HEARING THE TYPE OF A DOMAIN IN $\mathbb{C}^2$ WITH THE $\overline{\partial}$-NEUMANN LAPLACIAN

SIQI FU

Contents

1. Introduction 1
2. Preliminaries 3
3. Special holomorphic coordinates and non-isotropic bidiscs 6
4. Rescale the $\overline{\partial}$-Neumann Laplacian 12
5. Auxiliary estimates 16
6. Estimates on the spectral kernel 19
7. Hearing a finite type property 23
References 25

1. Introduction

Motivated by Mark Kac’s famous question “Can one hear the shape of a drum?” (see [Kac66, GWW92]), we study the interplays between the geometry of a bounded domain in $\mathbb{C}^n$ and the spectrum of the $\overline{\partial}$-Neumann Laplacian. Since the work of Kohn [Ko63, Ko64], it has been discovered that regularity of the $\overline{\partial}$-Neumann Laplacian is intimately connected to the boundary geometry. (See, for example, the surveys [BS99, Ch99, DK99, FS01] and the books [FoK72, H90, Kr92, CS99, O02].) It is then natural to expect that one can “hear” more about the geometry of a bounded domain in $\mathbb{C}^n$ with the $\overline{\partial}$-Neumann Laplacian than with the usual Dirichlet/Neumann Laplacians.

For bounded domains in $\mathbb{C}^n$, it follows from Hörmander’s $L^2$-estimates of the $\overline{\partial}$-operator [H65] that pseudoconvexity implies positivity of the spectrum of the $\overline{\partial}$-Neumann Laplacian on all $(0,q)$-forms, $1 \leq q \leq n - 1$. The converse is also true (under the assumption that the interior of the closure of the domain is the domain itself). This is a consequence of the sheaf cohomology theory dated back to Oka and H. Cartan—it is proved, implicitly, in [Se53, L66]. (See [Fu05] for a discussion and proofs of this and other facts without the sheaf cohomology theory.) Therefore, in Kac’s language, we can “hear” pseudoconvexity via the $\overline{\partial}$-Neumann Laplacian.

Regularity and spectral theories of the $\overline{\partial}$-Neumann Laplacian closely intertwine. For example, on the one hand, by a classical theorem of Hilbert in general operator theory, compactness of the $\overline{\partial}$-Neumann operator is equivalent to emptiness of the essential spectrum of the $\overline{\partial}$-Neumann Laplacian. On the other hand, by a result of Kohn and Nirenberg [KN65],

The author was supported in part by NSF grants DMS 0070697/0500909 and by an AMS Centennial Research Fellowship.
compactness of the $\overline{\partial}$-Neumann operator implies exact global regularity of the $\overline{\partial}$-Neumann Laplacian on $L^2$-Sobolev spaces. It was shown in [FS98] that for a bounded convex domain in $\mathbb{C}^n$, the essential spectrum of the $\overline{\partial}$-Neumann Laplacian on $(0,q)$-form is empty if and only if the boundary contains no $q$-dimensional complex varieties. However, such characterization does not hold even for pseudoconvex Hartogs domains in $\mathbb{C}^2$ (see [Ma97], also [FS01]). Recently, it was proved in [CF05] that for smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$, emptiness of the essential spectrum of $\overline{\partial}$-Neumann Laplacian on $(0,1)$-forms implies that the boundary contains no pluripotentials (more precisely, it satisfies property $(P)$ in the sense of Catlin [Ca84b] or equivalently is $B$-regular in the sense of Sibony [Si87]). This, together with an earlier result of Catlin [Ca84b] (cf. [St97]), shows that one can determine whether the boundary of a Hartogs domain in $\mathbb{C}^2$ contains pluripotentials via the spectrum of the $\overline{\partial}$-Neumann Laplacian.

In this paper, we continue our study of the spectral theory of the $\overline{\partial}$-Neumann Laplacian. Our main result can be stated as follows.

**Theorem 1.1.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Let $\mathcal{N}(\lambda)$ be the number of eigenvalues of the $\overline{\partial}$-Neumann Laplacian that are less than or equal to $\lambda$. Then $\partial \Omega$ is of finite type if and only if $\mathcal{N}(\lambda)$ has at most polynomial growth.

Recall that a smooth domain is strictly pseudoconvex if for each boundary point, there is a local change of holomorphic coordinates which makes the boundary near this point strictly convex (in the sense that the real Hessian of the defining function is positive definite on the real tangent space). It is pseudoconvex if it can be exhausted by strictly pseudoconvex ones. (We refer the reader to the textbook [Kr01] for a treatise of these concepts.) The type of a smooth boundary $\partial \Omega$ (in the sense of Kohn [Ko72]) is the maximal order of contact of a (regular) complex variety with $\partial \Omega$. (See [D82, Ca84a, D93] for more information on this and other notions of finite type.)

Theorem 1.1 consists of two parts. More precisely, for the sufficiency, we establish the following result.

**Theorem 1.2.** Let $\Omega \subset \mathbb{C}^2$ be smooth bounded pseudoconvex domain of finite type $2m$. Then $\mathcal{N}(\lambda) \lesssim \lambda^{m+1}$.

The Weyl type asymptotic formula for $\mathcal{N}(\lambda)$ for strictly pseudoconvex domains in $\mathbb{C}^n$ was established in [Me81] by Metivier via an analysis of the spectral kernel of the $\overline{\partial}$-Neumann Laplacian. The heat kernel of the $\overline{\partial}$-Neumann Laplacian on strictly pseudoconvex domains, as well as that of the Kohn Laplacian on the boundary, were studied in a series of papers [SS1, BGSS1, STS1, BeSS7, BeSS8] by various authors. Metivier’s formula was recovered as a consequence. Recently, the heat kernel of the Kohn Laplacian on finite type boundaries in $\mathbb{C}^2$ was studied by Nagel and Stein [NS01], from which one could also deduce a result similar to Theorem 1.2 for the Kohn Laplacian on the boundary.

We follow Metivier’s approach in proving Theorem 1.2 by studying the spectral kernel. We are also motivated by the work on the Bergman kernel by Nagel et al [NRSW89] and by Catlin [Ca89] and McNeal [Me89] as well as related work of Christ [Chi88] and Fefferman and Kohn [FeK88]. There is one important distinction between the spectral kernel and the Bergman kernel: While the Bergman kernel transforms well under biholomorphic mappings, the spectral kernel does not. It does not transform well even under non-isotropic dilations. To overcome this difficulty, instead of (locally) rescaling the domain to unit scale and studying the $\overline{\partial}$-Neumann Laplacian on the rescaled domain as in the Bergman kernel case, we rescale both the domain and the $\overline{\partial}$-Neumann Laplacian as in [Me81] (see Section 4). In
doing so, we are led to study non-isotropic bidiscs that have larger radii in the complex normal direction. Roughly speaking, at a boundary point of type $2m$, the quotient of the radii in the complex tangential and normal directions for the bidiscs used here is $\tau : \tau^m$ while in the Bergman kernel case it is $\tau : \tau^{2m}$ ($\tau > 0$ is small). To establish desirable properties, such as doubling and engulfing properties, for these non-isotropic bidiscs, we employ both pseudoconvexity and the finite type condition (see Section 3). Note that only the finite type condition was used in establishing these properties for the smaller bidiscs used in the Bergman kernel case. Here in our analysis of these bidiscs, we make essential use of an observation by Fornæss and Sibony [FoSi89]. Also crucial to our analysis is a uniform Kohn type estimate on the rescaled $\partial$-Neumann Laplacian. Here the sesquilinear form that defines the rescaled $\overline{\partial}$-Neumann Laplacian controls not only the tangential Sobolev norm of some positive order $\varepsilon$ of any $(0, 1)$-form $u$ compactly supported on the larger bidiscs but also the square of the quotient of the radii times the tangential Sobolev norm of order $-1 + \varepsilon$ of the bar derivative of $u$ in the complex normal direction (see Lemma 4.5 below). By carefully flattening the boundary, we then reduce the problem to estimating eigenvalues of auxiliary operators on the half-space (see Section 5). The final steps of the proof of Theorem 1.2 is given in Section 6.

For the necessity, we prove the following slightly more general result.

**Theorem 1.3.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $N_q(\lambda)$ be the number of eigenvalues of the $\overline{\partial}$-Neumann Laplacian on $(0, q)$-forms that are less than or equal to $\lambda$. If $N_q(\lambda)$ has at most polynomial growth for some $q$, $1 \leq q \leq n - 1$, then $b\Omega$ is of finite $D_{n-1}$-type.

Recall that the $D_{n-1}$-type of $b\Omega$ is the maximal order of contact of $(n - 1)$-dimensional (regular) complex varieties with $b\Omega$. It was observed by D’Angelo [D87] that the $D_{n-1}$-type is identical to the second entry in Catlin’s multitype. An ingredient in the proof of Theorem 1.3 is a wavelet construction of Lemarié and Meyer [LM86] (see Section 7). A result similar to Theorem 1.3 for the Kohn Laplacian on the boundaries in $\mathbb{C}^2$ is also known to M. Christ [Ch].

Throughout the paper, we use $C$ to denote positive constants which may be different in different appearances. For the reader’s convenience, we make an effort to have our presentation self-contained.

**Acknowledgment:** The author is indebted to Professors M. Christ, H. Jacobowitz, J. J. Kohn, T. Ohsawa, N. Stanton, and Y.-T. Siu for stimulating conversations and kind encouragement.

### 2. Preliminaries

We first recall several relevant facts from the general operator theory. Let $H_1$, $H_2$ be complex Hilbert spaces. Let $T$ be a compact operator from $H_1$ into $H_2$. Then it follows from the min-max principle that the singular values of $T$ (i.e., the non-zero eigenvalues of $|T| = (T^*T)^{1/2}$) is given by

$$\lambda_j(T) = \inf_{g_1, \ldots, g_{j-1} \in H_1} \sup\{\|Tf\| : f \perp g_1, \ldots, g_{j-1}; \|f\| = 1\}, \quad j = 1, 2, \ldots,$$

where the singular values are arranged in decreasing order and repeated according to multiplicity (e.g., [W80]). (Throughout this paper, we will use $\lambda_j(T)$ to denote the eigenvalues/singular values of a compact operator $T$ arranged in this order as well as the eigenvalues...
of an unbounded operator $T$ with compact resolvent arranged in the reverse order.) It follows that for compact operators $T_1, T_2: H_1 \to H_2$ and $T_3: H_2 \to H_3$,

\begin{equation}
\lambda_{j+k+1}(T_1 + T_2) \leq \lambda_{j+1}(T_1) + \lambda_{j+1}(T_2)
\end{equation}

and

\begin{equation}
\lambda_{j+k+1}(T_3 \circ T_1) \leq \lambda_{j+1}(T_1)\lambda_{k+1}(T_3).
\end{equation}

Let $Q$ be a non-negative, densely defined, and closed sesquilinear form on a complex Hilbert space $H$. Then $Q$ uniquely determines a non-negative, densely defined, and self-adjoint (unbounded) operator $S$ such that $\text{Dom}(S^{1/2}) = \text{Dom}(Q)$ and

\[ Q(u, v) = (Su, v) = (S^{1/2}u, S^{1/2}v) \]

for all $u \in \text{Dom}(S)$ and $v \in \text{Dom}(Q)$. (See, for example, [Dav95] for related material.) For any subspace $L \subset \text{Dom}(Q)$, let $\lambda(L) = \sup\{|Q(u, u)| : u \in L, \|u\| = 1\}$. For any positive integer $j$, let

\begin{equation}
\lambda_j(Q) = \inf\{\lambda(L) | L \subset \text{Dom}(Q), \dim(L) = j\}.
\end{equation}

It follows that the associated operator $S$ has compact resolvent if and only if $\lambda_j(Q) \to \infty$ as $j \to \infty$. In this case, $\lambda_j(Q)$ equals $\lambda_j(S)$, the $j^{th}$ eigenvalue of $S$.

We now recall the setup for the $\overline{\partial}$-Neumann Laplacian (e.g., [FoK72, CS99]). Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. For $1 \leq q \leq n$, let $L^2_{(0,q)}(\Omega)$ denote the space of $(0,q)$-forms with square integrable coefficients and with the standard Euclidean inner product whose norm is given by

\[ \|\sum' a_J d\bar{z}_J\|^2 = \sum' \int_{\Omega} |a_J|^2 dV(z), \]

where the prime indicates the summation over strictly increasing $q$-tuples $J$. (We consider $a_J$ to be defined on all $q$-tuples, antisymmetric with respect to $J$.) For $0 \leq q \leq n - 1$, let $\overline{\partial}_q: L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ be the $\overline{\partial}$-operator defined in the sense of distribution. This is a closed and densely defined operator. Let $\overline{\partial}^*_q$ be its adjoint. For $1 \leq q \leq n - 1$, let

\[ Q_q(u, v) = (\overline{\partial}_q u, \overline{\partial}_q v) + (\overline{\partial}^*_{q-1} u, \overline{\partial}^*_{q-1} v) \]

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with $\text{Dom}(Q_q) = \text{Dom}(\overline{\partial}_q) \cap \text{Dom}(\overline{\partial}^*_{q-1})$. It is evident that $Q_q$ is non-negative, densely defined, and closed. The operator associated with $Q_q$ is the $\overline{\partial}$-Neumann Laplacian $\Box_q$ on $L^2_{(0,q)}(\Omega)$. The following lemma is a simple consequence of the min-max principle \cite{2.4}.

**Lemma 2.1.** Suppose $\lambda_j(Q) \geq C_1 j^\varepsilon$ for some constants $C_1 > 0$ and $\varepsilon > 0$. If $u_k \in \text{Dom}(Q)$, $1 \leq k \leq j$, satisfy

\[ \|\sum_{k=1}^j c_k u_k\|^2 \geq C_2 \sum_{k=1}^j |c_k|^2 \]

for some constant $C_2 > 0$ and for all $(c_1, \ldots, c_j) \in \mathbb{C}^j$, then

\[ \max_{1 \leq k \leq j} Q(u_k, u_k) \geq C_1 C_2 j^\varepsilon / (1 + \varepsilon). \]
Proof. Let $\hat{\lambda}_k, 1 \leq k \leq j$, be the eigenvalues of the Hermitian matrix $M = (Q(u_k, u_l))_{1 \leq k, l \leq j}$. Then by the min-max principle,

$$\hat{\lambda}_k = \inf \{ \lambda(\bar{L}) ; \bar{L} \subset \mathbb{C}^j, \dim(\bar{L}) = k \}$$

where

$$\lambda(\bar{L}) = \sup \{ \sum_{k, l=1}^{j} c_k \bar{c}_l Q(u_k, u_l) ; (c_1, \ldots, c_j) \in \bar{L}, \sum_{k=1}^{j} |c_k|^2 = 1 \}.$$

Let $L = \{ \sum_{j=1}^{j} c_j u_l ; (c_1, \ldots, c_j) \in \bar{L} \}$. Then $\hat{\lambda}(\bar{L}) \geq C_2 \lambda(L)$. Hence $\hat{\lambda}_k \geq C_2 \lambda_k(Q)$ for all $1 \leq k \leq j$. Therefore,

$$j \max_{1 \leq k \leq j} Q(u_k, u_k) \geq \text{tr}(M) = \sum_{k=1}^{j} Q(u_k, u_k) = \sum_{k=1}^{j} \hat{\lambda}_k \geq C_2 \sum_{k=1}^{j} \lambda_k(Q) \geq C_1 C_2 \sum_{k=1}^{j} k^\varepsilon \geq C_1 C_2 \int_{0}^{j} x^\varepsilon \, dx = C_1 C_2 j^{\varepsilon+1}/(\varepsilon + 1).$$

Dividing both sides by $j$, we obtain the lemma. \qed

**Proposition 2.2.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Then $\lambda_j(\Box_q) \leq \lambda_{nj}(\Box_{q+1})$ for all $1 \leq q \leq n-1$ and $j$. In particular, if $\Box_q$ has compact resolvent, so is $\Box_{q+1}$.

