FRONT BLOCKING IN THE PRESENCE OF GRADIENT DRIFT

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Abstract. In this paper we derive quantitative conditions under which a compactly supported drift term blocks the propagation of a traveling wave in a straight cylinder in dimension \( n \geq 3 \) under the condition that the drift has a potential.

1. Introduction

This paper is an extension of paper [7] where the author has given an explicit a-priori condition for blocking of traveling waves in a cylindrical domain subject to (compactly supported) drift to occur in one spatial dimension. In this article we are going to discuss the case \( n \geq 3 \). The main assumption we make on the drift term is that it has a potential.

To the best knowledge of the author there are no quantitative results on that matter available apart from [7]. So we hope to contribute to the understanding of blocking of traveling waves.

The investigation of traveling waves in cylinders, also subject to drift, has been done in depth in the seminal paper [5]. However the drift term has been required to be independent of the direction of propagation, in order to allow for classical traveling waves. Since then the notion of traveling waves has been broadened to more general media, i.e. pulsating fronts for periodic media [2] and the very general transition fronts for very general media [3]. In recent years there have been investigations of existence and non existence of transition fronts in outer domains with a compactly supported obstacle [4], in cylinders with varying nonlinearity [11, 8] and, with respect to this work especially interesting, in opening or closing cylinders [1, 6, 9].

The subject of this paper are entire solutions of the generalized initial value problem

\[
\begin{align*}
\partial_t u - \Delta u + k(x) \cdot \nabla u &= f(u) & \text{in } D, \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial D, \\
u(t, x) - \phi(x_1 + ct) &\to 0 \text{ as } t \to -\infty & \text{uniformly in } D,
\end{align*}
\]

(1.1)

where \( D := \mathbb{R} \times \Omega, \Omega \subset \mathbb{R}^{n-1} \) a smooth domain, \( k \in C_c(\overline{D}, \mathbb{R}^n) \), \( \supp k \subset [-x_0, 0] \times \Omega, x_0 > 0 \) and \( f \) is a bistable nonlinearity (for details see section 2). Furthermore the major assumption will be that \( k \) has a potential, i.e. there is \( H \in C^1(D) \) s.t.

\[
\nabla H = k \quad \text{in } D.
\]

We are able to give an explicit criterion for blocking involving the net drift and some term that takes into account the concentrations of \( k \) as formulated in the following theorem.

\[1^{1}\]

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\[ k(x) \]

\[ -x_0 \quad 0 \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Infinite cylinder with transition zone}
\end{figure}

**Theorem 1.1.** Let \( n \geq 3 \). Then there is a constant \( C(f,n) > 0 \) only depending on \( f \) and \( n \) and constants \( C_i(\partial \Omega, f, n) \) only depending on the Lipschitz norm of \( \partial \Omega \), \( f \) and \( n \) (all of them explicit) such that if

\[
C_1(n,f) > \exp\left( - \int_{-x_0}^{0} \int_{\Omega} k_1(x_1,y) \, dx_1 \, dy \right) \max\{C_2(n,f,\partial \Omega), C_3(n,f,\partial \Omega) \}
\]

\[
\left\| \exp\left( - \int_{-x_0}^{x_1} k_1(\zeta,y) \, d\zeta \right) \right\|_{L^\infty(D^n_{-x_0})}^{n-1} \left( \int_{-x_0}^{x_1} \exp\left( - \int_{-x_0}^{\zeta} k_1(\zeta,y) \, d\zeta \right) \, dx_1 \, dy \right)^{n+1} \right\}^{\frac{n}{n+1}}
\]

holds, the unique solution of (1.1) is blocked to the left, i.e. there exists a stationary supersolution \( w : D \rightarrow \mathbb{R} \) of (1.1) such that

\[ u(t,x) \leq w(x) \quad \text{for all } t \in \mathbb{R}, x \in D \]

and \( w(x) \to 0 \) as \( x_1 \to -\infty \).

The strategy of the proof of Theorem 1.1 is to construct the stationary supersolution \( w \) as local minimizer of an appropriate functional in some weighted Sobolev space. The main observation is that (1.1) becomes variational if the drift term is encoded in some weight. With this ‘trick’ we are able to use ideas from [1], where the authors show that a neck can be introduced into a given tube in such a way that propagation gets blocked by constructing a stationary supersolution that vanishes behind the neck.

The paper is organized as follows. First we clarify assumptions and notation. Then, for the sake of completeness we shortly address the question of existence and uniqueness in section 3. In section 4 we give the strategy of the proof and the proof of Theorem 1.1.

**Remark 1.2.** The case \( n = 2 \) can be handled by extending the solution to three dimensions, i.e.

\[ \tilde{u}(x_1, x_2, x_3) := u(x_1, x_2), \quad \tilde{k}(x_1, x_2, x_3) := (k_1(x_1, x_2), k_2(x_1, x_2), 0) \]

and \( \tilde{H}(x_1, x_2, x_3) := H(x_1, x_2) \)

and assuming without loss of generality that \( \Omega = (0, r) \), by setting (e.g.) \( \tilde{\Omega} = (0, r) \times (0, 1) \). Then Theorem 1.1 provides a criterion for blocking that can be projected back onto \( n = 2 \).

