Co-ideal quantum affine algebra and boundary scattering of the deformed Hubbard chain

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Abstract

We consider boundary scattering for a semi-infinite one-dimensional deformed Hubbard chain with boundary conditions of the same type as for the $Y = 0$ giant graviton in the AdS\textsubscript{5}×S\textsubscript{5} correspondence. We show that the recently constructed quantum affine algebra of the deformed Hubbard chain has a co-ideal subalgebra which is consistent with the reflection (boundary Yang–Baxter) equation. We derive the corresponding reflection matrix and furthermore show that the aforementioned algebra in the rational limit specializes to the (generalized) twisted Yangian of the $Y = 0$ giant graviton.

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1. Introduction

The Hubbard model was introduced in order to study strongly correlated electrons [1], and later due to many generalizations (e.g. [2–4]) it grew into a large family of models (see e.g. [5, 6]). Recently, the interest in the Hubbard model has been renewed due to remarkable successes in solving similar models appearing in the context of the AdS\textsubscript{5}×S\textsubscript{5} correspondence (see for instance [7] and references therein).

An interesting relation between the Hubbard model and the AdS\textsubscript{5}×S\textsubscript{5} string was found by studying the centrally extended $U_q(\mathfrak{su}(2|2))$ algebra $Q$ [8]. The integrable model with this underlying algebra turns out to describe a variety of quantum deformed Hubbard models as well as the AdS\textsubscript{5}×S\textsubscript{5} superstring in the rational $q \rightarrow 1$ limit. This makes it an interesting
model to study since it offers a unified description of all of these systems. We will simply refer to it as the deformed Hubbard model.

It was found that the deformed Hubbard model is actually invariant under the affine extension, $\hat{Q}$, of the symmetry algebra $[9]$. In the rational $q \to 1$ limit, this algebra becomes the Yangian symmetry of the $\text{AdS}_5 \times S^5$ superstring $[10]$. The fundamental $R$-matrix can be found by requiring invariance under $Q$ alone, but the affine extension plays a crucial role in the determination of $S$-matrices in higher representations. The fundamental $S$-matrix was found in $[8]$ and in the rational limit is equivalent to Shastry’s $R$-matrix $[11–13]$. The $S$-matrix describing bound state scattering has also recently been derived $[14]$. Its construction relies heavily on the affine generators in $\hat{Q}$. This is similar to the situation in $\text{AdS}_5 \times S^5$, where this part was played by the Yangian charges instead $[15]$.

When studying integrable models with periodic boundary conditions, the spectrum is governed by the $S$-matrix and thus indirectly through the underlying (bulk) symmetry algebra. However, for integrable systems with boundaries, there is another object, called the reflection matrix, which describes the scattering of excitations off the boundary. Generically, boundaries preserve a subalgebra of the bulk Lie algebra and this subalgebra then determines the corresponding reflection matrix. However, this is usually not enough to determine the bound state reflection matrix and a co-ideal subalgebra of the corresponding bulk Yangian or quantum affine algebra is required.

Open boundary conditions for the one-dimensional Hubbard model have received less attention than their closed chain counterpart, but they exhibit a rich variety of structures (see e.g. $[16–18]$). The reflection matrices for open boundary conditions for the deformed Hubbard chain have been studied in $[19]$. Drawing from similarities with open spin chains in the context of $\text{AdS/CFT} [20]$, two types of boundary conditions were formulated corresponding to $Y = 0$ and $Z = 0$ brane configurations. In this paper, we will study boundary conditions corresponding to the $Y = 0$ system applied to bound states. For this configuration, the boundary representation is a singlet and the boundary conditions preserve half of the supersymmetries.

The aim of this paper is twofold. Firstly, we want to identify the symmetry algebra that governs boundary scattering. For the $\text{AdS}_5 \times S^5$ superstring, this algebra is a (generalized) twisted Yangian $[21, 22]$. Here, we find it to be an affine co-ideal subalgebra $\hat{B}$ of $\hat{Q}$. Its structure turns out to be governed by the notion of quantum symmetric pairs which have been heavily studied in the context of semi-simple Lie algebras $[24–27]$. Inspired by these results we explicitly construct the boundary algebra $\hat{B}$.

Secondly, having found the symmetry algebra $\hat{B}$ we use it to compute the reflection matrix of arbitrary bound states and show that it satisfies the reflection (also called boundary Yang–Baxter) equation. Conversely, we explicitly solve the reflection equation and find that our reflection matrix is the unique solution, thus proving that $\hat{B}$ is indeed the correct and unique symmetry algebra. Finally, we also show that in the $q \to 1$ limit we reproduce the results for the $\text{AdS}_5 \times S^5$ superstring found in $[28]$.

It is worth noting that somewhat similar boundary scattering problems for quantum affine algebras of the Lie algebras of a classical type have been considered in $[29, 30]$, where again the scattering is governed by some co-ideal subalgebra $B$, which in some cases is called a $q$-Yangian $[31]$. Boundary scattering has also been intensively studied for sine–Gordon and affine Toda field theories $[32–35]$. The investigation of the reflection equation bearing on the quantum symmetric pairs constructed by Letzter was considered in $[36, 37]$.

This work is organized as follows. In section 2, we discuss the relevant notation and definitions of the bulk algebra and its bound state representations. In section 3, we present the required axiomatic formulation of the co-ideal subalgebras and quantum symmetric pairs. Then in section 4 we present the general form of the reflection matrix for arbitrary bound states.
and discuss its properties. We end with some concluding remarks. The appendix is reserved for the $q \to 1$ limit of $\widehat{\mathcal{Q}}$ and also for a brief review of the twisted Yangian of the $Y = 0$ giant graviton.

2. Deformed quantum affine algebra

In this section, we shall review the quantum affine algebra constructed in [9] and its bound state representation constructed in [14].

Quantum affine algebra $\widehat{\mathcal{Q}}$. The algebra $\widehat{\mathcal{Q}}$ of the quantum deformed one-dimensional Hubbard chain was recently constructed in [9] and is a deformation of the centrally extended affine algebra $\widehat{\mathfrak{sl}}(2|2)$. It is generated by four sets of the Chevalley–Serre generators $V_k \equiv q^{H_k}$, $E_i, F_i$ ($i = 1, 2, 3, 4$) and two sets of the central elements $U_k$ and $V_k$ ($k = 2, 4$), with $U_k$ being responsible for the deformation of the co-product.

Let us start by recalling the symmetric matrix $DA$ and the normalization matrix $D$ associated with the Cartan matrix $A$ for $\widehat{\mathfrak{sl}}(2|2):$

$$DA = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad D = \text{diag}(1, -1, -1, -1).$$

The algebra is then defined accordingly by the following commutation relations:

$$K_i E_j = q^{D_{ij}} E_j K_i, \quad K_i F_j = q^{-D_{ij}} F_j K_i,$$

$$\{E_2, F_4\} = -\tilde{g}_a^{-1}(K_4 - U_2 U_4^4 K_2^{-1}), \quad \{E_4, F_2\} = \tilde{g}_a^{+1}(K_2 - U_2 U_4^4 K_4^{-1}),$$

$$\{E_j, F_j\} = D_{jj} K_j = K_j^{-1}, \quad \{E_i, F_j\} = 0, \quad i \neq j, \quad i + j \neq 6.$$ 

These are supplemented by a set of Serre relations ($j = 1, 3$):

$$\{E_j, [E_j, E_k]\} - (q - 2 + q^{-1}) E_j E_k E_j = 0, \quad \{E_1, E_3\} = E_2 E_2 = E_3 E_4 = (E_2, E_4) = 0,$$

$$\{F_j, [F_j, F_k]\} - (q - 2 + q^{-1}) F_j F_k F_j = 0, \quad \{F_1, F_3\} = F_2 F_2 = F_3 F_4 = (F_2, F_4) = 0.$$ 

Central elements are related to the quartic Serre relations (for $k = 2, 4$) as follows:

$$\{[E_1, E_k], [E_3, E_k]\} - (q - 2 + q^{-1}) E_1 E_k E_3 E_k = g_k a_k (1 - V_k^2 U_k^2),$$

$$\{[F_1, F_k], [F_3, F_k]\} - (q - 2 + q^{-1}) F_1 F_k F_3 F_k = g_k a_k (V_k^2 - U_k^2).$$

This algebra has three central charges

$$C_1 = K_1 K_2 K_3,$$

$$C_2 = \{[E_2, E_1], [E_2, E_3]\} - (q - 2 + q^{-1}) E_2 E_1 E_3 E_2,$$

$$C_3 = \{[F_2, F_1], [F_2, F_3]\} - (q - 2 + q^{-1}) F_2 F_1 F_3 F_2.$$ 

The central elements $V_k$ are constrained by the relation $K_1^{-1} K_2^{-1} K_3^{-1} = V_k^2$.

