Comment on Cherenkov radiation in magnetized vacuum

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December 30, 2021

Abstract

The Cherenkov radiation in the vacuum with constant magnetic background is denied within quantum kinematics in spite of the refractive index of that equivalent medium being greater than unity.

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1 Introduction

It was noted rather long ago [1] that there is no Cherenkov effect in the vacuum filled with a strong magnetic field, in spite of the fact that the latter supplies it with nontrivial dielectric permeability [2] and hence with the possibility that a charged particle may propagate with the speed exceeding that of light in this equivalent optically dense medium. On the other hand, recently [3], the opposite was claimed, and Cherenkov radiation (ChR) due to classical motion of a charge across the magnetic field was estimated, the circular orbit being approximated by a straight line for this purpose. The effect was also recognized in a number of subsequent publications, e.g. [4], [5].

In the present Letter I reinforce the denial of ChR in the magnetized vacuum. I state that the full quantum treatment of the charge motion provides a kinematic ban against ChR. Note that in the standard Cherenkov effect the quantum treatment of its kinematics was proposed by V.L. Ginzburg in 1940 [7]. The corresponding formulas are equivalent to the classical ones derived in the pioneering work [8] on ChR only when the energy of the irradiated photon and, correspondingly, the energy of the irradiating charge are much smaller than its mass of rest. Evidently, interests of nowadays extend far beyond this restriction.

I shall refer to the quantum relativistic electron-positron gas in the magnetic field that exhibits both the synchrotron and Cherenkov radiation united into their common Compton scattering matrix element $e \rightarrow e\gamma$. To separate these different types of radiation I define ChR as the one that has its kinematic border at a certain speed of the charged particle. I demonstrate that in the limit where the plasma is absent, ChR inherent to it disappears, while the synchrotron radiation remains.

2 Quantum kinematics of the process $e \rightarrow e\gamma$ and bans on the Cherenkov radiation in vacuum with magnetic field

In an external magnetic field the process of a photon emission by an electron $e \rightarrow e\gamma$ exists on the mass shell of the involved particles already in the vacuum.

Let, in any reference frame, where only the magnetic part $\mathbf{B}$ of a constant external field exists, the electron be characterized by its momentum component $p_\parallel$ along $\mathbf{B}$ and by the Landau quantum number $n$ presenting partially the transverse degree of freedom of the electron. The energy of the electron is $\varepsilon(p_\parallel, n) = \sqrt{m^2 + p_\parallel^2 + 2enB}$, where $m$ is the electron mass and $e$ its charge ($c = \hbar = 1$).

There is another continuous quantum number responsible for the center-of-orbit coordinate in the plane $(X, Y)$ orthogonal to $\mathbf{B}$ (the axis directed along it is understood as $Z$). Choosing the vector-potential of the external field to be in the linear gauge $A_x = -BY$, $A_y = A_z = 0$ and directing the $y$-axis so that it pass through the center of orbit, $X = 0$, $Y = Y_0$, we may write the corresponding ”momentum” component in the plane orthogonal

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1 This objection does not relate to ChR in a plane-wave field, which is another subject in [3], as well as in [6], [4].
to the magnetic field as \( p_x = Y_0 eB \). The energy does not depend on this quantum number, however.

Let the electron produce a photon with the energy \( k_0 \) and its momentum components along and across the field \( B \) being \( k_\parallel \) and \( k_\perp \), resp. Once the vector-potential is independent of time and of the coordinate \( X \) the conserving components of the energy-momentum are the energy and the momenta along the magnetic field and \( p_x \)

\[
p_\parallel = k_\parallel + p'_\parallel, \quad \varepsilon(p_\parallel, n) = k_0 + \varepsilon(p'_\parallel, n'), \quad p_x = (k_\perp)_x + p'_x
\]  

(1)

Primes here mark the final state. The third conservation law here will not play a role within the present consideration.

