Entanglement in fermionic Fock space

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Abstract

We propose a generalization of the usual stochastic local operations and classical communications (SLOCC) and local unitary (LU) classifications of entangled pure state fermionic systems based on the spin group. Our generalization relies on the fact that there is a representation of this group acting on the fermionic Fock space, which, when restricted to fixed particle number subspaces, naturally recovers the usual SLOCC transformations. The new ingredient is the occurrence of Bogoliubov transformations of the whole Fock space, which change the particle number. The classification scheme built on the spin group naturally prohibits entanglement between states containing even and odd numbers of fermions. In our scheme the problem of the classification of entanglement types boils down to the classification of spinors where totally separable states are represented by so-called pure spinors. We construct the basic invariants of the spin group and show how some of the known SLOCC invariants are just their special cases. As an example we present the classification of fermionic systems with a Fock space based on six single particle states; an intriguing duality between two different possibilities for embedding three-qubit systems inside the fermionic ones is revealed. This duality is elucidated via an interesting connection to the configurations of wrapped membranes reinterpreted as qubits.

Keywords: quantum entanglement, Fock space, spin group, fermions

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1. Introduction

In quantum information theory the concept of entanglement is regarded as a resource for completing various tasks which are otherwise unachievable or uneffective by means of classical methods [1]. In order to use entanglement in this way one first needs to classify quantum states according to the type of entanglement they possess. A physically well-motivated classification scheme of multipartite quantum systems is based on protocols employing local operations
and classical communications (LOCC) [2]. Two states are said to be LOCC equivalent if there is a reversible LOCC transformation between them. Obviously reversible LOCC transformations form a group and the aforementioned classification problem boils down to the task of identifying the orbits of a group on one of its particular representations. In the case of pure state systems with distinguishable constituents represented by the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ and the composite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ the LOCC group is simply the group of local unitary transformations

$$U(\mathcal{H}_1) \otimes U(\mathcal{H}_2) \otimes \cdots \otimes U(\mathcal{H}_n).$$

(1)

For practical purposes it is sometimes more convenient to consider the classification of states under reversible stochastic local operations and classical communications (SLOCC) [3, 4]. For pure state multipartite systems the SLOCC group of such transformations is just the one of invertible local operators

$$GL(\mathcal{H}_1) \otimes GL(\mathcal{H}_2) \otimes \cdots \otimes GL(\mathcal{H}_n).$$

(2)

For systems with distinguishable constituents on pure state SLOCC classification there is a great variety of results available in the literature. A somewhat more recent question is the entanglement classification for pure state systems with indistinguishable constituents [5–16]. For these the Hilbert space is just the symmetric (bosons) or antisymmetric (fermions) tensor power of the single particle Hilbert space $\mathcal{H}$ i.e. for an $n$ particle system $\vee^n \mathcal{H}$ or $\wedge^n \mathcal{H}$ respectively. The indistinguishable SLOCC group is again the linear group $GL(\mathcal{H})$, but this time with the $n$-fold diagonal action on $n$ particle states.

For low dimensional systems with just few constituents the SLOCC classes are well known. These results were originally obtained by mathematicians and rediscovered later in the context of entanglement by physicists. For bipartite bosonic and fermionic systems a method similar to the one based on the usual Schmidt decomposition provides the SLOCC and LU classification [5, 7]. For tripartite pure state fermionic systems the SLOCC classification is available up to the case of a nine-dimensional single particle Hilbert space (with the six-dimensional case being the first non-trivial one) [12, 17–23]. Some interesting results concerning fourpartite fermionic systems with eight single particle states have appeared recently [16]. We note that most of the cases, where the complete SLOCC classification is known, correspond to the prehomogeneous vector spaces classified by Sato and Kimura [24, 25].

In this work we introduce a classification scheme which is valid for fermionic states based on the natural action of the spin group on the corresponding Fock space. Physically the Fock space of identical particles is a Hilbert space which describes processes where the number of constituents (particles) is not conserved. In the case of fermions with a finite-dimensional single particle Hilbert space $\mathcal{H}$ the Fock space is just the vector space underlying the exterior algebra over $\mathcal{H}$ and hence is also of a finite dimension. It is well known to mathematicians that the spin group can be represented in this exterior algebra [26]. We show that when restricted to fixed particle number subspaces this group action is just the usual one of the indistinguishable SLOCC group. The transformations mixing these subspaces are the so-called Bogoliubov transformations [27], which are well known in condensed matter physics where these transformations relate different ground states of a system usually corresponding to a phase transition. One such example is the usual mean-field treatment of BCS superconductivity [28] where the BCS ground state is related by a Bogoliubov transformation to the Fermi-sea state. A characteristic feature of our new classification scheme is that it distinguishes between the even and odd particle states of the Fock space hence such ‘fermionic’ and ‘bosonic’ states cannot be in the same class. The novelty in our approach is unlike that of the usual SLOCC scheme as now particle number changing protocols are also allowed.
We emphasize that our proposed classification scheme is precisely the one well-known to mathematicians as ‘classification of spinors’. In this terminology a spinor is just an element of the vector space on which the two-sheeted cover of the orthogonal group (i.e. the spin group) is represented. Two spinors are equivalent if and only if there exists an element of the spin group which transforms one spinor to the other. Now the word ‘classification’ means the decomposition of the space of spinors into equivalence classes (orbits) and determining the stabilizer of each orbit, a problem already solved by mathematicians up to dimension twelve [29]. From a physical point of view, this means that a full classification of fermionic systems, according to our proposed scheme, is available up to single particle Hilbert spaces of dimension six. Apart from generalizing the SLOCC scheme, one of the aims of the present paper was to communicate these important results to the physics and quantum information community in an accessible way.

The organization of this paper is as follows. In section 2 we introduce the actions of the spin group on the fermionic Fock space and describe the above ideas in detail. We then introduce the notion of pure spinors and argue that these are generalized separable states. In section 3 we show that the unitary (probability conserving) subgroup of the complex spin group is in its compact real form and that it naturally incorporates the LU group. In section 4 we introduce some tools from the theory of spinors such as the Mukai pairing [30], the moment map [31] and the basic invariants under the spin group. We argue that these tools are useful to obtain orbit classification. In section 5 we discuss some simple examples and show how the above machinery works in practice. The first non-trivial example is the fermionic Fock space with a six-dimensional single particle Hilbert space where a full classification is presented according to the systems equivalence [23] to a particular Freudenthal triple system [32, 33]. In closing, an intriguing duality between two different possibilities for embedding three-qubit systems inside our fermionic ones is revealed. This duality is elucidated via an interesting connection to the configurations of wrapped membranes reinterpreted as qubits [34–36].

2. Generalization of SLOCC classification

Let $\mathcal{H}$, dim $\mathcal{H} = d$ be the (finite dimensional) one particle Hilbert space. We define the fermionic Fock space of $\mathcal{H}$ by

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus \wedge^2 \mathcal{H} \oplus \ldots \oplus \wedge^d \mathcal{H}. \quad (3)$$

Obviously dim $\mathcal{F} = \sum_{k=0}^d \binom{d}{k} = 2^d$. Let us introduce the vector space of the creation operators $W = \{ f^+ : \wedge^k \mathcal{H} \to \wedge^{k+1} \mathcal{H} | f^+ \text{is linear} \}$ and define the vector space isomorphism between $W$ and $\mathcal{H}$ via the exterior product:

$$|v⟩ \wedge |φ⟩ = (f_v)^+ |φ⟩, \quad |v⟩ \in \mathcal{H}, \quad |φ⟩ \in \mathcal{F}, \quad (f_v)^+ \in W. \quad (4)$$

Obviously $(f_v)^+ (f_u)^+ = -(f_u)^+ (f_v)^+$ Also the space of annihilation operators $W^* = \{ f : \wedge^k \mathcal{H} \to \wedge^{k-1} \mathcal{H} | f \text{ is linear} \}$ is isomorphic to the dual space, $\mathcal{H}^*$ via the interior product:

$$i_w |φ⟩ = f_w |φ⟩, \quad w \in \mathcal{H}^*, \quad |φ⟩ \in \mathcal{F}, \quad f_w \in W^*. \quad (5)$$

