Aximorphic Perspective Projection Model for Immersive Imagery

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Abstract
A wide choice of cinematic lenses enables motion-picture creators to adapt image visual-appearance to their creative vision. Such choice does not exist in the realm of real-time computer graphics, where only one type of perspective projection is widely used, a linear perspective. This paper presents an extended perspective imaging model, which can represent distortion and FoV parameters of entire variety of film and photographic lenses (e.g., wide-angle, fisheye, anamorphic), while preserving parametrization in an artistically convincing manner. Self-experimentation with the model revealed that each projection type provides accurate perception of a different aspect of depicted space (e.g., speed, distance, shape). Presented model, enables combination of multiple projections, each on a different axis of the image, to achieve optimal perception for a given scenario. This new projection, named aximorphic, was made available here, under an open license (CC BY-SA 3.0), for a wide and easy adoption.

CCS Concepts:
• Computing methodologies → Perception; Ray tracing; • Human-centered computing → Media arts;

Keywords: Curvilinear Perspective, Panini, Anamorphic, Cylindrical, Fisheye, lens Map

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1 Introduction
The perspective in the computer real-time graphics hasn’t changed since the dawn of CGI. It is based on a concept as old as the fifteenth-century Renaissance, a linear projection [Alberti 1435; Argan and Robb 1946; McArdle 2013]. This situation is similar to the beginnings of photography, where only one type of lens was widely used, an Anno Domini 1866 Rapid Rectilinear [Kingslake 1989] lens. Linear perspective even at the time of its advent, 500 years ago, has been criticized for distorting proportions [Da Vinci 1632]. In a phenomenon known today as Leonardo’s paradox [Dixon 1987], in which figures further away but at the periphery appear larger, than those located near the optical center. Computer graphics really skipped the artistic achievements of the last five centuries in regard to perspective. This includes the cylindrical perspective of Panini [Sharpless et al. 2010], Barker [Wikipedia, contributors 2019], and anamorphic lenses used in cinematography [Giardina 2016; Neil 2004; Sasaki 2017a,b]. The situation is even more critical, as there is no mathematical model for generating anamorphic projection in an artistically-convincing manner [Yuan and Sasian 2009]. Some attempts were made at alternative projections for computer graphics, with fixed cylindrical or spherical geometry [Baldwin et al. 2014; Sharpless et al. 2010]. A parametrized perspective model was also proposed as a new standard [Correia and Romão 2007], but wasn’t adopted. It included interpolation states in between rectilinear/equidistant and spherical/cylindrical projection. The cylindrical parametrization of this solution was merely an interpolation factor, where intermediate values did not correspond to any common projection type. Therefore, it was not well-suitable for artistic or professional use.

1.1 Break from Axiom of Pyramid Frustum
The notion of sphere as a projective surface that incorporates cartographic mapping to produce a perspective picture became popularized [German et al. 2007; Peñaranda et al. 2015; Williams 2015]. Also, perspective parametrization that transitions according to the content (by the view-tilt angle) has been developed, as a modification to the computer game MINECRAFT [Williams 2017]. But the results of these solutions were more a gimmick and have not found practical use. The linear perspective projection was still the way-to-go for most digital content. Some state-of-the-art video games incorporated limited lens distortion, like RESIDENT EVIL series (after 2017), Alan Wake 2 and more strongly Unrecord.

One of the reasons for a limited adoption of a non-linear projection was the fixed-function architecture of GPUs in regard to rasterization. But with the advent of real-time ray-tracing and variable shading-rate, more exotic projections could become widely adopted and integrated into the tools.

1.2 Presented New Model
This paper aims to provide a perspective model with a mathematical parametrization that allows artistic-style interaction with image geometry. Similar in a way film directors choose lenses for each scene based on aesthetics [Giardina...
This naming convention simplifies the process of translating this document uses the following naming convention: from axis-, line of symmetry, and *morphé, shape; “varying shape across axes”).