Calculation of matrix elements in \([10]\) shows a profound difference between transitions with \( n \neq n' \) and \( n = n' \) in what concerns the threshold behavior of the corresponding matrix elements: it is infinite in the first case and finite in the second. It is natural to refer to processes with \( n \neq n' \) as the synchrotron radiation and attribute the term "ChR" to the process where electron emits a photon without changing its Landau quantum number. The physical ground for this classification will become clear later below.

2.1 Radiation of a photon by electron without changing its Landau quantum number, \( n = n' \)

Now I set \( n = n' \) and denote \( m_n^2 = m^2 + 2eB \). By squaring equations (1) and solving them with respect to \( p_\parallel \) we obtain

\[
p^{(1,2)}_\parallel = \frac{k_\parallel}{2} \pm \frac{k_0}{2} \sqrt{1 + \frac{4m_n^2}{k_\parallel^2 - k_0^2}},
\]  

(2)

This is real in two regions. One is \( k_0^2 - k_\parallel^2 \geq 4m_n^2 \). In this domain Eqs. (2) are redundant solutions acquired by the squaring. They correspond to the minus sign in front of \( \varepsilon(p'_\parallel, n') \) in (1) and serve the production by a photon of an electron-positron pair, which is not the process under consideration. Therefore, we are left with the other possible region

\[
z = k_\parallel^2 - k_0^2 \geq 0, \quad z < k_\parallel^2.
\]  

(3)

Only this domain is kinematically admitted for the reaction considered with \( n = n' \). However, even within this domain not the both values (2) are solutions to the set (1) with \( n = n' \). To select the genuine solutions, let us substitute (2) back into (1). We obtain for the energies of the initial and final electron

\[
\varepsilon^{(1,2)} = \sqrt{m_n^2 + (p^{(1,2)})^2} = \left| \frac{k_0}{2} \pm \frac{k_\parallel}{2} \sqrt{1 + \frac{4m_n^2}{z}} \right|,
\]

\[
\varepsilon'^{(1,2)} = \sqrt{m_n^2 + (p'^{(1,2)}_\parallel - k_\parallel)^2} = \left| -\frac{k_0}{2} \pm \frac{k_\parallel}{2} \sqrt{1 + \frac{4m_n^2}{z}} \right|.
\]
Since \( k_0 > 0 \), we have to choose the upper sign if \( k_0 > 0 \) and lower sign if \( k_0 < 0 \), and then the second equation in (1) \( \varepsilon(p_0, n) = k_0 + \varepsilon(p'_0, n) \) is fulfilled for each of these two choices. Finally, the genuine solution to the set (1) in the domain (3) is

\[
p_\parallel = \frac{k_\parallel}{2} + \frac{k_0}{2} \sqrt{1 + \frac{4m_n^2}{z} \sgn k_\parallel}, \quad \varepsilon = \frac{1}{2} k_\parallel \sqrt{1 + \frac{4m_n^2}{z}}
\]

\[
p'_\parallel = -\frac{k_\parallel}{2} + \frac{k_0}{2} \sqrt{1 + \frac{4m_n^2}{z} \sgn k_\parallel}, \quad \varepsilon' = \frac{1}{2} k_\parallel \sqrt{1 + \frac{4m_n^2}{z}}
\]

(4)

It is seen that due to the inequality (3), the emitted photon momentum component \( k_\parallel \) along the magnetic field is directed in the same way as that of the emitting electron.

Introducing the photon emission angle \( \theta \) respective to the magnetic field via the relation \( \cos \theta = \frac{k_\parallel}{|k|} \), and the refraction index as \( N(k_0, k) = \frac{|k|}{k_0} \) (the phase velocity of the emitted photon is the inverse of it, \( v_{ph} = \frac{1}{N} \)), let us consider the case \( k_0 > 0 \), i.e. the photon emitted ahead of the electron under the angle \( \pi/2 > \theta > -\pi/2 \). In this case, in the domain (3) it holds that \( \infty > \frac{k_\parallel}{k_0} = N \cos \theta \geq 1 \). For the speed of the irradiating electron we have the following expression

\[
v_{ch} = \frac{p_\parallel}{\varepsilon} = \frac{k_\parallel + k_0 \sqrt{1 + \frac{4m_n^2}{z}}}{k_0 + k_\parallel \sqrt{1 + \frac{4m_n^2}{z}}} = \frac{\lambda N \cos \theta + 1}{\lambda + N \cos \theta},
\]

