Connections are found between the two-component percolation problem and the conductor/insulator percolation problem. These produce relations between critical exponents, and suggest formulae connecting the conductivity exponents in different dimensions. Values for the critical exponents are obtained from calculations on the incipient infinite cluster in two and three dimensions.

I. INTRODUCTION

Percolation is a prototypical example of a critical phenomenon \[1\]. In particular, a percolating system is characterized by a correlation length $\xi$ that diverges as the percolation threshold $p_c$ is approached. At the critical point $p_c$, the geometric and dynamic attributes of the infinite, percolating cluster (termed the “incipient cluster”) are identified with a set of critical exponents whose values collectively constitute a universality class; that is, the set of exponent values is particular to the dimension of the Euclidean space rather than the underlying (regular) lattice structure. Because microscopic details of the system near $p_c$ are not important, percolation serves as a useful model for a variety of natural phenomena \[2\] where a dynamical process is affected by the connectivity of the system. In this paper, however, uncorrelated, isotropic systems are considered, where the interest is in the values of the critical exponents and the relations between them.

Two different approaches to the critical point (at which $\xi$ becomes infinite) are taken by the two-component percolation problem and the more-familiar conductor/insulator percolation problem. These two systems have no geometric attributes in common, but are related by their dynamic exponents.

The two-component percolation problem \[3\] involves a two-component material system of infinite extent. The higher conductivity phase, having conductivity $\sigma_1$, is randomly mixed with the lower conductivity phase ($\sigma_2$); further, the volume fraction $p$ of the higher conductivity phase is precisely at the percolation threshold $p_c$. It is reasonable to expect the effective conductivity $\sigma$ of the system to exhibit critical behavior as the conductivity value $\sigma_2$ approaches zero. Indeed, the power-law relation

$$\sigma \sim \sigma_1 r^\nu$$  \hspace{1cm} (1)

is found to hold for the 2D square bond lattice \[3\] and square site lattice \[4\] over $0 < r < 1$, where ratio $r \equiv \sigma_2/\sigma_1$.

The conductor/insulator percolation problem involves an insulator phase randomly mixed with a conducting phase of volume fraction $p > p_c$. The effective conductivity $\sigma$ exhibits the asymptotic behavior

$$\sigma \sim (p - p_c)^t$$  \hspace{1cm} (2)

as $p$ approaches $p_c$ from above.

[Some comments on notation: The symbol $\sim$ (“asymptotically equal to”) is not to be confused with $\propto$, which means “proportional to”. The letter $t$ is used both for the conductivity exponent (as in the equation above) and for the variable “time”; it should be clear from the context, and placement, what meaning should be assumed for $t$. In parts of this paper it is convenient to denote an effective conductivity in a more particular way than is done above. For example, $\sigma(p, \sigma_1; (1-p), \sigma_2)$ is the effective conductivity of an uncorrelated system comprised of volume fraction $p$ of sites having conductivity $\sigma_1$, and volume fraction $(1-p)$ of sites having conductivity $\sigma_2$.]

The following section presents the Walker Diffusion Method by which many of the analytical and numerical results in this paper are obtained. Subsequent sections are devoted to the two-component percolation problem, the conductor/insulator percolation problem, and numerical methods and results.

II. WALKER DIFFUSION METHOD

The WDM was developed to calculate effective transport coefficients (e.g., conductivity) of composite materials and systems \[4, 5\]. This method exploits the isomorphism between the transport equations and the diffusion equation for a collection of non-interacting walkers (hence the name). Accordingly, the phase domains in a composite correspond to distinct populations of walkers, where the walker density of a population is given by the value of the transport coefficient of the corresponding phase domain. The principle of detailed balance ensures that the population densities are maintained, and provides the following rule for walker diffusion over a digitized (pixelated) composite: a walker at site (or pixel) $i$ attempts a move to a randomly chosen adjacent site $j$ during the time interval $\tau = (4d)^{-1}$, where $d$ is the Euclidean dimension of the space; this move is successful with probability $p_{ij} = \sigma_j / (\sigma_i + \sigma_j)$, where $\sigma_i$ and $\sigma_j$ are the transport coefficients for the phases comprising sites $i$ and $j$, respectively. (In practice, the unsuccessful moves inherent in this rule are eliminated by use of the variable residence time algorithm \[4\].) The path of a walker thus reflects the composition and morphology of the domains that are encountered. Over distances greater than the correlation length $\xi$, the walker diffusion is characterized by the
diffusion coefficient $D_w$, which is related to the effective transport coefficient $\sigma$ by

