ON THE EXISTENCE AND CONSTRUCTION OF PROPER COSTRATIFYING SYSTEMS

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In this article we further study the notion of proper costratiﬁng systems, deﬁned in [10]. We give sufﬁcient conditions for their existence, and investigate the relation between the stratifying systems deﬁned by K. Erdmann and C. Sáenz in [7] and the proper costratiﬁng systems.

Key Words: Artin algebras; Modules; Proper costratiﬁng systems.

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INTRODUCTION

In [10], we deﬁne and study the notion of a proper costratiﬁng system, which is a generalization of the so-called proper costandard modules to the context of stratifying systems.

In this article we investigate the relation between stratifying systems and proper costratiﬁng systems. Moreover, we study sufﬁcient conditions for their existence, as we explain next.

Let $P(1), \ldots, P(n)$ be an ordered sequence of the non-isomorphic indecomposable projective modules over an artin algebra $\Lambda$. By deﬁnition, the standard module $\Delta(i)$ is the largest factor module of $P(i)$ with composition factors only amongst $S(1), \ldots, S(i)$, where $S(j)$ is the simple top of $P(j)$. Denote by $\text{mod}(\Lambda)$ the category of ﬁnitely generated left $\Lambda$-modules. Let $\mathcal{F}(\Delta)$ denote the subcategory of $\text{mod}(\Lambda)$ consisting of the $\Lambda$-modules having a ﬁltration with factors isomorphic to standard modules. The algebra $\Lambda$ is said to be standardly stratified if all projective $\Lambda$-modules belong to $\mathcal{F}(\Delta)$ (see [1, 2, 4, 7, 12, 14, 15]).

For an artin algebra $\Lambda$ there exists another family of modules which plays an important role: the proper standard (respectively, the proper costandard) modules $\Delta(i)$ (respectively, $\nabla(i)$), deﬁned as some appropriate factors of the $\Delta(i)$ (respectively, submodules of the $\nabla(i)$). These modules were deﬁned by V. Dlab and

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have the property that $\Lambda$ is a standardly stratified algebra, that is, all projective $\Lambda$-modules belong to $\mathcal{F}(\Delta)$, if and only if all injective $\Lambda$-modules belong to $\mathcal{F}(\Delta)$ (see [5, 8]). Here $\mathcal{F}(\Delta)$ denotes the subcategory of $\text{mod}(\Lambda)$ consisting of the $\Lambda$-modules having a filtration with factors isomorphic to proper costandard modules.

In connection with the study of standardly stratified algebras, I. Agoston, D. Happel, E. Lukács, and L. Unger showed, for such an algebra $(\Lambda, \preceq)$, that $\mathcal{F}(\Delta) = \mathcal{F}(\Delta)^{-1} = \{ M \in \text{mod}(\Lambda) : \text{Ext}^1(\mathcal{F}(\Delta), M) = 0 \}$ (see [2]). In addition, they proved that there exists a tilting $\Lambda$-module $T = \{ T(i) \}_{i=1}^n$, called the characteristic tilting module, such that $\text{add}(T) = \mathcal{F}(\Delta) \cap \mathcal{F}(\Delta)^{-1}$ (see also [12]). Moreover, $(\Delta, \{ T(i) \}_{i=1}^n, \preceq)$ is a stratifying system and $(\Delta, \{ T(i) \}_{i=1}^n, \preceq)$ is a proper costratifying system. This motivates the following question: given a set $Q = \{ Q(i) \}_{i=1}^t$ of pairwise non-isomorphic indecomposable $\Lambda$-modules and a linear order $\preceq$ on $[1, 2, \ldots, t]$, how does the existence of a proper costratifying system $(\Psi, Q, \preceq)$ relate to the existence of a stratifying system $(\Theta, Q, \preceq)$? In Section 5 we study this question, proving two theorems. The first one gives, for a given proper costratifying system $(\Psi, Q, \preceq)$, necessary and sufficient conditions for the existence of a family $\Theta = \{ \Theta(i) \}_{i=1}^t$ in $\text{mod}(\Lambda)$ such that $(\Theta, Q, \preceq)$ is a stratifying system. The second one is the reciprocal result.

K. Erdmann and C. Sáenz defined in [7] the notion of a stratifying system $(\Theta = \{ \Theta(i) \}_{i=1}^t, \preceq, \leq)$, and proved that such a system satisfies that each $\Theta(i)$ is indecomposable and $\text{Ext}^1(\Theta(i), \Theta(j)) = 0$ for $i \geq j$. Reciprocally, they also showed that a given family of indecomposable $\Lambda$-modules $\Theta = \{ \Theta(i) \}_{i=1}^t$ satisfying that $\text{Ext}^1(\Theta(i), \Theta(j)) = 0$ for $i \geq j$, and such that $\text{Hom}_{\Lambda}(\Theta(i), \Theta(j)) = 0$ for $i > j$, there is a stratifying system $(\Theta, \preceq, \leq)$.

For proper costratifying systems, the situation is different. On the one hand, $\text{Ext}^1(\Psi(i), \Psi(j)) = 0$ for $i < j$ (see [10, Lemma 3.8]). However, the existence of a family $\Psi = \{ \Psi(i) \}_{i=1}^t$ such that $\text{Hom}_{\Lambda}(\Psi(i), \Psi(j)) = 0 = \text{Ext}^1(\Psi(i), \Psi(j))$ for $i < j$, does not ensure the existence of a proper costratifying system $(\Psi, Q, \preceq)$, even if we assume that $\text{End}_{\Lambda}(\Psi(i))$ is a division ring for all $i \in [1, t]$.

In this article, we prove an existence result for proper costratifying systems in this direction. To this end, we assume as an additional hypothesis that the length of the indecomposable modules filtered by $\Psi$ is uniformly bounded.

We describe briefly the content of the article section by section.

After a section of preliminaries, Sections 2 and 3 are devoted to give sufficient conditions for the existence of a proper costratifying system. Section 4 studies homological properties which will be applied to prove the mentioned results in Section 5.

1. PRELIMINARIES

Throughout this article, $\Lambda$ is an artin $R$-algebra, where $R$ is a commutative artinian ring. The term '$_{\Lambda}$module' means finitely generated left $\Lambda$-module. We denote by $\text{mod}(\Lambda)$ the category of finitely generated left $\Lambda$-modules and by $\text{proj}(\Lambda)$ the full subcategory $\text{mod}(\Lambda)$ consisting of the finitely generated projective $\Lambda$-modules. Let $\text{ind}(\Lambda)$ denote the full subcategory of $\text{mod}(\Lambda)$ whose objects are chosen representatives of the isomorphism classes of indecomposable modules in $\text{mod}(\Lambda)$. For $\Lambda$-modules $M$ and $N$, $\text{Tr}_M(N)$ is the trace of $M$ in $N$, that is, $\text{Tr}_M(N)$ is the
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2. PROPER PRE-COSTRATIFYING SYSTEMS

Let \( \Lambda \)-submodule of \( N \) generated by the images of all morphisms from \( M \) to \( N \). Let \( D : \text{mod}(\Lambda) \to \text{mod}(\Lambda^{op}) \) denote the usual duality for artin algebras, and \( * \) denote the functor \( \text{Hom}_\Lambda(\_, \_ : \text{mod}(\Lambda) \to \text{mod}(\Lambda^{op}) \). Then \( * \) induces a duality from \( \text{proj}(\Lambda) \to \text{proj}(\Lambda^{op}) \). For a given natural number \( t \), we set \( [1, t] = \{1, 2, \ldots, t\} \).

Let \( \mathfrak{C} \) be a class of objects in \( \text{mod}(\Lambda) \). For each natural number \( n \), we set \( \mathfrak{C}^n = \{ M \in \text{mod}(\Lambda) : \text{Ext}^n_\Lambda(M, -) = 0 \} \) and \( \mathfrak{C}^n = \bigcap_{n=0}^{\infty} \mathfrak{C}^n \). The notions of \( \mathfrak{C}^n \) and \( \mathfrak{C}^n \) are introduced similarly.

In addition, we recall that the \( \Lambda \)-length of the class \( \mathfrak{C} \) is \( \ell_\Lambda(\mathfrak{C}) = \sup \{ \ell_\Lambda(C) : C \in \mathfrak{C} \} \), where \( \ell_\Lambda(C) \) stands for the length of the \( \Lambda \)-module \( C \).

For a \( \Lambda \)-module \( M \), \( \text{add}(M) \) denotes the full subcategory of \( \text{mod}(\Lambda) \) consisting of the direct summands of direct sums of copies of \( M \).

Let \( \Lambda \) be an algebra, and let \( n \) be the rank of the Grothendieck group \( K_0(\Lambda) \). We fix a linear order \( \leq \) on \( [1, n] \) and a representative set \( \Lambda = \{ \Lambda(i) : i \in [1, n] \} \) containing one module of each iso-class of indecomposable projective \( \Lambda \)-modules. The injective envelope of the simple \( \Lambda \)-module \( \Lambda S \) is denoted \( \Lambda I(i) \). For the opposite algebra \( \Lambda^{op} \), we always consider the representative \( \Lambda^{op} = \{ \Lambda^{op}(i) : i \in [1, n] \} \) of indecomposable projective \( \Lambda^{op} \)-modules, where \( \Lambda^{op}(i) = (\Lambda(i))^* \) for all \( i \in [1, n] \).

The set of standard \( \Lambda \)-modules is \( \Lambda \Delta = \{ \Lambda \Delta(i) : i \in [1, n] \} \), where \( \Lambda \Delta(i) = \Lambda P(i)/\text{rad}_{\Lambda^{op}}(\Lambda P(i)) \). Then, \( \Lambda \Delta(i) \) is the largest factor module of \( \Lambda P(i) \) with composition factors only amongst \( \Lambda S(j) \) for \( j \leq i \). The set of costandard \( \Lambda \)-modules is \( \Lambda \nabla = \{ \Lambda \nabla(i) : i \in [1, n] \} \), where \( \Lambda \nabla(i) \) is the largest submodule of \( \Lambda I(i) \) with composition factors only amongst \( \Lambda S(j) \) for \( j \leq i \). Clearly, \( \Lambda \nabla = D(\Lambda^{op} \Delta) \), where the pair \( (\Lambda^{op} P, \leq) \) is used to compute \( \Lambda^{op} \Delta \).

The set of proper standard \( \Lambda \)-modules is \( \Lambda \Delta = \{ \Lambda \Delta(i) : i \in [1, n] \} \), where \( \Lambda \Delta(i) = \Lambda P(i)/\text{rad}_{\Lambda^{op}}(\Lambda P(i)) \). Then, \( \Lambda \Delta(i) \) is the largest factor module of \( \Lambda \Delta(i) \) such that the multiplicity \( [\Lambda \Delta(i) : S(i)] \) of \( S(i) \) as a composition factor of \( \Lambda \Delta(i) \) is 1. The set of proper costandard \( \Lambda \)-modules is \( \Lambda \nabla = D(\Lambda^{op} \Delta) \), where the pair \( (\Lambda^{op} P, \leq) \) is used to compute \( \Lambda^{op} \Delta \).

