ON ELLIPTIC SOLUTIONS OF THE ASSOCIATIVE YANG–BAXTER EQUATION

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Abstract. We give a direct proof of the fact that elliptic solutions of the associative Yang–Baxter equation arise from an appropriate spherical order on an elliptic curve.

1. Introduction

Let \( \mathfrak{A} = \text{Mat}_n(\mathbb{C}) \) be the algebra of square matrices of size \( n \in \mathbb{N} \) and \( (\mathbb{C}^3,0) \to \mathfrak{A} \otimes \mathfrak{A} \) be the germ of a meromorphic function. The following version of the associative Yang–Baxter equation (AYBE) with spectral parameters was introduced by Polishchuk in [9]:

\[
(1) \quad r(u; x_1, x_2)^{12} r(u + v; x_2, x_3)^{23} = r(u + v; x_1, x_3)^{13} r(-v; x_1, x_2)^{12} + r(v; x_2, x_3)^{23} r(u; x_1, x_3)^{13}.
\]

The upper indices in this equation indicate the corresponding embeddings of \( \mathfrak{A} \otimes \mathfrak{A} \) into \( \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} \). For example, the germ \( r^{13} \) is defined as

\[
r^{13} : \mathbb{C}^3 \to \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\sigma} \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A},
\]

where \( r^{13}(x \otimes y) = x \otimes 1 \otimes y \). Two other germs \( r^{12} \) and \( r^{23} \) are defined in a similar way. We are interested in those solutions of AYBE, which are non-degenerate, skew-symmetric (meaning that \( r(v; x_1, x_2) = -r^{21}(-v; x_2, x_1) \)) and which admit a Laurent expansion of the form

\[
(2) \quad r(v; x_1, x_2) = \frac{1}{v} + r_0(x_1, x_2) + vr_1(x_1, x_2) + v^2 r_2(x_1, x_2) + \ldots
\]

All elliptic and trigonometric solutions of AYBE satisfying (2) were classified in [9, 10].

Recall the description of elliptic solutions of AYBE. Let \( \varepsilon = \exp\left(\frac{2\pi i d}{n}\right) \), where \( 0 < d < n \) is such that \( \text{gcd}(d, n) = 1 \). We put

\[
X = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \varepsilon & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \varepsilon^{n-1}
\end{pmatrix}
\]

and

\[
Y = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{pmatrix}.
\]

For any \( (k, l) \in I := \{1, \ldots, n\} \times \{1, \ldots, n\} \) denote \( Z_{(k,l)} = Y^k X^{-l} \) and \( Z_{(k,l)}^\vee = \frac{1}{n} X^l Y^{-k} \).

Then the following expression

\[
(4) \quad r_{(n,d)}(v; x_1, x_2) = \sum_{(k,l) \in I} \exp\left(\frac{2\pi i d}{n} k x\right) \sigma\left(v + \frac{d}{n} (k \tau + l), x\right) Z_{(k,l)}^\vee \otimes Z_{(k,l)}
\]

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is a solution of AYBE satisfying (2), where \( x = x_2 - x_1 \) and

\[
\sigma(a, z) = 2\pi i \sum_{n \in \mathbb{Z}} \frac{\exp(-2\pi inz)}{1 - \exp(-2\pi i(a - 2\pi in\tau))}
\]

is the Kronecker elliptic function \([12]\) for \( \tau \in \mathbb{C} \) such that \( \text{Im}(\tau) > 0 \). See also [8, Section III] for a direct proof of this fact.

In his recent work [11] Polishchuk showed that non-degenerate skew-symmetric solutions of AYBE satisfying (2) can be obtained from appropriate triple Massey products in the perfect derived category of coherent sheaves \( \text{Perf}(E) \) on a non-commutative projective curve \( E = (E, A) \), where \( E \) is an irreducible projective curve over \( \mathbb{C} \) of arithmetic genus one and \( A \) is a symmetric spherical order on \( E \). A simplest example of such an order is given by \( A = \text{End}_E(F) \), where \( F \) is a simple vector bundle on \( E \). Let \( E = E_\tau := \mathbb{C}/(1, \tau) \) be the elliptic curve determined by \( \tau \in \mathbb{C} \) and \( F \) be a simple vector bundle of rank \( n \) and degree \( d \) on \( E \). It follows from results of Atiyah [1] that such \( F \) exists and the sheaf of algebras \( A = A_{(n,d)} := \text{End}_E(F) \) does not depend on the choice of \( F \). We show that the solution of AYBE arising from the non-commutative projective curve \( (E_\tau, A_{(n,d)}) \) is given by the formula (4). In [9, Section 2], the corresponding computations were performed using the homological mirror symmetry and explicit formulae for triple Massey products in the Fukaya category of a torus. The expression for the resulted solution of AYBE (see [9, formula (2.3)]) was different from (4). Our computations are straightforward and based by techniques developed in the articles [6, 5].

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2. Symmetric spherical orders on curves of genus one and AYBE

In this section we make a brief review of Polishchuk’s construction [11]. Let \( E \) be an irreducible projective curve over \( \mathbb{C} \) of arithmetic genus one, \( \tilde{E} \) its smooth part, \( O \) its structure sheaf, \( K \) the sheaf of rational functions on \( E \) and \( \Omega \) the sheaf of regular differential one-forms on \( E \). There exists a regular differential one-form \( \omega \in \Gamma(E, \Omega) \) such that \( \Gamma(E, \Omega) = \mathbb{C} \omega \). Such \( \omega \) also defines an isomorphism \( O \cong \Omega \). If \( E \) is singular then it is rational. In this case, let \( \mathbb{P}^1 \xrightarrow{\nu} E \) be the normalization morphism and \( \tilde{O} = \nu_*(O_{\mathbb{P}^1}) \).

Let \( A \) be a sheaf of orders on \( E \). By definition, \( A \) is a torsion free coherent sheaf of \( O \)-algebras on \( E \) such that \( A \otimes_O K \cong \text{Mat}_n(K) \) for some \( n \in \mathbb{N} \). For any order \( A \) we have the canonical trace morphism \( A \xrightarrow{t} \tilde{O} \), which coincides with the restriction of the trace morphism \( A \hookrightarrow A \otimes_O K \cong \text{Mat}_n(K) \xrightarrow{t} K \) (if \( E \) is smooth then \( \tilde{O} = O \)). Following [11], the order \( A \) is called symmetric spherical if the following conditions are fulfilled:

- The image of the trace morphism \( t \) is \( O \) and the induced morphism of coherent sheaves \( A \xrightarrow{t} A' := \text{Hom}_E(A, O) \) is an isomorphism.
- We have: \( \Gamma(E, A) \cong \mathbb{C} \).

