CARLESON MEASURES, BMO SPACES AND BALAYAGES ASSOCIATED TO 
SCHRÖDINGER OPERATORS 

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Abstract. Let $\mathcal{L}$ be a Schrödinger operator of the form $\mathcal{L} = -\Delta + V$ acting on $L^2(\mathbb{R}^n)$, $n \geq 3$, where the nonnegative potential $V$ belongs to the reverse Hölder class $B_q$ for some $q \geq n$. Let $\text{BMO}_\mathcal{L}(\mathbb{R}^n)$ denote the BMO space associated to the Schrödinger operator $\mathcal{L}$ on $\mathbb{R}^n$. In this article we show that for every $f \in \text{BMO}_\mathcal{L}(\mathbb{R}^n)$ with compact support, then there exist $g \in L^\infty(\mathbb{R}^n)$ and a finite Carleson measure $\mu$ such that 

$$f(x) = g(x) + S_{\mu, \mathcal{L}}(x),$$

with $\|g\|_\infty + \|\mu\|_c \leq C\|f\|_{\text{BMO}_\mathcal{L}(\mathbb{R}^n)}$, where 

$$S_{\mu, \mathcal{L}} = \int_{\mathbb{R}^{n+1}} \mathcal{P}_t(x, y) d\mu(y, t),$$

and $\mathcal{P}_t(x, y)$ is the kernel of the Poisson semigroup $\{e^{-t\sqrt{\mathcal{L}}}\}_{t > 0}$ on $L^2(\mathbb{R}^n)$. Conversely, if $\mu$ is a Carleson measure, then $S_{\mu, \mathcal{L}}$ belongs to the space $\text{BMO}_\mathcal{L}(\mathbb{R}^n)$. This extends the result for the classical John–Nirenberg BMO space by Carleson [1] (see also [13, 7, 14]) to the BMO setting associated to Schrödinger operators.

1. Introduction

Consider the Schrödinger operator with the non-negative potential $V$:

$$(1.1) \quad \mathcal{L} = -\Delta + V(x) \quad \text{on} \quad L^2(\mathbb{R}^n), \quad n \geq 3,$$

where $V$ belongs to the reverse Hölder class $B_q$ for some $q \geq n/2$, which by definition means that $V \in L^q_{\text{loc}}(\mathbb{R}^n)$, $V \geq 0$, and there exists a constant $C > 0$ such that the reverse Hölder inequality

$$(1.2) \quad \left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq C \frac{1}{|B|} \int_B V(y) \, dy$$

holds for all balls $B$ in $\mathbb{R}^n$.

The operator $\mathcal{L}$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$, and $\mathcal{L}$ generates the Poisson semigroup $\{e^{-t\sqrt{\mathcal{L}}}\}_{t > 0}$ on $L^2(\mathbb{R}^n)$. Since the potential $V$ is non-negative, the semigroup kernels $\mathcal{P}_t(x, y)$ of the operators $e^{-t\sqrt{\mathcal{L}}}$ satisfy

$$0 \leq \mathcal{P}_t(x, y) \leq p_t(x - y)$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$, where

$$(1.3) \quad p_t(x - y) = c_n \frac{t}{(t^2 + |x|^2)^{n/2}}, \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}$$

is the kernel of the classical Poisson semigroup $\{\mathcal{P}_t\}_{t > 0} = \{e^{-t\Delta}\}_{t > 0}$ on $\mathbb{R}^n$.

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Following [5], a locally integrable function $f$ belongs to $\text{BMO}_L(\mathbb{R}^n)$ whenever there is constant $C \geq 0$ so that

\begin{equation}
\frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \leq C,
\end{equation}

where $f_Q$ stands for the average of $f$ over the cube $Q$ on $\mathbb{R}^n$, i.e., $f_Q = \frac{1}{|Q|} \int_Q f(y) dy$, and

\begin{equation}
\frac{1}{|Q|} \int_Q |f(y)| dy \leq C
\end{equation}

for every cubes $Q$ with $\ell(Q) > \rho(x)$, where $\ell(Q)$ is the sidelength of $Q$ and $x$ is the centre of $Q$, and the function $\rho(x) = \rho(x; V)$ takes the explicit form

\begin{equation}
\rho(x) = \sup \{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \}.
\end{equation}

Throughout the article we assume that $V \not\equiv 0$ so that $0 < \rho(x) < \infty$ (see [11]). We define $\|f\|_{\text{BMO}_L(\mathbb{R}^n)}$ to be the smallest $C$ in the right hand sides of (1.4) and (1.5). Because of (1.5), this $\text{BMO}_L(\mathbb{R}^n)$ space is in fact a proper subspace of the classical $\text{BMO}$ space of John-Nirenberg.

Consider the function

\begin{equation}
S_{\mu, P}(x) = \int_{\mathbb{R}^{n+1}} P_t(x, y) d\mu(y,t),
\end{equation}

where $\mu$ is a Borel measure on $\mathbb{R}_{n+1}^n$. It is easy to see that if $\mu$ is finite then the integral in (1.7) – called the sweep or balayage of $\mu$ with respect $P$ – converges absolutely for a.e. $x \in \mathbb{R}^n$, and $\|S_{\mu, P}\|_1 \leq C(\eta)\|\mu\|_c$.

The aim of this article is to prove the following result.

**Theorem 1.1.** Suppose $V \in B_q$ for some $q \geq n$. Then we have

(i) If $\mu$ is a Carleson measure, then $S_{\mu, P} \in \text{BMO}_L(\mathbb{R}^n)$ with $\|S_{\mu, P}\|_{\text{BMO}_L(\mathbb{R}^n)} \leq C(n)\|\mu\|_c$.

(ii) Let $f \in \text{BMO}_L(\mathbb{R}^n)$ have compact support. There exist $g \in L^\infty(\mathbb{R}^n)$ and a finite Carleson measure $\mu$ such that

\begin{equation}
f(x) = g(x) + S_{\mu, P}(x),
\end{equation}

where $\|g\|_{L^\infty(\mathbb{R}^n)} + \|\mu\|_c \leq C(n)\|f\|_{\text{BMO}_L(\mathbb{R}^n)}$.

We would like to mention that for the case that $P_t(x, y)$ is the classical Poisson kernel $p_t(x - y)$ in (1.3), the proof of (i) of Theorem 1.1 of the classical BMO is quite standard (see [6]); and the result (ii) of Theorem 1.1 is proved by an iteration argument in the work of L. Carleson [1]; and also in the work of A. Uchiyama [13], and by Garnett and Jones [7]. Later, J.M. Wilson [14] gives a new proof by using the Poisson semigroup property and Green’s theorem to avoid the iteration to make the construction much more explicit. Our Theorem 1.1 extends the result of the classical BMO to the space $\text{BMO}_L(\mathbb{R}^n)$ associated with the Schrödinger operators. The proof of Theorem 1.1 follows the idea of [14] i.e., by using the Poisson semigroup property, and Green’s theorem, but differs from it in method and techniques since the kernel for the operator $L$ is not translation invariant and several techniques for the classical Possion kernel are not applicable here. The standard preservation condition of the semigroup $\{e^{-tL}\}_{t>0}$ fails, i.e.,

\[ e^{-tL} 1 \neq 1. \]

This is indeed one of the main obstacles in this article and makes the theory quite subtle and delicate. We overcome this problem in the proof by making use of the estimates on the kernel of
the Poisson semigroup under the assumption on \( V \in B_q \) for some \( q \geq n \), and some techniques to estimate the norm of the space \( \text{BMO}_L(\mathbb{R}^n) \) associated with the Schrödinger operators.

Throughout the article, the letter “\( C \)” will denote (possibly different) constants that are independent of the essential variables.

