Trimaximal-Cabibbo neutrino mixing: A parametrization in terms of deviations from tri-bimaximal mixing

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Abstract

In this paper we study a parametrized description of neutrino mixing from a phenomenological point of view. We concentrate on the parametrization in terms of higher order corrections to the leading order mixing matrix. A method to describe subleading contributions and its applications to tri-bimaximal mixing are discussed. We show that mixing matrices similar to tri-bimaximal-Cabibbo mixing can be obtained by straightforward choices of parameters. To achieve better agreement with the experimental data without increasing the number of free parameters, we impose a simple phenomenological relation from which a trimaximal-like mixing matrix, parametrized by $U_{e3} = \sin \theta_{13} e^{-i\varphi}$, can be derived straightforwardly without imposing additional requirements. It can describe the current global fit to three-neutrino mixing with good accuracy. Its theoretical explanation and phenomenological applications are discussed briefly.

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I. INTRODUCTION

Neutrino mixing is one of the most extensively studied topics in neutrino physics. The mixing pattern observed in neutrino oscillation experiments provides clear evidence, implying a non-trivial but perhaps simple flavor structure of the lepton sector. Many interesting mixing patterns are proposed to describe the mixing data, including tri-bimaximal mixing (TBM) \cite{1}, bimaximal mixing \cite{2}, golden-ratio mixings \cite{3}, democratic mixing \cite{4} and hexagonal mixing \cite{5}, etc. Some of these often appear as exact mixing matrices 1 in models where neutrino mixing is determined by underlying discrete flavor symmetries (see, e.g. \cite{8, 9}).

This paradigm worked quite well before the recent discovery of a relatively large $\theta_{13}$ (for recent global fits, see \cite{10–12}), which signals a deviation from those exact mixing patterns. For example, a recent global fit by M. C. Gonzalez-Garcia \textit{et al}. \cite{12} gives

$$
\sin^2 \theta_{12} = 0.30 \pm 0.013,
\sin^2 \theta_{23} = 0.41^{+0.037}_{-0.025} \oplus 0.59^{+0.021}_{-0.022},
\sin^2 \theta_{13} = 0.023 \pm 0.0023, \tag{1}
$$

where two 1σ ranges for $\sin^2 \theta_{23}$ are given because the present data cannot resolve the $\theta_{23}$ octant degeneracy \cite{12}. In the global fit given above, $\theta_{ij}$ are the mixing angles defined in the standard parametrization. From Eq. (1), the squared mixing matrix elements $|\langle \nu_e \rangle_{ij}|^2$ can be calculated and given collectively in a matrix as

$$
\begin{pmatrix}
0.684^{+0.013}_{-0.013} & 0.293^{+0.013}_{-0.013} & 0.023^{+0.0023}_{-0.0023} \\
\cdots & \cdots & 0.401^{+0.036}_{-0.024} \oplus 0.576^{+0.021}_{-0.022} \\
\cdots & \cdots & 0.576^{+0.024}_{-0.036} \oplus 0.401^{+0.022}_{-0.021}
\end{pmatrix} \tag{2}
$$

in which $|\langle \nu_e \rangle_{ij}|^2$ with $i \neq e$ and $j \neq 3$ are omitted because they are affected by the Dirac CP phase whose experimental value has a relatively large error. In our discussion they are determined by other parameters and, hence, the CP phase can be extracted by standard formalism.

Although most exact mixing patterns including those mentioned above are not in precise agreement with the experimental data, improvement can be made by introducing small

\footnote{By "exact mixing matrix", we mean the mixing matrix that does not depend on other parameters, including lepton masses. The corresponding effective neutrino mass matrix is sometimes called the "form diagonalizable matrix" \cite{4, 7}.}
but non-negligible subleading contributions, which can be induced by radiative corrections, charged lepton corrections, etc. (see, e.g., [13]). With this in mind, the parametrization of neutrino mixing by deviations from leading order (LO) exact mixing patterns has been studied extensively. For some relevant works, see [14–17] and other references given in this paper.

In the existing literature, a common choice of parameters is the deviation of mixing angles and the Dirac CP phase. In this paper, we use a different method where the mixing matrix is given by the product of a matrix describing deviations and the LO mixing matrix. In some cases it is simpler than the method dealing with mixing angles, and the physical relevance is more transparent. Although this method can be used for any LO mixing, we shall concentrate on tri-bimaximal mixing in this paper. We show that mixing matrices similar to tri-bimaximal-Cabibbo (TBC) mixing [14] can be obtained by this method with straightforward choices of parameters.

