Optimal Distributed Weighted Set Cover Approximation

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Abstract

We present a time-optimal deterministic distributed algorithm for approximating a minimum weight vertex cover in hypergraphs of rank $f$. This problem is equivalent to the Minimum Weight Set Cover Problem in which the frequency of every element is bounded by $f$. The approximation factor of our algorithm is $(f + \epsilon)$. Let $\Delta$ denote the maximum degree in the hypergraph. Our algorithm runs in the congest model and requires $O(\log \Delta / \log \log \Delta)$ rounds, for constants $\epsilon \in (0, 1]$ and $f \in \mathbb{N}^+$. This is the first distributed algorithm for this problem whose running time does not depend on the vertex weights or the number of vertices. Thus adding another member to the exclusive family of provably optimal distributed algorithms.

For constant values of $f$ and $\epsilon$, our algorithm improves over the $(f + \epsilon)$-approximation algorithm of [KMW06] whose running time is $O(\log \Delta + \log W)$, where $W$ is the ratio between the largest and smallest vertex weights in the graph.

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1 Introduction

In the Minimum Weight Vertex Cover (MWVC) problem, we are given an undirected graph \( G = (V, E) \) with vertex weights \( w : V \rightarrow \{1, \ldots, W\} \), for \( W = n^{O(1)} \). The goal is to find a minimum weight cover \( U \subseteq V \) such that \( \forall e \in E : e \cap U \neq \emptyset \). This problem is one of the classical NP-hard problems presented in \[\text{Kar72}].

In this paper, we consider the Minimum Weight Hypergraph Vertex Cover (MWHVC) problem, a generalization of the MWVC problem to hypergraphs of rank \( f \). In a hypergraph, \( G = (V, E) \), each edge is a nonempty subset of the vertices. A hypergraph is of rank \( f \) if the size of every hyperedge is bounded by \( f \). The MWVC problem naturally extends to MWVC using the above definition. Note that MWVC is equivalent to the Minimum Weight Set Cover problem with element frequencies bounded by \( f \).

We consider the MWVC problem in the distributed setting, where the communication network is a bipartite graph \( H(E \cup V, \{\{e, v\} \mid v \in e\}) \). We refer to the network vertices as nodes and network edges as links. The nodes of the network are the hypergraph vertices on one side and hyperedges on the other side. There is a network link between vertex \( v \in V \) and hyperedge \( e \in E \) if \( v \in e \). The computation is performed in synchronous rounds, where messages are sent between neighbors in the communication network. As for message size, we consider the CONGEST model where message sizes are bounded to \( O(\log |V|) \). This is more restrictive than the LOCAL model where message sizes are unbounded.

Denoting by \( \Delta \) the maximum vertex degree in \( G \), any distributed constant-factor approximation algorithm requires \( O(\log \Delta / \log \log \Delta) \) rounds to terminate, even for unweighted graphs and \( f = 2 \) \[\text{KMW16}\]. Two results match the lower bound. For the Minimum Weight Vertex Cover Problem in graphs \( (f = 2) \), the lower bound was matched by \[\text{BCS17}\] with a \((2 + \epsilon)\)-approximation algorithm (BCS algorithm) with optimal round complexity for every \( \epsilon = \Omega(\log \log \Delta / \log \Delta) \). The progress of the BCS algorithm is analyzed via a trade-off between reducing the degree of the vertices and reducing the weight of the vertices. We do not know how to generalize the BCS algorithm and its analysis to hypergraphs. For the Minimum Cardinality Vertex Cover in Hypergraphs Problem, the lower bound was matched by \[\text{EGM18}\] with an \((f + \epsilon)\)-approximation algorithm. The round complexity in \[\text{EGM18}\] is \( O\left(f/\epsilon \cdot \log(f \cdot \Delta) / \log \log(f \cdot \Delta)\right) \), which is optimal for constant \( f \) and \( \epsilon \). The algorithm in \[\text{EGM18}\] and its analysis is a deterministic version of the maximal independent set algorithm of \[\text{Gha16}\]. We do not know how to generalize the algorithm in \[\text{EGM18}\] and its analysis to hypergraphs with vertex weights.