**Remark 1.3.** An analogous criterion for almost unchanged propagation as in the one-dimensional case in [7] does also hold in the \( n \)-dimensional case, i.e.
Theorem 1.4 (almost unchanged propagation). There is a constant $C(f,x_0) > 0$ (only depending on $f$ and $x_0$ which is explicitly given) such that if $K := \max\{\max_{x \in D} k_1(x), 0\}$ is small enough to satisfy

$$K \leq C(f,x_0),$$

then the unique solution $u$ of (1.1) converges to a traveling wave with profile $\phi$ and speed $c$, i.e.

$$u(t,x) - \phi(x_1 + ct + \beta) \to 0 \text{ as } t \to +\infty \text{ uniformly in } D,$$

where $\beta \in \mathbb{R}$ is a constant shift.

Remark 1.5. For $n = 1$ even though we did only derive the criterion for $n \geq 3$ the expression behaves as in [7]. But due to the Sobolev-embeddings becoming weaker in increasing dimensions, this generalized criterion is weaker, in the sense that the criterion for $n > m$ might not be satisfied for an $m$-dimensional drift term constantly extended to $n$ dimensions, that satisfies the criterion in $m$ dimensions.

Remark 1.6. It has been proven in [6] that in three dimensions abrupt opening of a channel that is also steep enough leads to blocking of a generalized solution of (1.1). This suggests that a very concentrated drift term with sufficiently big integral should satisfy criterion (1.2). Indeed, if we set

$$k^{x,C}(x) := \frac{C}{\varepsilon} \chi_{[-\varepsilon,0]}(x_1)e_1$$

for $\varepsilon, C > 0$. Then criterion (1.2) holds if $C$ is big enough such that

$$\frac{C_1(n,f)}{C_3(n,f,\partial\Omega)} \geq \exp\left( - \int_{-x_0}^0 \int_{\Omega} k^{x,C} \right) = \exp(-C)$$

and $\varepsilon$ is small enough such that

$$\frac{C_2(n,f,\partial\Omega)}{C_3(n,f,\partial\Omega)} > \left( \int_{\Omega} \exp\left( \int_{-x_0}^{x_1} k^{x,C}(\zeta,y) \, d\zeta \right) \, dx_1 \, dy \right)^{\frac{n+1}{n}}$$

$$= \left( \frac{|\Omega|}{(1+n)C_3(\exp((1+n)C) - 1)} \right)^{\frac{n}{n+1}} \varepsilon^{\frac{1}{n+1}}.$$

(Note that here $x_0(\varepsilon) = \varepsilon$.)

Acknowledgement

We thank Prof. Dr. G.S. Weiss for fruitful discussions.

2. Notation and assumptions

In this section we clarify the assumptions we make and give the notation that shall be used in the following. Let $D$ be the cylindrical domain as specified above. The nonlinearity $f$ shall be of bistable type and obey

$$f \in C^2([0,1]),$$

$$f(0) = 0, f(1) = 0,$$

$$f'(0) < 0, f'(1) < 0,$$

$$f < 0 \text{ on } (0,\theta), f > 0 \text{ on } (\theta,1) \text{ for some } \theta \in (0,1),$$

$$\int_0^1 f(u) \, du > 0.$$
Then there is a unique speed $c > 0$ and unique (up to translation) traveling wave profile $\phi$ for the nonlinearity $f$, i.e. there are speeds $c > 0$ and wave profile $\phi : \mathbb{R} \to \mathbb{R}$, such that (see e.g. [10])

\[
\begin{align*}
\phi''(z) - c\phi'(z) + f(\phi(z)) &= 0 \text{ for all } z \in \mathbb{R}, \\
\phi(-\infty) &= 0, \phi(+\infty) = 1, \\
0 < \phi(z) < 1 &\text{ for all } z \in \mathbb{R}, \\
\phi'(z) > 0 &\text{ for all } z \in \mathbb{R}.
\end{align*}
\]

3. Existence and uniqueness

For the sake of completeness and to ensure the reader that we are not investigating the empty set of solutions or assume uniqueness of solutions of (1.1) without justification, let us mention that existence and uniqueness for solutions of (1.1) can be obtained almost literally copying the proof of Theorem 2.1 in [4] or Appendix A in [8].

4. A necessary condition for propagation / A sufficient condition for blocking

The objective of this section shall be the proof of Theorem 1.1. The proof of our result on blocking will rely mainly on the observation that the stationary version of problem (1.1)

\[-\Delta w + k \cdot \nabla w = f(w) \text{ in } A \subset D\]

is variational with functional

\[J_A(w) = \int_A \left( \frac{1}{2} |\nabla w|^2 + F(w) \right) \psi(x) \, dx,\]

where $F(t) := \int_1^t f(s) \, ds$ and $\psi$ is given by

\[\psi(x) := \exp(-H(x)) \quad \text{in } D.\]

Therefore $\psi(x) > 0$ for all $x \in D$ and $\psi$ is qualified as weight function. With this trick of encoding the drift term in a weight function we are now in the position to use variational techniques to construct a local minimizer of the functional $J$ that will then be extended to a stationary supersolution. The strategy of this proof is inspired by the strategy used in [1] where the authors show that a thin neck can be introduced into a given channel in such a way that a traveling wave gets blocked.

