Complex Scalar fields in Scalar-Tensor and Scalar-Torsion theories

Andronikos Paliathanasis$^{1,2,3,*}$

$^1$Institute of Systems Science, Durban University of Technology,
PO Box 1334, Durban 4000, South Africa
$^2$Instituto de Ciencias Físicas y Matemáticas,
Universidad Austral de Chile, Valdivia 5090000, Chile
$^3$Mathematical Physics and Computational Statistics Research Laboratory,
Department of Environment, Ionian University, Zakinthos 29100, Greece

(Dated: October 11, 2022)

We investigate the cosmological dynamics in a spatially flat Friedmann–Lemaître–Robertson–Walker geometry in scalar-tensor and scalar-torsion theories where the nonminimally coupled scalar field is a complex field. We derive the cosmological field equations and we make use of dimensionless variables in order to determine the stationary points and determine their stability properties. The physical properties of the stationary points are discussed while we find that the two-different theories, scalar-tensor and scalar-torsion theories, share many common features in terms of the evolution of the physical variables in the background space.

PACS numbers: 98.80.-k, 95.35.+d, 95.36.+x

Keywords: Scalar field; Complex field; Scalar-tensor; Scalar-torsion; Dynamical analysis.

1. INTRODUCTION

Scalar fields in gravitational theory is a simple mechanism to introduce new degrees of freedom which can play an important role in the description of observable cosmological phenomena [1]. The early acceleration phase of the universe is attributed to a scalar field known as inflaton [2–5]. In addition, scalar fields have been used to describe also the late-time acceleration phase of the universe attributed to dark energy, or other matter components

*Electronic address: anpaliat@phys.uoa.gr
such as dark matter, see for instance [6–15] and references therein.

Multi-scalar fields have been widely studied in the literature. Some well-known two-scalar field models are the quintom [16] or the Chiral model which leads to hyperbolic inflation [17], while other proposed multi-scalar fields theories can be found for instance in [18–23] and references therein. A simple mechanism to introduce a multi-scalar field theory is to consider the existence of a complex scalar field, the real and imaginary parts of which give the equivalent of a two scalar-field theory [24, 25].

An inflationary model with a complex scalar field was proposed in [26]. Specifically it was found that, when inflation occurs, the imaginary component of the complex scalar field does not contribute in the cosmological fluid, that is, the phase of the complex scalar field is constant. The cosmological perturbations with a complex scalar field were investigated in [27]. Furthermore, in [28] the authors used cosmological observations to reconstruct the quintessence potential for a complex scalar field. A nonminimally coupled scalar field cosmological model has been studied in [29], while for some recent studies of complex scalar field cosmological models we refer the reader to [30–37].

In this study we consider the scalar-tensor and the scalar-torsion theories with a complex scalar field [38]. In scalar-tensor theory, the scalar field is minimally coupled to gravity. The scalar field interacts with the gravitational Action Integral of Einstein’s General Relativity, that is, the Ricci-scalar of the Levi-Civita connection, the scalar-tensor theory satisfies the Machian Principle [39] and the theory is defined in the so-called Jordan frame [40]. The Brans-Dicke Lagrangian [39] is the most common scalar-tensor theory. On the other hand, the scalar-torsion theory is the equivalent of scalar-tensor model in teleparallelism. In the latter, the fundamental geometric invariant is the torsion scalar determined by the curvatureless Weitzenböck connection [41, 42]. There are many important results in the literature on the cosmological studies of the scalar-tensor and scalar-torsion theories, for an extended discussion we refer the reader to [43–54] and references therein.

The purpose of this study is to investigate the effects of a complex scalar field in the evolution of the cosmological dynamics for the background space for these two different gravitational theories and to compare the results. Such analysis provides us with important results in order to understand the differences between the use of the Ricci-scalar and of the torsion scalar in the background geometry. We use dimensionless variables to perform a detailed analysis of the dynamical systems which describe the evolution of the physical
variables. Such an approach has been widely studied before with many interesting results about the viability of proposed gravitational theories, see for instance [55–60]. The plan of the paper is as follows.

In Section 2 we consider a spatially flat Friedmann–Lemaître–Robertson–Walker geometry in scalar-tensor gravitational theory with complex scalar field. We derive the field equations and we write the point-like Lagrangian. Moreover, we determined the stationary points for the field equations and we investigate their stability properties. In Section 3 we perform a similar analysis but now in the context of scalar-tensor theory. Finally, in Section 4 we summarize our results and compare the physical results provided by the two theories and draw our conclusions.