Hopf algebra. The group-like elements $X \in \{1, K_j, U_k, V_k\}$ ($j = 1, 2, 3, 4$ and $k = 2, 4$) have the co-product $\Delta$ defined in a usual way, $\Delta(X) = X \otimes X$, while for the remaining Chevalley–Serre generators they are deformed by the central elements $U_k$. Similar considerations work for the antipode $S$ and co-unit $\varepsilon$. Summarizing, we have

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U_2^{+3/4} U_4^{+3/4} \otimes E_j, \quad \Delta(F_j) = F_j \otimes K_j + U_2^{-3/4} U_4^{-3/4} \otimes F_j.$$ 

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U_2^{+3/4} U_4^{+3/4} \otimes E_j, \quad \Delta(F_j) = F_j \otimes K_j + U_2^{-3/4} U_4^{-3/4} \otimes F_j.$$
Representation. We shall be using the $q$-oscillator representation (for any complex $q$ not a root of unity) constructed in [14]. The bound state representation is defined on vectors

$$|m, n, k, l\rangle = (a_1^\dagger)^m (a_2^\dagger)^n (a_3^\dagger)^k (a_4^\dagger)^l |0\rangle,$$

where the indices 1 and 2 denote bosonic and 3 and 4 denote fermionic oscillators; the total number of excitations $k + l + m + n = M$ is the bound state number and the dimension of the representation is $\dim = 4M$. This representation constrains the central elements as $U := U_2 = U_4^{-1}$ and $V := V_2 = V_4^{-1}$ and describes a spin-chain excitation with quasi-momentum $p$ related to the deformation parameter as $U = e^{ip}$.

The triples corresponding to the bosonic and fermionic $sl_q(2)$ in this representation are given by

$$H_1|m, n, k, l\rangle = (l - k)|m, n, k, l\rangle, \quad H_3|m, n, k, l\rangle = (n - m)|m, n, k, l\rangle,$$

$$E_1|m, n, k, l\rangle = [k]_q |m, n, k - 1, l + 1\rangle, \quad E_3|m, n, k, l\rangle = |m + 1, n - 1, k, l\rangle,$$

$$F_1|m, n, k, l\rangle = [l]_q |m, n, k + 1, l - 1\rangle, \quad F_3|m, n, k, l\rangle = |m - 1, n + 1, k, l\rangle.$$  

(8)

The supercharges act on the basis states as

$$H_2|m, n, k, l\rangle = -\left\{ C - \frac{k - l + m - n}{2} \right\} |m, n, k, l\rangle,$$

$$E_2|m, n, k, l\rangle = a(-1)^m [l]_q |m, n, k - 1, l + 1\rangle + b|m - 1, n, k + 1, l\rangle,$$

$$F_2|m, n, k, l\rangle = c[l]_q |m + 1, n, k - 1, l\rangle + d(-1)^m |m - 1, n, k, l + 1\rangle.$$  

(9)

Here $|n\rangle_q = (q^n - q^{-n})/(q - q^{-1})$ and $C$ is the $q$-factor of the central element $V = q^C$ representing the energy of the state. The representation labels $a, b, c$ and $d$ satisfy constraints

$$ad = \frac{q^aV - q^{-a}V^{-1}}{q^{2CM} - q^{-2CM}}, \quad bc = \frac{q^bV - q^{-b}V^{-1}}{q^{2CM} - q^{-2CM}},$$

$$ab = \frac{g_{ac}}{[M]_q}(1 - U^2V^2), \quad cd = \frac{g_{ac}}{[M]_q}(V^2 - U^{-2}),$$

(10)

which altogether give the multiplet shortening (mass-shell) condition

$$\frac{q^2}{[M]_q^2}(V^{-2} - U^{-2})(1 - U^2V^2) = \frac{(V - q^{CM}V^{-1})(V - q^{-CM}V^{-1})}{(q^{CM} - q^{-CM})^2}.\quad(11)$$

The explicit $x^\pm$ parametrization of the representation labels is

$$a = \sqrt{\frac{g}{[M]_q}} \gamma, \quad b = \sqrt{\frac{g}{[M]_q}} \frac{x^- - x^+}{\gamma x^+},$$

$$c = \sqrt{\frac{g}{[M]_q}} \frac{i\bar{g}q^\frac{\gamma}{2}}{\alpha V g(x^+ + \xi)} \gamma x^+, \quad d = \sqrt{\frac{g}{[M]_q}} \frac{\bar{g}q^\frac{\gamma}{2}}{\xi x^+ + 1}.\quad(12)$$

The central elements in this parametrization read as

$$U_1^2 = \frac{1}{q^{CM}} x^+ + \xi = q^{CM} x^+ x^- + 1\xi x^+ + 1, \quad V_2^2 = \frac{1}{q^{CM}} \xi x^+ + 1 = q^{CM} x^+ x^- + \xi,$$

while the shortening condition (11) becomes

$$\frac{1}{q^{CM}} \left( x^+ + \frac{1}{x^-} \right) - q^{CM} \left( x^- + \frac{1}{x^+} \right) = \left( q^{CM} - \frac{1}{q^{CM}} \right) \left( \xi + \frac{1}{\xi} \right),$$

(14)

where $\xi = -i\bar{g}(q - q^{-1})$ and $\bar{g}^2 = g^2/(1 - g^2(q - q^{-1})^2)$.

The action of the affine charges $H_3$, $E_3$ and $F_3$ is defined in exactly the same way as for the regular supercharges subject to the following substitutions $C \to -C$ and
The affine labels $\tilde{a}, \tilde{b}, \tilde{c}$ and $\tilde{d}$ are acquired from (12) by replacing
\[ V \to V^{-1}, \quad x^\pm \to \frac{1}{x^\pm}, \quad \gamma \to i\tilde{\gamma}, \quad \alpha \to \tilde{\alpha}^2, \quad \tilde{a} \to -\frac{1}{\tilde{a}}. \tag{15} \]
Finally, we introduce the multiplicative spectral parameter of the algebra
\[ z = \frac{1 - U^2V^2}{V^2 - U^2} = q^{-M} \bar{\vartheta}(x^+) = q^{+M} \bar{\vartheta}(x^-) \quad \text{with} \quad \vartheta(x) = -\frac{(x + \xi)(1 + 1/(\xi x))}{\xi - \xi^{-1}}, \tag{16} \]
which will play an important role in describing the reflection algebra.

3. Co-ideal quantum affine algebra

We define the boundary conditions for the deformed one-dimensional Hubbard chain to be of the same type as those of the $Y = 0$ giant graviton [19, 20] (see appendix B for details). This kind of boundary conditions is described by a quantum version of the symmetric pair $(g, g^\theta)$ and the boundary scattering is governed by a co-ideal subalgebra. The axiomatic formulation and classification of co-ideal subalgebras and quantum symmetric pairs for semi-simple Lie algebras were done by Letzter in a series of works [24–27]. We shall explicitly construct the quantum affine co-ideal subalgebra $\hat{B}$ of $\hat{Q}$ relying on Letzter’s construction.

3.1. Boundary algebra and symmetric pairs

Before moving on to quantum deformed (affine) algebras, let us first briefly recall the algebraic structure for boundaries with Yangian algebras. Consider an integrable model with the symmetry algebra described by the Yangian $Y(g)$ for some Lie algebra $g$. Suppose that the boundary module respects a subalgebra $a \subset g$ and that there is an involution $\theta : g \mapsto g$ such that $a = g^\theta$ is the $\theta$-fixed subalgebra of $g$. Then $a$ and subset $b = g \setminus a$ respecting
\[ [a, a] \subset a, \quad [a, b] \subset b, \quad [b, b] \subset a \tag{17} \]
are positive and negative eigenspaces of $\theta$, namely $\theta(a) = +a$ and $\theta(b) = -b$. Thus, if the underlying symmetry algebra in the bulk is the Yangian $Y(g)$, then the associated symmetry algebra respected by the boundary is the so-called (generalized) twisted Yangian $Y(g, a)$ [23] generated by the level-0 charges $\tilde{\gamma}^i$ and twisted level-1 charges$^7$
\[ \tilde{J}^p := \tilde{\gamma}^p + \frac{\alpha}{4} J^p (\tilde{\gamma}^q \tilde{\gamma}^r + \tilde{\gamma}^q J^r) = \tilde{\gamma}^p - \frac{\alpha}{4} [T^a, J^p], \tag{18} \]
where indices $i, j, k, \ldots$ run over the $a$-indices and $p, q, r, \ldots$ over the $b$-indices. $\alpha$ is a formal deformation parameter conventionally set to $1$ and $T^a$ is the quadratic Casimir operator of $g$ restricted to the subalgebra $a$. The Yangian $Y(g, a)$ is a left co-ideal subalgebra,
\[ \Delta Y(g, a) \subset Y(g) \otimes Y(g, a), \tag{19} \]
and is invariant under the extension $\tilde{\theta}$ of the involution $\theta$ acting on Yangian charges as
\[ \tilde{\theta}(\tilde{J}^p_a) = (-1)^p \tilde{J}^p_a, \quad \tilde{\theta}(\tilde{J}^p_n) = (-1)^{p+1} \tilde{J}^p_n, \quad \tilde{\theta}(\alpha) = -\alpha, \tag{20} \]
where $n$ is the level of the charge; thus it is easy to see that $\tilde{\gamma}^p$ is invariant under $\tilde{\theta}$ which acts as a filtration on $Y(g)$.

