Establishing the Uniqueness of the Connection between AdS$_5$ and Conformally Invariant Relativistic Systems: A Group/Field Theoretical Approach

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Abstract

Adopting as working assumption that the conformal group O(4,2) of Minkowski space, being the largest symmetry group which respects its light cone structure, is the appropriate global symmetry underlying the description of relativistic systems, it is shown that AdS$_5$ uniquely emerges as the space on the boundary of which a corresponding relativistic field system should be accommodated. The basic mathematical tools employed for establishing this result are (a) Cartan’s theory of spinors and (b) group contraction methods. Extending our considerations to supersymmetry it is demonstrated how an $N=1$ SUSY YM field system can emerge as a broken version of an $N=4$ SUSY YM field system. An especially important feature of the presentation is the ‘unearting’ of seminal, independent from each other, works of I. Segal and of S. Fubini which give a purely field theoretical perspective on the intimate relation between conformally invariant relativistic field theories and AdS$_5$ including, in particular, the warping phenomenon.
1 Introductory Remarks

The conformal group of Minkowski space-time is the largest symmetry group which preserves its light cone structure. In this sense, it can be said that conformal invariance is the maximal symmetry compatible with a four-dimensional spacetime which does not admit absolute simultaneity. Generally speaking, any $n$-dimensional (pseudo)Euclidean, space $E_{m,n-m}$, $m \leq n$, has $O(m+1,n-m+1)$ as its corresponding conformal symmetry group$^1$. A given physical system, formulated in $E_{m,m-n}$ and so constructed as to be symmetric under transformations induced by its conformal group, has extremely stringent properties the most characteristic aspect of which is that it does not allow, by definition, the introduction of any a priori given scale(s). Conformal symmetry has proven itself a valuable tool in specific situations such as the theoretical analyses of scattering processes at very high energies, but more importantly, it plays a central role in efforts to attack fundamental theoretical issues from a global perspective. Historically speaking, perhaps the best known mathematical construction which admits conformal flatness is that of Penrose [1], which is expressed in the language of twistors. A lesser known space-time scheme that adopts conformal invariance (in four space-time dimensions) is Segal's chronogeometry [2], which is formulated in a more conventional language. In our times, string theory dominates efforts aiming at the unified description of our physical cosmos at a fundamental, microscopic level. In this context, the game widens so that in addition to space-time the so-called, in the old language, internal type symmetries are included as well. The AdS/CFT conjecture [3], in particular, relates certain conformally invariant field systems, defined on the (four dimensional) boundary of AdS$_5$ space, to corresponding (super)string theories defined inside AdS$_5$, modulo an ‘internal’ space. A primary example is the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, in the large N limit, on the boundary of AdS$_5$, being dual to a Type IIB superstring theory ‘living’ inside AdS$_5(\times S^5)$.

The central objective in this work goes, in a sense, the opposite way. The idea is to explore the geometrical profile of a, generic, conformally symmetric field system in Minkowski space

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$^1$Given the fact that spinors enter our analysis in a basic way, we shall, throughout this exposition, refer to full orthogonal groups instead of their simply connected components; e.g. $O(3)$ instead of $SO(3)$, unless we are explicitly referring to the proper part and/or its covering.
and determine, constructively, how such a field system finds a natural accommodation on the boundary of AdS$_5$. The relevant analysis will heavily rely on group theory, especially through the utilization of the method of group contractions. At the same time, it will extensively employ Cartan’s theory of spinors [4] as a fundamental mathematical tool. According to Cartan, spinors, for a given (pseudo)Euclidean space, are described by coordinates which can be viewed, in a sense, as ‘square roots’ of a set of tensors$^2$ of various degrees belonging to a Clifford algebra associated with the (pseudo)Euclidean space. In different words, the said tensors can be expressed as bilinears in spinorial coordinates, an occurrence which, among other things, can be used as a criterion for identifying positive definite elements of the Clifford algebra. Utilizing these tools the connection between AdS$_5$ and conformal field systems will emerge in a natural manner and new insights regarding the AdS/CFT duality conjecture will be gained.

The exposition in this paper is organized as follows. In section 2 we shall direct our thinking towards mapping a course whose starting point is the adoption of the conformal symmetry group O(4,2) of the Minkowski spacetime as the underlying symmetry characterizing a given relativistic (field theoretical) system of interest and subsequently devises a systematic reduction procedure, guided by the following requirement: The proper relativistic Hamiltonian, equivalently, time development operator of the constructed system is the maximally positive generator of the group$^3$. As it will turn out, via the utilization of Cartan’s theory of spinors, the realization of such a goal invariably passes through the anti de Sitter space AdS$_5$ which, once compactified, acquires a local Minkowskian structure at its boundary. The intermediary role of AdS$_5$ as fundamental component of relativistic descriptions which recognizes the necessity to distinguish between the adequacy of the Poincaré group for ‘local’ relativistic descriptions and a ‘takeover’ by a maximally positive generator of the conformal group O(4,2), at very large distances, was recognized long time ago by Segal[2] whose, relevant, chronogeometric theory will be discussed in Section 3. Operating independently, in a field theoretical context, Fubini [5] came into exactly the same realization guided by his

$^2$This is related to the fact that spinors are fundamentally associated with reflections and two reflections amount to a rotation.

