COMMUTING MAPS WITH THE MEAN TRANSFORM UNDER JORDAN PRODUCT

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ABSTRACT. In this article, we give a complete characterization of the bijective maps which commute with the mean transform under Jordan product. The main result is the following: Let $H, K$ be two complex Hilbert spaces and $\Phi : B(H) \to B(K)$ be a bijective map, then

$$\mathcal{M}(\Phi(A) \circ \Phi(B)) = \Phi(\mathcal{M}(A \circ B)) \text{ for all } A, B \in B(H)$$

if and only if there exists a unitary or anti-unitary operator $U : H \to K$ such that,

$$\Phi(T) = UTU^* \text{ for all } T \in B(H).$$

1. INTRODUCTION

Let $H$ and $K$ be two complex Hilbert spaces and let $B(H, K)$ be the Banach space of all bounded linear operators from $H$ into $K$. In the case $K = H$, $B(H, H)$ is simply denoted by $B(H)$ and it is a Banach algebra.

For an arbitrary operator $T \in B(H, K)$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and $T^*$ the range, the null subspace and the adjoint operator of $T$, respectively. For $T \in B(H)$, the spectrum of $T$ is denoted by $\sigma(T)$.

An operator $T \in B(H, K)$ is a partial isometry when $T^*T$ is an orthogonal projection (or, equivalently $TT^*T = T$). In particular $T$ is an isometry if $T^*T = I$, and $T$ is unitary if it is a surjective isometry. An operator $T \in B(H)$ is said to be normal if $T^*T = TT^*$, and quasi-normal if $TT^*T = TT^*T$. As usual, we denote the module of $T \in B(H)$ by $|T| = (T^*T)^{1/2}$, and $T = V|T|$ is the unique polar decomposition of $T$, where $V$ is a partial isometry satisfying $\mathcal{N}(V) = \mathcal{N}(T)$. In general $V$ and $|T|$ does not commute and it is the case if $T$ is quasi-normal.

From the polar decomposition, the Aluthge transform is defined in ([14]), by

$$\Delta(T) = |T|^1V|T|^1.$$

The Aluthge transform has been well studied by many authors, it is a good tool for studying sum class of operators (see [3, 4, 9, 6, 17]). In the same way, the mean transform of the operator $T$ was introduced in [8, 13], by

$$\mathcal{M}(T) := \frac{1}{2}(V|T| + |T|V).$$

2010 Mathematics Subject Classification. 47A05, 47A10, 47B49, 46L40.

Key words and phrases. normal, quasi-normal operators, polar decomposition, mean transform, Jordan product.
The mean transform has also well been studied in many articles, it is the arithmetic mean of \( T = V|T| \) and its Duggal transform \( \tilde{T} = |T|V \), it has nice properties, for example we can cite \([5, 13, 10]\). The fixed point of mean transform are the quasi-normal operator as the Aluthge transform.

The main result of the present paper is the following theorem which gives a nice characterization of the commuting maps with the mane transform under the Jordan product.

**Theorem 1.1.** Let \( H \) and \( K \) be two complex Hilbert space, with \( \dim H \geq 3 \). Let \( \Phi : \mathcal{B}(H) \to \mathcal{B}(K) \) be a bijective map. Then \( \Phi \) satisfies the condition

\[
M(\Phi(A) \circ \Phi(B)) = \Phi(M(A \circ B)) \quad \text{for all} \quad A, B \in \mathcal{B}(H),
\]

if and only if there exists a unitary or anti-unitary operator \( U : H \to K \), such that

\[
\Phi(T) = U T U^* \quad \text{for every} \quad T \in \mathcal{B}(H).
\]

**Remark 1.1.** Observe that, even if the hypothesis on the map \( \Phi \) is purely algebraic, the conclusion gives automatically the continuity of the map. Also, the linearity of \( \Phi \) is not assumed, we get it automatically.

2. **Auxiliary results of the mean transform**

For \( x, y \in H \) we denote by \( x \otimes y \) the rank one operator (or 0) defined by

\[
(x \otimes y)u = \langle u, y \rangle x \quad \text{for} \quad u \in H.
\]

Every rank one operator has the previous form, and \( x \otimes y \) is an orthogonal projection (i.e. \( T^2 = T = T^* \)) if and only if \( x = y \) and \( \|x\| = 1 \).

The following results can be found in \([5]\).

**Proposition 2.1.** \([5]\) Let \( x, y \in H \) be two non-zero vectors. Let \( T = x \otimes y \), then

\[
M(T) = M(x \otimes y) = \frac{1}{2}(x + \frac{\langle x, y \rangle}{\|y\|^2}y) \otimes y.
\]

**Proposition 2.2.** \([5]\) Let \( T \in \mathcal{B}(H) \). Then

\[
M(T) = 0 \iff T = 0.
\]

Next lemma gives a characterization of the nilpotent operator of order two.

