Abstract

Upper Confidence Bound (UCB) method is arguably the most celebrated one used in online decision making with partial information feedback. Existing techniques for constructing confidence bounds are typically built upon various concentration inequalities, which thus lead to over-exploration. In this paper, we propose a non-parametric and data-dependent UCB algorithm based on the multiplier bootstrap. To improve its finite sample performance, we further incorporate second-order correction into the above construction. In theory, we derive both problem-dependent and problem-independent regret bounds for multi-armed bandits with symmetric rewards under a much weaker tail assumption than the standard sub-Gaussianity. Numerical results demonstrate significant regret reductions by our method, in comparison with several baselines in a range of multi-armed and linear bandit problems.

1 Introduction

In artificial intelligence, learning to make decisions online plays a critical role in many fields, such as personalized news recommendation [Li et al., 2010a], robotics [Kober et al., 2013] and the game of Go [Silver et al., 2016]. To learn to make optimal decisions as soon as possible, the decision-makers must carefully design an algorithm to balance the trade-off between the exploration and exploitation [Sutton and Barto, 2018, Lattimore and Szepesvári, 2018]. Over-exploration could be expensive and unethical in practice, e.g., medical decision making [Bastani and Bayati, 2015, Bastani et al., 2017, Bird et al., 2016]. On the other hand, insufficient exploration tends to make an algorithm stuck at a sub-optimal solution. The delicate design of exploration methods stands in the heart of online learning and decision making.

Upper Confidence Bound (UCB) [Auer, 2002, Auer et al., 2002, Dani et al., 2008, Li et al., 2010b, Abbasi-Yadkori et al., 2011] is a class of highly effective algorithms in dealing with the exploration-exploitation trade-off in bandits and reinforcement learning. The tightness of confidence bound, as is known, is the key ingredient to achieve the optimal degree of explorations. To the best of our knowledge, nearly all the existing works construct confidence bounds based on various concentration inequalities, e.g. Hoeffding-type [Auer et al., 2002], empirical Bernstein type [Mnih et al., 2008] or self-normalized type [Abbasi-Yadkori et al., 2011]. Those concentration-based confidence bounds, however, are typically conservative since they are data-independent. Concentration inequalities only exploit tail information, e.g., bounded or sub-Gaussian, rather than the whole distribution.
knowledge. In general, the loose constant factor may result in confidence bounds that are too wide to be informative [Russo and Van Roy, 2014].

In this paper, we propose a non-parametric and data-dependent UCB algorithm based on the multiplier bootstrap [Rubin, 1981, Wu et al., 1986, Arlot et al., 2010, Chernozhukov et al., 2014, Spokoiny et al., 2015], called bootstrapped UCB. The principle is to use the multiplier bootstrapped quantile as the confidence bound to enforce the exploration. Inspired by recent advances on non-asymptotic guarantee and non-asymptotic inference such as [Arlot et al., 2010, Chernozhukov et al., 2014, Spokoiny et al., 2015, Yang et al., 2017], we develop an explicit second-order correction for the multiplier bootstrapped quantile that ensures the non-asymptotic validity. Our algorithm is easy to implement and has the potential to be generalized to more complicated models such as structured contextual bandits.

In theory, we develop both problem-dependent and problem-independent regret bounds for multi-armed bandits with symmetric rewards under a much weaker tail assumption, i.e., sub-Weibull distribution, than the classical sub-Gaussianity. In this case, it is proven that the mean estimator can still achieve the same problem-independent regret bound as the one under the sub-Gaussian assumption. Note that our result does not rely on other sophisticated approaches such as median-of-means or Catoni’s M-estimator in [Bubeck et al., 2013]. A key technical tool we propose is a new concentration inequality for the sum of sub-Weibull random variables. Empirically, we evaluate our method in several multi-armed and linear bandit models. When the exact posterior is unavailable or the noise variance is mis-specified, the bootstrapped UCB demonstrates superior performance over variants of Thompson sampling and concentration-based UCB due to its non-parametric and data-dependent nature.

Recently, an increasing number of works [Elmachtoub et al., 2017, Osband et al., 2016, Tang et al., 2015, Eckles and Kaptein, 2014] study bootstrap methods for multi-armed and contextual bandits as an alternative to Thompson sampling. Most treat the bootstrap just as a way to randomize historical data (without any theoretical guarantee). One exception is [Kveton et al., 2018] who derive a regret bound for Bernoulli bandit by adding pseudo observations. However, their method cannot be easily extended to unbounded cases, and their analyses heavily limit to the Bernoulli assumption. In contrast, our method applies to a broader class of bandit models with rigorous regret analysis.

The rest of the paper is organized as follows. Section 2 introduces the basic setup and our bootstrapped UCB algorithm. Section 3 provides the regret analysis and Section 4 conducts several experiments.

Notations. Throughout the paper, we denote \(P_w(\cdot), E_w(\cdot)\) as the probability and expectation operator with respect to the distribution of the vector \(w\) only, conditioning on other random variables. We use similar notations for \(P_y(\cdot), E_y(\cdot)\) with respect to \(y\) only. \([n]\) means the set \(\{1, 2, \ldots, n\}\). We denote boldface lower letters (e.g. \(x, y\)) as a vector. For a set \(E\), we define its complement as \(E^c\).

### 2 Bootstrapped UCB

**Problem setup.** As a fruit fly, we illustrate our idea on the stochastic multi-armed bandit problem [Lai and Robbins, 1985, Lattimore and Szepesvári, 2018]. In detail, the decision-makers interact with an environment for \(T\) rounds. In round \(t \in [T]\), the decision-makers pull an arm \(I_t \in [K]\) and observes its reward \(y_{I_t}\), which is drawn from a distribution associated with the arm \(I_t\), denoted by \(P_{I_t}\) with an unknown mean \(\mu_{I_t}\). Without loss of generality, we assume arm 1 is the optimal arm, that is, \(\mu_1 = \max_{k \in [K]} \mu_k\). In multi-armed bandit problems, the objective is to minimize the expected cumulative regret, defined as,

\[
R(T) = T \mu_1 - E \left[ \sum_{t=1}^{T} y_{I_t} \right] = \sum_{k=2}^{K} \Delta_k E \left[ \sum_{t=1}^{T} I\{I_t = k\} \right],
\]

(2.1)

where \(\Delta_k = \mu_1 - \mu_k\) is the sub-optimality gap for arm \(k\), and \(I\{\cdot\}\) is an indicator function. Here, the second equality is from the regret decomposition Lemma (Lemma 4.5 in [Lattimore and Szepesvári, 2018]).
We call an upper bound of $R(T)$ problem-independent if the bound only depends on the distributional assumption and not on the specific bandit problem, say the gap $\Delta_k$.

**Upper Confidence Bound.** The upper confidence bound (UCB) algorithm [Auer et al., 2002] is based on the principle of optimism in the face of uncertainty. The key idea is to act as if the environment (parameterized by $\mu_k$ in multi-armed bandits) is as nice as plausibly possible. Concretely, a plausible environment refers to an upper confidence bound $G(y_n, 1 - \alpha)$ for the true mean $\mu$, of the form

$$G(y_n, 1 - \alpha) = \{ x \in \mathbb{R}, x - \bar{y}_n \leq h_\alpha(y_n) \},$$

(2.2)

where $y_n = (y_1, \ldots, y_n)^\top$ is the sample vector, $\bar{y}_n$ is the empirical mean, $\alpha \in (0, 1)$ is the confidence level, and $h_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a threshold that could be either data-dependent or data-independent.

**Definition 2.1.** We define $G(y_n, 1 - \alpha)$ as a non-asymptotic upper confidence bound if for any sample size $n \geq 1$, the following inequality holds

$$\mathbb{P}\left( \mu \in G(y_n, 1 - \alpha) \right) \geq 1 - \alpha.$$  

(2.3)

In bandit problems, a non-asymptotic control on the confidence level is more commonly used. This is rather different from the asymptotic validity of confidence bound in statistics literature [Casella and Berger, 2002].

A generic UCB algorithm will select the action based on its UCB index $\bar{y}_n + h_\alpha(y_n)$ for different arms. As is well known, the sharper the threshold is, the better exploration and exploitation trade-off one can achieve [Lattimore and Szepesvári, 2018]. By the definition of quantile, the sharpest threshold in (2.2) is the $(1 - \alpha)$-quantile of the distribution of $\bar{y}_n - \mu$. However, this quantile relies on the knowledge of the exact reward distribution and is therefore itself unknown. To evaluate this value, we construct a data-dependent confidence bound based on the multiplier bootstrap.

### 2.1 Confidence Bound Based on Multiplier Bootstrap

**Multiplier Bootstrap.** Multiplier bootstrap is a fast and easy-to-implement alternative to the standard bootstrap, and has been successfully applied in various statistical contexts [Arlot et al., 2010, Chernozhukov et al., 2014, Spokoiny et al., 2015]. Its goal is to approximate the distribution of the target statistic by reweighing its summands with random multipliers independent of the data. For instance, in a mean estimation problem, we define a multiplier bootstrapped estimator of the target statistic by reweighing its summands with random multipliers independent of the data. For instance, in a mean estimation problem, we define a multiplier bootstrapped estimator $\bar{y}_n + h_\alpha(y_n)$ where $\bar{y}_n$ is the empirical mean, $\alpha \in (0, 1)$ is the confidence level, and $h_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a threshold that could be either data-dependent or data-independent.

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- **Efron’s bootstrap weights.** $(w_1, \ldots, w_n)$ is a multinomial random vector with parameters $(n; n^{-1}, \ldots, n^{-1})$. This is the standard nonparametric bootstrap [Efron, 1982].
- **Gaussian weights.** $w_i$’s are i.i.d standard Gaussian random variables. This is closely related to Gaussian approximation in statistics [Chernozhukov et al., 2014].
- **Rademacher weights.** $w_i$’s are i.i.d Rademacher variables. This is closely related to symmetrization in learning theory.

The bootstrap principle suggests that the $(1 - \alpha)$-quantile of the distribution of $n^{-1} \sum_{i=1}^n w_i(y_i - \bar{y}_n)$ conditionally on $y_n$, could be used to approximate the $(1 - \alpha)$-quantile of the distribution of $\bar{y}_n - \mu$. As the first building block, the multiplier bootstrapped quantile is defined as,

$$q_\alpha(y_n - \bar{y}_n) := \inf \left\{ x \in \mathbb{R} \mid \mathbb{P}_w \left( \frac{1}{n} \sum_{i=1}^n w_i(y_i - \bar{y}_n) > x \right) \leq \alpha \right\}.$$  

(2.4)

The question is whether $q_\alpha(y_n - \bar{y}_n)$ is a valid threshold for any sample size $n \geq 1$. 

3
2.2 Second-order Correction

Most statistical theories guarantee the asymptotic validity of \( q_\alpha(y_n - \bar{y}_n) \) by the multiplier central limit theorem [Van der Vaart, 2000]. However, we show that such a claim is valid non-asymptotically at the cost of adding a second-order correction. Next theorem rigorously characterizes this phenomenon under a symmetric assumption on the reward. Moreover, in Section A in the supplement, we show that without the second-order correction, a naive bootstrapped UCB will result in linear regret.

**Theorem 2.2** (Non-asymptotic Second-order Correction). Suppose \( \{y_i\}_{i=1}^n \) are i.i.d symmetric random variables with respect to its mean \( \mu \), and the bootstrap weights \( \{w_i\}_{i=1}^n \) are i.i.d Rademacher random variables. For two arbitrary parameters \( \alpha, \delta \in (0, 1) \), the following inequality holds for any sample size \( n \geq 1 \),

\[
P_y \left( \bar{y}_n - \mu \geq q_{\alpha(1-\delta)}(y_n - \bar{y}_n) + \sqrt{\frac{\log(2/\alpha\delta)}{n}} \phi(y_n) \right) \leq 2\alpha, \tag{2.5}
\]

where \( \phi(y_n) \) is a non-negative function satisfying \( P_y(|\bar{y}_n - \mu| \geq \phi(y_n)) \leq \alpha \).