However, these TBC-like mixing patterns agree with the data only marginally. To achieve better agreement, sizable corrections to $\theta_{12}$ and $\theta_{23}$ must be taken into account. To do that, one can introduce more parameters. Nevertheless, parametrizations with fewer parameters can lead to simplified descriptions of neutrino phenomenologies and may provide clues to underlying physics. Therefore, it is also worthwhile to look for and study simple descriptions of neutrino mixing suggested by the experimental data. In this respect, phenomenological or empirical relations are very useful, e.g., the relation between $\theta_{13}$ and the Cabibbo angle $\theta_C$: $\theta_{13} \approx \theta_C / \sqrt{2}$ [14], the quark-lepton complementarity [19, 20], self-complementarity [21, 22], bi-large mixing [23], bi-pair mixing [24], etc. Besides their theoretical implications, they may also give rise to economical but rather accurate descriptions of neutrino mixing and phenomenologies.

In this paper, we introduce a relation, i.e., $| (U_\nu)_{e1} |^2 - | (U_\nu)_{e3} |^2 = 2/3$, which agrees with the data quite well, as can be seen from Eq. (2). We show that this relation leads to a parametrization that can describe the current global fit with good accuracy. It is derived from the result obtained in Sec. II and parametrized by $U_{e3} = \sin \theta_{13} e^{-i\varphi}$. Hence it can be regarded as an improved TBC mixing. Moreover, it is also a trimaximal-like mixing with the TM$_2$ trimaximal condition [15] being perturbed by a small correction. Hence, we refer to it as trimaximal-Cabibbo mixing.

\footnote{Majorana phases can be ignored since they do not affect neutrino oscillations [18].}
In summary, the paper is organized as follows. In Sec. II, we introduce a method to describe small deviations from LO mixing matrices. Several cases are discussed in detail. In Secs. III and IV, the results obtained in Sec. II are used to derive TBC-like mixing and trimaximal-Cabibbo mixing. In particular, the latter and its derivation are discussed in greater detail in Sec. IV. In Sec. V, we summarize and discuss briefly the theoretical explanation and phenomenological applications of trimaximal-Cabibbo mixing.

II. DEVIATIONS FROM LEADING ORDER MIXING MATRIX

In this section we discuss a method to describe deviations from LO mixing matrices. Note that the lepton mixing matrix $U_\nu$ can always be written as a product of two unitary matrices, i.e.,

$$U_\nu = U_\nu^0 T$$  \hspace{1cm} (3)

When $T = I$, the identity matrix, one has $U_\nu = U_\nu^0$. Hence, we use $U_\nu^0$ to denote the LO mixing matrix and the matrix $T$ to describe the deviations of $U_\nu$ from $U_\nu^0$. Just for convenience, in the following the matrix $T$ will be referred to as a perturbation matrix, although it is appropriate only when the deviations are very small. The formalism developed in the following can also be used for the case where the matrix $T$ multiplies $U_\nu^0$ from the left, i.e., $U_\nu = T U_\nu^0$, which will be discussed briefly at the end of this section.

In Eq. (3), the mixing matrix $U_\nu$ and the LO mixing matrix $U_\nu^0$ can be written as

$$U_\nu = (U_1, U_2, U_3), \quad U_\nu^0 = (K_1, K_2, K_3).$$  \hspace{1cm} (4)

where $U_i$ and $K_i$ for $i = 1, 2, 3$ are column vectors of $U_\nu$ and $U_\nu^0$, i.e.

$$U_i = \begin{pmatrix} (U_\nu)_{ei} \\ (U_\nu)_{\mu i} \\ (U_\nu)_{\tau i} \end{pmatrix} \quad K_i = \begin{pmatrix} (U_\nu^0)_{ei} \\ (U_\nu^0)_{\mu i} \\ (U_\nu^0)_{\tau i} \end{pmatrix}.$$  \hspace{1cm} (5)

The perturbation matrix $T$ in Eq. (3) is a unitary matrix that can be parametrized by three angles denoted by $\varsigma_{ij}$ and six phases. As usual, angle $\varsigma_{ij}$ corresponds to the rotation angle in the $(i, j)$ plane.

We consider first the simplest case where $T$ has two vanishing angles. For example, when
$\varsigma_{12} = \varsigma_{23} = 0$ and $\varsigma_{13} \neq 0$, $T$ can be written as
\begin{equation}
T = e^{i\omega} \begin{pmatrix}
\cos \varsigma_{13} & 0 & \sin \varsigma_{13} e^{-i\alpha} \\
0 & 1 & 0 \\
-\sin \varsigma_{13} e^{i\alpha} & 0 & \cos \varsigma_{13}
\end{pmatrix} P_\omega
\end{equation}
where $P_\omega = \text{diag}\{e^{i\omega_1}, 1, e^{i\omega_2}\}$. Since $P_\omega$ and $e^{i\omega}$ can be adsorbed by Majorana phases or charged leptons, we will ignore them in the following discussions. Thus, from Eqs. (6), (7), and (8) one has
\begin{equation}
U_1 = (K_1 - x^* K_3)/f_1, \quad U_2 = K_2, \quad U_3 = (K_3 + x K_1)/f_3
\end{equation}
where
\begin{equation}
f_1 = f_3 = \sqrt{1 + |x|^2}, \quad x = \tan \varsigma_{13} e^{-i\alpha}
\end{equation}
As we can see, instead of angles and phases, one can also use $x$ to parametrize $U_\nu$.

