ON THE HILBERT VECTOR OF THE JACOBIAN MODULE OF A PLANE CURVE

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Abstract. We identify several classes of complex projective plane curves $C : f = 0$, for which the Hilbert vector of the Jacobian module $N(f)$ can be completely determined, namely the 3-syzygy curves, the maximal Tjurina curves and the nodal curves, having only rational irreducible components. A result due to Hartshorne, on the cohomology of some rank 2 vector bundles on $\mathbb{P}^2$, is used to get a sharp lower bound for the initial degree of the Jacobian module $N(f)$, under a semistability condition.

Introduction

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in three variables $x, y, z$ with complex coefficients. Let $C : f = 0$ be a reduced curve of degree $d$ in the complex projective plane $\mathbb{P}^2$. We denote by $J_f = (f_x, f_y, f_z)$ the Jacobian ideal, i.e. the homogeneous ideal in $S$ spanned by the partial derivatives $f_x, f_y, f_z$ of $f$. Since $C$ is reduced, the singular subscheme of $C$, which is defined by the Jacobian ideal $J_f$, is 0-dimensional, and its degree is denoted by $\tau(C)$, and is called the global Tjurina number of $C$. Consider the graded $S$–module of Jacobian syzygies of $f$, namely

$$\text{Syz}(J_f) = \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}.$$ 

Let $\text{mdr}(f) := \min\{k : \text{Syz}(J_f)_k \neq (0)\}$ be the minimal degree of a Jacobian syzygy for $f$; in this paper we will assume $\text{mdr}(f) \geq 1$ unless otherwise specified. In fact, if $\text{mdr}(f) = 0$, the curve $C$ is a pencil of lines. We say that $C : f = 0$ is a $m$–syzygy curve if the $S$–module $\text{Syz}(J_f)$ is minimally generated by $m$ homogeneous syzygies, $r_1, r_2, \ldots, r_m$, of degree $d_i = \deg r_i$, ordered such that

$$1 \leq d_1 \leq d_2 \leq \ldots \leq d_m.$$ 

The multiset $(d_1, d_2, \ldots, d_m)$ is called the exponents of the plane curve $C$ and $(r_1, r_2, \ldots, r_m)$ is said to be a minimal set of generators for the $S$–module $\text{Syz}(J_f)$. Some of the $m$–syzygy curves have been carefully studied. We recall that:

- a 2–syzygy curve $C$ is said to be free, since then the $S$–module $\text{Syz}(J_f)$ is a free module of the rank 2, see [8, 14, 27, 28, 29];
- a 3–syzygy curve is said to be nearly free when $d_3 = d_2 = d_1 + d_2 = d$, see [11, 2, 3, 5, 6, 23];
- a 3–syzygy line arrangement is said to be a plus-one generated line arrangement of level $d_3$ when $d_1 + d_2 = d$ and $d_3 \geq d_2$, see [11]. By extension, a 3–syzygy curve $C$ is said to be a plus-one generated curve of level $d_3$ when $d_1 + d_2 = d$ and $d_3 \geq d_2$, see [14].

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The Jacobian module of \( f \), or of the plane curve \( C : f = 0 \), is the quotient module \( N(f) = J_f/J_f \), with \( J_f \) the saturation of the ideal \( J_f \) with respect to the maximal ideal \( \mathfrak{m} = (x, y, z) \) in \( S \). The Jacobian module \( N(f) \) coincides with \( H^0_{\mathfrak{m}}(S/J_f) \), see [20]. Let \( n(f)_j = \dim N(f)_j \), \( T = 3(d - 2) \) and recall that the Jacobian module \( N(f) \) enjoys a weak Lefschetz type property, see [7] for this result, and [19, 20, 22] for Lefschetz properties of Artinian algebras in general. More precisely, we have

\[
\begin{align*}
1 & \quad n(f)_0 \leq n(f)_1 \leq \ldots \leq n(f)_{\lfloor \frac{d}{2} \rfloor - 1} \leq n(f)_{\lfloor \frac{d}{2} \rfloor} \geq n(f)_{\lfloor \frac{d}{2} \rfloor + 1} \geq \ldots \geq n(f)_T. 
\end{align*}
\]

We consider the following two invariants for a curve \( C : f = 0 \)

\[
\sigma(C) := \min \{ j : n(f)_j \neq 0 \} = \text{indeg}(N(f)), \quad \nu(C) := \max \{ n(f)_j \}_j.
\]

The self duality of the graded \( S \)-module \( N(f) \), see [21, 26, 30], implies that

\[
\begin{align*}
n(f)_j = n(f)_{T-j},
\end{align*}
\]

for any integer \( j \), in particular \( n(f)_k \neq 0 \) exactly for \( k = \sigma(C), \ldots, T - \sigma(C) \).

