On the maximum principles and the quantitative version of
the Hopf lemma for uniformly elliptic integro-differential
operators

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Abstract

In the present paper we prove estimates on subsolutions of the equation
\[-Av + c(x)v = 0, \; x \in D,\]
where \(D \subset \mathbb{R}^d\) is a domain (i.e. an open and connected set) and
\(A\) is an integro-differential operator of the Waldenfels type, whose differential
part satisfies the uniform ellipticity condition on compact sets. In general, the
coefficients of the operator need not be continuous but only bounded and Borel
measurable. Some of our results may be considered "quantitative" versions of the
Hopf lemma, as they provide the lower bound on the outward normal directional
derivative at the maximum point of a subsolution in terms of its value at the
point. We shall also show lower bounds on the subsolution around its maximum
point by the principal eigenfunction associated with \(A\) and the domain. Additional
results, among them the weak and strong maximum principles, the weak Harnack
inequality are also proven.

1 Introduction

The maximum principle and the Hopf lemma express some of the most fundamental
properties of solutions of elliptic partial differential equations. Consider a differential
operator

\[ Lu(x) := \frac{1}{2} \sum_{i,j=1}^{d} q_{i,j}(x) \partial_{x_i,x_j}^2 u(x) + \sum_{i=1}^{d} b_i(x) \partial_{x_i} u(x), \quad u \in C^2(\mathbb{R}^d) \]  

that is uniformly elliptic on compact sets, i.e. the coefficients \(q_{i,j}, b_j, i, j = 1, \ldots, d\) are bounded, Borel measurable and the matrix \(Q(x) = [q_{i,j}(x)]_{i,j=1}^{d}\) is uniformly positive definite on compact sets, see (2.3) below.

The strong maximum principle, see e.g. [19 Theorem 9.6, p. 225], states that if \(D\)
is a bounded domain (an open and connected set) and \(u \in W^{2,d}_{\text{loc}}(D)\) (i.e. it has two
generalized derivatives that locally belong to \(L^d(D)\)) satisfying

\[ Lu - c(x)u \geq 0, \quad \text{where } c(x) \geq 0, \; x \in D, \]  

admits a non-negative maximum in \(D\), then it has to be constant.

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eigenvalue and eigenvector
If \( \partial D \) - the boundary of the domain - is sufficiently regular, say \( C^2 \) smooth and \( u \) is non-constant, then the Hopf lemma, see e.g. [27] Theorem 7, p. 65], asserts that \( \partial_n u(\hat{x}) \) - the directional derivative of \( u \) in the outward normal to \( \partial D \) - at the maximal point \( \hat{x} \) has to be strictly positive.

It has been shown in [7] that when \( \partial D \) is \( C^2 \) smooth and \( \underline{c} := \inf_{x \in D} c(x) > 0 \), there exists a constant \( a > 0 \), depending only on the ellipticity constant on \( D \), \( \underline{c} \) the \( L^\infty \) bound on the coefficients of \( L \) and \( c \), and \( D \) such that

\[
\partial_n u(\hat{x}) > au(\hat{x}) \tag{1.3}
\]

for any \( u \) satisfying \((1.2)\) that is not identically equal to 0. The above estimate could be viewed as a form of a quantitative Hopf lemma. An elementary example, using harmonic functions, shows that \((1.3)\) is false if \( c \equiv 0 \).

One of the objectives of the present paper is to investigate an analogue of \((1.3)\) and its generalizations for non-local operators. More precisely, we consider non-divergence form integro-differential operators

\[
Au(x) = Lu(x) + Su(x), \quad u \in C^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \tag{1.4}
\]

where \( L \) is given by \((1.1)\) and

\[
Su(x) := \int_{\mathbb{R}^d} \left( u(x + y) - u(x) - \sum_{i=1}^d \frac{y_i \partial_{x_i} u(x)}{1 + |y|^2} \right) N(x, dy), \quad x \in \mathbb{R}^d. \tag{1.5}
\]

The coefficients of the operator \( L \) are as in the foregoing and the (Lévy) kernel \( N \) assigns to each \( x \in \mathbb{R}^d \) a \( \sigma \)-finite Borel measure \( N(x, \cdot) \), which satisfies

\[
N_* := \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \min\{1, |y|^2\} N(x, dy) < +\infty. \tag{1.6}
\]

Such operators are sometimes called of the Waldenfels type, see e.g. [30] Chapter 10].

Let

\[
S(D) := \bigcup_{x \in D} S_x, \quad \text{where } S_x := [x + y : y \in \text{supp } N(x, \cdot)]. \tag{1.7}
\]

The set \( S_x \) describes the "range of non-locality" of \( A \) at \( x \), i.e. the evaluation of \( Au(x) \) depends on the values of \( u \) in an arbitrarily small neighborhood of \( x \), and in \( S_x \).

Then, the following version of the "quantitative Hopf lemma" holds, see Theorem 3.5 below for a precise statement.

**Theorem 1.1.** Suppose that \( \underline{c} > 0 \), the domain \( D \) is bounded and satisfies the uniform interior ball condition (see Definition 2.2) and \( u \), such that \( Au - c(x)u \geq 0 \), \( x \in D \), attains its maximum over \( S(D) \cup \overline{D} \) at some \( \hat{x} \in \partial D \) and is not identically equal to 0. Then, there exists \( a > 0 \) that depends only on \( D \), ellipticity constant, \( \underline{c} \), the \( L^\infty \) norm of the coefficients and \( N_* \) such that estimate \((1.3)\) holds

We can relax the assumption that \( \underline{c} > 0 \). Namely, see Theorem 3.6 below, we admit that \( \underline{c} = 0 \) but \( c(x) \) does not vanish a.e. in \( D \). However, in that case we have to allow \( a > 0 \) in \((1.3)\) to depend also on the operator \( A \) (in particular on the lower bound of its Green function), not just on some characteristics of its coefficients as in Theorem 1.1.

In our further generalizations of estimate \((1.3)\), see Theorems 5.1 and 5.2 below, we compare the increment \( u(\hat{x}) - u(x) \) with the value of the principal eigenfunction \( \varphi_D(x) \).
associated with the operator $A$ on the domain $D$, i.e. the unique eigenfunction that is strictly positive in $D$, see Theorem 4.7. It turns out (see Theorem 5.2) that, with no assumption on the regularity of the domain $D$, the following bound holds

$$u(\hat{x}) - u(x) \geq \frac{\varphi_D(x)}{2e\|\varphi_D\|_\infty} \left[ \frac{c u(\hat{x})}{\lambda_D + \varphi} + \text{essinf}_D (Au - cu) \right], \quad x \in D. \quad (1.8)$$

Here $\lambda_D$ is the principal eigenvalue corresponding to $\varphi_D$ and $\| \cdot \|_\infty$ is the usual supremum norm.

In fact, instead of the essential infimum of $Au - cu$ in (1.8), we can use its average over $D$ with respect to some measure equivalent with the Lebesgue measure. Namely, see Theorem 5.1, there exist $\psi(x)$ continuous, strictly positive in $D$ and $\chi(x)$ strictly positive, Borel measurable such that any subsolution $u$, described in the foregoing, satisfies

$$u(\hat{x}) - u(x) \geq \frac{c \varphi_D(x)u(\hat{x})}{2e\|\varphi_D\|_\infty(\lambda_D + \varphi)} + \psi(x) \int_D (Au - cu) \chi \, dx, \quad x \in D. \quad (1.9)$$

We have already mentioned that estimates (1.8) and (1.9) are obtained without making assumptions on the regularity of the domain $D$. However, if we suppose that both the uniform interior and exterior ball conditions hold, see Definitions 2.2 and 2.5, then we can choose $\psi(x) = \delta_D(x)$, where $\delta_D(x) := \text{dist}(x, D^c)$, see Theorem 5.4. Furthermore, if some regularity of the coefficients of $A$ is allowed, then, in addition to the above choice of $\psi(x)$, we can admit also $\chi(x) = a\delta_D(x)$, with some $a > 0$, see Theorems 5.6 and 5.8.

It is also worthwhile to mention that the estimates (1.8) and (1.9) hold even without the uniform ellipticity hypothesis. They are related to ergodic properties of the Markov process, associated with the operator $A$ via the respective martingale problem, see Section 4 below, e.g. its uniform conditional ergodicity, or intrinsic ultracontractivity of its transition semigroup. This shall be the topic of our forthcoming paper [21].

In addition to the results mentioned above, in the present paper we also obtain a weak maximum principle for operators of the form (1.4), see Proposition 3.1. Namely, if $Au - c(x)u \geq 0$ on $D$, where $c(x) \geq 0$, then

$$\sup_{x \in D} u(x) \leq \max \left\{ \sup_{y \in \partial S(D) \setminus D} u^+(y), \sup_{y \in \partial D} u^+(y) \right\}. \quad (1.10)$$

This result holds without assuming that $D$ is connected. Using the weak form of the maximum principle we establish its strong version and the Hopf lemma, see Theorem 3.4. In addition we prove also the Bony type maximum principle and the weak Harnack inequality, see Theorems 3.3 and 5.9. Recall here that Harnack inequality in general does not hold for operators (1.4) without additional restrictions on the Lévy kernel $N$, see [18] Section 7.

Let us briefly discuss the relation between our work and some results existing in the literature. The maximum principles both weak and strong, together with the Hopf lemma, see Theorem 3.4. In addition we prove also the Bony type maximum principle and the weak Harnack inequality, see Theorems 3.3 and 5.9. Recall here that Harnack inequality in general does not hold for operators (1.4) without additional restrictions on the Lévy kernel $N$, see [18] Section 7.

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to part 1) of our Theorem 3.4, that we have encountered, can be found in the recent paper of Taira [32], where the author has additionally assumed that $S_x(D) = \overline{D}$ for $x \in D$ (no jumps outside $\overline{D}$), there exists a reference measure, with respect to which, all the Lévy kernels are absolutely continuous. The regularity of the resulting densities is also assumed. The domain $D$ is to be of class $C^{1,1}$ and $u \in W^{2,p}(D)$ for some $p > d$.

Our formulation of the Hopf lemma, see part 2) of Theorem 3.4 and in particular Corollary 3.10, resembles somewhat Proposition 1.34 of [20], which actually uses the phrase "quantitative Hopf lemma" in this context. The result of ibid. holds for harmonic functions on a ball $B$ and asserts that there exists $a > 0$ such that for any bounded harmonic function $u$ in $B$, we have

$$\partial_n u(\hat{x}) \geq a(u(\hat{x}) - u(x_0))$$

with $\hat{x} \in \partial B$ a maximum point of $u$ in $\bar{B}$ and $x_0$ - the center of the ball. The counterpart of this estimate for $A$-harmonic functions, provided that the operator has the Harnack property (see Definition 3.8), is stated in (3.8) below.

Theorem 5.1 can be compared with the Morel and Oswald version of the Hopf lemma proved by Brezis and Cabré in [11], see Lemma 3.2. It asserts that for a smooth and bounded domain $D$ there exists $a > 0$ such that for any superharmonic function $u$, vanishing on the boundary, we have

$$u(x) \geq a\delta_D(x) \int_D (-\Delta u)(y)\delta_D(y) dy, \quad x \in D.$$

(1.12)

The estimate is sometimes referred to as the uniform Hopf lemma (see e.g. [16, 14, 26]).

We prove an analogue of (1.12), see Theorem 5.8 for $C^2$-regular domains, function $u(x)$ satisfying $Au - c(x)u \geq 0$ on $D$, where $c(x) \geq 0$ and operator $A$ of the form (1.4), whose coefficients of the second derivative, as well as the respective coefficients of its formal adjoint belong to the VMO class, see Definition 5.7. From (1.12) one can easily conclude the weak Harnack inequality (see e.g. [19, Theorem 9.22]) that is shown in the present paper in Theorem 5.9.

The principal tools used in our analysis are: the martingale problem associated with operator $A$, see Section 4.1 and the Feynman-Kac formula for the respective canonical process $(X_t)_{t \geq 0}$, see Proposition 4.6. They allow us to obtain a probabilistic representation for a subsolution of the equation $-Av + c(x)v = 0$, see Definition 2.1, in terms of the expectation of an appropriate functional of the process, see (4.10). We exploit various properties the process $(X_t)_{t \geq 0}$, such as its irreducibility and the strong Feller property, to derive our results.

The organization of the paper is as follows. In Section 2 we introduce the basic notation and definitions. The first and second parts of the set of formulations of our main results are contained in Sections 3 and 5. In Section 4 we introduce basic probabilistic tools used throughout the remained of the paper, such as: the already mentioned martingale problem associated with $A$, the (probabilistic) semigroup and resolvent corresponding to the canonical process, the gauge functional and the principal eigenvalue and eigenfunction associated with $A$ and domain $D$ that are used in formulations of the results of Section 5. Finally, Section 6 contains the proofs of the announced results.
2 Preliminaries

2.1 Basic notation

Given a metric space $E$ we denote by $B(E)$ its Borel algebra. Let $B_b(E)$ ($B_b^+(E)$) be the space of all (non-negative) bounded Borel measurable functions and let $C_b(E)$ ($C_c(E)$) be the space of all bounded continuous (compactly supported) functions on $E$. Furthermore by $\mathcal{M}(E)$ we denote the set of all Borel measures on $E$. Suppose that $\mu, \nu \in \mathcal{M}(E)$. We say that $\mu$ dominates $\nu$ and write $\nu \ll \mu$ if all null sets for $\mu$ are also null for $\nu$. We say that measures are equivalent and write $\mu \sim \nu$ iff $\mu \ll \nu$ and $\nu \ll \mu$.

Given a point $x \in E$ and $r > 0$ we let $B(x, r)$ be the open ball of radius $R$ centered at $x$ and $\bar{B}(x, r)$ its closure. As it is customary for a given function $f : E \to \mathbb{R}$ we denote $\|f\|_\infty = \sup_{x \in E} |f(x)|$. For a subset $B \subset E$ we let $B^c := E \setminus B$ and $\bar{B}$ be its closure.