1.3 Document Naming Convention
This document uses the following naming convention:

- A left-handed coordinate system is used.
- Vectors are presented in column format.
- Matrices use row-major order and are denoted as “M_{row \times col}”.
- Matrix multiplication is denoted as “[column]_{a} [row]_{b} = M_{a b}”.
- A single bar enclosure “|u|” represents the absolute value of a scalar.
- A single bar enclosure “|v|” represents the length of a vector.
- Vectors with an arithmetic sign, or without, are calculated component-wise to form another vector.
- Centered dot “.” represents the dot product of two vectors.
- Square brackets with a comma “[x,y]” denotes an interval.
- Square brackets with blanks “[x|y]” denotes a vector or a matrix.
- The power of “−1” implies the reciprocal of the value.
- The QED symbol “□” marks the final result or output.

This naming convention simplifies the process of translating formulas into shadier code.

2 Aximorphic Primary-ray Map
If we assume that a projective visual space is spherical [Fleck 1994; McArdle 2013], one can define perspective picture as an array of rays pointing to the surface of visual sphere. This is how the algorithm described below will output a viewing-ray map (aka lens map). Lens map is a two-dimensional R^3 vector field representing viewing rays, where each ray is assigned to a screen pixel. Visual sphere as the image model enables wider angle of view, beyond the limit for planar projection of 180°. Such vector field can be easily converted to a cube UV map, ST map, or other screen distortion format.

Here, the procedural algorithm for lens map uses two types of input values from the user; distortion parameter for two principal axes and focal-length or angle-of-view (aka FoV). Two distinct principal axes define the aximorphic projection type. Each axis distortion profile is expressed by the azimuthal projection factor k [Bettonvil 2005; Fleck 1994; Krause 2019]. Both principal axes share the same focal-length value f. The evaluation of each principal-axis distortion profile produces spherical angles θx and θy. These angles are then combined to form the aximorphic azimuthal-projection angle θ'. The interpolation of θ components is achieved through aximorphic weights φx and φy, which are derived from the spherical angle φ of the azimuthal projection.

Note. Calculation of φ angle is omitted here, as view-coordinate v alone allows for direct calculation of φ weights.

\[ r = |\vec{v}| = \sqrt{\vec{v}_x^2 + \vec{v}_y^2} \] (1a)

\[ \vec{\theta}_{x,y} = \begin{cases} \arctan\left(\frac{\vec{v}_y}{\vec{v}_x}\right), & \text{if } \vec{v}_x > 0 \\ \frac{\vec{v}_x}{\vec{v}_y}, & \text{if } \vec{v}_x = 0 \\ \arcsin\left(\frac{\vec{v}_y}{\vec{v}_x}\right), & \text{if } \vec{v}_x < 0 \end{cases} \] (1b)

\[ \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \begin{bmatrix} \cos^2 \phi \\ \sin^2 \phi \end{bmatrix} = \begin{bmatrix} 1 + \cos(2\phi) \\ 2 \end{bmatrix} = \begin{bmatrix} \cos^2 \phi' \\ \sin^2 \phi' \end{bmatrix}, \quad (1c)\]

where \( r \in \mathbb{R}_{>0} \) is the view-coordinate radius (vector magnitude). Vector \( \vec{\theta} \in [0, \pi]^2 \) contains two incidence angles (measured from the optical axis) of two azimuthal projections determined by two distinct k parameters. Vector \( \vec{\phi} \in [0, 1]^2 \) contains the aximorphic interpolation weights, which are linear \( \vec{\phi}_x + \vec{\phi}_y = 1 \), but exhibit spherical distribution (see Figure 1 on the facing page). Vector \( \vec{k} \in [-1, 1]^2 \) (also \([-1, 1]^3 \) in a variant from subsection 2.2) describes two power axes of aximorphic projection. The algorithm is evaluated per-pixel for position \( \vec{v} \in \mathbb{R}^2 \) in view-space coordinates, centered at the optical axis and normalized at the chosen angle-of-view (horizontal or vertical). The final aximorphic incidence angle θ' is obtained through interpolation of θ components by φ weights.

