On the emergence of flavor mixing through interaction with an external vector field

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Abstract. We consider the possibility that two-flavor neutrino mixing can be viewed as resulting from the interaction of two non-mixed flavor neutrinos with an external vector field. Two distinct scenarios of the origin of such a vector field are presented. First we argue that the vector field might be understood as an $su(2)$ gauge field. In the second scenario we put forward the idea that the vector field can be identified with a Nambu–Goldstone vector field which results from a spontaneous symmetry breakdown of the Lorentz symmetry. Phenomenological implications of these pictures are briefly discussed.

1. Introduction

Particle mixing and neutrino oscillations in particular [1] are currently rapidly developing areas in both theoretical and experimental physics. Much of what is known about neutrino oscillations comes from the many experiments [2] about solar, atmospheric or reactor neutrinos. This has, in turn, produced a vast literature trying to work out possible extensions of the original Standard Model which does not accommodate non-zero neutrino masses and mixings. In spite of this, the true origin of the mixing is still rather elusive, although it is generally believed that it is the result of physics occurring at much higher energies with respect to the electroweak scale. An interesting alternative proposal was put forward recently in Ref. [3], considering mixing as arising from the low energy behavior of the interacting fermion fields.

Here we discuss another scenario which originates in the context of the quantum-field theoretical treatment of particle mixing presented in Refs. [4, 5]. There a nontrivial vacuum structure has been discovered in connection with mixing. In this approach, flavor states for mixed particles are consistently defined as eigenstates of the flavor charges [6, 7]. This leads to the derivation of exact oscillation formulas [8] which exhibit corrections with respect to the usual ones. One of the difficulties of this approach resides in the fact that the above flavor states are not representations of the Poincaré group [9]. This in turn leads to problems in formulating a perturbative expansion [10].

Interestingly enough, these difficulties can be bypassed by considering mixing not as an inherent property of the fields (or their associated particles), but as the result of an interaction of non-oscillatory flavor fields with an additional background vector field [5, 11]. In this paper we put forward two possible scenarios in which such a background field can find a natural basis.
First we argue that two-flavor neutrino oscillations can be equivalently phrased in terms of a minimally coupled \( su(2) \)-valued vector field which interacts with two non-oscillatory flavor fields. This reveals a non Abelian gauge-like structure in the mixing of fields. We show that in this way mixed particles can be considered to be on-shell just like ordinary particles. This treatment also removes an arbitrariness present in the formalism of quantum field mixing [12].

The second possibility is to identify the background field with a Nambu–Goldstone vector field which results from a spontaneous symmetry breakdown of the Lorentz symmetry down to its rotation subgroup. This is, in a sense, a natural choice in the light of both the time-like and fixed-norm nature of the involved vector field. Models with such type of vector fields are known as \textit{aether} \( \sigma \)-models [13] and they constitute a typical framework in which one investigates violations of Lorentz invariance at low energies.

The structure of the paper is as follows. To set the stage we recall in the following section some fundamentals of neutrino mixing. Section 3 is devoted to reformulation of usual two-flavor mixing in terms of the \( su(2) \)-valued vector field which interacts via minimal coupling with on-shell like (i.e., non-oscillatory) flavor neutrinos. This leads to a picture in which flavor mixing is viewed as a non-abelian gauge theory. Peculiarities of the Lorentz and Poincaré symmetries that are related with this gauge theory are relegated to Section 4. Some of the ensuing phenomenological implications are discussed in Section 5. In Section 6 we promote the idea that the vector field is in fact the Nambu–Goldstone vector field. Since Nambu–Goldstone fields naturally contribute into \textit{dark matter}, the neutrinos serve as a probe of such a dark matter. Various remarks and generalizations are proposed in the concluding section. For the reader’s convenience the paper is supplemented with one appendix which clarify some finer technical details.

2. Neutrino mixing: essentials

Let us start with the Lagrange density describing two mixed neutrino fields. This can be written in the form

\[
\mathcal{L} = \bar{\nu}_e (i \not\partial - m_e) \nu_e + \bar{\nu}_\mu (i \not\partial - m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \\
= \bar{\nu}_f (i \not\partial - M) \nu_f .
\]

(1)

Here the flavor doublet is \( \nu_f = (\nu_e, \nu_\mu)^T \) and the mass matrix is \( M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix} \). The standard treatment of the problem is based on the observation that this Lagrangian, being quadratic, can be diagonalized by means of the \textit{mixing transformation} as

\[
\nu_e = \nu_1 \cos \theta + \nu_2 \sin \theta , \\
\nu_\mu = -\nu_1 \sin \theta + \nu_2 \cos \theta .
\]

(2)

The latter allows to rewrite (1) as the sum of two free Dirac Lagrangians:

\[
\mathcal{L} = \bar{\nu}_m (i \not\partial - M_d) \nu_m .
\]

(3)

Here \( \nu_m = (\nu_1, \nu_2)^T \), \( M_d = \text{diag}(m_1, m_2) \), \( \theta \) is the mixing angle and

\[
m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta , \\
m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta , \\
m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta .
\]

