the other hand, for any $e \in \Omega_k$, the unit segment $T_e \subset X$ is well covered by the balls $\{B_i\}_{i \in I_k}$.

Claim 2. There exists $k \geq k_0$ and $\Omega_k \subset \Omega$ such that for any $e \in \Omega_k$,
\[ |T_e \cap Y_k|_1 \geq \frac{1}{k^2} \]
and also $\mu(\Omega_k) > \frac{1}{k^2}$.

Proof. If this is not the case, then for any $k \geq k_0$ we have
\[ \mu\left( \{e \in \Omega : |T_e \cap Y_k|_1 \geq \frac{1}{k^2}\} \right) \leq \frac{1}{k^2}. \]
Hence we have
\[ \mu\left( \{e \in \Omega : \exists k \geq k_0, |T_e \cap Y_k|_1 \geq \frac{1}{k^2}\} \right) \leq \sum_{k \geq k_0} \frac{1}{k^2} < \mu(\Omega) \]
and so there is $e \in \Omega$ such that $|T_e \cap Y_k|_1 < \frac{1}{k^2}$ for any $k \geq k_0$. This is not possible since $Y$ covers $X$ and so $T_e$ in particular.

step 2: slicing $\mathbb{R}^n$ at two scales

We fix such a $k$ and we let $\delta := \delta_k$. Recall that since $k \geq k_0$ and that we can choose $k_0$ arbitrarily large, the same is true for $k$ i.e. the integer $k$ can be chosen arbitrarily large. Also observe that by definition we have
\[ k \simeq \log \log(\delta^{-\eta}). \]

Now we fix two integers $N, M \in \mathbb{N}$ such that
\[ (N, M) \simeq (\delta^{\eta-1}, \delta^{-\eta}). \]
We are going to slice $\mathbb{R}^n$ at two different scales ($\delta$ and $\delta^\eta$) along the vector $(0, \ldots, 1)$. Precisely for $j \leq N$ and $m \leq M$, we define
\[ A_{j,m} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : j\delta + mN\delta \leq x_n \leq (j+1)\delta + mN\delta\} \]
and $A_j := \bigcup_{m \leq M} A_{j,m}$.

Claim 3. For any $e \in \Omega_k$ and $j \leq N$, we have $|T_e \cap A_j|_1 \simeq M\delta \simeq \frac{1}{N}$.

Proof. The claim comes from the fact that we have supposed that for any $e = (e_1, \ldots, e_n) \in \Omega$ we have $e_n > \frac{1}{N}$.
By definition of $\Omega_k$, we also have the following estimate
\[
\frac{1}{k^4} \leq \frac{\mu(\Omega_k)}{k^2} \leq \int_{\Omega_k} |Y_k \cap T_e|_1 d\mu(e) = \sum_{j \leq N} \int_{\Omega_k} |Y_k \cap T_e \cap A_j|_1 d\mu(e).
\]
We define then the subset $J \subset \{0, \ldots, N\}$ as
\[
J = \{j \leq N : \int_{\Omega_k} |Y_k \cap T_e \cap A_j|_1 \geq \frac{1}{2Nk^4}\}.
\]
The following claim states that this set $J$ is not too small in $\{0, \ldots, N\}$.

**Claim 4.** We have $\#J \gtrsim \delta^q N$.

**Proof.** The proof comes from a reverse Markov inequality. \hfill \Box

Now for each $j \in J$, we extract a subset $\Omega_{k,j}$ from $\Omega_k$ in the same fashion that we have extracted $\Omega_k$ from $\Omega$. The proof is the same than for Claim 2.

**Claim 5.** For any $j \in J$, there exists $\Omega_{k,j} \subset \Omega_k$ such that for any $e \in \Omega_{k,j}$,
\[
|T_e \cap Y_k \cap A_j| > \frac{|T_e \cap A_j|_1}{4k^4}
\]
and also $\mu(\Omega_{k,j}) > \frac{\mu(\Omega_k)}{k^2}$.

**step 3 : conclusion**

Observe that for $|j - j'| > 2$, the sets $Y_k \cap A_j$ and $Y_k \cap A_{j'}$ are separated by a distance greater than $2\delta$. Hence, on one hand we have
\[
\sum_{j \in J} C(Y_k \cap A_j, \delta) \lesssim C(Y_k, \delta) \lesssim \delta^{-s-\epsilon}.
\]

On the other hand, suppose that for any $j \in J$ we have
\[
C(Y_k \cap A_j, \delta) \gtrsim \delta^{\frac{4}{3}(2\eta-1)d}.
\]
In this case we obtain
\[
\sum_{j \in J} C(Y_k \cap A_j, \delta) \gtrsim \#J \times \delta^{\frac{4}{3}(2\eta-1)d} \gtrsim \delta^{\frac{4}{3}(2\eta-1)d + (2\eta-1)}.
\]
Since $\delta$ is small enough, we get $\frac{6}{11}(1 - 2\eta)d - 2\eta + 1 \leq s + \epsilon$. Taking $\epsilon$ and $\eta$ arbitrarily small we conclude that
\[
s \geq \frac{6}{11}d + 1.
\]

**Lower bound for $C(Y_k \cap A_j, \delta)$**

Hence we are left to prove that we have, for any $j \in J$, the following bound
\[
C(Y_k \cap A_j, \delta) \gtrsim \delta^{\frac{4}{3}(2\eta-1)d}.
\]
We will start by applying Heath-Brown’s Theorem and we will use thereafter the Sum-difference Theorem to obtain a bound on $C(Y_k \cap A_j, \delta)$. 

