SENSOR FUSION FOR BIMODAL GENERALIZED LIKELIHOOD RATIO TEST WITH UNKNOWN NOISE VARIANCES

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In this paper we address the problem of sensor fusion. We formulate the joint detection problem using a general linear observation model and inter-modality independence assumption for noises. We derive the fusion architecture based on the generalized likelihood ratio principle and calculate the expressions for the distributions of the test statistic under the signal present and the null hypotheses. To obtain these results we develop a methodology for the joint detection algorithm analysis based on the theory of the Meijer G-function.

1. Introduction. This paper focuses on joint detection with parameter uncertainty. Joint detection involves fusion of data from several sensors (measurement modalities) and is often necessary, because a single sensor has too low detection probability (high false alarm rate). Joint detection has wide range of applications. For example, it is used in landmine detection [3], multimodality breast cancer detection [7] and multisite radar [2]. In this paper we significantly generalize and extend the statistical analysis developed by Kirshin et al. [7], where a joint breast cancer detection system using two sensor modalities (ultrawide-band radar and microwave-induced thermoacoustics) was presented. The main focus of the paper was on numerical experiments showing the potential of the joint detection system. Our current paper focuses on developing a general data-level fusion rule based on the generalized maximum likelihood (GLR) approach and on thoroughly analyzing the distributions of the resulting test statistic. The contribution of our paper is thus (i) the development of a new class of GLR based probabilistic fusion rules, (ii) theoretical analysis of their detection performance, (iii) methodology for the analysis of the fusion rules based on the theory of the Meijer G-function. Although our study was motivated by the concrete application described in [7], we believe that the results presented in this paper have much more general applicability. They can be used for the statistical analysis and design of a wide range of sensor fusion systems that

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can be described by the general signal model presented in section 2.

The rest of the paper is organized as follows. Section 2 formally defines the signal models and the problem to be solved. Section 3 describes the GLR based fusion rule and Section 4 analyzes the distributions of the GLR based fused test statistic. Section 5 provides discussion of our results and Section 6 concludes the paper.

2. Problem Statement. In this paper we consider the classical linear observation model resulting in the following quasi-deterministic signal description under signal present hypothesis $H_1$:

\begin{align}
(2.1) & \quad x = H_x \theta_x + \xi, \\
(2.2) & \quad y = H_y \theta_y + \zeta.
\end{align}

Here $x = [x_1, \ldots, x_N]^T$ and $y = [y_1, \ldots, y_M]^T$ are waveforms observed by two different sensors when hypothesis $H_1$ (signal present) holds true. These waveforms consist of the signal contributions given by the observation matrices $H_x$ and $H_y$ and two sets of unknown deterministic parameters $\theta_x$ and $\theta_y$; and interfering Gaussian noises $\xi$ and $\zeta$ with zero mean and covariance matrices $\sigma_\xi^2 R_\xi$ and $\sigma_\zeta^2 R_\zeta$. Note that the adopted general classical linear observation model contains many important detection problems as special cases. For example, signals with unknown amplitude and/or phase, signal with unknown arrival time and/or frequency, signals received by an antenna array can all be represented using this model via proper choice of observation matrix $H$ and parametrization $\theta$.

In this paper we assume that noises $\xi$ and $\zeta$ are independent and that the $H_0$ hypothesis corresponds to the noise only observation scenario: $x = \xi$, $y = \zeta$. The independency assumption can be justified in many practical situations. For example, when physics that govern measurement process are significantly different for the two sensors or measurements are significantly separated in space, time, or frequency domains, this assumption holds. In fact, from the system design perspective that would be the best sensor configuration, when sensor fusion has potential to provide significant information gain. On the contrary, little fusion gain is to be expected when sensor noises are strongly correlated. We also assume that the noise covariance matrices are known up to the scaling factors $\sigma_\xi^2$ and $\sigma_\zeta^2$ and we treat these as the nuisance parameters.

The goal of this paper is to derive the fusion rule for the Generalized Likelihood Ratio Test (GLRT) based detector and to obtain the exact non-asymptotic expressions for the test statistic probability density functions (PDFs) under both $H_0$ and $H_1$. 
3. GLRT based fusion rule. The GLRT performs the comparison of the GLR $L_G(x, y)$ against the threshold $\gamma$:

$$L_G(x, y) \overset{H_1}{\gtrless} \gamma. \tag{3.1}$$

The GLR is obtained by plugging the maximum likelihood estimates (MLEs) of unknown parameters under each hypothesis into the likelihood ratio [5]. Under the assumptions stipulated in Section 2, $L_G(x, y)$ can be factorized as follows:

$$L_G(x, y) = \frac{p(x|\hat{\theta}_x, \hat{\sigma}_x^2; H_1)}{p(x|\hat{\sigma}_x^2; H_0)} \frac{p(y|\hat{\theta}_y, \hat{\sigma}_y^2; H_1)}{p(y|\hat{\sigma}_y^2; H_0)}. \tag{3.2}$$

MLEs of the unknown parameters are presented in Appendix A.1. Substituting them into (3.2) results in:

$$L_G(x, y) = \left( \frac{x^T R_{\xi}^{-1} x}{x^T R_{\xi}^{-1/2} P_x R_{\xi}^{-1/2} x} \right)^\frac{N}{2} \left( \frac{y^T R_{\zeta}^{-1} y}{y^T R_{\zeta}^{-1/2} P_y R_{\zeta}^{-1/2} y} \right)^\frac{M}{2}. \tag{3.3}$$

The test statistic $L_G(x, y)$ can thus be represented as the product of two exponentiated random variables, $Z = Z_x^N Z_y^M$, of the form

$$Z_x = \frac{x^T R_{\xi}^{-1} x}{x^T R_{\xi}^{-1/2} P_x R_{\xi}^{-1/2} x} \text{ and } Z_y = \frac{y^T R_{\zeta}^{-1} y}{y^T R_{\zeta}^{-1/2} P_y R_{\zeta}^{-1/2} y}. \tag{3.4}$$

Using the properties of signal projection matrices $P_x$ and $P_y$ outlined in Appendix A.1, random variables $Z_x$ and $Z_y$ can be further represented as the following configuration of independent random variables:

$$Z_x = \frac{S_x + R_x}{S_x}, Z_y = \frac{S_y + R_y}{S_y}. \tag{3.5}$$

Here $S_x = x^T R_{\xi}^{-1/2} P_x R_{\xi}^{-1/2} x$, $R_x = x^T R_{\zeta}^{-1/2} P_x R_{\zeta}^{-1/2} x$ and $S_y = y^T R_{\zeta}^{-1/2} P_y R_{\zeta}^{-1/2} y$, $R_y = y^T R_{\xi}^{-1/2} P_y R_{\xi}^{-1/2} y$.