2. SCALAR-TENSOR COSMOLOGY

Consider the complex scalar field $\psi$ in the case of scalar-tensor theory [38], for which the gravitational Action Integral is defined as

$$S_{\text{Tensor}} = \int d^4x \sqrt{-g} \left[ F(|\psi|) R + \frac{1}{2} g^{\mu\nu} \psi_\mu \psi^*_\nu - V(|\psi|) \right],$$

(1)

where $R$ is the Ricci scalar related to the Levi-Civita connection for the metric tensor $g_{\mu\nu}$; $|\psi|$ is the norm of the complex field $\psi$, that is, $|\psi|^2 = \psi \psi^*$, $F(|\psi|)$ is the coupling function between the gravitational and the scalar field $\psi$ and $V(|\psi|)$ is the potential function which drives the dynamics. The Action Integral (1) admits the $U(1)$ symmetry.

The Brans-Dicke theory with a complex field is recovered for $F(|\psi|) = F_0 |\psi|^2$, that is, the Action Integral (1) is [38]

$$S_{BD} = \int d^4x \sqrt{-g} \left[ F_0 |\psi|^2 R + \frac{1}{2} g^{\mu\nu} \psi_\mu \psi^*_\nu - V(|\psi|) \right].$$

(2)

In the case of a spatially flat FLRW universe with line element,

$$ds^2 = -N^2(t) dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),$$

(3)

we derive for the Ricci scalar

$$R = 6 \left( \frac{1}{N} \dot{H} + 12 H^2 \right),$$

(4)

where $H = \frac{1}{N} \frac{\dot{a}}{a}$, $\dot{a} = \frac{da}{dt}$, is the Hubble function.
We substitute (4) into (1) which by integration by parts gives the point-like Lagrangian
\[ L_{\text{Tensor}}(N, a, \dot{a}, \psi, \dot{\psi}) = \frac{1}{N} \left( 6F(|\psi|) a\dot{a}^2 + 6\dot{F}(|\psi|) a^2\dot{a} + \frac{1}{2} a^3 \left( \dot{\psi}\psi^* \right) \right) - a^3 NV(|\psi|), \] (5)
or in the case of Brans-Dicke
\[ L_{BD}(N, a, \dot{a}, \psi, \dot{\psi}) = \frac{1}{N} \left( 6F_0 |\psi|^2 a\dot{a}^2 + 6F_0 (|\psi|^2) a^2\dot{a} + \frac{1}{2} a^3 \left( \dot{\psi}\psi^* \right) \right) - a^3 NV(|\psi|). \] (6)

We focus now in the Brans-Dicke theory in which the field equations are described by the point-like Lagrangian (6). Moreover, the complex scalar field is written with the use of the polar form, \( \psi(t) = \phi(t) e^{i\theta(t)} \), such that the Lagrangian (6) becomes
\[ L_{BD}(N, a, \dot{a}, \phi, \dot{\phi}, \theta, \dot{\theta}) = \frac{1}{N} \left( 6F_0 \phi^2 a\dot{a}^2 + 12F \phi a^2\dot{\phi} + \frac{1}{2} a^3 \left( \dot{\phi}^2 + \phi^2 \dot{\theta}^2 \right) \right) - a^3 NV(\phi). \] (7)

It is obvious that the Lagrangian function (7) describes a multi-scalar field cosmological model, where \( \phi \) is the Brans-Dicke field and \( \theta \) is a second-scalar field minimally coupled to gravity but coupled to the Brans-Dicke field \( \phi \). The \( U(1) \) for the point-like Lagrangian (7) provides the invariant transformation \( \theta = \theta + \varepsilon \), and the conservation law \( I_0 = \frac{1}{N} a^3 \phi^2 \dot{\theta} \).

Variation with respect to the dynamical variables \( \{N, a, \phi, \theta\} \) of the Lagrangian (7) provides the cosmological field equations which are
\[ 0 = 6F_0 \phi^2 H^2 + 12F \phi H \dot{\phi} + \frac{1}{2} \left( \dot{\phi}^2 + \phi^2 \dot{\theta}^2 \right) + V(\phi), \] (8)
\[ 0 = 2F_0 \phi^2 \left( 2\dot{H} + 3H^2 \right) + 8F_0 H \phi \dot{\phi} - \frac{1}{2} \phi^2 + 4F_0 \dot{\phi}^2 - \frac{1}{2} \phi^2 \theta^2 + 4F_0 \phi \dot{\theta} + V(\phi), \] (9)
\[ 0 = \ddot{\phi} + \phi \left( 12F_0 \dot{H} - \dot{\theta}^2 \right) + 3\dot{\phi} \dot{H} + 12F_0 \phi H^2 + V_{,\phi} \] (10)
and
\[ 0 = \phi \ddot{\theta} + \left( 2\dot{\phi} + 3H \phi \right) \dot{\theta} \text{ or } I_0 = a^3 \phi^2 \dot{\theta}, \] (11)
where without loss of generality we have assumed \( N(t) = 1 \).

