HIGHLY SYMMETRIC MATROIDS, THE STRONG RAYLEIGH PROPERTY, AND SUMS OF SQUARES

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Abstract. We investigate the strong Rayleigh property of matroids for which the basis enumerating polynomial is invariant under a Young subgroup of the symmetric group on the ground set. In general, the Grace-Walsh-Szegő theorem can be used to simplify the problem. When the Young subgroup has only two orbits, such a matroid is strongly Rayleigh if and only if an associated univariate polynomial has only real roots. When this polynomial is quadratic we get an explicit structural criterion for the strong Rayleigh property. Finally, if one of the orbits has rank two then the matroid is strongly Rayleigh if and only if the Rayleigh difference of any two points on this line is in fact a sum of squares.

1. Introduction.

The strong Rayleigh property of a matroid is a real semi-algebraic condition which is motivated by its connection with abstractions of physical properties of electrical networks [1, 2, 3, 4, 5, 8, 13, 15]. Brändén [1] shows that this condition is equivalent to “stability” or the “half-plane property” studied in the references above, from which we take the following facts. Binary matroids are strongly Rayleigh if and only if they are regular. GF(3)- or GF(4)-representable matroids are strongly Rayleigh if and only if they are sixth-root of unity matroids. Uniform matroids and the Vámos matroid are strongly Rayleigh, and a few more examples are known. Minors, duality, free extensions, and two-sums preserve the property. Determining whether or not a given matroid is strongly Rayleigh is often a challenging problem. Here we provide infinitely many new examples of strongly Rayleigh matroids, although they all have a very simple structure.

2010 Mathematics Subject Classification. 05B35; 14P10, 05E18, 05E30.
Key words and phrases. matroid; half-plane property; stable polynomial; Rayleigh difference, positive semidefinite form; Johnson scheme.
Research supported by an NSERC Undergraduate Research Award.
Research supported by NSERC Discovery Grant OGP0105392.
Given a matroid $\mathcal{M}$ on the ground set $E$, let $y = \{y_h : h \in E\}$ be algebraically independent commuting indeterminates, and for $S \subseteq E$ let $y^S = \prod_{h \in S} y_h$. The basis enumerator of $\mathcal{M}$ is the polynomial $M(y) = \sum_B y^B$ with the sum over the set of all bases of $\mathcal{M}$. For distinct elements $e, f \in E$, one can write $M(y) = M_{ef} + y_e M^e_f + y_f M^f_e + y_e y_f M_{ef}$ uniquely, in which the polynomials $M_{ef}, M^e_f, M^f_e, M_{ef}$ do not involve the variables $y_e$ or $y_f$. The Rayleigh difference of $e$ and $f$ in $\mathcal{M}$ is $\Delta M\{e, f\} = M^f_e M^e_f - M_{ef} M_{ef}$.

The matroid $\mathcal{M}$ has the strong Rayleigh property provided that for every pair of distinct elements $e, f \in E$, then $\Delta M\{e, f\}(a) \geq 0$ for all $a \in \mathbb{R}^{E \setminus \{e, f\}}$. This definition extends naturally to any multiaffine polynomial $Z(y) = \sum_{S \subseteq E} \varphi(S) y^S$ with real coefficients.

Let $\pi$ be a partition of the set $E$, and let $\mathfrak{S}_\pi$ be the Young subgroup of all permutations of $E$ which leave each block of $\pi$ invariant as a set. We consider the class of matroids which are invariant under some Young subgroup of the ground set. The restriction of such a matroid to any orbit is thus a uniform matroid. As seen in Proposition 5, the Grace-Walsh-Szegő theorem allows us to reduce the number of variables in the basis enumerator from $|E|$ to $|\pi|$, substantially simplifying the problem. When $\pi$ has only two blocks, we obtain the following.

**Theorem 1.** Let $\mathcal{M}$ be the matroid of rank $r$ with point set $E = S \cup T$ partitioned into disjoint flats $S$ of rank $s$ and $T$ of rank $t$, and with $|S| = a$ and $|T| = b$, with points in the most general position possible. Then $\mathcal{M}$ is strongly Rayleigh if and only if the polynomial $P_M(x) = \sum_{i=r-t}^s \binom{a}{i} \binom{b}{r-i} x^{i-r+t}$ has only real (nonpositive) roots.

In Theorem 1 if $s + t = r$ then $\mathcal{M}$ is a direct sum, and if $s + t = r + 1$ then $\mathcal{M}$ is a two-sum, of uniform matroids. These cases were already known to be strongly Rayleigh. When $s + t \geq r + 2$ the condition in Theorem 1 is nontrivial, and for $s + t = r + 2$ we get the following explicit criterion from the discriminant of a quadratic univariate polynomial.

**Corollary 2.** Adopt the notation of Theorem 1 and assume that $s + t = r + 2$. Then $\mathcal{M}$ is strongly Rayleigh if and only if
\[
\frac{(a - s + 2)(b - t + 2)}{(a - s + 1)(b - t + 1)} \geq \frac{4(s - 1)(t - 1)}{st}.
\]
Finally, we consider whether a Rayleigh difference in \( M \) is in fact a sum of squares of polynomials.

**Theorem 3.** Let \( r \geq 3 \), \( \ell \geq 1 \), and \( a \geq r - 2 \) be integers. Let \( M = M(r, \ell, a) \) be the rank \( r \) simple matroid with \( \ell + 2 \) points on a line \( L \cup \{e, f\} \) and a set \( A \) of \( a \) points in general position relative to this line. Then \( M(r, \ell, a) \) is strongly Rayleigh if and only if either \((r - 2)\ell \leq 2\) or 

\[
a \leq A(r, \ell) = r + \frac{2(r + \ell + 1)}{(r - 2)\ell - 2}.
\]

Moreover, the Rayleigh difference \( \Delta M\{e, f\} \) is a sum of squares if and only if this condition holds.

Table 1 indicates the upper bound of Theorem 3 for small values of \( r \) and \( \ell \).

We assume familiarity with matroid theory [10] and the rudiments of symmetric functions [12]. In Section 2 we discuss some preliminary material, prove Theorem 1 and apply our method in the case of uniform matroids for later use. In Section 3 we apply the method to prove Theorem 3. In Section 4 we discuss some further potential applications which might be tractable.

