Measurements of $H_0$ in modified gravity theories

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In this work, we obtain measurements of the Hubble constant in the context of modified gravity theories. We set up our theoretical framework by considering viable cosmological $f(R)$ and $f(T)$ models, and we analyzed them through the use of geometrical data sets obtained in model-independently way, namely, gravitationally lensed quasars with measured time delays, standard candles from cosmic chronometers, and standard candles from the Pantheon Supernovae Ia sample. We find $H_0 = (72.4 \pm 1.4) \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $H_0 = (71.5 \pm 1.3) \text{ km s}^{-1} \text{ Mpc}^{-1}$ for the $f(R)$ and $f(T)$ models, respectively. Our results represent 1.9% and 1.8% measurements of the Hubble constant, which are fully consistent with the local estimate of $H_0$ by the Hubble Space Telescope. We do not find significant departures from general relativity, as our study shows that the characteristic parameters of the extensions of gravity beyond general relativity are compatible with the $\Lambda$CDM cosmology. Moreover, within the standard cosmological framework, our joint analysis suggests that it is possible to measure the dark energy equation of state parameter at 1.2% accuracy, although we find no statistical evidence for deviations from the cosmological constant case.

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I. INTRODUCTION

Several astronomical observations predict that the Universe is currently in an accelerated expansion phase [1,3]. The theoretical modelling that explains such evidence is certainly one of the biggest open problem in contemporary physics and astronomy. Over the last two decades, the $\Lambda$CDM model has been shown to explain with great precision the observations in the most different scales and cosmic distances. Due to this great success, such a scenario is considered the standard cosmological model.

Nowadays, we have increasingly accurate measurements of the cosmological parameters that challenge the consensus on the $\Lambda$CDM model. Certainly, the most significant tension with the standard model prediction is the observed value of the present cosmic expansion rate, quantified by the Hubble constant, $H_0$. Analyses of the Cosmic Microwave Background (CMB) observations by the Planck collaboration, assuming the $\Lambda$CDM baseline as input scenario, obtained $H_0 = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [4]. On the other hand, model-independent local measurements by the Hubble Space Telescope (HST) showed that $H_0 = (74.03 \pm 1.42) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [5], which is in 4.4$\sigma$ tension with Planck’s estimate. Moreover, the H0LiCOW collaboration has revealed its measurement of $H_0$ from its blind (i.e. model-independent) analysis of gravitationally lensed quasars with measured time delays, showing $H_0 = (73.3_{-1.8}^{+1.7}) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [6]. This value is in 3.1$\sigma$ tension with the Planck CMB data, increasing 5.3$\sigma$ when combined with the HST result. Obviously, such a large discrepancy in the $H_0$ measurements has led to examine the model-dependency of the CMB data or possible underestimated systematic effects in the analysis of the $H_0$ parameter. Therefore, it has been widely discussed in the literature whether a new physics beyond the standard cosmological model can solve the $H_0$ tension (see [7–16] for a short list).

Extensions of General Relativity (GR) have been proposed (see [17–20] for a review) and exhaustively investigated to explain the observational data at both cosmological and astrophysical levels. The additional gravitational degree(s) of freedom from the modified gravity models quantify extensions of the $\Lambda$CDM cosmology and can drive the accelerating expansion of the Universe at late times. Several of these extensions have shown to fit the data well, leading to a possible theoretical degeneracy [21,22]. Among viable candidate for modified gravity theories, two classes of theories have been well accepted and investigated in literature, namely, the $f(R)$ gravity and $f(T)$ gravity. The $f(R)$ scenarios are gravitational modifications that add higher-order corrections to the Einstein-Hilbert action, extending the Ricci scalar $R$ to an arbitrary function $f(R)$. We refer to [26] for a review on the $f(R)$ gravity. The $f(R)$ gravity has been tested against several different data and some viable $f(R)$ models have been constrained at different cosmological scales [27,54]. However, one can equally construct the gravitational modifications starting from the torsion-based for-

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1 It has been argued and explored that information from gravitational wave (GW) observations, in particular measures on the propagation speed of GWs, can strongly discriminate among possible extensions of GR. See [23,24] for discussions in this regard.
mulation, and specifically from the Teleparallel Equivalent of General Relativity (TEGR) \cite{37}. In this theory, the Lagrangian is the torsion scalar $T$, and its simplest generalization is represented by the $f(T)$ gravity (see \cite{38} for a review). Also, the $f(T)$ theories have been shown to be a strong and viable modified gravity candidate in alternative to GR \cite{38}.

