Density operator approach for Landau problem quantum Hamiltonians

Isiaka Aremua\textsuperscript{a,\dagger}, Mahouton Norbert Hounkonnou\textsuperscript{b,‡} and Ezinvi Baloïtcha\textsuperscript{b,⋆}

\textsuperscript{a}Université de Lomé (UL)
Faculté Des Sciences (FDS), Département de Physique
B.P. 1515 Lomé TOGO
\textsuperscript{b}International Chair of Mathematical Physics and Applications
ICMPA-UNESCO Chair
University of Abomey-Calavi
072 B.P. 50 Cotonou, Republic of Benin
E-mail: \textsuperscript{dagger}claudisak@gmail.com, \textsuperscript{‡}norbert.hounkonnou@cipma.uac.bj, \textsuperscript{⋆}ezinvi.baloitcha@cipma.uac.bj.

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Abstract

In this work, the definition of the density operator on quantum states in Hilbert spaces and some of its aspects relevant in thermodynamics and information-theoretical entropy calculations are given. In this framework, a physical model describing an electron in a magnetic field is investigated. The so-called exotic Landau problem in noncommutative plane is also considered. Then, a model related to the fractional quantum Hall effect is revisited. Thanks to the completeness relations verified by the coherent states (CS) in these models, the thermodynamics is discussed by using the diagonal $P$-representation of the density operator. Specifically, the $Q$-Husimi distribution and the Wehrl entropy are determined.
1 Introduction

In quantum mechanics, the thermal density operator is powerfully used in order to represent ensembles of pure or mixed quantum states. See for e.g., [37, 21], and references therein. For the usual treatment of equilibrium in statistical mechanics using the Gibbs’s canonical distribution ([37, 14]), the normalized density operator is given by

\[ \rho = \frac{1}{Z} e^{-\beta H}, \]

where \( Z = \text{Tr}(e^{-\beta H}) \) is the partition function, \( \beta = 1/k_B T \); \( T \) is the temperature, and \( k_B \) the Boltzmann constant which, in SI units, has the value \( 1.3806503 \times 10^{-23} \text{J/K} \). The diagonal expansion of the density operator, known as the Glauber-Sudarshan (GS)-P-representation, was introduced independently by Glauber [22] and Sudarshan [43] for the harmonic oscillator coherent states (CS). It is given in GS CS \(|\alpha\rangle\) framework by [22, 13, 12]

\[ \rho = \int d^2\alpha \ P(\alpha)|\alpha\rangle\langle\alpha| \quad (1) \]

with \( P(\alpha) \) a quasiprobability distribution function.

In quantum information, for an ensemble, e.g., of qubits, the density operator was used to describe the informational content of the ensemble [37]. The quantum-mechanical phase-space distributions of the harmonic oscillator CS were shown to be useful in different situations. See for e.g., [22, 27, 14]. Particularly, in [5], the concepts of Husimi distribution [25] and Wehrl [46] entropy, needed in the generalized, Fisher, and Shannon informations measures [14], were discussed.

In [37], the density operator was built for photon-added Barut-Girardello CS in the cases of pseudoharmonic oscillator and generalized hypergeometric thermal CS, with the relevant statistical properties. In [35], a \( q \)-analogue of the diagonal representation of the density matrix, using \( q \)-boson CS, was derived, and the \( q \)-generalization of the density matrix self-reproducing property was discussed. Besides, a Glauber-Sudarshan \( P \)-representation of the density matrix and relevant issues related to the properties of the reproducing kernel were investigated for a construction of a dual pair of nonlinear CS for a model obeying a \( f \)-deformed Heisenberg algebra [7]. More recently [42], the density matrix of a quantum canonical ideal gas of a system in thermodynamic equilibrium was given in the generalized photon-added associated hypergeometric CS (GPAH-CS), with a discussion of statistical analysis in the context of photon-added CS for shape invariant potentials [42].

The behavior of an electron in an external magnetic field was extensively studied [29]-[9]. This implied a great interest to other similar physical systems describing, for instance, the quantum Hall effect [23]. In some previous works, this physical model was also proved to show an interesting application [1] of the Tomita-Takesaki modular theory [11]-[15]. In [8], the density operator was achieved in the Barut-Girardello CS representation for Landau levels of a gas of spinless charged particles, subject to a perpendicular magnetic field confined in a harmonic potential; the Husimi distribution and Wehrl entropy were investigated. Recently [9], the Hilbert-Schmidt operators and the Tomita-Takesaki modular theories were recast for noncommutative quantum mechanics formulation, and the density matrix formalism related to von Neumann algebras, displayed by the Landau problem [1], was revisited.

Our present contribution paper is organized as follows:

- First, we recast the density operator theory in the framework of CS by giving some basic preliminaries and usual definitions.
- Then, we apply the density operator approach to a physical model describing the motion of a charged particle on the flat plane \( xy \) in the presence of a constant magnetic field along the \( z \)-axis. First, the study is performed in the noncommutative quantum mechanics formalism [39]. Next, it is achieved in the context of modular theory based on Hilbert-Schmidt operators for which thermal CS were constructed [1].
2 Preliminaries-Density operator and coherent states

This section is devoted to some basic facts about the density operator in the CS setting, the Wehrl entropy and the Husimi distribution. More details can be found in [14, 46, 5, 12, 25, 37].

2.1 Coherent states

Coherent states (CS) were introduced for the first time by Schrödinger [41] in 1926 for the quantum harmonic oscillator as the specific quantum state which has dynamical behavior similar to that of the classical harmonic oscillator. They were rediscovered by Klauder [26] in a mathematical physics application, and by Glauber [22] and Sudarshan [43] in the context of quantum optics at the beginning of the 1960’s. CS are useful in condensed matter physics, quantum optics, quantum field theory, quantization problems, quantum information, etc. [28]-[21, 6]. Apart from the canonical or harmonic oscillator CS, CS are also generated as the lowerin g operator eigenstate (the so-called CS of the Barut-Girardello kind) or by applying the displacement operator on a ground state (Klauder-Perelomov CS) or CS of the Gazeau-Klauder kind, including the nonlinear CS, squeezed states and deformed CS, see [27, 3, 16, 21]. In the group theory approach, they are built as the orbit under the action of a group representation [36, 3].

CS can be defined over complex domains in the Hilbert space $\mathcal{H} = \text{span}\{\phi_m, m \in \mathbb{N}\}$, which realizes at the mathematical side, the skeleton of quantum theories, as [3]

$$|z\rangle = (\mathcal{N}|z\rangle)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} |\phi_m\rangle, \quad z = re^{i\theta}$$

(2)

where $\{\rho(m)\}_{m=0}^{\infty}$ is a sequence of non-zero positive numbers chosen so as to ensure the convergence of the sum in a non-empty open subset $\mathcal{D}$ of the complex plane, $\mathcal{N}|z\rangle$ is the normalization factor ensuring that $\langle z|z\rangle = 1$.

The resolution of the identity is given by

$$\int_{\mathcal{D}} |z\rangle\langle z|d\mu = I_{\mathcal{H}},$$

(3)

where $d\mu$ is an appropriate chosen measure and $I_{\mathcal{H}}$ the identity operator on the Hilbert space $\mathcal{H}$.