(4)
Solar neutrino experiments combined with KamLAND data allow to evaluated the mixing angle to be approximately $\theta = 33.9^\circ$. From the Lagrangian (1) one can derive the canonical energy-momentum tensor:

\[
T_{\rho\sigma} = \frac{\partial L}{\partial (\partial_{\rho} \nu_{f})} \partial_{\sigma} \nu_{f} - \eta_{\rho\sigma} L
\]

\[
= \bar{\nu}_{e} i\gamma_{\rho} \partial_{\nu} \nu_{e} - \eta_{\rho\sigma} \bar{\nu}_{e} (i \gamma^{\lambda} \partial_{\lambda} - m_{e}) \nu_{e} + \bar{\nu}_{\mu} i\gamma_{\rho} \partial_{\sigma} \nu_{\mu} - \eta_{\rho\sigma} \bar{\nu}_{\mu} (i \gamma^{\lambda} \partial_{\lambda} - m_{\mu}) \nu_{\mu} + \eta_{\rho\sigma} m_{e\mu}(\bar{\nu}_{e} \nu_{\mu} + \bar{\nu}_{\mu} \nu_{e})
\]

\[
= \nu_{1} i\gamma_{\rho} \partial_{\sigma} \nu_{1} - \eta_{\rho\sigma} \nu_{1} (i \gamma^{\lambda} \partial_{\lambda} - m_{1}) \nu_{1} + \nu_{2} i\gamma_{\rho} \partial_{\sigma} \nu_{2} - \eta_{\rho\sigma} \nu_{2} (i \gamma^{\lambda} \partial_{\lambda} - m_{2}) \nu_{2},
\]

where $\eta_{\rho\sigma} = \text{diag}(1, -1, -1, -1)$ is the usual Minkowski metric tensor. From Eq. (5) follows that the total Hamiltonian $H = \int d^{3}x T_{00}$ is just the sum of the two free field Hamiltonians, i.e., $H = H_{1} + H_{2}$. The same situation holds also for the total 3-momentum operator $P_{i} = \int d^{3}x T_{i0}$.

The conserved (Noether) charges associated with the global symmetry of the free fields $\nu_{1}$ and $\nu_{2}$ are:

\[
Q_{j} = \int d^{3}x \nu_{j}^{\dagger}(x) \nu_{j}(x), \quad j = 1, 2,
\]

with the total charge $Q = Q_{1} + Q_{2}$. The analysis of symmetries of the Lagrangian in the flavor basis Eq. (1) leads to the identification of the (non conserved) flavor charges [6]:

\[
Q_{\sigma}(x_{0}) = \int d^{3}x \nu_{\sigma}^{\dagger}(x) \nu_{\sigma}(x), \quad \sigma = e, \mu,
\]

with $Q_{e}(x_{0}) + Q_{\mu}(x_{0}) = Q$. Here it should be stressed that it is only the sum of the charges which is the genuine integral of the motion. The respective flavor charges in the sum are not.

3. Flavor mixing as a non-abelian gauge theory

In this section we show that the mixing interaction can be alternatively understood as the interaction of the flavor neutrino fields with a constant minimally coupled external vector field. The case of a dynamical vector field will be discussed in Section 6.

The most direct way to seeing this goes through the Euler–Lagrange equations corresponding to the Lagrangian (1). These can be compactly written as [11]

\[
iD_{0} \nu_{f} = (-i \alpha \cdot \nabla + \beta \tilde{M}_{d}) \nu_{f},
\]

where $\nu_{f} = (\nu_{e}, \nu_{\mu})^{T}$ is the flavor doublet and $\tilde{M}_{d} = \text{diag}(m_{e}, m_{\mu})$ is a diagonal mass matrix. In Eq. (8) we have defined the (non-abelian) covariant derivative:

\[
D_{0} \equiv \partial_{0} + i m_{e\mu} \beta \sigma_{1},
\]

where $m_{e\mu} = \frac{1}{2} \tan 2 \theta \delta m$ and $\delta m = m_{\mu} - m_{e}$ is the mass differences parameter.

So flavor mixing can be seen as an interaction of the flavor fields with an $su(2)$ valued constant vector field having the following structure:

\[
A_{\mu} = \frac{1}{2} A_{\mu}^{a} \sigma_{a} = n_{\mu} \delta m \frac{\sigma_{1}}{2}, \quad n^{\mu} = (1, 0, 0, 0)^{T}.
\]

The covariant derivative is then:

\[
D_{\mu} = \partial_{\mu} + i g \beta A_{\mu},
\]
where \( g = \tan 2 \theta \) acts as the coupling constant. Note that in the case of maximal mixing (\( \theta = \pi/4 \)), the coupling constant grows to infinity while \( \delta m \) goes to zero. We thus see some hints of an approximate non-abelian gauge symmetry hidden in neutrino mixing. For this reason we will often refer to the vector field just introduced as a gauge field. Since the vector field is a constant, with just one non-zero component in group space, its field strength vanishes identically:

\[
F_{\mu\nu}^a = \epsilon^{abc} A^b \mu A^c \nu = 0,
\]

with \( a, b, c = 1, 2, 3 \). The fact that, despite \( F_{\mu\nu} \) vanishes identically, the gauge field has physical effects, suggests an analogy with the Bohm–Aharonov effect [14].

