On the classification of finite dimensional irreducible modules for affine BMW algebras

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Abstract In this paper, we classify the finite dimensional irreducible modules for affine BMW algebra over an algebraically closed field with arbitrary characteristic.

1 Introduction

In [19, Definition 4], Haering-Oldenburg introduced a class of associative algebras called affine Birman–Murakami–Wenzl (BMW for brevity) algebras in order to study knot invariants. These algebras, which can be considered as an affinization of BMW algebras in [6,21], have been studied extensively by many authors in [10–12,14–16,23,26–30,32–35] etc.

The representation theory of an affine BMW algebra can be studied via its cyclotomic quotients, called cyclotomic BMW algebras. In order to get a well-behaved representation theory of cyclotomic BMW algebras, one needs to impose some additional conditions on the parameters of the ground ring. The suitable condition, which is called admissibility, was found in [33, Definition 3.1] and [30, Definition 2.27] via explicit but complicated relations on the parameters. Wilcox and Yu [33] showed their relations are equivalent to a very simple module theoretic statement, which we take as the definition of admissibility; see Definition 2.4 for details.

Recently, Goodman [14] introduced the notion of d-semi-admissibility, and used it to study the representations of cyclotomic BMW algebras with arbitrary parameters over a field, by relating them to representations of cyclotomic BMW algebras with admissible parameters, see [29,30]. Combining this with [36, Corollary 3.14] and [30, Theorem 5.3], one obtains all finite dimensional irreducible modules for affine BMW algebras over an algebraically closed field $\kappa$ with arbitrary characteristic.

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In order to classify finite dimensional irreducible modules for an affine BMW algebra over an algebraically closed field $\kappa$, we have to determine whether two irreducible modules for different cyclotomic BMW algebras are isomorphic as modules for the affine BMW algebra. For this, we need a similar result for the extended affine Hecke algebra $\mathcal{H}_n$ of type $A_{n-1}$ as follows.

The first result on the classification of irreducible $\mathcal{H}_n$-modules is due to Bernstein and Zelevinsky [7,38], who classified the irreducible $\mathcal{H}_n$-modules over $\mathbb{C}$ when the defining parameter $q$ is not a root of unity. In this case, they used multisegments of length $n$ to index a complete set of non-isomorphic irreducible $\mathcal{H}_n$-modules. In [24], Rogawski gave a different method to reprove Bernstein and Zelevinsky’s result. Note that Kazhdan-Lusztig [20] and Xi [37] classified the finite dimensional irreducible modules for affine Hecke algebras in any type. This includes the extended affine Hecke algebra of type $A_{n-1}$ as a special case.

On the other hand, any irreducible $\mathcal{H}_n$-module over an algebraically closed field $\kappa$ can be realized as an irreducible module for an Ariki-Koike algebra [2]. In the later case, Ariki [1, Theorem 4.2] showed its irreducible modules are indexed by Kleshchev multipartitions in the sense of the definition given in page 605 of [4]. In [31, Corollary 3.6], Vazirani gave an explicit relationship between the set of Kleshchev multipartitions and the set of multisegments when the ground field is $\mathbb{C}$ and $q$ is not a root of unity. If $q$ is a root of unity, the irreducible $\mathcal{H}_n$-modules have been classified via aperiodic multisegments in [8] (resp. [4, Theorem B]) over $\mathbb{C}$ (resp. over $\kappa$), and the comparison of irreducible modules between Ariki–Koike algebras and the affine Hecke algebras is established in [3, Theorem 6.2]. This is the result that we need for classifying finite dimensional irreducible modules for affine BMW algebras over $\kappa$.

Let $e$ be the smallest positive integer such that

$$1 + q^2 + q^4 + \cdots + q^{2(e-1)} = 0$$

in $\kappa$. We remark that the current $q^2$ is $q$ in [4]. If there is no such a positive integer, then we set $e = \infty$. A segment $\Delta$ of length $j = |\Delta|$ is a sequence of consecutive residues $[i, i+1, \ldots, i+j-1]$ where $i, i+1, \ldots, i+j-1 \in \mathbb{Z}_e$. A multisegment $\Delta$ is an unordered collection of segments $\Delta_i$ with length $\sum_i |\Delta_i|$. Following statements in page 602 in [4], we says that $\Delta$ is aperiodic if for every $j$, there is an $i \in \mathbb{Z}_e$ such that $[i, i+1, \ldots, i+j-1]$ does not appear in $\Delta$.

For any rational number $a$, let $\lfloor a \rfloor$ be the maximal integer which is less than $a$. The aim of this paper is to classify the finite dimensional irreducible modules for affine BMW algebras $\tilde{\mathcal{H}}_n$ over $\kappa$ via pairs $(f, \Delta)$ with $0 \leq f \leq \lfloor n/2 \rfloor$ and a family of aperiodic multisegments $\Delta$. See Theorem 3.10 for details.

We organized our paper as follows. We recall some results on affine BMW algebras and its cyclotomic quotients in Sect. 2 and classify finite dimensional irreducible modules for affine BMW algebras in Sect. 3.

### 2 Affine BMW algebras and cyclotomic BMW algebras

Throughout, let $\kappa$ be an algebraically closed field which contains non-zero elements $q, \varrho$, and a family of elements $\Omega = \{\omega_i \mid i \in \mathbb{N}\}$ such that $q - q^{-1} \neq 0$ and $\omega_0 = 1 - (q - q^{-1})^{-1}(\varrho - \varrho^{-1})$. Let $n$ be a positive integer with $n \geq 2$.

In this section, we recall some results on affine BMW algebras and cyclotomic BMW algebras over $\kappa$; some of the results are valid over an integral domain.
Definition 2.1 [19, Definition 4] The affine BMW algebra \( \hat{\mathcal{B}}_n(\Omega, q, \rho) \) is the unital associative \( \kappa \)-algebra generated by \( g_i, e_i, x_i^{\pm 1} \), \( 1 \leq i \leq n-1 \) subject to the following relations:

1. \( x_1 x_1^{-1} = x_1^{-1} x_1 = 1 \) and \( g_i g_i^{-1} = g_i^{-1} g_i = 1 \), for \( 1 \leq i \leq n \),
2. \( g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \), for \( 1 \leq i < n - 1 \),
3. \( g_i g_j = g_j g_i \) if \( |i - j| > 1 \),
4. \( x_1 g_i x_1 g_i = g_i x_1 g_1 x_1, \) and \( x_1 g_j = g_j x_1 \) for \( j \geq 2 \),
5. \( e_i^2 = \omega_0 e_i \), for \( 1 \leq i < n \),
6. \( e_i x_i^a e_i = \omega_a e_i \), for \( a \in \mathbb{Z}^{>0} \),
7. \( x_1 g_j = g_j x_1 \), for \( 2 \leq j \leq n - 1 \),
8. \( g_i e_j = e_j g_i \), and \( e_i e_j = e_j e_i \) if \( |i - j| > 1 \),
9. \( e_i g_i = g_i e_i \), for \( 1 \leq i \leq n - 1 \),
10. \( e_i g_i \pm e_i = \omega e_i \), \( e_i e_i \pm e_i = e_i \),
11. \( g_i g_i \pm e_i = e_i g_i \pm e_i \) and \( e_i g_i \pm e_i = e_i e_i \),
12. \( g_i - g_i^{-1} = (q - q^{-1})(1 - e_i) \), for \( 1 \leq i \leq n \),
13. \( e_i x_1 g_i x_1 g_i = e_1 = g_1 x_1 g_1 x_1 e_1 \).

