Light-mediated quantum phase transition and manipulations of the quantum states of arrayed two-level atoms

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Abstract. In a quantum electrodynamics (QED) system of arrayed two-level atoms interacting with light, because the energy of the photon is around the spacing between two atomic energy levels, the photon will be absorbed and is not in the propagating mode but the attenuated mode. Our theoretical study shows that coherent phasing of atomic states occurs when the photons are in the attenuated mode. The existence of this ordered state is a result of a quantum phase transition induced by the mediation of the attenuated photonic field. By tuning the intensity of the light, the ordered state can be manipulated so that two-level atoms can be in an arbitrary uniform linear combination of the single-atom ground state and the excited state. Potential applications of this phenomenon are quantum computation, lasing physics, optical lattices and related subjects in nanotechnology. Similar phase transitions induced by the mediation of transmuted intermediate boson-like phonons, magnons or other fields may also occur in condensed-matter systems.

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Due to current research into semiconducting [1–4], superconducting [5–7], and other condensed-matter systems [8–10], matter–light interactions have attracted considerable interest recently. Many experimental and theoretical works are currently studying the interactions between atoms and photons, investigating fundamental physics, and aiming to perform certain operations on atomic and/or photonic states. New research across several fields, including artificial atom lasing in quantum optics [11], quantum bits in quantum computation [6, 7, 10], and optical lattices, etc, has been stimulated by manipulation of electronic states by the emission and absorption of photons. Also, the physics of one artificial atom coupled with a photonic field has been intensively explored [11, 12]. Recently, a growing number of studies have been conducted to explore the interactions between a small number of artificial atoms and a photonic field [10, 13].

We therefore consider a situation in which a photonic field generated from an external source interacts with \( N \) two-level atoms on a one-dimensional lattice. The atomic states can be changed by the emission and absorption of photons. When the photons are absorbed by the atoms, they are annihilated and this would have a great impact on the quantum states of the photonic field so that its dispersion relation would be different from that of free photons. It can be seen that as the energy of the photon is around the spacing between two atomic energy levels, the photon will be absorbed and it is not in the propagating mode but in the attenuated mode. Therefore, an energy gap will appear in the photonic dispersion relation. Due to the close interactions and phase couplings between the photonic field and atoms, the influence of the two modes of photonic field on atomic states will be quite different.

In this paper, we report that quantum state ordering of two-level atoms in an array occurs when illuminated by an electromagnetic (EM) field at certain frequencies in the attenuated mode. Without the light, due to quantum fluctuations and low dimensionality, the ordering of states of one-dimensional two-level atoms would not occur even if we lower the temperature to 0 K to remove thermal fluctuations. It is therefore a quantum phase transition induced by the mediation of an attenuated photonic field between arrayed two-level atoms. This may be interesting in fundamental research into condensed-matter physics, especially the collective phenomena in artificial atoms.

On the other hand, the photonic field in the arrayed two-level atoms can be controlled to be either attenuated or propagating, and the corresponding atomic states are either ordered or disordered. Further, we will show that the ordered quantum states of two-level atoms can be manipulated to be an arbitrary combination of the ground state and the excited state of single
atoms by adjusting the intensity of the EM field. This may be worth further investigation and research into artificial atoms, quantum computation, photo-electronics and other related fields.

Similar phase transitions in condensed-matter systems induced by the mediation of transmuted intermediate bosonlike phonons, magnons, or other fields may also be worth investigating.

2. Model

We consider a model of $N$ two-level atoms ($\sigma_i$'s) on a linear lattice ($x$-direction) interacting with a quantized EM field $\hat{A}$ through the Jaynes–Cummings Hamiltonian. To avoid unnecessary complications, we restrict our studies to zero temperature throughout this paper. Assuming that the EM wave is moving in the $x$-direction and is uniform along $\hat{y}$, $\hat{z}$, we can then write the vector potential $\hat{A}$ as $A(x) = (0, A_x(x), A_z(x))$, by adopting the radiation gauge ($\nabla \cdot \hat{A} = 0$). The Hamiltonian $H_{\text{em}}$ for the EM field is

$$H_{\text{em}} = \frac{1}{2} \int \text{d}x \left[ \frac{\hbar^2}{2 m} (\hat{A}^x + (\nabla \times \hat{A})^2) \right] = \int \text{d}x \left( \hat{A}^x \hat{A}^+ + \nabla \hat{A}^+ \cdot \nabla \hat{A} \right),$$

where the field operators $A(x)$ and $A^+(x)$ are defined as $A = (A_x + iA_z)/\sqrt{2}$, $A^+ = (A_x - iA_z)/\sqrt{2}$ and which describe the annihilation and creation of one photon, respectively. To describe transitions between the ground state $|g\rangle_j$ and the excited state $|e\rangle_j$ of the two-level atom on the $j$th site, the raising and lowering operators are defined,

$$\sigma_j^+ \equiv |e\rangle_j \langle g|_j, \quad \sigma_j^- \equiv |g\rangle_j \langle e|_j.$$