The main aim of this paper is to identify classes of curves \( C : f = 0 \) for which the Hilbert vector \( (n(f)_j) \) of the Jacobian module \( N(f) \) can be completely determined. In [13, Theorem 3.1, Theorem 3.2], recalled below in Theorem 1.3, there is a description of the dimensions \( n(f)_j \) for a certain range of \( j \). Moreover, in [13, Theorem 3.9, Corollary 3.10], recalled below in Theorem 1.3 and Corollary 1.3 there are descriptions of the minimal resolution of \( N(f) \), when \( C : f = 0 \) is a \( 3 \)-syzygy curve, and respectively a plus-one generated curve of degree \( d \geq 3 \). Using these results, we first give a general formula for the Hilbert vector \( (n(f)_j) \) of the Jacobian module of a \( 3 \)-syzygy curve in Theorem 2.1 as well as a graphic representation of its behavior. Then we determine the Hilbert vector \( (n(f)_j) \) of the Jacobian module \( N(f) \) when \( C : f = 0 \) is a maximal Tjurina curve, see Proposition 3.1. Next we get some information on the Hilbert vector \( (n(f)_j) \) when \( C : f = 0 \) is a nodal curve, which is complete if in addition all the irreducible components of \( C \) are rational, see Theorem 3.2.

In the final section we use a result due to Hartshorne, see [15, Theorem 7.4], to relate the cohomology of some rank 2 vector bundles on \( \mathbb{P}^2 \) to the Hilbert vector \( (n(f)_j) \) of the Jacobian module \( N(f) \). More precisely, we get in this way a sharp lower bound for the initial degree \( \sigma(C) \) of the Jacobian module \( N(f) \), under the condition \( \text{mdr}(f) \geq (d - 1)/2 \), see Theorem 4.2.

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1. Preliminaries

We recall some notations and results. Let \( C : f = 0 \) be a reduced complex plane curve of \( \mathbb{P}^2 \), assumed not free, and consider the Milnor algebra \( M(f) = S/J_f \), where \( J_f = (f_x, f_y, f_z) \) is the Jacobian ideal. The general form of the minimal resolution for the Milnor algebra \( M(f) \) of such a curve \( C : f = 0 \) is

\[
\begin{align*}
0 \to \bigoplus_{i=1}^{m-2} S(-e_i) \to \bigoplus_{i=1}^{m-1} S(1 - d - d_i) \to S^1(1 - d) \to S, 
\end{align*}
\]

with \( e_1 \leq e_2 \leq \ldots \leq e_{m-2} \) and \( 1 \leq d_1 \leq d_2 \leq \cdots \leq d_m \). It follows from [21, Lemma 1.1] that one has

\[
\begin{align*}
e_j = d + d_{j+2} - 1 + \epsilon_j, \quad \text{for} \; j = 1, \ldots, m - 2 \quad \text{and some integers} \; \epsilon_j \geq 1. \quad \text{The minimal resolution of} \; N(f) \; \text{obtained from [3]}, \; \text{by [21, Proposition 1.3]}, \; \text{is}
\end{align*}
\]

\[
\begin{align*}
0 \to \bigoplus_{i=1}^{m-2} S(-e_i) \to \bigoplus_{i=1}^{m-1} S(-e_i) \to \bigoplus_{i=1}^{m-1} S(d_i - 2(d - 1)) \to \bigoplus_{i=1}^{m-2} S(e_i - 3(d - 1)),
\end{align*}
\]
where $\ell_i = d + d_i - 1$. It follows that
\[ \sigma(C) = 3(d - 1) - \epsilon_{m-2} = 2(d - 1) - d_m - \epsilon_{m-2}. \]
The following result describes the central part of the Hilbert vector of $N(f)$.

**Theorem 1.1.** Let $C : f = 0$ be a reduced, non free curve of degree $d$ and set $r = \text{mdr}(f)$. Then one has the following.