The equations of motion for the mixed fields in a covariant form are:

\[
(i \gamma^\mu D_\mu - \bar{M}_d) \nu_f = 0,
\]

and the Lagrangian density (1) has the form of the one describing a doublet of Dirac fields in interaction with an external gauge field:

\[
\mathcal{L} = \bar{\nu}_f (i \gamma^\mu D_\mu - \bar{M}_d) \nu_f.
\]

The energy momentum tensor associated with the flavor neutrino fields in interaction with the external gauge field is given by [11]:

\[
\bar{T}_{\rho\sigma} = \bar{\nu}_f i \gamma^\rho D_\rho \nu_f - \eta_{\rho\sigma} \bar{\nu}_f (i \gamma^\lambda D_\lambda - \bar{M}_d) \nu_f.
\]

By comparing with the canonical energy momentum tensor given in Eq. (5) we see that the difference between the two is just the presence of the interaction terms in the “00” component, i.e., \( T_{00} - \bar{T}_{00} = m_{\nu_e} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \), while we have \( T_{0i} = \bar{T}_{0i} \), \( T_{ij} = \bar{T}_{ij} \).

Following the usual procedure, we now define a 4-momentum operator \( \bar{P}^\mu \) from \( \bar{T}^{\mu\nu} \). We obtain a conserved 3–momentum operator:

\[
\bar{P}^i = \int d^3 x \bar{T}^{0i} = i \int d^3 x \nu_f^\dagger \partial^i \nu_f \equiv \bar{P}_e^i(x_0) + \bar{P}_\mu^i(x_0), \quad i = 1, 2, 3,
\]

and a non conserved Hamiltonian operator:

\[
\bar{P}^{0}(x_0) \equiv \bar{H}(x_0) = \int d^3 x \bar{T}^{00} = \int d^3 x \bar{\nu}_f (i \gamma_0 D_0 - i \gamma^\mu D_\mu + \bar{M}_d) \nu_f \\
\equiv \bar{H}_e(x_0) + \bar{H}_\mu(x_0).
\]

We remark that both the Hamiltonian and the momentum operators split into two contributions, one involving only the electron neutrino field and the other one only the muon neutrino field. We have thus the Hamiltonian and the momentum operators for each flavor field. Notice that the tilde Hamiltonian is not the generator of time translations. This role competes to the complete Hamiltonian \( H = \int d^3 x T^{00} \), which includes the interaction term.

4. Flavor neutrino states and Lorentz invariance

We are now in the position to construct flavor neutrino states which are simultaneous eigenstates of the above 4–momentum operators and of the flavor charges. Of course, this is a highly nontrivial task. We will see, however, that such states can indeed be constructed, but it will involve a nontrivial redefinition of the flavor vacuum which will also erase any reference to the \( \nu_1 \) and \( \nu_2 \) fields.
As shown in Appendix, the flavor neutrino field operators can be expanded in the same basis as the free fields with masses \( m_1 \) and \( m_2 \). With the convention \((\sigma, j) = (\{e, 1\}, \{\mu, 2\})\) we can write

\[
u_\sigma(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_r \left[ u^r_{k,j}(x_0)\alpha^r_{k,\sigma}(x_0) + v^r_{-k,j}(x_0)\beta^r_{-k,\sigma}(x_0) \right] e^{i k \cdot x},
\]

(18)

where \( \alpha_{k,\sigma} \) and \( \beta_{-k,\sigma} \) are the flavor ladder operators [4]. In the same Appendix we show that flavor neutrino states, defined as \( |\nu^r_{k,\sigma}\rangle = \alpha^r_{k,\sigma} |0\rangle_{e,\mu} \), are eigenstates of the flavor charge operators \( Q_\sigma \), at a given time. They turn out also to be eigenstates of the momentum operators \( P^i = \int d^3x T^{0i} \). However, since the Hamiltonian operator \( H \) does not commute with the charges \( Q_\sigma \), the above flavor states do not have definite energies.

We will now show that this problem can be solved by noting that the expansion (18) actually relies on a special choice of the bases of spinors, namely those referring to the free field masses \( Q_i \). It seems natural to expand the flavor fields in the bases corresponding to the couple of masses \( m_1 \) and \( m_2 \).

The original flavor vacuum is of course the one associated with the couple \((\mu, \mu)\). Let \( \mu_\sigma, \sigma = e, \mu \) be such a couple of arbitrary parameters.

The Bogoliubov transformation to be performed is the following:

\[
\begin{pmatrix}
\tilde{\alpha}^r_{k,\sigma}(x_0) \\
\tilde{\beta}^r_{-k,\sigma}(x_0)
\end{pmatrix} = J^{-1}_\mu(x_0) \begin{pmatrix}
\alpha^r_{k,\sigma}(x_0) \\
\beta^r_{-k,\sigma}(x_0)
\end{pmatrix} J_\mu(x_0),
\]

(19)

whose generator is [12]:

\[
J_\mu(x_0) = \prod_{k,\sigma} \exp \left\{ i \sum_{(\sigma,j)} \xi_{\sigma,j} \left[ \alpha^r_{k,\sigma}(x_0)\beta^r_{-k,\sigma}(x_0) + \beta^r_{-k,\sigma}(x_0)\alpha^r_{k,\sigma}(x_0) \right] \right\},
\]

(20)

and \((\sigma,j) = (\{e, 1\}, \{\mu, 2\})\), \( \xi_{\sigma,j} = (\chi_\sigma - \chi_j)/2 \) and \( \chi_\sigma = \arctan(\mu_\sigma/|k|) \), \( \chi_j = \arctan(m_j/|k|) \).

We have thus a whole family of flavor vacua, denoted with a tilde and parameterized by the couples \((\mu_e, \mu_\mu)\):

\[
|\tilde{0}(x_0)\rangle_{e\mu} = J^{-1}_\mu(x_0)|0(x_0)\rangle_{e\mu}.
\]

(21)

The original flavor vacuum is of course the one associated with the couple \((m_1, m_2)\).