2.2 The $P$-distribution function

The diagonal expansion [12] of the normalized canonical density operator is

$$\rho = \int_{\mathcal{D}} d\mu|z\rangle P(|z|^2) \langle z|, \quad \int_{\mathcal{D}} d\mu P(|z|^2) = 1$$

(4)

where the $P$-distribution function $P(|z|^2)$ satisfying the normalization to unity condition must be determined.

2.3 The $Q$-Husimi function or distribution

The Husimi distribution [25] as a function in the phase space, in general viewed as Gaussian smoothing of the Wigner function, is derived by use of the expected value of the density operator in a basis of CS [27].

Taking the normalized density operator $\rho$ given in the basis $\{|n\rangle\}_{n=0}^{\infty}$ by

$$\rho = \frac{1}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta E_n}|n\rangle \langle n|$$

(5)

with $Z(\beta)$ the partition function, the $Q$-Husimi function or distribution [25] is provided as

$$Q(|z|^2) = \langle z|\rho|z\rangle, \quad \text{with} \quad \int_{\mathcal{D}} d\mu Q(|z|^2) = 1$$

(6)
assuring the normalization to unity of the $Q$-Husimi function.

The normalization of the density operator in the CS basis leads to

$$\text{Tr} \rho = \int_D d\mu \langle z|\rho|z \rangle = 1. \tag{7}$$

### 2.4 The Wehrl entropy

The Wehrl entropy or the “classical” entropy associated with a quantum system is the entropy of the probability distribution in phase-space, corresponding to the Husimi $Q$-function in terms of CS (see also [5, 30, 14, 37] and references therein). It is of great importance for the measure of localization in the phase-space and constitutes a powerful tool in statistical physics. It is given by

$$W := -\int \frac{dx dp}{2\pi \hbar} \mu(x, p) \ln \mu(x, p) \tag{8}$$

where $\mu(x, p)$ in the CS basis is such that: $\mu(x, p) = \langle z|\rho|z \rangle$ being the $Q$-Husimi function or distribution corresponding to “semi-classical” phase-space distribution function associated to the density matrix $\rho$, with $z = \sqrt{\frac{m\omega}{2\hbar}}x + i\sqrt{\frac{m\omega}{2\hbar}}p$.

From (3), taking $D = \mathbb{C}, d\mu = \frac{dz}{\pi}$, we have

$$I_H = \int_D |z\rangle\langle z| \frac{dz}{\pi} = \int \frac{dx dp}{2\pi \hbar} |x, p\rangle\langle x, p| \tag{9}$$

such that (8) takes the form

$$W = \int_D \frac{dz}{\pi} \langle z|\rho|z \rangle \ln(\langle z|\rho|z \rangle). \tag{10}$$

Both $Q$-Husimi function and Wehrl entropy are used in information-theoretical entropy of some quantum oscillators, and also in Fisher’s and Shannon information measures in statistical mechanics [14, 37].

### 3 Density operator approach in CS for the exotic Landau problem

The Landau problem [29, 17, 9] is related to the motion of a charged particle on the flat plane $xy$ in the presence of a constant magnetic field along the $z$-axis. In metals, the electrons occupy many Landau levels $E_n = \hbar \omega_c (n + \frac{1}{2})$, each level being infinitely degenerate, with $\omega_c = eB/Mc$, the cyclotron frequency, which correspond to the kinetic energy levels of electrons, and are those of the one-dimensional harmonic oscillator. Here, we deal with the Landau exotic problem [48]. We use the formalism developed in [24, 39] to construct CS for this model. From this setup, we derive the related density operator.

#### 3.1 The model

Let us first make a brief review of the main features of “exotic” particles. An exotic particle is a particle moving in a planar electromagnetic field $E$ and $B$ [assumed static for simplicity] described by the equations

$$M^* \dot{x}^i = p^i - M\epsilon\epsilon^{ij}E^j, \quad \dot{p}^i = eB\epsilon^{ij}\dot{x}^j + eE^i \tag{11}$$
with $\varepsilon^{ij}$ the components of the antisymmetric tensor normalized by $\varepsilon^{12} = 1$, where $M, e$ and $\theta$ are the mass, charge and noncommutative parameter, respectively, and $M^* = (1 - eB\theta)M$ is the effective mass. The equations (11) derive from the symplectic form and Hamiltonian [48],

$$\Omega = dp^i \land dx^i + \frac{\theta}{2} \varepsilon^{ij} dp^i \land dp^j + \frac{eB}{2} \varepsilon^{ij} dx^i \land dx^j, \quad \mathcal{H} = \frac{p^2}{2M} + V(x)$$  \hspace{1cm} (12)$$

respectively, through the “exotic” Poisson brackets

$$\{x^i, x^j\} = \frac{\theta}{1 - eB\theta} \varepsilon^{ij}, \quad \{x^i, p^j\} = \frac{\delta^{ij}}{1 - eB\theta}, \quad \{p^i, p^j\} = \frac{eB}{1 - eB\theta} \varepsilon^{ij}$$  \hspace{1cm} (13)$$

where the non-critical regime, $eB\theta \neq 1$, is assumed. When the magnetic field takes the critical value

$$B = B_c = \frac{1}{e\theta}$$  \hspace{1cm} (14)$$

the system becomes singular: the determinant of the symplectic matrix is given by $\det(\Omega_{\alpha\beta}) = (M^*/M)^2 = 0$, and consistency requires the Hall law,

$$p^i = Me\theta \varepsilon^{ij} E^j, \quad \dot{x}^i = \varepsilon^{ij} E^j B,$$  \hspace{1cm} (15)$$

Consider the physical Hamiltonian, given in [12], related to the exotic Landau problem [48], where the “exotic” Poisson brackets (13) with $i,j = 1,2$ hold. The Hamiltonian (12) writes with chiral coordinates $X_{\pm}$ which describe the system and given by [4,48]:

$$\{X_{\pm}^i, X_{\pm}^j\} = \frac{1}{eB} \varepsilon^{ij}, \quad \{X_{\pm}^i, X_{\mp}^j\} = 0, \quad \{X_{\mp}^i, X_{\mp}^j\} = \frac{1}{eB(1 - eB\theta)} \varepsilon^{ij}$$  \hspace{1cm} (16)$$

where the extension of the original coordinates can be performed as

$$X_{\pm}^i = x^i \pm \frac{1}{eB} \varepsilon^{ij} \lambda_j^\pm, \quad \dot{X}_{\pm}^i = x^i \pm \frac{1}{eB} \varepsilon^{ij} \lambda_j^\mp$$  \hspace{1cm} (17)$$

with $\lambda_j^\pm = p_i - m_i v_i$ measuring the difference between the canonical ($p_i$) and the mechanical ($m_i v_i$) momenta [48] and satisfying

$$\{\lambda_j^+, \lambda_j^+\} = \frac{eB}{1 - eB\theta} \varepsilon^{ij}, \quad \{\lambda_j^-, \lambda_j^-\} = -eB \varepsilon^{ij}.$$  \hspace{1cm} (18)$$

