LINEAR FAMILY OF LIE BRACKETS ON THE SPACE OF MATRICES \textit{Mat}(n \times m, \mathbb{K}) AND ADO’S THEOREM

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Abstract. In this paper we classify a linear family of Lie brackets on the space of rectangular matrices \textit{Mat}(n \times m, \mathbb{K}) and we give an analogue of the Ado’s Theorem. We give also a similar classification on the algebra of the square matrices \textit{Mat}(n, \mathbb{K}) and as a consequence, we prove that we can’t built a faithful representation of the \((2n + 1)\)-dimensional Heisenberg Lie algebra \(\mathfrak{o}_n\) in a vector space \(V\) with \(\text{dim } V \leq n+1\). Finally, we prove that in the case of the algebra of square matrices \textit{Mat}(n, \mathbb{K}), the corresponding Lie algebras structures are a contraction of the canonical Lie algebra \(\mathfrak{gl}(n, \mathbb{K})\).

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0. Introduction

We begin by setting some notations which will be used throughout the paper. Let \(\mathbb{K}\) be a field with characteristic \(p = 0\), \textit{Mat}(n \times m, \mathbb{K}) be the linear space of \(n \times m\) rectangular matrices with coefficients in \(\mathbb{K}\) and \textit{Mat}(n, \mathbb{K}) is the associative algebra of square matrices with coefficients in \(\mathbb{K}\).

When dealing with the problem of representation of finite dimensional Lie algebras two presentations leap into mind: a presentation by matrices and a presentation by an array of structure constants. In the first presentation the Lie algebra is given by a finite set of matrices \(\{A_1, \ldots, A_n\}\) that form a basis of the Lie algebra \(\mathfrak{g}\). If \(A\) and \(B\) are two elements of the space spanned by the \(A_i\), then their Lie product is defined as \([A, B] = A.B - B.A\) (where the \(\cdot\) denotes the ordinary matrix multiplication). The second approach is more abstract. The Lie algebra is a (abstract) vector space over a field \(\mathbb{K}\) with basis \(\{x_1, \ldots, x_n\}\) and the Lie multiplication is determined by

\[
[x_i, x_j] = \sum_{k=1}^{n} c_{i,j}^k x_k.
\]

Here \((c_{i,j}^k) \in \mathbb{K}^3\) is an array of \(n^3\) structure constants that determines the Lie multiplication completely. By linear algebra it is seen that from a presentation of a Lie algebra by matrices, we easily can obtain a presentation of it by structure constants. Ado’s theorem states that the opposite direction is also possible.
Theorem 0.1. (Ado) Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero can be viewed as a Lie algebra of square matrices under the commutator bracket. More precisely $\mathfrak{g}$ has a linear representation $\rho$ over $\mathbb{K}$, on a finite dimensional vector space $V$, that is a faithful representation, making $\mathfrak{g}$ isomorphic to a subalgebra of the endomorphisms of $V$.

If a such representation of $\mathfrak{g}$ is built on a vector space $V$, we can request if we can built such representation in a lower dimensional vector space $V$.

In the other hand, let $\mathcal{A}$ be a finite-dimensional associative algebra over a field $\mathbb{K}$ of a characteristic $p = 0$, then $\mathcal{A}$ is canonically equipped with Lie algebra structure given by the following bracket

$$[u, v] = u.v - v.u, \quad u, v \in \mathcal{A}.$$  

For instance, if $\mathcal{A} = \text{End}(V)$ the set of linear transformations on $V$ where $V$ is a finite dimensional vector space over $\mathbb{K}$. In equivalent way, if we consider $\text{Mat}(n, \mathbb{K})$ the set of square matrices with coefficients in $\mathbb{K}$, then we get a canonical Lie algebra structure with the above commutator bracket, this algebra is denoted by $\mathfrak{gl}(n, \mathbb{K})$.

However the structure of associative algebra over $\mathcal{A}$ is not unique [4], indeed if we fix $w \in \mathcal{A}$ then we can define the product $(u \circ v)_w = u.w.v$ and with respect $\mathcal{A}$ is again an associative algebra. This induces a new Lie algebra structure, defined by the bracket

$$[u, v]_w = u.w.v - v.w.u, \quad u, v \in \mathcal{A} \quad (1).$$

Thus we obtain a family of Lie brackets, labelled by the element $w$. It is readily seen that we actually have a linear space of Lie brackets, since the sum of two such brackets is also a Lie bracket of the same type, and a natural question is to classify these structures. For instance if $w = 0$ then the Lie algebra $(\mathcal{A}, [\ , \ ]_0)$ is just the abelian Lie algebra $\mathbb{K}^{n^2}$. The above construction can be applied to the algebra of square matrices $\text{Mat}(n, \mathbb{K})$. It can be applied also even if $u, v, w$ are not $n \times n$ matrices, since (1) makes sense when $u, v \in \text{Mat}(n \times m, \mathbb{K})$-the linear space of $n \times m$ matrices and $w \in \text{Mat}(m \times n, \mathbb{K})$- the linear space of $m \times n$ matrices.

