Two-photon exchange at low $Q^2$

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We study two-photon exchange for elastic electron-proton scattering at low $Q^2$. Compact approximate formulae for the amplitudes are obtained. Numerical calculations are done for $Q^2 \leq 0.1$ GeV$^2$ with several realistic form factor parameterizations, yielding similar results. They indicate that the corrections to magnetic form factor can visibly affect cross-section and proton radii. For low-$Q^2$ electron-neutron scattering two-photon exchange corrections are shown to be negligibly small.

During last few years, two-photon exchange (TPE) in elastic electron-proton scattering draw a lot of attention. The study of TPE was inspired by the problem in the proton form factor (FF) measurements [1]. However, this is not the only field where TPE effects can manifest oneself. The others include single-spin asymmetries [2], radiative corrections to parity-violating observables [3] and determination of proton radii [4, 5]. The latter is done via study of elastic electron-proton scattering at low $Q^2$. The TPE corrections are known to be important here, but the whole TPE amplitude was never calculated in closed analytic form.

The TPE amplitude can be split into elastic (with proton intermediate state) and inelastic contributions. Naturally, the elastic part should be dominant at low energy. In Ref.[6] the elastic contribution was reduced to the twofold integral containing FFs from the space-like region only. Nevertheless, the functions under the integral are complicated and possess singularities, thus it is not so easy to integrate numerically. At low energies, further simplification is possible, yielding compact analytic expressions for the TPE amplitudes.

We follow the notations of Ref.[6]. The particle momenta are defined according to Eq.-(2) and everywhere below their real parts are understood. The invariant amplitudes $\tilde{F}_i$ are functions of $\nu$ and $t$. In the Born approximation (BA) $\tilde{F}_1(\nu,t) = F_1(t)$, $\tilde{F}_2(\nu,t) = F_2(t)$, $\tilde{F}_3(\nu,t) = 0$, where $F_{1,2}(t)$ are usual Dirac and Pauli proton FFs. Alternatively, we may use any three independent linear combinations of $\tilde{F}_i$; e.g. instead of $\tilde{F}_1$ and $\tilde{F}_2$ one may define $\tilde{G}_M = \tilde{F}_1 + \tilde{F}_2$ and $\tilde{G}_E = \tilde{F}_1 + \frac{\nu}{4M^2} \tilde{F}_2$, such that in BA $\tilde{G}_E$ and $\tilde{G}_M$ are equal to electric and magnetic FFs.

The difference between $\tilde{F}_1$ and their Born values is proportional to $\alpha \approx \frac{1}{137}$. Neglecting the terms of order $\alpha^2$ w.r.t. the leading one, the cross-section for unpolarized particles is

$$d\sigma = d\sigma_0 \left( \varepsilon \mathcal{G}_E^2 - \frac{t}{4M^2} \mathcal{G}_M^2 \right),$$

where

$$\mathcal{G}_M = \tilde{G}_M + \varepsilon \frac{\nu}{4M^2} \tilde{F}_3, \quad \mathcal{G}_E = \tilde{G}_E + \frac{\nu}{4M^2} \tilde{F}_3,$$

and

$$d\sigma_0 = \frac{8\pi\alpha^2}{1 - \varepsilon (\nu - t)^2} \frac{4M^2}{t} dt.$$

The invariant amplitudes are complex; in Eq.(2) and everywhere below their real parts are understood.

Eq.(2) looks like Rosenbluth formula (certainly the experimental separation of $\mathcal{G}_M$ and $\mathcal{G}_E$ is not possible, since both are $\varepsilon$-dependent beyond BA). Because of this simple form, we adopt the following set of invariant amplitudes: $\mathcal{G}_E$, $\mathcal{G}_M$, and $\tilde{G}_M$.

In this paper we are interesting in TPE contributions $\delta \mathcal{G} = (\delta \mathcal{G}_E, \delta \mathcal{G}_M, \delta \tilde{G}_M)$ at low $t$, $-t \ll 4M^2$. On the other hand, we neglect the electron mass $m$, so it should be $4m^2 \ll -t$.

