Ginzburg–Landau Expansion and the Upper Critical Field in the Disordered Attractive Hubbard Model (Brief Review)

N. A. Kuleeva*, E. Z. Kuchinskii**, and M. V. Sadovskii***

*a Institute for Electrophysics, Russian Academy of Sciences, Ural Branch, Yekaterinburg, 620016 Russia
**e-mail: strigina@iep.uran.ru
***e-mail: sadovski@iep.uran.ru

Received September 24, 2020; revised September 24, 2020; accepted September 29, 2020

We present a brief review of our studies of disorder influence upon Ginzburg–Landau expansion coefficients in Anderson–Hubbard model with attraction in the framework of the generalized DMFT + Σ approximation. A wide range of attractive potentials $U$ is considered from weak coupling limit, where superconductivity is described by BCS model, to the limit of very strong coupling, where superconducting transition is related to the Bose–Einstein condensation of compact Cooper pairs, which are formed at temperatures significantly higher than the superconducting transition temperature, as well as the wide range of disorders from weak to strong, when the system is in the vicinity of Anderson transition. For the same range of parameters we study in detail the temperature behavior of orbital and paramagnetic upper critical field $H_{c2}(T)$, which demonstrates the anomalies due both to the growth of attractive potential and to the effects of strong disordering.

DOI: 10.1134/S002136402021002X

1. INTRODUCTION

The studies of disorder influence upon superconductivity have rather long history. In classic papers by Abrikosov and Gor’kov [1–4], the weak disorder limit ($p_F l \ll 1$, where $p_F$ is Fermi momentum and $l$ is the mean free path) was considered for the case of weak coupling superconductivity, which is well described by BCS theory. The notorious “Anderson theorem” on the critical temperature $T_c$ of superconductors with “normal” (nonmagnetic) disorder [5, 6] is also related to this limit. The generalization of the theory of “dirty” superconductors to the case of strong enough disorder ($p_F l \sim 1$) (and further, up to the vicinity of Anderson transition) was made in [7–10], where superconductivity was also considered in the weak coupling limit.

The problem of BCS theory generalization to the region of very strong coupling is also analyzed for a long time. Significant progress in this direction was achieved in a paper by Nozieres and Schmitt-Rink [11], who proposed an effective method to study the crossover from BCS behavior in the weak coupling limit to Bose–Einstein condensation (BEC) in the region of strong coupling. At the same time, the problem of superconductivity in disordered systems in the limit of strong coupling and in the region of BCS–BEC crossover is poorly studied.

One of the simplest models to study the BCS–BEC crossover is the Hubbard model with attraction. The most successful approach to study Hubbard model, both to describe the strongly correlated systems for the case of repulsive interactions and to study BCS–BEC crossover, is the dynamical mean field theory (DMFT) [12–14]. In recent years we have developed the generalized DMFT + Σ approach to Hubbard model [15–21], which is quite convenient for the studies the role of different external (with respect to those taken into account by DMFT) interactions. In [22], we used this approach to analyze the single-particle properties and optical conductivity of the Hubbard model with attraction. Further, on, we used the DMFT + Σ method in [23, 24] to study disorder influence on the superconducting transition temperature, which was calculated within Nozieres–Schmitt–Rink approach.

Starting with the classic paper by Gor’kov [3] it is well known that the Ginzburg–Landau expansion is of fundamental importance in the theory of dirty superconductors, allowing the effective studies of the behavior of different physical properties dependences close to critical temperature on disorder [6]. The generalization of this theory to the region of strong disorder (up to Anderson metal–insulator transition) was also based on microscopic derivation of the coefficients of this expansion [7–10]. However, this analy-
sis, as noted above, was always done in the weak coupling limit of BCS theory.

In this paper we shall present a short review of the results obtained in [25–27], devoted to microscopic derivation of the coefficients of Ginzburg–Landau expansion, taking into account the role of disorder in the wide region of BCS–BEC crossover and including the region of strong disorder in the vicinity of Anderson transition. We shall also review the closely related results of [28, 29] on the temperature dependence of superconducting transition. We shall also review the closely related results of [28, 29] on the temperature dependence of superconducting transition. We shall also review the closely related results [28, 29] on the temperature dependence of superconducting transition.

2. SUPERCONDUCTING TRANSITION TEMPERATURE

Consider disordered nonmagnetic Hubbard model with attraction and the Hamiltonian:

$$H = -t \sum_{\langle i,j \rangle} a_{i\sigma}^\dagger a_{j\sigma} + \sum_{i} \varepsilon_i n_{i\sigma} - U \sum_{i} n_{i\uparrow} n_{i\downarrow},$$

(1)

where $t > 0$ is the transfer amplitude between the nearest neighbors, $U$ is onsite attraction potential, $n_{i\sigma} = a_{i\sigma}^\dagger a_{i\sigma}$ is onsite number of electrons operator, $a_{i\sigma} (a_{i\sigma}^\dagger)$ is annihilation (creation) operator of an electron with spin $\sigma$. Local energies $\varepsilon_i$ are assumed to be independent random variables on different lattice sites. We assume the Gaussian distribution of energy levels $\varepsilon_i$:

$$\mathcal{P}(\varepsilon) = \frac{1}{\sqrt{2\pi}W} \exp\left(-\frac{\varepsilon^2}{2W^2}\right).$$

(2)

Parameter $W$ here serves as the measure of disorder strength and the Gaussian random field of energy levels creates “impurity” scattering, which is considered within the standard approach, based upon calculations of the averaged Green’s functions [30, 31].

The generalized DMFT + $\Sigma$ approach [15–17, 20] extends the standard dynamical mean field theory (DMFT) [12–14] by the addition of “external” self-energy part ($\Sigma$) (in general momentum dependent), which is due to any interaction outside DMFT and provides an effective calculation method both for single-particle and two-particle properties [18, 19].

For an external SEP entering DMFT + $\Sigma$ loop, for the case of scattering by disorder analyzed here, we use the simplest self-consistent Born approximation, neglecting the “crossing” diagrams for impurity scattering:

$$\Sigma_{\text{imp}}(\varepsilon) = W^2 \sum_p G(\varepsilon, p),$$

(3)

where $G(\varepsilon, p)$ is the full single-electron Green’s function in the DMFT + $\Sigma$ approximation.