We thank Petter Brändén, Chris Godsil, Mario Kummer, Levent Tunçel, and Cynthia Vinzant for interesting and helpful conversations and correspondence.
2. Preliminaries.

2.1. The strong Rayleigh property. A form is a homogeneous polynomial; a $d$-form has degree $d$; a polynomial is multiaffine if each variable occurs to at most the first power. A polynomial $F(y) \in \mathbb{R}[y]$ is positive semidefinite (PSD) when $F(a) \geq 0$ for all $a \in \mathbb{R}^E$. A matroid is strongly Rayleigh exactly when every Rayleigh difference is a PSD form. A sum-of-squares (SOS) polynomial $F(y)$ is one for which there are polynomials $p_i(y) \in \mathbb{R}[y]$ for $1 \leq i \leq n$ such that $F(y) = \sum_{i=1}^{n} p_i(y)^2$. It is not hard to see that a SOS $2d$-form is a sum of squares of $d$-forms. Clearly SOS polynomials are PSD, but the converse is false. The relationship between these concepts is the source of Hilbert’s 17th problem, and is a subject of continuing interest [11].

In the following proposition, a polynomial $Z(y_1, \ldots, y_m)$ is stable provided that for any $w_i \in \mathbb{C}$ with $\text{Im}(w_i) > 0$ for all $1 \leq i \leq m$, then $Z(w_1, \ldots, w_m) \neq 0$. Note that a univariate real polynomial is stable if and only if it has only real roots.

Proposition 4. Let $Z(y_1, \ldots, y_m)$ be a multiaffine polynomial with real coefficients, and let $E = \{1, \ldots, m\}$. The following conditions are equivalent:

(a) $Z$ is stable.
(b) $Z$ has the strong Rayleigh property.
(c) for every index $g \in E$, both $Z_g$ and $Z^g$ have the strong Rayleigh property, and either $m \leq 1$ or for some pair of indices $\{e, f\} \subseteq E$, $\Delta Z\{e, f\}(a) \geq 0$ whenever $a \in \mathbb{R}^{E \setminus \{e, f\}}$.
(d) Either $m \leq 1$ or there exists a pair of indices $\{e, f\}$ such that $Z_e$, $Z^e$, $Z_f$, $Z^f$ are all strongly Rayleigh, and $\Delta Z\{e, f\}(a) \geq 0$ whenever $a \in \mathbb{R}^{E \setminus \{e, f\}}$.

Proof. The equivalence of (a) and (b) is Theorem 5.6 of Brändén [1]. That (b) is equivalent to (c) is proved in Theorem 3 of [15]. Condition (c) clearly implies (d). That (d) implies (b) is part of Theorem 3.1 of [14], but the argument there is sketchy. Here we show that (d) implies (c), bridging this gap. First, it follows from (2.1) below that if $Z$ is a strongly Rayleigh multiaffine polynomial, then every deletion $Z^g$ and every contraction $Z_g$ is also strongly Rayleigh.

We prove that (d) implies (c) by induction on $m$, with the basis of induction $m \leq 2$ being trivial. For the induction step assume that (d) holds, let $m \geq 3$, let $\{e, f\}$ be a pair of indices as in (d), and let $g \not\in \{e, f\}$ be any third index. To prove (c) we need only show that $Z_g$ and $Z^g$ are strongly Rayleigh.
Expanding $\Delta Z\{e, f\}$ as a quadratic in $y_g$, we have

\begin{equation}
\Delta Z\{e, f\} = y_g^2 \Delta Z_g\{e, f\} + y_g Q + \Delta Z^g\{e, f\}
\end{equation}

for some polynomial $Q$ not involving $y_e$, $y_f$, or $y_g$. Setting $y_g = 0$ in (2.1), condition (d) implies that $\Delta Z^g\{e, f\}(a) \geq 0$ whenever $a \in \mathbb{R}^{E \setminus \{e, f, g\}}$. Condition (d) also implies that all of $Z^g$, $Z^g_{e, g}$, $Z^g_{f, g}$, $Z^g_{E \setminus \{e, f, g\}}$ are strongly Rayleigh. So $Z^g$ satisfies (d), and so by induction on $m$, condition (c) holds for $Z^g$, so that $Z^g$ is strongly Rayleigh since (c) implies (b). By considering the limit of $y_g^{-2} \Delta Z\{e, f\}$ as $y_g \to \infty$, a similar argument shows that $Z_g$ is strongly Rayleigh. Thus, condition (c) holds.

This completes the induction step, and the proof. \hfill \Box

If every Rayleigh difference of $M$ is a square of a polynomial, then $M$ is certainly strongly Rayleigh. (Regular matroids have this property.) By Theorem 5.5 of [8], the basis enumerator $M(y)$ of such a matroid has a “definite determinantal representation”. The Vámos matroid $V_8$ is known to be strongly Rayleigh [15], but its basis enumerator does not have a definite determinantal representation [2]. In fact, for some pair of elements of $V_8$ the Rayleigh difference is a PSD form but not a SOS form [7, 8].

2.2. Highly symmetric matroids. The following follows from the Grace-Walsh-Szego theorem; see [3, 14].

**Proposition 5.** Let $Z(y) \in \mathbb{R}[y]$ be a polynomial and $\pi$ a partition of the set $E$. Assume that $Z$ is invariant under every permutation in $\mathfrak{S}_\pi$. Let $x = \{x_B : B \in \pi\}$ be indeterminates indexed by the blocks of $\pi$, and define $\beta_\pi : \mathbb{R}[y] \to \mathbb{R}[x]$ by setting $\beta_\pi(y_B) = x_B$ for all $B \in \pi$ and $h \in B$, and algebraic extension. Then $Z(y)$ is stable if and only if $\beta_\pi Z(x)$ is stable.

For an integer partition $\lambda$ and subset $S \subseteq E$, let $e_\lambda(S)$ be the elementary symmetric function of shape $\lambda$ in the variables $\{y_h : h \in S\}$, and similarly for monomial symmetric functions $m_\lambda(S)$. Integer partitions with parts of size at most two will occur frequently; it is convenient to use the notation $[n, i] = 2^i 1^{n-2i}$ for the partition of $n$ with $i$ parts of size 2 and $n-2i$ parts of size 1.