(5)

where \( \lambda = \sqrt{\frac{z}{z + 4m_n^2}} < 1 \). It is readily seen that this speed is certainly not larger than unity (the speed of light in the vacuum), but not smaller than the phase speed of light \( v_{ph} = \frac{1}{N} \). The first claimed inequality \( 1 > v_{ch} \) reads \( \lambda + N \cos \theta \geq (\lambda N \cos \theta + 1) \), or \( (1 - \lambda) N \cos \theta \geq (1 - \lambda) \). This relation is evident since \( N \cos \theta \geq 1 \), while \( 1 - \lambda \) is positive and can therefore be cancelled from it. The second claimed inequality \( v_{ch} \geq v_{ph} \) reads \( (\lambda N \cos \theta + 1) N \geq \lambda + N \cos \theta \). This is definitely fulfilled, too, since \( N^2 \cos \theta \geq 1 \) and \( N \geq N \cos \theta \).

The same conclusions are achieved for the backward radiation.

The above analysis reveals every feature of the Cherenkov effect, but it realizes only if the reaction takes place in a material medium, but not in a magnetized vacuum, though \( N \) may be greater than unity there. The point is that the produced Cherenkov photon should fit a dispersion law, but no dispersion curve is admitted in domain (3) by the relativistic covariance and the causality principle as far as the vacuum with a constant magnetic field is concerned.

### 2.2 Incompatibility with Lorentz covariance and causality

In the vacuum filled with a constant external field there exist only two independent Lorentz invariants that carry dependence on the photon momentum. These are \( k^\mu F^2_{\mu\nu} k^\nu \) and \( k^\mu \tilde{F}^2_{\mu\nu} k^\nu \) (The other invariant \( k^2 = k^0_0 - k^2 \) is related to the previous two as \( k^\mu \tilde{F}^2_{\mu\nu} k^\nu + k^\mu F^2_{\mu\nu} k^\nu = 2 F^2 k^2 \)). Here \( F^2_{\mu\nu} \) and \( \tilde{F}^2_{\mu\nu} \) are matrix squares of the external field tensor and of its dual, and the only nonvanishing invariant of the external field is denoted as \( \mathcal{F} = (F^2)^\mu_\mu \). In any of the special
Lorentz frames where the external field does not contain an electric part, the invariants become $\mathcal{F} = \frac{E^2}{2}$, $\frac{k^\mu F^\mu_{\nu}}{2\mathcal{F}} = -k^2$, $\frac{k^\mu F^\mu_{\nu}}{2\mathcal{F}} = k^2_0 - k^2_\parallel$. The dispersion laws for each of the three photon polarization modes, numbered by $i$, are to be found as solutions to the equations $k^2 = \kappa_i$, where the Lorentz scalars $\kappa_i$ are eigenvalues of the polarization $4 \times 4$ tensor $[11], [12]$. They depend on the above invariants. Hence in the special frames the photon dispersion laws may be written in the form

$$k^2_0 - k^2_\parallel = f_i(k^2_\perp, B).$$

We shall omit an explicit indication of the argument $B$ in what follows. It follows from (6) that the refraction index $N = \frac{|k_0|}{k_0}$ for each mode $i$ can be expressed as $N = \left(\frac{k^2_0 + k^2_\perp}{k^2_0 + f_i(k^2_\perp, B)}\right)^{1/2}$. Depending on the momentum range, $N$ can be either larger or smaller than unity. Anyway, below the first pair creation threshold $0 < k^2_0 - k^2_\parallel < 4m^2$ and, for larger energies, in the regions adjacent from below to higher thresholds of creation of pairs occupying excited Landau levels, $k^2_0 - k^2_\parallel < (m_n + m_{n'})^2$, the dispersion curves are known $[13], [1]$ to lie outside the light cone, $k^2 = k^2_0 - k^2 > 0$, i.e. $f_i(k^2_\perp) < k^2_\perp$, hence $N > 1$. This circumstance provokes an (unsound) conclusion that ChR should exist in the vacuum.