To solve the effective single Anderson impurity problem of DMFT we used the numerical renormalization group (NRG) [32].

Below, we consider the “bare” band with semi-elliptic density of states (per unit cell and per single spin projection):

$$N_0(\varepsilon) = \frac{2}{\pi D^2} \sqrt{D^2 - \varepsilon^2},$$

(4)

where $D$ defines the half-width of the conduction band, which is a good approximation for the three-dimensional case. In [24] we have shown that in DMFT + $\Sigma$ approach for the model with semi-elliptic density of states all the influence of disorder upon single-particle properties is reduced to the disorder-induced broadening of the band, i.e., to the substitution $D \to D_{\text{eff}}$, where $D_{\text{eff}}$ is the effective half-width of the bare band in the absence of electronic correlations ($U = 0$), widened by disorder:

$$D_{\text{eff}} = D \sqrt{1 + 4 \left(\frac{W^2}{D^2}\right)}.$$  

(5)

The bare (in the absence of $U$) density of states, “dressed” by disorder:

$$\tilde{N}_0(\varepsilon) = \frac{2}{\pi D_{\text{eff}}^2} \sqrt{D_{\text{eff}}^2 - \varepsilon^2},$$

(6)

remains semi-elliptic in the presence of disorder.

All calculations below were done for the case of quarter-filled band (number of electrons per lattice site $n = 0.5$).

To consider superconductivity in a wide interval of the pairing interaction $U$, following [22, 24], we use the Nozieres–Schmitt–Rink approximation [11], which allows a qualitatively correct (though approximate) description of the BCS–BEC crossover region. In this approach, the critical temperature $T_c$ is determined [24] by the usual BCS-like equation:

$$1 = \frac{\mu}{2} \int_{-\infty}^{\infty} d\varepsilon \tilde{N}_0(\varepsilon) \frac{\tanh \frac{\varepsilon - \mu}{2T_c}}{\varepsilon - \mu},$$

(7)

where the chemical potential $\mu$ for different values of $U$ and $W$ is obtained from the standard equation for the number of electrons (band filling), determined from the full Green’s function, calculated in the DMFT + $\Sigma$ approximation. This allows finding $T_c$ for a wide interval of the values of parameters of the theory, including the BCS–BEC crossover region and the limit of strong coupling, as well as for different levels of disorder. It is the essence of the Nozieres–Schmitt–Rink interpolation scheme: the transition temperature in the weak coupling region is controlled by the equation for Cooper instability (7), while in the strong coupling limit it is determined as the temperature of BEC, which is controlled by the chemical potential. In [24], we have shown that disorder influence on the critical
The Ginzburg–Landau expansion for the difference of free energies in superconducting and normal states can be written in the standard form [31]

\[ F_s - F_n = A|\Delta_q|^2 + q^2C|\Delta_q|^2 + B|\Delta_q|^4, \]

where \( \Delta_q \) is the amplitude of the Fourier component of order parameter. Expansion (8) is determined by diagrams of the loop expansion for free energy in the field of fluctuations of order parameter (denoted by dashed lines) with small wave vector \( q \) [31], shown in Fig. 2 [31].

Within the Nozieres–Schmitt–Rink approach [11], we use the weak coupling approximation to analyze Ginzburg–Landau coefficients, so that the loops with two and four Cooper vertexes shown in Fig. 2 do not contain contributions from Hubbard attraction and are dressed only by impurity scattering. However, as in the case of calculation of \( T_c \), the chemical potential, which is essentially dependent on coupling strength and in the strong coupling limit determines the condition of the Bose–Einstein condensation of Cooper pairs, should be calculated within the full DMFT + \( \Sigma \) procedure. In [25] we have shown that in this approach the coefficients \( A \) and \( B \) are given by the expressions

\[ A(T) = \frac{1}{U} - \int_{-\infty}^{\infty} d\varepsilon \tilde{N}_0(\varepsilon) \tanh \frac{\varepsilon - \mu}{2T}, \]

\[ B = \int_{-2(\varepsilon - \mu)}^{\infty} d\varepsilon \left( \frac{\tanh \frac{\varepsilon - \mu}{2T} - (\varepsilon - \mu)/2T}{\cosh^2 \frac{\varepsilon - \mu}{2T}} \right) \tilde{N}_0(\varepsilon). \]

For \( T \rightarrow T_c \), the coefficient \( A(T) \) acquires the form

\[ A(T) \equiv \alpha(T - T_c). \]

In the BCS limit, we obtain the standard result for the coefficients \( \alpha \) and \( B \) [31]:

\[ \alpha_{BCS} = \frac{\tilde{N}_0(\mu)}{T_c}, \quad B_{BCS} = \frac{7\zeta(3)}{8\pi^2T_c^2} \tilde{N}_0(\mu). \]

Thus, the coefficients \( A \) and \( B \) are determined only by the density of states \( \tilde{N}_0(\varepsilon) \) widened by disorder and by the chemical potential. For the semi-elliptic bare density of states the dependence of these coefficients on disorder is due only to substitution \( \tilde{N}_0(\varepsilon) = \frac{\tilde{N}_0(\mu)}{T_c} \), and in the strong coupling region the critical temperature is mainly determined by BEC condition for Cooper pairs and drops with the growth of \( U \) as \( T \sim tU/\Delta + \Delta^2 + \Delta^4 \), passing the maximum at \( U/2D_{eff} \sim 1 \). The review of these and some other results obtained for disordered Hubbard model in DMFT + \( \Sigma \) approximation can be found in [21].

3. GINZBURG–LANDAU EXPANSION

The Ginzburg–Landau expansion for the difference of free energies in superconducting and normal states can be written in the standard form [31]

\[ F_s - F_n = A|\Delta_q|^2 + q^2C|\Delta_q|^2 + B|\Delta_q|^4, \]

where \( \Delta_q \) is the amplitude of the Fourier component of order parameter. Expansion (8) is determined by diagrams of the loop expansion for free energy in the field of fluctuations of order parameter (denoted by dashed lines) with small wave vector \( q \) [31], shown in Fig. 2 [31].