However, this construction does not straightforwardly extend to the quantum deformed algebras. Of course, at the level of the algebra, one can again define the involution $\theta$ that will specify the preserved subalgebra. But this cannot be extended to $U_q(g)$, since $U_q(g^\theta)$ in the general case need not be a Hopf subalgebra of $U_q(g)$. This complicates identifying the symmetry algebra of the boundary which should clearly be a subalgebra of the full symmetry algebra.

$^7$ This construction is not valid when $\theta$ is trivial, $g^\theta = g$. For this case, we refer to [38].
3.2. Co-ideal subalgebras and quantum symmetric pairs

We continue by giving the formulation of co-ideal subalgebras for semi-simple algebras as described in [24–27]. The results presented here allow for a generalization consistent with the affine structure presented in section 2. We shall follow [26] quite closely and for the reader’s convenience we shall try to give all the necessary constructions that will be used later on in the explicit construction of the co-ideal subalgebra of $\hat{Q}$.

**Setting.** Consider a Lie algebra $g$ with the Cartan matrix $(a_{ij})$ and a diagonal normalization matrix $(d_{ij})$ (here and further $1 \leq i, j \leq n$) such that $(d_{i}a_{ij})$ is a symmetric matrix. Let the triangular decomposition of the algebra be $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and let $\Phi$ denote the root system of $g$ and $\Phi^+$ the set of the positive roots. Let $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple positive roots and $(\alpha_i, \alpha_j) = d_{ij}a_{ij}$ denote the Cartan inner product on $\mathfrak{h}^*$. Then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ have a basis of root vectors $\{e_\beta | \beta \in \Phi^+\}$ and $\{f_\beta | \beta \in \Phi^+\}$, respectively. Let $h_1, \ldots, h_n$ be the basis of $\mathfrak{h}$. Then the standard Chevalley–Serre basis of $g$ is given by $\{e_\beta, f_\beta | \beta \in \Phi^+\} \cup \{h_1, \ldots, h_n\}$.

Let the quantized universal enveloping algebra $U_q(\mathfrak{g})$ be generated over $\mathbb{C}[q]$ by the elements $x_i, y_i, t_i^{\pm 1}$ that correspond to the standard Chevalley–Serre basis. The algebra $U_q(\mathfrak{g})$ becomes a Hopf algebra $H$ when equipped with the co-product $\Delta$, co-unit $\epsilon$ and antipode $\sigma$ given by\(^8\)

\[
\Delta(x_i) = x_i \otimes 1 + t_i^{-1} \otimes x_i, \quad \epsilon(x_i) = 0, \quad \sigma(x_i) = -t_i x_i, \\
\Delta(y_i) = y_i \otimes t_i + 1 \otimes y_i, \quad \epsilon(y_i) = 0, \quad \sigma(y_i) = -y_i t_i^{-1}, \\
\Delta(t_i) = t_i \otimes t_i, \quad \epsilon(t_i) = 1, \quad \sigma(t_i) = t_i^{-1}.
\]

Being a Hopf algebra, $U_q(\mathfrak{g})$ admits left and right adjoint actions making $U_q(\mathfrak{g})$ into a left or right module. The (twisted) adjoint action is defined as

\[
(\mathrm{ad}_x)_b = x_ib - (-1)^{|b||x|}t_i^{-1}bt_ix_i, \quad (\mathrm{ad}_y)_b = t_ibx_i - (-1)^{|b||x|}t_ix_ib, \\
(\mathrm{ad}_y)_b = y_it_i^{-1}b - (-1)^{|b||y|}_{t_i}b_{y_i}t_i^{-1}, \quad (\mathrm{ad}_y)_b = y_{t_i}b - (-1)^{|b||y|}_{t_i}y_{t_i}b_{t_i}^{-1},
\]

for all $b \in U_q(\mathfrak{g})$. Here, $(-1)^{|b||x|}$ represents the grading factor of supercharges. We shall also be using the shorthand notation $\mathrm{ad}_y_1 \cdots y_n = \mathrm{ad}_y_1 \cdots \mathrm{ad}_y_n$ and similarly for $\mathrm{ad}_x$. Finally, let us introduce the Abelian subgroup $T$ of $U_q(\mathfrak{g})$ generated by the elements $t_i^{\pm 1}$. Let $Q(\pi)$ be equal to the integral lattice generated by $\pi$, i.e. $Q(\pi) = \sum_{1 \leq i \leq n} N\alpha_i$; here, $\mathbb{N}$ is the set of non-negative integers. Then there is an isomorphism $\tau$ of Abelian groups from $Q(\pi)$ to $T$ defined by $\tau(\alpha_i) = t_i$, and thus for every $\lambda \in \Phi$ there is an image $\tau(\lambda) \in T$.

**Co-ideal subalgebra.** A vector subspace $I$ of the Hopf algebra $H$ is called a left co-ideal if

\[
\Delta(I) \subset H \otimes I.
\]

(23)

In the same way the right co-ideal may be defined if $\Delta(I) \subset I \otimes H$. Let $M$ be a Hopf subalgebra of $H$ such that $\Delta M \subset M \otimes M$. Then $(\mathrm{ad}_M) I$ (resp. $(\mathrm{ad}_M) I$) is an $(\mathrm{ad}_M)$ (resp. $(\mathrm{ad}_M)$) invariant co-ideal of $H$.

We shall be considering the scattering off the right boundary; thus we shall be interested in left co-ideals only. However, all considerations we shall present may be straightforwardly extended for right co-ideals (scattering off the left boundary).

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\(^8\) Note that $t_i^{\pm 1} = t_i^{\mp 1}$. This is due to the consistency with (6).
Quantum symmetric pairs. Let $\theta$ be a non-trivial involution of $\mathfrak{g}$. It defines a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$, where $\mathfrak{g}^\theta$ is the $\theta$-fixed subalgebra of $\mathfrak{g}$. We shall assume that $\theta$ is maximally split with respect to the Cartan algebra, i.e. it satisfies the following three conditions:\(^9\)

(a) $\theta(h) = h$,
(b) if $\theta(h_i) = h_i$, then $\theta(e_i) = e_i$ and $\theta(f_i) = f_i$,
(c) if $\theta(h_i) \neq h_i$, then $\theta(e_i) \in n^-$ and $\theta(f_i) \in n^+$.

Quantum symmetric pairs are the quantum analogs of the pair of enveloping algebras $U_q(\mathfrak{g})$, $U(\mathfrak{g}^\theta)$ and consists of a pair of algebras $U_q(\mathfrak{g})$ and $\mathcal{B}$, where that $\mathcal{B}$ is a left co-ideal subalgebra

$$\Delta \mathcal{B} \subset U_q(\mathfrak{g}) \otimes \mathcal{B}. \quad (25)$$

The pair $(\mathcal{B}, U_q(\mathfrak{g}))$ is then called a quantum symmetric pair if $\mathcal{B}$ is the unique\(^{10}\) maximal co-ideal subalgebra which specialized to $U(\mathfrak{g}^\theta)$ as $q \to 1$.

Construction of $\mathcal{B}$. The explicit construction of $\mathcal{B}$ is given in [26]. It is formulated in terms of the root vectors of $\mathfrak{g}$.

The involution $\theta$ of $\mathfrak{g}$ has an associated automorphism $\Theta$ on the root system $\Phi$. Let $\pi_{\Theta} = \{ \Theta(\alpha_i) = \alpha_i \mid \alpha_i \in \pi \} = \Theta(\pi) \cap \pi$, then (24) tells that $\Theta(-\alpha_i) \in \Phi^+$ for all $\alpha_i \notin \pi_{\Theta}$.

More precisely,

$$\Theta(-\alpha_i) = \sum_{n \in \mathbb{N}} n \alpha_j + Q^+(\pi_{\Theta}), \quad \text{for} \quad \forall \alpha_i \notin \pi_{\Theta}. \quad (26)$$

This implies that there exists a permutation $p$ on the set $\{ i \mid \alpha_i \in \pi \setminus \pi_{\Theta} \}$ such that $\forall \alpha_i \in \pi \setminus \pi_{\Theta} \Rightarrow \Theta(-\alpha_i) - \alpha_{p(i)} \in Q^+(\pi_{\Theta})$. Let $\pi^\ast$ be a maximal subset of $\pi \setminus \pi_{\Theta}$ such that $\alpha_i \in \pi^\ast$ if $i = p(i)$ or $\alpha_{p(i)} \notin \pi^\ast$. Then for given $i$, such that $\alpha_i \in \pi^\ast$, there exists a sequence $\{ \alpha_i, \ldots, \alpha_i \}_1 \notin \pi_{\Theta}$ and a set of positive integers $m_1, \ldots, m_i$, such that the involution $\theta$ may be lifted to the quantum involution $\tilde{\theta}$ of $U_q(\mathfrak{g})$ subject to the following properties:

(a) $\tilde{\theta}(q) = q^{-1}$,
(b) $\tilde{\theta}(\tau(\lambda)) = \tau(-\Theta(\lambda))$ for all $\tau(\lambda) \in T$,
(c) $\tilde{\theta}(x_i) = x_i$ and $\tilde{\theta}(y_i) = y_i$ for all $\alpha_i \in \pi_{\Theta}$,
(d) $\tilde{\theta}(x_i) = (\text{ad}_{x_i^{(m_i)}} \cdots x_i^{(m_1)}) x_{p(i)}$ and $\tilde{\theta}(y_{p(i)}) = (-1)^{m(p(i))} (\text{ad}_{y_{p(i)}^{(m_1)}} \cdots y_{p(i)}^{(m_1)}) y_i$ for all $\alpha_i \in \pi^\ast$,
(e) $\tilde{\theta}(x_i) = (\text{ad}_{x_i^{(m_1)}} \cdots y_i^{(m_1)}) y_{p(i)}$ and $\tilde{\theta}(y_{p(i)}) = (-1)^{m(p(i))} (\text{ad}_{y_{p(i)}^{(m_1)}} \cdots x_i^{(m_1)}) y_i$ for all $\alpha_i \in \pi^\ast$. \quad (27)

Here, $x_i = t_i x_i$ and $x_i^{(m)} = x_i^m/[m]_{q_i}$, and $y_i^{(m)} = y_i^m/[m]_{q_i}$, where $q_i = q^{\Theta(\alpha_i)/2}$.

