**B-Meson Observables in the Maximally CP-Violating MSSM with Minimal Flavour Violation**

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**ABSTRACT**

Additional sources of CP violation in the MSSM may affect $B$-meson mixings and decays, even in scenarios with minimal flavour violation (MFV). We formulate the maximally CP-violating and minimally flavour-violating (MCPMFV) variant of the MSSM, which has 19 parameters, including 6 phases that violate CP. We then develop a manifestly flavour-covariant effective Lagrangian formalism for calculating Higgs-mediated FCNC observables in the MSSM at large $\tan\beta$, and analyze within the MCPMFV framework FCNC and other processes involving $B$ mesons. We include a new class of dominant subleading contributions due to non-decoupling effects of the third-generation quarks. We present illustrative numerical results that include effects of the CP-odd MCPMFV parameters on Higgs and sparticle masses, the $B_s$ and $B_d$ mass differences, and on the decays $B_s \rightarrow \mu^+\mu^-$, $B_u \rightarrow \tau\nu$ and $b \rightarrow s\gamma$. We use these results to derive illustrative constraints on the MCPMFV parameters imposed by D0, CDF, BELLE and BABAR measurements of $B$ mesons, demonstrating how a potentially observable contribution to the CP asymmetry in the $b \rightarrow s\gamma$ decay may arise in the MSSM with MCPMFV.

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1 Introduction

Models incorporating supersymmetry (SUSY), such as the Minimal Supersymmetric Standard Model (MSSM), contain many possible sources of flavour and CP violation. In particular, the soft SUSY-breaking sector in general introduces many new sources of flavour and CP violation, giving rise to effects that may exceed the experimental limits by several orders of magnitude. The unitarity of the Cabibbo–Kobayashi–Maskawa (CKM) quark mixing matrix suppresses flavour-changing-neutral currents (FCNC) and CP violation somewhat, thanks to the Glashow–Iliopoulos–Maiani (GIM) mechanism [1], to the extent that the soft SUSY-breaking scalar masses are universal. One possible solution to the flavour and CP problems is to ensure that the soft SUSY-breaking sector is fully protected by the GIM mechanism. This can be achieved within the so-called framework of minimal flavour violation (MFV), where all flavour and CP effects are mediated by the superpotential interactions corresponding to the ordinary Yukawa couplings of the Higgs bosons to quarks and leptons. In this framework, FCNC and CP-violating observables depend only on the fermion masses and their mixings, and hence the CKM mixing matrix $V$ [2]. In such a scenario, all FCNC and CP violation observables would vanish in the MSSM if $V$ were equal to the unit matrix $1$.

A minimal realization of MFV in the MSSM is obtained by assuming that all soft SUSY-breaking bilinear masses for the scalar particles, such as squarks, sleptons and Higgs bosons, are equal to a common value $m_0$ at the gauge coupling unification point $M_{\text{GUT}}$, where $M_{\text{GUT}}$ might be the threshold for some underlying grand unified theory (GUT) based, e.g., on SU(5) or SO(10). Likewise, the soft masses of the fermionic SUSY partners of the gauge fields, the gauginos, might also be equal to a common value $m_{1/2}$ at $M_{\text{GUT}}$ and, in the same spirit, all soft trilinear Yukawa couplings of the Higgs bosons to squarks and sleptons could be real and equal to a universal parameter $A$ times the corresponding Higgs-fermion-antifermion couplings. The Higgs supermultiplet mixing parameter $\mu$ and the corresponding soft SUSY-breaking term $B\mu$ introduce two additional mass scales in the theory. However, minimization conditions on the Higgs potential can be used to eliminate these two last mass scales in favour of the electroweak scale $M_Z$ and $\tan \beta \equiv v_u/v_d$, where $v_{u,d}$ are the vacuum expectation values (VEVs) of the two Higgs doublets $H_u,d$ in the MSSM.

It is well known that a minimal expansion of the above MFV framework is to allow the soft SUSY-breaking mass parameters $m_{1/2}$ and $A$ to be complex with CP-odd phases, thereby introducing two additional sources of CP violation in the theory. In this case, all FCNC observables, whether CP-conserving or not, still depend on the CKM mixing matrix $V$ in such a way that they vanish if $V$ is assumed to be diagonal, i.e., equal to...
the unit matrix. However, the two new phases introduce the possibility of CP violation in flavour-conserving processes even if $V$ is real, and in general CP violation in FCNC processes may differ from CKM predictions.

Here we go one step further, and ask the following question. What is the maximal number of additional CP-violating parameters and extra flavour-singlet mass scales that could be present in the MSSM, for which the above notion of MFV remains still valid, i.e., all FCNC effects vanish in the limit of a diagonal $V$? We call this scenario the maximally CP-violating MSSM with minimal flavour violation, or in short, the MSSM with MCPMFV. As we will see in Section 2, there are a total of 19 parameters in the MSSM with MCPMFV, including 6 CP-violating phases and 13 real mass parameters. The purposes of this paper are to formulate the MSSM with MCPMFV, calculate the most relevant $B$-meson observables, and explore the experimental constraints on the MCPMFV theoretical parameters, exploiting a manifestly flavour-covariant effective Lagrangian formalism for calculating Higgs-mediated FCNC observables at large $\tan \beta$ that we develop here.

At large values of $\tan \beta$, e.g. $\tan \beta \gtrsim 40$, one-loop threshold effects on Higgs-boson interactions to down-type quarks get enhanced [3–5], and so play an important role in FCNC processes, such as the $K^0$-$\bar{K}^0$ mass difference, $B_s$-$\bar{B}_s$ and $B_d$-$\bar{B}_d$ mixings, and the decays $B \rightarrow X_s \gamma$, $B \rightarrow K l^+ l^-$, $B_{s,d} \rightarrow \mu^+ \mu^-$ [6–15], and $B \rightarrow \tau \nu$ [16, 17]. We present in this paper a manifestly flavour-covariant effective Lagrangian formalism for calculating FCNC processes that follows the lines of the effective Lagrangian approach given in [12]. In addition, we include here the dominant subleading contributions to the one-loop Higgs-mediated FCNC interactions due to non-decoupling large Yukawa-coupling effects of the third-generation quarks. Based on this improved formalism, we compute FCNC observables in constrained versions of the MSSM, where MFV has been imposed on the soft SUSY-breaking mass parameters as a boundary condition at the scale $M_{\text{GUT}}$. We present numerical results for $B$-meson observables in one example of the MCPMFV framework, from which illustrative constraints on the basic theoretical parameters are derived, after incorporating the recent experimental results from D0 and CDF [18].

The paper is organized as follows: in Section 2, after briefly reviewing the MFV framework, we derive the maximal number of flavour-singlet mass parameters that can be present in the MSSM with MCPMFV at the GUT scale. All relevant one-loop RGEs are given in Appendix A. In Section 3, we present an effective Lagrangian formalism for Higgs-mediated FCNC interactions that respects flavour covariance. We also discuss the dominant subleading effects at large $\tan \beta$, due to the large Yukawa couplings of the third generation. Useful relations which result from Ward identities (WIs) that involve the $Z$ and $W$-boson interactions to quarks are derived in Appendix B. Section 4 summarizes all relevant analytic
results pertinent to FCNC B-meson observables. In Section 5 we exhibit numerical estimates and predictions for various FCNC processes, including the $B_s\bar{B}_s$ and $B_d\bar{B}_d$ mixings, and the decays $B_{s,d} \rightarrow \mu^+ \mu^-$, $B \rightarrow X_s \gamma$, and $B \rightarrow \tau \nu$. We also illustrate the combined constraints on the theoretical parameters imposed by data from D0, CDF, BELLE and BABAR in one sample MCPMFV model. We summarize our conclusions in Section 6.

2 Maximal CP and Minimal Flavour Violation

In this section we derive the maximal number of CP-violating and real flavour-singlet mass parameters that can be present in the CP-violating MSSM and satisfy the property of MFV as described in the Introduction.

The superpotential defining the flavour structure of the MSSM may be written as

$$W_{\text{MSSM}} = \hat{U}^C h_u \hat{Q} \hat{H}_u + \hat{D}^C h_d \hat{H}_d \hat{Q} + \hat{E}^C h_e \hat{H}_d \hat{L} + \mu \hat{H}_u \hat{H}_d,$$  \hspace{1cm} (2.1)

where $\hat{H}_{u,d}$ are the two Higgs chiral superfields, and $\hat{Q}, \hat{L}, \hat{U}^C, \hat{D}^C$ and $\hat{E}^C$ are the left- and right-handed superfields related to up- and down-type quarks and charged leptons. The Yukawa couplings $h_{u,d,e}$ are $3 \times 3$ complex matrices describing the charged-lepton and quark masses and their mixings. The superpotential (2.1) contains one mass parameter, the $\mu$ parameter that mixes the Higgs supermultiplets, which has to be of the electroweak order for a natural realization of the Higgs mechanism.

In an unconstrained version of the MSSM, there is a large number of different mass parameters present in the soft SUSY-breaking Lagrangian

$$-\mathcal{L}_{\text{soft}} = \frac{1}{2} \left( M_1 \hat{B} \hat{B} + M_2 \hat{W}^i \hat{W}^i + M_3 \hat{g}^a \hat{g}^a + \text{h.c.} \right) + \hat{Q}^\dagger \tilde{M}_Q^2 \hat{Q} + \hat{L}^\dagger \tilde{M}_L^2 \hat{L} + \hat{U}^\dagger \tilde{M}_U^2 \hat{U}$$
$$+ \tilde{D}^\dagger \tilde{M}_D^2 \tilde{D} + \tilde{E}^\dagger \tilde{M}_E^2 \tilde{E} + M_{H_u}^2 H_u H_u + M_{H_d}^2 H_d H_d + \left( B_\mu H_u H_d + \text{h.c.} \right)$$
$$+ \left( \tilde{U}^\dagger a_u \hat{Q} H_u + \tilde{D}^\dagger a_d H_d \hat{Q} + \tilde{E}^\dagger a_e H_d \hat{L} + \text{h.c.} \right).$$  \hspace{1cm} (2.2)

Here $M_{1,2,3}$ are the soft SUSY-breaking masses associated with the $U(1)_Y$, $SU(2)_L$ and $SU(3)_c$ gauginos, respectively. In addition, $M_{H_u,d}^2$ and $B_\mu$ are the soft masses related to the Higgs doublets $H_{u,d}$ and their bilinear mixing. Finally, $\tilde{M}_{Q,L,D,U,E}$ are the $3 \times 3$ soft mass-squared matrices of squarks and sleptons, and $a_{u,d,e}$ are the corresponding $3 \times 3$ soft Yukawa mass matrices $^1$. Hence, in addition to the $\mu$ term, the unconstrained CP-violating MSSM contains 109 real mass parameters.

$^1$Alternatively, the soft Yukawa mass matrices $a_{u,d,e}$ may be defined by the relation: $(a_{u,d,e})_{ij} = (h_{u,d,e})_{ij} \left( A_{u,d,e} \right)_{ij}$, where the parameters $(A_{u,d,e})_{ij}$ are generically of order $M_{\text{SUSY}}$ in gravity-mediated SUSY breaking models. In our paper, both definitions for the soft SUSY-breaking Yukawa couplings will be used, where convenient.
One frequently considers the constrained MSSM (CMSSM), which has a common gaugino mass \( m_{1/2} \), a common soft SUSY-breaking scalar mass \( m_0 \) and a common soft trilinear Yukawa coupling \( A \) for all squarks and sleptons at the GUT scale. The number of independent mass scales is greatly reduced since, even allowing for maximal CP violation, the free parameters are just \( m_{1/2}, \mu, m_0, A, B, \) where all but \( m_0 \) are complex variables. The phase \( \arg \mu \) may be removed by means of a global Peccei–Quinn (PQ) symmetry under which \( H_u \) and \( H_d \) have the same charges. Imposing the two CP-even tadpole conditions on the Higgs potential, one may replace \( \mu = |\mu| \) and \( \text{Re}(B\mu) \) by the \( Z \)-boson mass \( M_Z \) and the ratio \( \tan \beta = v_u/v_d \) of the VEVs of the Higgs doublets \( H_{u,d} \) in the phase convention where \( v_{u,d} \) are real and positive. Linked to this, there is one extra CP-odd tadpole condition which can be used to eliminate \( \text{Im}(B\mu) \) in favour of maintaining the same phase convention for the VEVs, order by order in perturbation theory [19]. Thus, a convenient set of input mass parameters of the constrained CP-violating MSSM is

\[
\tan \beta(m_t), \ m_{1/2}(M_{\text{GUT}}), \ m_0(M_{\text{GUT}}), \ A(M_{\text{GUT}}), \ (2.3)
\]

where the relative sign of \( \mu \) can always be absorbed into the phase definition of the complex parameters \( m_{1/2} \) and \( A \). Thus, in addition to \( \tan \beta \), this CP-violating CMSSM has just 5 real mass parameters, two more than in its CP-conserving counter-part, namely the CP-odd parameters: \( \text{Im} m_{1/2} \) and \( \text{Im} A \).

How can the general notion of MFV be extended to this constrained CP-violating MSSM? In such a constrained model, the physical FCNC observables remain independent of details of the Yukawa texture chosen at the GUT scale. They depend only on the CKM mixing matrix \( V \), the fermion masses, \( \tan \beta \) and the 5 real mass parameters mentioned above. If the CKM matrix \( V \) were equal to the unit matrix \( 1 \), the FCNC observables would vanish, but flavour-conserving, CP-violating effects would still be present, associated with \( \text{Im} m_{1/2} \) and \( \text{Im} A \). Moreover, these parameters also contribute to CP-violating FCNC observables in the presence of non-trivial CKM mixing. Most noticeably, \( \text{Im} m_{1/2} \) and \( \text{Im} A \) cannot generically mimic the effects of the usual CKM phase \( \delta \).

We now consider how the above notion of MFV can be further extended within the more general CP-violating MSSM. To address this question, we first notice that under the unitary flavour rotations of the quark and lepton superfields,

\[
\hat{Q}' = U_Q \hat{Q}, \quad \hat{L}' = U_L \hat{L}, \quad \hat{U}^C = U_U^* \hat{U}^C, \quad \hat{D}^C = U_D^* \hat{D}^C, \quad \hat{E}^C = U_E^* \hat{E}^C, \quad (2.4)
\]

the complete MSSM Lagrangian of the theory remains invariant provided the model parameters are redefined as follows:

\[
\begin{align*}
\mathbf{h}_{u,d} & \to U^\dagger_{U,D} h_{u,d} U_Q, & \mathbf{h}_e & \to U^\dagger_E h_e U_L,
\end{align*}
\]
The remaining mass scales, $\mu$, $M_{1,2,3}$, $M_{H_{u,d}}^2$, and $B\mu$, do not transform under the unitary flavour rotations (2.4). In fact, it is apparent that the one-loop RGEs presented in Appendix A are invariant under the redefinitions in (2.5), provided the unitary flavour matrices $U_{Q,L,U,D,E}$ are taken to be independent of the RG scale. The effective Lagrangian formalism we describe in Section 3 respects manifestly the property of flavour covariance under the unitary transformations (2.4).

It is apparent from (2.5) that the maximal set of flavour-singlet mass scales includes:

$$M_{1,2,3}, \ M_{H_{u,d}}^2, \ \tilde{M}_{Q,L,U,D,E}^2 = \tilde{M}_{Q,L,U,D,E}^2 \ 1_3, \ \mathbf{A}_{u,d,e} = A_{u,d,e} \ 1_3, \ (2.6)$$

where the mass parameters $\mu$ and $B\mu$ can be eliminated by virtue of a global PQ symmetry and by the CP-even and CP-odd minimization conditions on the Higgs potential. The scenario (2.6) has a total of 19 mass parameters that respect the general MFV property, 6 of which are CP-odd, namely $\text{Im} M_{1,2,3}$ and $\text{Im} A_{u,d,e}$.