The detailed proof is deferred to Section B.1 in the supplement. In (2.5), the bootstrapped threshold may be interpreted as a main term, i.e., \( q_{\alpha(1-\delta)}(y_n - \bar{y}_n) \) (at a shrunk confidence level), plus a second-order correction term, i.e., \( (\log(2/\alpha\delta)/n)^{1/2} \phi(y_n) \). The latter is added to guarantee the non-asymptotic validity of the bootstrapped threshold. In the above, \( \phi(y_n) \) could be any preliminary upper bound on \( \bar{y}_n - \mu \). Hence, Theorem 2.2 transforms a possibly coarse prior bound \( \phi(y_n) \) on quantiles into a more accurate version that is based on a main term estimated by multiplier bootstrap plus a second-order correction term based on \( \phi(y_n) \) multiplied by a \( O(n^{-1/2}) \) factor.

**Remark 2.3** (Choice of \( \phi(y_n) \)). If \( \{y_i\}_{i=1}^n \) are independent 1-sub-Gaussian random variables, a natural choice of \( \phi(y_n) \) is \( (2 \log(1/\alpha)/n)^{1/2} \) by Hoeffding’s inequality (Lemma 2). Plugging it into (2.5) and letting \( \delta = 1/2 \), the bootstrapped threshold in (2.5) becomes

\[
\frac{q_{\alpha/4}(y_n - \bar{y}_n)}{\text{main term}} + \frac{2 \log(8/\alpha)}{n} \text{ second order correction}. \tag{2.6}
\]

Lemma B.3 in the supplement shows that the main term is of order at least \( O(n^{-1/2}) \) as \( n \) grows, which implies the second order correction is just a remainder term. We emphasize that the reminder term is obviously not sharp and will be sharpened as a future work.

**Remark 2.4.** Existing works on UCB-type algorithms typically utilized various concentration inequalities, e.g. Hoeffding’s inequality [Auer et al., 2002] or empirical Bernstein’s inequality [Mnih et al., 2008], to find a valid threshold \( h_\alpha(y_n) \). However, they are not data-dependent and only use the tail information, rather than fully exploit the whole distribution knowledge. This is typically conservative, and leads to over-exploration.

**Remark 2.5.** Empirical KL-UCB [Cappé et al., 2013] used empirical likelihood to build confidence intervals for general distributions that have support in \([0, 1]\). Although empirical KL-UCB is also data-dependent, our proposed method is from a very different non-parametric perspective and uses different tools by bootstrap. In practice, resampling tends to be more efficient computationally, without solving a convex optimization each round like empirical KL-UCB. Moreover, our method can work with unbounded rewards and we believe it is easier to generalize to structured bandits, e.g. linear bandit.

In Figure 1, we compare different approaches to calculate 95\% confidence bound for the population mean based on samples from a truncated-normal distribution. When the sample size is extremely small (\( \leq 10 \)), the naive bootstrap (without any correction) cannot output a valid threshold since the bootstrapped quantile is smaller than the true 95\% quantile. This confirms the necessity of the
second-order correction. When the sample size increases, our bootstrapped threshold converges to
the truth rapidly. This confirms the correction term is just a small remainder term. Additionally, the
bootstrapped threshold is shown to be sharper than Hoeffding’s bound and empirical Bernstein bound
when sample size is large (see the right panel of Figure 1).

Figure 1: 95% confidence bound of the sample mean.

2.3 Main Algorithm: Bootstrapped UCB

Based on the above theoretical findings, we conclude that bootstrapped UCB will select the arm
according to its UCB index defined as below:

\[
\text{UCB}_k(t) = \bar{y}_{n_k,t} + \alpha (1 - \delta) (y_{n_k,t} - \bar{y}_{n_k,t}) + \sqrt{\frac{\log(2/\alpha \delta)}{n_k,t}} \varphi(y_{n_k,t}),
\]

where \(n_k,t\) is the number of pulls for arm \(k\) until time \(t\). Practically, we may use Monte Carlo
quantile approximation to get an approximated bootstrapped quantile \(\tilde{q}_{\alpha(1-\delta)}(y_{n_k,t} - \bar{y}_{n_k,t}, w^B)\)
and corresponding theorem for the control of the approximation of the bootstrapped quantile is also
derived (see Section D in the supplement for details). The algorithm is summarized in Algorithm 1.

The computational complexity at step \(t\) is \(\tilde{O}(Bt) \leq \tilde{O}(BT)\). Comparing with vanilla UCB, the extra
\(Bt\) is due to resampling. In practice, the choice of \(B\) is seldom treated as a tuning parameter, but
usually determined by the available computational resource.

Algorithm 1 Bootstrapped UCB

**Input:** the number of bootstrap repetitions \(B\), hyper-parameter \(\delta\).

**for** \( t = 1 \) to \( K \) \n
Pull each arm once to initialize the algorithm.

**end**

**for** \( t = K + 1 \) to \( T \) \n
Set confidence level \(\alpha = 1/(t + 1)\).
Calculate the bootstrapped quantile \(\tilde{q}_{\alpha(1-\delta)}(y_{n_k,t} - \bar{y}_{n_k,t}, w^B)\).
Pull the arm

\[ I_t = \arg\max_{k \in [K]} (\bar{y}_{n_k,t} + \tilde{q}_{\alpha(1-\delta)}(y_{n_k,t} - \bar{y}_{n_k,t}, w^B) + (\log(2/\alpha \delta)/n_k,t)^{1/2} \varphi(y_{n_k,t})). \]

Receive reward \(y_{I_t}\).

**end**

3 Regret Analysis

In Section 3.1, we derive regret bounds for bootstrapped UCB. Moreover, we show that naive
bootstrapped UCB will result in linear regret in some cases in Section A in the supplement.
3.1 Regret Bound for Bootstrapped UCB

For multi-armed bandit problems, most literature [Lattimore and Szepesvári, 2018] consider sub-Gaussian rewards. In this work, we move beyond sub-Gaussianity and consider the reward under a much weaker tail assumption, so-called sub-Weibull distribution. As shown in [Kuchibhotla and Chakrabortty, 2018, Vladimirova and Arbel, 2019], it is characterized by the right tail of the Weibull distribution and generalizes sub-Gaussian and sub-exponential distributions.

**Definition 1** (Sub-Weibull Distribution). We define $y$ as a sub-Weibull random variable if it has a bounded $\psi_\beta$-norm. The $\psi_\beta$-norm of $y$ for any $\beta > 0$ is defined as

$$
\|y\|_{\psi_\beta} := \inf\left\{ C \in (0, \infty) : E[\exp(|y|/C^{\beta})] \leq 2 \right\}.
$$

Particularly, when $\beta = 1$ or 2, sub-Weibull random variables reduce to sub-exponential or sub-Gaussian random variables, respectively. It is obvious that the smaller $\beta$ is, the heaver tail the random variable has. Next theorem provides a corresponding concentration inequality for the sum of independent sub-Weibull random variables.

**Theorem 3.1** (Concentration Inequality for Sub-Weibull Distribution). Suppose $\{y_i\}_{i=1}^n$ are independent sub-Weibull random variables with $\|y_i\|_{\psi_\beta} \leq \sigma$. Then there exists an absolute constant $C_{\beta}$ only depending on $\beta$ such that for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $0 < \alpha < 1/e^2$,

$$
\left| \sum_{i=1}^n a_i y_i - E\left( \sum_{i=1}^n a_i y_i \right) \right| \leq C_{\beta} \sigma \left( \|a\|_2(\log \alpha^{-1})^{1/2} + \|a\|_\infty(\log \alpha^{-1})^{1/\beta} \right)
$$

with probability at least $1 - \alpha$.

The proof relies on a precise characterization of $p$-th moment of a Weibull random variable and standard symmetrization arguments. Details are deferred to Section B.2 in the supplement. This theorem generalizes the Hoeffding-type concentration inequalities for sub-Gaussian random variables (see, e.g. Proposition 5.10 in Vershynin [2012]), and Bernstein-type concentration inequalities for sub-exponential random variables (see, e.g. Proposition 5.16 in Vershynin [2012]) up to some constants.

In Theorem 3.2, we provide both problem-dependent and problem-independent regret bounds.

**Theorem 3.2.** Consider a stochastic $K$-armed sub-Weibull bandit, where the noise follows a symmetric sub-Weibull distribution with its $\psi_\beta$-norm upper bounded by $\sigma$. Denote $n_{k,t}$ as the number of pulls for arm $k$ until time $t$. We choose $\varphi$ according to Theorem 3.1 as follows

$$
\varphi(y_{n_{k,t}}) = C_{\beta} \sigma \left( \frac{\log 1/\alpha}{n_{k,t}} + \frac{(\log 2/\alpha)^{1/\beta}}{n_{k,t}} \right), \quad (3.1)
$$

and let the confidence level $\alpha = 1/T^2$. For any round $T$, the problem-dependent regret of bootstrapped UCB is upper bounded by

$$
R(T) \leq \sum_{k: \Delta_k > 0} 128C_{\beta}^2 \sigma^2 \frac{\log T}{\Delta_k} + 2^\beta + 1/\beta C_{\beta} \sigma K (\log T)^{1/\beta} + 4 \sum_{k=2}^{K} \Delta_k, \quad (3.2)
$$

where $C_{\beta}$ is some absolute constant from Theorem 3.1, and $\Delta_k$ is the sub-optimality gap. Moreover, if the round $T \geq 2^{2L-1} K (\log T)^{2L-1}$, the problem-independent regret of bootstrapped UCB is upper bounded by

$$
R(T) \leq 32\sqrt{2} C_{\beta} \sigma \sqrt{TK \log T} + 4K \mu_1^* . \quad (3.3)
$$

The main proof structure follows the standard analysis of UCB [Lattimore and Szepesvári, 2018] and relies on a sharp upper bound for the (data-dependent) bootstrapped quantile term by Theorem 3.1. Details are deferred to Section B.3 in the supplement. When $\beta \geq 1$, (3.2) provides a logarithm regret that matches the state-of-art result [Lattimore and Szepesvári, 2018]. When $\beta < 1$, we have a non-negligible term $(\log T)^{1/\beta}$ that is the price paid for heavy-tailedness. However, this term does not depend on the gap $\Delta_k$. Therefore, we have an optimal problem-independent regret bound.
Remark 3.3. The choice of $\alpha = 1/T^2$ led to an easy analysis. Using similar techniques in Chapter 8.2 of [Lattimore and Szepesvári, 2018], we can achieve a similar regret bound by setting $\alpha_t = 1/(t \log^+ (t))$ for any $\tau > 0$.

Remark 3.4. [Bubeck et al., 2013] consider bandit with heavy-tail (moment of order $(1 + \varepsilon)$) based on a median-of-means estimator. As mentioned in Chen and Zhou [2019], there are two disadvantages for median-of-means approach: (a) it involves an additional tuning parameter; (b) it is numerically unstable for small sample size. In contrast, we identify a class of heavy-tailed bandits (sub-Weibull bandit) where mean estimators can still achieve regret bounds of the same order as those under sub-Gaussian reward distributions. The reason is that although sub-Weibull r.v. has heavier tail than sub-Gaussian r.v., its tail still has an exponential-like decay.

4 Experiments

In Section 4.1, we consider multi-armed bandits with both symmetric and asymmetric rewards. In Section 4.2, we extend our method to linear bandits. Implementation details and some additional experimental results are deferred to Section E in the supplement.

4.1 Multi-armed Bandit

In this section, we compare bootstrapped UCB (Algorithm 1) with three baselines: Upper Confidence Bound based on concentration inequalities (Vanilla UCB), Thompson sampling with normal Jeffery prior [Korda et al., 2013] (Jeffery-TS) and Thompson sampling with Beta prior [Agrawal and Goyal, 2013a] (Bernoulli-TS). For bounded rewards, we also compare with Giro [Kveton et al., 2018], that is a sampling-based exploration method by adding artificial pseudo observations $\{0, 1\}$ to escape from local optima, and empirical KL-UCB [Garivier and Cappé, 2011] using package: PymaBandits.

