Numerical Methods for Distributed Stochastic Compositional Optimization Problems with Aggregative Structure

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Abstract. The paper studies the distributed stochastic compositional optimization problems over networks, where all the agents’ inner-level function is the sum of each agent’s private expectation function. Focusing on the aggregative structure of the inner-level function, we employ the hybrid variance reduction method to obtain the information on each agent’s private expectation function, and apply the dynamic consensus mechanism to track the information on each agent’s inner-level function. Then by combining with the standard distributed stochastic gradient descent method, we propose a distributed aggregative stochastic compositional gradient descent method. When the objective function is smooth, the proposed method achieves the optimal convergence rate $O(K^{-1/2})$. We further combine the proposed method with the communication compression and propose the communication compressed variant distributed aggregative stochastic compositional gradient descent method. The compressed variant of the proposed method maintains the optimal convergence rate $O(K^{-1/2})$. Simulated experiments on decentralized reinforcement learning verify the effectiveness of the proposed methods.

Key words. Distributed stochastic compositional optimization, aggregative structure, hybrid variance reduction technique, dynamic consensus mechanism, communication compression

1 Introduction

Stochastic compositional optimization problem has been widely studied due to its extensively emerging applications in machine learning [3, 4, 6, 7], risk-averse portfolio optimization [24] and adaptive simulation [13]. In this paper, we consider the distributed stochastic compositional optimization problem

$$\min_{x \in \mathbb{R}^d} h(x) := \frac{1}{n} \sum_{j=1}^{n} f_j \left( \frac{1}{n} \sum_{j=1}^{n} g_j(x) \right)$$  \hspace{1cm} (1)$$

over networks, where $g_j(x) := \mathbb{E}[G_j(x; \phi_j)]$ is the private expectation function and $f_j(y) := \mathbb{E}[F_j(y; \zeta_j)]$ is the private outer-level function of agent $j \in [1, 2, \cdots, n]$, $G_j(\cdot; \phi_j) : \mathbb{R}^d \rightarrow \mathbb{R}^p$ and $F_j(\cdot; \zeta) : \mathbb{R}^p \rightarrow \mathbb{R}$ are measurable functions parameterized by random variables $\phi_j$ and $\zeta_j$ respectively. Since the inner level function $\frac{1}{n} \sum_{j=1}^{n} g_j(x)$ aggregates each agent’s private function $g_j(\cdot), j = 1, \cdots, n$, we call problem (1) distributed stochastic compositional optimization.
problem with aggregative structure \cite{16} and DSCA for short.

In the past decades, stochastic gradient decent methods have been well studied for solving stochastic compositional optimization problem, such as, two-timescale scheme method \cite{28,29}, sing-timescale scheme method \cite{4,12} and variance reduction based method \cite{14,17,25}. More recently, Gao and Huang \cite{10} study distributed stochastic compositional optimization problem over undirected communication networks, where a gossip-based distributed stochastic gradient descent method and its gradient-tracking version are proposed. Zhao and Liu \cite{37} propose a push-pull based distributed stochastic gradient descent method for distributed stochastic compositional optimization problem over directed communication networks. Both of works \cite{10,37} achieve the optimal convergence rate $O\left( K^{-1/2} \right)$. The deterministic distributed optimization problems with the aggregative structure have been studied in \cite{16,22}. Li et al. \cite{16} consider the distributed aggregative optimization problem, which allows each agent’s objective function to be dependent not only on their own decision variables, but also on the average of summable functions of decision variables of all other agents. Ram et al. \cite{22} model the regression problem as the distributed constrained optimization problem, where all agents cooperatively minimize a nonlinear function of the sum of the individual agent’s objective functions over the constraint set. Yang et al. \cite{32} consider the distributed bilevel stochastic optimization problem, where each agent’s objective depends not only on its own decision variable but also on the optimal solution of an sum of the expect-valued functions held privately by all agents. The authors \cite{32} propose a gossip-based distributed bilevel learning algorithm that allows networked agents to solve both the inner and outer optimization problems in a single timescale and share information via network propagation.

In this paper, we focus on the numerical methods for distributed stochastic compositional optimization problems with aggregative structure \cite{1}. We propose a distributed aggregative stochastic compositional gradient descent method (D-ASCGD for short), which combines the distributed stochastic gradient descent method with the hybrid variance reduction method \cite{5} and the dynamic average consensus mechanism \cite{21,38}. Specifically, we first construct the estimators of the values and the gradients of each agent’s private expectation function with diminishing bias via the hybrid variance reduction technique. Then we track the values and the gradients of the inner-level function by employing dynamic average consensus mechanism. Finally, combined with the standard distributed stochastic gradient method, we arrive at the D-ASCGD. The proposed D-ASCGD achieves the convergence rate $O\left( K^{-1/2} \right)$, which matches the optimal convergence rate of stochastic gradient descent method \cite{11}. We further combine the D-ASCGD with the compress procedure considered in \cite{20}, which induces the communication compressed variant of distributed stochastic compositional gradient descent method (CD-ASCGD for short). The CD-ASCGD compresses the decision variables, the trackers of the inner-level function value and its corresponding gradient to provide a communication-efficient implementation. CD-ASCGD also achieves the optimal convergence rate $O\left( K^{-1/2} \right)$.

The rest of this paper is organized as follows. Section 2 introduces the proposed D-ASCGD and some standard assumptions on problem \cite{1}, communication graphs and weighted matrices. Section 3 focuses on the convergence analysis of D-ASCGD. Section 4 presents the communication compressed variant CD-ASCGD and analyzes its convergence rate. Preliminary numerical test is provided in section 5.
In Algorithm 1, the key issue is to estimate the stochastic gradient. Throughout this paper, we use the following notation. $\mathbb{R}^d$ denotes the $d$-dimension Euclidean space endowed with norm $\|x\| = \sqrt{\langle x, x \rangle}$. $\|\cdot\|_F$ and $\|\cdot\|$ denote the Frobenius norm and matrix norm respectively. $\text{col}(x_1, x_2, \cdots, x_n)$ denotes the column vector by stacking up vectors $x_1, x_2, \cdots, x_n$, $1 := (1 \ 1 \ldots 1)^T \in \mathbb{R}^n$ and $0 := (0 \ 0 \ldots 0)^T \in \mathbb{R}^d$. $I_d \in \mathbb{R}^{d \times d}$ stands for the identity matrix and $A \otimes B$ denotes the Kronecker product of matrices $A$ and $B$. For any positive sequences $\{a_k\}$ and $\{b_k\}$, $a_k = O(b_k)$ if there exists $c > 0$ such that $a_k \leq cb_k$. The communication relationship between agents is characterized by a directed graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. For any $i \in \mathcal{V}$, $P_{\phi_i}$ and $P_{\zeta_i}$ are the distributions of random variables $\phi_i$ and $\zeta_i$ respectively. For a set $S$, $|S|$ denotes its cardinality.

## 2 D-ASCGD method

As we discussed in the introduction, the D-ASCGD uses hybrid variance reduction technique $[5]$ to estimate the value and the gradient of each agent’s private expectation function, and uses the dynamic average consensus mechanism $[21, 38]$ to track the value and the gradient of the inner-level function, which reads as follows.

**Algorithm 1** Distributed Aggregative Stochastic Compositional Gradient Descent (D-ASCGD):

**Input:** initial values $x_{i,1} \in \mathbb{R}^d$, $y_{i,1} = G_{i,1} \in \mathbb{R}^p$, $z_{i,1} = \hat{G}_{i,1} \in \mathbb{R}^{d \times p}$; stepsizes $\alpha_k > 0$, $\beta_k > 0$, $\gamma_k > 0$; nonnegative weight matrix $W = \{w_{ij}\}_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n}$.

1: for $k = 1, 2, \ldots$

2: for $i = 1, \ldots, n$ in parallel do

3: Draw $\phi_{i,k+1} \overset{i.i.d.}{\sim} P_{\phi_i}$, $\zeta_{i,k+1} \overset{i.i.d.}{\sim} P_{\zeta_i}$. Compute function values $G_i(x_{i,k}; \phi_{i,k+1})$, $G_i(x_{i,k+1}; \phi_{i,k+1})$ and gradients $\nabla F_i(y_{i,k}; \zeta_{i,k+1})$, $\nabla G_i(x_{i,k}; \phi_{i,k+1})$, $\nabla G_i(x_{i,k+1}; \phi_{i,k+1})$.

4: $x_{i,k+1} = \sum_{j=1}^{n} w_{ij} x_{j,k} - \alpha_k z_{i,k} \nabla F_i(y_{i,k}; \zeta_{i,k+1})$.

5: $G_{i,k+1} = (1 - \beta_k) (G_i(x_{i,k}; \phi_{i,k+1})) + G_i(x_{i,k+1}; \phi_{i,k+1})$.

6: $\hat{G}_{i,k+1} = (1 - \gamma_k) (\hat{G}_i(x_{i,k}; \phi_{i,k+1})) + \nabla G_i(x_{i,k+1}; \phi_{i,k+1})$.

7: $y_{i,k+1} = \sum_{j=1}^{n} w_{ij} y_{i,k} + G_{i,k+1} - G_{i,k}$.

8: $z_{i,k+1} = \sum_{j=1}^{n} w_{ij} z_{j,k} + \hat{G}_{i,k+1} - \hat{G}_i$.

9: end for

10: end for

In Algorithm 1, the key issue is to estimate the stochastic gradient

$$
\frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x) \nabla F_i \left( \frac{1}{n} \sum_{j=1}^{n} g_j(x); \zeta_{i,k+1} \right)
$$

for each agent $j$. We employ the hybrid variance reduction technique $[5]$ to estimate local function value $g_i(x)$ and gradient $\nabla g_i(x)$ in Lines 3 and 6. Taking $G_{i,k+1}$ as an example, we have

$$
G_{i,k+1} = (1 - \beta_k) (G_i(x_{i,k}; \phi_{i,k+1})) + G_i(x_{i,k+1}; \phi_{i,k+1}) + \beta_k G_i(x_{i,k+1}; \phi_{i,k+1})
$$
where the first term plays the role of variance reduction and the second term is the stochastic function value. The convex combination of the two terms may reduce the variance of estimating $g(x)$ gradually. In Lines 7 and 8 we utilize the dynamic average consensus mechanism to design the trackers of the inner level function. As a result of $W$ being doubly stochastic, we have

$$\frac{1}{n} n_j=1^ny_{j,k} = \frac{1}{n} n_j=1^ng_{j,k}, \quad \frac{1}{n} n_j=1^nz_{j,k} = \frac{1}{n} n_j=1^g\hat{g}_{j,k}.$$ 

Then if $G_{i,k}$ and $\hat{G}_{i,k}$ converge to $g_i(x)$ and $\nabla g_i(x)$ respectively, $y_{i,k}$ and $z_{i,k}$ could track $\frac{1}{n} n_j=1^ng_j(x)$ and $\frac{1}{n} n_j=1^\nabla g_j(x)$ successfully.

For the sake of notational convenience, we denote

$$G_{k+1,k+1} = col(G_1(x_{1,k+1};\phi_{1,k+1}),G_2(x_{2,k+1};\phi_{2,k+1}),\ldots, G_n(x_{n,k+1};\phi_{n,k+1}),$$

$$G_{k+1,k} = col(G_1(x_{1,k};\phi_{1,k+1}),G_2(x_{2,k};\phi_{2,k+1}),\ldots, G_n(x_{n,k};\phi_{n,k+1}),$$

$$\nabla G_{k+1,k+1} = (\nabla G_1(x_{1,k+1};\phi_{1,k+1})^T, \nabla G_2(x_{2,k+1};\phi_{2,k+1})^T,\ldots, \nabla G_n(x_{n,k+1};\phi_{n,k+1})^T)^T,$$

$$\nabla G_{k+1,k} = (\nabla G_1(x_{1,k};\phi_{1,k+1})^T, \nabla G_2(x_{2,k};\phi_{2,k+1})^T,\ldots, \nabla G_n(x_{n,k};\phi_{n,k+1})^T)^T,$$

$$G_k = col(G_1, G_2, \ldots, G_n), \quad \hat{G}_k = (\hat{G}^1, \hat{G}^2, \ldots, \hat{G}^n)^T,$$

$$U_{k+1} = col(z_{1,k}^1\nabla F_1(y_{1,k};\zeta_{1,k+1}), \ldots, z_{2,k}^n\nabla F_n(y_{n,k};\zeta_{n,k+1}),$$

$$y_k = col(y_{1,k}, y_{2,k}, \ldots, y_{n,k}), \quad z_k = (z_{1,k}^1, z_{2,k}^2, \ldots, z_{n,k}^n)^T,$$

$$\nabla g_k = (\nabla g_1(x_{1,k})^T, \ldots, \nabla g_n(x_{n,k})^T)^T,$$

$$\bar{W} := W \otimes I, \quad \bar{y}_k = \frac{1}{n} n_j=1^ny_{j,k}, \quad \bar{z}_k = \frac{1}{n} n_j=1^nz_{j,k}, \quad \bar{x}_k = \frac{1}{n} n_j=1^nx_{j,k}.$$ 

Then steps 4-8 in Algorithm 1 can be rewritten as

$$x_{k+1} = \bar{W}x_k - \alpha_k U_{k+1} \quad (2)$$

$$\hat{G}_{k+1} = (1-\beta_k)(G_k - G_{k+1,k}) + G_{k+1,k+1}, \quad (3)$$

$$\hat{G}_{k+1} = (1-\gamma_k)(\hat{G}_k - \nabla G_{k+1,k}) + \nabla G_{k+1,k+1}, \quad (4)$$

Throughout our analysis in the paper, we make the following two assumptions on objective functions, networks and weight matrices.