Throughout, we use \( \hat{\mathcal{B}}_n \) instead of \( \hat{\mathcal{B}}_n(\Omega, q, \rho) \). In other words, \( \hat{\mathcal{B}}_n \) always defines the affine BMW algebra with these parameters.

There is a slight difference between Definition 2.1 and that given in [19, Definition 4]. Haering-Oldenburg did not assume that \( x_1 \) is invertible.

Because of the symmetry of the relations in Definition 2.1, there is a unique algebra anti-involution * : \( \hat{\mathcal{B}}_n \rightarrow \hat{\mathcal{B}}_n \) which fixes generators \( g_i, e_i \) and \( x_1 \), \( 1 \leq i \leq n - 1 \). By [16, Remark 3.7(4)] there exist elements \( \omega_{-a} \in \kappa \) for all positive integers \( a \) such that \( e_1 x_1^{-a} e_1 = \omega_{-a} e_1 \). Further, \( \omega_{-a} \) is a polynomial in \( \omega_b \)'s, \( 1 \leq b \leq a \). Therefore, \( \omega_a \) is well-defined for all \( a \in \mathbb{Z} \).

The following result is due to Goodman and Hauschild Mosley. In fact, their result is valid over an integral domain.

Theorem 2.2 [16, Theorem 6.13] \( \hat{\mathcal{B}}_n \) is of infinite dimension over \( \kappa \).

It is well-known that an affine Wenzl algebra in [22, (4.1–4.4)] can be considered as a degenerate affine BMW algebra. Ariki, Mathas and Rui [5, Example 2.17] constructed an infinite dimensional irreducible module for affine Wenzl algebra. Mimicking this construction, we know that \( \hat{\mathcal{B}}_n \) has infinite dimensional irreducible modules over a field. In other words, \( \hat{\mathcal{B}}_n \) is not finitely generated over its center. For the description of the center of \( \hat{\mathcal{B}}_n \), see [10, Theorem 4.5].

Definition 2.3 [19, Definition 4] Let \( I \) be the two-sided ideal of \( \hat{\mathcal{B}}_n \) generated by the cyclotomic polynomial

\[
f(x_1) = (x_1 - u_1)(x_1 - u_2) \cdots (x_1 - u_r),
\]

where \( u_i \in \kappa^* \), \( 1 \leq i \leq r \). The cyclotomic BMW algebra \( \mathcal{B}_{r,n}(\Omega, q, \rho, u) \) is the quotient algebra \( \hat{\mathcal{B}}_n/I \) where \( u = (u_1, u_2, \cdots, u_r) \).

Note that the algebra anti-involution * on \( \hat{\mathcal{B}}_n \) induces an algebra anti-involution * on \( \mathcal{B}_{r,n}(\Omega, q, \rho, u) \).

Throughout, we use \( \mathcal{B}_{r,n}(u) \) instead of \( \mathcal{B}_{r,n}(\Omega, q, \rho, u) \). In other words, \( \mathcal{B}_{r,n}(u) \) always means the cyclotomic BMW algebra with these parameters. We remark that Haering-Oldenburg [19, Definition 4] defined \( \mathcal{B}_{r,n}(u) \) without assuming \( u_i \in \kappa^* \), \( 1 \leq i \leq r \) since he did not assume \( x_1 \) is invertible in \( \hat{\mathcal{B}}_n \). When \( r = 1 \), \( \mathcal{B}_{1,n}(u) \) is the usual BMW algebra, which was introduced by Birman–Wenzl [6] and independently by Murakami [21].
It is known that $B_{r,n}(u)$ can be used to study finite dimensional irreducible $\hat{B}_n$-modules over $\kappa$. Pick a finite dimensional irreducible $\hat{B}_n$-module $M$ over $\kappa$. Let $f(x_1)$ be the characteristic polynomial of $x_1$ with respect to $M$. Since $\kappa$ is an algebraically close field, there exists $u = (u_1, u_2, \ldots, u_r)$ with $u_i \in \kappa$, $1 \leq i \leq r$ such that $f(x_1)$ factors as in (2.1). Further, $u_i \in \kappa^*$ since $x_1$ is invertible in $B_{r,n}(u)$. So, $M$ is an irreducible $B_{r,n}(u)$-module where $B_{r,n}(u)$ is given in Definition 2.3. Therefore, we will get all finite dimensional irreducible $\hat{B}_n$-modules over $\kappa$ if we classify the irreducible $B_{r,n}(u)$-modules for all $u \in (\kappa^*)^r$ and $r \geq 1$.

In order to study representations of $B_{r,n}(u)$ over $\kappa$, we need the notion of admissibility from [33, Corollary 4.5].

**Definition 2.4** [33, Corollary 4.5] We say that $B_{r,n}(u)$ is admissible or has admissible parameters if $\{e_1, e_1x_1, \ldots, e_1x_1^{r-1}\}$ is linear independent in $B_{r,2}(u)$.

Explicit conditions on the parameters which are equivalent to admissibility have been given by Rui and Xu in [30, Definition 2.27] and by Wilcox and Yu in [33, Definition 3.1]. The equivalence of admissibility, as defined in Definition 2.4, with the explicit conditions in [30, Definition 2.27] and [33, Definition 3.1] was shown in [33, Corollary 4.5] and [15, Theorem 4.4].

The following definition is due to Goodman [14].

**Definition 2.5** [14, Definition 5.2] We say that $B_{r,n}(u)$ is $d$–semi-admissible or has $d$–semi-admissible parameters if $d$ is the minimal integer such that $\{e_1, e_1x_1, \ldots, e_1x_1^d\}$ is linear dependent in $B_{r,2}(u)$.

Evidently, for any cyclotomic BMW algebra $B_{r,n}(u)$, there is a unique $d$ with $0 \leq d \leq r$ such that $B_{r,n}(u)$ is $d$-semi-admissible. We have $e_1 = 0$ if $d = 0$. In this case, there is no restriction on $u_i$’s. Further, when $e_1 = 0$, $B_{r,n}(u)$ is the Ariki-Koike algebra $H_{r,n}(u)$ [2] whose irreducible modules have been classified in [1, Theorem 4.2]. Note that $r$-semi-admissibility is the same as admissibility; in particular, $B_{1,n}(u)$ is always admissible if $e_1 \neq 0$.

