Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup.

Convergencia no tangencial para el semigrupo de Ornstein-Uhlenbeck

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Abstract
In this paper we are going to get the non tangential convergence, in an appropriated parabolic “gaussian cone”, of the Ornstein-Uhlenbeck semigroup in providing two proofs of this fact. One is a direct proof by using the truncated non tangential maximal function associated. The second one is obtained by using a general statement. This second proof also allows us to get a similar result for the Poisson-Hermite semigroup.

Key words and phrases: Non tangential convergence, Ornstein-Uhlenbeck semigroup, Poisson-Hermite semigroup, Hermite expansions.

Resumen
En este artículo vamos a obtener la convergencia no tangencial, en un “cono gaussiano” parabólico apropiado, del semigrupo de Ornstein-Uhlenbeck dando dos pruebas diferentes de ello. La primera es una prueba directa usando la función maximal no tangencial truncada asociada. La segunda prueba se obtiene usando principios generales. Esta última prueba nos permite obtener un resultado análogo para el semigrupo de Poisson-Hermite.

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1 Introduction

Let us consider the Gaussian measure
\[ \gamma_d(x) = \frac{e^{-|x|^2}}{\pi^d} \] with \( x \in \mathbb{R}^d \) and the Ornstein-Uhlenbeck differential operator
\[ L = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle. \] (1)

Let \( \beta = (\beta_1, ..., \beta_d) \in \mathbb{N}^d \) be a multi-index, let \( \beta! = \prod_{i=1}^{d} \beta_i! \), \( |\beta| = \sum_{i=1}^{d} \beta_i \), \( \partial_i = \frac{\partial}{\partial x_i} \), for each \( 1 \leq i \leq d \) and \( \partial^{\beta} = \partial^{\beta_1}_1 ... \partial^{\beta_d}_d \).

Let us consider the normalized Hermite polynomial of order \( \beta \), in \( d \) variables
\[ h_{\beta}(x) = \frac{1}{(2^{\beta!} \beta! \pi^d)^{1/2}} \prod_{i=1}^{d} (-1)^{\beta_i} \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}} \left( e^{-x_i^2} \right), \] (2)

then, since the one dimensional Hermite polynomials satisfies the Hermite equation, see [7], then the the normalized Hermite polynomial \( h_{\beta} \) is an eigenfunction of \( L \), with eigenvalue \( -|\beta| \),
\[ Lh_{\beta}(x) = -|\beta| h_{\beta}(x). \] (3)

Given a function \( f \in L^1(\gamma_d) \) its \( \beta \)-Fourier-Hermite coefficient is defined by
\[ \hat{f}(\beta) = \langle f, h_{\beta} \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_{\beta}(x) \gamma_d(dx). \]

Let \( C_n \) be the closed subspace of \( L^2(\gamma_d) \) generated by the linear combinations of \( \{h_{\beta} : |\beta| = n\} \). By the orthogonality of the Hermite polynomials with respect to \( \gamma_d \) it is easy to see that \( \{C_n\} \) is an orthogonal decomposition of \( L^2(\gamma_d) \),
\[ L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n \]
which is called the Wiener chaos.

Let \( J_n \) be the orthogonal projection of \( L^2(\gamma_d) \) onto \( C_n \). If \( f \) is a polynomial,
\[ J_n f = \sum_{|\beta|=n} \hat{f}(\beta) h_{\beta}. \]
The Ornstein-Uhlenbeck semigroup \( \{ T_t \}_{t \geq 0} \) is given by
\[
T_t f(x) = \frac{1}{(1 - e^{-2t})^{d/2}} \int \frac{e^{-2t|x|^2 - |y|^2 - 2e^{-t}(x,y)}}{1 - e^{-2t}} f(y) \gamma_d(dy) \\
= \frac{1}{\pi^{d/2}(1 - e^{-2t})^{d/2}} \int e^{-\frac{|y-x|^2}{1 - e^{-2t}}} f(y) dy.
\]
(4)

\( \{ T_t \}_{t \geq 0} \) is a strongly continuous Markov semigroup of contractions on \( L^p(\gamma_d) \), with infinitesimal generator \( L \). Also, by a change of variable we can write,
\[
T_t f(x) = \int f(\sqrt{1 - e^{-2t}u + e^{-t}x}) \gamma_d(du).
\]
(5)

**Definition 1.1.** The maximal function for the Ornstein-Uhlenbeck semigroup is defined as
\[
T^* f(x) = \sup_{t > 0} |T_t f(x)| \\
= \sup_{0 < r < 1} \frac{1}{\pi^{d/2}(1 - r^2)^{d/2}} \int \frac{e^{-\frac{|y-x|^2}{1 - r^2}} f(y) dy}{\gamma_d(dy)}.
\]
(6)

In [4] C. Gutiérrez and W. Urbina obtained the following inequality for the maximal function \( T^* f \),
\[
T^* f(x) \leq C_d M_{\gamma_d} f(x) + (2 \vee |x|)^d e^{\|x\|^2} \|f\|_{L^1(\gamma_d)},
\]
(7)

where \( M_{\gamma_d} f \) is the Hardy-Littlewood maximal function of \( f \) with respect to the gaussian measure \( \gamma_d \).

\[
M_{\gamma_d} f(x) = \sup_{r > 0} \frac{1}{\gamma_d(B(x, r))} \int_{B(x, r)} |f(y)| \gamma_d(dy).
\]
(8)

Unfortunately, this inequality only allows to get the weak (1,1) continuity of \( T^* f \) in the one dimensional case, \( d = 1 \), but allows to get a pointwise convergence result. Several results of this paper, see Lemma 1.1 and Theorem 1.2, use techniques contained in that paper.

