NON-NEGATIVE PERTURBATIONS OF NON-NEGATIVE
SELF-ADJOINT OPERATORS

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ABSTRACT. Let $A$ be a non-negative self-adjoint operator in a Hilbert space $\mathcal{H}$
and $A_0$ be some densely defined closed restriction of $A_0$, $A_0 \subseteq A \neq A_0$. It is of
interest to know whether $A$ is the unique non-negative self-adjoint extensions of $A_0$ in $\mathcal{H}$. We give a natural criterion that this is the case and if it fails, we
describe all non-negative extensions of $A_0$. The obtained results are applied to
investigation of non-negative singular point perturbations of the Laplace and
poly-harmonic operators in $L_2(\mathbb{R}_n)$.

1. INTRODUCTION

In this paper we deal with a non-negative self-adjoint operator $A$ in a Hilbert
space $\mathcal{H}$, some densely defined not essentially self-adjoint restriction $A_0$ of $A$ and
again with self-adjoint extensions of $A_0$ in $\mathcal{H}$, which following [1] we call here
singular perturbations of $A$. For quick getting onto the matter of main problem
let us compare the point perturbations of self-adjoint Laplace operators $-\Delta$ in
three and two dimensions acting in $L_2(\mathbb{R}_3)$ and $L_2(\mathbb{R}_2)$, respectively, that is let us
consider the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H^2_0(\mathbb{R}_i \setminus \{0\})$, $i = 3, 2$
and self-adjoint extensions $-\Delta_{\alpha}$, $\alpha \in \mathbb{R}$ of $-\Delta^0$ in $L_2(\mathbb{R}_i)$ with domains

\begin{equation}
D^{(3)}_\alpha := \{ f : f \in H_0^2(\mathbb{R}_3), \lim_{|x|\to 0} \frac{|\alpha|}{|x|^2} (|x| f(x)) - \alpha |x| f(x) = 0 \},
\end{equation}

\begin{equation}
D^{(2)}_\alpha := \{ f : f \in H_0^2(\mathbb{R}_2), \lim_{|x|\to 0} \left(\frac{2\pi \alpha}{|x|^2} + 1\right) f(x) - \lim_{|x'|\to 0} \ln |x| f(x') = 0 \}.
\end{equation}

The self-adjoint operators $-\Delta_{\alpha}$ are just mentioned above singular perturbations of
$-\Delta$. Resolvents $(-\Delta_{\alpha} - z)^{-1}$, $z \in \rho(-\Delta_{\alpha})$, of operators $-\Delta_{\alpha}$ act in the corre-
sponding spaces $L_2$ as integral operators with kernels (Green functions) [1]:

\begin{equation}
G^{(3)}_{\alpha,z}(x, x') = \begin{cases}
G^{(0)}_z(x, x') + (\alpha - i\sqrt{2}/4\pi)^{-1} G^{(0)}_z(x, 0) G^{(0)}_z(0, x'),
\end{cases}
\end{equation}

\begin{equation}
G^{(2)}_{\alpha,z}(x, x') = \begin{cases}
G^{(0)}_z(x, x') + 2\pi (2\pi \alpha - \psi(1) + \ln \sqrt{2}/2\pi)^{-1} G^{(0)}_z(x, 0) G^{(0)}_z(0, x'),
\end{cases}
\end{equation}

\begin{equation}
G^{(2)}_z(x, x') = \left(\frac{i}{4}\right) H_0^{(1)}(i\sqrt{2}|x - x'|) \quad \text{(two dimension)}.
\end{equation}

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By (1.2) the Green function $G_{\alpha,z}(x,x')$ of self-adjoint operator $-\Delta_{\alpha}$ in $L^2(\mathbb{R}^3)$ is holomorphic on the half-axis $(-\infty,0)$ for $\alpha \geq 0$ and has on this half-axis a simple pole for $\alpha < 0$. Hence in the case of three dimensions self-adjoint extensions $-\Delta_{\alpha}$ are non-negative for all ($\alpha \geq 0$) and non-positive for $\alpha < 0$.

