On Boole-Type Combinatorial Numbers and Polynomials

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Abstract. The aim of this paper is to construct generating functions for Boole-type combinatorial numbers and polynomials. Using these generating functions, we derive not only fundamental properties of these numbers and polynomials, but also some identities and formulas. Finally, we present a brief historical remarks and observations on our generating functions and Peters and Boole-type numbers and polynomials.

1. Introduction, Definitions and Notations

Throughout this paper, we need the following definitions and notations. \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) denotes the set of integers, the set of real numbers and the set of complex numbers, respectively.

Generating functions for some special numbers and polynomials are given below:

The Apostol-Bernoulli polynomials and the Apostol-Euler polynomials are defined by, respectively:

\[
F_B(t, x; \lambda) = \frac{1}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!}
\]

(1)

and

\[
F_E(t, x; \lambda) = \frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n(x; \lambda) \frac{t^n}{n!}
\]

(2)

which, for \( x = 0 \), are reduced to the Apostol-Bernoulli numbers \( B_n(\lambda) = B_n(0; \lambda) \) and the Apostol-Euler numbers \( E_n(\lambda) = E_n(0; \lambda) \). For special value of the parameter \( \lambda \), we also get the Bernoulli numbers and the Euler numbers (cf. \[2\], \[7\], \[8\]).

The Stirling numbers of the first kind and the second kind are defined by, respectively:

\[
F_{S_1}(t, k) = \left(\log(1 + t)\right)^k k!
\]

(3)

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\[(x)_n = x(x-1) \ldots (x-n+1) = \sum_{j=0}^{n} x^j s(n, j) \tag{4}\]

and

\[F_{S_2}(t, k) = \left(\frac{(e^t-1)^k}{k!}\right) = \sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} \tag{5}\]

(cf. [1], [2], [5], [8]).

Integral representation of the Cauchy numbers \(C_n\) is given by

\[C_n = \int_{0}^{1} (x)_n dx \tag{6}\]

(cf. [5]).

The Peters polynomials are defined by

\[F_{P}(t, x; \lambda, \mu) = \frac{(1+t)^x}{(1+(1+t)^{1})^\mu} = \sum_{n=0}^{\infty} s_n(x; \lambda, \mu) \frac{t^n}{n!} \tag{7}\]

(cf. [3], [4], [5]). Setting \(x = 0\) in (7), the polynomials \(s_n(x; \lambda, \mu)\) are reduced to the Peters numbers \(s_n(\lambda, \mu) = s_n(0; \lambda, \mu)\). When \(\mu = 1\), equation (7) is reduced to the generating function for the Boole polynomials \(\xi_n(x; \lambda) = s_n(x; \lambda, 1)\) (cf. [3], [5]). Substituting \(x = 0\) and \(\mu = 1\) into (7), the Peters polynomials are reduced to the Boole numbers \(\xi_n(\lambda) = s_n(0; \lambda, 1)\) (cf. [3]) and also \(Ch_n = 2\xi_n(1) = 2s_n(0; 1, 1)\) denotes the Changhee numbers (cf. [4]).

We [7] defined the following combinatorial numbers and polynomials:

The numbers \(Y_n(\lambda)\) and the polynomials \(Y_n(x; \lambda)\) are defined by, respectively:

\[\mathcal{F}(t; \lambda) = \frac{2}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda) \frac{t^n}{n!} \tag{8}\]

and

\[\mathcal{F}(t, x; \lambda) = (1+\lambda t)^x \mathcal{F}(t; \lambda) = \sum_{n=0}^{\infty} Y_n(x; \lambda) \frac{t^n}{n!} \tag{9}\]

(cf. [7]).

Observe that

\[Y_n(-1) = (-1)^{n+1} Ch_n \tag{10}\]

(cf. [9], Lemma 2).