$$\sigma = \langle \sigma(r) \rangle D_w$$

(3)

where $\langle \sigma(r) \rangle$ is the volume average of the constituent transport coefficients. The diffusion coefficient $D_w$ is calculated from the equation

$$D_w = \frac{\langle R(t)^2 \rangle}{2dt}$$

(4)

where the set $\{R\}$ of walker displacements, each occurring over the time interval $t$, comprises a Gaussian distribution that must necessarily be centered well beyond $\xi$. (For practical purposes, the correlation length $\xi$ is the length scale above which the "effective", or macroscopic, value of a transport property is obtained.)

[It should be clear that the WDM as described here is a mathematical method—not a model of a physical process. To this point, the local transport coefficients, which in this paper are local conductivity values $\sigma$, may be local values of fluid permeability $k$ or thermal conductivity $\kappa$, for example.]

For displacements $R < \xi$, the walker diffusion is anomalous rather than Gaussian due to the heterogeneity of the composite at length scales less than $\xi$. There is, however, an additional characteristic length $\xi_0 < \xi$ below which the composite is effectively homogeneous; this may correspond, for example, to the average phase domain size. A walker displacement of $\xi$ requiring a travel time $t_\xi = \xi^2/(2dD_w)$ is then comprised of $(\xi/\xi_0)^{d_w}$ segments of length $\xi_0$, each requiring a travel time of $t_0 = \xi_0^2/(2dD_0)$, where $D_0$ is the walker diffusion coefficient calculated from displacements $R < \xi_0$. Setting $t_\xi = (\xi/\xi_0)^{d_w} t_0$ gives the relation

$$D_w = D_0 \left( \frac{\xi}{\xi_0} \right)^{2-d_w} = \left( \frac{\xi_0^d}{2dD_0} \right) \xi^{2-d_w}$$

(5)

between the walker diffusion coefficient $D_w$, the fractal dimension $d_w$ of the walker path, and the correlation length $\xi$.

III. TWO-COMPONENT PERCOLATION PROBLEM

From the point of view of the WDM, the two-component percolation problem differs from the conductor/insulator percolation problem mainly by the fact that walkers are never "stranded" on finite clusters of conductor sites (until precisely $r = 0$). Thus the approach to the endpoint, which in both cases is percolation only via the incipient cluster, reflects that difference and so produces a different set of critical exponents.

Combining Eqs. (14), (15) and (16) gives the relation

$$r^u = \left( \frac{\langle \sigma \rangle}{\sigma_1} \right) \left( \frac{\xi_0^d}{2dD_0} \right) \xi^{2-d_w}$$

(6)

which upon rearrangement produces

$$\xi = \left( \frac{\langle \sigma \rangle}{\sigma_1} \right)^{1/(d_w-2)} \left( \frac{\xi_0^d}{2dD_0} \right)^{1/(d_w-2)} r^{-u/(d_w-2)}$$

(7)

Thus the correlation length $\xi$ diverges as

$$\xi \sim r^{-u/(d_w-2)}$$

(8)

near $r = 0$. The exponent $d_w$ is the limit of the walker path dimension in this paper are local conductivity values $\sigma$, may be local values of fluid permeability $k$ or thermal conductivity $\kappa$, for example.]
tor sites. As the walker diffusion coefficient $D$ is the component percolation problem.

$\sigma$ with the conjectured result $\langle \rangle = 3 - \frac{p}{10}$

Figure 1. Data supporting the conjecture that the 3D conductor/insulator system has effective conductivity $D_w$ for two-component percolation problem. The analytical results $A$ numerical value for the exponent $u_3$ is obtained by the WDM (details of this sort of calculation are given in Sec. V). Figure 1 shows calculated points (in $t$, $\ln(R(t)^2)$) for two-component systems with $r = 0.1$, $10^{-2}$, $10^{-3}$, $10^{-5}$, in order from left to right in the event that $u_3 = \frac{3}{4}$. The points that lie above the straight lines are affected by their proximity to the correlation length $\xi$ (which increases with decreasing $r$).

