Exciting flavored bound states

E. Rojas,1 B. El-Bennich,1 and J. P. B. C. de Melo1

1Laboratório de Física Teórica e Computacional, Universidade Cruzeiro do Sul,
Rua Galvão Bueno, 868, 01506-000 São Paulo, SP, Brazil

We study ground and radially excitations of flavor-singlet and flavored pseudoscalar mesons within the framework of the rainbow-ladder truncation using an infrared massive and finite interaction in agreement with recent results for the gluon-dressing function from lattice QCD and Dyson-Schwinger equations. Whereas the ground-state masses and decay constants of the light mesons as well as charmonia are well described, we confirm previous observations that this truncation is inadequate to provide realistic predictions for the spectrum of excited and exotic states. Moreover, we find a complex conjugate pair of eigenvalues for the excited $D_{s}(s)$ mesons, which indicates a non-hermiticity of the interaction kernel in the case of heavy-light systems and the present truncation. Nevertheless, limiting ourselves to the leading contributions of the Bethe-Salpeter amplitudes, we find a reasonable description of the charmed ground states and their respective decay constants.

PACS numbers: 12.38.-t 11.10.St 11.15.Tk, 14.40.Lb, 14.40.Df

I. INTRODUCTION

Bound states are a fascinating subject as they teach us a great deal about the nature of interactions between constituents of a given field theory. In the case of Quantum Chromodynamics (QCD), this fascination is augmented by the empirical fact that no asymptotically free states, i.e. colored quarks and gluons, are observed. While studies of the analytic structure of colored Green functions are very instructive and were indeed crucial to get insight on their infrared behavior [1–23] and associated confinement mechanisms, they do not offer direct access to experimental tests. Nonetheless, their precise form may be studied indirectly via electromagnetic probes of the bound states formed by the theory’s constituents [24–27] and many dedicated experiments at existing and future accelerator facilities will serve exactly this purpose.

In QCD, the simplest possible bound state is given by the pion which plays a pivotal role in the understanding of the low-energy domain, being the lightest strongly bound antiquark-quark state as well as the Goldstone bosons associated with chiral symmetry breaking. Model-independent properties of the Goldstone boson were derived long ago [28] and express the intimate connection between the pion’s Bethe-Salpeter amplitude (BSA) and the quark propagator in the chiral limit.

Heavy mesons, on the other hand, provide a formidable opportunity to study additional nonperturbative features of QCD and can be used to test simultaneously all manifestations of the Standard Model, namely the interplay between electroweak and strong interactions. In the infinite-heavy quark limit, the heavy-quark velocity becomes a conserved quantity and the momentum exchange with surrounding light degrees of freedom is predominantly soft. Since the heavy-quark spin decouples in this limit, light quarks are blind to it. In essence, they do not experience any different interactions with a much heavier quark in a pseudoscalar or vector meson. In practice, however, heavy-flavor and -spin breaking effects are important and while dynamical chiral symmetry breaking (DCSB) hardly plays a role in charmonium and bottomonium states, it cannot be ignored in heavy-light systems, such as in $D$ and $B$ mesons. A daunting challenge is presented by the disparate energy scales and asymmetric momentum distribution within these flavor non-singlet mesons, i.e. the simultaneous treatment of heavy-quark symmetry breaking effects and DCSB, as the interactions of a heavy with a light quark are governed by the non-perturbative dynamics of the order $\Lambda_{QCD}$. Incidentally, this is also the heavy meson’s size scale.

Numerical solutions of the heavy meson’s BSA with renormalization-group improved ladder truncation, following work on the kaon BSA [29, 30], proved to be unsuccessful: the truncations do not yield the Dirac equation when one of the quark masses is much larger. To circumvent these problems, the Dyson-Schwinger equation (DSE) for the heavy quark was solved for an infrared-suppressed gluon momentum as described in Ref. [31]; no such infrared suppression was applied to the dressing of the light quark and the binding kernel. This approach reproduces well the masses for ground state pseudoscalar and vector heavy-light mesons, but overestimates experimental weak decay constants by $\sim 20\%$ in case of the $D$ meson and $\sim 40\%$ for the $B$ meson. Alternatively, in the heavy-quark propagation the mass function is approximated by a momentum-independent constituent mass. This is justified for the $b$ quark and reasonable for the $c$ quark given the modest variation in its functional behavior over a wide momentum range and the constituent-quark mass is fitted to the lightest meson, the $D$ and $B$ mesons in case of pseudoscalars [31, 32]. The resulting BSA yields again ground-state pseudoscalar and vector meson masses which compare very well with experimental values but strongly underestimate the leptonic decay constants. The situation improves when one uses a complex conjugate pole representation (with three poles) instead of a heavy-quark constituent propagator to compute the normalization and weak decay constant, though omitting a simultaneous application to the Bethe-Salpeter kernel seems inconsistent [32]. Nevertheless, in the latter case
the decay constants are overestimated by about 15% for the $D$, $D_s$ and $B$ mesons compared with experimental averages.

Aside of an unfavorable comparison with experimental data, one may object that the constituent-quark approach neglects valuable information of the quark’s dressed mass function, namely its imaginary part, $\Im M(p^2) \sim 0$ [33–36]. Thus, in solving the Bethe-Salpeter equation, one approximates the complex mass function by a flat surface on the complex plane. Our motivation is to verify whether a fully consistent, simultaneous numerical solution of the heavy-quark DSE and heavy-light pseudoscalar meson BSE can be obtained with a modern approach to the rainbow-ladder (RL) truncation based on the interaction proposed by Qin et al. [37]. This ansatz produces an infrared behavior of the interaction, commonly described by a “dressing function” $G(k^2)$ [28], congruent with the decoupling solution found in DSE and lattice studies of the gluon propagator [7–18]. Indeed, the gluon propagator is found to be a bounded function of space-like momenta with a maximum value at $k^2 = 0$.

