NULLIFICATION WRITHE AND
CHIRALITY OF ALTERNATING LINKS

CORINNE CERF*
Département de Mathématiques, CP 216
Université Libre de Bruxelles
Boulevard du Triomphe
B-1050 Bruxelles, Belgium
(mailing address)
and
Department of Chemistry
Princeton University
Princeton, NJ 08544, U.S.A.

Published in J. Knot Theory Ramifications 6 (1997) 621-632.

ABSTRACT

In this paper, we show how to split the writhe of reduced projections of oriented alternating links into two parts, called the nullification writhe \( w_x \), and the remaining writhe \( w_y \), such that the sum of these quantities equals the writhe \( w \) and each quantity remains an invariant of isotopy. The chirality of oriented alternating links can be detected by a non-zero \( w_x \) or \( w_y \), which constitutes an improvement compared to the detection of chirality by a non-zero \( w \). An interesting corollary is that all oriented alternating non-split links with an even number of components are chiral, a result that also follows from properties of the Conway polynomial.

Keywords: Link, chirality, writhe, nullification writhe, remaining writhe.

1. Introduction

Determining whether knots and links are equivalent or not to their mirror images, i.e., whether they are achiral or chiral, has been a longstanding question in knot theory. Tait pioneered this field in the nineteenth century by developing empirical methods to this end [1]. Subsequently, more and more sophisticated methods capable of detecting the chirality of links (essentially numerical and polynomial invariants) were elaborated by numerous mathematicians. A critical review of these methods can be found in [2, 3].

In 1991, Sola introduced a numerical invariant for oriented alternating links, called the nullification number [4]. This invariant is, however, insensitive to chirality.

*The author is “Chargé de Recherches” of the Belgian “Fonds National de la Recherche Scientifique”. E-mail: ccerf@ulb.ac.be
We show here that by a modification of the nullification number, a new numerical invariant capable of detecting chirality can be derived. It is called the nullification writhe.

2. Concept of nullification

Let us begin with some definitions.

**Definition 1.** The nullification of a crossing of an oriented link projected on $S^2$ is the process described in Fig. 1, where $\times$ means $\times$ or $\times$.

![Fig. 1](image)

**Definition 2.** The nullification of an oriented link projection on $S^2$ consists in nullifying a series of crossings until an unknot (or unlink in the case of a split link) is reached, while preventing at each step the disconnection of the link (or link components in the case of a split link).

Let us observe that when the stage of the unknot (or unlink) is reached, all remaining crossings are nugatory crossings, so that nullifying any one of them would disconnect the unknot (or add a disconnected component to the unlink). Conversely, any projection composed of nugatory crossings only, is necessarily an unknot or an unlink.

We will call the set of crossings so nullified the nullification set. Sola proves that the cardinality of this set, called the nullification number and denoted by $o$, is well defined and invariant for reduced alternating link projections [4]:

$$o(K) = n(K) - s(K) + 1 \quad (2.1)$$

where $n(K)$ is the number of crossings of a reduced alternating projection of link $K$ and $s(K)$ is the number of Seifert circles of a reduced alternating projection of link $K$ (obtained by simultaneously nullifying all crossings).

We would like to prove that the writhe of the nullification set (the nullification writhe) is also well defined and invariant for reduced alternating link projections.

**Definition 3.** The nullification writhe $w_x$ of an oriented link projection on $S^2$ is the sum of the signs of the crossings extracted during the nullification of the projection.
**Definition 4.** The remaining writhe \( w_y \) of an oriented link projection on \( S^2 \) is the sum of the signs of the crossings of the unknot (or unlink in the case of a split link) remaining after nullification of the projection.

It follows immediately that for a given projection of a link \( K \),

\[
w(K) = w_x(K) + w_y(K).
\]  

(2.2)

We will make use of two other descriptions of the link projection. The first one is called *Seifert diagram*. It is obtained by simultaneously nullifying all crossings (this gives rise to a collection of Seifert circles) and by replacing each crossing of the link projection with an arm, called *connection*, connecting both involved Seifert circles. Positive and negative connections are distinguished and represented by a solid bar and a dashed bar, respectively. The second description is called *Seifert graph*. It is obtained from the Seifert diagram by crushing each Seifert circle to a point, that becomes a vertex of the graph. Connections become positive and negative edges of the graph. An example is shown in Fig. 2.