(i) if $r \geq \frac{d}{2}$ and $2d - 4 - r \leq j \leq d - 2 + r$, then
\[ n(f)_j = \begin{cases} 3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) & \text{for } d = 2d' + 1 \\ 3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) & \text{for } d = 2d' \end{cases} \]

(ii) if $r < \frac{d}{2}$ and $d + r - 3 \leq j \leq 2d - r - 3$, then $n(f)_j = \nu(C)$. Moreover
\[ n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) - 1. \]

**Proof.** See [13, Theorem 3.1 and Theorem 3.2]. \(\square\)

By Theorem [11] in case (i), the points $(j, n(f)_j)$ lie on an upward pointing parabola. Moreover, using the formulas [11] and [11] and Remark 4.1, the claim (5) can be written:
\[ n(f)_j = \begin{cases} \nu(C) - (j - \lfloor \frac{T}{2} \rfloor)(j - \lfloor \frac{T}{2} \rfloor) & \text{for } d = 2d' + 1 \\ \nu(C) - (j - \lfloor \frac{T}{2} \rfloor)^2 & \text{for } d = 2d' \end{cases} \]

with $T = 3(d - 2)$ as above. On the other hand, in the case (ii), the points $(j, n(f)_j)$ lie on a horizontal line segment, with a one-unit drop at the extremities, as represented in Figure 1 below.

![Figure 1](image_url)

**Figure 1.** The case $r < \frac{d}{2}$.

Recall the following definition, see [5, 10].

**Definition 1.2.** For a plane curve $C : f = 0$, the coincidence threshold of $f$ is the integer
\[ \text{ct}(f) = \max \{ q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q \}, \]

with $f_s$ a homogeneous polynomial in $S$ of the same degree $d$ as $f$ and such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

It is known that $\text{ct}(f) \geq d - 2 + \text{mdr}(f)$.

Note that $C : f = 0$ is a free curve if and only if $\nu(C) = 0$, hence $N(f) = 0$. In other words, the Jacobian ideal is saturated in this case, i.e. $J_f = J_f$, see [8, 29]. The first nontrivial case is that of nearly free curves; indeed, by [11, Corollary 2.17], for a nearly free curve $C : f = 0$, one has $N(f) \neq 0$ and $\nu(C) = 1$. Moreover $\sigma(C) = d + d_1 - 3$ and this describes completely the Hilbert vector of the Jacobian module of a nearly free curve. In particular it has the shape described on the left side of Figure 2. Recall that a nearly free curve is exactly a plus-one generated curve with exponents satisfying $d_2 = d_3$. For the more general case of the 3−syzygy curves, we recall the following result.
Theorem 1.3. [14] Theorem 3.9] Let $C : f = 0$ be a $3$–syzygy curve with exponents $(d_1, d_2, d_3)$ and set $e = d_1 + d_2 + d_3$. Then the minimal free resolution of $N(f)$ as a graded $S$–module has the form
\[ 0 \to S(-e) \to \oplus_{i=1}^3 S(1-d - d_i) \to \oplus_{i=1}^3 S(d_i + 2 - 2d) \to S(e + 3 - 3d), \]
where the leftmost map is the same as in the resolution \[3\], when $m = 3$. In particular,
\[ \sigma(C) = 3(d-1) - (d_1 + d_2 + d_3). \]

This implies the following.

Corollary 1.4. [14] Corollary 3.10] Let $C : f = 0$ be a plus-one generated curve of degree $d \geq 3$ with $(d_1, d_2, d_3)$, which is not nearly free, i.e. $d_2 < d_3$. Set $k_j = 2d - d_j - 3$ for $j = 1, 2, 3$. Then one has the following minimal free resolution of $N(f)$ as a graded $S$–module:
\[ 0 \to S(-d - d_3) \to S(-d - d_2 + 1) \oplus S(-k_1 - 2) \oplus S(-k_2 - 2) \to \]
\[ \to S(-k_1 - 1) \oplus S(-k_2 - 1) \oplus S(-k_3 - 1) \to S(-k_3). \]

In particular $\sigma(C) = k_3 < k_2 \leq \frac{T}{2}$ and the Hilbert vector of $N(f)$ is given by following formulas:
\begin{enumerate}
  \item $n(f)_j = 0$ for $j < k_3$;
  \item $n(f)_j = j - k_3 + 1$ for $k_3 \leq j \leq k_2$;
  \item $n(f)_j = d_3 - d_2 + 1 = \nu(C)$ for $k_2 \leq j \leq \frac{T}{2}$.
\end{enumerate}

By above corollary, the Hilbert vector of the Jacobian module of a plus-one generated curve of degree $d$ and level $d_3$ has the shape given on the right hand side of Figure 2, where we have drawn only the part corresponding to $j \leq \frac{T}{2}$, due to the symmetry [2].

