Regularization and Anomalies in Gauge Theory

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Abstract

Some of the basic issues related to the regularization and anomalies in gauge theory are reviewed, with particular emphasis on the recent development in lattice gauge theory. The generalized Pauli-Villars regularization is discussed from a viewpoint of the covariant regularization of currents, and the construction of a regularized effective action in terms of covariant currents is compared with the lattice formulation of chiral Abelian theory.

1 Introduction

The regularization is a fundamentally important issue of field theory with an infinite number of degrees of freedom. A closely related issue in field theory is quantum anomaly, though the anomaly itself is perfectly finite. The anomaly is more closely related to “conditional convergence” in a loose sense, the boundary between divergence and convergence. For this reason, the treatment of anomaly becomes rather subtle in a finite theory such as the lattice theory. In this talk, I briefly review some of the fundamental issues related to the regularization and anomalies from my own viewpoint. I will discuss the continuum regularization as well as the lattice regularization, with particular emphasis on the recent exciting development in lattice gauge theory[1][2][3][4].

2 Brief review of continuum path integral

We start with a brief summary of the continuum path integral approach to chiral anomaly[5] and a regularization which may be called “mode cut-off”.

1Talk given at NATO Advanced Research Workshop “Lattice Fermions and Structure of the Vacuum”, October 5-9, 1999, at Dubna, Russia (To be published in the Proceedings)
We study the QCD-type Euclidean path integral with
\[ D \equiv \gamma^\mu (\partial_\mu - igA_\mu^a T^a) = \gamma^\mu (\partial_\mu - igA_\mu) , \]

\[ \int D \bar{\psi} D\psi [DA_\mu] \exp \left[ \int \bar{\psi} (i D - m) \psi d^4 x + S_{YM} \right] \]

(2.1)

where \( \gamma^\mu \) matrices are anti-hermitian with
\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^\mu\nu = -2\delta^\mu\nu , \]
and \( \gamma_5 = -\gamma^1 \gamma^2 \gamma^3 \gamma^4 \) is hermitian. \( S_{YM} \) stands for the Yang-Mills action and \( [DA_\mu] \) contains a suitable gauge fixing.

To analyze the chiral Jacobian we expand the fermion variables \[ \psi(x) = \sum_n a_n \varphi_n(x) \]
\[ \bar{\psi}(x) = \sum_n \bar{b}_n \varphi_n^\dagger(x) \]

(2.2)
in terms of the eigen-functions of hermitian \( \slashed{D} \)

\[ \slashed{D} \varphi_n(x) = \lambda_n \varphi_n(x) \]
\[ \int d^4 x \varphi_n^\dagger(x) \varphi_l(x) = \delta_{n,l} \]

(2.3)
which diagonalize the fermionic action in (2.1). The fermionic path integral measure is then written as

\[ D\bar{\psi}D\psi = \lim_{N \to \infty} \prod_{n=1}^N d\bar{b}_n d\alpha_n \]

(2.4)

Under an infinitesimal global chiral transformation
\[ \delta \psi = i \alpha \gamma_5 \psi , \quad \delta \bar{\psi} = \bar{\psi} i \alpha \gamma_5 \]

(2.5)
we obtain the Jacobian factor

\[ J = \exp \left[ -2i \alpha \lim_{N \to \infty} \sum_{n=1}^N \int d^4 x \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) \right] \]
\[ = \exp \left[ -2i \alpha (n_+ - n_-) \right] \]

(2.6)
where \( n_\pm \) stand for the number of eigenfunctions with vanishing eigenvalues and \( \gamma_5 \varphi_n = \pm \varphi_n \), respectively, in (2.3). We here used the relation
\[ \int d^4 x \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) = 0 \] for \( \lambda_n \neq 0 \). The Atiyah-Singer index theorem \( n_+ - n_- = \nu \) with Pontryagin index \( \nu \), which was confirmed for one-instanton sector in \( R^4 \) space by Jackiw and Rebbi[6], shows that the chiral Jacobian contains the correct information of chiral anomaly.
To extract a local version of the index (i.e., anomaly), we start with the expression

\[ n_+ - n_- = \lim_{N \to \infty} \sum_{n=1}^{N} \int d^4x \varphi_n^\dagger(x) \gamma_5 f((\lambda_n)^2/M^2) \varphi_n(x) \]

\[ = \lim_{N \to \infty} \sum_{n=1}^{N} \int d^4x \varphi_n^\dagger(x) \gamma_5 f(\nabla^2/M^2) \varphi_n(x) \]

\[ = Tr \gamma_5 f(\nabla D/M^2) \]  