**Assumption 1 (Objective function).** Let $C_g, C_f, V_g, L_g$ and $L_f$ be positive scalars. For $\forall i \in \mathcal{V}$, $\forall x, x' \in \mathbb{R}^d$, $\forall y, y' \in \mathbb{R}^p$,

(a) functions $G_i(\cdot;\phi_i)$ and $F_i(\cdot;\zeta_i)$ are smooth, that is,

$$\|\nabla G_i(x;\phi_i) - \nabla G_i(x';\phi_i)\| \leq L_g\|x - x'\|$$

and

$$\|\nabla F_i(y;\zeta_i) - \nabla F_i(y';\zeta_i)\| \leq L_f\|y - y'\|;$$
Lemma 1. Under Assumptions 1 and 2, \( \text{lemmas are relegated to Appendix A.} \)
\[
\begin{align*}
\text{Throughout the paper, the proof of all the} & \quad \text{gence rate of D-ASCGD in Theorem 1, we first quantify the estimating errors for the value and} \\
\text{d} & \quad \text{and} \quad \text{are analogous to the unbiasedness and bounded variance assumptions for} \\
\text{non-compositional stochastic optimization problems.} \\
\text{Assumption 1 is standard for stochastic compositional optimization problem} & \quad \text{Assumption 2 (weight matrices and networks). The directed graph} \ G & \quad \text{admits a strongly connected and permits a nonnegative doubly stochastic weight matrix} \ W & \quad \text{W1 = 1 and} \ 1^T \ W = 1^T. \\
\text{Assumption 2 implies that} & \quad \frac{1}{n} \ 11^T W = W \left( \frac{1}{n} \ 11^T \right) = \frac{1}{n} \ 11^T \text{and the spectral norm} \ \rho := & \quad \text{satisfies} \ \rho < 1. \\
\end{align*}
\]

3 Convergence analysis for D-ASCGD

In this section, we study the convergence of D-ASCGD. Before the presentation of the convergence rate of D-ASCGD in Theorem 1, we first quantify the estimating errors for the value and gradient of each agent’s private expected function. Throughout the paper, the proof of all the lemmas are relegated to Appendix A.

Lemma 1. Under Assumptions 1 and 2,
\[
\begin{align*}
\mathbb{E} \left[ \left\| G_{k+1} - g_{k+1} \right\|^2 \right] & \leq (1 - \beta_k)^2 \mathbb{E} \left[ \left\| G_k - g_k \right\|^2 \right] + 48(1 - \beta_k)^2 \mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] \\
& \quad + 12(1 - \beta_k)^2 C_g \alpha_k^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] + 3\beta_k^2 V_g \\
\end{align*}
\]

and
\[
\begin{align*}
\mathbb{E} \left[ \left\| \hat{g}_{k+1} - \nabla g_{k+1} \right\|^2 \right] & \leq (1 - \gamma_k)^2 \mathbb{E} \left[ \left\| \hat{g}_k - \nabla g_k \right\|^2 \right] + 48(1 - \gamma_k)^2 pL_p^2 \mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] \\
& \quad + 12(1 - \gamma_k)^2 pL_p^2 \alpha_k^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] + 3\gamma_k^2 V_g. \\
\end{align*}
\]

The next lemma studies the asymptotic consensus of D-ASCGD.

Lemma 2. Suppose (a) Assumptions 1\(2\) hold; (b) \(\{\beta_k\}\) and \(\{\gamma_k\}\) are nonincreasing sequences such that \(\beta_1 \leq 1\) and \(\gamma_1 \leq 1\). Then
\[
\begin{align*}
\mathbb{E} \left[ \left\| x_{k+1} - 1 \otimes \bar{x}_{k+1} \right\|^2 \right] & \leq \frac{1 + p^2}{2} \mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] + \frac{1 + p^2}{1 - p^2} \alpha_k^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right],
\end{align*}
\]


\[ \mathbb{E} \left[ \| y_{k+1} - 1 \otimes \tilde{y}_{k+1} \|^2 \right] \leq \frac{1 + \rho^2}{2} \mathbb{E} \left[ \| y_k - 1 \otimes \tilde{y}_k \|^2 \right] + 4 \frac{1 + \rho^2}{1 - \rho^2} \left( \beta_k^2 \mathbb{E} \left[ \| G_k - g_k \|^2 \right] \right) + 2C_g \left( 4 \mathbb{E} \left[ \| x_k - 1 \otimes \tilde{x}_k \|^2 \right] + \alpha_k^2 C_f \mathbb{E} \left[ \| z_k \|^2_F \right] \right) + \beta_k^2 V_g \] (8)

and

\[ \mathbb{E} \left[ \| z_{k+1} - 1 \otimes \tilde{z}_{k+1} \|^2 \right] \leq \frac{1 + \rho^2}{2} \mathbb{E} \left[ \| z_k - 1 \otimes \tilde{z}_k \|^2 \right] + 3 \frac{1 + \rho^2}{1 - \rho^2} \left( \gamma_k^2 \mathbb{E} \left[ \| \tilde{G}_k - \nabla g_k \|^2_F \right] \right) + 2L_g^2 \left( 4 \mathbb{E} \left[ \| x_k - 1 \otimes \tilde{x}_k \|^2 \right] + \alpha_k^2 C_f \mathbb{E} \left[ \| z_k \|^2_F \right] \right) + \gamma_k^2 V_g'. \] (9)

**Lemma 3.** Suppose (a) Assumptions 1-2 hold; (b) \{\beta_k\} and \{\gamma_k\} are nonincreasing sequences such that \( \beta_1 \leq 1 \) and \( \gamma_1 \leq 1 \). Denote

\[ c_1 = \frac{1 - \rho^2}{24}, \quad c_2 = \frac{1 - \rho^2}{24}, \quad c_3 = \frac{1 - \rho^2}{48C_g}, \quad c_4 = \frac{1 - \rho^2}{24}, \]
\[ c_5 = 12C_gC_f c_3 + 12pL_g^2C_f c_4 + 8 \frac{1 + \rho^2}{1 - \rho^2} C_gC_f c_1 + 8 \frac{1 + \rho^2}{1 - \rho^2} L_g^2 C_f c_2, \]
\[ a_k = \max \left\{ \frac{2 + \rho^2}{3}, \left( \frac{1 + \rho^2}{2} + \frac{3pc_5}{c_2} \alpha_k^2 \right), \left( (1 - \beta_k)^2 + 6\beta_k^2 \right) \right\}, \]
\[ b_k = 3np c_5 C_g \alpha_k^2 + \left( \frac{4 + \rho^2}{1 - \rho^2} V_g c_1 + 3V_g c_3 \right) \beta_k^2 + \left( \frac{4 + \rho^2}{1 - \rho^2} 2V_g' c_2 + 3V_g' c_4 \right) \gamma_k^2. \]
\[ V_k = \mathbb{E} \left[ \| x_k - 1 \otimes \tilde{x}_k \|^2 \right] + c_1 \mathbb{E} \left[ \| y_k - 1 \otimes \tilde{y}_k \|^2 \right] + c_2 \mathbb{E} \left[ \| z_k - 1 \otimes \tilde{z}_k \|^2 \right] + c_3 \mathbb{E} \left[ \| G_k - g_k \|^2 \right] + c_4 \mathbb{E} \left[ \| \tilde{G}_k - \nabla g_k \|^2_F \right]. \] (12)

Then

\[ V_{k+1} \leq a_k V_k + b_k. \] (13)

Lemma 3 is a technical result, which characterizes a recursive inequality relation of the estimating errors of hybrid variance reduction method and the dynamic average consensus method.

We are ready to present the convergence rate of D-ASCGD.

**Theorem 1.** Let \( c_1, \cdots, c_5 \) be defined in (10). Suppose that (a) Assumptions 1-2 hold, (b) \( \alpha_k = \frac{s_1}{\sqrt{K}}, \beta_k = \frac{s_2}{\sqrt{K}}, \gamma_k = \frac{s_3}{\sqrt{K}} \), where positive constants \( s_1, s_2, s_3 \) are small enough such that \( \alpha_k < 1, \beta_k < 1, \gamma_k < 1 \) and

\[ \frac{s_1^2}{K} < \frac{(1 - \rho^2)c_2}{6pc_5}, \quad \frac{s_2}{\sqrt{K}} < \frac{2(1 + \rho^2)}{7 - 5\rho^2}, \quad \frac{s_3}{\sqrt{K}} < \frac{3pc_5c_4}{c_3}, \quad \frac{s_1^2}{s_1 \sqrt{K}} < 2. \] (14)

Then for any \( i \in V \),

\[ \frac{1}{K} \sum_{k=1}^{K} \mathbb{E} \left[ \| \nabla h(x_{i,k}) \|^2 \right] \leq \frac{8 \mathbb{E} [h(\tilde{x}_i)] - \mathbb{E} [h(\tilde{x}_{K+1})]}{s_1 \sqrt{K}} + \frac{24pLC_f C_g s_1}{n\sqrt{K}} + \frac{8c_6 (V_i + b)}{K(1 - a_K)}, \] (15)

6
where \( a_k \) is defined in \((11)\), \( L = C_gL_f + C_f^{1/2}L_g \) and

\[
\begin{align*}
b &= 3nc_5C_g s_1^2 + 3 \left( \frac{1 + \rho^2}{1 - \rho^2} V_g c_1 + V_g c_3 \right) s_2^2 + 3 \left( \frac{1 + \rho^2}{1 - \rho^2} 2V_g' c_2 + V_g' c_4 \right) s_3^2, \\
c_6 &= \max \left\{ 2 \left( \frac{C_g^2 L_f^2}{n} + \frac{C_f L_g^2}{n} + \frac{L^2}{8} \right), \frac{2C_g^3 L_f^2}{n \min \{c_1, c_3\}}, \frac{(4C_f + 3LC_f) p}{2 \min \{c_2, c_4\} n} \right\}.
\end{align*}
\]

**Proof.** We first provide an upper bound for \( \mathbb{E} \left[ \| \nabla h(\tilde{x}_k) \|^2 \right] \). Noting that \( \nabla h(x) \) is \( L \left( := C_gL_f + C_f^{1/2}L_g \right) \)-smooth \( 35 \),

\[
\begin{align*}
h(\tilde{x}_{k+1}) &\leq h(\tilde{x}_k) + \langle \nabla h(\tilde{x}_k), \tilde{x}_{k+1} - \tilde{x}_k \rangle + \frac{L}{2} \| \tilde{x}_{k+1} - \tilde{x}_k \|^2 \\
&= h(\tilde{x}_k) - \langle \nabla h(\tilde{x}_k), \alpha_k \tilde{U}_{k+1} \rangle + \frac{L}{2} \| \alpha_k \tilde{U}_{k+1} \|^2 \\
&= h(\tilde{x}_k) - \alpha_k \| \nabla h(\tilde{x}_k) \|^2 + \frac{L}{2} \| \alpha_k \tilde{U}_{k+1} \|^2 + \langle \nabla h(\tilde{x}_k), \alpha_k \left( \nabla h(\tilde{x}_k) - \tilde{U}_{k+1} \right) \rangle,
\end{align*}
\]

where \( \tilde{U}_{k+1} = \left( \frac{1}{n} \otimes \textbf{I}_d \right) \textbf{U}_{k+1} \), the first equality follows from the fact that \( \tilde{x}_{k+1} = \tilde{x}_k - \alpha_k \tilde{U}_{k+1} \). Take expectation on both sides of above inequality,

\[
\mathbb{E} \left[ h(\tilde{x}_{k+1}) \right] \leq \mathbb{E} \left[ h(\tilde{x}_k) \right] - \alpha_k \mathbb{E} \left[ \| \nabla h(\tilde{x}_k) \|^2 \right] + \frac{L}{2} \mathbb{E} \left[ \| \alpha_k \tilde{U}_{k+1} \|^2 \right] \\
+ \mathbb{E} \left[ \langle \nabla h(\tilde{x}_k), \alpha_k \left( \nabla h(\tilde{x}_k) - \tilde{U}_{k+1} \right) \rangle \right].
\]

For the third term on the right hand side of \((17)\),

\[
\frac{L}{2} \mathbb{E} \left[ \| \alpha_k \tilde{U}_{k+1} \|^2 \right] \leq \frac{L \alpha_k^2}{2n} \mathbb{E} \left[ \| \textbf{U}_{k+1} \|^2 \right] \leq \frac{L \alpha_k^2}{2n} C_f \mathbb{E} \left[ \| \textbf{z}_k \|^2_F \right],
\]

where the second inequality follows from the definition of \( \textbf{U}_{k+1} \) and Assumption \((1)\). For the term \( \mathbb{E} \left[ \| \textbf{z}_k \|^2_F \right] \), it is easy to observe that

\[
\mathbb{E} \left[ \| \textbf{z}_k \|^2_F \right] = \mathbb{E} \left[ \left\| \textbf{z}_k - 1 \otimes \tilde{z}_k + 1 \otimes \tilde{z}_k - 1 \otimes \left( \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_{j,k}) \right) + 1 \otimes \left( \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_{j,k}) \right) \right\|^2_F \right]
\leq 3 \left( \mathbb{E} \left[ \| \textbf{z}_k - 1 \otimes \tilde{z}_k \|^2_F \right] + n \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_{j,k}) \right\|^2_F \right] \right)
\leq 3 \left( \mathbb{E} \left[ \| \textbf{z}_k - 1 \otimes \tilde{z}_k \|^2_F \right] + \mathbb{E} \left[ \| \textbf{G}_k - \nabla g_k \|^2 \right] + npC_g \right)
\leq 3p \left( \mathbb{E} \left[ \| \textbf{z}_k - 1 \otimes \tilde{z}_k \|^2 \right] + \mathbb{E} \left[ \| \textbf{G}_k - \nabla g_k \|^2 \right] + nC_g \right).
\]