### 2. Preliminaries

Throughout the article, we may sometimes use capital letters to denote points in \( \mathbb{R}^{n+1} \), e.g., \( X = (x, t) \), and set

\[
\nabla_X u(X) = (\nabla_x u, \partial_t u) \quad \text{and} \quad |\nabla_X u| = |\nabla_x u| + |\partial_t u|.
\]

For simplicity we will denote by \( \nabla \) the full gradient \( \nabla_X \) in \( \mathbb{R}^{n+1} \). For every cube \( Q \) on \( \mathbb{R}^n \), we set \( \ell(Q) \) the sidelength of \( Q \), and let \( CQ \) denote the cube concentric with \( Q \) and with sidelength \( C \) times as big. For \( Q \) as above, we write \( \widehat{Q} = \{(x, t) \in \mathbb{R}^{n+1} : x \in Q, 0 < t < \ell(Q)\} \). Recall that a measure \( \mu \) defined on \( \mathbb{R}^{n+1} \) is said to be a Carleson measure if

\[
|||\mu|||_c =: \sup_{Q \subset \mathbb{R}^n} \frac{|\mu|}{|Q|} < \infty.
\]

#### 2.1. Basic properties of the heat and Poisson semigroups of Schrödinger operators

Let us recall some basic properties of the critical radii function \( \rho(x) \) under the assumption (1.2) on \( V \) (see Section 2, [5]). Suppose \( V \in B_q \) for some \( q > n/2 \). There exist \( C > 0 \) and \( k_0 \geq 1 \) such that for all \( x, y \in \mathbb{R}^n \)

\[
C^{-1} \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq C \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{k_0}.
\]

In particular, \( \rho(x) \sim \rho(y) \) when \( y \in B(x, r) \) and \( r \leq c \rho(x) \).

The following estimates on the heat kernel of \( L \) are well known.

**Proposition 2.1** ([5]). Let \( L = -\Delta + V \) with \( V \in B_q \) for some \( q \geq n/2 \). Then for each \( N > 0 \) there exists \( C_N > 0 \) such that the heat kernel \( \mathcal{H}_t(x, y) \) of the semigroup \( \{e^{-Lt}\}_{t \geq 0} \) satisfies

\[
\mathcal{H}_t(x, y) \leq C_N \frac{e^{-|x-y|^2/ct}}{t^{n/2}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}
\]

and

\[
|\mathcal{H}_t(x, y) - \mathcal{H}_t(x', y)| \leq C_N \frac{|x - x'|^\beta}{\sqrt{t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}
\]

whenever \( |x - x'| \leq \sqrt{t} \) and for any \( 0 < \beta < \min\{1, 2 - n/q\} \).

Through Bochner’s subordination formula (see [12]), the Poisson semigroup associated to \( L \) can be obtained from the heat semigroup:

\[
e^{-t\sqrt{t}} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} e^{-u L} f(x) du = \frac{t}{2 \sqrt{\pi}} \int_0^\infty e^{-u^2/2} e^{-s L} f(x) ds,
\]

which connects the Poisson kernel and the heat kernel as follows:

\[
\mathcal{P}_t(x, y) = \frac{t}{2 \sqrt{\pi}} \int_0^\infty e^{-u^2/2} \mathcal{H}_t(x, y) ds.
\]
Lemma 2.2 ([4, 5, 10]). Suppose \( V \in B_q \) for some \( q \geq n \). For any \( 0 < \beta < \min \{ 1, 2 - n/q \} \) and every \( N > 0 \), there exists a constant \( C = C_N \) such that for \( x, y \in \mathbb{R}^n \) and \( t > 0 \),

\[
(i) \quad |P_t(x, y)| \leq C \frac{t}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}} \left(1 + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(x)} + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(y)}\right)^{-N};
\]

(ii) The kernel \( \nabla P_t(x, y) \) satisfies

\[
|t\nabla P_t(x, y)| \leq \frac{Ct}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}} \left(1 + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(x)} + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(y)}\right)^{-N};
\]

(iii) \( |t\nabla e^{-t\sqrt{V}}(1)(x)| \leq C \left(\frac{t}{\rho(x)}\right)^\beta \left(1 + \frac{t}{\rho(x)}\right)^{-N}. \)

It follows from Lemmas 1.2 and 1.8 in [11] that there is a constant \( C > 0 \) such that for a nonnegative Schwartz class function \( \varphi \) there exists a constant \( C \) such that

\[
\int_{\mathbb{R}^n} \varphi_t(x - y)V(y)dy \leq \begin{cases} C t^{-1}(\varphi(\sqrt{t})^\delta & \text{for } t \leq \rho(x)^2; \\
C(\varphi(\sqrt{t})^{C_{0}+2-n} & \text{for } t > \rho(x)^2, \end{cases}
\]

where \( \varphi_t(x) = t^{-n/2}\varphi(x/\sqrt{t}) \), and \( \delta = 2 - n/q > 0 \). From (2.2), we can follow the proof of (d) of Proposition 3.6, [11] to show that for \( V \in B_q, q \geq n/2 \), there is some \( \delta \in (0, 1) \) and \( N > \max\{4, 2 \log(C_0 + 2 - n)\} \) with \( C_0 \) the doubling constant of with respect to the potential \( V \) as in (1.1), such that for \( t > 0 \) and \( x \in \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} \left(1 + \left|\log\frac{\rho(y)}{t}\right|\right) |P_t(x, y)V(y)dy \leq C t^{-1} \min\left\{ \left(\frac{t}{\rho(x)}\right)^\delta, \left(\frac{t}{\rho(x)}\right)^{-\frac{N}{2}}\right\}. \tag{2.8}
\]

Lemma 2.3. Suppose \( V \in B_q \) for some \( q \geq n/2 \). Let \( f \in \text{BMO}_\mathcal{L}(\mathbb{R}^n) \). Then we have

(i) There exists \( C > 0 \) such that for all \( Q \) with \( \ell(Q) < \rho(x) \), then

\[
|f|_{L^p(Q)} \leq C \left(1 + \log \frac{\rho(x)}{\ell(Q)}\right) \|f\|_{\text{BMO}_\mathcal{L}(\mathbb{R}^n)}.
\]

(ii) For every \( p \in [1, \infty) \), there exists \( C = C(p, \rho) > 0 \) such that

\[
\left( \frac{1}{|Q|} \int_Q |f - f_Q|^p dx \right)^{1/p} \leq C \|f\|_{\text{BMO}_\mathcal{L}(\mathbb{R}^n)}, \quad \text{for all cubes } Q,
\]

and for every \( Q \) with \( \ell(Q) \geq \rho(x) \):

\[
\left( \frac{1}{|Q|} \int_Q |f|^p dx \right)^{1/p} \leq C \|f\|_{\text{BMO}_\mathcal{L}(\mathbb{R}^n)}.
\]

Proof. For the proof of (i), we can obtain it by making minor modification with that of Lemma 2, [5], and we skip it here. For the proof of (ii), we refer it to Corollary 3, [5]. \( \square \)
2.2. The Balayage associated with generalized approximation to the identity. In this section we will work with a class of integral operators \( \{ \mathcal{A}_t \}_{t > 0} \), which plays the role of generalized approximations to the identity. Assume that the operators \( \mathcal{A}_t \) are defined by kernels \( a_t(x, y) \) in the sense that

\[
\mathcal{A}_t f(x) = \int_{\mathbb{R}^n} \mathcal{A}_t(x, y) f(y) dy,
\]

where the kernels \( \mathcal{A}_t(x, y) \) satisfy the estimate

\[
|\mathcal{A}_t(x, y)| \leq C t^{-n} \left( 1 + \frac{|x-y|}{t} \right)^{-(n+\epsilon)}
\]

for some \( \epsilon > 0 \) and for every function \( f \) which satisfies some suitable growth condition.