Notice that the flavor charges, as well as the exact oscillation formulae are invariant under the above Bogoliubov transformations [15], i.e., \( \tilde{Q}_\sigma = Q_\sigma \), with:

\[
\tilde{Q}_\sigma(x_0) = \sum \int d^3k \left( \tilde{\alpha}_{k,\sigma}^r(x_0)\tilde{\alpha}_{k,\sigma}^r(x_0) - \tilde{\beta}_{-k,\sigma}^r(x_0)\tilde{\beta}_{-k,\sigma}^r(x_0) \right).
\]

(22)

In the context of the above reformulation of mixing as the interaction with an external field, it seems natural to expand the flavor fields in the bases corresponding to the couple of masses \((m_e, m_\mu)\). We will see that with this choice the flavor states will be eigenstates of the Hamiltonian operator.

The new spinors are defined as the solutions of the equations:

\[
(-\alpha \cdot k + m_\sigma \beta) u^r_{k,\sigma} = \omega_{k,\sigma} u^r_{k,\sigma},
\]

\[
(-\alpha \cdot k + m_\sigma \beta) v^r_{-k,\sigma} = -\omega_{k,\sigma} v^r_{-k,\sigma},
\]

(23)

where \( \omega_{k,\sigma} \) is the \( c \)-representation of the \( \tilde{Q}_\sigma \) generator.
where $\omega_{k,\sigma} = \sqrt{k^2 + m^2_{\sigma}}$. These are the momentum-space version of the free Dirac equation with mass $m_{\sigma}$. The flavor field operators are then expanded as follows:

$$\nu_\sigma(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \sum_r \left[ u^r_{k,\sigma}(x_0) \tilde{\alpha}^r_{k,\sigma}(x) + v^r_{-k,\sigma}(x_0) \tilde{\beta}^r_{k,\sigma}(x) \right] e^{i k \cdot x}, \quad \sigma = e, \mu, \quad (24)$$

with

$$u^r_{k,\sigma}(x_0) = u^r_{k,\sigma} e^{-i \omega_{k,\sigma} x_0} \quad \text{and} \quad v^r_{-k,\sigma}(x_0) = v^r_{-k,\sigma} e^{i \omega_{k,\sigma} x_0}. \quad (25)$$

Here and throughout the tilde operators are those corresponding to the specific couple $(m_e, m_\mu)$. With these definitions all the calculations at a fixed instant of time $x_0$ can be performed in exactly the same way they are done in the free field case. The explicit time dependence of the creation and destruction operators is due to the interaction with the external field and it does not create problems as the states which are acted upon by the operators are evaluated at the same time as the operators themselves and the commutators are all considered at equal times.

In terms of the tilde flavor ladder operators, the Hamiltonian and momentum operators Eqs. (16), (17) read:

$$\tilde{P}_\sigma(x_0) = \sum_r \int d^3 k \, k \left( \tilde{\alpha}^r_{k,\sigma}(x_0) \tilde{\alpha}^r_{k,\sigma}(x) + \tilde{\beta}^r_{k,\sigma}(x_0) \tilde{\beta}^r_{k,\sigma}(x) \right), \quad (26)$$

$$\tilde{H}_\sigma(x_0) = \sum_r \int d^3 k \omega_{k,\sigma} \left( \tilde{\alpha}^r_{k,\sigma}(x_0) \tilde{\alpha}^r_{k,\sigma}(x) - \tilde{\beta}^r_{k,\sigma}(x_0) \tilde{\beta}^r_{k,\sigma}(x) \right). \quad (27)$$

The new flavor states are defined by the action of the tilde creation operator on the tilde flavor vacuum:

$$|\tilde{\nu}^r_{k,\sigma}(x_0)\rangle = \tilde{\alpha}^r_{k,\sigma}(x_0) |\tilde{0}(x_0)\rangle_{e\mu}. \quad (28)$$

We easily find the result that these single particle states are eigenstates of both the Hamiltonian and the momentum operator:

$$\left( \frac{\tilde{H}_\sigma(x_0)}{\tilde{P}_\sigma(x_0)} \right) |\tilde{\nu}^r_{k,\sigma}(x_0)\rangle = \left( \frac{\omega_{k,\sigma}}{k} \right) |\tilde{\nu}^r_{k,\sigma}(x_0)\rangle, \quad (29)$$

making explicit the 4-vector structure.

The flavor charges commute with the tilde Hamiltonian operator: $[\tilde{Q}_\sigma(x_0), \tilde{H}(x_0)] = 0$, as a consequence of $[\tilde{Q}_\sigma(x_0), \tilde{H}_\sigma(x_0)] = 0, \sigma, \sigma' = e, \mu$, and of the absence of an interaction term in $\tilde{H}$. This ensures the existence of a common set of eigenstates of these operators. In this set we find the flavor states (28), which are straightforwardly seen to be also eigenstates of the flavor charges:

$$\tilde{Q}_\sigma(x_0) |\tilde{\nu}^r_{k,\sigma}(x_0)\rangle = |\tilde{\nu}^r_{k,\sigma}(x_0)\rangle. \quad (30)$$

