An Identity relating Eisenstein series on general linear groups

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Abstract

We give a general identity relating Eisenstein series on general linear groups. We do it by constructing an Eisenstein series, attached to a maximal parabolic subgroup and a pair of representations, one cuspidal and the other a character, and express it in terms of a degenerate Eisenstein series. In the local fields analogue, we prove the convergence in a half plane of the local integrals, and their meromorphic continuation. In addition, we find that the unramified calculation gives the Godement-Jacquet zeta function. This realizes and generalizes the construction proposed by Ginzburg and Soudry in Section 3 in [GS19].

1 Introduction

1.1 Statement of the main results

Global theory

Let \( k \) be a number field, and let \( \mathbb{A} \) be its ring of adeles. Let \( m \geq n \) be two positive integers. Let \( \pi \) be an irreducible, automorphic, cuspidal representation of \( \text{GL}_n(\mathbb{A}) \) and assume that its central character \( \omega_\pi \) is unitary. Let \( \varphi_\pi \) be a cusp form in the space of \( \pi \). Consider, for a complex number \( s \), the Eisenstein series \( E_{mn}(f_{\omega_\pi,s}) \) on \( \text{GL}_m(\mathbb{A}) \), attached to a smooth, holomorphic section \( f_{\omega_\pi,s} \) of the normalized parabolic induction \( \text{Ind}_{\text{GL}_{mn}(\mathbb{A})}^{\text{GL}_{m,n}(\mathbb{A})} (1 \otimes \omega_{\pi}^{-1}) \delta_{P_{mn-1,1}}^s \). We set

\[
E(f_{\omega_\pi,s}, \varphi_\pi)(h) = \int_{Z_n(\mathbb{A})\text{GL}_m(k)\setminus\text{GL}_m(\mathbb{A})} \varphi_\pi(g)E_{mn}(f_{\omega_\pi,s})(t(h,g)) \, dg. \tag{1.1}
\]

Here, \( Z_n(\mathbb{A}) \) denotes the center of \( \text{GL}_n(\mathbb{A}) \) and the map \( t : \text{GL}_m(\mathbb{A}) \times \text{GL}_n(\mathbb{A}) \rightarrow \text{GL}_{mn}(\mathbb{A}) \) denotes the Kronecker product (see Section 1.3.3). Our main theorem is the identity

**Theorem 1.** \( E(f_{\omega_\pi,s}, \varphi_\pi) \) is an Eisenstein series on \( \text{GL}_m(\mathbb{A}) \), corresponding to the normalized parabolic induction \( \text{Ind}_{\text{GL}_{mn}(\mathbb{A})}^{\text{GL}_{m,n}(\mathbb{A})} (1 \otimes \pi) \delta_{P_{mn,n-1}}^s \). In more details, there exists an explicit meromorphic section \( \xi(f_{\omega_\pi,s}, \varphi_\pi) \) of this parabolic induction, given by

\[Key words and phrases.\] general linear group, p-adic group, adeles, automorphic forms, Eisenstein series, L-functions, representation theory.
the following integral which converges absolutely for \( \Re(s) \) sufficiently large and admits a meromorphic continuation to \( \mathbb{C} \),

\[
\xi(f_{\omega_\nu, s}, \varphi_\pi)(h) = \int_{Z_n(k) \backslash GL_n(k)} \varphi(g) f_{\omega_\nu, s}(\tilde{\varepsilon} t(h, g)) \, dg,
\]

where \( \tilde{\varepsilon} \) is a specific element of \( GL_{mn}(k) \). For \( \Re(s) \) sufficiently large,

\[
\mathcal{E}(f_{\omega_\nu, s}, \varphi_\pi)(h) = \sum_{\gamma \in P_{mn-1,1}(k) \backslash GL_{mn}(k)} \xi(f_{\omega_\nu, s}, \varphi_\pi)(\gamma h).
\]

The R.H.S of eq. (1.3) continues to a meromorphic function in the whole plane. Denote it by \( E_m(\xi(f_{\omega_\nu, s}, \varphi_\pi)) \). Then, as meromorphic functions on \( GL_m(k) \),

\[
\mathcal{E}(f_{\omega_\nu, s}, \varphi_\pi) = E_m(\xi(f_{\omega_\nu, s}, \varphi_\pi)).
\]

**Remark 1.1.** The identity (1.2) immediately gives a realizations of \( \pi \), in the case \( m = n \).

The proof of this theorem occupies the next two sections. In Section 2 we formally show that the global identity eq. (1.3) holds. In Section 3 we make sense of the integral in eq. (1.2). This allows us to complete the proof of Theorem 1.

**Local theory**

Let \( \nu \) be a finite place of \( k \). Denote by \( k_\nu \) the completion of \( k \) with respect to \( \nu \). Denote by \( O_\nu \) its ring of integers. Let \( \pi_\nu \) be a smooth, irreducible, generic representation of \( GL_n(k_\nu) \) (with central character \( \omega_{\pi_\nu} \)). Our global integral gives rise to the local integral at \( \nu \):

\[
I(f_{\omega_\nu, s}, v_{\pi_\nu}) = \int_{Z_n(k_\nu) \backslash GL_n(k_\nu)} f_{\omega_\nu, s}(\tilde{\varepsilon} t(I_m, g)) \pi_\nu(g) v_{\pi_\nu} dg_\nu,
\]

where \( v_{\pi_\nu} \) is in the space of \( \pi_\nu \), and \( f_{\omega_\nu, s} \) is a smooth, holomorphic section of the induced representation \( Ind_{P_{mn-1,1}(k_\nu)}^{GL_{mn}(k_\nu)}(1 \otimes \omega_{\pi_\nu}^{-1}) \delta_{P_{mn-1,1}}^s \).

In Section 3.2 we prove

**Theorem 2.** Assume \( \Re(s) \gg 0 \). If \( \nu \) is an Archimedean place, then \( I(f_{\omega_\nu, s}, v_{\pi_\nu})(h_\nu) \) is a meromorphic function. If \( \nu \) is a ramified non-Archimedean place, then

\[
I(f_{\omega_\nu, s}, v_{\pi_\nu})(h_\nu) = P_\nu(q_\nu^{-s}),
\]

where \( P_\nu(q_\nu^{-s}) \) is a rational function in \( \mathbb{C}(q_\nu^{-s}) \). Moreover, \( I(f_{\omega_\nu, s}, v_{\pi_\nu})(h_\nu) \) extends to a meromorphic function on all \( \mathbb{C} \), for all place \( \nu \).

Assume that \( \pi_\nu \) is unramified (this is the case for all but finitely many places \( \nu \)). Fix a spherical vector \( \tilde{v}_\nu^0 \) in \( V_{\pi_\nu}^s \). Then, there exists a unique spherical vector \( v_\nu^0 \) in \( V_{\pi_\nu} \), such that \( \langle v_\nu^0, \tilde{v}_\nu^0 \rangle = 1 \). Similarly, there exists a unique unramified section \( f_{\omega_\nu, s}^0 \) normalized by the condition \( f_{\omega_\nu, s}^0(I_{mn}) = 1 \).
Given a matrix coefficient \( c_{v,\tilde{w}}(g) = \langle \pi(g)v, \tilde{w} \rangle \) of \( \pi_\nu \), where \( v \in V_\nu \) and \( \tilde{w} \in V_\tilde{\nu} \), and given a Schwartz-Bruhat function \( \Phi \in S(M_n(k_\nu)) \), we recall the Godement-Jacquet zeta integral [GJ72]

\[
Z_{GJ}(s,c_{v,\tilde{w}},\Phi) = \int_{GL_n(k_\nu)} \Phi(g_\nu)c_{v,\tilde{w}}(g_\nu) |\det g_\nu|^{s+\frac{n-1}{2}} dg_\nu,
\]

which is absolutely convergent for \( \Re(s) \) sufficiently large. In Section 3.1 we prove

**Theorem 3.**

\[
I\left( f_{\omega_{\pi_\nu},s},v_{\pi_\nu}^0 \right) = \frac{Z_{GJ}\left( m(s + \frac{1}{2}) - \frac{n-1}{2},c_{v_{\tilde{\pi}_\nu},\tilde{v}_{\tilde{\pi}_\nu}},\Phi_0 \right)}{L(m(s + \frac{1}{2}),\omega_{\pi_\nu})} v_{\pi_\nu}^0,
\]

where \( \Phi_0 \) is the characteristic function of \( M_n(O_\nu) \).

By the test vector lemma [GJ72, Lemma 6.10] we have \( Z_{GJ}(s,c_{v_0,\tilde{w}},\Phi_0) = L(s,\pi_\nu) \). Thus, Theorem 3 immediately implies

**Theorem 4.**

\[
I\left( f_{\omega_{\pi_\nu},s},v_{\pi_\nu}^0 \right) = \frac{L\left( m(s + \frac{1}{2}) - \frac{n-1}{2},\pi_\nu \right)}{L(m(s + \frac{1}{2}),\omega_{\pi_\nu})} v_{\pi_\nu}^0.
\]

### 1.2 Background and motivation

Eisenstein series are key objects in the theory of automorphic forms. They are an important tool in the study of automorphic \( L \)-functions, and they figure out in the spectral decomposition of the \( L^2 \)-space of automorphic forms. In recent years, new constructions of global integrals generating identities relating Eisenstein series were discovered.

In [GS18] Ginzburg and Soudry introduced two general identities relating Eisenstein series on split classical groups, as well as double covers of symplectic groups. Basically, the idea of their theorem, for example, in case of symplectic group, is as follows. Let \( \tau \) be an irreducible, automorphic, cuspidal representation of \( GL_n(\mathbb{A}) \). Denote by \( \Delta(\tau,i) \) the Speh representation of \( GL_{ni}(\mathbb{A}) \) of “length” \( i \) corresponding to \( \tau \). Let \( \sigma \) be an irreducible, automorphic, cuspidal representation of \( Sp_{ni}(\mathbb{A}) \), and \( \sigma^t \) a certain outer conjugation of \( \sigma \) by an element of order 2. Their theorem shows that Eisenstein series parabolically induced from \( \Delta(\tau,i)|\det|^{\frac{s}{2}} \times \sigma^t \) can be expressed in terms of “more degenerate” Eisenstein series, namely Eisenstein series parabolically induced from the Speh representation \( \Delta(\tau,m+i)|\det|^{\frac{s}{2}} \). In particular they get that any Eisenstein series attached to an irreducible, cuspidal representation on a maximal parabolic subgroup can be expressed in terms of an Eisenstein series, attached to a Speh representation on a Siegel parabolic subgroup. They generalized Mœglin’s work [Mœg97] which is an extension of [GPSR97]. The identity of Ginzburg and Soudry can also be viewed as an extension of the doubling construction introduced in [CFGK16]. The second identity in [GS18] generalizes Ikeda’s work [Ike94] and can be viewed as a generalization of the descent construction studied in [GRS11].

In this work we demonstrate the principle above by proving an identity relating Eisenstein series on general linear groups. Namely, we express an Eisenstein series,
attached to a maximal parabolic subgroup and a pair of representations, one cuspidal and the other a character, in terms of a degenerate Eisenstein series.

The identity (1.4) has further applications. In an ongoing project, we explore several of these. For example, we use it to provide another proof that the Eisenstein series $E_m(\xi(f_\omega,s,\varphi_\pi))$ is holomorphic, and also apply the aforementioned realization (see Remark 1.1) to certain Rankin-Selberg integrals.

1.3 Preliminaries and notation

1.3.1 The groups

Recall that $k$ denotes a number field, and $A$ its ring of adeles. We consider general linear groups and their parabolic subgroups, as algebraic groups over $k$. Let $\ell, r \geq 0$ be two integers. We write $P_{\ell,r}$ for the block upper-triangular maximal parabolic subgroup of $GL_{\ell+r}$ with Levi part $S_{\ell,r} \cong GL_{\ell} \times GL_r$. Its Levi decomposition is

\begin{align*}
P_{\ell,r} &= S_{\ell,r} \ltimes U_{\ell,r}, \\
S_{\ell,r} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in GL_\ell, B \in GL_r \right\}, \\
U_{\ell,r} &= \left\{ \begin{pmatrix} I_\ell & X \\ 0 & I_r \end{pmatrix} \mid X \in M_{\ell,r} \right\}.
\end{align*}

We denote by $H_{\ell,1}$ (or $H_{1,r}$) the subgroup of $P_{\ell,1}$ (or $P_{1,r}$) such that $B = 1$ (or $A = 1$) in eq. (1.11). The standard Borel subgroup of $GL_\ell$ is denoted by $P_\ell = S_\ell U_\ell$, and the Weyl group of $GL_\ell$ by $W_\ell$.

For a place $\nu$, we let $k_\nu$ be the completion of $k$ with respect to the absolute value $|\cdot|_\nu$. Let $\ell \geq 1$ be an integer. We denote the maximal compact subgroup of $GL_\ell(k_\nu)$ by $K_{\ell,\nu}$. For $\nu = \mathbb{R}$, we have $K_{\ell,\nu} = O_\ell$, the orthogonal subgroup, and for $\nu = \mathbb{C}$, we have $K_{\ell,\nu} = U_\ell(\mathbb{C})$, the unitary subgroup (not to be confused with the unipotent radical). For $\nu < \infty$, we denote the ring of integers of $k_\nu$ by $O_\nu$ and its maximal ideal by $P_\nu$. We denote a uniformizer of $\mathcal{P}_\nu$ by $\varpi$, and the cardinality of the residue field by $q_\nu$. In this case, $K_{\ell,\nu} = GL_\ell(O_\nu)$. We denote $K_\ell := \prod_\nu K_{\ell,\nu}$.

Let $P = SU$ be a standard parabolic subgroup of $G = GL_\ell$. Let $T_S = Z(S)$ be the center of $S$. For simplicity we denote $T = T_{S_\ell}$. Let $\Sigma$ be the set of all roots corresponding to the pair $(G,T)$. i.e., the non-trivial eigencharacters of the adjoint action of $T$ on the lie algebra $\mathfrak{g}$ of $G$. Let $\Sigma^+$ be the subset of positive roots determined by $P_\ell$, so that $u = \bigoplus_{\alpha \in \Sigma^+} u_\alpha$, where $\mathfrak{g}_\alpha$ is the eigenspace of $\alpha$, and $u$ is the lie algebra of $U_\ell$. Let $\Delta = \Delta_{P_\ell}$ be the basis of $\Sigma^+$, so that every root in $\Sigma^+$ is a sum of roots in $\Delta$.

1.3.2 Sections and Eisenstein series

We recall the definition of a holomorphic section of a parabolically induced representation, parameterized by an unramified character of the Levi part (see [Kap13, §2.4]). For a thorough treatment of this subject refer to [Wal03, §IV] and [Mui08]. Let $\ell$ and $r$ be two positive numbers. Let $\tau$ an automorphic representation of $GL_\ell(A)$. Consider, for a complex number $s$, the normalized parabolic induction

$$\rho_{s} = \text{Ind}^{GL_{\ell+r}(A)}_{P_{\ell,r}(A)} (1 \otimes \tau) \delta_{P_{\ell,\nu}}.$$
Here, $\delta_{P_{\ell,r}}$ stands for the modulus character of $P_{\ell,r}(\mathbb{A})$, i.e.,

$$\delta_{P_{\ell,r}} \left( \begin{pmatrix} A & X \\ B \end{pmatrix} \right) = |\det A|^r |\det B|^{-\ell},$$

where $A \in \text{GL}_\ell(\mathbb{A})$, and $B \in \text{GL}_r(\mathbb{A})$, and $X \in M_{\ell,r}(\mathbb{A})$.

For a given complex number $s$, the representation $\rho_{\tau,s}$ acts in the space of all smooth, holomorphic functions $\tilde{f}_{\tau,s} : \text{GL}_{\ell+r}(\mathbb{A}) \times \text{GL}_\ell(\mathbb{A}) \times \text{GL}_r(\mathbb{A}) \to \mathbb{C}$ that satisfy

I. For all $g \in \text{GL}_{\ell+r}(\mathbb{A})$, $a \in \text{GL}_\ell(\mathbb{A})$, $b \in \text{GL}_r(\mathbb{A})$, and $X \in M_{\ell,r}(\mathbb{A})$,

$$\tilde{f}_{\tau,s} \left( \begin{pmatrix} A & X \\ B \end{pmatrix} g; a, b \right) = \delta_{P_{\ell,r}}^{s+1/2} \left( \begin{pmatrix} A \\ B \end{pmatrix} \right) \tilde{f}_{\tau,s} (g; aA, bB).$$

II. For fixed $g \in \text{GL}_{\ell+r}(\mathbb{A})$,

$$[(a, b) \mapsto \tilde{f}_{\tau,s} (g; a, b)] \in \tau,$$

where $a \in \text{GL}_\ell(\mathbb{A})$ and $b \in \text{GL}_r(\mathbb{A})$.

By smooth we mean that there is some compact open subgroup $Y \leq \text{GL}_{\ell+r}(\mathbb{A})$ such that $\tilde{f}_{\tau,s}(ng) = \tilde{f}_{\tau,s}(g)$ for all $n \in Y$ and $g \in \text{GL}_{\ell+r}(\mathbb{A})$. We realize the space of $\rho_{\tau,s}$ as smooth, holomorphic functions from $\text{GL}_{\ell+r}(\mathbb{A})$ to $\mathbb{C}$ by setting

$$f_{\tau,s} (g) = \tilde{f}_{\tau,s} (g; I_\ell, I_r).$$

The function $f_{\tau,s}$ satisfies

$$f_{\tau,s} \left( \begin{pmatrix} A & X \\ B \end{pmatrix} g \right) = \delta_{P_{\ell,r}}^{s+1/2} \left( \begin{pmatrix} A \\ B \end{pmatrix} \right) \tilde{f}_{\tau,s} (g; A, B).$$

In particular,

$$f_{\tau,s} : \text{St}_{\ell,r}(k) U_{\ell,r}(\mathbb{A}) \setminus \text{GL}_{\ell+r}(\mathbb{A}) \to \mathbb{C}.$$

There is a bijection between $\rho_{\tau,s}|_{K_{\ell+r}}$ and $\text{Ind}^{K_{\ell+r}}_{P_{\ell,r}(\mathbb{A}) \cap K_{\ell+r}} (1 \otimes \tau)|_{P_{\ell,r}(\mathbb{A}) \cap K_{\ell+r}}$. Let $g = gpgK$ were $gp \in P_{\ell,r}(\mathbb{A})$ and $gK \in K_{\ell+r}$ be the Iwasawa decomposition of $g \in \text{GL}_{\ell+r}(\mathbb{A})$. This bijection is given by mapping $\varphi_\tau \in \text{Ind}^{K_{\ell+r}}_{P_{\ell,r}(\mathbb{A}) \cap K_{\ell+r}} (1 \otimes \tau)|_{P_{\ell,r}(\mathbb{A}) \cap K_{\ell+r}}$ to $f_{\varphi_\tau,s}$:

$$f_{\varphi_\tau,s} (g) = \delta_{P_{\ell,r}} (gp) (1 \otimes \tau) (gp) \varphi_\tau (gK).$$

A section of the form $f_{\varphi_\tau,s} (g)$ is called a standard section (when restricted to the maximal compact subgroup, it does not depend on $s$). The space of holomorphic sections equals to the space of all linear combinations of standard sections over $\mathbb{C}[q_{\nu}^{-s}, q_{\nu}^s]$. Hence, a holomorphic section $f_{\tau,s}$ can be written as

$$f_{\tau,s} = \sum_{i=1}^N P_i(q_{\nu}^{-s}, q_{\nu}^s) f_{\varphi_{\nu,i,s}^s}.$$

where for all $1 \leq i \leq N$, $P_i \in \mathbb{C}[q_{\nu}^{-s}, q_{\nu}^s]$ and $f_{\varphi_{\nu,i,s}^s}$ is a standard section.