The main aim of this work is to use the gravitationally lensed quasars with measured time-delays compiled by the H0LiCOW collaboration to obtain new observational constraints on both $f(R)$ gravity and $f(T)$ viable models. In particular, these frameworks have proven to be important for measuring $H_0$ parameter with excellent accuracy, as we shall discuss in the following. Hence, it is interesting to check whether alternative gravitational models could provide an explanation to the standing $H_0$ tension. To do that, we will complement the time-delay distance data with other geometrical probes such as standard candles from type Ia Supernovae (SN Ia), and standard clocks from cosmic chronometers, which are available without assuming a cosmological model. Employing these data, we will be able to obtain accurate estimates of the free parameters of the theories, specially the $H_0$ parameter, and check the feasibility of the models. For the quantitative discussion, we will also analyze the $\Lambda$CDM and $wCDM$ models in light of these data and, through a statistical Bayesian comparison, we will interpret the evidence for all the models beyond the $\Lambda$CDM scenario under consideration.

The manuscript is organized as follows. In Sec. II, we provide a brief description of the cosmological dynamics of the $f(R)$ and $f(T)$ gravity theories. In Sec. III we present the data sets and our methodology to analyze them, whereas in Section IV we present our main results. In Sec. V we statistically compare the predictions of the different theoretical scenarios, and finally, in Sec. VI we summarize our conclusions and indicate the perspectives of our work.

Throughout the text, we use units such that $c = \hbar = 1$, and the notation $\kappa \equiv 8\pi G = M_P^{-2}$, where $M_P$ is the reduced Planck mass, and $G$ is the gravitational constant. As usual, the symbol dot indicates derivative with respect to the cosmic time, and a subscript zero refers to any quantity evaluated at the present time.

II. THEORETICAL FRAMEWORK

In what follows, we describe in a nutshell the theoretical framework of our study.

A. The $f(R)$ gravity

We start with a brief review of the $f(R)$ cosmology. The $f(R)$ gravitational theories consist in extending the Einstein-Hilbert action in the form

$$S = \int d^4x\sqrt{-g} \left(\frac{M_P^2}{2} f(R) + S_m\right),$$

where $g$ is the determinant of the metric tensor, $f(R)$ is a generic function of the Ricci scalar, and $S_m$ is the action of matter fields. For $f(R) = R$, the GR case is recovered.

Let us now consider a spatially flat FLRW Universe dominated by pressureless matter (baryonic plus dark matter) and radiation with energy densities $\rho_m, \rho_r$ and pressures $P_m, P_r$, respectively. The modified Friedmann equations in the metric formalism are given by \cite{20}

$$3FH^2 = 8\pi G (\rho_m + \rho_r),$$

$$-2FH = 8\pi G (\rho_m + \rho_r + P_r) + \ddot{F} - H\dot{F},$$

where $F \equiv \frac{\kappa}{M_P^2}$. Moreover, one obtains the following useful relation:

$$R = 6 \left(2\dot{H}^2 + \dot{H}\right).$$

In order to move on, we need to specify some $f(R)$ function. Adopting the formalism presented in \cite{50, 51}, one can write

$$f(R) = R - 2\Lambda y(R, b),$$

where the function $y(R, b)$ quantifies the deviation from Einstein’s gravity, i.e. the effect of the $f(R)$ modification, through the parameter $b$.

We thus consider viable models that have up to two parameters, where the $f(R)$ function is given by Eq. (5). This methodology has been used earlier to investigate the observational constraints on $f(R)$ gravity in \cite{50, 51}. In this respect, one of the most well-known scenario in the modified gravity theory literature is the Hu-Sawicki (HS) model \cite{52}, which satisfies all the dynamics conditions required for a given $f(R)$ function. The function $y(R, b)$ for the HS model reads

$$y(R, b) = 1 - \frac{1}{1 + \left(\frac{R}{\Lambda b}\right)^n},$$

where $b > 0$ and we assume $n = 1$. We refer to \cite{50, 51} for more details.

