New definite integrals and a two-term dilogarithm identity

F. M. S. Lima

Institute of Physics, University of Brasília, P.O. Box 04455, 70919-970, Brasília, DF, Brazil

Abstract
Among the several proofs known for \( \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6 \), the one by Beukers, Calabi, and Kolk involves the evaluation of \( \int_0^1 \int_0^1 \frac{1}{(1-x^2 y^2)} \, dx \, dy \). It starts by showing that this double integral is equivalent to \( \frac{1}{2} \sum_{n=1}^{\infty} 1/n^2 \), and then a non-trivial trigonometric change of variables is applied which transforms that integral into \( \int_T 1 \, du \, dv \), where \( T \) is a triangular domain whose area is simply \( \pi^2/8 \). Here in this note, I introduce a hyperbolic version of this change of variables and, by applying it to the above integral, I find exact closed-form expressions for \( \int_0^\infty [\sinh^{-1} (\cosh u) - u] \, du \), \( \int_0^\infty [u - \cosh^{-1} (\sinh u)] \, du \), and \( \int_0^\infty \ln (\tanh u) \, du \), where \( \alpha = \sinh^{-1}(1) \). From the latter integral, I also derive a two-term dilogarithm identity.

Keywords: Euler sums, Riemann zeta function, Double integrals, Hyperbolic functions, Dilogarithm identities
2010 MSC: 40C10, 11M06, 33B30

1. Introduction
The Riemann zeta function \( \zeta(s) \) is defined, for complex values of \( s \), \( \Re(s) > 1 \), by \( \zeta(s) := \sum_{n=1}^{\infty} 1/n^s \). For integer values of \( s \), \( s > 1 \), the first zeta value is \( \zeta(2) = \sum_{n=1}^{\infty} 1/n^2 \). Euler was the first to estimate the numerical value of this series to more than 5 decimal places (1735), as well as to determine an exact closed-form expression for it (1740), as given by \( \frac{\pi^2}{6} \):

\textbf{Lemma 1 (Euler’s result for} \( \zeta(2) \)). \( \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6 \).

From Euler’s time onward, this result has been proved in several forms, from elementary to complex ones \( \boxed{2} \). Among these proofs, at least two involve the evaluation of a double integral over the unit square and these are the ones in

\[ \textit{Email address: fabio@fis.unb.br} \] (F. M. S. Lima)
which we are interested here. Let us briefly describe these two proofs for a
better exposition of the hyperbolic approach and its consequences.

**Proof (Apostol’s proof).** The short proof by Apostol (1983) can be found
in Ref. [4]. He starts by expanding the integrand of Beukers’ integral
$I := \int_0^1 \int_0^1 \frac{1}{1 - x y} \, d x \, d y$ as a geometric series, in order to show that it is equivalent to

$$\int_0^1 \int_0^1 (1 + x y + x^2 y^2 + \ldots) \, d x \, d y = \int_0^1 \left(1 + \frac{y}{2} + \frac{y^2}{3} + \ldots\right) \, d y = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

the interchange of integrals and sums being justifiable by the fact that each inte-
grand is nonnegative and the sums converge absolutely (see the only corollary of
Theorem 7.16 in Ref. [5]).

By applying a simple linear change of variables cor-
responding to the rotation of the coordinate axes through the an-
gle $\pi/4$ radians,

$$x = \frac{u - v}{\sqrt{2}} \quad \text{and} \quad y = \frac{u + v}{\sqrt{2}},$$

directly to the Beukers’ integral, one finds [4, 6]:

$$I = 4 \int_0^{\sqrt{2}/2} \frac{\arctan \left( \frac{u}{\sqrt{2} - u^2} \right)}{\sqrt{2} - u^2} \, d u + 4 \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{\arctan \left( \frac{\sqrt{2} - u}{\sqrt{2} - u^2} \right)}{\sqrt{2} - u^2} \, d u.$$

Now, the substitution $u = \sqrt{2} \sin \theta$, followed by the use of the trigonometric
identity $\arctan \left( \frac{\sec \theta - \tan \theta}{1 - \tan \theta} \right) = \pi/4 - \theta/2$ on the second integral, yields $I = \pi^2/9 + \pi^2/9 = \pi^2/6$.