![Projection Seifert diagram Seifert graph](image)

Fig. 2

The process of nullification of a link projection transposed in the Seifert graph representation consists in removing the maximum number of edges without disconnecting the graph (or without adding any additional disconnected component to the graph, in the case of a split link). This process leaves a spanning tree (respectively, a spanning forest). The starting graph possesses \( s \) vertices (the number of Seifert circles) and \( n \) edges (the number of crossings). It is well known that the number of edges of a spanning forest for a split graph with \( s \) vertices and \( k \) components is \( s - k \) (see [5] p. 45 for example). It follows that the nullification number, i.e., the number of edges that have been removed, is \( n - s + k \), which confirms Sola’s result (Sola assumes implicitly that the link is non-split, so \( k = 1 \)).

### 3. Well-definedness

In order to prove that \( w_x \) and \( w_y \) are well defined, we will need the following lemma:
Lemma 5. In the Seifert diagram corresponding to an alternating link projection, all connections on the same side of a Seifert circle have the same sign, and all connections on opposite sides of a Seifert circle have opposite signs.

Proof. Two adjacent connections on the same side of a Seifert circle can never have opposite signs, because the corresponding link projection would be non-alternating (Fig. 3). Two adjacent connections, one on one side and one on the other side of a Seifert circle can never have the same sign, because the corresponding link projection would be non-alternating (Fig. 4).

We can now prove

Proposition 6. The nullification writhe and the remaining writhe of a reduced projection of an oriented alternating link are independent of the choice of the nullification set.

Proof. We know that any two nullification sets of a reduced projection of an oriented alternating link have the same cardinality. Do they have the same writhe (nullification writhe)? In the Seifert graph description, the nullification set contains all edges to remove in order to obtain a spanning tree (or spanning forest if the link is split), that is, all edges but one between each pair of vertices linked by multiple edges, and one edge from each cycle of edges. By Lemma 5 all multiple edges in
a Seifert graph (corresponding to multiple connections in a Seifert diagram) have
the same sign, so keeping any one of them will add the same contribution to the
nullification writhe. As far as cycles are concerned, because of Lemma 5 and of the
Jordan Curve Theorem (see [6] for example), all cycles consist of edges having
the same sign, so removing any edge will add the same contribution to the nullification
writhe. The nullification writhe is thus independent of the choice of the nullification
set. From Eq. 2.2 (or simply for complementarity reasons), it follows that the
remaining writhe is also independent of the choice of the nullification set. □

4. Invariance

Let us begin by proving the following lemma that will simplify the proof of
invariance:

Lemma 7. Given any crossing of a reduced projection of an oriented alternating
link, there exists a nullification set containing this crossing.

Proof. The fact that the projection is reduced means that there is no nugatory
crossing in it. In the Seifert diagram description, this means that there is no
connection whose removal would disconnect the diagram (or add a disconnected
component in the case of a split link). In the nullification process, we may thus
choose to begin by the extraction of any crossing, that will become part of the
nullification set. □

Proposition 8. Any two reduced projections of an oriented alternating link
have the same nullification writhe and the same remaining writhe.

Proof. As asserted by the Tait Flyping Conjecture [1] proved by Menasco and
Thistlethwaite [7], any two reduced projections of an alternating link are related by
a sequence of moves called flypes. It remains to prove that two oriented reduced
alternating projections related by a flype have the same nullification writhe and the
same remaining writhe.

Two projections related by a flype can be represented as in Fig. 5, where R and
S are tangles, i.e., parts of a link projection with four emerging arcs. The depicted
crossing may be positive or negative. The sign of the crossing is unchanged by
the flyping operation. Let us now represent this operation in the description of
Seifert diagrams. Depending on the orientation, two situations occur, shown in
Figs. 6 and 8. The highlighted connection is represented as a positive connection,
but it could also be a negative connection, for the same reason as above.