$^3$Clearly, this Hamiltonian should tend, for any local measurement, to the conventional relativistic one, i.e., in group theoretical terms, to the Poincaré generator $P_0$. 

3
interest to determine the appropriate manner by which an, originally adopted, conformally symmetric relativistic field system should break spontaneously in order to accommodate realistic descriptions of physical processes associated, e.g., with particle masses. This approach will be considered in Section 4. An extension to a, corresponding, supersymmetric scenario will be subsequently presented in Section 5, where, following Fubini’s spontaneous symmetry breaking approach, an explicit construction will be presented which demonstrates how a, conformally invariant, \( N=4 \) supersymmetric Yang Mills system naturally breaks into, for example, \( N=1 \) super YM one.

2 Some Mathematical Preliminaries

Consider some \( n \)-dimensional (pseudo)Euclidean, space \( E_{m,n-m}, m \leq n \). One associates with it a \( 2^n(2^{n-1}) \) dimensional spinor space \( S \) for \( n \) even(odd). At the same time a corresponding Clifford algebra \( \mathcal{C}_n \) can be constructed whose only non-trivial, finite dimensional irreducible representation is given in terms of \( 2^n \times 2^n \left( 2^{\frac{n-1}{2}} \times 2^{\frac{n-1}{2}} \right) \) matrices. The \( 2^n((2^{\frac{n-1}{2}})-\)dimensional algebra \( \mathcal{C}_n \) is so organized as to contain the unit scalar, \( n \) one-vectors, \( \frac{n(n-1)}{2} \) two-vectors, etc. up to and including the unit pseudoscalar (\( n \)-vector). The one-vectors of \( \mathcal{C}_n \) are in a 1-1 correspondence with the elements of a vector base of the underlying (pseudo)Euclidean space \( E_{m,n-m} \) in the spinorial representation. The 2-vectors correspond to rank 2 tensors and so on up to and including the unit pseudoscalar, which is identified with the \( n \)-vector. The \( 2^n \times 2^n \left( 2^{\frac{n-1}{2}} \times 2^{\frac{n-1}{2}} \right) \) irreducible representation of the Clifford algebra elements establishes a common language between spinors and tensors. Finally, the 2-vectors of \( \mathcal{C}_n \) are in one to one correspondence with the generators of the rotation group of the underlying (pseudo)Euclidean space \( E_{m,n-m} \) (spinorial representation thereof).

In Ref. [6] a lemma was proved, for the particular case of the group \( O(4,2) \), according to which given two spinors \( \xi, \xi' \) in \( S \) there always exist bilinear forms \( Y_{AB \ldots C}(\xi, \xi') \), indices running through the values\(^4\) 0,1,2,3,5,6 and such that \( A < B < \cdots < C \), which form the components of some \( p \)-vector. Furthermore, it has been shown that, if instead of two different spinors one uses components of a single spinor \( \xi \) to form bilinear expressions, one

\(^4\)Spinor component indices are represented by capital Latin letters.
can only form components of a 2-vector, a 3-vector and a 6-vector (unit pseudoscalar). Of crucial importance to the proof of the lemma is the involvement of the conjugation matrix \( J \), the analogue of \( \gamma_0 \) for the Minkowski case, which takes spinor \( \xi \) to its conjugate spinor \( \bar{\xi}(\equiv \xi^TJ, \text{for real spinors}) \). As Cartan establishes, \( J \) is a \( p \)-vector formed by the Clifford product among all the 1-vectors which correspond to the (pseudo)Euclidean directions with positive signature, \textit{i.e.} the ‘time’ directions\(^5\) in physics language. In the case of \( O(4,2) \) \( J = \beta_0\beta_6 \equiv J_{06} \), where \( \beta_0 \) and \( \beta_6 \) are the Clifford 1-vectors assigned to the 0- and 6-direction, respectively.