**Lemma 2.1.** \([5]\) Let \( T \in \mathcal{B}(H) \). Then

\[
M(T) = \frac{T}{2} \quad \text{if and only if} \quad T^2 = 0.
\]

**Theorem 2.1.** Let \( T \in \mathcal{B}(H) \). Then

\[
M(T^2) = T \quad \text{if and only if} \quad T \text{ is an orthogonal projection}.
\]
Proof. It is clear that, if $T$ is a projection, then $\mathcal{M}(T^2) = T = T^*$. So we need to show the direct meaning.

Let $T^2 = V_2|T^2|$ be the polar decomposition of $T^2$ and suppose that $\mathcal{M}(T^2) = T$. We will divided the proof on a few steps.

**Step 1**: The operator $|T^2|V_2|T^2|$ is positive. From the assumption, we have

\[
T^2 + |T^2|V_2 = T \quad \text{and} \quad \frac{(T^2)^* + V_2|T^2|}{2} = T^*.
\]

Thus

\[
\frac{|T^2| + V_2|T^2|V_2}{2} = V_2^*T \geq 0,
\]

is fact that the operators $|T^2|$ and $V_2|T^2|$ are positive. Therefore

\[
T^*|T^2| = T^*V_2T^2 = T^*(V_2^*T)T \geq 0.
\]

In particular $T^*|T^2|$ is self adjoint. Hence $T^*|T^2| = |T^2|T$. Multiplying on the left by $T^*$, we get that

\[
|T^2|V_2|T^2| = |T^2|TT = T^*|T^2|T \geq 0.
\]

**Step 2**: $2I - T^*$ is injective.

Let $x \in H$ such that $(2I - T^*)x = 0$. Then $T^*x = 2x$ and thus $(T^2)^*x = 4x$. By the assumption we have $|T^2|V_2 = 2T - T^2$. We take the adjoint, we get that

\[
V_2^*|T^2| = 2T^* - (T^2)^*.
\]

It follows that $V_2^*|T^2|x = 2T^*x - (T^2)^*x = 0$. Thus $|T^2|V_2^*|T^2|x = 0$. Since

\[
|T^2|V_2^*|T^2| = |T^2|V_2|T^2| \geq 0 \quad \text{(see Step 1)},
\]

then

\[
|T^2|V_2^*|T^2|x = 0.
\]

Multiplying the preceding equality on the left by $V_2$, we get that

\[
T^4x = V_2^*|T^2|V_2^*|T^2|x = 0.
\]

It follows that

\[
16\|x\|^2 = \langle (T^4)^*x, x \rangle = \langle x, T^4x \rangle = 0.
\]

By consequence $x = 0$ and then $2I - T^*$ is injective.

**Step 3**: The inclusion $\mathcal{N}(T^2) \subseteq \mathcal{N}(T^*) \subseteq \mathcal{N}((T^2)^*)$ holds.

Let $x \in \mathcal{N}(T^2)$, we have $T^2x = |T^2|x = 0$, hence by equation (2.1)

\[
V_2^*|T^2|x = (2I - T^*)T^*x = 0,
\]

and since $2I - T^*$ is injective then $T^*x = 0$, thus $x \in \mathcal{N}(T^*) \subseteq \mathcal{N}((T^2)^*)$.

**Step 4**: $T$ is a projection.
Now we are in position to prove that $T$ is a projection. First we show that $V_2 \geq 0$. Let $x \in H$ and consider the following decomposition $H = N(|T|^2) \oplus \mathcal{R}(|T|^2)$. Put $x = x_1 + x_2$ with $x_1 \in N(|T|^2)$, $x_2 \in \mathcal{R}(|T|^2)$. Then we have

$$
\langle V_2 x, x \rangle = \langle V_2 x_1 + V_2 x_2, x_1 + x_2 \rangle = \langle V_2 x_2, x_2 \rangle + \langle V_2 x_2, x_1 \rangle = < V_2 x_2, x_2 > + < x_2, V_2^* x_1 > = < V_2 x_2, x_2 > \quad (\text{since } N(V_2) \subseteq N(V_2^*)).
$$

Since $|T|^2 |V_2| |T|^2 \geq 0$ (see Step 1), $< V_2 x, x > = < V_2 x_2, x_2 > \geq 0$. Thus $V_2$ a positive partial isometry, so $V_2$ is an orthogonal projection and $T^2 = V_2 |T|^2 = V_2^* V_2 |T|^2 = |T|^2$ is positive. Therefore $\mathcal{M}(T^2) = T^2 = T = T^*$. This completes the proof.

**Corollary 2.1.** Let $T \in \mathcal{B}(H)$. Then the following equivalence holds:

$$
T \text{ is a projection } \iff (\mathcal{M}(T))^2 = T.
$$

**Proof.** We need to show the second direction. Suppose that $(\mathcal{M}(T))^2 = T$. Then $\mathcal{M}((\mathcal{M}(T))^2) = \mathcal{M}(T)$. By Theorem 2.1 replacing $T$ by $\mathcal{M}(T)$, we get that $\mathcal{M}(T)$ is a projection. Hence $\mathcal{M}(T) = (\mathcal{M}(T))^2 = T$ is also a projection.

