DIVISOR CLASS GROUPS OF RATIONAL TRINOMIAL VARIETIES

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ABSTRACT. We give an explicit description of the divisor class groups of rational trinomial varieties. As an application we show the connection between the iteration of Cox rings of varieties with torus action of complexity one of arbitrary dimension to the iteration of Cox rings of the Du Val surfaces.

1. Introduction

This article contributes to the explicit calculation of divisor class groups of affine varieties; see [Fle81, Lan83, SSS84, SS07] for some previous work and Remark 2.10 for the relations to our results. We consider affine algebraic varieties; see [Fle81, Lan83, SS84, SS07] for some previous work and Remark 2.19. Let

\[T_i := \gcd(l_1, l_2, l_3),\]

for each exponent vector \(T = (l_1, l_2, l_3)\) with monomials \(T_i l_i = T_1 l_1 \cdots T_m l_m\) and pairwise different \(\theta_i \in \mathbb{C}^*\). We call such a variety a trinomial variety. Our first main result describes explicitly the divisor class groups of rational non-factorial trinomial varieties. For each exponent vector \(l_i := \gcd(l_1, \ldots, l_m)\), denote \(l := \gcd(l_0, l_1, l_2)\) and define

\[c(0) := \gcd(l_1, l_2),\quad c(1) := \gcd(l_0, l_2),\quad c(2) := \gcd(l_0, l_1),\]

\[c(i) := \frac{1}{i} \gcd(l_1, l_2) \gcd(l_0, l_2) \gcd(l_0, l_1)\quad \text{for}\quad i \geq 3.\]

Note that due to [ABHW18 Cor. 5.8] one can easily decide if a given trinomial variety is rational or factorial just in terms of the numbers \(l_i\), see also Remark 2.2.

Theorem 1.1. Let \(X\) be an affine, rational, non-factorial trinomial variety and set \(n := \sum_{i=0}^m (c(i) - 1)n_i - c(i) + 1\).

(i) If \(c := \gcd(l_0, l_1) > 1\) and \(\gcd(l_1, l_j) = 1\) holds whenever \(j \notin \{0, 1\}\), then the divisor class group \(\mathcal{C}(X)\) is isomorphic to

\[(\mathbb{Z}/l_2 \mathbb{Z})^{c-1} \times \cdots \times (\mathbb{Z}/l \mathbb{Z})^{c-1} \times \mathbb{Z}^n.\]

(ii) If \(\gcd(l_0, l_1) = \gcd(l_1, l_2) = \gcd(l_0, l_2) = 2\) and \(\gcd(l_1, l_j) = 1\) holds whenever \(j \notin \{0, 1, 2\}\), then the divisor class group \(\mathcal{C}(X)\) is isomorphic to

\[\mathbb{Z}/(l_0 l_1 l_2/4) \mathbb{Z} \times (\mathbb{Z}/l_2 \mathbb{Z})^3 \times \cdots \times (\mathbb{Z}/l \mathbb{Z})^3 \times \mathbb{Z}^n.\]

In order to prove this result we make use of the fact that rational trinomial varieties are \(T\)-varieties of complexity one, i.e., they are endowed with an effective torus action \(\mathbb{T} \times X \to X\) such that \(\dim(\mathbb{T}) = \dim(X) - 1\) holds. We use the description of their total coordinate spaces, i.e., the spectrum of their Cox rings, given in [HW18 Prop. 2.6] to prove the above theorem and obtain as a by-product an explicit description of the divisor class group grading on the Cox ring of a rational trinomial variety; see Corollary 2.20.

Using Corollary 2.20, we can give a new perspective on the iteration of Cox rings for \(T\)-varieties of complexity one. For this let \(X\) be a hyperplatonic trinomial...
variety, i.e., \( t_0^{-1} + \ldots + t_r^{-1} > r - 1 \) holds. This means that after reordering \( t_0, \ldots, t_r \) decreasingly, \( t_i = 1 \) holds for all \( i \geq 3 \) and \((t_0, t_1, t_2)\) is a platonic triple, i.e., one of the triples \((5, 3, 2), (4, 3, 2), (3, 3, 2), (x, 2, 2), (x, y, 1)\), where \( x, y \in \mathbb{Z}_{\geq 1} \). We call this triple the basic platonic triple of \( X \). Note that these varieties comprise all total coordinate spaces of affine log terminal varieties of complexity one; see [ABHW18] for the precise statement. Due to [HW18, Thm. 1.1] a hyperplatonic variety \( X \) admits iteration of Cox rings, i.e., there exists a chain

\[
X_p \xrightarrow{\delta_{H_{p-1}}} X_{p-1} \xrightarrow{\delta_{H_{p-2}}} \ldots \xrightarrow{\delta_{H_2}} X_2 \xrightarrow{\delta_{H_1}} X_1 := X,
\]

where \( X_p \) is a factorial affine variety, and in each step, \( X_{i+1} \) is the total coordinate space of \( X_i \) and \( H_i := \text{Spec} \mathbb{C}[\text{Cl}(X_i)] \). Moreover any of the occurring total coordinate spaces is again hyperplatonic and there are exactly the following possible sequences of basic platonic triples arising from Cox ring iterations of hyperplatonic varieties, see [HW17, Cor. 1.4]:

1. \((1, 1, 1) \rightarrow (2, 2, 2) \rightarrow (3, 3, 2) \rightarrow (4, 3, 2)\),
2. \((1, 1, 1) \rightarrow (x, x, 1) \rightarrow (2x, 2, 2)\),
3. \((1, 1, 1) \rightarrow (x, x, 1) \rightarrow (x, 2, 2)\),
4. \((t_0^{-1}t_1, t_0^{-1}t_1, 1) \rightarrow (t_0, t_1, 1)\), where \( t_0 := \text{gcd}(t_0, t_1) > 1 \).