B. The $f(T)$ gravity

Inspired by the $f(R)$ extensions of GR, we can generalize $T$ to a function $T + f(T)$, constructing the action of $f(T)$ gravity as \cite{53}

$$S = \frac{1}{16\pi G} \int d^4x \; e [T + f(T)] + S_m,$$

with $e = \det(e^A_\mu) = \sqrt{-g}$ and $S_m$ is the action for matter fields. We note that the TEGR is restored when

\footnote{We use the vierbein fields $e^\mu_A$, which form an orthonormal base on the tangent space at each manifold point $x^\mu$. The metric then reads $g_{\mu\nu} = g_{AB} e^A_\mu e^B_\nu$.}
f(T) = 0, whereas, for f(T) = const, we recover GR with a cosmological constant, i.e. the ΛCDM model. In the action above, the torsion scalar T is constructed by contractions of the torsion tensor T^{\mu\nu} as \[ T = \frac{1}{4} T^{\mu\nu} T_{\mu\nu} + \frac{1}{2} R^{\mu\nu\rho\sigma} T_{\mu\rho\sigma} - T_{\mu\nu} T^{\mu\nu} \] \[ (8) \]

Variation of the action (7) with respect to the vierbeins provides the field equations:

\[ e^{-1} \partial_{\mu}(e_A^{\nu\rho} S_{\rho \mu})[1 + f_T] + e_A^{\rho \nu} \partial_{\mu}(T) f_{TT} = \frac{1}{2} e^{\rho \nu} (T + f(T)) ] \]

\[ = \frac{4 \pi G e^A}{T^{(m) \nu} + T^{(r) \nu}} \]

where \( f_T = \partial f / \partial T \), \( f_{TT} = \partial^2 f / \partial T^2 \), while \( T^{(m) \nu} \) and \( T^{(r) \nu} \) are the matter and radiation energy-momentum tensors, respectively.

We then focus on homogeneous and isotropic space-time. Thus, the flat FLRW background metric corresponds to the following choice for the vierbeins:

\[ e^A_{\mu} = \text{diag}(1, a, a, a) \]

where \( a \) is the cosmic scale factor. Inserting the vierbein (10) into the field equations (9), we obtain the Friedmann equations:

\[ H^2 = \frac{8 \pi G}{3} (\rho_m + \rho_r) - \frac{f}{6} T f_T \]

\[ \frac{\dot{H}}{H} = -\frac{4 \pi G (\rho_m + P_m + \rho_r + P_r)}{1 + f_T + 2 T f_{TT}} \]

where \( H \equiv \dot{a}/a \) is the Hubble parameter. In the above relations, we have used the relation

\[ T = -6H^2 \]

which arises straightforwardly from the FLRW metric through Eq. (8).

Defining the quantity \( E \equiv H / H_0 \), one can thus rewrite Eq. (11) as

\[ E^2(z, r) = \Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_{F0} y(z, r) \]

where we have introduced the redshift \( z \equiv a^{-1} - 1 \) and

\[ \Omega_{F0} = 1 - \Omega_{m0} - \Omega_{r0} \]

being \( \Omega_{m0} = \frac{8 \pi G \rho_m}{H_0^2} \) the corresponding density parameters at present. In this case, the effect of the \( f(T) \) modification is encoded in the function \( y(z, r) \) (normalized to unity at present time), which depends on \( \Omega_{m0}, \Omega_{r0} \) and on the \( f(T) \)-form parameters \( \psi_i, \psi_j, \psi_k, \ldots \), namely \( \Omega_{F0} \) and

\[ y(z, r) = \frac{1}{T_0 \Omega_{F0}} [f - 2 T f_T] \]

We note that, due to (13), the additional term (16) is a function of the Hubble parameter only.