It is interesting to note that $Z_x$ and $Z_y$ are the GLRT test statistics for the individual samples $x$ and $y$ respectively. Since transformation $(\cdot)^{1/(N/2+M/2)}$ is a monotonically increasing function, it is not hard to see that the test statistic $L_G(x, y)^{1/(N/2+M/2)}$ is equivalent to (3.3). The GLRT based joint processing thus leads to the weighted geometric mean based fusion architecture. This statement can be straightforwardly generalized to the multi-sensor setting with samples $x, y, z, \ldots$
4. Distributions of the fused test statistic. In this section we derive the distributions of the test statistic under hypotheses $H_0$ and $H_1$. We will rely heavily on the apparatus of Meijer G-functions introduced and studied by the Dutch mathematician C. S. Meijer [8] and defined as Mellin-Barnes integrals of the form [1]:

\[
G_{p,q}^{m,n}(x | a_p, b_q) = \frac{1}{2\pi i} \int_C g(a_p, b_q, \eta)x^{-\eta}d\eta,
\]

where

\[
g(a_p, b_q, \eta) = \frac{\prod_{j=1}^m (b_j + \eta) \prod_{j=1}^n (1 - a_j - \eta)}{\prod_{j=n+1}^p (a_j + \eta) \prod_{j=m+1}^q (1 - b_j - \eta)}.
\]

For the convenience of the reader in Appendix A.2 we provide some key identities and the G-function related notation that will be further used in the proofs.

4.1. Fused test statistic under $H_0$. Under $H_0$ we have that the components of the fused test statistic: $S_x, R_x$ and $S_y, R_y$, defined in (3.5), are central chi-square distributed random variates with $c_x = N - \text{rank}(P_x)$, $d_x = \text{rank}(P_x)$ and $c_y = M - \text{rank}(P_y)$, $d_y = \text{rank}(P_y)$ degrees of freedom respectively [5]. In this section we are interested in the $H_0$ distribution of the derived test statistic represented as the random variable $Z = \frac{Z^N_x}{Z^M_y}$.

**Theorem 4.1.** The PDF and the CDF of the random variable $Z$ under hypothesis $H_0$ have the following expressions, respectively:

\[
p_{Z|H_0}(z|H_0) = \frac{2\Gamma(c_x + d_x)\Gamma(c_y + d_y)}{N^{\frac{c_x}{2}}M^{\frac{d_y}{2}}\Gamma(c_y/2)\Gamma(d_y/2)}z^{\frac{c_x - c_y}{2} - 1} \times G^{0,M+N}_{M+N,M+1}(z^2 | \Delta(N, -\frac{c_x}{2}, \Delta(M, 1 - \frac{c_y}{2}, -\frac{M}{N})
\]

\[
- \Delta(N, -\frac{c_y}{2}, \Delta(M, 1 - \frac{c_x}{2}, -\frac{d_y}{2} - \frac{M}{N})}
\]

\]

\[
P_{Z|H_0}(z|H_0) = \frac{\Gamma(c_x + d_x)\Gamma(c_y + d_y)}{N^{\frac{c_x}{2}}M^{\frac{d_y}{2}}\Gamma(c_y/2)\Gamma(d_y/2)}z^{\frac{c_x - c_y}{2} - 1} \times G^{0,M+N+1}_{M+N+1,M+N+1}(z^2 | \Delta(N, -\frac{c_x}{2}, \Delta(M, 1 - \frac{c_y}{2}, -\frac{M}{N}), 1 - \frac{1}{N})
\]

\[
- \Delta(N, -\frac{c_y}{2}, \Delta(M, 1 - \frac{c_x}{2}, -\frac{d_y}{2} - \frac{M}{N}), -\frac{1}{N})}
\]
Proof. The outline of the proof that appears in Appendix B is as follows: 1) represent $Z_y$ via joint distribution of $R_y + S_y$ and $S_y$ in terms of H-function of two variables 2) find the distribution of $Z_y^{-1}$ using Theorem 4.1, case IV from Kellogg and Barnes [6] 3) apply random variable transformation to calculate the distribution of $Z_y^{M/N}$, 4) repeat these steps for $Z_x$, 5) find the distribution of $Z_x Z_y^{M/N}$ via multiplicative convolution, 6) find the distribution of $(Z_x Z_y^{M/N})^{N/2} = Z_x^{N/2} Z_y^{M/2}$ via Jacobian method for random variable transformations.

Note that $1 - P_{Z|H_0}(z|H_0)$ provides us with the expression for the probability of false alarm for the fused test statistic (3.3).

4.2. Fused test statistic under $H_1$. Under $H_1$ we have that the components of the fused test statistic $S_x$ and $S_y$ are central chi-square distributed random variates with $c_x = N - \text{rank}(P_x)$ and $c_y = M - \text{rank}(P_y)$ degrees of freedom respectively. The components $R_x$, $R_y$ are non-central chi-square variates with degrees of freedom $d_x = \text{rank}(P_x)$, $d_y = \text{rank}(P_y)$ and non-centrality parameters $\lambda_x = \theta_x^T H_x R_x^{-1} H_x \theta_x$, $\lambda_y = \theta_y^T H_y R_y^{-1} H_y \theta_y$ respectively [5].