A result of utmost importance from Cartan is the following: Among all elements of \( C_n \) the one which is given as a maximal, positive definite bilinear expression in terms of spinorial components is precisely \( J \). The proof [6], basically rests on the fact that the bilinear form \((-1)^j\xi^T JX_{(p)}\xi\), which applies to any \( p \)-vector \( X_{(p)} \) and where \((-1)^j\) is a phase-factor associated with the reflection of \( X_{(p)} \) with respect to a given \( E(4,2) \) vector \( \vec{a} \), becomes positive definite only when \( J \) is substituted for \( X_{(p)} \), as it so happens [4,6] that \( J^2 = (-1)^j \) for the pseudo-Euclidean space \( E(4,2) \). This means that the maximal positive definite element happens to be a 2-vector \textit{hence a generator of the group} \( O(4,2) \). Note, in passing, that neither for \( O(5,1) \) nor for \( O(3,3) \) is the maximal positive definite bilinear a generator of the corresponding “rotation” group. Given that for \( O(4,2) \) the generator \( J_{06} \) has a (maximally) positive definite spectrum, we shall adopt, as a working hypothesis, that \textit{it} should be one’s choice for representing the energy operator of a conformally invariant system, a choice which, of course, is subject to falsification. Let us also mention that, as it turns out (see relevant footnote in section 5), the conventional Minkowski space energy operator \( P_0 \) is also positive definite, but it is only a part of \( J_{06} \) in the sense that whereas the latter generator is a positive definite bilinear (in spinorial coordinates) composed of eight terms, \( P_0 \) is given by a subset of only four of them. From a, local, field theoretical point of view the motivation for testing the viability of \( J_{06} \) as the energy operator of a relativistic system will be based on evidence, see, for example, Ref. [2], that its physical descriptions locally coincide with those associated with \( P_0 \). At very large distances, on the other hand, it takes over -being a basic ingredient of conformal

\(^5\)Our convention for the Minkowski metric, which appropriately adjusts to the other, higher dimensional spaces entering our analysis, is +,-,-,-.
symmetry— as the proper energy operator, equivalently, time development, generator. An
observation of note is that $J_{06}$ enters as a generator of a homogeneous group, as opposed to
$P_0$ which belongs to the ‘inhomogeneous’ sector of the Poincaré group.

Our findings, to this point, suggest that the energy operator associated with confor-
mally invariant descriptions in Minkowski space-time should correspond to the generator of
rotations with respect to the 0-6 directions in a five dimensional ‘sphere’ $S_{4,2}$ specified by

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_5^2 + \eta_6^2 = \text{const} \ (1)$$

and as such it generates the $O(2)$ factor of the maximally compact connected subgroup of
$O(4,2)$, which is isomorphic to $O(4) \times O(2)$. Now, the rank of the group $O(4,2)$ is 3. This
entails the presence of three Casimir operators. Consistency with the, local, Minkowski
space instruction that elementary particle entities need two Casimir operators for their full
specification calls for a reduction from $O(4,2)$ to a rank two subgroup which properly char-
acterizes particle entities in a given local measurement. To this end, we shall enlist the
aid of the method of group contractions from a (pseudo)orthogonal $O(p,q)$ to an inhomo-
geneous (pseudo)orthogonal group acting on a space with one less homogeneous dimension.
Specifically, one has

$$O(p, q) \rightarrow IO(p, q - 1) \text{ or } IO(p - 1, q), \quad p \neq q. \ (2)$$

We recall that the process of contraction has the following picture. Given an $O(p,q)$-
variant hypersphere one imagines a locally perpendicular patch to a given direction, stretch-
ing to infinity so that the whole, $O(p,q)$-invariant configuration tends to a $(p+q-1)$-dimensional
flat space with $IO(p,q-1)$, or $IO(p-1,q)$, its group of isometries. The particular outcome de-
pends on the orientation of the patch being stretched. In this way some of the rotation
generators become translational ones. In the present case the requirement that the genera-
tor $J_{06}$ remains intact uniquely points to the contraction $O(4,2) \rightarrow IO(3,2)$. It will be now
demonstrated that this commitment will be realized once the constant appearing on the rhs
of Eq. (1) has a negative value, i.e. the original $O(4,2)$-invariant hypersphere is given by

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_5^2 + \eta_6^2 = -R^2. \ (3)$$
To this end, let us recall that the contraction process results through the following procedure. One considers standing in the vicinity of the “north pole”, \((0, 0, 0, 0, R, 0)\), where \(\eta_5\) has been chosen as the “north” direction\(^6\).

Given, now, the generators \(J_{AB} = i \left( g_{AC} \eta^C \frac{\partial}{\partial \eta^B} - g_{BC} \eta^C \frac{\partial}{\partial \eta^A} \right)\) of \(O(4,2)\) one redefines them by setting \(Y_{\alpha\beta} = J_{\alpha\beta}\), if \(J_{\alpha\beta}\) does not involve \(\eta_5\) and \(P_\alpha = \frac{1}{R} J_{\alpha\beta} \eta_5\), with \(\alpha = 0, 1, 2, 3, 6\) and \(\beta = 5\). Upon taking the limit \(R \to \infty\), the hypersurface tends towards a flat, 5-dimensional pseudo-Euclidean space \((\text{AdS}_5)\) and one obtains the algebra of \(IO(3,2)\):

\[
[Y_{\alpha\beta}, Y_{\gamma\delta}] = i\{g_{\alpha\delta} Y_{\beta\gamma} - g_{\alpha\gamma} Y_{\beta\delta} + g_{\beta\gamma} Y_{\alpha\delta} - g_{\beta\delta} Y_{\alpha\gamma}\}
\]

\[
[P_\alpha, P_\beta] = 0
\]

\[
[Y_{\alpha\beta}, P_\gamma] = i\{g_{\beta\gamma} P_\alpha - g_{\alpha\gamma} P_\beta\},
\]

with \(g_{\alpha\beta} = \text{diag}(+,-,-,-,+)\). In other words, by having set \(\text{const} = -R^2\) in Eq. (1) it has been ascertained that one of the negative signature directions, \(\eta_5\) in our case, has been eliminated\(^7\). Finally, note should be taken of the fact that mathematical consistency requires that the \(O(4,2)\)-invariant sphere should, in its Euclidean version, be ‘punctured’ at a point, e.g. “south pole” in order for our construction to achieve the asymptotic flatness.