Let us analyze the first situation (Fig. 6). The flype transforms each connection
into a connection of the same sign. We now nullify both diagrams of Fig. 6.
For the left-hand diagram, we choose a nullification set containing the highlighted
connection (which is legitimate because of Lemma 7). It remains a connected sum
of two tangles $R'$ and $S'$ (Fig. 5 left). Knowing that it is an unknot (or an unlink),
tangles $R'$ and $S'$ are necessarily unknots (or unlinks). Then, for the diagram on
On the right, we choose as nullification set the image of the left-hand nullification set. Is this really a nullification set? What is left is a connected sum of the same tangents $R'$ and $S'$ with a different relative orientation (Fig. 7, right). Since $R'$ and $S'$ are unknots (or unlinks), the right-hand diagram is an unknot (or unlink). This completes the proof for the first situation.

Let us now analyze the second situation (Fig. 8). Again, each connection is transformed into a connection of the same sign. We nullify both diagrams (Fig. 3). For the left-hand diagram, we choose a nullification set containing the highlighted connection (which is legitimate because of Lemma 7). For the right-hand diagram, we use the image of this set.

Four subcases arise, represented in Figs. 10-13. In the first two subcases (Figs. 10-11), we know that at the end of the nullification process, there must remain only one connection (direct or indirect) between both displayed circles. One of $\{R', S'\}$ may thus be split into 2 tangles (horizontal splitting in the first subcase, vertical splitting in the second subcase). The diagrams become connected sums of 3 tangles. Because the left-hand diagrams are unknots (or unlinks), the 3 tangles are necessarily unknots (or unlinks). Therefore, the right-hand diagrams, that are connected sums of the same 3 tangles with different orientations, also are unknots (or unlinks). The image of a nullification set is thus a nullification set.

Let us finally consider the last two subcases (Figs. 12-13). By the definition of a connection, there cannot exist a connection connecting a circle to itself. Therefore, both $R'$ and $S'$ may be split into 2 tangles (in each subcase, one horizontal and one vertical splitting). Let us split one tangle only. This leads again to connected sums of 3 tangles. Because the left-hand diagrams are unknots (or unlinks), the 3 tangles are necessarily unknots (or unlinks). Therefore, the right-hand diagrams, that are connected sums of the same 3 tangles with different orientations, also are unknots (or unlinks). The image of a nullification set is thus a nullification set.

This completes the proof of Proposition 8. □

5. Detection of Chirality

As was known for long, an easy way to detect the chirality of an oriented alternating link is to look at the writhe of a reduced projection of the link. If it is non-zero, then the link is chiral. Here is a very simple proof of this fact.

**Proposition 9.** If an oriented alternating link $K$ is achiral, then the writhe of a reduced projection of $K$ is equal to zero.

**Proof.** Flypes preserve the writhe of reduced projections of oriented alternating links. If a link is achiral, the proven Tait Flyping Conjecture says that its mirror image can be obtained by a series of flypes. Thus a projection and its mirror image have at the same time identical writhes (because of the Tait Flyping Conjecture) and opposite writhes (because in the mirror image, all crossings have changed sign). The writhe of the link must then be equal to zero. □
The chirality of an alternating link can thus be detected by a non-zero writhe. Unfortunately, this is not a universal way to detect chirality: there exist many chiral links whose writhe is equal to zero too.

The same problem of zero-writhe chiral links has been encountered in a previous study aimed at partitioning chiral knots and links into $D$ and $L$ classes, and was overcome by defining a property called *writhe profile* [8, 9]. This property is of no help here, because it is a “chirality-classifier” and not a “chirality-detector”. The chirality of the link is a prerequisite in order to apply the method. In contrast, the new invariants introduced in this paper, namely the nullification writhe $w_x$ and the remaining writhe $w_y$, are capable of detecting chirality, as will be proved below.