**Proof of Theorem 7** The basis enumerator of $M$ is $M = \sum_{i,j} e_i(S)e_j(T)$, with the sum over all pairs $(i, j)$ with $i + j = r$ and $0 \leq i \leq s$ and $0 \leq j \leq t$. Since $s+t \geq r$ and $j = r-i$, the summation can be replaced with the sum over $r - t \leq i \leq s$. By Proposition 5, $M$ is stable if and only if the bivariate $r$-form $F(\alpha, \beta) = \sum_{i=r-t}^{s} \binom{a}{i} \binom{b}{r-i} \alpha^i \beta^{r-i}$ is stable.
Thus, $F(\alpha, \beta) = \beta^r x^{r-t} P_M(x)$. As $\alpha$ and $\beta$ vary over all complex numbers with positive imaginary part, $x$ varies over all complex numbers except for nonpositive real numbers. Thus, $F(\alpha, \beta)$ is stable if and only if $P_M(x)$ has only real nonpositive roots. (Since the coefficients of $P_M(x)$ are positive, it has no positive real roots.)

Corollary 2 follows immediately by applying the quadratic formula to $P_M(x)$.

The first claim of Theorem 3 follows from Corollary 2, since $M(r, \ell, a)$ is the case of $\mathcal{M}$ in Theorem 1 in which $S = A$ and $s = r$, and $T = L \cup \{e, f\}$ and $t = 2$ and $b = \ell + 2$. Some routine calculation shows that the condition in Corollary 2 holds if and only if either $(r - 2)\ell \leq 2$ or $a \leq \mathcal{A}(r, \ell)$.

By Proposition 4, a matroid is strongly Rayleigh if and only if every Rayleigh difference is a PSD form. In the rest of the paper, we begin to address the question of when these Rayleigh differences are SOS forms.

**Proposition 6.** For uniform matroids, every Rayleigh difference is a SOS form.

**Proof.** Let $\mathcal{M} = \mathcal{U}_{r,m}$ be the uniform matroid of rank $r$ on a set $E$ of size $m$; its basis enumerator is $M = e_r(E)$. By 2-transitivity of $\mathcal{G}_E$, only one Rayleigh difference $\Delta M\{e, f\}$ needs to be checked. Fix $e, f \in E$, let $H = E \setminus \{e, f\}$, let $e_\lambda = e_\lambda(H)$, and let $d = r - 1$. Since $M'_e = M'_f = e_d$ and $M_{ef} = e_{d-1}$ and $M_{e^f} = e_{d+1}$, it follows that $\Delta M\{e, f\} = e_d^2 - e_{d-1}e_{d+1}$. For $0 \leq r \leq 2$ this is easily seen to be a sum of squares, so assume that $d \geq 2$. We claim that

$$e_d^2 - e_{d-1}e_{d+1} = \frac{1}{d + 1} \sum_{j=0}^{d} \binom{d}{j}^{-1} \psi_{d,j}(H)$$

in which $\psi_{d,j}(H)$ is defined for $0 \leq j \leq d$ by

$$\psi_{d,j}(H) = \sum_{J \subseteq H: |J| = j} (y^J)^2 e_{d-j}(H \setminus J)^2.$$ 

This expresses $\Delta M\{e, f\}$ as a SOS form. Note that each $2d$-form $\psi_{d,j}(H)$ is a symmetric function of $\{y_h : h \in H\}$.

The first step is to express both sides of (2.2) in terms of the monomial basis $\{m_{[2d,d-k]} : 0 \leq k \leq d\}$. One sees that $e_d^2 = \sum_{k=0}^{d} \binom{2k}{k} m_{[2d,d-k]}$ and that $e_{d-1}e_{d+1} = \sum_{k=1}^{d} \binom{2k}{k-1} m_{[2d,d-k]}$, and it follows that $e_d^2 - e_{d-1}e_{d+1} = \sum_{k=0}^{d} \binom{2k}{k} m_{[2d,d-k]}$. On the RHS of (2.2), the coefficient of $m_{[2d,d-k]}$ in $\psi_{d,j}(H)$ is $\binom{d-k}{j}$.

To see this, let $U, V \subseteq H$ be
disjoint sets with \(|U| = d - k|\) and \(|V| = 2k|; the coefficient in question is the coefficient of \((y^U)^2y^V\) in \(\psi_{d,j}(H)\). This monomial is constructed in \(\psi_{d,j}(H)\) as the product \((y^J)^2y^S\) by choosing a \(j\)-subset \(J \subseteq U\) and a \(k\)-subset \(K \subseteq V\), and setting \(S = (U \setminus J) \cup K\) and \(T = (U \setminus J) \cup (V \setminus K)\). This construction is bijective: given such a pair \((S,T)\) we recover \(J = U \setminus (S \cap T)\) and \(K = S \setminus T\). There are \({d-j \choose j}\) choices for \(J\) and \({2k \choose k}\) choices for \(K\), establishing the formula.

Thus, equation (2.2) is equivalent to the statement that for all \(0 \leq k \leq d\),

$$
\frac{1}{d+1} \sum_{j=0}^{d} \binom{d}{j}^{-1} \binom{d-k}{j} \binom{2k}{k} = \frac{1}{k+1} \binom{2k}{k}.
$$

Multiply both sides by \((d+1)\binom{d}{d-k}^{-1}\) and use \(d+1 \binom{d}{d-k} = \binom{d+1}{d-k+1}\) and \((d)\binom{d-k}{j} = \binom{d-j}{j}\) to see that this is equivalent to \(\sum_{j=0}^{d} \binom{d-j}{k} = \binom{d+1}{k+1}\). This is equivalent to \(\sum_{j=0}^{d-k} \binom{d-j}{k} = \binom{d+1}{k+1}\), since if \(d-k < j \leq d\) then \(\binom{d-j}{k} = 0\). This well-known binomial identity enumerates lattice paths from \((0, 0)\) to \((k+1, d-k)\), partitioned according to which of the edges \((k, d-k-j) \to (k+1, d-k-j)\) is crossed, for each \(0 \leq j \leq d - k\).

2.3. Sums of squares. Consider an arbitrary matroid \(M\) of rank \(r = d + 1\) on a set \(E\) of size \(m\), let \(\{e, f\} \subseteq E\), and let \(H = E \setminus \{e, f\}\) and \(F(y) = \Delta M\{e, f\}(y)\). This \(F(y)\) is a 2d-form and each variable \(\{y_h : h \in H\}\) occurs at most to the second power. Thus, one can write \(F(y) = \sum_{\alpha} F_{\alpha} y^\alpha\) for some integers \(F_{\alpha} \in \mathbb{Z}\) indexed by the functions \(\alpha : H \to \{0, 1, 2\}\) for which \(|\alpha| = \sum_{h \in H} \alpha(h) = 2d\), and in which \(y^\alpha = \prod_{h \in H} y_{h}^{\alpha(h)}\).