The homogeneous part of the contracted group acts naturally on an \(SO(3,2)\)-invariant, four-dimensional ‘hypersphere’ \(S_{3,2}\). The latter can be projected onto the original \(O(4,2)\)-invariant ‘hypersphere’ \(S_{4,2}\) anywhere on a locus which will appear as a “trajectory” of \(S_{3,2}\) in \(S_{4,2}\)\(^8\).

The inhomogenous part of the contraction pertains to translational generators in \(\text{AdS}_5\). Of utmost importance is the fact that the rank-2 symmetry subgroup has two Casimir operators and contains \(J_{06}\) as one of its generators. The original \(O(4,2)\) symmetry, is expected to still be operational in one way or other and this matter will draw a considerable portion of our attention throughout this work.

For now let us make a first connection with the \(\text{AdS/CFT}\) duality scenario according to which the conformal field theory component ‘lives’ on the, four dimensional, boundary

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\(^6\)The patch is locally tangential to the pole.

\(^7\)In the sense that the group contraction process reduces the homogeneous dimension of the space-time manifold by one, while compensating via the introduction of space-time translation generators.

\(^8\)In the sense of the Euclidean analogue of \(S_n/S_{n-1} \simeq S_1\)
of AdS$_5$. Accordingly, a compactification procedure is called for. This matter has been
given special attention by Witten in [7] (see also [8]) who, working in Euclidean formalism,
has dealt with the issue by adding a point at infinity which accomplishes the task. This
act, in the presently advocated scheme, we interpret as ‘putting back’ the point that was
‘taken out’ during the contraction procedure. Finally, the inhomogeneous part of IO(3,2)
commutation relations refer, for the compact version of AdS$_5$, to translations in the interior
of the ball, while O(3,2) is associated with rotations of a five-dimensional sphere. Finally, the
local Minkowski character of $S^{3,2}$ emerges through the contraction $O(3,2) \to IO(3,1)$. This
implies that in the flat limit, which is equivalent to saying ‘locally’, our universe becomes
Minkowski space and our geometrical group contracts to that of Poincaré$^9$.

In closing this section let us make a quantitative remark relating the O(3,2)-invariant
hypersphere to the O(4,2)-invariant one with which we started. As already pointed out,
for a fixed value of $|R^2|$ an O(3,2)-invariant hypersphere $S_4$, in Euclidean version, can be
placed anywhere on a one dimensional circular trajectory in $S_5$. A given choice, of course,
fixes a specific 4-dimensional ‘sphere’ $S^{3,2}$. Let, now, $r'$ be the radius of the aforementioned
‘trajectory’ corresponding to the case where the ‘pole point’ $\eta_5$ is inserted on the rhs of
(1), while $r$ the respective radius when $R(< \eta_5)$ is inserted. We then have that $r/\eta_5 = R/\eta_5$.
Substituting into the equation which defines the hypersphere for the arbitrary value of $\eta_5$
one obtains

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 + \eta_6^2 = -R^2 + \frac{r'^2}{r^2}R^2. \quad (5)$$

Introducing the set of $\zeta$-coordinates, where $\zeta_a = \eta_a \frac{r}{\eta_5}$ one writes

$$\zeta_0^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2 + \zeta_6^2 = R^2 \left(1 - \frac{r'^2}{r^2}\right) \equiv a^2. \quad (6)$$

Notice that the positive definite character of $a^2$, which can be surmised from the fact that
$r < r'$ and $\text{sign} r = \text{sign} r'$, confirms that it is one of the directions 0 or 6 which ‘flattens up’
in the limit.

$^9$In fact, the contraction $O(3,2) \to IO(3,1)$ involves the mapping of one of the generators $J_{ab}$ with $b = 0$
or 6 into the translation operators $P_{\mu}$. 

8
3 Segal’s chronogeometry

Our discussion, to this point, has followed a general line of reasoning, which promoted a mathematical scenario according to which it is the conformal group of Minkowski space that describes globally, the spacetime symmetries of our world, while it is expected to approach Poincaré group based descriptions at local level. Our immediate obligation is to demonstrate that the differences between $J_{06}$ and $P_0$ are unobservably small for sufficiently local Minkowskian regions. To this end, we now turn our attention to Segal’s chronogeometry [2], which was developed by the author for, among other things, ‘rationalizing’ observational data regarding motions of stars at extragalactic distances. The, expected, significant departures between $J_{06}$- and $P_0$-based estimations of the velocity of distant stars turns out to overwhelmingly favor the former over the latter.

