On Uniqueness of Complete Ricci Flow Solution with Curvature Bounded from Below

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Abstract

Let \((M, g)\) be a complete noncompact non-collapsing \(n\)-dimensional riemannian manifold, whose complex sectional curvature is bounded from below and scalar curvature is bounded from above. Then ricci flow with above as its initial data, has at most one solution in the class of complete riemannian metric with complex sectional curvature bounded from below.

1 Introduction

Let \(M\) be a differentiable manifold, \(I\) be an interval, and \(g(t) \equiv g_t \equiv g(x, t)\), \((x, t) \in M \times I\), be a family of complete riemannian metrics on \(M\), which is parameterized by \(t \in I\). Let \(\text{rc}_t\) be ricci curvature of \(g_t\). The ricci flow is a weakly parabolic system of partial differential equations defined by

\[
\frac{\partial}{\partial t} g_t = -2 \text{rc}_t.
\]

It was first introduced by Richard Hamilton in his celebrated 1982 paper[12] and used to deform riemannian metrics. Thereafter, as a powerful tool, ricci flow helped changing the landscape of differential geometry greatly. For

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example, it is a crucial tool to prove Poincaré conjecture and differentiable sphere theorem. See, for example, [19] and [1].

Ricci flow as a system of partial differential equations, the short time existence and uniqueness of the solution to initial value problem are among the most fundamental questions. When underlying manifold is compact without boundary, they were proved by Hamilton in the same paper mentioned above. Shortly after, DeTurck [9] introduced a technique now commonly named after himself, to simplify the proof of existence. Together with the help from harmonic heat flow, Hamilton used DeTurck’s trick to give a new proof to the uniqueness in [14].

For the case of compact manifold with boundary, which we will not discuss in this paper, interested readers may consult [10] and bibliography therein.

When the manifold is complete non-compact without boundary, Shi [23] first constructed a complete solution to every initial metric with bounded riemannian curvature. If the curvature bound is $k_0 > 0$, then there exists a $T = T(n, k_0)$, such that his solution exists on $[0, T]$ and satisfies

$$|
abla^m r|^2 \leq \frac{c_m}{t^n}$$

for some constant $c_m > 0$. Cabezas-Rivas and Wilking [2] improved Shi’s result by removing the curvature upper bound for the initial data, and only assuming that it is non-collapsing and has nonnegative complex sectional curvature (see below for definition). Furthermore, their solution preserves nonnegativity of complex sectional curvature.

For the uniqueness of solution to classic heat equation, in contrast, one cannot take the uniqueness of solution for granted. It is Tychonoff who had not only discovered a non-zero solution to heat equation on $\mathbb{R} \times [0, \infty)$ with vanishing initial data, but also proved uniqueness by assuming solutions at most exponential growth (see e.g. John [15] page 211 and 217). However, in the case of ricci flow, the situation is revealed to be subtle by the discovery that, in dimension two and three, the only solution to ricci flow with euclidean initial data is euclidean. We will discuss this later.

Nonetheless, following spirit of Tychonoff, Chen and Zhu established a uniqueness theorem in the class of metrics with bounded curvature.

**Theorem 1.1** (Chen-Zhu [6]). Let $(M^n, g)$ be a complete noncompact riemannian manifold of dimension $n$ with bounded curvature. Let $g_1(t)$ and $g_2(t)$ be two solutions to the ricci flow on $M \times [0, T]$ with $g$ as their common initial data and with bounded curvatures. Then $g_1 = g_2$ for all $(x, t) \in M^n \times [0, T]$. 

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Also along this spirit, Kotschwar [16] weakened the condition to curvature growth rate being at most quadratic and equivalence of $g(t)$ to $g(0)$.

In lower dimensions, however, the ricci flow is “better” than the classic heat equation. Topping [11] [26] proved that for any initial metric on two dimensional manifold, which could be noncomplete and/or with unbounded curvature, ricci flow has one and only one instantaneously complete solution. Readers may also see [5] for related discussion on surface.

In dimension three, Chen [5] proved that for any complete noncompact initial metric, if it has nonnegative and bounded sectional curvature, then on some time interval depending only on initial curvature bound, there exists a unique complete ricci flow solution. As a corollary, the only complete ricci flow solution on $\mathbb{R}^3$ with euclidean initial metric is euclidean. His idea of the proof is first to prove that the curvature is bounded over some time interval, then the uniqueness follows from the theorem of Chen-Zhu.

Another way to weaken the condition is replacing the bilateral curvature bound by unilateral. We first consider the case with curvature upper bound. On riemann surface, Chen and Yan [7] proved that curvature upper bound is sufficient. It is worth noting that their theorem does not require curvature lower bound for the initial metric.

In general dimension, when we assume in addition that the initial metric is with bounded curvature, one can prove the uniqueness by assuming only the upper bound of sectional curvature. In fact, this is a straight forward consequence of Corollary 2.3 in [5], which says when scalar curvature is bounded from below initially, it is preserved along the flow. So we can apply theorem [17] to obtain uniqueness.