Proof. Let $u = \sum_{|j|=q+1}^n u_j d\bar{z}_j \in C^\infty(\Omega) \cap \text{Dom}(Q_{q+1})$. Write

$$u = \frac{1}{(q+1)!} \sum_{|j|=q+1}^n u_j d\bar{z}_j = \frac{1}{q+1} \sum_{j=1}^{n} \left( \frac{(-1)^q}{q!} \sum_{|K|=q} u_j K d\bar{z}_K \right) \wedge d\bar{z}_j = \frac{1}{q+1} \sum_{j=1}^{n} u_j \wedge d\bar{z}_j$$

where the $u_j$’s are $(0, q)$-forms defined by the expression in the parenthesis in the above equalities. It is easy to see that $u_j \in C^\infty(\Omega) \cap \text{Dom}(Q_q)$ and $\sum_{j=1}^{n} \|u_j\|^2 = (q + 1)\|u\|^2$. Moreover, by the Kohn-Morrey formula, we have

$$\sum_{j=1}^{n} Q_q(u_j, u_j) = (q + 1) \sum_{|j|=q+1}^{\prime} \sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial u_j}{\partial \bar{z}_j} \right|^2 dV + q \sum_{|K|=q} \sum_{j=1}^{n} \int_{\partial \Omega} \frac{\partial^2 \rho(z)}{\partial z \partial \bar{z}_j} u_j K d\bar{z}_K dS$$

(2.5)

$$\leq (q + 1) Q_{q+1}(u, u)$$

where $\rho$ is any defining function of $\Omega$ whose gradient has unit length on $\partial \Omega$. Consider $Q(u, u) = \sum_{j=1}^{n} Q_q(u_j, u_j)$ as a quadratic form on $\oplus_{j=1}^{n} L^2_{(0,q)}(\Omega)$. The associated self-adjoint operator is then $\Box = \oplus_{j=1}^{n} \Box_q$. Let $\hat{\lambda}_j$ be the number defined by (2.4) with $Q$ replaced by $\hat{Q}$. We then have $\lambda_j(\Box_{q+1}) \geq \hat{\lambda}_j$. If $\Box_q$ has compact resolvent, so does $\Box$. In this case, $\hat{\lambda}_{nj} = \lambda_j(\Box_q)$. If $\Box_q$ does not have compact resolvent, let $a$ be the bottom of its essential spectrum. If $\lambda_j(\Box_q) < a$ for all positive integer $j$, then the $\lambda_j(\Box_q)$’s are again eigenvalues of finite multiplicity and $\hat{\lambda}_{nj} = \lambda_j(\Box_q)$. Otherwise, let $j_0$ be the smallest integer such that $\lambda_{j_0}(\Box_q) = a$. In this case, $\lambda_j(\Box_q), 1 \leq j < j_0$, are eigenvalues and $\lambda_j(\Box_q) = a$ for $j \geq j_0$. Hence $\lambda_{nj} = \lambda_j(\Box_q)$ for $1 \leq j < j_0$ and $\hat{\lambda}_j = a$ for $j > n j_0$. Therefore, for all cases, we have $\lambda_j(\Box_q) = \hat{\lambda}_{nj}$. We thus conclude the proof of the lemma. \qed
Next we recall some elements of a wavelet construction of Lemarié and Meyer \((\text{LMS}6; \text{see also Dau88, HG96})\). Let \(a(t) \in C_0^\infty(\mathbb{R})\) be the cut-off function defined by \(a(t) = \exp(1/(t - 1) - 2/(2t - 1))\) on \((1/2, 1)\) and \(a(t) = 0\) elsewhere. Let
\[
b(t) = \begin{cases} 
(\int_{1/2}^{1} a(s) \, ds / \int_{1/2}^{1} a(s) \, ds)^{1/2}, & \text{if } t < 0; \\
(\int_{t}^{1} a(s) \, ds / \int_{t}^{1} a(s) \, ds)^{1/2}, & \text{if } t \geq 0.
\end{cases}
\]
Then \(b(t)\) is a smooth function supported in \([-1/2, 1]\) and satisfying \(b(t) \equiv 1\) on \([0, 1/2]\) and \(b^2(t) + b^2(t - 1) \equiv 1\) on \([1/2, 1]\). Let \(c_0\) be the \(L^2\)-norm of \(b(t)\). Let
\[
g_{k,c}(t) = c_0^{-1} c^{1/2} b(ct)e^{2\pi kct}
\]
where \(c > 0\) and \(k \in \mathbb{Z}\).

**Lemma 2.3.** For any given \(c > 0\), \(\{g_{k,c}(t)\}_{k \in \mathbb{Z}}\) is an orthonormal sequence in \(L^2\).

**Proof.** We provide the proof for completeness. It is easy to see that the \(L^2\)-norm of \(g_{k,c}(t)\) is 1. For distinct \(k, k' \in \mathbb{Z}\), we have
\[
\langle g_{k,c}, g_{k',c} \rangle = c_0^{-2} \int_{-\infty}^{\infty} b^2(t) e^{2\pi(k-k')ti} \, dt
\]
\[
= c_0^{-2} \left( \int_{-1/2}^{0} + \int_{0}^{1/2} + \int_{1/2}^{1} \right) b^2(t) e^{2\pi(k-k')ti} \, dt
\]
\[
= c_0^{-2} \left( \int_{1/2}^{1} (b^2(t - 1) + b^2(t)) e^{2\pi(k-k')ti} \, dt + \int_{0}^{1/2} b^2(t) e^{2\pi(k-k')ti} \, dt \right)
\]
\[
= c_0^{-2} \int_{0}^{1} e^{2\pi(k-k')ti} \, dt = 0.
\]

\(\square\)

3. Special holomorphic coordinates and non-isotropic bidiscs

The non-isotropic geometry of finite type boundaries in \(\mathbb{C}^2\) has been studied in depth in \(\text{NSW}5, \text{NRSW}89, \text{Ca}89\). For completeness we provide details below. The key difference here is, as noted above, that by using a result of Fornæss and Sibony \(\text{FoSS}9\), we establish desirable properties for non-isotropic bidiscs of larger size.

Let \(\Omega = \{z \in \mathbb{C}^2 \mid r(z) < 0\}\) be a smooth bounded domain with a defining function \(r \in C^\infty(\mathbb{C}^2)\). Assume that \(|dr| = 1\) on \(b\Omega\). Let
\[
L = \frac{\partial r}{\partial z_1} \frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_2} \frac{\partial}{\partial z_2}.
\]

Let \(z' \in b\Omega\). For \(j, k \geq 1\), let
\[
\mathcal{L}_{jk} \partial \bar{\partial} r(z') = \underbrace{\mathcal{L} \ldots \mathcal{L}}_{j \text{ times}} \underbrace{\overline{\mathcal{L}} \ldots \overline{\mathcal{L}}}_{k \text{ times}} \partial \bar{\partial} r(L, \overline{L})(z').
\]

Let \(m\) be any positive integer. For any \(2 \leq l \leq 2m\), let
\[
A_l(z') = \left( \sum_{j+k \leq l \atop j,k \geq 0} |\mathcal{L}_{jk} \partial \bar{\partial} r(z')|^2 \right)^{1/2}.
\]

(3.1)
Let $\tilde{r}$ be any defining function for $\Omega$ and let $\tilde{L}$ be any non-vanishing complex tangential vector field of $b\Omega$. Let $\tilde{A}_l(z')$ be defined by \[ (3.1) \] with $r$ replaced by $\tilde{r}$ and $L$ by $\tilde{L}$, then it is easy to check that $A_l(z') \approx \tilde{A}_l(z')$. Thus, whether or not $A_l(z')$ vanishes is a property that is independent of the choice of either the defining function or the complex tangential vector field of $b\Omega$. Furthermore, $b\Omega$ is of finite type $2m'$ at $z'$ if and only if $A_l(z')$ is 0 for all $2 \leq l \leq 2m' - 1$ but is positive for $l = 2m'$. For any $\tau > 0$, let

\[ (3.2) \quad \delta(z', \tau) = \sum_{l=2}^{2m} A_l(z')\tau^l. \]

Evidently,

\[ (3.3) \quad \delta(z', \tau) \lesssim \tau^2 \quad \text{and} \quad c^{2m}\delta(z', \tau) \leq \delta(z', c\tau) \leq c^2\delta(z', \tau), \]

for any $\tau$ and $c$ such that $0 < \tau, c < 1$. Furthermore, $b\Omega$ is of finite type $2m$ if and only if $\delta(z', \tau) \gtrsim \tau^{2m}$ uniformly for all $z' \in b\Omega$ and $\delta(z_0', \tau) \lesssim \tau^{2m}$ for some $z_0' \in b\Omega$.

Let $U$ be a neighborhood of a boundary point. Assume without loss of generality that $|\partial r/\partial z_2| \gtrsim 1$ on $U$. After a change of (global) holomorphic coordinates of the form

\[ (3.4) \quad (\xi_1, \xi_2) = \Phi'(z_1, z_2) = (z_1 - z_1', 2\partial_r/\partial z_2(z)(z_2 - z_2') + \sum_{k=1}^{2m} \alpha_k(z')(z_1 - z_1')^k), \]

we have that the image $\Omega' = \Phi'(\Omega)$ is defined by

\[ (3.5) \quad \rho(\xi) = r((\Phi')^{-1}(\xi)) = \text{Re} \xi_2 + \sum_{2 \leq j+k \leq 2m} a_{jk}(z')\xi_1^j\xi_2^k + O(|\xi_1|^{2m+1} + |\xi_2||\xi_1|). \]

The $\alpha_k(z')$'s and $a_{jk}(z')$'s depend smoothly on $z'$, and they are unique in the sense that if after a holomorphic change of coordinates $\tilde{\Phi}'$ of the form \[ (3.3) \] but with possible different $\alpha_k(z')$'s, $r((\tilde{\Phi}')^{-1}(\xi))$ is in the form of \[ (3.5) \] but with possible different $a_{jk}(z')$'s, then $\tilde{\Phi}' = \Phi'$. (See [Ca89] for a detailed discussion on the above coordinates.) Solving $\text{Re} \xi_2$ in terms of the other variables, we have that $b\Omega'$ is defined near the origin by $\bar{\rho}(\xi) = \text{Re} \xi_2 + \bar{h}(\xi_1, \text{Im} \xi_2)$, where

\[ (3.6) \quad \bar{h}(\xi_1, \text{Im} \xi_2) = \sum_{2 \leq j+k \leq 2m} \bar{a}_{jk}(z')\xi_1^j\xi_2^k + O(|\xi_1|^{2m+1} + |\text{Im} \xi_2||\xi_1| + |\text{Im} \xi_2|^2). \]

It is easy to see that

\[ (3.7) \quad A_l(z') \approx \sum_{j,k \leq l, j,k > 0} |a_{jk}(z')| \approx \sum_{j,k \leq l, j,k > 0} |\bar{a}_{jk}(z')|, \]

for $2 \leq l \leq 2m$.

Write

\[ (3.8) \quad \bar{h}(\xi_1, \text{Im} \xi_2) = \bar{P}(\xi_1) + (\text{Im} \xi_2)\bar{Q}(\xi_1) + O(|\xi_1|^{2m+1} + |\text{Im} \xi_2|^2 + |\text{Im} \xi_2||\xi_1|^{m+1}), \]

where

\[ \bar{P}(\xi_1) = \sum_{2 \leq j+k \leq 2m} \bar{a}_{jk}(z')\xi_1^j\xi_2^k, \quad \bar{Q}(\xi_1) = \sum_{1 \leq j+k \leq m} \bar{b}_{jk}(z')\xi_1^j\xi_2^k. \]
The harmonic terms can be expunged from the polynomial $\tilde{Q}$ without introducing harmonic terms into the $\tilde{P}$ term by a change of (local) holomorphic coordinates of the form

$$(\tilde{\xi}_1, \tilde{\xi}_2) = \Phi^*(\xi_1, \xi_2) = (\xi_1, \xi_2 \prod_{j=1}^{m} (1 - \beta_j(z')\xi_1^j)).$$

(See [FoS89].) Finally, after another change of coordinates of the form

$$(\xi_1, \xi_2) = \tilde{\Phi}(\xi_1, \xi_2) = (\tilde{\xi}_1, \tilde{\xi}_2 - \gamma(z')\tilde{\xi}_2^2),$$

we can also eliminate the term $\gamma(z')(\mbox{Im} \tilde{\xi}_2)^2$ from the remainder of the Taylor expansion without introducing harmonic terms into the $\tilde{P}$ and $\tilde{Q}$ terms, and obtain that $b\Omega$ is defined near $z'$ in the new $(\tilde{\xi}_1, \tilde{\xi}_2)$-coordinates by $\hat{\rho} = \mbox{Re} \tilde{\xi}_2 + \hat{h}(\tilde{\xi}_1, \mbox{Im} \tilde{\xi}_2) = 0$ with

$$(3.9) \quad \hat{h}(\tilde{\xi}_1, \mbox{Im} \tilde{\xi}_2) = \tilde{P}(\tilde{\xi}_1) + (\mbox{Im} \tilde{\xi}_2)\tilde{Q}(\tilde{\xi}_1) + O(|\tilde{\xi}_1|^{m+1} + |\mbox{Im} \tilde{\xi}_2|\tilde{\xi}_1|m+1 + |\mbox{Im} \tilde{\xi}_2^2|\tilde{\xi}_1|),$$

where

$$\tilde{P}(\tilde{\xi}_1) = \sum_{l=2}^{2m} \tilde{P}_l(\tilde{\xi}_1) = \sum_{l=2}^{2m} \sum_{j+k=l} \tilde{a}_{jk}(z')\tilde{\xi}_1^j\tilde{\xi}_1^k$$

and

$$\tilde{Q}(\tilde{\xi}_1) = \sum_{l=2}^{m} \tilde{Q}_l(\tilde{\xi}_1) = \sum_{l=2}^{m} \sum_{j+k=l} \tilde{b}_{jk}(z')\tilde{\xi}_1^j\tilde{\xi}_1^k.$$ 

Let

$$\tilde{A}_l(z') = \left( \sum_{j+k \leq l, j,k>0} |\tilde{a}_{jk}(z')|^2 \right)^{1/2} \quad \text{and} \quad \tilde{B}_l(z') = \left( \sum_{j+k \leq l, j,k>0} |\tilde{b}_{jk}(z')|^2 \right)^{1/2}.$$ 

We now summarize what we have obtained from these changes of holomorphic coordinates. For any $z' \in U \cap b\Omega$, there exists a neighborhood $U_{z'}$ of $z'$ and a biholomorphic map $\zeta = \tilde{\Psi}_{z'}(z) = \tilde{\Phi} \circ \Phi^* \circ \Phi'(z)$ from $U_{z'}$ onto a ball $B(0, \varepsilon_0)$ of uniform radius $\varepsilon_0$ such that

(A-1) $\tilde{\Psi}_{z'}$ depends smoothly on $z'$ and its components are holomorphic polynomials of degrees $\leq m^2 + 5m$ for each $z'$. Moreover, the Jacobian determinant $J\Phi_{z'}$ of $\Phi_{z'}$ is uniformly bounded from above and below on $U_{z'}$.