In terms of the photonic field operators $A$ and $A^+$, and the raising and lowering operators for the two-level atoms, $\sigma^+$ and $\sigma^-$, the Hamiltonian of the whole system can be expressed as,

$$H = H_{\text{em}}(A^+, A) + H_{\text{2LS}}(\{\sigma_j\}) + H_{\text{int}}(A, \{\sigma\}),$$

$$H_{\text{2LS}}(\{\sigma_j\}) = \sum_j (v_j + i\delta) \frac{1 + \sigma_{jz}}{2}, \quad \sigma_{jz} \equiv |e\rangle_j \langle e|_j - |g\rangle_j \langle g|_j,$$

$$H_{\text{int}}(A, \{\sigma\}) = g \sum_j [\sigma_j^+ A(x_j) + A^+(x_j) \sigma_j^-],$$

where $H_{\text{2LS}}(\{\sigma_j\})$ is the Hamiltonian of $N$ two-level atoms with excitation energy $v_j$ of each atom (ground state energy is 0); and $H_{\text{int}}(A, \{\sigma\})$ is the Jaynes–Cummings type Hamiltonian for the atom–photon interaction with the coupling constant $g \sim e \sim \sqrt{1/137}$ ($\hbar \equiv c \equiv 1$). In equation (1), $\delta$ is a positive finite number and $1/\delta$ is proportional to the relaxation time of the excited state as will be shown later. For simplicity, we assume $\delta$ a constant, and thus a uniform relaxation time for the system. As $v_j = v_0 + \delta v_j$ in equation (1) follows a Gaussian distribution with peak $v_0$ and fluctuation $\mu$; i.e. the width of the excitation energy is proportional to $2\mu$. In equation (2), the first term on the rhs, $\sigma_j^+ A(x_j)$ describes the process of excitation of the $j$th atom by absorbing a photon, and the 2nd term $A^+(x_j) \sigma_j^-$ is for the downward transition of the $j$th atom from the excited state to the ground state by emitting a photon. For example, spontaneous emission of the photon from the $j$th atom, when no external light source is present,
can be included in our model with the interaction $A^\dagger(x_j)\sigma^j_-$ as $\int \langle 1, g | e^{-i/ht} | 0, e \rangle_j$ with $\omega = -igt/h - j(1, g) A^\dagger(x_j) \sigma^j_0 | 0, e \rangle_j$ as the lowest order amplitude.

The propagator of the two-level atom alone (with no atom–photon interaction taken in) would be

$$\Delta_j(t) = \int d\omega \tilde{\Delta}_j(\omega) e^{-i\omega t}, \quad \tilde{\Delta}_j(\omega) = \frac{i}{\omega - \nu_j + i\delta},$$

which gives the well-known amplitude $e^{-i\omega t}$ times a factor $e^{-\delta t}$, and $1/\delta$ is therefore proportional to the relaxation time of the excited state.

The Green’s function of the EM field $G(x, t; x', t')$ satisfies the Dyson’s equation as,

$$G(x, t; x', t') = G_0(x, t; x', t') + g^2 \sum_j G(x, t; x_j, t') \Delta_j(t'' - t') G_0(x_j, t''; x', t'),$$

or can be expressed in the following way in the momentum space:

$$\tilde{G}(k, \omega; k', \omega')^{-1} = \tilde{G}(k + h, \omega; k', \omega) - \delta(\omega' - \omega) \delta_{k' + h, k}$$

$$\quad = [\tilde{G}_0(k, \omega)^{-1} + 2i \Pi(\omega) \delta_{k' + h, k}] \cdot \delta(\omega' - \omega),$$

where $\tilde{G}_0(k, \omega) = i/(\omega^2 - k^2 + i\epsilon)$ is the free propagator of the EM field ($\epsilon \to 0^+$), $h$’s are the reciprocal lattice vectors ($h = 2\pi \tau/a$, $a$ is the lattice constant), and $\Pi(\omega)^2$ is

$$\Pi(\omega)^2 = -\frac{ig^2}{2Na} \sum_j \tilde{\Delta}_j(\omega),$$

which represents the modification to the propagator (self-energy) of the EM wave due to atom–photon interaction.

