TANGENT BUNDLE OF $\mathbb{P}^2$ AND MORPHISM FROM $\mathbb{P}^2$ TO $\text{Gr}(2, \mathbb{C}^4)$

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Abstract. In this note we study the image of $\mathbb{P}^2$ in $\text{Gr}(2, \mathbb{C}^4)$ given by tangent bundle of $\mathbb{P}^2$. We show that there is component $\mathcal{H}$ of the Hibert scheme of surfaces in $\text{Gr}(2, \mathbb{C}^4)$ with no point of it corresponds to a smooth surface.

Keywords: Projective plane; Tangent bundle; Morphisms; Grassmannian.

1. Introduction

Let $\mathbb{P}^2$ denote the projective plane over the field of complex numbers $\mathbb{C}$ and $\text{Gr}(2, \mathbb{C}^4)$ Grassman variety of two dimensional quotients of the vector space $\mathbb{C}^4$.

The aim of this paper is to study the image of $\mathbb{P}^2$ by non constant morphisms $\mathbb{P}^2 \to \text{Gr}(2, \mathbb{C}^4)$ obtained by tangent bundle $T_{\mathbb{P}^2}$ of $\mathbb{P}^2$. The bundle $T_{\mathbb{P}^2}$ is generated by sections and hence it is generated by four (= rank($T_{\mathbb{P}^2}$) + dim($\mathbb{P}^2$)) independent global sections. Any set $S$ of four independent generating sections of $T_{\mathbb{P}^2}$ defines a morphism $\phi_S : \mathbb{P}^2 \to \text{Gr}(2, \mathbb{C}^4)$, such that the $\phi_S^*(Q) = T_{\mathbb{P}^2}$, where $Q$ is the universal rank two quotient bundle on $\text{Gr}(2, \mathbb{C}^4)$:

$$\mathcal{O}^4_{\text{Gr}(2, \mathbb{C}^4)} \to Q \to 0.$$

According to a result of Tango [4], if $\phi : \mathbb{P}^2 \to \text{Gr}(2, \mathbb{C}^4)$ is an imbedding then the pair of Chern classes $(c_1(\phi^*(Q)), c_2(\phi^*(Q)))$ is equal to one of the following pairs: $(H, 0), (H, H^2), (2H, H^2)$ or $(2H, 3H^2)$, where $H$ is the ample generator of $H^2(\mathbb{P}^2, \mathbb{Z})$.

Since $c_1(T_{\mathbb{P}^2}) = 3H$ and $c_2(T_{\mathbb{P}^2}) = 3H^2$, the morphism $\phi_S$ defined by a set $S$ of four independent generating sections of $T_{\mathbb{P}^2}$ is not an imbedding. Thus, it is natural to ask, does there exists a set $S$ of four independent generating sections of $T_{\mathbb{P}^2}$ for which $\phi_S$ is generically injective? In this direction we have the following (Theorem 3.6):

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Theorem 1.1. For general choice of an ordered set \( S \) of four independent generating sections of \( T_{\mathbb{P}^2} \) the morphism

\[
\phi_S : \mathbb{P}^2 \to \text{Gr}(2, \mathbb{C}^4)
\]

is generically injective.

We also show that in fact one can find an ordered set of generators \( S \) of \( T_{\mathbb{P}^2} \) the morphism is an immersion i.e., the morphism induces an injection on all the tangent spaces.

As by product of our result we obtain the following (Theorem 4.2):

Theorem 1.2. There is an irreducible component \( \mathcal{H} \) of the Hilbert scheme of surfaces in \( \text{Gr}(2, \mathbb{C}^4) \) no point which corresponds to a smooth surface.

2. The Tangent bundle of \( \mathbb{P}^2 \).

The tangent bundle of \( \mathbb{P}^2 \) fits in an exact sequence called the “Euler sequence”:

\[
(1) \quad 0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2} \to 0.
\]

This exact sequence together with the fact \( H^1(\mathcal{O}_{\mathbb{P}^n}) = 0 \), implies that \( \dim H^0(T_{\mathbb{P}^2}) = 8 \), where \( H^i \) denotes the \( i \) th sheaf cohomology group. Since the rank two bundle \( T_{\mathbb{P}^2} \) on \( \mathbb{P}^2 \) is ample and generated by sections, a minimal generating set of independent sections has cardinality four. Any set \( S \) of four independent generators of \( T_{\mathbb{P}^2} \) gives to an exact sequence:

\[
0 \to E_S \to \mathcal{O}_{\mathbb{P}^2}^4 \to T_{\mathbb{P}^2} \to 0.
\]

This in turn corresponds to a morphism \( \phi_S : \mathbb{P}^2 \to \text{Gr}(2, \mathbb{C}^4) \), where \( \phi_S(x) = \{ \mathbb{C}^4 = \mathcal{O}_{\mathbb{P}^2}^4|_x \to T_{\mathbb{P}^2}|_x \to 0 \} \).