(A-2) $\tilde{\Psi}_{z'}(z') = 0$ and $\Phi_{z'}(U_{z'} \cap \Omega) = \{ \zeta \in B(0, \varepsilon_0) \mid \tilde{\rho}(\zeta) = \mbox{Re} \tilde{\xi}_2 + \hat{h}(\tilde{\xi}_1, \mbox{Im} \tilde{\xi}_2) < 0 \}$, where $\hat{h}(\tilde{\xi}_1, \mbox{Im} \tilde{\xi}_2)$ is in the form of (3.9).

(A-3) There exist positive constants $C_1$ and $C_2$ independent of $z'$ such that $C_1A_l(z') \leq \tilde{A}_l(z') \leq C_2A_l(z')$ for $2 \leq l \leq 2m$.

Notice that these properties hold for any smooth bounded domain $\Omega$. From now on, we will assume that $\Omega$ is pseudoconvex of finite type $2m$. When these assumptions come into play, then it follows from [FoS89] that

$$(3.10) \quad \sum_{l=2}^{m} \|\tilde{Q}_l\|_{\infty}|\xi_1|^l \lesssim |\xi_1|(\sum_{l=2}^{2m} \|\tilde{P}_l\|_{\infty}|\xi_1|^l)^{1/2},$$

where $\|P\|_{\infty}$ denotes the sup-norm of $P(\xi_1)$ on $|\xi_1| = 1$. It is easy to see that

$$\| \sum_{j+k=l} c_{jk} \xi_1^j \xi_1^k \|_{\infty} \approx \sum_{j+k=l} |c_{jk}|.$$
Therefore, in light of (A-3) and (3.10), we have

\[
(3.11) \quad \sum_{l=2}^{m} \hat{B}_l(z') \tau^l \lesssim \tau (\delta(z', \tau))^{1/2}
\]

for \(0 < \tau < 1\).

Let \(\tilde{P}_\tau(z') = \{|\zeta_1| < \tau, |\zeta_2| < \delta(z', \tau)^{1/2}\}\), and let

\[
R_\tau(z') = (\tilde{\Psi}_{z'})^{-1}(\tilde{P}_\tau(z')).
\]

We now study the non-isotropic “bidiscs” \(R_\tau(z')\). Notice that the size of \(\tilde{P}_\tau(z')\) is different from those used to study the Bergman kernel in \[Ca89\], \[Mc89\], \[NR\]. Here we have \(|\zeta_2| < (\delta(z', \tau))^{1/2}\) instead of \(|\zeta_2| < \delta(z', \tau)\). This seems to be crucial in our analysis. Let \(\tau_0\) be a sufficiently small positive constant such that \(\tilde{P}_{\tau_0}(z') \subset B(0, \varepsilon_0)\) for all \(z' \in b\Omega\). Let

\[
(3.12) \quad \tilde{S}^{a,b}_{z'} = \{ f \in C^\infty(B(0, \varepsilon_0)) \mid \forall j, k \geq 0, \exists C_{jk} > 0 \text{ such that } \|D^j f\|_{C_{jk}} \leq C_{jk} \tau^{a-j}(\delta(z', \tau))^{b-k/2}, \forall \zeta \in \tilde{P}_\tau(z'), \forall \tau \in (0, \tau_0) \}.
\]

(Here \(D^j f\) denotes the partial derivatives of order \(j\) with respect to \(\zeta_1\) or \(\zeta_2\).) The following facts can be checked easily:

(C-1) If \(f \in \tilde{S}^{a,b}_{z'}\), then \(D^j f \in \tilde{S}^{a-j, b-k}_{z'}\).

(C-2) If \(f \in \tilde{S}^{a,b}_{z'}\) and \(g \in \tilde{S}^{a-d, b-d}_{z'}\), then \(fg \in \tilde{S}^{a+c, b+d}_{z'}\).

**Lemma 3.1.** Under the pseudoconvexity and finite type assumptions on \(\Omega\), \(
\hat{h}(\zeta_1, \text{Im } \zeta_2) \in \tilde{S}^{0,1}_{z'}.
\)

**Proof.** Write \(\delta = \delta(z', \tau)\) and let \(\hat{h}_\tau(w_1, w_2) = (1/\delta)\hat{h}(\tau w_1, \delta^{1/2} \text{Im } w_2)\). Since \(\tau^{2m} \lesssim \delta \lesssim \tau^2\), it follows that the Taylor expansion of \(\hat{h}_\tau\) at the origin has the form

\[
(3.13) \quad \hat{h}_\tau(w_1, w_2) = \sum_{l=2}^{2m} \frac{\tau^l \hat{a}_{jik}(z')}{\delta} w_j^i w_k^j + (\text{Im } w_2) \sum_{l=2}^{m} \frac{\tau^l \hat{b}_{jik}(z')}{\delta^{1/2}} w_j^i w_k^j + O(\tau(|w_1|^{2m+1} + |\text{Im } w_2||w_1|^{m+1} + |\text{Im } w_2|^2)).
\]

Notice that the Taylor coefficients in the first sum above have modulus \(\lesssim 1\) by property (A-3) and those in the second sum have modulus \(\lesssim \tau\) by (3.11). The coefficients of the Taylor expansion of the remainder are also \(\lesssim \tau\) as shown above. Therefore, \(\hat{h}(\zeta_1, \text{Im } \zeta_2) \in \tilde{S}^{0,1}_{z'}\). \(\square\)

The next two lemmas establish the doubling and engulfing properties for the non-isotropic bidiscs \(R_\tau(z')\) (cf. \[Ca89\]).

**Lemma 3.2.** Under the same assumptions, if \(z'' \in R_\tau(z') \cap b\Omega\), then \(\delta(z'', \tau) \approx \delta(z', \tau)\).

**Proof.** We shall use the above lemma. (Compare the proof of Proposition 1.3 in \[Ca89\].) Details are provided for the reader’s convenience. Let \(\hat{f}(z) = \rho(\tilde{\Psi}_{z'}(z))\). Let

\[
L^* = \frac{\partial}{\partial \zeta_1} - \frac{\partial \hat{\rho}}{\partial \zeta_1} \left( \frac{\partial \hat{\rho}}{\partial \zeta_2} \right)^{-1} \frac{\partial}{\partial \zeta_2} = \frac{\partial}{\partial \zeta_1} - \frac{\partial \hat{h}}{\partial \zeta_1} \left( \frac{1}{2} + \frac{\partial \hat{h}}{\partial \zeta_2} \right)^{-1} \frac{\partial}{\partial \zeta_2}.
\]
Notice that the coefficient of $\partial/\partial \zeta_2$ in the above expression of $L^*$ belongs to $\mathcal{S}_{z'}^{-1,1}$. Let $L' = (\Psi_{z'}^{-1})_*(L^*)$. Write
\[
\mathcal{L}'_{j,k} \partial \Phi \tilde{r}(z) = \sum_{j=1}^{\infty} L' \sum_{k=1}^{\infty} \partial \Phi \tilde{r}(L', L')
\]
and
\[
\mathcal{L}^*_{j,k} \partial \Phi \tilde{r}(z) = \sum_{j=1}^{\infty} L^* \sum_{k=1}^{\infty} \partial \Phi \tilde{r}(L^*, L^*).
\]
Then by functoriality, for $z \in U_{z'}$,
\[
\mathcal{L}'_{j,k} \partial \Phi \tilde{r}(z) = \mathcal{L}^*_{j,k} \partial \Phi \tilde{r}(z),
\]
where $\zeta = \tilde{\Psi}_{z'}(z)$. It is easy to see that $\mathcal{L}^*_{j,k} \partial \Phi \tilde{r}(z) \in \mathcal{S}_{z'}^{j-k,1}$; in fact,
\[
(3.14)
\mathcal{L}^*_{j,k} \partial \Phi \tilde{r}(z) = \frac{\partial^{j+k} \hat{h}(\bar{\zeta})}{\partial i_{j} \partial k_{1}} + s_{j-k+1}(\zeta)
\]
for some $s_{j-k+1}(\zeta) \in \mathcal{S}_{z'}^{j-k,1}$. It follows that when $z'' \in R_\tau(z') \cap b\Omega$,
\[
A_1(z'') \approx \max \{|\mathcal{L}'_{j,k} \partial \Phi \tilde{r}(z'')|; \ 2 \leq j + k \leq l\}
\]
\[
= \max \{|\mathcal{L}^*_{j,k} \partial \Phi \tilde{r}(z'')|; \ 2 \leq j + k \leq l\}
\]
\[
\lesssim \tau^{-l} \delta(z', \tau).
\]
Therefore, $\delta(z'', \tau) \lesssim \delta(z', \tau)$. We now prove the estimate in the opposite direction. From the definition of $\delta(z', \tau)$ we know that there exist $j_0, k_0 > 0$ with $j_0 + k_0 = l_0 \leq 2m$ such that
\[
|\mathcal{L}'_{j_0, k_0} \partial \Phi \tilde{r}(0)| \gtrsim \tau^{-l_0} \delta(z', \tau),
\]
where the constant in the above estimate depending only on $m$. Now let $z'' \in R_\varepsilon(z')$ where $\varepsilon$ is a sufficiently small constant to be determined. By (3.14) and (3.3), we have
\[
|\mathcal{L}^*_{j_0, k_0} \partial \Phi \tilde{r}(z'') - \mathcal{L}_j \partial \Phi \tilde{r}(0)| \lesssim \tau^{-l_0} \delta(z', \tau)
\]
\[
\lesssim (\varepsilon + \tau) \tau^{-l_0} \delta(z', \tau).
\]
Therefore when both $\tau$ and $\varepsilon$ are sufficiently small, we have
\[
\delta(z'', \tau) \gtrsim \tau^{-l_0} \tau \delta(\tilde{\Psi}(z''), \tau) \gtrsim \delta(z', \tau).
\]
We then conclude the proof by replacing $\varepsilon \tau$ by $\tau$ and using (3.3).

\[\square\]

**Lemma 3.3.** If $z'' \in R_\tau(z') \cap b\Omega$, then there exists a positive constant $C$ such that
\[
R_\tau(z') \subset R_C(z'') \quad \text{and} \quad R_\tau(z'') \subset R_C(z').
\]

**Proof.** It follows from $\Phi'(R_\tau(z')) = (\Phi \circ \Phi^*)^{-1}((\tilde{\Phi}_r(z'))$ that
\[
\{|\xi_1| < C^{-1} \tau, \ |\xi_2| \leq \delta(z', C^{-1} \tau)\} \subset \Phi'(R_\tau(z')) \subset \{|\xi_1| < C \tau, \ |\xi_2| \leq \delta(z', C \tau)\}
\]
for some constant $C > 0$. Thus $\xi'' = \Phi(z'') \in \{|\xi_1| < C \tau, \ |\xi_2| \leq \delta(z', C \tau)\}$. After a change of coordinates of form
\[
(3.16) \quad (\hat{\xi}_1, \hat{\xi}_2) = \Psi''(\xi_1, \xi_2) = (\xi_1 - \xi''_1, 2 \frac{\partial \tilde{r}}{\partial \xi_2}(\xi_2 - \xi''_2) + \sum_{k=1}^{2m} e_k(z'')(\xi_1 - \xi''_1)^k),
\]
we have

\[r((\Psi'' \circ \Phi')^{-1}(\xi)) = \sum_{2 \leq j + k < 2m, j, k \geq 0} a_{jk}(z'')\tilde{\xi}^j_1\tilde{\xi}^k + O(|\tilde{\xi}_1|^{2m+1} + |\tilde{\xi}_1||\tilde{\xi}_1|).\]

The \(e_k(z'')\)'s are determined inductively as follows.

\[e_1(z'') = 2 \frac{\partial \rho}{\partial \xi_1}((\xi''), \ e_k(z'') = \frac{2}{k!} \frac{\partial^k \rho}{\partial \xi_1^k}(0), \ k \geq 2,\]

where

\[\rho_1 = \rho, \ \rho_k = \rho_{k-1} \circ (\phi_{k-1})^{-1}, \ k \geq 2,\]

and

\[\phi_1 = (\xi_1 - \xi_1', 2 \frac{\partial \rho}{\partial \xi_2}(\xi_2 - \xi_2') + e_1(z'')(\xi_1 - \xi_1'')), \ \phi_k = (\xi_1, \xi_2 + e_k(z'')(\xi_1^k)), \ \forall k \geq 2.\]

It follows from (3.17) that

\[|D^k_{\xi_1}\rho_k(\xi)| \lesssim \tau^{-l}(\delta(z', \tau))^{1/2}, \ \text{for} \ \ |\xi_1| \lesssim \tau, \ |\xi_2| \lesssim (\delta(z', \tau))^{1/2}.\]

By induction on \(k\), we obtain that

\[(3.17) \quad |D^k_{\xi_1}\rho_k(\xi)| \lesssim \tau^{-l}(\delta(z', \tau))^{1/2} \quad \text{and} \quad |e_k(z'')| \lesssim \tau^{-l}(\delta(z', \tau))^{1/2}.\]

By the uniqueness in the sense noted after (3.15), we obtain as above that

\[\{||\xi_1| < C^{-1}\tau, \ |\xi_2| < \delta(z'', C^{-1})^{1/2}\} \subset (\Psi'' \circ \Phi')(R_\tau(z'')) \subset \{||\xi_1| < C\tau, \ |\xi_2| < \delta(z'', C\tau)^{1/2}\}.\]

It follows from Lemma 3.2 and 3.17 that if \(\xi \in \Phi'(R_\tau(z'))\), then

\[|\xi_1| \lesssim \tau \quad \text{and} \quad |\xi_2| \lesssim (\delta(z'', \tau))^{1/2}.\]

Thus, \(R_\tau(z') \subset R_{C\tau}(z'').\) Similarly, \(R_\tau(z'') \subset R_{C\tau}(z').\)

Denote by \(d(z)\) the Euclidean distance from \(z\) to \(b\Omega\). Let \(\pi(z)\) be the projection from a neighborhood of \(b\Omega\) onto \(b\Omega\) such that \(|z - \pi(z)| = d(z) \approx r(z)\). Denote by \(\chi_A\) the characteristic function for a set \(A\). Let

\[A_\tau = \{z \in \Omega \mid d(z) < (\delta(\tau, \tau))^{1/2}\}.