Now assume that \(F(y) = \sum_{i=1}^{n} p_i(y)^2\) is a SOS form. It is not hard to see that each of the polynomials \(p_i(y)\) must be a multiaffine \(d\)-form. Thus, each \(p_i(y)\) for \(1 \leq i \leq n\) can be written

$$
p_i(y) = \sum_{S \subseteq H: |S| = d} c_{(S,i)} y^S
$$

for some real coefficients \(c_{(S,i)} \in \mathbb{R}\). For each \(\alpha : H \to \{0, 1, 2\}\), let \(\mathcal{P}(\alpha)\) be the set of pairs \((S,T)\) such that \(S,T \subseteq H\), \(|S| = |T| = d\), \(S \cap T = \alpha^{-1}(2)\), and \(S \triangle T = \alpha^{-1}(1)\). (Here \(\triangle\) denotes symmetric difference of sets.) In other words, \((S,T) \in \mathcal{P}(\alpha)\) if and only if \(|S| = |T| = d\) and \(y^S y^T = y^\alpha\). Note that if \(y^\alpha\) is a monomial of \(m_{[2d,d-k]}\) then
\[ |\mathcal{P}(\alpha)| = \binom{2k}{k}. \]

It follows that for all \(1 \leq i \leq n\) and \(\alpha : H \to \{0, 1, 2\}\),
\[ [\mathbf{y}^\alpha]p_i(\mathbf{y})^2 = \sum_{(S,T) \in \mathcal{P}(\alpha)} c_{(S,i)} c_{(T,i)}. \]

Consequently, for all \(\alpha : H \to \{0, 1, 2\}\),
\[ F_\alpha = [\mathbf{y}^\alpha]F(\mathbf{y}) = \sum_{i=1}^{n} \sum_{(S,T) \in \mathcal{P}(\alpha)} c_{(S,i)} c_{(T,i)}. \]

For each \(S \subseteq H\) with \(|S| = d\), define the vector \(\mathbf{c}_S \in \mathbb{R}^n\) by \((\mathbf{c}_S)_i = c_{(S,i)}\) for each \(1 \leq i \leq n\), and equip \(\mathbb{R}^n\) with its usual Euclidean (dot) inner product. The previous equation becomes
\[ \sum_{(S,T) \in \mathcal{P}(\alpha)} \langle \mathbf{c}_S, \mathbf{c}_T \rangle = F_\alpha. \]

Thus, the existence of a SOS expression for \(F(\mathbf{y})\) is equivalent to the existence of a set of vectors \(\{\mathbf{c}_S \in \mathbb{R}^n : S \subseteq H\ and |S| = d\}\) such that the inner products \(\langle \mathbf{c}_S, \mathbf{c}_T \rangle\) satisfy a certain system \(\mathcal{L}\) of linear equations \((2.3)\) for each \(\alpha\), with the RHSs of these equations determined by the coefficients of \(F(\mathbf{y})\).

For a 2\(d\)-form \(F(\mathbf{y}) = \Delta M\{e, f\}(\mathbf{y})\) as above, the system \(\mathcal{L}\) has an unwieldy number of variables: \(\binom{t+1}{2}\), in which \(t = \binom{m-2}{d}\). However, when \(F(\mathbf{y})\) has a large group of symmetries, as it does in our case, there is an enriched system which is consistent if and only if \(\mathcal{L}\) is consistent, and which has significantly fewer free parameters.

**Lemma 7.** Let \(\mathbf{v}_1, \ldots, \mathbf{v}_t\) be finitely many vectors in a Euclidean space \(V\), and let \(\Gamma \subseteq S_t\) be a group of permutations of \(\{1, \ldots, t\}\). Let \(\mathcal{L}\) be a system of linear equations satisfied by the inner products \(\langle \mathbf{v}_i, \mathbf{v}_j \rangle\) for \(1 \leq i, j \leq t\) that is invariant under the action of \(\Gamma\). Then there is a set of vectors \(\mathbf{w}_1, \ldots, \mathbf{w}_t\) in a Euclidean space \(W\) such that

(i) the inner products \(\langle \mathbf{w}_i, \mathbf{w}_j \rangle\) satisfy \(\mathcal{L}\), and

(ii) for all \(1 \leq i, j \leq t\) and \(\sigma \in \Gamma\), \(\langle \mathbf{w}_{\sigma(i)}, \mathbf{w}_{\sigma(j)} \rangle = \langle \mathbf{w}_i, \mathbf{w}_j \rangle\).

**Proof.** For each \(1 \leq i, j \leq t\) let \(\alpha_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle\) and let \(A = (\alpha_{ij})\) be the Gram matrix of the vectors \(\{\mathbf{v}_i\}\). Let \(\beta_{ij} = |\Gamma|^{-1} \sum_{\sigma \in \Gamma} \alpha_{\sigma(i), \sigma(j)}\), and let \(B = (\beta_{ij})\) be the corresponding matrix. With \(A_\sigma = (\alpha_{\sigma(i), \sigma(j)})\) for each \(\sigma \in \Gamma\) we have \(B = |\Gamma|^{-1} \sum_{\sigma \in \Gamma} A_\sigma\), in which each matrix \(A_\sigma\) is positive semidefinite, and so \(B\) is also positive semidefinite. Therefore \(B\) is the Gram matrix of some set of vectors \(\{\mathbf{w}_i\}\) in some Euclidean space \(W\).
For any \( \sigma \in \Gamma \), \( \beta_{\sigma(i),\sigma(j)} = \beta_{ij} \), and so the vectors \( \{w_i\} \) satisfy condition (ii). Condition (i) follows from the \( \Gamma \)-invariance of \( L \); for any equation \( \sum_{ij} c_{ij} \alpha_{ij} = \eta \) in \( L \) and \( \sigma \in \Gamma \), the equation \( \sum_{ij} c_{ij} \alpha_{\sigma(i),\sigma(j)} = \eta \) is also in \( L \); it follows that

\[
\sum_{ij} c_{ij} \beta_{ij} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \sum_{ij} c_{ij} \alpha_{\sigma(i),\sigma(j)} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \eta = \eta,
\]

as required. \( \square \)