In this paper we will deal with the algebra of the endomorphisms of a finite-dimensional vector space $V$ or in equivalent way, the algebra of square matrices $\text{Mat}(n, \mathbb{K})$ and we will deal also with $\text{Mat}(n \times m, \mathbb{K})$-the linear space of $n \times m$ matrices.

Our first goal in this paper is to give a complete classification of these Lie algebra structures, on $\text{Mat}(n, \mathbb{K})$ (respectively on $\text{Mat}(n \times m, \mathbb{K})$), labelled by the Lie brackets $[\ , \ ]_J$ with $J \in \text{Mat}(n, \mathbb{K})$ (respectively in $\text{Mat}(m \times n, \mathbb{K})$).

The paper is organized as follows. First we begin by recalling some classical results of linear algebra and matrix properties. Secondly, we classify the Lie algebra structures on $\text{Mat}(n, \mathbb{K})$ and $\text{Mat}(n \times m, \mathbb{K})$. As a consequence of these classifications, we prove an analogue of the Ado’s Theorem, namely, we prove that any finite dimensional Lie algebra $\mathfrak{g}$ can be viewed as a Lie sub-algebra $(\text{Hom}(\mathbb{K}^m, \mathbb{K}^n), [\ , \ ]_J)$ for some
positive integers $n, m$ and some matrix $J \in \text{Mat}(m \times n, \mathbb{K})$. We prove also that there is no faithful linear representation of the classical $(2n + 1)$-Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ in $\mathfrak{gl}(V)$ with $\dim V \leq n + 1$. Finally we prove that in the case of the algebra of square matrices, the Lie algebra structure labelled by the family of brackets $([\cdot, \cdot], J)$ constructed on $\text{Mat}(n, \mathbb{K})$ are a contraction of the canonical structure $\mathfrak{gl}(n, \mathbb{K})$.

1. Preliminaries

Let $n, m$ be integers in $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and let $\text{Mat}(n \times m, \mathbb{K})$ be the linear vector space of $n \times m$ rectangular matrices. We denote its canonical basis $(E_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ with $E_{i,j} = (\delta_{p,i} \delta_{q,j})_{1 \leq p \leq n, 1 \leq q \leq m}$, where $\delta_{p,i}$ is the Kronecker symbol.

**Definition 1.1.** Let $J$ and $J'$ be in $\text{Mat}(m \times n, \mathbb{K})$. We say that $J$ is equivalent to $J'$ if and only if there exist two invertible matrices $P \in \text{GL}(n, \mathbb{K})$ and $Q \in \text{GL}(m, \mathbb{K})$ such that

$$J = QJ'P.$$  

**Remark 1.2.**

1. The matrices $Q$ and $P$ are not unique.
2. The above property of equivalent matrices in $\text{Mat}(m \times n, \mathbb{K})$ is an equivalence relation.
3. $J$ and $J'$ are equivalent if and only if they represent the same endomorphism in different bases.

The characterization of two equivalent matrices is given by the following

**Lemma 1.3.** Let $J, J' \in \text{Mat}(m \times n, \mathbb{K})$. $J$ and $J'$ are equivalent if and only if they have the same rank.

From the above Lemma, we can conclude the following

**Lemma 1.4.** Let $J$ a matrix in $\text{Mat}(m \times n, \mathbb{K})$ of rank $r \leq \min(m, n)$, then there exist two invertible matrices $Q \in \text{GL}(m, \mathbb{K})$ and $P \in \text{GL}(n, \mathbb{K})$ such that

$$J = Q \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} P,$$

where $0_{p,q}$ is the zero matrix of $\text{Mat}(p \times q, \mathbb{K})$ and $I_r$ is the identity matrix of $\text{Mat}(r, \mathbb{K})$.

We will denote the matrix $\begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}$ by $J_{m,n,r}$ and when $n = m$ we simply denote it by $J_{n,r}$. Note that $J_{n,n} = I_n$ the identity matrix.

Consider now the linear space $\text{Mat}(n \times m, \mathbb{K})$ and for $J \in \text{Mat}(m \times n, \mathbb{K})$ put

$$[A, B]_J = AJB - BJA, \quad A, B \in \text{Mat}(n \times m, \mathbb{K}),$$

then we have
Lemma 1.5. (i) $[A, B]_{J + , J'} = [A, B]_J + \alpha [A, B]_J$, $\forall \alpha \in \mathbb{K}$, $J, J' \in \text{Mat}(n \times m, \mathbb{K})$.

(ii) $(\text{Mat}(n \times m, \mathbb{K}), [\ , \])$ is a Lie algebra.

The proof is a simple verification of the anti-symmetry of the bracket $[\ , \]$ and the Jacobi identity.

Now we give the following

Proposition 1.6. If $J$ and $J'$ are equivalent matrices in $\text{Mat}(m \times n, \mathbb{K})$, then the corresponding Lie algebra structures on $\text{Mat}(n \times m, \mathbb{K})$ are isomorphic.