(2.7)

for any smooth function \( f(x) \) which rapidly goes to zero for \( x = \infty \) with \( f(0) = 1 \). Since \( \gamma_5 f(\nabla^2/M^2) \) is a well-regularized operator, we may now use the plane wave basis of fermionic variables to extract an explicit gauge field dependence, and we define a local version of the index as

\[ \lim_{M \to \infty} tr \gamma_5 f(\nabla D/M^2) \]

\[ \equiv \lim_{M \to \infty} \sum_{n=1}^{\infty} \varphi_n^\dagger(x) \gamma_5 f(\nabla^2/M^2) \varphi_n(x) \]

\[ = \lim_{M \to \infty} tr \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\nabla^2/M^2) e^{ikx} \]

(2.8)

\[ = \lim_{M \to \infty} tr M^4 \int \frac{d^4k}{(2\pi)^4} \gamma_5 f\{ (ik_\mu + D_\mu)^2/M^2 - \frac{i g}{4} [\gamma_\mu, \gamma_\nu] F_{\mu\nu}/M^2 \} \]

where the remaining trace stands for Dirac and Yang-Mills indices. We also used the relation

\[ \nabla^2 = D_\mu D^\mu - \frac{i g}{4} [\gamma_\mu, \gamma_\nu] F_{\mu\nu} \]  

(2.9)

and the rescaling of the variable \( k_\mu \to M k_\mu \).

By noting \( tr \gamma_5 = tr \gamma_5 [\gamma_\mu, \gamma_\nu] = 0 \), the above expression (after expansion in powers of \( 1/M \)) is written as (with \( \epsilon^{1234} = 1 \))

\[ \lim_{M \to \infty} tr \gamma_5 f(\nabla^2/M^2) = Tr \gamma_5 \frac{1}{2!} \left\{ \frac{-ig}{4} [\gamma_\mu, \gamma_\nu] F_{\mu\nu} \right\}^2 \int \frac{d^4k}{(2\pi)^4} f''(-k_\mu k^\mu) \]

\[ = \frac{g^2}{32\pi^2} tr \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \]  

(2.10)
where we used
\[
\int \frac{d^4 k}{(2\pi)^4} f''(-k_{\mu} k^\mu) = \frac{1}{16\pi^2} \int_0^\infty f''(x) x dx = \frac{1}{16\pi^2}
\]
with \( x = -k_{\mu} k^\mu > 0 \) in our metric. One can confirm that any finite interval \(-L \leq k_{\mu} \leq L\) of momentum variables in (2.8) before the rescaling \( k_{\mu} \rightarrow M k_{\mu} \) gives rise to a vanishing contribution to (2.10). In this sense, the short distance contribution determines the anomaly.

When one combines (2.7) and (2.10), one establishes the Atiyah-Singer index theorem (in \( R^4 \) space). We note that the local version of the index (anomaly) is valid for Abelian theory also. The global index (2.7) as well as a local version of the index (2.10) are both independent of the regulator \( f(x) \) provided[5]
\[
f(0) = 1, \quad f(\infty) = 0, \quad f'(x)x|_{x=0} = f'(x)x|_{x=\infty} = 0 \quad (2.12)
\]
If one chooses a smooth function \( f(x) \) such that
\[
f(x) \simeq 1, \quad 0 \leq x \leq 1 \quad (2.13)
\]
and \( f(x) \) goes to 0 very rapidly for \( x > 1 \), one has
\[
\lim_{N \to \infty} \sum_{n=1}^N \varphi_n^\dagger(x) \gamma_5 f((\lambda_n)^2/M^2) \varphi_n(x) \simeq \sum_{|\lambda_n| \leq M} \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) \quad (2.14)
\]
The essence of the present regularization may thus be called gauge invariant “mode cut-off”, following the terminology of Zinn-Justin.