Then, the Hamiltonian (12) splits into two uncoupled systems both with 2d phase spaces as follows [4,48]:

$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- = \{-eE.X_+\} + \left\{\frac{(eB)^2}{2M} X_+^2 - eE.X_-\right\}$$  \hspace{1cm} (19)$$

where the symplectic form writes as

$$\Omega = \Omega_+ + \Omega_- = \left\{\frac{eB}{2} (\varepsilon^{ij} dX_+^i \land dX_+^j)\right\} - \left\{(1 - eB\theta) \frac{eB}{2} (\varepsilon^{ij} dX_-^i \land dX_-^j)\right\}.$$  \hspace{1cm} (20)$$

### 3.2 The quantum Hamiltonian

We investigate the quantum Hamiltonian in the purely magnetic case $E = 0$ [10]. Let us assume that $V$ (but not $B$) vanishes, so that the Hamiltonian (12) reduces to

$$\mathcal{H} = \frac{P^2}{2M}$$  \hspace{1cm} (21)$$
where we deal with the non-critical regime, $eB\theta \neq 1$.

The commutators associated to the relations (13) are given by

$$
[x^i, x^j] = \frac{i\theta}{1-eB\theta} \epsilon^{ij}, \quad [x^i, p^j] = \frac{i\beta^{ij}}{1-eB\theta}, \quad [p^i, p^j] = i\frac{eB}{1-eB\theta} \epsilon^{ij}.
$$

(22)

In order to quantize the Hamiltonian physical model, we consider, by assuming the relation $1-eB\theta > 0$, a more convenient system of coordinates by introducing a set of chiral complex coordinates $Z_{\pm}$ defined, by setting

$$
\mathcal{X}_+^i = \sqrt{\frac{eB\theta}{1-eB\theta}} X_+^i, \quad \mathcal{X}_-^i = \sqrt{eB\theta} X_-^i, \quad i = 1, 2,
$$

(23)
as

$$
Z_+ = \sqrt{\frac{1-eB\theta}{2\theta}} [\mathcal{X}_+^1 - i\mathcal{X}_+^2], \quad Z_- = \sqrt{\frac{1-eB\theta}{2\theta}} [\mathcal{X}_+^1 + i\mathcal{X}_+^2].
$$

(24)

Then, from the relations (16) satisfied by the set $\{X_\pm^i, X_\pm^j, i, j = 1, 2\}$, it follows

$$
\{Z_+, \tilde{Z}_+\} = -i, \quad \{Z_+, \tilde{Z}_-\} = 0 = \{\tilde{Z}_+, Z_-\}, \quad \{\tilde{Z}_-, Z_-\} = -i.
$$

(25)

Next, denote by $\{\hat{\mathcal{Z}}_\pm, \hat{\mathcal{Z}}_\pm^\dagger\}$, where $\hat{\mathcal{Z}}_+^\dagger \equiv \tilde{Z}_+$, the corresponding operators of the chiral coordinates $\{Z_\pm, \tilde{Z}_\pm\}$. Then, the classical structure (25) is replaced by commutators as follows:

$$
\{A, B\} \rightarrow \frac{1}{i}\{\hat{A}, \hat{B}\}, \quad \hbar = 1
$$

(26)

where $\hat{A}, \hat{B}$ denote the quantized variables such that

$$
[\hat{\mathcal{Z}}_+, \hat{\mathcal{Z}}_+^\dagger] = 1 = [\hat{\mathcal{Z}}_-, \hat{\mathcal{Z}}_-^\dagger], \quad [\hat{\mathcal{Z}}_-, \hat{\mathcal{Z}}_+^\dagger] = 0 = [\hat{\mathcal{Z}}_+, \hat{\mathcal{Z}}_-^\dagger], \quad [\hat{\mathcal{Z}}_+, \hat{\mathcal{Z}}_-] = 0.
$$

(27)

The relations (27) displaying that $\{\hat{\mathcal{Z}}_\pm, \hat{\mathcal{Z}}_\pm^\dagger\}$ form an irreducible set of operators on the chiral boson Fock spaces $\mathcal{F}_\pm = \{|n_{\pm}\rangle\}_{n_{\pm}=0}^{\infty}$, suggest to identify the system $\{\hat{\mathcal{Z}}_\pm, \hat{\mathcal{Z}}_\pm^\dagger\}$ with the annihilation and creation operators relevant in the formalism of noncommutative quantum mechanics developed in [39] and acting on the states $|n_+, n_-\rangle = |n_+\rangle|n_-\rangle$, $n_{\pm} = 0, 1, 2, \ldots$, as

$$
\tilde{Z}_+|n_+, n_-\rangle = \sqrt{n_+|n_- - 1, n_-\rangle} \quad \tilde{Z}_+^\dagger|n_+, n_-\rangle = \sqrt{n_++1}|n_+ + 1, n_-\rangle, \quad (28)
$$

$$
\tilde{Z}_-|n_+, n_-\rangle = \sqrt{n_--|n_- - 1, n_-\rangle} \quad \tilde{Z}_-^\dagger|n_+, n_-\rangle = \sqrt{n_-+1}|n_+, n_- + 1\rangle. \quad (29)
$$

We have

$$
|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!n_-!}} (\tilde{Z}_+^\dagger)^{n_+} (\tilde{Z}_-^\dagger)^{n_-} |0\rangle\langle 0|
$$

(30)

where $\tilde{Z}_+^\dagger$ may have an action on the right by $\tilde{Z}_-$ on $|0\rangle\langle 0|$. $||n_+, n_-|| = 1$ and $|0\rangle\langle 0|$ stands for the vacuum state on $\mathcal{H}_q$ i.e. the space of Hilbert-Schmidt operators acting on the noncommutative configuration (Hilbert) space $\mathcal{H}_c$ (isomorphic to the boson Fock space). $\mathcal{H}_q$ is defined as:

$$
\mathcal{H}_q = \{\psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}(\mathcal{H}_c), \text{tr}_c(\psi(\hat{x}_1, \hat{x}_2)^\dagger, \psi(\hat{x}_1, \hat{x}_2)) < \infty\}
$$

(31)

$\hat{x}_1, \hat{x}_2$ being the coordinates of noncommutative configuration space. $\mathcal{H}_q$ is endowed with the following inner product

$$
(\psi(\hat{x}_1, \hat{x}_2), \phi(\hat{x}_1, \hat{x}_2)) = \text{tr}_c(\psi(\hat{x}_1, \hat{x}_2)^\dagger, \phi(\hat{x}_1, \hat{x}_2))
$$

(32)
where $tr_c$ stands for the trace over $\mathcal{H}_c$, $\mathcal{B}(\mathcal{H}_c)$ being the set of bounded operators on $\mathcal{H}_c$.