Obviously, the cause for the difficulties encountered in solving the BSE for heavy-light mesons may not solely lie in the infrared behavior of the interaction but also in the truncation employed. In particular, hadron properties are not much affected by the deep infrared of the interaction; rather, they seem to be sensitive to the support and strength of the interaction at the scales of a few hundred MeV [25, 38]. As known from a long series of BSE studies, the RL truncation is successful for equal-mass mesons, for instance light $q\bar{q}$ mesons, such as the pion, kaon and $\rho$ [28], but also for $Q\bar{Q}$ charmonia and bottomonium [39–42] where subtle cancellations between contributions from the quark-gluon and antiquark-gluon vertices are in order. Due to the very different momentum and energy-scale distributions, the dressing of the quark-gluon vertex becomes important for heavy-light systems. Progress to go beyond RL truncation has been made and is underway [43–49] and the consistent inclusion of a dressed quark-gluon vertex which satisfies the relevant Ward-Takahashi identities (WTI) can be realized by a general form of the BSE [50].

Nonetheless, we here focus on the interaction proposed in Ref. [37] within the RL approximation to describe the properties of the light, strange and charmed flavor singlet and non-singlet pseudoscalar mesons, such as their mass spectrum, that of the first radial excitation and related weak decay constants. We show that the particular infrared behavior of this interaction in conjunction with recent improved numerical techniques [51, 52] is able to satisfactorily reproduce the aforementioned observables if particular care is taken in the numerical treatment of the flavored DSE and BSE. In the case of the $D$ and $D_s$ mesons, however, the truncation leads to complex eigenvalues if higher moments are included in the Chebyshev expansion of the BSA. As discussed in Section IV, a successful description of the heavy-light flavored mesons requires more efforts which implies BSE solutions beyond the RL approximation.

## II. Pseudoscalar Bound States

### A. Dyson-Schwinger equation

We work in a continuum approach of QCD based on the RL truncation of the quark DSE and of the quark-antiquark kernel of the meson’s BSE, which is the leading term in a systematic symmetry-preserving truncation scheme. The quark’s gap equation is described by the following DSE,

$$S^{-1}(p) = Z_2(i\gamma_5 + m_{\text{hm}}) + \Sigma(p^2),$$

where the dressed-quark self-energy contribution is $(q = k - p)$,

$$\Sigma(p^2) = Z_1 g^2 \int_k D^\mu\nu(q) \frac{\lambda^a}{2} \gamma_\mu S(k) \Gamma^a(k, p).$$

The mnemonic shorthand $\int_k D^\mu\nu \equiv \int^4 d^4k/(2\pi)^4$ represents a Poincaré invariant regularization of the integral with the regularization mass scale, $\Lambda$, and where $Z_{1,2}(\mu, \Lambda)$ are the vertex and quark wave-function renormalization constants. The nonperturbative contributions contribute to the self-energy, $\Sigma(p^2)$, which corrects the current-quark bare mass, $m_{\text{hm}}(\Lambda)$. The integral is over the dressed gluon propagator, $D^\mu\nu(q)$, the dressed quark-gluon vertex, $\Gamma^a(k, p)$, and the color matrices $\lambda^a$ are in the fundamental representation of $SU(3)$. In Landau gauge the gluon propagator is purely transversal,

$$D^\mu\nu_{\text{ab}}(q) = \delta^{\mu\nu} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{q^2} \right) \frac{\Delta(q^2)}{q^4},$$

where $\Delta(k^2)$ is the gluon-dressing function. The quark-gluon vertex in this truncation is simply given by,

$$\Gamma^a_{\mu}(k, p) = \frac{\lambda^a}{2} \gamma_\mu.$$

Following Ref. [37] and suppressing color indices, we write

$$Z_1 g^2 D^\mu\nu_{\text{ab}}(q) \Gamma^a_{\mu}(k, p) \rightarrow Z_2^2 \frac{G(q^2)}{q^2} T^\mu\nu_{\text{ab}}(q) \gamma_\mu,$$

with the transverse projection operator $T^\mu\nu_{\text{ab}}(q) := g^\mu\nu - q^\mu q^\nu/q^2$ and the effective coupling is the sum of two

---

1 We employ throughout a Euclidean metric in our notation: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$; $\gamma^\mu = \gamma_\mu$; $\gamma_5 = \gamma_4 \gamma_1 \gamma_2 \gamma_3$; $\text{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = -4 \sigma_{\mu\nu}\rho\sigma\tau; \sigma_{\mu\nu} = (1/2)[\gamma_\mu, \gamma_\nu]; a \cdot b = \sum_{i=1}^4 a_i b_i$; and $F_\mu$ timelike $\Rightarrow P^2 < 0$. 
The renormalisation-group invariant current-quark mass
where one particular deep spacelike point, \( p \),
else but the dressed-quark mass function evaluated at \( Z \),
to realize DCSB and/or confinement. For momenta \( k^2 \gtrsim 2 \text{ GeV}^2 \),
the perturbative contributions markedly dominate the interaction.

With this, the solutions to the gap equation (1) for
a given flavor, \( f \), are generally,
\[
S_f(p) = \left[ i \not{p} A_f(p^2) + \Gamma_D B_f(p^2) \right]^{-1},
\] (7)
where one imposes the renormalization condition,
\[
Z_f(p^2) = 1/A_f(p^2)|_{p^2 = \mu^2} = 1,
\] (8)
at large spacelike \( \mu^2 \gg \Lambda^2_{QCD} \). The mass function,
\( M_f(p^2) = B_f(p^2, \mu^2)/A_f(p^2, \mu^2) \), is independent of the
renormalization point \( \mu \). In order to make quantitative
matching with pQCD, another renormalization condition,
\[
S_f^{-1}(p)|_{p^2 = \mu^2} = i \not{p} + m_f(\mu) \Gamma_D,
\] (9)
is imposed, where \( m_f(\mu) \) is the renormalized running quark mass,
\[
Z^f_m(\mu, \Lambda) m_f(\mu) = m^\text{fm}(\Lambda),
\] (10)
where \( Z^f_m(\mu, \Lambda) = Z^f(\mu, \Lambda)/Z^f(\mu, \Lambda) \) is the flavor
dependent mass-renormalization constant and \( Z^f_m(\mu, \Lambda) \) is the
renormalization constant associated with the Lagrangian’s mass term. In particular, \( m_f(\mu) \) is nothing else but the dressed-quark mass function evaluated at one particular deep spacelike point, \( p^2 = \mu^2 \), namely:
\[
m_f(\mu) = M_f(\mu).
\] (11)
The renormalisation-group invariant current-quark mass can be inferred via,
\[
\tilde{m}_f = \lim_{p^2 \to \infty} M_f(p^2) \left[ 1 - \ln \left( \frac{p^2}{\Lambda^2_{QCD}} \right) \right]^{\gamma_m}.
\] (12)