The space \( \text{BMO}_A(\mathbb{R}^n) \) associated with a generalized approximation to the identity \( \{ \mathcal{A}_t \}_{t > 0} \) was introduced and studied in [2]. In the sequel, \( \epsilon \) is a constant as in (2.9).

**Definition 2.4.** We say that \( f \in L^1((1+|x|)^{-(n+\epsilon)}) \) is in \( \text{BMO}_A(\mathbb{R}^n) \), the space of functions of bounded mean oscillation associated with a generalized approximation to the identity \( \{ \mathcal{A}_t \}_{t > 0} \), if there exists some constant \( C \) such that for any ball \( B \),

\[
(2.10) \quad \frac{1}{|Q|} \int_Q |f(x) - \mathcal{A}_t(Q, f(x))| dx \leq C,
\]

where \( \ell(Q) \) is the sidelength of the cube \( Q \). The smallest bound \( c \) for which (2.10) is satisfied is then taken to be the norm of \( f \) in this space, and is denoted by \( \|f\|_{\text{BMO}_A} \).

Note first that \( \text{BMO}_A, \| \cdot \|_{\text{BMO}_A} \) is a semi-normed vector space, with the seminorm vanishing on the space \( \mathcal{K}_A \), defined by

\[
\mathcal{K}_A = \left\{ f \in L^1((1+|x|)^{-(n+\epsilon)}): \mathcal{A}_t f(x) = f(x) \text{ for almost all } x \in \mathbb{R}^n \text{ for all } t > 0 \right\}.
\]

The space \( \text{BMO}_A \) is understood to be modulo \( \mathcal{K}_A \). It is easy to check that \( L^\infty(\mathbb{R}^n) \subset \text{BMO}_A(\mathbb{R}^n) \) with \( \|f\|_{\text{BMO}_A} \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \).

In this section, we assume that

i) \( A_0 \) is the identity operator and the operators \( \{ \mathcal{A}_t \}_{t > 0} \) form a semigroup, that is, for any \( t, s > 0 \) and \( f \in \mathcal{M} \), \( \mathcal{A}_t \mathcal{A}_s f(x) = \mathcal{A}_{t+s} f(x) \) for almost all \( x \in \mathbb{R}^n \).

ii) There exists some \( 0 < \epsilon' \leq \epsilon \) such that the kernels \( \mathcal{A}_t \) satisfy the estimate

\[
(2.11) \quad \left| \frac{\partial}{\partial t} \mathcal{A}_t(x, y) \right| \leq C t^{-n} \left( 1 + \frac{|x-y|}{t} \right)^{-(n+\epsilon')}.
\]

Examples of operators \( \{ \mathcal{A}_t \}_{t > 0} \) which satisfy conditions i) and ii) above include the Poisson and heat kernels of certain operators including Schrödinger operators with nonnegative potentials and second order divergence form elliptic operators (see for example, [2] [3] [4] [5] [10] [11] [12]).

**Proposition 2.5.** Assume that \( \{ \mathcal{A}_t \}_{t > 0} \) is a generalized approximation to the identity with properties i) and ii) above. If \( \mu \) is a Carleson measure on \( \mathbb{R}^{n+1}_+ \), then the function

\[
(2.12) \quad S_{\mu, \mathcal{A}}(x) = \int_{\mathbb{R}^{n+1}_+} \mathcal{A}_t(x, y) d\mu(y, t)
\]

belongs to \( \text{BMO}_A(\mathbb{R}^n) \) with \( \|S_{\mu, \mathcal{A}}\|_{\text{BMO}_A(\mathbb{R}^n)} \leq C(n)\|\mu\|_C \).
Proof. Let \( Q \) be a cube with the center \( x_Q \) and its sidelength \( \ell(Q) \). Let \( Q_k = 2^k Q \). It follows from the assumption (i) of \( \{ A_t \}_{t \geq 0} \) that

\[
\int_Q |S_{\mu_{\mathcal{A}}}(x) - A_{t(Q)}S_{\mu_{\mathcal{A}}}(x)|dx \leq \int_Q \int_{\mathbb{R}^n} |A_t(x, z) - A_{t(Q)}A_t(x, z)|d\mu(z, t)dx
\]

\[
\leq \int_Q \int_{\mathcal{Q}_1} |A_t(x, z) - A_{t+\ell(Q)}(x, z)|d\mu(z, t)dx
\]

\[
+ \int_Q \int_{\mathbb{R}^n \setminus \mathcal{Q}_1} |A_t(x, z) - A_{t+\ell(Q)}(x, z)|d\mu(z, t)dx = I + II.
\]

For the term \( I \), one easily sees that

\[
I = \int_{\mathcal{Q}_1} \int_Q |A_t(x, z) - A_{t+\ell(Q)}(x, z)|dx d\mu(z, t) \leq C|\mathcal{Q}_1| \leq C|Q|.
\]

Consider the term \( II \). We apply the formula:

\[
A_t(x, z) - A_{t+\ell(Q)}(x, z) = -\int_0^{\ell(Q)} \partial_s A_{s+t}(x, z)dz.
\]

This, together with the assumption (ii) of \( \{ A_t \}_{t \geq 0} \), yields that for some \( 0 < \epsilon' \leq \epsilon \),

\[
II \leq \int_Q \int_{0}^{\ell(Q)} \int_{\mathbb{R}^n \setminus \mathcal{Q}_1} |\partial_s A_{s+t}(x, z)|d\mu(z, t)dsdx
\]

\[
= \sum_{k=1}^{\infty} \int_Q \int_{0}^{\ell(Q)} \int_{\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k} |\partial_s A_{s+t}(x, z)|d\mu(z, t)dsdx
\]

\[
\leq C \sum_{k=1}^{\infty} \int_Q \int_{0}^{\ell(Q)} \int_{\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k} \frac{1}{s + t (s + t + |x-z|)^{n+\epsilon'}} d\mu(z, t)dsdx
\]

\[
=: \sum_{k=1}^{\infty} II_k.
\]

Notice that for every \( x \in Q, 0 < t < \ell(Q) \) and \( (s, z) \in \mathcal{Q}_{k+1} \setminus \mathcal{Q}_k, k = 1, 2, \cdots \), we have that \( s + t + |x-z| \geq 2^k \ell(Q) \). It tells us that

\[
II_k \leq C(2^k \ell(Q))^{-(n+\epsilon')} \int_Q \int_{\mathcal{Q}_{k+1} \setminus \mathcal{Q}_k} (s + t)^{\epsilon'-1} d\mu(z, t)dsdx
\]

\[
\leq C|Q|(2^k \ell(Q))^{-(n+\epsilon')} \ell(Q)^{\epsilon'} |2^k Q|
\]

\[
\leq 2^{-k\epsilon'} |Q|,
\]

which shows that \( II \leq \sum_{k=1}^{\infty} 2^{-k\epsilon'} |Q| \leq C|Q| \). This, in combination with \((2.14)\), yields the desired result. This completes the proof. \( \square \)

3. Proof of Theorem \[1.1\]

Let \( \text{BMO}_p(\mathbb{R}^n) \) denote the \( \text{BMO} \) space associated with the semigroup \( \{ e^{-t\sqrt{\Delta}} \}_{t > 0} \) defined in Definition \[2.4\]. Under the assumption of \( V \in B_q \) for some \( q \geq n/2 \), it is known (see \[3\]) that the spaces \( \text{BMO}_p(\mathbb{R}^n) \) coincide, and their norms are equivalent. From this, (i) of Theorem \[1.1\] is a straightforward consequence of Proposition \[2.5\].