It can be obtained that

$$\Pi(\omega)^2 = \frac{g^2}{2a} \left( \omega - \nu_0 + i\delta - \frac{\alpha}{\mu^2} \frac{\mu^2}{3 - \frac{\alpha^2}{2} \omega - \nu_0 + i\delta} \right), \quad \alpha \approx 1.33,$$

as is shown in appendix A. It contains both a real part and an imaginary part which originates from $\delta$.

From the Green’s function in equation (4), as is shown in appendix B, the dispersion relation of the photon can be calculated ($c \equiv 1$)

$$a^2 \Pi(\omega)^2 \frac{\sin \omega a}{\omega a} + \cos \omega a - \cos ka = 0,$$

and depicted in figure 1(a) in the reduced zone scheme. Here, in order to focus on studying manipulations of atomic states, we shall consider the situation where an external light source exists with tuneable frequency and its intensity is strong enough to serve as the only dominating light source. Therefore, we shall take $\omega$ as an independent real variable in equation (7). Since $\Pi(\omega)^2$ (equation (6)) is complex, the wave number $K$ satisfying the dispersion relation is also complex and will be written as $K_\omega = k_\omega + ik_\omega$. Figure 1(a) shows the first three Brillouin zones (BZ) (in which $\kappa_\omega = 0$) separated by two energy gaps (EG) of outgoing waves. Between $\omega_1^{(1)}/\pi$ and 0.428, $K_\omega = \pi/a + ik_\omega$, and between 0.428 and $\omega_2^{(1)}/\pi$, $K_\omega = i\kappa_\omega$, here $\kappa_\omega > 0$. Similarly, between $\omega_1^{(2)}/\pi$ and 0.573, $K_\omega = \pi/a + i\kappa_\omega$, and between 0.573 and $\omega_2^{(2)}/\pi$, $K_\omega = i\kappa_\omega$, here.
Figure 1. (a) Dispersion relations (the real part $k_\omega a/\pi$ versus $\omega/\pi$ (solid line), and the imaginary part $\kappa_\omega a/\pi$ versus $\omega/\pi$ (dashed line)) of the photonic field propagating (outgoing) in an array of two-level atoms in the reduced zone scheme. Here we choose $g_2/a = 1$, $\nu_0/\pi = 0.5$, $\delta = \nu_0/40$ and $\mu = \nu_0/7$. The parameters $\omega_1^{(1)}/\pi \sim 0.334$, $\omega_2^{(1)}/\pi \sim 0.473$, and $\omega_2^{(2)}/\pi \sim 0.529$, $\omega_3^{(2)}/\pi \sim 0.623$. (b) Dispersion relations of the photonic field propagating (outgoing) in an array of two-level atoms with the same excitation energies. Here, we use the same parameters as in figure 1(a) except that $\mu = 0$. The parameters $\omega_1^{g1}/\pi \sim 0.361$, and $\omega_2^{g2}/\pi \sim 0.596$. 

$\kappa_\omega > 0$. In the BZ, wave vector $K_\omega$ is real and it corresponds to the propagating wave; while in the energy gaps, wave vector $K_\omega$ is complex and it corresponds to the attenuated wave ($e^{iK_\omega x} = \pm e^{-\kappa_\omega |x|}$). We found that if the width of the excitation energy vanishes ($\mu = 0$), the BZ between the two energy gaps disappears and the two gaps merge together as shown in figure 1(b).