If \( f \in L^1(\gamma_d) \), \( u(x, t) = T_t f(x) \) is a solution of the initial value problem
\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = Lu(x, t) \\
u(x, 0) = f(x)
\end{cases}
\]
where \( u(x,0) = f(x) \) means that

\[
\lim_{t \to 0^+} u(x,t) = f(x), \text{ a.e. } x
\]

We want to prove that this convergence is also non-tangential in the following sense. Let

\[
\Gamma^p_\gamma(x) = \left\{ (y,t) \in \mathbb{R}_{+}^{d+1} : |y - x| < t^{1/2} \land \frac{1}{|x|} \land 1 \right\}
\]

be a parabolic “gaussian cone”. We want to prove that

\[
\lim_{(y,t) \to x, (y,t) \in \Gamma^p_\gamma(x)} T_t f(y) = f(x), \text{ a.e. } x
\]

Using the Bochner subordination formula (see [6]),

\[
e^{-\lambda} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \sqrt{u} e^{-u^2/4u} du,
\]

we define the Poisson-Hermite semigroup \( \{P_t\}_{t \geq 0} \) as

\[
P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du.
\]

\( \{P_t\}_{t \geq 0} \) is also a strongly continuous semigroup on \( L^p(\gamma_d) \), with infinitesimal generator \(-(-L)^{1/2}\). From (10) we obtain, after the change of variable \( r = e^{-t^2/4u} \),

\[
P_t f(x) = \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 \frac{t \exp(t^2/4 \log r)}{(-\log r)^{3/2}} \frac{\exp(-u - 1/4 \log r)}{(1-r^2)^{d/2}} \frac{dr}{r} f(y) dy.
\]

**Definition 1.2.** The maximal function for the Poisson-Hermite semigroup is defined as

\[
P^* f(x) = \sup_{t \geq 0} |P_t f(x)|.
\]

If \( f \in L^1(\gamma_d) \), \( u(x,t) = P_t f(x) \) is solution of the initial value problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2}(x,t) = -Lu(x,t) \\
u(x,0) = f(x)
\end{cases}
\]
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where \( u(x, 0) = f(x) \) means that

\[
\lim_{t \to 0^+} u(x, t) = f(x), \text{ a.e. } x
\]

We want to prove that this convergence, for the Poisson-Hermite semigroup, is also non-tangential in the following sense. Let

\[
\Gamma_\gamma(x) = \left\{ (y, t) \in \mathbb{R}_{+}^{d+1} : |y - x| < t \wedge \frac{1}{|x|} \wedge 1 \right\},
\]

be a "gaussian cone". Also we want to prove that

\[
\lim_{(y, t) \to x, (y, t) \in \Gamma_\gamma(x)} P_t f(y) = f(x), \text{ a.e. } x
\]

In order to study the non-tangential convergence for the Ornstein-Uhlenbeck semigroup we are going to consider the following maximal function, that was defined by L. Forzani and E. Fabes [3].

**Definition 1.3.** The non tangential maximal function associated to the Ornstein-Uhlenbeck semigroup is defined as

\[
\mathcal{T}^*_{\gamma} f(x) = \sup_{(y, t) \in \Gamma_\gamma^*(x)} | T_t f(y) |.
\]

Using an inequality for a generalized maximal function, obtained by L. Forzani in [2] (for more details see [3] pag 65–73 and 88–92), it can be proved that \( \mathcal{T}^*_{\gamma} f \) is weak \((1, 1)\) and strong \((p, p)\) for \( 1 < p < \infty \), with respect to the Gaussian measure.

Actually for the non-tangential convergence for the Ornstein-Uhlenbeck semigroup it is enough to consider a "truncated" maximal function. Let

\[
\Gamma^p(x) = \left\{ (y, t) \in \mathbb{R}_{+}^{d+1} : |y - x| < t^{\frac{1}{p}}, 0 < t < \frac{1}{|x|^{2}} \wedge \frac{1}{4} \right\},
\]

be a truncated parabolic "gaussian cone".

**Definition 1.4.** The truncated non-tangential maximal function associated to the Ornstein-Uhlenbeck semigroup is defined as

\[
\mathcal{T}^* f(x) = \sup_{(y, t) \in \Gamma^p(x)} | T_t f(y) |.
\]
2 Non-tangential convergence of the Ornstein-Uhlenbeck semigroup: direct proof.

As we have mentioned already, the main result of this paper is to prove the non tangential convergence, in an appropriated parabolic “gaussian cone”, of the Ornstein-Uhlenbeck semigroup. Let us see a direct proof by using the truncated non tangential maximal function associated.

In the next lemma we are going to get a inequality better than (8) for the truncated non tangential maximal function \( T^* f \), which implies, immediately, that \( T^* f \) is weak \((1,1)\) and strong \( (p,p) \) for \( 1 < p < \infty \), with respect to the gaussian measure.

Lemma 2.1.

\[
T^* f(x) \leq C_d M_{\gamma_d} f(x),
\]

for all \( x \in \mathbb{R}^d \)

\[
 \text{Proof.}
\]

Let us take \( u(y,t) = T_t f(y) \) and without loss of generality let us assume \( f \geq 0 \).