Following Segal we associate the generator of time development with that of rotations in the 0-6 plane. The corresponding ‘time’ parameter $\tau$ is thereby identified with the angle of such rotations. One writes

$$\xi_0 = \tan \tau,$$

with $-\pi < \tau < \pi$, a periodicity which brings to surface the well known problem regarding conformal invariance and causality. Its confrontation calls for reverting to the universal covering of the proper group SO(4,2), namely SU(2,2). In doing so the one parameter subgroup $\{ T_t \}$ generated by $J_{06}$ is covered an infinite number of times, equivalently, the SU(2,2) chronometric world becomes an infinite-sheeted four dimensional manifold $\tilde{M}$. A point on $\tilde{M}$ is described by a set of coordinates $(\tau, u_1, u_2, u_3, u_4)$, where $u = (u_1, u_2, u_3, u_4)$ is a point on a four-dimensional Euclidean sphere, a specification implied by the, local projective identification of the SO(3,2)-invariant hypersphere and the Minkowski space. Explicitly, the projective identification of the five-dimensional coordinates $(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_6)$, transforming like the components of an SO(3,2)-vector and the Minkowski coordinates is given by the relations

$$\zeta_\mu = \frac{2a^2x_\mu}{a^2 + x^2}, \quad \zeta_6 = \frac{a(a^2 - x^2)}{a^2 + x^2},$$

where $a$ is a fixed quantity with the dimension of length. It immediately follows that

$$\tan \tau = \frac{ax_0}{a^2 - x^2}.$$
Upon introducing
\[ u_j = \frac{a\zeta_j}{\zeta_0 + \zeta_0^2}, \quad j = 1, 2, 3 \] (10)
and
\[ u_4 = \frac{a^2}{\zeta_0 + \zeta_0^2}, \quad j = 1, 2, 3, \] (11)
one immediately obtains that \( \sum_{j=1}^{4} u_j^2 = a^2 \), i.e. \( a \) is the radius of a 4-dimensional Euclidean sphere\(^{10}\). Furthermore, one obtains the relations
\[ u_j = 2\lambda x_j, \quad u_4 = \frac{\lambda(a^2 + x^2)}{a}, \] (12)
where
\[ \lambda = \frac{a^2}{[(a^2 - x^2)^2 + 4a^2x_0^2]^{1/2}}, \] (13)
with \( x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \).

In this way one is able to relate the Minkowski coordinates to Segal’s chronometric ones \((\tau, u_j)\) on the, 4-dimensional O(3,2)-invariant, hypersphere, i.e. \( \text{AdS}_5 \) space. More precisely, once the measure of \( a \) is set to unity, the above relations give Segal’s mapping effecting the embedding of the Minkowski space in \( \tilde{M} \). The interested reader regarding basic issues such as causality, simultaneity, quantization, masses, etc. the is referred to Segal’s papers. As a specific example we here outline Segal’s derivation of a “red shift phenomenon” associated with distant observations in the “chronometric universe” \( \tilde{M} \) [2]. One starts by observing that the ‘time displacement’ (dually energy) operator \( J_{06} \equiv H \) can be split into two parts, \( H = H_0 + H_1 \). As it turns out, \( H_0 \), is scale covariant, while the second is anti-scale covariant and respectively identify with the generators of \( P_0 \) and \( K_0 \), i.e. the zero components of the translation and special conformal transformations. Each one is given as a positive definite quantity, i.e. as sum of four square terms (in spinorial coordinates). Between them they share the eight terms entering the expression for \( J_{06} \).

As it turns out, for local events, with respect to a given observer, the effects ‘evolving’ through \( H_1 \) are negligible. For large distances, on the other hand, there arise notable differences. Suppose, for example, that a photon has been emitted from a distant star. Its energy

\(^{10}\) Corresponding to the four spacelike directions.
will be measured locally and at the time of its emission is determined by the operator \( \hat{H}_0 \).

The development in chronometric time is given, in the Heisenberg picture, by

\[
H_0(\tau) = e^{-i\hat{H}_0\tau}H_0e^{i\hat{H}_0\tau}.
\]

(14)

Given the non vanishing of the commutator \([H, H_0]\), one writes, group theoretically,

\[
H_0(\tau) = \alpha H_0 + \beta H_1 + \gamma [H_0, H_1],
\]

(15)

where \( \alpha, \beta \) and \( \gamma \) are functions of \( \tau \). A red shift factor \( Z \) emerges once the expectation value of \( H_0(\tau) \) is compared with that of \( H_0 \). The following result is obtained [2]

\[
\langle H_0(\tau) \rangle = \frac{1}{1 + Z} \langle H \rangle,
\]

(16)

where \( Z = \tan^2 \frac{\xi}{2} \).

Plotting \( \log Z \) against cosmographical parameters, Segal obtains remarkable agreements with existing data. A similar red shift factor, going by the name of ‘warping’, rises in connection with the AdS/CFT duality scheme [8].