Consider the non-commutative projective curve \( E = (E, A) \). Let \( \text{Coh}(E) \) be the category of coherent sheaves on \( E \) (these are sheaves of \( A \)-modules which are coherent as \( O \)-modules)
and $\text{Perf}(\mathcal{E})$ be the corresponding perfect derived category. Recall that $\Omega_\mathcal{E} := \text{Hom}_\mathcal{E}(A, \Omega)$ is a dualising bimodule of $\mathcal{E}$. If $A$ is symmetric then $\Omega_\mathcal{E} \cong A$ as $A$-bimodules and $\text{Perf}(\mathcal{E})$ is a triangulated 1-Calabi–Yau category. The last assertion means that for any pair of objects $\mathcal{G}^*, \mathcal{H}^*$ in $\text{Perf}(\mathcal{E})$ there is an isomorphism of vector spaces

$$(6) \quad \text{Hom}_\mathcal{E}(\mathcal{G}^*, \mathcal{H}^*) \cong \text{Hom}_\mathcal{E}(\mathcal{H}^*, \mathcal{G}^*[1])^*$$

which is functorial in both arguments.

Let $P = \text{Pic}^0(\mathcal{E})$ be the Jacobian of $\mathcal{E}$ and $\mathcal{L} \in \text{Pic}(P \times \mathcal{E})$ be a universal line bundle. For any $v \in P$, let $\mathcal{L}^v := \mathcal{L}|_{\{v\} \times \mathcal{E}} \in \text{Pic}^0(\mathcal{E})$ and $\mathcal{A}^v := A \otimes_\mathcal{O} \mathcal{L}^v \in \text{Coh}(\mathcal{E})$.

**Lemma 2.1.** The coherent sheaf $A$ is semi-stable of slope zero. Moreover,

$$(7) \quad \Gamma(\mathcal{E}, \mathcal{A}^v) = 0 = H^1(\mathcal{E}, \mathcal{A}^v)$$

for all but finitely many points $v \in P$.

**Proof.** Let $\mathcal{B}$ be the kernel of the trace morphism $\mathcal{A} \xrightarrow{\iota} \mathcal{O}$. It follows from the long exact cohomology sequence of $0 \to \mathcal{B} \to \mathcal{A} \xrightarrow{\iota} \mathcal{O} \to 0$ that $H^0(\mathcal{E}, \mathcal{B}) = 0 = H^1(\mathcal{E}, \mathcal{B})$. Hence, $\mathcal{B}$ is a semi-stable coherent sheaf on $\mathcal{E}$ of slope zero and $\mathcal{A} \cong \mathcal{B} \oplus \mathcal{O}$. It follows that $A$ is semi-stable, too. The latter fact also implies the vanishing $\Gamma(\mathcal{E}, \mathcal{A}^v) = 0 = H^1(\mathcal{E}, \mathcal{A}^v)$ for all but finitely many $v \in P$. \hfill \Box

**Corollary 2.2.** There exists a proper closed subset $D \subset P \times P$ such that

$$(8) \quad \text{Hom}_\mathcal{E}(\mathcal{A}^{v_1}, \mathcal{A}^{v_2}) = 0 = \text{Ext}_\mathcal{E}^1(\mathcal{A}^{v_1}, \mathcal{A}^{v_2})$$

for all $v_1, v_2 \in (P \times P) \setminus D$.

**Proof.** This statement follows from the isomorphisms

$$(9) \quad \text{Ext}_\mathcal{E}^i(\mathcal{A}^{v_1}, \mathcal{A}^{v_2}) \cong H^i(\mathcal{E}, \text{End}_\mathcal{E}(\mathcal{A}^{v_1}, \mathcal{A}^{v_2})) \cong H^i(\mathcal{E}, \mathcal{A}^{v_2-v_1}), \quad i = 0, 1$$

and the vanishing (7). \hfill \Box

Recall that for any $x, y \in \tilde{\mathcal{E}}$ we have the following standard short exact sequences

$$(10) \quad 0 \to \Omega \to \Omega(x) \xrightarrow{\text{res}_x} \mathbb{C}_x \to 0 \quad \text{and} \quad 0 \to \mathcal{O}(-y) \to \mathcal{O} \xrightarrow{\text{ev}_y} \mathbb{C}_y \to 0,$$

where $\text{res}_x$ and $\text{ev}_y$ are the residue and evaluation morphisms, respectively. Using the isomorphism $\mathcal{O} \xrightarrow{\omega} \Omega$, we can rewrite the first short exact sequence as

$$(11) \quad 0 \to \mathcal{O} \to \mathcal{O}(x) \xrightarrow{\text{res}_x} \mathbb{C}_x \to 0.$$ 

For any $\mathcal{H} \in \text{Coh}(\mathcal{E})$ denote $\mathcal{H}|_x := \mathcal{H} \otimes_\mathcal{O} \mathbb{C}_x \in \text{Coh}(\mathcal{E})$. Tensoring (11) by $\mathcal{A}^v$ (where $v \in P$ is an arbitrary point), we get the following short exact sequence in $\text{Coh}(\mathcal{E})$:

$$0 \to \mathcal{A}^v \to \mathcal{A}^v(x) \xrightarrow{\text{res}_x} \mathcal{A}^v|_x \to 0.$$ 

Next, for any $(u, v) \in P \times P$ we have the induced long exact sequence of vector spaces

$$0 \to \text{Hom}_\mathcal{E}(\mathcal{A}^u, \mathcal{A}^v) \to \text{Hom}_\mathcal{E}(\mathcal{A}^u, \mathcal{A}^v(x)) \to \text{Hom}_\mathcal{E}(\mathcal{A}^u|_x, \mathcal{A}^v|_x) \to \text{Ext}_\mathcal{E}(\mathcal{A}^u, \mathcal{A}^v).$$

It follows from (7) that the linear map

$$(12) \quad \text{Hom}_\mathcal{E}(\mathcal{A}^u, \mathcal{A}^v(x)) \xrightarrow{\text{res}^A(u,v;x)} \text{Hom}_\mathcal{E}(\mathcal{A}^v|_x, \mathcal{A}^v|_x)$$
is an isomorphism if \((u, v) \in (P \times P) \setminus D\).