**Proposition 10.** If $K$ is an oriented alternating link represented by a reduced projection and if $K^*$ is its mirror image, then $w_x(K) = -w_x(K^*)$ and $w_y(K) = -w_y(K^*)$.

**Proof.** Given a nullification set for $K$, let us choose for $K^*$ the corresponding nullification set, i.e., the crossings located at the same position and having the opposite sign. The writhes of these two sets are obviously opposite, so $w_x(K) = -w_x(K^*)$. Similarly, the sums of the signs of the remaining crossings in $K$ and $K^*$ are opposite, so $w_y(K) = -w_y(K^*)$.

**Corollary 11.** If an oriented alternating link $K$ is achiral, then $w_x(K)$ and $w_y(K)$, computed on a reduced projection of $K$, are equal to zero.

**Proof.** As we proved in Proposition 8 flypes preserve $w_x$ and $w_y$ for reduced projections of oriented alternating links. If a link $K$ is achiral, the Tait Flyping Conjecture says that its mirror image, $K^*$, can be obtained by a series of flypes. Thus $K$ and $K^*$ have at the same time identical $w_x$ (respectively, $w_y$) because of the Tait Flyping Conjecture, and opposite $w_x$ (respectively, $w_y$) because of Proposition 10. It implies that $w_x(K) = 0$ and $w_y(K) = 0$.

We have thus the following implication: if $w_x(K) \neq 0$ or $w_y(K) \neq 0$, then $K$ is chiral. What are the consequences of this? For each oriented alternating link represented by a reduced projection, the writhe $w$ is split into two parts $w_x$ and $w_y$. If $w$ is different from zero, $w_x$ and/or $w_y$ are different from zero and the link is chiral. In this case, $w_x$ and $w_y$ are thus no better than $w$ in detecting chirality. In contrast, if $w$ equals zero but the link is chiral, then in some cases $w_x$ and $w_y$ might be different from zero ($w_y = -w_x$), which would give a way to detect the chirality of the link even if it is not detectable by $w$.

We have applied this procedure to all chiral oriented alternating “classical” knots and links, i.e., prime knots with up to 10 crossings and non-split prime links with up to 9 crossings and 4 components, that have a writhe of zero. The results are shown in Tables 1 and 2 and confirm that chirality is detected in several cases where it was not detectable by the writhe only.
Table 1. Chiral oriented alternating prime knots with up to 10 crossings, and \( w = 0 \) (\( w_y = -w_x \)).

| knot\(^{1,2} \) | \( 8_4 \) | \( 10_{15} \) | \( 10_{19} \) | \( 10_{31} \) | \( 10_{42} \) | \( 10_{48} \) | \( 10_{52} \) | \( 10_{54} \) | \( 10_{71} \) | \( 10_{93} \) | \( 10_{104} \) | \( 10_{108} \) |
|----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( w_x \)     | \(-2\) | \(+2\) | \(-2\) | \(0\)  | \(0\)  | \(+2\) | \(+2\) | \(0\)  | \(0\)  | \(-2\) | \(0\)  | \(+2\) |

Table 2. Chiral oriented alternating non-split prime links with up to 9 crossings and 4 components, and \( w = 0 \) (\( w_y = -w_x \)).

| link\(^{1,2} \) | \( 8_6^{\pm} \) | \( 8_7^{\pm} \) | \( 8_7^{\mp} \) | \( 8_8^{\pm} \) | \( 8_9^{\pm} \) | \( 8_9^{\mp} \) | \( 8_9^{\pm} \) | \( 8_9^{++} \) | \( 8_9^{+-} \) | \( 8_9^{++} \) | \( 8_9^{+-} \) |
|----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( w_x \)     | \(-1\) | \(+1\) | \(-1\) | \(+1\) | \(-1\) | \(-2\) | \(0\)  | \(-1\) | \(-1\) | \(-1\) | \(-1\) |

6. Links with an even number of components

This last section will be devoted to the case of oriented alternating non-split links with an even number of components, for which the properties of the nullification writhe lead to a very nice proposition:

**Proposition 12.** All oriented alternating non-split links with an even number of components are chiral.

**Proof.** Let us nullify an oriented alternating non-split link \( K \) with an even number of components. Each step in the nullification process changes the parity of the number of components of the link, as can be easily understood by looking at Fig. 14 where \( \times \) means \( \checkmark \) or \( \times \). If the two arrows of a crossing are part of the same component, the nullification of the crossing increases by one the number of components. If the two arrows belong to two different components, the nullification of the crossing decreases by one the number of components. On the other hand, at the end of the nullification process, we get an unknot (with one component) because \( K \) is non-split. This implies that the nullification process contains an odd number of steps. Since each step contributes \( \pm 1 \) to \( w_x \), \( w_x \) can never be equal to zero, which implies that \( K \) is chiral. \( \square \)

John Conway pointed out that this also follows from properties of his \( \nabla \) polynomial \(^{12} \), which satisfies

\[
\nabla(K^*) = (-1)^{c+1} \nabla(K),
\]

\(^{1}\)The enantiomorphs are those represented in \(^{10} \), oriented either way.

\(^{2}\)This list should also contain the non-invertible oriented knots whose non-oriented version is achiral. Because of non-invertibility, these knots are chiral when oriented. They have been omitted since the nullification writhe is insensitive to non-invertibility.

\(^{3}\)The enantiomorphs are those represented in \(^{11} \), using the same convention of labeling and orientation of components.
where $K^*$ is the mirror image of $K$ and $c$ is the number of components of $K$. This implies that an oriented link with an even number of components can only be achiral if its $\nabla$ polynomial vanishes identically. This includes our corollary, since it is known that all oriented alternating non-split links have non-zero Alexander polynomials and thus non-zero $\nabla$ polynomials ($\nabla$ polynomials are normalized Alexander polynomials).

Acknowledgements

Anna-Barbara Berger, Ines Stassen, and Urs Burri are gratefully acknowledged for a crucial discussion held in Bern. This work has benefited from many fruitful conversations with Chengzhi Liang and Kurt Mislow. Also Erica Flapan is heartily thanked for her critical reading of the manuscript and her invaluable comments. Finally, I would like to express my gratitude to John Conway for his helpful suggestions, and to one of the referees for his/her detailed and illuminating review. This work was supported in part by Grant No CHE-9401774 to K. Mislow.

References

[1] P. G. Tait, On knots I, II, III, in: Scientific Papers Vol. I, Cambridge University Press, London (1898) 273-347.

[2] E. Flapan, Topological techniques to detect chirality, in: New Developments in Molec-
ular Chirality, ed. P. G. Mezey, Kluwer Academic Publishers, Dordrecht (1991) 209-239.

[3] C. Liang and K. Mislow, Topological chirality and achirality of links, J. Math. Chem. 18 (1995) 1-24.

[4] D. Sola, Nullification number and flyping conjecture, Rend. Sem. Mat. Univ. Padova 86 (1991) 1-16.

[5] R. J. Wilson, Introduction to Graph Theory, Longman and Wiley, New York (1985)

[6] M. A. Armstrong, Basic Topology, Springer-Verlag, New York (1983)

[7] W. W. Menasco and M. B. Thistlethwaite, The Tait flyping conjecture, Bull. Am. Math. Soc. 25 (1991) 403-412.

[8] C. Liang and K. Mislow, A left-right classification of topologically chiral knots, J. Math. Chem. 15 (1994) 35-62.

[9] C. Liang, C. Cerf and K. Mislow, Specification of chirality for links and knots, J. Math. Chem. 19 (1996) 241-263.

[10] D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, 1976; second printing with corrections: Publish or Perish, Houston, 1990) Appendix C: Table of knots and links, 388-429.

[11] H. Doll and J. Hoste, A tabulation of oriented links, Math. Comput. 57 (1991) 747-761. Links are depicted in the Appendix A of the microfiche supplement.

[12] J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, in: Computational Problems in Abstract Algebra, ed. J. Leech, Pergamon Press, Oxford (1970) 329-358.

[13] R. H. Crowell, Genus of alternating link types, Ann. of Math. 69 (1959) 258-275.