In this section we are interested in the $H_1$ distribution of the fused test statistic represented, as before, as the random variable $Z = Z_x Z_y^{M/N}$. Under $H_1$, $Z_x$ and $Z_y$ contain non-centrally distributed components and approach we used in section 4.1 does not seem to be applicable. We thus exploit a different technique to identify the fused test statistic distribution under $H_1$. This technique is summarized in the outline of the proof of Theorem 4.2.

**Theorem 4.2.** The PDF and the CDF of the random variable $Z$ under hypothesis $H_1$ have the following expressions, respectively:

$$p_{Z|H_1}(z|H_1) = \frac{2C_x C_y}{N z^{\frac{N}{2} + 1}} \sum_{k=0}^{\infty} \frac{z^{-\frac{N}{2}}}{k!} G_{2,1}^{2,1} \left( \frac{\lambda_x}{2} \left| \frac{a_x}{k}, b_x \right. \right) G_{3,4}^{1,2} \left( \frac{\lambda_y}{2} \left| 0, a_y \right., \frac{b_y}{k'}, \frac{b_y}{k'} \right) \Gamma(-k')$$

$$+ \frac{2C_x C_y}{M z^{\frac{M}{2} + 1}} \sum_{m=0}^{\infty} \frac{z^{-\frac{M}{2}}}{m!} G_{2,1}^{2,1} \left( \frac{\lambda_x}{2} \left| \frac{a_x}{m}, b_y \right. \right) G_{3,4}^{1,2} \left( \frac{\lambda_y}{2} \left| 0, a_x \right., \frac{b_x}{m}, \frac{b_x}{m'} \right) \Gamma(-m').$$

(4.5)
\[ P_{Z|\mathcal{H}_1}(z|\mathcal{H}_1) = 1 - \frac{C_x C_y}{z^{\frac{M}{2}}} \sum_{k=0}^{\infty} \frac{z^{-\frac{2k}{M}}}{k!} G_{3,4}^{2,1} \left( \frac{-a_x}{2}, 0 \bigg| \frac{\lambda_x}{2}, k, b_x \right) G_{3,4}^{1,2} \left( \frac{-a_y}{2}, 0 \bigg| \frac{\lambda_y}{2}, b_y, k' \right) \frac{\Gamma(-k')}{\frac{c_y}{2} + k} \]

(4.6) \[ - \frac{C_x C_y}{z^{\frac{M}{2}}} \sum_{m=0}^{\infty} \frac{z^{-\frac{2m}{M}}}{m!} G_{3,4}^{2,1} \left( \frac{-a_y}{2}, 0 \bigg| \frac{\lambda_y}{2}, m, b_y \right) G_{3,4}^{1,2} \left( \frac{-a_x}{2}, 0 \bigg| \frac{\lambda_x}{2}, b_x, m' \right) \frac{\Gamma(-m')}{\frac{c_x}{2} + m} \]

With indices \( m' = m + \frac{N}{M} \), \( C_x = \frac{\pi^2}{2} e^{-\frac{\lambda y}{2}} \), \( C_y = \frac{\pi^2}{2} e^{-\frac{\lambda x}{2}} \), \( k' = k + \frac{M}{N} \), \( \lambda_y = \frac{\lambda}{N} \), \( \lambda_x = \frac{\lambda}{N} \); constants \( C_y \), \( C_x \), \( \lambda_y \), \( \lambda_x \), \( k \), \( k' \), \( m \), \( m' \), \( y \), \( x \), \( N \), \( M \), \( \lambda \). \( a_x \), \( b_x \), \( a_y \), \( b_y \) are coefficient vectors.

**Proof.** The outline of the proof that appears in Appendix C is as follows: 1) construct the joint distribution of \( U = R_y + S_y \) and \( W = S_y \), 2) find moment generating function \( \mathcal{M}_W(s) \) of random variable \( W/U \), 3) find the PDF of \( Z^{-1}_y = W/U \) using inverse Laplace transform of \( \mathcal{M}_W \), 4) apply random variable transformation to calculate the distribution of \( Z^{-M/2}_x \), 5) repeat these steps for \( Z^{-N/2}_x \), 6) find the distribution of \( Z^{-N/2}_x Z^{-M/2}_y \) via multiplicative convolution and reciprocal transformation. \( \square \)

Note that \( 1 - P_{Z|\mathcal{H}_1}(z|\mathcal{H}_1) \) gives us the probability of detection for the fused decision rule.

5. **Discussion of Results.** In the previous section we have derived the expressions for the probability of false alarm and the probability of detection for the fused GLR based decision rule developed in Section 3. The expression for the probability of false alarm (see eq. (4.4)) can be used to set the detection threshold for the test (3.1) using Neyman-Pearson criterion. The expression for the probability of detection (see eq. (4.6)) can be used to analyze the performance of the fused decision rule.

Note that the probability of false alarm of a single sensor can be calculated using expression (B.9) for sensor \( x \) (and similar one for sensor \( y \)). Using technique developed in Section 4.2 we can also derive the expression for the \( \mathcal{H}_1 \) CDF for a single sensor:

\[ P_{Z_x|\mathcal{H}_1}(z|\mathcal{H}_1) = 1 - \frac{C_x}{z^{\frac{M}{2}}} \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} G_{3,4}^{2,1} \left( \frac{-a_x}{2}, 0 \bigg| \frac{\lambda_x}{2}, k, b_x \right) \frac{\Gamma(-k')}{\frac{c_y}{2} + k} \]

(5.1) \[ \frac{C_x}{z^{\frac{M}{2}}} \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} G_{3,4}^{2,1} \left( \frac{-a_x}{2}, 0 \bigg| \frac{\lambda_x}{2}, k, b_x \right) \frac{\Gamma(-k')}{\frac{c_y}{2} + k} \]

Next we provide the following illustrative performance analysis example. We fix the number of samples for sensor outputs, \( N = 6 \) and \( M = 16 \). We fix
the signal subspace degrees of freedom $c_x = 2$ and $c_y = 3$ (sensor $x$ has two unknown parameters and sensor $y$ has three unknown parameters). We vary the noncentrality parameters $\lambda_x$ and $\lambda_y$ in the range $[0; 30]$. The resulting detection probability curves for the probability of false alarm fixed at the level $0.01$ obtained using equations (4.4), (4.6), (B.9) and (5.1) are shown in Fig. 1.