Then we turn to the case with curvature lower bound, which is the main purpose of this paper. Before stating our main result, we first remind readers of the definition of complex sectional curvature.

Let $(M^n, g)$ be a n-dimensional riemannian manifold and $T^c M = TM \otimes \mathbb{C}$ be complexified tangent bundle. Let $\nabla$ be Levi-Civita connection, and $rm$ be riemannian curvature defined by

$$rm(u, v, w, z) = g(\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w, z)$$

where $u, v, w, z \in T_p M$ for any $p \in M$. And

$$\mathcal{R} : \wedge^2 TM \mapsto \wedge^2 TM$$

is the riemannian curvature operator induced by $rm$. Let $sc$ be the scalar curvature. We extend both metric $g$ and curvature tensor $rm$ to be $\mathbb{C}$-multilinear

\[ \text{3} \]
maps
\[ g : S^2 T^c M \to \mathbb{C} \quad \text{rm} : S^2 (\wedge^2 T^c M) \to \mathbb{C}. \]

**Definition 1.2.** For any 2-dimensional complex subspace \( \sigma \in T^c M \), the complex sectional curvature is defined by

\[ \sec^c(\sigma) := \text{rm}(u, v, \bar{v}, \bar{u}) = g(\mathfrak{R}(u \wedge v), u \wedge v), \]

for any unitary orthogonal vectors \( u, v \in \sigma \) i.e. \( g(u, \bar{u}) = g(v, \bar{v}) = 1 \) and \( g(u, \bar{v}) = 0 \), where, e.g. \( \bar{u} \) is the complex conjugate of \( u \).

We can now state the main result.

**Theorem 1.3.** Let \((M^n, g)\) be a complete noncompact \( n \)-dimensional riemannian manifold, with complex sectional curvature \( \sec^c \geq -1 \) and \( s^c \leq s_0 \) for some \( s_0 > 0 \). In addition, the volume of every unit ball is bounded from below uniformly by some constant \( v_0 > 0 \). Let \( g_i(t) \) \( i = 1, 2 \) be two solutions to the ricci flow on \( M \times [0, T] \) with same initial data \( g_i(0) = g \). If both \( g_i(t) \) are complete riemannian metrics with \( \sec^c(g_i(t)) \geq -1 \), for all \( t \in [0, T] \), then there exists a constant \( C_n > 0 \) depending only on dimension \( n \), s.t. \( g_1(t) = g_2(t) \) on \( M \times [0, \min\{\epsilon, T\}] \), where \( \epsilon = \min\{v_0/2C_n, 1/(n-1)\} \).

We need the volume lower bound for unit ball of initial metric to obtain injectivity radius lower bound of \( g(t) \) when \( t > 0 \). This is crucial to establish Cheeger-Gromov convergence of ricci flows in our proof. Actually, the curvature condition alone cannot guarantee the volume lower bound, which is shown by the following example. Consider a rotationally symmetric metric \( g = dr^2 + e^{-2r} ds^2 \) on a half-cylinder. Its curvature is \(-\frac{e^{-r} r''}{e^{-r}} = -1\). Closing up one end at \( r = 0 \) by putting on a cap, smoothing if necessary, we get a complete metric with bounded curvature. But the volume of unit ball goes to zero when \( r \) increases to infinity. By crossing with \( \mathbb{R}^n \) we get examples for higher dimensions.

With Theorem 1.3 above theorem is an immediate corollary of the following one.

**Theorem 1.4.** Let \((M^n, g)\) be as in above theorem. Let \( g(t) \) be a ricci flow solution on \( M^n \times [0, T] \) with \( g(0) = g \). If \( \sec_t \geq -1 \) and \( g(t) \) is complete for any \( t \in [0, T] \), then \( \text{rm}_t \) is uniformly bounded on \( M \times [0, \min\{\epsilon, T\}] \) for the same \( \epsilon \) in above theorem.
Remark 1.5. Readers may wonder if the uniqueness is true on $[0,T]$. Unfortunately, our method is not strong enough to prove it. Actually, Cabezas-Rivas and Wilking had constructed an immortal complete Ricci flow solution with positive curvature operator (hence with positive complex sectional curvature) which is bounded if and only if $t \in [0,1)$ (See [2] Theorem 4b). And by a Toponogov’s theorem below (Theorem 4.1), it has a uniform lower bound for volume of any unit ball in initial manifold. So it does satisfy assumptions of Theorem 1.3. However, our method to prove uniqueness is to show that curvature is bounded, which is not true general for the whole interval of $[0,T]$ as shown by above example. Nonetheless, the question is still open.

Remark 1.6. Authors are in debt to Cabezas-Rivas and Wilking [2] for following their use of complex sectional curvature and adapting many of their arguments to serve their own purpose.

2 Preliminaries

In this section, we list a few facts for later use.