We term this scenario the maximally CP-violating and minimally flavour-violating (MCPMFV) variant of the MSSM, or in short, the MSSM with MCPMFV.

It is worth noting that, in addition to the flavour-singlet mass scales mentioned above, there may exist flavour non-singlet mass scales in the MSSM. For example, one could impose an unconventional boundary condition on the left-handed squark mass matrix $\tilde{M}_Q^2$, such that

$$\tilde{M}_Q^2 (M_X) = \tilde{M}_Q^2 \ 1_3 + \tilde{m}_1^2 (h_d^\dagger h_d) + \tilde{m}_2^2 (h_u^\dagger h_u) + \tilde{m}_3^2 (h_d^\dagger h_d)(h_u^\dagger h_u) + \ldots, \ (2.7)$$

where $M_X$ could be $M_{\text{GUT}}$ or some other scale. Evidently, there are in principle a considerable number of extra mass parameters $\tilde{m}_n^2$ that could also be present in $\tilde{M}_Q^2 (M_X)$, beyond the flavour-singlet mass scale $\tilde{M}_Q^2$. In fact, these additional flavour non-singlet mass parameters $\tilde{m}_n^2$ can be as many as 9 (including $\tilde{M}_Q^2$), as determined by the dimensionality of the $3 \times 3$ hermitian matrix $\tilde{M}_Q^2 (M_X)$. The generalized boundary condition (2.7) on $\tilde{M}_Q^2 (M_X)$ is in agreement with the notion of MFV for solving the flavour problem by suppressing the GIM-breaking effects, provided the hierarchy $\tilde{m}_n^2 \ll \tilde{M}_Q^2$ is assumed. In particular, if these flavour-non-singlet mass parameters $\tilde{m}_n^2$ are induced by RG running, they may be generically much smaller than $\tilde{M}_Q^2$. In this case, the $\tilde{m}_n^2$ will not all be independent of each other, e.g., in our MCPMFV scenario, the RG-induced flavour-non-singlet scales $\tilde{m}_n^2$ would be functionals of the 19 flavour-singlet mass parameters stated in (2.6). In general, a non-singlet mass parameter could either be introduced by hand or induced by RG running of a theory beyond the MSSM with more flavour-singlet mass scales [20]. However,
since introducing $\tilde{m}_n^2 \ll \tilde{M}_Q^2$ by hand has no strong theoretical motivation, we focus our attention here on the flavour-singlet MSSM framework embodied by the MCPMFV.

Before calculating FCNC observables in the MSSM with MCPMFV, we first develop in the next section an effective Lagrangian approach to the computation of Higgs-mediated effects, which play an important role in our analysis.

### 3 Effective Lagrangian Formalism

Here we present a manifestly flavour-covariant effective Lagrangian formalism. This formalism enables one to show the flavour-basis independence of FCNC observables in general soft SUSY-breaking scenarios of the MSSM. It will also be used in Section 4 to calculate FCNC processes in the MSSM with MCPMFV.

To make contact between our notation and that used elsewhere in the literature [21], we redefine the Higgs doublets $H_{u,d}$ as $H_u \equiv \Phi_2$ and $H_d \equiv i\tau_2\Phi_1^*$, where $\tau_{1,2,3}$ are the usual Pauli matrices. We start our discussion by considering the effective Lagrangian that describes the $\tan\beta$-enhanced supersymmetric contributions to the down-type quark self-energies as shown in Fig. 1. The effective Lagrangian can be written in gauge-symmetric and flavour-covariant form as follows:

$$-\mathcal{L}_{\text{eff}}^d[\Phi_1, \Phi_2] = \mathcal{D}_{iR}^0 \left( h_d \Phi_1^\dagger + \Delta h_d[\Phi_1, \Phi_2] \right) Q_{jL}^0 + \text{h.c.},$$

where the superscript ‘0’ indicates weak–eigenstate fields. In (3.1), the first term denotes the tree-level contribution and $\Delta h_d$ is a $3 \times 3$ matrix which is a Coleman–Weinberg–type [22] effective functional of the background Higgs doublets $\Phi_{1,2}$. We note that the one-loop effective functional $\Delta h_d[\Phi_1, \Phi_2]$ has the same gauge and flavour transformation properties as $h_d\Phi_1^\dagger$. Its analytic and flavour-covariant form may be calculated via

$$(\Delta h_d)_{ij} = \int \frac{d^n k}{(2\pi)^n} \left[ P_L \left( \frac{2g_s^2C_FM_3^*}{k^2 - |M_3^2|} \right) \frac{1}{k^21_{12} - \tilde{M}^2} \tilde{D}_{i}\tilde{Q}_j^\dagger \right. + P_L \left( \frac{1}{k1_8 - M_CP_L - M_CP_R^*} \right) \tilde{H}_{u}\tilde{H}_d \left( \frac{1}{k^21_{12} - \tilde{M}^2} \right) \tilde{Q}_i \tilde{Q}_k^\dagger \left( h_u \right)_{kj} \right. \left. + P_L \left( \frac{1}{k1_8 - M_CP_L - M_CP_R^*} \right) \tilde{H}_d\tilde{B} \left( \frac{1}{k^21_{12} - \tilde{M}^2} \right) \tilde{Q}_i \tilde{Q}_j \left( \sqrt{2g'} \right) \right. + \sum_{k=1}^3 P_L \left( \frac{1}{k1_8 - M_CP_L - M_CP_R^*} \right) \tilde{H}_{d}\tilde{Q}_k \left( \frac{1}{k^21_{12} - \tilde{M}^2} \right) \tilde{Q}_i \tilde{Q}_j \left( g'_{\gamma k} \right) \right],$$

where $n = 4 - 2\epsilon$ is the usual number of analytically–continued dimensions in dimensional regularization (DR), $1_N$ stands for the $N \times N$-dimensional unit matrix, $P_L(R) = \frac{1}{2} \left[ 1 - (+) \gamma_5 \right]$.
are the standard chirality–projection operators, and \( C_F = 4/3 \) is the quadratic Casimir invariant of QCD in the fundamental representation. The \( 8 \times 8 \) and \( 12 \times 12 \)-dimensional matrices \( M_C \) and \( \tilde{M}^2 \) describe the squark and chargino-neutralino mass spectrum in the background of non-vanishing Higgs doublets \( \Phi_{1,2} \).

It proves convenient to express the \( 8 \times 8 \)-dimensional chargino-neutralino mass matrix \( M_C \) in the Weyl basis \((\tilde{B}, \tilde{W}^{1,2,3}, \tilde{H}_u, \tilde{H}_d)\), where \( \tilde{H}_{u,d} \) are SU(2)\(_L\) doublets: \( \tilde{H}_u = (\tilde{h}_u^+, \tilde{h}_u^0)^T \) and \( \tilde{H}_d = (\tilde{h}_d^0, \tilde{h}_d^-)^T \). In this weak basis, the Higgs-field-dependent chargino-neutralino mass matrix \( M_C[\Phi_1, \Phi_2] \) reads:

\[
M_C[\Phi_1, \Phi_2] = \begin{pmatrix}
M_1 & 0 & -\frac{1}{\sqrt{2}} g' \Phi_2^\dagger (i\tau_2) \\
0 & M_2 & \frac{1}{\sqrt{2}} g' \Phi_2^\dagger \tau_i \\
-\frac{1}{\sqrt{2}} g' \Phi_2^* (i\tau_2) & \frac{1}{\sqrt{2}} g' \Phi_2^* \tau_i & 0_2 \\
-\frac{1}{\sqrt{2}} (i\tau_2) g' \Phi_1 & \frac{1}{\sqrt{2}} g' \tau_i (i\tau_2) \Phi_1 & -\mu (i\tau_2) & 0_2
\end{pmatrix}, \tag{3.3}
\]

where \( g \) and \( g' \) are the SU(2)\(_L\) and U(1)\(_Y\) gauge couplings, respectively. Correspondingly, in the presence of non-vanishing Higgs doublets \( \Phi_{1,2} \), the \( 12 \times 12 \)-dimensional squark mass matrix \( \tilde{M}^2[\Phi_1, \Phi_2] \) is given by

\[
\tilde{M}^2[\Phi_1, \Phi_2] = \begin{pmatrix}
(\tilde{M}_Q^2)_{Q_1 Q_2} & (\tilde{M}_Q^2)_{Q_1 U} & (\tilde{M}_Q^2)_{Q_1 D} \\
(\tilde{M}_U^2)_{U_1 Q} & (\tilde{M}_U^2)_{U_1 U} & (\tilde{M}_U^2)_{U_1 D} \\
(\tilde{M}_D^2)_{D_1 Q} & (\tilde{M}_D^2)_{D_1 U} & (\tilde{M}_D^2)_{D_1 D}
\end{pmatrix}, \tag{3.4}
\]

with

\[
(\tilde{M}_Q^2)_{Q_1 Q_2} = (\tilde{M}_Q^2)_{ij} 1_2 + (h_u^i h_d^j) \Phi_1 \Phi_2^\dagger + (h_u^i h_u^j) \left( \Phi_2^2 1_2 - \Phi_1 \Phi_2 \right)
\]
\[
- \frac{1}{2} \delta_{ij} g^2 \left( \Phi_1 \Phi_1^T - \Phi_2 \Phi_2^T \right) + \delta_{ij} \left( \frac{1}{4} g^2 - \frac{1}{12} g'^2 \right) \left( \Phi_1 \Phi_1 - \Phi_2 \Phi_2 \right) 1_2,
\]
\[
(\tilde{M}^2)_{i\bar{i}} \bar{q}_j = (\tilde{M}^2)_{\bar{i}j} = -(a_u)_{ij} \Phi_2^T i\tau_2 + (h_u)_{ij} \mu^* \Phi_1^T i\tau_2,
\]
\[
(\tilde{M}^2)_{ij} \bar{d}_j = (\tilde{M}^2)_{ij} = (a_d)_{ij} \Phi_1^T - (h_d)_{ij} \mu^* \Phi_2^T,
\]
\[
(\tilde{M}^2)_{i\bar{i}} \tilde{u}_j = (\tilde{M}^2)_{\bar{i}j} = (h_u)_{ij} \Phi_1^T \Phi_2 + \frac{1}{3} \delta_{ij} g^2 \left( \Phi_1 \Phi_1 - \Phi_2 \Phi_2 \right),
\]
\[
(\tilde{M}^2)_{ij} \tilde{d}_j = (\tilde{M}^2)_{ij} = (h_d)_{ij} \Phi_1^T \Phi_1 - \frac{1}{6} \delta_{ij} g^2 \left( \Phi_1 \Phi_1 - \Phi_2 \Phi_2 \right),
\]
\[
(\tilde{M}^2)_{i\bar{i}} \tilde{d}_j = (\tilde{M}^2)_{\bar{i}j} = (h_u)_{ij} \Phi_1^T i\tau_2 \Phi_2 \, , \tag{3.5}
\]

where \(\delta_{ij}\) is the usual Kronecker symbol.

The form of the derived effective Lagrangian depends, to some extent, on the choice of renormalization scheme. As usual, one may adopt the \(\overline{\text{MS}}\) or \(\overline{\text{DR}}\) schemes of renormalization. In general, the different schemes affect the holomorphic part of the Lagrangian at the one-loop level. Thanks to the non-renormalization theorems of SUSY, the Yukawa couplings \(h_u,d\) are not renormalized, and the wave functions of \(\Phi_1,2, Q_{iL}, u_{iR}\) and \(d_{iR}\) remove the ultraviolet (UV) divergences of the one-loop corrections to the Yukawa couplings \(\tilde{d}_{iR} \Phi_1^T Q_{jL}\) and \(\bar{u}_{iR} \Phi_2 Q_{jL}\). The left-over UV-finite terms are not \(\tan \beta\)-enhanced and can be absorbed into the definition of \(h_u,d\), up to higher-order scheme-dependent corrections. Although the latter could be consistently included in our gauge-symmetric and flavour-covariant formalism, we ignore these small UV-finite holomorphic terms as they are higher-order effects beyond the one-loop approximation of our interest.

By analogy, the gauge- and flavour-covariant effective Lagrangian for the up-type quark self-energies may be written down as follows:

\[
- L_{\text{eff}}^u[\Phi_1, \Phi_2] = \tilde{u}_{iR}^0 \left( h_u \Phi_2^T (-i\tau_2) + \Delta h_u[\Phi_1, \Phi_2] \right) Q_{jL}^0 + \text{h.c.} , \tag{3.6}
\]

where \(\Delta h_u[\Phi_1, \Phi_2]\) may be calculated from Feynman diagrams analogous to Fig. 1. As opposed to the down-type quark self-energy case, these radiative corrections are not enhanced for large values of \(\tan \beta\) and so are ignored in our numerical analysis in Section 5.

The weak quark chiral states, \(u_{iL,R}^0\) and \(d_{iL,R}^0\), are related to their respective mass eigenstates, \(u_{iL,R}\) and \(d_{iL,R}\), through the unitary transformations:

\[
u_{iL}^0 = U_{L}^Q u_{iL} \, , \quad d_{iL}^0 = U_{L}^Q V d_{iL} \, , \quad u_{iR}^0 = U_{R}^u u_{iR} \, , \quad d_{iR}^0 = U_{R}^d d_{iR} , \tag{3.7}
\]

where \(U_{L}^Q, U_{R}^u,d\) are \(3 \times 3\) unitary matrices and \(V\) is the CKM mixing matrix. All these unitary matrices are determined by the simple mass renormalization conditions:

\[
\left< L_{\text{eff}}^d[\Phi_1, \Phi_2] \right> = - \bar{d}_{iR} \tilde{M}_d d_{iL} + \text{h.c.} , \quad \left< L_{\text{eff}}^u[\Phi_1, \Phi_2] \right> = - \bar{u}_{iR} \tilde{M}_u u_{iL} + \text{h.c.} , \tag{3.8}
\]
where $\langle \ldots \rangle$ denotes the value when the Higgs doublets $\Phi_{1,2}$ acquire their VEVs, and $\hat{M}_{u,d}$ are the physical diagonal mass matrices for the up- and down-type quarks. Imposing the conditions (3.8) yields [12]

$$U_R^d h_d U_L^Q = \frac{\sqrt{2}}{v_1} \hat{M}_d V_{R_d}^{-1}, \quad U_R^u h_u U_L^Q = \frac{\sqrt{2}}{v_2} \hat{M}_u V_{R_u}^{-1},$$

where

$$R_d = 1 + \frac{\sqrt{2}}{v_1} U_L^Q \left\langle h_d^{-1} \Delta h_d [\Phi_1, \Phi_2] \right\rangle U_L^Q,$$

$$R_u = 1 + \frac{\sqrt{2}}{v_2} U_L^Q \left\langle h_u^{-1} \Delta h_u [\Phi_1, \Phi_2] \right\rangle U_L^Q.$$ (3.10)

In (3.10) and in the following, the symbol $1$ without a subscript will always denote the $3 \times 3$ unit matrix. We observe that the unitary matrices $U_L^Q$, $U_R^{u,d}$ can all be set to $1$ by virtue of the flavour transformations given in (2.4). The Yukawa couplings $h_{u,d}$ are determined by the physical mass conditions (3.9). It is important to remark here [12] that these conditions form a coupled system of non-linear equations with respect to $h_{u,d}$, since the Yukawa couplings also enter the right sides of (3.9) through the expressions $R_{d,u}$ in (3.10). In addition, one should notice that the physical CKM mixing matrix $V$ remains unitary throughout our effective Lagrangian approach. As we will see below and more explicitly in Appendix B, the unitarity of $V$ throughout the renormalization process is a crucial property for maintaining the gauge symmetries through the Ward identities (WIs) in our effective Lagrangian formalism.