Figure 1 demonstrates that fusion provides significant gain in terms of detection reliability even in the case when one of the fused sources has significantly better detection characteristics than the other. In other words, it seems that adding even a relatively weak detector to the fusion rule may result in significant improvement in joint detection performance.

6. Concluding Remarks. In this paper we developed the fusion rule for joint detection with parametric signal uncertainty and noise nuisance parameters. We considered classical linear observation model that includes many practical detection problems as special cases. In our model we also incorporated uncertainty regarding noise variance. Within this framework we derived the fusion rule based on the generalized likelihood ratio paradigm and obtained the expressions characterizing probability of false alarm and probability of detection for the derived fusion rule. Analytical expressions developed in this paper provide important research tools. From the theoretical standpoint, they form a basis for analytical manipulation and general study of fused distributions. From the practical point of view, our expressions provide guidelines for building fusion architecture and tools for direct numerical evaluation of detection performance of this architecture in a sit-
uation with concrete fixed parameters of the individual sensors constituting the joint detection system. In the future we would like to extend our current results by considering the joint GLRT detection problem with completely unknown covariance matrices.

APPENDIX A: USEFUL FORMULAE

A.1. Expressions for the ML estimators. The MLEs of the unknown parameters can be shown to be:

(A.1) \( \hat{\theta}_x = \left( H_x^T R_\xi^{-1} H_x \right)^{-1} H_x^T R_\xi^{-1} x; \) \( \hat{\theta}_y = \left( H_y^T R_\xi^{-1} H_y \right)^{-1} H_y^T R_\xi^{-1} y; \)

(A.2) \( \hat{\sigma}^2_{\xi,1} = \frac{1}{N} x^T R_\xi^{-1/2} P_x^\dagger R_\xi^{-1/2} x; \) \( \hat{\sigma}^2_{\xi,0} = \frac{1}{N} x^T R_\xi^{-1} x; \)

(A.3) \( \hat{\sigma}^2_{\gamma,1} = \frac{1}{M} y^T R_\gamma^{-1/2} P_y^\dagger R_\gamma^{-1/2} y; \) \( \hat{\sigma}^2_{\gamma,0} = \frac{1}{M} y^T R_\gamma^{-1} y; \)

In the expressions above \( P_x^\dagger = I - P_x, P_y^\dagger = I - P_y \) and the signal projection matrices \( P_x \) and \( P_y \) are given by \( P_x = R_\xi^{-1/2} H_x \left( H_x^T R_\xi^{-1} H_x \right)^{-1} H_x^T R_\xi^{-1/2} \)

and \( P_y = R_\gamma^{-1/2} H_y \left( H_y^T R_\gamma^{-1} H_y \right)^{-1} H_y^T R_\gamma^{-1/2} \) respectively. It is straightforward to verify that \( P_x^\dagger P_x = P_x^\dagger, P_y^\dagger P_y = P_y^\dagger; P_x P_x = P_x, P_y P_y = P_y; \)

and \( P_x^\dagger P_x = 0, P_y^\dagger P_y = 0. \)

A.2. Meijer-G function identities. These and many other identities can be readily found in [1, 9, 4].

(A.4) \( x^T G_{p,q}^{m,n} \left( x \begin{array}{c} a_p \\ b_q \end{array} \right) = G_{p,q}^{m,n} \left( x \begin{array}{c} a_p + t \\ b_q + t \end{array} \right), \)

(A.5) \( G_{p,q}^{m,n} \left( \begin{array}{c} 1 \\ x \end{array} \begin{array}{c} a_p \\ b_q \end{array} \right) = G_{q,p}^{m,m} \left( x \begin{array}{c} 1 - b_p \\ 1 - a_q \end{array} \right), \)

\[
\int_0^\infty x^{\alpha-1} G_{u,v}^{s,t} \left( \sigma x \begin{array}{c} c_u \\ d_v \end{array} \right) G_{p,q}^{m,n} \left( \omega x \begin{array}{c} a_p \\ b_q \end{array} \right) dx = \frac{k^{1+\alpha(u-v)-1} \sigma^{-\alpha}}{(2\pi)^{b^*(\ell-1)} + c^* (k-1)} \\
\times G_{k+k(p+q), k+k}, \quad c^* = m + n - (p + q)/2, \quad \rho = \frac{\sum_{j=1}^d d_j - \sum_{j=1}^u c_j + (u + v)/2 + 1,}{\sum_{j=1}^p a_j + (u + v)/2 + 1, \quad e_{kp+\ell v} =}
\]
We can now find the distribution of random variable $V$ if we set the parameters of this random variable transformation one can show that

$$
\int_0^y x^{a-1} G_{p,q}^{m,n} \left( \omega x \begin{bmatrix} a_p \\ b_q \end{bmatrix} \right) dx = y^a G_{p+1,q+1}^{m,n+1} \left( \omega y \begin{bmatrix} a_1, \ldots, a_n, 1 - \alpha, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, -\alpha, b_{m+1}, \ldots, b_q \end{bmatrix} \right).
$$

**APPENDIX B: PROOF OF THEOREM 4.1**

First denote $U = R_y + S_y$ and $W = S_y$. Using the Jacobian method for random variable transformation one can show that

$$
p_{W,U}(w, u) = p_X(u - w)p_Y(w).
$$

Taking into account the fact that $p_{R_y}(x) = \frac{1}{\Gamma(\frac{d_y}{2}) \frac{d_y}{2}^{\frac{d_y}{2}-1}} x^{\frac{d_y}{2}-1} e^{-\frac{x}{2}}$ and $p_{S_y}(x) = \frac{1}{\Gamma(\frac{c_y}{2}) \frac{c_y}{2}^{\frac{c_y}{2}-1}} y^{\frac{c_y}{2}-1} e^{-\frac{y}{2}}$ are central chi-square distributions with degrees of freedom $c_y, d_y$ and substituting these into the previous expression we obtain:

$$
p_{W,U}(w, u) = \frac{(u - w)^{\frac{d_y}{2}-1} w^{\frac{c_y}{2}-1} e^{-\frac{u}{2}}}{\Gamma(\frac{d_y}{2}) \Gamma(\frac{c_y}{2}) \frac{d_y}{2}^{\frac{d_y}{2}+\frac{c_y}{2}}} , \quad 0 < w < u.
$$