The other proof, in which we are more interested here, was given by Beukers,
Calabi, and Kolk (1993) [7].

**Proof (BCK proof).** Similarly to Apostol’s proof, Beukers, Calabi, and Kolk
(BCK) start by showing that

$$K := \int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} \, d x \, d y = \int_0^1 \int_{n=0}^{\infty} \frac{1}{2} \sum_{n=0}^{\infty} x^{2n} y^{2n} \, d x \, d y$$

$$= \sum_{n=0}^{\infty} \int_0^1 \int_{n=0}^{\infty} x^{2n} y^{2n} \, d x \, d y = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}. \quad (1)$$

Again, as each integrand is nonnegative the absolute convergence allows the
interchange of the sum and the integral. Since $\sum_{n=1}^{\infty} 1/2n^2 = 1/4 \sum_{n=1}^{\infty} 1/n^2$,
then $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. They then evaluate the unit square integral that defines $K$ by applying the following non-trivial, trigonometric change of variables [7]:

$$x = \frac{\sin u}{\cos v}, \quad y = \frac{\sin v}{\cos u}.$$ 

For this change of variables, one easily finds that $|J| = 1 - \tan^2 u \tan^2 v = 1 - x^2 y^2$, which reduces the integral for $K$ to $\int_0^\pi \int_0^\pi 1 \, du \, dv$ where $T$ is the domain in the $uv$-plane defined by $\{(u, v) : u \geq 0, v \geq 0, u + v \leq \pi/2\}$ [2, 6, 7]. This double integral exactly evaluates to $\frac{\pi^2}{8}$, i.e. the area of the triangle corresponding to $T$, which implies Lemma 1. □

Here in this work, I introduce a hyperbolic version of BCK change of variables for double integrals over the unit square. By applying this new change of variables to $\int_0^1 \int_0^1 1/(1 - x^2 y^2) \, dx \, dy$, I derive exact closed-form expressions for some definite integrals. From one of these integrals, I deduce a closed-form expression for $\text{Li}_2(\sqrt{2} - 1) + \text{Li}_2(1 - \sqrt{2}/2)$.

2. A hyperbolic version of BCK change of variables

Let us introduce the following hyperbolic version of BCK change of variables for double integrals over the unit square:

$$x = \frac{\sinh u}{\cosh v} \quad \text{and} \quad y = \frac{\sinh v}{\cosh u}. \quad (2)$$

The corresponding Jacobi determinant is:

$$J = \begin{vmatrix} \cosh u & -\sinh u \sinh v \\ \cosh v & \cosh^2 v \\ -\sinh u \sinh v & \cosh v \\ \cosh^2 u & \cosh u \end{vmatrix} = 1 - \tanh^2 u \tanh^2 v. $$

As $0 \leq x y \leq 1$ in the unit square, then $1 - x^2 y^2 \geq 0$, then $|J|$ is also equal to $1 - \tanh^2 u \tanh^2 v$. Therefore

$$\int_0^1 \int_0^1 F(x, y) \, dx \, dy = \int_S \int F(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv$$

$$= \int_S \int \tilde{F}(u, v) \left(1 - \tanh^2 u \tanh^2 v\right) \, du \, dv, \quad (3)$$

where $S$ is the cusped hyperbolic ‘quadrilateral’ on the right-hand side of Fig. 4. We are assuming that this change of variables is a $C^1$ diffeomorphism, which is reasonable since the functions $x = x(u, v)$ and $y = y(u, v)$ given in Eq. (2), as well as

$$u = u(x, y) = \cosh^{-1} \sqrt{\frac{1 + x^2}{1 - x^2 y^2}}$$

3
\[ v = v(x, y) = \cosh^{-1} \sqrt{\frac{1 + y^2}{1 - x^2 y^2}}, \]

which define the inverse change are continuously differentiable in all points of their domains (see the hachured regions in Fig. 1), the only exception being point \((x, y) = (1, 1)\). As indicated there in Fig. 1 \(S\) is bounded by the axes \(u\) and \(v\), the curve \(v = f(u) = \sinh^{-1} (\cosh u)\), above the diagonal \(v = u\), and the curve \(v = g(u) = \cosh^{-1} (\sinh u)\), below the diagonal. The parameter \(\alpha\) can be readily evaluated by noting that it is the abscissa of the point where \(v = g(u)\) intersects the \(u\)-axis. There in that point, one has \(\cosh^{-1}(\sinh \alpha) = 0\), thus \(\sinh \alpha = 1\). Since \(\sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right)\), \(\forall x \in \mathbb{R}\), then

\[ \alpha = \sinh^{-1}(1) = \ln \left( 1 + \sqrt{2} \right). \]  

(4)

3. New definite integrals and a dilogarithm identity

Let us now apply our hyperbolic change of variables to the unit square integral \(K\), as defined in Eq. 1.