In this work, we consider the parametric form given by the power-law model \[ f(T) = \alpha (-T)^b \]

where \( \alpha \) and \( b \) are the free parameters of the model. Inserting this \( f(T) \) form into the Friedmann equation (11) evaluated at present, we find

\[ \alpha = (6H_0^2)^{1-b} \frac{\Omega_{F0}}{2b - 1} \]

while (16) yields

\[ y(z, b) = E^{2b}(z, b) \]

Clearly, for \( b = 0 \) the present scenario reduces to the ΛCDM cosmology. Finally, we mention that one needs \( b < 1 \) in order to obtain an accelerating expansion. We refer to \( [54, 55] \) for more details.

### III. DATA SETS AND METHODOLOGY

Here, we briefly describe the observational data sets and the statistical methods that we use to explore the parameter space of the modified background dynamics presented above.

#### A. H0LiCOW

A powerful geometric method to measure \( H_0 \) is offered by the gravitational lensing. The time delay between multiple images, produced by a massive object (lens) and the gravitational potential between a light-emitting source and an observer, can be measured by looking for flux variations that correspond to the same source event. This time delay depends on the mass distribution along the line of sight and in the lensing object, and it represents a complementary and independent approach with respect to the CMB and the distance ladder. Due to their variability and brightness, lensed quasars have been widely used to determine \( H_0 \) through this method (see \( [58, 59] \) and references therein). One can calculate the time delay between two images \( i \) and \( j \) as

\[ \Delta t_{ij} = D_{\Delta t} \left[ \frac{(\theta_i - \beta)^2}{2} - \psi(\theta_i) - \frac{(\theta_i - \beta)^2}{2} + \psi(\theta_j) \right] \]

where \( \theta_{i,j} \) are the angular positions of the images, \( \beta \) is the angular position of the source, and \( \psi(\theta_{i,j}) \) is the lens potentials at the image positions. Here, \( D_{\Delta t} \) is the ‘time-delay distance’, which is given by \( [61] \)

\[ D_{\Delta t} = (1 + z_l) \frac{D_l D_s}{D_{ls}} \]

where \( z_l \) is the redshift of the lens, while \( D_l, D_s \) and \( D_{ls} \) are the angular diameter distances to the lens, to the
source, and between the lens and the source, respectively. The quantity \( D_{\Delta i} \) is highly sensitive to \( H_0 \), with a weak dependence on other cosmological parameters.

In the present work, we use the six systems of strongly lensed quasars analyzed by the H0LiCOW collaboration (we refer to \[66, 67\] for the details). The likelihood probability function for the \( D_{\Delta i} \) data points reads

\[
\mathcal{L}_{\text{H0LiCOW}} \propto \exp \left\{ -\frac{1}{2} \sum_{i=1}^{6} \left[ \frac{D_{\Delta o,i} - D_{\Delta th,i}}{\sigma_{D_{\Delta i}}} \right]^2 \right\},
\]

(22)

B. Pantheon

We also take into account the Pantheon sample \[62\] of 1048 SN Ia in the redshift region \( z \in [0.01, 2.3] \), whose distance moduli are standardized through the SALT-2 light-curve fitter (see \[63, 64\] for details).

As shown in \[62\], under the only assumption of a flat Universe, the full Pantheon catalogue can be compressed into six model-independent \( E^{-1}(z) \) measurements. Therefore, consistently with the assumptions of our theoretical framework, we use these measurements correlated among them according to the covariance matrix \( C_{ij} \) given in \[62\]. In this case, the likelihood probability function can be written as

\[
\mathcal{L}_{\text{Pantheon}} \propto \exp \left\{ -\frac{1}{2} V^\dagger C^{-1} V \right\},
\]

(23)

where \( V = E^{-1}_{\text{obs}} - E^{-1}_{\text{th}} \) measures the differences between the observed values and the theoretical expectations.