From Figure 2, bootstrapped UCB outperforms Jeffery-TS and Vanilla-UCB for truncated-normal bandit and has comparable or sometimes better performance over empirical KL-UCB. It’s obvious that if the reward distribution is exactly Gaussian and the plug-in estimate for the noise standard deviation is the truth, Jeffery-TS should be the best. However, when the posterior (plots (a),(b)) or noise standard derivation (plot (c)) are mis-specified, the performance of TS deteriorates fast. Since (concentration-based) Vanilla UCB only uses the tail information (bounded or sub-Gaussian), it is very conservative and results in bad regret as expected.

Second, we consider asymmetric rewards with a mean parameter $\mu_k$ generated from Uniform$(0.25, 0.75)$. For Bernoulli bandit, the reward follows $\text{Ber}(\mu_k)$; for Beta bandit, the reward follows $\text{Beta}(v\mu_k, v(1 - \mu_k))$ for $v = 8$. From Figure 3, bootstrapped UCB outperforms Vanilla UCB and Giro in both cases, and outperforms Bernoulli-TS for Beta bandit. In fact, we are supposed not to beat Bernoulli-TS for Bernoulli bandit since TS fully makes use of the distribution knowledge in this case. One possible explanation is that our method is non-parametric.

1We have implemented Giro in the unbounded reward case, which could result in linear regret in most cases. See Figure 7 in the supplement. So, it’s unclear what is the best way to add pseudo observations in this case.

2We adopt the technique in Agrawal and Goyal [2013a] to run Thompson Sampling with $[0, 1]$ rewards. In particular, for any reward $y_i \in [0, 1]$, we draw pseudo reward $\hat{y}_i \sim \text{Ber}(y_i)$, and then use $\hat{y}_i$ instead of $y_i$ in the algorithm.
Third, we demonstrate that the robustness of bootstrapped UCB over mis-specifications of the noise standard deviation. In the left panel of Figure 4, we consider the cumulative regret at round \( T = 2000 \) of standard Gaussian bandit. As one can see, when we increase the plug-in upper bound of the standard deviation of the noise, bootstrapped UCB is more robust than Bernoulli-TS and Vanilla UCB.

Figure 4: The left panel is the cumulative regret over noise levels while the right panel is the instance-dependent regret of various algorithms as a function of gaps. The results are averaged over 200 realizations.
Last, we present a frequentist instance-dependent regret curve for truncated-normal bandit and the experiment set up follows Lattimore [2018]. We plot cumulative regrets at $T = 2000$ of various algorithms with respect to the instance gap $\Delta$ and the mean vector $\mu = (\Delta, 0, 0, 0, 0)$. The results are summarized in the right panel of Figure 4.

4.2 Linear Bandit

We extend our method to linear bandit case. The basic set up follows the one in Russo and Van Roy [2014]. In detail, $\theta^* \in \mathbb{R}^d$ is drawn from a multivariate Gaussian distribution with mean vector $\mu = 0$ and covariance matrix $\Sigma = 10I_d$. The noise follows a standard Gaussian distribution. There are 100 actions with feature vector components drawn uniformly at random from $[-1/\sqrt{10}, 1/\sqrt{10}]$. We consider two state-of-art methods: Thompson sampling for linear bandit [Agrawal and Goyal, 2013b] (TSL) and optimism in the face of uncertainty for linear bandits [Abbasi-Yadkori et al., 2011] (OFUL). Following the principle of constructing second-order correction in mean problems (Theorem 2.2), we construct the bootstrapped UCB for linear bandit (BUCBL) as follows: At each round $t$, the action is selected as $\arg\max_x (x^\top \hat{\theta}_t + \beta_{BUCBL} t, 1 - \delta \|x\| V_{t}^{-1})$, where $\beta_{BUCBL} = q_0 (\hat{\theta}_t - \tilde{\theta}_t) + \beta_{OFUL} / \sqrt{n}$.

The formal definition of $\hat{\theta}_t, \tilde{\theta}_t, \beta_{OFUL}$ and some basic setups are given in Section E.2 in the supplement. To be fair, the confidence level for all methods is set to be $\delta = 1/(1 + t)$ and we plug in the true standard deviation of the noise for each method. From Figure 5, we can see that bootstrapped UCB greatly improves the cumulative regret over TSL and OFUL.

5 Conclusion

In this paper, we propose a novel class of non-parametric and data-driven UCB algorithms based on multiplier bootstrap. It is easy to implement and has the potential to be generalized to other complex structured problems. As future works, we will evaluate our idea on other structured contextual bandits and reinforcement learning problems.

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References

Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In Advances in Neural Information Processing Systems, pages 2312–2320, 2011.
Radoslaw Adamczak, Alexander E Litvak, Alain Pajor, and Nicole Tomczak-Jaegermann. Restricted isometry property of matrices with independent columns and neighborly polytopes by random sampling. *Constructive Approximation*, 34(1):61–88, 2011.

Shipra Agrawal and Navin Goyal. Further optimal regret bounds for thompson sampling. In *Artificial intelligence and statistics*, pages 99–107, 2013a.

Shipra Agrawal and Navin Goyal. Thompson sampling for contextual bandits with linear payoffs. In *International Conference on Machine Learning*, pages 127–135, 2013b.

Noga Alon and Joel H Spencer. *The probabilistic method*. John Wiley & Sons, 2004.

Sylvain Arlot, Gilles Blanchard, Etienne Roquain, et al. Some nonasymptotic results on resampling in high dimension, i: confidence regions. *The Annals of Statistics*, 38(1):51–82, 2010.

Peter Auer. Using confidence bounds for exploitation-exploration trade-offs. *Journal of Machine Learning Research*, 3(Nov):397–422, 2002.

Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.

Hamsa Bastani and Mohsen Bayati. Online decision-making with high-dimensional covariates. *Available at SSRN 2661896*, 2015.

Hamsa Bastani, Mohsen Bayati, and Khashayar Khosravi. Mostly exploration-free algorithms for contextual bandits. *arXiv preprint arXiv:1704.09011*, 2017.

Sarah Bird, Solon Barocas, Kate Crawford, Fernando Diaz, and Hanna Wallach. Exploring or exploiting? social and ethical implications of autonomous experimentation in ai. In *Workshop on Fairness, Accountability, and Transparency in Machine Learning*, 2016.

Robert Bogucki. Suprema of canonical weibull processes. *Statistics & Probability Letters*, 107:253–263, 2015.

Sébastien Bubeck, Nicolo Cesa-Bianchi, and Gábor Lugosi. Bandits with heavy tail. *IEEE Transactions on Information Theory*, 59(11):7711–7717, 2013.

Olivier Cappé, Aurélien Garivier, Odalric-Ambrym Maillard, Rémi Munos, Gilles Stoltz, et al. Kullback–leibler upper confidence bounds for optimal sequential allocation. *The Annals of Statistics*, 41(3):1516–1541, 2013.

George Casella and Roger L Berger. *Statistical inference*, volume 2. Duxbury Pacific Grove, CA, 2002.

Xi Chen and Wen-Xin Zhou. Robust inference via multiplier bootstrap. *The Annals of Statistics, to appear*, 2019.

Victor Chernozhukov, Denis Chetverikov, Kengo Kato, et al. Gaussian approximation of suprema of empirical processes. *The Annals of Statistics*, 42(4):1564–1597, 2014.

Varsha Dani, Thomas P Hayes, and Sham M Kakade. Stochastic linear optimization under bandit feedback. 2008.

Victor De la Pena and Evarist Giné. *Decoupling: from dependence to independence*. Springer Science & Business Media, 2012.

Dean Eckles and Maurits Kaptein. Thompson sampling with the online bootstrap. *arXiv preprint arXiv:1410.4009*, 2014.

Bradley Efron. *The jackknife, the bootstrap, and other resampling plans*, volume 38. Siam, 1982.
Adam N. Elmachtoub, Ryan McNellis, Sechan Oh, and Marek Petrik. A practical method for solving contextual bandit problems using decision trees. In Proceedings of the Thirty-Third Conference on Uncertainty in Artificial Intelligence, UAI 2017, Sydney, Australia, August 11-15, 2017, 2017.

Aurélien Garivier and Olivier Cappé. The kl-ucb algorithm for bounded stochastic bandits and beyond. In Proceedings of the 24th annual Conference On Learning Theory, pages 359–376, 2011.

P Hitzенko, SJ Montgomery-Smith, and K Oleszkiewicz. Moment inequalities for sums of certain independent symmetric random variables. Studia Math, 123(1):15–42, 1997.

Jens Kober, J Andrew Bagnell, and Jan Peters. Reinforcement learning in robotics: A survey. The International Journal of Robotics Research, 32(11):1238–1274, 2013.

Tze Leung Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. Advances in applied mathematics, 6(1):4–22, 1985.

Tae-Sung Lee and Michael Wolf. Exact and approximate stepdown methods for multiple hypothesis testing. Journal of the American Statistical Association, 100(469):94–108, 2005.

Donald B Rubin. The bayesian bootstrap. The annals of statistics, pages 130–134, 1981.

Daniel Russo and Benjamin Van Roy. Learning to optimize via posterior sampling. Mathematics of Operations Research, 39(4):1221–1243, 2014.

Michel Ledoux and Michel Talagrand. Probability in Banach Spaces: isoperimetry and processes. Springer Science & Business Media, 2013.
David Silver, Aja Huang, Chris J Maddison, Arthur Guez, Laurent Sifre, George Van Den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, et al. Mastering the game of go with deep neural networks and tree search. *Nature*, 529(7587):484, 2016.

Vladimir Spokoiny, Mayya Zhilova, et al. Bootstrap confidence sets under model misspecification. *The Annals of Statistics*, 43(6):2653–2675, 2015.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

Michel Talagrand. The supremum of some canonical processes. *American Journal of Mathematics*, 116(2):283–325, 1994.

Liang Tang, Yexi Jiang, Lei Li, Chunqiu Zeng, and Tao Li. Personalized recommendation via parameter-free contextual bandits. In *Proceedings of the 38th international ACM SIGIR conference on research and development in information retrieval*, pages 323–332. ACM, 2015.

Aad W Van der Vaart. *Asymptotic statistics*, volume 3. Cambridge university press, 2000.

Sharan Vaswani, Branislav Kveton, Zheng Wen, Anup Rao, Mark Schmidt, and Yasin Abbasi-Yadkori. New insights into bootstrapping for bandits. *arXiv preprint arXiv:1805.09793*, 2018.

Roman Vershynin. *Compressed sensing*, chapter Introduction to the non-asymptotic analysis of random matrices, pages 210–268. Cambridge Univ. Press, 2012.

Mariia Vladimirova and Julyan Arbel. Sub-weibull distributions: generalizing sub-gaussian and sub-exponential properties to heavier-tailed distributions. *arXiv preprint arXiv:1905.04955*, 2019.

Chien-Fu Jeff Wu et al. Jackknife, bootstrap and other resampling methods in regression analysis. *the Annals of Statistics*, 14(4):1261–1295, 1986.

Yun Yang, Zuofeng Shang, and Guang Cheng. Non-asymptotic theory for nonparametric testing. *arXiv preprint arXiv:1702.01330*, 2017.
Supplement to “Bootstrapping Upper Confidence Bound”

In this supplement, we provide linear regret result in Section A, major proofs in Sections B and C. Some implementation details are in Sections D and E. In the end, we provide several supporting lemmas in Section F.