In this paper, we present a deterministic distributed \((f + \epsilon)\)-approximation algorithm for minimum weight vertex cover in \( f \)-rank hypergraphs, which completes in \( O(\log \Delta / \log \log \Delta) \) rounds in the CONGEST model, for any constants \( \epsilon \in (0, 1) \) and \( f \in \mathbb{N}^+ \). Our running time is optimal according to a lower bound by \[\text{KMW16}\]. This is the first distributed algorithm for this problem whose round complexity does not depend on the node weights. For constant values of \( f \), Astrand et al. \[\text{AS10}\] present an \( f \)-approximation algorithm whose running time is \( O(\Delta^2 + 2 \cdot \log^* W) \) (\( W \) is the ratio between the largest and smallest weights in the graph). Kuhn et al. \[\text{Kuhn05} \text{, KMW06}\] present an \((f + \epsilon)\)-approximation algorithm that terminates in \( O(\log \Delta + \log W) \) rounds. To the best of our knowledge, these are the only works that deal with the Minimum Weight Hypergraph Vertex Cover Problem (MWVC) in the distributed setting.

\[1\] Recently, the range of \( \epsilon \) for which the runtime is optimal was improved to \( \Omega(\log^{-c} \Delta) \) for any \( c = O(1) \) \[\text{BEKS18}\].

\[2\] The dependence of the round complexity of our algorithm on \( \epsilon \) and \( f \) is given by \( O\left(\log \Delta / \log \log \Delta + \left(\frac{\Delta}{2}\right)^{1/\gamma} \cdot \log \log \Delta\right) \), for every constant \( \gamma \in (0, 1) \) (Theorem \[\text{C1}\].

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Our algorithm is one of a handful of distributed algorithms for local problems which are provably optimal \[ \text{BCGS17, BCS17, CKP16, CV86, GS17, EGM18}. \] Among these are the classic Cole-Vishkin algorithm [CV86] for 3-coloring a ring, the more recent results of \[ \text{BCGS17} \] and \[ \text{BCS17} \] for MWVC and Maximum Matching, and the very recent result of \[ \text{EGM18} \] for Minimum Cardinality Hypergraph Vertex Cover.

1.1 Tools and techniques

Our solution employs the Primal-Dual schema. The Primal-Dual approach introduces, for every hyperedge \( e \in E \), a dual variable denoted by \( \delta(e) \). The dual edge packing constraints are \( \forall v \in V, \sum_{e \in E} \delta(e) \leq w(v) \). If for some \( \beta \in [0, 1] \) it holds that \( \sum_{e \in E} \delta(e) \geq (1 - \beta) \cdot w(v) \), we say the \( v \) is \( \beta \)-tight. Let \( \beta = \varepsilon/(f + \varepsilon) \). For every feasible dual solution, the weight of the set of \( \beta \)-tight vertices is at most \((f + \varepsilon)\) times the weight of an optimal (fractional) solution. The algorithm terminates when the set of \( \beta \)-tight edges constitutes a vertex cover.

The challenge in designing a distributed algorithm is in controlling the rate at which we increase the dual variables. On the one hand, we must grow them rapidly to reduce the number of communication rounds. On the other hand, we may not violate the edge packing constraints. The algorithm proceeds in iterations, each of which requires a constant number of communication rounds. On the one hand, we must grow them rapidly to reduce the number of communication rounds. On the other hand, we may not violate the edge packing constraints. The algorithm proceeds in iterations, each of which requires a constant number of communication rounds. We initialize the dual variables in a 'safe' way so that feasibility is guaranteed. We refer to the additive increase of the dual variable \( \delta(e) \) in iteration \( i \) by \( \text{deal}_i(e) \). Loosely speaking, the algorithm increases the increments deal\(_i\)(e) exponentially (multiplication by \( \alpha \)) provided that no vertex \( v \in e \) is \((\beta/\alpha)\)-tight with respect to the deals of the previous iteration. Otherwise, the increment deal\(_i\)(e) equals the previous increment deal\(_{i-1}\)(e). The analysis builds on two observations: (1) The number of times that the increment deal\(_i\)(e) is multiplied by \( \alpha \) is bounded by \( \log_\alpha \Delta \). (2) The number of iterations in which a vertex is \((\beta/\alpha)\)-tight with respect to the deals of the previous iteration is at most \( \alpha/\beta \). Hence the total number of iterations is bounded by \( \log_\alpha \Delta + f \cdot \alpha/\beta \). Setting \( \alpha = \log \Delta / \log \log \Delta \) implies that the number of iterations is \( O(\log \Delta / \log \log \Delta) \).