Then, the eigenvalues of the Hamiltonian $\hat{H}$ with the following chiral decomposition
\[
\hat{H} = \hat{H}_+ \otimes I_{F_-} + I_{F_+} \otimes \hat{H}_-
\]  
where $I_{F_\pm}$ are the identity operators on the chiral boson Fock spaces $F_\pm = \{|n_\pm\rangle\rangle_0$, respectively, are given by
\[
E_{n_+,n_-} = \hbar \omega_c \left( n_+ + \frac{1}{2} \right) + \hbar \omega_c \left( n_- + \frac{1}{2} \right)
\]  
where \( \omega_c = \frac{\epsilon_B}{M^*} \), \( \omega_c^* = \frac{\epsilon_B}{M^*} \), $M^* = (1 - eB\theta)M$, with associated eigenstates given in (30).

With the help of the operators $\{\hat{Z}_\pm, \hat{\bar{Z}}_\pm\}$ provided in [27]-[30], since there exists a set of eigenstates $|z_\pm\rangle$ satisfying
\[
\hat{Z}_\pm|z_\pm\rangle = z_\pm|z_\pm\rangle, \quad \langle z_\pm|\hat{\bar{Z}}_\pm = \langle z_\pm|\bar{z}_\pm
\]  
having complex eigenvalues $z_\pm$ with
\[
|z_\pm\rangle = e^{-\frac{|z_\pm|^2}{2}} e^{z_\pm \hat{\bar{Z}}_\pm} |0\rangle
\]  
given in terms of the chiral Fock basis and provided by the Baker-Campbell-Hausdorff identity
\[
e^{\{\hat{Z}_\pm, \hat{\bar{Z}}_\pm\}} = e^{-\frac{|z_\pm|^2}{2}} e^{z_\pm \hat{\bar{Z}}_\pm} e^{-\bar{z}_\pm \hat{Z}_\pm},
\]  
the CS of the noncommutative plane related to the quantum Hamiltonian (33) are given by
\[
|z_\pm\rangle = |z_+\rangle|z_-\rangle = e^{-\frac{1}{2}(|z_+|^2+|z_-|^2)} \sum_{n_+,n_-=0}^{\infty} \frac{z_+^{n_+}z_-^{n_-}}{\sqrt{n_+!n_-!}} |n_+\rangle|n_-\rangle.
\]  
The identity operator $I_q$ on $\mathcal{H}_q$ writes in terms of the states $|n_+,n_-\rangle$ as follows:
\[
I_q = \sum_{n_+,n_-=0}^{\infty} |n_+\rangle|n_-\rangle \langle n_-| \langle n_+|.
\]  

**Proposition 3.1** The CS $|z_\pm\rangle$ satisfy the resolution of the identity [24]
\[
\frac{1}{\pi^2} \int_{\mathbb{C}^2} |z_\pm\rangle(z_\pm|d^2z_+d^2z_- \equiv I_q
\]  
where the identity operator on $\mathcal{H}_q$ is given by [39]
\[
I_q = \frac{1}{\pi} \int_{\mathbb{C}} dz \bar{z}|z\rangle e^{\bar{z} \bar{\partial}_z} |z\rangle.
\]  

**Proof.** See [24].

\[\square\]

### 3.3 Statistical properties

In this section, we will carry out a discussion on the statistical properties of the CS $|z_+,z_-\rangle$ for the Landau exotic problem. We use the results issued from the paragraph 3.2 as key ingredients to construct the diagonal $P$-representation, derive diagonal elements of the density operator $\rho$ characterizing the probability distribution on the states of a physical system, and examine its physical properties (see for e.g. [8] and references listed therein).

Considering that the quantum system obeys the canonical distribution, let us take the partition function $Z$ as that of a composite system made of two independent systems such that is the product of the partition functions of the components, i.e. $Z = Z_+ Z_-$. 

6
Proposition 3.2  The diagonal elements of the normalized density operator $\rho = \frac{1}{Z} e^{-\beta H}$ in the CS $|z_+, z_-\rangle$ representation, also known as the $Q$-distribution function or the $Q$-Husimi distribution [32], are derived as

$$
(z_+, z_- | z_+, z_-) = \frac{1}{\bar{n} + 1} e^{-\frac{1}{\bar{n}}} |z_+|^2 \times \frac{1}{\bar{n}^* + 1} e^{-\frac{1}{\bar{n}^*}} |z_-|^2
$$

$$
= Q(|z_+|^2)Q(|z_-|^2)
$$

(42)

with $\bar{n} = [e^{\beta \omega_c} - 1]^{-1}$ and $\bar{n}^* = [e^{\beta \omega_c^*} - 1]^{-1}$ being the corresponding thermal expectation values of the number operator (the Bose-Einstein distribution functions for oscillators with angular frequencies $\omega_c$ and $\omega_c^*$, respectively) or the thermal mean occupancy for harmonic oscillators with the angular frequencies $\omega_c$ and $\omega_c^*$, respectively.

The quantity $|(n_+, n_- | z_+, z_-)|^2$ is such that

$$
|(n_+, n_- | z_+, z_-)|^2 = e^{-|z_+|^2} \frac{|z_+|^{2n_+}}{n_+!} e^{-|z_-|^2} \frac{|z_-|^{2n_-}}{n_-!}
$$

(43)

displaying that the CS $|z_+, z_-\rangle$ obey the photon-number Poisson distribution corresponding to a Mandel parameter $Q = 0$ [32]. The right-hand side of (42) corresponds to the product of two harmonic oscillators Husimi distributions.

Proof. We have

$$
(z_+, z_- | z_+, z_-) = \frac{1}{Z} \sum_{n_+, n_-=0}^{\infty} e^{-\beta H} |(n_+, n_- | z_+, z_-)|^2
$$

$$
= \frac{1}{Z} \sum_{n_+, n_-=0}^{\infty} e^{-\beta H} e^{-\beta H} e^{-|z_+|^2} \frac{|z_+|^{2n_+}}{n_+!} e^{-|z_-|^2} \frac{|z_-|^{2n_-}}{n_-!}, \quad \text{with } Z = Z_+ Z_-\n$$

$$
= \left\{ \frac{1}{Z_+} e^{-\frac{\beta \omega_c}{2}} e^{-|z_+|^2} \left[ e^{-\beta \omega_c} |z_+|^2 \right] \right\} \left\{ \frac{1}{Z_-} e^{-\frac{\beta \omega_c^*}{2}} e^{-|z_-|^2} \left[ e^{-\beta \omega_c^*} |z_-|^2 \right] \right\}
$$

(44)

where

$$
\frac{1}{Z_+} = \left[ \frac{e^{-\beta \omega_c}}{1 - e^{-\beta \omega_c}} \right]^{-1}, \quad \text{and} \quad \frac{1}{Z_-} = \left[ \frac{e^{-\beta \omega_c^*}}{1 - e^{-\beta \omega_c^*}} \right]^{-1}.
$$

(45)

Thereby,

$$
(z_+, z_- | z_+, z_-) = \left[ 1 - e^{-\beta \omega_c} \right] e^{-(1-e^{-\beta \omega_c})|z_+|^2} \times \left[ 1 - e^{-\beta \omega_c^*} \right] e^{-(1-e^{-\beta \omega_c^*})|z_-|^2}
$$

$$
= Q(|z_+|^2)Q(|z_-|^2).
$$

(46)

The variables changes $r_+ = \left[ 1 - e^{-\beta \omega_c} \right]^{1/2} |z_+|$ and $r_- = \left[ 1 - e^{-\beta \omega_c^*} \right]^{1/2} |z_-|$ with $d^2r = rd^r d\varphi$, $r \in [0, \infty)$, $\varphi \in (0, 2\pi)$ and the help of the resolution of the identity [10], lead to

$$
Tr\rho = \frac{1}{\pi^2} \int_{C^2} d^2z_+ d^2z_- (z_+, z_- | z_+, z_-) = 1,
$$

(47)

where we have used the following integral

$$
\int_0^\infty \frac{1}{n_{\pm}!} 2^{n_{\pm}+1} n_{\pm}! e^{-n_{\pm}^2} = 1,
$$

(48)

ensuring that the normalization condition of the density matrix is accomplished.

