Global Navier-Stokes flows for non-decaying initial data with slowly decaying oscillation

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Abstract

Consider the Cauchy problem of incompressible Navier-Stokes equations in $\mathbb{R}^3$ with uniformly locally square integrable initial data. If the square integral of the initial datum on a ball vanishes as the ball goes to infinity, the existence of a time-global weak solution has been known. However, such data do not include constants, and the only known global solutions for non-decaying data are either for perturbations of constants, or when the velocity gradients are in $L^p$ with finite $p$. In this paper, we construct global weak solutions for non-decaying initial data whose local oscillations decay, no matter how slowly.

Keywords: incompressible Navier-Stokes equations, non-decaying initial data, oscillation decay, global existence, local energy solution

Mathematics Subject Classification (2010): 35Q30, 76D05, 35D30

1 Introduction

In this paper, we consider the incompressible Navier-Stokes equations

$$\begin{cases}
\partial_t v - \Delta v + (v \cdot \nabla)v + \nabla p = 0 \\
\text{div} v = 0 \\
v|_{t=0} = v_0
\end{cases} \tag{NS}$$

in $\mathbb{R}^3 \times (0, T)$ for $0 < T \leq \infty$. These equations describe the flow of incompressible viscous fluids, so the solution $v : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \times (0, T) \to \mathbb{R}$ represent the flow velocity and the pressure, respectively.

For an initial datum with finite kinetic energy, $v_0 \in L^2(\mathbb{R}^3)$, the existence of a time-global weak solution dates back to Leray [23]. This solution has a finite global energy, i.e., it satisfies the energy inequality:

$$\|v(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2 \|\nabla v\|_{L^2(0, t; L^2(\mathbb{R}^3))}^2 \leq \|v_0\|_{L^2(\mathbb{R}^3)}^2, \quad \forall t > 0. \tag{1.1}$$

In Hopf [11], this result is extended to smooth bounded domains with the Dirichlet boundary condition. We say $v$ is a Leray-Hopf weak solution to (NS) in $\Omega \times (0, T)$ for a domain $\Omega \subset \mathbb{R}^3$, if

$$v \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_{0,\sigma}(\Omega)) \cap C_{wk}([0, T); L^2_\sigma(\Omega))$$

satisfies the weak form of (NS) and the energy inequality (1.1).

However, when a fluid fills an unbounded domain, it is possible to have finite local energy but infinite global energy. One such example is a fluid with constant velocity. There are also many interesting non-decaying infinite energy flows like time-dependent spatially
periodic flows (flows on torus) and two-and-a-half dimensional flows; see [27, Section 2.3.1] and [7]. Can we get global existence for such data? To analyze the motion of such fluids, one may consider the class $L^q_{uloc}$ for velocity field $v_0$ in $\mathbb{R}^3$ whose kinetic energy is uniformly locally bounded. Here, for $1 \leq q \leq \infty$, we denote by $L^q_{uloc}$ the space of functions in $\mathbb{R}^3$ with

$$\|v_0\|_{L^q_{uloc}} := \sup_{x_0 \in \mathbb{R}^3} \|v_0\|_{L^q(B(x_0,1))} < \infty.$$ 

We also denote its subspace with spatial decay

$$E^q = \{ v_0 \in L^q_{uloc} : \lim_{|x_0| \to \infty} \|v_0\|_{L^q(B(x_0,1))} = 0 \}.$$ 

In [20], Lemarié-Rieusset introduced the class of local energy solutions for initial data $v_0 \in L^2_{uloc}$ (see Section 3 for details). He proved the short time existence for initial data in $L^2_{uloc}$, and the global in time existence for $v_0 \in E^2$, those initial data in $L^2_{uloc}$ which further satisfy the spatial decay condition

$$\lim_{|x_0| \to \infty} \int_{B(x_0,1)} |v_0|^2 dx = 0. \quad (1.2)$$

Then, Kikuchi-Seregin [17] added more details to the results in [20], especially the careful treatment of the pressure. They also allowed a force term $g$ in (NS) which satisfies $\text{div} \, g = 0$ and

$$\lim_{|x_0| \to \infty} \int_0^T \int_{B(x_0,1)} |g(x,t)|^2 dx dt = 0, \quad \forall T > 0.$$ 

Recently, Maekawa-Miura-Prange [25] generalized this result to the half-space $\mathbb{R}^3_+$. The treatment of the pressure in [25] is even more complicated.

One key difficulty in the study of infinite energy solutions is the estimates of the pressure. While finite energy solutions have enough decay at spatial infinity and one may often get the pressure from the equation $p = (-\Delta)^{-1} \partial_i \partial_j (v_i v_j)$, this is not applicable to infinite energy solutions because of their slow (or no) spatial decay.

To estimate the pressure, the definition of a local energy solution in [17] includes a locally-defined pressure decomposition near each point in $\mathbb{R}^3$, see condition (v) in Definition 3.1. (It is already in [20] but not part of the definition.) In [12]-[13], on the other hand, Jia and Šverák use a slightly different definition by replacing the decomposition condition by the spatial decay of the velocity

$$\lim_{|x_0| \to \infty} \int_{0}^{\infty} \int_{B(x_0,R)} |v(x,t)|^2 dx dt = 0, \quad \forall R > 0. \quad (1.3)$$

Under the decay assumption (1.2) on initial data, these two definitions can be shown to be equivalent; see [25, 14]. However, for general non-decaying initial data, the decay condition (1.3) is not expected, while the decomposition condition still works. For this reason, we follow the definition of Kikuchi-Seregin [17] in this paper.

A new feature in the study of infinite energy solutions with non-decaying initial data is the abundance of parasitic solutions,

$$v(x,t) = f(t), \quad p(x,t) = -f'(t) \cdot x$$

for a smooth vector function $f(t)$. They solve the Navier-Stokes equations with initial data $f(0)$. If we choose $f_1(t) \neq f_2(t)$ with $f_1(0) = f_2(0)$, the corresponding parasitic solutions...
give two different local energy solutions with the same initial data. Such solutions have non-decaying initial data, and can be shown to fail the pressure decomposition condition. More generally, if \((v,p)\) is a solution to \((NS)\), then the following parasitic transform
\[
  u(x,t) = v(y,t) + q'(t), \quad \pi(x,t) = p(y,t) - q''(t) \cdot y, \quad y = x - q(t) \tag{1.4}
\]
gives another solution \((u,\pi)\) to \((NS)\) with the same initial data \(v_0\) for any vector function \(q(t)\) satisfying \(q(0) = q'(0) = 0\).

We now summarize the known existence results in \(\mathbb{R}^3\). In addition to the weak solution approach based on the a priori bound (1.1) following Leray and Hopf, another fruitful approach is the theory of mild solutions, treating the nonlinear term as a source term of the nonhomogeneous Stokes system. In the framework of \(L^q(\mathbb{R}^3)\), there exist short time mild solutions in \(L^q(\mathbb{R}^3)\) when \(3 \leq q \leq \infty\) \((6, 15, 9)\). When \(q = 3\), these solutions exist for all time for sufficiently small initial data in \(L^3(\mathbb{R}^3)\); see \([15]\). Similar small data global existence results hold for many other spaces of similar scaling property, such as \(L^3_{\text{weak}},\) Morrey spaces \(M_{p,3-p},\) negative Besov spaces \(B_{3/q-1}^3,\) \(3 < q < \infty,\) and the Koch-Tataru space \(\text{BMO}^{-1};\) See e.g. \([10, 16, 19, 1, 5, 2, 18]\).

For any data \(v_0 \in L^q(\mathbb{R}^3),\) \(2 < q < 3,\) Calderón \([4]\) constructed a global solution. His strategy is to first decompose \(v_0 = a_0 + b_0\) with small \(a_0 \in L^3(\mathbb{R}^3)\) and large \(b_0 \in L^2(\mathbb{R}^3)\). A solution is then obtained as \(u = a + b\), where \(a\) is a global small mild solution of \((NS)\) in \(L^3(\mathbb{R}^3)\) with \(a(0) = a_0\), and \(b\) is a global weak solution of the \(a\)-perturbed Navier-Stokes equations in the energy class with \(b(0) = b_0\).

This idea is then used by Lemarié-Rieusset \([20]\) to construct global local energy solutions for \(v_0 \in E^2;\) also see Kikuchi-Seregin \([17]\).

We now summarize the known existence results for non-decaying initial data. For the local existence, many mild solution existence theorems mentioned earlier allow non-decaying data. The most relevant to us are Giga-Inui-Matsui \([9]\) for initial data in \(L^\infty(\mathbb{R}^3)\) and \(BUC(\mathbb{R}^3),\) and Maekawa-Terasawa \([26]\) for initial data in the closure of \(\bigcup_{p > 3} L^p_{uloc} in L^3_{uloc}\)-norm, and any small initial data in \(L^3_{uloc}.\) Smallness is needed for \(L^3_{uloc}\) data even for short time existence.

When it comes to the global existence for non-decaying data, a solution theory for perturbations of constant vectors seems straightforward. Lemarié-Rieusset \([21, \text{Theorem 1}(C)]\) constructed global weak solutions for \(u_0\) in Morrey space \(M^{2,1},\) which contains non-decaying functions, e.g.
\[
  v_0(x) = \sum_{k \in \mathbb{N}} \zeta(x - x_k)
\]
with \(|x_k| \rightarrow \infty\) rapidly as \(k \rightarrow \infty.\) Here \(\zeta\) is any smooth divergence free vector field with compact support.

The only other result we are aware of is the recent paper Maremonti-Shimizu \([28,\text{which}\) proved the global existence of weak solutions for initial data \(v_0 \in L^\infty(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)^{1/q},\) \(3 < q < \infty.\) In particular, they assume \(\nabla v_0 \in L^q(\mathbb{R}^3).\) Their strategy is to decompose the solution \(v = U + w,\) \(U = \sum_{k=1}^n v^k,\) where \(v^1\) solves the Stokes equations with the given initial data, and \(v^{k+1}, k \geq 1,\) solves the linearized Navier-Stokes equations with the force \(f^k = -v^k \cdot \nabla v^k\) and homogeneous initial data. The force \(f^1 \in L^q(0,T;L^q(\mathbb{R}^3))\) thanks to the assumption on \(v_0.\) In each iteration, we get an additional decay of the force \(f^k.\) The perturbation \(w\) is then solved in the framework of weak solutions. The paper \([28]\) motivated this paper.
We now state our main theorem. Denote the average of a function \( v \) in a set \( O \subset \mathbb{R}^3 \) by 
\[
(v)_O = \frac{1}{|O|} \int_O v(x) \, dx.
\]
We denote \( w \in E^2_\sigma \) if \( w \in E^2 \) and \( \text{div} \, w = 0 \).

**Theorem 1.1.** For any vector field \( v_0 \in E^2_\sigma + L^3_{uloc} \) satisfying \( \text{div} \, v_0 = 0 \) and

\[
\lim_{|x_0| \to \infty} \int_{B(x_0,1)} |v_0 - (v_0)_{B(x_0,1)}| \, dx = 0,
\]

we can find a time-global local energy solution \((v,p)\) to the Navier-Stokes equations (NS) in \( \mathbb{R}^3 \times (0,\infty) \), in the sense of Definition 3.1.

Our main assumption is the “oscillation decay” condition (1.5). Note that all \( v_0 \in L^2_{uloc} \) satisfying (1.2) also satisfy (1.5). Furthermore, for \( v_0 \in L^2_{uloc} \), either \( v_0 \in E^1 \) or \( \nabla v_0 \in E^1 \) implies the condition (1.5). Recall \( E^q \) for \( 1 \leq q \leq \infty \) is the space of functions in \( L^q_{uloc} \) whose \( L^q \)-norm in a ball \( B_1(x_0) \) goes to zero as \( |x_0| \) goes to infinity. In particular, our result generalizes the global existence for decaying initial data \( v_0 \in E^2 \) in [20] and [17]. It also extends [28] for \( v_0 \in L^\infty \) and \( \nabla v_0 \in L^q \).

**Example 1.2.** Consider

\[
v_0 = v_1 + v_2, \quad v_1 = \frac{(-x_2, x_1, 0)}{\sqrt{|x|^2 + 1}}, \quad v_2 = \frac{(-x_2, x_1, 0)}{|x|^2 + 1} \sin \left((x^2 + 1)^{100}\right).
\]

We have \( \text{div} \, v_1 = \text{div} \, v_2 = 0 \), \( v_0 \notin E^2 \), \( v_0 \) satisfies the oscillation decay condition, and

\[
\limsup_{|x_0| \to \infty} \int_{B_1(x_0)} |
\nabla v_0| = \infty.
\]

In particular, \( v_0 \in L^\infty \) but \( \nabla v_0 \notin L^q \) for any \( q \leq \infty \). Moreover, \( v_0 \) is not a perturbation of constant, although it converges to a constant along each direction.

The condition \( v_0 \in E^2_\sigma + L^3_{uloc} \) gives us more regularity on the nondecaying part of \( v_0 \). We do not know if it is necessary for the global existence, but it is essential for our proof, and enables us to prove that for small \( t > 0 \),

\[
\|w(t)\chi_R\|_{L^2_{uloc}} \lesssim (t^{\frac{1}{20}} + \|w_0\chi_R\|_{L^2_{uloc}}),
\]

where \( \chi_R(x) \) is a cut-off function supported in \( |x| > R \), we decompose \( v_0 = w_0 + u_0 \) with \( w_0 \in E^2_\sigma \) and \( u_0 \in L^3_{uloc} \), and \( w(t) = v(t) - e^{t \Delta} u_0 \) with \( w(0) = w_0 \). This estimate shows that \( \|w(t)\chi_R\|_{L^2_{uloc}} \) vanishes as \( t \to 0+ \) and \( R \to \infty \).

The idea of our proof is as follows. First, we construct a local energy solution in a short time. For \( v_0 \in L^2_{uloc} \), this is done in [20] but not in [17]. However, we use a slightly revised approximation scheme to make all statements about the pressure easy to verify. In our scheme, we not only mollify the non-linear term as in [23] and [20], but also insert a cut-off function, so that the non-linear term \((v \cdot \nabla) v\) is replaced by \((J_\epsilon(v) \cdot \nabla)(v \Phi_\epsilon)\), where \( J_\epsilon \) is a mollification of scale \( \epsilon \) and \( \Phi_\epsilon \) is a radial bump function supported in the ball \( B(0, 2\epsilon^{-1}) \).

Once we have a local-in-time local energy solution, we need some smallness to extend the solution globally in time. To this end, we decompose the solution as \( v = V + w \) where \( V(t) = e^{t \Delta} u_0 \) solves the heat equation. The main effort is to show that \( w(t) \in E^2 \) for all \( t \) and \( w(t) \in E^6 \) for almost all \( t \). The proof is similar to the decay estimates in [20, 17] and we try to do local energy estimate for \( w \chi_R \). The background \( V \) has no spatial decay, but
we can show the decay of $\nabla V(x,t)$ in $L^\infty(B_R^c(0) \times (t_0, \infty))$ as $R \to \infty$ for any $t_0 > 0$. This decay is not uniform up to $t_0 = 0$ as $u_0$ is rather rough. We need a new decomposition formula of the pressure, so that in the intermediate regions we can show the decay of the pressure using the decay of $\nabla V$. Because the decay of $\nabla V$ is not up to $t_0 = 0$, we need to do the local energy estimate in the time interval $[t_0, T)$, $0 < t_0 \ll 1$. This forces us to prove the estimate (1.6), and the strong local energy inequality for $w$ away from $t = 0$.

Once we have shown $w(t) \in E^6$ for almost all $t < T$, we can extend the solution as in [20] and [17]. However, we avoid using the strong-weak uniqueness as in [20, 17], and choose to verify the definition of local energy solutions directly as in [25].

The rest of the paper consists of the following sections. In Section 2, we discuss the properties of the heat flow $e^{t\Delta}u_0$, especially the decay of its gradient at spatial infinity assuming (1.5). In Section 3, we recall the definition of local energy solutions as in [17] and use our revised approximation scheme to find a local energy solution local-in-time. In Section 4, we find a new pressure decomposition formula suitable of using the decay of $\nabla V$, prove the estimate (1.6) and the strong local energy inequality, and then do the local energy estimate of $w\chi_R$, which implies $w(t) \in E^6$ for almost all $t$. In Section 5, we construct the desired time-global local energy solution. In Section 6, by a similar and easier proof, we additionally obtain perturbations of time-global solutions with no spatial oscillation decay.

2 Notations and preliminaries

2.1 Notation

Given two comparable quantities $X$ and $Y$, the inequality $X \lesssim Y$ stands for $X \leq CY$ for some positive constant $C$. In a similar way, $\gtrsim$ denotes $\geq C$ for some $C > 0$. We write $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. Furthermore, in the case that a constant $C$ in $X \leq CY$ depends on some quantities $Z_1, \cdots, Z_n$, we write $X \lesssim Z_1, \cdots, Z_n Y$. The notations $\gtrsim Z_1, \cdots, Z_n$ and $\sim Z_1, \cdots, Z_n$ are similarly defined.

For a point $x \in \mathbb{R}^3$ and a positive real number $r$, $B(x,r)$ is the Euclidean ball in $\mathbb{R}^3$ centered at $x$ with a radius $r$,

$$B(x,r) = B_r(x) = \{ y \in \mathbb{R}^3 : |y - x| < r \}.$$ 

When $x = 0$, we denote $B_r = B(0,r)$. For a point $x \in \mathbb{R}^3$ and $r > 0$, we denote the open cube centered at $x$ with a side length $2r$ as

$$Q(x,r) = Q_r(x) = \left\{ y \in \mathbb{R}^3 : \max_{i=1,2,3} |y_i - x_i| < r \right\}.$$ 

We denote the mollification $J_\epsilon(v) = v * \eta_\epsilon$, $\epsilon > 0$, where the mollifier is $\eta_\epsilon(x) = \epsilon^{-3} \eta(\frac{x}{\epsilon})$ and $\eta$ is a fixed nonnegative radial bump function in $C^\infty_c(\mathbb{R}^3)$ supported in $B(0,1)$ satisfying $\int \eta dx = 1$.

Various test functions in this paper are defined by rescaling and translating a nonnegative radially decreasing bump function $\Phi$ satisfying $\Phi = 1$ on $B(0,1)$ and supp($\Phi$) $\subset B(0,\frac{3}{2})$.

For $k \in \mathbb{N} \cup \{0, \infty\}$, let $C^k_c(\mathbb{R}^3)$ be the subset of functions in $C^k(\mathbb{R}^3)$ with compact supports, and

$$C^k_{c,\sigma}(\mathbb{R}^3) = \left\{ u \in C^k_c(\mathbb{R}^3, \mathbb{R}^3) : \text{div} \ u = 0 \right\}.$$ 

5
2.2 Uniformly locally integrable spaces

To consider infinite energy flows, we work in the spaces $L^q_{uloc}$, $1 \leq q \leq \infty$, and $U^{s,p}(t_0, t)$ for $1 \leq s, p \leq \infty$ and $0 \leq t_0 < t \leq \infty$, defined by

$$L^q_{uloc} = \left\{ u \in L^1_{loc}(\mathbb{R}^3) : \|u\|_{L^q_{uloc}} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^q(B_1(x_0))} < +\infty \right\}$$

and

$$U^{s,p}(t_0, t) = \left\{ u \in L^1_{loc}(\mathbb{R}^3 \times (t_0, t)) : \|u\|_{U^{s,p}(t_0, t)} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^s(t_0, t; L^p(B_1(x_0)))} < +\infty \right\}.$$

When $t_0 = 0$, we simply use $U^{s,p}_T = U^{s,p}(0, T)$. Note that $U^{\infty,p}(t_0, t) = L^\infty(t_0, t; L^p_{uloc})$, $1 \leq p \leq \infty$, but for general $1 \leq s < \infty$ and $1 \leq p \leq \infty$, $U^{s,p}(t_0, t)$ and $L^s(t_0, t; L^p_{uloc})$ are not equivalent norms. Indeed, we can only guarantee that

$$\|u\|_{U^{s,p}(t_0, t)} \leq \|u\|_{L^s(t_0, t; L^p_{uloc})},$$

but not the inequality of the other direction.

**Example 2.1.** Fix $1 \leq s < \infty$ and $p \in [1, \infty]$. Let $x_k$ be a sequence in $\mathbb{R}^3$ with disjoint $B_1(x_k)$, $k \in \mathbb{N}$, and let $t_k = t_0 + 2^{-k}$. Define a function $u$ by $u(x, \tau) = 2^{k/s}$ on $B_1(x_k) \times (t_0, t_k)$, $k \in \mathbb{N}$, and $u(x, \tau) = 0$ otherwise. It is defined independently of $p$. We have $u \in U^{s,p}(t_0, t)$, but

$$\int_{t_0}^{t_1} \int_{t_0}^{t_k} c_p 2^{k} d\tau = \sum_{k=1}^{\infty} \frac{1}{2} c_p = \infty,$$

and hence $u \notin L^s(t_0, t; L^p_{uloc})$. \hfill \Box

We define a local energy space $E(t_0, t)$ by

$$E(t_0, t) = \left\{ u \in L^1_{loc}([t_0, t] \times \mathbb{R}^3; \mathbb{R}^3) : \text{div} \ u = 0, \ \|u\|_{E(t_0, t)} < +\infty \right\},$$

where

$$\|u\|_{E(t_0, t)} := \|u\|_{U^{\infty,2}(t_0, t)} + \|\nabla u\|_{U^{2,2}(t_0, t)}.$$

When $t_0 = 0$, we use the abbreviation $E_T = E(0, T)$.

The spaces $E^p$ and $G^p(t_0, t)$, $1 \leq p \leq \infty$, are defined by an additional decay condition at infinity,

$$E^p := \{ f \in L^p_{uloc} : \|f\|_{L^p(B(x_0, 1))} \to 0, \ \text{as} \ |x_0| \to \infty \},$$

and

$$G^p(t_0, t) := \{ u \in U^{p,p}(t_0, t) : \|u\|_{L^p([t_0, t] \times B(x_0, 1))} \to 0, \ \text{as} \ |x_0| \to \infty \}.$$

We let $L^p_{uloc, \sigma}$, $E^p_\sigma$ and $G^p_\sigma(t_0, t)$ denote divergence-free vector fields with components in $L^p_{uloc}$, $E^p$ and $G^p(t_0, t)$, respectively.

The space $E^p$, $1 \leq p < \infty$, can be characterized as $C_c^\infty(\mathbb{R}^3)^{L^p_{uloc}}$. The analogous statement for $E^p_\sigma$ is true.

**Lemma 2.2.** ([17, Appendix]) Suppose that $f \in E^p_\sigma$ for some $1 \leq p < \infty$. Then, for any $\varepsilon > 0$, we can find $f^\varepsilon \in C_c^\infty(\mathbb{R}^3)$ such that

$$\|f - f^\varepsilon\|_{L^p_{uloc}} < \varepsilon.$$
2.3 Heat and Oseen kernels on $L^q_{uloc}$

Now, we study the operators $e^{t\Delta}$ and $e^{t\Delta}P\nabla$ on $L^q_{uloc}$. Here $P$ denotes the Helmholtz projection in $\mathbb{R}^3$. Both are defined as convolution operators

$$e^{t\Delta}f = H_t \ast f, \quad \text{and} \quad e^{t\Delta}P\nabla \cdot f = \partial_k S_{ij} \ast F_{jk},$$

where $H_t$ and $S_{ij}$ are the heat kernel and the Oseen tensor, respectively,

$$H_t(x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|x|^2}{4t} \right),$$

and

$$S_{ij}(x,t) = H_t(x)\delta_{ij} + \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} \frac{H_t(y)}{|x-y|} \, dy.$$ 

In this note, we use $(\text{div} \; F)_i = (\nabla \cdot F)_i = \partial_j F_{ji}$. Note that the Oseen tensor satisfies the following pointwise estimates

$$|\nabla_{x}^{l} \partial_{t}^{k} S(x,t)| \leq C_{k,l}(\sqrt{t}/|x| + 1)^{-3l-2k}. \quad (2.3)$$

We have the following estimates.