In the above iterations, the steps corresponding to \((1, 1, 1) \rightarrow (x, x, 1)\) as well as the step of Case (iv) are exactly those steps, where \( H_i \) is a torus. The remaining parts of the iteration chains can be represented by Cox ring iterations of Du Val surfaces: Any platonic triple \((a, b, c)\) defines a Du Val singularity by

\[
Y(a, b, c) := V(T^a_1 + T^b_2 + T^c_3) \subseteq \mathbb{C}^3.
\]

Case (i) corresponds to the chain \( \mathbb{C}^2 \rightarrow A_1 \rightarrow D_4 \rightarrow E_6 \) and \((x, x, 1) \rightarrow (2x, 2, 2)\) resp. \((x, x, 1) \rightarrow (2x, 2, 2)\) correspond to the chains \( \mathbb{C}^2 \rightarrow A_n \) resp. \( \mathbb{C}^2 \rightarrow A_{2n} \) with \( n > 0 \) odd. Overall we obtain the following structural result.

**Corollary 1.2.** Let \( X \) be a hyperplatonic variety with basic platonic triple \((t_0, t_1, t_2)\). Denote by \((t'_0, t'_1, t'_2)\) the basic platonic triple of the total coordinate space \( X' \) of \( X \). Then there is a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\text{TCS}} & X \\
\downarrow{\#T'} & & \downarrow{\#T} \\
Y(t'_0, t'_1, t'_2) & \xrightarrow{\text{TCS}} & Y(t_0, t_1, t_2),
\end{array}
\]

where the horizontal arrows labelled "TCS" are total coordinate spaces and the downward arrows are good quotients by torus actions.

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**2. Proof of the main results**

We work in the notation of [HH13, HW17], where the Cox rings of rational \( T \)-varieties of complexity one are described. Note that the trinomial varieties defined in the introduction arise as the spectrum of these rings. We briefly recall the necessary results and constructions here. For a general introduction to the theory of Cox rings see e.g. [ADHL15].
Construction 2.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_0 + \ldots + n_r$ with positive integers $n_i$. For every $i = 0, \ldots, r$, fix a tuple $l_i \in \mathbb{Z}_{\geq 0}^{n_i}$ and define a monomial
\[ T_i^{l_i} := T_{i_1}^{l_{i_1}} \cdots T_{i_{n_i}}^{l_{i_{n_i}}} \in \mathbb{C}[T_{i_1}, S_k]; \quad 0 \leq i \leq r, \; 1 \leq j \leq n_i, \; 1 \leq k \leq m. \]

We will also write $\mathbb{C}[T_{ij}, S_k]$ for the above polynomial ring. Let $A := (a_0, \ldots, a_r)$ be a $2 \times (r + 1)$ matrix with pairwise linearly independent columns $a_i \in \mathbb{C}^2$. For every $i = 0, \ldots, r - 2$ we define
\[ b_i := \det \begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix} \in \mathbb{C}[T_{ij}, S_k]. \]

We build up an $r \times (n + m)$ matrix from the exponent vectors $l_0, \ldots, l_r$ of these polynomials:
\[ P_0 := \begin{bmatrix} -l_0 & l_1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -l_0 & 0 & l_r & 0 & \ldots & 0 \end{bmatrix}. \]

Denote by $P_0^*$ the transpose of $P_0$ and consider the projection $Q: \mathbb{Z}^{n+m} \to K_0 := \mathbb{Z}^{n+m}/\text{im}(P_0^*)$. Denote by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables $T_{ij}, S_k$. Define a $K_0$-grading on $\mathbb{C}[T_{ij}, S_k]$ by setting
\[ \deg(T_{ij}) := Q(e_{ij}) \in K_0, \quad \deg(S_k) := Q(e_k) \in K_0. \]

This is the finest possible grading of $\mathbb{C}[T_{ij}, S_k]$ leaving the variables and the $g_i$ homogeneous and any other such grading coarsens this maximal one. In particular, we have a $K_0$-graded factor algebra $R(A, P_0) := \mathbb{C}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2} \rangle$.

By the results of [HH13] [HW17] the rings $R(A, P_0)$ are normal complete intersections and admit only constant homogeneous units. We use the following rationality criterion from [ABHW18] Cor. 5.8 for the spectrum of a ring $R(A, P_0)$ as above:

Remark 2.2. Let $R(A, P_0)$ be a ring as in Construction 2.1 and set $l_i := \gcd(l_{i_1}, \ldots, l_{i_{n_i}})$. Then Spec $R(A, P_0)$ is rational if and only if one of the following conditions holds:

(i) We have $\gcd(l_i, l_j) = 1$ for all $0 \leq i < j \leq r$, in other words, $R(A, P_0)$ is factorial.
(ii) There are $0 \leq i < j \leq r$ with $\gcd(l_i, l_j) > 1$ and $\gcd(l_i, l_k) = 1$ whenever $v \notin \{i, j\}$.
(iii) There are $0 \leq i < j < k \leq r$ with $\gcd(l_i, l_j) = \gcd(l_i, l_k) = \gcd(l_j, l_k) = 2$ and $\gcd(l_k, l_k) = 1$ whenever $v \notin \{i, j, k\}$.

Definition 2.3. Let $R(A, P_0)$ be as above such that Spec $R(A, P_0)$ is rational. We say that $P_0$ is gcd-ordered if it satisfies the following two properties

(i) $\gcd(l_i, l_j) = 1$ for all $i = 0, \ldots, r$ and $j = 3, \ldots, r$,
(ii) $\gcd(l_i, l_j) = \gcd(l_0, l_1, l_2)$.

If Spec $R(A, P_0)$ is rational, one can always achieve that $P_0$ is gcd-ordered by suitably reordering $l_0, \ldots, l_r$, which does not affect the $K_0$-graded algebra $R(A, P_0)$ up to isomorphism.

