On the Nilpotent Multiplier of a Free Product *

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Abstract

In this paper, using a result of J. Burns and G. Ellis (Math. Z. 226(1997) 405-28.), we prove that the $c$-nilpotent multiplier (the Baer-invariant with respect to the variety of nilpotent groups of class at most $c$, $N_c$.) does commute with the free product of cyclic groups of mutually coprime order.

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1. Introduction and Motivation

I. Schur [12], in 1904, using projective representation theory of groups, introduced the notion of a multiplier of a finite group. It was known later that the Schur multiplier had a relation with homology and cohomology of groups. In fact, if $G$ is a finite group, then

$$M(G) \cong H^2(G, \mathbb{C}^*) \quad \text{and} \quad M(G) \cong H_2(G, \mathbb{Z}) ,$$

where $M(G)$ is the Schur multiplier of $G$, $H^2(G, \mathbb{C}^*)$ is the second cohomology of $G$ with coefficient in $\mathbb{C}^*$ and $H_2(G, \mathbb{Z})$ is the second internal homology of $G$ [see 7]. In 1942, H. Hopf [6] proved that

$$M(G) \cong H^2(G, \mathbb{C}^*) \cong \frac{R \cap F'}{[R, F]} ,$$

where $G$ is presented as a quotient $G = F/R$ of a free group $F$ by a normal subgroup $R$ in $F$. He also proved that the above formula is independent of the presentation of $G$.

R. Baer [1], in 1945, using the variety of groups, generalized the notion of the Schur multiplier as follows.

Let $\mathcal{V}$ be a variety of groups defined by the set of laws $V$ and let $G$ be a group with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$. Then the Baer-invariant of $G$ with respect to the variety $\mathcal{V}$ is defined to be

$$\mathcal{V}M(G) := \frac{R \cap V(F)}{[RV^*F]} ,$$

where $V(F)$ is the verbal subgroup of $F$ with respect to $\mathcal{V}$ and

$$[RV^*F] = \langle v(f_1, \ldots, f_{i-1}, f_r, f_{i+1}, \ldots, f_n)v(f_1, \ldots, f_i, \ldots f_n)^{-1} \mid r \in R, \ 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbb{N} \rangle .$$

It is known that the Baer-invariant of a group $G$ is always abelian and independent of the choice of the presentation of $G$. (See C. R. Leedham-Green and S. McKay [8], from which our notation has been taken, and H.
Neumann [10] for the notion of variety of groups.) Note that if \( V \) is the variety of abelian groups, \( \mathcal{A} \), then the Baer-invariant of \( G \) will be

\[ \mathcal{A}M(G) = \frac{R \cap F'}{[R, F]} , \]

which is the Schur multiplier of \( G \), \( M(G) \). Also if \( V = \mathcal{N}_c \) is the variety of nilpotent groups of class at most \( c \geq 1 \), then the Baer-invariant of the group \( G \) with respect to \( \mathcal{N}_c \) will be

\[ \mathcal{N}_cM(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]} , \]

where \( \gamma_{c+1}(F) \) is the \((c + 1)\)-st term of the lower central series of \( F \) and \( [R, F] = [R, F] , [R, cF] = [[R, c-1 F], F] \). According to J. Burns and G. Ellis’ paper [2] we shall call \( \mathcal{N}_cM(G) \) the \( c \)-nilpotent multiplier of \( G \) and denote it by \( \mathcal{M}(c)(G) \). It is easy to see that 1-nilpotent multiplier is actually the Schur multiplier.

**Theorem 1.1**

Let \( V \) be a variety of groups, then \( VM( - ) \) is a covariant functor from the category of all groups, \( \text{Groups} \), to the category of all abelian groups, \( \text{Ab} \).

**Proof.** See [8] page 107.

Now with regards to the above theorem, we are going to concentrate on the relation between the functors, \( \mathcal{M}(c)( - ) \), \( c \geq 1 \), and the free product as follows.

