Young differential inclusions

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Abstract. We define in this note a notion of Young differential inclusion and give an existence result for such a differential system driven by mildly rough signals. As a by-product of our proof, we show that a bounded, compact-valued, \( \gamma \)-Hölder continuous set-valued map on the interval \([0,1]\) has a selection with finite \( p \)-variation, for \( p > 1/\gamma \).

1 – Introduction

A set valued application \( F \) is a map from \([0, T] \times \mathbb{R} \) into the power set of \( \mathbb{R}^d \). A Caratheodory solution of the differential inclusion

\[
\dot{x}(t) \in F(t, x(t)), \ t \in [0, T], \ x(0) = \xi \in \mathbb{R}^d
\]

is an absolutely continuous path \( x \) whose derivative \( \dot{x} \) satisfies

\[
\dot{x}(t) \in F(t, x(t))
\]

at almost all times \( t \in [0, T] \). Existence of solutions of differential inclusions and their properties were widely studied; see for instance J.P. Aubin and A. Cellina’s book [1] for an authoritative pedagogical treatment. Equation (1.1) has at least one solution under two kinds of assumptions on the set valued map \( F \), supposing either that it is upper semicontinuous with compact, convex values, or that \( F \) is lower semicontinuous, and the existence proof for solutions of the differential inclusion (1.1) generically rely on either approximation schemes or a fixed point reformulation.

Some extensions to stochastics cases were also studied, after the pioneering works of Aubin-da Prato [2, 3, 4, 5] and Kisielwicz. The former were essentially motivated by studying viability questions in a stochastic setting. M. Kisielwicz in [12] define the notion of solution and obtain the existence of stochastic differential inclusions on the form

\[
X_t - X_s = \int_s^t \mathbb{F}(r, X_r)dr + \int_s^t \mathbb{G}(r, X_r)dW_r
\]

where \( W \) is an \( \mathbb{R}^d \)-valued Brownian motion, \( \mathbb{F} \) is a lower semicontinuous set-valued random map with values in \( \mathbb{R}^d \) and \( \mathbb{G} \) is a lower semicontinuous set-valued map with values in \( \mathbb{R}^{d \times d} \). In [13], M. Kisielwicz also studied the case with in addition a compound Poisson measure; the case of semimartingal drivers was also investigated. All proofs fundamentally rely on the isometry property of stochastic integration with respect to Brownian motion or compound Poisson measures. There are however a number of interesting non-semimartingale random processes, such as Mandelbrojt’s fractional Brownian motion [17]. Its sample paths are \( \alpha \)-Hölder continuous, for some \( \alpha \in (0,1) \). One can use T. Lyons’ work [17] to solve solve differential equations driven by a fractional Brownian motion, for \( \alpha > 1/2 \); it relies on the notion of Young integral [20]. The case \( 1/4 \leq \alpha \leq 1/2 \) is much more involved and can be handled using Lyons’ theory of rough paths [18]. In [16], A. Levakov and M. Vas’kovskii obtain the existence of solutions to stochastic differential inclusions of the form

\[
X_t - X_s = \int_s^t \mathbb{F}(r, X_r)dr + \int_s^t \mathbb{G}(r, X_r)dW_r + \int_s^t \mathbb{G}_{FBM}(r, X_r)dW_{FBM}^\alpha
\]

where \( W_{FBM} \) is a fractional Brownian motion with \( \alpha \)-Hölder continuous sample path, with \( \alpha > 1/2 \), and the set valued map \( \mathbb{G}_{FBM} \) takes nonempty compact convex values and satisfies the local \( \delta \)-Hölder
condition as a function of time, with \( \delta > 1 - \alpha \), and \( \mathcal{G}_{FBM} \) is globally Lipschitz condition as a function of \( x \).

The aim of this article is to define and prove the existence of Young differential inclusions

\[
y_t = \xi + \int_0^t x_r dw_r,
\]

(1.3)

where \( w \) is a, \( \mathbb{R}^d \)-valued deterministic \( \alpha \)-Hölder continuous function, with \( \alpha > 1/2 \), and \( F \) is a \( \gamma \)-Hölder continuous set valued map with compact values, for some regularity exponent \( \gamma \in (\frac{1}{\alpha} - 1, 1) \).

We do not require that \( F \) takes values in convex sets.

The notion of a solution to a Young differential inclusion involves some elementary results on Young integrals that are recalled in Appendix A. Given a set \( X \), we denote by \( 2^X \) the power set of \( X \), and write \( 2^X \) for \( 2^X \setminus \{\emptyset\} \).

**Definition** \(-\) Let \( F : \mathbb{R}^d \mapsto 2^{L(\mathbb{R}^d, \mathbb{R}^d)} \) be a set-valued map, and \( x \) be an \( \alpha \)-Hölder \( \mathbb{R}^d \)-valued path defined on the time interval \([0, T] \). A solution to the Young differential inclusion

\[
dz_t \in F(z_t)dt,
\]

(1.4)

is a pair of paths \((z, v)\), defined on the time interval \([0, T] \),

- with \( v \) with finite \( p \)-variation, for some positive \( q \) with \( \alpha + \frac{1}{q} > 1 \),
- with \( v_t \in F(z_t) \) and

\[
z_t = z_0 + \int_0^t v_s \, dx_s,
\]

for all \( 0 \leq t \leq T \).