According to the normalized density operator expression

$$
\rho = \left\{ \frac{e^{-\beta \omega_c}}{Z_+ Z_-} \right\} \left\{ \sum_{n_+=0}^{\infty} e^{-\beta \omega_c n_+} \sum_{n_-=0}^{\infty} e^{-\beta \omega_c^* n_-} \right\} |n_+, n_-\rangle \langle n_+, n_-|
$$

(49)
we have
\[
(n_+, n_- | \rho | n_+, n_-) = \frac{1}{\tilde{n} + 1} \left( \frac{\tilde{n}}{\tilde{n} + 1} \right)^{n_+} \times \frac{1}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-}.
\] (50)

The diagonal expansion of the density operator \(\rho\) in the CS \(|z_+, z_-\rangle\) is given with the help of the probability \(P(|z_+|^2, |z_-|^2)\) as follows:
\[
\rho = \frac{1}{\pi^2} \int_{C_2} d^2z_+ d^2z_- P(|z_+|^2, |z_-|^2) |z_+, z_-\rangle\langle z_+, z_-|.
\] (51)

For the Glauber CS \(|\alpha\rangle\) of the harmonic oscillator the expansion (51) is called the Glauber-Sudarshan \(P\)-representation of the density operator [13, 21].

**Proposition 3.3** The \(P\)-distribution function \(P(|z_+|^2, |z_-|^2) := P(|z_+|^2)P(|z_-|^2)\), is given, by taking \(n_+ = s_+ - 1, n_- = s_- - 1\) and using the resolution of the identity (50), as
\[
P(|z_+|^2, |z_-|^2) = \frac{1}{\tilde{n}} \frac{\tilde{n}}{\tilde{n} + 1} \frac{\tilde{n}^*}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-} = \frac{1}{\tilde{n}^*} e^{-\frac{1}{\tilde{n}^*}|z_+|^2} \frac{\tilde{n}}{\tilde{n} + 1} e^{-\frac{1}{\tilde{n}}|z_-|^2}
\] (52)

where the following Meijer’s \(G\)-function and Mellin inversion theorem [22, 23]
\[
\int_0^\infty dxx^{-1} G_{m,n}^{p,q} \left( ax \begin{pmatrix} a_1, \ldots, a_n; a_{n+1}, \ldots, a_p \end{pmatrix} b_1, \ldots, b_m; b_{m+1}, \ldots, b_q \right) = \frac{1}{\alpha^n} \prod_{j=m+1}^p \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s) \prod_{j=m+1}^p \Gamma(a_j + s)
\] have been used.

**Proof.** Starting from (51), and using the results (50) and (13) together, we obtain, by setting
\[
\mathcal{P}(|z_+|^2, |z_-|^2) = P(|z_+|^2, |z_-|^2) G_{0,1}^1 \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-} = \int_{C_2} d^2z_+ d^2z_- \frac{\tilde{n}^*}{\pi^2} \frac{\tilde{n}^*}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-} = \int_{C_2} d^2z_+ d^2z_- \frac{\tilde{n}^*}{\pi^2} \frac{\tilde{n}^*}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-} = \int_{C_2} d^2z_+ d^2z_- \frac{\tilde{n}^*}{\pi^2} \frac{\tilde{n}^*}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-}.
\] (53)

Then, taking \(n_+ = s_+ - 1, n_- = s_- - 1\), we get, after performing the angular integration,
\[
P(|z_+|^2, |z_-|^2) = \frac{1}{\tilde{n}} \frac{\tilde{n}^*}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-} = \int_{C_2} d^2z_+ d^2z_- \frac{\tilde{n}^*}{\pi^2} \frac{\tilde{n}^*}{\tilde{n}^* + 1} \left( \frac{\tilde{n}^*}{\tilde{n}^* + 1} \right)^{n_-}.
\] (54)

Thus, using the connection between the generalized hypergeometric function and the Meijer’s \(G\)-function [34, 37] leads to
\[
P(|z_+|^2, |z_-|^2) = \frac{1}{\tilde{n}^*} F_0(\begin{pmatrix} -\tilde{n}^* - 1 \end{pmatrix} |z_+|^2) \times \frac{1}{\tilde{n}^*} F_0(\begin{pmatrix} -\tilde{n}^* - 1 \end{pmatrix} |z_-|^2)
\]
\[
P(|z_+|^2, |z_-|^2) = \frac{1}{\tilde{n}^*} e^{-\frac{1}{\tilde{n}^*}|z_+|^2} \frac{1}{\tilde{n}^*} e^{-\frac{1}{\tilde{n}^*}|z_-|^2}.
\] (56)
Proposition 3.4  The related Wehrl entropy given by
\[ W = -\frac{1}{\pi} \left\{ -\frac{1}{\pi} \int_{\mathbb{C}^2} (z_+, z_- | \rho | z_+, z_-) \ln \{(z_+, z_- | \rho | z_+, z_-)\} d^2z_+ d^2z_- \right\} \] (57)
is obtained as
\[ W = \left[ 1 - \ln \left( 1 - e^{-\beta \hbar \omega_c^z} \right) \right] \left[ 1 - \ln \left( 1 - e^{-\beta \hbar \omega_c^z} \right) \right]. \] (58)

Proof. From the Husimi distribution, we get
\[ W = -\frac{1}{\pi} \left\{ -\frac{1}{\pi} \int_{\mathbb{C}^2} (z_+, z_- | \rho | z_+, z_-) \ln \{(z_+, z_- | \rho | z_+, z_-)\} d^2z_+ d^2z_- \right\} \]
\[ = \left\{ -\frac{1}{\pi} \int_{\mathbb{C}} Q(|z_+|^2) \ln \left( Q(|z_+|^2) \right) d^2z_+ \right\} \]
\[ = \left[ 1 + \ln \left( \frac{1}{e^{\beta \hbar \omega_c^z} - 1} \right) \right] \left[ 1 + \ln \left( \frac{1}{e^{\beta \hbar \omega_c^z} - 1} \right) \right] \] (59)

where the following variables changes \( u_+ = \frac{1}{n+1} r_+^2 \) and \( u_- = \frac{1}{n+1} r_-^2 \), have been performed. Thereby,
\[ W = \left[ 1 - \ln \left( 1 - e^{-\beta \hbar \omega_c^z} \right) \right] \left[ 1 - \ln \left( 1 - e^{-\beta \hbar \omega_c^z} \right) \right]. \] (60)

\[ \square \]

3.4 A bit on the Landau diamagnetism

By following the approach of [15], we take the expression of the partition function for a cylindrical body of length \( L \), radius \( R \) and of volume \( V \), which is oriented along the \( z \)-direction in noncommutative coordinates, to be
\[ Z = \frac{V \beta \hbar \omega_c^z}{\lambda^3} \frac{1}{2 \sinh \{\beta \hbar \omega_c^z/2\}} \] (61)

where \( \lambda = (2\pi \hbar^2/M)^{1/2} \) is the thermal wavelength.