Let \( \mathcal{F}(\Lambda \Delta) \) be the subcategory of \( \text{mod}(\Lambda) \) consisting of the \( \Lambda \)-modules having a \( \Lambda \)-filtration, that is, a filtration \( 0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_s = M \) with factors \( M_{i+1}/M_i \) isomorphic to a module in \( \Lambda \Delta \) for all \( i \). The algebra \( \Lambda \) is a standardly stratified algebra with respect to the linear order \( \leq \) on the set \( [1, n] \), if \( \text{proj}(\Lambda) \subseteq \mathcal{F}(\Lambda \Delta) \) (see [1, 4, 5]).

2. PROPER PRE-COSTRATIFYING SYSTEMS

We begin by recalling the definition of proper costratifying systems introduced in [10].

Definition 2.1. Let \( \Lambda \) be an artin \( R \)-algebra. A proper costratifying system \( (\Psi, Q, \leq) \) of size \( t \) in \( \text{mod}(\Lambda) \) consists of two families of \( \Lambda \)-modules \( \Psi = \{ \Psi(i) \}_{i=1}^{t} \) and \( Q = \{ Q(i) \}_{i=1}^{t} \), with \( Q(i) \) indecomposable for all \( i \), and a linear order \( \leq \) on the set \( [1, t] \), satisfying the following conditions:

(a) \( \text{End}_\Lambda(\Psi(i)) \) is a division ring for all \( i \in [1, t] \);
(b) \( \text{Hom}_\Lambda(\Psi(i), Q(j)) = 0 \) if \( i < j \);
(c) For each \( i \in [1, t] \), there is an exact sequence

\[ e_i : 0 \rightarrow Z(i) \rightarrow Q(i) \rightarrow \Psi(i) \rightarrow 0, \]

with \( Z(i) \in \mathcal{T}(\{\Psi(j) : j \leq i\}) \);

(d) \( Q \subseteq \mathcal{T} \Psi \), that is, \( \text{Ext}_A(\Psi(i), -)|_{\Psi} = 0 \) for any \( i \in [1, n] \).

Our aim is to show an existence result for proper costratifying systems along the lines of the existence result of K. Erdmann and C. Sáenz for stratifying systems, mentioned in the introduction of this article. As we said there, we have a different situation for proper costratifying systems. In fact, the existence of a family \( \Psi = \{\Psi(i)\}_{i=1}^n \) such that \( \text{Hom}_A(\Psi(i), \Psi(j)) = 0 = \text{Ext}_A(\Psi(i), \Psi(j)) \) for \( i < j \), does not guarantee the existence of a proper costratifying system \( (\Psi, Q, \leq) \), even if we assume that \( \text{End}_A(\Psi(i)) \) is a division ring for all \( i \in [1, t] \), as we show in the following example.

**Example 2.2.** Let \( A \) be the path algebra of the quiver

\[ 1 \rightarrow 2 \]

with the natural order \( 1 \leq 2 \). Consider \( \Psi = \{\Psi(1) = k, \Psi(2) = k^{J_{i,1}}\} \). Then \( \Psi(1) \) belongs to a tube \( \mathcal{T} \) in the AR quiver of \( A \). It follows from the structure of the tube that \( \mathcal{T} \subseteq \mathcal{T}(\Psi) \). So \( \mathcal{T}(\Psi) \) contains modules of arbitrary length. Let now \( Q(1) \in \mathcal{T}(\Psi) \). Then, there exists a monomorphism \( \Psi(1) \rightarrow Q(1) \), so that \( Q(1) \in \mathcal{T} \) or \( Q(1) \) is preinjective. If there is, moreover, a map \( Q(1) \rightarrow \Psi(1) \), then \( Q(1) \in \mathcal{T} \). So, if \( Q(1) \) satisfies (c) in Definition 2.1, then \( Q(1) = k^{J_{i,1}} \) for some \( i \geq 1 \) where \( J_{i,1} \) denotes the Jordan block corresponding to the eigenvalue 1. In this case \( \text{Ext}_A(\Psi(1), \Psi(1)) \neq 0 \) because there is a nonsplit exact sequence

\[ 0 \rightarrow \Psi(1) \rightarrow \left( \begin{array}{c} k^{J_{i,1}} \\ \hline \end{array} \right) \rightarrow Q(1) \rightarrow 0. \]

This proves that there is no module \( Q(1) \in \mathcal{T}(\Psi) \) satisfying (c) and (d) in Definition 2.1. Thus, there is no proper costratifying system \( (\Psi, \{Q(1)\}, \leq) \) for the family \( \Psi \).

With the previous example in mind, to prove our desired existence result for proper costratifying systems, we will assume as an additional hypothesis that the length \( \ell_A(X) \) of the indecomposable modules \( X \) in \( \mathcal{T}(\Psi) \) is uniformly bounded. We also introduce the following notion.

**Definition 2.3.** Let \( A \) be an artin \( R \)-algebra. A proper pre-costratifying system \( (\Psi, \leq) \) of size \( t \) in \( \text{mod}(A) \) consists of a family of \( A \)-modules \( \Psi = \{\Psi(i)\}_{i=1}^t \) and a linear order \( \leq \) on the set \([1, t]\), satisfying the following conditions:

(a) \( \text{End}_A(\Psi(i)) \) is a division ring for all \( i \in [1, t] \);

(b) \( \text{Hom}_A(\Psi(i), \Psi(j)) = 0 \) for \( i < j \);

(c) \( \text{Ext}_A(\Psi(i), \Psi(j)) = 0 \) for \( i < j \).
Remark 2.4. It is not hard to prove, by using an inductive argument, that if (b) in the above definition holds for the family \( \Psi \), then \( \text{Hom}_A(X, Y) = 0 \) for \( X \in \mathcal{F}(\{\Psi(j) : j < s\}) \), \( Y \in \mathcal{F}(\{\Psi(j) : j \geq s\}) \) and \( s \in [1, r] \).

Definition 2.5. Let \((\Psi, \leq)\) be a proper pre-costratifying system of size \( t \) in \( \text{mod}(A) \), and let \( X \in \mathcal{F}(\Psi) \). An ordered \( \Psi \)-filtration of \( X \) of length \( n \) is a \( \Psi \)-filtration

\[
\mathcal{C}_{n,X} : 0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X,
\]

with factors \( X_k/X_{k-1} \cong \Psi(i_k) \) for \( 1 \leq k \leq n \), where \( i_1 \leq i_2 \leq \cdots \leq i_n \). In this case, we also say that \( \text{min}(\mathcal{C}_{n,X}) = i_k \).

Remark 2.6. The hypothesis about \( \text{Ext}^1 \) assumed in Definition 2.3 (c) ensures the existence of ordered \( \Psi \)-filtrations (see [10, Lemma 3.10]). Moreover, any \( \Psi \)-filtration of \( X \) can be rearranged to an ordered one with the same \( \Psi \)-composition factors.

We aim to show that \( \text{min}(\mathcal{C}_{n,X}) \) depends only on \( X \), and not on the chosen ordered \( \Psi \)-filtration. In order to do that, we start with the following lemma.

Lemma 2.7. Let \((\Psi, \leq)\) be a proper pre-costratifying system of size \( t \) in \( \text{mod}(A) \), and let

\[
\cdots \xrightarrow{p} X \xrightarrow{\Psi(s)} \xrightarrow{\Psi(\hat{s})} E \xrightarrow{\Psi(\tilde{s})} Y \xrightarrow{\Psi(s)} X \xrightarrow{\Psi(s)} \cdots
\]

be a diagram in \( \text{mod}(A) \) such that the sequence \( \varepsilon \) is exact. If \( X \in \mathcal{F}(\{\Psi(j) : j \geq s\}) \) and \( q \neq 0 \), then \( \hat{s} \geq s \).

Proof. Let \( X \in \mathcal{F}(\{\Psi(j) : j \geq s\}) \) and \( q \neq 0 \). Consider the following pull back diagram:

\[
\varepsilon' : 0 \longrightarrow \Psi(s) \longrightarrow E \xrightarrow{\Psi(\tilde{s})} \Psi(\tilde{s}) \longrightarrow 0
\]

\[
\varepsilon : 0 \longrightarrow \Psi(s) \longrightarrow X \xrightarrow{\Psi(s)} Y \longrightarrow 0.
\]

If \( \varepsilon' \) does not split, then \( \text{Ext}^1(\Psi(\tilde{s}), \Psi(s)) \neq 0 \), and so \( \hat{s} \geq s \). Assume now that \( \varepsilon' \) splits. Therefore, there exists \( q' : \Psi(\tilde{s}) \to X \) such that \( pq' = q \), and since \( q \neq 0 \), it follows that \( 0 \neq q' \in \text{Hom}_A(\Psi(\tilde{s}), X) \). Thus, from Remark 2.4, we conclude that \( \hat{s} \geq s \) since \( X \in \mathcal{F}(\{\Psi(j) : j \geq s\}) \). \( \square \)

Proposition 2.8. Let \((\Psi, \leq)\) be a proper pre-costratifying system of size \( t \) in \( \text{mod}(A) \), and let \( X \in \mathcal{F}(\Psi) \). If \( \mathcal{O}_{m,X} \) and \( \mathcal{O}'_{n,X} \) are ordered \( \Psi \)-filtrations of \( X \), then \( \text{min}(\mathcal{O}_{m,X}) = \text{min}(\mathcal{O}'_{n,X}) \).

Proof. Let \( \text{min}(\mathcal{O}_{m,X}) = i_1 \) and \( \text{min}(\mathcal{O}_{n,X}) = j_1 \). So, we have exact sequences \( \varepsilon_1 : 0 \to \Psi(i_1) \to X \to Y \to 0 \) and \( \varepsilon_2 : 0 \to \Psi(j_1) \to X \to Z \to 0 \) in \( \text{mod}(A) \). Let
The above proposition allows us to introduce the following definition.

**Definition 2.9.** Let $(\Psi, \preceq)$ be a proper pre-costratisfying system of size $t$ in mod$(\Lambda)$ and let $X \in \mathcal{T}(\Psi)$. Then $\min(X) := \min(\mathcal{C}_{n,X})$ for any ordered $\Psi$-filtration $\mathcal{C}_{n,X}$ of $X$ of length $n$.

It follows directly from the above definition that the following statement holds.

**Remark 2.10.** Let $(\Psi, \preceq)$ be a proper pre-costratisfying system of size $t$ in mod$(\Lambda)$, and let $X \in \mathcal{T}(\Psi)$. Then, there exists an exact sequence $0 \to \Psi(\min(X)) \to X \to X' \to 0$ in $\mathcal{T}(\Psi)$, satisfying the following two conditions:

(a) $\min(X') \geq \min(X)$;
(b) If $X$ has an ordered $\Psi$-filtration of length $n$, then $X'$ has an ordered $\Psi$-filtration of length $n - 1$.