Then

\[
\frac{L}{2} \mathbb{E} \left[ \| \alpha_k \tilde{U}_{k+1} \|^2 \right] \leq 3p \frac{L \alpha_k^2}{2n} C_f \left( \mathbb{E} \left[ \| \textbf{z}_k - 1 \otimes \tilde{z}_k \|^2 \right] + \mathbb{E} \left[ \| \textbf{G}_k - \nabla g_k \|^2 \right] + nC_g \right).
\]
For the fourth term on the right hand side of (17),

\[ E \left[ \langle \nabla h(\bar{x}_k), \alpha_n (\nabla h(\bar{x}_k) - \bar{U}_{k+1}) \rangle \right] = E[\langle \alpha_n \nabla h(\bar{x}_k), P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 \rangle] \]

\[ \leq \alpha_k^2 E \left[ \| \nabla h(\bar{x}_k) \|^2 \right] + 3\tau \left( E \left[ \| P_1 \|^2 \right] + E \left[ \| P_2 \|^2 \right] + E \left[ \| P_3 \|^2 \right] + E \left[ \| P_4 \|^2 \right] + E \left[ \| P_5 \|^2 \right] + E \left[ \| P_6 \|^2 \right] \]

\[ + E \left[ \langle \nabla h(\bar{x}_k), P_7 \rangle \right], \]

where \( \tau \) is any positive scalar,

\[ P_1 = \nabla h(\bar{x}_k) - \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_k) \right) \left( \nabla f_j \left( \frac{1}{n} \sum_{j=1}^{n} g_j(x_k) \right) \right), \]

\[ P_2 = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_k) - \nabla g_j(x_{j,k}) \right) \left( \nabla f_j \left( \frac{1}{n} \sum_{j=1}^{n} g_j(x_{j,k}) \right) \right), \]

\[ P_3 = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_{j,k}) \right) \left( \nabla f_j(y_k) - \nabla f_j(y_{j,k}) \right), \]

\[ P_4 = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \nabla g_j(x_{j,k}) \right) \nabla f_j(y_{j,k}), \]

\[ P_5 = \frac{1}{n} \sum_{j=1}^{n} \left( \nabla f_j(y_{j,k}) - \nabla f_j(y_{j,k};\zeta_{j,k+1}) \right), \]

\[ P_6 = \frac{1}{n} \sum_{j=1}^{n} \left( \zeta_k - z_{j,k} \right) \nabla f_j(y_{j,k}), \]

\[ P_7 = \frac{1}{n} \sum_{j=1}^{n} \left( \nabla f_j(y_{j,k}) - \nabla f_j(y_{j,k};\zeta_{j,k+1}) \right), \]

the inequality follows from Cauchy-Schwarz inequality and the fact \( ab \leq \frac{1}{2\tau}a^2 + \frac{\tau}{2}b^2 \). Defining

\[ F_1 = \sigma \left( x_{i,1}, y_{i,1,1}, z_{i,1}, G_{i,1}, \tilde{G}_{i,1} : i \in \mathcal{V} \right), \]

\[ F_k = \sigma \{ x_{i,1}, y_{i,1,1}, z_{i,1}, G_{i,1}, \tilde{G}_{i,1}, \phi_{i,t}, \zeta_{i,t} : i \in \mathcal{V}, 2 \leq t \leq k \} (k \geq 2), \]

we have \( E \left[ \nabla F_j(y_{j,k};\zeta_{j,k+1}) | F_k \right] = \nabla f_j(y_{j,k}) \) and then the third term on the right hand side of inequality (19) is equal to 0. Moreover, by Assumption (a) and (c),

\[ E \left[ \langle \nabla h(\bar{x}_k), \alpha_n (\nabla h(\bar{x}_k) - \bar{U}_{k+1}) \rangle \right] \]

\[ \leq \alpha_k^2 E \left[ \| \nabla h(\bar{x}_k) \|^2 \right] + 3\tau \left( \frac{C_1^2 L_1^2}{n} + \frac{C_f L_2^2}{n} \right) E \left[ \| x_k - 1 \otimes \bar{x}_k \|^2 \right] + \frac{3\tau C_2 L_3^2}{n} E \left[ \| g_k - G_k \|^2 \right] \]

\[ + \frac{3\tau C_3 L_4^2}{n} E \left[ \| y_k - 1 \otimes \bar{y}_k \|^2 \right] + \frac{3\tau C_f L_5}{n} E \left[ \| V g_k - \tilde{G}_k \|^2 \right] + \frac{3\tau C f L_6}{n} E \left[ \| z_k - 1 \otimes \bar{z}_k \|^2 \right]. \]
Plug (18), (20) into (17) and set $\tau = \frac{2n}{3}$.

$$\mathbb{E}[h(\bar{x}_{k+1})]$$

\begin{align*}
&\leq \mathbb{E}[h(\bar{x}_k)] - \frac{\alpha_k}{4} \mathbb{E}[\|\nabla h(\bar{x}_k)\|^2] + 2\alpha_k \left( \frac{C_g^2 L_f^2}{n} + \frac{C_f L_g^2}{n} + \frac{L^2}{8} \right) \mathbb{E}[\|x_k - 1 \otimes \bar{x}_k\|^2] \\
&\quad + \frac{2\alpha_k C_g^2 L_f^2}{n} \mathbb{E}[\|g_k - G_k\|^2] + \frac{2\alpha_k C_g L_f^2}{n} \mathbb{E}[\|y_k - 1 \otimes \bar{y}_k\|^2] + 3p \frac{L_0^2}{n} C_f C_g \\
&\quad + \frac{(4C_f + 3pLC_f\alpha_k)\alpha_k}{2n} \mathbb{E}\left[\|\nabla g_k - \hat{G}_k\|^2 \right] + \frac{(4C_f + 3LC_f\alpha_k)p\alpha_k}{2n} \mathbb{E}[\|z_k - 1 \otimes \bar{z}_k\|^2] .
\end{align*}

(21)

Next, we provide an upper bound of $\mathbb{E}[\nabla h(x_{i,k})]$. By the Lipschitz continuity of $\nabla h(\cdot)$,

$$\frac{1}{2} \mathbb{E}[\|\nabla h(x_{i,k})\|^2] \leq \mathbb{E}[\|\nabla h(x_{i,k}) - \nabla h(\bar{x}_k)\|^2] + \mathbb{E}[\|\nabla h(\bar{x}_k)\|^2]$$

\begin{align*}
&\leq L^2 \mathbb{E}[\|x_{i,k} - \bar{x}_k\|^2] + \mathbb{E}[\|\nabla h(\bar{x}_k)\|^2] \\
&\leq L^2 \mathbb{E}[\|x_k - 1 \otimes \bar{x}_k\|^2] + \mathbb{E}[\|\nabla h(\bar{x}_k)\|^2] ,
\end{align*}

where the second inequality follows from the Lipschitz continuity of $\nabla h(\cdot)$. Then

$$-\mathbb{E}[\|\nabla h(\bar{x}_k)\|^2] \leq \frac{1}{2} \mathbb{E}[\|\nabla h(x_{i,k})\|^2] + L^2 \mathbb{E}[\|x_k - 1 \otimes \bar{x}_k\|^2] .$$

Substitute the above inequality into (21),

$$\mathbb{E}[h(\bar{x}_{k+1})]$$

\begin{align*}
&\leq \mathbb{E}[h(\bar{x}_k)] - \frac{\alpha_k}{8} \mathbb{E}[\|\nabla h(x_{i,k})\|^2] + 2\alpha_k \left( \frac{C_g^2 L_f^2}{n} + \frac{C_f L_g^2}{n} + \frac{L^2}{8} \right) \mathbb{E}[\|x_k - 1 \otimes \bar{x}_k\|^2] \\
&\quad + \frac{2\alpha_k C_g^2 L_f^2}{n} \mathbb{E}[\|g_k - G_k\|^2] + \frac{2\alpha_k C_g L_f^2}{n} \mathbb{E}[\|y_k - 1 \otimes \bar{y}_k\|^2] + 3p \frac{L_0^2}{n} C_f C_g \\
&\quad + \frac{(4C_f + 3pLC_f\alpha_k)\alpha_k}{2n} \mathbb{E}\left[\|\nabla g_k - \hat{G}_k\|^2 \right] + \frac{(4C_f + 3LC_f\alpha_k)p\alpha_k}{2n} \mathbb{E}[\|z_k - 1 \otimes \bar{z}_k\|^2] \\
&\leq \mathbb{E}[h(\bar{x}_k)] - \frac{\alpha_k}{8} \mathbb{E}[\|\nabla h(x_{i,k})\|^2] + 3p \frac{L_0^2}{n} C_f C_g + c_6\alpha_k V_k ,
\end{align*}

(22)

where

$$c_6 = \max\left\{ 2 \left( \frac{C_g^2 L_f^2}{n} + \frac{C_f L_g^2}{n} + \frac{L^2}{8} \right), \frac{2C_g^2 L_f^2}{n \min\{c_1, c_3\}}, \frac{(4C_f + 3LC_f)p}{2 \min\{c_2, c_4\} n} \right\} ,$$

$c_1, c_3, c_3, c_4$ are defined in (10) and $V_k$ is defined in (12). Reordering the terms of (22) and summing over $k$ from 1 to $K$,

$$\sum_{k=1}^{K} \frac{\alpha_k}{8} \mathbb{E}[\|\nabla h(x_{i,k})\|^2] \leq \mathbb{E}[h(\bar{x}_1)] - \mathbb{E}[h(\bar{x}_{K+1})] + \frac{3pLC_fC_g}{n} \sum_{k=1}^{K} \alpha_k^2 + c_6 \sum_{k=1}^{K} \alpha_k V_k .$$

(23)

Note that definitions of $\alpha_k$, $\beta_k$, $\gamma_k$ guarantee $a_k = a < 1$\footnote{For any fixed $K$, $a_k$ is a constant dependent on $K$.} $b_k = b/K$ where $b$ is defined in (16). Then by Lemma [3]

$$V_k \leq a_{k-1}V_{k-1} + b_{k-1} = aV_{k-1} + b/K \leq \cdots \leq a^{k-1}V_1 + (b/K) \sum_{t=0}^{k-2} a^t \leq a^{k-1}V_1 + \frac{b}{K(1-a)} .$$
Substitute the above inequality into (23) and multiply both sides of (23) by \( \frac{8}{s_1 \sqrt{K}} \),
\[
\frac{1}{K} \sum_{k=1}^{K} \mathbb{E} \left[ \| \nabla h(x_{i,k}) \|^2 \right] \leq \frac{8\mathbb{E} \left[ h(\hat{x}_1) \right] - \mathbb{E} \left[ h(\hat{x}_{K+1}) \right]}{s_1 \sqrt{K}} + \frac{24pLC_f C_y s_1}{n \sqrt{K}} + \frac{8\epsilon_0}{K} \sum_{k=1}^{K} \left( a^{k-1} V_1 + \frac{b}{K(1-a)} \right)
\]
\[
\leq \frac{8\mathbb{E} \left[ h(\hat{x}_1) \right] - \mathbb{E} \left[ h(\hat{x}_{K+1}) \right]}{s_1 \sqrt{K}} + \frac{24pLC_f C_y s_1}{n \sqrt{K}} + \frac{8\epsilon_0 (V_1 + b)}{K(1-a)}. \tag{24}
\]

The proof is complete. \( \square \)

By the definitions of \( a, b, V_1 \), the third term on the right hand side of (24) could be expressed more exactly,
\[
O \left( \max \left\{ \frac{1}{K^{\rho}}, \frac{1}{\sqrt{K}} \right\} \right).
\]

Then similar to the more recent work [32], the order of magnitude of inequality (24) is \( O(1/\sqrt{K}) \), which achieves the optimal convergence rate of stochastic gradient descent method [1]. The DSBO method [32] combines the gossip communication with weighted average stochastic approximation and D-ASCGD combines the distributed stochastic gradient descent method with hybrid variance reduction method and dynamic consensus mechanism.

### 4 Compressed D-ASCGD method

In recent years, various techniques have been developed to reduce communication costs [27, 31]. They are extensively incorporated into centralized optimization methods [1, 23, 30] and decentralized methods [8, 9, 18, 20, 33]. This motivates us to provide an extension of D-ASCGD by combining it with communication compressed method, which reads as follows.

**Algorithm 2 Compressed Distributed Aggregative Stochastic Compositional Gradient Descent (CD-ASCGD)\(^{[3]}\)**

**Input:** initial values \( x_1, G_1, G_1, H^x_1, H^y_1, H^z_1, y_1 = G_1, z_1 = G_1 \); stepsizes \( \alpha_k > 0, \beta_k > 0, \gamma_k > 0 \); scaling parameters \( \alpha_w \in (0,1) \); nonnegative weight matrix \( W \in \mathbb{R}^{n \times n}, W = W \odot I_d \)

1. \( H^x_{1,w} = WH^x_1, H^y_{1,w} = WH^y_1, H^z_{1,w} = WH^z_1 \)
2. for \( k = 1, 2, \ldots \) do
3. \( \hat{x}_k, \hat{x}_{k+1}', H^{x,\hat{x}}_{k+1} = \text{COMM} (x_k, H^x_k, H^x_{k+1}) \)
4. \( \hat{y}_k, \hat{y}_{k+1}', H^{y,\hat{y}}_{k+1} = \text{COMM} (y_k, H^y_k, H^y_{k+1}) \)
5. \( \hat{z}_k, \hat{z}_{k+1}', H^{z,\hat{z}}_{k+1} = \text{COMM} (z_k, H^z_k, H^z_{k+1}) \)
6. For any \( i \in V \), draw \( \phi_{i,k+1} \overset{iid}{\sim} P_{\phi_i}, \zeta_{i,k+1} \overset{iid}{\sim} P_{\zeta_i} \), and compute function values \( G_i(x_{i,k}; \phi_{i,k+1}), G_i(x_{i,k+1}; \phi_{i,k+1}) \) and gradients \( \nabla F_i(y_{i,k}; \zeta_{i,k+1}), \nabla G_i(x_{i,k}; \phi_{i,k+1}) \) and \( \nabla G_i(x_{i,k+1}; \phi_{i,k+1}) \)
7. \( x_{k+1} = x_k - \alpha_w (\hat{x}_k - \hat{x}_{k}') - \alpha_k U_{k+1} \)
8. \( G_{k+1} = (1 - \beta_k) (G_k - G_{k+1,k}) + G_{k+1,k+1} \)
9. \( \hat{G}_{k+1} = (1 - \gamma_k) (\hat{G}_k - \nabla \hat{G}_{k+1,k}) + \nabla G_{k+1,k+1} \)
10. \( y_{k+1} = y_k - \alpha_w (\hat{y}_k - \hat{y}_{k}') + G_{k+1} - G_k \)
11. \( z_{k+1} = z_k - \alpha_w (\hat{z}_k - \hat{z}_{k}') + G_{k+1} - \hat{G}_k \)
12. end for
*In Lines 3-5 of Algorithm 2 set scaling parameter $\alpha$ and compressor $C$ as $\alpha_z$ and $C_1$ for $x_k$, $\alpha_y$ and $C_2$ for $y_k$, $\alpha_z$ and $C_3$ for $z_k$ in procedure COMM ($v, H, H^w$).