Similarly, for any \(v \in P\) and \(x \neq y \in \hat{E}\) we have the following short exact sequence

\[
0 \longrightarrow A^v(x - y) \longrightarrow A^v(x) \longrightarrow A^v(x) \mid_y \longrightarrow 0
\]

in Coh\((\hat{E})\), which is obtained by tensoring the evaluation sequence \((10)\) by \(A^v(x)\). After applying to it the functor \(\text{Hom}_E(A^u, -)\) and using a canonical isomorphism \(A^v \mid_y \cong A^v(x) \mid_y\), we obtain a linear map

\[
\text{Hom}_E(A^u, A^v(x)) \xrightarrow{ev^A(u, v; x, y)} \text{Hom}_E(A^u \mid_y, A^v \mid_y).
\]

Let \(\text{Hom}_E(A^u \mid_x, A^v \mid_x) \xrightarrow{\alpha(u, v; x, y)} \text{Hom}_E(A^u \mid_y, A^v \mid_y)\) be (the unique) linear map making the following diagram of vector spaces commutative. Let \(\gamma(u, v; x, y) \in \text{Hom}_E(A^u \mid_x, A^v \mid_x) \otimes \text{Hom}_E(A^u \mid_y, A^v \mid_y)\) be the image of \(\alpha(u, v; x, y)\) under the composition of the following canonical isomorphisms of vector spaces:

\[
\text{Lin}\left(\text{Hom}_E(A^u \mid_x, A^v \mid_x), \text{Hom}_E(A^u \mid_y, A^v \mid_y)\right) \cong \text{Hom}_E(A^u \mid_x, A^v \mid_x) \ast \text{Hom}_E(A^u \mid_y, A^v \mid_y)
\]

\[
\cong \text{Hom}_E(A^u \mid_x, A^v \mid_x) \otimes \text{Hom}_E(A^u \mid_y, A^v \mid_y),
\]

where the last isomorphism is induced by the trace morphism \(t\).

Let \(P \times P \xrightarrow{\eta} P\) be the group operation on \(P\) and \(o \in P\) be the corresponding neutral element (i.e. \(O \cong \mathcal{L}^o\)). Consider the canonical projections \(P \times P \xrightarrow{\pi} P \times E, (x_1, x_2; x) \mapsto (x_i, x)\) for \(i = 1, 2\) and \(P \times P \xrightarrow{\eta} P \times E, (x_1, x_2; x) \mapsto (x_1, x_2)\). Then there exists \(\mathcal{S} \in \text{Pic}(P \times P)\) such that

\[
(\eta \times 1)\ast \mathcal{L} \cong \pi_1^\ast \mathcal{L} \otimes \pi_2^\ast \mathcal{L} \otimes \pi_3^\ast \mathcal{S}.
\]

In particular, \(\mathcal{L}^{v_1} \otimes \mathcal{L}^{v_2} \cong \mathcal{L}^{v_1+v_2}\), where \(v_1 + v_2 = \eta(v_1, v_2)\).

For any type of \(E\) (elliptic, nodal or cuspidal) there exists a complex analytic covering map \((\mathbb{C}, +) \xrightarrow{\chi} (P, \eta)\), which is also a group homomorphism. In this way we get a local coordinate on \(P\) in a neighbourhood of \(o\). Next, we put: \(\mathcal{T} := (\chi \times 1)\ast \mathcal{L}\). Since any line bundle on \(\mathbb{C} \times \mathbb{C}\) is trivial, we get from \((15)\) an induced isomorphism

\[
(\tilde{\eta} \times 1)\ast \mathcal{T} \cong \pi_1^\ast \mathcal{T} \otimes \pi_2^\ast \mathcal{T},
\]

where \(\tilde{\eta}\) (respectively, \(\tilde{\pi}_i\)) is the composition of \(\eta\) (respectively, \(\pi_i\)) with \(\chi \times \chi\). It follows that we have isomorphisms

\[
\mathcal{O}_E \xrightarrow{\alpha} \mathcal{T} \big|_{0 \times E} \quad \text{and} \quad \mathcal{O}_{\mathbb{C} \times \mathbb{C} \times E} \xrightarrow{\beta} \tilde{\eta}^\ast \mathcal{T}' \otimes \tilde{\pi}_1^\ast \mathcal{T} \otimes \tilde{\pi}_2^\ast \mathcal{T}.
\]
Let $U \subset \tilde{E}$ be an open subset for which there exists an isomorphism of $\Gamma(U, \mathcal{O}_E)$-algebras
\begin{equation}
\Gamma(U, \mathcal{A}) \xrightarrow{\xi} \mathfrak{A} \otimes \mathcal{C} \Gamma(U, \mathcal{O}_E)
\end{equation}
as well as a trivialization
\begin{equation}
\Gamma(\mathbb{C} \times U, \mathcal{T}) \xrightarrow{\tilde{\xi}} \Gamma(\mathbb{C} \times U, \mathcal{O}_{\mathbb{C} \times E}),
\end{equation}
which identify the sections $\alpha$ and $\beta$ from [17] with the identity section. Since $\eta$ is a complex analytic covering map, we get from $\zeta$ a local trivialization $\zeta$ of the universal family $L$. Then such trivializations $\xi$ and $\zeta$ allow to identify $\gamma(u, v; x, y)$ with a tensor $\rho(u, v; x, y) \in \mathfrak{A} \otimes \mathfrak{A}$. Note that by the construction the tensor $\rho(u, v; x, y)$ depends only the difference $w := u - v \in P$ with respect to the group law on the Jacobian $P$.

**Theorem 2.3** (Polishchuk [11]). The constructed tensor $\rho(w; x, y) = \rho(u, v; x, y)$ is a non-degenerate skew-symmetric solution of the associative Yang–Baxter equation (1).