Along Ricci flow, the volume form satisfies the following evolution equation.

$$\frac{\partial}{\partial t} \text{dvol} = -\text{sc} \text{ dvol}$$

We also write down the rescaling relations for later use. If $\tilde{g} = k^2 g$, $\tilde{\text{sec}}^c = k^{-2} \text{sec}^c$.

Remember that sectional curvature of a tangent plane $\sigma$ spanned by $u, v \in T_p M$ is defined by

$$\text{sec}(\sigma) = \frac{\text{rm}(u,v,v,u)}{|u|^2 |v|^2 - g(u,v)^2}.$$  

We have the following relations between complex sectional curvature and other curvatures.

Lemma 2.1. If $\text{sec}^c > (\geq)C$ for any constant $C$, then sectional curvature $\text{sec} > (\geq)C$.

Proof. Let $u, v \in T_p M \subset T_p^c$ be any pair of orthogonal unit vectors and $\sigma$ be the complex plane spanned by them. Then sectional curvature on the real plane spanned by them is $\text{sec}(u,v) = \text{rm}(u,v,v,u) = \text{sec}^c(\sigma) > C$. \qed
Lemma 2.2. If sectional curvature is bounded from below and scalar curvature is bounded from above, then the sectional curvature is bounded from both sides, as is the curvature operator.

Proof. Fix a point \( p \in M \). Suppose for any plane \( P \in T_pM \), the sectional curvature \( \text{sec}(P) \geq c \); and the scalar curvature \( \text{sc}(p) \leq C \) for some fixed constants \( c \leq C \).

Then for any o.n. basis \( \{e_i\} \) of \( T_pM \),

\[
C > \text{sc} = \sum_{i,j} \text{rm}(e_i, e_j, e_j, e_i) = 2 \sum_{i<j} \text{rm}(e_i, e_j, e_j, e_i) = 2 \sum_{i<j} \text{sec}(P_{ij})
\]

where \( P_{ij} \) is the plane spanned by \( \{e_i, e_j\} \). Because sec is bounded from below, by above inequality, \( \text{sec}(P_{ij}) \) is bounded from above by a constant depending on only \( c \), \( C \), and dimension. The above orthonormal basis is arbitrary, so sectional curvature at \( p \) is bounded from above.

Any component of curvature operator is a linear combination of sectional curvature for various planes in tangent space where the coefficients depend only on dimension (see e.g. Cheeger and Ebin [3] page 14). Thus curvature operator is bounded. 

\( \square \)

3 Proof of Theorem 1.4

Proof of theorem 1.4. The key to the proof is establishing the following estimate

\[
\text{sc}_t \leq \frac{C}{t}.
\]

Then, using Theorem 3.6 in B.L.Chen [5] (see also Simon [24]), we get a uniform bound for \( |\text{rm}| \). The arguments here are adaptations of those in Cabezas-Rivas and Wilking [2].

3.1 Estimate Volume Lower Bound

We first establish uniform volume lower bound for unit balls along ricci flow.

Lemma 3.1. Let \((M, g)\) be a complete riemannian n-manifold whose unit balls have volume lower bound \( v_0 > 0 \). Let \((M, g(t))\) be a solution of ricci flow on \( t \in [0, T] \) with \( g(0) = g \), and its sectional curvature is bounded from below by \(-1\) for any \( t \in [0, T] \), then there exists a constant \( C_n > 0 \) depending only
on dimension \( n \), s.t. The volume of any ball \( B_t(p,e) \subset (M, g(t)) \) bounded from below by \( v_0/2 \) for \( t \in [0, \varepsilon] \), where \( \varepsilon = \min\{v_0/2C_n, 1/(n-1)\} \).

**Proof of Lemma 3.1.** Because sectional curvature is bounded uniformly from below by \(-1\), \( \mathrm{rc}_t \geq -(n-1)g(t) \), we have the following estimate for distance function.

**Lemma 3.2.** For any fixed points \( p, q \in M \),

\[
\text{dist}_t(p, q) \leq \text{dist}_0(p, q) e^{(n-1)t}.
\]

**Proof.** Let \( \gamma : [0, 1] \to M \) be a length minimizing geodesic with respect to \( g(0) \) such that \( \gamma(0) = p \) and \( \gamma(1) = q \). Let \( |\gamma|_t \) be the length of the curve \( \gamma \) with respect to \( g(t) \). Then

\[
\frac{d}{dt} |\gamma|_t = \int_0^1 \frac{d}{dt} \sqrt{g_t(\dot{\gamma}(s), \dot{\gamma}(s))} \, ds = \int_0^1 \frac{-\mathrm{rc}(\dot{\gamma}(s), \dot{\gamma}(s))}{\sqrt{g_t(\dot{\gamma}(s), \dot{\gamma}(s))}} \, ds \\
\leq \int_0^1 (n-1)\|\dot{\gamma}(s)\|_t \, ds = (n-1)|\gamma|_t.
\]