In 1952, C. Miller [9] proved that \( M(G) \cong H(G) \), where \( H(G) \) is the group of all commutator relations of \( G \), taken modulo universal commutator relations. He also showed that

**Theorem 1.2** (C. Miller [9])

Let \( G_1 \) and \( G_2 \) be two arbitrary groups, then \( H(G_1 * G_2) \cong H(G_1) \oplus H(G_2) \), where \( G_1 * G_2 \) is the free product of \( G_1 \) and \( G_2 \).

By the above theorem we can conclude the following corollary.
Corollary 1.3

The Schur multiplier functor, \( M(-) : \text{Groups} \rightarrow \text{Ab} \), is coproduct-preserving. (Note that coproduct in \( \text{Groups} \) is free product and in \( \text{Ab} \) is direct sum.)

In view of homology and cohomology of groups, we have the following theorem.

**Theorem 1.4**

Let \( A \) be a \( G \)-module, then \( H^n(-, A) \), \( H_n(-, A) \) are coproduct-preserving functors from \( \text{Groups} \) to \( \text{Ab} \), for \( n \geq 2 \), i.e

\[
H^n(G_1 * G_2, A) \cong H^n(G_1, A) \oplus H^n(G_2, A) \quad \text{for all } n \geq 2 ,
\]

\[
H_n(G_1 * G_2, A) \cong H_n(G_1, A) \oplus H_n(G_2, A) \quad \text{for all } n \geq 2 .
\]

**Proof.** See [5, page 220].

Note that the above theorem does also confirm that the functor

\[
M(-) = H_2(-, \mathbb{Z}) = H^2(-, C^*),
\]

is coproduct-preserving.

Now, with regards to the above theorems, it seems natural to ask whether the \( c \)-nilpotent multiplier functors \( M^{(c)}(-) \), \( c \geq 2 \), are coproduct-preserving or not. To answer the question, first we state an important theorem of J. Burns and G. Ellis [2, Proposition 2.13 & Erratum at http://hamilton.ucg.ie/] which is proved by a homological method.

**Theorem 1.5** (J. Burns and G. Ellis [2])

Let \( G \) and \( H \) be two arbitrary groups, then there is an isomorphism

\[
M^{(2)}(G*H) \cong M^{(2)}(G) \oplus M^{(2)}(H) \oplus M(G) \otimes H^{ab} \oplus M(H) \otimes G^{ab} \oplus \text{Tor}(G^{ab}, H^{ab}) ,
\]

where \( G^{ab} = G/G' \), \( H^{ab} = H/H' \) and \( \text{Tor} = \text{Tor}_1^\mathbb{Z} \).
Now, we are ready to show that the second nilpotent multiplier functor $M^{(2)}(-)$, is not coproduct-preserving, in general.

**Example 1.6**

Let $D_\infty = \langle a, b | a^2 = b^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$ be the infinite dihedral group. Then

$$M^{(2)}(D_\infty) \ncong M^{(2)}(\mathbb{Z}_2) \oplus M^{(2)}(\mathbb{Z}_2).$$

**Proof.** By Theorem 1.5 we have

$$M^{(2)}(D_\infty) = M^{(2)}(\mathbb{Z}_2 * \mathbb{Z}_2)$$

$$\cong M^{(2)}(\mathbb{Z}_2) \oplus M^{(2)}(\mathbb{Z}_2) \oplus \mathbb{Z}_2 \otimes M(\mathbb{Z}_2) \oplus M(\mathbb{Z}_2) \otimes \mathbb{Z}_2 \oplus \text{Tor}(\mathbb{Z}_2, \mathbb{Z}_2).$$

Clearly $M^{(2)}(\mathbb{Z}_2) = 0 = M(\mathbb{Z}_2)$. Also it is well-known that $\text{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$ (see [11]). Therefore

$$M^{(2)}(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

but

$$M^{(2)}(\mathbb{Z}_2) \oplus M^{(2)}(\mathbb{Z}_2) \cong 1.$$

Hence the result holds. □

In spite of the above example, using Theorem 1.5, we can show that the second nilpotent multiplier functor, $M^{(2)}(-)$, preserves the coproduct of a finite family of cyclic groups of mutually coprime order.