2 Problem Formulation

Let \( G = (V, E) \) denote a hypergraph. Vertices in \( V \) are equipped with nonnegative weights \( w(v) \). For a subset \( U \subseteq V \), let \( w(U) \triangleq \sum_{v \in U} w(v) \). Let \( E(U) \) denote the set of hyperedges that are incident to some vertex in \( U \) (i.e., \( E(U) \triangleq \{ e \in E \mid e \cap U \neq \emptyset \} \)).

The Minimum Weight Hypergraph Vertex Cover Problem \((\text{mwhvc})\) is defined as follows.

**Input:** Hypergraph \( G = (V, E) \) with vertex weights \( w(v) \).

**Output:** A subset \( C \subseteq V \) such that \( E(C) = E \).

**Objective:** Minimize \( w(C) \).

The \text{mwhvc} Problem is equivalent to the Weighted Set Cover Problem. Consider a set system \((X, U)\), where \( X \) denotes a set of elements and \( U = \{ U_1, \ldots, U_m \} \) denotes a collection of subsets of \( X \). The reduction from the set system \((X, U)\) to a hypergraph \( G = (V, E) \) proceeds as follows. The set of vertices is \( V \triangleq \{ u_1, \ldots, u_m \} \) (one vertex \( u_i \) per subset \( U_i \)). The set of edges is \( E \triangleq \{ e_x \}_{x \in X} \) (one hyperedge \( e_x \) per element \( x \)), where \( e_x \triangleq \{ u_i : x \in U_i \} \). The weight of vertex \( u_i \) equals the weight of the subset \( U_i \).
3 Distributed \((f + \varepsilon)\)-Approximation Algorithm for \textsc{mwhvc}

3.1 Input

The input is a hypergraph \(G = (V, E)\) with non-negative vertex weights \(w : V \to \mathbb{R}^+\) and an approximation ratio parameter \(\varepsilon \in (0, 1]\). We denote the rank of \(G\) by \(f\) (i.e., each hyperedge contains at most \(f\) vertices) and the maximum degree of \(G\) by \(\Delta\) (i.e., each vertex belongs to at most \(\Delta\) edges).

Assumptions. We assume that (i) Vertex weights are polynomial in \(n = |V|\) so that sending a vertex weight requires \(O(\log n)\) bits. (ii) Vertex degrees are polynomial in \(n\) (i.e., \(|E(v)| = n^{O(1)}\)) so that sending a vertex degree requires \(O(\log n)\) bits. Since \(|E(v)| \leq n^f\), this assumption trivially holds for constant \(f\). (iii) The maximum degree is at least 3 so that \(\log \log \Delta > 0\).

3.2 Output

A vertex cover \(C \subseteq V\). Namely, for every hyperedge \(e \in E\), the intersection \(e \cap C\) is not empty. The set \(C\) is maintained locally in the sense that every vertex \(v\) knows whether it belongs to \(C\) or not.

3.3 Communication Network

The communication network \(N(E \cup V, \{\{e, v\} | v \in e\})\) is a bipartite graph. There are two types of nodes in the network: servers and clients. The set of servers is \(V\) (the vertex set of \(G\)) and the set of clients is \(E\) (the hyperedges in \(G\)). There is a link \((v, e)\) from server \(v \in V\) to a client \(e \in E\) if \(v \in e\). We note that the degree of the clients is bounded by \(f\) and the degree of the servers is bounded by \(\Delta\).

3.4 Parameters and Variables

- The approximation factor parameter \(\varepsilon \in (0, 1]\) and the rank \(f\) determine the parameter \(\beta\) defined by \(\beta \triangleq \varepsilon / (f + \varepsilon)\).

- The parameter \(\alpha\) is set to \(\log \Delta / \log \log \Delta\) and determines the factor by which “deals” are multiplied\(^3\). See Section \(\ref{sec:alpha}\) in the Appendix for a setting of \(\alpha\) that reduces the dependency of the running time on \(\varepsilon\) and \(f\).

