Quandle and hyperbolic volume

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1. Introduction

- quandle
  - an algebraic system
  - homology theory

- hyperbolic volume (of 3-manifolds)
  - hyperbolic geometry
  - cohomology class of $H^3(\text{PSL}(2; \mathbb{C}); \mathbb{R})$.

We will show that hyperbolic volume is a quandle 2-cocycle.

Further, we will show that the quandle 2-cocycle gives us a criterion for determining invertibility and amphicheirality of hyperbolic knots.
2. Preliminaries

Definition (quandle)

\( X \) : a non-empty set,

\( \ast : X \times X \rightarrow X \) a binary operation.

\((X, \ast) : a \text{ quandle} \)

\( \ast \) satisfies the following three axioms:

(Q1) \( \forall x \in X, \ x \ast x = x. \)

(Q2) \( \forall y \in X, \ast y : X \rightarrow X \ (x \mapsto x \ast y) \) a bijection.

(Q3) \( \forall x, y, z \in X, \ (x \ast y) \ast z = (x \ast z) \ast (y \ast z) \).
Example (conjugation quandle)

$G$: a group,

$X \subset G$: a subset closed under conjugations.

$\forall x, y \in X, x \ast y := y^{-1}xy$.

(Q1) $\forall x \in X, x \ast x = x^{-1}xx = x$.

(Q2) $\forall y, z \in X, \exists! x \in X \text{ s.t. } x \ast y = z$ ($x = yzyy^{-1}$).

(Q3) $\forall x, y, z \in X$,

$$(x \ast y) \ast z = z^{-1}(y^{-1}xy)z = z^{-1}y^{-1}xyz,$$

$$(x \ast z) \ast (y \ast z) = (z^{-1}yz)^{-1}(z^{-1}xz)(z^{-1}yz) = z^{-1}y^{-1}xyz.$$
Definition (associated group)

$X$ : a quandle.

$G_X := \langle x \in X \mid x \ast y = y^{-1}xy \ (\forall x, y \in X) \rangle$

: the associated group of $X$.

More precisely,

$\mathcal{F}(X) :$ a free group generated by the elements of $X$,

$\mathcal{N}(X) :$ a subgroup of $\mathcal{F}(X)$ normally generated by

$y^{-1}xy(x \ast y)^{-1} \ (\forall x, y \in X)$.

$G_X = F(X)/\mathcal{N}(X)$. 
$X$: a quandle, $Y$: a set equipped with right action of $G_X$.
$K$: an oriented knot, $D$: a diagram of $K$.

**Definition (shadow coloring)**

- $A : \{\text{arcs of } D\} \rightarrow X$ an arc coloring

  $\xRightarrow{\text{def}} A$ satisfies the condition at each crossing.

- $R : \{\text{regions of } D\} \rightarrow Y$ a region coloring

  $\xRightarrow{\text{def}} R$ satisfies the condition around each arc.

- $S := (A, R)$: a shadow coloring.
A finite sequence of Reidemeister moves transforms a shadow coloring $\mathcal{S}$ into a unique shadow coloring $\mathcal{S}'$.

Therefore, the multi-set

$$\{ \mathcal{S} : \text{a shadow coloring of } D \text{ w.r.t. } X \text{ and } Y \}$$

does not depend on the choice of a diagram $D$ of $K$.

Using quandle 2-cocycle, we can refinement this set.
Quandle homology

$X$ : a quandle.

$C_n^R(X) := \text{span}_{\mathbb{Z}[G_X]}\left\{ (x_1, x_2, \cdots, x_n) \in X^n \right\}$.

Define $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

$$
\partial(x_1, \cdots, x_n) = \sum_{i=1}^{n} (-1)^i \{(x_1, \cdots, \hat{x}_i, \cdots, x_n)
- x_i (x_1 \ast x_i, \cdots, x_{i-1} \ast x_i, x_{i+1}, \cdots, x_n)\}.
$$

$\rightsquigarrow (C_n^R(X), \partial) : a chain complex.$
\[ \partial(x_1, \cdots, x_n) = \sum_{i=1}^{n} (-1)^i \left\{ (x_1, \cdots, \widehat{x_i}, \cdots, x_n) \right. \]
\[ - x_i \left( (x_1 \ast x_i, \cdots, x_{i-1} \ast x_i, x_{i+1}, \cdots, x_n) \right) \].
$C^R_n(X) = \text{span}_{\mathbb{Z}[G_X]} \{(x_1, x_2, \cdots, x_n) \in X^n \}.$

$\cup$

$C^D_n(X) := \text{span}_{\mathbb{Z}[G_X]} \{(x_1, x_2, \cdots, x_n) \in X^n \mid \exists i \text{ s.t. } x_i = x_{i+1} \}.$

$C^Q_n(X) := C^R_n(X)/C^D_n(X).$

**Definition (quandle homology group)**

$M$ : a right $\mathbb{Z}[G_X]$-module,

$C^Q_n(X; M) := M \otimes_{\mathbb{Z}[G_X]} C^Q_n(X).$

$H^Q_n(X; M) := H_n(C^Q_*(X; M))$ : the quandle homology group.