For a monochromatic point source with a frequency of $\omega_s$ located at origin $\xi(x, t) \equiv \xi_0 \delta(x) e^{-i\omega_s t}$ ($\tilde{\xi}(k, \omega) = \xi_0 \delta(\omega - \omega_s)$), the expected value of the EM field at the $j$th lattice site is

$$ \langle A(x_j, t) \rangle = \int dx' dt' G(x_j, t; x', t') \tilde{\xi}(x', t') $$

$$ = \xi_0 e^{-i\omega_s t} \int \frac{dK}{2\pi} \frac{dK'}{2\pi} e^{iK_x x_j} \tilde{G}(K, \omega_s; K', \omega_s). $$

Assuming that $K = K_{\omega_s}$ satisfies equation (7) and is a simple pole of $\tilde{G}(K, \omega_s; K', \omega_s)$, then $K' = K_{\omega_s} + h$'s are simple poles, too. By taking residues at the poles $K_{\omega_s} + h$'s (noting that exp ($ih \cdot x_j$)=1), the above integration can be shown as,

$$ \int \frac{dK}{2\pi} \frac{dK'}{2\pi} e^{iK_x x_j} \tilde{G}(K, \omega_s; K', \omega_s) = e^{iK_{\omega_s} |x_j|} \cdot C(\omega_s), $$

where $C(\omega_s)$ is a function which depends only on the incoming frequency $\omega_s$, and $\langle A(x_j, t) \rangle$ becomes,

$$ \langle A(x_j, t) \rangle = \xi_0 C(\omega_s) e^{iK_{\omega_s} |x_j|} e^{-i\omega_s t}. $$

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Figure 2. Amplitudes of the photonic field (solid line) and a two-level atom in the excited state (dashed line) at $x = 5a$ and $t = 0$ versus $\omega$. The vertical axis is an arbitrary unit. Here, we use the same parameters as in figure 1 (a).

In the attenuated mode ($K_{\omega_s} = i\kappa_s$ or $\pi/a + i\kappa_s$, $\kappa_s > 0$), the expected value of the photonic field at lattice site $j$ is

$$\langle A(x_j, t) \rangle = \pm \xi_0 C(\omega_s) e^{-\kappa_s|x_j|} e^{-i\omega_s t}. \quad (8)$$

On the other hand, the probability amplitude $b_j(t)$ that the $j$th atom stays in the excited state is

$$b_j(t) = g \int dx' dt'' dt' \Delta_j(t-t'') G(x_j, t''; x', t') \xi(x', t')$$

$$= g \xi_0 C(\omega_s) i \Delta_j(\omega_s) e^{i\kappa_s|x_j|} e^{-i\omega_s t} = g i \Delta_j(\omega_s) \langle A(x_j, t) \rangle, \quad (9)$$

and that it is in phase with the local EM field. Amplitudes $\langle A(x_j, t) \rangle$ and $b_j(t)$ at a typical site as a function of $\omega$ are solved numerically and shown in figure 2. In EG1, $\kappa_s$ goes to 0 at the edges ($\omega_1/\pi$ & $\omega_2/\pi$), and it increases toward the middle (0.428) (and drops abruptly to 0) (figure 1(a); and both amplitudes decrease exponentially with $\kappa_s$ (except at $\omega/\pi = 0.428$). A similar situation occurs in EG2, also.

We can write $b_j(t)$ in terms of two real parameters $\Psi_j$ and $\phi_j$ as $b_j(t) \equiv \Psi_j e^{i\phi_j}$, where $\Psi_j^2$ is the probability that the $j$th atom is in the excited state, and $\phi_j$ is the relative phase difference between the amplitudes that the $j$th atom is in, in the excited state and the ground state. With $\Psi_j$ and $\phi_j$ given, the quantum state of the $j$th atom can be determined uniquely.

In the attenuated mode of the photon, the light wave is attenuated (then by equations (8) and (9), $b_j(t)$’s are in phase with each other, or $\phi_j$ is spatially uniform ($\phi_j = \phi$). As $\kappa_s$ decreases, the attenuated light wave becomes broader, and the correlation between the two atoms increases. With $\kappa_s \rightarrow 0$, the spatial distribution of the attenuated light wave becomes flat (figure 3), and $\Psi_j$ is nearly uniform ($\Psi_j \approx \Psi$). Hence, an ordered state of the arrayed two-level atoms exists in such conditions. Further, a natural order parameter for this state would be $\Psi e^{i\phi}$. On the other hand, if not for the mediation of the attenuated photonic field, the quantum fluctuation alone would destroy the order of the state even if the temperature is lowered to 0 K.
Figure 3. Spatial distributions of the occupation number of hard-core bosons at three different frequencies of the photonic field: $\omega_1 = 1.471 \times 0.99\omega_2^{(1)}$ (full circles), and $\omega_2 = 1.456 \times 0.98\omega_2^{(1)}$ (open circles) are in EG1; here $\omega_3 = 0.03$ (full square) is in the 1st BZ. Here, we use the same parameters as in figure 1(a).

