ON A VARIATIONAL APPROACH TO THE NAVIER-STOKES EQUATIONS

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a domain. The initial-boundary value problem for the incompressible Navier-Stokes Equations is the following one,

\begin{align}
(i) & \quad \frac{\partial v}{\partial t} + \text{div}_x (v \otimes v) + \nabla_x p = \nu \Delta_x v + f \quad \forall (x,t) \in \Omega \times (0,T), \\
(ii) & \quad \text{div}_x v = 0 \quad \forall (x,t) \in \Omega \times (0,T), \\
(iii) & \quad v = 0 \quad \forall (x,t) \in \partial \Omega \times (0,T), \\
(iv) & \quad v(x,0) = v_0(x) \quad \forall x \in \Omega.
\end{align}

Here $v = v(x,t) : \Omega \times (0,T) \to \mathbb{R}^N$ is an unknown velocity, $p = p(x,t) : \Omega \times (0,T) \to \mathbb{R}$ is an unknown pressure, associated with $v$, $\nu > 0$ is a given constant viscosity, $f : \Omega \times (0,T) \to \mathbb{R}^N$ is a given force field and $v_0 : \Omega \to \mathbb{R}^N$ is a given initial velocity. The existence of weak solution to (1.1) satisfying the Energy inequality was first proved in the celebrating works of Leray (1934). There are many different procedures for constructing weak solutions (see Leray [9],[10] (1934); Kiselev and Ladyzhenskaya [8] (1957); Shinbrot [12] (1973)). The most common methods are based on the so called Faedo-Galerkin approximation process. Application of Faedo-Galerkin method for (1.1) was first considered by Hopf in [7]. We also refer to Masuda [11] for the problem in higher dimension. In this paper we present a variational method to investigate the Navier-Stokes equations that we thought to be completely new, see however the remarks below. As an application of this method we give a relatively simple proof of the existence of weak solutions to the problem (1.1).

Let us briefly describe our method. Consider for simplicity $f = 0$ in (1.1). For every smooth $u : \bar{\Omega} \times [0,T] \to \mathbb{R}^N$ satisfying conditions $\text{(ii)} - \text{(iv)}$ of (1.1) define the

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energy functional

\[(1.2)\quad E(u) := \frac{1}{2} \int_0^T \int_\Omega \left( \nu |\nabla u|^2 + \frac{1}{\nu} |\nabla \bar{H} u|^2 \right) \, dxdt + \frac{1}{2} \int_\Omega |u(x,T)|^2 \, dx,\]

where $\bar{H}(x,t)$ solves the following Stokes system for every $t \in (0, T),$

\[(1.3)\]
\[
\begin{cases}
\Delta_x \bar{H} = \left( \frac{\partial u}{\partial t} + \div_x (u \otimes u) \right) + \nabla_x p, & x \in \Omega, \\
\div_x \bar{H} = 0, & x \in \Omega, \\
\bar{H} = 0, & \forall x \in \partial \Omega.
\end{cases}
\]

A simple integration by parts gives

\[(1.4)\quad E(u) = \frac{1}{2\nu} \int_0^T \int_\Omega \left( |\nu \nabla u - \nabla \bar{H} u|^2 \right) \, dxdt + \frac{1}{2} \int_\Omega |v_0(x)|^2 \, dx.
\]

Therefore, if there exists at least a smooth solution to (1.1) (with $f = 0$) then a smooth function $u : \Omega \times (0, T) \to \mathbb{R}^N$ will be a solution to (1.1) (with $f = 0$) if and only if it is a minimizer of the functional in (1.2) among all smooth divergence free vector fields satisfying the boundary and the initial value conditions of (1.1). For the rigorous formulations and statements, see Section 5. This remark relates the problem of existence of solutions of the Navier-Stokes equations to that of minimizing the energy $E(u)$.

Unfortunately, when applying this method to the Navier-Stokes Equation one meets certain difficulties, for example in proving the existence of minimizers to $E$. But we can apply this method to a suitable approximation of problem (1.1). We approximate (1.1) by replacing the nonlinear term $\div_x (v \otimes v)$ with the terms $\div_x \{ f_n(|v|^2)(v \otimes v) \}$, where $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ are regular cutoff functions satisfying $f_n(s) = 1$ for $s \leq n$ and $f_n(s) = 0$ for $s > 2n$. The approximating problems are simpler than (1.1), since the nonlinear term has higher integrability. Next we consider the energies $E_n$ corresponding to the approximating problems and investigate the Euler-Lagrange equations of $E_n$ and the existence of minimizers. In this way we get solutions to the approximating problems which satisfy the energy equality (in fact these solutions will be regular if the initial data and the domain are). Next we pass to the limit for $n \to \infty$ and obtain a weak solution to (1.1). For the details see Section 3.

After completing the first version of this paper I learned that an energy-functional, very similar to (1.2), was used by Ghoussoub and his coauthors, see [3, 4, 5, 6], to prove existence of weak solutions for (1.1) and to study many other evolution equations. The basic variational principle behind this approach was first introduced...
by Brezis and Ekeland, see [1] (I wasn’t aware of this article as well). The main new feature of our method is that, unlike the previous works mentioned above, we manage to deduce directly from the Euler-Lagrange equation associated with (1.2) that the minimizer is a solution of the original problem (1.1).

We shall now demonstrate our method in the simple example of the heat equation. In this case, the energy-functional takes the form

\begin{equation}
E(u) := \frac{1}{2} \int_0^T \int_\Omega \left( |\nabla_x u|^2 + |\nabla_x \{\Delta^{-1}_x (\partial_t u)\}|^2 \right) dx dt + \frac{1}{2} \int_\Omega |u(x, T)|^2 dx,
\end{equation}

where \(\Delta^{-1} f\) is the solution of

\[
\begin{aligned}
\Delta y &= f \\
y &= 0 \quad \forall x \in \partial \Omega.
\end{aligned}
\]

The variational functional of type (1.5) was first considered by Brezis and Ekeland [1] in the more general case of gradient flows. Let us investigate the Euler-Lagrange equation for (1.5). If \(u\) satisfies \(u(x, t) = 0\) for every \((x, t) \in \partial \Omega \times (0, T)\) and \(u(x, 0) = u_0(x)\), then, as before,

\[
E(u) := \frac{1}{2} \int_0^T \int_\Omega \left( |\nabla_x \{u - \Delta^{-1}_x (\partial_t u)\}|^2 \right) dx dt + \frac{1}{2} \int_\Omega |u_0|^2(x) dx,
\]

Set \(W_u := u - \Delta^{-1}_x (\partial_t u)\). Then, for every minimizer \(u\) and for every smooth test function \(\delta(x, t)\) satisfying \(\delta(x, t) = 0\) for every \((x, t) \in \partial \Omega \times (0, T)\) and \(\delta(x, 0) = 0\), we obtain

\[
0 = \lim_{s \to 0} \frac{1}{2s} \int_0^T \int_\Omega (\Delta_x W_{u + s\delta} - \Delta_x W_u) \cdot (W_{u + s\delta} + W_u) = \lim_{s \to 0} \frac{1}{2s} \int_0^T \int_\Omega (\Delta_x W_{u + s\delta} - \Delta_x W_u) \cdot (W_{u + s\delta} + W_u) = \int_0^T \int_\Omega (\Delta_x W_u \cdot \nabla_x \delta + W_u \cdot \partial_t \delta).
\]

Since \(\delta\) was arbitrary (in particular \(\delta(x, T)\) is free) we deduce that \(\Delta_x W_u + \partial_t W_u = 0\), \(W_u(x, T) = 0\) and \(W_u = 0\) if \(x \in \partial \Omega\). Changing variables \(\tau := T - t\) gives

\[
\begin{aligned}
\partial_\tau W_u &= \Delta_x W_u \quad \forall (x, \tau) \in \Omega \times (0, T), \\
W_u(x, 0) &= 0, \\
W_u(x, \tau) &= 0 \quad \forall (x, \tau) \in \partial \Omega \times (0, T).
\end{aligned}
\]

Therefore \(W_u = 0\) and then \(\Delta_x u = \partial_t u\), i.e., \(u\) is the solution of the heat equation.
2. Preliminaries

For two matrices $A, B \in \mathbb{R}^{p \times q}$ with $ij$-th entries $a_{ij}$ and $b_{ij}$ respectively, we write $A : B := \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ij}$.