3. **CONSTRUCTING THE FAMILY $Q = \{Q(i)\}^t_{i=1}$**

Let $(\Psi, \preceq)$ be a proper pre-costratisfying system of size $t$ in mod$(\Lambda)$. The aim of this section is to prove the existence of a family of $\Lambda$-modules $Q = \{Q(i)\}^t_{i=1}$ such that $(\Psi, Q, \preceq)$ is a proper costratisfying system. The proof will be done under the hypothesis that $\ell_{\Lambda}(\ind(\mathcal{T}(\Psi))) < \infty$. The following lemmas will be used in the sequel.

**Lemma 3.1.** Let $\varepsilon: 0 \to X \xrightarrow{(z_1)} Y_1 \oplus Y_2 \xrightarrow{(\beta_1\beta_2)} Z \to 0$ be a nonsplit exact sequence in mod$(\Lambda)$, where $X$ and $Z$ are indecomposable $\Lambda$-modules, $Y_1 \neq 0$ and $Y_2 \neq 0$. Then $\beta_1z_1 = -\beta_2z_2 \neq 0$.

**Proof.** Suppose that $\beta_1z_1 = -\beta_2z_2 = 0$. Then, it can be proven that $X = \ker(z_1) \oplus \ker(z_2)$. Hence, using the fact that $X$ is indecomposable, it follows that either $\ker(z_1) = 0$ or $\ker(z_2) = 0$. We may assume that $\ker(z_1) = 0$. Therefore $X = \ker(z_2)$ and so $z_2 = 0$. Thus $Z \cong \corker((\gamma_n)) = Y_1/z_1(X) \oplus Y_2$. Furthermore, since $Z$ is indecomposable and $Y_2 \neq 0$, we have that $z_1(X) = Y_1$. Hence $z_1$ is an isomorphism and $(z_1^{-1}0)((\gamma_n)) = 1_X$. Therefore $\varepsilon$ splits, which is a contradiction. □

**Lemma 3.2.** Let $\mathcal{C}$ be a class of objects in mod$(\Lambda)$ closed under extensions, and let $\{\beta_i : X_i \to X_{i-1}\}^{n}_{i=1}$ be a family of epimorphisms in mod$(\Lambda)$. If $\ker(\beta_i) \in \mathcal{C}$ for all $i \in [1, n]$, then $\ker(\beta_1 \cdots \beta_n) \in \mathcal{C}$.

**Proof.** We proceed by induction on $n$. It is immediate that the lemma holds for $n = 1$. 

$q := p_1v_2 : \Psi(j_1) \to Y$. If $q \neq 0$, then from Lemma 2.7 it follows that $j_1 \geq i_1$ since $X \in \mathcal{T}([\Psi(j) : j \geq i_1])$. On the other hand, if $q = 0$, there is some $v : \Psi(j_1) \to \Psi(i_1)$ such that $v_1v = v_2 \neq 0$. Thus $0 \neq v \in \text{Hom}_A(\Psi(j_1), \Psi(i_1))$ and therefore $j_1 \geq i_1$. Similarly, it can be seen that $j_1 \leq i_1$, so the equality holds. □
Let $n > 1$. We consider the following exact and commutative diagram:

\[
\begin{array}{ccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{Ker}(\beta_n) & \text{Ker}(\beta_n) & & & & & \\
0 & \text{Ker}(\beta_1 \cdots \beta_n) & \longrightarrow & X_n & \longrightarrow & X_0 & \longrightarrow 0 \\
\downarrow & \downarrow & \beta_n & & & \| & \\
0 & \text{Ker}(\beta_1 \cdots \beta_{n-1}) & \longrightarrow & X_{n-1} & \longrightarrow & X_0 & \longrightarrow 0 \\
\downarrow & \downarrow & & & & \downarrow & \\
0 & 0, & & & & & \\
\end{array}
\]

By hypothesis $\text{Ker}(\beta_n) \in \mathcal{C}$, and by induction $\text{Ker}(\beta_1 \cdots \beta_{n-1}) \in \mathcal{C}$. Thus $\text{Ker}(\beta_1 \cdots \beta_n) \in \mathcal{C}$ since $\mathcal{C}$ is closed under extensions, and this ends the proof of the proposition. □

To carry on the construction of the family $\mathcal{Q} = \{\mathcal{Q}(i)\}_{i=1}^{\infty}$, let us start with a family $\Psi = \{\Psi(1)\}$ having just one element, such that $\text{End}_\lambda(\Psi(1))$ is a division ring. We need to construct an indecomposable module $\mathcal{Q}(1)$ and an exact sequence $\varepsilon_1 : 0 \to Z(1) \to \mathcal{Q}(1) \to \Psi(1) \to 0$, with $Z(1) \in \mathcal{T}(\{\Psi(1)\})$, and such that $\text{Ext}^1(\mathcal{Q}(1), \Psi(1)) = 0$. We will do this through successive extensions in the following way:

1) If $\text{Ext}^1(\Psi(1), \Psi(1)) = 0$, we choose $\mathcal{Q}(1) = \Psi(1)$;

2) If $\text{Ext}^1(\Psi(1), \Psi(1)) \neq 0$, we consider a nonsplit exact sequence $0 \to \Psi(1) \to X_1 \to X_0 \to 0$, where $X_1 = \Psi(1)$. In case $\text{Ext}^1(X_1, \Psi(1)) = 0$, we choose $\mathcal{Q}(1) = X_1$. Otherwise, we iterate the above procedure and we find nonsplit exact sequences $0 \to \Psi(1) \to X_i \to X_0 \to 0$ for $i \geq 1$. Since all $X_i \in \mathcal{T}(\Psi)$, $\ell_\lambda(X_{i+1}) > \ell_\lambda(X_i)$, and we are assuming that $\ell_\lambda(\text{ind}(\mathcal{T}(\Psi))) < \infty$, this process must stop. Thus, there is a natural $n$ such that $\text{Ext}_\lambda^n(X_n, \Psi(1)) = 0$, and we choose $\mathcal{Q}(1) = X_n$. If $X_n$ is indecomposable, the exact sequence $0 \to Z(1) \to X_n \xrightarrow{\beta_n} \Psi(1) \to 0$, with $Z(1) = \text{Ker}(\beta_1 \cdots \beta_n)$, satisfies the required conditions because $Z(1) \in \mathcal{T}(\Psi)$ (see Lemma 3.2).

Let $\mathcal{T}'(\{\Psi(1)\})$ be the family of all the modules $X_i$ which occur in the above procedure. Then, we have actually proved the following result.

**Lemma 3.3.** Let $\Psi(1) \in \text{mod}_(\Lambda)$ be such that $\text{End}_\lambda(\Psi(1))$ is a division ring and $\ell_\lambda(\text{ind}(\mathcal{T}(\{\Psi(1)\}))) < \infty$. Then, there exists a nonsplit exact sequence $0 \to Z(1) \to \mathcal{Q}(1) \to \Psi(1) \to 0$ such that $Z(1) \in \mathcal{T}(\{\Psi(1)\})$ and $\mathcal{Q}(1) \in \mathcal{T}'(\{\Psi(1)\})$.

So, we only need to show that the module $X_n$ constructed above is indecomposable. To do this, we will prove that each $X_i$ in $\mathcal{T}'(\{\Psi(1)\})$ is indecomposable. More generally, for a family $\Psi = \{\Psi(1), \ldots, \Psi(i)\}$, we next define
the class $\mathcal{F}(\Psi)$ and later, in Proposition 3.10, we will prove that the modules in $\mathcal{F}(\Psi)$ are indecomposable.

**Definition 3.4.** Let $(\Psi, \leq)$ be a proper pre-costratifying system of size $\ell$ in $\text{mod}(\Lambda)$ and $n$ a natural number. We inductively define the class $\mathcal{F}_n(\Psi)$ as follows:

(a) $\mathcal{F}_1(\Psi) = \Psi$;
(b) Suppose $n > 1$ and $\mathcal{F}_{n-1}(\Psi)$ is already defined. Then $X \in \mathcal{F}_n(\Psi)$ if and only if either $X \in \mathcal{F}_{n-1}(\Psi)$, or $X$ admits an ordered $\Psi$-filtration of length $n$ and a nonsplit exact sequence

$$e_n : 0 \rightarrow \Psi(\min(X)) \rightarrow X \rightarrow X' \rightarrow 0 \text{ with } X' \in \mathcal{F}_{n-1}(\Psi).$$

We set $\mathcal{F}(\Psi) = \cup_{i \geq 1} \mathcal{F}_i(\Psi)$.

**Remark 3.5.**

(1) Observe that $\mathcal{F}_1(\Psi) \subseteq \mathcal{F}_2(\Psi) \subseteq \cdots \subseteq \mathcal{F}_n(\Psi) \subseteq \mathcal{F}(\Psi)$, for any $n$.

(2) In the above definition $e_n$ is a sequence in $\mathcal{F}([\Psi(j) : j \geq \min(X)])$, and $X'$ satisfies $\min(X') \geq \min(X)$, as follows from Lemma 2.7.

For a proper pre-costratifying system $(\Psi, \leq)$ in $\text{mod}(\Lambda)$, we will prove that any module $X$ in $\mathcal{F}(\Psi)$ is indecomposable. The proof will be done by induction on $n$ such that $X$ admits an ordered $\Psi$-filtration of length $n$. To do this, we will use that $\text{Coker}(\eta)$ belongs to $\mathcal{F}([\Psi(j) : j \geq \min(X)]) \cup \{0\}$, for any $Y \in \mathcal{F}([\Psi(j) : j \geq \min(X)])$ and any arbitrary monomorphism $\eta : \Psi(\min(X)) \rightarrow Y$. So, we start by studying further properties of monomorphisms with domain $\Psi(i)$ for some $i \in [1, \ell]$.

**Definition 3.6.** For any $K \in \text{mod}(\Lambda)$, we consider the class $\mathfrak{M}_K$ of all $X \in \text{mod}(\Lambda)$ satisfying the following property:

For all $f \in \text{Hom}_\Lambda(K, X)$, $f$ is either zero or a monomorphism.

**Proposition 3.7.** Let $0 \rightarrow K \xrightarrow{\alpha} X \xrightarrow{\beta} Z \rightarrow 0$ be an exact sequence in $\text{mod}(\Lambda)$, and let $L \in \text{mod}(\Lambda)$. Then, the following statements hold:

(a) If $K \in \mathfrak{M}_L$ and $Z \in \mathfrak{M}_L$, then $X \in \mathfrak{M}_L$;
(b) If $\text{Hom}_\Lambda(L, K) = 0$ and $\eta : L \rightarrow X$ is nonzero, then $\beta \eta \neq 0$;
(c) If $Z \in \mathfrak{M}_L$ and $\eta : L \rightarrow X$ is such that $\beta \eta \neq 0$, then $\eta(L) \cap \zeta(K) = 0$.