This expression exactly corresponds to the McKay’s bivariate gamma distribution ([Kellogg and Barnes, [6, p. 213]]) if we set the parameters of this distribution $a = 1/2$, $p = \frac{c_y}{2}$, $q = \frac{d_y}{2}$ (here we refer to the Kellogg and Barnes’ original notation). It thus can be represented as the bivariate $H$-function ([6, p. 213])

$$
p_{W,U}(w, u) = \frac{(1/2)^2}{\Gamma(\frac{d_y}{2})} H_{1,0,0,0,1,0}^{1,0,0,1,0,1} \left( \begin{bmatrix} 1/2 w \\ 1/2 u \end{bmatrix} \begin{bmatrix} (\frac{c_y}{2} + \frac{d_y}{2} - 2, 1) \\ (\frac{c_y}{2} + \frac{d_y}{2} - 1, 1) \end{bmatrix}, \begin{bmatrix} (\frac{c_y}{2} + \frac{d_y}{2} - 2, 1) \\ (\frac{c_y}{2} - 1, 1) \end{bmatrix} \right).
$$

We can now find the distribution of random variable $V = WU^{-1}$ using Theorem 4.1, case IV ([Kellogg and Barnes [6, p. 213]]):

$$
p_V(v) = \frac{\Gamma(\frac{c_y}{2} + \frac{d_y}{2})}{\Gamma(\frac{c_y}{2})} H_{2,1}^{1,0,1} \left( v \begin{bmatrix} (\frac{c_y}{2} + \frac{d_y}{2} - 1, 1) \end{bmatrix} \right).
$$
Using the relationship between the H-function and the G-function [9, p. 531] we can further simplify this expression:

\[
p_V(v) = \frac{\Gamma\left(\frac{c_y}{2} + \frac{d_y}{2}\right)}{\Gamma\left(\frac{c_y}{2}\right)} G_{1,1}^{1,0}\left(\frac{c_y}{2} + \frac{d_y}{2}, \frac{c_y}{2} - 1\right).
\]

The last expression gives the pdf of \(S_y R_y + S_y\). To find the pdf of \(Z_{y}^{M/N}\) we use the fact that \(Z_{y}^{M/N} = \frac{1}{V_{M/N}}\) and apply the Jacobian transformation method:

\[
p_{Z_{y}^{M/N}}(z) = \left| \frac{\partial V_{M/N}}{\partial v} \right| \left| v = \frac{1}{z_{N/M}} \right| = \frac{N}{M} \frac{p_V\left(\frac{1}{z_{N/M}}\right)}{(z_{N/M})^{M/N+1}}
\]

This results in the following expression:

\[
p_{Z_{y}^{M/N}}(z) = \frac{N}{M} \frac{\Gamma\left(\frac{c_y}{2} + \frac{d_y}{2}\right)}{\Gamma\left(\frac{c_y}{2}\right)} G_{1,1}^{0,1}\left(\frac{N}{2}, \frac{1}{2} \left| 1 - \frac{c_y}{2} - \frac{d_y}{2} - \frac{M}{N}\right.\right)
\]

Similarly, the distribution of \(Z_x\) appears to be:

\[
p_{Z_x}(z) = \frac{\Gamma\left(\frac{c_x}{2} + \frac{d_x}{2}\right)}{\Gamma\left(\frac{c_x}{2}\right)} G_{1,1}^{0,1}\left(\frac{N}{2}, \frac{1}{2} \left| -\frac{c_x}{2} - \frac{d_x}{2}\right.\right)
\]

The next step is to find the PDF of \(Z = Z_x Z_{y}^{M/N}\). Using the Jacobian technique again one can show that

\[
p_{Z_{x}Z_{y}^{M/N}}(z) = \int_0^z \frac{p_{Z_x}(z/v)p_{Z_{y}^{M/N}}(v)}{v} dv
\]

Substituting the PDFs \(p_{Z_x}\) and \(p_{Z_{y}^{M/N}}\) obtained in the previous steps, utilizing the fact that \(p_{Z_x}(z/v) = 0\) for \(v > z\) and using the G-function iden-
tity (A.5) and integration formula (A.6) we obtain:

\[
p_{Z_{X_{Y}}^{M/N}}(z) = \frac{N \Gamma\left(\frac{c}{2} + \frac{d}{2}\right) \Gamma\left(\frac{c}{2} + \frac{d}{2}\right)}{M \Gamma\left(\frac{c}{2}\right) \Gamma\left(\frac{c}{2}\right)} \int_{0}^{\infty} G_{1,1}^{0,1} \left(\frac{v}{z} \left| \begin{array}{c} 1 + \frac{c}{2} + \frac{d}{2} \end{array} \right. \right) \times G_{1,1}^{0,1} \left(\frac{v}{z} \left| \begin{array}{c} 1 - \frac{c}{2} - \frac{M}{N} \end{array} \right. \right) dv
\]

\[
\times \Gamma\left(\frac{c}{2} + \frac{d}{2}\right) \Gamma\left(\frac{c}{2}\right) \Gamma\left(\frac{c}{2}\right)
\]

\[
\times G_{M+N,M+N}^{0,M+N} \left( z^{N} \left| \begin{array}{c} \Delta(N, -\frac{c}{2}), \Delta(M, 1 - \frac{c}{2} - \frac{d}{2} - \frac{M}{N}) \\ \Delta(N, -\frac{c}{2} - \frac{d}{2} - \frac{M}{N}) \end{array} \right. \right).
\]

The final step of the proof is to find the PDF of \(Z = Z_{X}^{N/2}Z_{Y}^{M/2}\) by applying the transformation \((\cdot)^{N/2}\) to the random variable \(Z_{X}Z_{Y}^{M/N}\). After some algebra, this results in expression (4.3).

