Embedding simply connected
2-complexes in 3-space
V. A refined Kuratowski-type characterisation

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Abstract

This paper is the last paper in a series of five papers. Building on earlier papers in this series, we prove an analogue of Kuratowski’s characterisation of graph planarity for three dimensions.

More precisely, a simply connected 2-dimensional simplicial complex embeds in 3-space if and only if it has no obstruction from an explicit list of obstructions. This list of obstructions is finite except for one infinite family.

1 Introduction

We assume that the reader is familiar with [1]. In that paper we prove that a locally 3-connected simply connected 2-dimensional simplicial complex has a topological embedding into 3-space if and only if it has no space minor from a finite explicit list $Z$ of obstructions. The purpose of this paper is to extend that theorem beyond locally 3-connected (2-dimensional) simplicial complexes to simply connected simplicial complexes in general.

The first question one might ask in this direction is whether the assumption of local 3-connectedness could simply be dropped from the result of [1]. Unfortunately this is not true. One new obstruction can be constructed from the Möbius-strip as follows.

Consider the central cycle of the Möbius-strip, see Figure 1. Now attach a disc at that central cycle. In a few lines we explain why this topological space $X$ cannot be embedded in 3-space. Any triangulation of $X$ gives an
obstruction to embeddability. It can be shown that such triangulations have no space minor in the finite list $\mathcal{Z}$.

Figure 1: The Möbius-strip. The central cycle is depicted in grey.

Why can $X$ not be embedded in 3-space? To answer this, consider a small torus around the central cycle. The disc and the Möbius-strip each intersect that torus in a circle. These circles however have a different homotopy class in the torus. Since any two circles in the torus of a different homotopy class intersect\(^1\) the space $X$ cannot be embedded in 3-space without intersections of the disc and the Möbius-strip. Obstructions of this type we call torus crossing obstructions. A precise definition is given in Section 2.

A refined question might now be whether the result of \cite{1} extends to simply connected simplicial complex if we add the list $\mathcal{T}$ of torus crossing obstructions to the list $\mathcal{Z}$ of obstructions. The answer to this question is ‘almost yes’. Indeed, we just need to add to the space minor operation the two simple operations of stretching defined in Section 3 and Section 4. These operations are illustrated in Figure 2 and Figure 3.

Figure 2: If we stretch the highlighted edge in the simplicial complex on the left, we obtain the one on the right. The newly added faces are depicted in grey.

\(^1\)A simple way to see this is to note that the torus with a circle removed is an annulus.
It is not hard to show that stretching preserves embeddability, see  \textbf{Lemma 3.1} and \textbf{Lemma 4.1} below. The main result of this paper is the following.

**Theorem 1.1.** Let $C$ be a simply connected simplicial complex. The following are equivalent.

- $C$ has a topological embedding in 3-space;
- $C$ has no stretching that has a space minor in $Z \cup T$.

We deduce \textbf{Theorem 1.1} from the results of \cite{1} in two steps as follows. The notion of ‘almost local 3-connectedness and stretched out’ is slightly more general and more technical than ‘local 3-connectedness’, see \textbf{Section 2} for a definition. First we extend the results of \cite{1} to almost locally 3-connected and stretched out simply connected simplicial complexes, see \textbf{Theorem 2.4} below.

By lemmas of \cite{2} it suffices to prove \textbf{Theorem 1.1} for simply connected simplicial complexes that are locally connected. We conclude the proof by showing that any such simplicial complex can be stretched to an almost locally 3-connected and stretched out one. More precisely:

**Theorem 1.2.** For any locally connected simplicial complex $C$, there is a stretching $C'$ of $C$ that is locally almost 3-connected and stretched out such that $C$ embeds in 3-space if and only if $C'$ embeds in 3-space.

Moreover $C$ is simply connected if and only if $C'$ is simply connected.

The paper is organised as follows. In \textbf{Section 2} we prove \textbf{Theorem 2.4}. In \textbf{Section 3} and \textbf{Section 4} we prove \textbf{Theorem 1.2}. We conclude the paper with the proof of \textbf{Theorem 1.1}.
For graph theoretic definitions we refer the reader to [3].

2 A Kuratowski theorem for locally almost 3-connected simply connected simplicial complexes

In this section we prove Theorem 2.4, which is used in the proof of the main theorem. First we define the list \( T \) of torus crossing obstructions.