Suppose that $B$ is an arbitrary set. For functions $f, g : B \to [0, +\infty)$ we write $f \preceq g$ on $B$ if there exists number $C > 0$, i.e., constant, such that

$$f(x) \leq Cg(x), \quad x \in B.$$  

Furthermore, we write $f \approx g$ if both $f \preceq g$ and $g \preceq f$.

Assume now that $D \subset \mathbb{R}^d$ and $V \subset D$ are open. We shall write $V \Subset D$ if $V$ is compact and $V \subset D$. Furthermore we let $\mathcal{C}^m(D)$, $m \geq 1$ be the class of $m$-times continuously differentiable functions in $D$. By $\mathcal{C}_0(D)$ we denote the subset of $\mathcal{C}(D)$ that consists of functions extending continuously to $\bar{D}$ by letting $f(x) \equiv 0$, $x \in \partial D$ - the boundary of $D$.

Denote by $m_d$ the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$. Sometimes we omit writing the subscript, when the dimension is obvious from the context. By $dx$ we denote the volume element corresponding to the Lebesgue measure.

For $p \in [1, +\infty)$ we denote by $L^p(D)$ ($L^p_{\text{loc}}(D)$) the space of functions that are integrable with their $p$-th power on $D$ (any compact subset of $D$). By $W^{k,p}(D)$ ($W^{k,p}_{\text{loc}}(D)$) we denote the Sobolev space of functions whose $k$ generalized derivatives belong to $L^p(D)$ ($L^p_{\text{loc}}(D)$). Given $f \in L^p(D)$ and $B \in \mathcal{B}(D)$ we denote by $\text{ess inf}_B f$ the usual essential infimum of $f$ in $B$. We define the essential limit inferior of $f$ at $x_0 \in D$ as

$$\liminf_{x \to x_0} \text{ess inf} f(x) := \lim_{r \to 0^+} \text{ess inf}_{B(x_0, r) \subset D} f. \quad (2.1)$$

2.2 Second-order, elliptic integro-differential operators

Consider an elliptic integro-differential operator of the form

$$ Au(x) = Lu(x) + Su(x), \quad u \in C^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \quad (2.2)$$

where $L, S$ are a differential and integro-differential operators, given by (1.1) and (1.5) respectively. Throughout the paper we shall always assume that the coefficients satisfy the following hypotheses:

A1) $q_{i,j} \in B_b(\mathbb{R}^d)$, $b_j \in B_b(\mathbb{R}^d)$, $i, j = 1, \ldots, d$ and we suppose that the differential operator $L$ is uniformly elliptic on compact sets, i.e. the matrix $Q(x) = [q_{i,j}(x)]_{i,j=1}^d$
is uniformly positive definite on compact sets. The latter means that for any compact set \( K \subset \mathbb{R}^d \) there exists \( \lambda_K > 0 \) such that

\[
\lambda_K |\xi|^2 \leq \sum_{i,j=1}^{d} q_{i,j}(x) \xi_i \xi_j, \quad \xi \in K, \; \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.
\]  \tag{2.3}

A2) \( N : \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d) \), called the Lévy kernel, is a \( \sigma \)-finite Borel measure valued function such that (1.6) holds.

We let

\[
M_A := \sum_{i,j=1}^{d} \|q_{i,j}\|_{\infty} + \sum_{i=1}^{d} \|b_i\|_{\infty} + N_* \quad \tag{2.4}
\]

Suppose that \( D \) is an open set. Obviously in order to define \( Au(x) \) for \( x \in D \) it suffices to assume that \( u \in C^2(D) \cap C_b(\mathbb{R}^d) \). In fact, \( Au \) can be defined as an element of \( L^p(D) \), using (1.4)–(1.5), even if \( u \in W^{2,p,\text{loc}}(D) \cap C_b(\mathbb{R}^d) \) when \( p > d \). This is possible due to a well known estimate, see [8, Lemme 1, p. 361]: for any \( p > d \) there exists \( C > 0 \) such that

\[
\|U[v]\|_{L^p(\mathbb{R}^d)} \leq C\|\nabla^2 v\|_{L^p(\mathbb{R}^d)}, \quad v \in W^{2,p}(\mathbb{R}^d). 
\]

Here

\[
U[v](x) := \sup_{|y|>0} \left| y \right|^{-2} \left| v(x+y) - v(x) - \sum_{i=1}^{d} y_i \partial_{x_i} v(x) \right|. 
\]

2.3 A subsolution, supersolution and solution of an elliptic type integro-differential equation

**Definition 2.1.** Suppose that \( c : \mathbb{R}^d \to \mathbb{R} \). A function \( u : \mathbb{R}^d \to \mathbb{R} \) is called a subolution of the equation

\[
(-A + c) f = 0 
\]  \tag{2.5}

on an open set \( D \subset \mathbb{R}^d \) if \( u \in W^{2,p,\text{loc}}(D) \cap C_b(\mathbb{R}^d) \) for some \( p > d \) and

\[
-Au(x) + c(x)u(x) \leq 0, \quad \text{for a.e. } x \in D. \tag{2.6}
\]

We say that \( u \) is a supersolution of equation (2.5) if \( -u \) is its subsolution. Furthermore, \( u \) is a solution if it is both sub- and supersolution.

Given a bounded and measurable function \( c : D \to \mathbb{R} \) we let

\[
\bar{c} = \sup_{x \in D} c(x), \quad \underline{c} := \inf_{x \in D} c(x) \quad \text{and} \quad \langle c \rangle_D := \int_D c \, dx. \tag{2.7}
\]

Concerning the coefficient \( c(\cdot) \) it is assumed to satisfy \( c \in B_b(D) \) and either one of the following hypotheses:

A3) it is non-negative on \( D \), or

A3’) it is both non-negative on \( D \) and \( \langle c \rangle_D > 0 \), (i.e. \( c \not\equiv 0 \) a.e.), or
Below we formulate some hypotheses concerning an open set $D$ that shall be used in various results presented in the paper. Throughout most of the paper, unless stated otherwise (see e.g. Proposition 3.1), it is supposed to be a domain, i.e. an open and connected subset of $\mathbb{R}^d$.

**Definition 2.2** (Interior ball condition). For a given domain $D$ and $\hat{x} \in \partial D$, we say that the **interior ball condition** is satisfied at $\hat{x}$, if there exists $B$ - an open ball of a positive radius - contained in $D$ and such that $\hat{x} \in \partial D \cap \bar{B}$. Any ball $B$ as above shall be called an **interior ball** for $D$ at $\hat{x}$. We furthermore say that $D$ satisfies the **interior ball condition** if the condition is satisfied at any point of its boundary. We let $r_D(\hat{x})$ denote the maximal radius, that is less than, or equal to 1, of an interior ball at the given $\hat{x} \in \partial D$. We say that $D$ satisfies the **uniform interior ball condition** if it satisfies the interior ball condition and there exists $r_D \in (0,1)$ such that $r_D(\hat{x}) \geq r_D$, for all $\hat{x} \in \partial D$. (2.8)

Suppose $D$ satisfies the interior ball condition at $\hat{x} \in \partial D$ and $B(y,r) \subset D$, with $r > 0$, is an interior ball at $\hat{x}$. Any vector of the form

$$n := \frac{1}{r}(\hat{x} - y)$$

shall be called a **generalized unit outward normal vector** to $\partial D$ at $\hat{x}$. Denote by $n(\hat{x})$ the set of all such vectors at $\hat{x} \in \partial D$.

**Definition 2.3** (Lower outward normal derivative at a boundary point). Suppose that $D$ satisfies the interior ball condition at $\hat{x} \in D$. Assume furthermore that $f : \mathbb{R}^d \to \mathbb{R}$. For any $n \in n(\hat{x})$ define the **lower outward normal derivative** at $\hat{x}$ as

$$\underline{\partial}_n f(\hat{x}) := \liminf_{h \to 0^+} \frac{f(\hat{x}) - f(\hat{x} - h n)}{h}.$$  (2.10)

**Remark 2.4.** If $\partial D$ is $C^2$ smooth, then for any $\hat{x} \in \partial D$ the set $n(\hat{x})$ is a singleton and consists only of the unit outward normal vector to $\partial D$ at $\hat{x}$. In addition, if $f : \mathbb{R}^d \to \mathbb{R}$ possesses a derivative at $\hat{x} \in \partial D$, then $\underline{\partial}_n f = \partial_n f$ - the outward normal derivative.

In some cases, see e.g. Theorem 5.4 below, we shall assume that the domain satisfies the exterior ball condition in the following sense.

**Definition 2.5.** For a given domain $D$ and $\hat{x} \in \partial D$, we say that the **exterior ball condition** is satisfied at $\hat{x}$, if there exists $B$ - an open ball of a positive radius - contained in $D^c$ and such that $\hat{x} \in \partial D \cap \bar{B}$. Any ball $B$ as above shall be called an **exterior ball** for $D$ at $\hat{x}$. We furthermore say that $D$ satisfies the **exterior ball condition** if the condition is satisfied at any point of its boundary.

### 3 Maximum principles and quantitative Hopf Lemmas

The present section is devoted to the formulation of the first part of the set of our main results. Their proofs are presented throughout Section 6.
3.1 Maximum principles and the Hopf lemma

Our first result is a form of a weak maximum principle.

Proposition 3.1. Suppose that the function $c$ satisfies A3). If $u$ is a subsolution to (2.5) on a bounded open set $D \subset \mathbb{R}^d$ then (cf. (1.7))

$$\sup_{x \in D} u(x) \leq \max \left\{ \sup_{y \in S(D) \setminus D} u^+(y), \sup_{y \in \partial D} u^+(y) \right\}. \quad (3.1)$$

Moreover, if $c \equiv 0$, then $u^+$ on the right-hand side can be replaced by $u$.

The proof of the proposition is shown in Section 6.2.

Remark 3.2. Observe that the right-hand side of (3.1) reduces to $\sup_{y \in D} c u^+(y)$ in the case $S(D) = \mathbb{R}^d$ and to $\sup_{y \in \partial D} u^+(y)$ when $S(D) = D$.

Theorem 3.3 (Bony maximum principle). Let $D$ be an open subset of $\mathbb{R}^d$. Assume that $S(D) \subset \overline{D}$. Suppose that $u \in W^{2,p}_{\text{loc}}(D) \cap C_0(\mathbb{R}^d)$ for some $p > d$ and $\hat{x} \in D$ is its maximum point in $D$. Then, cf. (2.1),

$$\liminf_{x \to \hat{x}} \text{ess inf} Au(x) \leq 0. \quad (3.2)$$

The proof of the theorem is presented in Section 6.7.

In an analogy to the case of elliptic differential operators, we have the strong maximum principle and Hopf lemma for a subsolution of (2.5). The formulation of the latter, given below, is of a bit more "quantitative" nature than it is usually encountered in the literature. To state it we introduce some notation. For $r > 0$ we let

$$D_r := \{ x \in D : \text{dist}(x, \partial D) > r \}. \quad (3.3)$$

Theorem 3.4. Suppose that the function $c$ satisfies A3).

1) (The strong maximum principle) If $u$ is a subsolution to (2.5) on a domain $D$ that attains its maximum $M \geq 0$ over $S(D) \cup \overline{D}$ at $\hat{x} \in D$, then $u \equiv M$ in $D$.

2) (The Hopf lemma) Assume that $D$ satisfies the uniform interior ball condition, with constant $r_D \in (0,1]$. Then, there exist $r \in (0,r_D), a > 0$ depending only on $M_A$ and $\lambda_D$ such that for any non-constant subsolution $u$ to (2.5) that attains its non-negative maximum over $S(D) \cup \overline{D}$ at $\hat{x} \in \partial D$, and any $n \in n(\hat{x})$ at $\hat{x} \in \partial D$ we have

$$\partial_n u(\hat{x}) \geq a \inf_{x \in D_r/2} (u(\hat{x}) - u(x)) > 0. \quad (3.4)$$

The proof of the theorem is given in Section 6.3.2.

3.2 Quantitative Hopf lemmas

Suppose that $u$ is a subsolution of (2.5). If $c$ satisfies either the hypothesis A3'), or A3''), then we can formulate a lower bound on the outward normal partial derivative of $u$ at the maximum point in terms of its maximal value.
Theorem 3.5 (Quantitative Hopf Lemma I.A). Suppose that function $c$ satisfies hypothesis A3). In addition, assume that the domain $D$ is bounded and satisfies the uniform interior ball condition. Then, there exists $a > 0$ that depends only on $r_D$ of (2.8), ellipticity constant $\lambda_D$, the lower bound $\underline{c}$ and $M_A + \|c\|_\infty$ (cf (2.4)) such that for any $u$ - a subsolution to (2.5) - that attains its maximum over $S(D) \cup \overline{D}$ at $\hat{x} \in \partial D$ and is not identically equal to 0 we have

$$\partial_n u(\hat{x}) > au(\hat{x}).$$

(3.5)

The proof of the theorem is presented in Section 6.3.4.

A version of the result in the case when $c$ is only assumed to be non-negative and not equal to 0 a.e. can be stated as follows.

Theorem 3.6 (Quantitative Hopf lemma I.B). Suppose that function $c$ satisfies A3'). In addition, assume that the interior ball condition for the domain $D$ is in force. Then, there exist a constant $r_0 > 0$ depending only on $D$, $\lambda_D$ and $M_A + \|c\|_\infty$ (cf (2.4)) and a nondecreasing function $\rho_{c,A,D} : (0, +\infty) \to (0, +\infty)$, depending only on $A, c$ and $D$, such that for $u$ and $\hat{x} \in \partial D$ as in Theorem 3.5 we have

$$\partial_n u(\hat{x}) > a\rho_{c,A,D}(r_0 \wedge \tau(\hat{x}))u(\hat{x}).$$

(3.6)

Here $\tau(\cdot)$ is as in Definition 2.2.