**Lemma 2.6** Suppose $B_{r,2}(u)$ is admissible. We have $\omega_i \neq 0$ for some $i$, $0 \leq i \leq r - 1$ if there is a $j \in \mathbb{Z}$ such that $\omega_j \neq 0$.

**Proof** The result follows from the fact that the cyclotomic condition $f(x_1) = 0$ in $B_{r,2}(u)$ implies a periodicity condition on the $\omega_i$’s. \qed

We recall the notion of cellular algebras from [17].

**Definition 2.7** [17, Definition 1.1] Assume that $R$ is a commutative ring with multiplicative identity 1. Let $A$ be an $R$–algebra with an $R$-linear algebra anti-involution $*: A \to A$. Fix a finite partially ordered set $\Lambda = (\Lambda, \geq)$ and for each $\lambda \in \Lambda$ let $T(\lambda)$ be a finite set. Finally, let

$$\mathcal{M} = \{ m_{s,t} | \lambda \in \Lambda \text{ and } s, t \in T(\lambda) \}$$

be a subset of $A$. Then the triple $(\Lambda, T, \mathcal{M})$ is a cell datum for $A$ if

(a) $\mathcal{M}$ is an $R$–basis for $A$;
(b) $(m_{s,t})^* = m_{t,s}$, for all $\lambda \in \Lambda$ and all $s, t \in T(\lambda)$;
(c) for all $\lambda \in \Lambda$, $s \in T(\lambda)$ and $a \in A$ there exist scalars $r_{tu}(a) \in R$ such that

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The bilinear form \( \phi_\lambda \) is an \( A \)-method to classify irreducible modules for cellular algebras. An algebra \( A \) is a cellular algebra if it has a cell datum and in this case we call \( \mathcal{M} \) a cellular basis of \( A \).

The notion of weakly cellular algebras in [13, Definition 2.9] is obtained from Definition 2.7 by using

\[
(m_{s\ell})^* \equiv m_{t_\ell} \pmod{A^{\triangleright \lambda}}
\]

\[\text{(2.2)}\]

instead of \((m_{s\ell})^* = m_{t_\ell}\) in Definition 2.7(b). The results and proofs of [17] are equally valid for weakly cellular algebras, so in the remainder of the paper we will not distinguish between cellular algebras and weakly cellular algebras.

Now, we briefly recall the representation theory of cellular algebras over a field as developed in [17]. We remark that all modules considered in this paper are right modules.

A cellular algebra \( A \) has a family of distinguished modules, called cell modules, described as follows: for each \( \lambda \in \Lambda \), let \( A^{\triangleright \lambda} \) be the \( R \)-span in \( A \) of \( \{ m_{uv} | \mu \triangleright \lambda \} \) and \( u, v \in T(\mu) \). For each \( \lambda \in \Lambda \) fix \( s \in T(\lambda) \) and let

\[ m_t = m_{s\ell} + A^{\triangleright \lambda}. \]

Then the cell module \( S^\lambda \) is the submodule of \( A^{\triangleright \lambda} / A^{\triangleright \lambda} \) spanned by \( \{ m_t : t \in T(\lambda) \} \). Evidently, \( S^\lambda \) is a free as an \( R \)-module with basis \( \{ m_t : t \in T(\lambda) \} \). The cell modules have the following relation to the simple \( A \)-modules: Every irreducible \( A \)-module arises in a unique way as the simple head of some cell module.

The cell module \( S^\lambda \) comes equipped with a natural symmetric bilinear form \( \phi_\lambda \) which is determined by the equation

\[ m_{s\ell} m_{t_\ell}^* \equiv \phi_\lambda (m_t, m_{t_\ell}) \cdot m_{s\ell} \pmod{A^{\triangleright \lambda}}. \]

The bilinear form \( \phi_\lambda \) is \( A \)-invariant in the sense that

\[ \phi_\lambda(xa, y) = \phi_\lambda(x, ya^*), \]

for \( x, y \in S^\lambda \) and \( a \in A \). Consequently,

\[ \text{Rad } S^\lambda = \{ x \in S^\lambda | \phi_\lambda(x, y) = 0 \text{ for all } y \in S^\lambda \} \]

is an \( A \)-submodule of \( S^\lambda \). It is show in [17] that \( D^\lambda = S^\lambda / \text{Rad } S^\lambda \) is either zero or absolutely irreducible.

Moreover, Graham and Lehrer [17, Theorem 3.4] show that the set of non-zero \( D^\lambda \) constitute a complete set of pairwise non-isomorphic irreducible \( A \)-modules. This gives a useful method to classify irreducible modules for cellular algebras.

Recall that a composition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( m \) is a sequence of non-negative integers with \( |\lambda| = \sum_i \lambda_i = m \). If \( \lambda \) is weakly decreasing, then \( \lambda \) is called a partition. Similarly, an \( r \)-partition of \( m \) is an ordered \( r \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) of partitions such that \( |\lambda| = \sum_{i=1}^{r} |\lambda^{(i)}| = m \). Such a \( \lambda \) will be called a multipartition if \( r \) is understood from context. Let \( \Lambda^n_r \) be the set of all multipartitions of \( n \). We say that \( \mu \) dominates \( \lambda \) and write \( \lambda \triangleright \mu \) if

\[ \sum_{j=1}^{i-1} \lambda^{(j)} + \sum_{k=1}^{l} \lambda_k^{(i)} \leq \sum_{j=1}^{i-1} \mu^{(j)} + \sum_{k=1}^{l} \mu_k^{(i)} \]
for $1 \leq i \leq r$ and $l \geq 0$. So, $(\Lambda^+_1(n), \leq)$ is a poset. If $\lambda \leq \mu$ and $\lambda \neq \mu$, we write $\lambda < \mu$.

Let

$$\Lambda_{r,n} = \{(k, \lambda) \mid 0 \leq k \leq \lfloor n/2 \rfloor, \lambda \in \Lambda^+_1(n - 2k)\}. \quad (2.3)$$

Then $\Lambda_{r,n}$ is a poset with $\leq$ as a partial order on it. More explicitly, $(\ell, \mu) \leq (k, \lambda)$ for $(k, \lambda), (\ell, \mu) \in \Lambda_{r,n}$ if either $\ell < k$ in the usual sense or $\ell = k$ and $\mu \leq \lambda$ in $\Lambda^+_1(n - 2k)$.

The following theorem has been proved independently by Goodman, Rui-Xu and Wilcox-Yu.