**Algorithm 3 procedure** COMM $[20]$

Input: $v, H, H^w$

1: $Q = \text{Compress} (v - H)$ \quad $\triangleright$ Compression

2: $\bar{v} = H + Q$

3: $\bar{w} = H^w + WQ$ \quad $\triangleright$ Communication

4: $H \leftarrow (1 - \alpha)H + \alpha\bar{v}$

5: $H^w \leftarrow (1 - \alpha)H^w + \alpha\bar{w}$

6: Return: $\bar{v}, \bar{w}, H, H^w$

We first explain how COMM ($v, H, H^w$) in Algorithm 3 works by taking decision variable $x_k$ as an example. With inputs including decision variable $x_k$ and its two auxiliary variables $H^x_k$ and $H^{x,w}_k$, COMM ($v, H, H^w$) compress $x_k - H^x_k$ as a low-bit variable $Q$. Next, all agents communicate low-bit variable $Q$ with their neighbors and obtain $\bar{x}^w_k = H^{x,w}_k + WQ$. Auxiliary variables $H^x_k$ and $H^{x,w}_k$ are updated in Lines 4 and 5 to improve the stability of the compression procedure.

Now we explain the relationship between the Algorithm 2 and Algorithm 1. Denoting $\mathbf{E}^x_{k+1} := x_{k+1} - x_k$, the iteration $x_{k+1}$ in line 7 can be reformulated as

$$x_{k+1} = \left[ (1 - \alpha_w)I_{nd} + \alpha_w W \right] x_k - \alpha_k U_{k+1} + \alpha_w \left[ I_{nd} - W \right] \mathbf{E}^x_{k+1}.$$  \hfill (25)

Compared with the iterations of $x_k$ in Algorithm 1, there is a term $\alpha_w \left[ I_{nd} - W \right] \mathbf{E}^x_{k+1}$ induced by the compress errors $\mathbf{E}^x_{k+1}$. Similarly, denote $\mathbf{E}^y_{k+1} := y_{k+1} - \bar{y}_k$ and $\mathbf{E}^z_{k+1} := z_{k+1} - \bar{z}_k$, the iterations $y_{k+1}$ and $z_{k+1}$ in lines 10 and 11 of Algorithm 2 can be reformulated as

$$y_{k+1} = \left[ (1 - \alpha_y)I_{nd} + \alpha_y W \right] y_k + G_{k+1} - G_k + \alpha_w \left[ I_{nd} - W \right] \mathbf{E}^y_{k+1}.$$  \hfill (26)

$$z_{k+1} = \left[ (1 - \alpha_z)I_{nd} + \alpha_z W \right] z_k + G_{k+1} - G_k + \alpha_w \left[ I_{nd} - W \right] \mathbf{E}^z_{k+1}.$$  \hfill (27)

Again, the difference between (26) and the iterations of $y_{k+1}, z_{k+1}$ in Algorithm 1 are the terms induced by compress errors $\mathbf{E}^y_{k+1}$ and $\mathbf{E}^z_{k+1}$.

In the next, we establish the convergence of CD-ASC GD. The following conditions on the compressors are needed.

**Assumption 3.** The compressors $C_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d, C_2 : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $C_3 : \mathbb{R}^{d \times p} \rightarrow \mathbb{R}^{d \times p}$ satisfy

$$\mathbb{E} \left[ \frac{\| C_1(x) - x \|^2}{r_1} \right] \leq (1 - \psi_1)\| x \|^2, \quad \forall x \in \mathbb{R}^d,$$  \hfill (28)

$$\mathbb{E} \left[ \frac{\| C_2(z) - z \|^2}{r_2} \right] \leq (1 - \psi_2)\| z \|^2, \quad \forall z \in \mathbb{R}^p,$$  \hfill (29)

* *A complete algorithm description from the agent’s perspective can be found in Appendix B.*
for some constants $\psi_1, \psi_2, \psi_3 \in (0, 1]$ and $r_1, r_2, r_3 \in (0, +\infty)$. Here $E[.]$ denotes the expectation over the internal randomness of the stochastic compression operator.

For vector $x \in \mathbb{R}^d$ or $z \in \mathbb{R}^p$, the class of compressors satisfying (28) or (29) is broad, such as random quantization [23, 26], sparsification [15, 19], the norm-sign compressor [18, 33]. For matrix $y \in \mathbb{R}^{d \times p}$, we may construct the dp-length vector by stacking up the columns of $y$ and then implement predetermined compressors to compress the new constructed vector.

Similar to the analysis of D-ASCSD, we first provide the upper bounds of the local inner-level function estimating errors and the consensus errors of the CD-ASCSD in Lemmas 4 and 5 respectively.

**Lemma 4.** Under Assumptions 2, 3

$$E\left[\left\|G_{k+1} - g_{k+1}\right\|^2\right] \leq (1 - \beta_k)^2 E\left[\left\|G_k - g_k\right\|^2\right] + 3\beta_k^2 V_g + 72(1 - \beta_k)^2 C_g \alpha_w^2 \bar{r}_1 E\left[\left\|x_k - H_k^2\right\|^2\right] + 18(1 - \beta_k)^2 C_g \alpha_k^2 C_f E\left[\left\|z_k\right\|^2_F\right] + 72(1 - \beta_k)^2 C_g \alpha_k^2 E\left[\left\|x_k - 1 \otimes \bar{x}_k\right\|^2\right]$$

and

$$E\left[\left\|G_{k+1} - \nabla g_{k+1}\right\|^2\right] \leq (1 - \gamma_k)^2 E\left[\left\|G_k - \nabla g_k\right\|^2\right] + 3\gamma_k^2 V_g + 72(1 - \gamma_k)^2 pL_g^2 \alpha_k^2 \bar{r}_1 E\left[\left\|x_k - H_k^2\right\|^2\right] + 18(1 - \gamma_k)^2 pL_g^2 \alpha_k^2 C_f E\left[\left\|z_k\right\|^2_F\right] + 72(1 - \gamma_k)^2 pL_g^2 \alpha_k^2 E\left[\left\|x_k - 1 \otimes \bar{x}_k\right\|^2\right],$$

where $\bar{r}_1 = 2r_1^2(1 - \psi_1) + 2(1 - r_1)^2$, $\psi_1$ and $r_1$ are defined in Assumption 3.

**Lemma 5.** Suppose (a) Assumptions 2, 3 hold; (b) $\{\beta_k\}$ and $\{\gamma_k\}$ are nonincreasing sequences such that $\beta_1 \leq 1$ and $\gamma_1 \leq 1$. Then

$$E\left[\left\|x_{k+1} - 1 \otimes \bar{x}_{k+1}\right\|^2\right] \leq \frac{1 + \rho_w^2}{2} E\left[\left\|x_k - 1 \otimes \bar{x}_k\right\|^2\right] + \frac{1 + \rho_w^2}{1 - \rho_w^2} \alpha_k^2 C_f E\left[\left\|z_k\right\|^2_F\right]$$

$$+ 4\frac{1 + \rho_w^2}{1 - \rho_w^2} \alpha_k^2 \bar{r}_1 E\left[\left\|x_k - H_k^2\right\|^2\right],$$

$$E\left[\left\|y_{k+1} - 1 \otimes \bar{y}_{k+1}\right\|^2\right] \leq \frac{1 + \rho_w^2}{2} E\left[\left\|y_k - 1 \otimes \bar{y}_k\right\|^2\right] + 4\frac{1 + \rho_w^2}{1 - \rho_w^2} \left(\beta_k^2 E\left[\left\|G_k - g_k\right\|^2\right] + \beta_k^2 V_g + 3C_g \left(4\alpha_w^2 E\left[\left\|x_k - 1 \otimes \bar{x}_k\right\|^2\right] + \alpha_k^2 C_f E\left[\left\|z_k\right\|^2_F\right]\right)\right)$$

$$+ 48C_g \frac{1 + \rho_w^2}{1 - \rho_w^2} \alpha_w^2 \bar{r}_2 E\left[\left\|x_k - H_k^2\right\|^2\right] + 4\frac{1 + \rho_w^2}{1 - \rho_w^2} \alpha_w^2 \bar{r}_2 E\left[\left\|y_k - H_k^2\right\|^2\right]$$
and
\[
\mathbb{E} \left[ \left\| \mathbf{z}_{k+1} - 1 \otimes \hat{z}_{k+1} \right\|^2 \right] \\
\leq \frac{1 + \rho_w^2}{2} \mathbb{E} \left[ \left\| \mathbf{z}_k - 1 \otimes \hat{x}_k \right\|^2 \right] + 4 \frac{1 + \rho_w^2}{1 - \rho_w} \left( \gamma_k^2 \mathbb{E} \left[ \left\| \mathbf{G}_k - \nabla \mathbf{g}_k \right\|_F^2 \right] + \gamma_k^2 \nu \right) + 3 L_g^2 \left( 4 \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{x}_k - 1 \otimes \hat{x}_k \right\|^2 \right] + \alpha_k^2 C_f \mathbb{E} \left[ \left\| \mathbf{z}_k \right\|_F^2 \right] \right) \\
+ 48 L_g^2 \frac{1 + \rho_w^2}{1 - \rho_w} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{x}_k - \mathbf{H}_k \right\|^2 \right] + 4 \frac{1 + \rho_w^2}{1 - \rho_w} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{z}_k - \mathbf{H}_k \right\|^2 \right],
\]

where \( \rho_w := \frac{(1 - \alpha_w) I_{nd} + \alpha_w \mathbf{W} - \frac{11}{n} \otimes I_d}{1} \), \( \hat{r}_1 = 2r_1^2 (1 - \psi_1) + 2 (1 - r_1)^2 \), \( \hat{r}_2 = 2r_2^2 (1 - \psi_2) + 2 (1 - r_2)^2 \), \( \hat{r}_3 = 2r_3^2 (1 - \psi_3) + 2 (1 - r_3)^2 \), and \( \psi_1, \psi_2, \psi_3, r_1, r_1, r_3 \) are defined in Assumption 3.

The following lemma characterizes the upper bound of compression errors.

**Lemma 6.** Let \( \alpha_x \in \left( 0, \frac{1}{r_1} \right) \), \( \alpha_y \in \left( 0, \frac{1}{r_2} \right) \), \( \alpha_z \in \left( 0, \frac{1}{r_3} \right) \) and
\[
\alpha_w \in \left( 0, \min \left\{ \frac{(\alpha_x r_1 \psi_1)^3}{24 \hat{r}_1 (1 + \alpha_x r_1 \psi_1)^2}, \frac{(\alpha_y r_2 \psi_2)^3}{24 \hat{r}_2 (1 + \alpha_y r_2 \psi_2)^2}, \frac{(\alpha_z r_3 \psi_3)^3}{24 \hat{r}_3 (1 + \alpha_z r_3 \psi_3)^2} \right\} \right),
\]

where \( \hat{r}_1, \hat{r}_2 \) and \( \hat{r}_3 \) are defined in Lemma 3. \( \psi_1, \psi_2, \psi_3, r_1, r_1, r_3 \) are defined in Assumption 3. Suppose (a) Assumptions 1-3 hold; (b) \( \{\beta_k\} \) and \( \{\gamma_k\} \) are nonincreasing sequences such that \( \beta_1 \leq 1 \) and \( \gamma_1 \leq 1 \). Then
\[
\mathbb{E} \left[ \left\| \mathbf{x}_{k+1} - \mathbf{H}_{k+1}^x \right\|^2 \right] \leq \left( 1 - \frac{(\alpha_x r_1 \psi_1)^2}{2} \right) \mathbb{E} \left[ \left\| \mathbf{x}_k - \mathbf{H}_k^x \right\|^2 \right] + \frac{1 + \alpha_x r_1 \psi_1}{\alpha_x r_1 \psi_1} 3 \alpha_x^2 C_f \mathbb{E} \left[ \left\| \mathbf{z}_k \right\|_F^2 \right] \\
+ 12 \frac{1 + \alpha_x r_1 \psi_1}{\alpha_x r_1 \psi_1} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{x}_k - 1 \otimes \hat{x}_k \right\|^2 \right],
\]

and
\[
\mathbb{E} \left[ \left\| \mathbf{y}_{k+1} - \mathbf{H}_{k+1}^y \right\|^2 \right] \leq \left( 1 - \frac{(\alpha_y r_2 \psi_2)^2}{2} \right) \mathbb{E} \left[ \left\| \mathbf{y}_k - \mathbf{H}_k^y \right\|^2 \right] + 9 C_g \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \alpha_y^2 C_f \mathbb{E} \left[ \left\| \mathbf{z}_k \right\|_F^2 \right] \\
+ 36 C_g \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{x}_k - 1 \otimes \hat{x}_k \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{x}_k - \mathbf{H}_k^x \right\|^2 \right] \\
+ 12 \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{y}_k - 1 \otimes \hat{y}_k \right\|^2 \right] + 36 C_g \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \alpha_y^2 C_f \mathbb{E} \left[ \left\| \mathbf{y}_k \right\|_F^2 \right] \\
+ 12 \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \beta_2^2 \mathbb{E} \left[ \left\| \mathbf{G}_k - \mathbf{g}_k \right\|^2 \right] + 12 \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \beta_3^2 \nu \mathbb{E} \left[ \left\| \mathbf{g}_k \right\|^2 \right],
\]

