Relating Field Theories via Stochastic Quantization

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Abstract

This note aims to subsume several apparently unrelated models under a common framework. Several examples of well-known quantum field theories are listed which are connected via stochastic quantization. We highlight the fact that the quantization method used to obtain the quantum crystal is a discrete analog of stochastic quantization. This model is of interest for string theory, since the (classical) melting crystal corner is related to the topological A–model. We outline several ideas for interpreting the quantum crystal on the string theory side of the correspondence, exploring interpretations in the Wheeler–De Witt framework and in terms of a non–Lorentz invariant limit of topological M–theory.
1 Introduction

In this note, we aim to connect several apparently unrelated theories by showing that they are related through a common quantization scheme, namely stochastic quantization [1]. It finds applications in field theory, the world of statistical and condensed matter physics, and via correspondences also in string theory.

Models that can be regarded as resulting from stochastic quantization all have a common structure. Starting from a (classical) model in \(d\) dimensions one obtains a quantum model in \(d+1\) dimensions as a Markov process that converges at thermal equilibrium to the minima of the classical action. The partition function of the classical precursor model is expressed as the norm square of a wave function, which is the ground state of the theory in \(d+1\) dimensions,

\[
Z_{\text{cl}}^d = \langle \psi_{\text{ground}}^{d+1} | \psi_{\text{ground}}^{d+1} \rangle. \tag{1.1}
\]

After summarizing the preliminaries of stochastic quantization, we collect several examples of theories which are related by stochastic quantization. The most basic and prototypical example is the one of zero dimensional field theory, whose stochastic quantization leads straight to supersymmetric quantum mechanics. In the case of our next example, the free boson, the naturally supersymmetric description of the stochastically quantized theory provides a true
advantage over its prior treatment in the literature [2]. We show furthermore that the quantization scheme employed to obtain the quantum crystal in [3], which is the same as the one used for the quantum dimer model [4], is nothing else but a discrete analog of stochastic quantization. In fact, both schemes are based on an underlying Markov process, which is a stochastic process for which the probability for the system to be in a state at time \( t \) depends only on the state at \( t - 1 \).

Even though stochastic quantization cannot be applied directly to the gauged WZW model, its partition function can be expressed as the norm square of a 3–dimensional wave function, which is the ground state of a strong coupling limit of topologically massive gauge theory (Chern–Simons plus Yang Mills). This example thus belongs to the same class of models. The example of the quantum crystal is interesting from the point of view of string theory, since a correspondence between the classical crystal melting configurations and the topological string A–model has been found [5, 6]. The quantum crystal, giving rise to a seven dimensional theory, suggests a connection to topological M–theory. We outline some ideas concerning the interpretation of the quantum crystal on the string theory side of the correspondence, such as a Wheeler–De Witt interpretation of the eigenvalue equation for the ground state. Inspired by the crystal/string correspondence, also the stochastic quantization of Kähler gravity [7] is sketched. Since Kähler gravity is the target space description of the topological A–model, its quantization should be related to the quantum crystal, and via Hitchin’s functional also to topological M–theory [8]. Given that quantum theories such as those arising from the stochastic quantization of Kähler geometry are all in need of regularization, the discrete framework of the quantum crystal ultimately seems, if anything, most suited for practical purposes.

The plan of this note is as follows. In Section 2 we summarize the technique of stochastic quantization in the Langevin and Fokker–Planck formulation and present its manifestly supersymmetric form, as first given in [9]. In Section 2.4 the discrete analog of the stochastic quantization scheme is discussed. In Section 3 several examples of theories that are connected via stochastic quantization are given. The prototypical example is the one of zero dimensional field theory, discussed in Section 3.1. The next example is the free boson, see Section 3.2. In Sec. 3.3 we discuss the relation of the gauged WZW model to a strong coupling limit of topologically massive gauge theory. In Sec. 3.4 we briefly mention the quantum dimer model which is a precursor of the quantum crystal. The latter is treated in Sec. 3.5 where also possible interpretations of the quantum system in string theory are outlined. The last example concerns the stochastic quantization of Kähler gravity, see Sec. 3.6. A brief summary and conclusion is given in Sec. 4.
2 Stochastic quantization revisited

Stochastic quantization is a quantization method for Euclidean field theories introduced in 1981 by Parisi and Wu [1]. It makes use of the fact that Euclidean Green’s functions can be interpreted as correlation functions of a statistical system in equilibrium. A good overview over this quantization scheme including also subsequent work can be found in [10].

Starting from a $d$ dimensional Euclidean field theory, the field is coupled to a white Gaussian noise which forces it to a random movement on its manifold, a continuum analog of a random walk. This stochastic system evolves along a new, fictitious, Euclidean time $t$. In the limit $t \to \infty$, where thermal equilibrium is reached, the $d$–dimensional correlation functions of the quantum field theory are recovered, thus the original $d$–dimensional field theory has been quantized. The equivalence of stochastic quantization and conventional path integral quantization has been shown. When considered purely as a way to quantize a $d$–dimensional field theory, the extra time dimension is but a computational device which allows us to compute the correlators of the $d$–dimensional quantum field theory after the $(d + 1)$–dimensional system has settled into equilibrium. In this paper, we will study the $(d + 1)$–dimensional quantum theory in its own right.