The integral \( \int_0^t v_s \, dx_s \) makes sense as a Young integral under the assumption \( \alpha + \frac{1}{q} > 1 \) – see the Appendix. Recall that endowing the space \( 2^L(\mathbb{R}^d, \mathbb{R}^d) \) with the Hausdorff pseudo-metric turns the space \( \mathcal{K}(L(\mathbb{R}^d, \mathbb{R}^d)) \) of non-empty compact subsets of \( L(\mathbb{R}^d, \mathbb{R}^d) \) into a complete metric space.

**1.1. Theorem** \(-\) Let a time horizon \( T \) be given, together with an \( \mathbb{R}^d \)-valued \( \alpha \)-Hölder path \( x : [0, T] \mapsto \mathbb{R}^d \), for some \( \alpha > 1/2 \). Let \( F : \mathbb{R}^d \mapsto \mathcal{K}(L(\mathbb{R}^d, \mathbb{R}^d)) \), be a set-valued map that is bounded with compact values and \( \gamma \)-Hölder, for some regularity exponent \( \gamma \in (\frac{1}{\alpha} - 1, 1) \). Then, for any initial condition \( z_0 \in \mathbb{R}^d \), the Young differential inclusion

\[
dz_t \in F(z_t)dt,
\]

has a solution path started from \( z_0 \), defined over the time interval \([0, T] \).

For a singleton-valued map, this is a consequence of Young’s original result [20] [17]; no uniqueness is to be expected in the present setting, unlike the setting of ordinary differential equations. Our regularity condition on the set-valued map \( F \) is the same as in the ordinary differential setting. However, there may be no Hölder, or even continuous, solution of \( F \) – see Proposition 8.2 in [6], so existence results for Young differential inclusions do not follow from existence results for Young differential equations. We refer the reader to Chapters 1 and 2 of [1] for the basics on differential inclusions. Note that since continuous paths with finite \( p \)-variation can be reparametrized into \( 1/p \)-Hölder paths, the result of Theorem 1.1 holds for continuous paths with finite \( p \)-variation.

The proof of Theorem 1.1 goes as follows. Recall from Appendix A that the space of paths with finite \( p \)-variation is endowed with the norm \( \| \cdot \|_{p\text{-var}, x} \), defined in (A.1). Given a finite time horizon \( T \), we define the dyadic partition \( z^{(m)} = \{s^m_i\} \) of the interval \([0, T] \), with \( s^m_i := 2^{-m}T, 0 \leq i \leq 2^m \). Pick \( \frac{\alpha}{\nu} > 1 \). We construct in Section 2.1 an approximate solution to the problem on a sufficiently small time interval \([0, T] \), under the form of a pair \((z^m, v^m)\) such that

- \( v^m \) has \( \| \cdot \|_{p\text{-var}, x} \)-norm uniformly bounded in \( m \), and is equioscillating,
• $v_i^m \in F(z_i^m)$ for all dyadic times, and

$$z_i^m = z_0 + \int_0^t v_u^m du, \quad 0 \leq t \leq T.$$ 

It follows then from the first item that the sequence $v^m$ has a converging subsequence $v^{m_k}$ with limit some $v$, for the $\| \cdot \|_{q-var, \infty}$-norm, for any $q > p$ with $\frac{2}{q} + \alpha > 1$. The continuity statement from Corollary $A.2$ implies then that $z^{m_k}$ converges in $\| \cdot \|_{C^0, \infty}$ to the path $z := z_0 + \int_0^t v_u du$. One gets the fact that $v_t \in F(z_t)$, for all $0 \leq t \leq T$, from the fact that $F$ is bounded and takes values in closed sets. The existence of a solution to the inclusion defined up to the initial time horizon is a consequence of the fact that the previous existence time does not depend on $z_0$.

As a by-product of our proof, we show in Appendix $B$ that a bounded, compact-valued, $\gamma$-Hölder continuous set-valued map on the interval $[0,1]$ has a selection with finite $p$-variation, for $p > 1/\gamma$.

Throughout the proof, we set $x_{st} := x_t - x_s$, for any $0 \leq s \leq t \leq T$. We denote by $\| F \|_{C^0}$ the $C^0$-norm of the map $F$, and write $\|x\|_{C^0}$ for the $C^0$-norm of a path on its domain of definition.

## 2 – Proof of Theorem 1.1

We fix $\alpha > 1/2$ once and for all and work in the setting of Theorem 1.1. We follow the strategy of proof sketched above.

### 2.1 – Construction of the approximate solution

For $t \in \bigcup_{m \geq 0} \pi^{(m)}$, set

$$M(t) := \min \left\{ j; t \in \pi^{(j)} \right\}$$

and define the ancestor $s(t)$ of $t$ as

$$s(t) := \max \left\{ s \in \pi^{M(t) - 1}; s < t \right\}. \quad (2.1)$$

For each $m \geq 0$, we construct the path $z^m$ on $[0,t_{i+1}^m]$, and $v^m$ on $[0,t_i^m]$, recursively on $0 \leq i \leq 2^m - 1$. The construction is not inductive on $m$.

