Schauder estimates for solutions of sub-Laplace equations with Dini terms

Tingxi Hu, Pengcheng Niu†
Department of Applied Mathematics,
Northwestern Polytechnical University, Xi’an 710129, China

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Abstract
In this paper we establish Schauder estimates for the sub-Laplace equation

$$\sum_{j=1}^{m} X_j^2 u = f,$$

where $X_1, X_2, \ldots, X_m$ is a system of smooth vector field which generates the first layer in the Lie algebra of a Carnot group. We drive the estimate for the second order derivatives of the solution to the equation with Dini continue inhomogeneous term $f$ by the perturbation argument.

Keywords: Carnot group; sub-Laplace; Schauder estimate; Dini continue; perturbation argument.

MSC2010: 35B65; 35R03.

1 Introduction
Schauder estimates play an important role in the theory of elliptic equations, see [6, 11]. For the second order uniformly elliptic equation in any bounded domain $\Omega \subset \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij}^2 u = f,$$

such estimates provide a bound of the Hölder norm in $\Omega$ of the second derivatives of the solution $u$ in terms of the Hölder norms in $\Omega$ of the coefficients $a_{ij}$ and $f$.

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†Corresponding author. pengchengniu@nwpu.edu.cn (P. Niu)
A sharper form of these estimates was introduced by Caffarelli [3] in the study of fully non-linear elliptic equations. He derived Schauder estimates for viscosity solutions by comparing the solutions with osculating quadratic polynomials in a neighborhood of a fixed point, and this method is called “perturbation argument”. In Caffarelli’s approach, the Hölder regularity of \( u \) at a point is basically determined by the Hölder regularity of \( a_{ij} \) and \( f \) at the same point, hence such estimates are said pointwise Schauder estimates. Wang in [23] compared the quadratic part of the solutions to the Laplace equation with solutions of approximate equations and proved the Hölder norm of \( D^2 u \) in terms of Dini continuous inhomogeneous term. Afterwords, the method was used to investigate the fully non-linear elliptic and parabolic equations, see Liu-Trudinger-Wang[16], Tian-Wang [22].

For degenerate elliptic equations constructed by left translation invariant vector fields, several authors derived Schauder estimates, see Lunardi in [19], Capogna-Han [5], Polidoro -Di Francesco [21] and Gutiérrez-Lanconelli [12]. Schauder estimates for heat type equations induced by smooth vector fields satisfying Hörmander’s finite rank condition were showed by Bramanti-Brandolini [2]. Recently, Jiang-Tian [15] showed Schauder estimates for the Kohn-Laplace equation with Dini continuous inhomogeneous term in the Heisenberg group in the spirit of [23]. We generalize the result in [15] to the sub-Laplace equations in Carnot groups.

In the present paper we consider the equation

\[
Lu \equiv \sum_{i,j=1}^m X_i^2 X_j u = f \quad \text{in } B_1(0), \quad (1.1)
\]

in which \( L \) is the sub-laplacian on a Carnot group \( G \) and the right hand term \( f \) is Dini continuous, i.e., \( f \) satisfies

\[
\int_0^1 \frac{\omega_f(r)}{r} dr < \infty,
\]

where \( \omega_f(r) = \sup_{d(\xi, \eta) < r} |f(\xi) - f(\eta)| \) and \( d(\xi, \eta) \) is the pseudo-distance (see next section) between \( \xi \) and \( \eta \), \( B_1(0) \) denotes the unit gauge ball centered at origin.

Our main result is the following

**Theorem 1.1.** Let \( u \in C^2(B_1(0)) \) be a solution of (1.1), then for any \( \xi, \eta \in B_{1/2}(0), \ d = d(\xi, \eta) \), there exists a positive constant \( C \) such that

\[
|X_i X_j u(\xi) - X_i X_j u(\eta)| \leq C \left( d \left( \sup_{B_1(0)} |u| + \|f\|_{L^\infty} \right) + \int_0^1 \frac{\omega_f(r)}{r^2} dr + \int_0^{\sqrt{d}} \omega_f(r) dr \right). \quad (1.2)
\]