Given a simplicial complex \( C \), a mega face \( F = (f_i | i \in \mathbb{Z}_n) \) is a cyclic orientation of faces \( f_i \) of \( C \) together with for every \( i \in \mathbb{Z}_n \) an edge \( e_i \) of \( C \) that is only incident with \( f_i \) and \( f_{i+1} \) such that the \( e_i \) and \( f_i \) are locally distinct, that is, \( e_i \neq e_{i+1} \) and \( f_i \neq f_{i+1} \) for all \( i \in \mathbb{Z}_n \). We remark that since in a simplicial complex any two faces can share at most one edge, the edges \( e_i \) are implicitly given by the faces \( f_i \). A boundary component of a mega face \( F \) is a connected component of the 1-skeleton of \( C \) restricted to the faces \( f_i \) after we delete the edges \( e_i \). Given a cycle \( o \) that is a boundary component of a mega face \( F \), we say that \( F \) is locally monotone at \( o \) if for every edge \( e \) of \( o \) and each face \( f_i \) containing \( e \), the next face of \( F \) after \( f_i \) that contains an edge of \( o \) contains the unique edge of \( o \) that has an endvertex in common with \( e \) and \( e_{i+1} \). Under these assumptions for each edge \( e \) of \( o \) the number of indices \( i \) such that \( e \) is incident with \( f_i \) is the same. This number is called the winding number of \( F \) at \( o \).

A torus crossing obstruction is a simplicial complex \( C \) with a cycle \( o \) (called the base cycle) whose faces can be partitioned into two mega faces that both have \( o \) has a boundary component and are locally monotone at \( o \) but with different winding numbers. We denote the set of torus crossing obstructions by \( T \).

Remark 2.1. The set of torus crossing obstructions is infinite. Indeed, it contains at least one member for every pair of distinct winding numbers. So it is not possible to reduce it to a finite set. However one can further reduce torus crossing obstruction as follows. First, by working with the class of 3-bounded 2-complexes as defined in [1] instead of simplicial complexes, one may assume that the cycle \( o \) is a loop. Secondly, one may introduce the further operation of gluing two faces along an edge if that edges is only incident with these two faces. This way one can glue the two mega faces into single faces. Thirdly, one can enlarge the holes of the mega faces to make them into one big hole (after contracting edges afterwards one may assume that this single hole is bounded by a loop). After all these steps we only have one torus crossing obstruction left for any pair of distinct winding
numbers. This obstruction consists of three vertex-disjoint loops and two faces, each incident with two loops. The loop contained by both faces is the base cycle $o$. Here the faces may have winding number greater than one. The faces have winding number precisely one at the other loops.

By $B_m$ we denote the graph consisting of $m$ edges in parallel. Given a simplicial complex $C$ and a cycle $o$ of $C$ and $m \geq 2$, we say that $o$ is a $B_m$-cycle if all link graphs at the vertices of $o$ are obtained from subdivisions of $B_m$ by adding paths at some vertices and the edges of $o$ are branching vertices\footnote{A branching vertex of $B_m$ with $m \geq 3$ is one of the two vertices that has degree at least three in $B_m$. By adding paths to some vertices of $B_m$, the degrees of the vertices may change. This addition of paths, however, is not taken into account in the definition of branching vertices. If $m = 2$, then $B_m$ is a cycle and any vertex is a branching vertex.} in the link graphs.

**Lemma 2.2.** Let $C$ be a simplicial complex. Assume that $C$ has a $B_m$-cycle $o$ such that for some edge $e$ of $o$ the link graph $L$ of the contraction $C/(o-e)$ at the vertex $o-e$ is not loop planar. Then a torus crossing obstruction can be obtained from $C$ by deleting faces.

**Proof.** Our aim is to define a torus crossing obstruction with base cycle $o$. For that we define a set of possible mega faces as follows.

The complex $C/(o-e)$ has only one loop and that is $e$. We denote the two vertices of $L$ corresponding to $e$ by $\ell_1$ and $\ell_2$. Since $o$ is a $B_m$-cycle, the link graph $L$ is (isomorphic to) a subdivision of $B_m$ with branching vertices $\ell_1$ and $\ell_2$ plus attached paths. We shall define mega faces such that every edge of $B_m$ incident with $\ell_1$ is a face of precisely one of these mega faces. We define these mega faces recursively. So let $f$ be an edge of $B_m$ incident with $\ell_1$ that is not already assigned to a mega face. Let $P$ be the paths of $B_m$ between $\ell_1$ and $\ell_2$ that contains $f$. The edges on that path after $f$ are its consecutives in its mega face. The last edge of that path is incident with $\ell_2$ and hence it also corresponds to an edge incident with $\ell_1$. If that face is equal to $f$ we stop. Otherwise we continue with that face as we did with $f$, see Figure 4.

Eventually, we will come back to the face $f$. This completes the definition of the mega face containing $f$. This defines a mega face as all interior vertices of these paths have degree two. It is clear from this definition that the mega faces partition the edges of $B_m$. Since $o$ is a $B_m$-cycle, these mega-faces are also mega-faces of $C$ and the cycle $o$ is a boundary component of each of them. It is straightforward to check that these mega-faces are monotone at $o$.\footnote{A branching vertex of $B_m$ with $m \geq 3$ is one of the two vertices that has degree at least three in $B_m$. By adding paths to some vertices of $B_m$, the degrees of the vertices may change. This addition of paths, however, is not taken into account in the definition of branching vertices. If $m = 2$, then $B_m$ is a cycle and any vertex is a branching vertex.}
It suffices to show that two of these mega faces have distinct winding number at \( o \). Suppose not for a contradiction. Then all mega faces have the same winding number.