Proof. Since $J$ and $J'$ are equivalent in $\text{Mat}(m \times n, \mathbb{K})$, then there exist two invertible matrices $Q \in GL(m, \mathbb{K})$ and $P \in GL(n, \mathbb{K})$ such that $J = QJ'P$. Consider now the map

$$\varphi : \text{Mat}(\times m, \mathbb{K}) \longrightarrow \text{Mat}(n \times m, \mathbb{K})$$

$$A \longmapsto PAQ$$

we verify, since $Q$ and $P$ are invertible, that $\varphi$ is an isomorphism of vector space and

$$\varphi ([A, B]_J) = P(AJB - BJA)Q$$

$$= P(AQJ'PB - BQJ'PA)Q$$

$$= [PAQ, PBQ]_{J'}$$

and thus $\varphi$ is a Lie algebra isomorphism. \qed

Let $J \in \text{Mat}(m \times n, \mathbb{K})$ with $rk(J) = r$, from Proposition 1.6 $J$ is equivalent to the following matrix

$$\begin{pmatrix}
I_r & 0_{r,n-r} \\
0_{m-r,r} & 0_{m-r,n-r}
\end{pmatrix}$$

we shall denote the last matrix by $J_{m,n,r}$. In the case when $m = n$, we simply denote it by $J_{n,r}$. Recall that $J_{n,n} = I_n$, the identity matrix of $\text{Mat}(n, \mathbb{K})$.

For ease of notations, we shall denote the Lie algebra $(\text{Mat}(n \times m, \mathbb{K}), [\ , \])$ by $\mathfrak{gl}(n, m, r, \mathbb{K})$ and in the case when $m = n$, we simply denote the Lie algebra $(\text{Mat}(n, \mathbb{K}), [\ , \])$ by $\mathfrak{gl}(n, r, \mathbb{K})$. In order to give a complete classification of these Lie algebras structures labelled by the linear family of Lie brackets $([\ , \])$, we first give the following

Lemma 1.7. Let $M \in \text{Mat}(n \times m, \mathbb{K})$ and put $M = \begin{pmatrix}
M_1 & M_3 \\
M_2 & M_4
\end{pmatrix}$ with $M_1 \in \text{Mat}(r, \mathbb{K})$, $M_2 \in \text{Mat}((n - r) \times r, \mathbb{K})$, $M_3 \in \text{Mat}(r \times (m - r), \mathbb{K})$, $M_4 \in \text{Mat}((n - r) \times (m - r), \mathbb{K})$. Then the Lie bracket $[A, B]_{J_{m,n,r}}$ is given by

$$[A, B]_{J_{m,n,r}} = \begin{pmatrix}
[A_1, B_1] & A_1B_3 - B_1A_3 \\
A_2B_1 - B_2A_1 & A_2B_3 - B_2A_3
\end{pmatrix}.$$
Then we have
\[ AJ_{m,n,r} = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} \] and \[ J_{m,n,r}A = \begin{pmatrix} A_1 & A_3 \\ 0 & 0 \end{pmatrix} \]. □

In order to give a classification of the Lie algebras structures on \( \text{Mat}(n \times m, \mathbb{K}) \) labelled by the family of Lie brackets \( ([ , ]_J)_J \), we shall distinguish the two possible cases \( n = m \) and \( n \neq m \).

2. The case of \( \text{Mat}(n, \mathbb{K}) \)

In this section we will deal with the linear vector space of square matrices \( \text{Mat}(n, \mathbb{K}) \). It is well known that it has a structure of associative algebra and thus a Lie algebra with the commutator
\[ [A, B] = AB - BA, \quad A, B \in \text{Mat}(n, \mathbb{K}). \]
The commutator of two matrices \( E_{i,j} \) and \( E_{k,\ell} \) of the canonical basis (\( 1 \leq i, j, k, \ell \leq n \)) is given by
\[ [E_{i,j}, E_{k,\ell}] = \delta_{j,k}E_{i,\ell} - \delta_{\ell,j}E_{k,j}, \]
where \( \delta \) denotes the Kronecker symbol.

Fix \( J \in \text{Mat}(n, \mathbb{K}) \) and consider the bracket
\[ [A, B]_J = AJB - BJ A, \quad A, B \in \text{Mat}(n, \mathbb{K}). \]

We first give the following

Lemma 2.1. Let \( \mathfrak{h} \) be a Lie sub-algebra of \( \mathfrak{gl}(n, \mathbb{K}) \), then \( \mathfrak{h} \) is a Lie sub-algebra of \( \mathfrak{gl}(p, n, \mathbb{K}) \) for any integer \( p \geq n \).

Proof. For any subset \( \mathfrak{h} \subset \text{Mat}(n, \mathbb{K}) \), and \( p \geq n \), we have
\[ [\mathfrak{h}, \mathfrak{h}]_{I_p, n} = [\mathfrak{h}, \mathfrak{h}]. \]
This completes the proof. □

In order to classify the Lie algebras structures on \( \text{Mat}(n, \mathbb{K}) \) labelled by the family of brackets \( [ , ]_J \), we consider a vector space \( V \) over \( \mathbb{K} \) with \( \dim V = n \) and put \( V = V_1 \oplus V_2 \). Next, define \( \mathfrak{n} \) to be the two step nilpotent Lie algebra constructed on the vector space \( \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1) \oplus \text{Hom}(V_2, V_2) \) equipped with the bracket
\[ [(A, B, C), (A', B', C')] = (0, 0, AB' - A'B). \]
Finally, we define the Lie algebra \( S(V_1, V_2) \) to be the semi direct product of \( \text{End}(V_1) \) with \( \mathfrak{n} \). Then we have the following