**Proof.** (a) Assume that $K \in \mathfrak{M}_L$ and $Z \in \mathfrak{M}_L$, and let $0 \neq f : L \rightarrow X$. We next prove that $f$ is a monomorphism. Indeed, if $\beta f = 0$, then there exists $f' : L \rightarrow K$ such that $f = \alpha f'$. Hence $f' \neq 0$ and so $f'$ is a monomorphism, getting in this case that $f$ is a monomorphism. On the other hand, in case $\beta f : L \rightarrow Z$ is nonzero, it follows that $\beta f$ is a monomorphism, and then $f$ is so.

(b) The proof is straightforward.
(c) Let $Z \in \mathcal{M}_L$ and $\eta : L \to X$ be such that $0 \neq \beta \eta : L \to Z$. Since $Z \in \mathcal{M}_L$, we know that $\beta \eta$ is a monomorphism.

We now prove that $\eta(L) \cap \alpha(K) = 0$. Let $\lambda = \eta(x) = \alpha(y)$ with $x \in L$ and $y \in K$. Then $\beta \eta(x) = \beta \alpha(y) = 0$, and since $\beta \eta$ is a monomorphism, we conclude that $x = 0$ and hence $\lambda = 0$. Therefore, $\eta(L) \cap \alpha(K) = 0$. \hfill $\square$

**Proposition 3.8.** Let $(\Psi, \leq)$ be a proper pre-costratifying system of size $t$ in mod($\Lambda$). If $X \in \mathcal{F}(\Psi)$ and $\lambda \leq \min(X)$, then $X \in \mathcal{M}_{\Psi(\lambda)}$.

**Proof.** Let $X \in \mathcal{F}(\Psi)$. We write $\min(X) = i_0$ for short and consider $\lambda \leq i_0$. We proceed by induction on $n$ such that $X$ admits an ordered $\Psi$-filtration of length $n$.

If $n = 0$, we have that $X = 0$ and so $X \in \mathcal{M}_{\Psi(i_0)}$. On the other hand, if $n = 1$, then $X \simeq \Psi(i_0)$ and $\lambda \leq i_0$. Hence, the result follows directly from (a) and (b) of Definition 2.3.

Let $n > 1$ and suppose that $X$ admits an ordered $\Psi$-filtration of length $n$. If $\lambda < i_0$, then $\text{Hom}_1(\Psi(\lambda), X) = 0$ by Remark 2.4, and so $X \in \mathcal{M}_{\Psi(\lambda)}$. Assume now that $\lambda = i_0$. By Remark 2.10, we know that there exists an exact sequence

$$
\varepsilon : 0 \to \Psi(i_0) \to X \to X' \to 0
$$

with $\min(X) \geq i_0$ and such that $X'$ has an ordered $\Psi$-filtration of length $n - 1$. Then, by induction, $X' \in \mathcal{M}_{\Psi(i_0)}$. Now, by applying Proposition 3.7 (a) to $\varepsilon$, we obtain that $X \in \mathcal{M}_{\Psi(i_0)}$. \hfill $\square$

**Proposition 3.9.** Let $(\Psi, \leq)$ be a proper pre-costratifying system of size $t$ in mod($\Lambda$) and $n \in \mathbb{N}$. Then, for any $X \in \mathcal{F}_{t}(\Psi)$ and any nonzero morphism $\eta : \Psi(\min(X)) \to X$, we have that $X/\eta(\Psi(\min(X))) \in \mathcal{F}_{t-1}(\Psi) \cup \{0\}$.

**Proof.** Let us consider $X \in \mathcal{F}_{t}(\Psi)$, $i_0 = \min(X)$ and a nonzero morphism $\eta : \Psi(i_0) \to X$. We proceed by induction on $n$.

If $n = 1$, then $X = \Psi(i_0)$ and hence $\eta$ is an isomorphism, so $X/\eta(\Psi(i_0)) = 0$.

Let $n > 1$. If $X \in \mathcal{F}_{n-1}(\Psi)$, then by induction, we get that $X/\eta(\Psi(\min(X))) \in \mathcal{F}_{n-2}(\Psi) \cup \{0\} \subseteq \mathcal{F}_{n-1}(\Psi) \cup \{0\}$. Otherwise, there exists a nonsplit exact sequence

$$
\varepsilon : 0 \to \Psi(i_0) \to X \to X' \to 0,
$$

with $X' \in \mathcal{F}_{n-1}(\Psi)$. By Remark 3.5 (2), we know that $\min(X') \geq i_0$. Thus, by Proposition 3.8, we conclude that $X' \in \mathcal{M}_{\Psi(i_0)}$. If $\alpha(\Psi(i_0)) = \eta(\Psi(i_0))$, then $X/\eta(\Psi(i_0)) \simeq X' \in \mathcal{F}_{n-1}(\Psi)$.

Let $\alpha(\Psi(i_0)) \neq \eta(\Psi(i_0))$. We assert that $0 \neq \beta \eta : \Psi(i_0) \to X'$. Indeed, if $\beta \eta = 0$, then $\eta = \alpha \eta'$ for some nonzero map $\eta' : \Psi(i_0) \to \Psi(i_0)$. Since we are assuming that $\text{End}(\Psi(i_0))$ is a division ring, we conclude that $\eta'$ is an isomorphism. Therefore, $\eta(\Psi(i_0)) = \alpha(\eta'(\Psi(i_0))) = \alpha(\Psi(i_0))$, contradicting our hypothesis and proving that $\beta \eta \neq 0$. So, from $X' \in \mathcal{M}_{\Psi(i_0)}$ and Proposition 3.7 (c), we conclude that $\beta \eta : \Psi(i_0) \to X'$.
is a monomorphism and $\pi(\Psi(i_0)) \cap \eta(\Psi(i_0)) = 0$. Therefore, we get the following exact and commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & 0 & \to & \Psi(i_0) & \to & \beta \eta(\Psi(i_0)) & \to & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\varepsilon: 0 & \to & \Psi(i_0) & \to & X & \to & X' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\varepsilon': 0 & \to & \Psi(i_0) & \to & X/\eta(\Psi(i_0)) & \to & X'/\beta \eta(\Psi(i_0)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

Since $X' \in \mathcal{F}_{n-1}'(\Psi)$, we get by induction that $X'/\beta \eta(\Psi(i_0)) \in \mathcal{F}_{n-2}'(\Psi) \cup \{0\}$. If $X'/\beta \eta(\Psi(i_0)) = 0$, then $X/\eta(\Psi(i_0)) \cong \Psi(i_0) \in \mathcal{F}_1'(\Psi) \subseteq \mathcal{F}_{n-1}'(\Psi)$. Otherwise, $X'/\beta \eta(\Psi(i_0))$ is in $\mathcal{F}_{n-2}'(\Psi)$ and has therefore an ordered $\Psi$-filtration of length $n-2$. The lower horizontal sequence $\varepsilon'$ in the diagram shows then that $X/\eta(\Psi(i_0))$ has an ordered $\Psi$-filtration of length $n-1$. Moreover, the fact that $\varepsilon$ is a non-split exact sequence implies that $\varepsilon'$ does not split, and therefore $X/\eta(\Psi(i_0)) \in \mathcal{F}_{n-1}'(\Psi)$. □

**Proposition 3.10.** Let $(\Psi, \leq)$ be a proper pre-costratifying system of size $t$ in $\text{mod}(\Lambda)$. Then, any $\Lambda$-module in $\mathcal{F}'(\Psi)$ is indecomposable.

**Proof.** Let $X \in \mathcal{F}_n'(\Psi)$ and $i_0 = \text{min}(X)$. We show, by induction on $n$, that $X$ is an indecomposable $\Lambda$-module. If $n = 1$, then $X = \Psi(i_0)$, and hence $X$ is indecomposable.

Let $n > 1$. If $X \in \mathcal{F}_{n-1}'(\Psi)$, then we get by induction that $X$ is indecomposable. Otherwise, there is a non-split exact sequence $\varepsilon: 0 \to \Psi(i_0) \to X \xrightarrow{\beta} Z \to 0$ with $Z \in \mathcal{F}_{n-1}_n'(\Psi)$. Hence $Z$ is indecomposable by induction.

Suppose that $X = X' \oplus X''$ with $X' \neq 0$ and $X'' \neq 0$. Then $\pi = (\pi', \pi'') : \Psi(i_0) \to X' \oplus X''$ and $\beta = (\beta', \beta'') : X' \oplus X'' \to Z$, and by Lemma 3.1, we have that $\beta' \pi'(\Psi(i_0)) = -\beta'' \pi''(\Psi(i_0)) \neq 0$. Thus, by Proposition 3.8, we have that $\beta' \pi': \Psi(i_0) \to Z$ and $\beta' \pi' : \Psi(i_0) \to Z$ are monomorphisms. In particular, $\pi'$ and $\pi''$ are monomorphisms. Moreover, from $\beta' \pi'(\Psi(i_0)) = -\beta'' \pi''(\Psi(i_0))$ it follows that the epimorphism $\beta : X \to Z$ induces an epimorphism

\[
X'/\pi'(\Psi(i_0)) \oplus X''/\pi''(\Psi(i_0)) \to Z/\beta' \pi'(\Psi(i_0)).
\]

The equalities

\[
\ell_A(X'/\pi'(\Psi(i_0)) \oplus X''/\pi''(\Psi(i_0))) = \ell_A(X) - \ell_A(\Psi(i_0)) - \ell_A(\Psi(i_0)) - \ell_A(Z/\beta' \pi'(\Psi(i_0)) = \ell_A(Z) - \ell_A(\Psi(i_0)) = \ell_A(\Psi(i_0))
\]

show that $\beta$ is an isomorphism.
Next we show that $Z/\beta'x'(\Psi(i_0))$ is indecomposable. Suppose first that $Z/\beta'x'(\Psi(i_0)) = 0$. Then $x' = x'(\Psi(i_0))$, because $\beta$ is an isomorphism, and the morphism $((x')^{-1} 0) : X' \oplus X'' \to \Psi(i_0)$ shows that the nonsplit exact sequence $\varepsilon$ splits. This contradiction proves that $Z/\beta'x'(\Psi(i_0)) \neq 0$. Now, we consider the exact sequence

$$0 \longrightarrow \Psi(i_0) \overset{\beta'x'}{\longrightarrow} Z \longrightarrow Z/\beta'x'(\Psi(i_0)) \longrightarrow 0.$$

Since $Z \in \mathcal{T}'_{n-1}(\Psi)$, it follows from Proposition 3.9 that the module $Z/\beta'x'(\Psi(i_0))$ belongs to $\mathcal{T}'_{n-2}(\Psi)$ and is therefore indecomposable, by induction.

Finally, using that the map $\beta$ is an isomorphism, we have that $x'(\Psi(i_0)) = X'$ or $x'(\Psi(i_0)) = X''$, and hence $\varepsilon$ splits, a contradiction. Therefore, $X$ is an indecomposable $\Lambda$-module. □

We are now in a position to prove the main result of this section.