2.1. Cosmological dynamics

In order to reconstruct the cosmological history provided by this specific complex scalar-tensor theory we make use of dimensionless variables in the context of \( H \)-normalization and we investigate the dynamical evolution of the field equations (8)-(11) by determining the stationary points and their stability properties.
We consider the new dimensionless dependent variables

\[
\dot{\phi} = 2\sqrt{3}H \phi x, \quad V(\phi) = 6H^2 \phi^2 y, \quad \dot{\psi} = 2\sqrt{3}Hz, \quad V_{,\phi} = \lambda \frac{V(\phi)}{\phi}.
\]  

(12)

Hence, for the new independent variable, \( \tau = \ln a \), the field equations \((8)-(11)\) are written as the following dynamical system

\[
2F_0(12F_0 - 1) \frac{dx}{d\tau} = 2\sqrt{3}(7 - 48F_0)x^2 + (3 - 24F_0)x^3 \quad (13)
\]

\[
+ 2\sqrt{3}F_0(F_0 + z^2 + (\lambda - 3)y) + 3x(F_0 + (4\lambda F_0 - 1)y + (1 - 8F_0)z^2),
\]

\[
F_0(12F_0 - 1) \frac{dy}{d\tau} = 2\sqrt{3}F_0x(4 - \lambda + 12F_0(\lambda - 2)) + (3 - 24F_0)x^2 \quad (14)
\]

\[
+ 3(F_0(16F_0 - 1) + (4\lambda F_0 - 1)y + (1 - 8F_0)z^2),
\]

\[
-2F_0(12F_0 - 1) \frac{dz}{d\tau} = 4\sqrt{3}F_0(8F_0 - 1)x + (8F_0 - 1)x^2 \quad (15)
\]

\[
+ (1 - 4\lambda F_0)y + (8F_0 - 1)(F_0 + z^2),
\]

\[
\frac{d\lambda}{d\tau} = 2\sqrt{3}\lambda x(1 - \lambda + \Gamma(\lambda)), \quad \Gamma(\lambda) = \phi \frac{V_{,\phi\phi}}{V_{,\phi}}
\]  

(16)

with algebraic constraint

\[
x^2 + z^2 + y + F_0(1 + 4\sqrt{3}x) = 0. \quad (17)
\]

For the scalar field potential we assume \( V(\phi) = V_0\phi^{\lambda_0}; \) such that \( \lambda = \lambda_0 \) is always a constant and the dimension of the dynamical system is two, after we apply the constraint equation \((17)\). From \((17)\) we substitute the parameter \( y \) and we end up with a two-dimensional system on the space of variables \( \{x, z\} \). Moreover, we observe that the dynamical system remains invariant under the change of variable \( z \rightarrow -z \). Thus, without loss of generality, we can restrict our analysis to the region with \( z \geq 0 \). Furthermore, for completeness of our analysis the scalar field potential can be positive or negative such that \( y \in \mathbb{R} \).

The stationary points of the reduced dynamical system \( A = (x(A), z(A)) \) are as follows

\[
A_1 = \left( -\frac{\sqrt{3}F_0(\lambda - 4)}{6F_0(\lambda + 2) - 3}, 0 \right)
\]

and

\[
A_2 = \left( x_2, \sqrt{-F_0 - x_2(4\sqrt{3}F_0 + x_2)} \right).
\]
Each stationary point describes an asymptotic solution for the field equations in which the effective fluid has the equation of state parameter
\[ w_{\text{eff}}(x, y, z) = \frac{3\sqrt{3}F_{0}x + (3 - 34F_{0})x^{2} + 3(4F_{0}^{2} + (4F_{0}\lambda - 1)y + (1 - 8F_{0})z^{2})}{3F_{0}(12F_{0} - 1)}. \] (18)

We proceed with the discussion of the physical properties and the stability properties for the admitted stationary points.

The stationary point \( A_{1} \) describes a universe dominated only by the scalar field \( \phi \) and its potential function \( V(\phi) \), because \( y(A_{1}) = \frac{F_{0}(12F_{0} - 1)(3 + F_{0}(\lambda - 10)(2 + \lambda))}{3(1 - 2F_{0}(2 + \lambda))^{3}} \). The effective equation of state parameter for the asymptotic solution is \( w_{\text{eff}}(A_{1}) = \frac{3 + 2F_{0}(2 + \lambda)(\lambda - 9)}{6F_{0}(2 + \lambda)^{3}} \), from which it follows that \( A_{1} \) describes an accelerated universe as is given in Fig. [1]. Moreover, the eigenvalues of the linearized system around the stationary point are derived, \( e_{1}(A_{1}) = \frac{3 + F_{0}(\lambda - 10)(2 + \lambda)}{2F_{0}(2 + \lambda) - 1} \), \( e_{2}(A_{1}) = \frac{3 + F_{0}(\lambda - 10)(2 + \lambda)}{2F_{0}(2 + \lambda) - 1} \). In Fig. [1] we present the region plot in the two-dimensional space of the free variables \( \{F_{0}, \lambda\} \), where the eigenvalues have negative real parts, that is, point \( A_{1} \) is an attractor.