### 3 Index theorem on the lattice

We now come to the recent interesting development in lattice gauge theory. This development is based on the so-called Ginsparg-Wilson relation[7]
\[
\gamma_5 D + D \gamma_5 = a D \gamma_5 D.
\]
where \( a \) stands for the lattice spacing. If one defines the operator
\[
\Gamma_5 \equiv \gamma_5 (1 - \frac{1}{2} a D) \quad (3.2)
\]
which is hermitian, the above relation is written as
\[ \Gamma_5 \gamma_5 D + \gamma_5 D \Gamma_5 = 0. \quad (3.3) \]
Namely, \( \Gamma_5 \) plays a role of \( \gamma_5 \) in continuum theory. An explicit example of the operator \( D \) which satisfies the Ginsparg-Wilson relation has been constructed by Neuberger[1] and it is known as the overlap operator.

All the finite dimensional representations of the Ginsparg-Wilson algebra (3.3) or the eigenstates \( \phi_n \) of the hermitian \( \gamma_5 D \)
\[ \gamma_5 D \phi_n = \lambda_n \phi_n \quad (3.4) \]
on a finite lattice are categorized into the following 3 classes:
(i) \( n_\pm \) states,
\[ \gamma_5 D \phi_n = 0, \quad \gamma_5 \phi_n = \pm \phi_n, \quad (3.5) \]
(ii) \( N_\pm \) states (\( \Gamma_5 \phi_n = 0 \)),
\[ \gamma_5 D \phi_n = \pm \frac{2}{a} \phi_n, \quad \gamma_5 \phi_n = \pm \phi_n, \quad \text{respectively}, \quad (3.6) \]
(iii) Remaining states with \( 0 < |\lambda_n| < 2/a \),
\[ \gamma_5 D \phi_n = \lambda_n \phi_n, \quad \gamma_5 D (\Gamma_5 \phi_n) = -\lambda_n (\Gamma_5 \phi_n), \quad (3.7) \]
and the sum rule \( n_+ + N_+ = n_- + N_- \) holds[8].

All the \( n_\pm \) and \( N_\pm \) states are the eigenstates of \( D \), \( D \phi_n = 0 \) and \( D \phi_n = (2/a) \phi_n \), respectively. If one denotes the number of states in (iii) by \( 2N_0 \), the total number of states (dimension of the representation) \( N \) is given by \( N = 2(n_+ + N_+ + N_0) \), which is expected to be a constant independent of background gauge field configurations.

The index theorem on the lattice formulated by Hasenfratz, Laliena and Niedermayer[2] is stated as the equality
\[ Tr \Gamma_5 = n_+ - n_- = \nu \quad (3.8) \]
in the continuum limit \( a \to 0 \). Here \( n_\pm \) stand for the number of zero eigenvalue states in (3.5) with \( \gamma_5 \phi_n = \pm \phi_n \), respectively, and \( \nu \) stands for the
Pontrygin index or integrated form of chiral anomaly. The proof of this index relation proceeds as follows:
We first evaluate by using the above classification of states
\[
Tr \Gamma_5 = \sum_{\lambda_n} \phi_n^\dagger \Gamma_5 \phi_n \\
= \sum_{\lambda_n=0} \phi_n^\dagger \Gamma_5 \phi_n \\
= \sum_{\lambda_n=0} \phi_n^\dagger \gamma_5 \phi_n = n_+ - n_- \tag{3.9}
\]
The explicit evaluation of \(Tr \Gamma_5\) has been performed by various authors by perturbative calculation. The result\cite{2,9} confirms the relation for \(a \to 0\)
\[
Tr \gamma_5 (1 - \frac{a}{2}D)(x) = \int d^4x \frac{g^2}{32\pi^2} tr \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \tag{3.10}
\]
The actual calculation is rather involved.