**Theorem 2.8** [13, Theorem 4.5], [30, Theorem 4.19], [34, Theorem 3.5] Suppose $B_{r,n}(u)$ is admissible over $\kappa$. Then $B_{r,n}(u)$ is (weakly) cellular with poset $\Lambda_{r,n}$. The dimension of $B_{r,n}(u)$ is $r^n(2n - 1)$!!.

In this paper, we do not need the explicit definition of the (weakly) cellular basis of $B_{r,n}(u)$. What we will need is some properties of cell modules $S^f, \lambda \in \Lambda_{r,n}$, determined by the cellular basis of $B_{r,n}(u)$ in Theorem 2.8. Since we are going to use some results on the representations of $B_{r,n}(u)$ from [30], we define $S^{f, \lambda}$ via the version of the cellular basis in [30, Theorem 4.19]. Let $\phi_{f, \lambda}$ be the invariant form on $S^{f, \lambda}$ with respect to $(f, \lambda) \in \Lambda_{r,n}$.

We have

$$B_{r,n}(u)/I \cong \mathcal{H}_{r,n}(u), \quad (2.4)$$

where $\mathcal{H}_{r,n}(u)$ is the Ariki-Koike algebra [2] and $I$ is the two-sided ideal of $B_{r,n}(u)$ generated by $e_1$. We remark that part of the cellular basis of $B_{r,n}(u)$ in Theorem 2.8 maps injectively onto the cellular basis of $\mathcal{H}_{r,n}(u)$ in [9, 3.26] and part is killed by the quotient map. The cell module of $\mathcal{H}_{r,n}(u)$ labelled by $\lambda \in \Lambda^+_1(n)$ is denoted by $S^\lambda$. Let $\phi_\lambda$ be the invariant form on $S^\lambda$.

The following result has been proved in [30, 5.2] for an arbitrary field.

**Proposition 2.9** [30, Lemma 5.2] Suppose $B_{r,n}(u)$ is admissible over $\kappa$. Assume that $(f, \lambda) \in \Lambda_{r,n}$.

(a) If $f \neq n/2$, then $\phi_{f, \lambda} \neq 0$ if and only if $\phi_\lambda \neq 0$.

(b) If $\omega_a \neq 0$ for some non-negative integer $a \leq r - 1$, then $\phi_{n/2, 0} \neq 0$.

(c) If $\omega_a = 0$ for all non-negative integers $a \leq r - 1$, then $\phi_{n/2, 0} = 0$.

Note that $\phi_\lambda \neq 0$ if and only if $D^\lambda \neq 0$ for $B_{r,n-2f}(u)$.

**Theorem 2.10** [1, Theorem 4.2] Let $D^\lambda$ be defined via the cellular basis of $\mathcal{H}_{r,k}(u)$ in [9, 3.26]. Then $D^\lambda \neq 0$ if and only if $\lambda$ is a Kleshchev multipartition of $n - 2f$ in the sense of the definition in page 605 in [4].

By Theorem 2.10, irreducible $B_{r,n}(u)$-modules are classified via Proposition 2.9. More explicitly, we have the following result in [36] for $r = 1$ and [30] for $r \geq 2$. We remark that $B_{1,n}(u)$ is always admissible if $e_1 \neq 0$.

**Theorem 2.11** [30, Theorem 5.3], [36, Corollary 3.14] Suppose $B_{r,n}(u)$ is admissible over $\kappa$.

(a) If either $\omega_a \neq 0$ for some non-negative integer $a \leq r - 1$ or $\omega_a = 0$ for all non-negative integers $a \leq r - 1$ and $2 \nmid n$, then the irreducible $B_{r,n}(u)$-modules are indexed by the set of $(f, \lambda)$ with $0 \leq f \leq \lfloor n/2 \rfloor$ and $\lambda$ a Kleshchev multipartition of $n - 2f$. 

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(b) If \( \omega_a = 0 \) for all non-negative integers \( a \leq r - 1 \) and \( 2 \mid n \), then the irreducible \( B_{r,n}(u) \)-modules are indexed by the set of \( (f, \lambda) \) with \( 0 \leq f < [n/2] \) and \( \lambda \) a Kleshchev multipartition of \( n - 2f \).

In the remainder of this section, we recall Goodman’s result \([14]\) concerning cyclotomic BMW algebras with \( d \)-semi-admissible parameters, with \( 0 < d < r \). Recall that \( d \) is the minimal integer such that \( \{e_1, e_1x_1, \ldots, e_1x_d^0\} \) is linear dependent in \( B_{r,2}(u) \). Goodman showed that there is a polynomial \( g(x) \in \kappa[x] \) with \( \deg g(x) = d \) such that \( e_1g(x_1) = 0 \) and \( e_1h(x_1) \neq 0 \) in \( B_{r,2}(u) \) for any polynomial \( h(x) \in \kappa[x] \) with \( \deg h(x) < d \). Further, since \( e_1f(x_1) = 0 \) in \( B_{r,2}(u) \), it is not difficult to see that \( g(x_1) \mid f(x_1) \). So, write

\[
g(x_1) = (x_1 - v_1)(x_1 - v_2) \cdots (x_1 - v_d),
\]

where \( \{v_1, v_2, \ldots, v_d\} \subset \{u_1, u_2, \ldots, u_r\} \). Let \( \langle e_1 \rangle_r \) (resp. \( \langle e_1 \rangle_d \)) be the two-sided ideal of \( B_{r,n}(u) \) (resp. \( B_{d,n}(v) \)) generated by \( e_1 \).

**Theorem 2.12** \([14, \, 5.11]\) Suppose that \( B_{r,n}(u) \) is \( d \)-semi-admissible over \( \kappa \), where \( 0 < d < r \).

(a) There is a subset \( \{v_1, v_2, \ldots, v_d\} \subset \{u_1, u_2, \ldots, u_r\} \) such that the corresponding cyclotomic BMW algebra \( B_{d,n}(v) \) is admissible over \( \kappa \).

(b) There is a surjective homomorphism \( \theta : B_{r,n}(u) \to B_{d,n}(v) \) such that the restriction of \( \theta \) to \( \langle e_1 \rangle_r \) gives rise to an isomorphism between \( \langle e_1 \rangle_r \) and \( \langle e_1 \rangle_d \).

Since \( B_{d,n}(v) \) is admissible, \( \langle e_1 \rangle_d \) is cellular with a basis which is given in Theorem 2.8 for \( B_{d,n}(v) \) with respect to the poset which consists of all pairs \( (f, \lambda) \in \Lambda_{d,n} \) with \( f \geq 1 \). Via the isomorphism \( \theta \), Goodman \([17]\) lifted a cellular basis of \( \langle e_1 \rangle_d \) to get the corresponding cellular basis of \( \langle e_1 \rangle_r \). Using the epimorphism \( \pi : B_{r,n}(u) \to H_{r,n}(u) \), Goodman showed the following result.