Let \( a_o = 0 \) and \( a_j = \sqrt{j}, \ j \in \mathbb{N} \), then \( a_j < a_{j+1} \ \forall j \in \mathbb{N} \), and let us denote

\[
A_j(y,t) = \{ u \in \mathbb{R}^d : a_{j-1}(1 - e^{-2t})^{\frac{1}{2}} \leq | e^{-t} y - u | < a_j(1 - e^{-2t})^{\frac{1}{2}} \},
\]

the annulus with center \( e^{-t} y \). Now consider for each \( j \in \mathbb{N} \) the ball with center \( e^{-t} y \), and radius \( a_j(1 - e^{-2t})^{\frac{1}{2}} \) and let us denote it by \( B_j(y,t) = B(e^{-t} y, a_j(1 - e^{-2t})^{\frac{1}{2}}) \), then

\[
A_j(y,t) = B_j(y,t) \setminus B_{j-1}(y,t)
\]

\[
u(y,t) = \frac{1}{\pi^{\frac{d}{2}}(1 - e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|e^{-t} y - u|^2}{1 - e^{-2t}}} f(u) du = \frac{1}{\pi^{\frac{d}{2}}(1 - e^{-2t})^{\frac{d}{2}}} \sum_{j=1}^{\infty} \int_{A_j(y,t)} e^{-\frac{|e^{-t} y - u|^2}{1 - e^{-2t}}} f(u) du
\]

Now if \( (y,t) \in \Gamma^p(x) \) and \( | e^{-t} y - u | < a_j(1 - e^{-2t})^{\frac{1}{2}} \) then,
\[ |e^{-t}x - u| = |e^{-t}x - e^{-t}y + e^{-t}y - u| \]
\[ \leq |e^{-t}(x - y)| + |e^{-t}y - u| \]
\[ < e^{-t\frac{t^2}{2}} + a_j(1 - e^{-2t})^\frac{1}{2} \]
\[ < t^{\frac{1}{2}} + a_j(1 - e^{-2t})^\frac{1}{2} \]
\[ < (1 + a_j)(1 - e^{-2t})^\frac{1}{2}, \]

since \( t < 1 - e^{-2t} \) if \( t < 0.8 \).

Considering \( C_j(x, t) = B(e^{-t}x, (1 + a_j)(1 - e^{-2t})^\frac{1}{2}) \), we have

\[ u(y, t) \leq \frac{1}{\pi^\frac{d}{2}(1 - e^{-2t})^\frac{d}{2}} \sum_{j=1}^{\infty} e^{-a_j^2 - 1} \int_{C_j(x, t)} f(u)du \]

Now,

\[ \int_{C_j(x, t)} f(u)du = \int_{C_j(x, t)} f(u)e^{|u|^2} e^{-|u|^2} du \]
\[ = \int_{C_j(x, t)} f(u)e^{u-e^{-t}x^2 + 2e^{-t}x(u-e^{-t}x) + |e^{-t}x|^2 - |u|^2} du \]
\[ \leq e^{(1+a_j)^2(1-e^{-2t})+2(1+a_j)(1-e^{-2t})^\frac{1}{2}|e^{-t}x|+|e^{-t}x|^2} e^{-|u|^2} \int_{C_j(x, t)} f(u)e^{-|u|^2} du \]

but, \( |e^{-t}x - u| < (1 + a_j)(1 - e^{-2t})^\frac{1}{2} \) and therefore,

\[ |x - u| = |x - e^{-t}x + e^{-t}x - u| < (1 - e^{-t}) |x| + (1 + a_j)(1 - e^{-2t})^\frac{1}{2}. \]

Taking

\[ D_j(x, t) = B(x, (1 - e^{-t}) |x| + (1 + a_j)(1 - e^{-2t})^\frac{1}{2}), \]
we get

\[ \int_{C_j(x,t)} f(u)e^{-|u|^2} du \leq \int_{D_j(x,t)} f(u)e^{-|u|^2} du \]

\[ \leq M_\gamma f(x) \int_{D_j(x,t)} e^{-|u-x|^2+2x(u-u)-|x|^2} du \]

\[ \leq M_\gamma f(x)e^{-|x|^2} \int_{D_j(x,t)} e^{-|u-x|^2+2|x-u||x-u|} du \]

\[ \leq M_\gamma f(x)e^{-|x|^2+2|x||x|+(1+a_j)(1-e^{-2t})^2} \int_{D_j(x,t)} e^{-|u-x|^2} du \]

\[ = M_\gamma f(x)e^{-|x|^2+2|x||x|+(1+a_j)(1-e^{-2t})^2} \int_{E_j(x,t)} e^{-|w|^2} dw \]

where \( E_j(x, t) = B(0, (1-e^{-t}) |x| + (1+a_j)(1-e^{-2t})^{1/2}) \).

Since \( \gamma_d \) is a \( d \)-dimensional measure, and using that \( t < \frac{1}{|x|^2} \wedge \frac{1}{4} \), we get

\[ \int_{C_j(x,t)} f(u)e^{-|u|^2} du \leq C_d M_\gamma f(x)e^{-|x|^2+2|x||x|+(1+a_j)(1-e^{-2t})^{1/2}} \]

\[ \times ((1-e^{-t}) |x| + (1+a_j)(1-e^{-2t})^{1/2})^d \]

\[ = C_d M_\gamma f(x)e^{-|x|^2+2|1-e^{-t}||x|+(1+a_j)(1-e^{-2t})^{1/2}} \]

\[ \times (1-e^{-t})^{1/2} ((1-e^{-t})^{1/2} |x| + (1+a_j)(1+e^{-t})^{1/2})^d \]