and
\[
\mathbb{E} \left[ \left\| \mathbf{z}_{k+1} - \mathbf{H}_{k+1}^z \right\|^2 \right] \leq \left( 1 - \frac{(\alpha_z r_3 \psi_3)^2}{2} \right) \mathbb{E} \left[ \left\| \mathbf{z}_k - \mathbf{H}_k^z \right\|^2 \right] + 9 L_g^2 \frac{1 + \alpha_z r_3 \psi_3}{\alpha_z r_3 \psi_3} \alpha_z^2 C_f \mathbb{E} \left[ \left\| \mathbf{z}_k \right\|_F^2 \right] \\
+ 9 L_g^2 \frac{1 + \alpha_z r_3 \psi_3}{\alpha_z r_3 \psi_3} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{z}_k - 1 \otimes \hat{z}_k \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{z}_k - \mathbf{H}_k^z \right\|^2 \right] \\
+ 12 \frac{1 + \alpha_z r_3 \psi_3}{\alpha_z r_3 \psi_3} \alpha_w^2 \mathbb{E} \left[ \left\| \mathbf{z}_k - 1 \otimes \hat{z}_k \right\|^2 \right] + 36 L_g^2 \frac{1 + \alpha_z r_3 \psi_3}{\alpha_z r_3 \psi_3} \alpha_z^2 C_f \mathbb{E} \left[ \left\| \mathbf{y}_k \right\|_F^2 \right] \\
+ 12 \frac{1 + \alpha_z r_3 \psi_3}{\alpha_z r_3 \psi_3} \gamma_2^2 \mathbb{E} \left[ \left\| \mathbf{G}_k - \nabla \mathbf{g}_k \right\|_F^2 \right] + 12 \frac{1 + \alpha_z r_3 \psi_3}{\alpha_z r_3 \psi_3} \gamma_2^2 \nu \mathbb{E} \left[ \left\| \mathbf{g}_k \right\|^2 \right].
\]
Lemma 7. Denote
\[ \hat{c}_1 = \frac{1 - \rho_w^2}{48} C_0 \alpha_w^2, \quad \hat{c}_2 = \frac{1 - \rho_w^2}{48} L_2 \alpha_w^2, \quad \hat{c}_3 = \frac{1 - \rho_w^2}{72 C_0 \alpha_w^2}, \quad \hat{c}_4 = \frac{1 - \rho_w^2}{72 p \mu^2}, \]
\[ \hat{c}_5 = \frac{1 - \rho_w^2}{12 \alpha_x \alpha_y \alpha_w^2}, \quad \hat{c}_6 = \frac{1 - \rho_w^2}{144 C_0 \alpha_x \alpha_y \alpha_w^2}, \quad \hat{c}_7 = \frac{1 - \rho_w^2}{144 L_2 \alpha_x \alpha_y \alpha_w^2}, \]
\[ \hat{c}_8 = \left( \frac{1}{1 - \rho_w^2} C_f + \frac{1}{1 - \rho_w^2} C_f \hat{c}_1 + \frac{1}{1 - \rho_w^2} L_2 \alpha_x \alpha_y \alpha_w^2 \right) \left( \frac{1}{1 - \rho_w^2} C_f \hat{c}_6 + 36 C_0 \alpha_x \alpha_y \alpha_w^2 \right), \]
\[ \hat{d}_k = \max \left\{ \frac{2 + \rho_w^2}{3} \hat{c}_2, (1 - \beta_k)^2 + 6 \beta_k^2, 1 - \frac{(a_x r_1 \psi_1)^2}{4}, (1 - \gamma_k)^2 + 6 \gamma_k^2 + \frac{3 \rho_w^2 \alpha_k^2}{\hat{c}_4}, 1 - \frac{(a_y r_2 \psi_2)^2}{4}, 1 - \frac{(a_x r_3 \psi_3)^2}{4} \right\}, \]
\[ \hat{b}_k = 3 \rho_w^2 \alpha_k \beta_k + \left( \frac{1 + \rho_w^2}{1 - \rho_w^2} \hat{c}_4 V_g + 3 \hat{c}_3 \beta_k \gamma_k + 12 \frac{\alpha_y r_2 \psi_2}{\alpha_x r_3 \psi_3} V_g \right) \gamma_k \]
\[ \hat{V}_k = \hat{c}_4 \hat{c}_2 \left( \left\| \mathbf{y}_k - 1 \hat{c}_4 \right\| \right), \]
Suppose (a) Assumptions 1-3 hold, (b) \( \alpha_x, \alpha_y, \alpha_k, \beta_k, \gamma_k \) and \( a_k \) are positive and satisfy \( \alpha_x < \frac{1}{r_1}, \alpha_y < \frac{1}{r_2}, \alpha_k < \frac{1}{r_3}, \beta_k < 1, \gamma_k < 1, a_k < 1, \)
(c) \( \alpha_w \) is positive and satisfies
\[ 4 \frac{1 + \rho_w^2}{1 - \rho_w^2} \alpha_w^2 \leq \frac{1 - \rho_w^2}{6}, \quad \alpha_w \leq \min \left\{ \sqrt{\frac{(a_x r_1 \psi_1)^3}{24 r (1 + \alpha_x r_1 \psi_1)}}, \sqrt{\frac{(a_y r_2 \psi_2)^3}{48 r (1 + \alpha_y r_2 \psi_2)}}, \sqrt{\frac{(a_x r_3 \psi_3)^3}{48 r (1 + \alpha_x r_3 \psi_3)}} \right\}, \]
\[ 4 \alpha_w^2 \left( \frac{1 + \rho_w^2}{1 - \rho_w^2} \hat{c}_4 + 12 \left( 1 - \beta_k \right)^2 C_0 \hat{c}_1 \right) \left( \frac{1 + \rho_w^2}{1 - \rho_w^2} \hat{c}_4 + 12 \left( 1 - \gamma_k \right)^2 L_2 \hat{c}_1 \right) \left( \frac{1 + \rho_w^2}{1 - \rho_w^2} \hat{c}_4 + 18 \left( 1 - \beta_k \right)^2 C_0 \hat{c}_1 \right) \]
\[ + 18 \left( 1 - \gamma_k \right)^2 L_2 \hat{c}_1 \leq \frac{(a_x r_1 \psi_1)^2}{4}. \]
Then
\[ \hat{V}_{k+1} \leq \hat{a}_k \hat{V}_k + \hat{b}_k. \]

Similar to Lemma 3, Lemma 6 is a technical result, which characterizes a recursive inequality relation of the compression errors and the estimating errors of hybrid variance reduction method and the dynamic average consensus method.

We are ready to study the convergence rate of CD-ASCGD.
Theorem 2. Let $\alpha_k = \frac{a_1}{\sqrt{K}}$, $\beta_k = \frac{a_3}{\sqrt{K}}$, $\gamma_k = \frac{a_4}{\sqrt{K}}$. Under the conditions of Lemma 7,

$$\frac{1}{K} \sum_{k=1}^{K} \mathbb{E} \left[ \| \nabla h(x_{i,k}) \|^2 \right] \leq \frac{8 \mathbb{E} [h(\bar{x}_1)] - \mathbb{E} [h(\bar{x}_{K+1})]}{s_1 \sqrt{K}} + \frac{24pLCr_1Cg s_1}{n \sqrt{K}} + \frac{8 \tilde{c}_9 (\tilde{V}_1 + \tilde{b})}{K(1 - \bar{a}_k)}, \forall i \in V,$$

where $L = C_g L_f + C_f^{1/2} L_g$,

$$\tilde{b} = 3 n p \bar{c}_4 C_g s_1^2 + \left( 4 \frac{1 + \rho_1^2}{1 - \rho_1^2} \bar{c}_1 V_g + 3 \bar{c}_3 V_2 + 12 \frac{1}{\alpha_9 r_2^2 \psi_2} \bar{V}_g \right) s_2^2
+ \left( 4 \frac{1 + \rho_1^2}{1 - \rho_1^2} \bar{c}_2 V_g + 3 \bar{c}_4 V_2 + 12 \frac{1}{\alpha_2 r_3^3 \psi_3} \bar{V}_g \right) s_3^2,$$

$$\tilde{c}_9 = \max \left\{ 2 \left( \frac{C_1^1 L_f^2}{n} + \frac{C_1^2 L_f^2}{n} + \frac{L_f^2}{8} \right), \frac{2C_1^2 L_f^2}{n \min \{ \bar{c}_1, \bar{c}_3 \}}, \frac{(4C_f + 3LC_f) p}{2 \min \{ \bar{c}_2, \bar{c}_4 \} n} \right\}.$$

Proof. Similar to the analysis of [22] in the proof of Theorem 1 we have

$$\mathbb{E} [h(\bar{x}_{k+1})] \leq \mathbb{E} [h(\bar{x}_k)] - \frac{\alpha_k}{8} \mathbb{E} \left[ \| \nabla h(x_{i,k}) \|^2 \right] + 3 \frac{pL \alpha_0^2}{n} C_f C_g + \tilde{c}_9 \alpha_k \tilde{V}_k,$$

where

$$\tilde{c}_9 = \max \left\{ 2 \left( \frac{C_1^1 L_f^2}{n} + \frac{C_1^2 L_f^2}{n} + \frac{L_f^2}{8} \right), \frac{2C_1^2 L_f^2}{n \min \{ \bar{c}_1, \bar{c}_3 \}}, \frac{(4C_f + 3LC_f) p}{2 \min \{ \bar{c}_2, \bar{c}_4 \} n} \right\},$$

$\bar{c}_1, \ldots, \bar{c}_4$ are defined in (35), $\tilde{V}_k$ is defined in (37). Reordering the terms of (40) and summing over $k$ from 1 to $K$,

$$\sum_{k=1}^{K} \frac{\alpha_k}{8} \mathbb{E} \left[ \| \nabla h(x_{i,k}) \|^2 \right] \leq \mathbb{E} [h(\bar{x}_1)] - \mathbb{E} [h(\bar{x}_{K+1})] + \frac{3pLCr_1Cg s_1}{n} \sum_{k=1}^{K} \alpha_k^2 + \tilde{c}_9 \sum_{k=1}^{K} \alpha_k \tilde{V}_k. \quad (41)$$

Note that definitions of $\alpha_k$, $\beta_k$, $\gamma_k$ could guarantee $\bar{a}_k = \bar{a} < 1$\footnote{$\bar{a}$ is dependent on $K$.}, $\tilde{b}_k = \tilde{b}/K$ where $\tilde{b}$ is defined in (38). Then by Lemma 3

$$\tilde{V}_k \leq \tilde{a}_{k-1} \tilde{V}_{k-1} + b_{k-1} = \tilde{a} \tilde{V}_{k-1} + \tilde{a}/K \leq \cdots \leq \tilde{a}^{k-1} \tilde{V}_1 + (\tilde{a}/K)^{k-1} \tilde{V}_1 + \frac{\tilde{a}}{K(1 - \bar{a})} + \frac{\tilde{b}}{K(1 - \bar{a})}.$$ Substituting the above inequality in to (41) and multiplying both sides of (41) by $\frac{8}{s_1 \sqrt{K}}$,

$$\frac{1}{K} \sum_{k=1}^{K} \mathbb{E} \left[ \| \nabla h(x_{i,k}) \|^2 \right] \leq \frac{8 \mathbb{E} [h(\bar{x}_1)] - \mathbb{E} [h(\bar{x}_{K+1})]}{s_1 \sqrt{K}} + \frac{24pLCr_1Cg s_1}{n \sqrt{K}} + \frac{8 \tilde{c}_9 \sum_{k=1}^{K} \bar{a}^{k-1} \tilde{V}_1 + \frac{\tilde{b}}{K(1 - \bar{a})}}{K(1 - \bar{a})} \leq \frac{8 \mathbb{E} [h(\bar{x}_1)] - \mathbb{E} [h(\bar{x}_{K+1})]}{s_1 \sqrt{K}} + \frac{24pLCr_1Cg s_1}{n \sqrt{K}} + \frac{8 \tilde{c}_9 (\tilde{V}_1 + \tilde{b})}{K(1 - \bar{a})}. \quad (42)$$

The proof is complete. \qed
Similar with D-ASCGD, CD-ASCGD achieves the optimal convergence rate $O\left(1/\sqrt{K}\right)$. The only difference is the third term \( \frac{8\tilde{c}_b(V_1+\delta)}{K(1-\alpha)} \) in (42) and the third term \( \frac{8c_b(V_1+\delta)}{K(1-\alpha)} \) in (24). By the definition of $\tilde{a}$, $\tilde{V}_1$, $\bar{b}$, and the fact $\rho_w < 1$, the order of magnitude of $\frac{8\tilde{c}_b(V_1+\delta)}{K(1-\alpha)}$ is

$$O \left( \max \left\{ \frac{1}{1-\rho^2} \cdot \frac{1}{\alpha \cdot \sigma^2}, \frac{1}{\alpha \cdot \gamma^2}, \frac{1}{\alpha \cdot (\gamma+\delta)^2} \right\}, \frac{\sqrt{K}}{K} \right).$$

By the similar analysis, the order of magnitude of $\frac{8c_b(V_1+\delta)}{K(1-\alpha)}$ is

$$O \left( \max \left\{ \frac{1}{1-\rho^2}, \frac{\sqrt{K}}{K} \right\} \right).$$

It is easy to observe that the order of magnitude of the two terms is $O\left(1/\sqrt{K}\right)$.