**Theorem 2.13** \([14, \, \text{Theorem 6.4}]\) Suppose \( B_{r,n}(u) \) is \( d \)-semi-admissible over \( \kappa \). Then \( B_{r,n}(u) \) is (weakly) cellular with partially ordered set

\[
\tilde{\Lambda}_{r,n} = \bigcup_{1 \leq f \leq [n/2]} \{(f, \lambda) \mid (f, \lambda) \in \Lambda^{+}_d(n - 2f) \} \cup \{(0, \lambda) \mid \lambda \in \Lambda^{+}_d(n) \}
\]

in the sense of Definition 2.7. Further, \( \dim_{\kappa} B_{r,n}(u) = d^n(2n - 1)!! + r^n n! - d^n n! \).

We remark that \( (f, \lambda) \leq (\ell, \mu) \) for \( (f, \lambda), (\ell, \mu) \in \tilde{\Lambda}_{r,n} \) if either \( f < \ell \) or \( f = \ell \) and \( \lambda < \mu \), where \( \leq \) is the dominance order on \( \Lambda^{+}_d(n - 2f) \), if \( f \) is positive, or on \( \Lambda^{+}_d(n) \), if \( f = 0 \).

For each \( (f, \lambda) \in \Lambda_{d,n} \) with \( f > 0 \), let \( S^{f,\lambda} \) (resp. \( D^{f,\lambda} \)) be the cell (resp. irreducible) module of \( B_{d,n}(v) \) with respect to the cellular basis in Theorem 2.8. Then \( S^{f,\lambda} \) (resp. \( D^{f,\lambda} \)) can be considered as the corresponding cell (resp. irreducible) module of \( B_{r,n}(u) \) with respect to \( (f, \lambda) \in \tilde{\Lambda}_{r,n} \) such that \( f > 0 \).

Let \( J \) be the two-sided ideal of \( \hat{\mathcal{H}}_n \) generated by \( e_1 \). Then

\[
\hat{\mathcal{H}}_n / J \cong \hat{\mathcal{H}}_n
\]

(2.5)

where \( \hat{\mathcal{H}}_n \) is the extended affine Hecke algebra \( \hat{\mathcal{H}}_n \) of type \( A_{n-1} \).

The following result plays an important role in the classification of finite dimensional irreducible \( \hat{\mathcal{H}}_n \)-modules over \( \kappa \). It follows from Goodman’s result in Theorem 2.13 etc.
Theorem 2.14 Suppose that $M$ is an irreducible $\hat{\mathcal{D}}_n$-module over $\kappa$.

(a) If $Me_1 = 0$, then $M$ is an irreducible $\mathcal{H}_{r,n}(\mathbf{u})$-module for some positive integer $r$ and some $\mathbf{u} = (u_1, u_2, \cdots, u_r) \in (\kappa^*)^r$.

(b) If $Me_1 \neq 0$, then $M$ is an irreducible $\mathcal{D}_{r,n}(\mathbf{u})$-module for some positive integer $r$ and some $\mathbf{u} = (u_1, u_2, \cdots, u_r) \in (\kappa^*)^r$. In this case, we may assume that $\mathcal{D}_{r,n}(\mathbf{u})$ is admissible.

Proof If $Me_1 = 0$, then $M$ can be considered as an irreducible module for $\mathcal{H}_n$. Let $f(x_1)$ be the characteristic polynomial of $x_1$ with respect to $M$. Since we are assuming that $\kappa$ is an algebraically closed field, there exist $u_1, u_2, \cdots, u_r \in \kappa$ such that $f(x_1)$ factors as in (2.1). Further, since $x_1$ is invertible in $\mathcal{H}_n, u_i \in \kappa^*$. Therefore, $M$ is an irreducible $\mathcal{H}_n / (f(x_1))$-module. Note that $\mathcal{H}_{r,n}(\mathbf{u}) \cong \mathcal{H}_n / (f(x_1))$. This proves (a). By similar arguments, when $Me_1 \neq 0$, $M$ is an irreducible $\mathcal{D}_{r,n}(\mathbf{u})$-module for some positive integer $r$ and some $\mathbf{u} = (u_1, u_2, \cdots, u_r) \in (\kappa^*)^r$. By Theorem 2.13 and statements above (2.5), we may assume that $\mathcal{D}_{r,n}(\mathbf{u})$ is admissible. This proves (b). \hspace{1cm} \Box

3 Classification of finite dimensional irreducible $\hat{\mathcal{D}}_n$-modules

In this section, we classify finite dimensional irreducible $\hat{\mathcal{D}}_n$-modules over an algebraically closed field $\kappa$ with arbitrary characteristic.

Lemma 3.1 Suppose $n > 2$. If $\omega_0 \neq 0$, we define $e = \omega_0^{-1}e_{n-1}$. Otherwise, we define $e = \rho^{-1}e_{n-1}g_{n-2}$. Then $e^2 = e$ and $e\hat{\mathcal{D}}_n e = \hat{\mathcal{D}}_{n-2}e \cong \hat{\mathcal{D}}_{n-2}$ as $\kappa$-algebras.

Proof It follows from Definition 2.1(5)(10) that $e^2 = e$. By [16, 3.17, 3.20], $e_{n-1}\hat{\mathcal{D}}_{n-1}e_{n-1} = e_{n-1}\hat{\mathcal{D}}_{n-2}$ and $\hat{\mathcal{D}}_n e_{n-1} = \hat{\mathcal{D}}_{n-1}e_{n-1}$. Therefore, $e_{n-1}\hat{\mathcal{D}}_{n-1} = e_{n-1}\hat{\mathcal{D}}_{n-2}$.

For any $x, y \in \hat{\mathcal{D}}_{n-2}$, since $e_{n-1}$ commutes with all elements in $\hat{\mathcal{D}}_{n-2}$, we have $xye = xye$ if $e = \omega_0^{-1}e_{n-1}$. If $e = \rho^{-1}e_{n-1}g_{n-2}$, by Definition 2.1(10), we have

\[
xeye = \rho^{-2}xe_{n-1}g_{n-2}y e_{n-1}g_{n-2} = \rho^{-2}xe_{n-1}g_{n-2}e_{n-1}yg_{n-2} = \\
= \rho^{-1}xe_{n-1}yg_{n-2} = xye.
\]

So, $\hat{\mathcal{D}}_{n-2} \cong \hat{\mathcal{D}}_{n-2}e$ and the required isomorphism sends $x \in \hat{\mathcal{D}}_{n-2}$ to $xe$ for all $x \in \hat{\mathcal{D}}_{n-2}$. One can verify the injectivity of this homomorphism by using the result on the basis of $\hat{\mathcal{D}}_n$ in [13, Section 3.2]. \hspace{1cm} \Box

Let $\hat{\mathcal{D}}_n$-mod be the category of finite dimensional right $\hat{\mathcal{D}}_n$-modules over $\kappa$. By Lemma 3.1 and standard result in [18, section 6.1, 6.2a], we have an exact functor $\mathfrak{F} : \hat{\mathcal{D}}_n$-mod $\rightarrow$ $\hat{\mathcal{D}}_{n-2}$-mod such that

\[(3.1) \quad \mathfrak{F}(M) = Me \]

for any object $M \in \hat{\mathcal{D}}_n$-mod. For any choice $\mathbf{u}$, we have an epimorphism $\phi : \hat{\mathcal{D}}_n \rightarrow \mathcal{D}_{r,n}(\mathbf{u})$. So, $\mathcal{D}_{r,n}(\mathbf{u})$-mod, the category of finite dimensional right $\mathcal{D}_{r,n}(\mathbf{u})$-modules, is a full subcategory of $\hat{\mathcal{D}}_n$-mod.