By integrating above differential inequality, we get

\[
\text{dist}_t(p, q) \leq |\gamma|_t \leq |\gamma|_0 e^{(n-1)t} = \text{dist}_0(p, q) e^{(n-1)t}.
\]

Therefore \( B_0(p, 1) \subset B_t(p, e) \) for any \( 0 \leq t \leq 1/(n-1) \) and any \( p \in M \). Then,

\[
\frac{d}{dt} \text{Vol}_t(B_0(p, 1)) = -\int_{B_0(p, 1)} \mathrm{sc}_t \, d\text{vol}_t \\
= -\int_{B_t(p,e)} \mathrm{sc}_t \, d\text{vol}_t + \int_{B_t(p,e)-B_0(p,1)} \mathrm{sc}_t \, d\text{vol}_t \\
\geq -C(n) + \int_{B_t(p,e)} -n(n-1) \, d\text{vol}_t.
\]

In last step we use \( \mathrm{sc}_t \geq -n(n-1) \) and Petrunin’s estimate as below.
Theorem 3.3 (Petrunin[22]). Let $M$ be a complete riemannian manifold whose sectional curvature is at least $-1$. Then
\[ \int_{B(p,1)} sc \leq C(n) \]
for any $p \in M$, where $B(p,1)$ is the unit ball centered in $p$ and $C(n)$ is a constant depending only on dimension $n$.

Then, applying Bishop-Gromov volume comparison,
\[ \frac{d}{dt} \text{Vol}_t(B_0(p,1)) \geq -C(n) - n(n-1) \text{Vol}_H(e) =: -C'(n), \]
where $\text{Vol}_H(r)$ is the volume of a ball with radius $r$ in simply connected hyperbolic space of constant curvature $-1$. Integrating above,
\[ \text{Vol}_t(B_t(p,e)) \geq \text{Vol}_0(B_0(p,1)) - C'(n)t \geq v_0 - C(n)t. \]
Hence, let $\epsilon = \min \{v_0/2C'(n), 1/(n-1)\}$,
\[ \text{Vol}_t(B_t(p,e)) \geq v_0/2 \text{ for any } t \in [0, \epsilon]. \]

3.2 Curvature Uniform Bound

Lemma 3.4. Let $(M^n, g(t))$ be a complete solution of ricci flow for $M \times [0, \epsilon]$ with $\sec \geq -1$. If there is a constant $v' > 0$ such that volume of balls $\text{Vol}_t(B_t(p,e)) > v'$, for any $(t,p) \in [0, \epsilon] \times M$, then there exists a constant $C > 0$, so that
\[ sc_t(p) \leq \frac{C}{t} \]
for any $(p,t) \in M \times (0, \epsilon]$.

Proof. Prove by contradiction. Suppose there is a sequence $(p_k, t_k) \subset M \times (0, \epsilon]$ such that,
\[ sc_k(p_k) > \frac{4^k}{t_k}. \] (2)
where and hereafter, $B_k = B_{g(t_k)}$, $B_t = B_{g(t)}$, $sc_t = sc_{g(t)}$, $sc_k = sc_{g(t_k)}$, $\text{dist}_t = \text{dist}_{g(t)}$, and $\text{dist}_k = \text{dist}_{g(t_k)}$. We can pick points in space-time to blow up by following trick first introduced by Perelman in [18].
Claim 3.5. For any sufficiently large integer \( k \), there exists \( \bar{p}_k \in M \) and \( \bar{t}_k \in (0, t_k] \) satisfy the following equations

\[
sc_t(x) \leq 2 sc_{\bar{t}_k}(\bar{p}_k) \quad \text{for all} \quad \begin{cases} 
    x \in B_{\bar{t}_k}(\bar{p}_k, \frac{k}{\sqrt{sc_{\bar{t}_k}(\bar{p}_k)}}) \quad \text{and}, \\
    t \in (\bar{t}_k - \frac{k}{sc_{\bar{t}_k}(\bar{p}_k)}, \bar{t}_k]
  \end{cases}
\]  

(3)

\[
sc_{\bar{t}_k}(\bar{p}_k) \geq \frac{4^k}{t_k}. 
\]  

(4)

Proof of Claim 3.5. We start searching for \((\bar{p}_k, \bar{t}_k)\) from \((p_k, t_k)\) — if it satisfies equation (3), we are done. Otherwise we can find \( x_1 \) and \( \tau_1 \) s.t.

\[
\text{dist}_{\bar{t}_k}(x_1, p_k) \leq \frac{k}{\sqrt{sc_k(p_k)}}, \quad \tau_1 \in (t_k - \frac{k}{sc_k(p_k)}, t_k], \quad \text{and} \quad sc_{\tau_1}(x_1) \geq 2 sc_k(p_k). 
\]

If \((x_1, \tau_1)\) is not what we are looking for, we can find \( x_2 \) and \( \tau_2 \), s.t.