The above procedure can be applied consecutively and iteratively. Below we consider two cases that are relevant to later discussions. In the first case the perturbation matrix $T$ is given by a rotation in $(1, 3)$ plane followed by a rotation in $(1, 2)$ plane. By applying Eq. (7) twice on these rotations, it is straightforward to find that
\begin{align*}
U_1 &= (K_1 - y^* f_3 K_2 - x^* K_3)/f_1, \\
U_2 &= (f_3 K_2 + y K_1 - x^* y K_3)/f_2, \\
U_3 &= (K_3 + x K_1)/f_3,
\end{align*}
where
\begin{equation}
f_1 = f_2 = \sqrt{(1 + |x|^2)(1 + |y|^2)}, \quad f_3 = \sqrt{1 + |x|^2}.
\end{equation}
Equations (8) and (9) can be expanded in terms of $x$ and $y$. When $|x|$ and $|y|$ are small, the expansions can be simplified by ignoring higher order terms. Since in our discussions, parameter $x$ and $y$ are, at most, of order $\mathcal{O}(\lambda_C)$ where $\lambda_C = 0.2253 \pm 0.0007$ is the Wolfenstein parameter, terms of order $\mathcal{O}(|x|^3)$, $\mathcal{O}(|y|^3)$, or higher can be ignored and, hence one has
\begin{align*}
U_1 &\approx (1 - a) K_1 - y^* K_2 - x^* K_3, \\
U_2 &\approx (1 - b) K_2 + y K_1 - x^* y K_3, \\
U_3 &\approx (1 - c) K_3 + x K_1,
\end{align*}
where $a, b, c$ are parameters to be determined.
where

\[ a = \left( |x|^2 + |y|^2 \right)/2, \ b = |y|^2/2, \ c = |x|^2/2. \quad (11) \]

As another example, we consider the case where the perturbation matrix \( T \) is given by a \((1, 3)\) rotation followed by a \((2, 3)\) rotation. Similarly, one finds that

\[
\begin{align*}
U_1 &\simeq (1 - c) K_1 - x^* K_3, \\
U_2 &\simeq (1 - b) K_2 - y^* K_3 - xy^* K_1, \\
U_3 &\simeq (1 - a) K_3 + y K_2 + x K_1,
\end{align*}
\]

(12)

where \( a, b, \) and \( c \) are given in Eq. (11).

The perturbation matrix \( T \) may depend on two or more different rotations. The corresponding mixing matrices can be obtained in the same way. This method can also be used in the case where the perturbation matrix \( T \) multiplies \( U^0_\nu \) from the left, i.e. \( U_\nu = TU^0_\nu \) by applying this method to its transpose, i.e. \( \tilde{U}_\nu = \tilde{U}^0_\nu \tilde{T} \).

Obviously, the method discussed above is different from the one dealing with mixing angles. It can provide a simple way to construct mixing matrices from LO mixing matrices when the deviations are small. It can also lead to some interesting results, including the trimaximal-Cabibbo mixing derived in Sec. IV. This method can be applied to any LO mixing matrix, but note that even if the LO mixing matrix \( U^0_\nu \) is in the standard parametrization, the mixing matrix \( U_\nu \) may not be in the standard parametrization. This is not physically significant, although it may require slightly more work to extract mixing parameters used in the global fits, especially the CP phase. In addition, in some cases, its physical relevance is more transparent. For example, when \( U_\nu \) is given by \( TU^0_\nu \), the perturbation matrix \( T \) can be related to charged lepton corrections.

III. TRI-BIMAXIMAL-CABIBBO MIXING

In the rest of this paper we consider deviations from tri-bimaximal mixing, i.e.

\[
U^0_\nu = U_{\text{TBM}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & -\sqrt{2} & -\sqrt{3} \\ -1 & \sqrt{2} & \sqrt{3} \end{pmatrix}.
\]

(13)

It may be instructive to show some simple applications of the method introduced in the previous section. We begin with the simplest case, where the perturbation matrix \( T \) is
described by a single rotation. From Eq. (7), one finds that the second column of $U^0_\nu$ remains intact. Hence $U_\nu$ is a trimaximal mixing matrix (see, e.g. [15]). Since this case is very simple, below we consider the other two cases discussed in the previous section.