**Lemma 2.3** (Remark 3.2 in [26]). For $1 \leq q \leq p \leq \infty$, the following holds. For any vector field $f$ and any 2-tensor $F$ in $\mathbb{R}^3$,

$$\left\| \partial^{\alpha} \partial_{x}^{\beta} e^{t\Delta} f \right\|_{L^p_{uloc}} \lesssim \frac{1}{t^{[\alpha] + [\beta]/2}} \left( 1 + \frac{1}{t^{\frac{3}{2} \left( \frac{1}{q} - \frac{1}{p} \right)}} \right) \|f\|_{L^q_{uloc}},$$

$$\left\| \partial^{\alpha} \partial_{x}^{\beta} e^{t\Delta} P \nabla \cdot F \right\|_{L^p_{uloc}} \lesssim \frac{1}{t^{[\alpha] + [\beta]/2 + 1/2}} \left( 1 + \frac{1}{t^{\frac{3}{2} \left( \frac{1}{q} - \frac{1}{p} \right)}} \right) \|F\|_{L^q_{uloc}}.$$ 

Note $p = \infty$ is allowed, with $L^\infty_{uloc} = L^\infty$.

**Lemma 2.4.** For any $T > 0$, if $f \in L^2_{uloc}$ and $F \in L^2_{T,2}$, then we have

$$\left\| e^{t\Delta} f \right\|_{\mathcal{E}_T} \lesssim (1 + T^{\frac{3}{2}}) \|f\|_{L^2_{uloc}},$$

$$\left\| \int_{0}^{t} e^{(t-s)\Delta} P \nabla \cdot F(s) \, ds \right\|_{\mathcal{E}_T} \lesssim (1 + T) \|F\|_{L^2_{T,2}}.$$ 

Recall $\|u\|_{\mathcal{E}_T} = \|u\|_{L^\infty_{T,2}} + \|\nabla u\|_{L^2_{T,2}}$. Similar estimates can be found in the proof of [22, Theorem 14.1]. We give a slightly revised proof here for completeness.

**Proof.** Fix $x_0 \in \mathbb{R}^3$ and let $\phi_{x_0}(x) = \Phi \left( \frac{x - x_0}{2} \right)$. We decompose $f$ and $F$ as

$$f = f \phi_{x_0} + f(1 - \phi_{x_0}) = f_1 + f_2$$

and

$$F = F \phi_{x_0} + F(1 - \phi_{x_0}) = F_1 + F_2.$$ 

7
Since $f_1 \in L^2(\mathbb{R}^3)$ and $F_1 \in L^2(0,T; L^2(\mathbb{R}^3))$, by the usual energy estimates for the heat equation and the Stokes system, we get

$$
\|e^{t\Delta} f_1\|_{\mathcal{E}_T} \lesssim \|f_1\|_2 \lesssim \|f\|_{L^2_{uloc}}
$$

(2.4)

and

$$
\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot F_1(s) ds \right\|_{\mathcal{E}_T} \lesssim \|F_1\|_{L^2(0,T; L^2(\mathbb{R}^3))} \lesssim \|F\|_{L^{2,2}_T}.
$$

(2.5)

On the other hand, by Lemma 2.3,

$$
\|e^{t\Delta} f_2\|_{U^{\infty,2}_T} = \|e^{t\Delta} f_2\|_{L^\infty(0,T; L^2_{uloc})} \lesssim \|f_2\|_{L^2_{uloc}} \lesssim \|f\|_{L^2_{uloc}}.
$$

Together with (2.4), we get

$$
\|e^{t\Delta} f\|_{U^{\infty,2}_T} \lesssim \|f\|_{L^2_{uloc}}.
$$

(2.6)

(This also follows from Lemma 2.3.) By the heat kernel estimates,

$$
\|\nabla e^{t\Delta} f_2\|_{L^2((0,T) \times B(x_0,1))} \lesssim T^{\frac{1}{2}} \|\nabla e^{t\Delta} f_2\|_{L^\infty((0,T) \times B(x_0,1))}
\lesssim T^{\frac{1}{2}} \int_{B(x_0,2^c)} \frac{1}{|x_0 - y|^4} |f_2(y)| dy
\lesssim T^{\frac{1}{2}} \sum_{k=1}^{\infty} \int_{B(x_0,2^{k+1}) \setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |f(y)| dy
\lesssim T^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \int_{B(x_0,2^{k+1})} |f(y)| dy.
$$

We may cover $B(x_0,2^{k+1})$ by $\bigcup_{j=1}^{J_k} B(x_j^k,1)$ with $J_k$ bounded by $C_0 2^{4k}$ for some constant $C_0 > 0$. Then

$$
\|\nabla e^{t\Delta} f_2\|_{L^2((0,T) \times B(x_0,1))} \lesssim T^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \sum_{j=1}^{J_k} \int_{B(x_j^k,1)} |f(y)| dy \lesssim T^{\frac{1}{2}} \|f\|_{L^2_{uloc}}.
$$

Together with (2.4), we get

$$
\|\nabla e^{t\Delta} f\|_{L^2((0,T) \times B(x_0,1))} \lesssim (1 + T^{\frac{1}{2}}) \|f\|_{L^2_{uloc}}.
$$

Taking supremum in $x_0$, we obtain

$$
\|\nabla e^{t\Delta} f\|_{U^{2,2}_T} \lesssim (1 + T^{\frac{1}{2}}) \|f\|_{L^2_{uloc}}.
$$

This and (2.6) show the first bound of the lemma, $\|e^{t\Delta} f\|_{\mathcal{E}_T} \lesssim (1 + T^{\frac{1}{2}}) \|f\|_{L^2_{uloc}}$. 

8
Denote $\Psi F(t) = \int_0^t e^{(t-s)\Delta} \nabla \cdot F(s) ds$. By the pointwise estimates (2.3) for Oseen tensor, we have
\[
\|\Psi F\|_{L^\infty(0,T;L^2(B(x_0,1)))} \lesssim \int_0^t \int_{B(x_0,2^j)} \frac{1}{|x_0 - y|^t} |F_2(y,s)| dy ds \\
\lesssim \sum_{k=1}^\infty \frac{1}{2^{4k}} \int_0^t \int_{B(x_0,2^{k+1})} |F(y,s)| dy ds \\
\lesssim \sum_{k=1}^\infty \frac{1}{2^{4k}} \sum_{j=1}^{J_k} \int_0^t \int_{B(x_0,2^{k+1})} |F(y,s)| dy ds \\
\lesssim \|F\|_{U^1_T} \lesssim T^\frac{1}{2} \|F\|_{U^2_T},
\]
and
\[
\|\nabla \Psi F\|_{L^2((0,T)\times B(x_0,1))} \lesssim T^\frac{1}{2} \|\nabla \Psi F\|_{L^\infty((0,T)\times B(x_0,1))} \\
\lesssim T^\frac{1}{2} \int_0^t \int_{B(x_0,2^j)} \frac{1}{|x_0 - y|^t} |F_2(y,s)| dy ds \\
\lesssim \sum_{k=1}^\infty \frac{1}{2^{4k}} \sum_{j=1}^{J_k} \int_0^t \int_{B(x_0,2^{k+1})} |F(y,s)| dy ds \\
\lesssim T \|F\|_{U^2_T}.
\]
Combined with (2.5), we have
\[
\|\Psi F\|_{L^\infty(0,T;L^2(B(x_0,1)))} \lesssim (1 + T^\frac{1}{2}) \|F\|_{U^2_T}
\]
and
\[
\|\nabla \Psi F\|_{L^2((0,T)\times B(x_0,1))} \lesssim (1 + T) \|F\|_{U^2_T}.
\]
Finally, we take suprema in $x_0$ to get
\[
\left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot F(s) ds \right\|_{L^\infty_T} \lesssim (1 + T) \|F\|_{U^2_T}.
\]
This is the second bound of the lemma.

### 2.4 Heat kernel on $L^1_{uloc}$ with decaying oscillation

In this subsection, we investigate how the decaying oscillation assumption (1.5) on initial data affects the heat flow. Recall
\[
(u)_{Q_r(x)} = \int_{Q_r(x)} u(y) dy = \frac{1}{|Q_r(x)|} \int_{Q_r(x)} u(y) dy.
\]

Lemma 2.5. Suppose that $u \in L^1_{uloc}(\mathbb{R}^3)$ satisfies
\[
\lim_{|x_0| \to \infty} \int_{Q_{1}(x_0)} |u - (u)_{Q_1(x_0)}| dx = 0. \tag{2.7}
\]
Then, for any \( r > 0 \), we have
\[
\lim_{|x_0| \to \infty} \int_{Q_r(x_0)} |u - (u)_{Q_r(x_0)}| \, dx = 0, \tag{2.8}
\]
and
\[
\lim_{|x_0| \to \infty} \sup_{y \in Q_{2r}(x_0)} |(u)_{Q_r(y)} - (u)_{Q_r(x_0)}| = 0. \tag{2.9}
\]

Proof. First note that \((u)_{Q_r(x)}\) is finite for any \( x \in \mathbb{R}^3 \) and \( r > 0 \). Indeed,
\[
|(u)_{Q_r(x)}| \leq C_r \|u\|_{L^1_{\text{loc}}}
\]
for a constant \( C_r \) independent of \( x \), \( C_r < C \) for \( r > 1 \), and \( C_r \sim r^{-3} \) for \( r \ll 1 \).

Fix \( x_0 \in \mathbb{R}^3 \) and \( r > 0 \). For any constant \( c \in \mathbb{R} \), we get
\[
\int_{Q_r(x_0)} |u - (u)_{Q_r(x_0)}| \, dx \leq \int_{Q_r(x_0)} |u - c| + |(u)_{Q_r(x_0)} - c| \, dx
\]
\[
= \int_{Q_r(x_0)} |u - c| \, dx + \int_{Q_r(x_0)} (u - c) \, dx
\]
\[
\leq 2 \int_{Q_r(x_0)} |u - c| \, dx.
\]

Then, for \( Q_r = Q_r(x_1) \subset Q_R(x_0), R > r \), we get
\[
\int_{Q_r} |u - (u)_{Q_r}| \, dx \leq 2 \int_{Q_r} |u - (u)_{Q_R(x_0)}| \, dx \leq \frac{2R^3}{r^3} \int_{Q_R(x_0)} |u - (u)_{Q_R(x_0)}| \, dx. \tag{2.10}
\]

With \( x_0 = x_1 \) and \( R = 1 \) in (2.10), (2.7) implies (2.8) for all \( r \in (0, 1) \).

If \( y \in Q_{2r}(x_0) \), then
\[
Q_r(x_0) \cup Q_r(y) \subset Q_R(x_1), \quad x_1 = \frac{1}{2}(x_0 + y), \quad R \geq 2r.
\]

Thus,
\[
|(u)_{Q_r(x_0)} - (u)_{Q_r(y)}| \leq \int_{Q_r(x_0)} u - (u)_{Q_R(x_1)} \, dx + \int_{Q_r(y)} u - (u)_{Q_R(x_1)} \, dx
\]
\[
\leq \int_{Q_r(x_0)} |u - (u)_{Q_R(x_1)}| \, dx + \int_{Q_r(y)} |u - (u)_{Q_R(x_1)}| \, dx
\]
\[
\leq \frac{2R^3}{r^3} \int_{Q_R(x_1)} |u - (u)_{Q_R(x_1)}| \, dx. \tag{2.11}
\]

With \( R = 1 \), this and (2.7) imply (2.9) for all \( r \in (0, \frac{1}{2}] \).

Now, for any \( Q_r(x_0) \) with \( r > 1 \), choose the smallest integer \( N > 2r \) and let \( \rho = r/N < \frac{1}{2} \).

We can find a set \( S = S_{x_0,r} \) of \( N^3 \) points such that \( \{Q_\rho(z) : z \in S\} \) are disjoint and
\[
\overline{Q_r(x_0)} = \bigcup_{z \in S} \overline{Q_\rho(z)}.
\]
For any \( z, z' \in S \), we can connect them by points \( z_j \) in \( S \), \( j = 0, 1, \ldots, N \), such that \( z_0 = z \), \( z_N = z' \), and \( z_j \in \overline{Q_2 \rho(z_j-1)} \), \( j = 1, \ldots, N \). We allow \( z_{j+1} = z_j \) for some \( j \). Thus

\[
|u \rho(z) - (u) \rho(z')| \leq \sum_{j=1}^{N} |(u) \rho(z_j) - (u) \rho(z_{j-1})|,
\]

and hence

\[
\max_{z,z' \in S_{x_0,r}} |u \rho(z) - (u) \rho(z')| = o(1) \quad \text{as } |x_0| \to \infty \tag{2.12}
\]

by (2.9) as \( \rho \in (0, \frac{1}{2}) \). We have

\[
\int_{Q_\rho(x_0)} |u - (u) \rho(x_0)| dx
\]

\[
= 3^{-3} \int_{Q_\rho(z)} |u - (u) \rho(z)| dx
\]

\[
\leq 3^{-3} \int_{Q_\rho(z)} |u - (u) \rho(z)| + |(u) \rho(z_0) - (u) \rho(z)| dx
\]

\[
\leq \max_{z,z' \in S} |u \rho(z) - (u) \rho(z')| + o(1) \quad \text{as } |x_0| \to \infty
\]

by (2.8) and (2.12) for \( \rho \in (0, \frac{1}{2}) \). This shows (2.8) for all \( r > 1 \).

Finally, (2.9) for \( r > 1/2 \) follows from (2.8) and (2.11).

The following lemma says that decaying oscillation over cubes is equivalent with decaying oscillation over balls.

**Lemma 2.6.** Suppose \( u \in L^1_{\text{uloc}} \). Then \( u \) satisfies (2.7) if and only if

\[
\lim_{|x_0| \to \infty} \int_{B_1(x_0)} |u - (u) B_1(x_0)| dx = 0. \tag{2.13}
\]

**Proof.** Let \( \rho = 3^{-1/2} \). We have \( \rho(x_0) \subset B_1(x_0) \subset Q_1(x_0) \). Similar to the proof of (2.10), we have

\[
\int_{B_1(x_0)} |u - (u) B_1(x_0)| dx \leq C \int_{Q_1(x_0)} |u - (u) Q_1(x_0)| dx
\]

and hence (2.13) follows from (2.7). Similarly, we also have

\[
\int_{Q_\rho(x_0)} |u - (u) \rho(x_0)| dx \leq C \int_{B_1(x_0)} |u - (u) B_1(x_0)| dx
\]

and hence (2.8) for \( r = \rho \) follows from (2.13). Then \( v(x) = u(\rho x) \) satisfies (2.7). By Lemma 2.5, \( v \) satisfies (2.8) for any \( r > 0 \), and we get (2.7) for \( u \).
Lemma 2.7. Suppose $v_0 \in L_{uloc}^1$ and

$$\int_{Q(x_0,1)} |v_0 - (v_0)_{Q(x_0,1)}| \to 0, \quad \text{as } |x_0| \to \infty.$$ 

Let $V = e^{\Delta}v_0$. Then $(\nabla V)(t_0) \in C_0(\mathbb{R}^3)$ for every $t_0 > 0$. Furthermore, for any $t_0 > 0$, we have

$$\sup_{t > t_0} \|\nabla V(\cdot, t)\|_{L^\infty(B(x_0,1))} \to 0, \quad \text{as } |x_0| \to \infty. \quad (2.14)$$

Proof. For $k \in \mathbb{Z}^3$, let $\Sigma_k$ denote the set of its neighbor integer points,

$$\Sigma_k = \mathbb{Z}^3 \cap Q(k,1.01) \setminus \{k\}.$$ 

Let

$$a_k = (v_0)_{Q_1(k)}, \quad b_k = \max_{k' \in \Sigma_k} |a_{k'} - a_k|, \quad c_k = \int_{Q_1(k)} |v_0(x) - a_k|dx.$$

By the assumption, $c_k \to 0$ as $|k| \to \infty$ and by Lemma 2.5, $b_k \to 0$ as $|k| \to \infty$.

Choose a nonnegative $\phi \in C^\infty_c(\mathbb{R}^3)$ with supp $\phi \subset Q_1(0)$ and

$$\sum_{k \in \mathbb{Z}^3} \phi_k(x) = 1 \quad \forall x \in \mathbb{R}^3, \quad \phi_k(x) = \phi(x - k).$$

Define

$$v_1(x) = \sum_{k \in \mathbb{Z}^3} a_k \phi_k(x).$$

Since $|a_k| \lesssim \|v_0\|_{L_{uloc}^1}$, $v_1$ is in $L^\infty(\mathbb{R}^3)$. For $x \in Q_1(k)$, it can be written as

$$v_1(x) = a_k + \sum_{k' \in \Sigma_k} (a_{k'} - a_k) \phi_{k'}(x).$$

Thus

$$\int_{Q_1(k)} |v_0(x) - v_1(x)|dx \leq \int_{Q_1(k)} |v_0(x) - a_k|dx + \sum_{k' \in \Sigma_k} \int_{Q_1(k)} |a_k - a_{k'}|\phi_{k'}(x)dx \quad (2.15)$$

and

$$\sup_{x \in Q_1(k)} |\nabla v_1(x)| \leq \sup_{x \in Q_1(k)} \sum_{k' \in \Sigma_k} |a_{k'} - a_k| \cdot |\nabla \phi_{k'}(x)| \leq Cb_k. \quad (2.16)$$

Let $\psi_R(x) = \Phi \left( \frac{x}{R} \right)$. We decompose

$$\nabla V(x,t) = \int \nabla H_t(x - y)v_0(y)(1 - \psi_R(x - y))dy$$

$$+ \int \nabla H_t(x - y)|v_0(y) - v_1(y)|\psi_R(x - y)dy$$

$$+ \int \nabla H_t(x - y)v_1(y)\psi_R(x - y)dy = I_1 + I_2 + I_3.$$
By integration by parts, we can rewrite $I_3$,

$$I_3 = \int H_t(x - y)\nabla v_1(y)\psi_R(x - y)dy - \int H_t(x - y)v_1(y)(\nabla \psi_R)(x - y)dy = I_{31} + I_{32}.$$  

Fix $\epsilon > 0$ and consider $t > t_0 > 0$. Since for any $t > 0$ and $x \in \mathbb{R}^3$, we have

$$|I_1| \lesssim \int_{B(x, R)^c} \frac{|x - y|^5}{t^2} e^{-\frac{|x - y|^2}{4t}} \frac{1}{|x - y|^2} |v_0(y)|dy \lesssim \frac{1}{R} \|v_0\|_{L^1_{uloc}},$$

and

$$|I_{32}| \lesssim \|H_t\|_1 \|v_1\|_\infty \|\nabla \psi_R\|_\infty \lesssim \frac{1}{R} \|v_0\|_{L^1_{uloc}},$$

we can choose sufficiently large $R > 0$ such that

$$|I_1, I_{32}| < \epsilon.$$

The integrands of both $I_2$ and $I_{31}$ are supported in $|y - x| \leq 2R$. If $|x| > 2\rho$ with $\rho > 2R$ and $|y - x| \leq 2R$, then $|y| \geq |x| - |x - y| > \rho$. Let

$$1_{>\rho}(y) = 1 \text{ for } |y| > \rho, \text{ and } 1_{>\rho}(y) = 0 \text{ for } |y| \leq \rho.$$

We have

$$|I_2| \lesssim \|\nabla H_t\|_1 \|v_0 - v_11_{>\rho}\|_{L^\infty(\mathbb{R}^3)} \lesssim t_0^{-\frac{1}{2}} \left(1 + t_0^{-\frac{3}{2}}\right) \|v_0 - v_11_{>\rho}\|_{L^1_{uloc}}$$

by Lemma 2.3, and

$$|I_{31}| \lesssim \|e^{t\Delta}(|\nabla v_1|1_{>\rho})\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\nabla v_1|1_{>\rho}\|_{L^\infty(\mathbb{R}^3)}.$$

If we take $\rho$ sufficiently large, by (2.15) and (2.16), we have $|I_2| + |I_{31}| \leq 2\epsilon$.

Since for any $t > t_0$ and $\epsilon > 0$, we can choose $\rho > 0$ such that

$$\sup_{t > t_0} \|\nabla V(\cdot, t)\|_{L^\infty(B(0, 2\rho)^c)} < 4\epsilon,$$

we get (2.14). $\square$

3 Local existence

In this section, we recall the definition of local energy solutions and prove their time-local existence using a revised approximation scheme. Note that we do not assume spatial decay of initial data for the time-local existence.

As mentioned in the introduction, we follow the definition in Kikuchi-Seregin [17].

**Definition 3.1** (local energy solution). Let $v_0 \in L^2_{uloc}$ with $\text{div} v_0 = 0$. A pair $(v, p)$ of functions is a local energy solution to the Navier-Stokes equations (NS) with initial data $v_0$ in $\mathbb{R}^3 \times (0, T)$, $0 < T < \infty$, if it satisfies the followings.
(i) $v \in \mathcal{E}_T$, defined in (2.2), and $p \in L^\frac{3}{2}(\mathbb{R}^3, \mathbb{R})$.

(ii) $(v, p)$ solves the Navier-Stokes equations (NS) in the distributional sense.

(iii) For any compactly supported function $\varphi \in L^2(\mathbb{R}^3)$, the function $\int_{\mathbb{R}^3} v(x,t) \cdot \varphi(x) \, dx$ of time is continuous on $[0,T]$. Furthermore, for any compact set $K \subset \mathbb{R}^3$,

$$\|v(\cdot,t) - v_0\|_{L^2(K)} \to 0, \quad \text{as } t \to 0^+.$$

(iv) $(v, p)$ satisfies the local energy inequality (LEI) for any $t \in (0,T)$:

$$\int_{\mathbb{R}^3} |v|^2 \xi(x,t) \, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \xi \, dx \, ds \leq \int_0^t \int_{\mathbb{R}^3} |v|^2 (\partial_s \xi + \Delta \xi) + (|v|^2 + 2p)(v \cdot \nabla) \xi \, dx \, ds,$$

for all non-negative smooth functions $\xi \in C^\infty_c((0,T) \times \mathbb{R}^3)$.

(v) For each $x_0 \in \mathbb{R}^3$, we can find $c_{x_0} \in L^{\frac{3}{2}}(0,T)$ such that

$$p(x,t) = \hat{p}_{x_0}(x,t) + c_{x_0}(t), \quad \text{in } L^{\frac{3}{2}}(B(x_0, \frac{3}{2}) \times (0,T)),$$

where

$$\hat{p}_{x_0}(x,t) = -\frac{1}{3} |v(x,t)|^2 + \text{p.v.} \int_{B(x_0,2)} K_{ij}(x-y) v_i v_j(y,t) \, dy + \int_{B(x_0,2)^c} (K_{ij}(x-y) - K_{ij}(x_0-y)) v_i v_j(y,t) \, dy$$

for $K(x) = \frac{1}{4\pi |x|}$ and $K_{ij} = \partial_{ij} K$.