For any object $M$ in $\mathcal{D}_{r,n}(\mathbf{u})$-mod, $\mathfrak{F}(M)$ is the same as $\mathcal{F}(M)$ where $\mathcal{F}$ is the exact functor from $\mathcal{D}_{r,n}(\mathbf{u})$-mod to $\mathcal{D}_{r-2}(\mathbf{u})$-module in [28, Proposition 3.29]. Therefore, we will not distinguish $\mathfrak{F}$ and $\mathcal{F}$.

As we explained in Sect. 2, we can always assume that $\mathcal{D}_{r,n}(\mathbf{u})$ is admissible when we discuss irreducible $\mathcal{D}_{r,n}(\mathbf{u})$-modules $D_{f,\lambda}$ with $f > 0$. \hspace{1cm} \Box

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Lemma 3.2 Suppose $\mathcal{B}_{r,n}(u)$ is admissible. For any $(f, \lambda) \in \Lambda_{r,n}$, let $S^{f,\lambda}$ denote the corresponding cell module, and let $D^{f,\lambda} = S^{f,\lambda}/\text{Rad}(S^{f,\lambda})$. Suppose $f > 0$.

(a) $\mathfrak{N}(S^{f,\lambda}) \cong S^{f-1,\lambda}$.

(b) If $D^{f,\lambda} \neq 0$, then $\mathfrak{N}(D^{f,\lambda}) \neq 0$ and $\mathfrak{N}(D^{f,\lambda}) \cong D^{f-1,\lambda}$.

Proof The first isomorphism has been proved in [28, Proposition 3.29a]. Suppose that $D^{f,\lambda} \neq 0$. By the exactness of $\mathfrak{N}$ and [28, Proposition 3.29e], there is a non-zero epimorphism from $S^{f-1,\lambda}$ to $\mathfrak{N}(D^{f,\lambda})$ forcing $\mathfrak{N}(D^{f,\lambda}) \neq 0$. By [18, 6.2b], $\mathfrak{N}(D^{f,\lambda})$ is an irreducible $\mathcal{B}_{r,n-2}(u)$-module. But a cell module of any cellular algebra has a unique simple quotient whenever the canonical bilinear form on the cell module is not identically zero [17, Proposition 3.2(ii)]. By Proposition 2.9, $D^{f-1,\lambda} \neq 0$. Hence $\mathfrak{N}(D^{f,\lambda}) \cong D^{f-1,\lambda}$. \hfill $\Box$

Lemma 3.3 For any $(0, \lambda) \in \Lambda_{r,n}$, let $S^{0,\lambda}$ denote the corresponding cell module, and let $D^{0,\lambda} = S^{0,\lambda}/\text{Rad}(S^{0,\lambda})$. Then $\mathfrak{N}(S^{0,\lambda}) = \mathfrak{N}(D^{0,\lambda}) = 0$.

Proof The result follows from the fact that $S^{0,\lambda}e_1 = D^{0,\lambda}e_1 = 0$. \hfill $\Box$

We do not assume that $\mathcal{B}_{r,n}(u)$ is admissible in Lemma 3.3.

Lemma 3.4 Let $(f, \lambda) \in \Lambda_{r,n}$ and suppose that the $\mathcal{B}_{r,n}(u)$-module $D^{f,\lambda}$ is non-zero. Likewise, let $(\ell, \mu) \in \Lambda_{s,n}$, and suppose that the $\mathcal{B}_{s,n}(v)$-module $D^{\ell,\mu}$ is non-zero. If $D^{f,\lambda} \cong D^{\ell,\mu}$ as $\hat{\mathcal{B}}_n$-modules, then $f = \ell$ and $D^{\lambda} \cong D^{\mu}$ as $\hat{\mathcal{H}}_{n-2f}$-modules.

Proof By Lemmas 3.2–3.3, $f$ is the maximal integer such that $\mathfrak{N}(D^{f,\lambda}) \neq 0$. It follows $D^{f,\lambda} \cong D^{\ell,\mu}$ as $\hat{\mathcal{B}}_n$-modules, then $f = \ell$ and $D^{0,\lambda} \cong D^{0,\mu}$ as $\hat{\mathcal{H}}_{n-2f}$-modules. In other words, $D^{\lambda} \cong D^{\mu}$ as $\hat{\mathcal{H}}_{n-2f}$-modules. \hfill $\Box$

Lemma 3.5 Let $(f, \lambda) \in \Lambda_{r,n}$ and suppose that the $\mathcal{B}_{r,n}(u)$-module $D^{f,\lambda}$ is non-zero. Likewise, let $(f, \mu) \in \Lambda_{s,n}$, and suppose that the $\mathcal{B}_{s,n}(v)$-module $D^{f,\mu}$ is non-zero. If $D^{\lambda} \cong D^{\mu}$ as $\hat{\mathcal{H}}_{n-2f}$-modules, then $D^{f,\lambda} \cong D^{f,\mu}$ as $\hat{\mathcal{B}}_n$-modules.

Proof First, we can assume $f \neq 0$. Otherwise, $D^{0,\lambda} \cong D^{0,\mu}$ as $\hat{\mathcal{B}}_n$-modules since $D^{\lambda}$ (resp. $D^{\mu}$) can be considered as $\hat{\mathcal{B}}_n$-module $D^{0,\lambda}$ (resp. $D^{0,\mu}$).