Therefore, a quantum phase transition of the arrayed two-level atoms is induced by the mediated attenuated photonic field.

3. Summary and discussions

We studied the system of a linear array of $N$ two-level atoms including the energy width of excited states interacting with a photonic field via a Jaynes–Cummings interaction. Taking photon-atom interactions into account, we calculated the photonic dispersion relation. We show that, as the energy of the photon is around the spacing between two atomic energy levels, the photon is absorbed and is in the attenuated mode. Due to many-body interactions, photons are absorbed within finite ranges of frequency. It follows that energy gaps appear in the dispersion relation.

When the frequency $\omega$ of the photonic field is within the energy gaps ($e^{iKx \omega} = \pm e^{-\kappa \omega|x|}$), there is no phase difference between all two-level atoms. Therefore, a quantum phase transition occurs and there exists an ordered state for the arrayed two-level atoms. Hence, we can correlate distant two-level atoms on an array by tuning the frequency of the EM wave. Our results are especially relevant in systems of the so-called quantum meta-materials [14] which consist of an array of Josephson qubits and effectively serve as two-level atoms.

In view of the rapid progress in experiments, it can be expected that light-mediated quantum phase transitions can be controlled precisely, and therefore it would provide significant opportunities for new studies of collective phenomena in artificial atoms.

Furthermore, by modulating the frequency and intensity of the external light source, a coherent state of the arrayed two-level atoms can be manipulated to be an arbitrary linear combination of the single atomic ground state and excited state. It may shed some light on research into quantum computation, photo-electronics, optical lattices and related fields.
On the other hand, we have found an example of phase transition induced by attenuated photons in arrayed two-level atoms. Similar phase transitions in condensed-matter systems induced by mediation of transmuted intermediate boson-like phonons, magnons or other fields may also be worth investigating.

During the revision of this paper, we were informed of the work of Nienhuisi and Schulleri in 1987 [15] studying the decay rate and spontaneous emission in an array of two-level atoms. Their study shows that under certain conditions, the decay rate disappears and spontaneous emission is trapped. We think that this phenomenon is closely related to our results. However, in [15] they took a density matrix approach, while we take the approach of the second quantized form of the Hamiltonian. Incidentally, Nienhuisi and Schulleri mainly studied decay rate and spontaneous emission; while our emphasis was on quantum phase transition and manipulations of atomic quantum states. Therefore, further study is needed to investigate the relationship between the results of the two works in detail.

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**Appendix A. Calculations of $\Pi(\omega)^2$**

By equation (5), we have

$$\Pi(\omega)^2 = \frac{g^2}{2Na} \sum_j \frac{1}{\omega - \nu_0 - \delta \nu_j + i\delta} = \frac{g^2}{2iN} \sum_j \int_0^\infty ds \ e^{i\delta (\nu_0 - \delta \nu_j + i\delta)}$$

$$= \frac{g^2}{2iN} \int_0^\infty ds \ e^{i\delta (\nu_0 + i\delta)} \sum_j e^{-i\delta \nu_j}.$$  

Since all $\delta \nu_j$'s follow the Gaussian distribution at center 0 and fluctuation $\mu$, the above summation can be converted to the following integration:

$$\sum_j e^{-i\delta \nu_j} \to N \int_{-\infty}^{\infty} d\nu \ e^{-\frac{(i\delta \nu)^2}{2\mu^2}} e^{-i\omega \delta \nu} = N e^{-\frac{\omega^2 \delta^2}{2}};$$

and we then have [16]

$$\Pi(\omega)^2 = \frac{g^2}{2ia} \int_0^\infty ds \ e^{i\delta (\nu_0 + i\delta)} e^{-\frac{\omega^2 \delta^2}{2}}$$

$$= -\frac{g^2}{\sqrt{2a\mu}} (\omega - \nu_0 + i\delta) \int_0^\infty dt \ e^{-\frac{t^2}{2\mu^2}} \frac{e^{-\frac{\omega^2 \delta^2}{2}}}{t^2 - (\omega - \nu_0 + i\delta)^2}.$$  