Let us briefly describe our strategy in the following. The goal is to construct $w$ such that

\[u(t, x) \leq w(x) \quad \text{for all } x \in D, t \in \mathbb{R}\]

and $w(x) \to 0$ as $x_1 \to -\infty$, which will be possible if condition (1.2) is met.

To make $J$ well defined and to ensure that $F$ grows quadratically at infinity $f$ shall be extended linearly to a function $f \in C^{1,1}(\mathbb{R})$. Let us introduce the following shorthand notation

\[D^b_a := \{ a < x_1 < b \} \cap D = (a, b) \times \Omega \text{ for } a < b \text{ and }\]

\[H^1(A, \psi) := H^1(A, \psi \, dx).\]

In order to construct such a supersolution we

1. first show that for any $R < -x_0 - 1$ (arbitrary but fixed) there is $\delta(f, \psi, k, a) > 0$ independent of $R$ such that

\[J_{D^b_a}(w) > J_{D^b_a}(w_0)\]
for all \( w \in H_{0,1}^1(D_R, \psi) \) such that \( \|w - w_0\|_{H^1(D_R, \psi)} = \delta \) and \( w_0(x) = w_0(x_1) := \frac{x_1}{a} \chi_{[0,a]}(x_1), a > 0 \) is an auxiliary constant that can be chosen in an optimal way (depending on \( f \)) and we understand

\[
H_{0,1}^1(D^d_c, \psi) := \{ v \in H^1(D^d_c, \psi) : v(c, y) = 0, v(d, y) = 1 \text{ for almost all } y \in \Omega \}
\]

(where boundary values are understood in the sense of traces). From this we can conclude by the direct method, that there is a local minimizer \( w_R \in H_{0,1}^1(D^R_{aR}, \psi) \cap \{ \|w - w_0\|_{H^1(D^R_{aR}, \psi)} \leq \delta \} \) that is a weak solution of

\[
-\Delta w_R - \frac{\nabla \psi}{\psi} \cdot \nabla w_R = f(w_R) \text{ in } D^R_{aR},
\]

\[
\frac{\partial w_R}{\partial \nu} = 0 \text{ on } (R, a) \times \partial \Omega,
\]

\[
w_R(R, y) = 0, w_R(a, y) = 1 \text{ for almost all } y \in \Omega
\]

with \( 0 < w_R < 1 \) in \( D^R_{aR} \) (by comparison principle).

2. In a next step we pass to the limit \( R \to -\infty \) exploiting that \( \delta \) is independent of \( R \) and show that the limit \( w_\infty \) solves

\[
-\Delta w_\infty - \frac{\nabla \psi}{\psi} \cdot \nabla w_\infty = f(w_\infty) \text{ in } D^\infty_{-\infty},
\]

\[
\frac{\partial w_\infty}{\partial \nu} = 0 \text{ on } (-\infty, a) \times \partial \Omega,
\]

\[
w_\infty(-\infty, y) = 0, w_\infty(a, y) = 1 \text{ for almost all } y \in \Omega
\]

and it follows for such a solution (by the strong maximum principle) that \( 0 < w_\infty < 1 \) in \((-\infty, a) \times \Omega\).

3. In the last step we show that if we extend \( w_\infty \) by 1 into \([a, \infty) \times \Omega\) it is a supersolution of (1.1).
Proposition 4.1. Assume that condition (1.2) holds, then for all \( R < -x_0 - 1 \) there is
\[
\delta := \left( \frac{\alpha}{2} \right)^{\frac{1}{2 \gamma}} \tilde{\psi}(-x_0) \max \left\{ \tilde{\gamma} \left( \frac{\alpha}{4} \right), C_2^m (\partial \Omega) \right\},
\]
(4.1)
\[
\left( \frac{\alpha}{2}, m \right) C_2^m (\partial \Omega) 2^{\frac{13-q}{2p}} \left\| \exp \left( - \int_{-x_0}^{x_1} k_1(\zeta, y) \, d\zeta \right) \right\|_{L^\infty(D_0^0)}
\]
\[
\left\{ \int_{D_0^0-x_0} \exp \left( \frac{j}{2} \int_{-x_0}^{x_1} k_1(\zeta, y) \, d\zeta \right) \, dx_1 \, dy \right\}^{\frac{m}{p}} \geq \frac{1}{2}\]
(4.3)

(independent of \( R \)) such that for any \( R < -x_0 - 1 \) there is a local minimizer \( w_R \in H_{0,1}^1(D_R^0, \psi) \cap \left\{ \| w_R - w_0 \|_{H^1(D_R^0, \psi)} < \delta \right\} \) of
\[
J_{D_R^0} \]
in \( H_{0,1}^1(D_R^0, \psi) \cap \left\{ \| w_R - w_0 \|_{H^1(D_R^0, \psi)} < \delta \right\} \). The constant
\[
\alpha := \min \left\{ \frac{1}{4}, \frac{-f'(0)}{4} \right\} > 0
\]
(4.2)

only depends on \( f, \tilde{\gamma} \) as defined in (4.3) depends on \( n \) and \( f \) and the other constants depend on \( n \) and are given by \( q := 2^* = \frac{2n}{n+1}, p = \frac{2n+1}{n+2}, m := p^* = \frac{2n}{n+1}, j := 2(n+1) \).

In order to prove this we will split up \( D_R^0 \) into the part \( D_0^0 \) where \( \psi \) is constant and \( w_0 \) is linear and \( D_R^0 \) where \( w_0 \equiv 0 \) and \( \psi \) does encode the behaviour of \( k \).