\[ \theta' = \begin{bmatrix} \vec{\theta}_x \\ \vec{\theta}_y \end{bmatrix} \begin{bmatrix} \vec{\phi}_x \\ \vec{\phi}_y \end{bmatrix} = \vec{\theta}_x \vec{\phi}_x + \vec{\theta}_y \vec{\phi}_y \] (2a)

\[ \begin{bmatrix} \vec{G}_x \\ \vec{G}_y \\ \vec{G}_z \end{bmatrix} = \begin{bmatrix} \sin \theta' \\ \cos \theta' \\ \cos \theta' \end{bmatrix} \] (2b)

here θ' ∈ (0, π] is the aximorphic incidence angle, measured from the optical axis. This measurement resembles azimuthal projection of a globe (here a visual sphere) [McArdle 2013]. The final incidence vector \( \vec{G} \in [-1, 1]^3 \) (aka viewing-ray) is obtained from the aximorphic angle θ'. Parameters r, v, φ are in view-space, while \( \vec{\theta}, \theta', \phi, \vec{G} \) are in visual-sphere space. Essentially, the aximorphic primary-ray map preserves the azimuthal angle φ while modulating only the radius of the image.
Inverse Mapping. It can be obtained through a lookup mesh shaped as a tessellated screen-plane, mapped to the visual-sphere surface by equation (2). This mesh can be directly sampled by a ray to read the UV coordinates, or rasterized to a UV coordinates map. Ray-mesh sampling offers the advantage of an unlimited angle-of-view compared to texture sampling, which is practically limited to around 140° Ω.

2.1 Focal Length and Angle-of-View
To enhance control over the image, a mapping between the angle-of-view Ω and the focal length f can be established. Here, the focal length is expressed in reciprocal format to optimize its use in equation (1).

\[
\begin{align*}
    f^{-1} &= \begin{cases} 
    \tan \left( \frac{\Omega}{2r_\Omega} \right) / r_\Omega, & \text{if } k_x > 0 \\
    \frac{\Omega}{2r}, & \text{if } k_x = 0 \\
    \sin \left( \frac{\Omega}{2r_\Omega} \right) / r_\Omega, & \text{if } k_x < 0,
    \end{cases} 
\end{align*}
\]

where \( \Omega_h \in (0, \pi] \) denotes horizontal angle of view, and \( r \) denotes radius at Ω. Similarly, vertical Ω \( \Omega_v \) can be obtained using \( k_y \) parameter instead. The resultant value \( 1/f \in \mathbb{R}_{>0} \) is the reciprocal focal-length.

Remark. The focal-length \( f \) value must be the same for both \( \tilde{k}_x \) and \( \tilde{k}_y \). Therefore, only one reference angle \( \tilde{\Omega} \) can be chosen, either horizontal or vertical.

Inverse function, to equation (3), for angle-of-view \( \tilde{\Omega} \), from focal-length \( f \), is obtained as follows:

\[
\begin{align*}
    \tilde{\Omega} &= \begin{cases} 
    2 \tan^{-1} \left( \frac{\tilde{k}_y}{k_y} \right), & \text{if } \tilde{k}_y > 0 \\
    \frac{\Omega}{2r}, & \text{if } \tilde{k}_y < 0 \\
    2 \arcsin \left( \frac{\tilde{k}_y}{k_y} \right), & \text{if } \tilde{k}_y < 0
    \end{cases} 
\end{align*}
\]

This formula can be used to obtain the actual vertical angle-of-view from a horizontally established focal length. Similarly horizontal angle \( \Omega_h \) can be obtained using the \( k_x \) parameter. Here input value \( r \) denotes radius at Ω.

2.2 Asymmetrical Aximorphism
Parameter \( \tilde{k}_y \) can be further augmented to produce an asymmetrical aximorphic projection by introducing a third input value denoted as \( \tilde{k}_z \). In such a case, the bottom and top halves of the image can present different azimuthal projections along the axis.

\[
\tilde{k}_y' = \begin{cases} 
    \tilde{k}_z, & \text{if } \tilde{\Omega}_y < 0 \\
    \tilde{k}_y, & \text{otherwise},
    \end{cases} 
\]

therefore \( \tilde{k}_y' \) replaces \( \tilde{k}_y \) in equations (1), (4) and (6).