The proof of the theorem is presented in Section 6.3.3.

3.3 Harnack property and the Hopf lemma

For operators $A$ having the Harnack property and $A$-harmonic functions we can replace the right hand of (3.4) by a constant multiplied by $u(\hat{x}) - u(x_0)$ for a fixed $x_0 \in D$. To rigorously formulate the result we start with the notion of an $A$-harmonic function.

Definition 3.7. Suppose that $p > d$ and $D \subset \mathbb{R}^d$ is open. We say that a function $u \in C_b(\mathbb{R}^d) \cap W^{2,p}_{loc}(D)$ is $A$-harmonic in $D$ iff $Au = 0$ a.e. in $D$.

Definition 3.8. We say that $A$ has the Harnack property in $D$ iff for any $r \in (0,1]$ there exists $C > 0$ such that for any $y_0 \in D$ satisfying $B(y_0, r) \subset D$ and a non-negative $u \in C_b(\mathbb{R}^d)$, which is $A$-harmonic in $B(y_0, r)$, we have

$$u(x) \leq Cu(y), \quad x, y \in B(y_0, r/2).$$

(3.7)

Remark 3.9. In general, the Harnack property need not hold for $A$. In fact, some additional hypotheses on the Lévy kernel $N$ are needed. Some sufficient conditions on the kernel guaranteeing the Harnack property of $A$ can be found in [18, Theorem 2.4, p. 25].

The following result is a straightforward conclusion from Theorem 3.4 and the Harnack property of $A$.

Corollary 3.10. In addition to the assumptions made in Theorem 3.4, suppose that the domain $D$ is bounded and $A$ has the Harnack property in $D$. Then, for any $x_0 \in D$ there exists $a > 0$ depending only on $M_A$, $\lambda_D$, $\text{diam}D$ and $\text{dist}(x_0, \partial D)$ such that for any non-negative $u$, which is $A$-harmonic in $D$ and attains its maximum in $S(D) \cup \overline{D}$ at $\hat{x} \in \partial D$, we have

$$\partial_n u(\hat{x}) \geq a(u(\hat{x}) - u(x_0)) > 0.$$
Proof. By Theorem 3.4 we have estimate (3.4). To replace its right hand side by the right hand side of (3.8) it suffices to apply the Harnack property of $A$ to the non-negative harmonic function $u(\hat{x}) - u(x)$ and use the fact that compact set $\bar{D}_{r/2}$ can be covered by finitely many balls of radius $r$. \hfill $\blacksquare$

4 Probabilistic preliminaries

Throughout this section we shall assume that the hypotheses A1) – A3) hold.

4.1 On the martingale problem associated with operator $A$

Let $D$ be the space consisting of all functions $\omega : [0, +\infty) \to \mathbb{R}^d$, that are right continuous and possess the left limits for all $t > 0$ (c\'{a}dl\'{a}gs), equipped with the Skorochod topology, see e.g. Section 12 of [3]. Define the canonical process $X_t(\omega) := \omega(t)$, $\omega \in D$ and its natural filtration $\mathcal{F}_t$, with $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$.

Definition 4.1 (A solution of the martingale problem associated with the operator $A$).

Suppose that $\mu$ is a Borel probability measure on $\mathbb{R}^d$. A Borel probability measure $P_\mu$ on $D$ is called a solution of the martingale problem associated with $A$ with the initial distribution $\mu$, cf [29], if

i) $P_\mu[X_0 \in Z] = \mu[Z]$ for any Borel measurable $Z \subset \mathbb{R}^d$.

ii) For every $f \in C_b^2(\mathbb{R}^d)$ - a $C^2$-smooth function bounded with its two derivatives on $\mathbb{R}^d$ - the process

$$M_t[f] := f(X_t) - f(X_0) - \int_0^t Af(X_r) \, dr, \ t \geq 0 \quad (4.1)$$

is a (c\'{a}dl\'{a}g) martingale under measure $P_\mu$ with respect to natural filtration $\mathcal{F}_{t \geq 0}$. As usual we write $P_x := P_{h_x}$, $x \in \mathbb{R}^d$ and say that $x$ is the initial condition. The expectations with respect to $P_\mu$ and $P_x$ shall be denoted by $E_\mu$ and $E_x$, respectively.

Definition 4.2 (A strong Markovian solution of the martingale problem). We say that a family of Borel probability measures $(P_x)_{x \in \mathbb{R}^d}$ on $D$ is a strong Markovian solution to the martingale problem associated with $A$ if:

i) each $P_x$ is a solution of the martingale problem associated with $A$, corresponding to the initial condition at $x$,

ii) the canonical process $(X_t)$ is strongly Markovian with respect to the natural filtration $\mathcal{F}_t$ and the family $(P_x)_{x \in \mathbb{R}^d}$,

iii) the mapping $x \to P_x[C]$ is measurable for any Borel $C \subset D$,

iv) for any Borel probability measure $\mu$ on $\mathbb{R}^d$ the probability measure

$$P_\mu(\cdot) := \int_{\mathbb{R}^d} P_x(\cdot) \mu(dx)$$

is a solution to the martingale problem associated with $A$ with the initial distribution $\mu$. 

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The following result concerning the existence of a strong Markov solution for a martingale problem has been proven in [1], see also [22].

**Theorem 4.3.** Suppose that conditions (2.3) and (2.4) are satisfied. Then, the martingale problem associated with the operator $A$ admits a strong Markovian solution.

### 4.2 Analytic description of the canonical process and weak subsolution to (2.5)

#### 4.2.1 Exit time, transition probability semigroup, resolvent operator

For a given domain $D$ define the exit time $\tau_D : D \to [0, +\infty]$ of the canonical process $(X_t)_{t \geq 0}$ from $D$ as

$$\tau_D := \inf\{t > 0 : X_t \notin D\}. \quad (4.2)$$

It is a stopping time, i.e. for any $t \geq 0$ we have $[\tau_D \leq t] \in \mathcal{F}_t$, see Theorem I.10.7, p. 54 of [5] and Theorem IV.3.12, p. 181 of [17].

**Proposition 4.4.** Suppose that $D$ is a bounded domain. Then,

$$\sup_{x \in D} \mathbb{E}_x \tau_D < +\infty. \quad (4.3)$$

**Proof.** See e.g. [18, Lemma 4]. \qed

We let

$$P_t^D f(x) = \mathbb{E}_x [f(X_t), t < \tau_D] \quad t \geq 0, \quad f \in B_b(D). \quad (4.4)$$

This is a probabilistic formula for the semigroup generated by operator (1.4) on $D$ with the null exterior condition.

For a positive $f \in B(D)$ and $\alpha \geq 0$ we let

$$R_\alpha^D f(x) := \int_0^\infty e^{-\alpha t} P_t^D f(x) \, dt = \mathbb{E}_x \left[ \int_0^{\tau_D} e^{-\alpha r} f(X_r) \, dr \right], \quad x \in D. \quad (4.5)$$

Set $R^D := R_0^D$. We have $R_\alpha^D f(x) < +\infty, x \in D$ for $f \in B_b(D)$ and $\alpha > 0$.

#### 4.2.2 The gauge function for $c(\cdot)$ and domain $D$

**Definition 4.5** (The gauge function for $c(\cdot)$ and domain $D$). The function

$$v_{c,D}(x) = \mathbb{E}_x e_c(\tau_D), \quad x \in \mathbb{R}^d, \quad (4.6)$$

where

$$e_c(t) := e^{-\int_0^t c(X_r) \, dr}, \quad t \geq 0, \quad (4.7)$$

is called the **gauge function** corresponding to $c(\cdot)$ and domain $D$, cf Section 4.3 of [12].

Let us denote

$$w_{c,D}(x) := 1 - v_{c,D}(x), \quad x \in \mathbb{R}^d. \quad (4.8)$$

Obviously $w_{c,D}(x) \geq 0, x \in \mathbb{R}^d$. In fact, as it turns out, under the assumptions made in the present paper $w_{c,D}(x)$ is strictly positive and continuous in $D$, see Lemma 6.14 below.
4.3 Feynman-Kac formula for $(X_t)$

**Proposition 4.6.**

1) If $u \in W^{2,p}_{\text{loc}}(D)$ and $p > d$, then for any $V \subset D$ and $t \geq 0$

$$u(x) = \mathbb{E}_x \left[ e_c(\tau_V \wedge t) u(X_{\tau_V \wedge t}) \right] + \mathbb{E}_x \left[ \int_0^{\tau_V \wedge t} e_c(s)(-A u + cu)(X_s) \, ds \right], \quad x \in V. \quad (4.9)$$

Moreover, $(4.9)$ holds also with $\tau_V$ in place of $\tau_V \wedge t$.

2) If $u$ is a subsolution of $(2.5)$, then

$$u(x) = \mathbb{E}_x \left[ e_c(\tau_D \wedge t) u(X_{\tau_D \wedge t}) \right] + \mathbb{E}_x \left[ \int_0^{\tau_D \wedge t} e_c(s)(-A u + cu)(X_s) \, ds \right], \quad x \in D. \quad (4.10)$$

for any $t \geq 0$. Moreover, $(4.10)$ holds also with $\tau_D$ in place of $\tau_D \wedge t$.

The proof of this result is given in Section 6.1 below.

4.4 Principal eigenvalue and eigenfunction associated with $A$

The following result is the version of the Krein-Rutman theorem for the operators considered in Section 2.2.

**Theorem 4.7.** Suppose that the domain $D$ is bounded. Then, there exist a unique pair $(\varphi_D, \lambda_D)$, where $\varphi_D : D \to (0, +\infty)$ - a strictly positive continuous function - and $\lambda_D > 0$ are such that

$$e^{-\lambda_D t} \varphi_D(x) = P^D_t \varphi_D(x), \quad t \geq 0, \quad x \in D. \quad (4.11)$$

The proof of the theorem is presented in Section 6.4.

**Definition 4.8.** The function $\varphi_D$ and $\lambda_D$ are called respectively: the principal eigenfunction and its associate principal eigenvalue for the operator $A$, with the Dirichlet exterior condition on $D^c$, see e.g. [6].

5 Generalizations of the quantitative Hopf lemmas and weak Harnack estimates

Throughout this section we shall assume that the domain $D$ is bounded.

5.1 Some generalizations of Theorems 3.5 and 3.6

Below, we formulate some generalizations of estimates (3.5) and (3.6). In general, we shall no longer assume that the boundary $\partial D$ is regular enough so that the outward normal, appearing in the left hand sides of the aforementioned bounds, can be defined. It shall be replaced by the ratio $[u(\hat{x}) - u(x)] / \psi(x)$, $x \in D$. Here $\hat{x} \in \partial D$ and $\psi : D \to (0, +\infty)$ are the maximum point of the subsolution $u$ and some suitably defined function (e.g. the principal eigenfunction associated with the operator), respectively. In case $\psi(x) \approx \delta_D(x)(:= \text{dist}(x, D^c))$, when $x \to \hat{x}$ (which may happen for sufficiently regular domains) these type of estimates lead to (3.5) and (3.6).
**Theorem 5.1** (Quantitative Hopf lemma II.A). Suppose that the assumption A3) holds. Then, there exist a continuous function \( \psi : D \to (0, +\infty) \) and a Borel function \( \chi : D \to (0, +\infty) \), such that for any subsolution \( u(\cdot) \) of (2.5) satisfying

\[ u(\hat{x}) = \max_{x \in \partial S(D) \cup \partial D} u(x) \geq 0 \text{ for some } \hat{x} \in \partial D \]

we have

\[ u(\hat{x}) - u(x) \geq \frac{\varphi_D(x)u(\hat{x})}{2e\varphi_D(\lambda_D + \mathcal{L})} + \psi(x) \int_D (Au - cu)\chi \, dx, \quad x \in D. \]  

(5.1)

The result is proved in Section 6.6.

Instead of the second term in the right hand side of (5.1) we can use a term depending on the principal eigenfunction, but then we have to replace the mean of \( Au - cu \), with respect to some measure, by its essential infimum.

**Theorem 5.2** (Quantitative Hopf lemma II.B). Suppose that \( u(\cdot) \) and \( \hat{x} \) are as in Theorem 5.1. Then, under the assumptions of the above theorem we have (cf (2.7))

\[ u(\hat{x}) - u(x) \geq \frac{\varphi_D(x)u(\hat{x})}{2e\varphi_D(\lambda_D + \mathcal{L})} + \psi(x) \int_D (Au - cu)\chi \, dx, \quad x \in D. \]

(5.2)

The result is proved in Section 6.5.

**Remark 5.3.** We stress the fact that in both Theorems 5.1 and 5.2 no assumption on \( \partial D \) is made, except that it is a boundary of a bounded domain.

**5.2 Remarks on Theorem 5.1**

It is interesting to compare Theorem 5.1 with Lemma 3.2 of [11], where a similar estimate is obtained for solutions of the homogeneous Dirichlet boundary problem for the Poisson equation, i.e.

\[ -\Delta u = f \geq 0 \quad \text{and} \quad u(x) = 0, \quad x \in \partial D. \] (5.3)

According to [11] if \( f \in L^\infty, \partial D \) is smooth, then there exists \( a > 0 \), depending only on \( D \), such that for any \( u \) satisfying (5.3) we have

\[ u(x) \geq \psi(x) \int_D f\chi \, dx, \quad x \in D, \] (5.4)

where

\[ \chi(x) = \delta_D(x) \quad \text{and} \quad \psi(x) = a\delta_D(x). \] (5.5)

Note that in Theorem 5.1 no assumptions on smoothness of \( \partial D \) is made. In addition, the operator \( A \) is no longer just the laplacian. Instead, it suffices that hypotheses A1) and A2) hold. Obviously, in such a generality one cannot hope to show an analogue of (5.1), with the functions \( \psi, \chi \) given by (5.5), see Remark 5.5 below. The estimate however holds, provided both the domain and the coefficients of \( A \) are sufficiently regular. We have the following.