The action of $\mathcal{H}^*$ on $\mathcal{H}$ maps to the canonical anticommutation relations:

$$\{ f_u, (f_v)^+ \} = v(u)I, \quad u \in \mathcal{H}, \quad v \in \mathcal{H}^*, \quad (6)$$

where $I$ denotes the identity operator on $\mathcal{F}$. Indeed, by the antiderivation property of the interior product, we have

$$i_w (|v⟩ \wedge |φ⟩) = (i_w |v⟩) \wedge |φ⟩ - |v⟩ \wedge i_w |φ⟩ = v(w) |φ⟩ - |v⟩ \wedge i_w |φ⟩, \quad (7)$$
It is not difficult to show that the vector space $V$ is a Clifford algebra $\text{Cliff}(V)$ if there is a multiplication satisfying
\[
\langle w^2 = (w, w)1, \quad \forall w \in V \rangle
\]
with $(\cdot, \cdot)$ an appropriate inner product on $V$. Now consider the vector space $V = W \oplus W^*$ with the usual operator multiplication and define the inner product using the anticommutator:
\[
(f_1 + f_2^+, f_3 + f_4^+) = \frac{1}{2}f_1f_3 + \frac{1}{2}f_2f_4,
\]
and denote the corresponding bases $\{f_i\}_{i=1}^4$. That keeps the anticommutator, i.e. $O : W \oplus W^* \rightarrow W \oplus W^*$ is a Bogoliubov transformation if
\[
\{O(f_1 + f_2^+), O(f_3 + f_4^+)\} = \{f_1 + f_2^+, f_3 + f_4^+\}.
\]
This means that with regard to our inner product $(O(f_1 + f_2^+), O(f_3 + f_4^+)) = (f_1 + f_2^+, f_3 + f_4^+)$, so the group of all Bogoliubov transformations is $SO(W \oplus W^*) \cong SO(2d, \mathbb{C})$. The Lie algebra of this group is defined by
\[
\mathfrak{so}(W \oplus W^*) = \{T | (Tx, y) + (x, Ty) = 0, \quad x, y \in W \oplus W^*\}.
\]
This can be satisfied with the parametrization
\[
T \left( \frac{(f^+_i)^*}{f_i} \right) = \begin{pmatrix}
A_{ik} & \beta_{ik} \\
B_{jk} & -A_{jk}^*
\end{pmatrix} \left( \frac{(f^+_i)^*}{f_i} \right), \quad B_{ij} = -B_{ji}, \quad \beta^{ij} = -\beta_{ji},
\]
where we have picked the bases $\{e_i\} \subset \mathcal{H}, \{e^i\} \subset \mathcal{H}^*$ and denoted the corresponding bases of creation and annihilation operators as $\{f_i^\dagger \equiv (f^+_i)^*\} \subset W$ and $\{f_e^\dagger \equiv f_i\} \subset W^*$. It is well known that we can embed the Lie algebra $\mathfrak{so}(W \oplus W^*)$ into $\text{Cliff}(W \oplus W^*)$.
\[
x, y \in W \oplus W^* \subset \text{Cliff}(W \oplus W^*) \Rightarrow \frac{1}{2}[x, y] \in \mathfrak{so}(W \oplus W^*).
\]
We can represent $\mathfrak{so}(W \oplus W^*)$ on $\mathcal{F}$ via this embedding and the Clifford algebra representation given in (10). In this manner the spinor representation [26] on $\mathcal{F}$ is defined by
\[
T \left( \frac{(f^+_i)^*}{f_i} \right) = \begin{pmatrix}
A_{ik} & \beta_{ik} \\
B_{jk} & -A_{jk}^*
\end{pmatrix} \left( \frac{(f^+_i)^*}{f_i} \right).
\]
It is not difficult to show that $T$ is implemented by
\[
T = -B - \beta + A - \frac{1}{2} \text{Tr} A \cdot I,
\]
where
\[
A \in \text{End}(W), \quad A = A_{ij}(f^+_j)^* f_i, \\
B \in \wedge^2 W : W \rightarrow W^*, \quad B = \frac{1}{2}B_{ij}(f^+_i)^* (f^+_j)^+, \\
\beta \in \wedge^2 W^* : W^* \rightarrow W, \quad \beta = \frac{1}{2} \beta_{ij} f_i f_j.
\]
This shows that $\mathfrak{so}(W \oplus W^*) = \text{End}(W) \oplus \wedge^2 W^* \oplus \wedge^2 W$. 

The whole spin group can be obtained from the Clifford algebra as follows

\[
\text{spin}(W \oplus W^*) = \{ O = x_1 \ldots x_r | x_i = f_{ui} + f_{ui}^* \in W \oplus W^*, \ (x_i, x_i) = \pm 1, \ r \text{ is even} \}.
\]  

(19)

The finite version of (16) is the usual double cover

\[
OxO^{-1} = O(x), \ x \in W \oplus W^*, \ O \in \text{spin}(W \oplus W^*), \ O \in SO(W \oplus W^*),
\]  

(20)

where (16) can be recovered by putting \( O = e^T \) and \( O = e^T \). We note that the exponential map gives only the identity component spin_0(W \oplus W^*). For clarity, we always denote matrices in the vector representation acting on \( W \oplus W^* \) with calligraphic letters (e.g. \( T, O \)) and operators in the Fock space spinor representation with roman letters (e.g. \( T, O \)).

It is known that the representation constructed in this way is reducible: \( \mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^- \), where \( \mathcal{F}^\pm \) are the \( \pm \) eigenspaces of the volume element \( v_\mathcal{W} = 2d \prod_{i=1}^{2d} ((f^i)^+ f_i - \frac{1}{2}) \) of the Clifford algebra (this is just the generalization of the usual \( \gamma_5 \) matrix). These are simply the even and odd particle subspaces of the Fock space:

\[
\mathcal{F}^+ = \wedge^e \mathcal{H} \otimes (\wedge^d \mathcal{H}^*)^{1/2}, \quad \mathcal{F}^- = \wedge^o \mathcal{H} \otimes (\wedge^d \mathcal{H}^*)^{1/2}.
\]  

(21)

From our perspective this reducibility corresponds to the superselection between ‘fermionic’ and ‘bosonic’ states. The appearance of the factors of \((\wedge^d \mathcal{H}^*)^{1/2}\) is explained before equation (24).

Let us now consider the Fock vacuum \( |0\rangle \). By the above it is easy to see that a finite \( O \in \text{spin}(W \oplus W^*) \) transformation acts on this as

\[
O|0\rangle = e^T |0\rangle = (\det e^T)^{-1/2} e^{-B} |0\rangle = (\det e^T)^{-1/2} \left( 1 + B + \frac{1}{2} B^2 + \cdots + \frac{1}{(d/2)!} B^{d/2} \right) |0\rangle,
\]  

(22)

since \( A \) and \( B \) annihilates \( |0\rangle \). Notice that if \( |0\rangle \) is a Fock vacuum of \( f \) then \( e^{-B} |0\rangle \) is a Fock vacuum of \( O f O^{-1} \). As a consequence the orbit \( \{ O|0\rangle | O \in \text{spin}(W \oplus W^*) \} \) contains possible Bogoliubov Fock vacuums in \( \mathcal{F} \). Moreover for an arbitrary state, obtained with a definite number of creation operators, acting on the Fock vacuum, we have

\[
O(f_1^+ \cdots f_n^+ |0\rangle) = O f_1^+ O^{-1} O f_2 O^{-1} \cdots O f_n O^{-1} |0\rangle = (\det e^T)^{-1/2} O(f_1^+) \cdots O(f_n^+) e^{-B} |0\rangle.
\]  

(23)

Notice that the transformations with \( B_{ij} = \beta^{ij} = 0 \) have the form of an ordinary SLOCC transformation on the creation operators. Indeed in this case \( O(f^i)^+ O^{-1} = G_k^i (f^k)^+ \) where \( G \in \text{GL}(W) \) is the exponential of the matrix \( \Lambda^f_{ij} \). However, because of the \( 1/2 \) TrA factor in equation (17) a fix particle number state picks up a determinant factor:

\[
(f^i)^+ \cdots (f^k)^+ |0\rangle \mapsto (\det G)^{-1/2} G_k^i (f^k)^+ \cdots G_k^k (f^k)^+ |0\rangle.
\]  

(24)

This justifies the factors of \((\wedge^d \mathcal{H}^*)^{1/2}\) in (21). Now let us introduce the analogues of separable states.