2.4. Uniform matroids and Johnson schemes. We return to the case of uniform matroids, for later use. Let \( M = \mathcal{U}_{r,m} \) be the uniform matroid of rank \( r = d + 1 \) on a set \( E \) of size \( m = v + 2 \). Adopting the notation above, \( \Delta M \{e, f\} = e_d^2 - e_{d-1} e_{d+1} \) is invariant under the symmetric group \( \mathfrak{S}_H \). The orbits of the induced action of \( \mathfrak{S}_H \) on pairs \( (S, T) \) of \( d \)-subsets of \( H \) are indexed by the integers \( 0 \leq k \leq s = \min\{d, v-d\} \), with the orbit indexed by \( k \) corresponding to those pairs \( (S, T) \) with \( |S \cap T| = d - k \). For each \( 0 \leq k \leq s \), let \( A_k \) be the square matrix indexed by \( d \)-subsets of \( H \), and with \( (S, T) \)-entry

\[
(A_k)_{S,T} = \begin{cases} 
1 & \text{if } |S \cap T| = d - k, \\
0 & \text{otherwise}. 
\end{cases}
\]

These are the adjacency matrices of the Johnson association scheme \( J(v, d) \); see \cite{6, 16}. They are symmetric and pairwise commuting, and hence simultaneously diagonalizable; \( A_0 = I \) is the identity matrix and \( \sum_k A_k = J \) is the all-ones matrix. For each \( 0 \leq k \leq s \), the all-ones vector \( 1 \) is an eigenvector of \( A_k \) with eigenvalue \( \binom{d}{d-k} \binom{v-d}{k} \).

By Proposition \( \ref{prop:sos} \) since \( M = \mathcal{U}_{r,m} \), \( \Delta M \{e, f\} \) is a SOS form. Let \( \{c_S \in \mathbb{R}^n : S \subseteq H \text{ and } |S| = d\} \) be the corresponding set of vectors as in Section 2.3. Consider any \( \alpha : H \to \{0, 1, 2\} \) with \( |\alpha| = 2d \), and let \( |\alpha^{-1}(2)| = d - k \) for some \( 0 \leq k \leq s \). Since \( e_d^2 - e_{d-1} e_{d+1} = \sum_{k=0}^s \frac{1}{k+1} \binom{2k}{k} m_{[2d, d-k]} \), it follows that

\[
\sum_{(S,T) \in \mathcal{P}(\alpha)} \langle c_S, c_T \rangle = [y^a] \Delta M \{e, f\} = \frac{1}{k+1} \binom{2k}{k}.
\]

This system of linear equations is invariant under the action of \( \mathfrak{S}_H \), and so by Lemma \( \ref{lem:sym} \) there is a solution \( \{c_S\} \) such that \( \langle c_S, c_T \rangle \) depends only on \( |S \cap T| = d - k \). There are \( \binom{2k}{k} \) terms on the LHS of (2.4), all in the same orbit of \( \mathfrak{S}_H \), and it follows that in this symmetrized solution \( \langle c_S, c_T \rangle = 1/(k+1) \) whenever \( |S \cap T| = d - k \). In other words, in terms of the matrices of the Johnson scheme, the Gram matrix of \( \{c_S\} \) is

\[
G = A_0 + \frac{1}{2} A_1 + \cdots + \frac{1}{s+1} A_s.
\]
Proposition 8. Let $0 \leq d \leq v$, let $s = \min\{d, v - d\}$, and let \( \{A_0, A_1, ..., A_s\} \) be the adjacency matrices of the Johnson scheme \( J(v, d) \). Then the matrix \( G = \sum_{k=0}^{s} \frac{1}{k+1} A_k \) is positive semidefinite and 1 is an eigenvector for \( G \) with eigenvalue \( \frac{1}{d+1} (v+1) \).

Proof. The preceding remarks of this section show that \( G \) is positive semidefinite and that 1 is an eigenvector for \( G \) with eigenvalue \( \sum_{k=0}^{s} \frac{1}{k+1} \binom{d}{d-k} \binom{v-d}{k} \). If \( v - d = s < k \leq d \) then \( \binom{v-d}{k} = 0 \), so this summation can be extended to \( 0 \leq k \leq d \) in either case. To complete the proof it suffices to show that
\[
\sum_{k=0}^{d} \frac{1}{k+1} \binom{d}{d-k} \binom{v-d}{k} = \frac{1}{d+1} \binom{v+1}{d}.
\]
Multiplying both sides by \( d+1 \) and using \( \frac{d+1}{k+1} \binom{d}{d-k} = \binom{d+1}{d-k} \), it suffices to show that \( \sum_{k=0}^{d} \binom{d+1}{d-k} \binom{v-d}{k} = \binom{v+1}{d} \). This well-known binomial identity enumerates lattice paths from \((0,0)\) to \((d,v+1-d)\), with the \( k \)-th summand enumerating those paths which pass through the point \((d-k,k+1)\), for each \( 0 \leq k \leq d \). Each lattice path from \((0,0)\) to \((d,v+1-d)\) passes through exactly one of these points, proving the result. \( \square \)

3. Proof of Theorem 1.

3.1. Analyzing the SOS equations. Let \( M = M(r, \ell, a) \) be the matroid in Theorem 3 with \( L \) and \( A \) as in the statement and with \( d = r-1 \) and \( H = E \setminus \{e, f\} = L \cup A \). The basis enumerator of \( M \) is
\[(3.1) \quad M = e_r(A) + e_1(L \cup \{e, f\}) e_{r-1}(A) + e_2(L \cup \{e, f\}) e_{r-2}(A).\]

Lemma 9. The Rayleigh difference of \( \{e, f\} \) in \( M \) is
\[
\Delta M\{e, f\} = \sum_{k=0}^{d} \frac{1}{k+1} \binom{2k}{k} m_{2d-k}(A)
+ m_1(L) \sum_{k=1}^{d} \binom{2k-1}{k} m_{2d-1-k}(A)
+ \left(m_2(L) + m_{11}(L)\right) \sum_{k=0}^{d-1} \binom{2k}{k} m_{2d-2-k-1}(A).
\]

