THE INTERSECTION POLYNOMIALS OF A VIRTUAL KNOT

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Abstract. We introduce three kinds of invariants of a virtual knot called
the first, second, and third intersection polynomials. The definition is based
on the intersection number of a pair of curves on a closed surface. We study
properties of the intersection polynomials and their applications concerning the
behavior on symmetry, the crossing number and the virtual crossing number, a
connected sum of virtual knots, characterizations of intersection polynomials,
finitive type invariants of order two, and a flat virtual knot.

1. Introduction

In classical knot theory, we treat a circle embedded in a 3-dimensional sphere $S^3$
under an ambient isotopy. It is equivalent to study a circle embedded in the product
$S^2 \times I$ of a 2-sphere $S^2$ and an interval $I$ instead of $S^3$. In this sense, it is natural
to treat a circle in the product $\Sigma_g \times I$ of a closed, connected, oriented surface $\Sigma_g$ of
genus $g$ and $I$. Kauffman [12] leads the unification of such knot theories in $\Sigma_g \times I$
for all $g \geq 0$ and introduces virtual knot theory.

A virtual knot is described by a diagram on $\Sigma_g$ for some $g \geq 0$ under the projection
$\Sigma_g \times I$ onto $\Sigma_g$. A virtual knot is regarded as an equivalence class of diagrams
on closed surfaces under the relation generated by three kinds of Reidemeister moves
for diagrams and (de)stabilizations for surfaces. See Figure 1.

Some invariants of a virtual knot are natural generalizations of those of a classical
knot such as knot groups and Jones polynomials [12], and some vanish for classical

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knots such as Sawollek polynomials \cite{26} and writhe polynomials. The invariants introduced in this paper are of the latter type.

The writhe polynomial \(W_K(t)\) of an oriented virtual knot \(K\) is a Laurent polynomial originally defined in \cite{2,5,14,25} independently, and is extensively studied (cf. \cite{4,16,20,22,24,29}). It can be described in terms of homological intersections on \(\Sigma_g\) as follows. Let \(c_1,\ldots,c_n\) be the crossings of an oriented diagram \(D \subset \Sigma_g\) of \(K\), and \(\varepsilon_i\) the sign of \(c_i\) \((1 \leq i \leq n)\). Splicing a crossing \(c_i\), we obtain two cycles \(\gamma_i, \overline{\gamma}_i\) on \(\Sigma_g\) such that \(\gamma_i\) is the part of \(D\) from the over-crossing to the under-crossing at \(c_i\), and \(\overline{\gamma}_i\) is the one from the under-crossing to the over-crossing. Then the writhe polynomial can be expressed by

\[
W_K(t) = \sum_{i=1}^n \varepsilon_i (t^{\gamma_i} - 1) \in \mathbb{Z}[t, t^{-1}],
\]

where \(\gamma_i \cdot \overline{\gamma}_i\) is the intersection number of homological cycles \(\gamma_i\) and \(\overline{\gamma}_i\) on \(\Sigma_g\). This invariant does not depend on the genus \(g\) of the supporting surface \(\Sigma_g\).

In Section \cite{2}, we modify the right hand side of this equation and consider four kinds of Laurent polynomials defined by

\[
\begin{align*}
I_{01}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1), \\
I_{10}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_j} - 1), \\
I_{00}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1), \\
I_{11}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\overline{\gamma}_j} - 1).
\end{align*}
\]

They are invariant under second and third Reidemeister moves, but not invariant under a first Reidemeister move generally. By studying the difference, we introduce three kinds of invariants of a virtual knot \(K\) as follows;

\[
\begin{align*}
I_K(t) &= I_{01}(D; t) - \omega_D W_K(t), \\
II_K(t) &= I_{00}(D; t) + I_{11}(D; t) - \omega_D \overline{W}_K(t), \text{ and} \\
III_K(t) &= I_{00}(D; t) \pmod{\overline{W}_K(t)},
\end{align*}
\]

where \(\omega_D = \sum_{i=1}^n \varepsilon_i\) is the writhe of \(D\), \(\overline{W}_K(t) = W_K(t) + W_K(t^{-1})\), and \(f(t) \equiv g(t) \pmod{h(t)}\) means \(f(t) - g(t) = mh(t)\) for some \(m \in \mathbb{Z}\). They are called the first, second, and third intersection polynomials of a virtual knot \(K\), respectively. We remark that \(I_K(t)\) and \(II_K(t)\) are Laurent polynomials in \(\mathbb{Z}[t, t^{-1}]\), and \(III_K(t)\) is an equivalence class of Laurent polynomials.

In Section \cite{3} we explain how to calculate the intersection polynomials through a Gauss diagram, which also presents a virtual knot as well as a diagram on a closed surface. We give the intersection polynomials for all the virtual knots up to crossing number four in Appendices \cite{A} and \cite{B}. By observing the obtained intersection polynomials, we see that \(I_K(t)\), \(II_K(t)\), \(III_K(t)\), and \(W_K(t)\) are independent of each other (Proposition \ref{thm:independence}). The Alexander polynomials \(\Delta_i(K)\) \((i \geq 0)\) of a virtual knot \(K\) are defined in \cite{27} as a generalization of the classical ones. The 0-th Alexander polynomial \(\Delta_0(K)\) is also called the Sawollek polynomial \cite{26}. It is known in \cite{20} that the writhe polynomial is obtained from \(\Delta_0(K)\). On the other hand, we see that the \(I_K(t)\), \(II_K(t)\), and \(III_K(t)\) are not obtained from \(\Delta_i(K)\) \((i \geq 0)\) generally (Remark \ref{rem:independence}).
In Section 4, we study fundamental properties of the intersection polynomials. The writhe polynomial satisfies \( W_K(1) = W'_K(1) = 0 \). We prove that the same property holds for the first intersection polynomial; that is, \( I_K(1) = I'_K(1) = 0 \) (Theorem 4.2). On the other hand, the second intersection polynomial is reciprocal with \( II_K(1) = 0 \) and \( II'_K(1) \equiv 0 \pmod{4} \) (Theorem 4.3). Furthermore, the third intersection polynomial is also reciprocal with \( III_K(1) = 0 \) and \( III'_K(1) \equiv W''_K(1) \pmod{4} \) (Theorem 4.6). The last congruence is proved by using the forbidden move which is an unknotting operation for a virtual knot \([11, 23]\). These fundamental properties characterize a Laurent polynomial to be the intersection polynomial of some virtual knot, which will be proved in Section 8.

In Section 5, we first study the behaviors of intersection polynomials on symmetry of a virtual knot. For a virtual knot \( K \), we denote by \(-K\), \( K^#\), and \( K^\ast\) the reverse, the vertical mirror image, and the horizontal mirror image of \( K \), respectively. Here, the vertical (or horizontal) mirror image of \( K \) presented by a circle in \( \Sigma_g \times I \) is obtained by taking \( \Sigma_g \times (-I) \) (or \( (-\Sigma_g) \times I \)). By using the intersection polynomials, we can construct an infinite family of virtual knots \( K \) such that \( K, -K, K^#, K^\ast, -K^#, K^{##}, \) and \(-K^{##}\) are mutually distinct (Theorem 5.4). In this section, we also give lower bounds of the crossing number \( c(K) \) and the virtual crossing number \( vc(K) \) of a virtual knot \( K \). It is known in \([25]\) that \( c(K) \geq \deg W_K(t) + 1 \) and \( vc(K) \geq \deg W_K(t) \).

Similarly to these inequations, we prove that

\[
\begin{align*}
    c(K) &\geq \deg I_K(t) + 1, \deg II_K(t) + 1, \text{ and } \deg III_K(t) + 1, \\
    vc(K) &\geq \deg I_K(t), \deg II_K(t), \text{ and } \deg III_K(t)
\end{align*}
\]

(Propositions 5.7 and 5.9). We also construct an infinite family of virtual knots \( K \) such that \( c(K) \) and \( vc(K) \) are determined by the intersection polynomials, but not by the writhe polynomial (Example 5.10).

In Section 6, we study a dotted virtual knot which is also called a long virtual knot. We introduce a pair of invariants \( W_0^0(t) \) and \( W_1^1(t) \) of a dotted virtual knot \( T \) in a similar way to the definition of the writhe polynomial. The virtual knot obtained from \( T \) by ignoring the base point is called the closure of \( T \) and denoted by \( \hat{T} \). We prove that for any virtual knot \( K \), there are infinitely many dotted virtual knots \( T \) with \( \hat{T} = K \) by using the invariants \( W_0^0(t) \) and \( W_1^1(t) \) (Proposition 6.4).

In Section 7, we give the connected sum formulae for the first and second intersection polynomials (Theorems 7.1 and 7.5). A connected sum \( K'' \) of virtual knots \( K \) and \( K' \) is defined to be the closure of the sum of some dotted virtual knots \( T \) and \( T' \) with \( \hat{T} = K \) and \( \hat{T'} = K' \). It is known in \([2, 5, 25]\) that the writhe polynomial is additive with respect to a connected sum; that is,

\[
W_{K''}(t) = W_{K}(t) + W_{K'}(t).
\]

In Section 8, we study the fundamental properties of the intersection polynomials. The writhe polynomial satisfies \( W_K(1) = W'_K(1) = 0 \). We prove that the same property holds for the first intersection polynomial; that is, \( I_K(1) = I'_K(1) = 0 \) (Theorem 4.2). On the other hand, the second intersection polynomial is reciprocal with \( II_K(1) = 0 \) and \( II'_K(1) \equiv 0 \pmod{4} \) (Theorem 4.3). Furthermore, the third intersection polynomial is also reciprocal with \( III_K(1) = 0 \) and \( III'_K(1) \equiv W''_K(1) \pmod{4} \) (Theorem 4.6). The last congruence is proved by using the forbidden move which is an unknotting operation for a virtual knot \([11, 23]\). These fundamental properties characterize a Laurent polynomial to be the intersection polynomial of some virtual knot, which will be proved in Section 8.
On the other hand, we prove that
\[ I_{K''}(t) = I_K(t) + I_{K'}(t) + W^0_T(t)W^1_T(t) + W^1_T(t)W^0_T(t) \text{ and} \]
\[ II_{K''}(t) = II_K(t) + II_{K'}(t) + W^0_T(t)W^0_T(t^{-1}) + W^1_T(t)W^1_T(t^{-1}) + W^0_T(t^{-1})W^0_T(t) + W^1_T(t^{-1})W^1_T(t). \]
Therefore, the first and second intersection polynomials are not additive generally. As an application of the connected sum formulae, we prove that for any virtual knots \( K \) and \( K' \), there are infinitely many connected sums of \( K \) and \( K' \) (Theorem 7.11).

In Section 8, we give characterizations of the intersection polynomials. It is known in \([25]\) that a Laurent polynomial \( f(t) \) satisfies \( f(t) = W_K(t) \) for some virtual knot \( K \) if and only if \( f(1) = f''(1) = 0 \). A characterization of the first intersection polynomial is exactly the same as above (Theorem 8.1). On the other hand, \( f(t) = II_K(t) \) holds for some \( K \) if and only if \( f(t) \) is reciprocal, \( f(1) = 0 \), and \( f''(1) \equiv 0 \pmod{4} \) (Theorem 8.5). Furthermore, a pair of Laurent polynomials \( f(t) \) and \( g(t) \) satisfies \( f(t) = W_K(t) \) and \( g(t) \equiv III_K(t) \pmod{f(t) + f(t^{-1})} \) for some \( K \) if and only if \( g(t) \) is reciprocal, \( f(1) = f'(1) = g(1) = 0 \), and \( f''(1) \equiv g''(1) \pmod{4} \) (Theorem 8.9). We also give characterizations of the intersection polynomials of a connected sum of two trivial knots (Propositions 8.11, 8.13).

In Section 9, we study a finite type invariant of a virtual knot. Dye \([4]\) proves that the writhe polynomial is a finite type invariant of order 1. On the other hand, we prove that the first and second intersection polynomials are finite type invariants of order 2 (Theorem 9.2). The definition of a finite type invariant adopted in this paper is a natural generalization of that in classical knot theory, which is different from the one given by Goussarov, Polyak, and Viro \([7]\).

Section 10 proves that \( I_K(t) + I_K(t^{-1}) \) is invariant under a crossing change (Theorem 10.3). The difference \( W_K(t) - W_K(t^{-1}) \) also has the same property \([25]\). Therefore these polynomials are regarded as invariants of a flat virtual knot, which is obtained from a virtual knot by ignoring the over/under-information of crossings. In particular, they are invariants of a homotopy class of a circle immersed in a closed surface. The supporting genus \( sg(K) \) of a virtual knot \( K \) is the minimal genus of the surface where a diagram of \( K \) exists. We see that if \( W_K(t) \neq W_K(t^{-1}) \) or \( I_K(t) + I_K(t^{-1}) \neq II_K(t) \), then we have \( sg(K) \geq 2 \) (Corollary 10.6).

In Appendix C we construct an infinite family of virtual knots such that the virtual unknotting number is equal to one, and the writhe, Jones, Miyazawa polynomials are all trivial, but the intersection polynomials are non-trivial.

2. Definitions

Let \( \Sigma_g \) be a closed, connected, oriented surface of genus \( g \), and \( \alpha \) and \( \beta \) closed, oriented curves on \( \Sigma_g \). We often regard these curves as homology cycles of \( H_1(\Sigma_g) \). The intersection number \( \alpha \cdot \beta \in \mathbb{Z} \) is defined to be the homology intersection of the ordered pair \( (\alpha, \beta) \). Geometrically it is calculated as follows. By perturbing \( \alpha \) and \( \beta \) if necessary, we may assume that \( \alpha \cap \beta \) consists of \( m \) transverse double points \( p_1, \ldots, p_m \). At a double point \( p_k \) (\( 1 \leq k \leq m \)), if \( \beta \) intersects \( \alpha \) from the left or right as we walk along \( \alpha \), we define \( e_k = +1 \) or \( -1 \), respectively. Then we have
\[ \alpha \cdot \beta = \sum_{k=1}^{n} e_k. \]  See Figure 2. We remark that \( \alpha \cdot \beta = -\beta \cdot \alpha \) and \( \alpha \cdot \alpha = 0 \) by definition.

We consider a circle embedded in \( \Sigma_g \times [0,1] \) for some \( g \geq 0 \). We identify two embedded circles up to ambient isotopies and (de)stabilizations. Such an equivalence class is called a virtual knot (cf. [3, 10, 12, 17]).

More precisely, a virtual knot is described by a diagram on \( \Sigma_g \) which is a projection image under the projection \( \Sigma_g \times [0,1] \to \Sigma_g \) equipped with over/under-information at double points. A double point with over/under information is called a crossing. Two diagrams \((\Sigma_g, D)\) and \((\Sigma_g', D')\) present the same virtual knot if and only if there is a finite sequence of diagrams

\[
(\Sigma_g, D) = (\Sigma_{g_1}, D_1), (\Sigma_{g_2}, D_2), \ldots, (\Sigma_{g_s}, D_s) = (\Sigma_{g'}, D')
\]
such that for each \( 1 \leq i \leq s - 1 \),

(i) \( g_{i+1} = g_i \) holds and \( (\Sigma_{g_{i+1}}, D_{i+1}) \) is obtained from \( (\Sigma_{g_i}, D_i) \) by an orientation-preserving homeomorphism of \( \Sigma_{g_i} = \Sigma_{g_{i+1}} \),

(ii) \( g_{i+1} = g_i \pm 1 \) holds and \( \Sigma_{g_{i+1}} \) is obtained from \( \Sigma_{g_i} \) by 1- or 2-handle surgery missing \( D_i = D_{i+1} \). Such a deformation is called a stabilization or destabilization, respectively.

Throughout this paper, we assume that all virtual knots are oriented.

Let \( D = (\Sigma, D) \) be a diagram of a virtual knot \( K \), and \( c_1, \ldots, c_n \) the crossings of \( D \). We denote by \( \gamma_D \) the closed, oriented curve on \( \Sigma \) obtained from \( D \) by ignoring over/under-information at \( c_i \)'s. Furthermore, we denote by \( \gamma_i \) \( (1 \leq i \leq n) \) the closed, oriented curve as a part of \( \gamma_D \) from the overcrossing to the undercrossing at \( c_i \), and by \( \overline{\gamma}_i \), the curve from the undercrossing to the overcrossing at \( c_i \). See Figure 3. These curves satisfy \( \gamma_i + \overline{\gamma}_i = \gamma_D \) and \( \gamma_i \cdot \overline{\gamma}_i = \gamma_i \cdot (\gamma_D - \gamma_i) = \gamma_i \cdot \gamma_D \) as homology cycles on \( \Sigma \). We call \( \gamma_i \) and \( \overline{\gamma}_i \) the cycles at \( c_i \) on \( \Sigma \).

**Figure 3**

**Definition 2.1** ([2, 3, 13, 23]). The Laurent polynomial

\[
W_D(t) = \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i} \overline{\gamma}_i - 1) = \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i \cdot \gamma_D} - 1) \in \mathbb{Z}[t, t^{-1}]
\]
Lemma 2.2. For any diagram $D$ on $\Sigma$, we have $W_{D^\#}(t) = -W_D(t^{-1})$.

Proof. Let $c_1^\#, \ldots, c_n^\#$ be the crossings of $D^\#$ such that $c_i^\#$ corresponds to $c_i$, $\varepsilon_i^\#$ the sign of $c_i$, and $\gamma_i^\#$ the cycle at $c_i^\#$ on $\Sigma$ $(1 \leq i \leq n)$. Then we have

$$\varepsilon_i^\# = -\varepsilon_i, \quad \gamma_i^\# = \gamma_i, \quad \text{and} \quad \tau_i^\# = \gamma_i.$$

Therefore it holds that

$$W_{D^\#}(t) = \sum_{i=1}^{n} \varepsilon_i^\#(t^{\gamma_i^\#} - 1) = -\sum_{i=1}^{n} \varepsilon_i(t^{\gamma_i} - 1) = -W_D(t^{-1}).$$

Now we consider four kinds of Laurent polynomials as follows:

$$f_{01}(D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_j} - 1), \quad f_{10}(D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1),$$

$$f_{00}(D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1), \quad f_{11}(D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_j} - 1).$$

Lemma 2.3. For any diagram $D$ on $\Sigma$, we have the following.

(i) $f_{10}(D; t) = f_{01}(D^\#; t)$.

(ii) $f_{11}(D; t) = f_{00}(D^\#; t)$.

Proof. Let $c_1^\#, \ldots, c_n^\#$ be the crossings of $D^\#$ such that $c_i^\#$ corresponds to $c_i$, $\varepsilon_i^\#$ the sign of $c_i$, and $\gamma_i^\#$ the cycle at $c_i^\#$ on $\Sigma$ $(1 \leq i \leq n)$. Then we have

$$\varepsilon_i^\# = -\varepsilon_i, \quad \gamma_i^\# = \gamma_i, \quad \text{and} \quad \tau_i^\# = \gamma_i.$$

Since it holds that

$$\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i^\# = \tau_i^\# \cdot \tau_j = \gamma_i^\# \cdot \gamma_j^\#$$

we have the conclusion by definition. \hfill $\Box$

In what follows, we often abbreviate $f_{pq}(D; t)$ to $f_{pq}(D)$ for $p, q \in \{0, 1\}$.

Lemma 2.4. If a diagram $D'$ is obtained from $D$ by a second or third Reidemeister move on $\Sigma$, then we have $f_{pq}(D) = f_{pq}(D')$ for any $p, q \in \{0, 1\}$.

Proof. Since $D^{\#}$ is obtained from $D^\#$ by a second or third Reidemeister move on $\Sigma$, it is sufficient to prove the invariance of $f_{01}(D)$ and $f_{00}(D)$ by Lemma 2.3.

A second Reidemeister move. Assume that $D'$ is obtained from $D$ by a second Reidemeister move removing a pair of crossings $c_1$ and $c_2$ of $D$. For $3 \leq i \leq n$, let $c_i'$ be the crossing of $D'$ corresponding to $c_i$, $\varepsilon_i'$ the sign of $c_i'$, and $\gamma_i'$ the cycle at $c_i'$ on $\Sigma$. Then it holds that

$$\varepsilon_1 = -\varepsilon_2, \quad \gamma_1 = \gamma_2, \quad \text{and} \quad \varepsilon_i' = \varepsilon_i, \quad \gamma_i' = \gamma_i \quad (3 \leq i \leq n).$$
See Figure 4. A disk on $\Sigma$ is bounded by two arcs on $D$ connecting $c_1$ and $c_2$. Since $\gamma_{D'} = \gamma_D$ and $\gamma_i' = \gamma_i$ $(3 \leq i \leq n)$, we have

\[
f_{01}(D) - f_{01}(D') = \varepsilon_1^2(t^{\gamma_1 \gamma_1} - 1) + \varepsilon_2 \varepsilon_1(t^{\gamma_2 \gamma_1} - 1) + \varepsilon_2^2(t^{\gamma_2 \gamma_2} - 1) + \sum_{j=3}^{n} \varepsilon_1 \varepsilon_j(t^{\gamma_1 \gamma_j} - 1) + \sum_{j=3}^{n} \varepsilon_2 \varepsilon_j(t^{\gamma_2 \gamma_j} - 1) + \sum_{i=3}^{n} \varepsilon_i \varepsilon_1(t^{\gamma_i \gamma_1} - 1) + \sum_{i=3}^{n} \varepsilon_i \varepsilon_2(t^{\gamma_i \gamma_2} - 1) + \sum_{i=3}^{n} \varepsilon_i \varepsilon_{i+1} = (t^{\gamma_1 \gamma_1} - 1) - (t^{\gamma_1 \gamma_1} - 1) + (t^{\gamma_1 \gamma_1} - 1) = 0.
\]

Therefore $f_{01}(D)$ is invariant under a second Reidemeister move. On the other hand, the invariance of $f_{00}(D)$ is proved by

\[
f_{00}(D) - f_{00}(D') = (t^{\gamma_1 \gamma_1} - 1) - (t^{\gamma_1 \gamma_1} - 1) + (t^{\gamma_1 \gamma_1} - 1) = 0.
\]

A third Reidemeister move. Assume that $D'$ is obtained from $D$ by a third Reidemeister move involving three crossings $c_1, c_2, c_3$ of $D$. For $1 \leq i \leq n$, let $c_i'$ be the crossing corresponding to $c_i$, $\varepsilon_i'$ the sign of $c_i'$, and $\gamma_i'$ the cycle at $c_i'$ on $\Sigma$. Then it holds that $\varepsilon_i' = \varepsilon_i$ and $\gamma_i' = \gamma_i$ $(1 \leq i \leq n)$.