We consider first the case where $U_\nu$ is given by Eqs. (12). From the last equation in (12) and Eq. (5) one has

$$U_3 = \begin{pmatrix} (U_\nu)_{e3} \\ (U_\nu)_{\mu3} \\ (U_\nu)_{\tau3} \end{pmatrix} \simeq (1 - a) K_3 + yK_2 + xK_1$$

(14)

where $a$ is given in Eq. (11). Note that, as discussed in the previous section, in Eq. (14), terms of order $O(|x|^3)$, $O(|y|^3)$ or higher are ignored. Since $U^0_\nu = U_{TBM}$, then from Eq. (5) and Eq. (13), one has

$$K_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad K_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad K_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$  

(15)

Substituting $K_i$ given above into Eq. (14) leads to

$$|y \frac{1}{\sqrt{3}} + x \frac{2}{\sqrt{6}}| = |(U_\nu)_{e3}| = \left| \frac{\lambda e^{-i\phi}}{\sqrt{2}} \right| = \frac{\lambda}{\sqrt{2}}$$

(16)

where $(U_\nu)_{e3}$ is parametrized as $\lambda e^{-i\phi}/\sqrt{2}$. Parameters $x$ and $y$ can be written as

$$x = x_0 \lambda e^{i\phi} \frac{\sqrt{3}}{2}, \quad y = y_0 \lambda e^{i\phi} \frac{\sqrt{3}}{2}. \quad (17)$$

Then one has

$$|y_0 + x_0| = 1$$

(18)

Because $U_\nu$ contains only one physical Dirac CP phase, for simplicity we require that $x_0$ and $y_0$ are real. In addition, we require that they do not depend on other parameters. In principal, one may use any $x_0$ and $y_0$ as long as $|x|$ and $|y|$ are small. As an example, let $y_0 = 1/3$ and $x_0 = 2/3$. Then from Eqs. (12) and (17), one has

$$U_\nu = \begin{pmatrix} \frac{\sqrt{2}}{3} \left( 1 - \frac{\lambda^2}{6} \right) & \frac{1}{\sqrt{3}} \left( 1 - \frac{5\lambda^2}{12} \right) & \frac{\lambda e^{-i\phi}}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \left( 1 - \lambda e^{i\phi} - \frac{\lambda^2}{6} \right) & \frac{1}{\sqrt{3}} \left( 1 + \frac{\lambda e^{i\phi}}{2} + \frac{\lambda^2}{12} \right) & -\frac{1}{\sqrt{2}} \left( 1 - \frac{\lambda^2}{4} \right) \\ -\frac{1}{\sqrt{6}} \left( 1 + \lambda e^{i\phi} - \frac{\lambda^2}{6} \right) & \frac{1}{\sqrt{3}} \left( 1 - \frac{\lambda e^{i\phi}}{2} + \frac{\lambda^2}{12} \right) & \frac{1}{\sqrt{2}} \left( 1 - \frac{\lambda^2}{4} \right) \end{pmatrix} + O(\lambda^3)$$
which is similar to the TBC mixing introduced in [14], which is also parametrized by \( \sin \theta_{13} \) whose global-fit value is in good agreement with the relation \( \sin \theta_{13} = \lambda C / \sqrt{2} \). Note that \( \lambda_C \approx 0.011 \) and, hence, terms of order \( \mathcal{O}(\lambda^3) \) or higher can be neglected. One may check explicitly that \( U_\nu \) is unitary up to \( \mathcal{O}(\lambda^3) \) corrections. The mixing angles extracted from \( U_\nu \) are given by

\[
\sin^2 \theta_{12} = \frac{1}{3} - \frac{\lambda^2}{9}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad \sin^2 \theta_{13} = \frac{\lambda^2}{2}
\]

Substituting \( \lambda_C \) for \( \lambda \), one finds that the deviation of \( \sin^2 \theta_{12} \) from its TBM or TBC value is \( \lambda^2 / 9 = 0.0056 \), which is much smaller than the experimental error. Note that this is the only difference between the mixing derived above and the TBC mixing proposed in [14].

The above is an example in which \( U_\nu = U_{TBM} T \). Below we consider another example in which \( U_\nu = T U_{TBM} \). Taking the transpose leads to \( \tilde{U}_\nu = \tilde{U}_{TBM} \tilde{T} \). Therefore, as discussed in Sec. II, one can use the method in the example above to obtain \( \tilde{U}_\nu \), the transpose of \( U_\nu \). For instance, one can use Eqs. (10) to construct \( \tilde{U}_\nu \). Denote \( \tilde{U}_{TBM} \) by \((K'_1, K'_2, K'_3)\) and substitute \( K'_i \) for \( K_i \) in Eqs. (10), from \( |(\tilde{U}_\nu)_{3e}| = |(U_{\nu})_{3e}| = \lambda / \sqrt{2} \) one finds that \( |y^* - x^*| = \lambda \). As above, one may set \( x = x_0 \lambda e^{i\varphi} \) and \( y = y_0 \lambda e^{i\varphi} \) and then one has \( |y_0 - x_0| = 1 \). Letting \( y_0 = -x_0 = 1/2 \) leads to another TBC-like mixing matrix with mixing angles given by

\[
\sin^2 \theta_{12} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2} - \frac{\lambda^2}{8}, \quad \sin^2 \theta_{13} = \frac{\lambda^2}{2}.
\]