We here present a somewhat simpler calculation\cite{10}, which is similar to the continuum calculation in Section 2. We start with
\[
Tr \{ \gamma_5 [1 - \frac{1}{2}aD] f(\frac{(\gamma_5 D)^2}{M^2}) \} = n_+ - n_- \tag{3.11}
\]
Namely, the index is not modified by any regulator \(f(x)\) with \(f(0) = 1\), as can be confirmed by using the basis in (3.5)-(3.7). The hermitian operator \(\gamma_5 D\) plays a privileged role in the present analysis of the index theorem. We then consider a local version of the index
\[
tr \{ \gamma_5 [1 - \frac{1}{2}aD] f(\frac{(\gamma_5 D)^2}{M^2}) \} \tag{3.12}
\]
where trace stands for Dirac and Yang-Mills indices. A local version of the index is not sensitive to the precise boundary condition, and one may take the infinite volume limit \(L = Na \to \infty\) in the above expression.

We now examine the continuum limit \(a \to 0\) of the above local expression (3.12).\footnote{This continuum limit corresponds to the so-called “naive” continuum limit in the context of lattice gauge theory.} We first observe that the term
\[
tr \{ \frac{1}{2}a \gamma_5 D f(\frac{(\gamma_5 D)^2}{M^2}) \} \tag{3.13}
\]
goes to zero in this limit. The large eigenvalues of $\gamma_5 D$ are truncated at the value $\sim M$ by the regulator $f(x)$ which rapidly goes to zero for large $x$. In other words, the global index of the operator $Tr \frac{a}{2} \gamma_5 D f \left( \frac{(\gamma_5 D)^2}{M^2} \right) \sim O(aM)$.

We thus examine the small $a$ limit of

$$tr\{\gamma_5 f \left( \frac{(\gamma_5 D)^2}{M^2} \right) \} \quad (3.14)$$

The operator appearing in this expression is well regularized by the function $f(x)$, and we evaluate the above trace by using the plane wave basis to extract an explicit gauge field dependence. We consider a square lattice where the momentum is defined in the Brillouin zone

$$-\frac{\pi}{2a} \leq k_\mu < \frac{3\pi}{2a} \quad (3.15)$$

We assume that the operator $D$ is free of species doubling; in other words, the operator $D$ blows up rapidly ($\sim \frac{1}{a}$) for small $a$ in the momentum region corresponding to species doublers. The contributions of doublers are eliminated by the regulator $f(x)$ in the above expression. We thus examine the above trace in the momentum range of the physical species

$$-\frac{\pi}{2a} \leq k_\mu < \frac{\pi}{2a} \quad (3.16)$$

We now obtain the limiting $a \to 0$ expression

$$\lim_{a \to 0} tr\{\gamma_5 f \left( \frac{(\gamma_5 D)^2}{M^2} \right) \}$$

$$= \lim_{a \to 0} tr \int_{\frac{\pi}{2a}}^{\frac{3\pi}{2a}} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f \left( \frac{(\gamma_5 D)^2}{M^2} \right) e^{ikx}$$

$$= \lim_{L \to \infty} \lim_{a \to 0} tr \int_{-L}^{L} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f \left( \frac{(\gamma_5 D)^2}{M^2} \right) e^{ikx}$$

$$= \lim_{L \to \infty} tr \int_{-L}^{L} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f \left( \frac{(\gamma_5 D)^2}{M^2} \right) e^{ikx}$$

$$= tr\{\gamma_5 f \left( \frac{D^2}{M^2} \right) \} \quad (3.17)$$

where we first take the limit $a \to 0$ with fixed $k_\mu$ in $-L \leq k_\mu \leq L$, and then take the limit $L \to \infty$. This procedure is justified if the integral is
well convergent[10]. We also assumed that the operator $D$ satisfies the following relation in the limit $a \to 0$

$$De^{ikx}g(x) \to e^{ikx}(\mathbf{k} - i\,\partial - \mathbf{A})g(x) = -i\mathcal{D}(e^{ikx}g(x))$$  \hspace{1cm} (3.18)$$

for any fixed $k_\mu$, $(-\frac{\pi}{2a} < k_\mu < \frac{\pi}{2a})$, and a sufficiently smooth function $g(x)$. The function $g(x)$ corresponds to the gauge potential in our case, which in turn means that the gauge potential $A_\mu(x)$ is assumed to vary very little over the distances of the elementary lattice spacing. It is shown that an explicit example of $D$ given by Neuberger[1] satisfies the property (3.18) without species doublers.