Points \( A_{2} \) describe a family of stationary points which are real and physical accepted when \( -F_{0} - x_{2}(4\sqrt{3}F_{0} + x_{2}) > 0 \). For the asymptotic solution we calculate \( w_{\text{eff}}(A_{2}) = 1 + \frac{8}{\sqrt{3}}x_{2} \),
which means that the family of points $A_2$ describe accelerated universes for $x_2 < -\frac{1}{2\sqrt{3}}$. Furthermore, the eigenvalues are calculated $e_1(A_2) = 6 + 2\sqrt{3}(2 + \lambda) x_2$ and $e_2(A_2) = 0$. Then for $6 + 2\sqrt{3}(2 + \lambda) x_2 < 0$, where $e_1(A_2) < 0$, the Center Manifold Theorem (CMT) can be applied in order to investigate for a possible stable submanifold and to infer about the stability. However, such an analysis does not contribute to the physical discussion of the present theory and we select to work numerically. In Fig. 2 we present the phase space portrait in the two-dimensional place $(x, z)$ from which we observe that the stationary points are always saddle points for $e_1(A_2) < 0$.

From the above analysis it is clear that various major eras of the cosmological evolution can be described by the stationary points, $A_1$ and $A_2$. For instance for $F_0 = -\frac{3}{2}(2 - 9\lambda + \lambda^2)^{-1}$, $w_{eff}(A_1) = 0$ which means that $A_1$ describes the matter dominated era, and points $A_2$ can describe the early and late acceleration phases of the universe. That is not the unique case, since, for $x_2 = -\frac{\sqrt{3}}{8}$, $w_{eff}(A_2) = 0$, and $A_1$ can describe the future acceleration phase of the universe.

2.2. Analysis at Infinity

Because variables $(x, z)$ are not bounded, they can take values at all the range of the real numbers, which means that they can take values at infinity. Until now we have investigated the stationary points at the finite regime. Hence in order to search for stationary points at infinity we consider the Poincaré map

$$x = \frac{X}{\sqrt{1 - X^2 - Z^2}}, \quad z = \frac{Z}{\sqrt{1 - X^2 - Z^2}}, \quad d\sigma = \sqrt{1 - X^2 - Z^2}d\tau. \quad (19)$$

Therefore, at infinity, $1 - X^2 - Z^2 = 0$, i.e. $Z = \sqrt{1 - X^2}$, the dynamical system is reduced to the single ordinary differential equation

$$\frac{dX}{d\sigma} = \frac{\sqrt{3}(\lambda - 4)}{12F_0 - 1} (X^2 - 1). \quad (20)$$

Consequently, the stationary points at the infinity $B = (X(B), Z(B))$ are

$$B_1 = (1, 0) \quad \text{and} \quad B_2 = (-1, 0). \quad (21)$$

Hence, at infinity only the scalar field $\phi$ contributes to the cosmological solution and the physical properties of the points are similar to those of $A_1$. 
FIG. 2: Phase-space portrait for the scalar-tensor theory on the two-dimensional plane \((x, z)\) for different values of the free parameters \(F_0\) and \(\lambda\). From the figure it is clear that the surface of points described by \(A_2\) are always saddle points or source points. With solid line we present the family of points \(A_2\) for \(e_1 (A_2) < 0\).

The eigenvalues of the two-dimensional dynamical system at the stationary points at infinity are

\[
e_1 (B_1) = \frac{\sqrt{3} (\lambda - 4)}{12F_0 - 1}, \quad \text{Re} (e_2 (B_1)) = 0
\]  

(22)
FIG. 3: Phase-space portrait for the scalar-tensor theory on the two-dimensional plane \((X, Z)\) for \(Z \geq 0\), for different values of the free parameters \(F_0\) and \(\lambda\). From the phase-space portraits we observe that points \(B_1\) and \(B_2\) can be attractors at the infinity regime for the dynamical system.

and

\[
e_1(B_2) = -\frac{\sqrt{3} (\lambda - 4)}{12F_0 - 1}, \quad \text{Re} \left( e_2 (B_2) \right) = 0.
\]  \hspace{1cm} (23)

From the phase-space portraits of Fig. 3 we can easily observe that stationary point \(B_1\) is an attractor when \(e_1(B_1) < 0\), while, when \(e_1(B_1) > 0\), point \(B_2\) is an attractor.
3. SCALAR-TORSION COSMOLOGY

Scalar-torsion theory can be seen as the extension of scalar-tensor theory in teleparallelism. The gravitational Action Integral is

$S_{\text{Torsion}} = \int d^4 x \sqrt{-e} \left[ F(|\psi|) T + \frac{1}{2} g^{\mu\nu} \psi_\mu \psi_\nu - V(|\psi|) \right], \quad (24)$

where $T$ is the torsion scalar of the antisymmetric Weitzenböck connection, $e = \sqrt{-g}$, $e_i$ describes the unholonomic frame, with $g(e_i, e_j) = e_i \cdot e_i = \eta_{ij}$ or in terms of coordinates, $e_i = h^\mu_i (x^k) \partial_i$, where now the Weitzenböck connection is expressed as