**Corollary 1.7**

Let $\{C_i | 1 \leq i \leq n\}$ be a family of cyclic groups of mutually coprime order. Then

$$M^{(2)}\left(\prod_{i=1}^n C_i\right) \cong \bigoplus_{i=1}^n M^{(2)}(C_i),$$

where $\prod_{i=1}^n C_i$ is the free product of $C_i$’s, $1 \leq i \leq n$.

**Proof.** We proceed by induction on $n$. If $n = 2$, then by Theorem 1.5 and
using the fact that the Baer-invariant of any cyclic group is trivial, we have

$$M^{(2)}(C_1 \ast C_2) \cong \text{Tor}(C_1, C_2).$$

Since $C_1$ and $C_2$ are finite abelian groups with coprime order, Tor$(C_1, C_2) \cong C_1 \otimes C_2 = 1$ (see [11]).

If $n = 3$, then similarly we have

$$M^{(2)}(C_1 \ast C_2 \ast C_3) \cong M^{(2)}(C_1 \ast C_2) \oplus M^{(2)}(C_3) \oplus M^{(1)}(C_1 \ast C_2) \otimes C_3$$

$$\oplus (C_1 \ast C_2)^{ab} \otimes M^{(1)}(C_3) \oplus \text{Tor}((C_1 \ast C_2)^{ab}, C_3)$$

$$\cong \text{Tor}(C_1 \oplus C_2, C_3) \cong (C_1 \oplus C_2) \otimes C_3 \cong (C_1 \otimes C_3) \oplus (C_2 \otimes C_3) = 1.$$

Note that $M^{(2)}(C_1 \ast C_2) = M^{(2)}(C_3) = M^{(1)}(C_1 \ast C_2) = 1$ By a similar procedure we can complete the induction. □

2. The Main Result

In this section, we are going to generalize the above corollary to the variety of nilpotent groups of class at most $c$, $N_c$, for all $c \geq 2$.

Notation 2.1

Let $C_i = \langle x_i | x_i^{r_i} \rangle \cong \mathbb{Z}_{r_i}$ be cyclic group of order $r_i$, $1 \leq i \leq t$ such that $(r_i, r_j) = 1$ for all $i \neq j$. Put $C = \Pi_{i=1}^t C_i$, the free product of $C_i$’s, $1 \leq i \leq t$, $F = \prod_{i=1}^t F_i$, where $F_i$ is the free group on $\{x_i\}$, $1 \leq i \leq t$, and $S = \langle x_i^{r_i} | 1 \leq i \leq t \rangle$, the normal closure of $\{x_i^{r_i} | 1 \leq i \leq t\}$ in $F$. Note that $F$ is free on $\{x_1, \ldots x_t\}$. It is easy to see that the following sequence is exact.

$$1 \longrightarrow S \longrightarrow F \longrightarrow \text{nat} \longrightarrow C \longrightarrow 1.$$

Define by induction $\rho_1(S) = S$, $\rho_{n+1}(S) = [\rho_n(S), F]$. Now by Theorems 1.2 and 1.5, we have the following corollary.

Corollary 2.2

By the above notation and assumption, we have

(i) $S \cap \gamma_2(F) = \rho_2(S)$. 

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(ii) $S \cap \gamma_3(F) = \rho_3(S)$ and hence $\rho_2(S) \cap \gamma_3(F) = \rho_3(S)$.

**Proof.** (i) By Corollary 1.3 $M(C) = M(\prod_{i=1}^t C_i) \cong \oplus \sum_{i=1}^t M(C_i) = 1$. On the other hand, $M(C) \cong S \cap \gamma_2(F)/[S,F]$. Thus $S \cap \gamma_2(F)/[S,F] = 1$ and so $S \cap \gamma_2(F) = [S,F] = \rho_2(S)$.