In particular, if \( f \in C^{0, \alpha}(B_1(0)) (0 < \alpha \leq 1) \), then

\[
|X_i X_j u(\xi) - X_i X_j u(\eta)| \leq C d^{\alpha/2} \left( \sup_{B_1(0)} |u| + \|f\|_{C^{0, \alpha}} \right), \ \alpha \in (0, 1), \quad (1.3)
\]
\[ |X_j X_j u(\xi) - X_j X_j u(\eta)| \leq C d^{1/2} \left( \sup_{B_1(0)} |u| + \|f\|_{C^{0,1}} \left( 1 + |\sqrt{d} \log \sqrt{d}| \right) \right), \quad \alpha = 1. \] (1.4)

The plan of the paper is as follows: in Section 2 we introduce knowledge related to Carnot groups and some preliminary lemmas. Also a maximum principle for (1.1) with Dirichlet boundary value problem is proved. Section 3 is devoted to the proof of Theorem 1.1. We mention that the treatment for the Taylor polynomials in the Carnot group is more complicated than in the Heisenberg group. Also necessary techniques to use the perturbation argument are given.

## 2 Preliminary results

We begin by describing several known facts on Carnot groups and refer to [1, 10] for more information. Especially, we provide a maximum principle (Lemma 2.5) for solutions of a boundary value problem to the sub-Laplace equation.

A Carnot group \( G \) of step \( s \) is a simply connected nilpotent Lie group such that its Lie algebra \( g \) admits a stratification \( g = \bigoplus_{i=1}^s V_i \), with \([V_1, V_i] = V_{i+1} \) \((i = 1, 2, \ldots, s - 1)\) and \([V_1, V_s] = \{0\}\). Denoting \( m_l = \dim V_l \), we fix on \( G \) a system of coordinates \( \xi = (z_1, z_2, \ldots, z_s) \), in which \( z_l = (x_{l,1}, x_{l,2}, \ldots, x_{l,m_l}) \in \mathbb{R}^{m_l} \).

Every Carnot group \( G \) is naturally equipped with a family of non-isotropic dilations defined by \( \delta_r: \)

\[ \delta_r(\xi) = (rz_1, r^2z_2, \ldots, r^s z_s), \xi \in G, r > 0, \]

and the homogeneous dimension of \( G \) is given by \( Q = \sum_{l=1}^{s} l m_l \). We express by \( dH(\xi) \) a fixed bi-invariant Haar measure on \( G \). One easily sees \( dH(\delta_r(\xi)) = r^Q dH(\xi) \). By the Baker-Campbell-Hausdorff formula, the group law on \( G \) is

\[ \xi \eta = \xi + \eta + \sum_{1 \leq l, k \leq s} Z_{l,k}(\xi, \eta), \quad \xi, \eta \in G, \]

where \( Z_{l,k}(\xi, \eta) \) is a fixed linear combination of iterated commutators containing \( l \) times \( \xi \) and \( k \) times \( \eta \).

The homogenous norm of \( \xi \) on \( G \) is defined by \( |\xi| = (\sum_{j=1}^{s} |z_j|^{2a_j/j})^{1/2a_j}, \) where \( |z_j| \) denotes the Euclidean norm of \( z_j \in \mathbb{R}^{m_j} \). Such homogenous norm on \( G \) can be used to define a pseudo-distance on \( G \) which is \( d(\xi, \eta) = |\xi^{-1} \eta| \). Denote the gauge ball of radial \( r \) centered at \( \xi \) by \( B_r(\xi) = \{ \eta \in G | d(\xi, \eta) < r \} \).

Let \( X = \{X_1, X_2, \ldots, X_m\} \) be a basis of \( V_1 \), then we can write \( X_i \) as

\[ X_i = \partial_{i,1} + \sum_{j=i+1}^{m} a_{ij}(\xi) \partial_{i,j} + \sum_{l=2}^{s} \sum_{k=1}^{m_l} b_{ilk}(\xi) \partial_{l,k}, \quad X_i(0) = \partial_{i,1}. \]

\[ 3 \]
distance. If 0 then we illustrate the Hölder space and Lipschitz space with respect to the pseudo-
space by

\[ \langle J \rangle \]

where \( J \) is a multi-index of the monomial \( x^I = \sum_{k=1}^s x^{i_k} \). The homogeneous degree of the monomial \( x^I \) is given by the sum \( |I| = \sum_{k=1}^s i_k \).