$\hat{\Gamma}^\lambda_{\mu\nu} = h^\lambda_a \partial_\mu h_a^\nu \quad (25)$

and $T = S_\beta^{\mu\nu} T^\beta_{\mu\nu}$, with $T^\beta_{\mu\nu}$ to be the torsion tensor $T^\beta_{\mu\nu} = \hat{\Gamma}^\beta_{\nu\mu} - \hat{\Gamma}^\beta_{\mu\nu}$, and

$S_\beta^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}_\beta + \delta^\mu_\beta T^\theta_{\theta\nu} - \delta^\nu_\beta T^\theta_{\theta\mu}), \quad (26)$

where $K^{\mu\nu}_\beta$ is the contorsion tensor

$K^{\mu\nu}_\beta = -\frac{1}{2} (T^{\mu\nu}_\beta - T^{\nu\mu}_\beta - T^\beta_{\mu\nu}). \quad (27)$

For the spatially flat FLRW spacetime with line element $[3]$ we consider the diagonal frame

$h^i_{\mu}(t) = \text{diag}(N(t), a(t), a(t), a(t)), \quad (28)$

from which we calculate $[61]$

$T = -6H^2. \quad (29)$

By replacing in the Action Integral $[24]$ and substituting the complex scalar field as $\psi = \phi e^{i\theta}$, we end up with the point-like Lagrangian function

$L_{\text{Torsion}}(N, a, \dot{a}, \phi, \dot{\phi}, \psi, \dot{\psi}) = -\frac{6}{N} F(\phi) a\dot{a}^2 + \frac{a^3}{2N} \left( \dot{\phi}^2 + \phi \ddot{\theta}^2 \right) - a^3 NV(\phi). \quad (30)$

In an analogue to the Brans-Dicke model we assume now that $F(\phi) = F_0 \phi^2$. This is not a random choice, the Dilaton field, or the Brans-Dicke field for the specific potential function $V(\phi) = V_0 \phi^2$ admits a discrete symmetry known as the Gasperini-Veneziano duality transformation. Similarly, the scalar-torsion theory for $F(\phi) = F_0 \phi^2$ and potential function $V(\phi) = V_0 \phi^2$ admits a discrete symmetry similar to the Gasperini-Veneziano duality transformation. Thus, the $F(\phi) = F_0 \phi^2$ can be seen as the teleparallel Brans-Dicke equivalent model.
Variation with respect to the dynamical variables in the Lagrangian function (30) gives the field equations

\[ 0 = -6F_0\phi^2H^2 + \frac{1}{2} \left( \dot{\phi}^2 + \phi^2\dot{\theta}^2 \right) + V(\phi), \]  

(31)

\[ 4F_0\phi^2 \left( 2\dot{H} + 3H^2 \right) + 16F_0\phi\dot{H} + \left( \dot{\phi}^2 + \phi^2\dot{\theta}^2 \right) - V(\phi) = 0, \]  

(32)

\[ \ddot{\phi} - \phi\theta^2 + 3H \left( 4F_0\phi\dot{\phi} + \dot{\phi} \right) + V_{,\phi} = 0 \]  

(33)

and

\[ \dot{\phi}\ddot{\theta} + \dot{\theta} \left( 3H\phi + 2\dot{\phi} \right) = 0. \]  

(34)

As in the case of the scalar-tensor theory we proceed with the analysis for the dynamics of the latter system of differential equations.

### 3.1. Cosmological dynamics

We make use of the dimensionless variables (12) and we write the field equations (31)-(34) in the equivalent dynamical system

\[ F_0 \frac{dx}{d\tau} = \left( 4\sqrt{3}F_0 + 3x \right) x^2 - 2\sqrt{3}F_0 \left( 2F_0 + \lambda y - 2z^2 \right) - 3x \left( F_0 + y - z^2 \right), \]  

(35)

\[ F_0 \frac{dy}{d\tau} = y \left( 2\sqrt{3}F_0 \left( 2 + \lambda \right) x + 3x^2 + 3 \left( F_0 - y + z^2 \right) \right), \]  

(36)

\[ F_0 \frac{dz}{d\tau} = \frac{3}{2} z \left( x^2 - y + z^2 - F_0 \right), \]  

(37)

\[ \frac{d\lambda}{d\tau} = 2\sqrt{3}\lambda x \left( 1 - \lambda + \Gamma(\lambda) \right), \quad \Gamma(\lambda) = \phi \frac{V_{,\phi\phi}}{V_{,\phi}} \]  

(38)

with constraint equation

\[ x^2 + y + z^2 - F_0 = 0, \]  

(39)

where now the parameter for the equation of state for the effective fluid is expressed as follows

\[ w_{eff}(x, y, z) = \frac{8}{\sqrt{3}}x + \frac{1}{F_0} \left( x^2 - y + z^2 \right). \]  