Asymmetrical aximorphism can be applied to any side of the principal axes. A use case for such a perspective could be in racing, where the bottom half of the screen contains an image of the road. Choosing equidistant projection (which preserves angular speed) would provide an accurate perception of velocity. The top half of the screen contains the image of opponent vehicles or the road ahead. Choosing stereographic projection (which preserves angles and proportions) would provide an enhanced perception for choosing the optimal trajectory. For the horizontal power axis, choosing equidistant projection (which preserves distance) would enhance the perception of distance to the turn for braking.

2.3 Vignetting Mask
Vignetting is a crucial visual symbol indicating the stretching of the visual sphere by the projection. Incorporating a vignetting effect with a custom projection enhances spatial perception.
Here, the vignetting mask is obtained in the same way for all aximorphic, anamorphic, and spherical projections. It is generated as the ratio of the circumference of a small circle \((2\pi \sin \theta')\) to the circumference of the image circle \((2\pi r_i)\), where the image circle’s radius \(r_i\) is obtained through the normalization of the picture-space radius \(r\) by the focal length \(f\).

\[
\Lambda = \begin{cases} 
1, & \text{if } r = 0 \\
\frac{\sin \theta'}{r}, & \text{else,} \\
\end{cases}
\]  

where \(\Lambda \in [0, 1]\) is the aximorphic vignetting mask value, which is inversely proportional to the scaling of visual sphere surface in image space.

**Note.** In the shader implementation, conditional branching can be omitted as equation \(\frac{\sin \theta'}{r}\) will automatically yield 1.

This vignetting model accounts for the natural falloff due to the stretching of the projected visual sphere. In real optical systems, the vignetting effect is often enhanced at the borders by the gradual occlusion of the aperture, especially at lower f-stops. At lower f-stops, some lens casing elements can block the entrance pupil at steep angles, usually when the aperture is wide-open. Therefore presented model is closest achieved at lowest apertures, when the entrance pupil is small and vignetting is closest to natural.

### 3 Converting Ray-map to ST-map

The ray/lens-map can be easily converted to the \(ST\)-map format for distorting a rectilinear source image, provided that the maximum view angle \(\Omega\) does not exceed or equal 180°.

\[
\begin{bmatrix} \tilde{a}_x \\ \tilde{a}_y \end{bmatrix} = \begin{cases} 
\begin{bmatrix} 1 \\ \frac{\nu}{\pi} \end{bmatrix}, & \text{if } \Omega_h \\
\begin{bmatrix} k \\ 1 \end{bmatrix}, & \text{if } \Omega_v \\
\end{cases}
\]  

\[
\begin{bmatrix} \tilde{r} \\ \tilde{f} \end{bmatrix} = \cot \frac{\Omega}{2G_z} \begin{bmatrix} \tilde{G}_x \\ \tilde{G}_y \end{bmatrix} \begin{bmatrix} \tilde{a}_x \\ \tilde{a}_y \end{bmatrix} + \frac{1}{2}. \\
\]
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(a) $k = 1$, Rectilinear (standard).

(b) $\tilde{k} = \begin{bmatrix} 1/2 & 2/5 \end{bmatrix}$, first-person (anamorphic style).

(c) $\tilde{k} = \begin{bmatrix} 1/2 \end{bmatrix}$, panini.

(d) $\tilde{k} = \begin{bmatrix} 1/2 & 0 & -1/2 \end{bmatrix}$, flying (asymmetrical).

(e) $\tilde{k} = \begin{bmatrix} -1/2 & 1/2 \end{bmatrix}$, racing (asymmetrical).