**Definition (quandle cohomology group)**

$N$ : a left $\mathbb{Z}[G_X]$-module,

$C^Q_n(X; N) := \text{Hom}_{\mathbb{Z}[G_X]}(C^Q_n(X), N).$

$H^Q_n(X; N) := H^n(C^*_Q(X; N))$ : the quandle cohomology group.
Remark

$X$ : a quandle,
$Y$ : a set equipped with a right action of $G_X$,
$A$ : an abelian group.

$\rightsquigarrow \mathbb{Z}[Y]$ : a right $\mathbb{Z}[G_X]$-module,
$\text{Hom}(\mathbb{Z}[Y], A)$ : a left $\mathbb{Z}[G_X]$-module.

$\rightsquigarrow H_Q^*(X; \mathbb{Z}[Y]), H_Q^*(X; \text{Hom}(\mathbb{Z}[Y], A))$.

Define $\langle \ , \ \rangle : H_Q^n(X; \text{Hom}(\mathbb{Z}[Y], A)) \otimes H_Q^n(X; \mathbb{Z}[Y]) \rightarrow A$ by

$$\langle f, r \otimes (x_1, x_2, \cdots, x_n) \rangle := f(x_1, x_2, \cdots, x_n)(r).$$
\( X \) : a quandle, \( Y \) : a set equipped with right action of \( G_X \).
\( K \) : an oriented knot, \( D \) : a diagram of \( K \).

**Definition (shadow coloring)**

- \( A : \{\text{arcs of } D\} \rightarrow X \) an arc coloring
  \[
  \text{def } A \text{ satisfies the condition at each crossing.}
  \]

- \( R : \{\text{regions of } D\} \rightarrow Y \) a region coloring
  \[
  \text{def } R \text{ satisfies the condition around each arc.}
  \]

- \( S := (A, R) \) : a shadow coloring.
$S$: a shadow coloring of $D$ w.r.t. $X$ and $Y$.

$$C(S) := \sum_{\text{crossings}} \pm r \otimes (x, y) \in C_2^Q(X; \mathbb{Z}[Y]).$$

**Proposition (Carter-Kamada-Saito ’01)**

$C(S) \in C_2^Q(X; \mathbb{Z}[Y])$ is a cycle.

**Definition (fundamental class)**

$[C(S)] \in H_2^Q(X; \mathbb{Z}[Y])$: the fundamental class of $S$. 
Theorem (Carter-Kamada-Saito '01)

$S, S'$: shadow colorings related with each other by Reidemeister moves.

$\Rightarrow [C(S)] = [C(S')]$.

$A$: an abelian group.

$\theta \in C^2_Q(X; \text{Hom}(\mathbb{Z}[Y], A))$: a cocycle, fix.

Then the multi-set

$$\{\langle \theta, [C(S)] \rangle \in A \mid S: \text{a shadow coloring of } D \text{ w.r.t. } X \text{ and } Y \}$$

does not depend on the choice of a diagram $D$ of $K$.

We call the multi-set a quandle cocycle invariant of $K$. 
3. Hyperbolic volume is a quandle 2-cocycle

\( K \): an oriented knot in \( S^3 \),

\( N(K) \): a regular n.b.h.d. of \( K \).

\( p \in S^3 \setminus K \) fix.

\( Q(K) := \{ x : \text{a path from } p \text{ to } N(K) \} / \text{homotopy} \).
$Q(K)$ is a quandle with $[x] * [y] := [x * y]$ \ ($\forall [x], [y] \in Q(K)$).

We call $Q(K)$ the knot quandle of $K$. 
\( q \in S^3 \setminus K \) fix.

\[ Z(K) := \{ r : \text{a path from } p \text{ to } q \} / \text{homotopy.} \]

\( G_{Q(K)}(= \pi_1(S^3 \setminus K)) \) acts on \( Z(K) \) from the right.

\[ Z(K) \ni r \quad g \in G_{Q(K)} \quad rg \in Z(K) \]
$K$ : an oriented hyperbolic knot in $S^3$, \\
$\Psi : \mathbb{H}^3 \rightarrow S^3 \setminus K$ the universal covering. \\
$\tilde{p} \in \Psi^{-1}(p)$ fix. \\
We have a map $Q(K) \rightarrow \partial \mathbb{H}^3$ which $x \mapsto x_{\infty}$. 