- We denote the dual variables at the end of iteration \(i\) by \(\delta_i(e)\) (see Appendix \(\ref{sec:lp}\) for a description of the dual edge packing linear program). The amount by which \(\delta_i(e)\) is increased in iteration \(i\) is denoted by deal\(_i\)(\(e\)). Namely, \(\delta_i(e) = \sum_{j \leq i} \text{deal}_j(e)\).

3.5 Notation

- We say that an edge \(e\) is \textit{covered} by \(C\) if \(e \cap C \neq \emptyset\).

- Let \(E(v) \triangleq \{e \in E | v \in e\}\) denote the set of hyperedges that contain \(v\).

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\(^3\) For simplicity, we assume that \(\Delta\) is known and that \(\Delta \geq 3\). The assumption that the maximal degree \(\Delta\) is known to all vertices is not required. Instead, each hyperedge \(e\) can compute a local maximum degree \(\Delta(e)\), where \(\Delta(e) \triangleq \max_{u \in e} |E(u)|\). The local maximum degree \(\Delta(e)\) can be used instead of \(\Delta\) to define local value of the multiplier \(\alpha = \alpha(e)\).
• For every vertex $v$, the algorithm maintains a subset $E'(v) \subseteq E(v)$ that consists of the uncovered hyperedges in $E(v)$ (i.e., $E'(v) = \{e \in E(v) \mid e \cap C = \emptyset\}$).

### 3.6 Algorithm MWHVC

1. Initialization. Set $C \leftarrow \emptyset$. For every vertex $v$, set $E'(v) \leftarrow E(v)$.

2. Iteration $i = 0$. The edge $e$ collects the weight $w(v)$ and degree $|E(v)|$ from every vertex $v \in e$, and sets: $\text{deal}_0(e) = \beta \cdot \min_{v \in e} \{w(v)/|E(v)|\}$. The value $\text{deal}_0(e)$ is sent to every $v \in e$. The dual variable is updated $\delta_0(e) \leftarrow \text{deal}_0(e)$.

3. For $i = 1$ to $\infty$ do:
   
   (a) Every vertex $v \notin C$ checks if it is $\beta$-tight. If $\sum_{e \in E(v)} \delta_{i-1}(e) \geq (1-\beta) \cdot w(v)$, then $v$ joins the cover $C$, sends a message to every $e \in E'(v)$ that $e$ is covered, and $(v)$ terminates.
   
   (b) For every uncovered edge $e$, if $e$ receives a message that it is covered, then it tells all its vertices that $e$ is covered, and $e$ terminates.
   
   (c) For every vertex $v \notin C$, if it receives a message from $e$ that $e$ is covered, then $E'(v) \leftarrow E'(v) \setminus \{e\}$. If $E'(v) = \emptyset$, then $v$ terminates (without joining the cover).
   
   (d) For every vertex $v \notin C$, if $\sum_{e \in E'(v)} \text{deal}_{i-1}(e) \leq \beta \cdot w(v)$, then send the message “raise” to every $e \in E'(v)$, else send the message “stuck” to every $e \in E'(v)$.
   
   (e) For every uncovered edge $e$. If $e$ received a “stuck” message then $\text{deal}_i(e) \leftarrow \text{deal}_{i-1}(e)$, else (if all incoming messages are “raise”) $\text{deal}_i(e) \leftarrow \alpha \cdot \text{deal}_{i-1}(e)$). Send $\text{deal}_i(e)$ to every $v \in e$, who updates $\delta_i(e) \leftarrow \delta_{i-1}(e) + \text{deal}_i(e)$.

**Termination** Every vertex $v$ terminates when either $v \in C$ or every edge $e \in E(v)$ is covered (i.e., $E'(v) = \emptyset$). Every edge $e$ terminates when it is covered (i.e., $e \cap C \neq \emptyset$).

**Execution in CONGEST.** See Section [3] in the Appendix for a discussion of how Algorithm MWHVC is executed in the CONGEST model.

### 4 Algorithm Analysis

#### 4.1 Approximation Ratio

The following claim states that, in each iteration, the sum of the deals of edges incident to a vertex $v$ is bounded by $\beta \cdot w(v)$.