Suppose

$$\mathcal{B}_{r,n}(u) = \hat{\mathcal{B}}_n/I$$

and

$$\mathcal{B}_{s,n}(v) = \hat{\mathcal{B}}_n/J,$$

where $I$ is the two-sided ideal of $\hat{\mathcal{B}}_n$ generated by $f(x_1)$ and $J$ is the two-sided ideal generated by $g(x_1)$, and

$$f(x_1) = (x_1 - u_1)(x_1 - u_2) \cdots (x_1 - u_r),$$

$$g(x_1) = (x_1 - v_1)(x_1 - u_2) \cdots (x_1 - v_s).$$

Let $h(x)$ be the least common multiple of $f(x)$ and $g(x)$ in $\kappa[x]$. Let $w = (w_1, w_2, \ldots, w_t)$ be the list of roots of $h(x)$ in $\kappa$. Let $\mathcal{B}_{t,n}(w) = \hat{\mathcal{B}}_n/K$ where $K$ is the two-sided ideal of $\hat{\mathcal{B}}_n$ generated by $h(x_1)$. Then there are two surjective homomorphisms

$$\phi : \mathcal{B}_{t,n}(w) \rightarrow \mathcal{B}_{r,n}(u), \quad \psi : \mathcal{B}_{t,n}(w) \rightarrow \mathcal{B}_{s,n}(v)$$

mapping generators to generators.
The simple $\mathcal{B}_{r,n}(u)$-module $D^{f,\lambda}$ pulls back via $\phi$ to a simple $\mathcal{B}_{r,n}(w)$-module. By Theorem 2.13, this simple $\mathcal{B}_{r,n}(w)$-module must be $D^{f,\alpha}$ for some $f \geq 0$ and some multipartition $\alpha$. By Lemma 3.4, $\ell = f$ and $D^\alpha \cong D^\lambda$ as $\mathcal{H}_{n-2f}$-modules. Similarly, the simple $\mathcal{B}_{r,n}(v)$-module $D^{f,\mu}$ pulls back via $\psi$ to a simple $\mathcal{B}_{r,n}(w)$-module $D^{f,\beta}$ with $D^\beta \cong D^\mu$ as $\mathcal{H}_{n-2f}$-modules. Note that both $D^\alpha$ and $D^\beta$ can be considered as irreducible modules for the same Ariki-Koike algebra $\mathcal{K}_{n-2f}(w)$. We have $\alpha = \beta$. So, $D^{f,\lambda} \cong D^{f,\alpha} \cong D^{f,\mu}$ as $\hat{\mathcal{H}}_n$-modules, proving the result. $\square$

We need some results on the representations of $\mathcal{H}_n$ and $\mathcal{H}_{r,n}(u)$ over $k$ from [3,4].

Following [4], write $u = u_1 \cup u_2 \cup \cdots \cup u_k$ for some $k$ such that $u_1, u_j \in u_i$ if and only if $u_i / u_j \in q^{2k}$. Let $\hat{u}_i$ be the $\langle q^2 \rangle$-orbit containing $u_i$. In other words, any $u \in \hat{u}_i$ is of form $u q^{2k}$ for some $k \in \mathbb{Z}$ and $v \in u_i$.

For any $i$, $2 \leq i \leq n$, let $x_i \in \mathcal{H}_n$ such that $x_i = g_{i-1} x_{i-1} g_{i-1}^{-1}$. By abusing of notation, we use $x_i$ to denote the image of $x_i$ in $\mathcal{H}_{r,n}(u)$. Since $x_1, x_2, \ldots, x_n$ generate an abelian subalgebra of $\mathcal{H}_{r,n}(u)$, its irreducible modules are of one dimensional and are labeled by the set $X$ of weights.

Following [4], for a fixed sequence of integers $n_1, n_2, \ldots, n_k$, let $X_{\text{red}}$ be the set of weights $\chi \in X$ such that $\chi(x_i) \in \hat{u}_j$ for $n_1 + \cdots + n_{j-1} + 1 \leq i \leq n_1 + \cdots + n_j$. Important point here is that they belong to different orbits whenever the intervals are different.

Suppose $M$ is an $\mathcal{H}_n$-module. Following [4], let $M^\chi_{\text{gen}}$ be the generalized eigenspace of $M$ with respect to $\chi \in X_{\text{red}}$ and define

$$M_{\text{red}} = \bigoplus_{\chi \in X_{\text{red}}} M^\chi_{\text{gen}}.$$

We consider $D^\lambda$, the irreducible $\mathcal{H}_{r,n}(u)$-module over $k$ with respect to the Kleshchev multipartition $\lambda = (\lambda(1), \lambda(2), \ldots, \lambda(r)) \in \Lambda_+(n)$ under the assumption $u = u_1 \cup u_2 \cup \cdots \cup u_k$. We remark that Ariki and Mathas [4] allowed $u_i$ to be zero in $k$ since they do not assume $x_1$ is invertible. Write $u_j = \{u_{j_1}, u_{j_2}, \ldots, u_{j_{r_j}}\}$ for $j = 1, 2, \ldots, k$. Let $n_j = \sum_{i=1}^{r_j} |\lambda(j_i)|$ and let $r_j$ be the cardinality of $u_i$. The following result is from [4].

**Proposition 3.6** [4, 1.3, 1.4] Let $D^\lambda$ be the irreducible $\mathcal{H}_{r,n}(u)$-module with respect to the Kleshchev multipartition $\lambda = (\lambda(1), \lambda(2), \ldots, \lambda(r))$ of $n$. Let $\lambda_j = (\lambda(j_1), \lambda(j_2), \ldots, \lambda(j_{r_j}))$ such that $u_j = \{u_{j_1}, u_{j_2}, \ldots, u_{j_{r_j}}\}$. Then

(a) $D^\chi_{\text{red}} = D^{(\lambda(1))} \otimes \cdots \otimes D^{(\lambda(k))}$ is an irreducible $\mathcal{H}_{r_1,n_1}(u_1) \otimes \cdots \otimes \mathcal{H}_{r_k,n_k}(u_k)$-module.

(b) $D^\lambda \cong D^\chi_{\text{red}} \otimes \mathcal{H}_{r,n}$.

**Corollary 3.7** Let $\lambda \in \Lambda_+(n)$ and suppose that the $\mathcal{H}_{r,n}(u)$-module $D^\lambda$ is non-zero. Likewise, let $\mu \in \Lambda_+(n)$, and suppose that the $\mathcal{H}_{r,n}(v)$-module $D^\mu$ is non-zero. If $D^\lambda \cong D^\mu$ as $\mathcal{H}_n$-modules, then there is a positive integer $k$ such that

(a) $u = u_1 \cup u_2 \cup \cdots \cup u_k$ and $v = v_1 \cup v_2 \cup \cdots \cup v_k$.

(b) $n_i = n_i^k$, $1 \leq i \leq k$ where $n_i^k$'s are defined for $D^\lambda$ (resp. $D^\mu$) as above Proposition 3.6.

(c) $D^\chi_{\text{red}} \cong D^\mu_{\text{red}}$ as $\mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2} \otimes \cdots \otimes \mathcal{H}_{n_k}$-modules.