• For $m = 0$, choose $v_t^0 \in F(z_0)$, and set

$$v_t^0 = v_0^0, \quad z_t^0 = \xi + v_0^0 x_0 t, \quad \forall t \in [0,T].$$

• Pick $m \geq 1$. For $0 \leq t \leq 2^{-m}T$, set

$$v_t^m := v_0^0, \quad z_t^m := \xi + v_0^0 x_0 t,$n

for $t \in [0,2T2^{-m})$ for $v^m$, and $t \in [0,2T2^{-m})$ for $z^m$. This starts the induction over $0 \leq i \leq 2^m - 1$. If $z^m : [0,t_i^m] \to \mathbb{R}^d$ and $v^m : [0,t_i^m] \to \mathbb{R}^d$, have been constructed, use the Hölder continuity of $F$ to choose $v_t^m \in F(z_t^m)$ such that

$$\left| v_{t_i}^m - v_{s(t_i)}^m \right| \leq \| F \|_{C^0} \| z_{t_i}^m - z_{s(t_i)}^m \|_2,$$

and set

$$v_t^m := v_{t_i}^m, \quad z_t^m := z_{t_i}^m + v_{t_i}^m x_i t,$n

for $t$ in $[t_i^m, t_i^m + T2^{-m})$ and $[t_i^m, t_i^m + T2^{-m})$, respectively. If $t_i^m + 2^{-m}T = T$, set

$$v_T^m := v_{t_i}^m.$$

We have $z^m = \xi + \int_0^t v_u^m du$, as a consequence of the fact that $v^m$ is constant along the intervals of the partition $\pi^{(m)}$ of $[0,T]$. The next section is dedicated to proving a uniform $p$-variation bound on the $v^m$ on a small time interval.
2.2 – Study of the $p$-variation norm of $v^m$ in a small time interval

2.1. Proposition – Pick $\frac{1}{1+\gamma} < \beta < \alpha$, and set

$$T_0 := \min \left\{ 1, \left( 2 \| F \|_{C^0} \right)^{\frac{1}{1+\gamma}}, \left( \frac{2 \| F \|_{C^0} \| x \|_{C^0}}{1 - 2^{-(\alpha + \beta - 1)}} \right)^{\frac{1}{1+\gamma}}, \left( \frac{\| F \|_{C^0}}{1 - 2^{-\gamma}} \right)^{\frac{1}{1+\gamma}} \right\}.$$ 

Pick $p > 1/(\gamma \beta)$. Then we have, for any $S \leq T_0$,

$$\| v^m \|_{p\text{-var},[0,S]} \leq \left( \frac{1}{1 - 2^{-\gamma \beta p}} \right)^{\frac{1}{\gamma \beta}} S^{(\alpha - \beta)/4} + \gamma \beta. \quad (2.2)$$

The proof of Proposition 2.1 proceeds in several steps. We first give a discrete integral representation of $z^m$ that proves useful.

2.2. Lemma – Pick $0 \leq n \leq m$, and two consecutive points $s, t$ in $\pi(n)$. Then, setting $s^k = s + i 2^{-n-k}T$, we have

$$z^m_{st} = v^m_{st} + \sum_{k=0}^{m-n-1} 2^k \sum_{i=0}^{2^k-1} v^m_{s^k+1, t^k+1} x^{k+1}_{s^k+1,s^k+1}.$$ 

Proof – For $0 \leq s \leq t \leq T$, set

$$\mu^m_{st} := \mu^m_{s,t} = v^m_{st}$$

and, for $k \geq 1$,

$$\mu^m_{s,t} = \mu^m_{s,t} + \mu^m_{s,t}.$$ 

• We first prove by descending induction on $n$ that for $0 \leq n \leq m$, and two consecutive points $s, t$ in $\pi(n)$, we have

$$z^m_{st} = \mu^m_{s,t} = \mu^m_{s,t}, \quad \forall k \geq m - n. \quad (2.3)$$

This identity holds true when $n = m$ as a consequence of the definition of the objects. Assume that (2.3) holds true for $n \leq m$, and let $s, t$ be two consecutive points in $\pi(n-1)$. Since $s$ and $s^k$, and $t$ and $t^k$, are consecutive points in $\pi(n)$, we have, from the definition of $\mu^m_{s,t}$ and induction,

$$z^m_{st} = \mu^m_{s,t} + \mu^m_{s,t} = \mu^m_{s,t} + \mu^m_{s,t}.$$ 

For $k \geq (m - n + 1)$, we have

$$\mu^m_{s,t} = \mu^m_{s,t} + \mu^m_{s,t} = \mu^m_{s,t} + \mu^m_{s,t}.$$ 

This closes the inductive proof of identity (2.3)

• One then sees by induction on $k$ that setting as in the statement of the Lemma

$$s_k = s + i 2^{-k}(t - s) = s + i 2^{-n-k}T,$$

one has

$$\mu^m_{s,t} = \sum_{i=0}^{2^k-1} \mu^m_{s^i, t^{i+1}}$$

and

$$\mu^m_{s,t} - \mu^m_{s,t} = \sum_{i=0}^{2^k-1} \mu^m_{s^i, t^{i+1}} x^{k+1}_{s^i, t^{i+1}}.$$ 

Summing equation (2.4) for $k$ from 0 to $m - n$, and using identity (2.3) gives the identity of the Lemma. 