A Linear Regret

Following the augments in Vaswani et al. [2018], Kveton et al. [2018], in this section, we show that UCB with a naive bootstrapped confidence bound will result in linear regret in two-armed Bernoulli bandit. At round $t + 1$, the UCB index without the correction term for arm $k$ can be written as

$$\text{UCB}_k(t) = \bar{y}_{n_k,t} + q_{\alpha(1-\delta)}(y_{n_k,t} - \bar{y}_{n_k,t}).$$

Consider the case where the first observation on the optimal arm is 0 but on the sub-optimal arm is 1. A key fact is that if the rewards are all zero, no matter how you bootstrap the data, the bootstrapped quantile is always zero. This will make the algorithm stuck into the sub-optimal arm.

**Theorem A.1.** Consider a stochastic 2-arm Bernoulli bandit with mean parameter $\mu_1, \mu_2$. The expected regret of the naive bootstrapped UCB can be lower bounded by

$$R(T) \geq \Delta_2 \left( (1 - \mu_1)\mu_2(T-2) + 1 \right). \quad (A.1)$$

**Proof.** Without loss of generality, we assume arm 1 is the optimal arm. Suppose at round $t = 1, 2$, we pull each arm once such that $y_1$ is with arm 1 and $y_2$ is with arm 2. Then we define a bad event as follows:

$$\mathcal{E} = \{y_1 = 0, y_2 = 1\}. \quad (A.2)$$

We know that under event $\mathcal{E}$, the decision-maker will never pull arm 1 any more starting from round $t = 3$. This is because if the rewards are all zero, no matter how you bootstrap the data, the bootstrapped quantile is always zero and then makes the decision-maker stuck into the sub-optimal arm. Finally, we can lower bound the cumulative regret by,

$$R(T) = \Delta_2 \mathbb{E}\left[ \sum_{t=1}^{T} I\{I_t = 2\} \right]$$

$$= \Delta_2 \mathbb{E}\left[ \sum_{t=3}^{T} I\{I_t = 2\} | \mathcal{E} \right] \mathbb{P}(\mathcal{E}) + \Delta_2 \mathbb{E}\left[ \sum_{t=3}^{T} I\{I_t = 2\} | \mathcal{E}^c \right] \mathbb{P}(\mathcal{E}^c) + \Delta_2$$

$$\geq \Delta_2 \mathbb{E}\left[ \sum_{t=3}^{T} I\{I_t = 2\} | \mathcal{E} \right] \mathbb{P}(\mathcal{E}) + \Delta_2$$

$$= \Delta_2 T \mathbb{P}(y_1 = 0) \mathbb{P}(y_2 = 1) + \Delta_2$$

$$= \Delta_2 \left( 1 - \mu_1 \right) \mu_2 (T-2) + 1.$$ 

This ends the proof.

We further demonstrate this phenomenon empirically for both Bernoulli bandit and Gaussian bandit in Figure 6.

B Proofs of Main Theorems

In this section, we provide detailed proofs of Theorems 2.2, 3.1 and 3.2.

B.1 Proof of Theorem 2.2

The proof borrows the analysis from Arlot et al. [2010] but with refined analysis and sharp large deviation bound for binomial random variables.
Step One. Recall that (2.4) can be seen as the multiplier bootstrapped quantile around its empirical mean. We first take advantage of the symmetry of each $y$ around its mean by connecting the true quantile of $\bar{y}_n - \mu$ and the multiplier bootstrapped quantile around the true mean. Define the multiplier bootstrapped quantile around the true mean as

$$q_\alpha(y_n - \mu) := \inf \left\{ x \in \mathbb{R} \mid P_w \left( \frac{1}{n} \sum_{i=1}^{n} w_i (y_i - \mu) > x \right) \leq \alpha \right\}.$$  \hspace{1cm} (B.1)

Since the probability operator $P_w$ is conditionally on $y_n$, all the randomness of $q_\alpha(y_n - \mu)$ come from $y_n$. By the symmetric assumption of the reward, the distribution of $y_i - \mu$ is exactly the same as the distribution of $w_i(y_i - \mu)$ for Rademacher r.v. $\{w_i\}$. Then we have

$$P(y_n - \mu > q_\alpha(y_n - \mu)) = \mathbb{E}_w \left[ P(y \left( \frac{1}{n} \sum_{i=1}^{n} w_i (y_i - \mu) > q_\alpha((y_n - \mu) \circ w_n)) \right) \right],$$  \hspace{1cm} (B.2)

where $\circ$ is the Hadamard product. By Fubini’s theorem, we can interchange the probability operator and expectation operator as follows

$$\mathbb{E}_w \left[ P(y \left( \frac{1}{n} \sum_{i=1}^{n} w_i (y_i - \mu) > q_\alpha((y_n - \mu) \circ w_n)) \right) \right] = \mathbb{E}_y \left[ P_w \left( \frac{1}{n} \sum_{i=1}^{n} w_i (y_i - \mu) > q_\alpha(y_n - \mu) \right) \right] \leq \alpha,$$  \hspace{1cm} (B.3)

where the first inequality is due to the fact that for any arbitrary sign reversal, $q_\alpha((y_n - \mu) \circ w_n) = q_\alpha(y_n - \mu)$ based on the definition of $q_\alpha$ and the last inequality is from the definition of quantile. Combining (B.2) and (B.3) together, we conclude that

$$P(y_n - \mu > q_\alpha(y_n - \mu) \leq \alpha.$$  \hspace{1cm} (B.4)

Step Two. We define a good event

$$\mathcal{E} = \left\{ y_n \mid q_\alpha(y_n - \mu) \leq q_{\alpha(1-\delta)}(y_n - \bar{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \right\}.$$  \hspace{1cm} (B.5)
Together with (B.4) and union event trick,
\[ P\left( \tilde{y}_n - \mu > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right) \]
\[ = P\left( \{ \tilde{y}_n - \mu > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \} \cap \{ y_n \in \mathcal{E} \} \right) \]
\[ = P\left( \{ \tilde{y}_n - \mu > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \} \cap \{ y_n \in \mathcal{E} \} \right) \]
\[ + P\left( \{ y_n - \mu > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) + (2 \log(2/\alpha \delta)/n)^{1/2} \phi(y_n) \} \cap \{ y_n \in \mathcal{E} \} \right) \]
\[ \leq P\left( \tilde{y}_n - \mu > q_{\alpha}(y_n - \mu) \right) + P\left( y_n \in \mathcal{E}^c \right) \]
\[ \leq \alpha + P(y_n \in \mathcal{E}^c). \]

To bound \( P(y_n \in \mathcal{E}^c) \), we first prove the following claim:

Claim: \( \mathcal{E}^c \subset \{ y_n \mid P_w\left( \tilde{w}_n(y_n - \mu) > \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right) \geq \alpha \delta \}, \quad (B.6) \)

where \( \tilde{w}_n = \sum_{i=1}^n w_i/n \). To show this, we have by the definition of \( q_{\alpha}(y_n - \mu) \) in (B.1),
\[ P_w\left( \frac{1}{n} \sum_{i=1}^n w_i(y_i - \mu) > q_{\alpha}(y_n - \mu) \right) = \alpha. \]

By some simple algebras, we have
\[ \frac{1}{n} \sum_{i=1}^n w_i(y_i - \mu) = \frac{1}{n} \sum_{i=1}^n w_i(y_i - \tilde{y}_n + \tilde{y}_n - \mu) = \frac{1}{n} \sum_{i=1}^n w_i(y_i - \tilde{y}_n) + \tilde{w}_n(\tilde{y}_n - \mu). \quad (B.7) \]

For any \( y_n \in \mathcal{E}^c \),
\[ \alpha = P_w\left( \frac{1}{n} \sum_{i=1}^n w_i(y_i - \mu) > q_{\alpha}(y_n - \mu) \right) \]
\[ \leq P_w\left( \frac{1}{n} \sum_{i=1}^n w_i(y_i - \mu) > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right) \text{ (by the definition of } \mathcal{E}^c \text{)} \]
\[ = P_w\left( \frac{1}{n} \sum_{i=1}^n w_i(y_i - \tilde{y}_n) + \tilde{w}_n(\tilde{y}_n - \mu) > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right) \text{ (by (B.7))} \]
\[ \leq P_w\left( \frac{1}{n} \sum_{i=1}^n w_i(y_i - \tilde{y}_n) > q_{\alpha(1-\delta)}(y_n - \tilde{y}_n) \right) + P_w\left( \tilde{w}_n(\tilde{y}_n - \mu) > \frac{2 \log(2/\alpha \delta)}{n} \phi(y_n) \right) \]
\[ \leq \alpha(1 - \delta) + P_w\left( \tilde{w}_n(\tilde{y}_n - \mu) > \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right). \]

This proves the claim of (B.6).

**Step Three.** We start to bound the second term above as follows,
\[ P_w\left( \tilde{w}_n(\tilde{y}_n - \mu) > \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right) \quad (B.8) \]
\[ \leq P_w\left( |\tilde{w}_n(\tilde{y}_n - \mu)| > \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \phi(y_n) \right) \]
\[ \leq P_w\left( |\tilde{w}_n| > \sqrt{2n \log(2/\alpha \delta)} \frac{\phi(y_n)}{y_n - \mu} \right), \quad (B.9) \]

where the last inequality is actually conditional on the event \( \{ \tilde{y}_n \neq \mu \} \) that holds with probability one. Note that \( (w_i + 1/2) \sim \text{Bernoulli}(1/2) \) and thus \( \sum_{i=1}^n (w_i + 1)/2 \sim \text{Binomial}(n, 1/2) \). Denote \( X_n \)
is a Binomial($n, 1/2$) random variable. Applying the sharp large deviation bound in Lemma 1 with $p_i = 1/2$, we have

\[
P_{\mathcal{X}} \left( \mathcal{X}_n - \frac{n}{2} > \sqrt{2n \log(2/\alpha \delta)} \varphi(y_n) \right) \leq 2 \exp \left( - \frac{2 \varphi(y_n)^2}{(y_n - \mu)^2} \frac{2n \log(2/\alpha \delta)}{n} \right)
\]

Putting (B.8) and (B.10) together,

\[
P_\omega \left( \bar{w}_n(y_n - \mu) > \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \varphi(y_n) \right) \leq 2 \exp \left( - \frac{4 \log(2/\alpha \delta) \varphi(y_n)^2}{(y_n - \mu)^2} \right).
\]

From (B.6), it remains to bound

\[
P \left( y_n \in \mathcal{E}^c \right) \leq P_y \left( P_\omega \left( \bar{w}_n(y_n - \mu) > \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \varphi(y_n) \right) \geq \alpha \delta \right)
\]

\[
\leq P_y \left( 2 \exp \left( - \frac{4 \log(2/\alpha \delta) \varphi(y_n)^2}{(y_n - \mu)^2} \right) \geq \alpha \delta \right)
\]

\[
= P_y \left( |y_n - \mu| \geq 2 \varphi(y_n) \right).
\]

This reaches

\[
P \left( \bar{y}_n - \mu > q_{\alpha(1-\delta)}(y_n - \bar{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \varphi(y_n) \right) \leq \alpha + P_y \left( |y_n - \mu| \geq \varphi(y_n) \right).
\]

(B.11)

Letting $\varphi(y_n)$ be a non-negative function such that

\[
P_y \left( |y_n - \mu| \geq \varphi(y_n) \right) \leq \alpha,
\]

we have

\[
P \left( \bar{y}_n - \mu > q_{\alpha(1-\delta)}(y_n - \bar{y}_n) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n}} \varphi(y_n) \right) \leq 2\alpha.
\]

Redefine $\varphi(y_n) = 2\varphi(y_n)$ with a little bit abuse of notations. This ends our proof.

\[\blacksquare\]

### B.2 Proof of Theorem 3.1

We start by an upper bound for the $p$-th moment of sum of sub-Weibull random variables with bounded $\psi_\beta$-norm. The proof of Lemma B.1 is deferred to Section C.