We enumerate the mega faces and let \( K \) be their total number. The winding number of a mega face is equal to the number of its traversals of the edge \( e \), that is, its number of faces that – when considered as edges of \( B_m \) – are incident with \( \ell_1 \). So by our assumption, there is a constant \( W \) such that all our mega faces contain precisely \( W \) faces incident with \( e \). We enumerate these faces in a subordering of the mega face. More precisely, by \( f[k, w] \) we denote the \( k \)-th face incident with \( e \) on the \( w \)-th mega face, where \( k \) and \( w \) are in the cyclic groups \( \mathbb{Z}_K \) and \( \mathbb{Z}_W \), respectively.

We will derive a contradiction by constructing a rotation system of the link graph \( L \) that is loop planar. Note that it suffices to show how to embed \( B_m \) in the plane. The paths can clearly be added afterwards. We embed \( B_m \) in the plane such that the rotation system at \( \ell_1 \) is \( f[1, 1], f[2, 1], \ldots, f[K, 1], f[1, 2], f[2, 2], \ldots, f[K, 2], f[1, 3], \ldots, f[K, W], f[1, 1] \).

Then the rotation system at \( \ell_2 \) is obtained from the that of \( \ell_1 \) by replacing each face \( f[k, w] \) by \( f[k, w + 1] \) and then reversing. Since this shift operation keeps this particular cyclic ordering invariant, the rotation systems at \( \ell_1 \) and \( \ell_2 \) are reverse. So this defines a loop planar embedding of \( B_m \). Hence \( L \) has a loop planar rotation system. This is the desired contradiction to our assumption. Hence two mega faces must have a different winding number. So \( C \) contains a torus crossing obstruction.

Next we define ‘stretched out’. This is a technical condition, which used only once in the argument, namely in the proof of Lemma 2.3 below. A simplicial complex is stretched out if
1. every edge incident with only two faces has an endvertex that is incident with precisely four faces;

2. if the link graph at a vertex \( v \) is obtained from a subdivision of a 3-connected graph or \( B_m \) by attaching paths, then no such path is attached at a subdivision vertex.

A path in a simplicial complex \( C \) is a \( B_m \)-path if

1. the link graphs at all interior vertices of \( P \) are subdivisions of graphs of the form \( B_k \) for some \( k \geq 3 \) plus possibly some attached paths;

2. the link graphs at the two endvertices of \( P \) are subdivisions of 3-connected graphs plus possibly some attached paths.

**Lemma 2.3.** Let \( C \) be a stretched out simplicial complex\(^3\) with a \( B_m \)-path \( P \). Then the complex \( C' \) obtained from \( C \) by contracting all edges of the path \( P \) has at most one loop which is incident with more than one face.

**Proof.** Let \( v \) and \( w \) be the endvertices of the path \( P \). Since \( C \) is a simplicial complex, there is at most one edge between \( v \) and \( w \). Our aim is to show that any other loop of \( C' \) is only incident with a single face.

So let \( e \) be a loop of \( C' \) such that in \( C \) the edge \( e \) has an endvertex \( u \) different from \( v \) and \( w \). As \( e \) is a loop in \( C' \), both endvertices of \( e \) in \( C \) are on \( P \). Thus the vertex \( u \) is an interior vertex of \( P \). So the link graph \( L(u) \) at \( u \) is obtained from a subdivision of \( B_m \) with \( m \geq 3 \) by attaching paths. Since \( C \) is stretched out, these paths can only be attached at the two vertices of \( B_m \) at degree at least three. As \( e \) is not on \( P \), in the link graph \( L(u) \) it is a vertex of degree at most two. Since in the simplicial complex \( C \) every vertex of \( P \) is incident with more than four faces, and \( e \) has both endvertices on \( P \), no endvertex of \( e \) can be incident with four faces. Since \( C \) is stretched out, the edge \( e \) cannot be incident with two faces. Thus \( e \) has degree one in \( L(u) \). That is, \( e \) is incident with only a single face in \( C \).

A graph is **almost 3-connected** if it is obtained from a 3-connected graph or \( B_m \) by subdividing edges and by attaching paths at some of the vertices. Note that since we allow \( m \) to be equal to two, all cycles are almost 3-connected. A simplicial complex is **locally almost 3-connected** if all its link graphs are almost 3-connected.

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\(^3\)In this paper we follow the convention that every edge of a simplicial complex is incident with some face.
Theorem 2.4. Let $C$ be a simplicial complex that is locally almost 3-connected and stretched out. The following are equivalent.

- $C$ has a planar rotation system;
- $C$ has no space minor in $Z \cup T$.

As a preparation for the proof of Theorem 2.4, we prove the following analogue of [1, Lemma 4.1].