**Theorem 3.11.** Let $(\Psi, \leq)$ be a proper pre-costatifying system of size $t$ in mod $(\Lambda)$. If $\ell_A(\operatorname{ind}(\mathcal{T}(\Psi))) < \infty$, then there exists a family $\mathcal{Q} = \{Q(i)\}_{i=1}^t$ of $\Lambda$-modules in $\mathcal{T}'(\Psi)$ such that $(\Psi, \mathcal{Q}, \leq)$ is a proper costatifying system of size $t$ in mod $(\Lambda)$.

**Proof.** Let $\ell_A(\operatorname{ind}(\mathcal{T}(\Psi))) < \infty$. Since modules in $\mathcal{T}'(\Psi)$ are indecomposable (see Proposition 3.10), the proof is completed by showing that there exists a family $\mathcal{Q} = \{Q(i)\}_{i=1}^t$ of $\Lambda$-modules in $\mathcal{T}'(\Psi)$ satisfying conditions (c) and (d) in Definition 2.1.

To prove this we proceed by induction on the size $t$ of $\Psi$. If $t = 1$ the result follows directly from Lemma 3.3.

Let $t > 1$. For the sake of simplicity, we may assume that the order $\leq$ in $[1, t]$ is the natural one. Then $(\Psi = \{\Psi(2), \ldots, \Psi(t)\}, \leq)$ is a proper pre-costatifying system of size $t - 1$ in mod $(\Lambda)$. Therefore, from the inductive hypothesis applied to this smaller system, we conclude the existence of a family $\mathcal{Q} = \{Q(i)\}_{i=2}^t$ in $\mathcal{T}'(\Psi) \subseteq \mathcal{T}'(\Psi)$ satisfying the following conditions:

(\hat{c}) For each $i \in [2, t]$, there is an exact sequence

$$\tilde{\varepsilon}_i : 0 \longrightarrow \tilde{Z}(i) \longrightarrow \tilde{Q}(i) \overset{\varepsilon_i}{\longrightarrow} \Psi(i) \longrightarrow 0,$$

with $\tilde{Z}(i) \in \mathcal{T}(\{\Psi(j) : 2 \leq j \leq i\})$; and

(\hat{d}) $\tilde{Q} \subseteq \varepsilon_{i-1}^{-1}\Psi$.

Now, consider the family with just one element $\{\Psi(1)\}$. In this case, we already proved the theorem. Thus, there exists an exact sequence

$$\varepsilon_1 : 0 \longrightarrow Z(1) \longrightarrow Q(1) \overset{\beta}{\longrightarrow} \Psi(1) \longrightarrow 0,$$

with $Z(1) \in \mathcal{T}(\{\Psi(1)\})$, $Q(1) \in \mathcal{T}'(\{\Psi(1)\}) \subseteq \mathcal{T}'(\Psi)$, and $\operatorname{Ext}_1^\Lambda(Q(1), \Psi(1)) = 0$. Then $\varepsilon_1$ satisfies (c) in Definition 2.1. Furthermore, since $Q(1) \in \mathcal{T}(\{\Psi(1)\})$ and $\operatorname{Ext}_1^\Lambda(\Psi(1), \Psi(j)) = 0$ for $j \geq 2$, we have that $\operatorname{Ext}_1^\Lambda(Q(1), \Psi) = 0$. Thus (d) in Definition 2.1 holds.

We next construct the required exact sequence $\varepsilon_i$ for each $i \in [2, t]$. If $\operatorname{Ext}_1^\Lambda(\tilde{Q}(i), \Psi(1)) = 0$, then we set $\tilde{\varepsilon}_i = \varepsilon_i$ and $\tilde{Q}(i) = Q(i)$.
Suppose that $\text{Ext}_A^1(\widetilde{Q}(i), \Psi(1)) \neq 0$. Then there exists a nonsplit exact sequence

$$\delta_1 : 0 \to \Psi(1) \to X_1 \xrightarrow{\beta_1} \widetilde{Q}(i) \to 0.$$  

Since $\widetilde{Q}(i) \in \mathcal{F}(\Psi)$, we have that $X_1 \in \mathcal{F}(\Psi)$. Moreover, $X_1 \in \mathcal{F}(\Psi)$, as follows by applying $\text{Hom}_A(-, \Psi(j))$ to $\delta_1$ with $j \in [2, i]$. On the other hand, we have the exact sequence $v_i : 0 \to K_1 \to X_1 \xrightarrow{\lambda_i \beta_1} \Psi(i) \to 0$ where $K_1 = \text{Ker}(\lambda_i \beta_1)$. Then, by Lemma 3.2, $K_1 \in \mathcal{F}([\Psi(j) : j \leq i])$. Thus, if $\text{Ext}_A^1(X_1, \Psi(1)) = 0$, we conclude that $v_i = v_1$, with $Q(i) = X_1$, satisfies the required conditions.

Assume now that $\text{Ext}_A^1(X_1, \Psi(1)) \neq 0$. Then there exists a nonsplit exact sequence $\delta_1 : 0 \to \Psi(1) \to X_2 \xrightarrow{\beta_2} X_1 \to 0$. Here $X_2 \in \mathcal{F}(\Psi)$ because $X_1 \in \mathcal{F}(\Psi)$. Analogously, from $X_1 \in \mathcal{F}(\Psi)$, we have that $X_2 \in \mathcal{F}(\Psi)$. Moreover, by Lemma 3.2, we have an exact sequence $v_2 : 0 \to K_2 \to X_2 \xrightarrow{\lambda_2 \beta_2} \Psi(i) \to 0$ where $K_2 \in \mathcal{F}([\Psi(j) : j \leq i])$. Iterating this procedure, we obtain nonsplit exact sequences $\delta_j : 0 \to \Psi(1) \to X_j \xrightarrow{\beta_j} X_{j-1} \to 0$ with $X_j \in \mathcal{F}(\Psi) \cap \mathcal{F}(\Psi)$, for $1 \leq j \leq k$. Since $\ell_A(X_j) < \ell_A(X_2) < \cdots < \ell_A(X_1)$ and $\ell_A(\text{ind}(\mathcal{F}(\Psi))) < \infty$, we eventually reach some $X_n \in \mathcal{F}(\Psi)$ such that $\text{Ext}_A^1(X_n, \Psi(1)) = 0$, and then $X_n \in \mathcal{F}(\Psi)$. Therefore, the exact sequence $0 \to Z(i) \to Q(i) \xrightarrow{\beta} \Psi(i) \to 0$ with $Q(i) = X_n$, $\beta = \lambda_1 \beta_1 \beta_2 \cdots \beta_n$, and $Z(i) = \text{Ker}(\beta)$ satisfies the required conditions. \hfill \Box

**Corollary 3.12.** Let $A$ be an artin algebra of finite representation type. Then any proper pre-costratiﬁng system $(\Psi, \leq)$ of size $t$ in $\text{mod}(A)$ deﬁnes a proper costratiﬁng system $(\Psi, Q, \leq)$ of size $t$ in $\text{mod}(A)$.

### 4. SOME HOMOLOGICAL CHARACTERIZATIONS

Throughout the rest of this article, for $M$ in $\text{mod}(A)$, we consider the artin algebra $\Gamma_M = \text{End}(A, M)^{op}$ and the functors

$$\text{mod}(A) \xrightarrow{\mathcal{F}_M} \text{mod}(\Gamma),$$

where $F_M = \text{Hom}_A(M, -)$ and $G_M = M \otimes \Gamma -$, as well as the functors

$$\text{mod}(A) \xrightarrow{\mathcal{G}_M} \text{mod}(\Gamma^{op}),$$

where $\mathcal{F}_M = \text{Hom}_A(-, M)$ and $\mathcal{G}_M = \text{Hom}_{\Gamma^{op}}(-, M)$. We also have the functor $\ast = \text{Hom}_A(-, \Gamma) : \text{mod}(\Gamma) \to \text{mod}(\Gamma^{op})$. This functor induces a duality $\ast : \text{proj}(\Gamma) \to \text{proj}(\Gamma^{op})$, whose quasi-inverse $\text{Hom}_{\Gamma^{op}}(-, \Gamma^{op})$ is also denoted by $\ast$. Finally, we denote by $D$ the usual duality for artin algebras.

We recall that the functors $F_M$ and $G_M$ induce, by restriction, inverse equivalences between add $(M)$ and $\text{proj}(\Gamma)$. Furthermore, the functors $\mathcal{F}_M$ and $\mathcal{G}_M$ induce, by restriction, inverse equivalences between add $(M)$ and $\text{proj}(\Gamma^{op})$.

**Remark 4.1.** Notice that the functors $\mathcal{F}_M$ and $\ast \circ F_M$ coincide in add $(M)$.
We next recall the definition of the class $C^\vee_2(M)$, which was introduced by Platzeck and Pratti in [11] (the notation, used in [11], for such a class is $C^M_2$). The objects in $C^\vee_2(M)$ are the $\Lambda$-modules $X$ admitting an exact sequence in $\text{mod}(\Lambda)$

$$M_2 \to M_1 \to M_0 \to X \to 0$$

with $M_i \in \text{add}(M)$, and such that the induced sequence

$$F_M(M_2) \to F_M(M_1) \to F_M(M_0) \to F_M(X) \to 0$$

is exact in $\text{mod}(\Gamma)$.

Dually, we define the class $C^\wedge_2(M)$, consisting of the $\Lambda$-modules $Z$ admitting an exact sequence in $\text{mod}(\Lambda)$

$$0 \to Z \to M_0 \to M_1 \to M_2,$$

with $M_i \in \text{add}(M)$, and such that the induced sequence

$$\overline{F}_M(M_2) \to \overline{F}_M(M_1) \to \overline{F}_M(M_0) \to \overline{F}_M(Z) \to 0$$

is exact in $\text{mod}(\Gamma^\text{op})$.

Remark 4.2. Let $M$ be in $\text{mod}(\Lambda)$. Then, since $F_{DM} \circ D \simeq \overline{F}_M$ and $D \circ G_{DM} \simeq \overline{G}_M$, we have that $D(C^\vee_2(M)) = C^\wedge_2(DM)$. Furthermore, the rings $\Gamma_M$ and $\Gamma_{DM}^\text{op}$ are isomorphic.

Proposition 4.3. Let $M$ be in $\text{mod}(\Lambda)$, and let $\mathcal{C}$ be a class of objects in $\text{mod}(\Lambda)$ such that $M \in \mathcal{C}$. If $\mathcal{F}(\mathcal{C}) \subseteq C^\vee_2(M)$, then the restriction $\overline{F}_M|_{\mathcal{F}(\mathcal{C})} : \mathcal{F}(\mathcal{C}) \to \overline{F}(\mathcal{C})$ is an exact duality with quasi inverse the restriction $\overline{G}_M|_{\mathcal{F}(\mathcal{C})} : \overline{F}(\mathcal{C}) \to \mathcal{F}(\mathcal{C})$.