B. Bethe-Salpeter equation

The homogeneous BSE for a \( \bar{q}q \) bound state with relative momentum \( p \) and total momentum \( P \) can be written as,
\[
\Gamma^{\bar{q}g}_{mn}(p, P) = \int_k K_{mn}^{kl}(p, k, P) \left[ \Gamma^{\bar{q}g}(k, P) S_y(k - \gamma) \right]_{kl},
\] (13)
where \( m, n, k, l \) collect Dirac and color indices, \( f, g \) are flavor indices and \( k_+ = k + \eta, k_- = k - \eta \). \( P, \eta_+ + \eta_- = 1 \). Since we work within the RL truncation, the BSE kernel is given by,
\[
K_{mn}^{kl}(p, k, P) = -\frac{Z^2}{q^2} \left( \frac{\lambda^a}{2} \gamma_\mu \right)_f \lambda_\mu(\rho) \left( \frac{\rho^a}{2} \gamma_\alpha \right)_m,
\] (14)
which satisfies the axialvector WTI [28] and therefore ensures a massless pion in the chiral limit. Eqs. (13) and (14) define an eigenvalue problem with physical solutions at the mass-shell points, \( P^2 = -M^2 \), where \( M \) is the bound-state mass.

In the following, we consider the spectrum of flavor singlet and nonsinglet pseudoscalar mesons, which are exhibited as a pole contribution to the axial-vector and pseudoscalar vertices (omitting regular terms in the neighborhood of the poles),
\[
\Gamma^{\bar{q}g}_{\rho n}(p, P)|_{p^2 + M^2 = 0} = \rho_{\rho n} \frac{P_{\rho n}}{P^2 + M_n^2} \Gamma^{\bar{q}g}_{\rho n}(p, P),
\] (15)
\[
i \Gamma^{\bar{q}g}_{\rho n}(p, P)|_{p^2 + M^2 = 0} = \rho_{\rho n} \frac{P_{\rho n}}{P^2 + M_n^2} \Gamma^{\bar{q}g}_{\rho n}(p, P),
\] (16)
where \( \Gamma^{\bar{q}g}_{\rho n}(p, P) \equiv \Gamma^{\bar{q}g}_{mn}(p, P) \rho_{mn} \) is the bound state’s BSA. The principal quantum number is \( n = 0 \) for the ground state and \( n \gtrsim 1 \) for the radial excitations with bound-state mass, \( M_n \). In Eqs. (15) and (16), the expressions for \( \rho_{\rho n} \) and \( f_{\rho n} \) are defined in Eqs. (A4) and (A5), respectively. The properties of the excited states are expected to be sensitive to details of the long-range component of the interaction in Eq. (6), which provides more support at large inter-quark separation than, e.g., the Maris-Tandy model [30]. The present study thus allows for a test of the interaction of strange- and charm-flavored \( 0^- \) mesons and their excited states, in addition to those considered in Ref. [37], via comparison with experimental values where available.

As mentioned in Section I, it is most likely that beyond RL contributions are important in heavy-light mesons. In fact, the RL truncation with the interaction in Eq. (6) should work best for heavy-heavy (QQ) systems. Consider for instance the Ball-Chiu ansatz to the nonperturbative quark-gluon vertex, \( \Gamma_{\mu}(k, p) \), which is fully determined with the knowledge of \( A(p^2) \) and \( B(p^2) \) [53, 54]: for heavier flavors, \( f = c, b \), the scalar function \( B_s(p^2) \) and even more so \( B_h(p^2) \) remain constant over a wide momentum range, while for the vector functions \( A_{c,b}(k^2) \approx A_{c,b}(p^2) \) \( \simeq 1 \). The heavy-quark gluon vertex can therefore reasonably be approximated by a bare vertex.

Thus, the RL approximation ought to work better for heavy mesons such as the charmonia and bottomonium, which is indeed confirmed in numerical studies [25, 40, 42]. The effects of DCSB and the importance of other tensor structures in the quark-gluon vertex
become increasingly more important for lighter quarks, where, for example, mass splittings between parity partners are only satisfactorily described by including the tensor structure which corresponds to the chromomagnetic moment [55]. Nonetheless, a hallmark of the RL is the successful description of the pion, the key point being that the axialvector WTI be preserved and likewise the Goldstone-boson character of the pion.

III. NUMERICAL IMPLEMENTATION

A. DSE on the complex plane

We follow Refs. [46, 51, 52] and choose the gluon momentum, \( q \), to be real. This implies that when the external DSE momentum (1) is complex, so is the internal quark’s momentum. It is the case in Eq. (13) which requires DSE solutions for \( S(p_\pm) \) in the complex plane with,

\[
p_\pm^2 = p^2 - \eta_\pm^2 M_n^2 \pm 2i\sqrt{p^2}\eta_\pm M_n ,
\]

where \( p \) is collinear with \( P = (\bar{0}, iM_n) \) in the meson’s rest frame and in the quark’s DSE we employ \( \eta_\pm = 1/2 \). As we add \( P \) to the quark’s external momentum, the internal quark momentum becomes \( k \pm p \), \( k \in \mathbb{R}^+ \), and it follows that \( q = k_\pm - p_\pm \) is real. The internal quark momentum is given by,

\[
k_\pm^2 = k^2 - \eta_\pm^2 M_n^2 \pm 2i\sqrt{k^2}\eta_\pm M_n z ,
\]

with the angle, \(-1 \leq z \leq +1\), and is also bounded by a parabola whose vertex is \((0, -\eta_\pm^2 M_n^2)\). Within the parabola, the real integration variable, \( k \), is limited by \( \Lambda \) as follows.