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Low \( t \) can be achieved in two ways: (i) at fixed energy and small scattering angle and (ii) at fixed angle (or fixed \( \varepsilon \), which is nearly the same) and low energy. We will consider the latter case. This implies

\[
\nu = \kappa \sqrt{-t(4M^2 - t)} \sim \sqrt{-t}, \quad \text{where} \quad \kappa = \frac{1 + \varepsilon}{1 - \varepsilon}.
\]

(5)

In the Breit frame \( \kappa = 1/\sin \frac{\theta}{2} \), this quantity was denoted \( x \) in Ref.[7]. Eq.(2) suggests that at low \( t \) one should primarily consider the amplitude \( G_2 \). However for generality we will study all three amplitudes. The role of amplitude \( G_M \) increases for backward angles (\( \varepsilon \approx 0 \)) and for neutron target (\( G_E \approx 0 \)). The amplitude \( G_M \) may contribute to polarization observables.

The TPE contribution to any of the invariant amplitudes can be written as [6]

\[
\delta G = \frac{i\alpha}{2\pi^3} \sum_n \int \sum_{i,j=1}^2 A_{n,ij}(t_1,t_2) F_i(t_1) F_j(t_2) \frac{d^4p''}{t_1 t_2 D_n},
\]

(6)

where \( n = 1, 2, 3, 4, 4x; \)

\[
\begin{align*}
D_1 &= 1, & D_2 &= (P - p'' + K)^2 - m^2, & D_3 &= p''^2 - M^2, \\
D_4 &= [(P - p'' + K)^2 - m^2][p''^2 - M^2], & D_{4x} &= [(P - p'' - K)^2 - m^2][p''^2 - M^2]
\end{align*}
\]

and \( t_1 = (p - p'')^2 \), \( t_2 = (p' - p'')^2 \). The factor \( \frac{d^4p''}{t_1 t_2 D_n} \) is introduced for convenience. \( A_{n,ij} \) are some polynomials in \( t_1, t_2 \), depending also on \( \nu \) and \( t \), and originate from the numerators of TPE diagrams. They are different for different amplitudes in the l.h.s., but to simplify notation we do not attach an index to indicate this. The crossing symmetry implies

\[
A_{1,2,3}(\nu) = -A_{2,1,3}(-\nu), \quad A_4(\nu) = -A_{4x}(-\nu).
\]

(8)

Following Ref.[6], the integrals over \( d^4p'' \) can be reduced to the twofold integrals over \( t_1, t_2 \):

\[
\delta G = \frac{\alpha}{2\pi^3} \sum_n \int \mathcal{K}_n(t_1,t_2) \sum_{i,j=1}^2 A_{n,ij}(t_1,t_2) F_i(t_1) F_j(t_2) \frac{dt_1 dt_2}{t_1 t_2},
\]

(9)

where \( \mathcal{K}_n \) are known functions, given in Eqs.(31-35) of Ref.[6]. It is convenient to define \( A_{4\pm} = A_4 \pm A_{4x}, \mathcal{K}_{4\pm} = \frac{1}{2}(\mathcal{K}_4 \pm \mathcal{K}_{4x}) \), then

\[
A_4 \mathcal{K}_4 + A_{4x} \mathcal{K}_{4x} = A_{4+} \mathcal{K}_{4+} + A_{4-} \mathcal{K}_{4-}.
\]

(10)

Now we will simplify all the terms of Eq.(9) in the case of low \( t \). Due to \( \theta \)-functions, contained in \( \mathcal{K}_n \), the integration in (9) is done over three regions: (I) \( x_{\infty} > 0 \), (II) \( x_{\infty} < 0 < x_M, t_1 + t_2 - t > 0 \) and (III) \( x_M < 0 < x_m, t_1 + t_2 - t > 0 \), see Fig. 1. In the regions II and III \( t_1, t_2 \sim t \) but in the region I \( t_1, t_2, t > 0 \) can be large. However the FFs entering (9) decrease rapidly as \( t_1, t_2 \rightarrow -\infty \), so the main contribution in the region I also comes from \( t_1, t_2 \sim t \). Assuming \( t_1, t_2 \sim t \), we obtain