3. The main result

Note that in the “Euler sequence” \((1)\) the injective map

\[
0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3
\]

is given by the section \( v = (X, Y, Z) \), where \( X, Y, Z \) is the standard basis of \( H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \). For a section \( v_i \) of \( \mathcal{O}_{\mathbb{P}^2}(1)^3 \) we denote \( w_i \) the image section of \( T_{\mathbb{P}^2} \) under the surjection

\[
\mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2} \to 0
\]

in \((1)\). Let \( \tilde{S} = (v_1, v_2, v_3, v_4) \) be an ordered set of four linearly independent sections of \( \mathcal{O}_{\mathbb{P}^2}(1)^3 \) and \( S = (w_1, w_2, w_3, w_4) \) be the corresponding
ordered set of sections of $T_{P^2}$. Clearly, the set $S$ is generating set of independent sections of $T_{P^2}$ if and only if $\tilde{S} \cup \{v\}$ is a generating set of independent sections of $O_{P^2}(1)^3$.

**Lemma 3.1.** Let $v_1 = (X, 0, 0), v_2 = (0, Y, 0), v_3 = (Y, Z, X), v_4 = (Z, X, Y)$ be four sections of $O_{P^2}(1)^3$ and $\tilde{S} = (v_i | 1 \leq i \leq 4)$. Then the ordered set $\tilde{S}$ with \{v = (X, Y, Z)\} is a generating set of independent sections of $O_{P^2}(1)^3$. Hence the corresponding ordered set of sections $S = (w_1, w_2, w_3, w_4)$ generate $T_{P^2}$, where $w_i$ is the image of $v_i$ under the map given in exact sequence (1).

**Proof:** Clearly, $v$ generate a subspace of $O_{P^2}(1)^3$ dimension one at every point of $P^2$. Hence $w_i, w_j$ is not independent at a point $p \in P^2$ if and only if the section $v_{ij} = v_i \wedge v_j \wedge v$ of $O_{P^2}(3)$ vanishes at $p$. Thus, if the six independent sections $\{v_i \wedge v_j \wedge v | 1 \leq i < j \leq 4\}$ has no common zero implies $\tilde{S}$ with $\{v = (X, Y, Z)\}$ is a generating set of independent sections of $O_{P^2}(1)^3$. Note that $v_{12} = XYZ, v_{13} = X(Z^2 - XY), v_{14} = X(XZ - Y^2), v_{23} = Y(X^2 - YZ), v_{24} = Y(YX - Z^2), v_{34} = 3XYZ - (X^3 + Y^3 + Z^3)$. It is easy to see that the set $\{v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}\}$ of sections of $O_{P^2}(3)$ has no common zero in $P^2$ and hence the ordered set $\tilde{S}$ with $\{v = (X, Y, Z)\}$ is a generating set of independent sections of $O_{P^2}(1)^3$. Hence the corresponding ordered set of sections $S = (w_1, w_2, w_3, w_4)$ generate $T_{P^2}$. 

**Lemma 3.2.** Let $f : P^2 \to P^n$ be a non constant morphism and $f^*(O_{P^n}(1)) = O_{P^2}(m)$. Assume that there exists a linear subspace $W$ of codimension two such that $W \cap f(P^2)$ consists of exactly $m^2$ points. Then the morphism $f$ is generically injective.

**Proof:** Note as the morphism $f$ is non constant $f^*(O_{P^n}(1)) = O_{P^2}(m)$ with $m > 0$ and hence is ample. This means $f$ is finite map. Set $r = \deg(f)$, the number of elements $f^{-1}(f(x))$ for a general $x \in P^2$. If $d$ to be the degree of $f(P^2)$ in $P^n$ then it is easy to see that $m^2 = d.r$. On the other hand the assumption, $W \cap f(P^2)$ consists of exactly $m^2$ points, implies $d \geq m^2$. Thus we must have $d = m^2$ and $r = \deg(f) = 1$. Thus $f$ is generically injective.