Figure 3. Example of various wide–angle ($\Omega \approx 110^\circ$) aximorphic-azimuthal projections with vignetting in 4/3 aspect-ratio. The checkerboard depicts a cube centered at the observation point, with each face colored according to the axis direction. Here, primary colors represent the positive axis, and neighboring complementary colors its negative equivalent (same as in the color-wheel), $\{Mg, Yl, Cy\} \leftrightarrow -\{X, Y, Z\} + \mapsto \{R, G, B\}$.

where $\tilde{a} \in \mathbb{R}^2$ is the square-mapping vector for both the horizontal and vertical angle of view. Values $w$ and $h$ represent picture width and height, respectively. $\Omega < \pi$ is the angle of view. $\tilde{f} \in [0, 1]^2$ represent the final ST-map vector. $\tilde{G} \in [0, 1]^3$ is the input viewing-ray map vector.

4 Aximorphic Lens Distortion

The presented perspective model can be used to mimic the real-world anamorphic lens and its effects. Effects such as disproportionate lens breathing using focal length-based parametrization, which are unique to anamorphic photography [Neil 2004; Sasaki 2017b]. Some additional lens-corrections may be added to the initial ray-map, to simulate more complex lens distortions and lens imperfections.

Below, an algorithm for aximorphic distortion of view coordinates is presented, which can be used as an input for
| Picture content type | Aximorphic $\tilde{k}$ values |
|----------------------|-------------------------------|
| Racing               | $\tilde{k} = \begin{bmatrix} -1/2 & 1/2 & 0 \end{bmatrix}$ |
| Flying               | $\tilde{k} = \begin{bmatrix} 1/2 & 0 & 0 \end{bmatrix}$ |
| First-person (generic) | $\tilde{k} = \begin{bmatrix} 1/2 & 22/25 & 22/25 \end{bmatrix}$ |
| First-person (aiming) | $\tilde{k} = \begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix}$ |
| Pan motion           | $\tilde{k}_x \neq \tilde{k}_y$ |
| Roll motion          | $\tilde{k}_y = \tilde{k}_x$ |
| Tilt motion\(^a\)   | $\tilde{k}_y \rightarrow \tilde{k}_x$ |

Source: Values determined empirically with self experimentation using various competitive video games, in accordance to data in Table 1b.\(^a\)Mapping of vertical distortion by a tilt motion introduced first in a Minecraft mod [Williams 2017].

Table 2. Recommended values of $\tilde{k}$, for various scenario and parameter behavior for a given camera motion type.

viewing-ray map algorithm (equation 1 on page 2). The algorithm is based on the Brown-Conrady lens-distortion model [Wang et al. 2008] in a division variant [Fitzgibbon 2001]. It is executed on a view-coordinate $\vec{v}$, forming alternative vector $\vec{v}'$.

$$\begin{bmatrix} \tilde{f}_x \\ \tilde{f}_y \end{bmatrix} = \begin{bmatrix} \tilde{v}_x - \tilde{c}_1 \\ \tilde{v}_y - \tilde{c}_2 \end{bmatrix}$$

(8a)

cardinal offset $a$

$$\begin{bmatrix} \tilde{v}_x' \\ \tilde{v}_y' \end{bmatrix} = \begin{bmatrix} \tilde{f}_x \\ \tilde{f}_y \end{bmatrix} \left( \begin{bmatrix} 1 + \tilde{k}_{x1} r^2 + \tilde{k}_{x2} r^4 + \cdots \\ 1 + \tilde{k}_{y1} r^2 + \tilde{k}_{y2} r^4 + \cdots \end{bmatrix} \cdot \begin{bmatrix} \tilde{v}_x \\ \tilde{v}_y \end{bmatrix} \right)^{-1}$$

(8b)

radial aximorphic

$$\begin{bmatrix} \tilde{v}_x' \\ \tilde{v}_y' \end{bmatrix} = \begin{bmatrix} \tilde{f}_x \\ \tilde{f}_y \end{bmatrix} \left( \begin{bmatrix} \tilde{f}_x \\ \tilde{f}_y \end{bmatrix} \cdot \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} + \tilde{r}^2 \tilde{q}_1 + \tilde{c}_1 \right) + \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix}$$