4 Dynamical aspects: Fubini’s field theoretical approach

The theoretical considerations developed in the previous section are, basically, of ‘geometrical’ nature. In this section we shall gain an alternative perspective on the general theme we have been consistently developing in the preceding considerations by taking a point of view which focuses on dynamical implications. In particular, we shall proceed to assess the viability of spontaneous symmetry breaking mechanisms operating on an, originally, conformally invariant field theoretical equations whose solutions exhibit a breakdown to conventional relativistic form, beyond a given energy regime. To this end we follow Fubini [5] by considering a (simple) system defined by the Lagrangian density

\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g\phi^4,
\]

(17)

which does not contain any dimensional parameter. We are interested in exploring nontrivial solutions for this system of the form

\[
\phi(x) = B(x) + \phi'(x),
\]

(18)
where $B(x)$ is a classical solution of the field equation and $\phi'(x)$ a small quantum disturbance such that $\langle 0 \mid \phi'(x) \mid 0 \rangle = 0$. Normalizing the vacuum state to unity we have

$$\langle 0 \mid \phi'(x) \mid 0 \rangle = B(x). \quad (19)$$

Now, being a classical solution, $B(x)$ satisfies the conformally invariant equation

$$\partial^\mu \partial_\mu B + 4gB^3 = 0. \quad (20)$$

In a search for particular, conformally invariant, solutions one is guided by symmetry considerations for the ground state. Specifically, given a generator $G_\kappa$, of the conformal group, expressed in differential form, a solution of (20) will be invariant under the action of $G_\kappa$, if

$$\langle 0 \mid [G_\kappa, B(x)] \mid 0 \rangle = 0 \quad (21)$$

holds true. One expects that only the trivial solution $B(x) = 0$ satisfies the above equation, if $G_\kappa$ runs through all generators of $O(4,2)$. If one requires that the invariance is with respect to all the Poincaré group generators, but not any of the rest, then the solution $B(x) =$const. is the most general one, as can be demonstrated by the action under $P_\mu$, i.e.

$$i \frac{\partial B(x)}{\partial x^\mu} = 0, \quad (22)$$

coinciding, as expected, with the solution of the free equation ($g = 0$).

Consider, now, the case where one demands invariance under the action of the generators $R_\mu = \frac{1}{2} \left( aP_\mu + \frac{1}{a} K_\mu \right)$, where $a$ enters as a fundamental length, necessary for balancing the units of the two terms entering the sum$^{11}$. It leads to the equation

$$\frac{a^2 + x^2}{2} \frac{\partial B(x^2)}{\partial x^\mu} + x_\mu B(x^2) = 0, \quad (23)$$

with the argument $x^2$ serving to take account of the fact that the Lorentz ‘rotation’ symmetry remains intact.

The solution of Eq. (22) is given by [5]

$$B(x^2) = \frac{1}{\sqrt{2g \ x^2 + a^2}}. \quad (24)$$

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$^{11}$Obviously the dimensional analysis pertains to the interpretation of the various operators in the context of Minkowski space.
In a set of six dimensional coordinates appropriate to \( \tilde{M} \) and associated with the covering group \( SU(2, 2) \), defined, in obvious notation, by 
\[
\begin{align*}
\nu_i v^i &= -2g, \\
\nu_\mu &= u_\mu, \\
u_5 &= \frac{1-x^2}{2}, \\
u_6 &= \frac{1+x^2}{2},
\end{align*}
\]
the above solution takes the simple form
\[
B(u) = (v_i u^i)^{-1}.
\tag{25}
\]

It turns out, that the six-vector \( v_i \) indicates the direction along which the \( O(4,2) \) symmetry breaks during the contraction to \( IO(3,2) \), e.g. the \( \eta_5 \)-direction for the procedure adopted in Section 2. One actually verifies [5] that for a positive value of \( g \) the six-vector \( v_i \) lies on the hyperboloid 
\[
v_i v^i = -R^2,
\]
respectively negative sign for the contraction to \( IO(4,1) \) (de Sitter space). A different way of assessing the situation we have just analyzed is to say that the breaking of the conformal symmetry towards the anti-deSitter vs. deSitter direction depends on the sign of the coupling constant \( g \).

It will now be demonstrated that the \( O(4,2) \) generator \( J_{06} \) coincides, via the contraction process, with the \( \frac{1}{2} \left( a P_0 + \frac{1}{a} K_0 \right) \) combination of \( IO(3,2) \) generators. To this end consider the relations in Eq. (8) which relate the Minkowski coordinates to the five coordinates \( \zeta_i \) for the \( O(4,2) \)-invariant sphere. In terms of the latter one writes
\[
J_{06} = i \left( \zeta_0 \frac{\partial}{\partial \zeta_6} - \zeta_6 \frac{\partial}{\partial \zeta_0} \right),
\tag{26}
\]
which yields
\[
J_{06} = \frac{i}{2} \left( a - \frac{x^2}{a} \right) \partial_0 + 2 \frac{\nu_0 x^\nu}{a} \partial_\nu \right].
\tag{27}
\]
This clearly coincides with \( \frac{1}{2} \left( a P_0 + \frac{1}{a} K_0 \right) \). It can be similarly shown that \( J_{6\mu} = \frac{1}{2} \left( a P_\mu + \frac{1}{a} K_\mu \right) \), while the remaining generators of \( O(3,2) \) coincide with the Lorentz ones, i.e. \( M_{\mu\nu} \).