**IV. TRIMAXIMAL-CABIBBO MIXING**

As we can see, for the TBC-like mixings discussed above, one has \( \sin^2 \theta_{12} \approx \sin^2 \theta_{12}^{TBM} = 1/3 \) and \( \sin^2 \theta_{23} \approx \sin^2 \theta_{23}^{TBM} = 1/2 \), which fit the data within the range between the 2\( \sigma \) and 3\( \sigma \) experimental bounds. To achieve a better agreement with the data, corrections to \( \theta_{12} \) and \( \theta_{23} \) should also be taken into account. As discussed in the first section, to avoid introducing additional parameters, we will impose a phenomenological relation. We note that, to some extent, several well-known mixing patterns such as TBM and BM can be regarded as phenomenological mixing patterns. Trimaximal mixing [15] and TBC mixing [14] can also be derived with certain phenomenological relations in mind. The relation used in our discussion is given by

\[
|\langle U_\nu \rangle_{e1}|^2 - |\langle U_\nu \rangle_{e3}|^2 = \frac{2}{3}
\]
which is consistent with the present data. Like most other phenomenological or empirical
relations, it can hardly be generated as an exact relation. Nevertheless, note that this
relation is also satisfied by TBM and, hence, it can be considered as a phenomenological
constraint on the deviations from TBM induced by higher order corrections. Its possible
theoretical explanation and phenomenological applications are discussed in the next section.

We discuss first its implication on mixing parameters. As in the previous section, we
denote \((U_\nu)_{e3}\) by \(\lambda e^{-i\varphi}/\sqrt{2}\). From Eq. (19) we have

\[
|\langle U_\nu \rangle_{e1}|^2 = \frac{2}{3} + \frac{\lambda^2}{2},
|\langle U_\nu \rangle_{e2}|^2 = \frac{1}{3} - \lambda^2, \quad |\langle U_\nu \rangle_{e3}|^2 = \frac{\lambda^2}{2}
\]

(20)

from which it follows that

\[
\sin^2 \theta_{13} = \frac{\lambda^2}{2}, \quad \sin^2 \theta_{12} \simeq \frac{1}{3} \left(1 - \frac{5}{2} \lambda^2\right)
\]

where higher order terms are neglected. Using the global-fit value of \(\sin \theta_{13}\) given in Eq. (11),
one finds that the deviation of \(\sin^2 \theta_{12}\) from \(\sin^2 \theta_{12}^{TBM} = 1/3\) is \(5 \sin^2 \theta_{13}/3 = 0.038 \pm 0.0038\),
which is much larger than the 1\(\sigma\) experimental error. Therefore, to satisfy Eq. (19),
the correction to \(\theta_{12}^{TBM}\) cannot be neglected. Also note that this relation does not constra
the atmospheric mixing angle \(\theta_{23}\). Nevertheless, below we will show that even without making a
particular choice of parameters, the mixing matrix given by Eqs. (10) leads straightforwardly
to an improved TBC or trimaximal-like mixing that can fit the data very well, including \(\theta_{23}\).

We consider first the case where the perturbation matrix \(T\) multiplies \(U_{TBM}\) from the right,
i.e. \(U_\nu = U_{TBM}T\). Since \(U_\nu^0 = U_{TBM}\), from Eq. (20) one finds that \(|\langle U_\nu \rangle_{e1}|^2 > |\langle U_\nu^0 \rangle_{e1}|^2 = 2/3\).
Because \(\langle U_\nu^0 \rangle_{e3} = 0\), \(\langle U_\nu \rangle_{e1}\) must receive a contribution from \(\langle U_\nu^0 \rangle_{e2}\), i.e.,
the first element of \(K_2\), which is the second column of \(U_\nu^0\). Therefore we use Eqs. (10) to construct \(U_\nu\),
which is the only choice made in this case. The mixing matrix \(U_\nu\) can then be derived
straightforwardly in a similar manner as in the example discussed in the previous section.
From the last equation in (10) and \(|\langle U_\nu \rangle_{e3}| = \lambda/\sqrt{2}\), one finds that \(|x| \simeq \sqrt{3}\lambda/2\). Then from
the first equation in (10) and Eq. (20) one has

\[
\left| \left(1 - \frac{|x|^2 + |y|^2}{2}\right) \sqrt{\frac{2}{3}} \left(- \frac{1}{\sqrt{3}} y^*\right) \right|^2 \simeq \left(\frac{2}{3} + \frac{\lambda^2}{2}\right)
\]

For simplicity, we assume that \(y\) is real. Since \(|x| \simeq \sqrt{3}\lambda/2\), it is straightforward to solve
the above equation for \(y\). Ignoring higher order terms, one has

\[
x = \frac{\sqrt{3}}{2} \lambda e^{-i\varphi}, \quad y = -\frac{3\sqrt{3}}{4} \lambda^2.
\]