Next, define a set $\mathcal{D} = \{ \alpha_i \in \pi^\ast \mid i \neq p(i) \text{ and } (\alpha_i, \Theta(\alpha_i)) \neq 0 \}$ and let $\mathcal{M}$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by $\{ x_i, y_i, t_i^{-1} \mid \alpha_i \in \pi_{\Theta} \}$. This setting allows us to define

\(^9\) If $\theta$ is not maximally split it is often possible to replace $\theta$ by a conjugate $\theta' = \psi \theta \psi^{-1}$ which is maximally split; here, $\psi$ is an automorphism of $\mathfrak{g}$. An example, let $\mathfrak{g} = \mathfrak{sl}(2)$. Then there is only one non-trivial involution $\theta(h) = h$, $\theta(e) = -e$, $\theta(f) = f$ leading to $\mathfrak{g}^\theta = [h]$. However, it is isomorphic to a maximally split involution $\theta'(h) = -h$, $\theta'(e) = -f$, $\theta'(f) = -e$ with $\mathfrak{g}^{\theta'} = [e - f]$.

\(^{10}\) In the sense of [26].
the co-ideal subalgebra $B$ of $U_q(g)$ generated by $M$, $T_0 = \{ \tau(\lambda) | \Theta(\lambda) = \lambda \}$ and the set of twisted charges $B = \{B_{\alpha,i}, \ \alpha \in \pi \setminus \pi_0\}$, defined by

$$B_{\alpha,i} = y t_i^{-1} + d_{\alpha,i} \theta(y_i) t_i^{-1} \quad \text{and} \quad B_{\alpha,i} = t_i^{-1} x_i^\prime + d_{\alpha,i} t_i^{-1} \tilde{\theta}(x_i).$$

(28)

with $d_{\alpha,i}, d_{\alpha,i} = 1$ for $\alpha \notin \mathcal{D}$ and $d_{\alpha,i}, d_{\alpha,i} \in \mathbb{C}$ otherwise. Note that $B_{\alpha,i}$ may be obtained from $B_{\beta,i}$ with the help of the anti-automorphism $\kappa$ of $U_q(g)$, which is defined as $\kappa(x_i) = y t_i^{-1}$, $\kappa(y_i) = t_i x_i$, $\kappa(t_i^{\pm 1}) = t_i^{\pm 1}$, and also if $a \in \mathbb{C}$ and $\bar{a}$ is a complex conjugate of $a$, then $\kappa(\bar{a} u) = \bar{a} u$ for $\forall u \in U_q(g)$, i.e. $\kappa$ is a conjugate linear map and gives $U_q(g)$ a structure of $H$-algebra. Then it is easy to see that $\kappa(B_{\alpha,i}) = B_{\alpha,i}$,

$$\kappa(t_i^{-1} x_i^\prime) = y t_i^{-1} \quad \text{and} \quad \kappa(d_{\alpha,i} t_i^{-1} \tilde{\theta}(x_i)) = \tilde{d}_{\alpha,i} t_i^{-1} \tilde{\theta}(x_i).$$

(29)

Furthermore, it implies that

$$d_{\alpha,i} = \tilde{d}_{\alpha,i}.$$  

(30)

3.3. Construction of the co-ideal subalgebra

Having all the algebraic structures presented we are ready to explicitly construct the quantum affine co-ideal subalgebra $\hat{B}$ of $\hat{Q}$ by generalizing the results derived in the previous section. Let us start from inspecting the charges of $\hat{Q}$. It has eight regular supercharges, namely

$$F_2, \ F_{21}, \ F_{32}, \ F_{321} \ \text{and} \ E_2, \ E_{21}, \ E_{32}, \ E_{321},$$

(31)

where we have used a shorthand notation $F_{ijk} = [F_i, [F_j, F_k]]$ and the same for $E_{ijk}$. By replacing $F_2 \rightarrow F_4$ and $E_2 \rightarrow E_4$ eight affine supercharges are obtained,

$$F_4, \ F_{41}, \ F_{43}, \ F_{341} \ \text{and} \ E_4, \ E_{41}, \ E_{34}, \ E_{341}.$$  

(32)

The replacement of the same type applied to $\hat{5}$ produces affine partners of the central charges, $\hat{C}_1, \hat{C}_2$ and $\hat{C}_3$. And finally, the affine partners of $\hat{F}_1$ and $\hat{E}_1$ and $\hat{F}_4$ and $\hat{E}_4$ are

$$\hat{F}_1 = E_{342}, \ \hat{E}_1 = F_{342} \ \text{and} \ \hat{F}_3 = E_{421}, \ \hat{E}_3 = F_{421}.$$  

(33)

The boundary we are considering does not respect bosonic symmetries $\hat{E}_1$ and $\hat{F}_1$, central charges $\hat{C}_2$ and $\hat{C}_3$ and affine charges $\hat{E}_4$ and $\hat{F}_4$ (let us name these charges as the broken, while the rest will be named as preserved); thus it breaks exactly half of the supercharges (31) and (32) with the broken regular supercharges being

$$E_{21}, \ E_{321} \ \text{and} \ F_{21}, \ F_{321}. $$

(34)

In other words, we consider the involution that simply acts like

$$\theta(X) = X, \quad \forall X \in \{E_{2}, E_{3}, F_{2}, F_{3}, K_{1}, K_{2}, K_{3}, K_{4}, C_{1}\}$$

$$\theta(X) = -X, \quad \forall X \in \{E_{1}, E_{4}, F_{4}, C_{2}, C_{3}\}.$$  

(35)

In the $q \rightarrow 1$ limit, this clearly gives rise to the symmetric pair

$$g^0 = \{\hat{5}_1, \ \hat{E}_2, \ \hat{F}_2, \ \hat{5}_2, \ \hat{E}_3, \ \hat{F}_3, \ \hat{5}_3, \ \hat{E}_4, \ \hat{F}_4, \ \hat{C}_1\},$$

$$g \setminus g^0 = \{E_1, \ \hat{5}_1, \ \hat{F}_2, \ \hat{E}_3\}.$$  

(36)

of the centrally extended $\mathfrak{sl}(2|2)$ algebra (see appendix A for details on the $q \rightarrow 1$ limit).

Furthermore, it is easy to see that $\theta$ is isomorphic to a maximally split symmetric pair. Thus the construction presented in section 3.2 and the relation to the algebraic structures of the $Y = 0$ giant graviton imply that for each broken regular charge, the algebra $\hat{B}$ must possess a corresponding twisted affine charge satisfying co-ideal property (25). We shall denote these charges as

$$B = \{\hat{F}_1, \ \hat{F}_{21}, \ \hat{F}_{321}, \ \hat{E}_1, \ \hat{E}_{21}, \ \hat{E}_{321}, \ \hat{C}_2, \ \hat{C}_3\}.$$  

(37)
**Co-ideal subalgebra.** The set of positive simple roots of $\hat{Q}$ is $\pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The boundary conditions imply that the corresponding root space automorphism $\Theta$ (26) acts on the simple roots as

$$
\Theta(\alpha_2) = \alpha_2, \quad \Theta(\alpha_1) = -\alpha_2 - \alpha_3 - \alpha_4,
$$

$$
\Theta(\alpha_3) = \alpha_3, \quad \Theta(\alpha_4) = -\alpha_1 - \alpha_2 - \alpha_3.
$$

(38)

Thus, $\pi_\Theta = \{\alpha_2, \alpha_3\}$ and it gives rise to a subalgebra $\mathcal{M}$ of $\hat{Q}$. Note that $\alpha_4 = \delta - \tilde{\theta}$ is the affine root, where $\tilde{\theta} = \alpha_1 + \alpha_2 + \alpha_3$ is the highest root of the non-affine algebra $Q$. However, we are interested in the finite-dimensional representations which are constructed by dropping all imaginary roots, thus giving the constraint $K_1 K_2 K_4 = 1$ [9, 14].