\[ \leq C_d M_\gamma f(x)e^{-|x|^2+2(1-e^{-t})^{1/2}+2(1+a_j)(1-e^{-2t})^{1/4}} \]

\[ \times (1-e^{-t})^{1/2} \left( \frac{(1-e^{-t})^{1/2}}{t^{1/2}} + (1+a_j)(1+e^{-t})^{1/2} \right)^d. \]
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Therefore

\[
\int_{C_t(x,t)} f(u) du \leq e^{1+a_j)^2(1-e^{-2t})+2(1+a_j)(1-e^{-2t})^\frac{j}{2}+e^{-2t}|x|^2} \int_{C_t(x,t)} f(u)e^{-|u|^2} du
\]

\[
\leq e^{(1+a_j)^2(1-e^{-2t})+2(1+a_j)(1-e^{-2t})^\frac{j}{2}+|x|^2} e^{2(1-e^{-t})}\frac{C_d M_{\gamma_d}}{(1 e^{-t})^\frac{j}{2}}\]

\[
\times (1 - e^{-t})^\frac{j}{2} + (1 + a_j)(1 + e^{-t})^\frac{j}{2})^d
\]

\[
\leq e^{(1+a_j)^2(1-e^{-2t})+4(1+a_j)(1-e^{-2t})^\frac{j}{2}+|x|^2} e^{2(1-e^{-t})}\frac{C_d M_{\gamma_d}}{(1 e^{-t})^\frac{j}{2}}\]

\[
\times (1 - e^{-t})^\frac{j}{2} + (1 + a_j)(1 + e^{-t})^\frac{j}{2})^d
\]

\[
\leq e^{(1+a_j)^2(1-e^{-2t})+4(1+a_j)(1-e^{-2t})^\frac{j}{2}+|x|^2} (1 - e^{-t})^\frac{j}{2} + (1 + a_j)(1 + e^{-t})^\frac{j}{2})^d C_d M_{\gamma_d} f(x),
\]

since \(0 < t < \frac{1}{4}\) and \(\frac{1 - e^{-t}}{t} < 1, 1 + e^{-t} < 2\), if \(t > 0\).

Thus,

\[
u(y, t) \leq \frac{1}{\pi^{\frac{j}{2}}(1 - e^{-2t})^\frac{j}{2}} \sum_{j=1}^{\infty} e^{-a_j^2 - 1} \int_{C_t(x,t)} f(u) du
\]

\[
\leq C_d M_{\gamma_d} f(x) \frac{1}{\pi^{\frac{j}{2}}(1 + e^{-t})^\frac{j}{2}(1 - e^{-t})^\frac{j}{2}}
\]

\[
\times \sum_{j=1}^{\infty} e^{-a_j^2 - 1} e^{(1+a_j)^2(1-e^{-\frac{j}{2}})+2(1+a_j)(1-e^{-t})}^2+4(1+a_j)(1-e^{-t})^\frac{j}{2}(1 + (1 + a_j)(1 + e^{-t})^\frac{j}{2})^d
\]

\[
\leq C_d M_{\gamma_d} f(x) \frac{1}{\pi^{\frac{j}{2}} \sum_{j=1}^{\infty} e^{-a_j^2 - 1} e^{(1+a_j)^2(1-e^{-\frac{j}{2}})+4(1+a_j)(1-e^{-t})^\frac{j}{2}(1 + (1 + a_j)(1 + e^{-t})^\frac{j}{2})^d},
\]

since \(1 + e^{-t} \geq 1\). Now it is easy to see that

\[
-a_j^2 - 1 + (1 + a_j)^2(1 - e^{-\frac{j}{2}}) + 4(1 + a_j)(1 - e^{-t})^\frac{j}{2} + 2
\]

\[
= 4 + 4\sqrt{2} - e^{-\frac{j}{2}} - [-((2(1 - e^{-\frac{j}{2}}) + 4\sqrt{2}) + e^{-\frac{j}{2}} \sqrt{2})\sqrt{j},
\]

which is negative for \(j\) sufficiently big, then

\[
\sum_{j=1}^{\infty} e^{-a_j^2 - 1} e^{(1+a_j)^2(1-e^{-\frac{j}{2}})+4(1+a_j)(1-e^{-t})^\frac{j}{2}(1 + (1 + a_j)(1 + e^{-t})^\frac{j}{2})^d} < \infty.
\]
Thus $u(y, t) \leq C_d M_{\gamma_d} f(x)$ and since $(y, t) \in \Gamma^p(x)$ is arbitrary

$$T^* f(x) = \sup_{(y, t) \in \Gamma^p(x)} u(y, t) \leq C_d M_{\gamma_d} f(x).$$

Now we are ready to establish the convergence result for the Ornstein-Uhlenbeck semigroup.

**Theorem 2.2.** The Ornstein-Uhlenbeck semigroup $\{T_t f\}$ converges in $L^1(\gamma_d)$ a.e if $t \to 0^+$, for any function $f \in L^1(\gamma_d)$.

$$\lim_{t \to 0^+} u(x, t) = f(x), \text{ a.e. } x \quad (18)$$

Moreover, if $u(y, t) = T_t f(y)$ then $u(y, t)$ tends to $f(x)$ non tangentially, i.e.

$$\lim_{(y,t) \to x, (y,t) \in \Gamma^p(x)} T_t f(y) = f(x), \text{ a.e. } x. \quad (19)$$

**Proof.** We have,

$$u(y, t) = \frac{1}{\pi^{d/2}(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} \frac{-|e^{-t}y - u|^2}{1 - e^{-2t}} f(u) du,$$

considering

$$\Omega f(x) = \lim_{\alpha \to 0^+} \left[ \sup_{(y,s) \in \Gamma^p(x), 0 < s < \alpha} |u(y, s) - f(x)| \right],$$

and let us set $f(x) = f(x)\chi_{(0,k)} + f(x)(I - \chi_{(0,k)}) = f_1(x) + f_2(x)$, for $k \in \mathbb{N}$ fix.