Let \( \Omega \subset G \) be an open set. If \( k \in \mathbb{N} \) and \( 1 \leq p < \infty \), we define the horizontal Sobolev space by

\[ \text{HW}^{k,p}(\Omega) = \{ f : |X^I f| \in L^p(\Omega), 0 \leq |I| \leq k \}. \]

Then we illustrate the Hölder space and Lipschitz space with respect to the pseudo-distance. If \( 0 < \alpha \leq 1 \) and \( f \) is a function defined in an open set \( \Omega \), let

\[ [f]_{C^{0,\alpha}} = \sup \left\{ \frac{|f(\xi) - f(\eta)|}{d(\xi, \eta)^\alpha} : \xi, \eta \in \Omega, \xi \neq \eta \right\}. \]

The Hölder space is defined by

\[ C^{0,\alpha}(\Omega) = \{ f : [f]_{C^{0,\alpha}} < \infty \}, \quad 0 < \alpha < 1 \]

and Lipschitz space by \( C^{0,1}(\Omega) = \{ f : [f]_{C^{0,1}} < \infty \} \). In addition we denote that \( ||f||_{C^{0,\alpha}} := [f]_{C^{0,\alpha}} + ||f||_{L^\infty} \), for \( 0 < \alpha \leq 1 \).

We introduce some known results that will be used in this paper.

**Lemma 2.1.** ([1, pp.390-391]) The gauge balls \( B_r(\xi)(\xi \in G, r > 0) \) are L-regular open sets, i.e., for any \( f \in C^\infty(B_r(\xi)) \), there exists a Perron-Wiener-Brelot generalized solution \( u \in C^\infty(B_r(\xi)) \cap C(\partial B_r(\xi)) \) to the boundary value problem

\[ \begin{cases} 
Lu = f & \text{in } B_r(\xi), \\
|u|_{\partial B_r(\xi)} = g & g \in C(\partial B_r(\xi)). 
\end{cases} \tag{2.1} \]
Lemma 2.2. (a priori estimates, [4]) Let $\Omega \subset G$ be an open set and $u$ be $L$ harmonic, i.e., $u$ satisfies $Lu = 0$, then for a given integer $k$ and any multiple-index $I$, $|I| \leq k$, there exists a constant $C$ depending on $G$ and $k$ such that if $B_r(\xi) \subset \Omega$, then

$$|X^I u(\eta)| \leq C r^{-k} \sup_{B_r(\xi)} |u|, \quad \eta \in B_r(\xi).$$

(2.2)

Lemma 2.3. (Folland-Stein [7]) Let $\Omega \subset G$ be an open set, then for any $1 < p < Q$, there exist a positive constant $S_\rho$ depending on $G$, such that for $f \in C_0^\infty(\Omega)$,

$$\left( \int_\Omega |f|^{p\ast} dH \right)^{1/p\ast} \leq S_\rho \left( \int_\Omega |X f|^p dH \right)^{1/p},$$

(2.3)

where $p\ast = \frac{pQ}{Q - p}$, $|X f| = (\sum_{j=1}^m |X_j f|^2)^{1/2}$.

The following technical lemma is adapted from Chen and Wu [6].

Lemma 2.4. (De Giorgi’s iteration lemma, [6]) Let $\phi(t)$ be a nonnegative and non-increasing function on $[k_0, +\infty)$ satisfying

$$\phi(h) \leq \frac{C}{(h - k)^\alpha} \phi(k)\beta, \quad h > k \geq k_0,$$

for some constant $C > 0$, $\alpha > 0$, $\beta > 1$. Then we have

$$\phi(k_0 + \tilde{d}) = 0,$$

(2.4)

in which $\tilde{d} = C^{1/\alpha} \phi(k_0)^{1/(\beta - 1)/\alpha 2^\beta/(\beta - 1)}$.

Following the method of proving a classical maximum principle in [6, Theorem 2.4], we can obtain the following result by combining Lemmas 2.3 and 2.4.