Proposition 2.2. (i) The Lie algebra \( S(V_1, V_2) \) is isomorphic to \( \mathfrak{gl}(n, r, \mathbb{K}) \).
(ii) \( \text{End}(V_1) \) is a Lie subalgebra of \( \mathfrak{gl}(n, s, \mathbb{K}) \) for any \( s \geq r \).
(iii) The reductive part of \( \mathfrak{gl}(n, r, \mathbb{K}) \) is \( \text{End}(V_1) \oplus \text{End}(V_2) \).
Proof. (i) First, we can simply verify that \( n \) is a two-step nilpotent Lie algebra. Now for each \( X \in \text{End}(V_1) \), consider
\[
\pi_X : n \longrightarrow n \\
(A, B, C) \longmapsto (-AX, XB, 0).
\]
Then we easily verify that \( X \longmapsto \pi_X \) is a Lie algebra homomorphism from \( \mathfrak{gl}(n, \mathbb{K}) \) into \( \partial(n) \) the Lie algebra of the derivations of \( n \).

Now consider the mapping \( \phi : S(V_1, V_2) \longrightarrow \mathfrak{gl}(n, r, \mathbb{K}) \)
\[
(X, A, B, C) \longmapsto (XB, A).
\]
\( \phi \) is a vector space isomorphism.

Let \( X, X' \in \text{End}(V_1) \) and \( (A, B, C), (A', B', C') \in \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1) \oplus \text{Hom}(V_2, V_2) \). Put \( n = (A, B, C), n' = (A', B', C') \) then
\[
\phi([([X, n], (X', n'))] = \phi(([X, X'], [n, n'] + \pi_{X'}(n) - \pi_X(n')))
= [\phi((X, n)), \phi((X', n'))]_{J_n,r}.
\]

(ii) The statement is a simple consequence of Lemma 1.7.

(iii) It is easy to verify that \( \text{End}(V_1) \) is reductive in \( \mathfrak{gl}(n, r, \mathbb{K}) \). Now, let \( g \) be a reductive subalgebra in \( \mathfrak{gl}(n, r, \mathbb{K}) \) which is maximal (for the inclusion). From [3] we can write \( g = \mathfrak{z} \oplus [g, g] \), where \( \mathfrak{z} \) is the center of \( g \). Since \( \text{End}(V_1) \subset g \) then we must have \([Z, X] = 0\), for any \( Z \in \mathfrak{z} \) and \( X \in \text{End}(V_1) \).

Put \( Z = \begin{pmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{pmatrix} \), \( X = \begin{pmatrix} X \\ 0 \end{pmatrix} \) then
\[
[Z, X]_{J_n,r} = \begin{pmatrix} [Z_1, X] & -XZ_3 \\ Z_2X & 0 \end{pmatrix} = 0,
\]
and thus \( Z = \begin{pmatrix} \lambda I_r & 0 \\ 0 & Z_4 \end{pmatrix} \) with \( \lambda \in \mathbb{K} \) and \( Z_4 \in \text{End}(V_2) \).

Finally, let \( Z = \begin{pmatrix} \lambda I_r & 0 \\ 0 & Z_4 \end{pmatrix} \in \mathfrak{z} \) and \( A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \in g \), then we have
\[
[Z, A]_{J_n,r} = \begin{pmatrix} 0 & \lambda A_3 \\ -\lambda A_2 & 0 \end{pmatrix} = 0.
\]
Thus \( g = \begin{pmatrix} \text{End}(V_1) & 0_{r,n-r} \\ 0_{n-r,r} & \text{End}(V_2) \end{pmatrix} \simeq \text{End}(V_1) \oplus \text{End}(V_2) \), which completes the proof. \( \square \)

Now we shall determine the center of the Lie algebra \( (\text{Mat}(n, \mathbb{K}), [, ]_J) \). Let us denote it by \( \mathcal{Z}_J \), then we have the following
Proposition 2.3. Put $r = \text{rank}(J)$, if $r < n$ then $\dim Z_J = (n - r)^2$ while if $r = n$, then $\dim Z_J = 1$.

Proof. Case 1: $\text{rank}(J) = n$

In this case, the mapping $\varphi : A \mapsto JA$ is a Lie algebra isomorphism from $(\text{Mat}(n, K), [\cdot, \cdot], J)$ onto $\mathfrak{gl}(n, K)$, and since the center $Z$ of $\mathfrak{gl}(n, K)$ is $Z = KI_n$ then $Z_J = \varphi^{-1}(Z) = KJ^{-1}$.