**Definition.** A spinor \(|\phi\rangle \in \mathcal{F} \) is said to be pure if the annhilator subspace

\[
E_\phi = \{ x \in W \oplus W^* | x|\phi\rangle = 0 \}
\]  

(25)

is a maximal isotropic subspace (i.e. has dimension equal to \( \text{dim} W \)).

Indeed, \( E_\phi \) has to be an isotropic subspace since for all \( x \in E_\phi \) we have \( 0 = x^2 |\phi\rangle = (x, x)|\phi\rangle \) thus \( x \) is null. Clearly the Fock vacuum is a pure spinor since its isotropic subspace consists of all the annihilation operators, \( E_0 = \{ f \in W^* \subset W \oplus W^* \} \). Since if an \( x \in W \oplus W^* \) annihilates \(|\phi\rangle \) we have \( x|\phi\rangle = 0 \Rightarrow OxO^{-1} |\phi\rangle = 0 \), so the annhilator subspace transforms as \( E_{O\phi} = OE_\phi \). As a consequence, states on the same spin \((W \oplus W^*) \)
orbit have the same dimensional annihilator subspace thus every state on the vacuum orbit \( |0\rangle |O \in \text{spin}(W \oplus W^*) \) is a pure spinor. Moreover all the separable states are pure spinors. Indeed, a state of the form \( |\phi\rangle = f_1^+ \ldots f_n^+ |0\rangle \) is annihilated by
\[
E_{\phi} = \{f_1^+ \ldots f_n^+ , f_{n+1}, \ldots, f_{dn}\}.
\] (26)
Notice that a 'bosonic' Fermi-sea state is in the same class as the Fock vacuum. More generally [26, 38] any pure spinor can be expressed in the form
\[
|\phi_0\rangle = \lambda e^B f_1^+ \ldots f_n^+ |0\rangle,
\] (27)
where \( \lambda \neq 0 \) is an arbitrary complex number, \( B = \frac{1}{2}B_{ij}(f^i)^+(f^j)^+ \in \wedge^2 W \) and \( v_1, \ldots, v_k \), \( k \leq d \) are some linearly independent vectors in \( \mathcal{H} \). The corresponding annihilator subspace is just
\[
E_{\phi_0} = \{e^B f_1^+ e^{-B} , \ldots, e^B f_n^+ e^{-B} , e^B f_{n+1} e^{-B} , \ldots, e^B f_{dn} e^{-B} |u_i(v_j) = 0,
1 \leq i \leq d, \quad 1 \leq j \leq k\},
\] (28)
where \( u_1, \ldots, u_{d-k} \) are linearly independent elements of \( \mathcal{H}^* \). Because of these properties we propose pure spinors to be the analogues of separable states of ordinary SLOCC classification in the fermionic Fock space.

3. Generalization of LOCC classification

Notice that if we have a Hermitian inner product \( \langle . , . \rangle \) on \( \mathcal{H} \), to maintain consistency with this, we require that a Bogoliubov transformed annihilation operator \( O f O^{-1} \) must stay the adjoint of the Bogoliubov transformed creation operator \( f^+ O^{-1} \). Thus, if we have \( f^+ = f^\dagger \) then we must have
\[
(O f O^{-1})^\dagger = (O f^+ O^{-1})^{-1}.
\] (29)
This constraint means that the spinor representation \( O = e^T = e^{-B - \beta + A - \frac{1}{2}T \lambda A^T} \) must be unitary with respect to this inner product. One can clarify that the matrix \( A_{ij}^T \) must be antiHermitian and the matrices \( B_{ij} \), \( \beta^i \) must satisfy \( (B^T)_{ij} = -\beta^j \), thus the group of admissible Bogoliubov transformations is restricted to \( SO(W \oplus W^*) \cap SU(W \oplus W^*) \). Notice that though the matrix
\[
T = \begin{pmatrix} A_{ik}^T & \beta^i \\ B_{jk} & -A_{ij}^T \end{pmatrix}
\] (30)
now has to be antiHermitian, it does not have to be real. Therefore the Bogoliubov transformation can still have complex coefficients. Indeed, one defines the block matrix
\[
g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
\] (31)
With this the \( SO \) property of \( O = e^T \) reads as \( O g O^T = g \) (see (12)). On the other hand \( T \) is antiHermitian so \( O^T = O^{-1} \). Combine the two to get the following reality condition
\[
O^* = g O g^\dagger.
\] (32)
Now we shall prove that the subgroup of \( SO(W \oplus W^*) = SO(2d, \mathbb{C}) \) satisfying the above reality condition is just the compact real form \( SO(2d, \mathbb{R}) \). Define the \( 2d \times 2d \) unitary matrix
\[
N = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}.
\] (33)
It is easy to check that \( N \) diagonalizes \( g \):
\[
g N = N g_0.
\] (34)
where we have defined
\[ g_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \] (35)

**Proposition.** The map \( O \mapsto S = N^T O N \) is a group isomorphism from the subgroup \( \{ O \in SO(2d, \mathbb{C}) | O^* = gOg \} \) to the group \( SO(2d, \mathbb{R}) \).

**Proof.** It is obvious that the map giving rise to \( S \) is a homomorphism and since \( N \) is invertible it is also an isomorphism. Now it is very easy to directly check that \( gN^* = N \) and hence \( N^T = N^T g \). We have
\[ SS^T = N^T ONN^T O^T N^* = N^T gN^* = I, \] (36)
hence for every unitary \( O \) satisfying \( O gO^T = g \), \( S \) is an element of \( SO(2d, \mathbb{R}) \). For the converse, we have to check whether \( O = NSN^T \) is a unitary matrix with the condition \( O gO^T = g \) satisfied for all \( S \in SO(2d, \mathbb{R}) \). It is obvious that \( O \) is unitary. For the other we write
\[ O gO^T = N S N^T g N^* S^T N^T, \] (37)
but again \( gN^* = N \) so
\[ O gO^T = N S N^T S N^T N^T = N S S^T N^T = NN^T = g. \] (38)
\[ \square \]

It is very important that the previously SLOCC type transformations with \( B = \beta = 0 \) are restricted in this case to simple LOCC \( U(d) \) because \( A \) has to be antiHermitian. Actually this seems quite intuitive. Recall that the extra constraints arose from requiring \( O = e^T \) to be unitary with respect to a fixed inner product. But we know that the role of the inner product of our Hilbert space is to associate probabilities to states. Henceforth the restricted Bogoliubov transformations corresponding to \( SO(2d, \mathbb{R}) \) preserve probabilities while the whole \( SO(2d, \mathbb{C}) \) does not. This is exactly the physical difference between LOCC and SLOCC transformations.

### 4. The moment map

There is a canonical antiautomorphism of the exterior algebra \( \wedge^* \mathcal{H} \) called the transpose, which is the linear extension of the map:
\[ t : (f_1)^+ \ldots (f_d)^+ |0\rangle \mapsto (f_d)^+ \ldots (f_1)^+ |0\rangle = (-1)^{\frac{d(d-1)}{2}} (f_1)^+ \ldots (f_d)^+ |0\rangle. \] (39)
Define a bilinear product on \( \mathcal{F} \) as
\[ (\phi, \psi) = (|\phi\rangle^T \wedge |\psi\rangle)_{\text{top}} \in \mathbb{C}, \] (40)
where the subscript top means that one has to take the coefficient of the top component i.e. the number multiplying \( (f_1)^+ (f_2)^+ \ldots (f_d)^+ |0\rangle \). In the following, the transformation properties of this product under spin(2d, \( \mathbb{C} \)) is of central importance to us so we shall examine it in detail here [26, 31].