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This universal dependence of specific heat discontinuity on $U/2D_{\text{eff}}$ is shown in Fig. 3. In BCS limit specific heat discontinuity grows with coupling strength, while in BEC limit it drops, passing through a maximum at $U/2D_{\text{eff}} = 0.55$. This behavior of specific heat discontinuity is determined mainly by the behavior of $T_c$ (cf. Fig. 1), while the ratio $\frac{\alpha^2}{B}$ in Eq. (13) smoothly depends on the coupling strength.

Now we shall follow [26, 27] to analyze the coefficient $C$. From diagrammatic representation of Ginzburg–Landau expansion, shown in Fig. 2, it is clear that $C$ is determined as a coefficient before $\Gamma^{0}$ in Cooper-like two-particle loop (first term in Fig. 2). Thus, we obtain the expression

$$C = -T \lim_{q \to 0} \sum_{\alpha \neq \beta} \frac{\Psi_{\alpha \beta}(\omega, q) - \Psi_{\alpha \beta}(\omega, 0)}{q^2}, \quad \quad \text{(14)}$$

where $\Psi_{\alpha \beta}(\omega, q)$ is the two-particle Green’s function in the Cooper channel, dressed (in the Nozieres–Schmitt–Rink approximation) only by impurity scattering.

In BCS limit and in the absence of disorder, the coefficient $C$ acquires the form [31]

$$C_{\text{BCS}} = \frac{7\zeta(3)}{16\pi^2 T_c} N_0(\mu) \frac{v_F^2}{d}, \quad \quad \text{(15)}$$

where $v_F$ is the velocity at the Fermi surface, $d$ is space dimensionality. Disorder influence on coefficient $C$ is not reduced only to the substitution $N_0 \to \bar{N}_0$, so that in the presence of disorder, in contrast to coefficients $\alpha$ and $B$ (cf. (12)), even in BCS limit we cannot obtain any compact expression for $C$ similar to Eq. (15). After rather cumbersome analysis [26, 27], we get the following general expression for the coefficient $C$:

$$C = -\frac{1}{8\pi} \int_{-\infty}^{\infty} d\epsilon \frac{\tanh \frac{\epsilon}{2T}}{\epsilon} \text{Im} \left( \frac{iD(2\epsilon)\sum_p \Delta G_p(\epsilon)}{\epsilon + i\delta} \right)$$

$$= -\frac{1}{8\pi} \int_{-\infty}^{\infty} d\epsilon \frac{\tanh \frac{\epsilon}{2T}}{\epsilon^2} \text{Re}(D(2\epsilon)\sum_p \Delta G_p(\epsilon))$$

$$- \frac{1}{16T} \text{Im} \left( D(0)\sum_p \Delta G_p(0) \right), \quad \quad \text{(16)}$$

where $\Delta G_p(\epsilon) = G^R_p(\epsilon, p) - G^A(-\epsilon, p)$ and $D(\omega)$ is the frequency dependent generalized diffusion coefficient [31, 33–39], which is determined within generalization of the self-consistent theory of localization by the self-consistency equation [19]

$$D(\omega) = i \langle \nu \rangle^2 \frac{d}{d} \left( \omega - \Delta \Sigma^{RA}(\omega) \right)$$

$$+ W^A(\Delta G_p(\epsilon))\sum_q \frac{1}{\omega + iD(\omega q^2)}^{-1}, \quad \quad \text{(17)}$$

where $\omega = 2\epsilon$, $\Delta \Sigma^{RA}(\omega) = \Sigma^{imp}(\epsilon) - \Sigma^{imp}(-\epsilon)$, $d$ is the dimension of space, and the velocity $\langle \nu \rangle$ is determined by the expression

$$\langle \nu \rangle = \frac{\sum_p \left| \nu_p \right| \Delta G_p(\epsilon)}{\sum_p \Delta G_p(\epsilon)}; \quad \nu_p = \frac{\partial \epsilon(p)}{\partial p}. \quad \quad \text{(18)}$$

Taking into account applicability limits of diffusion approximation, summation over $q$ in Eq. (17) should be limited by [31, 38]:

$$q < k_0 = \min\{l^{-1}, p_F\}, \quad \quad \text{(19)}$$

where $l$ is the mean free path due to elastic scattering by disorder, $p_F$ is the Fermi momentum.

Thus, we obtain an interpolation scheme to determine the coefficient $C$, which in the weak disorder limit reproduces the results of the "ladder" approximation, while in the strong disorder limit it takes into account the effects of Anderson localization (in the framework of self-consistent theory of localization).

It was shown [19, 20] that in DMFT + $\Sigma$ approximation for Anderson–Hubbard model the critical disorder for Anderson metal–insulator transition $W/2D = 0.37$ (for the choice of cutoff as in Eq. (19)), so that in this approximation it does not depend on the value of Hubbard interaction $U$. The approach developed above allows determination of coefficient $C$ including the region of Anderson insulator with disorder $W/2D > 0.37$. 

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\textbf{Fig. 3.} The universal dependence of specific heat discontinuity on $U/2D_{\text{eff}}$ for different levels of disorder.
4. PHYSICAL PROPERTIES NEAR THE SUPERCONDUCTING TRANSITION TEMPERATURE

The coherence length at a given temperature $\xi(T)$ determines the characteristic scale of inhomogeneities of superconducting order parameter:

$$\xi^2(T) = \frac{C}{A}. \quad (20)$$

From Eq. (11), we have $A = \alpha(T - T_c)$; then,

$$\xi(T) = \frac{\xi}{\sqrt{1 - T/T_c}}, \quad (21)$$

where we introduce the coherence length of a superconductor as:

$$\xi = \frac{C}{\alpha T_c}. \quad (22)$$

which in the weak coupling limit and in the absence of disorder has the standard form [31]:

$$\xi_{BCS} = \frac{C_{BCS}}{\alpha_{BCS} T_c} = \frac{7\xi(3) v_F}{\sqrt{16\pi^2 d T_c}}. \quad (23)$$

The penetration depth of magnetic field into superconductor is defined as:

$$\lambda^2(T) = \frac{c^2 B}{32\pi e^2 AC}. \quad (24)$$

Thus,

$$\lambda(T) = \frac{\lambda}{\sqrt{1 - T/T_c}}, \quad (25)$$

where we have introduced:

$$\lambda^2 = \frac{c^2 B}{32\pi e^2 \alpha C T_c}, \quad (26)$$

which in the absence of disorder and in the weak coupling limit is:

$$\lambda^2_{BCS} = \frac{c^2 B_{BCS}}{32\pi e^2 \alpha_{BCS} C_{BCS} T_c} = \frac{c^2 B}{16\pi e^2 N_0(\mu) v_F^2}. \quad (27)$$

Note that $\lambda_{BCS}$ does not depend on $T_c$, i.e., on the coupling strength, and it can be conveniently used to normalize penetration depth $\lambda$ (26) at arbitrary $U$ and $W$.