We use similar notations over $k_{\nu}$. Let $\tau_{\nu}$ be a representation of $\text{GL}_\ell(k_{\nu})$. Consider, for a complex number $s$, the normalized parabolic induction

$$\rho_{\tau_{\nu},s} = \text{Ind}^{\text{GL}_{\ell+r}(k_{\nu})}_{P_{\ell,r}(k_{\nu})} (\tau_{\nu}) \delta_{P_{\ell,r}}^s,$$
where \( \delta_{P_t,r} \) is the modulus character of \( P_{t,r}(k) \). We use analogous notation realize the space of \( \rho_{\tau,s} \) as functions \( f_{\tau,s} \) from \( U_{t,r}(k) \backslash GL_{t+r}(k) \) to \( V_{\tau} \). Similarly we can write a holomorphic section \( f_{\tau,s} \) as

\[
 f_{\tau,s} = \sum_{i=1}^{N} P_i(q_{\nu}^{-s}, q_{\nu}) f_{\phi_{i,s}},
\]

where for all \( 1 \leq i \leq N \), \( P_i \) is holomorphic function in \( q_{\nu}^{+s} \) (if \( \nu \) is non-Archimedean then \( P_i \in \mathbb{C}[q_{\nu}^{+s}, q_{\nu}^{-s}] \)) and \( f_{\phi_{i,s}} \) is a standard section in \( \rho_{\tau,s} \).

Then, \( \tau, s \in P_{t,s} \backslash GL_{t+r}(k) \) the Eisenstein series on \( GL_{t+r}(\mathbb{A}) \), attached to \( f_{\tau,s} \). For \( \Re(s) \) sufficiently large, it is given by the following (absolutely convergent) series

\[
 E_{t+r}(f_{\tau,s})(h) = \sum_{\gamma \in P_{t,r}(k) \backslash GL_{t+r}(k)} f_{\tau,s}(\gamma h).
\]

### 1.3.3 Kronecker product

Let \( F \) be a field. We realize the tensor product map \( t_F : GL_{t}(F) \times GL_{r}(F) \rightarrow GL_{t+r}(F) \) as follows. Let \( h \) and \( g \) be two square matrices of sizes \( \ell \) and \( r \), respectively. Then, \( t_F(h, g) \) is the \( \ell r \) square block matrix

\[
 t_F(h, g) = \begin{pmatrix}
 h_{1,1}g & \cdots & h_{1,\ell}g \\
 \vdots & \ddots & \vdots \\
 h_{\ell,1}g & \cdots & h_{\ell,\ell}g
 \end{pmatrix},
\]

where \( h = (h_{i,j})_{1 \leq i, j \leq \ell} \).

By eq. (1.14) we immediately get that \( \ker t_F = \{ (\lambda I_{\ell}, \lambda^{-1}I_r) | \lambda \in F^\times \} \). Therefore,

\[
 T_{t,r}(F) := \text{Im} t_F \cong F^\times \backslash (GL_{\ell}(F) \times GL_{r}(F)).
\]

It is convenient to simply denote \( t_F = t \).

We denote the transpose of the matrix \( X \) by \( X^T \). For a square matrix \( Y \), we denote its determinant by \( |Y| \). If \( Y \) is also invertible we set \( Y^* := (Y^T)^{-1} \). The Kronecker product satisfies

\[
 t(h, g) = t(h, I_r)t(I_{\ell}, g) = t(I_{\ell}, g)t(h, I_r),
\]

\[
 (t(h, g))^T = t(h^T, g^T), \quad |t(h, g)| = |h|^r |g|^\ell, \text{ and } (t(h, g))^* = t(h^*, g^*).
\]

### 2 Proof of Theorem 1 - global theory

In this section we prove the global unfolding part of Theorem 1. First, we note that the integral of eq. (1.1) is absolutely convergent. This is due to the rapid decrease of the cusp form, the moderate growth of the Eisenstein series and the fact that the domain of integration is of finite measure.

We start with unfolding the Eisenstein series \( E_{mn}(f_{\omega,s}) \) for \( \Re(s) >> 0 \) in eq. (1.1), where it is defined by

\[
 E_{mn}(f_{\omega,s})(t(h, g)) = \sum_{\varepsilon \in P_{mn-1,1}(k) \backslash GL_{mn}(k)} f_{\omega,s}(\varepsilon t(h, g)).
\]
The group $T_{m,n}(k)$ acts on the set of cosets $P_{mn-1,1}(k)\backslash \operatorname{GL}_{mn}(k)$ from the right. We split the sum as

$$ E_{mn}(f_{\omega_{n,s}})(t(h,g)) = \sum_{\varepsilon \in P_{mn-1,1}(k)\backslash \operatorname{GL}_{mn}(k)/T_{m,n}(k)} \sum_{\gamma \in Q^\varepsilon(k)\backslash T_{m,n}(k)} f_{\omega_{n,s}}(\varepsilon \gamma t(h,g)), $$

(2.2)

where $Q^\varepsilon := P_{mn-1,1}^\varepsilon \cap T_{m,n}$ and $P_{mn-1,1}^\varepsilon := \varepsilon^{-1} P_{mn-1,1} \varepsilon$.

In Section 2.1 we show that the set $P_{mn-1,1}(k)\backslash \operatorname{GL}_{mn}(k)/T_{m,n}(k)$ is finite and find an explicit set of representatives. In Section 2.2 we find for each representative $\varepsilon$, the stabilizer $Q^\varepsilon$. In Section 2.3 we show that the representative corresponding to the open cell is the only one that contributes to the integral eq. (1.1). Then, we rewrite the integral in eq. (1.1) in the form of an Eisenstein series on $\operatorname{GL}_{mn}(k)$ attached to a section $\xi(f_{\omega_{n,s}}, \varphi_\pi)$ which we write explicitly.

## 2.1 The double cosets $P_{mn-1,1}(k)\backslash \operatorname{GL}_{mn}(k)/T_{m,n}(k)$

As mentioned above, in this section we show that the set $P_{mn-1,1}(k)\backslash \operatorname{GL}_{mn}(k)/T_{m,n}(k)$ is finite and find an explicit set of representatives. Namely, we prove

**Theorem 5.** There are exactly $n$ double cosets in $P_{mn-1,1}(k)\backslash \operatorname{GL}_{mn}(k)/T_{m,n}(k)$. We list the following set of representatives: for $0 \leq r \leq n - 1$

$$ \varepsilon_r := \begin{pmatrix} I_{(m-r)n-1} & & 1 \\ & I_{rn} & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_{(m-r)n-1} & 0 & 0 \\ & 1 & b_r \\ & & I_{rn} \end{pmatrix}, $$

(2.3)

where $b_r := (e_{n-1}^T, e_{n-2}^T, \ldots, e_{n-r}^T)$.

The rest of this section is devoted to the proof of Theorem 5.

We begin as follows. Using the Bruhat decomposition in $\operatorname{GL}_{mn}(k)$, we have

$$ \operatorname{GL}_{mn}(k) = \bigcup_{1 \leq j \leq mn} \bigcup_{u \in U_{mn}^{(j)}(k)} P_{mn-1,1}(k)w_j u T_{m,n}(k), $$

(2.4)

where $U_{mn}^{(j)}(k) := w_j^{-1} P_{mn-1,1}(k)w_j \cap U_{mn}(k) \backslash U_{mn}(k)$, and for $1 \leq j \leq mn$, $w_j$ are the representatives of $\left( \begin{array}{c} W_{mn-1} \\ 1 \end{array} \right) \backslash W_{mn}$, i.e.

$$ w_j = \begin{pmatrix} I_{j-1} & & 1 \\ & I_{mn-j} & \\ & & 1 \end{pmatrix}. $$

(2.5)

The following lemma gives an explicit form of the representatives in eq. (2.4).

**Lemma 2.1.** Let $1 \leq j \leq mn$. The elements

$$ u_j(v_j) := \begin{pmatrix} I_{j-1} & 0 & 0 \\ 1 & v_j & \\ & & I_{mn-j} \end{pmatrix}, \quad v_j \in k^{mn-j}. $$

form a set of representatives of $U_{mn}^{(j)}(k)$.  

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Proof. Let
\[ u = \begin{pmatrix} u_{j-1} & X_1 & X_2 \\ 1 & v_j & u_{mn-j} \end{pmatrix} \in U_{mn}(k). \]
Then,
\[ w_j \begin{pmatrix} u_{j-1} & X_1 & X_2 \\ 1 & v_j & u_{mn-j} \end{pmatrix} w_j^{-1} = \begin{pmatrix} u_{j-1} & X_2 & X_1 \\ u_{mn-j} & 0 & v_j \\ v_j & 1 \end{pmatrix}. \]
Thus, \( w_j u w_j^{-1} \in P_{mn-1,1}(k) \) iff \( v_j = 0 \). This gives
\[ U_{mn}(k) \cap w_j^{-1} P_{mn-1,1}(k) w_j = \left\{ \begin{pmatrix} u_{j-1} & X_1 & X_2 \\ 1 & 0 & u_{mn-j} \end{pmatrix} \in U_{mn}(k) \right\} \]
and the lemma follows.

We denote
\[ C_j(v_j) := P_{mn-1,1}(k) w_j (v_j) T_{m,n}(k). \]
In this notation we can rewrite the decomposition in eq. (2.4) as follows.
\[ \text{GL}_{mn}(k) = \bigcup_{1 \leq j \leq m} \bigcup_{v_j \in k^{mn-j}} C_j(v_j). \quad (2.6) \]

We now note that in particular, for \( r = 0 \) we have \( \varepsilon_0 = I_{mn} \) and \( C_{mn}(b_0) = P_{mn-1,1}(k) T_{m,n}(k). \) For \( 0 \leq r \leq n - 1 \) the representative \( \varepsilon_r \) corresponds to the double coset \( C_{(m-r)n}(b_r) \). Therefore, Theorem 5 reduces the decomposition in eq. (2.6) to the disjoint union
\[ \text{GL}_{mn}(k) = \bigcup_{0 \leq r \leq n-1} C_{(m-r)n}(b_r). \quad (2.7) \]

We continue by viewing the first row of \( u_1(v_1) \) as \( m \) vectors in \( k^n \), \( (a_1, \ldots, a_m) \in k^{mn} \), where \( a_j \in k^n \) for all \( 1 \leq j \leq m \) and \( a_1 = (1, a'_1) \) with \( a'_1 \in k^{n-1} \). Denote
\[ R_{m,n}(v_1) = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in M_{m,n}(k). \]
Denote also \( u_1(R_{m,n}(v_1)) := u_1(v_1) \). The following lemma describes the orbit of the right action of \( H_{1,m-1}(k) \otimes H_{1,n-1}(k) \) on the coset \( P_{mn-1,1}(k) w_1 u_1 \left( R_{m,n}(v_1) \right) \).

Lemma 2.2. Let \( h \in H_{1,m-1}(k) \) and \( g \in H_{1,n-1}(k) \). Then,
\[ P_{mn-1,1}(k) w_1 u_1 \left( R_{m,n}(v_1) \right) t(h, g) = P_{mn-1,1}(k) w_1 u_1 \left( h^T R_{m,n}(v_1) g \right). \quad (2.8) \]
Proof. By the Levi decomposition, we can write \( h = h_U h_S \) and \( g = g_U g_S \). We denote
\[ h_U := \begin{pmatrix} 1 & x \\ 0 & I_{m-1} \end{pmatrix}, \quad g_U := \begin{pmatrix} 1 & y \\ 0 & I_{n-1} \end{pmatrix}, \]

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where \( x = (x_2, \ldots, x_m) \), \( y = (y_2, \ldots, y_n) \), and \( h_S := \text{diag}(1, h') \), \( g_S := \text{diag}(1, g') \), where \( h' = (h_{i,j})_{2 \leq i, j \leq m} \in \text{GL}_{m-1}(k) \), \( g' = (g_{i,j})_{2 \leq i, j \leq n} \in \text{GL}_{n-1}(k) \). We have

\[
t(h, g) = t(h_U, I_n) t(I_m, g_U) t(h_S, I_n) t(I_m, g_S).
\] (2.9)

We first prove eq. (2.8) for each one of the matrices on the right hand side of eq. (2.9).

Case \( t(h_U, I_n) \):

\[
u_1(R_{m,n}(v_1)) t(h_U, I_n) = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ I_n & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & I_n \end{pmatrix} \begin{pmatrix} I_n & x_2 I_n & \cdots & x_m I_n \\ I_n & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & I_n \end{pmatrix},
\]

where

\[
A_1 = \begin{pmatrix} 1 & a_1' \\ 0 & I_{n-1} \end{pmatrix} \in \text{GL}_n(k) \quad \text{and} \quad \forall 2 \leq m, A_i = \begin{pmatrix} a_i \\ 0 \end{pmatrix} \in M_n(k).
\]

Thus,

\[
u_1(R_{m,n}(v_1)) t(h_U, I_n) = \begin{pmatrix} A_1 & x_2 A_1 + A_2 & \cdots & x_m A_1 + A_m \\ I_n & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & I_n \end{pmatrix} = u_1(h_U^T R_{m,n}(v_1)).
\]

Case \( t(I_m, g_U) \):

\[
u_1(R_{m,n}(v_1)) t(I_m, g_U) = \begin{pmatrix} A_1 g_U & A_2 g_U & \cdots & A_m g_U \\ I_n & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & I_n \end{pmatrix}.
\]

By Lemma 2.1 \( \text{diag}(I_n, g_U, \ldots, g_U) \in w_1^{-1} P_{mn-1,1}(k) w_1 \cap U_{mn}(k) \). Therefore,

\[
P_{mn-1,1}(k) w_1 u_1(R_{m,n}(v_1)) t(I_m, g_U) = P_{mn-1,1}(k) w_1 u_1(R_{m,n}(v_1) g_U).
\]

Case \( t(h_S, I_n) \):

\[
u_1(R_{m,n}(v_1)) t(h_S, I_n) = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ I_n & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & h_{2,2} I_n & \cdots & h_{2,m} I_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{m,2} I_n & \cdots & h_{m,m} I_n \end{pmatrix}.
\]

We get that,

\[
u_1(R_{m,n}(v_1)) t(h_S, I_n) = t(h_S, I_n) \begin{pmatrix} A_1 & \sum_{j=2}^m h_{j,2} A_j & \cdots & \sum_{j=2}^m h_{j,m} A_j \\ I_n & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & I_n \end{pmatrix}.
\]
Therefore,

\[ u_1 \left( R_{m,n}(v_1) \right) t(h_S, I_n) = t(h_S, I_n)u_1 \left( h_S^T R_{m,n}(v_1) \right). \]

**Case** \( t(I_m, g_S) \):

\[ u_1 \left( R_{m,n}(v_1) \right) t(I_m, g_S) = t(I_m, g_S) \left( g_S^{-1} A_1 g_S \quad g_S^{-1} A_2 g_S \quad \cdots \quad g_S^{-1} A_m g_S \right) \quad I_n. \]

Again, by the fact that the first column of \( g_S \) is \((1, 0, \ldots, 0) \in k^n \) and for \( 1 \leq i \leq m \), only the first row of \( A_i \) is nonzero, we have \( g_S^{-1} A_i g_S = A_i g_S \). Thus,

\[ u_1 \left( R_{m,n}(v_1) \right) t(I_m, g_S) = t(I_m, g_S)u_1 \left( R_{m,n}(v_1) g_S \right). \]

Let \( D \) be one of the matrices \( t(h_S, I_n) \) or \( t(I_m, g_S) \). Then, by the fact that the first column of \( D \) is \((1, 0, \ldots, 0) \in k^{mn} \), we have \( w_1 D w_1^{-1} \in P_{mn-1,1}(k) \). Hence, eq. (2.8) is true for the last two cases as well.

All in all, together with eq. (2.9), we get

\[ P_{mn-1,1}(k)w_1 u_1 \left( R_{m,n}(v_1) \right) t(h, g) = P_{mn-1,1}(k)w_1 u_1 \left( h_S^T h_T R_{m,n}(v_1) g v g_S \right), \]

and the lemma follows.

\[ \square \]

Next, we use Lemma 2.2 to show that for each \( v_1 \in k^{mn-1} \), the double coset \( C_1(v_1) \) equals to one of the double cosets in eq. (2.7).

**Lemma 2.3.** There exists \( 0 \leq r \leq n - 1 \) such that \( C_1(v_1) = C_{(m-r)n}(b_r) \).

**Proof.** Let \( r := \text{rk}(R_{m,n}(v_1)) - 1 \), i.e. the dimension of the space spanned by \( a_1, \ldots, a_m \) minus 1. There exist \( h_0 \in GL_m(k) \) and \( g_0 \in GL_n(k) \) such that \( h_0^T R_{m,n}(v_1) g_0 = R_{m,n}(b_r) \), where \( b_r \) corresponds to

\[ R_{m,n}(b_r) = \begin{pmatrix} I_{r+1} & 0 \\ 0 & 0 \end{pmatrix}. \]

In fact, in order to preserve the 1 at the top left corner of \( R_{m,n}(v_1) \), we must have \( h_0 \in H_{1,m-1}(k) \) and \( g_0 \in H_{1,n-1}(k) \). Hence, by Lemma 2.2, we have

\[ C_1(v_1) = P_{mn-1,1}(k)w_1 u_1 \left( R_{m,n}(v_1) \right) t(h_0, g_0) T_{m,n}(k) = C_1(b_r). \]

Now,

\[ C_1(b_r) = P_{mn-1,1}(k)w_1 E \left( E^{-1} u_1 \left( b_r \right) E \right) T_{m,n}(k), \]

where

\[ E := t \left( \begin{pmatrix} 1 & 1 \\ 1 & I_{m-r-2} \\ & 1 \\ & & \ddots \\ & & & 1 \end{pmatrix} \right). \]

On the one hand, \( E^{-1} u_1 \left( b_r \right) E = u_{(m-r)n} \left( b_r \right) \), and on the other, \( P_{mn-1,1}(k)w_1 E = P_{mn-1,1}(k)w_{(m-r)n} \). Thus, \( C_1(b_r) = C_{(m-r)n}(b_r) \) as requested. \( \square \)
Now, we conclude that for each $1 \leq j \leq mn$ and $v_j \in k^{mn-j}$, the double coset $C_j(v_j)$ equals to one of the double cosets in eq. (2.7).