**Theorem 1 (First definite integral).**

\[ \int_0^\infty \left[ \sinh^{-1} (\cosh u) - u \right] \, du = \frac{\pi^2}{16}. \]

**Proof.** From the BCK proof, one knows that \(\int_0^1 \frac{1}{(1 - x^2 y^2)} \, dx \, dy = \frac{\pi^2}{8}\). By combining this result with (3), one has

\[ \frac{\pi^2}{8} = \int \int_{S} \frac{1}{1 - \tanh^2 u \tanh^2 v} \left( 1 - \tanh^2 u \tanh^2 v \right) \, du \, dv \]

\[ = \int \int_{S} 1 \, du \, dv = \text{area of } S. \]

The area of the domain \(S\) can be equalized to a single definite integral by noting that \(S\) is symmetric with respect to the diagonal \(v = u\) (see Fig. 1). Then, by taking only the upper half of \(S\) into account, one has

\[ \text{area of } S = 2 \int_0^\infty \int_{u}^{f(u)} 1 \, dv \, du = 2 \int_0^\infty \left[ \sinh^{-1} (\cosh u) - u \right] \, du, \]

which leads to the desired result. \(\square\)

\(\footnote{Note that the real function \(\sinh^{-1} x\) is the inverse of the \textit{bijective} function \(\sinh x\) for all \(x \in \mathbb{R}\), as usual. However, since the function \(\cosh x\) is not bijective over the entire real domain, we can reduce its domain to the nonnegative reals and then \textit{define} its inverse as \(\cosh^{-1} x := \ln \left( x + \sqrt{x^2 - 1} \right)\), \(\forall x \geq 1\).}
Theorem 2 (Second definite integral).

\[ \int_{\alpha}^{\infty} \left[ u - \cosh^{-1} (\sinh u) \right] \, du = \frac{\pi^2}{16} - \frac{1}{2} \ln^2 \left( 1 + \sqrt{2} \right). \]

Proof. Similarly to the previous proof, by taking the lower half of \( S \) into account and noting that it is bounded by the \( u \)-axis, the diagonal \( v = u \), and the curve \( v = g(u) \), as seen in Fig. 1, one has, from Eq. (3), that:

\[
\text{area of } S = 2 \left[ \frac{\alpha^2}{2} + \int_{\alpha}^{\infty} \int_{g(u)}^{u} 1 \, dv \, du \right] = \alpha^2 + 2 \int_{\alpha}^{\infty} \left[ u - \cosh^{-1} (\sinh u) \right] \, du.
\]

Again, since the area of \( S \) is \( \frac{\pi^2}{8} \), one finds

\[ \int_{\alpha}^{\infty} \left[ u - \cosh^{-1} (\sinh u) \right] \, du = \frac{1}{2} \left( \frac{\pi^2}{8} - \alpha^2 \right) = \frac{\pi^2}{16} - \frac{\alpha^2}{2}. \]

\[ \square \]

Theorem 3 (Third definite integral).

\[ \int_{\alpha/2}^{\infty} \ln (\tanh z) \, dz = \frac{1}{4} \ln^2 \left( 1 + \sqrt{2} \right) - \frac{\pi^2}{16}. \]