Given a vector valued function $f(x) = (f_1(x), \ldots, f_k(x)) : \Omega \rightarrow \mathbb{R}^k$ ($\Omega \subset \mathbb{R}^N$) we denote by $\nabla_x f$ the $k \times N$ matrix with $ij$-th entry $\frac{\partial f_i}{\partial x_j}$.

For a matrix valued function $F(x) := \{ F_{ij}(x) \} : \mathbb{R}^N \rightarrow \mathbb{R}^{k \times N}$ we denote by $\text{div} F$ the $\mathbb{R}^k$-valued vector field defined by $\text{div} F := (l_1, \ldots, l_k)$ where $l_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$. Throughout the rest of the paper we assume that $\Omega$ is domain in $\mathbb{R}^N$.

Definition 2.1. We denote:

- By $\mathcal{V}_N$ the space $\{ \varphi \in C_c^\infty(\Omega, \mathbb{R}^N) : \text{div} \varphi = 0 \}$ and by $L_N$ the space, which is the closure of $\mathcal{V}_N$ in the space $L^2(\Omega, \mathbb{R}^N)$, endowed with the norm $\| \varphi \| := \left( \int_{\Omega} |\varphi|^2 \, dx \right)^{1/2}$.
- By $\dot{H}^1_0(\Omega, \mathbb{R}^N)$ the closure of $C_c^\infty(\Omega, \mathbb{R}^N)$ with respect to the norm $||| \varphi ||| := \left( \int_{\Omega} |\nabla \varphi|^2 \, dx \right)^{1/2}$. This space differ from $H^1_0(\Omega, \mathbb{R}^N)$ only in the case of unbounded domain.
- By $V_N$ the closure of $\mathcal{V}_N$ in $\dot{H}^1_0(\Omega, \mathbb{R}^N)$.
- By $V_N^{-1}$ the space dual to $V_N$.
- By $\mathcal{Y}$ the space $\mathcal{Y} := \{ \varphi(x,t) \in C_c^\infty(\Omega \times [0,T], \mathbb{R}^N) : \text{div}_x \varphi = 0 \}$.

Remark 2.1. It is obvious that $u \in \mathcal{D}'(\Omega, \mathbb{R}^N)$ (rigorously the equivalence class of $u$, up to gradients) belongs to $V_N^{-1}$ if and only if there exists $w \in V_N$ such that

$$\int_{\Omega} \nabla w : \nabla \delta \, dx = - < u, \delta > \quad \forall \delta \in V_N.$$

In particular $\Delta w = u + \nabla p$ as a distribution and

$$||| w ||| = \sup_{\delta \in V_N, ||\delta|| \leq 1} < u, \delta > = ||| u |||_{-1}.$$

Definition 2.2. We will say that the distribution $l \in \mathcal{D}'(\Omega \times (0,T), \mathbb{R}^N)$ belongs to $L^2(0,T; V_N^{-1})$, if there exists $v(\cdot, t) \in L^2(0,T; V_N^{-1})$, such that for every $\psi(x,t) \in C_c^\infty(\Omega \times (0,T), \mathbb{R}^N)$, satisfying $\text{div}_x \psi = 0$, we have

$$< l(\cdot, \cdot), \psi(\cdot, \cdot) > = \int_{0}^{T} < v(\cdot, t), \psi(\cdot, t) > \, dt.$$
Remark 2.2. Let \( v(\cdot, t) \in L^2(0, T; V_N^{-1}) \). For a.e. \( t \in [0, T] \) consider \( V_v(\cdot, t) \) as in Remark 2.1 corresponding to \( v(\cdot, t) \), i.e.
\[
\int_{\Omega} \nabla_x V_v(x, t) : \nabla_x \delta(x) \, dx = - < v(\cdot, t), \delta > \quad \forall \delta \in V_N.
\]
Then it is clear that \( V_v(\cdot, t) \in L^2(0, T; V_N) \) and
\[
\| V_v \|_{L^2(0, T; V_N)} = \| v \|_{L^2(0, T; V_N^{-1})}.
\]

In the sequel we will need several lemmas. In all of them \( \Omega \subset \mathbb{R}^N \) is a bounded domain. The following Lemma can be proved in the same way as Lemmas 2.1 and 2.2 in [2].

**Lemma 2.1.** Let \( u \in L^2(0, T; V_N) \cap L^\infty(0, T; L_N) \) be such that \( \partial_t u \in L^2(0, T; V_N^{-1}) \). Consider \( V_0(\cdot, t) \in L^2(0, T; V_N) \) as in Remark 2.2 corresponding to \( \partial_t u \). Then we can redefine \( u \) on a subset of \([0, T]\) of Lebesgue measure zero, so that \( u(\cdot, t) \) will be \( L_N \)-weakly continuous in \( t \) on \([0, T]\). Moreover, for every \( 0 \leq a < b \leq T \) and for every \( \psi(x, t) \in \mathcal{Y} \) (see Definition 2.1) we will have

\[
(2.1) \quad \int_a^b \int_{\Omega} \nabla_x V_0 : \nabla_x \psi \, dxdt - \int_a^b \int_{\Omega} u \cdot \partial_t \psi \, dxdt = \int_{\Omega} u(x, a) \cdot \psi(x, a) \, dx - \int_{\Omega} u(x, b) \cdot \psi(x, b) \, dx.
\]

**Remark 2.3.** Let \( F \in Lip(\mathbb{R}^N, \mathbb{R}^{N \times N}) \) satisfying \( F(0) = 0 \). Then for every \( u \in L^\infty(0, T; L_N) \) we have \( F(u) \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^{N \times N})) \) and therefore \( \text{div}_x F(u) \in L^2(0, T; V_N^{-1}) \). If in addition \( \partial_t u \in L^2(0, T; V_N^{-1}) \) then we obtain \( \partial_t u + \text{div}_x F(u) \in L^2(0, T; V_N^{-1}) \).

We have then the following Corollary to Lemma 2.1

**Corollary 2.1.** Let \( u \) be as in Lemma 2.1 and let \( F \in Lip(\mathbb{R}^N, \mathbb{R}^{N \times N}) \) satisfying \( F(0) = 0 \). Assume, in addition, that \( u(\cdot, t) \) is \( L_N \)-weakly continuous in \( t \) on \([0, T]\) (see Lemma 2.1). Consider \( V(\cdot, t) \in L^2(0, T; V_N) \) as in Remark 2.2 corresponding to \( \partial_t u + \text{div}_x F(u) \). Then for every \( 0 \leq a < b \leq T \) and for every \( \psi(x, t) \in \mathcal{Y} \) we have

\[
(2.2) \quad \int_a^b \int_{\Omega} \nabla_x V : \nabla_x \psi \, dxdt - \int_a^b \int_{\Omega} (u \cdot \partial_t \psi + F(u) : \nabla_x \psi) \, dxdt = \int_{\Omega} u(x, a) \cdot \psi(x, a) \, dx - \int_{\Omega} u(x, b) \cdot \psi(x, b) \, dx.
\]

We will need in the sequel the following compactness result.
Lemma 2.2. Let \( \{u_n\} \subset L^2(0,T;V_N) \cap L^\infty(0,T;L_N) \) be a subsequence, bounded in \( L^\infty(0,T;L_N) \) and such that

\[
(2.3) \quad u_n \rightharpoonup u_0 \quad \text{weakly in } L^2(0,T;V_N),
\]

and

\[
(2.4) \quad u_n(\cdot,t) \rightharpoonup u_0(\cdot,t) \quad \text{weakly in } L_N \quad \forall t \in (0,T).
\]

Then

\[
(2.5) \quad u_n \to u_0 \quad \text{strongly in } L^2(0,T;L_N).
\]

We will give the proof of this Lemma in the Appendix.

3. Existence of the weak solution to the Navier-Stokes Equations

Throughout this section we assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain.

Definition 3.1. Let \( F(v) = \{F_{ij}(v)\} \in C^1(\mathbb{R}^N, \mathbb{R}^{N \times N}) \cap \text{Lip} \) satisfy \( F(0) = 0 \) and \( \frac{\partial F_{ij}}{\partial v_m}(v) = \frac{\partial F_{mj}}{\partial v_i}(v) \) for all \( v \in \mathbb{R}^N \) and \( m,i,j \in \{1,\ldots,N\} \). Denote the class of all such \( F \) by \( \mathcal{F} \).