We [7, Eq. (2.3)] constructed the following \(p\)-adic integral representation for the \(p\)-adic meromorphic function as follows:

\[
\int_{\mathfrak{X}} \lambda^{x}(1+\lambda t)^x \chi(x) d\mu_p(x) = \frac{[2]}{(\lambda q)^d(1+\lambda t)^d + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j)(\lambda q)^j(1+\lambda t)^j,
\]

where \(\mu_p(x) = \mu_p(x + p^d \mathbb{Z}_p) = \frac{q^x}{\prod_{q \neq 1} x_q}, \chi = \chi_d = \lim_{N \to \infty} \mathbb{Z}_p / dp^N \mathbb{Z}_p, \chi_1 = \mathbb{Z}_p\) denotes the set of \(p\)-adic integers and \(d\) is an odd positive integer and \(\lambda \in \mathbb{Z}_p\) with \(\lambda \neq 1, \chi\) is the Dirichlet character with odd conductor \(d\).
By the above $p$-adic integral representation, we constructed the following generating function for the so-called generalized Apostol-Changhee numbers and polynomials, respectively:

$$F_C(t; \lambda, q, \chi) = \sum_{d=0}^{\infty} \frac{(-1)^d \chi(d) \lambda q^d (1 + \lambda t)^d}{n!}$$

and

$$F_C(t, z; \lambda, q, \chi) = F_C(t; \lambda, q, \chi)(1 + \lambda t)^z = \sum_{n=0}^{\infty} \frac{F(n, \lambda, q)}{n!}.$$

2. Generating functions for combinatorial type numbers

In this section, with the aid of (10), we derive the following generating function

$$G_{y_7}(t, \lambda, q) = \frac{[2](\lambda q)(1 + \lambda t)^2}{(\lambda q)(1 + \lambda t) + 1} = \sum_{n=0}^{\infty} \frac{y_{7,n}(\lambda, q)}{n!}$$

and

$$F_{y_7}(t, z; \lambda, q, \chi) = G_{y_7}(t, \lambda, q)(1 + \lambda t)^z = \sum_{n=0}^{\infty} \frac{y_{7,n}(z; \lambda, q)}{n!}.$$

By using the above equations, we derive various kind of identities, relations and formulas for the polynomials $y_{7,n}(z; \lambda, q)$ and the numbers $y_{7,n}(\lambda, q)$.

2.1. Identities and relations for the numbers $y_{7,n}(\lambda, q)$

In this section, by using equation (11) with its functional equations, we provide some identities and relations for not only the numbers $y_{7,n}(\lambda, q)$, but also the Stirling numbers, Apostol-type numbers and also combinatorial numbers.

**Theorem 2.1.** Let $n \in \mathbb{N}_0$. Then we have

$$y_{7,n}(\lambda, q) = \frac{[2](\lambda q) \gamma^n}{(\lambda q + 1)^{n+1}}.$$

**Proof.** By (11), we have

$$[2] \sum_{n=0}^{\infty} \frac{(-1)^n \lambda q^n n!}{(\lambda q + 1)^{n+1}} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{y_{7,n}(\lambda, q)}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we get the derived result. \( \square \)

**Theorem 2.2 (Recurrence Relation).** Let

$$y_{7,0}(\lambda, q) = \frac{1 + q}{1 + \lambda q}.$$

Then we have

$$y_{7,n}(\lambda, q) = -\frac{n \lambda q}{\lambda q + 1} y_{7,n-1}(\lambda, q)$$

where $n \in \mathbb{N}$. 


Proof. By (11), we have
\[[2] = (\lambda q + 1) \sum_{n=0}^{\infty} y_{7,n}(\lambda, q) \frac{t^n}{n!} + q\lambda^2 \sum_{n=0}^{\infty} ny_{7,n-1}(\lambda, q) \frac{t^n}{n!}.
\]
Therefore we get
\[[2] = (\lambda q + 1) y_{7,0}(\lambda, q)
\]
and
\[0 = (\lambda q + 1) y_{7,n}(\lambda, q) + q\lambda^2 ny_{7,n-1}(\lambda, q).
\]
Thus we get the result of theorem.