A numerical value for the exponent $u_3$ was obtained by the WDM (details of this sort of calculation are given in Sec. V). Figure 1 shows calculated points (in $t$, $\ln(R(t)^2)$) for two-component systems with $r = 0.1$, $10^{-2}$, $10^{-3}$, $10^{-5}$. The four straight lines of slope 1 represent the relation $D_w = \sigma / (\sigma)$ and so correspond to equations

$$y = x + \ln[2 d D_w(r)] = x + \ln \left[ \frac{2 d r u}{pc + (1 - pc) r} \right]$$

for the four values of $r$, with the exponent $u_3$ set to the value 0.75. The coincidence of the points and the lines support a previous conjecture that $u_3 = \frac{3}{4}$.

The analytical results $u_d < 1$ and $u_2 = \frac{1}{2}$ together with the conjectured result $u_3 = \frac{3}{4}$ suggest the relations $u_{d+1} = (u_2 + 1)/2$ and

$$u_d = 1 - (1 - u_2)^{d-1}$$

between the conductivity exponents of the two-component percolation problem.

IV. CONDUCTOR/INSULATOR PERCOLATION PROBLEM

The conductor/insulator system has effective conductivity $\sigma = \sigma_1 p D_w$ where $p$ is the fraction of conductor sites. As the walker diffusion coefficient $D_w = \langle R(t)^2 \rangle / (2dt)$ with walk time $t \gg t_c$ is obtained from walkers on all conductor sites, not just those on the percolating cluster, the conductor/insulator problem is recast as a two-component problem. Namely, the insulator sites become conductor sites with very low conductivity value $\sigma_2 \ll \sigma_1$. Then the conductivity exponent $t$ is obtained in the limit $\sigma_2 = 0$ (that is, $r = 0$) at $p = p_c$. Thus the correlation length for this two-component system is

$$\xi = \left( \frac{c d_w}{2d \sigma_0} \right)^{1/(d_w-2)} D_w^{1/(d_w-2)}$$

$$\sim p^{1/(d_w-2)}(p - p_c)^{-t/(d_w-2)}.$$

It is also the case that $\sigma = \sigma_1 p^{\prime} D_w^\prime$, where $p^{\prime}$ is the fraction of system sites comprising the percolating cluster, and $D_w^\prime$ is the diffusion coefficient for walkers on the percolating cluster. Additionally, it is known that $p^{\prime} \sim (p - p_c)^{\beta}$ for $p > p_c$. Thus

$$\xi \sim (p - p_c)^{\beta/(d_w^* - 2)}(p - p_c)^{-t/(d_w^* - 2)}$$

$$\sim (p - p_c)^{-t/\nu}.$$  

Here the exponent relation $\nu = (t - \beta)/(d_w^* - 2)$ is obtained, where

$$d_w^* = 2 + (t - \beta)/\nu$$

is the limit of the walker path dimension $d_w$ at $p = p_c$. (Thus $d_w^*$ is the fractal dimension of the walker path on the incipient infinite cluster.) Note that the walker path dimensions $d_w$ and $d_w^*$ are related by $d_w^* - d_w = \beta/\nu$, and that $d_w^* = 2 + t/\nu$.

[A more succinct derivation of the exponent relation Eq. (15) is $\sigma(\xi) = \sigma_1 p^{\prime}(\xi) D_w^\prime(\xi)$ implies $\xi^{-1/\nu} \propto \xi^{\beta/\nu} \xi^{2-d_w^*}$]

The exponents pertaining to the incipient cluster are additionally connected by a hyperscaling law (a relation that includes the dimension $d$ of the system). This follows from the asymptotic relation $p^{\prime} \sim \xi^{-\beta/\nu}$ and the observation that

$$p^{\prime} \sim \frac{\xi^D}{\xi^d}$$

where the right-hand side is the volume fraction occupied by the incipient cluster, the exponent $D$ being the fractal "mass dimension" of that cluster. Thus $\beta = -\nu(D - d)$ at the percolation threshold.