2.3. Corollary – Pick \( \frac{1}{1+\gamma} < \beta < \alpha \), and set

\[
T_1 := \min \left\{ 1, (2\|F\|_{\infty} \|x\|_{C^0})^{\frac{2}{1+\gamma}}, \left( \frac{2\|F\|_{\infty} \|x\|_{C^0}}{1 - 2^{-(n+\beta+1)}} \right)^{-\frac{1}{1+\gamma}} \right\}.
\]

Then we have for any \( 0 \leq S \leq T_1 \) and any \( 0 \leq n \leq m \), the \( m \)-uniform bound

\[
\sup_{[s,t] \in \pi^{(n)}} |z^m_{st}| \leq S^{\frac{\alpha-\beta}{\gamma}} (S2^{-n})^\beta;
\]

the supremum is over consecutive points \( s, t \) of \( \pi^{(n)} \).

Proof – The proof is again by descending induction on \( n \in [0, m] \). We first have for two consecutive points \( s, t \) of \( \pi^{(m)} \) the estimate

\[
\sup_{[s,t] \in \pi^{(n)}} |z^m_{st}| \leq \|F\|_{\infty} \|x\|_{C^0} (S2^{-m})^\alpha,
\]

so (2.5) holds true for \( m = 0 \) since \( 0 \leq S \leq T_1 \). Assume now that (2.6) has been proved for \( n \leq m \), and let \( s, t \) be two consecutive points of \( \pi^{(n-1)} \). We use the representation formula

\[
z^m_{st} = x^m_{st} + \sum_{k=0}^{m-n+1} \sum_{i=0}^{2^k-1} v^m_{x^k_{s+2i+1}} S^{\frac{k}{2^k}} (S2^{-n-k})^\beta,
\]

from Lemma 2.2, with \( s^k_i = s + i2^{-n-k+1}S \). Note that \( M(s^k_{2i+1}) = n + k \), here, and the ancestor \( s(s^k_{2i+1}) \) of \( s^k_{2i+1} \) is \( s^k_i \). Then, using (2.5) for \( n + k \), we have

\[
\sup_{[s,t] \in \pi^{(n)}} |z^m_{st}| \leq \|F\|_{C^0} \left( \sum_{k=0}^{m-n+1} S^{\frac{k}{2^k}} (S2^{-n-k})^\beta \right)^\gamma \leq \|F\|_{C^0} \sup_{[s,t] \in \pi^{(n)}} \|x\|_{C^\gamma} \sup_{[s,t] \in \pi^{(n)}} \|F\|_{C^0} \sup_{[s,t] \in \pi^{(n)}} \|x\|_{C^\gamma} \frac{S^{\alpha+\gamma}}{1 - 2^{-(n+\beta+1)}}.
\]

The choice of \( S \leq T_1 \) ensures that

\[
\|F\|_{\infty} \|x\|_{C^0} (2^{-n+1}S)^\alpha \leq \frac{1}{2} S^{\alpha+\beta} (2^{-1})^\beta
\]

and since \( \alpha + \gamma > 1 > \alpha \) and \( S < 1 \)

\[
\frac{2^{-(n+\beta+1)}}{1 - 2^{-(n+\beta+1)}} \leq \frac{1}{2} S^{\alpha+\beta} (2^{-1})^\beta;
\]

this closes the descending induction step. \( \triangleright \)

Recall the oscillation \( \text{Osc}(v, I) \) of a function \( v : I \mapsto \mathbb{R}^d \), is defined by the formula

\[
\text{Osc}(v, I) := \sup_{a \in I} (v(b) - v(a)).
\]

The uniform control of the oscillation of the \( v^m \) provided by the next statement is necessary to use the compactness result on the set of bounded functions equipped with uniform norm stated in Theorem 5, Section 4, Chapter 0 of Aubin and Cellina’s book [1]. Recall the definition of \( T_0 \leq T_1 \) from the statement of Proposition 2.1. The notation \([s, t] \in \pi^{(n)}\) used below stands for two consecutive points \( s, t \) in \( \pi^{(n)} \).

2.4. Corollary – For any \( 0 \leq S \leq T_0 \), we have, for any \( 0 \leq n \leq m \),

\[
\sup_{[s,t] \in \pi^{(n)}} \text{Osc}(v^m, [s, t]) \leq S^{\frac{(\alpha-\beta)}{\gamma}} (S2^{-n})^\beta.
\]

(2.6)
We define a partition of \([a,b]\), since it is constant on the intervals of the partition \(\pi(m)\). Let then take \(n \leq (m-1)\). Set \(s_0 = s\) and define a finite sequence \((s_i)_{i=0,\ldots,m-r}\) setting \(s_{i+1} = s_i + 2^{-r(i+1)}\) for \(s_i \leq t\), and \(s_{i+1} = s_i\), otherwise. Then \(s_i \in (\pi(i))\), for any \(i\), and either \(s_{i+1} = s_i\) or its ancestor \(s(s(i+1)) = s_i\). We then have from the uniform estimate (2.3) on \(s_{i+1}^m\) the bound

\[
|v_{s_i,s_{i+1}}^m| \leq \|F\|_{C^j} |v_{s_i,s_{i+1}}^m| \leq \|F\|_{C^j} \left( S^{\frac{\alpha}{2} - \frac{1}{\nu}} (S 2^{-r})^\gamma \right).
\]