\[
\begin{align*}
\mathcal{K}_1 &\sim 1, \quad \mathcal{K}_2 \sim 1/t, \quad \mathcal{K}_{4-} \sim 1/t\sqrt{-t}, \\
\mathcal{K}_3 &\approx -\frac{\pi^2 i}{4M\sqrt{-t}} \theta(x_{\infty}) + 2\theta(-x_{\infty})\theta(x_M)\theta(t_1 + t_2 - t) \approx -\frac{\pi^2 i}{4M\sqrt{-t}} \theta(x_{\infty}) \sim 1/\sqrt{-t}, \\
\mathcal{K}_{4+} &\approx -\frac{\pi^2 i}{2\sqrt{R}} \text{sign}(t_1 + t_2 - t) \theta(x_{\infty}) + 2\theta(-x_{\infty})\theta(x_M)\theta(t_1 + t_2 - t) \approx -\frac{\pi^2 i}{2\sqrt{R}} \theta(x_{\infty}) \text{sign}(t_1 + t_2 - t) \sim 1/\sqrt{-t},
\end{align*}
\]

(11)

where \( R \approx \nu^2 \left( \frac{t_1 + t_2 - t}{2} \right)^2 - t_1 t_2 (\nu^2 + 4M^2 t) \), and the square root means arithmetic value. Since the area of region II is small as \( O(t) \), the term \( \theta(-x_{\infty})\theta(x_M)\theta(t_1 + t_2 - t) \) brings, after the integration, an additional factor of \( t \). Therefore we neglect it w.r.t. \( \theta(x_{\infty}) \). The \( \mathcal{K}_n \) which are not written explicitly will give negligible contribution, see below.

It is convenient to introduce new variables \( a \) and \( b \) such that

\[
t_1 = t(a + 1/4 + b), \quad t_2 = t(a + 1/4 - b),
\]

(12)

then

\[
x_{\infty} = t(a - b^2), \quad dt_1 dt_2 = 2t^2 da db,
\]

(13)
the condition $x_{\infty} \geq 0$ is equivalent to $a \geq 0$, $b^2 \leq a$ and $t_{1,2} \sim t$ means $a, b \sim 1$.

Consider the quantities $A_{n,ij}$. Taking into account (5), the low-$t$ behaviour of $A_{n,ij}$, for all three amplitudes, is

$$A_1 \sim \sqrt{-t}, \quad A_2 \sim t\sqrt{-t}, \quad A_{4 \pm} \sim t^2, \quad A_3 \sim \sqrt{-t}, \quad A_{4 \pm} \sim t\sqrt{-t}.$$  \hfill (14)

Comparing with (11), we conclude that $A_1 K_1$, $A_2 K_2$ and $A_{4-} K_{4-}$ give a contribution of order $O(\sqrt{-t})$ w.r.t. $A_3 K_3$ and $A_{4+} K_{4+}$, so further we will neglect the former and consider the latter. The explicit expressions for the leading terms in $A_{n,ij}$ are,

for the amplitude $\hat{G}_M$:

$$A_3 = \frac{\nu(a - 1/4)}{x^2 - 1} \left( \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right), \quad A_{4+} = \nu t \left[ a + 1/4 - \frac{(a - 1/4)^2}{x^2 - 1} \right] \left( \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right) - \nu t b \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),$$  \hfill (15)

for the amplitude $\hat{G}_R$:

$$A_3 = \frac{\nu}{x^2 + 1} \left( \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right), \quad A_{4+} = \frac{\nu t}{x^2 + 1} \left[ \frac{2x^2 + 1}{4} + b^2 \right] \left( \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right) - \nu t b \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$  \hfill (16)

and for $\hat{G}_E$:

$$A_3 = \frac{\nu}{x^2 - 1} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad A_{4+} = \nu t \left[ 1 - \frac{a - 1/4}{x^2 - 1} \right] \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),$$  \hfill (17)

where we use matrix notation $A_n = \left( \begin{array}{cc} A_{n,11} & A_{n,12} \\ A_{n,21} & A_{n,22} \end{array} \right)$ for compactness.