We now consider the effective FCNC Lagrangian related to Higgs interactions to down-type quarks. From (3.1), we find that

$$-L_{\text{eff}}^{d,H} = \bar{d}_R \frac{h_d}{\sqrt{2}} \left[ \phi_1 \left( 1 + \Delta_{d_1}^\phi \right) - ia_1 \left( 1 + \Delta_{a_d}^\phi \right) + \phi_2 \Delta_{d_2}^\phi - i a_2 \Delta_{a_d}^\phi \right] V d_L$$

$$+ \bar{d}_R h_d \left[ \phi_1^\dagger \left( 1 + \Delta_{d_1}^\phi \right) + \phi_2^\dagger \Delta_{d_2}^\phi \right] u_L + \text{h.c.},$$

(3.11)

where the individual components of the Higgs doublets $\Phi_{1,2}$ are given by

$$\Phi_{1,2} = \begin{pmatrix} \frac{\phi_{1,2}^+}{\sqrt{2}} \\ v_{1,2} + \phi_{1,2} + ia_{1,2} \end{pmatrix}.$$ (3.12)

Moreover, the $3 \times 3$ matrices $\Delta_{d_1}^\phi$, $\Delta_{a_d}^\phi$ and $\Delta_{d_2}^\phi$ are given by

$$\Delta_{d_1}^\phi = \sqrt{2} \left\langle \frac{\delta}{\delta \phi_{1,2}} \Delta_d \right\rangle, \quad \Delta_{a_d}^\phi = i \sqrt{2} \left\langle \frac{\delta}{\delta a_{1,2}} \Delta_d \right\rangle, \quad \Delta_{d_2}^\phi = \left\langle \frac{\delta}{\delta \phi_{1,2}^+} \Delta_d \right\rangle,$$

(3.13)
where we have used the short-hand notation, $\Delta_d \equiv h_d^{-1} \Delta h_d[\Phi_1, \Phi_2]$, and suppressed the vanishing iso-doublet components on the LHS’s of (3.13). In the CP-violating MSSM, the weak-state Higgs fields $\phi_{1,2}$, $a_{1,2}$ and $\phi_{1,-2}$ are related to the neutral CP-mixed mass eigenstates $H_{1,2,3}$ [21,23], the charged Higgs boson $H^-$ and the would-be Goldstone bosons $G^0$ and $G^-$, associated with the $Z$ and $W^-$ bosons, through:

$$
\begin{align*}
\phi_1 &= O_{1i} H_i, & \phi_2 &= O_{2i} H_i, \\
a_1 &= c_\beta G^0 - s_\beta O_{3i} H_i, & a_2 &= s_\beta G^0 + c_\beta O_{3i} H_i, \\
\phi_- &= c_\beta G^- - s_\beta H^- , & \phi_+ &= s_\beta G^- + c_\beta H^- ,
\end{align*}
\tag{3.14}
$$

where $s_\beta \equiv \sin \beta$, $c_\beta \equiv \cos \beta$ and $O$ is an orthogonal $3 \times 3$ Higgs-boson-mixing matrix.

One may now exploit the properties of gauge- and flavour-covariance of the effective functional $\Delta_d[\Phi_1, \Phi_2]$ to obtain useful relations in the large-$\tan \beta$ limit. Specifically, $\Delta_d[\Phi_1, \Phi_2]$ should have the form:

$$
\Delta_d[\Phi_1, \Phi_2] = \Phi_1^i f_1 + \Phi_2^i f_2 ,
\tag{3.15}
$$

where $f_{1,2}\left(\Phi_1^i \Phi_1, \Phi_2^i \Phi_2, \Phi_1^i \Phi_2, \Phi_2^i \Phi_1\right)$ are calculable $3 \times 3$-dimensional functionals which transform as $h_d^i h_d$ or $h_u^i h_u$ under the flavour rotations (2.4). Given the form (3.15), it is then not difficult to show that in the infinite-$\tan \beta$ limit ($v_1 \to 0$),

$$
\lim_{v_1 \to 0} \frac{i}{\sqrt{2}} \left\langle \frac{\delta}{\delta a_2} \Delta_d \right\rangle = \frac{\sqrt{2}}{v_2} \left\langle \Delta_d \right\rangle , \quad \lim_{v_1 \to 0} \left\langle \frac{\delta}{\delta \phi_2} \Delta_d \right\rangle = \frac{\sqrt{2}}{v_2} \left\langle \Delta_d \right\rangle .
\tag{3.16}
$$

Very similar relations may be derived for the up-type quark sector, but in the limit of vanishing $\tan \beta$. As we show in Appendix B, Ward identities (WIs) involving the $W^-$ and $Z$-boson couplings to quarks give rise to the following exact relations:

$$
\Delta_d^{G_0} \equiv i \sqrt{2} \left\langle \frac{\delta}{\delta G^0} \Delta_d \right\rangle = \frac{\sqrt{2}}{v} \left\langle \Delta_d \right\rangle , \quad \Delta_d^{G^-} \equiv \left\langle \frac{\delta}{\delta G^-} \Delta_d \right\rangle = \frac{\sqrt{2}}{v} \left\langle \Delta_d \right\rangle ,
\tag{3.17}
$$

where $v = \sqrt{v_1^2 + v_2^2}$ is the VEV of the Higgs boson in the SM. Relations very analogous to those stated in (3.17) hold true for the up-type sector as well, i.e. $\Delta_u^{G_0} = \Delta_u^{G^+} = -\sqrt{2} \left\langle \Delta_u \right\rangle / v$, where the extra minus sign comes from the opposite isospin of the up-type quarks with respect to the down-type quarks.

For our phenomenological analysis in Section 4, we may conveniently express the general flavour-changing (FC) effective Lagrangian for the interactions of the neutral and charged Higgs fields to the up- and down-type quarks $u, \ d$ in the following form:

$$
\mathcal{L}_{FC} = - \frac{g}{2M_W} \left[ H_i \bar{d} \left( \tilde{M}_d^L g_{H_{i,d}^d}^L P_L + g_{H_{i,d}^d}^R \tilde{M}_d^R P_R \right) d + G^0 \bar{d} \tilde{M}_d^i \gamma_5 d \right]
$$
where the Higgs couplings in the flavour basis $U \leq U \to U_L = U_R^u = U_R^d = 1$ are given by

$$
g^L_{H,d d} = \frac{O_{i1}}{c_\beta} V^\dagger R^{-1}_d \left(1 + \Delta^\phi_d\right) V + \frac{O_{2i}}{c_\beta} V^\dagger R^{-1}_d \Delta^\phi_d \ V$$

$$+ iO_{3i} t_\beta V^\dagger R^{-1}_d \left(1 + \Delta^\alpha_d - \frac{1}{t_\beta} \Delta^\phi_d\right) V,$$

$$
g^R_{H,d d} = (g^L_{H, d d})^\dagger,$$

$$
g^L_{H, d u} = \frac{O_{i1}}{s_\beta} R^{-1}_u \Delta^\phi_u + \frac{O_{2i}}{s_\beta} R^{-1}_u \left(1 + \Delta^\phi_u\right)$$

$$+ iO_{3i} t_\beta^{-1} R^{-1}_u \left(1 - \Delta^\phi_u\right),$$

$$
g^R_{H, d u} = (g^L_{H, d u})^\dagger,$$

$$
g^L_{H, d u} = -t_\beta V^\dagger R^{-1}_d \left(1 + \Delta^\phi_d\right) + V^\dagger R^{-1}_d \Delta^\phi_d,$$

$$
g^R_{H, d u} = -t_\beta^{-1} V^\dagger \left(1 - \Delta^\phi_u\right) (R^{-1}_u)^\dagger - V^\dagger \left(\Delta^\phi_u\right)^\dagger (R^{-1}_u)^\dagger,$$

and $t_\beta \equiv \tan \beta$. We note that the Higgs-boson vertex-correction matrices for the up-type quarks, $\Delta^1_u, \Delta^2_u$ and $\Delta^3_u$, are defined as in (3.13).

The above general form of the effective Lagrangian $L_{FC}$ extends the one derived in [12] in several aspects. First, it consistently includes all higher-order terms of the form $(t_\beta m_\mu/M_{\text{SUSY}}^{n-1})$, which can become important in scenarios with large bottom-squark mixing [5]. Secondly, it does not suffer from the limitation that the soft SUSY-breaking scale should be much higher than the electroweak scale $M_Z$. Specifically, SM electroweak corrections may be included in the Coleman–Weinberg-type effective functionals $\Delta_{d,u}[\Phi_1, \Phi_2]$, provided the theory is quantized in non-linear gauges [24] that preserve the Higgs-boson low-energy theorem (HLET) [25]. Finally, the effective Lagrangian $L_{FC}$ implements properly all the gauge symmetries through the WIs as discussed in Appendix B.

The general FC effective Lagrangian (3.18) takes on the form presented in [12] in the single-Higgs-insertion approximation. In this case, the $\tan \beta$-enhanced threshold corrections
Figure 2: Two-Higgs-doublet model (2HDM) contribution to the one-loop self-energy graphs for down-type quarks in the single-Higgs-insertion approximation.

\[ \Delta d^a, \Delta \phi_d^2, \Delta \phi_d^2 \] and \( \langle \Delta_d \rangle \) are inter-related as follows:

\[
\frac{\sqrt{2}}{v_2} \langle \Delta_d \rangle = \Delta d^a = \Delta \phi_d^2 = \Delta \phi_d^2 = \left( \Delta \phi_d^+ \right)^\dagger, \tag{3.25}
\]

where \( \langle \Delta_d \rangle \) is given in the MSSM with MCPMFV by

\[
\frac{\sqrt{2}}{v_2} \langle \Delta_d \rangle = 1 \frac{2\alpha_3}{3\pi} \mu^M_3 I \left( \tilde{M}_Q^2, \tilde{M}_D^2, |M_3|^2 \right) + \frac{h_u^h}{16 \pi^2} \mu^A_u I \left( \tilde{M}_Q^2, \tilde{M}_U^2, |\mu|^2 \right) + \ldots, \tag{3.26}
\]

and \( I(x, y, z) \) is the one-loop function:

\[
I(x, y, z) = \frac{xy \ln(x/y) + yz \ln(y/z) + xz \ln(z/x)}{(x - y) (y - z) (x - z)}. \tag{3.27}
\]

The ellipses in (3.26) denote the small contributions coming from the Feynman diagram in Fig. 1(c), which has the same flavour structure as the gluino-mediated graph in Fig. 1(a), i.e., this contribution is flavour-singlet in the single-Higgs-insertion approximation. We remark, finally, that in writing down (3.26) we have not considered the RG-running effects on the squark mass matrices between \( M_{\text{GUT}} \) and \( M_{\text{SUSY}} \). These effects are important, and are taken into account in our numerical analysis in Section 5.

In addition to graphs involving SUSY particles, the two-Higgs-doublet model (2HDM) sector of the MSSM may also contribute significantly to the one-loop self-energy graphs of the down quarks. This contribution is shown in Fig. 2 and is formally enhanced at large \( \tan \beta \), since it is proportional to \( h_d \). In the single-Higgs-insertion approximation, the 2HDM contribution is given by

\[
\frac{\sqrt{2}}{v_2} \langle \Delta_d^{2\text{HDM}} \rangle = \frac{h_u^h}{16 \pi^2} \frac{B^* \mu^*}{M_{H_d}^2 - M_{H_u}^2} \ln \left| \frac{M_{H_d}^2 + |\mu|^2}{M_{H_u}^2 + |\mu|^2} \right|. \tag{3.28}
\]
This contribution turns out to be subleading with respect to the Feynman diagram 1(b) and exhibits a very similar flavour structure. Beyond the single-Higgs-insertion approximation, the effective functional $\Delta h_d^{2\text{HDM}}[\Phi_1, \Phi_2]$ is calculated as

$$
(\Delta h_d^{2\text{HDM}})_{ij} = \int \frac{d^4k}{(2\pi)^n} \left( b_d \right)_{ij} P_L \left( \frac{1}{k^n M q P_L - M q P_R} \right) Q i a_k P_L (h_d)_{kj} \times \left( \frac{1}{k^2 1_4 - M_H^2} \right)_{\phi_1, \phi_2} .
$$

(3.29)

where $M q[\Phi_1, \Phi_2]$ and $M_H^2[\Phi_1, \Phi_2]$ are the $6 \times 6$- and $4 \times 4$-dimensional quark and Higgs-boson mass matrices in the background of non-zero $\Phi_{1,2}$. The $6 \times 6$-dimensional quark mass matrix is given by

$$
M_q[\Phi_1, \Phi_2] = \begin{pmatrix}
(M_q)_{\bar{u}, Q_1} \\
(M_q)_{\bar{d}, Q_1}
\end{pmatrix} = \begin{pmatrix}
(h_u)_{ij} \Phi_2^T (-i\tau_2) \\
(h_d)_{ij} \Phi_1^T
\end{pmatrix} .
$$

(3.30)

The Higgs-boson background mass matrix $M_H^2[\Phi_1, \Phi_2]$ receives appreciable radiative corrections beyond the tree level [19, 21, 23, 26]. At the tree level, the $4 \times 4$-dimensional matrix $M_H^2[\Phi_1, \Phi_2]$ is given in the weak basis $(\Phi_1, \Phi_2)$ by

$$
M_H^2[\Phi_1, \Phi_2] = \begin{pmatrix}
(M_H^2)_{\phi_1^\dagger \phi_1} & (M_H^2)_{\phi_1^\dagger \phi_2} \\
(M_H^2)_{\phi_2^\dagger \phi_1} & (M_H^2)_{\phi_2^\dagger \phi_2}
\end{pmatrix} ,
$$

(3.31)

where

$$
(M_H^2)_{\phi_1^\dagger \phi_1} = \left( \frac{M_H^2}{2} + |\mu|^2 + \frac{g^2 + g^2}{2} \Phi_1^\dagger \Phi_1 + \frac{g^2 - g^2}{4} \Phi_2^\dagger \Phi_2 \right) 1_2 + \frac{g^2}{2} \Phi_2 \Phi_2^\dagger ,
$$

$$
(M_H^2)_{\phi_2^\dagger \phi_2} = \left( \frac{M_H^2}{2} + |\mu|^2 + \frac{g^2 + g^2}{2} \Phi_2^\dagger \Phi_2 + \frac{g^2 - g^2}{4} \Phi_1^\dagger \Phi_1 \right) 1_2 + \frac{g^2}{2} \Phi_1 \Phi_1^\dagger ,
$$

$$
(M_H^2)_{\phi_1^\dagger \phi_2} = \left( \frac{M_H^2}{2} + |\mu|^2 + \frac{g^2 + g^2}{2} \Phi_1^\dagger \Phi_1 + \frac{g^2 - g^2}{4} \Phi_2^\dagger \Phi_2 \right) 1_2 + \frac{g^2 - g^2}{4} \Phi_1 \Phi_2^\dagger .
$$

(3.32)

In the one-loop effective Lagrangian $\mathcal{L}_{\text{FC}}$ given in (3.18), the couplings of the Goldstone bosons $G^0$ and $G^\pm$ to quarks retain their tree-level form. This result is not accidental, but a consequence of the Goldstone theorem, which applies when the momenta of the external particles are all set to zero. However, the tree-level form of the Goldstone couplings gets modified when momentum-dependent (derivative) terms are considered. To leading order in a derivative expansion, one would have to consider the effective Lagrangian

$$
\mathcal{L}_D = i \bar{Q} L \left[ Z_Q \bar{\psi} + A_Q^{(i,j)} \left( \Phi_i^\dagger (\bar{\psi} \Phi_j) - (\bar{\psi} \Phi_j)^\dagger \Phi_i \right) + B_Q^{(i,j)} \left( \Phi_i (\bar{\psi} \Phi_j)^\dagger - (\bar{\psi} \Phi_j) \Phi_i^\dagger \right) \right] Q_L + \ldots ,
$$

(3.33)
where the dots denote analogous terms for the right-handed up- and down-type quarks $u_R$ and $d_R$. The first term depending on $Z_Q$ is a functional of $\Phi_{1,2}$ for the left-handed quarks $Q_L$. Such a term is not $\tan\beta$-enhanced and renormalization-scheme dependent. As mentioned above, these terms can be neglected to a good approximation. The effective functionals $A_Q^{(i,j)}[\Phi_1, \Phi_2]$ and $B_Q^{(i,j)}[\Phi_1, \Phi_2]$ are UV finite and include large Yukawa-coupling effects due to $h_t$. In particular, this is the case for the effective functionals with $i = j = 2$. One typical graph of such a contribution is displayed in Fig. 3. Because of gauge invariance, analogous contributions will be present in the one-loop $Z$- and $W$-boson couplings. All these effects are not enhanced by $\tan\beta$, and can be consistently neglected without spoiling the gauge symmetries of the effective Lagrangian $\mathcal{L}_{\text{FC}}$.