We obtain (2.4) summing these inequalities for \(i\) from 0 to \((m-n)\), and from the definition of \(T_0\). \(\Box\)

**Proof of Proposition 2.1** - Take 0 \(\leq S \leq T_0\), and let \(\pi = (s_i)_{i=0}^N\) be a partition of the interval \([0, S]\). The following partition is a kind of greedy partition of the set \(\{0, \ldots, N-1\}\), in terms of the size of corresponding increments in the above formula. Let then set

\[
I_1 := \left\{ \ell \in \{0, \ldots, N]\}; 2^{-1} \in (s_\ell, s_{\ell+1}] \right\},
\]

and, for \(2 \leq j \leq m\), set

\[
I_j := \left\{ \ell \in \{0, \ldots, N]\}; \exists t \in \pi(j) \setminus \pi(j-1) \cap (s_\ell, s_{\ell+1}] \right\}.
\]

We define a partition of \([0, N-1]\) setting \(K_0 := I_0\) and

\[
K_j := I_j \setminus \bigcup_{k=0}^{j-1} I_k,
\]

for \(2 \leq j \leq m\). Note that \(K_j\) has at most \(2^j\) elements.

- For \(\ell \in K_1\), we know from Corollary 2.2 on oscillations of \(v^m\), with \(r = 0\), that

\[
|v_{s_\ell,s_{\ell+1}}^m - v_{s_\ell}^m| \leq S^{\frac{\alpha}{2} - \frac{1}{\nu}} + \gamma^\beta.
\]

- For \(\ell \in K_j\), there exists \(t \in \pi(j) \setminus \pi(j-1)\) such that \(s_{\ell-1} \leq t \leq s_\ell\), and for all \(u \in \pi(j-1)\), one has either \(u \leq s_{\ell-1}\), or \(u > s_\ell\).

Then \([s_\ell, s_{\ell+1}] \subset [t - S 2^{-j}, t + S 2^j]\), and using Corollary 2.2 with \(j = 1\), one gets

\[
|v_{s_\ell,s_{\ell+1}}^m - v_{s_\ell}^m| \leq S^{\frac{\alpha}{2} - \frac{1}{\nu}} (S 2^{-j})^\gamma^\beta.
\]

Taking \(p > \gamma^\beta\), on then has

\[
\sum_{i=0}^{N-1} |v_{s_i,s_{i+1}}^m - v_{s_i}^m| \leq 2S^{\frac{\alpha}{2} - \frac{1}{\nu}} \sum_{j=0}^{m} 2^j (S 2^{-j})^\gamma^p \leq 2S^{\frac{\alpha}{2} - \frac{1}{\nu}} S^{\gamma^\beta p} \frac{1}{1 - 2^{-\gamma^\beta p}},
\]

from which (2.2) follows

\[
|v^m|_{p-var,[0,S]} \leq \left( \frac{1}{1 - 2^{-\gamma^\beta p}} \right)^{1/p} S^{\frac{\alpha}{2} - \frac{1}{\nu}} + \gamma^\beta.
\]
2.3 – Local and global existence for solutions

We give the details of the proof of Theorem \([\text{1.1}]\) in this section. We first prove the existence of a solution to the differential inclusion \([\text{1.3}]\) on the time interval \([0, T_0]\); since the definition of \(T_0\) does not involve the initial condition \(z_0\) of the dynamics, we obtain by concatenation a solution to the inclusion defined over the whole interval \([0, T]\).

- Recall that a sequence of bounded functions \((y^m)_{m \geq 0}\) from compact segment \([a, b]\) into a compact set \(K\) is said to be **equioscillating** if, and only if, one can associate to any positive \(\varepsilon\) a finite partition \((J_k)_{0 < k < r}\) of \([a, b]\) into subintervals such that \(\text{Osc}(y^m, J_k) \leq \varepsilon\), uniformly in \(k, m\). The Ascoli-Arzela-type convergence theorem from Theorem 5 of section 4 of chap 0 of Aubin and Cellina’s book \([\mathbb{I}]\), ensure the existence of a subsequence \((y^{m_k})_{k \geq 0}\) which converges uniformly to some limit function \(y\). Since the family \((y^m)_{m \geq 0}\) is bounded by \(\|F\|_x\), and equioscillating, from Corollary \([\text{2.3}]\) it has a uniformly converging subsequence, with limit \(v\).

- Pick then \(q > p\) such that \(\frac{1}{q} + \alpha > 1\). Since all the \(v^m\) have the same starting point, we have the elementary interpolation bound

\[
\|v^m - v^n\|_{q-\text{var},[0,T]} \leq (2\|v^m - v^n\|_{x,[0,T_0]} + \|v^n\|_{p-\text{var},[0,T_0]})^{\frac{p}{q}}
\]

which, by Corollary 2.4 and Corollary 2.3, we get from the uniform bound from Proposition 2.1. The convergence to \(v\) of this subsequence is thus in the sense of the \(\|\cdot\|_{x,q-\text{var},[0,T_0]}\)-norm.

