Decomposing a Matrix into Two Submatrices with Extremely Small Operator Norm

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The paper deals with estimates of operator norms of submatrices. At present, this field of research is being actively developed and finds various applications. The paper continues the study in [1], where the case of $(2, 1)$-norm was considered. For matrices orthogonal with respect to columns, this case was previously studied in [2], where it was shown that there exists an analog of partition (2) (see below) with extremely small $(2, 1)$-norms of the corresponding submatrices. In what follows, we significantly strengthen Statement 4 in [1] and generalize Statement 3 to the case of $(X, q)$-norm with $1 ≤ q < ∞$; the case of $(1, q)$-norm is considered separately.

Given an $N × n$ matrix $A$ considered as an operator from $l^p$ to $l^q$, we define its $(p, q)$-norm as

$$\|A\|(p, q) = \sup_{\|x\|_p ≤ 1} \|Ax\|_q, \quad 1 ≤ p, q ≤ ∞.$$  (1)

Statement 2 is proved below in the case of the more general $(X, q)$-norm, where $X$ is an arbitrary $n$-dimensional normed space.

We use the following notation:

- $\text{rk}(A)$ is the rank of a matrix $A$;
- $\langle N \rangle$ is the set of positive integers $1, 2, \ldots, N$;
- the $v_i, i ∈ \langle N \rangle$, are the rows of $A$;
- the $w_j, j ∈ \langle n \rangle$, are the columns of $A$;
- $a_{ij}$ is the element of $A$ in the $i$th row and the $j$th column.

For a subset $ω ⊂ \langle N \rangle$, $A(ω)$ is the submatrix of $A$ formed by the rows $v_i, i ∈ ω$, and $\overline{ω} = \langle N \rangle \setminus ω$. We denote the inner product in $\mathbb{R}^n$ by $(\cdot, \cdot)$ and the norm of a vector $x ∈ \mathbb{R}^N$ in the space $l^q$, $1 ≤ q ≤ ∞$, by $∥x∥_q$. For the normed space $X$, $∥·∥_X$ is the norm on $X$.

The following condition is a direct analog of the condition imposed in [1] on a matrix in the case of arbitrary $1 ≤ q < ∞$:

$$\forall x ∈ \mathbb{R}^n \quad \forall i_0 ∈ \langle N \rangle \quad |(v_{i_0}, x)| ≤ ε \left( \sum_{i=1}^{N} |(v_i, x)|^q \right)^{1/q}. \quad (1)$$

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Statement 1. Assume that, for an \( N \times n \) matrix \( A \), condition (1) is satisfied for some \( \varepsilon \) and \( q \) with \( 0 < \varepsilon \leq (\text{rk}(A))^{-1/q} \) and \( 1 \leq q < \infty \). Then there exists a partition

\[
\langle N \rangle = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \emptyset,
\]

such that, for any \( x \in \mathbb{R}^n \) and \( k = 1, 2 \),

\[
\|A(\Omega_k)x\|_q \leq \gamma \|Ax\|_q, \quad \gamma = \frac{1}{2}^{1/q} + \frac{2 + 3 \cdot 2^{-1/q}}{q} \left( \text{rk}(A)\varepsilon^q \ln \frac{6q}{(\text{rk}(A)\varepsilon^q)^{1/3}} \right)^{1/3}.
\]

Sketch of the Proof. First, we prove the statement for a matrix \( A \) with \( \text{rk}(A) = n \). We write

\[
\delta = \frac{(n\varepsilon^q)^{1/3}}{q}.
\]

Let \( X \) be the space \( \mathbb{R}^n \) with norm \( \|x\|_X = \|Ax\|_q \) (this is a norm on \( \mathbb{R}^n \), because \( \text{rk}(A) = n \)). Let \( S_X = \{x \in \mathbb{R}^n : \|x\|_X = 1\} \) be the unit sphere in the space \( X \). Let \( Y \) be a \( \delta \)-net in the metric of \( X \) on \( S_X \) containing \( (3/\delta)^n \) elements. Assume that the required assertion is false, i.e., for any partition (2), there is either a vector \( x_1 \in S_X \) such that

\[
\|A(\Omega_1)x_1\|_q > \gamma \|Ax_1\|_q
\]