However, causality implies that the group velocity of electromagnetic radiation $v_{gr} = \frac{d\omega}{dk}$ should be smaller than 1

$$|v_{gr}|^2 = \left(\frac{\partial k_0}{\partial k_\parallel}\right)^2 + \left|\frac{\partial k_0}{\partial k_\perp}\right|^2 = \frac{k^2_0 + k^2_\perp}{k^2_0} \left(\frac{f_i(k^2_\perp)}{k^2_0}\right)^2 \leq 1,$$

the prime indicates differentiation over $k^2_\perp$. Eq. (7) written in the form $k^2_0 \left(\frac{f_i(k^2_\perp)}{k^2_0}\right)^2 \leq k^2_0 - k^2_\parallel$ excludes the negativity of the difference $k^2_0 - k^2_\parallel$ prescribed by the kinematical inequality (8). Consequently, ChR could only be a super-luminal tachyon, if dynamics of a theory admits its appearance.

The above general conclusion that the photon dispersion curves do not get into the domain (8) is confirmed by all available calculations within quantum electrodynamics of strong magnetic field, using both the polarization tensor obtained by field-differentiation from the Euler-Heisenberg effective Lagrangian (valid, according to $[13]$), in the infrared region under the anisotropic conditions: $k^2_0 - k^2_\parallel$ should be much less than the inverse Compton length squared $m^2$, and $k^2_\perp$ much less than the Larmour magnetic length $\sqrt{eB}$, or calculated as the one-loop Furry diagram with the fermion propagators in it taken as solutions of the Dirac equation with an external magnetic field $[11], [16], [12], [13]$ (this result is not subject to any frequency-momentum restrictions).

$^2$The inequality $N \leq 1$ also changes to the opposite when poles of the polarization operator corresponding to creation by a photon of mutually bound state of the electron-positron pair are crossed by the dispersion curve$[14]$.

$^3$Note, that what matters for causality (7) is not the position of a dispersion curve inside or outside the light cone, but its slope $|f_i(k^2_\perp)| \leq \left(\frac{k^2_0 - k^2_\parallel}{k^2_\perp}\right)^{1/2}$. The right-hand side in this inequality is smaller than unity in the domains, where the refractive index $N$ exceeds unity (outside of the light cone), and it is greater than unity otherwise.
2.3 Contrasting to the case of medium with the magnetic field

When there is a homogenous medium, whose anisotropy is solely owing to the magnetic field imposed on it, the Cherenkov radiation by an electron without change of its Landau quantum number does occur along the same kinematical lines as described in Subsect. 2.1. This time, however, the forbiddance pointed in Subsect. 2.2 does not act, because of the additional violation of the Lorentz invariance due to the medium, since its rest frame is specialized. As an example of the needed medium let me refer to the electron-positron gas at finite temperature and finite chemical potential placed in a magnetic field. To achieve the Lorentz-covariant formulation of this system one should introduce an extra, time-like, vector of four velocity of the medium $u^\mu$, $u^2 = 1$. Now, apart from the invariants $k^\mu \tilde{F}_{\mu\nu}^2 k^\nu$ and $k^\mu F_{\mu\nu}^2 k^\nu$, the three eigenvalues of the polarization tensor $\kappa^i$ may depend on other momentum-containing Lorentz invariants: $(u^k)_{\mu}, u^\mu F_{\mu\nu}^2 k^\nu, u^\mu \tilde{F}_{\mu\nu}^2 k^\nu, u^\mu F_{\mu\nu} k^\nu, u^\mu \tilde{F}_{\mu\nu} k^\nu$. Then the form of the dispersion curves (6) is no longer true and the principle of causality cannot be given the form (7). Thereby the forbiddance proof of ChR given above is invalidated.