**Proposition.** Let \( x \in W \oplus W^* \), \( |\phi\rangle, |\psi\rangle \in \mathcal{F} \). We have
\[ (x \phi, x \psi) = (x, x) (\phi, \psi). \] (41)
where the inner product \( (x, x) \) is the one defined in (9).
\textbf{Proof.} Fix an element \( \Omega = f_1 f_2 \cdots f_d \) in the one-dimensional space \( \wedge^d W^* \). This is unique up to a scale. It is not difficult to check that
\[
\Omega (f^1)^+ (f^2)^+ \cdots (f^d)^+ |0\rangle = (-1)^{\frac{d(d-1)}{2}} |0\rangle. \tag{42}
\]
Taking the coefficient of the top form is equivalent to writing
\[
((\phi)^I \wedge |\psi\rangle)_{\text{exp}} |0\rangle = (-1)^{\frac{d(d-1)}{2}} \Omega ((\phi)^I \wedge |\psi\rangle). \tag{43}
\]
Now any \( \phi \) element of \( \mathcal{F} \) can be written as a Clifford algebra element (such as \( \phi^{(0)} I + \phi^{(1)} (f^1)^+ + \phi^{(2)} (f^1)^+ (f^2)^+ + \cdots \)) acting on the vacuum \( |0\rangle \). So we choose \( \Phi, \Psi \in Cliff(W \oplus W^*) \) so that \( |\phi\rangle = \Phi |0\rangle \) and \( |\psi\rangle = \Psi |0\rangle \). This way we have
\[
(\phi, \psi) |0\rangle = (-1)^{\frac{d(d-1)}{2}} \Omega \Phi^I \Psi |0\rangle. \tag{44}
\]

Now since \( x \in W \oplus W^* \) by definition we have \((x\Phi)^I = \Phi^I x\)
\[
(\phi, x\psi) |0\rangle = (-1)^{\frac{d(d-1)}{2}} \Omega (x\Phi)^I x |0\rangle = (-1)^{\frac{d(d-1)}{2}} \Omega \Phi^I x^2 |0\rangle = (x, x)(\phi, \psi |0\rangle). \tag{45}
\]
where we used the defining relation of the Clifford algebra for \( x^2 \).
\[\Box\]

Using this result and (19) it is straightforward to see that for \( O \in \text{spin}(W \oplus W^*) = \text{spin}(2d, \mathbb{C}) \) we have
\[
(O \phi, O \psi) = \pm (\phi, \psi). \tag{46}
\]
If \( d = \dim \mathcal{H} \) is even, this is the so-called Mukai pairing on the irreducible subspaces \( \mathcal{F}^\pm \) which is symmetric if \( d = 4k \) and antisymmetric if \( d = 4k + 2 \). Let \( |\phi_n\rangle \) be the part of \( |\phi\rangle \in \mathcal{F} \) that lies in \( \wedge^d \mathcal{H} \). Then the Mukai pairing explicitly reads as
\[
(\phi, \psi) = \sum_m (-1)^m |\phi_{2m}\rangle \wedge |\psi_{d-2m}\rangle \in \wedge^d \mathcal{H} \otimes \wedge^d \mathcal{H}^* = \mathbb{C}, \tag{47}
\]
if \( |\phi\rangle, |\psi\rangle \in \mathcal{F}^+ \) and
\[
(\phi, \psi) = \sum_m (-1)^m |\phi_{2m+1}\rangle \wedge |\psi_{d-2m-1}\rangle \in \wedge^d \mathcal{H} \otimes \wedge^d \mathcal{H}^* = \mathbb{C}, \tag{48}
\]
if \( |\phi\rangle, |\psi\rangle \in \mathcal{F}^- \). As already mentioned, this satisfies \((O \phi, O \psi) = \pm (\phi, \psi)\) for any \( O \in \text{spin}(2d, \mathbb{C}) \). When \( d = 4k + 2 \) with the use of this invariant bilinear product one can associate elements of the Lie algebra \( \mathfrak{so}(2d, \mathbb{C}) \) to the elements of the Fock space, which can be used for classification of orbits and the construction of invariants. So define the moment map
\[
\mu : \mathcal{F} \rightarrow \mathfrak{so}(2d, \mathbb{C}),
\]
\[|\phi\rangle \mapsto \mathcal{T}_\phi, \tag{49}\]
as
\[g(\mathcal{T}_\phi, \mathcal{T}) = \frac{1}{2} (T \phi, \phi), \quad \forall T \in \mathfrak{so}(2d, \mathbb{C}), \tag{50}\]
where \( g \) is the Killing form. On \( \mathfrak{so}(2d, \mathbb{C}) \) we have \( g(\mathcal{T}_\phi, \mathcal{T}) = 2(d - 1) \text{Tr}(\mathcal{T}_\phi \mathcal{T}) \). Note that the invariance of the bilinear product requires \((T \phi, \phi) + (\phi, T \phi) = 0\), thus in the case of \( d = 4k \), where the product is symmetric, the moment map vanishes identically. However, if \( d = 4k \) the product \((\phi, \phi)\) is nonvanishing and is a good quadratic invariant. Nevertheless, when \( d = 4k + 2 \) the moment map is a useful tool because \( \mathcal{T}_\phi \) has good transformation properties.
After matching coefficients of $T_{O\phi}$, the element associated to $O|\phi\rangle$. By definition for every $T$ we have
\[
2(d - 1)\text{Tr}(T_{O\phi}T) = \frac{1}{2}(T O \phi, O \phi) = \frac{1}{2}(O^{-1} T O \phi, O \phi) = \frac{1}{2}(O^{-1} T O \phi, \phi) = 2(d - 1)\text{Tr}(T_{O\phi}O T O^{-1}) = 2(d - 1)\text{Tr}(O^{-1} T_{O\phi}O T), \tag{51}
\]
thus $T_{O\phi} = O^{-1} T_{O \phi} O$. As a consequence the rank of $T_\phi$ is invariant under the action of spin(2$d$, $\mathbb{C}$). Moreover the quantities
\[
q_k(\phi) = \frac{8^k(d - 1)^2}{2} \text{Tr}(T_k^k), \quad k \in \mathbb{N}
\tag{52}
\]
are invariant homogeneous degree 2$k$ polynomials in the coefficients of $|\phi\rangle$.

It is helpful to work out the explicit form of $T_\phi$. For this, put in the definition (50) $T$ and $T$ from (14) and (17) respectively and use the parametrization
\[
T_\phi = \frac{1}{8(d - 1)} \begin{bmatrix} [A_\phi]_{ij} & [B_\phi]_{jk} \\ [B_\phi]_{ki} & -[A_\phi]_{ij} \end{bmatrix} \tag{53}
\]
After matching coefficients of $A_{ik}, B_{jk}$ and $\beta_{il}$ one gets
\[
[A_\phi]_{ij} = ((f')^i f_j \phi, \phi),
[B_\phi]_{jk} = (f_j f_k \phi, \phi),
[\beta_{il}] = ((f')^i (f')^l \phi, \phi). \tag{54}
\]
Now write $|\phi\rangle \in F^+$ as
\[
|\phi\rangle = \sum_m \frac{1}{(2m)!} \phi^{(2m)}_{i_1 \ldots i_2m} (f')^+ \ldots (f')^+ |0\rangle. \tag{55}
\]
Also define the dual amplitudes $\tilde{\phi}_{(2m)}$ through
\[
\phi^{(d - 2m)}_{i_1 \ldots i_{d - 2m}} = \frac{1}{(2m)!} \tilde{\phi}^{(2m)}_{i_1 \ldots i_{d - 2m}} \epsilon_{j_1 \ldots j_{2m} i_1 \ldots i_{d - 2m}}. \tag{56}
\]
With these notations we have
\[
[A_\phi]_{ij} = \sum_m \frac{(-1)^m}{(2m - 1)!} \phi^{(2m)}_{i_j \ldots j_{2m}} \tilde{\phi}^{(2m)}_{i_j \ldots j_{2m}},
[B_\phi]_{jk} = \sum_m \frac{(-1)^m}{(2m)!} \phi^{(2m + 2)}_{i_j \ldots j_{2m + 2}} \tilde{\phi}^{(2m)}_{i_j \ldots j_{2m + 2}},
[\beta_{il}] = \sum_m \frac{(-1)^m}{(2m - 2)!} \phi^{(2m - 2)}_{i_j \ldots j_{2m - 2}} \tilde{\phi}^{(2m)}_{i_j \ldots j_{2m - 2}}. \tag{57}
\]
Formulas for $F^-$ can be easily obtained by replacing every $2m$ with $2m + 1$ whilst leaving the $(-1)^m$ factors unchanged.