(ii) By Corollary 1.7 $M^{(2)}(C) = M^{(2)}(\prod_{i=1}^t C_i) \cong \oplus \sum_{i=1}^t M^{(2)}(C_i) = 1$. Also by definition $M^{(2)}(C) \cong S \cap \gamma_3(F)/[S,F]$, so $\cap \gamma_3(F) = [S,F] = \rho_3(S)$. Moreover $\rho_3(S) \subseteq \rho_2(S) \cap \gamma_3(F) \subseteq S \cap \gamma_3(F) = \rho_3(S)$ and hence $\rho_2(S) \cap \gamma_3(F) = \rho_3(S)$. \qed

Now we consider the following two technical lemmas.

**Lemma 2.3**

By the Notation 2.1 $\rho_n(S) \cap \gamma_{n+1}(F) = \rho_{n+1}(S)$, for all $n \geq 1$.

**Proof.** We proceed by induction on $n$. The assertion holds for $n = 1, 2$, by Corollary 2.2.

Now in order to avoid a lot of commutator manipulations, we prove the result for $n = 3$ in the special case $t = 2$. Put $x = x_1$, $y = x_2$, $r = r_1$, $s = r_2$. So $F$ is free on $\{x, y\}$ and $S = \langle x^r, y^s \rangle^F$.

Let $g$ be a generator of $\rho_3(S)$, then

$$g = [(x^r)^{a_1}, y^{a_2}, x^{a_3}] \text{ or } [(x^r)^{a_1}, y^{a_2}, y^{a_3}] \text{ or } [(y^s)^{a_1}, x^{a_2}, y^{a_3}] \text{ or } [(y^s), x^{a_2}, x^{a_3}],$$

where $a_i \in \mathbb{Z}$. Clearly modulo $\rho_4(S)$ we have

$$g \equiv [x^r, y, x]^\alpha \text{ or } [x^r, y, y]^\alpha \text{ or } [y^s, x, y]^\alpha \text{ or } [y^s, x, x]^\alpha, \text{ where } \alpha \in \mathbb{Z}.$$

Now, let $z \in \rho_3(S) \cap \gamma_4(F)$, then $z \in \rho_3(S)$. By the above fact and using a collecting process similar to basic commutators (see [3]) we can obtain the following congruence modulo $\rho_4(S)$

$$z \equiv [y^s, x, y]^{\alpha_1} [y, x^r, y]^{\beta_1} [y^s, x, x]^{\alpha_2} [y, x^r, x]^{\beta_2}$$

$$\equiv [y, x, y]^{\alpha_1 + r \beta_1} [y, x, x]^{\alpha_2 + r \beta_2} \pmod{\gamma_4(F)}, \text{ where } \alpha_i, \beta_i \in \mathbb{Z}.$$
Note that we consider the order on \( \{x, y\} \) as \( x < y \).

Since \( z \in \rho_3(S) \cap \gamma_4(F) \) and \( \rho_4(S) \subseteq \gamma_4(F) \), we have
\[
[y, x, y]^{s\alpha_1+r\beta_1}[y, x, x]^{s\alpha_2+r\beta_2} \in \gamma_4(F).
\]

It is a well-known fact, by P. Hall [3, 4], that \( \gamma_3(F)/\gamma_4(F) \) is the free abelian group on \( \{[y, x, y], [y, x, x]\} \). Therefore we conclude that \( s\alpha_i + r\beta_i = 0 \), for \( i = 1, 2 \).