(21)
Then from Eqs. (10) it follows that

\[ U_\nu \simeq \begin{pmatrix} \frac{\sqrt{2}}{3} \left( 1 + \frac{3\lambda^2}{8} \right) & -\frac{1}{\sqrt{3}} \left( 1 - \frac{3\lambda^2}{8} \right) & \frac{\lambda e^{-i\varphi}}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \left( 1 - \frac{3\lambda e^{i\varphi}}{2} - \frac{15\lambda^2}{8} \right) & \frac{1}{\sqrt{3}} \left( 1 + \frac{3\lambda^2}{8} \right) & -\frac{1}{\sqrt{2}} \left( 1 + \frac{\lambda e^{-i\varphi}}{2} - \frac{3\lambda^2}{8} \right) \\ -\frac{1}{\sqrt{6}} \left( 1 + \frac{3\lambda e^{i\varphi}}{2} - \frac{15\lambda^2}{8} \right) & \frac{1}{\sqrt{3}} \left( 1 + \frac{3\lambda^2}{4} \right) & -\frac{1}{\sqrt{2}} \left( 1 - \frac{\lambda e^{-i\varphi}}{2} - \frac{3\lambda^2}{8} \right) \end{pmatrix} + O(\lambda^3). \]  

(22)

which is slightly more complicated than the TBC mixing matrix \([14]\) but in a better agreement with the data. Hence it can be considered as an improved TBC mixing. For the reason discussed in Sec. III, terms of order \(O(\lambda^3)\) or higher can be neglected. One can also check explicitly that \(U_\nu\) is unitary up to \(O(\lambda^3)\) corrections.

From Eq. (22) a simple relation between \(\sin \theta_{23}\), \(\sin \theta_{13}\), and the CP phase \(\varphi\) can be derived:

\[ \sin^2 \theta_{23} \simeq \frac{1}{2} \left( 1 + \lambda \cos \varphi + \frac{\lambda^3}{8} \cos \varphi \right). \]

where the \(\lambda^3\) term can also be neglected. The Jarlskog invariant is given by

\[ J \simeq -(\lambda \sin \varphi)/6 + O(\lambda^3). \]

Note that \(\varphi\) is not the CP phase in the standard parametrization, but their difference is very small, as will be shown in the next section.

In the case with vanishing CP phase, substituting \(\sqrt{2} \sin^2 \theta_{13}\) for \(\lambda\) and using the global-fit value for \(\sin^2 \theta_{13}\) given in Eq. (1), one has

\[ \sin^2 \theta_{12} \simeq 0.295^{+0.004}_{-0.004}, \quad \sin^2 \theta_{23} \simeq 0.607^{+0.005}_{-0.006} \]

which agree with the data within 1\(\sigma\) range. Note that by switching \(\lambda \rightarrow -\lambda\) or \(\varphi \rightarrow \varphi + \pi\), \(\sin^2 \theta_{23}\) can be brought into the 1\(\sigma\) range in the first octant given in Eq. (1). For the nonvanishing phase, one has

\[ 0.39 \lesssim \sin^2 \theta_{23} \lesssim 0.61 \]

which is within the 3\(\sigma\) experimental range [12].

In addition, it is interesting to see that \(U_\nu\) given in Eq. (22) may also be regarded as a variant of the TM\(_2\) trimaximal mixing [15] with the TM\(_2\) condition \(|(U_\nu)_{\alpha\beta}|^2 = 1/3 \) (\(\alpha = e, \mu, \tau\)) being perturbed by small corrections of order \(O(\lambda^2)\). Hence, we refer to it as trimaximal-Cabibbo mixing since substituting \(\lambda_C\) for \(\lambda\) in Eq. (22) leads to

\[ \sin^2 \theta_{12} \simeq 0.29, \quad \sin^2 \theta_{23} \simeq 0.61, \quad \sin^2 \theta_{13} \simeq 0.025 \]
which also agree with the data within $1\sigma$ range.

Now we consider another case where the perturbation matrix $T$ multiplies $U_{\text{TBM}}$ from the left, i.e. $U_{\nu} = TU_{\text{TBM}}$. As discussed in previous sections, in this case, one can move the perturbation matrix to the right by taking transpose. Since the discussion is similar, we just give the result. From Eqs. (10) we find that $x$ and $y$ should satisfy

$$|-x^* + y^*|^2 \simeq \lambda^2,$$

$$\left|1 - \frac{|x|^2 + |y|^2}{2} - x^* - y^*\right|^2 \simeq 1 - 3\lambda^2.$$