5. Examples

5.1. Two state system

It is also helpful to work out the trivial example where $\mathcal{H} = \mathbb{C}^2$. The full Fock space is
\[
\mathcal{F}^+ = \mathbb{C} \oplus \mathbb{C}^2 \oplus \Lambda^2 \mathbb{C} \text{ with even and odd components:}
\]
\[
\mathcal{F}^+ = \mathbb{C} \oplus \Lambda^2 \mathbb{C},
\mathcal{F}^- = \mathbb{C}^2. \tag{58}
\]

which are simply one-qubit Hilbert spaces. Take
\[ |\phi\rangle = \phi_0|0\rangle + \phi_1 (f^1)^+ (f^2)^+ |0\rangle \in \mathcal{F}^+ \]
\[ |\psi\rangle = \psi_0 (f^1)^+ |0\rangle \in \mathcal{F}^- \]

An easy calculation shows that the moment maps for these states are
\[
T_\phi = \frac{1}{8} \begin{pmatrix}
-\psi \phi \phi_1 & 0 & 0 & -\phi_1^2 \\
0 & -\psi \phi \phi_1 & \phi_0 & 0 \\
0 & -\psi_2^2 & \phi_0 \phi_1 & 0 \\
\phi_0^2 & 0 & 0 & \phi_0 \phi_1 \\
\end{pmatrix},
\]
\[
T_\psi = \frac{1}{8} \begin{pmatrix}
\psi_1 \psi_2 & 0 & 0 & -\psi_1^2 \\
-\psi_1^2 & \psi_1 \psi_2 & 0 & 0 \\
0 & -\psi_1 \psi_2 & \psi_2^2 & 0 \\
0 & 0 & \psi_2^2 & \psi_1 \psi_2 \\
\end{pmatrix}.
\]

Of course both of these square to zero, giving \( q_k = 0 \) for all \( k \). Moreover, both matrices are either rank two or zero corresponding to the fact that we only have two group orbits: the trivial one with the zero vector and the rest. In the case of \( |\psi\rangle \) we have \( \beta |\psi\rangle = B |\psi\rangle = 0 \) thus only the \( A \) generators act non-trivially on \( \mathcal{F}^- \). This means that we only have to consider the action of \( GL(2, \mathbb{C}) \subset \text{spin}(4, \mathbb{C}) \) which is just the SLOCC group of the trivial one-qubit system.

Note that both \( |\phi\rangle \) and \( |\psi\rangle \) are pure spinors. Indeed, we see that \( |\psi\rangle \) is manifestly of the form (27), while \( |\phi\rangle \) can be written as
\[
|\phi\rangle = \phi_0 \exp \left( \frac{\phi_1}{\phi_0} (f^1)^+ (f^2)^+ \right) |0\rangle.
\]

We mention that the three-state system based on \( \mathcal{H} = \mathbb{C}^3 \) is also one spin orbit [29] thus contains only pure spinor states.

5.2. Four state system

Here \( \mathcal{H} = \mathbb{C}^4 \) and because the dimension is divisible by four we do not have a moment map. In the case of \( \mathcal{F}^+ \) we parametrize a state as
\[
|\phi\rangle = \eta |0\rangle + \frac{1}{2} \xi_{ij} (f^j)^+ (f^i)^+ |0\rangle + \frac{1}{4!} \rho \epsilon_{ijkl} (f^i)^+ (f^j)^+ (f^k)^+ (f^l)^+ |0\rangle.
\]

We have a quadratic invariant
\[
(\phi, \phi) = 2\eta \rho - 2 \text{Pf}(\xi),
\]
where the Pfaffian of the antisymmetric matrix \( \xi \) is \( \text{Pf}(\xi) = \frac{1}{4!} \epsilon^{ijkl} \xi_{ij} \xi_{kl} \). Pure spinors are the ones with [29, 31] \( (\phi, \phi) = 0 \). In particular for two fermion states \( \eta = \rho = 0 \) the relation \( (\phi, \phi) = -2 \text{Pf}(\xi) = 0 \) gives the Plücker relations which are necessary and sufficient conditions for a fixed fermion number state to be separable [37]. Notice that for normalized states the quantity \( 0 \leq 64|\text{Pf}(\xi)| \leq 1 \) is just the canonical entanglement measure used for two fermions with four single particle states [7, 8]. Notice that an arbitrary two-qubit state
\[
|x\rangle = \sum_{i,j=\{0,1\}} x_{ij} |i\rangle \otimes |j\rangle
\]
can be embedded into this fermionic system as [13, 14]
\[
|\xi_i\rangle = \sum_{i,j=\{0,1\}} \xi_{ij} (f^{i+1})^+ (f^{j+3})^+ |0\rangle.
\]

Under this embedding the entanglement measure 64|\text{Pf}(\xi)| boils down to the pure state version of the usual concurrence [39, 40].
Now take an element $|\psi\rangle$ of $\mathcal{F}^+$ parametrized as

$$|\psi\rangle = v_i (f_i')^+ |0\rangle + \frac{1}{3!} P_{ijk} (f_j')^+ (f_j')^+ (f_k')^+ |0\rangle.$$  

(66)

For this we have

$$(\psi, \psi) = \frac{1}{3} v_i P_{ijk} \epsilon^{ijkl},$$

(67)

showing us in particular that for a three-fermion state with $v_i = 0$ no entanglement can occur because of the duality $\wedge^3 \mathbb{C}^4 \cong \wedge^1 \mathbb{C}$. The space $\mathcal{F}^+$ contains two Spin orbits other than the zero vector [29]; one with $(\psi, \psi) = 0$ and one with $(\psi, \psi) \neq 0$.

5.3. Five state system

In the case of an odd dimensional single particle space $\mathcal{F}^+$ is dual to $\mathcal{F}^-$ so it is only necessary to consider one of them. In this case the pairing (40) is only nonvanishing between $\mathcal{F}^+$ and $\mathcal{F}^-$. This allows a slightly different construction than the moment map. We associate an element of $W \oplus W^*$ with an element of $\mathcal{F}^+$ parametrized as

$$|\phi\rangle = \eta |0\rangle + \frac{1}{2!} \xi_{ij}(f'_i)^+ (f'_j)^+ |0\rangle + \frac{1}{4!} \chi^a \epsilon_{ajijkl} (f'_i)^+ (f'_j)^+ (f'_k)^+ (f'_l)^+ |0\rangle,$$

(68)

in the following way.

$$(\eta, v_\phi) \implies (\eta \phi, \phi), \quad \forall \eta \in W \oplus W^*, \quad v_\phi \in W \oplus W^*,$$

(69)

where the product on the left is the one defined in (9), whilst the one on the right is the one defined in (40). A short calculation shows that the components of $v_\phi$ are

$$(v_\phi)_i = (f'_i \phi, \phi) = 2 \xi_{ik} \chi^k, \quad (v_\phi)_i' = ((f'_i)^+ \phi, \phi) = \frac{1}{12} \eta \chi^i - \frac{1}{2} \xi_{ik} \epsilon^{ijklm},$$

(70)

and by construction it transforms as an $SO(10, \mathbb{C})$ vector under spin(10, C). A quartic invariant can be constructed as $(v_\phi, v_\phi)$ but this turns out to be identically zero. However, $\mathcal{F}^+$ consists of two orbits [29] one with $v_\phi = 0$ and one with $v_\phi \neq 0$.

5.4. Six state system

Here $\mathcal{H} = \mathbb{C}^6$ and the Fock space is of the dimension 64. We begin with the 32 dimensional even particle subspace $\mathcal{F}^+$.