Figure 5 shows $\gamma_i' = \gamma_i$ for $i = 1, 2, 3$. A disk on $\Sigma$ is bounded by three arcs of $D$ connecting $c_1$ and $c_2$, $c_1$ and $c_3$, and $c_2$ and $c_3$. Therefore both $f_{01}(D)$ and $f_{00}(D)$ are invariant under a third Reidemeister move. We remark that $\gamma_1 + \gamma_2 = \gamma_3$ holds in this case.

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**Figure 4**

**Figure 5**
We use the notation \( \mathcal{W}_K(t) = \mathcal{W}_D(t) = W_D(t) + W_D(t^{-1}) \). It follows by Lemma 2.2 that
\[
\mathcal{W}_D^\#(t) = W_D^\#(t) + W_D^\#(t^{-1}) = -W_D(t^{-1}) - W_D(t) = -\mathcal{W}_D(t).
\]

**Lemma 2.5.** If a diagram \( D' \) is obtained from \( D \) by a first Reidemeister move on \( \Sigma \) as shown in Figure 6(a)–(d), then the difference \( f_{pq}(D) - f_{pq}(D') \) is given as shown in Table 1.

![Diagram](image)

**Table 1**

|   | (a) \( \varepsilon_1 = +1 \) | (b) \( \varepsilon_1 = -1 \) | (c) \( \varepsilon_1 = -1 \) | (d) \( \varepsilon_1 = +1 \) |
|---|-----------------|-----------------|-----------------|-----------------|
| \( f_{01}(D) - f_{01}(D') \) | \( \mathcal{W}_K(t) \) | \(-\mathcal{W}_K(t) \) | \(-\mathcal{W}_K(t) \) | \( \mathcal{W}_K(t) \) |
| \( f_{10}(D) - f_{10}(D') \) | \( \mathcal{W}_K(t^{-1}) \) | \(-\mathcal{W}_K(t^{-1}) \) | \(-\mathcal{W}_K(t^{-1}) \) | \( \mathcal{W}_K(t^{-1}) \) |
| \( f_{00}(D) - f_{00}(D') \) | \( 0 \) | \( 0 \) | \(-\mathcal{W}_K(t) \) | \( \mathcal{W}_K(t) \) |
| \( f_{11}(D) - f_{11}(D') \) | \( \mathcal{W}_K(t) \) | \(-\mathcal{W}_K(t) \) | \( 0 \) | \( 0 \) |

**Proof.** Assume that \( c_1 \) is removed from \( D \) by a first Reidemeister move. For \( 2 \leq i \leq n \), let \( c'_i \) be the crossing of \( D' \) corresponding to \( c_i \), \( \varepsilon'_i \) the sign of \( c'_i \), and \( \gamma'_i \) the cycle at \( c'_i \) on \( \Sigma \). Then it holds that \( \gamma'_i = \gamma_i \) and \( \varepsilon'_i = \varepsilon_i \) (\( 2 \leq i \leq n \)). By definition, we have \( \gamma_1 = 0 \) and \( \tau_1 = \gamma_D \) for (a) and (b), and \( \gamma_1 = \gamma_D \) and \( \tau_1 = 0 \) for (c) and (d).

\( (p, q) = (0, 1) \). It holds that
\[
f_{01}(D) - f_{01}(D') = \varepsilon_1^2 (t^{\gamma_1 \tau_1} - 1) + \sum_{j=2}^{n} \varepsilon_1 \varepsilon_j (t^{\gamma_j \tau_j} - 1) + \sum_{i=2}^{n} \varepsilon_i \varepsilon_1 (t^{\gamma_i \tau_i} - 1).
\]

For (a) and (b), we have
\[
f_{01}(D) - f_{01}(D') = \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_i \tau_i} - 1) = \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_i \tau_i} - 1) = \varepsilon_1 \mathcal{W}_K(t).
\]

For (c) and (d), by using the equation \( \gamma_D \cdot \tau_j = (\gamma_j + \tau_j) \cdot \tau_j = \gamma_j \cdot \tau_j \), we have
\[
f_{01}(D) - f_{01}(D') = \varepsilon_1 \sum_{j=2}^{n} \varepsilon_j (t^{\gamma_j \tau_j} - 1) = \varepsilon_1 \sum_{j=2}^{n} \varepsilon_j (t^{\gamma_j \tau_j} - 1) = \varepsilon_1 \mathcal{W}_K(t).
\]
\( (p, q) = (1, 0) \). If \( D' \) is obtained from \( D \) by (a), (b), (c), or (d), then \( D'' \) is obtained from \( D \) by (c), (d), (a), or (b), respectively. By Lemmas 2.2, 2.3, and the equation for \( (p, q) = (0, 1) \), it holds that

\[
f_{10}(D) - f_{10}(D') = f_{01}(D'') - f_{01}(D') = (-\varepsilon_1) \cdot W_D(t) = \varepsilon_1 W_D(t^{-1}).
\]

\( (p, q) = (0, 0) \). It holds that

\[
f_{00}(D) - f_{00}(D') = \varepsilon_1^2 (t^{\gamma_1 - \gamma_1} - 1) + \sum_{j=2}^{n} \varepsilon_j (t^{\gamma_j - \gamma_j} - 1) + \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_i - \gamma_i} - 1).
\]

For (a) and (b), we have \( f_{00}(D) - f_{00}(D') = 0 \) by \( \gamma_1 = 0 \). For (c) and (d), by using \( \gamma_1 = \gamma_D \), it holds that

\[
f_{00}(D) - f_{00}(D') = \varepsilon_1 \cdot W_K(t^{-1}) + \varepsilon_1 W_K(t) = \varepsilon_1 W_D(t).
\]

\( (p, q) = (1, 1) \). The proof is similar to the case \( (p, q) = (1, 0) \). For (a) and (b), \( D'' \) is obtained from \( D \) by (c) and (d), respectively. By the equation for \( (p, q) = (0, 0) \), we have

\[
f_{11}(D) - f_{11}(D') = f_{00}(D'') - f_{00}(D') = (-\varepsilon_1) \cdot W_D(t) = \varepsilon_1 W_D(t).
\]

For (c) and (d), since \( D'' \) is obtained from \( D \) by (a) and (b), respectively, it holds that \( f_{11}(D) - f_{11}(D') = f_{00}(D'') - f_{00}(D') = 0 \).

The writhe of a diagram \( D \) is the sum of the signs of crossings of \( D \), and denoted by \( \omega_D = \sum_{i=1}^{n} \varepsilon_i \). We consider two kinds of Laurent polynomials

\[
I_D(t) = f_{01}(D; t) - \omega_D W_K(t) \quad \text{and} \quad II_D(t) = f_{00}(D; t) + f_{11}(D; t) - \omega_D W_K(t).
\]

**Theorem 2.6.** The Laurent polynomials \( I_D(t) \) and \( II_D(t) \) \( \in \mathbb{Z}[t, t^{-1}] \) do not depend on a particular choice of a diagram \( D \) of a virtual knot \( K \).

**Proof.** Since the intersection numbers among \( \gamma_i \)'s and \( \pi_i \)'s \( (1 \leq i \leq n) \) do not change by a (de)stabilization, it is sufficient to consider the case that a diagram \( D' \) is obtained from \( D \) by a Reidemeister move on \( \Sigma \).

Assume that \( D' \) is obtained from \( D \) by a second or third Reidemeister move on \( \Sigma \). Since \( \omega_{D'} = \omega_D \) holds, we have \( I_{D'}(t) = I_D(t) \) and \( II_{D'}(t) = II_D(t) \) by Lemma 2.4.

Assume that \( D' \) is obtained from \( D \) by a first Reidemeister move such that a crossing \( c_1 \) with the sign \( \varepsilon_1 \) is removed from \( D \). Since it holds that \( \omega_D = \omega_{D'} + \varepsilon_1 \) and \( f_{01}(D) - f_{01}(D') = \varepsilon_1 W_K(t) \) by Lemma 2.5, we have

\[
I_D(t) = f_{01}(D) - \omega_D W_K(t) = f_{01}(D') + \varepsilon_1 W_K(t) - (\omega_{D'} + \varepsilon_1) W_K(t) = f_{01}(D') - \omega_{D'} W_K(t) = I_{D'}(t).
\]
Similarly, since it holds that
\[(f_{00}(D) + f_{11}(D)) - (f_{00}(D') + f_{11}(D')) = \varepsilon_1 \overline{W}_K(t)\]
by Lemma 2.5 we have \(II_D(t) = II_{D'}(t)\).

**Definition 2.7.** The Laurent polynomials \(I_D(t)\) and \(II_D(t)\) \(\in \mathbb{Z}[t, t^{-1}]\) are called the first and second intersection polynomials of a virtual knot \(K\), and denoted by \(I_K(t)\) and \(II_K(t)\), respectively.

**Remark 2.8.** We can also consider the polynomial \(f_{10}(D; t) - \omega_D W_K(t^{-1})\) which defines an invariant of \(K\) by Lemmas 2.4 and 2.5. However, this is coincident with the first intersection polynomial \(I_K(t^{-1})\). In fact, we have
\[
 f_{10}(D; t) - \omega_D W_K(t^{-1}) = \sum_{1 \leq i, j \leq n} \varepsilon_i (t^{\gamma_i} - 1) - \omega_D W_K(t^{-1}) \\
 = \sum_{1 \leq i, j \leq n} \varepsilon_i (t^{-\gamma_i} - 1) - \omega_D W_K(t^{-1}) = I_K(t^{-1}).
\]

For a Laurent polynomial \(h(t)\), we consider an equivalence relation in \(\mathbb{Z}[t, t^{-1}]\) such that \(f(t) \equiv g(t) \pmod{h(t)}\) holds if and only if \(f(t) - g(t) = mh(t)\) for some \(m \in \mathbb{Z}\). In the case of \(h(t) = 0\), this equivalence relation gives \(f(t) = g(t)\) only. For a diagram \(D\) of a virtual knot \(K\) on \(\Sigma\), we consider the equivalence class \(f_{00}(D) \pmod{W_K(t)}\). The invariance follows by Lemmas 2.4 and 2.5 immediately.

**Definition 2.9.** The equivalence class \(f_{00}(D) \pmod{W_K(t)}\) is called the third intersection polynomial of a virtual knot \(K\), and denoted by \(III_K(t)\).

**Remark 2.10.** We can also consider the equivalence class \(f_{11}(D) \pmod{W_K(t)}\) as an invariant of \(K\). However, since \(II_K(t) \equiv f_{00}(D) + f_{11}(D) \pmod{W_K(t)}\) holds by definition, we have
\[
f_{11}(D) \equiv II_K(t) - III_K(t) \pmod{W_K(t)}.
\]

A virtual knot is classical if it is presented by a diagram on \(S^2\). By definition, the writhe polynomial \(W_K(t)\) vanishes for any classical knot \([2, 14]\). The intersection polynomials satisfy the same property as follows.

**Lemma 2.11.** Any classical knot satisfies
\[I_K(t) = II_K(t) = III_K(t) = 0.\]

**Proof.** All intersection numbers between two cycles on \(S^2\) are zero. \(\square\)

3. Calculations

Let \(C\) be a closed, oriented curve on \(\Sigma\) with a finite number of crossings. When we consider \(C\) as the image of an immersion \(S^1 \rightarrow \Sigma\), the curve \(C\) is presented by a Gauss diagram \(G\) consisting of the circle \(S^1\) equipped with chords each of which connects the preimage of a crossing of \(C\). The endpoints of chords admit signs with respect to the orientation of \(C\) as shown in Figure 7.

The endpoints of a chord of \(G\) divide the circle \(S^1\) into two arcs. Let \(\alpha \subset S^1\) be such an arc, and \(P(\alpha)\) the set of endpoints of the chords of \(G\) in the interior of \(\alpha\). For an endpoint \(x \in P(\alpha)\), we denote by \(\text{sgn}(x)\) the sign of \(x\), and by \(\tau(x)\) the other endpoint of the chord incident to \(x\). The arc \(\alpha \subset S^1\) presents a cycle on \(\Sigma\), which is also denoted by \(\alpha \subset \Sigma\). See Figure 8.
Lemma 3.1. Let $\overline{\alpha}$ be the complementary arc of $\alpha \subset S^1$. Then we have
\[
\alpha \cdot \overline{\alpha} = \sum_{x \in P(\alpha)} \text{sgn}(x).
\]

Proof. Any chord whose endpoints both lie on $\alpha$ does not contribute to the sum in the right hand side of the equation. Therefore it holds that
\[
\sum_{x \in P(\alpha)} \text{sgn}(x) = \sum_{x \in P(\alpha), \tau(x) \in P(\overline{\alpha})} \text{sgn}(x) = \alpha \cdot \overline{\alpha}.
\]

\[\square\]

Let $\alpha$ and $\beta \subset S^1$ be arcs for distinct chords $a$ and $b$ of $G$, respectively. We consider an integer
\[
S(\alpha, \beta) = \sum_{x \in P(\alpha), \tau(x) \in P(\beta)} \text{sgn}(x).
\]
It follows by definition that $S(\alpha, \beta) = -S(\beta, \alpha)$. We say that the chords $a$ and $b$ of $G$ are linked if their endpoints appear on $S^1$ alternately, and otherwise unlinked. The number $S(\alpha, \beta)$ is equal to the sum of the signs of the endpoints indicated by dots as shown in Figure 9.

Then the intersection number $\alpha \cdot \beta$ of the cycles $\alpha$ and $\beta \subset \Sigma$ is calculated as follows.

Lemma 3.2. (i) If $a$ and $b$ are unlinked, then $\alpha \cdot \beta = S(\alpha, \beta)$. 
(ii) Assume that $a$ and $b$ are linked as shown in Figure 10 where $\varepsilon, \delta \in \{\pm\}$. Then it holds that

$$\alpha \cdot \beta = S(\alpha, \beta) + \frac{1}{2}(\varepsilon + \delta) \quad \text{and} \quad \beta \cdot \alpha = S(\beta, \alpha) - \frac{1}{2}(\varepsilon + \delta).$$

\[\text{Figure 10}\]

Proof. We prove two cases $(\varepsilon, \delta) = (+, +)$ and $(+, -)$ in (ii) as shown in Figure 11. Other cases are similarly proved.

$(\varepsilon, \delta) = (+, +)$. We take a parallel copy of the curve $\alpha \subset \Sigma$ (or $\beta$) which lies on the left (or right) side of the original curve. See the left of Figure 11. Then the intersections between $\alpha$ and $\beta$ except one point near the crossing $b$ correspond to the endpoints $x \in P(\alpha)$ with $\tau(x) \in P(\beta)$. Since the sign of the exceptional intersection is equal to $+$, we obtain $\alpha \cdot \beta = S(\alpha, \beta) + 1$.

$(\varepsilon, \delta) = (+, -)$. Similarly to the above case, we consider parallel copies of $\alpha$ and $\beta$. In this case, there is no exceptional intersection near $b$. See the right of Figure 11. Therefore we have $\alpha \cdot \beta = S(\alpha, \beta)$. \[\square\]

\[\text{Figure 11}\]

Example 3.3. We consider three arcs $\alpha$, $\beta$, and $\gamma$ of the Gauss diagram as shown in Figure 12. We have

$$S(\alpha, \beta) = -2, \quad S(\alpha, \gamma) = 1, \quad \text{and} \quad S(\beta, \gamma) = 0.$$ Since $\alpha$ and $\beta$ are unlinked, it holds that $\alpha \cdot \beta = S(\alpha, \beta) = -2$ by Lemma 3.2(i). On the other hand, since $\alpha$ and $\gamma$, $\beta$ and $\gamma$ are linked, respectively, it holds that $\alpha \cdot \gamma = S(\alpha, \gamma) = 1$ and $\beta \cdot \gamma = S(\beta, \gamma) + 1 = 1$ by Lemma 3.2(ii). \[\square\]
Let $D \subset \Sigma$ be a diagram of a virtual knot $K$ with $n$ crossings $c_1, \ldots, c_n$, and $G$ the Gauss diagram of $D$. We also denote by $c_i$ the chord of $G$ corresponding to a crossing $c_i$ of $D$. Each chord of $G$ is oriented from the over-crossing to the under-crossing, and equipped with the same sign as that of the corresponding crossing of $D$. Then we see that if the sign of a chord is equal to $\varepsilon$, then the initial and terminal endpoints of the chord have the sign $-\varepsilon$ and $\varepsilon$, respectively. See Figure 13.

The endpoints of an oriented chord $c_i$ of $G$ divide the circle $S^1$ into two arcs. The arc from the initial endpoint to the terminal corresponds to the cycle $\gamma_i \subset \Sigma$ at the crossing $c_i$, and the other arc corresponds to $\overline{\gamma_i}$. Therefore we see that

$$\gamma_i \cdot \overline{\gamma_i} = \gamma_i \cdot \gamma_D = \sum_{x \in \mathcal{P}(\gamma_i)} \text{sgn}(x).$$

If two chords $c_i$ and $c_j$ are unlinked, then it follows by Lemma 3.2(i) that

$$\gamma_i \cdot \overline{\gamma_j} = S(\gamma_i, \overline{\gamma_j}), \quad \gamma_i \cdot \gamma_j = S(\gamma_i, \gamma_j), \quad \text{and} \quad \overline{\gamma_i} \cdot \overline{\gamma_j} = S(\overline{\gamma_i}, \overline{\gamma_j}).$$

We have similar equations in the case that $c_i$ and $c_j$ are linked by Lemma 3.2(ii). For example, we consider the case as shown in Figure 14. Then we have

$$\gamma_i \cdot \overline{\gamma_j} = S(\gamma_i, \overline{\gamma_j}) + (\varepsilon_i - \varepsilon_j)/2,$$

$$\gamma_i \cdot \gamma_j = S(\gamma_i, \gamma_j) - (\varepsilon_i + \varepsilon_j)/2, \quad \text{and} \quad \overline{\gamma_i} \cdot \overline{\gamma_j} = S(\overline{\gamma_i}, \overline{\gamma_j}) + (\varepsilon_i + \varepsilon_j)/2.$$

Let $c(K)$ denote the crossing number of $K$, which is the minimal number of crossings for all diagrams of $K$. The virtual knots up to crossing number four are given by Green [8]. In what follows, the labels of virtual knots are due to Green's table.

**Example 3.4.** We consider the Gauss diagram $G$ of a virtual knot $K = 4.39$ as shown in Figure 15. Table 2 shows the intersection numbers $\gamma_i \cdot \overline{\gamma_j}$, $\gamma_i \cdot \gamma_j$, and
Let $\gamma_i, \gamma_j$ for $1 \leq i, j \leq 4$. Since we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -$ and $\varepsilon_4 = +$, it holds that

$$W_K(t) = \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i} - 1) = -t^3 + t^2 + 1 - t^{-1}$$
and

$$\overline{W}_K(t) = -t^3 + t^2 - t - 2 - t^{-1} + t^{-2} - t^{-3}.$$  

On the other hand, we have

$$f_{01}(D) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1) = 2t^2 - 2t - 2 + 2t^{-1},$$

$$f_{00}(D) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1) = t^3 - t^2 - t^{-2} + t^{-3},$$

$$f_{11}(D) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1) = t^2 - t - t^{-1} + t^{-2}.$$  

Since $\omega_D = -2$ holds, we obtain

$$I_K(t) = -2t^3 + 4t^2 - 2t,$$

$$II_K(t) = -t^3 + 2t^2 - 3t + 4 - 3t^{-1} + 2t^{-2} - t^{-3},$$
and

$$III_K(t) \equiv -t + 2 - t^{-1} \pmod{-t^3 + t^2 - t + 2 - t^{-1} + t^{-2} - t^{-3}}.$$  

\[\square\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure14}
\caption{}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure15}
\caption{}
\end{figure}

**Theorem 3.5.** For the virtual knots $K$ with $c(K) \leq 4$, the intersection polynomials $I_K(t)$, $II_K(t)$, and $III_K(t)$ are given in Appendices A and B.\[\square\]

By observing the calculations in Appendices A and B, we see that the writhe polynomial and the intersection polynomials are independent of each other in the following sense.
Proposition 3.6. There are four pairs of virtual knots $K_i$ and $K'_i$ (i = 1, 2, 3, 4) which satisfy the following.

(i) For the virtual knots $K_1(t) \neq W_{K'_1}(t)$ and $X_{K_1}(t) = X_{K'_1}(t)$ ($X = I, II, III$).
(ii) $I_{K_2}(t) \neq I_{K'_2}(t)$ and $X_{K_2}(t) = X_{K'_2}(t)$ ($X = W, II, III$).
(iii) $II_{K_3}(t) \neq II_{K'_3}(t)$ and $X_{K_3}(t) = X_{K'_3}(t)$ ($X = W, I, III$).
(iv) $III_{K_4}(t) \neq III_{K'_4}(t)$ and $X_{K_4}(t) = X_{K'_4}(t)$ ($X = W, I, II$).