In the next two sections, we present analytic and numerical results related to FCNC $B$-meson observables, using the effective Lagrangian (3.18) and including the 2HDM contribution (3.29).

4 FCNC $B$-Meson Observables

In this section, our interest will be in FCNC $B$-meson observables, such as the $B_{d,s}^0 - \bar{B}_{d,s}^0$ mass differences $\Delta M_{B_{d,s}}$, and the decays $B_{s,d} \to \mu^+\mu^-$, $B_u \to \tau\nu$ and $B \to X_s\gamma$. These effects have first been identified and studied in [27] within the Standard Model.
4.1 $\Delta M_{B_d,s}$

Our discussion and conventions here follow closely [12]. In the approximation of equal $B$-meson lifetimes, the SM and SUSY contributions to $\Delta M_{B_d,s}$ may be written separately, as follows:

$$
\Delta M_{B_q} = 2 \left| \langle \bar{B}_q^0 | H_{\text{eff}}^{\Delta B=2} | B_q^0 \rangle \right|_{\text{SM}} + \langle \bar{B}_q^0 | H_{\text{eff}}^{\Delta B=2} | B_q^0 \rangle |_{\text{SUSY}},
$$

(4.1)

where $q \equiv d, s$ and $H_{\text{eff}}^{\Delta B=2}$ is the effective $\Delta B = 2$ Hamiltonian. Neglecting the subdominant SM contribution, the SUSY contributions to the $\Delta B = 2$ transition amplitudes are given by

$$
\langle \bar{B}_d^0 | H_{\text{eff}}^{\Delta B=2} | B_d^0 \rangle |_{\text{SUSY}} = 1711 \text{ ps}^{-1} \left( \frac{\hat{B}_{B_d}^{1/2} F_{B_d}}{230 \text{ MeV}} \right)^2 \left( \frac{\eta_B}{0.55} \right)
\times \left[ 0.88 \left( C_2^{\text{LR(DP)}} + C_2^{\text{LR(2HDM)}} \right) - 0.52 \left( C_1^{\text{SLL(DP)}} + C_1^{\text{SRR(DP)}} \right) \right],
$$

$$
\langle \bar{B}_s^0 | H_{\text{eff}}^{\Delta B=2} | B_s^0 \rangle |_{\text{SUSY}} = 2310 \text{ ps}^{-1} \left( \frac{\hat{B}_{B_s}^{1/2} F_{B_s}}{265 \text{ MeV}} \right)^2 \left( \frac{\eta_B}{0.55} \right)
\times \left[ 0.88 \left( C_2^{\text{LR(DP)}} + C_2^{\text{LR(2HDM)}} \right) - 0.52 \left( C_1^{\text{SLL(DP)}} + C_1^{\text{SRR(DP)}} \right) \right],
$$

(4.2)

where DP stands for the Higgs-mediated double-penguin effect. In addition, we have used the next-to-leading order QCD factors determined in [28–32], along with their hadronic matrix elements at the scale $\mu = 4.2$ GeV:

$$
P_1^{\text{LR}} = -0.58, \quad P_2^{\text{LR}} = 0.88, \quad P_1^{\text{SLL}} = -0.52, \quad P_2^{\text{SLL}} = -1.1.
$$

(4.3)

The Wilson coefficients occurring in (4.2) are given by

$$
C_1^{\text{SLL(DP)}} = - \frac{16\pi^2 m_b^2}{\sqrt{2} G_F M_W^2} \sum_{i=1}^3 \frac{g_{H_i b q}^L g_{H_i b q}^L}{M_{H_i}^2},
$$

$$
C_1^{\text{SRR(DP)}} = - \frac{16\pi^2 m_q^2}{\sqrt{2} G_F M_W^2} \sum_{i=1}^3 \frac{g_{H_i b q}^R g_{H_i b q}^R}{M_{H_i}^2},
$$

$$
C_2^{\text{LR(DP)}} = - \frac{32\pi^2 m_b m_q}{\sqrt{2} G_F M_W^2} \sum_{i=1}^3 \frac{g_{H_i b q}^L g_{H_i b q}^R}{M_{H_i}^2},
$$

(4.4)

where the $\tan^2 \beta$-enhanced couplings $g_{H_i b d}^{L,R}$ may be obtained from (3.18). Hence, the DP Wilson coefficients in (4.4) have a $\tan^4 \beta$ dependence and, although two-loop suppressed, they become significant for large values of $\tan \beta \gtrsim 40$. 
There are two relevant one-loop contributions to $\langle \bar{B}_0 | H^{\Delta B=2}_\text{eff} | B^0 \rangle_{\text{SUSY}}$ at large $\tan \beta$: (i) the $t-H^\pm$ box contribution to $C_2^{LR}$ of the 2HDM type, and (ii) the one-loop chargino-stop box diagram contributing to $C_1^{SLL}$. To a good approximation, $C_2^{LR(2\text{HDM})}$ may be given by [32]

$$C_2^{LR(2\text{HDM})} \approx \frac{-2m_b m_q}{M_W^2} (V^*_{tb} V_{tq})^2 \tan^2 \beta.$$  \hfill (4.5)

In the kinematic region $M_{H^\pm} \approx m_t$, the above contribution can amount to as much as 10% of the DP effects mentioned above. This estimate is obtained by noticing that the light-quark masses in (4.4) and (4.5) are running and are evaluated at the top-quark mass scale, i.e., $m_s(m_t) \approx 90$ MeV, $m_d(m_t) \approx 4$ MeV [33]. The second contribution (ii) turns out to be non-negligible only for small values of the $\mu$-parameter [32], i.e., for $|\mu| \lesssim 200$ GeV.

### 4.2 $\bar{B}_{d,s}^0 \to \mu^+ \mu^-$

The leptonic decays of neutral $B$ mesons, $\bar{B}_{d,s}^0 \to \mu^+ \mu^-$, are enhanced at large values of $\tan \beta$ [6–15]. Neglecting contributions proportional to the lighter quark masses $m_{d,s}$, the relevant effective Hamiltonian for $\Delta B = 1$ FCNC transitions is given by

$$H^{\Delta B=1}_{\text{eff}} = -2 \sqrt{2} G_F V_{tb} V^*_{tq} \left( C_S \mathcal{O}_S + C_P \mathcal{O}_P + C_{10} \mathcal{O}_{10} \right),$$  \hfill (4.6)

where

$$\mathcal{O}_S = \frac{e^2}{16\pi^2} m_b \langle \bar{q} P_R b \rangle (\bar{\mu} \mu),$$

$$\mathcal{O}_P = \frac{e^2}{16\pi^2} m_b \langle \bar{q} P_R b \rangle (\bar{\mu} \gamma_5 \mu),$$

$$\mathcal{O}_{10} = \frac{e^2}{16\pi^2} (\bar{q} \gamma^\mu P_L b) (\bar{\mu} \gamma_\mu \gamma_5 \mu).$$ \hfill (4.7)

Using the resummed FCNC effective Lagrangian (3.18), the Wilson coefficients $C_S$ and $C_P$ in the region of large values of $\tan \beta$ are given by

$$C_S = \frac{2 \pi m_\mu}{\alpha_{em}} \frac{1}{V_{tb} V^*_{tq}} \sum_{i=1}^3 g^{R}_{H_i, q b} g^{S}_{H_i, \bar{\mu} \mu} \frac{M^2_{H_i}}{M^2_{H_i}},$$

$$C_P = i \frac{2 \pi m_\mu}{\alpha_{em}} \frac{1}{V_{tb} V^*_{tq}} \sum_{i=1}^3 g^{R}_{H_i, q b} g^{P}_{H_i, \bar{\mu} \mu} \frac{M^2_{H_i}}{M^2_{H_i}},$$ \hfill (4.8)

where $C_{10} = -4.221$ denotes the leading SM contribution. In addition, the reduced scalar and pseudoscalar Higgs couplings to charged leptons $g^{S,P}_{H_i, \bar{\mu} \mu}$ in (4.8) are given by

$$g^{S}_{H_i, \bar{\mu} \mu} = \frac{O_{1i}}{\cos \beta}, \quad g^{P}_{H_i, \bar{\mu} \mu} = -\tan \beta O_{3i}.$$ \hfill (4.9)
Here we neglect the non-holomorphic vertex effects on the leptonic sector since they are unobservably small.

Taking into consideration the aforementioned approximations, the branching ratio for $\bar{B}_q^0 \to \mu^+ \mu^-$ is found to be [8]

$$B(\bar{B}_q^0 \to \mu^+ \mu^-) = \frac{G_F^2 \alpha_{em}^2}{16\pi^3} M_{B_q} \tau_{B_q} |V_{tb} V_{ts}^*|^2 \sqrt{1 - \frac{4m_{\mu}^2}{M_{B_q}^2}} \left[ \left( 1 - \frac{4m_{\mu}^2}{M_{B_q}^2} \right) |F_S^q|^2 + |F_F^q|^2 \right],$$

where $q = d, s$ and $\tau_{B_q}$ is the total lifetime of the $B_q$ meson. Moreover, the form factors $F_{S,P,A}^q$ are given by

$$F_{S,P}^q = -\frac{i}{2} M_{B_q}^2 \frac{m_b}{m_b + m_q} C_{S,P}, \quad F_A^q = -\frac{i}{2} F_{B_q} C_{10}. \quad (4.11)$$

Although the Wilson coefficient $C_{10}$ is subdominant for $\tan \beta > 40$, its effect has been included in our numerical estimates.

### 4.3 $B_u \to \tau \nu$

There is an important tree-level charged-Higgs boson contribution to $B_u \to \tau \nu$ decay [16, 17]. It is not helicity suppressed and interferes destructively with the SM contribution [34]. The ratio of the branching ratio to the SM value is given by

$$R_{B\tau \nu} = \frac{B(B^- \to \tau^- \bar{\nu})}{B^{SM}(B^- \to \tau^- \bar{\nu})} = \left| 1 + \tan \beta \left( \frac{g_{H^- da}^L}{V_{13}} \right) \left( \frac{M_{B^+}}{M_{H^+}} \right)^2 \right|^2, \quad (4.12)$$

where $g_{H^- da}^L = -\tan \beta V^\dagger_{13}$ at tree level [cf. (3.23)], leading to the negative interference with the SM contribution.

### 4.4 $B \to X_s \gamma$

The relevant effective Hamiltonian for $B \to X_s \gamma$ is given by

$$H_{eff}^{b \to s \gamma} = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \left\{ \sum_{i=2,7,8} C_i(\mu_b) O_i(\mu_b) + C_7'(\mu_b) O_7'(\mu_b) + C_8'(\mu_b) O_8'(\mu_b) \right\}, \quad (4.13)$$

with

$$O_2 = \bar{s} L \gamma_\mu c_L \bar{c} L \gamma^{\mu} b_L,$$

$$O_7 = \frac{e m_b}{16 \pi^2} \bar{s}_L \sigma_{\mu \nu} F^{\mu \nu} b_R; \quad O_7' = \frac{e m_b}{16 \pi^2} \bar{s}_R \sigma_{\mu \nu} F^{\mu \nu} b_L,$$

$$O_8 = \frac{g_s m_b}{16 \pi^2} \bar{s}_L \sigma_{\mu \nu} F^{\mu \nu} b_R; \quad O_8' = \frac{g_s m_b}{16 \pi^2} \bar{s}_R \sigma_{\mu \nu} F^{\mu \nu} b_L. \quad (4.14)$$
We closely follow the calculations of Refs. [35] for the branching ratio $B(B \to X_s \gamma)$ and the direct CP asymmetry in the decay. For the running $c$ quark mass, we use $m_c(m_c^{pole})$ to capture a part of NNLO corrections [36]. We refer to, for example, Appendix B of Ref. [37] for the detailed expression of the branching ratio in terms of the Wilson coefficients which we are going to present below.

The LO charged-Higgs contribution is given by

$$C_{7,8}^{(0) H^\pm} (M_W) = \frac{1}{3} \frac{g_{H^- d_u}^{R^+} (g_{H^- d_u}^{R^+} + g_{H^+ d_u}^{L^+} + g_{H^+ d_u}^{L^+})}{V_{33} V_{23}^\dagger} F_{7,8}^{(1)} (y) + \frac{g_{H^- d_u}^{L^+}}{V_{33} V_{23}^\dagger} F_{7,8}^{(2)} (y),$$

where $y \equiv m_t^2 / M_{H^\pm}$, the ratio of the top-quark running mass at the scale $M_W$ to the charged Higgs-boson pole mass. In the numerical analysis, we include the NLO contribution. Note that $g_{H^\pm d_u}^{R^+} = -t_{\beta} \mathbf{V}^\dagger$ and $g_{H^\pm d_u}^{L^+} = -t_{\beta} \mathbf{V}^\dagger$ at tree level, see Eqs. (3.23) and (3.24). The functions $F_{7,8}^{(1),(2)}$ can be found in Ref. [37, 38].

The chargino contributions are

$$C_{7,8}^{\chi^\pm (\mu_{SUSY})} = \sum_{i=1, 2} \left\{ \frac{2}{3} \frac{M_{W_i}^2}{m_{\tilde{q}}^2} |(C_R)_{i1}|^2 F_{7,8}^{(1)} (x_{\chi_i^\pm}) - \frac{(V_{R_d}^{\dagger})_{i3} V_{21}^\dagger + (V_{L_d}^{\dagger})_{i3} V_{22}^\dagger}{c_\beta V_{33} V_{23}^\dagger} \frac{(C_L)_{i2} (C_R)_{i1} M_W}{\sqrt{2} m_{\chi_i^\pm}} F_{7,8}^{(3)} (x_{\chi_i^\pm}) \right\},$$

$$+ \frac{2}{3} \sum_{j=1, 2} \left\{ (C_R)_{i1} (U_{j1}^\dagger)^* - \frac{\langle M_u R_u \rangle_{i33}}{\sqrt{2} s_\beta M_W} (C_R)_{i1} (U_{j2}^\dagger)^* \right\} \frac{M_{W_j}^2}{m_{\tilde{t}_j}} F_{7,8}^{(4)} (x_{\tilde{t}_j \chi_i^\pm}) + \frac{(V_{R_d}^{\dagger})_{i33}}{c_\beta V_{33}} \sum_{j=1, 2} \left\{ (C_L)_{i2} (C_R)_{i1} M_W \right\} \left\{ \frac{U_{j1}^\dagger}{\sqrt{2} m_{\chi_i^\pm}} \right\} \frac{M_{W_j}^2}{m_{\tilde{t}_j}} F_{7,8}^{(3)} (x_{\tilde{t}_j \chi_i^\pm}) \right\},$$

where $x_{ij} \equiv m_{\tilde{t}_j}^2 / m_{\tilde{t}_j}^2$. We refer to [39] for the functions $F_{7,8}^{(3)}$ and to [40] for the chargino mixing matrices $C_{L,R}$ and the stop mixing matrix $U_{t1}^\dagger$.