Now let us consider the FFs $F_i(t_1) F_j(t_2) \equiv F_i(ta + t/4 + tb) F_j(ta + t/4 - tb)$. For $a \gg 1$ we have $b \leq \sqrt{a} \ll a$, so $F_i(t_1) F_j(t_2) \approx F_i(ta + t/4) F_j(ta + t/4)$. In the main region of interest $a \sim 1$ we expand $F$ in Taylor series around $ta + t/4$ and obtain

$$F_i(t_1) F_j(t_2) = F_i(ta + t/4) F_j(ta + t/4) \approx 2 F_i F_j + O(t/t_0)^2,$$  \hfill (18)

$$F_i(t_1) F_j(t_2) - F_i(ta + t/4) F_j(ta + t/4) \approx t b \left( F_i' F_j - F_i F_j' \right) + O(t/t_0)^3,$$  \hfill (19)

where FFs and their derivatives in the r.h.s are taken at $ta + t/4$, and $t_0$ is characteristic scale for FFs,

$$t_0 = 6/\langle r^2 \rangle \approx 0.3 \text{ GeV}^2 \text{ for proton.}$$  \hfill (20)

Note that $t_0 \ll 4M^2$ (this is because proton radius is related to pion rather than proton mass), and thus an additional requirement arises: $(t/t_0)^2 \ll 1$.

The r.h.s. of Eq. (19) is of order $t/t_0$. In the presence of FF scaling, $F_2/F_1 \approx \text{const}$, it becomes much smaller, since $F_i' F_j - F_i F_j' = F_i F_j (\ln F_i/F_j)'$, so we will assume (19) to be negligible and thus put everywhere

$$F_i(t_1) F_j(t_2) = F_i(ta + t/4) F_j(ta + t/4).$$  \hfill (21)
Replacing $F(ta + t/4)$ by just $F(0)$ would be too rough an approximation, since the FFs suppress the integrand at $a \gg 1$.

After all the approximation made the FFs do not depend on $b$ and the integration over $b$ can be done analytically:

\[
J_3 = -\frac{2Mt^2\sqrt{-t}}{\pi^2} \int dt \mathcal{K}_3 = \frac{1}{a + 1/4} \ln \left| \frac{\sqrt{a} + 1/2}{\sqrt{a} - 1/2} \right|, \\
J_4 = -\frac{2Mt^2\sqrt{-t}}{\pi^2} \ln \left(\frac{a + 1/4 + \sqrt{a}}{a + 1/4 - \sqrt{a}}\right), \\
J'_4 = -\frac{2Mt\sqrt{-t}}{\pi^2} \ln \left(\frac{a - 1/4 + \sqrt{a}}{a - 1/4 - \sqrt{a}}\right).
\]

Collecting all the expressions together, we obtain finally

\[
\delta \hat{G}_M = \frac{2\alpha}{\pi} \int_0^\infty \left\{ \frac{a - 1/4}{\kappa^2 - 1} J_3 + \left[ a + 1/4 - \frac{(a - 1/4)^2}{\kappa^2 - 1} \right] J_4 \right\} F_3(F_1 + F_2) da, \\
\delta G_M = \frac{2\alpha}{\pi} \int_0^\infty \left\{ \frac{1}{2} J_4 + \frac{1}{\kappa^2 + 1} \left[ J_3 + (a - 1/4)(a + 3/4) J_4 - J'_4 \right] \right\} F_1(F_1 + F_2) da, \\
\delta G_E = \frac{\alpha}{\pi} \int_0^\infty \left\{ \frac{1}{\kappa^2 - 1} J_3 + \left( 1 - \frac{a - 1/4}{\kappa^2 - 1} \right) J_4 \right\} F_1^2 da,
\]

where $F_1, 2 \equiv F_1, 2(ta + t/4)$.

The approximate formulae (25-27) should work well for

\[ \frac{-t/4m^2}{\gg 1}, \quad \sqrt{-t/2M} \ll 1, \quad (t/t_0)^2 \ll 1. \]

The infrared problems would arise at $t_1 = 0, t_2 = t$ or $t_1 = t, t_2 = 0$, that is, at $a = 1/4$. However the singularity in (25-27) at $a = 1/4$ is integrable, so the IR divergent terms do not appear. Actually such terms have the form of $\ln \frac{\nu}{\kappa}$ and are negligible in our approximation.