Let us now consider the algebra of the generators descending from the energy-momentum tensor (15). All the generators are defined in the usual way. Besides the translation generators defined by Eqs. (16) and (17), we have the Lorentz generators, defined as:

$$\tilde{M}^{\lambda\rho}(x_0) = \int d^3 x \left( \tilde{T}^{0\rho} x^\lambda - \tilde{T}^{0\lambda} x^\rho \right) + \frac{1}{2} \int d^3 x \nu^\rho_f \sigma^{\lambda\rho} v_f = \tilde{M}_\rho^{\lambda\rho}(x_0) + \tilde{M}^{\lambda\rho}(x_0), \quad (31)$$
where $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The algebra of (equal-time) commutators of these generators will be just the direct sum of two Poincaré algebras (we omit the specification of the instant of time):

$$[\tilde{P}^\mu_\sigma, \tilde{P}^\nu_{\sigma'}] = 0, \quad [\tilde{M}^{\mu\nu}_\sigma, \tilde{P}^\lambda] = i \delta_{\sigma\sigma'} (\eta^{\mu\lambda} \tilde{P}_\sigma - \eta^{\nu\lambda} \tilde{F}_\sigma), \quad \sigma, \sigma' = e, \mu,$$

$$[\tilde{M}^{\mu\nu}_\sigma, \tilde{M}^\lambda_{\sigma'}] = i \delta_{\sigma\sigma'} \left( \eta^{\mu\lambda} \tilde{M}^{\nu\nu}_\sigma - \eta^{\nu\lambda} \tilde{M}^{\mu\mu}_\sigma - \eta^{\mu\nu} \bar{M}^\nu_{\sigma'} + \eta^{\nu\mu} \bar{M}^\mu_{\sigma'} \right). \quad (32)$$

Note that the above construction and the consequent Poincaré invariance, holds at a given time $x_0$. Thus, for each different time, we have a different Poincaré structure. The relation of neutrino mixing with the breakdown of Lorentz symmetry will be further discussed in Section 6.

5. Phenomenological implications

The above results reflect into a number of phenomenological consequences which could as well serve for falsifying the proposed scenario. In Ref. [16] some of them have been discussed: in particular, it has been pointed out that very precise direct measurements of neutrino masses, as those done by looking at tritium decay, could reveal if the mass eigenstates $\nu_i$ enter in fundamental interactions, or rather the flavor eigenstates are the basic objects, as here is proposed. In the latter case, the measured masses should be the average masses $m_e$ and $m_\mu$ above introduced.

Another test can be done by looking at possible violations of lepton charge conservation at tree level. This effect is by definition absent when considering neutrinos as flavor charge eigenstates as we have done above. However, the usual Pontecorvo states:

$$|\nu_{k,e}^r\rangle_P = \cos \theta |\nu_{k,1}^r\rangle + \sin \theta |\nu_{k,2}^r\rangle \quad (33)$$

$$|\nu_{k,m}^r\rangle_P = -\sin \theta |\nu_{k,1}^r\rangle + \cos \theta |\nu_{k,2}^r\rangle \quad (34)$$

are not eigenstates of the flavor charges, as can be easily checked.

Thus, if we consider the expectation values of the above defined flavor charges onto the Pontecorvo states, we see how much the lepton charge is violated (at tree level) by the usual quantum mechanical states. One finds [10]:

$$\rho^r_{|\nu_{k,e}^r\rangle_P} = \cos^4 \theta + \sin^4 \theta + 2|U_k| \sin^2 \theta \cos^2 \theta + \sum_r \int d^3k, \quad (35)$$

and

$$1,2 \langle 0 |\tilde{Q}_e(x_0) |0\rangle_{1,2} = \sum_r \int d^3k. \quad (36)$$

The diverging terms can be removed by normal ordering the flavor charges with respect to the mass vacuum $|0\rangle_{1,2}$. One has then

$$\rho^r_{|\nu_{k,e}^r\rangle} : \tilde{Q}_e(x_0) : |\nu_{k,e}^r\rangle_P = \cos^4 \theta + \sin^4 \theta + 2|U_k| \sin^2 \theta \cos^2 \theta < 1, \quad (37)$$

$\forall \theta \neq 0, m_1 \neq m_2, k \neq 0$, and

$$1,2 \langle 0 |\tilde{Q}_e(x_0) : |0\rangle_{1,2} = 0, \quad (38)$$
i.e. a violation of lepton charge of the order of \(1 - |U_k|\) \(\sin^2(2\theta)\) is present in the Pontecorvo flavor neutrino states.

We also note that the corresponding quantum fluctuations are divergent:

\[
1,2(0)(\tilde{Q}_e(x_0):2|0)_{1,2} = 4 \sin^2 \theta \cos^2 \theta \int d^3k|V_k|^2,
\]

\[
P\langle \nu_{k,e}\rangle(\tilde{Q}_e(x_0):2|\nu_{k,e}\rangle_P = \cos^6 \theta + \sin^6 \theta + \sin^2 \theta \cos^2 \theta \left[2|U_k| + |U_k|^2 + 4 \int d^3k|V_k|^2\right].
\]