Let us prove that

$$\Omega f(x) \leq C_d M_{\gamma_d} f_2(x), \text{ a.e.},$$

for $|x| \leq k - 1$.

Let us consider $x$ a Lebesgue’s point for $f \in L^1(\gamma_d)$, i.e. $x$ verifies

$$\lim_{r \to 0^+} \frac{1}{\gamma_d(B(x;r))} \int_{B(x;r)} |f(u) - f(x)\gamma_d(du) = 0$$

Then given $\epsilon > 0$ there exists $0 < \delta < 1$ such that

$$\frac{1}{\gamma_d(B(x;r))} \int_{B(x;r)} |f(u) - f(x)\gamma_d(du) < \epsilon,$$
for $0 < r < \delta$. Let us define $g$ as $g(u) = \begin{cases} f(u) - f(x) & \text{if } |u - x| \leq \delta \\ 0 & \text{if } |u - x| > \delta \end{cases}$

Thus $g$ depends on $x$ and $M_{\gamma} g(x) < \epsilon$.

On the other hand, since $u(y, t) - f(x) = u^1(y, t) - f_1(x) + u^2(y, t) - f_2(x)$

where

$$u^i(y, t) = \frac{1}{\pi^{\frac{d}{2}}(1 - e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{- \frac{|e^{-t}_y - u|^2}{1 - e^{-2t}}} f_i(u) du \quad i = 1, 2,$$

then we get,

$$u^1(y, t) - f_1(x) = \frac{1}{\pi^{\frac{d}{2}}(1 - e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{- \frac{|e^{-t}_y - u|^2}{1 - e^{-2t}}} (f_1(u) - f_1(x)) du$$

$$= \frac{1}{\pi^{\frac{d}{2}}(1 - e^{-2t})^{\frac{d}{2}}} \int_{|x - u| \leq \delta} e^{- \frac{|e^{-t}_y - u|^2}{1 - e^{-2t}}} (f_1(u) - f_1(x)) du$$

$$+ \frac{1}{\pi^{\frac{d}{2}}(1 - e^{-2t})^{\frac{d}{2}}} \int_{|x - u| > \delta} e^{- \frac{|e^{-t}_y - u|^2}{1 - e^{-2t}}} (f_1(u) - f_1(x)) du.$$ 

Now we have that if $|x| \leq k - 1$ and $(y, t) \in \Gamma_\gamma(x)$ with $t < \frac{1}{|x|} \wedge \frac{1}{4}$, then $(y, t) \in \Gamma^\gamma(x)$. Thus $|u - x| \leq \delta$ implies

$$|u| = |u - x + x| \leq |u - x| + |x| < \delta + k - 1 < 1 + k - 1 = k$$
and then, \( f_1(u) = f(u) \land f_1(x) = f(x) \). Therefore

\[
\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \left| \int_{|x-u| \leq \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} (f_1(u) - f_1(x)) du \right|
\]

\[
= \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \left| \int_{|x-u| \leq \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} (f(u) - f(x)) du \right|
\]

\[
= \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \left| \int_{|x-u| \leq \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} g(u) du \right|
\]

\[
\leq T^*g(x) \leq C_d M_n g(x) \leq C_d \epsilon.
\]

Now observe that if \((y,t) \in \Gamma_r(x)\) and \(t^{\frac{d}{2}} \leq \frac{\delta}{2}\) then, \(|u - x| > \delta\) implies \(\delta < |u - y| + |y - x|\) and thus

\[
\delta < |u - y| + |y - x| < |u - y| + t^{\frac{d}{2}} \leq |u - y| + \delta^{\frac{\delta}{2}},
\]

thus \(|u - y| > \delta^{\frac{\delta}{2}}\). Therefore,

\[
\frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \left| \int_{|u-x| > \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} (f_1(u) - f_1(x)) du \right|
\]

\[
\leq \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|u-x| > \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} |f_1(u)| du
\]

\[
+ \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|u-x| > \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} |f_1(x)| \left( \int_{|u-x| > \delta} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} du \right)
\]

\[
\leq \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|u-y| > \delta^{\frac{\delta}{2}}} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} |f_1(u)| du
\]

\[
+ \frac{1}{\pi^{\frac{d}{2}}(1-e^{-2t})^{\frac{d}{2}}} \int_{|u-x| > \delta^{\frac{\delta}{2}}} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} |f_1(x)| \left( \int_{|u-x| > \delta^{\frac{\delta}{2}}} \frac{-|e^{-t}y-u|^2}{1-e^{-2t}} du \right).
\]
Now, we have

\[
\frac{1}{\pi^2 (1 - e^{-2t})^2} \int_{|u-y| > \frac{\delta}{2}, |u| < k} \frac{- |e^{-t}y - u|^2}{e^{1 - e^{-2t}} |f_1(u)| du}
\]

\[
= \frac{1}{\pi^2 (1 - e^{-2t})^2} \int_{|u-y| > \frac{\delta}{2}, |u| < k} \frac{- |e^{-t}y - u|^2}{e^{1 - e^{-2t}} |f(u)| du}
\]

\[
= \frac{1}{\pi^2 (1 - e^{-2t})^2} \int_{|u-y| > \frac{\delta}{2}, |u| < k} \frac{- |e^{-t}y - u|^2}{e^{1 - e^{-2t}} e^{2|u|^2} |f(u)| |e^{-|u|^2} du}
\]

\[
\leq \frac{1}{\pi^2 (1 - e^{-2t})^2} e^{k^2} \int_{|u-y| > \frac{\delta}{2}, |u| < k} \frac{- |e^{-t}y - u|^2}{e^{1 - e^{-2t}} |f(u)| |e^{-|u|^2} du}.
\]