We will use two facts about the space \( BMO_{L^2} \) with \( \text{supp} f \subseteq \{ x = (x_1, \cdots, x_n) : |x_i| \leq 1, i = 1, \cdots, n \} \) and set \( Q_0 = \{ x = (x_1, \cdots, x_n) : |x_i| \leq 2, i = 1, \cdots, n \} \). Write
\[
  u(x, t) = \mathcal{P}_t f(x) = e^{-t \nabla f} f(x).
\]

We will use two facts about the space \( BMO_{L^2} \) (see [4, 5, 10]) under the assumption of \( V \in B_q, q \geq n \):

1. Let \( Q = Q(x, t) \) denote the cube with center \( x \) and sidelength \( t \). Then we have
   \[
   |\mathcal{P}_t f - f_Q(x)| = |e^{-t \nabla} (f - f_Q)(x)| \leq C \| f \|_{BMO_{L^2}(\mathbb{R}^n)}.
   \]

2. Let \( \nabla \) denote the full gradient in \( \mathbb{R}^{n+1} \). We have that for all \( (x, t) \in \mathbb{R}^{n+1} \),
   \[
   |t \nabla u(x, t)| = |t \nabla e^{-t \nabla} f(x)| \leq C \| f \|_{BMO_{L^2}(\mathbb{R}^n)}
   \]

In the sequel, for a cube \( Q \subset \mathbb{R}^n \) we let \( z_Q = (x_Q, t_Q) \) with \( t_Q = \ell(Q) \), where \( x_Q \) is the center of \( Q \) \((z_Q \text{ is the center of the top face of } \hat{Q})\). We define generations, \( G_k \), of subcubes of \( Q_0 \) as follows:
\[
  G_0 = \{ Q_0 \} ;
\]
\[
  G_{k+1} = \{ Q' \subset Q \in G_k : Q' \text{ maximal dyadic such that } |u(x_Q, t_Q) - u(x_{Q'}, t_{Q'})| > A \}
\]
for \( k \geq 0 \), where \( A \) is a large constant to be chosen later.

In the sequel, for every \( Q \in G_k, k \geq 0 \) set
\[
  \Sigma_Q = \hat{Q} \setminus \bigcup_{Q' \in G_k \cap Q' \in G_{k+1}} \hat{Q}',
\]
and \( \partial \Sigma_Q \) denotes the boundary of \( \Sigma_Q \). Then the following result holds.

**Lemma 3.1.** We have the properties:

(i) There exists a constant \( C > 0 \) such that for every \( (x, t) \in \Sigma_Q \cup \partial \Sigma_Q \),
   \[
   |u(x, t) - u(x_Q, t_Q)| \leq A + C \| f \|_{BMO_{L^2}(\mathbb{R}^n)}.
   \]

(ii) For every \( Q \in G_k, k \geq 0 \),
   \[
   \sum_{Q' \subset Q \in G_k \cap Q' \in G_{k+1}} |Q'| \leq CA^{-1} |Q| \| f \|_{BMO_{L^2}(\mathbb{R}^n)}.
   \]

As a consequence, if \( A \) is large enough, then \( \sum_{Q' \subset Q \in G_k} \sum_{Q' \in G_{k+1}} |Q'| \leq |Q|/2 \).

**Proof.** For a cube \( J \), we set \( T(J) \) the top half of \( \hat{J} \), i.e., \( T(J) = \{ (x, t) : x \in J, \ell(J)/2 \leq t < \ell(J) \} \).

We notice that, for every dyadic cube \( I \),
\[
  \hat{T} = \bigcup_{J \subset I} T(J),
\]
and therefore, if \( Q \in G_k, \Sigma_Q \) is the union of all the sets \( T(J) \) such that \( J \subset Q \) is dyadic and \( J \) is not a subset of any \( Q' \in G_{k+1} \). If \( (x, t) \in \Sigma_Q \), then \( (x, t) \) lies in some \( T(J) \) as described above. But then \( |u(x_Q, t_Q) - u(x_J, t_J)| \leq A \) because if it were \( |u(x_Q, t_Q) - u(x_J, t_J)| > A \), \( J \) would belong to \( G_{k+1} \) or would be contained in some cube in \( G_{k+1} \), i.e., a maximal cube for the property. But then \( (x, t) \) would not line in \( \Sigma_Q \). On the other hand, the fact \( (3.2) \) implies that for every \( (x, t) \in T(J), \)
\[
  |u(x, t) - u(x_J, t_J)| \leq C \| f \|_{BMO_{L^2}(\mathbb{R}^n)}.
\]
All together, yields the desired result for $(x, t) \in \Sigma_Q$. Taking limits, this also holds for $(x, t) \in \partial \Sigma_Q$. This proves (i).

Let us show (ii). For every $k$, it follows from the definition of $G_k$ that

$$\sum_{Q' \subset Q \in G_k} |Q'| \leq \frac{1}{A} \sum_{Q' \subset Q \in G_{k+1}} |Q' u(x_Q, t_Q) - u(x_{Q'}, t_{Q'})|$$

Observe that $u(x_Q, t_Q) = P_{t_Q} f(x_Q) = P_{t_Q} (f - f_Q(x_Q)) + P_{t_Q} (1)(x) f_Q$. We obtain

RHS of (3.4) $\leq \frac{1}{A} \sum_{Q' \subset Q \in G_k} |Q'| \left( |P_{t_Q} (f - f_Q(x_Q))| + |P_{t_Q} (f - f_{Q'}(x_Q))| + |P_{t_Q} (1)(x_Q)| f_Q - f_{Q'} | \right)

= I + II + III.$

We first note that $I \leq \frac{C}{A} |Q| \|f\|_{BMO_{\mathbb{R}^n}}$. In fact, this follows from the estimate in (3.1) and from the facts that for every $t > 0$ and $x \in \mathbb{R}^n$, $|P_t(1)(x)| \leq C$ and that for every $Q' \subset Q$, $|f_Q - f_{Q'}| \leq |Q'|^{-1} \int_{Q'} |f - f_Q| dx \leq C \|f\|_{BMO_{\mathbb{R}^n}}$.

To estimate the term $II$, we first assume that $t_Q \geq \rho(x_Q)$. Then from the facts that for every $t > 0$ and $x \in \mathbb{R}^n$, $|P_t(1)(x)| \leq C$ and that $|f_Q| \leq C \|f\|_{BMO_{\mathbb{R}^n}}$, the term $II$ is bounded by

$$\frac{1}{A} \sum_{Q' \subset Q \in G_k} |Q'| \|f\|_{BMO_{\mathbb{R}^n}} \leq \frac{C}{A} |Q| \|f\|_{BMO_{\mathbb{R}^n}},$$

where the last inequality follows from the fact that these $Q'$s are pairwise disjoint.

Next we consider the case $t_Q < \rho(x_Q)$. From (2.2), we have that $\rho(x) \sim \rho(x_Q)$ for $x \in Q$. By (iii) of Lemma 2.2

$$|P_{t_Q} (1)(x_Q) - P_{t_Q} (1)(x_Q)| = \left| \int_{t_Q}^{t_Q} s \partial_s e^{-s \sqrt{L}} (1)(x_Q) \frac{ds}{s} \right|\]

$$\leq C \int_{t_Q}^{t_Q} \left( \frac{s}{\rho(x_Q)} \right)^{\beta} \frac{ds}{s} \leq C \left( \frac{t_Q}{\rho(x_Q)} \right)^{\beta},$$

which, together with (i) of Lemma 2.3 implies

$$\sum_{Q' \subset Q \in G_k} \frac{1}{A} |Q'| |f_Q| |P_{t_Q} (1)(x_Q) - P_{t_Q} (1)(x_Q)| \leq \frac{C}{A} \left( \frac{t_Q}{\rho(x_Q)} \right)^{\beta} \sum_{Q' \subset Q \in G_k} \int_{Q'} dy \leq \frac{C}{A} |Q| |f_Q| \left( \frac{t_Q}{\rho(x_Q)} \right)^{\beta}$$
From this, we rewrite (3.10) defined in (3.3). Observe that if $Q$ where the improper integral converges in (3.9), by (3.8), it follows that (3.7)

$$\sum_{Q' \subseteq Q \in G} |Q'| \leq \frac{C}{A} |Q| \|f\|_{BMO^2(R^n)} \leq \frac{1}{2} |Q|,$$

by choosing $A$ large enough. This ends the proof of (ii) of Lemma 3.1. The proof is complete. □