**Theorem 5.4.** Assume that \( D \) satisfies both the uniform interior ball and exterior ball conditions. Then, there exists \( a > 0 \) such that (5.1) holds with \( \psi \) replaced by \( a\delta_D \).
The proof of the corollary can be found in Section 6.12.1.

**Remark 5.5.** An example of a domain that satisfies both the interior and exterior ball conditions, but not in the uniform sense, see [2, Theorem 4], shows that even in the case when $A$ is the laplacian one cannot use $\psi = a\delta_D$ in (5.1) (thus also in (5.4), with $\psi$ as above) for any $a > 0$. On the other hand, the Hopf lemma is then satisfied.

We can further strengthen Theorem 5.4 allowing $\psi$ and $\chi$ as in (5.5), if some additional assumptions on $A$ are made. Consider the operator $\hat{A}$ that is the formal adjoint (w.r.t. the Lebesgue measure) to $A$, i.e.

$$\int_{\mathbb{R}^d} vAu \, dx = \int_{\mathbb{R}^d} u\hat{A}v \, dx, \quad u, v \in C^2_c(\mathbb{R}^d). \tag{5.6}$$

Assume that it is of the form $\hat{A}_0 - \hat{c}$, where

$$\hat{A}_0 u(x) = \frac{1}{2} \sum_{i,j=1}^{d} \hat{q}_{i,j}(x) \partial^2_{x_i,x_j} u(x) + \sum_{i=1}^{d} \hat{b}_i(x) \partial_{x_i} u(x)$$

$$+ \int_{\mathbb{R}^d} \left( u(x+y) - u(x) - \sum_{i=1}^{d} \frac{y_i \partial_{x_i} u(x)}{1+|y|^2} \right) \hat{N}(x,dy), \quad u \in C^2(D) \cap C_b(\mathbb{R}^d), \tag{5.7}$$

with the coefficients $\hat{q}_{i,j}, \hat{b}_i, \hat{N}$ satisfying A1), A2) and $\hat{c} \in B_b(\mathbb{R}^d)$. Define $\hat{S}(D)$ using (1.7) for $\hat{N}$.

Let $(\hat{P}_x)$ be a strong Markovian solution to the martingale problem corresponding to $\hat{A}_0$. Then, by the strong Markov property

$$\hat{P}_x^\alpha f(x) := \hat{E}_x [f(X_t)e^\alpha(t), t < \tau_D], \quad x \in D, t \geq 0, f \in B_b(D) \tag{5.8}$$

defines a semigroup of operators $(\hat{P}_x^\alpha)_{t \geq 0}$ on $B_b(D)$. Let $\hat{R}_x^\alpha$ be the respective resolvent operator, defined by (4.5), where $P_t^D$ is replaced by $\hat{P}_t^D$. It can be defined for all $\alpha \geq \|\hat{c}^+\|_\infty$. Assume that

$$\int_D \hat{R}_x^\alpha fg \, dx = \int_D f R_x^\alpha g \, dx, \quad f, g \in B^+(D), \alpha \geq \|\hat{c}^+\|_\infty. \tag{5.9}$$

The following result can be inferred from Theorem 5.1. Its proof is given in Section 6.12.1.

**Theorem 5.6.** Assume that $D$ satisfies the assumptions made in Theorem 5.4 and $\hat{A}$ is as described above. Then, there exists $\alpha > 0$ such that (5.1) holds with $\psi$ and $\chi$ as in (5.5).

In order to use Theorem 5.6 we need to verify that the resolvent $\hat{R}_x^\alpha$ satisfies (5.9). The equality holds, provided that both $\partial D$ and coefficients of $\hat{A}$ are sufficiently regular. Before we formulate some sufficient condition let us first recall the notion of a function of vanishing mean oscillation.

**Definition 5.7** (Functions of vanishing mean oscillation). Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $r > 0$ let us define

$$\eta(r) := \sup_{B \subset \mathbb{R}^d, m_d(B) \in \mathbb{N}} \frac{1}{m_d(B)} \int_B |f(x) - f_B| \, dx.$$
Here \( \mathfrak{B}_r \) denotes the family of all balls of radius less than, or equal to \( r \) and \( f_B := (1/m_d(B)) \int_B f(x) \, dx \). We say that a function is of vanishing mean oscillation (VMO) if \( \sup_{r>0} \eta(r) < +\infty \), i.e. it is of bounded mean oscillation (BMO), and \( \lim_{r \to 0} \eta(r) = 0 \). Denote by VMO the class of all such functions.

The following result is shown in Section 6.12.2.

**Theorem 5.8.** Suppose that \( \partial D \) is of \( C^2 \) class, the coefficients of both \( A \) and \( \hat{A}_0 \) satisfy hypotheses A1) and A2) and \( \hat{c} \in B_b(\mathbb{R}^d) \). Furthermore, assume that \( q_{i,j}, \hat{q}_{i,j} \) belong to VMO for all \( i, j = 1, \ldots, d \) and \( S(D) \cup \hat{S}(D) \subset \overline{D} \). Then (5.9) holds.

### 5.3 Weak Harnack inequality

We end this section with the result that can be interpreted as a weak form of the Harnack inequality.

**Theorem 5.9.** Let \( D \) be a bounded domain in \( \mathbb{R}^d \) and \( c \) satisfies hypothesis A3). Then, there exists a bounded Borel measurable function \( \chi : D \to (0, +\infty) \), such that for any \( V \subset D \) we can find a constant \( C > 0 \) such that for any \( u \) - a non-negative supersolution to (2.5) - we have

\[
\inf_{x \in V} u(x) \geq C \int_V u \chi \, dx. \tag{5.10}
\]

The proof of the result is shown in Section 6.8.

A direct consequence of part 2) of Theorem 3.4 and Theorem 5.9 is the following.

**Corollary 5.10.** Under the assumptions of part 2) of Theorem 3.4 and Theorem 5.9 for any \( r > 0 \) there exist \( C > 0 \) such that any non-constant subsolution \( u \) to (2.5), that attains its non-negative maximum over \( S(D) \cup \hat{S}(D) \) at \( \hat{x} \in \partial D \), and any \( n \in n(\hat{x}) \) at \( \hat{x} \in \partial D \) satisfies (cf (3.3))

\[
\partial_n u(\hat{x}) \geq C \int_{D_r} (u(\hat{x}) - u(x)) \chi(x) \, dx > 0. \tag{5.11}
\]

Here \( \chi \) is as in the statement of Theorem 5.9.

### 6 Proofs

#### 6.1 Proof of Proposition 4.6

Suppose that \( u \in W^{2p}_{\text{loc}}(D) \cap C_b(\mathbb{R}^d) \) for some \( p > d \) is a subsolution of (2.5). We can find \( (u_n) \subset C^2_c(\mathbb{R}^d) \) such that \( \|u - u_n\|_{W^{2,p}(V)} \) for any \( V \subset D \). This in particular implies that \( u_n \to u \) and \( \nabla u_n \to \nabla u \) uniformly and \( \|D^2 u_n - D^2 u\|_{L^p(V)} \to 0 \) for any \( V \subset D \), as \( n \to +\infty \). As a result for any such \( V \), we have

\[
\lim_{n \to +\infty} \|A u_n - A u\|_{L^p(V)} = 0. \tag{6.1}
\]

By [29] Theorem 4.3,

\[
M^u_t := u_n(X_t) - u_n(x) - \int_0^t A u_n(X_r) \, dr, \quad t \geq 0 \tag{6.2}
\]
is a martingale under the measure $P_x$ for every $x \in \mathbb{R}^d$. By the Itô formula, applied to $e_c(t)u_n(X_t)$, for any $V \subseteq D$ we get

$$u_n(x) = e_c(\tau_V \wedge t)u_n(X_{\tau_V \wedge t}) + \int_0^{\tau_V \wedge t} e_c(r)(-Au_n + cu_n)(X_r) \, dr$$

\[ - \int_0^{\tau_V \wedge t} e_c(r) \, dM^u_r, \quad t \geq 0, \quad P_x\text{-a.s.} \]

Hence, we conclude that

$$u_n(x) = \mathbb{E}_x\left[e_c(\tau_V \wedge t)u_n(X_{\tau_V \wedge t})\right] + \mathbb{E}_x\left[\int_0^{\tau_V \wedge t} e_c(s)(-Au_n + cu_n)(X_s) \, ds\right], \quad x \in V. \quad (6.3)$$

Recall the Krylov estimate

$$\sup_{x \in V} \mathbb{E}_x\left[\int_0^{\tau_V} |f(X_t)| \, dt\right] \leq C\|f\|_{L^d(V)}, \quad f \in L^d(V) \quad (6.4)$$

for some $C > 0$, see [11, Corollary p. 143], or [22, Thm III14, (53)]. Letting $n \to \infty$ in (6.3) and using (6.4) we get

$$u(x) = \mathbb{E}_x\left[e_c(\tau_V \wedge t)u(X_{\tau_V \wedge t})\right] + \mathbb{E}_x\left[\int_0^{\tau_V \wedge t} e_c(s)(-Au + cu)(X_s) \, ds\right], \quad x \in V. \quad (6.5)$$

Letting $t \to \infty$ and using (6.4) and the fact that $P_x(\tau_V < \infty) = 1$, we get (1). Let $\{D_n\}$ be an increasing sequence of relatively compact open subsets of $D$ such that $D_n \subseteq D_{n+1} \subseteq D$ and $\bigcup_{n \geq 1} D_n = D$. We also have $\tau_{D_n}$ increases to $\tau_D$, as $n \to +\infty$. Taking $D_n$ in place of $V$ in (6.3) and letting $n \to \infty$, we get (using quasi-left continuity of the process, see Theorem IV.3.12, p. 181 of [17]) formula (4.10) for $x \in D$. Next, using the fact that $Au - cu \geq 0$ on $D$ (since $u$ is a subsolution of (2.3)), the fact that $P_x(\tau_D < \infty) = 1$, and the monotone convergence theorem, we may let $t \to \infty$ in (4.10), where $\tau_D$ replaces $\tau_D \wedge t$. \qed

### 6.2 Proof of Proposition 3.1

By Proposition 4.6, we can write

$$u(x) \leq \mathbb{E}_x\left[e_c(\tau_D)u(X_{\tau_D})\right] \leq \mathbb{E}_xu^+(X_{\tau_D}), \quad x \in D. \quad (6.6)$$

Note that $X_{\tau_D}$ is well defined a.s., thanks to the fact that $D$ is bounded, cf (4.3).

Using the Ikeda-Watanabe formula, see [10, Remark 2.46, page 65], we obtain

$$\mathbb{E}_{x_0}\left[\sum_{0 < s \leq \tau_D} 1_D(X_{s-})1_{S(D)^c \setminus D}(X_s)\right]$$

\[ = \mathbb{E}_{x_0}\left[\int\int_0^{\tau_D} 1_D(X_{s-})1_{S(D)^c \setminus D}(y)N(X_{s-}, dy - X_{s-}) \, ds\right]. \quad (6.7)\]

From the definition of $S(D)$, cf (1.7), we conclude that the right-hand side of the above equation vanishes. Thus, $X_{\tau_D} \in (S(D) \setminus D)$, if $X_{\tau_D} \neq X_{\tau_D-}$, or $X_{\tau_D} \in \partial D$, if otherwise. Summarizing, we have shown that

$$P_x(X_{\tau_D} \in (S(D) \setminus D) \cup \partial D) = 1, \quad x \in D$$

and estimate (3.1) follows. When $c \equiv 0$ we get (6.6) with $u^+$ replaced by $u$. \qed
6.3 Proofs of Theorems 3.4, 3.5 and 3.6

6.3.1 Some auxiliaries

Given \( \bar{y} \in \mathbb{R}^d \) and \( \alpha > 0 \) define
\[
\eta(x; \alpha, \bar{y}, r) = e^{-\alpha |x-\bar{y}|^2} - e^{-\alpha r^2}, \quad x \in \mathbb{R}^d.
\] (6.8)

Define also open annuli
\[
V_\epsilon(\bar{y}; r) = B(\bar{y}, r) \setminus \bar{B}(\bar{y}, r/2), \quad V^\epsilon(\bar{y}; r) = B(\bar{y}, 3r/2) \setminus \bar{B}(\bar{y}, r/2).
\] (6.9)

We shall need the following auxiliary facts.

**Lemma 6.1.** Suppose that \( D \) satisfies the interior ball condition with \( \tau(\cdot) \) as in Definition 2.2. Then, for any \( K > 0 \) there exist constants \( r_0, C > 0 \) depending only on \( \lambda_\tau, \|N\|_\infty, \|Q\|_\infty, \|b\|_\infty, \|c\|_\infty \) such that for every \( \hat{x} \in \partial D \), any interior ball \( B(\bar{y}, r) \) at \( \hat{x} \) (cf Definition 2.2) with \( r \in (0, r_0 \wedge \tau(\hat{x})) \), we have
\[
(A - c)\eta(x; Cr^{-2}, \bar{y}, r) \geq K, \quad x \in V^\epsilon(\bar{y}; r).
\] (6.10)

**Lemma 6.2.** Suppose that assumption \( A3') \) is fulfilled and \( D \) satisfies the interior ball condition. Let \( r_0, C \) be constants from Lemma 6.1 for \( K = 1 \). Then, there exists a nondecreasing, strictly positive continuous function \( \rho_{c,A,D} : (0, +\infty) \to (0,1] \) depending only on \( c \), of operator \( A \) and the domain \( D \) such that for any \( \hat{x} \in \partial D \) and interior ball \( B(\bar{y}, r) \) at \( \hat{x} \) (cf Definition 2.2) with \( r \in (0, r_0 \wedge \tau(\hat{x})) \), we have
\[
\eta(x; Cr^{-2}, \bar{y}, r) \rho_{c,A,D}(r/2) \leq w_{c,D}(x), \quad x \in V_\epsilon(\bar{y}; r).
\] (6.11)

**Lemma 6.3.** Suppose that assumption \( A3'') \) is fulfilled and \( D \) satisfies the interior ball condition. Then, for any \( \hat{x} \in \partial D \) and any interior ball \( B(\bar{y}, r) \) at \( \hat{x} \) there exists \( C > 0 \), depending only on \( \lambda_\tau, \|N\|_\infty, \|Q\|_\infty, \|b\|_\infty, \|c\|_\infty \) and \( a > 0 \) depending on the above parameters and \( c > 0 \) such that
\[
\alpha \eta(x; Cr^{-2}, \bar{y}, r) \leq w_{c,D}(x), \quad x \in V_\epsilon(\bar{y}; r).
\] (6.12)

The respective proofs of the above lemmas are given in Sections 6.9, 6.10 and 6.11 below.