Lemma 2.5. Let $C$ be a simplicial complex that is locally almost 3-connected. Then $C$ has a planar rotation system unless

1. $C$ is not locally planar;
2. there is a $B_m$-path $P$ such that $C/P$ is not locally planar at the vertex $P$;
3. the contraction $C/(o - e)$ is not locally planar, where $o$ is a cycle and $e$ is an edge of $o$ and either $o$ is chordless and not a loop or else $o$ is a $B_m$-cycle.

Proof. For simplicity, we first give a proof, where we strengthen the assumption of ‘locally almost 3-connectedness’ to ‘all of whose link graphs are subdivisions of either 3-connected graphs or graphs of the form $B_m$’. We stress that we allow that $m = 2$, which allows the link graphs to be cycles of arbitrary length.

We obtain $H$ from the 1-skeleton of $C$ by deleting all edges of $C$ that are incident with precisely two faces. In order to show that $C$ has a planar rotation system it suffices to construct for each connected component $H'$ of $H$ a rotation system of $C$ that is planar at all vertices of $H'$. Indeed, since the rotators at vertices of degree two are unique, we can combine these rotation systems for the different components of $H$ to a planar rotation system of $C$. We call such a rotation system planar at $H'$. So now let $H'$ be a connected component of $H$.

First assume that $H'$ just consists of a single vertex. Either $C$ has a rotation system that is planar at $H'$ or the link graph of $C$ at the single vertex of $H'$ is not loop planar. That is, we have the first outcome of the lemma.

Next assume that all link graphs at vertices of $H'$ are subdivisions of $B_m$. Since we may assume that $H'$ contains at least two vertices, $m$ is at least three and each vertex of $H'$ is incident with precisely two edges (which are the branching vertices in its link graph). So the connected graph $H'$ is
a cycle $o$. In fact, it is a $B_m$-cycle. Similarly as [1, Lemma 2.2] one proves that there is a rotation system planar at $H'$ unless there is an edge $e$ of $o$ such that $C/(o-e)$ is not loop planar at $o-e$. That is, we have the third outcome of the lemma.

Thus we may assume that $H'$ contains a vertex whose link graph is a subdivision of a 3-connected graph. Let $W$ be the set of vertices of $H'$ whose link graphs are subdivisions of a graph $B_m$. We shall prove by induction on the size of $W$ that there is a rotation system planar at $H'$. Since this induction involves contractions of edges and the class of simplicial complexes is not closed under contractions, we work inside the slightly larger class of 3-bounded 2-complexes, see [1]. The base case is proved as [1, Lemma 4.1] (there is a slight shift of language. Instead of ‘$C/e$ is not loop planar at $e$’ we say in the more general context of this proof that ‘$e$ is a $B_m$-path without interior vertices satisfying 2’).

Now assume that we constructed for all $H'$ with smaller sets $W$ rotation systems that are planar at $H'$. By the base case, $W$ contains a vertex $w$, which has degree two in $H'$. Let $x$ be an edge of $H'$ incident with $w$. Clearly the 3-bounded 2-complex $C/x$ has one vertex less whose link graph is of the form $B_m$ in the component $H'/x$. Similarly as [1, Lemma 2.2] one proves that $C/x$ has a rotation system planar at $H'/x$ if and only if $C$ has a rotation system planar at $H'$ [4]. So we can apply the induction hypothesis. That is, there is a rotation system planar at $H'$ or there is some vertex $v$ of $H'/x$ such that one of $C/x$, $C/(P+x)$ or $C/(o-e+x)$ is not planar at $v$ (with $P$, $o$ and $e$ as in the statement of Lemma 2.5).

If $C$ is not locally planar, or $P$ is a $B_m$-path in $C$ satisfying 2 of Lemma 2.5 or $o$ is a cycle in $C$ satisfying 3, we are done. Hence we may assume that $w$ is contracted onto $v$ in $H'/x$ and that the vertex $w$ has not degree one in the contraction set $x$, $P+x$ or $o-e+x$, respectively, since the link graph at $w$ is of the form $B_m$ and so whether we contract $x$ or not would then not affect whether 1, 2 or 3 is satisfied. So $w$ is incident with an edge aside from $x$ in the contraction set. Since $x$ is a serial edge of $H'$, it cannot be in parallel to any edge of $P$ or $o$. Hence $P+x$ is a $B_m$-path or $o+x$ is a cycle, respectively. So we have outcome 2 or 3. This completes the induction step. Hence by induction there is a rotation system planar at $H'$.

Having finished the proof under the stronger assumption that all link graphs are subdivisions of either 3-connected graphs or graphs of the form

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4[1, Lemma 2.2] proves this statement if $C = H'$. The proof of the more general statement needed here can be proved precisely the same way as [1, Lemma 2.2].
$B_m$, we now explain how this proof can be modified to give a proof under the weaker assumption that all link graphs are almost 3-connected. That is, they are of the above form with some paths attached to some of the vertices.