On the second subset we will exploit the following Lemma.

Lemma 4.2. With \( \delta \) and \( \alpha \) given as in (4.1), (4.2) in Proposition 4.1, it holds that for all \( w \in H_{0,1}^1(D_R^0, \psi) \cap \left\{ \| w - w_0 \|_{H^1(D_R^0, \psi)} = \| w \|_{H^1(D_R^0, \psi)} \leq \delta \right\} \)
\[
J_{D_R^0} (w) \geq J_{D_0^0} (w_0) + \alpha \| w \|^2_{H^1(D_R^0, \psi)}
\]
where \( H_{0,1}^1(D_R^0, \psi) := \left\{ w \in H^1(D_R^0, \psi) : w(R, y) = 0 \right\} \) for almost all \( y \in \Omega \).

Proof of the Lemma. First by a Taylor expansion of \( F \) we find that
\[
J_{D_R^0} (w) = \int_{D_R^0} \left( \frac{1}{2} |\nabla w|^2 + F(0) + F'(0) w + \frac{1}{2} F''(0) w^2 + \eta(w) w^2 \right) \psi \, dx.
\]
We rewrite this as
\[
J_{D_R^0} (w) = J_{D_0^0} (w_0) + \int_{D_R^0} \left( \frac{1}{2} |\nabla w|^2 - \frac{1}{2} F'(0) w^2 + \eta(w) w^2 \right) \psi.
\]
It is immediate that
\[
\int_{D_R^0} \left( \frac{1}{2} |\nabla w|^2 - \frac{1}{2} F'(0) w^2 \right) \psi \geq \min \left\{ \frac{1}{2}, \frac{-f'(0)}{2} \right\} \| w \|^2_{H^1(D_R^0, \psi)}.
\]
It remains to absorb the last term in the Taylor expansion into this. In order to do so we will use that
\[ \int_{P^0_n} \eta(w)w^2 \leq \sigma(\psi, k, x_0, \|w\|_{H^1(D^0_\infty, \psi)}) \|w\|_{H^1(D^0_\infty, \psi)}^2, \]
where \( \sigma \) is independent of \( R \) and \( \sigma \to 0 \) as \( \|w\|_{H^1(D^0_\infty, \psi)} \to 0 \).

First of all we estimate the error-term in the Taylor expansion \( \eta(w)w^2 \). By definition
\[
\eta(s)s^2 = F(s) - \left( F(0) + F'(0)s + \frac{1}{2}F''(0)s^2 \right).
\]
- for \( s \in (-\infty, 0] \):
  \[
  \eta(s)s^2 = \int_s^1 f(t) \; dt - \left( F(0) + F'(0)s + \frac{1}{2}F''(0)s^2 \right)
  = \int_s^1 f(t) \; dt + \int_0^s f'(0)t \; dt - \left( F(0) - \frac{1}{2}f'(0)s^2 \right) = 0
  \]
- for \( s \in [0, 1] \): Since \( f \in C^2([0, 1]) \) hence \( F \in C^3([0, 1]) \) from Taylor’s Theorem we know that
  \[ \eta(s) \leq \|f''\|_{L^\infty} s \text{ hence in this regime } \eta(s)s^2 \leq \|f''\|_{L^\infty} s^3. \]
- for \( s \in [1, \infty) \) we have
  \[
  \eta(s)s^2 = \int_s^1 f'(1)(t-1) \; dt - \left( F(0) - \frac{1}{2}f'(0)s^2 \right)
  = \left. -\frac{1}{2}f'(1)(s-1)^2 - F(0) + \frac{1}{2}f'(0)s^2 \right|_{s=0}^{s=1}.
  \]
This implies that
\[
|\eta(s)s^2| \leq 2\left( \frac{F(0)}{2} - \frac{1}{2}(f'(0) + f'(1)) \right) s^2 \leq \mu(f) s^2
\]
Putting everything together we find that
\[ |\eta(s)s^2| \leq \|f''\|_{L^\infty} s^3 \chi_{(0, 1)}(s) + \mu(f)s^2 \chi_{(1, \infty)}(s). \]
We claim furthermore that for any \( \gamma > 0 \) and \( q > 2 \) there is \( \tilde{\gamma}(\gamma, q, f, n) \geq 0 \) s.t.
\[ |\eta(s)s^2| \leq \gamma s^2 + \tilde{\gamma}|s|^q \text{ for all } s \in \mathbb{R}. \]
- \( s \in (-\infty, 0] \): nothing is to do.
- \( s \in [0, 1] \):
  - if furthermore \( \|f''\|_{L^\infty} s^3 \leq \gamma s^2 \) nothing is to do.
\[ \text{If otherwise } \|f''\|_{L^\infty} s^3 > \gamma s^2 \text{ then} \]
\[ |\eta(s)|^2 \leq \|f''\|_{L^\infty} s^3 = \|f''\|_{L^\infty} s^{3-q} s^q \]
\[ \leq \begin{cases} \|f''\|_{L^\infty} s^q & \text{if } 3 - q \geq 0 \\ \|f''\|_{L^\infty} \left( \frac{\gamma}{\|f''\|_{L^\infty}} \right)^{3-q} s^q & \text{if } 3 - q < 0 \end{cases} \]

- \( s \in [1, \infty) \): Since \( q > 2 \) choosing \( \bar{\gamma} := \mu(f) \) does it in this regime.