Proof. (i) For the virtual knots $K_1 = 4.36$ and $K'_1 = 4.65$, it holds that
\[ W_{K_1}(t) = t^2 - 2 + t^{-2} \text{ and } W_{K'_1}(t) = -t^2 + 2 - t^{-2}. \]

On the other hand, we have
\[
\begin{align*}
I_{K_1}(t) &= I_{K'_1}(t) = -t^2 + 2 - t^{-2}, \\
II_{K_1}(t) &= II_{K'_1}(t) = -2t^2 + 4 - 2t^{-2}, \text{ and} \\
III_{K_1}(t) &= III_{K'_1}(t) \equiv t^2 - 2 + t^{-2} \pmod{2t^2 - 4 + 2t^{-2}}.
\end{align*}
\]

(ii) For the trivial knot $K_2 = O$ and the virtual knot $K'_2 = 4.16$, it holds that
\[ I_{K_2}(t) = 0 \text{ and } I_{K'_2}(t) = -t^2 + 3t - 3 + t^{-1}. \]

On the other hand, we have
\[
\begin{align*}
W_{K_2}(t) &= W_{K'_2}(t) = 0, \\
II_{K_2}(t) &= II_{K'_2}(t) = 0, \text{ and} \\
III_{K_2}(t) &= III_{K'_2}(t) = 0.
\end{align*}
\]

(iii) For the virtual knots $K_3 = 3.2$ and $K'_3 = 4.33$, it holds that
\[ II_{K_3}(t) = -2t + 4 - 2t^{-1} \text{ and } II_{K'_3}(t) = t^2 - 6t + 10 - 6t^{-1} + t^{-2}. \]

On the other hand, we have
\[
\begin{align*}
W_{K_3}(t) &= W_{K'_3}(t) = -t + 2 - t^{-1}, \\
I_{K_3}(t) &= I_{K'_3}(t) = -t + 2 - t^{-1}, \text{ and} \\
III_{K_3}(t) &= III_{K'_3}(t) \equiv t - 2 + t^{-1} \pmod{2t - 4 + 2t^{-1}}.
\end{align*}
\]

(iv) For the virtual knots $K_4 = O$ and $K'_4 = 4.13$, it holds that
\[ III_{K_4}(t) = 0 \text{ and } III_{K'_4}(t) = 2t - 4 + 2t^{-1}. \]

On the other hand, we have
\[
\begin{align*}
W_{K_4}(t) &= W_{K'_4}(t) = 0, \\
I_{K_4}(t) &= I_{K'_4}(t) = 0, \text{ and} \\
II_{K_4}(t) &= II_{K'_4}(t) = 0.
\end{align*}
\]
The Gauss diagrams of the knots are illustrated in Figure 16.

\[ \text{Figure 16} \]

Remark 3.7. For a virtual knot $K$, Silver and Williams [27] define a sequence of Alexander polynomials $\Delta_i(K)$ as an extension of the classical Alexander polynomial. The zero-th Alexander polynomial $\Delta_0(K)(u,v)$ is also defined by Sawollek [26]. Mellor [20] proves that the writhe polynomial is obtained from $\Delta_0(K)(u,v) = (1 - uv)\tilde{\Delta}_0(K)(u,v)$ by the equation

$$W_K(t) = -\tilde{\Delta}_0(K)(t, t^{-1}).$$

It is natural to ask whether the intersection polynomials are also obtained from $\Delta_i(K)$. However this does not hold generally; in fact, for the virtual knot $K = 4.8$, we have $\Delta_0(K) = 0$ and $\Delta_i(K) = 1$ ($i \geq 1$) which are coincident with those of the trivial knot. On the other hand, it holds that the intersection polynomials $I_K(t)$, $II_K(t)$, and $III_K(t)$ are all non-trivial.

4. FUNDAMENTAL PROPERTIES

The writhe polynomial is characterized by the following property.

Theorem 4.1 ([25]). Any virtual knot $K$ satisfies $W_K(1) = W'_K(1) = 0$. Conversely, if a Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ satisfies $f(1) = f'(1) = 0$, then there is a virtual knot $K$ with $f(t) = W_K(t)$.

We remark that the equation $W'_K(1) = 0$ is equivalent to

$$\sum_{i=1}^n \varepsilon_i(\gamma_i \cdot \tau_i) = \sum_{i=1}^n \varepsilon_i(\gamma_i \cdot \gamma_D) = 0$$

by definition.

The first intersection polynomial $I_K(t)$ satisfies the same property as above, which characterizes a Laurent polynomial to be some first intersection polynomial. The characterization will be proved in Section 8.

Theorem 4.2. Any virtual knot $K$ satisfies $I_K(1) = I'_K(1) = 0$. 
Proof. We have $I_K(1) = 0$ by definition. Since $W_K'(1) = 0$ holds by Theorem 4.1 it holds that

$$I_K(1) = f_{01}'(D; 1) - \omega_D W_K'(1) = f_{01}'(D; 1)$$

$$= \sum_{1 \leq i, j \leq n} \epsilon_i \epsilon_j (\gamma_i \cdot \gamma_j) = \left( \sum_{i=1}^{n} \epsilon_i \gamma_i \right) \cdot \left( \sum_{j=1}^{n} \epsilon_j \gamma_j \right)$$

$$= \left( \sum_{i=1}^{n} \epsilon_i \gamma_i \right) \cdot \left( \omega_D \gamma_D - \sum_{j=1}^{n} \epsilon_j \gamma_j \right)$$

$$= \omega_D \sum_{i=1}^{n} \epsilon_i (\gamma_i \cdot \gamma_D) - \left( \sum_{i=1}^{n} \epsilon_i \gamma_i \right) \cdot \left( \sum_{j=1}^{n} \epsilon_j \gamma_j \right)$$

$$= \omega_D W_K'(1) = 0.$$

□

A Laurent polynomial $f(t)$ is reciprocal if it satisfies $f(t^{-1}) = f(t)$. The second intersection polynomial $I_K(t)$ satisfies different properties from $I_K(t)$ as follows. These properties characterize a Laurent polynomial to be some second intersection polynomial as shown in Section 8.

**Theorem 4.3.** For any virtual knot $K$, $I_K(t)$ is reciprocal with

$$I_K(1) = 0 \text{ and } I_K''(1) \equiv 0 \pmod{4}.$$

To prove this theorem, we prepare the following two lemmas.

**Lemma 4.4.** Let $f(t) = \sum_{k \in \mathbb{Z}} a_k t^k$ be a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$.

(i) If $f'(1) = 0$, then $f''(1) \equiv \sum_{k \text{ odd}} a_k \pmod{4}$.

(ii) If $f(t)$ is reciprocal, then $f''(1) \equiv \sum_{k \text{ odd}} a_k \pmod{4}$.

Proof. (i) It holds that

$$f'(t) = \sum_{k \in \mathbb{Z}} k a_k t^{k-1} \text{ and } f''(t) = \sum_{k \in \mathbb{Z}} k(k-1) a_k t^{k-2}.$$

Then we have

$$f''(1) = \sum_{k \in \mathbb{Z}} k(k-1) a_k = \sum_{k \in \mathbb{Z}} k^2 a_k - f'(1) = \sum_{k \in \mathbb{Z}} k^2 a_k \equiv \sum_{k \text{ odd}} a_k \pmod{4}.$$

(ii) We may take $f(t) = \sum_{k \geq 1} a_k (t^k + t^{-k}) + a_0$. Since $f'(t) = \sum_{k \geq 1} a_k (t^k - t^{-k})$ holds, we have $f'(1) = 0$. By (i), we have the conclusion. □

**Lemma 4.5.** For a Laurent polynomial $f(t)$, the following are equivalent.

(i) $f(t)$ is reciprocal, $f(1) = 0$, and $f''(1) \equiv 0 \pmod{4}$.

(ii) $f(t) = \sum_{k \geq 1} a_k (t^k + t^{-k} - 2)$ for some $a_k \in \mathbb{Z}$ ($k \geq 1$) with $\sum_{k \text{ odd} \geq 1} a_k \equiv 0 \pmod{2}$.

(iii) There is a Laurent polynomial $g(t) \in \mathbb{Z}[t, t^{-1}]$ such that $g(1) = g'(1) = 0$ and $f(t) = g(t) + g(t^{-1})$. 

Proof. \( (i) \Rightarrow (ii) \). Since \( f(t) \) is reciprocal, we may take \( f(t) = \sum a_k(t^k + t^{-k}) + a_0 \). Since \( f(1) = 0 \), we have \( a_0 = -2\sum_{k \geq 1} a_k \) to obtain \( f(t) = \sum_{k \geq 1} a_k(t^k + t^{-k} - 2) \). Furthermore, the sum of the coefficients of odd terms of \( f(t) \) is equal to \( 2\sum_{k \text{odd} \geq 1} a_k \), it follows by Lemma 4.4 that
\[
2 \sum_{k \text{odd} \geq 1} a_k = f''(1) \equiv 0 \pmod{4}.
\]

\( (ii) \Rightarrow (iii) \). We have
\[
f(t) = \sum_{k \geq 1} a_k(t^k + t^{-k} - 2)
= \sum_{k \geq 2} a_k(t^k - kt + k - 1) + \sum_{k \geq 2} a_k(t^{-k} - kt^{-1} + k - 1)
+ \sum_{k \geq 1} ka_k(t + t^{-1} - 2).
\]

By assumption, we may put \( \sum_{k \geq 1} ka_k = 2m \) for some \( m \in \mathbb{Z} \). Consider the Laurent polynomial
\[
g(t) = \sum_{k \geq 1} a_k(t^k - kt + k - 1) + m(t + t^{-1} - 2).
\]

Then it satisfies that \( g(1) = g'(1) = 0 \) and \( f(t) = g(t) + g(t^{-1}) \).

\( (iii) \Rightarrow (i) \). Since \( g(1) = g'(1) = 0 \), we can take \( g(t) = (t - 1)^2h(t) \) for some \( h(t) \in \mathbb{Z}[t, t^{-1}] \). Then the reciprocal polynomial \( f(t) = g(t) + g(t^{-1}) \) satisfies
\[
f(1) = 2g(1) = 0 \quad \text{and} \quad f''(1) = 2g''(1) = 4h(1) \equiv 0 \pmod{4}.
\]

\[\square\]

Proof of Theorem 4.3. Since \( f_{00}(D; t) \), \( f_{11}(D; t) \), and \( W_K(t) \) are reciprocal, so is \( II_K(t) \). We have \( II_K(1) = 0 \) by definition.

We will prove \( II_K'(1) \equiv 0 \pmod{4} \). Since \( W_K(t) = W_K(t) + W_K(t^{-1}) \) with \( W_K(1) = W'_K(1) = 0 \), we have \( W'_K(1) \equiv 0 \pmod{4} \) by Lemma 4.5. Let \( S \) be the sum of the coefficients of odd terms of \( f_{00}(D; t) + f_{11}(D; t) \). Since \( f_{00}(D; t) + f_{11}(D; t) \) is reciprocal, it is sufficient to prove that \( S \equiv 0 \pmod{4} \) by Lemma 4.4.

By definition, we have
\[
S = \sum_{\gamma_i, \gamma_j \text{odd}} \varepsilon_i \varepsilon_j + \sum_{\tau_i, \tau_j \text{odd}} \varepsilon_i \varepsilon_j = 2 \left( \sum_{\gamma_i, \gamma_j \text{odd}, i < j} \varepsilon_i \varepsilon_j + \sum_{\tau_i, \tau_j \text{odd}, i < j} \varepsilon_i \varepsilon_j \right)
\equiv 2 \left( \sum_{1 \leq i < j \leq n} \gamma_i \cdot \gamma_j + \sum_{1 \leq i < j \leq n} \tau_i \cdot \tau_j \right) \pmod{4}.
\]
On the other hand, we have
\[
\sum_{1 \leq i < j \leq n} \gamma_i \cdot \gamma_j + \sum_{1 \leq i < j \leq n} \overline{\gamma}_i \cdot \overline{\gamma}_j = \sum_{1 \leq i < j \leq n} \left( \gamma_i \cdot \gamma_j + (\gamma_D - \gamma_i) \cdot (\gamma_D - \gamma_j) \right)
\]
\[
= \sum_{1 \leq i < j \leq n} \left( 2\gamma_i \cdot \gamma_j - \gamma_i \cdot \gamma_D - \gamma_D \cdot \gamma_j \right)
\]
\[
= \sum_{1 \leq i < j \leq n} \left( \gamma_i \cdot \gamma_D + \gamma_j \cdot \gamma_D \right) \pmod{2}
\]
\[
= \sum_{i=1}^{n} (n-i)(\gamma_i \cdot \gamma_D) + \sum_{j=1}^{n} (j-1)(\gamma_j \cdot \gamma_D)
\]
\[
= (n-1) \sum_{i=1}^{n} \gamma_i \cdot \gamma_D
\]
\[
\equiv (n-1) \sum_{i=1}^{n} \varepsilon_i (\gamma_i \cdot \gamma_D) \pmod{2}
\]
\[
= (n-1) W'_K(1) = 0.
\]
Therefore we have \( S \equiv 0 \pmod{4} \). \( \square \)

As seen in the proof of Theorem 4.3, we have \( W''_K(1) \equiv 0 \pmod{4} \). Therefore if two Laurent polynomials \( f(t) \) and \( g(t) \) satisfy \( f(t) \equiv g(t) \pmod{W_K(t)} \), then it holds that \( f''(1) \equiv g''(1) \pmod{4} \), which induces the well-definedness of \( III'_K(1) \pmod{4} \). Then the third intersection polynomial \( III_K(t) \) satisfies the following properties. The characterization of \( III_K(t) \) will be given in Section 8.

**Theorem 4.6.** For any virtual knot \( K \), \( III_K(t) \) is reciprocal with

\[
III_K(1) = 0 \quad \text{and} \quad III'_K(1) \equiv W''_K(1) \pmod{4}.
\]

To prove this theorem, we prepare the following two lemmas. An upper (or lower) forbidden move changes the position of consecutive over-crossings (or under-crossings), respectively, which is known as an unknotting operation \([11,23]\). Assume that a Gauss diagram \( G' \) is obtained from \( G \) by an upper forbidden move involving a pair of chords \( c_1 \) and \( c_2 \) of \( G \) as shown in Figure 17.

![Figure 17](image)

For \( 1 \leq i \leq n \), let \( c'_i \) be the chord of \( G' \) corresponding to \( c_i \), \( \varepsilon'_i \) the sign of \( c'_i \), and \( \gamma'_i \) the cycle at \( c'_i \). Let \( x_1 \) and \( x_2 \) be the terminal endpoints of \( c_1 \) and \( c_2 \),
respectively. We classify the chords $c_3, \ldots, c_n$ of $G$ into four sets such that
\[
\begin{align*}
P &= \{c_i \mid \text{neither } x_1 \text{ nor } x_2 \text{ lies on } \gamma_i\}, \\
Q &= \{c_i \mid \text{both } x_1 \text{ and } x_2 \text{ lie on } \gamma_i\}, \\
R &= \{c_i \mid x_1 \text{ does not lie on } \gamma_i \text{ and } x_2 \text{ lies on } \gamma_i\}, \\
S &= \{c_i \mid x_1 \text{ lies on } \gamma_i \text{ and } x_2 \text{ does not lie on } \gamma_i\}.
\end{align*}
\]

Lemma 4.7. (i) $\sum_{1 \leq i < j \leq n} (\gamma'_i \cdot \gamma'_j - \gamma_i \cdot \gamma_j) \equiv \#R + \#S + (\varepsilon_1 + \varepsilon_2)/2 \pmod{2}$.
(ii) $\#Q + \#S \equiv \gamma_1 \cdot \gamma_1 + \gamma_2 \cdot \gamma_2 \pmod{2}$.
(iii) $\sum_{\gamma_i \cdot \gamma_j \text{ odd}} \varepsilon_i \varepsilon_j = \sum_{\gamma_i \cdot \gamma_j \text{ odd}} \varepsilon_i \varepsilon_j \equiv \gamma_1 \cdot \gamma_1 + \gamma_2 \cdot \gamma_2 + \varepsilon_1 + \varepsilon_2 \pmod{4}$.

Proof. (i) Since it holds that $\gamma'_i \cdot \gamma'_j = \gamma_i \cdot \gamma_j$ (3 ≤ $i < j$ ≤ $n$), we have
\[
\sum_{1 \leq i < j \leq n} (\gamma'_i \cdot \gamma'_j - \gamma_i \cdot \gamma_j) = \sum_{3 \leq i < j \leq n} (\gamma'_i \cdot \gamma'_j - \gamma_i \cdot \gamma_j) + \sum_{3 \leq j \leq n} (\gamma'_1 \cdot \gamma'_j - \gamma_1 \cdot \gamma_j)
\]
\[
+ (\gamma'_1 \cdot \gamma'_2 - \gamma_1 \cdot \gamma_2).
\]

The first sum in the right hand side have the same parity as $\#Q + \#R$. In fact, if $\gamma_j \in P \cup S$, then we have $\gamma'_i \cdot \gamma'_j = \gamma_i \cdot \gamma_j$. On the other hand, if $\gamma_j \in Q \cup R$, it holds that $\gamma'_1 \cdot \gamma'_2 = \gamma_1 \cdot \gamma_2 \pm \varepsilon_2$.

Similarly, the second sum have the same parity as $\#Q + \#S$. In fact, if $\gamma_j \in P \cup R$, then we have $\gamma'_1 \cdot \gamma'_2 = \gamma_2 \cdot \gamma_j$. On the other hand, if $\gamma_j \in Q \cup S$, it holds that $\gamma'_1 \cdot \gamma'_2 = \gamma_2 \cdot \gamma_j \pm \varepsilon_2$.

Finally it holds that $\gamma'_1 \cdot \gamma'_2 = \gamma_1 \cdot \gamma_2 - (\varepsilon_1 + \varepsilon_2)/2$ by Lemma 3.2.

(ii) Let $m_1$, $m_2$, and $m_3$ be the numbers of endpoints of chords as shown in Figure 18. Since it holds that $\gamma_1 \cdot \gamma_1 \equiv m_1$, $\gamma_2 \cdot \gamma_2 \equiv m_2$, and $m_1 + m_2 + m_3 \equiv 0 \pmod{2}$, we see that $\gamma_1 \cdot \gamma_1 + \gamma_2 \cdot \gamma_2$ has the same parity as $m_3$.

On the other hand, a chord belongs to $R \cup S$ if and only if it is linked with exactly one of $c_1$ and $c_2$. Since the number of such chords has the same parity as $m_3$, we have the conclusion.

\[
\begin{align*}
\sum_{\gamma_i \cdot \gamma_j \text{ odd}} \varepsilon_i \varepsilon_j &= \sum_{\gamma_i \cdot \gamma_j \text{ odd}, i < j} \varepsilon_i \varepsilon_j \equiv \sum_{1 \leq i < j \leq n} \gamma_i \cdot \gamma_j \pmod{4}.
\end{align*}
\]

Then we have the conclusion by (i) and (ii). □

---

**Figure 18**
Lemma 4.8. \( \sum_{i} \gamma_i \tau_i \cdot \sum_{\text{odd}} \varepsilon_i' - \sum_{\gamma_i \text{ odd}} \varepsilon_i = (-1)^{\gamma_1 \tau_1 + (-1)^{\gamma_2 \tau_2 \cdot \varepsilon_2}}. \)

\textbf{Proof.} We see that \( \gamma_i \cdot \tau_i \) and \( \gamma_i' \cdot \tau_i \) have opposite parity for \( i = 1, 2 \) and coincide for \( 3 \leq i \leq n \). Since \( \varepsilon_i = \varepsilon_i' \) holds, we have

\[
\sum_{\gamma_i \text{ odd}} \varepsilon_i' - \sum_{\gamma_i \text{ odd}} \varepsilon_i = \begin{cases} 
\varepsilon_1 + \varepsilon_2 & \text{for } \gamma_1 \cdot \tau_1 \equiv \gamma_2 \cdot \tau_2 \equiv 0 \pmod{2}, \\
\varepsilon_1 - \varepsilon_2 & \text{for } \gamma_1 \cdot \tau_1 \equiv 0, \gamma_2 \cdot \tau_2 \equiv 1 \pmod{2}, \\
-\varepsilon_1 + \varepsilon_2 & \text{for } \gamma_1 \cdot \tau_1 \equiv 1, \gamma_2 \cdot \tau_2 \equiv 0 \pmod{2}, \\
-\varepsilon_1 - \varepsilon_2 & \text{for } \gamma_1 \cdot \tau_1 \equiv \gamma_2 \cdot \tau_2 \equiv 1 \pmod{2}.
\end{cases}
\]

\( \square \)

\textbf{Proof of Theorem 4.6.} Since \( f_{00}(D; t) \) and \( \overline{W}_K(t) \) are reciprocal, so is \( \overline{III}_K(t) \). We have \( \overline{III}_K(1) = 0 \) by \( f_{00}(D; 1) = \overline{W}_K(1) = 0 \).

Assume that a Gauss diagram \( G' \) is obtained from \( G \) by an upper forbidden move. Then it follows by Lemmas 4.7(iii) and 4.8 that

\[
\left( \sum_{\gamma_i \text{ odd}} \varepsilon_i' \epsilon_j' - \sum_{\gamma_i \text{ odd}} \varepsilon_i' \right) - \left( \sum_{\gamma_i \text{ odd}} \varepsilon_i \epsilon_j - \sum_{\gamma_i \text{ odd}} \varepsilon_i \right) = 2\gamma_1 \cdot \tau_1 + 2\gamma_2 \cdot \tau_2 + \varepsilon_1 + \varepsilon_2 - (-1)^{\gamma_1 \cdot \tau_1 \cdot \varepsilon_1} - (-1)^{\gamma_2 \cdot \tau_2 \cdot \varepsilon_2} = 0 \pmod{4}.
\]

In other words, \( \sum_{\gamma_i \text{ odd}} \varepsilon_i \epsilon_j - \sum_{\gamma_i \text{ odd}} \varepsilon_i \) (mod 4) is invariant under an upper forbidden move. The invariance under a lower forbidden move can be proved similarly. Since the forbidden move is an unknotting operation for a virtual knot, we obtain

\[
\sum_{\gamma_i \text{ odd}} \varepsilon_i \epsilon_j = \sum_{\gamma_i \text{ odd}} \varepsilon_i \pmod{4}.
\]

Since \( \overline{III}_K(t) \) is reciprocal and \( W''_K(1) = 0 \), this congruence is equivalent to

\[
\overline{III}_K''(1) = f_{00}(D; 1) \equiv W''_K(1) \pmod{4}
\]

by Lemma 4.4. \( \square \)

\textbf{Remark 4.9.} The \textit{odd writhe} \[13\] of a virtual knot \( K \) is the sum of the coefficients of odd terms of \( W_K(t) \), and denoted by \( J(K) \in \mathbb{Z} \). We have \( W_K''(1) = J(K) \pmod{4} \) by Lemma 4.4. It is known that \( J(K) \) is always even \[13\], which is also obtained from \( W_K''(1) = 0 \) immediately.

5. Symmetries and crossing numbers

For a diagram \( D \) on \( \Sigma \) of a virtual knot \( K \), let \( -D \) be the diagram by reversing the orientation of \( D \), \( \overline{D} \) the one by changing over/under-information at every crossing of \( D \), and \( D^* \) the one obtained by an orientation-reversing homeomorphism of \( \Sigma \). The virtual knots presented by \( -D \), \( \overline{D} \), and \( D^* \) are called the reverse, the vertical mirror image, and the horizontal mirror image of \( K \), and denoted by \( -K \), \( K \), and \( K^* \), respectively. The writhe polynomials of these knots are given as follows.