The appearance of the critical exponent $d_w$ in both the two-component percolation problem and the conductor/insulator percolation problem points to a fundamental connection between the two systems. Very near the percolation threshold, the effective conductivity of the conductor/insulator system exhibits critical behavior according to the equation

$$\sigma(p > p_c, 1; (1 - p), 0) \sim (p - p_c)^t \sim \xi^{-t/\nu}$$

while the effective conductivity of the conductor/superconductor system exhibits critical behavior described by

$$\sigma(p < p_c, \infty; (1 - p), 1) \sim |p - p_c|^{-s} \sim \xi^{s/\nu}.$$  

The exponents $t$ and $s$ can be related to $u$ and $u - 1$ from the two-component percolation problem by noting
that the conductivities of the two conducting systems 
\((p > p_c, 1; (1-p), 0)\) and \((p_c, 1; (1-p_c), r)\) are identical, and 
the conductivities of the two superconducting systems 
\((p < p_c, \infty; (1-p), 1)\) and \((p_c, r^{-1}; (1-p_c), 1)\) are 
identical, when the parameters \(p\) and \(r\) are very close to 
\(p_c\) and 0, respectively. These asymptotic relations are 
\[ \sigma(p_c, 1; (1-p_c), r) \sim \sigma(p > p_c, 1; (1-p), 0) \sim \xi^{-t/\nu} \quad (22) \]
and 
\[ \sigma(p_c, r^{-1}; (1-p_c), 1) \sim \sigma(p < p_c, \infty; (1-p), 1) \sim \xi^{s/\nu}. \quad (23) \]
Note that Eq. (10) implies 
\[ \sigma(p_c, 1; (1-p_c), r) = [\sigma(p_c, r^{-1}; (1-p_c), 1)]^{u/(u-1)}. \quad (24) \]
The asymptotic version of this is 
\[ \xi^{-t/\nu} = [\xi^{s/\nu}]^{u/(u-1)}, \]
which reveals the exponent relation 
\[ \frac{t}{s} = \frac{u}{1-u} \quad (25) \]
or equivalently \(u = t/(s + t)\), in all dimensions.

In fact the four conductivities in Eqs. (22) and (23) 
have in common the asymptotic relationship 
\(D_w \sim \xi^{-d^w/u}\). This is because very near \(r = 0\) and \(p = p_c\), 
the four systems have (statistically) identical morphologies, 
and identical phase conductivity ratios. Thus a diffusing 
walker finds the four systems identical. For example, the 
effective conductivity 
\[ \sigma(p < p_c, \infty; (1-p), 1) \sim \sigma(p < p_c, r^{-1}; (1-p), 1) \sim r^{-1} \xi^{2-d^w/u} \sim \xi^{(t/\nu)/u} \xi^{-t/\nu} = \xi^{s/\nu}. \quad (26) \]

Similarly, the conductivities in Eqs. (20) and (21) have in common the asymptotic relationship 
\(D^w_i \sim \xi^{-d^w/u}\).

Given the exponent relation Eq. (25), a consequence of Eq. (19) is 
\[ \frac{t_d}{s_d} = 2^{d-1} - 1. \quad (27) \]
Using the value for exponent \(t_d\) calculated in the following section, 
the value \(s_d = 0.67787(105)\) is a prediction.

It is interesting to consider a counterpart to Eq. (19) for the conductor/insulator system. In this case the 
conductivity exponent \(t_d\) increases towards 3 as the dimension increases. \[ t_d = 3 \left[ 1 - \left(1 - \frac{t_2}{3}\right)^{d-1} \right]. \quad (28) \]

Given the generally accepted value \(t_2 = 1.30\), this equation produces \(t_3 = 2.03667\) and similarly 
reasonable values for higher dimensions.

The diffusivity counterpart to the asymptotic relation 
\(\sigma \sim (p - p_c)^{t-\beta}\) for conductivity is 
\(D^w_i \sim (p - p_c)^{t-\beta}\). Then 
\[ t_d - \beta_d \approx 1.91 \left[ 1 - \left(1 - \frac{t_2 - \beta_2}{1.91}\right)^{d-1} \right]. \quad (29) \]

which also produces very reasonable results. Note that 
the factor 1.91 is not optimized to give best results. Perhaps these relations connecting dynamic exponents 
across dimensions indicate a broader concept of “universality”.

V. NUMERICAL APPROACH AND RESULTS

Because the critical exponents are obtained from the incipient infinite cluster, it is important to ensure that 
the diffusing walkers, which perform the calculations, are 
indeed on that cluster. To start, a walker is placed on a 
conductor site at the center of a vast volume of “unde-
finite” sites. Then each neighboring site is defined to be 
conducting (with probability \(p_c\)) or is otherwise insulating. Rather than have the walker then attempt a move 
to a randomly chosen neighboring site (which may not be 
successful), it is more efficient to utilize the variable 
residence time algorithm, which takes advantage of the 
statistical nature of the diffusion process.