The following lemma is an easy consequence of Lemma 3.2 and Lemma 3.3.

**Lemma 3.4.** For any \(\alpha \in \mathbb{R}\), there exists a sufficiently large constant \(C > 0\) such that for any sufficiently small \(\tau > 0\) and for any \(z \in A_\tau\),

\[\chi_{A_{C^{-1}\tau}}(z) \lesssim \tau^{-2}(\delta(\tau, \tau))^{-\alpha/2} \int_{b\Omega} \chi_{R_\tau(z')} \xi_{\Omega}(z)(\delta(z', \tau))^\alpha \ dS(z') \lesssim \chi_{A_{C\tau}}(z).\]

**Proof.** It is easy to see that if \(z \in R_\tau(z')\) then \(\pi(z) \in R_{C\tau}(z')\). Thus by Lemma 3.2 \(\delta(z', \tau) \approx \delta(\pi(z), \tau)\), and by Lemma 3.3 \(z' \in R_{C\tau}(\pi(z))\). It follows that

\[A_{C^{-1}\tau} \subset \cup_{z' \in \Omega} R_{C\tau}(z') \cap \Omega \subset A_{C\tau},\]

and for any \(z \in A_\tau\),

\[b\Omega \cap R_{C^{-1}\tau}(\pi(z)) \subset \{z' \in b\Omega \mid z \in R_\tau(z')\} \subset b\Omega \cap R_{C\tau}(\pi(z)).\]

Thus

\[\int_{b\Omega} \chi_{R_\tau(z')} \xi_{\Omega}(z)(\delta(z', \tau))^\alpha \ dS(z') \lesssim (\delta(\tau, \tau))^{\alpha} \text{Area}(b\Omega \cap R_{C\tau}(\pi(z))) \chi_{A_{C\tau}}(z) \lesssim \tau^{2}(\delta(\tau, \tau))^{\alpha+1/2} \chi_{A_{C\tau}}(z).\]
The other estimate in Lemma 4.1 follows similarly.

4. RESCALE THE $\bar{\partial}$-NEUMANN LAPLACIAN

We will keep the notations from the previous section. Let $\tilde{\Omega}_{z'} = \tilde{\Phi}_{z'}(\Omega \cap U_{z'})$ and write
\[
\bar{h}(\zeta_1, \text{Im } \zeta_2) = f(\zeta_1) + (\text{Im } \zeta_2)g_1(\zeta_1) + (1/2)(\text{Im } \zeta_2)^2g_2(\zeta_1) + \sigma_3(\zeta_1, \text{Im } \zeta_2),
\]
where $\sigma_3(\zeta_1, \text{Im } \zeta_2) = O(|\text{Im } \zeta_2|^3)$. Then
\[
f(\zeta_1) = \tilde{P}(\zeta_1) + O(|\zeta_2|^{2m+1}), \quad g_1(\zeta_1) = \tilde{Q}(\zeta_1) + O(|\zeta_1|^{m+1}), \quad g_2(\zeta_1) = O(|\zeta_1|).
\]
It is evident that $f \in \mathcal{S}^{0,1}$, $g_1 \in \mathcal{S}^{1,1/2}$, and $g_2 \in \mathcal{S}^{1,0}$.

We flatten the boundary before the rescaling. Let
\[
(\eta_1, \eta_2) = \tilde{\Phi}_{z'}(\zeta_1, \zeta_2) = (\zeta_1, \zeta_2 + \hat{h}(\zeta_1, \text{Im } \zeta_2) - F(\zeta_1, \zeta_2)),
\]
where $F(\zeta_1, \zeta_2) = g_2(\zeta_1)(\text{Re } \zeta_2 + \hat{h}(\zeta_1, \text{Im } \zeta_2))^2/2 + i(g_1(\zeta_1)(\text{Re } \zeta_2) + g_2(\zeta_1)(\text{Re } \zeta_2)(\text{Im } \zeta_2))$. Of course it is not possible to flatten the boundary with a holomorphic change of variables: The term $F(\zeta_1, \zeta_2)$ is added to ensure that $\partial \eta_2/\partial \zeta_2$ vanishes to a desirable higher order at the origin. Note that $F \in \mathcal{S}^{1,1}$. Let
\[
\tilde{P}_r(z') = \{ |\eta_1| < r, |\eta_2| < (\delta, r)^{1/2} \}.
\]
Let $\tilde{\mathcal{S}}_{z'}^{a,b}$ be the class of smooth functions in $\eta$ on a neighborhood of the origin defined as in (8.12) but with $\zeta$ replaced by $\eta$ and $\tilde{P}_r(z')$ replaced by $\tilde{P}_r(z')$.

**Lemma 4.1.** There exists a constant $C > 0$ such that
\[
\tilde{P}_{C-1,r}(z') \subset \tilde{\Phi}_{z'}(\tilde{P}_r(z')) \subset \tilde{P}_{C/r}(z').
\]

**Proof.** The inclusion $\tilde{\Phi}_{z'}(\tilde{P}_r(z')) \subset \tilde{P}_{C/r}(z')$ is evident. Now if $\eta \in \tilde{P}_r(z')$, then
\[
|\zeta_2| = |\eta_2 - \hat{h}(\eta_1, \text{Im } \zeta_2) - F(\eta_1, \zeta_2)|
\]
\[
\leq \delta^{1/2} + |\hat{h}(\eta_1, \text{Im } \zeta_2)| + |F(\eta_1, \zeta_2)|
\]
\[
\lesssim \delta^{1/2} + \tau \delta^{1/2}|\zeta_2| + \tau|\zeta_2|^2 + |\zeta_2|^3.
\]
Thus $|\zeta_2| \lesssim \delta^{1/2}$. The other inclusion then follows. \qed

Let $\tilde{\rho}(\zeta) = \tilde{\rho}(\zeta) - (1/2)g_2(\zeta_1)(\tilde{\rho}(\zeta))^2$. Then $\tilde{\rho}(\zeta)$ is a defining function for $b\tilde{\Omega}_{z'}$ near the origin. Let $\tilde{r}(z) = \tilde{\rho}(\tilde{\Phi}_{z'}(z))$. Then $\tilde{r}(z)$ is a defining function for $b\Omega \cap U_{z'}$ (shrinking $U_{z'}$ if necessary). Let
\[
(\partial \tilde{r}/\partial \bar{z}_1) \partial \bar{z}_1 = \frac{\partial \tilde{r}}{\partial \bar{z}_1} \partial \bar{z}_1 + \frac{\partial \tilde{r}}{\partial \bar{z}_2} \partial \bar{z}_2,
\]
and let
\[
\omega_1 = \frac{\partial \tilde{r}}{\partial \bar{z}_1} d\bar{z}_1 - \frac{\partial \tilde{r}}{\partial \bar{z}_2} d\bar{z}_2 \quad \text{and} \quad \omega_2 = \frac{\partial \tilde{r}}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial \tilde{r}}{\partial \bar{z}_2} d\bar{z}_2.
\]
Then $\{L_1, L_2\}$ forms an orthogonal basis for $T^{1,0}(\mathbb{C}^2)$ and $\{\omega_1, \omega_2\}$ for $\Lambda^{1,0}(\mathbb{C}^2)$ on $U_{z'}$. Denote by $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{\omega}_1$, and $\tilde{\omega}_2$ the vectors and forms defined as above by replacing $\tilde{r}$ by $\tilde{\rho}$, and $z_1$, $z_2$ by $\zeta_1$, $\zeta_2$ respectively. Let $\tilde{L}_k = (\tilde{\Phi}_{z'})_*(\tilde{L}_k)$, $k = 1, 2$. Write $\zeta_2 = \bar{s} + i\tau$ and $\eta_2 = \bar{s} + i\bar{\tau}$. 


Lemma 4.2. With above notations,
\[ \overline{L}_1 = \left( \frac{1}{2} + \alpha_1 \right) \frac{\partial}{\partial \eta_1} + \left( - \frac{i}{2} \frac{\partial \hat{h}}{\partial \zeta_1} + \beta_1 \right) \frac{\partial}{\partial t}, \quad \overline{L}_2 = \left( \frac{1}{2} + \alpha_2 \right) \frac{\partial}{\partial \eta_2} + \alpha_3 \frac{\partial}{\partial \eta_1} + \beta_2 \frac{\partial}{\partial s} + \beta_3 \frac{\partial}{\partial t}, \]
where the \( \alpha \)'s are in \( \hat{S}^{a_{1/2}}_z \) and the \( \beta \)'s in \( \hat{S}^{a_1}_z \).

Proof. By direct computations, we have
\[
\overline{L}_1 = \frac{\partial \tilde{\rho}}{\partial \zeta_2} \frac{\partial}{\partial \eta_1} + \left( \frac{\partial \tilde{\rho}}{\partial \zeta_2} \frac{\partial \hat{t}}{\partial \zeta_1} - \frac{\partial \tilde{\rho}}{\partial \zeta_1} \frac{\partial \hat{t}}{\partial \zeta_2} \right) \frac{\partial}{\partial t},
\]
\[
\overline{L}_2 = \left[ \frac{\partial \tilde{\rho}}{\partial \zeta_2} \left( 1 + \frac{\partial \hat{h}}{\partial \zeta_2} - \frac{\partial F}{\partial \zeta_1} \right) + \frac{\partial \tilde{\rho}}{\partial \zeta_1} \left( \frac{\partial \hat{h}}{\partial \zeta_1} - \frac{\partial F}{\partial \zeta_1} \right) \right] \frac{\partial}{\partial \eta_2} + \frac{\partial \tilde{\rho}}{\partial \zeta_1} \frac{\partial}{\partial \eta_1}.
\]
Note that \( \tilde{\Phi}^*_z \) is an isomorphism from \( \hat{S}^{a_{1/2}}_{z'} \) onto \( \hat{S}^{a_{1/2}}_{z'} \). The lemma then follows from the facts that \( h \in \hat{S}^{a_{1/2}}_{z'}, \tilde{\rho} - \text{Re} \, \zeta_2 \in \hat{S}^{a_{1/2}}_{z'}, F \in \hat{S}^{a_{1/2}}_{z'}, \) and
\[
\frac{\partial \hat{h}}{\partial \zeta_2} - \frac{\partial F}{\partial \zeta_2} = -i \frac{g_2}{2} (\hat{s} + \hat{h}) \frac{\partial \hat{h}}{\partial t} - \frac{1}{2} \frac{\partial \hat{h}}{\partial t} + O(|\hat{t}|^2) \in \hat{S}^{a_{1/2}}_{z'}.
\]

We now proceed with the rescaling. For any positive \( \tau \), let \( \delta = \delta(z', \tau) \) be defined by \( (3.2) \). Let
\[
(w_1, w_2) = D_{z', \tau}(\eta_1, \eta_2) = (\eta_1 / \tau, \eta_2 / \delta).
\]
Let \( \tilde{\Psi}^{z'}_{z', \tau} = D_{z', \tau} \circ \tilde{\Phi}^*_z \) and let \( \tilde{\Omega}^{z', \tau} = \tilde{\Psi}^{z'}_{z', \tau}(\tilde{\Omega}_z) \). (In what follows, we sometimes suppress the subscript \( z' \) for economy of notations when there is no confusion.) Let
\[
P_\tau(z') = \{ ||w_1| < 1, \ |w_2| < \delta^{-1/2} \}.
\]
Let \( S_{z'}^{a_{1/2}} \) be the class of functions \( f \) depending smoothly on \( w \in \mathbb{C}^2 \) and \( \tau > 0 \) such that for any \( j, k \geq 0 \), there exists constants \( C_{jk} > 0 \), independent of \( \tau \), such that
\[
|D_{w_1}^j D_{w_2}^k f| \leq C_{jk} \tau^a \delta^{b+k/2}
\]
on \( P_\tau(z') \) for sufficiently small \( \tau > 0 \). Here, as before, \( D_{w_i}^j \) denotes the partial derivatives of order \( j \) with respect to \( w_i \) or \( \tilde{w_i} \). Clearly, if \( g \in \hat{S}_{z'}^{a_{1/2}} \), then \( (D_\tau^{-1})^*(g) \in S_{z'}^{a_{1/2}} \).

Write \( w_1 = x + iy \) and \( w_2 = s + it \). Let
\[
\overline{L}_{1, \tau} = \tau D_{\tau \tau} \overline{L}_1 = \left( \frac{1}{2} + \alpha_1 \right) \frac{\tau}{\partial \eta_1} + \left( - \frac{i}{2} \frac{\partial \hat{h}}{\partial \zeta_1} + \beta_1 \right) \frac{\tau}{\partial t},
\]
\[
\overline{L}_{2, \tau} = \tau D_{\tau \tau} \overline{L}_2 = \left( \frac{1}{2} + \alpha_2 \right) \frac{\tau}{\partial \eta_2} + \alpha_3 \frac{\tau}{\partial \eta_1} + \beta_2 \frac{\tau}{\partial s} + \beta_3 \frac{\tau}{\partial t}.
\]
Write \( L^0 = \overline{L}_{1, \tau} \) and \( L^1 = \overline{L}_{1, \tau} \). For any tuple \( (i_1, \ldots, i_l) \) of 0’s and 1’s, define \( L^{(i_1, \ldots, i_l)} \) inductively by
\[
L^{(i_1, \ldots, i_l)} = [L^{i_1} L^{(i_1, \ldots, i_{l-1})}],
\]
Write
\[
L^{(i_1, \ldots, i_l)} = l^{i_1, \ldots, i_l} \frac{\partial}{\partial t} + a^{i_1, \ldots, i_l} L^0 + b^{i_1, \ldots, i_l} L^1.
\]
Lemma 4.3. With the above notations,

1. \( \lambda^{i_1 \cdots i_l} \in S_x^{0,0}, \ a^{i_1 \cdots i_l} \in S_x^{1,0}, \) and \( b^{i_1 \cdots i_l} \in S_x^{1,0}. \)
2. \( (L^{(i_1 \cdots i_l)})^* = -L((1-i_1) \cdots (1-i_l)) + \sigma \) for some \( \sigma \in S_x^{1,0}. \)
3. There exists a tuple \( (i_1 \cdots i_l_0) \) of length \( l_0 \leq 2m \) such that \( |\lambda^{i_1 \cdots i_l_0}| \gtrsim 1 \) on \( P_\tau(z'). \)

Proof. A direction calculation yields that

\[
L^{(10)} = \left( -\frac{i}{4} \left( \frac{\partial}{\partial w_1} \left( \frac{\partial h}{\partial \xi_1} \right) + \frac{\partial}{\partial \bar{w}_1} \left( \frac{\partial \bar{h}}{\partial \xi_1} \right) \right) \frac{\tau}{\delta} + \sigma \right) \frac{\partial}{\partial t} + a^{10}L^0 + b^{10}L^1
\]

with \( a^{10}, b^{10}, \sigma \in S_x^{1,0}. \) (Here and in what follows, \( \sigma \) could be different in different appearances, but is always in \( S_x^{1,0}. \) ) It is also easy to see that

\[
[L^{(i_1 \cdots i_l+1)}, \frac{\partial}{\partial t}] = \sigma \frac{\partial}{\partial t} \mod (L^0, L^1)
\]

where the modulus is with coefficients in \( S_x^{1,0}. \) Thus,

\[
L^{(i_1 \cdots i_l+1)} = (L^{(i_1 \cdots i_l+1)} + \lambda^{0i_1 \cdots i_l} + \lambda^{i_1 \cdots i_l+1} + \sigma \lambda^{i_1 \cdots i_l}) \frac{\partial}{\partial t} \mod (L^0, L^1).
\]