(40)

We observe that the field equations are invariant under the change of variable \( z \rightarrow -z \). Thus without loss of generality we select to work in the region \( z \geq 0 \). Moreover, from the constraint equation (39) the dimension of the dynamical system is reduced by one, while, for the power-law potential function \( V(\phi) = V_0\phi^\lambda \), the field dimension of the dynamical system is two.
The stationary points on the two-dimensional plane \( \{ x, z \} \), are of the form \( C = (x(C), z(C)) \), they are

\[
C_1 = \left( \frac{F_0}{\sqrt{3}} (2 + \lambda), 0 \right), \\
C_2 = \left( x_2, \sqrt{F_0 - (x_2)^2} \right).
\]

Point \( C_1 \) describes a universe in which the field \( \phi \) and the potential function contribute to the cosmological solution. The physical parameter \( w_{\text{eff}} \) is calculated to be \( w_{\text{eff}} (C_1) = \frac{1}{3} (2F_0 (\lambda^2 - 4) - 3) \) from which it easily follows that \( C_1 \) describes an accelerated asymptotic cosmological solution for \( \{ |\lambda| < 2, F_0 > \frac{1}{x^2 - 4} \} \) or \( \{ |\lambda| > 2, F_0 < \frac{1}{x^2 - 4} \} \) while, for \( |\lambda| = 2 \), the de Sitter Universe is recovered. The eigenvalues of the linearized system near to the stationary point are

\[
e_1 (C_1) = -3 + F_0 (\lambda^2 + 2), \quad e_2 (C_1) = -3 + F_0 (\lambda^2 + 2).
\]

Thus \( C_1 \) is an attractor when \(-3 + F_0 (\lambda^2 + 2) < 0\).

The family of points \( C_2 \) exists when \( F_0 > 0 \) and describes asymptotic solutions with \( w_{\text{eff}} (C_2) = 1 + \frac{8}{\sqrt{x^2}} x_2 \) and eigenvalues

\[
e_1 (C_2) = 6 + 2\sqrt{3} (2 + \lambda) x_2, \quad e_2 (C_2) = 0.
\]

Because one of the eigenvalues is zero, in Fig. [4] we present the phase-space portraits for the dynamical system in the two-dimensional plane \( (x, z) \) for positive values of \( F_0 \) and different parameters of \( \lambda \). From the diagrams we observe that when \( e_1 (C_2) < 0 \) the family of points \( C_2 \) describes attractors, otherwise the points are sources.

### 3.2. Analysis at Infinity

We consider now the Poincaré map [19] and we write the two-dimensional system in the equivalent form

\[
F_0 \frac{dX}{d\sigma} = \left( F_0 - (1 + F_0) (X^2 + Z^2) \right) \left( \sqrt{3}F_0 (2 + \lambda) (X^2 - 1) - 3X\sqrt{1 - X^2 - Z^2} \right), \quad (42)
\]

\[
F_0 \frac{dZ}{d\sigma} = \left( F_0 - (1 + F_0) (X^2 + Z^2) \right) \left( \sqrt{3}F_0 (2 + \lambda) X - 3\sqrt{1 - X^2 - Z^2} \right). \quad (43)
\]

The stationary points at infinity are \( D = (X(D), Z(D)) \) with \( 1 - X(D)^2 - Z(D)^2 = 0 \). We find the points

\[
D_1 = (1, 0) \quad \text{and} \quad D_2 = (-1, 0), \quad (44)
\]
FIG. 4: Phase-space portrait for the scalar-torsion theory on the two-dimensional plane $(x, z)$, for different values of the free parameters $F_0$ and $\lambda$ in order to investigate the stability properties of the family of points $C_2$ (solid lines). From the plots we observe that, when $e_1 (C_2) < 0$, the points are attractors, otherwise they are source points.

From which it is clear that only the scalar field $\phi$ contributes in the cosmological solution.

The eigenvalues of the linearized system are

$$e_1 (D_1) = -\sqrt{3} (2 + \lambda) \quad , \quad \text{Re} (e_2 (D_1)) \simeq -\text{sign} (F_0)$$

(45)
and

\[ e_1(D_2) = \sqrt{3} (2 + \lambda) \quad \text{Re} \left( e_2(D_2) \right) \simeq -\text{sign} \left( F_0 \right). \] (46)

Therefore point \( D_1 \) is an attractor when \( F_0 > 0 \) and \( 2 + \lambda > 0 \), while \( D_2 \) is attractor for \( F_0 > 0 \) and \( 2 + \lambda < 0 \).

In Fig. 5 we present phase-space portraits for the dynamical system of scalar-torsion theory on the Poincaré variables.