\[
\text{dist}_{\tau_1}(x_1, x_2) \leq \frac{k}{\sqrt{sc_{\tau_1}(x_1)}}, \quad \tau_2 \in (\tau_1 - \frac{k}{sc_{\tau_1}(x_1)}, \tau_1], \quad \text{and} \quad sc_{\tau_2}(x_2) \geq 2 sc_{\tau_1}(x_1). 
\]

We claim that, along this way only for finite steps, we can find desired \((\bar{t}_k, \bar{p}_k)\). Otherwise, we get a sequence of \((x_k, \tau_k)\) and \( sc_{\tau_k}(x_k) \to \infty \) as \( k \to \infty \). Here readers may worry that \( \tau_i \) would go below 0 thus ill-defined. However, by our construction, not only this cannot happen, but also, the above sequence is well bounded in both space and time, as shown in the following discussion. For the convenience, let \( \tau_0 := t_k \) and \( x_0 := p_k \).

Firstly,

\[
t_k \geq \tau_{i+1} \geq t_k - \sum_{l=0}^{i} \frac{k}{sc_{\tau_l}(x_l)} \geq t_k - t_k \sum_{l=0}^{i} \frac{k}{2^l sc_k(p_k)} \geq t_k \left(1 - \frac{2k}{sc_k(p_k)}\right) =: \epsilon_k > 0.
\]

for sufficiently large \( k \), where \( \epsilon_k \) is a constant independent of \( i \).

Secondly, \( x_i \) is also bounded with respect to a background metric \( g_k \). Based on assumption, ricci curvature is bounded from below by \(-(n-1)\),
and by definition $\tau_i < t_k$, then by equation (1), for any integer $l \geq 1$
\[\text{dist}_k(p_k, x_l) \leq e^{(n-1)t_k} \text{dist}_{\tau_{l-1}}(x_{l-1}, x_l)\]
Thus
\[\text{dist}_k(p_k, x_l) \leq e^{(n-1)t_k} \sum_{l=1}^i \text{dist}_{\tau_{l-1}}(x_{l-1}, x_l)\]
\[\leq e^{(n-1)t_k} \sum_{l=1}^\infty \frac{k}{\sqrt{\text{sc}_{\tau_{l-1}}(x_{l-1})}}\]
\[\leq e^{(n-1)t_k} \sum_{l=1}^\infty \frac{1}{(\sqrt{2})^{l-1} \sqrt{\text{sc}_k(p_k)}}\]
\[=: C_k \leq \infty,\]
where $C_k$ is a constant independent of $i$.

Therefore, $(x_i, \tau_i)$ subconverges to a limit, say $(x_\infty, \tau_\infty)$. Then, by continuity of scalar curvature, $\text{sc}_{\tau_\infty}(x_\infty) = \infty$, which is absurd. \(\square\)

Now we consider the volume ratios. By Bishop-Gromov relative volume comparison theorem, for any $0 < r \leq e$,
\[\frac{\text{Vol}_k(p_k, r)}{r^n} = \frac{\text{Vol}_k(p_k, r)}{\text{Vol}_H(r)} \frac{\text{Vol}_H(r)}{\text{Vol}_H(e)} \frac{\text{Vol}_H(e)}{r^n} \geq \frac{v'}{\text{Vol}_H(e)} \frac{1}{c(n)} =: v'' > 0\]
where $v'' = v'(v_0, n)$ is a constant depending only on $v_0$ and $n$.

Let $Q_k = \text{sc}_{\tau_k}(p_k)$. We rescale metric $g_k$ on $B_{t_k}(\bar{p}_k, k/\sqrt{Q_k})$ for $k$ large. In this case, $k/\sqrt{Q_k} < e$, so above lower bound for volume ratio is true for $B_{t_k}(\bar{p}_k, r)$ with any $r < k/\sqrt{Q_k}$. We define a new metric by parabolic rescaling
\[\tilde{g}_k(x, s) := Q_k g(x, t_k + Q_k^{-1} s)\]
Hereafter $\tilde{\text{sc}}_k$ is the scalar curvature of $\tilde{g}_k$, so are the other $\tilde{\text{ed}}$ quantities. Then by definition, $\tilde{g}_k$ is a solution to ricci flow on $(-k, 0] \times B_{\tilde{g}_k(0)}(\bar{p}_k, k)$.\[10\]
Note by equation (3), scalar curvature of $\tilde{g}_k$ is bounded from above by 2. And the complex sectional curvature

$$\tilde{\text{sec}}^c \geq -\frac{1}{Q_k} > -1.$$  \hspace{1cm} (5)

Consequently, by Lemma 2.1 and 2.2 for some constant $c(n) > 0$ depending only on dimension,

$$c(n) |\tilde{\text{ric}}| \leq \tilde{\text{sc}} \leq 2.$$ \hspace{1cm} (6)