One can verify that

$$x = -\frac{\lambda e^{-i\varphi}}{2} + \frac{5\lambda^2}{8}, \quad y = \frac{\lambda e^{-i\varphi}}{2} + \frac{5\lambda^2}{8}$$

satisfy the above equations. Then from Eqs. (10) one has

$$U_{\nu} \simeq \begin{pmatrix}
\sqrt{\frac{2}{3}}\left(1 + \frac{3\lambda^2}{8}\right) & \frac{1}{\sqrt{3}}\left(1 - \frac{3\lambda^2}{2}\right) & \frac{\lambda e^{-i\varphi}}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}}\left(1 - \lambda e^{i\varphi} - \frac{9\lambda^2}{8}\right) & \frac{1}{\sqrt{3}}\left(1 + \frac{\lambda e^{-i\varphi}}{2} + \frac{3\lambda^2}{4}\right) & -\frac{1}{\sqrt{2}}\left(1 - \frac{3\lambda^2}{8}\right) \\
-\frac{1}{\sqrt{6}}\left(1 + \lambda e^{i\varphi} - \frac{11\lambda^2}{8}\right) & \frac{1}{\sqrt{3}}\left(1 - \frac{\lambda e^{-i\varphi}}{2} + \frac{\lambda^2}{2}\right) & \frac{1}{\sqrt{2}}\left(1 - \frac{\lambda^2}{8}\right)
\end{pmatrix} + O(\lambda^3).$$

(23)

which leads to a nearly maximal $\theta_{23}$, i.e.

$$\sin^2 \theta_{23} \simeq \frac{1}{2}(1 - \frac{\lambda^2}{4}).$$

Although acceptable, it does not fit the data as well as the trimaximal-Cabibbo mixing derived above. One may improve that by adjusting $x$ and $y$, which is possible for this case, but the choice of parameters is not very straightforward, so we will leave that for future considerations when more experimental data are available.

V. SUMMARY AND DISCUSSIONS

In this paper a general method to parametrize the neutrino mixing matrix in terms of deviations from leading order mixing is discussed. Using this method, we show that mixing matrices similar to tri-bimaximal-Cabibbo mixing can be derived by straightforward choices of parameters. However, these mixing matrices fit the data only marginally. To improve that without increasing the number of free parameters, we introduce a phenomenological relation, i.e., $| (U_{\nu})_{e1} |^2 - | (U_{\nu})_{e3} |^2 = 2/3$. Two mixing matrices satisfying this relation are constructed. The one referred to as trimaximal-Cabibbo mixing provides a good two-parameter
description of the present mixing data and, hence, can serve as a useful parametrization as long as future experimental data do not change the current global fit significantly. Below we discuss briefly its phenomenological applications and possible theoretical explanations.

Since the trimaximal-Cabibbo mixing given in Eq. (22) involves only two free parameters, when it is used to parametrize the lepton mixing matrix, the expressions for many phenomenological quantities can be greatly simplified. As an interesting application, we consider the neutrino mixing probabilities for phase-averaged propagation with oscillation phase \((\Delta m^2)/4E \gg 1\). To simplify our discussion, we use the results given in [16] in which more details can be found. Because in [16] the neutrino mixing probabilities are expressed in terms of a set of parameters different from those in trimaximal-Cabibbo mixing, one needs to find first the relations between them. Substituting Eq. (22) into the formalism in [16] one finds that the parameters we need can be written as

\[
e_{21} \simeq -\frac{5\lambda^2}{4\sqrt{2}}, \quad e_{32} \simeq \frac{\lambda}{2} \cos \varphi, \quad e_{13} \simeq \frac{\lambda}{\sqrt{2}},
\]

(24)

where \(\lambda\) and \(\varphi\) are the two parameters in trimaximal-Cabibbo mixing. In addition, one also needs the Dirac CP phase in the standard parametrization which is denoted by \(\varphi_D\). From Eq. (22), one can derive the relation between \(\varphi_D\) and \(\varphi\), which is given by

\[
\cos \varphi_D \simeq \cos \varphi - \frac{1}{2} \lambda^2 \cos \varphi \sin^2 \varphi.
\]

For phenomenological applications, the second term can be ignored since it is roughly of order \(O(\lambda^3)\) and, hence, one may set \(\varphi_D = \varphi\). From Eq. (24) and the results given in [16], one finds that flavor mixing probabilities can be expressed in terms of \(\epsilon \equiv \lambda \cos \varphi\), i.e.

\[
\begin{align*}
P_{\nu_e\leftrightarrow\nu_e} &= 5/9, \quad P_{\nu_\mu\leftrightarrow\nu_\mu} = (7 + 2\epsilon + 3\epsilon^2) / 18, \quad P_{\nu_\tau\leftrightarrow\nu_\tau} = (7 - 2\epsilon + 3\epsilon^2) / 18, \\
P_{\nu_e\leftrightarrow\nu_\mu} &= (2 - \epsilon) / 9, \quad P_{\nu_e\leftrightarrow\nu_\tau} = (2 + \epsilon) / 9, \quad P_{\nu_\mu\leftrightarrow\nu_\tau} = (7 - 3\epsilon^2) / 18.
\end{align*}
\]