Lemma 2.5. (Maximum principle) Let $\Omega \subset G$ be an open set, $f \in L^\infty(\Omega)$, $u \in C^2(\Omega)$ solves (2.1), then

$$\sup_\Omega |u| \leq \sup_{\partial \Omega} |g| + C ||f||_{L^\infty(\Omega)} |\Omega|^{2/Q 2^{(Q+2)/4}}.$$  

(2.5)

Proof. Notice that for every $\phi \in C^2_0(\Omega)$, we have

$$\int_\Omega \sum_{i=1}^m X_i u X_i \phi dH = - \int_\Omega f \phi dH.$$

Set $k_0 = \sup_{\partial \Omega} |g|$ and $\phi = (u - k)_+$ with $k > k_0$, and denote $A(k) = \{ \xi \in \Omega | u > k \}$. It is easy to know $X_i \phi = X_i u$ in $A(k)$. Then

$$\int_{A(k)} \sum_{i=1}^m |X_i \phi|^2 dH = \int_{A(k)} \sum_{i=1}^m X_i u X_i \phi dH = \int_{A(k)} f \phi dH.$$  

(2.6)
By Lemma 2.3, it obtains
\[
\left( \int_{A(k)} |\phi|^{2^*} dH \right)^{2/2^*} \leq C \int_{A(k)} \sum_{i=1}^{m} |X_i\phi|^2 dH.
\] (2.7)

On the other hand,
\[
\int_{A(k)} f \phi dH \leq \left( \int_{A(k)} |\phi|^{2^*} dH \right)^{1/2^*} \left( \int_{A(k)} |f|^{2^*(2+Q)/(2^*+Q)} dH \right)^{(2+Q)/2Q}.
\] (2.8)

Since \( A(h) \subset A(k) \) and \( \phi \geq h - k \) in \( A(h) \) if \( k < h \), it follows
\[
(h - k)^{2^*} |A(h)| \leq \int_{A(h)} |\phi|^{2^*} dH \leq \int_{A(k)} |\phi|^{2^*} dH.
\] (2.9)

Combining (2.6)-(2.9), it yields
\[
|A(h)| \leq \frac{(C||f||_{L^\infty(\Omega)})^{2^*}}{(h - k)^{2^*}} |A(k)|^{(Q + 2)/(Q - 2)}.
\]

By Lemma 2.4 we get (2.5). \( \Box \)

We will need the following three Lemmas referring to [1, 8], which are important in applying the perturbation argument.

**Lemma 2.6.** (Taylor polynomial) Let \( f \in C^\infty(G) \), then for every integer \( n \), there exists a unique polynomial \( P_n(f,0) \) homogenous of degree at most \( n \), such that
\[
X^I P_n(f,0)(0) = X^I f(0),
\] (2.10)
for all multiple-index \( I \) satisfying \( |I| \leq n \).

**Lemma 2.7.** (Remainder in Taylor formula) Let \( f \in C^n+1(G) \), \( \xi \in G \), then
\[
f(\eta) - P_n(f,\xi)(\eta) = O_{\eta \to \xi} (d^{n+1}(\xi^{-1}\eta)).
\] (2.11)

**Lemma 2.8.** (Mean value theorem) There exist absolute constants \( b, C > 0 \), depending only on \( G \) and the homogenous norm \( |\cdot| \), such that
\[
|f(\xi \eta) - f(\xi)| \leq C|\eta| \sup_{B_{b|\eta|}(\xi)} |Xf|,
\] (2.12)
for all \( f \in C^1(G) \) and every \( \xi, \eta \in G \).

**Remark 1.** The constant \( b \) in Lemma 2.8 can be taken 1 when the homogenous norm \( |\cdot| \) is changed by the Carnot-Carathéodory distance, see [1] for detail. In the sequel we always suppose \( b \geq 1 \) without loss of generality.
3 Proof of main result

Proof of Theorem 1.1. We divide the proof into three steps.