- The continuity result on Young integrals recalled in Corollary \([\text{A.2}]\) from Appendix \([A]\) implies then the convergence in the \(\|\cdot\|_{x,C^\alpha,[0,T_0]}\)-norm of \(z^{m_k}\) to the path \(z\) defined by the equation

\[
z_t = z_0 + \int_0^t v_u \, dx_u.
\]

- It remains to prove that \(v_t \in F(z_t)\), for all times \(0 \leq t \leq T_0\). For a dyadic time \(t \in \bigcup_{m \geq 0} \pi^{(m)}\), then for \(k\) big enough, one has \(v^{m_k}_t \in F(z^{m_k}_t)\). Then, since \(F\) is \(\gamma\)-Hölder, one has

\[
d(v_t, F(z_t)) \leq \|v_t - v^{m_k}_t\| + d(F(z^{m_k}_t), F(z_t)) \leq \|v_t - v^{m_k}_t\| + \|F\|_{C^\gamma} \|z^{m_k}_t - z_t\|^\gamma.
\]

so one gets \(d(v_t, F(z_t)) = 0\), and \(v_t \in F(z_t)\), since \(F(z_t)\) is closed. For an non-dyadic time \(t \in [0, T_0]\), there exists two consecutive points in \(\pi^{(m)}\) such that \(t \in [u, v]\), for every \(m \geq 0\). Then,

\[
d(v_t, F(z_t)) \leq \|v_t - v^u_t\| + \|v^u_t - v^m_t\| + d(v^m_t, F(z^m_t)) + d(F(z^m_t), F(z_t))
\]

while we have from Corollary \([\text{2.4}]\) and Corollary \([\text{2.3}]\)

\[
d(v_t, F(z_t)) \leq \|v - v^{m_k}\|_{x,[0,T]} + (T_0/2)^{-m_k\gamma}\beta + \|F\|_{C^\gamma} \|z^{m_k}_t - z_t\|^\gamma + \|F\|_{C^\gamma} \|z_t\|^\gamma
\]

and

\[
d(v_t, F(z_t)) \leq \|v - v^{m_k}\|_{x,[0,T]} + (T_0/2)^{-m_k\gamma}\beta + \|F\|_{C^\gamma} \|z^{m_k}_t - z_t\|^\gamma + \|z\|_{C^\alpha} (T_0)^{-m_k\gamma\alpha}.
\]

So one gets \(d(v_t, F(z_t)) = 0\), and \(v_t \in F(z_t)\), since \(F(z_t)\) is closed.

### A – Basics on Young integrals

Let \(p > 1\) be given. Recall that an \(\mathbb{R}^d\)-valued path \(x\) defined on the time interval \([a, b]\) has finite \(p\)-variation over that interval if

\[
\|y\|_{p-\text{var}} := \|y\|_{p-\text{var},[a,b]} := \sup_{i} \|y_{i+1} - y_i\|^p < \infty.
\]
where the supremum runs over the set of finite partitions \( \{ s_i \} \) of \([a, b]\). We denote by \( V_p([a, b], \mathbb{R}^d) \) the set of \( \mathbb{R}^d \)-valued path with finite \( p \)-variation over \([a, b]\). Note that an element of \( V_p([a, b], \mathbb{R}^d) \) need not be continuous, while \( C^{1/p}([a, b], \mathbb{R}^d) \subset V_p([a, b], \mathbb{R}^d) \). Also, a continuous path with finite \( p \)-variation has a reparametrized version that is \( \frac{1}{p} \)-Hölder over its interval of definition. Refer to \( [6] \) for a reference. We endow the space of paths with finite \( p \)-variation with the norm

\[
\|y\|_{p\text{-var}, \infty} := \|y\|_{p\text{-var}} + \|y\|_\infty. \tag{A.1}
\]

Similarly, we define a norm on the set of \( \mathbb{R}^d \)-valued \( \alpha \)-Hölder functions defined on the interval \([a, b]\), setting

\[
|x|_{C^{\alpha}, \infty} := |x|_{C^{\alpha}([a,b],\mathbb{R}^d)} + |x|_\infty.
\]

Given \( a \leq b \), set \( \Delta_{a,b} := \{(s,t); a \leq s \leq t \leq b \} \). A \textbf{control} over the interval \([a,b]\) is a map \( \omega : \Delta_{a,b} \mapsto \mathbb{R}^+ \) that is null on the diagonal, is non-increasing, resp. non-decreasing, as a function of each of its first, resp. second, argument, and is sub-additive

\[
\omega(s,u) + \omega(u,t) \leq \omega(s,t), \quad \forall a \leq s \leq u \leq t \leq b.
\]

Denote by \( \omega(s,t^-) \) the left limit in \( t \) of the non-decreasing function \( \omega(s, \cdot) \). A control is said to be \textbf{regular} if it is continuous in a neighbourhood of the diagonal. As an example, for any path \( y \in V_p([a, b], \mathbb{R}^d) \), the function \( \omega_y(s,t) := \|y\|_{p\text{-var},[s,t]} \) is a control. For an \( \mathbb{R}^d \)-valued \( \alpha \)-Hölder path \( x \), with \( 0 < \alpha < 1 \), the function \( \omega_x(s,t) := \|x\|_{C^{\alpha}([s,t],\mathbb{R}^d)} \) is a regular control.