Proof. It follows from Remark 4.2 that this result is dual to the statement of Theorem 2.10 in [10]. □

For a given family $\overline{M} = \{M(i)\}_{i=1}^t$ of objects in $\text{mod}(\Lambda)$, we set $M = \bigoplus_{i=1}^t M(i)$. We now recall the definition of an Ext-injective stratifying system.

Definition 4.4 ([7, Definition 1.1]). An Ext-injective stratifying system $(\Theta, \leq)$ of size $t$ in $\text{mod}(\Lambda)$ consists of two families of nonzero $\Lambda$-modules $\Theta = \{\Theta(i)\}_{i=1}^t$ and $\mathcal{Y} = \{Y(i)\}_{i=1}^t$, with $Y(i)$ indecomposable for all $i$, and a linear order $\leq$ on the set $[1, t]$, satisfying the following conditions:

(a) $\text{Hom}_\Lambda(\Theta(i), \Theta(j)) = 0$ if $i > j$;
(b) For each $i \in [1, t]$, there is an exact sequence

$$0 \to \Theta(i) \to Y(i) \to Z(i) \to 0,$$

with $Z(i) \in \mathcal{F}(\{\Theta(j) : j < i\})$;
(c) $\text{Ext}_\Lambda^1(-, Y)_{|_0} = 0$. 
Throughout the rest of the section, $\mathcal{Y}$ denotes the family of indecomposable $\Lambda$-modules $\mathcal{Y} = \{Y(i)\}_{i=1}^{r}$, and we consider $\Gamma = \text{End}(\mathcal{Y})^{\text{op}}$ and the functors $F = F_{\mathcal{Y}}$, $G = G_{\mathcal{Y}}$, $\mathcal{F} = \mathcal{F}_{\mathcal{Y}}$, and $\mathcal{G} = \mathcal{G}_{\mathcal{Y}}$. We also consider the representative set $\Gamma \mathcal{P} = \{\Gamma(i) : i \in [1, r]\}$ of indecomposable projective $\Gamma^{\text{op}}$-modules, where $\Gamma(i) = \mathcal{F}(Y(i))$ for all $i$.

**Remark 4.5.** Notice that we have two representative sets of indecomposable projective $\Gamma^{\text{op}}$-modules: $\Gamma \mathcal{P}$ (defined above) and $\Gamma \mathcal{P} = \{\Gamma(i) = F(Y(i))^{\ast} : i \in [1, r]\}$. By Remark 4.1, it follows that $\Gamma(i) \simeq \Gamma(i)$ for all $i$. In particular, if $\leq$ is a linear order on $[1, r]$ and $\leq^{\text{op}}$ the opposite order to $\leq$, then there is no difference (up to isomorphism) between the family of standard (proper standard) $\Gamma^{\text{op}}$-modules computed by using either $(\Gamma \mathcal{P}, \leq)$ or $(\Gamma \mathcal{P}, \leq^{\text{op}})$.

We will need the following result, proven by E. Marcos, O. Mendoza, and C. Sáenz in [9], which gives necessary and sufficient conditions for a family of indecomposable $\Lambda$-modules $\mathcal{Y}$ to admit an Ext-injective stratifying system $(\Theta, \mathcal{Y}, \leq)$.

**Theorem 4.6** ([9, Theorem 2.3]). Let $\mathcal{Y} = \{Y(i)\}_{i=1}^{r}$ be a set of pairwise non-isomorphic indecomposable $\Lambda$-modules and be a linear order on $[1, r]$. Let $\Gamma \Delta$ be the family of standard $\Gamma^{\text{op}}$-modules corresponding to the pair $(\Gamma \mathcal{P}, \leq^{\text{op}})$, where $\leq^{\text{op}}$ is the opposite order to $\leq$. Then the following statements, (I) and (II), are equivalent:

(I) There exists a family $\Theta = \{\Theta(i)\}_{i=1}^{r}$ in mod($\Lambda$) such that $(\Theta, \mathcal{Y}, \leq)$ is an Ext-injective stratifying system;

(II) (a) $\text{Ext}^{1}_{\Gamma \Delta}(\Gamma \mathcal{P}, \mathcal{Y}) = 0$ and the pair $(\Gamma^{\text{op}}, \leq^{\text{op}})$ is a standardly stratified algebra;

(b) There is a full subcategory $\mathcal{A}$ of mod($\Lambda$), closed under extensions, and such that $\mathcal{Y} \subseteq \mathcal{A} \cap \mathcal{A}^{\text{op}}$;

(c) The restriction $F|_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}(\Gamma \Delta)$ is an exact duality with quasi inverse $G|_{\mathcal{F}(\Gamma \Delta)} : \mathcal{F}(\Gamma \Delta) \to \mathcal{A}$.

Moreover, if one of these equivalent conditions holds, then $\mathcal{A}$ is uniquely determined (up to equivalences) by the family $\mathcal{Y}$. More precisely, $\mathcal{A} \simeq \mathcal{F}(\Theta)$ and $\Theta(i) \simeq G|_{\mathcal{F}(\Gamma \Delta)}$ for all $i \in [1, r]$.

If we only assume that condition (a) in (II) holds then condition (b) in the definition of an Ext-injective stratifying system holds, as we prove next.

**Lemma 4.7.** With the hypothesis and notations of Theorem 4.6, let $\Phi = \{\Phi(i)\}_{i=1}^{r}$ where $\Phi(i) = G|_{\mathcal{F}(\Gamma \Delta)}$ for all $i$. If the pair $(\Gamma^{\text{op}}, \leq^{\text{op}})$ is a standardly stratified algebra and $\text{Ext}^{1}_{\Gamma \Delta}(\Gamma \mathcal{P}, \mathcal{Y}) = 0$, then the following conditions hold:

(a) For each $i \in [1, r]$, there is an exact sequence

$$0 \to \Phi(i) \to Y(i) \to Z(i) \to 0,$$

with $Z(i) \in \mathcal{F}(\{\Phi(j) : j < i\})$;

(b) If, moreover, $\text{Ext}^{1}_{\Lambda}(\Phi, Y) = 0$ then, for each $M \in \mathcal{F}(\Phi)$, there exists an exact sequence

$$0 \to M \to Y' \to M' \to 0$$

in $\mathcal{F}(\Phi)$ with $Y' \in \text{add}(Y)$. In particular $\mathcal{F}(\Phi) \subseteq C^{2}_{1}(Y)$. 


ON THE EXISTENCE OF PROPER COSTRATIFYING SYSTEMS

Proof. (a) Suppose that \( \operatorname{Ext}^{1\gammaop}_{\Gamma\Delta}(\Gamma\gammaop Y) = 0 \). Then, \( \overline{G}|_{\overline{\mathcal{F}}(\Gamma\gammaop \Delta)} : \overline{\mathcal{F}}(\Gamma\gammaop \Delta) \to \operatorname{mod}(\Lambda) \) is exact on \( \overline{\mathcal{F}}(\Gamma\gammaop \Delta) \). Since \((\Gamma^{\gammaop}, \leq^{\gammaop})\) is a standardly stratified algebra, we have for each \( i \in [1, t] \) an exact sequence in \( \overline{\mathcal{F}}(\Gamma\gammaop \Delta) \):

\[
0 \to U(i) \to \Gamma\gammaop P(i) \to \Gamma\gammaop \Delta(i) \to 0,
\]

with \( U(i) \in \mathcal{F}(\{1\gammaop \Delta(j) : j >^{\gammaop} i\}) \). By applying \( \overline{G} \) to the above sequence, we get the following exact sequence in \( \operatorname{mod}(\Lambda) \):

\[
0 \to \overline{G}(\Gamma\gammaop \Delta(i)) \to \overline{G}(\Gamma\gammaop P(i)) \to \overline{G}(U(i)) \to 0.
\]

We know that \( \Phi(i) = \overline{G}(\Gamma\gammaop \Delta(i)) \), \( \overline{G}(\Gamma\gammaop P(i)) = \overline{G}(\mathcal{F}(Y(i))) \cong Y(i) \) and also \( \overline{G}(U(i)) \in \mathcal{F}(\{\Phi(j) : j < i\}) \). Then condition (a) follows.

(b) Let \( M \in \mathcal{F}(\Phi) \). We consider a \( \Phi \)-filtration of \( M \)

\[
0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M,
\]

where \( M_k/M_{k-1} \cong \Phi(i_k) \), for all \( 1 \leq k \leq n \).

Using (a), Snake’s Lemma, and the fact that \( \Lambda Y \in \mathcal{F}(\Phi)^{\Delta_1} \), we get an exact and commutative diagram in \( \mathcal{F}(\Phi) \)

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 \to \Phi(i_1) \to M_2 \to \Phi(i_2) \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 \to Y(i_1) \to Y(i_1) \oplus Y(i_2) \to Y(i_2) \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 \to Z(i_1) \to Z_2 \to Z(i_2) \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

In particular, the middle vertical sequence \( 0 \to M_2 \to Y(i_1) \oplus Y(i_2) \to Z_2 \to 0 \) gives us the required exact sequence for \( n = 2 \). The result follows by iterating this argument. \( \square \)

Using Proposition 4.3 and Lemma 4.7 (b) we can prove the following result, where for a given family \( \underline{Y} = \{Y(i)\}_{i=1}^{t} \) of \( \Lambda \)-modules, we write \( \Gamma = \operatorname{End}(\Lambda Y)^{\gammaop} \), \( \mathcal{F} = \mathcal{F}_Y \), and \( \overline{G} = \overline{G}_Y \).

Theorem 4.8. Let \( \underline{Y} = \{Y(i)\}_{i=1}^{t} \) be a set of pairwise non-isomorphic indecomposable \( \Lambda \)-modules, and let \( \leq \) be a linear order on \([1, t]\). Let \( \Gamma\gammaop \Delta \) be the family of standard \( \Gamma^{\gammaop} \)-modules, computed by using the pair \( (\Gamma\gammaop P, \leq^{\gammaop}) \), where \( \leq^{\gammaop} \) is the opposite order to \( \leq \) and \( \Gamma\gammaop P(i) = \mathcal{F}(Y(i)) \) for all \( i \). Then there exists a family \( \Theta = \{\Theta(i)\}_{i=1}^{t} \) in \( \operatorname{mod}(\Lambda) \).
such that \((\Theta, Y, \leq)\) is an Ext-injective stratifying system if and only if the following conditions hold:

(a) The pair \((\Gamma^{\text{op}}, \leq^{\text{op}})\) is a standardly stratified algebra;
(b) \(\text{Ext}^1_{\Gamma^{\text{op}}} (\Gamma^{\text{op}} \Delta, \Gamma^{\text{op}} Y) = 0 = \text{Ext}^1 (\overline{G}(\Gamma^{\text{op}} \Delta), \lambda Y)\).