When considering in [?] the mechanisms of absorption of a photon via creation of an electron-positron pair, both belonging to the plasma and occupying the same Landau levels, $n = n'$, it was found that the antiHermitean parts of the polarization tensor, which are responsible for this absorption, calculated using the temperature Green function method [17] within one-loop approximation, do exist in the domain (3). The calculated matrix element squared for this process $\gamma \to e^+e^-$, indicated as inverse Cherenkov radiation in [?], is the same as that for the process under consideration.

2.4 Does the synchrotron radiation, $n \neq n'$ include a Cherenkov component?

For $n \neq n'$ equation (1)

$$p_\parallel = k_\parallel + p'_{\parallel}, \quad \varepsilon(p_\parallel, n) = k_0 + \varepsilon(p'_\parallel, n'), \quad p_x = (k_\perp)_x + p'_x,$$

when squared, has two solutions

$$p^{(1,2)}_\parallel = \frac{-k_\parallel J_{nn'} \pm k_0 \Lambda_{nn'}}{2z} + k_\parallel = \frac{k_\parallel J_{nn'} \pm k_0 \Lambda_{nn'}}{2z},$$

where $J_{nn'} = z + 2eB(n' - n)$ and $\Lambda_{nn'} = (J^2_{nn'} + 4m_n^2 z)^{\frac{1}{2}}$. In the limit $n = n'$ the values (9) coincide with (1). In the domain

$$-(m_n - m_{n'})^2 < z = k_\parallel^2 - k_0^2 < 0$$

both of them are solutions to equation (1) $n > n'$, whereas equation (1) has no solutions in this domain for $n < n'$. There is no solution in the adjacent domain $-(m_n - m_{n'})^2 > z = k_\parallel^2 - k_0^2$ in either case. As for the domain (3), it is not interesting as far as the vacuum is concerned, since as it was argued previously, the photon dispersion curve does not get there,
no matter whether this is a Cherenkov photon nor a cyclotron photon. The energies of the irradiating electron having the momenta (9) are

\[ \varepsilon^{(1,2)} = \frac{-k_0 J_{n'n'} \pm k_0 \Lambda_{n'n'}}{2z} + k_0 = \frac{k_0 J_{n'n'} \pm k_0 \Lambda_{n'n'}}{2z}. \]

The corresponding speeds of the irradiating electrons are

\[ v_{\text{sync}}^{(1,2)} = \frac{p_{\parallel}^{(1,2)}}{\varepsilon^{(1,2)}} = \frac{k_0 J_{n'n'} \pm k_0 \Lambda_{n'n'}}{k_0 J_{n'n'} \pm k_0 \Lambda_{n'n'}} = \frac{-\lambda_{n'n'} N \cos \theta \mp 1}{-\lambda_{n'n'} \mp N \cos \theta}, \]

where

\[ \lambda_{n'n'} = \frac{J_{n'n'}}{\Lambda_{n'n'}} = \frac{z + 2eB (n' - n)}{((z + 2eB (n - n'))^2 + 4m_n'^2 z)^{\frac{1}{2}}} > 0. \]

It holds that \( \lambda_{n'n'} < 0 \), since \( n > n' \) and \( z < 0 \) in (10). On the other hand \( |z + 2eB (n' - n)| > |z + 2eB (n - n')| \). To decide, whether \( |\lambda_{n'n'}| \) is smaller or larger than unity, consider its values at the borders of the domain (10). Evidently at \( z = 0 \) we have \( \lambda_{n'n'} = -1 \). At \( z = -2(m_n - m_{n'})^2 \) the denominator is zero., hence \( \lambda_{n'n'} = -\infty \).