In order to prove our main results we make use of the explicit description of the total coordinate space of a rational trinomial variety given in [HW18]. We state the two necessary results here:
Lemma 2.4. [HW18] Lemma 2.5] Let $R(A, P_0)$ be a ring as in Construction 2.1 and $X := \text{Spec } R(A, P_0)$ be rational. Assume that $P_0$ is gcd-ordered. Then, with $l := \gcd(l_0, l_1, l_2)$, the number $c(i)$ of irreducible components of $V(X, T_{ij})$, where $j = 1, \ldots, n_i$, is given by

| $i$ | 0 | 1 | 2 | $\geq 3$ |
|-----|---|---|---|---------|
| $c(i)$ | $\gcd(l_1, l_2)$ | $\gcd(l_0, l_2)$ | $\gcd(l_0, l_1)$ | $\frac{1}{l} \gcd(l_1, l_2) \gcd(l_0, l_2) \gcd(l_0, l_1)$ |

Proposition 2.5. [HW18] Prop. 2.6] Let $R(A, P_0)$ be non-factorial with Spec $R(A, P_0)$ rational. Assume that $P_0$ is gcd-ordered and set

$$P_1 := \begin{bmatrix}
\frac{1}{\gcd(l_0, l_2)} l_0 & \frac{1}{\gcd(l_0, l_2)} l_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
n & 0 & \frac{1}{\gcd(l_0, l_2)} l_2 & 0 & 0 \\
- l_0 & 0 & l_3 & 0 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
- l_0 & 0 & \ldots & 0 & l_r & 0 & 0 & 0
\end{bmatrix}.$$ 

Moreover, let $c(i)$ be as above and define numbers $n' := c(0)n_0 + \ldots + c(r)n_r$ and $n_{i, 1}, \ldots, n_{i, c(i)} := n_i, l_{ij, 1}, \ldots, l_{ij, c(i)} := \gcd((P_1)_{1, ij}, \ldots, (P_1)_{r, ij}).$

Then the vectors $l_{i, \alpha} := (l_{i, 1}, \ldots, l_{i, c(i)}) \in \mathbb{Z}_n$ build up an $r' \times (n' + m)$ matrix $P_0^\prime$ with $r' = c(0) + \ldots + c(r) - 1$. With a suitable matrix $A'$, the affine variety Spec $R(A', P_0^\prime)$ is the total coordinate space of the affine variety Spec $R(A, P_0)$.

Construction 2.6. Let $R(A, P_0)$ be a ring as in Construction 2.1. Choose an integral $s \times (n + m)$ matrix $d$ and build the $(r + s) \times (n + m)$ stack matrix

$$P := \begin{bmatrix} P_0 \\ d \end{bmatrix}.$$ 

We require the columns of $P$ to be pairwise different primitive vectors generating $Q^{r+s}$ as a vector space. Let $P^*$ denote the transpose of $P$ and consider the projection

$$Q: \mathbb{Z}^{n+m} \rightarrow K := \mathbb{Z}^{n+m}/\text{im}(P^*).$$

Denoting as before by $e_{ij}, e_{k} \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables $T_{ij}$ and $S_k$, we obtain a $K$-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}) \in K, \quad \deg(S_k) := Q(e_{k}) \in K.$$ 

This $K$-grading coarsens the $K_0$-grading of $\mathbb{K}[T_{ij}, S_k]$ given in Construction 2.1 and thus defines a grading on $R(A, P_0)$.

Now, consider a rational trinomial variety $X := \text{Spec } R(A, P_0)$. Let $\text{Spec } R(A', P_0^\prime)$ be its total coordinate space and denote by $\mathcal{R}(X)$ its Cox ring. Then there exists a $K'$-grading on $R(A', P_0^\prime)$ such that $R(A', P_0^\prime) \cong \mathcal{R}(X)$ as graded rings. In particular $K' \cong \text{Cl}(X)$ holds and there exists a good quotient

$$\text{Spec } R(A', P_0^\prime) \overset{\text{Hilb}}{\rightarrow} \text{Spec } R(A, P_0)$$

with respect to the corresponding group action of $H' := \text{Spec } \mathbb{C}[K']$. Moreover, due to [HW17] Thm. 1.7 we find a description of this grading via a stack matrix

$$P' := \begin{bmatrix} P_0' \\ d \end{bmatrix}.$$
with $K' = \mathbb{Z}^{n'+m}/\text{im}(P')^\ast$ as in Construction 2.8. In particular the transpose $(P')^\ast$ defines an injective map. Now consider the group $K'_0 := \mathbb{Z}^{n'+m}/\text{im}((P'_0)^\ast)$ and denote by $(K'_0)^\text{tors}$ the torsion subgroup of $K'_0$. Then
\[
(K'_0)^\text{tors} \subseteq \mathbb{Z}^{n'+m}/\text{im}((P')^\ast) = K' \cong \text{Cl}(X)
\]
holds and we call $\text{Cl}(X)^\text{tors} := (K'_0)^\text{tors}$ the compulsory torsion of the divisor class group of $X$.

**Lemma 2.7.** Let $R(A, P_0)$ be a non factorial ring such that $X := \text{Spec } R(A, P_0)$ is rational and assume that $P_0$ is gcd-ordered.

(i) If $c := \gcd(t_0, l_1) > 1$ and $\gcd(t_1, l_j) = 1$ holds whenever $j \notin \{0, 1\}$, then the compulsory torsion of the divisor class group of $X$ is
\[
(Z/l_2Z)^{c-1} \times \cdots \times (Z/l_cZ)^{c-1}.
\]

(ii) If $\gcd(t_0, l_1) = \gcd(t_1, l_2) = \gcd(l_0, l_2) = 2$ and $\gcd(l_1, l_j) = 1$ holds whenever $j \notin \{0, 1, 2\}$, then the compulsory torsion of the divisor class group of $X$ is
\[
Z/(l_0/2Z) \times Z/(l_1/2Z) \times Z/(l_2/2Z) \times (Z/l_3Z)^3 \times \cdots \times (Z/l_cZ)^3.
\]

**Proof.** We prove (i). With our subsequent considerations we obtain that the divisor class group of $X$ is given as $\mathbb{Z}^{n'+m}/\text{im}((P')^\ast)$, where $P'$ is some $(r' + s') \times (n' + m)$ stack matrix
\[
\begin{bmatrix}
P_0 \\
d'
\end{bmatrix},
\]
of full row rank, and with Proposition 2.8 we get that $P'_0$ is the $r' \times (n' + m)$ matrix build up by the exponent vectors $c^{-1}l_0, c^{-1}l_1$ and $c$ copies $l_{i,1}, \ldots, l_{i,c}$ of $l_i$ for $i \geq 2$.

Thus, to obtain the assertion, we compute the elementary divisors of $P'_0$: Suitable elementary column operations transform $P'_0$ into
\[
\begin{bmatrix}
c^{-1}l_0 & c^{-1}l_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
c^{-1}l_0 & 0 & l_{2,1} & \cdots & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
c^{-1}l_0 & 0 & \cdots & l_{r,c} & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

As $\gcd(t_i, l_j) = 1$ holds for $i, j \notin \{0, 1\}$ we obtain for $1 \leq t \leq c$ that the $(r' - t + 1)$-th determinantal divisor of $P'_0$ equals $l_0^{c-t} \cdots l_c^{c-1}$. The assertion follows.