4. CONCLUSIONS

In this piece of work we considered a complex scalar field in the context of scalar-tensor and scalar-torsion theories in a spatially flat FLRW background geometry. For these two gravitational models we derived the field equations and we investigated the dynamical evolution of the physical quantities by using dimensional variables. With the use of the latter variables the field equations for both theories are reduced to a two-dimensional dynamical system of first-order ordinary differential equations. In order to construct the cosmological history as provided by the given models we determined the stationary points and we investigated their stability properties.

For the two different dynamical systems which correspond to the scalar-tensor and scalar-torsion theories respectively, we found that the stationary points for the two dynamical systems are four. They are two points at the finite regime and two points at infinity. The stationary points can describe accelerated asymptotic solutions which can describe the early and late-time acceleration phases of the universe. The stationary points and their stability properties are summarized in Table II.

We can easily conclude that the two theories, that is, the scalar-torsion theory and the scalar-tensor theory have the same number of stationary points and similar stability properties. The physical variables of the background geometry at the stationary points have similar properties with a different functional dependence on the free variables \( F_0 \) and \( \lambda \). The dynamical equivalence of the two theories is an expected result. Torsion is not dynamical and there are no new degrees of freedom in scalar-torsion cosmology with respect to scalar-tensor cosmology.

From the above analysis it is clear that the two theories cannot be distinguished from the evolution of the background geometry and the analysis of the perturbations should be
performed. However, that overpasses the scopus of this work and will be published elsewhere.

Finally, as far as the complex scalar field is concerned, we remark that there only at
the (families) stationary points $A_2$ and $C_2$ the imaginary part of the complex scalar field
contributes in the physical quantities of the asymptotic solutions. However, these families
of points can describe important eras of the cosmological history, for instance, the early
inflationary era, the radiation era or the matter dominated era. What will be of special

FIG. 5: Phase-space portraits for the scalar-torsion theory on the two-dimensional plane $(X, Z)$,
for different values of the free parameters $F_0$ and $\lambda$. 
TABLE I: Stationary points and their stability properties for the cosmological models of our consideration.

| Theory       | Point   | Finite/Infinity | Stability                  |
|--------------|---------|----------------|----------------------------|
| Scalar-Tensor|         |                |                            |
| $A_1$        | Finite  | Attractor see Fig. 1 |
| $A_2$        | Finite  | Saddle         |
| $B_1$        | Infinity| Attractor $\frac{\sqrt{3}(\lambda - 4)}{12F_0 - 1} < 0$ |
| $B_2$        | Infinity| Attractor $-\frac{\sqrt{3}(\lambda - 4)}{12F_0 - 1} < 0$ |
| Scalar-Torsion|      |                |                            |
| $C_1$        | Finite  | Attractor $-3 + F_0 (\lambda^2 + 2) < 0$ |
| $C_2$        | Finite  | Attractor $6 + 2\sqrt{3} (2 + \lambda) x_2 < 0$ |
| $D_1$        | Infinity| Attractor $F_0 > 0, 2 + \lambda > 0$ |
| $D_2$        | Infinity| Attractor $F_0 > 0, 2 + \lambda > 0$ |

interest is to investigate the existence of a scalar field potential which does not depend only on the norm of the complex scalar field.

Acknowledgments

This work was partially supported by the National Research Foundation of South Africa (Grant Numbers 131604). The author thanks Dr. G. Anargirou for the hospitality provided while part of this work carried out.