6. Flavor mixing and Spontaneous Symmetry Breakdown of the Lorentz symmetry

In this section we provide yet another interpretation of the static vector field \(A^\mu\). We have seen, the structure of \(A^\mu\) explicitly violates the Lorentz symmetry. It is well known that in a non-gravitational setting, it suffices to specify fixed background fields violating Lorentz symmetry in order to formulate the Lorentz violating (LV) matter dynamics. However, fixed background fields break general covariance. If one wishes to preserve the successes of general relativity in accounting for gravitational phenomena, breaking general covariance is not an option. The obvious alternative is to promote the LV background fields to dynamical fields, governed by a generally covariant action. By allowing the vector field to vary over space-time, the mixing and ensuing flavor oscillations will not be necessarily uniform over space-time. There are several candidates for the covariant kinetic term. At this stage we realize (taking into account Eq. (10)) that in the general frame \(n^\mu n_\mu = 1, n_0 > 0\). So the dynamics of \(A^\mu\) takes place on the Lobachevsky 4D target space. This, in fact, corresponds to a quotient \(M = SO(3,1)/SO(3)\) which effectively describes the target space of soft modes (Nambu-Goldstone modes) after the Lorentz symmetry is broken to the rotational symmetry. The latter target-space constraint might be introduced as a constraint in the path-integral measure, such that

\[
\mathcal{D}n^\mu \delta[n^\mu n_\mu - 1] = \mathcal{D}\lambda \mathcal{D}n^\mu \exp \left(i \int d^4x \lambda(x)(n^\mu(x)n_\mu(x) - 1)\right)
\]

\[
= \mathcal{D}\lambda \mathcal{D}n^\mu \exp \left(i \int d^4x \mathcal{L}_{\text{constr.}}\right).
\]

Massless field theories where the target space is the group coset space are commonly known as non-linear \(\sigma\) models. Non-linear \(\sigma\) models typically describe the dynamics of Nambu–Goldstone bosons. The sigma model in question is sometimes called the \textit{space-time \(\sigma\)-model} [19] or \textit{Einstein aether \(\sigma\)-model} [13]. It is well known, that with a suitable choice of the interaction Hamiltonian the generalized coherent state functional integrals describe low-energy effective field theories — Nambu–Goldstone fields, including their mutual interactions [5, 18].

At this stage it should be reminded that in contrast to the spontaneous symmetry breakdown (SSB) of an internal symmetry which produces Nambu–Goldstone scalar fields, the corresponding SSB of Lorentz symmetry yields Nambu–Goldstone vector fields [20]. Thus here we do not regard the above field as a \(SU(2)\) gauge field, as done in the previous section, but simply as a massless vector field.

Let us now consider the corresponding \(SO(3,1)\) coherent-state functional. This represents a natural playground for discussing the quantum dynamics of Nambu–Goldstones [5, 18]. This can be done in terms of the unit-vector parameters \(n_\mu\). The kinetic term for the gauge field (now dynamically epitomized via \(n^\mu\)) is a Wess–Zumino–Witten term in \(3 + 1\) dimensions. This term can be written in the form (see, e.g., Ref. [13])

\[
\mathcal{L}_{\text{WZW}} = \frac{1}{2} (\nabla_\mu n_\nu)(\nabla^\mu n^\nu).
\]
Here the gradient $\nabla_\mu$ should not be mistaken with a covariant derivative. The ensuing total Lagrangian has thus the form

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_f + \mathcal{L}_{\text{WZW}} + \mathcal{L}_{\text{constr}},$$

(43)

where $\mathcal{L}_f$ is the field Lagrangian (14). It can be shown [13, 21] that this version of the aether $\sigma$-theory is stable under small perturbations and the Hamiltonian is globally bounded.

So finally we see that the original Lagrangian (1) for oscillating neutrinos (where mass and flavor states are simultaneously not well defined) can be equally formulated as the Lagrangian for definite mass and definite flavor neutrinos interacting with the Nambu–Goldstone vector field. The natural way to proceed with a quantization of this model would be to consider the corresponding $SO(3,1)$ coherent-state functional. Here we shall refrain from doing this. More detailed discussion of this issue will be published elsewhere.

7. Conclusions

In the framework of the quantum field theoretical formalism for flavor mixing, we have discussed a vector-field-like structure associated to two-flavor neutrino mixing. In this connection two natural strategies could be identified. In the first case, flavor neutrino fields are assumed to be primitive entities describing on-shell particles with definite masses $m_e$ and $m_\mu$, which are different from those of the mass eigenstates $m_1$ and $m_2$ of the standard approach. Flavor oscillations then arise as a consequence of the interaction with the external $su(2)$ gauge field, which acts as a sort of refractive medium. An interesting consequence of such an analysis is that we recover, locally in time, a Poincaré structure for the flavor neutrino states. Several phenomenological implications can be worked out from such a picture [16].

In the second case, we put forward a proposal in which the vector field is a Nambu–Goldstone vector field which naturally emerges when the Lorentz symmetry $SO(3,1)$ is broken down to the rotation subgroup $SO(3)$. To furnish a dynamics to such a vector field one can take advantage of the fact that Nambu–Goldstone fields are described via non-linear $\sigma$-models. The non-linear $\sigma$-models have a naturally induced kinetic term — a Wess–Zumino–Witten term. In the present case the $\sigma$-model is also known as aether $\sigma$-model and the Hamiltonian is globally bounded when only two-derivative terms are included in the action. We have also presented the ensuing Lagrangian which can serve as a starting point for a path-integral quantization. Further work in this direction is in progress.

It would be definitely important to generalize our two approaches also to bosonic systems, such as $K^0 - \bar{K}^0$, $D^0 - \bar{D}^0$ or $B^0 - \bar{B}^0$ flavor oscillations. The extension to 3-flavor mixing is still an open issue, though preliminary results are already available [22].

Finally, we comment on the recent data from the OPERA collaboration [23], exhibiting an apparent superluminal behaviour of neutrinos. In Ref. [24] it has been shown that such a feature can be accomodated in the framework of flavor neutrino states satisfying modified dispersion relations as discussed in Refs. [9], provided at least one of the mass eigenstates being tachyonic. A nice feature of this approach is that one does not encounter the usual runaway pathologies of tachyons. In the present approach, it is not clear yet if superluminal neutrino propagation can arise. Further study in this direction is in progress.