Remark 3.1. Let \( F \in \mathcal{F} \). Then it is clear that there exists \( G(v) = (G_1(v), \ldots, G_N(v)) \in C^2(\mathbb{R}^N, \mathbb{R}^N) \), such that \( \frac{\partial G_i}{\partial v_i}(v) = F_{ij}(v) \) i.e. \( \nabla_v G(v) = (F(v))^T \).

Using our variational approach, we will prove in the sequel the existence of a solution of the following problem

\[
(3.1) \quad \begin{cases}
\frac{\partial v}{\partial t} + \text{div}_x F(v) + \nabla_x p = \Delta_x v & \forall (x,t) \in \Omega \times (0,T), \\
\text{div}_x v = 0 & \forall (x,t) \in \Omega \times (0,T), \\
v = 0 & \forall (x,t) \in \partial \Omega \times (0,T), \\
v(x,0) = v_0(x) & \forall x \in \Omega,
\end{cases}
\]

for every \( F \in \mathcal{F} \), which in addition satisfies the Energy Equality (see Theorem 4.4).

But first of all, in the proof of the following theorem we would like to explain how this fact implies the existence of weak solution to the Navier-Stokes Equation.

Theorem 3.1. Let \( v_0(x) \in L_N \). Then there exists \( u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N) \) satisfying

\[
(3.2) \quad \int_\Omega v_0(x) \cdot \psi(x,0) \, dx + \int_0^T \int_\Omega (u \cdot \partial_t \psi + (u \otimes u) : \nabla_x \psi) = \int_0^T \int_\Omega \nabla_x u : \nabla_x \psi,
\]
for every \( \psi(x,t) \in C^\infty_c(\Omega \times [0,T),\mathbb{R}^N) \) such that \( \text{div}_x \psi = 0 \), i.e.

\[
\Delta_x u = \partial_t u + \text{div}_x (u \otimes u) + \nabla_x \phi , \quad \text{and } u(x,0) = v_0(x).
\]

Moreover, for a.e. \( \tau \in [0,T] \) we have

\[
\int_0^\tau \int_\Omega |\nabla_x u|^2 \, dx \, dt \leq \frac{1}{2} \left( \int_\Omega v_0^2(x) \, dx - \int_\Omega u^2(x,\tau) \, dx \right).
\]

**Proof.** Fix some \( h(s) \in C^\infty(\mathbb{R},[0,1]) \), satisfying \( h(s) = 1 \) \( \forall s \leq 1 \) and \( h(s) = 0 \) \( \forall s \geq 2 \). For every \( n \in \mathbb{N} \) define \( f_n(s) := h(s/n) \). Consider

\[
F_n(v) := f_n(|v|^2)(v \otimes v) + g_n(|v|^2)I_N ,
\]

where \( I_N \) is a \( N \times N \)-unit matrix and \( g_n(r) := \frac{1}{2} \int_0^r f_n(s) \, ds \). Then for every \( n \) we have \( F_n \in \mathcal{F} \) and there exists \( A > 0 \) such that \( |F_n(v)| \leq A|v|^2 \) for every \( v \) and \( n \). Fix also some sequence \( \{v_0^{(n)}\}_{n=1}^\infty \subset \mathcal{V}_N \) such that \( v_0^{(n)} \to v_0 \) strongly in \( L_N \) as \( n \to \infty \). By Theorem \ref{thm-conv}, for every \( n \) there exist a function \( u_n \in L^2(0,T;\Omega^N) \cap L^\infty(0,T;L_N) \), such that \( \partial_t u_n \in L^2(0,T;\Omega^N) \) and \( u_n(\cdot,t) \) is \( L_N \)-weakly continuous in \( t \) on \([0,T]\), which satisfy

\[
\int_0^T \int_\Omega v_0^{(n)}(x) \cdot \psi(x,0) + \int_0^\tau \int_\Omega (u_n \cdot \partial_t \psi + F_n(u_n) : \nabla_x \psi) = \int_0^T \int_\Omega \nabla_x u_n : \nabla_x \psi ,
\]

for every \( \psi(x,t) \in C^\infty_c(\Omega \times [0,T),\mathbb{R}^N) \), such that \( \text{div}_x \psi = 0 \). Moreover, by the same Theorem, for every \( \tau \in [0,T] \) we obtain

\[
\frac{1}{2} \int_\Omega u_n^2(x,\tau) \, dx + \int_0^\tau \int_\Omega |\nabla_x u_n|^2 \, dx \, dt = \frac{1}{2} \int_\Omega (v_0^{(n)})^2(x) \, dx.
\]

Therefore, since \( v_0^{(n)} \) is bounded in \( L_N \) we obtain that there exists \( C > 0 \) independent of \( n \) and \( t \) such that

\[
\|u_n(\cdot,t)\|_{L_N} \leq C \quad \forall n \in \mathbb{N}, t \in [0,T].
\]

Moreover, \( \{u_n\} \) is bounded in \( L^2(0,T;\Omega^N) \). By \( \text{(3.5)} \) and \( \text{(2.2)} \), for every \( t \in [0,T] \) and for every \( \phi \in \mathcal{V}_N \), we have

\[
\int_\Omega v_0^{(n)}(x) \cdot \phi(x) \, dx - \int_0^t \int_\Omega \nabla_x u_n : \nabla_x \phi + \int_0^t \int_\Omega F_n(u_n) : \nabla_x \phi
\]

\[
= \int_\Omega u_n(x,t) \cdot \phi(x) \, dx.
\]

Since \( |F_n(u_n)| \leq C|u_n|^2 \), by \( \text{(3.7)} \),

\[
\|F_n(u_n(\cdot,t))\|_{L^1(\Omega,\mathbb{R}^N \times \mathbb{R}^N)} \leq C \quad \forall n \in \mathbb{N}, t \in [0,T].
\]
In particular \( \{F_n(u_n)\} \) is bounded in \( L^1(\Omega \times (0, T), \mathbb{R}^{N \times N}) \). Therefore, there exists a finite Radon measure \( \mu \in \mathcal{M}(\Omega \times (0, T), \mathbb{R}^{N \times N}) \), such that, up to a subsequence, \( F_n(u_n) \rightharpoonup \mu \) weakly as a sequence of finite Radon measures. Then for every \( \psi \in C_0^\infty(\Omega \times (0, T), \mathbb{R}^{N \times N}) \) we have

\[
\lim_{n \to \infty} \int_0^T \int_\Omega F_n(u_n) : \psi \, dx \, dt = \int_{\Omega \times (0, T)} \psi : d\mu.
\]

Moreover, by (3.9), we obtain

\[
\lim_{n \to \infty} \int_0^T \int_\Omega |F_n(u_n)| \, dx \, dt \leq C(b - a).
\]

Then, by (3.9), (3.10) and (3.11), for every \( \phi \in \mathcal{V}_N \) and every \( t \in [0, T] \) we obtain

\[
\lim_{n \to \infty} \int_0^t \int_\Omega F_n(u_n(x, s)) : \nabla \phi(x) \, dx \, ds = \int_{\Omega \times (0, t)} \nabla \phi(x) : d\mu(x, s).
\]

But since \( u_n \) is bounded in \( L^2(0, T; \mathcal{V}_N) \), up to a subsequence, it converge weakly in \( L^2(0, T; \mathcal{V}_N) \) to the limit \( u_0 \). We also know that \( u_n(\cdot, 0) \rightharpoonup v_0(\cdot) \) weakly in \( L_N \). Plugging these facts and (3.12) into (3.8), for every \( t \in [0, T] \) and every \( \phi \in \mathcal{V}_N \) we infer

\[
\lim_{n \to \infty} \int_\Omega u_n(x, t) \cdot \phi(x) \, dx = \int_\Omega v_0(x) \cdot \phi(x) \, dx - \int_0^t \int_\Omega \nabla u_0 : \nabla \phi + \int_{\Omega \times (0, t)} \nabla \phi : d\mu(x, s) \cdot dxds.
\]

Since \( \mathcal{V}_N \) is dense in \( L_N \), by (3.7), and (3.13), for every \( t \in [0, T] \) there exists \( u(\cdot, t) \in L_N \) such that

\[
u_n(\cdot, t) \rightharpoonup u(\cdot, t) \text{ weakly in } L_N \quad \forall t \in [0, T],
\]