Our final expression (3.17) in the limit $M \to \infty$ thus reproduces the index theorem in the continuum formulation, (2.10), by using the quite general properties of the basic operator $D$ only: The basic relation (3.1) with hermitian $\gamma_5 D$ and the continuum limit property (3.18) without species doubling in the limit $a \to 0$.

### 3.1 Modified chiral transformation

Utilizing the notion of the index on the lattice, L"uscher introduced a new kind of chiral transformation[3]

$$\delta\psi = i\alpha\gamma_5(1 - \frac{1}{2}aD)\psi, \quad \delta\bar{\psi} = \bar{\psi}i\alpha(1 - \frac{1}{2}aD)\gamma_5$$  \hspace{1cm} (3.19)$$

with an infinitesimal constant parameter $\alpha$. This transformation leaves the action invariant due to the property (3.1), and gives rise to the chiral Jacobian factor

$$J = \exp\{-2i\alpha Tr\gamma_5(1 - \frac{1}{2}aD)\}$$  \hspace{1cm} (3.21)$$

The index theorem (3.8) shows that this Jacobian factor indeed carries the correct chiral anomaly.

As a generalization of the vector-like(QCD-type) theory discussed so far, L"uscher[11] showed that a chiral Abelian gauge theory can be consistently defined on a lattice. In particular, the anomaly in the fermion
number current, which generally appears in chiral gauge theory, arises as a result of the non-vanishing index of a rectangular $m \times n$ matrix $M$

$$dimker M - dimker M^\dagger = m - n$$

(3.22)

Namely, the regularized Lagrangian for chiral fermion is characterized by a rectangular matrix instead of a naive square matrix. A further comment on the Abelian chiral theory on the lattice will be given later.

The lattice regularization of chiral non-Abelian gauge theory has not been formulated so far[12].

4 Generalized Pauli-Villars regularization

A Lagrangian level regularization of chiral non-Abelian gauge theory in continuum has been formulated by Frolov and Slavnov[13], and Narayanan and Neuberger[14]. This scheme is based on a generalization of the Pauli-Villars regularization. To regularize one chiral fermion, one needs to introduce an infinite number of fermions and unphysical bosonic fermions. This regularization is applicable to anomaly-free gauge theory only.

Instead of writing the regularized Lagrangian, we here discuss the generalized Pauli-Villars regularization from a viewpoint of the regularization of currents[15]. The essence of the generalized Pauli-Villars regularization is summarized in terms of regularized currents as follows:

$$<\overline{\psi}(x) T^a \gamma^\mu \left(\frac{1 + \gamma_5}{2}\right) \psi(x)>_{PV}$$

$$= - \lim_{y \rightarrow x} \left\{ \frac{1}{2} Tr \left[ T^a \gamma^\mu f(D^2/\Lambda^2) \frac{1}{i\not{D}} \delta(x - y) \right] ight.$$

$$+ \frac{1}{2} Tr \left[ T^a \gamma^\mu \gamma_5 \frac{1}{i\not{D}} \delta(x - y) \right]\}

$$<\overline{\psi}(x) \gamma^\mu \left(\frac{1 + \gamma_5}{2}\right) \psi(x)>_{PV}$$

$$= - \lim_{y \rightarrow x} \left\{ \frac{1}{2} Tr \left[ \gamma^\mu f(D^2/\Lambda^2) \frac{1}{i\not{D}} \delta(x - y) \right] ight.$$