5 Experimental results

We consider a multi-agent Markov Decision Processes problem arising in reinforcement learning [32]

$$\min_{x \in \mathbb{R}^d} \frac{1}{2|S|} \sum_{s=1}^{|S|} \left( \phi_s^T x - \frac{1}{|S|} \sum_{s'} \mathbb{E}_{s'} \left[ r_{s,s'} + \gamma \phi_{s'}^T x \right] \right)^2 + \frac{\lambda}{2} \|x\|^2,$$  
(43)

where $S$ is a finite state space, $\phi_s \in \mathbb{R}^d$ is the state feature, $r_{s,s'}$ denotes the random reward of transition from $s$ to $s'$ for agent $j$, $\gamma \in (0, 1)$ is a discount factor, $\lambda$ is the coefficient for the $l_2$-regular. In the setting of federated learning, each agent $j$ has access to its own data with heterogeneous random reward $r_{s,s'}$ for any $s, s' \in S$. Denote

$$g_j(x) = \left( x^T, \mathbb{E}_{s,j} \left[ r_{s,s'}^j + \gamma \phi_{s'}^T x \right] | s = 1 \right), \cdots, \mathbb{E}_{s,j} \left[ r_{s,s'}^j + \gamma \phi_{s'}^T x \right] | s = |S| \right)^T,$$

$$f_j(y) = \frac{1}{2|S|} \sum_{s=1}^{|S|} \left( \phi_s^T y(1 : d) - y(d + s) \right)^2 + \frac{\lambda}{2} \|y(1 : d)\|^2,$$

where $y(d)$ and $y(1 : d)$ are the $(d)$-th component and the first to $d$-th components of vector $y$ respectively. Then problem [43] can be reformulated as the form of problem [1].

Similar to [32], we set regular parameter $\lambda = 1$, the number of states $|S| = 100$. For any state $s \in S$, the feature $\phi_s \in [0, 1]^d$ and the mean of rewards $\bar{r}_{s,s'} \in [0, 1]$ are uniformly distributed. Moreover, the transition probabilities $P_{s,s'}$ are uniformly generated and are standardized such that $\sum_{s'} P_{s,s'} = 1$. In each simulation, we simulate a random transition from state $s$ to state $s'$ using the transition probability $P_{s,s'}$. generate a random reward $r_{s,s'}^i \sim N(\bar{r}_{s,s'}, 1)$ for agent $i$.

We compare D-ASCGD with centralized algorithm SCSC [1] and distributed algorithm DSBO [32] with stepsizes $\alpha_k = 0.01$, $\beta_k = 0.01$ and $\gamma_k = 0.01$. In each iteration, D-ASCGD and DSBO utilize 5 samples to calculate gradients and function values for each agent and SCSC
Figure 1: Evolutions of $\frac{1}{n} \sum_{j=1}^{n} (h(x_{j,k}) - h(x^*))$ w.r.t the number of iterations.

Figure 2: Evolutions of $\frac{1}{n} \sum_{j=1}^{n} \|\nabla h(x_{j,k})\|^2$ w.r.t to the number of iterations.
Figure 3: Evolutions of $\frac{1}{n} \sum_{j=1}^{n} (h(x_{j,k}) - h(x^*))$ and $\frac{1}{n} \sum_{j=1}^{n} \|\nabla h(x_{j,k})\|^2$ w.r.t to the number of iterations (ring network, n=24).

Figure 4: Evolutions of $\frac{1}{n} \sum_{j=1}^{n} (h(x_{j,k}) - h(x^*))$ w.r.t the number of transmitted bits.

utilizes 5n samples to calculate gradients and function value. We test D-ASCGD and DSBO over exponential networks and ring networks with varying network sizes $n = 6, 12, 24$. The weight matrices related to the exponential networks are generated by the way described in [34].
We record the performance on the average of the norm square of gradients $\frac{1}{n} \sum_{j=1}^{n} \|\nabla h(x_{j,k})\|^2$ and the averaged residual $\frac{1}{n} \sum_{j=1}^{n} (h(x_{j,k}) - h(x^*))$ in Figures 1 and 2, where the optimal solution $x^*$ is obtained by standard gradient descent method. Similar to [32], we run the simulations 10 times for the algorithms and report the average performance. We can observe from Figures 1 and 2 that the three algorithms converge fast and have the comparable residuals of optimal values and gradients. Note also that SCSC uses the information of global function, its performance is slightly better than the distributed algorithms D-ASCGD and DSBO. For the two distributed algorithms D-ASCGD and DSBO, they have the comparable convergence over the networks with different types and sizes, which matches the conclusion of Theorem 1.

For CD-ASCGD, we consider the following two compressors.

- **$l$-bits quantizer** [36]:

  $\mathcal{C}(x) = x_{\text{max}} \left( \frac{1}{2^{l-1}} \text{sign}(x) \odot \left[ \frac{2^{l-1} |x|}{x_{\text{max}}} + u \right] \right),$

  \( h(x) \) denotes the objective function of problem (43).
where \( \text{sign}(x) \) is the sign function, \( \odot \) is the Hadamard product, \( |x| \) is the element-wise absolute value of \( x \), and \( u \) is a random perturbation vector uniformly distributed in \([0, 1]^d\), \( x_{\text{max}} \) refers to the largest absolute value of the elements of \( x \). In the test, we choose \( l = 2, 4 \) and \( b = 64 \).

- Top-\( t \) sparsifier [2]:

\[
C(x) = \sum_{l=1}^{t} |x|_{i_l} e_{i_l},
\]

where \( \{e_1, \ldots, e_d\} \) is the standard basis of \( \mathbb{R}^d \) and \( i_1, \ldots, i_t \) are the indices of largest \( t \) coordinates in magnitude of \( x \). We consider the cases that \( t = d/2, 10 \) (\( d > 20 \)).

In Figure 3 we record the average performance on \( \frac{1}{n} \sum_{j=1}^{n} \left\| \nabla h(x_{j,k}) \right\|^2 \) and \( \frac{1}{n} \sum_{j=1}^{n} (h(x_{j,k}) - h(x^*)) \) with 10 times simulation. As shown in Figure 3, CD-ASCGD has comparable convergence speeds with D-ASCGD, which is consistent with our theoretical results. In Figures 4 and 5, we record the convergence performances of CD-ASCGD and D-ASCGD with respect to the number of bits transmitted between agents. It is easy to observe that CD-ASCGD transmits less bits than D-ASCGD, which becomes more obvious as the size of network increasing and the needed transmitted bits of compressors decreasing. Moreover, the CD-ASCGD needs to transmit more bits between agents with the increasing of the network size.

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References

[1] D. Alistarh, D. Grubic, J. Li, R. Tomioka, and M. Vojnovic, Qsgd: Communication-efficient sgd via gradient quantization and encoding, in Advances in Neural Information Processing Systems, vol. 30, Curran Associates, Inc., 2017.

[2] A. Beznosikov, S. Horváth, P. Richtárik, and M. Safaryan, On biased compression for distributed learning, arXiv preprint arXiv:2002.12410, (2020).

[3] L. Bottou, F. E. Curtis, and J. Nocedal, Optimization methods for large-scale machine learning, SIAM Review, 60 (2018), pp. 223–311.

[4] T. Chen, Y. Sun, and W. Yin, Solving stochastic compositional optimization is nearly as easy as solving stochastic optimization, IEEE Transactions on Signal Processing, 69 (2021), pp. 4937–4948.

[5] A. Cutkosky and F. Orabona, Momentum-based variance reduction in non-convex sgd, in Advances in Neural Information Processing Systems, vol. 32, Curran Associates, Inc., 2019.

[6] B. Dai, N. He, Y. Pan, B. Boots, and L. Song, Learning from Conditional Distributions via Dual Embeddings, in Proceedings of the 20th International Conference on Artificial Intelligence and Statistics, vol. 54, PMLR, 20–22 Apr 2017, pp. 1458–1467.

[7] B. Dai, A. Shaw, L. Li, L. Xiao, N. He, Z. Liu, J. Chen, and L. Song, SBEED: Convergent reinforcement learning with nonlinear function approximation, in Proceedings of the 35th International Conference on Machine Learning, vol. 80 of Proceedings of Machine Learning Research, PMLR, 10–15 Jul 2018, pp. 1125–1134.

[8] T. T. Doan, S. T. Maguluri, and J. Romberg, Convergence rates of distributed gradient methods under random quantization: A stochastic approximation approach, IEEE Transactions on Automatic Control, 66 (2021), pp. 4469–4484.

[9] ——, Fast convergence rates of distributed subgradient methods with adaptive quantization, IEEE Transactions on Automatic Control, 66 (2021), pp. 2191–2205.
[10] H. Gao and H. Huang, Fast training method for stochastic compositional optimization problems, in Advances in Neural Information Processing Systems, vol. 34, 2021.

[11] S. Ghadimi and G. Lan, Stochastic first- and zeroth-order methods for nonconvex stochastic programming, SIAM Journal on Optimization, 23 (2013), pp. 2341–2368.

[12] S. Ghadimi, A. Ruszczyński, and M. Wang, A single timescale stochastic approximation method for nested stochastic optimization, SIAM Journal on Optimization, 30 (2020), pp. 960–979.

[13] J. Hu, E. Zhou, and Q. Fan, Model-based annealing random search with stochastic averaging, ACM Transactions on Modeling and Computer Simulation (TOMACS), 24 (2014), pp. 1–23.

[14] Z. Huo, B. Gu, J. Liu, and H. Huang, Accelerated method for stochastic composition optimization with nonsmooth regularization, in Proceedings of the 32nd AAAI Conference on Artificial Intelligence, 2018, pp. 3287–3294.

[15] N. Ivkin, D. Rothchild, E. Ullah, V. Braer, V. Mo, and R. Arora, Communication-efficient distributed sgd with sketching, in Advances in Neural Information Processing Systems, vol. 32, Curran Associates, Inc., 2019.

[16] X. Li, L. Xie, and Y. Hong, Distributed aggregative optimization over multi-agent networks, IEEE Transactions on Automatic Control, 67 (2022), pp. 3165–3171.

[17] X. Lian, M. Wang, and J. Liu, Finite-sum Composition Optimization via Variance Reduced Gradient Descent, in Proceedings of the 20th International Conference on Artificial Intelligence and Statistics, vol. 54, PMLR, 20–22 Apr 2017, pp. 1159–1167.

[18] Y. Liao, Z. Li, K. Huang, and S. Pu, A compressed gradient tracking method for decentralized optimization with linear convergence, IEEE Transactions on Automatic Control, (2022), pp. 1–8.

[19] Y. Lin, S. Han, H. Mao, Y. Wang, and B. Dally, Deep gradient compression: Reducing the communication bandwidth for distributed training, in International Conference on Learning Representations, 2018.

[20] X. Liu, Y. Li, R. Wang, J. Tang, and M. Yan, Linear convergent decentralized optimization with compression, in International Conference on Learning Representations, 2021.

[21] R. Olfati-Saber and J. Shamma, Consensus filters for sensor networks and distributed sensor fusion, in Proceedings of the 44th IEEE Conference on Decision and Control, 2005, pp. 6698–6703.

[22] S. S. Ram, A. Nedić, and V. V. Veeravalli, A new class of distributed optimization algorithms: application to regression of distributed data, Optimization Methods and Software, 27 (2012), pp. 71–88.

[23] F. Seide, H. Fu, J. Droppo, G. Li, and D. Yu, 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns, in Fifteenth Annual Conference of the International Speech Communication Association, 2014.

[24] A. Shapiro, D. Dentcheva, and A. Ruszczyński, Lectures on stochastic programming. Modeling and theory., Philadelphia, PA:SIAM, 2009.

[25] S. Shen, L. Xu, J. Liu, J. Guo, and Q. Ling, Asynchronous stochastic composition optimization with variance reduction, arXiv preprint arXiv:1811.06396, (2018).

[26] A. T. Suresh, F. X. Yu, S. Kumar, and H. B. McMahan, Distributed mean estimation with limited communication, in Proceedings of the 34th International Conference on Machine Learning, vol. 70 of Proceedings of Machine Learning Research, PMLR, 06–11 Aug 2017, pp. 3329–3337.

[27] Z. Tang, S. Shi, X. Chu, W. Wang, and B. Li, Communication-efficient distributed deep learning: A comprehensive survey, arXiv preprint arXiv:2003.06307, (2020).

[28] M. Wang, E. X. Fang, and H. Liu, Stochastic compositional gradient descent: algorithms for minimizing compositions of expected-value functions, Mathematical Programming, 161 (2017), pp. 419–449.

[29] M. Wang, J. Liu, and E. Fang, Accelerating stochastic composition optimization, Journal of Machine Learning Research, 18 (2017), pp. 1–23.