Moreover, \( \|u(\cdot, t)\|_{L_N} \leq C \). But we have \( u_n \rightharpoonup u_0 \) in \( L^2(0, T; \mathcal{V}_N) \), therefore \( u = u_0 \) and so \( u \in L^2(0, T; \mathcal{V}_N) \cap L^\infty(0, T; L_N) \). Then we can use (3.7), (3.14) and Lemma 2.2 to deduce that \( u_n \to u \) strongly in \( L^2(0, T; L_N) \). Then, up to a subsequence, we have \( u_n(x, t) \to u(x, t) \) almost everywhere in \( \Omega \times (0, T) \). In particular \( f_n(|u_n(x, t)|^2) \to 1 \) almost everywhere in \( \Omega \times (0, T) \). Then,

\[
\lim_{n \to \infty} \int_0^T \int_\Omega |f_n(|u_n|^2)(u_n \otimes u_n) - (u \otimes u)| \, dx \, dt \\
\leq \lim_{n \to \infty} \int_0^T \int_\Omega |f_n(|u_n|^2)| - (u \otimes u) \, dx \, dt + \lim_{n \to \infty} \int_0^T \int_\Omega u^2|f_n(|u_n|^2) - 1| \, dx \, dt = 0.
\]

Therefore, letting \( n \) tend to \( \infty \) in (3.5), we obtain (3.2). Moreover, by (3.6), for a.e. \( t \in [0, T] \) we obtain (3.3). This completes the proof.  

\[ \square \]
4. Proof of the existence of solutions to \((3.1)\)

Throughout this section we assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain. The following Lemma can be proved in the same way as Theorem 4.1 in [2].

**Lemma 4.1.** Let \(u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)\) be such that \(\partial_t u \in L^2(0,T;V_N^{-1})\) and let \(u(\cdot,t)\) be \(L_N\)-weakly continuous in \(t\) on \([0,T]\) (see Lemma [2.7]). Consider \(V_0(\cdot,t) \in L^2(0,T;V_N)\) as in Remark [2.2] corresponding to \(\partial_t u\). Then for every \(t \in [0,T]\) we have
\[
\int_0^\tau \int_\Omega \nabla_x u : \nabla_x V_0 \, dxdt = \frac{1}{2} \left( \int_\Omega u^2(x,0) \, dx - \int_\Omega u^2(x,\tau) \, dx \right).
\]

**Corollary 4.1.** Let \(u \in L^2(0,T;V_N)\) be such that \(\partial_t u \in L^2(0,T;V_N^{-1})\). Then \(u \in L^\infty(0,T;L_N)\).

We will give the proof of this Corollary in the Appendix.

Next we have the second Corollary to Lemma [4.1].

**Corollary 4.2.** Let \(F \in \mathfrak{F}\) and let \(u \in L^2(0,T;V_N) \cap L^\infty(0,T;L_N)\) be such that \(\partial_t u \in L^2(0,T;V_N^{-1})\) and let \(u(\cdot,t)\) be \(L_N\)-weakly continuous in \(t\) on \([0,T]\) (see Lemma [2.7]). Consider \(V(\cdot,t) \in L^2(0,T;V_N)\) as in Remark [2.2] corresponding to \(\partial_t u + \text{div}_x F(u)\) (see Remark [2.3]). Then for every \(\tau \in [0,T]\) we have
\[
(4.1) \quad \int_0^\tau \int_\Omega \nabla_x u : \nabla_x V \, dxdt = \frac{1}{2} \left( \int_\Omega u^2(x,0) \, dx - \int_\Omega u^2(x,\tau) \, dx \right).
\]

**Proof.** By Lemma [4.1] for every \(\tau \in [0,T]\) we obtain
\[
(4.2) \quad \int_0^\tau \int_\Omega \nabla_x V : \nabla_x u \, dxdt - \int_0^\tau \int_\Omega F(u) : \nabla_x u \, dxdt = \frac{1}{2} \left( \int_\Omega u^2(x,0) \, dx - \int_\Omega u^2(x,\tau) \, dx \right).
\]
But for almost every \(t \in [0,T]\) \(u(\cdot,t) \in V_N\), therefore, for every such fixed \(t\) there exists a sequence \(\{\delta_n(\cdot)\}_{n=1}^\infty \in \mathcal{V}_N\), such that \(\delta_n(\cdot) \to u(\cdot,t)\) in \(V_N\). But for every \(\delta \in V_N\) we obtain
\[
\int_\Omega F(\delta) : \nabla_x \delta = \int_\Omega \sum_{i=1}^N \sum_{j=1}^N : F_{ij}(\delta) \frac{\partial \delta_i}{\partial x_j} = \int_\Omega \sum_{i=1}^N \sum_{j=1}^N \frac{\partial G_i}{\partial x_j}(\delta) \frac{\partial \delta_i}{\partial x_j} = \int_\Omega \text{div}_x G(\delta) = 0,
\]
where \(G\) is as in Remark [3.1]. Therefore, since \(F\) is Lipshitz function, we obtain
\[
\int_\Omega F(u(x,t)) : \nabla_x u(x,t) \, dx = \lim_{n \to \infty} \int_\Omega F(\delta_n(x)) : \nabla_x \delta_n(x) \, dx = 0.
\]
Therefore, using (4.2), we obtain (4.1) and the result follows. \(\square\)
Definition 4.1. Let \( u \in L^2(0, T; V_N) \cap L^\infty(0, T; L_N) \) be such that \( \partial_t u \in L^2(0, T; V_N^{-1}) \) and such that \( u(\cdot, t) \) is \( L_N \)-weakly continuous in \( t \) on \([0, T]\). Denote the set of all such functions \( u \) by \( \mathcal{R} \). For a fixed \( F \in \mathcal{G} \) and for every \( u \in \mathcal{R} \) let \( H_u(\cdot, t) \in L^2(0, T; V_N) \) be as in Remark 2.2 corresponding to \( \partial_t u + \text{div}_x F(u) \). That is for every \( \psi(x, t) \in C^\infty_c(\Omega \times (0, T), \mathbb{R}^N) \) such that \( \text{div}_x \psi = 0 \) we have

\[
\int_0^T \int_{\Omega} (u \cdot \partial_t \psi + F(u) : \nabla_x \psi) \, dx \, dt = \int_0^T \int_{\Omega} \nabla_x H_u : \nabla_x \psi \, dx \, dt.
\]

Define a functional \( I_F(u) : \mathcal{R} \to \mathbb{R} \) by

\[
I_F(u) := \frac{1}{2} \left( \int_0^T \int_{\Omega} (|\nabla_x u|^2 + |\nabla_x H_u|^2) \, dx \, dt + \int_{\Omega} |u(x, T)|^2 \, dx \right),
\]

and for every \( v_0 \in V_N \) consider the minimization problem

\[
(4.4) \quad \inf \{ I_F(u) : u \in \mathcal{R}, u(\cdot, 0) = v_0(\cdot) \}.
\]

Remark 4.1. Since by Corollary 4.2 we have

\[
\int_0^T \int_{\Omega} \nabla_x u : \nabla_x H_u \, dx \, dt = \frac{1}{2} \left( \int_{\Omega} |u(x, 0)|^2 \, dx - \int_{\Omega} |u(x, T)|^2 \, dx \right),
\]

we can rewrite the definition of \( I_F \) in (4.3) by

\[
(4.5) \quad I_F(u) := \frac{1}{2} \left( \int_0^T \int_{\Omega} |\nabla_x u - \nabla_x H_u|^2 \, dx \, dt + \int_{\Omega} |u(x, 0)|^2 \, dx \right) \quad \forall u \in \mathcal{R}.
\]