Proof. From Theorem 1, we know that the area of \( S \) evaluates to

\[ 2 \int_{0}^{\alpha/2} \int_{u}^{\infty} 1 \, dv \, du = \frac{\pi^2}{8}, \]

where \( f(u) = \sinh^{-1} (\cosh u) \). Let us rotate the coordinate axes by the angle \( \pi/4 \) rad, as in Apostol’s proof. Since “pure” rotations about the origin always yield an unitary Jacobi determinant, one has:

\[
\frac{\pi^2}{8} = 2 \int_{0}^{\alpha/\sqrt{2}} \int_{0}^{X} 1 \, dY \, dX + 2 \int_{\alpha/\sqrt{2}}^{\infty} \int_{0}^{h(X)} 1 \, dY \, dX
\]

\[ = \alpha^2 \frac{1}{2} + 2 \int_{\alpha/\sqrt{2}}^{\infty} h(X) \, dX, \tag{5} \]

where

\[ u = \frac{X - Y}{\sqrt{2}}, \quad v = \frac{X + Y}{\sqrt{2}}, \]

and \( h(X) \) is determined by changing the variables \( u \) and \( v \) on the equation \( v = f(u) \) to the new variables \( X \) and \( Y \), as follows. Since \( v = f(u) = \sinh^{-1} (\cosh u) \), then \( \sinh v = \cosh u \), so

\[ \sinh \left( \frac{X + Y}{\sqrt{2}} \right) = \cosh \left( \frac{X - Y}{\sqrt{2}} \right). \]
By making use of the addition and subtraction formulas for hyperbolic functions, as well as of the substitutions

\[ X' := \frac{X}{\sqrt{2}} \quad \text{and} \quad Y' := \frac{Y}{\sqrt{2}}, \]

the above equality expands to

\[ \sinh X' \cosh Y' + \sinh Y' \cosh X' = \cosh X' \cosh Y' - \sinh X' \sinh Y'. \]

By dividing both sides by \( \cosh X' \cosh Y' \), one finds

\[ \tanh X' + \tanh Y' = 1 - \tanh X' \cdot \tanh Y', \]

which can be written as

\[ \tanh Y' = \frac{1 - \tanh X'}{1 + \tanh X'}. \]  \hspace{1cm} (6)

Now, for simplicity, let us define \( t := \tanh X' \). After some algebra, Eq. (6) becomes

\[ Y' = \frac{1}{2} \ln \left( \frac{1 + t}{1 - t} \right) = \frac{1}{2} \ln \left( \frac{1}{t} \right) = -\frac{1}{2} \ln t, \]

which means that

\[ Y = h(X) = -\frac{\sqrt{2}}{2} \ln \left( \tanh \left( \frac{X}{\sqrt{2}} \right) \right). \]

By putting this function on the last integral of Eq. (5), above, one has

\[ \frac{\pi^2}{8} = \frac{\alpha^2}{2} - \sqrt{2} \int_{\alpha/\sqrt{2}}^{\infty} \ln \left( \tanh \left( \frac{X}{\sqrt{2}} \right) \right) dX. \]  \hspace{1cm} (7)

The simple substitution \( z = X/\sqrt{2} \) promptly reduces this equation to

\[ \int_{\alpha/2}^{\infty} \ln (\tanh z) \, dz = \frac{\alpha^2}{4} - \frac{\pi^2}{16}. \]

After this proof, we make the substitution \( t = \tanh z \) in the integral in Eq. (7), above. When the result is expressed in terms of the dilogarithm function, an interesting identity arises.

**Theorem 4 (Dilogarithm identity).** Let us define \( \text{Li}_2(z) := \sum_{k=1}^{\infty} z^k/k^2 \), which is valid for all complex \( z \) with \( |z| \leq 1 \). Then

\[ \text{Li}_2 \left( \sqrt{2} - 1 \right) + \text{Li}_2 \left( 1 - \frac{1}{\sqrt{2}} \right) = \frac{\pi^2}{8} - \frac{\ln^2(1 + \sqrt{2})}{2} - \frac{1}{8} \ln^2 2. \]
Proof. By applying the substitution \( t = \tanh z \) \((dt = \text{sech}^2 z \, dz)\) to the definite integral of Theorem 3, in the previous section, one finds

\[
\frac{\alpha^2}{4} - \frac{\pi^2}{16} = \int_{\tanh(\alpha/2)}^{1} \frac{\ln t}{1 - t^2} \, dt = \int_{\sqrt{2} - 1}^{1} \frac{\ln t}{1 - t^2} \, dt,
\]

since

\[
\tanh\left(\frac{\alpha}{2}\right) = \tanh\left(\ln \sqrt{1 + \sqrt{2}}\right) = \frac{\sqrt{2} - 1}{\sqrt{1 + \sqrt{2}}} = \sqrt{2} - 1.
\]