Recall the key steps of the proof of this result. For any $x \in \tilde{E}$, let $S^x \in \text{Coh}(\mathbb{E})$ be a simple object of finite length supported at $x$ (which is unique, up to an isomorphism). For any $(u, v) \in (P \times P) \setminus D$ and $(x, y) \in (\tilde{E} \times \tilde{E})$, consider the triple Massey product
\begin{equation}
\text{Hom}_{\mathbb{E}}(\mathcal{A}^u, S^x) \otimes \text{Ext}^1_{\mathbb{E}}(S^x, \mathcal{A}^v) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^v, S^y) \xrightarrow{m_3(u, v; x, y)} \text{Hom}_{\mathbb{E}}(\mathcal{A}^u, S^y)
\end{equation}
in the triangulated category $\text{Perf}(\mathbb{E})$. Since $\text{Ext}^1_{\mathbb{E}}(S^x, \mathcal{A}^v)^* \cong \text{Hom}_{\mathbb{E}}(\mathcal{A}^v, S^x)$ (see [6]), we get from $m_3(u, v; x, y)$ a linear map
\begin{equation}
\text{Hom}_{\mathbb{E}}(\mathcal{A}^u, S^x) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^v, S^y) \xrightarrow{m_{x, y}^{u, v}} \text{Hom}_{\mathbb{E}}(\mathcal{A}^u, S^x) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^v, S^y).
\end{equation}
The constructed family of maps $m_{x, y}^{u, v}$ satisfies the identity
\begin{equation}
(m_{x_1, x_2}^{v_1, v_2})^{12}(m_{x_1, x_3}^{v_1, v_3})^{13} - (m_{x_2, x_3}^{v_1, v_3})^{23}(m_{x_1, x_2}^{v_1, v_2})^{12} + (m_{x_1, x_3}^{v_1, v_2})^{13}(m_{y_2, y_3}^{v_2, v_3})^{23} = 0,
\end{equation}
both sides of which are viewed as linear maps
\[
\text{Hom}_{\mathbb{E}}(\mathcal{A}^{v_1}, S^{x_1}) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v_2}, S^{x_2}) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v_3}, S^{x_3}) \longrightarrow \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v_2}, S^{x_1}) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v_3}, S^{x_2}) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v_1}, S^{x_3}).
\]
Moreover, $m_{x, y}^{u, v}$ is non-degenerate and skew-symmetric:
\begin{equation}
\iota(m_{x, y}^{u, v}) = -m_{y, x}^{v, u},
\end{equation}
where $\iota$ is the isomorphism
\[
\text{Hom}_{\mathbb{E}}(\mathcal{A}^{u}, S^{x}) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v}, S^{y}) \longrightarrow \text{Hom}_{\mathbb{E}}(\mathcal{A}^{v}, S^{y}) \otimes \text{Hom}_{\mathbb{E}}(\mathcal{A}^{u}, S^{x})
\]
given by $\iota(f \otimes g) = g \otimes f$. Both identities (21) and (22) are consequences of existence of an $A_\infty$-structure on $\text{Perf}(\mathbb{E})$ which is cyclic with respect to the Serre duality [6]. Applying appropriate canonical isomorphisms, one can identify $m_{x, y}^{u, v}$ with the linear map $\alpha(u, v; x, y)$ from the commutative diagram [14]. See also [6, Theorem 2.2.17] for a detailed exposition in a similar setting. \qed
3. Solutions of AYBE as a section of a vector bundle

Following the work [6], we provide a global version of the commutative diagram (14). Let

\[ B := P \times P \times \tilde{E} \times \tilde{E} \setminus (D \times \tilde{E} \times \tilde{E}) \cup (P \times P \times \Xi), \]

where \( D \subset P \times P \) is the locus defined by \( \Theta \) and \( \Xi \subset \tilde{E} \times \tilde{E} \) is the diagonal. Let \( X := B \times E \). Then the canonical projection \( X \xrightarrow{\pi} B \) admits two canonical sections \( B \xrightarrow{\sigma_i} X \) given by \( \sigma_i(v_1, v_2; x_1, x_2) := (v_1, v_2; x_1, x_2, x_i) \) for \( i = 1, 2 \). Let \( \Sigma_i := \sigma_i(B) \subset X \) be the corresponding Cartier divisor. Note that \( \Sigma_1 \cap \Sigma_2 = \emptyset \).

Similarly to (11), we have the following short exact sequence in the category \( \text{Coh}(X) \):

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(\Sigma_1) \xrightarrow{\res^\omega_{\Sigma_1}} \mathcal{O}_{\Sigma_1} \rightarrow 0.
\]

Here, for a local section \( g(v_1, v_2; x_1, x_2; x) = \frac{f(v_1, v_2; x_1, x_2; x)}{x - x_1} \) of the line bundle \( \mathcal{O}_X(\Sigma_1) \) we put:\n
\[
\res^\omega_{\Sigma_1}(g) = \res_{x=x_1}(g\omega_x),
\]

where \( \omega_x \) is the pull-back of \( \omega \) under the canonical projection \( X \xrightarrow{\pi_{\Sigma_2}} E \).

Consider the non-commutative scheme \( \mathcal{X} = (X, \pi^*_5(\mathcal{A})) \) as well as coherent sheaves \( \mathcal{A}(i) := \pi^*_5(\mathcal{A}) \otimes \pi_{i,5}^*(\mathcal{L}) \in \text{Coh}(\mathcal{X}) \), where \( X \xrightarrow{\pi_{i,5}} P \times E \) is the canonical projection for \( i = 1, 2 \). Tensoring (23) by \( \mathcal{A}(2) \), we get a short exact sequence

\[
0 \rightarrow \mathcal{A}(2) \rightarrow \mathcal{A}(2)(\Sigma_1) \rightarrow \mathcal{A}(2)|_{\Sigma_1} \rightarrow 0
\]

in the category \( \text{Coh}(\mathcal{X}) \). Since \( \mathcal{A}(1) \) is a locally projective \( \mathcal{O}_X \)-module, applying the functor \( \text{Hom}_X(\mathcal{A}(1), -) \) to (24), we get an induced short exact sequence

\[
0 \rightarrow \text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)) \rightarrow \text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)(\Sigma_1)) \rightarrow \text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)|_{\Sigma_1}) \rightarrow 0
\]

in the category \( \text{Coh}(X) \). Base-change isomorphism combined with the vanishing (8) imply that \( R\pi_*\left(\text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2))\right) = 0 \), where \( R\pi_* : D^b(\text{Coh}(X)) \rightarrow D^b(\text{Coh}(X)) \) is the derived direct image functor. Applying the functor \( \pi_* \) to the short exact sequence (25), we get the following isomorphism

\[
\pi_*\left(\text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)(\Sigma_1))\right) \cong \pi_*\text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)|_{\Sigma_1})
\]

of coherent sheaves on \( B \). Since \( \text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)|_{\Sigma_1}) \cong \text{Hom}_X(\mathcal{A}(1)|_{\Sigma_1}, \mathcal{A}(2)|_{\Sigma_1}) \), we get an isomorphism \( \res^\mathcal{A}_{\Sigma_1} \) of coherent sheaves on \( B \) (which are even locally free) given as the composition

\[
\pi_*\left(\text{Hom}_X(\mathcal{A}(1), \mathcal{A}(2)(\Sigma_1))\right) \cong \pi_*\text{Hom}_X(\mathcal{A}(1)|_{\Sigma_1}, \mathcal{A}(2)|_{\Sigma_1}) \xrightarrow{\cong} \pi_*\text{Hom}_X(\mathcal{A}(1)|_{\Sigma_1}, \mathcal{A}(2)|_{\Sigma_1})
\]