Taking the expressions for the free energy, the magnetization and the susceptibility in the standard definitions, i.e.
\[ F = -\frac{n}{\beta} \ln Z, \quad \mathcal{M} = -\frac{\partial F}{\partial B}, \quad \chi = \frac{1}{n} \frac{\partial \mathcal{M}}{\partial B}, \] (62)
respectively, we get the modified quantities for the exotic Landau model as follows:

\[ F = -\frac{n}{\beta} \left[ \ln \frac{V}{\lambda^3} + \ln \frac{\beta \hbar \omega_c^z}{2} (1 - eB\theta) - \ln \left( \sinh \left\{ \frac{\beta \hbar \omega_c^z}{2} (1 - eB\theta) \right\} \right) \right], \]
\[ \mathcal{M} = \frac{\hbar e}{Mc} \left( 1 - 2eB\theta \right) \left[ \frac{1}{\beta \hbar \omega_c^z (1 - eB\theta)} - \frac{1}{2} \coth \left\{ \frac{\beta \hbar \omega_c^z}{2} (1 - eB\theta) \right\} \right], \] (63)

\[ \chi = -\frac{2 \hbar e}{Mc} e\theta \left[ \frac{1}{\beta \hbar \omega_c^z (1 - eB\theta)} - \frac{1}{2} \coth \left\{ \frac{\beta \hbar \omega_c^z}{2} (1 - eB\theta) \right\} \right] \]
\[ \times \left[ \frac{1}{(\beta \hbar \omega_c^z (1 - eB\theta))^2} + \frac{1}{4} \left( 1 - \coth^2 \left\{ \frac{\beta \hbar \omega_c^z}{2} (1 - eB\theta) \right\} \right) \right]. \] (64)

In the high temperature limit, \( \beta \ll 1 \) i.e. \( x = \beta \hbar \omega_c^z (1 - eB\theta) \ll 1 \), we obtain
\[ \chi = -\frac{1}{3} \left( \frac{\hbar e}{2Mc} \right)^2 [1 + 6x + 6x^2] \] (65)
with $\kappa = -eB\theta$. Since $1 - eB\theta > 0$ in our model, taking the noncommutativity parameter $\theta$ positive such that $eB\theta < 1$, the system is then diamagnetic, except for the values $-0.8 < \kappa < -0.2$ where $\chi$ becomes positive.

The free energy, magnetization and susceptibility derived in (63) and (65), respectively, are found to be standard ones when

1. $\kappa = -eB\theta \ll 1$, i.e.

$$F \approx -\frac{n}{\beta} \left[ \ln \frac{V}{\lambda^3} + \ln \frac{\beta \hbar \omega_c}{2} - \ln \left( \sinh \left( \frac{\beta \hbar \omega_c}{2} \right) \right) \right], \quad M \approx n \frac{\hbar e}{M_c} \left[ \frac{1}{\beta \hbar \omega_c} - \frac{1}{2} \coth \left( \frac{\beta \hbar \omega_c}{2} \right) \right];$$

(66)

2. $\kappa = -eB\theta \rightarrow 0$, i.e.

$$\chi \approx -\frac{1}{3} \left( \frac{\hbar e}{2M_c} \right)^2 \beta$$

(67)

which is the usual Landau diamagnetism.

4 Density operator approach in CS related to the harmonic oscillator thermal state

We start by sketching key ingredients from [11, 1, 44] as needed for this section.

**Definition 4.1** Consider the unitary operator $U(x, y)$ on $\mathcal{H}$ given by

$$(U(x, y)\Phi)(\xi) = e^{-ix(\xi - y/2)}\Phi(\xi - y),$$

(68)

for $x, y, \xi \in \mathbb{R}$, where $U(x, y) = e^{-i(xQ + yP)}$, with $[Q, P] = i\hbar \mathcal{I}$, and the Wigner transform given by

$$W : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(\mathbb{R}^2, dxdy) = \tilde{\mathcal{H}}, \quad (WX)(x, y) = \frac{1}{(2\pi)^{1/2}} Tr[(U(x, y))^*X],$$

(69)

where $X \in \mathcal{B}_2(\mathcal{H})$, $x, y \in \mathbb{R}$. $W$ is unitary.

**Definition 4.2** Let $\mathcal{B}_2(\mathcal{H})$ be the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H} = L^2(\mathbb{R})$, with the scalar product

$$\langle X|Y \rangle_2 = Tr[X^*Y] = \sum_{k=0}^{\infty} \langle \Phi_k|X^*Y\Phi_k \rangle,$$

(70)

where $\{\Phi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $\mathcal{H}$.

$$\mathcal{B}_2(\mathcal{H}) \simeq \mathcal{H} \otimes \mathcal{H}$$

(71)

where $\mathcal{H}$ denotes the dual of $\mathcal{H}$, and basis vectors of $\mathcal{B}_2(\mathcal{H})$ are given by

$$\Phi_{nl} := |\Phi_n\rangle \langle \Phi_l|, \quad n, l = 0, 1, 2, \ldots, \infty.$$

**Definition 4.3** Let $A$ and $B$ two operators on $\mathcal{H}$. The operator $A \vee B$ is such that

$$A \vee B(X) = AXB^*, \quad X \in \mathcal{B}_2(\mathcal{H}).$$

(72)

For $A$ and $B$, both bounded operators, $A \vee B$ defines a linear operator on $\mathcal{B}_2(\mathcal{H})$.

**Kubo-Martin-Schwinger (KMS) state**

Let $\alpha_i$, $i = 1, 2, \ldots, N$ be a sequence of non-zero, positive numbers, satisfying $\sum_{i=1}^{N} \alpha_i = 1$. Let

$$\Phi = \sum_{i=1}^{N} \alpha_i^* \xi_i \in \mathcal{B}_2(\mathcal{H})$$

with $X_{ii} \in \mathcal{B}_2(\mathcal{H})$. Then, we have the following properties:
1. Proposition 4.4 Φ defines a vector state φ on the von Neumann algebra \( \mathfrak{A}_i \), corresponding to the operators given with \( A \) in the left of the identity operator \( I_H \) on \( \mathfrak{H} \), i.e., \( \mathfrak{A}_i = \{ A_I = A \vee I \mid A \in \mathcal{L}(\mathfrak{H}) \} \).