$$+ \frac{1}{2} Tr \left[ \gamma^\mu \gamma_5 \frac{1}{i\not{D}} \delta(x - y) \right]\}$$
\[ <\overline{\psi}(x)\gamma^\mu\gamma_5(\frac{1+\gamma_5}{2})\psi(x)>_{PV} \]
\[ = -\lim_{y\to x} \left\{ \frac{1}{2} Tr \left[ \gamma^\mu\gamma_5 f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y) \right] \right. \]
\[ \left. + \frac{1}{2} Tr \left[ \gamma^\mu\frac{1}{i\not{D}}\delta(x-y) \right] \right\} \]  
\[ (4.1) \]

where the regularization function is defined by
\[ f(\not{D}^2/\Lambda^2) = \sum_{n=-\infty}^{\infty} (-1)^n \not{D}^2/[(\not{D}^2 + (n\Lambda)^2] = \frac{\pi(\not{D}/\Lambda)}{\sinh(\pi\not{D}/\Lambda)} \]  
\[ (4.2) \]

Note that \( f(x) \) satisfies all the properties in (2.12). In the left-hand sides of these equations (4.1), the currents are defined in terms of the fields appearing in the original chiral Lagrangian, which one wants to regularize, while the right-hand sides of these equations stand for the regularized expressions. The axial-vector and vector \( U(1) \) currents in terms of the chiral fermion fields in the original Lagrangian are identical if one note \( \gamma_5^2 = 1 \), but the regularized versions (i.e. the last two equations in (4.1)) are different. In particular, the vector \( U(1) \) current (i.e., the second equation in (4.1)) is not completely regularized. We emphasize that all the one-loop diagrams are generated from the (partially) regularized currents in (4.1), as will be discussed later in connection with the effective action; in other words, (4.1) retains all the information of the generalized Pauli-Villars regularization[13][14]. It is interesting that this regularization is implemented in the Lagrangian level.

### 4.1 Covariant regularization

A closely related regularization of chiral currents is known as the covariant regularization, which regularizes all the currents (and consequently all the one-loop fermionic diagrams) and reproduces the so-called covariant anomalies[15]. This covariant regularization is, however, not implemented in the Lagrangian level, in general. The currents in the covariant regularization are written as
\[ <\overline{\psi}(x)T^a\gamma^\mu(\frac{1+\gamma_5}{2})\psi(x)>_{cov} \]
\[ = -\lim_{y\to x} \left\{ Tr \left[ T^a\gamma^\mu(\frac{1+\gamma_5}{2})f(\not{D}^2/\Lambda^2)\frac{1}{i\not{D}}\delta(x-y) \right] \right\} \]
\[ \langle \overline{\psi}(x)\gamma^\mu\left(\frac{1 + \gamma_5}{2}\right)\psi(x) \rangle_{\text{cov}} = -\lim_{y \to x} \left\{ Tr \left[ \gamma^\mu\left(\frac{1 + \gamma_5}{2}\right) f(\mathcal{D}^2/\Lambda^2) \frac{1}{i} \mathcal{D} \delta(x - y) \right] \right\} \]

(4.3)

The difference of this regularization from the generalized Pauli-Villars regularization in (4.1) is that all the components (either vector or axial-vector) are well-regularized. All the fermionic one-loop diagrams are thus regularized. The price we have to pay for this is that this regularization is not implemented in the Lagrangian level.

The anomaly in the gauge current is given by

\[ D_\mu \langle \overline{\psi}(x) T^a \gamma^\mu\left(\frac{1 + \gamma_5}{2}\right)\psi(x) \rangle_{\text{cov}} = -D_\mu \sum_n \varphi_n(x) \mid T^a \gamma^\mu\left(\frac{1 + \gamma_5}{2}\right) f(\lambda_n^2/\Lambda^2) \frac{1}{i\lambda_n} \varphi_n(x) \]

\[ = \sum_n (\mathcal{D}\varphi_n(x)) \mid T^a \gamma^\mu\left(\frac{1 + \gamma_5}{2}\right) f(\lambda_n^2/\Lambda^2) \frac{1}{i\lambda_n} \varphi_n(x) \]

\[ - \sum_n \varphi_n(x) \mid T^a \gamma^\mu\left(\frac{1 - \gamma_5}{2}\right) f(\lambda_n^2/\Lambda^2) \frac{1}{i\lambda_n} \mathcal{D} \varphi_n(x) \]

\[ = -i \sum_n \varphi_n(x) \mid T^a \gamma_5 f(\lambda_n^2/\Lambda^2) \varphi_n(x) \]

(4.4)

where we used the eigenfunctions

\[ \mathcal{D}\varphi_n = \lambda_n \varphi_n \]

(4.5)

We thus recover the Jacobian factor corresponding to the covariant anomaly.