**Lemma 2.2.** [5] Let $T \in \mathcal{B}(H)$ and $T = V|T|$ be the polar decomposition of $T$. The following equivalence holds:

$$
\mathcal{M}(T) \text{ is self-adjoint } \iff V \text{ is self-adjoint}.
$$

**Lemma 2.3.** Let $T \in \mathcal{B}(H)$. Then $\mathcal{M}(T)$ is a projection if and only if $T$ is also a projection. In this case $\mathcal{M}(T) = T$.

**Proof.** It is clear that we need to show the first direction. Let $T = V|T|$ be the polar decomposition of $T$ and suppose that $\mathcal{M}(T)$ is a projection. In particular is a self-adjoint. By the preceding lemma, $V$ is a self adjoint partial isometry. Hence $V^3 = V$ and

$$
\mathcal{R}(T) = \mathcal{R}(V) = \mathcal{R}(V^* V) = \mathcal{R}(T^*) = \mathcal{R}(|T|).
$$

Hence $V^2$ is the projection on $\mathcal{R}(V) = \mathcal{R}(V^2)$.

On the other hand, we have also $\mathcal{N}(\mathcal{M}(T)) = N(V) = N(V^2)$. Since $\mathcal{M}(T)$ and $V^2$ are both projections with the same kernel, then $V^2 = \mathcal{M}(T)$ and thus

$$
V = V^3 = V \mathcal{M}(T) = V \left( \frac{|T| + V|T|V}{2} \right) = \frac{|T| + V|T|V}{2}.
$$

From this equality, we get $V = \frac{|T| + V|T|V}{2} \geq 0$ as sum of two positive operators. It follows that $V$ is positive partial isometry. In particular $V$ is also projection and then $V = V^2$. Now from the polar decomposition of $T$, we have $T = V|T| = V^2|T| = |T|$ is positive. This implies that $\mathcal{M}(T) = T$ is also a projection.

**Lemma 2.4.** Let $T \in \mathcal{B}(H)$. Then

$$
\mathcal{M}(T \circ P) = P \quad \text{for all rank one projection } P, \quad \text{if and only if,} \quad T = I.
$$
Proof. Clearly, if $T = I$ then for every rank one projection $P$, we have $\mathcal{M}(T \circ P) = P$. So we need to show the direct implication. Indeed, let $x \in H$ be a unit vector, and $P = x \otimes x$. Let $S = T \circ P$, it is clear that $S$ is an operator of at most of rank two. So $\mathcal{M}(S)$ is at most of rank four. Using the functional trace $\text{Tr}$ well defined on the set of finite rank operator, then we get $\text{Tr}(S) = \text{Tr}(\mathcal{M}(S)) = \text{Tr}(P) = 1$, thus $\langle Tx, x \rangle = \text{Tr}(T \circ P) = \text{Tr}(S) = 1$. Hence we conclude that the numeral range $\langle Tx, x \rangle = \|x\|^2$ for all $x \in H$, and thus $T = I$. \hfill $\square$

3. The proof of main theorem

The proof of Theorem 1.1 will be divided in many intermediary lemmas and it will be given at the end of this section. Along of this section, we assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map, which satisfies the condition :

$$M(\Phi(A) \circ \Phi(B)) = \Phi(M(A \circ B)) \quad \text{for all } A, B \in \mathcal{B}(H).$$

(3.1)

As an immediate consequence of (3.1) and Lemma 2.1 and Theorem 2.4, we derive the following result :

Lemma 3.1. The following two statements are hold :

(i) $\Phi(0) = 0$,

(ii) $\Phi(I) = I$.

Proof. (i). Since $\Phi$ is onto, there is $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. By (3.1) we have

$$0 = M((\Phi(0) \circ \Phi(A)) = \Phi(0).$$

Therefore, $\Phi(0) = 0$.

(ii) For simplicity, let us denote $T = \Phi(I)$. We take $A = B = I$ in (3.1), we get that

$$M(T^2) = \Phi(I) = T.$$ 

(3.2)

Using Theorem 2.1 we get that $T^2 = T = T^*$. 

To complete the proof, it must to show $T$ is injective. Pick a $y \in K$ such that $Ty = 0$, we have also $T^*y = 0$. Since $\Phi$ is onto, there exists $B \in \mathcal{B}(H)$ such that $\Phi(B) = y \otimes y$. By (3.1), we get

$$0 = M(\frac{1}{2}(Ty \otimes y + y \otimes T^*y) = M(\Phi(I)\Phi(B)) = \Phi(M(B)).$$

Since $\Phi$ is bijective and $\Phi(0) = 0$, we have $M(B) = 0$. By Proposition 2.2 we get that $B = 0$. Therefore $y \otimes y = \Phi(B) = 0$ and $y = 0$. Hence $T$ is an injective projection, then $\Phi(I) = T = I$. \hfill $\square$

As a consequence of the previous lemma, we get the following result.

