DYNAMIC LOCALIZATION EFFECTS IN L-RING CIRCUIT

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Using suitable magnetic flux operators established in terms of discrete derivatives leads to quantum-mechanical descriptions of LC-circuits with an external time dependent periodic voltage. This leads to second order discrete Schrödinger equations provided by discretization conditions of the electric charge. Neglecting the capacitance leads to a simplified description of the L-ring circuit threaded by a related time dependent magnetic flux. The equivalence with electrons moving on one dimensional (1D) lattices under the influence of time dependent electric fields can then be readily established. This opens the way to derive dynamic localization conditions serving to applications in several areas, like the time dependent electron transport in quantum wires or the generation of higher harmonics by 1D conductors. Such conditions, which can be viewed as an exact generalization of the ones derived before by Dunlap and Kenkre [Phys. Rev. B 34, 3625(1986)], proceed in terms of zero values of time averages of related persistent currents over one period.

**Keywords**: Charge discretization; Quantum L-ring circuits; Persistent currents; Dynamic localization conditions; Quantum wires

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I. INTRODUCTION

Quantum circuits with charge discreteness have been the focus considerable interest during the last decade [1-5]. It has been realized that quantum circuits are able to provide useful developments in the field of nanodevices, transmission lines as well as of molecular electronic circuits [6-8]. The chargediscreteness referred to above gets incorporated in the eigenvalue equation

\[ Q|n> = n q_e |n> \]  \hspace{1cm} (1)

where \( n \) is an integer and where \( Q \) denotes the Hermitian operator of the electric charge. The elementary charge is denoted by \( q_e \). The common choice is to put \( q_e = e \), where \( e \) is the charge of the electron. The charge eigenfunctions are orthonormalized, in which case the time dependent wavefunction describing the quantum circuit can be expressed as

\[ |\psi(t)> = \sum_n C_n(t)|n> \]  \hspace{1cm} (2)

where \( n \in (-\infty, \infty) \) We have also to keep in mind that usual derivatives have to be replaced by discrete ones when dealing with \( n \) dependent functions. Proceeding in this manner leads to a discrete Schrödinger equation, as shown before [1-3]. On the other hand, the influence of the capacitance can be disregarded, which results in a discrete Schrödinger-equation concerning the L-ring circuit. The interesting point is that this latter equation is equivalent, under suitable matching conditions, to the one describing the 1D conductor, i.e. to the electron on the 1D lattice under the influence of a time dependent electric field. This opens the way to a general derivation of dynamic localization conditions (DLC’s) for periodic modulations of the voltage.

II. PRELIMINARIES AND NOTATIONS

One starts from the classical description of Kirchhoff’s law for the LC-circuit with a voltage source in terms of Hamilton’s equations. This amounts to consider a classical Hamiltonian like \( H(\Phi/c, Q) \), where \( \Phi \) denotes the magnetic flux. Accordingly \( Q \) and \( \Phi/c \) play the role of the coordinate and momentum, respectively. The next step is to perform the quantization of canonically conjugated variables. For this purpose, the usual canonical commutation relations could be applied [9, 10]. However, we have to account for the discreteness of the electric charge, which means that the appropriate quantum mechanical description of the magnetic flux should proceed in terms of discrete derivatives [2, 3]. Under such conditions one gets faced with a deformation of the Heisenberg algebra emphasized usually. This results in the Hamiltonian

\[ H \left( \frac{\Phi}{c}, Q \right) = \frac{1}{2Lc^2} \Phi^+ \Phi + \frac{Q^2}{2C} - QV_s(t) \]  \hspace{1cm} (3)

where \( V_s(t) \) denotes the external time dependent voltage. Now the magnetic flux is described by the non-Hermitian operator

\[ \Phi = -i \frac{h}{q_e} \Delta \]  \hspace{1cm} (4)

so that \( \Phi^+ = -i(hc/q_e)\Delta \), where the “+” superscript stands for the Hermitian conjugation. The right and left-hand discrete derivatives displayed above act as \( \Delta f(n) = f(n+1) - f(n) \) and \( \nabla f(n) = f(n) - f(n-1) \), respectively. The present quantization condition is then given by the deformed relationship.
\[ [\Phi, Q] = i\hbar c \left(1 + i\frac{q_e}{\hbar c}\Phi \right) \] (5)

which differs from the usual canonical commutation relation by virtue of the deformation rule \(1 \rightarrow 1 + (iq_e\Phi)/\hbar c\), such as displayed by the r.h.s. in (5). Resorting to (2), then yields the discrete time dependent Schrödinger-equation [1]

\[-\frac{\hbar^2}{2Lq_e^2} (C_{n+1}(t) + C_{n-1}(t)) + MC_n(t) = i\hbar \frac{\partial}{\partial t} C_n(t) \] (6)

in which the \((\hbar^2c^2)/(Lq_e^2)\)-term in (6) can be ruled out by virtue of the gauge transformation

\[ C_n(t) \rightarrow C_n(t) e^{-i\frac{\hbar^2}{Lq_e}t} \] (7)

where \(M = \left(\frac{\hbar^2}{2c^2}n^2 - nq_eV_s(t) + \frac{\hbar^2c^2}{Lq_e^2}\right)\).