**Lemma 2.4.** Let $1 \leq j \leq mn$ and $v_j \in k^{mn-j}$. There exists $0 \leq r \leq n-1$ such that $C_j(v_j) = C_{(m-r)n}(b_r)$.

**Proof.** If $j = 1$ then by Lemma 2.3 there exists $0 \leq r \leq n-1$ such that $C_1(v_1) = C_{(m-r)n}(b_r)$ as requested. For all other cases $1 < j \leq mn$, we denote $j = \alpha n + \beta$, where $0 \leq \alpha \leq m-1$ and $1 \leq \beta \leq n$. We have

$$C_j(v_j) = P_{m(n-1)}(k)w_jE \left(E^{-1}u_j \left(v_j \right) \right) T_{m,n}(k),$$

where

$$E := t \begin{pmatrix} I_\alpha & 0 \\ 0 & I_{m-\alpha-1} \end{pmatrix}, \quad \begin{pmatrix} I_{\beta-1} \\ 0 & I_{n-\beta} \end{pmatrix}.$$

On the one hand, $E^{-1}u_j \left(v_j \right) E = u_1 \left(v_j \right)$, where $v_j = (0,v_j) \in k^{mn-1}$, and on the other, $P_{m(n-1)}(k)w_jE = P_{m(n-1)}(k)w_1$. Thus, $C_j(v_j) = C_1(v_1)$, and the proof follows from the $j = 1$ case.

**Lemma 2.4** covers all the possibilities for the double cosets that appear in eq. (2.6), i.e.

$$\bigcup_{1 \leq j \leq mn} \bigcup_{v_j \in k^{mn-j}} C_j(v_j) = \bigcup_{0 \leq r \leq n-1} C_{(m-r)n}(b_r).$$

It is left to show that the double cosets in eq. (2.7) are pairwise disjoint. This is done in following lemma.

**Lemma 2.5.** Let $0 \leq r \neq \ell \leq n-1$ be two integers. Then, $C_{(m-r)n}(b_r) \neq C_{(m-\ell)n}(b_\ell)$.

**Proof.** Assume that $C_{(m-r)n}(b_r) = C_{(m-\ell)n}(b_\ell)$. i.e.,

$$P_{m(n-1)}(k)w_{(m-r)n}(b_r)T_{m,n}(k) = P_{m(n-1)}(k)w_{(m-\ell)n}(b_\ell)T_{m,n}(k).$$

We show that $r = \ell$. Equation (2.10) gives that there exist $p \in P_{m(n-1)}(k)$, $h \in \text{GL}_m(k)$, and $g \in \text{GL}_n(k)$ such that

$$w_{(m-\ell)n}^{-1}p\left(u_{(m-\ell)n}(b_\ell)\right) = u_{(m-\ell)n}(b_\ell) t(h, g).$$

Denote

$$p := \begin{pmatrix} A_1 & A_2 & y_1 \\ A_3 & A_4 & y_2 \\ 0 & 0 & d \end{pmatrix},$$

where $A_4$ is a $\ell n$ by $rn$ matrix and $d \in k^\times$. Then, the left hand side of eq. (2.11) equals

$$w_{(m-\ell)n}^{-1} \begin{pmatrix} A_1 & A_2 & y_1 \\ A_3 & A_4 & y_2 \\ 0 & 0 & d \end{pmatrix} w_{(m-r)n}(b_r) = \begin{pmatrix} A_1 & y_1 \cdot b_r + A_2 \\ 0 & d b_r \end{pmatrix} \left( A_3 & y_2 \cdot b_r + A_4 \right).$$

(2.12)
Denote $h = (h_{i,j})_{1 \leq i, j \leq m}$ and $g = (g_{i,j})_{1 \leq i, j \leq n}$. Then, the right hand side of eq. (2.11) equals

$$u_{(m-\ell)n} (b_\ell) t(h, g) = \left( \begin{array}{c|c} I_{(m-\ell)n-1} & 0 \\ \hline 0 & b_\ell \\ \hline I_{\ell n} & h_{1,1}g & \cdots & h_{1,\ell}g \\ \vdots & \vdots \\ h_{\ell,1}g & \cdots & h_{\ell,\ell}g \end{array} \right).$$  \hspace{1cm} (2.13)

We see that for $1 \leq i \leq (m-r)n-1$ the coordinates $((m-r)n,i)$ of the matrix in the right hand side of eq. (2.12) are all zero. Hence, by comparing eqs. (2.12) and (2.13), we get the following system of equations. For all $1 \leq j \leq n$ and all $1 \leq j' \leq m-r-1$

$$\sum_{i=0}^\ell h_{m-\ell+i,j}g_{n-i,j} = 0.$$  

The rows of $g$ are linearly independent, so we conclude that for $m-\ell \leq i \leq m$ and $1 \leq j \leq m-r-1$ we have $h_{i,j} = 0$. This gives

$$h = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where $D \in M_{\ell+1, r+1}(k)$. i.e. $D$ has $\ell + 1$ linearly independent rows in $k^{r+1}$. This implies $\ell \leq r$. From symmetry of $r$ and $\ell$ in eq. (2.10) we get that $r \leq \ell$ as well. Thus, $r = \ell$ as requested.  \hfill \Box

### 2.2 The stabilizers

In this section we compute $Q^{\varepsilon_r} = P^{\varepsilon_{r-1}}_{m,n-1} \cap T_{m,n}$ for all $0 \leq r < n$, where $\varepsilon_r$ are the different representatives in eq. (2.3). We denote the following maximal parabolic subgroups of $\text{GL}_m(k)$ and $\text{GL}_n(k)$ by $P^m_{m-r+1} := P_{m-r-1,m+r}(k)$ and $P^n_{r+1} := P_{n-r-1,n+r}(k)$, respectively. Consider the following subgroup of $t \left( \frac{P^m_{r+1}(k)}{P^n_{r+1}(k)} \right) \leq T_{m,n}(k),$ 

$$t_\Delta \left( \frac{P^m_{r+1}(k)}{P^{n}_{r+1}(k)} \right) := \left\{ t \left( \begin{pmatrix} A & B \\ 0 & \lambda d^* \end{pmatrix} \right), \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in T_{m,n}(k) \bigg | d \in \text{GL}_{r+1}(k), \lambda \in k^* \right\},$$

such that $d^* := \tilde{w}_r^{-1}(d^T)^{-1}\tilde{w}_r$, where 

$$\tilde{w}_r = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \text{GL}_r(k).$$

**Proposition 2.6.** Let $0 \leq r \leq n-1$. Then, $Q^{\varepsilon_r}(k) = t_\Delta \left( \frac{P^m_{r+1}(k)}{P^{n}_{r+1}(k)} \right).$

Before proving Proposition 2.6, we need the following two lemmas. Generally, we have,

**Lemma 2.7.** Let $V, W$ be vector spaces of dimension $\ell$ over $k$, with bases $\{v_1, v_2, \ldots, v_\ell\}$ and $\{w_1, w_2, \ldots, w_\ell\}$, respectively. Let $B : V \times W \to k$ be a non-degenerate bilinear form, such that $B(v_i, w_j) = \delta_{i,j}$ for all $1 \leq i, j \leq \ell$. Let $S : V \to V$ and $R : W \to W$ be linear transformations. Then, for $\alpha \in k$,

$$\left( S \otimes R \right) \left( \sum_{j=1}^\ell v_j \otimes w_j \right) = \alpha \sum_{j=1}^\ell v_j \otimes w_j.$$  \hspace{1cm} (2.14)
iff $S \circ R^T = \alpha \cdot \text{id}_V$. Therefore, in case $\alpha \neq 0$, eq. (2.14) holds iff $S,R$ are invertible and $S = \alpha (R^T)^{-1}$, where $R^T : V \to V$ is the transformation adjoint to $R$ via the non-degenerate bilinear form $B$.

**Proof.** The bilinear form $B$ defines an isomorphism $\iota : V \otimes W \to \text{Hom}_k(V,V)$ by sending $v \otimes w$ to the linear map $\iota(v \otimes w) = i_{v \otimes w} : V \to V$ defined by $i_{v \otimes w}(x) = B(x,w)v$. Under this isomorphism, the inverse image of $\text{id}_V$ is $\sum_{j=1}^r v_j \otimes w_j$. We have $(S \circ i_{v \otimes w})(x) = B(x,w)S(v) = i_{S(v) \otimes w}$, and $(i_{v \otimes w} \circ S)(x) = B(S(x),w)v = B(x,S^T(w))v = i_{v \otimes S^T(w)}$. Thus, we get that $S \otimes \text{id}_V$ corresponds to precomposition by $S$, and that $\text{id}_V \otimes R$ corresponds to precomposition by $R^T$. Hence, eq. (2.14) is equivalent under the isomorphism $\iota$ to $S \circ R^T = \alpha \cdot \text{id}_V$, as requested. \hfill \Box

We now make use of Lemma 2.7 to prove the following lemma,

**Lemma 2.8.** Let $h \in \text{GL}_m(k)$, $g \in \text{GL}_n(k)$, and

$$\underline{v} := (0_n, \ldots, 0_n, e_n^T, e_{n-1}^T, \ldots, e_1^T) \quad (2.15)$$

where $0_n$ is the row vector of $n$ zeros (it appears $m - r - 1$ times in $\underline{v}$).

Then, there is $\lambda \in k^\times$ such that $\underline{vt}(h,g) = \lambda \underline{v}$ iff $t(h,g) \in t_\Delta \left( P_{r+1}^m(k), P_{r+1}^n(k) \right)$.

**Proof.** Let $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ be the bases of standard row vectors of $k^m$ and $k^n$, respectively. We take the basis of $k^m \otimes k^n$ in lexicographic order, i.e.

$$\{v_1 \otimes w_1, \ldots, v_1 \otimes w_n, \ldots, v_m \otimes w_1, \ldots, v_m \otimes w_n\}. \quad (2.16)$$

We have

$$\underline{v} = \sum_{j=0}^r v_{m-j} \otimes w_{n-r+j}.$$

Now, we write

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $D \in \text{M}_{r+1}(k)$ and $d \in \text{M}_{r+1}(k)$. Denote $E_0 = \text{Sp}\{v_{m-r}, \ldots, v_m\}$, $E'_0 = \text{Sp}\{w_{n-r}, \ldots, w_n\}$, $E_1 = \text{Sp}\{v_1, \ldots, v_{m-r+1}\}$, and $E'_1 = \text{Sp}\{w_1, \ldots, w_{n-r+1}\}$. The matrices $C, D$, and $c, d$ act from right on the space $k^{r+1}$. Let us denote the corresponding linear transformations $T_C, T_c, T_D$ and $T_d$, respectively, such that

$$T_C : E_0 \to E_1, \quad T_D : E_0 \to E_0,$$

$$T_c : E'_0 \to E'_1, \quad T_d : E'_0 \to E'_0.$$

With this notation, the equation $\underline{vt}(h,g) = \lambda \underline{v}$ takes the form

$$\sum_{j=0}^r (T_C (v_{m-j}) + T_D (v_{m-j})) \otimes (T_c (w_{n-r+j}) + T_d (w_{n-r+j})) = \lambda \sum_{j=0}^r v_{m-j} \otimes w_{n-r+j}.$$
The tensor product is bilinear, \( v \in E_0 \otimes E'_0 \) and we have

\[
\sum_{j=0}^{r} T_C(v_{m-j}) \otimes T_c(w_{n-r+j}) \in E_1 \otimes E'_1,
\]

\[
\sum_{j=0}^{r} T_C(v_{m-j}) \otimes T_d(w_{n-r+j}) \in E_1 \otimes E'_0,
\]

\[
\sum_{j=0}^{r} T_C(v_{m-j}) \otimes T_d(w_{n-r+j}) \in E_0 \otimes E'_1,
\]

\[
\sum_{j=0}^{r} T_D(v_{m-j}) \otimes T_c(w_{n-r+j}) \in E_0 \otimes E'_0.
\]

We conclude, by comparing the coefficients of the basis, that eq. (2.17) is equivalent to

\[
\begin{align*}
\sum_{j=0}^{r} T_C(v_{m-j}) \otimes T_c(w_{n-r+j}) &= 0, \\
\sum_{j=0}^{r} T_C(v_{m-j}) \otimes T_d(w_{n-r+j}) &= 0, \\
\sum_{j=0}^{r} T_D(v_{m-j}) \otimes T_c(w_{n-r+j}) &= 0, \\
\sum_{j=0}^{r} T_D(v_{m-j}) \otimes T_d(w_{n-r+j}) &= \lambda \sum_{j=0}^{r} v_{m-j} \otimes w_{n-r+j}.
\end{align*}
\]

We apply Lemma 2.7 by taking \( V = E_0, \; W = E'_0, \; \ell = r + 1, \; S = T_D, \; R = T_d \). We identify \( E_0 \) and \( E'_0 \) with \( k^{r+1} \) by the linear isomorphisms \( \psi_{E_0} \) and \( \psi_{E'_0} \), defined by \( \psi_{E_0}(v_{m-j}) = e_{j+1} \) and \( \psi_{E'_0}(w_{n-r+j}) = e_{j+1} \) for all \( 0 \leq j \leq r \), where \( \{e_1, \ldots, e_{r+1}\} \) is the standard basis of \( k^{r+1} \). Then, \( T_D(v) = \psi_{E_0}^{-1}(D\psi_{E_0}(v)) \) for all \( v \in E_0 \) and \( T_d(w) = \psi_{E'_0}^{-1}(d\psi_{E'_0}(w)) \) for all \( w \in E'_0 \). Now, \( B \) in Lemma 2.7 is given by

\[
B(v, w) = (\psi_{E_0}(v))^T \tilde{w}_r \psi_{E'_0}(w),
\]

where \( v \in E_0 \) and \( w \in E'_0 \). Therefore, we have

\[
B(T_D(v), w) = (\psi_{E_0}(v))^T D^T \tilde{w}_r \psi_{E'_0}(w) = (\psi_{E_0}(v))^T \tilde{w}_r \psi_{E'_0}(w).
\]

The matrix \( \tilde{D} := \tilde{w}_r^{-1}D^T \tilde{w}_r \) defines a linear transformation \( T_{\tilde{D}} \), such that, \( T_{\tilde{D}}(w) = \psi_{E_0}^{-1}(\tilde{D}\psi_{E_0}(w)) \) for all \( w \in E'_0 \). Thus,

\[
B(T_D(v), w) = B \left( v, \psi_{E_0}^{-1} \left( \tilde{D}\psi_{E_0}(w) \right) \right) = B \left( v, T_{\tilde{D}}(w) \right).
\]

Hence, \( T_{\tilde{D}}^T(w) = T_D(w) \). By Lemma 2.7, the last equation in eq. (2.18) is equivalent to \( T_D \circ T_d^T = \lambda \text{id}_{E_0} \) (by taking \( \alpha = \lambda \)), i.e., \( T_D \) and \( T_d \) are invertible, and \( T_D = \lambda(T_d^T)^{-1} = \lambda T_d^{-1} \). So,

\[
D = \lambda \tilde{d}^{-1} = \lambda \tilde{w}_r^{-1}(d^T)^{-1} \tilde{w}_r = \lambda d^*,
\]

as requested. Similarly, by Lemma 2.7 (where we take \( \alpha = 0 \)), the second and the third equations in eq. (2.18) are equivalent to \( T_D \circ T_d^T = 0 \) and \( T_D \circ T_c^T = 0 \). These last equations, in turn, are equivalent to say that both \( T_C \) and \( T_c \) are zero transformations, as \( T_d \) and \( T_D \) are invertible. i.e. \( C = 0 \) and \( c = 0 \), as requested.
Proof of Proposition 2.6. We first note that

$$Q^{\varepsilon_r} = P_{m,n-1,1}^{(r)} \cap T_{m,n} = \left( P_{m,n-1,1} \cap T_{m,n}^{(r)} \right)^{-1} \varepsilon_r.$$  

So, we find $P_{mn-1,1}(k) \cap T_{m,n}(k)^{\varepsilon_r^{-1}}$ and conjugate the result by $\varepsilon_r$. Let $X \in P_{mn-1,1}(k) \cap T_{m,n}(k)^{\varepsilon_r^{-1}}$. The condition $X \in T_{m,n}(k)^{\varepsilon_r^{-1}}$ implies that there exist $h \in \text{GL}_m(k)$ and $g \in \text{GL}_n(k)$ such that

$$X = \varepsilon_r t(h,g) \varepsilon_r^{-1}. \quad (2.19)$$

On the other hand, the condition $X \in P_{mn-1,1}(k)$ yield that there exists $\lambda \in k^\times$ such that

$$e_{mn}X = \lambda e_{mn}, \quad (2.20)$$

where $e_{mn} = (0, \ldots, 0, 1)$. Plugging eq. (2.19) to eq. (2.20) gives

$$e_{mn}x = \lambda e_{mn} \varepsilon_r. \quad (2.21)$$

Recall that by Theorem 5, $\varepsilon_r = w(m-r)n u(m-r)n(b_r)$, where $b_r := (e_{n-1}^T, e_{n-2}^T, \ldots, e_{n-r}^T)$. Hence, eq. (2.21) becomes,

$$e_{mn}w(m-r)n u(m-r)n(b_r) t(h,g) = \lambda e_{mn}w(m-r)n u(m-r)n(b_r). \quad (2.22)$$