Lemma 3.2. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map satisfying (3.1). Then

(i) $M(\Phi(B)) = \Phi(M(B))$, for all $B \in \mathcal{B}(H)$.

In particular, $\Phi$ preserves the set of quasi-normal operators in both directions.

(ii) $\Phi(A^2) = (\Phi(A))^2$ for all $A$ quasi-normal.
(iii) \( \Phi \) preserves the set of orthogonal projections in both directions.

(iv) \( \Phi \) preserves the orthogonality between the projections:

\[
P \perp Q \iff \Phi(P) \perp \Phi(Q).
\]

(v) \( \Phi \) preserves the order relation between projections in both directions:

\[
Q \leq P \iff \Phi(Q) \leq \Phi(P).
\]

(vi) \( \Phi(P + Q) = \Phi(P) + \Phi(Q) \) for all orthogonal projections \( P, Q \) such that \( P \perp Q \).

(vii) \( \Phi \) preserves the set of rank one projections in both directions.

Proof. (i). Taking \( B = I \) in (3.1), then we get \( \mathcal{M}(\Phi(B)) = \Phi(\mathcal{M}(B)) \). Since the quasi-normal operators are exactly the fixed point of the mean transform, then (i) holds.

(ii). Let \( A \) be a quasi-normal operator. Since \( \Phi \) preserves the set of quasi-normal operators, then \( \Phi(A), \Phi(A^2) \) and \( (\Phi(A))^2 \) are also quasi-normal. By (3.1) with \( B = A \), we get \( \mathcal{M}(\Phi(A))^2 = \Phi(\mathcal{M}(A^2)) \). Hence \( (\Phi(A))^2 = \Phi(A^2) \), it is also the fact that the quasi-normal operators are the fixed point of \( \mathcal{M} \).

(iii). It is an immediate consequence of (ii) and the fact that a idempotent quasi-normal is orthogonal projection.

Throughout the remaining of the proof \( P \) and \( Q \) are orthogonal projections.

(iv). Assume that \( P, Q \) are orthogonal (i.e. \( PQ = 0 \) this equivalent also to \( P \circ Q = 0 \)). Since \( \Phi \) preserves the set of orthogonal projections, then \( \Phi(P), \Phi(Q) \) are orthogonal projection. By condition (3.1), we get

\[
\mathcal{M}(\Phi(P) \circ \Phi(Q)) = \Phi(\mathcal{M}(P \circ Q)) = \Phi(0) = 0.
\]

Thus \( \Phi(Q) \circ \Phi(P) = 0 \). The converse holds since the inverse \( \Phi^{-1} \) satisfies the same condition as \( \Phi \).

(v). Suppose that \( Q \leq P \) or equivalently \( Q \circ P = P \circ Q = Q \). Then by (3.1), we get

\[
\Phi(Q) \circ \Phi(P) = \mathcal{M}(\Phi(P) \circ \Phi(Q)) = \Phi(Q).
\]

Since \( \Phi^{-1} \) has the same assumption as \( \Phi \), then we deduce that \( \Phi \) preserves the order relation between the orthogonal projections in both directions.

(vi). We have \( P, Q \leq P + Q \). Then \( \Phi(P), \Phi(Q) \leq \Phi(P + Q) \). So \( \Phi(P) + \Phi(Q) \leq \Phi(P + Q) \). Since \( \Phi \) and \( \Phi^{-1} \) both satisfy the same conditions, it follows that \( \Phi(P) + \Phi(Q) = \Phi(P + Q) \).

(vii). Let \( P = x \otimes x \) be a rank one projection. Then \( \Phi(P) \) is a non zero projection. Let \( y \in K \) be a unit vector such that \( y \otimes y \leq \Phi(P) \). Thus \( \Phi^{-1}(y \otimes y) \leq P \). Since \( P \) is a minimal projection and \( \Phi^{-1}(y \otimes y) \) is a non zero projection, then \( \Phi^{-1}(y \otimes y) = P \). Therefore \( \Phi(P) = y \otimes y \) is a rank one projection. This complete the proof. □

Lemma 3.3. There exists a bijective multiplicative function \( h : \mathbb{C} \to \mathbb{C} \) such that for every quasi-normal \( A \in \mathcal{B}(H) \) and \( \alpha \in \mathbb{C} \), we have \( \Phi(\alpha A) = h(\alpha)\Phi(A) \).