Finally, we find the corresponding CDF by using the change of variable \(y = x^2\) in the integral below and by applying the indefinite G-function integration formula (A.7), resulting in (4.4).

**APPENDIX C: PROOF OF THEOREM 4.2**

First denote, as previously, \(U = R_{Y} + S_{Y}\) and \(W = S_{Y}\). Now recall that the PDFs of \(R_{Y}\) and \(S_{Y}\) under \(H_{1}\) can be written as follows:

\[
p_{R_{Y}}|H_{1}(x|H_{1}) = \frac{1}{2} e^{-\frac{x + \lambda_{y}}{\sqrt{\lambda_{y}}} - \frac{x}{\lambda_{y}}} \left(\frac{x}{\lambda_{y}}\right)^{-\frac{1}{2} - \frac{1}{2}} I_{\frac{1}{2} - 1}^{0} (\sqrt{\lambda_{y}x})
\]

(C.1)

\[
p_{S_{Y}}|H_{1}(x|H_{1}) = \frac{1}{2} e^{-\frac{x}{\sqrt{\lambda_{y}}} - \frac{x}{\lambda_{y}}}.
\]

Here \(I_{\frac{1}{2} - 1}^{0} (\cdot)\) is the modified Bessel function of the first kind. Using the relationship between this function and the Meijer G-function \([10]\) we can write the \(H_{1}\)-hypothesis joint distribution of \(U\) and \(W\) as follows:

\[
p_{U,W}(u, w) = p_{R_{Y}}|H_{1}(u) = w) p_{S_{Y}}|H_{1}(w), \quad 0 \leq w < u < \infty
\]

(C.2)

\[
= C_{1} e^{-\frac{u}{\sqrt{\lambda_{y}}} - \frac{w}{\lambda_{y}}} G_{1,0}^{1,0} \left(\frac{\lambda_{y}(u - w)}{4} \left| \begin{array}{c} \frac{dy}{2} - 1, 0, \frac{dy}{2} - \frac{1}{2} \end{array} \right. \right).
\]
Where $C_1 = \frac{\pi^2 d_x - \frac{c_y}{2}}{e^{\frac{c_y}{2} \Gamma(\frac{c_y}{2})} d_x - 1}$. Using the definition of Meijer G-function we can write the following integral representation of the G-function above:

\[(C.3)\]

\[G_{1,3}^{1,0} \left( \frac{\lambda_y (u - w)}{4} \right| \frac{d_x - 1}{2}, 1, 0, \frac{d_x - 1}{2} \right) = \frac{1}{2\pi i} \int_C g_1(\eta) \left( \frac{\lambda_y}{4} \right)^{-\eta} (u - w)^{-\eta} d\eta.\]

Since $0 \leq w < u$ this integral can be expanded in the uniformly convergent series:

\[(C.4)\]

\[\int_C g_1(\eta) \left( \frac{\lambda_y}{4} \right)^{-\eta} (u - w)^{-\eta} d\eta = \sum_{k=0}^{\infty} \int_C \frac{u^{-k-\eta} w^k}{k!} g_1(\eta) \frac{\Gamma(\eta + k)}{\Gamma(\eta)} \left( \frac{\lambda_y}{4} \right)^{-\eta} d\eta.\]

Where the order of integration and summation can be interchanged because of the uniform convergence.

We can now write down the expression for the moment generating function of random variable $\frac{W}{U}$

\[(C.5)\]

\[M_{\frac{W}{U}}(s) = \int_0^{\infty} \int_0^u e^{-\frac{w}{u} s} p_{W,U}(w, u) du dw.\]

The application of (C.4) and some reorganization of the above formula lead to

\[(C.6)\]

\[M_{\frac{W}{U}}(s) = \frac{C_1}{2\pi i} \sum_{k=0}^{\infty} \int_C \int_0^{\infty} e^{-\frac{u}{2} s} k! \int_0^{\infty} \frac{u^{-k-\eta}}{k!} J_k(u, s) du g_1(\eta) \frac{\Gamma(\eta + k)}{\Gamma(\eta)} \left( \frac{\lambda_y}{4} \right)^{-\eta} d\eta.\]

Here we have

\[(C.7)\]

\[J_k(u, s) = \int_0^u e^{-\frac{w}{u} s} w^{k+\frac{c_y}{2}} dw = \left[ \Gamma(k + \frac{c_y}{2}) - \Gamma(k + \frac{c_y}{2}, s) \right] \left( \frac{s}{u} \right)^{-k-\frac{c_y}{2}}.\]

Furthermore, since

\[(C.8)\]

\[\int_0^{\infty} e^{-\frac{s}{u}} \left( \frac{s}{u} \right)^{-k-\eta} du = s^{-k-\frac{c_y}{2}} \Gamma(1 - \eta + \frac{c_y}{2}) \Gamma(1 - \eta + \frac{c_y}{2}),\]
we can denote $C_2 = 2^{1+\frac{c_y}{2}}C_1$ and simplify (C.6) as follows:

$$M_{\frac{W}{U}}(s) = \frac{C_2}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{s^{-k-\frac{c_y}{2}}}{k!} \left[ \Gamma(k + \frac{c_y}{2}) - \Gamma(k + \frac{c_y}{2}, s) \right] \times \int g_1(\eta)\Gamma(1 - \eta + \frac{c_y}{2}) \frac{\Gamma(\eta + k)}{\Gamma(\eta)} \left( \frac{\lambda_y}{2} \right)^{-\eta} d\eta.$$  

(C.9)

Next we can write down the expression for PDF of random variable $W/U$ using the fact that it is equal to the inverse Laplace transform $L^{-1}$ of $M_{\frac{W}{U}}$ and noting that for $0 \leq z \leq 1$ we have

$$L^{-1}\left\{s^{-k-\frac{c_y}{2}} \left[ \Gamma(k + \frac{c_y}{2}) - \Gamma(k + \frac{c_y}{2}, s) \right]\right\} = z^{k+\frac{c_y}{2}-1},$$

that leads to the following expression:

(C.11)

$$p_{\frac{W}{U}}(z) = \frac{C_2}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{z^{k+\frac{c_y}{2}-1}}{k!} \frac{\Gamma(\eta + k)}{\Gamma(\eta)} g_1(\eta)\Gamma(1 - \eta + \frac{c_y}{2}) \left( \frac{\lambda_y}{2} \right)^{-\eta} d\eta.$$

Transforming this back to the G-function domain, recalling that $Z_Y = \frac{U}{W}, 1 \leq Z_Y \leq \infty$ and denoting $C_Y = \frac{\pi \frac{d_Y}{2} - 1 e^{-\frac{\lambda_y}{2}}}{\Gamma(\frac{c_y}{2})\lambda_y^{\frac{c_y}{2}-1}}$ results in:

(C.12)

$$p_{Z_Y^{-1}}(z) = C_Y z^{\frac{c_y}{2}-1}G_{2,3}^{1,1}\left( \frac{\lambda_Y}{2} (1 - z) \left| \begin{array}{c} -\frac{c_y}{2}, \frac{d_Y-1}{2} \\ \frac{d_Y}{2} - 1, 0, -\frac{d_Y-1}{2} \end{array} \right. \right).$$

By analogy, we have for $Z_X^{-1}$:

(C.13)

$$p_{Z_X^{-1}}(z) = C_X z^{\frac{c_x}{2}-1}G_{2,3}^{1,1}\left( \frac{\lambda_X}{2} (1 - z) \left| \begin{array}{c} -\frac{c_x}{2}, \frac{d_X-1}{2} \\ \frac{d_X}{2} - 1, 0, -\frac{d_X-1}{2} \end{array} \right. \right),$$

where $C_X = \frac{\pi \frac{d_X}{2} - 1 e^{-\frac{\lambda_x}{2}}}{\Gamma(\frac{c_x}{2})\lambda_x^{\frac{c_x}{2}-1}}.$

Using the Jacobian method for random variable transformation one can further show that the PDFs for $Z_X^{-\frac{N}{2}}$ and $Z_Y^{-\frac{M}{2}}$ take on the form:

(C.14)

$$p_{Z_X^{-\frac{N}{2}}}(z) = \frac{2C_X}{N} z^{\frac{c_x}{2}-1}G_{2,3}^{1,1}\left( \frac{\lambda_X}{2} (1 - z^{\frac{2}{N}}) \left| \begin{array}{c} -\frac{c_x}{2}, \frac{d_X-1}{2} \\ \frac{d_X}{2} - 1, 0, -\frac{d_X-1}{2} \end{array} \right. \right),$$

(C.15)

$$p_{Z_Y^{-\frac{M}{2}}}(z) = \frac{2C_Y}{M} z^{\frac{c_y}{2}-1}G_{2,3}^{1,1}\left( \frac{\lambda_Y}{2} (1 - z^{\frac{2}{M}}) \left| \begin{array}{c} -\frac{c_y}{2}, \frac{d_Y-1}{2} \\ \frac{d_Y}{2} - 1, 0, -\frac{d_Y-1}{2} \end{array} \right. \right).$$
We can now find the PDF of random variable \( V = \frac{X^N}{X^M} \frac{Y^M}{Y^N} \) using formula analogous to (B.10) and using the fact that \( 0 \leq \frac{X^M}{X^N} \leq 1 \) and \( 0 \leq \frac{Y^N}{Y^M} \leq 1 \) results in \( V \leq \frac{X^M}{X^N} \frac{Y^M}{Y^N} \leq 1 \). Upon denoting \( a_x = [-\frac{a_x}{2}, \frac{a_x}{2} - 1] \), \( b_x = [\frac{a_x}{2} - 1, 0, \frac{a_x}{2}] \) and \( a_y = [-\frac{a_y}{2}, \frac{a_y}{2} - 1] \), \( b_y = [\frac{a_y}{2} - 1, 0, \frac{a_y}{2}] \) we have

\[
\begin{align*}
P_{Z_x \frac{X^N}{X^M} Z_y \frac{Y^M}{Y^N}}(z) &= \frac{4C_x C_y}{MN} \int_{z}^{1} \frac{u^{\frac{a_y}{2}}}{u} \left( \frac{z}{u} \right)^{\frac{a_y}{2} - 1} G_{1,1}^{1,1} \left( \frac{\lambda_x}{2} \left( 1 - \left( \frac{z}{u} \right)^{\frac{a_x}{2}} \right) \right) \left[ \frac{a_x}{b_x} \right] du \\
&\times G_{1,3}^{1,1} \left( \frac{\lambda_y}{2} \left( 1 - u^{\frac{a_y}{2}} \right)^{-\omega} \right) I(z, \eta, \omega) d\eta d\omega
\end{align*}
\]

(C.16)

Using the integral representation of the Meijer G-function we can further write it as

\[
\begin{align*}
P_{Z_x \frac{X^N}{X^M} Z_y \frac{Y^M}{Y^N}}(z) &= \frac{4C_x C_y}{MN} \int_{z}^{1} \frac{u^{\frac{a_y}{2}}}{u} \left( \frac{z}{u} \right)^{\frac{a_y}{2} - 1} g(\eta, a_x, b_x) g(\omega, a_y, b_y) \\
&\times \left( \frac{\lambda_x}{2} \right)^{-\eta} \left( \frac{\lambda_y}{2} \right)^{-\omega} I(z, \eta, \omega) d\eta d\omega
\end{align*}
\]

(C.17)

Where the integrand in \( I(z, \eta, \omega) \):

\[
I(z, \eta, \omega) = \int_{z}^{1} \frac{u^{\frac{a_y}{2} - \frac{a_x}{2} - 1}}{u} \left[ 1 - \left( \frac{z}{u} \right)^{\frac{a_y}{2}} \right]^{-\eta} \left[ 1 - u^{\frac{a_y}{2}} \right]^{-\omega} du.
\]