Case 2: $\text{rank}(J) = r < n$

Let $Q, P \in \text{GL}(n, K)$ such that $J = QJ_{n,r}P$ and let $\varphi$ be the isomorphism mapping defined in Proposition 1.6, then

$$Z_J = \varphi^{-1}(Z_{J_{n,r}}),$$

then we shall compute $Z_{J_{n,r}}$ the center of $\mathfrak{gl}(n, r, K)$.

$$Z_{J_{n,r}} = \{ A \in \text{Mat}(n, K), [A, B]_{J_{n,r}} = 0 \ \forall B \in \text{Mat}(n, K) \}.$$ 

Put $A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}$, with $A_1 \in \text{Mat}(r, K), A_2 \in \text{Mat}((n - r) \times r, K), A_3 \in \text{Mat}(r \times (n - r), K), A_4 \in \text{Mat}(n - r, K)$, using Lemma 1.7 we get

$$[A, B]_{J_{n,r}} = \begin{pmatrix} [A_1, B_1] & A_1B_3 - B_1A_3 \\ A_2B_1 - B_2A_1 & A_2B_3 - B_2A_3 \end{pmatrix}.$$ 

Then $A \in Z_{J_{n,r}}$ if and only if the bracket $[A, B]_{J_{n,r}}$ is identically zero for any matrix $B$ in $\text{Mat}(n, K)$ which implies in particular $[A_1, B_1] = 0, \ \forall B_1 \in \text{Mat}(r, K)$ and thus $A_1$ is in the center of $\mathfrak{gl}(r, K)$, that is

$$A_1 = \lambda I_r, \quad \lambda \in K.$$ 

Now with the equations

$$A_1B_3 - B_1A_3 = A_2B_1 - B_2A_1 = A_2B_3 - B_2A_3 = 0, \ \forall B \in \text{Mat}(n, K),$$

we obtain

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}, \text{ with } A_4 \in \text{Mat}(n - r, K).$$ 

Thus

$$Z_{J_{n,r}} = K\begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix} \cong \text{End}(V_2), \text{ with } A_4 \in \text{Mat}(n - r, K),$$

and

$$Z_J = P^{-1}Z_{J_{n,r}}Q^{-1}. \quad \Box$$

Now we are able to give the following
Theorem 2.4. Let \( n \) be an integer with \( n \geq 2 \). The Lie algebras \((\text{Mat}(n, \mathbb{K}), [ , ]_{J})\) and \((\text{Mat}(n, \mathbb{K}), [ , ]_{J'})\) are isomorphic if and only if the matrices \( J \) and \( J' \) are equivalent.

Proof. Let \( J, J' \in \text{Mat}(n, \mathbb{K}) \). If \( J \) and \( J' \) are equivalent then from Proposition 1.6 the corresponding Lie algebras are isomorphic. Now we shall prove the converse.

Let \( J, J' \in \text{Mat}(n, \mathbb{K}) \) with \( \text{rk}(J) = r \) and \( \text{rk}(J') = s \) with \( r \neq s \).

Let \( V \) be a vector space over \( \mathbb{K} \) with dim \( V = n \) and put \( V = V_{1} \oplus V_{2} = W_{1} \oplus W_{2} \) with \( \text{dim} V_{1} = r \) and \( \text{dim} W_{1} = s \). From Proposition 1.6, we can assume that \( J = J_{n,r} \) and \( J' = J_{n,s} \) and from Proposition 2.2, the reductive part of \( \mathfrak{gl}(n, r, \mathbb{K}) \) is \( \text{End}(V_{1}) \oplus \text{End}(V_{2}) \) while the reductive part of \( \mathfrak{gl}(n, s, \mathbb{K}) \) is \( \text{End}(W_{1}) \oplus \text{End}(W_{2}) \). These reductive parts have the same dimension if and only if \( r + s = n \) but in this case they don’t have the same dimensionality of their center. Thus they are not isomorphic and this completes the proof. \( \square \)

As a consequence of this classification, we have the following theorem.

Theorem 2.5. Let \( \mathfrak{h}_{n} \) be the \( (2n + 1) \)-dimensional Heisenberg algebra. Then there is no faithful finite dimensional linear representation of \( \mathfrak{h}_{n} \) on a vector space \( V \) with dim \( V \leq n + 1 \).

Proof. Suppose that there exists a faithful linear representation \( \rho \) of \( \mathfrak{h}_{n} \), on a vector space \( V \) with dim \( V \leq n + 1 \). Let \( \rho \) such mapping, then we have \( \rho : \mathfrak{h}_{n} \rightarrow \mathfrak{gl}(r, \mathbb{K}) \) which an injective Lie algebra morphism.

Put \( \mathcal{H}_{n} = \text{Span}\{X_{i} = E_{1,i+1}, Y_{i} = E_{i+1,n+1}, Z = E_{1,n+2}, \ i = 1, \ldots, n\} \). We simply check that \( \mathcal{H}_{n} \) is a subalgebra of \( \mathfrak{gl}(n + 2, n + 1, \mathbb{K}) \), and \( [X_{i}, Y_{j}]_{J_{n+1,n+1}} = \delta_{i,j} Z \), it is the \( (2n + 1) \)-dimensional Heisenberg algebra. Then we can take \( \mathcal{H}_{n} \) instead of \( \mathfrak{h}_{n} \). If such mapping \( \rho \) exists, then we must have

\[
\rho([X_{i}, Y_{i}]_{J_{n+2,n+1}}) = [\rho(X_{i}), \rho(Y_{i})],
\]

\[
\rho(Z) = [\rho(X_{i}), \rho(Y_{i})],
\]

but since \( \rho \) is an injective Lie algebra morphism, then \( \rho(Z) = \lambda I_{r} \) with \( \lambda \neq 0 \) wick is impossible because \( I_{r} \) is traceless matrix. \( \square \)