According to this algorithm, the actual behavior of 
the walker is well approximated by a sequence of moves in 
which the direction of the move from a site \(i\) is determined 
randomly by the set of probabilities \(\{P_{ij}\}\), where 
\(P_{ij}\) is the probability that the move is to adjacent site \(j\) which 
has conductivity \(\sigma_j\) and is given by the equation 
\[ P_{ij} = \frac{\sigma_j}{\sigma_i + \sigma_j} \left[ \sum_{k=1}^{2d} \left( \frac{\sigma_k}{\sigma_i + \sigma_k} \right) \right]^{-1}. \quad (30) \]
The sum is over all sites adjacent to site \(i\). The time 
interval over which the move occurs is 
\[ T_i = \left[ \sum_{k=1}^{2d} \left( \frac{\sigma_k}{\sigma_i + \sigma_k} \right) \right]^{-1}. \quad (31) \]

Note that this version of the variable residence time al-
gorithm is intended for orthogonal systems (meaning a site in a 3D system has six neighbors, for example).

After each move, any “undefined” neighboring sites are 
converted to conducting or insulating. In this way the 
cluster grows. A walk is complete when the sum of move 
times \(T_i\) reaches or exceeds a preset walk time \(T\).

Of course, many of those clusters turn out to be finite 
and so clearly are not part of the incipient cluster. In-
deed, the larger the preset walk time \(T\), the greater the 
likelihood that a nascent cluster will turn out to be finite. 
Finite clusters are identified by the fact that all conduc-
tor sites comprising the cluster have been visited by time 
\(T\) (so the cluster is completely surrounded by insulator 
sites). An “infinite” (or percolating) cluster includes at 
least one conductor site on the boundary that was “cre-
ated” by the walker (in the manner described above) but 
ever actually visited in time \(T\).

In general, \(n \times 10^5\) “infinite” clusters for each walk time 
\(T\) were used to determine the value of a critical exponent.
or a ratio of exponents. These represent $n \times 10^5$ different pieces, each of size corresponding to the walk time $T$, of the incipient cluster. It doesn’t matter that a cluster still “infinite” at time $T$ might turn out to be finite were the walk extended to longer times, since every finite cluster at the percolation threshold resembles the incipient cluster (which is statistically self-similar over all length scales) over length scales up to the size of the cluster.

The numerical data recorded for the incipient cluster was, for each of several preset walk times $T$, the following:

1. The number $N_{pc} = 10^5$ of percolating (“infinite”) clusters over which most other quantities are averaged.
2. The average number $\langle n_m(t) \rangle$ of independent walks from which the average value is obtained.
3. The actual (averaged) walk time $t$ (very slightly larger than $T$).
4. The average walker displacement $\langle R(t) \rangle$.
5. The average walker displacement-squared $\langle R(t)^2 \rangle$.
6. The average number $\langle n_m(t) \rangle$ of visits over which most other quantities are averaged.
7. The average number $\langle n_m(t) \rangle$ of visited sites.

The percolation threshold values used in the calculations are $p_c = 0.592746$ (2D) and $p_c = 0.311607$ (3D). The ‘standard’ values for $\beta$, $\nu$, and $D$ referred to below are $\beta_2 = 5/36$, $\nu_2 = 4/3$, $D_2 = 91/48$ [4]; and $\beta_3 = 0.41810(57)$, $\nu_3 = 0.87642(115)$, $D_3 = 2.52295(15)$, derived from values $1/\nu_3 = 1.1410(15)$ and $\beta_3/\nu_3 = 0.47705(15)$ [5].

A. Comment on average value $\langle R(t)^2 \rangle$

Most calculations of interest require arguably correct (as well as accurate) values for the average walker displacement-squared $\langle R(t)^2 \rangle$. In particular it is important that a sufficient number of independent walks (i.e., walks over a sufficient number of distinct sections of a percolating system) be taken in order that a mean value for $\langle R(t)^2 \rangle$ with reasonably narrow bounds is obtained. Figures 2 and 3 are instructive on this point.