(in this case, we set \( \omega' = \Omega_1 \) and \( x_{\omega'} = x_1 \)) or a vector \( x_2 \in S_X \) such that

\[
\|A(\Omega_2)x_2\|_q > \gamma \|Ax_2\|_q
\]

(in this case, we set \( \omega' = \Omega_2 \) and \( x_{\omega'} = x_2 \)). For each pair \((\omega, \bar{\omega})\), we find \( \omega' \) and \( x_{\omega'} \). Let \( y_{\omega'} \) be one of the vectors in \( Y \) that are nearest to \( x_{\omega'} \). The total number of distinct partitions of the set \( \langle N \rangle \) into two nonempty parts is equal to \( 2^{N-1} - 1 \). Therefore, there exists a \( y_0 \in Y \) such that the set \( K = \{\omega' : y_0 = y_{\omega'}\} \) is sufficiently large:

\[
|K| \geq (2^{N-1} - 1) \left( \frac{\delta}{3} \right)^n \geq 2^N \left( \frac{\delta}{6} \right)^n
\]

(we assume that \( n > 1 \); otherwise, the statement is obvious). Thus, there exists a \( y_0 \in S_X \) and at least \( 2^N (\delta/6)^n \) subsets \( \omega' \subset \langle N \rangle \) for which \( \|A(\omega')x_{\omega'}\|_q \geq \gamma \|Ax_{\omega'}\|_q \) and \( \|y_0 - x_{\omega'}\|_X < \delta \).

We note that \( \|A(\omega)x\|_q \leq \|A(\omega)\|_{(X,q)} \leq \|A\|_{(X,q)} \) for \( x \in S_X \) and \( \omega \subset \langle N \rangle \).

We assume that \( \gamma < 1 \); otherwise, estimate (3) is obvious. For \( \omega' \in K \), taking into account the fact that \( \gamma < 1 \), we obtain

\[
\|A(\omega'y_0)\|_q \geq \|A(\omega')x_{\omega'}\|_q - \|A(\omega')(x_{\omega'} - y_0)\|_q > \gamma \|Ax_{\omega'}\|_q - \delta \left\| \frac{x_{\omega'} - y_0}{\|x_{\omega'} - y_0\|_X} \right\|_q
\]

\[
\geq (\gamma - \delta) A_{y_0} - \delta \left\| \frac{x_{\omega'} - y_0}{\|x_{\omega'} - y_0\|_X} \right\|_q
\]

\[
\geq (\gamma - \delta) A_{y_0} - 2\delta \geq (\gamma - 2\delta) A_{y_0} = \|A_{y_0}\|_q (\gamma - 2\delta).
\]

In the penultimate inequality, we have used the fact that \( \|A_{y_0}\|_q = \|y_0\|_X = 1 \) for \( y_0 \in S_X \).

We let \( R \) denote the number of subsets \( \omega \subset \langle N \rangle \) for which

\[
\|A(\omega)y_0\|_q > (\gamma - 2\delta) \|A_{y_0}\|_q = (\gamma - 2\delta).
\]

Let \( K_1 \) be the set of such subsets. We shall show that \( R < 2^N (\delta/6)^n \) and thus obtain a contradiction, which will complete the proof of Statement 1 for the case \( \text{rk}(A) = n \). We write

\[
M = 3 \cdot 2^{-1/q}, \quad \phi(n, \varepsilon) = \frac{n \varepsilon^q \ln (6q/((n\varepsilon)^{1/3}))^{1/3}}{q}.
\]
Since $\delta \leq \phi(n, \varepsilon)$, for $\omega' \in K_1$, we have

$$\sum_{i \in \omega'} |(v_i, y_0)|^q > (\gamma - 2\delta)^q > \left( \frac{1}{2^{1/q}} + M\phi(n, \varepsilon) \right)^q \geq \frac{1}{2} + q\frac{1}{2^{(q-1)/q}} M\phi(n, \varepsilon) = \frac{1}{2} + q\frac{2^{1/q}}{2} M\phi(n, \varepsilon).$$

Now the quantity $R$ can be estimated just as in the proof of Statement 3 in [1].