![Diagram showing a hyperbolic knot and its universal covering space.](image-url)
\(\forall r \in Z(K), \forall x, y \in Q(K),\)

we can construct an oriented tetrahedron \(\Delta(\tilde{p}, \tilde{r}(1), x_\infty, y_\infty)\).
∀r ∈ Z(K), ∀x, y ∈ Q(K),
\[ \forall r \in Z(K), \forall x, y \in Q(K), \]

\[
\begin{align*}
\text{positive crossing} & \\
\text{negative crossing}
\end{align*}
\]
∀r ∈ Z(Γ), ∀x, y ∈ Q(Γ),
∀r ∈ Z(K), ∀x, y ∈ Q(K),

\[
\text{vol} : Q(K) \times Q(K) \to \text{Hom}(\mathbb{Z}[Y], \mathbb{R})
\]

\[
\text{vol}(x, y)(r) := \text{algvol}(\Delta(\tilde{p}, \tilde{r}(1), x_\infty, y_\infty)) + \text{algvol}(\Delta(\tilde{p}, \tilde{r}x(1), y_\infty, x_\infty)) + \text{algvol}(\Delta(\tilde{p}, \tilde{r}xy(1), (x * y)_\infty, y_\infty)) + \text{algvol}(\Delta(\tilde{p}, \tilde{r}y(1), y_\infty, (x * y)_\infty)).
\]

**Theorem**

vol ∈ \(C^2_Q(Q(K); \text{Hom}(\mathbb{Z}[Z(K)], \mathbb{R}))\) is a cocycle.
Theorem

$K$ : an oriented hyperbolic knot in $S^3$.

$\text{vol} \in \mathcal{C}_Q^2(Q(K); \text{Hom}(\mathbb{Z}[Z(K)], \mathbb{R}))$ the above cocycle.

$K'$ : an oriented knot in $S^3$,

$D'$ : a diagram of $K'$.

$\forall S :$ a shadow coloring of $D'$ w.r.t. $Q(K)$ and $Z(K)$,

$\exists f_S : S^3 \setminus K' \to S^3 \setminus K$ a continuous map s.t.

$\langle [\text{vol}], [C(S)] \rangle = \text{deg}(f_S) \cdot \text{vol}(S^3 \setminus K)$.
Outline of the proof
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\[ r \in \mathbb{Z}(K) \]

\[ p \quad q \]

\[ K \]

\[ K' \]

\[ Q(K) \ni y \]

\[ x \in Q(K) \]

\[ f_S \]

\[ \Psi(\Delta(\tilde{p}, \tilde{r}(1), x_\infty, y_\infty)) \]

\[ Q(K) \ni y \]

\[ x \in Q(K) \]

\[ r \in Z(K) \]
4. Invertibility and amphicheirality of hyperbolic knots

\( K \): an oriented hyperbolic knot in \( S^3 \).
\( \text{vol} \in \mathbb{Z}_\mathbb{Q}^2(Q(K); \text{Hom}(\mathbb{Z}[Z(K)], \mathbb{R})) \) as above.

**Proposition**

\( K' = K \) or \( -K \),
\( D' \): a diagram of \( K' \).
\( \forall S : \text{a shadow coloring of } D' \text{ w.r.t. } Q(K) \text{ and } Z(K), \langle [\text{vol}], [C(S)] \rangle = \pm \text{vol}(S^3 \setminus K) \) or 0.

**Proposition**

\( D \): a diagram of \( K \).
\( \exists S : \text{a shadow coloring of } D \text{ w.r.t. } Q(K) \text{ and } Z(K) \text{ s.t. } \langle [\text{vol}], [C(S)] \rangle = \text{vol}(S^3 \setminus K). \)
Theorem

\( K \): an oriented hyperbolic knot in \( S^3 \),
\( D \): a diagram of \( K \), \( -D \): a diagram of \(-K\).

(i) \( \exists S \): a shadow coloring of \( D \) w.r.t. \( Q(K) \) and \( Z(K) \) s.t.
\[
\langle [\text{vol}], [C(S)] \rangle = -\text{vol}(S^3 \setminus K)
\]
\( \iff K \) is negative amphicheiral (i.e. \( K \cong -K^* \)).

(ii) \( \exists S \): a shadow coloring of \( -D \) w.r.t. \( Q(K) \) and \( Z(K) \) s.t.
\[
\langle [\text{vol}], [C(S)] \rangle = \text{vol}(S^3 \setminus K)
\]
\( \iff K \) is invertible (i.e. \( K \cong -K \)).

(iii) \( \exists S \): a shadow coloring of \( -D \) w.r.t. \( Q(K) \) and \( Z(K) \) s.t.
\[
\langle [\text{vol}], [C(S)] \rangle = -\text{vol}(S^3 \setminus K)
\]
\( \iff K \) is positive amphicheiral (i.e. \( K \cong K^* \)).