Finally, the gluino contributions to the Wilson coefficients $C_{7,8}$ are given by

$$C_{7,8}^{g (\mu_{SUSY})} = - \frac{8 \pi \alpha_s}{9 \sqrt{2} G_F |M_3|^2 \lambda_t} \sum_{i=1}^6 x_i (G_{L}^{d})_{i2}^* \times \left\{ (G_{L}^{d})_{i3} f_2 (x_i) + (G_{R}^{d})_{i3} \frac{M_3}{m_b} f_4 (x_i) \right\},$$

$$C_{7,8}^{g (\mu_{SUSY})} = - \frac{\pi \alpha_s}{\sqrt{2} G_F |M_3|^2 \lambda_t} \sum_{i=1}^6 x_i (G_{L}^{d})_{i2}^* \left\{ (G_{L}^{d})_{i3} \left[ 3 f_1 (x_i) + \frac{1}{3} f_2 (x_i) \right] \right\} + (G_{R}^{d})_{i3} \frac{M_3}{m_b} \left[ 3 f_3 (x_i) + \frac{1}{3} f_4 (x_i) \right],$$

(4.17)
where $\lambda_t \equiv V_{33} V_{23}^\dagger = V_{tb} V_{ts}^*$ and $x_i \equiv |M_3|^2/m_{\tilde{d}_i}^2$. The loop functions $f_{1,2,3,4}(x_i)$ may be found in Ref. [41]. The Wilson coefficients for the primed operators $O_{7,8}'$ can be obtained by the exchange $L \leftrightarrow R$ and $M_3 \rightarrow M_3'$:

$$C_{7}''(\mu_{\text{SUSY}}) = - \frac{8 \pi \alpha_s}{9 \sqrt{2} G_F |M_3|^2 \lambda_t} \sum_{i=1}^{6} x_i (G_R^d)_{i2}^* \times \left[ (G_R^d)_{i3} f_2(x_i) + (G_L^d)_{i3} \frac{M_3^*}{m_b} f_4(x_i) \right],$$

$$C_{8}''(\mu_{\text{SUSY}}) = - \frac{\pi \alpha_s}{\sqrt{2} G_F |M_3|^2 \lambda_t} \sum_{i=1}^{6} x_i (G_R^d)_{i2}^* \left\{ (G_R^d)_{i3} \left[ 3 f_1(x_i) + \frac{1}{3} f_2(x_i) \right] \\
+ (G_L^d)_{i3} \frac{M_3^*}{m_b} \left[ 3 f_3(x_i) + \frac{1}{3} f_4(x_i) \right] \right\}. \quad (4.18)$$

In the above, $C_{7,8}''$, the down-type squark-gluino-quark couplings $G_{L,R}^d$ are defined through the interaction Lagrangian (suppressing the colour indices)

$$\mathcal{L}_{d\tilde{g}d} = -\sqrt{2} g_s \left\{ \tilde{d}_i^* t^a \tilde{g}^a \left[ (G_L^d)_{i\alpha} P_L + (G_R^d)_{i\alpha} P_R \right] d_{\alpha} \right.\\
\left. + \tilde{d}_\alpha \left[ (G_L^d)^*_{i\alpha} P_R + (G_R^d)^*_{i\alpha} P_L \right] \tilde{g}^a t^a \tilde{d}_i \right\}, \quad (4.19)$$

where $t^a$ are the usual Gell-Mann matrices, $i = 1, 2, \ldots, 6$ label the mass eigenstates of down-type squarks, and $\alpha = 1, 2, 3$ label the mass eigenstates of down-type quarks. The couplings are given by the down-type squark mixing matrix as

$$(G_L^d)_{i\alpha} = \left( U^{\dagger} \tilde{d} \right)_{i\alpha}, \quad (G_R^d)_{i\alpha} = - \left( U^{\dagger} \tilde{d} \right)_{i\alpha + 3}. \quad (4.20)$$

The $6 \times 6$ unitary matrix $U^{\dagger} \tilde{d}$ diagonalizes the down-type squark mass matrix as

$$U^{\dagger} \tilde{M}_d^2 U^{\dagger} \tilde{d} = \text{diag}(m_{\tilde{d}_1}^2, m_{\tilde{d}_2}^2, \ldots, m_{\tilde{d}_6}^2), \quad (4.21)$$

where $\tilde{d}_1$ is the lightest and $\tilde{d}_6$ the heaviest. In the super-CKM basis, in which the down squarks are aligned with the down quarks and $U_L^Q = U_R^q = U_R^d = 1$, the $6 \times 6$ down-type squark mass matrix $\tilde{M}_d^2$ takes on the form

$$\tilde{M}_d^2 = \begin{pmatrix} \tilde{V}^\dagger \tilde{M}_{LL}^2 \tilde{V} & \tilde{V}^\dagger \tilde{M}_{LR}^2 \\
\tilde{M}_{RL}^2 \tilde{V} & \tilde{M}_{RR}^2 \end{pmatrix}, \quad (4.22)$$

where the $3 \times 3$ submatrices are given by

$$\tilde{M}_{LL}^2 = \tilde{M}_{QQ}^2 + \frac{1}{2} \left( h_d^* h_d \right) + c_{2\beta} M_Z^2 \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) 1, \quad (4.23)$$

where $M_{\tilde{Q}}$ is the seesaw scale of the squarks, $h_{\tilde{d}}$ is the Majorana mass of the down-type squarks, $c_{2\beta}$ is the cosine of twice the mixing angle, and $M_Z$ is the electroweak gauge boson mass.
\[ \tilde{M}_{LR}^2 = \frac{1}{\sqrt{2}} a_d^i v_1 - \frac{1}{\sqrt{2}} h_d^i \mu v_2, \]
\[ \tilde{M}_{RL}^2 = \frac{1}{\sqrt{2}} a_d v_1 - \frac{1}{\sqrt{2}} h_d^{*} v_2, \]
\[ \tilde{M}_{RR}^2 = \tilde{M}_D^2 + \frac{v_1^2}{2} (h_d h_d^*) + c_{2\beta} M_Z^2 \left( -\frac{1}{3} s_W^2 \right) 1, \]

with \( h_d = \frac{\sqrt{2}}{v_1} \tilde{M}_d V_d^{-1} \). As a byproduct of the chosen super-CKM basis, we observe the absence of flavour mixing in \( M_d^2 \), for all \( h_d \)-dependent terms, when \( R_d \propto 1 \).

### 5 Numerical Examples

For our numerical estimates of FCNC observables at large \( \tan \beta \), we take the GUT scale to be the same as in the usual CMSSM with MFV, and a dedicated program has been developed to calculate the RG evolution from the GUT scale to the low-energy SUSY scale in the MCPMFV framework of the MSSM. For the Higgs mass spectrum and the mixing matrix \( O_{ai} \) at the \( M_{\text{SUSY}} \) scale, the code \texttt{CPsuperH} [40] has been used. In the calculation of the flavour-changing effective couplings, only the leading contributions have been kept in the single-Higgs-insertion approximation, neglecting the EW corrections and the generically small flavour-off-diagonal elements of the squark mass matrices.

In order to study the effects of CP-violating phases in the MCPMFV framework, we consider a CP-violating variant of a typical CMSSM scenario:

\[
|M_{1,2,3}| = 250 \text{ GeV}, \\
M_{H_u}^2 = M_{H_d}^2 = \tilde{M}_Q^2 = \tilde{M}_U^2 = \tilde{M}_D^2 = \tilde{M}_L^2 = (100 \text{ GeV})^2, \\
|A_u| = |A_d| = |A_e| = 100 \text{ GeV},
\]

at the GUT scale with \( \tan \beta (M_{\text{SUSY}}) = 10 \), which corresponds to \( \tan \beta (m_t^{\text{pole}}) \approx 10.2 \). As for the CP-violating phases, we adopt the convention that \( \Phi_\mu = 0^\circ \), and we vary the following three phases:

\[
\Phi_{12} \equiv \Phi_1 = \Phi_2; \quad \Phi_3; \quad \Phi_{A}^{\text{GUT}} \equiv \Phi_{A_u} = \Phi_{A_u} = \Phi_{A_e},
\]

where, for simplicity, common phases \( \Phi_{12} \) and \( \Phi_{A}^{\text{GUT}} \) are taken for the phases of \( M_{1,2}(M_{\text{GUT}}) \) and \( A_{u,d,e}(M_{\text{GUT}}) \), respectively. We note that the phases of the gaugino mass parameters, \( \Phi_{1,2,3} \), and the \( \mu \) parameter, \( \Phi_\mu \), are unchanged by the RG evolution, whilst the phases of the elements of the matrix \( A_{u,d,e} \) could be significantly different at low scales from the values given at the GUT scale. This scenario becomes the SPS1a point [42] when \( \Phi_{1,2,3} = 0^\circ \)
and $\Phi_A^{\text{GUT}} = 180^\circ$. We have found that $M_{\text{SUSY}}$ varies between 530 GeV and 540 GeV, and $M_{\text{GUT}}/10^{16}$ GeV between 1.825 and 1.838 depending on the values of the CP-violating phases.

We do not consider in this section the electric dipole moment constraints [43] on the MCPMFV parameter space of the MSSM. A systematic implementation of these constraints and their impact on the FCNC observables will be given in a forthcoming communication.

### 5.1 Phases and Masses

We first consider the $(3,3)$ elements $A_f^3 \equiv (a_f)_{33}/(h_f)_{33}$ at $M_{\text{SUSY}}$ with $f = u, d, e$ and $f_3 = t, b, \tau$. We find that the complex quantity $A_f^3$ can be written in terms of the complex $A_f$ and $M_j$ at the GUT scale as:

$$A_f^3(M_{\text{SUSY}}) \approx C_{f_3}^{A_f} A_f(M_{\text{GUT}}) - C_{f_3}^{M_i} M_i(M_{\text{GUT}}),$$

where the real coefficients $C_{f_3}^{A_f}$ and $C_{f_3}^{M_i}$ are functions of the Yukawa and gauge couplings. This expression is similar to that found in Ref. [44]. In general, $C_{t,b}^{A_u,d}$ are much smaller than $C_{t,b}^{M_{1,2}}$. Indeed, they are even smaller than $C_{t,b}^{M_1,2}$ with $C_{t,b}^{A_u} < C_{b}^{A_d}$. For $A_{\tau}$, $C_{\tau}^{A_e}$ is not so much smaller than $C_{\tau}^{M_1,2}$, whilst $C_{\tau}^{M_3}$ is negligible. This is because the strong coupling amplifies the influence of $M_3$, while the large Yukawa couplings suppress those of the $A$ terms via renormalization effects [44]. For the parameter set (5.1) with $\tan \beta = 10$, we observe that the phases $\Phi_{A_t}(M_{\text{SUSY}})$ and $\Phi_{A_b}(M_{\text{SUSY}})$ are largely determined by $\Phi_3$, whereas the phase $\Phi_{A_{\tau}}(M_{\text{SUSY}})$ is more affected by $\Phi_{1,2}$ than by $\Phi_A^{\text{GUT}}$. This situation becomes different for larger values of $\tan \beta$, i.e. we find that $C_{\tau}^{M_3}$ becomes significant and $C_{b}^{A_d}$ decreases when $\tan \beta$ increases.

In Fig. 4 we show $\sin \Phi_{A_t}$, $\sin \Phi_{A_b}$, and $\sin \Phi_{A_{\tau}}$ for the parameter set (5.1) with $\tan \beta(M_{\text{SUSY}}) = 10$. In the left frames, we observe that $\Phi_{A_t,b}$ and $\Phi_{A_{\tau}}$ can be fully generated from $\Phi_3$ and $\Phi_{1,2}$, respectively, even when $A_{u,d,e}$ at the GUT scale are real, $\Phi_A^{\text{GUT}} = 180^\circ$. Whilst the dependence of $\Phi_{A_{\tau}}$ on $\Phi_3$ is negligible (solid line in the left-lower frame), the dependences of $\Phi_{A_{t,b}}$ on $\Phi_{1,2}$ can be sizeable (dashed lines in the left-upper and left-middle frames). In the right frames, the cases with $\Phi_3 = 0^\circ$ ($\Phi_{A_{t,b}}$) and $\Phi_{12} = 0^\circ$ ($\Phi_{A_{\tau}}$) are considered, showing how large the $A$-term phases may become at the $M_{\text{SUSY}}$ scale for real $M_3$ and/or real $M_1$ and $M_2$. When the gaugino masses are all real, $|\sin \Phi_{A_t}|$ and $|\sin \Phi_{A_b}|$ turn out to be 0.06 and 0.12, respectively, whereas $|\sin \Phi_{A_{\tau}}|$ can be as large as 0.55. Somewhat larger CP-violating phases are possible for $\Phi_{A_t}$ and $\Phi_{A_b}$ when $M_1$ and $M_2$ are pure imaginary (see dashed and dash-dotted lines in the right-upper and right-middle frames of Fig. 4). Finally, there are no visible effects of $\Phi_3$ on $\Phi_{A_{\tau}}$. 
Figure 4: In the left frames, taking \( \Phi_A^{\text{GUT}} = 180^\circ \), \( \sin \Phi_A \) (upper), \( \sin \Phi_A \) (middle), and \( \sin \Phi_{A_r} \) (lower) are shown as functions of \( \Phi_3 \) taking \( \Phi_{12} = 0^\circ \) (solid lines) and \( \Phi_{12} \) taking \( \Phi_3 = 0^\circ \) (dashed lines). In the right frames they are shown as functions of \( \Phi_A^{\text{GUT}} \) taking \( \Phi_{12} = 0^\circ \) or \( \Phi_{12} = 0^\circ \). For \( \sin \Phi_A \) and \( \sin \Phi_A \), three cases are shown: \( \Phi_{12} = 270^\circ \) (blue dash-dotted lines), \( 0^\circ \) (black solid lines), and \( 90^\circ \) (red dashed lines). For \( \sin \Phi_{A_r} \), we set \( \Phi_3 = 0^\circ \) as well. The parameters are taken as in Eq. (5.1) with \( \tan \beta(M_{\text{SUSY}}) = 10 \).

We now discuss the effects of CP-violating phases on the masses of Higgs bosons, third-generation squarks and heavy neutralinos and chargino. In the upper-left frame of Fig. 5, we show the absolute values of \( A_{t,b,\tau} \) as functions of a common phase \( \Phi_M \equiv \Phi_1 = \Phi_2 = \Phi_3 \)
Figure 5: The absolute values of $A_{t,b,\tau}$ (upper-left) and the masses of the heavy Higgs bosons (upper right), sbottoms and stops (lower left), and charginos and neutralinos (lower right) as functions of a common phase $\phi_M \equiv \Phi_1 = \Phi_2 = \Phi_3$. The solid lines are for $\phi_{A}^{GUT} = 180^\circ$ and the dashed lines for $\phi_{A}^{GUT} = 0^\circ$. The parameters are listed in Eq. (5.1).

for two values of $\phi_{A}^{GUT}$: $0^\circ$ (dashed lines) and $180^\circ$ (solid lines). In this case, one can show the absolute values squared depend only on the difference $\phi_{A}^{GUT} - \phi_M$:

$$|A_f|^2 \approx \alpha_f - \beta_f \cos(\phi_{A}^{GUT} - \phi_M),$$

(5.4)

using Eq. (5.3), with $\alpha_f, \beta_f > 0$. From Fig. 5, we observe that there is strong correlation
between $|A_{t,b,\tau}|$ and the particle mass spectrum. This correlation is due to the phase-dependent terms $\text{Tr}(a^\dagger_i a_u)$ and $\text{Tr}(a^\dagger_i a_d)$ in $dM^2_{H_u,H_d}/dt$ and $dM^2_{\tilde{Q},U,D}/dt$. The fact that $|M^2_{H_u}|$ decreases (increases) when $\text{Tr}(a^\dagger_i a_u)$ decreases (increases) explains the CP-odd phase dependence of heavier Higgs-boson masses, as can be seen from the upper-right frame of Fig. 5. The same correlation is observed for the heavy chargino and neutralinos in the lower-right frame of Fig. 5, since a decreased (increased) value of $|\mu|$ leads to smaller (larger) values of $|\mu|$. We find that the variations in the masses of the lightest Higgs boson $H_1$ and the lightest neutralino $\tilde{\chi}^0_1$ amount to 2 GeV and 3 GeV, respectively. The CP-odd phase dependences of $\tilde{M}^2_Q$, $\tilde{M}^2_U$, and $\tilde{M}^2_D$ at the scale $M_{\text{SUSY}}$ can be understood similarly. Here the (3,3) components of the mass matrices decrease (increase) when $\text{Tr}(a^\dagger_i a_u)$ decreases (increases). For the chosen value of $\tan \beta(M_{\text{SUSY}}) = 10$, the (3,3) component of $\tilde{M}^2_U$ shows the largest effect, since $d\tilde{M}^2_U/\!\!\!dt$ contains $2\text{Tr}(a^\dagger_i a_u)$ compared to $\text{Tr}(a^\dagger_i a_u) + \text{Tr}(a^\dagger_i a_d)$ in $d\tilde{M}^2_Q/\!\!\!dt$ and $2\text{Tr}(a^\dagger_i a_d)$ in $d\tilde{M}^2_D/\!\!\!dt$. Furthermore, we note that $\tilde{t}_1 \sim \tilde{t}_R$ and $\tilde{b}_1 \sim \tilde{b}_L$. From these observations, one can understand the qualitative CP-odd phase dependence of the stop and sbottom masses, as shown in the lower-left frame of Fig. 5.