Now assume that the rank of the matrix is $r < n$. To be definite, assume that $w_1, \ldots, w_r$ are linearly independent. It is clear that, for the matrix $A$ composed of the first $r$ columns of the matrix $A$, condition (1) is satisfied and $\text{rk} A = r$. Therefore, there exists a partition of the form (2) such that estimate (3) holds. Let $w_j = \sum_{i=1}^r x_i^j w_i$. For $x \in \mathbb{R}^n$, we construct the vector $\tilde{x} \in \mathbb{R}^r$ with coordinates $\tilde{x}_i = x_i + \sum_{j=r+1}^n x_i^j x_j$. Then $Ax = \tilde{A}x$ and $A(\Omega_k)x = \tilde{A}(\Omega_k)\tilde{x}$ for $k = 1, 2$, and hence, for the obtained partition, the matrix $A$ has property (3) as well.

The following statement is a simple consequence of Statement 1.

**Statement 2.** Assume that, for an $N \times n$ matrix $A$, condition (1) is satisfied for some $\varepsilon$ and $q$ with $0 < \varepsilon \leq (\text{rk}(A))^{-1/q}$ and $1 \leq q < \infty$. Then there exists a partition of the form (2) such that, for $k = 1, 2$,

$$\|A(\Omega_k)\|_{(x,q)} \leq \gamma \|A\|_{(x,q)},$$

where $\gamma$ is defined in Statement 1.

The following assertion is an analog of Statement 2 for the $(1,q)$-norm. Let $e_j, j \in \langle n \rangle$, be the standard basis in $\mathbb{R}^n$.

**Statement 3.** Assume that, for an $N \times n$ matrix $A$, the inequality

$$|a_{ij}^n| \leq \varepsilon \|w_j\|_q$$

holds for some $1 \leq q < \infty$ and $0 < \varepsilon < 1$ and for any $i \in \langle N \rangle$ and $j \in \langle n \rangle$. Then there exists a partition of the form (2) such that, for $k = 1, 2$,

(a) $\|A(\Omega_k)\|_{(1,q)} \leq \left( \frac{1}{2} + \frac{3}{2} \varepsilon^{q/3} \ln^{1/3} (4n) \right)^{1/q} \|A\|_{(1,q)},$

(b) $\|A(\Omega_k)\|_{(1,q)} \leq \left( \frac{1}{2} + \frac{1}{2} \varepsilon^{q} \sqrt{N} \left( 1 + \log \left( \frac{n}{N} + 1 \right) \right)^{1/2} \right)^{1/q} \|A\|_{(1,q)},$

(c) $\|A(\Omega_k)\|_{(1,q)} \leq \left( \frac{1 + n\varepsilon^q}{2} \right)^{1/q} \|A\|_{(1,q)}.$

**Remark.** For Statement 3 to hold, it is sufficient to impose a significantly weaker condition on the elements of the matrix than that in Statement 2.

**Proof.** Since the function $\|Ax\|_q$ is convex, it follows that the $(1,q)$-norm of the matrix is attained at an element of the standard basis.

The proof of assertion (a) is similar to the above argument; therefore, we only outline it here. We assume that this assertion is false, that is, for any partition of the form (2), there exists a $k$ such that

$$\|A(\Omega_k)\|_{(1,q)} > \left( \frac{1}{2} + \frac{3}{2} \varepsilon^{q/3} \ln^{1/3} (4n) \right)^{1/q} \|A\|_{(1,q)}.$$
We write \( \omega' = \Omega_k \). The \((1,q)\)-norm of the matrix \( A_{\omega'} \) is attained at a certain vector \( e_{j_{\omega'}} \), \( j_{\omega'} \in \langle n \rangle \), which implies

\[
\sum_{i \in \omega'} |a_{i,j_{\omega'}}|^q > \left( \frac{1}{2} + \frac{3}{2} \varepsilon q/3 \ln^{1/3} (4n) \right) \| w_{j_{\omega'}} \|^q_q.
\]