Properties (1) and (2) in the lemma then follow from an easy inductive argument on \( l. \) To prove (3), one notices from the above formulas that

\[
\lambda^{10i_3 \cdots i_l} = -\frac{i}{2^{l+1}} \frac{\partial^2 \hat{h}}{\partial \xi_1^j \partial \xi_1^k} \frac{\tau^l}{\delta} + \sigma,
\]

where \( j \) and \( k \) are the numbers of the 0’s and 1’s in \( (10i_3 \cdots i_l) \) respectively. It follows from the proof of Lemma 3.2 that there exists \( j_0, k_0 > 0 \) with \( j_0 + k_0 = l_0 \leq 2m \) such that

\[
\left| \frac{\partial^2 \hat{h}}{\partial \xi_1^j \partial \xi_1^k} \frac{\tau^l}{\delta} \right| \approx 1
\]

on \( R_\tau(z'). \) This then implies the last part of the lemma. \( \square \)

We now define the rescaled \( \overline{\partial}-\text{Neumann Laplacian.} \) Let \( \overline{\partial}_\tau : (L^2(\overline{\Omega}_\tau))^2 \rightarrow L^2_{(0,1)}(\overline{\Omega}) \) be the transformation defined by

\[
\overline{\partial}_\tau (u_1, u_2) = |\det d\overline{\Psi}_\tau|^{1/2} (u_1(\overline{\Psi}_\tau) \overline{w}_1 + u_2(\overline{\Psi}_\tau) \overline{w}_2),
\]

and let \( \overline{G}_\tau : L^2_{(0,1)}(\overline{\Omega}) \rightarrow L^2_{(0,1)}(\Omega \cap \overline{\Omega}) \) be defined likewise by

\[
\overline{G}_\tau (u_1 \overline{w}_1 + u_2 \overline{w}_2) = J\overline{\Psi}(z) (u_1(\overline{\Psi}) \overline{w}_1 + u_2(\overline{\Psi}) \overline{w}_2),
\]

where \( J\overline{\Psi} \) is the Jacobian determinant of \( \overline{\Psi}. \) Evidently, \( \|\overline{\partial}_\tau u\|_{\overline{\Omega}} \approx \|u\|_{\overline{\Omega}_\tau} \) and \( \|\overline{G}_\tau u\|_{\overline{\Omega}} \approx \|u\|_{\Omega \cap \overline{\Omega}}. \) Let \( \overline{\partial}_\tau = \overline{G} \circ \overline{G}_\tau \) and let \( \overline{\partial}_\tau (u) = (\delta \tau)^{-2} u \circ D_\tau \) be the unitary transformation on \( L^2 \)-spaces associated with the dilation \( D_\tau. \) Let

\[
Q_\tau (u, v) = \tau^2 Q(\overline{\partial}_\tau u, \overline{\partial}_\tau v)
\]

be the densely defined, closed sesquilinear form on \( (L^2(\overline{\Omega}_\tau))^2 \) with \( \text{Dom}(Q_\tau) = \{ \overline{G}_\tau^{-1} (u); \ u \in \text{Dom}(Q) \}. \) Here \( Q(\cdot, \cdot) \) is the sesquilinear form associated with the \( \overline{\partial}-\text{Neumann Laplacian on } L^2_{(0,1)}(\Omega). \)
Lemma 4.4. For any $u \in \text{Dom}(Q) \cap C_0^\infty(\Omega \cap U)$,
\[
Q(u, u) \approx \tilde{Q}(\tilde{\Lambda}^{-1}u, \tilde{\Lambda}^{-1}u),
\]
where $\tilde{Q}$ is the sesquilinear form associated with the $\tilde{\partial}$-Neumann Laplacian on $L^2(0,1)(\tilde{\Omega})$.

Proof. From [Ko72], we know that
\[
Q(u, u) \approx \|\tilde{\Lambda}^{-1}u\|_{\tilde{\Omega}}^2 + \|L_1\tilde{\Lambda}^{-1}u\|_{\tilde{\Omega}}^2 + \|\tilde{\Lambda}^{-1}L_2u\|_{\tilde{\Omega}}^2 + \int_{\tilde{\Omega}} (\partial\tilde{\partial}\tilde{\rho}(\tilde{L}_1, \tilde{L}_1)(z))|u|^2\,dS(z).
\]
It follows from (A-1) that
\[
Q(u, u) \approx \|\tilde{\Lambda}^{-1}u\|_{\tilde{\Omega}}^2 + \|\tilde{L}_1\tilde{\Lambda}^{-1}u\|_{\tilde{\Omega}}^2 + \|\tilde{\Lambda}^{-1}L_2u\|_{\tilde{\Omega}}^2 + \int_{\tilde{\Omega}} (\tilde{\partial}\tilde{\partial}\tilde{\rho}(\tilde{L}_1, \tilde{L}_1)(z))|\tilde{\Lambda}^{-1}u|^2\,dS(z).
\]
Thus $Q(u, u) \approx \tilde{Q}(\tilde{\Lambda}^{-1}u, \tilde{\Lambda}^{-1}u)$. \hfill \Box

Let $\tilde{u}(\xi', s) = (\mathcal{F}_{\text{tan}}(u))(\xi', s)$ be the tangential Fourier transform of $u$ in the $x' = (x, y, t)$ variables. Recall that the tangential Laplacian $\Lambda^s$ is defined by
\[
\mathcal{F}_{\text{tan}}(\Lambda^s u)(\xi', s) = (1 + |\xi'|^2)^{s/2}\tilde{u}(\xi', s)
\]
and the tangential $L^2$-Sobolev norm of order $s$ by
\[
\|u\|_{L^{2,s}(\tilde{\Omega})}^2 = \int_0^\infty \int_{\mathbb{R}^3} (1 + |\xi'|^2)^s|\tilde{u}(\xi', s)|^2\,d\xi'\,ds.
\]

Lemma 4.5. There exists an $\varepsilon > 0$ such that for any sufficiently small $\tau > 0$,
\[
Q_\tau(u, u) \geq \|u\|_{\varepsilon}^2 + \tau^2\delta^{-2}\|\partial\tilde{\partial}u\|_{-1+\varepsilon}^2,
\]
for all $u \in \text{Dom}(Q_\tau) \cap C_0^\infty(\mathcal{P}_\tau(z'))$.

Proof. By Lemma 4.4,
\[
Q_\tau(u, u) \approx \tau^2\tilde{Q}(\tilde{\Lambda}^s u, \tilde{\Lambda}^s u)
\]
\[
\approx \tau^2(\|\tilde{\Lambda}^s u\|_{\tilde{\Omega}}^2 + \|\tilde{L}_1\tilde{\Lambda}^s u\|_{\tilde{\Omega}}^2 + \|\tilde{L}_2\tilde{\Lambda}^s u\|_{\tilde{\Omega}}^2)
\]
\[
\approx \tau^2\|u\|_{\varepsilon}^2 + \|\tilde{L}_1\tau u\|_{\tilde{\Omega}}^2 + \|\tilde{L}_2\tau u\|_{\tilde{\Omega}}^2.
\]
We first prove that there exists an $\varepsilon > 0$ such that
\[
\|\partial\tilde{\partial}u\|_{-1+\varepsilon} \lesssim Q_\tau(u, u).
\]
This is a direct consequence of Kohn’s method [Ko72], in light of (A.7) and Lemma 4.3. Since we need to keep track that the constant in (A.8) is independent of $\tau$, we sketch the proof for completeness. By Lemma 4.3 (1) and (3), we have
\[
\|\partial\tilde{\partial}u\|_{-1} \lesssim \|\Lambda^{i_1}u\|_{-1} \lesssim \|L^{i_1-i_0}u\|_{-1} + Q_\tau(u, u).
\]
It remains to estimate $\|L^{i_1-i_0}u\|_{-1}$, which equals
\[
\langle L^{(i_1-i_0-1)}u, L^{i_0}u \rangle - \langle L^{i_0}u, L^{(i_1-i_0)}u \rangle.
\]
The first term above equals
\[
(L^{(i_1\ldots i_{i_0-1})} u, \Lambda^{2(\varepsilon-1)} L^{(i_1\ldots i_{i_0})} (L^{i_0})^* u) + (L^{(i_1\ldots i_{i_0-1})} u, [(L^{i_0})^*, \Lambda^{2(\varepsilon-1)}] L^{(i_1\ldots i_{i_0})} u)
\]
\[
+ (L^{(i_1\ldots i_{i_0-1})} u, \Lambda^{2(\varepsilon-1)} [(L^{i_0})^*, L^{(i_1\ldots i_{i_0})}] u) = I + II + III.
\]
We have
\[
|I| = |\langle \Lambda^{2(\varepsilon-1)} L^{(i_1\ldots i_{i_0-1})} u, (-L^{(1-i_{i_0})} + \sigma) u \rangle|
\]
\[
\lesssim \|L^{(i_1\ldots i_{i_0-1})} u\|_{2\varepsilon-1}^2 + Q_\varepsilon(u, u),
\]
because \((\varepsilon, \Lambda^{(\varepsilon-1)} L^{(i_1\ldots i_{i_0})})^*\) is a tangential pseudodifferential operator of order \(2\varepsilon - 1\). Also, since \([(L^{i_0})^*, \Lambda^{2(\varepsilon-1)}]\) is of order \(2(\varepsilon - 1)\), we have
\[
|II| \lesssim C \|L^{(i_1\ldots i_{i_0-1})} u\|_{2\varepsilon-1}^2 + (1/C) \|L^{(i_1\ldots i_{i_0})} u\|_{2\varepsilon-1}^2.
\]
Furthermore, as for (4.9), it follows from Lemma 4.3 that
\[
|III| = |\langle L^{(i_1\ldots i_{i_0-1})} u, \Lambda^{2(\varepsilon-1)} (-L^{(1-i_{i_0})} + \sigma, L^{(i_1\ldots i_{i_0})}] u \rangle|
\]
\[
\lesssim C \|L^{(i_1\ldots i_{i_0-1})} u\|_{2\varepsilon-1}^2 + (1/C) \|L^{(i_1\ldots i_{i_0})} u\|_{2\varepsilon-1}^2 + Q_\varepsilon(u, u).
\]
The second term in (4.10) is estimated similarly and is left to the reader. From these estimates, we then have
\[
\|L^{(i_1\ldots i_{i_0})} u\|_{2\varepsilon-1}^2 \lesssim \|L^{(i_1\ldots i_{i_0-1})} u\|_{2\varepsilon-1}^2 + Q_\varepsilon(u, u).
\]
Let \(\varepsilon = 2^{-2\delta m}\). Repeating the above arguments, we then obtain (4.8).

Since \(\partial / \partial s\) is a linear combination of \(L^0, L^1\), and \(\overline{\mathcal{L}}_2\) with coefficients in \(S^{1,0}_{\varepsilon}\), it follows from (4.8) that
\[
(4.11) \quad \| \frac{\partial u}{\partial s} \|_{2-1+\varepsilon}^2 \lesssim Q_\varepsilon(u, u).
\]
Combining (4.7), (4.8), (4.11), and Lemma 4.2, we then have
\[
\| \nabla u \|_{2-\varepsilon}^2 + \tau^2 \delta^{-2} \| \frac{\partial u}{\partial \tilde{w}_2} \|_{2-1+\varepsilon}^2 \lesssim Q_\varepsilon(u, u).
\]
By applying the Poincaré inequality to \(\tilde{u}(\xi', \cdot)\), we know that the left-hand side above dominates \(\|u\|^2\). We thus conclude the proof of the lemma.  

5. Auxiliary estimates

For any \(\varepsilon\) such that \(0 < \varepsilon \leq 1/2\) and any \(\delta > 0\), let \(W_{\varepsilon, \delta}\) be the space of all \(u \in L^2(\mathbb{C}^2)\) such that
\[
(5.1) \quad \|u\|_{\varepsilon, \delta}^2 = \|u\|_{\varepsilon}^2 + \delta^{-1} \|\nabla u\|_{\varepsilon}^2 < \infty.
\]
Let \(\tilde{Q}_{\varepsilon, \delta}\) be the sesquilinear form on \(L^2(\mathbb{C}^2)\) associated with the above norm with \(\text{Dom}(\tilde{Q}_{\varepsilon, \delta}) = W_{\varepsilon, \delta}\). Let \(\Delta_{\varepsilon, \delta}\) be the associated densely defined, self-adjoint operator on \(L^2(\mathbb{C}^2)\) and let \(\tilde{N}_{\varepsilon, \delta}\) be its inverse. Let \(\chi(w_1, w_2)\) be a smooth cut-off function supported on \(\{|w_1| < 1, |w_2| < 1\}\) and identically 1 on \(\{|w_1| < 1/2, |w_2| < 1/2\}\). Let \(\chi_\delta(w_1, w_2) = \chi(w_1, \delta^{1/2} w_2)\). We now study the spectral behavior of \(\chi_\delta \tilde{N}_{\varepsilon, \delta}\) as \(\delta \to 0^+\). Let \(P\) be the orthogonal projection from \(L^2(\mathbb{C}^2)\) onto \(H = \{u \in L^2(\mathbb{C}^2) \mid \partial u / \partial \tilde{w}_2 = 0\}\). Namely, \(P\) is the partial Bergman projection in the \(w_2\)-variable.
Lemma 5.1. For all $\delta > 0$ and $u \in W_{\varepsilon,\delta}$,

\[ \|(I-P)u\|_{\varepsilon,\delta} \lesssim \|u\|_{\varepsilon,\delta}; \quad \| \frac{\partial}{\partial s} (I-P)u \|_{-1+\varepsilon} + \| \frac{\partial}{\partial t} (I-P)u \|_{-1+\varepsilon} \leq \| \frac{\partial u}{\partial \bar{w}_2} \|_{-1+\varepsilon}. \]

**Proof.** This lemma follows from standard elliptic theory (cf. [Me81]). We provide the proof for completeness. Recall that $I-P = 4\frac{\partial}{\partial \bar{w}_2} G \frac{\partial}{\partial \bar{w}_2}$, where $G$ is the Green’s operator in $w_2$-variable (i.e., the inverse of $-\Delta_{w_2}$). Throughout this section, we will use $\zeta_1 = \xi + i\eta$ and $\zeta_2 = \mu + iv$ to denote the dual variables of $w_1 = x + iy$ and $w_2 = s + it$ in the Fourier transform. Recall that $\tilde{u}$ denotes the tangential Fourier transform of $u$ in the $(x,y,t)$ variables. We have

\[ \| (I-P)u \|_{\varepsilon,\delta}^2 = \int_{-\infty}^{0} ds \int_{\mathbb{R}^4} (1 + |\zeta_1|^2 + \nu^2)^{\varepsilon} \frac{1}{2} (|\frac{\partial}{\partial s} + \nu^2)G(\frac{\partial}{\partial \bar{w}_2})^2 d\xi d\eta dt. \]