6.3.2 Proof of Theorem 3.4

**Proof of part 1): the strong maximum principle**

We start with the argument for the strong maximum principle. Recall that \( u \in W^{2,p}_{K0}(D) \cap C_b(\mathbb{R}^d) \). The proof is carried out by a contradiction. Suppose that there exist \( P, Q \in D \) such that \( u(P) = M \geq 0 \) and \( u(Q) < M \). In fact, see the proof of Theorem 3.5, p. 61 of [27], we can assume that there exists a ball \( B(\bar{y}, 3r/2) \subset D \) such that \( u(\hat{x}) = M \) for some \( \hat{x} \in \partial B(\bar{y}, r) \) and \( u(x) < M \) for all \( x \in B(\bar{y}, r) \). By virtue of Lemma 6.1 we can assume that there exists also a function \( \eta(\cdot; \alpha, \bar{y}, r) \) such that (6.10) holds on \( V^\epsilon := V^\epsilon(\bar{y}; r) \), (cf (6.9)). We can choose \( \epsilon > 0 \) sufficiently small so that
\[
v(x) := u(x) + \epsilon \eta(x; \alpha, \bar{y}, r) < M \quad \text{for } x \in B(\bar{y}, r/2).
\] (6.13)
We also have \( v(x) \leq M \) on \( \bar{B}^c(\bar{y},r) \) (since then \( \eta(x;\alpha,\bar{y},r) < 0 \)). Since both \( u(\cdot) \) and \( \eta(\cdot) \) are subsolutions to (2.5) on \( V^* \) so is \( v(\cdot) \). Let \( \tilde{v}(x) := v(x) - M \). It is a subsolution to (2.5) on \( V^* \). Therefore by Proposition 3.1 we have

\[
\sup_{x \in V^*} \tilde{v}(x) \leq \sup_{x \in (V^*)^c} \tilde{v}(x) < 0,
\]

which leads to a contradiction, as \( 0 = \tilde{v}(\hat{x}) \) at \( \hat{x} \in V^* \).

**Proof of part 2): the Hopf lemma**

Concerning the proof of (3.4) we essentially follow the classical proof of the Hopf lemma for a subsolution of a uniformly elliptic equation, as presented in e.g. [27], see Theorem 3.7, p. 65. Since \( D \) satisfies the interior ball condition, thanks to Lemma 6.1, we can find an interior ball \( B(\bar{y};r) \) at \( \hat{x} \) and \( \alpha > 0 \) such that

\[
(A - c)\eta(x;\alpha,\bar{y},r) > 0, \quad x \in \bar{V}_*,
\]

where \( V_* := V_*(\bar{y};r) \). Thanks to the already proved strong maximum principle for \( u \) and our assumption that \( u \not\equiv M \) on \( D \), we know that \( u(x) < M \) for \( x \in B(\bar{y};r/2) \).

Since \( \eta \leq 1 \) on \( B(\bar{y};r/2) \subset D_{r/2} \), see (3.3), we have

\[
v(x) := u(x) + \varepsilon \eta(x;\alpha,\bar{y},r) < M \quad \text{for } x \in B(\bar{y};r/2).
\]

with

\[
\varepsilon := \inf_{y \in D_{r/2}} (M - u(y)).
\]

We also have \( v(x) < M \) for \( x \in \bar{B}^c(\bar{y},r) \) (as then \( \eta \leq 0 \)). Therefore, by Proposition 3.1 applied to \( v \) on \( \bar{V}_* \), we conclude that

\[
v(x) \leq M = u(\hat{x}) = v(\hat{x}), \quad x \in \bar{V}_*,
\]

which yields \( \partial_n v(\hat{x}) \geq 0 \). Using (6.14) we get

\[
\partial_n u(\hat{x}) \geq 2\alpha \varepsilon r e^{-\alpha r^2} > 0.
\]

By the construction of \( \eta(\cdot;\alpha,\bar{y},r) \), the parameters \( \alpha, r \) depend only on \( \lambda_{D^*}. \) This ends the proof of Theorem 3.4.

**Remark 6.4.** We follow the notation of the proof of Theorem 3.4. From the proof of part 2) of the theorem we conclude that

\[
u(\hat{x}) - u(x) \geq \eta(x;\alpha,\bar{y},r) \inf_{y \in D_{r/2}} (M - u(y)), \quad x \in \bar{V}_*.
\]

Recalling the definition of \( \eta(\cdot;\alpha,\bar{y},r) \) (see (6.3)) we infer that for \( r \in (0,r_0] \), where \( 2\alpha r_0^2 < 1 \),

\[
\frac{u(\hat{x}) - u(\hat{x} - t n(\hat{x},\bar{y}))}{t} \geq \alpha t e^{-\alpha r^2} \inf_{y \in D_{r/2}} (M - u(y)), \quad t \in (0,1],
\]

where \( n(\hat{x},\bar{y}) := (\hat{x} - \bar{y})/|\hat{x} - \bar{y}|. \)
6.3.3 Proof of Theorem 3.6

Suppose that \( \hat{x} \) is as in the statement of Theorem 3.6 and \( u \) is a subsolution of (2.5). The conclusion of the theorem is obvious if \( u(\hat{x}) < 0 \) and follows directly from (3.4) if \( u(\hat{x}) = 0 \). Assume therefore that \( u(\hat{x}) > 0 \). Using the conclusion of Proposition 4.6 we infer that for any \( x \in D \)

\[
u(\hat{x}) - u(x) \geq u(\hat{x}) - E_x[c_{(\tau_D)} u(X_{\tau_D})] \geq u(\hat{x}) (1 - v_{c,D}(x)) = u(\hat{x}) w_{c,D}(x). \tag{6.18}
\]

Invoking Lemma 6.2 for any interior ball \( B(\bar{y}, r) \) for \( D \) at \( \hat{x} \) with \( r \in (0, r_0 \wedge \tau(\hat{x})] \), (6.11) holds. Therefore, for any \( n \) that is the outward unit normal to \( \partial B(\bar{y}, r) \) at \( \hat{x} \), we have

\[
\partial_n u(\hat{x}) \geq -\rho_{c,A,D}(r/2) u(\hat{x}) \partial_n \exp \left\{ -\frac{C}{r^2} |x - \bar{y}|^2 \right\} \bigg|_{x = \hat{x}} = \frac{2C}{r} e^{-C \rho_{c,A,D}(r/2)} u(\hat{x}). \tag{6.19}
\]

The conclusion of Theorem 3.6 then follows.

\[\square\]

6.3.4 Proof of Theorem 3.5

We start with (6.18). Invoking Lemma 6.3 we can find \( B(\bar{y}, r) \) - an interior ball for \( D \) at \( \hat{x} \) - and \( C, a > 0 \) as in the statement of the lemma such that (6.12) holds. Therefore, for any \( n \) that is the outward unit normal to \( \partial B(\bar{y}, r) \) at \( \hat{x} \), we have

\[
\partial_n u(\hat{x}) \geq -au(\hat{x}) \partial_n \exp \left\{ -\frac{C}{r^2} |x - \bar{y}|^2 \right\} \bigg|_{x = \hat{x}} = \frac{2Ca}{r^2} e^{-C u(\hat{x})}. \tag{6.20}
\]

The conclusion of Theorem 3.5 then follows.

6.4 Proof of Theorem 4.7

6.4.1 Properties of the resolvent

Recall that \( R^D_\alpha f \) is defined in (4.5) for all \( \alpha \geq 0 \) and \( f \in B^+(D) \). Using condition (4.3) we infer that the operator \( R^D_\alpha : L^\infty(D) \to L^\infty(D) \) is bounded for any \( \alpha \geq 0 \). The operators \( (R^D_\alpha)_{\alpha \geq 0} \) satisfy the resolvent identity

\[
R^D_\alpha - R^D_\beta = (\beta - \alpha) R^D_\alpha R^D_\beta, \quad \alpha, \beta \geq 0. \tag{6.21}
\]

Thanks to assumptions A1)-A2) there exists \( C > 0 \) such that

\[
\sup_{x \in \bar{D}} \mathbb{E}_x \left[ \int_0^{\tau_D} |f(X_t)| dt \right] \leq C \| f \|_{L^1(D)}, \quad f \in L^1(D). \tag{6.22}
\]

see (53) of [22], or Corollary 2 of [24]. This estimate implies that in fact \( R^D_\alpha \) extends to the operator \( R^D_\alpha : L^d(D) \to L^\infty(D) \) for each \( \alpha \geq 0 \). This in turn allows us to conclude that for any \( \alpha \geq 0 \) there exists \( r^D_\alpha : D \times D \to [0, +\infty) \) such that

\[
R^D_\alpha f(x) = \int_D r^D_\alpha(x, y) f(y) dy, \quad x \in D \tag{6.23}
\]

and

\[
esup_{x \in D} \| r^D_\alpha(x, \cdot) \|_{L^d/d-1(D)} \leq \| R_\alpha \|_{L^d \to L^\infty}. \tag{6.24}
\]
The above implies that $R^D_\alpha : L^d(D) \to L^d(D)$ is compact for any $\alpha \geq 0$. This can be easily seen, as from any bounded sequence $(f_n) \subset L^d(D)$ we can select a subsequence $(f_{n_k})$ weakly converging to some $f$. Using (6.23) we conclude that $\lim_{k \to +\infty} R^D_\alpha f_{n_k}(x) = R^D_\alpha f(x)$ for a.e. $x \in D$. Since $(\|R^D_\alpha f_{n_k}\|_{L^\infty})$ is bounded, this implies the strong convergence of $(R^D_\alpha f_{n_k})$ to $R^D_\alpha f$.

By the support theorem of [18, Section 4], or by [23, Corollary 2] for any $x \in D$,

$$r^D_\alpha(x, \cdot) > 0, \quad \text{m.a.e. on } D, \alpha \geq 0. \quad (6.25)$$

By [24, Theorem 1]

$$R^D_\alpha(B_\beta(D)) \subset C_\beta(D), \quad \alpha \geq 0. \quad (6.26)$$

In other words resolvent $R^D_\alpha$ is strongly Feller for each $\alpha \geq 0$.

### 6.4.2 The end of the proof of Theorem 4.7

We have already concluded that $R^D_\alpha : L^d(D) \to L^d(D)$ is compact for any $\alpha \geq 0$. In addition $\rho(R^1_\alpha)$ - the spectral radius of $R^1_\alpha$ - belongs to $[0,1]$. Moreover, by (6.25), $R^D_1$ is irreducible, i.e.

for any $f \in B^+(D)$ such that $\int_D f \, dx > 0$ we have $R^D_1 f(x) > 0$, $x \in D. \quad (6.27)$

By virtue of the Jentzsch Theorem, see Theorem 6.6. of [28], the spectral radius $\rho(R^D_1)$ is positive and it equals to the (unique) simple eigenvalue of $R^D_1$. The corresponding eigenvector $\varphi_D$ is strictly positive on $D$. With the help of the resolvent identity (6.21) one can easily check that

$$e^{-t} P^D_t \varphi_D(x) = \varphi_D(x) - \frac{1}{\rho(R^D_1)} \int_0^t e^{-s} P^D_s \varphi_D(x) \, ds, \quad t \geq 0,$$

which in turn leads to (4.11), with $\lambda_D := \rho^{-1}(R^D_1) - 1$. Clearly, $\lambda_D > 0$ as otherwise, this would contradict (4.23).