### 5.2 Effects on $\Delta M_{B_s}$ and $\Delta M_{B_d}$

In the upper-left frame of Fig. 6, we show the SUSY contribution to $\Delta M_{B_s}$ in units of ps$^{-1}$ as a function of $\tan \beta(M_{\text{SUSY}})$ for three values of the common phase, namely $\Phi_M = 0^\circ$ (solid line), $90^\circ$ (dashed line), and $180^\circ$ (dash-dotted line). The horizontal line is for the measured value: $\Delta M^{\text{EXP}}_{B_s} = 17.77 \pm 0.10$ (stat.) $\pm 0.07$ (syst.) ps$^{-1}$ [18]. We observe that the SUSY contribution can be larger than the current observed value for $\Phi_M = 180^\circ$ when $\tan \beta$ is large. Indeed, for $\Phi_M = 180^\circ$ ($90^\circ$), we find $\tan \beta < 44$ (48), whereas there is no restriction on $\tan \beta$ for $\Phi_M = 0^\circ$.

The SUSY contribution $C^{\text{SRR(DP)}}_1$ is suppressed by $m^2_H/m^2_\beta$ with respect to $C^{\text{SLL(DP)}}_1$ [see Eq. (4.4)]. The $|C^{\text{LR(DP)}}_2|$ is comparable to $|C^{\text{SLL(DP)}}_1|$, while the 2HDM contribution, $C^{\text{LR(2HDM)}}_2$, becomes less important as $\tan \beta$ increases. The dip of the coupling $|C^{\text{SLL(DP)}}_1|$ for $\Phi_M = 180^\circ$ (upper-right frame) at $\tan \beta \approx 45$ is due to the fact that the three Higgs bosons become degenerate and cancel other contributions. Beyond this point, $M_{H_1} \sim M_{H_2}$ decreases rapidly while $M_{H_3} \sim 110$ GeV remains nearly unchanged.

In the upper-left frame of Fig. 7, we show the SUSY contribution to $\Delta M_{B_d}$ in units of ps$^{-1}$ as a function of $\tan \beta(M_{\text{SUSY}})$, using the same line conventions as in Fig. 6. The horizontal line is for the measured value: $\Delta M^{\text{EXP}}_{B_d} = 0.507 \pm 0.005$ ps$^{-1}$ [45]. We observe that the SUSY contribution is always smaller than the measured value, although it does exhibit a strong dependence on the CP-violating phase $\Phi_M$. The dips at $\tan \beta \approx 45$
Figure 6: The SUSY contribution to $\Delta M_{B_s}$ in units of $\text{ps}^{-1}$ (upper-left) and the relevant couplings in the other three frames, as functions of $\tan \beta (M_{\text{SUSY}})$, for three values of the common phase: $\Phi_M = 0^\circ$ (solid lines), $90^\circ$ (dashed lines), and $180^\circ$ (dash-dotted lines). We fix $\Phi_G^\text{GUT} = 0^\circ$ and the parameters are taken as in Eq. (5.1), except that here we choose $\tilde{M}_{L,E} = 200 \text{ GeV}$ so as to avoid a very light or tachyonic $\tilde{\tau}_1$ state for large $\tan \beta$. In the upper-left frame, we show the currently measured value as the horizontal line.

$(\Phi_M = 180^\circ)$ and $\tan \beta \simeq 49$ ($\Phi_M = 90^\circ$) arise for the same reason as in the $\Delta M_{B_s}$ case. The dominant contribution comes from $C_1^{\text{SLL(DP)}}$, and $C_1^{\text{SRR(DP)}}$ is suppressed by $m_d^2/m_b^2$. The value of $|C_2^{\text{LR(DP)}}|$ is smaller than that of $|C_1^{\text{SLL(DP)}}|$. Finally, as before, the 2HDM

\[\text{(102x502)}\]
contribution $C_{2}^{LR(2HDM)}$ becomes less significant for large values of $\tan\beta$.

5.3 Effects on $B_s \rightarrow \mu^+ \mu^-$

In the upper-left frame of Fig. 8, we show the branching ratio $B(B_s \rightarrow \mu^+ \mu^-)$ as a function of $\tan\beta(M_{SUSY})$ using the same line conventions as in Fig. 6 for three values of the common
Figure 8: The branching ratio $B(B_s \rightarrow \mu^+ \mu^-)$ in the upper-left frame and the relevant couplings in the other three frames, in units of GeV$^{-1}$ as functions of $\tan \beta (M_{\text{SUSY}})$. The line conventions and the parameters chosen are the same as in Fig. 6, except that the two horizontal lines in the upper-left frame are for the SM prediction and the current upper limit at 90 % C.L. We observe that the branching ratio changes substantially as $\Phi_M$ varies. Specifically, for $\Phi_M = 180^\circ$ (90$^\circ$) 0$^\circ$, we find that the present
The phase dependence of the branching ratio comes from that of the couplings $C_S$ and $C_P$ [see (4.8)], which are shown in the upper-right and the lower-left frames, respectively. We find that $|C_S| \simeq |C_P|$, since $O_{11} \sim O_{a1} \sim 0$ and $M_{H_2} \sim M_{H_3}$ [cf. (4.8) and (4.9)]. We note that, for $\Phi_M = 180^\circ$, $B(B_s \to \mu^+\mu^-)$ can be smaller than the SM prediction for $\tan \beta \sim 24$. This is because the Higgs-mediated contribution $C_P$ cancels the SM one $C_{10}$, as shown in the lower-right frame of Fig. 8, in which the factor $m_b/(m_b + m_s)$ [cf. (4.11)] has been suppressed in the label of the $y$-axis.

5.4 Effects on $B^u \to \tau \nu$

The recent BELLE and BABAR results for the branching ratio $B(B^- \to \tau^-\bar{\nu})$ are [46,47]

\[
B(B^- \to \tau^-\bar{\nu})^{\text{BELLE}} = (1.79^{+0.56}_{-0.49} \text{ (stat)} + ^{+0.46}_{-0.51} \text{ (syst)}) \times 10^{-4},
\]

\[
B(B^- \to \tau^-\bar{\nu})^{\text{BABAR}} = (1.2 \pm 0.4 \text{ (stat)} \pm 0.3 \text{ (bkg syst)} \pm 0.2 \text{ (other syst)}) \times 10^{-4},
\]

which lead to $B(B^- \to \tau^-\bar{\nu})^{\text{EXP}} = (1.4 \pm 0.43) \times 10^{-4}$. Combining the BELLE and BABAR results with the SM value $B(B^- \to \tau^-\bar{\nu})^{\text{SM}} = (1.41 \pm 0.33) \times 10^{-4}$ obtained by the global fit without using $B(B^- \to \tau^-\bar{\nu})$ as an input [48], we have the following 1 $\sigma$ range for the ratio to the SM prediction $^3$:

\[
R_{B^\tau\nu}^{\text{EXP}} = 1.0 \pm 0.38.
\]

In the upper-left frame of Fig. 9, we show possible values of this ratio in the MSSM with MCPMFV, together with the experimental range given in (5.6), as functions of $\tan \beta$ for three representative values of the common phase $\Phi_M$ and for $\Phi_A^{\text{GUT}} = 0$. The three thin arrows at the bottom indicate the positions where the ratio vanishes at the tree level without including threshold corrections for $\Phi_M = 180^\circ$, $90^\circ$, and $0^\circ$ (from left to right). Beyond the minimum point, the charged Higgs-boson contribution dominates over the SM one. It rapidly grows as $\tan^4 \beta$ initially and then goes over to $\tan^2 \beta$ due to the threshold corrections. For each displayed value of $\Phi_M$, we find two regions of $\tan \beta$ where the experimental value of $B(B^- \to \tau^-\bar{\nu})$ is obtained. One region is at $\tan \beta < 25 (27) 29$ for $\Phi_M = 180^\circ (90^\circ) 0^\circ$, and corresponds to the case where the charged Higgs-boson contribution is a small `correction' to the SM term. The second region is at $\tan \beta \sim 41 (46) 48$, for $\Phi_M = 180^\circ (90^\circ) 0^\circ$, and corresponds to the case where the charged Higgs-boson contribution dominates over the SM term. We note that the locations of these second allowed regions would not be estimated correctly if the threshold corrections were

\[^3\text{This range is different from that used in [49] due to the new BABAR result [47].}\]
Figure 9: The ratio $R_{B\tau\nu}$ (upper-left), the charged-Higgs boson mass in GeV (upper-right), and the real (lower-left) and imaginary (lower-right) parts of the coupling $(g_{L^+_{H^-}d_u})_{31}/V_{13} = (g_{H^-_{d_u}}^L)^{31}_{V_{ub}}$ as functions of $\tan\beta$ for three or four values of $\Phi_M$, taking $\Phi_A^{\text{GUT}} = 0^\circ$. The experimentally allowed 1-$\sigma$ region is bounded with two horizontal lines in the upper-left frame. The straight line with a tag ‘Tree’ in the lower-left frame shows the tree-level coupling. The parameters are the same as in Fig. 6.

The tree-level vanishing points are also indicated in the upper-right frame as intersec-

not included. These regions are actually excluded by the $B_s \to \mu^+\mu^-$ constraint discussed previously.
tions of the $M_{H^\pm}$ and $\tan \beta \times M_{B^\pm}$ lines. We observe that the resummed threshold effects enhance the charged Higgs-boson contribution when $\Phi_M = 180^\circ$ and suppress it when $\Phi_M = 0^\circ$. As can be seen from the lower-left frame of Fig. 9, for $\Phi_M = 90^\circ$, the $\tan \beta$-dependence of $R_{B_{\tau \nu}}$ becomes rather similar to the tree-level one. However, as displayed in the lower-right frame of Fig. 9, there is a non-vanishing contribution from the imaginary part of the coupling $(g_{H^-_d_u}^{L_{\tau}})^{13}/V_{13}$.

5.5 Effects on $B \to X_s \gamma$

The current experimental bound on $B(B \to X_s \gamma)$ with a photon energy cut of $E_\gamma > E_{\text{cut}} = 1.6$ GeV is [50]

$$B(B \to X_s \gamma)^{\text{EXP}} = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}.$$  (5.7)

Our estimate of the SM prediction based on the NLO calculation is $3.35 \times 10^{-4}$, which is about 1 $\sigma$ larger than the NNLO result, $(3.15 \pm 0.23) \times 10^{-4}$ [36]. In Fig. 10 we show the branching ratio $B(B \to X_s \gamma)$ and the direct CP asymmetry $A_{\text{CP}}^{\text{dir}}(B \to X_s \gamma)$ as functions of $\tan \beta$. In the upper-left frame, we include only the charged-Higgs contribution, which increases the branching ratio. The larger contribution in the high-$\tan \beta$ region is due to the decrease of the charged Higgs-boson mass. In the upper-right frame of Fig. 10, we add the contribution from the chargino-mediated loops. This contribution largely cancels the charged-Higgs contribution, when $\Phi_M < \sim 90^\circ$. Instead, if $\Phi_M$ is larger than $\sim 90^\circ$, the chargino contribution interferes constructively with the SM one, resulting in a rapid increase of the branching ratio as $\tan \beta$ grows. This behaviour can be understood from the fact that the dominant contribution to $C_7^{X_8}$ comes from the last term of Eq. (4.16), which is proportional to $e^{i \Phi_A}/c_\beta$, and the branching ratio is proportional to its real part, namely $\cos \Phi_A/c_\beta$. We recall that the phase $\Phi_A$ at the low-energy scale can largely be induced by non-vanishing $\Phi_M$ even when $\Phi_A^{\text{GUT}}$ vanishes (see the upper frames of Fig. 4). In the lower-left frame of Fig. 10, we show the full result including the contribution of the gluino-mediated loops, which is non-vanishing in the presence of flavour mixing in the down-type squark mass matrix. We find that it is numerically negligible for the parameters chosen. In the same frame, as well as in the upper-right one, we show the case of the common phase $\Phi_M = 60^\circ$, in which there is a nearly exact cancellation between the chargino and charged-Higgs contributions, and all the $\tan \beta$ region considered is compatible with the current experimental bound. This observation is also apparent in the left panel of Fig. 11. In the lower-right frame of Fig. 10, we show the direct CP asymmetry for several combinations of $(\Phi_A^{\text{GUT}}, \Phi_M)$, finding that it can be as large as $\sim -4 \%$, when $\Phi_M = 60^\circ$. 

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Figure 10: The branching ratio $B(B \to X_s \gamma)$ as a function of $\tan \beta$ for several values of the common phase $\Phi_M = \Phi_1 = \Phi_2 = \Phi_3$ and $\Phi_A^{\text{GUT}}$. The region allowed experimentally at the 2-$\sigma$ level is bounded by two horizontal lines. In the upper-left frame, only the charged-Higgs contribution is added to the SM prediction. In the upper-right and lower-left frames, the SUSY contributions are included. The direct CP asymmetry $A_{\text{CP}}^{\text{dir}}(B \to X_s \gamma)$ is also shown in the lower-right frame for several combinations of $(\Phi_A^{\text{GUT}}, \Phi_M)$. The parameters are the same as in Fig. 6.

To illustrate the strong dependences of the branching ratio and the CP asymmetry on the common phase $\Phi_M$, we show them as functions of $\Phi_M$ for four values of $\tan \beta$ in Fig. 11.
The region allowed experimentally at the 2-\(\sigma\) level is bounded by two horizontal lines in the left frame. In the right frame, points within this region are denoted with open squares.

We observe that the branching ratio is quite insensitive to \(\tan \beta\) around \(\Phi_M = 60^\circ\), whereas the CP asymmetry can be as large as \(\pm 5\%\) for points within the current 2-\(\sigma\) bound on the branching ratio. For comparison, we note that the experimental range currently allowed is 0.4\(\pm 3.7\%\) [50], implying that the new contribution in the MSSM with MCPMFV could be comparable to the present experimental error, and much larger than the SM contribution, which is expected to be below 1\%. Finally, it is important to remark that, in the absence of any cancellation mechanism [43], EDM constraints severely restrict the soft CP-odd phases in constrained models of low-scale SUSY, such as the constrained MSSM. In a forthcoming paper, however, we will demonstrate in detail, how these constraints can be considerably relaxed in the MSSM with MCPMFV.
6 Conclusions

In this paper we have formulated the maximally CP-violating version of the MSSM with minimal flavour violation, the MSSM with MCPMFV, showing that it has 19 parameters, including 6 additional CP-violating phases beyond the CKM phase in the SM. As preparation for our discussion of $B$-meson observables, we have developed a manifestly flavour-covariant effective Lagrangian formalism, including a new class of dominant sub-leading contributions due to non-decoupling effects of the third-generation quarks. We have presented analytical results for a range of different $B$-meson observables, including the $B_s$ and $B_d$ mass differences, and the decays $B_s \rightarrow \mu^+\mu^-$, $B_u \rightarrow \tau\nu$ and $b \rightarrow s\gamma$. We have presented numerical results for these observables in one specific MCPMFV scenario. This serves to demonstrate that the experimental constraints on $B$-meson mixings and their decays impose constraints, e.g., on $\tan\beta$, that depend strongly on the CP-violating phases in the MCPMFV model, most notably on the soft gluino-mass phase in the specific example studied.