2. Results on the Hilbert vector of $N(f)$ for $3$–syzygy curves

As a simple example of a $3$–syzygy curve which is not a plus-one generated curve, let $C : f = 0$ be a smooth curve of degree $d \geq 3$, where $d_1 = d_2 = d_3 = d - 1$. It is known that the Hilbert function of the Milnor algebra $M(f)$ is in this case \[\left(\frac{1-t^{d-1}}{1-t}\right)^3.\] For a smooth curve we have $N(f) = M(f)$, hence $n(f)_j = \dim M(f)_j$ and the Hilbert vector of the Jacobian module $N(f)$ has the shape described in Figure 3. It is interesting to notice the change in convexity when we pass through the value $j = d - 1$.

For a general $3$–syzygy curve, we have the following result.

Theorem 2.1. Let $C : f = 0$ be a $3$–syzygy curve of degree $d$, not plus-one generated, with exponents $d_1 \leq d_2 \leq d_3$. Set $e = d_1 + d_2 + d_3$ and $k_i = 2(d - 1) - d_3$.

\[ \sigma(C) = k_3 \leq k_2 \leq \frac{T}{2} \]

Figure 2. The case of plus-one generated curves

\[ d_3 - d_2 + 1 = \nu(C) \]
for \( i = 1, 2, 3 \). Then the following hold.

\[
n(f)_k = \begin{cases} 
0 & \text{for } k < \sigma \\
\left(\frac{k-\sigma+2}{2}\right) & \text{for } \sigma \leq k < k_3 \\
\left(\frac{k-\sigma+2}{2}\right) - \left(\frac{k-k_3+2}{2}\right) & \text{for } k_3 \leq k < k_2 \\
\left(\frac{k-\sigma+2}{2}\right) - \left(\frac{k-k_3+2}{2}\right) - \left(\frac{k-k_2+2}{2}\right) & \text{for } k_2 \leq k < T_0,
\end{cases}
\]

where \( \sigma = \sigma(C) = 3(d-1) - e \) and

\[
T_0 = \begin{cases} 
k_1 - 1 & \text{if } d_1 \geq \frac{d}{2} \\
d + d_1 - 2 & \text{if } d_1 < \frac{d}{2}.
\end{cases}
\]

Note that \( n(f)_k \) is known for \( T_0 \leq k \leq \frac{T}{2} \) in view of Theorem 14, hence the information on the Hilbert vector of \( N(f) \) is complete in this situation.

**Proof.** Note that \( \sigma \geq 0 \), since, by [14, Theorem 2.4] we have \( d_j \leq d-1 \) for \( j = 1, 2, 3 \). Then, by [14, Theorem 2.3] we have \( d_1 + d_2 > d > d - 1 \) and hence

\[
\sigma = 3(d-1) - (d_1 + d_2 + d_3) < 3(d-1) - (d-1) - d_3 = 2(d-1) - d_3 = k_3.
\]

By Theorem 13, the minimal resolution of \( N(f) \) is

\[
0 \to S(-e) \to \oplus_{j=1}^{3} S(-\ell_j) \to \oplus_{i=1}^{3} S(-k_i) \to S(e - 3(d-1)),
\]

where \( \ell_j = d - 1 + d_1 \). We note that \( k_3 \leq k_2 \leq k_1 \) and also \( k_2 \leq T_0 \). If we fix \( k \) with \( \sigma \leq k < k_3 \), the minimal resolution of \( N(f) \) above yields

\[
n(f)_k = \dim S_{k-\sigma} = \left(\frac{k-\sigma+2}{2}\right).
\]

Now we consider the case \( k_3 \leq k < k_2 \). We have \( \ell_1 > k_2 \), since \( d_1 + d_2 > d \) as we have seen above. It follows that

\[
n(f)_k = \dim S_{k-\sigma} - \dim S_{k-k_3} = \left(\frac{k-\sigma+2}{2}\right) - \left(\frac{k-k_3+2}{2}\right).
\]

This difference is a linear form in \( k \), and the coefficient of \( k \) is given by \((k_3 - \sigma)\). Note that \( k_3 - \sigma = 2(d-1) - d_3 = 3(d-1) + e = d_1 + d_2 - (d-1) \geq 2 \). To continue, we need to discuss the position of \( \ell_1 \) with respect to \( k_1 \). Note that \( \ell_1 > k_1 \) if and only if \( d_1 \geq d/2 \). Hence we have to consider two cases.