\textbf{Lemma 5.1} \[2,13,25\]. Any virtual knot \( K \) satisfies

\[
W_{-K}(t) = W_K(t^{-1}) \text{ and } W_{K^*}(t) = W_{K^*}(t) = -W_K(t^{-1}).
\]

Therefore we have

\[
\overline{W}_{-K}(t) = \overline{W}_K(t) \text{ and } \overline{W}_{K^*}(t) = \overline{W}_{K^*}(t) = -\overline{W}_K(t).
\]
The intersection polynomials of $-K$, $K^*$, and $K^*$ are given as follows.

**Lemma 5.2.** For a virtual knot $K$, we have the following.

(i) $I_{-K}(t) = I_{K^*}(t) = I_{K^*}(t) = I_K(t^{-1})$.

(ii) $II_{-K}(t) = II_{K^*}(t) = II_{K^*}(t) = II_K(t)$.

(iii) $III_{-K}(t) = III_{K^*}(t) = II_K(t) - III_K(t)$ and $III_{K^*}(t) = III_K(t)$.

**Proof.** $I_{-K}(t)$, $II_{-K}(t)$, and $III_{-K}(t)$. Let $c_i^*$ be the crossing of $-D$ corresponding to $c_i$, $\gamma_i^*$ the cycle at $c_i^*$ on $\Sigma$, and $\varepsilon_i^*$ the sign of $\gamma_i^*$ ($1 \leq i \leq n$). Then it holds that $\gamma_i^* = -\tau_i$ and $\varepsilon_i^* = \varepsilon_i$. By definition, we have

$$f_{01}(-D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i^* \varepsilon_j^*(t^{\gamma_i^* \gamma_j^*} - 1) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j(t^{\gamma_i \gamma_j} - 1)$$

$$f_{00}(-D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j(t^{\gamma_i \gamma_j} - 1) = f_{11}(D; t),$$

$$f_{11}(-D; t) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j(t^{\gamma_i \gamma_j} - 1) = f_{00}(D; t).$$

Since $\omega_{-D} = \omega_D$ holds, we have

$$I_{-K}(t) = f_{01}(-D; t) - \omega_D W_D(t)$$

$$f_{01}(D; t^{-1}) - \omega_D W_D(t^{-1}) = I_K(t^{-1}),$$

$$II_{-K}(t) = f_{00}(-D; t) + f_{11}(-D; t) - \omega_D(W_D(t) + W_D(t^{-1}))$$

$$f_{01}(D; t) + f_{00}(D; t) - \omega_D(W_D(t^{-1}) + W_D(t)) = II_K(t)$$

$$III_{-K}(t) = f_{00}(-D; t) = II_{K^*}(t) = II_{K^*}(t) = II_K(t) - III_K(t).$$

$II_K(t)$, $II_{K^*}(t)$, and $III_{K^*}(t)$. We use the notations in the proof of Lemma 2.2

By the lemma, we have $f_{00}(D^*; t) = f_{11}(D; t)$ and hence $f_{11}(D^*; t) = f_{00}(D; t)$.

Furthermore it holds that

$$f_{01}(D^*; t) = \sum_{1 \leq i, j \leq n} (-\varepsilon_i)(-\varepsilon_j)(t^{\gamma_i \gamma_j} - 1) = f_{01}(D; t^{-1}).$$

Since it holds that $\omega_{D^*} = -\omega_D$ and $W_{D^*}(t) = -W_D(t^{-1})$, we have

$$I_{K^*}(t) = f_{01}(D^*; t) - \omega_{D^*} W_{D^*}(t)$$

$$f_{01}(D; t^{-1}) - \omega_D W_D(t^{-1}) = I_K(t^{-1}),$$

$$II_{K^*}(t) = f_{00}(D^*; t) + f_{11}(D^*; t) - \omega_{D^*}(W_{D^*}(t) + W_{D^*}(t^{-1}))$$

$$f_{11}(D; t) + f_{00}(D; t) - \omega_D(W_D(t^{-1}) + W_D(t)) = II_K(t)$$

$$III_{K^*}(t) = f_{00}(D^*; t) = II_{K^*}(t) - f_{00}(D; t) = II_K(t) - III_K(t).$$

$II_K(t)$, $II_{K^*}(t)$, and $III_{K^*}(t)$. Let $c_i^*$ be the crossing of $D^*$ corresponding to $c_i$, $\gamma_i^*$ the cycle at $c_i^*$ on $\Sigma$, and $\varepsilon_i^*$ the sign of $\gamma_i^*$ ($1 \leq i \leq n$). Then it holds that

$$\gamma_i^* \cdot \gamma_j^* = -\gamma_i \cdot \gamma_j, \quad \gamma_i^* \cdot \gamma_j^* = -\gamma_i \cdot \gamma_j, \quad \gamma_i \cdot \gamma_j = -\gamma_i \cdot \gamma_j$$

and $\varepsilon_i^* = -\varepsilon_i$. Since $f_{00}(D; t)$ and $f_{11}(D; t)$ are reciprocal, we have

$$f_{01}(D^*; t) = f_{01}(D; t^{-1}), \quad f_{00}(D^*; t) = f_{00}(D; t), \quad f_{11}(D^*; t) = f_{11}(D; t).$$
Since it holds that $\omega_D = -\omega_D$ and $W_D(t) = -W_D(t^{-1})$, we have

\[
I_K(t) = f_{01}(D; t) - \omega_D W_D(t) = f_{01}(D; t^{-1}) - \omega_D W_D(t^{-1}) = I_K(t^{-1}),
\]

\[
II_K(t) = f_{00}(D; t) + f_{11}(D; t) - \omega_D (W_D(t) + W_D(t^{-1})) = f_{00}(D; t) + f_{11}(D; t) - \omega_D (W_D(t^{-1}) + W_D(t)) = II_K(t),
\]

\[
III_K(t) = f_{00}(D; t) = f_{00}(D; t) \equiv III_K(t).
\]

\[\square\]

**Proposition 5.3.** If a virtual knot $K$ satisfies

(i) $2III_K(t) \not\equiv II_K(t) \pmod{W_K(t)}$,

(ii) $W_K(t) \not\equiv W_K(t^{-1})$, and

(iii) $W_K(t) \not\equiv -W_K(t^{-1})$ or $I_K(t) \not\equiv I_K(t^{-1})$,

then the eight virtual knots

\[K, -K, K^\#, K^*, -K^#, -K^*, K^{\#*}, \text{ and } -K^{\#*}\]

are mutually distinct.

**Proof.** By Lemma 5.2(iii), the virtual knots $K, K^*, -K^#, \text{ and } -K^{\#*}$ have the same third intersection polynomial $III_K(t)$, and $-K, -K^*, K^#$, and $K^{\#*}$ have $II_K(t) = III_K(t)$. Therefore, by the condition (i), it holds that

\[
\{K, K^*, -K^#, -K^{\#*}\} \cap \{-K, -K^*, K^#, K^{\#*}\} = \emptyset.
\]

Furthermore, by Lemma 5.1, the first four virtual knots $K, K^*, -K^#, \text{ and } -K^{\#*}$ have the writhe polynomials $W_K(t), -W_K(t^{-1}), -W_K(t), \text{ and } W_K(t^{-1})$, respectively. Since $W_K(t) \not\equiv 0$ follows by the condition (ii), we have

\[
\{K, K^*\} \cap \{-K^#, -K^{\#*}\} = \emptyset.
\]

Finally, each of the pairs $K$ and $K^*$, and $-K^#$ and $-K^{\#*}$ can be distinguished by the condition (iii) and Lemma 5.2(i). We can prove that the latter four virtual knots $-K, -K^*, K^#$, and $K^{\#*}$ are mutually distinct similarly. \[\square\]

**Theorem 5.4.** There are infinitely many virtual knots $K$ such that

\[K, -K, K^\#, K^*, -K^#, -K^*, K^{\#*}, \text{ and } -K^{\#*}\]

are mutually distinct.
Proof. Let $K_n \ (n \geq 1)$ be the virtual knot presented by the Gauss diagram $G_n$ as shown in Figure 19. It holds that
\[
W_{K_n}(t) = t^{n+1} - (n+1)t + (n+1)t^{-1} - t^{-n-1},
\]
\[
I_{K_n}(t) = -t^{n+2} - nt^{n+1} + t - (2n - 1)(n+1)t^{-n-1} + 2 \sum_{i=1}^{n} (t^i + t^{-i}),
\]
\[
II_{K_n}(t) = -(t^{2n+2} + t^{-2n-2}) - (t^{n+1} + t^{-n-1}) - (2n + 1)(t^n + t^{-n}) - 2n(2n + 3)(t + t^{-1}) + 2(n^2 + n + 2)
\]
\[+ 4 \sum_{i=1}^{n+1} (t^i + t^{-i}), \text{ and}
\]
\[
III_{K_n}(t) = -(t^{n+1} + t^{-n-1}) - (n+1)(t + t^{-1}) + 2 \sum_{i=1}^{n+1} (t^i + t^{-i}) - 2n,
\]
where we have $\overline{W}_{K_n}(t) = 0$. Since these invariants of $K_n$ satisfy the conditions (i), (ii), and (iii) $I_{K_n}(t) \neq I_{K_n}(t^{-1})$ in Proposition 5.3, the eight kinds of virtual knots associated with $K_n$ are mutually distinct. Furthermore $K_n \neq K_m \ (n \neq m)$ holds by $\text{deg} W_{K_n}(t) = n + 1$. We remark that $K_n$ satisfies $W_{K_n}(t) = -W_{K_n}(t^{-1})$. □

\[\text{Figure 19}\]

Example 5.5. We can construct an infinite family of virtual knots $K$ satisfying the conditions (i), (ii), and (iii) $W_K(t) \neq W_K(t^{-1})$ in Proposition 5.3.

Let $K' \ (n \geq 3)$ be the virtual knot presented by the Gauss diagram $G_n'$ as shown in Figure 20. We have
\[
W_{K'_n}(t) = -t^{n-1} + nt - n + t^{-1},
\]
\[
I_{K'_n}(t) = -t^n + (n-2)t^{n-1} - 2 \sum_{i=1}^{n-2} t^i + (2n-1) - nt^{-1},
\]
\[
II_{K'_n}(t) = -(t^n + t^{-n}) + (n-1)(t^{n-1} + t^{-n+1})
\]
\[\quad - 2 \sum_{i=1}^{n-2} (t^i + t^{-i}) - (n^2 - n + 1)(t + t^{-1}) + 2n^2 - 2, \text{ and}
\]
\[
III_{K'_n}(t) = - \sum_{i=1}^{n-2} (t^i + t^{-i}) - (t + t^{-1}) + 2n - 2 \pmod{W_{K'_n}(t)},
\]
where $W_{K_n}(t) = -(t^{-n}+t+n^{-1})+(n+1)(t+t^{-1})-2n$. Since these invariants of $K'_n$ satisfy the conditions (i), (ii), and (iii) $W_{K_n}(t) \neq W_{K_n}(t^{-1})$ in Proposition 5.3, the eight kinds of virtual knots associated with $K'_n$ are mutually distinct. Furthermore $K'_n \neq K'_m$ $(n \neq m)$ holds by $\deg W_{K_n}(t) = n-1$. We remark that $K'_n$ satisfies (iii) $I_{K'_n}(t) \neq I_{K'_n}(t^{-1})$ in Proposition 5.3.

For a Laurent polynomial $f(t)$, let $\deg f(t)$ denote the maximal degree of $f(t)$. The writhe polynomial $W_K(t)$ gives a lower bound of the crossing number $c(K)$ as follows.

**Lemma 5.6 ([25]).** Any non-trivial virtual knot $K$ satisfies $c(K) \geq \deg W_K(t) + 1$.

We remark that the minimal degree of $W_K(t)$ also gives a lower bound of $c(K) = c(-K)$ by the equation to $W_K(t^{-1}) = W_{-K}(t)$. The span of $W_K(t)$ is the difference of the maximal and minimal degrees of $W_K(t)$, and denoted by $\text{span} W_K(t)$. Then Lemma 5.6 induces a weaker inequation

$$c(K) \geq \frac{1}{2} \text{span} W_K(t) + 1$$

immediately. The intersection polynomials also gives lower bounds of $c(K)$ as follows. Here, $\deg \text{III}_K(t)$ denotes the maximal number of $\deg f(t)$ for all $f(t)$ with $f(t) \equiv \text{III}_K(t) \mod W_K(t)$.

**Proposition 5.7.** Let $K$ be a non-trivial virtual knot.

(i) $c(K) \geq \deg I_K(t) + 1$.
(ii) $c(K) \geq \deg II_K(t) + 1$.
(iii) $c(K) \geq \deg III_K(t) + 1$.

**Proof.** Assume that a diagram $D$ of $K$ satisfies $c(D) = c(K)$. Since $K$ is non-trivial, it holds that $c(D) \geq 2$. For $(\alpha, \beta) = (\gamma_i, \gamma_j), (\gamma_i, \gamma_j)$, and $(\gamma_i, \gamma_j)$ with $i \neq j$, the intersection number $\alpha \cdot \beta$ is equal to $S(\alpha, \beta) - 1$, $S(\alpha, \beta)$, or $S(\alpha, \beta) + 1$ by Lemma 3.2 so that we obtain $\alpha \cdot \beta \leq S(\alpha, \beta) + 1$.

Since there are $c(D) - 2$ chords other than $c_i$ and $c_j$ in the Gauss diagram of $D$, we have $S(\alpha, \beta) \leq c(D) - 2$. Therefore it holds that

$$\deg f_{pq}(D) \leq c(D) - 1 = c(K) - 1$$

for $(p, q) = (0, 1), (0, 0)$, and $(1, 1)$. Since $\deg W_K(t) \leq c(K) - 1$, we have the conclusion.

\[\square\]
As well as a diagram on $\Sigma$ or a Gauss diagram, a virtual knot is also presented by a **virtual diagram in** $\mathbb{R}^2$. It is an immersed circle in $\mathbb{R}^2$ with real and virtual crossings $[12]$. Here, the real crossings correspond to the crossings on $\Sigma$, and the virtual crossings are surrounded by small circles. The **virtual crossing number** of a virtual knot $K$ is the minimal number of virtual crossings for all virtual diagrams of $K$, and denoted by $vc(K)$.

The intersection polynomials are also calculated from a virtual diagram. For example, we consider the virtual diagram with three real crossings $c_1, c_2,$ and $c_3$ and two virtual crossings as shown in the leftmost of Figure 21 which presents the virtual knot $K = 3.4$. To calculate $\gamma_1 \cdot \overline{\gamma}_2$, we draw the curves $\gamma_1$ and $\overline{\gamma}_2$ equipped with virtual crossings, and then take the sum of signs of two intersections with ignoring virtual crossings to obtain $\gamma_1 \cdot \overline{\gamma}_2 = 2$. See the second from the left in the figure. Similarly we obtain $\gamma_1 \cdot \gamma_2 = 0$ and $\overline{\gamma}_1 \cdot \overline{\gamma}_2 = -1$ as shown in the third and fourth from the left.

The writhe polynomial $W_K(t)$ gives a lower bound of the virtual crossing number $vc(K)$ as follows.

**Lemma 5.8 ([25]).** Any non-trivial virtual knot $K$ satisfies $vc(K) \geq \deg W_K(t)$.

We remark that Lemma 5.8 induces a weaker inequation

$$vc(K) \geq \frac{1}{2} \text{span} W_K(t)$$

immediately, which is proved in [20]. The intersection polynomials also gives lower bounds of $vc(K)$ as follows.

**Proposition 5.9.** Let $K$ be a virtual knot.

(i) $vc(K) \geq \deg I_K(t)$.

(ii) $vc(K) \geq \deg II_K(t)$.

(iii) $vc(K) \geq \deg III_K(t)$.

**Proof.** Let $D$ be a virtual diagram of $K$ in $\mathbb{R}^2$, and $\alpha$ and $\beta$ cycles on $D$ with corners at (possibly the same) real crossings of $D$. By a slight perturbation of $\beta$ if necessary, we may assume that $\alpha$ and $\beta$ intersect in a finite number of double points near real and virtual crossings of $D$ as explained as above. By Lemma 5.8 it is sufficient to prove that if the intersection number restricted to the real crossings between $\alpha$ and $\beta$ in $\mathbb{R}^2$ is equal to $n$, then the number of virtual crossings of $D$ is greater than or equal to $|n|$. 

![Figure 21](image-url)
Since the total intersection number between $\alpha$ and $\beta$ in $\mathbb{R}^2$ is equal to zero, the intersection number restricted to the virtual crossings between $\alpha$ and $\beta$ is equal to $-n$.

Let $v$ be a virtual crossing of $D$ where two short paths $\lambda$ and $\lambda' \subset D$ intersect. If $\alpha$ and $\beta$ intersect in virtual crossings near $v$, then there are four cases as follows.

(i) $\alpha \supset \lambda$, $\alpha \not\supset \lambda'$, and $\beta \not\supset \lambda$, $\beta \supset \lambda'$.
(ii) $\alpha \supset \lambda$, $\alpha \supset \lambda'$, and $\beta \not\supset \lambda$, $\beta \supset \lambda'$.
(iii) $\alpha \supset \lambda$, $\alpha \not\supset \lambda'$, and $\beta \supset \lambda$, $\beta \supset \lambda'$.
(iv) $\alpha \supset \lambda$, $\beta \supset \lambda'$ and $\beta \supset \lambda$, $\lambda'$.

See Figure 22.

In the case (iv), the pair of virtual crossings between $\alpha$ and $\beta$ does not contribute to the intersection number; in fact, they have opposite signs. On the other hand, each case of (i)–(iii) contains a single virtual crossing. It follows that the number of virtual crossings of $D$ in the cases (i)–(iii) is greater than or equal to $|n|$.

**Example 5.10.** (i) Let $K_n$ ($n \geq 1$) be the virtual knot presented by the Gauss diagram and the virtual diagram as shown in Figure 23. We see that $c(K_n) = n + 3$ and $vc(K_n) = n + 2$ can be detected by $I_{K_n}(t)$ but not by $W_{K_n}(t)$, $II_{K_n}(t)$, and $III_{K_n}(t)$. In fact, we have
\[
W_{K_n}(t) = t^{n+1} - t^2 - nt + n + 1 - t^{-1},
\[
I_{K_n}(t) = -t^{n+2} + (n + 1)t^{n+1} - t^n + (n + 1)t - 2 \sum_{i=1}^{n} t^i,
\[
II_{K_n}(t) = n(t^{n+1} + t^{-n-1}) - (n^2 + n + 2)(t + t^{-1}) + 2(n^2 + 2n + 2)
\[
\quad - 2 \sum_{i=1}^{n} (t^i + t^{-i}),
\]
and
\[
III_{K_n}(t) \equiv n(t^2 + t^{-2}) - 2(t + t^{-1}) + 4 - \sum_{i=1}^{n} (t^i + t^{-i}) \pmod{W_{K_n}(t)},
\]
where $W_{K_n}(t) = (t^{n+1} + t^{-n-1}) - (t^2 + t^{-2}) - (n + 1)(t + t^{-1}) + 2n + 2$.

(ii) Let $K'_n$ ($n \geq 1$) be the virtual knot presented by the Gauss diagram and the virtual diagram as shown in Figure 24. We see that $c(K'_n) = n + 3$ and $vc(K'_n) = n + 2$ can be detected by $II_{K'_n}(t)$ but not by $W_{K'_n}(t)$, $I_{K'_n}(t)$, and $III_{K'_n}(t)$.
In fact, we have
\[
W_{K_n'}(t) = t^{n+1} - nt + n - 1 - t^{-1} + t^{-2},
\]
\[
I_{K_n'}(t) = (n - 1)t^{n+1} + (2n + 1)t - n - 1 - (n - 2)t^{-1} + (n - 1)t^{-2} - 2 \sum_{i=1}^{n} t^i,
\]
\[
II_{K_n'}(t) = (t^{n+2} + t^{-n-2}) + (n - 2)(t^{n+1} + t^{-n-1})
+ (n - 1)(t^2 + t^{-2}) - n(n + 1)(t + t^{-1}) + 2(n^2 + n + 2)
- \sum_{i=1}^{n} (t^i + t^{-i}) - \sum_{i=1}^{n} (t^{i-1} + t^{-i+1}), \text{ and}
\]
\[
III_{K_n'}(t) = -n(t + t^{-1}) + 4n - \sum_{i=1}^{n} (t^{i-1} + t^{-i+1}) \pmod{W_{K_n'}(t)},
\]
where \( \overline{W}_{K_n'}(t) = (t^{n+1} + t^{-n-1}) + (t^2 + t^{-2}) - (n - 1)(t + t^{-1}) + 2n - 2. \)

6. DOTTED VIRTUAL KNOTS

Let \((D, p)\) be a pair of a diagram \(D \subset \Sigma\) and a point \(p\) on \(D\) except the crossings of \(D\). We consider an equivalence relation among all \((D, p)\)'s generated by stabilizations, destabilizations, and Reidemeister moves away from \(p\). Such an equivalence class is called a dotted virtual knot. A dotted virtual knot \(T\) can be regarded as a long virtual knot or a 1-string virtual tangle by cutting a diagram open at \(p\). In
The Laurent polynomials

Lemma 6.1. The Laurent polynomials $W^{0}_{D,p}(t)$ and $W^{1}_{D,p}(t) \in \mathbb{Z}[t, t^{-1}]$ do not depend on a particular choice of a diagram of a dotted virtual knot $T$.

Proof. Since a (de)stabilization does not change the intersection number $\gamma_i \cdot \tau_i$, $W^{0}_{D,p}(t)$ and $W^{1}_{D,p}(t)$ are invariant under a (de)stabilization.

Assume that $(D', p')$ is obtained from $(D, p)$ by a Reidemeister move on $\Sigma$. If a crossing $c_1$ of $D$ is removed by a first Reidemeiste move, then we have $\gamma_1 = 0$ or $\tau_1 = 0$ and hence $\varepsilon_1(t^{\gamma_1} \tau_1 - 1) = 0$.

If a pair of crossings $c_1$ and $c_2$ of $D$ is removed by a second Reidemeister move, then we have $\gamma_1 = \gamma_2$ and $\varepsilon_1 = -\varepsilon_2$ as shown in the proof of Lemma 2.4. Furthermore the point $p$ lies on $\gamma_1$ if and only if $p'$ lies on $\gamma_2$. Therefore the sum $\varepsilon_1(t^{\gamma_1} \tau_1 - 1) + \varepsilon_2(t^{\gamma_2} \tau_2 - 1) = 0$ appears in one of $W^{0}_{D,p}(t)$ or $W^{1}_{D,p}(t)$.