### 6.5 Proof of Theorem 5.2

By Proposition [4.6] we can write

$$u(\hat{x}) - u(x) \geq (1 - E_x e_c(\tau_D \wedge t)) u(\hat{x}) + E_x \left[ \int_0^{\tau_D \wedge t} e_c(s)(Au - cu)(X_s) \, ds \right], \quad t > 0. \quad (6.28)$$

Observe that

$$\left(1 - E_x e_c(\tau_D \wedge t)\right) u(\hat{x}) \geq (1 - e^{-\lambda_D t}) P_x(\tau_D > t) u(\hat{x}). \quad (6.29)$$

From (4.11) we have

$$e^{-\lambda_D t} \varphi_D(x) = E_x [\varphi_D(X_t), \tau_D > t] \leq \|\varphi_D\|_{\infty} P_x(\tau_D > t). \quad (6.30)$$

Substituting into (6.29) the lower bound on $P_x(\tau_D > t)$ obtained from (6.30) and maximizing over $t > 0$ we conclude that

$$u(\hat{x}) - u(x) \geq \frac{u(\hat{x}) \varphi_D(x)}{\|\varphi_D\|_{\infty}} \cdot \frac{\zeta/\lambda_D}{(1 + \zeta/\lambda_D)^1 + \lambda_D/\zeta} \geq \frac{u(\hat{x}) \varphi_D(x)}{\|\varphi_D\|_{\infty}} \cdot \frac{\zeta}{\lambda_D + \zeta}. \quad (6.31)$$

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In the last inequality we have used an elementary bound \((1 + s)^{1/s} \leq e\), for \(s \in (0, 1)\).
Furthermore, since \(e(x) \geq 0\), we have \(\bar{e} := \|e\|_{\infty}\) (cf (2.7)) and
\[
\mathbb{E}_x \left[ \int_0^{\tau_D \wedge t} e_c(r)(Au - cu)(X_r) \, dr \right] \geq \text{essinf}_D (Au - cu) \mathbb{E}_x \left[ \int_0^{\tau_D \wedge t} e^{-cr} \, dr \right] \geq \frac{1}{\bar{e}} (1 - e^{-c}) \text{essinf}_D (Au - cu) P_x (\tau_D > t).
\] (6.32)

Using again (6.30) to estimate \(P_x (\tau_D > t)\) from below and maximizing over \(t > 0\) we conclude that
\[
u(\hat{x}) - u(x) \geq \frac{\varphi_D(x)}{\|\varphi_D\|_{\infty}} \cdot \frac{\bar{e}/\lambda_D}{(1 + \bar{e}/\lambda_D)^{1+\lambda_D/\bar{e}}} \text{essinf}_D (Au - cu)
\] (6.33)

Estimate (5.2) follows easily from (6.31) and (6.33).

\[\mathbf{6.6 \ Proof \ of \ Theorem \ 5.1}\]

We start with the estimate (6.28). The first term in the right hand side is bounded in the same way as in Section 6.5. To estimate the second term we note that, thanks to Proposition 4.4, (6.28) follows easily from (6.31) and (6.33).

We shall prove estimate (6.35) momentarily, but first use it to finish the proof of Theorem 5.1. Thanks to (6.25) and (6.26) we conclude that \(\psi := R_{\epsilon+1}^{D} \hat{\psi}\) is strictly positive and continuous in \(D\). By (6.25), measure \(\nu(A) = \int_{D} \bar{\varphi} R_{\epsilon+1}^{D} 1_A \, dx\), \(A \in \mathcal{B}(D)\) is equivalent to \(m\). Thus, \(\chi := dv/dx\) is strictly positive in \(D\). Applying (6.35) we obtain
\[
R_{\epsilon}^{D} ((A - cu)(x)) \geq \psi(x) \int_{D} (Au - cu) \chi \, dx, \quad x \in D
\] (6.36)

and (5.1) follows. The only thing that yet needs to be shown is (6.35).

\[\mathbf{Proof \ of \ estimate \ (6.35)}\]

We use the notion of an irreducibility measure associated with a non-negative kernel. Given \(\alpha > 0\) we let \(K_{\alpha} : D \times \mathcal{B}(D) \to [0, +\infty)\) be a substochastic kernel defined by
\[
K_{\alpha}(x, B) := \alpha R_{\alpha}^{D} 1_B(x) \quad \text{for} \quad (x, B) \in D \times \mathcal{B}(D).
\]

\[\text{Definition 6.5.} \quad (\text{cf [25, Section 2.2]})\]

Suppose that \(\nu\) is a non-trivial Borel measure on \(D\). We say that the kernel \(K_{\alpha}\) is \(\nu\)-irreducible, if for any \(B \in \mathcal{B}(D)\) such that \(\nu(B) > 0\)
and \(x \in D\) we have \(K_{\alpha}^n(x, B) := (\alpha R_{\alpha}^{D})^n 1_B(x) > 0\) for some \(n \geq 1\). Measure \(\nu\) is called an irreducibility measure for \(K_{\alpha}\). The kernel is said to be irreducible if it is irreducible for some measure.
\( \textbf{Lemma 6.6.} \text{ For each } \alpha \geq 0 \text{ there exists a pair of non-trivial functions } \overline{\psi}_\alpha, \overline{\phi}_\alpha \in B^+_b(D) \text{ such that} \)

\[
R^D_\alpha f(x) \geq R^D_{\alpha+1} \overline{\psi}_\alpha(x) \int_D R^D_{\alpha+1} f \overline{\phi}_\alpha \, dx, \quad x \in D, \quad f \in B^+_b(D).
\]  

\( (6.37) \)

\textbf{Proof.} Estimate (6.27) implies that the kernel \( K_\alpha(\cdot, \cdot) \) is \( m \)-irreducible. Here, as we recall \( m \) is the \( d \)-dimensional Lebesgue measure. By \cite{25} Theorem 2.1 there exist a non-trivial function \( \overline{\psi}_\alpha : D \to [0, \infty) \) and a non-trivial Borel measure \( \nu_\alpha \) on \( D \), the so-called \textit{small function} and \textit{small measure}, such that

\[
R^D_\alpha f(x) \geq \overline{\psi}_\alpha(x) \int_D f \, d\nu_\alpha, \quad f \in B^+(D), \quad x \in D.
\]  

\( (6.38) \)

By the resolvent identity, see (6.21),

\[
R^D_\alpha R^D_{\alpha+1} f = R^D_{\alpha+1} R^D_\alpha f = R^D_\alpha f - R^D_{\alpha+1} f \quad \text{for any } f \in B^+_b(D).
\]  

\( (6.39) \)

Therefore, applying (6.38) for \( R^D_{\alpha+1} f \) in place of \( f \), we get

\[
R^D_\alpha f(x) \geq \overline{\psi}_\alpha(x) \int_D f \, d\overline{\nu}_\alpha, \quad f \in B^+(D), \quad x \in D,
\]  

\( (6.40) \)

where \( \overline{\nu}_\alpha(B) := \int_B R^D_{\alpha+1} 1_B \, d\nu_\alpha, \ B \in B(D). \) Applying \( R^D_{\alpha+1} \) to both sides of (6.40) and using again (6.39), we get

\[
R^D_\alpha f(x) \geq R^D_{\alpha+1} \overline{\psi}_\alpha(x) \int_D f \, d\overline{\nu}_\alpha, \quad f \in B^+(D), \quad x \in D.
\]  

\( (6.41) \)

Invoking (6.27) we conclude also that \( \overline{\nu}_\alpha \) is equivalent to \( m \). Thus, there exists a strictly positive \( \phi_\alpha \in B_0^+(D) \) such that \( d\overline{\nu}_\alpha = \phi_\alpha \, dx \). Therefore, applying (6.41) with \( f \) replaced by \( R^D_{\alpha+1} f \) and using one more time (6.39) yields (6.37).

\section{6.7 Proof of Theorem 3.3}

Let \( \varepsilon_0 > 0 \) be such that \( B(\hat{x}, 2\varepsilon_0) \subset D \). Set \( V_\varepsilon = B(\hat{x}, \varepsilon) \). By Proposition 4.6, for any \( \varepsilon \in (0, \varepsilon_0), \)

\[
u(\hat{x}) - \mathbb{E}_{\hat{x}} \left[ u(X_{\tau_{V_\varepsilon}}) \right] = -\mathbb{E}_{\hat{x}} \left[ \int_{0}^{\tau_{V_\varepsilon}} (Au)(X_s) \, ds \right].
\]  

\( (6.42) \)

Using the Ikeda Watanabe formula, see \cite{10} Remark 2.46, page 65, we get

\[
\mathbb{E}_{\hat{x}} \left[ \sum_{0 \leq s \leq \tau_D} 1_D(X_{s-}) 1_{\overline{D}}(X_s) \right] = \mathbb{E}_{\hat{x}} \left[ \int_{\mathbb{R}^d} \int_{0}^{\tau_D} 1_D(X_{s-}) 1_{\overline{D}}(y) N(X_{s-}, dy - X_{s-}) \, ds \right] = 0.
\]  

\( (6.43) \)

The last equality follows from the fact that \( S(D) \subset \overline{D} \). This implies that \( \mathbb{P}_{\hat{x}}(X_{\tau_{V_\varepsilon}} \in \overline{D}) = 1 \).

Since \( \hat{x} \in D \) is a global maximal point of \( u \) in \( \overline{D} \) (by continuity of \( u \), from (6.42), we obtain

\[
\mathbb{E}_{\hat{x}} \left[ \int_{0}^{\tau_{V_\varepsilon}} (Au)(X_s) \, ds \right] \leq 0, \quad \varepsilon \in (0, \varepsilon_0),
\]  

\( 22 \)
Combining the above with (6.23) (here \(D := V_\varepsilon\) and \(\alpha = 0\),
\[
\left(\text{ess inf}_{V_\varepsilon} Au\right) \int_{V_\varepsilon} r_{V_0}^{V_\varepsilon}(\hat{x}, y) \, dy \leq \mathbb{E}_{\hat{x}} \left[ \int_{0}^{\tau_{V_\varepsilon}} (Au)(X_s) \, ds \right] \leq 0, \ \varepsilon \in (0, \varepsilon_0].
\]
By (6.25) we have \(\int_{V_\varepsilon} r_{V_0}^{V_\varepsilon}(\hat{x}, y) \, dy > 0\). Therefore, \(\text{ess inf}_{V_\varepsilon} Au \leq 0, \ \varepsilon \in (0, \varepsilon_0]\), which proves the assertion of the theorem, cf (2.1).

6.8 Proof of Theorem 5.9

By Lemma 6.6, there exist non-trivial functions \(\bar{\psi}, \bar{\phi} \in B^+_b(D)\) such that
\[
R^{\bar{c}+1}_e f(x) \geq R^{\bar{c}+2}_e \bar{\psi}(x) \int_D R^{\bar{c}+2}_e f \bar{\phi} dx, \ x \in D, \ f \in B^+_b(D).
\]
Applying this inequality, with \(f\) replaced by \(R^{\bar{c}} f\), and using (6.39) we get
\[
R^{\bar{c}} f(x) \geq R^{\bar{c}+2}_e \bar{\psi}(x) \int_D R^{\bar{c}} f d\nu, \ x \in D, \ f \in B^+_b(D),
\]
where \(\nu(B) = \int_D \bar{\phi} R^{\bar{c}+2}_e 1_B dx, \ B \in \mathcal{B}(D)\). By (6.25) and (6.26) we infer that \(R^{\bar{c}+2}_e \bar{\psi}\) is continuous and strictly positive on \(D\), and \(\nu\) is equivalent to \(m\).

In the next step, we show that \(R^{\bar{c}} f(x)\) in (6.45) can be replaced by \(w\) - any \(\bar{c}\)-excessive function with respect to \((P^D_t)_{t \geq 0}\), i.e. a non-negative function satisfying
\[
P^D_t w(x) \leq e^{\bar{c} t} w(x), \ t \geq 0, \ x \in D
\]
and
\[
\lim_{t \to 0^+} e^{-\bar{c} t} P^D_t w(x) = w(x), \ x \in D.
\]
More precisely, for any \(w\), that is \(\bar{c}\)-excessive, we have
\[
w(x) \geq R^{\bar{c}+2}_e \bar{\psi}(x) \int_D w d\nu, \ x \in D.
\]
We prove (6.48) momentarily, but first let us show how to finish the proof of (5.10). The latter follows from (6.48), provided we prove that \(u\) is \(\bar{c}\)-excessive. Since \(u\) is a supersolution to (2.5), by Proposition 4.6 we have
\[
u(x) \geq \mathbb{E}_{x} e_c(\tau_D \wedge t) u(X_{\tau_D \wedge t}), \ t \geq 0, \ x \in D.
\]
Thus, by non-negativity of \(u\),
\[
u(x) \geq e^{-\bar{c} t} \mathbb{E}_x [u(X_t), t < \tau_D] = e^{-\bar{c} t} P^D_t u(x), t \geq 0
\]
and (6.46) holds. Equality (6.47) is a consequence of continuity of \(u\), in \(\bar{D}\). This ends the proof of Theorem 5.9 provided we show (6.48).
Proof of (6.48)

Let \( w \) be \( \bar{c} \)-excessive. From [5, Proposition II.2.6] we conclude that, there exists a sequence of non-negative Borel functions \((f_n)_{n \geq 1}\) such that \( R_{\bar{c}} f_n(x) \) is monotone increasing and

\[
\lim_{n \to +\infty} R_{\bar{c}} f_n(x) = w(x), \quad x \in D. \quad (6.49)
\]

We can write inequality (6.45) for each \( R_{\bar{c}} f_n(x) \). Estimate (6.48) then follows by an application of the monotone convergence theorem and (6.49).

Remark 6.7. Under the assumptions of Theorem 5.4 (resp. Theorem 5.6), using a similar arguments as in its proof, we can prove that in (5.10), \( C := a \text{dist}(V,D) \) (resp. \( \nu := a \delta_D dx, \) and \( C := \text{dist}(V,D) \)) for some \( a > 0 \), depending only on \( D \).