Close to $T_c$ the upper critical magnetic field $H_{c2}$ is defined via Ginzburg–Landau coefficients as:

$$H_{c2} = \frac{\Phi_0}{2\pi \xi^2(T)} = -\frac{\Phi_0}{2\pi C} A, \quad (28)$$

where $\Phi_0 = \pi e$ is the magnetic flux quantum. Then the slope of the upper critical field close to $T_c$ is given by:

$$\frac{dH_{c2}}{dT} = \frac{\Phi_0 \alpha}{2\pi C}. \quad (29)$$

Coefficient $C$ is essentially a two-particle entity, thus it is not universally dependent on disorder in contrast to coefficients $A$ and $B$ and disorder influence upon it does not reduce only to effective band broadening by disorder. Let us now discuss the main results of our calculations for this coefficient (for more details cf. [26, 27]). The coefficient $C$ decreases rapidly with the growth of coupling strength. Especially strong drop is observed in the weak coupling region. Localization corrections become important in the limit of strong enough disorder ($W/2D > 0.25$). For such disorder level localization corrections significantly suppress the coefficient $C$ in the weak coupling region, while in the strong coupling region for $U/2D > 1$ localization corrections in fact do not change the value of the coefficient, even in the limit of strong disorder with $W/2D > 0.37$, when the system becomes Anderson insulator. This occurs apparently because the (pseudo)gap in the region of strong coupling is opened in the density of states at the Fermi level [22], so that there are no states to localize near the Fermi level at all. In Fig. 4, we show the dependences of coefficient $C$ on disorder strength for different values of coupling $U/2D$. In this figure (and all that follow in this section) the filled symbols and continuous lines correspond to calculations taking into account localization corrections, while the empty symbols and dashed lines correspond to the ladder approximation. In the weak coupling limit ($U/2D = 0.1$) we observe fast enough drop of the coefficient $C$ with disorder growth in the region of weak impurity scattering. At the same time in the region of strong enough disorder in the ladder approximation we can observe the increase in the coefficient $C$ with the growth of disorder, which is mainly due the noticeable broadening of the band by this disorder and corresponding drop of the effective coupling strength $U/2D_{eff}$. However, localization corrections, which become important for strong disorder $W/2D > 0.25$, lead to suppression of $C$ while disorder grows, also in the limit of strong impurity scattering. In the region of intermediate coupling ($U/2D = 0.4–0.6$) coefficient $C$ in the ladder approximation is rather insignificantly increasing with disorder growth. In BEC limit ($U/2D > 1$) coefficient $C$ in fact is independent of impurity scattering both in the ladder approximation and with the account of localization corrections. Localization corrections in the BEC limit in fact do not change the value of coefficient $C$ as compared to the ladder approximation. As the Ginzburg–Landau coefficients $\alpha$ and $B$ are universally dependent on disorder, Anderson localization has no influence upon them at all, and coefficient $C$, which is strongly dependent on localization correction in the weak coupling limit, in the BEC limit is in fact independent of these corrections. Correspondingly, the physical properties depending on the coefficient $C$ are also significantly dependent on localization corrections in the weak
In the weak coupling limit, but in fact do not feel Anderson localization in the BEC limit.

In Fig. 5, we show the dependence of coherence length $\xi$ on the level of disorder for different values of coupling strength. In the weak coupling limit, the coherence length $\xi$ drops rapidly with the growth of $U$ at any disorder level, reaching the values about the lattice parameter $a$ in the intermediate coupling region of $U/2D \sim 0.4–0.6$. Further growth of the coupling strength only slightly changes the coherence length. In BCS limit, i.e., for the weak coupling and weak enough impurity scattering we observe (cf. the inset of Fig. 5) the standard dependence for dirty superconductors $\xi \sim l^{1/2}$; i.e., the coherence length rapidly drops with the growth of disorder. However, at strong enough disorder in ladder approximation (dashed lines) coherence length grows with disorder, which is mainly due to noticeable broadening of the bare band and corresponding suppression of $U/2D_{\text{eff}}$. Localization corrections are important only for large disorder ($W/2D > 0.25$) and lead to significant drop of the coherence length in the BCS limit of weak coupling and practically does not change the coherence length in the BEC limit. Taking into account localization corrections leads to noticeable drop of coherence length as compared to the ladder approximation in the limit of strong disorder restoring the suppression of $\xi$ with the growth of disorder in this limit. In the standard BCS model with the bare band of infinite width in the limit weak disorder the coherence length drops with disorder $\xi \sim l^{1/2}$, and close to the Anderson transition $\xi$ drops even faster as $\xi \sim l^{2/3}$ [7–9], in contrast to our model, where close to the Anderson transition the coherence length rather weakly depends on disorder, which is related to a significant broadening of the band by disorder. With the growth of the coupling strength $U/2D \gtrsim 0.4–0.6$ the coherence length $\xi$ becomes of the order of lattice parameter and becomes almost disorder independent. In particular, in BEC limit of very strong coupling $U/2D = 1.4$, 1.6, the growth of disorder up to very strong values ($W/2D = 0.5$) leads to a factor two drop of coherence length, so that in the limit of strong coupling the account of localization corrections becomes irrelevant.