Assume that a triplet of crossings $c_1$, $c_2$, and $c_3$ of $D$ is involved in a third Reidemeister move. Let $c'_1$, $c'_2$, and $c'_3$ be the corresponding crossings of $D'$. Then we have $\gamma_i = \gamma_i'$ and $\varepsilon_i = \varepsilon_i'$ ($i = 1, 2, 3$) as shown in the proof of Lemma 2.4. Furthermore for each $i$, the point $p$ lies on $\gamma_i$ if and only if $p'$ lies on $\gamma_i'$.

In all cases, $W^{0}_{D,p}(t)$ and $W^{1}_{D,p}(t)$ are invariant under a Reidemeister move. □

Definition 6.2. The Laurent polynomials $W^{0}_{D,p}(t)$ and $W^{1}_{D,p}(t)$ are called the writhe polynomials of a dotted virtual knot $T$, and denoted by $W^{0}_{T}(t)$ and $W^{1}_{T}(t)$, respectively.

Example 6.3. A diagram of a dotted virtual knot has a Gauss diagram with a point on the circle naturally. Let $(G, p)$ and $(G, p')$ be Gauss diagrams with a point as shown in Figure 25 and $T$ and $T'$ the dotted virtual knots presented by $(G, p)$ and $(G, p')$, respectively.
For the chords \( c_i \ (i = 1, 2, 3, 4) \) of \((G, p)\), it holds that \( M_0(D) = \{2, 3, 4\} \) and \( M_1(D) = \{1\} \). Since \( \gamma_1 \cdot \tilde{\gamma}_1 = \gamma_2 \cdot \tilde{\gamma}_2 = 2 \) and \( \gamma_3 \cdot \tilde{\gamma}_3 = \gamma_4 \cdot \tilde{\gamma}_4 = 0 \), we have \( W^0_T(t) = t^2 - 1 \) and \( W^1_T(t) = -t^2 + 1 \).

On the other hand, since \((G, p')\) is related to the one with no chord by first and second Reidemeister moves, \( T' \) is trivial and satisfies \( W^0_{T'}(t) = W^1_{T'}(t) = 0 \). Therefore \( T \) and \( T' \) are different dotted virtual knots. \( \square \)

By definition, it holds that \( W^0_T(t) + W^1_T(t) = \hat{W}_T(t) \) and \( W^0_T(1) = W^1_T(1) = 0 \) for any dotted virtual knot \( T \). Conversely we have the following.

**Proposition 6.4.** Let \( K \) be a virtual knot, and \( f(t) \) a Laurent polynomial with \( f(1) = 0 \). Then there is a dotted virtual knot \( T \) such that

1. \( \hat{\hat{T}} = K \),
2. \( W^0_T(t) = f(t) \), and
3. \( W^1_T(t) = W_K(t) - f(t) \).

Therefore, for any virtual knot \( K \), there are infinitely many dotted virtual knot \( T \) with \( \hat{\hat{T}} = K \).

**Proof.** We take a dotted virtual knot \( T_0 \) with \( \hat{\hat{T_0}} = K \) arbitrarily. Since it holds that \( f(1) - W^0_{T_0}(1) = 0 \), there are integers \( a_n \ (n \neq 0) \) such that

\[
  f(t) = W^0_{T_0}(t) + \sum_{n \neq 0} a_n(t^n - 1)
\]

Therefore it is sufficient to prove that for any \( n \neq 0 \) and \( \varepsilon \in \{\pm\} \), there is a dotted virtual knot \( T_1 \) with \( \hat{\hat{T_1}} = K \) and \( W^0_{T_1}(t) = W^0_{T_0}(t) + \varepsilon(t^n - 1) \).

We take a Gauss diagram \((G_0, p_0)\) of \( T_0 \). We consider the Gauss diagram \((G_1, p_1)\) as shown in the middle of Figure 26 for \( n > 0 \) and in the right for \( n < 0 \). Let \( T_1 \) be the dotted virtual knot presented by \((G_1, p_1)\). Since \( G_1 \) without \( p_1 \) is related to \( G_0 \) by first and second Reidemeister moves, we have \( \hat{T_1} = \hat{T_0} = K \). Furthermore we see that \( W^0_{T_1}(t) = W^0_{T_0}(t) + \varepsilon(t^n - 1) \). \( \square \)

![Figure 26](image-url)
7. Connected sums of virtual knots

Let $T$ and $T'$ be dotted virtual knots, and $(D, p)$ and $(D', p')$ diagrams of $T$ and $T'$ on $\Sigma$ and $\Sigma'$, respectively. We remove a small open disk neighborhood $\Delta$ of $p$ in $\Sigma$ (or $\Delta'$ of $p'$ in $\Sigma'$) and identify their boundaries to obtain a diagram $(D'', p'')$ on $\Sigma'' = (\Sigma \setminus \Delta) \cup (\Sigma' \setminus \Delta')$, where the point $p''$ is taken such that $D'$ follows after $D$ with respect to the orientation. See Figure 27. We denote by $D$ on $\Sigma$ and $\Sigma'$ by $D_T$ in $\Sigma$ (or $\Delta_T$ in $\Sigma'$).

For any dotted virtual knots $T$ and $T'$ (cf. [2, 5, 25]). On the other hand, the first intersection polynomial is not additive in general as follows.

**Theorem 7.1.** For dotted virtual knots $T$ and $T'$, it holds that

$$I_{T+T'}(t) = I_T(t) + I_{T'}(t) + W_T^0(t)W_{T'}^0(t) + W_T^1(t)W_{T'}^0(t).$$

To prove this theorem, we prepare several lemmas. Let $c_1, \ldots, c_n$ be the crossings of $(D, p)$ and $\gamma_i (1 \leq i \leq n)$ the cycle at $c_i$ on $\Sigma$. Similarly, let $c_1', \ldots, c_m'$ be the crossings of $(D', p')$, and $\gamma_j' (1 \leq j \leq m)$ the cycle at $c_j'$ on $\Sigma'$. The crossings of $(D'', p'')$ are identified with $c_1, \ldots, c_n$ and $c_1', \ldots, c_m'$. Let $\Gamma_i (1 \leq i \leq n)$ be the cycle at $c_i$ on $\Sigma''$, and $\Gamma_j' (1 \leq j \leq m)$ the one at $c_j'$ on $\Sigma''$.

**Lemma 7.2.** (i) $\Gamma_i \cdot \Gamma_j' = \gamma_i \cdot \gamma_j' (1 \leq i, j \leq n)$.

(ii) $\Gamma_i' \cdot \Gamma_j' = \gamma_i' \cdot \gamma_j' (1 \leq i, j \leq m)$.

**Proof.** We prove (i) only. The equation (ii) is similarly proved. By definition, it holds that

$$\Gamma_i = \begin{cases} \gamma_i & \text{for } i \in M_0(D), \\ \gamma_i + \gamma_{D'} & \text{for } i \in M_1(D), \end{cases} \quad \Gamma_j' = \begin{cases} \gamma_j' & \text{for } j \in M_0(D), \\ \gamma_j' + \gamma_D & \text{for } j \in M_1(D), \end{cases}$$

Since $\gamma_i \cdot \gamma_{D'} = \gamma_{D'} \cdot \gamma_j = \gamma_D \cdot \gamma_{D'} = 0$ hold, we have the equation (i). $\square$

The intersection number $\Gamma_i \cdot \Gamma_j' (1 \leq i \leq n, 1 \leq j \leq m)$ is given as follows.

**Lemma 7.3.** For any $1 \leq i \leq n$ and $1 \leq j \leq m$, we have the following.

(i) If $i \in M_0(D)$ and $j \in M_0(D')$, then $\Gamma_i \cdot \Gamma_j' = \gamma_i \cdot \gamma_D$.

(ii) If $i \in M_0(D)$ and $j \in M_1(D')$, then $\Gamma_i \cdot \Gamma_j' = 0$.

(iii) If $i \in M_1(D)$ and $j \in M_0(D')$, then $\Gamma_i \cdot \Gamma_j' = \gamma_i \cdot \gamma_D + \gamma_j' \cdot \gamma_{D'}$.

(iv) If $i \in M_1(D)$ and $j \in M_1(D')$, then $\Gamma_i \cdot \Gamma_j' = \gamma_j' \cdot \gamma_{D'}$. 

\[\]
Proof. By definition, it holds that

\[
\Gamma_i = \begin{cases} 
\gamma_i & \text{for } i \in M_0(D), \\
\gamma_i + \gamma_{D'} & \text{for } i \in M_1(D),
\end{cases}
\]
and

\[
\Gamma_j = \begin{cases} 
\gamma_j' + \gamma_D & \text{for } j \in M_0(D'), \\
\gamma_j' & \text{for } j \in M_1(D').
\end{cases}
\]

(i), (ii) Since \(\gamma_i \cdot \gamma_j' = 0\) holds, we have the equations.

(iii), (iv) Since it holds that \(\gamma_{D'} \cdot \gamma_j' = -(\gamma_{D'} - \gamma_j') \cdot \gamma_{D'} = \gamma_j' \cdot \gamma_{D'}\) and \(\gamma_{D'} \cdot \gamma_D = 0\), we have the equations. \(\Box\)

The intersection number \(\Gamma_j, \Gamma_i (1 \leq i \leq n, 1 \leq j \leq m)\) is given as follows. Since the proof is similar to that of Lemma 7.3, we leave it to the reader.

Lemma 7.4. For any \(1 \leq i \leq n\) and \(1 \leq j \leq m\), we have the following.

(i) If \(i \in M_0(D)\) and \(j \in M_0(D')\), then \(\Gamma_j \cdot \Gamma_i = \gamma_j' \cdot \gamma_D\).

(ii) If \(i \in M_0(D)\) and \(j \in M_1(D')\), then \(\Gamma_j \cdot \Gamma_i = \gamma_i \cdot \gamma_D + \gamma_j' \cdot \gamma_{D'}\).

(iii) If \(i \in M_1(D)\) and \(j \in M_0(D')\), then \(\Gamma_j \cdot \Gamma_i = 0\).

(iv) If \(i \in M_1(D)\) and \(j \in M_1(D')\), then \(\Gamma_j \cdot \Gamma_i = \gamma_i \cdot \gamma_D\). \(\Box\)

Proof of Theorem 7.1. Let \(\varepsilon_i\) be the sign of \(c_i \ (1 \leq i \leq n)\), and \(\varepsilon_j'\) the sign of \(c_j' \ (1 \leq j \leq m)\). Put

\[
\omega_{D}^{k} = \sum_{i \in M_0(D)} \varepsilon_i \quad \text{and} \quad \omega_{D'}^{k} = \sum_{j \in M_0(D')} \varepsilon_j' \quad (k = 0, 1).
\]

By Lemmas 7.2, 7.3 and 7.4 it holds that

\[
f_{01}(D'') - f_{01}(D) - f_{01}(D') = \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\Gamma_i, \Gamma_j} - 1) + \sum_{1 \leq i,j \leq m} \varepsilon_i \varepsilon_j' (t^{\Gamma_i, \Gamma_j'} - 1)
\]

\[
\quad + \sum_{1 \leq i \leq n, 1 \leq j \leq m} \varepsilon_i \varepsilon_j' (t^{\Gamma_i, \Gamma_j'} - 1) + \sum_{1 \leq i \leq n, 1 \leq j \leq m} \varepsilon_i \varepsilon_j' (t^{\Gamma_i', \Gamma_j} - 1)
\]

\[
\quad - \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i, \gamma_D} - 1) - \sum_{1 \leq i,j \leq m} \varepsilon_i \varepsilon_j' (t^{\gamma_i', \gamma_{D'}} - 1)
\]

\[
\quad = \sum_{i \in M_0(D), j \in M_0(D')} \varepsilon_i \varepsilon_j' (t^{\gamma_i, \gamma_D} - 1) + \sum_{i \in M_1(D), j \in M_0(D')} \varepsilon_i \varepsilon_j' (t^{\gamma_i', \gamma_D} - 1)
\]

\[
\quad + \sum_{i \in M_0(D), j \in M_0(D')} \varepsilon_i \varepsilon_j' (t^{\gamma_i, \gamma_{D'} + \gamma_j', \gamma_{D'}} - 1)
\]

\[
\quad + \sum_{i \in M_0(D), j \in M_0(D')} \varepsilon_i \varepsilon_j' (t^{\gamma_i', \gamma_{D'}} - 1) + \sum_{i \in M_1(D), j \in M_0(D')} \varepsilon_i \varepsilon_j' (t^{\gamma_i, \gamma_D} - 1)
\]

\[
\quad + \sum_{i \in M_0(D), j \in M_0(D')} \varepsilon_i \varepsilon_j' (t^{\gamma_i, \gamma_{D'} + \gamma_j', \gamma_{D'}} - 1).
\]

The sum of the first and third lines is equal to

\[
\omega_{D}^{0}W_{D}^{0}(t) + \omega_{D}^{1}W_{D}^{1}(t) + \omega_{D'}^{0}W_{D'}^{0}(t) + \omega_{D'}^{1}W_{D'}^{1}(t).
\]

On the other hand, it holds that

\[
t^{\gamma_i, \gamma_D} + t^{\gamma_i', \gamma_{D'}} - 1 = (t^{\gamma_i, \gamma_D} - 1)(t^{\gamma_i', \gamma_{D'}} - 1) + (t^{\gamma_i, \gamma_D} - 1) + (t^{\gamma_i', \gamma_{D'}} - 1).
\]

Therefore the sum of the second and fourth lines is equal to

\[
W_{D}^{0}(t)W_{D'}^{0}(t) + W_{D'}^{1}(t)W_{D}^{1}(t) + \omega_{D'}^{0}W_{D'}^{0}(t) + \omega_{D'}^{1}W_{D'}^{1}(t) + \omega_{D}^{0}W_{D}^{0}(t) + \omega_{D}^{1}W_{D}^{1}(t).
\]
Since it holds that \( \omega^0_D + \omega^1_D = \omega_D \) and \( \omega^0_{D'} + \omega^1_{D'} = \omega_{D'} \), we have

\[
\begin{align*}
I_{T\vee T'}(t) - I_T(t) - I_{T'}(t) &= f_{01}(D'') - f_{01}(D) - f_{01}(D') \\
&= \omega_D W_T(t) + \omega_D W_{\hat{T}}(t) + W_T^0(t) W_{T'}^1(t) + W_T^1(t) W_{T'}^0(t).
\end{align*}
\]

Thus we obtain

\[
I_{T\vee T'}(t) - I_T(t) - I_{T'}(t) = \omega_D W_T(t) + \omega_D W_{\hat{T}}(t) + W_T^0(t) W_{T'}^1(t) + W_T^1(t) W_{T'}^0(t).
\]

\[
\square
\]

Similarly to the first intersection polynomial, the second intersection polynomial satisfies the following.

**Theorem 7.5.** For dotted virtual knots \( T \) and \( T' \), it holds that

\[
II_{T\vee T'}(t) = II_T(t) + II_{\hat{T}}(t) + W_T^0(t) W_{T'}^1(t) + W_T^1(t) W_{T'}^0(t).
\]

To prove this theorem, we prepare the following lemmas. Since the proofs are similar to those of Lemmas 7.2, 7.3 and 7.4 we omit them.

**Lemma 7.6.** For any \( 1 \leq i, j \leq n \), we have the following.

(i) \( \Gamma_i \cdot \Gamma_j = \gamma_i \cdot \gamma_j \).

(ii) \( \Gamma_i \cdot \Gamma_j = \gamma_i \cdot \gamma_j \). \( \square \)

**Lemma 7.7.** For any \( 1 \leq i, j \leq m \), we have the following.

(i) \( \Gamma'_i \cdot \Gamma'_j = \gamma'_i \cdot \gamma'_j \).

(ii) \( \Gamma'_i \cdot \Gamma'_j = \gamma'_i \cdot \gamma'_j \). \( \square \)

**Lemma 7.8.** For any \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), we have the following.

(i) \( \text{If } i \in M_0(D) \text{ and } j \in M_0(D'), \text{ then } \Gamma_i \cdot \Gamma'_j = 0. \)

(ii) \( \text{If } i \in M_0(D) \text{ and } j \in M_1(D'), \text{ then } \Gamma_i \cdot \Gamma'_j = \gamma_i \cdot \gamma_D. \)

(iii) \( \text{If } i \in M_1(D) \text{ and } j \in M_0(D'), \text{ then } \Gamma_i \cdot \Gamma'_j = -\gamma'_i \cdot \gamma_{D'}. \)

(iv) \( \text{If } i \in M_1(D) \text{ and } j \in M_1(D'), \text{ then } \Gamma_i \cdot \Gamma'_j = \gamma_i \cdot \gamma_D - \gamma'_i \cdot \gamma_{D'}. \)

The intersection number \( \Gamma'_i \cdot \Gamma_j \) is given by \( \Gamma'_i \cdot \Gamma_j = -\Gamma_i \cdot \Gamma'_j. \) \( \square \)

**Lemma 7.9.** For any \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), we have the following.

(i) \( \text{If } i \in M_0(D) \text{ and } j \in M_0(D'), \text{ then } \Gamma'_i \cdot \Gamma'_j = -\gamma_i \cdot \gamma_D + \gamma'_i \cdot \gamma_{D'}. \)

(ii) \( \text{If } i \in M_0(D) \text{ and } j \in M_1(D'), \text{ then } \Gamma'_i \cdot \Gamma'_j = \gamma'_i \cdot \gamma_{D'}. \)

(iii) \( \text{If } i \in M_1(D) \text{ and } j \in M_0(D'), \text{ then } \Gamma'_i \cdot \Gamma'_j = -\gamma_i \cdot \gamma_D. \)

(iv) \( \text{If } i \in M_1(D) \text{ and } j \in M_1(D'), \text{ then } \Gamma'_i \cdot \Gamma'_j = 0. \)

The intersection number \( \Gamma'_j \cdot \Gamma_i \) is given by \( \Gamma'_j \cdot \Gamma_i = -\Gamma'_i \cdot \Gamma'_j. \) \( \square \)
Proof of Theorem 7.5. Since the proof is similar to that of Theorem 7.1, we just give an outline of it. By Lemmas 7.6,7.9, it holds that
\[
f_{00}(D'') - f_{00}(D) - f_{00}(D') = \omega_D(W_T(t) + W_{\hat{T}}(t) + W_{\hat{T}}(t^{-1})) + W_T^1(t)W_T^1(t^{-1})W_{\hat{T}}^1(t), \text{ and}
\]
f_{11}(D'') - f_{11}(D) - f_{11}(D') = \omega_D(W_{\hat{T}}(t^{-1}) + W_{\hat{T}}(t)) + \omega_D(W_{\hat{T}}(t^{-1}) + W_{\hat{T}}(t)) + W_T^0(t^{-1})W_T(t) + W_T^0(t)W_T^0(t^{-1}).
\]
Therefore we have
\[
\begin{align*}
\mathcal{II}_{\hat{T}^{-1}\hat{T}}(t) - \mathcal{II}_T(t) - \mathcal{II}_{\hat{T}}(t) &= f_{00}(D'') - f_{00}(D) - f_{00}(D') + f_{11}(D'') - f_{11}(D) - f_{11}(D') \\
&- (\omega_D + \omega_{\hat{D}})(W_T(t) + W_{\hat{T}}(t) + W_{\hat{T}}(t^{-1}) + W_{\hat{T}}(t^{-1})) \\
+ \omega_D(W_{\hat{T}}(t) + W_{\hat{T}}(t^{-1})) + \omega_{\hat{D}}(W_{\hat{T}}(t) + W_{\hat{T}}(t^{-1})) \\
&= W_T^0(t)W_T^0(t^{-1}) + W_T^1(t)W_T^1(t^{-1}) + W_T^0(t^{-1})W_T(t) + W_T^1(t^{-1})W_T(t).
\end{align*}
\]

Definition 7.10. For virtual knots \( K \) and \( K' \), we consider the set of virtual knots
\[
\mathcal{C}(K, K') = \{ \hat{T} + T' \mid T, T' \text{ dotted virtual knots with } \hat{T} = K \text{ and } \hat{T}' = K' \}.
\]
A virtual knot in this set is called a connected sum of \( K \) and \( K' \).

Theorem 7.11. For any virtual knots \( K \) and \( K' \), the set \( \mathcal{C}(K, K') \) is infinite; that is, there are infinitely many connected sums of \( K \) and \( K' \).

Proof. Let \( n \) be an integer. By Proposition 6.4, there are dotted virtual knots \( T_n \) and \( T'_n \) such that
\[
\begin{align*}
\hat{T}_n &= K, \quad W_{T_n}^0(t) = t^n - 1, \quad W_{T_n}^1(t) = W_K(t) - t^n + 1, \text{ and} \\
\hat{T}'_n &= K', \quad W_{T'_n}^0(t) = t^n - 1, \quad W_{T'_n}^1(t) = W_{K'}(t) - t^n + 1.
\end{align*}
\]
We denote by \( K_n \) the virtual knot \( \hat{T}_n + T'_n \) which is a connected sum of \( K \) and \( K' \). Then by Theorem 7.1, it holds that
\[
I_{K_n}(t) = I_K(t) + I_{K'}(t) + (t^n - 1)(W_{K'}(t) - t^n + 1) + (W_K(t) - t^n + 1)(t^n - 1) \\
= I_K(t) + I_{K'}(t) + (W_K(t) + W_{K'}(t))(t^n - 1) - 2(t^n - 1)^2.
\]
For any sufficiently large \( n \), we have \( \deg I_{K_n}(t) = 2n \) and \( K_n \)'s are mutually distinct. Therefore \( \mathcal{C}(K, K') \) is an infinite set.

8. Characterizations of intersection polynomials

In this section, we give characterizations of the intersection polynomials \( I_K(t) \), \( \mathcal{II}_K(t) \), and \( \mathcal{III}_K(t) \).

Let \( \mathcal{P}_1 \) denote the set of Laurent polynomials defined by \( \mathcal{P}_1 = \{ I_K(t) \mid K \text{ virtual knots} \} \). The characterization is exactly the same as that of the writhe polynomial as given in Theorem 1.1

Theorem 8.1. For \( f(t) \in \mathbb{Z}[t, t^{-1}] \), the following are equivalent.
Theorem 4.2. This follows by

Proof. For any $f(t) \in \mathcal{P}_1$.