Note that $e_{mn}w(m-r)n = e_{(m-r)n}^T$ and that $e_{(m-r)n}u(m-r)n(b_r) = v$, where $v$ is as in eq. (2.15). Therefore, eq. (2.22) can be written as

$$vt(h,g) = \lambda v.$$

The proof now follows from Lemma 2.8. \hfill \Box

2.3 Analysis of the contributions of the double cosets

We now plug eq. (2.2) to our construction in eq. (1.1). By Theorem 5, we get that for $\Re(s) > 0$

$$\mathcal{E}(f_{\omega,s}, \varphi_\pi)(h) = \sum_{r=0}^{n-1} Z_{n(A)} GL_{n}(k) \setminus GL_{n}(A) \int \varphi_\pi(g) \sum_{\gamma \in Q^{\varepsilon_r}(k) \setminus T_{m,n}(k)} f_{\omega,s}(\varepsilon_{r} \gamma t(h,g)) \, dg. \quad (2.23)$$

For all $0 \leq r \leq n - 1$ and all $n$, $\sum_{\gamma} |f_{\omega,s}(\varepsilon_{r} \gamma t(h,g))|$ is of moderate growth in $g$, $\varphi_\pi$ is rapidly decreasing, and since $Z_{n(A)} GL_{n}(k) \setminus GL_{n}(A)$ is of finite measure, the integral in eq. (2.23) is absolutely convergent. In this section, we find the contribution of each double coset to the integral eq. (2.23). In Proposition 2.9 we prove that all double cosets, except the one that corresponds to the open cell, contribute zero. We then analyze, in Proposition 2.10, the contribution of the open cell, and show that it allows us to rewrite eq. (2.23) in a form of an Eisenstein series on $\text{GL}_m(A)$. In Proposition 2.11 we formally show that this series is the Eisenstein series on $\text{GL}_m(A)$ corresponding to a section of $\text{Ind}_{\text{GL}_{m,n}}^{A}(\text{GL}_m(A))$ corresponding to $\delta_{\pi}^s$. In Proposition 2.12 we state the last argument required for completing the proof of Theorem 1.
Proposition 2.9. Let $0 \leq r \leq n-2$ and assume that $\Re(s) >> 0$. Then,
\[
\int_{Z_n(k)\text{GL}_n(k)\backslash GL_n(k)} \varphi_{\pi}(g) \sum_{\gamma \in Q^r(k) \backslash T_{m,n}(k)} f_{\omega, s}(\varepsilon_t \gamma t(h, g)) \, dg = 0. \tag{2.24}
\]

Proof. Recall that Proposition 2.6 gives
\[
Q^r = t\Delta \left( P^n_{r+1}(k), P^n_{r+1}(k) \right) \leq t \left( P^n_{r+1}(k), P^n_{r+1}(k) \right).
\]

Therefore, we split the summation in eq. (2.24) as follows,
\[
\sum_{\gamma \in t \left( P^n_{r+1}(k), P^n_{r+1}(k) \right) \backslash T_{m,n}(k)} \sum_{\gamma' \in Q^r(k) \backslash t \left( P^n_{r+1}(k), P^n_{r+1}(k) \right)} f_{\omega, s}(\varepsilon_t \gamma' \gamma t(h, g)). \tag{2.25}
\]

We can write the representatives of $t \left( P^n_{r+1}(k), P^n_{r+1}(k) \right) \backslash T_{m,n}(k)$ as $\gamma = t(\gamma_1, \gamma_2)$, where $\gamma_1 \in P^n_{r+1}(k) \backslash \text{GL}_m(k)$, and $\gamma_2 \in P^n_{r+1}(k) \backslash \text{GL}_n(k)$. Similarly, we can write the representatives of $Q^r(k) \backslash t \left( P^n_{r+1}(k), P^n_{r+1}(k) \right)$ as $\gamma' = t(\gamma_3, I_n)$, where
\[
\gamma_3 = \begin{pmatrix} I_{m-r-1} & 0 \\ 0 & d \end{pmatrix},
\]
such that $d \in Z_{r+1}(k) \backslash \text{GL}_{r+1}(k)$.

With this notation, we can write the left hand side of eq. (2.24) as follows.
\[
\int_{Z_n(k)\text{GL}_n(k)\backslash GL_n(k)} \varphi_{\pi}(g) \sum_{\gamma_1} \sum_{\gamma_2} f_{\omega, s}(\varepsilon_t \gamma_1 \gamma_2 \gamma t(h, g)) \, dg. \tag{2.26}
\]

By the absolute convergence of the integral, we may switch the order of the $dg$-integral with the summations over $\gamma_1$ and $\gamma_3$. By the automorphic property of $\varphi_{\pi}$, we can write $\varphi_{\pi}(g) = \varphi_{\pi}(\gamma g)$, which allows us to collapse the sum over $\gamma_2$ and the integral. Thus, eq. (2.26) becomes
\[
\sum_{\gamma_1} \sum_{\gamma_3} \int_{Z_n(k)P^n_{r+1}(k)\backslash \text{GL}_n(k)} \varphi_{\pi}(g) f_{\omega, s}(\varepsilon_t \gamma_1 \gamma_3 h, g)) \, dg. \tag{2.27}
\]

We now focus on the inner integral in eq. (2.27). We unfold it using the Levi decomposition $P^n_{r+1} = S^n_{r+1} \times U^n_{r+1}$,
\[
\int_{Z_n(k)S^n_{r+1}(k)U^n_{r+1}(k)\backslash \text{GL}_n(k)U^n_{r+1}(k)\backslash \text{GL}_n(k)U^n_{r+1}(k)} \varphi_{\pi}(ug) f_{\omega, s}(\varepsilon_t \gamma_1 \gamma_3 h, ug)) \, dudg.
\]

Notice that, $t (\gamma_1 \gamma_3 h, ug) = t (I_m, u) t (\gamma_1 \gamma_3 h, g)$. Now, $t (I_m, u)$ commutes with $\varepsilon_t$, as it is in the stabilizer $Q^r$, and the section $f_{\omega, s}$ is invariant under left multiplication by unipotent elements. So, we can extract an inner integration
\[
\int_{U^n_{r+1}(k)\backslash U^n_{r+1}(k)} \varphi_{\pi}(ug) \, du = 0, \tag{2.28}
\]
which vanishes by the cuspidality of $\varphi_{\pi}$, for all $0 \leq r \leq n-2$. \hfill \square
Proposition 2.10. Assume that $\Re(s) >> 0$. Then,

$$\mathcal{E}(f_{\omega,s}, \varphi) (h) = \sum_{\gamma \in P_{m-n,n} \backslash \text{GL}_m} \xi(f_{\omega,s}, \varphi) (\gamma h),$$  \hspace{1cm} (2.29)

where

$$\xi(f_{\omega,s}, \varphi) (h) = \int_{Z_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi(g) f_{\omega,s}(\tilde{\varepsilon} t (h, g)) \, dg.$$  \hspace{1cm} (2.30)

The integral eq. (2.30) converges absolutely for $\Re(s) >> 0$.

*Proof.* By now, eq. (2.23) is simplified to

$$\mathcal{E}(f_{\omega,s}, \varphi) (h) = \int_{Z_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi(g) \sum_{\gamma \in Q^\mathfrak{f}(k) \backslash T_{m,n}(k)} f_{\omega,s}(\tilde{\varepsilon} \gamma t (h, g)) \, dg,$$  \hspace{1cm} (2.31)

where $\tilde{\varepsilon} := \varepsilon_{n-1}$, and $Q^\mathfrak{f}(k) = t_\Delta (P^m_n(k), \text{GL}_{n}(k))$. As in the beginning of the proof of Proposition 2.9, we split the summation in eq. (2.31) as follows,

$$\sum_{\gamma \in t(P^m_n(k), \text{GL}_{n}(k)) \backslash T_{m,n}(k)} \sum_{\gamma' \in Q^\mathfrak{f}(k) \backslash t(P^m_n(k), \text{GL}_{n}(k))} f_{\omega,s}(\tilde{\varepsilon} \gamma' \gamma t (h, g)).$$  \hspace{1cm} (2.32)

As in the last proof, we can write the representatives of $t(P^m_n(k), \text{GL}_{n}(k)) \backslash T_{m,n}(k)$ as $\gamma = t(\gamma_1, I_n)$, where $\gamma_1 \in P^m_n(k) \backslash \text{GL}_{m}(k)$. We can also write the representatives of $Q^\mathfrak{f}(k) \backslash t(P^m_n(k), \text{GL}_{n}(k))$ as $\gamma' = t(\gamma_3, I_n)$, where

$$\gamma_3 = \begin{pmatrix} I_{m-n} & 0 \\ 0 & d \end{pmatrix},$$

such that $d \in Z_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})$. With this notation, we can write eq. (2.32) as follows.

$$\int_{Z_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi(g) \sum_{\gamma_1} \sum_{\gamma_3} f_{\omega,s}(\tilde{\varepsilon} t (\gamma_3 \gamma_1 h, g)) \, dg.$$  \hspace{1cm} (2.33)

Let

$$\gamma_3^{-1} := \begin{pmatrix} I_{m-n} & 0 \\ 0 & (d^{-1})^* \end{pmatrix}.$$  

Then, $f_{\omega,s}$ is invariant under left multiplication by the element $t \left( \gamma_3^{-1}, (\gamma_3^{-1})^* \right)$, which also commutes with $\tilde{\varepsilon}$. So,

$$f_{\omega,s}(\tilde{\varepsilon} t (\gamma_3 \gamma_1 h, g)) = f_{\omega,s}(\tilde{\varepsilon} t (\gamma_1 h, (\gamma_3^{-1})^* g)).$$

As in the last proof, we switch the order of the $dg$-integration with the summation over $\gamma_1$, and then collapse the sum over $\gamma_3$ and the integral. Thus, eq. (2.33) gets a form of an Eisenstein series. i.e.

$$\mathcal{E}(f_{\omega,s}, \varphi) (h) = \sum_{\gamma_1 \in P^m_n(k) \backslash \text{GL}_m(k)} \int_{Z_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi(g) f_{\omega,s}(\tilde{\varepsilon} t (\gamma_1 h, g)) \, dg.$$  \hspace{1cm} (2.34)
By writing $\gamma_1 = \gamma$ we get the requested result. We note that for $\Re(s) > 0$,
\[
\sum_{\gamma_1 \in \mathcal{P}_{2n}(k) \backslash \text{GL}_m(k) \mathcal{Z}_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \int |\varphi_\pi(g) f_{\omega_{s}, s}(\tilde{\varepsilon} t (\gamma_1 h, g))| \, dg < \infty
\]
\[
\square
\]

The integral in eq. (2.30) lies in the space of $\text{Ind}^{\text{GL}_m(\mathbb{A})}_{\mathcal{P}_{m-n,n}(\mathbb{A})} (1 \otimes \tilde{\pi}) \delta^{s}_{m-n,n}$. The next proposition prove this fact at the formal level. The precise meaning of this integral is shown in Section 3.

**Proposition 2.11.** Assume that $\Re(s) > 0$. Let $p \in \mathcal{P}_{m-n,n}(\mathbb{A})$ be of the form
\[
p = \begin{pmatrix} A & B \\ D & \end{pmatrix},
\]
where $A \in \text{GL}_{m-n}(\mathbb{A})$, $D \in \text{GL}_n(\mathbb{A})$ and $B \in \text{M}_{m-n,n}(\mathbb{A})$. Then
\[
\xi (f_{\omega_{s}, s}, \varphi_\pi) (ph) = \delta^{s+1/2}_{\mathcal{P}_{m-n,n}} (p) \int_{\mathcal{Z}_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi_\pi(D^* g) f_{\omega_{s}, s}(\tilde{\varepsilon} t (h, g)) \, dg. \tag{2.35}
\]

**Proof.** We begin by the variables change $g \mapsto D^* g$.
\[
\xi (f_{\omega_{s}, s}, \varphi_\pi) (ph) = \int_{\mathcal{Z}_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi_\pi(D^* g) f_{\omega_{s}, s}(\tilde{\varepsilon} t (ph, D^* g)) \, dg.
\]

By the definition of $Q^\xi$, the matrix $t(p, D^*)$ conjugated by $\tilde{\xi}$, is an element in $\mathcal{P}_{mn-1,1}$. Hence, we denote it by
\[
\begin{pmatrix} R & Y \\ 0 & \alpha \end{pmatrix} := \tilde{\varepsilon} t (p, D^*) \tilde{\varepsilon}^{-1},
\]
where $R \in \text{GL}_{mn-1}(\mathbb{A})$, $Y \in k^{mn-1}$, and $\alpha \in k^\times$.
\[
\xi (f_{\omega_{s}, s}, \varphi_\pi) (ph) = \omega^{-1}_\pi(\alpha) \delta^{s+1/2}_{\mathcal{P}_{mn-1,1}} \begin{pmatrix} R & Y \\ 0 & \alpha \end{pmatrix} \cdot \int_{\mathcal{Z}_n(\mathbb{A}) \backslash \text{GL}_n(\mathbb{A})} \varphi_\pi(D^* g) f_{\omega_{s}, s}(\tilde{\varepsilon} t (h, g)) \, dg.
\]

By eq. (2.20) we have $e^T_{mn} \tilde{\varepsilon} t (p, D^*) \tilde{\varepsilon}^{-1} = e^T_{mn}$. i.e., $\alpha e^T_{mn} = e^T_{mn}$. Thus, $\alpha = 1$ and $\omega^{-1}_\pi(\alpha) = 1$. This also gives, $|\det(R)| = |\det (\tilde{\varepsilon} t (p, D^*) \tilde{\varepsilon}^{-1})|$. Therefore,
\[
\delta^{s+1/2}_{\mathcal{P}_{mn-1,1}} \begin{pmatrix} R & Y \\ 0 & \alpha \end{pmatrix} = |\det(R)| = |\det (p)|^n |\det (D^*)|^m = |\det (A)|^n |\det (D)|^{-(m-n)}.
\]

This implies
\[
\delta^{s+1/2}_{\mathcal{P}_{mn-1,1}} (\tilde{\varepsilon} t (p, D^*) \tilde{\varepsilon}^{-1}) = \delta^{s+1/2}_{\mathcal{P}_{m-n,n}} (p).
\]

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The point now is that the function

$$D \mapsto \int_{\Gamma \backslash \mathbb{H}} \varphi_\pi(D^* g) f_{\omega_0, s} (\xi t (h, g)) \, dg$$

is formally a cusp form in the space of $\tilde{\pi}$, that is of the form $\varphi_\pi'(D^*)$ where $\varphi_\pi'$ is in the space of $\pi$. □

In order to complete the proof of Theorem 1, we need the following proposition.

**Proposition 2.12.** The function on $\text{GL}_m(k)$, $\xi (f_{\omega_0, s}, \varphi_\pi)$, defined for $\Re(s)$ sufficiently large by the integral eq. (2.30), admits an analytic continuation to a meromorphic function of $s$ in the whole plane. It defines a smooth meromorphic section of

$$\rho_{\tilde{\pi}, s} := \text{Ind}_{\pi(m-n, m)}^{\text{GL}_m(k)} (1 \otimes \tilde{\pi}) \delta_{\text{r}, m-n, n}.$$  \hspace{1cm} (2.36)

Thus, by eq. (2.29), $\mathcal{E}(f_{\omega_0, s}, \varphi_\pi)(h)$ is the Eisenstein series on $\text{GL}_m(A)$, corresponding to the section $\xi (f_{\omega_0, s}, \varphi_\pi)$ of $\rho_{\tilde{\pi}, s}$.

The proof of Proposition 2.12 is given at the end of Section 3, where we first make sense of the integral in eq. (2.30).

## 3 Local result

We begin this section by explaining how the global integral in eq. (1.2) gives rise to corresponding local integrals. Then, in Section 3.1 we explicitly compute these local integrals at the unramified places. In Section 3.2 we show that, in the “bad” places, Archimedean or ramified non-Archimedean, these local integrals are meromorphic functions (rational functions in $q_v^{-r}$'s in the $p$-adic case).

In the course of proving Theorem 1, we showed that, for $\Re(s) > 0$, $h \in \text{GL}_n(A)$

$$\mathcal{E}(f_{\omega_0, s}, \varphi_\pi)(h) = \sum_{\gamma \in P_{m-n, n}(k) \backslash \text{GL}_m(k)} \xi (f_{\omega_0, s}, \varphi_\pi)(\gamma h).$$

where (for $\Re(s) > 0$),

$$\xi (f_{\omega_0, s}, \varphi_\pi)(h) = \int_{\Gamma \backslash \mathbb{H}} \varphi_\pi(g) f_{\omega_0, s} (\xi t (h, g)) \, dg.$$  \hspace{1cm} (3.1)

We have seen formally that $\xi (f_{\omega_0, s}, \varphi_\pi) \in \text{Ind}_{\pi(m-n, m)}^{\text{GL}_m(k)} (1 \otimes \tilde{\pi}) \delta_{\text{r}, m-n, n}$. In order to substantiate this statement, we carry out a local study at each place of eq. (3.1). We denote by $\pi_\nu$ the generic, irreducible, smooth representation of $\text{GL}_n(k_\nu)$ (with central character $\omega_0$), which is the local factor of $\pi$ at $\nu$. We write $\pi$ as a restricted tensor product of the local representations $\pi_\nu$, $\pi = \ell_\pi (\otimes_\nu' \pi_\nu)$, where $\ell_\pi$ is a fixed isomorphism. Assume that the cusp form $\varphi_\pi$ is decomposable, i.e. $\varphi_\pi = \ell_\pi (\otimes_\nu' v_{\pi_\nu})$, where $v_{\pi_\nu}$ lies in the space of $\pi_\nu$. So,

$$\varphi_\pi(g) = (\pi(g) \varphi_\pi)(I_n) = (\pi(g) \ell_\pi (\otimes_\nu' v_{\pi_\nu}))(I_n) = \ell_\pi (\otimes_\nu' \pi_\nu(g_\nu) v_{\pi_\nu})(I_n),$$  \hspace{1cm} (3.2)
where \( g = \prod_{\nu} g_{\nu} \in \text{GL}_n(A) \). Assume that \( f_{\omega, s} \) is decomposable and corresponds to a tensor product of local sections \( f_{\omega_{\nu, s}} \) of

\[
p_{\omega_{\nu, s}} := \text{Ind}_{P_{mn-1,1}(k_{\nu})}^{\text{GL}_{mn}(k_{\nu})} (1 \otimes \omega_{\nu}^{-1}) \delta_{P_{mn-1,1}}^{\omega_{\nu}}.
\]  

(3.3)

Let us fix a finite set of places \( S_0 \), containing the Archimedean places, for which, \( \pi_{\nu} \) is unramified. For all \( \nu \notin S_0 \), we fix a spherical vector \( \nu_{\pi_{\nu}} \in V_{\pi_{\nu}} \) and \( f_{\omega_{\nu, s}} \) unramified and normalized, such that for \( p \in P_{mn-1,1}(k_{\nu}) \) and \( x \in K_{n, \nu}, f_{\omega_{\nu, s}}(I_n) = 1 \).