It is easy to check that $\tilde{G}(u) = -E_+ E_- u$, where

\[ (E_- u)(\zeta_1, s, \nu) = \int_{-\infty}^{s} e^{-|\nu| (s-s')} \tilde{u}(\zeta_1, s', \nu) ds', \]

and

\[ (E_+ u)(\zeta_1, s, \nu) = -\int_{s}^{0} e^{\nu (s-s')} \tilde{u}(\zeta_1, s', \nu) ds'. \]

(See, e.g., Chapter III in [Y75].) Using the identities $\frac{\partial (E_+ \tilde{u})}{\partial s} = |\nu| E_+ \tilde{u} + \tilde{u}$ and $E_- (\frac{\partial}{\partial s}) = -|\nu| E_- (\tilde{u}) + \tilde{u}$, we obtain that

\[ 2(\frac{\partial}{\partial s} + \nu^2)G(\frac{\partial}{\partial \bar{w}_2}) = (\nu + |\nu|^2) E_+ E_- \tilde{u} - (\nu + |\nu|)(E_+ \tilde{u} - E_- \tilde{u}) - \tilde{u}. \]

Since by the Minkowski inequality,

\[ \int_{-\infty}^{0} |E\tilde{u}(\zeta_1, s, \nu)|^2 \leq |\nu|^{-2} \int_{-\infty}^{0} |\tilde{u}(\zeta_1, s, \nu)|^2 ds \]

holds for both $E_+$ and $E_-$, we obtain that $\| (I-P)u \|_{\varepsilon,\delta} \lesssim \| u \|_{\varepsilon,\delta}$. The first inequality then follows. The second inequality is treated similarly and its proof is left to the reader. \(\Box\)

Lemma 5.2. For sufficiently small $\delta > 0$ and sufficiently large $j$,

\[ \lambda_j(\chi_{\delta} N_{1/2}^{\varepsilon,\delta}) \lesssim (1 + j \delta^{3/2})^{-\varepsilon/4}. \]

**Proof.** For $u \in L^2(\mathbb{C}^\pm)$, we write $N_{1/2}^{\varepsilon,\delta} u = (I-P)N_{1/2}^{\varepsilon,\delta} u + P N_{1/2}^{\varepsilon,\delta} u = v_1 + v_2$. We first study $\chi_{\delta}(I-P)N_{1/2}^{\varepsilon,\delta}$. We extend $v_1$ evenly to $s > 0$ by letting $v_1(w_1, s + it) = v_1(w_1, -s + it)$. Denote by $\tilde{v}_1$ the Fourier transform of the extended $v_1$ in all variables. Then by Lemma 5.1,

\[ \| N_{1/2}^{\varepsilon,\delta} u \|_{\varepsilon,\delta}^2 \gtrsim \int_{\mathbb{R}^4} (1 + |\zeta_1|^2 + \nu^2)^{\varepsilon} |\tilde{v}_1|^2 dV(\zeta) \]

\[ \gtrsim \int_{\mathbb{R}^4} (1 + |\zeta_1|^2 + \delta^{-1} |\zeta_2|^2)^{\varepsilon} |\tilde{v}_1|^2 dV(\zeta) \equiv \| v_1 \|_{\varepsilon,\delta}^2, \]

where in the last estimate we use the following simple inequality: $a^2 b^{1-\varepsilon} \leq \varepsilon a + (1-\varepsilon)b$. Let $\Delta_{\varepsilon,\delta}'$ be the Dirichlet realization of the self-adjoint operator associated with the sesquilinear form that defines the norm $\| \cdot \|_{\varepsilon,\delta}'$ on \{ $|w_1| < 1$, $|w_2| < \delta^{-1/2}$\}. As such, we have $\| v_1 \|_{\varepsilon,\delta}' = \| \Delta_{\varepsilon,\delta}' v_1 \|_{\varepsilon,\delta}'$, and
\[
\| (\Delta'_{\varepsilon, \delta})^{1/2} v \| \leq \delta^{1/2} \| u \|_{L^2(\mathbb{C}^2)},
\]
Let \( S_{\delta} u(w_1, w_2) = \delta^{1/2} u(w_1, \delta^{1/2} w_2) \). Then \( S_{\delta} \) is an isometry on \( L^2(\mathbb{C}^2) \). Furthermore, it is easy to see that \( \| v \|_{\varepsilon, \delta} = \| S_{\delta} v \|_{\varepsilon, 1} \). It follows that
\[
(5.4) \quad \lambda_j(\Delta'_{\varepsilon, \delta}) = \lambda_j(\Delta'_{\varepsilon, 1}) \approx \lambda_j(\Delta^\varepsilon) \approx j^{\varepsilon/2},
\]
where \( \Delta \) is the usual Dirichlet Laplacian on \( \{ |w_1| < 1, |w_2| < 1 \} \) and the last estimate follows from the classical Weyl formula. Thus it follows from (5.3) that
\[
\| (\Delta'_{\varepsilon, \delta})^{1/2} \chi_\varepsilon (I - P) \tilde{N}_{\varepsilon, \delta} u \|^2 = \| \chi_\varepsilon v_1 \|^2_{\varepsilon, \delta} + \| v_2 \|^2_{\varepsilon, \delta} \lesssim \| \tilde{N}_{\varepsilon, \delta} u \|^2_{\varepsilon, \delta} = \| u \|^2.
\]
Therefore, by (2.3) and (5.4), we have
\[
(5.5) \quad \lambda_j(\chi_\varepsilon (I - P) \tilde{N}_{\varepsilon, \delta}) \lesssim \lambda_j((\Delta'_{\varepsilon, \delta})^{-1/2}) \approx (1 + j)^{-\varepsilon/4}.
\]
We now study the eigenvalues of \( \chi_\delta P \tilde{N}_{\varepsilon, \delta} \). Let \( \mathcal{R} : L^2(\mathbb{C} \times (0, \infty)) \to H \) be defined by
\[
\mathcal{R} \phi(w_1, w_2) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{w_2 \nu} \phi(w_1, \nu) \sqrt{\nu} \, d\nu,
\]
and let \( \mathcal{R}^* : L^2(\mathbb{C} \times (0, \infty)) \to L^2(\mathbb{C} \times (0, \infty)) \) be defined by
\[
\mathcal{R}^* u(w_1, \nu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 (\mathcal{F}_t u)(w_1, s, \nu) e^{s \nu} \, ds,
\]
where, as usual, \( \mathcal{F} \) is the Fourier transform in the \( t \)-variable. It is easy to see that \( \mathcal{R} \) is isometric and onto. Furthermore, \( \mathcal{R}^* \mathcal{R} = I \) and \( \mathcal{R} \mathcal{R}^* = P \).

For any \( \lambda > 1 \), let \( \mathcal{E}_\lambda : L^2(\mathbb{C} \times (0, \infty)) \to L^2(\mathbb{C} \times (0, \infty)) \) be defined by
\[
\mathcal{E}_\lambda \phi(w_1, \nu) = (\mathcal{F}_{w_1}^{-1} \chi_{\{1+|\xi_1|^2+\nu^2<\lambda^2\}} \mathcal{F}_{w_1} \phi)(w_1, \nu),
\]
where \( \chi_A \) is the characteristic function for the set \( A \) as before and \( \mathcal{F}_{w_1} \) is the Fourier transform in the \( x \) and \( y \) variables. (Recall that \( w_1 = x + iy \), \( w_2 = s + it \), and their dual variables are \( \xi_1 = \xi + i\eta \) and \( \xi_2 = \mu + i\nu \).) Let \( \mathcal{M}_\lambda = \mathcal{R} \mathcal{E}_\lambda \mathcal{R}^* : L^2(\mathbb{C}^2) \to H \). Then \( \mathcal{M}_\lambda \) is an orthogonal project into \( H \). A straightforward calculation yields that the kernel \( \mathcal{M}_\lambda \) of \( \mathcal{M}_\lambda \) is given by
\[
M_\lambda(w, w') = \frac{1}{4\pi^3} \int_0^\infty \int_{\mathbb{C}} \nu e^{w_2 \nu} \chi_{\{1+|\xi_1|^2+\nu^2<\lambda^2\}} \nu \, d\xi \, d\eta \, d\nu.
\]
Thus,
\[
M_\lambda(w, w) = \frac{1}{4\pi^3} \int_0^{\sqrt{\lambda^2-1}} \nu e^{2\nu} (\lambda^2 - 1 - \nu^2) \, d\nu.
\]
The square of the Hilbert-Schmidt norm of \( \chi_{\delta} \mathcal{M}_\lambda \) equals
\[
\int_{\mathbb{C}^2} |\chi_{\delta}(w) M_\lambda(w, w')|^2 \, dV(w, w') = \int_{\mathbb{C}^2} |\chi_{\delta}|^2 M_\lambda(w, w) \, dV(w)
\]
\[
= \frac{1}{4\pi^{d+3}} \int_{\mathbb{C}^2} |\chi(w_1, w_2)|^2 M_\lambda(w_1, \delta^{-1/2} w_2) \, dV(w)
\]
\[
\lesssim \delta^{-1} \int_{-\infty}^0 d\nu \int_0^{\sqrt{\lambda^2-1}} \nu e^{2\delta^{-1/2} \nu} (\lambda^2 - 1 - \nu^2) \, d\nu
\]
\[
\lesssim \delta^{-1/2} \int_0^{\sqrt{\lambda^2-1}} (\lambda^2 - 1 - \nu^2) \, d\nu \lesssim \delta^{-1/2} \lambda^3.
\]
Thus, on the one hand, we have
\[
(5.6) \quad \lambda_j(\chi_{\delta} \mathcal{M}_\lambda) \lesssim (\delta^{-1/2} \lambda^3/j^{1/2}).
\]
On the other hand, we have
\[
\|v_2 - \mathcal{M}_\lambda v_2\|^2 = \|\mathcal{R}R^*v_2 - \mathcal{R}E_\lambda R^*v_2\|^2 = \|(I - E_\lambda)R^*v_2\|^2 = \\
\leq \int_0^\infty dv \int_{C} \chi_{\{1 + |\zeta|^2 + v^2 \geq \lambda^2\}} d\xi d\eta \sqrt{\nu} \left| \int_{-\infty}^0 (\mathcal{F}_{u_1} F_{v_2})(\zeta_1, \nu, s) e^{st} ds \right|^2 \\
\leq \int_0^\infty dv \int_{C} \chi_{\{1 + |\zeta|^2 + v^2 \geq \lambda^2\}} d\xi d\eta \int_{-\infty}^0 |\mathcal{F}_{u_1} F_{v_2}|^2 ds \\
\lesssim \lambda^{-2\epsilon} \|v_2\|^2 \lesssim \lambda^{-2\epsilon} \|\tilde{N}_{\epsilon,\delta}^{1/2} u\|_\epsilon,\delta^2 = \lambda^{-2\epsilon} \|u\|^2.
\]
(Here we have used Lemma 5.1 in the last estimate.) From (5.6), (5.7), and (2.1), we obtain
\[
\lambda_j(\chi_3 P \tilde{N}_{\epsilon,\delta}^{1/2}) \lesssim (\delta^{-1/2} \lambda^3 j^{-1})^{1/2} + \lambda^{-\epsilon} \\
\lesssim (j\delta^{1/2})^{-1/8} + (j\delta^{1/2})^{-\epsilon/4} \lesssim (j\delta^{1/2})^{-\epsilon/4},
\]
by taking \(\lambda = (j\delta^{1/2})^{1/4}\). This estimate, combined with (5.5), then gives us the desired estimate. \(\square\)

6. Estimates on the spectral kernel

Let \(E(\lambda)\) be the spectral resolution of the \(\overline{T}\)-Neumann Laplacian \(\square\) on \(L^2_{(0,1)}(\Omega)\) and let \(e(\lambda; z, \zeta)\) be its kernel. By the classical elliptic theory we know that
\[
\lim_{\lambda \to -1}\lambda^{-2} \text{tr} e(\lambda; z, z) = (2\pi)^{-2},
\]
where the limit is uniform on any compact subset of \(\Omega\) (e.g., [G53]). In fact, \(\text{tr} e(\lambda; z, z) \lesssim \lambda^2\) for all \(z \in \Omega\) with \(d(z) \geq \lambda^{-1/2}\) (e.g., [M08]).

**Lemma 6.1.** Let \(\tau = 1/\sqrt{\lambda}\). For sufficiently large \(\lambda > 0\),
\[
\text{tr} e(\lambda; z, z) \lesssim \lambda(\delta(\pi(z), \tau))^{-1},
\]
for all \(z \in \Omega\) with \(d(z) \geq (\delta(\pi(z), \tau))^{1/2}\).

**Proof.** In light of the above remarks, it suffices to prove (6.2) when \((\delta(\pi(z), \tau))^{1/2} \lesssim \lambda^{-1/2}\). We will use a global rescaling scheme which is slightly different from the local rescaling scheme introduced in Section 4.