We say the pair $(v, p)$ is a local energy solution to (NS) in $\mathbb{R}^3 \times (0, \infty)$ if it is a local energy solution to (NS) in $\mathbb{R}^3 \times (0,T)$ for all $0 < T < \infty$.

For an initial data $v_0 \in L^2_{uloc}$ whose local kinetic energy is uniformly bounded, we reprove the local existence of a local energy solution of [20, Chapt 32].

**Theorem 3.2 (Local existence).** Let $v_0 \in L^2_{uloc}$ with $\text{div} \, v_0 = 0$. If

$$T \leq \frac{\varepsilon_1}{1 + \|v_0\|_{L^2_{uloc}}^4},$$

for some small constant $\varepsilon_1 > 0$, we can find a local energy solution $(v, p)$ on $\mathbb{R}^3 \times (0,T)$ to Navier-Stokes equations (NS) for the initial data $v_0$, satisfying $\|v\|_{\mathcal{E}_T} \leq C \|v_0\|_{L^2_{uloc}}$.

Note that we do not assume $v_0 \in E^2$, i.e., we do not assume spatial decay of $v_0$. Although the local existence theorem is proved in [20, Chapt 32], a few details are missing there, in particular those related to the pressure. These details are given in [17] for the case $v_0 \in E^2$. Here we treat the general case $v_0 \in L^2_{uloc}$.
Recall the definitions of \( J_\epsilon(\cdot) \) and \( \Phi \) in Section 2 and let \( \Phi_\epsilon(x) = \Phi(\epsilon x) \), \( \epsilon > 0 \). To prove Theorem 3.2, we consider approximate solutions \((v^\epsilon, p^\epsilon)\) to the localized-mollified Navier-Stokes equations

\[
\begin{aligned}
\partial_t v^\epsilon - \Delta v^\epsilon + (J_\epsilon(v^\epsilon) \cdot \nabla)(v^\epsilon \Phi_\epsilon) + \nabla p^\epsilon &= 0 \\
\text{div } v^\epsilon &= 0 \\
v^\epsilon|_{t=0} &= v_0
\end{aligned}
\]

in \( \mathbb{R}^3 \times (0, T) \).

Since \( v_0 \in L^2_{\text{uloc}} \) has no decay, it cannot be approximated by \( L^2 \)-functions, as was done in [17] when \( v_0 \in E^2 \). Hence the approximation solution \( v^\epsilon \) cannot be constructed in the energy class \( L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \), and has to be constructed in \( \mathcal{E}_T \) directly.

Compared to [20, 17], our mollified nonlinearity has an additional localization factor \( \Phi_\epsilon \). It makes the decay of the Duhamel term apparent when the approximation solutions have no decay.

We first construct a mild solution \( v^\epsilon \) of (3.4) in \( \mathcal{E}_T \).

**Lemma 3.3.** For each \( 0 < \epsilon < 1 \) and \( v_0 \) with \( \|v_0\|_{L^2_{\text{uloc}}} \leq B \), if \( 0 < T < \min(1, \epsilon^3 B^{-2}) \), we can find a unique solution \( v = v^\epsilon \) to the integral form of (3.4)

\[
v(t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta} \Phi_\epsilon \cdot (J_\epsilon(v) \otimes v \Phi_\epsilon)(s)ds
\]

satisfying

\[
\|v\|_{\mathcal{E}_T} \leq 2C_0 B,
\]

where \( c > 0 \) and \( C_0 > 1 \) are absolute constants and \((a \otimes b)_{jk} = a_j b_k\).

**Proof.** Let \( \Psi(v) \) be the map defined by the right side of (3.5) for \( v \in \mathcal{E}_T \). By Lemma 2.4 and \( T \leq 1 \),

\[
\|\Psi(v)\|_{\mathcal{E}_T} \leq \|v_0\|_{L^2_{\text{uloc}}} + \|J_\epsilon(v) \otimes v \Phi_\epsilon\|_{L^2_{\text{uloc}}} \\
\leq \|v_0\|_{L^2_{\text{uloc}}} + \|J_\epsilon(v)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \|v\|_{L^2_{\text{uloc}}} \\
\leq \|v_0\|_{L^2_{\text{uloc}}} + \epsilon^{-\frac{3}{2}} \sqrt{T} \|v\|_{L^2_{\text{uloc}}}^2.
\]

Thus

\[
\|\Psi(v)\|_{\mathcal{E}_T} \leq C_0 \|v_0\|_{L^2_{\text{uloc}}} + C_1 \epsilon^{-\frac{3}{2}} \sqrt{T} \|v\|_{\mathcal{E}_T}^2,
\]

for some constants \( C_0, C_1 > 0 \). Similarly, for \( v, u \in \mathcal{E}_T \),

\[
\|\Psi(v) - \Psi(u)\|_{\mathcal{E}_T} \leq C_1 \epsilon^{-\frac{3}{2}} \sqrt{T} \left(\|v\|_{\mathcal{E}_T} + \|u\|_{\mathcal{E}_T}\right) \|v - u\|_{\mathcal{E}_T}.
\]

By the Picard contraction theorem, if \( T \) satisfies

\[
T < \frac{\epsilon^3}{64(C_0 C_1 B)^2} = \epsilon^3 B^{-2},
\]

then we can always find a unique fixed point \( v \in \mathcal{E}_T \) of \( v = \Psi(v) \), i.e., (3.5), satisfying

\[
\|v\|_{\mathcal{E}_T} \leq 2C_0 B.
\]
Lemma 3.4. Let $v_0 \in L^2_{uloc}$ with $\text{div} v_0 = 0$. For each $\epsilon \in (0, 1)$, we can find $v^\epsilon$ in $\mathcal{E}_T$ and $p^\epsilon$ in $L^\infty(0, T; L^2(\mathbb{R}^3))$ for some positive $T = T(\epsilon, \|v_0\|_{L^2_{uloc}})$ which solves the localized-mollified Navier-Stokes equations (3.4) in the sense of distributions, and $\lim_{t \to 0^+} \|v^\epsilon(t) - v_0\|_{L^2(E)} = 0$ for any compact subset $E$ of $\mathbb{R}^3$.

Proof. By Lemma 3.3, there is a mild solution $v^\epsilon \in \mathcal{E}_T$ of (3.5) for some $T = T(\epsilon, \|v_0\|_{L^2_{uloc}})$. Apparently,

$$\|v^\epsilon - e^{t\Delta} v_0\|_{U^1_t;L^2} = \left\|\int_0^t e^{(t-s)\Delta} \mathbb{P} \cdot (J_\epsilon(v) \otimes v \Phi_\epsilon)(s) \, ds\right\|_{U^1_t;L^2} \lesssim \|J_\epsilon(v) \otimes v \Phi_\epsilon\|_{U^2_t;L^2} \lesssim \epsilon^{-\frac{3}{2}}\sqrt{T} \|v\|^2_{U^\infty_t;L^2}.$$ 

Also, for any compact subset $E$ of $\mathbb{R}^3$, we have $\|e^{t\Delta} v_0 - v_0\|_{L^2(E)} \to 0$ as $t$ goes to 0; by Lebesgue’s convergence theorem

$$\|e^{t\Delta} v_0 - v_0\|_{L^2(E)} \lesssim \frac{1}{(4\pi)^{\frac{3}{2}}} \int e^{-|z|^2/4} \left|v_0(\cdot - \sqrt{t}z) - v_0\right|_{L^2(E)} \, dz \to 0,$$

as $t \to 0^+$. Then, it follows that $\lim_{t \to 0^+} \|v^\epsilon(t) - v_0\|_{L^2(E)} = 0$ for any compact subset $E$ of $\mathbb{R}^3$.

Note that $e^{t\Delta} v_0$ with $v_0 \in L^2_{uloc}$ solves the heat equation in the distributional sense. Also, using $\text{div} v_0 = 0$, we can easily see that $\text{div} e^{t\Delta} v_0 = 0$.

On the other hand, $J_\epsilon(v^\epsilon) \in L^\infty(\mathbb{R}^3 \times [0, T])$ and $v^\epsilon \in \mathcal{E}_T$ imply

$$J_\epsilon(v^\epsilon) \otimes v^\epsilon \Phi_\epsilon \in L^\infty(0, T; L^2(\mathbb{R}^3))$$

and hence by the classical theory, $w^\epsilon = v^\epsilon - V$ and $p^\epsilon$ defined by

$$p^\epsilon = (-\Delta)^{-1} \partial_i \partial_j (J_\epsilon(v^\epsilon) v^\epsilon_i \Phi_\epsilon) \in L^\infty(0, T; L^2(\mathbb{R}^3)).$$

(3.6)

solves Stokes system with the source term $\nabla \cdot (J_\epsilon(v^\epsilon) \otimes v^\epsilon \Phi_\epsilon)$ in the distributional sense.

By adding the heat equation for $V$ with $\text{div} V = 0$ and the Stokes system for $(w^\epsilon, p^\epsilon)$, $v^\epsilon = V + w^\epsilon$ satisfies

$$\partial_t v^\epsilon - \Delta v^\epsilon + (J_\epsilon(v^\epsilon) \cdot \nabla)(v^\epsilon \Phi_\epsilon) + \nabla p^\epsilon = 0$$

in the sense of distribution.

To extract a limit solution from the family $(v^\epsilon, p^\epsilon)$ of approximation solutions, we need a uniform bound of $(v^\epsilon, p^\epsilon)$ on a uniform time interval $[0, T]$, $T > 0$.

Lemma 3.5. For each $\epsilon \in (0, 1)$, let $(v^\epsilon, p^\epsilon)$ be the solution on $\mathbb{R}^3 \times [0, T_\epsilon]$, for some $T_\epsilon > 0$, to the localized-mollified Navier-Stokes equations (3.4) constructed in Lemma 3.4. There is a small constant $\varepsilon_1 > 0$, independent of $\epsilon$ and $\|v_0\|_{L^2_{uloc}}^2$, such that, if $T_\epsilon \leq T_0 = \varepsilon_1(1 + \|v_0\|_{L^2_{uloc}}^4)^{-1}$, then $v^\epsilon$ is uniformly bounded

$$\|v^\epsilon\|_{L^\infty_t;L^2_E} \leq C \|v_0\|_{L^2_{uloc}}^2,$$

(3.7)

where the constant $C$ on the right hand side is independent of $\epsilon$ and $T_\epsilon$. 

16
Proof. Let $\phi_{x_0} = \Phi(\cdot - x_0)$ be a smooth cut-off function supported around $x_0$. For the convenience, we drop the index $x_0$. Starting from $v^\epsilon \in \mathcal{E}_T$ and $p^\epsilon \in L^\infty_T L^2$, and using the interior regularity theory for perturbed Stokes system with smooth coefficients, we have

$$\|v^\epsilon, \partial_t v^\epsilon, \nabla v^\epsilon, \Delta v^\epsilon\|_{L^\infty((\delta, T_\epsilon) \times \mathbb{R}^3)} < +\infty$$

for any $\delta \in (0, T_\epsilon)$. Using $2v^\epsilon \psi$ with $\psi \in C^\infty_c((0, T_\epsilon) \times \mathbb{R}^3)$ as a test function in (3.4), we get

$$2 \int_0^T \int |\nabla v^\epsilon|^2 \psi \, dx \, ds = \int_0^T \int |v^\epsilon|^2 (\partial_s \psi + \Delta \psi) \, dx \, ds + \int_0^T \int |v^\epsilon|^2 \Phi \epsilon (J^\epsilon (v^\epsilon) \cdot \nabla) \psi \, dx \, ds$$

$$+ 2 \int_0^T \int p^\epsilon v^\epsilon \cdot \nabla \psi \, dx \, ds - \int_0^T \int |v^\epsilon|^2 \psi (J^\epsilon (v^\epsilon) \cdot \nabla) \Phi \epsilon \, dx \, ds.$$  

Using $\lim_{t \to 0^+} \|v^\epsilon (t) - v_0\|_{L^2(B_n)} = 0$ for any $n \in \mathbb{N}$ (Lemma 3.4), we can show

$$\int |v^\epsilon|^2 \psi (x, t) \, dx + 2 \int_0^t \int |\nabla v^\epsilon|^2 \psi \, dx \, ds = \int |v_0|^2 \psi (\cdot, 0) \, dx$$

$$+ \int_0^t \int |v^\epsilon|^2 (\partial_s \psi + \Delta \psi) \, dx \, ds + \int_0^t \int |v^\epsilon|^2 \Phi \epsilon (J^\epsilon (v^\epsilon) \cdot \nabla) \psi \, dx \, ds$$

$$+ 2 \int_0^t \int p^\epsilon v^\epsilon \cdot \nabla \psi \, dx \, ds - \int_0^t \int |v^\epsilon|^2 \psi (J^\epsilon (v^\epsilon) \cdot \nabla) \Phi \epsilon \, dx \, ds \quad (3.8)$$

for any $\psi \in C^\infty_c((0, T_\epsilon) \times \mathbb{R}^3)$ and $0 < t < T_\epsilon$.

We suppress the index $\epsilon$ in $v^\epsilon$ and $p^\epsilon$, and take $\psi(x, s) = \phi(x) \theta(s)$ where $\theta(s) \in C^\infty_c([0, T_\epsilon])$ and $\theta(s) = 1$ on $[0, t]$ to get

$$\|v(t) \phi\|_2^2 + 2 \|\nabla v \phi\|_{L^2(0, t) \times \mathbb{R}^3}^2$$

$$\leq \|v_0\|_{L^2_{t, x} \text{loc}}^2 + \int_0^t \int |v^2| \Delta \phi^2 \, dx \, ds + \int_0^t \int |v^2 \phi (J^\epsilon (v) \cdot \nabla) \Phi \epsilon \, dx \, ds$$

$$+ \int_0^t \int |v^2 \Phi \epsilon (J^\epsilon (v) \cdot \nabla) \phi^2 \, dx \, ds + \int_0^t \int 2 \hat{p} (v \cdot \nabla) \phi^2 \, dx \, ds \quad (3.9)$$

$$= \|v_0\|_{L^2_{t, x} \text{loc}}^2 + I_1 + I_2 + I_3 + I_4,$$

where $\hat{p} = \hat{p}_0$ will be defined later in (3.11) as a function satisfying $\nabla (p - \hat{p}) = 0$ on $B(\frac{x_0}{2}) \times (0, T)$.

The bounds of $I_1, I_2$ and $I_3$ can be easily obtained by Hölder inequalities,

$$I_1 \lesssim \|v\|_{U^2_t}^2, \quad \text{and} \quad I_2, I_3 \lesssim \|v\|_{U^{3,3}_t}^3.$$

(3.10)

Here we have used $|\nabla \Phi \epsilon| \lesssim \epsilon \leq 1$.

On the other hand, $I_4$ can be estimated as

$$I_4 \lesssim \|\tilde{p}\|_{L^2(0, t) \times (B(\frac{x_0}{2}))} \|v\|_{U^{3,3}_t}^3.$$

Now, we define $\tilde{p}^\epsilon$ on $B(\frac{x_0}{2}) \times [0, T]$ by

$$\tilde{p}^\epsilon (x, t) = -\frac{1}{3} J^\epsilon (v^\epsilon) \cdot v^\epsilon \Phi \epsilon (x, t) + n_{v^\epsilon} \int_{B(\frac{x_0}{2})} K_{ij} (x - y) \sigma_i^\epsilon (v^\epsilon_j (y, t) \Phi \epsilon (y) dy$$

$$+ \int_{B(\frac{x_0}{2})} (K_{ij} (x - y) - K_{ij} (x_0 - y)) \sigma_i^\epsilon (v^\epsilon_j (y, t) \Phi \epsilon (y) dy$$

$$= \tilde{p} + \tilde{p}^2 + \tilde{p}^3.$$

17
Comparing the above with (3.6) for \( p^\epsilon \), which has the singular integral form

\[
p^\epsilon(x, t) = -\frac{1}{3} \mathcal{J}_\epsilon(v^\epsilon) \cdot v^\epsilon(x, t) \Phi(x) + \text{p.v.} \int K_{ij}(x - y) \mathcal{J}_\epsilon(v^\epsilon) v^\epsilon_j(y, t) \Phi(y) dy,
\]

we see that \( p - \hat{p} \) depends only on \( t \), and hence \( \nabla \hat{p} = \nabla p \) on \( B(x_0, \frac{3}{2}) \times [0, T] \).

Then, using the interpolation inequality and Young’s inequality, we take \( L^2(0, t) \times B(x_0, \frac{3}{2}) \)-norm for each term to get

\[ \| \hat{p}^2 \|_{L^2(0, t) \times B(x_0, \frac{3}{2})} \lesssim \| v \|_{U_t^{3,3}}^2, \]

and

\[ \| \hat{p}^2 \|_{L^2(0, t) \times B(x_0, \frac{3}{2})} \lesssim \| \mathcal{J}_\epsilon(v_1) v_2 \Phi \|_{L^2(0, t) \times B(x_0, 2)} \lesssim \| v \|_{U_t^{3,3}}^2. \]

The second inequality for \( \hat{p}^2 \) follows from Calderon-Zygmund theorem. Finally, using

\[ |K_{ij}(x - y) - K_{ij}(x_0 - y)| \lesssim \frac{|x - x_0|}{|x_0 - y|^4} \]

for \( x \in B(x_0, \frac{3}{2}) \) and \( y \in B(x_0, 2)^c \), we have

\[
\| \hat{p}^2 \|_{L^2(0, t) \times B(x_0, \frac{3}{2})} \lesssim \left\| \int_{B(x_0, 2)^c} \frac{1}{|x_0 - y|^4} \mathcal{J}_\epsilon(v_1) v_j(y, s) \Phi(y) dy \right\|_{L^2(0, t)}
\leq \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \left\| \int_{B(x_0, 2k+1)} |\mathcal{J}_\epsilon(v_i) v_j(y, s)| dy \right\|_{L^2(0, t)}
\leq \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \left\| \mathcal{J}_\epsilon(v_i) v_j \right\|_{U_t^{3,3}} \lesssim \| v \|_{U_t^{3,3}}^2.
\]

Above we have taken \( B(x_0, 2k+1) \subset \bigcup_{j=1}^{J_k} B(x_j^k, 1) \) with \( J_k \lesssim 2^{3k} \).

Therefore, we get

\[ \| \hat{p} \|_{L^2(0, t) \times B(x_0, \frac{3}{2})} \lesssim \| v \|_{U_t^{3,3}}^2 \] \tag{3.12}

and

\[ I_4 \lesssim \| v \|_{U_t^{3,3}}^3. \]

Combining this with (3.10) and taking supremum on (3.9) over \( \{x_0 \in \mathbb{R}^3\} \), we have

\[ \| v(t) \|_{L^2_{\text{aloc}}}^2 + 2 \| \nabla v \|_{U_t^{3,3}} \lesssim \| v_0 \|_{L^2_{\text{aloc}}}^2 + \int_0^t \| v(s) \|_{L^2_{\text{aloc}}}^2 ds + \| v \|_{U_t^{3,3}}^3. \]

Then, using the interpolation inequality and Young’s inequality,

\[
\| v \|_{U_t^{3,3}}^3 \lesssim \| v \|_{U_t^{3,3}}^{3/2} \| v \|_{U_t^{3,6}}^{3/2} \lesssim \| v \|_{L^2(0, t; L^2_{\text{aloc}})}^6 + \| v \|_{L^2(0, t; L^2_{\text{aloc}})}^2 + \| \nabla v \|_{U_t^{2,2}}^2,
\]

18
we get
\[
\|v(t)\|_{L^2_{uloc}}^2 + \|\nabla v\|_{L^2_{uloc}}^2 \leq \|v_0\|_{L^2_{uloc}}^2 + \int_0^t \|v(s)\|_{L^2_{uloc}}^2 \, ds + \int_0^t \|v(s)\|_{L^2_{uloc}}^6 \, ds. \tag{3.13}
\]

Finally, we apply the Grönwall inequality, so that there is a small \( \varepsilon_1 > 0 \) such that, if \( v^\epsilon \) exists on \([0,T]\) for \( T \leq T_0, T_0 = \varepsilon_1 \left( 1 + \|v_0\|_{L^2_{uloc}}^4 \right)^{-1} \), then we have
\[
\sup_{0< t < T} \|v^\epsilon(t)\|_{L^2_{uloc}} \leq \|v_0\|_{L^2_{uloc}} \left( 1 - \frac{Ct \|v_0\|_{L^2_{uloc}}^4}{\min(1, \|v_0\|_{L^2_{uloc}}^4)} \right)^{-\frac{1}{4}} \leq \|v_0\|_{L^2_{uloc}} (1 - C\varepsilon_1)^{-\frac{1}{4}}.
\]
Together with \((3.13)\), this completes the proof. \( \square \)

**Lemma 3.6.** The distributional solutions \( \{(v^\epsilon, p^\epsilon)\}_{0< \varepsilon < 1} \) of \((3.4)\) can be extended to the uniform time interval \([0,T_0]\), where \( T_0 \) is as in Lemma 3.5.

**Proof.** We will prove it by iteration. For the convenience, we fix \( 0 < \epsilon < 1 \) and drop the index \( \varepsilon \) in \( v^\epsilon \) and \( p^\epsilon \). Denote the uniform bound in Lemma 3.5 by
\[
B = C(\|v_0\|_{L^2_{uloc}}), \quad B \geq \|v_0\|_{L^2_{uloc}}.
\]
If an initial data \( v(t_0) \) satisfies \( \|v(\cdot, t_0)\|_{L^2_{uloc}} \leq B \), by Lemma 3.4, we get \( S = S(\epsilon, B) > 0 \) and a unique solution \( v(x, t + t_0) \) on \( \mathbb{R}^3 \times [0,S] \) to \((3.5)\) satisfying
\[
\|v(t + t_0)\|_{E_S} \leq 2C_0B.
\]
Now, we start the iteration scheme. Since \( \|v_0\|_{L^2_{uloc}} \leq B \), a unique solution \( v \) exists in \( E_S \) to \((3.5)\). By Lemma 3.4 and Lemma 3.5, \( v \) satisfies
\[
\|v\|_{E_S} \leq B.
\]
Then, we choose \( \tau \in (\frac{3}{4}S, S) \), so that \( \|v(\tau)\|_{L^2_{uloc}} \leq B \), and hence we obtain a solution \( \tilde{v} \in E(\tau, \tau + S) \) to
\[
\tilde{v}(t) = e^{(t-\tau)\Delta}v|_{t=\tau} + \int_\tau^t e^{(t-s)\Delta}P\nabla \cdot N^\varepsilon(\tilde{v})(s) \, ds,
\]
where we denote \( N^\varepsilon(v) = J_\varepsilon(v) \otimes v\Phi_\varepsilon \).