Assuming $\mu$ is a small parameter, the integrand in the above integration is significantly nonzero in a small interval between 0 and $\alpha \mu$ with $\alpha \sim 1$, and its exact value is fixed by the condition that our results would reduce to those when all $\nu_j = \nu_0$ (i.e., $\mu = 0$ with no fluctuation). Thus
we have,
\[
\Pi(\omega)^2 \approx -\frac{g^2}{\sqrt{2}a} \int_0^{\alpha \mu} dt \frac{1 - \frac{t^2}{2\mu^2}}{t^2 - (\omega - \nu_0 + i\delta)^2}
\]
\[
\approx \frac{g^2}{\sqrt{2}a} (\omega - \nu_0 + i\delta) \left[ \frac{\alpha \mu}{(\omega - \nu_0 + i\delta)^2} - \frac{\alpha^3 \mu}{6(\omega - \nu_0 + i\delta)^2} + \frac{\alpha^3 \mu^3}{3(\omega - \nu_0 + i\delta)^4} \right]
\]
\[
\approx \sqrt{2} \left( 1 - \frac{\alpha^2}{6} \right) \frac{g^2}{2a} \frac{1}{\omega - \nu_0 + i\delta} - \frac{\alpha}{3 - \frac{a^2}{2}} \frac{\mu^2}{\omega - \nu_0 + i\delta}.
\]
As \( \mu = 0 \) (\( \nu_j = \nu_0 \)), by equations (3) and (5), we have
\[
\Pi(\omega)^2 = g^2 \frac{1}{2a} \frac{1}{\omega - \nu_0 + i\delta}. \tag{A.1}
\]
Comparing the above two equations, we show that \( \alpha = \sqrt{6 - 3\sqrt{2}} \approx 1.33 \).

Appendix B. Dispersion relation \( \omega_k \)

To find the dispersion relation of the photonic field is the same as to obtain the poles of the Green’s function \( \tilde{G}(k, k'; \omega) \), or the eigenvalues \( \omega_k \) of the following eigenvalue equation,
\[
\sum_{k'} \tilde{G}(k, k'; \omega)^{-1} \Phi(k'; \omega) = 0,
\]
or, by equation (4),
\[
(\omega^2 - k^2 + i\epsilon) \Phi(k; \omega) - \sum_{h=2n\pi/a} 2\Pi(\omega)^2 \Phi(k + h; \omega) = 0.
\]
Then we have
\[
\Phi(k; \omega) = \frac{2\Pi(\omega)^2}{\omega^2 - k^2 + i\epsilon} \sum_{h=2n\pi/a} \Phi(k + h; \omega). \tag{B.1}
\]
Define \( F(k; \omega) = \sum_{h=2n\pi/a} \Phi(k + h; \omega) \), and thus \( F(k + 2m\pi/a; \omega) = F(k; \omega) \).

Setting \( k = k + h' \) in equation (B.1) and then summing over \( h' \), we obtain,
\[
\sum_{h'=2n'\pi/a} \Phi(k + h'; \omega) = 2\Pi(\omega)^2 \sum_{h'=2n'\pi/a} \frac{F(k + h'; \omega)}{\omega^2 - (k + h')^2 + i\epsilon}.
\]
In the above equation, the lhs is exactly \( F(k; \omega) \), and, on the rhs, \( F(k + h'; \omega) = F(k; \omega) \).
Therefore, remembering \( \epsilon \) being infinitesimal, we have,
\[
1 = 2\Pi(\omega)^2 \sum_n \frac{1}{\omega^2 - (k + 2n\pi/a)^2 + i\epsilon} \]
\[
= \frac{\Pi(\omega)^2}{\omega} \sum_n \left[ \frac{1}{k + 2n\pi/a + \omega + i\epsilon} - \frac{1}{k + 2n\pi/a - \omega - i\epsilon} \right] \]
\[
= \frac{a \Pi(\omega)^2}{2\omega} \left[ \cot \frac{(k + \omega)a}{2} - \cot \frac{(k - \omega)a}{2} \right],
\]
by using the identity \( \cot x = \sum_{n} \frac{1}{n\pi + x} \). In terms of sine and cosine functions, the above equation can be rewritten as,

\[
1 = \frac{a \Pi(\omega)^2}{2\omega} \frac{\sin \omega a}{\sin \frac{(k + \omega) a}{2}} \frac{\sin \frac{(k - \omega) a}{2}}{\omega a} = \frac{a^2 \Pi(\omega)^2}{\omega a} \frac{\sin \omega a}{\cos ka - \cos \omega a},
\]

or,

\[
a^2 \Pi(\omega)^2 \frac{\sin \omega a}{\omega a} + \cos \omega a - \cos ka = 0.
\]

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