We shall build $\tilde{B}$ based on the affine extension; hence, we set $\pi^* = \{\alpha_4\}$. This fixes the permutation map $p$ to act as $p(4) = 1$. Next, with the help of (28) and (38), we define the twisted affine charges to be\(^{11}\)

$$
\tilde{E}_{321} = F_4 K_4^{-1} + d_4 \hat{\theta}(F_4) K_4^{-1}, \quad \hat{\theta}(F_4) = (\text{ad}, E_3\text{ad}, E_2) E'_4,
$$

(39)

$$
\tilde{F}_{321} = E'_4 K_4^{-1} + d_4 \hat{\theta}(E'_4) K_4^{-1}, \quad \hat{\theta}(E'_4) = (\text{ad}, F_3\text{ad}, F_2) F_1.
$$

(40)

Then with the help of the right adjoint action $\text{ad}_R\mathcal{M}$, we construct the rest of the twisted affine charges

$$
\tilde{E}_{21} = (\text{ad}, F_3) \tilde{E}_{321}, \quad \tilde{F}_{21} = (\text{ad}, E_3) \tilde{F}_{321},
$$

(41)

$$
\tilde{E}_1 = (\text{ad}, F_2\text{ad}, F_3) \tilde{E}_{321}, \quad \tilde{F}_1 = (\text{ad}, E_2\text{ad}, E_3) \tilde{F}_{321},
$$

(42)

$$
\tilde{E}_2 = (\text{ad}, E_2) \tilde{E}_{321}, \quad \tilde{F}_2 = (\text{ad}, F_2) \tilde{F}_{321}.
$$

(43)

Let us show the co-ideal property for these charges explicitly. However, it is enough to show this property for the charges (39) and (40) only.

$$
\Delta \tilde{E}_{321} = F_4 K_4^{-1} \otimes 1 + U K_4^{-1} \otimes \tilde{E}_{321} + d_4 \hat{\theta}(F_4) K_4^{-1} \otimes K_5
$$

$$
+ d_4 (q^2 - 1) \left(q^{-1} K_4^{-1} \otimes \text{ad}_R\mathcal{M} E_4' \otimes K_5 E_3 - U E_1' K_4^{-1} \otimes K_1 K_4^{-1} \text{ad}_R\mathcal{M} E_2'\right)
$$

$$
\in \hat{Q} \otimes \tilde{B},
$$

(44)

and

$$
\Delta \tilde{F}_{321} = E'_4 K_4^{-1} \otimes 1 + U^{-1} K_4^{-1} \otimes \tilde{F}_{321} + d_4 \hat{\theta}(E'_4) K_4^{-1} \otimes K_5
$$

$$
- d_4 (q^2 - 1) \left(K_4^{-1} \otimes \text{ad}_R\mathcal{M} F_1 \otimes K_3^{-1} K_4^{-1} F_3 - U^{-1} K_1 K_4^{-1} F_1 \otimes \text{ad}_R\mathcal{M} F_2 K_1 K_4^{-1}\right)
$$

$$
\in \hat{Q} \otimes \tilde{B}.
$$

(45)

Here, $K_5 = K_1 K_2 K_4^{-1}$ and the co-ideal property is satisfied provided $K_1 K_4^{-1} \in T_\Theta$. This is easy to see, because the linear combination $\alpha_1 - \alpha_4$ is invariant under the automorphism (38) as $\Theta(\alpha_1 - \alpha_4) = \alpha_1 - \alpha_4$. The co-ideal property for the rest of the charges, (41), (42) and (43), is obvious since $\tilde{B}$ is invariant under the adjoint action of $\mathcal{M}$.

**Reflection algebra.** In order to build the boundary scattering theory we need to have representations of $\hat{Q}$ corresponding to incoming and reflected states. We shall be bearing on the reflection Hopf algebra constructed in [38].

Let the representation defined in section 2 describe incoming states carrying momentum $p$. It is related to the deformation parameter as $U = e^{i\theta}$. Then the representation corresponding to the reflected states with momentum $-p$ shall have deformation parameter equal to $e^{-i\theta} = U^{-1}$.

\(^{11}\) Equivalently, one could choose $\pi^* = \{\alpha_1\}$ as a starting point giving $\tilde{E}_1 = E'_1 K_1^{-1} + d_1 \hat{\theta}(E'_1) K_1^{-1}$ and $\tilde{F}_1 = F_1 K_1^{-1} + d_1 \hat{\theta}(F_1) K_1^{-1}$, where $\hat{\theta}(E'_1) = (\text{ad}, F_2 \text{ad}, F_3) F_4$ and $\hat{\theta}(F_1) = (\text{ad}, E_2 \text{ad}, E_3) E'_4$. 
Next, the total fermion and boson number conservation together with the energy conservation constrains central elements $V$ and $K_j$ to be invariant under the reflection.

This implies that there is a reflection automorphism $\kappa$ of the algebra defined as

$$\kappa: (V, U) \mapsto (V, U) \quad \text{and} \quad \kappa: (E_j, F_j, K_j) \mapsto (E_j, F_j, K_j).$$

where the underlined charges describe the representation of reflected states and the constraints

$$U = U^{-1}, \quad V = V, \quad K_j = K_j$$

(46) define the representation uniquely. The representation labels $a, b, c$ and $d$ associated with the charges $E_j$ and $F_j$ may be obtained from (10) by replacing $U \mapsto U^{-1}$ and similarly for the affine ones. Then, we can express the labels of the reflected charges in terms of the initial ones as

$$a = \frac{\gamma}{\gamma}a, \quad b = \frac{\gamma}{\gamma}a^2 cdV^2, \quad c = \frac{\gamma}{\gamma}abV^{-2}, \quad d = \frac{\gamma}{\gamma}d,$$  

(48)
giving

$$a = \sqrt{\frac{g}{|M|}}a, \quad b = \sqrt{\frac{g}{|M|}}\gamma g^2(x^+ - x^-), \quad c = \sqrt{\frac{g}{|M|}}g^2(1 + \xi x^+)(\xi + x^+), \quad d = \sqrt{\frac{g}{|M|}}g^2gV x^+ - x^-.$$  

(49)
The extension to the affine case is obvious. Here, we have chosen $a = \frac{\gamma}{\gamma}a$ as an initial constraint with $\gamma$ being the reflected version of $\gamma$, i.e. $\kappa(\gamma) = \gamma$. By comparing (49) with (12), we find the reflection map for the $x^\pm$ parameterization to be

$$\kappa : x^\pm \mapsto -x^\mp + \xi.$$  

(50)

It is in agreement with the one conjectured in [19], in the $q \rightarrow 1$ limit, this map reduces to the usual reflection map $\kappa : x^\pm \mapsto -x^\mp$.

Let us also introduce the reflected co-products of $E_j$ and $F_j$ associated with the reflection Hopf algebra [38]. They are

$$\Delta \text{ref}(E_j) = E_j \otimes 1 + K_j^{-1}U^{-d_j,1} \otimes E_j, \quad \Delta \text{ref}(F_j) = F_j \otimes K_j + U^{d_j,1} \otimes F_j.$$  

(51)

These shall play an important role in finding the explicit form of the reflection matrix.

The expressions in (48) may be casted in a matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} D = T \begin{pmatrix} a & b \\ c & d \end{pmatrix} T^{-1} \quad \text{with} \quad D = \begin{pmatrix} \gamma / \gamma & 0 \\ 0 & \gamma / \gamma \end{pmatrix}, \quad T = \begin{pmatrix} U^{-2} & 0 \\ 0 & -z \end{pmatrix}.$$  

(52)

revealing the explicit relation between two isomorphic representations of $\hat{\mathfrak{g}}$. Here, $\gamma$ and $\gamma'$ are unconstrained parameters defining the representations of incoming and reflected states. Thus the matrix $D$ may be understood as a matrix relating two different bases, while the matrix $T$ expresses the reflection automorphism of the algebra. Indeed, physical intuition tells that the representations of incoming and reflected states should be related via the spectral and deformation parameters only.

Finally, we want to perform some checks of our constructions. Firstly, the twisted affine central charges $\tilde{C}_2$ and $\tilde{C}_3$ (43) must be conserved under the reflection. Thus requiring $\tilde{C}_2 = \tilde{C}_2$ and $\tilde{C}_3 = \tilde{C}_3$, we find

$$d_j = \frac{g}{\gamma a} \frac{g}{\gamma a} \frac{g}{\gamma a} \quad \text{and} \quad d_j = -a a \frac{g}{\gamma a} \frac{g}{\gamma a}.$$  

(53)

The authors of [19] are using the $x^\pm$ parametrization of [8], while we use the one of [9]. The map between these two is $x_{\text{ref}}^\pm = g g^{-1}(x_{\text{HOM}} + \xi)$. 

12
Let us make a direct link to the constraint (30) arising from the Hopf $*$-algebra. Requiring $\tilde{g}/g$ to be real, we find $(\alpha\tilde{\alpha})^2 = -1$ having a solution $\alpha = 1$ and $\alpha = i$, which corresponds to the usual setting of unitary representations.