Because volume ratio is scaling-invariant,

$$\frac{\text{Vol}_{\tilde{g}_k(0)}(B_{\tilde{g}_k(0)}(\bar{p}_k, r))}{r^n} \geq \nu'' > 0$$ \hspace{1cm} (7)

for any $0 < r \leq k$. Therefore, combining equation (6) and (7) with injectivity radius estimate of Cheeger-Gromov-Taylor [4] (see also Cabezas-Rivas and Wilking [2] Theorem C.3), we have

$$\text{inj}_{\tilde{g}(0)}(\bar{p}_k) \geq c(\nu'', n) > 0.$$

Together with equation (6), applying Hamilton’s compactness theorem (see [13] also [8]), the sequence of pointed ricci flow solution $(B_{\tilde{g}_k(\bar{p}_k, k)}, \tilde{g}_k(s), \bar{p}_k)$ subconverge to a complete ancient solution $(M_\infty, g_\infty(s), p_\infty)$ for $s \in (-\infty, 0]$ in the pointed Cheeger-Gromov sense. From the convergence and $\tilde{\text{sc}}(\bar{p}_k) = 1$, $\text{sec}_{g_\infty(0)}(p_\infty) = 1$, hence the limit metric is non-flat. For the same reason, its riemannian curvature is bounded from equation (6) and its complex sectional curvature is nonnegative from equation (5). It is also noncompact for every $s \in (-\infty, 0]$. We first observe the diameter of $g_\infty(0)$ is infinity. And we pick a sequence of points $y_i \in M_\infty$ for $i = 1, 2, \ldots$, s.t. $\text{dist}_{g_\infty(0)}(p_\infty, y_i) = i$. Because ricci curvature is nonnegative, using Lemma 3.2 again, we show $\text{dist}_{g_\infty(-s)}(p_\infty, y_i) \geq i \rightarrow \infty$. With above observations, we can apply Lemma 4.5 in Cabezas-Rivas and Wilking [2] to this ancient solution, and get, for any $s \in (-\infty, 0]$,

$$\lim_{r \rightarrow \infty} \frac{\text{Vol}_{g_\infty(s)}(B_{g_\infty(s)}(\cdot, r))}{r^n} = 0.$$  \hspace{1cm} (8)

However, this contradicts to the fact that, for any $r > 0$,

$$\frac{\text{Vol}_{g_\infty(0)}(B_{g_\infty(0)}(p_\infty, r))}{r^n} > 0$$  \hspace{1cm} (9)

from equation (7).
Now we are ready to use following theorem to conclude the proof.

**Theorem 3.6** (B.L.Chen[5] or M.Simon[24]). *There is a constant \( C = C(n) \) with the following property. Suppose we have a smooth solution to the ricci flow on \( M^n \times [0, T] \) such that \( B_t(x_0, r_0), 0 \leq t \leq T, \) is compactly contained in \( M \) and*

1. \(|rm| \leq r_0^{-2} \) on \( B_0(x_0, r_0) \) at \( t = 0; \)

2. \[ |rm|_t(x) \leq \frac{K}{t} \]

*where \( K \geq 1, \) \( \text{dist}_t(x_0, x) < r_0, \) whenever \( 0 \leq t \leq T. \)

*Then we have* \[ |rm|_t(x) \leq e^{CK} \left( r_0 - \text{dist}_t(x_0, x) \right)^{-2} \]

*whenever \( 0 \leq t \leq T, \) \( \text{dist}_t(x_0, x) < r_0. \)

Let \( x_0 \) be any point in \( M. \) By our assumption, the ricci flow is complete for any \( t \) hence any ball in \( M \) is compactly contained. Again by assumption, \( |rm_0|_0 \) is bounded. Thus we may choose \( r_0 \) small enough so that condition 1 is satisfied. Finally, lemma 3.4 gives condition 2, i.e.

\[ c(n) |rm|_t(x) \leq sc_t \leq \frac{C}{t}. \]

Consequently, there exists a \( C \) depending on only dimension \( n, \) such that \[ |rm_t(x)| \leq C \]

whenever \( 0 \leq t \leq \epsilon \) and \( \text{dist}_t(x_0, x) \leq \frac{C'}{r_0}. \) For \( x_0 \in M \) is arbitrary, we have \( rm \) is uniformly bounded on \( M \times [0, \epsilon]. \)

\[ \square \]

### 4 Corollaries

We conclude the paper by a few corollaries of the main theorem.
4.1 Solutions with Nonnegative Curvature

If we assume further, that \((M^n, g)\) has nonnegative complex sectional curvature, together with upper bound for scalar curvature, it has nonnegative sectional curvature by Lemma 2.1 and 2.2 stated in the next section. Then \(g\) has a positive lower bound for injectivity radius by using Toponogov’s injectivity radius estimate as follows.