If these conditions hold, the Ext-injective stratifying system \((\Theta, Y, \leq)\) is uniquely determined (up to isomorphism) and \(\Theta(i) \simeq \overline{G}(\Gamma^{\text{op}} \Delta(i))\) for all \(i \in [1, t]\).

**Proof.** Suppose that there exists a family \(\Theta = \{\Theta(i)\}_{i=1}^t\) in \(\text{mod}(\Lambda)\) such that \((\Theta, Y, \leq)\) is an Ext-injective stratifying system. Then, by Theorem 4.6, we get that there exists a full subcategory \(\mathcal{A}\) of \(\text{mod}(\Lambda)\) such that \(\overline{G}_{|\Gamma^{\text{op}} \Delta} \subseteq \mathcal{A}\) and \(Y \subseteq \mathcal{A}^{+1}\). Hence, \(\text{Ext}^1 (\overline{G}(\Gamma^{\text{op}} \Delta), \lambda Y) = 0\). The rest of the proof of (a) and (b) follows immediately from the same theorem.

Assume now that (a) and (b) hold. Since \((\Gamma^{\text{op}}, \leq^{\text{op}})\) is a standardly stratified algebra and \(\text{Ext}^1_{\Gamma^{\text{op}}} (\Gamma^{\text{op}} \Delta, \Gamma^{\text{op}} Y) = 0\), the proof is completed by showing that (b) and (c) in Theorem 4.6 hold. Let \(\mathcal{A} = \Phi (\overline{G}(\Gamma^{\text{op}} \Delta))\). By applying Lemma 4.7 (a) and the equality \(\text{Ext}^1 (\overline{G}(\Gamma^{\text{op}} \Delta), \lambda Y) = 0\), we have that \(Y \subseteq \mathcal{A} \cap \mathcal{A}^{+1}\). On the other hand, it follows from Lemma 4.7 (b) that \(\mathcal{A} \subseteq \mathcal{C}^2 (Y)\). Then (c) in Theorem 4.6 holds by Proposition 4.3. \(\square\)

An algebra \(\Lambda\) is standardly stratified if and only if \(\lambda \Lambda\) is filtered by the standard modules. This is the case if and only if \(D(\Lambda, \lambda)\) is filtered by the corresponding proper costandard modules (see [5], Proposition 2.2). This fact, together with Remark 4.5 allows us to adapt the proof of Lemma 4.7 and prove the following result about proper costandard modules instead of standard modules.

**Lemma 4.9.** Let \(Y = \{Y(i)\}_{i=1}^t\) be a set of pairwise non-isomorphic indecomposable \(\Lambda\)-modules and \(\leq\) be a linear order on \([1, t]\). Let \(\overline{\nabla}\) be the family of proper costandard \(\Gamma^{\text{op}}\)-modules computed by using the pair \((\Gamma^{\text{op}} I, \leq^{\text{op}})\), where \(\Gamma^{\text{op}} I\) is the representative set of injective \(\Gamma^{\text{op}}\)-modules defined by \(\Gamma^{\text{op}} I(i) = D(F(Y(i)))\) for all \(i\). If \((\Gamma^{\text{op}}, \leq^{\text{op}})\) is a standardly stratified algebra, \(\text{Tor}^1_i (Y, \overline{\nabla}) = 0\) and \(\Phi(i) = G_1 (\overline{\nabla}(i))\) for all \(i\), then the following conditions hold:

(a) For each \(i \in [1, t]\), there is an exact sequence

\[
0 \rightarrow Z(i) \rightarrow Y(i) \rightarrow \Phi(i) \rightarrow 0,
\]

with \(Z(i) \in \Phi (\{\Phi(j) : j \leq i\})\);

(b) If, moreover, \(\text{Ext}^1 (Y, \Phi) = 0\) then, for each \(M \in \Phi (\Phi)\), there exists an exact sequence

\[
0 \rightarrow M' \rightarrow Y' \rightarrow M \rightarrow 0
\]

in \(\Phi (\Phi)\) with \(Y' \in \text{add} (Y)\). In particular, \(\Phi (\Phi) \subseteq \mathcal{C}^2 (Y)\).
5. RELATION BETWEEN PROPER COSTRATIFYING SYSTEMS AND EXT-INJECTIVE STRATIFYING SYSTEMS

Generalizing results of C. M. Ringel for quasi-hereditary algebras, it is proven in [2, 12] that, for a standardly stratified algebra \((\Lambda, \leq)\), there exists a tilting \(\Lambda\)-module \(T = \{T(i)\}_{i=1}^{n}\), called the characteristic tilting module, such that \(\text{add}(T) = \mathcal{F}(\Delta) \cap \mathcal{F}(\Delta)^{+1}\) and the pair \((\text{End}_\Lambda(T), \leq^{op})\) is again a standardly stratified algebra. Moreover, \((\bar{\Lambda}, \{\mathcal{T}(i)\}_{i=1}^{n}, \leq)\) is an Ext-injective stratifying system and \((\bar{\Lambda}, \{\mathcal{T}(i)\}_{i=1}^{n}, \leq)\) is a proper costratiﬁng system. This raises the following question: Given a set \(Q = \{Q(i)\}_{i=1}^{n}\) of pairwise non-isomorphic indecomposable \(\Lambda\)-modules and a linear order \(\leq\) on \([1, t]\), how does the existence of a proper costratiﬁng system \((\Psi, Q, \leq)\) relate to the existence of an Ext-injective stratifying system \((\Theta, Q, \leq)\)?

This section is devoted to answering this question.

We use throughout the following notation. Let \(Q = \{Q(i)\}_{i=1}^{n}\) be a set of pairwise non-isomorphic indecomposable \(\Lambda\)-modules, \(\leq\) be a linear order on \([1, t]\), and \(Q = \bigoplus_{i=1}^{n} Q(i)\). We consider the algebra \(\Gamma = \text{End}_\Lambda(Q)^{op}\) and the functors \(F = F_Q, \ G = G_Q, \ F = \bar{F}_Q, \ \text{and} \ G = \bar{G}_Q\). According with Remark 4.5, the family \(\Gamma, \Delta\) can be computed by using either \((\Gamma, \bar{\Delta}, \leq^{op})\) or \((\Gamma, \bar{\Delta}, \leq^{op})\), where \(\Gamma(i) = \bar{F}(Q(i))\) and \(\Gamma(i) = F(Q(i))\) for all \(i\). We also consider the family \(\Gamma, \Delta\) which is computed with the order \(\leq^{op}\) on \([1, t]\), by using the representative set \(\Gamma(i)\) of projective indecomposable \(\Gamma\)-modules, where \(\Gamma(i) = F(Q(i))\) for all \(i\).

5.1. From Proper Costratiﬁng Systems to Ext-Injective Stratifying Systems

Let \((\Psi, Q, \leq)\) be a proper costratiﬁng system of size \(t\) in \(\text{mod}(\Lambda)\). We recall from [10, Theorem 4.3], that \((\Gamma, \leq^{op})\) is a standardly stratified algebra and the restriction \(F|_{\mathcal{F}(\Psi)} : \mathcal{F}(\Psi) \to \mathcal{F}(\Delta)\) is an equivalence with quasi inverse \(G|_{\mathcal{F}(\Delta)} : \mathcal{F}(\Delta) \to \mathcal{F}(\Psi)\).

In our next theorem we state, for a given proper costratiﬁng system \((\Psi, Q, \leq)\), necessary and suﬃcient conditions for the existence of a family \(\Theta = \{\Theta(i)\}_{i=1}^{n}\) in \(\text{mod}(\Lambda)\) such that \((\Theta, Q, \leq)\) is an Ext-injective stratifying system.

**Theorem 5.1.** Let \((\Psi, Q, \leq)\) be a proper costratiﬁng system of size \(t\) in \(\text{mod}(\Lambda)\). Then the following conditions are equivalent:

(a) There exists a family \(\Theta = \{\Theta(i)\}_{i=1}^{n}\) in \(\text{mod}(\Lambda)\) such that \((\Theta, Q, \leq)\) is an Ext-injective stratifying system;

(b) \(\text{Ext}^1_{\Gamma, \Delta}(\Gamma, Q) = 0 = \text{Ext}^1_{\Lambda}(\bar{\mathcal{G}}(\Gamma, \Delta), \Lambda Q)\).

If these conditions hold, the system \((\Theta, Q, \leq)\) is uniquely determined (up to isomorphism) and \(\Theta(i) \cong \bar{\mathcal{G}}(\Gamma, \Delta(i))\) for all \(i\).

**Proof.** The proof follows directly from Theorem 4.8, since by [10, Theorem 4.3] we know that \((\Gamma, \leq^{op})\) is a standardly stratified algebra.

**Example 5.2.** In the following example, we give a proper costratiﬁng system \((\Psi, Q, \leq)\) and apply Theorem 5.1 to get an Ext-injective stratifying system \((\Theta, Q, \leq)\).
Let $\Lambda$ be the path algebra given by the quiver

$$
\begin{array}{c}
\circ_1 \\
\circ_2 \\
\circ_3 \\
\end{array}
\xrightarrow{\varepsilon} \xrightarrow{\mu} \xrightarrow{\eta}
$$

with the natural order $1 \leq 2 \leq 3$. Consider $\Psi = \{\Psi(1) = 3, \Psi(2) = 1, \Psi(3) = \frac{1}{2}\}$ and $Q = \{(Q(1) = 3, Q(2) = 1, Q(3) = \frac{1}{2}\}$. Then $(\Psi, Q, \leq)$ is a proper costratifying system of size 3 in mod($\Lambda$). In this case, the algebra $\Gamma^{op} = \text{End}_\Lambda(Q)$ is given by the quiver

$$
\begin{array}{c}
\circ_1 \\
\circ_2 \\
\circ_3
\end{array}
\xrightarrow{\varepsilon} \xrightarrow{\mu}
$$

with the relation $\mu \varepsilon = 0$. We consider $(\Gamma^{op}, \leq^{op})$, where $3 \leq^{op} 2 \leq^{op} 1$. Then the corresponding standard modules are $\Gamma^{\theta} \Delta = \{\Gamma^{\theta} \Delta(1) = \frac{1}{3}, \Gamma^{\theta} \Delta(2) = 2, \Gamma^{\theta} \Delta(3) = 3\}$, and $\Gamma^{\theta} Q = \frac{1}{2} \oplus 3 \oplus \frac{1}{2}$. Using the fact that $\Gamma^{\theta} \Delta(1)$ and $\Gamma^{\theta} \Delta(2)$ are projective modules, that $\Gamma^{\theta} Q(1)$ and $\Gamma^{\theta} Q(3)$ are injective modules, and that $\text{Ext}^1_{\Gamma^{\theta}}(\Gamma^{\theta} \Delta(1), \Gamma^{\theta} Q(2)) = 0$, we get that $\text{Ext}^1_{\Gamma^{\theta}}(\Gamma^{\theta} \Delta, \Gamma^{\theta} Q) = 0$. It remains to check that $\text{Ext}^1_{\Gamma^{\theta}}(\Gamma^{\theta} \Delta(1), Q) = 0$. Since $\mathcal{G}(\Gamma^{\theta} \Delta(1)) = \mathcal{G}(\Gamma^{\theta} Q(3)) = \Lambda P(3)$ and $\mathcal{G}(\Gamma^{\theta} \Delta(3)) = \mathcal{G}(\Gamma^{\theta} Q(2)) = \Lambda P(2)$ are projective modules, then the required condition holds. On the other hand, a computation shows that $\mathcal{G}(\Gamma^{\theta} \Delta(2)) = 1$ and $\text{Ext}^1_{\Lambda}(\mathcal{G}(\Gamma^{\theta} \Delta(2)), Q) = 0$. Thus, by Theorem 5.1, $(\Theta, Q, \leq)$ is an Ext-injective stratifying system with $\Theta(1) \simeq \mathcal{G}(\Gamma^{\theta} \Delta(1)) = 3, \Theta(2) \simeq \mathcal{G}(\Gamma^{\theta} \Delta(2)) = 1$, and $\Theta(3) \simeq \mathcal{G}(\Gamma^{\theta} \Delta(3)) = \frac{1}{2}$.