**Lemma B.1.** Suppose \(\{y_i\}_{i=1}^n\) are $n$ independent sub-Weibull random variables satisfying $\|y_i\|_{\psi_\beta} \leq \sigma$ with $\beta > 0$. Then for all $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $p \geq 2$, we have

\[
\left( \mathbb{E} \left| \sum_{i=1}^n a_i y_i - \mathbb{E} \left( \sum_{i=1}^n a_i y_i \right) \right|^p \right)^{1/p} \leq \left\{ \begin{array}{ll}
C_\beta \sigma \left( \sqrt{p} \|a\|_2 + p^{1/\beta} \|a\|_\infty \right), & \text{if } 0 < \beta < 1; \\
C_\beta \sigma \left( \sqrt{p} \|a\|_2 + p^{1/\beta} \|a\|_\infty \right), & \text{if } \beta \geq 1.
\end{array} \right.
\]

(B.12)

where $1/\beta^* + 1/\beta = 1$, $C_\beta$ are some absolute constants only depending on $\beta$.

**Remark B.2.** If $0 < \beta < 1$, (B.12) is a combination of Theorem 6.2 in Hitczenko et al. [1997] and the fact that the $p$-th moment of a Weibull variable with parameter $\beta$ is of order $p^{1/\beta}$. If $\beta \geq 1$, (B.12) follows from a combination of Corollaries 2.9 and 2.10 in Talagrand [1994]. Continuing with standard symmetrization arguments, we reach the conclusion for general random variables. When $\beta = 1$ or $2$, (B.12) coincides with standard moment bounds for a sum of sub-Gaussian and sub-exponential random variables in Vershynin [2012].
After we get the $p$-th moment bound in Lemma B.1, we can use Markov’s inequality to transfer it to a high-probability as follows. For any $t > 0$, by Markov’s inequality,
\[
\mathbb{P}\left(\left| \sum_{i=1}^{n} a_i y_i - \mathbb{E}\left(\sum_{i=1}^{n} a_i y_i\right) \right| \geq t \right) = \mathbb{P}\left(\left| \sum_{i=1}^{n} a_i y_i - \mathbb{E}\left(\sum_{i=1}^{n} a_i y_i\right) \right|^p \geq t^p \right)
\]
\[
\leq \frac{\mathbb{E}\left(\sum_{i=1}^{n} a_i y_i - \mathbb{E}\left(\sum_{i=1}^{n} a_i y_i\right) \right)|^p}{t^p} \leq \frac{C_\beta \sigma^p \left(\sqrt{p} \|\mathbf{a}\|_2 + p^{1/\beta} \|\mathbf{a}\|_\infty \right)^p}{t^p},
\]
where the last inequality is from Lemma B.1. By setting $t$ such that
\[
\exp(-p) = C_\beta \sigma^p \left(\sqrt{p} \|\mathbf{a}\|_2 + p^{1/\beta} \|\mathbf{a}\|_\infty \right)^p / t^p,
\]
we have for $p \geq 2$,
\[
\left| \sum_{i=1}^{n} a_i y_i - \mathbb{E}\left(\sum_{i=1}^{n} a_i y_i\right) \right| \leq C_\beta \sigma \left(\sqrt{p} \|\mathbf{a}\|_2 + p^{1/\beta} \|\mathbf{a}\|_\infty \right)
\]
holds with probability at least $1 - \exp(-p)$. Letting $\alpha = \exp(-p)$, we have that for any $0 < \alpha < 1/e^2$, where
\[
\left| \sum_{i=1}^{n} a_i y_i - \mathbb{E}\left(\sum_{i=1}^{n} a_i y_i\right) \right| \leq C_\beta \sigma \left(\|\mathbf{a}\|_2 (\log \alpha)^{1/2} + \|\mathbf{a}\|_\infty (\log \alpha)^{1/\beta} \right),
\]
holds with probability at least $1 - \alpha$. This ends the proof.

**B.3 Proof of Theorem 3.2**

We first prove a problem-dependent bound then a problem-independent bound.

**Problem-Dependent Bound.** Recall that at round $t + 1$, the UCB index used in our algorithm is
\[
\text{UCB}_k(t) = \bar{y}_{n_{k,t}} + h_\alpha(y_{n_{k,t}}),
\]
where $n_{k,t}$ is the number of pulls until round $t + 1$ for arm $k$ and
\[
h_\alpha(y_{n_{k,t}}) = q_{\alpha/2}(y_{n_{k,t}} - \bar{y}_{n_{k,t}}) + \sqrt{\frac{2 \log(4/\alpha)}{n_{k,t}}} \varphi(y_{n_{k,t}}),
\]
where
\[
\varphi(y_{n_{k,t}}) = C_\beta \sigma \left(\sqrt{\frac{\log 1/\alpha}{n_{k,t}}} + \frac{\left(\log 2/\alpha\right)^{1/\beta}}{n_{k,t}} \right).
\]
From Theorem 3.1, for any fixed $n_{k,t} = s$, we know that
\[
\mathbb{P}\left(\bar{y}_s - \mu_k \geq \varphi(y_{s}) \right) \leq \alpha.
\]
From Theorem 2.2, for any fixed $n_{k,t} = s$, we have
\[
\mathbb{P}\left(\mu_k - \bar{y}_s > h_\alpha(y_{s}) \right) \leq 2\alpha, \ k \in [K].
\]
The basic idea is to bound the expected number of pulls $\mathbb{E}(n_{k,t})$ for sub-optimal arms. To decouple the randomness from the behavior of the UCB algorithm, we define a good event as follows,
\[
\mathcal{E}_k = \{\mu_1 < \min_{t \in [T]} \text{UCB}_1(t)\} \cap \{\bar{y}_{b_k} + h_\alpha(y_{b_k}) < \mu_1\}, \ k \in [K],
\]
where $b_k \in [T]$ is a constant to be chosen later.

First, we want to prove the following claim: if event $\mathcal{E}_k$ happens, then $n_{k,t} \leq b_k$. To show this, we use a contradiction argument. If $n_{k,t} > b_k$, then arm $k$ was pulled more than $b_k$ times over the first $T$ rounds, and so there must exist a round $t \in [T]$ such that $n_{k,t} = b_k$ and $I_t = k$. This implies
\[
\text{UCB}_k(t) = \bar{y}_{n_{k,t}} + h_\alpha(y_{n_{k,t}}) = \bar{y}_{b_k} + h_\alpha(y_{b_k}).
\]
From the definition of $\mathcal{E}_k$, we have
\[
\bar{y}_{b_k} + h_\alpha(y_{b_k}) < \mu_1 < \min_{t \in [T]} \text{UCB}_1(t) \leq \text{UCB}_1(t).
\]
This results in a contradiction. Then we can decompose $\mathbb{E}[n_{k,t}]$ with respect to the event $\mathcal{E}_k$,
\[
\mathbb{E}[n_{k,t}] = \mathbb{E}[I(\mathcal{E}_k)n_{k,t}] + \mathbb{E}[I(\mathcal{E}_k^c)n_{k,t}] \leq b_k + \mathbb{P}(\mathcal{E}_k^c)T. \quad (B.16)
\]
Second, we will derive an upper bound for $\mathbb{P}(\mathcal{E}_k^c)T$. By definition,
\[
\mathbb{P}(\mathcal{E}_k^c) = \mathbb{P}\left(\left\{ \mu_1 \geq \min_{t \in [T]} \text{UCB}_1(t) \right\} \cup \left\{ \bar{y}_{b_k} + h_\alpha(y_{b_k}) \geq \mu_1 \right\} \right) \leq \mathbb{P}\left(\left\{ \mu_1 \geq \min_{t \in [T]} \text{UCB}_1(t) \right\} \right) + \mathbb{P}\left(\left\{ \bar{y}_{b_k} + h_\alpha(y_{b_k}) \geq \mu_1 \right\} \right). \quad (B.17)
\]
To bound $I_1$, we apply the union bound trick as follows,
\[
\left\{ \mu_1 \geq \min_{t \in [T]} \text{UCB}_1(t) \right\} \subset \left\{ \mu_1 \geq \min_{s \in [T]} \bar{y}_s + h_\alpha(y_s) \right\} = \cup_{s \in [T]} \left\{ \mu_1 \geq \bar{y}_s + h_\alpha(y_s) \right\}.
\]
By B.14, it implies
\[
\mathbb{P}\left(\mu_1 \geq \min_{t \in [T]} \text{UCB}_1(t) \right) \leq \sum_{s=1}^{T} \mathbb{P}\left(\mu_1 \geq \bar{y}_s + h_\alpha(y_s) \right) \leq 2\alpha T. \quad (B.18)
\]
To bound $I_2$, the key step is to derive an sharp upper bound for threshold $h_\alpha(y_{b_k})$. Next lemma presents an upper bound for the multiplier bootstrapped quantile which is the main part of $h_\alpha(y_{b_k})$. The proof is deferred to Section C.2.

**Lemma B.3.** Suppose $\{y_i - \mu\}_{i=1}^{n}$ follows sub-Weibull distribution with $\|y_i - \mu\|_{\psi_\beta} \leq \sigma$ and $\{w_i\}_{i=1}^{n}$ are i.i.d Rademacher random variables independent of $y_i$. Then we have
\[
\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} \left( w_i - \bar{w} \right) (y_i - \mu) \leq C_{\beta} \sigma \left( \sqrt{\frac{\log(1/\alpha)}{n}} + \frac{(\log(1/\alpha))^{1/\beta}}{\alpha} \right) \right) \geq 1 - \alpha. \quad (B.19)
\]
By the definition of $q_{\alpha/2}(y_{b_k} - \bar{y}_{b_k})$ in (2.4), we have
\[
q_{\alpha/2}(y_{b_k} - \bar{y}_{b_k}) \leq C_{\beta} \sigma \left( \sqrt{\frac{\log(2/\alpha)}{b_k}} + \frac{(\log(2/\alpha))^{1/\beta}}{b_k} \right), \quad (B.20)
\]
with probability at least $1 - \alpha/2$. Recall that
\[
\sqrt{\frac{2 \log(4/\alpha)}{b_k}} \varphi(y_{b_k}) = \sqrt{\frac{2 \log(4/\alpha)}{b_k}} \left( \sqrt{\frac{\log(1/\alpha)}{b_k}} + \frac{(\log(1/\alpha))^{1/\beta}}{b_k} \right). \quad (B.21)
\]
Overall, we have
\[
h_\alpha(y_{b_k}) = q_{\alpha/2}(y_{b_k} - \bar{y}_{b_k}) + \sqrt{\frac{2 \log(4/\alpha)}{b_k}} \varphi(y_{b_k}) \leq 2C_{\beta} \sigma \left( \sqrt{\frac{\log(2/\alpha)}{b_k}} + \frac{(\log(2/\alpha))^{1/\beta}}{b_k} \right), \quad (B.23)
\]
with probability $1 - \alpha/2$ as long as $b_k \geq 2 \log(4/\alpha)/(C_{\beta}^2 \sigma^2)$.