In the sequel, we fix a constant $A > 0$ large enough so that (3.7) holds. By (ii) of Lemma 2.3, we know that $f \in L^2_{loc}(R^n)$. Since $\text{supp } f \subseteq Q_0 = \{ x = (x_1, \cdots, x_n) : |x_i| \leq \sqrt{2}, i = 1, \cdots, n \}$, $f$ belongs to $L^2(R^n)$. It follows by the spectral theory ([12]) that

$$f(x) = 2 \int_0^\infty \sqrt{t} e^{-2t} \sqrt{t} f(x) dt,$$

where the improper integral converges in $L^2(R^n)$. Note that both functions $P_t(x, y)$ and $u(y, t)$ are $C^1(R^n)$ in $y$. Since $-u_t + Lu = 0$, we apply Green’s theorem to obtain

$$0 = - \iint_{R_n} \frac{d^2}{dt^2} P_t(x, y) u(y, t) dy dt + \iint_{R_n} tP_t(x, y) (-\Delta + V) u(y, t) dy dt$$

$$= \iint_{R_n} \frac{d}{dt} P_t(x, y) u(y, t) dy dt + \iint_{R_n} t\nabla P_t(x, y) \nabla u(y, t) dy dt$$

$$+ \iint_{R_n} tP_t(x, y) V(u(y, t) dy dt.$$ By (3.8), it follows that

$$f(x) = 2 \iint_{R_n} (t \nabla P_t(x, y) \nabla u(y, t) + tP_t(x, y) V(y) u(y, t)) dy dt,$$

where the improper integral converges in $L^2(R^n)$.

We are now going to cut up the integral in (3.8). For every $Q \in G_k, k \geq 0$, we recall that $\Sigma_Q$ is defined in (3.3). Observe that if $Q \neq \Sigma_Q'$, then $\Sigma_Q \cap \Sigma_{Q'} = \emptyset$. It follows that

$$R_n^+ = \bar{Q}_0 \cup \left( R_n^+ \setminus \bar{Q}_0 \right) = \left( \bigcup_{k \geq 0} Q \subseteq G_k \right) \bigcup \left( R_n^+ \setminus \bar{Q}_0 \right).$$

From this, we rewrite

$$f(x) = \sum_{k \geq 0} \left( \sum_{Q \subseteq G_k} 2 \iint_{\Sigma_Q} (t \nabla P_t(x, y) \nabla u(y, t) + tP_t(x, y) V(y) u(y, t)) dy dt$$

$$+ 2 \iint_{R_n^+ \setminus \bar{Q}_0} (t \nabla P_t(x, y) \nabla u(y, t) + tP_t(x, y) V(y) u(y, t)) dy dt$$

$$= \left( \sum_{k \geq 0} \sum_{Q \subseteq G_k} f_Q(x) \right) + f_2(x) =: f_1(x) + f_2(x).$$
We will deal with $f_1(x)$ first. Since $-u_t + Lu = 0$, we apply Green’s theorem to each $f_Q$ of the summands in $f_1$ to obtain

$$f_Q(x) = u(y_Q, t_Q) \int_{\Sigma_Q} t\mathcal{P}_t(x, y)V(y)dydt$$

(3.12)\[+ \left\{ \int_{\partial\Sigma_Q} \frac{\partial}{\partial \nu} \mathcal{P}_t(x, y) \left( u(y, t) - u(y_Q, t_Q) \right) d\sigma_Q + \int_{\partial\Sigma_Q} \mathcal{P}_t(x, y) \frac{\partial}{\partial \nu} u(y, t) d\sigma_Q - \int_{\partial\Sigma_Q} \mathcal{P}_t(x, y) \left( u(y, t) - u(y_Q, t_Q) \right) \frac{\partial}{\partial t} d\sigma_Q \right\} =: I_Q(x) + I\bar{I}_Q(x).

**Lemma 3.2.** There exists a constant $C > 0$ such that for all $x \in \mathbb{R}^n$,

$$\sum_Q |I_Q(x)| \leq CA + C\|f\|_{BMO_{\mathbb{R}^n}}.$$

**Proof.** To estimate the term

$$\int_{\Sigma_Q} u(y_Q, t_Q)\mathcal{P}_t(x, y)V(y)dydt.$$

We write $u(y_Q, t_Q) = \mathcal{P}_{t_Q}f(y_Q) = \mathcal{P}_{t_Q}(f - f_{Q(y, t_Q)})(y_Q) + \mathcal{P}_{t_Q}(1)(y_Q)f_{Q(y, t_Q)}$ where $(y, t) \in \Sigma_Q$. This, in combination with (3.1) and (2.8), shows that

(3.13)\[\sum_Q |I_Q(x)| \leq \sum_Q \int_{\Sigma_Q} |\mathcal{P}_{t_Q}(f - f_{Q(y, t_Q)})(y_Q)|t\mathcal{P}_t(x, y)V(y)dydt + \sum_Q \int_{\Sigma_Q} |\mathcal{P}_{t_Q}(1)(y_Q)f_{Q(y, t_Q)}|t\mathcal{P}_t(x, y)V(y)dydt \leq C\|f\|_{BMO_{\mathbb{R}^n}} + C \sum_Q \bar{I}_Q(x).

By Lemma 2.3

$$|f_{Q(y, t_Q)}| \leq \begin{cases} C\|f\|_{BMO_{\mathbb{R}^n}} \left(1 + \log \frac{\rho(y)}{t_Q} \right), & \text{if } t_Q \leq \rho(y); \\ C\|f\|_{BMO_{\mathbb{R}^n}}, & \text{if } t_Q > \rho(y). \end{cases}$$

It follows from (2.8) that

$$\sum_Q \bar{I}_Q(x) \leq C\|f\|_{BMO_{\mathbb{R}^n}} \sum_Q \int_{\Sigma_Q} \left(1 + \max \left\{ \log \frac{\rho(y)}{t_Q}, 1 \right\} \right)t\mathcal{P}_t(x, y)V(y)dydt \leq C\|f\|_{BMO_{\mathbb{R}^n}} \int_{\Sigma_Q} \left(1 + \left| \log \frac{\rho(y)}{t} \right| \right)t\mathcal{P}_t(x, y)V(y)dydt \leq C\|f\|_{BMO_{\mathbb{R}^n}} \int_0^\infty \min \left\{ \left( \frac{t}{\rho(x)} \right) ^{\delta}, \left( \frac{t}{\rho(x)} \right)^{-\frac{\gamma}{2}+2} \right\} \frac{dt}{t} \leq C\|f\|_{BMO_{\mathbb{R}^n}},$$

which, together with (3.13), shows that $\sum_Q |I_Q(x)| \leq C\|f\|_{BMO_{\mathbb{R}^n}}$. The proof is complete. \qed
Next we estimate the term $II_Q(x)$. Following [14], one writes
\[
F(Q : t, x, y) = t \left( \mathcal{P}(x, y) \frac{\partial}{\partial y} u(y, t) + \frac{\partial}{\partial y} \mathcal{P}(x, y) (u(y, t) - u(y_Q, t_Q)) \right) - \mathcal{P}(x, y) (u(y, t) - u(y_Q, t_Q)) \frac{\partial t}{\partial y}
\]
and
\[
II_Q(x) = \int_{\partial Q} F(Q : t, x, y) d\sigma_Q
\]
(3.14)
\[
= \int_{\partial Q \cap \mathbb{R}^n} F(Q : t, x, y) d\sigma_Q + \int_{\partial Q \cap \{t > 0\}} F(Q : t, x, y) d\sigma_Q =: II_Q^{(1)}(x) + II_Q^{(2)}(x).
\]
It can be verified that $II_Q^{(1)}(x)$ is equal to
\[
(f(x) - u(y_Q, t_Q)) \chi_{\partial Q \cap \mathbb{R}^n}(x) = h_Q(x).
\]
The supports of the different $h_Q$'s are easily seen to be disjoint, and so we may set
\[
h(x) = \sum_{Q \in \bigcup_{i=0}^{\infty} G_i} h_Q(x).
\]
Then the following result holds.