(8c)

decentering thin prism cardinal $b$

$$\begin{bmatrix} \tilde{v}_x' \\ \tilde{v}_y' \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{G}_x \\ \tilde{G}_y \end{bmatrix}$$

(8d)

decentering thin prism cardinal $b$

where $\tilde{c}_1$, $\tilde{c}_2$ are the cardinal-offset parameters, $\tilde{q}_1$, $\tilde{q}_2$ are the thin-prism distortion parameters and $\tilde{p}_1$, $\tilde{p}_2$ are the decentering parameters. A set of $\tilde{k}$ parameters define radial distortion for each aximorphic power axis. $\vec{v}$ is the input view-coordinate, and $\vec{v}'$ is the view coordinate with applied lens-transformation. $\vec{v} \in [0, 1]^2$ is the aximorphic interpolation weight, defined in section 2 on page 2.

Figure 4. Mapping of $t \in [0, 1]$ to spectral color $\vec{x} \in [0, 1]^3$, for emulation of chromatic aberration. This is an output of periodic function, found in equation (9). Distribution of the values ensures proper color order and sum-of-samples with guaranteed neutral-white tint.

5 Aximorphic Chromatic Aberration

A chromatic aberration effect can be achieved with multi-sample blur, where each sample layer is colored with the corresponding spectral-value [Gilcher 2015]. Presented periodic function for spectral color $\vec{x}$ produces samples that always add up to 1 (neutral white) when number of samples is even. It also exhibits the correct order of spectrum colors.

$$\begin{bmatrix} \tilde{x}_r \\ \tilde{x}_g \\ \tilde{x}_b \end{bmatrix} = \frac{1}{0} \left( \frac{3}{2} - |4 \bmod (t + \begin{bmatrix} 1/4 \\ 0 \\ 1/4 \end{bmatrix})| \right)$$

(9)

where $\tilde{x} \in [0, 1]^3$ is the spectral-color value for position $t \in [0, 1]$ (see figure 4 for more information).

Performing a spectral blur on an image involves the sum of multiple spectrum-colored layers. Scalar $t$ (here replaced by sample progress) should never reach 1, which ensures preservation of picture’s white-balance. The number of samples $n$ must be an even number, and no less than 2 for a correct, neutral-white result.

$$\begin{bmatrix} \tilde{x}_r' \\ \tilde{x}_g' \\ \tilde{x}_b' \end{bmatrix} = \frac{2}{n} \sum_{i=0}^{n-1} \begin{bmatrix} \tilde{f}_r \\ \tilde{f}_g \\ \tilde{f}_b \end{bmatrix} \left( \begin{bmatrix} 3/2 - |4 \bmod \begin{bmatrix} i/4 \\ 0 \\ 3/4 \end{bmatrix}| \right)$$

(10)

where $n \in 2\mathbb{N}_1$ is the even number of samples for the chromatic aberration color-split. $\tilde{f} \in [0, 1]^3$ is the current-sample position color-value. $\tilde{f}' \in [0, 1]^3$ is the final spectral-blurred color value.
Figure 5. Example of aximorphic lens distortion with chromatic aberration, where $k_{x1} = -0.25$, $k_{y1} = 0.04$, $d = 0.5$, with 64–spectral samples.

The equation for spectral color $\tilde{c}$ can be rewritten to a more optimized form, for parallel computation.

$$
\begin{bmatrix}
\hat{X}_t \\
\hat{Y}_t \\
\hat{Z}_t
\end{bmatrix} = \begin{bmatrix}
\text{clamp}_0^1 (\sqrt{2} - |4t - 1|) \\
\text{clamp}_0^1 (\sqrt{2} - |4t - 2|) \\
- \text{clamp}_0^1 (\sqrt{2} - |4t - 1|)
\end{bmatrix} + \begin{bmatrix}
\text{clamp}_0^1 (4t - 7/2) \\
0 \\
1 - \text{clamp}_0^1 (4t - 7/2)
\end{bmatrix} \Delta \tilde{d},
$$

(11)

where $\tilde{c} \in [0, 1]^3$ is the spectral color at position $t \in [0, 1]$. See the figure 4 for visualization.