As in the proof of Statement 1, there exists a \( j_0 \in \langle n \rangle \) such that the set \( K = \{ \omega' : j_{\omega'} = j_0 \} \) is sufficiently large:

\[
|K| \geq \frac{(2N-1) - 1}{n} \geq \frac{2N-2}{n}.
\] (6)

In this case, it is clear that, for any \( \omega \in K \),

\[
\sum_{i \in \omega} |a_{i,j_0}|^q > \left( \frac{1}{2} + \frac{3}{2} \varepsilon q/3 \ln^{1/3} (4n) \right) \| w_{j_0} \|^q_q.
\] (7)

Therefore, to prove assertion (a), it suffices to verify that the number \( R \) of subsets \( \omega \subset \langle N \rangle \) for which inequality (7) holds is less than the right-hand side of (6). The quantity \( R \) can be estimated just as in the proof of Statement 3 in [1].

To prove assertion (b), we use Corollary 5 in [3]. We let \( \tilde{w}_j = (|a_{j,1}|^q, \ldots, |a_{j,N}|^q) \) denote the vector obtained from the \( j \)th column of the matrix \( A \) by raising the absolute values of coordinates to the power \( q \). For all \( j \in \langle n \rangle \), we have \( \| w_j \|^q_q \leq \| A \|^q_{(1,q)} \), and hence it follows from (5) that \( \| \tilde{w}_j \|_\infty \leq \varepsilon q \| A \|^q_{(1,q)} \). Then the above-cited corollary in [3] implies the existence of a vector \( \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \) with unit (in absolute value) coordinates which satisfies, for all \( j \in \langle n \rangle \), the inequality

\[
|\langle \tilde{w}_j, \xi \rangle| \leq \varepsilon q \sqrt{N} \left( 1 + \log \left( \frac{n}{N} + 1 \right) \right)^{1/2} \| A \|^q_{(1,q)}.
\]

Assume that \( \Omega_1 = \{ i \in \langle N \rangle : \xi_i = 1 \} \) and \( \Omega_2 = \langle N \rangle \setminus \Omega_1 = \{ i \in \langle N \rangle : \xi_i = -1 \} \). Let us verify that assertion (b) holds for the partition thus constructed. We write

\[
\theta = \sqrt{N} \left( 1 + \log \left( \frac{n}{N} + 1 \right) \right)^{1/2}.
\]

For \( k = 1, 2 \), there exists a \( j_0^k \in \langle n \rangle \) such that

\[
\| A(\Omega_k) \|^q_{(1,q)} = \sum_{i \in \Omega_k} |a_{i,j_0^k}|^q \leq \frac{1}{2} \left( \sum_{i \in \langle N \rangle} |a_{i,j_0}|^q + \varepsilon q \theta \| A \|^q_{(1,q)} \right) \leq \left( \frac{1}{2} + \frac{1}{2} \varepsilon q \theta \right) \| A \|^q_{(1,q)},
\]

as required.

To prove assertion (c), we apply the following statement (generalized ham sandwich theorem).

**Theorem** ([4, p. 287]). Let \( A_1, \ldots, A_n \) be sets of finite Lebesgue measure in \( \mathbb{R}^n \). Then there exists a hyperplane \( \pi \) that divides the measure of each set in half.

We write \( M = \max_{i,j} \{|a_{ij}|^q\} + 1 \). In \( \mathbb{R}^n \), we place \( N \) cubes with sides equal to \( M \) and parallel to the coordinate axes so that any hyperplane in \( \mathbb{R}^n \) passes through at most \( n \) cubes. It is easy to construct such cubes from \( N \) point in general position in \( \mathbb{R}^n \) by using the general equation of the plane that passes through \( n \) given points. Let us number the obtained cubes. For \( i \in \langle N \rangle \), we let \( u_i \) denote the vertex of the \( i \)th cube with the least coordinates. For each element of the matrix \( a_{ij} \), we define the parallelepiped

\[
\hat{P}_j = [0,1]^{j-1} \times [1,1 + |a_{ij}|^q] \times [0,1]^{n-j}.
\]