6.9 Proof of Lemma 6.1

![Figure 1](image)

Let \( 0 < r \leq 1 \), \( B(\bar{y}, r) \) be an interior ball in \( D \) at \( \hat{x} \), and \( V^* := V^*(\bar{y}; r) \) (cf (6.9)), see Fig. 1. Let \( \eta(x) := \eta(x; \alpha, \bar{y}, r), \ x \in \mathbb{R}^d \) be given by (6.8). Then,

\[
\nabla \eta(x) = -2\alpha(x - \bar{y}) e^{-\alpha |x - \bar{y}|^2},
\]

\[
\nabla^2 \eta(x) = e^{-\alpha |x - \bar{y}|^2} \left( 4\alpha^2(x - \bar{y})^2 - 2\alpha I_d \right).
\]

With the notation \( \text{Tr}(Q(x)) \) for the trace of \( Q(x) \), we can write

\[
\sum_{i,j=1}^{d} \frac{1}{2} q_{i,j}(x) \eta_{x_i x_j}(x) = \alpha e^{-\alpha |x - \bar{y}|^2} \left[ 2\alpha(x - \bar{y})^T Q(x)(x - \bar{y}) - \text{Tr}(Q(x)) \right]
\]

and

\[
\sum_{i=1}^{d} b_i(x) \eta_{x_i} = -2\alpha e^{-\alpha |x - \bar{y}|^2} b^T(x) \cdot (x - \bar{y}).
\]

Using the uniform ellipticity assumption (2.3), we get

\[
L\eta(x) \geq \alpha e^{-\alpha |x - \bar{y}|^2} \left\{ 2\lambda_{\text{tr}} \alpha |x - \bar{y}|^2 - (\text{Tr}(Q(x)) - 2b^T(x) \cdot (x - \bar{y})) \right\}
\]

\[
\geq \alpha e^{-9\alpha r^2/4} \left\{ \frac{\lambda_{\text{tr}} \alpha r^2}{2} - \left( \|\text{Tr}Q\|_{\infty} + \frac{3r}{2} \|b\|_{\infty} \right) \right\}, \quad x \in V^*. \quad (6.51)
\]
We shall choose $\alpha > 0$, $r \in (0, 1]$ in such a way that

$$\alpha r^2 = \gamma_* := \frac{4}{\lambda_V} \left( \| \text{Tr} \, Q \|_\infty + \frac{3}{2} \| b \|_\infty \right) =: C. \quad (6.52)$$

Then, remembering that $r \in (0, 1]$, we get

$$L \eta(x) \geq \frac{\alpha \gamma_*}{2 \lambda_V} e^{-9 \gamma_* / 4}, \quad x \in V^*. \quad (6.53)$$

Next, using (1.5), for any $M > 10$ we can write

$$S \eta = \sum_{j=1}^{4} S_j \eta, \quad (6.54)$$

where, cf (6.50),

$$S_1 \eta(x) := \int_{\mathbb{R}^d} \frac{|y|^2 (\eta(x+y) - \eta(x))}{1 + |y|^2} N(x, dy), \quad (6.55)$$

$$S_2 \eta(x) := -2 \alpha \int_{0}^{1} d\theta_1 \int_{0}^{\theta_1} d\theta_2 \int_{|y| \leq Mr} e^{-\alpha |x+\theta_2 y - \bar{y}|^2} \frac{|y|^2}{1 + |y|^2} N(x, dy),$$

$$S_3 \eta(x) := -2 \alpha \int_{0}^{1} d\theta_1 \int_{0}^{\theta_1} d\theta_2 \int_{|y| > Mr} e^{-\alpha |x+\theta_2 y - \bar{y}|^2} \frac{|y|^2}{1 + |y|^2} N(x, dy),$$

$$S_4 \eta(x) := 4 \alpha \int_{0}^{1} d\theta_1 \int_{0}^{\theta_1} d\theta_2 \int_{\mathbb{R}^d} e^{-\alpha |x+\theta_2 y - \bar{y}|^2} \frac{(x + \theta_2 y - \bar{y})^2}{1 + |y|^2} N(x, dy), \quad x \in V^*. \quad (6.58)$$

We have $S_4 \eta(x) \geq 0$, therefore

$$S \eta(x) \geq \sum_{j=1}^{3} S_j \eta(x), \quad x \in V^*. \quad (6.56)$$

Furthermore, since $\| \eta \|_\infty = 1$ we get

$$|S_1 \eta(x)| \leq 2 \| N_{\mathbb{R}^d} \|_\infty, \quad x \in V^*. \quad (6.57)$$

In addition,

$$|S_2 \eta(x)| \leq \alpha n(Mr), \quad x \in V^*, \quad (6.58)$$

where

$$n(r) := \sup_{x \in V} N_{\mathbb{R}^d}(x).$$

Using assumption A2) and Dini’s uniform convergence theorem we conclude that

$$\lim_{r \to 0+} n(r) = 0. \quad (6.59)$$

Finally, we can write

$$|S_3 \eta(x)| \leq 2 \alpha \chi(x, Mr), \quad (6.60)$$

where

$$\chi(x, r) := \int_{0}^{1} d\theta_2 \int_{0}^{1} d\theta_1 \int_{|y| > r} e^{-\alpha |x+\theta_2 y - \bar{y}|^2} \frac{|y|^2}{1 + |y|^2} N(x, dy).$$
We have
\[ |x + \theta_2 y - \bar{y}| \geq \left( \theta_2 M - \frac{3}{2} \right) r \geq \frac{M r}{2}, \]
provided that
\[ \frac{r}{2} + \frac{3}{2M} =: \theta_2 \leq \theta_2 \leq 1. \]

We can estimate
\[
\chi(x, Mr) \leq \int_0^{\theta_*} d\theta_2 \int_{\theta_2}^1 d\theta_1 \int_{|y| > Mr} \frac{|y|^2}{1 + |y|^2} N(x, dy)
+ \int_{\theta_*}^1 d\theta_2 \int_{\theta_2}^1 d\theta_1 e^{-\alpha(Mr)^2/4} \int_{|y| > Mr} \frac{|y|^2}{1 + |y|^2} N(x, dy)
\leq \left( \theta_* + e^{-\alpha(Mr)^2/4} \right) \|N_{\mathbb{R}^d}\|_{\infty} \leq \left( \frac{r}{2} + \frac{3}{2M} + e^{-M^2 \gamma_*/4} \right) \|N_{\mathbb{R}^d}\|_{\infty}
= \left( \frac{\sqrt[2]{\gamma_*}}{2\sqrt{\alpha}} + \frac{3}{2M} + e^{-M^2 \gamma_*/4} \right) \|N_{\mathbb{R}^d}\|_{\infty}.
\]

Using estimates (6.53), (6.57) – (6.60) we get
\[
(A - c)\eta(x) \geq \frac{\alpha \gamma_*}{2\sqrt{\alpha}} e^{-9\gamma_*/4} - 2\|N_{\mathbb{R}^d}\|_{\infty} - \alpha n \left( \frac{M \sqrt{\gamma_*}}{\sqrt{\alpha}} \right) - 2 \left( \frac{\sqrt{\gamma_*}}{2\sqrt{\alpha}} + \frac{3}{2M} + e^{-M^2 \gamma_*/4} \right) \|N_{\mathbb{R}^d}\|_{\infty}
= \alpha \left\{ \frac{\gamma_*}{2\sqrt{\alpha}} e^{-9\gamma_*/4} - n \left( \frac{M \sqrt{\gamma_*}}{\sqrt{\alpha}} \right) - 2 \left( \frac{\sqrt{\gamma_*}}{2\sqrt{\alpha}} + \frac{3}{2M} + e^{-M^2 \gamma_*/4} \right) \|N_{\mathbb{R}^d}\|_{\infty} \right\}
- 2\|N_{\mathbb{R}^d}\|_{\infty} - \|c\|_{\infty}.
\]

Therefore, for a \( \alpha_0 \gg M^2 \gg 1 \), we get that the right-hand side of the above inequality is greater then \( K \) on \( V^* \). By (6.52) we conclude (6.10) with \( r_0 = \sqrt{C}/\alpha_0 \).

6.10 Proof of Lemma 6.2

6.10.1 Extended generator and the notion of a weak subsolution

**Definition 6.8** (Extended generator of the canonical process). Suppose that \( D \subset \mathbb{R}^d \) is open. Define \( D(\tilde{A}|_D) \) as the subset of \( B(\mathbb{R}^d) \) that consists of functions \( u \in B(\mathbb{R}^d) \) for which there exists \( f \in B(D) \) such that
\[
P_x \left[ \int_0^{\tau_D} |f(X_r)| \, dr < \infty \right] = 1, \quad x \in D,
\]
and the process
\[
\tilde{M}_t[u] := u(X_{t \wedge \tau_D}) - u(x) - \int_0^{t \wedge \tau_D} f(X_s) \, ds, \quad t \geq 0,
\]
is a \( P_x \)-local martingale for any \( x \in D \). The set of all such functions \( f \) shall be denoted by \( \tilde{A}|_D u \) and is called the *extended generator* of the canonical process, cf Section 4 of [15].

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Definition 6.9 (Weak subsolution, supersolution and solution of (2.5)). Suppose that \(c, g : \mathbb{R}^d \to \mathbb{R}\). A function \(u \in B_b(\mathbb{R}^d)\) is called a weak subsolution to (2.5) in \(D\) if \(u \in \mathcal{D}(\bar{A}_D)\) and there exists \(f \in \bar{A}_D u\) such that
\[
-f(x) + c(x)u(x) \leq 0, \quad \text{for all } x \in D. \tag{6.62}
\]

The notions of a supersolution and solution can be then introduced analogously as in Definition 2.1.

Remark 6.10. In what follows, when speaking about a subsolution \(u\), we shall denote by \(\bar{A}_Du\) an arbitrary \(f\) that appears Definition 6.9.

Remark 6.11. By Proposition 4.6 if \(u \in C_b(\mathbb{R}^d) \cap W_{\text{loc}}^{2, d}(D)\) is a subsolution to (2.5) in the sense of Definition 2.1 then \(Au \in \bar{A}_Du\). Thus \(u\) is a weak subsolution to (2.5).

The following form of a weak maximum principle follows by the same argument as Proposition 3.1, see Section 6.2.

Proposition 6.12. If \(u\) is a weak subsolution of (2.5), then (3.1) is in force.

It follows from our arguments used in the proofs of Theorems 3.4 – 3.6 and Theorems 5.1 – 5.2 that their corresponding versions can be formulated for weak subsolutions of (2.5). This is a consequence of the fact that our method is based on the Feynman-Kac representation of solutions, see Proposition 4.6. This representation holds for weak subsolutions by their very definition. In particular, we have the following result.

Proposition 6.13. Assume that a domain \(D\) satisfies the uniform interior ball condition. Suppose that non-trivial \(f \in B^+(D)\) is such that \(R^D_\alpha f \in C_b(D)\) for some \(\alpha \geq 0\). Then, there exists \(a > 0\) such that
\[
R^D_\alpha f(x) \geq a\delta_D(x), \quad x \in D. \tag{6.63}
\]

Proof. It can be easily seen, by an application of the Itô formula, that \(u = -R^D_\alpha f(x)\) is a weak subsolution of \((A - \alpha)v = 0\), with \(\alpha u - f \in \bar{A}_Du\). By virtue of (6.25) we have \(u(x) < 0, x \in D\). We can invoke an analogue of (6.16) that also holds for weak subsolutions and conclude that there exists \(a > 0\) such that for any \(\hat{x} \in \partial D\)
\[
R^D_\alpha f(\hat{x} - tn(\hat{x}, \bar{y})) = -u(\hat{x} - tn(\hat{x}, \bar{y})) \geq at \inf_{y \in D_{r/2}} (-u(y)) = at \inf_{y \in D_{r/2}} R^D_\alpha f(y), \quad t \in (0, 1], \tag{6.64}
\]
where \(n(\hat{x}, \bar{y}) := (\hat{x} - \bar{y})/|\hat{x} - \bar{y}|\). This, in turn, implies (6.63). \(\square\)

6.10.2 Properties of \(w_{c, D}\)
Recall that \(w_{c, D}(\cdot)\) is given by (4.8). Obviously \(0 \leq w_{c, D}(x) \leq 1, x \in D\), and \(w_{c, D}(x) = 0, x \in D^c\). Define a martingale
\[
M_t := E_x(e_c(\tau_D)|\mathcal{F}_t), \quad t \geq 0.
\]
where \(e_c(t)\) is defined in (4.7). Suppose that \(x \in D\). By the Markov property we have
\[
M_t = e_c(\tau_D)1_{[\tau_D \leq t]} + v_{c, D}(X_t)e_c(t)1_{[\tau_D > t]},
\]
cf (4.6). Hence,
\[ w_{c,D}(X_{t \wedge \tau_D}) = 1 - e_c^{-1}(t \wedge \tau_D)M_{t \wedge \tau_D}, \quad t \geq 0. \] (6.65)

Applying Itô’s formula to \( e_c^{-1}(t)M_t \) yields
\[ e_c^{-1}(t)M_t = M_0 + \int_0^t e_c^{-1}(s) dM_s + \int_0^t e_c^{-1}(s)M_sc(X_s) ds. \] (6.66)

Let \( N_t \) be the second term in the right-hand side of the above equality. It is a local martingale. Thus, substituting from (6.66) into (6.65), we get
\[ w_{c,D}(X_{t \wedge \tau_D}) = 1 - M_0 - N_{t \wedge \tau_D} - \int_0^{t \wedge \tau_D} M_se_c^{-1}(s)c(X_s) ds = w_{c,D}(X_0) - N_{t \wedge \tau_D} - \int_0^{t \wedge \tau_D} (cw_{c,D})(X_s) ds, \quad t \geq 0. \]

Therefore, cf Definition 6.8, \(-cw_{c,D} \in \tilde{A}_D w_{c,D}\). Thus, \( w_{c,D} \) is a weak solution to
\[ -\tilde{A}_D w_{c,D} + cw_{c,D} = c. \] (6.67)

Since \( w_{c,D} \) is bounded, a simple calculation shows that
\[ w_{c,D}(x) = R^D(cv_{c,D})(x) = R^D(c - cw_{c,D})(x), \quad x \in D. \] (6.68)

By the very definition of \( w_{c,D} \) we have
\[ w_{c,D}(x) > 0 \quad \text{iff} \quad P_x \left( \int_0^{\tau_D} c(X_r) dr > 0 \right), \]
which in turn is equivalent to \( R^Dc(x) > 0 \). The latter however is a direct consequence of (6.25). Thanks to (6.26) we conclude also that \( w_{c,D} \) is continuous in \( D \). Summarizing we have shown the following.