In summary, on the one hand, our paper introduces a new class of MSSM models of potential phenomenological interest and develops an appropriate formalism for analyzing them, and on the other, it presents exploratory numerical studies of the constraints imposed by experimental limits on $B$-meson observables. In view of the large number of the theoretical parameters in the MSSM with MCPMFV, we leave for future work a more complete exploration of its parameter space, including the correlation with other experimental constraints, e.g. those imposed by limits on electric dipole moments.

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A Renormalization Group Equations

Here we list all relevant one-loop renormalization group equations (RGEs) for the gauge and Yukawa couplings [51], as well as for the soft SUSY-breaking mass parameters of the general MSSM [52, 53]. Defining the RG evolution parameter $t = \ln(Q^2/M_{GUT}^2)$, we may write down the one-loop RGEs as follows:

$$\frac{dg_{1,2,3}}{dt} = \frac{1}{32\pi^2} \left\{ \frac{33}{5} g_1^3, \ g_2^3, \ -3g_3^3 \right\}, \quad (A.1)$$

$$\frac{dM_{1,2,3}}{dt} = \frac{1}{16\pi^2} \left\{ \frac{33}{5} g_1^2 M_1, \ g_2^2 M_2, \ -3g_3^2 M_3 \right\}, \quad (A.2)$$

$$\frac{dh_u}{dt} = \frac{h_u}{32\pi^2} \left( -\frac{13}{15} g_1^2 - 3g_2^2 - \frac{16}{3} g_3^2 + 3 \text{Tr}(h_u^\dagger h_u) + 3 \text{Tr}(h_d^\dagger h_d) \right), \quad (A.3)$$

$$\frac{dh_d}{dt} = \frac{h_d}{32\pi^2} \left( -\frac{7}{15} g_1^2 - 3g_2^2 - \frac{16}{3} g_3^2 + 3 \text{Tr}(h_d^\dagger h_d) + \text{Tr}(h_u^\dagger h_u) \right) \quad (A.4)$$

$$\frac{da_u}{dt} = \frac{1}{32\pi^2} \left[ \left( \frac{26}{15} g_1^2 M_1 + 6g_2^2 M_2 + \frac{32}{3} g_3^2 M_3 \right) h_u - \left( \frac{13}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) a_u ight.
+ 4 h_u h_u^\dagger a_u + 5 a_d h_u^\dagger h_u + 6 \text{Tr}(h_u^\dagger a_u) h_u + 3 \text{Tr}(h_u^\dagger h_u) a_u + 2 h_u h_d^\dagger a_d
+ a_u h_d^\dagger h_d \right], \quad (A.5)$$

$$\frac{da_d}{dt} = \frac{1}{32\pi^2} \left[ \left( \frac{14}{15} g_1^2 M_1 + 6g_2^2 M_2 + \frac{32}{3} g_3^2 M_3 \right) h_d - \left( \frac{7}{15} g_1^2 + 3g_2^2 + \frac{16}{3} g_3^2 \right) a_d ight.
+ 4 h_d h_d^\dagger a_d + 5 a_d h_d^\dagger h_d + 6 \text{Tr}(h_d^\dagger a_d) h_d + 3 \text{Tr}(h_d^\dagger h_d) a_d + 2 h_d h_u^\dagger a_u
+ a_d h_u^\dagger h_u + 2 \text{Tr}(h_u^\dagger a_u) h_d + \text{Tr}(h_u^\dagger h_u) a_d \right], \quad (A.6)$$

$$\frac{da_e}{dt} = \frac{1}{32\pi^2} \left[ \left( 6g_1^2 M_1 + 6g_2^2 M_2 \right) h_e - \left( 3g_1^2 + 3g_2^2 \right) a_e ight.
+ 4 h_e h_e^\dagger a_e + 5 a_e h_e^\dagger h_e + 2 \text{Tr}(h_e^\dagger a_e) h_e + \text{Tr}(h_e^\dagger h_e) a_e
+ 6 \text{Tr}(h_d^\dagger a_d) h_e + 3 \text{Tr}(h_d^\dagger h_d) a_e \right], \quad (A.7)$$

Our results are in agreement with [53].
\[
\frac{dB}{dt} = \frac{3}{16\pi^2} \left( \frac{1}{5} g_1^2 M_1 + g_2^2 M_2 + \text{Tr}(h_u^\dagger a_u) + \text{Tr}(h_d^\dagger a_d) + \frac{1}{3} \text{Tr}(h_e^\dagger a_e) \right), \quad (A.9)
\]
\[
\frac{d\mu}{dt} = \frac{3 \mu}{32\pi^2} \left( -\frac{1}{5} g_1^2 - g_2^2 + \text{Tr}(h_u^\dagger h_u) + \text{Tr}(h_d^\dagger h_d) + \frac{1}{3} \text{Tr}(h_e^\dagger h_e) \right), \quad (A.10)
\]
\[
\frac{dM_{H_u}^2}{dt} = \frac{3}{16\pi^2} \left( -\frac{1}{5} g_1^2 |M_1|^2 - g_2^2 |M_2|^2 + \text{Tr}(h_u^\dagger \tilde{M}_Q^2 h_u) + \text{Tr}(h_d^\dagger \tilde{M}_Q^2 h_d) 
+ M_{H_u}^2 \text{Tr}(h_u^\dagger h_u) + \text{Tr}(a_u^\dagger a_u) + \frac{1}{10} g_1^2 \text{Tr}(YM^2) \right), \quad (A.11)
\]
\[
\frac{dM_{H_d}^2}{dt} = \frac{3}{16\pi^2} \left( -\frac{1}{5} g_1^2 |M_1|^2 - g_2^2 |M_2|^2 + \text{Tr}(h_d^\dagger \tilde{M}_Q^2 h_d) + \text{Tr}(h_d^\dagger \tilde{M}_Q^2 h_d) 
+ M_{H_d}^2 \text{Tr}(h_d^\dagger h_d) + \text{Tr}(a_d^\dagger a_d) + \frac{1}{3} \text{Tr}(h_e^\dagger \tilde{M}_Q^2 h_e) + \frac{1}{3} \text{Tr}(h_e^\dagger \tilde{M}_Q^2 h_e) 
+ \frac{1}{3} M_{H_d}^2 \text{Tr}(h_e^\dagger h_e) + \frac{1}{3} \text{Tr}(a_e^\dagger a_e) - \frac{1}{10} g_1^2 \text{Tr}(YM^2) \right), \quad (A.12)
\]
\[
\frac{d\tilde{M}_Q^2}{dt} = \frac{1}{16\pi^2} \left[ -\left( \frac{1}{15} g_1^2 |M_1|^2 + 3g_2^2 |M_2|^2 + \frac{16}{3} g_3^2 |M_3|^2 \right) 1_3 + \frac{1}{2} h_u^\dagger h_u \tilde{M}_Q^2 
+ \frac{1}{2} \tilde{M}_Q^2 h_u^\dagger h_u + M_{H_u}^2 h_u^\dagger h_u + a_u^\dagger a_u + \frac{1}{2} h_d^\dagger h_d \tilde{M}_Q^2 + \frac{1}{2} \tilde{M}_Q^2 h_d^\dagger h_d 
+ h_d^\dagger \tilde{M}_D^2 h_d + M_{H_d}^2 h_d^\dagger h_d + a_d^\dagger a_d + \frac{1}{10} g_1^2 \text{Tr}(YM^2) 1_3 \right], \quad (A.13)
\]
\[
\frac{d\tilde{M}_L^2}{dt} = \frac{1}{16\pi^2} \left[ -\left( \frac{3}{5} g_1^2 |M_1|^2 + 3g_2^2 |M_2|^2 \right) 1_3 + \frac{1}{2} h_e^\dagger h_e \tilde{M}_L^2 + \frac{1}{2} \tilde{M}_L^2 h_e^\dagger h_e 
+ h_e^\dagger \tilde{M}_E^2 h_e + M_{H_e}^2 h_e^\dagger h_e + a_e^\dagger a_e - \frac{3}{10} g_1^2 \text{Tr}(YM^2) 1_3 \right], \quad (A.14)
\]
\[
\frac{d\tilde{M}_U^2}{dt} = \frac{1}{16\pi^2} \left[ -\left( \frac{16}{15} g_1^2 |M_1|^2 + \frac{16}{3} g_3^2 |M_3|^2 \right) 1_3 + h_u^\dagger h_u \tilde{M}_U^2 + \tilde{M}_U^2 h_u^\dagger h_u 
+ 2 h_u^\dagger \tilde{M}_Q^2 h_u^\dagger + 2 M_{H_u}^2 h_u^\dagger h_u + 2 a_u^\dagger a_u - \frac{2}{5} g_1^2 \text{Tr}(YM^2) 1_3 \right], \quad (A.15)
\]
\[
\frac{d\tilde{M}_D^2}{dt} = \frac{1}{16\pi^2} \left[ -\left( \frac{4}{15} g_1^2 |M_1|^2 + \frac{16}{3} g_3^2 |M_3|^2 \right) 1_3 + h_d^\dagger h_d \tilde{M}_D^2 + \tilde{M}_D^2 h_d^\dagger h_d 
+ 2 h_d^\dagger \tilde{M}_Q^2 h_d^\dagger + 2 M_{H_d}^2 h_d^\dagger h_d + 2 a_d^\dagger a_d + \frac{1}{5} g_1^2 \text{Tr}(YM^2) 1_3 \right], \quad (A.16)
\]
\[
\frac{d\tilde{M}_E^2}{dt} = \frac{1}{16\pi^2} \left( -\frac{12}{5} g_1^2 |M_1|^2 1_3 + h_e^\dagger \tilde{M}_E^2 + \tilde{M}_E^2 h_e^\dagger + 2 h_e^\dagger \tilde{M}_Q^2 h_e^\dagger \right)
\]
+ 2 M^2_{H_d} h_d h_d^\dagger + 2 a_e a_e^\dagger + \frac{3}{5} g^2 \text{Tr} (Y M^2) 1_3 \right), \quad (A.17)

where \( g_1 \) is the GUT-normalized gauge coupling, which is related to the \( U(1)_Y \) gauge coupling \( g' \) of the SM through \( g_1 = \sqrt{5/3} g' \). In addition, the expression

\[
\text{Tr} (Y M^2) = M^2_{H_u} - M^2_{H_d} + \text{Tr} \left( \tilde{M}_Q^2 - \tilde{M}_L^2 - 2 \tilde{M}_U^2 + \tilde{M}_D^2 + \tilde{M}_E^2 \right) \quad (A.18)
\]

is the Fayet–Iliopoulos D-term contribution to the one-loop RGEs. It can be shown that \( d \text{Tr} (Y M^2)/dt \propto \text{Tr} (Y M^2) \), i.e., the expression \( \text{Tr} (Y M^2) \) is multiplicatively renormalizable. As usual, the GUT scale is determined by the boundary condition: \( g_1(M_{\text{GUT}}) = g_2(M_{\text{GUT}}) = g_3(M_{\text{GUT}}) \). We note, finally, that the one-loop RGEs listed above are invariant under the unitary flavour transformations given in (2.5).

### B Z- and \( W^\pm \)-Boson Ward Identities

In the absence of gauge quantum corrections, the Z- and \( W^\pm \) boson couplings to quarks obey the following tree-level WIs \([54]\):

\[
\frac{q^\mu}{M_Z} i \Gamma^{Zff'}_\mu(q, p, p - q) + \Gamma^{G^0ff'}_\mu(q, p, p - q) = \quad (B.1)
\]

\[
\frac{q^\mu}{M_Z} \frac{i g_w}{c_w} \left[ \left( T^f_{z'P_L} - 2 Q f s^2_w \right) \Sigma_{ff'}(p) - \left( T^f_{zP_R} - 2 Q f s^2_w \right) \Sigma_{ff'}(p - q) \right],
\]

\[
\frac{q^\mu}{M_W} i \Gamma^{W^+ud}_{\mu}(q, p, p - q) + i \Gamma^{G^+ud}_{\mu}(q, p, p - q) = \quad (B.2)
\]

\[
- \frac{i g_w}{M_W} \left[ V_{u'd} \Sigma_{uu'}(p) P_L - V_{ud'} P_R \Sigma_{dd'}(p - q) \right],
\]

where \( c_w = \sqrt{1 - s^2_w} \) is the cosine of the weak mixing angle and \( T^u_{z(x)} = \frac{1}{2} (-\frac{1}{2}) \) and \( Q_{u(d)} = \frac{2}{3} (-\frac{4}{3}) \) are the weak isospin and electric charge quantum numbers for the \( u \) and \( d \) quarks. In (B.1) and (B.2), \( \Sigma_{ff'}(p) \) are quark self-energies describing the fermionic transition \( f' \rightarrow f \), with \( f = u, d \) and \( f' = u', d' \). In addition, \( \Gamma^{Zff'}_{\mu}(q, p, p - q) \) and \( \Gamma^{W^+ud}_{\mu}(q, p, p - q) \) are vertex functions that describe the interaction of the Z- and \( W^+ \)-boson to quarks, respectively. The momenta \( q^\mu \) of the gauge bosons are defined as flowing into the vertex, while the momentum flow of the quarks follows the fermion arrow, where \( p^\mu \) always denotes the outgoing momentum.

In general, virtual strong and electroweak gauge corrections to the Z- and \( W^\pm \)-boson vertices usually distort these identities, through terms that depend on the gauge-fixing parameter, e.g., \( \xi \). One possible framework in which these identities can be enforced is
the pinch technique [55], leading to analytic results that are independent of $\xi$. Recently, this approach has been extended to super Yang-Mills theories [56]. We ignore the gauge quantum corrections in our phenomenological analysis, since they are rather small.

In the limit $q^\mu \to 0$, the WIs (B.1) and (B.2) simplify considerably. Let us first consider the WI involving the $Z$ boson and its associated would-be Goldstone boson $G^0$. Since the vertex function $\Gamma^{Z,ff'}_\mu(q,p,p-q)$ has no IR singularities in the limit $q^\mu \to 0$, the WI (B.1) takes on the much simpler form

$$\Gamma^{G^0,ff'}(0,p,p) = \frac{ig_w}{M_Z} T^f_\mu \left[ \Sigma_{ff'}(p)_L - P_R \Sigma_{ff'}(p)_R \right].$$  (B.3)

Decomposing the quark self-energies $\Sigma_{ff'}(p)$ with respect to their spinorial structure,

$$\Sigma_{ff'}(p) = \Sigma^L_{ff'}(p^2)_\mu P_L + \Sigma^R_{ff'}(p^2)_\mu P_R + \Sigma^{D}_{ff'}(p^2)_\mu P_L + \Sigma^{D*}_{ff'}(p^2)_\mu P_R,$$  (B.4)

we may rewrite (B.3) as follows:

$$\Gamma^{G^0,ff'}(0,p,p) = \frac{ig_w}{M_W} T^f_\mu \left[ \Sigma^D_{ff'}(p^2)_\mu P_L - P_R \Sigma^{D*}_{ff'}(p^2)_\mu \right].$$  (B.5)

Considering the proper normalizations determined by the relations given in (3.13), it is possible to make the following identifications in the effective potential limit $p^\mu \to 0$:

$$\Sigma^D_{ff'}(0) = U^{f^t}_R h_f \langle \Delta_f \rangle U^f_L,$$

$$\frac{i}{\sqrt{2}} U^{f^t}_R h_f \Delta^G_{ff'}(0)_L P_L,$$  (B.6)

where the unitary matrices $U^{u,d}_{L,R}$ take care of the weak to the mass basis transformations as given in (3.7), with $U^u_L = U^Q_L$ and $U^d_L = U^Q_L V$. Then, the simplified WI (B.5) implies that

$$\Delta^G_{ff'} = -\frac{\sqrt{2}}{v} T^f_\mu \langle \Delta_f \rangle,$$  (B.7)

which is the relation assumed in Section 3 [cf. (3.17)].