Then for \(0 < t < \log \left( \frac{4k + 2\delta}{4k + \delta} \right)\); \(|u - y| > \frac{\delta}{2}, \ |u| < k\) implies that

\[
|e^{-t}y - u| = |e^{-t}y - e^{-t}u + e^{-t}u - u| = |e^{-t}(y - u) - (u - e^{-t}u)|
\]

\[
\geq e^{-t} |y - u| - |u - e^{-t}u| = e^{-t} |y - u| - (1 - e^{-t}) |u|
\]

\[
\geq e^{-t} \frac{\delta}{2} - k(1 - e^{-t}) = e^{-t} \left( \frac{\delta}{2} + k \right) - k,
\]

but \(0 < t < \log \left( \frac{4k + 2\delta}{4k + \delta} \right)\) and therefore \(e^{-t} > \frac{4k + \delta}{4k + 2\delta}\), then,

\[
e^{-t} \left( \frac{\delta}{2} + k \right) - k > \frac{4k + \delta}{4k + 2\delta} \left( \frac{\delta + 2k}{2} \right) - k
\]

\[
= \frac{4k + \delta}{4(2k + \delta)} (2k + \delta) - k
\]

\[
= \frac{4k + \delta}{4} - k = \frac{4k + \delta - 4k}{4} = \frac{\delta}{4}.
\]
Therefore \( |u - y| > \frac{\delta}{2} \), \( |u| < k \) implies \( |e^{-t}y - u| > \frac{\delta}{4} \) and thus

\[
\frac{1}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} \int_{|u-y| > \frac{\delta}{2}} \frac{-|e^{-t}y - u|^2}{e^{-1 - e^{-2t}}} |f_1(u)| \, du
\leq \frac{1}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} e^{k^2} \int_{|u-y| > \frac{\delta}{2}, |u| < k} e^{-\frac{\delta^2}{16(1 - e^{-2t})}} |f(u)| e^{-|u|^2} \, du
\leq \frac{e^{-\frac{k^2}{16(1 - e^{-2t})}} + k^2}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(u)| e^{-|u|^2} \, du = \frac{e^{-\frac{k^2}{16(1 - e^{-2t})}} + k^2}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} \|f\|_{1,\gamma,d}
\]

On the other hand, taking the change of variable \( s = u - e^{-t}y \), we have

\[
\frac{1}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} |f_1(x)| \int_{|u-x| > \delta} \frac{-|e^{-t}y - u|^2}{e^{-1 - e^{-2t}}} d\mu
= \frac{|f_1(x)|}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} \int_{|x-s-e^{-t}y| > \delta} e^{-\frac{|s|^2}{1 - e^{-2t}}} ds
= \frac{|f(x)|}{\pi^{\frac{d}{2}} (1 - e^{-2t})^{\frac{d}{2}}} \int_{|x-s-e^{-t}y| > \delta} e^{-\frac{|s|^2}{1 - e^{-2t}}} ds,
\]

since \( f_1(x) = f(x) \) as \( |x| \leq k - 1 < k \).

Thus taking \( 0 < t < \log \left( \frac{k - 1 - \delta/2}{k - 1 - 3\delta/4} \right) \), \( |x - s - e^{-t}y| > \delta \) implies

\[
|s| = |s - x + e^{-t}y + x - e^{-t}y| = |s - x + e^{-t}y - (e^{-t}y - x)|
\geq |s - x + e^{-t}y| - |e^{-t}y - x|.
\]

But

\[
|e^{-t}y - x| = |e^{-t}y - e^{-t}x + e^{-t}x - x| \leq e^{-t} |y - x| + (1 - e^{-t}) |x|
\leq e^{-t}t^{\frac{d}{2}} + (1 - e^{-t})(k - 1).
\]
Thus, since \( t^{\frac{1}{2}} \leq \frac{\delta}{2} \),

\[
|s - x + e^{-t}y| - |e^{-t}y - x| > \delta - e^{-t}t^{\frac{1}{2}} - (1 - e^{-t})(k - 1)
\]

\[
\geq \delta - e^{-t}\frac{\delta}{2} - (k - 1)(1 - e^{-t})
\]

\[
= \delta - (k - 1) + (k - 1 - \frac{\delta}{2})e^{-t},
\]

and as \( 0 < t < \log \left( \frac{k - 1 - \frac{\delta}{2}}{k - 1 - 3\delta/4} \right) \), then \( e^{-t} > \frac{k - 1 - 3\delta/4}{k - 1 - \frac{\delta}{2}} \). Hence,

\[
|s| > \delta - (k - 1) + (k - 1 - \frac{\delta}{2})e^{-t}
\]

\[
> \delta - (k - 1) + (k - 1 - \frac{\delta}{2})\frac{k - 1 - 3\delta/4}{k - 1 - \frac{\delta}{2}}
\]

\[
= \delta - (k - 1) + k - 1 - 3\delta/4 = \delta - 3\delta/4 = \frac{\delta}{4}.
\]

Then \( |x - s - e^{-t}y| > \delta \) implies \( |s| > \frac{\delta}{4} \) if \( 0 < t < \log \left( \frac{k - 1 - \frac{\delta}{2}}{k - 1 - 3\delta/4} \right) \).