In particular, \( h(0) = 0, h(1) = 1 \) and \( h(-1) = -1 \).
Proof. Let $x \in H$ be a unit vector and $\alpha \in \mathbb{C}$. Let $T = \Phi(\alpha x \otimes x)$. By the assumption and condition (3.1), we have
\[
\mathcal{M}(\Phi(\alpha x \otimes x) \circ x \otimes x)) = \Phi(\alpha x \otimes x).
\]
Therefore
\[
(3.4) \quad \mathcal{M}(Tx \otimes x + x \otimes T^*x) = 2T.
\]
Taking the norm and using the triangular inequality, we get
\[
2\|T\| = \|\mathcal{M}(Tx \otimes x + x \otimes T^*x)\| \leq \|Tx \otimes x + x \otimes T^*x\| \leq \|Tx\| + \|T^*x\| \leq 2\|T\|.
\]
Which implies the following,
\[
2\|T\| = \|Tx \otimes x + x \otimes T^*x\|, \quad \text{and} \quad \|T\| = \|Tx\| = \|T^*x\|.
\]
Let denote $S = Tx \otimes x + x \otimes T^*x$, form the preceding equality we get
\[
4\|T\|^2 = \|S\|^2 = \|S^*S\| = r(S^*S) \leq Tr(S^*S).
\]
On the other hand, we calculate $S^*S$ then
\[
S^*S = (\|Tx\|^2 x \otimes x + x, Tx > x \otimes T^*x < x, Tx > x \otimes T^*x < x, Tx > T^*x \otimes x + T^*x \otimes T^*x).
\]
It follows that
\[
4\|T\|^2 \leq Tr(S^*S)
\]
\[
= Tr(\|Tx\|^2 x \otimes x + x, Tx > x \otimes T^*x < x, Tx > x \otimes T^*x < x, Tx > T^*x \otimes x + T^*x \otimes T^*x)
\]
\[
= \|Tx\|^2 + 2\|Tx, x > x \otimes T^*x < x, Tx > T^*x \otimes x + T^*x \otimes T^*x\|^2
\]
\[
= 2\|T\|^2 + 2\|Tx, x > x \otimes T^*x < x, Tx > T^*x \otimes x + T^*x \otimes T^*x\|^2 \leq 4\|T\|^2.
\]
Therefore
\[
\|T\|^2 \leq |<Tx, x>|^2, \quad \text{and} \quad \|T\| = |<Tx, x>| = \|Tx\| = \|T^*x\|.
\]
By consequence,
\[
\|Tx, x > x\|^2 = \|Tx\|^2 - |<Tx, x>|^2 = 0,
\]
and thus $Tx =<Tx, x > x$. With the same way, $T^*x =<T^*x, x > x$. By equation (3.4), it follows that
\[
\Phi(\alpha x \otimes x) = T =<Tx, x > x \otimes x = h_x(\alpha)x \otimes x,
\]
where $h_x(\alpha) =<Tx, x > x$ for all $\alpha \in \mathbb{R}$. In particular $h_x(0) = 0$ and $h_x(1) = 1$.

Now, by condition (3.1), we have for all unit vector $x \in H$,
\[
\mathcal{M}(\Phi(\alpha I) \circ x \otimes x) = \Phi(\alpha x \otimes x) = h_x(\alpha)x \otimes x.
\]
By Lemma 2.4, $\Phi(\alpha I)x \circ x = h_x(\alpha)x \otimes x$, this implies that $\Phi(\alpha I)x$ and $x$ are colinear for all unit vector $x$ of $H$. By classical arguments, $\Phi(\alpha I)$ is a scalar multiple of the identity, so there exists a function $h : \mathbb{C} \rightarrow \mathbb{C}$ such that $\Phi(\alpha I) = h(\alpha)I$ for all $\alpha \in \mathbb{C}$. By the assumption on $\Phi$, the function $h$ is bijective and multiplicative. Moreover $\Phi(\alpha A) = h(\alpha)\Phi(A)$ for every quasi-normal operator $A$. \qed
Lemma 3.4. Let $P, Q$ two orthogonal projections and $\alpha, \beta \in \mathbb{C}$, then we have
\[ \Phi(\alpha P + \beta Q) = h(\alpha)\Phi(P) + h(\beta)\Phi(Q). \]

Proof. If $\alpha = 0$ or $\beta = 0$, then the result follows from preceding lemma. So suppose that $\alpha \neq 0$ and $\beta \neq 0$. By Lemma 3.2, $\Phi(P)$ and $\Phi(Q)$ are also two orthogonal projections, using condition (3.1), we get the following:
\[ M(\Phi(\alpha P + \beta Q) \circ \Phi(P)) = \Phi(M(\alpha P + \beta Q) \circ P)) = \Phi(\alpha P) = h(\alpha)\Phi(P). \]

Therefore
\[ M(\Phi(\alpha P + \beta Q) \circ \Phi(P)) = h(\alpha)\Phi(P). \]

By Lemma 3.2
\[ (3.5) \quad \Phi(\alpha P + \beta Q) \circ \Phi(P) = \Phi(\alpha P + \beta Q)\Phi(P) = \Phi(P)\Phi(\alpha P + \beta Q) = h(\alpha)\Phi(P). \]