5.1 Chromatic Aberration through Lens Distortion

Chromatic aberration can be integrated into lens distortion with spectral blurring, through the lens-transformation vector $\Delta \tilde{d}$. Below, the equation for the spectral-blur displacement vector $\tilde{s}$ is presented, calculated per sample at position $t$.

$$
\begin{bmatrix}
\Delta \tilde{d}_x \\
\Delta \tilde{d}_y
\end{bmatrix} = \begin{bmatrix}
\tilde{s}_x - \tilde{u}_x \\
\tilde{s}_y - \tilde{u}_y
\end{bmatrix}
$$

(12a)

$$
\begin{bmatrix}
\tilde{s}_x \\
\tilde{s}_y
\end{bmatrix} = \left( 1 + \left( t - \frac{1}{2} \right) d \right) \begin{bmatrix}
\Delta \tilde{d}_x \\
\Delta \tilde{d}_y
\end{bmatrix},
$$

(12b)

where $\tilde{s} \in \mathbb{R}^2$ is the spectral blur sample-offset vector at position $t \in [0, 1]$. Value $d \in \mathbb{R}$ denotes the lens dispersion-scale. For a visually pleasing result, additional blur pass can be applied, which direction is perpendicular to the $\tilde{s}$ vector, and smaller in magnitude, as below:

$$
\begin{bmatrix}
\tilde{s}'_x \\
\tilde{s}'_y
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
-\tilde{s}_y \\
\tilde{s}_x
\end{bmatrix},
$$

(13)

where $\tilde{s}'$ is the second-pass blur direction vector.

Figure 5 presents the final effect of two-pass blur, where the first pass is spectral, along lens-distortion $\Delta \tilde{d}$, and the second-pass (of quarter-magnitude) is perpendicular $\perp \Delta \tilde{d}$. A combination of both adds a defocusing effect to the image distortion.

6 Final Thoughts

In this paper, a mathematical model for generating various asymmetrical aximorphic perspective projections has been provided, along with perception-driven distortion-design recommendations. In the model, each principal axis of the image resembles some azimuthal projection exactly, while regions in-between are transitional, creating a hybrid projection. This way, advantage can be taken of many projection types, to create an optimal, tailored view for a given specific scenario. Such perception-driven parametrization enables the picture’s geometry to adapt to the context, allowing dynamic adjustments on-the-fly in an artistically convincing manner.

In addition to aximorphic perspective, this paper presents vignetting effects and lens distortion with integrated chromatic aberration. The selection of these key features, enables almost complete digital lens simulation, for an immersive-imagery production.

Additional shader implementation of this technique can be found in the open-source PerfectPerspective.fx shader available on GitHub or via ReShade platform.

6.1 Prospects for Future Improvement

In the future, wide choice of cinematic lenses may be totally replaced by a post-production technique or an in-camera special effect, performed on-stage. Such technical solution would require several new technologies, like perhaps universal camera system, consisting of:

- Universal lens:
  - With parallax aberration correction and carefully mapped distortion.

- Universal camera sensor:
  - In correlation with the lens would produce distortion-free picture. Perhaps a light-field capturing sensor.

- Universal perspective algorithm:
  - Satisfied with this document.

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(a) $\mathbf{k} = [1/2, 1], f = 0.6, \Omega_v \approx 159^\circ$, panini preset, which along vertical axis, preserves straight lines, and along horizon preserves proportions.

(b) $\mathbf{k} = [1/2, 0 - 1/2], f = 0.6, \Omega_v \approx 159^\circ$, flying (asymmetrical), where bottom-half preserves distance, horizon preserves shape, and top-half preserves speed.

(c) $\mathbf{k} = [0, 1 \ 4/5], \Omega_v = 180^\circ$, artistic projection where background architecture points straight-up. Attention is focused at the center figure by compression of periphery, and proper composition is achieved by expansion of the bottom field.