Step 1. Denote $B_k = B_{\rho^k}(0)$, $\rho = \frac{1}{2}$. By Lemma 2.1, there exists a solution $u_k \in C^\infty(B_k) \cap C(\bar{B}_k)$ to the boundary value problem
\[
\begin{cases}
Lu_k = f_0 = f(0) & \text{in } B_k, \\
u_k = u & \text{on } \partial B_k.
\end{cases}
\]
Then $v_k = u - u_k$ satisfies the Dirichlet boundary value problem
\[
\begin{cases}
Lv_k = f - f_0 & \text{in } B_k, \\
v_k = 0 & \text{on } \partial B_k.
\end{cases}
\]
By Lemma 2.5, we have
\[\sup_{B_k} |v_k| \leq C \rho^2 \omega f(\rho^k).\] (3.1)

Since $w_k = u_k - u_{k+1}$ is $L$-harmonic in $B_{k+2}$, we have by Lemma 2.2 and (3.1) that
\[\sup_{B_{k+2}} |X_i w_k| \leq C \rho^{-k-2} \sup_{B_{k+1}} |w_k| \leq C \rho^{-k} \left( \sup_{B_{k+1}} |v_k| + \sup_{B_{k+1}} |v_{k+1}| \right) \leq C \rho^k \omega f(\rho^k).\] (3.2)

and
\[\sup_{B_{k+2}} |X_i X_j w_k| \leq C \rho^{-2k-4} \sup_{B_{k+1}} |w_k| \leq C \rho^{-2k} \left( \sup_{B_{k+1}} |v_k| + \sup_{B_{k+1}} |v_{k+1}| \right) \leq C \omega f(\rho^k).\] (3.3)

Applying Lemma 2.6 to $u \in C^2(B_1(0))$, it gets a homogenous polynomial $P_2(u,0)$ of degree 2 such that for $1 \leq i, j \leq m$,
\[X_i P_2(u,0)(0) = X_i u(0)\]
and
\[X_i X_j P_2(u,0)(0) = X_i X_j u(0)\]
By (3.1) and Lemma 2.7, we have
\[\sup_{B_k} |u_k - P_2(u,0)| \leq \sup_{B_k} |u - u_k| + \sup_{B_k} |u - P_2(u,0)| \leq C \omega f(\rho^k) \rho^{2k} + o(\rho^{2k}) \leq o(\rho^k).\] (3.4)

Noting $LP_2(u,0) = Lu(0) = f(0) = Lu_k$, it sees that $u_k - P_2(u,0)$ is $L$-harmonic, and follows by Lemma 2.2 and (3.4) that
\[\sup_{B_k} |X_i u_k - X_i P_2(u,0)| \leq C \rho^{-k} o(\rho^{2k}) = o(\rho^k)\]
Since by Lemma 2.5 and (3.1), we have

\[
sup_{B_k} |X_i X_j u_k - X_i X_j P_2(u, 0)| \leq C \rho^{-2k} o(\rho^{2k}) = o(1),
\]

hence

\[
\lim_{k \to \infty} X_i u_k(0) = X_i P_2(u, 0)(0) = X_i u(0),
\]

(3.5)

\[
\lim_{k \to \infty} X_i X_j u_k(0) = X_i X_j P_2(u, 0)(0) = X_i X_j u(0).
\]

(3.6)

For any point \( \xi_0 \) near the origin satisfying \(|\xi_0| \leq 1/4b^2 \), we have

\[
|X_i X_j u(\xi_0) - X_i X_j u(0)| \leq |X_i X_j u(\xi_0) - X_i X_j u_k(\xi_0)| + |X_i X_j u_k(\xi_0) - X_i X_j u_k(0)| + |X_i X_j u_k(0) - X_i X_j u(0)|
\]

\[
= I_1 + I_2 + I_3.
\]

(3.7)

**Step 2.** We now estimate \( I_1, I_2 \) and \( I_3 \), respectively, to prove (1.2).

To estimate \( I_3 \), let \( k \) satisfy \( \rho^{2k+4} \leq |\xi_0| := d_0 \leq \rho^{2k+3} \). It shows by (3.3) and (3.6) that

\[
I_3 \leq \sum_{i=k}^{\infty} |X_i X_j u_i(0) - X_i X_j u_{i+1}(0)| \leq C \sum_{i=k}^{\infty} \frac{|X_i X_j u_i(0)|}{\rho^i} \rho^i \leq C \int_0^{\sqrt{d_0}} \frac{\omega(r)}{r} dr.
\]

(3.8)