Lemma 4.2. For every \( u \in \mathcal{R} \) and every \( \delta(x, t) \in \mathcal{Y} \), such that \( \delta(x, 0) = 0 \), we have

\[
(4.6) \quad \lim_{s \to 0} \frac{I_F(u + s\delta) - I_F(u)}{s} = \int_0^T \int_{\Omega} \left\{ \nabla_x W_u : \nabla_x \delta + \partial_t \delta \cdot W_u - \left( \sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \right\} \, dx \, dt,
\]

where we denote \( W_u := u - H_u \).
Proof. We have

\[ \frac{1}{2s} \int_0^T \int_{\Omega} \left( |\nabla_x W_{(u+s\delta)}|^2 - |\nabla_x W_u|^2 \right) = \]

\[ = \frac{1}{2s} \int_0^T \int_{\Omega} \left( \nabla_x W_{(u+s\delta)} - \nabla_x W_u \right) : \left( \nabla_x W_{(u+s\delta)} + \nabla_x W_u \right) = \]

\[ \frac{1}{2s} \int_0^T \int_{\Omega} \left( \nabla_x W_{(u+s\delta)} + \nabla_x W_u \right) : \left( \nabla_x W_{(u+s\delta)} + \nabla_x W_u \right) = \]

\[ = \int_0^T \int_{\Omega} \left\{ \frac{1}{2} \left( \nabla_x W_{(u+s\delta)} + \nabla_x W_u \right) : \nabla_x \delta + \partial_t \delta (x,t) \cdot \frac{1}{2} \left( W_{(u+s\delta)} (x,t) + W_u (x,t) \right) \right\} \]

\[ - \int_0^T \int_0^T \frac{1}{s} \left( F(u + s\delta) - F(u) \right) : \frac{1}{2} \left( \nabla_x W_{(u+s\delta)} + \nabla_x W_u \right) \, dx \, dt. \]

Since \( F \) is Lipschitz and \( C^1 \), we obtain

\[ \frac{1}{s} (F(u + s\delta) - F(u)) \to \sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j} (u) \quad \text{as} \quad s \to 0 \quad \text{strongly in} \quad L^2(\Omega \times (0,T), \mathbb{R}^{N \times N}). \]

On the other hand, for every \( h(x,t) \in L^2(0,T; V_N) \) we obtain

\[ \lim_{s \to 0} \int_0^T \int_{\Omega} \left( \nabla_x W_{(u+s\delta)} - \nabla_x W_u \right) : \nabla_x h(x,t) = \]

\[ \lim_{s \to 0} \left( \int_0^T \int_{\Omega} \left( \partial_t \delta - \Delta_x \delta \right) \cdot h \, dx \, dt - \int_0^T \int_{\Omega} \left( F(u + s\delta) - F(u) \right) : \nabla_x h \, dx \, dt \right) = 0. \]

Therefore

\[ W_{(u+s\delta)} \rightharpoonup W_u \quad \text{weakly in} \quad L^2(0,T; V_N). \]

In particular \( W_{(u+s\delta)} \) remains bounded in \( L^2(0,T; V_N) \) as \( s \to 0 \). Therefore, by \( (4.7) \), we obtain

\[ \lim_{s \to 0} \int_0^T \int_{\Omega} \left( |\nabla_x W_{(u+s\delta)}|^2 - |\nabla_x W_u|^2 \right) = 0. \]

So

\[ W_{(u+s\delta)} \to W_u \quad \text{strongly in} \quad L^2(0,T; V_N). \]

Therefore, using \( (4.11) \) and \( (4.8) \) in \( (4.7) \), we infer

\[ \lim_{s \to 0} \frac{1}{2s} \int_0^T \int_{\Omega} \left( |\nabla_x W_{(u+s\delta)}|^2 - |\nabla_x W_u|^2 \right) = \]

\[ \int_0^T \int_{\Omega} \left\{ \nabla_x W_u : \nabla_x \delta + \partial_t \delta \cdot W_u - \left( \sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j} (u) \right) : \nabla_x W_u \right\} \, dx \, dt. \]
So, by (4.5) and (4.12), we obtain that for every \( \delta(x,t) \in \mathcal{Y} \), such that \( \delta(x,0) = 0 \), we must have (4.6). \( \square \)

**Lemma 4.3.** Let \( u \in \mathcal{R} \) be a minimizer to (4.4). Then \( H_u = u \), i.e.
\[
\Delta_x u = \partial_t u + \text{div}_x F(u) + \nabla_x p.
\]

**Proof.** Let \( \delta(x,t) \in \mathcal{Y} \) be such that \( \delta(x,0) = 0 \). Then for every \( s \in \mathbb{R} \) \( (u + s\delta) \in \mathcal{R} \) and \( (u + s\delta)(\cdot,0) = v_0(\cdot) \). Therefore,
\[
\lim_{s \to 0} \frac{I_F(u + s\delta) - I_F(u)}{s} = 0.
\]

So, by (4.6) in Lemma 4.2, for every \( \delta(x,t) \in \mathcal{Y} \) such that \( \delta(x,0) = 0 \) we must have
\[
\int_0^T \int_{\Omega} \nabla_x W_u : \nabla_x \delta + \partial_t \delta \cdot W_u - \left( \sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx \, dt = 0,
\]
where \( W_u \in L^2(0,T;V_N) \) defined by \( W_u = u - H_u \). Since \( \frac{\partial F}{\partial u_j} \in L^\infty \), we obtain that the functional \( L(\phi) : V_N \to \mathbb{R} \) defined by
\[
L(\phi) := \int_{\Omega} \left( \sum_{j=1}^N \phi_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx
\]
begins to \( V_N^{-1} \) for a.e. \( t \in (0,T) \). Moreover there exists \( Q(x,t) \in L^2(0,T;V_N) \) such that for a.e. \( t \in (0,T) \) we have
\[
L(\phi) := \int_{\Omega} \left( \sum_{j=1}^N \phi_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx = \int_{\Omega} \nabla_x Q(x,t) : \nabla_x \phi(x) \, dx \quad \forall \phi \in V_N.
\]

Then from (4.14) we obtain that \( \partial_t W_u \in L^2(0,T;V_N^{-1}) \) and we have
\[
< \partial_t W_u(\cdot,\cdot),\psi(\cdot,\cdot)> = -\int_0^T \int_{\Omega} \nabla_x (Q - W_u) : \nabla_x \psi \, dx \, dt
\]
\[
\forall \psi \in C^\infty_c(\Omega \times (0,T),\mathbb{R}^N) \text{ s.t. div}_x \psi = 0.
\]

Therefore, by Corollary 4.1 \( W_u \in L^\infty(0,T;L_N) \) and by Lemma 2.1 we can redefine \( W_u(\cdot,t) \) on a set of Lebesgue measure zero on \([0,T]\) so that \( W_u(\cdot,t) \) be \( L_N \)-weakly continuous in \( t \) on \([0,T]\). From now we consider such \( W_u \). Moreover, by (2.2) and (4.15), for every \( \delta \in \mathcal{Y} \), such that \( \delta(x,0) = 0 \), we obtain
\[
\int_0^T \int_{\Omega} \nabla_x (Q - W_u) : \nabla_x \delta \, dx \, dt - \int_0^T \int_{\Omega} W_u \cdot \partial_t \delta \, dx \, dt = -\int_{\Omega} W_u(x,T) \cdot \delta(x,T) \, dx,
\]

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or in the another form

\begin{equation}
(4.16) \quad \int_0^T \int_\Omega \nabla_x W_u : \nabla x \delta \, dx \, dt - \int_0^T \int_\Omega \left( \sum_{j=1}^N \delta_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx \, dt \\
+ \int_0^T \int_\Omega W_u \cdot \partial_t \delta \, dx \, dt - \int_\Omega W_u(x, T) \cdot \delta(x, T) \, dx = 0.
\end{equation}

Comparing (4.16) with (4.14), we obtain that \( W_u(\cdot, T) = 0 \). Therefore, by Corollary 4.1 and Lemma 4.1, for every \( t \in [0, T] \) we obtain

\[ \int_t^T \int_\Omega \nabla_x W_u : \nabla x (Q - W_u) \, dx \, ds = \frac{1}{2} \int_\Omega W_u^2(x, t) \, dx, \]

or in the equivalent form

\begin{equation}
(4.17) \quad \int_t^T \int_\Omega |\nabla_x W_u|^2 \, dx \, ds + \frac{1}{2} \int_\Omega W_u^2(x, t) \, dx = \int_t^T \int_\Omega \left( \sum_{j=1}^N (W_u)_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx \, ds.
\end{equation}

In particular there exists \( C > 0 \), independent of \( t \), such that

\[ \int_t^T \int_\Omega |\nabla_x W_u|^2 \, dx \, ds \leq \int_t^T \int_\Omega \left( \sum_{j=1}^N (W_u)_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx \, ds \leq C \left( \int_t^T \int_\Omega |\nabla_x W_u|^2 \, dx \, ds \cdot \int_t^T \int_\Omega |W_u|^2 \, dx \, ds \right)^{1/2}. \]