(i) $f(t) = f(t^{-1}) \in \mathcal{P}_1$.

(ii) $f(1) = f'(1) = 0$.

(iii) $f(t) = \sum_{k \neq 0,1} a_k (t^k - kt + k - 1)$ for some $a_k \in \mathbb{Z}$ ($k \neq 0,1$).

To prove this theorem, we prepare several lemmas.

Lemma 8.2. For any $f(t), g(t) \in \mathcal{P}_1$, we have the following.

(i) $f(t^{-1}) \in \mathcal{P}_1$.

(ii) $f(t) + g(t) \in \mathcal{P}_1$.

Proof. Let $K$ and $K'$ be virtual knots with $I_K(t) = f(t)$ and $I_{K'}(t) = g(t)$.

(i) It holds that $I_{-K}(t) = f(t^{-1}) \in \mathcal{P}_1$ by Lemma 5.2

(ii) By Proposition 6.4 there are dotted virtual knots $T$ and $T'$ such that

\[
\begin{align*}
\hat{T} &= K, & W_{T}^0(t) &= W_K(t), & W_{T}^1(t) &= 0, & \text{and} \\
\hat{T}' &= K', & W_{T'}^0(t) &= W_{K'}(t), & W_{T'}^1(t) &= 0.
\end{align*}
\]

Then we have $I_{\hat{T}+\hat{T}'}(t) = I_K(t) + I_{K'}(t) = f(t) + g(t) \in \mathcal{P}_1$ by Theorem 7.1. □

Lemma 8.3. Let $n \geq 2$ be an integer.

(i) There are integers $a_k$ ($0 \leq k \leq n-1$) such that $t^n + \sum_{k=0}^{n-1} a_k t^k \in \mathcal{P}_1$.

(ii) There are integers $a'_k$ ($0 \leq k \leq n-1$) such that $-t^n + \sum_{k=0}^{n-1} a'_k t^k \in \mathcal{P}_1$.

Proof. (i) For $n = 2$, we have $I_{4,4}(t) = (t-1)^2 \in \mathcal{P}_1$.

Assume that $n \geq 3$. We consider the trivial virtual knot $O$ and the virtual knot $3.4$ with $W_{3.4}(t) = (t-1)^2$ and $I_{3.4}(t) = 0$. By Proposition 6.4 there are dotted virtual knots $T$ and $T'$ such that

\[
\begin{align*}
\hat{T} &= O, & W_{T}^0(t) &= -(t-1)t^{n-3}, & W_{T}^1(t) &= (t-1)t^{n-3}, & \text{and} \\
\hat{T}' &= 3.4, & W_{T'}^0(t) &= (t-1)^2, & W_{T'}^1(t) &= 0.
\end{align*}
\]

Then it holds that $I_{\hat{T}+\hat{T}'}(t) = (t-1)^3t^{n-3} \in \mathcal{P}_1$.

(ii) For $n = 2$, we have $I_{3,1}(t) = -(t-1)^2 \in \mathcal{P}_1$. Assume that $n \geq 3$. We consider the trivial knot $O$ and the virtual knot $3.4^\#$. We remark that $W_{-3.4^\#}(t) = -W_{3.4}(t) = -(t-1)^2$ and $I_{-3.4^\#}(t) = I_{3.4}(t) = 0$. Then similarly to the proof of (i), we have $-(t-1)^3t^{n-3} \in \mathcal{P}_1$. □

Lemma 8.4. Let $n \leq -1$ be an integer.

(i) There are integers $a_k$ ($n+1 \leq k \leq 1$) such that $t^n + \sum_{k=n+1}^{1} a_k t^k \in \mathcal{P}_1$.

(ii) There are integers $a'_k$ ($n+1 \leq k \leq 1$) such that $-t^n + \sum_{k=n+1}^{0} a'_k t^k \in \mathcal{P}_1$.

Proof. Assume that $n \leq -2$. By Lemmas 8.2(i) and 8.3, we have $t^n + \sum_{k=n+1}^{0} a_k t^k \in \mathcal{P}_1$ and $-t^n + \sum_{k=n+1}^{0} a'_k t^k \in \mathcal{P}_1$ for some $a_k, a'_k \in \mathbb{Z}$ ($n+1 \leq k \leq 0$).

For $n = -1$, we have $I_{4,9}(t) = t^{-1} - 2 + t \in \mathcal{P}_1$ and $I_{2,1}(t) = -t^{-1} + 2 - t \in \mathcal{P}_1$. □

Proof of Theorem 8.1 (i)⇒(ii). This follows by Theorem 4.2

(ii)⇒(iii). Assume that $f(t) = \sum_{k \in \mathbb{Z}} a_k t^k$ satisfies $f(1) = f'(1) = 0$. Then we have $a_0 = \sum_{k \neq 0,1} (k-1) a_k$ and $a_1 = -\sum_{k \neq 0,1} k a_k$.

(iii)⇒(i). By Lemmas 8.2(ii), 8.3 and 8.4 there are integers $a'_0$ and $a'_1$ such that $g(t) = \sum_{k \neq 0,1} a'_k t^k + a'_1 + a'_0 \in \mathcal{P}_1$. Since $g(1) = g'(1) = 0$ holds by Theorem 4.2 we have $a'_0 = \sum_{k \neq 0,1} (k-1) a_k$ and $a'_1 = -\sum_{k \neq 0,1} k a_k$. Therefore we have $g(t) = f(t) = f(t) \in \mathcal{P}_1$. □
Let $\mathcal{P}_2$ denote the set of Laurent polynomials defined by $\mathcal{P}_2 = \{ II_K(t) \mid K : \text{virtual knots} \}$.

**Theorem 8.5.** For $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.

(i) $f(t) \in \mathcal{P}_2$.

(ii) $f(t)$ is reciprocal, $f(1) = 0$, and $f''(1) \equiv 0 \pmod{4}$.

To prove this theorem, we prepare several lemmas.

**Lemma 8.6.** For any $f(t), g(t) \in \mathcal{P}_2$, we have $f(t) + g(t) \in \mathcal{P}_2$.

*Proof.* Let $K$ and $K'$ be virtual knots with $II_K(t) = f(t)$ and $II_{K'}(t) = g(t)$.

By Proposition 6.4, there are dotted virtual knots $T$ and $T'$ such that

\[
\begin{aligned}
\hat{T} &= K, & W^0_T(t) &= 0, & W^1_T(t) &= W_K(t), & W^2_T(t) &= 0, \\
\hat{T}' &= K', & W^2_{T'}(t) &= W_{K'}(t), & W^1_{T'}(t) &= 0, & W^0_{T'}(t) &= 0.
\end{aligned}
\]

Then we have $II_{\hat{T} + \hat{T}'}(t) = II_K(t) + II_{K'}(t) = f(t) + g(t) \in \mathcal{P}_2$ by Theorem 7.5. □

**Lemma 8.7.** Let $n \geq 2$ be an integer.

(i) There are integers $a_k$ ($0 \leq k \leq n - 1$) such that $(t^n + t^{-n}) + \sum_{k=1}^{n-1} a_k(t^k + t^{-k}) + a_0 \in \mathcal{P}_2$.

(ii) There are integers $a'_k$ ($0 \leq k \leq n - 1$) such that $-(t^n + t^{-n}) + \sum_{k=1}^{n-1} a'_k(t^k + t^{-k}) + a'_0 \in \mathcal{P}_2$.

*Proof.* (i) We consider the trivial virtual knot $O$ and the virtual knot 4.20 with $W_{4.20}(t) = (t-1)^2$ and $II_{4.20}(t) = 0$. By Proposition 6.4, there are dotted virtual knots $T$ and $T'$ such that

\[
\begin{aligned}
\hat{T} &= O, & W^0_T(t) &= (t-1)t^{n-1}, & W^1_T(t) &= -(t-1)t^{n-1}, & W^2_T(t) &= 0, \\
\hat{T}' &= 4.20, & W^0_{T'}(t) &= (t-1)^2, & W^1_{T'}(t) &= 0.
\end{aligned}
\]

Then it holds that

\[
II_{\hat{T} + \hat{T}'}(t) = (t-1)(t^{-1}-1)^2t^{n-1} + (t-1)^2(t^{-1}-1)t^{-n+1} = (t-1)^3t^{n-3} + (t^{-1}-1)^3t^{-n+3} \in \mathcal{P}_2.
\]

(ii) We consider the trivial knot $O$ and the virtual knot $-4.20\#$. We remark that $W_{-4.20\#}(t) = -W_{4.20}(t) = -(t-1)^2$ and $II_{-4.20\#}(t) = II_{4.20}(t) = 0$. Then similarly to the proof of (i), we have $-(t-1)^3t^{n-3} - (t^{-1}-1)^3t^{-n+3} \in \mathcal{P}_2$. □

**Lemma 8.8.** $2t - 4 + 2t^{-1} \in \mathcal{P}_2$ and $-2t + 4 - 2t^{-1} \in \mathcal{P}_2$.

*Proof.* We have $II_{2.1}(t) = -2t + 4 - 2t^{-1} \in \mathcal{P}_2$. Furthermore, since $II_{4.56} = 4t - 8 + 4t^{-1} \in \mathcal{P}_2$, we have $2t - 4 + 2t^{-1} = (-2t + 4 - 2t^{-1}) + (4t - 8 + 4t^{-1}) \in \mathcal{P}_2$ by Lemma 8.6. □

*Proof of Theorem 8.5.* (i)⇒(ii). This follows by Theorem 4.3.

(ii)⇒(i). By Lemma 4.5, we may take $f(t) = \sum_{k \geq 1} a_k(t^k + t^{-k} - 2)$ for some $a_k \in \mathbb{Z}$ ($k \geq 1$) with $\sum_{k \text{ odd} \geq 1} a_k \equiv 0 \pmod{2}$. By Lemmas 8.6 and 8.7, there are integers $a'_0$ and $a'_1$ such that

\[
g(t) = \sum_{k \geq 2} a_k(t^k + t^{-k}) + a'_1(t + t^{-1}) + a'_0 \in \mathcal{P}_2.
\]
Since \( \sum_{k \geq d \geq 3} a_k + a_1' \equiv 0 \pmod{2} \) by Theorem 4.1, we obtain \( a_1' \equiv a_1 \pmod{2} \). Therefore, by Lemmas 8.6 and 8.8, there is an integer \( a_0'' \) such that

\[
h(t) = \sum_{k \geq 1} a_k (t^k + t^{-k}) + a_0'' \in \mathcal{P}_2.
\]

Since \( h(1) = 0 \) holds, we have \( a_0'' = -2 \sum_{k \geq 1} a_k \) and \( h(t) = f(t) \in \mathcal{P}_2 \). \( \qed \)

Let \( \mathcal{P}_3 \) denote the set of pairs of Laurent polynomials defined by

\[
\mathcal{P}_3 = \{ (f(t), g(t)) \mid f(t) = W_K(t) \text{ and } g(t) \equiv III_K(t) \text{ for some } K \}.
\]

A pair of Laurent polynomials in the set \( \mathcal{P}_3 \) is characterized as follows.

**Theorem 8.9.** For \( f(t) \) and \( g(t) \in \mathbb{Z}[t, t^{-1}] \), the following are equivalent.

(i) \( (f(t), g(t)) \in \mathcal{P}_3 \).

(ii) \( g(t) \) is reciprocal, \( f(1) = f'(1) = g(1) = 0 \), and \( f''(1) \equiv g''(1) \pmod{4} \).

To prove this theorem, we prepare the following lemma.

**Lemma 8.10.** Let \( T \) and \( T_0 \) be dotted virtual knots with \( \hat{T} = K \) and \( \hat{T}_0 = O \), respectively. Then we have

\[
III_{\hat{T} + \hat{T}_0} (t) \equiv III_K (t) + W^1_{\hat{T}} W^1_{\hat{T}_0} (t^{-1}) + W^1(t) W^1_{\hat{T}_0} (t) \pmod{\bar{W}_K(t)}.
\]

Proof. Let \( (D, p) \) and \( (D_0, p_0) \) be diagrams of \( T \) and \( T_0 \), respectively. Since \( \bar{W}_O(t) = 0 \) holds, we have \( f_{00}(D_0) = III_O(t) = 0 \). Let \( (D', p') \) be the diagram of \( T + T_0 \) obtained by taking the sum of \( (D, p) \) and \( (D_0, p_0) \). By the equation in the proof of Theorem 7.5 it holds that

\[
f_{00}(D') = f_{00}(D) + f_{00}(D_0) + \omega_{D_0} \bar{W}_K(t) + \omega_{D} \bar{W}_O(t) + W^1(t) W^1_{\hat{T}_0} (t) + W^1(t) W^1_{\hat{T}_0} (t) + W^1(t) W^1_{\hat{T}_0} (t).
\]

Therefore we have we have the conclusion. \( \qed \)

Proof of Theorem 8.9. (i) \( \Rightarrow \) (ii). This follows by Theorems 4.1 and 4.6.

(ii) \( \Rightarrow \) (i). By Theorem 4.1 there is a virtual knot \( K \) with \( W_K(t) = f(t) \). We take a Laurent polynomial \( h(t) \) with \( III_K(t) \equiv h(t) \pmod{\bar{W}_K(t)} \). By Theorem 4.6 \( h(t) \) is reciprocal, \( h(1) = 0 \), and \( h''(1) \equiv f''(1) \pmod{4} \).

Consider the Laurent polynomial \( p(t) = g(t) - h(t) \). Then \( p(t) \) is reciprocal, \( p(1) = g(1) - h(1) = 0 \), and \( p''(1) = g''(1) - h''(1) \equiv 0 \pmod{4} \). By Lemma 4.5 there is a Laurent polynomial \( q(t) \) such that

\[
p(t) = (t - 1)(t^{-1} - 1)q(t) + (t^{-1} - 1)(t - 1)q(t^{-1}).
\]

It follows by Proposition 6.4 that there are dotted virtual knots \( T \) and \( T_0 \) such that

\[
\begin{align*}
\hat{T} & = K, & W^1_{\hat{T}} (t) & = W_K(t) - (t - 1)q(t), & W^1(t) & = (t - 1)q(t), \text{ and} \\
\hat{T}_0 & = O, & W^0_{\hat{T}_0} (t) & = -(t - 1), & W^1_{\hat{T}_0} (t) & = t - 1.
\end{align*}
\]

Then we have \( p(t) = W^1(t) W^1_{\hat{T}_0} (t^{-1}) + W^1(t) W^1_{\hat{T}_0} (t) \).

By Lemma 8.10, the virtual knot \( T + T_0 \) satisfies

\[
\begin{align*}
III_{T + T_0} (t) & = W_K(t) + W_O(t) = f(t) \text{ and} \\
W^1_{\hat{T} + \hat{T}_0} (t) & = h(t) + p(t) = g(t).
\end{align*}
\]
By Theorem 7.11, there are infinitely many connected sums of two trivial virtual knots. In particular, we have the following, where $2P_1 = \{2f(t) \mid f(t) \in P_1\}$ and $2P_2 = \{2f(t) \mid f(t) \in P_2\}$.

**Proposition 8.11.** For $f(t) \in \mathbb{Z}[t,t^{-1}]$, the following are equivalent.

(i) There is a virtual knot $K \in \mathcal{C}(O,O)$ with $f(t) = I_K(t)$.

(ii) $f(t) \in 2P_1$.

**Proof.** (i)$\Rightarrow$(ii). Let $T$ and $T'$ be dotted virtual knots with $K = \widehat{T} + \widehat{T'}$ and $\widehat{T} = \widehat{T'} = O$. By Theorem 4.2, we have $f(1) = f'(1) = 0$. Furthermore, since

$$W_T^0(t) = -W_T^1(t)$$

and

$$W_T^0(t) = -(t-1), \quad W_T^1(t) = t-1,$$

we have $I_K(t) = -2W_T^0(t)W_T^0(t)$ by Theorem 7.1. Therefore all the coefficients of $f(t)$ are even.

(ii)$\Rightarrow$(i). By Theorem 8.1, there is a Laurent polynomial $g(t) \in \mathbb{Z}[t,t^{-1}]$ with $f(t) = 2(t-1)^2g(t)$. By Proposition 6.4, there are dotted virtual knots $T$ and $T'$ such that

$$\begin{cases} 
\widehat{T} = O, & W_T^0(t) = -(t-1), \quad W_T^1(t) = t-1, \\
\widehat{T'} = O, & W_{T'}^0(t) = (t-1)g(t), \quad W_{T'}^1(t) = -(t-1)g(t). 
\end{cases}$$

Then the connected sum $K = \widehat{T} + \widehat{T'} \in \mathcal{C}(O,O)$ satisfies

$$I_K(t) = -2W_T^0(t)W_{T'}^0(t) = 2(t-1)^2g(t) = f(t).$$

\[\square\]

**Proposition 8.12.** For $f(t) \in \mathbb{Z}[t,t^{-1}]$, the following are equivalent.

(i) There is a virtual knot $K \in \mathcal{C}(O,O)$ with $f(t) = II_K(t)$.

(ii) $f(t) \in 2P_2$.

**Proof.** (i)$\Rightarrow$(ii). Let $T$ and $T'$ be dotted virtual knots with $K = \widehat{T} + \widehat{T'}$ and $\widehat{T} = \widehat{T'} = O$. By Theorem 4.3, $f(t)$ is reciprocal, $f(1) = 0$, and $f''(1) \equiv 0$ (mod 4). Furthermore, since we may take

$$W_T^0(t) = -W_T^1(t) = (t-1)p(t) \quad \text{and} \quad W_T^1(t) = -W_T^0(t) = (t-1)q(t)$$

for some $p(t), \quad q(t) \in \mathbb{Z}[t,t^{-1}]$, we have

$$f(t) = II_K(t) = 2(W_T^0(t)W_T^0(t^{-1}) + W_T^0(t^{-1})W_T^0(t)) = 2(t-1)(t^{-1} - 1)p(t)q(t^{-1}) + p(t^{-1})q(t)).$$

Therefore all the coefficients of $f(t)$ are even.

(ii)$\Rightarrow$(i). Put $\tilde{f}(t) = f(t)/2 \in \mathbb{Z}[t,t^{-1}]$. By Theorem 4.3, it satisfies that $\tilde{f}(t)$ is reciprocal, $\tilde{f}(1) = 0$, and $f''(1) \equiv 0$ (mod 4).

By Lemma 4.5, there is a Laurent polynomial $\tilde{g}(t)$ such that

$$\tilde{f}(t) = (t-1)(t^{-1} - 1)\tilde{g}(t) + (t^{-1} - 1)(t-1)\tilde{g}(t^{-1}).$$

By Proposition 6.4, there are dotted virtual knots $T$ and $T'$ such that

$$\begin{cases} 
\widehat{T} = O, & W_T^0(t) = (t-1)\tilde{g}(t), \quad W_T^1(t) = -(t-1)\tilde{g}(t), \\
\widehat{T'} = O, & W_{T'}^0(t) = t-1, \quad W_{T'}^1(t) = -(t-1). 
\end{cases}$$

\[\square\]
Then the connected sum \( K = \hat{T} + T' \in \mathcal{C}(O, O) \) satisfies
\[
II_K(t) = 2(t - 1)(t^{-1} - 1)(\tilde{g}(t) + \tilde{g}(t^{-1})) = f(t).
\]

\( \square \)

**Proposition 8.13.** For \( f(t) \in \mathbb{Z}[t, t^{-1}] \), the following are equivalent.

(i) There is a virtual knot \( K \in \mathcal{C}(O, O) \) with \( f(t) = III_K(t) \).

(ii) \( f(t) \in P_2 \).

**Proof.** By Proposition 6.4 and Lemma 8.10, the condition (i) is equivalent to
\[
f(t) \in \{ p(t)q(t) + p(t^{-1})q(t^{-1}) \mid p(t), q(t) \in \mathbb{Z}[t, t^{-1}], p(1) = q(1) = 0 \}.
\]

This set is coincident with
\[
\{ g(t) + g(t^{-1}) \mid g(t) \in \mathbb{Z}[t, t^{-1}], g(1) = g'(1) = 0 \}.
\]

Therefore (i) is equivalent to (ii) by Lemma 4.5 and Theorem 8.5. \( \square \)

**Example 8.14.** Let \( T \) and \( T' \) be the dotted virtual knots presented by Gauss diagrams as shown in Figure 28. The closures \( \hat{T} \) and \( \hat{T}' \) are trivial virtual knots. Consider the virtual knot \( K = \hat{T} + \hat{T}' \) which is a connected sum of two trivial knots. In [15], Kishino and the fourth author study the virtual knot \( K = 4.5.5 \) as shown in Figure 28 and prove that \( K \) is not classical by using computer calculation of the Jones polynomial of the 3-parallel of \( K \). Since we have
\[
W^0_T(t) = -W^1_T(t) = -(t - 1) \quad \text{and} \quad W^0_T'(t) = -W^1_T'(t) = -(t^{-1} - 1),
\]
it holds that
\[
I_K(t) = 2t - 4 + 2t^{-1},
\]
\[
II_K(t) = 2t^2 - 4t + 4 - 4t^{-1} + 2t^{-2}, \quad \text{and}
\]
\[
III_K(t) = t^2 - 2t + 2 - 2t^{-1} + t^{-2}
\]
by Theorems 7.4, 7.5, and Lemma 8.10. This also induces that \( K \) is not classical by Lemma 2.11.

![Figure 28](image-url)
9. Finite type invariants of order 2

Let \( D \) be a diagram on a closed, connected, oriented surface \( \Sigma \) with \( m \) double points (with no over/under-information) and some crossings. For \( \varepsilon_i \in \{ \pm 1 \} \) \((1 \leq i \leq m)\), we denote by \( D_{\varepsilon_1...\varepsilon_m} \) the diagram by giving over/under-information at \( p_i \) with sign \( \varepsilon_i \), and \( K_{\varepsilon_1...\varepsilon_m} \) the virtual knot presented by \( D_{\varepsilon_1...\varepsilon_m} \). An invariant \( v(K) \) of a virtual knot \( K \) is called a finite type invariant of order \( m \) \([12]\) if it satisfies

\[
(i) \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 ... \varepsilon_{m+1} v(K_{\varepsilon_1...\varepsilon_{m+1}}) = 0 \text{ for any diagram } D \text{ with } m + 1 \text{ double points, and}
\]

\[
(ii) \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 ... \varepsilon_m v(K_{\varepsilon_1...\varepsilon_m}) \neq 0 \text{ for some diagram } D \text{ with } m \text{ double points.}
\]

We remark that this definition of a finite type invariant is different from the one given by Goussarov, Polyak, and Viro \([7]\).