We denote the regularization mass scale in Eq. (13) by \( \Lambda_{BSE} \) and relate it to the one in Eq. (1) with,

\[
\Lambda_{DSE}^2 = \Lambda_{BSE}^2 + \frac{1}{4} M_f^2 = \Lambda_{BSE}^2 + \eta_\pm^2 M_P^2 ,
\]

for \( f = u, d, s, c \). This procedure is best illustrated with an example, e.g. for a quark with strangeness, \( \bar{s} \); we determine \( \eta_s \) and \( \eta_d \) pushing in each case \( M_{f=u,s} \) to the limit at which we encounter poles while solving the DSE on the complex plane. By equating both sides in Eq. (19), we obtain \( \eta_s \) and \( \eta_d \) and imposing \( \eta_s + \eta_d = 1 \) we find,

\[
\frac{1}{2} M_s = \eta_s M_P ; \quad \frac{1}{2} M_u = \eta_u M_P \Rightarrow \frac{M_s + M_u}{2} = M_P , \tag{20}
\]

where \( M_P \) is the “maximal meson mass” — not to be confused with \( M_\pi \) — for which the DSE can be solved for each flavor and respective parameters, \( \eta_x = M_x/(M_\pi + M_x) \) and \( \eta_n = M_n/(M_\pi + M_n) \). The latter are used as momentum partitioning parameters \( \eta_\pm \) in the BSE. Of course, this is done for computational convenience and we have checked that our results are stable under variations of \( \eta_\pm \), as expected from a Poincaré invariant approach. In solving the BSE, Eq. (20) imposes an upper boundary on the mass of the meson and its excited states: \( M_s < M_P \). For flavor-singlet equal-mass \( \bar{q}q \) mesons we simply choose \( \eta_s = \eta_u = \eta_\pi = 1/2 \).

The solution of Eq. (1) for a complex value of \( p_\pm^2 \) requires the simultaneous iteration on a complex 2d grid bounded by the parabola of Eq. (18). Alternatively, we can exploit the information from the contour of the parabola via the Cauchy theorem [52]. To proceed with the integration, we parametrize the contour (counterclockwise) in the upper complex plane, \( c_+ \): \( p_+^2 \), with \( p := z_+(t) = t \Lambda_{DSE} + (1 - t) \Lambda_{BSE} \); in the lower plane, \( c_- \): \( p_-^2 \), with \( p := z_-(t) = t \Lambda_{DSE} + (1 - t) \Lambda_{BSE} \); and we close the parabola with the path \( c_y \): \( p_y^2 := \Lambda_{DSE}^2 - y^2(t) + 2i y(t) \Lambda_{DSE} \) with \( y(t) = M_n t - \eta_\pi M_n \), \( t \in [0, 1] \). More precisely, we use a linear parametrization given by:

\[
\begin{align*}
dp_+^2 & = 2( z_+(t) + i \eta_+ M_n) \, dz_+(t) ; \quad 0 \leq z_+(t) < \Lambda_{DSE} ,
\end{align*}
\]

\[
\begin{align*}
dp_-^2 & = 2( -y(t) + i \Lambda_{DSE}) \, dy(t) ; \quad -\eta_\pi M_n \leq y(t) \leq \eta_\pi M_n ,
\end{align*}
\]

\[
\begin{align*}
dp_y^2 & = 2( z_+(t) - i \eta_+ M_n) \, dz_-(t) ; \quad 0 \leq z_-(t) < \Lambda_{DSE} .
\end{align*}
\]

The DSE for the quark is solved for 128 complex momenta, \( p_\pm \), on each contour section using 128 momenta, \( k_\pm \), and 128 angles, \( z \), in the interior of the parabola parametrized by Eq. (21), where the values for \( A(k_\pm^2) \) and \( B(k_\pm^2) \) are obtained via the Cauchy theorem and simultaneous iteration of their values on the contour. For numerical precision, the points on the contour are skewed towards the vertex of the parabola near \( k_\pm^2 \sim -\eta_\pm^2 M_n^2 \).

We use a quadrature method for the integral evaluation and cross-check the results with the numerical integration library CUBA [56]. The BSE (13) is solved for 64 external momenta, \( p \), 64 internal momenta, \( k \), and 20 angles.

The renormalization condition in Eqs. (8) and (9) are imposed on the DSE solutions at the space-like real-axis point, \( \mu = 19 \) GeV, a value chosen to match the bulk of extant studies [39, 57]. The current-quark masses are fixed by requiring that the pion and kaon BSEs produce \( m_\pi \approx 0.138 \) GeV and \( m_K \approx 0.493 \) GeV. We thus use \( m_u,d(\mu) = 3.4 \) MeV, \( m_s(\mu) = 82 \) MeV [37] and \( m_c(\mu) = 0.905 \) GeV [66]. However, in computing the values of \( A(p^2) \) and \( B(p^2) \) for the momenta on the parabola (17), we are numerically limited by the appearance of conjugate poles and cannot use an arbitrarily large cutoff \( \Lambda \) in Eq. (1). This implies that we impose the renormalization conditions, Eqs. (8) and (9), at a lower momentum value, \( \xi < \mu \), and thus obtain new renormalization constants, \( Z^A_\eta \) and \( Z^B_\eta \). The details of our renormalization procedure are discussed in Appendix A and our complex solutions for \( A(p^2) \) and \( B(p^2) \) with the interaction in Eq. (6) are plotted for the \( u \) quark in Fig. 3 of Ref. [58].

Alternatively, we cross-checked our renormalization procedure by setting \( A(\mu^2) = 1 \) for \( \mu = 2 \) GeV and imposing in the ultraviolet the condition (12) following Refs. [37, 57]. We also reproduce our results using a
complex pole representation for the quarks with three poles [76] to which we fit the DSE solutions on the real axis renormalized at 19 GeV. The numerical differences with the masses and decay constants obtained with the solutions of the DSE on the complex plane are negligible.