[30] W. Wen, C. Xu, F. Yan, C. Wu, Y. Wang, Y. Chen, and H. Li, Terngrad: Ternary gradients to reduce communication in distributed deep learning, in Advances in Neural Information Processing Systems, 2018, pp. 3287–3294.
Proof. By the iteration (3),
\[
\|G_{k+1} - g_{k+1}\|^2 = \|(1 - \beta_k)(G_k - g_k) + (G_{k+1,k+1} - g_{k+1}) + (1 - \beta_k)(g_k - G_{k+1,k})\|^2 \\
= (1 - \beta_k)^2\|G_k - g_k\|^2 + \|G_{k+1,k+1} - g_{k+1}\|^2 (1 - \beta_k)(g_k - G_{k+1,k})
\]
\[
+ 2 \langle (1 - \beta_k)(G_k - g_k), (G_{k+1,k+1} - g_{k+1}) + (1 - \beta_k)(g_k - G_{k+1,k}) \rangle.
\] (44)

Note that
\[
E[(G_{k+1,k+1} - g_{k+1}) + (1 - \beta_k)(g_k - G_{k+1,k})] = 0,
\]
where
\[
\mathcal{F}_1 = \sigma(x_{i,1}, y_{i,1}, z_{i,1}, G_{i,1}, \hat{G}_{i,1} : i \in \mathcal{V}),
\]
\[
\mathcal{F}_k = \sigma(x_{i,1}, y_{i,1}, z_{i,1}, G_{i,1}, \hat{G}_{i,1}, \phi_{i,t}, \zeta_{i,t} : i \in \mathcal{V}, 2 \leq t \leq k | k \geq 2).
\] (45)

Then, taking expectation on both sides of (44),
\[
E[\|G_{k+1} - g_{k+1}\|^2] = (1 - \beta_k)^2 E[\|G_k - g_k\|^2] + E[\|G_{k+1,k+1} - g_{k+1}\|^2 + (1 - \beta_k)(g_k - G_{k+1,k})|^2].
\]
By Assumption 1(c) and (d),
\[
\mathbb{E} \left[ \left\| (G_{k+1,k+1} - g_{k+1}) + (1 - \beta_k)(g_k - G_{k+1,k}) \right\|^2 \right]
\]
\[
= \mathbb{E} \left[ \left\| (1 - \beta_k)(G_{k+1,k+1} - G_{k+1,k}) + \beta_k(G_{k+1,k+1} - g_{k+1}) + (1 - \beta_k)(g_k - g_{k+1}) \right\|^2 \right]
\]
\[
\leq 3(1 - \beta_k)^2 \mathbb{E} \left[ \left\| G_{k+1,k+1} - G_{k+1,k} \right\|^2 \right] + 3\beta_k \mathbb{E} \left[ \left\| G_{k+1,k+1} - g_{k+1} \right\|^2 \right] + 3(1 - \beta_k)^2 \mathbb{E} \left[ \left\| g_k - g_{k+1} \right\|^2 \right]
\]
\[
\leq 6(1 - \beta_k)^2 C_g \mathbb{E} \left[ \left\| x_{k+1} - x_k \right\|^2 \right] + 3\beta_k^2 V_g.
\]
Note that
\[
\mathbb{E} \left[ \left\| x_{k+1} - x_k \right\|^2 \right] = \mathbb{E} \left[ \left( \mathbf{W} - \mathbf{I}_n \right) (x_k - 1 \otimes \bar{x}_k) - \alpha_k U_{k+1} \right]^2
\]
\[
\leq 2 \left( 4\mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] + \alpha_k^2 \mathbb{E} \left[ \left\| U_{k+1} \right\|^2 \right] \right)
\]
\[
\leq 2 \left( 4\mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] + \alpha_k^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] \right),
\]
where the equality follows from the row stochasticity of \( \mathbf{W} \), the first inequality follows from the facts \( \left\| \mathbf{W} - \mathbf{I}_n \right\| \leq 2 \) and \( \left\| \mathbf{W} \right\| = 1 \), the second inequality follows from the definition of \( U_{k+1} \) and Assumption 1(c). Then
\[
\mathbb{E} \left[ \left\| G_{k+1} - g_{k+1} \right\|^2 \right] \leq (1 - \beta_k)^2 \mathbb{E} \left[ \left\| G_k - g_k \right\|^2 \right] + 6(1 - \beta_k)^2 C_g \mathbb{E} \left[ \left\| x_{k+1} - x_k \right\|^2 \right] + 3\beta_k^2 V_g
\]
\[
\leq (1 - \beta_k)^2 \mathbb{E} \left[ \left\| G_k - g_k \right\|^2 \right] + 48(1 - \beta_k)^2 C_g \mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right]
\]
\[
+ 12(1 - \beta_k)^2 C_g \alpha_k^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] + 3\beta_k^2 V_g,
\]
which verifies (5).

Inequality (3) could be obtained by the similar analysis of (5). The proof is complete.

**Proof of Lemma 2**

**Proof.** We first provide the upper bound of consensus error \( \mathbb{E} \left[ \left\| x_{k+1} - 1 \otimes \bar{x}_{k+1} \right\|^2 \right] \). By the definition of \( \bar{x}_{k+1} \) and the row stochasticity of weight matrix \( \mathbf{W} \),
\[
\bar{x}_{k+1} = \left( \frac{1}{n} \otimes \mathbf{I}_d \right) x_{k+1} = \left( \frac{1}{n} \otimes \mathbf{I}_d \right) \left( \mathbf{W} x_k - \alpha_k U_{k+1} \right) = \left( \frac{1}{n} \otimes \mathbf{I}_d \right) (x_k - \alpha_k U_{k+1}).
\]
Combining the above equality with the iteration (2),
\[
\left\| x_{k+1} - 1 \otimes \bar{x}_{k+1} \right\|^2
\]
\[
= \left\| \left( \mathbf{W} - \frac{11^T}{n} \otimes \mathbf{I}_d \right) x_k - \alpha_k \left( \mathbf{I}_{dn} - \frac{11^T}{n} \otimes \mathbf{I}_d \right) U_{k+1} \right\|^2
\]
\[
= \left\| \left( \mathbf{W} - \frac{11^T}{n} \otimes \mathbf{I}_d \right) (x_k - 1 \otimes \bar{x}_k) - \alpha_k \left( \mathbf{I}_{dn} - \frac{11^T}{n} \otimes \mathbf{I}_d \right) U_{k+1} \right\|^2
\]
\[
\leq (1 + \tau) \left\| \left( \mathbf{W} - \frac{11^T}{n} \otimes \mathbf{I}_d \right) (x_k - 1 \otimes \bar{x}_k) \right\|^2 + \left( 1 + \frac{1}{\tau} \right) \left\| \alpha_k \left( \mathbf{I}_{dn} - \frac{11^T}{n} \otimes \mathbf{I}_d \right) U_{k+1} \right\|^2
\]
\[
\leq (1 + \tau)^2 \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 + \left( 1 + \frac{1}{\tau} \right) \alpha_k^2 \left\| \mathbf{I}_n - \frac{11^T}{n} \right\|^2 \left\| U_{k+1} \right\|^2
\]
\[
= (1 + \tau)^2 \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 + \left( 1 + \frac{1}{\tau} \right) \alpha_k^2 \left\| U_{k+1} \right\|^2,
\]
(47)
where \( \rho := \| W - \frac{111}{n} I \| < 1 \) [Lemma 4], the second equality follows from the row stochasticity of \( W \), the first inequality follows from that \( (a+b)^2 \leq (1+\tau)a^2 + (1+1/\tau)b^2 \) for any \( \tau > 0 \), the last equality follows from the fact \( \| I_n - \frac{111}{n} I \| = 1 \). Taking expectation on both sides of inequality (47),

\[
\mathbb{E} \left[ \| x_{k+1} - 1 \otimes \bar{x}_{k+1} \|^2 \right] \leq (1 + \tau) \rho^2 \mathbb{E} \left[ \| x_k - 1 \otimes \bar{x}_k \|^2 \right] + \left( 1 + \frac{1}{\tau} \right) \alpha_k^2 \mathbb{E} \left[ \| U_{k+1} \|^2 \right]
\]

\[
\leq (1 + \tau) \rho^2 \mathbb{E} \left[ \| x_k - 1 \otimes \bar{x}_k \|^2 \right] + \left( 1 + \frac{1}{\tau} \right) \alpha_k^2 C_f \mathbb{E} \left[ \| z_k \|^2 \right],
\]

where the last inequality follows from the definition of \( U_{k+1} \) and Assumption 1(c). Setting \( \tau = \frac{1-\rho^2}{2\rho^2} \), we obtain (7).

Next, we provide the upper bound of consensus error \( \mathbb{E} \left[ \| y_{k+1} - 1 \otimes \bar{y}_{k+1} \|^2 \right] \). Similar to the analysis of (47), it follows from the definition of \( \bar{y}_k \) and iteration (4) that

\[
\| y_{k+1} - 1 \otimes \bar{y}_{k+1} \|^2 \leq (1 + \tau) \rho^2 \| y_k - 1 \otimes \bar{y}_k \|^2 + \left( 1 + \frac{1}{\tau} \right) \| G_{k+1} - G_k \|^2.
\]

(48)

By iteration (3),

\[
G_{k+1} - G_k = -\beta_k G_k + (1 - \beta_k) (G_{k+1,k+1} - G_{k+1,k}) + \beta_k G_{k+1,k+1}
\]

\[
= -\beta_k (G_k - g_k) + (1 - \beta_k) (G_{k+1,k+1} - G_{k+1,k}) + \beta_k (G_{k+1,k+1} - g_{k+1}) + \beta_k (g_{k+1} - g_k).
\]

Then

\[
\| y_{k+1} - 1 \otimes \bar{y}_{k+1} \|^2 \leq (1 + \tau) \rho^2 \| y_k - 1 \otimes \bar{y}_k \|^2 + 4 \left( 1 + \frac{1}{\tau} \right) \beta_k^2 \| G_k - g_k \|^2
\]

\[
+ (1 - \beta_k)^2 \| G_{k+1,k+1} - G_{k+1,k} \|^2 + \beta_k^2 \| G_{k+1,k+1} - g_k \|^2 + \beta_k^2 \| g_{k+1} - g_k \|^2.
\]

Taking expectation on both sides of the inequality above,

\[
\mathbb{E} \left[ \| y_{k+1} - 1 \otimes \bar{y}_{k+1} \|^2 \right] \leq (1 + \tau) \rho^2 \mathbb{E} \left[ \| y_k - 1 \otimes \bar{y}_k \|^2 \right] + 4 \left( 1 + \frac{1}{\tau} \right) \left( \beta_k^2 \mathbb{E} \left[ \| G_k - g_k \|^2 \right] + \| G_{k+1,k+1} - g_k \|^2 + \beta_k^2 \| g_{k+1} - g_k \|^2 \right)
\]

\[
\leq (1 + \tau) \rho^2 \mathbb{E} \left[ \| y_k - 1 \otimes \bar{y}_k \|^2 \right] + 4 \left( 1 + \frac{1}{\tau} \right) \left( \beta_k^2 \mathbb{E} \left[ \| G_k - g_k \|^2 \right] + \| G_{k+1,k+1} - g_k \|^2 + \beta_k^2 \| g_{k+1} - g_k \|^2 \right)
\]

\[\quad + C_g \mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right] + \beta_k^2 V_g \]

\[
\leq (1 + \tau) \rho^2 \mathbb{E} \left[ \| y_k - 1 \otimes \bar{y}_k \|^2 \right] + 4 \left( 1 + \frac{1}{\tau} \right) \left( \beta_k^2 \mathbb{E} \left[ \| G_k - g_k \|^2 \right] + \| g_{k+1} - g_k \|^2 \right)
\]

\[\quad + 2C_g \left( 4 \mathbb{E} \left[ \| x_k - 1 \otimes \bar{x}_k \|^2 \right] + \alpha_k^2 C_f \mathbb{E} \left[ \| z_k \|^2 \right] \right) + \beta_k^2 V_g ,
\]

(50)
where the second inequality follows from Assumption 1(c), (d) and the fact $\beta_k \geq 1$, the last inequality follows from (46). Setting $\tau = \frac{1 - \rho^2}{2 \rho^2}$, we obtain (8).

Inequality (9) could be obtained by the similar analysis of (8). The proof is complete.

**Proof of Lemma 3**

Proof. Multiplying $c_1, \cdots, c_4$ on both sides of (8), (9), (5) and (6) respectively, and then adding up them to (7), we have

$$V_{k+1} \leq \frac{2 + \rho^2}{3} E \left[ \| x_k - 1 \odot x_k \|^2 \right] + \frac{1 + \rho^2}{2} c_1 E \left[ \| y_k - 1 \odot y_k \|^2 \right] + \frac{1 + \rho^2}{2} c_2 E \left[ \| z_k - 1 \odot z_k \|^2 \right]$$

$$+ \left( (1 - \beta_k)^2 + 6 \frac{1 - \rho^2}{1 + \rho^2} \beta_k^2 \right) c_3 E \left[ \| G_k - g_k \|^2 \right] + \left( (1 - \gamma_k)^2 + 6pLg \frac{1 - \rho^2}{1 + \rho^2} \gamma_k^2 \right) c_4 E \left[ \| \hat{G}_k - \nabla g_k \|^2_F \right]$$

$$+ c_5 \alpha_k^2 E \left[ \| z_k \|^2_F \right] + 3 \left( 1 + \rho^2 \frac{1 - \rho^2}{1 + \rho^2} V_g c_1 + V_g c_3 \right) \beta_k^2 + 3 \left( 1 + \rho^2 \frac{1 - \rho^2}{1 + \rho^2} 2V'_g c_2 + V'_g c_4 \right) \gamma_k^2.$$ 

Note that

$$c_5 \alpha_k^2 E \left[ \| z_k \|^2_F \right] = c_5 \alpha_k^2 E \left[ \left\| z_k - 1 \odot z_k + 1 \odot z_k - 1 \odot \left( \frac{1}{n} \sum_{j=1}^n \nabla g_j(x_{j,k}) \right) + 1 \odot \left( \frac{1}{n} \sum_{j=1}^n \nabla g_j(x_{j,k}) \right) \right\|^2_F \right]$$

$$\leq 3c_5 \alpha_k^2 \left( E \left[ \| z_k - 1 \odot z_k \|^2_F \right] + nE \left[ \left\| z_k - 1 \odot \left( \frac{1}{n} \sum_{j=1}^n \nabla g_j(x_{j,k}) \right) \right\|^2_F \right] + \sum_{j=1}^n E \left[ \| \nabla g_j(x_{j,k}) \|^2_F \right] \right)$$

$$\leq 3c_5 \alpha_k^2 \left( E \left[ \| z_k - 1 \odot z_k \|^2_F \right] + E \left[ \left\| \hat{G}_k - \nabla g_k \right\|^2_F \right] + npC_g \right)$$

$$\leq 3pc_5 \alpha_k^2 \left( E \left[ \| z_k - 1 \odot z_k \|^2 \right] + E \left[ \left\| \hat{G}_k - \nabla g_k \right\|^2 \right] + nC_g \right).$$

Then

$$V_{k+1} \leq \frac{2 + \rho^2}{3} E \left[ \| x_k - 1 \odot x_k \|^2 \right] + \frac{1 + \rho^2}{2} c_1 E \left[ \| y_k - 1 \odot y_k \|^2 \right]$$

$$+ \left( 1 + \rho^2 + \frac{3pc_5 \alpha_k^2}{c_2} \right) c_2 E \left[ \| z_k - 1 \odot z_k \|^2 \right] + \left( (1 - \beta_k)^2 + 6 \frac{1 - \rho^2}{1 + \rho^2} \beta_k^2 \right) c_3 E \left[ \| G_k - g_k \|^2 \right]$$

$$+ \left( (1 - \gamma_k)^2 + 6pLg \frac{1 - \rho^2}{1 + \rho^2} \gamma_k^2 + \frac{3pc_5 \alpha_k^2}{c_4} \right) c_4 E \left[ \| \hat{G}_k - \nabla g_k \|^2_F \right]$$

$$+ 3pc_5 \alpha_k^2 + 3 \left( 1 + \rho^2 \frac{1 - \rho^2}{1 + \rho^2} V_g c_1 + V_g c_3 \right) \beta_k^2 + 3 \left( 1 + \rho^2 \frac{1 - \rho^2}{1 + \rho^2} 2V'_g c_2 + V'_g c_4 \right) \gamma_k^2$$

$$\leq a_k V_k + b_k.$$

The proof is complete.