Therefore, taking \( w = \frac{s}{\sqrt{1 - e^{-2t}}} \),

\[
\frac{1}{\pi^\frac{3}{2}(1 - e^{-2t})^\frac{3}{4}} \int_{|u - y| > \frac{\delta}{4}} e^{\frac{-|e^{-t}y - u|^2}{1 - e^{-2t}}} |f_1(u)| \, du
\]

\[
\leq \frac{|f(x)|}{\pi^\frac{3}{2}(1 - e^{-2t})^\frac{3}{4}} \int_{|s| > \frac{\delta}{4}} e^{\frac{-|s|^2}{1 - e^{-2t}}} \, ds
\]

\[
= \frac{|f(x)|}{\pi^\frac{3}{2}} \int_{|w| > \frac{\delta}{4\sqrt{1 - e^{-2t}}}} e^{-|w|^2} \, dw.
\]

Now since, \( |x| \leq k - 1 < k \), then \( f_2(x) = 0 \). Hence

\[
|u^2(y, t) - f_2(x)| = |u^2(y, t)| \leq T^*f_2(x) \leq C_dM_{\gamma_d}f_2(x)
\]
for \((y, t) \in \Gamma^p(x)\). Therefore,

\[
| u(y, t) - f(x) | \leq | u^1(y, t) - f_1(x) | + | u^2(y, t) - f_2(x) |
\]

\[
= | u^1(y, t) - f_1(x) | + | u^2(y, t) |
\]

\[
\leq C_d \epsilon + \frac{e^{\frac{-d(x_2 - t_2)^2}{2(1 - e^{-2t})^2}}}{\pi} \| f \|_{1, \gamma_d} + \frac{| f(x) |}{\pi^{\frac{d}{2}}} \int_{|w| > \frac{d}{\sqrt{1 + 2t}}} e^{-|w|^2} dw
\]

\[
+ C_d M_{\gamma_d} f_2(x),
\]

if \((y, t) \in \Gamma^p(x)\) and

\[
0 < t < \min \left\{ \log \left( \frac{4k + 2\delta}{4k + \delta} \right), \log \left( \frac{k - 1 - \delta/2}{k - 1 - 3\delta/4} \right), \frac{1}{|x|^2} \land \frac{1}{4} \right\} =: \alpha.
\]

Thus taking supremum on \((y, t) \in \Gamma^p(x)\), \(0 < t < \alpha < a\) and then taking \(\alpha \to 0^+\) we obtain,

\[
\Omega f(x) \leq C_d (\epsilon + M_{\gamma_d} f_2(x))
\]

for all \(\epsilon > 0\) and almost every \(x\) with \(|x| \leq k - 1\).

Given \(\epsilon > 0\), let us take \(k\) sufficiently large such that

\[
\| f_2 \|_{1, \gamma_d} \leq C_d \epsilon^2,
\]

then by the estimation of \(\Omega\) and the weak continuity of \(M_{\gamma_d}\) we get

\[
\gamma_d(\{x \in \mathbb{R}^d : |x| \leq k - 1, \Omega f(x) > \epsilon \}) \leq \epsilon
\]

and that implies that \(\Omega f(x) = 0\) a.e. \(\square\)

A similar proof for the Poisson-Hermite semigroup, using the non-tangential maximal function defined as

\[
P_{\gamma}^* f(x) = \sup_{(y, t) \in \Gamma_\gamma(x)} | P_t f(y) |,
\]

and its analogous truncated version, should be possible but it has some technical difficulties that we have been unable to overcome so far.
3 Non-tangential convergence of the Ornstein-Uhlenbeck semigroup: alternative proof.

Let us now prove a general statement for families of linear operators that will allow us to get a simpler proof of the non-tangential convergence, both for the Ornstein-Uhlenbeck semigroup and for the Poisson-Hermite semigroup. It is a generalization of Theorem 2.2 of J. Duoandikoetxea's book [1].

**Theorem 3.1.** Let \( \{T_t\}_{t \geq 0} \) be a family of linear operators on \( L^p(\mathbb{R}^d, \mu) \) and for any \( x \in \mathbb{R}^d \), let \( \Gamma(x) \) be a subset of \( \mathbb{R}^{d+1}_+ \) such that \( x \) is in \( (\Gamma(x))' \), that is to say \( x \) is an accumulation point of \( \Gamma(x) \). Let us define

\[
T^*f(x) = \sup_{(y, t) \in \Gamma(x)} \{|T_t f(y)| : (y, t) \in \Gamma(x)\},
\]

for \( f \in L^p(\mathbb{R}^d, \mu) \) and \( x \in \mathbb{R}^d \). If \( T^* \) is weak \((p, q)\) then the set

\[
S = \left\{ f \in L^p(\mathbb{R}^d, \mu) : \lim_{(y, t) \to x, (y, t) \in \Gamma(x)} T_t f(y) = f(x) \text{ a.e.} \right\}
\]

is closed in \( L^p(\mathbb{R}^d, \mu) \).