5.4.1. Even particle subspace. We parameterize a general state with two complex scalars $\eta$, $\xi$ and two antisymmetric $6 \times 6$ complex matrices $y$ and $x$ in the following way

$$|\phi\rangle = \eta |0\rangle + \frac{1}{2!} Y_{ab} (f'_a)^+ (f'_b)^+ |0\rangle + \frac{1}{24!} \chi^{abdef} (f'_a)^+ (f'_b)^+ (f'_c)^+ (f'_d)^+ |0\rangle$$

$$+ \frac{1}{6!} \xi_{abcdef} (f'_a)^+ (f'_b)^+ (f'_c)^+ (f'_d)^+ (f'_e)^+ (f'_f)^+ |0\rangle.$$  

(71)

Using (57) we can calculate the elements of the moment map

$$[A_\phi]_{ik} = 2 \xi_{ik} y_{ab} - \left( \frac{1}{2} \text{Tr}(xy) + \eta \xi \right) \delta_{ik},$$

$$[B_\phi]_{jk} = \frac{1}{2} \chi^{ab} x_{ab} \epsilon_{abcdj} - 2 \xi y_{jk},$$

$$[\beta_\phi]_{il} = \frac{1}{2} Y_{ab} y_{cd} \epsilon_{abcdil} - 2 \eta x_{il}.$$  

(72)

An easy calculation shows that we have a non-trivial quartic invariant

$$\frac{1}{6} q_2(\eta, y, x, \xi) = (\eta \xi + \frac{1}{2} \text{Tr}(xy))^2 + 4 \eta \text{Pf}(x) + 4 \xi \text{Pf}(y) - \frac{1}{4} \left( (\text{Tr}(xy))^2 - 2 \text{Tr}(xy xx) \right).$$  

(73)
Table 1. Canonical forms of the orbits of $\mathcal{F}^+$ in six dimensions. The parametrization is given in (75).

| Rank $|\phi|$ | Rank $T_\phi$ | $a$ | $b$ | $c$ | $d$ |
|----------|-------------|-----|-----|-----|-----|
| 4        | 12          | 1   | 1   | 1   | 1   |
| 3        | 6           | 1   | 1   | 1   | 0   |
| 2        | 2           | 1   | 1   | 0   | 0   |
| 1        | 0           | 1   | 0   | 0   | 0   |
| 0        | 0           | 0   | 0   | 0   | 0   |

where we have introduced the Pfaffian of an antisymmetric matrix $\text{Pf}(x) = \frac{1}{3!2^3} \epsilon_{abcdef} x^{ab} x^{cd} x^{ef}$.

We have shown in a previous paper [23] that the action of spin$(12, \mathbb{C})$ on $\mathcal{F}^+$ can be described in the language of the so-called Freudenthal triple systems [32, 33] which are widely known in mathematical and supergravity literature. Particularly $\mathcal{F}^+$, as a vector space, is isomorphic to the Freudenthal triple system over the biquaternions. An element of this is parameterized by two complex scalars and two biquaternion entry quaternion-Hermitian $3 \times 3$ matrices. The action of spin$(12, \mathbb{C})$ on $\mathcal{F}^+$ is then just the action of the conformal group of the cubic Jordan algebra $\text{Herm}(3, \mathbb{H}) \otimes \mathbb{C}$ on the Freudenthal system. Any Freudenthal triple system admits a quartic invariant and an antisymmetric bilinear product. The quartic invariant is just $\frac{1}{6} q_2(\phi)$, the bilinear product is just the pairing $(\phi, \psi)$. Every element of a Freudenthal triple system has a so-called Freudenthal dual which is cubic in the original parameters. This dual is just mapped to $T_\phi|\phi\rangle$. And finally a Freudenthal system always has five orbits under the action of its conformal group [32] thus we can deduce that $\mathcal{F}^+$ is split into five orbits under the action of spin$(12, \mathbb{C})$. These are:

1. rank $|\phi| = 4$ if $q_2(\phi) \neq 0$,
2. rank $|\phi| = 3$ if $q_2(\phi) = 0$ but $T_\phi|\phi\rangle \neq 0$,
3. rank $|\phi| = 2$ if $T_\phi|\phi\rangle = 0$ but $T_\phi \neq 0$,
4. rank $|\phi| = 1$ if $T_\phi \neq 0$ but $|\phi\rangle \neq 0$,
5. rank $|\phi| = 0$ if $|\phi\rangle = 0$.

The canonical form of an element of the GHZ-like first orbit is

$$|\phi_0\rangle = |0\rangle + (f^1)^+ (f^2)^+ (f^3)^+ (f^4)^+ (f^5)^+ (f^6)^+ |0\rangle,$$

i.e. $\eta = \xi = 1$ and $x = y = 0$. For this we have $\frac{1}{6} q_2(\phi_0) = 1$. The fourth class is the one of pure spinors. One can easily check that, for example, a state of the form $e^{-B} |0\rangle$ has a vanishing moment map.

We can list representatives from all of the classes. Consider a state parametrized by four complex numbers $a$, $b$, $c$, $d$ defined by

$$\eta = 0, y = \begin{pmatrix}
0 & a & 0 & 0 & 0 & 0 \\
-a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 \\
0 & 0 & -b & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & -c & 0 \\
\end{pmatrix}, \quad x = 0, \quad \xi = d. \quad (75)$$

For these states we have $\frac{1}{6} q_2(\phi_0) = 4\xi \text{Pf}(x) = 4abcd$. The values of the four parameters for the different classes can be found in table 1.
5.4.2. Odd particle subspace. Now consider \( \mathcal{F}^- \). A general state can be parametrized as
\[
|\psi\rangle = u_0 (f^a)^+|0\rangle + \frac{1}{3!} P_{abc} (f^b)^+ (f^c)^+ |0\rangle + \frac{1}{5!} w_{ij} \varepsilon_{abcd} (f^a)^+ (f^b)^+ (f^c)^+ (f^d)^+ |0\rangle,
\]
where \( u \) and \( w \) are six-dimensional complex vectors and \( P \) is a rank 3 antisymmetric tensor. Using (57) the moment map reads as
\[
[A_\phi]_k^i = 2w^i u_k - (K_\phi)^j_k w^a u_i b^j_k,
[B_\phi]_k^i = 2P_{akj} w^a,
[\beta_\phi]^{ij}_k = \frac{2}{3!} u_t P_{bcde} \varepsilon^{iabcdef},
\]
where we have defined the matrix
\[
(K_\phi)^j_k = \frac{1}{3!2!} P_{abc} P_{cde} \varepsilon^{iabcdef},
\]
which is important in the ordinary SLOCC classification of the three particle subspace. The non-trivial quartic invariant is
\[
q_2(\psi) = 6(w^a u_i)^2 - 4w^a u_j (K_\phi)^j_i + \text{Tr} K_\phi^2.
\]
We note that for simple three-fermion states of the form
\[
|\psi_0\rangle = \frac{1}{3!} P_{abc} (f^a)^+ (f^b)^+ (f^c)^+ |0\rangle,
\]
we have \( q_2(\psi_0) = \text{Tr} K_\phi^2 \) which is just the usual quartic invariant of three fermions with six single particle states [12]. Moreover rank \( \mathcal{F}_{\psi_0} = 2 \cdot \text{rank} K_\phi \) and rank \( K_\phi \) are enough to resolve all the SLOCC classes, namely if rank \( K_\phi = 6, 3, 1, \) or 0 then \( |\psi_0\rangle \) belongs to the GHZ, W, biseparable or separable class respectively [12, 17].

5.4.3. Embedded three-qubit system. Recall that an unnormalized three-qubit pure state is an element of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) and can be written with the help of eight complex amplitudes \( \Phi_{ijk} \) as
\[
|\Phi\rangle = \sum_{i,j,k \in \{0,1\}} \Phi_{ijk} |i\rangle \otimes |j\rangle \otimes |k\rangle,
\]
where \( |0\rangle, |1\rangle \) are the basis vectors of \( \mathbb{C}^2 \). The SLOCC group of the distinguishable three-qubit system [3, 4] is \( \text{GL}(2, \mathbb{C}) \otimes \text{GL}(2, \mathbb{C}) \otimes \text{GL}(2, \mathbb{C}) \) which acts on the amplitudes as
\[
\Phi_{ijk} \mapsto (G_1)_i^j (G_2)_j^k (G_3)_k^l \Phi_{ijlk},
\]
\( G_1 \otimes G_2 \otimes G_3 \in \text{GL}(2, \mathbb{C}) \otimes \text{GL}(2, \mathbb{C}) \otimes \text{GL}(2, \mathbb{C}) \).

In this section we show that this system can be embedded in both the even and odd particle subspaces of the six single particle state Fock space with the SLOCC group being a subgroup of the spin(12, \( \mathbb{C} \)) group. However, unlike in the odd particle subspaces, where the SLOCC group shows up as the expected subgroup of \( \text{GL}(6, \mathbb{C}) \subset \text{spin}(12, \mathbb{C}) \), in the even particle subspace this group also arises from taking into account the Bogoliubov transformations not belonging to the trivial \( \text{GL}(6, \mathbb{C}) \) subgroup.