For the proof of (ii) we note that in this case $P'_0$ is built up by 2 copies of $1/2l_0, 1/2l_1$ and 1/2l_2 and 4 copies of each term $l_i$ for $i \geq 3$. Then, applying the same arguments as above, we obtain the assertion. \qed

**Construction 2.8.** Let $X$ be an irreducible, normal variety with $\Gamma(X, \mathcal{O}_X) = \mathbb{C}^\times$ and finitely generated divisor class group. Denote by WDiv($X$) the group of Weil-divisors of $X$ and fix a finitely generated subgroup $\mathbb{Z}^n \cong \langle D_1, \ldots, D_n \rangle \leq \text{WDiv}(X)$ such that the map $\pi: \mathbb{Z}^n \to \text{Cl}(X)$ sending each Weil divisor $D$ to its class $[D] \in \text{Cl}(X)$ is surjective. Let $f_1, \ldots, f_r$ be any linear relations between the the classes of $D_1, \ldots, D_r$ with
\[
f_j([D_1], \ldots, [D_n]) = \sum_{i=1}^n \alpha_{ij}[D_i] = [0] \in \text{Cl}(X)
\]
and set
\[
P := \begin{bmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{r1} & \cdots & \alpha_{rn}
\end{bmatrix}.
\]
Then there is a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{\pi} & \text{Cl}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}^n/\text{im}(P^*) & & \\
\end{array}
\]

In particular Cl(X) is a factor group of \( \mathbb{Z}^n/\text{im}(P^*) \).

**Lemma 2.9.** Let \( l_i \in \mathbb{Z}_{\geq 0}^n \) be any tuple, \( k \in \mathbb{Z}_{\geq 1} \) and consider the matrix

\[
A(k, l_i) := \begin{bmatrix}
  l_i & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & l_i \\
E_{n_i} & \cdots & E_{n_i}
\end{bmatrix} \in \text{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}),
\]

where \( E_{n_i} \) denotes the identity matrix of size \( n_i \). Then \( A(k, l_i) \) has rank \( n_i - 1 + k \) and the \( (n_i - 1 + k) \)-th determinantal divisor divides \( \text{gcd}(l_{i1}, \ldots, l_{in_i}) \).

**Proof.** Choose for any \( 2 \leq t \leq k \) an integer \( 1 \leq j_t \leq n_i \) and denote by \( e_{j_t} \) the column vector having 1 as \( j_t \)-th entry and all other entries equal zero. Consider the following \( (n_i - 1 + k) \times (n_i - 1 + k) \) square matrix obtained by deleting the first row and several of the last \( (k - 1) \cdot n_i \) columns of \( A(k, l_i) \):

\[
\begin{bmatrix}
  0 & \cdots & 0 & l_{ij_2} & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \cdots & \vdots \\
  0 & \cdots & 0 & 0 & \cdots & l_{ij_k} \\
E_{n_i} & e_{j_2} & \cdots & e_{j_k}
\end{bmatrix}
\]

The determinant of this matrix equals up to sign \( l_{ij_2} \cdots l_{ij_k} \).

With \( l_i = \text{gcd}(l_{i1}, \ldots, l_{in_i}) \) we obtain

\[
\text{gcd}\left( \prod_{t=2}^k l_{ij_t} ; j_t \in \{1, \ldots, n_i\} \right) = l_i^{k-1}.
\]

This shows that the \( (n_i - 1 + k) \)-th determinantal divisor divides \( l_i^{k-1} \). Moreover, as \( A(k, l_i) \) is obviously not of full rank this proves the assertions. \( \square \)

The rings \( R(A, P_0) \) as defined in Construction 2.1 are in general not unique factorization domains but have a similar property that will play an important role in our further considerations:

**Definition 2.10.** Let \( K \) be an abelian group and \( R = \oplus_{w \in K} R_w \) a finitely generated integral \( K \)-graded \( \mathbb{C} \)-algebra. Set \( H := \text{Spec} \mathbb{C}[K] \) and \( X := \text{Spec} R \).

(i) A homogeneous element \( 0 \neq f \in R \setminus R^* \) is called \( K \)-prime if whenever \( fgh \) holds for homogeneous elements \( g, h \in R \) we have \( f|g \) or \( f|h \).

(ii) We call \( R \) factorially \( K \)-graded if every homogeneous \( 0 \neq f \in R \setminus R^* \) is a product of \( K \)-prime elements.

(iii) An \( H \)-prime divisor on \( X \) is a Weil divisor \( 0 \neq \sum a_D D \), where \( a_D \in \{0, 1\} \), the \( D \) are prime and those with \( a_D = 1 \) are transitively permuted by \( H \).

**Remark 2.11.** Let \( R(A, P_0) \) be as in Construction 2.1. Then due to [ADHL15 Thm. 3.4.2.3] \( R(A, P_0) \) is factorially \( K_0 \)-graded and the variables \( T_{ij} \) and \( S_k \) are \( K_0 \)-prime. Due to [ADHL15 Prop. 1.5.3.3] this implies that the divisors \( \text{div}(T_{ij}) \) and \( \text{div}(S_k) \) are \( H_0 \)-prime, where \( H_0 := \text{Spec} \mathbb{C}[K_0] \) holds.
Moreover, due to Lemma 2.13 the defining relations of $R_{2.13.1}$ implies $T_{2.11}$ as

As $S_k$ is $K_0$-prime we conclude

Note that all free variables of $R(A', P'_0)$ arise this way.

Lemma 2.13. Let $R(A, P_0)$ be a ring defining a rational variety $X := \text{Spec } R(A, P_0)$. Assume that $P_0$ is $\text{gcd}$-ordered and $\text{gcd}(0, 1) > 1$ holds, whenever $j \notin \{0, 1\}$. Then the defining relations of the Cox ring $R(A', P'_0)$ of $X$ have $\text{Cl}(X)$-degree zero.