Proof. From (3.1), we see that
\[
M_e^f = M_f^e = e_{r-1}(A) + e_1(L)e_{r-2}(A),
M_{ef} = e_{r-2}(A),
\]
and \( M^{ef} = e_r(A) + e_1(L)e_{r-1}(A) + e_2(L)e_{r-2}(A) \).
It follows that $M_f^e M_f^e - M_e f M_e f$ equals
\[ e_d(A)^2 - e_1(A)e_{d+1}(A) + e_1(L)e_d(A)e_{d-1}(A) + (e_1(L)^2 - e_2(L))e_{d-1}(A)^2. \]
Arguments analogous to those in the proof of Proposition 6 finish the calculation. \hfill \Box

Now $\Delta M\{e, f\} = \sum_{i=1}^n p_i(y)^2$ is a SOS form if and only if there is a corresponding set of vectors $\{c_S \in \mathbb{R}^n : S \subseteq H$ and $|S| = d\}$ as in Section 2.3. Let $\mathcal{L}$ denote the system of linear equations (2.3) induced by comparison of coefficients, with RHSs given by $F_\alpha = [y^\alpha] \Delta M\{e, f\}$.

Consider any $\alpha : H \to \{0, 1, 2\}$ with $|\alpha| = 2d$, and let $U = \alpha^{-1}(2)$ and $V = \alpha^{-1}(1)$. Let $|U| = d - k$, so that $0 \leq k \leq d$ and $|V| = 2k$. From Lemma 9, the equations $\sum_{(X, Y) \in \mathcal{P}(\alpha)} \langle c_X, c_Y \rangle = F_\alpha$ of $\mathcal{L}$ fall into several cases, as follows.

(3.2) If $U \subseteq A$ and $V \subseteq A$, then $F_\alpha = \frac{1}{(2k)^{k+1}} \binom{2k}{1} \binom{2k}{k}$.

(3.3) If $U \subseteq A$ and $|V \cap L| = 1$, then $F_\alpha = \frac{1}{(2k)^{k+1}} \binom{2k-1}{k} \binom{2k-1}{k}$.

(3.4) If $U \subseteq A$ and $|V \cap L| = 2$, then $F_\alpha = \frac{1}{(2k)^{k+1}} \binom{2k-2}{k-1} \binom{2k-2}{k-1}$.

(3.5) If $|U \cap L| = 1$ and $V \subseteq A$, then $F_\alpha = \frac{1}{(2k)^{k+1}} \binom{2k}{k}$.

(3.6) In all remaining cases, $F_\alpha = 0$.

One sees that $\Delta M\{e, f\}$ is invariant under the Young subgroup $\Gamma = \mathcal{S}_L \times \mathcal{S}_A$ of $\mathcal{S}_H$. If $(U, V)$ is a pair as in cases (3.2) to (3.6), and $\sigma \in \Gamma$, then $(\sigma(U), \sigma(V))$ is another such pair, and is in the same one of these cases as is $(U, V)$. It follows that the system $\mathcal{L}$ of linear equations is invariant under the action of $\Gamma$. By Lemma 7 we may enrich $\mathcal{L}$ by the requirement that $\langle c_S, c_T \rangle$ depends only on the orbit of $(S, T)$ in the action of $\Gamma$ on pairs of $d$-subsets of $H$ without introducing a new inconsistency. Since $\langle c_S, c_T \rangle = \langle c_T, c_S \rangle$, this common value on the orbit of $(S, T)$ is the same as the common value on the orbit of $(T, S)$. For a pair $(S, T)$ of $d$-subsets of $H$, let
\[ \omega(S, T) = (|S \cap L|, |T \cap L|, |S \cap T \cap L|, |S \cap T \cap A|). \]
Two such pairs $(S, T)$ and $(X, Y)$ are in the same orbit of $\Gamma$ if and only if $\omega(S, T) = \omega(X, Y)$.

3.2. The putative Gram matrix. We continue with the notation of Section 3.1. Also, the notation $\mathcal{S}(H, d) = \{ S \subseteq H : |S| = d \}$ will be convenient.

Lemma 10. Let $\{c_S \in \mathbb{R}^n : S \in \mathcal{S}(H, d) \}$ be a set of vectors solving the equations $\mathcal{L}$ of (3.2) to (3.6). Let $S, T \in \mathcal{S}(H, d)$ and let $p \in L$. 
(a) Then $\langle c_S, c_S \rangle = 1$ if $|S \cap L| \leq 1$, and $c_S = 0$ if $|S \cap L| \geq 2$.
(b) If $S \cap L = T \cap L = \{p\}$, then $c_S = c_T$.

Proof. For (a), the inner product $\langle c_S, c_S \rangle$ corresponds to $y^\alpha = (y^S)^2$ and $P(\alpha) = \{(S,S)\}$. This corresponds to $U = S$ and $V = \emptyset$ and $k = 0$. If $S \cap L = \emptyset$ then case (3.2) applies and $\langle c_S, c_S \rangle = \frac{1}{0+1}(0) = 1$. If $|S \cap L| = 1$ then case (3.5) applies and $\langle c_S, c_S \rangle = \big(\frac{0}{0}\big) = 1$. If $|S \cap L| \geq 2$ then case (3.6) applies and $\langle c_S, c_S \rangle = 0$, so that $c_S = 0$.

For (b), let $y^\alpha = y^S y^T$, and define $U$, $V$, and $k$ accordingly from $\alpha$; this is in case (3.5) above. In the equation $\sum_{(X,Y) \in P(\alpha)} \langle c_X, c_Y \rangle = \binom{2k}{k}$ the LHS has $\binom{2k}{k}$ terms, all in the same orbit of $\Gamma$ as $(S,T)$. It follows that $\langle c_S, c_T \rangle = 1$. Since $\langle c_S, c_S \rangle = \langle c_T, c_T \rangle = 1$, it follows that $c_S = c_T$.

For each $p \in L$, denote by $c_p \in \mathbb{R}^n$ the vector such that $c_S = c_p$ for all $S \in S(H,d)$ for which $S \cap L = \{p\}$.

Proposition 11. With the notation above, $\Delta M\{e,f\}$ is a SOS form if and only if there are unit vectors $\{c_S \in \mathbb{R}^n : S \in S(A,d)\}$ and $\{c_p \in \mathbb{R}^n : p \in L\}$ such that the following hold.
(a) For $S,T \in S(A,d)$ with $|S \cap T| = d - k$, $\langle c_S, c_T \rangle = 1/(k+1)$.
(b) For $S \in S(A,d)$ and $p \in L$, $\langle c_S, c_p \rangle = 1/2$.
(c) For $p,q \in L$ with $p \neq q$, $\langle c_p, c_q \rangle = 1/2$.