Now we can interpret eq. (3.1) via a product over the places of \( k \). Our assumptions together with eq. (3.2) give

\[
\xi(f_{\omega, s}, \varphi_{\pi})(h) = \int_{Z_n(\mathbb{A})/\text{GL}_n(\mathbb{A})} \ell(\otimes_{\nu} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}})dg
\]

\[
\xi(f_{\omega, s}, \varphi_{\pi})(h) = \int_{Z_n(\mathbb{A})/\text{GL}_n(\mathbb{A})} \ell(\otimes_{\nu \notin S} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}})dg,
\]

(3.4)

where \( h = \prod_{\nu} h_{\nu} \). Equation (3.4) is defined by the following limit.

\[
\xi(f_{\omega, s}, \varphi_{\pi})(h) = \lim_{S_0 \subseteq S \nearrow} \ell(\otimes_{\nu \in S} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}})dg
\]

\[
\otimes_{\nu \notin S} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}})(I_n),
\]

(3.5)

where the limit is taken over increasing finite sets of places \( S \supseteq S_0 \). For all \( g_{\nu} \in K_{n, \nu} \) we have

\[
f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) = f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, I_n)).
\]

This gives

\[
\int_{\mathcal{O}_{n}^{\nu}/K_{n, \nu}} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}}dg_{\nu} = v_{\pi_{\nu}}f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, I_n)).
\]

Therefore, we can rewrite eq. (3.5) as

\[
\xi(f_{\omega, s}, \varphi_{\pi})(h) = \lim_{S_0 \subseteq S \nearrow} \ell(\otimes_{\nu \in S} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}})dg
\]

\[
\otimes_{\nu \notin S} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, I_n))(I_n).
\]

(3.6)

All in all, the inner local integrals in eq. (3.6) are the corresponding local integrals to our global construction. We denote them by

\[
I(f_{\omega_{\nu, s}}, \varphi_{\pi})(h_{\nu}) = \int_{Z_n(k_{\nu})/\text{GL}_n(k_{\nu})} f_{\omega_{\nu, s}}(\tilde{\varepsilon}(h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu})v_{\pi_{\nu}}dg_{\nu}.
\]

(3.7)
In this section we use the Cartan decomposition (modulo the center). We can write \( g_{\nu} \in Z_n(k_{\nu})\backslash GL_n(k_{\nu}) \) as
\[
g_{\nu} = A_lB,
\] (3.8)
where \( A, B \in K_{n,\nu} \) and \( \mathfrak{t} = \text{diag} (t_1, \ldots, t_{n-1}, 1) \) such that for \( \nu < \infty \),
\[
\mathfrak{t} \in Z_n(k_{\nu}) \backslash T^- := \{ \text{diag} (\varpi^1, \ldots, \varpi^{n-1}, 1) \in GL_n(k_{\nu}) | r_1 \geq \ldots \geq r_{n-1} \geq 0 \}\]
and for \( \nu = \infty \),
\[
\mathfrak{t} \in Z_n(k_{\nu}) \backslash T^- := \{ \text{diag} (e^{r_1}, \ldots, e^{r_{n-1}}, 1) \in GL_n(k_{\nu}) | r_1 \geq \ldots \geq r_{n-1} \geq 0 \}.
\]
In both cases \( \mathfrak{t} \) satisfies \( |t_1| \leq \ldots \leq |t_{n-1}| \leq 1 \).

We also make use of the following integral formula. We refer to the paper [II10] for the statement of this claim in this explicit form.

**Claim 3.1** ([II10] p.12). Let \( G \) be a reductive algebraic group defined over \( k_{\nu} \), and \( G = KT^-K \) its Cartan decomposition. Let \( f \in L^1(G) \). Then,
\[
\int_{G} f(g)dg = \int_{T^-} \mu(a) \int_{K^2} f(k_1ak_2)dk_1dadk_2,
\]
where \( \mu(a) = \mu(KaK)/\mu(K) \) for non-Archimedean places \( \nu \) and \( \mu(a) \geq 0 \) for the Archimedean places.

Moreover, we have the following formulae. By [Mac98, Chapter V, eq. (2.9)],

**Claim 3.2.** Let \( \nu < \infty \). Let \( \mathfrak{t} = \text{diag} (\varpi^{r_1}, \ldots, \varpi^{r_n}) \in T^- \). Suppose that the integers \( n_i \) are the lengths of constant runs in the sequence \( (r_i) \), so that \( n_1 + \ldots + n_\ell = n \) and
\[
r_1 = \ldots = r_{n_1} > r_{n_1+1} = \ldots = r_{n_1+n_2} > r_{n_1+n_2+1} = \ldots
\]
Then,
\[
\mu(\mathfrak{t}) = q^n \sum_{i=1}^{n_\ell} (n-2i+1)r_i \frac{\varphi_n(q^{-1})}{(1-q^{-1})^n} \prod_{j=1}^{\ell} \frac{(1-q^{-1})^{n_j}}{\varphi_{n_j}(q^{-1})},
\]
where as usual \( \varphi_j(t) = (1-t)(1-t^2)\ldots(1-t^j) \).

By [Hel84, Section I.5, Theorem 5.8] or by [Kna01, Proposition 5.28],

**Claim 3.3.** Let \( \nu = \infty \). Let \( a \in T^- \). Set \( H \) in the Lie algebra of \( T^- \) such that \( a = e^H \). Then,
\[
\mu(a) = \sum_{\alpha \in \Sigma_+} |\sinh \alpha(H)|^{\dim \mathfrak{g}_\alpha}.
\]

### 3.1 Unramified computation

In this section we prove Theorem 3 by computing the local integrals eq. (3.7) at the unramified places \( \nu \). i.e.
\[
I \left( f^0_{\omega_{\nu},s}, v^0_{\nu} \right) (h_{\nu}) = \int_{Z_n(k_{\nu}) \backslash GL_n(k_{\nu})} f^0_{\omega_{\nu},s} (\mathfrak{t} (h_{\nu}, g_{\nu})) \pi_{\nu}(g_{\nu}) v^0_{\nu} dg_\nu.
\] (3.9)

As part of the proof of Theorem 3, we also conclude by the end of this section that
Lemma 3.4. \( I \left( f^o_{\omega_{u,v},s}, v^o_{\pi v} \right) \) is an unramified section of \( \text{Ind}^{GL_m(k_\nu)}_{P_{m-n,n}(k_\nu)} (1 \otimes \pi_\nu) \delta^s_{P_{m,n,n}}. \)

In the meantime, we note that \( I \left( f^o_{\omega_{u,v},s}, v^o_{\pi v} \right) \) is left-invariant under \( K_{m,\nu} \) by the fact that \( f^o_{\omega_{u,v}} \) is unramified. In addition, a similar (local) computation to Proposition 2.11 formally gives that \( I \left( f^o_{\omega_{u,v},s}, v^o_{\pi v} \right) \) lies in \( \text{Ind}^{GL_m(k_\nu)}_{P_{m-n,n}(k_\nu)} (1 \otimes \pi_\nu) \delta^s_{P_{m,n,n}}. \) Therefore, by writing the Iwasawa decomposition \( h_\nu = h_P h_K \), where \( h_P \in P_{m-n,n}(k_\nu) \) and \( h_K \in K_m \), we get that it is sufficient to evaluate \( I \left( f^o_{\omega_{u,v},s}, v^o_{\pi v} \right) \) at \( I_m \).

Before proving Theorem 3, we find in Lemma 3.5 and Proposition 3.6 the value of \( f^o_{\omega_{u,v},s} (\tilde{t} (I_m, g_\nu)) \), as an expression of \( t \).

Lemma 3.5. Let \( g_\nu = A t B \), be the Cartan decomposition (modulo the center) of \( g_\nu \) as in eq. (3.8). Then,

\[
\tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, g_\nu)) = \tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, t)).
\]

Proof. The section \( f^o_{\omega_{u,v},s} \) is unramified, so, in particular, it is right-invariant under \( t(\text{diag}(I_{m-n}, x^*), I_n) \), where \( x \in K_{n,\nu} \). Thus,

\[
\tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, x g_\nu)) = \tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (\text{diag}(I_{m-n}, x^*), x g_\nu)).
\]

On the other hand, \( t(\text{diag}(I_{m-n}, x^*), x) \in Q^\nu(k_\nu) \), so its conjugation with \( \tilde{\epsilon} \) lies in \( P_{m-1,1}(k_\nu) \cap K_{mn,\nu} \). Since \( f^o_{\omega_{u,v},s} \) is left invariant under \( P_{m-1,1}(k_\nu) \cap K_{mn,\nu} \), we conclude that

\[
\tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, x g_\nu)) = \tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, g_\nu)) \quad \text{(3.10)}
\]

We conclude that the function \( g_\nu \mapsto f^o_{\omega_{u,v},s} (\tilde{t} (I_m, g_\nu)) \) is bi-invariant under \( K_{m,\nu} \).

Therefore,

\[
\tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, g_\nu)) = \tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, A t B)) = \tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, t)) \quad \square
\]

Proposition 3.6. In the notation above,

\[
\tilde{f}^o_{\omega_{u,v},s} (\tilde{t} (I_m, t)) = |\det t|^{ms + \frac{m}{2}}.
\]

Proof. We denote \( t^\Delta := t (I_m, t) \). Recall that \( \tilde{\epsilon} = \tilde{\omega} u \) where we write \( \tilde{\omega} := w(m-n+1)n \) and \( \tilde{u} := u(m-n+1)n(b_{n-1}) \). In this notation we have,

\[
\tilde{t}^\Delta = \tilde{w} t^\Delta \tilde{w}^{-1} \tilde{u} = (t^\Delta)^{-1} \tilde{u} t^\Delta \tilde{w}^{-1} = (t^\Delta)^{-1} \tilde{u} t^\Delta \tilde{w}^{-1} \tilde{u} u(m-n+1)n(b_{n-1}(t)),
\]

where \( b_{n-1}(t) = (t_{n-1} \cdot e_{n-1}^T, \ldots, t_1 \cdot e_1^T) \). All the elements above the diagonal of the unipotent matrix \( u(m-n+1)n(b_{n-1}(t)) \) are bounded in absolute value by 1. Therefore, \( \tilde{w} u(m-n+1)n(b_{n-1}(t)) \in K_{mn,\nu} \), which implies

\[
f^o_{\omega_{u,v},s} (\tilde{\epsilon}) = f^o_{\omega_{u,v},s} (\tilde{w} t^\Delta \tilde{w}^{-1}) \quad \text{(3.11)}
\]

Since \( \tilde{w} t^\Delta \tilde{w}^{-1} \) equals \( t^\Delta \) up to the order of the elements on the diagonal

\[
f^o_{\omega_{u,v},s} (\tilde{w} t^\Delta \tilde{w}^{-1}) = \omega^{-1}_{\pi,\nu} (\alpha) \delta^{s+\frac{1}{2}}_{P_{m-1,1}} (\tilde{w} t^\Delta \tilde{w}^{-1}) \quad \text{ (3.12)}
\]
where $\alpha$ is the $(mn, mn)$-th coordinate of $\tilde{w}^\Delta \bar{w}^{-1}$. It is given by
\[
e^T_{mn} (\tilde{w}^\Delta \bar{w}^{-1}) e_{mn} = e^T_{(m-n+1)n} t^\Delta e_{(m-n+1)n},
\]
i.e. it is the $((m - n + 1)n, (m - n + 1)n)$-th coordinate of $t^\Delta$, which equals 1. So $\alpha = 1$ and
\[
\omega^{-1}(\alpha) = 1. \tag{3.13}
\]
This also gives that
\[
\delta_{P_{mn-1,1}} (\tilde{w}^\Delta \bar{w}^{-1}) = \left| \det \tilde{L}^\Delta \right| = \left| \det L^m \right| = \prod_{i=1}^{n-1} |t_i|^m. \tag{3.14}
\]
We now plug eq. (3.13) and eq. (3.14) in eq. (3.12). The proposition now follows by plugging this result in eq. (3.11).

We are now ready to prove Theorem 3.

*Proof of Theorem 3.* We normalize the measure such that
\[
\mu(K_{n,\nu}) = \int_{K_{n,\nu}} dx = 1. \tag{3.15}
\]
Let $g_\nu = A\bar{t}B$, be the Cartan decomposition (modulo the center) of $g_\nu$ as in eq. (3.8). By Claim 3.1 we can write eq. (3.9) as
\[
I \left( f^\circ_{\omega_{n,v}, s}, v^\circ_{\pi_\nu} \right) (I_m) = \int_{Z_n(k_\nu) \setminus T^-} \mu(t) \int_{K_{n,\nu}} f^\circ_{\omega_{n,v}, s} (\hat{\omega} (I_m, A\bar{t}B)) \pi_\nu(A\bar{t}B) v^\circ_{\pi_\nu} dAdB dt, \tag{3.16}
\]
where $\mu(t) = \mu(K_{n,\nu} A K_{n,\nu})$. We assume that $\Re(s) >> 0$. By Lemma 3.5 we have $f^\circ_{\omega_{n,v}, s} (\hat{\omega} (I_m, A\bar{t}B)) = f^\circ_{\omega_{n,v}, s} (\hat{\omega} (I_m, \hat{t}))$. In addition, $v^\circ_{\pi_\nu}$ is spherical so $\pi_\nu(A\bar{t}B) v^\circ_{\pi_\nu} = \pi_\nu(A\tilde{t}) v^\circ_{\pi_\nu}$. Therefore, eq. (3.9) equals
\[
I \left( f^\circ_{\omega_{n,v}, s}, v^\circ_{\pi_\nu} \right) (I_m) = \int_{Z_n(k_\nu) \setminus T^-} \mu(t) f^\circ_{\omega_{n,v}, s} (\hat{\omega} (I_m, \tilde{t})) \int_{K_{n,\nu}} \pi_\nu(A\tilde{t}) v^\circ_{\pi_\nu} dAd\tilde{t}. \tag{3.17}
\]
Applying Proposition 3.6 gives
\[
I \left( f^\circ_{\omega_{n,v}, s}, v^\circ_{\pi_\nu} \right) (I_m) = \int_{Z_n(k_\nu) \setminus T^-} \mu(t) \left| \det \tilde{t} \right|^{ms+\frac{n}{2}} \int_{K_{n,\nu}} \pi_\nu(A\tilde{t}) v^\circ_{\pi_\nu} dAd\tilde{t}. \tag{3.18}
\]
The inner integral is an unramified functional on $V_{\pi}$. Hence, there exists a constant $c(\tilde{t})$ such that
\[
\int_{K_{n,\nu}} \pi_\nu(A\tilde{t}) v^\circ_{\pi_\nu} dA = c(\tilde{t}) v^\circ_{\pi_\nu}. \tag{3.19}
\]
We apply the unique vector $\tilde{v}^\circ_{\pi_\nu} \in V_{\pi}$, such that $\langle v^\circ_{\pi_\nu}, \tilde{v}^\circ_{\pi_\nu} \rangle = 1$, on eq. (3.19)
\[
c(\tilde{t}) = \int_{K_{n,\nu}} \langle \pi_\nu(A\tilde{t}) v^\circ_{\pi_\nu}, \tilde{v}^\circ_{\pi_\nu} \rangle dA = \int_{K_{n,\nu}} \langle \pi_\nu(\tilde{t}) v^\circ_{\pi_\nu}, \pi_\nu(A^{-1}) \tilde{v}^\circ_{\pi_\nu} \rangle dA = \langle \pi_\nu(\tilde{t}) v^\circ_{\pi_\nu}, \tilde{v}^\circ_{\pi_\nu} \rangle.
\]
Therefore, by eq. (3.15) we conclude that \( c(t) = c_{v_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}} (t) \) (the matrix coefficient of \( \pi_{\nu} \)).

Thus, for \( \Re(s) > 0 \)

\[
I \left( f_{s}^{0}, v_{\pi_{\nu}}^{0} \right) (I_{m}) = \int_{Z_{n}(k_{\nu}) \setminus T^{-}} \mu(t) c_{\nu_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}} (t) v_{\pi_{\nu}}^{0} \left| \det t \right|^{ms + \frac{m}{2}} dt. \tag{3.20}
\]

By the fact that \( I \left( f_{s}^{0}, v_{\pi_{\nu}}^{0} \right) (I_{m}) \) is a \( K_{n,\nu} \)-invariant vector, there exists a constant \( c(I, s) \) such that

\[
I \left( f_{s}^{0}, v_{\pi_{\nu}}^{0} \right) (I_{m}) = c(I, s) v_{\pi_{\nu}}^{0}. \tag{3.21}
\]

Applying \( \tilde{v}_{\pi_{\nu}}^{0} \) on eq. (3.21) gives

\[
c(I, s) = \int_{Z_{n}(k_{\nu}) \setminus T^{-}} \mu(t) c_{\nu_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}} (t) \left| \det t \right|^{ms + \frac{m}{2}} dt. \tag{3.22}
\]

We now consider the Godement-Jacquet zeta integral eq. (1.7). We evaluate it at the normalized matrix coefficient, which correspond to the spherical vectors \( v_{\pi_{\nu}}^{0}, \tilde{v}_{\pi_{\nu}}^{0} \), and at \( \Phi_{0} \), the characteristic function of \( M_{n} (\mathcal{O}_{\nu}) \),

\[
Z_{GJ} \left( s, c_{v_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}}, \Phi_{0} \right) = \int_{\text{GL}_{n}(k_{\nu})} \Phi_{0}(g_{\nu}) c_{v_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}} (g_{\nu}) \left| \det g_{\nu} \right|^{s + \frac{n-1}{2}} dg_{\nu}. \tag{3.23}
\]

Notice that the integrand is bi-invariant under \( K_{n,\nu} \). Therefore, by writing the Cartan decomposition (modulo the center) of \( g_{\nu} \) as in eq. (3.8), applying Claim 3.1 on eq. (3.23), and splitting the integral through the center, we have

\[
Z_{GJ} \left( s, c_{v_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}}, \Phi_{0} \right) = \int_{Z_{n}(k_{\nu}) \setminus T^{-}} \mu(t) F_{s, \Phi_{0}} (t) c_{v_{\pi_{\nu}}, \tilde{v}_{\pi_{\nu}}} (t) \left| \det t \right|^{s + \frac{m}{2}} dt \tag{3.24}
\]

where

\[
F_{s, \Phi_{0}} (t) := \int_{k_{\nu}^{\times}} \Phi_{0}(at) \left| a \right|^{n \left( s + \frac{m}{2} \right)} \omega_{\pi_{\nu}} (a) d^{\times} a. \tag{3.25}
\]

For \( a \in k_{\nu}^{\times} \),

\[
a t \in M_{n} (\mathcal{O}_{\nu}) \iff |a| \leq 1,
\]

i.e.

\[
\Phi_{0}(at) = \begin{cases} 1, & |a| \leq 1 \\ 0, & |a| > 1. \end{cases} \tag{3.26}
\]

By eq. (3.26), we find that eq. (3.25) equals

\[
F_{s, \Phi_{0}} (t) = \int_{|a| \leq 1} \left| a \right|^{n \left( s + \frac{m}{2} \right)} \omega_{\pi_{\nu}} (a) d^{\times} a = \sum_{j=0}^{\infty} q_{\nu}^{-jn \left( s + \frac{n-1}{2} \right)} \omega_{\pi_{\nu}}^{j} (\varpi). \tag{3.27}
\]

Hence,

\[
F_{s, \Phi_{0}} (t) = \frac{1}{1 - \omega_{\pi_{\nu}} (\varpi) q_{\nu}} = L \left( \left( s + \frac{n-1}{2} \right), \omega_{\pi_{\nu}} \right). \tag{3.28}
\]
We now plug eq. (3.28) in eq. (3.24).
\[
\frac{Z_{GJ} (s, c_{\rho_0, \omega_{\pi_\nu}^0}, \Phi_0)}{L (n (s + \frac{n-1}{2}), \omega_{\pi_\nu})} \ = \ \int_{Z_n(k_\nu) \setminus T^-} \mu (t) c_{\rho_0, \omega_{\pi_\nu}^0} (t) \left| \det (t) \right|^{s + \frac{n-1}{2}} \, dt, \tag{3.29}
\]
where we divided the $L$-function of $\omega_{\pi_\nu}$ from both sides. Now by taking $s \mapsto m(s + \frac{1}{2}) - \frac{n-1}{2}$ we get that the integral in eq. (3.29) equals to $c(I, m(s + \frac{1}{2}))$ in eq. (3.22). Now we get eq. (1.8) as requested.