B. Solving the BSE with Arnoldi factorization

The general Poincaré invariant form of the solutions of Eq. (13) for the pseudoscalar channel $J^P = 0^-$ and the trajectory, $P^2 = -M_n^2$, in a non-orthogonal base with respect to the Dirac trace, $\mathcal{A}^\alpha(p,P) = \gamma_5 \{ i 1_D, P, \not{p}(p \cdot P), \sigma_{\mu\nu} P_\mu P_\nu \}$, is given by,

$$\Gamma_{P_n}(p,P) = \gamma_5 \{ i 1_D E_{P_n}(p,P) + P F_{P_n}(p,P) + \not{p}(p \cdot P) G_{P_n}(p,P) + \sigma_{\mu\nu} P_\mu P_\nu H_{P_n}(p,P) \}$$

where we have suppressed color, Dirac and flavor indices for the sake of visibility. The functions $F_{P_n}^\alpha(p,P) = \{ E_{P_n}(p,P), F_{P_n}(p,P), G_{P_n}(p,P), H_{P_n}(p,P) \}$ are Lorentz-invariant scalar amplitudes. We solve the BSE by projecting on the functions $F_{P_n}^\alpha(p,P)$ with the appropriate projectors,

$$P^\alpha(p,P) = \sum_{\beta=1}^{4} P^{\alpha\beta}(p,P) \mathcal{A}^\beta(p,P)$$

$$\frac{1}{4} \sum_{\beta=1}^{4} P^{\alpha\beta}(p,P) \operatorname{Tr}_{\mathcal{D}} [ A^{\alpha}(p,P) \mathcal{A}^\gamma(p,P) ] = \delta_{\alpha\gamma}$$

which allows for an extraction of $F_{P_n}^\alpha(p,P)$ from the BSA,

$$F_{P_n}^\alpha(p,P) = \frac{1}{4} \operatorname{Tr}_{\mathcal{D}} [ P^\alpha(p,P) \Gamma_p(p,P) ]$$

This leads to the eigenvalue equation ($\alpha, \beta = 1, \ldots, 4$),

$$\lambda_n(p^2) F_{P_n}^\alpha(p,P) = \int_k \mathcal{K}^{\alpha\beta}(p,k,P) F_{P_n}^\beta(k,P)$$

where $\mathcal{K}^{\alpha\beta}(p,k,P)$ is obtained using Eq. (14):

$$\mathcal{K}^{\alpha\beta}(p,k,P) = -\frac{Z_2^2 G(q^2)}{4 q^2} T_{\mu\nu}(q) \operatorname{Tr}_{\mathcal{D}} \left[ P^\alpha(p,P) \right]$$

$$\times \gamma_\mu \gamma_\nu S(k_+) \mathcal{A}^\beta(k,P) S(k_-) \gamma_\mu \gamma_\nu$$

The propagators and functions $F_{P_n}^\alpha(p,P)$ in Eq. (25) are expanded in Chebyshev polynomials of the 2nd kind, $U_m(z_k)$ and $U_m(z_p)$, with the angles, $z_k = P \cdot k/(\sqrt{P^2} \sqrt{k^2})$ and $z_p = P \cdot p/(\sqrt{P^2} \sqrt{p^2})$, and the momenta, $k$ and $p$, are discretized, e.g.,

$$F_{P_n}^\alpha(p_i,P) = \sum_{m=0}^{\infty} F_{P_n,m}^\alpha(p_i,P) U_m(z_p)$$

where we define $F_{m1}^\alpha \equiv F_{P_n,m}^\alpha(p_i,P)$. We employ 3 Chebyshev polynomials for the ground states and 5 Chebyshev polynomials for the excited states.

We solve the eigenvalue problem posed in Eq. (25) by means of Arnoldi factorization implemented in the ARPACK library [59, 60] which computes the eigenvalue spectrum for a given $N \times N$ matrices. Practical implementation implies a transcription of the BSE kernel $\mathcal{K}^{\alpha\beta}(p,k,P)$ in Eq. (26) as,

$$F_{m1}^\alpha = (\mathcal{K}^{\alpha\beta})_{m1}^{n1} F_{n1}^\beta$$

which depends on 6 indices. In order to cast Eq. (28) as an $N \times N$ matrix,

$$F_I = K_{IJ} F_J, \quad I, J = 1, ..., N$$

we relate the two sets of indices, $I$ and $\alpha, \beta, m, n, i, j$, with the condition,

$$I(\alpha, m, i) = \alpha(m + 1)i$$

$$J(\beta, n, j) = \beta(n + 1)j$$

To recover the components of the BSA, $F_{P_n}^\alpha(p,P)$, one simply identifies $F_{m1}^\alpha = F_I$ where $I$ is given by Eq. (30). As mentioned, the factorization with ARPACK yields the eigenvalue spectrum, $\lambda_n(P^2)$, of the kernel, $K_{IJ}$, and therefore we obtain eigenvectors, $F_I$, or equivalently the BSA, $F_{P_n}^\alpha(p,P)$, for each $\lambda_n(P^2) = -M_n^2$. These numerical results are presented and discussed in Section IV.

For sake of completeness, we mention that all BSA are normalized canonically as,

$$2P_\mu = \int_k \operatorname{Tr}_{\mathcal{D}} \left[ \Gamma_{P_n}(k,-P) \frac{\partial S(k_+)}{\partial P_\mu} \Gamma_{P_n}(k,P) S(k_-) \right] + \frac{\partial \Gamma_{P_n}(k,-P) S(k_+)}{\partial P_\mu} \Gamma_{P_n}(k,P) S(k_-)$$

where we have omitted a third term that stems from the derivative of the kernel, $\partial \mathcal{K}^{\alpha\mu}_{mn}(p,k,P)/\partial P_\mu$, as it does not contribute in the RL truncation of Eq. (14).

In Eq. (32), the charge-conjugated BSA is defined as $\Gamma(k,-P) := C \Gamma^T(-k,-P) C^T$, where $C$ is the charge conjugation operator. We shall discuss the peculiarities of charge symmetry in case of heavy-light mesons in Section IV.

IV. DISCUSSION OF RESULTS

We summarize our results for the mass spectrum and weak decay constants of the pseudoscalar flavor singlet.
and non-singlet mesons in Tables I, II and III, where the DSE and BSE are solved for two parameter sets of the interaction in Eq. (6), denoted by Model 1 and 2 in all tables, also discussed in Ref. [57]. In Table I, we list the masses and decay constants, adopting the Particle Data Group (PDG) conventions [63], of the \( \pi, K, \pi(1300), K(1460), \eta_c(1S) \) and \( \eta_c(2S) \). We also include the mass and decay constant of a pure \( \bar{s}s \) state which cannot be related to the \( \eta \) or \( \eta' \) but whose study is worthwhile to elucidate the trajectory of the weak decay constants as a function of the current quark mass. We remind that the \( \eta \) and \( \bar{s}s \) are neutral particles which do not undergo a purely leptonic decay as in Eq. (A5), yet they are useful quantities to calculate [64].