Proof. Note that due to Lemma 2.4 there is at least one integer $i \in \{0, 1, 2\}$ such that $V(X, T_{ij}) = D_{ij,1}$ is irreducible for $j = 1, \ldots, n_i$. As $R(A, P_0)$ is $K_0$-factorial, $K_0$-primeness of the variable $T_{ij}$ implies that $D_{ij,1}$ is a principal divisor for $j = 1, \ldots, n_i$; see Remark 2.11. We conclude

As $T_{ij,1}$ occurs as a term in at least one defining relation of $R(A', P'_0)$ and all of the defining relations have the same degree, the assertion follows.

Proof of Theorem 2.7. Case (i). Set $H_0^0 := H_0/H_0^{\text{tors}}$. We recall that the $H_0^{\text{tors}}$-invariant prime divisors with finite isotropy generate the divisor class group of $X = \text{Spec } R(A, P_0)$ and those are exactly the irreducible components of $V(X, T_{ij})$, where $i = 0, \ldots, r$ and $1 \leq j \leq n_i$. Our aim is to determine some relations between the $\text{Cl}(X)$-degrees of the divisors arising this way. Using Construction 2.8 this gives rise to an abelian group having $\text{Cl}(X)$ as a factor group.

Let $D_{ij,1} \cup \cdots \cup D_{ij,c(i)}$ be the decomposition of $V(X, T_{ij})$ into prime divisors. As $R(A, P_0)$ is $K_0$-factorial and $T_{ij}$ is $K_0$-prime, [ADHL15] Prop. 1.5.3.3, see Remark 2.11 implies

Moreover, due to Lemma 2.13 the defining relations of $R(A', P'_0)$ have degree zero. In particular, due to Proposition 2.9 for every $i = 0, \ldots, r$ and $1 \leq t \leq c(i)$ we obtain a term $T_{ij,1}^{t, i} = T_{i1,1}^{t, i} \cdots T_{in_i,1}^{t, i}$ of degree zero occurring in the relations of
$R(A', P'_0)$. This gives rise to relations

\[(2.13.2) \quad \sum_{j=1}^{n_1} l_{ij,t} [D_{ij,t}] = [0] \in \text{Cl}(X),\]

where $i = 0, \ldots, r$ and $t = 1, \ldots, c(i)$. As $l_{i,1} = \cdots = l_{i,c(i)}$ holds for any $i = 0, \ldots, r$, the relations \[(2.13.1)\] and \[(2.13.2)\] give rise to block matrices $A(c(i), l_{i,1})$ in a matrix $P$ as in Construction 2.8. In particular we get an $m' \times n'$ matrix with $m' := \sum_{i=0}^{r}(n_i + c(i))$ and $n' := \sum_{i=0}^{r} c(i) \cdot n_i$ of the following form

\[(2.13.3) \quad P := \begin{bmatrix} A(c(0), l_{0,1}) & 0 & \cdots & 0 \\ 0 & A(c(1), l_{1,1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(c(r), l_{r,1}) \end{bmatrix}.\]

Note that $P$ is of rank $\sum_{i=0}^{r}(n_i - 1 + c(i))$ and the $\text{rk}(P)$-th determinantal divisor of $P$ equals the product of the $(n_i - 1 + c(i))$-th determinantal divisors of the block matrices $A(c(i), l_{i,1})$. With Lemma 2.15 we conclude that the divisor class group of $X$ is isomorphic to a factor group of the group

\[(2.13.4) \quad \mathbb{Z}^{n'}/\text{im}(P^*) \cong \mathbb{Z}^{n'-\text{rk}(P)} \times G\]

with some finite abelian group $G$ of order $k$ with $k|(c_{i,0}^{(0)} - 1) \cdots c_{r,1}^{(r)-1})$.

We show that $\mathbb{Z}^{n'}/\text{im}(P^*) \leq \text{Cl}(X)$ and therefore equality holds. For this purpose we compare the dimensions of $X = \text{Spec} R(A, P_0)$ and $\overline{X} = \text{Spec} R(A', P'_0)$:

$$\dim(\overline{X}) - \dim(X) = n' - (r'^{-1} - 1) - (n - (r - 1)) = n' - \sum_{i=0}^{r} c(i) + 2 - \sum_{i=0}^{r} n_i + (r - 1) = n' - \text{rk}(P).$$

With $X = \overline{X} \# \text{Spec} \mathbb{C}[\text{Cl}(X)]$ we conclude $\mathbb{Z}^{n'-\text{rk}(P)} \leq \text{Cl}(X)$. Using Lemma 2.7 we obtain

$$|\text{Cl}(X)^{\text{tors}}| \leq |G| \leq |\text{Cl}(X)^{\text{tors}}|$$

and the assertion follows. $\square$

We turn towards the proof of the second assertion of Theorem 1.1.

**Definition 2.14.** Let $X$ be an irreducible normal variety and $Y \subseteq X$ a prime divisor. Let furthermore $\mathfrak{A} := \langle f_1, \ldots, f_r \rangle \leq \mathcal{O}(X)$ be any ideal. Then we define the order of $\mathfrak{A}$ along $Y$ to be $\text{min}(\text{ord}_Y(f_i))$: $i = 1, \ldots, r$.

**Lemma 2.15.** Let $X$ be an irreducible normal variety, $\mathfrak{A} := \langle f_1, \ldots, f_r \rangle \leq \mathcal{O}(X)$ any ideal and $f \in \mathcal{O}(X)$. Then the following statements are equivalent:

(i) $\text{ord}_Y(\mathfrak{A}) = \text{ord}_Y(f)$ holds for all prime divisors $Y \subseteq X$.

(ii) $\langle f \rangle = \mathfrak{A}$ holds, i.e. $\mathfrak{A}$ is a principal ideal.

In particular the Weil-divisor $D := \sum \text{ord}_Y(\mathfrak{A})$, where the sum runs over all prime divisors $Y \subseteq X$, is principal if and only if $\mathfrak{A}$ is a principal ideal.