As for the fermion number anomaly, we have similarly

\[ \partial_\mu \langle \overline{\psi}(x) \gamma^\mu\left(\frac{1 + \gamma_5}{2}\right)\psi(x) \rangle_{\text{cov}} = -i \sum_n \varphi_n(x) \mid \gamma_5 f(\lambda_n^2/\Lambda^2) \varphi_n(x) \]

\[ = \partial_\mu \langle \overline{\psi}(x) \gamma^\mu\gamma_5\left(\frac{1 + \gamma_5}{2}\right)\psi(x) \rangle_{\text{PV}} \]

(4.6)

This shows that one can reproduce the correct fermion number anomaly by using the axial \( U(1) \) current in (4.1) in the generalized Pauli-Villars regularization[16].
From this analysis, one can see that the generalized Pauli-Villars regularization is closely related to the covariant regularization. Since the covariant regularization is applicable to any chiral gauge theory, it is useful to decide if any theory is anomalous or not. However, the covariant current as it stands does not generate the integrable (or consistent) anomaly. This issue is discussed in the next Section.

5 Definition of effective action in terms of covariant currents

It is known that the effective action for the fermion is written in terms of the current. By using this fact, it has been proposed by H.Banerjee, R.Banerjee and P.Mitra to write the regularized effective action in terms of the regularized covariant current[17]. As a simplest example, we discuss the Abelian chiral gauge theory defined by

\[ Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \mathcal{D}(1 + \gamma_5 / 2) \bar{\psi} \gamma^\mu (1 + \gamma_5 / 2) \psi} \]

\[ W = \ln Z = \ln \det [\mathcal{D}(1 + \gamma_5 / 2)] \]

\[ \mathcal{D} = \gamma^\mu (\partial^\mu - igA_\mu) \] (5.1)

We then obtain

\[ \frac{\partial W}{\partial g} = \langle \int d^4x A_\mu(x) \bar{\psi}(x) \gamma^\mu (1 + \gamma_5 / 2) \psi(x) \rangle \] (5.2)

The regularized effective action may be defined in terms of the covariant current by

\[ W_{\text{reg}} \equiv \int_0^g \int d^4x A_\mu(x) \langle \bar{\psi}(x) \gamma^\mu (1 + \gamma_5 / 2) \psi(x) \rangle_{\text{cov}} \] (5.3)

The consistent current is then derived from this regularized effective action as

\[ j^\mu(x)_{\text{cons}} \equiv \frac{\delta}{\delta A_\mu(x)} W_{\text{reg}} \]

\[ = \int_0^g dg j^\mu(x)_{\text{cov}} + \int_0^g dg \int dy A_\nu(y) \frac{\delta j^\nu(y)_{\text{cov}}}{\delta A_\mu(x)} \]

\[ = j^\mu(x)_{\text{cov}} - \int_0^g dgg \frac{\partial}{\partial g} j^\mu(x)_{\text{cov}} + \int_0^g dg \int dy A_\nu(y) \frac{\delta j^\nu(y)_{\text{cov}}}{\delta A_\mu(x)} \] (5.4)
Also
\[ \frac{\partial}{\partial g} j^\mu(x)_{\text{cov}} = \int dy A_\nu(y) \frac{\delta}{\delta (gA_\nu(y))} j^\mu(x)_{\text{cov}} \] (5.5)
as \( j^\mu(x)_{\text{cov}} \) depends on \( g \) only through the combination \( aA_\nu(y) \). We thus obtain
\[ j^\mu(x)_{\text{cons}} = j^\mu(x)_{\text{cov}} + \int_0^g dg \int dy A_\nu(y) \{ \frac{\delta j^\nu(y)_{\text{cov}}}{\delta A_\mu(x)} - \frac{\delta j^\mu(x)_{\text{cov}}}{\delta A_\nu(y)} \} \] (5.6)