A direct comparison of the mass and decay constant entries in both model columns reveals that the values obtained with Model 1 are in much better agreement with experimental values of the 0\(^-\) ground states, namely the \( \pi \), \( K \) and \( \eta_c \), whose \( \omega \)-dependence in the range, \( \omega \in [0.4, 0.6] \) GeV is rather weak. Conversely, Model 2 reproduces well the masses of the radially excited states which hints at the insufficiency of this truncation. However, Model 2 uniformly overestimates and excited states which are more "inflated" for the pion and the kaon. It is plausible that radiative and hadronic corrections return them to their appearance in the spectrum is likely an artifact of the current truncation; see, e.g., discussion in Ref. [65].

In summary, we do not find a parameter set that describes equally well the entire mass spectrum of ground and excited states which hints at the insufficiency of this truncation. However, Model 2 uniformly overestimates all masses by \( 6 - 10\% \) in Table I, where the masses are more "inflated" for the pion and the kaon. It is plausible that radiative and hadronic corrections return them to the observed values [44]. In the mass spectrum, we also find states with unnatural time parity, \( J = 0^+ \), referred to as exotics and presented in Table II. The mass difference pattern of these states parallels the ones observed in Table I. However, as they are not experimentally observed and expected to have masses above 2 GeV [63], their appearance in the spectrum is likely an artifact of the current truncation; see, e.g., discussion in Ref. [65].

The weak decay constants are obtained via Eq. (A5) and verified by means of the Gell-Mann-Oakes-Renner (GMOR) relation (A3), a relation valid for every 0\(^-\) meson irrespective of the magnitude of the current-quark

|                | Model 1 | Model 2 | Reference |
|----------------|---------|---------|-----------|
| \( m_{\pi^-} \) | 0.138   | 0.153   | 0.139 [63]|
| \( f_{\pi^-} \)  | 0.139   | 0.189   | 0.1304 [63]|
| \( m_{\pi(1300)} \) | 0.990   | 1.414   | 1.30±0.10 [63]|
| \( f_{\pi(1300)} \) | \(-1.1 \times 10^{-3}\) | \(-8.3 \times 10^{-4}\) |
| \( f_{\pi(1300)}^{\text{GMOR}} \) | \(-1.4 \times 10^{-3}\) | \(-4.0 \times 10^{-4}\) |
| \( m_K \)          | 0.493   | 0.541   | 0.493 [63]|
| \( f_K \)           | 0.164   | 0.214   | 0.156 [63]|
| \( f_K^{\text{GMOR}} \) | 0.162   | 0.214   |
| \( m_{K(1460)} \)   | 1.158   | 1.580   | 1.460 [63]|
| \( f_{K(1460)} \)   | \(-0.018\) | \(-0.017\) |
| \( f_{K(1460)}^{\text{GMOR}} \) | \(-0.018\) | \(-0.017\) |
| \( m_{\bar{s}s} \)  | 1.287   | 1.702   |           |
| \( f_{\bar{s}s} \)  | \(-0.0214\) | \(-0.0216\) |
| \( f_{\bar{s}s}^{\text{GMOR}} \) | \(-0.0215\) | \(-0.0218\) |
| \( m_{\eta_c(1S)} \) | 3.065   | 3.210   | 2.984 [63]|
| \( f_{\eta_c(1S)} \) | 0.389   | 0.464   | 0.395 [64]|
| \( f_{\eta_c(1S)}^{\text{GMOR}} \) | 0.380   | 0.451   |
| \( m_{\eta_c(2S)} \) | 3.402   | 3.784   | 3.639 [63]|
| \( f_{\eta_c(2S)} \) | 0.089   | 0.105   |
| \( f_{\eta_c(2S)}^{\text{GMOR}} \) | 0.088   | 0.103   |

mass, \( m_f^\eta(\mu) \). In the chiral limit,

\[
\rho_{\eta_n}^0(\mu) := \lim_{m \to 0} \rho_{\eta_n}^0(\mu) < \infty \, , \, \forall n ,
\]

owing to the ultraviolet behavior of the quark-antiquark scattering kernel in QCD which guarantees that \( \rho_{\eta_n}(\mu) \) in Eq. (A4) is cutoff independent. A necessary corollary is that in the chiral limit the decay constant of excited states vanishes identically [66]:

\[
f_{\eta_n}^\mu(\mu) \equiv 0 \, , \, n \geq 1 .
\]

We provide numerical verification of Eq. (34) in the case of the first radial excitation of the pion, i.e. the parity partner \( \pi(1300) \), in Table I from which it can also be read that the decay constant of the parity partner of the kaon, the \( K(1460) \), is strongly suppressed. Analogously, the decay constant value of the \( \bar{s}s \) state is very small. On the other hand, we note that all weak decay constants of the excited states but the \( \eta_c(2S) \) are negative with Model 1, slightly less so when using Model 2, which is consistent with the discussion in Ref. [57]. As it becomes
TABLE II: List of exotic states with “unnatural time parity”, \( J = 0^- \). In case of the \( \bar{u}s \) meson, \( C = -1 \), must be understood as approximate, since \( [\bar{u}s] \) is not an eigenstate of charge conjugation. Model 1 and 2 are as in Table I and dimensioned quantities in GeV. The decay constants of flavorless states are of the order \( 10^{-6} \) GeV.