Next, we have the following short exact sequence of coherent sheaves on \( X \):

\[
0 \rightarrow \mathcal{O}_X(-\Sigma_2) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\Sigma_2} \rightarrow 0.
\]
Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, the canonical morphism $O_{\Sigma_2} \to O(\Sigma_1)|_{\Sigma_2}$ is an isomorphism. Tensoring (26) by $A^{(2)}(\Sigma_1)$, we get a short exact sequence
\[
0 \to A^{(2)}(\Sigma_1 - \Sigma_2) \to A^{(2)}(\Sigma_1) \to A^{(2)}|_{\Sigma_2} \to 0
\]
in the category $\text{Coh}(\mathbb{X})$. Applying to (27) the functor $\text{Hom}_{\mathbb{X}}(A^{(1)}, -)$, we get an induced short exact sequence
\[
0 \to \text{Hom}_{\mathbb{X}}(A^{(1)}, A^{(2)}(\Sigma_1 - \Sigma_2)) \to \text{Hom}_{\mathbb{X}}(A^{(1)}, A^{(2)}(\Sigma_1)) \to \text{Hom}_{\mathbb{X}}(A^{(1)}, A^{(2)}|_{\Sigma_1}) \to 0
\]
in the category $\text{Coh}(\mathbb{X})$. Applying the functor $\pi_*$, we get a morphism of locally free sheaves $\text{ev}_{\Sigma_2}^{\mathcal{A}}$ on $\mathcal{B}$ given as the composition
\[
\pi_*\left(\text{Hom}_{\mathbb{X}}\left(A^{(1)}, A^{(2)}(\Sigma_1)\right)\right) \to \pi_*\text{Hom}_{\mathbb{X}}\left(A^{(1)}, A^{(2)}\right)|_{\Sigma_2} \to \pi_*\text{Hom}_{\mathbb{X}}\left(A^{(1)}|_{\Sigma_2}, A^{(2)}|_{\Sigma_2}\right).
\]
In other words, we get the following global version
\[
\pi_*\left(\text{Hom}_{\mathbb{X}}\left(A^{(1)}, A^{(2)}(\Sigma_1)\right)\right) \to \pi_*\text{Hom}_{\mathbb{X}}\left(A^{(1)}|_{\Sigma_1}, A^{(2)}|_{\Sigma_1}\right) \to \alpha^\mathcal{A} \to \pi_*\text{Hom}_{\mathbb{X}}\left(A^{(1)}|_{\Sigma_2}, A^{(2)}|_{\Sigma_2}\right)
\]
of the commutative diagram (14), where $\alpha^\mathcal{A} := \text{ev}_{\Sigma_2}^{\mathcal{A}} \circ (\text{res}_{\Sigma_1}^{\mathcal{A}})^{-1}$.

For any $1 \leq i, j \leq 2$, consider the canonical projection
\[
P \times P \times \tilde{E} \times \tilde{E} \xrightarrow{\kappa_{ij}} P \times E, \ (v_1, v_2; x_1, x_2) \mapsto (v_j, x_i).
\]
Then we have the following canonical isomorphism of coherent sheaves on $\mathcal{B}$:
\[
\pi_*\text{Hom}_{\mathbb{X}}\left(A^{(1)}|_{\Sigma_1}, A^{(2)}|_{\Sigma_1}\right) \cong A^{(i)} \otimes \text{Hom}_{\mathcal{B}}(\kappa_{11}^*(\mathcal{L}), \kappa_{21}^*(\mathcal{L}))
\]
where $A^{(i)}$ is the pull-back of $A$ on $\mathcal{B}$ via the projection morphism
\[
P \times P \times \tilde{E} \times \tilde{E} \to E, \ (v_1, v_2; x_1, x_2) \mapsto x_i
\]
for $i = 1, 2$. The morphism of locally free $O_B$-modules
\[
A^{(1)} \otimes \text{Hom}_{\mathcal{B}}(\kappa_{11}^*\mathcal{L}, \kappa_{21}^*\mathcal{L}) \xrightarrow{\alpha^\mathcal{A}} A^{(2)} \otimes \text{Hom}_{\mathcal{B}}(\kappa_{12}^*\mathcal{L}, \kappa_{22}^*\mathcal{L})
\]
determines a distinguished section
\[
\gamma^\mathcal{A} \in \Gamma\left(B, A^{(1)} \otimes A^{(2)} \otimes \kappa_{11}^*\mathcal{L} \otimes \kappa_{21}^*\mathcal{L}^\vee \otimes \kappa_{22}^*\mathcal{L} \otimes \kappa_{12}^*\mathcal{L}^\vee\right).
\]
For $i = 1, 2$ consider the canonical projections $P \times P \times E \xrightarrow{\psi_1} P \times E, \ (v_1, v_2; x) \mapsto (v_i; x)$ as well as $P \times P \times E \xrightarrow{\psi} P \times P, \ (v_1, v_2; x) \mapsto (v_1, v_2)$. Then there exists $\mathcal{S} \in \text{Pic}(P \times P)$ such that
\[
\psi_1^*(\mathcal{L}) \otimes \psi_2^*(\mathcal{L}^\vee) \cong (\mu \times 1)^* \otimes \psi^*(\mathcal{S}),
\]
where \( P \times P \mathrel{\overset{\mu}{\longrightarrow}} P, (v_1, v_2) \mapsto v_1 - v_2 \). Finally, for \( i = 1, 2 \) consider the morphism

\[
P \times P \times \tilde{E} \times \tilde{E} \mathrel{\overset{\mu_i}{\longrightarrow}} P \times \tilde{E}, (v_1, v_2; x_1, x_2) \mapsto (v_1 - v_2; x_i).
\]

Then we have an isomorphism of locally free sheaves

\[
\kappa_{11}^*(\mathcal{L}) \otimes \kappa_{22}^*(\mathcal{L}^\vee) \otimes \kappa_{12}^*(\mathcal{L}) \otimes \kappa_{21}^*(\mathcal{L}^\vee) \cong \mu_1^*(\mathcal{L}) \otimes \mu_2^*(\mathcal{L}^\vee).
\]

In these terms, we can regard \( \gamma^A \) from (28) as a section

\[
\gamma^A \in \Gamma \left( B, \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)} \otimes \mu_1^*(\mathcal{L}) \otimes \mu_2^*(\mathcal{L}^\vee) \right).
\]

Applying trivialisations \( \xi \) of \( \mathcal{A} \) (see (18)) and \( \zeta \) of \( \mathcal{L} \) (see (19)), we obtain from \( \gamma^A \) a tensor-valued function

\[
\rho^A_{\xi, \zeta} : V \times V \times U \times U \longrightarrow \mathfrak{A} \otimes \mathfrak{A},
\]

which satisfies the translation property

\[
\rho^A_{\xi, \zeta}(v_1 + u, v_2 + u; x_1, x_2) = \rho^A_{\xi, \zeta}(v_1, v_2; x_1, x_2).
\]

Recall that for all types of the genus one curve \( E \) (smooth, nodal or cuspidal) we have a group homomorphism \((\mathbb{C}, +) \longrightarrow (P, +)\), which is locally a biholomorphic map.