In Fig. 6, we show the dependence of penetration depth, normalized by its BCS value in the absence of disorder (27) on Hubbard attraction strength $U$ for different levels of disorder. In the absence of impurity scattering, the penetration depth grows with coupling strength. In the weak coupling limit, in accordance with the usual theory of dirty superconductors, disorder leads to a fast growth of penetration depth $(\lambda \sim l^{1/2})$, where $l$ is the mean free path). With increase in the coupling strength the growth of penetration depth with disorder slows down and in the limit of very strong coupling for $U/2D = 1.4$, 1.6, the penetration depth even slightly decreases with the growth of disorder. Thus, in presence of disorder we observe the drop of penetration depth with the growth of Hubbard attraction in the region of relatively weak coupling and the growth of $\lambda$ with $U$ in BEC strong coupling limit. The account of localization corrections is relevant only in the limit of strong disorder ($W/2D > 0.25$) and leads significant growth of penetration depth as compared with results of the ladder approximation in weak coupling limit. However, qualitatively the dependence of penetration depth on disorder does not change. In BEC limit localization influence on penetration depth is insignificant. Similar dependence on disorder is
observed also for dimensionless Ginzburg–Landau parameter \( \kappa = \lambda / \xi \). In weak coupling limit Ginzburg–Landau parameter rapidly grows with disorder in accordance with the theory of dirty superconductors, where \( \kappa \sim \Gamma^{-1} \). With the increase in the coupling strength the growth of Ginzburg–Landau parameter with disorder slows down and in the strong coupling limit of \( U/2D > 1 \) parameter \( \kappa \) is practically disorder independent. The account of localization corrections leads quantitatively to a noticeable increase in Ginzburg–Landau parameter in Anderson insulator phase \(( W/2D \geq 0.37 )\) for the weak coupling. In the limit of strong coupling, the account of localization is again irrelevant.

In Fig. 7, we show the dependence of the slope of the upper critical field on disorder. In the weak coupling limit, we again observe the behavior typical of dirty superconductors: the slope of the upper critical field grows with disorder (cf. inset of Fig. 7). Taking into account localization corrections in the weak coupling limit greatly increases the slope of the upper critical field as compared with the ladder approximation. As a result, the slope of the upper critical field in the Anderson insulator grows with impurity scattering much faster, than in the ladder approximation. At intermediate couplings \( (U/2D = 0.4–0.8) \) the slope of the upper critical field is practically independent of impurity scattering at weak disorder. In the ladder approximation, this behavior is conserved also in the region of strong disorder. However, the account of localization corrections leads to significant growth of the slope with disorder in Anderson insulator phase. In the limit of very strong coupling, the slope of the upper critical field can even slightly decreases with disorder, but for strong disorder, the slope grows with the growth of impurity scattering. In the BEC limit, the account of localization corrections becomes irrelevant and only slightly changes the slope of the upper critical field as compared with the ladder approximation.

5. TEMPERATURE DEPENDENCE OF THE ORBITAL UPPER CRITICAL FIELD

Most vividly, the influence of disordering is manifested in the behavior of the upper critical field in the theory of dirty superconductors. As disorder grows both the slope of the temperature dependence of the upper critical field at \( T_c \) [6] and \( H_{c2}(T) \) at all temperatures increase [40, 41]. Effects of Anderson localization in the limit of strong disorder also are most explicit in the temperature dependence of the upper critical field. Precisely at the point of Anderson metal–insulator transition localization effects lead to lead to sharp increase in \( H_{c2} \) at low temperatures and the temperature dependence of \( H_{c2}(T) \) is qualitatively different from Werthamer, Helfand, Hohenberg (WHH) dependence [40, 41], which is characteristic of the theory of dirty superconductors: \( H_{c2}(T) \) dependence becomes concave [7–9].

Let us consider disorder influence on the temperature dependence of the upper critical field \( H_{c2}(T) \) in a wide region of attraction strength \( U \), including the BCS–BEC crossover region, as well as for the wide interval of disorders, up to the vicinity of Anderson transition [28]. In Nozieres–Schmitt-Rink approach used here the critical superconducting transition temperature is determined by a joint solution of equation for Cooper instability in Cooper particle–particle channel in weak coupling approximation and equation...
for the chemical potential of the system, which is defined for the whole interval of the values of Hubbard interaction from the condition of quarter-filling of the band within DMFT + Σ approximation. The usual condition for Cooper instability has the form

\[ 1 = -U \chi(q). \]  

where \( \chi(q) \) is the Cooper susceptibility, determined by the loop in the Cooper channel. In the presence of an external magnetic field, the total momentum \( q \) in the Cooper channel acquires the additional contribution from vector potential \( A \) [6, 40]

\[ q \to q = \frac{2e}{c} A. \]

As our model assumes an isotropic spectrum, Cooper instability \( \chi(q) \) depends on \( q \) only through \( q^2 \). The minimum eigenvalue of the operator \( \left(q - \frac{2e}{c} A\right)^2 \), defining the upper critical magnetic field \( H = H_{c2} \) is [42]

\[ q_0^2 = 2\pi \frac{H}{\Phi_0}, \]

where \( \Phi_0 = \frac{e}{2}\hbar = \pi\hbar \) is the magnetic flux quantum. Then, the equation for \( T_c(H) \) or \( H_{c2}(T) \) remains as usual:

\[ 1 = -U \chi(q^2 = q_0^2). \]

In further analysis we shall neglect the relatively weak influence of magnetic field on diffusion (non-invariance with respect to time reversal), which is reflected in nonequality of the loops in Cooper and diffusion channels. This influence of magnetic field was analyzed in [9, 10, 43, 44], where it was demonstrated, that the account of this, even close to Anderson metal–insulator transition, only slightly decreases the value of \( H_{c2}(T) \) in low temperature region. Under the condition of invariance to time reversal and equivalence of the loops in Cooper and diffusion channels, Cooper instability is determined by the loop in diffusion channel. As a result, Eq. (33) for the orbital critical field \( H_{c2}(T) \) acquires the form [28]

\[ 1 = -\frac{U}{2\pi} \int d\epsilon \text{Im} \left\{ \sum_x \frac{\Delta G_{\mu}(\epsilon)}{2\epsilon + iD(2\epsilon)2\pi \hbar^2} \right\} \tanh \frac{\epsilon}{2T}. \]

The generalized diffusion coefficient is again determined in the framework of self-consistent theory of localization as described above.