So

\begin{equation}
(4.18) \quad \int_t^T \int_\Omega |\nabla_x W_u|^2 \, dx \, ds \leq C^2 \int_t^T \int_\Omega |W_u|^2 \, dx \, ds.
\end{equation}

Then, using (4.17) and (4.18) we obtain

\begin{equation}
(4.19) \quad \frac{1}{2} \int_\Omega W_u^2(x, t) \, dx \leq \int_t^T \int_\Omega \left( \sum_{j=1}^N (W_u)_j \frac{\partial F}{\partial u_j}(u) \right) : \nabla_x W_u \, dx \, ds \\
\leq C \left( \int_t^T \int_\Omega |\nabla_x W_u|^2 \, dx \, ds \cdot \int_t^T \int_\Omega |W_u|^2 \, dx \, ds \right)^{1/2} \leq C^2 \int_t^T \int_\Omega |W_u|^2 \, dx \, ds.
\end{equation}

Then by Gronwall’s Lemma \( \int_\Omega W_u^2(x, t) \, dx = 0 \). So, by definition of \( W_u \) we obtain \( H_u = u \). This completes the proof. \( \Box \)

**Theorem 4.1.** For every \( v_0(\cdot) \in V_N \) there exists a minimizer \( u \) to (4.4). It satisfies \( H_u = u \), i.e.

\[ \Delta_x u = \partial_t u + div_x F(u) + \nabla_x p, \]
\[ u(x, 0) = v_0(x) \] and

\[ \frac{1}{2} \int_{\Omega} u^2(x, \tau) dx + \int_0^\tau \int_{\Omega} |\nabla_x u|^2 \, dx \, dt = \frac{1}{2} \int_{\Omega} v_0^2(x) dx \quad \forall \tau \in [0, T]. \tag{4.20} \]

Moreover if \( v \in \mathcal{R} \) satisfy \( v(\cdot, 0) = v_0(\cdot) \) and \( H_v = v \), i.e. \( \Delta_x v = \partial_t v + \text{div}_x F(v) + \nabla_x p \), then \( v \) is a minimizer to \( (4.4) \).

**Proof.** First of all we want to note that the set \( A_{v_0} := \{ u \in \mathcal{R} : u(\cdot, 0) = v_0(\cdot) \} \) is not empty. In particular the function \( u_0(\cdot, t) := v_0(\cdot) \) belongs to \( A_{v_0} \). Let

\[ K := \inf_{u \in A_{v_0}} I_F(u). \]

Then \( K \geq 0 \). Consider the minimizing sequence \( \{ u_n \} \subset A_{v_0} \), i.e. the sequence such that \( \lim_{n \to \infty} I_F(u_n) = K \). Then, by the definition of \( I_F \) in \( (4.3) \), we obtain that there exists \( C > 0 \), independent of \( n \), such that

\[ \int_0^T \int_{\Omega} (|\nabla_x u_n|^2 + |\nabla_x H_{u_n}|^2) \, dx \, dt \leq C. \tag{4.21} \]

Then up to a subsequence,

\[ u_n \rightharpoonup u_0 \text{ weakly in } L^2(0, T; V_N) \quad \text{and} \quad H_{u_n} \rightharpoonup \bar{H} \text{ weakly in } L^2(0, T; V_N). \tag{4.22} \]

From the other hand, by Corollary \( 4.2 \), for every \( t \in [0, T] \) we have

\[ \int_{\Omega} u^2_n(x, t) dx = \int_{\Omega} u^2_n(x, 0) dx - 2 \int_0^t \int_{\Omega} \nabla_x u_n : \nabla_x H_{u_n}. \]

Therefore, since, \( u_n \) and \( H_{u_n} \) are bounded in \( L^2(0, T; V_N) \) by \( (4.21) \) and \( u_n(\cdot, 0) = v_0(\cdot) \) we obtain that there exists \( C > 0 \) independent of \( n \) and \( t \) such that

\[ \| u_n(\cdot, t) \|_{L^N} \leq C \quad \forall n \in \mathbb{N}, t \in [0, T]. \tag{4.23} \]

In particular, up to a further subsequence \( F(u_n) \rightharpoonup \bar{F} \) weakly in \( L^2(\Omega \times (0, T), \mathbb{R}^{N\times N}) \).

Then by \( (4.22) \) and \( (2.2) \), for every \( t \in [0, T] \) and for every \( \phi \in \mathcal{V}_N \), we have

\[ \lim_{n \to \infty} \int_{\Omega} u_n(x, t) \cdot \phi(x) dx = \]

\[ = \int_{\Omega} v_0(x) \cdot \phi(x) dx - \int_0^t \int_{\Omega} \nabla_x \bar{H} : \nabla_x \phi + \int_0^t \int_{\Omega} \bar{F} : \nabla_x \phi. \]
Since $\mathcal{V}_N$ is dense in $L_N$, by (4.23), and (4.24), for every $t \in [0,T]$ there exists $u(\cdot, t) \in L_N$ such that

$$\tag{4.25} u_n(\cdot, t) \rightharpoonup u(\cdot, t) \text{ weakly in } L_N \quad \forall t \in [0,T].$$

Moreover, $\|u(\cdot, t)\|_{L_N} \leq C$. But we have $u_n \rightharpoonup u_0$ weakly in $L^2(0,T; L_N)$, therefore $u = u_0$ a.e. and so $u \in L^2(0,T; V_N) \cap L^\infty(0,T; L_N)$. Then using (4.23), (4.22), (4.25) and Lemma 2.2 we deduce that

$$\tag{4.26} u_n \rightharpoonup u \text{ strongly in } L^2(0,T; L_N).$$

Moreover, by (4.24) we obtain that $u(\cdot, t)$ is $L_N$-weakly continuous in $t$ on $[0,T]$. Therefore, by (4.25) and (4.22),

$$\tag{4.27} \int_0^T \int_\Omega |\nabla_x u|^2 \, dxdt + \int_\Omega |u(x,T)|^2 \, dx \leq \lim_{n \to \infty} \left( \int_0^T \int_\Omega |\nabla_x u_n|^2 \, dxdt + \int_\Omega |u_n(x,T)|^2 \, dx \right).$$

Next for every $\psi(x,t) \in C^\infty_c(\Omega \times (0,T), \mathbb{R}^N)$ such that $\text{div}_x \psi = 0$ we obtain

$$\tag{4.28} \lim_{n \to \infty} \int_0^T \int_\Omega \left( u_n \cdot \partial_t \psi + F(u_n) : \nabla_x \psi \right) \, dxdt = \lim_{n \to \infty} \int_0^T \int_\Omega \nabla_x H_{u_n} : \nabla_x \psi \, dxdt = \int_0^T \int_\Omega \nabla_x \bar{H} : \nabla_x \psi \, dxdt.$$

But since $F$ is a Lipschitz function, by (4.26) we obtain

$$\lim_{n \to \infty} \int_0^T \int_\Omega \left( u_n \cdot \partial_t \psi + F(u_n) : \nabla_x \psi \right) \, dxdt = \int_0^T \int_\Omega \left( u \cdot \partial_t \psi + F(u) : \nabla_x \psi \right) \, dxdt$$

So, by (4.28), for every $\psi(x,t) \in C^\infty_c(\Omega \times (0,T), \mathbb{R}^N)$ such that $\text{div}_x \psi = 0$ we deduce

$$\tag{4.29} \int_0^T \int_\Omega \left( u \cdot \partial_t \psi + F(u) : \nabla_x \psi \right) \, dxdt = \int_0^T \int_\Omega \nabla_x \bar{H} : \nabla_x \psi \, dxdt.$$

In particular $\partial_t u + \text{div}_x F(u) \in L^2(0,T; V_N^{-1})$. Therefore $\partial_t u \in L^2(0,T; V_N^{-1})$ and then $u \in A_{v_0} = \{ u \in \mathcal{R} : u(\cdot, 0) = v_0(\cdot) \}$. Moreover, by (4.29), we obtain that $H_u = \bar{H}$. So, as before,

$$\tag{4.30} \int_0^T \int_\Omega |\nabla_x H_{u_n}|^2 \, dxdt \leq \lim_{n \to \infty} \int_0^T \int_\Omega |\nabla_x H_{u_n}|^2 \, dxdt.$$

Combining (4.30) with (4.27), we infer

$$I_F(u) \leq \lim_{n \to \infty} I_F(u_n) = K.$$

Therefore, $u$ is a minimizer to (4.4). By Lemma 4.3 it satisfies $H_u = u$, i.e.