Contrary to this by (1.3) in the case of two dimensions for any $\alpha \in \mathbb{R}$ the Green function $G_{\alpha,z}$ has a simple pole on the half-axis $(-\infty,0)$. Hence all singular perturbations $-\Delta_{\alpha}$ of the two-dimensional Laplace operators have one negative eigenvalue. In other words the standardly defined Laplace operator $-\Delta$ is the unique non-negative self-adjoint extension in $L^2(\mathbb{R}_2)$ of the symmetric operator $-\Delta^0$ in $L^2(\mathbb{R}_2)$.

In this note we try to reveal the underlying cause of such discrepancy. Remind that each densely defined non-negative symmetric operator has at least one non-negative canonical self-adjoint extension (Friedrichs extension). In more general setting we try to understand here why in some cases the non-negative extension appears to be unique. Actually this questions is embedded into the framework of the general extension theory for semi-bounded symmetric operators developed in the famous paper of M.G. Krein [2]. Naturally, there is a criterium of uniqueness of non-negative extension in [2]. In the next Section using only approaches of [2] we find another form of this criterium directly facilitated to investigation of singular perturbations and for cases where conditions of these criterium fail describe all non-negative singular perturbations of a given non-negative self-adjoint operator $A$ associated with some its densely defined non-self-adjoint restriction $A_0$. In fact we give here a parametrization of the operator interval $[A_\mu,A_M]$ of all canonical non-negative self-adjoint extensions of a given densely defined non-negative operator. The third Section illustrates obtained results by the example of singular perturbations of Laplace and poly-harmonic operators in $L^2(\mathbb{R}_n)$.

Note that very close results were obtained recently in somewhat different way in [3], where in terms of this note were described singular perturbations of the Friedrichs extension of a given densely defined non-negative operator and also with illustration by the example of singular perturbations of the Laplace operator in $L^2(\mathbb{R}_3)$.

2. Uniqueness criterium and parametrization of non-negative singular perturbations

Let $A$ be a non-negative self-adjoint operator acting in the Hilbert space $\mathcal{H}$ and $A_0$ be a densely defined closed operator, which is a restriction of $A$ onto a subset $\mathcal{D}(A_0)$ of the domain $\mathcal{D}(A)$ of $A$. Let us consider the subspaces $\mathcal{M} := (I + A_0 \mathcal{D}(A_0))$ and $\mathcal{N} := \mathcal{H} \ominus \mathcal{M}$. We will assume that

$$\begin{align*}
1) \quad & \mathcal{M} \neq \mathcal{H}, \\
2) \quad & \mathcal{N} \cap \mathcal{D}(A) = \{0\}.
\end{align*}$$

We call all self-adjoint extensions of $A_0$ in $\mathcal{H}$ other than the given $A$ singular perturbations of $A$. It is of interest to know whether there are non-negative operators among singular perturbations of $A$. In this section we try to find a convenient criterium that such singular perturbations of $A$ does not exist. In other words we look for a criterium that $A$ is one and only non-negative operator among all self-adjoint extensions of $A_0$. Following the approach developed in the renowned paper of M.G. [2],

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1The attention of author to this phenomenon was drawn by Sergey Gredeskul.
Krein [2] let us consider the operator from $K_0 : \mathcal{M} \to \mathcal{H}$ defined by relations
\begin{equation}
(2.2) \quad f = (I + A_0)x, \quad K_0 f = A_0 x, \quad x \in \mathcal{D}(A_0).
\end{equation}

It is easy to see that $K_0$ is a non-negative contraction:
\begin{equation}
(2.3) \quad (K_0 f, f) \geq 0, \quad \|K_0 f\|^2 \leq \|f\|^2, \quad f \in \mathcal{M}.
\end{equation}

Let $A_1$ be any non-negative self-adjoint extension of $A_0$ in $\mathcal{H}$. Then $K_1 := A_1 (A_1 + I)^{-1}$ is a non-negative operator, which is a contractive extension of $K_0$ from the domain $\mathcal{M}$ onto the whole $\mathcal{H}$, $K_1 f = K_0 f, f \in \mathcal{M}$.