We note that by using (5.6)
\[ W_{\text{reg}} \equiv \int_0^g \int d^4x A_\mu(x) j^\mu(x)_{\text{cov}} = \int_0^g \int d^4x A_\mu(x) j^\mu(x)_{\text{cons}} \] (5.7)
namely, the regularized effective action is independent of whether the regularized covariant current or regularized consistent current is used in its construction. All the naive properties are reproduced by our definition of \( W_{\text{reg}} \).

As for the chiral anomaly, we have (by noting that the Abelian covariant current is gauge invariant)
\[ W(A_\mu + \partial_\mu \omega)_{\text{reg}} = \int_0^g \int d^4x (A_\mu(x) + \partial_\mu \omega(x)) j^\mu(x)_{\text{cov}} \]
\[ = W_{\text{reg}} - \int_0^g \int d^4x \omega(x) \partial_\mu j^\mu(x)_{\text{cov}} \] (5.8)
and if one lets the cut-off parameter \( \Lambda \to \infty \) in the last covariant current, we generate the covariant anomaly
\[ W(A_\mu + \partial_\mu \omega)_{\text{reg}} = W_{\text{reg}} - \frac{1}{16\pi^2} \int_0^g \int d^4x \omega(x) F(gA_{\mu u}) \tilde{F}(gA_\mu) \]
\[ = W_{\text{reg}} - \frac{1}{3} \frac{1}{16\pi^2} \int d^4x \omega(x) F(gA_{\mu u}) \tilde{F}(gA_\mu) \] (5.9)
and we reproduce the consistent anomaly with the correct Bose symmetrization factor \( 1/3 \). It is known that this scheme works for the non-Abelian theory also[17].

5.1 Application to lattice gauge theory

It has been pointed out by H. Suzuki[18] that the basic aspect of the above construction of the regularized effective action in terms of the covariant
current works for the lattice theory also, and one in fact obtains a formula closely related to the construction of the Abelian chiral theory given by Lüscher[11].

The starting expression is

\[ W = \int_0^g dg Tr \frac{\partial D}{\partial g} \left( \frac{1 + \gamma_5}{2} \right) D^{-1} \] (5.10)

where \( D \) stands for the lattice Dirac operator which satisfies the Ginsparg-Wilson relation, and

\[ \hat{\gamma}_5 \equiv \gamma_5 (1 - aD) \] (5.11)

with \((\hat{\gamma}_5)^2 = 1\). A naive continuum limit of (5.10) is

\[ W_{naive} = \int_0^g dg Tr \left[ A_\mu \gamma_\mu \left( \frac{1 + \gamma_5}{2} \right) \frac{1}{i D} \right] \] (5.12)

and \( W \) is gauge invariant in Abelian theory. Thus \( W \) in (5.10) is a counter part of \( W_{reg} \) in continuum theory.

It has been shown by Suzuki[18] that \( W \) in (5.10) for lattice theory gives the first term of the consistent lattice Abelian anomaly

\[ \frac{1}{3} \frac{1}{16\pi^2} F \tilde{F}_{lattice} + \partial_\mu K^\mu \] (5.13)

where the second term is a “lattice artifact” found by Lüscher[11]. Namely, \( K^\mu \) is gauge invariant and goes to 0 in the naive continuum limit \( a \to 0 \). An improvement of the above \( W \) by using this \( K^\mu \) has been shown [18] to be identical to the result in Ref.[11].

6 Conclusion

The remarkable development in lattice theory enriched our understanding of the regularization of fermions and the basic aspects of chiral symmetry and anomalies in gauge theory. It is interesting to see that the covariant current and consistent current play mutually complementary roles in these constructions[19].

The interesting notion of index on the lattice[2] deserves further investigation. The lattice formulation of chiral non-Abelian theory (and eventually supersymmetric theory) remains as a challenging problem[12].
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