Similarly,
\[ (3.6) \quad \Phi(\alpha P + \beta Q) \circ \Phi(Q) = \Phi(\alpha P + \beta Q)\Phi(Q) = \Phi(Q)\Phi(\alpha P + \beta Q) = h(\beta)\Phi(Q). \]

Again the condition (3.1) implies that
\[ \Phi(\alpha P + \beta Q) = \Phi(M((\alpha P + \beta Q) \circ (P + Q))) = M(\Phi(\alpha P + \beta Q) \circ \Phi(P + Q)) = M(\Phi(\alpha P + \beta Q) \circ \Phi(P) + \Phi(\alpha P + \beta Q) \circ \Phi(Q)) = M(h(\alpha)\Phi(P) + h(\beta)\Phi(Q)) = h(\alpha)\Phi(P) + h(\beta)\Phi(Q). \]

Lemma 3.5. The function $h : \mathbb{C} \to \mathbb{C}$ is a automorphism of complex fields $\mathbb{C}$.

Proof. Let $x, y$ be two unit and orthogonal vectors, and let $A = x \otimes y + y \otimes x$. First, note that $A$ is self adjoint operator of rank two, and we have
\[ A^2 = x \otimes x + y \otimes y, \]
which is a non-trivial orthogonal projection. Since the dimensional of $H$ is greater than $3$, therefore the spectrum of $A$ is $\sigma(A) = \{-1, 0, 1\}$. Hence we can find $f_1, f_2$ two unit and orthogonal vectors from $H$ (\|f_1\| = \|f_2\| = 1$ and $<f_1, f_2>=0$) such that
\[ A = f_1 \otimes f_1 - f_2 \otimes f_2. \]
In particular
\[ \Phi(A) = \Phi(f_1 \otimes f_1 - f_2 \otimes f_2) = \Phi(f_1 \otimes f_1) - \Phi(f_2 \otimes f_2), \]
this implies that $\Phi(A)$ is self-adjoint as sum of two rank one projection.
Now, let us consider the rank one projections \( P = x \otimes x, \ Q = y \otimes y \). For \( \alpha, \beta \in \mathbb{C} \), we denote by \( B = \alpha P + \beta Q \). Clearly

\[
A \circ P = \frac{1}{2} A \quad \text{and} \quad A \circ Q = \frac{1}{2} A,
\]

by condition (3.1) we get the following equalities:

\[
h(\frac{1}{2}) \Phi(A) = \Phi(\frac{1}{2} A) = \mathcal{M}(\Phi(A) \circ \Phi(P)) = \Phi(A) \circ \Phi(P).
\]

And

\[
h(\frac{1}{2}) \Phi(A) = \Phi(\frac{1}{2} A) = \mathcal{M}(\Phi(A) \circ \Phi(Q)) = \Phi(A) \circ \Phi(Q).
\]

Since \( A \circ B = \frac{\alpha + \beta}{2} A \) then,

\[
h(\frac{\alpha + \beta}{2}) \Phi(A) = \Phi(\mathcal{M}(A \circ B))
\]

\[
= \mathcal{M}(\Phi(A) \circ \Phi(B))
\]

\[
= \mathcal{M}(\Phi(A) \circ \Phi(\alpha P + \beta Q))
\]

\[
= \mathcal{M}(\Phi(A) \circ (h(\alpha) \Phi(P) + h(\beta) \Phi(Q)))
\]

\[
= \mathcal{M}(h(\alpha) \Phi(A) \circ \Phi(P) + h(\beta) \Phi(A) \circ \Phi(Q))
\]

\[
= \mathcal{M}(h(\alpha) h(\frac{1}{2}) \Phi(A) + h(\beta) h(\frac{1}{2}) \Phi(A))
\]

\[
= \mathcal{M}(h(\frac{\alpha}{2} + h(\frac{\beta}{2}) \Phi(A))
\]

\[
= (h(\frac{\alpha}{2}) + h(\frac{\beta}{2})) \Phi(A).
\]

Consequently

\[
h(\frac{\alpha}{2}) + h(\frac{\beta}{2}) = h(\frac{\alpha + \beta}{2}),
\]

and thus

\[
h(\alpha' + \beta') = h(\alpha') + h(\beta'), \quad \text{for all} \ \alpha', \beta' \in \mathbb{C}.
\]

\[\square\]

The following result has been chowed by Uhlhorn, it gives a nice characterization of a bijective map \( \Psi : P_1(H) \rightarrow P_1(H) \) which preserves the orthogonality. This result can be reformulate as:

**Theorem 3.1** (Uhlhorn’s Theorem). Let \( \Phi : P_1(H) \rightarrow P_1(K) \) be a bijective map, with \( \dim H \geq 3 \). Assume that \( \Phi \) satisfies the following property

\[
PQ = 0 \iff \Phi(P) \Phi(Q) = 0 \ (P, Q \in P_1(H)).
\]
Then there exists a unitary or anti-unitary operator $U : H \to K$, such that $\Phi$ is of the form:

\begin{equation}
\Phi(P) = UPU^* \quad \text{for all} \quad P \in P_1(H).
\end{equation}

**Lemma 3.6.** The function $h$ is the identity or the complex conjugate. Moreover for every unit vector $y \in K$ and $x \in H$ such that $\Phi(x \otimes x) = y \otimes y$, and for every self-adjoint operator $A \in \mathcal{B}(H)$, we have

\begin{equation}
\langle \Phi(A)y, y \rangle = \langle Ax, x \rangle.
\end{equation}

**Proof.** Let $A \in \mathcal{B}(H)$ be a self-adjoint operator, for an arbitrary unit vectors $x \in H$ and $y \in K$, such that $\Phi(x \otimes x) = y \otimes y$, put $P = x \otimes x$, then $A \circ P$ is also self-adjoint of rank less than 2. And So there exists $\alpha, \beta \in \mathbb{C}$ and two orthogonal projections $P_1$ and $P_2$ such that $A \circ P = \alpha P_1 + \beta P_2$. In particular we have

\[ Tr(A \circ P) = \alpha + \beta = \langle Ax, x \rangle. \]

By the Lemma 3.4, we have

\[ \Phi(A \circ P) = \Phi(\alpha P_1 + \beta P_2) = h(\alpha)\Phi(P_1) + h(\beta)\Phi(P_2), \]

so

\[ \Phi(A \circ P) = h(\alpha) + h(\beta) = h(\alpha + \beta) = h(\langle Ax, x \rangle). \]

On the other hand,

\[ Tr(\Phi(A) \circ \Phi(P)) = \langle \Phi(A)y, y \rangle \]

and by the assumption we have

\[ \langle \Phi(A)y, y \rangle = Tr(M(\Phi(A) \circ \Phi(P))) = Tr(M(\Phi(A \circ P))) = Tr(\Phi(A \circ P)) = h(\langle Ax, x \rangle). \]

From the preceding argument, it follows that for any self-adjoint operator $A \in \mathcal{B}(H)$ we have

\begin{equation}
h(W(A)) = W(\Phi(A)).
\end{equation}

Now, let $a, b \in \mathbb{R}$ such that $a < b$, and let $A \in \mathcal{B}(H)$ be a self-adjoint operator such that the numerical range $W(A) = \{\langle Au, u \rangle : \|u\| = 1\} = [a, b]$, for example we take $A = \alpha P + \beta Q$ where $P, Q$ are two orthogonal projections. By the preceding (3.9) $h([a, b]) = h(W(A)) = W(\Phi(A))$, is bounded, according to the numerical range of an operator $T \in \mathcal{B}(K)$ is always bounded. Which implies that the automorphism $h : \mathbb{C} \to \mathbb{C}$ is bounded on all segment $[a, b] \subset \mathbb{R}$ and also bounded in all rectangle set $\mathcal{R} = [a, b] + i[c, d]$. By Proposition 1.1 in [11] $h$ is either the identity $h(z) = Id_{\mathbb{C}}(z) = z$, for all $z \in \mathbb{C}$ or the conjugate complex $h(z) = Id_{\overline{\mathbb{C}}}(z) = \overline{z}$ for all $z \in \mathbb{C}$.

In particular, we conclude that for every self-adjoint operator $A \in \mathcal{B}(H)$ and for every unit vector $y \in K$ and $x \in H$ such that $\Phi(x \otimes x) = y \otimes y$ we have

\[ \langle \Phi(A)y, y \rangle = \langle Ax, x \rangle \in \mathbb{R}. \]

□
Corollary 3.1. The maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserves the set self adjoint operator. Moreover, the exist an unitary or anti-unitary operator $U : H \to K$ such that

$$\Phi(A) = U A U^* \quad \text{for all } A \in \mathcal{B}_s(H).$$

Proof. The map $\Phi$ satisfies the conditions of Uhlhorn’s, hence $\Phi$ takes the form on the set $P_1(H)$ of rank one operators.

Let $A \in \mathcal{B}(H)$ be a self-adjoint operator, and let $x \in H$ be an arbitrary unit vectors from $H$, we have such that $\Phi(x \otimes x) = U x \otimes U x$, and thus by equation (3.8)

$$\langle U^* \Phi(A) U x, x \rangle = \langle \Phi(A) U x, U x \rangle$$

$$\langle Ax, x \rangle$$

Therefore $U^* \Phi(A) U = A$ and this completes the proof. \hfill \square

In the rest of the manuscript, we replace the map $\Phi$ by the map $T \to U^* \Phi(T) U$ which satisfies the same conditions as $\Phi$ and noted it too by $\Phi$. Then we can suppose that $\Phi(A) = A$ for every self-adjoint operator $A \in \mathcal{B}(H)$. To finish the proof, it must to show that $\Phi$ is the identity on $\mathcal{B}(H)$.