To estimate \( I_1 \), we consider the boundary value problem

\[
\begin{cases}
Lu' = f_\xi(\xi) & \text{in } B_k(\xi_0), \\
u' = u & \text{on } \partial B_k(\xi_0).
\end{cases}
\]

Similarly to (3.3) and (3.6), it follows

\[
\sup_{B_{k+2}(\xi_0)} |X_i X_j u_i(\xi_0) - X_i X_j u_{i+1}(\xi_0)| \leq C \omega_f(\rho^k),
\]

(3.9)

\[
\lim_{k \to \infty} X_i X_j u_k(\xi_0) = X_i X_j u(\xi_0).
\]

(3.10)

Since \( L(u_k' - u_k) = f_\xi - f_0 \) in \( B_{k+2}(\xi_0) \), it implies

\[
L[u_k' - u_k - \frac{1}{2}(f_\xi - f_0)\eta^2] = 0, \text{ in } B_{k+2}(\xi_0).
\]

By Lemma 2.5 and (3.1), we have

\[
|X_i X_j u_k(\xi_0) - X_i X_j u_k(\xi_0)|
\]

\[
\leq |(f_\xi - f_0)| + |X_i X_j u_k(\xi_0) - X_i X_j u_k(\xi_0) - \frac{1}{2} X_i X_j (f_\xi - f_0)\eta^2|
\]

\[
\leq C \omega_f(\rho^k) + C \rho^{2k} \sup_{B_{k+2}(\xi_0)} |u_k' - u_k| + C \sup_{B_{k+2}(\xi_0)} |(f_\xi - f_0)\eta^2|
\]

\[
\leq C \omega_f(\rho^k) + C \rho^{2k} \left( \omega_f(\rho^k) + \sup_{\partial B_{k+2}(\xi_0)} |u - u_k| \right) + C \rho^{2k} \omega_f(\rho^k)
\]

\[
\leq C \omega_f(\rho^k).
\]

(3.11)
With a similar process to (3.8), one has by (3.9), (3.10) and (3.11) that

\[
I_1 \leq |X_iX_j u(\xi_0) - X_iX_j u_k(\xi_0)| + |X_iX_j u_k(\xi_0) - X_iX_j u_k(\xi_0)| + C\omega_j(\rho^k)
\]
\[
\leq C \int_0^{\sqrt{d_0}} \frac{\omega(r)}{r} dr.
\] (3.12)

Finally, let us estimate $I_2$. Since $w_k \in C^\infty(B_{k+2})$, we have by Lemma 2.8 that

\[
|X_iX_j w_k(\xi_0) - X_iX_j w_k(0)| \leq C_{d_0} \sup_{|\eta| < \xi_0 + \rho^{k+2}} |X_iX_j w_k(\eta)| \leq C_{d_0} \rho^{-k} \omega_j(\rho^k).
\] (3.13)

On the other hand, it derives

\[
|X_iX_j u_1(\xi_0)| \leq C \sup_{|\eta| < \xi_0 + \rho^{k+2}} |X_iX_j (u_1(\eta) - P(u_1,0)(\eta))|
\]
\[
\leq C \left( \sup_{B_1} |u_1| + \sup_{B_1} |P(u_1,0)| \right)
\]
\[
\leq C \left( \sup_{B_1} |u| + \|f\|_\infty + \sum_{l=1}^{k-1} \|X^l(u_1 - \frac{1}{2} f_0 x_1^2)\| + \sum_{l=2}^{k-1} \|X^l(f_0 x_1^2)\| \right)
\]
\[
\leq C \left( \sup_{B_1} |u| + \|f\|_\infty \right).
\] (3.14)

Then we get by (3.13) and (3.14) that

\[
I_2 \leq |X_iX_j u_{k-1}(\xi_0) - X_iX_j u_{k-1}(0)| + |X_iX_j w_{k-1}(\xi_0) - X_iX_j w_{k-1}(0)|
\]
\[
\leq |X_iX_j u_1(\xi_0)| - \sum_{l=1}^{k-1} |X_iX_j w_l(\xi_0) - X_iX_j w_l(0)|
\]
\[
\leq C \left( \sup_{B_1} |u| + \|f\|_\infty + \sum_{l=1}^{k-1} \frac{\omega_j(\rho^l)}{\rho^{2l}} \rho^l \right)
\]
\[
\leq C \left( \sup_{B_1} |u| + \|f\|_\infty + \int_0^{\sqrt{d_0}} \frac{\omega_j(r)}{r^2} dr \right).
\] (3.15)