**Proof.** We prove (i) $\Rightarrow$ (ii). Observe that $f | f_i$ holds for $i = 1, \ldots, r$ as $\text{div}(f) \leq \text{div}(f_i)$ by construction. In particular $\langle f \rangle \supseteq \mathfrak{A}$. We prove the other inclusion. Consider the covering $\bigcup_{i=1}^{r} U_i$ of $X$ where

$$U_i := X \setminus (Y_{i,1} \cup \cdots \cup Y_{i,n_i}),$$

where all prime divisors $Y$ with $\text{ord}_Y(f_i) \neq \text{ord}_Y(\mathfrak{A})$ occur among the $Y_{i,1}$. Then inside $U_i$ we have $f_i | f$. We obtain $c_i \cdot f_i = f$ with $c_i \in \mathcal{O}(U)^*$. Considering the associated sheaf $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ we obtain $f \in \overline{\mathfrak{A}}(X) = \mathfrak{A}$. The other implication is clear. $\square$
Lemma 2.16. Let \( R(A, P_0) \) be a ring as in Construction \[2.1\] with \( g_0 \) of the form \( T_0^0 + T_1^1 + T_2^2 \) and assume gcd\( (l_0, l_1) = g_0 \) and gcd\( (l_0, l_2) = 2 \) and gcd\( (l_1, l_1) = 1 \) holds. Fix an integer \( y \in \mathbb{Z}_{\geq 0} \) with \( y \mid l_0 \) and set \[
\mathfrak{A}_y := (T_1^{1/2l_1} + i \cdot T_2^{1/2l_2}, T_0^{1/y} l_0) \leq R(A, P_0).
\]
Then \( \mathfrak{A} \) is a principal ideal if and only if \( y = 1 \) holds.

Proof. Note that \( \mathfrak{A}_1 = (T_1^{1/2l_1} + i T_2^{1/2l_2}) \) holds in \( R(A, P_0) \). So let \( y \neq 1 \) and assume there is an \( f \in \mathfrak{A}_0 \) with \( (f) = \mathfrak{A} \). Then there exist \( g_1, g_2, h_1, h_2 \in \mathbb{K}[T_{ij}, S_k] \) with \( g_1 \cdot f + I = T_0^{1/y} l_0 + I \) and \( g_2 \cdot f + I = T_1^{1/2l_1} + i T_2^{1/2l_2} + I \) and \[
 h_1 \cdot T_0^{1/y} l_0 + h_2 \cdot (T_1^{1/2l_1} + i T_2^{1/2l_2}) + I = f + I. \]

Inserting the third formula into the first one we obtain \[
T_0^{1/y} l_0 + I = g_1 \cdot h_1 \cdot T_0^{1/y} l_0 + g_1 \cdot h_2 \cdot (T_1^{1/2l_1} + i T_2^{1/2l_2}) + I
\]
and so in particular
\[
(2.16.1) \quad h := (g_1 \cdot h_1 - 1) \cdot T_0^{1/y} l_0 + g_1 \cdot h_2 \cdot (T_1^{1/2l_1} + i T_2^{1/2l_2}) \in I.
\]

As there can not occur any term \( T_0^{1/y} l_0 \) in \( I \) for \( y \neq 1 \), we conclude that \( g_1 \) and \( h_1 \) each have a constant term. Inserting the third formula above into the second, we obtain a constant term in \( g_2 \) and \( h_2 \) with similar arguments. But this leads to a term \( \lambda \cdot (T_1^{1/2l_1} + i \cdot T_2^{1/2l_2}) \) with \( \lambda \neq 0 \) in \( \mathfrak{A}_0 \); a contradiction to \( h \in I \).

Proof of Theorem \[1.3]\ Case (ii). With the same arguments as in the Case (i) we get relations of the form \( (2.13.1) \). Moreover since the degrees of the relations and thus all terms occurring in the Cox ring \( R(A', P_0') \) of \( X = \text{Spec } R(A, P_0) \) coincide, we obtain
\[
(2.16.2) \quad \sum_{j=1}^{n_0} l_{0j,1} [D_{0j,1}] = \sum_{j=1}^{n_0} l_{ij,t(i)} [D_{ij,t(i)}] \in \text{Cl}(X),
\]
where \( i = 0, \ldots, r \) and \( 1 \leq t(i) \leq c(i) \). Those replace the relations \( (2.13.2) \). Suitably ordered, this gives rise to a matrix
\[
(2.16.3) \quad P := \begin{bmatrix}
-l_{0,1} & l_{0,2} & 0 & \cdots & 0 \\
E_{n_0} & E_{n_0} & 0 & \cdots & 0 \\
* & 0 & A(c(1), l_{1,1}) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & 0 & 0 & \cdots & A(c(r), l_{r,1})
\end{bmatrix},
\]
where we use \( c(0) = 2 \) and the * indicates that there might be some non-zero entries. By suitably swapping columns, applying elementary row operations and using \( l_{0,1} = l_{0,2} \) one achieves a matrix
\[
P' := \begin{bmatrix}
-2l_{0,1} & 0 & \cdots & 0 \\
* & A(c(1), l_{1,1}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & 0 & \cdots & A(c(r), l_{r,1}) \\
E_{n_0} & 0 & \cdots & 0
\end{bmatrix}.
\]
The rank of \( P' \) equals \( \sum_{i=0}^r (n_i - 1 + c(i)) \). Using \( l_{i,1} = l_{i,2} = l_{i}/2 \) for \( i = 0, 1, 2 \), we obtain with Lemma \[2.9\] that the \((n_i - 1 + c(i))\)-th determinantal divisors of \( A(c(i), l_{i,1}) \) divides \( l_{i,2} \) for \( i = 1, 2 \). Using \( l_{i,1} = \ldots = l_{i,3} = l_i \) for \( i \geq 3 \) we obtain that the \((n_i - 1 + c(i))\)-th determinantal divisors of \( A(c(i), l_{i,1}) \) divides \( l_i^r \) for \( i \geq 3 \). Thus considering the maximal square submatrices just including one of the first \( n_0 \) columns, Laplace expansion with respect to the first row shows that the \( \text{rk}(P') \)-th
determinantal divisor of $P'$ divides $l_0$. If we delete all of the first $n_0$ columns we observe that the $(\text{rk}(P')-1)$-th determinantal divisor of $P'$ divides 1, i.e., it equals 1 up to sign. Thus $\text{Cl}(X)$ is a factor group of
\[ \mathbb{Z}^{n_0}/\text{im}(P') \cong \mathbb{Z}^{n_0-\text{rk}(P')} \times G, \]
where $G$ is a finite group of order $k$ with $k|(l_0(l_1/2)l_2/2 \ldots l_i)$. We show equality of these groups. Observe that we may assume the relation $g_0$ of $R(A, P_0)$ to be of the form $T_0^2 + T_1^{l_1} + T_2^{l_2}$. In particular the irreducible components $D_{0j,1}$ and $D_{0j,2}$ of $V(X; D_{0j})$ are of the form
\[ D_{0j,1} = V(T_{0j}, T_1^{l_1/2} + i \cdot T_2^{l_2/2}) \quad \text{and} \quad D_{0j,2} = V(T_{0j}, T_1^{l_1/2} - i \cdot T_2^{l_2/2}). \]
We conclude that for $y \in \mathbb{Z}_{>0}$ with $y | l_0$
\[ D := \sum_{j=1}^{n_0} \frac{1}{y} D_{0j,1} = \sum_{y} \text{ord}_y(\mathfrak{A}_y) \]
holds with $\mathfrak{A}_y$ as in Lemma 2.10. As $\mathfrak{A}_y$ is principal if and only if $y = 1$ holds, we obtain $\mathbb{Z}/l_0\mathbb{Z}$ as a factor of the divisor class group of $X$. Calculating the difference between the dimensions of $\text{Spec} \ R(A, P_0)$ and $\text{Spec} \ R(A', P_0)$ as in the proof of the case (i) we conclude $\mathbb{Z}^{n_0-\text{rk}(P')} \cong \text{Cl}(X)$. As due to Lemma 2.7 and the assumption that $\gcd(l_0, l_1) = \gcd(l_1, l_2) = \gcd(l_0, l_2) = 2$ and $\gcd(l_1, l_j) = 1$ whenever $j \notin \{0, 1, 2\}$ holds, $l_0$ does not divide $|\text{Cl}(X)|$ but $l_0/2$ does, we obtain
\[ 2 \cdot |\text{Cl}(X)| \leq |G| \leq 2 \cdot |\text{Cl}(X)| \]
and the assertion follows. \hfill \Box