To sum it all up we set
\[ \bar{\gamma}(\gamma, q, f, n) := \begin{cases} \max\{\|f''\|_{L^\infty}, \mu(f)\} & \text{if } 3 - q \geq 0, \\ \max\{\|f''\|_{L^\infty} \left( \frac{\gamma}{\|f''\|_{L^\infty}} \right)^{3-q}, \mu(f)\} & \text{if } 3 - q < 0. \end{cases} \]

(For the following we are going to suppress the dependence on \( f \) and \( n \).) We know that in the case of \( q := 2s = \frac{m}{n-2} > 2 \) the Sobolev-embedding
\[ H^1(D) \hookrightarrow L^q(D), \quad \|w\|_{L^q(D)} \leq C(\text{Lip}(\partial D)) \|w\|_{H^1(D)} \]
does only depend on the Lipschitz-norm of the boundary of \( D \) but is independent of the measure of the set \( D \), we have that
\[ \|w\|_{L^q(D_{R^{-\alpha}})} \leq C_1(\text{Lip}(\partial \Omega)) \|w\|_{H^1(D_{R^{-\alpha}})}, \]
since \( D \) is a cylindrical domain.

Using this Sobolev-embedding we calculate (exploiting that \( \psi \) is constant in \( D_{R^{-\alpha}} \))
\[ \int_{D_{R^{-\alpha}}^0} |\eta(w)w^2| \psi \leq \int_{D_{R^{-\alpha}}^0} (\gamma w^2 + \bar{\gamma}(\gamma, q, \psi) w) \psi \]
\[ \leq \gamma \|w\|_{L^2(D_{R^{-\alpha}}, \psi)}^2 + \bar{\gamma}(\gamma, q, \psi)(-x_0) \|w\|_{L^q(D_{R^{-\alpha}})}^q \]
\[ \leq \gamma \|w\|_{L^2(D_{R^{-\alpha}}^0, \psi)}^2 + \bar{\gamma}(\gamma, q, \psi)(-x_0)C^q(\text{Lip}(\partial \Omega)) \|w\|_{H^1(D_{R^{-\alpha}})}^q \]
\[ \leq \gamma \|w\|_{L^2(D_{R^{-\alpha}}^0, \psi)}^2 + \bar{\gamma}(\gamma, q, \psi)^{1 - \frac{2}{q}}(\psi)(-x_0)C^q(\text{Lip}(\partial \Omega)) \|w\|_{H^1(D_{R^{-\alpha}}, \psi)}^q \]

For the part where \( \psi \) is not constant, i.e. in \( D_{R^{-\alpha}}^0 \) we use a different embedding
\[ \|w\|_{L^q(D_{R^{-\alpha}}^0, \psi)} \leq C_2(\partial \Omega) \|w\|_{W^{1, p}(D_{R^{-\alpha}}^0)}, \]
where we set \( p := 2\frac{n+1}{n+2} \in (1, 2) \) and \( m := p^* = \frac{np}{n-p} = 2 \frac{n}{n-1} > 2 \). Then we get
\[ \int_{D_{R^{-\alpha}}^0} \eta(w)w^2 \psi \leq \gamma \int_{D_{R^{-\alpha}}^0} w^2 \psi + \bar{\gamma}(\gamma, m) \int |w|^m \psi \]
\[ \leq \gamma \int_{D_{R^{-\alpha}}^0} w^2 \psi + \bar{\gamma}(\gamma, m) \|w\|_{L^\infty(D_{R^{-\alpha}}^0)} C^m(\partial \Omega) \|w\|_{W^{1, p}(D_{R^{-\alpha}}^0)}^m \]
\[ \leq \gamma \|w\|_{L^2(D_{R^{-\alpha}}^0, \psi)}^2 + \bar{\gamma}(\gamma, m) \|w\|_{L^\infty(D_{R^{-\alpha}}^0)} C^m(\partial \Omega) \left( \|\psi\|_{L^2(D_{R^{-\alpha}}^0)} \|\psi\|_{W^{1, p}(D_{R^{-\alpha}}^0)} \right)^m \]

where we again used Hölder’s inequality with \( \frac{1}{p} = \frac{1}{2} + \frac{1}{f} \) (recall that \( p \in (1, 2) \)).

Since \( m < q \) for all \( b > 0 \) arbitrary but fixed, it holds that for any \( z \in \mathbb{R} \):
\[ |z|^m \leq b |z|^2 + b^\frac{m-2}{m} |z|^q. \]
Note that \( \frac{m-\frac{3}{2}}{m} < 0 \). Using this estimate we get for arbitrary but fixed \( b > 0 \):