From the other hand for any contractive extension $K_1$ from $\mathcal{M}$ onto $\mathcal{H}$ such that the unity is not its eigenvalue the non-negative self-adjoint operator $A_1 = K_1 (I - K_1)^{-1}$ is a self-adjoint extension in $\mathcal{H}$ if and only if $K_0$ admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which is $1$, that is $K = A(I + A)^{-1}$. So the uniqueness of $A$ as non-negative extension of $A_0$ is equivalent to uniqueness of $K_0$ as non-negative contractive extension of $K_0$.

From now on we will denote by G the set consisting of $A$ and all its singular perturbations and by $C$ the set of non-negative contractions obtained from $G$ by transformation $A_1 \to A_1 (A_1 + I)^{-1}, A_1 \in G$. Let us denote by $P_M$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$ and let $P_N = I - P_M$. With respect to representation of $\mathcal{H}$ as the orthogonal sum $\mathcal{M} \oplus \mathcal{N}$ we can represent each operator from $C$ as $2 \times 2$ block operator matrix
\begin{equation}
(2.4) \quad K_X = \begin{pmatrix} T & \Gamma^* \\ \Gamma & X \end{pmatrix}
\end{equation}

Here
\begin{align*}
T &= P_M K_0|_M, \quad \Gamma = P_M K_0|_M, \\
\end{align*}

and $X$ is some non-negative contraction in $\mathcal{N}$, which distinguishes different elements from $C$. Since each $K_X \in C$ is non-negative and contractive then
\begin{equation}
(2.5) \quad T \geq 0; \quad T^2 + \Gamma^* \Gamma \leq I
\end{equation}

Note further that
\begin{equation}
(2.6) \quad K_X + \varepsilon I \geq 0; \quad (1 + \varepsilon)I - K_X \geq 0
\end{equation}

for any $\varepsilon > 0$.

The block matrix representation of $K_X$ and the Schur-Frobenius factorization formula transform $2.6$ into the following block matrix inequalities:
\begin{equation}
(2.7) \quad \begin{pmatrix} I & 0 \\ \Gamma(T + \varepsilon)^{-1} & I \end{pmatrix} \begin{pmatrix} T + \varepsilon & 0 \\ 0 & X + \varepsilon - \Gamma(T + \varepsilon)^{-1}\Gamma^* \end{pmatrix} \begin{pmatrix} I \\ (T + \varepsilon)^{-1}\Gamma^* \end{pmatrix} \geq 0,
\end{equation}

\begin{equation}
(2.8) \quad \begin{pmatrix} I & 0 \\ -\Gamma(I + \varepsilon - T)^{-1} & I \end{pmatrix} \begin{pmatrix} 1 + \varepsilon - T & 0 \\ 0 & 1 + \varepsilon - X - \Gamma(1 + \varepsilon - T)^{-1}\Gamma^* \end{pmatrix} \begin{pmatrix} I \\ -(1 + \varepsilon - T)^{-1}\Gamma^* \end{pmatrix} \geq 0.
\end{equation}
By our assumptions $T \geq 0$ and $I - T \geq 0$. Therefore block matrix inequalities (2.7) and (2.8) are reduced to

\[
\begin{align*}
X + \varepsilon I - \Gamma(T + \varepsilon I)^{-1}\Gamma^* & \geq 0, \\
(1 + \varepsilon)I - X - \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* & \geq 0, \quad \varepsilon > 0.
\end{align*}
\]

Observe that operator functions of $\varepsilon$ in the left hand sides of inequalities (2.9) are monotone. Setting

\[
Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T + \varepsilon I)^{-1}\Gamma^*
\]

we conclude from (2.9) that

\[
KX \in C \text{ if and only if } (2.10) 0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \{\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^*\}.
\]

Hence the equality

\[
(2.11) \quad I - \lim_{\varepsilon \downarrow 0} \{\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^*\} = 0
\]

is the criterium that there are no contractive non-negative extension of $K_0$ in $\mathcal{H}$ other than $K$.