In Fig. 8, we show temperature dependences of the upper critical field for different values of disorder in three regions of coupling strength of interest to us: in BCS weak coupling limit \( (U/2D = 0.2) \), in the BCS–BEC crossover region (intermediate coupling \( U/2D = 1.0 \)) and in BEC limit of strong coupling \( (U/2D = 1.6) \).

In the weak coupling limit (Fig. 8a), the growth of disorder leads to increase in the upper critical field at all temperatures in weak disorder limit \( (W/2D < 0.19) \), in
this case the temperature dependences have negative curvature and are close in form to the standard WHH dependence [40, 41]. With further growth of disorder and without account of localization corrections, the upper critical field at all temperatures starts to decrease. However, the account of localization corrections in weak coupling limit at strong disorder \((W/2D \geq 0.37)\) significantly increases the upper critical field and qualitatively changes its temperature behavior, so that the dependences of \(H_{c2}(T)\) acquire the positive curvature. The upper critical field increases rapidly with disorder at all temperatures.

For intermediate coupling (Fig. 8b) in the limit of weak disorder, the temperature dependence of the upper critical field becomes practically linear. The upper critical field at all temperatures increases with the growth of disorder. In the limit of strong disorder \((W/2D \geq 0.37)\) localization corrections, as in the weak coupling limit, increase the upper critical field at all temperatures. The dependences of \(H_{c2}(T)\) acquire positive curvature. However, in the region of intermediate coupling the influence of localization effects is significantly weaker, than in the limit of weak coupling being relevant only in the low temperature region.

In the BEC limit of strong coupling (Fig. 8c) in the weak disorder region, \(H_{c2}(T)\) dependences are in fact linear. The upper critical filed grows with increasing disorder at all temperatures. In the limit of strong disorder at the point of Anderson transition itself \((W/2D = 0.37)\) the dependence \(H_{c2}(T)\) remains linear and taking into account localization corrections in fact does not change the temperature dependence of the upper critical field. Further increase in disorder leads to the growth of \(H_{c2}(T)\). Deep in the Anderson insulator phase \((W/2D = 0.5)\), the \(H_{c2}(T)\) dependence acquires positive curvature and the account of Anderson localization increases \(H_{c2}(T)\) in low temperature region, while close to \(T_c\), localization corrections become irrelevant even at such a strong disorder. Thus, the strong coupling significantly decreases the influence of localization effects on the temperature dependence of the upper critical field.

Thus, the increase in coupling strength \(U\) leads to a rapid growth of \(H_{c2}(T)\), especially in low temperature region. In the BEC limit and in the BEC–BCS crossover region, the \(H_{c2}(T)\) dependence becomes practically linear. Disordering at any coupling strength also leads to the growth of \(H_{c2}(T)\). In the BCS limit of weak coupling, increasing disorder leads to the growth of both the slope of the upper critical field close to \(T = T_c\) and \(H_{c2}(T)\) in low temperature region. In the limit of strong disorder, localization corrections near the Anderson transition lead to the additional sharp increase in the upper critical field in low temperature region, so that the \(H_{c2}(T)\) dependence becomes concave, acquiring the positive curvature. In BCS–BEC crossover region and in BEC limit weak disorder influence on the slope of the upper critical field at \(T_c\) is negligible, though strong disorder in the vicinity of Anderson transition leads to noticeable increase in the slope of the upper critical field with disorder. In low temperature region \(H_{c2}(T)\) significantly grows with increasing disorder, especially near the Anderson transition, where localization corrections noticeably increase \(H_{c2}(T = 0)\) and the \(H_{c2}(T)\) dependence becomes concave, instead of linear, characteristic of the strong coupling at weak disorder.

In the model under discussion, the upper critical field at low temperatures can reach extreme values, up to (or even formally exceeding) \(\Phi_0/2\pi a^2\). This requires further analysis of the model, both taking into account inevitable quantization of electronic spectrum in magnetic field and paramagnetic effect.

6. TEMPERATURE DEPENDENCE OF THE PARAMAGNETIC CRITICAL FIELD

In the weak coupling region and for weak disorder, the upper critical magnetic field of a superconductor is determined by orbital effects and is usually much lower than paramagnetic limit. However, the growth of the coupling strength and disorder, as was shown above, lead to a rapid increase in the orbital \(H_{c2}\) possibly exceeding the paramagnetic limit. In this section, we shall consider the behavior of paramagnetic critical field for a wide region of coupling strength \(U\), including the region of BCS–BEC crossover and the limit of very strong coupling, with the account of disorder (including rather strong one).

It is well known that in BCS weak coupling limit paramagnetic effects (spin splitting effects) lead to the existence at low temperatures a region on the phase diagram of a superconductor in magnetic field, where paramagnetic critical magnetic field \(H_{c_p}\) decreases with lowering temperature. This behavior is an evidence of instability, leading to a first order phase transition to the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) phase [45–47] with Cooper pairs with finite momentum \(q\) and the order parameter periodic in space. Further on, our analysis will be limited only to a second order phase transition and superconducting order parameter will be assumed spatially homogeneous, allowing us to determine the border of instability towards first order transition in the regions of BCS–BEC crossover and strong coupling, also for different levels of disorder. The problem of stability of FFLO state under these conditions will not be considered.
Within Nozieres–Schmitt–Rink approach the critical temperature in the presence of spin splitting in external magnetic field (neglecting orbital effects) or paramagnetic critical magnetic field $H_{cp}$ at temperature $T < T_c$ is determined by the BCS-like equation [29]

$$1 = \frac{U}{4} \int_{-\infty}^{\infty} d\varepsilon \frac{N_0(\varepsilon)}{\varepsilon - \mu} \left( \tanh \frac{\varepsilon - \mu - \mu_B H_{cp}}{2T} ight.$$

$$\left. + \tanh \frac{\varepsilon - \mu + \mu_B H_{cp}}{2T} \right),$$

(35)

where the chemical potential $\mu$ for different values of $U$ and $W$ is determined from DMFT + $\Sigma$ calculations, i.e., from the standard equation for the number of electrons in the band. It should be noted that Eq. (35) is obtained from the exact Ward identity [29] and remains valid in the presence of strong disorder, including the vicinity of the Anderson transition. Equation (35) demonstrates, that all of disorder influence on $H_{cp}$ reduces to renormalization of the bare semi-elliptic density of states by disorder, that is for bare band with semi-elliptic density of states the influence of disorder on $H_{cp}$ is universal and reduces only to band broadening by disorder, i.e., to the substitution $D \rightarrow D_{eff}$. It is clear that paramagnetic critical field will be in general rising with the growth of coupling strength $U$ as it becomes more and more difficult for magnetic field to break pairs of strongly coupled electrons [29].