\[
\int_{D_b^0} \eta(w) w^2 \psi \leq \gamma \|w\|^2 \|\bar{z}(\Omega)\|_{L^2(\Omega)} \frac{\tilde{\gamma}(\gamma, q) \psi(-x_0)^{1 - \frac{2}{p}} C_2^m(\partial \Omega)}{\|w\|_{L^1(D_{-x_0})}^{\frac{m}{2}} b^{\frac{m}{2}}} \cdot \left( \|w\|_{H^1(D_{-x_0})}^2 + b \|w\|_{H^1(D_{-x_0})}^2 \right) \leq \left( \gamma + \tilde{\gamma}(\gamma, m) \|\psi\|_{L^\infty(D_{-x_0})} C_2^m(\partial \Omega) \|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})}^{\frac{m}{2}} b \|w\|_{H^1(D_{-x_0})} \right) \cdot \left( \lambda + \tilde{\gamma}(\gamma, m) \|\psi\|_{L^\infty(D_{-x_0})} C_2^m(\partial \Omega) \|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})}^{\frac{m}{2}} b \|w\|_{H^1(D_{-x_0})} \right) \leq \left( \gamma + \tilde{\gamma}(\gamma, m) \|\psi\|_{L^\infty(D_{-x_0})} C_2^m(\partial \Omega) \|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})}^{\frac{m}{2}} b \right) \cdot \left( \lambda + \tilde{\gamma}(\gamma, m) \|\psi\|_{L^\infty(D_{-x_0})} C_2^m(\partial \Omega) \|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})}^{\frac{m}{2}} b \right)
\]

In order to finish the proof we require

I) \( \gamma \leq \frac{\alpha}{4} \)

II) \( \tilde{\gamma}(\gamma, m) \|\psi\|_{L^\infty(D_{-x_0})} C_2^m(\partial \Omega) \|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})}^{\frac{m}{2}} b \leq \frac{\alpha}{4} \)

III) \( \max \left\{ \tilde{\gamma}(\gamma, q) \psi(-x_0)^{1 - \frac{2}{p}} C_2^m(\partial \Omega), \right\} \leq \frac{\alpha}{4} \)

Let us now set

\[
\gamma := \frac{\alpha}{4} \quad \quad \quad \quad b := \frac{\alpha}{4} \left( \tilde{\gamma}(\gamma, m) \|\psi\|_{L^\infty(D_{-x_0})} C_2^m(\partial \Omega) \|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})}^{\frac{m}{2}} b \right)^{-1}
\]

Let us note that

\[
\psi(x_1, y) = \psi(-x_0) \exp \left( \int_{x_0}^{x_1} -k_1(\zeta, y) \, d\zeta \right)
\]

and using that

\[
\|\psi\|_{L^\infty(D_{-x_0})} = \psi(-x_0) \exp \left( - \int_{x_0}^{x_1} k_1(\zeta, y) \, d\zeta \right)
\]

\[
\|\psi^{\frac{1}{2}}\|_{L^m(D_{-x_0})} = \psi^{\frac{1}{2}}(-x_0) \left[ \int_{D_{-x_0}} \exp \left( \frac{j}{2} \int_{x_0}^{x_1} k_1(\zeta, y) \, d\zeta \right) \, dx \, dy \right]^{\frac{1}{2}}
\]
Putting this into III) we get
\[ \|w\|_{H^1(D_{dR}^n, \psi)} \leq \left( \frac{\alpha}{2} \right)^{\frac{1}{2m}} \psi \left( -x_0 \right) \max \left\{ \tilde{\gamma} \left( \frac{\alpha}{2}, q \right) C_1^q(\partial \Omega), \left( \gamma \frac{\alpha}{2}, m \right) C_2^m(\partial \Omega) 2 \frac{2^{q-1}m}{m} \left\| \exp \left( - \int_{-x_0}^{x} k_1(\zeta, y) d\zeta \right) \right\|_{L^\infty(D_{dR}^n, \psi)} \right\} \]
\[ \left\| \int_{D_{dR}^n, \psi} \exp \left( \frac{i}{2} \int_{-x_0}^{x} k_1(\zeta, y) d\zeta \right) dx_1 dy \right\| \leq \tilde{\delta}, \]
\[ = \delta. \]

With Lemma 4.2 being proved we are in the position to prove Proposition 4.1.

**Proposition 4.1.** The strategy of this proof is to show that for all \( w \in H^1_{dR} \) such that \( \|w - w_0\|_{H^1(D_{dR}^n, \psi)} = \delta \) it holds that
\[ J_{D_{dR}^n}(w) > J_{D_{dR}^n}(w_0). \]
Since \( J_{D_{dR}^n} \) is weakly lower semicontinuous, bounded below and coercive, \( J_{D_{dR}^n} \) has a local minimizer among all the functions \( w \in H^1_{dR} \) such that \( \|w - w_0\|_{H^1(D_{dR}^n, \psi)} \leq \delta \). And since we have a local minimizer that does not lie on the boundary of \( \|w - w_0\|_{H^1(D_{dR}^n, \psi)} = \delta \) we derive an Euler-Lagrange-equation. Let us first note that from \( \|w - w_0\|_{H^1(D_{dR}^n, \psi)} \leq \delta \) it follows that \( \|w - w_0\|_{H^1(D_{dR}^n, \psi)} \leq \delta \).