We note that eq. (3.21) shows in particular that $I \left( f_{\omega_{\pi_\nu^0}, \nu}^0 \right)$ lies in the space of $\pi_{\nu}$, which is Lemma 3.4. Also, Theorem 4 follows, and, in particular, $I \left( f_{\omega_{\pi_\nu^0}, \nu}^0 \right)$ continues to a meromorphic function in $\mathbb{C}$, which is a rational function in $q_{\nu}^{-s}$.

### 3.2 The “bad” places

#### 3.2.1 Common statements

Let $\nu$ be an Archimedean or a ramified non-Archimedean place. Recall that for a section $f_{\omega_{\pi_\nu}}$ in the induced space (given in eq. (3.3)):
\[
\rho_{\omega_{\pi_\nu}, \nu} = \text{Ind}_{P_{m-1,1}(k_\nu)}^{GL_{mn}(k_\nu)} \left( 1 \otimes \omega_{\pi_\nu}^{-1} \right) \delta_{P_{m-1,1}, s}^s,
\]
we consider the local integrals (given in eq. (3.7)):
\[
I \left( f_{\omega_{\pi_\nu}, \nu}^0, v_{\pi_\nu} \right) (h_\nu) = \int_{Z_n(k_\nu) \setminus GL_n(k_\nu)} \pi_\nu (g_\nu v_{\pi_\nu}) f_{\omega_{\pi_\nu}, \nu} (\hat{t} (h_\nu, g_\nu)) \, dg_\nu.
\]

This section is dedicated to prove Theorem 2.

First, we make some reductions. We first note that
\[
f_{\omega_{\pi_\nu}, \nu} (\hat{t} (h_\nu, g_\nu)) = \left[ \rho_{\omega_{\pi_\nu}, \nu} (t(h_\nu, I_n)) f_{\omega_{\pi_\nu}, \nu} \right] (\hat{t} (I_m, g_\nu)).
\]
Hence, by replacing $f_{\omega_{\pi_\nu}, \nu}$ by the smooth section $\rho_{\omega_{\pi_\nu}, \nu} (t(h_\nu, I_n)) f_{\omega_{\pi_\nu}, \nu}$, we can assume $h_\nu = I_m$ in eq. (3.7). Now, we prove

**Lemma 3.7.** $I \left( f_{\omega_{\pi_\nu}, \nu}^0, v_{\pi_\nu} \right) (I_m)$ is a finite sum of integrals of the form
\[
\alpha (\tau, \phi, \phi') \int_{Z_n(k_\nu) \setminus GL_n(k_\nu)} P_t (q_{\nu}^{-s}, v_{\nu}^0) f_{\omega_{\pi_\nu}, \nu} (\hat{t} (I_m, g_\nu)) (\pi_\nu (g_\nu v_{\pi_\nu}, \phi)) \, dg_\nu \cdot \phi', \tag{3.30}
\]
where $f_{\omega_{\pi_\nu}, \nu}$ is a standard section, $\tau$ is an irreducible representation of the compact subgroup $t (\text{diag}(I_{m-n}, K_{n,0}), I_n) \leq K_{mn, \nu}$, $\phi$ and $\phi'$ are vectors in (an orthonormal basis of) the finite dimensional isotypic subspace $V_{\pi} (\tau)$, and $\alpha (\tau, \phi, \phi') \in \mathbb{C}$.

We note that Lemma 3.7 also proves that $I \left( f_{\omega_{\pi_\nu}, \nu}^0, v_{\pi_\nu} \right)$ is a section in
\[
\text{Ind}_{P_{m-n,n}(k_\nu)}^{GL_m(k_\nu)} (1 \otimes \pi_\nu) \delta_{P_{m-n,n}}^s.
\]

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Proof. We normalize the measure such that

\[
\mu(K_{n,\nu}) = \int_{K_{n,\nu}} dx = 1.
\] (3.31)

We begin by writing \(f_{\omega_{\nu},s}\) in terms of standard sections as in eq. (1.13):

\[
f_{\omega_{\nu},s} = \sum_{i=1}^{N} P_i(q_{\nu}^{-s}, q_{\nu}^s) f_{\phi(i)_{\omega_{\nu},s}},
\]

where for all \(1 \leq i \leq N\), \(P_i \in \mathbb{C}[q_{\nu}^{-s}, q_{\nu}^s]\) and \(f_{\phi(i)_{\omega_{\nu},s}}\) is a standard section in \(\rho_{\omega_{\nu},s}\).

Therefore, it suffices to prove the proposition for \(I \left( P(q_{\nu}^{-s}, q_{\nu}^s) f_{\phi_{\omega_{\nu},s},\upsilon_{\nu}} \right) (I_m)\), where \(P(q_{\nu}^{-s}, q_{\nu}^s)\) is a holomorphic function (polynomial for non-Archimedean \(\nu\)) and \(f_{\phi_{\omega_{\nu},s}}\) is a standard section. Let \(x \in K_{n,\nu}\) and \(g_{\nu} \in \text{GL}_n(k_{\nu})\). We now show that

\[
f_{\phi_{\omega_{\nu},s}} (\tilde{\epsilon} t (I_m, x g_{\nu})) = \sum_{j=1}^{N} \langle \xi_{\tau_j}, \tilde{\gamma}_{\tau_j} (x^*) \tilde{\xi}_{\tilde{\tau}_j} \rangle \ f_{\phi(i)_{\omega_{\nu},s}} (\tilde{\epsilon} t (I_m, g_{\nu})),
\] (3.32)

where for all \(1 \leq j \leq N\), \(\tau_j\) is an irreducible representation of the compact subgroup \(t(\text{diag}(I_{m-n}, K_{n,\nu}), I_n) \leq K_{mn,\nu}\), \(\xi_{\tau_j} \in V_{\tau_j}\) and \(\tilde{\xi}_{\tilde{\tau}_j} \in V_{\tilde{\tau}_j}\). Indeed, generally, by the \(K_{mn,\nu}\)-finiteness of \(f_{\phi_{\omega_{\nu},s}}\), there exist \(N \in \mathbb{N}\) and \(\{ f_{\phi(j)_{\omega_{\nu},s}} \}_{j=1}^{N} \subseteq \rho_{\omega_{\nu},s}\) such that for all \(A \in \text{GL}_{mn}(k_{\nu})\) and \(X \in K_{mn,\nu}\) we have

\[
f_{\phi_{\omega_{\nu},s}} (AX) = \sum_{j=1}^{N} c_j(X) f_{\phi(j)_{\omega_{\nu},s}} (A),
\] (3.33)

where

\[
c_j(X) = \left( \rho_{\omega_{\nu},s} (X) f_{\phi_{\omega_{\nu},s}^{(j)}}, f_{\phi_{\omega_{\nu},s}^{(j)}} \right),
\] (3.34)

and \((,\)\) is a \(K_{mn,\nu}\)-invariant inner product of the space of \(\rho_{\omega_{\nu},s}\). We note that for all \(1 \leq j \leq N\), \(c_j(X)\) is independent of \(s\), as \(f_{\phi_{\omega_{\nu},s}}\) is a standard section. Let \(x \in K_{n,\nu}\). Then,

\[
f_{\phi_{\omega_{\nu},s}} (\tilde{\epsilon} t (I_m, x g_{\nu})) = f_{\phi_{\omega_{\nu},s}} (\tilde{\epsilon} t (\text{diag}(I_{m-n}, x^*) \text{diag}(I_{m-n}, (x^*-1), x g_{\nu}))).
\]

On the other hand, \(t(\text{diag}(I_{m-n}, x^*), x) \in Q^x(k_{\nu})\), so its conjugation with \(\tilde{\epsilon}\) lies in \(P_{mn-1,1}(k_{\nu}) \cap K_{mn,\nu}\). Since \(f_{\phi_{\omega_{\nu},s}}\) is \(P_{mn-1,1}(k_{\nu}) \cap K_{mn,\nu}\)-left invariant,

\[
f_{\phi_{\omega_{\nu},s}} (\tilde{\epsilon} t (I_m, x g_{\nu})) = f_{\phi_{\omega_{\nu},s}} (\tilde{\epsilon} t (I_m, g_{\nu}) t (\text{diag}(I_{m-n}, (x^*-1), I_n))).
\]

By eq. (3.33),

\[
f_{\phi_{\omega_{\nu},s}} (t (I_m, x g_{\nu})) = \sum_{j=1}^{N} c_j^{(j)} (x^*-1) f_{\phi(i)_{\omega_{\nu},s}} (\tilde{\epsilon} t (I_m, g_{\nu})),
\] (3.35)

where

\[
c_j^{(j)} (x^*-1) = c_j (t (\text{diag}(I_{m-n}, (x^*-1), I_n))).
\]
We can assume that $c_j'$ are matrix coefficients of an irreducible representation, say $\tau_j$, of the compact subgroup $t(\text{diag}(I_{m-n}, K_{n,\nu}), I_n) \leq K_{mn,\nu}$. I.e. by Riesz representation theorem there exist $\xi_{\tau_j} \in V_{\tau_j}$ and $\tilde{\xi}_{\tau_j} \in V_{\tilde{\tau}_j}$ such that

$$c_j' ((x)^{-1}) = \langle \tau_j \left( \vec{w} x^T \vec{w} \right) \xi_{\tau_j}, \tilde{\xi}_{\tau_j} \rangle = \langle \xi_{\tau_j}, \tilde{\tau}_j (x^*) \tilde{\xi}_{\tau_j} \rangle. \quad (3.36)$$

Plugging eq. (3.36) to eq. (3.35) gives eq. (3.32).

Now, we write $1 \left( P(q_{\nu}^{-s}, q_{\mu}^{s}) f_{\omega_{\pi,\nu}, s}, v_{\pi,\nu} \right) (I_{m})$ as

$$\frac{1}{\mu(K_{n,\nu})} \int_{Z_n(k) \setminus \text{GL}_n(k)} \int_{K_{n,\nu}} P(q_{\nu}^{-s}, q_{\mu}^{s}) f_{\omega_{\pi,\nu}, s} (\varepsilon \in (I_{m}, x g_{\nu})) \pi_{\nu}(x g_{\nu}) v_{\pi,\nu} dx dg_{\nu}. \quad (3.37)$$

Applying eq. (3.32) implies that $I \left( P(q_{\nu}^{-s}, q_{\mu}^{s}) f_{\omega_{\pi,\nu}, s}, v_{\pi,\nu} \right) (I_{m})$ is a finite sum of integrals of the form

$$\int_{Z_n(k) \setminus \text{GL}_n(k)} \int_{K_{n,\nu}} P(q_{\nu}^{-s}, q_{\mu}^{s}) f_{\omega_{\pi,\nu}, s} (\varepsilon \in (I_{m}, x g_{\nu})) \pi_{\nu}(x g_{\nu}) v_{\pi,\nu} dx dg_{\nu}. \quad (3.38)$$

Let $\tau = \tau_\ell$ for some $1 \leq \ell \leq N$. Let $P_{\xi_\tau, \hat{\xi}_\tau} : V_\pi \rightarrow V_\tau$ be the following map

$$P_{\xi_\tau, \hat{\xi}_\tau} := \int_{K_{n,\nu}} \langle \xi_\tau, \hat{\tau} (x^*) \hat{\xi}_\tau \rangle \pi_{\nu}(x) dx, \quad (3.39)$$

We note that the inner integral over $K_{n,\nu}$ in eq. (3.38) equals $P_{\xi_\tau, \hat{\xi}_\tau} (\pi_{\nu}(g_{\nu}) v_{\pi,\nu})$. Let $B := \{ \phi_1, \ldots, \phi_r \}$ be an orthonormal basis of of the finite dimensional isotypic subspace $V_\pi(\tau)$. Then,

$$P_{\xi_\tau, \hat{\xi}_\tau} (\pi_{\nu}(g_{\nu}) v_{\pi,\nu}) = \sum_{1 \leq i, j \leq r} (\pi_{\nu}(g_{\nu}) v_{\pi,\nu}, \phi_j) \int_{K_{n,\nu}} \langle \xi_\tau, \hat{\tau} (x^*) \hat{\xi}_\tau \rangle (\pi_{\nu}(x^{-1}) \phi_i, \phi_j) dx \cdot \phi_i. \quad (3.40)$$

Indeed, the map $P_{\xi_\tau, \hat{\xi}_\tau}$ is a projection to $V_\pi(\tau)$. Then,

$$P_{\xi_\tau, \hat{\xi}_\tau} (\pi_{\nu}(g_{\nu}) v_{\pi,\nu}) = \sum_{i=1}^{r} e_i(g_{\nu}) \phi_i, \quad (3.41)$$

where

$$e_i(g_{\nu}) = \left( P_{\xi_\tau, \hat{\xi}_\tau} (\pi_{\nu}(g_{\nu}) v_{\pi,\nu}), \phi_i \right) = \int_{K_{n,\nu}} \langle \xi_\tau, \hat{\tau} (x^*) \hat{\xi}_\tau \rangle (\pi_{\nu}(x g_{\nu}) v_{\pi,\nu}, \phi_i) dx. \quad (3.42)$$

By the fact that $\pi_{\nu}$ is unitary we have $(\pi_{\nu}(x g_{\nu}) v_{\pi,\nu}, \phi_i) = (\pi_{\nu}(g_{\nu}) v_{\pi,\nu}, \pi_{\nu}(x^{-1}) \phi_i)$. The vector $\phi_i$ is in the finite dimensional space $V_\pi(\tau)$. Hence,

$$\pi_{\nu}(x^{-1}) \phi_i = \sum_{j=1}^{r} (\pi_{\nu}(x^{-1}) \phi_i, \phi_j) \phi_j. \quad (3.43)$$

We now plug eq. (3.43) to eq. (3.42). Then, we apply the result to eq. (3.41) and get eq. (3.40).
The result now followed immediately by denoting
\[
\alpha(\tau, \phi, \phi') := \int_{K_{n,T}} \langle \xi_{x}, \pi_{\nu}(x)\xi_{y} \rangle \left(\pi_{\nu}(x^{-1})\phi, \phi \right) dx,
\]
which absolutely converges as it is an integral of continuous function on a compact subgroup.

We state two corollaries obtained from this proof,

**Corollary 3.8.** Let \( g_\nu \in \text{GL}_n(k_\nu) \), \( x \in K_\nu \), and \( f_{\omega_{\nu,s}} \) a standard section. Then, \( f_{\omega_{\nu,s}}(\bar{\epsilon}(I_m, g_\nu x)) \) equals to a finite sum of elements of the form \( c(x)f_{\omega_{\nu,s}}(\bar{\epsilon}(I_m, g_\nu)) \), where \( c(x) \) is independent of \( s \).

*Proof.* This is eq. (3.33).

**Corollary 3.9.** Let \( g_\nu = A \tilde{t}B \), be the Cartan decomposition (modulo the center) of \( g_\nu \) as in eq. (3.8), and \( f_{\omega_{\nu,s}} \) a standard section. Then, \( f_{\omega_{\nu,s}}(\bar{\epsilon}(I_m, g_\nu)) \) equals to a finite sum of elements of the form \( c(A,B)f_{\omega_{\nu,s}}(\bar{\epsilon}(I_m, t)) \), where \( c(A,B) \) is independent of \( s \).

*Proof.* This follows immediately from eq. (3.32) and corollary 3.8.

In the next subsections we split to the cases of non-Archimedean and Archimedean places.

### 3.2.2 Non-Archimedean places

Let \( \nu < \infty \). We now denote \( \bar{t}^\Delta := t(I_m, t) \). Recall that \( \bar{\epsilon} = \bar{w}\bar{u} \) where we write \( \bar{w} := \bar{w}(m-n+1)_n \) and \( \bar{u} := u(m-n+1)_n(b_{n-1}) \). We also denote \( \bar{u}(t) := u(m-n+1)_n(b_{n-1}(t)) \), where
\[
b_{n-1}(t) = (t_{n-1} \cdot e_{n-1}^T, \ldots, t_1 \cdot e_1^T) .
\]

**Proposition 3.10.** Let \( \bar{t} \in Z_n(k_\nu) \setminus T^- \). Then, for a standard section \( f_{\omega_{\nu,s}} \) we have
\[
f_{\omega_{\nu,s}}(\bar{\epsilon}\bar{t}^\Delta) = \sum_{j=1}^{N} c_j(\bar{u}\bar{u}(\bar{t})) |\text{det }\bar{t}|^{m_{\nu}+\frac{m}{2}},
\]
where for \( 1 \leq j \leq N \), \( c_j \) is the inner product of the space of \( \rho_{\omega_{\nu,s}} \) given in eq. (3.34).