Let us express now (2.10) and (2.11) in terms of given $K$ and $A$. To this end we use the following proposition.

**Proposition 2.1.** Let $L$ be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as $2 \times 2$ block operator matrix,

\[
L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},
\]

where $R$ and $S$ are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and $U, V$ act between $\mathcal{M}$ and $\mathcal{N}$. If $R$ is invertible operator in $\mathcal{M}$, then

\[
\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1}P_N\Lambda^{-1}P_NL^{-1}, \quad \Lambda = P_NL^{-1}|_N.
\]

Setting

\[
\begin{align*}
\Lambda_1, \varepsilon &= P_N(K + \varepsilon I)^{-1}|_N, \\
\Lambda_2, \varepsilon &= P_N[(1 + \varepsilon)I - K]^{-1}|_N
\end{align*}
\]

and applying (2.12) with $L = K + \varepsilon I$ and

\[
\begin{align*}
R &= T + \varepsilon I, \\
U &= \Gamma^* = P_MK|_N = P_M[K + \varepsilon I]|_N, \\
V &= \Gamma = P_NK|_M = P_N[K + \varepsilon I]|_M, \\
S &= P_N\Lambda|_N + \varepsilon I
\end{align*}
\]

yields

\[
\Gamma(T + \varepsilon I)^{-1}\Gamma^* = P_NK|_N + \varepsilon I - \Lambda_1^{-1}.
\]

In the same fashion we get

\[
\Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* = P_N[I - K]|_N + \varepsilon I - \Lambda_2^{-1}.
\]

Hence

\[
(2.14) \quad I - \lim_{\varepsilon \downarrow 0} \{\Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^*\} = \lim_{\varepsilon \downarrow 0} \Lambda_1^{-1} + \lim_{\varepsilon \downarrow 0} \Lambda_2^{-1}.
\]

Combining (2.10), (2.11) and (2.14) results in the following theorem.
**Theorem 2.2.** Let $K$ be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, $K_0$ is the restriction of $K$ onto the subspace $\mathcal{M} = \mathcal{M} \oplus \{0\}$ and

$$G_1 = \lim_{\epsilon \downarrow 0} (P_N[K + \epsilon I]_{|\mathcal{N}})^{-1} \quad G_2 = \lim_{\epsilon \downarrow 0} (P_N[I - K + \epsilon I]_{|\mathcal{N}})^{-1}$$

Then the set $C$ of all non-negative contractive extensions $K_X$ of $K_0$ in $\mathcal{H}$ is described by expression

$$(2.15) \quad K_X = \begin{pmatrix} P_M K_{|\mathcal{M}} & P_M K_{|\mathcal{N}} \cr P_M K_{|\mathcal{N}} & X \end{pmatrix},$$

where $X$ runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$(2.16) \quad P_N K_{|\mathcal{N}} - G_1 \leq X \leq P_N K_{|\mathcal{N}} + G_2.$$ 

In particular, $K$ is the unique non-negative contractive extension of $K_0$ if and only if $G_1 = G_2 = 0$.

**Remark 2.3.** The set $C$ of non-negative contractive of $K_0$ contains the minimal extension $K_{X_{\mu}}$ with $X_{\mu} = P_N K_{|\mathcal{N}} - G_1$ in (2.15) and the maximal extension $K_{X_{M}}$ with $X_{M} = P_N K_{|\mathcal{N}} + G_2$ in (2.15). If $G_1 = 0 (G_2 = 0)$, then $K$ is the minimal (maximal) element of $C$.

Theorem 2.2 can be formulated in terms of non-negative self-adjoint operator $A$ and its non-negative singular perturbations.