Let \(z' \in b\Omega\) and let \(\Omega' = \Phi'(\Omega)\) as in Section 3 where \(\Phi'\) is given by \(6.24\). Let \(\delta = \delta(z', \tau)\). For any \(\sigma\) such that \(\sqrt{\delta} \lesssim \sigma \lesssim \tau\), let \((\xi_1', \xi_2') = D_{\tau,\sigma'}(\xi_1, \xi_2) = (\xi_1/\tau, \xi_2/\sigma).\) Let \(\Omega'_{\tau,\sigma} = D_{\tau,\sigma}(\Omega)\). Let
\[
L_1' = \frac{1}{|\partial r|} \left( \frac{\partial r}{\partial z_2} \frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1} \frac{\partial}{\partial z_2} \right), \quad L_2' = \frac{1}{|\partial r|} \left( \frac{\partial r}{\partial z_1} \frac{\partial}{\partial z_1} + \frac{\partial r}{\partial z_2} \frac{\partial}{\partial z_2} \right).
\]
We extend the vector fields \(L_1'\) and \(L_2'\) to form an orthonormal basis for \(T^{1,0}(\mathbb{C})\) over \(\overline{\Omega}\). Let \(\omega_1'\) and \(\omega_2'\) be the dual basis. Denote by \(L_1'^{\tau,\sigma}, L_2'^{\tau,\sigma}, \omega_1'^{\tau,\sigma}, \omega_2'^{\tau,\sigma}\) the vector fields and forms defined as above but with \(z\) replaced by \(\xi'\) and \(r\) replaced by \(r_{\tau,\sigma}(\xi') = (1/\sigma)\rho(\tau\xi_1', \sigma\xi_2')\) where \(\rho\) is given by (5.5). Let \(H_{\tau,\sigma} : L^2_{(0,1)}(\Omega'_{\tau,\sigma}) \to L^2_{(0,1)}(\Omega)\) be the unitary transformation defined by
\[
H_{\tau,\sigma}(v_1\omega_1'^{\tau,\sigma} + v_2\omega_2'^{\tau,\sigma}) = (\tau\sigma)^{-1} J\Phi'(z)(v_1(D_{\tau,\sigma}' \circ \Phi')\omega_1' + v_2(D_{\tau,\sigma}' \circ \Phi')\omega_2'),
\]
where as before $\mathcal{J}\Phi'$ is the Jacobian determinant of $\Phi'$. Let

$$Q^*_{\tau,\sigma}(u, v) = \tau^2 Q(\mathcal{H}_{\tau,\sigma} u, \mathcal{H}_{\tau,\sigma} v)$$

with $\text{Dom}(Q^*_{\tau,\sigma}) = \text{Dom}(Q'_{\tau,\sigma})$ where $Q$ is the sesquilinear form associated with the $\overline{\partial}$-Neumann Laplacian $\square$ on $\Omega$ as before, and $Q'_{\tau,\sigma}$ is associated with the $\overline{\partial}$-Neumann Laplacian $\square'_{\tau,\sigma}$ on $\Omega'_{\tau,\sigma}$. Let $\square_{\tau,\sigma}$ be the self-adjoint operator defined by $Q^*_{\tau,\sigma}$ and let $e^*_{\tau,\sigma}(\lambda; \xi')$ be the kernel of the spectral resolution of $\square_{\tau,\sigma}$. Then

$$e(\lambda; z, \zeta) = (\tau \sigma)^{-2} J\Phi'(z) J\Phi'(\zeta) e^*_{\tau,\sigma}(\tau^2 \lambda; D'_{\tau,\sigma} \circ \Phi'(z), D'_{\tau,\sigma} \circ \Phi'(\zeta)).$$

Let $P' = \{\xi' \in \mathbb{C}^2 \mid |\xi'_1| < c, |\xi'_2 + 1/2| < c\}$. Then for sufficiently small $c > 0$, $P'$ is relatively compact subset of $\Omega'_{\tau,\sigma}$. Furthermore, if $u$ is supported in $P'$, then

$$Q^*_{\tau,\sigma}(u, u) \gtrsim \tau^2 (|\tau \sigma|)^{-1} |\nabla u(\tau^{-1} \xi'_1, \sigma^{-1} \xi'_2)|^2 \gtrsim \|\nabla \xi' u\|_{L^2}^2,$$

where the rescaling of $u$ is component-wise and last estimate follows from $\sigma \lesssim \tau$. In (6.4), we take $z = (\Phi')^{-1}(0, -\sigma/2)$. Then $d(z) \approx \sigma$. From (6.4), we have

$$\text{tr } e(\lambda; z, z) \lesssim (\tau \sigma)^{-2} \text{tr } e^*_{\tau,\sigma}(1; (0, -1/2), (0, -1/2)).$$

By (5.5) and the Sobolev lemma, we have for any $k > 2$ and $\xi' \in P'$

$$\|\text{tr } e^*_{\tau,\sigma}(\tau^2 \lambda; \cdot, \xi')\|_{L^\infty(P')} \lesssim \|\square^k_{\tau,\sigma} e^*_{\tau,\sigma}(\tau^2 \lambda; \cdot, \xi')\|_{L^2(\Omega'_{\tau,\sigma})} + \|\text{tr } e^*_{\tau,\sigma}(\tau^2 \lambda; \cdot, \xi')\|_{L^2(\Omega'_{\tau,\sigma})}$$

$$\lesssim (1 + (\tau \sigma)^k)(\text{tr } e^*_{\tau,\sigma}(\tau^2 \lambda; \xi', \xi'))^{1/2}.$$
\[ R_{C_\tau}(z') \] for some sufficiently large constant \( C > 0 \). By Lemma 4.5 and using the fact that \( \delta = \delta(z', \tau) \leq \tau^2 \), we have
\[ Q_\tau(u, u) \gtrsim \|u\|^{2}_{\varepsilon, C\delta}, \]
for \( u \in \text{Dom}(\Box^{1/2}) \), where \( C > 0 \) is any constant. Therefore,
\[ \|u\|^2 = Q_\tau(N_{\tau}^{1/2}u, N_{\tau}^{1/2}u) \gtrsim \|\Box^{1/2}N_{\tau}^{1/2}u\|^2. \]
Choose \( C > 2 \). Then \( N_{\tau}^{1/2} = \chi_{C_\delta}N_{\tau}^{1/2} = \chi_{C_\delta}\tilde{N}_{\tau}^{1/2} \). It follows from (2.3) and Lemma 5.2 that
\[ (6.10) \lambda_j(N_{\tau}^{1/2}) \lesssim \lambda_j(\chi_{C_\delta}\tilde{N}_{\tau}^{1/2}) \lesssim (j\delta^{1/2})^{-\varepsilon/4}. \]

Let \( K \) be any positive integer such that \( K > 4/\varepsilon \). Let \( \chi^{(k)} \), \( k = 0, 1, \ldots, K \), be a family of cut-off functions supported in \( \{|w_1| < 1, |w_2| < 1\} \) such that \( \chi^{(0)} = \chi \) and \( \chi^{(k+1)} = 1 \) on \( \text{Supp} \chi^{(k)} \). Let
\[ (6.11) E^{(0)}_\tau(\lambda) = E_\tau(\lambda) \]
and
\[ \|E^{(0)}_\tau(1)\| = \|\chi^{(0)}(\cdot)\| \]
It is easy to check the following commutating identity:
\[ (6.12) Q(\theta u, \theta u) = \text{Re}(\theta \Box u, \theta u) + (1/2)(u, [\theta, A] u) \]
where \( \theta \) is any smooth function and \( A = [\overline{\nabla}, \theta][\overline{\nabla} + \overline{\theta}[\overline{\nabla}, \theta] + \overline{\theta}[\overline{\theta}, \theta] + [\overline{\theta}, \theta][\overline{\theta}] \). Note that \( [\theta, A] \) is of zero order. Using the above identity and the Schwarz inequality, we obtain that for any \( u \in (L^2(\Omega_{\tau}))^2 \),
\[ \|\Box^{1/2}\chi^{(k)}(\cdot)E^{(1)}_\tau(u)\|^2 = Q_\tau(\chi^{(k)}(\cdot)E^{(1)}_\tau(u), \chi^{(k)}(\cdot)E^{(1)}_\tau(u)) \]
\[ = \tau^2 Q(\chi^{(k)}(\cdot)(\chi^{(k)}(\cdot))\Box^{1/2}E^{(1)}_\tau(\cdot), \chi^{(k)}(\cdot)(\chi^{(k)}(\cdot))\Box^{1/2}E^{(1)}_\tau(\cdot)) \]
\[ \lesssim \|\chi^{(k)}(\cdot)(\cdot)^{\Box^{1/2}}\|^{2} + \|\chi^{(k+1)}(\cdot)(\cdot)^{\Box^{1/2}}\|^{2} = \|\chi^{(k)}(\cdot)(\cdot)^{\Box^{1/2}}\|^{2} + \|\chi^{(k+1)}(\cdot)(\cdot)^{\Box^{1/2}}\|^{2}. \]
By (2.4) and (2.3), we have
\[ (6.13) \lambda_{3j+1}(\chi^{(k)}_\delta(\cdot)E^{(1)}_\tau(\cdot)) \leq \lambda_{j+1}(N_{\tau}^{1/2})\lambda_{2j+1}(\Box^{1/2}\chi^{(k)}_\delta(\cdot)E^{(1)}_\tau(\cdot)) \]
\[ \leq \lambda_{j+1}(N_{\tau}^{1/2}) \left( \lambda_{j+1}(\chi^{(k)}_\delta(\cdot)E^{(j+1)}_\tau(\cdot)) + \lambda_{j+1}(\chi^{(k+1)}_\delta(\cdot)E^{(j+1)}_\tau(\cdot)) \right). \]
Using (6.10), (6.11), and (6.13), we then obtain by an inductive argument on \( K - (k + l) \) that
\[ (6.14) \lambda_j(\chi^{(k)}_\delta E^{(1)}_\tau(\cdot)) \lesssim (j\delta^{1/2})^{-(K-(k+l))\varepsilon/4} \]
for any pair of non-negative integers \( k, l \) such that \( 0 \leq k + l \leq K \). In particular,
\[ \lambda_j(\chi^{(k)}_\delta E^{(1)}_\tau(\cdot)) \lesssim (j\delta^{1/2})^{-K\varepsilon/4}. \]
Since \( E^{(1)}_\tau(\cdot) \) is a contraction, we also have that \( \lambda_j(\chi^{(k)}_\delta E^{(1)}_\tau(\cdot)) \leq 1 \). The trace norm of \( \chi^{(k)}_\delta E^{(1)}_\tau(\cdot) \) is then given by
\[ \sum_{j \leq \delta^{-1/2}} \lambda_j(\chi^{(k)}_\delta E^{(1)}_\tau(\cdot)) + \sum_{j > \delta^{-1/2}} \lambda_j(\chi^{(k)}_\delta E^{(1)}_\tau(\cdot)) \lesssim \delta^{-1/2} + \sum_{j > \delta^{-1/2}} (j\delta^{-1/2})^{-K\varepsilon/4} \lesssim \delta^{-1/2}. \]
Inequality (6.9) is now an easy consequence of the above estimate.

Lemma 6.3. For sufficiently large $\lambda > 0$,

\[
\int_{A_r} \mathrm{tr} \, e(\lambda; z, z) \, dV(z) \lesssim \tau^{-2} \int_{b\Omega} (\delta(z', \tau))^{-1} \, dS(z') \lesssim \lambda^{1+m}.
\]

Proof. By Lemma 6.2, we have

\[
(\delta(z', \tau))^{-1/2} \int_{\Omega} \chi_{\Omega \cap R_{\tau}(z')} \, \mathrm{tr} \, e(\lambda; z, z) \, dV(z) \lesssim (\delta(z', \tau))^{-1}.
\]

Integrating both sides with respect to $z' \in b\Omega$ and using the Fubini-Tonelli Theorem, we have

\[
\int_{\Omega} \mathrm{tr} \, e(\lambda; z, z) \, dV(z) \int_{z' \in b\Omega} \chi_{\Omega \cap R_{\tau}(z')} \, (\delta(z', \tau))^{-1/2} \, dS(z') \lesssim \int_{b\Omega} (\delta(z', \tau))^{-1} \, dS(z').
\]

By Lemma 6.4 and the fact that $\delta(z', \tau) \gtrsim \tau^{2m}$, we then obtain

\[
\int_{A_r} \mathrm{tr} \, e(\lambda; z, z) \, dV(z) \lesssim \tau^{-2} \int_{b\Omega} (\delta(z', \tau))^{-1} \, dS(z') \lesssim \lambda^{1+m}.
\]

Lemma 6.3 then follows from a rescaling of $\tau$ in the above arguments.

We are now in position to prove the following variation of Theorem 1.2.

Proposition 6.4. Let $\Omega$ be a smooth bounded pseudoconvex domain of finite type $2m$. Then

\[
\limsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{m+1}} \lesssim \limsup_{\lambda \to \infty} \frac{1}{\lambda^m} \int_{b\Omega} (\delta(z', 1/\sqrt{\lambda}))^{-1} \, dS(z') \lesssim 1.
\]

Proof. Note that

\[
\frac{N(\lambda)}{\lambda^{1+m}} = \lambda^{-1-m} \int_{\Omega} \mathrm{tr} \, e(\lambda; z, z) \, dV(z)
\]

\[
= \lambda^{-1-m} \int_{A_r} \mathrm{tr} \, e(\lambda; z, z) \, dV(z) + \lambda^{-1-m} \int_{\Omega \setminus A_r} \mathrm{tr} \, e(\lambda; z, z) \, dV(z).
\]

By Lemma 6.3, the first term in the last expression is bounded above by

\[
\lambda^{-m} \int_{b\Omega} (\delta(z', 1/\sqrt{\lambda}))^{-1} \, dV(z).
\]

By (6.1), Lemma 6.2, and the Lebesgue dominated convergence theorem, we have

\[
\lim_{\lambda \to \infty} \lambda^{-1-m} \int_{\Omega \setminus A_r} \mathrm{tr} \, e(\lambda; z, z) \, dV(z) = \begin{cases} (2\pi)^{-2} \text{vol}(\Omega), & \text{if } m = 1 \\ 0, & \text{if } m > 1. \end{cases}
\]

We then conclude the proof of the proposition by noting that $\lambda^{-m} \lesssim \delta(z', 1/\sqrt{\lambda}) \lesssim \lambda^{-1}$.

Remarks. It follows from the above proof that $\limsup_{\lambda \to \infty} N(\lambda)/\lambda^{m+1} = 0$ when $m > 1$. 
7. Hearing a finite type property

We prove Theorem 1.3 in this section. The following lemma is well-known (see [PoS89] for the two dimensional case and [Yau95] for the general case). It can be proved along the lines of the arguments in Section 3. Throughout this section, we will use $z'$ to denote the first $(n-1)$-tuple of $z \in \mathbb{C}^n$.

**Lemma 7.1.** Let $\Omega$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that the $D_{n-1}$-type of $b\Omega$ at $z^0$ is $\geq 2m$. Then there exists a neighborhood $U$ of $z^0$ and a biholomorphic map $w = \Psi(z)$ from $U$ into $\mathbb{C}^n$ such that $\Psi(z^0) = 0$ and

$$\Psi(\Omega \cap U) = \{ w \in \mathbb{C}^n \mid |w'| < 1, |\text{Im} \ w_n| < 1, \rho(w) = \text{Re} \ w_n + h(w', \text{Im} \ w_n) < 0 \},$$

where $h(w', \text{Im} \ w_n) = f(w') + (\text{Im} \ w_n) \cdot g(w') + \sigma(w', \text{Im} \ w_n)$ with $|f(w')| \leq |w'|^{2m}$, $|g(w')| \leq |w'|^{m+1}$, and $|\sigma(w', \text{Im} \ w_n)| \leq (\text{Im} \ w_n)^2$.

We now prove Theorem 1.3. Assume that the $D_{n-1}$-type of $b\Omega$ at $z^0$ is $\geq 2m$. We apply Lemma 7.1 and keep its notations.

Write $w_n = s + it$. Let $b(t)$ be the cut-off function constructed in the paragraph preceding Lemma 2.3. We first extend $b(t)$ to the whole complex plane as follows. Let $\chi$ be any smooth cut-off function supported on $(-2, 2)$ and identically 1 on $(-1, 1)$ and let

$$B(w_n) = b(t) - ib'(t)s - b''(t)s^2/2)\chi(s/(1 + |t|^2)).$$

Then $B(0, t) = b(t)$ and $|\partial B(w_n)/\partial \bar{w}_n| \leq |s|^2$.