Let $A \in \mathcal{B}(H)$ and let $x \in H$ be an arbitrary unit vector from $H$. Put

$$T = 2A \circ x \otimes x = Ax \otimes x + x \otimes A^* x,$$

thus $T^* = x \otimes Ax + A^* x \otimes x$, in particular $T$ and $T^*$ are of rank two, moreover there image

$$\mathcal{R}(T) \subseteq \text{span}\{x, Ax\}, \quad \text{and} \quad \mathcal{R}(T^*) \subseteq \text{span}\{x, A^* x\}.$$ 

First we have

$$M(\Phi(T)) = 2M(\Phi(A \circ x \otimes x)) = 2M(\Phi(A) \circ \Phi(x \otimes x)) = 2M(\Phi(A) \circ x \otimes x).$$

Hence

$$Tr(\Phi(T)) = 2\langle \Phi(A) x, x \rangle$$

On the other hand, we have

$$T \circ (x \otimes x - \frac{I}{2}) = \langle Ax, x \rangle x \otimes x.$$

By the assumption on $\Phi$ and $h$, we get that

$$M(\Phi(T) \circ (x \otimes x - \frac{I}{2})) = h(\langle Ax, x \rangle) x \otimes x.$$

By Lemma 2.3

$$\Phi(T) \circ (x \otimes x - \frac{I}{2}) = h(\langle Ax, x \rangle) x \otimes x.$$ 

Therefore, from the equation (3.13),

$$\Phi(T) = 2\Phi(T) \circ (x \otimes x) - 2h(\langle Ax, x \rangle) x \otimes x$$

$$= \Phi(T) x \otimes x + x \otimes \Phi(T)^* x - 2h(\langle Ax, x \rangle) x \otimes x.$$


Multiplying the last equation on the right by \( x \otimes x \), then we get
\[
\Phi(T)x \otimes x = \Phi(T)x \otimes x + (\Phi(T)x, x)x \otimes x - 2h(\langle Ax, x \rangle)x \otimes x.
\]
This implies the following equation
\[
\langle \Phi(T)x, x \rangle = 2h(\langle Ax, x \rangle).
\]
Now, from equation (3.13) we get
\[
h(\langle Ax, x \rangle) = Tr(\Phi(T) \circ (x \otimes x - \frac{I}{2}))
\]
\[
= Tr(\Phi(T)(x \otimes x - \frac{I}{2}))
\]
\[
= \langle \Phi(T)x, x \rangle - \frac{Tr(\Phi(T))}{2}.
\]

Then, from equations (3.12) we get
\[
Tr(\Phi(T)) = 2\langle \Phi(T)x, x \rangle - 2h(\langle Ax, x \rangle) = 2h(\langle Ax, x \rangle).
\]
From equation (3.12) and (3.15), we conclude that
\[
\langle \Phi(A)x, x \rangle = h(\langle Ax, x \rangle).
\]
The last equality holds for every unit vector \( x \in H \) and every \( A \in \mathcal{B}(H) \). Let us distinct the following two cases:

**Case 1:** If the function \( h = Id_C \), then from the equation (3.16) we have,
\[
\langle \Phi(A)x, x \rangle = \langle Ax, x \rangle, \text{ for all unit vector } x \text{ and } A \in \mathcal{B}(H).
\]
This show that the map \( \Phi \) is the identity.

**Case 2:** Suppose that the function \( h = \overline{Id}_C \) is the complex conjugate. Hence from equation (3.16), we get
\[
\langle \Phi(A)x, x \rangle = \langle A^*x, x \rangle, \text{ for all unit vector } x \text{ and } A \in \mathcal{B}(H).
\]
Which implies that \( \Phi(A) = A^* \) for all \( A \in \mathcal{B}(H) \).

Now, let us consider \( A = x \otimes x' \) with \( x, x' \) are unit, independent and non-orthogonal vectors in \( H \). Then \( A^* = x' \otimes x \). By Proposition 2.1, we have
\[
\mathcal{M}(\Phi(A)) = \mathcal{M}(A^*) = \frac{1}{2}(x' + <x', x> x) \otimes x,
\]
and
\[
\Phi(\mathcal{M}(A)) = (\mathcal{M}(A))^* = \frac{1}{2}(x' \otimes (x + <x', x > x')).
\]
which contradicts with the fact that \( \Phi \) commute with the mean transform. Hence, we conclude that the function \( h \) must be the identity, and \( \Phi \) must be linear and of the form
\[
\Phi(T) = UTU^*, \text{ for all } T \in \mathcal{B}(H),
\]
where $U : H \rightarrow K$ is a unitary or anti-unitary operator.

**Acknowledgments.**
I wish to thank Professor Mostafa Mbekhta for the interesting discussions as well as his useful suggestions for the improvement of this paper.

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