**Lemma 3.3.** With the notations of Lemma 3.1, the surjection of vector bundles on $P^2$

$$O_{P^2}^1 \to T_{P^2}$$

given by $S$ defines a generically injective morphism

$$\phi_S : P^2 \to Gr(2, C^4).$$
Proof: Let $p : \text{Gr}(2, \mathbb{C}^4) \to \mathbb{P}^5$ be the Plucker imbedding given by the determinant of the universal quotient bundle. Then $p \circ \phi_S$ is given by
\[(x; y; z) \mapsto (xyz; x(z^2 - xy); x(xz - y^2); y(x^2 - yz); y(xy - z^2); 3xyz - (x^3 + y^3 + z^3)).\]
To prove the map $\phi_S$ is generically injective it is enough to prove the map $p \circ \phi_S$ is so. Set $(Z_0, \ldots, Z_5)$ as the homogeneous coordinates of $\mathbb{P}^5$ and $W$ be the codimension two subspace of $\mathbb{P}^5$ defined by $Z_0 = 0 = Z_5$. Then $W \cap p \circ \phi_S(\mathbb{P}^2)$ is equal to
\[\{(0, -\omega^i, 1, 0, 0, 0); (0, 0, 0, -\omega^i, 1, 0); (0, \omega^i, 1, \omega^i, 1, 0)|1 \leq i \leq 3\},\]
where $\omega$ is a primitive cube root of unity. Note that
\[(p \circ \phi_S)^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^2}(3).\]
Hence, the required result follows from Lemma 3.2.

Remark 3.4. We show (see Lemma 4.7) that $p \circ \phi_S$ is an immersion, i.e., the induced linear map on the tangent space at every point of $\mathbb{P}^2$ is injective and one to one except finitely many points.

Next we recall the following [See, Lemma(3.13)[1]]:

Lemma 3.5. Let $X$ and $Y$ be two irreducible projective varieties. Let $T$ be an irreducible quasi-projective variety and $t_0 \in T$ be a point. Let
\[F : X \times T \to Y\]
be a morphism. Assume that $F_t := F|_{X \times t} : X \to Y$ is finite for all $t \in T$ and $F_{t_0}$ is a birational onto its image. Then there is an open subvariety $U$ of $T$ such that $t_0 \in U$ and for $t \in U$ the morphism $F_t$ is birational onto its image.

Proof: For the sake of completeness we reproduce the proof here. Consider the morphism $G = F \times 1_{d_T} : X \times T \to Y \times T$. Then the assumption $F_t$ is finite implies the morphism $G$ is finite and proper. Hence $G = G_*\mathcal{O}_{X \times T}$ is coherent sheaf of $\mathcal{O}_{Y \times T}$ modules. Let $Z \subset Y \times T$ be the subvariety on which the sheaf $G_*\mathcal{O}_{X \times S}$ is supported. Then clearly the map $p : Z \to T$, restriction of the natural projection, is surjective. The section 1 $\in \mathcal{O}_{X \times T}$ gives an inclusion of $\mathcal{O}_Z$ in $G$. Let $\mathcal{F} = G/\mathcal{O}_Z$. Let $Z_1 \subset Y \times T$ be the subvariety on which the sheaf $\mathcal{F}$ supported. Let $q : Z_1 \to T$ be the natural projection and let $U = \{ t \in T | \dim q^{-1}(t) < \dim(X) \}$ then we see that by semi continuity [See, page 95, Exercise (3.22)[3]], $U$ is an open subset and is non-empty as $t_0 \in U$. For $t \in U$ the morphism $F_t$ is an isomorphism on $X \times t - G^{-1}(q^{-1}(t)).$ Since $G$ is finite $G^{-1}(q^{-1}(t))$ is proper closed subset.
of $X \times t$ and hence the morphism $F_t$ is birational onto its image. This proves the Lemma.

\[\square\]

**Theorem 3.6.** For a generic choice of an ordered set $S$ of four independent generating sections of $T_{\mathbb{P}^2}$ the morphism

$$\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$$

is generically injective.

**Proof:** It is easy to see that the ordered set of four sections $S$ generating $T_{\mathbb{P}^2}$ is an irreducible quasi projective variety. In fact it is an open subvariety of the affine space $V^4$, where $V = H^0(T_{\mathbb{P}^2})$. The theorem at once follows from Lemma(3.5), if we show the existence of one $S$ for which $\phi_S$ is generically injective. But the existence of one such $S$ follows from Lemma(3.3).

\[\square\]

4. An example

The result of the previous section can be used give an example of a component of a Hilbert Scheme of $Gr(2, \mathbb{C}^4)$ without any point corresponding to a smooth surface.

**Lemma 4.1.** The morphism $p \circ \phi_S : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ of Lemma(3.2) is an immersion i.e., the induced linear map on the tangent space at every point of $\mathbb{P}^2$ is injective. Moreover, $p \circ \phi_S$ one to one except

$$S_1 = \{(1; 0; 0), (0; 1; 0), (0; 0; 1)\} \mapsto (0; 0; 0; 0; 0; 1)$$

and

$$S_2 = \{(1; 1; 1), (\omega; \omega^2; 1), (\omega^2; \omega; 1)\} \mapsto (1; 0; 0; 0; 0; 0),$$

where $\omega$ is a primitive cube root of unity.