Denote the glued solution by \( u(x,t) = v(x,t)1_{[0,\tau]}(t) + \tilde{v}(x,t)1_{(\tau,\tau+S]}(t) \), where \( 1_E \) is a characteristic function of a set \( E \subset [0,\infty) \). We claim that it solves \((3.5)\) in \((0, \tau + S)\); it is obvious for \( t \in (0, \tau] \), and for \( t \in (\tau, \tau + S] \),
\[
u(t) = \tilde{v}(t)
\]
\[
eq e^{(t-\tau)\Delta} \left( e^{\tau\Delta}v_0 + \int_0^\tau e^{(\tau-s)\Delta}P\nabla \cdot N^\varepsilon(v)(s) \, ds \right) + \int_\tau^t e^{(t-s)\Delta}P\nabla \cdot N^\varepsilon(\tilde{v})(s) \, ds
\]
\[
eq e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}P\nabla \cdot N^\varepsilon(v)(s) \, ds + \int_\tau^t e^{(t-s)\Delta}P\nabla \cdot N^\varepsilon(\tilde{v})(s) \, ds
\]
\[
eq e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}P\nabla \cdot N^\varepsilon(u)(s) \, ds.
\]
By Lemma 3.5 again, it satisfies
\[ \|u\|_{L^1(0,T+S)} \leq B. \]
By uniqueness, we get \( u = v \) for \( 0 \leq t \leq S \). In other words, \( u \) is an extension of \( v \).
Repeat this until the extended solution exists on \( [0,T_0] \). Since at each iteration, we can extend the time interval by at least \( \frac{2}{3}S \), in finite numbers of iterations, we have a distributional solution \( (v^\varepsilon, p^\varepsilon) \) of (3.4) on \( \mathbb{R}^3 \times [0,T_0] \).

**Proof of Theorem 3.2.** For \( 0 < \varepsilon \ll 1 \), let \( (v^\varepsilon, p^\varepsilon) \) be the distributional solution to the localized-mollified Navier-Stokes equations (3.4) on \( \mathbb{R}^3 \times [0,T] \) constructed in Lemmas 3.4 and 3.6, where \( T = T(\|v_0\|_{L^2_{uloc}}) \) is independent of \( \varepsilon \). By Lemma 3.5,
\[ \|v^\varepsilon\|_{L^2_T} \leq C(\|v_0\|_{L^2_{uloc}}). \]

We then define \( p^\varepsilon \in L^2_{loc}([0,T] \times \mathbb{R}^3) \) by
\[
p^\varepsilon(x,t) = -\frac{1}{3} \mathcal{J}_\varepsilon(v^\varepsilon) \cdot v^\varepsilon(x,t) \Phi_\varepsilon(x) + \text{p.v.} \int_{B_2} K_{ij}(x-y)N^\varepsilon_{ij}(y,t)dy \]
\[ + \text{p.v.} \int_{B_2^c} (K_{ij}(x-y) - K_{ij}(-y))N^\varepsilon_{ij}(y,t)dy, \tag{3.14} \]
\[ N^\varepsilon_{ij}(y,t) = \mathcal{J}_\varepsilon(v^\varepsilon)v^\varepsilon_j(y,t)\Phi_\varepsilon(y). \]

Because \( N^\varepsilon_{ij} \in L^\infty(0,T; L^2(\mathbb{R}^3)) \), the right side of (3.14) is defined in \( L^\infty(0,T; L^2(\mathbb{R}^3)) + L^\infty(0,T) \). Note that \( \nabla(p^\varepsilon - p^\varepsilon_0) = 0 \) because
\[
(p^\varepsilon - p^\varepsilon_0)(t) = \int_{B_2^c} K_{ij}(-y)\mathcal{J}_\varepsilon(v^\varepsilon_0)v^\varepsilon_j(y,t)\Phi_\varepsilon(y)dy \in L^2_T(0,T).
\]

Therefore, \( (v^\varepsilon, p^\varepsilon) \) is another distributional solution to the localized-mollified equations (3.4). We will show that for each \( n \in \mathbb{N} \), \( p^\varepsilon \) has a bound independent of \( \varepsilon \) in \( L^2_T([0,T] \times B_{2^n}) \). We drop the index \( \varepsilon \) in \( v^\varepsilon \) and \( p^\varepsilon \) for a moment.

For \( n \in \mathbb{N} \), we rewrite (3.14) for \( x \in B_{2^n} \) as follows.
\[
p(x,t) = -\frac{1}{3} \mathcal{J}_\varepsilon(v) \cdot v(x,t) \Phi_\varepsilon(x) + \text{p.v.} \int_{B_2} K_{ij}(x-y)N_{ij}(y,t)dy \]
\[ + \left( \text{p.v.} \int_{B_{2^n+1}\setminus B_2} + \text{p.v.} \int_{B_{2^n+1}} \right) (K_{ij}(x-y) - K_{ij}(-y))N_{ij}(y,t)dy \]
\[ = p_1 + p_2 + p_3 + p_4. \]

All \( p_i \) are defined in \( L^\infty(0,T; L^2) + L^\infty(0,T) \).

By Lemma 3.5, we have
\[
\|N_{ij}\|_{L^\frac{3}{2}_T} \lesssim \|\mathcal{J}_\varepsilon(v)\|_{U^{1,3}_T} \|v\|_{U^{3,3}_T} \leq C(\|v_0\|_{L^2_{uloc}}), \tag{3.15} \]
and
\[
\|N_{ij}\|_{L^\frac{3}{2}_T([0,T] \times B_{2^n})} \lesssim 2^{2n} \|\mathcal{J}_\varepsilon(v)\|_{U^{1,3}_T} \|v\|_{U^{3,3}_T} \leq C(n, \|v_0\|_{L^2_{uloc}}), \quad \forall n \in \mathbb{N}. \tag{3.16} \]
Then, the bound of $p_1$ can be obtained since
\[
\|p_1\|_{L^2_T([0,T] \times B_{2^n})} \lesssim \sum_{i=1}^{3} \|N_{ij}^\varepsilon\|_{L^2_T([0,T] \times B_2)}.
\]
Using Calderon-Zygmund theorem, we get
\[
\|p_2\|_{L^2_T([0,T] \times B_{2^n})} \lesssim \|N_{ij}^\varepsilon\|_{L^2_T([0,T] \times B_2)},
\]
and
\[
\|p_{31}\|_{L^2_T([0,T] \times B_{2^n})} \lesssim \|N_{ij}^\varepsilon\|_{L^2_T([0,T] \times B_{2^{n+1}})}
\]
where
\[
p_{31}(x,t) = p.v. \int_{B_{2^{n+1}} \setminus B_2} K_{ij}(x-y) J_i(v_i) v_j(y,t) \Phi(y) dy.
\]
On the other hand, $p_{32} = p_3 - p_{31}$ satisfies
\[
\|p_{32}\|_{L^2_T([0,T] \times B_{2^n})} \lesssim 2^{2n} \left\| \frac{1}{|y|^3} \right\|_{L^3(B_{2^{n+1}} \setminus B_2)} \|N_{ij}^\varepsilon\|_{L^2_T([0,T] \times B_{2^{n+1}})}
\]
\[
\lesssim 2^{2n} \|N_{ij}^\varepsilon\|_{L^2_T([0,T] \times B_{2^{n+1}})}.
\]
Since for $x \in B_{2^n}$ and $y \in B_{2^{n+1}}$, we have
\[
|K_{ij}(x-y) - K_{ij}(-y)| \lesssim \frac{|x|}{|y|^4} \lesssim \frac{2^n}{|y|^4},
\]
the bound of $p_4$ can be obtained as
\[
\|p_4\|_{L^2_T([0,T] \times B_{2^n})} \lesssim 2^{2n} \|p_{4}^\varepsilon\|_{L^2_T([0,T],L^\infty(B_{2^n}))} \lesssim 2^{3n} \left\| \int_{B_{2^{n+1}}} \frac{1}{|y|^4} |N_{ij}^\varepsilon|(y,t) dy \right\|_{L^2_T([0,T])}
\]
\[
\lesssim 2^{3n} \sum_{k=n+1}^{\infty} \frac{1}{2^{nk}} \|N_{ij}^\varepsilon\|_{L^4_T([0,T];L^1(B_{2^{k+1}}))} \lesssim n \|N_{ij}^\varepsilon\|_{L^4_T([0,T];L^1)}.
\]
Adding the estimates and using (3.15)-(3.16), we get for each $n \in \mathbb{N}$,
\[
\|p^\varepsilon\|_{L^2_T([0,T] \times B_{2^n})} \leq C(n, \|v_0\|_{L^2_{\text{loc}}}). \tag{3.17}
\]
Now, we find a limit solution of $(v^\varepsilon, p^\varepsilon)$ up to subsequence on each $[0,T] \times B_{2^n}, n \in \mathbb{N}$. First, construct the solution $v$ on the compact set $[0,T] \times B_2$. By uniform bounds on $v^\varepsilon$ and the compactness argument, we can extract a sequence $v^{1,k}$ from $\{v^\varepsilon\}$ satisfying
\[
v^{1,k} \rightharpoonup v^1 \quad \text{in } L^\infty(0,T;L^2(B_2)),
\]
\[
v^{1,k} \rightarrow v^1 \quad \text{in } L^2(0,T;H^1(B_2)),
\]
\[
v^{1,k} \rightarrow v^1 \quad \text{in } L^3(0,T;L^3(B_2)),
\]
\[
J_{1,k}(v^{1,k}) \rightarrow v^1 \quad \text{in } L^3(0,T;L^3(B_{2^{-k}})),
\]
as $k \rightarrow \infty$. Let $v = v^1$ on $[0,T] \times B_2$. 21
Then, we extend \( v \) to \([0, T] \times B_4 \) as follows. In a similar way to getting \( v^1 \), we can find a subsequence \( \{(v_{2k}^{1}, p_{2k}^{1})\}_{k \in \mathbb{N}} \) of \( \{(v_{1k}^{1}, p_{1k}^{1})\}_{k \in \mathbb{N}} \) which satisfies the following convergence:

\[
\begin{aligned}
    v_{2k}^{2} & \xrightarrow{\ast} v^2 & \text{ in } L^\infty(0, T; L^2(B_4)), \\
    v_{2k}^{2} & \to v^2 & \text{ in } L^2(0, T; H^1(B_4)), \\
    v_{2k}^{2} & \to v^2 & \text{ in } L^3(0, T; L^3(B_4)), \\
    J_{2k}(v_{2k}^{2}) & \to v^2 & \text{ in } L^3(0, T; L^3(B_4-)),
\end{aligned}
\]

as \( k \to \infty \). Here, we can easily check that \( v^2 = v^1 \) on \([0, T] \times B_2 \), so that \( v = v^2 \) is the desired extension. By repeating this argument, we can construct a sequence \( \{v_n^k\} \) and its limit \( v \). Indeed, by the diagonal argument, \( v \) can be approximated by

\[
v_{(k)} = \begin{cases} v_{k,k} & \text{ in } [0, T] \times B_{2^k}, \\
0 & \text{ otherwise}
\end{cases}, \quad \forall k \in \mathbb{N}
\]

More precisely, on each \([0, T] \times B_{2^n}, \{v_{(k)}\}_{k=n}^\infty \) enjoys the same convergence properties as above. This follows from that \( \{v_{m,j}\}_{j \in \mathbb{N}}, m \geq n \) is a subsequence of \( \{v_{n,j}\}_{j \in \mathbb{N}} \). Indeed, for each \( v_{k,k}, k \geq n \), we can find \( j_k \geq k \) such that

\[
v_{k,k} = v_{n,j_k}.
\]

Then, by its construction, for each \( n \in \mathbb{N}, \{v_{(k)}\}_{k=n}^\infty \) satisfies

\[
\begin{aligned}
    v_{(k)} & \xrightarrow{\ast} v & \text{ in } L^\infty(0, T; L^2(B_{2^n})), \\
    v_{(k)} & \to v & \text{ in } L^2(0, T; H^1(B_{2^n})), \\
    v_{(k)} & \to v & \text{ in } L^3(0, T; L^3(B_{2^n})), \\
    J_{(k)}(v_{(k)}) & \to v & \text{ in } L^3(0, T; L^3(B_{2^n-}))
\end{aligned}
\]

as \( k \to \infty \). Furthermore, since \( \nu^s \) are uniformly bounded in \( \mathcal{E}_T \), we can easily see that \( v \in \mathcal{E}_T \) and \( v \in U_{T}^{3,3} \),

\[
\|v\|_{\mathcal{E}_T} + \|v\|_{U_{T}^{3,3}} \leq C(\|v_0\|_{L^2_{\text{loc}}}).
\]

Now, we construct a pressure \( p \) corresponding to \( v \). Using (3.14), we define \( p_{(k)} \) by

\[
\begin{aligned}
p_{(k)}(x, t) &= -\frac{1}{3} J_{(k)}(v_{(k)}) \cdot v_{(k)}(x, t) \Phi_{(k)}(x) \\
&\quad + \text{p.v.} \int_{B^2} K_{ij}(x-y)J_{(k)}(v_{i}^{(k)})v_{j}^{(k)}(y, t)\Phi_{(k)}(y)dy \\
&\quad + \text{p.v.} \int_{B^2} (K_{ij}(x-y)-K_{ij}(-y))J_{(k)}(v_{i}^{(k)})v_{j}^{(k)}(y, t)\Phi_{(k)}(y)dy,
\end{aligned}
\]

where \( \Phi_{(k)} = \Phi_{\epsilon_k} \) for \( \epsilon_k \) satisfying \( v_{k,k}^{\epsilon_k} = \nu^s \). Also define

\[
p(x, t) = \lim_{n \to \infty} \tilde{p}_n(x, t)
\]

where \( \tilde{p}_n(x, t) \) is defined for \( |x| < 2^n \) by

\[
\tilde{p}_n(x, t) = -\frac{1}{3} |v(x, t)|^2 + \text{p.v.} \int_{B^2} K_{ij}(x-y)v_i v_j(y, t)dy + \tilde{p}_n^3 + \tilde{p}_4^n,
\]
with
\[ p_3^n(x, t) = \text{p.v.} \int_{B_{2^n+1} \setminus B_2} (K_{ij}(x - y) - K_{ij}(-y))v_iv_j(y, t)\,dy, \]
\[ p_4^n(x, t) = \int_{B_{2^n+1}} (K_{ij}(x - y) - K_{ij}(-y))v_iv_j(y, t)\,dy. \]

The first two terms in \( p^n \) are defined in \( U_T^{3,2} \) and independent of \( n \). Among the last two terms, \( p_3^n \) converges absolutely but \( p_3^n \) only in \( U_T^{3,2} \). By estimates similar to those for \( p^\epsilon \), we get \( p_3^n, p_4^n \in L^{3/2}((0, T) \times B_{2^n}) \) and
\[ p_3^n + p_4^n = p_{3,n+1} + p_{4,n+1}, \text{ in } L^{3/2}((0, T) \times B_{2^n}) \]

Thus \( p^n(x, t) \) is independent of \( n \) for \( n > \log_2|x| \).

Our goal is to show that the strong convergences (3.20)-(3.21) of \( \{v^{(k)}\} \) gives
\[ p^{(k)} \to p \quad \text{in } L^\frac{3}{2}([0, T] \times B_{2^n}), \quad \text{for each } n \in \mathbb{N}, \quad (3.25) \]

Let \( N_{ij}^{(k)} = J^{(k)}(v_i^{(k)})v_j^{(k)}\Phi^{(k)} \) and \( N_{ij} = v_i v_j \). For any fixed \( R > 0 \), we have
\[ \left\| N_{ij}^{(k)} - N_{ij} \right\|_{L^\frac{3}{2}([0, T] \times B_R)} \leq \left\| J^{(k)}(v_i^{(k)}) - v_i \right\|_{L^\frac{3}{2}([0, T] \times B_R)} \left\| v_j^{(k)} \right\|_{L^3([0, T] \times B_R)} + \left\| v_i v_j (1 - \Phi^{(k)}) \right\|_{L^\frac{3}{2}([0, T] \times B_R)} + \left\| v_i v_j \right\|_{L^\frac{3}{2}([0, T] \times B_R)} \]
\[ \leq \left\| J^{(k)} - v \right\|_{L^3([0, T] \times B_R)} \left\| v^{(k)} \right\|_{L^3([0, T] \times B_R)} + \left\| v^{(k)} \right\|_{L^3([0, T] \times B_R)} + \left\| v^2 (1 - \Phi^{(k)}) \right\|_{L^\frac{3}{2}([0, T] \times B_R)} \to 0 \]

by (3.20), (3.21), and Lebesgue dominated convergence theorem. Then, it provides the convergence of \( p^{(k)} \) to \( p \): On \([0, T] \times B_{2^n}\), for \( m > n \),
\[ p^{(k)} - p = -\frac{1}{3} \text{tr}(N^{(k)} - N) + \text{p.v.} \int_{B_2} K_{ij}(\cdot - y)(N_{ij}^{(k)} - N_{ij})(y)\,dy \]
\[ + \left[ \text{p.v.} \int_{B_{2^n+1} \setminus B_2} \int_{B_{2m} \setminus B_{2m+1}} \int_{B_{2m}^c} (K_{ij}(\cdot - y) - K_{ij}(-y))(N_{ij}^{(k)} - N_{ij})(y)\,dy \right] \]
\[ = q_1 + q_2 + q_3 + q_4 + q_5. \]

In a similar way to getting (3.17), we have
\[ \|q_1, q_2, q_3\|_{L^\frac{2}{3}([0, T] \times B_{2^n})} \lesssim_n \|N^{(k)} - N\|_{L^\frac{2}{3}([0, T] \times B_{2^n+1})}, \]
and
\[ \|q_4\|_{L^\frac{2}{3}([0, T] \times B_{2^n})} \lesssim \|N^{(k)} - N\|_{L^\frac{2}{3}([0, T] \times B_{2^n})}. \]

On the other hand, using
\[ |K_{ij}(x - y) - K_{ij}(-y)| \lesssim \frac{|x|}{|y|^1}, \]

23
we obtain

\[\|q_5\|_{L^3_T([0,T] \times B_{2^n})} \leq \frac{2^{3n}}{2^m} \left( \|v\|_{L^{3,3}_T}^2 + \left\| J_k(v^{(k)}) \right\|_{L^{3,3}_T} + \left\| v^{(k)} \right\|_{L^{3,3}_T} \right) \leq C(n, \|v_0\|_{\text{uloc}}; T) \frac{1}{2^m}.
\]

Therefore, for fixed \(n\), if we choose sufficiently large \(m\), we can make \(q_5\) very small in \(L^3_T([0,T] \times B_{2^n})\) and then for sufficiently large \(k\), \(q_1, q_2, q_3,\) and \(q_4\) also become very small in \(L^3_T([0,T] \times B_{2^n})\) because of (3.26). This gives the desired convergence (3.25) of \(p^{(k)}\) to \(p\).

Now, we check that \((v, p)\) is a local energy solution. It is easy to prove that \((v, p)\) solves the Navier-Stokes equation in distributional sense by using the distributional form of (3.4) for \((v^{(k)}, p^{(k)})\) and the convergence (3.18)-(3.21) and (3.25)-(3.26). For example, for any \(\xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}^3)\),

\[
\begin{align*}
\int_0^T \int_0^T \int v^{(k)} \cdot \partial_t \xi dx &\to \int_0^T \int v \cdot \partial_t \xi dx \\
\int_0^T \int J_k(v^{(k)}) (v^{(k)} \Phi_{(k)}) : \nabla \xi dx &\to \int_0^T \int v \otimes v : \nabla \xi dx
\end{align*}
\]

as \(k \to \infty\).

Since we have

\[
\int_0^T \int (\Delta v - (v \cdot \nabla)v - \nabla p) \cdot \phi dx dt \leq \left| \int_0^T \int \nabla v \cdot \nabla \phi dx dt \right| + \left| \int_0^T \int v (v \cdot \nabla) \phi dx dt \right| + \left| \int_0^T \int p \div \phi dx dt \right|
\]

\[
\leq \left\| \nabla v \right\|_{L^2(0,T;L^2(B_{2^n}))} \left\| \nabla \phi \right\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \left( \left\| v \right\|_{L^3(0,T;L^3(B_{2^n}))} + \left\| p \right\|_{L^2(0,T;L^2(B_{2^n}))} \right) \left\| \nabla \phi \right\|_{L^3(0,T;L^3(\mathbb{R}^3))}
\]

\[
\leq C(n, T, \|v_0\|_{\text{uloc}}) \left\| \nabla \phi \right\|_{L^3(0,T;L^3(\mathbb{R}^3))},
\]

for any \(\phi \in C_c^\infty([0, T] \times B_{2^n}), n \in \mathbb{N}\), it follows that

\[\partial_t v = \Delta v - (v \cdot \nabla)v - \nabla p \in X_n\]

for any \(n \in \mathbb{N}\), where \(X_n\) is the dual space of \(L^2(0,T;W^{1,3}_0(B_{2^n}))\).

With this bound of \(\partial_t v\), for each \(n \in \mathbb{N}\), we may redefine \(v(t)\) on a measure-zero subset \(\Sigma_n\) of \([0, T]\) such that the function

\[t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot \zeta(x) dx\]

is continuous for any vector \(\zeta \in C_c^\infty(B_{2^n})\). Redefine \(v(t)\) recursively for all \(n\) so that (3.27) is true for any \(\zeta \in C_c^\infty(\mathbb{R}^3)\). It is then true for any \(\zeta \in L^2(\mathbb{R}^3)\) with a compact support using \(v \in L^\infty(0,T;L^2_{\text{uloc}})\).

Furthermore, consider the local energy equality (3.8) for \((v^{(k)}, p^{(k)})\) on the time interval \((0, T)\) for a non-negative \(\psi \in C_c^\infty([0, T] \times \mathbb{R}^3)\). The first term \(\int |v^{(k)}|^2 \psi(x, T) dx\) vanishes. Taking limit infimum as \(k\) goes to infinity, and using the weak convergence (3.19) and the strong convergence (3.20)-(3.21) and (3.25)-(3.26), we get

\[
2 \int_0^T \int |\nabla v|^2 \psi dx ds \leq \int |v_0|^2 \psi(\cdot, 0) dx
\]

\[
+ \int_0^T \int \left| v \right|^2 (\partial_s \psi + \Delta \psi) + (|v|^2 + 2\bar{p})(v \cdot \nabla) \psi dx ds,
\]
for any non-negative \( \psi \in C_c^\infty([0,T) \times \mathbb{R}^3) \).

Then, for any \( t \in (0, T) \) and non-negative \( \varphi \in C_c^\infty([0,T) \times \mathbb{R}^3) \), take \( \psi(x,s) = \varphi(x,s)\theta_\varepsilon(s) \), \( \varepsilon \ll 1 \), where \( \theta_\varepsilon(s) = \theta \left( \frac{s-t}{\varepsilon} \right) \) for some \( \theta \in C_c^\infty(\mathbb{R}) \) such that \( \theta(s) = 1 \) for \( s \leq 0 \) and \( \theta(s) = 0 \) for \( s \geq 1 \), and \( \theta'(s) \leq 0 \) for all \( s \). Note that \( \theta_\varepsilon(s) = 1 \) for \( s \leq t \) and \( \theta_\varepsilon(s) = 0 \) for \( s \geq t + \varepsilon \). Sending \( \varepsilon \to 0 \) and using

\[
\int |v(t)|^2 \varphi \, dx \leq \liminf_{\varepsilon \to 0} \int_0^t \int |v|^2 \varphi(-\theta_\varepsilon') \, dx \, ds
\]
due to the weak local \( L^2 \)-continuity (3.27), we get

\[
\int |v(t)|^2 \varphi \, dx + 2 \int_0^t \int |\nabla v|^2 \varphi \, dx \, ds \leq \int |v_0|^2 \varphi(\cdot,0) \, dx + \int_0^t \int \{|v|^2(\partial_s \varphi + \Delta \varphi) + (|v|^2 + 2\tilde{p})(v \cdot \nabla)\varphi\} \, dx \, ds
\]

(3.29)

for any \( t \in (0, T) \) and non-negative \( \varphi \in C_c^\infty([0,T) \times \mathbb{R}^3) \). The local energy inequality (3.1) is a special case of (3.29) for test functions vanishing at \( t = 0 \).