**Theorem 4.1** (Toponogov [25] also Maeda [17]). If \(M\) is a complete non-compact riemannian manifold and for all tangent two-plane \(\sigma\) its sectional curvature \(K_\sigma\) satisfies the inequality \(0 \leq K(\sigma) \leq \lambda\) then there exists a constant \(i_0 > 0\) such that for all \(p \in M\) the injectivity radius \(i(p)\) satisfies

\[
i(p) \geq i_0.
\]

Further, if \(0 < K(\sigma) \leq \lambda\) for all \(\sigma\), then for all \(p \in M\),

\[
i(p) \geq \frac{\pi}{\sqrt{\lambda}}.
\]

**Remark 4.2.** Beware the lower bound \(i_0\) in the first part, depends not only \(\lambda\) but also on \(g\) in a general way. In fact, a sequence of “thinner” and “thiner” flat tori whose injetivity radii approach 0 is an example. We only use this part of the theorem in this paper.

Because the sectional curvature is bounded from above, say by a positive constant \(C_0\), by volume comparison, the volume of any ball with fixed radius \(i_0\) a lower bound. Rescaling the metric if necessary, by Theorem 1.4, we get the following corollary.

**Corollary 4.3.** Let \((M^n, g)\) be a complete noncompact \(n\)-dimensional riemannian manifold, with complex sectional curvature \(\sec^c \geq 0\) and \(sc \leq s_0\) for some \(s_0 > 0\). Let \(g_i(t)\) \(i = 1, 2\) be two ricci flow solutions on \(M \times [0, T]\), with same initial data \(g_i(0) = g\). If both of them are complete riemannian metrics and \(\sec^c (g_i(t)) \geq 0\) for every \(t \in [0, T]\), then there exists a constant \(\epsilon > 0\), s.t. \(g_1(t) = g_2(t)\) on \(M \times [0, \min\{\epsilon, T\}]\).

4.2 Solutions on Three Manifolds

Next let us consider the three manifold. In this case sectional curvature, complex sectional curvature, and curvature operator bounded from below are all equivalent. Then we have the following corollary.
Corollary 4.4. Let \((M, g)\) be a smooth complete riemannian three manifold with bounded sectional curvature. And the volume of unit balls in \((M, g)\) is uniformly bounded by \(v_0 > 0\). Let \(g_i(t), \ i = 1, 2, \ t \in [0, T] \) be two complete solutions of ricci flow with \(g_i(0) = g\). If their sectional curvatures are both bounded from below for any \(t \in [0, T]\), then there exists a constant \(C_n\) depending on dimension \(n\), s.t. \(g_1(t) = g_2(t)\) on \(M \times [0, \min\{\epsilon, T\}]\), where \(\epsilon = \min\{v_0/2C_n, 1/(n - 1)\}\).

Remark 4.5. Readers may notice that above corollary together with the fact that nonnegativity of sectional curvature is preserved by 3-dimensional along ricci flow(Corollary 2.3 in [5]), imply B-L Chen’s strong uniqueness theorem (Theorem 1.1 in same paper above). However, this in essential is not a different proof because two crucial components in it are from B-L Chen’s paper. One is the preservation of nonnegative sectional curvature, the other one is Theorem 3.6 which we use to prove our main theorem.

4.3 Warped Product Solution

If we consider the ricci flow solutions in only the class of rotationally symmetric metrics, the curvature condition can be weaken to ricci bounded from below. Actually we can slightly enlarge the class. For convenience of stating our corollary, we need the following definition.

Definition 4.6. We call \(g\) a warped product metric on a manifold \(M\) if they satisfy the following conditions:

1. \(M\) is diffeomorphic to \(I \times N\) for some interval \(I\) and manifold \(N\).

2. There exist a smooth function \(f : I \mapsto \mathbb{R}^+\) and a riemannian metric \(h\) on \(N\) s.t. \(g = dr^2 + f^2(r)h\) on coordinate system \((r, p) \in I \times N\).

We call \((N, h)\) the warped component and \(\nabla r\) the radial direction.

Corollary 4.7. Let \((M, g)\) be a smooth complete noncompact riemannian manifold satisfying the following conditions:

1. \((M, g)\) has bounded sectional curvature.

2. Volume of any unit ball in \((M, g)\) is bounded from below uniformly by a positive constant.
3. $g$ is a warped product metric and its warped component has nonnegative curvature operator.

Let $g_i(t), i = 1, 2, t \in [0, T]$ be two complete solutions of ricci flow with $g_i(0) = g$. In addition, for any fixed $t \in [0, T]$, they satisfy the following conditions:

1. $g_i(t)$ is a warped product metric whose warped component is complete and has nonnegative curvature operator.

2. Ricci curvature of $g_i$ is bounded from below in radial direction by a uniform constant independent of $t$.

Then there exists an $\epsilon > 0$ such that $g_1(t) = g_2(t)$ for any $t \in [0, \min\{\epsilon, T\}]$.