2.1. Example. Let \( \mathfrak{h} = \text{span}\{Z,Y,X\} \) with the only non vanishing bracket is \( [X,Y] = Z \) then the classical representation of \( \mathfrak{h} \) is

\[
X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
From the last corollary, we can’t built a faithful homomorphism \( \mathfrak{h} \) in \( \mathfrak{gl}(2, \mathbb{R}) \) but we can built such homomorphism in \( \mathfrak{gl}(2, 1, \mathbb{R}) \) with the following realization

\[
X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

3. The case of \( \text{Mat}(n \times m, \mathbb{K}) \)

In this section we will deal with the linear space of strict rectangular matrices \( \text{Mat}(n \times m, \mathbb{K}) \) which is not an associative algebra. Fix \( J \in \text{Mat}(m \times n, \mathbb{K}) \) and put

\[
[A, B]_J = AJB - BJA, \quad A, B \in \text{Mat}(n \times m, \mathbb{K}).
\]

Recall that we have denoted the matrix

\[
\begin{pmatrix}
I_r & 0_{r,n-r} \\
0_{m-r,r} & 0_{m-r,n-r}
\end{pmatrix}
\]

by \( J_{m,n,r} \) and we will denotes the Lie algebra \( (\text{Mat}(n \times m, \mathbb{K}), [\ , ]_{J_{m,n,r}}) \) by \( \mathfrak{gl}(n, m, r, \mathbb{K}) \). In the case when \( n = m \), \( \mathfrak{gl}(n, n, r, \mathbb{K}) \) is simply \( \mathfrak{gl}(n, r, \mathbb{K}) \).

**Lemma 3.1.** Let \( \mathfrak{h} \) be a Lie sub-algebra of \( \mathfrak{gl}(n, \mathbb{K}) \), then \( \mathfrak{h} \) is a Lie sub-algebra of \( \mathfrak{gl}(p, q, n, \mathbb{K}) \) for any integers \( p, q \geq n \).

**Proof.** For any subset \( \mathfrak{h} \subset \text{Mat}(n, \mathbb{K}) \) and \( q, p \geq n \), we have

\[
[\mathfrak{h}, \mathfrak{h}]_{J_{q,p,n}} = [\mathfrak{h}, \mathfrak{h}].
\]

And thus the conclusion holds. \( \square \)

Now let \( J \in \text{Mat}(m \times n, \mathbb{K}) \), and let’s denotes by \( \mathcal{Z}_J \) the center of the Lie algebra \( (\text{Mat}(n \times m, \mathbb{K}), [\ , ]_J) \), then we have

**Proposition 3.2.** Let \( J \in \text{Mat}(m \times n, \mathbb{K}) \) with \( \text{rk}(J) = r \). Then the center of \( (\text{Mat}(n \times m, \mathbb{K}), [\ , ]_J) \) is \((n-r)(m-r)\) dimensional. In particular, if \( \mathcal{Z}_{J_{m,n,r}} \) denotes the center of \( (\text{Mat}(n \times m, \mathbb{K}), [\ , ]_{J_{m,n,r}}) \), then \( \mathcal{Z}_{J_{m,n,r}} \) is spanned by the matrices of the form

\[
A = \begin{pmatrix}
0_r & 0_{r,n-r} \\
0_{m-r,r} & A_4
\end{pmatrix}, \quad \text{with} \ A_4 \in \text{Mat}((m-r) \times (n-r), \mathbb{K}).
\]

**Proof.** The proof of this proposition is similar to the proof of the Proposition \[2.3\]. \( \square \)

Now we can give the following

**Theorem 3.3.** Let \( J, J' \in \text{Mat}(m \times n, \mathbb{K}) \) with \( \text{min}(m, n) \geq 2 \). Then the Lie algebras \( (\text{Mat}(n \times m, \mathbb{K}), [\ , ]_J) \) and \( (\text{Mat}(n \times m, \mathbb{K}), [\ , ]_{J'}) \) are isomorphic if and only if \( J \) and \( J' \) are equivalent in \( \text{Mat}(m \times n, \mathbb{K}) \).
Theorem 3.5. Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic zero can be viewed as a Lie sub-algebra of $\mathfrak{gl}(m, n, \mathbb{K})$. More precisely $\mathfrak{g}$ has a linear representation $\rho$ over $\mathbb{K}$ on $\mathfrak{gl}(m, n, \mathbb{K})$, that is a faithful representation, making $\mathfrak{g}$ isomorphic to a subalgebra of $(\text{Hom}(\mathbb{K}^n, \mathbb{K}^m), [\ , ]_{m,n,r})$.

Proof. By Ado’s Theorem, $\mathfrak{g}$ can be viewed as a subalgebra of $\mathfrak{gl}(p, \mathbb{K})$ and the last algebra is also a subalgebra of $\mathfrak{gl}(n, m, q, \mathbb{K})$ for any $q \geq p$ and $n, m \geq q$.  