**Theorem 2.4.** Let $A$ be a non-negative self-adjoint operator in the Hilbert space $\mathcal{H}$, $A_0$ is a densely defined closed symmetric operator, which is a restriction of $A$ onto a linear subset $\mathcal{D}(A_0) \subset \mathcal{D}(A)$ such that $\mathcal{N} = (I + A)\mathcal{D}(A_0) \neq \{0\}$ and let

$$G_1 = \lim_{\epsilon \downarrow 0} (P_N[I + A][A + \epsilon I]_{|\mathcal{N}})^{-1} \quad G_2 = \lim_{\epsilon \downarrow 0} (P_N[I + A][I + \epsilon A]_{|\mathcal{N}})^{-1}$$

Then the set of all non-negative singular perturbations $A_Y$ of $A$ is described by the formula

$$(2.17) \quad \left\{ \begin{array}{l} f = g - Y(I + A)g, \\ A_Y f = Ag + Y(I + A)g, \end{array} \right.$$ 

where $g \in \mathcal{D}(A)$ and $Y$ runs the set of non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$(2.18) \quad -G_1 \leq Y \leq G_2.$$ 

$A$ has no singular non-negative perturbations if and only if $G_1 = G_2 = 0$.

**Remark 2.5.** The set of all non-negative singular perturbations of $A$ contains the minimal perturbation $A_{\mu}$ with and the maximal perturbation $A_{M}$ such that any non-negative perturbation $A_X$ satisfies inequalities

$$(I + A_M)^{-1} \leq A_Y \leq (I + A_{\mu})^{-1}.$$ 

The corresponding values of parameters $Y$ in Theorem 2.4 are

$$(2.19) \quad Y_{\mu} = -G_1 \quad Y_{M} = G_2$$

If $G_1 = 0 (G_2 = 0)$, then the minimal (maximal) perturbation coincides with $A$.

By simple calculation we get from (2.17) the following version of the M.G. Krein resolvent formula.
Proposition 2.6. The set of resolvents of all non-negative singular perturbations $A_Y$ of $A$ is described by the M.G. Krein formula

\[(A_Y - zI)^{-1} = (A - zI)^{-1}\]

\[-(1 + z)(A + I)(A - zI)^{-1} Y \left[ (1 + z)P_N(A + I)(A - zI)^{-1} Y \right]^{-1} \times \]

\[P_N(A + I)(A - zI)^{-1},\]

where $Y$ runs contractions in $N$ satisfying inequalities $-G_1 \leq Y \leq G_2$.

3. Application to some differential operators

Let us consider the multiplication operator $A$ in $L^2(\mathbb{R}^n)$ by the continuous function $\varphi(k)$, $k^2 = k_1^2 + \ldots + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

\[\int_0^\infty \frac{1}{(1 + \varphi(k))^2} k^{n-1} dk < \infty.\]

$A$ is a non-negative self-adjoint operator,

\[\mathcal{D}(A) = \left\{ f : \int_{\mathbb{R}^n} |1 + \varphi(k)|^2 |f(k)|^2 dk < \infty, \; f \in L^2(\mathbb{R}^n) \right\}.\]

In the sequel $\hat{\delta}$ stands for the unbounded linear functional in $L^2(\mathbb{R}^n)$, formally defined as follows:

\[\hat{\delta}(f) = \int_{\mathbb{R}^n} f(k) dk.\]

Note that the domain of $\hat{\delta}$ contains $\mathcal{D}(A)$. Let us denote by $A_0$ the restriction of $A$ onto linear set

\[\mathcal{D}_0(A) := \left\{ f : f \in \mathcal{D}(A), \; \hat{\delta}(f) = 0. \right\} \]

The closure of $A_0 \neq A$ and

\[\mathcal{N} = (L^2(\mathbb{R}^n) \ominus (I + A)\mathcal{D}(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)}, \; \xi \in \mathbb{C} \right\}.\]

Applying Theorem 2.4 yields

Proposition 3.1. $A$ is the unique non-negative self-adjoint extension of $A_0$ that is $A$ has no non-negative singular perturbations if and only if

\[\int_0^\infty \frac{1}{\varphi(k)(1 + \varphi(k))} k^{n-1} dk = \infty \quad \text{and} \quad \int_0^\infty \frac{1}{(1 + \varphi(k))} k^{n-1} dk = \infty.\]