The proof is the same except that we make the following more general definition of the graph $H$. Given a face $f$ incident with an edge $e$, we say that $f$ is proper at $e$ if in both link graphs containing $e$ the unique edge corresponding to $f$ is not contained in any attached path. We obtain $H$ from the 1-skeleton of $C$ by deleting all edges $e$ that are incident with less than three faces proper at $e$.

**Proof of Theorem 2.4.** By Lemma 2.2 we may assume that $C$ has no $B_m$-cycle $o$ such that for some edge $e$ of $o$ the contraction $C/(o-e)$ is not loop planar at the vertex $o-e$.

Next we treat the case that $C$ has a $B_m$-path $P$ such that the link graph $L(P)$ of $C/P$ at $P$ is not loop planar. Let $e$ be a loop of $C/P$ such that there is a single face $f$ incident with $e$. Then $e$ is incident with the vertex $P$. Since $L(P)$ is a connected graph the face $f$ can only be incident with a single loop of $C/P$. In particular, there are only two edges of $L(P)$ corresponding to $f$ and their endvertices corresponding to the loop $e$ have degree one. Let $L'$ be the graph obtained from $L(P)$ by deleting all such faces. As $L(P)$ is not loop planar, also $L'$ is not loop planar. Moreover, $L'$ is the link graph in the complex $C'$ obtained from $C/P$ by deleting all faces $f$ such that there is a loop of $C/P$ that is only incident with $f$. By Lemma 2.3, $C'$ has at most one loop. Hence by [1 Lemma 6.4] or [1 Lemma 6.7] $C'$ has a space minor that is a generalised cone or a looped generalised cone that is not loop planar at its top, respectively. In the first case we deduce by [1 Lemma 6.6] that $C'$ has a space minor in $Z_1$. In the second case we deduce similarly as in the last paragraph of the proof of [1 Theorem 6.8] that $C'$ has a space minor in $Z_2$.

Having treated the above cases the rest of the proof of Theorem 2.4 is analogue to the proof of [1 Theorem 6.8] except that we refer to Lemma 2.5 instead of [1 Lemma 4.1].

### 3 Streching local 1-separators

Given a simplicial complex $C$ with a vertex $v$ and an edge $e$ incident with $v$ that is a cutvertex of the link graph $L(v)$, the simplicial complex $C_1$ obtained from $C$ by stretching $e$ at $v$ is defined as follows, see Figure 2.

Let $\Delta_n$ be the simplicial complex obtained by gluing $n$ triangles together at a single edge, see Figure 3.
Informally, we obtain $C_1$ from $C$ by replacing the edge $e$ by $\Delta_n$, where $n$ is the number of components of $L(v) - e$. More precisely, the simplicial complex $C_1$ is defined as follows. We denote the gluing edge of $\Delta_n$ by $\bar{e}$. We label the vertices of $\Delta_n$ not incident with $\bar{e}$ by the components of $L(v) - e$. The vertex set of $C_1$ is the vertex set of $C$ together with these new vertices for the components of $L(v) - e$. In our notation we suppress a bijection between the endvertices of $e$ in $C$ and the endvertices of $\bar{e}$ in $\Delta_n$ and will treat them as identical. The edge set of $C_1$ is that of $C$ with $e$ replaced by the edges of $\Delta_n$. The incidences between vertices and edges are as in $C$ or $\Delta_n$ except that an edge $x$ of $C - e$ incident with $v$ is now instead incident with the vertex of $\Delta_n$ corresponding to the component containing $x$.

The faces of $C_1$ are the faces of $C$ together with the faces of $\Delta_n$. Let $w$ be the endvertex of $e$ different from $v$. The incidences between edges and faces are as in $C$ or $\Delta_n$ except that a face $f$ that is incident with $e$ in $C$ now is incident an edge of $\Delta_n$; more precisely, it is the edge $wx$, where $x$ is the component of $L(v) - e$ such that in $L(v)$ the edge $f$ joins $v$ with a vertex of $x$. This completes the definition of stretching $e$ at $v$.

Note that the link graph at $w$ of $C$ is obtained from the link graph at $w$ in $C_1$ by contracting all edges incident with the vertex $\bar{e}$.

For the rest of this section, we fix a simplicial complex $C$ with a vertex $v$ and an edge $e$ incident with $v$ that is a cutvertex of the link graph $L(v)$. Let $C_1$ be obtained from $C$ by stretching $e$ at $v$.

**Lemma 3.1.** $C$ embeds in 3-space if and only if $C_1$ embeds in 3-space.

**Proof sketch.** This follows from combining the following two simple facts.

1. Let $C$ be a simplicial complex and $e$ be an edge of $C$ that is not a loop. Then $C$ is embeddable in 3-space if and only if $C/e$ is;

2. let $C$ be a simplicial complex with a face $f$ just consisting of the two edges $e_1$ and $e_2$. If $C$ is embeddable in 3-space, then so is the
contraction $C/f$. Conversely, if $C/f$ is embeddable in 3-space such that the faces incident with $e_1$ form an interval in the cyclic orientation of the edge of $C/f$ that corresponds to $f$, then also $C$ is embeddable in 3-space.