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Appendix A. Mixing of quantum neutrino fields

In this Appendix we briefly recall some fundamentals of the quantization for mixed Fermi fields, as given in Refs. [4, 6, 8]. We start from the free fields $\nu_1$ and $\nu_2$, whose Fourier expansions are

$$\nu_j(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_r \left[ u^r_{k,j}(x_0) \alpha^r_{k,j} + v^r_{-k,j}(x_0) \beta^r_{-k,j} \right] e^{ik \cdot x}, \quad j = 1, 2, \quad (A.1)$$

with

$$u^r_{k,j}(x_0) = u^r_{k,j} e^{-i\omega_{k,j} x_0}, \quad v^r_{-k,j}(x_0) = v^r_{-k,j} e^{i\omega_{k,j} x_0}, \quad (A.2)$$

and $\omega_{k,j} = \sqrt{k^2 + m^2}$. The operators $\alpha^r_{k,j}$ and $\beta^r_{-k,j}$, with $j = 1, 2$ and $r = 1, 2$ that appear in the mode expansion (A.1) are the annihilation operators for the vacuum state $|0\rangle_{1,2} = |0\rangle_1 \otimes |0\rangle_2$, i.e.,

$$\alpha^r_{k,j} |0\rangle_{1,2} = 0, \quad \beta^r_{-k,j} |0\rangle_{1,2} = 0. \quad (A.3)$$

The canonical anticommutation rules assume the form

$$\{\nu^\alpha_i(x), \nu^\beta_j(y)\}_{x=y_0} = \delta^\beta(y-x)\delta_{\alpha\beta}\delta_{ij}, \quad \alpha, \beta = 1, \ldots, 4. \quad (A.4)$$

In terms of the creation and annihilation operators these read as

$$\{\alpha^r_{k,i}, \alpha^s_{q,j}\} = \delta_{kq}\delta_{rs}\delta_{ij}, \quad \{\beta^r_{k,i}, \beta^s_{q,j}\} = \delta_{kq}\delta_{rs}\delta_{ij}, \quad i, j = 1, 2, \quad (A.5)$$

with all other anticommutators being zero. The corresponding orthonormality and completeness relations are

$$u^r_{k,j} u^r_{k,j} = \delta_{rs}, \quad v^r_{-k,j} v^s_{-k,j} = 0, \quad \sum_r (u^r_{k,j} v^r_{k,j} + v^r_{-k,j} u^r_{-k,j}) = 1. \quad (A.6)$$

We construct the generator for the mixing transformations (2) as:

$$\nu^\sigma_\theta(x) = G^\sigma_\theta(x_0) \nu^\sigma \theta G(x_0), \quad (\sigma, j) = (\{e, 1\}, \{\mu, 2\}), \quad (A.7)$$

$$G^\sigma \theta(x_0) = \exp \left[ \theta \int d^3x \left( \nu^1_1(x) \nu^2_2(x) - \nu^1_2(x) \nu^2_1(x) \right) \right].$$

At finite volume $G^\sigma_\theta$ is a unitary operator: $G^\sigma_\theta^{-1}(x_0) = G_{-\theta}(x_0) = G^\sigma_\theta(x_0)$, preserving the canonical anticommutation relations. $G^\sigma_\theta^{-1}(x_0)$ maps the Hilbert space for the free fields $\mathcal{H}_{1,2}$ to the Hilbert space for the mixed fields $\mathcal{H}_{e,\mu}$: $G^\sigma_\theta^{-1}(x_0) : \mathcal{H}_{1,2} \rightarrow \mathcal{H}_{e,\mu}$. In particular, the vacuum $|0\rangle_{1,2}$ is mapped into a new state:

$$|0(x_0)\rangle_{e,\mu} = G^\sigma_\theta^{-1}(x_0) |0\rangle_{1,2}, \quad (A.8)$$

which is referred to as the flavor vacuum. In the limit $V \rightarrow \infty$, the flavor vacuum becomes orthogonal to the vacuum of the free fields, which means that the two Hilbert spaces are unitarily inequivalent. From (A.8) we can deduce that the annihilating operators for the flavor vacuum are

$$\alpha^r_{k,\sigma}(x_0) = G^{-1}_\theta(x_0) \alpha^r_{k,j} G_\theta(x_0), \quad (\sigma, j) = (\{e, 1\}, \{\mu, 2\}). \quad (A.9)$$
The Hermitian conjugation of the former yields the corresponding relation for antiparticles.
Even so, the representation (A.7) (or A.9) of the canonical transformation loses its meaning for an (infinite) field systems, in as much as the exponential operator occurring in it has no domain on the representation space involved. This fact, however, has no direct bearing on our approach which uses the well-defined form (2). It just states that the operator-algebra representation which is used in the particle mixing is unitarily inequivalent to the Fock-space representation. In other words, the flavor vacuum \(|0(x_0)\rangle\) and the ordinary vacuum \(|0\rangle_{1,2}\) do not belong to the same Hilbert space.