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step 1: application of Heath-Brown’s Theorem

For any $j \in J$ and any $e \in \Omega_{k,j}$, we consider the following subset of $\{0, \ldots, M\}$

$$K(e, j) := \{m \leq M : Y_k \cap T_e \cap A_{j,m} \neq \emptyset\}.$$  

The following claim states that $K(e, j)$ contains a lot of element in $\{0, \ldots, M\}$.

**Claim 6.** We have $\#K(e, j) \gtrsim \frac{M}{\log \log(M)}$.

**Proof.** We have

$$\#K(e, j) \times \delta \gtrsim |Y_k \cap T_e \cap A_j| > \frac{1}{4Nk^4}.$$  

Hence

$$\#K(e, j) \gtrsim \frac{M}{k^4} \simeq \frac{M}{\log \log(M)}.$$  

\qed

Since we can take $M$ arbitrary large and that $K(e, j)$ is quite large in $\{0, \ldots, M\}$, we are able to exhibit arithmetic progressions of three terms in $K(e, j)$ using the Theorem of Heath-Brown.

**Claim 7.** For any $j \in J$ and $e \in \Omega_{k,j}$, the set $K(e, j)$ contains a subset of the form

$$\{m, m + m', m + 2m'\} \subset K(e, j).$$  

Hence for any $j \in J$ and any $e \in \Omega_{k,j}$, there exists $a_e, b_e \in Y_k(n\delta) \cap T_e(n\delta) \cap A_j \cap \delta \mathbb{Z}^n$ such that

$$\frac{a_e + b_e}{2} \in Y_k(n\delta) \cap T_e(n\delta) \cap A_j \cap \delta \mathbb{Z}^n$$

and $a_e, b_e$ belong to different sets $A_{j,m}$. Observe also that the sets $A_{j,m}$ for different indices $m$ are at least distant of $\simeq \delta^q$. Thus if $\delta$ is small enough we have $|a_e - b_e| \gtrsim \delta^q$. Finally, we consider the sets $A = \{a_e : e \in \Omega_{k,j}\}$, $B = \{b_e : e \in \Omega_{k,j}\}$ and

$$G := \{(a_e, b_e) : e \in \Omega_{k,j}\} \subset A \times B.$$  

Recall that we have

$$A, B \subset \delta \mathbb{Z}^n.$$  

step 2: upper bound for $\#\{(a - b : (a, b) \in G\}$

We are going to give an upper bound and lower bound on $\#\{(a - b : (a, b) \in G\}$. Observe that the cardinal of $A, B$ and $\{a + b : (a, b) \in G\}$ is controlled by the covering number $C(Y_k \cap A_j, \delta)$. Hence a direct application of the Sum-difference Theorem yields an upper bound on $\#\{(a - b : (a, b) \in G)\}$

**Claim 8.** We have $\#\{(a - b : (a, b) \in G) \lesssim C(Y_k \cap A_j, \delta)^{\frac{1}{2}}.$
step 3: lower bound for \( \# \{ a - b : (a, b) \in G \} \)

Finally we are also able to provide a lower bound on \( \# \{ a - b : (a, b) \in G \} \) using the measure \( \mu \).

**Claim 9.** We have \( \delta^{(2\eta-1)d} \lesssim \# \{ a - b : (a, b) \in G \} \).

**Proof.** Since \( a_e \) and \( b_e \) are in the \( n\delta \)-neighbourhood of \( T_e \) and \( |a_e - b_e| > \frac{\delta}{\eta^k} \) for \( e \in \Omega_{k,j} \), it follows that balls roughly of radius \( \delta^{1-\eta} \) centred at the unit vectors \( \frac{a_e - b_e}{|a_e - b_e|} \) (for \( e \in \Omega_{k,j} \)) cover \( \Omega_{k,j} \). As \( \mu(\Omega_{k,j}) > \frac{1}{4^k} \), this implies

\[
\#(A - B) \gtrsim \frac{\delta^{(\eta-1)d}}{k^4} \gtrsim \delta^{(2\eta-1)d}.
\]

Hence for any \( j \in J \), we have

\[
C(\Lambda_k \cap A_j, \delta) \gtrsim \delta^{(2\eta-1)d}.
\]

This concludes the proof of Theorem 4.

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