From the present day perspective, Gubinelli’ sewing lemma \( [11] \) offers an easy road to constructing the Young integral – see also \( [9] \). We give here a variation due to Friz and Zhang \( [10] \), tailor-made to our needs. Given a map \( \mu : \Delta_{a,b} \mapsto \mathbb{R}^d \) and a finite partition \( \pi = \{ s_i \} \) of the interval \([a, b] \), set

\[
\mu_\pi := \sum_i \mu_{s+1,i}.
\]

\begin{proposition}
Let a map \( \mu : \Delta_{a,b} \mapsto \mathbb{R}^d \) be given. If there exists positive exponents \( \alpha_1, \alpha_2 \), with \( \alpha_1 + \alpha_2 > 1 \) and controls \( \omega_1, \omega_2 \), with

\[
|\mu_{ts} - (\mu_{tu} + \mu_{us})| \leq \omega_1(s,u)\alpha_1 \omega_2(u,t)\alpha_2, \quad \forall a \leq s \leq u \leq t \leq b,
\]

with \( \omega_2 \) regular, then, for any interval \([s,t] \subset [a,b] \), the \( \mu_\pi_{s,t} \) converge to some limit \( I^*_\mu(s,t) \) as the mesh of the partition \( \pi_{s,t} \) of \([s,t] \) tends to 0, and one has

\[
|I^*_\mu(s,t) - \mu_\pi_{s,t}| \leq C \omega_1(s,t^-)^\alpha_1 \omega_2(s,t)^\alpha_2,
\]

for some positive constant \( C \) depending only on \( \alpha_1, \alpha_2 \).

Given \( x \in C^\alpha([a,b],\mathbb{R}^d) \) and \( y \in V_p([a,b], L(\mathbb{R}^d, \mathbb{R}^d)) \), set

\[
\mu_{st} := y_u(x_t - x_s),
\]

and note that

\[
\mu_{ts} - (\mu_{tu} + \mu_{us}) = (y_u - y_0)(x_t - x_u), \quad s \leq u \leq t,
\]

so one has for any interval \([s,t] \) the estimate

\[
|\mu_{ts} - (\mu_{tu} + \mu_{us})| \leq \|y\|_{p\text{-var},[s,t]} \|x\|_{C^{1/\alpha}([s,t],\mathbb{R}^d)}^\alpha.
\]

\end{proposition}

\begin{corollary}
If \( \frac{1}{q} + \alpha > 1 \), the Riemann sums \( \mu_\pi_{s,t} \) associated with the preceding two-index map \( \mu \) converge to some limit which we denote by \( \int s^t y_u dx_u \). One has

\[
\left\| \int_0^s y_u dx_u \right\|_{C^\alpha([s,t],\mathbb{R}^d)} \leq \|y\|_{p\text{-var},[s,t]} \|x\|_{C^\alpha([s,t],\mathbb{R}^d)},
\]

\end{corollary}
and for any \(\varepsilon > 0\) with \(\frac{1-\varepsilon}{\gamma} + \alpha > 1\), and \(y, y' \in \mathbb{V}_{q}(\{a, b\}, L(\mathbb{R}, \mathbb{R}^d))\), one has
\[
\left\| \int_0^y y_a dx_u - \int_0^{y'} y_a dx_u \right\|_{C^\alpha([s,t], \mathbb{R}^d)} \leq \left\| y - y' \right\|_{\mathbb{V}_{q}(\mathbb{R}, \mathbb{R}^d)} \left(\left\| y \right\|_{C^{rac{1}{\gamma} - \varepsilon \varreg \nu_{\gamma}}([s,t], \mathbb{R}^d)} + \left\| y' \right\|_{C^{rac{1}{\gamma} - \varepsilon \varreg \nu_{\gamma}}([s,t], \mathbb{R}^d)} \right) \left\| x \right\|_{C^\alpha([s,t], \mathbb{R}^d)}.
\]

**B – Selection results**

As a by product of the proof of Theorem 8.1, we obtain the following Selection result which partially answers to Remark 8.1 of Chistyakov and Galkin’s work [6].

**Proposition B.1.** Let \(F : [0, 1] \mapsto \mathbb{R}^d\), be a bounded, \(\gamma\)-Hölder, set valued map with values in compact subsets of \(\mathbb{R}^d\). Then, for any \(p > \frac{1}{\gamma}\) and any \(x_0 \in F(0)\), there exists a map \(f : [0, 1] \mapsto \mathbb{R}^d\), of finite \(p\)-variation, such that \(f(t) \in F(t)\), for all \(0 \leq t \leq 1\), which furthermore satisfies the estimate
\[
\|f\|_{p\text{-var}} \leq \|F\|_{C^\gamma}.
\]

**Proof –** We use the same notations as in the body of the text. Define for each non-negative integer \(m\) the partition \(\pi^m := \{t_i^m\}_{i=0,2m}\) of the interval \([0, 1]\), with \(t_i^m := i2^{-m}\). We define as follows a path \(x^m : [0, 1] \mapsto \mathbb{R}^d\), on each sub-interval \([0, t_i^m]\), recursively over \(i\).