In Fig. 9, we show the results on disorder influence of temperature dependence of paramagnetic critical magnetic field. In BCS weak coupling limit (Fig. 9a) disorder growth leads both to the decrease in critical temperature in the absence of magnetic field $T_{c0}$ (cf. [23, 24]) and to the decrease in the critical magnetic field at all temperatures. The instability region corresponding to a first-order transition is conserved also in the presence of disorder. In fact, as noted above, disorder influence upon $H_{cp}(T)$ is universal and related only to the substitution $D \rightarrow D_{eff}$. As a result, the growth of disorder leads to the decrease in effective coupling strength, which is determined by the dimensionless parameter $U/2D_{eff}$. This leads to a substantial broadening of a relative temperature $T/T_c(H)$ region of the first order transition.

For intermediate coupling ($U/2D = 0.8$) in the region of the BCS–BEC crossover (Fig. 9b), the growth of disorder only weakly changes the critical temperature $T_{c0}$ (cf. [23, 24]), leading to some increase in $H_{cp}(T)$. As the entire influence of disorder is related only to the substitution $D \rightarrow D_{eff}$, the increase in disorder here again leads to the decrease in effective coupling strength $U/2D_{eff}$ and restoration of instability region of the first order transition.

In BEC limit of strong coupling the growth of disorder leads to significant growth the critical temperature $T_{c0}$ (cf. [23, 24]). At the same time, the critical magnetic field in low temperature region is rather weakly increasing with disorder. In the BEC limit, the instability region of first order transition does not
appear even in the presence of very strong disorder ($W/2D = 0.5$). In fact in BEC limit the influence of disorder is also universal and related only to the substitution $D \rightarrow D_{\text{eff}}$. As a result, if we normalize spin splitting and temperature by effective bandwidth $2D_{\text{eff}}$ and fix the effective coupling strength $U/2D_{\text{eff}}$, we shall obtain the universal temperature dependence of paramagnetic critical magnetic field. In Fig. 10, we show examples of such universal behavior for typical cases of weak and strong coupling both in presence and in the absence of disorder.

In the absence of disorder in the BEC limit of strong coupling $U/2D = 1.6$ for $T \rightarrow 0$, we have $2\mu_{\text{B}}H_{\text{cp}}/2D = 0.125$, which for a characteristic bandwidth $2D \sim 1$ eV gives $H_{\text{cp}} \sim 10^7$ G. For the orbital critical magnetic field (cf. [28]) in the same model, for the same coupling strength and $T \rightarrow 0$, for a characteristic lattice parameter $a = 3.3 \times 10^{-8}$ cm, we obtain $H_{\text{c2}} \approx 1.6 \times 10^8$ G. Thus, the orbital critical magnetic field at low temperatures increases with coupling strength much faster than paramagnetic field and in BEC limit the main contribution to the upper critical magnetic field at low temperatures will be due to paramagnetic effect. The growth of disorder leads to a large increase in orbital critical magnetic field [28], while $H_{\text{cp}}(T \rightarrow 0)$ in the BCS–BEC crossover region and in the BEC limit is relatively weakly dependent on disorder. Then, also in the presence of disorder in the BEC limit, the main contribution to the upper critical magnetic field at low temperatures will come essentially from the paramagnetic effect.

Thus, the growth of the coupling strength $U$ leads to a rapid increase in $H_{\text{cp}}(T)$ and disappearance of the instability region of first order transition at low temperatures in BCS–BEC crossover region and in BEC limit, which appears at low temperatures in BCS limit of weak coupling. Physically, this is related to the fact that it is more difficult for magnetic field to break strongly coupled pairs. The growth of disorder on the BCS limit of weak coupling leads both to the decrease in critical temperature and to the decrease in $H_{\text{cp}}(T)$. The instability region of the first order transition at low temperatures in the presence of disorder is conserved. In the region of intermediate coupling ($U/2D = 0.8$), disorder influence on both critical temperature and $H_{\text{cp}}(T)$ is rather weak. However, the growth of disorder leads to restoration of low temperature region of instability of the first order transition, which is not observed in the absence of disorder. This rather unexpected conclusion is due to specifics of attractive Hubbard model, where the effective dimensionless parameter $U/2D_{\text{eff}}$ controls the coupling strength in disordered case.

In BEC limit at low temperatures, for reasonable parameters of the model, paramagnetic critical magnetic field is noticeably lower than the orbital one, so that the upper critical field in this region is determined essentially by paramagnetic critical field. In the presence of disorder, this conclusion is also even more valid, as the orbital critical field rapidly grows with disorder, while paramagnetic critical field in this limit only weakly dependent on disorder.

7. CONCLUSIONS

In this work, in the Nozieres–Schmitt-Rink approximation and DMFT + $\Sigma$ generalization of the dynamic mean field, we have studied the influence of disordering, including the strong one (Anderson localization region), on Ginzburg–Landau expansion and the behavior of related physical properties near $T_c$, and also the upper critical magnetic field (both orbital and paramagnetic) in disordered Anderson–Hubbard model with attraction, for a wide range of the values of attraction potential $U$, from the region of weak coupling, where instability of the normal phase and super-
conductivity are well described by the BCS model, up to the limit of strong coupling, where the superconducting transition is related to the Bose–Einstein condensation of compact Cooper pairs, which are formed at temperatures much higher than the superconducting transition temperature.

Due to size limitations of this review, we have presented above only a part of our results. Further details, as well as more detailed derivations of the main equations, can be found in original papers [25–29].