C. Cosmic Chronometers

The late expansion history of the Universe can be studied in a model-independent fashion by measuring the age difference of cosmic chronometers (CC), such as old and passively evolving galaxies that act as standard clocks \[66, 67\]. From the spectroscopic measurements of the redshifts between pairs of these galaxies and their differential age, one can obtain an estimate of the Hubble parameter through the relation

\[
H(z) = \frac{1}{1 + z} \frac{dz}{dt}.
\]

(24)

In our analysis we consider the 31 uncorrelated measurements of \( H(z) \) in the redshift range \( 0 < z < 2 \) tabulated in \[62\]. Confronting these values with the corresponding Hubble expansion rates predicted by the theoretical scenarios, one can construct the likelihood function as

\[
\mathcal{L}_{\text{CC}} \propto \exp \left\{ -\frac{1}{2} \sum_{i=1}^{31} \left[ \frac{H^{\text{obs}}_{\text{th},i} - H^{\text{th},i}}{\sigma_{H,i}} \right]^2 \right\},
\]

(25)

D. Monte Carlo method

We perform a statistical analysis of the data sets presented above through a Markov chain Monte Carlo (MCMC) method, based on the Metropolis-Hastings algorithm \[69\]. Specifically, we analyze the HS \( f(R) \) model and the \( f(T) \) power-law model by assuming the following flat priors on the cosmological parameters: \( H_0 \in [50, 90] \) km s\(^{-1}\) Mpc\(^{-1}\), \( \Omega_m \in [0, 1] \) and \( b \in [0, 1] \). In our study, we neglect the late-time contribution of radiation (\( \Omega_r \approx 0 \)). Moreover, for comparison, we also consider the standard \( \Lambda \)CDM model and its one-parameter extension, namely the \( w \)-CDM model, characterized by a constant equation of state parameter for the dark energy fluid (\( w \)). In this case, we assume the flat prior \( w \in [-0.3, -1.5] \).

Our analysis consists in two steps. We first combine the Pantheon + CC data to constrain the cosmological parameters of the theoretical scenarios under consideration, and we then compare these results with the outcomes of the full joint likelihood analysis \[P\] (Pantheon + CC + H0LiCOW), to check the effects of the time-delay quasars measurements on the \( H_0 \) value. We present our main results in what follows.

IV. RESULTS

In this section, we present our main results on the cosmological scenarios previously introduced, using different data combinations. We note that, in principle, one could choose other parametric \( f(R) \) and \( f(T) \) functions, but significant differences among parametric models should only have impact when analyzed at the perturbation level. Since the data analyzed here are all from geometrical origin, different functions should in fact not change the main results on the modified gravity scenarios. Therefore, without loss of generality, we focus on the most viable and studied models in the literature, which have been described in the previous sections.

In Table \[II\] we summarize the main results from the statistical analyses of the \( f(R) \) gravity and \( f(T) \) gravity models. For comparison, in Table \[III\] we also show the results concerning the \( \Lambda \)CDM and \( w \)-CDM models.

For the \( f(R) \) gravity, we find \( H_0 = (69.5 \pm 2.0) \) km s\(^{-1}\) Mpc\(^{-1}\) and \( H_0 = (72.4 \pm 1.4) \) km s\(^{-1}\) Mpc\(^{-1}\) at the 68% confidence level (C.L.) from Pantheon + CC and Pantheon + CC + H0LiCOW data, respectively. These estimates represent 2.8% (Pantheon + CC) and 1.9% (Pantheon + CC + H0LiCOW) precision measurements. The local measurement obtained by Riess et al. \[P\] from observations of long-period Cepheids in the Large Magellanic Cloud (LMC) is \( H_0 = (74.03 \pm 1.42) \) km s\(^{-1}\) Mpc\(^{-1}\).

\[3\] The joint likelihood is obtained as the product of the individual likelihoods: \( \mathcal{L}_{\text{joint}} = \mathcal{L}_{\text{Pantheon}} \times \mathcal{L}_{\text{CC}} \times \mathcal{L}_{\text{H0LiCOW}} \).
Thus, the result of our joint analysis is in full agreement with the local measurement of $H_0$, and in tension at 3.4σ with the most recent CMB estimate from Planck. On the other hand, the constraint from Pantheon + CC data is 1.8σ in tension with the local $H_0$ measurement and compatible at $\sim 1\sigma$ with Planck’s estimate. Regarding to possible deviations from the standard cosmological model, we find that $b$ is non-null at the 68% C.L. in both analyses, and $b < 0.77(0.48)$ at the 95% C.L. from Pantheon + CC (CC + Pantheon + H0LiCOW). Therefore, adding the H0LiCOW data in the analysis produces a significant improvement in the constraints of the additional parameter of the theory that quantifies deviations from the ΛCDM cosmology. In the left panel of Fig. 1 we show the parameter space of the $f(R)$ model at the 68% and 95% C.L. In particular, focusing on the $(b - H_0)$ plane, we can see that these parameters are not strongly correlated. Similar considerations apply also to the $(b - \Omega_m)$ plane.