**Claim 4.1.** If $v \notin C$, then $\sum_{e \in E'(v)} \text{deal}_i(e) \leq \beta \cdot w(v)$.

**Proof.** The proof is by induction on $i$. The induction basis, for $i = 0$, holds because $\text{deal}_0(e) \leq \beta \cdot w(v)/|E(v)|$ for every edge $e \in v$. The induction step, for $i \geq 1$, considers two cases. If $\text{deal}_{i+1}(e) = \text{deal}_i(e)$ for every $e \in E'(v)$, then the induction step follows from the induction hypothesis. If there exists an edge $e \in E'(v)$ such that $\text{deal}_{i+1}(e) = \alpha \cdot \text{deal}_i(e)$, then Step 3(b) implies that $\sum_{e \in E'(v)} \text{deal}_i(e) \leq \alpha \cdot \sum_{e \in E'(v)} \text{deal}_{i-1}(e) \leq \beta \cdot w(v)$, as required. \[\square\]

If an edge $e$ is covered in iteration $j$, then $e$ terminates and $\delta_i(e)$ is not set for $i \geq j$. In this case, we define $\delta_j(e) = \delta_{j-1}(e)$, namely, the last value assigned to a dual variable.
Claim 4.2. For every $i \geq 0$ the dual variables $\delta_i(e)$ constitute a feasible edge packing. Namely,

$$
\sum_{e \in E(v)} \delta_i(e) \leq w(v) \quad \text{for every vertex } v \in V,
$$

$$
\delta_i(e) \geq 0 \quad \text{for every edge } e \in E.
$$

Proof. Nonnegativity follows from the initialization and the positive increases by deals. The packing constraints are proved by induction on the number of iterations. The induction basis, for $i = 0$, holds because $\sum_{e \in E(v)} \delta_0(e) = \sum_{e \in E(v)} \text{deal}_0(e) = \beta \cdot w(v)$. (Recall that $\beta = \varepsilon/(f + \varepsilon) < 1$.) The induction step is proved as follows. By Step 3c, if $e \in E'(v)$, then $\delta_i(e) = \delta_{i-1}(e) + \text{deal}_i(e)$, otherwise $\delta_i(e) = \delta_{i-1}(e)$. By Step 3a, $\sum_{e \in E(v)} \delta_{i-1}(e) < (1 - \beta) \cdot w(v)$. By Claim 4.1 in Appendix A, $\sum_{e \in E'(v)} \text{deal}_i(e) \leq \beta \cdot w(v)$, and the claim follows.

Let $\text{opt}$ denote the cost of an optimal (fractional) weighted vertex cover of $G$.

Corollary 4.3. Upon termination, the approximation ratio of Algorithm MWHVC is $f + \varepsilon$.

Proof. Throughout the algorithm, the set $C$ consists of $\beta$-tight vertices. By Claim A.1, $w(C) \leq (f + \varepsilon) \cdot \text{opt}$. Upon termination, $C$ constitutes a vertex cover, and the corollary follows.

4.2 Communication Rounds Analysis

In this section, we prove that the number of communication rounds of Algorithm MWHVC is bounded by $O\left(\frac{f^2 \log \Delta}{\varepsilon \log \log \Delta}\right)$. It suffices to bound the number of iterations because each iteration consists of a constant number of communication rounds.

4.2.1 Raise or Stuck Iterations

Definition 4.4. An iteration $i \geq 1$ is an $e$-raise iteration if $\text{deal}_i(e) = \alpha \cdot \text{deal}_{i-1}(e)$. An iteration $i$ is a $v$-stuck iteration if $v$ sent the message “stuck” in iteration $i$.

Note that if iteration $i$ is a $v$-stuck iteration and $v \in e$, then $\text{deal}_i(e) = \text{deal}_{i-1}(e)$ and $i$ is not an $e$-raise iteration.

We bound the number of $e$-raise iterations as follows.

Lemma 4.5. The number of $e$-raise iterations is bounded by $\log_\alpha \Delta$.