By a routine commutator calculation we have
\[
[y^s, x, y]^{\alpha_1}[y, x^r, y]^{\beta_1} \equiv [[y^s, x]^{\alpha_1}[y, x^r]^{\beta_1}, y] \pmod{\rho_4(S)}
\]
\[
[y^s, x, x]^{\alpha_2}[y, x^r, x]^{\beta_2} \equiv [[y^s, x]^{\alpha_2}[y, x^r]^{\beta_2}, x] \pmod{\rho_4(S)}.
\]

Also
\[
[y, x]^{s\alpha_i+r\beta_i} \equiv [y^s, x]^{\alpha_i}[y, x^r]^{\beta_i} \in \rho_2(S), \text{ for } i = 1, 2 \pmod{\gamma_3(F)}.
\]

since \( s\alpha_i + r\beta_i = 0 \), \( i = 1, 2 \), we have
\[
[y^s, x]^{\alpha_1}[y, x^r]^{\beta_1} \in \rho_2(S) \cap \gamma_3(F), \text{ for } i = 1, 2.
\]

By corollary 2.2 \((ii)\) \( \rho_2(S) \cap \gamma_3(F) = \rho_3(S) \), thus
\[
[y^s, x]^{\alpha_i}[y, x^r]^{\beta_i} \in \rho_3(S), \text{ for } i = 1, 2.
\]

Therefore by \((*)\)
\[
[y^s, x, y]^{\alpha_1}[y, x^r, y]^{\beta_1}, \ [y^s, x, x]^{\alpha_2}[y, x^r, x]^{\beta_2} \in \rho_4(S).
\]

Hence \( z \in \rho_4(S) \), and then \( \rho_3(S) \cap \gamma_4(F) = \rho_4(S) \).

Note that by a similar method we can obtain the result for \( n \), using induction hypothesis. \( \square \)
Lemma 2.4

By the above notation and assumption, $S \cap \gamma_n(F) = \rho_n(S)$, for all $n \geq 1$.

**Proof.** We proceed by induction on $n$. For $n = 1, 2$ Corollary 2.2 gives the result. Now, suppose $S \cap \gamma_n(F) = \rho_n(S)$ for a natural number $n$. We show that $S \cap \gamma_{n+1}(F) = \rho_{n+1}(S)$.

Clearly $\rho_{n+1}(S) \subseteq S \cap \gamma_{n+1}(F)$, also $S \cap \gamma_{n+1}(F) \subseteq S \cap \gamma_n(F) = \rho_n(S)$, by induction hypothesis. Therefore by Lemma 2.3

$$\rho_{n+1}(S) \subseteq S \cap \gamma_{n+1}(F) \subseteq \rho_n(S) \cap \gamma_{n+1}(F) = \rho_{n+1}(S).$$

Hence the result holds. \(\Box\)

Now, we are ready to show that the $c$-nilpotent multiplier functors, $\mathcal{N}_c M(-)$, preserve the coproduct of cyclic groups of mutually coprime order, for all $c \geq 1$.

**Theorem 2.5**

By the above notation and assumption,$$
M^{(c)}(\prod_{i=1}^t C_i) \cong \bigoplus_{i=1}^t M^{(c)}(C_i) = 1 , \ for \ all \ c \geq 1.
$$

**Proof.** By Lemma 2.4 and the definition of $c$-nilpotent multiplier, we have

$$M^{(c)}(\prod_{i=1}^t C_i) = \frac{S \cap \gamma_{c+1}(F)}{[S,F]} = \frac{S \cap \gamma_{c+1}(F)}{\rho_{c+1}(S)} = 1 , \ for \ all \ c \geq 1 .$$

On the other hand, since $C_i$’s are cyclic, $M^{(c)}(C_i) = 1$, so $\bigoplus_{i=1}^t \mathcal{N}_c M(C_i) = 1$, for all $c \geq 1$. Hence the result holds. \(\Box\)

**Remark**

In [2] it can be found some relations between the $c$-nilpotent multiplier and the $c$-isoclinism theory of P. Hall and also the notion of $c$-capable groups. Moreover, one may find in [2, page 423] a topological and also a homological interpretation of the $c$-nilpotent multiplier. Thus our result, Theorem 2.5, can be expressed and used in the above mentioned areas.
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