And to make use of Lemma 4.2 we split the functional as follows. For any \( w \in H^1_{dR} \) it holds that
\[ J_{D_{dR}^n}(w) - J_{D_{dR}^n}(w_0) = J_{D_{dR}^n}(w) - J_{D_{dR}^n}(w_0) + J_{D_{dR}^n}(w) - J_{D_{dR}^n}(w_0). \]

From Lemma 4.2 it follows for the first term I
\[ J_{D_{dR}^n}(w) - J_{D_{dR}^n}(w_0) \geq \alpha \|w\|_{H^1(D_{dR}^n, \psi)}^2. \]

For the second term II we use the observation that there is \( K > 0 \) such that for all \( s \in \mathbb{R} \)
\[ F(s) \geq K(s - 1)^2. \] (4.4)

With this observation we conclude that for any measurable set \( A \subset D_{dR}^n \) it holds that
\[ J_A(w) \geq \nu \|w - 1\|_{H^1(A, \psi)}^2, \]
where \( \nu := \min \{ K, \frac{1}{2} \} \). Furthermore we estimate
\[ J_{D_2^n}(w_0) = \int_{D_2^n} \left( \frac{1}{2} \left( \frac{1}{a} \right)^2 + F(w_0) \right) \psi \leq \left( \frac{1}{2} \right) \left( \frac{1}{a} + a \max_{s \in [0, 1]} F(s) \right) \Omega \psi(0) \]
\[ := \beta \]

Together with (4.4) we get
\[ J_{D_2^n}(w) - J_{D_2^n}(w_0) \geq \nu \|w - 1\|_{H^1(D_{dR}^n, \psi)}^2 - \beta \psi(0). \]

In order to use the assumption that \( \|w - w_0\|_{H^1(D_{dR}^n, \psi)} = \delta \) we estimate that
\[ \nu \|w - 1\|_{H^1(D_{dR}^n, \psi)}^2 \geq \frac{\nu}{2} \|w - w_0\|_{H^1(D_{dR}^n, \psi)}^2 - \nu \|w_0 - 1\|^2_{(D_{dR}^n, \psi)} \]
using Young’s inequality. Furthermore by a direct calculation
\[
\|w_0 - 1\|^2_{H^1(D_{\delta}^0, \psi)} = \int_{D_{\delta}^0} \left( \frac{1}{a} \right)^2 + \left( \frac{x_1}{a} - 1 \right)^2 \psi = \psi(0) \left( \frac{1}{a} + \frac{1}{3}a \right) |\Omega|.
\]

Putting these estimations together we get
\[
J_{D_R^\alpha}(w) - J_{D_R^\alpha}(u_0) \geq \frac{\nu}{2} \|w - w_0\|^2_{H^1(D_{\delta}^0, \psi)} - \nu \psi(0) - \beta \psi(0) + \alpha \|w - w_0\|_{H^1(D_R^\alpha, \psi)}^2
\]
\[
\geq \min\left\{ \frac{\nu}{2}, \alpha \right\} \|w - w_0\|^2_{H^1(D_R^\alpha, \psi)} - \nu \psi(0) - \beta \psi(0).
\]

So the proposition is proved if this is positive. This is the case if
\[
\eta \beta^2 - (\nu \gamma + \beta) \psi(0) > 0.
\]

Exploiting that
\[
\frac{\psi(0)}{\psi(-x_0)} = \exp \left( - \int_{-x_0}^0 \int_{\Omega} k_1(x, y) \, dx \, dy \right)
\]
we arrive at the condition
\[
\frac{\eta}{\nu \gamma + \beta} \left( \frac{\alpha}{4} \right)^{\frac{1}{\beta}} > \exp \left( - \int_{-x_0}^0 \int_{\Omega} k_1(x, y) \, dx \, dy \right)
\]
\[
\max \left\{ \frac{\eta}{\nu \gamma + \beta} \left( \frac{\alpha}{4} \right)^{\frac{1}{\beta}}, \left[ \left( \frac{\alpha}{4}, m \right) C_2(\partial \Omega) \right]^{\frac{1}{\beta}} \right\}
\]
\[
\left\| \exp \left( - \int_{-x_0}^{x_1} k_1(\zeta, y) \, d\zeta \right) \right\|_{L^\infty(D_{\delta}^0, \psi)} \left( \int_{D_{\delta}^0, \psi} \exp \left( \frac{1}{2} \int_{-x_0}^{x_1} k_1(\zeta, y) \, d\zeta \right) \, dx \, dy \right)^{\frac{\beta - 2}{\beta}} \right\|
\]
\[
\frac{\beta - 2}{\beta}
\]

From Proposition 4.1 we get for any \( R < -x_0 - 1 \) existence of a local minimizer
\( w_R \in H^1_b(D_R^\alpha, \psi) \) of the functional \( J_{D_R^\alpha} \) such that \( \|w_R - w_0\|_{H^1(D_R^\alpha, \psi)} \leq \delta \). From this it follows that \( w_R \) is a weak solution of
\[
\begin{cases}
-\Delta w_R + k \cdot \nabla w_R = f(w_R) & \text{in } D_{\delta}^\alpha, \\
w_R(a, y) = 1 & \text{for almost all } y \in \Omega, \\
w_R(R, y) = 0 & \text{for almost all } y \in \Omega, \\
\frac{\partial w_R}{\partial n} = 0 & \text{on } \partial D \cap D_{\delta}^\alpha.
\end{cases}
\]