Substituting (3.8), (3.12), (3.15) into (3.7), we conclude that for every $\xi_0$ satisfying $d_0 = |\xi_0| \leq 1/4b^2$, it holds

\[
|X_iX_j u(\xi_0) - X_iX_j u(0)|
\]
\[
\leq C \left( \sup_{B_1(0)} |u| + \|f\|_\infty + \int_0^{\sqrt{d_0}} \frac{\omega_j(r)}{r^2} dr + \int_0^{\sqrt{d_0}} \frac{\omega_j(r)}{r} dr \right).
\] (3.16)
For any $\xi$ and $\eta$ in $B_{1/2}(0)$, $d = d(\xi, \eta)$, let us choose $\xi = \xi_1, \ldots, \xi_n = \eta$ such that

$$d(\xi_i, \xi_{i+1}) = d', d' < d_0, (n-1)d' = d, \text{ for } 1 \leq i \leq n-1.$$  

By applying (3.16) to those points, we get (1.2).

**Step 3.** If $f \in C^{0, \alpha}(B_1(0))$, $\alpha \in (0, 1)$, then

$$|f(\xi) - f(\eta)| \leq [f]_{C^{0, \alpha}} d(\xi, \eta)^\alpha,$$

thus

$$\omega_f(r) = \sup_{d(\xi, \eta) < r} |f(\xi) - f(\eta)| \leq [f]_{C^{0, \alpha}} r^\alpha.$$  

Hence it yields from the right side of (1.2) that

$$d \left( \sup_{B_1(0)} |u| + \|f\|_{L^\infty} + \int_0^{\sqrt{d}} \frac{\omega_f(r)}{r^2} dr \right) + \int_0^{\sqrt{d}} \frac{\omega_f(r)}{r} dr \leq d \left( \sup_{B_1(0)} |u| + \|f\|_{L^\infty} + [f]_{C^{0, \alpha}} \int_0^{\sqrt{d}} \frac{1}{r^{2-\alpha}} dr \right) + [f]_{C^{0, \alpha}} \int_0^{\sqrt{d}} \frac{1}{r^{1-\alpha}} dr$$

$$\leq d \sup_{B_1(0)} |u| + \frac{d}{2 - \alpha} [f]_{C^{0, \alpha}} \left( \frac{1}{(\sqrt{d})^{1-\alpha}} - 1 \right) + \frac{1}{\alpha} [f]_{C^{0, \alpha}} \left( \sqrt{d} \right)^\alpha$$

$$\leq Cd^{\alpha/2} \left( \sup_{B_1(0)} |u| + \|f\|_{C^{0, \alpha}} \right).$$

and proves (1.3).

If $f \in C^{0, 1}(B_1(0))$, then

$$\omega_f(r) = \sup_{d(\xi, \eta) < r} |f(\xi) - f(\eta)| \leq [f]_{C^{0, 1}} r$$

and

$$d \left( \sup_{B_1(0)} |u| + \|f\|_{L^\infty} + \int_0^{\sqrt{d}} \frac{\omega_f(r)}{r^2} dr \right) + \int_0^{\sqrt{d}} \frac{\omega_f(r)}{r} dr \leq d \left( \sup_{B_1(0)} |u| + \|f\|_{L^\infty} + [f]_{C^{0, 1}} \int_0^{\sqrt{d}} \frac{1}{r} dr \right) + [f]_{C^{0, 1}} \sqrt{d}$$

$$\leq d \left( \sup_{B_1(0)} |u| + \|f\|_{L^\infty} \right) + [f]_{C^{0, 1}} \sqrt{d} \left( 1 + |\sqrt{d} \log \sqrt{d}| \right)$$

$$\leq d^{1/2} \left( \sup_{B_1(0)} |u| + \|f\|_{C^{0, 1}} \left( 1 + |\sqrt{d} \log \sqrt{d}| \right) \right),$$

thus (1.4) is obtained. □
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