Put $\varphi(k) = k^2$ and let $n = 2$. Then the both integrals in Proposition 3.1 are divergent. Hence the restriction $A_0$ of the operator $A$ of multiplication by $k^2$ in $L^2(\mathbb{R}^2)$ onto the linear set (3.2) has unique non-negative self-adjoint extension in $L^2$. Note that the multiplication operator by $k^2$ in $L^2(\mathbb{R})$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $L^2(\mathbb{R})$ and its concerned here restriction $A_0$ is isomorphic to the restriction $-\Delta$ onto the Sobolev subspace $H^2_0(\mathbb{R} \setminus \{0\})$. As follows, the self-adjoint Laplace operator in $L^2(\mathbb{R}^2)$ has no non-negative singular perturbations with support at one point of $\mathbb{R}^2$.

However, the non-negative singular perturbations of $-\Delta$ in $L^2(\mathbb{R}^2)$ with support at two or more points do already exist. For example, let us consider there the
Hence \( G \) is one-dimensional and consists of functions collinear to \( N \) operator \( A \) extensions. It remains to note that \( A \) subset of function \( f \) \( L \) extensions in \( A \) in this case self-adjoint Laplace operator in \( A \) 3.1 is convergent while the second one as before divergent. Hence the restriction \( R \) has infinitely many non-negative self-adjoint extension in \( L \) the maximal element in the set of this perturbation.

Proposition 3.2. assuming that \( 4 \) \((perturbations of \( -\Delta \) polyharmonic operator \( ) \) is non-negative symmetric operator which is isomorphic to the restrictions \( A \) of the in \( -\Delta \)), Let us consider the restriction \( R \) has no such perturbations in \( L \). As the next example we consider the multiplication operator \( A \) by \( k^2 \) if \( n > 2l \) then there is the infinite set of non-negative singular perturbations of \( -\Delta \) associated with \( A_0 \) and for those as non-negative extensions of \( A_0 \) in the set of the in \( H^2 \) \( (R_2 \setminus \{0\}) \) the operator \( -\Delta \) is the maximal element.

Put now as above \( \varphi(k) = k^2 \) and let \( n = 3 \). Then the first integral in Proposition 3.1 is convergent while the second one as before divergent. Hence the restriction \( A_0 \) of the operator \( A \) of multiplication by \( k^2 \) in \( L_2(R_3) \) onto the linear set \( 3.2 \) has infinitely many non-negative self-adjoint extension in \( L_2(R_3) \). As follows, the self-adjoint Laplace operator in \( L_2(R_3) \) has infinitely many non-negative singular perturbations with support at one point of \( R_3 \) and the standardly defined Laplace the maximal element in the set of this perturbation.

As the next example we consider the multiplication operator \( A \) by \( k^{2l} \) in \( L_2(R_n) \) assuming that \( 4l \leq n + 1 \). \( A \) is isomorphic to the polyharmonic operator \( (-\Delta)^l \) in \( L_2(R_n) \). Let us consider the restriction \( A_0 \) of \( A \) with the domain \( 3.2 \) that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator \( (-\Delta)^l \) onto the Sobolev subspace \( H^2 \) \( (R_2 \setminus \{0\}) \). Applying Theorem 2.3 and Proposition 3.1 results in the following proposition.

Proposition 3.2. If \( n < 2l \) then there are infinitely many non-negative singular perturbations of \( (-\Delta)^l \) associated with the one-point symmetric restriction \( A_0 \) and \( (-\Delta)^l \) is the minimal element in the set of the non-negative extensions of \( A_0 \) in \( H^2 \) \( (R_n \setminus \{0\}) \).

If \( n = 2l \) then \( (-\Delta)^l \) has no such perturbations in \( H^2 \) \( (R_n \setminus \{0\}) \).

If \( n > 2l \) then there is the infinite set of non-negative singular perturbations of \( (-\Delta)^l \) associated with \( A_0 \) and for those as non-negative extensions of \( A_0 \) in the set of the in \( H^2 \) \( (R_n \setminus \{0\}) \) the operator \( -\Delta \) is the maximal element.
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