Figure 2 shows five sets of points (distinguished by color) pertaining to walker diffusion on the incipient infinite cluster in 2D. Consider one of these sets. The coordinates of the points are $(N_{pc}, \langle R(t)^2 \rangle)$, where the average of the points is $\langle R(t)^2 \rangle$ is obtained from $N_{pc}$ percolating clusters (that is, from $N_{pc}$ independent walks). As more walks are taken (i.e., as $N_{pc}$ increases), the average value $\langle R(t)^2 \rangle$ fluctuates less and flattens out. Then by creating several sets and reproducing this behavior, a set size $N_{pc}$ is found (105 in this case) that permits a mean value for $\langle R(t)^2 \rangle$ to be obtained with reasonably narrow bounds.

Similarly, Fig. 3 shows five sets of points pertaining to walker diffusion on the incipient cluster in 3D. Again, sets of size $N_{pc} = 10^5$ appear to be sufficient to obtain a defensible value for $\langle R(t)^2 \rangle$ for use in calculations. (Larger sets may naturally reduce the bounds, but at the cost of significantly increased computer time.)

Data from Figs. 2 (walk time $t = 10^7$) and 3 ($t = 10^6$) are used (together with additional sets of size $10^5$) in the calculations of $d_w^*$ below.

B. Walker path dimension $d_w^*$

For percolating systems of size $L < \xi$, the equivalent of Eq. (5) is

$$D_w(L) = D_w(\xi) \left( \frac{L}{\xi} \right)^{2-d_w} = \left( \frac{c_{d_w}}{2d_0} \right) L^{2-d_w}. \quad (32)$$

In the case of the incipient cluster, which is statistically self-similar over all length scales, this relation can be expressed in terms of the computable variable $\langle R(t)^2 \rangle$, namely,

$$\frac{\langle R(t)^2 \rangle}{2dt} = \left( \frac{c_{d_w}}{2d_0} \right) \langle R(t)^2 \rangle^{1-d_w/2} = \langle R(t)^2 \rangle^{1-d_w^*/2}. \quad (33)$$
The last equality comes about because the correlation length $\xi_0$ is the size of a single conductor site; that is, $\xi_0 = 1$. This Gaussian regime corresponds to walkers diffusing within the conductor site for walk times $t < t_0$. Then the diffusion coefficient $D_0 = 1$ and so the travel time $t_0 = (2d)^{-1}$. Thus

$$\langle R(t)^2 \rangle = (2dt)^{2/d_w^*}$$

or equivalently,

$$\ln \langle R(t)^2 \rangle = \frac{2}{d_w^*} \ln t + \frac{2}{d_w^*} \ln(2d). \tag{35}$$

This last equation produces the straight lines in Fig. 4. The line of greater (lesser) slope, running through the point corresponding to largest walk time $t$, has slope inversely proportional to the walker path dimension $d_w^*$ for 2D (3D) percolation. Note that in both cases, points corresponding to shorter walk times lie below the straight lines, due to the effect of the finite (not infinitesimal) size of the conductor sites. A more precise explanation is as follows: Walker diffusion on the incipient cluster comprised of conductor sites is Gaussian ($d_w = 2$) for walk times $t < t_0$, and anomalous ($d_w = d_w^* > 2$) for walk times $t > t_0$, so that lines of slope 1 and slope $2/d_w^*$ meet at the point $(\ln t_0, \ln \langle R(t_0)^2 \rangle) = (-\ln 2d, \ln 1)$. Points in the anomalous regime near $t = t_0$ are thus affected by the presence of the Gaussian regime and so lie below the slope $2/d_w^*$ line. Note that this behavior contrasts with that exhibited by bond systems at $p_c$. In the bond case the walkers reside at the zero-dimensional nodes $\mathbb{Z}^0$, so there is no Gaussian regime near $t = 0$. Thus all points lie on the slope $2/d_w^*$ line.

For 2D percolation, the value $d_w^* = 2.87038(60)$ was obtained from $10^6$ walks, each of duration $T = 10^7$, over ten sets of $10^5$ clusters (representing $10^6$ distinct sections of the incipient infinite cluster). The average number of moves per walk $\langle n_m \rangle > 25 \times 10^6$, and the average number of visited sites per walk $\langle n_s \rangle > 68 \times 10^3$.

For 3D percolation, the value $d_w^* = 3.84331(193)$ was obtained from $8 \times 10^5$ walks, each of duration $T = 10^6$, over eight sets of $10^5$ clusters (representing $8 \times 10^5$ distinct sections of the incipient infinite cluster). The average number of moves per walk $\langle n_m \rangle > 2.3 \times 10^6$, and the average number of visited sites per walk $\langle n_s \rangle > 12 \times 10^4$.