It is known in \([4]\) that the writhe polynomial is a finite type invariant of order 1. Namely any diagram with two double points satisfies

\[
W_{K_{++}}(t) - W_{K_{+-}}(t) - W_{K_{-+}}(t) + W_{K_{--}}(t) = 0.
\]

**Lemma 9.1.** For any diagram \( D \) with three double points and an integer \( \omega \), we have

\[
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} (\omega + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) \varepsilon_1 \varepsilon_2 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) = 0.
\]

**Proof.** Since the writhe polynomial is a finite type invariant of order 1, it holds that

\[
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \omega \varepsilon_1 \varepsilon_2 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) = \omega \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \left( \sum_{\varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_2 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) \right) = 0
\]

and

\[
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \varepsilon_1 \varepsilon_2 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t)
\]

\[
= \sum_{\varepsilon_1 = \pm 1} \left( \sum_{\varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_2 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) \right) + \sum_{\varepsilon_2 = \pm 1} \left( \sum_{\varepsilon_1, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) \right) + \sum_{\varepsilon_3 = \pm 1} \left( \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) \right) = 0.
\]

\(\square\)

Let \( D \) be a diagram with three double points \( p_1, p_2, p_3 \) and \( n - 3 \) crossings \( c_4, ..., c_n \). For each \( p_i \) and \( \varepsilon_i \in \{ \pm 1 \} \) \((i = 1, 2, 3)\), we denote by \( \gamma_i(\varepsilon_i) \) the cycle at \( p_i \) in the diagram \( D_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \) on \( \Sigma \). The cycle \( \gamma_1(\varepsilon_1) \) on \( \Sigma \) is determined independently of \( \varepsilon_2 \) and \( \varepsilon_3 \) and so on. It holds that \( \gamma_i(+) + \gamma_i(-) = \gamma_D \) \((i = 1, 2, 3)\) by definition. For \( 4 \leq i \leq n \), the cycle \( \gamma_i \) is defined at \( c_i \) on \( \Sigma \) independently of \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \).

**Theorem 9.2.** The first and second intersection polynomials are finite type invariants of order 2.
Proof. $I_K(t)$. Let $\omega$ be the sum of signs of $c_4, \ldots, c_n$ of $D$. Since the writhe of $D_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$ is equal to $\omega + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, we have
\[
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 I_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 f_{01}(D_{\varepsilon_1 \varepsilon_2 \varepsilon_3})
- \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} (\omega + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) \varepsilon_1 \varepsilon_2 \varepsilon_3 W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t).
\]

By Lemma 9.1, the second sum is equal to zero. On the other hand, the first sum is divided into five terms as follows:
\[
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{i=1}^{3} \varepsilon_i^2 (t^{\gamma_i(\varepsilon_i)} \bar{\gamma}_i(\varepsilon_i)) - 1 \right)
+ \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{1 \leq i \neq j \leq 3} \varepsilon_i \varepsilon_j (t^{\gamma_i(\varepsilon_i)} \bar{\gamma}_j(\varepsilon_j)) - 1 \right)
+ \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{1 \leq i \leq 3, 4 \leq j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i(\varepsilon_i)} \gamma_j(\varepsilon_j)) - 1 \right)
+ \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{4 \leq i \leq n, 1 \leq j \leq 3} \varepsilon_i \varepsilon_j (t^{\gamma_i(\varepsilon_i)} \gamma_j(\varepsilon_j)) - 1 \right)
+ \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{4 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i(\varepsilon_i)} \gamma_j(\varepsilon_j)) - 1 \right).
\]

We see that each of the five terms is equal to zero. For example, the second term is equal to
\[
\sum_{1 \leq i \neq j \leq 3} \left( \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_i \varepsilon_j (t^{\gamma_i(\varepsilon_i)} \bar{\gamma}_j(\varepsilon_j)) - 1 \right)
= \sum_{1 \leq i \neq j \leq 3} \left( \sum_{\varepsilon_p = \pm 1} \varepsilon_p \left( \sum_{\varepsilon_i, \varepsilon_j = \pm 1} (t^{\gamma_i(\varepsilon_i)} \bar{\gamma}_j(\varepsilon_j)) - 1 \right) \right) = 0,
\]
where $p$ is taken satisfying $\{i, j, p\} = \{1, 2, 3\}$. Similarly, the third term is equal to
\[
\sum_{1 \leq i \leq 3, 4 \leq j \leq n} \left( \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_i \varepsilon_j (t^{\gamma_i(\varepsilon_i)} \gamma_j(\varepsilon_j)) - 1 \right)
= \sum_{1 \leq i \leq 3, 4 \leq j \leq n} \left( \sum_{\varepsilon_p, \varepsilon_q = \pm 1} \varepsilon_p \varepsilon_q \left( \sum_{\varepsilon_i = \pm 1} \varepsilon_i (t^{\gamma_i(\varepsilon_i)} \gamma_j(\varepsilon_j)) - 1 \right) \right) = 0,
\]
where $p, q$ are taken satisfying $\{i, p, q\} = \{1, 2, 3\}$. The first, fourth, and fifth terms are similarly calculated to be zero.
Similarly to the case of $I_K(t)$, we have

$$
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 H_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t)
$$

\begin{align*}
&= \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 f_{00}(D_{\varepsilon_1 \varepsilon_2 \varepsilon_3}) + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 f_{00}(D_{\varepsilon_1 \varepsilon_2 \varepsilon_3}^\#) \\
&\quad - \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} (\omega + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) \varepsilon_1 \varepsilon_2 \varepsilon_3 (W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}}(t) - W_{K_{\varepsilon_1 \varepsilon_2 \varepsilon_3}^\#}(t)).
\end{align*}

By Lemma 9.1, the third sum is equal to zero. On the other hand, the first sum is equal to

$$
\sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{i=1}^{3} \varepsilon_i^2 (t^{\gamma_i} - 1) \right)
$$

\begin{align*}
&\quad + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{1 \leq i < j \leq 3} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1) \right) \\
&\quad + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{1 \leq i < j \leq n, 1 \leq j \leq 3} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1) \right) \\
&\quad + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \left( \sum_{1 \leq i \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1) \right).
\end{align*}

Each of the five terms is equal to zero. We omit the proof which is similar to that for $I_K(t)$. The second sum is obtained from the first by replacing $D_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$ with $D_{\varepsilon_1 \varepsilon_2 \varepsilon_3}^\#$ and is also equal to zero.

On the other hand, we consider the diagram $D$ with two double points as shown in Figure 29. It is easy to see that

$$
\sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 I_{K_{\varepsilon_1 \varepsilon_2}}(t) = I_{2.1}(t) - I_{O}(t) - I_{O}(t) + I_{2.1#}(t) = -2t + 4 - 2t^{-1}
$$

and

$$
\sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 II_{K_{\varepsilon_1 \varepsilon_2}}(t) = II_{2.1}(t) - II_{O}(t) - II_{O}(t) + II_{2.1#}(t) = -4t + 8 - 4t^{-1}.
$$

Since they are not equal to zero, we have the conclusion. \qed
10. Flat virtual knots

Let $D$ be a diagram with a double point $p_1$ and $n - 1$ crossings $c_2, \ldots, c_n$. For $\varepsilon_1 \in \{\pm 1\}$, we denote by $\gamma_1(\varepsilon_1)$ the cycle at $p_1$ in the diagram $D_{\varepsilon_1}$ on $\Sigma$.

Lemma 10.1. It holds that

$$\sum_{\varepsilon_1 = \pm 1} \varepsilon_1(f_{01}(D_{\varepsilon_1}) + f_{10}(D_{\varepsilon_1}) - f_{00}(D_{\varepsilon_1}) - f_{11}(D_{\varepsilon_1})) = 0.$$ 

Proof. Since $\gamma_1(-\varepsilon_1) = \overline{\gamma_1(\varepsilon_1)}$ and $\overline{\gamma_1(-\varepsilon_1)} = \gamma_1(\varepsilon_1)$ hold by definition, we have

$$\sum_{\varepsilon_1 = \pm 1} \varepsilon_1 f_{01}(D_{\varepsilon_1})$$

$$= \sum_{\varepsilon_1 = \pm 1} \varepsilon_1^2 (t^{\gamma_1(\varepsilon_1)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i \varepsilon_1 (t^{\gamma_i(\varepsilon_i)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1)$$

$$= \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 (t^{\gamma_1(\varepsilon_1)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_1(\varepsilon_1)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1),$$

$$\sum_{\varepsilon_1 = \pm 1} \varepsilon_1 f_{10}(D_{\varepsilon_1})$$

$$= \sum_{\varepsilon_1 = \pm 1} \varepsilon_1^2 (t^{\overline{\gamma_1(\varepsilon_1)} - 1})$$

$$+ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i \overline{\varepsilon_1} (t^{\overline{\gamma_i(\varepsilon_i)} - 1})$$

$$+ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1)$$

$$= -\sum_{\varepsilon_1 = \pm 1} \varepsilon_1 (t^{\overline{\gamma_1(\varepsilon_1)}} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \sum_{i=2}^{n} \varepsilon_i \overline{\varepsilon_1} (t^{\overline{\gamma_1(\varepsilon_1)}} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1),$$

$$\sum_{\varepsilon_1 = \pm 1} \varepsilon_1 f_{00}(D_{\varepsilon_1})$$

$$= \sum_{\varepsilon_1 = \pm 1} \varepsilon_1^2 (t^{\gamma_1(\varepsilon_1)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i \varepsilon_1 (t^{\gamma_i(\varepsilon_i)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1)$$

$$= \sum_{\varepsilon_1 = \pm 1} \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_1(\varepsilon_1)} - 1)$$

$$+ \sum_{\varepsilon_1 = \pm 1} \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i \gamma_j} - 1),$$

and
Lemma 10.2. Therefore we have the conclusion.

\[ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 f_{11}(D_{\varepsilon_1}) \]
\[= \sum_{\varepsilon_1 = \pm 1} \varepsilon_1^3 (\gamma_1(\varepsilon_1) - \gamma_1(\varepsilon_1)) - 1) + \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{j=2}^n \varepsilon_1 \varepsilon_j (\gamma_1(\varepsilon_1) - 1) + \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \sum_{2 \leq i, j \leq n} \varepsilon_i \varepsilon_j (\gamma_i - 1) \]
\[= \sum_{\varepsilon_1 = \pm 1} \sum_{j=2}^n \varepsilon_j (\gamma_1(\varepsilon_1) - 1) \]

Therefore we have the conclusion. \(\square\)

Lemma 10.2 ([25]). The Laurent polynomial \(W_K(t) - W_K(t^{-1})\) is invariant under a crossing change. \(\square\)

For the intersection polynomials, we have the following.

Theorem 10.3. The Laurent polynomial \(I_K(t) + I_K(t^{-1}) - 2I_K(t)\) is invariant under a crossing change.

Proof. Let \(\omega\) be the sum of signs of \(c_2, \ldots, c_n\) of \(D\). By Lemma 10.1

\[ \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 (I_{D_{\varepsilon_1}}(t) + I_{D_{\varepsilon_1}}(t^{-1}) - 2I_{D_{\varepsilon_1}}(t)) \]
\[= \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \{ f_{01}(D_{\varepsilon_1}) - (\omega + \varepsilon_1) W_{D_{\varepsilon_1}}(t) + f_{10}(D_{\varepsilon_1}) - (\omega + \varepsilon_1) W_{D_{\varepsilon_1}}(t^{-1}) \]
\[\quad - f_{00}(D_{\varepsilon_1}) - f_{11}(D_{\varepsilon_1}) + (\omega + \varepsilon_1)(W_{D_{\varepsilon_1}}(t) + W_{D_{\varepsilon_1}}(t^{-1})) \} = 0. \]

This induces \(I_{K_+}(t) + I_{K_+}(t^{-1}) - 2I_{K_+}(t) = I_{K_-}(t) + I_{K_-}(t^{-1}) - 2I_{K_-}(t) \). \(\square\)

A flat virtual knot \(F\) is an equivalence class of virtual knots under a crossing change. Equivalently, it is presented by an immersed circle in a closed, connected, oriented surface which has double points with no over/under-information up to stabilizations, destabilizations, and flattened Reidemeister moves [12, 28]. Let \(c(F)\) denote the crossing number of \(F\), which is the minimal number of double points for all immersed circles presenting \(F\). It is known that there are one flat virtual knot \(F(3, 1)\) of \(c(F) = 3\) and eleven flat virtual knots \(F(4, 1) - F(4, 11)\) of \(c(F) = 4\) up to mirror images and reversion (cf. [10, 18]), where Gauss diagrams of these flat virtual knots are illustrated in Figure 30.

There is a natural map \(\varphi : \{\text{virtual knots}\} \rightarrow \{\text{flat virtual knots}\}\) defined by ignoring the over/under-information of crossings. For each flat virtual knot \(F\) of \(c(F) \leq 4\), we give the list of virtual knots \(K\) of \(c(K) \leq 4\) which satisfy \(\varphi(K) = F\) as shown in Table 4 where we abbreviate \(K = 4n\) to \(n\) for \(1 \leq n \leq 108\). We remark that \(c(\varphi(K)) \leq c(K)\) holds for any virtual knot \(K\).

Lemma 10.2 and Theorem 10.3 imply that the Laurent polynomials \(W_K(t) - W_K(t^{-1})\) and \(I_K(t) + I_K(t^{-1}) - 2I_K(t)\) are invariants of a flat virtual knot \(F = \varphi(K)\).

Theorem 10.4. For the flat virtual knots \(F\) of \(c(F) \leq 4\), the Laurent polynomials \(W_K(t) - W_K(t^{-1})\) and \(I_K(t) + I_K(t^{-1}) - 2I_K(t)\) are given as in Table 4 where \(K\) is a virtual knot with \(F = \varphi(K)\). \(\square\)
Here, we use the following notations:

\[
\langle a_1 + a_2 + \cdots \rangle = a_1(t - t^{-1}) + a_2(t^2 - t^{-2}) + \cdots \quad \text{and}
\]

\[
[b_0 + b_1 + b_2 + \cdots] = b_0 + b_1(t + t^{-1}) + b_2(t^2 + t^{-2}) + \cdots.
\]
By observing Table 4, we see that these invariants classify the flat virtual knots $F$ of $c(F) \leq 4$ except two pairs \{O, $F(4.7)$\} and \{F(3.1), $F(4.6)$\}.

The supporting genus of a virtual knot $K$ is the minimal genus of $\Sigma$ for all diagrams $(\Sigma, D)$ of $K$, and denoted by $sg(K)$. We remark that $sg(K) = 0$ holds if and only if $K$ is a classical knot. Similarly, we can define the supporting genus $sg(F)$ of a flat virtual knot $F$.

**Lemma 10.5** ([9, 19]). There is no flat virtual knot $F$ of $sg(F) = 1$.

**Proof.** Let $C$ be an immersed circle in the torus $\Sigma = T^2$. By using flat Reidemeister moves and orientation-preserving homeomorphisms of $T^2$, $C$ can be deformed into the curve $C_n$ for some $n \geq 0$ as shown in Figure 31. We can apply a destabilization for $T^2$ so that $C_n$ presents the trivial flat virtual knot. \qed

**Figure 31**

\[ \begin{array}{ccc}
F(3.1) & 2 - 1 & -6 + 4 - 1 \\
F(4.1) & -1 + 2 - 1 & -4 + 1 + 2 - 1 \\
F(4.2) & 3 + 0 - 1 & -16 + 9 + 0 - 1 \\
F(4.3) & -1 - 1 + 1 & -4 + 1 + 2 - 1 \\
F(4.4) & 2 - 1 & 0 \\
F(4.5) & 0 & -8 + 2 + 4 - 2 \\
F(4.6) & 2 - 1 & -6 + 4 - 1 \\
F(4.7) & 0 & 0 \\
F(4.8) & 0 & -10 + 5 + 1 - 1 \\
F(4.9) & 0 & -6 + 4 - 1 \\
F(4.10) & 0 & -12 + 8 - 2 \\
F(4.11) & 0 & 12 - 8 + 2 \\
\end{array} \]

**Table 4**

**Corollary 10.6.** If a virtual knot $K$ satisfies $W_K(t) - W_K(t^{-1}) \neq 0$ or $I_K(t) + I_K(t^{-1}) - II_K(t) \neq 0$, then we have $sg(K) \geq 2$.

**Proof.** If $sg(K) \leq 1$ holds, then the flat virtual knot $F = \varphi(K)$ satisfies $sg(F) \leq 1$. By Lemma 10.5, $F$ is the trivial flat virtual knot and hence the invariants vanish. \qed
Example 10.7. We consider the virtual knot $K'_n$ $(n \geq 3)$ given in Example 5.5. It holds that

$$W_{K'_n}(t) - W_{K'_n}(t^{-1}) = -(t^{n-1} - t^{-n+1}) + (n - 1)(t - t^{-1})$$

and

$$I_{K'_n}(t) + I_{K'_n}(t^{-1}) - II_{K'_n}(t) = -(t^{n-1} + t^{-n+1}) + (n - 1)^2(t + t^{-1}) - 2n^2 + 4n.$$

Then we have $sg(K'_n) \geq 2$ by Corollary 10.6. On the other hand, since $G'_n$ is realized by a diagram on a closed surface of genus two, we have $sg(K'_n) \leq 2$, and hence $sg(K'_n) = 2$.

Remark 10.8. It is known that the writhe polynomial is invariant under a Delta move \cite{25}. We see that the first, second, and third intersection polynomials are also invariant under a Delta move. In fact, the proof is the same as that of the invariance under a third Reidemeister move in Lemma 2.4.

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APPENDIX A. Table of $W_K(t)$, $I_K(t)$, and $II_K(t)$.

Table 5 shows $W_K(t)$, $I_K(t)$, and $II_K(t)$ of a virtual knot $K$ up to crossing number four according to Green’s table [8] with a choice of orientations. We use the following notations:

\[
\begin{align*}
\{n\}(a_0 + a_1 + \cdots + a_m) &= a_0t^n + a_1t^{n+1} + \cdots + a_m t^{n+m} \quad \text{and} \\
|b_0 + b_1 + b_2 + \cdots| &= b_0 + b_1(t + t^{-1}) + b_2(t^2 + t^{-2}) + \cdots,
\end{align*}
\]

where $m \geq 1$ and $a_0 \neq 0$.