Consider first the odd particle subspace. If we restrict ourselves to states of the form (80) and consider only the particle number conserving subgroup \( \text{GL}(6, \mathbb{C}) \subset \text{spin}(12, \mathbb{C}) \) we will end up with the usual SLOCC classification of three fermions with six single particle states. Now it is well known that three qubits can be embedded in this system [12, 13, 15, 16] as
\[
|P_0\rangle = \sum_{i,j,k \in \{0,1\}} \Phi_{ijk} (f^{i+1})^+ (f^{j+3})^+ (f^{k+5})^+ |0\rangle \in \mathcal{A}^3 \mathcal{H}.
\]
Now the three-qubit SLOCC group as a subgroup of $GL(6, \mathbb{C})$ is parametrized as

$$G = \left( \begin{array}{c|c|c} G_1 & G_2 & G_3 \\ \hline \end{array} \right) \in GL(6, \mathbb{C}).$$  \hfill (84)

This way the five entanglement classes of three qubits [3] are mapped exactly into the five entanglement classes of three fermions [12, 13, 15, 16].

Now we will also consider the even particle subspace. Our aim is to present the ‘bosonic’ (even number of particles) analogue $|\phi_0\rangle$ of the ‘fermionic’ state (odd number of particles) $|\Phi\rangle$. When considered alongside our previous case it will give a dual description of our three-qubit state $|\Phi\rangle$. As a consequence of these considerations we will see that the usual SLOCC group on three-qubits can be recovered from two very different physical situations.

With this aim in view we first calculate the general form of the spin transformations generated by (17) with only $B, \beta$ or $A$ not being zero on a state parametrized as in equation (71). For $e^{-B}$ we have

$$e^{-B} : \left( \begin{array}{c} \eta \\ y_{cd} \\ \xi \end{array} \right) \mapsto \left( \begin{array}{c} \eta \\ y_{cd} + \frac{1}{2} \eta B_{cd} \\ \xi - \eta Pf(B) - \frac{1}{2} Tr((B \times B)y) + \frac{1}{2} Tr(Bx) \end{array} \right).$$  \hfill (85)

and for $e^{-\beta}$ we have

$$e^{-\beta} : \left( \begin{array}{c} \eta \\ y_{cd} \\ \xi \end{array} \right) \mapsto \left( \begin{array}{c} \eta + \frac{1}{2} \beta Pf(\beta) - \frac{1}{2} Tr((B \times B)x) - \frac{1}{2} Tr(\beta B) \\ y_{cd} + \frac{1}{2} \beta (B \times B)_{cd} + (B \times B)_{ab} \\ \xi - \frac{1}{2} \beta B_{cd} \end{array} \right).$$  \hfill (86)

Here we have introduced the notation $(C \times D)^{ab} = \frac{1}{2} \epsilon^{abcde} C_{cd} D_{ef}$. Finally, using (24) and the formula $\epsilon^{abcde} G^{ab} \cdots G^{ef} = (\det G)\epsilon^{efgde}cdef$ we see that the transformation $e^{A - \frac{1}{2} Tr A}$ acts as

$$e^{A - \frac{1}{2} Tr A} : \left( \begin{array}{c} \eta \\ y_{cd} \\ \xi \end{array} \right) \mapsto \left( \begin{array}{c} (\det G)^{-\frac{1}{2}} \eta \\ (\det G)^{-\frac{1}{2}} G_{cd}^{\mu \nu} \eta^{\mu \nu} \\ (\det G)^{-\frac{1}{2}} G_{cd}^{\mu \nu} \eta^{\mu \nu} x_{cd} \end{array} \right).$$  \hfill (87)

where the $6 \times 6$ matrix $G^{\mu \nu}_b$ is the matrix exponential of the coefficient matrix $A^{\mu \nu}_b$.

Let us now define the state $|\phi_0\rangle$. For this we input, in equation (71), the eight amplitudes of the three-qubit state $|\Phi\rangle$ of equation (81) as

$$\eta = \Phi_{000}, \quad y = \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & \Phi_{100} & 0 \\ 0 & 0 & 0 & 0 & \Phi_{010} & 0 \\ -\Phi_{100} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Phi_{010} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Phi_{001} & 0 & 0 & 0 \end{array} \right).$$

$$\xi = -\Phi_{111}, \quad x = \left( \begin{array}{cccccc} 0 & 0 & 0 & \Phi_{111} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi_{011} & 0 \\ 0 & 0 & 0 & 0 & \Phi_{101} & 0 \\ 0 & 0 & 0 & 0 & -\Phi_{111} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{110} \\ \Phi_{011} & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$  \hfill (88)

A very important feature of this embedding is that the mapping between the five well-known three-qubit SLOCC classes, namely the GHZ, W, Bisep, Sep and Null [3] and the spin group...
orbits of table 1 is one to one. Now with the use of equations (85)–(87) one can easily check that the $B$-transformation generated by

$$B = \begin{pmatrix}
0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & 0 & 0 & -b & 0 \\
a & 0 & 0 & 0 & 0 & -c \\
b & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0 & 0
\end{pmatrix},$$

simply implements the three-qubit SLOCC transformation

$$G_B = \left( \begin{array}{c}
a \\
0 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
b \\
0 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
c \\
0 \\
1
\end{array} \right),$$

while the $\beta$-transformation generated by

$$\beta = \begin{pmatrix}
0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & c \\
-\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma & 0 & 0 & 0
\end{pmatrix},$$

implements

$$G_{\beta} = \left( \begin{array}{c}
a \\
0 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
b \\
0 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
c \\
0 \\
1
\end{array} \right).$$

Finally the $A$-transformation generated by

$$A = \begin{pmatrix}
\log a^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & \log b^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \log c^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & \log a^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & \log b^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \log c^{-1}
\end{pmatrix},$$

implements the SLOCC transformation

$$G_A = \left( \begin{array}{c}
a \\
0 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
b \\
0 \\
1
\end{array} \right) \otimes \left( \begin{array}{c}
c \\
0 \\
1
\end{array} \right).$$

This way with successive $B, \beta$ and $A$ transformations we can generate the group $\text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C})$ acting properly on three-qubit amplitudes [3]. Note however, that in order to reproduce the full SLOCC group $\text{GL}(2, \mathbb{C}) \otimes \text{GL}(2, \mathbb{C}) \otimes \text{GL}(2, \mathbb{C})$ in this dual picture we need to extend the spin group to $\mathbb{C}^\times \times \text{spin}(12, \mathbb{C})$.

It is also important to realize that, when evaluated at $|\phi\rangle$, the quartic invariant of (73) is just Cayley’s hyperdeterminant appearing in the definition of the three-tangle well known from studies concerning three-qubit entanglement [39]. Notice also that although the $B$ and $\beta$ transforms are of Bogoliubov type, and hence they change the particle number in the original picture, they still implement ordinary SLOCC transformations. For these Bogoliubov type transformations this observation gives rise to a conventional entanglement based interpretation.

Let us illustrate this interesting duality between the two different representations $|P_{\beta}\rangle$ and $|\phi\rangle$ of our three-qubit state $|\Phi\rangle$ by a set-up borrowed from solid state physics [8, 14]. Suppose we have three nodes where spin 1/2 fermions can be localized. The states associated to these nodes will be denoted by $|r_1\rangle, |r_2\rangle, |r_3\rangle$. They are representing a fermion localized on the first, second or the third node. These span a three-dimensional complex Hilbert space.
Figure 1. Three-qubit states \(|000\rangle, |001\rangle\) and \(|011\rangle\) as single occupancy states in a three-fermion three node system (left) and as double occupancy states in an even fermion number three node system (right).

denoted by \(\mathcal{H}_{\text{node}}\). The two-dimensional Hilbert space of a spin 1/2 is spanned by \(|\uparrow\rangle, |\downarrow\rangle\) and is denoted by \(\mathcal{H}_{\text{spin}}\). Thus the single particle Hilbert space is the six-dimensional one: \(\mathcal{H} = \mathcal{H}_{\text{node}} \otimes \mathcal{H}_{\text{spin}}\).