*Proof.* We have
\[
\bar{\epsilon}\bar{t}^\Delta = (\bar{w}\bar{t}^\Delta\bar{w}^{-1}) \bar{w} (\bar{t}^\Delta)^{-1} \bar{u}\bar{t}^\Delta = (\bar{w}\bar{t}^\Delta\bar{w}^{-1}) \bar{w}\bar{u}(\bar{t}).
\]

All the elements above the diagonal of the unipotent matrix \( u(m-n+1)_n(b_{n-1}(t)) \) are bounded in absolute value by 1. Therefore, \( \bar{w}\bar{u}(\bar{t}) \in K_{mn,T} \). Corollary 3.8 implies that \( f_{\omega_{\nu,s}}(\bar{\epsilon}\bar{t}^\Delta) \) equals to a finite sum of elements of the form \( c_j(\bar{w}\bar{u}(\bar{t}))f_{\omega_{\nu,s}}(\bar{w}\bar{t}^\Delta\bar{w}^{-1}) \).

Now, by the same arguments as in the proof of Proposition 3.6 starting with eq. (3.11), we find that \( f_{\omega_{\nu,s}}(\bar{w}\bar{t}^\Delta\bar{w}^{-1}) = \prod_{i=1}^{n-1} |t_i|^{m_{\nu}+\frac{m}{2}} \) and the proposition follows. □
Corollary 3.11. Assume that \( R(s) \gg 0 \). In the notation of Lemma 3.7 and proposition 3.10, \( I \left( f_{\omega_{\nu}, s}, v_{\tau} \right) (I_m) \) is a finite sum of integrals of the form

\[
\alpha (\tau, \phi, \phi') \beta \int_{Z_{n}(k_v) \setminus T^{-}} P(q_{\nu}^{-s}, q_{\nu}') \mu(t) c(X(\bar{\tau}, \bar{\mu})) \| \det \mathbb{L} \|^{m s + \frac{m}{2}} (\pi_{\nu}(t)v_{\tau}, \phi) dt \cdot \phi',
\]

where \( P(q_{\nu}^{-s}, q_{\nu}') \) is a holomorphic function (polynomial for non-Archimedean \( \nu \)), \( \mu(t) = \mu(K_{n, \nu}^{\mathbb{L}}K_{n, \nu}) \), and \( c \) is the inner product of the space of \( \rho_{\omega_{\nu}, s} \) given in eq. (3.34).

Proof. By Lemma 3.7, \( I \left( f_{\omega_{\nu}, s}, v_{\tau} \right) (I_m) \) is a finite sum of integrals of the form

\[
\alpha (\tau, \phi, \phi') \int_{Z_{n}(k_v) \setminus GL_{n}(k_v)} P(q_{\nu}^{-s}, q_{\nu}') f_{\omega_{\nu}, s} (\bar{\tau} (I_m, g_{\nu})) (\pi_{\nu}(g_{\nu})v_{\tau}, \phi) dg_{\nu} \cdot \phi'.
\]

By Corollary 3.9, \( f_{\omega_{\nu}, s} (\bar{\tau} (I_m, g_{\nu})) \) equals to a finite sum of \( c(A, B) f_{\omega_{\nu}, s} (\bar{\tau} (I_m, \mathbb{L})) \), where \( c(A, B) \) is independent of \( s \). Since \( v_{\tau} \) and \( \phi \) are \( K_{n, \nu}^{\mathbb{L}} \)-finite, we may replace \( (\pi_{\nu}(g_{\nu})v_{\tau}, \phi) \) by \( (\pi_{\nu}(I_{\nu})v_{\tau}, \phi) \). Hence, by Claim 3.1 we get that \( I \left( f_{\omega_{\nu}, s}, v_{\tau} \right) (I_m) \) is a finite sum of integrals of the form

\[
\alpha (\tau, \phi, \phi') \int_{K_{n, \nu}^{\mathbb{L}}} c(A, B) dAdB \int_{Z_{n}(k_v) \setminus T^{-}} P(q_{\nu}^{-s}, q_{\nu}') \mu(t) f_{\omega_{\nu}, s} (\bar{\tau} (I_m, \mathbb{L})) (\pi_{\nu}(I_{\nu})v_{\tau}, \phi) dt \cdot \phi',
\]

where \( \mu(t) = \mu(K_{n, \nu}^{\mathbb{L}}K_{n, \nu}) \). The integral \( \beta := \int_{K_{n, \nu}^{\mathbb{L}}} c(A, B) dAdB \) absolutely converges.

Finally, Proposition 3.10 implies that eq. (3.46) equals to a finite sum of terms of the form

\[
\alpha (\tau, \phi, \phi') \beta \int_{Z_{n}(k_v) \setminus T^{-}} P(q_{\nu}^{-s}, q_{\nu}') \mu(t) c(X(\bar{\tau}, \bar{\mu})) \| \det \mathbb{L} \|^{m s + \frac{m}{2}} (\pi_{\nu}(t)v_{\tau}, \phi) dt \cdot \phi',
\]

where \( c \) is the inner product of the space of \( \rho_{\omega_{\nu}, s} \) given in eq. (3.34). \( \square \)

Next, we make use of the asymptotic behavior of matrix coefficients due to [Cas95] (quoted from [Haz22] for convenience). For each \( \Theta \subseteq \Delta \) and \( 0 < \varepsilon \leq 1 \), we define

\[
T_{\Theta} (\varepsilon) = \left\{ a \in T \mid |a| (a) \leq \varepsilon \forall a \in \Delta \setminus \Theta, |a| (a) < \varepsilon \forall a \in \Theta \right\}.
\]

This is a subset of \( T^{-} \). For each \( \Theta \subseteq \Delta \) we denote by \( P_{\Theta} = M_{\Theta} N_{\Theta} \) the standard parabolic subgroup corresponding to \( \Theta \).

**Theorem** (Theorem 1 in [Haz22]). Let \( v \in V \) and \( \bar{v} \in \bar{V} \). There exist \( \varepsilon > 0 \) and finite sets of vectors, that depend on \( \{ \pi_{\nu}, v, \bar{v} \} \), \( p' = (p'_{1}, \ldots, p'_{r}) \in \mathbb{R}^{r'} \), \( p = (p_{1}, \ldots, p_{r}) \in \mathbb{Z}_{\geq 0}^{r} \), and \( \chi = (\chi_{1}, \ldots, \chi_{r}) \) where for all \( 1 \leq i \leq r \), \( \chi_{i} : k^{\times} \to \mathbb{C}^{\times} \) are unitary characters, such that for all \( a \in T^{-} \), on has

\[
\langle \pi_{\nu}(a)v, \bar{v} \rangle = \sum_{\Theta \subseteq \Delta} \chi_{\Theta}(a) \alpha_{\Theta}^{p', p} \prod_{i=1}^{r_{\Theta}} \chi_{i}(a_{i}) |a_{i}|^{p'_{i} \log p_{i}} |a_{i}|, \quad (3.47)
\]

where \( \chi_{\Theta}(a) \) is the indicator function of \( T_{\Theta}^{-}(\varepsilon) \), \( r_{\Theta} \) is such that \( T_{M_{\Theta}} \cong (k^{\times})^{r_{\Theta}} \) by the map \( a \mapsto (a_{1}, \ldots, a_{r_{\Theta}}) \), and \( \alpha_{\Theta}^{p', p} \in \mathbb{C} \) are such that \( \alpha_{\Theta}^{p', p} = 0 \) for all but finitely many \( p', p, \chi \).
Let \( g = (g_{i,j})_{1 \leq i,j \leq n} \in \text{GL}_n(k_v) \). Define \( \|g\| = \max_{1 \leq i,j \leq n} \{ |g_{i,j}|, |g_{i,j}^{-1}| \} \). Then, eq. (3.47) gives in particular,

**Corollary 3.12.** There exist constants \( \lambda, m_\alpha \in \mathbb{R}_+ \) such that

\[
|\langle \pi(a)\nu, \tilde{\nu} \rangle| \leq \lambda\|a\|^{m_\alpha}.
\]

**Proposition 3.13.** The integral \( I\left(f_{\omega_\nu, s}, \nu_{\pi_\nu}\right)(I_m) \) absolutely converges in \( \Re(s) >> 0 \).

**Proof.** For any \( 0 < \varepsilon \leq 1 \), \( T^- \) is the disjoint union of \( T^- \Theta(\varepsilon) \) as \( \Theta \) ranges over all subsets of \( \Delta \). Thus, together with Corollary 3.11, \( I\left(f_{\omega_\nu, s}, \nu_{\pi_\nu}\right)(I_m) \) is a finite sum of integrals of the form

\[
\alpha (\tau, \phi, \phi') \beta \int_{Z_n(k_v) \setminus T^- \Theta(\varepsilon)} P(q_{-s}^{-s}, q_{-s}^\nu) \mu(\xi) c(X(\xi, \xi)) |\det \xi|^{ms+\frac{m}{2}} |\pi_\nu(\xi) \nu_{\pi_\nu}, \phi| dt \cdot \phi'.
\]

where, \( P(q_{-s}^{-s}, q_{-s}^\nu) \) is polynomial, \( c(X(\xi, \xi)) \) does not depend on \( s \), \( \mu(\xi) = \frac{\mu_M}{\mu_G} \delta^{-1}_A(a) \) (by Claim 3.2). We have \( |\xi| = |t_1|^{-1} \). Therefore, each of the integrals above are bounded in absolute value by

\[
\int_{\sum |t_{n-1}|^{n-3+ms+m/2} d^x t_1 \cdots d^x t_{n-1}} |t_1|^{1-n-m_a+ms+m/2} |t_2|^{3-n+ms+m/2} \cdots |t_{n-1}|^{n-3+ms+m/2} d^x t_1 \cdots d^x t_{n-1}.
\]

By separate variables we find that the last integral equals

\[
\int_{|t_1| \leq 1} |t_1|^{1-n-m_a+ms+m/2} d^x t_1 \prod_{j=2}^{n-1} \int_{|t_j| \leq 1} |t_j|^{j+1-n+ms+m/2} d^x t_j.
\]

Hence, it is absolutely convergent iff \( \Re(1-n-m_a+ms+m/2) \geq 0 \). \( \square \)

Moreover, the integral \( I\left(f_{\omega_\nu, s}, \nu_{\pi_\nu}\right)(I_m) \) has meromorphic continuation. We show it by first applying eq. (3.47) to Corollary 3.11.

**Corollary 3.14.** Assume that \( \Re(s) >> 0 \). In the notation of Lemma 3.7 and proposition 3.10, \( I\left(f_{\omega_\nu, s}, \nu_{\pi_\nu}\right)(I_m) \) is a finite sum of integrals of the form

\[
\alpha (\tau, \phi, \phi') \beta \int_{Z_n(k_v) \setminus T^- \Theta(\varepsilon)} P(q_{-s}^{-s}, q_{-s}^\nu) \mu(\xi) c(X(\xi, \xi)) |\det \xi|^{ms+\frac{m}{2}} \\
\cdot \alpha_{p^{\nu}\chi} \prod_{i=1}^{\text{re}} \chi_i(t_i) |t_i|^{p_i} \log^{p_i} |t_i| dt \cdot \phi'.
\]

where \( P(q_{-s}^{-s}, q_{-s}^\nu) \) is a holomorphic function (polynomial for non-Archimedean \( \nu \)), \( \mu(\xi) = \mu(K_{n, p} \xi K_{n, \nu}) \), and \( c \) is the inner product of the space of \( \rho_{\omega_\nu, s} \) given in eq. (3.34).

**Proposition 3.15.** The integral \( I\left(f_{\omega_\nu, s}, \nu_{\pi_\nu}\right)(I_m) \) has meromorphic continuation for \( \Re(s) >> 0 \).
Proof. By Corollary 3.11, \( I(f_{\omega_{\nu}, s}, \nu_{\nu}) (I_m) \) is a finite sum of integrals of the form

\[
\alpha(\tau, \phi, \phi') \beta \int_{Z_n(k_{\nu}) \setminus T_\nu} P(q_{\nu}^{-s}, q_{\nu}^s) \mu(t) c(X(\tilde{\varepsilon}, \tilde{L})) |\text{det} \tilde{L}|^{m_s + m_m} \cdot \alpha_{p, p', \chi} \prod_{i=1}^{r_{\nu}} \chi_i(t_i) |t_i|^{p_i} \log |t_i| |\text{det} t|^{m_s + m_m} \cdot |t_i| dt \cdot \phi',
\]

where, \( P(q_{\nu}^{-s}, q_{\nu}^s) \) is polynomial, \( c(X(\tilde{\varepsilon}, \tilde{L})) \) does not depend on \( s \), and by Claim 3.2,

\[
\mu(t) = \sum_{n=1}^{\infty} (n-2i+1)r_i \frac{\varphi_n(q^{-1})}{(1-q^{-1})^n} \prod_{j=1}^{\ell} \frac{(1-q^{-1})^{n_j}}{\varphi_{n_j}(q^{-1})}.
\]

We have \( ||t|| = |t_1|^{-1} \). Therefore, each of the integrals above can be brought to the form

\[
\int_{|t_1| \leq |t_2| \leq \ldots \leq |t_{n-1}| \leq 1} \prod_{i=1}^{r_{\nu}} \chi_i(t_i) |t_i|^{p_i} \log |t_i| |\text{det} t|^{m_s + m_m} dt_1 \ldots dt_{n-1}.
\]

By separating variables we find that the last integral equals

\[
\prod_{j=1}^{n-1} \int_{|t_j| \leq 1} \chi_j(t_j) |t_j|^{\alpha_j + m_s} \log |t_j| |\text{det} t|^{m_s + m_m} dt_j. \tag{3.49}
\]

We can assume \( \chi_i(t_i) = 0 \) for all \( i \), otherwise the integral will be zero. Therefore, eq. (3.49) equals

\[
\prod_{j=1}^{n-1} \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} i q_{\nu}^{-i(\alpha_j + m_s)}.
\]

The last sum is a derivative of a rational function, and the meromorphic continuation follows.

### 3.2.3 Archimedean places

In this section we briefly provide Archimedean analogues of the results in the Section 3.2.2. Let \( \nu = \infty \).

**Proposition 3.16.** Let \( \tilde{L} \in Z_n(k_{\nu}) \setminus T^- \). Then, for a standard section \( f_{\omega_{\nu}, s} \) we have

\[
f_{\omega_{\nu}, s}(\tilde{\varepsilon}, \tilde{L}) = c(\tilde{\omega}, \tilde{L}) |\text{det} \tilde{L}|^{m_s + m_m} \omega_{\pi_{\nu}} \left( 1 + \sum_{i=1}^{n-1} t_i^2 \right) \left( 1 + \sum_{i=1}^{n-1} t_i^2 \right)^{-m_n(s+\frac{1}{2})},
\]

where \( c(\tilde{\omega}, \tilde{L}) \) is the inner product of the space of \( \rho_{\omega_{\nu}, s} \) given in eq. (3.34) evaluated on a matrix in \( K_{mn} \) that depends on \( \tilde{\omega} \) and \( \tilde{L} \).
Proof. In any case $\nu = \mathbb{R}$ or $\nu = \mathbb{C}$, for all $1 \leq j \leq n - 1$, we can write $t_j = r_j e^{i\theta_j}$, where $0 < r_1 \leq r_2 \leq \ldots \leq r_{n-1} \leq 1$ and $-\pi \leq \theta_j < \pi$. We write

$$t = \text{diag}(r_1, \ldots, r_{n-1}) \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}}),$$

By Corollary 3.8 we have

$$f_{\omega_{\nu,s}}(\tilde{t}^\Delta) = c(\theta) f_{\omega_{\nu,s}}(\tilde{t}^\Delta),$$

where $\theta = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}})$ and $c(\theta)$ is a holomorphic function in $q_{\nu}^{+s}$. Therefore, we assume that $0 < t_1 \leq t_2 \leq \ldots \leq t_{n-1} \leq 1$. Following the proof of Proposition 3.10 we get that all the arguments hold except that this time $\tilde{w} \tilde{t}^\Delta \in K_{mn,\nu}$. Hence,

$$f_{\omega_{\nu,s}}(\tilde{t}^\Delta) = |\text{det} \tilde{t}^{|ms|+\frac{m}{2}}| f_{\omega_{\nu,s}}(\tilde{w} \tilde{t}^\Delta \tilde{w}^{-1} \tilde{w}).$$

(3.50)

We apply Corollary 3.8 again

$$f_{\omega_{\nu,s}}(\tilde{w} \tilde{t}^\Delta) = c(\tilde{w}) f_{\omega_{\nu,s}}(\tilde{w} \tilde{t}^\Delta \tilde{w}^{-1}),$$

(3.51)

where $c(\tilde{w})$ is a holomorphic function in $q_{\nu}^{+s}$. Now

$$\tilde{w} \tilde{t}^\Delta \tilde{w}^{-1} = \begin{pmatrix} I_{mn-1} & 0 \\ x(t) & 1 \end{pmatrix},$$

where the last row equals

$$(x(t), 1) = (0,(m-n+1)n, t_{n-1}e_{n-2}, \ldots, t_3 e_2, t_2 e_1 + t_1 e_n, e_n).$$

(3.52)

We would like to find the Iwasawa decomposition $\tilde{w} \tilde{t}^\Delta \tilde{w}^{-1} = x_P x_K$, such that $x_P \in P_{mn-1,1}(k_{\nu})$ and $x_K \in K_{mn,\nu}$. This implies

$$f_{\omega_{\nu,s}}(\tilde{w} \tilde{t}^\Delta \tilde{w}^{-1}) = c(x_K)(1 \otimes \omega_{\nu}) (x_P) \delta_{P_{mn-1,1}}^{s+\frac{1}{2}} (x_P).$$

(3.53)

Most of the rows and columns of $\tilde{w} \tilde{t}^\Delta \tilde{w}^{-1}$ are already the standard orthonormal basis of $k_{\nu}^{mn}$. Therefore, it is sufficient to find the Iwasawa decomposition

$$\begin{pmatrix} I_{n-1} & 0 \\ y(t) & 1 \end{pmatrix} = y_P y_K,$$

(3.54)

where the last row equals

$$(y(t), 1) = (t_{n-1}, \ldots, t_1, 1).$$

(3.55)

We denote

$$y_P^{-1} = \begin{pmatrix} X_{n,n-1} & \ldots & X_{n,0} \\ \vdots & \ddots & \vdots \\ X_{1,n-1} & X_{1,0} \end{pmatrix}, \quad \begin{pmatrix} I_{n-1} & 0 \\ y(t) & 1 \end{pmatrix} = \begin{pmatrix} -u_1^- \\ \vdots \\ -u_n^- \end{pmatrix}, \quad y_K = \begin{pmatrix} -v_n^- \\ \vdots \\ -v_1^- \end{pmatrix}.$$