We now turn our attention to the WI involving the $W^+$ boson and the associated would-be Goldstone boson $G^+$. In the effective potential limit $q^\mu, p^\mu \to 0$, we obtain

$$i\Gamma^{G^+,ud}(0,0,0) = -\frac{ig_w}{\sqrt{2} M_W} \left[ V_{u'd} \Sigma^D_{uu'}(0)_L P_L - V_{ud} \Sigma^{D*}_{dd'}(0)_R P_R \right].$$  (B.8)

Employing the definitions (3.13) and taking the weak-to-mass basis rotations of the quark states into account, we find the relations:

$$P_L \Gamma^{G^+,ud}(0,0,0) = U^{u^t}_R h_u \Delta^G_u U^d_L P_L,$$  (B.9)
From the simplified WI (B.8) and its Hermitean conjugate, we then derive that
\[ \Delta^+_G = -\sqrt{2} v \langle \Delta_u \rangle, \quad \Delta^-_G = \sqrt{2} v \langle \Delta_d \rangle, \] (B.10)
which is in agreement with (3.17) and the discussion given below. We note that the unitarity of the radiatively-corrected CKM matrix \(V\) lies at the heart of deriving the relations (B.7) and (B.10).

C CPsuperH Interface

To solve the RGEs given in Appendix A, we have considered the following input parameters:

- The gauge couplings at the scale \(M_Z\):
  \[ \alpha_1(M_Z) = \frac{5}{3} \frac{g^2(M_Z)}{4\pi}; \quad \alpha_2(M_Z) = \frac{g^2(M_Z)}{4\pi}; \quad \alpha_3(M_Z), \] (C.1)
where \(g(M_Z) = e(M_Z)/s_W\) and \(g'(M_Z) = e(M_Z)/c_W\) with \(\alpha_{em}(M_Z) = e^2(M_Z)/4\pi\).

The evolutions of \(\alpha_{1,2}\) from \(M_Z\) to \(m_t^{pole}\) are determined by [57]
\[ \alpha_{1,2}^{-1}(m_t^{pole}) = \alpha_{1,2}^{-1}(M_Z)^{-1} + \frac{53}{30\pi} \ln(M_Z/m_t^{pole}), \]
\[ \alpha_{2}^{-1}(m_t^{pole}) = \alpha_{2}^{-1}(M_Z)^{-1} - \frac{11}{6\pi} \ln(M_Z/m_t^{pole}). \] (C.2)
On the other hand, \(\alpha_3(m_t^{pole})\) has been obtained by solving the following equation iteratively [58]
\[ \alpha_3^{-1}(m_t^{pole}) = \alpha_3^{-1}(M_Z) - b_0 \ln \left( \frac{m_t^{pole}}{M_Z} \right) - \frac{b_1}{b_0} \ln \left( \frac{\alpha_3(m_t^{pole})}{\alpha_3(M_Z)} \right) - \left( \frac{b_2 b_0 - b_1^2}{b_0^2} \right) \left[ \alpha_3(m_t^{pole}) - \alpha_3(M_Z) \right] + \mathcal{O}(\alpha_3^2) \] (C.3)
where \(b_0 = -(11 - 2 N_F/3)/2\pi\), \(b_1 = -(51 - 19 N_F/3)/4\pi^2\), and \(b_2 = -(2857 - 5033 N_F/9 + 325 N_F^2/27)/64\pi^3\) with \(N_F = 5\).

- The masses of the quarks and the charged leptons at the top-quark pole-mass scale \(m_t^{pole}\). In particular, the top-quark running mass at \(m_t^{pole}\) is obtained from:
  \[ m_t(m_t^{pole}) = m_t^{pole}/\left[1 + 4\alpha_3(m_t^{pole})/3\pi \right]. \]
The CKM matrix \(V\) is assumed to be given at the same scale \(m_t^{pole}\). Then, in general, the complex \(3 \times 3\) Yukawa matrices at \(m_t^{pole}\) are given by
\[ h_{u,e}(m_t^{pole}) = \sqrt{2} \tilde{M}_{u,e}(m_t^{pole}), \quad h_d(m_t^{pole}) = \sqrt{2} \tilde{M}_d(m_t^{pole}) V^\dagger(m_t^{pole}) \] (C.4)
in the flavour basis $U^Q_L = U^u_R = U^d_R = 1_3$. The diagonal quark and charged-lepton mass matrices are given by

$$\tilde{M}_u(m^\text{pole}_t) = \text{diag} \left[ m_u(m^\text{pole}_t), m_c(m^\text{pole}_t), m_t(m^\text{pole}_t) \right],$$

$$\tilde{M}_d(m^\text{pole}_t) = \text{diag} \left[ m_d(m^\text{pole}_t), m_s(m^\text{pole}_t), m_b(m^\text{pole}_t) \right],$$

$$\tilde{M}_e(m^\text{pole}_t) = \text{diag} \left[ m_e(m^\text{pole}_t), m_\mu(m^\text{pole}_t), m_\tau(m^\text{pole}_t) \right]. \quad (C.5)$$

Given $\alpha_{1,2,3}(m^\text{pole}_t)$ and $h_{u,d,e}(m^\text{pole}_t)$, the evolution from $m^\text{pole}_t$ to the scale $M_{\text{SUSY}}$ have been obtained by solving the SM RGEs. Here the SUSY scale $M_{\text{SUSY}}$ has been determined by solving

$$Q^2 \left|_{Q= M_{\text{SUSY}}} = \max [m^2_t(Q^2), m^2_b(Q^2)] \right. \quad (C.6)$$

iteratively, where $m^2_t \equiv \max (m^2_{Q_3} + m^2_{\nu_3} + m^2_b)$ and $m^2_b \equiv \max (m^2_{Q_3} + m^2_{\nu_3} + m^2_\nu + m^2_\tau)$. For $m^2_{Q_3, \nu_3, D_3, L_3, E_3}(Q^2)$, we have taken the $(3, 3)$ component of the corresponding mass matrix as

$$m^2_{Q_3, \nu_3, D_3, L_3, E_3}(Q^2) = \left[ \tilde{M}^2_{Q,D,L,E}(Q^2) \right]_{(3,3)} \quad (C.7)$$

At the scale $M_{\text{SUSY}}$, the Yukawa matrices match as

$$h_u(M^\text{SUSY}_{\text{SUSY}}) = h_u(M^\text{SUSY}) / \sin \beta(M^\text{SUSY}),$$

$$h_{d,e}(M^\text{SUSY}_{\text{SUSY}}) = h_{d,e}(M^\text{SUSY}) / \cos \beta(M^\text{SUSY}), \quad (C.8)$$

and, finally, the evolution from $M_{\text{SUSY}}$ to $M_{\text{GUT}}$ have been obtained by solving the MSSM RGEs.

- The 19 flavour-singlet mass scales of the MSSM with MCPMFV, which are parameterized as follows:

$$|M_{1,2,3}| e^{\Phi_{1,2,3}}, \quad |A_{u,d,e}| e^{\Phi_{A_{u,d,e}}}, \quad \tilde{M}^2_{Q,D,L,E}, \quad M^2_{R_{u,d}}. \quad (C.9)$$

These are inputed at the GUT scale $M_{\text{GUT}}$, which is defined as the scale where the couplings $g_1$ and $g_2$ meet. Any difference between $g_3(M_{\text{GUT}})$ and $g_1(M_{\text{GUT}})$ may be attributed to some unknown threshold effect at the GUT scale.

By solving the RGEs from the GUT scale $M_{\text{GUT}}$ to the SUSY scale $M_{\text{SUSY}}$, we obtain:

- Three complex gaugino masses, $|M_i| e^{\Phi_i(Q = M_{\text{SUSY}})}$.
- Three $3 \times 3$ complex Yukawa coupling matrices, $h_{u,d,e}(Q = M_{\text{SUSY}})$. 

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Three $3 \times 3$ complex $a$-term matrices, $a_{u,d,e}(Q = M_{\text{SUSY}})$.

The soft Higgs masses, $M_{H_u,H_d}^2(Q = M_{\text{SUSY}})$.

The complex $3 \times 3$ sfermion mass matrices, $\tilde{M}_{Q,U,D,L,E}^2(Q = M_{\text{SUSY}})$.

The inputs for the code $\text{CPsuperH}$ are:

$$\tan \beta(m_t^\text{pole}), \ M_{H^\pm}^\text{pole}, \ \mu(M_{\text{SUSY}}), \ M_{1,2,3}(M_{\text{SUSY}}),$$
$$m_{\tilde{Q}_3,\tilde{U}_3,\tilde{D}_3,\tilde{L}_3,\tilde{E}_3}(M_{\text{SUSY}}), \ A_{t}(M_{\text{SUSY}}), \ A_{b}(M_{\text{SUSY}}), \ A_{\tau}(M_{\text{SUSY}}). \ (C.10)$$

The ratio of the vacuum expectation values at $m_t^\text{pole}$ is related to that at $M_{\text{SUSY}}$ by [21]

$$\tan \beta(m_t^\text{pole}) = \frac{\xi^-(m_t^\text{pole})}{\xi^+(m_t^\text{pole})} \tan \beta(M_{\text{SUSY}}) \quad (C.11)$$

with

$$\xi^{+(-)}_{1(2)}(m_t^\text{pole}) = 1 + \frac{3|\overline{h}_{b(t)}|^2}{32\pi^2} \ln \frac{M_{\text{SUSY}}^2}{m_t^\text{pole}^2}. \ (C.12)$$

The gaugino mass parameters are directly read from the results of the RG running, the sfermion masses are given by

$$m_{\tilde{Q}_3,\tilde{U}_3,\tilde{D}_3,\tilde{L}_3,\tilde{E}_3}(M_{\text{SUSY}}) = \left\{ \tilde{M}_{Q,U,D,L,E}^2(M_{\text{SUSY}}) \right\}^{1/2} \ (3,3) \ (C.13)$$

and the $A$ parameters, including their CP-violating phases, by

$$A_f(M_{\text{SUSY}}) = \frac{[a_f(M_{\text{SUSY}})]_{(3,3)}}{[h_f(M_{\text{SUSY}})]_{(3,3)}}. \quad (C.14)$$

The $\mu$ parameter and charged Higgs-boson pole mass $M_{H^\pm}^\text{pole}$ can be obtained from $M_{H_u}^2(M_{\text{SUSY}})$ and $M_{H_d}^2(M_{\text{SUSY}})$ by imposing the two CP-even tadpole conditions, $T_{\phi_1} = T_{\phi_2} = 0$ [21]. The tadpoles can be cast into the form

$$T_{\phi_1(\phi_2)} = v_{1(2)} \overline{m}_{1(2)}^2 + v_{2(1)} \Re \overline{m}_{12}^2 + v_{1(2)} \left[ \overline{\lambda}_{1(2)} v_{1(2)}^2 + \frac{1}{2} (\overline{\lambda}_3 + \overline{\lambda}_4) v_{2(1)}^2 \right] + v_{1(2)} X_{1(2)} \quad (C.15)$$

where

$$X_{1(2)} = \frac{3}{8\pi^2} \left[ |\overline{h}_{b(t)}|^2 m_{b(t)}^2 \left( \ln \frac{m_t^\text{pole}^2}{m_t^\text{pole}^2} - 1 \right) \right]. \quad (C.16)$$

The quantities $\overline{m}_{1,2}^2$ and $\overline{\lambda}_i$ are given by

$$\overline{m}_{1,2}^2 = -M_{H_u,H_d}^2 - |\mu|^2 + \mu_{1,2}^{2(1)}(m_t^\text{pole}), \quad \overline{\lambda}_i = \lambda_i + \lambda_i^{(1)}(m_t^\text{pole}) + \lambda_i^{(2)}(m_t^\text{pole}), \quad (C.17)$$
where

\[
\mu_1^{(2)}(m_{t}^{\text{pole}}) = -\frac{3}{16\pi^2} \left[ |h_t|^2|\mu|^2 \ln \frac{M_t^2}{m_t^{\text{pole}2}} + |h_b|^2|A_b|^2 \ln \frac{M_b^2}{m_t^{\text{pole}2}} \right],
\]

\[
\mu_2^{(2)}(m_{t}^{\text{pole}}) = -\frac{3}{16\pi^2} \left[ |h_t|^2|A_t|^2 \ln \frac{M_t^2}{m_t^{\text{pole}2}} + |h_b|^2|\mu|^2 \ln \frac{M_b^2}{m_t^{\text{pole}2}} \right].
\]

The couplings \(\lambda_i, \lambda_i^{(1)}(m_{t}^{\text{pole}})\) and \(\lambda_i^{(2)}(m_{t}^{\text{pole}})\) may be found in Ref. [21]. The squared absolute value \(|\mu|^2\) can be determined from \((T_{\phi_1}/v_2 - T_{\phi_2}/v_1) = 0\), which does not depend on Re \(m_{12}^2\), since

\[
|\mu|^2 = \frac{(M_{H_d}^2 - M_{H_u}^2 t_\beta^2) - (\lambda_1 v_1^2 - \lambda_2 v_2^2 t_\beta^2) + X_A - (X_1 - t_\beta^2 X_2)}{(t_\beta^2 - 1) + X_{tb}}
\]

with

\[
X_A \equiv \frac{3}{16\pi^2} \left( |h_t|^2|A_t|^2 \ln \frac{M_t^2}{m_t^{\text{pole}2}} - t_\beta^2|h_t|^2|A_t|^2 \ln \frac{M_t^2}{m_t^{\text{pole}2}} \right),
\]

\[
X_{tb} \equiv -\frac{3}{16\pi^2} \left( |h_t|^2 \ln \frac{M_t^2}{m_t^{\text{pole}2}} - t_\beta^2|h_b|^2 \ln \frac{M_b^2}{m_t^{\text{pole}2}} \right).
\]

We note that the phase of the \(\mu\) parameter, \(\Phi_\mu\), is not renormalized.

Once \(|\mu|^2\) is found, Re \(m_{12}^2\) can be obtained from \(T_{\phi_1} = 0\) or \(T_{\phi_2} = 0\). With Re \(m_{12}^2\) known, the charged Higgs-boson pole mass can be obtained by solving the following equation iteratively:

\[
(M_{H^\pm}^{\text{pole}})^2 = \frac{\text{Re} m_{12}^2}{s_{\beta} c_{\beta}} + \frac{1}{2} \lambda_4 v^2 - \text{Re} \hat{\Pi}_{H^{+H^-}}(\sqrt{s} = M_{H^\pm}^{\text{pole}}).
\]

For the explicit form of \(\hat{\Pi}_{H^{+H^-}}\), we refer to Ref. [59]. We note that, for large \(\tan \beta\), Re \(m_{12}^2/s_{\beta} c_{\beta} \approx M_{H_d}^2 - M_{H_u}^2 - M_Z^2\) at the tree level. Finally, after imposing the CP-odd tadpole condition \(\text{Im} (B\mu) = 0\), we use \(B\mu = \text{Re} m_{12}^2\) to calculate the 2HDM contribution (3.28), by noting \(H_u H_d = -\Phi_1^* \Phi_2\).
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