- For \(m = 0\), set \(x_0(t) = x_0\), for all \(0 \leq t \leq 1\).
- For \(m \geq 1\), set first \(x^m(t) = x_0\), on \([0, t_0^m]\), and assuming \(x^m\) has been constructed on the time interval \([0, t_i^m]\), set \(x^m_{t_i^m} = x_i^m\), if \(t = T\), otherwise choose \(x_i^m \in F(t_i^m)\) such that
\[
d \left( x_i^m, x_s(t_i^m) \right) \leq \|F\|_{C^\gamma, \gamma} \left(2^{-M(r+1)}\right)\gamma
\]
and set \(x_i^m = x_i^m\), for \(t_i^m \leq t < t_{i+1}^m\).

We first prove that we have for all \(r, m\) the estimate
\[
\max_{t \in [0,1]} \sup_{s(t_i)} \left| x_i^m \right| \leq \frac{\|F\|_{C^\gamma, \gamma}}{1 - 2^{-\gamma}} 2^{-r\gamma}. \tag{B.1}
\]
By construction, \(x^m\) is constant on the interval of \(\pi^m\), so if \(r \geq m\) and \(t \in \pi^r\) and \(s \in [t, t+2^{-r})\), then \(x_s^m = 0\) and (B.1) holds true. Let then consider the case where \(r < m\). We define a finite sequence of times \((s_i)_{i=0, \ldots, m-r}\), with \(s_0 = t\), such that for \(0 \leq i \leq (m-r-1)\), we have \(s_{i+1} = s_i + 2^{-r-i}\), if \(s_i + 2^{-r-i} \leq s\), and \(s_{i+1} = s_i\) otherwise. Then, for \(s_i \in \pi^{r+1}\), for \(i \geq 1\), and \((s_i)_{i=0, \ldots, m-r}\), with possibly empty sets.

**Summing these estimates for \(0 \leq i \leq m-r\), gives (B.1), and**
\[
\frac{\max_{t \in [0,1]} \text{osc} (x^m, [t, t+2^{-r}))}{1 - 2^{-\gamma}} \leq \frac{\|F\|_{C^\gamma, \gamma}}{1 - 2^{-\gamma}} 2^{-r\gamma}. \tag{B.2}
\]
Since \(x^m\) is constant along the sub-intervals of the partition \(\pi^m\) it is enough, to compute the \(p\)-variation of \(x^m\) along any partition \(\pi\) of \([0, 1]\), to assume that all the partition times \(s_i \in \pi^m\). Let then define as follows a finite partition \((K_j)_{j=0, \ldots, m}\) of \([0, N - 1]\), with possibly empty sets.

- \(I_1 := \{ \ell \in [0, N] ; 2^{-1} \in (s_{\ell}, s_{\ell+1}) \}\),
- \(I_j := \{ \ell \in [0, N] ; \exists t \in \pi^{j-1} \setminus \pi^j, t \in (s_{\ell}, s_{\ell+1}) \}, \quad 2 \leq j \leq m\)
\[ K_0 := I_0, \]
\[ K_{j+1} := I_{j+1} \left( \bigcup_{k=1}^{j} I_k \right). \]

For \( \ell \in K_1 \), using (B.2) for \( r = 0 \) we bound
\[ |x_{s_{t+1}}^m - x_{s_t}^m| \leq \frac{2\|F\|_{C^\gamma}}{1 - 2^{-\gamma}}. \]  
(B.3)

For \( \ell \in K_j \), then there exists \( t \in \pi^j \setminus \pi^{j-1} \) such that
\[ s_{\ell-1} < t \leq s_{\ell}, \]
and any \( u \in \pi^{j-1} \) satisfies either \( u \leq s_{\ell-1} \) or \( u > s_{\ell} \). Then \( [s_{\ell}, s_{\ell+1}] \subset [t - 2^{-j}, t + 2^j] \), and using Lemma (B.2) for \( j \) = 1, we see that
\[ |x_{s_{t+1}}^m - x_{s_t}^m| \leq \frac{2\|F\|_{C^\gamma}}{1 - 2^{-\gamma}} 2^{-(j-1)\gamma}. \]  
(B.4)

The number of indices \( \ell \in K_j \) is at most \( 2^j \). Using the fact that the family \( \{K_j\}_{j \leq m} \) defines a partition of \([1, N]\), and adding inequalities (B.3) and (B.4) for \( q > 1/\gamma \), we then have
\[ \sum_{i=0}^{N-1} |x_{s_{i+1}}^m - x_{s_i}^m|^q \leq \frac{2\|F\|_{C^\gamma} 2^{\gamma}}{1 - 2^{-\gamma}} \sum_{j=0}^{m} 2^{j2^{-(j-1)\gamma}q} \]
\[ \leq \frac{2\|F\|_{C^\gamma} 2^{\gamma}}{1 - 2^{-\gamma}} \frac{1}{1 - 2^{-\gamma q + 1}} \]
and we derive
\[ \|x^m\|_{\text{var}} \leq \frac{2\|F\|_{C^\gamma}}{1 - 2^{-\gamma}} \frac{1}{1 - 2^{-\gamma q + 1} \gamma q}. \]

From Helly’s selection principle, Theorem 6.1 in [6], the sequence \( x^m \) has a convergent subsequence in \( p \)-variation, for any \( p > q \). We identify the path \( f \) from the statement as such a limit. One proves that \( f \) is a selection of \( F \) in the same way as we proved that \( v_t \in F(z_t) \) in Section 2.23 using the fact that \( F(t) \) is closed for all times, and the regularity properties of \( F \).

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