For two events $\mathcal{A}$ and $\mathcal{B}$, we have
\[
\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B}^c) + \mathbb{P}(\mathcal{A} \cap \mathcal{B}) \leq \mathbb{P}(\mathcal{A} \cap \mathcal{B}) + \mathbb{P}(\mathcal{B}^c). \quad (B.24)
\]
Next we define an event $B_k = \{ h_\alpha(y_{b_k}) \leq \Delta_k / 2 \}$, where $\Delta_k = \mu_1 - \mu_k$. We decompose $I_2$ with respect to $B_k$ following the union event rule (B.24),

\[
\begin{align*}
\mathbb{P} \left( \bar{y}_{b_k} + h_\alpha(y_{b_k}) \geq \mu_1 \right) &= \mathbb{P} \left( \bar{y}_{b_k} + h_\alpha(y_{b_k}) \geq \mu_k \geq \mu_1 - \mu_k \right) \\
&\leq \mathbb{P} \left( \bar{y}_{b_k} - \mu_k \geq \Delta_k - h_\alpha(y_{b_k}) \cap B_k \right) + \mathbb{P}(B_k^c) \\
&\leq \mathbb{P} \left( \bar{y}_{b_k} - \mu_k \geq \frac{\Delta_k}{2} \cap B_k \right) + \mathbb{P}(B_k^c) \\
&\leq \mathbb{P} \left( \bar{y}_{b_k} - \mu_k \geq \Delta_k / 2 \right) + \mathbb{P}(B_k^c).
\end{align*}
\]

To bound the first part, we reuse the concentration inequality in Theorem 3.1 such that,

\[
\mathbb{P} \left( \bar{y}_{b_k} - \mu_k \geq \frac{\Delta_k}{2} \right) \leq \exp \left( - \min \left[ \left( \frac{\Delta_k}{C_\beta \sigma} \right)^2 b_k, \left( \frac{\Delta_k b_k}{4C_\beta \sigma} \right)^{\beta} \right] \right). \tag{B.25}
\]

To bound the second part, we bound $\mathbb{P}(B_k^c)$ in three steps,

1. By (B.22), we have

\[
\mathbb{P}(B_k^c) = \mathbb{P} \left( h_\alpha(y_{b_k}) > \Delta_k / 2 \right) \\
\leq \mathbb{P} \left( 2C_\beta \sigma \left( \sqrt{\log(2/\alpha)} b_k + (\log(2/\alpha))^{1/\beta} \right) > \Delta_k / 2 \right) + \alpha / 2. \tag{B.26}
\]

2. To ensure that $2C_\beta \sigma \sqrt{\log(2/\alpha)} / b_k \leq \Delta_k / 4$, we need

\[
b_k \geq \left( \frac{8C_\beta \sigma}{\Delta_k} \right)^2 \log(2/\alpha).
\]

To ensure that $2C_\beta \sigma (\log(2/\alpha))^{(1/\beta)} / b_k \leq \Delta_k / 4$, we need

\[
b_k \geq \frac{8C_\beta \sigma (\log(2/\alpha))^{(1/\beta)}}{\Delta_k}.
\]

3. Then if we choose $b_k$ as

\[
b_k = \left( \frac{8C_\beta \sigma}{\Delta_k} \right)^2 \log(2/\alpha) + \frac{8C_\beta \sigma (\log(2/\alpha))^{1/\beta}}{\Delta_k}, \tag{B.27}
\]

we have

\[
\mathbb{P} \left( 2C_\beta \sigma \left( \sqrt{\log(2/\alpha)} b_k + (\log(2/\alpha))^{1/\beta} \right) > \Delta_k / 2 \right) = 0. \tag{B.28}
\]

Combining (B.26) and (B.28), we conclude that when $b_k$ is choose as in (B.27), we have

\[
\mathbb{P}(B_k^c) \leq \alpha / 2. \tag{B.29}
\]

Combing (B.25) and (B.29), we have

\[
\mathbb{P} \left( \bar{y}_{b_k} + h_\alpha(y_{b_k}) \geq \mu_1 \right) \leq \exp \left( - \min \left[ \left( \frac{\Delta_k}{C_\beta \sigma} \right)^2 b_k, \left( \frac{\Delta_k b_k}{4C_\beta \sigma} \right)^{\beta} \right] \right) + \alpha / 2, \tag{B.30}
\]

when $b_k$ is chosen as below

\[
b_k = \left( \frac{8C_\beta \sigma}{\Delta_k} \right)^2 \log(1/\alpha) + \frac{8C_\beta \sigma (\log(2/\alpha))^{1/\beta}}{\Delta_k}.
\]
Combining (B.17), (B.18) and (B.30) together,
\[
P(\mathcal{E}_k^c) \leq 2T\alpha + \exp\left(-\min\left[\left(\frac{\Delta_k}{C_\beta \sigma}\right)^2 b_k, \left(\frac{\Delta_k b_k}{4C_\beta \sigma}\right)^2\right]\right) + \alpha/2
\]
\[
\leq 2T\alpha + \exp\left(-\min\left[\left(\frac{\Delta_k}{C_\beta \sigma}\right)^2 \left(\frac{8C_\beta \sigma (\log(2/\alpha))^{1/\beta} b_k}{\Delta_k}\right)^2\right]\right) + \alpha/2
\]
\[
= 2T\alpha + \exp\left(-\min(64, 2^\beta) \log(2/\alpha)\right) + \alpha/2. \tag{B.31}
\]
Plugging (B.27), (B.31) into (B.16),
\[
E[n_{k,t}] \leq b_k + P(\mathcal{E}_k^c)T
\]
\[
= \left(\frac{8C_\beta \sigma}{\Delta_k}\right)^2 \log(2/\alpha) + \frac{8C_\beta \sigma (\log(2/\alpha))^{1/\beta}}{\Delta_k} + 2T^2 \alpha + T\alpha^{\min(64, 2^\beta)} + T\alpha/2.
\]
By choosing \(\alpha = 2/T^2\), we have
\[
E[n_{k,t}] \leq \left(\frac{8C_\beta \sigma}{\Delta_k}\right)^2 \log(T) + \frac{8C_\beta \sigma (2 \log(T))^{1/\beta}}{\Delta_k} + 4,
\]
\[
\text{(B.32)}
\]
since \(1 - 2 \min(64, 2^\beta) < 0\) for \(\beta > 0\). Finally, the cumulative regret is upper bounded by
\[
R(T) = \sum_{k=2}^K \Delta_k E[n_{k,t}] \tag{B.33}
\]
\[
\leq \sum_{k=2}^K 128(C_\beta \sigma)^2 T \frac{\log T}{\Delta_k} + 8C_\beta \sigma K (2 \log T)^{1/\beta} + 4 \sum_{k=2}^K \Delta_k. \tag{B.34}
\]
This ends the proof.

**Problem-Independent Bound.** First we let \(\Delta > 0\) as a threshold which will be specified later. Then we decompose \(R(T)\) with respect to the value of \(\Delta\) as follows,
\[
R(T) = \sum_{k=2}^K \Delta_k E[n_{k,t}]
\]
\[
= \sum_{k; \Delta_k < \Delta} \Delta_k E[n_{k,t}] + \sum_{k; \Delta_k \geq \Delta} \Delta_k E[n_{k,t}]
\]
\[
\leq T\Delta + \sum_{k; \Delta_k \geq \Delta} \left(128(C_\beta \sigma)^2 T \frac{\log T}{\Delta_k} + 8C_\beta \sigma (2 \log T)^{1/\beta} + 4\Delta_k\right)
\]
\[
\leq 8C_\beta \sigma K (2 \log T)^{1/\beta} + 4 \sum_{k=2}^K \Delta_k + 128(C_\beta \sigma)^2 \frac{K \log T}{\Delta} + T\Delta, \tag{B.35}
\]
where the first inequality is from (B.32). Letting \(128(C_\beta \sigma)^2 \frac{K \log T}{\Delta} = T\Delta\), we have
\[
\Delta = \left(128C_\beta^2 \frac{2K \log T}{T}\right)^{1/2}. \tag{B.36}
\]
Plugging (B.36) back into (B.35), we have
\[
R(T) \leq 2 \cdot 128^{1/2} C_\beta \sigma \sqrt{TK \log T} + 4 \sum_{k=1}^K \Delta_k + 8C_\beta \sigma K (2 \log T)^{1/\beta}.
\]
When \(T \geq 2^{2/\beta - 3} K (\log T)^{2/\beta - 1}\), we have
\[
R(T) \leq 32\sqrt{2} C_\beta \sigma \sqrt{TK \log T} + 4 \sum_{k=2}^K \Delta_k
\]
\[
\leq 32\sqrt{2} C_\beta \sigma \sqrt{TK \log T} + 4K \mu_1^*.
\]
This ends the proof.
C Proofs of Main Lemmas

In this section, we provide the proofs of Lemmas B.1 and B.3.

C.1 Proof of Lemma B.1

Without loss of generality, we assume \( \|x_i\|_{\phi_\beta} = 1 \) and \( \mathbb{E}x_i = 0 \) throughout this proof. Let \( \beta = (\log 4)^{1/\beta} \). For notation simplicity, we define \( \|x\|_p = (\mathbb{E}|x|^p)^{1/p} \) for a random variable \( X \). The following step is to estimate the moment of linear combinations of variables \( \{x_i\}_i^\infty \).

According to the symmetrization inequality (e.g., Proposition 6.3 of Ledoux and Talagrand [2013]), we have

\[
\left\| \sum_{i=1}^n a_i x_i \right\|_p \leq 2 \left\| \sum_{i=1}^n a_i \varepsilon_i x_i \right\|_p = 2 \left\| \sum_{i=1}^n a_i \varepsilon_i x_i \right\|_p , \tag{C.1}
\]

where \( \{\varepsilon_i\}_i^\infty \) are independent Rademacher random variables and we notice that \( \varepsilon_i x_i \) and \( \varepsilon_i |x_i| \) are identically distributed. By triangle inequality,

\[
2 \left\| \sum_{i=1}^n a_i \varepsilon_i |x_i| \right\|_p \leq 2 \left\| \sum_{i=1}^n a_i \varepsilon_i |x_i - \beta| \right\|_p \\
\leq 2 \left\| \sum_{i=1}^n a_i \varepsilon_i |x_i - \beta| \right\|_p + 2 \left\| \sum_{i=1}^n a_i \varepsilon_i \beta \right\|_p . \tag{C.2}
\]

Next, we will bound the second term of the RHS of (C.2). In particular, we will utilize Khinchin-Kahane inequality, whose formal statement is included in Lemma 5 for the sake of completeness. From Lemma 5 we have

\[
\left\| \sum_{i=1}^n a_i \varepsilon_i \beta \right\|_p \leq \left( \frac{p-1}{2-1} \right)^{1/2} \left\| \sum_{i=1}^n a_i \varepsilon_i \beta \right\|_2 \\
\leq \beta \sqrt{p} \left\| \sum_{i=1}^n a_i \varepsilon_i \right\|_2 . \tag{C.3}
\]

Since \( \{\varepsilon_i\}_i^\infty \) are independent Rademacher random variables, some simple calculations implies

\[
\left( \mathbb{E} \left( \sum_{i=1}^n \varepsilon_i a_i \right)^2 \right)^{1/2} = \left( \mathbb{E} \left( \sum_{i=1}^n \varepsilon_i^2 a_i^2 + 2 \sum_{1 \leq i < j \leq n} \varepsilon_i \varepsilon_j a_i a_j \right) \right)^{1/2} \\
= \left( \sum_{i=1}^n a_i^2 \mathbb{E} \varepsilon_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j \mathbb{E} \varepsilon_i \varepsilon_j \right)^{1/2} \\
= \left( \sum_{i=1}^n a_i^2 \right)^{1/2} = \|a\|_2 . \tag{C.4}
\]

Combining inequalities (C.2)-(C.4),

\[
2 \left\| \sum_{i=1}^n a_i \varepsilon_i |x_i| \right\|_p \leq 2 \left\| \sum_{i=1}^n a_i \varepsilon_i |x_i - \beta| \right\|_p + 2 \beta \sqrt{p} \|a\|_2 . \tag{C.5}
\]

Let \( \{y_i\}_i^\infty \) be independent symmetric random variables satisfying \( \mathbb{P}(|y_i| \geq t) = \exp(-t^\beta) \) for all \( t \geq 0 \). Then we have

\[
\mathbb{P}(|x_i - \beta| \geq t) \leq \mathbb{P}(x_i \geq t + \beta) + \mathbb{P}(x_i \leq \beta - t) \\
\leq 2 \mathbb{P}(\exp(|x_i|^\beta) \geq \exp((t + \beta)^\beta)) \\
\leq 2(\mathbb{E}|x_i|^\beta) \cdot \exp(-(t + \beta)^\beta) \\
\leq 4 \exp(-(t + \beta)^\beta) \\
\leq 4 \exp(-t^\beta - \beta^\beta) = \mathbb{P}(|y_i| \geq t) ,
\]

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which implies
\[
\left\| \sum_{i=1}^{n} a_i \varepsilon_i |x_i - \beta| \right\|_p \leq \left\| \sum_{i=1}^{n} a_i \varepsilon_i y_i \right\|_p = \left\| \sum_{i=1}^{n} a_i y_i \right\|_p, \tag{C.6}
\]

since \( \varepsilon_i y_i \) and \( y_i \) have the same distribution due to symmetry. Combining (C.5) and (C.6) together, we reach
\[
\left\| \sum_{i=1}^{n} a_i x_i \right\|_p \leq 2\beta \sqrt{p} \|a\|_2 + 2 \left\| \sum_{i=1}^{n} a_i y_i \right\|_p. \tag{C.7}
\]

For \( 0 < \beta < 1 \), it follows Lemma 4 that
\[
\left\| \sum_{i=1}^{n} a_i y_i \right\|_p \leq C_\beta (\sqrt{p} \|a\|_2 + p^{1/\beta} \|a\|_\infty), \tag{C.8}
\]

where \( C_\beta \) is some absolute constant only depending on \( \beta \).