[1] B. Ratra and P.J.E Peebles, Phys. Rev. D 37 3406 (1988)
[2] J. Rubio, Front. Astron. Space Sci. 5, 50 (2019)
[3] L.N. Granda and D.F. Jimenez, EPJC 79, 772 (2019)
[4] P. Parsons and J.D. Barrow, Phys. Rev. D 51, 6757 (1995)
[5] J.D. Barrow and A. Paliathanasis, Gen. Rel. Grav. 50, 82 (2018)
[6] E.V Linder, Phys. Rev. D. 70 023511 (2004)
[7] S. Barshay and G. Kreyerhoff, EPJC 5, 369 (1998)
[8] T. Matos, J.-R. Luevano, I. Quiros, L.A. Urena-Lopez and J.A. Vazquez, Phys. Rev. D 80, 123521 (2009)
[9] A.R. Liddle, C. Pahud and L.A. Urena-Lopez, Phys. Rev. D 77, 121301 (2008)
[10] L.P. Chimento, V. Mendez and N. Zuccala, Class. Quantum Grav. 16, 3749 (1999)
[11] C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D 80, 064015 (2009)
[12] D. Bertacca, N. Bartolo and S. Matarrese, Adv. Astron. 2010, 904379 (2010)
[13] D. Bertacca, A. Raccanelli, O.F. Piatella, D. Pietrobon, N. Bartolo, S. Matarrese and T. Giannantonio, JCAP 03, 039 (2011)
[14] A. Suárez and T. Matos, MNRAS 416, 87 (2011)
[15] C. Cosme, J.G. Rosa and O. Bertolami, Phys. Lett. B 781, 639 (2018)
[16] W. Hu, Phys. Rev. D 71, 047301 (2005)
[17] D. Wands, Lect. Notes Phys. 738, 275 (2008)
[18] A.A. Coley and R.J. van den Hoogen, Phys. Rev. D 62, 023517 (2000)
[19] A. Arbey, Phys. Rev. D 74, 043516 (2006)
[20] C.-J. Gao and Y.-G. Shen, Phys. Lett. B 541, 1 (2002)
[21] D.S.M. Alves and G.M. Kremer, JCAP 10, 009 (2004)
[22] V. Sivanesan, Phys. Rev. D 90, 104006 (2014)
[23] A. Paliathanasis, Universe 8, 325 (2022)
[24] I.M. Khalatnikov and A. Mezhlumian, Phys. Lett. A 169, 308 (1992)
[25] I.M. Khalatnikov, Lect. Notes Phys. 455, 343 (1995)
[26] D. Scialom, Helv. Phys. Acta 69, 190 (1996)
[27] P. Jetzer and D. Scialom, Phys. Rev. D 55, 7440 (1997)
[28] J.-A. Gu and W.-Y. Hwang, Phys. Lett. B 517, 1 (2001)
[29] A.Yu.Kamenshchik, I.M.Khalatnikov and A.V.Toporensky, Phys. Lett. B 357, 36 (1995)
[30] M. Arik, M. Calik and N. Katirci, Central Eur. J. Phys. 9, 1465 (2011)
[31] G. Rosen, EPL 89, 19002 (2010)
[32] Y.-G. Shen and X.-H. Ge, Int. J. Theor. Phys. 45, 17 (2006)
[33] A. Arbey, Phys. Rev. D 74, 043516 (2006)
[34] R.C.G. Landim, EPJC 76, 1 (2016)
[35] B. Li, P.R. Shapiro and T. Rindler-Daller, PoS BASH2015, 028 (2016)
[36] A. Bevilacqua, J. Kowalski-Glikman and W. Wislicki, Phys. Rev. D 105, 105004 (2022)
[37] H. Foidl and T. Rindler-Daller, Phys. Rev. D 105, 123534 (2022)
[38] V. Faraoni, Cosmology in Scalar-Tensor Gravity, Fundamental Theories of Physics, Springer Dordrecht (2004)
[39] C. Brans and R.H. Dicke, Phys. Rev. 124, 195 (1961)
[40] P. Jordan, Nature 164, 637 (1937) and Schwerkraft und Weltfall, 2nd ed., Vieweg und Sohn, Braunschweig, (1955)
[41] K. Hayashi and T. Shirafuji, Phys. Rev. D 19, 3524 (1979)
[42] M. Tsamparlis, Phys. Lett. A 75, 27 (1979)
[43] V. Faraoni, Phys. Rev. D 59, 084021, (1999)
[44] J. Garcia-Bellido, D. Wands, Phys. Rev. D 52, 6739 (1995)
[45] S. Sen and A.A. Sen, Late time acceleration in Brans Dicke Cosmology, Phys. Rev. D 63, 124006 (2001)
[46] M. Artymowski, Z. Lalak and M. Lewicki, Inflation and dark energy from the Brans-Dicke theory, JCAP 06, 031 (2015)
[47] M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993)
[48] L. Jarv and J. Lember, Universe 7, 178 (2021)
[49] M. Hohmann, Phys. Rev. D 98, 064002 (2018)
[50] M. Hohmann and C. Pfeifer, Phys. Rev. D 98, 064003 (2018)
[51] H. Wei, Phys. Dynamics of Teleparallel Dark Energy, Lett. B 712, 430 (2012)
[52] C.-Q. Geng, C.-C. Lee, E.N. Saridakis and Y.-P. Wu, ”Teleparallel” Dark Energy, Phys. Lett. B 704, 384 (2011)
[53] C. Xu, E.N. Saridakis and G. Leon, Phase-Space analysis of Teleparallel Dark Energy, JCAP 07, 005 (2012)
[54] A. Paliathanasis, Eur. Phys. J. Plus 136, 674 (2021)
[55] E.J. Copeland, A.R. Liddle and D. Wands, Phys. Rev. D 57, 4686 (1998)
[56] . Wainwright and G.F.R Ellis, Cambridge University Press, Cambridge (1997)
[57] A.A. Coley, Dynamical Systems and Cosmology, Springer, Dordrecht (2003)
[58] L. Amendola and S. Tsujikawa, Dark Energy, Cambridge University Press, Cambridge (2010)
[59] C.R. Fadragas and G. Leon, Class. Quantum Grav. 31, 195011 (2014)
[60] O. Hrycyna and M. Szydlowski, JCAP 1312, 016 (2013)

[61] S. Bahamonte, K.F. Dialektopoulos, C. Escamilla-Rivera, V. Gakis, M. Hendry, J.L. Said, J. Mifsud and E. Di Valentino, Teleparallel Gravity: From Theory to Cosmology, arXiv:2106.13793 (2021)