   **Proof.** See [9].

2. Proposition 4.5 The state \( \varphi \) is faithful and normal.

   **Proof.** See [9].

3. Proposition 4.6 The vector \( \Phi \) is cyclic and separating for \( \mathfrak{A}_i \).

   **Proof.** See [9].

The fact that \( \Phi \) is separating for \( \mathfrak{A}_i \) is obtained through the relation

\[
(A \vee I) \Phi = (B \vee I) \Phi \iff A \vee I = B \vee I, \quad \forall A, B \in \mathfrak{A}_i.
\]

**Proof.** See [9].

4.1 Thermal state

Here, we give two examples of thermal states as known from the literature. For more details, see [45, 44, 11, 1, 9].

1. Let \( \alpha_i, i = 1, 2, \ldots, N \) be a sequence of non-zero, positive numbers, satisfying \( \sum_{i=1}^{N} \alpha_i = 1 \). Then the thermal state is defined as:

\[
\Phi := \sum_{i=1}^{N} \alpha_i X_{ii} = \sum_{i=1}^{N} \alpha_i |\zeta_i \rangle \langle \zeta_i|.
\]

   **Proof.** See [9].

2. The thermal equilibrium state \( \Phi \) at inverse temperature \( \beta \), corresponding to the harmonic oscillator Hamiltonian \( H_{OSC} = \frac{1}{2}(P^2 + Q^2) \), with \( H_{OSC} \phi_n = \omega(n + \frac{1}{2}) \phi_n, n = 0, 1, 2, \ldots, \) where the density matrix is

\[
\rho_\beta = \frac{e^{-\beta H_{OSC}}}{\text{Tr}[e^{-\beta H_{OSC}}]} = (1 - e^{-\omega \beta}) \sum_{n=0}^{\infty} e^{-n\omega \beta} |\phi_n \rangle \langle \phi_n|, \quad \text{Tr}[e^{-\beta H_{OSC}}] = \frac{e^{-\frac{\omega}{1 - e^{-\beta}}}}{1 - e^{-\beta \omega}}.
\]

   is

\[
\Phi = (1 - e^{-\omega \beta})^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{-\frac{n}{2} \omega \beta} |\phi_n \rangle \langle \phi_n|.
\]

4.2 Coherent states built from the harmonic oscillator thermal state

Take the cyclic vector \( \Phi \) of the von Neumann algebra \( \mathfrak{A}_1 \) generated by the unitary operator

\[
U_1(x, y) = \mathcal{W} [U(x, y) \vee I_\mathfrak{H}] \mathcal{W}^{-1}, \quad \text{with } \mathcal{W} \text{ given in [69].}
\]
such that $\Phi = \Phi_\beta$ with the $\lambda_n$ corresponding to the thermal state $\Phi_\beta$

$$\Phi_\beta = [1 - e^{-\omega \beta}]^\frac{1}{2} \sum_{n=0}^{\infty} e^{-n\omega \beta} \Psi_{nn}, \text{ i.e., } \lambda_n = (1 - e^{-\omega \beta})e^{-n\omega \beta}. \quad (79)$$

The CS, denoted $|z, \bar{z}, \beta\rangle^{\text{KMS}}$, built from the thermal state $\Phi_\beta$, are given by

$$|z, \bar{z}, \beta\rangle^{\text{KMS}} = U_1(z)|\Phi_\beta\rangle = e^{zA_1^\dagger - \bar{z}A_1}|\Phi_\beta\rangle.$$ \quad (80)

with $U_1(z) := U_1(x,y) = e^{zA_1^\dagger - \bar{z}A_1}$, where the actions of the annihilation and creation operators, $A_1$ and $A_1^\dagger$ are given by

$$A_1^\dagger|\Psi_{nl}\rangle = \sqrt{n + 1}\Psi_{n+1l}, \quad A_1|\Psi_{nl}\rangle = \sqrt{n}\Psi_{n-1l}. \quad (81)$$

**Proposition 4.7** From the fact that the states $\phi_i$, $i = 0, 1, 2, \ldots, \infty$, form a basis of $\mathcal{H} = L^2(\mathbb{R})$, the following equalities

$$U_1(x,y)|\Phi_\beta\rangle = (2\pi)^{\frac{1}{2}} \sum_{i,j=0}^{\infty} \lambda_i^j \Psi_{ji}(x,y)|\Psi_{ji}\rangle, \quad U_1(x,y)^*|\Phi_\beta\rangle = (2\pi)^{\frac{1}{2}} \sum_{i,j=0}^{\infty} \lambda_j^i \Psi_{ji}(x,y)|\Psi_{ji}\rangle \quad (82)$$

hold.

**Proof.** See [9]. \hfill \Box

**Proposition 4.8** (11)

From the isometry $\mathcal{W}$, the CS $|z, \bar{z}, \beta\rangle^{\text{KMS}}$ satisfy the resolution of the identity condition

$$\frac{1}{2\pi} \int_\mathbb{C} |z, \bar{z}, \beta\rangle^{\text{KMS}}\langle z, \bar{z}, \beta|d\bar{z}dy = I_{\mathcal{B}}, \quad \mathcal{B} = L^2(\mathbb{R}^2, d\bar{z}dy). \quad (83)$$

**Proof.** By definition of the states $|z, \bar{z}, \beta\rangle^{\text{KMS}}$ and Proposition 4.7, we have:

$$|z, \bar{z}, \beta\rangle^{\text{KMS}} = (2\pi)^{\frac{1}{2}} \sum_{i,j=0}^{\infty} \lambda_i^j \Psi_{ji}(x,y)|\Psi_{ji}\rangle, \quad \text{KMS}\langle z, \bar{z}, \beta| = (2\pi)^{\frac{1}{2}} \sum_{l,k=0}^{\infty} |\Psi_{lk}\rangle |\lambda_k^l\Psi_{lk}(x,y). \quad (84)$$

Thereby

$$|z, \bar{z}, \beta\rangle^{\text{KMS}}\langle z, \bar{z}, \beta| = 2\pi \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda_i^j \lambda_k^l \mathcal{W}\phi_{ji}(x,y)|\mathcal{W}\phi_{lk}(x,y)|\Psi_{ji}\rangle \langle\Psi_{lk}|. \quad (85)$$

By integrating the two members of the Eq. (85) over $\mathbb{C}$, we get by using the Wigner map $\mathcal{W}$,

$$\frac{1}{2\pi} \int_\mathbb{C} |z, \bar{z}, \beta\rangle^{\text{KMS}}\langle z, \bar{z}, \beta|d\bar{z}dy = \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda_i^j \lambda_k^l \mathcal{W}\phi_{ji}(x,y)|\mathcal{W}\phi_{lk}(x,y)|\Psi_{ji}\rangle \langle\Psi_{lk}| \int_{\mathbb{R}^2} \mathcal{W}\phi_{ji}(x,y)|\mathcal{W}\phi_{lk}(x,y)d\bar{z}dy = \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda_i^j \lambda_k^l |\Psi_{ji}\rangle \langle\Psi_{ji}| \delta_{ij} \delta_{kl} = I_{\mathcal{B}}. \quad (86)$$