$$\Delta_x u = \partial_t u + \text{div}_x F(u) + \nabla_x p.$$
Moreover, by Lemma 4.1, for every $t \in [0, T]$ we have
\[
\int_0^t \int_\Omega \nabla_x u : \nabla_x H_u = \frac{1}{2} \left( \int_\Omega v_0^2(x) dx - \int_\Omega u^2(x, t) dx \right).
\]
Therefore we obtain (4.20). Moreover, $I_F(u) = \frac{1}{2} \int_\Omega v_0^2(x) dx$. Finally if $v \in \mathcal{R}$ satisfy $v(\cdot, 0) = v_0(\cdot)$ and $H_v = v$ then by (4.5) we have $I_F(v) = \frac{1}{2} \int_\Omega v_0^2(x) dx = I_F(u)$. So $v$ is a minimizer to (4.4). □

Remark 4.2. For a fixed $r(x, t) \in L^2(0, T; V_N)$ we can define a functional $\tilde{I}_{\{F, r\}}(u) : \mathcal{R} \to \mathbb{R}$ by
\[
\tilde{I}_{\{F, r\}}(u) := \frac{1}{2} \left( \int_0^T \int_\Omega \left( |\nabla_x u + \nabla_x r|^2 + |\nabla_x H_u - \nabla_x r|^2 \right) dx dt + \int_\Omega |u(x, T)|^2 dx \right),
\]
and for every $v_0 \in V_N$ we can consider the minimization problem
\[
\inf \{ \tilde{I}_{\{F, r\}}(u) : u \in \mathcal{R}, u(\cdot, 0) = v_0(\cdot) \}.
\]
Then similarly to the proof of Theorem 4.1 we can prove that there exists a minimizer $u$ to (4.32) and it satisfies $H_u = u + r$, i.e.
\[
\Delta_x u + \Delta_x r = \partial_t u + \text{div}_x F(u) + \nabla_x p.
\]
Then, using this fact, as in the proof of Theorem 3.1 we can deduce the existence of a weak solution to (1.1) with $f \in L^2(0, T; V_N^{-1})$.

Remark 4.3. Similar method as in the proof of Theorem 3.1 we can apply to the unbounded domain $\Omega$. In this case we consider a sequence of smooth bounded domains $\{\Omega_n\}$, such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$, and a sequence $v_0^{(n)} \to v_0$ in $L_N$, such that supp $v_0^{(n)} \subset \Omega_n$. Consider $u_n(x, t) \in \mathcal{R}(\Omega_n)$, such that $u_n(\cdot, 0) = v_0^{(n)}(\cdot)$ and for every $\psi(x, t) \in C_c^\infty(\Omega_n \times (0, T), \mathbb{R}^N)$, satisfying $\text{div}_x \psi = 0$, we have (3.5), where $F_n$ is defined by (3.4). Then we can deduce that there exists $u \in L^2(0, T; V_N) \cap L^\infty(0, T; L_N)$ such that, up to a subsequence, $u_n \to u$ strongly in $L^2_{\text{loc}}(\Omega \times (0, T), \mathbb{R}^N)$. Then $u$ will satisfy conditions of Theorem 3.1.

5. Variational principle for more regular solutions of the Navier-Stokes Equations

Let $\Omega \subset \mathbb{R}^N$ be a domain with Lipschitz boundary (not necessarily bounded). We denote by $H_N$ the closure of $V_N$ in $H^1_0(\Omega, \mathbb{R}^N)$ (the spaces $H_N$ and $V_N$ differ only in the case of unbounded domain). For every $u \in L^4(\Omega \times (0, T), \mathbb{R}^N)$ we have
\((u \otimes u) \in L^2(0, T; L^2(\Omega, \mathbb{R}^{N \times N}))\) and therefore \(\text{div}_x(u \otimes u) \in L^2(0, T; V_N^{-1})\). If in addition \(\partial_t u \in L^2(0, T; V_N^{-1})\) then we obtain \(\partial_t u + \text{div}_x(u \otimes u) \in L^2(0, T; V_N^{-1})\).

**Definition 5.1.** Let \(u \in L^2(0, T; H_N) \cap L^\infty(0, T; L_N)\) be such that \(\partial_t u \in L^2(0, T; V_N^{-1})\) and such that \(u(\cdot, t)\) is \(L_N\)-weakly continuous in \(t\) on \([0, T]\). Denote the set of all such functions \(u\) by \(\mathcal{R}'\). Denote the set \(\mathcal{R}' \cap L^4(\Omega \times (0, T), \mathbb{R}^N)\) by \(\mathcal{P}\). For every \(u \in \mathcal{P}\) let \(\tilde{H}_u(\cdot, t) \in L^2(0, T; V_N)\) be as in Remark 2.2 corresponding to \(\partial_t u + \text{div}_x(u \otimes u)\).

That is for every \(\psi(x, t) \in C_c^\infty(\Omega \times (0, T), \mathbb{R}^N)\) such that \(\text{div}_x \psi = 0\) we have
\[
\int_0^T \int_\Omega \left( \partial_t \psi + (u \otimes u) : \nabla_x \psi \right) dx dt = \int_0^T \int_\Omega \nabla_x \tilde{H}_u : \nabla_x \psi \, dx dt .
\]

For a fixed \(r(x, t) \in L^2(0, T; V_N)\) define a functional \(J_{\{\varphi, r\}}(u) : \mathcal{P} \to \mathbb{R}\) by
\[
J_{\{\varphi, r\}}(u) := \frac{1}{2} \left( \int_0^T \int_\Omega \left( |\nabla_x u + \nabla_x r|^2 + |\nabla_x \tilde{H}_u - \nabla_x r|^2 \right) dx dt + \int_\Omega |u(x, T)|^2 dx \right) .
\]

**Theorem 5.1.** Let \(v_0 \in L_N\) and \(r(x, t) \in L^2(0, T; V_N)\). Assume that there exists \(u \in \mathcal{P}\) which satisfies \(u(x, 0) = v_0(x)\), and
\[
\int_0^T \int_\Omega \left( \partial_t \psi + (u \otimes u) : \nabla_x \psi \right) dx dt = \int_0^T \int_\Omega \left( \nabla_x u + \nabla_x r \right) : \nabla_x \psi \, dx dt
\]
for every \(\psi(x, t) \in C_c^\infty(\Omega \times (0, T), \mathbb{R}^N)\), such that \(\text{div}_x \psi = 0\), i.e.
\[
\Delta_x u = \partial_t u + \text{div}_x(u \otimes u) + \nabla_x p - \Delta_x r .
\]

Then \(u\) is a minimizer of the following problem
\[
\inf \{ J_{\{\varphi, r\}}(u) : u \in \mathcal{P}, u(\cdot, 0) = v_0(\cdot) \} .
\]

Moreover if \(\bar{u}\) is a minimizer to \((5.3)\), then \(\bar{u}\) is a solution to \((5.2)\).

**Proof.** In the same way as in the proof of Theorem 4.1 in [2] we obtain that for every \(\bar{u} \in \mathcal{P}\) we must have
\[
\int_0^T \int_\Omega \nabla_x \bar{u} : \nabla_x \tilde{H}_{\bar{u}} \, dx dt = \frac{1}{2} \left( \int_\Omega \bar{u}^2(x, 0) \, dx - \int_\Omega \bar{u}^2(x, T) \, dx \right) .
\]

Therefore,
\[
J_{\{\varphi, r\}}(\bar{u}) = \frac{1}{2} \int_0^T \int_\Omega \left( |\nabla_x \bar{u} + \nabla_x r - \nabla_x \tilde{H}_{\bar{u}}|^2 + |\nabla_x r|^2 \right) dt + \frac{1}{2} \int_\Omega \bar{u}^2(x, 0) \, dx .
\]

Therefore, \(u \in \mathcal{P}\) which satisfy \(u(x, 0) = v_0(x)\) and \(\nabla_x u + \nabla_x r = \nabla_x \tilde{H}_u\) will be the minimizer to \((5.3)\). Then also every minimizer \(\bar{u}\) will satisfy \(\nabla_x \bar{u} + \nabla_x r = \nabla_x \tilde{H}_{\bar{u}}\), i.e. will satisfy \((5.2)\). \(\square\)
Appendix A.