Sending \( t \to 0_+ \) in (3.29) we get \( \limsup_{t \to 0_+} \int |v(t)|^2 \varphi \, dx \leq \int |v_0|^2 \varphi(\cdot,0) \, dx \) for any non-negative \( \varphi \in C_c^\infty \). Together with the weak continuity (3.27), we get \( \lim_{t \to 0_+} \int_{B_n} |v(x,t) - v_0(x)|^2 \, dx = 0 \) for any \( n \in \mathbb{N} \).

Finally, we consider the decomposition of the pressure. Recall that the pressure \( p \) is defined recursively by (3.23)-(3.24). For any \( x_0 \in \mathbb{R}^3 \) define \( \tilde{p}_{x_0} \in L^2([0,T] \times B(x_0, \frac{3\varepsilon}{2})) \) by (3.3), i.e.,

\[
\tilde{p}_{x_0}(x,t) = \frac{1}{3} |v|^2(x,t) + \text{p.v.} \int_{B(x_0,2)} K_{ij}(x-y)v_i v_j(y,t) \, dy
\]

(3.29)

\[
= \int_{B(x_0,2)^c} (K_{ij}(x-y) - K_{ij}(x_0-y))v_i v_j(y,t) \, dy.
\]

Let \( c_{x_0} = p - \tilde{p}_{x_0} \). If \( B(x_0, \frac{3\varepsilon}{2}) \subset B_{2^n} \), then

\[
c_{x_0}(t) = \int_{B_{2^n+1} \setminus B(x_0,2)} K_{ij}(x_0-y)v_i v_j(y,t) \, dy
\]

\[
- \int_{B_{2^n+1} \setminus B_2} K_{ij}(-y)v_i v_j(y,t) \, dy
\]

(3.30)

\[
+ \int_{B_{2^n+1}^c} (K_{ij}(x_0-y) - K_{ij}(-y))v_i v_j(y,t) \, dy.
\]

Note that \( c_{x_0} \in L^{3/2}(0,T) \), and \( c_{x_0}(t) \) is independent of \( x \in B(x_0, \frac{3\varepsilon}{2}) \), \( n \), and \( T \). Therefore, we get the desired decomposition (3.2) of the pressure.

\[\square\]

**Remark 3.1.** Our approach in this section is similar to that in Kikuchi-Seregin [17]. However, there are two significant differences:

1. Since we include initial data \( v_0 \) not in \( E^2 \), we add an additional localization factor \( \Phi(x) \) to the nonlinearity in the localized-mollified equations (3.4). Our approximation solutions \( \nu \) live in \( L^2_{\text{uloc}} \) and are no longer in the usual energy class.

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25
2. The pressure \( p \) and \( c_{x_0} \) are implicit in [17], but are explicit in this paper. We first specify the formula (3.23) of the pressure and then justify the strong convergence and decomposition. In particular, our \( c_{x_0}(t) \) is given by (3.30) and independent of \( T \).

Remark 3.2. Estimate (3.12) and its proof for \( \hat{p}_x^{x_0} \) are not limited to our approximation solutions. They are in fact also valid for any local energy solution \((v,p)\) in \((0,T)\) with local pressure \( \hat{p}_x \) given by (3.3), that is,

\[
\|\hat{p}_x\|_{L^2_t([0,T] \times B(x_0,\frac{\pi}{2}))} \leq C \|v\|_{U^3_t}^2, \quad \forall t < T, \tag{3.31}
\]

with a constant \( C \) independent of \( t, T \).

4 Spatial decay estimates

Recall that our initial data \( v_0 \in E_{\sigma}^2 + L^3_{uloc,\sigma} \). In Sections 4 and 5, we decompose

\[
v_0 = w_0 + u_0, \quad w_0 \in E_{\sigma}^2, \quad u_0 \in L^3_{uloc,\sigma}. \tag{4.1}
\]

Our goal in this section is to show that, although the solution \( v \) has no spatial decay, its difference from the linear flow, \( w = v - V \), \( V(t) = e^{t\Delta}u_0 \), does decay due to the decay of the oscillation of \( u_0 \). Here, the oscillation decay of \( u_0 \) follows from that of \( v_0 \) and \( w_0 \in E^2 \). The main task is to show that the contribution from the nonlinear source term

\[(V \cdot \nabla)V = \nabla \cdot (V \otimes V)\]

has decay, although \( V \) itself does not. On the other hand, we also need the decay of the pressure. However, \( \hat{p}_x \) given by (3.3) does not decay. Thus we need a different decomposition of the pressure \( p \) near each point \( x_0 \in \mathbb{R}^3 \).

Lemma 4.1 (New pressure decomposition). Let \( v_0 = w_0 + u_0 \) with \( w_0 \in E_{\sigma}^2 \) and \( u_0 \in L^3_{uloc,\sigma} \). Let \((v,p)\) be any local energy solution of (NS) with initial data \( v_0 \) in \( \mathbb{R}^3 \times (0,T) \), \( 0 < T < \infty \). Then, for each \( x_0 \in \mathbb{R}^3 \), we can find \( q_{x_0} \in L^3(0,T) \) such that

\[p(x,t) = \hat{p}_{x_0}(x,t) + q_{x_0}(t) \quad \text{in} \quad L^3((0,T) \times B(x_0,\frac{3}{2}))\]

where

\[
\hat{p}_{x_0} = -\frac{1}{3}(|w|^2 + 2w \cdot V) + \text{p.v.} \int_{B(x_0,2)} K_{ij}(\cdot - y)(w_iw_j + V_iw_j + w_iV_j)(y)dy \\
+ \int_{B(x_0,2)} (K_{ij}(\cdot - y) - K_{ij}(x_0 - y))(w_iw_j + V_iw_j + w_iV_j)(y)dy \\
+ \int K_i(\cdot - y)[(V \cdot \nabla)V_i]\rho_2(y)dy \\
+ \int (K_{ij}(\cdot - y) - K_{ij}(x_0 - y))V_iv_j(1 - \rho_2)(y)dy \\
+ \int (K_i(\cdot - y) - K_i(x_0 - y))V_i\rho_2(y)dy.
\tag{4.2}
\]

Here, \( w = v - V \), \( V(t) = e^{t\Delta}u_0 \), \( K_i = \partial_i K \), \( K_{ij} = \partial_{ij} K \), \( K(x) = \frac{1}{4\pi|x|} \), and \( \rho_2 = \Phi(\frac{-x}{2}) \).
Proof. Consider \((x,t) \in B(x_0, \frac{2}{3}) \times (0,T)\). Let \(F_{ij} = w_i w_j + V_i w_j + w_i V_j\) and \(G_{ij} = V_i V_j\). Substituting \(v = V + w\) in \((3.3)\), we get

\[
\begin{aligned}
\bar{p}_{x_0} &= p_{x_0}^F + p_{x_0}^G \\
p_{x_0}^F &= -\frac{1}{3} \text{tr } F + \text{p.v.} \int_{B(x_0,2)} K_{ij}(\cdot - y) F_{ij}(y) dy \\
&\quad + \int_{B(x_0,2)^c} (K_{ij}(\cdot - y) - K_{ij}(x_0 - y)) F_{ij}(y) dy \\
\end{aligned}
\tag{4.3}
\]

and

\[
\begin{aligned}
p_{x_0}^G &= -\frac{1}{3} \text{tr } G + \text{p.v.} \int_{B(x_0,2)} K_{ij}(\cdot - y) G_{ij}(y) dy \\
&\quad + \int_{B(x_0,2)^c} (K_{ij}(\cdot - y) - K_{ij}(x_0 - y)) G_{ij}[\rho_2 + (1 - \rho_2)](y) dy \\
&= -\frac{1}{3} \text{tr } G + \text{p.v.} \int_{B(x_0,2)} K_{ij}(\cdot - y) G_{ij}\rho_2(y) dy + p_{x_0,\text{far}} + \tilde{q}_{x_0}(t),
\end{aligned}
\]

where

\[
p_{x_0,\text{far}} = \int_{B(x_0,2)} (K_{ij}(\cdot - y) - K_{ij}(x_0 - y)) G_{ij}(1 - \rho_2)(y) dy,
\]

\[
\tilde{q}_{x_0}(t) = -\int_{B(x_0,2)^c} K_{ij}(x_0 - y) G_{ij}\rho_2(y) dy.
\]

Integrating by parts the principle value integral, we get

\[
p_{x_0}^G = \int K_i(\cdot - y) \partial_j [G_{ij}\rho_2(y)] dy + p_{x_0,\text{far}} + \tilde{q}_{x_0}(t).
\]

Note \(\partial_j [G_{ij}\rho_2] = (V \cdot \nabla V_i)\rho_2 + G_{ij} \partial_j \rho_2\). Denote

\[
\tilde{q}_{x_0}(t) = \int K_i(x_0 - y)V_i V_j(\partial_j \rho_2)(y) dy.
\]

We get

\[
\begin{aligned}
\bar{p}_{x_0}(x,t) &= p_{x_0}^F + \int K_i(\cdot - y)(V \cdot \nabla)V_i\rho_2(y) dy + p_{x_0,\text{far}} + \tilde{q}_{x_0}(t) \\
&\quad + \int (K_i(\cdot - y) - K_i(x_0 - y))V_i V_j(\partial_j \rho_2)(y) dy + \tilde{q}_{x_0}(t) \\
&= \bar{p}_{x_0}(x,t) + q_{x_0}(t) + \tilde{q}_{x_0}(t).
\end{aligned}
\tag{4.4}
\]

Thus we have \(p(x,t) = \bar{p}_{x_0}(x,t) + q_{x_0}(t)\) with

\[
q_{x_0}(t) = c_{x_0}(t) + \tilde{q}_{x_0}(t) + \bar{q}_{x_0}(t).
\]

Note that using \(\|G\|_{L^\infty_T} \leq \|V\|_{L^\infty_T}^2\) and \(|x_0 - y| > 2\) for \(y \in \text{supp}(\partial_j \rho_2)\), we have

\[
\|\bar{q}_{x_0}(t)\|_{L^\infty(0,T)} + \|\tilde{q}_{x_0}(t)\|_{L^\infty(0,T)} \lesssim \left\| \int_{B(x_0,3) \setminus B(x_0,2)} |G_{ij}|(y) dy \right\|_{L^\infty(0,T)}
\]

\[
\lesssim \|G\|_{L^\infty(0,T;L^1(B(x_0,3)))} \lesssim \|V\|_{L^\infty_T}^2.
\tag{4.5}
\]

Since \(\bar{q}_{x_0}(t) + \tilde{q}_{x_0}(t)\) is in \(L^{3/2}(0,T)\), so is \(q_{x_0}(t)\).
Although $\nabla V$ has spatial decay, it is not uniform in $t$. Thus, to show the spatial decay of $w$, we will first show (1.6), i.e., the smallness of $w$ in $L^2_{uloc}$ at far distance for a short time in Lemma 4.5. For that we need Lemmas 4.2, 4.3 and 4.4.

**Lemma 4.2.** For $u_0 \in L^3(\mathbb{R}^3)$, if $\frac{2}{s} + \frac{3}{q} = 1$ and $3 \leq q < 9$, then

$$\|e^{t\Delta}u_0\|_{L^3(0,\infty;L^3(\mathbb{R}^3))} \leq C_q \|u_0\|_{L^3(\mathbb{R}^3)}.$$  

This is proved in Giga [8, 196–197]. The case $q = 9$ is also true according to [8, Acknowledgment], but there is no detailed proof.

**Lemma 4.3.** Suppose $u_0 \in L^2_{uloc}$ and $u_0 \in L^3(B(x_0,3))$. Then, $V = e^{t\Delta}u_0$ satisfies

$$\|V\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} \lesssim \|u_0\|_{L^3(B(x_0,3))} + T^\frac{2}{s} \|u_0\|_{L^2_{uloc}}. \quad (4.6)$$

**Proof.** Let $\phi(x) = \Phi(\frac{x-x_0}{2})$. Decompose

$$u_0 = u_0\phi + u_0(1-\phi) =: u_1 + u_2.$$  

By Lemma 4.2,

$$\|e^{t\Delta}u_1\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} \lesssim \|e^{t\Delta}u_1\|_{L^3(0,T;L^3(\mathbb{R}^3))} \lesssim \|u_1\|_{L^3(\mathbb{R}^3)} \lesssim \|u_0\|_{L^3(B(x_0,3))}. \quad (4.7)$$

On the other hand, we have

$$\|e^{t\Delta}u_2\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} \lesssim \|\nabla e^{t\Delta}u_2\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} + \|e^{t\Delta}u_2\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))}.$$

Obviously,

$$\|e^{t\Delta}u_2\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} \lesssim T^\frac{2}{s} \|e^{t\Delta}u_2\|_{L^3(0,T;L^2_{uloc})} \lesssim T^\frac{2}{s} \|u_2\|_{L^2_{uloc}}.$$

Using $\text{supp}(u_2) \subset B(x_0,2)^c$ and heat kernel estimate, we get

$$\|\nabla e^{t\Delta}u_2\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} \lesssim T^\frac{2}{s} \|\nabla e^{t\Delta}u_2\|_{L^3((0,T)\times B(x_0,\frac{3}{2}))}$$

$$\lesssim T^\frac{1}{s} \int_{B(x_0,2)^c} \frac{1}{|x_0-y|^s} |u_0(y)| dy$$

$$\lesssim T^\frac{2}{s} \sum_{k=1}^{\infty} \int_{B(x_0,2^{k+1})\setminus B(x_0,2^k)} \frac{1}{2^{ks}} |u_0(y)| dy$$

$$\lesssim T^\frac{4}{s} \|u_0\|_{L^2_{uloc}}.$$  

Therefore, we obtain

$$\|e^{t\Delta}u_2\|_{L^3(0,T;L^3(B(x_0,\frac{3}{2})))} \lesssim T^\frac{1}{s} \|u_0\|_{L^2_{uloc}}.$$

Together with (4.7), we get (4.6). \qed
The perturbation \( w = v - V \), \( V(t) = e^{t\Delta} u_0 \), satisfies the perturbed Navier-Stokes equations in the sense of distributions,

\[
\begin{aligned}
\partial_t w - \Delta w + (V + w) \cdot \nabla (V + w) + \nabla p &= 0 \\
\text{div } w &= 0 \\
w|_{t=0} &= w_0.
\end{aligned}
\]  

(4.8)

It also satisfies the following local energy inequality for test functions supported away from \( t = 0 \).

**Lemma 4.4** (Local energy inequality for \( w \)). Let \( v_0, u_0 \in L^2_{\text{uloc}, \sigma} \). Let \((v, p)\) be any local energy solution of (NS) with initial data \( v_0 \) in \( \mathbb{R}^3 \times (0, T) \), \( 0 < T < \infty \). Then \( w(t) = v(t) - e^{t\Delta} u_0 \) satisfies

\[
\int |w|^2 \varphi(x, t) \, dx + 2 \int_0^t \int |\nabla w|^2 \varphi \, dx \, ds 
\leq \int_0^t \int |w|^2 (\partial_t \varphi + \Delta \varphi + v \cdot \nabla \varphi) \, dx \, ds 
+ \int_0^t \int 2p \cdot \nabla \varphi \, dx \, ds + \int_0^t \int 2V \cdot (v \cdot \nabla)(w \varphi) \, dx \, ds,
\]

for any non-negative \( \varphi \in C^\infty_0((0, T) \times \mathbb{R}^3) \) and any \( t \in (0, T) \).

**Proof.** Recall that we have the local energy inequality (3.1) for \((v, p)\). The equivalent form for \((w, p)\) is exactly (4.9). Indeed, (3.1) and (4.9) are equivalent because they differ by an equality which is the sum of the weak form of \( V \)-equation with \( 2v \varphi \) as the test function and the weak form of the \( w \)-equation (4.8) with \( 2V \varphi \) as the test function, after suitable integration by parts. This equality can be proved because \( V \) and \( \nabla V \) are in \( L^\infty(0, T; L^\infty(\mathbb{R}^3)) \), and \( \varphi \) has a compact support in space-time. \( \square \)

For \( r > 0 \), let

\[
\chi_r(x) = 1 - \Phi \left( \frac{x}{r} \right),
\]

so that \( \chi_r(x) = 1 \) for \( |x| \geq 2r \) and \( \chi_r(x) = 0 \) for \( |x| \leq r \).

**Lemma 4.5.** Let \( v_0 = w_0 + u_0 \) with \( w_0 \in E^2_\sigma \) and \( u_0 \in L^2_{\text{uloc}, \sigma} \). Let \((v, p)\) be any local energy solution of (NS) with initial data \( v_0 \) in \( \mathbb{R}^3 \times (0, T) \), \( 0 < T < \infty \). Then, there exist \( T_0 = T_0(\|v_0\|_{L^2_{\text{uloc}, \sigma}}) \) \in \( (0, 1) \) and \( C_0 = C_0(\|w_0\|_{L^2_{\text{uloc}, \sigma}}, \|u_0\|_{L^2_{\text{uloc}, \sigma}}) \) \( > 0 \) such that \( w(t) = v(t) - e^{t\Delta} u_0 \) satisfies

\[
\|w(t)\chi_R\|_{L^2_{\text{uloc}}} \leq C_0(t^{\frac{R}{2}} + \|w_0\chi_R\|_{L^2_{\text{uloc}}},
\]

(4.10)

for any \( R > 0 \) and any \( t \in (0, T_1) \), \( T_1 = \min(T_0, T) \).

In this lemma, we do not assume the oscillation decay.
Proof. By Lemma 2.4 and similar to (3.7), we can find $T_0 = T_0(\|v_0\|_{L^2_{\text{uloc}}}^2) \in (0, 1)$ such that, for $T_1 = \min(T_0, T)$,

$$\|w\|_{\mathcal{E}_{T_1}} + \|V\|_{\mathcal{E}_{T_1}} \lesssim \|w_0\|_{L^2_{\text{uloc}}} + \|u_0\|_{L^2_{\text{uloc}}}.$$ 

By interpolation, it follows that for any $2 \leq s \leq \infty$, and $2 \leq q \leq 6$ satisfying $\frac{2}{s} + \frac{3}{q} = \frac{3}{2}$, we have

$$\|w\|_{U^s_{T_1}} + \|V\|_{U^s_{T_1}} \lesssim \|w_0\|_{L^2_{\text{uloc}}} + \|u_0\|_{L^2_{\text{uloc}}}.$$ 

On the other hand, by Lemma 4.3, for any $t \in (0, 1)$,

$$\|V\|_{U^t_{s,4}} \lesssim \|u_0\|_{L^3_{\text{uloc}}}.$$ 

Let $A = \|w_0\|_{L^2_{\text{uloc}}} + \|u_0\|_{L^3_{\text{uloc}}}$. Then, both inequalities can be combined for $t \leq T_1$ as

$$\|w\|_{\mathcal{E}_t} + \|V\|_{\mathcal{E}_t} + \|w\|_{U^t_{s,4}} + \|V\|_{U^t_{s,4}} + \|V\|_{U^t_{s,4}} \lesssim A. \quad (4.11)$$

Fix $x_0 \in \mathbb{R}^3$ and $R > 0$, and let

$$\phi_{x_0} = \Phi(\cdot - x_0), \quad \xi = \phi_{x_0}^2 \chi_R.$$ 

Fix $\Theta \in C^\infty(\mathbb{R})$, $\Theta' \geq 0$, $\Theta(t) = 1$ for $t > 2$, and $\Theta(t) = 0$ for $t < 1$. Define $\theta_\epsilon \in C^\infty_c(0, T)$ for sufficiently small $\epsilon > 0$ by

$$\theta_\epsilon(s) = \Theta \left( \frac{s}{\epsilon} \right) - \Theta \left( \frac{s - T + 3\epsilon}{\epsilon} \right). \quad (4.13)$$

Thus $\theta_\epsilon(s) = 1$ in $(2\epsilon, T - 2\epsilon)$ and $\theta_\epsilon(s) = 0$ outside of $(\epsilon, T - \epsilon)$. We now consider the local energy inequality (4.9) for $w$ with $\varphi(x, s) = \xi(x)\theta_\epsilon(s)$. We may replace $p$ by $\tilde{p}_{x_0}$ in (4.9) as $\text{supp} \xi \subset B(x_0, \frac{3}{2})$ and $\int c_{x_0}(t)w \cdot \nabla \xi \, dx \, dt = 0$. We now take $\epsilon \to 0_+$. Since $\|v(t) - v_0\|_{L^2(B_{2}(x_0))} \to 0$ and $\|V(t) - u_0\|_{L^2(B_{2}(x_0))} \to 0$ as $t \to 0^+$, we get

$$\int_{0}^{2\epsilon} \int |w|^2 \xi \theta_\epsilon' \, dx \, ds \to \int |w_0|^2 \xi \, dx. \quad (4.14)$$

The last term in (4.9) converges by Lebesgue dominated convergence theorem using

$$\int_{0}^{t} \int |V \cdot (v \cdot \nabla)(w \xi)| \, dx \, ds \lesssim \|V\|_{L^s(0, T; L^1(B_{x_0, \frac{3}{2}))}) \|v\|_{U^t_{s,3/4}} \|\nabla w\|_{U^2_{t,2}}^2 + \|w\|_{L^2_{t,2}}^2,$$

where the right hand side of the inequality is bounded independently of $\epsilon$.