As far as the $f(T)$ gravity is concerned, at the 68% C.L. we find $H_0 = (69.1 \pm 1.9)$ km s$^{-1}$ Mpc$^{-1}$ from Pantheon + CC, and $H_0 = (71.5 \pm 1.3)$ km s$^{-1}$ Mpc$^{-1}$ from Pantheon + CC + H0LiCOW, which represent 2.7% and 1.8% precision estimates, respectively. Our joint analysis result is thus away by only 1.3σ from the local $H_0$ measurement by Riess at al., but almost 3σ from the CMB estimate. As also observed in $f(R)$ gravity, within the $f(T)$ gravity framework we note that $b$ is non-null at the 68% C.L., although fully compatible with GR at a larger statistical significance. On the other hand, when the H0LiCOW data are added in the analysis, the constraints on $b$ are not improved in the same efficient way as in $f(R)$ gravity. The right panel of Fig. 1 shows the parameter space of the $f(T)$ model at the 68% and 95% C.L. In this case, we can see a strong anti-correlation in the plane $(b - \Omega_m)$. We also note less amount of (dark) matter density at late times with respect to the amount predicted by the ΛCDM cosmology (cf. Table I).

![Figure 1](image.png)

**FIG. 1.** 2D parameter regions and 1D posterior distributions for the Hu-Sawicki $f(R)$ model (left panel) and $f(T)$ power-law model (right panel) as results of the MCMC analysis of different combinations of data.

| Model          | Data                    | $H_0$       | $\Omega_m$   | $b$         |
|----------------|-------------------------|-------------|--------------|-------------|
| $f(R)$         | Pantheon+CC             | 69.5 ± 2.0(3.9) | 0.289 ± 0.025(0.053) | 0.32 ± 0.17(0.45) |
|                | Pantheon+CC+H0LiCOW     | 72.4 ± 1.4(2.8) | 0.267 ± 0.023(0.045) | 0.19 ± 0.10(0.29) |
| $f(T)$         | Pantheon+CC             | 69.1 ± 1.9(3.8) | 0.251 ± 0.050(0.084) | 0.30 ± 0.16(0.49) |
|                | Pantheon+CC+H0LiCOW     | 71.5 ± 1.3(2.6) | 0.233 ± 0.044(0.072) | 0.27 ± 0.16(0.49) |

**TABLE I.** 68% (95%) C.L. constraints on the Hu-Sawicki $f(R)$ model and the $f(T)$ power-law model from different combinations of data. $H_0$ is measured in units of km s$^{-1}$ Mpc$^{-1}$.
TABLE II. 68% (95%) C.L. constraints on the ΛCDM and wCDM models from different combinations of data. $H_0$ is measured in units of km s$^{-1}$ Mpc$^{-1}$.

| Model        | Data                          | $H_0$   | $\Omega_{m0}$ | $w$         |
|--------------|-------------------------------|---------|----------------|-------------|
| ΛCDM         | Pantheon+CC                   | 69.2 ± 1.9(3.7) | 0.296$^{+0.026(0.056)}_{-0.029(0.051)}$ | $-1$       |
|              | Pantheon+CC+H0LiCOW           | 71.8 ± 1.3(2.5) | 0.272$^{+0.021(0.046)}_{-0.023(0.043)}$ | $-1$       |
| wCDM         | Pantheon+CC                   | 69.2$^{+2.0(3.9)}_{-2.0(3.8)}$ | 0.329$^{+0.045(0.087)}_{-0.045(0.094)}$ | $-1.15^{+0.18(0.33)}_{-0.16(0.35)}$ |
|              | Pantheon+CC+H0LiCOW           | 72.2$^{+1.5(2.9)}_{-1.5(2.8)}$ | 0.289$^{+0.040(0.073)}_{-0.035(0.077)}$ | $-1.09^{+0.13(0.26)}_{-0.13(0.27)}$ |