It might be of interest, at this point to reproduce an argument by Segal [2], which confirms the positive definiteness of \( J_{06} \): As is well known, \( K_\mu = IP_\mu I \), where \( I \) is the inversion operator, which induces the, Minkowski space transformation \( x_\mu \rightarrow \frac{x_\mu}{x^2} \), on \( \tilde{M} \) space, where it appears as the singularity free transformation \( (\tau, u) \rightarrow (\pi - \tau, u) \). Hence, \( I \) is continuously connected with the time reversal transformation, i.e. it is represented by an antiunitary operator. Thus, \( IP_0 I \) has a positive spectrum when \( P_0 \) does.

The energy density in Fubini’s scheme is given by the expression
\[
E = \frac{1}{2} \left( a T_{00} + \frac{1}{a} K_{00} \right),
\tag{28}
\]

13
where $T_{\mu\nu}$ is the energy momentum tensor and $K_{\mu\nu}$ a local tensor current associated with the conformal charges. For the particular model under consideration it is given by

$$K_{\mu\nu} = 2x^\rho x^\nu T_{\mu\rho} - x^2 T_{\mu\nu} + 2x_\nu (\partial_\mu \phi) \phi.$$  

(29)

It is a straightforward task to calculate $E$ for the classical solution given by (23). One obtains

$$E = \frac{2a(x_E^2 + a^2)(a^2 - x^2)}{g(a^2 + x^2)^4},$$  

(30)

where $x_E^2$ denotes the Euclidean magnitude $x_0^2 + x_1^2 + x_2^2 + x_3^2$.

The corresponding relativistic expression, i.e. the one formulated in Minkowski space, gives

$$E' = \frac{2a^3(x_E^2 + a^2)}{g(a^2 + x^2)^4}.$$  

(31)

The difference $\Delta E = E - E'$ is seen to be positive on a space-like surface, another verification of the maximality of the energy associated with this approach. This red shift effect reproduces, once again the warping associated with the AdS/CFT duality scheme.

5 Supersymmetry Considerations

In this section we extend our considerations to supersymmetry. To begin, let us recall that the original version of supersymmetry [9] contained the conformal algebra of $\text{SO}(4,2)$ as an integral part, along with eight spinorial charges. We shall refer to this as the Wess-Zumino algebra and denote it by $\mathcal{W}$. The particular subalgebra of $\mathcal{W}$ which contains the Poincaré generators and only four spinorial charges was originally proposed by Volkov and Akulov [10] and will be denoted by $\mathcal{V}$. In the framework of the basic theme of this work we shall proceed to investigate possible advantages of the former over the latter. Now, in Ref [6] it was shown that the conformal algebra has a unique extension to the $\mathcal{W}$ supersymmetry, an extension which does not seem to hold between the Poincaré algebra and $\mathcal{V}$. Moreover, according to Haag et al [11], see also Ref. [6], it is only within the framework of $\mathcal{W}$ that it becomes possible to intertwine internal-type symmetries (R-symmetries in current language) with supersymmetry in a non-trivial way. A complete listing of all the supersymmetry algebras can be found in the classic work of Nahm [12].
Let us recall that \( \mathcal{W} \) is a 24-generator graded algebra spanned by the set of generators 
\( \{K_\mu(2), Q_\alpha(1), M_{\mu\nu}, D, \Pi(0), Q^0_\alpha(-1), P_\mu(-2)\} \), where the numbers in parentheses give corresponding grades. The immediate question is whether \( \mathcal{W} \) can be reorganized so as to define, along with the generators 
\( \{M_{\mu\nu}, R_\mu = \frac{1}{2} (aP_\mu + \frac{1}{a}K_\mu)\} \), a self-consistent algebraic structure. To this end, we introduce a new set of spinorial charges, \( \Xi_\alpha \), given by 
\[
\Xi_\alpha = \frac{1}{2} \left( \sqrt{a}Q^0_\alpha + \frac{1}{\sqrt{a}}Q^1_\alpha \right). 
\] (32)
It is a matter of simple algebra to show that 
\[
[\Xi_\alpha, R_\mu] = (\gamma_\mu)^{\beta}_\alpha \Xi_\beta \\
\{\Xi_\alpha, \Xi_\beta\} = \left[ (\gamma^{\mu\nu})^{\alpha\beta}R_\mu - \frac{1}{2} (\gamma^{\mu\nu}(\gamma_0)_{\alpha\beta}M_{\mu\nu}) \right], 
\] (33)
with the \( \gamma \)'s in the Majorana representation. Consequently, a graded algebra, to be referred to as \( \mathcal{Z} \), spanned by the set \( \{M_{\mu\nu}, R_\mu, \Xi_\alpha\} \) is formed, which is a subalgebra of \( \mathcal{W} \), non-isomorphic to \( \mathcal{V} \). This supersymmetric algebra has \( \text{O}(3,2) \) as its spacetime component and admits, according to Nahm’s classification \( \text{o}(N) \), \( N = 1, 2, \cdots \), as its ‘internal’ symmetry algebra. In connection with QCD and taking into account the analysis of Polchinski and Strassler [13], one scenario of considerable interest arises from the breaking of the, \( N=4 \), conformal super Yang Mills system to a corresponding \( N=1 \) supersymmetric one at some scale slightly above \( \Lambda_{\text{QCD}} \). Such a scheme is naturally accommodated by the \( \text{O}(4,2) \mapsto \text{O}(3,2) \) supersymmetry breaking procedure advocated in this section.