**Proposition 4.9** The components of the KMS CS $|z, \bar{z}, \beta\rangle^{\text{KMS}}$ in the states $|\phi_n\rangle$ are derived as

$$|\langle \phi_n |z, \bar{z}, \beta\rangle^{\text{KMS}}|^2 = [1 - e^{-\omega \beta}] \sum_{s,l=0}^{\infty} \sum_{i,j=0}^{\infty} e^{-(t+j)\frac{\omega}{2}} \times \left\{ 2\pi \langle \phi_n |\Psi_{ij}\rangle \mathcal{W}(\langle \phi_t\rangle(\phi_i)(x,y)) \mathcal{W}(\langle \phi_s\rangle(\phi_i)(x,y)\Psi_{st}|\phi_n) \right\}. \quad (87)$$
Proof. According to (79), it follows that
\[
|\langle \phi_n | z, \bar{z}, \beta \rangle|_{KMS}^2 = \left[ 1 - e^{-\omega \beta} \right]^{\infty} \sum_{s,t=0}^{\infty} \sum_{i,j=0}^{\infty} e^{-(t+j)\frac{\omega}{2}} \times 
\left\{ 2\pi \langle \phi_n | \Psi_{ij} \rangle \mathcal{V}(|\phi_i\rangle|\phi_j\rangle)(x,y)\mathcal{V}(|\phi_s\rangle)(x,y)|\Psi_{st}\rangle|\phi_n\rangle \right\}.
\]
(88)

Next, from the Proposition 4.9, the Q-Husimi distribution is performed as
\[
KMS(z, \bar{z}, \beta | \rho(z, \bar{z}, \beta)|_{KMS} = \left[ 1 - e^{-\omega \beta} \right]^{\infty} \sum_{n=0}^{\infty} \sum_{s,t=0}^{\infty} \sum_{i,j=0}^{\infty} e^{-n\omega \beta} e^{-(t+j)\frac{\omega}{2}} \times 
\left\{ 2\pi \langle \phi_n | \Psi_{ij} \rangle \mathcal{V}(|\phi_i\rangle|\phi_j\rangle)(x,y)\mathcal{V}(|\phi_s\rangle)(x,y)|\Psi_{st}\rangle|\phi_n\rangle \right\}.
\]
(89)

The \(P\)-distribution function is found from the following result:

**Proposition 4.10** From the Glauber-Sudarshan \(P\)-distribution function in the KMS CS \(z, \bar{z}, \beta\)_{KMS},
\[
\rho = \frac{1}{2\pi} \int_C dxdy P(|z|^2) |z, \bar{z}, \beta\rangle_{KMS}^{|z, \bar{z}, \beta}\]
(90)
the diagonal elements of the normalized density operator \(\rho\) in the basis of the states \(\{|\phi_n\rangle\}_{n=0}^{\infty}\), by using the thermal state (79) and the resolution of the identity (83), are linked to the \(P\)-distribution function through the following equality
\[
\langle \phi_n | \rho | \phi_n \rangle = \frac{1}{\bar{n}_0 + 1} \left( \frac{\bar{n}_0}{\bar{n}_0 + 1} \right)^n = \sum_{k,l=0}^{\infty} \lambda_k | \langle \phi_n | \Psi_{lk} \rangle|^2 \left\{ \int_C \frac{d^2z}{\pi} P(|z|^2) \right\}
\]
(91)
where \(\bar{n}_0 = [1 - e^{-\omega \beta}]^{-1}\) is the thermal occupancy for the harmonic oscillator with the angular frequency \(\omega\).

Proof. Let us start with the definition of Glauber-Sudarshan \(P\)-distribution function [22, 43],
\[
\rho = \frac{1}{2\pi} \int_C dxdy P(|z|^2) |z, \bar{z}, \beta\rangle_{KMS}^{|z, \bar{z}, \beta}\]
(92)
such that we get
\[
\langle \phi_n | \rho | \phi_n \rangle = \frac{1}{2\pi} \int_C dxdy P(|z|^2) |\langle \phi_n | z, \bar{z}, \beta \rangle|_{KMS}^2
\]
\[
= \left[ 1 - e^{-\omega \beta} \right]^{\infty} \sum_{s,t=0}^{\infty} \sum_{i,j=0}^{\infty} e^{-(t+j)\frac{\omega}{2}} \times 
\left\{ \int_C P(|z|^2) |\langle \phi_n | \Psi_{ij} \rangle \mathcal{V}(|\phi_i\rangle|\phi_j\rangle)(x,y)\mathcal{V}(|\phi_s\rangle)(x,y)|\Psi_{st}\rangle|\phi_n\rangle dxdy \right\}.
\]
(93)

Then, introducing the differential element of area in the \(z\) plane [22], \(\frac{d^2z}{\pi} = d(Rez)d(Imz) = \frac{dxdy}{2\pi^2}(\hbar = 1)\), it follows that
\[
\int_C \frac{d^2z}{\pi} \langle \phi_n | \rho | \phi_n \rangle = \sum_{k,l=0}^{\infty} \sum_{k,l=0}^{\infty} \lambda_k^2 \lambda_l^2 \left\{ \int_C \frac{d^2z}{\pi} P(|z|^2) \right\} \left\{ \delta_{ij} \delta_{lk} \langle \Psi_{lk} | \phi_n \rangle \langle \phi_n | \Psi_{ij} \rangle \right\}.
\]
(94)

Thus
\[
\langle \phi_n | \rho | \phi_n \rangle = \frac{1}{\bar{n}_0 + 1} \left( \frac{\bar{n}_0}{\bar{n}_0 + 1} \right)^n = \sum_{k,l=0}^{\infty} \lambda_k | \langle \phi_n | \Psi_{lk} \rangle|^2 \left\{ \int_C \frac{d^2z}{\pi} P(|z|^2) \right\}
\]
(95)
where the thermal occupancy for the harmonic oscillator $\bar{n}_0 = \left[1 - e^{-\omega \beta}\right]^{-1}$ with the angular frequency $\omega$, has been introduced.

Using the $Q$-Husimi distribution expression, the Wehrl entropy is deduced as

$$W = \int \frac{d^2z}{\pi} KMS \langle z, \bar{z}, \beta | \rho | z, \bar{z}, \beta \rangle^KMS \ln \{KMS \langle z, \bar{z}, \beta | \rho | z, \bar{z}, \beta \rangle^KMS\}. \quad (96)$$

5 Concluding remarks

In this work, the definition of the density operator on quantum states in Hilbert spaces and some of its features relevant in thermodynamics and information-theoretical entropy calculations have been provided. As application, the physical model describing an electron in a magnetic field has been studied. The exotic Landau problem in noncommutative plane has been investigated and the related CS have been constructed. In addition, the quantum model for which modular structures emerging for two underlying von Neumann algebras have been provided, has been revisited. The resolution of the identity satisfied by the CS built out of Kubo-Martin-Schwinger (KMS) state has been achieved. Thanks to the completeness relations verified by the CS in these two examples, the thermodynamics has been discussed, using the diagonal $P$-representation of the density operator in the constructed CS and in the Hilbert space basis. Besides, the $Q$-Husimi distribution and the Wehrl entropy have been determined.

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