**Proof:** Let $X, Y, Z$ be the homogeneous coordinates functions on $\mathbb{P}^2$ and $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$ be the homogeneous coordinates functions on $\mathbb{P}^5$. Clearly under the morphism $p \circ \phi_S : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ the set $S_1$ maps to $(0; 0; 0; 0; 0; 1)$ and the set $S_2$ maps to $(1; 0; 0; 0; 0; 0)$. Note that the lines $X = 0, Y = 0, Z = 0$ mapped to nodal cubics $Z_0 = Z_1 = Z_2 = Z_3^2 + Z_4^3 - z_3 z_4 z_5 = 0, Z_0 = Z_3 = Z_4 = Z_1^3 + Z_3^3 + Z_1 z_2 z_5 = 0,$ and $Z_0 = Z_1 - Z_4 = Z_2 + Z_3 = Z_1^3 + Z_2^3 - Z_1 z_2 z_5 = 0$ respectively. Thus we can conclude that the morphism $p \circ \phi_S$ is an immersion on these three lines. On the complement of these lines the morphism $p \circ \phi_S$ can be described as

$$(x, y) \mapsto \left(1/y - x, x/y - y, x - y/x, y - 1/x, 3 - x^2/y - y^2/x - 1/xy\right)$$
from $\mathbb{C}^2 - \{xy = 0\} \to \mathbb{C}^4$. If $(x, y)$ and $(x_1, y_1)$ maps to the same point then we get the following equations:

(2) \quad 1/y - x = 1/y_1 - x_1

(3) \quad x/y - y = x_1/y_1 - y_1

(4) \quad y/x - x = y_1/x_1 - x_1

(5) \quad 1/x - y = 1/x_1 - y_1.

The equations (2) and (5) gives us

\[ \frac{xy - 1}{y} = \frac{x_1 y_1 - 1}{y_1} ; \quad \frac{xy - 1}{x} = \frac{x_1 y_1 - 1}{x_1}. \]

Since $xy \neq 0$ and $x_1 y_1 \neq 0$ we see that either $xy - 1 \neq 0$ and $(x, y) = (x_1, y_1)$ or $xy - 1 = 0 = x_1 y_1 - 1$. Hence we get $(p \circ \phi_S)$ is one to one outside the set $\{(1, 1), (\omega, \omega^2), (\omega^2, \omega)\}$ and this is mapped to $(0, 0, 0, 0, 0)$. The assertion about the immersion of the given morphism $\mathbb{C}^2 - \{xy = 0\} \to \mathbb{C}^4$

can be checked by looking at the two by two minors of the below jacobian matrix of the morphism:

\[
\begin{pmatrix}
-1 & 1/y & 1 + y/x^2 & 1/x^2 & -2x/y + y^2/x^2 + 1/x^2y \\
-1/y^2 & -x/y^2 - 1 & 1/x & x^2/y^2 - 2y/x + 1/xy^2
\end{pmatrix}
\]

\[ \square \]

**Theorem 4.2.** Let $\mathcal{H}$ be the irreducible component of the Hilbert scheme of $\text{Gr}(2, \mathbb{C}^4)$ containing the point corresponding to the image surface of the morphism $\phi_S : \mathbb{P}^2 \to \text{Gr}(2, \mathbb{C}^4)$ of 3.3 Then no point of $\mathcal{H}$ corresponds to a smooth surface.

**Proof:** Let $p : \text{Gr}(2, \mathbb{C}^4) \to \mathbb{P}^5$ be the Plucker imbedding. Since $(p \circ \phi_S)^* (\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^2}(3)$ and by Lemma 4.1 the morphism $p \circ \phi_S$ is an imbedding outside finite set of points. Moreover, general hyperplane section of $(p \circ \phi_S)(\mathbb{P}^2)$ in $\mathbb{P}^5$ is smooth curve of genus one. If a point of the irreducible $\mathcal{H}$ of corresponds to a smooth surface $Y$ then it has to have the same cohomology class as that of $(p \circ \phi_S)(\mathbb{P}^2)$ namely $(3, 6) \in H^4(\text{Gr}(2, \mathbb{C}^4), \mathbb{Z})$. Also, the general hyperplane section of $p(Y)$ has to be a smooth curve of genus one. But according to the classification of smooth surfaces of type $(3, 6)$ in $\text{Gr}(2, \mathbb{C}^4)$ (see, [2, Theorem 4.2]) implies that there are no smooth surface of type $(3, 6)$ with hyperplane section a smooth curve of genus one. This contradiction proves that no point of $\mathcal{H}$ corresponds to a smooth surface. \[ \square \]
Remark 4.3. The component $\mathcal{H}$ of the Hilbert Scheme in Theorem 4.2 is reduced irreducible of dimension 23. In fact computing the normal sheaf associated to the morphism $\phi_S$ of Lemma 3.3 and counting the dimension of space of all such morphisms we see that $\mathcal{H}$ is a reduced irreducible of dimension 23.

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