### 5.2. From Ext-Injective Stratifying Systems to Proper Costratifying Systems

The aim of this subsection is to find, for a given Ext-injective stratifying system $(\Theta, Q, \leq)$, necessary and sufficient conditions for the existence of a family $\Psi = \{\Psi(i)\}_{i=1}^t$ in mod($\Lambda$) such that $(\Psi, Q, \leq)$ is a proper costratifying system. We first consider the more general problem of studying the existence of such a proper costratifying system $(\Psi, Q, \leq)$ assuming only that $Q = \{Q(i)\}_{i=1}^t$ is a family of pairwise non-isomorphic indecomposable $\Lambda$-modules. In analogy with [9, Theorem 2.3] and Theorem 4.8, we prove the following result.

**Theorem 5.3.** Let $Q = \{Q(i)\}_{i=1}^t$ be a set of pairwise non-isomorphic indecomposable $\Lambda$-modules. Then, the following conditions, (I), (II), and (III) are equivalent:

(I) (a) $(\Gamma^{op}, \leq^{op})$ is a standardly stratified algebra and $\text{Tor}_1^\Gamma(Q, \Gamma^{\eta} \Delta) = 0$.

(b) There is a full subcategory $\mathcal{B}$ of mod($\Lambda$), closed under extensions, and such that $Q \subseteq \mathcal{B} \cap \mathcal{B}^{\perp} \mathcal{B}$.

(c) The restriction $F|_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{F}(\Gamma^{\eta} \Delta)$ is an exact equivalence of categories with quasi inverse $G|_{\mathcal{F}(\Gamma^{\eta} \Delta)} : \mathcal{F}(\Gamma^{\eta} \Delta) \rightarrow \mathcal{B}$.

(II) There exists a family $\Psi = \{\Psi(i)\}_{i=1}^t$ in mod($\Lambda$) such that $(\Psi, Q, \leq)$ is a proper costratifying system.

(III) (a) $(\Gamma^{op}, \leq^{op})$ is a standardly stratified algebra.

(b) $\text{Tor}_1^\Gamma(Q, \Gamma^{\eta} \Delta) = 0 = \text{Ext}_1^\Gamma(Q, G(\Gamma^{\eta} \Delta))$.

Moreover, if these conditions hold, then $\mathcal{B}$ is uniquely determined by the family $Q$. More precisely, $\mathcal{B} \simeq \mathcal{F}(\Psi)$ and $\Psi(i) \simeq G(\Gamma^{\eta} \Delta(i))$ for all $i \in [1, t]$. 

**Theorem 4.8** can be adapted to this case, by using Lemma 4.9 and [10, Theorem 2.10].

We are now in a position to show the main result of this subsection, which we state in the next theorem and is an analogue of Theorem 5.1.

**Theorem 5.4.** Let \((\Theta, \mathcal{Q}, \leq)\) be an Ext-injective stratifying system of size \(t\) in \(\mod \Lambda\). Then, the following conditions are equivalent:

(a) There exists a family \(\Psi = \{\Psi(i)\}_{i=1}^{t}\) in \(\mod \Lambda\) such that \((\Psi, \mathcal{Q}, \leq)\) is a proper costratifying system;

(b) \(\text{Tor}_1^\Lambda(Q, \Gamma(i) \overline{\Delta}) = 0 = \text{Ext}_1^\Lambda(Q, \Gamma(i) \overline{\Delta})\).

If these conditions hold, the system \((\Psi, \mathcal{Q}, \leq)\) is uniquely determined (up to isomorphism) and \(\Psi(i) \simeq \Gamma(i) \overline{\Delta}(i)\) for all \(i \in [1, t]\).

**Proof.** The proof follows immediately from Theorem 5.3 and Remark 4.5, since \((\Gamma^{op}, \leq^{op})\) is a standardly stratified algebra ([7, Proposition 1.3]).

The following result shows that, for a given family \(\mathcal{Q}\) of \(\Lambda\)-modules over an algebra \(\Lambda\) not necessarily standardly stratified, the simultaneous existence of \(\Theta\) and \(\Psi\) such that \((\Theta, \mathcal{Q}, \leq)\) is an Ext-injective stratifying system and \((\Psi, \mathcal{Q}, \leq)\) is a proper costratifying system, implies that \(\Gamma^{op} = \text{End}(\mathcal{Q})\) is a standardly stratified algebra and \(\Gamma^{op} \overline{T}\) coincides with the characteristic tilting module associated to \(\Gamma^{op}\).

**Corollary 5.5.** Let \((\Psi, \mathcal{Q}, \leq)\) be a proper costratifying system of size \(t\) in \(\mod \Lambda\). Then, the following conditions are equivalent, where \(\Gamma^{op} \overline{T}\) is the characteristic tilting module associated to the standardly stratified algebra \((\Gamma^{op}, \leq^{op})\).
(a) There exists a family \( \Theta = \{\Theta(i)\}_{i=1}^t \) in \( \text{mod}(\Lambda) \) such that \( (\Theta, Q, \leq) \) is an Ext-injective stratifying system;
(b) \( \Gamma^0 Q \simeq \Gamma^0 T \) and \( \text{Ext}^1_\Lambda(G(\Gamma, \Delta), _\Lambda Q) = 0 \).

**Proof.** (a) \( \Rightarrow \) (b) Suppose that \( (\Theta, Q, \leq) \) is an Ext-injective stratifying system. Since \( (\Psi, Q, \leq) \) is a proper costratifying system, it follows from Theorem 5.1 that \( \text{Ext}^1_\Lambda(G(\Gamma, \Delta), _\Lambda Q) = 0 \). To prove that \( \Gamma^0 Q \simeq \Gamma^0 T \), it is enough to show that \( \Gamma^0 Q \in \mathcal{F}(\Gamma, \Delta) \cap \mathcal{P}(\Gamma, \Delta)^{\pm 1} = \text{add}(\Gamma^0 T) \) (see [2, Proposition 2.2]). From Theorem 5.4 (b), we have that \( \text{Tor}^1_\Lambda(Q, \Gamma \Delta) = 0 \). Then, by [3, p. 120], \( \text{Ext}^1_\Lambda(\Gamma^0 Q, \Gamma^0 \Psi) \simeq D(\text{Tor}^1_\Psi(Q, \Gamma \Delta)) = 0 \), that is, \( Q \in \mathcal{P}(\Gamma, \Delta) \) (see [2, Theorem 1.6]). On the other hand, we get from Theorem 5.1 (b) that \( \text{Ext}^1_\Lambda(\Gamma^0 \Delta, \Gamma^0 Q) = 0 \), that is, \( Q \in \mathcal{F}(\Gamma, \Delta)^{\pm 1} \). Thus (b) holds.

(b) \( \Rightarrow \) (a) Since \( \Gamma^0 T \) is the characteristic tilting module, we get \( \text{Ext}^1_\Lambda(\Gamma^0 \Delta, \Gamma^0 Q) \simeq \text{Ext}^1_\Lambda(\Gamma^0 \Delta, \Gamma^0 T) = 0 \). Therefore, (a) follows from Theorem 5.1.

We illustrate this result with the following example.

**Example 5.6.** Let \( \Lambda \) be given by the quiver

\[\begin{array}{c}
1 \\
\alpha \\
2 \\
\beta \\
3
\end{array}\]

with the relations \( \beta^2 = 0, \alpha \beta = 0, \) and \( \beta_2 = 0 \). Consider the natural order \( 1 \leq 2 \leq 3 \), and the sets \( \Psi = \{\Psi(1) = 2, \Psi(2) = \frac{1}{3}, \Psi(3) = \frac{4}{3}\} \) and \( Q = \{Q(1) = \frac{1}{3}, Q(2) = \frac{5}{3}, Q(3) = 1^{1/3}\} \). Then \( (\Psi, Q, \leq) \) is a proper costratifying system of size 3 in \( \text{mod}(\Lambda) \). In this case, \( \Gamma^0 = \text{End}(Q) \) is given by the quiver

\[\begin{array}{c}
1 \\
\epsilon \\
2 \\
\delta \\
3
\end{array}\]

with the relations \( \epsilon \mu = 0 \) and \( \mu \delta \mu = 0 \). It is easy to check that \( (\Gamma^0, \leq^0) \) is a standardly stratified algebra. The characteristic tilting module is \( \Gamma^0 T = \begin{align*} 
1 \\
\frac{1}{3} \\
\frac{1}{2} \\
\frac{1}{2} \\
1 
\end{align*} \oplus \begin{align*} 
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} 
\end{align*} \oplus \begin{align*} 
1 \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} 
\end{align*} \). Hence it follows from Corollary 5.5 (b) that there exists no family \( \Theta = \{\Theta(i)\}_{i=1}^t \) in \( \text{mod}(\Lambda) \) such that \( (\Theta, Q, \leq) \) is an Ext-injective stratifying system.

The following result can be proved by arguments similar to those used in the proof of Corollary 5.5.

**Corollary 5.7.** Let \( (\Theta, Q, \leq) \) be an Ext-injective stratifying system of size \( t \) in \( \text{mod}(\Lambda) \). Then the following conditions are equivalent, where \( \Gamma^0 T \) is the characteristic tilting module associated to the standardly stratified algebra \( (\Gamma^0, \leq^0) \):

(a) There exists a family \( \Psi = \{\Psi(i)\}_{i=1}^t \) in \( \text{mod}(\Lambda) \) such that \( (\Psi, Q, \leq) \) is a proper costratifying system;
(b) \( \Gamma^0 Q \simeq \Gamma^0 T \) and \( \text{Ext}^1_\Lambda(Q, G(\Gamma, \Delta)) = 0 \).
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