Theorem 2.3. Let \(m \geq 1\). Then we have
\[B_m \left( \frac{q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) = \sum_{n=0}^{m-1} S(m-1,n) y_{7,n}(\lambda, q) \cdot \frac{t^n}{n!}.
\]
Proof. Replacing \(t\) by \(e^t - 1\) in (11), and by using (1), we get
\[\sum_{n=0}^{\infty} y_{7,n}(\lambda, q) \frac{(e^t - 1)^n}{n!} = \left[2\right] \frac{\lambda q + 1}{(q\lambda^2 - q\lambda - 1)} \sum_{n=0}^{m-1} S(m-1,n) y_{7,n}(\lambda, q) \cdot \frac{t^n}{n!}.
\]
After some elementary calculations, we get
\[\frac{(q\lambda^2 - q\lambda - 1)}{[2]} \sum_{n=0}^{m} \sum_{n=0}^{m-1} S(m-1,n) y_{7,n}(\lambda, q) \cdot \frac{t^n}{n!} = \sum_{m=0}^{\infty} B_m \left( \frac{q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) \frac{t^n}{n!}.
\]
Comparing the coefficients of \(\frac{t^n}{n!}\) on the both sides of the above equation, we get the derived result.

By using same computation of equation (15), we also derive the following theorem:

Theorem 2.4. Let \(m \in \mathbb{N}_0\). Then we have
\[\mathcal{E}_m \left( \frac{-q\lambda^2}{q\lambda^2 - q\lambda - 1} \right) = -\frac{2}{[2]} (q\lambda^2 - q\lambda - 1) \sum_{n=0}^{m} S(m,n) y_{7,n}(\lambda, q).
\]

2.2. Logarithm functions associated with integral representation of the numbers

Here, we give integral representation of the numbers \(y_{7,n}(\lambda, q)\) and also give some integral formulas.

Integrating equation (11) with respect to \(t\) from 0 to 1, we get
\[\sum_{n=0}^{\infty} y_{7,n}(\lambda, q) \frac{t^n}{n!} dt = \left[2\right] \frac{\lambda q + 1}{\lambda^2 q + 1} \sum_{n=0}^{\infty} \frac{t^n}{(\lambda^2 q + 1)^n} dt.
\]
Hence, we have
\[\sum_{n=0}^{\infty} y_{7,n}(\lambda, q) \frac{1}{(n+1)!} = \left[2\right] \frac{\lambda^2 q}{\lambda^2 q + 1} \left( \frac{\lambda^2 q}{\lambda^2 q + 1} + 1 \right).
\]
Thus, we arrive at the following theorem:
Theorem 2.5.
\[
\sum_{n=0}^{\infty} \frac{y_{\gamma,n}(\lambda, q)}{(n+1)!} = \frac{[2]}{\lambda^2 q} \ln \left( \frac{\lambda^2 q + \lambda q + 1}{\lambda q + 1} \right).
\] (16)

Combining (16) with (13), we get
\[
\sum_{n=0}^{\infty} \frac{[2](-1)^n(\lambda^2 q)^n}{(n+1)!} = \frac{[2]}{\lambda^2 q} \ln \left( \frac{\lambda^2 q + \lambda q + 1}{\lambda q + 1} \right).
\]

Therefore, after some elementary calculations, we arrive at a series representation of \( \ln \) function by the following corollary:

Corollary 2.6.
\[
\ln \left( \frac{\lambda^2 q}{\lambda q + 1} + 1 \right) = \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{(n+1)} \frac{\lambda^2 q}{\lambda q + 1} \right)^{n+1}.
\]

We remark that the above series has also been studied in [6].

3. A new polynomials \( y_{\gamma,n}(x; \lambda, q) \)

In this section, we give some properties of the polynomials \( y_{\gamma,n}(x; \lambda, q) \). By using equation (12), we derive formulas for these polynomials.