Indeed, we obtain $C$ from $C'$ by first contracting all edges incident with the new endvertex of $\bar{e}$ and then contracting all faces of $\Delta_n$. □

**Lemma 3.2.** $C$ is simply connected if and only if $C_1$ is simply connected.

*Proof sketch.* It is easy to derive this lemma from the fact that $C$ can be obtained from $C_1$ by contracting edges that are not loops and contracting faces of size two in the sense of [1]. □

A graph is *almost 2-connected* if it is obtained from a 2-connected graph by attaching paths at some of the vertices. A simplicial complex is *locally almost 2-connected* if all its link graphs are almost 2-connected.

**Lemma 3.3.** To any locally connected simplicial complex $C$ we can apply stretchings at edges such that the resulting simplicial complex is locally almost 2-connected.

*Proof.* We prove this by induction on the number of vertices of $C$ whose link graph is not almost 2-connected. We may assume that there is a vertex $v$ whose link graph is not almost 2-connected. We consider the block-cutvertex tree of the link graph $L(v)$ and successively stretch $C$ at all edges that are cutvertices of $L(v)$. If $e$ is such an edge an $w$ is the endvertex of $e$ different from $w$, it is straightforward to check that if $L(w)$ is almost 2-connected, the same is true after the stretching. Thus the resulting simplicial complex has one vertex less whose link graph is not almost 2-connected. Hence we can apply induction. □

The lemmas of this section cumulate in the following.

**Theorem 3.4.** For any locally connected simplicial complex $C$, there is a stretching $C_1$ of $C$ that is locally almost 2-connected such that $C$ embeds in 3-space if and only if $C_1$ embeds in 3-space.

Moreover $C$ is simply connected if and only if $C_1$ is simply connected.

*Proof.* We construct $C_1$ as in **Lemma 3.3**. By **Lemma 3.1** $C$ embeds in 3-space if and only if $C_1$ embeds in 3-space. By **Lemma 3.2** $C$ is simply connected if and only if $C_1$ is simply connected. □

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5The contraction $C/f$ is obtained from $C$ by identifying the two edges $e_1$ and $e_2$ along $f$, see [1]. The new edge is labelled $f$. 

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4 Streching local 2-separators

This section is analogue to Section 3 but in parts slightly more complicated. A 2-separator in a (multi-) graph \( L \) is a pair of vertices \((a, b)\) such that \( L - a - b \) has at least two proper components or else only one proper component and at least two edges between the vertices \( a \) and \( b \). In the later case we call the 2-separator artificial.

Given a simplicial complex \( C \) with a vertex \( v \) and 2-separator \((a, b)\) of the link graph \( L(v) \), the simplicial complex \( C_2 \) obtained from \( C \) by stretching \( \{a, b\} \) at \( v \) is defined as follows. See Figure 3.

Let \( \Delta^+_n \) be the simplicial complex obtained by gluing \( n \) copies of \( \Delta_2 \) together at a path of length 2 whose endvertices have degree two in \( \Delta_2 \) (this is uniquely defined up to isomorphism), see Figure 6.

The simplicial complex \( \Delta^+_3 \) with the gluing edges labelled \( \bar{a} \) and \( \bar{b} \).

Informally, we obtain \( C_2 \) from \( C \) by replacing the edges \( a \) and \( b \) by \( \Delta^+_n \), where \( n \) is the number of proper components of \( L(v) - a - b \). More precisely, the simplicial complex \( C_2 \) is defined as follows. We denote the gluing edges of \( \Delta^+_n \) by \( \bar{a} \) and \( \bar{b} \). We label the vertices of \( \Delta^+_n \) incident with neither \( \bar{a} \) nor \( \bar{b} \) by the proper components of \( L(v) - a - b \). The vertex set of \( C_2 \) is that of \( C \) together with these new vertices for the proper components of \( L(v) - a - b \). In our notation we suppress a bijection between the endvertices of \( a \) in \( C \) and the endvertices of \( \bar{a} \) in \( \Delta^+_n \) and will treat them as identical. Similarly, we suppress a bijection between the endvertices of \( b \) and \( \bar{b} \). Both these bijections agree at the common endvertex \( v \) of \( a \) and \( b \).

The edge set of \( C_2 \) is that of \( C \) with \( a \) and \( b \) replaced by the edges of \( \Delta^+_n \). The incidences between vertices and edges are as in \( C \) or \( \Delta^+_n \) except that

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6To be very precise, if the graph only consists of the two vertices \( a \) and \( b \) and has at least three edges in parallel, then \((a, b)\) is also a 2-separator. However all 2-separators we consider in this paper will be within graphs of at least three vertices.