Corollary 2.17. Let $X$ be an affine, rational, trinomial variety. Then the divisor class group of $X$ is free abelian if and only if $X$ is factorial or after reordering decreasingly we have $l_0 \geq l_1 \geq l_2 = \ldots = l_r = 1$.

Proof. Assume the divisor class group of $X$ is free abelian. Then either $X$ is factorial and thus $\text{Cl}(X) = \{0\}$ holds or we may apply Theorem 1.1 and conclude $\gcd(l_0, l_1) > 1$ and $l_2 = \ldots = l_r = 1$ holds. The other direction is a direct consequence of Theorem 1.1. \hfill \Box

As an application, we consider trinomial varieties with an isolated singularity; recall that [LS13] Thm. 6.5] gives a complete description of all those with trivial divisor class group.

Corollary 2.18. Let $X$ be an affine, trinomial variety with an isolated singularity. Then $\dim(X) \leq 5$ holds and we are in one the following cases:

(i) If $\dim(X) = 2$ holds and $X$ is rational then its divisor class group is a torsion group.

(ii) If $\dim(X) = 3$ holds then $X$ is rational and its divisor class group is free abelian.

(iii) If $\dim(X) \geq 4$ holds then $X$ is factorial.

Proof. Assume $X$ is two-dimensional. Then $n_i = 1$ holds for all $i = 0, \ldots, r$ and $X$ has an isolated singularity at zero. Thus if $X$ is rational, Theorem 1.1 implies that its divisor class group is a torsion group.

Assume $\dim(X) \geq 3$ holds. Then, considering the Jacobian of $X$, we conclude that $X$ has an isolated singularity at zero if and only if $X$ is a hypersurface with defining relation $g = r_0^{l_0} + r_1^{l_1} + r_2^{l_2}$, where $1 \leq n_0 \leq n_1 \leq n_2 = 2$ and $r_j = 1$ whenever $n_j = 2$, see also [LS13]. In particular $\dim(X) \leq 5$ holds and $X$ is rational due to Remark 2.2. If $n_0 = n_1 = 1$ holds, i.e. $X$ is of dimension three, we obtain $l_0, l_1 \geq l_2 = 1$. Applying Corollary 2.17 we conclude that $X$ is free abelian. In the
Remark 2.19. We compare our results with the existing works already stated in the introduction.

In [Fle81] H. Flenner shows that rational three-dimensional quasihomogeneous complete intersections over algebraically closed fields of arbitrary characteristic with an isolated singularity have a free abelian divisor class group. Corollary 2.18 shows that this is as well true for all trinomial varieties with isolated singularity of dimension at least three.

Using Corollary 2.17 one can construct examples of affine, rational, trinomial varieties $X$ with free abelian divisor class group having a higher dimensional singular locus: The three-dimensional variety

$$V(T_1^4 + T_1^2 + T_2^2 + T_3^2) \subseteq \mathbb{C}^4$$

has divisor class group $\mathbb{Z}$ and a one-dimensional singular locus. Note that not any three-dimensional trinomial variety has a free abelian divisor class group as for instance, we obtain divisor class group $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for the hypersurface

$$V(T_1^4 + T_1^2 + T_2^3) \subseteq \mathbb{C}^4.$$  

In [Lan83, SSS84, SS07] J. Lang, A. Singh, S. Spiroff, G. Scheja and U. Storch present divisor class group computations for hypersurfaces of the form $K[z, x_1, \ldots, x_d]/(z^n - g)$, where $g$ is a weighted homogeneous polynomial in $x_1, \ldots, x_d$ of degree relatively prime to $n$ are treated. In particular using various methods they give explicit descriptions of the divisor class groups in any characteristic. In particular the divisor class groups of trinomial hypersurfaces of the form $V(T_1^2 + T_1 + T_2) \subseteq \mathbb{C}^3$ with gcd$(l_0, l_1) = 1 = \text{gcd}(l_0, l_2)$ can be calculated with their results and are regained as part of our Theorem 1.1 (i). Note that any rational trinomial variety fulfilling Remark 2.2 (iii) leaves the framework of [Lan83, SS07, SSS84] but can be treated via Theorem 1.1 explicit examples are the two hypersurfaces given above.