After making these identifications, we get the germ of a meromorphic function

\[
(C^3, 0) \mathrel{\overset{\varrho}{\longrightarrow}} \mathfrak{A} \otimes \mathfrak{A}, \quad \text{where} \quad \varrho(v_1 - v_2; x_1, x_2) := \rho^A_{\xi, \zeta}(v_1, v_2; x_1, x_2).
\]

This function is a non-degenerate skew-symmetric solution of AYBE.

**Summary.** Let \( \mathbb{E} = (E, \mathcal{A}) \) be a non-commutative projective curve, where \( E \) is an irreducible projective curve of arithmetic genus one and \( \mathcal{A} \) be a symmetric spherical order on \( E \). Let \( P \) be the Jacobian of \( E \) and \( \mathcal{L} \) be a universal family of degree zero line bundles on \( E \). Then we have a distinguished section \( \gamma^A \in \Gamma \left( B, \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)} \otimes \mu_1^*(\mathcal{L}) \otimes \mu_2^*(\mathcal{L}^\vee) \right) \).

Choosing trivializations \( \xi \) of \( \mathcal{A} \) (see (18)) and \( \zeta \) of \( \mathcal{L} \) (see (19)), we get the germ of a meromorphic function \((\mathbb{C}^3, 0) \mathrel{\overset{\varrho}{\longrightarrow}} \mathfrak{A} \otimes \mathfrak{A}, \) which is a non-degenerate skew-symmetric solution of AYBE. A different trivialization \( \xi \) of \( \mathcal{A} \) leads to a gauge-equivalent solution \((\varphi(x_1) \otimes \varphi(x_1)) \varrho(v; x_1, x_2), \) where \((\mathbb{C}, 0) \mathrel{\overset{\varphi}{\longrightarrow}} \text{Aut}_\mathbb{C}(\mathfrak{A})\) is the germ of \( \xi \xi^{-1} \). Analogously, another choice of a trivialization \( \zeta \) leads to an equivalent solution \(\varrho(v; x_1, x_2) \) for some holomorphic \((\mathbb{C}, 0) \mathrel{\overset{\beta}{\longrightarrow}} \mathbb{C} \).

**Remark 3.1.** The simplest example of a symmetric spherical order is \( \mathcal{A} = \text{End}_E(\mathcal{F}) \), where \( \mathcal{F} \) is a simple vector bundle on \( E \) of rank \( n \) and degree \( d \). It follows from [11, 11, 11] that such \( \mathcal{F} \) exists if and only if \( n \) and \( d \) are coprime and the sheaf of algebras \( \mathcal{A} = \mathcal{A}_{(n, d)}^E = \text{End}_E(\mathcal{F}) \) does not depend on the choice of \( \mathcal{F} \). Moreover, according to [11, Proposition 1.8.1], any symmetric spherical order on an elliptic curve \( E \) is isomorphic to \( \mathcal{A}_{(n, d)} \) for some \( 0 < d < n \) mutually prime.

**Remark 3.2.** Let \((u, v) \in P \times P \setminus D \) and \((x, y) \in \tilde{E} \times \tilde{E} \setminus \Xi \). Then we have canonical isomorphisms

\[
\text{Hom}_E \left( A^u |_x, A^v |_x \right) \cong H^0(E, \text{Hom}_E \left( A^u |_x, A^v |_x \right)) \cong H^0 \left( E, A^{v-u}([x]) \right).
\]
Analogously, we have canonical isomorphisms
\[ \text{Hom}_E\left(A^u|_x, A^v|_x\right) \cong A^{v-u}|_x \] and \[ \text{Hom}_E\left(A^u|_y, A^v|_y\right) \cong A^{v-u}|_y \]
such that the following diagram
\[
\begin{array}{ccc}
\text{Hom}_E\left(A^u|_x, A^v|_x\right) & \xrightarrow{\text{res}^A(u,v;x)} & \text{Hom}_E\left(A^u, A^v(x)\right) \\
\downarrow{\cong} & & \downarrow{\cong} \\
A^{v-u}|_x & \xrightarrow{\text{ev}^v_{y-u}} & H^0\left(E, A^{v-u}(x)\right) \\
\end{array}
\]
\[ (31) \]
is commutative. Here, the linear maps \( \text{res}^u_{x-u} \) and \( \text{ev}^v_{y-u} \) are induced by the standard short exact sequences \[ (10) \].

4. Elliptic solutions of AYBE

Let \( \tau \in \mathbb{C} \) be such that \( \text{Im}(\tau) > 0 \), \( \mathbb{C} \supset \Lambda = \langle 1, \tau \rangle \cong \mathbb{Z}^2 \) and \( E = E_\tau = \mathbb{C}/\Lambda \). Recall some standard techniques to deal with holomorphic vector bundles on complex tori. An \textit{automorphy factor} is a pair \((A, V)\), where \( V \) is a finite dimensional vector space over \( \mathbb{C} \) and \( A : \Lambda \times \mathbb{C} \to \text{GL}(V) \) is a holomorphic function such that \( A(\lambda + \mu, z) = A(\lambda, z + \mu)A(\mu, z) \) for all \( \lambda, \mu \in \Lambda \) and \( z \in \mathbb{C} \). Such \((A, V)\) defines the following holomorphic vector bundle on the torus \( E \):

\[ \mathcal{E}(A, V) := \mathbb{C} \times V / \sim, \text{ where } (z, v) \sim (z + \lambda, A(\lambda, z)v) \text{ for all } (\lambda, z, v) \in \Lambda \times \mathbb{C} \times V. \]

Given two automorphy factors \((A, V)\) and \((B, V)\), the corresponding vector bundles \( \mathcal{E}(A, V) \) and \( \mathcal{E}(B, V) \) are isomorphic if and only if there exists a holomorphic function \( H : \mathbb{C} \to \text{GL}(V) \) such that

\[ B(\lambda, z) = H(z + \lambda)A(\lambda, z)H(z)^{-1} \text{ for all } (\lambda, z) \in \Lambda \times \mathbb{C}. \]

Let \( \Phi : \mathbb{C} \to \text{GL}_n(\mathbb{C}) \) be a holomorphic function such that \( \Phi(z + 1) = \Phi(z) \) for all \( z \in \mathbb{C} \). Then one can define the automorphy factor \((A, \mathbb{C}^n)\) in the following way.