Consider the limit $t = 0$. In this case $F_1 = 1, F_2 = \mu - 1$, where $\mu$ is magnetic moment. In other words, the result will be the same as for point-like particle. Taking into account that $\nu > 0$, we perform the integration and obtain for $t = 0$

\[
\delta \hat{G}_M = \mu \alpha \pi \frac{\kappa + 1/2}{\kappa + 1}, \quad \delta G_M = \mu \alpha \pi \frac{\kappa + 1/2}{\kappa^2 + 1}, \quad \delta G_E = \alpha \pi \frac{1/2}{\kappa + 1}.
\]

The last result for $\delta G_E$ corresponds to well-known second-order correction to Coulomb scattering [8].

One point is to be clarified here. The TPE contributions should be odd functions of $\nu$ [7]. From this the authors of Ref.[7] make a (erroneous) conclusion, that TPE amplitudes should have the form

\[
\delta \hat{G} = \kappa \sum_{k=0}^\infty c_k(t) \kappa^{2k},
\]

which is apparently not the case for (29). The source of error is that the series (30) converges only in certain circle around $\nu = 0$ in the complex $\kappa$ plane. The radius of the circle is the distance to the nearest singularity of $\delta \hat{G}(\nu)$. Such singularities are branching points at $s = (m + M)^2$ and at $u = (m + M)^2$, that is, at $\nu = s - u = \pm(4Mm + t)$; neglecting $m$, this yields $s = u = \sqrt{(4Mm + t)} < 1$. But the physical values are $\nu \geq 1$, see Eq.(5). They therefore lie outside the convergence circle, and Eq.(30) is not valid for these $\nu$.

The results of numerical evaluation of TPE amplitudes according to Eqs.(25-27) are shown in Fig. 2 (the amplitudes $\delta \hat{G}_M$ and $\delta \hat{G}_M$ are divided by $\mu = 2.79$). The calculation is done with the dipole FFs. To check the sensitivity to the FF parameterization, we repeated the calculation using FF fits [9]; the resulting changes in $\delta \hat{G}$ were small.

The TPE amplitudes are falling rapidly near $t = 0$; especially this pertains to the “magnetic” corrections $\delta G_M$ and $\delta \hat{G}_M$. Since the determination of rms radius involves FF derivative rather than FF itself, our results suggest that magnetic TPE correction may be important here. Roughly speaking, TPE corrections are to be subtracted from the experimentally measured FFs, resulting in slower decrease of “corrected” FFs. So we expect that inclusion of TPE should lessen proton rms radii. Surprisingly, Refs.[4, 5], where TPE effects were calculated merely numerically, claim the increase of charge radius, by $(0.008 - 0.013)$ fm [4] and by 0.0015 fm [5].
FIG. 2: TPE amplitudes at $\varepsilon = 0.5$ (a) and at $Q^2 = 0.05$ GeV$^2$ (b); $\delta G_E$ (solid), $\delta G_M/\mu$ (dashed) and $\delta \tilde{G}_M/\mu$ (dotted).

FIG. 3: TPE corrections to the cross-section at $\varepsilon = 0.5$, electric (dash-dotted), magnetic (dashed) and total (solid).

In Fig. 3 we show the TPE corrections to the cross-section at $\varepsilon = 0.5$, electric (coming from $\delta G_E$), magnetic (coming from $\delta G_M$) and total. We see that magnetic contribution is of the same order as the electric one and represents an essential piece of the total correction. At lower $\varepsilon$ it becomes even more important.

The formalism developed here can be applied to electron-neutron scattering as well. In this case $G_E, F_1 \sim t$, and we see from (25-27) that $\delta G_M \sim t$ and $\delta G_E \sim t^2$. In other words, the TPE amplitudes are suppressed, with respect to the Born amplitude, by a factor of $\alpha t$, which is much smaller than just $\alpha$. Consequently, TPE corrections to the elastic $en$ scattering are negligibly small at low $t$.

In summary, we obtained simple approximate formulae for TPE amplitudes of elastic electron-proton scattering at low $Q^2$. Numerical calculations indicate that the magnetic TPE corrections are relatively large and have a sharp $Q$-dependence, which can affect the determination of proton rms radii.

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