Proof. Let $v^*$ denote a vertex with minimum normalized weight in $e$. The first deal satisfies $\text{deal}_0(e) = \beta \cdot w(v^*)/|E(v^*)| \geq \beta \cdot w(v^*)/\Delta$. By Claim 4.1, $\text{deal}_i(e) \leq \beta \cdot w(v^*)$. Since the deal is multiplied by $\alpha$ in each $e$-raise iteration, the lemma follows.

We bound the number of $v$-stuck as follows.

Lemma 4.6. The number of $v$-stuck iterations is bounded by $\frac{\Delta}{f^2}$.

Proof. Suppose that iteration $i$ (for $i \geq 1$) is a $v$-stuck iteration. This implies that $\sum_{e \in E'(v)} \text{deal}_{i-1}(e) > \frac{\Delta}{\alpha} \cdot w(v)$. Thus $\delta_{i-1}(e) - \delta_{i-2}(e) > \frac{\Delta}{\alpha} \cdot w(v)$. Had there been more than $\frac{\Delta}{f^2}$ iterations that are $v$-stuck, then the dual variable $\delta(e)$ would be larger than $w(v)$, contradicting Claim 4.1.2.
4.2.2 Putting it Together

**Theorem 4.7.** Fix some $\alpha > 1$, the number of iterations of Algorithm MWHVC is

$$O\left(\log_\alpha \Delta + f \cdot \frac{\alpha}{\beta}\right)$$

**Proof.** Fix an edge $e$. We bound the number of iterations until $e$ is covered as follows. Every iteration is either an $e$-raise iteration or a $v$-stuck iteration for some $v \in e$. Since $e$ contains at most $f$ vertices, we conclude that the number of iterations is bounded by the number of $e$-stuck iterations plus the sum over $v \in e$ of the number of $v$-stuck iterations. The theorem follows from Lemmas 4.5 and 4.6. \qed

Finally, by setting $\alpha$ appropriately, we bound the running time as follows.

**Corollary 4.8.** If $\alpha = \log \Delta / \log \log \Delta$, then the round complexity of Algorithm MWHVC is $O\left(\frac{f^2}{\epsilon} \cdot \frac{\log \Delta}{\log \log \Delta}\right)$.

A refined assignment of $\alpha$ that leads to a reduced dependency of the running time on $\epsilon$ and $f$ is presented in Section C in the Appendix.

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A Primal-Dual Approach

The fractional LP relaxation of mwhvc is defined as follows.

\[ \text{minimize: } \sum_{v \in V} w(v) \cdot x(v) \]
\[ \text{subject to: } \]
\[ \sum_{v \in e} x(v) \geq 1, \text{ } \forall e \in E \]
\[ x(v) \geq 0, \text{ } \forall v \in V \]

The dual LP is an Edge Packing problem defined as follows:

\[ \text{maximize: } \sum_{e \in E} \delta(e) \]
\[ \text{subject to: } \]
\[ \sum_{e \ni v} \delta(e) \leq w(v), \text{ } \forall v \in V \]
\[ \delta(e) \geq 0, \text{ } \forall e \in E \]

The following claim is used for proving the approximation ratio of the mwhvc algorithm.

Claim A.1. Let \( \text{opt} \) denote the value of an optimal fractional solution of the primal LP \( (P) \). Let \( \{\delta(e)\}_{e \in E} \) denote a feasible solution of the dual LP \( (D) \). Let \( \varepsilon \in (0,1) \) and \( \beta \triangleq \varepsilon/(f + \varepsilon) \). Define the \( \beta \)-tight vertices by:
\[ T_\varepsilon \triangleq \{ v \in V \mid \sum_{e \ni v} \delta(e) \geq (1 - \beta) \cdot w(v) \} \]

Then \( w(T_\varepsilon) \leq (f + \varepsilon) \cdot \text{opt} \).
Proof.

\[ w(T_ε) = \sum_{v \in T_ε} w(v) \]

\[ \leq \frac{1}{1 - \beta} \cdot \left( \sum_{v \in T_ε} \sum_{e \ni v} \delta(e) \right) \]

\[ \leq \frac{f}{1 - \beta} \sum_{e \in E} \delta(e) \leq (f + \varepsilon) \cdot \text{opt}. \]

The last transition follows from \( f/(1 - \beta) = f + \varepsilon \) and by weak duality. The claim follows. \( \square \)

B Adaptation to the CONGEST model

To complete the discussion, we need to show that the message lengths in Algorithm \textsc{mwhvc} are \( O(\log n) \).