**Case 1:** \( d_1 \geq d/2 \). In this case, we can compute the value \( n(f)_k \) for \( k \leq k_1 - 1 \) exactly as above, and we get

\[
n(f)_k = \left(\frac{k-\sigma+2}{2}\right) - \left(\frac{k-k_3+2}{2}\right) - \left(\frac{k-k_2+2}{2}\right).
\]
Note that in this case \( T_0 = k_1 - 1 = 2(d - 1) \) and hence all the Hilbert vector \((n(f))\) is known by using Theorem 1.1 (i).

**Case 2:** \( d_1 < d/2 \). In this case \( \ell_1 \leq k_1 \), and we can compute the value \( n(f)_{k_1} \) for \( k \leq \ell_1 - 1 \) exactly as above, obtaining the same formula. Note that in this case \( T_0 = \ell_1 - 1 = d - 2 + d_1 > d - 3 + d_1 \), and hence again all the Hilbert vector \((n(f))\) is known by using Theorem 1.1 (ii).

□

**Example 2.2.** Let \( C : f = (x^9 + y^4 z^5)^7 + x z^{62} = 0 \), a singular curve of degree \( d = 63 \). It is a 3-syzygy curve, not plus-one generated, with \( d_1 = 9 \), \( d_2 = 56 \) and \( d_3 = 62 \). We have

\[
e = \sum_{i=1}^{3} d_i = 127, \quad \sigma = 59, \quad k_3 = 62, \quad k_2 = 68.
\]

Since \( d_1 < \frac{d}{2} \), \( T_0 = d + d_1 - 2 = 70 \). The first quadratic part is for \( k \in [59, 61] \), the middle linear part is for \( k \in [62, 67] \), and the second quadratic part is for \( k \in [68, 70] \). This second quadratic part is too short, containing only 3 points \((j, n(f)_j)\), to be seen in a graphical representation of the corresponding Hilbert vector. Note also that one has \( n(f)_{k_3} = \nu(C) = 27 \) for \( k \in [69, 91] \), where \( 91 = \left\lceil T/2 \right\rceil \). In particular, the last two points on the second quadratic part are in fact situated on this horizontal line segment.

**Example 2.3.** Let \( C : f = (x + y)^2(x - y)^2(x + 2y)^2(x - 2y)^2(x + 3y)^2(x - 3y)^2(x + 4y)^2(x - 4y)^2(x + 5y)^2(x - 5y)^2 + z^{20} = 0 \), a singular curve of degree \( d = 20 \). It is a 3-syzygy curve not plus-one generated, with \( d_1 = 9 \) and \( d_2 = d_3 = 19 \). We have

\[
e = \sum_{i=1}^{3} d_i = 47, \quad \sigma = 10, \quad k_3 = k_2 = 19.
\]

Since \( d_1 < \frac{d}{2} \), \( T_0 = d + d_1 - 2 = 27 = T/2 \). The first quadratic part is for \( k \in [10, 18] \), the middle linear part is missing since \( k_2 = k_3 \), the second quadratic part is for \( k \in [19, 27] \) and one has \( n(f)_{k_3} = \nu(C) = 81 \) for \( k \in [26, 27] \), where \( 27 = T/2 = T_0 \).

![Figure 4. The Hilbert vector for Example 2.3](image)
3. Maximal Tjurina curves and nodal curves

We assume in this section that \( r = d_1 \geq d/2 \).

A reduced plane curve \( C : f = 0 \) of degree \( d \) is called a maximal Tjurina curve if the global Tjurina number \( \tau(C) \) equals the du Plessis-Wall upper bound, namely if

\[
\tau(C) = (d-1)(d-r-1) + r^2 - \left(2r - d + \frac{5}{2}\right),
\]

see [15, 16, 17]. We know that a reduced plane curve \( C : f = 0 \) of degree \( d \) is a maximal Tjurina curve if and only if one has \( d_1 = d_2 = \cdots = d_m = r \), \( e_1 = e_2 = \cdots = e_{m-2} = d + r \) and \( m = 2r - d + 3 \), see [15 Theorem 3.1]. Using now the equality (4), it follows that in this case

\[
\sigma(C) = 2d - r - 3.
\]

Theorem 1.1 yields then the following result.