III. THE DUALITY BETWEEN THE L-RING CIRCUIT AND THE ELECTRON ON THE 1D LATTICE

Neglecting the capacitance and applying the \(C \rightarrow \infty\) - limit, the presence of the harmonic oscillator term in (6) is ruled out. This results in an explicit discrete Schrödinger equation for the L-ring circuit only. On the other hand the electron on the 1D lattice under the influence of an external time dependent electric field like \(E_e(t) = (\hbar E_F f(t))/(ca)\), where \(a\) denotes the lattice spacing, is characterized by the evolution equation [11]

\[ \hbar V (C_{n+1}(t) + C_{n-1}(t)) - \hbar E_F n f(t) C_n(t) = i\hbar \frac{\partial}{\partial t} C_n(t) \] (8)

Keeping in mind that \(C \rightarrow \infty\), one sees that (6) reproduces precisely (8) in terms of the matching conditions

\[ V_s(t) = \frac{\hbar}{q_e} E_F f(t) = -\frac{d}{dt}\Phi_e(t) \] (9)

and \(\hbar V = -\hbar^2/(2Lq_e^2)\). One remarks, of course, that (9) incorporates Faraday’s law. Accordingly, there is a duality between the L-ring circuit and the electron on the 1D-lattice, which is useful for the study of dynamic localization effects [11].

IV. DYNAMIC LOCALIZATION EFFECTS

Let us consider a sinusoidal modulation of the magnetic flux like

\[ \Phi_e(t) = \Phi_0 \sin(wt) \] (10)

This yields a time dependent electric field modulated by the characteristic function

\[ f(t) = \cos(wt) \] (11)

so that the field amplitude is given by

\[ E_F = -\frac{q_e}{\hbar c}\Phi_0 w \] (12)

Now we have to remember that there is a periodic return of the electron to the initially occupied site if the quotient \(E_F/w\) is a root of the Bessel function of order zero and of the first kind [11]:

\[ J_0 \left(\frac{E_F}{w}\right) = 0 \] (13)

Accordingly, the mean square displacement (MSD)

\[ < n^2 > = \sum_{n=-\infty}^{n=\infty} |C_n(t)|^2 n^2 \] (14)

Remains bounded in time, which is synonymous to the onset of the dynamic localization. We then have to realize that this latter effect should also concern the carriers of the discretized charge in the L-ring circuit. Indeed, (13) can be rewritten equivalently as

\[ J_0 \left(\frac{q_e\Phi_0}{\hbar c}\right) = 0 \] (15)

whereas the corresponding MSD has been established [3] in terms of velocity autocorrelation functions [12]. Dynamic localization conditions like (13) and/or (15) are of interest to the study of time dependent electron transport through quantum wires [13, 14], of the generation of higher harmonics by 1D conductors [15] as well as of other areas [16]. The dynamic localization has been observed in the linear optical absorption spectra of quantum dot superlattices [17]. Negative conductances observed in photon assisted tunneling effects can also be invoked [18].

V. PERSISTENT CURRENTS

In order to proceed further, let us remember that the persistent current carried by a flux dependent energy level \(E = E(k, \Phi_e)\) at is [19]

\[ I_k(\Phi_e) = -c\frac{\partial}{\partial\Phi_e} E(k, \Phi_e) \] (16)
where the presence of the wavenumber $k$ accounts for the existence of energy bands. We shall also assume that $f(t)$ in (9) is periodic in time with period $T$. On the other hand, the energy dispersion law characterizing (8) reads

$$E(k) = 2\hbar V \cos(ka)$$

such as produced by inserting the free-field amplitude

$$C_n(t) = e^{-\frac{i}{\hbar}E(k)t} e^{ikan}$$

into the $E_F = 0$ form of (8). What then remains is to account for the external time dependent electric field

$$E_e(t) = -\frac{1}{c} \frac{\partial}{\partial t} A_1(t)$$

in terms of the minimal substitution

$$\hbar k \rightarrow \hbar k + \frac{e}{c} A_1(t)$$

in which

$$A_1(t) = -\frac{\hbar c}{ea} E_F \eta(t)$$

and

$$\eta(t) = \int_0^t f(s) \, ds$$

Accordingly, (17) becomes

$$E(k) \rightarrow E(k, \Phi_e) = 2\hbar V \cos \left( ka + \frac{q_e}{\hbar c} \Phi_e(t) \right)$$

which yields the persistent current

$$I_k(\Phi_e) = -\frac{\hbar}{q_e L} \sin \left( ka + \frac{q_e}{\hbar c} \Phi_e \right)$$

by virtue of (16). Similar results have been written down before [3] by resorting to the -wavenumber representation of (6).