In both cases Fig. 4 shows that these walks are of sufficient length (sufficient walk time) that finite-site-size effects on these $d_w^*$ values are negligible, and Figs. 2 and 3 show that a sufficient number of randomly selected sections of the incipient cluster are explored to give exponent values within meaningful brackets.

Table 1 presents values of critical exponents calculated from these WDM values for $d_w^*$.

|                  | $d_w^*$ | $t/\nu$ | $t$ | $d_s$ |
|------------------|---------|---------|-----|-------|
| 2D               | 2.87038(60) | 0.974542(600) | 1.29939(80) | 1.32097(28) |
| 3D               | 3.84331(193) | 2.32036(193) | 2.6336(32) | 1.3129(7) |

A lower bound $D_s$ on the mass dimension $D$ of the incipient cluster is found by considering the number $S(t)$ of distinct sites visited during a walk to be proportional to $R_c(t)^D$, where $R_c(t)$ is the crude radius of the cluster of visited sites. This cluster radius can be related to the walker displacement $R(t)$ by noting that the walker is essentially equilibrated after many moves over the cluster of visited sites ($n_m/n_s \gg 1$). Then the displacement $R(t)$ finds the walker at any site of the cluster with equal probability. For example, in the case of a walker confined to a 3D spherical cluster of conductor sites, the average value $\langle r \rangle$ is given by

$$\langle r \rangle = \frac{4}{3} \pi R_c^3 \int_{r=0}^{R_c} r \cdot 4\pi r^2 \; dr = \frac{3}{4} R_c \tag{36}$$

since $r$, that is $R(t)$, is measured from the origin of the cluster (the original site from which the cluster grew). More generally, $R_c \propto \langle R(t) \rangle$ and therefore

$$\langle S(t) \rangle \propto \langle R(t) \rangle^{D_s} \tag{37}$$

with the averages obtained from a very large number of clusters and walks.
This relation produces the straight lines in Figs. 5 and 6 which describe the growth of the cluster of visited sites produced by walkers confined to the incipient cluster. In Fig. 5 the slope $D_s = 1.89503$ is obtained for 2D percolation; this $D_s$ value is slightly less than the fractal dimension $D = 91/48 = 1.89583$ of the incipient cluster. In Fig. 6 the slope $D_s = 2.49848$ is obtained for 3D percolation; similarly, this $D_s$ value is slightly less than the standard value $D = 2.52295(15)$ for the incipient cluster. In both cases the line was fit to the two largest-walk-time points (each point obtained from eight or more sets of $10^5$ independent walks) in order to minimize the effects of the finite (not infinitesimal) size of the conductor sites apparent at shorter times $t$.

While the value $D_s$ may be very close to $D$, it will always be smaller since the cluster $S(t)$ will never completely fill the section of the incipient cluster explored by the walker over time $t$ (the walker will never visit every accessible site in that section). An extreme example of this effect is walker diffusion over a homogeneous 2D system: the path dimension $d_w$ is found to be precisely 2, but $D_s \approx 1.885$ (far less than $D = d = 2$) since the cluster $S(t)$ in that case grows in a non-compact way and so suggests a system with dimension less than 2.

Note that a variation on Eq. (37) is

$$S(t) \propto \langle R(t)^2 \rangle^{D/2} = \left[ (2dt)^2/d_w^2 \right]^{D/2} \propto t^{d_w/2} \quad (38)$$

where the equality is obtained from Eq. (34), and the spectral dimension $d_s = 2D/d_w$. However, this approach is discouraged as $\langle R(t)^s \rangle^{1/2}$ is a very poor approximation of $\langle R(t) \rangle$.

D. Fraction $p'$

The fraction $p'$ of system sites comprising the percolating cluster appears in the expression for conductivity

$$\sigma = \sigma_1 p' D'_w$$

for systems with $p > p_c$, and in the relation

$$p' \sim (p - p_c)^\beta$$

for infinite systems very close to the percolation threshold. The function $p'$ is derived here, as it is used in calculations below.