3.1. Examples. (a) Let $V = \mathbb{R}^2$, we identify $V$ with $\text{Mat}(2 \times 1, \mathbb{R})$, we choose a basis $(e_1, e_2)$ in $V$ with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we can check that $[e_2, e_1]_J = e_1$ with $J = (1, 0)$. Then the two-dimensional affine Lie algebra can be viewed as $(\text{Hom}(\mathbb{R}^2, \mathbb{R}), [\ , ]_{2,1})$.

These constructions can be generalized for $V = \mathbb{R}^n$ with $n \geq 2$.

(b) Let $V = \mathbb{R}^3 \simeq \text{Mat}(3 \times 1, \mathbb{R})$. Put $e_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then we verify that $[e_1, e_2]_J = e_2, [e_1, e_3]_J = e_3, [e_2, e_3]_J = 0$ where $J = (1, 0, 0)$. It is the Lie algebra $\mathfrak{g}_{3,2}(1)$ (see \cite{1} for notations).

(c) Let $V = \mathbb{R}^4$, then we can identify $V$ with $\text{Mat}(4 \times 1, \mathbb{R})$ and in this case we take $J = (1, 0, 0, 0)$ or with $\text{Mat}(2, \mathbb{R})$ and in this case we can choose $J = I_2$ or $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

4. Contractions and Extensions of Lie Algebras

Let $n, r \in \mathbb{N}$ with $n \geq r, n \geq 2$ and put $J = J_{n,r}$, then we have the following

Proposition 4.1. The Lie algebra $(\text{Mat}(n, \mathbb{K}), [\ , ]_{n,r})$ is a contraction of $\mathfrak{gl}(n, \mathbb{K})$. 

Proof. Let $J$ and $J'$ two equivalent matrices in $\text{Mat}(m \times n, \mathbb{K})$ then from Proposition 1.6 the corresponding Lie algebras structures on $\text{Mat}(n \times m, \mathbb{K})$ are isomorphic.

Conversely, let $J, J'$ be in $\text{Mat}(m \times n, \mathbb{K})$ with $\text{rk}(J) \neq \text{rk}(J')$, then from Proposition 2.3 we have $\dim \mathcal{Z}_J \neq \dim \mathcal{Z}_{J'}$, since

$$\dim \mathcal{Z}_J = (n - \text{rk}(J))(m - \text{rk}(J)) \quad \text{and} \quad \dim \mathcal{Z}_{J'} = (n - \text{rk}(J'))(m - \text{rk}(J')).$$

□

Corollary 3.4. Let $V$ be a vector space over $\mathbb{K}$ with $\dim V \geq 2$. Then $V$ can be equipped with a non trivial Lie algebra structure.

Proof. Let $n$ be the dimension of $V$. We identify $V$ with $\text{Mat}(n \times 1, \mathbb{K})$ and put $J = (1, 0, \ldots, 0)$, then $(V, [\ , ]_J)$ is a non trivial Lie algebra.  

□

Now we can give an analogue of the Ado’s Theorem
Proof. Let \((E_{i,j})\) the canonical basis of Mat\((n, \mathbb{K})\), and define
\[ g = \text{Span}\{E'_{i,j}, \ i, j = 1, \ldots, n\}, \]
with
\[ E'_{i,j} = \begin{cases} E_{i,j} & \text{if } i \leq r \text{ and } j \leq r, \\ \varepsilon E_{i,j} & \text{if } i > r \text{ and } j \leq r \text{ or } j > r \text{ and } i \leq r, \\ \varepsilon^2 E_{i,j} & \text{if } i > r \text{ and } j > r. \end{cases} \]
where \(\varepsilon \in \mathbb{K}\). Then we have
\[
[E'_{i,j}, E'_{k,l}] = \begin{cases} \\
\delta_{j,k}E'_{i,l} - \delta_{l,i}E'_{k,j} & \text{if } i, j, k, l \leq r, \\
\delta_{j,k}E'_{i,l} & \text{if } i, j, l \leq r \text{ and } k > r, \\
-\delta_{l,i}E'_{k,j} & \text{if } i, j, l \leq r \text{ and } k > r, \\
0 & \text{if } i, j \leq r \text{ and } k, l > r, \\
0 & \text{if } i, k \leq r \text{ and } l, j > r, \\
\delta_{j,k}\varepsilon^2 E'_{i,l} & \text{if } i \leq r \text{ and } j, k, l > r, \\
\delta_{j,k}\varepsilon^2 E'_{i,l} - \delta_{l,i}\varepsilon^2 E'_{k,j} & \text{if } i, j, k, l > r. \\
\end{cases} \]
And when \(\varepsilon \to 0\), then we obtain
\[
\lim_{\varepsilon \to 0}[E'_{i,j}, E'_{k,l}] = [E_{i,j}, E_{k,l}]_{J_{n,r}}.
\]
□

If we repeat again this contraction, we get a new algebra which can be directly obtained from the first Lie algebra since \(J_{n,r}J_{n,s} = J_{n,t}\) with \(t = \min(r, s)\).