1. In round 0, every vertex \( v \) sends its weight \( w(v) \) and degree \( |E(v)| \) to every hyperedge in \( e \in E(v) \). We assume that the weights and degrees are polynomial in \( n \), hence the length of the binary representations of \( w(v) \) and \( |E(v)| \) is \( O(\log n) \).

Every hyperedge \( e \) sends back to every \( v \in e \) the pair \( (w(v_e), |E(v_e)|) \), where \( v_e \) has the smallest normalized weight, i.e., \( v_e = \arg\min_{v \in e} \{w(v)/|E(v)|\} \).

Every vertex \( v \in e \) locally computes \( \text{deal}_0(e) = \beta \cdot w(v_e)/|E(v_e)| \) and \( \delta_0(e) = \text{deal}_0(e) \).

2. In round \( i \geq 1 \), the following types of messages are sent: “\( e \) is covered”, “raise”, or “stuck”. These messages require only a constant number of bits. The decision whether \( \text{deal}_i(e) = \text{deal}_{i-1}(e) \) or \( \text{deal}_i(e) = \alpha \cdot \text{deal}_{i-1}(e) \) requires a single bit.

3. Finally, if \( \alpha = \alpha(e) \) is set locally based on the local maximum degree \( \max_{v \in e} |E(v)| \), then every vertex \( v \) sends its degree to all the edges \( e \in E(v) \). The local maximum degree for \( e \) is sent to every vertex \( v \in V \), and this parameter is used to compute \( \alpha(e) \) locally.

C Improved Running Time

In this section, we present a modified definition of the multiplier \( \alpha \) that leads to an improved dependence of the running time on \( f \) and \( \varepsilon \).

Let \( \gamma \in (0, 1) \) denote a constant. Set the multiplier \( \alpha \) as follows:

\[
\alpha \triangleq \begin{cases} 
\left( \frac{\log \Delta}{\log \log \Delta} \right)^{(1-\gamma)} & \text{if } \frac{f}{\beta} < \left( \frac{\log \Delta}{\log \log \Delta} \right)^{\gamma} \\
2 & \text{otherwise.}
\end{cases}
\]

Note that in the following, the round complexity is monotonically nonincreasing in \( \gamma \), so it may be chosen arbitrarily close to 1.

Theorem C.1. For every constant \( \gamma \in (1, 0) \), by setting \( \alpha \) according to Eq. 1, the round complexity of Algorithm \textsc{mwhvc} is bounded by

\[ O \left( \frac{\log \Delta}{\log \log \Delta} + \left( \frac{f^2 \varepsilon}{\varepsilon} \right)^{1/\gamma} \cdot \log \log \Delta \right) \]
Proof. By Theorem 4.7, the number of iterations is bounded by $\log_\alpha \Delta + f \cdot \frac{\alpha}{\beta}$. We consider two cases.

1. Suppose that $\frac{f}{\beta} < \left( \frac{\log \Delta}{\log \log \Delta} \right) ^\gamma$. In this case, $\alpha = \left( \frac{\log \Delta}{\log \log \Delta} \right) ^{(1-\gamma)}$. The terms in the bound on the number of iterations satisfy:

$$\log_\alpha \Delta = O \left( \frac{\log \Delta}{\log \log \Delta} \right)$$

$$f \cdot \frac{\alpha}{\beta} \leq \frac{\log \Delta}{\log \log \Delta}.$$

2. Suppose that $\frac{f}{\beta} \geq \left( \frac{\log \Delta}{\log \log \Delta} \right) ^\gamma$. In this case $\alpha = 2$, and hence

$$\log_\alpha \Delta \leq \left( \frac{f}{\beta} \right)^{1/\gamma} \cdot \log \log \Delta$$

$$f \cdot \frac{\alpha}{\beta} \leq O \left( \left( \frac{f}{\beta} \right)^{1/\gamma} \right).$$

In both cases, the bound on the number of iterations is bounded by the expression in Eq. 2 and the theorem follows. □