Proof. We have seen that $\Delta M\{e,f\}$ is a SOS form if and only if $L$ has a solution that is constant on orbits of $\Gamma$ acting on pairs of $d$-subsets of $H$. By Lemma 9 such a solution must consist of unit vectors $\{c_S\}$ and $\{c_p\}$ indexed as in the statement. The remaining equations from $L$ are equivalent to (a), (b), and (c), as follows.

For (a), let $y^\alpha = y^S y^T$, and define $U$, $V$, and $k$ accordingly from $\alpha$; this is in case (3.2) above. In the equation $\sum_{(X,Y) \in P(\alpha)} \langle c_X, c_Y \rangle = \frac{1}{k+1}\binom{2k}{k}$ the LHS has $\binom{2k}{k}$ terms, all in the same orbit of $\Gamma$ as $(S,T)$. It follows that $\langle c_S, c_T \rangle = 1/(k+1)$.

For (b), let $T \subseteq H$ be such that $T \cap L = \{p\}$. Let $y^\alpha = y^S y^T$, and define $U$, $V$, and $k$ accordingly from $\alpha$; this is in case (3.3) above. In the equation $\sum_{(X,Y) \in P(\alpha)} \langle c_X, c_Y \rangle = \binom{2k-1}{k}$ the LHS has $\binom{2k}{k}$ terms, all in the same orbit of $\Gamma$ as either $(S,T)$ or $(T,S)$. It follows that $\langle c_S, c_T \rangle = \binom{2k-1}{k} \binom{2k}{k} = 1/2$. This is independent of the choice of $T$, so $\langle c_S, c_p \rangle = 1/2$ is self-consistent.

For (c), let $S,T \subseteq H$ be such that $S \cap L = \{p\}$ and $T \cap L = \{q\}$. Let $y^\alpha = y^S y^T$, and define $U$, $V$, and $k$ accordingly from $\alpha$; this is in case (3.4) above. In the equation $\sum_{(X,Y) \in P(\alpha)} \langle c_X, c_Y \rangle = \binom{2k-2}{k-1}$ the LHS has $\binom{2k}{k}$ terms, but if $\{p,q\} \subseteq X$ or $\{p,q\} \subseteq Y$ then $c_X = 0$ or
c_Y = 0, so that ⟨c_X, c_Y⟩ = 0. There are \(2^{(2k-2)_{k-1}}\) other terms, obtained by choosing a \((k - 1)\)-subset \(X' \subseteq V \cap A\), letting \(Y' = (V \cap A) \setminus X'\), and considering the pairs \((X' \cup \{p\}, Y' \cup \{q\})\) and \((X' \cup \{q\}, Y' \cup \{p\})\). Each of these pairs is in the same orbit of \(\Gamma\) as \((S, T)\), and it follows that 
\[\langle c_S, c_T \rangle = \frac{2^{(2k-2)_{k-1}}}{2^{(2k-2)_{k-1}}} = 1/2.\]
This is independent of the choice of \(S\) and \(T\), so \(\langle c_p, c_q \rangle = 1/2\) is self-consistent.

This shows that (a), (b), and (c) are necessary. Conversely, assume that \(\{c_S : S \in S(A, d)\}\) and \(\{c_p : p \in L\}\) are unit vectors as in the statement of the proposition, and for \(S \in S(H, d) \setminus S(A, d)\) let \(c_S = c_p\) if \(S \cap L = \{p\}\) and let \(c_S = 0\) if \(|S \cap L| \geq 2\). This set \(\{c_S : S \in S(H, d)\}\) is a solution to \(L\), as is easily checked. As in the previous three paragraphs, cases (3.2), (3.3), and (3.4) follow from (a), (b), and (c). Case (3.5) follows from \(c_S = c_T\) whenever \(S \cap L = T \cap L = \{p\}\) as in the proof of Lemma 10, and case (3.6) follows from \(c_S = 0\) when \(|S \cap L| \geq 2\).

This completes the proof. □

Imagine the vectors \(\{c_S \in \mathbb{R}^n : S \in S(A, d)\}\) in any order, followed by the vectors \(\{c_p : p \in L\}\) in any order, and form the putative Gram matrix \(\mathcal{G}\) of their inner products. By Proposition 11 this matrix has the block form

\[(3.7)\]

\[\mathcal{G} = \frac{1}{2} \begin{bmatrix}
2G_t & J_{t \times t} \\
J_{t \times t} & I_t + J_t
\end{bmatrix}\]

in which \(t = \binom{n}{a}\). The upper-left block is 2 times the \(t\)-by-\(t\) square matrix \(G_t\) of Proposition 3 (with \(v = a\)) for the Johnson scheme \(J(a, d)\). The lower-right block is \(\ell\)-by-\(\ell\) square, \(I_\ell\) is the identity matrix, and the various \(J\) matrices are all-ones matrices of the appropriate shapes. We have reduced the problem to the following.

**Corollary 12.** With the notation above, \(\Delta M\{e, f\}\) is a SOS form if and only if the matrix \(\mathcal{G}\) of (3.7) is positive semidefinite.

### 3.3. Dénouement

The matrix \(I_\ell + J_\ell\) has eigenvalues \(\ell + 1\) of multiplicity one and 1 of multiplicity \(\ell - 1\), and thus is positive definite and hence invertible. The matrix \(\mathcal{G}\) is thus positive semidefinite if and only if the Schur complement

\[C_t = G_t - \frac{1}{2} J_{t \times t} (I_\ell + J_\ell)^{-1} J_{t \times t}\]

is positive semidefinite (see item (0.8.5) of [9]). One easily checks that \((I_\ell + J_\ell)^{-1} = I_\ell - (\ell + 1) J_\ell,\) and that the Schur complement in question
is

\[(3.8) \quad C_t = G_t - \frac{\ell}{2\ell + 2} J_t.\]

**Lemma 13.** With the notation above, \(\Delta M\{e, f\}\) is a SOS form if and only if \(\frac{1}{d+1} \binom{a+1}{d} \geq \frac{\ell}{2\ell + 2} \binom{a}{d}\).