4.1. Example. Let \(\mathfrak{sl}(2, \mathbb{R})\) be the semi-simple (real) Lie algebra spanned by \(\{H, X, Y\}\) where
\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
The (usual) brackets are:
\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.
\]
Pick \(J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\), then we obtain the following brackets:
\[
[H, X]_J = X, \quad [H, Y]_J = -Y, \quad [X, Y]_J = 0.
\]
The new Lie algebra is completely solvable and thus non isomorphic to \(\mathfrak{sl}(2, \mathbb{R})\) since \(\mathfrak{sl}(2, \mathbb{R})\) is semi-simple.

Let us restrict ourself to the Lie algebra \(\mathfrak{gl}(n, \mathbb{K})\). Then we have the following

**Lemma 4.2.** (\([4], [5]\)) The map \((A, B) \mapsto [A, B]_J\) is a two-coboundary for the adjoint representation of \(\mathfrak{gl}(n, \mathbb{K})\).
Proof. We can easily check that
\[ [A, B]_J = ad_A\alpha(B) - ad_B\alpha(A) - \alpha([A, B]) = (d\alpha)(A, B). \]
where
\[ \alpha(X) = \frac{1}{2}(XJ + JX). \]

Let \( t \in [0, 1] \) and \( J = J_{n,r} \) (with \( r < n \)), then we can write
\[ [A, B]_{(1-t)I+J} = [A, B] + t[A, B]_{J-I}, \]
thus since \( [A, B]_{J-I} \) is a coboundary, \( t \mapsto [A, B]_{(1-t)I+J} \) is a deformation infinitesimally trivial at \( t = 0 \) from \( \mathfrak{gl}(n, \mathbb{K}) \) to \( \mathfrak{gl}(n, r, \mathbb{K}) \). For any \( t < 1 \), the Lie algebra \((\text{Mat}(n, \mathbb{K}), [\cdot, \cdot]_{(1-t)I+J})\) is isomorphic to \( \mathfrak{gl}(n, \mathbb{K}) \) but for \( t = 1 \) we get a completely different structure which is labelled by the bracket \([\cdot, \cdot]_J\). The mapping
\[ \Psi_t: (\text{Mat}(n, \mathbb{K}), [\cdot, \cdot]_{(1-t)I+J}) \rightarrow \mathfrak{gl}(n, \mathbb{K}), \quad \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) \mapsto \left( \begin{array}{cc} X_1 & (1-t)X_2 \\ X_3 & (1-t)X_4 \end{array} \right) \]
is for any \( t \in [0, 1] \) invertible and \([X, Y]_{(1-t)I+J} = \Psi_t^{-1}(\Psi_t(X), \Psi_t(Y))\), and we have \( \frac{\partial}{\partial t}(\cdot, \cdot}_{(1-t)I+J})|_{t=0} \) is the 2-coboundary: \((A, B) \mapsto [A, B]_{J-I} \).

**Lemma 4.3.** The vector space \( \text{Mat}(n, \mathbb{K}) \) is a \((\text{Mat}(n, \mathbb{K}), [\cdot, \cdot])\)-module.

Proof. We can verify that the map
\[ \varphi: (\text{Mat}(n, \mathbb{K}), [\cdot, \cdot]) \rightarrow \mathfrak{gl}(\text{Mat}(n, \mathbb{K})), \quad A \mapsto ad^J_A, \]
where
\[ ad^J_A(B) = [A, B]_J, \]
is a Lie algebra homomorphism since \([\cdot, \cdot]_J \) is a Lie bracket on \( \text{Mat}(n, \mathbb{K}) \). □

Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{K} \) and \((X_1, \ldots, X_r)\) with the following commutations roles \([X_i, X_j] = \sum c_{i,j}^k X_k \). By the Ado’s theorem, there is a faithful representation \( \rho \) of \( \mathfrak{g} \) in \( \mathfrak{gl}(n, \mathbb{K}) \) for some integer \( n \). Let us identify \( \mathfrak{g} \) with its image under \( \rho \). Then we have the following

**Proposition 4.4.** Let \( \mathfrak{h} = \text{Mat}(r \times (n - r), \mathbb{K}), \mathfrak{h}' = \text{Mat}((n - r) \times r, \mathbb{K}) \) and \( \mathfrak{h}'' = \text{Mat}(n - r, \mathbb{K}) \). Then we have the following subalgebras inclusions
\[ \left( \begin{array}{c} \mathfrak{g} \\ \mathfrak{h} \end{array} \right) \subset \left( \begin{array}{c} \mathfrak{g} \\ \mathfrak{h}' \end{array} \right) \subset \mathfrak{gl}(n, r, \mathbb{K}) \]
and
\[ \left( \begin{array}{c} \mathfrak{g} \\ \mathfrak{h}' \end{array} \right) \subset \left( \begin{array}{c} \mathfrak{g} \\ \mathfrak{h}'' \end{array} \right) \subset \mathfrak{gl}(n, r, \mathbb{K}). \]
LINEAR FAMILY OF LIE BRACKETS ON THE SPACE OF MATRICES $\text{Mat}(n \times m, \mathbb{K})$ AND ADO’S THEOREM

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