7A component \( K \) of \( L(v) - a - b \) is proper if it has both \( a \) and \( b \) as a neighbour.
an edge $x$ of $C$ incident with $v$ is incident with the vertex for the (proper) component of $L(v) - a - b$ that contains $x$.

The faces of $C_2$ are the faces of $C$ together with the faces of $\Delta_a^n$. The incidences between edges and faces are as in $C$ or $\Delta_a^n$ except that a face $f$ that is incident with precisely one of $a$ or $b$ in $C$ now is incident with an edge of $\Delta_a^n$; more precisely, if $f$ is incident with $a$, then in $C_2$ it is incident with the edge $w_ax$, where $w_a$ is the endvertex of $a$ different from $v$ and $x$ is the component of $L(v) - a - b$ such that in $L(v)$ the edge $f$ joins $a$ with a vertex of $x$. If $f$ is incident with $b$ this is defined the same with ‘$b’ in place of ‘$a’$. This completes the definition of stretching a 2-separator at a vertex.

Let $w_a$ be the endvertex of $a$ different from $v$. The link graph at $w_a$ of $C$ is obtained from the link graph at $w_a$ in $C_2$ by contracting all edges incident with the vertex $\bar{a}$. Note that $w_a$ cannot be incident with $b$ as $C$ is a simplicial complex.

For the rest of this section, we fix a simplicial complex $C$ with a vertex $v$ and edges $a$ and $b$ incident with $v$ such that $L(v) - a - b$ has at least two proper components. Let $C_2$ be obtained from $C$ by stretching $\{a,b\}$ at $v$. The following two lemmas are proved analogously to Lemma 3.1 and Lemma 3.2, respectively.

**Lemma 4.1.** $C$ embeds in 3-space if and only if $C_2$ embeds in 3-space. \[\square\]

**Lemma 4.2.** $C$ is simply connected if and only if $C_2$ is simply connected. \[\square\]

**Observation 4.3.** If $C$ is locally almost 2-connected, then so is $C'$. \[\square\]

**Lemma 4.4.** To any locally almost 2-connected simplicial complex $C$ we can apply stretching at pairs of edges such that the resulting simplicial complex is locally almost 3-connected.

**Remark 4.5.** The proof idea of this lemma is quite simple. If a simplicial complex $C$ has a link graph that is not of the desired type, then we find a suitable 2-separation of that link graph. Then we stretch $C$ along that 2-separator. In order to finish the argument, it suffices to define a way in which this new graph is smaller than $C$ and apply induction. Our definition of such a parameter is quite technical; the first part of our proof defines it.

**Proof of Lemma 4.4.** We shall prove this by induction. First we need some definitions in order to define the parameter we would like to apply induction on. We denote by $\leq_{\text{lex}}$ the lexicographical ordering on the set of finite sequences of natural numbers that are at least three. An example of such
a sequence is an *(abbreviated) degree sequence* of a graph $G$: a sequence of the numbers of vertex degrees of $G$ that are at least three. Here we allow multiplicities. We denote the degree sequence of a (labelled) graph $G$ by $\gamma(G)$. In this proof the ordering of the sequence will not matter: in a slight abuse of grammar, we shall be talking about ‘the’ degree sequence of an unlabelled graph. We denote by $\leq$ the lexicographical ordering on the multi-set of finite sets of finite decreasing sequences. That is: given two such finite multi-set $X$ and $Y$. If the $\leq_{lex}$-largest element of $X$ is strictly smaller than the $\leq_{lex}$-largest element of $Y$, then $X \leq Y$. Otherwise we compare the second largest elements and so forth.

An example of such a multi-set is the multi-set of degree sequences of link graphs of a simplicial complex $C$ (ordered by $\leq_{lex}$). We denote that parameter by $\beta(C)$. Now let $C$ be a locally almost 2-connected simplicial complex and assume by induction that we already proved the lemma for all locally almost 2-connected simplicial complexes $C'$ with $\beta(C') < \beta(C)$. If all link graphs of $C$ are almost 3-connected, we are done. Hence we may assume that there is a vertex $v$ of $C$ whose link graph $L(v)$ is not of that form. Let $L_1$ be the (unique) 2-connected graph obtained from $L(v)$ by deleting paths attached at vertices. Let $L_2$ be the graph obtained from $L_1$ by suppressing subdivision vertices. By assumption $L_2$ is neither 3-connected, nor a cycle nor a graph $B_n$. We apply the Tutte decomposition to $L_2$. It has a 2-separator $\{a,b\}$ (we stress that $(a,b)$ is allowed to be an artificial 2-separator). Then $\{a,b\}$ is also a 2-separator of $L(v)$.

Let $C'$ be the simplicial complex obtained from $C$ by stretching the pair $\{a,b\}$. By Observation 4.3 $C'$ is locally almost 2-connected. Our aim is to show that $\beta(C') < \beta(C)$ in order to apply induction on $\beta$.