The flavor fields can be expanded in the same bases as the fields \(\nu_i\):

\[
\nu_\sigma(x) = \int \frac{d^3k}{(2\pi)^3} \sum_{r} \left[ v^r_{k,j}(x_0) \alpha^r_{k,\sigma}(x_0) + v^r_{-k,j}(x_0) \beta^r_{-k,\sigma}(x_0) \right] e^{ik\cdot x},
\]

where \((\sigma, j) = \{(\sigma, 1), (\mu, 2)\}\). With the help of Eq. (7) it can be seen [6] that the flavor charges have the following expansion

\[
Q_\sigma(x_0) = \sum_r \int d^3k \left( \alpha^r_{k,\sigma}(x_0) \alpha^r_{k,\sigma}(x_0) - \beta^r_{-k,\sigma}(x_0) \beta^r_{-k,\sigma}(x_0) \right).
\]

The flavor neutrino states are then defined as eigenstates of the flavor charges. In particular, we can write that \(|\nu^k_\sigma(x_0)\rangle = \alpha^r_{k,\sigma}(x_0)|0(x_0)\rangle_{e,\mu} + \text{similar for antiparticles}.

Note that the flavor charges do not commute with the Hamiltonian of the system

\[
[H, Q_\sigma(x_0)] \neq 0, \quad \sigma = e, \mu
\]

with the consequence that they are not conserved by time evolution. This is of course the hallmark of the flavor oscillation phenomenon. The corresponding flavor oscillation formulas are derived by computing the expectation value of the flavor charge operators in the flavor state [8].

References
[1] Bilenky S M and Pontecorvo B 1978 Phys. Rept. 41 225
[2] Sny M B (Super-Kamiokande) 2003 Nucl. Phys. Proc. Suppl. 118 25 (2003)
[3] Hallin A L et al. 2003 Nucl. Phys. Proc. Suppl. 118 3
[4] Eguchi K et al. (KamLAND) 2003 Phys. Rev. Lett. 90 021802
[5] Hannel W et al. (GALLEX) 1999 Phys. Lett. B 447 127
[6] Altmann M et al. (GNO) 2000 Phys. Lett. B 490 16
[7] Abdurashitov J N et al. (SAGE) 2002 J. Exp. Theor. Phys. 95 181
[8] Terranova F 2011 Int. J. Mod. Phys. A 26 4739
[9] Blasone M and Vitiello G 1997 Annals Phys. 244 283
[10] Blasone M, Jizba P and Vitiello G 2010 Quantum Field Theory and its Macroscopic Manifestations (London: World Scientific & ICP)
[11] Blasone M, Jizba P and Vitiello G 2001 Phys. Lett. B 517 471
[12] Blasone M, Capolupo A, Ji C R and Vitiello G 2010 Int. J. Mod. Phys. A 25 4179
[13] Blasone M, Henning P A and Vitiello G 1999 Phys. Lett. B 451 140
[14] Blasone M, Magnejo J and Pires-Pacheco P 2005 Europhys. Lett. 70 600
[15] Blasone M, Magnejo J and Pires-Pacheco P 2005 Braz. J. Phys. 35 447
[16] Blasone M, Capolupo A, Terranova F and Vitiello G 2005 Phys. Rev. D 72 013003.
[17] Blasone M, Di Mauro M and Vitiello G 2010 (Preprint hep-ph/1003.5812)
[18] Fujii K, Habe C and Yabuki T 2001 Phys. Rev. D 59 113003
Fujii K, Habe C and Yabuki T 2001 Phys. Rev. D 59 013011
[19] Carroll S M, Dulaney T R, Gresham M I, and Tann M H 2009 Phys. Rev. D 79 065012
[20] Aharonov Y and Bohm D 1959 Phys. Rev. 115 48
[21] Blasone M and Vitiello G 1999 Phys. Rev. D 60 111302
[22] Blasone M 2011 J. Phys. Conf. Ser. 306 012037
[17] Pakvasa S 2001 (Preprint hep-ph/0110175)
Lambiase G 2003 Phys. Lett. B 560 1
Kostelecky V A and Mewes M 2004 Phys. Rev. D 69 (2004) 016005
Kostelecky V A and Mewes M 2004 (ibid.) Phys. Rev. D 70 031902
Klinkhamer F R 2006 Int. J. Mod. Phys. A 21 161
Di Grezia E, Esposito S and Salesi G 2006 Mod. Phys. Lett. A 21 349
Hooper D, Morgan D and Winstanley E 2005 Phys. Rev. D 72 065009
Ellis J R, Harries N, Meregaglia A, Rubbia A and Sakharov A 2008 Phys. Rev. D 78 033013
Diaz J S, Kostelecky V A and Mewes M 2009 Phys. Rev. D 80 076007
[18] Blasone M and Jizba P 2011 (Preprint hep-th/1111.3228)
[19] Froggatt C D and Nielsen H B 1991 Origin of Symmetries (Singapore: World Scientific)
[20] Chkareuli J L, Froggatt C D and Nielsen H B 2001 Phys. Rev. Lett. 87 091601
Chkareuli J L, Froggatt C D and Nielsen H B 2001 Nucl. Phys. B 609 46
[21] Kostelecky V A and Lehnert R 2001 Phys. Rev. D 63 065008
Elliott J W, Moore G D and Stoica H 2005 J. High Energy Phys. 08 066
Dulaney T R, Gresham M I and Wise M B 2008 Phys. Rev. D 77 083510
[22] Blasone M, Terranova F and Vitiello G, work in progress.
[23] Adam T et al. [OPERA Collaboration] 2011 (Preprint hep-ex/1109.4897)
[24] Magueijo J 2011 (Preprint hep-ph/1109.6055)