The ratio \(\Phi_\mu/\Phi_\tau\) of the \(\nu_\mu\) flux to the \(\nu_\tau\) flux arriving at earth, which measures the violation of \(\mu - \tau\) symmetry, can be written as

\[
\Phi_\mu/\Phi_\tau = 1 + 26\epsilon^2 / 9.
\]

Comparing with [16], one finds that the expressions for mixing probabilities are considerably simplified. For more details about mixing probabilities, see [16] and references therein.
Before ending this paper, we briefly discuss possible theoretical explanations for trimaximal-Cabibbo mixing. As discussed in Sec. IV, one may begin with a trimaximal-mixing model. For example, consider the one proposed in [26]. It is shown that an $A_4$ model with a $1'$ (and/or a $1''$) flavon can lead to an effective neutrino mass matrix given by

$$ M_\nu^0 = U^T_{\text{TBM}} \begin{pmatrix} a + c - \frac{d}{2} & 0 & \frac{\sqrt{3} d}{2} \\ 0 & a + 3b + c + d & 0 \\ \frac{\sqrt{3} d}{2} & 0 & a - c + \frac{d}{2} \end{pmatrix} \tilde{U}_{\text{TBM}} \quad (25) $$

where $a$, $b$, $c$, and $d$ depend on model parameters. One can show that $M_\nu^0$ leads to TM$_2$ trimaximal mixing with

$$ \begin{pmatrix} |(U_\nu)e_2| \\ |(U_\nu)\mu_2| \\ |(U_\nu)\tau_2| \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}. $$

Based on this model, one may introduce higher order corrections or additional contributions to produce the trimaximal-Cabibbo mixing given by Eq. (22).

Before we proceed, one can compare $M_\nu^0$ with the effective neutrino mass matrix corresponding to trimaximal-Cabibbo mixing, which is given by

$$ M^\text{TMC}_\nu \simeq U^T_{\text{TBM}} \begin{pmatrix} m_1 + \frac{3}{4} \Delta_{31} \lambda^2 & -\frac{3\sqrt{2}}{4} \Delta_{21} \lambda^2 & \frac{\sqrt{3}}{2} \Delta_{31} \\ -\frac{3\sqrt{2}}{4} \Delta_{21} \lambda^2 & m_2 & 0 \\ \frac{\sqrt{3}}{2} \Delta_{31} & 0 & m_3 - \frac{3}{4} \Delta_{31} \lambda^2 \end{pmatrix} \tilde{U}_{\text{TBM}} \quad (27) $$

where $m_i$ are neutrino masses and $\Delta_{ij} \equiv m_i - m_j$. It can also be written as

$$ M^\text{TMC}_\nu = U^T_{\text{TBM}} \begin{pmatrix} a + c - \frac{d}{2} + \frac{e}{3} & -\frac{\sqrt{2}}{3} e & \frac{\sqrt{3}}{2} d \\ -\frac{\sqrt{2}}{3} e & a + 3b + c + d + \frac{2e}{3} & 0 \\ \frac{\sqrt{3}}{2} d & 0 & a - c + \frac{d}{2} - e \end{pmatrix} \tilde{U}_{\text{TBM}} \quad (28) $$

where $a$, $b$, $c$, $d$, and $e$ can be determined by comparing the two equations above. One can show that $M^\text{TMC}_\nu$ can be decomposed as

$$ M^\text{TMC}_\nu = M^0_\nu + M^1_\nu = M^0_\nu + e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (29) $$
where $M^0_{\nu}$ is the mass matrix given by Eq. (25) or Eq. (26). Note that a vanishing CP phase is assumed for simplicity and the second term in Eq. (29), i.e., $M^1_{\nu}$, can be replaced by a combination of $(1 \ 0 \ 0)$, $(0 \ 1 \ 0)$, and $(0 \ 0 \ 1)$, e.g. $(0 \ 1 \ 1)$, $(1 \ 0 \ 0)$, or $(0 \ 0 \ 0)$.

Now one can see that, in the $A_4$ model discussed above, if certain additional contributions can be introduced to account for the difference between $M^0_{\nu}$ and $M^{\text{TMC}}_{\nu}$, trimaximal-Cabibbo mixing can then be obtained from the latter. For instance, from Eq. (29) it follows that $M^{\text{TMC}}_{\nu}$ can be produced by introducing higher order corrections that contribute dominantly to the $(2, 3)$ element of the effective neutrino mass matrix. On the other hand, by comparing Eq. (28) with Eq. (25), one finds that it can also be generated by an additional nonvanishing $(1, 2)$ mass term in the TBM basis, which can be obtained by, e.g. adding Higgs triplets (see, e.g. [27]). Nevertheless, in both cases, to suppress other possible contributions, a certain amount of fine-tuning might be necessary. It would be interesting to see a more concrete model that can lead to trimaximal-Cabibbo mixing naturally. We leave that for future work.

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