**VI. DYNAMIC LOCALIZATION CONDITIONS FOR ARBITRARY TIME PERIODIC MODULATIONS**

Performing the time average over one period of (24) yields the averaged current

$$J_k(T) = -\frac{\hbar}{q_e L T} \int_0^T \sin \left( ka + \frac{q_e}{\hbar c} \Phi_e(t) \right) dt$$

which deserves further attention. Indeed, accounting for (11), one finds that the DLC (13) gets reproduced whenever

$$J_k(T) = 0$$

irrespective of $k$. For this purpose we have to resort to the generating function of Bessel functions of the first kind [20]

$$e^{iz \sin \varphi} = \sum_{m=-\infty}^{m=+\infty} J_m(z) e^{im \varphi}$$

It is also clear that integrating (9) yields

$$\Phi_e(t) - \Phi_e(0) = -\frac{\hbar c}{q_e} E_F \eta(t)$$

so that gets specified in terms of and conversely.

The dc-ac electric field $E(t) = E_0 + E_1 \cos (wt)$ for which

$$f(t) = \frac{w_B}{E_F} + \cos(wt)$$

can also be readily discussed. Now one has $E_F = (E_1 e a) / \hbar$, while denotes the Bloch frequency. Inserting (29) into (26) yields the DLC

$$J_n \left( \frac{E_F}{w} \right) = 0$$

if $w_B = nw$, where $n$ is a positive integer, such as established before within the quasi-energy description [21,22]. This latter equation can be viewed as an exact $n \neq 0$ counterpart of (13). Rational realizations of the matching ratio ($w_B/\omega$) require, however, a special treatment [21, 22].

Other cases can be treated in a similar manner.

**VII. CONCLUSIONS**

In this paper the quantum L-ring circuit threaded by a time dependent magnetic flux has been discussed with a special emphasis on the equivalence with the electron on the 1D lattice under the influence of a periodic time dependent electric field. Proceeding in this manner leads to (26), which has the meaning of an exact generalization of (13). Such results are useful for applications in
several areas already referred to above. Generalizations going beyond the nearest neighbor description deserve further attention, too [23].

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**References**

[1] Y. Li and B. Chen, Phys. Rev. B 53, 4027 (1996)
[2] T. Lu and Y. Q. Li, Mod. Phys. Lett. B 16, 975 (2002)
[3] B. Chen, X. Dai and R. Han, Phys. Lett. A 302, 325 (2002)
[4] B. Chen et al, Phys. Lett. A 335, 103 (2005)
[5] J. C. Flores, M. Bologna, K.J. Chandia and C. A. Utreras Diaz, Phys. Rev. B 74, 193319 (2006)
[6] S. Ani and C. Joachim, Phys. Rev. B 65, 155419 (2002)
[7] A. Wacker, Phys. Rep. 357, 1 (2002)
[8] M. Lieberman (Ed.), Molecules as Components of Electronic Devices (Oxford University Press, Oxford, 2003).
[9] W. H. Louisell, Quantum Statistical Properties of Radiation (John Willey, New York, 1973)
[10] J. C. Flores and C. A. Utreras Diaz, Phys. Rev. B 66, 153410 (2002)
[11] D. H. Dunlap and V. M. Kenkre, Phys. Rev. B 34, 3625 (1986)
[12] V. M. Kenkre, R. Kühne and P. Reineker, Z. Phys. B 41, 177 (1981)
[13] T. Kwapinski, Phys. Rev. B 69, 153303 (2004)
[14] I. Tralle, Physica E 9, 275 (2001)
[15] K. A. Pronin, A. D. Bandrauk and A. A. Ovchinnikov, Phys. Rev. B 50, 3473 (1994)
[16] V. M. Kenkre and S. Raghavan, J. Opt. B: Quantum Semiclass. Opt. 2, 686 (2000)
[17] J. R. Madureira, P. A. Schulz and M. Z. Maialle, Phys. Rev. B 70, 033309 (2004)
[18] B. J. Keay, S. Zeuner, S. J. Allen Jr., K. D. Maranowski, A. C. Gossard, U. Bhattacharya and M. J. Rodwell, Phys. Rev. Lett. 75, 4102 (1995)
[19] H. F. Cheung, Y. Gefen, E. K. Riedel and W. H. Shih, Phys. Rev. B 37, 6050 (1988)
[20] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press, New York, 1980)
[21] A. Z. Zhang, P. Zhang, D. Suqing, X. G. Zhao and J. Q. Liang, Phys. Lett. A 292, 275 (2002)
[22] X. G. Zhao, R. Jahnke and Q. Niu, Phys. Lett. A 202, 297 (1995)
[23] M. A. Jivulescu and E. Papp, J. Phys. Condens. Matter 18, 6853 (2006)

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