To consider inequalities that the speed (??) of the synchrotron-radiating electron satisfies we start with the inequality

\[ -1 > \lambda_{n'n'} > -\infty \]

and note that in the domain (10) the ratio \( \frac{|k_{\parallel}|}{k_0} = N |\cos \theta| \) lies within the limits

\[ \frac{|k_{\parallel}|}{\sqrt{(m_n - m_{n'})^2 + k_0^2}} < N |\cos \theta| < 1, \]

therefore the refraction index may be either larger or smaller than unity for the synchrotron radiation to be possible.

Bearing in mind that due to (13), (12) the denominator in (11) is positive \(-\lambda_{n'n'} \mp N \cos \theta > 0 \), the condition that the speed of the emitting electron should not exceed that of light in the vacuum, \( 1 \geq |v_{\text{sync}}| \), becomes \( \lambda_{n'n'} \pm N \cos \theta \leq -\lambda_{n'n'} N \cos \theta \mp 1 \leq -\lambda_{n'n'} \mp N \cos \theta \). The right-hand inequality may be written as \( \lambda_{n'n'}(1 - N \cos \theta) \leq \pm 1 \mp N \cos \theta \), which is equivalent to \( \lambda_{n'n'} \leq \pm 1 \). This is fulfilled due to (12). The left-hand inequality may be written as \( \lambda_{n'n'}(1 + N \cos \theta) \leq \mp 1 \mp N \cos \theta \), or \( \lambda_{n'n'} \leq \mp 1 \) which is true again. We conclude that the both radiating electrons are slower than light, as they should be.

It is not demonstrated that the generic property of the Cherenkov radiation that the emitting electron be faster than the phase speed of light \( |v_{\text{sync}}| \geq v_{\text{ph}} = \frac{1}{N} \) cannot be fulfilled in any case. But there is no sign that the equality \( |v_{\text{sync}}| = v_{\text{ph}} \), even if possible, might be a threshold for any new radiation.

We conclude that there is no Cherenkov part in the cyclotron radiation.
3 Conclusion

Based on the classical theory of electromagnetic radiation, some authors have recently come out with a statement, taken for granted also in a number of subsequent works, that in an external magnetic field an electric charge moving faster than the phase velocity of light \( v_{ph} \) can generate (Cherenkov) radiation because the vacuum with a background magnetic field forms an optically dense medium with a refractive index greater than unity, so that the phase velocity \( v_{ph} = N^{-1} \) is less than the speed of light in vacuum, \( v_{ph} < 1 \). Although the last statement is a well-known result of quantum electrodynamics, the process of radiation itself in a quantum medium fixed in this way is considered in these works classically, in particular, the trajectory of a charge in a magnetic field is classical.

On the contrary, the quantum kinetic analysis undertaken in our work showed that Cherenkov radiation in a magnetic field in a vacuum does not take place. An electron occupying a Landau quantum state and having a certain momentum along the magnetic field creates synchrotron radiation, as it passes into a state with a lower quantum Landau number, but its initial velocity may be both greater and lesser than the phase velocity of light, so that the phase speed is not a threshold for any radiation. In other words, in this radiation there is no generic sign of Cherenkov radiation.

This statement applies to the radiation of an electron both in vacuum and in an electron-positron plasma in a magnetic field — it is a synchrotron radiation.

Cherenkov radiation could occur, according to kinematics, in a process with conservation of the Landau quantum number in the electron-positron plasma. However, in a vacuum, it is prohibited, as shown in the work, by the principle of causality. This result is independent of the approximation used for the photon polarization operator. The principle of causality requires that the group velocity of a photon be limited to a value less than unity. This requirement excludes the Cherenkov photon from falling into the region allowed for it by the relativistic covariance in the external field. In the presence of plasma, an additional violation of the Lorentz invariance arises in comparison with that introduced by the external field, therefore the proof presented in the work is destroyed and the prohibition on Cherenkov radiation does not apply.

Acknowledgements

Supported by RFBR project No. 20-02-00193.

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