**Lemma 3.3.** The function $h$ satisfies
\[
\|h\|_{\infty} \leq A + C\|f\|_{\text{BMO}_L(\mathbb{R}^n)}
\]
with $\text{supp } h \subseteq \{x = (x_1, x_2, \ldots, x_n) : |x_i| \leq 2, i = 1, 2, \ldots, n\}$.

**Proof.** By (ii) of Lemma 2.3 we know that $f \in L^2_{\text{loc}}(\mathbb{R}^n)$. Since $f \in \text{BMO}_L(\mathbb{R}^n)$ have compact support, $f$ belongs to $L^2(\mathbb{R}^n)$. Then by (b) of Maximal Theorem on Stein’s book [12, Section 3 of Chapter III],
\[
\lim_{t \to 0} e^{-t \sqrt{\mathcal{T}}} f(x) = f(x), \quad \text{a.e.}
\]
By (i) of Lemma 3.1 it follows that for every $(x, t) \in \Sigma_Q$,
\[
|e^{-t \sqrt{\mathcal{T}}} f(x_Q) - e^{-t \sqrt{\mathcal{T}}} f(x)| \leq A + C\|f\|_{\text{BMO}_L(\mathbb{R}^n)}.
\]
Taking limits, this holds for each $h_Q(x), x \in \partial \Sigma_Q \cap \mathbb{R}^n$. This proves (3.15). \qed

We will finished with the term $f_1$ once we show that
\[
\sum_Q II_Q^{(2)}(x) = S_{\mu, \mathcal{P}}
\]
(3.16)
for some $\mu$ with $\|\mu\|_c \leq C(n)\|f\|_{\text{BMO}_L(\mathbb{R}^n)}$. We proceed to do this now. Write
\[
II_Q^{(2)}(x) = \int_{\partial Q \cap \{t \geq 0\}} t \left( \mathcal{P}(x, y) \frac{\partial}{\partial y} u(y, t) + \frac{\partial}{\partial y} \mathcal{P}(x, y) (u(y, t) - u(y_Q, t_Q)) \right) d\sigma_Q
\]
\[
- \int_{\partial Q \cap \{t > 0\}} \mathcal{P}(x, y) (u(y, t) - u(y_Q, t_Q)) \frac{\partial t}{\partial y} d\sigma_Q
\]
(3.17)
\[
=: II_Q^{(21)}(x) + II_Q^{(22)}(x).
\]
By fact (3.2) and the way we chose the $Q$'s,
\[
\left| t \frac{\partial}{\partial y} u(y, t) - (u(y, t) - u(y_Q, t_Q)) \frac{\partial t}{\partial y} \right| \leq C(n)\|f\|_{\text{BMO}_L(\mathbb{R}^n)}
\]
when \((y, t) \in \partial \Sigma_Q \cap \{ t > 0 \}\). Hence,
\[
\int_{\partial \Sigma_Q \cap \{ t > 0 \}} \left( t \frac{\partial}{\partial \nu} u(y, t) - (u(y, t) - u(y_Q, t_Q)) \right) \mathcal{P}_i(x, y) d\sigma_Q = \int_{\partial \Sigma_Q \cap \{ t > 0 \}} \mathcal{P}_i(x, y) h_Q(y, t) d\sigma_Q
\]
for some \(h_Q(y, t)\) with \(|h_Q| \leq C(n)\|f\|_{\text{BMO}_c(\mathbb{R}^n)}\). By (ii) of Lemma 3.1 it is well known (see [6, Chapter VIII], [14]) that
\[
||| \sum_{i \geq 0} \sum_{Q \in G_i} d\sigma_Q |||_c \leq C(n),
\]
which, together with the condition \(|h_Q| \leq C(n)\|f\|_{\text{BMO}_c(\mathbb{R}^n)}\), implies that
\[
||| \sum_{i \geq 0} \sum_{Q \in G_i} h_Q(y, t) d\sigma_Q |||_c \leq C(n)\|f\|_{\text{BMO}_c(\mathbb{R}^n)}.
\]
Thus, to obtain (3.16), we only need to estimate the integrals
\[
\int_{\partial \Sigma_Q \cap \{ t > 0 \}} t \frac{\partial}{\partial \nu} \mathcal{P}_i(x, y) (u(y, t) - u(y_Q, t_Q)) d\sigma_Q.
\]
For this, we need the following lemma.

**Lemma 3.4.** Let \(\{z_i\} = \{(x_i, t_i)\} \subset \mathbb{R}^{n+1}\) be points and let \(\delta_{z_i}\) be the Dirac mass at \(z_i\). Assume that
\[
||| \sum_{i} t_i^p \delta_{z_i} |||_c \leq 1.
\]
Let \(ds_i\) denote \(n\)-dimensional Lebesgue measure on the hyperplane \(\{ t = t_i \} \subset \mathbb{R}^{n+1}\) and let \(\Phi_i(x, y) = t/(t + |x - y|)^{n+1}\). Set
\[
\mu(x, t) = \sum_{i} t_i^p \Phi_i(x, x_i) ds_i.
\]
Then \(\|\mu\|_c \leq C(n)\).

**Proof.** Its proof is similar to that of LEMMA of [14], and we skip it here. \(\square\)

To estimate (3.19), we follow [14] to write
\[
\partial \Sigma_Q \cap \{ t > 0 \} = \bigcup_i E_i.Q
\]
where each \(E_i.Q\) is of the form
\[
E_i.Q = \begin{cases} 
\{(x, t) : x \in \partial Q_i, \ t(\ell(Q_i)) \leq t \leq 2\ell(Q_i)\} \cap \partial \Sigma_Q \quad \text{or} \\
\{(x, t) : x \in \overline{Q}_i, \ t = \ell(Q_i)\}
\end{cases}
\]
for some dyadic cube \(Q_i\). These \(E_i.Q\) make a tiling of \(\partial \Sigma_Q \cap \{ t > 0 \}\), with the size of the tiles going to zero as \(t \to 0\). Let \((x_i.Q, \tilde{t}_i.Q)\) be the centroid of \(E_i.Q\) in \(\mathbb{R}^{n+1}\) and let \(\xi_i.Q = (x_i.Q, \frac{1}{2}\tilde{t}_i.Q) \equiv (x_i.Q, t_i.Q)\). It is easy to see that, for any cube \(Q' = 2Q\),
\[
\int_{Q'} \sum_i \tilde{t}_i.Q \delta_{\xi_i.Q} dy dt \leq \int_{4Q} d\sigma_Q.
\]
Therefore, by (3.18),

\[
(3.20) \quad \|\sum_{i \in \mathcal{G}_k} \sum_{Q \in \mathcal{G}_k} \sum_{t} t^k_i \delta_{E_{i, Q}}\|_{L^1} \leq C(n)\|f\|_{\text{BMO}_\mathbb{R}(\mathbb{R}^n)}.
\]

Observe that

\[
t \frac{\partial}{\partial y} \mathcal{P}_i(x, y) = \int_{\mathbb{R}^n} \mathcal{P}_{b, Q}(x, z) \left(t \frac{\partial}{\partial y} \mathcal{P}_{t-i, Q}(z, y)\right) dz.
\]