**Proof of Lemma 4**

Proof. The proof is similar to Lemma 1.
**Proof of Lemma 5**

*Proof.* The proof is similar to Lemma 2. 

**Proof of Lemma 6**

*Proof.* We first provide the upper bound of compression error $\mathbb{E} \left[ \| x_{k+1} - H_{k+1}^F \|^2 \right]$. By the Line 4 of Algorithm 3,

$$H_{k+1}^F = H_k^F + \alpha_x C (x_k - H_k^F),$$

and then

$$\mathbb{E} \left[ \| x_{k+1} - H_{k+1}^F \|^2 \right] = \mathbb{E} \left[ \| x_{k+1} - x_k + x_k - (H_k^F + \alpha_x C (x_k - H_k^F)) \|^2 \right] \leq (1 + \tau) \mathbb{E} \left[ \| x_k - (H_k^F + \alpha_x C (x_k - H_k^F)) \|^2 \right] + (1 + \tau) \mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right]$$

$$= (1 + \tau) \mathbb{E} \left[ \alpha_x r_1 \left( x_k - H_k^F - \frac{C (x_k - H_k^F)}{r_1} \right) \right] + (1 - \alpha_x r_1) \mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right]$$

$$\leq (1 + \tau) \mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right] \leq (1 + \tau) \mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right] + \left( 1 + \frac{1}{\tau} \right) \mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right]$$

where the first inequality follows from that $(a + b)^2 \leq (1 + \tau)a^2 + (1 + 1/\tau)b^2$ for any $\tau > 0$, the second inequality follows from the convexity of $\| \cdot \|^2$ and the last inequality follows from Assumption 3. For the term $\mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right]$ on the right hand side of above inequality,

$$\mathbb{E} \left[ \| x_{k+1} - x_k \|^2 \right] = \mathbb{E} \left[ \left\| \alpha_w \left( \bar{W} - I_{nd} \right) (x_k - 1 \otimes \bar{x}_k) - \alpha_k U_{k+1} + \alpha_w \left( I_{nd} - \bar{W} \right) E_k^F \right\|^2 \right] \leq 3 \left\{ 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] + \alpha_k^2 \mathbb{E} \left[ \left\| U_{k+1} \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| E_{k+1} \right\|^2 \right] \right\} \leq 3 \left\{ 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - 1 \otimes \bar{x}_k \right\|^2 \right] + \alpha_k^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - \bar{H}_k^F \right\|^2 \right] \right\},$$

where the first equality follows from [25] and the row stochasticity of $W$, the first inequality follows from the fact $\| W - I_n \| \leq 2$, the second inequality follows from Assumption 3(c), the fact $E_{k+1}^F = x_k - H_k^F - C (x_k - H_k^F)$ and [28]. Substituting the above inequality into (51) and
setting $\tau = \alpha_y r_1 \psi_1$,

$$
\mathbb{E} \left[ \left\| x_{k+1} - H_{k+1}^x \right\|^2 \right] \leq \left( 1 - \left( \alpha_x r_1 \psi_1 \right)^2 + \frac{1 + \alpha_x r_1 \psi_1}{\alpha_x r_1 \psi_1} \right) \mathbb{E} \left[ \left\| x_k - H_k^x \right\|^2 \right] + \frac{1 + \alpha_x r_1 \psi_1}{\alpha_x r_1 \psi_1} \left( 12 \alpha_w x_2 \psi_1 \mathbb{E} \left[ \left\| x_k - 1 \times \bar{x}_k \right\|^2 \right] + 3 \alpha_w^2 \alpha_g C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] \right).
$$

Noting that $\alpha_w^2 \leq \frac{(\alpha_x r_1 \psi_1)^3}{24 r_1^2 (1 + \alpha_x r_1 \psi_1)}$, (32) holds.

Next, we provide the upper bound of compression error $\mathbb{E} \left[ \left\| y_{k+1} - H_{k+1}^y \right\|^2 \right]$. By the Line 4 of Algorithm 3,

$$
H_{k+1}^y = H_k^y + \alpha_y C \left( y_k - H_k^y \right),
$$

and then

$$
\mathbb{E} \left[ \left\| y_{k+1} - H_{k+1}^y \right\|^2 \right] = \mathbb{E} \left[ \left\| y_{k+1} - y_k + y_k - \left( H_k^y + \alpha_y C \left( y_k - H_k^y \right) \right) \right\|^2 \right]
$$

$$
\leq \left( 1 + \tau \right) \mathbb{E} \left[ \left\| y_k - \left( H_k^y + \alpha_y C \left( y_k - H_k^y \right) \right) \right\|^2 \right] + \left( 1 + \frac{1}{\tau} \right) \mathbb{E} \left[ \left\| y_{k+1} - y_k \right\|^2 \right]
$$

$$
= \left( 1 + \tau \right) \mathbb{E} \left[ \left\| \alpha_y r_1 \left( y_k - H_k^y \right) - \frac{\alpha_g C_y \left( y_k - H_k^y \right)}{r_1} \right\|^2 \right] + \left( 1 + \frac{1}{\tau} \right) \mathbb{E} \left[ \left\| y_{k+1} - y_k \right\|^2 \right]
$$

$$
\leq \left( 1 + \tau \right) \mathbb{E} \left[ \left\| \alpha_y r_1 \left( y_k - H_k^y \right) - \frac{\alpha_g C_y \left( y_k - H_k^y \right)}{r_1} \right\|^2 \right] + \left( 1 - \alpha_y r_1 \psi_1 \right) \mathbb{E} \left[ \left\| y_k - H_k^y \right\|^2 \right] + \left( 1 + \frac{1}{\tau} \right) \mathbb{E} \left[ \left\| y_{k+1} - y_k \right\|^2 \right],
$$

(53)

where the first inequality follows from that $(a + b)^2 \leq (1 + \tau) a^2 + (1 + 1/\tau) b^2$ for any $\tau > 0$, the second inequality follows from the convexity of $\left\| \cdot \right\|^2$ and the last inequality follows from Assumption 3. Similar to (52),

$$
\mathbb{E} \left[ \left\| y_{k+1} - y_k \right\|^2 \right] \leq 3 \left( 4 \alpha_w^2 \mathbb{E} \left[ \left\| y_k - 1 \times \bar{y}_k \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| y_k - H_k^x \right\|^2 \right] \right) + 9 C_g \left( 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - 1 \times \bar{x}_k \right\|^2 \right] + \alpha_w^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - H_k^x \right\|^2 \right] \right),
$$

(54)

Substituting (54) into (53) and setting $\tau = \alpha_y r_2 \psi_2$,

$$
\mathbb{E} \left[ \left\| y_{k+1} - H_{k+1}^y \right\|^2 \right] \leq \left( 1 - \left( \alpha_y r_2 \psi_2 \right)^2 + \frac{1 + \alpha_y r_1 \psi_1}{\alpha_y r_1 \psi_1} \right) \mathbb{E} \left[ \left\| y_k - H_k^y \right\|^2 \right] + 9 \alpha_g \left( \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - 1 \times \bar{x}_k \right\|^2 \right] + 4 \alpha_w^2 \mathbb{E} \left[ \left\| x_k - H_k^x \right\|^2 \right] \right)
$$

$$
+ 12 \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \alpha_w^2 \mathbb{E} \left[ \left\| y_k - 1 \times \bar{y}_k \right\|^2 \right] + 9 \alpha_g \left( \frac{1 + \alpha_y r_2 \psi_2}{\alpha_y r_2 \psi_2} \alpha_w^2 C_f \mathbb{E} \left[ \left\| z_k \right\|^2 \right] \right),
$$

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Noting that $\alpha_w^2 \leq \frac{(\alpha_y r_2 \psi_2)^3}{24r_2^2 (1 + \alpha_y r_2 \psi_2)},$ (33) holds.

Inequality (34) could be obtained by the similar analysis of (33).

Proof of Lemma 7

Proof. The proof is similar to Lemma 3.
Here we present CD-ASCGD method from each agent’s perspective.

**Algorithm 4** D-ASCGD method from agents’ view:

**Input:** initial values $x_{i,1}, H_{i,1}^x \in \mathbb{R}^d$, $H_{i,1}^y \in \mathbb{R}^p$, $H_{i,1}^z \in \mathbb{R}^{d \times p}$, $y_{i,1} = G_{i,1} \in \mathbb{R}^p$, $z_{i,1} = \hat{G}_{i,1} \in \mathbb{R}^{d \times p}$; stepsizes $\alpha_k > 0$, $\beta_k > 0$, $\gamma_k > 0$; scaling parameters $\alpha_w \in (0, 1)$; nonnegative weight matrix $W = \{w_{ij}\}_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$

1. $H_{i,1}^{x,w} = \sum_{j=1}^n w_{ij} H_{j,1}^x$, $H_{i,1}^{y,w} = \sum_{j=1}^n w_{ij} H_{j,1}^y$, $H_{i,1}^{z,w} = \sum_{j=1}^n w_{ij} H_{j,1}^z$

2. for $k = 1, 2, \ldots$ do

3. $q_{i,k}^x = \text{Compress} \left( x_{i,k} - H_{i,k}^x \right)$ \quad $\triangleright$ Compression

4. $\tilde{x}_{i,k} = H_{i,k}^x + q_{i,k}^x$

5. $\tilde{x}_{i,k}^w = H_{i,k}^{x,w} + \sum_{j=1}^n w_{ij} q_{j,k}^x$ \quad $\triangleright$ Communication

6. $H_{i,k+1}^{x,w} = (1 - \alpha_k) H_{i,k+1}^{x,w} + \alpha_k \tilde{x}_{i,k}$

7. $H_{i,k+1}^{y,w} = (1 - \alpha_k) H_{i,k+1}^{y,w} + \alpha_k \tilde{y}_{i,k}$

8. $q_{i,k}^y = \text{Compress} \left( y_{i,k} - H_{i,k}^y \right)$ \quad $\triangleright$ Compression

9. $\tilde{y}_{i,k} = H_{i,k}^y + q_{i,k}^y$

10. $\tilde{y}_{i,k}^w = H_{i,k}^{y,w} + \sum_{j=1}^n w_{ij} q_{j,k}^y$ \quad $\triangleright$ Communication

11. $H_{i,k+1}^{y,w} = (1 - \alpha_k) H_{i,k+1}^{y,w} + \alpha_k \tilde{y}_{i,k}$

12. $H_{i,k+1}^{z,w} = (1 - \alpha_k) H_{i,k+1}^{z,w} + \alpha_k \tilde{z}_{i,k}$

13. $q_{i,k}^z = \text{Compress} \left( z_{i,k} - H_{i,k}^z \right)$ \quad $\triangleright$ Compression

14. $\tilde{z}_{i,k} = H_{i,k}^z + q_{i,k}^z$

15. $\tilde{z}_{i,k}^w = H_{i,k}^{z,w} + \sum_{j=1}^n w_{ij} q_{j,k}^z$ \quad $\triangleright$ Communication

16. $H_{i,k+1}^{z,w} = (1 - \alpha_k) H_{i,k+1}^{z,w} + \alpha_k \tilde{z}_{i,k}$

17. Draw $\phi_{i,k+1} \overset{iid}{\sim} P_{\phi_i}$, $\zeta_{i,k+1} \overset{iid}{\sim} P_{\zeta_i}$, and compute function values $G_i(x_{i,k}; \phi_{i,k+1})$, and gradients $\nabla G_i(x_{i,k}; \zeta_{i,k+1})$, $\nabla G_i(x_{i,k}; \phi_{i,k+1})$ and $\nabla G_i(x_{i,k+1}; \phi_{i,k+1})$

18. $x_{i,k+1} = x_{i,k} - \alpha_w (\tilde{x}_{i,k} - \tilde{x}_{i,k}^w) - \alpha_k z_{i,k} \nabla F_i(y_{i,k}; \zeta_{i,k+1})$

19. $G_{i,k+1} = (1 - \beta_k) (G_{i,k} - G_i(x_{i,k}; \phi_{i,k+1})) + G_i(x_{i,k+1}; \phi_{i,k+1})$

20. $\hat{G}_{i,k+1} = (1 - \gamma_k) \left( \hat{G}_{i,k} - \nabla G_i(x_{i,k}; \phi_{i,k+1}) \right) + \nabla G_i(x_{i,k+1}; \phi_{i,k+1})$

21. $y_{i,k+1} = y_{i,k} - \alpha_w (\tilde{y}_{i,k} - \tilde{y}_{i,k}^w) + G_{i,k+1} - G_{i,k}$

22. $z_{i,k+1} = z_{i,k} - \alpha_w (\tilde{z}_{i,k} - \tilde{z}_{i,k}^w) + G_{i,k+1} - \hat{G}_{i,k}$

23. end for