Above corollary is a consequence of facts below. For convenience, we introduce $\phi = \log f$. Then $f''/f = \phi'' + (\phi')^2$ and $f'/f = \phi'$. Because $f > 0$ is a smooth function, $\phi$ is smooth wherever $f$ is defined. Then the curvatures (see e.g. Petersen[21])

$$\mathcal{R}(\nabla r \wedge X) = -(\phi'' + (\phi')^2)\nabla r \wedge X.$$ 

where $X$ is any vector field on $N$. Let $\{E_a\}, 1 \leq a \leq n(n - 1)/2$ be a o.n. frame of $\wedge^2T_xN$, which diagonalize the curvature operator $\mathcal{R}^h$, with correspondent eigenvalues $\lambda_a$. It is easy to see

$$\mathcal{R}(E_a) = (\lambda_a e^{-2\phi} - (\phi')^2)E_a.$$ 

Let $e_{\alpha} 0 \leq \alpha \leq n + 1$ be an o.n. frame for $g$ and $e_0 = \nabla r$. Hence $e_i, i = 1, 2, \cdots, n$ are orthogonal frame on $(N,h)$. We can choose $e_i$ further s.t. they diagonalize $rc^h$, then for any $i \neq j$,

$$rc(e_0, e_0) = -n(\phi'' + (\phi')^2) \quad rc(e_0, e_i) = 0 = rc(e_i, e_j).$$

Then let us prove the following lemma.

**Lemma 4.8.** Let $a : [0, +\infty) \to \mathbb{R}$ be a smooth function s.t. $a' + a^2 \leq C^2$ for some constant $C \geq 0$. Then, when $C > 0$, $-C \leq a \leq \max\{a(0), C\}$, when $C = 0$, $0 \leq a \leq a(0)$. 15
Proof. The case \( C > 0 \). Consider the correspondent ODE
\[
A' + A^2 = C^2.
\]
When \( A(0) = \pm C, A \equiv \pm C \). When \( A(0) \neq \pm C \), its solution is
\[
A = C + \frac{2C}{Ke^{2Cr} - 1} \quad \text{where } K \text{ is a constant satisfying } A(0) = C(1 + \frac{2}{K - 1}).
\]
Then let us have a closer look of \( A \) with different initial values other than \( \pm C \).

1. When \( A(0) > C \). Then \( K > 1 \). Hence \( A(r) \) is decreasing and \( A \to C \) as \( r \to \infty \). So \( A(0) \geq A \geq C \) as long as \( r \geq 0 \).

2. When \( -C < A(0) < C \). Then \( K < 0 \). So \( A(r) \) is increasing and \( A \to C \) as \( r \to \infty \). So \( C \geq A \geq -C \) as long as \( r \geq 0 \).

3. When \( A(0) < -C \). Then \( 0 < K < 1 \). So \( A(r) \) is decreasing and \( A \to -\infty \) as \( r \to r_0 \) for some \( 0 < r_0 < \infty \).

Let \( A(0) = a(0) \), by comparison principle of ordinary differential equations, \( a(r) \leq A(r) \) as long as \( A(r) \) exists. So, when \( a(0) < -C \), \( a(r) \) becomes discontinuous at some finite \( r \), which contradicts to our assumption on \( a \). Consequently, \( a(0) \geq -C \), thus \( a(r) \leq A(r) \leq \max\{a(0), C\} \). Not only so, we can further conclude that \( a(r) \geq -C \). Otherwise, there exists some \( \tilde{r} > 0 \) s.t. \( a(\tilde{r}) < -C \). Due to the fact that the differential inequality satisfied by \( a \) is translation invariant, it becomes discontinuous for the same reason mention as above.

Conclusion for \( C = 0 \) follows a similar argument. \( \Box \)

We have the following fact for warped product metrics.

**Proposition 4.9.** Let \( (N, h) \) be a complete riemannian \( n \)-manifold with nonnegative curvature operator. Let \( g = dr^2 + f^2(r)h \) be a complete noncompact metric on \( \mathbb{R} \times N \) or \( \mathbb{R}^+ \times N \). If the ricci curvature of \( g \) \( \text{rc}(\nabla r, \nabla r) \geq -C^2 \) for some constant \( C \geq 0 \), then the curvature operator \( \mathcal{R} \geq -C' I \) for constant \( C' \).

**Remark 4.10.** When \( f(r) \) is defined on \( \mathbb{R}^+ \times N \), \( g \) is complete if and only if \( f(r) \) is with certain restrictions and \( (N, h) \) is a standard sphere (up to scaling). Details can be found in Petersen[20].
Proof. Without loss of generality, assume $f > 0$ is a smooth function on $[0, +\infty)$. Since $\text{rc}(\nabla r, \nabla r) \geq -C^2$,

$$\phi'' + (\phi')^2 \leq \frac{C^2}{n}.$$  

By Lemma 4.8, $\phi'$ is bounded. And by assumption $\mathcal{H}^h$ is nonnegative, $\mathcal{H}(E_a) \geq - (\phi')^2 E_a$ is bounded from below. 

Proof of Corollary 4.7. By lemma above, we get $g_i(t)$ are both with curvature operators bounded from below, hence with complex sectional curvature bounded from below. Then the corollary follows from Theorem 1.3.

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