Secondly, the spectral parameter $z$ associated with the algebra is required to transform as $\kappa : z \mapsto z^{-1}$ under the reflection map. This is indeed true and follows straightforwardly when applying map $\kappa$ to (16).

**Yangian limit.** The algebra $\hat{Q}$ in the $q \to 1$ limit has no singular elements and the naive $q \to 1$ limit leads to the undeformed universal enveloping algebra. The relation to the associated Yangian algebra was explicitly shown in [9] by considering the specific combinations of charges of $\hat{Q}$ that are singular in the $q \to 1$ limit. The construction presented in [9] is very closely related to the so-called Drinfeldian [39]. However, the twisted affine charges ((39)–(43)) are already of the required form. Thus the algebra $\hat{B}$ in the rational $q \to 1$ limit is isomorphic to the associated twisted Yangian (B.3) proposed by [21, 22]. The explicit relations between the quantum affine and Yangian charges are

$$
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} E_{321}}{2(q-1)} = -\tilde{\delta}_{321} - \frac{g}{\alpha} \delta_2,
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} E_{231}}{2(q-1)} = \tilde{\delta}_{231} + \frac{g}{\alpha} \delta_2,
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} E_{321}}{2(q-1)} = -\tilde{\delta}_{231} - \frac{g}{\alpha} \delta_2,
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} E_{12}}{2(q-1)} = \tilde{\delta}_1,
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} \tilde{C}_{1}}{2(q-1)} = -\tilde{\delta}_1 - \frac{g}{\alpha} \delta_2.
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} \tilde{C}_{2}}{2(q-1)} = -\tilde{\delta}_2 + \frac{g}{\alpha} \delta_2,
\lim_{q \to 1} \frac{\alpha \tilde{\alpha} \tilde{C}_{3}}{2(q-1)} = \tilde{\delta}_2 - \frac{g}{\alpha} \delta_2.
$$

We have checked that these relations hold at both algebra and co-algebra levels. Note that the contributions of the extended central charges for the twisted Yangian generators are also recovered in the above limit (see also appendix B).

**4. Boundary scattering**

In this section, we consider the boundary scattering theory for the deformed Hubbard model and find the explicit form of the bound state reflection matrix. Moreover, we explicitly solve the reflection equation and show that the reflection matrix $K$ is indeed invariant under the co-ideal subalgebra $\hat{B}$.

**Reflection matrix.** The boundary we are considering is a singlet with respect to the boundary algebra $\hat{B}$; thus it may be represented via the boundary vacuum state $|0\rangle_B$. It is annihilated by all charges of it, with the exception of the generators $K_i$, which actually keep the boundary invariant,

$$
K_i |0\rangle_B = d_i |0\rangle_B = |0\rangle_B.
$$

We define the reflection matrix to be the intertwining matrix

$$
\mathcal{K} |m, n, k, l\rangle \otimes |0\rangle_B = K_{(m,n,k,l)}^{(a,b,c,d)} |a, b, c, d\rangle \otimes |0\rangle_B.
$$
The space of states $|m,n,k,l\rangle$ is $4M$ dimensional and can be decomposed into four $4M = (M+1) + (M-1) + M + M$ subspaces that have the orthogonal basis

\begin{align*}
|k\rangle^1 &= |0,0,k,M-k\rangle, \quad k = 0, \ldots, M, \\
|k\rangle^2 &= |1,1,k-1,M-k-1\rangle, \quad k = 1, \ldots, M-1, \\
|k\rangle^3 &= |1,0,k,M-k-1\rangle, \quad k = 0, \ldots, M-1, \\
|k\rangle^4 &= |0,1,k,M-k-1\rangle, \quad k = 0, \ldots, M-1.
\end{align*}

Symmetry constraints. The reflection matrix (56) is required to be invariant under the co-products of the boundary algebra [22]

$$\mathcal{K} \Delta(J) - \Delta^{\text{ref}}(J) \mathcal{K} = 0, \quad \forall J \in \hat{B}. \quad (58)$$

The form of the reflection matrix is constrained by the bosonic charges $E_3$ and $F_3$ to five independent sets of coefficients

\begin{align*}
\mathcal{K} |k\rangle^1 &= A_k |k\rangle^1 + D_k |k\rangle^2, \\
\mathcal{K} |k\rangle^2 &= B_k |k\rangle^2 + E_k |k\rangle^1, \\
\mathcal{K} |k\rangle^\alpha &= C_k |k\rangle^\alpha,
\end{align*}

where $\alpha = 3,4$ and we have dropped the boundary vacuum state. We note that the basis (57) was chosen in such a way that the reflection matrix would act diagonally on the quantum number $k$. Also we are working in an orthogonal, but not orthonormal basis in order to avoid having normalization factors appearing in explicit expressions. However, switching to the orthonormal basis is rather easy and requires only extra factors of $([k]|M-k|)^2$ and $([k]|M-k|)^{-2}$ to be added to $D_k$ and $E_k$, respectively.

We start by determining the limiting conditions—the constraints for reflection coefficients $A_0$, $D_0$ and $C_0$ and $A_M$, $D_M$ and $C_M$. This can be achieved by considering the reflection of the lowest state $|0\rangle^1$:

$$\mathcal{K} |0\rangle^1 = A_0 |0\rangle^1, \quad \text{and thus} \quad D_0 = 0. \quad (60)$$

Then the invariance condition (58) for the charge $E_2$,

$$(\mathcal{K} E_2 - E_2 \mathcal{K}) |0\rangle^1 = 0, \quad \text{gives} \quad C_0 = \frac{a}{\alpha} A_0 = \frac{\gamma}{\gamma'} A_0. \quad (61)$$

We choose the overall normalization to be $A_0 = 1$. The same constraint may be found by considering the reflection of states $|0\rangle^\alpha$ and the charge $F_2$. Similar considerations for the highest state $|M\rangle^1$ give

$$D_M = 0 \quad \text{and} \quad A_M = \frac{c}{\xi} C_{M-1} = -\frac{\gamma}{\gamma' zU^2} C_{M-1}. \quad (62)$$

Next we turn to the states $|k\rangle^\alpha$ as they scatter from the boundary diagonally. The twisted affine charge $\tilde{F}_1$ acts on these states as a raising operator

$$\tilde{F}_1 |k\rangle^\alpha = f_k(z) |k+1\rangle^\alpha, \quad \tilde{F}_1 |k\rangle^\alpha = f_k(1/z) |k+1\rangle^\alpha,$$

with $f_k(z) = d_k [M-k-1] q^{-M/2-k-1} (q^M - q^{2k+2} z V^{-1})$. The invariance condition then straightforwardly gives

$$C_{k+1} f_k(z) - f_k(1/z) C_k = 0, \quad (64)$$
leading to the iterative relation
\[ C_k = \frac{f_{k-1}(1/z)}{f_k(z)} C_{k-1} = \frac{q^M - q^{2k}/z}{q^M - q^{2z}} C_{k-1}. \] (65)
This relation is then simply solved by
\[ C_k = C_0 \prod_{n=1}^{k} \frac{q^M - q^{2n}/z}{q^M - q^{2n}z}. \] (66)

The coefficients \( C_k \) are (anti)symmetric up to a factor of \( z \) under the interchange \( k \to M-k-1 \) for \( M \) being (even)odd,
\[ z^k C_k = -z^{M-k-1} C_{M-k-1} \quad \text{for } M = \text{even} \quad \text{and} \quad k = 0, \ldots, M/2 - 1, \]
\[ z^k C_k = z^{M-k-1} C_{M-k-1} \quad \text{for } M = \text{odd} \quad \text{and} \quad k = 0, \ldots, (M-1)/2 - 1. \] (67)

This symmetry comes from the requirement that the reflection is covariant under the renaming of bosonic indices \( 1 \leftrightarrow 2 \) as the reflection is of a diagonal type for the states \( |k\rangle \). However, this is not the case for the states \( |k\rangle^{1,2} \), and thus there is no such symmetry for the rest of the reflection coefficients. The factors of \( z \) in (67) arise due to the non-commutative nature of the model. In the \( q \to 1 \) limit, this (anti)covariance specializes to (anti)symmetry for \( M \) being (even)odd, as observed in [28].