**Proof** Both $D^\lambda$ and $D^\mu$ have the same set of weights since $D^\lambda \cong D^\mu$ as $\mathcal{H}_n$-modules. This proves (a) and (b). Let $\phi : D^\lambda \rightarrow D^\mu$ be an $\mathcal{H}_n$-isomorphism. Then the restriction of $\phi$ on $D^\chi_{\text{red}}$ gives rise to the required isomorphism between $D^\lambda_{\text{red}}$ and $D^\mu_{\text{red}}$. $\square$

In [3, Theorem 6.2], Ariki–Jacon–Lecouvey defined a map, say $\phi$, from the set of Kleshchev multipartitions of $n$ and the set of aperiodic multisegments of length $n$. 

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**Theorem 3.8** [3, Theorem 6.2] Let \( \mathcal{H}_{r,n}(u) \) be the Ariki-Koike algebra over \( \mathbb{Z} \) with \( u = u_1 \). For \( \lambda \in \Lambda_r^+(n) \), suppose that the \( \mathcal{H}_{r,n}(u) \)-modules \( D^\lambda \) is non-zero. Then \( D^\lambda \) is isomorphic to the irreducible \( \mathcal{H}_n \)-module \( D^\phi(\lambda) \), where \( \phi(\lambda) \) is the aperiodic multisegment of length \( n \) with respect to \( \lambda \).

**Corollary 3.9** Let \( \lambda \in \Lambda_r^+(n) \) and suppose that the \( \mathcal{H}_{r,n}(u) \)-module \( D^\lambda \) is non-zero. Likewise, let \( \mu \in \Lambda_s^+(n) \), and suppose that the \( \mathcal{H}_{s,n}(v) \)-module \( D^\mu \) is non-zero. If \( D^\lambda \cong D^\mu \) as \( \mathcal{H}_n \)-modules, then \( \phi_j(\lambda_j) = \phi_j(\mu_j) \) for \( 1 \leq j \leq k \), where \( k \) is defined as Corollary 3.7(a).

**Proof** The result follows from Corollary 3.7 and Theorem 3.8.

The following result gives a classification of finite dimensional irreducible \( \mathcal{B}_n \)-modules over \( \kappa \).

**Theorem 3.10** Let \( \mathcal{B}_n \) be the affine BMW algebra over \( \kappa \).

(a) Any finite dimensional irreducible \( \mathcal{B}_n \)-module \( M \) factors through some cyclotomic BMW algebra \( \mathcal{B}_{r,n}(u) \) with admissible parameters \([4] \).

(b) If \( \mathcal{B}_{r,n}(u) \) is admissible, then the complete set of non-isomorphic irreducible \( \mathcal{B}_{r,n}(u) \)-modules is indexed by (\( f, \lambda \))’s in \( \Lambda_r \) such that \( 0 \leq f \leq \lfloor n/2 \rfloor \) and \( \lambda \) ranges over all Kleshchev multipartition of \( n - 2f \). Further, if \( a_0 = 0 \) for all \( a \in \mathbb{Z} \) and if \( n \not\equiv 0 \pmod{2} \), then \( f \equiv n/2 \).

(c) Let \( f, \lambda \in \Lambda_r \), and suppose \( D^{f,\lambda} \) is an irreducible \( \mathcal{B}_{r,n}(u) \)-module. Likewise, let \( f, \mu \in \Lambda_s \) and suppose \( D^{f,\mu} \) is an irreducible \( \mathcal{B}_{s,n}(v) \)-module. Then \( D^{f,\lambda} \cong D^{f,\mu} \) as \( \mathcal{B}_n \)-modules if and only if \( f = \ell \) and \( D^\lambda \cong D^\mu \) as \( \mathcal{H}_{n-2f} \)-modules.

(d) Let \( \lambda \in \Lambda_r^+(n-2f) \) and suppose that the \( \mathcal{H}_{r,n-2f}(u) \)-module \( D^\lambda \) is non-zero. Likewise, let \( \mu \in \Lambda_s^+(n-2f) \), and suppose that the \( \mathcal{H}_{s,n-2f}(v) \)-module \( D^\mu \) is non-zero. Then \( D^\lambda \cong D^\mu \) as \( \mathcal{H}_{n-2f} \)-modules if and only if there is a positive integer \( k \) such that \( u = u_1 \cup u_2 \cup \cdots \cup u_k \), \( v = v_1 \cup v_2 \cup \cdots \cup v_k \) and \( \phi_j(\lambda_j) = \phi_j(\mu_j) \) for \( 1 \leq j \leq k \).

**Proof** (a) is Theorem 2.14 which follows from Goodman’s result given in Theorem 2.13. (b) is Theorem 2.11 which has been proved in [36, Corollary 3.14] for \( r = 1 \) and [30, Theorem 5.3] for \( r \geq 2 \). (c) follows from Lemmas 3.4–3.5. Finally, (d) is Corollary 3.9, which follows from Ariki–Jacon–Lecouvey’s deep result in [3, Theorem 6.2] and some arguments in [4].

By previous work of Goodman [14] and Rui-Xu [30], we know that every simple module of an affine BMW algebra factors through either the extended affine Hecke algebra \( \hat{H}_n \) of type \( A_{n-1} \) or a cyclotomic BMW algebra \( \mathcal{B}_{r,n}(u) \) with admissible parameters, and the irreducible modules for \( \mathcal{H}_n \) (resp. \( \mathcal{B}_{r,n}(u) \) with admissible parameters) are classified in [4] (resp. [30]). For \( r = 1 \), see [36]. The new result here is an effective procedure for determining when two of these modules are isomorphic as modules of the affine BMW algebra. There is still an unresolved problem for the classification of simple modules of an affine BMW algebras with given parameters \( \Omega, \rho, q \) over an algebraically closed field. The problem is that we have no efficient way to actually find all (or any) of the parameters \( u = (u_1, u_2, \ldots, u_r) \) such that \( \mathcal{B}_{r,n}(u) \) is admissible. From [30, Definition 2.27] and [33, Definition 3.1], we have explicit relations between \( \Omega, \rho, q \), and \( u \), but it is hard to solve these relations for \( u \) given \( \Omega, \rho, q \).

We close our paper by giving the following remark.

**Remark 3.11** We can prove the result similar to Theorem 3.10(c) for affine Wenzl algebra over an algebraically closed field \( \kappa \). Theorem 3.10(a) for affine Wenzl algebras follows from
Goodman’s result on degenerate cyclotomic BMW algebra (or cyclotomic Nazarov–Wenzl algebra in [5]) in [14]. Theorem 3.10(b) for cyclotomic Nazarov–Wenzl algebra follows from [5, Theorem 8.5] for $\omega_0 \neq 0$ and [25, Theorem 3.12] for $\omega_0 = 0$. We do not know whether Theorem 3.10(d) is available for degenerate affine Hecke algebra of type $A_{n-1}$ and degenerate cyclotomic Hecke algebras.

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