By (12), we get
\[
\sum_{n=0}^{\infty} y_{\gamma,n}(x; \lambda, q) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (x)^n \frac{(\lambda t)^n}{n!} \sum_{n=0}^{\infty} y_{\gamma,n}(\lambda, q) \frac{t^n}{n!}.
\]

Therefore
\[
\sum_{n=0}^{\infty} y_{\gamma,n}(x; \lambda, q) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (x)^{n-j} \lambda^{n-j} y_{\gamma,j}(\lambda, q) \frac{t^j}{j!}.
\]

Comparing the coefficients of \( \frac{t^j}{j!} \) on the both sides of the above equation, we get the following theorem.

Theorem 3.1.
\[
y_{\gamma,n}(x; \lambda, q) = \sum_{j=0}^{n} \binom{n}{j} (x)^{n-j} \lambda^{n-j} y_{\gamma,j}(\lambda, q).
\] (17)

By (17), we see that
\[
y_{\gamma,n}(x; \lambda, q) = [2] \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n-j} \frac{q}{(\lambda q + 1)^{n-j}}}{(\lambda q + 1)^{n-j}}.
\]

Integrating the above equation from 0 to 1 with respect to \( x \), we get
\[
\int_0^1 y_{\gamma,n}(x; \lambda, q) dx = [2] \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n+j} \frac{q}{(\lambda q + 1)^{n+j}}}{(\lambda q + 1)^{n+j}} \int_0^1 (x)^{n-j} dx.
\]
By (6), we derive
\[
\int_0^1 y_{7,n}(x; \lambda, q) dx = [2 \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n+j} q^j}{(\lambda q + 1)^{j+1}} C_{n-j}].
\] (18)

By (4), we also derive
\[
\int_0^1 y_{7,n}(x; \lambda, q) dx = [2 \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n+j} q^j}{(\lambda q + 1)^{j+1}} \sum_{k=0}^{n-j} s(n - j, k) \frac{k+1}{k+1}].
\] (19)

Combining (18) and (19), we get the following theorem:

**Theorem 3.2.**
\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n+j} q^j}{(\lambda q + 1)^{j+1}} C_{n-j} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n+j} q^j}{(\lambda q + 1)^{j+1}} \sum_{k=0}^{n-j} s(n - j, k) \frac{k+1}{k+1}.
\] (20)

By using (20), we have
\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\lambda^{n+j} q^j}{(\lambda q + 1)^{j+1}} (C_{n-j} - \sum_{k=0}^{n-j} s(n - j, k) \frac{k+1}{k+1}) = 0.
\]

We observe from the above equation that the well-known formula for the Cauchy numbers is given by
\[
C_{n-j} = \sum_{k=0}^{n-j} s(n - j, k) \frac{k+1}{k+1}
\]
(c.f. [5]).

4. Further remarks and observations

Motivation of the numbers \(y_{7,n}(\lambda, q)\) is briefly given by
\[
y_{7,n}(\lambda, q) = (-1)^{n+1} \frac{q + 1}{2q^n} Y_n (-q\lambda).
\] (21)

In addition, substituting \(q = 1\) into (12), we have
\[
\frac{2(1 + \lambda t)^2}{\lambda(1 + \lambda t) + 1} = \sum_{n=0}^{\infty} y_{7,n}(x; \lambda, 1) \frac{t^n}{n!}.
\]

When \(\lambda = 1\), we obtain
\[
\frac{2(1 + t)^2}{t + 2} = \sum_{n=0}^{\infty} y_{7,n}(x; 1, 1) \frac{t^n}{n!}
\]

Hence, we have the following relations between the polynomials \(y_{7,n}(x; \lambda, q)\), Peters polynomials and Boole polynomials:
\[
s_n(x; 1, 1) = \frac{1}{2} y_{7,n}(x; 1, 1).
\]

Substituting \(x = 0, \lambda = q = 1\), we see that
\[
s_n(0; 1, 1) = \xi_n(1) = \frac{1}{2} Ch_n = \frac{1}{2} y_{7,n}(0; 1, 1).
\]
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