By expanding the integrand in Eq. (8) in partial fractions, one has

\[
\frac{\alpha^2}{4} - \frac{\pi^2}{16} = \frac{1}{2} \int_{1}^{\sqrt{2} - 1} \frac{\ln t}{1 + t} \, dt - \frac{1}{2} \int_{1}^{\sqrt{2} - 1} \frac{\ln t}{1 - t} \, dt.
\]

These integrals can be reduced to special values of the dilogarithm function if one adopts the more general integral definition for this function \([8]\), namely

\[
\text{Li}_2(z) := -\int_{0}^{z} \frac{\ln(1-s)}{s} \, ds,
\]

valid for \( z \in \mathbb{C}\setminus[1, \infty) \). Note that, by expanding the logarithm in powers of \( z \) in this integral definition, one finds \( \text{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2 \), which agrees with the basic definition stated in the theorem for all complex \( z \) with \( |z| \leq 1 \) [8]. By taking into account this summation form, it is clear that \( \text{Li}_2(1) = \pi^2/6 \), which implies that

\[
\text{Li}_2(z) = \frac{\pi^2}{6} - \int_{1}^{z} \frac{\ln(1-s)}{s} \, ds.
\]

From this integral form, it is easy to deduce that

\[
\int_{1}^{z} \frac{\ln t}{1 - t} \, dt = \text{Li}_2(1 - z)
\]

and

\[
\int_{z}^{1} \frac{\ln t}{1 + t} \, dt = -\text{Li}_2(-z) - \ln z \ln (z + 1) - \frac{\pi^2}{12}.
\]

By putting \( z = \sqrt{2} - 1 \) in these integrals and then substituting the resulting dilogarithms in Eq. (9), one finds

\[
\frac{\alpha^2}{4} - \frac{\pi^2}{16} = \frac{1}{2} \left[ -\text{Li}_2\left(1 - \sqrt{2}\right) - \ln\left(\sqrt{2} - 1\right) \ln \sqrt{2} - \frac{\pi^2}{12} \right] - \frac{1}{2} \text{Li}_2\left(2 - \sqrt{2}\right)
\]

\[
= -\frac{\text{Li}_2\left(1 - \sqrt{2}\right)}{2} - \frac{\text{Li}_2\left(2 - \sqrt{2}\right)}{2} - \frac{1}{4} \ln\left(\sqrt{2} - 1\right) \ln 2 - \frac{\pi^2}{24}.
\]

\[4\text{We are considering here the principal branch of the dilogarithm function } \text{Li}_2(z), \text{ defined by taking the principal branch of } \ln z, \text{ for which it has a cut along the negative real axis, with } |\arg(z)| < \pi. \text{ This defines the principal branch of } \text{Li}_2(z) \text{ as a single-valued function in the complex plane cut along the real axis, from } 1 \text{ to } +\infty.]
Now, by applying the Euler reflection formula for \( \text{Li}_2(z) \) about \( z = \frac{1}{2} \), which reads (see Eq. (1.11) in Ref. [8])

\[
\text{Li}_2(1 - z) = -\text{Li}_2(z) - \ln z \ln (1 - z) + \frac{\pi^2}{6},
\]
to \( \text{Li}_2(2 - \sqrt{2}) \) in Eq. (10), one finds

\[
\text{Li}_2\left(\sqrt{2} - 1\right) - \text{Li}_2\left(1 - \sqrt{2}\right) = \frac{\alpha^2}{2} - \frac{\pi^2}{8} + \frac{\pi^2}{4} - \ln\left(\sqrt{2} - 1\right). \tag{11}
\]

By applying the Landen’s formula, namely (see Eq. (1.12) in Ref. [8])

\[
\text{Li}_2(z) = -\text{Li}_2\left(\frac{z}{z - 1}\right) - \frac{1}{2} \ln^2 (1 - z),
\]
which is valid for all complex \( z, \ z \notin [1, \infty) \), to \( \text{Li}_2(1 - \sqrt{2}) \), one has

\[
\text{Li}_2\left(1 - \sqrt{2}\right) = -\text{Li}_2\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{8} \ln^2 2.
\]