To relate this construction to the chronogeometric scheme one observes that\(^{12}\)
\[
H = 1/2(aP_0 + \frac{1}{a}K_0) = \frac{1}{4} \left[ \sum_{\alpha=1}^{4} \Xi^2 - \frac{1}{4} \sum_{\alpha=1}^{4} \{Q^0_\alpha, Q^1_\alpha\} \right]. 
\] (34)
But 
\[
\{Q^0_\alpha, Q^1_\beta\} = -2[(\gamma^{\mu\nu}(\gamma_0)_{\alpha\beta}M_{\mu\nu} - (\gamma_0)_{\alpha\beta}D + (\gamma_5\gamma_0)_{\alpha\beta}\Pi]. 
\] (35)
\(^{12}\)In terms of the spinorial charges \( Q^0_\alpha \) and \( Q^1_\alpha \) of \( \mathcal{W} \), \( P_0 \) and \( K_0 \) acquire the form 
\[
P_0 = 1/8 \text{Tr} \left[ (|Q^0_\alpha\rangle\langle Q^0_\alpha|)_{\alpha\beta} + (|Q^0_\alpha\rangle\langle Q^0_\alpha|)_{\beta\alpha} \right] \\
K_0 = 1/8 \text{Tr} \left[ (|Q^1_\alpha\rangle\langle Q^1_\alpha|)_{\alpha\beta} + (|Q^1_\alpha\rangle\langle Q^1_\alpha|)_{\beta\alpha} \right].
\]
However, in the Majorana representation $\gamma^{\mu\nu}\gamma_0$ is symmetric whereas $\gamma_0$ and $\gamma_5$ and $\gamma_0$ are antisymmetric. Hence

$$\{Q^0_\alpha, Q^1_\alpha\} = -2[(\gamma^{\mu\nu}\gamma_0)_{\alpha\alpha}]M_{\mu\nu}. \tag{36}$$

Consequently, the vanishing of the vacuum expectation value of $H$ leads to

$$0 = \langle 0 | \sum_{\alpha=1}^{4} \Xi^2_\alpha + \frac{1}{2}\text{Tr}(\gamma^{\mu\nu}\gamma_0)M_{\mu\nu} | 0 \rangle. \tag{37}$$

Imposing Lorentz invariance on the theory, \textit{i.e.} setting $M_{\mu\nu} | 0 \rangle = 0$, one obtains $\langle 0 | \Xi^2_\alpha | 0 \rangle = 0$, which implies that $\Xi_\alpha | 0 \rangle = 0$, \textit{i.e.} all elements of $Z$ annihilate the vacuum. This conclusion, on the other hand, does not necessarily imply imply that $Q^0_\alpha | 0 \rangle = 0$. If, in fact, it did, then the theory would be symmetric under the whole of $W$, an occurrence which goes against the pattern that has been established so far, \textit{i.e.} by the (non-supersymmetric) physical implications of the Segal/Fubini schemes.

### 6 Summary and Comments

The analysis carried out in this paper has explored the relationship between the AdS$_5$ space and conformal descriptions of relativistic quantum field system. The unique way by which the anti-de Sitter space emerged, from different perspectives, by utilizing group contraction strategies, as well as enlisting the aid of Cartan’s theory of spinors, establishes in a unique manner the connection between AdS$_5$ and conformally invariant field systems thereby lending further credibility to the AdS/CFT conjecture. From a general viewpoint one might further assess the situation by asking oneself whether the length $a$, or its inverse (momentum), associated with the breaking of conformal invariance, for the non-supersymmetric and -more importantly- the supersymmetric versions) open the way for realistic applications. In fact, if conformal invariance is only \textit{spontaneously} broken, then it remains in the background as an overall fundamental symmetry of a given system (in reality it is simply hidden) degenerating to Poincaré-based descriptions, \textit{in a limiting way}, when it comes to local observations. Especially significant, in this respect, is the analysis conducted by Polchinski and Strassler [13] which pertains to, dynamical, QCD processes.
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