**Proposition 3.1.** Let \( C : f = 0 \) be a maximal Tjurina curve of degree \( d \) with \( r = d_1 \geq d/2 \). Then the Hilbert vector of the Jacobian module \( N(f) \) is given by the following

\[
n(f)_j = \begin{cases} 
3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) & \text{for } d = 2d' + 1 \\
3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) & \text{for } d = 2d' 
\end{cases}
\]

for \( 2d - 3 < j \leq d - 3 + r \) and \( n(f)_j = 0 \) otherwise.

Consider now an arbitrary nodal curve \( C : f = 0 \) of degree \( d \) in \( \mathbb{P}^2 \). Let \( N \) denote the set of nodes of the curve \( C \) and \( n(C) \) the number of irreducible components of \( C \). For such curves we have the following result.

**Theorem 3.2.** Let \( C : f = 0 \) is a nodal curve in \( \mathbb{P}^2 \) of degree \( d \geq 4 \). Then one has the following, with \( f_s \) as in Definition 1.2

\[
n(f)_k = \begin{cases} 
m(f_s)_k - |N| & \text{for } d - 3 < k \leq T/2 \\
m(f_s)_k - |N| + n(C) - 1 & \text{for } k = d - 3. 
\end{cases}
\]

Moreover, when all the irreducible components of \( C \) are rational, one has in addition \( n(f)_k = 0 \) for \( k \leq d - 3 \).

**Proof.** For any reduced plane curve \( C : f = 0 \), one clearly has

\[
n(f)_k = m(f)_k - d(f)_k,
\]

where \( m(f)_k = \dim M(f)_k \) and \( d(f)_k = \dim S_k/(J_f)_k \). Since we have to determine \( n(f)_k \) only for \( k \leq T/2 \) by symmetry, and since \( \text{ct}(f) \geq d - 2 + r > T/2 \) when \( r = d_1 \geq d/2 \), it follows that

\[
n(f)_k = m(f_s)_k - d(f)_k,
\]

with \( f_s \) as in Definition 1.2 and \( k \leq T/2 \). In particular, for such curves, we have to determine only the values \( d(f)_k \) for \( k \leq T/2 \). On the other hand, we know that

\[
d(f)_k = \tau(C),
\]

for \( k \geq T - \text{ct}(C) \), see [4, Proposition 2]. In particular, this equality holds for \( k \geq 3(d - 2) - (d - 2 + r) = 2d - 4 - r \), see also the proof of [13 Theorem 3.1].

Assume now that \( C : f = 0 \) is a nodal curve in \( \mathbb{P}^2 \). Then \( r = d_1 \geq d - 2 \geq d/2 \) for \( d \geq 4 \), see [8 Example 2.2 (i)]. Let \( \text{def} S_k(N) \) denote the defect of the set of nodes \( N \) with respect to the linear system \( S_k \). Then it is known that

\[
d(f)_k = |N| - \text{def} S_k(N),
\]

see [4]. On the other hand, [4 Corollary 1.6] implies that \( \text{def} S_k(N) = 0 \) for \( k > d - 3 \) and \( \text{def} S_k(N) = n(C) - 1 \) for \( k = d - 3 \). If all the irreducible components of \( C \) are
rational, then [12] Theorem 2.7 shows that \( n(f)_k = 0 \) for \( k \leq d - 3 \). These facts imply our claims. \( \square \)

4. Relation to a result by Hartshorne

Let \( C : f = 0 \) be a curve of degree \( d \) in \( \mathbb{P}^2 \), and let \( r = \text{mdr}(f) \) be the minimal degree of a Jacobian syzygy for \( f \). In this section we give some informations about the invariant \( \sigma(C) \), using a result by Hartshorne, namely [15] Theorem 7.4. We recall that the sheafification of \( \text{Syz}(J_f) \), denoted by \( E_C := \text{Syz}(J_f) \), is a rank two vector bundle on \( \mathbb{P}^2 \), see [2] [25] [26]. We set

\[
e(f)_m = \dim \text{Syz}(J_f)_m = \dim H^0(\mathbb{P}^2, E_C(m)),
\]

for any integer \( m \). Associated to the vector bundle \( E_C \) there is the normalized vector bundle \( \mathcal{E}_C \), which is the twist of \( E_C \) such that \( c_1(\mathcal{E}_C) \in \{-1, 0\} \). More precisely,