We denote the endvertex of $a$ different from $v$ by $w_a$; and the endvertex of $b$ different from $v$ by $w_b$. Note that $w_a \neq w_b$ as $C$ is a simplicial complex. Let $S$ be the multi-set consisting of the degree sequences of the multi-set of link graphs $L(v)$, $L(w_a)$ and $L(w_b)$ of $C$. Let $S'$ be the multi-set consisting of the degree sequences of the multi-set of link graphs $L'(w_a)$, $L'(w_b)$, $L(\bar{v})$ at the vertices, $w_a$, $w_b$ and $\bar{v}$, respectively – together with the link graphs $L'(K)$ for a proper component $K$ of $L(v) - a - b$.

It suffices to show the following.

**Sublemma 4.6.** $\beta(S') < \beta(S)$.

**Proof.** Since the link graph $L(w_a)$ is obtained from the link graph $L'(w_a)$ by

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8A *multi-set* is the same as a set except that elements are allowed to be contained with a multiplicity greater than one.
contracting all edges incident with the vertex \(a\), and this can only increase the abbreviated degree sequence, we have that \(\gamma(L'(w_a)) \leq \gamma(L(w_a))\); and the same inequation with '\(b'\) in place of '\(a'\).

Hence it remains to show that the degree sequences \(\gamma(L'(K))\) and \(\gamma(L'(\bar{v}))\) are all strictly less than \(\gamma(L(v))\).

First we show that \(\gamma(L'(\bar{v}))\) is strictly less than \(\gamma(L(v))\). Each of the degrees of \(a\) and \(b\) in \(L'(\bar{v})\) are at most their degree in \(L(v)\). Furthermore since \(L_2\) is not a graph of the form \(B_m\), \(L(v)\) contains a vertex of degree at least three different from \(a\) and \(b\). Thus \(\gamma(L'(\bar{v}))\) is strictly less than \(\gamma(L(v))\).

For any proper component \(K\) of \(L(v) - a - b\), the graph \(L'(K)\) is obtained from \(K\) by adding a path of length two between the vertices \(a\) and \(b\). So \(\gamma(L'(K))\) is at most \(\gamma(L(v))\).

Since \(L_2\) is not a graph of the form \(B_m\), each proper component of \(L_2 - a - b\) contains a vertex of degree at least three. Thus if \((a, b)\) is not artificial, then each \(\gamma(L'(K))\) is strictly less than \(\gamma(L(v))\).

If \((a, b)\) is artificial, it remains to show that the unique proper component \(K_1\) has \(\gamma(L'(K_1))\) strictly less than \(\gamma(L(v))\). This is clear as \(a\) and \(b\) have a strictly smaller degree in \(L'(K_1)\) than in \(L(v)\).

\[\square\]

\textbf{Lemma 4.7.} Let \(C\) be locally almost 3-connected simplicial complex. Then there is a stretching of \(C\) that has additionally the property that it is stretched out.

\textit{Proof.} Assume there is a vertex \(v\) of \(C\) such that the link graph \(L(v)\) is obtained from a subdivision of a 3-connected graph or of a graph \(B_m\) by attaching paths at some subdivision vertex \(u\). Then we stretch the two edges of \(C\) that are neighbours of \(u\) in the subdivision. This reduces the total number of such vertices \(u\) and clearly preserves being locally almost 3-connected. Hence we may assume that \(C\) has no such vertex \(u\).

Assume that \(C\) has an edge \(e\) only incident with two faces and that both endvertices of that face are incident with more than four faces. Then we pick an endvertex \(v\) of \(e\) arbitrarily and stretch at \(v\) the two edges incident with \(v\) that share faces with \(e\). This reduces the total number of such edges \(e\) and preserves all the above properties of \(C\). Hence there is a stretching of \(C\) that is locally almost 3-connected and stretched out.

\[\square\]

The lemmas of this section cumulate in the following.

\[\footnote{Here we suppress a bijection between the vertex sets of these two graphs.}\]
Theorem 4.8. For any locally almost 2-connected simplicial complex $C$, there is a stretching $C_2$ of $C$ that is locally almost 3-connected and stretched out such that $C$ embeds in 3-space if and only if $C_2$ embeds in 3-space.

Moreover $C$ is simply connected if and only if $C_2$ is simply connected.

Proof. We construct $C_2'$ as in Lemma 4.4. Applying Lemma 4.7 to $C_2'$ yields a simplicial complex $C_2$ that is locally almost 3-connected and stretched out. By Lemma 4.1, $C$ embeds in 3-space if and only if $C_2$ embeds in 3-space. By Lemma 4.2, $C$ is simply connected if and only if $C_2$ is simply connected. \qed

Proof of Theorem 1.2. Combine Theorem 3.4 and Theorem 4.8. \qed

Proof of Theorem 1.1. By [2, Lemma 5.1] and [2, Lemma 5.2] it suffices to prove Theorem 1.1 for simply connected simplicial complexes that are locally connected. So it follows by combining Theorem 2.4, Theorem 1.2 and [2, Lemma 1.1]. \qed

References

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