In the \( i \)th cube, we place \( n \) rectangular parallelepipeds constructed from the elements in the row \( v_i \) (\( P_j^i = u_i + \hat{P}_j \)). We note that \( \mu(P_j^i) = |a_{ij}|^q \). The set of parallelepipeds \( P_j^i, j \in \langle n \rangle \), for a fixed \( i \), will be called the \( i \)th corner.
For $j \in \langle n \rangle$, we write $A_j = \bigcup_{i \in \langle N \rangle} P^i_j$. Now we apply the theorem to the sets $A_j$ thus constructed. We obtain a hyperplane $\pi$ that divides the volume of each $A_j$ in half. Let $P_1$ and $P_2$ be the half-spaces into which $\pi$ divides the space $\mathbb{R}^n$. It follows from the construction that $\pi$ intersects at most $n$ of the constructed cubes and, therefore, at most $n$ corners. Now it is clear how to obtain a partition of the form (2): we put the numbers of the corners that are completely contained in the half-spaces $P_1$ and $P_2$ in $\Omega_1$ and $\Omega_2$, respectively, and the numbers of the corners intersecting both half-spaces in $\Omega_1$ (we denote the set of numbers of such corners by $G$). Let us show that, for any $j \in \langle n \rangle$, the $l_q^N$-norm of the column $w_j$ decreases at least by a factor of $((1 + ne^q)/2)^{1/q}$ for the partition described above. This immediately implies the desired assertion. Since $\pi$ divides the measure $A_j$ in half, we have

$$\sum_{i \in \Omega_1 \setminus G} |a^i_j|^q + V_1 = \sum_{i \in \Omega_2} |a^i_j|^q + V_2,$$

where the $V_k$, $k = 1, 2$, denote the volumes of $\bigcup_{i \in \langle G \rangle} P^i_j \cap P_k \setminus P_k$. Condition (5) and the fact that $\pi$ intersects at most $n$ corners imply

$$V_1 + V_2 \leq ne^q \sum_{i \in \langle N \rangle} |a^i_j|^q;$$

therefore, for $k = 1, 2$,

$$\sum_{i \in \Omega_k} |a^i_j|^q \leq \frac{1 + ne^q}{2} \sum_{i \in \langle N \rangle} |a^i_j|^q.$$

This completes the proof of Statement 3. \qed

The following statement shows that condition (5) with $\varepsilon < 1$ does not always imply the existence of a partition of a matrix into two submatrices with lesser $(1, q)$-norms.

**Statement 4.** For $n = 2^{2k-1}$, there exists a $2k \times n$ matrix $A$ for which condition (5) with $\varepsilon^q \log_2 2n \geq 2$ is satisfied, and the following relation holds for any partition of the form (2):

$$\max\{\|A(\Omega_1)\|_{1,q}, \|A(\Omega_2)\|_{1,q}\} = \|A\|_{1,q}.$$

**Proof.** For each pair of subsets $\omega$ and $\langle 2k \rangle \setminus \omega$ of the set $\langle 2k \rangle$, we choose any subset of the greatest cardinality. We number the chosen subsets as $B_1, \ldots, B_{2^{2k-1}}$ and use them to construct a matrix $A$ as follows: the element $a^i_j$ is equal to $1/|B_j|^{1/q}$ if $i \in B_j$ and to 0 otherwise. Condition (5) and the fact that $\|A(\Omega_1)\|_{1,q} = \|A\|_{1,q}$ or $\|A(\Omega_2)\|_{1,q} = \|A\|_{1,q}$ for an arbitrary partition of the form (2) can be verified directly. \qed

For $q = \infty$, there does not exist a partition into two submatrices with lesser $(X, \infty)$-norms for any matrix $A$, since, in this case, the matrix $A$ has a row $v_{\sup}$ such that $\|A\|_{(X, \infty)} = \sup_{\|x\|_X \leq 1} \langle x, v_{\sup} \rangle$, and the norm of the submatrix containing the row $v_{\sup}$ is equal to the norm of the matrix $A$.

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