In the limit $\epsilon \to 0_+$, for any $t \in (0, T)$, we get

$$\int w^2(x, t) \xi(x) \, dx + 2 \int_{0}^{t} \int |\nabla w|^2 \xi \, dx \, ds$$

$$\leq \int |w_0|^2 \xi \, dx + \int_{0}^{t} \int |w|^2 (\Delta \xi + v \cdot \nabla \xi) \, dx \, ds \quad (4.15)$$

$$+ \int_{0}^{t} \int 2\tilde{p}_{x_0} w \cdot \nabla \xi \, dx \, ds + \int_{0}^{t} \int 2V \cdot (v \cdot \nabla)(w \xi) \, dx \, ds,$$
for \( \xi \) given by (4.21). Now, we consider \( t \leq T_1 \). Using (4.11), we have

\[
\int_0^t \int |w|^2 \Delta \xi dxds \lesssim \|w\|_{U^2_t}^2 \lesssim A^2 t,
\]

\[
\int_0^t \int |w|^2 (v \cdot \nabla) \xi dxds \lesssim \|v\|_{U^3_t} \|w\|_{U^3_t} \lesssim A^3 t^{\frac{1}{10}}.
\]

For the convenience, we suppress the indexes \( x_0 \) and \( R \) in \( \phi_{x_0}, \hat{\phi}_{x_0} \) and \( \chi_R \). By additionally using (3.31),

\[
\int_0^t \int \hat{\varphi} \cdot \nabla \xi dxds \lesssim \int_0^t \int_{B(x_0, \frac{3}{2})} |\hat{\varphi}| |dxds \lesssim \|\hat{\varphi}\|_{L^2_t([0,t] \times B(x_0, \frac{3}{2}))} \|w\|_{U^3_t} \lesssim A^3 t^{\frac{1}{10}}.
\]

To estimate the last term in (4.15), we decompose it as

\[
\int_0^t \int V \cdot (v \cdot \nabla)(w\xi) dxds = I_1 + I_2 + I_3
\]

\[
= \int_0^t \int \xi V \cdot (v \cdot \nabla) w dxds + \int_0^t \int \xi V \cdot (w \cdot \nabla) w dxds + \int_0^t \int V \cdot w (v \cdot \nabla) \xi dxds.
\]

We have

\[
|I_1| \lesssim \|V\|_{L^4(0,T;L^4(supp(\xi)))}^2 \|\nabla w\|_{U^2_t} \lesssim A^3 t^{\frac{1}{4}}.
\]

On the other hand, by Poincaré inequality, we have

\[
\int_0^t \|w\phi\|_{L^6}^2 ds \lesssim \int_0^t \|\nabla (w\phi)\|_{L^2}^2 ds + \int_0^t \|\nabla \phi\|_{L^2}^2 ds \lesssim \int_0^t \|\nabla w\|_{L^2}^2 ds + \|w\|_{L^2}^2,
\]

which follows that (using Young’s inequality \( abc \leq \varepsilon a^2 + \varepsilon b^{8/3} + C(\varepsilon)c^8 \))

\[
|I_2| \leq \int_0^t \|\nabla w\phi\|_{L^2} \|w\phi\|_{L^4} \|V\|_{L^4(supp(\xi))} ds
\]

\[
\leq \int_0^t \|\nabla w\phi\|_{L^2} \|w\phi\|_{L^2}^\frac{3}{4} \|w\phi\|_{L^2}^\frac{1}{4} \|V\|_{L^4(supp(\xi))} ds
\]

\[
\leq \varepsilon \int_0^t \left( \|\nabla w\phi\|_{L^2}^2 + \|w\phi\|_{L^6}^2 \right) ds
\]

\[
+ C(\varepsilon) \int_0^t \|V\|_{L^4(supp(\xi))}^8 \|w\phi\|_{L^2}^2 ds
\]

\[
\leq \frac{1}{100} \int_0^t \|\nabla w\phi\|_{L^2}^2 ds + A^2 t + C \int_0^t \|V\|_{L^4(supp(\xi))}^8 \|w\phi\|_{L^2}^2 ds
\]

by choosing suitable \( \varepsilon \). It is easy to control \( I_3 \):

\[
|I_3| \lesssim t^{\frac{1}{10}} \|V\|_{U_{1\cdot 3}^{\frac{10}{3}}} \|w\|_{U_{1\cdot 3}^{\frac{10}{3}}} \lesssim A^3 t^{\frac{1}{4}}.
\]
Therefore, we obtain
\[
\left| \int_0^t V \cdot (v \cdot \nabla)(w \xi) \, dx \, ds \right| \leq \| \nabla w | \phi \chi \|_{L^2_2([0,t] \times \mathbb{R}^3)}^2 \\
+ C(1 + A^3) \left( t^{\frac{1}{10}} + \int_0^t \| V \|_{L^2_{4}(\text{supp}(\xi))} \| w \phi \chi \|_{L^2_2} \, ds \right),
\]
for some absolute constant \( C \). Finally, we combine all the estimates to get from (4.15) that
\[
\| w(t) \phi \chi \|_{L^2_2(\mathbb{R}^3)}^2 + \| \nabla w \phi \chi \|_{L^2_2([0,t] \times \mathbb{R}^3)}^2 \\
\lesssim \| w_0 \chi R \|_{L^2_{uloc}}^2 + (1 + A^3) \left( t^{\frac{1}{10}} + \int_0^t \| V \|_{L^2_4(\text{supp}(\xi))} \| w \phi \chi \|_{L^2_2} \, ds \right)
\]

Note that \( \| w(t) \phi \chi \|_{L^2_2(\mathbb{R}^3)} \) is lower semicontinuous in \( t \) as \( w \phi \) is weakly \( L^2 \)-continuous in \( t \). By Grönwall’s inequality and (4.11), we have
\[
\| w(t) \phi \chi \|_{L^2_2(\mathbb{R}^3)} \leq C_0^2 (\| w_0 \chi R \|_{L^2_{uloc}}^2 + t^{\frac{1}{10}}),
\]
for some \( C_0 = C_0(A) > 0 \). Taking supremum in \( x_0 \), we get
\[
\| w(t) \chi R \|_{L^2_{uloc}} \leq C_0 (t^{\frac{1}{10}} + \| w_0 \chi R \|_{L^2_{uloc}}).
\]
This finishes the proof of Lemma 4.5.

\[\Box\]

**Lemma 4.6 (Strong local energy inequality).** Let \((v, p)\) be a local energy solution in \( \mathbb{R}^3 \times (0, T) \) to Navier-Stokes equations (NS) for the initial data \( v_0 \in L^2_{uloc} \) constructed in Theorem 3.2, as the limit of approximation solutions \((v^{(k)}, p^{(k)})\) of (3.4). Then there is a subset \( \Sigma \subset (0, T) \) of zero Lebesgue measure such that, for any \( t_0 \in (0, T) \setminus \Sigma \) and any \( t \in (t_0, T) \), we have
\[
\int |v|^2 \varphi(x, t) \, dx + 2 \int_{t_0}^t \int \nabla |v|^2 \varphi \, dx \, ds \\
\leq \int |v|^2 \varphi(x, t_0) \, dx + \int_{t_0}^t \int \left\{ |v|^2 (\partial_t \varphi + \Delta \varphi) + (|v|^2 + 2p) v \cdot \nabla \varphi \right\} \, dx \, ds,
\]
for any \( \varphi \in C_c^\infty(\mathbb{R}^3 \times [t_0, T)) \). If, furthermore, for some \( u_0 \in L^2_{uloc, \sigma} \), \( V(t) = e^{t \Delta} u_0 \) and \( w = v - V \), then for any \( t_0 \in (0, T) \setminus \Sigma \) and any \( t \in (t_0, T) \), we have
\[
\int |w|^2 \varphi(x, t) \, dx + 2 \int_{t_0}^t \int |\nabla w|^2 \varphi \, dx \, ds \\
\leq \int |w|^2 \varphi(x, t_0) \, dx + \int_{t_0}^t \int |w|^2 (\partial_t \varphi + \Delta \varphi + v \cdot \nabla \varphi) \, dx \, ds \\
+ \int_{t_0}^t \int 2pw \cdot \nabla \varphi \, dx \, ds - \int_{t_0}^t \int 2(v \cdot \nabla) V \cdot w \varphi \, dx \, ds,
\]
for any \( \varphi \in C_c^\infty(\mathbb{R}^3 \times [t_0, T)) \).

This lemma is not for general local energy solutions, but only for those constructed by the approximation (3.4). Also note that (4.16) is true for \( t_0 = 0 \) since it becomes (3.1), but (4.17) is unclear for \( t_0 = 0 \) since the last integral in (4.17) may not be defined without further assumptions; Compare it with (4.15).
Proof. For any \( n \in \mathbb{N} \), the approximation \( v^{(k)} \) satisfy

\[
\lim_{k \to \infty} \left\| v^{(k)} - v \right\|_{L^2(0,T;L^2(B_n))} = 0.
\]

Thus there is a set \( \Sigma_n \subset (0,T) \) of zero Lebesgue measure such that

\[
\lim_{k \to \infty} \left\| v^{(k)}(t) - v(t) \right\|_{L^2(B_n)} = 0, \quad \forall t \in [0,T) \setminus \Sigma_n.
\]

Let

\[
\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n, \quad |\Sigma| = 0.
\]

We get

\[
\lim_{k \to \infty} \left\| v^{(k)}(t) - v(t) \right\|_{L^2(B_n)} = 0, \quad \forall t \in [0,T) \setminus \Sigma, \quad \forall n \in \mathbb{N}. \tag{4.18}
\]

The local energy equality of \( (v^{(k)}, p^{(k)}) \) in \([t_0, T]\) is derived similarly to (3.8)

\[
2 \int_{t_0}^{T} \int |\nabla v^{(k)}|^2 \psi dx ds = \int \left| v^{(k)} \right|^2 \psi(x,t_0) dx + \int_{t_0}^{T} \int \left| v^{(k)} \right|^2 (\partial_s \psi + \Delta \psi) dx ds
\]

\[
+ \int_{t_0}^{T} \int \left| v^{(k)} \right|^2 \Phi_{(k)}(J_{(k)}(v^{(k)}) \cdot \nabla) \psi dx ds
\]

\[
+ \int_{t_0}^{T} \int 2p^{(k)} v^{(k)} \cdot \nabla \psi dx ds
\]

\[
- \int_{t_0}^{T} \int \left| v^{(k)} \right|^2 \psi(J_{(k)}(v^{(k)}) \cdot \nabla) \Phi_{(k)} dx ds,
\]

for any \( \psi \in C_c^\infty(\mathbb{R}^3 \times [0,T]) \). By (4.18), we have

\[
\lim_{k \to \infty} \int \left| v^{(k)} \right|^2 \psi(x,t_0) dx = \int \left| v \right|^2 \psi(x,t_0) dx
\]

for \( t_0 \in [0,T) \setminus \Sigma \). Taking limit infimum \( k \to \infty \) in (4.19), we get

\[
2 \int_{t_0}^{T} \int |\nabla v|^2 \psi dx ds
\]

\[
\leq \int \left| v \right|^2 \psi(x,t_0) dx + \int_{t_0}^{T} \int \left\{ \left| v \right|^2 (\partial_s \psi + \Delta \psi) + (\left| v \right|^2 + 2p) v \cdot \nabla \psi \right\} dx ds.
\]

By the same argument for (3.29), we get (4.16) from the above.

Finally, inequality (4.17) for \( t_0 > 0 \) is equivalent to (4.16) by the same argument of Lemma 4.4. We have integrated by parts the last term in (4.17), which is valid since \( \nabla V \in L^\infty(\mathbb{R}^3 \times (t_0, t)) \). \(\square\)

We now prove the main result of this section.

**Proposition 4.7 (Decay of \( w \) and \( \tilde{p} \)).** Let \( v_0 = w_0 + u_0 \) with \( w_0 \in E_{\sigma}^2 \) and \( u_0 \in E_{uloc,\sigma}^3 \), and

\[
\lim_{|x_0| \to \infty} \int_{B(x_0,1)} |v_0 - (v_0)_{B(x_0,1)}| dx = 0.
\]
Let \((v,p)\) be a local energy solution in \(\mathbb{R}^3 \times (0,T)\) to Navier-Stokes equations (NS) for the initial data \(v_0 \in L^2_{uloc}\) constructed in Theorem 3.2, as the limit of approximation solutions \((v^{(k)}, p^{(k)})\) of (3.4). Let \(w = v - V\) for \(V(t) = e^{t \Delta} u_0\). Then, \(w\) and \(\tilde{p}_{x_0}\), defined in Lemma 4.1, decay at spatial infinity: For any \(t_1 \in (0,T)\),

\[
\lim_{|x_0| \to \infty} \left(\|w\|_{L^\infty_t L^2_x} + \|\nabla w\|_{L^2_t(Q_{x_0})} + \|\tilde{p}_{x_0}\|_{L^2_t(Q_{x_0})}\right) = 0,
\]

(4.20)

where \(Q_{x_0} = B(x_0, \frac{3}{2}) \times (t_1, T)\).

Note that we do not assert uniform decay up to \(t_1 = 0\). We assume the approximation (3.4) only to ensure the conclusion of Lemma 4.6, the strong local energy inequality.

**Proof.** Choose \(A = A(\|w_0\|_{L^2_{uloc}}, \|u_0\|_{L^2_{uloc}}, T)\) such that

\[
\|w\|_{\mathcal{E}_T} + \|V\|_{\mathcal{E}_T} + \|w\|_{U^s,q} + \|V\|_{U^s,q} \lesssim A,
\]

for any \(2 \leq s \leq \infty\), and \(2 \leq q \leq 6\) satisfying \(\frac{5}{2} + \frac{3}{q} = \frac{3}{2}\).

Fix \(x_0 \in \mathbb{Z}^3\) and \(R \in \mathbb{N}\). Let \(\phi_{x_0} = \Phi(\cdot - x_0)\), \(\chi_R(x) = 1 - \Phi\left(\frac{x}{R}\right)\), and

\[
\xi = \phi_{x_0}^2 \chi_R^2.
\]

(4.21)

For the convenience, we suppress the indexes \(x_0\) and \(R\) in \(\phi_{x_0}\), \(\tilde{p}_{x_0}\) and \(\chi_R\).

Let \(\Sigma\) be the subset of \((0,T)\) defined in Lemma 4.6. For any \(t_0 \in (0,t_1)\setminus \Sigma\) and \(t \in (t_0,T)\), choose \(\theta(t) \in C^\infty_{\text{c}}(0,T)\) with \(\theta = 1\) on \([t_0,t]\). Let \(\varphi(x,t) = \xi(x) \theta(t)\). By (4.17) of Lemma 4.6, using \(t_0 \not\in \Sigma\), we have

\[
\int |w(x,t)|^2 \xi(x) dx + 2 \int_{t_0}^t \int |\nabla w|^2 \xi dx ds \\
\leq \int |w(x,t_0)|^2 \xi(x) dx + \int_{t_0}^t \int |w|^2 (\Delta \xi + (v \cdot \nabla)\xi) dx ds \\
+ 2 \int_{t_0}^t \int \tilde{p}_{x_0} w \cdot \nabla \xi dx ds - 2 \int_{t_0}^t \int (v \cdot \nabla)V \cdot w \xi dx ds.
\]

(4.22)

Above we have replaced \(p\) by \(\tilde{p}_{x_0}\) using \(\int q_{x_0}(t) w \cdot \nabla \xi dx ds = 0\).

By the choice of \(\xi\), we can easily see that

\[
\int |w(\cdot,t)|^2 \xi dx + 2 \int_{t_0}^t \int |\nabla w|^2 \xi dx ds \geq \|w(\cdot,t)\chi\|^2_{L^2_t(B(x_0,1))} + 2 \|\nabla w\|_{L^2_t([t_0,t] \times B(x_0,1))}^2,
\]

\[
\int |w(\cdot,t_0)|^2 \xi dx \lesssim \|w(\cdot,t_0)\chi\|^2_{L^2_{uloc}},
\]

\[
\int_{t_0}^t \int |w|^2 \Delta \xi dx ds \lesssim \|w\chi\|^2_{L^2_{t_2}(t_0,t)} + \frac{1}{R} \|w\|^2_{U^s_2},
\]

and

\[
\int_{t_0}^t \int |w|^2 (v \cdot \nabla)\xi dx ds \lesssim \|v\|^2_{U^s_3,3} \|w\chi\|^2_{L^3_{t_0,t}} + \frac{1}{R} \|v\|^2_{U^s_3,3} \|w\|^2_{U^s_3,3}
\]

\[
\lesssim_A \|w\chi\|^2_{L^3_{t_0,t}} + \frac{1}{R}.
\]

34
The last term can be also estimated by

\[
\left| \int_{t_0}^t \int (v \cdot \nabla) V \cdot w \xi dxds \right| \lesssim \| \nabla V \|_{U^{\infty,3}(t_0,T)} \| v \|_{L^{2,6}_x} \| w \chi \|_{U^{2,2}(t_0,T)} \lesssim \Delta \| w \chi \|_{U^{2,2}(t_0,T)}^2 + \| \nabla V \|_{U^{\infty,3}(t_0,T)}^2.
\]

The only remaining term is the one with pressure. Note

\[
\int_{t_0}^t \int \tilde{p} w \cdot \nabla \xi dxds \lesssim \int_{t_0}^t \int_{B(x_0,3)} |\tilde{p}| |w| \chi^2 dxds + \frac{1}{R} \int_{t_0}^t \int_{B(x_0,3)} |\tilde{p}| |w| \chi dxds
\]
\[
\lesssim \| \tilde{p} \chi \|_{L^2_x([t_0,T] \times B(x_0,3))} \| w \chi \|_{U^{3,3}(t_0,T)} + \frac{1}{R} \| \tilde{p} \|_{L^2_x([t_0,T] \times B(x_0,3))} \| w \chi \|_{U^{3,3}}.
\]

For the second term, we can use a bound uniform in \( x_0 \)

\[
\| \tilde{p}_{x_0} \|_{L^2([t_0,T] \times B(x_0,3))} \leq C \| v \|_{U^{3,3}-2} + C(T) \| V \|_{U^{\infty,-2}}^2,
\]
which follows from (3.31), (4.4) and (4.5). For the first term, although the other factor \( \| w \chi \|_{U^{3,3}(t_0,T)} \) also has decay, it is larger than the left side of (4.22) by itself. Hence we need to estimate \( \| \tilde{p} \chi \|_{L^2_x([t_0,T] \times B(x_0,3))} \) and show its decay.

Let \( F_{ij} = wt_{ij} + w_i V_j + w_j V_i \) and \( G_{ij} = V_i V_j \). The local pressure \( \tilde{p} \) defined in Lemma 4.1 can be further decomposed as

\[
\tilde{p}(x, t) = p^F + p^{G,1} + p^{G,2} + p^{G,3}
\]

where \( p^F = p^F_{x_0} \) is defined as in (4.3),

\[
p^{G,1} = \int K_i(\cdot - y) [\partial_j G_{ij}] \rho_2(y) dy,
\]
\[
p^{G,2} = \int (K_{ij}(\cdot - y) - K_{ij}(x_0 - y)) G_{ij}(\rho_r - \rho_2)(y) dy
\]
\[
- \int (K_i(\cdot - y) - K_i(x_0 - y)) G_{ij} \partial_j (\rho_r - \rho_2)(y) dy,
\]
for \( \rho_r = \Phi\left(\frac{x_0}{\tau}\right) \), \( \tau > 4 \), and

\[
p^{G,3} = \int (K_{ij}(\cdot - y) - K_{ij}(x_0 - y)) G_{ij}(1 - \rho_r)(y) dy
\]
\[
+ \int (K_i(\cdot - y) - K_i(x_0 - y)) G_{ij} \partial_j \rho_r(y) dy.
\]

Recall \( p^F = p^F_{x_0} \)

\[
p^F = -\frac{1}{3} \operatorname{tr} F + \text{p.v.} \int_{B(x_0,2)} K_{ij}(\cdot - y) F_{ij}(y) dy
\]
\[
+ \int_{B(x_0,2)^c} (K_{ij}(\cdot - y) - K_{ij}(x_0 - y)) F_{ij}(y) dy
\]
\[
= p^{F,1} + p^{F,2} + p^{F,3}.
\]

35
We estimate $p^{F_i} \chi$, $i = 1, 2, 3$. Obviously, we have

$$\left\| p^{F,1} \chi \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))} \lesssim \left\| F x \right\|_{U^{\frac{5}{2}}((t_0,t))}.$$ 

Using $L^p$-norm preservation of Riesz transforms and $\left\| \nabla \chi \right\|_{\infty} \lesssim \frac{1}{R}$,

$$\left\| p^{F,2} \chi \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))} \leq \left\| \text{p.v.} \int_{B(x_0,2)} K_{ij} (\cdot - y) F_{ij} (y) \chi (y) dy \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))}$$

$$+ \left\| \text{p.v.} \int_{B(x_0,2)} K_{ij} (\cdot - y) F_{ij} (y) (\chi (\cdot) - \chi (y)) dy \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))}$$

$$\lesssim \left\| F \chi \right\|_{U^{\frac{5}{2}}((t_0,t))} + \frac{1}{R} \left\| \int_{B(x_0,2)} \left| \frac{1}{\cdot - y} \right|^{\frac{3}{2}} |F_{ij} (y)| dy \right\|_{L^2 (t_0,t; L^3 (\mathbb{R}^3))}$$

$$\lesssim \left\| F \chi \right\|_{U^{\frac{5}{2}}((t_0,t))} + \frac{1}{R} \left\| F \right\|_{U^{\frac{5}{2}}((t_0,t))}.$$ 

The last inequality follows from the Riesz potential estimate. Since

$$| \chi (x) - \chi (y) | \leq \left\| \nabla \chi \right\|_{\infty} | x - y | \lesssim \frac{1}{\sqrt{R}}$$

for $x \in B(x_0, \frac{3}{2})$ and $y \in B(x_0, \sqrt{R})$, and

$$| x - y | \geq | x_0 - y | - | x - x_0 | \geq \frac{1}{4} | x_0 - y |$$

for $x \in B(x_0, \frac{3}{2})$ and $y \in B(x_0, 2)^c$, we get

$$\left\| p^{F,3} \chi \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))} \leq \left\| \int_{B(x_0,\sqrt{R}) \setminus B(x_0,2)} \frac{1}{\cdot - y} |F_{ij} (y)| dy \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))}$$

$$+ \left\| \int_{B(x_0,\sqrt{R}) \setminus B(x_0,2)} \frac{1}{\cdot - y} |F_{ij} (y)| (\chi (\cdot) - \chi (y)) dy \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))}$$

$$+ \left\| \int_{B(x_0,\sqrt{R})^c} \frac{1}{\cdot - y} |F_{ij} (y)| dy \chi \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))}.$$ 

Thus

$$\left\| p^{F,3} \chi \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))} \lesssim \sum_{k=1}^{\infty} \left\| \int_{B(x_0,2^{k+1}) \setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |F_{ij} (y)| dy \right\|_{L^2_p ([t_0,t]; L^\infty (B(x_0, \frac{3}{2})))}$$

$$+ \frac{1}{\sqrt{R}} \sum_{k=1}^{\infty} \left\| \int_{B(x_0,2^{k+1}) \setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |F_{ij} (y)| dy \right\|_{L^\infty ([t_0,t] \times B(x_0, \frac{3}{2})))}$$

$$+ \sum_{k=1}^{\log_2 \sqrt{R}} \left\| \int_{B(x_0,2^{k+1}) \setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |F_{ij} (y)| dy \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))}$$

$$\lesssim \left\| F \chi \right\|_{L^2_p ([t_0,t] \times B(x_0, \frac{3}{2}))} + \frac{1}{\sqrt{R}} \left\| F \right\|_{U^{\infty,1}((t_0,t))}.$$
Combining the estimates for $p^{F_i} \chi$, $i = 1, 2, 3$, we obtain
\[
\|p^{F_i} \chi\|_{L^2([t_0, t] \times B(x_0, \frac{3}{2}))} \lesssim_T \|F\chi\|_{U^2_2(L^2(t_0, t))} + \frac{1}{R} \|F\|_{U^2_2(L^2(t_0, t))} + \frac{1}{\sqrt{R}} \|F\|_{U^\infty, 1(t_0, t)}.
\]
\[
\lesssim_{A, T} \|w\chi\|_{U^{3, 3}(t_0, t)} + \frac{1}{\sqrt{R}}.
\]