In this notation eq. (3.54) can be written as the following system of equation. For all $1 \leq i \leq n$,

$$v_i = \sum_{r=0}^{n-1} X_{i,r} u_{n-r}.$$
The Gram-Schmidt process gives for all $1 \leq i \leq n$

$$v_i' = \sum_{r=0}^{n-1} X_{i,r} u_{n-r}. $$

This provides $y_K$ by $v_i = \frac{v_i'}{||v_i'||}$, and $y_{P^{-1}}$ by $X_{i,r} = \frac{X_{i,r}}{||v_i'||}$. We set $v_1' = u_n$. For all $1 \leq i \leq n$ we denote

$$\varphi_i = 1 + \sum_{j=i}^{n-1} t_j^2. $$

We note that $\varphi_1 = ||u_n||^2 = ||v_1'||^2$, $\varphi_n = 1$, and

$$\varphi_1 + t_{i-1}^2 = \varphi_{i-1}. \quad (3.56)$$

We show by induction that for $2 \leq i \leq n$ we get

$$v_i' = u_{n-i+1} + \frac{t_{i-1}}{\varphi_{i-1}} \left( \sum_{r=1}^{i-2} t_r u_{n-r} - u_n \right), \quad (3.57)$$

$$||v_i'||^2 = \frac{\varphi_i}{\varphi_{i-1}}. \quad (3.58)$$

Indeed, by applying the Gram-Schmidt process for $i = 2$ we have

$$v_2' = u_{n-1} - \frac{\langle u_{n-1}, v_1' \rangle}{||v_1'||^2} v_1' = u_{n-1} - \frac{t_1}{\varphi_1} u_n. $$

This also implies,

$$v_2' = e_{n-1} - \frac{t_1}{\varphi_1} \left( \sum_{r=1}^{n-1} t_r e_{n-r} + e_n \right) = \left( 1 - \frac{t_2^2}{\varphi_1} \right) e_{n-1} - \frac{t_1}{\varphi_1} \left( \sum_{r=2}^{n-1} t_r e_{n-r} + e_n \right).$$

So,

$$||v_2'||^2 = \left( 1 - \frac{t_2^2}{\varphi_1} \right)^2 + \frac{t_2^2}{\varphi_1} \left( \sum_{r=2}^{n-1} t_r^2 + 1 \right) = \frac{\varphi_2}{\varphi_1} + \frac{t_2^2}{\varphi_1} \varphi_2 = \frac{\varphi_2 + t_2^2}{\varphi_1} = \frac{\varphi_2}{\varphi_1}. \quad (3.59)$$

Let $2 < i \leq n$ and assume the induction hypothesis eqs. (3.57) and (3.58) is true for all $2 \leq j < i$. Gram-Schmidt process gives

$$v_i' = u_{n-i+1} - \frac{\langle u_{n-i+1}, v_1' \rangle}{||v_1'||^2} v_1' - \sum_{\ell=2}^{i-1} \frac{\langle u_{n-i+1}, v_\ell' \rangle}{||v_\ell'||^2} v_\ell'. \quad (3.59)$$

Now,

$$\frac{\langle u_{n-i+1}, v_1' \rangle}{||v_1'||^2} v_1' = \frac{t_{i-1}}{\varphi_1} u_n, \quad (3.60)$$

and by the induction hypothesis we have for all $1 < \ell < i$

$$\langle u_{n-i+1}, v_\ell' \rangle = \langle u_{n-i+1}, u_{n-\ell+1} \rangle + \frac{t_{\ell-1}}{\varphi_{\ell-1}} \left( \sum_{r=1}^{\ell-2} t_r \langle u_{n-i+1}, u_{n-r} \rangle - \langle u_{n-i+1}, u_{n} \rangle \right).$$
Thus, the sum in the parenthesis in eq. (3.59)

\[ v_i' = u_{n-i+1} - \frac{t_i-1}{\varphi_1} u_n - \sum_{\ell=2}^{i-1} \frac{t_{\ell-1}t_i-1}{\varphi_\ell} u_{n-\ell+1} + \sum_{\ell=2}^{i-1} \frac{t_{\ell-1}^2t_i-1}{\varphi_\ell\varphi_{\ell-1}} \left( \sum_{r=1}^{\ell-2} t_r u_{n-r} - u_n \right). \]

By rearranging and changing the order of summation, we have

\[ v_i' = u_{n-i+1} - \left( \frac{t_i-1}{\varphi_1} + \sum_{\ell=2}^{i-1} \frac{t_{\ell-1}^2t_i-1}{\varphi_\ell\varphi_{\ell-1}} \right) u_n - \sum_{r=1}^{i-2} \frac{t_r}{\varphi_{r+1}} u_{n-r} + \sum_{r=1}^{i-3} t_r u_{n-r} \sum_{\ell=r+2}^{i-1} \frac{t_{\ell-1}^2t_i-1}{\varphi_\ell\varphi_{\ell-1}}. \]

We use eq. (3.62) to find the coefficients $X'_{i,r}$ for all $0 \leq r \leq n-1$. It is immediate to see that

\[ \begin{cases} X'_{i,r} = 0, & i \leq r \leq n-1 \\ X'_{i,i-1} = 1, \\ X'_{i,i-2} = \frac{t_i-1}{\varphi_{i-1}}. \end{cases} \]

For $1 \leq r \leq i-3$,

\[ X'_{i,r} = t_r t_{i-1} \left( \frac{1}{\varphi_{i+1}} + \sum_{\ell=r+2}^{i-1} \frac{t_{\ell-1}^2}{\varphi_\ell\varphi_{\ell-1}} \right). \]

Generally, for $1 \leq a \leq n-2$,

\[ \frac{1}{\varphi_a} + \frac{t_a^2}{\varphi_a \varphi_{a+1}} = \frac{1}{\varphi_a} \left( 1 + \frac{t_a^2}{\varphi_{a+1}} \right) = \frac{1}{\varphi_a \varphi_{a+1}} = \frac{1}{\varphi_{a+1}}. \]

Thus, the sum in the parenthesis in eq. (3.63) equals $\frac{1}{\varphi_{i-1}}$ and $X'_{i,r} = t_r t_{i-1} \frac{1}{\varphi_{i-1}}$, for all $1 \leq r \leq i-3$. This applies to $r = 0$ as well, with

\[ X'_{i,0} = t_{i-1} \left( \frac{1}{\varphi_1} + \sum_{\ell=2}^{i-1} \frac{t_{\ell-1}^2}{\varphi_\ell\varphi_{\ell-1}} \right) = \frac{t_{i-1}}{\varphi_{i-1}}. \]

This implies that eq. (3.57) holds true for all $2 \leq i \leq n$. We now plug $u_j = e_j$ for all $1 \leq j \leq n-1$ and eq. (3.55) in eq. (3.57):

\[ v_i' = e_{n-i+1} + \frac{t_i-1}{\varphi_{i-1}} \left( \sum_{r=1}^{i-2} t_r e_{n-r} - \sum_{r=1}^{n-1} t_r e_{n-r} + e_n \right), \]

and rearrange

\[ v_i' = \left( 1 - \frac{t_i^2}{\varphi_{i-1}} \right) e_{n-i+1} - \frac{t_{i-1}}{\varphi_{i-1}} \sum_{r=i}^{n-1} t_r e_{n-r} + \frac{t_{i-1}}{\varphi_{i-1}} e_n. \]

Therefore,

\[ ||v_i'||^2 = \left( 1 - \frac{t_i^2}{\varphi_{i-1}} \right)^2 + \frac{t_{i-1}^2}{\varphi_{i-1}^2} \sum_{r=i}^{n-1} t_r^2 + \frac{t_{i-1}^2}{\varphi_{i-1}^2} \sum_{r=i}^{n-1} t_r \sum_{r=i}^{n-1} \frac{t_{r-1}^2}{\varphi_r^2} \sum_{r=i}^{n-1} \frac{t_{r-1}^2}{\varphi_{r-1}^2} = \frac{\varphi_{i-1}}{\varphi_i}, \]

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and
\[ v_i = e_{n-i+1} - t_{i-1} \frac{1}{\varphi_i} \sum_{r=1}^{n-1} t_re_{n-r} + t_{i-1} e_n. \]

Thereby, \( y_P^{-1} \) is an upper triangular matrix with diagonal
\[ \text{diag} \left( X_{n-1}, \ldots, X_{1,0} \right) = \text{diag} \left( \frac{1}{||v'_1||}, \ldots, \frac{1}{||v'_1||} \right) = \text{diag} \left( \sqrt[2n]{\varphi_{n-1}}, \ldots, 1 \right). \]

Thus, \( y_P \) is also an upper triangular matrix with diagonal
\[ \text{diag} \left( \frac{\sqrt[n]{\varphi_n}}{\varphi_{n-1}}, \ldots, \sqrt[2]{\varphi_1} \right). \]

Now, by eq. (3.52) we get that \( x_P \) is an upper triangular matrix with diagonal
\[ \text{diag} \left( 0_{(m-n+1)n}, \sqrt[n]{\varphi_n}, \ldots, \sqrt[n]{\varphi_1} e_n, \ldots, \right) = \text{diag} \left( \sqrt[n]{\varphi_n}, \ldots, \sqrt[n]{\varphi_1} e_n, \sqrt[n]{\varphi_1} e_n \right). \]

Now we can use eq. (3.66) to evaluate eq. (3.53).
\[ f_{\omega_{\pi, \nu}} \left( \tilde{w} t^\Delta \tilde{w}^{-1} \right) = c(x_K) \omega_{\pi, \nu} \left( \sqrt[n]{\varphi_1} \right) \left| \sqrt[n]{\varphi_1} \right|^{-mn(s+\frac{1}{2})}. \]

Plugging this result to eq. (3.51) and then to eq. (3.50) gives
\[ f_{\omega_{\pi, \nu}} \left( \tilde{e} t^\Delta \right) = \left| \det \hat{L} \right|^{\frac{mn}{t}} c(\tilde{w}) c(x_K) \omega_{\pi, \nu} \left( \sqrt[n]{\varphi_1} \right) \left| \sqrt[n]{\varphi_1} \right|^{-mn(s+\frac{1}{2})}. \]

Recall that \( x_K \) is the compact part in the Iwasawa decomposition of \( \tilde{w} t^\Delta \tilde{w}^{-1} \). So, by denoting \( c(\tilde{w}, \hat{L}) := c(\hat{L}) c(\tilde{w}) c(x_K) \) the proof is done.

**Corollary 3.17.** Assume that \( R(s) >> 0 \). In the notation of Lemma 3.7 and proposition 3.10, \( I \left( f_{\omega_{\pi, \nu}}, v_{\pi, \nu} \right) (I_m) \) is a finite sum of integrals of the form
\[ \alpha \left( \tau, \phi, \phi' \right) \int_{Z_n(k_o) \setminus T^-} P(q_{\omega}^{-}, q_{\omega}^{+}) \mu(t) c(\tilde{w}, \hat{L}) \left| \det \hat{L} \right|^{\frac{mn}{t}} \omega_{\pi, \nu} \left( 1 + \sum_{i=1}^{n-1} t_j^2 \right)^{-mn(s+\frac{1}{2})} \left( \pi_{\nu} \left( \hat{L} \nu_{\pi, \nu}, \phi \right) \right) \hat{L} \cdot \phi', \]

where \( P(q_{\omega}^{-}, q_{\omega}^{+}) \) is a holomorphic function, \( \mu(t) = \mu(K_n, L K_n, L) \), and \( c \) is the inner product of the space of \( \rho_{\omega_{\pi, \nu}} \) given in eq. (3.34).

**Proof.** The proof is the same as the proof of Corollary 3.11, except that this time we apply Proposition 3.16 instead of Proposition 3.10.

Next, we make use of the asymptotic behavior of matrix coefficients due to [Cas80]. Embed \( T^- \) in \( \mathbb{C}^\Delta \): \( L \mapsto (\alpha(L))_{\alpha \in \Delta} \). For each \( s \in \mathbb{C}^\Delta \) define functions which are single-valued on \( T^- \), multivalued on the complement of coordinate hyperplanes in \( \mathbb{C}^\Delta \):
\[ a^s \log^c a = \prod_{\alpha \in \Delta} \alpha(a)^{s_\alpha} \log^{c_\alpha} \alpha(a), \]

where \( c \in \mathbb{N}^\Delta \subseteq \mathbb{C}^\Delta \), the set of integer vectors in \( \mathbb{C}^\Delta \).
**Proposition** (Casselman, 1978). There exist finite sets $S \subseteq \mathbb{C}^N, \mathcal{M} \subseteq \mathbb{N}^N$ such that for every $v \in V, \tilde{v} \in \tilde{V}$, there exist functions $h_{s,c}$ ($s \in S, c \in \mathcal{M}$) holomorphic in $T^-$ with
\[
\langle \pi(a)v, \tilde{v} \rangle = \sum_{s \in S, c \in \mathcal{M}} h_{s,c} a^s \log^c a,
\] (3.68)
for $a \in T^-$.

Similarly to the non-Archimedean case we can deduce in particular,

**Corollary 3.18.** There exist $\lambda, m_a \in \mathbb{R}_+$ such that
\[
|\langle \pi(a)v, \tilde{v} \rangle| \leq \lambda |a|^{m_a}.
\]

**Proposition 3.19.** The integral $I(f_{\omega_v,s}, v_{\pi_v}) (I_m)$ absolutely converges in $\Re(s) >> 0$.

**Proof.** By Corollary 3.17, the integral $I(f_{\omega_v,s}, v_{\pi_v}) (I_m)$ is a finite sum of integrals of the form eq. (3.67). By Claim 3.3 we have $\mu(t) = \sum_{\alpha \in \Sigma_+} |\sinh \alpha(H)|^{\dim \mathfrak{g}_\alpha}$. In addition, $||\xi|| = |t_1|^{-1}$. Thus, by Corollary 3.18, each of the integrals above is bounded in absolute value by
\[
\int_{|t_1| \leq |t_2| \leq \ldots \leq |t_{n-1}| \leq 1} \prod_{i=1}^{n-1} e^{c_i t_i} |t_1|^{1-n-m_a+ms+m/2} |t_2|^{3-n+ms+m/2} \ldots |t_{n-1}|^{n-3+ms+m/2} dt_1 \ldots dt_{n-1}.
\]
By separating variables, the last integral can be brought to the form
\[
\int_{|t_1| \leq 1} e^{c_1 t_1} |t_1|^{1-n-m_a+ms+m/2} dt_1 \prod_{j=2}^{n-1} \int_{|t_j| \leq 1} e^{c_j t_j} |t_j|^{j+1-n+ms+m/2} dt_j.
\]
Hence, it is absolutely convergent iff $\Re(1 - n - m_a + ms + m/2) \geq 0$. $\square$

Moreover, the integral $I(f_{\omega_v,s}, v_{\pi_v}) (I_m)$ has meromorphic continuation. In order to see that we first apply eq. (3.68) to Corollary 3.17 and obtain:

**Corollary 3.20.** Assume that $\Re(s) >> 0$. In the notation of Lemma 3.7 and proposition 3.16, $I(f_{\omega_v,s}, v_{\pi_v}) (I_m)$ is a finite sum of integrals of the form
\[
\alpha(\tau, \phi, \phi') \int_{Z_n(k_v) \backslash T^-} P(q_v^{-s}, q_v^s) \mu(t) c(\tilde{\omega}, t) |\det t|^{ms+m} \omega_{\pi_v} \left(1 + \sum_{i=1}^{n-1} t_i^2\right)
\]
\[
\cdot \left|1 + \sum_{i=1}^{n-1} t_i^2\right|^{mn(s+\frac{3}{2})} \sum_{s \in S, c \in \mathcal{M}} h_{s,c} a^s \log^c t dt \cdot \phi',
\] (3.69)
where $P(q_v^{-s}, q_v^s)$ is a holomorphic function, $\mu(t) = \sum_{\alpha \in \Sigma_+} |\sinh \alpha(H)|^{\dim \mathfrak{g}_\alpha}$, and $c$ is the inner product of the space of $\rho_{\omega_v,s}$ given in eq. (3.34).

**Proposition 3.21.** The integral $I(f_{\omega_v,s}, v_{\pi_v}) (I_m)$ has meromorphic continuation for $\Re(s) >> 0$.  

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Proof. By Corollary 3.11, $I \left( f_{\omega_{\nu}, s}, v_{\pi_{\nu}} \right) (I_m)$ is a finite sum of integrals of the form

$$\alpha(\tau, \phi, \phi') \beta \int_{Z_n(k_n)/T_\theta(\epsilon)} P(q_{\nu}, q_{\nu}') \mu(\xi)c(X(\xi, \xi)) |\det(\xi)|^{ms+a} \sum_{s \in S, c \in M} h_s c \log^c t d \xi \cdot \phi'.$$

where, $P(q_{\nu}, q_{\nu}')$ is a holomorphic function, $c(X(\xi, \xi))$ does not depend on $s$, $\mu(\xi) = \sum_{\alpha \in \Sigma_+} |\sinh(\alpha(H))|^{\dim(H)}$ (by Claim 3.3). We have $||\xi|| = |\xi_1|^{-1}$. Therefore, each of the integrals above equals

$$\int_{|t_1| \leq |t_2| \leq \ldots \leq |t_{n-1}| \leq 1} \prod_{i=1}^{n-1} e^{c_i t_i} |t_1|^{1-n+ms+ms/2} |t_2|^{3-n+ms+ms/2} \ldots |t_{n-1}|^{n-3+ms+ms/2}$$

$$\cdot \prod_{i=1}^{n-1} \chi_i(t_i) |t_i|^{p_i \log^p t_i} |t_i|^{d^\chi t_1 d^\chi t_2 \ldots d^\chi t_{n-1}}.$$

By separating variables we find that the last integral equals

$$\prod_{j=1}^{n-1} \int_{|t_j| \leq 1} e^{c_j t_j} \chi_j(t_j) |t_j|^{1+n+ms} \log^p t_j |d^\chi t_j. \quad (3.70)$$

Each of the classic integrals in eq. (3.70) has meromorphic continuation and so does $I \left( f_{\omega_{\nu}, s}, v_{\pi_{\nu}} \right) (I_m)$.

Acknowledgment

I am grateful to my advisor, David Soudry, for his guidance, patience, support, and for many insightful discussions. I would also like to thank David Ginzburg for suggesting the integral construction in eq. (1.1), and for his kind help and advice. My sincere gratitude goes to Solomon Friedberg and Elad Zelingher for their kind encouragement and practical suggestions. This work was supported by the Israel science foundation.

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