Let

$$L_j = \frac{\partial \rho}{\partial w_n} \frac{\partial}{\partial w_j} - \frac{\partial \rho}{\partial w_j} \frac{\partial}{\partial w_n}, \quad 1 \leq j \leq n - 1,$$

and

$$L_n = \sum_{j=1}^n \frac{\partial \rho}{\partial w_j} \frac{\partial}{\partial w_n}.$$

Let

$$\tilde{L}_j = (\Psi^{-1})^*_j(L_j), \quad 1 \leq j \leq n - 1,$$

and

$$\tilde{L}_n = \sum_{j=1}^n \frac{\partial \rho}{\partial \bar{w}_j} \frac{\partial}{\partial \bar{z}_j},$$

where $\tilde{\rho}(z) = \rho \circ \Psi(z)$. Then $\tilde{L}_j, 1 \leq j \leq n$, form a basis for $T_{1,0}(\mathbb{C}^n)$ in a neighborhood of $z^0$. Replacing $\tilde{L}_j$ by the product of $\tilde{L}_j$ with an appropriate cut-off function, we may assume that $\tilde{L}_j$ is supported in $U$. Let $\tilde{L}_j, 1 \leq j \leq n$, be the vector fields obtained after performing the Gram-Schmidt process on $\tilde{L}_j$. Let $\tilde{\omega}_j, 1 \leq j \leq n$, be the dual basis of $\tilde{L}_j$. Let $a(w')$ be a smooth function identically 1 near the origin and compactly supported in the unit ball in $\mathbb{C}^n$. For any positive integers $j$ and for any positive integer $k$ such that $2^{mj-1}/j \leq k \leq 2^{mj}/j$, let

$$f_{j,k}(w) = k^8(m+n-1)!a(8^j w')B(8^{mj} w_n)e^{2\pi k^2 8^{mj} w_n}.$$

Let $g_{j,k}(z) = f_{j,k}(\Psi(z)) \cdot (J \Psi(z))$. Let

$$u_{j,k} = g_{j,k}(z)\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_{n-1}.$$

Then for any sufficiently large $j$, $u_{j,k}$ is a compactly supported smooth $(0, n-1)$-form in $\text{Dom}(Q_{n-1})$. Moreover,

$$\|u_{j,k}\|_{\tilde{H}}^2 = \int_{\mathbb{C}^{n-1}} dV(w') \int_{\mathbb{R}} dt \int_{-\infty}^{-h(w', t)} |f_{j,k}(w)|^2 ds$$

$$= k^2 \int_{\mathbb{C}^{n-1}} |a(\tilde{w})|^2 dV(\tilde{w}') \int_{\mathbb{R}} d\tilde{t} \int_{-\infty}^{-8^{mj}h(8^{-j} \tilde{w}', 8^{-mj} \tilde{t})} |B(\tilde{s}, \tilde{t})|^2 e^{4\pi k^2 \tilde{s}} d\tilde{s}. $$
(After the substitution $\tilde{w}' = 8^j w'$, $\tilde{w}_n = 8^{mj} w_n$.) Since $|8^{mj} h(8^{-j} \tilde{w}', 8^{-mj} \tilde{t})| \lesssim 8^{-mj}$ and $k^2 8^{-mj} \lesssim j^{-2} 2^{-mj}$, we have

$$
\|u_{j,k}\|_{\Omega}^2 \lesssim k^2 \int_{C_{n-1}} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\mathbb{R}} d\tilde{t} \int_{-\infty}^{C 8^{-mj}} |B(\tilde{s}, \tilde{t})|^2 e^{4\pi k^2 \tilde{s} + d\tilde{s}}
\lesssim \int_{C_{n-1}} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{|\tilde{t}| < 1} e^{C k^2 8^{-mj}} d\tilde{t} \lesssim 1.
$$

Similarly, $\|u_{j,k}\|_{\Omega} \gtrsim 1$. Therefore, $\|u_{j,k}\|_{\Omega}^2 \approx 1$. Furthermore, after a substitution as above, we have that for any $k, l$ such that $2mj^{-1}/j \leq k, l \leq 2mj/j$, $\langle u_{j,k}, u_{j,l} \rangle$ equals

$$
kl \int_{C_n} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\mathbb{R}} d\tilde{t} \int_{C_{n-1}} |B(\tilde{s}, \tilde{t})|^2 e^{2\pi((k^2 + l^2)\tilde{s} + i(k^2 - l^2)\tilde{t})} d\tilde{s}
$$

Let $A$ be the above expression with the upper limit in the last integral replaced by 0 and let $B$ likewise be the above expression with the lower limit of the last integral replaced by 0. Thus $\langle u_{j,k}, u_{j,l} \rangle = A + B$. It is easy to see that

$$
|B| \leq \frac{k^l \int_{C_{n-1}} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\mathbb{R}} d\tilde{t} \int_{0}^{C 8^{-mj}} |B(\tilde{s}, \tilde{t})|^2 e^{2\pi((k^2 + l^2)\tilde{s} + i(k^2 - l^2)\tilde{t})} d\tilde{s}}{k^l + l^2 (1 - e^{C k^2 8^{-mj}})} \lesssim j^{-2} 2^{-mj}.
$$

To estimate $|A|$, we first observe that by Lemma 2.3, for $k \neq l$,

$$
A = kl \int_{C_{n-1}} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\mathbb{R}} d\tilde{t} \int_{-\infty}^{0} \left( |B(\tilde{s}, \tilde{t})|^2 - |B(0, \tilde{t})|^2 \right) e^{2\pi((k^2 + l^2)\tilde{s} + i(k^2 - l^2)\tilde{t})} d\tilde{s}.
$$

Hence

$$
|A| \lesssim kl \int_{|\tilde{s}| < 1} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{-\infty}^{0} d\tilde{t} \int_{0}^{1} \tilde{s} e^{2\pi((k^2 + l^2)\tilde{s} + i(k^2 - l^2)\tilde{t})} d\tilde{s} \lesssim kl/(k^2 + l^2)^2.
$$

Therefore, for sufficiently large $j$ and for any $k, l$ such that $2mj^{-1}/j \leq k, l \leq 2mj/j$, $k \neq l$, we have,

$$
|\langle u_{j,k}, u_{j,l} \rangle| \lesssim j^{-2} 2^{-mj}.
$$

For any $k$ such that $2mj^{-1}/j \leq k \leq 2mj/j$ and for any $c_k \in \mathbb{C}$, we have

$$
\| \sum_k c_k u_{j,k} \|^2 = \sum_k |c_k|^2 \|u_{j,k}\|^2 - \sum_{k,l} c_k \overline{c}_l \langle u_{j,k}, u_{j,l} \rangle 
\geq \sum_k |c_k|^2 \|u_{j,k}\|^2 - j^{-2} 2^{-mj} \sum_k |c_k|^2 
\geq (1 - j^{-4}) \sum_k |c_k|^2 \geq \sum_k |c_k|^2,
$$

where the summations are taken over all integers between $2mj^{-1}/j$ and $2mj/j$. 
Since each $\hat{L}_k$, $1 \leq k \leq n-1$, is a linear combination of $\hat{L}_1, \ldots, \hat{L}_k$, and $\hat{L}_n$ is just the normalization of $\hat{L}_n$, it follows that

$$Q_{n-1}(u_{j,k}, u_{j,k}) \lesssim \|g_{j,k}\|_{1,\Omega}^2 + \sum_{l=1}^{n-1} (\|L_l g_{j,k}\|_{1,\Omega}^2 + \|\tilde{L}_l g_{j,k}\|_{1,\Omega}^2) + \|L_n g_{j,k}\|_{1,\Omega}^2$$

$$\lesssim \|f_{j,k}\|_{\Psi(\Omega \cap U)}^2 + \sum_{l=1}^{n-1} \|L_l f_{j,k}\|_{\Psi(\Omega \cap U)}^2 + \|\overline{\nabla} f_{j,k}\|_{\Psi(\Omega \cap U)}^2.$$  

For $1 \leq l \leq n-1$, $\|L_l f_{j,k}\|_{\Psi(\Omega \cap U)}^2$ is bounded above by

$$2\left(\left\|\frac{\partial \rho}{\partial w_n} \frac{\partial f_{j,k}}{\partial w_l}\right\|_{\Psi(\Omega \cap U)}^2 + \left\|\frac{\partial \rho}{\partial w_l} \frac{\partial f_{j,k}}{\partial w_n}\right\|_{\Psi(\Omega \cap U)}^2\right)$$

$$\lesssim 8^{2j} + \int_{|\tilde{w}'| < 1} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\tilde{w}'}^{1} dt \int_{-\infty}^{-8^{m_j}h(8^{-j}\tilde{w}', 8^{-m_j}\tilde{t})} \left(8^{m_j}|\nabla B| + k^2 8^{m_j}|B|\right)(8^{-j}\tilde{w}'|2^{m-1} + 8^{-m_j}\tilde{t} \cdot |8^{-j}\tilde{w}'|^m_1 + |8^{-m_j}\tilde{t}|^2)^2 2^{j} t^2 e^{\pi k^2 t^2} d\tilde{s}$$

$$\lesssim 8^{2j}.$$  

Furthermore, $\|\overline{\nabla} f_{j,k}\|_{\Psi(\Omega \cap U)}^2$ is

$$\lesssim \sum_{l=1}^{n} \left\|\frac{\partial f_{j,k}}{\partial w_l}\right\|_{\Psi(\Omega \cap U)}^2$$

$$\lesssim 8^{2j} + 8^{2mj} k^2 \int_{|\tilde{w}'| < 1} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\tilde{w}'}^{1} dt \int_{-\infty}^{-8^{m_j}h(8^{-j}\tilde{w}', 8^{-m_j}\tilde{t})} \left|\frac{\partial B}{\partial w_n}\right|^2 e^{\pi k^2 t^2} d\tilde{s}$$

$$\lesssim 8^{2j} + 8^{2mj} k^2 \int_{|\tilde{w}'| < 1} |a(\tilde{w}')|^2 dV(\tilde{w}') \int_{\tilde{w}'}^{1} dt \int_{-\infty}^{-8^{m_j}h(8^{-j}\tilde{w}', 8^{-m_j}\tilde{t})} \tilde{s}^4 e^{\pi k^2 \tilde{s}^2} d\tilde{s}$$

$$\lesssim 8^{2j} + 8^{2mj} k^{-8} \lesssim 8^{2j}.$$  

Therefore, we have

$$Q_{n-1}(u_{j,k}, u_{j,k}) \lesssim 8^{2j}.$$  

We now invoke the hypothesis of Theorem 1.3. Since $\mathcal{N}_q(\lambda)$ has at most polynomial growth, $\lambda_j(\Box_q) \gtrsim j^\varepsilon$ for some $\varepsilon > 0$. It follows from Proposition 2.2 that $\lambda_j(\Box_{n-1}) \gtrsim j^\varepsilon$. By Lemma 2.1 for all sufficiently large $j$, there exists an integer $k_0 \in \left\{2^{m_j-1}/j, 2^{m_j}/j\right\}$ such that

$$Q_{n-1}(u_{j,k_0}, u_{j,k_0}) \gtrsim (2^{m_j}/j)^\varepsilon.$$  

Therefore, $8^{2j} \gtrsim (2^{m_j}/j)^\varepsilon$. Hence $m \leq 6/\varepsilon$. We thus conclude the proof of Theorem 1.3.

REFERENCES

[BGS84] Richard Beals, Peter C. Greiner, and Nancy K. Stanton, The heat equation on a CR manifold, J. Differential Geometry 20 (1984), no. 2, 343–387.

[BeS87] Richard Beals and Nancy K. Stanton, The heat equation for the $\overline{\partial}$-Neumann problem. I., Comm. Partial Differential Equations 12 (1987), no. 4, 351–413.

[BeS88] Richard Beals and Nancy K. Stanton, The heat equation for the $\overline{\partial}$-Neumann problem. II. Canad. J. Math. 40 (1988), no. 2, 502–512.

[BSt99] Harold P. Boas and Emil J. Straube, Global regularity of the $\overline{\partial}$-Neumann problem: a survey of the $L^2$-Sobolev theory, Several Complex Variables (M. Schneider and Y.-T. Siu, eds.), MSRI Publications, vol. 37, 79-112, 1999.
[Ko63] J. J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds, I, Ann. of Math. (2) 78 (1963), 112–148.

[Ko64] ——, Harmonic integrals on strongly pseudo-convex manifolds, II, Ann. of Math. (2) 79 (1964), 450–472.

[Ko72] ——, Boundary behavior of $\overline{\partial}$ on weakly pseudo-convex manifolds of dimension two, J. of Differential Geometry 6 (1972), 523-542.

[KN65] J. J. Kohn and L. Nirenberg, Non-coercive boundary value problems, Comm. on Pure and Applied Mathematics 18 (1965), 443–492.

[Kr92] Steven G. Krantz, Partial differential equations and complex analysis, CRC Press, Boca Raton, FL, 1992.

[Kr01] ——, Function theory of several complex variables, second ed., AMS Chelsea Publishing, Providence, RI, 2001.

[L66] H. Laufer, On sheaf cohomology and envelopes of holomorphy, Ann. of Math. 84 (1966), 102-118.

[Ma97] P. Matheos, A Hartogs domain with no analytic discs in the boundary for which the $\overline{\partial}$-Neumann problem is not compact, UCLA Ph.D. thesis, 1997.

[Me81] Guy Metivier, Spectral asymptotics for the $\overline{\partial}$-Neumann problem, Duke Math. J. 48 (1981), 779-806.

[Mc89] Jeffery D. McNeal, Boundary behavior of the Bergman kernel function in $C^2$, Duke Math. J. 58 (1989), no. 2, 499–512.

[NRSW95] Alexander Nagel, J. P. Rosay, E. M. Stein, and S. Wainger, Estimates for the Bergman and Szegö kernels in $C^2$, Ann. of Math. 129(1989), 113-149.

[NS01] Alexander Nagel and Elias M. Stein, The $\Box_b$-heat equation on pseudoconvex manifolds of finite type in $C^2$, Math. Z. 238 (2001), no. 1, 37–88.

[NSW85] Alexander Nagel, Elias M. Stein, and Stephen Wainger, Balls and metrics defined by vector fields. I. Basic properties, Acta Math. 155 (1985), no. 1-2, 103–147.

[O02] Takeo Ohsawa, Analysis of several complex variables, Translations of Mathematical Monographs, vol. 211, American Mathematical Society, Providence, RI, 2002.

[Se53] J.-P. Serre, Quelques problèmes globaux relatifs aux variétés de Stein, Colloque sur les Fonctions de Plusieurs Variables, 57-68, Brussels, 1953.

[Si87] Nessim Sibony, Une classe de domaines pseudoconvexes, Duke Math. J. 55 (1987), no. 2, 299–319.

[St97] E. J. Straube, Plurisubharmonic functions and subellipticity of the $\overline{\partial}$-Neumann problem on non-smooth domains, Math. Res. Lett. 4 (1997), 459-467.

[S84] Nancy K. Stanton, The heat equation in several complex variables, Bull. Amer. Math. Soc. (N.S.) 11 (1984), no. 1, 65–84

[ST84] Nancy K. Stanton and David S. Tartakoff, The heat equation for the $\overline{\partial}_b$-Laplacian, Comm. Partial Differential Equations 9 (1984), no. 7, 597–686

[Tr75] François Trèves, Basic linear partial differential equations, Pure and Applied Mathematics, vol. 62. Academic Press, New York-London, 1975.

[W80] Joachim Weidmann, Linear operators in Hilbert spaces, Graduate texts in mathematics, vol. 68, Springer-Verlag, 1980.

[Yu95] Jiye Yu, Weighted boundary limits of the generalized Kobayashi-Royden metrics on weakly pseudoconvex domains, Trans. of Amer. Math. Soc. 347(1995), 587-614.