- \( A(0, z) = I_n \) (the identity \( n \times n \) matrix).
- For any \( k \in \mathbb{N}_0 \) we set:
  \[ A(k\tau, z) = \Phi(z + (k - 1)\tau) \ldots \Phi(z) \text{ and } A(-k\tau, z) = A(k\tau, z - k\tau)^{-1}. \]

For a proof of the following result, see \[ 5 \] Proposition 5.1.

\textbf{Proposition 4.1.} Let \( 0 < d < n \) be coprime. Then the sheaf of orders \( A = A_{(n,d)} \) has the following description:

\[ (32) \quad A \cong \mathbb{C} \times A / \sim, \text{ where } (z, Z) \sim (z + 1, \text{Ad}_X(Z)) \sim (z + \tau, \text{Ad}_Y(Z)), \]

\( X \) and \( Y \) are matrices given by \[ 6 \] and \( \text{Ad}_T(Z) = TZT^{-1} \) for \( T \in \{X, Y\} \) and \( Z \in \mathfrak{A} \).
For any \((k, l) \in I := \{1, \ldots, n\} \times \{1, \ldots, n\}\) denote \(Z_{(k, l)} = Y^k X^{-l}\) and \(Z'_{(k, l)} = \frac{1}{n} X^l Y^{-k}\). Note that the operators \(\text{Ad}_X, \text{Ad}_Y \in \text{End}_\mathbb{C}(\mathfrak{A})\) commute. Moreover,

\[
\text{Ad}_X(Z_{(k, l)}) = \varepsilon^k Z_{(k, l)} \quad \text{and} \quad \text{Ad}_Y(Z_{(k, l)}) = \varepsilon^l Z_{(k, l)}
\]

for any \((k, l) \in I\). As a consequence \((Z_{(k, l)})_{(k, l) \in I}\) is a basis of \(\mathfrak{A}\).

Let \(\text{can} : \mathfrak{A} \otimes \mathfrak{A} \rightarrow \text{End}_\mathbb{C}(\mathfrak{A})\) be the canonical isomorphism sending a simple tensor \(Z' \otimes Z''\) to the linear map \(Z \mapsto \text{tr}(Z' \cdot Z) \cdot Z''\). Then we have:

\[
\text{can}(Z_{(k, l)}') \otimes Z_{(k, l)}(Z_{(k', l')}) = \begin{cases} Z_{(k, l)} & \text{if } (k', l') = (k, l) \\ 0 & \text{otherwise.} \end{cases}
\]

Recall the expressions for the first and third Jacobian theta-functions (see e.g. [7]):

\[
\begin{align*}
\bar{\theta}(z) &= \theta_1(z|\tau) = 2q^{1/2} \sum_{n=0}^{\infty} (-1)^n q^n (n+1) \sin((2n+1)\pi z), \\
\theta(z) &= \theta_3(z|\tau) = 1 + 2 \sum_{n=1}^{\infty} q^n \cos(2\pi nz),
\end{align*}
\]

where \(q = \exp(\pi i \tau)\). They are related by the following identity:

\[
\theta \left( z + \frac{1 + \tau}{2} \right) = i \exp(-\pi i (z + \frac{\tau}{4})) \bar{\theta}(z).
\]

**Lemma 4.2.** For any \(x \in \mathbb{C}\) consider the function \(\varphi_x(w) = -\exp(-2\pi i (w + \tau - x))\). Then the following results are true.

- **The vector space**

\[
\mathbb{C} \xrightarrow{f} \mathbb{C} \quad f \text{ is holomorphic} \quad \begin{cases} f(w + 1) = f(w) \\ f(w + \tau) = \varphi_x(w) f(w) \end{cases}
\]

is one-dimensional and generated by \(\theta_x(w) := \theta(w + \frac{1 + \tau}{2} - x)\).

- **We have:** \(\mathcal{E}(\varphi_x) \cong \mathcal{O}_E([x])\).

- **For** \(a, b \in \mathbb{R}\) let \(v = \sigma + b \in \mathbb{C}\) and \([v] = v(v) \in E\). **Then we have:**

\[
\mathcal{E}(\exp(-2\pi iv)) \cong \mathcal{O}_E([0] - [v]).
\]

In these terms we also get a description of a universal family \(\mathcal{L}\) of degree zero line bundles on \(E\).

A proof of these statements can be for instance found in [17] or [18] Section 4.1.

Let \(U \subset \mathbb{C}\) be a small open neighborhood of \(0\) and \(O = \Gamma(U, \mathcal{O}_\mathbb{C})\) be the ring of holomorphic functions on \(U\). Let \(z\) be a coordinate on \(U\), \(\mathbb{C} \xrightarrow{\pi} E\) be the canonical covering map, \(\omega = dz \in H^0(E, \Omega)\) and \(\Gamma(U, \mathcal{A}) \xrightarrow{\xi} \mathfrak{A} \otimes \mathbb{C} O\) be the standard trivialization induced by the automorphy data \((\text{Ad}_X, \text{Ad}_Y)\). One can also define a trivialization \(\zeta\) of the universal family \(\mathcal{L}\) of degree zero line bundles on \(E\) compatible with the isomorphisms \([37]\).

Consider the following vector space

\[
\text{Sol}((n, d), v, x) = \mathbb{C} \xrightarrow{F} \mathfrak{A} \quad \begin{cases} F \text{ is holomorphic} \\ F(w + 1) = \text{Ad}_X(F(w)) \\ F(w + \tau) = \varphi_{x-v}(w) \text{Ad}_Y(F(w)) \end{cases}.
\]
Proposition 4.3. The following diagram

\[
\begin{array}{ccc}
\mathcal{A}^v|_x & \xrightarrow{\text{res}^v} & H^0(\mathcal{A}^v(x)) \\
\downarrow \quad \text{res} & \quad & \downarrow \quad \text{ev}_y \\
\mathcal{A} & \xrightarrow{\text{Sol}} & \mathcal{S}ol((n,d),v,x) \\
\end{array}
\]

(38)

is commutative, where for \( F \in \text{Sol}((n,d),v,x) \) we have:

\[
\text{res}_x(F) = \frac{F(x)}{\theta'(1+\tau)}, \quad \text{and} \quad \text{ev}_y(F) = \frac{F(y)}{\theta(y-x+1+\tau)}.
\]

The isomorphisms of vector spaces \( \iota^v_x, \iota^v_y \) and \( \iota \) are induced by the trivializations \( \xi \) and \( \zeta \) as well as the pull-back functor \( \eta^* \).