**Lemma 6.14.** \( w_{c,D} \) is strictly positive and continuous on \( D \).

Recall that \( D_r = \{ x \in D : \text{dist}(x, \partial D) > r \} \) for \( r > 0 \), cf (3.3). Let \( r_* := \inf[r > 0 : D_r = \emptyset] \). Define
\[ \rho_{c,A,D}(r) := \inf_{x \in D_r} w_{c,D}(x), \quad r \in (0, r_*). \] (6.69)

We adopt the convention that \( \rho_{c,A,D}(r) := \lim_{u \to r_*^-} \rho_{c,A,D}(u) \), if \( r \geq r_* \). Clearly, \( \rho_{c,A,D} \) is nondecreasing and takes values in \((0, 1]\). Since \( w_{c,D} \) is a strictly positive and continuous on \( D \) we conclude that \( \rho_{c,A,D}(\cdot) \) is also strictly positive and continuous on \((0, +\infty)\).

### 6.10.3 The end of the proof of Lemma 6.2

By Lemma 6.1 for given \( \hat{x} \in \partial D \), and any interior ball \( B(\bar{y}, r) \) at \( \hat{x} \) (cf Definition 2.2) with \( r \in (0, \text{dist}(\hat{x}, \partial D) \wedge r_0) \), function \( \eta(\cdot) := \eta(\cdot; Cr^{-2}, \bar{y}, r) \) satisfies (6.10). This in particular means that \( g := -A\eta < 0 \) in \( V_*(\bar{y}; r) \). For any \( \varepsilon \in (0, 1] \) we let \( \eta_\varepsilon := \varepsilon\eta \). Clearly, \( \eta_\varepsilon \leq 1 \) (as \( \eta \leq 1 \)) and
\[ -A\eta_\varepsilon + c\eta_\varepsilon = c\eta_\varepsilon + \varepsilon g, \quad \text{in} \ V_*(\bar{y}; r). \]
By (6.67) we have \(-cw_{c,D} \in \tilde{A}_Dw_{c,D}\) and for this particular choice of the extended generator
\[-\tilde{A}_Dw_{c,D} + cw_{c,D} = c, \quad \text{in } \tilde{V}_s(y;r).\]
Therefore, \(w_{c,D} - \eta_c\) is a weak subsolution of (2.5) on \(\tilde{V}_s(y;r)\) in the sense of Definition 6.9. Since \(\eta \leq 1\) we have
\[\eta_c(x) \leq w_{c,D}(x), \quad x \in \tilde{B}(y,r/2),\]
with (cf (6.69))
\[\varepsilon := \inf_{D^c/2} w_{c,D} = \rho_{c,A,D}(r/2). \tag{6.70}\]
We also have \(\eta_c \leq w_{c,D}\) on \(\tilde{B}(y,r)\), therefore by Proposition 6.12 we have
\[\eta_c(x) - w_{c,D}(x) \leq \sup_{y \in \tilde{V}_{c}(y;r)} [\eta_c(y) - w_{c,D}(y)] \leq 0, \quad x \in \tilde{V}_s(y;r)\]
and the conclusion of the lemma follows. \(\Box\)

6.11 Proof of Lemma 6.3

We start with the following.

**Lemma 6.15.** Suppose that A3") holds. Then for any open set \(V\), compactly embedded in \(D\), there exists \(a_s > 0\) depending only on \(\|N\|_{\infty} ; \|Q\|_{\infty} ; \|b\|_{\infty} ; \ell > 0\) and \(\tau^* := \text{dist}(V,D^c) > 0\) such that
\[\inf_{x \in V} w_{c,D}(x) \geq a_s. \tag{6.71}\]

**Proof.** Let \(\bar{x} \in D\) and \(r > 0\) be such that \(\bar{B}(\bar{x}, r) \subset D\). Observe that for any \(t > 0\),
\[w_{c,D}(\bar{x}) \geq 1 - \mathbb{E}_x e^{-cT} \geq 1 - \mathbb{E}_x e^{-cB(x,r)} \geq (1 - e^{-c}) P_{\bar{x}}(\tau_{B}(\bar{x}, r) > t).\]
Let \(\rho_{\bar{x}, r} \in C^2_b([0,1])\) be an arbitrary function taking values in \([0,1]\) and satisfying
\[\rho_{\bar{x},r}(\bar{x}) = 1, \quad 1 - \rho_{\bar{x},r}(x) \geq a > 0, \quad x \in B^c(\bar{x}, r).\]
Since the process
\[M_t := 1 - \rho_{\bar{x},r}(X_{t \wedge \tau_{B}(\bar{x}, r)}) + \int_0^{t \wedge \tau_{B}(\bar{x}, r)} A\rho_{\bar{x},r}(X_s) \, ds, \quad t \geq 0\]
is a zero mean martingale we can write
\[P_{\bar{x}}(\tau_{B}(\bar{x}, r) \leq t) \leq \frac{1}{a} \mathbb{E}_{\bar{x}} \left[(1 - \rho_{\bar{x},r})(X_{t \wedge \tau_{B}(\bar{x}, r)})\right] = -\frac{1}{a} \mathbb{E}_{\bar{x}} \left[\int_0^{t \wedge \tau_{B}(\bar{x}, r)} A\rho_{\bar{x},r}(X_s) \, ds\right] \leq \frac{1}{a} \mathbb{E}_{\bar{x}}(t \wedge \tau_{B}(\bar{x}, r)) \sup_{B(\bar{x}, r)} (A\rho_{\bar{x},r})^- \leq \frac{t}{a} \sup_{B(\bar{x}, r)} (A\rho_{\bar{x},r})^- \tag{6.72}\]
for some constant \(C_A\) depending on the coefficients of the operator \(A\).

Taking \(r > 0\) sufficiently small, so that \(V \Subset D_r\) and letting \(\rho_{\bar{x},r}(x) = e^{-|x-\bar{x}|^2/r^2}\) for any \(\bar{x} \in V\), we can choose \(t\) sufficiently small so that (6.71) is in force. \(\Box\)

Going back to the proof of Lemma 6.3, suppose that \(r > 0\) is as in the proof of Lemma 6.2. Thanks to Lemma 6.15 we can choose \(a_s > 0\), depending on the same parameters as in the statement of the lemma, such that \(\varepsilon \geq a_s\), where \(\varepsilon\) is defined in (6.70). This allows us to conclude (6.12) and the assertion of Lemma 6.3 follows. \(\Box\)
6.12 Proofs of Theorems 5.4, 5.6 and 5.8

6.12.1 Proof of Theorems 5.4 and 5.6

Invoking the argument from the proofs of Theorem 5.1 and Lemma 6.6 we conclude that there exist non-trivial \( \bar{\psi}_0, \tilde{\phi}_0 \in B_b^+(D) \) such that

\[
u(x) = \frac{c \varphi_D(x) u(\bar{x})}{2c \| \varphi_D \|_{\infty} (\lambda_D + C)} + R_{\hat{c}+1}^D \psi_0(x) \int_D R_{\hat{c}+1}^D (Au-cu) \tilde{\phi}_0 \, dx, \quad x \in D. \tag{6.73}
\]

By (6.25), measure \( \nu(A) = \int_D \bar{\psi}_0 R_{\hat{c}+1}^D \chi_A \, dx, \quad A \in \mathcal{B}(D) \) is equivalent to \( m \). Thus, \( \phi_0 := d\nu/dx \) is strictly positive and

\[
\int_D R_{\hat{c}+1}^D (Au-cu) \tilde{\phi}_0 \, dx = \int_D (Au-cu) \phi_0 \, dx.
\]

By (6.25), measure \( \nu(A) = \int_D \bar{\psi}_0 R_{\hat{c}+1}^D \chi_A \, dx, \quad A \in \mathcal{B}(D) \) is equivalent to \( m \). Thus, \( \phi_0 := d\nu/dx \) is strictly positive and

\[
\int_D R_{\hat{c}+1}^D (Au-cu) \tilde{\phi}_0 \, dx = \int_D (Au-cu) \phi_0 \, dx.
\]

Since \( D \) satisfies the exterior ball condition and \( R_{\hat{c}+1}^D \) is strongly Feller (see (6.26)), we have \( R_{\hat{c}+1}^D \bar{\psi}_0 \in C_0(D) \), see Section 6.13 below. Using Proposition 6.13 we conclude that there exists \( a_1 > 0 \) such that \( R_{\hat{c}+1}^D \psi_0 \geq a_1 \delta_D \). This finishes the proof of Theorem 5.4.

Under assumptions of Theorem 5.6, we may write

\[
\int_D R_{\alpha}^D (Au-cu) \tilde{\phi}_0 \, dx = \int_D (Au-cu) \hat{\phi}_{\alpha^*} \, dx,
\]

where \( \alpha_* := (\bar{c} + 1) \land (\| \bar{\psi} \|_{\infty} + 1) \). Using Proposition 6.13 again, we conclude that there exists \( a_2 > 0 \) such that \( \hat{\phi}_{\alpha^*} \tilde{\phi}_0 \geq a_2 \delta_D \) and the conclusion of the theorem follows. \( \square \)

6.12.2 Proof of Theorem 5.8

With no loss of generality we may and shall assume that \( \bar{c} \leq 0 \), as otherwise we could have considered \( A - \alpha \) for a sufficiently large \( \alpha > 0 \).

Suppose that \( p > d \). By [31, Theorem 1.2], the operators \( (A, \mathcal{D}(A)) \) and \( (\hat{A}, \mathcal{D}(\hat{A})) \) generate \( C_0 \)-semigroups of contractions on \( C_0(D) \), where

\[
\mathcal{D}(A) = \{ u \in W^{2,p}(D) \cap C_0(D) : Au \in C_0(D) \},
\]

\[
\mathcal{D}(\hat{A}) = \{ u \in W^{2,p}(D) \cap C_0(D) : \hat{A}u \in C_0(D) \}. \]

By the standard approximation we conclude that (5.6) holds for \( u \in \mathcal{D}(A), v \in \mathcal{D}(\hat{A}). \) Let \( (G_\alpha)_{\alpha > 0}, (\hat{G}_\alpha)_{\alpha > 0} \) denote the semigroup resolvents generated by \( A, \hat{A} \), respectively.

Using the martingale problem we conclude that \( G_\alpha = R_{\alpha}^D \) and \( \hat{G}_\alpha = \hat{R}_{\alpha}^D \), where as we recall \( R_{\alpha}^D, \hat{R}_{\alpha}^D \) are the resolvent corresponding to the probability transition semigroups (4.1) and (5.8). Thus,

\[
R_{\alpha}^D (C_0(D)) \subset C_0(D) \cap W^{2,p}(D) \quad \text{and} \quad \hat{R}_{\alpha}^D (C_0(D)) \subset C_0(D) \cap W^{2,p}(D)
\]

for any \( p > d \) and \( \alpha \geq 0 \). Let \( (R_{\alpha}^D)^* \) denote the adjoint operator to \( R_{\alpha}^D \) in \( L^2(D; m) \).

Then for any \( w \in C_0(D) \), there exists \( \eta \in \mathcal{D}(A) \) such that \( (-A + \alpha)\eta = w \). By using (5.6), for any \( u \in C_0(D) \)

\[
\int_D w \hat{R}_{\alpha}^D u \, dx = \int_D (-A + \alpha)\eta \hat{R}_{\alpha}^D u \, dx
\]

\[
= \int_D \eta (-\hat{A} + \alpha) \hat{R}_{\alpha}^D u \, dx = \int_D w \eta \, dx.
\]
On the other hand
\[
\int_D u\eta \, dx = \int_D uR^D_\alpha (-A + \alpha)\eta \, dx = \int_D (R^D_\alpha)^* u(-A + \alpha)\eta \, dx \\
= \int_D (R^D_\alpha)^* u\alpha \, dx = \int_D uR^D_\alpha \omega \, dx.
\]
Since \(u, w \in C_0(\bar{D})\) were arbitrary, we get (5.9).

6.13 Proof of the continuity of the resolvent to the boundary

Recall that a point \(\hat{x} \in \partial D\) is called Dirichlet regular if \(P_\hat{x}(\tau_D > 0) = 0\). If all the points of the boundary of \(D\) are Dirichlet regular, we say that \(D\) is Dirichlet regular. It is well known (see [13, Theorem 1.12]) that if \((R^D_\alpha)_{\alpha > 0}\) is strongly Feller, i.e. (6.26) holds, and \(\hat{x} \in \partial D\) is Dirichlet regular, then

\[
\lim_{D \ni x \to \hat{x}} E_x \tau_D = 0.
\]

Therefore, for Dirichlet regular \(D\) and \(g \in B^+_b(D)\), \(R^D_\alpha g\) is continuous in \(\bar{D}\), and

\[
\lim_{D \ni x \to \hat{x}} R^D_\alpha g(x) = 0, \quad \text{for any } \hat{x} \in \partial D.
\]

Proposition 6.16. Suppose that \(D\) is bounded and conditions A1), A2) are in force. If \(D\) satisfies the exterior ball condition (see Definition 2.5), then \(D\) is Dirichlet regular.

Proof. By [4, Proposition VII.3.3], to prove that \(\hat{x} \in \partial D\) is Dirichlet regular it is enough to find a function \(\psi\) - the so called barrier - that is strictly positive, and superharmonic on \(D\), i.e. \(-A\psi \geq 0\) on \(D\), such that

\[
\lim_{x \to \hat{x}} \psi(x) = 0.
\]

For \(\hat{x} \in \partial D\) let \(B(y_0, r)\) be the exterior ball to \(D\) at this point. Define a compactly supported, smooth function, such that

\[
\psi(x; \hat{x}) := c \left( \frac{1}{r^\sigma} - \frac{1}{|x - y_0|^\sigma} \right), \quad \frac{r}{10} < |x - y_0| < M
\]

where \(c, \sigma > 0\), and \(M\) is large that \(D \subset B(y_0, M)\). It is elementary to check that \(\psi\) fulfills the requirements for a barrier, provided \(c, \sigma > 0\) are sufficiently large.

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