Using the maximum principle we conclude that for all \( R < -x_0 - 1 : 0 \leq w_R \leq 1 \) in \( D_{\delta}^\alpha \). From this we construct a supersolution to
\[
\begin{cases}
\partial_t u - \Delta u + k \cdot \nabla u = f(u) & \text{for all } (t, x) \in \mathbb{R} \times D \\
u(t, x) - \phi(x + ct) \to 0 & \text{as } t \to -\infty \text{ uniformly in } D
\end{cases}
\]
by passing to the limit \( R \to -\infty \) and extending by 1 onto \( D_{\delta}^\alpha \).
Proposition 4.3 (Existence of a stationary supersolution). Assume that condition (1.2) holds and let \(w_R\) be the local minimizer of the energy functional \(J_{D_R}\) as in Proposition 4.1, then \((w_R)_{R < x_0 - 1}\) converges up to a subsequence in \(C^2_{\text{loc}}(D_{a - \infty})\) to a solution \(w_\infty\) of

\[
\begin{align*}
-\Delta w_\infty + k \cdot \nabla w_\infty &= f(w_\infty) & \text{in } D_a, \\
w(a, y) &= 1 & \text{for almost all } y \in \Omega,
\end{align*}
\]

such that \(w_\infty(x) \to 0\) as \(x_1 \to -\infty\).

Proof. As \(0 \leq w_R \leq 1\) for all \(R > -x_0 - 1\) and using Schauder estimates there exists a subsequence \((R_n)_{n \in \mathbb{N}}\) with \(R_n \to -\infty\) as \(n \to \infty\) such that \(w_{R_n} \to w_\infty\) in \(C^2_{\text{loc}}(D_{a - \infty})\) as \(n \to \infty\). It remains to prove that the limit \(w_\infty\) satisfies \(w_\infty \to 0\) as \(x_1 \to -\infty\). By Fatou’s Lemma we find

\[
\|w_0 - w_\infty\|_{L^1(D_{a - \infty}, \psi)} \leq \liminf_{n \to \infty} \|w_0 - w_{R_n}\chi\{R_n < x_1 < a\}\|_{L^1(D_{a - \infty}, \psi)} \leq \delta^2.
\]

Then arguing by contradiction, we assume that there exists \(\eta > 0\) and a sequence \((x_n)_{n \in \mathbb{N}}\), such that \((x_n) \to -\infty\) as \(n \to \infty\) and \(w_\infty(x_n) > \eta\) for all \(n \in \mathbb{N}\). Since \(w_\infty \in C^2_{\text{loc}}\) and \(w_\infty\) is a bounded solution of (4.5), by standard parabolic estimates we know that \(|\nabla w_\infty| \leq C\) for some constant \(C > 0\). It follows that for all \(x \in B_{\frac{a}{2n}}(x_n)\)

\[
|w_\infty(x) - w_\infty(x_n)| \leq \max_{x \in B_{\frac{a}{2n}}(x_n)} |\nabla w_\infty||x - x_n|
\]

Hence \(w_\infty(x) \geq \frac{\eta}{2}\) for all \(x \in B_{\frac{a}{2n}}(x_n)\) and all \(n \in \mathbb{N}\). This yields that

\[
\|w_\infty - w_0\|_{L^2(D_{a - \infty}, \psi)}^2 \geq \psi(-x_0) \sum_{I} \left(\frac{n}{2}\right)^2 \frac{\eta}{C} = \infty,
\]

where \(I := \{i \in \mathbb{N} : |x_i - x_{i-1}| > \frac{a}{2n}\}\) for all \(\mathbb{N} \setminus \{i\}\) and \((x_i)_{i < -x_0}\) and obviously \(|I| = \infty\). But this is a contradiction to (4.6) and thereby the Proposition is proved. \(\square\)

The proof of Theorem 1.1 is now nothing but applying Proposition 4.3 and a comparison principle.

Proof of Theorem 1.1. Let us now take \(w_\infty\) as in Proposition 4.3 and let us extend \(w_\infty\) by 1 onto all of \(R\). We set

\[
\tilde{w}_\infty(x) := \begin{cases} 
  w_\infty(x), & \text{if } x_1 \leq a \\
  1, & \text{else}.
\end{cases}
\]

Thus \(\tilde{w}_\infty(x)\) is a supersolution of the parabolic problem

\[
\partial_t u - \Delta u + k(x) \cdot \nabla u = f(u) \text{ for } (t, x) \in \mathbb{R} \times D,
\]

\[
\frac{\partial u}{\partial \nu} = 0 \text{ on } \mathbb{R} \times \partial D.
\]

Furthermore it holds that

\[
\lim_{t \to -\infty} \inf_{x \in D} (\tilde{w}_\infty(x) - \phi(x_1 + ct)) \geq 0.
\]

Indeed

for \(x_1 \geq a\), for all \(t \in \mathbb{R} : \tilde{w}_\infty(x) - \phi(x_1 + ct) = 1 - \phi(x_1 + ct) \geq 0\)

for \(x_1 < a\), for all \(t \in \mathbb{R} : \tilde{w}_\infty(x) - \phi(x_1 + ct) \geq \tilde{w}_\infty(x) - \phi(a + ct) \to \tilde{w}_\infty(x) \geq 0\)

uniformly in \(D_{a - \infty}\) as \(t \to -\infty\).
Using the generalized maximum principle (Lemma 3.2 in [1]), we conclude that
\[ u(t, x) \leq \bar{w}_\infty(x) \] for all \( t \in \mathbb{R} \) and \( x \in D \).
Hence the stationary supersolution \( \bar{w}_\infty \) blocks the invasion of the stationary state 1 into the right.

\[ \square \]

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