For every \((y, t) \in E_{i, Q}\), it is easy to see that \(|y - x_{i, Q}| \leq t\). Also we have that \(t - t_i, Q > t_i, Q/3\), \(t \leq 8t_i, Q/3\), and so \(t_i, Q > t - t_i, Q\). Then there exists a constant \(C > 0\) uniformly for \((y, t) \in E_{i, Q}\) such that

\[
\left| t \frac{\partial}{\partial y} \mathcal{P}_{t-i, Q}(z, y) \right| \leq C \frac{t_i, Q}{(t_i, Q + |z - x_{i, Q}|)^{n+1}}.
\]

It then follows that

\[
(3.21) \quad \sum_{k \geq 0} \sum_{Q \in \mathcal{G}_k} \mathcal{H}^{(22)}_Q(x) = \sum_{k \geq 0} \sum_{Q \in \mathcal{G}_k} \sum_{i} \int_{E_{i, Q}} t \frac{\partial}{\partial y} \mathcal{P}_i(x, y) (u(y, t) - u(y_Q, t_Q)) \, d\sigma_Q
\]

\[
= \sum_{k \geq 0} \sum_{Q \in \mathcal{G}_k} \sum_{i} t^k_i \int_{\mathbb{R}^n} \mathcal{P}_{b, Q}(x, z) \Lambda_{t-i, Q}^i (z, x_{i, Q}) d\bar{z}_{i, Q}
\]

\[
= \int_{\mathbb{R}^{n+1}} \mathcal{P}_i(x, y) d\rho_Q(y, t),
\]

where

\[
d\rho_Q = \sum_{k \geq 0} \sum_{Q \in \mathcal{G}_k} \sum_{i} t^k_i \Lambda_{t-i, Q}^i (z, x_{i, Q}) d\bar{z}_{i, Q}
\]

and

\[
(3.23) \quad \left| \Lambda_{t-i, Q}^i (z, x_{i, Q}) \right| = \left| t^k_i \int_{E_{i, Q}} (u(y, t) - u(y_Q, t_Q)) \left( t \frac{\partial}{\partial y} \mathcal{P}_{i, t-i, Q}(z, y) \right) d\sigma_Q \right|
\]

\[
\leq C \frac{t_i, Q}{(t_i, Q + |z - x_{i, Q}|)^{n+1}}.
\]

Here we used the fact that for every \((y, t) \in E_{i, Q} \subset \partial \Sigma_Q\), it follows by (i) of Lemma 3.1 that \(|u(y, t) - u(y_Q, t_Q)| \leq A + C \|f\|_{\text{BMO}_\mathbb{R}(\mathbb{R}^n)}\). Hence, from estimates (3.20), (3.21), (3.22) and (3.23), we apply Lemma 3.4 to finish the proof of the term \(f_1\).

Now for the term \(f_2(x)\) in (3.11), Green’s theorem gives us

\[
(3.24) \quad f_2(x) = u(y_{Q_0}, t_{Q_0}) \int_{\mathbb{R}^{n+1}} t \mathcal{P}_i(x, y) V(y) dy dt
\]

\[
+ \left\{ \int_{\partial \tilde{Q}_0 \cap \mathbb{R}^{n+1}} t \frac{\partial}{\partial y} \mathcal{P}_i(x, y) (u(y, t) - u(y_{Q_0}, t_{Q_0})) d\sigma_{\tilde{Q}_0}
\]

\[
+ \int_{\partial \tilde{Q}_0 \cap \mathbb{R}^{n+1}} t \mathcal{P}_i(x, y) \frac{\partial}{\partial y} u(y, t) d\sigma_{\tilde{Q}_0}
\]

\[
- \int_{\partial \tilde{Q}_0 \cap \mathbb{R}^{n+1}} \mathcal{P}_i(x, y) (u(y, t) - u(y_{Q_0}, t_{Q_0})) \frac{\partial}{\partial y} d\sigma_{\tilde{Q}_0} \right\}
\]

\[
= III_{Q_0}(x) + IV_{Q_0}(x),
\]
where $d\sigma_{Q_0}$ is $n$-dimensional surface measure on $\partial \widehat{Q}_0$ and $\partial / \partial \nu$ now denotes the normal derivative into $\widehat{Q}_0$.

To estimate the term $III_{Q_0}(x)$, we write $u(y_{Q_0}, t_{Q_0}) = (u(y_{Q_0}, t_{Q_0}) - f_{Q_0}) + f_{Q_0}$, it follows by Lemma 2.3 that $|u(y_{Q_0}, t_{Q_0})| \leq C\|f\|_{\text{BMO}_c(\mathbb{R}^n)}$. We apply an argument as in $I_Q(x)$ to show that
\[
|III_{Q_0}(x)| \leq C\|f\|_{\text{BMO}_c(\mathbb{R}^n)} \int_{\mathbb{R}^{n+1}} t \mathcal{P}_f(x, y)V(y) dy dt
\]
\[
\leq C\|f\|_{\text{BMO}_c(\mathbb{R}^n)} \int_0^\infty \min \left\{ \left( \frac{t}{\rho(x)} \right)^\delta, \left( \frac{t}{\rho(x)} \right)^{-\frac{n+2}{2}} \right\} dt
\]
\[
\leq C\|f\|_{\text{BMO}_c(\mathbb{R}^n)}.
\]

For the term $IV_{Q_0}(x)$, we notice that $\partial \widehat{Q}_0 \cap \mathbb{R}^{n+1}$ is away from the support of $f$, $|u(y, t) - u(y_{Q_0}, t_{Q_0})| \leq C\|f\|_{\text{BMO}_c(\mathbb{R}^n)}$ and $|\nabla u(y, t)| \leq C\|f\|_{\text{BMO}_c(\mathbb{R}^n)}$ on $\partial \widehat{Q}_0 \cap \mathbb{R}^{n+1}$. So, since $d\sigma_{Q_0}$ is a Carleson measure with norm $\leq C$, these terms present no problem. We now handle the other term by cutting $\partial \widehat{Q}_0$ into tiles, just as we did for $f_1(x)$, to obtain estimate for $IV_{Q_0}(x)$. This finishes the proof of $f_2(x)$.

Finally, we collect estimates (3.11), (3.12), (3.14) for $f_1(x)$ and (3.24) for $f_2(x)$ to write
\[
f = g(x) + S_{\mu, \mathcal{H}}(x)
\]
where
\[
g(x) = \sum_{k \geq 0} \sum_{Q \in G_k} I_Q(x) + \sum_{k \geq 0} \sum_{Q \in G_k} II^{(1)}_Q(x) + III_{Q_0}(x) \in L^\infty(\mathbb{R}^n)
\]
and a finite Carleson measure $\mu$ such that
\[
S_{\mu, \mathcal{H}}(x) = \sum_{k \geq 0} \sum_{Q \in G_k} II^{(2)}_Q(x) + IV_{Q_0}(x) = \int_{\mathbb{R}^{n+1}} \mathcal{P}_f(x, y)d\mu(y, t)
\]
with $\|g\|_{L^\infty(\mathbb{R}^n)} + \|u\|_c \leq C(n)\|f\|_{\text{BMO}_c(\mathbb{R}^n)}$. The proof of Theorem 1.1 is complete.

Consider $\mathcal{L} = -\Delta + V(x)$, where $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a non-negative function on $\mathbb{R}^n$. Let $\{e^{-t\mathcal{L}}\}_{t>0}$ be the heat semigroup associated to $\mathcal{L}$:
\begin{align}
(3.25) \quad e^{-t\mathcal{L}} f(x) &= \int_{\mathbb{R}^n} \mathcal{H}_t(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad t > 0.
\end{align}

Since the potential $V$ is nonnegative, the kernel $\mathcal{H}_t(x, y)$ of the semigroup $e^{-t\mathcal{L}}$ satisfies
\begin{align}
(3.26) \quad 0 &\leq \mathcal{H}_t(x, y) \leq h_t(x - y)
\end{align}
for all $x, y \in \mathbb{R}^n$ and $t > 0$, where
\begin{align}
(3.27) \quad h_t(x - y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}
\end{align}
is the kernel of the classical heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$ on $\mathbb{R}^n$. Then we have the following result.