**Proof.** By Corollary 12, it suffices to determine when the matrix \(G\) of \((3.7)\) is positive semidefinite. By the remarks of this section, it suffices to determine when the matrix \(C_t\) in \((3.8)\) is positive semidefinite. Let \(\{u_1, ..., u_t\}\) be an orthogonal basis of \(\mathbb{R}^t\) consisting of eigenvectors for \(G_t\), with \(u_1 = 1\), and let \(\theta_1 = \frac{1}{d+1} \binom{a+1}{d}, \theta_2, ..., \theta_t\) be the corresponding eigenvalues; by Proposition 8, \(\theta_i \geq 0\) for all \(1 \leq i \leq t\). Since this basis is orthogonal, the \(\{u_i\}\) are eigenvectors of \(J_t\) as well, with corresponding eigenvalues \(\xi_1 = \binom{t}{0}\) and \(\xi_i = 0\) for all \(2 \leq i \leq t\). Thus, the \(\{u_i\}\) are an orthogonal basis of \(\mathbb{R}^t\) consisting of eigenvectors of \(C_t\), with corresponding eigenvalues \(\theta_i - t\xi_i/(2\ell + 2)\) for \(1 \leq i \leq t\). These eigenvalues are all nonnegative if and only if \(\theta_1 \geq t\xi_1/(2\ell + 2)\). This proves the result. \(\square\)

**Proof of Theorem 3.** Using \(\binom{a+1}{d} = \frac{a+1}{a+1-d} \binom{a}{d}\) and \(a + 1 - d \geq 0\), elementary calculation shows that the inequality of Lemma 13 is equivalent to

\[a((r - 2)\ell - 2) \leq r((r - 2)\ell - 2) + 2(r + \ell + 1).\]

When \((r, \ell)\) is one of \((3, 1), (3, 2),\) or \((4, 1)\) the factor \((r - 2)\ell - 2 \leq 0\) is nonpositive and the inequality holds. In all other cases \((r - 2)\ell - 2 > 0\) is positive, and the inequality is equivalent to \(a \leq A(r, \ell)\), as claimed. By Lemma 13 this establishes the first claim of the theorem.

We prove that \(M(r, \ell, a)\) is strongly Rayleigh when either \((r - 2)\ell \leq 2\) or \(a \leq A(r, \ell)\) by induction on \(\ell\). For the basis of induction it is convenient to take the degenerate case \(\ell = 0\); then \(M(r, 0, a) = U_{r,a+2}\) is a uniform matroid, which is strongly Rayleigh by Proposition 0. For the induction step, assume that \(\ell \geq 1\), and that if \(M(r, \ell - 1, a)\) satisfies either \(\ell - 1 = 0, (r - 2)(\ell - 1) \leq 2,\) or \(a \leq A(r, \ell - 1)\), then \(M(r, \ell - 1, a)\) is strongly Rayleigh. Let \(M = M(r, \ell, a)\) be such that either \((r - 2)\ell \leq 2\) or \(a \leq A(r, \ell)\).

By the first claim of the theorem, \(\Delta M\{e, f\}\) is a SOS form, hence PSD. By Proposition 11(d), to complete the proof it suffices to show that \(M_e, M^e, M_f,\) and \(M^f\) are strongly Rayleigh. By symmetry, it suffices to show that \(M_e\) (the basis enumerator of the contraction \(M/e\)) and \(M^e\) (the basis enumerator of the deletion \(M \setminus e\)) are strongly Rayleigh.

The contraction \(M/e\) is the uniform matroid \(U_{r-1,a+1}\) with the point corresponding to the image of \(f\) fattened to a parallel class of size \(\ell + 1\).
The simplification of $M/e$ is thus $U_{r-1,a+1}$, which is strongly Rayleigh, with basis enumerator $e_{r-1}(A \cup \{f\})$. The basis enumerator $M_e$ of $M/e$ is obtained from $e_{r-1}(A \cup \{f\})$ by substituting $y_f = y_1 + \cdots + y_{e+1}$ for new variables $\{y_1, \ldots, y_{e+1}\}$. One can check that this operation preserves the strong Rayleigh property. (In general, a matroid is strongly Rayleigh if and only if its simplification is strongly Rayleigh.) Thus, $M_e$ is strongly Rayleigh.

For the deletion $M \setminus e$, a short calculation shows that

$$\frac{\partial}{\partial \ell} A(r, \ell) = \frac{-2r(r - 1)}{(r - 2)\ell - 2} < 0$$

since $r \geq 3$. Therefore, $M \setminus e = M(r, \ell - 1, a)$ satisfies either $\ell - 1 = 0$, $(r - 2)(\ell - 1) \leq 2$, or $a \leq A(r, \ell) < A(r, \ell - 1)$. In any case the induction hypothesis applies, so that $M_e$ is strongly Rayleigh.

This completes the induction step, and the proof. \square

4. Concluding Remarks.

For any matroid, stability of the basis enumerator is a complex analytic criterion for all the Rayleigh differences to be PSD forms, while the method of Section 2.3 is a geometric criterion for some Rayleigh difference to be a SOS form. It is a fascinating interaction.

The strategy of our proof can naturally be extended to more complicated cases. Among these, the following simple matroids are perhaps tractable.

(i) $M$ consists of $\ell + 2 \geq t + 1$ points in general position on a flat of rank $t$, extended by $a \geq r - t$ points in general position (relative to the flat) in rank $r$. (The case we consider is $t = 2$.) The Young subgroup has two orbits on points, but the putative Gram matrix as in Section 3.2 has a more complicated block structure. Determining whether the basis enumerator is stable, as in Theorem 1, involves determining whether or not a particular univariate polynomial of degree $t$ has only real roots.

(ii) Represented over the reals, $M$ consists of $a \geq r$ points in a subspace $U$ of dimension $r - 1$, and $b \geq s$ points in a subspace $V$ of dimension $s - 1$, in as general position as possible subject to $\dim(U \cap V) = c$. (When $c \in \{0, 1\}$ this is a direct sum or 2-sum of uniform matroids, and hence is strongly Rayleigh. When $c = s - 1 = t$ this reduces to (i) above.)

(iii) $M$ consists of $c \geq r$ copunctal lines in rank $r \geq 3$, with lines and points in as general position as possible. In the case $c = r = 3$ this is known to be strongly Rayleigh – see Corollary 10.3 and Example 10.4 of [4].
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