**Proof.** Let us consider a sequence \( (f_n) \) in \( S \) such that \( f_n \to f \) en \( L^p(\mathbb{R}^d, \mu) \), then

\[
|T_t f(y) - f(x)| - |T_t f_n(y) - f_n(x)| \leq |T_t(f - f_n)(y) - (f(x) - f_n(x))|,
\]

this implies that for each \( n \in \mathbb{N} \), for almost every \( x \),

\[
\limsup_{(y, t) \to x, (y, t) \in \Gamma(x)} |T_t f(y) - f(x)| \leq \limsup_{(y, t) \to x, (y, t) \in \Gamma(x)} |T_t(f - f_n)(y) - (f(x) - f_n(x))| \\
\leq \limsup_{(y, t) \to x, (y, t) \in \Gamma(x)} |T_t(f - f_n)(y)| + \limsup_{(y, t) \to x, (y, t) \in \Gamma(x)} |f(x) - f_n(x)| \\
\leq T^*(f - f_n)(x) + |f(x) - f_n(x)|.
\]

On the other hand, if we know that \( a \leq b + c \) then \( a > \lambda \) implies \( b > \frac{\lambda}{2} \lor c > \frac{\lambda}{2} \).

Then, given \( \lambda > 0 \) and \( n \in \mathbb{N} \), \( \limsup_{(y, t) \to x, (y, t) \in \Gamma(x)} |T_t f(y) - f(x)| > \lambda \) implies

\[
T^*(f - f_n)(x) > \frac{\lambda}{2} \lor |f(x) - f_n(x)| > \frac{\lambda}{2} \text{ a.e.}
\]
and this implies that, given \( \lambda > 0 \),

\[
\mu \left( \left\{ x : \limsup_{(y,t) \to x, (y,t) \in \Gamma(x)} |T_t f(y) - f(x)| > \lambda \right\} \right)
\leq \mu \left( \left\{ x : T^*(f - f_n)(x) > \frac{\lambda}{2} \right\} \right)
\leq \mu \left( \left\{ x : f(x) - f_n(x) > \frac{\lambda}{2} \right\} \right)
\leq \left( \frac{2C}{\lambda} \|f - f_n\|_p \right)^q + \left( \frac{2}{\lambda} \|f - f_n\|_p \right)^p,
\]

for all \( n \in \mathbb{N} \). Therefore,

\[
\mu \left( \left\{ x : \limsup_{(y,t) \to x, (y,t) \in \Gamma(x)} |T_t f(y) - f(x)| > \lambda \right\} \right) = 0
\]

and since this is true for all \( \lambda > 0 \), we get that

\[
\mu \left( \left\{ x : \limsup_{(y,t) \to x, (y,t) \in \Gamma(x)} |T_t f(y) - f(x)| > 0 \right\} \right) = 0,
\]

as

\[
\left\{ x : \limsup_{(y,t) \to x, (y,t) \in \Gamma(x)} |T_t f(y) - f(x)| > 0 \right\}
= \bigcup_{n=1}^{\infty} \left\{ x : \limsup_{(y,t) \to x, (y,t) \in \Gamma(x)} |T_t f(y) - f(x)| > \frac{1}{n} \right\}.
\]

Thus

\[
\lim_{(y,t) \to x, (y,t) \in \Gamma(x)} T_t f(y) = f(x) \text{ a.e.}
\]

and then \( f \in S \). Therefore \( S \) is a closed set in \( L^p(\mathbb{R}^d, \mu) \).

Finally, as a consequence of this result, we get the non-tangential convergence for the Ornstein-Uhlenbeck semigroup \( \{T_t\}_{t>0} \) and the Poisson-Hermite semigroup \( \{P_t\}_{t>0} \). \( \square \)
Corollary 3.2. The Ornstein-Uhlenbeck semigroup \( \{T_t\}_{t>0} \) and the Poisson-Hermite semigroup \( \{P_t\}_{t>0} \) verify

\[
\lim_{(y,t) \to x, (y,t) \in \Gamma^p_{\gamma}(x)} T_t f(y) = f(x) \ a.e. \ x,
\]

\[
\lim_{(y,t) \to x, (y,t) \in \Gamma^p_{\gamma}(x)} P_t f(y) = f(x) \ a.e. \ x.
\]

Proof. Let us discuss the proof for the the Ornstein-Uhlenbeck semigroup \( \{T_t\}_{t>0} \). The proof for the Poisson-Hermite semigroup \( \{P_t\}_{t>0} \) is totally similar.

It is immediate that for any given polynomial \( f(x) = \sum_{k=0}^{n} J_k f(x) \), since 

\[
T_t f(y) = T_t(\sum_{k=0}^{n} J_k f(y)) = \sum_{k=0}^{n} e^{-tk} J_k f(y),
\]

we have the non-tangential convergence,

\[
\lim_{(y,t) \to x, (y,t) \in \Gamma^p_{\gamma}(x)} T_t f(y) = f(x),
\]

for all \( x \in \mathbb{R}^d \). Now considering the set

\[
S = \left\{ f \in L^p(\gamma_d) : \lim_{(y,t) \to x, (y,t) \in \Gamma^p_{\gamma}(x)} T_t f(y) = f(x) \ a.e. \right\},
\]

corresponding to the Ornstein-Uhlenbeck semigroup, then the polynomials are in \( S \). From the previous result, since non-tangential maximal function for the Ornstein-Uhlenbeck semigroup \( T_t^* f \) is weak \((1,1)\) with respect to the Gaussian measure, we get that the set \( S \) is closed in \( L^p(\gamma_d) \) and since the polynomials are dense in \( L^p(\gamma_d) \) then \( S = L^p(\gamma_d) \).

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