|                | Model 1 | Model 2 |
|----------------|---------|---------|
| \( m_{\bar{u}u} \) | 0.733   | 1.049   |
| \( m_{\bar{s}s} \) | 0.856   | 1.351   |
| \( m_{c\bar{c}} \) | 3.243   | 3.515   |
| \( m_{\bar{u}s} \) | 0.917   | 1.225   |
| \( f_{\bar{u}s} \) | 0.0152  | 0.0098  |
| \( f_{\bar{u}s}^{\text{GMOR}} \) | 0.0150  | 0.0097  |

clear from Fig. 1, for \( \omega \geq 0.5 \) and \( p^2 \geq 1 \) GeV\(^2\), the leading amplitude’s lowest Chebyshev projection, \( \omega D_1(p^2) \), is negative-definite in case of the first radial excitations, \( \pi(1300) \) and \( \bar{s}s \), whereas it remains positive for the ground states, which parallels the pattern of wave functions in quantum mechanics. A necessary consequence is then, \( f_\pi > 0 \) and \( f_{\pi(1300)} < 0 \), also observed for the kaon and its first radial excitation and for the \( \eta(1760) \). The negative value for \( f_{\pi(1300)} \) is consistent with lattice-QCD simulations [67]. The equality in Eq. (34) is only valid in the chiral limit and for increasing current-quark masses the excited pseudoscalar meson’s decay constant first remains negative, but after a minimum, which occurs between the strange and charm-quark mass, steadily increases and becomes positive before reaching the charm quark mass. We therefore find for the \( \eta_c(2S) \) a positive decay constant for both model parameters. We note that our turning point is below that observed in Ref. [68], which is about the charm-quark mass.

We now turn our attention to the charmed pseudoscalars, in particular the \( D \) and \( D_s \) mesons. Before discussing the results, a few technical comments are in order. Besides being bounded from above, the eigenvalue spectrum of the BSE should be positive definite [69] owing to the hermiticity requirement of physical operators. However, this is not generally true for flavored mesons with unequal masses, in particular heavy-light systems in the rainbow-ladder truncation. For example, in solving the BSE for the \( D \) and \( D_s \) mesons, we find a complex conjugate pair of eigenvalues for the excited \( D_{(s)} \) states if we include contributions from higher Chebyshev polynomials, \( U_{m>1}(z_p) \). This occurs for either parameter set of the interaction and indicates an non-hermiticity of the hamiltonian in the case of \( [c\bar{u}] \) and \( [c\bar{s}] \) bound states.\(^4\)

\(^4\) A discussion of complex eigenvalues and their origin in the crossing of normal and abnormal (which have vanishing binding energy) eigenstates can be found in Ref. [69]. Note that in quantum mechanics even non-hermitian operators yield real and positive eigenvalues provided that PT symmetry is conserved, whereas if it is broken the eigenvalue spectrum is complex [70]. Yet, an analogical observation in Quantum Field Theory and more precisely in nonperturbative QCD is not straightforward.
this approximation bears similarity with common phenomenological ansätze to the BSA for $D$ mesons [71–75] and this is exactly what the values in Table III represent: masses and decay constants obtained with a model based on the lowest-order approximation. The mass difference, $\Delta m \simeq 15 – 30$ MeV, between the $D$ and $D_s$ mesons is smaller than experimentally observed, i.e. $\Delta m \approx 100$ MeV. On the other hand, the weak decay constants are in good agreement with experimental averages for Model 1 which again provides the preferred interaction for the ground states. We also stress that the results in Table III were obtained without any modification of the interaction (6) in the RL truncation, such as suppression of the infrared domain [31] or use of a constituent-quark mass [32]. Nevertheless, as has been realized previously [31], the RL approximation is an inadequate truncation scheme for heavy-light mesons.

V. CONCLUSION

We computed the BSAs for the ground and first excited states of the flavor-singlet and flavored pseudoscalar mesons with an interaction ansatz that is massive and finite in the infrared and massless in the ultraviolet domain. This interaction is qualitatively in accordance with the so-called decoupling solutions of the gluon’s dressing function and thus represents an improvement on the Maris-Tandy model [30]. In conjunction with the RL truncation, the latter proves to be a successful interaction model for the light meson spectrum, $M \lesssim 1$ GeV, but fares less well in applications to heavy-light systems where infrared-modifications of the interaction are required in order to obtain numerical results which are still not satisfactory [31].

Motivated by the successful application of the interaction ansatz of Eq. (6) to the mass spectrum of light mesons as well as some of their excited states in Ref. [57], we extend this study to the strange and charm sector and obtain the masses of ground states and resonances as well as decay constants as presented in Table I. These numerical results are in good agreement with experimental averages for the ground states, yet we confirm the earlier observation that no single parametrization of Eq (6) is able to reproduce the mass spectrum of both, the ground and excited states in the RL truncation. The case of the $D$ and $D_s$ mesons remains equally unresolved and clearly shows that the mass asymmetry and disparate scales in heavy-light systems require corrections beyond the leading truncation. We postpone the treatment of heavy-flavored mesons with a BSE ansatz valid for any symmetry-preserving dressed quark-gluon vertex [50] to a future publication. We stress that the merit of BSE solutions for heavy-light systems goes beyond a successful description of the heavy meson’s mass spectrum and decay constants and that the light-front projection of the meson’s Bethe-Salpeter wave function allows to extract its light cone distribution amplitudes [76, 77]. The latter, commonly expanded in Gegenbauer polynomials, encode the relevant nonperturbative information in QCD factorization of heavy-meson weak decays and their calculation represents a big step beyond the usually employed asymptotic form, $\phi(x) \propto 6x(1-x)$. Indeed, in the case of $D$ and $B$ mesons, very little is known about the nonperturbative nature of these distribution amplitudes.

Acknowledgments

This work was supported by the São Paulo Research Foundation (FAPESP) and benefitted from access to the computing facility of the Centro Nacional de Supercomputação at the Federal University of Rio Grande do Sul (UFRGS). We appreciated insightful discussions with and comments by Lei Chang, Gernot Eichmann, Andreas Krassnigg, Craig Roberts and Peter Tandy.