Comment on the proof. Since an analogous result is proven in [6, Corollary 4.2.1], we omit details here. \( \square \)

Now we are prepared to prove the main result of this work.

Theorem 4.4. Let \( r_{((n,d),\tau)}(v;x,y) \) be the solution of AYBE corresponding to the datum \((E_\tau, \mathcal{A}_{(n,d)})\) with respect to the trivializations \( \xi \) (respectively, \( \zeta \)) of \( \mathcal{A} \) (respectively, \( \mathcal{L} \)) introduced above. Then it is given by the expression (4).

Proof. We first compute an explicit basis of the vector space \( \text{Sol}((n,d),v,x) \). Let

\[
F(w) = \sum_{(k,l) \in I} f_{(k,l)}(w) Z_{(k,l)}.
\]

The condition \( F \in \text{Sol}((n,d),v,x) \) yields the following constraints on the coefficients \( f_{(k,l)} \):

\[
\begin{align*}
\{ f_{(k,l)}(w+1) &= \varepsilon^k f_{(k,l)}(w) \\
f_{(k,l)}(w+\tau) &= \varepsilon^l \varphi_{z-v} f_{(k,l)}(w).
\end{align*}
\]

(39)

It follows from Lemma 4.2 that the vector space of holomorphic solutions of the system (39) is one-dimensional and generated by the function

\[
f_{(k,l)}(w) = \exp\left(-\frac{2\pi i d}{n} kw\right) \theta\left(w + \frac{1+\tau}{2} + v - x - d \frac{(k\tau - l)}{n}\right).
\]

From Proposition 4.3 and formula (33) it follows that \( r_{((n,d),\tau)}(v;x,y) \) is given by the following expression:

\[
r_{((n,d),\tau)}(v;x,y) = \sum_{(k,l) \in I} r_{(k,l)}(v;z) Z_{(k,l)} \otimes Z_{(k,l)},
\]

where \( z = y - x \) and

\[
r_{(k,l)}(v;z) = \exp\left(-\frac{2\pi i d}{n} k z\right) \theta'\left(1+\tau\right) \theta\left(z + v + \frac{1+\tau}{2} - d \frac{(k\tau - l)}{n}\right) \theta\left(v + \frac{1+\tau}{2} - d \frac{(k\tau - l)}{n}\right) \theta\left(z + \frac{1+\tau}{2}\right).
\]
Relation (3.5) implies that
\[ \frac{\theta'(\frac{1 + \tau}{2}) \theta\left(z + v + \frac{1 + \tau}{2} - \frac{d}{n}(k\tau - l)\right)}{\theta\left(v + \frac{1 + \tau}{2} - \frac{d}{n}(k\tau - l)\right) \theta\left(z + \frac{1 + \tau}{2}\right)} = \frac{\theta'(\frac{1 + \tau}{2}) \theta\left(z + v - \frac{d}{n}(k\tau - l)\right)}{\theta\left(v - \frac{d}{n}(k\tau - l)\right) \theta(z)} \]

Moreover, it follows from (3.5) that \( \theta'(\frac{1 + \tau}{2}) = i \exp\left(-\frac{\pi i r}{4}\right) \theta'(0) \). Hence, we get:
\[ r_{(k,l)}(v;z) = \exp\left(-\frac{2\pi i d}{n} k z\right) \frac{\theta'(0) \theta\left(z + v - \frac{d}{n}(k\tau - l)\right)}{\theta\left(v - \frac{d}{n}(k\tau - l)\right) \theta(z)} \]
\[ = \exp\left(-\frac{2\pi i d}{n} k z\right) \sigma(v - \frac{d}{n}(k\tau - l), z). \]

Here we use the fact that the Kronecker elliptic function \( \sigma(u,z) \) defined by (5) satisfies the formula: \( \sigma(u,z) = \frac{\theta'(0) \theta_1(u + z)}{\theta(u) \theta(z)} \) (see for instance [13, Section 3]). We have a bijection \( \{1, \ldots, n\} \rightarrow \{0, \ldots, n - 1\}, k \mapsto (n - k) \). Using this substitution as well as the identity \( \sigma(u - d\tau, x) = \exp(2\pi i dz) \sigma(u, z) \), we end up with the expression (4), as asserted.

\[ \square \]

**Remark 4.5.** Let \( r(u; x_1, x_2) \) be a non-degenerate skew-symmetric solution of AYBE (1) satisfying (2). Let \( g = \mathfrak{sl}_n(\mathbb{C}) \) and \( \mathfrak{sch} \xrightarrow{\pi} g, Z \mapsto Z - \frac{1}{n} \text{tr}(Z) I_n \). Then
\[ \tilde{r}(x_1, x_2) = (\pi \otimes \pi)(r_1(x_1, x_2)) \]
is a solution of the classical Yang–Baxter equation
\[ \left\{ [\tilde{r}^{12}(x_1, x_2), \tilde{r}^{13}(x_1, x_3)] + [\tilde{r}^{13}(x_1, x_3), \tilde{r}^{23}(x_2, x_3)] + [\tilde{r}^{12}(x_1, x_2), \tilde{r}^{23}(x_2, x_3)] = 0 \right\} \]
\[ \tilde{r}^{12}(x_1, x_2) = -\tilde{r}^{21}(x_2, x_1), \]
see [9, Lemma 1.2]. Under certain additional assumptions (which are fulfilled provided \( \tilde{r}(x_1, x_2) \) is elliptic or trigonometric), the function \( R(x_1, x_2) = r(u; x_1, x_2) \) (where \( u = u_0 \) from the domain of definition of \( r \) is fixed) satisfies the quantum Yang–Baxter equation
\[ R(x_1, x_2)^{12} R(x_1, x_3)^{13} R(x_2, x_3)^{23} = R(x_2, x_3)^{23} R(x_1, x_3)^{13} R(x_1, x_2)^{12}, \]
see [10, Theorem 1.5]. In fact, the expression (4) is a well-known elliptic solution of Belavin of the quantum Yang–Baxter equation; see [2].

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