**Theorem 3.5.** Suppose $V \in B_q$ for some $q \geq n$. Then for every $f \in \text{BMO}_L(\mathbb{R}^n)$ with compact support, there exist $g \in L^\infty(\mathbb{R}^n)$ with compact support and a finite Carleson measure $\mu$ such that
\[
f(x) = g(x) + \int_{\mathbb{R}^{n+1}} \mathcal{H}_t(x, y)d\mu(y, t),
\]
where \( \|g\|_{L^p(\mathbb{R}^n)} + \|\mu\|_c \leq C(n)\|f\|_{\text{BMO}_L(\mathbb{R}^n)} \).

Proof. From Theorem \[1.1\] we know that for every \( f \in \text{BMO}_L(\mathbb{R}^n) \) with compact support, there exist \( g \in L^\infty(\mathbb{R}^n) \) and a finite Carleson measure \( \mu \) such that \( f(x) = g(x) + \sum \mu(\{x\}) \) \( \mu \in L^1(\mathbb{R}^n) \) and \( \mu \) is a finite Carleson measure. From this, and the subordination formula (2.5), we rewrite

\[
S_{\mu, \mathcal{F}}(x) = \int_{\mathbb{R}^{n+1}} \mathcal{P}(x, y) d\mu(y, t)
\]

\[
= \int_{\mathbb{R}^{n+1}} \mathcal{H}_z(x, y) \left( \frac{1}{\sqrt{\pi}} \int_0^\infty t \frac{e^{-\frac{t^2}{2}}}{s^2} d\mu(y, t) ds \right)
\]

\[
= \int_{\mathbb{R}^{n+1}} \mathcal{H}_z(x, y) d\mu(y, s).
\]

The proof reduces to show that

\[
d\mu(y, s) = \frac{1}{\sqrt{\pi}} \int_0^\infty t \frac{e^{-\frac{t^2}{2}}}{s^2} d\mu(y, t) ds
\]

is a Carleson measure on \( \mathbb{R}^{n+1} \). Indeed, for each cube \( Q \) on \( \mathbb{R}^n \),

\[
\int_0^\ell(Q) \int_Q \left( \int_0^\infty t \frac{e^{-\frac{t^2}{2}}}{s^2} d\mu(y, t) ds \right) = \int_0^\ell(Q) \int_Q \left( \int_0^\infty \frac{e^{-\frac{s^2}{2}}}{s^2} d\mu(y, t) ds \right)
\]

\[
+ \sum_{k=1}^\infty \int_0^\ell(Q) \int_{2^k(Q)} \left( \int_{2^{k-1}(Q)} t \frac{e^{-\frac{t^2}{2}}}{s^2} d\mu(y, t) ds \right)
\]

\[
\leq \int_Q d\mu(y, t) \left( \int_0^\infty \frac{e^{-\frac{s^2}{2}}}{s^2} ds \right) + \sum_{k=1}^\infty \int_{2^k(Q)} d\mu(y, t) \left( \int_0^\ell(Q) \frac{e^{-\frac{s^2}{2}}}{s^2} ds \right)
\]

\[
\leq C|Q| + C \sum_{k=1}^\infty 2^{-k(n+1)}|2^k Q|
\]

\[
\leq C|Q|.
\]

This completes the proof. \( \square \)

Finally, we apply Theorem 1.1 to discuss the dual theory of the spaces \( H^1_L(\mathbb{R}^n) \) and \( \text{BMO}_L(\mathbb{R}^n) \) associated to the Schrödinger operator. Let \( L = -\Delta + V \) with \( V \in B_q \) for some \( q \geq n \). Recall that a Hardy-type space associated to \( L \) was introduced by J. Dziubanski et al. (see [5]), defined by

\[
H^1_L(\mathbb{R}^n) := \{ f \in L^1(\mathbb{R}^n) : \mathcal{P}^* f(x) = \sup_{|x-y| \leq r} |e^{-i\sqrt{L}(x-y)}| \in L^1(\mathbb{R}^n) \}
\]

with \( \|f\|_{H^1_L(\mathbb{R}^n)} := \|\mathcal{P}^* f\|_{L^1(\mathbb{R}^n)} \).

For such class of potentials, \( H^1_L(\mathbb{R}^n) \) admits an atomic characterization, where cancellation conditions are only required for atoms with small supports. It is known that if \( V \in B_q \) for some \( q > n/2 \), then the dual space of \( H^1_L(\mathbb{R}^n) \) is \( \text{BMO}_L(\mathbb{R}^n) \), i.e.,

\[
(H^1_L(\mathbb{R}^n))^* = \text{BMO}_L(\mathbb{R}^n).
\]
The proof of \((3.29)\) was given in [5, Theorem 4]. See also [3]. Now, we can derive the half of the duality result \((3.29)\) from Theorem 1.1, see the proposition below. We note that our proof here is independent of atomic decomposition of the Hardy space \(H^1_L(\mathbb{R}^n)\) ([9, Theorem 6.2]).

**Proposition 3.6.** Suppose \(V \in B_q\) for some \(q \geq n\). Then \(\text{BMO}_L(\mathbb{R}^n)\) is in the dual space of \(H^1_L(\mathbb{R}^n)\), i.e.,

\[
\text{BMO}_L(\mathbb{R}^n) \subseteq (H^1_L(\mathbb{R}^n))^*.
\]

**Proof.** Let \(g \in \text{BMO}_L(\mathbb{R}^n)\) with compact support and \(\|g\|_{\text{BMO}_L(\mathbb{R}^n)} = 1\). By Theorem 1.1 there exist \(h \in L^\infty(\mathbb{R}^n)\) and a finite Carleson measure \(\mu\) such that \(g(x) = h(x) + S_{\mu,P}(x)\), where \(\|h\|_{L^\infty(\mathbb{R}^n)} + \|\mu\|_c \leq C(n)\). Then for every \(f \in H^1_L(\mathbb{R}^n)\),

\[
\left| \int f(x)g(x)dx \right| \leq \left| \int f(x)h(x)dx \right| + \left| \int f(x)S_{\mu,P}(x)dx \right| \leq C\|f\|_{L^1(\mathbb{R}^n)} + \int \mathcal{P}_tf(x)d\mu(x,t) \leq C\|f\|_{L^1(\mathbb{R}^n)} + C\|P^tf\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{H^1_L(\mathbb{R}^n)}.
\]

The desired result follows by the standard density argument.

**Remarks.** We would like to comment on the possibility of several generalizations and open problems related to Theorem 1.1:

1. The first one is the extension of (ii) of Theorem 1.1 for the Schrödinger operators \(-\Delta + V\) with the nonnegative potential \(V \in B_q\) for some \(q \geq n/2\), or assuming merely that the potential \(V\) of \(L\) is a locally nonnegative integrable function on \(\mathbb{R}^n\), and this question will be considered in the future.

2. The proof of Theorem 1.1 uses the Poisson semigroup property, and Green’s theorem in \(\mathbb{R}^n\). We may ask whether (ii) of Theorem 1.1 still holds for the space \(\text{BMO}_L(X)\) associated to the Poisson semigroup \(\{e^{-t\sqrt{L}}\}\) of abstract selfadjoint operators \(L\) on metric measure space \(X\) with certain proper assumptions, along which direction there have been lots of success in the last few years, see for example, in [2, 3, 4, 5, 9, 10, 11] and the references therein.

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