For \( \beta \geq 1 \), we will combine Lemma 3 and the method of the integration by parts to pass from tail bound result to moment bound result. Recall that for every non-negative random variable \( x \), integration by parts yields the identity
\[
\mathbb{E} x = \int_{0}^{\infty} \mathbb{P}(x \geq t) dt.
\]

Applying this to \( x = | \sum_{i=1}^{n} a_i y_i |^p \) and changing the variable \( t = t^p \), then we have
\[
\mathbb{E} \left| \sum_{i=1}^{n} a_i y_i \right|^p = \int_{0}^{\infty} \mathbb{P} \left( \left| \sum_{i=1}^{n} a_i y_i \right| \geq t \right) t^{p-1} dt 
\leq \int_{0}^{\infty} 2 \exp \left( - \frac{t}{c \|a\|_2^2} \right) \left( \frac{t^{\beta}}{\|a\|_2^\beta} \right) t^{p-1} dt, \tag{C.9}
\]

where the inequality is from Lemma 3 for all \( p \geq 2 \) and \( 1/\beta + 1/\beta^* = 1 \). In this following, we bound the integral in three steps:

1. If \( \frac{t^2}{\|a\|_2^2} \leq \frac{t^\beta}{\|a\|_2^\beta} \), (C.9) reduces to
\[
\mathbb{E} \left| \sum_{i=1}^{n} a_i y_i \right|^p \leq 2p \int_{0}^{\infty} \exp \left( - \frac{t^2}{c \|a\|_2^2} \right) t^{p-1} dt.
\]

Letting \( t' = c t^2 / \|a\|_2^2 \), we have
\[
2p \int_{0}^{\infty} \exp \left( - \frac{t^2}{c \|a\|_2^2} \right) t^{p-1} dt = \frac{p \|a\|_2^p}{c^{p/2}} \int_{0}^{\infty} e^{-t'} t'^{p/2-1} dt' = \frac{p \|a\|_2^p}{c^{p/2}} \Gamma \left( \frac{p}{2} \right) \leq \frac{p \|a\|_2^p}{c^{p/2}} \left( \frac{p}{2} \right)^{p/2},
\]

where the second equation is from the density of Gamma random variable. Thus,
\[
\left( \mathbb{E} \left| \sum_{i=1}^{n} a_i y_i \right|^p \right)^{1/p} \leq \frac{p^{1/p}}{(2c)^{1/2} \sqrt{p} \|a\|_2} \leq \frac{\sqrt{2}}{\sqrt{c}} \sqrt{p} \|a\|_2. \tag{C.10}
\]

2. If \( \frac{t^2}{\|a\|_2^2} > \frac{t^\beta}{\|a\|_2^\beta} \), (C.9) reduces to
\[
\mathbb{E} \left| \sum_{i=1}^{n} a_i y_i \right|^p \leq 2p \int_{0}^{\infty} \exp \left( - \frac{t^\beta}{c \|a\|_2^\beta} \right) t^{p-1} dt.
\]
Letting \( t' = ct^\beta / \|a\|_{\beta} \), we have
\[
2p \int_0^\infty \exp \left( -c t^\beta \left/ \|a\|_{\beta} \right. \right) t^{p-1} dt = \frac{2p\|a\|_{\beta}^p}{\beta c^p} \int_0^\infty e^{-ct^\beta} t^{p-1} dt = \frac{2p\|a\|_{\beta}^p}{\beta c^p} \Gamma \left( \frac{p}{\beta} \right) \leq \frac{2p\|a\|_{\beta}^p}{\beta c^p} \left( \frac{p}{\beta} \right)^{\beta/p}.
\]
Thus,
\[
\left( \mathbb{E} \sum_{i=1}^n |a_i y_i|^p \right)^{\frac{1}{p}} \leq \frac{2p^{1/p}}{(c\beta)^{1/\beta}} \|a\|_{\beta} \leq \frac{4}{(c\beta)^{1/\beta}} p^{1/\beta} \|a\|_{\beta}.
\]
(C.11)

3. Overall, we have the following by combining (C.10) and (C.11),
\[
\left( \mathbb{E} \sum_{i=1}^n a_i y_i \right) \leq \max \left( \sqrt{\frac{2}{c \beta}}, \frac{4}{(c\beta)^{1/\beta}} \right) \left( \sqrt{p\|a\|_2} + p^{1/\beta} \|a\|_{\beta} \right).
\]
After denoting \( C_\beta = \max \left( \sqrt{\frac{2}{c \beta}}, \frac{4}{(c\beta)^{1/\beta}} \right) \), we reach
\[
\left\| \sum_{i=1}^n a_i y_i \right\|_p \leq C_\beta \left( \sqrt{p\|a\|_2} + p^{1/\beta} \|a\|_{\beta} \right).
\]
(C.12)

Since \( 0 < \beta < 1 \), the conclusion can be reached by combining (C.7),(C.8) and (C.12).

\[\square\]

C.2 Proof of Lemma B.3

Note that with probability one,
\[
\sum_{i=1}^n (w_i - \bar{w})^2 = \sum_{i=1}^n w_i^2 - n\bar{w}^2 - n(1 - \bar{w}) \leq n, \quad \max_{i} (w_i - \bar{w}) \leq 1.
\]

We define a good event \( \mathcal{E} \) as follows
\[
\mathcal{E} = \{ \sum_{i=1}^n (w_i - \bar{w})^2 \leq n \} \cup \{ \max_{i} (w_i - \bar{w}) \leq 1 \}.
\]
(C.13)

Then we decompose (B.19) conditional on \( \mathcal{E} \),
\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n (w_i - \bar{w})(y_i - \mu) \geq C_\beta \sigma \left( \sqrt{\frac{\log 1/\alpha}{n} + \frac{(\log 1/\alpha)^{1/\beta}}{n}} \right) \right)
\]
\[
= \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n (w_i - \bar{w})(y_i - \mu) \geq C_\beta \sigma \left( \sqrt{\frac{\log 1/\alpha}{n} + \frac{(\log 1/\alpha)^{1/\beta}}{n}} \right) \right| \mathcal{E} \right)
\]
\[
+ \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n (w_i - \bar{w})(y_i - \mu) \geq C_\beta \sigma \left( \sqrt{\frac{\log 1/\alpha}{n} + \frac{(\log 1/\alpha)^{1/\beta}}{n}} \right) \right| \mathcal{E}^c \right)
\]
\[
\leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n (w_i - \bar{w})(y_i - \mu) \geq C_\beta \sigma \left( \sqrt{\frac{\log 1/\alpha}{n} + \frac{(\log 1/\alpha)^{1/\beta}}{n}} \right) \right)
\]
\[
\leq \alpha,
\]
where the first inequality is from \( \mathbb{P}(\mathcal{E}^c) = 0 \), the second inequality is from the independence of \( w_i \) and \( y_i \), the third inequality is from the concentration inequality in Theorem 3.1. This ends the proof. \[\square\]
D Monte Carlo Approximations

Suppose $n_{k,t}$ is the number of rewards associated with arm $k$ until round $t$. Practically, we could use Monte Carlo quantile approximation to calculate the multiplier bootstrapped quantile $q_\alpha(y_{n,k,t} - \bar{y}_{n,k,t})$. Let $\{w_n^{(1)}, \ldots, w_n^{(B)}\}$ denote $B$ sets of independent random weight vectors and define

$$\tilde{q}_\alpha(y_n - \bar{y}_n, w^B) := \inf \left\{ x \in \mathbb{R} \left| \frac{1}{B} \sum_{b=1}^{B} I\left\{ \frac{1}{n} \sum_{i=1}^{n} w_i^{(b)}(y_i - \bar{y}_n) \geq x \right\} \leq \alpha \right. \right\}, \quad \text{(D.1)}$$

where $B$ is the number of bootstrap repetitions and $w^B = (w_n^{(1)}, \ldots, w_n^{(B)})$. Then the UCB index for arm $k \in [K]$ can be written as

$$\text{UCB}_k(t) = \bar{y}_{n,k,t} + \tilde{q}_\alpha(1-\delta)(y_{n,k,t} - \bar{y}_{n,k,t}, w^B) + \sqrt{\frac{2 \log(2/\alpha \delta)}{n_{k,t}} \varphi(y_{n,k,t})}. \quad \text{(D.2)}$$

The decision-makers choose to pull arm $I_{t+1} = \arg\max_{k \in [K]} \text{UCB}_k(t)$. If $\text{UCB}_k(t) = \text{UCB}_{k'}(t)$ for $k \neq k'$, the tie is broken by a fixed rule that is chosen randomly in advance. Next theorem controls the approximation error of the bootstrapped quantile.

**Theorem D.1** (Monte Carlo Quantile Approximation). Suppose the same conditions in Theorem 2.2 hold. We have

$$\mathbb{P}_{y,w}(\bar{y}_n - \mu > \tilde{q}_\alpha(y_n - \bar{y}_n, w^B) + \sqrt{\log(2/\alpha \delta)/n \varphi(y_n)}) \leq \alpha + \frac{|B\alpha| + 1}{B + 1} \leq 2 \alpha + \frac{1}{B + 1},$$

where $\tilde{q}_\alpha(y_n - \bar{y}_n, w^B)$ is the Monte Carlo approximated quantile defined in (D.1).

By replacing the true quantile $q_\alpha$ by a MC quantile $\tilde{q}_\alpha^B$ based on $B$ i.i.d bootstrapped weights, we lose at most $1/(B + 1)$ for the confidence level.

**Proof Sketch.** The proof is similar to the proof of Theorem 2.2 except for the control of i.i.d approximation error. First, we define

$$\tilde{q}_\alpha(y_n - \mu, w^B) := \inf \left\{ x \in \mathbb{R} \left| \frac{1}{B} \sum_{b=1}^{B} I\left\{ \frac{1}{n} \sum_{i=1}^{n} w_i^{(b)}(y_i - \mu) \geq x \right\} \leq \alpha \right. \right\}.$$

By using the similar symmetry properties as we did in (B.2) and (B.3), we have

$$\mathbb{E}_{w^B}\mathbb{P}_y\left( \frac{1}{n} \sum_{i=1}^{n} w_i(y_i - \mu) > \tilde{q}_\alpha(y_n - \mu, w^B) \right)$$

$$= \mathbb{E}_{w^B}\mathbb{E}_{w^B} F_y\left( \frac{1}{n} \sum_{i=1}^{n} w_i(y_i - \mu) > \tilde{q}_\alpha((y_n - \mu) \circ w_n, w^B) \right)$$

$$= \mathbb{E}_{y}\mathbb{E}_{w^B} F_{w^B}\left( \frac{1}{n} \sum_{i=1}^{n} w_i(y_i - \mu) > \tilde{q}_\alpha(y_n - \mu, w^B \cdot \text{diag}(w_n)) \right)$$

$$= \mathbb{E}_{y}\mathbb{E}_{w^B} F_{w^B}\left( \frac{1}{n} \sum_{i=1}^{n} w_i(y_i - \mu) > \tilde{q}_\alpha(y_n - \mu, w^B) \right)$$

$$= \mathbb{E}_{y}\mathbb{E}_{w^B} F_{w^B}\left( \frac{1}{n} \sum_{i=1}^{n} w_i(y_i - \mu) > \tilde{q}_\alpha(y_n - \mu, w^B) \right)$$

$$= \mathbb{E}_{y}\mathbb{E}_{w^B} F_{w^B}\left( \frac{1}{n} \sum_{i=1}^{n} w_i(y_i - \mu) > \tilde{q}_\alpha(y_n - \mu, w^B) \right)$$

where the last inequality can be derived from Lemma 1 in [Romano and Wolf, 2005]. The rest of the proof will follow step two in the proof of Section B.1.
E Additional Experimental Results and Implementation Details

In Section E.1, we present the implementation details for multi-armed bandits. In Section E.2, we present the implementation details for linear bandits. In Section E.3, we present formal definitions for logistic distribution and truncated-normal distribution.