The remaining reflection coefficients, as we shall show, will be expressed in terms of \( C_k \) and \( C_{k-1} \). Requiring the reflection matrix to be invariant under the charges \( E_2 \) and \( E_3 \) on the bosonic states \( |k\rangle^{1,2} \), we obtain the following set of separable equations:
\[
\begin{align*}
D_j b - [M - k]_q (C_j a - A_j d) &= 0, & C_j b - [M - k]_q E_j a - B_j b &= 0, \\
D_j d + [k]_q (C_{k-1} c - A_k c) &= 0, & C_{k-1} d + [k]_q E_k c - B_k d &= 0,
\end{align*}
\] (68)
with the unique solution
\[
\begin{align*}
A_k &= (C_{k-1} [k]_q b c + C_k [M - k]_q a d) / N, & D_k &= [k]_q [M - k]_q (C_k a - C_{k-1} c) / N, \\
B_k &= (C_k [k]_q b c + C_{k-1} [M - k]_q a d) / N, & E_k &= (C_k b d - C_{k-1} b d) / N,
\end{align*}
\] (69)
where the normalization factor \( N \) is
\[ N = [k]_q d c + [M - k]_q a d = V q^{M/2-k} - V^{-1} q^{-M/2+k} / (q - q^{-1}). \] (70)

Writing the coefficients explicitly in terms of \( x^\pm \) parametrization, we then finally obtain
\[
\begin{align*}
A_k &= y g^{W} (x^- - x^+) (g^{2} [k]_q C_{k-1} - g^{2} [M - k]_q C_k (\xi + x^+)^2) V / \gamma g^{2} [M]_q (\xi + x^+) (1 + \xi x^+) N, \\
B_k &= y g^{W} (x^- - x^+) (g^{2} [M - k]_q C_{k-1} (\xi + x^-)^2 - g^{2} [M]_q C_k (1 + \xi x^-)^2) / \gamma g^{2} [M]_q (\xi + x^-) (1 + \xi x^-) V N, \\
D_k &= y g^{W} [k]_q [M - k]_q (g^{2} C_{k-1} (\xi - x^-)^2 + g^{2} C_k (1 + \xi x^-) (\xi + x^+)) / \gamma g^{2} [M]_q (\xi + x^-) (1 + \xi x^-) V N, \\
E_k &= y g^{W} (x^- - x^+) (g^{2} C_{k-1} (\xi + x^-)^2 + g^{2} C_k (\xi + x^+) (1 + \xi x^-)) / \gamma g^{2} [M]_q (\xi + x^-) (1 + \xi x^-) V N.
\end{align*}
\] (71)

**Unitarity.** The reflection matrix satisfies the unitarity constraint
\[ \| \zeta(p) \| \zeta(-p) = 1. \] (72)
Rational limit. In the $q \to 1$ limit, the reflection coefficients get reduced to
\[
A_k = \frac{\gamma x^-}{\gamma x^+ N}((M-k)C_k(x^+)^2 - kC_k), \quad B_k = \frac{\gamma x^+}{\gamma x^- N}((M-k)C_k(kx^-)^2 - kC_k),
\]
\[
D_k = \frac{\gamma \gamma}{\alpha} k(M-k)(C_kx^+ + C_{k-1}x^-), \quad E_k = \frac{\alpha x^- - x^+}{\gamma \gamma} (C_kx^+ + C_{k-1}x^-), \quad (73)
\]
where the coefficients $C_k$ and the normalization factor $N$ are given by
\[
C_k = \frac{2\mathrm{ig}u - M + 2k}{-2\mathrm{ig}u - M + 2k} C_{k-1}, \quad N = k + (M-k)x^- x^+.
\]
By choosing $\gamma = \sqrt{1(x^- - x^+)}$ and $\upsilon = \gamma$, these are in agreement with the ones found in [28].

Fundamental representation. In this case, $M = 1$ and the state $|k\rangle^2$ is absent; thus, the reflection matrix is purely diagonal. The charges $E_2$ and $E_3$ constrain the reflection coefficients to be
\[
A_0 = -\frac{\alpha}{\alpha} C_0 = \gamma C_0, \quad (75)
\]
\[
A_1 = -\frac{\gamma}{\zeta} C_0 = -\frac{\gamma}{\zeta} \frac{\gamma}{\zeta} x^- C_0. \quad (76)
\]
Again, choosing the normalization to be $A_0 = 1$, this is in agreement with [19] and with [20] in the rational limit.

Reflection equation. In order to show the integrability of the model, we have to show that the reflection matrix is a solution of the reflection equation (boundary Yang–Baxter equation). In fact, we shall explicitly derive the coefficient $C_k$ by solving the reflection equation. The unique solution we find agrees perfectly with the coefficients that are derived from the symmetry considerations. This explicitly proves that $K$ respects the co-ideal subalgebra $\hat{B}$.

Consider two states with bound state numbers $z_1$ and $z_2$. Let us denote $K_\alpha = K_{M}(z_\alpha)$ and $S_{ij} = S_{MM}(z_\alpha, z_\beta)$ and also let the underscored index indicate that the corresponding representation is reflected. Then the reflection equation is given by
\[
|K_{2} \Sigma_{21} K_{1} \Sigma_{12} = S_{21} K_{1} \Sigma_{12} K_{2}, \quad (77)
\]
which when explicitly written out in components reads as
\[
K^\alpha_{\beta}(z_2) S^{\gamma \delta}_{\mu \nu}(z_2, z_1^{-1}) K^\gamma_{\delta}(z_1) S^{\phi \Psi}_{\mu \nu}(z_1, z_2) = S^{\phi \Psi}_{\mu \nu}(z_2^{-1}, z_1^{-1}) K^\phi_{\Psi}(z_2) S^{\gamma \delta}_{\mu \nu}(z_1, z_2^{-1}) K^\gamma_{\delta}(z_1), \quad (78)
\]
where we have used Roman and Greek letters to distinguish indices of the first and second states, respectively.

Let us first consider states of the form $|k_1\rangle^\alpha \otimes |k_2\rangle^\alpha$ because the reflection matrix acts diagonally on these states. This corresponds to the subspace I case in terms of the analysis performed in [14]. Then the reflection equation becomes
\[
\sum_{n=0}^{k_1+k_2} C_m(z_1) \mathcal{F}_{m}^{K-n,n}(z_2, z_1^{-1}) C_n(z_2) \mathcal{F}_{n}^{k_1,k_2}(z_1, z_2)
\]
\[
= \sum_{n=0}^{k_1+k_2} \mathcal{F}_{m}^{n,K-n}(z_2^{-1}, z_1^{-1}) C_m(z_1) \mathcal{F}_{n}^{k_1,k_2}(z_1, z_2^{-1}) C_k(z_2). \quad (79)
\]

13 Here we have rescaled the normalization factor as $N(J_{0}) \to N(J_{0}/(x^+ x^- - 1))$ in the $q \to 1$ limit.
14 Up to some factors due to a different choice of the basis (57) with respect to the one in [28].
We will now proceed with the derivation of $C_k$. For $k_1 = k_2 = 0$, we easily find that the reflection equation is satisfied. Next we consider the state where $k_1 = 1$ and $k_2 = 0$. In this case, the reflection equation is satisfied provided that $C_1$ satisfies the following relation:

$$C_1(z_1) = C_0(z_1) \left[ 1 - \frac{z_1^2 - 1}{z_1 \left( \frac{q^{M_1} - q^{M_2} C_1(z_2) - G_1(z_2)}{z_1 C_1(z_2) - G_1(z_2)} \right) + z_1} \right].$$

The right-hand side is allowed to depend solely on $z_1$; thus there are two solutions, a trivial one $C_1 = C_0$ and

$$C_1 = \frac{z_{-1} - A q^M}{z - A q^M} C_0.$$

The latter solution has an undetermined constant $A$. This coefficient may be determined by either considering the rational limit, or by studying the reflection equation involving states from the subspace II of [14]. Both arguments lead to $A = q^{-2}$. Finally, by studying a state with $k_1 = 2$ and $k_2 = 0$, we can solve for $C_2$ and so on. This leads to the following solution:

$$C_k = C_0 \prod_{i=1}^{k} \frac{q_i^M - q_i^{2i}/z}{q_i^M - q_i^{-2i}/z},$$

which perfectly agrees with (66). As expected, the trivial solution does not solve the reflection equation in the general case.

5. Discussion

In this work, we have considered open boundary conditions for the deformed Hubbard model of the same type as for the $Y = 0$ giant graviton in the AdS/CFT correspondence. In this situation, exactly half of the supersymmetries are broken. The symmetry algebra compatible with these boundary conditions is a twisted co-ideal quantum affine algebra $\hat{B}$ of the quantum affine deformed algebra $\hat{Q}$.

Inspired by results obtained for semi-simple Lie algebras [24–27], we provide an explicit construction of $\hat{B}$ that is of a co-ideal form. We then find the corresponding reflection matrix for arbitrary bound states and show that it satisfies the reflection equation (boundary Yang–Baxter equation). Conversely, we explicitly solve the reflection equation and find that our reflection matrix corresponds to the unique solution compatible with the boundary conditions. This proves that the reflection matrix indeed respects $\hat{B}$ as a symmetry algebra. However, the boundary algebra defines the reflection matrix up to the overall dressing phase only. This phase may be obtained by considering the crossing equation (see e.g. [20]), although it requires the dressing phase of the bulk $S$-matrix which at the moment is also not known.

Finally, we show that the twisted quantum affine algebra and the reflection matrix we have found in the rational $q \to 1$ limit specialize exactly to the twisted Yangian of the $Y = 0$ giant graviton and the reflection matrix associated with it, which for the reflection of arbitrary bound states was found by L Palla [28]. Furthermore, we have casted the twisted Yangian of the $Y = 0$ giant graviton in a very compact form (B.3) with the help of the outer automorphism $U$ of the extended $su(2|2)$ algebra. However, the explicit form (54) of the rational $q \to 1$ limit of $\hat{B}$ was somewhat surprising. Bearing on the Yangian limit of $\hat{Q}$ found in [9], we were expecting to obtain the twisted secret charges constructed in [40]. However, it turned out not to be the case; thus the role of the twisted secret charges of [40] remains unknown.
Due to the high complexity of the bulk $S$-matrix we were unable to check the reflection equation in complete generality; however, we did check a wide number of generic cases numerically and all of them were satisfied.