As a direct consequence of the proof of Theorem 1.1 we obtain the following description of the divisor class group grading on the Cox ring $R(A', P_0')$ of a rational trinomial variety $\text{Spec } R(A, P_0)$:

**Corollary 2.20.** Let $X := \text{Spec } R(A, P_0)$ be a rational trinomial variety and assume that $P_0$ is gcd-ordered. Then the divisor class group grading on the Cox ring $R(A', P_0')$ is given as

$$\text{deg}(T_{i,j,k}) = Q(e_{i,j,k}), \quad \text{with } Q : \mathbb{Z}^{n'+m} \to \mathbb{Z}^{n'+m}/\text{im}(P^*)$$

where $P$ is one of the following:

(i) If $c := \text{gcd}(l_0, l_1) > 1$ and gcd$(l_1, l_j) = 1$ holds whenever $j \notin \{0, 1\}$, then $P$ is built up as in (2.15.5).

(ii) If gcd$(l_0, l_1) = \text{gcd}(l_1, l_2) = \text{gcd}(l_0, l_2) = 2$ and gcd$(l_1, l_j) = 1$ holds whenever $j \notin \{0, 1, 2\}$, then $P$ is built up as in (2.16.5).

**Remark 2.21.** As a direct consequence of Theorem 1.1 we can compute the divisor class groups of all affine varieties arising from a hyperplatonic Cox ring. We list the basic platonic tuple (bpt) of $R(A, P_0)$ and the divisor class group of $X := \text{Spec } R(A, P_0)$ in a table:
| Case | bpt of $R(A, P_0)$ | divisor class group |
|------|-------------------|-------------------|
| (i)  | $(4, 3, 2)$       | $\mathbb{Z}^{n_1 + n_3 + \cdots + n_r - (r-1)} \times \mathbb{Z}/3\mathbb{Z}$ |
| (ii) | $(3, 3, 2)$       | $\mathbb{Z}^{2(n_2 + \cdots + n_r - (r-1))} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ |
| (iii)| $(x, y, 1)$       | $\mathbb{Z}^{\gcd(x, y) - 1}(n_2 + \cdots + n_r - (r-1))$ |
| (iv) | $(x, 2, 2)$ and $2 | x$ | $\mathbb{Z}^{n_0 + n_3 + \cdots + n_r - (r-1)} \times \mathbb{Z}/x\mathbb{Z}$ |
| (v)  | $(x, 2, 2)$ and $2 | x$ | $\mathbb{Z}^{n_0 + n_1 + n_2 + 3(n_3 + \cdots + n_r - (r-1))} \times \mathbb{Z}/x\mathbb{Z}$ |

With the explicit description of the grading of the Cox ring of a rational trinomial variety given via the matrices $P$ as described in Corollary 2.20, we are able to prove our second main result.

Proof of Corollary 2.20. In a first step we show that for any hyperplatonic ring $R$ with basic platonic triple $(l_0, l_1, l_2)$, there exists a good quotient $\mathbb{C}^{n+m} \supseteq \text{Spec } R \to Y(l_0, l_1, l_2)$ with respect to some grading group $G$. Setting

$$\hat{P} := \begin{bmatrix} 1/l_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/l_r \end{bmatrix},$$

the map $Q: \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}/\text{im}(\hat{P}^\ast)$ defines a grading on $R$, which coarsens the grading given by $P_0$ as in Construction 2.1. Moreover the Veronese subalgebra $S$ with respect to the degree zero is generated by the monomials $T_0^{l_0/l_0}, \ldots, T_r^{l_r/l_r}$ and we conclude $\text{Spec } S \cong Y(l_0, l_1, l_2)$.

Now denote by $R'$ resp. $S'$ the Cox rings of $\text{Spec } R$ resp. $\text{Spec } S$ as given in Proposition 2.5 with the grading given by matrices $P(R)$ resp. $P(S)$ as in Corollary 2.20. We claim that we obtain the following commutative diagram

$$\begin{array}{ccc} R' & \rightarrow & R \\ \uparrow & & \uparrow \\ S' & \rightarrow & S \end{array}$$

where the upward arrow on the r.h.s. is the embedding of a Veronese subalgebra with respect to some grading group $\mathbb{Z}^k$ and the other arrows denote the embeddings of the Veronese subalgebras as defined above. This proves the assertion as considering the grading given by $P(S)$ on $S'$ one directly checks that the isomorphism $S' \to \mathbb{C}[T_0, T_1, T_2]/(T_0^{l_0} + T_1^{l_1} + T_2^{l_2})$ deleting the redundant relations is a graded isomorphism with respect to the Cox ring grading on the latter ring.

To prove our result it is now only necessary to show that the composition of the embeddings $S \to S' \to R'$ given by the matrices $\hat{P}$ and $P(S)$ factorizes over the embedding $R \to R'$ given by $P(R)$. Note that the grading giving rise to the composed Veronese embedding $S \to S' \to R'$ can be represented by a matrix of the same shape and with the same number of columns as $P(R)$ but replacing the matrices $A(c(i), l_{i,1})$ by matrices of the following form:

$$B(c(i), l_{i,1}) := \begin{bmatrix} l_{i,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_{i,1} \\ l_{i,1}/l_{i,1} & \cdots & l_{i,1}/l_{i,1} \end{bmatrix} \in \text{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}),$$
and in case of $P$ as in (2.16.3) additionally replacing the rows $2$ to $n_0 + 1$ with one row $(l_{0,1}/l_{0,1}, l_{0,1}, 0, \ldots, 0)$. In particular the row lattice of this matrix is a sublattice of the row lattice of $P(R)$ and we only have to show that it is a saturated sublattice. By the structure of the occurring matrices this means that the row lattice generated by the matrix $B(c(i), l_{i,1})$ is a saturated sublattice of the row lattice of the matrix $A(c(i), l_{i,1})$. Note that the row lattice of $A(c(i), l_{i,1})$ is generated by the rows of

$$
\begin{bmatrix}
  l_{i,1} & \cdots & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & l_{i,1} & 0 \\
  E_{n_i} & \cdots & E_{n_i} & \cdots & E_{n_i}
\end{bmatrix} \in \text{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}).
$$

In particular the last $n_i$ rows span a saturated sublattice of this row lattice. As the lattice generated by $(l_i/l_i, \ldots, l_i/l_i)$ lies saturated in this sublattice, the assertion follows.

□

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