| $W_K(t)$ | $I_K(t)$ | $II_K(t)$ |
|-----------|----------|-----------|
| 2.1       | [2, 1]   | [2, 1]    | 4 - 2     |
| 3.1       | [−1](−1 + 1 + 1 − 1) | [0](−1 + 2 − 1) | 4 - 2     |
| 3.2       | [2, 1]   | [2, 1]    | 4 - 2     |
| 3.3       | [−1](−2 + 3 + 0 − 1) | [−1](-2 + 2 + 2 − 2) | 10 - 4 - 1 |
| 3.4       | [0](1 - 2 + 1) | 0          | 6 - 4 + 1 |
| 3.5       | [2, 0 + 1] | [0 + 2 − 2] | 0 + 4 - 4 |
| 3.6       | 0        | 0          | 0         |
| 3.7       | [2 + 0 − 1] | [4, 2]    | 8 - 4     |
| 4.1       | [4, 2]   | [4, 2]    | 20 - 12 + 2 |
| 4.2       | 0        | [4, 2]    | -4 + 4 - 2 |
| 4.3       | [4, 2]   | [−1](-3 + 7 + 5 − 1) | 14 - 8 + 1 |
| 4.4       | [2, 1]   | [0](-1 + 2 − 1) | 10 - 6 + 1 |
| 4.5       | [2, 1]   | [−1](-2 + 5 + 4 − 1) | -2 + 2 - 1 |
| 4.6       | 0        | [−1](-1 + 1 + 1 − 1) | 2 + 0 - 1 |
| 4.7       | [4, 2]   | [10 - 6 + 1] | 8 - 4     |
| 4.8       | 0        | [−2 + 2 − 1] | 8 - 4     |
| 4.9       | [2, 1]   | [−2 + 1]   | 2 + 2 + 1 |
| 4.10      | [−1](−1 + 1 + 1 − 1) | [−1](−1 + 0 + 3 − 2) | 0 + 2 - 2 |
| 4.11      | [−1](−2 + 3 + 0 − 1) | [−1](-2 + 2 + 2 − 2) | 10 - 4 - 1 |
| 4.12      | 0        | 0          | 0         |
| 4.13      | 0        | 0          | 0         |
| 4.14      | [0 - 1 + 1] | [−2 + 1] | 6 - 3 - 1 + 1 |
| 4.15      | [−1](-2 + 3 + 0 − 1) | [−2](1 - 4 + 2 + 4 - 3) | 4 + 0 - 2 |
| 4.16      | 0        | [−1](1 - 3 + 3 − 1) | 0         |
| 4.17      | [−1](-1 + 1 + 1 − 1) | [0](-1 + 2 - 1) | 4 - 2     |
| 4.18      | [2, 1]   | [2 - 1]   | 4 - 2     |
| 4.19 | \{-1\}(-1 + 1 + 1 - 1) | \{0\}(-1 + 2 - 1) | 4 - 2 |
| 4.20 | \{0\}(1 + 2 + 1) | 0 | 0 |
| 4.21 | \{-2\}0 + 1 | \{-2\}(-2 + 3 - 1 + 1 - 1) | 8 - 1 - 4 + 1 |
| 4.22 | \{-1\}(-1 + 1 + 0 + 1) | 0 | 4 - 1 - 2 + 1 |
| 4.23 | \{-1\}(-1 + 1 + 1 - 1) | \{-1\}(-1 + 0 + 3 - 2) | 0 + 2 - 2 |
| 4.24 | \{-2\}(1 + 0 - 1 + 1) | \{-2\}(-1 + 1 - 2 + 3 + 1 - 2) | 0 + 3 - 2 - 1 |
| 4.25 | \{4 - 2\} | \{-1\}(-3 + 7 - 5 + 1) | 14 - 8 + 1 |
| 4.26 | \{-1\}(-1 + 0 + 2 + 0 - 1) | \{-1\}(1 - 3 + 3 - 1) | 10 - 5 - 1 + 1 |
| 4.27 | \{2 - 1\} | \{-1\}(-2 + 3 + 0 - 1) | 6 - 2 - 1 |
| 4.28 | \{-1\}(2 - 2 - 1 + 0 + 1) | \{0\}(-3 + 4 + 1 - 2) | 10 - 5 + 1 - 1 |
| 4.29 | \{-1\}(-2 + 3 + 0 - 1) | \{-1\}(-2 + 2 + 2 - 2) | 10 - 4 - 1 |
| 4.30 | \{2 - 1\} | \{1\}(-1 + 2 - 1) | 10 - 6 + 1 |
| 4.31 | 0 | 0 | 0 |
| 4.32 | \{-1\}(-1 + 1 + 1 - 1) | \{2 - 1\} | 4 - 2 |
| 4.33 | \{2 - 1\} | \{2 - 1\} | 10 - 6 + 1 |
| 4.34 | \{0\}(1 + 2 + 1) | 0 | 6 - 4 + 1 |
| 4.35 | \{-1\}(-1 + 1 + 1 - 1) | \{2 - 1\} | 4 - 2 |
| 4.36 | \{-2\} + 0 + 1 | \{2 + 0 - 1\} | 4 + 0 - 2 |
| 4.37 | \{4 - 1 - 1\} | \{6 + 1 - 2\} | 12 - 2 - 4 |
| 4.38 | \{0\}(1 + 2 + 1) | \{1\}(2 - 4 + 2) | 4 - 4 + 2 |
| 4.39 | \{-1\}(-1 + 1 + 1 - 1) | \{1\}(-2 + 4 - 2) | 4 - 3 + 2 - 1 |
| 4.40 | \{2 - 1\} | \{2 - 1\} | 4 - 2 |
| 4.41 | 0 | 0 | 0 |
| 4.42 | \{0\}(1 + 1 + 1 - 1) | \{1\}(-1 + 2 - 1) | 4 - 2 |
| 4.43 | \{4 - 2\} | \{8 - 4\} | 16 - 8 |
| 4.44 | \{2 - 1\} | \{0\}(1 + 2 + 1) | 2 - 2 + 1 |
| 4.45 | \{-1\}(-2 + 2 + 1 + 0 - 1) | \{0\}(-2 + 3 + 0 - 1) | 12 - 6 |
| 4.46 | 0 | \{-1\}(-1 + 1 + 1 - 1) | 2 + 0 - 1 |
| 4.47 | \{-1\}(1 + 0 - 2 + 0 + 1) | \{-1\}(1 + 4 + 4 + 0 - 1) | 8 - 4 |
| 4.48 | \{4 - 1 - 1\} | \{2 + 1 - 2\} | 14 - 3 - 5 + 1 |
| 4.49 | \{0\}(1 - 2 + 1) | 0 | 6 - 4 + 1 |
| 4.50 | \{-1\}(-1 + 1 + 1 - 1) | \{-1\}(-1 + 0 + 3 - 2) | 0 + 2 - 2 |
| 4.51 | 0 | 0 | 0 |
| 4.52 | \{2 - 1\} | \{-2\} + 1 | \{2 - 1 + 2\} |
| 4.53 | \{4 - 2\} | \{6 - 4 + 1\} | 12 - 8 + 2 |
| 4.54 | \{2 - 1\} | \{-1\}(-2 + 3 + 0 - 1) | 6 - 2 - 1 |
| 4.55 | 0 | \{-4 + 2\} | \{4 - 4 + 2\} |
| 4.56 | 0 | \{0\}(2 - 4 + 2) | \{-8 + 4\} |
| 4.57 | \{-1\}(-1 + 1 + 1 - 1) | \{0\}(1 - 2 + 2 - 1) | \{4 - 2\} |
| 4.58 | 0 | \{-1\}(-1 + 1 + 1 - 1) | \{8 - 4\} |
| 4.59 | 0 | \{0\}(-1 + 1 + 1 - 1) | \{8 - 4\} |
| 4.60 | \{2 - 1\} | \{2 - 1\} | \{4 - 2\} |
| 4.61 | \{2 - 1\} | \{2 - 1\} | \{4 - 2\} |
| 4.62 | \{-2\}(-1 - 1 + 3 + 0 + 0 - 1) | \{-1\}(-3 + 4 + 0 + 0 - 1) | \{12 - 4 - 2\} |
| 4.63 | \{-1\}(-2 + 3 + 0 - 1) | \{-2\}(1 - 4 + 4 + 0 - 1) | \{8 - 4\} |
| 4.64 | \{0 - 1 + 1\} | \{2 - 1\} | \{4 - 2\} |
|   | 4.65 | 4.66 | 4.67 | 4.68 | 4.69 | 4.70 | 4.71 | 4.72 | 4.73 | 4.74 | 4.75 | 4.76 | 4.77 | 4.78 | 4.79 | 4.80 | 4.81 | 4.82 | 4.83 | 4.84 | 4.85 | 4.86 | 4.87 | 4.88 | 4.89 | 4.90 | 4.91 | 4.92 | 4.93 | 4.94 | 4.95 | 4.96 | 4.97 | 4.98 | 4.99 | 4.100 | 4.101 | 4.102 | 4.103 | 4.104 | 4.105 | 4.106 | 4.107 | 4.108 |
|---|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|   | [2 + 0 − 1] | [2 + 0 − 1] | [2 + 0 − 1] | 0 | 2 − 1 | [−2](1 + 0 − 1 + 0 + 1) | 0 | [−1](1 + 0 − 1 + 1) | 0 | 2 − 1 | 0 | 0 | 2 − 1 | [−2](1 − 3 + 2 + 1 − 1) | 0 | [−1](−1 + 1 + 1 − 1) | 0 | 0 | 2 − 1 | [−2](−1 + 2 − 1 + 2 + 1) | 0 | [−1](−1 + 1 + 1 − 1) | 0 | [−2](−1 + 1 + 0 + 0 + 0 + 1) | [−2](−1 + 1 + 0 + 1 + 3 − 2) | 0 | 2 + 0 − 1 | [−2](−1 + 1 + 0 + 0 + 1 − 1) | 0 | 2 + 0 − 1 | 0 | 0 | 4 + 2 | 4 + 2 | 8 + 4 | 0 | 0 |
|   | 4 + 0 − 2 | 8 − 3 + 2 + 1 | 4 − 2 | 4 − 2 | 8 − 4 | 4 − 2 | 8 − 4 | 16 − 8 | 2 − 2 + 1 | 4 + 0 − 2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

**Table 5**

Appendix B. Table of $W_K(t)$, $f_{00}(D;t)$, $f_{11}(D;t)$, and $III_K(t)$.

Table B shows $W_K(t)$, $f_{00}(D;t)$, $f_{11}(D;t)$, and $III_K(t)$ of a virtual knot $K$ up to crossing number four with a choice of orientations. We remark that these polynomials are all reciprocal.

|   | $W_K(t)$ | $f_{00}(D;t)$ | $f_{11}(D;t)$ | $III_K(t)$ |
|---|---------|-------------|-------------|---------|
| 2.1 | 4 − 2   | −2 + 1      | −2 + 1      | −2 + 1   |
| 3.1 | 2 + 0 − 1 | 2 − 2 + 1  | 0           | 4 − 2    |
| 3.2 | 4 − 2   | 2 − 1       | −2 + 1      | 2 − 1    |
| 3.3 | 6 − 2 − 1 | −4 + 1 + 1 | −4 + 1 + 1  | 2 − 1    |
| 3.4 | 2 − 2 + 1 | 2 − 1       | 2 − 1       | 2 − 1    |
| 3.5 | 4 + 0 − 2 | −6 + 2 + 1  | −6 + 2 + 1  | −6 + 2 + 1 |
| 3.6 | 0       | 0           | 0           | 0        |
| 3.7 | 4 + 0 − 2 | 2 − 2 + 1  | 2 − 2 + 1   | 2 − 2 + 1 |
| 4.1 | 8 − 4   | −8 + 2 + 2  | −4 + 2      | 0 − 2 + 2 |
| 4.2 | 0       | 0 + 2 − 2   | −4 + 2      | 0 + 2 − 2 |
| 4.3 | 8 − 4   | −8 + 4      | −10 + 4 + 1 | 0        |
| 4.4 | 4 − 2   | 2 − 1       | 0 − 1 + 1   | 2 − 1    |
| 4.5 | 4 − 2   | −4 + 3 − 1  | −6 + 3      | 0 + 1 − 1 |
| 4.6 | 0       | 2 + 0 − 1   | 0           | 2 + 0 − 1 |
| 4.7 | 8 − 4   | −12 + 6     | −12 + 6     | −4 + 2   |
| 4.8 | 0       | 4 − 2       | 4 − 2       | 4 − 2    |
| 4.9 | 4 − 2   | −10 + 5     | −4 + 1 + 1  | −2 + 1   |
| 4.10 | 2 + 0 − 1 | 0          | −4 + 2      | 0        |
| 4.11 | 6 − 2 − 1 | 0 − 1 + 1  | −2 + 1      | 6 − 3    |
| 4.12 | 0       | 0           | 0           | 0        |
| 4.13 | 0       | −4 + 2      | 4 − 2       | −4 + 2   |
| 4.14 | 0 − 2 + 2 | 2 − 1      | 4 − 2 − 1 + 1 | 2 − 1 |
| 4.15 | 6 − 2 − 1 | −8 + 3 + 1 | −12 + 5 + 1 | −2 + 1   |
| 4.16 | 0       | 0           | 0           | 0        |
| 4.17 | 2 + 0 − 1 | 0          | 4 − 2       | 0        |
| 4.18 | 4 − 2   | −2 + 1      | −2 + 1      | −2 + 1   |
| 4.19 | 2 + 0 − 1 | 0          | 4 − 2       | 0        |
| 4.20 | 2 − 2 + 1 | −2 + 1      | 2 − 1       | −2 + 1   |
| 4.21 | −4 + 0 + 2 | −2 + 0 + 1 | 2 − 1 − 1 + 1 | −2 + 0 + 1 |
| 4.22 | 2 − 1 + 1 − 1 | 2 + 0 − 2 + 1 | 2 − 1 | 4 − 1 − 1 |
| 4.23 | 2 + 0 − 1 | −4 + 2      | 0           | −4 + 2   |
| 4.24 | −2 − 1 + 1 + 1 | −2 + 1 − 1 + 1 | −2 + 0 + 1 | 0 + 2 − 2 |
| 4.25 | 8 − 4   | −10 + 4 + 1 | −8 + 4      | −2 + 0 + 1 |
| 4.26 | 0 + 1 + 0 − 1 | 6 − 3 − 1 + 1 | 4 − 2 | 6 − 2 − 1 |
| 4.27 | 4 − 2   | −2 + 1      | 0 − 1 − 1   | −2 + 1   |
| 4.28 | −4 + 1 + 0 + 1 | 2 − 2 + 0 + 1 | 0 − 1 + 1 | 6 − 3 |
| 4.29 | 6 − 2 − 1 | −6 + 1 + 2  | −8 + 3 + 1  | 6 − 3    |
| 4.30 | 4 − 2   | 0 − 1 + 1   | 2 − 1       | 0 − 1 + 1 |
| 4.31 | 0       | 4 − 2       | −4 + 2      | 4 − 2    |
| 4.32 | 2 + 0 − 1 | 4 − 2       | 0           | 4 − 2    |
| 4.33 | 4 - 2 | -2 + 1 | 4 - 3 + 1 | -2 + 1 |
| 4.34 | 2 - 2 + 1 | 4 - 3 + 1 | 2 - 1 | 2 - 1 |
| 4.35 | 2 + 0 - 1 | 0 | 4 - 2 | 0 |
| 4.36 | -4 + 0 + 2 | -2 + 0 + 1 | -2 + 0 + 1 | -2 + 0 + 1 |
| 4.37 | 8 - 2 - 2 | -10 + 3 + 2 | -10 + 3 + 2 | -2 + 1 |
| 4.38 | 2 - 2 + 1 | -2 + 1 | 2 - 1 | -2 + 1 |
| 4.39 | 2 - 1 + 1 - 1 | 0 + 0 - 1 + 1 | 0 - 1 + 1 | 2 - 1 |
| 4.40 | 4 - 2 | 2 - 1 | 2 - 1 | 2 - 1 |
| 4.41 | 0 | 0 | 0 | 0 |
| 4.42 | 2 - 1 - 1 + 1 | 0 | 4 - 2 | 0 |
| 4.43 | 8 - 4 | -8 + 4 | -8 + 4 | 0 |
| 4.44 | 4 - 2 | -4 + 1 + 1 | -2 + 1 | 0 - 1 + 1 |
| 4.45 | 4 - 1 + 0 - 1 | 2 - 2 + 0 + 1 | 2 - 2 + 0 + 1 | 6 - 1 - 2 |
| 4.46 | 0 | 0 | 2 + 0 - 1 | 0 |
| 4.47 | 0 - 1 + 0 + 1 | 4 - 2 | 4 - 2 | 4 - 2 |
| 4.48 | 8 - 2 - 2 | -6 + 2 + 0 + 1 | -12 + 3 + 3 | -6 + 2 + 0 + 1 |
| 4.49 | 2 - 2 + 1 | 0 + 1 - 1 | 2 - 1 | 2 - 1 |
| 4.50 | 2 + 0 - 1 | 0 | -4 + 2 | 0 |
| 4.51 | 0 | -4 + 2 | 4 - 2 | -4 + 2 |
| 4.52 | 4 - 2 | -2 + 1 | 4 - 3 + 1 | -2 + 1 |
| 4.53 | 8 - 4 | -10 + 4 + 1 | -10 + 4 + 1 | -2 + 0 + 1 |
| 4.54 | 4 - 2 | 0 + 1 - 1 | -2 + 1 | 0 + 1 - 1 |
| 4.55 | 0 | 2 - 2 + 1 | 2 - 2 + 1 | 2 - 2 + 1 |
| 4.56 | 0 | -4 + 2 | -4 + 2 | -4 + 2 |
| 4.57 | 2 + 0 - 1 | -2 + 0 + 1 | 2 - 2 + 1 | 0 |
| 4.58 | 0 | 4 - 2 | 4 - 2 | 4 - 2 |
| 4.59 | 0 | 4 - 2 | 4 - 2 | 4 - 2 |
| 4.60 | 4 - 2 | 2 - 1 | 2 - 1 | 2 - 1 |
| 4.61 | 4 - 2 | -6 + 3 | -6 + 3 | -2 + 1 |
| 4.62 | 6 - 1 - 1 - 1 | -2 + 0 + 0 + 1 | 2 - 2 + 0 + 1 | 4 - 1 - 1 |
| 4.63 | 6 - 2 - 1 | 0 - 1 + 1 | -4 + 1 + 1 | 6 - 3 |
| 4.64 | 0 - 2 + 2 | 2 - 1 | 2 - 1 | 2 - 1 |
| 4.65 | 4 + 0 - 2 | -2 + 0 + 1 | 2 - 2 + 0 + 1 | -2 + 0 + 1 |
| 4.66 | -2 + 1 + 1 + 1 | 4 - 1 - 2 + 1 | 4 - 2 | 6 + 0 - 3 |
| 4.67 | -2 + 0 + 1 | 4 - 2 | 0 | 4 - 2 |
| 4.68 | 0 | 0 | 0 | 0 |
| 4.69 | 4 - 2 | -6 + 3 | -6 + 3 | -2 + 1 |
| 4.70 | 2 + 0 - 1 | 2 - 2 + 1 | -2 + 0 + 1 | 4 - 2 |
| 4.71 | 0 | 4 - 2 | 4 - 2 | 4 - 2 |
| 4.72 | 0 | 4 - 2 | 4 - 2 | 4 - 2 |
| 4.73 | 8 - 4 | -8 + 4 | -8 + 4 | 0 |
| 4.74 | 4 - 2 | -4 + 1 + 1 | -2 + 1 | 0 - 1 + 1 |
| 4.75 | 0 | 2 + 0 - 1 | 2 + 0 - 1 | 2 + 0 - 1 |
| 4.76 | 0 | -2 + 2 - 1 | -2 + 2 - 1 | -2 + 2 - 1 |
| 4.77 | 0 | 4 - 2 | 4 - 2 | 4 - 2 |
| 4.78 | 6 - 1 - 1 - 1 | -12 + 4 + 1 + 1 | -8 + 2 + 1 + 1 | -6 + 3 |
Recall the virtual knot $K = 4.55$ in Example 8.14. It is known that $K$ satisfies the properties

(i) $W_K(t) = 0$,
(ii) $V_K(t) = 1$, where $V_K(t)$ is the Jones polynomial of $K$, and
(iii) $u^v(K) = 1$, where $u^v(K)$ is the virtual unknotting number of $K$.

Ohyama and Sakurai [24] construct an infinite family of virtual knots with the same properties (i)–(iii) as $K$, which can be distinguished by using their Miyazawa polynomials [21]. In this appendix, we construct another infinite family of virtual knots with the property

(iv) $R_K(A, \bar{x}) = -A^2 - A^{-2}$, where $R_K(A, \bar{x})$ is the Miyazawa polynomial of $K$.

### Table 6

| 4.79 | $2 - 1 + 1$ | $-2 + 2 - 1$ | $2 + 0 - 1$ | $-2 + 2 - 1$ |
|------|-------------|---------------|-------------|---------------|
| 4.80 | $8 - 3 + 0 - 1$ | $-6 + 1 + 1 + 1$ | $-6 + 1 + 1 + 1$ | $2 - 2 + 1$ |
| 4.81 | $4 - 3 + 0 + 1$ | $4 - 1 - 1$ | $4 - 1 - 1$ | $4 - 1 - 1$ |
| 4.82 | $8 - 2 - 2$ | $-8 + 3 + 1$ | $-12 + 3 + 3$ | $-8 + 3 + 1$ |
| 4.83 | $4 - 1 + 0 - 1$ | $-2 + 0 + 0 + 1$ | $-2 + 1$ | $2 - 1$ |
| 4.84 | $0 + 2 - 2$ | $4 - 1 - 1$ | $0 - 1 + 1$ | $4 - 1 - 1$ |
| 4.85 | $4 + 0 - 2$ | $0 - 1 + 0 + 1$ | $0$ | $0 - 1 + 0 + 1$ |
| 4.86 | $4 + 0 - 2$ | $6 - 2 - 1$ | $2 - 2 + 1$ | $6 - 2 - 1$ |
| 4.87 | $8 - 1 - 2 - 1$ | $-10 + 2 + 2 + 1$ | $-10 + 2 + 2 + 1$ | $-2 + 1$ |
| 4.88 | $0 + 1 - 2 + 1$ | $2 - 1$ | $2 - 1$ | $2 - 1$ |
| 4.89 | $8 + 0 - 4$ | $-8 + 1 + 2 + 1$ | $-8 + 1 + 2 + 1$ | $0 + 1 - 2 + 1$ |
| 4.90 | $0$ | $4 - 2$ | $4 - 2$ | $4 - 2$ |
| 4.91 | $8 - 2 + 0 - 2$ | $-12 + 3 + 2 + 1$ | $-12 + 3 + 2 + 1$ | $-12 + 3 + 2 + 1$ |
| 4.92 | $4 + 0 + 0 - 2$ | $-10 + 2 + 2 + 1$ | $-10 + 2 + 2 + 1$ | $-10 + 2 + 2 + 1$ |
| 4.93 | $4 - 1 - 2 + 1$ | $0$ | $0$ | $0$ |
| 4.94 | $4 - 2$ | $-6 + 3$ | $-6 + 3$ | $-2 + 1$ |
| 4.95 | $4 + 0 + 0 - 2$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ |
| 4.96 | $4 + 0 - 2$ | $0$ | $-4 + 2$ | $0$ |
| 4.97 | $0 - 1 + 0 + 1$ | $2 + 0 - 1$ | $2 - 1 + 1$ | $2 + 0 - 1$ |
| 4.98 | $0$ | $4 - 2$ | $4 - 2$ | $4 - 2$ |
| 4.99 | $0$ | $8 - 4$ | $8 - 4$ | $8 - 4$ |
| 4.100 | $8 - 4$ | $-8 + 4$ | $-8 + 4$ | $0$ |
| 4.101 | $4 + 0 + 0 - 2$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ |
| 4.102 | $0 - 2 + 0 + 2$ | $4 - 1 - 2 + 1$ | $4 - 1 - 2 + 1$ | $4 - 1 - 2 + 1$ |
| 4.103 | $4 + 1 - 2 - 1$ | $0 - 1 + 0 + 1$ | $0 - 1 + 0 + 1$ | $4 + 0 - 2$ |
| 4.104 | $-4 + 0 + 0 + 2$ | $6 - 2 - 2 + 1$ | $6 - 2 - 2 + 1$ | $6 - 2 - 2 + 1$ |
| 4.105 | $0$ | $-8 + 4$ | $-8 + 4$ | $-8 + 4$ |
| 4.106 | $4 + 0 - 2$ | $-6 + 2 + 1$ | $-2 + 2 - 1$ | $-6 + 2 + 1$ |
| 4.107 | $0$ | $0 + 2 - 2$ | $0 + 0 - 2 + 2$ | $0 + 2 - 2$ |
| 4.108 | $0$ | $0$ | $0$ | $0$ |
in addition to (i)–(iii). We will distinguish the virtual knots by using their intersection polynomials. Let $K_n$ ($n \geq 0$) be the virtual knot presented by the Gauss diagram $G_n$ as shown in Figure 32.

Figure 32

The writhe polynomial and the intersection polynomials of $K_n$ are given as follows:

\[
W_{K_n}(t) = 0,
\]

\[
I_{K_n}(t) = -t^{2n+2} - t^{2n+1} + 2t + 2 - t^{-2n} - t^{-2n-1},
\]

\[
II_{K_n}(t) = -(t^{2n+2} + t^{-2n-2}) - 2(t^{2n+1} + t^{-2n-1}) - (t^{2n} + t^{-2n}) + 2(t + t^{-1}) + 4,
\]

where we have $W_{K_n}(t) = 0$. Therefore $K_n$ satisfies the property (i) and $K_n \neq K_m$ for any $n \neq m$.

By removing the horizontal chord of $G_n$, the obtained diagram presents the trivial knot so that $K_n$ satisfies the property (iii).

Since the Jones polynomial is derived from the Miyazawa polynomial, it is sufficient to prove the property (iv). By applying a skein relation for the chord in the middle of the Gauss diagram $G_n$, we have $R_{K_n}(A, \vec{x}) = -A^2 - A^{-2}$ immediately.

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