It is reasonable to assume that a created cluster of size greater than the correlation length $\xi$ (which occurs when the created cluster is “infinite” at preset walk time $T \gg t_c$) is part of the infinite percolating cluster. A very large number $N_{pc}$ of such “infinite” clusters are needed in the calculation of $D'_w$. In the process of creating these $N_{pc}$ percolating clusters, a number $N_{tc}$ of smaller, “finite” clusters are generated that cannot be used in the calculation of $D'_w$. Recall that creation of each cluster ($N_{pc} + N_{tc}$ in total) begins at a randomly selected conductor site (in practice, at a conductor site completely surrounded by a sea of “undefined” sites). As this random selection of conductor sites will produce a fraction $p'/p$ that belong to the percolating cluster, the fraction $N_{pc}/(N_{pc} + N_{tc}) = p'/p$; that is,

$$p' = p \left( 1 + \frac{N_{tc}}{N_{pc}} \right)^{-1}. \quad (39)$$

E. Exponent ratio $\beta/\nu$

The asymptotic relations $p' \sim (p - p_c)^\beta$ and $\xi \sim (p - p_c)^{-\nu}$ produce $p' \sim \xi^{-\beta/\nu}$. This inspires a finite-size scaling relation $p'(L) \propto L^{-\beta/\nu}$ that gives the fraction of
sites in an arbitrary portion of size \( L \) of an infinite system at \( p = p_c \), that belong to the cluster that percolates the size \( L \) volume. The equivalent scaling relation for \( p'(t) \) is \( p'(t) \sim \langle R(t) \rangle^{-\beta/\nu} \) or \( p'(t) \sim \langle R(t)^2 \rangle^{-\beta/2\nu} \). These are asymptotic relations because the expression \( p'(t) = p_c/(1 + N_{iC}/N_{pc}) \) [Eq. (38)] with \( p = p_c \) is valid only for systems of size larger than the correlation length \( \xi \).

The first formulation produces the straight lines (with slope approximating \( -\beta/\nu \)) in Figs. 7 and 8. In both the 2D (Fig. 7) and 3D (Fig. 8) cases the fits are to the points for the two largest walk times (each point obtained from eight or more sets of \( 10^5 \) independent walks). These produce values \( \beta_2/\nu_2 = 0.101027 \) (compare to the exact value \( 5/48 = 0.104167 \)) and \( \beta_3/\nu_3 = 0.454446 \) (compare to the value 0.47705(15)). The second formulation gives very similar values: \( \beta_2/\nu_2 = 0.100952 \) and \( \beta_3/\nu_3 = 0.453645 \).

As points are obtained at ever-larger walk times, the slopes of the fitted lines will increase in magnitude, giving values for the exponent ratio \( \beta/\nu \) closer to the true ones.

**VI. CONCLUDING REMARKS**

The intent of this research was to clarify the relationship between the two-component percolation problem and the familiar conductor/insulator percolation problem. The Walker Diffusion Method provided a new conceptual, analytical, and numerical approach to this task.

An important achievement is the introduction of a new critical exponent \( d_w^c \) that connects the two types of percolating systems. This is the fractal dimension of the walker path in the two-component system at the endpoint \( r = 0 \). It is also the limit of the walker path dimension \( d_w \) in the conductor/insulator system when all conductor clusters are connected by an extremely low conductivity “background” (replacing the insulator phase), attained at \( p = p_c \) and background conductivity reduced to zero.
The connection made apparent by $d^\dagger_w$ leads to Eq. (25), relating the conductivity exponent $t$ and superconductivity exponent $s$, and the corresponding exponents $u$ and $1-u$.

The value $d^\dagger_w$ is best calculated from the exponent relation $d^\dagger_w = 2 + t/\nu$ derived in Sec. IV. Use of the calculated value for $t_2$ and the standard value for $\nu_2$ produce $d^\dagger_w = 2.97454(60)$ for 2D systems. In principle $d^\dagger_w$ may also be obtained via the relation

$$\langle R(t)^2 \rangle = (2dt)^{2/d^\dagger_w}$$

(40)

describing walks over the conductor/insulator system at $p = p_c$, where walkers on the finite clusters (in addition to those on the incipient cluster) are included in the calculation. Those trapped walkers diffuse according to the variable residence time algorithm during the walk time $T$, and so contribute to the average displacement-squared $\langle R(t)^2 \rangle$ (hence $d^\dagger_w > d_*^w$).

Additionally, very good values for the critical exponent $d^*_w$ in two and three dimensions are obtained, which enable calculation of accurate values for the conductivity exponents $t_2$ and $t_3$. WDM calculations also support the conjectured value $u_3 = 3/4$, which motivates a proposed set of equations connecting conductivity exponents across dimensions.

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