Moving to multi-particle states, the creation operator associated to the basis \(|r_i\rangle \otimes |\sigma\rangle\) is \((f_{r_i,\sigma})^+\). Let us introduce a short-hand notation. We will use simple numbers to denote the nodes with spin up and numbers with bars to denote nodes with spin down, i.e. \((f_{i,\uparrow})^+ = (f_i^+)^+\) and \((f_{i,\downarrow})^+ = (f_i^\bar{t})^+\), \(i = 1, 2, 3\). The connection with the labeling used in (71) is simply \(1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 3, 4 \leftrightarrow 1, 5 \leftrightarrow 2, 6 \leftrightarrow 3\). With this notation, when embedding three qubits in a three-fermion system as in equation (83) and after a relabelling of indices as \((i + 1, j + 3, k + 5) \mapsto (3i + 1, 3j + 2, 3k + 3)\), one can set up the following correspondence between basis states:

\[
|000\rangle \leftrightarrow (f_{1}^+)^+ (f_{2}^+)^+ (f_{3}^+)^+ |0\rangle, \quad |001\rangle \leftrightarrow (f_{1}^+)^+ (f_{2}^+)^+ (f_{3}^\bar{t})^+ |0\rangle, \\
|011\rangle \leftrightarrow (f_{1}^+)^+ (f_{2}^\bar{t})^+ (f_{3}^\bar{t})^+ |0\rangle, \quad \text{etc.}
\]

This clearly shows that the three-qubit states are mapped to the single occupancy states of the three-fermion system [14]. In this case on every site we can find at most one fermion (see: left side of figure 1.). On the other hand when the embedding is made into the even particle Fock space as in (88) the correspondence between states reads as

\[
|000\rangle \leftrightarrow |0\rangle, |001\rangle \leftrightarrow [(f_{3}^+)^+ (f_{3}^\bar{t})^+] |0\rangle, \\
|011\rangle \leftrightarrow [(f_{2}^+)^+ (f_{2}^\bar{t})^+] [(f_{3}^+)^+ (f_{3}^\bar{t})^+] |0\rangle, \quad \text{etc.}
\]

This shows that the three-qubit states are mapped to the double occupancy states of the even particle number Fock space. In this case only empty and twice occupied nodes are allowed (see: right side of figure 1.).

5.4.4. Relation to wrapped membrane configurations in string theory. Let us also mention that the duality, as described above, has a particularly nice realization in string theory. Indeed, after invoking some ideas of the recently discovered Black-Hole/Qubit Correspondence [36] one can also map the two possible embeddings \(|P_{\Phi}\rangle\) and \(|\phi_{\Phi}\rangle\) of a three-qubit state \(|\Phi\rangle\) to the so-called IIA and IIB duality frames of toroidal compactifications of type IIA and IIB string theory. These theories are consistent merely in ten dimensions [41], hence to account
for the four space-time dimensions we see six dimensions have to be compactified to tiny six-dimensional tori $T^6$. One can regard this six-torus as $T^6 = \mathbb{T}_2 \times \mathbb{T}_2 \times \mathbb{T}_2$. Hence the three sites of figure 1 in this picture correspond to three 2-tori. Now recall that these theories are featuring extended objects called (mem)branes that can be interpreted as qubits [34]. In the case of IIB string theory the situation is depicted by the left-hand side of figure 1. In this case, the theory contains 3-branes which correspond to our three fermions. These branes wrap around the noncontractible cycles of $T^2 \times T^2 \times T^2$. A single $T^2$ contains two basic noncontractible loops. These loops correspond to the two possibilities of spin up and spin down. Hence the basis states on the left-hand side of figure 1 give a nice mnemonic for the basic wrapping configurations of the 3-branes in the IIB picture. The three-qubit amplitudes multiplying these basis states are integers corresponding to the winding numbers. In the usual four space-time dimensional physics they are reinterpreted as charges of electric and magnetic type. If we also allow the shapes of these tori to fluctuate in such a way that the volumes are unchanged the natural basis states will be modified. In this case the amplitudes of the three-qubit states turn out to be complex, depending on the deformation parameters and the winding numbers [35]. Hence one can summarize this interpretation of the left-hand side of figure 1 as: sites with possible spin projections correspond to 2-tori taken together with their basic loops, and that the three fermions correspond to 3-branes.

In the case of IIA string theory we have 0-branes, 2-branes, 4-branes and 6-branes. Now they correspond to states containing 0, 2, 4, 6 fermions in a Fock space. Clearly, unlike in the previous case, now the number of fermions is not fixed. The membranes of different dimensions can wrap different dimensional subspaces of $T^2 \times T^2 \times T^2$. Clearly the vacuum state corresponds to a 0-brane not wrapping any volume. 2-branes are wrapping any of the volumes of the three 2-tori $T^2$ etc. Generally we have $2n$-branes wrapping the $2n$-volumes of $T^{2n}$ tori with $n = 0, 1, 2, 3$. Some of the basis configurations are illustrated by the states on the right-hand side of figure 1. Again the amplitudes multiplying these basis states are integers corresponding to the wrapping numbers, facilitating an interpretation of electric and magnetic charges in the usual four-dimensional space-time picture. Note that in this case the volumes of the sites i.e. the 2-tori are subject to fluctuations [41]. Again in a convenient basis, adapted to volume fluctuations, we will be given three-qubit states with the complex amplitudes instead depending on the deformation parameters and wrapping numbers.

The two different embeddings of the same SLOCC subgroup (the so-called U-duality group) $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ provide a nice example of the two wildly different physical scenarios that are amenable to a three-qubit interpretation. It is also important to realize that under duality the deformation parameters of shape and volume of the tori are also exchanged. This is an example of the famous mirror symmetry [41] well known in string theory. Is there an analogue of this phenomenon in solid state physics? For the implementation of this idea clearly the amplitudes of the states $|P\rangle$ and $|\phi\rangle$ should depend somehow on additional parameters with peculiar properties specified by some extra physical constraints.

6. Conclusions

In this paper we proposed a generalization of the usual SLOCC and LU classification of entangled quantum systems represented by fermionic Fock spaces. Our approach is based on the representation theory of the spin group. In particular for classifying the entanglement types of fermionic systems with $d$ single particle states and an indefinite number of particles; we suggested the group $\text{spin}(2d, \mathbb{C})$. Our new classification scheme naturally incorporates the usual one, based on fixed particle number subspaces. However, our approach also gives rise
to the notion of equivalence under transformations that are changing the particle number. As far as entanglement is concerned, the new ingredient in our scheme is the identification of states that can be obtained from each other by Bogoliubov transformations. There are always at least two orbits corresponding to the ‘fermionic’ (odd number of particles) and ‘bosonic’ (even number of particles) subspaces of the full Fock space. Thus entanglement is prohibited between these different kinds of states.

The totally unentangled states are represented by the pure spinors. They incorporate all the possible vacua and the states that can be written in terms of a single Slater determinant. Local processes that conserve probabilities are described by the (fermionic) LU group $U(d)$. The generalization of this group should be the compact real subgroup $SO(2d, \mathbb{R})$. This group also has the properties mentioned previously, but in addition it only incorporates the LOCC group and it conserves the probabilities.

In order to enable an explicit construction, in the second half of this paper we have presented some useful mathematical tools. Namely, we have introduced notions such as the Mukai pairing and the moment map. We have constructed the basic invariants of the spin group and have shown how some of the known SLOCC invariants are just their special cases. As an example, we have presented the classification of fermionic systems based on the group $\text{spin}(12, \mathbb{C})$ where the underlying single particle Hilbert space has dimension-six. An intriguing duality between the two different possibilities for embedding three-qubit systems inside this fermionic system has been revealed. We have elucidated this duality by establishing an interesting connection between such embedded three-qubit systems and configurations of wrapped membranes reinterpreted as qubits. Finally we mention that for the treatment of the bosonic case it is clear that the group $SO(2d, \mathbb{C})$ has to be replaced by $Sp(2d, \mathbb{C})$ but the Fock space, in this case, is infinite dimensional which questions the possibility of further development.

Despite how natural this generalization is, currently it is not known to us whether there exist *realistic* physical scenarios where our newly proposed classification scheme turns out to be useful. Such possibilities certainly deserve attention to be fully explored and will be subject to further investigations.

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