(C.18)

can be expanded into the double uniformly convergent series since \( z \leq u \leq 1 \) and \( 0 \leq z \leq 1 \). After changing the order of integration and summation (valid due to the uniform convergence) and evaluating the integral this yields:

\[
I(z, \eta, \omega) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k! m!} \frac{\Gamma(\eta + k) \Gamma(\omega + m)}{\Gamma(\eta) \Gamma(\omega)} \int_{z}^{1} \frac{u^{\frac{a_y}{2}}}{u^{\frac{a_x}{2} - \frac{a_y}{2} - 1}} \left( \frac{z}{u} \right)^{\frac{a_x}{2}} u^{\frac{mN}{2}} d\eta d\omega
\]

(C.19)

Changing the order of summation, substituting the following two expressions

\[
\begin{align*}
\sum_{m=0}^{\infty} \frac{\Gamma(\omega + m)}{m! \Gamma(\omega)} \frac{M_N}{2} (2k + c_x) - \frac{N}{2} (2m + c_y) &= M \Gamma(1 - \omega) \Gamma\left( \frac{c_y}{2} - k \frac{M}{2} - \frac{c_x M}{2N} \right) \\
&\quad \Gamma\left( \frac{c_y}{2} - k \frac{M}{2} - \frac{c_x M}{2N} \right) \\
\sum_{k=0}^{\infty} \frac{\Gamma(\eta + k)}{k! \Gamma(\eta)} \frac{M_N}{2} (2k + c_x) - \frac{N}{2} (2m + c_y) &= N \Gamma(1 - \eta) \Gamma\left( \frac{c_x}{2} - m \frac{N}{2} - \frac{c_y N}{2M} \right) \\
&\quad \Gamma\left( \frac{c_x}{2} - m \frac{N}{2} - \frac{c_y N}{2M} \right)
\end{align*}
\]

(C.20)
into (C.19) and returning to the G-function representation of the Mellin-Barnes integrals we have the following expression for the PDF of random variable $Z_y \overset{\mathcal{N}}{\overset{\mathcal{M}}{\times}} Z_x \overset{\mathcal{M}}{\overset{\mathcal{N}}{\times}}$:

$$p_{Z_y \overset{\mathcal{N}}{\overset{\mathcal{M}}{\times}} Z_x \overset{\mathcal{M}}{\overset{\mathcal{N}}{\times}}}(z) = \frac{2C_x C_y}{N} z^{\frac{\lambda_y}{2} - 1} \sum_{k=0}^{\infty} \frac{z^{\frac{2k}{M}}}{k!} G_{3,4}^{1,2} \left( \frac{\lambda_y}{2} \left| \begin{array}{c} \alpha_y, 0, a_x, 0 \\ b_y, k \frac{M}{N} + \frac{c_y M}{2N} - \frac{c_y}{2} \end{array} \right. \right) \Gamma \left( \frac{c_y}{2} - k \frac{M}{N} - \frac{c_x M}{2N} \right)$$

$$\times G_{3,4}^{1,2} \left( \frac{\lambda_x}{2} \left| \begin{array}{c} 0, a_x, 0, a_x \\ b_x, m \frac{N}{M} + \frac{c_x N}{2M} - \frac{c_x}{2} \end{array} \right. \right) \Gamma \left( \frac{c_x}{2} - m \frac{N}{M} - \frac{c_y N}{2M} \right).$$

(C.21)

Finally, using the reciprocal transformation of the random variable $Z_y \overset{\mathcal{N}}{\overset{\mathcal{M}}{\times}} Z_x \overset{\mathcal{M}}{\overset{\mathcal{N}}{\times}}$ and denoting $m' = m \frac{N}{M} + \frac{c_y N}{2M} - \frac{c_y}{2}$, $k' = k \frac{M}{N} + \frac{c_x M}{2N} - \frac{c_x}{2}$ we obtain the desired PDF $p_{Z_y \overset{\mathcal{N}}{\overset{\mathcal{M}}{\times}} Z_x \overset{\mathcal{M}}{\overset{\mathcal{N}}{\times}}}(z)$ in expression (4.5).

The CDF follows straightforwardly via the term-wise integration of (4.5) using the fact that $Z \geq 1$, which implies

$$P_{Z|\mathcal{H}_1}(z|\mathcal{H}_1) = \int_1^z p_{Z|\mathcal{H}_1}(u|\mathcal{H}_1)du.$$  

(C.22)

and evaluation of the last integral leads to (4.6).

REFERENCES

[1] Adamchik, V. and Marichev, O. I. (1990). The Algorithm for Calculating Integrals of Hypergeometric Type Functions and Its Realization in REDUCE System. In Proc. ISSAC 1990 212–224.
[2] Chernyak, V. S. (1998). Fundamentals of Multisite Radar Systems. Gordon and Breach Science Publishers.
[3] Cremer, F., Schutte, K., Schavemaker, J. G. M. and den Breijen, E. (2001). A comparison of decision-level sensor-fusion methods for anti-personnel landmine detection. Information Fusion 2 187–208.
[4] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G. and et al., (1953). Higher transcendental functions, Vol. I. McGraw-Hill, New York.
[5] Kay, S. M. (1998). Fundamentals of Statistical Signal Processing, Volume 2: Detection Theory. Prentice Hall PTR.
[6] Kellogg, S. D. and Barnes, J. W. (1987). The distribution of products, quotients, and powers of two dependent H-function variates. Mathematics and Computers in Simulation 29 209–221.
Kirshin, E., Oreshkin, B. N., Zhu, G. K., Popovic, M. and Coates, M. J. (2011). Microwave breast cancer detection: optimal detection rule for joint microwave radar and microwave-induced thermoacoustics modalities. In Proc. Int. Symp. Biomedical Imaging 2011.

Meijer, C. S. (1946). On the G-function. I–VIII. Proc. Nederl. Akad. Wetensch. 49.

Prudnikov, A. P., Marichev, O. I. and Brychkov, Y. A. (2003). Integraly i ryady. Tom 3. Īptitsal’nye funktsii. Dopolnitel’nye glavy. Bibfizmat.

Wolfram, (2011). The Wolfram’s functions cite. available online: http://functions.wolfram.com/03.02.26.0005.01.