Finally, by substituting this result on Eq. (11), one finds

\[
\text{Li}_2\left(\sqrt{2} - 1\right) + \text{Li}_2\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{8} - \frac{\alpha^2}{2} - \frac{1}{8} \ln^2 2.
\]

\[\square\]

It should be mentioned that neither Mathematica (release 7) nor Maple (release 13) are able to generate any closed-form expression for this sum. Indeed, this identity is not found in standard bibliographic sources for dilogarithms [8, 9], though it resembles the identity

\[
\mathcal{L}\left(\sqrt{2} - 1\right) + \mathcal{L}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{3}{4},
\]

where \( \mathcal{L}(x) := \frac{6}{\pi^2} \left[ \text{Li}_2(x) + \frac{1}{2} \ln x \ln (1 - x) \right] \) is the normalized Roger’s dilogarithm, as found by Bytsko (1999) [10].

It seems plausible to extend the results obtained here in this paper to other known unit square integrals, such as

\[
\int_0^1 \int_0^1 \ln(xy)/(1 - x^2y^2) \, dx \, dy = -\frac{3}{4} \zeta(3),
\]

\[
\int_0^1 \ln x/(1 - x^2y^2) \, dx \, dy = -\frac{3}{8} \zeta(3),
\]

\[
\int_0^1 \ln(1 - xy)/(1 - xy) \, dx \, dy = -\zeta(3),
\]

and many others (see, e.g., Ref. [11]). Similarly, it should also be interesting to include a third dimension in the hyperbolic change of variables in order to investigate some triple integrals over the unit cube, such as

\[
\int_0^1 \int_0^1 \frac{1}{1 - xyz} \, dx \, dy \, dz = \zeta(3) \quad \text{and} \quad \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - x^2y^2z^2} \, dx \, dy \, dz = \frac{7}{8} \zeta(3).
\]

Extensive work in both directions is already ongoing by the author.
Acknowledgments

The author acknowledges a postdoctoral fellowship from CNPq (Brazilian agency) during the course of this work.

References

[1] L. Euler, De summis serierum reciprocarum, Opera Omnia 14 (1740) 73–86.

[2] D. Kalman, Six ways to sum a series, Coll. Math. J. 24 (1993) 402–421.

[3] R. Chapman, Evaluating $\zeta(2)$. Available at:
\protect\vrule width0pt\protect\href{http://secamlocal.ex.ac.uk/people/staff/rjchapma/etc}{

[4] T. M. Apostol, A proof that Euler missed: evaluating $\zeta(2)$ the easy way, Math. Intelligencer 5:3 (1983) 59–60.

[5] W. Rudin, Principles of Mathematical Analysis, 3rd ed., McGraw-Hill, New York, 1976.

[6] M. Aigner and G. M. Ziegler, Proofs from THE BOOK, 4th ed., Springer, New York, 2010. Chap. 8.

[7] F. Beukers, J. A. C. Kolk, and E. Calabi, Sums of generalized harmonic series and volumes, Nieuw Arch. Wiskd. 11 (1993) 217–224.

[8] L. Lewin, Polylogarithms and Associated Functions, Elsevier, New York, 1981.

[9] D. Zagier, The Dilogarithm Function. In: Frontiers in Number Theory, Physics, and Geometry II, Ed. by P. Cartier, P. Moussa, B. Julia, and P. Vanhove. Springer, New York, 2007. Pages 3–61.

[10] A. G. Bytsko, Two-term dilogarithm identities related to conformal field theory, Lett. Math. Phys. 50 (1999) 213–228.

[11] E. W. Weisstein, Unit square integrals. From MathWorld, a Wolfram Web resource, available at:
\protect\vrule width0pt\protect\href{http://mathworld.wolfram.com/UnitSquareIntegral.html}{

9
Figure 1: Mapping the unit square in the $xy$-plane (left-hand side) onto the hyperbolic ‘quadrilateral’ in the $uv$-plane (right-hand side). The symmetry about the diagonal $v = u$, represented by the inclined dashed line, is exploited in the text. Note that the hyperbolic region formed in the $uv$-plane is bounded by the coordinate axes and the curves $v = f(u)$ and $v = g(u)$. Both these functions, as well as the expressions for the change of variables indicated in the central part, are stated elsewhere in the text. According to Eq. (4), $\alpha$ evaluates to $\sinh^{-1}(1) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$. 