Note that all results obtained in this work implicitly used an assumption of self-averaging superconducting order parameter entering Ginzburg–Landau expansion. It is well known [9] that this assumption becomes, in general case, invalid close to Anderson metal–insulator transition, which is due to development in this region of strong fluctuations of the local density of states, leading to strong spatial fluctuations of the order parameter [48] and inhomogeneous picture of superconducting transition [49]. This problem is of great interest in the context of superconductivity in BCS–BEC crossover and in the region of strong coupling, and deserves further studies.

FUNDING

This work was supported in part by the Russian Foundation for Basic Research, project no. 20-02-00011.

REFERENCES

1. A. A. Abrikosov and L. P. Gor'kov, Sov. Phys. JETP 9, 220 (1959).
2. A. A. Abrikosov and L. P. Gor'kov, Sov. Phys. JETP 9, 1090 (1959).
3. L. P. Gor'kov, Sov. Phys. JETP 36, 1364 (1959).
4. A. A. Abrikosov and L. P. Gor'kov, Sov. Phys. JETP 12, 1243 (1961).
5. P. W. Anderson, J. Phys. Chem. Solids 11, 26 (1959).
6. P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, New York, 1966).
7. L. N. Bulaevskii and M. V. Sadovskii, JETP Lett. 39, 640 (1984).
8. L. N. Bulaevskii and M. V. Sadovskii, J. Low. Temp. Phys. 59, 89 (1985).
9. M. V. Sadovskii, Phys. Rep. 282, 226 (1997).
10. M. V. Sadovskii, *Superconductivity and Localization* (World Scientific, Singapore, 2000).
11. P. Nozieres and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985).
12. Th. Pruschke, M. Jarrell, and J. K. Freericks, Adv. Phys. 44, 187 (1995).
13. A. Georges, G. Kotliar, W. Krauth, and M. J. Rozenberg, Rev. Mod. Phys. 68, 13 (1996).
14. D. Vollhardt, in *Lectures on the Physics of Strongly Correlated Systems XIV*, Ed. by A. Avella and F. Mancini, AIP Conf. Proc. 1297, 339 (2010); arXiv: 1004.5069.
15. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, JETP Lett. 82, 198 (2005).
16. M. V. Sadovskii, I. A. Nekrasov, E. Z. Kuchinskii, Th. Pruschke, and V. I. Anisimov, Phys. Rev. B 72, 155105 (2005).
17. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Low Temp. Phys. 32, 398 (2006).
18. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Phys. Rev. B 75, 115102 (2007).
19. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, J. Exp. Theor. Phys. 106, 581 (2008).
20. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Phys. Usp. 53, 325 (2012).
21. E. Z. Kuchinskii and M. V. Sadovskii, J. Exp. Theor. Phys. 122, 509 (2016).
22. N. A. Kuleeva, E. Z. Kuchinskii, and M. V. Sadovskii, J. Exp. Theor. Phys. 119, 264 (2014).
23. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, JETP Lett. 100, 192 (2014).
24. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, J. Exp. Theor. Phys. 120, 1055 (2015).
25. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, J. Exp. Theor. Phys. 122, 375 (2016).
26. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, Low Temp. Phys. 42, 17 (2017).
27. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, J. Exp. Theor. Phys. 125, 111 (2017).
28. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, J. Exp. Theor. Phys. 125, 1127 (2017).
29. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, J. Exp. Theor. Phys. 127, 753 (2018).
30. A. A. Abrikosov, L. P. Gor’kov, and I. E. Dzyaloshinskii, Quantum Field Theoretical Methods in Statistical Physics (Pergamon, Oxford, 1965).
31. M. V. Sadovskii, *Diagrammatics* (World Scientific, Singapore, 2019).
32. R. Bulla, T. A. Costi, and T. Pruschke, Rev. Mod. Phys. 60, 395 (2008).
33. D. Vollhardt and P. Wölfle, Phys. Rev. B 22, 4666 (1980).
34. D. Vollhardt and P. Wölfle, Phys. Rev. Lett. 48, 699 (1982).
35. P. Wölfle and D. Vollhardt, in *Anderson Localization*, Ed. by Y. Nagaoka and H. Fukuyama, Springer Ser. Solid State Sci. 39, 26 (1982).
36. A. V. Myasnikov and M. V. Sadovskii, Sov. Phys.-Solid State 24, 2033 (1982).
37. E. A. Kotov and M. V. Sadovskii, Zs. Phys. B 51, 17 (1983).
38. M. V. Sadovskii, in *Soviet Scientific Reviews — Physics Reviews*, Ed. by I. M. Khalatnikov (Harwood Academic, New York, 1986), Vol. 7, p. 1.
39. D. Vollhardt and P. Wölfle, in *Electronic Phase Transitions*, Ed. by W. Hanke and Yu. V. Kopaev (North-Holland, Amsterdam, 1992), Vol. 32, p. 1.
40. N. R. Werthamer and E. Helfand, Phys. Rev. 147, 288 (1966).
41. N. R. Werthamer, E. Helfand, and P. C. Hohenberg, Phys. Rev. 147, 295 (1966).
42. E. M. Lifshits and L. P. Pitaevskii, *Course of Theoretical Physics*, Vol. 9: Statistical Physics, Part 2 (Nauka, Moscow, 1978; Pergamon, New York, 1980), Chap. 5.
43. E. Z. Kuchinskii and M. V. Sadovskii, Sverkhprovodim., Fiz., Khim., Tekh. 4, 2278 (1991).
44. E. Z. Kuchinskii and M. V. Sadovskii, Phys. C (Amsterdam, Neth.) 185–189, 1477 (1991).
45. P. Fulde and R. A. Ferrell, Phys. Rev. A 135, 550 (1964).
46. A. I. Larkin and Yu. N. Ovchinnikov, Sov. Phys. JETP 20, 762 (1964).
47. D. Saint-James, G. Sarma, and E. J. Thomas, *Type II Superconductivity* (Pergamon, Oxford, 1969).
48. L. N. Bulaevskii and M. V. Sadovskii, JETP Lett. 43, 99 (1986).
49. L. N. Bulaevskii, S. V. Panyukov, and M. V. Sadovskii, Sov. Phys. JETP 65, 380 (1987).