E.1 Multi-armed Bandit

For UCB1, at each round, the action is selected as

$$\text{argmax}_{k \in [K]} \frac{1}{n_k} \sum_{s=1}^{n_k} y^k_s + \hat{\sigma} \sqrt{\frac{2 \log(1/\alpha)}{n_k}}.$$  

For Jeffery-TS, at each round, the parameter is sampled from $\mathcal{N}(\frac{1}{n_k} \sum_{s=1}^{n_k} y^k_s, \hat{\sigma}^2/n_k)$.

Here, $\hat{\sigma}$ is the upper bound on the estimator of standard deviation. $\{y^k_s\}$ are the reward associated with arm $k$ and $n_k$ is the number of reward associated with arm $k$. For notation simplicity, we ignore their dependency on round $t$.

In addition to Gaussian bandit and truncated-normal bandit, we also consider logistic bandit with parameter $(\mu = 0, s = 0.5)$. The formal definition of logistic distribution and truncated-normal distribution. The results are summarized in Figure 7. Giro is almost failed.

E.2 Linear Bandit

Setup. We particularly consider the following linear bandit setup. Let $\mathcal{D}_t \subset \mathbb{R}^d$ be an arbitrary (finite or infinite) set of arms. When an arm $x \in \mathcal{D}_t$ is pulled, the agent receives a reward

$$y(x) = x^\top \theta^* + \epsilon,$$  

(E.1)

where $\theta^* \in \mathbb{R}^d$ is the true reward parameter and $\epsilon$ is a zero-mean random noise with variance $\sigma^2$. We assume $\|\theta^*\|_2 \leq S$. An arm $x \in \mathcal{D}_t$ is evaluated according to its expected reward $x^\top \theta^*$ and for any $\theta \in \mathbb{R}^d$, we denote the optimal arm and its value by

$$x^*(\theta) = \arg\min_{x \in \mathcal{D}_t} x^\top \theta, \quad J(\theta) = \sup_{x \in \mathcal{D}_t} x^\top \theta.$$  

Thus $x^* = x^*(\theta^*)$ is the optimal arm for $\theta^*$ and $J(\theta^*)$ is its optimal value. At each round $t$, the agent selects an arm $x_t \in \mathcal{D}_t$ based on past observations. Then, it observes the reward $y_t = x_t^\top \theta^* + \epsilon_t$. 

Figure 7: Cumulative regret for logistic bandit. The left panel is for $\hat{\sigma} = 1$, and the right panel is for $\hat{\sigma} = 2$. 

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and it suffers a regret equal to the difference in expected reward between the optimal arm $x^*$ and the arm $x_t$. The objective of the agent is to minimize the cumulative regret up to round $t$,

$$R(T) = \sum_{t=1}^{T} (x^* - x_t, \theta^*),$$

where $T$ is the time horizon. Note that the regret holds with high probability and thus is slightly from the standard notion of pseudo regret [Abbasi-Yadkori et al., 2011].

Denote $X_t = (x_1, \ldots, x_t)^T \in \mathbb{R}^{t \times d}$, $y_t = (y_1, \ldots, y_t)^T \in \mathbb{R}^{t \times 1}$. At round $t + 1$, consider a ridge estimator

$$\hat{\theta}_t = (X_t^T X_t + \lambda I_d)^{-1} X_t y_t.$$ (E.2)

Let us denote $V_t = \sum_{s=1}^{t} x_s x_s^T \in \mathbb{R}^{d \times d}$ as the empirical covariance matrix.

**Algorithms.** For TSL: Thompson sampling for linear bandit [Agrawal and Goyal, 2013b], at each round $t$, the parameter is sampled as $\theta_t = \hat{\theta}_t + \bar{\sigma} \sqrt{d \log(1/\delta)} V_t^{-1/2} \eta$ with $\eta \sim \mathcal{N}(0, I_d)$, where $\bar{\sigma}$ is a standard deviation estimator. [Agrawal and Goyal, 2013b] suggests an even larger constant for the bonus term to enforce over exploration in theory. In practice, it will make the regret exploding. So we remove that large constant in our simulation.

For OFUL: optimism in the face of uncertainty for linear bandits [Abbasi-Yadkori et al., 2011], at each round $t$, the action is selected as $\arg\max_x (x^T \hat{\theta}_t + \beta_{t,1-\delta,\sigma}^{\text{OFUL}} \|x\|_{V_t^{-1}})$, where

$$\beta_{t,1-\delta,\sigma}^{\text{OFUL}} = \bar{\sigma} \sqrt{2 \log \left( \frac{|\det(V_t)|^{1/2} \det(\lambda I_d)^{1/2}}{\delta} \right)} + \lambda^{1/2} S.$$ (E.3)

For BUCBL: bootstrapped UCB for linear bandit, we consider multinomial weights which is equivalent to sample with replacement. In detail, we generate $B$ sets of bootstrap repetitions $\{X_t^{(b)}, y_t^{(b)}\}$ from $\{X_t, y_t\}$ by sample with replacement, and calculate corresponding bootstrapped estimator

$$\hat{\theta}_t^{(b)} = (X_t^{(b)^T} X_t^{(b)} + \lambda I_d)^{-1} X_t^{(b)^T} y_t^{(b)},$$ (E.4)

and $V_t^{(b)} = \sum_{s=1}^{t} x_s^{(b)} x_s^{(b)^T}$. Define the bootstrapped weighted $\ell_2$-norm as follow

$$\|\hat{\theta}_t^{(b)} - \hat{\theta}_t\|_{V_t^{(b)} + \lambda I_d} = \sqrt{(\hat{\theta}_t^{(b)} - \hat{\theta}_t)^T (V_t^{(b)} + \lambda I_d) (\hat{\theta}_t^{(b)} - \hat{\theta}_t)}.$$

For each set of bootstrap repetitions, we could calculate the $\|\hat{\theta}_t^{(b)} - \hat{\theta}_t\|_{V_t^{(b)} + \lambda I_d}$ accordingly. Therefore, the bootstrapped threshold is defined as

$$q_{\alpha}(\hat{\theta}_t^{(b)} - \hat{\theta}_t) := (1 - \alpha)\text{-quantile of } \left\{ \|\hat{\theta}_t^{(1)} - \hat{\theta}_t\|_{V_t^{(1)} + \lambda I_d}, \ldots, \|\hat{\theta}_t^{(B)} - \hat{\theta}_t\|_{V_t^{(B)} + \lambda I_d} \right\}.$$ (E.5)

At each round $t$, the action is selected as $\arg\max_x (x^T \hat{\theta}_t + (q_{\alpha}(\hat{\theta}_t^{(b)} - \hat{\theta}_t) + \beta_{t,1-\delta,\sigma}^{\text{OFUL}}/\sqrt{B}) \|x\|_{V_t^{-1}})$.

### E.3 Logistic Distribution and Truncated-Normal Distribution

**Logistic Distribution** In probability theory and statistics, the logistic distribution is a continuous probability distribution. Its cumulative distribution function is the logistic function, which appears in logistic regression and feed forward neural networks. It resembles the normal distribution in shape but has heavier tails.

**Definition E.1.** The probability density function (pdf) of the logistic distribution $(\mu, s)$ is given by:

$$f(x) = \frac{\exp(-(x - \mu)/s)}{s(1 + \exp(-(x - \mu)/s))^2},$$

where $\mu$ is a location parameter and $s > 0$ is a scale parameter. The mean is $\mu$ and the variance is $s^2 \pi^2 / 3$. 

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Truncated-normal Distribution In probability and statistics, the truncated normal distribution is the probability distribution derived from that of a normally distributed random variable by bounding the random variable from either below or above (or both).

Definition E.2. Suppose $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$ and lies within the interval $(a, b)$. Then $X$ conditional on $a < X < b$ has a truncated normal distribution $(\mu, a, b)$. Its probability density function $f$ is given by

$$f(x) = \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)},$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function.

F Supporting Lemmas

Lemma 1 (Large Deviation Bound, Theorem A.1.4 in [Alon and Spencer, 2004]). Suppose $x_1, \ldots, x_n$ are mutually independent random variables with distribution

$$P(x_i = 1 - p_i) = p_i, \ P(x_i = -p_i) = 1 - p_i,$$

where $p_i \in [0, 1]$. For any $\alpha > 0$, we have

$$P\left(\sum_{i=1}^{n} x_i > a\right) < \exp(-2a^2/n).$$

When all $p_i = p$, the sum $\sum_{i=1}^{n} X_i$ has distribution Binomial$(n, p) - np$ where $B(n, p)$ is the Binomial distribution.

Lemma 2 (Hoeffding’s inequality, Proposition 5.10 in [Vershynin, 2012]). Let $X_1, \ldots, X_n$ be independent centered sub-Gaussian random variables, and let $K = \max_i \|X_i\|_{\alpha,2}$. Then for any $\alpha = (a_1, \ldots, a_n)^T$ and any $t > 0$, we have

$$P\left(\left|\sum_{i=1}^{n} a_i X_i\right| > t\right) \leq e \exp\left(-\frac{ct^2}{K^2\|\alpha\|_2^2}\right).$$

Lemma 3 (Tail Probability for the Sum of Weibull Distributions (Lemma 3.6 in Adamczak et al. [2011])). Let $\alpha \in [1, 2]$ and $Y_1, \ldots, Y_n$ be independent symmetric random variables satisfying $P(|Y_i| \geq t) = \exp(-t^\alpha)$. Then for every vector $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and every $t \geq 0$,

$$P\left(\left|\sum_{i=1}^{n} a_i Y_i\right| \geq t\right) \leq 2 \exp\left(-c \min\left(\frac{t^2}{\|\alpha\|_2^2}, \frac{t^\alpha}{\|\alpha\|_\alpha^\alpha}\right)\right).$$

Lemma 4 (Moments for the Sum of Weibull Distributions (Corollary 1.2 in Bogucki [2015])). Let $X_1, X_2, \ldots, X_n$ be a sequence of independent symmetric random variables satisfying $P(|Y_i| \geq t) = \exp(-t^\alpha)$, where $0 < \alpha < 1$. Then, for $p \geq 2$ and some constant $C(\alpha)$ which depends only on $\alpha$,

$$\left\|\sum_{i=1}^{n} a_i X_i\right\|_p \leq C(\alpha)(\sqrt{p}\|\alpha\|_2 + p^{1/\alpha}\|\alpha\|_\infty).$$

Lemma 5 (Khinchin-Kahane Inequality (Theorem 1.3.1 in De la Pena and Giné [2012])). Let $\{a_i\}_{i=1}^{n}$ a finite non-random sequence, $\{\varepsilon_i\}_{i=1}^{n}$ be a sequence of independent Rademacher variables and $1 < p < q < \infty$. Then

$$\left\|\sum_{i=1}^{n} \varepsilon_i a_i\right\|_q \leq \left(\frac{q-1}{p-1}\right)^{1/2} \left\|\sum_{i=1}^{n} \varepsilon_i a_i\right\|_p.$$