Now, we consider $p^{G, i}$'s. Since for $x \in B(x_0, \frac{3}{2})$, $p^{G, 1}$ satisfies
\[
|p^{G, 1} \chi(x, t)| \leq \int_{|x - y| \leq 3} \left| (\nabla K)(x - y) \right| |V||\nabla V|(y, t)(|\chi(y)| + |\chi(x) - \chi(y)|) dy
\]
\[
\lesssim \int_{B_3(x_0)} \frac{1}{|x - y|^2} |V||\nabla V|(y, t)\chi(y) dy + \frac{1}{R} \int_{B_3(x_0)} \frac{1}{|x - y|} |V||\nabla V|(y, t)| dy
\]
using $|\chi(x) - \chi(y)| \lesssim \|\nabla \chi\|_{\infty} |x - y|$, the estimate for $p^{G, 1} \chi$ can be obtained from Young's convolution inequality;
\[
\|p^{G, 1} \chi\|_{L^2([t_0, t] \times B(x_0, \frac{3}{2}))} \lesssim \left\| \int_{|x - y| \leq 3} \frac{1}{|y - y|^2} |V||\nabla V|(y, t)\chi(y) dy \right\|_{L^2([t_0, t] \times \mathbb{R}^3)} + \frac{1}{R} \left\| \int_{B_3(x_0)} \frac{1}{|x - y|^2} |V||\nabla V|(y, t)dy \right\|_{L^2_2(t_0, t; L^{30}(\mathbb{R}^3))}
\]
\[
\lesssim \left\| \frac{1}{|x|} \right\|_{L^{\frac{3}{2}, \infty}} \left\| |\nabla V|\chi\right\|_{L^\infty(t_0, T; L^2(B(x_0, 3)))} \left\| |V|\right\|_{L^2_2(t_0, T; L^\frac{3}{2}(B(x_0, 3)))} + \frac{1}{R} \left\| \frac{1}{|x|} \right\|_{L^{3, \infty}} \left\| |V|\right\|_{L^2_2(t_0, T; L^\frac{3}{2}(B(x_0, 3)))}
\]
\[
\lesssim_{A, T} \left\| |\nabla V|\chi\right\|_{U^{3, \frac{3}{2}}(t_0, T)} + \frac{1}{R}.
\]

By integration by parts, for $x \in B(x_0, \frac{3}{2})$, $p^{G, 2}$ can be rewritten as
\[
p^{G, 2} = \int (K_i(x - y) - K_i(x_0 - y))V_i \partial_j V_j(y, t)(\rho_\tau - \rho_2)(y) dy
\]
and then it satisfies
\[
|p^{G, 2} \chi(x, t)| \lesssim \int_{2 < |x_0 - y| \leq 2\tau} \frac{1}{|x_0 - y|^3} |V||\nabla V|(y, t)(|\chi(y)| + |\chi(x) - \chi(y)|) dy
\]
\[
\lesssim \sum_{m_r} \int_{B_{i+1} \setminus B_i} \frac{1}{|x_0 - y|^3} |V||\nabla V|(y, t)\left(|\chi(y)| + \frac{\tau}{R}\right) dy,
\]
where $m_r = \lfloor \ln(2\tau)/\ln 2 \rfloor$ and $B_i = B(x_0, 2^i)$. Taking $L^2(t_0, t)$ on it, we have
\[
\|p^{G, 2} \chi\|_{L^2(t_0, t; L^\infty(B(x_0, \frac{3}{2}))}) \lesssim \sum_{m_r} \int_{B_{i+1} \setminus B_i} \frac{1}{|x_0 - y|^3} |V||\nabla V|(y, t)\left(|\chi(y)| + \frac{\tau}{R}\right) dy \|_{L^2(t_0, t)}
\]
\[
\lesssim \sum_{m_r} \frac{1}{2^{m_r}} \left( \|V||\nabla V|\chi\|_{L^2(t_0, t; L^1(B_{i+1}))} + \frac{\tau}{R} \||V||\nabla V||_{L^2(t_0, t; L^1(B_{i+1}))}\right)
\]
\[
\lesssim \sum_{m_r} \left( \|V||\nabla V|\chi\|_{U^2, 1(t_0, T)} + \frac{\tau}{R} \||V||\nabla V||_{U^2, 1}\right)
\]
\[
\lesssim T \ln \tau \|V\|_{U^\infty, 2} \|\nabla V|\chi\|_{U^\infty, 2(t_0, T)} + \frac{\tau \ln \tau}{R} \|V\|_{U^\infty, 2} \|\nabla V\|_{U^2, 2}.
\]
Lastly,
\[ |p^{G,3}(x, t)| \leq \int_{|x_0 - y| \geq \tau} \frac{|V(y, t)|^2}{|x_0 - y|^4} dy + \frac{1}{\tau} \int_{\tau \leq |x_0 - y| \leq 2\tau} \frac{|V(y, t)|^2}{|x_0 - y|^3} dy \leq \frac{1}{\tau} \|V\|_{L^2_{t,x}}^2. \]

Hence
\[ \|p^{G,3}\|_{L^2_{(t,x)}}(t_0, x_0, \frac{3}{2}) \leq \|p^{G,3}\|_{L^2_{(t,x)}}(t_0, x_0, \frac{3}{2}) \lesssim A, T \frac{1}{\tau}. \]

To summarize, we have shown
\[ \sum_{i=1}^{3} \|p^{G,3}\|_{L^2_{(t,x)}}(t_0, x_0, \frac{3}{2}) \lesssim A, T \ln \tau \|\nabla V|\|_{U^3(t_0, T)} + \frac{\tau \ln \tau}{R} + \frac{1}{\tau}, \]

and therefore
\[ \|\tilde{p}\|_{L^2_{(t,x)}}(t_0, x_0, \frac{3}{2}) \lesssim A, T \|w\|_{U^3(t_0, T)} + \frac{1}{\sqrt{R}} + \ln \tau \|\nabla V|\|_{U^2(t_0, T)} + \frac{\tau \ln \tau}{R} + \frac{1}{\tau}. \]  

(4.23)

Finally, combining all estimates and then taking supremum on (4.22) over \( x_0 \in \mathbb{R}^3 \), we obtain
\[ \|w(\cdot, t)\|_{L^2_{\text{uloc}}}^2 + 2 \|\nabla w|\|_{U^2(t_0, t)}^2 \lesssim A, T \|w(\cdot, t)\|_{L^2_{\text{uloc}}}^2 + \|w\|_{U^2(t_0, t)} + \|w\|_{U^3(t_0, t)} \begin{align*}
+ (\ln \tau)^2 \|\nabla V|\|_{U^3(t_0, T)}^2 + \frac{(\tau \ln \tau)^2}{R^2} + \frac{1}{\tau^2} + \frac{1}{R}. \end{align*} \]  

(4.24)

Using the estimates
\[ \|w\|_{U^3(t_0, t)}^2 \lesssim \|w\|_{U^2(t_0, t)}^2 \begin{align*}
+ \|\nabla w|\|_{U^2(t_0, t)}^2 + \frac{1}{R} \|w\|_{U^2(t_0, t)}^2 \end{align*}, \]  

and Lemma 4.5, it becomes
\[ \|w(\cdot, t)\|_{L^2_{\text{uloc}}}^2 + \|\nabla w|\|_{U^2(t_0, t)}^2 \lesssim A, T, C_0 \frac{t_{0^+}}{t_0^+} + \|w_0\|_{L^2_{\text{loc}}}^2 + \|w\|_{L^6(t_0, t; L^2_{\text{loc}})}^2 \begin{align*}
+ (\ln \tau)^2 \|\nabla V|\|_{U^3(t_0, T)}^2 + \frac{(\tau \ln \tau)^2}{R^2} + \frac{1}{\tau^2} + \frac{1}{R}, \end{align*} \]  

(4.26)

where \( C_0 \) is defined as in Lemma 4.5.

Note that \( \|w(\cdot, t)\|_{L^2_{\text{loc}}}^2 \) is lower semicontinuous in \( t \) as \( w \) is weakly \( L^2(B_n) \)-continuous in \( t \) for any \( n \). By Grönwall inequality, we have
\[ \|w\|_{L^6(t_0, T; L^2_{\text{loc}})}^2 \lesssim A, T, C_0 \frac{t_{0^+}}{t_0^+} + \|w_0\|_{L^2_{\text{loc}}}^2 + (\ln \tau)^2 \|\nabla V|\|_{U^3(t_0, T)}^2 + \frac{(\tau \ln \tau)^2}{R^2} + \frac{1}{\tau^2} + \frac{1}{R}. \]  

(4.27)

We now prove (4.20). Fix \( t_1 \in (0, T) \). For every \( n \in \mathbb{N} \) we can choose \( t_0 = t_0(n) \in (0, t_1) \setminus \Sigma \) satisfying
\[ \frac{t_{0^+}}{t_0^+} < \frac{1}{n}. \]

38
At the same time, we pick $\tau = \tau(n) > 4$ satisfying $\tau^{-2} \leq 1/n$. After $t_0$ and $\tau$ are fixed, we can make all the remaining terms small by choosing $R = R(n, \|v_0\|_{L^2_{\text{loc}}}, t_0, \tau)$ sufficiently large:

$$\|w_0 \chi_R\|^2_{L^2_{\text{loc}}} + (\ln \tau)^2 \|\nabla V \chi_R\|^2_{L^2_{\text{loc}, (t_0, T)}} + \frac{(\tau \ln \tau)^2}{R^2} + \frac{1}{R} \leq \frac{1}{n}.$$ 

Here, the smallness of the second term follows from $\nabla V$ decay (Lemma 2.7), using the oscillation decay of $v_0$. In conclusion, by (4.27), for each $n \in \mathbb{N}$, we can find $t_0$, $\tau$ and $R \gg 1$ so that

$$\|w \chi_R\|^2_{L^6((t_0, T); L^2_{\text{loc}})} \lesssim_A T, C_0 \frac{1}{n}.$$ 

By (4.26),

$$\|w \chi_R\|^2_{L^\infty((t_0, T); L^2_{\text{loc}})} + \|\nabla w \chi_R\|^2_{L^2_{\text{loc}, (t_0, T)}} \lesssim_A T, C_0 \frac{1}{n}.$$ 

By (4.25),

$$\|w \chi_R\|^2_{L^{3,3}((t_0, T))} \lesssim_A T, C_0 \frac{1}{n}.$$ 

Restricted to the original time interval $(t_1, T)$, the perturbation $w$ satisfies

$$\lim_{R \to \infty} \|w \chi_R\|^2_{L^{3,3}(t_1, T)} = 0,$$

$$\lim_{R \to \infty} \|w \chi_R\|^2_{L^{\infty}(t_1, T; L^2_{\text{loc}})} + \|\nabla w \chi_R\|^2_{L^2(t_1, T)} = 0.$$ 

Using (4.23), we also have

$$\lim_{R \to \infty} \sup_{x_0 \in \mathbb{R}^3} \|\tilde{p}_{x_0} \chi_R\|^2_{L^2(\overline{B(x_0, \frac{1}{2})} \times (t_1, T))} = 0.$$ 

This completes the proof of Proposition 4.7. \qed

**Corollary 4.8.** Under the same assumptions of Proposition 4.7, the perturbed Navier-Stokes flow $w = v - e^{t \Delta} u_0$ satisfies $w(t) \in E^p(\mathbb{R}^3)$ for almost all $t \in (0, T]$ for any $3 \leq p \leq 6$.

**Proof.** By Proposition 4.7, for any fixed $x_0 \in \mathbb{R}^3$ and $t_1 \in (0, T)$, the perturbed local energy solution $w$ to Navier-Stokes equations satisfies

$$\|w\|_{L^3(B_{3/2}(x_0) \times (t_1, T))} + \|\tilde{p}_{x_0}\|_{L^{3/2}(B_{3/2}(x_0) \times (t_1, T))} \to 0 \quad \text{as} \quad |x_0| \to \infty.$$ 

Recall that $V \in C^1([\delta, \infty) \times \mathbb{R}^3)$ for any $\delta > 0$. Then, by Caffarelli-Kohn-Nirenberg criteria [3], for any $t_2 \in (t_1, T)$, we can find $R_0 > 0$ such that if $|x_0| \geq R_0$,

$$\|w\|_{L^\infty([t_2, T] \times B_1(x_0))} \lesssim \|w\|_{L^3(B_{3/2}(x_0) \times (t_1, T))} + \|\tilde{p}_{x_0}\|_{L^{3/2}(B_{3/2}(x_0) \times (t_1, T))}^{1/2},$$

and the constant in the inequality is independent of $x_0$. Moreover, $\|w\|_{L^\infty([t_2, T] \times B_1(x_0))} \to 0$ as $|x_0| \to \infty$. Although the system (4.8) satisfied by $w$ is not the original (NS), similar proof works since $V \in C^1$. See [13, Theorem 2.1] for more singular $V \in L^m$, $m > 1$, but without the source term $V \cdot \nabla V$.

On the other hand, $w \in E_T$ implies that

$$w \in L^s(0, T; L^p(B_{R_0}))$$

for any $s \in [2, \infty]$ and $p \in [2, 6]$ with $\frac{2}{s} + \frac{2}{p} = \frac{3}{2}$, and therefore $w(t) \in E^p$ for a.e. $t \in (0, T)$. \qed
5 Global existence

In this section, we prove Theorem 1.1. We first give the following decay estimates.

Lemma 5.1. Let \((v, p)\) be a local energy solution in \(\mathbb{R}^3 \times [t_0, T]\), \(0 < t_0 < T < \infty\), to the Navier-Stokes equations (NS) for the initial data

\[ v|_{t=t_0} = w_* + e^{t_0}u_0 \]

where \(w_* \in E^2\) and \(u_0 \in L^3_{uloc,\sigma}\) satisfies the oscillation decay (1.5). Let \(V(t) = e^{t}u_0\). Then, the perturbation \(w = v - V\) also decays at infinity:

\[ \|w\|_{L^3([t_0, T] \times B(x_0, 1))} + \|\tilde{p}_{x_0}\|_{L^3([t_0, T] \times B(x_0, 1))} \to 0, \]

and

\[ \|w\|_{L^\infty(t_0, T; L^2(B(x_0, 1)))} + \|\nabla w\|_{L^2(t_0, T; L^2(B(x_0, 1)))} \to 0, \quad \text{as } |x_0| \to \infty. \]

Remark. This \(T\) is arbitrarily large, unlike the existence time given in the local existence theorem, Theorem 3.2. We assume \(w_* \in E^2\), and we have \(V \in C^1(\mathbb{R}^3 \times [t_0, T])\). We no longer need Lemma 4.5 nor the strong local energy inequality.

Proof. The proof is almost the same as that of Proposition 4.7 except for the way to estimate \(\|w(\cdot, t_0)\chi_R\|_{L^3_{uloc}}\) in (4.24). Indeed, \(\lim_{R \to \infty} \|w(\cdot, t_0)\chi_R\|_{L^3_{uloc}} = 0\) by the assumption \(w(\cdot, t_0) = w_* \in E^2\). \(\square\)

Now, we prove the main theorem.

Proof of Theorem 1.1. Let \((v, p)\) be a local energy solution to the Navier-Stokes equations in \(\mathbb{R}^3 \times [0, T_0]\), \(0 < T_0 < \infty\), for the initial data \(v|_{t=0} = v_0\), constructed in Theorem 3.2. By Corollary 4.8, there exists \(t_0 \in (0, T_0)\), arbitrarily close to \(T_0\), with \(w(t_0) = v(t_0) - e^{t_0}u_0 \in E^4\). Then, by Lemma 2.2, for any small \(\delta > 0\), we can decompose

\[ w(t_0) = W_0 + h_0, \]

where \(W_0 \in C^\infty_c(\mathbb{R}^3)\) and \(h_0 \in E^4(\mathbb{R}^3)\) with \(\|h_0\|_{L^4_{uloc}} < \delta\).

To construct a local energy solution \((\tilde{v}, \tilde{p})\) to (NS) for \(t \geq t_0\) with initial data \(\tilde{v}|_{t=t_0} = v(t_0)\), we decompose \((\tilde{v}, \tilde{p})\) as

\[ \tilde{v} = V + h + W, \quad \tilde{p} = p_h + p_W \]

where \(V(t) = e^{t}v_0\), \((h, p_h)\) satisfies

\[
\begin{align*}
\partial_t h - \Delta h + \nabla p_h &= -H \cdot \nabla H, \quad H = V + h, \\
\text{div } h &= 0, \quad h|_{t=t_0} = h_0,
\end{align*}
\]

so that \(H\) solves (NS) with \(H(t_0) = e^{t_0}u_0 + h_0\), and \((W, p_W)\) satisfies

\[
\begin{align*}
\partial_t W - \Delta W + \nabla p_W &= -[(H + W) \cdot \nabla] W - (W \cdot \nabla) H, \\
\text{div } W &= 0, \quad W|_{t=t_0} = W_0.
\end{align*}
\]
Our strategy is to first find, for each \( \epsilon > 0 \), a distributional solution \((h^\epsilon, p^\epsilon_t)\) and a Leray-Hopf weak solution \((W^\epsilon, p^\epsilon_t)\) to \(\epsilon\)-approximations of (5.1) and (5.2) for \( t \in I \) for some \( S = S(\delta, V) > 0 \) uniform in \( \epsilon \). Then, we prove that they have a limit \((\tilde{v}, \tilde{p})\) which is a desired local energy solution to (NS) on \( I \). By gluing two solutions \( v \) and \( \tilde{v} \) at \( t = t_0 \), we can get the extended local energy solution on the time interval \([0, t_0 + S]\). Repeating this process, we get a time-global local energy solution. The detailed proof is given below.

**Step 1. Construction of approximation solutions**

Let \( I = (t_0, t_0 + S) \) for some small \( S \in (0, 1) \) to be decided. For \( 0 < \epsilon < 1 \), we first consider the fixed point problem for

\[
\Psi(h) = e^{(t-t_0)\Delta}h_0 - \int_{t_0}^{t} e^{(t-s)\Delta} F_e(s) ds, \quad H = V + h, \tag{5.3}
\]

where \( F_e(t) = H \ast \eta_\epsilon \) is mollification of scale \( \epsilon \) and \( \Phi_\epsilon(x) = \Phi(\epsilon x) \) is a localization factor of scale \( \epsilon^{-1} \). We will solve for a fixed point \( h = h^\epsilon \) in the Banach space

\[
\mathcal{F} = \mathcal{F}_{t_0, S} := \{ h \in U^{\infty,4}(I) : (t-t_0)\hat{\Phi} h(\cdot, t) \in L^{\infty}(I \times \mathbb{R}^3) \}
\]

for some \( S > 0 \) with

\[
\|h\|_{\mathcal{F}} := \|h\|_{U^{\infty,4}(I)} + \left\| (t-t_0)\hat{\Phi} h(t) \right\|_{L^{\infty}(I \times \mathbb{R}^3)}.
\]

Denote \( M = \|V\|_{L^{\infty}(I \times \mathbb{R}^3)} \lesssim (1 + t_0^{-4/3}) \|v_0\|_{L^2_\text{uloc}} \). By Lemma 2.3, we have

\[
\|\Psi h\|_{U^{\infty,4}(I)} \lesssim \|h_0\|_{L^{4}_{\text{uloc}}} + S^{\frac{1}{2}} \|h\|_{U^{\infty,4}}^2 + S^{\frac{1}{2}} M \|h\|_{U^{\infty,4}} + S^{\frac{3}{2}} \|V\|_{L^{\infty}(I; L^8_{\text{uloc}})}^2 \lesssim \|h_0\|_{L^{4}_{\text{uloc}}} + S^{\frac{1}{2}} \|h\|_{\mathcal{F}}^2 + S^{\frac{1}{2}} M \|h\|_{\mathcal{F}} + S^{\frac{3}{2}} M^2,
\]

and for \( t \in I \),

\[
\|\Psi h(t)\|_{L^{\infty}(\mathbb{R}^3)} \lesssim (t-t_0)^{-\frac{3}{8}} \|h_0\|_{L^{4}_{\text{uloc}}} + \int_{t_0}^{t} |t-s|^{-1/2} \left( \|h(s)\|_{L^\infty}^2 + M^2 \right) ds \lesssim (t-t_0)^{-\frac{3}{8}} \|h_0\|_{L^{4}_{\text{uloc}}} + (t-t_0)^{-1/4} \|h\|_{\mathcal{F}}^2 + (t-t_0)^{1/2} M^2.
\]

Therefore, we get

\[
\|\Phi h\|_{\mathcal{F}} \lesssim \|h_0\|_{L^{4}_{\text{uloc}}} + S^{\frac{1}{2}} \|h\|_{\mathcal{F}}^2 + S^{\frac{1}{2}} M \|h\|_{\mathcal{F}} + S^{\frac{3}{2}} M^2.
\]

Similarly we can show

\[
\|\Phi h_1 - \Phi h_2\|_{\mathcal{F}} \lesssim \left\{ S^{\frac{3}{2}} (\|h_1\|_{\mathcal{F}} + \|h_2\|_{\mathcal{F}}) + S^{\frac{1}{2}} M \right\} \|h_1 - h_2\|_{\mathcal{F}}.
\]

By the Picard contraction theorem, we can find \( S = S(\delta, \|V\|_{L^{\infty}(I \times \mathbb{R}^3)}) \in (0, 1) \) such that a unique fixed point (mild solution) \( h^\epsilon \) to (5.3) exists in \( \mathcal{F}_{t_0, S} \) with

\[
\|h^\epsilon\|_{\mathcal{F}} \leq C\delta, \quad \forall 0 < \epsilon < 1. \tag{5.4}
\]

We also have the uniform bound

\[
\|h^\epsilon\|_{\mathcal{E}(I)} \lesssim \|F_e H^\epsilon \otimes H^\epsilon P_e\|_{U^{2,2}(I)} \lesssim \|h^\epsilon\|_{\mathcal{F}}^2 + \|V\|_{U^{4,4}(I)}^2 \lesssim \delta^2 + M^2. \tag{5.5}
\]
Now, we define \( H^\epsilon = V + h^\epsilon \) and the pressure \( p^\epsilon_h \) by
\[
p^\epsilon_h = -\frac{1}{3} \mathcal{J}_\epsilon H^\epsilon \cdot H^\epsilon \Phi_\epsilon + \text{p.v.} \int_{B_2} K_{ij}(\cdot - y)(\mathcal{J}_\epsilon H^\epsilon)_i H^\epsilon_j \Phi_\epsilon(y, t) \, dy
+ \text{p.v.} \int_{B_2^*} (K_{ij}(\cdot - y) - K_{ij}(-y)) (\mathcal{J}_\epsilon H^\epsilon)_i H^\epsilon_j \Phi_\epsilon(y, t) \, dy.
\]

It is well defined thanks to the localization factor \( \Phi_\epsilon \). For each \( R > 0 \), we have a uniform bound
\[
\|p^\epsilon_h\|_{L^2(I \times B_R)} \leq C(R) \tag{5.6}
\]
in a similar way to getting (3.17). The pair \((h^\epsilon, p^\epsilon_h)\) solves, with \( H^\epsilon = V + h^\epsilon \),
\[
\begin{align*}
\partial_t h^\epsilon - \Delta h^\epsilon + \nabla p^\epsilon_h &= - (\mathcal{J}_\epsilon H^\epsilon \cdot \nabla)(H^\epsilon \Phi_\epsilon), \\
\text{div} \, h^\epsilon &= 0, \quad h^\epsilon|_{t=0} = h_0 \in L^4_{uloc}
\end{align*}
\]
in \( \mathbb{R}^3 \times I \) in the distributional sense.