The H0LiCOW collaboration \cite{6} reported $H_0 = 73.6^{+1.6}_{-1.8}$ km s$^{-1}$ Mpc$^{-1}$ and $H_0 = 74.9^{+2.2}_{-2.4}$ km s$^{-1}$ Mpc$^{-1}$ using Pantheon + H0LiCOW data for the ΛCDM and wCDM models, respectively. For a direct comparison, we added the CC data in our analysis and we note that these constraints can be improved (see Table III). With regard to the dark energy equation of state, we did not find any significant deviations from $w = -1$ in both analyses (see Fig. 2). From our joint analysis, we find that $w$ is measured at 1.2% accuracy, and $H_0$ at 2% accuracy.

V. STATISTICAL COMPARISON

Finally, we perform a statistical comparison of the different scenarios with the ΛCDM model by using the well-known Akaike information criterion (AIC) \cite{70} and Bayesian information criterion (BIC) \cite{71}. The AIC is defined through the relation

$$AIC = -2\ln L_{\text{max}} + 2N = \chi^2_{\text{min}} + 2N ,$$

(26)

where $L_{\text{max}}$ is the maximum likelihood function of the model, and $N$ is the total number of free parameters in the model. For the statistical comparison, the AIC difference between the model under study and the reference model is calculated. This difference in AIC values can be interpreted as the evidence in favour of the model under study over the reference model. It has been argued in \cite{72} that one model can be preferred with respect to another if the AIC difference between the two models is greater than a threshold value $\Delta_{\text{threshold}}$. As a rule of thumb, $\Delta_{\text{threshold}} = 5$ can be considered the minimum value to assert a strong support in favour of the model with a smaller AIC value, regardless of the properties of the models under comparison \cite{72}.

The BIC is defined as

$$BIC = -2\ln L_{\text{max}} + N \ln(k) = \chi^2_{\text{min}} + N \ln(k) ,$$

(27)

where $k$ is the sample size. The strength of the evidence against the model with higher BIC value can be summa-
rized as follows: for $0 \leq \Delta \text{BIC} < 2$, there is not enough evidence; for $2 \leq \Delta \text{BIC} < 6$, there exists a moderate evidence; for $\Delta \text{BIC} > 6$, there is a strong evidence. Here, we compare the $f(R)$ model, the $f(T)$ model and the $\omega$CDM model with the reference scenario represented by the ΛCDM model.

Table III summarizes our results from the AIC and BIC analysis comparison of different cosmological models from different combinations of data investigated in this work. Due to a minimal number of free parameters, the ΛCDM cosmology is the statistically preferred scenario to best-fit the data, whereas there are weak-to-moderate evidences against the alternative scenarios. Thus, we found no significant support for deviations from GR.

VI. FINAL REMARKS

Using geometric model-independent low and intermediate redshift data, we obtained the most up-to-date robust measurements of the Hubble constant in the context of modified background dynamics beyond GR. At the same time, we found new constraints on the free parameters of such theories. Particular attention was given to cosmologically viable $f(R)$ and $f(T)$ gravity models, for which we showed that $H_0$ can be measured with an accuracy of 1.9% and 1.8%, respectively. Including the time-delays observations from strong gravitationally lensed quasars in our Monte Carlo statistical analysis, our results appear consistent with the local (direct) measurement of $H_0$ from the LMC Cepheid standards, while they are $\gtrsim 3\sigma$ in tension with the CMB estimate based on the ΛCDM cosmology.

Although the free parameters of the theories analyzed here are constrained in a precise and robust way, we found no significant deviations from GR, and the dynamics of the Universe is compatible with that of the ΛCDM model at the background level. Finally, it would be interesting to implement a cosmographic analysis of the time-delay quasars measurements and obtain model-independent constraints on kinematic parameters, using machine learning methods in order to verify their compatibility with the predictions of a given theoretical scenario.

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