In this work, we have used the reflection Hopf algebra formalism introduced in [38]; however, we have not stated explicitly the reflection automorphism of the algebra. We have only constructed the map between the representations of incoming and reflected states. This was sufficient for our purpose—finding an explicit form of the reflection matrix. Construction of the reflection automorphism requires a detailed analysis of the outer-automorphism group of $\hat{Q}$ and thus is beyond the scope of this paper. Nevertheless, it is a very important question and deserves to be explored.

This work has revealed one more algebraic structure related to the deformed quantum affine algebra $\hat{Q}$. The natural next step would be to explore boundary scattering for other boundary conditions, most notably the one of the same type as that of the $Z = 0$ giant graviton. It would also be interesting to apply the Bethe ansatz for these systems and derive their transfer matrices [41]. Furthermore, there has recently been a rapid development in the $q$-deformed Pohlmeyer-reduced version of the AdS$_5 \times S^5$ superstring theory [42, 43]; however, the presence of boundaries in the Pohlmeyer reduced theories has not been much investigated. Thus, it would be very interesting to see the effects of boundaries in such theories and find plausible links to the algebraic constructions considered in this work.

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Appendix A. Algebra maps

The $q \to 1$ limit map. Algebra $\hat{Q}$ in the conventional $q \to 1$ limit specializes to the centrally extended $\hat{sl}(2|2)$. Central elements $U$ and $V$ of the algebra in this limit become $U \to U$ and $V \to 1$. The representation labels $(a, b, c, d)$ (12) specialize to the usual non-deformed labels $(a, b, c, d)$ of [12], while the affine ones become

$$(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \to \left(\alpha \tilde{a} c, a \tilde{a} d, -\frac{a}{\alpha \tilde{a}}, -\frac{b}{\alpha \tilde{a}}\right). \tag{A.1}$$

The natural choice of normalization for $\tilde{a}$ is 1. The explicit map between the generators of the algebras is

$$E_i \to \mathcal{E}_i, \quad F_i \to \mathcal{F}_i, \quad H_i \to \mathcal{H}_i \quad \text{for} \quad i = 1, 2, 3, \tag{A.2}$$

and

$$E_4 \to \alpha \mathcal{F}_{321}, \quad F_4 \to -\alpha^{-1} \mathcal{E}_{321}, \quad H_4 \to -\mathcal{H}_1 - \mathcal{H}_2 - \mathcal{H}_3. \tag{A.3}$$

The central charges are being mapped as

$$C_1 \to \mathcal{C}, \quad C_2 \to \mathcal{P}, \quad C_3 \to \mathcal{R}. \tag{A.4}$$
The $\mathfrak{sl}(2|2) \rightarrow \mathfrak{su}(2|2)$ map. The map between these algebras reads as
\begin{align}
\mathcal{E}_1 & \rightarrow L_1^1, & \mathcal{G}_1 & \rightarrow L_1^1, & \mathcal{S}_1 & \rightarrow -2L_1^1, \\
\mathcal{E}_2 & \rightarrow Q^2_2, & \mathcal{G}_2 & \rightarrow G^2_2, & \mathcal{S}_2 & \rightarrow -L_1^1 - R^3_3 + \frac{1}{2}H, \\
\mathcal{E}_3 & \rightarrow R^3_3, & \mathcal{G}_3 & \rightarrow R^3_3, & \mathcal{S}_3 & \rightarrow -2R^3_3.
\end{align}
(A.5)

Their commutators are
\begin{align}
\mathcal{E}_{32} & \rightarrow Q^2_4, & \mathcal{E}_{21} & \rightarrow Q^1_4, & \mathcal{E}_{321} & \rightarrow Q^1_4, \\
\mathcal{G}_{32} & \rightarrow G^2_4, & \mathcal{G}_{12} & \rightarrow G^4_1, & \mathcal{G}_{321} & \rightarrow G^1_1.
\end{align}
(A.6)

Finally, the central charges are being mapped as
\begin{align}
\mathcal{C} & \rightarrow 2H, & \mathcal{P} & \rightarrow \mathcal{C}, & \mathcal{R} & \rightarrow \mathcal{C}^t.
\end{align}
(A.7)

For details on $\mathfrak{su}(2|2)$ see e.g. [44].

Appendix B. Symmetries of $Y = 0$ giant graviton

The $Y = 0$ giant graviton preserves the $\mathfrak{su}(2|1)$ subalgebra of the $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ and has no degrees of freedom attached to the end of the spin chain [20]. The algebra $a = \mathfrak{su}(2|1)$ is obtained from the $g = \mathfrak{psu}(2|2) \times \mathbb{R}^3$ by dropping the generators with bosonic indices $a, b, c, \ldots = 1$ (or equivalently with $a, b, c, \ldots = 2$). Thus, the surviving (preserved) charges form a subgroup $a = \{R^a, L_1^1, Q^a_2, G^a_2, H\}$, while the broken charges form a subset $b = \mathfrak{psu}(2|2) \times \mathbb{R}^3 \setminus \mathfrak{su}(2|1)$ consisting of $\{L_1^2, L_2^1, Q^1_1, G^1_3, C, \mathcal{C}^t\}$.

Since the Cartan–Killing form of the extended $\mathfrak{su}(2|2)$ algebra is degenerate, the algebra does not have a well-defined Casimir operator. Thus we cannot apply the formula (18) naively. In order to use (18), the Casimir operator of $\mathfrak{su}(2|1)$ has to be enhanced by $\mathfrak{u}(1)$ outer automorphism $\mathcal{U}$,
\begin{equation}
T^a = -R^a_1 R^a_3 + 2L^1_1 L^1_1 + Q^2_2 G^2_2 - G^1_1 Q^1_2 - 2H U.
\end{equation}
(B.1)

This extra charge is the Cartan generator of $\mathfrak{su}(2)$ outer automorphism of the extended $\mathfrak{su}(2|2)$ algebra and it serves the hypercharge for generators,
\begin{align}
[U, Q^a_2] & = +\frac{1}{2}Q^a_2, & [U, G^a_2] & = -\frac{1}{2}G^a_2, & [U, C] & = +C, & [U, \mathcal{C}^t] & = -\mathcal{C}^t, \\
[U, L^a_1] & = [U, R^a_3] = [U, H] = 0.
\end{align}
(B.2)

Then (18) implies that the twisted Yangian charges governing the scattering of the $Y = 0$ giant graviton are
\begin{align}
\bar{Q}^a_1 & = \bar{Q}^a_1 - \frac{1}{4}[T^a, Q^a_2], & \bar{L}^1_1 & = \bar{L}^2_1 - \frac{1}{4}[T^a, L^1_1], & \bar{C} & = \bar{C} - \frac{1}{4}[T^a, C], \\
\bar{G}^a_1 & = \bar{G}^a_1 - \frac{1}{4}[T^a, G^a_2], & \bar{L}^2_1 & = \bar{L}^1_1 - \frac{1}{4}[T^a, L^1_1], & \bar{C}^t & = \bar{C}^t - \frac{1}{4}[T^a, C^t].
\end{align}
(B.3)

The explicit forms of their co-products are
\begin{align}
\Delta \bar{Q}^a_1 & = \bar{Q}^a_1 \otimes 1 + 1 \otimes \bar{Q}^a_1 + Q^a_2 \otimes R^a_3 - L^1_1 \otimes Q^a_2 + Q^a_1 \otimes K^a_2 + \varepsilon_{a\beta} C \otimes G^2_2, \\
\Delta \bar{G}^a_1 & = \bar{G}^a_1 \otimes 1 + 1 \otimes \bar{G}^a_1 - G^a_2 \otimes R^a_3 + L^1_1 \otimes G^a_2 - G^a_1 \otimes K^a_2 - \varepsilon_{a\beta} C^t \otimes Q^a_2, \\
\Delta \bar{L}^1_1 & = \bar{L}^1_1 \otimes 1 + 1 \otimes \bar{L}^1_1 - 2L^1_1 \otimes L^1_1 - G^1_1 \otimes Q^2_2, \\
\Delta \bar{L}^2_1 & = \bar{L}^2_1 \otimes 1 + 1 \otimes \bar{L}^2_1 + 2L^1_1 \otimes L^1_1 - Q^1_1 \otimes G^2_2, \\
\Delta \bar{C} & = \bar{C} \otimes 1 + 1 \otimes \bar{C} + C \otimes H, \\
\Delta \bar{C}^t & = \bar{C}^t \otimes 1 + 1 \otimes \bar{C}^t - C^t \otimes H.
\end{align}
(B.4)

Here, $K^a_2 = R^a_3 + L^1_1 + \frac{3}{2}H$ and $4 = 3, 3 = 4$.

\textsuperscript{15} In the case of bulk scattering, one of the regularizations is to use the $\mathfrak{su}(2)$ outer automorphism [10].
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