Appendix A: Renormalization conditions

In order to eliminate the cutoff dependence it is necessary to impose renormalization conditions. The usual procedure is to relate bare and renormalized masses as in Eq. (10) and for the fermion fields one writes analogously,

$$ q_f^{\text{bm}}(p^2) = \left( Z_2^f(\mu, \Lambda) \right)^2 q_f(p^2, \mu). \quad (A1) $$

The renormalization constants $Z_m$ and $Z_2$ are determined by imposing the conditions,

$$ B(\mu, \mu) = m_f(\mu) \quad \text{and} \quad A(\mu, \mu) = 1. \quad (A2) $$
For arbitrary flavors and as usual Eqs. (10) and (A2), the GMOR expression can be written as,
\[ f_{P_{\mu}}M_{P_{\mu}}^2 = (m_{fm}^{4\mu} + m_{gf}^{4\mu})\rho_{P_\mu}^4 \]
\[ = (m_{fm}^{4\mu} + m_{gf}^{4\mu})\left(0|\bar{q}g\gamma_5 q_f|P_{\mu}\right) \]
\[ = (Z^4_{m} m_{4\mu}) + \left(0\bar{q}g\gamma_5 q_f\right)\rho_{P_{\mu}}^4 , \]
(A3)

where we have,
\[ \rho_{P_{\mu}}^4 = (Z^4_{m} Z^4_{P_{\mu}})\left(0\bar{q}g\gamma_5 q_f\right)\]
\[ = -i\left(Z^4_{m} Z^4_{P_{\mu}}\right)\text{TrCD}\int_{k}^{A} \gamma_5 S_f(k_+)\Gamma^{fg}_{P_{\mu}}(k, P)S_g(k_-) , \]
with \(Z^4_{m}(\mu, \Lambda) = \rho_{P_{\mu}}^4(\mu, \Lambda)\). The weak decay constant can, of course, be directly inferred from,
\[ f_{P_{\mu}} = -i\left(0\bar{q}g\gamma_5 q_f\right)\rho_{P_{\mu}}^4 , \]
(A4)

\[ = (Z^4_{m} Z^4_{P_{\mu}})\text{TrCD}\int_{k}^{A} \gamma_5 \mu \text{S}_f(k_+)\Gamma^{fg}_{P_{\mu}}(k, P)S_g(k_-) . \]
(A5)

In solving the DSE within a complex parabola as discussed in Section III A, it is convenient to compute the renormalization constants at a momentum scale, \(\xi\), different from \(\mu > \xi\) while still demanding that Eq. (A2) be satisfied. Once we know the solutions of and \(A(p^2)\) and \(B(p^2)\) renormalized at \(p^2 = \mu^2\), the values for \(B(\xi, \mu)\) and \(A(\xi, \mu)\) are readily inferred. One may as well calculate \(Z^4_{m}(\mu, \Lambda)\) and \(Z^4_{P_{\mu}}(\mu, \Lambda)\) at a point, \(p^2 = \xi^2\), using the condition,
\[ B(\xi, \mu) = Z^4_{m}(\mu, \Lambda) \rho_{P_{\mu}}^4 + \Pi_B(\xi, \Lambda) , \]
\[ A(\xi, \mu) = Z^4_{m}(\mu, \Lambda) + \Pi_A(\xi, \Lambda) , \]
(A6)

where \(\Pi_B\) and \(\Pi_A\) are the expressions inferred from the appropriate projections of the quark self-energy correction in Eq. (2). Solving the DSE starting by imposing new renormalization conditions, namely \(B(p^2, \xi)\) and \(A(p^2, \xi)\), in Eq. (A6), one verifies that the conditions in Eq. (A2) are satisfied with high precision. We check the stability of all results from \(\Lambda = 4\) GeV up to \(\Lambda = 10\) GeV, in which case \(Z_2(\xi, \Lambda)\) and \(Z_4(\xi, \Lambda)\) are modified in order to absorb the cutoff dependence; our results prove to be stable and show no cutoff dependence. The GMOR relation (A3) provide us with an additional cross-check.

[1] G. Krein, C. D. Roberts and A. G. Williams, Int. J. Mod. Phys. A 7, 5607 (1992).
[2] P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly and A. G. Williams, Phys. Rev. D 70, 034509 (2004).
[3] P. O. Bowman, U. M. Heller and A. G. Williams, Phys. Rev. D 66, 014505 (2002).
[4] M. B. Parappilly, P. O. Bowman, U. M. Heller, D. B. Leinweber, A. G. Williams and J.-B. Zhang, Phys. Rev. D 73, 054504 (2006).
[5] S. Furui and H. Nakajima, Phys. Rev. D 73, 074503 (2006).
[6] A. Cucchieri, T. Mendes, O. Oliveira and P. J. Silva, Phys. Rev. D 76, 114507 (2007).
[7] A. Cucchieri and T. Mendes, PoS QCD -TNT09, 026 (2009).
[8] A. Bogolubsky, E. M. Ilgenfritz, M. Müller-Preussker and A. Sternbeck, Phys. Lett. B 676, 69 (2009).
[9] A. Sternbeck and M. Müller-Preussker, Phys. Lett. B 726, 396 (2013).
[10] O. Oliveira and P. J. Silva, Phys. Rev. D 86, 114513 (2012).
[11] A. Ayala, A. Bashir, D. Binosi, M. Cristoforetti and J. Rodríguez-Quintero, Phys. Rev. D 86, 074512 (2012).
[12] P. Boucaud, J.-P. Leroy, A. L. Yaouanc, J. Micheli, O. Pène and J. Rodríguez-Quintero, JHEP 0806, 012 (2008).
[13] P. Boucaud, J. P. Leroy, A. L. Yaouanc, J. Micheli, O. Pène and J. Rodríguez-Quintero, Few Body Syst. 53, 387 (2012).
[14] A. C. Aguilar and J. Papavassiliou, JHEP 0612, 012 (2006).
[15] G. Krein, C. D. Roberts and A. G. Williams, Int. J. Mod. Phys. A 7, 5607 (1992).
[16] P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly and A. G. Williams, Phys. Rev. D 70, 034509 (2004).
[17] M. B. Parappilly and A. G. Williams, Int. J. Mod. Phys. E 1 (2014).
[18] A. C. Aguilar, D. Binosi and J. Papavassiliou, Phys. Rev. D 78, 025010 (2008).
[19] M. R. Pennington and D. J. Wilson, Phys. Rev. D 84, 119901 (2011).
[20] S. Strauss, C. S. Fischer and C. Kellermann, Phys. Rev. Lett. 109, 252001 (2012).
[21] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[22] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[23] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[24] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[25] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[26] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[27] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[28] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[29] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
[30] A. Cucchieri, T. Mendes and O. Oliveira, Phys. Rev. D 81, 074505 (2010).