We next consider the equation for \( W = W^\epsilon \),
\[
\begin{align*}
\partial_t W - \Delta W + \nabla f_W &= f^\epsilon_W \\
f^\epsilon_W := -\mathcal{J}_\epsilon (H^\epsilon + W) \cdot \nabla W - \mathcal{J}_\epsilon W \cdot \nabla H^\epsilon, \\
\text{div} \, W &= 0, \quad W|_{t=0} = W_0 \in C_\infty^\epsilon.
\end{align*}
\]

Note that (5.8) is a mollified and perturbed \((\text{NS})\), and has no localization factor \( \Phi_\epsilon \).

Using \( W^\epsilon \) itself as a test function, we can get an a priori estimate: for \( t \in I \),
\[
\|W(t)\|_{L^2(\mathbb{R}^3)}^2 + 2 \|\nabla W\|_{L^2(t_0, t \times \mathbb{R}^3)}^2 \leq \|W_0\|_{L^2(\mathbb{R}^3)}^2 + \iint f^\epsilon_W \cdot W.
\]

Note that \( \iint \mathcal{J}_\epsilon (H + W) \cdot \nabla W \cdot W = 0 \) and \( -\iint (\mathcal{J}_\epsilon W \cdot \nabla) h^\epsilon \cdot W = \iint (\mathcal{J}_\epsilon W \cdot \nabla) W \cdot h^\epsilon \). Also recall that
\[
\|h^\epsilon W\|_{L^2(Q)} \lesssim \|h^\epsilon\|_{L^\infty(I; L^3_{uloc})} (\|\nabla W\|_{L^2(Q)} + \|W\|_{L^2(Q)})
\]
for \( Q = [t_0, t] \times \mathbb{R}^3 \). Its proof can be found in [17, page 162]. Thus
\[
\begin{align*}
\iint f^\epsilon_W \cdot W &= \iint (\mathcal{J}_\epsilon W \cdot \nabla) W \cdot h^\epsilon - \iint (\mathcal{J}_\epsilon W \cdot \nabla) V \cdot W \\
&\leq C \|\nabla W\|_{L^2(Q)} \delta(\|\nabla W\|_{L^2(Q)} + \|W\|_{L^2(Q)}) + M_1 \|W\|_{L^2(Q)}^2
\end{align*}
\]
where \( M_1 = \|\nabla V\|_{L^\infty(I \times \mathbb{R}^3)} \). By choosing \( \delta \) sufficiently small, we conclude
\[
\|W(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla W\|_{L^2([t_0, t] \times \mathbb{R}^3)}^2 \leq \|W_0\|_{L^2(\mathbb{R}^3)}^2 + C(1 + M_1) \|W\|_{L^2(Q)}^2.
\]

By Grönwall inequality (using that \( \|W(t)\|_{L^2(\mathbb{R}^3)}^2 \) is lower semicontinuous), we obtain
\[
\|W^\epsilon\|_{L^\infty(I; L^2(\mathbb{R}^3))}^2 + \|\nabla W^\epsilon\|_{L^2(I \times \mathbb{R}^3)}^2 \leq C(M_1) \|W_0\|_{L^2(\mathbb{R}^3)}^2 \tag{5.9}
\]
With this uniform a priori bound, for each \( 0 < \epsilon < 1 \), we can use Galerkin method to construct a Leray-Hopf weak solution \( W^\epsilon \) on \( I \times \mathbb{R}^3 \) to (5.8).
Define \( F^\varepsilon_{ij} = \mathcal{J}_\varepsilon(W^\varepsilon + H^\varepsilon)_iW^\varepsilon_j + (\mathcal{J}_\varepsilon W^\varepsilon)_iH^\varepsilon_j \). We have the uniform bound
\[
\|F^\varepsilon_{ij}\|_{L^{3/2,3/2}(I)} \leq C \|V\| + \|H^\varepsilon\| + \|W^\varepsilon\|_{L^{3/2,3/2}(I)} \leq C(M, M_1, \|W_0\|_{L^2(\mathbb{R}^3)}).
\]

Define \( p^\varepsilon_n(x, t) = \lim_{n \to \infty} p^\varepsilon_{W}(x, t) \), and \( p^\varepsilon_{W}(x, t) \) is defined for \(|x| < 2^n\) by
\[
p^\varepsilon_{W}(x, t) = -\frac{1}{3} \text{tr} F^\varepsilon_{ij}(x, t) + \text{p.v.} \int_{B_2(0)} K_{ij}(x - y) F^\varepsilon_{ij}(y, t) dy + \left( \text{p.v.} \int_{B_{2n+1} \setminus B_2} + \int_{B_{2n+1}} \right) (K_{ij}(x - y) - K_{ij}(-y)) F^\varepsilon_{ij}(y, t) dy.
\]

For each \( R > 0 \), we have a uniform bound
\[
\|p^\varepsilon_W\|_{L^{\frac{3}{2}}(I \times B_R)} \leq C(R, M, M_1, \|W_0\|_{L^2(\mathbb{R}^3)}). \tag{5.10}
\]

By the usual theory for the nonhomogeneous Stokes system in \( \mathbb{R}^3 \), the pair \((W^\varepsilon, p^\varepsilon_W)\) solves \((5.8)\) in distributional sense.

We now define
\[
v^\varepsilon = H^\varepsilon + W^\varepsilon = V + h^\varepsilon + W^\varepsilon, \quad p^\varepsilon = p^\varepsilon_h + p^\varepsilon_W.
\]

Summing \((5.7)\) and \((5.8)\), the pair \((v^\varepsilon, p^\varepsilon)\) solves in distributional sense
\[
\begin{aligned}
\partial_t v^\varepsilon - \Delta v^\varepsilon + \nabla p^\varepsilon &= -\mathcal{J}_\varepsilon v^\varepsilon \cdot \nabla v^\varepsilon + E^\varepsilon, \\
E^\varepsilon &= \mathcal{J}_\varepsilon H^\varepsilon \cdot \nabla (H^\varepsilon(1 - \Phi_\varepsilon)), \\
\text{div} v^\varepsilon &= 0, \\
v^\varepsilon|_{t = t_0} &= v(t_0).
\end{aligned} \tag{5.11}
\]

Thanks to the mollification, \( h^\varepsilon \) and \( W^\varepsilon \) have higher local integrability by the usual regularity theory. Thus we can test \((5.11)\) by \(2v^\varepsilon \xi, \xi \in C^\infty_c([t_0, t_0 + S] \times \mathbb{R}^3)\), and integrate by parts to get the identity
\[
2 \int_I \int_I |\nabla v^\varepsilon|^2 \xi dx ds = \int_I |v^\varepsilon|^2 \xi(x, t_0) dx \\
+ \int_I \int_I |v^\varepsilon|^2 (\partial_t \xi + \Delta \xi) + (|v^\varepsilon|^2 \mathcal{J}_\varepsilon v^\varepsilon + 2p^\varepsilon v^\varepsilon) \cdot \nabla \xi + E^\varepsilon \cdot 2v^\varepsilon \xi dx ds. \tag{5.12}
\]

Note that \( v \) in \( \int |v|^2 \xi(x, t_0) dx \) is the original solution in \([0, T)\).

**Step 2.** A local energy solution on \( I = (t_0, t_0 + S) \)

We now show that \((v^\varepsilon, p^\varepsilon)\) has a weak limit \((\tilde{v}, \tilde{p})\) which is a local energy solution on \( I \). Recall the uniform bounds \((5.4), (5.5), (5.6), (5.9)\), and \((5.10)\) for \( h^\varepsilon, p^\varepsilon_h, W^\varepsilon \) and \( p^\varepsilon_W \). As in the proof of Theorem 3.2, from the uniform estimates and the compactness argument, we can find a subsequence \((v^{(k)}, p^{(k)})\), \( k \in \mathbb{N} \), from \((v^\varepsilon, p^\varepsilon)\) which converges to some \((\tilde{v}, \tilde{p})\) in the following sense: for each \( n \in \mathbb{N} \),
\[
\begin{align*}
v^{(k)} &\rightharpoonup \tilde{v} \quad \text{in } L^\infty(I; L^2(B_{2^n})), \\
v^{(k)} &\to \tilde{v} \quad \text{in } L^2(I; H^1(B_{2^n})), \\
v^{(k)}, \mathcal{J}(k)v^{(k)} &\to \tilde{v} \quad \text{in } L^3(I \times B_{2^n}), \\
p^{(k)} &\to \tilde{p} \quad \text{in } L_{\text{loc}}^\frac{3}{2}(I \times B_{2^n}),
\end{align*}
\]
where $\tilde{p}(x,t) = \lim_{n \to \infty} \tilde{p}^n(x,t)$, and $\tilde{p}^n(x,t)$ is defined for $|x| < 2^n$ by

$$
\tilde{p}^n(x,t) = -\frac{1}{3}|\tilde{v}(x,t)|^2 + \text{p.v.} \int_{B_2} K_{ij}(x-y) \tilde{v}_i \tilde{v}_j dy
+ \left( \text{p.v.} \int_{B_{2n+1} \setminus B_2} + \int_{B_{2n+1}^c} \right) (K_{ij}(x-y) - K_{ij}(-y)) \tilde{v}_i \tilde{v}_j(y,t) dy.
$$

Taking the limit of the weak form of (5.11), we obtain that $(\tilde{v}, \tilde{p})$ satisfies the weak form of (NS) for the initial data $\tilde{v}|_{t=t_0} = v(t_0)$. Furthermore, the limit of (5.12) gives us the local energy inequality: For any $\xi \in C^\infty_c([t_0, t_0 + S] \times \mathbb{R}^3)$, $\xi \geq 0$, we have

$$
2 \int_I \int |\nabla \tilde{v}|^2 \xi dx ds \leq \int |v|^2 \xi(x, t_0) dx
+ \int_I \int |\tilde{v}|^2 (\partial_s \xi + \Delta \xi) + (|\tilde{v}|^2 + 2\tilde{p}) \tilde{v} \cdot \nabla \xi dx ds.
$$

(5.13)

Here we have used that $\int E^{(k)} \cdot v^{(k)} \xi = \int f^{(k)} \cdot \nabla (H^{(k)}(1-\Phi^{(k)}) \cdot v^{(k)} \xi = 0$ for $k$ sufficiently large. In a way similar to the proof of Theorem 3.2, we get the local pressure decomposition for $\tilde{p}$, weak local $L^2$-continuity of $\tilde{v}(t)$, and local $L^2$-convergence to initial data. We also get (5.13) with the time interval $I$ replaced by $[t_0, t]$ and an additional term $\int |\tilde{v}|^2 \xi(x, t) dx$ in the left side.

We have shown that $(\tilde{v}, \tilde{p})$ is a local energy solution on $\mathbb{R}^3 \times I$ with initial data $\tilde{v}|_{t=t_0} = v(t_0)$.

**Step 3.** To extend to a time-global local energy solution.

We first prove that the combined solution

$$
u = v1_{[0,t_0]} + \tilde{v}1_I, \quad q = p1_{[0,t_0]} + \tilde{p}1_I
$$

is a local energy solution on the extended time interval $[0, T_1] = [0, t_0 + S]$. It is obvious that $u$ and $q$ are bounded in $E_{T_1}$ and $L^2_{\text{loc}}([0, T_1] \times \mathbb{R}^3)$, respectively and $q$ satisfies the decomposition at each point $x_0 \in \mathbb{R}^3$. Since we have for any $\xi \in C^\infty_c([t_0, T_1) \times \mathbb{R}^3; \mathbb{R}^3)

$$
\int_{t_0}^{T_1} - (\nabla \tilde{v} \cdot \nabla \xi) + (\tilde{v} \cdot \nabla \xi) + (\tilde{p}, \text{div} \xi) dt = (\tilde{v}, \zeta(t_0)) = (v, \zeta(t_0)),
$$

and for any $\xi \in C^\infty_c((0, t_0] \times \mathbb{R}^3; \mathbb{R}^3)$

$$
\int_{0}^{t_0} - (v, \partial_s \zeta) + (\nabla v \cdot \nabla \zeta) + (v, (v \cdot \nabla) \zeta) + (p, \text{div} \zeta) dt = -(v, \zeta)(t_0),
$$

from the weak continuity of $\tilde{v}$ at $t_0$ from the right and that of $v$ at $t_0$, we can prove that $(u, p)$ satisfies (NS) in the distribution sense: For any $\zeta \in C^\infty_c((0, T_1) \times \mathbb{R}^3; \mathbb{R}^3)$

$$
\int_{0}^{T_1} - (u, \partial_s \zeta) + (\nabla u \cdot \nabla \zeta) + (u, (u \cdot \nabla) \zeta) + (q, \text{div} \zeta) dt = 0.
$$

Also, since we already have local $L^2$-weak continuity of $u$ on $[0, T_1] \setminus \{t_0\}$, it is enough to check it at $t_0$; for any $\varphi \in L^2(\mathbb{R}^3)$ with a compact support,

$$
\lim_{t \to t_0} (u, \varphi)(t) = \lim_{t \to t_0} (v, \varphi)(t) = (v, \varphi)(t_0) = \lim_{t \to t_0^+} (\tilde{v}, \varphi)(t) = \lim_{t \to t_0^+} (u, \varphi)(t).
$$
Finally, we prove the local energy inequality (3.1). Indeed, for any $t \in (0, t_0]$, the inequality follows from the one of $v$. For $t \in (t_0, T_1)$, we add the inequality of $v$ in $[0, t_0]$ to the one of $\tilde{v}$ in $[t_0, t]$ to get, for any non-negative $\xi \in C^\infty_c((0, T_1) \times \mathbb{R}^3)$,

$$\int |u|^2 \xi(t) dx + 2 \int_0^t \int |\nabla u|^2 \xi dx ds \leq \int |\tilde{v}|^2 \xi(t) dx + 2 \int_0^t \int |\nabla \tilde{v}|^2 \xi dx ds + \int_0^t \int |\nabla \tilde{v}|^2 \xi dx ds$$

$$\leq \int_0^t \int |v|^2 (\partial_x \xi + \Delta \xi) + (|v|^2 + 2p)(v \cdot \nabla) \xi dx ds + \int_0^t \int |\tilde{v}|^2 (\partial_x \xi + \Delta \xi) + (|\tilde{v}|^2 + 2\tilde{p})(\tilde{v} \cdot \nabla) \xi dx ds$$

$$= \int_0^t \int |u|^2 (\partial_x \xi + \Delta \xi) + (|u|^2 + 2q)(u \cdot \nabla) \xi dx ds.$$

Therefore, $(u, q)$ is a local energy solution on $[0, T_1]$ and is an extension of $(v, p)$.

Then, by Lemma 5.1 and the proof of Corollary 4.8, we can find $t_1 \in (t_0 + \frac{1}{8}S, t_0 + S)$ such that $u(t_1) - V(t_1) \in E^4$. Repeating the above argument with new initial time $t_1$, we can get a local energy solution in $[0, t_1 + S)$. Iterating this process, we get a local energy solution global in time. Note that $\|V\|_{L^\infty([t_1, \infty) \times \mathbb{R}^3)} \leq \|V\|_{L^\infty([t_0, \infty) \times \mathbb{R}^3)}$ whenever $t_1 > t_0$, so that on each step, we can extend the time interval for the existence by at least $\frac{7}{8}S$. □

6 Perturbations of global solutions with no spatial oscillation decay

As mentioned in the introduction, there are many known non-decaying flows like constant flows, spatially periodic flows (flows on torus) and two-and-a-half dimensional flows. The last two do not have oscillation decay in general. We do not have a general existence theory for initial data with no oscillation decay. However, the method of this paper can be used to construct perturbations of global solutions with no spatial oscillation decay. The perturbation of a constant flow is already covered by Theorem 1.1. The perturbation of spatially periodic flows and two-and-a-half dimensional flows are covered by the following theorem, which does not assume spatial decay or spatial oscillation decay of initial data.

**Theorem 6.1.** Let $V(x, t)$ be a global in time local energy solution of (NS) with $V \in L^\infty(0, \infty; L^q_{uloc})$, $V|_{t=0} = V_0 \in L^q_{uloc, \sigma}$, for some $q > 3$. Then for any $w_0 \in E^2_\sigma$, there is a global-in-time local energy solution $v$ of (NS) with initial data $v_0 = V_0 + w_0$.

**Proof.** We may assume $3 < q < \infty$. Let $P$ be an associated pressure of $V$. Let $w = v - V$ and $q = p - P$. If $(v, p)$ is a solution of (NS), then $(w, q)$ should satisfy the perturbed equation

$$\begin{cases}
\partial_t w - \Delta w + (V + w) \cdot \nabla w + w \cdot \nabla V + \nabla q = 0, & \text{div } w = 0 \\
w|_{t=0} = w_0,
\end{cases}$$

(6.1)
which is (4.8) without the source term $V \cdot \nabla V$. As a result, we don’t need the spatial decay of $\nabla V$, the strong local energy inequality (4.16), or the spatial decay estimate (4.24) with $\nabla V$. Hence, the proof is much easier.

Since $v_0 \in L_{uloc}^2$, a local energy solution $v$ to (NS) exists on the time interval $[0, T]$ for some $T > 0$ by Theorem 3.2. Using Lemma 4.4, we have the local energy estimate for $w$

$$
\int |w|^2(x, t)\xi(x)\, dx + 2 \int_0^t \int |w\nabla\xi|^2 \, dxds
\leq \int |w_0|^2\xi(x)\, dx + \int_0^t \int |w|^2(\Delta\xi + v \cdot \nabla\xi)\, dxds
+ \int_0^t \int 2q_{x_0}w \cdot \nabla\xi\, dxds + \int_0^t \int 2V \cdot (w \cdot \nabla)(w\xi)\, dxds,
$$

(6.2)

for any $\xi \in C_c^\infty(\mathbb{R}^3)$, $\xi \geq 0$. Here $q_{x_0}$ is defined by

$$
q_{x_0}(x, t) = -\frac{1}{3}(|w(x, t)|^2 + 2w \cdot V) + \text{p.v.} \int_{B(x_0, 2)} K_{ij}(x - y)(w_iw_j + V_iw_j + w_iV_j)(y, t)dy
+ \int_{B(x_0, 2)} (K_{ij}(x - y) - K_{ij}(x_0 - y))(w_iw_j + V_iw_j + w_iV_j)(y, t)dy,
$$

where $K_{ij}(x) = \partial_{ij}\frac{1}{|x|^2}$. Let $\phi_{x_0}$ and $\chi_R$ be defined as in (4.21). We first derive an a priori bound from (6.2) taking $\xi = \phi_{x_0}^2$ and taking sup over $x_0 \in \mathbb{R}^3$ using $q > 3$ (compare (6.5) below for the last term of (6.2))

$$
\sup_{0 < t < T} \|w(\cdot, t)\|^2_{L_{uloc}^2} + \|\nabla w\|^2_{U_T^{2, 2}} + \|w\|^2_{U_T^{3, 3}} \leq A,
$$

(6.3)

where $A = A(T, \|w_0\|^2_{L_{uloc}^2}, \|V\|^2_{L_\infty L_{uloc}^q})$. Next, by the proof of [17, Section 2] with $\xi = \phi_{x_0}^2 \chi_R$, we can prove a spatial decay estimate (easier than (4.24))

$$
\sup_{0 < t < T} \|w(\cdot, t)\chi_R\|^2_{L_{uloc}^2} + \|\nabla w\chi_R\|^2_{U_T^{2, 2}} \leq C \left(\|w_0\chi_R\|^2_{L_{uloc}^2} + R^{-\frac{2}{3}}\right),
$$

(6.4)

where $C = C(T, A, q, \|V\|^2_{L_\infty L_{uloc}^q})$. Indeed, all terms in (6.2) except the last one can be estimated in the same way. For the last term,

$$
\int_0^T \int V(w \cdot \nabla)(w\phi_{x_0}^2 \chi_R^2)\, dxdt
\lesssim \int_0^T \int |V||w||\nabla w|\phi_{x_0}^2 \chi_R^2 + |w|\phi_{x_0} \chi_R^2 + \frac{1}{R}|w|\phi_{x_0}^2\, dxdt
\lesssim_{A, T, q} \|V\|^2_{L_\infty(0, \infty; L_{uloc}^q)} \left[\|w\chi_R\|_{U_T^{2, \left(\frac{3}{2} - \frac{1}{4}\right)}}^{-1} \|\nabla w\chi_R\|_{U_T^{2, 2}} + \|w\chi_R\|^2_{U_T^{3, 3}} + \frac{1}{R}\right].
$$

Then, we use the Gagliardo-Nirenberg interpolation inequality to get

$$
\|w\chi_R\|_{U_T^{2, \left(\frac{3}{2} - \frac{1}{4}\right)}}^{-1} \lesssim \|\nabla(w\chi_R)\|^\frac{3}{7}_{U_T^{2, 2}} \|w\chi_R\|^\frac{1}{7}_{U_T^{2, 2}} + \|w\chi_R\|_{U_T^{2, 2}}
\lesssim \|\nabla w\chi_R\|^\frac{3}{4}_{U_T^{2, 2}} \|w\chi_R\|^\frac{1}{4}_{U_T^{2, 2}} + \|w\chi_R\|_{U_T^{2, 2}} + \frac{C_q(A, T)}{R^\frac{2}{7}},
$$

46
and hence (using $q > 3$ to get a small constant)

$$\int_0^T \int V(w \cdot \nabla)(w \partial^2_{xx} R) dx dt \leq \frac{1}{99} \| \nabla w \|_{L^{2,2}}^2 + C_q(A,T) \left( \| w \chi R \|_{U^3_T}^2 + \frac{1}{R^3} \right). \quad (6.5)$$

This is enough to complete the proof for (6.4). Finally, as in Corollary 4.8, it implies

$$w(t) \in E^p(\mathbb{R}^3), \quad \text{for almost all } t \in (0,T] \quad (6.6)$$

for any $3 \leq p \leq 6$.

Now, we repeat the extension argument in Section 5 with the replacement of the heat equation solution by the time-global solution $V$ given in Theorem 6.1. Assume that a local energy solution $(v,p)$ to (NS) for initial data $v_0 \in L^2_{uloc}(\mathbb{R}^2)$ exists on $[0,T_0]$, $T_0 \in (0,\infty)$. Then, by (6.6), we can find $t_0 \in (0,T_0)$, arbitrarily close to $T_0$, such that $w(t_0) = W_0 + h_0$ where $W_0 \in C^c_{\sigma}(\mathbb{R}^3)$ and $h_0 \in E^4(\mathbb{R}^4)$ with $\| h_0 \|_{L^4_{uloc}} < \delta$. The construction of a local energy solution $(\bar{v},\bar{p})$ after time $t_0$ proceeds as follows. We decompose the solution

$$\bar{v} = V + h + W, \quad \bar{p} = p_V + p_h + p_W,$$

where $V$ is the given solution with pressure $p_V$, $(h,p_h)$ solves

$$\begin{cases}
\partial_t h - \Delta h + \nabla p_h = -(V + h) \cdot \nabla h - (h \cdot \nabla) V \\
\text{div } h = 0, \quad h|_{t=t_0} = h_0,
\end{cases} \quad (6.7)$$

and $(W,p_W)$ satisfies (5.2) with the given solution $V$. The only difference with (5.1) is that (6.7) excludes the term $(V \cdot \nabla) V$. With the interior regularity (see e.g. [24, Theorem A1])

$$\| V \|_{L^\infty(\mathbb{R}^3 \times (t_0,\infty))} \leq C(t_0, \| V \|_{L^\infty(0,\infty;L^a_{uloc})}),$$

(we need the strict inequality $q > 3$ for this uniform estimate), the rest of the proof is the same as in Section 5.

\[\square\]

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