(d) $\mathbf{k} = [1/2 \ 0 \ 5/8], \Omega_v = 195^\circ$, artistic projection with wide vertical field and limited bottom field, preserving proportion on the horizontal axis.

Figure 6. Examples of super wide-angle views in aximorphic projection with natural vignetting. Mapped from various 360° panoramas in equirectangular projection.
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float get_fov(float k, float focal)
{
    if (k > 0.0) return degrees(2 * atan(k / focal) / k); // stereographic, rectilinear projections
    else if (k < 0.0) return degrees(2 * asin(k / focal) / k); // equisolid, orthographic projections
    else /* k == 0 */ return degrees(2 / focal); // equidistant projection
}

Listing 1. Mapping function for field of view from focal length and k parameter in GLSL.

float get_rcp_focal(float halfOmega, float radiusOfOmega, float k)
{
    if (k > 0.0) return tan(k * halfOmega) / (radiusOfOmega * k); // stereographic, rectilinear projections
    else if (k < 0.0) return sin(k * halfOmega) / (radiusOfOmega * k); // equisolid, orthographic projections
    else /* k == 0 */ return halfOmega / radiusOfOmega; // equidistant projection
}

Listing 2. Mapping function for focal length from field of view and k parameter in GLSL.

float get_theta(float radius, float rcpFocal, float k)
{
    if (k > 0.0) return atan(radius * rcpFocal * k) / k; // stereographic, rectilinear projections
    else if (k < 0.0) return asin(radius * rcpFocal * k) / k; // equisolid, orthographic projections
    else /* k == 0 */ return radius / rcpFocal; // equidistant projection
}

Listing 3. Mapping function for θ angle of azimuthal projection in GLSL.

float get_radius(float theta, float rcpFocal, float k)
{
    if (k > 0.0) return tan(k * theta) / (rcpFocal * k); // stereographic, rectilinear projections
    else if (k < 0.0) return sin(k * theta) / (rcpFocal * k); // equisolid, orthographic projections
    else /* k == 0 */ return theta / rcpFocal; // equidistant projection
}

Listing 4. Mapping function for azimuthal projection radius from θ angle in GLSL.

float get_vignette(float theta, float radius, float rcpFocal)
    { return sin(theta) / (radius * rcpFocal); }

Listing 5. Function for vignetting mask from projection radius and incidence angle θ in GLSL.

vec2 get_phi_weights(vec2 viewCoord)
{
    viewCoord *= viewCoord; // squared vector coordinates
    return viewCoord / (viewCoord.x + viewCoord.y); // [cos²φ sin²φ] vector
}

Listing 6. Function for aximorphic interpolation φ weights in GLSL.

vec3 spectrum(float hue)
{
    hue *= 4.0;
    vec3 hueColor;
    hueColor.rg = hue - vec2(1.0, 2.0);
    hueColor.rg = clamp(1.5 - abs(hueColor.rg), 0.0, 1.0);
    hueColor.r += clamp(hue - 3.5, 0.0, 1.0);
    hueColor.b = 1.0 - hueColor.r;
    return hueColor;
}

Listing 7. Mapping function for chromatic aberration spectrum, from hue value in GLSL.
Listing 8. Aximorphic and other lens distortion functions in GLSL.

// Radial distortion-division model (add to coordinates)
vec2 radial(float radius2, vec2 viewCoord, vec2 kx, vec2 ky)
{
    float radius4 = radius2 * radius2; // r^4
    vec2 phiWeights = viewCoord * viewCoord;
    phiWeights /= phiWeights.x + phiWeights.y;
    vec2 kVec = vec2(1.0 + kx[0] * radius2 + kx[1] * radius4,
                    1.0 + ky[0] * radius2 + ky[1] * radius4);
    return viewCoord / dot(kVec, phiWeights) - viewCoord;
}

// Decentering distortion model (add to coordinates)
vec2 decentering(vec2 viewCoord, vec2 p)
{
    return viewCoord * dot(viewCoord, p);
}

// Thin prism distortion model (add to coordinates)
vec2 thinPrism(float radius2, vec2 q)
{
    return radius2 * q;
}

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