Proof of Lemma 2.2. Since for every \( n = 0, 1, \ldots \) and every \( t \in (0, T) \) the functional
\[
l_{n,t}(\phi) := \int_{\Omega} u_n(x, t) \cdot \phi(x) \, dx
\]
is bounded in \( H^1_0(\Omega, \mathbb{R}^N) \), by Riesz Representation Theorem for every \( n = 0, 1, \ldots \) and every \( t \in [0, T] \) there exists \( w_n(\cdot, t) \in H^1_0(\Omega, \mathbb{R}^N) \) such that
\[
(A.1) \quad l_{n,t}(\phi) = \int_{\Omega} u_n(x, t) \cdot \phi(x) \, dx = \int_{\Omega} \nabla_x w_n(x, t) : \nabla_x \phi(x) \, dx \quad \forall \phi \in H^1_0(\Omega, \mathbb{R}^N),
\]
Equation (A.1) gives in particular
\[
(A.2) \quad \int_{\Omega} u_n(x, t) \cdot w_n(x, t) \, dx = \int_{\Omega} |\nabla_x w_n(x, t)|^2 \, dx.
\]
Then we obtain that there exist \( C_0, C > 0 \), independent of \( n \) and \( t \), such that
\[
(A.3) \quad \int_{\Omega} |\nabla_x w_n(x, t)|^2 \, dx \leq C_0 \int_{\Omega} |u_n(x, t)|^2 \, dx \leq C.
\]
Moreover, using (2.4), (A.2) and the compact embedding of \( H^1_0(\Omega, \mathbb{R}^N) \) into \( L^2(\Omega, \mathbb{R}^N) \), we obtain
\[
(A.4) \quad w_n(\cdot, t) \to w_0(\cdot, t) \quad \text{strongly in } H^1_0(\Omega, \mathbb{R}^N) \quad \forall t \in (0, T).
\]
We have \( w_n(\cdot, \cdot) \in L^2(0, T; H^1_0(\Omega, \mathbb{R}^N)) \) and moreover, by (A.4) and (A.3), we obtain
\[
(A.5) \quad w_n(\cdot, \cdot) \to w_0(\cdot, \cdot) \quad \text{strongly in } L^2(0, T; H^1_0(\Omega, \mathbb{R}^N)).
\]
But, by (A.1), we have,
\[
(A.6) \quad \int_0^T \int_{\Omega} |u_n(x, t)|^2 \, dx \, dt = \int_0^T \int_{\Omega} |\nabla_x w_n(x, t)|^2 \, dx \, dt,
\]
and since \( u_n \rightharpoonup u_0 \) in \( L^2(0, T; V_N) \), by (A.5), we obtain
\[
(A.7) \quad \lim_{n \to \infty} \int_0^T \int_{\Omega} |u_n(x, t)|^2 \, dx \, dt = \lim_{n \to \infty} \int_0^T \int_{\Omega} |\nabla_x w_n(x, t)|^2 \, dx \, dt
\]
\[
= \int_0^T \int_{\Omega} |\nabla_x w_0(x, t)|^2 \, dx \, dt = \int_0^T \int_{\Omega} |u_0(x, t)|^2 \, dx \, dt.
\]
But by (2.3) we have
\[
(A.8) \quad u_n \rightharpoonup u_0 \quad \text{weakly in } L^2(0, T; L_N).
\]
Therefore, by (A.8) and (A.7) we obtain (2.5). □
Proof of Corollary 4.1. Let \( \eta \in C_c^\infty(\mathbb{R}, \mathbb{R}) \) be a mollifying kernel, satisfying \( \eta \geq 0 \), \( \int_\mathbb{R} \eta(t) dt = 1 \), supp \( \eta \subset [-1, 1] \) and \( \eta(-t) = \eta(t) \) \( \forall t \). Given small \( \varepsilon > 0 \) and \( \psi(x, t) \in C_c^\infty(\Omega \times (2\varepsilon, T - 2\varepsilon), \mathbb{R}^N) \) such that \( \text{div}_x \psi = 0 \), define

\[
(A.9) \quad \psi_x(x, t) := \frac{1}{\varepsilon} \int_0^T \eta\left(\frac{s - t}{\varepsilon}\right) \psi(x, s) ds .
\]

Then \( \psi_x(x, t) \in C_c^\infty(\Omega \times (0, T), \mathbb{R}^N) \) and satisfies \( \text{div}_x \psi_x = 0 \). Therefore we obtain

\[
(A.10) \quad \int_0^T \int_\Omega u \cdot \partial_t \psi_x dxdt = \int_0^T \int_\Omega \nabla_x V_u : \nabla_x \psi_x dxdt ,
\]

where \( V_u(\cdot, t) \in L^2(0, T; V_N) \) is as in Remark 2.2 corresponding to \( \partial_t u \). But

\[
\int_0^T \int_\Omega u \cdot \partial_t \psi_x dxdt = \int_0^T \int_\Omega \frac{1}{\varepsilon} \int_0^T \eta\left(\frac{s - t}{\varepsilon}\right) \partial_s \psi(x, s) ds \ dxdt = \int_0^T \int_\Omega \partial_t \psi(x, t) \cdot \frac{1}{\varepsilon} \int_0^T \eta\left(\frac{s - t}{\varepsilon}\right) u(x, s) ds \ dxdt = \int_0^T \int_\Omega \partial_t \psi(x, t) \cdot u_\varepsilon(x, t) dxdt ,
\]

where \( u_\varepsilon(x, t) = \frac{1}{\varepsilon} \int_0^T \eta((s - t)/\varepsilon) u(x, s) ds \). By the other hand

\[
\int_0^T \int_\Omega \nabla_x V_u : \nabla_x \psi_x dxdt = \int_0^T \int_\Omega \nabla_x V_u(x, t) : \left(\frac{1}{\varepsilon} \int_0^T \eta\left(\frac{s - t}{\varepsilon}\right) \nabla_x \psi(x, s) ds \right) dxdt = \int_0^T \int_\Omega \nabla_x \psi(x, t) : \nabla_x \left(\frac{1}{\varepsilon} \int_0^T \eta\left(\frac{s - t}{\varepsilon}\right) V_u(x, s) ds \right) dxdt = \int_0^T \int_\Omega \nabla_x \psi(x, t) : \nabla_x (V_u)_\varepsilon(x, t) dxdt ,
\]

where \( (V_u)_\varepsilon(x, t) = \frac{1}{\varepsilon} \int_0^T \eta((s - t)/\varepsilon) V_u(x, s) ds \). Therefore, by (A.10), we infer

\[
(A.11) \quad \int_0^T \int_\Omega u_\varepsilon \cdot \partial_t \psi dxdt = \int_0^T \int_\Omega \nabla_x (V_u)_\varepsilon : \nabla_x \psi dxdt .
\]

So \( \partial_t u_\varepsilon \in L^2(2\varepsilon, T - 2\varepsilon; V_N^{-1}) \). Moreover \( u_\varepsilon \in L^2(0, T; V_N) \cap L^\infty(0, T; L_N) \). We have \( u_\varepsilon \rightarrow u \) and \( (V_u)_\varepsilon \rightarrow V_u \) strongly in \( L^2(0, T; V_N) \) as \( \varepsilon \rightarrow 0 \). Moreover, up to a subsequence \( \varepsilon_n \rightarrow 0 \), we have \( u_{\varepsilon_n}(\cdot, t) \rightarrow u(\cdot, t) \) strongly in \( L_N \) a.e. in \( [0, T] \). In addition, by Lemma 4.1 for every \( a, b \in [2\varepsilon, T - 2\varepsilon] \) we have

\[
(A.12) \quad \int_a^b \int_\Omega \nabla_x u_\varepsilon : \nabla_x (V_u)_\varepsilon dxdt = \frac{1}{2} \left( \int_\Omega u_\varepsilon^2(x, a) dx - \int_\Omega u_\varepsilon^2(x, b) dx \right) .
\]

Then letting \( \varepsilon \rightarrow 0 \) in (A.12), we obtain that for almost every \( a \) and \( b \) in \( (0, T) \) we have

\[
\int_a^b \int_\Omega \nabla_x u : \nabla_x V_u dxdt = \frac{1}{2} \left( \int_\Omega u^2(x, a) dx - \int_\Omega u^2(x, b) dx \right) .
\]

So \( u \in L^\infty(0, T; L_N) \). \( \square \)
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