**when \( d = 2d' + 1 \) is odd:**

\[
\mathcal{E}_C = E_C(d'), \quad c_1(\mathcal{E}_C) = 0, \quad c_2(\mathcal{E}_C) = 3(d')^2 - \tau(C),
\]

and

**when \( d = 2d' \) is even:**

\[
\mathcal{E}_C = E_C(d' - 1), \quad c_1(\mathcal{E}_C) = -1, \quad c_2(\mathcal{E}_C) = 3(d')^2 - 3d' + 1 - \tau(C),
\]

see [13] Section 2.

**Remark 4.1.** The vector bundle \( E_C \) is stable if and only if \( \mathcal{E}_C \) has no sections, see [24] Lemma 1.2.5. This is equivalent to \( r = \text{mdr}(f) \geq \frac{d}{2} \), see [26] Proposition 2.4. Moreover by [13] Theorem 2.2 and using the formulas \( 9 \) and \( 10 \), we have that for a stable vector \( E_C \), \( c_2(\mathcal{E}_C) = \nu(C) \). Moreover, the vector bundle \( E_C \) is semistable if and only if \( r = \text{mdr}(f) \geq (d - 1)/2 \), see again [24] Lemma 1.2.5, a condition that occurs in our Theorem 4.2 below.

The important key point is the identification

\[
H^1(C, E_C(k)) = N(f)_{k+d-1}
\]

for any integer \( k \), see [26] Proposition 2.1]. Hence the study of the Hilbert vector of the Jacobian module \( N(f) \) is equivalent to the study of the dimension of \( H^1(C, E_C(k)) \).

**Theorem 4.2.** Let \( C : f = 0 \) be a curve of degree \( d \), and let \( r = \text{mdr}(f) \) be the minimal degree of a Jacobian syzygy for \( f \). Assume that \( r \geq (d - 1)/2 \), in other words that the rank 2 vector bundle \( E_C \) is semistable. Then we have the following.

1. If \( d = 2d' + 1 \) is odd, then
   \[
   \sigma(C) \geq \tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 1.
   \]

2. If \( d = 2d' \) is even, then
   \[
   \sigma(C) \geq \tau(C) - 2(d')^2 - 2rd' + r^2 + 5d' + r - 3.
   \]

The above inequalities are sharp, in particular they are equalities when \( C \) is a maximal Tjurina curve with \( r \geq d/2 \).

**Proof.** We discuss only the case \( d = 2d' + 1 \), the other case being completely similar. One has

\[
n(f)_k = h^1(\mathbb{P}^2, \mathcal{E}_C(k - 3d')).
\]
Moreover \( h^0(P^2, E_C'(t)) = h^0(P^2, E_C(t + d')) \neq 0 \) if and only if \( t + d' \geq r \). Hence the minimal \( t \) satisfying this condition is \( t_m = r - d' \geq 0 \). Then [18, Theorem 7.4] implies that \( n(f)_k = 0 \) when
\[
k - 3d' \leq -c_2(E) + t_m^2 - 2.
\]
Using the formula for \( t_m \) above, and the formula for \( c_2(E) \) given in the equations (8), we get that \( n(f)_k = 0 \) when
\[
k \leq \tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 2,
\]
which clearly implies our claim (1). The fact that the inequality in (1) is in fact an equality when \( C \) is a maximal Tjurina curve with \( r \geq d/2 \) follows by a direct computation. Indeed, using the above definition of a maximal Tjurina curve of degree \( d = 2d' + 1 \), namely the equality (7), we see that
\[
\tau(C) = 2(d')^2 + 2rd' - r^2 - r + d'.
\]
Hence
\[
\tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 1 = 2d - r - 3 = \sigma(C),
\]
where the last equality follows from (8).

**Example 4.3.** Let \( C : f = 0 \) be a curve of degree \( d = 2d' + 1 \), having a unique node as singularities. Then it is known that \( r = d - 1 = 2d' \), and \( \tau(C) = \sigma(C) = 1 \).

The inequality in Theorem 4.2 (1) is in this case
\[
1 \geq d'(3 - 2d'),
\]
and hence the two terms in this inequality can be far apart in some cases.

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