In the following, we will give a very brief introduction to the subject, referring the reader to the literature for the formal definitions of all the concepts used. We follow the notation used by [10].

2.1 Langevin formulation

A stochastic variable $X$, defined by a range of values $x$ and a probability distribution $P(x)$ over these values, is generally used to describe the random fluctuations of a heat reservoir background. A stochastic process is a process depending on the time $t$ and a stochastic variable. A Markov process is a stochastic process in which the conditional probability for the system to be in a certain state at time $t$ only depends on its state at $t - \delta t$ and not on its earlier history. The simplest example of a Markov process process is Brownian motion. Take a particle of mass $m$ moving in a liquid with friction coefficient $\alpha$. Its motion obeys the Langevin equation

$$m \frac{d}{dt} \vec{v} = -\alpha \vec{v}(t) + \vec{f}(t),$$

where $\vec{f}(t)$ is the stochastic force vector representing the collisions between the particle and the molecules of the fluid. $\vec{f}(t)$ is a stochastic variable and can be assumed to have a Gaussian distribution.

Stochastic quantization of a Euclidean field theory works as follows. We supplement the field $\phi(x)$ with an extra time dimension $t$ (which must not be confused with the Euclidean time $x_0$). Then we demand that the time evolution of $\phi(x,t)$ obeys a stochastic differential
equation such as the Langevin equation (2.1), which allows the relaxation to equilibrium:

\[ \frac{\partial \phi(x,t)}{\partial t} = -\frac{1}{2} \delta S_{\text{cl}} \frac{\delta \phi}{\delta \phi} + \eta(x,t), \tag{2.2} \]

with \( S_{\text{cl}} \) the Euclidean action. The correlations of \( \eta \), which is a white Gaussian noise, are given by

\[ \langle \eta(x,t) \rangle = 0, \quad \langle \eta(x_1,t_1)\eta(x_2,t_1) \rangle = 2\delta(t_1 - t_2)\delta^d(x_1 - x_2). \tag{2.3} \]

Equation (2.2) has to be solved given an initial condition at \( t = t_0 \) leading to an \( \eta \)-dependent solution \( \phi_\eta(x,t) \). As a consequence, also \( \phi_\eta(x,t) \) is now a stochastic variable. Its correlation functions are defined by

\[ \langle \phi_\eta(x_1,t_1) \ldots \phi_\eta(x_k,t_k) \rangle_\eta = \int \mathcal{D}\eta \exp \left( -\frac{1}{2} \int \int dx dt \eta^2(x,t) \right) \phi_\eta(x_1,t_1) \ldots \phi_\eta(x_k,t_k). \tag{2.4} \]

The central points for stochastic quantization to make sense is that equilibrium is reached for \( t \to \infty \), and that

\[ \lim_{t \to \infty} \langle \phi_\eta(x_1,t) \ldots \phi_\eta(x_k,t) \rangle_\eta = \langle \phi(x_1) \ldots \phi(x_k) \rangle, \tag{2.5} \]

i.e. that the equal time correlators for \( \phi_\eta \) tend to the corresponding quantum Green’s functions.

### 2.2 Fokker–Planck formulation

The conditional probability distribution of the speed of a particle executing a Brownian motion depends on the initial conditions \( P(v,t_0) \) and satisfies the Fokker–Planck (FP) equation:

\[ \frac{\partial}{\partial t} P(v,t) = \frac{\partial}{\partial v} \left( v + \frac{\partial}{\partial v} \right) P(v,t), \tag{2.6} \]

which can be derived from the Langevin equation (2.1) in a standard way (after setting \( m = \alpha = 1 \)). For stochastic quantization, the FP formulation can be used as an alternative approach.

Starting from (2.2), the evolution of the probability \( P[\phi(x),t] \) in the extra time direction \( t \) is given by the FP equation

\[ \frac{\partial}{\partial t} P[\phi,t] = \int \delta \phi \frac{\delta}{\delta \phi} \left( \delta S_{\text{cl}} \frac{\delta \phi}{\delta \phi} + \delta \phi \right) P[\phi,t], \tag{2.7} \]
together with an initial condition. One easily verifies that for $t \to \infty$,

$$\lim_{t \to \infty} P[\phi, t] = P^{eq}[\phi] = \frac{e^{-S_{cl}[\phi]}}{\int D\phi e^{-S_{cl}[\phi]}}. \quad (2.8)$$

The FP equation is a so-called *master equation* (which means in particular that its kernel is not Hermitian). To bring it into Hamiltonian form we introduce

$$\psi[\phi, t] = P[\phi, t] e^{S_{cl}[\phi]/2}. \quad (2.9)$$

This function $\psi$ satisfies the equation

$$\frac{\partial}{\partial t} \psi[\phi, t] = -\int d^d x \mathcal{H}[\phi] \psi[\phi, t], \quad (2.10)$$

where

$$\mathcal{H}[\phi] = -\frac{1}{2} \frac{\delta^2}{\delta \phi^2} + U[\phi], \quad \text{and} \quad U[\phi] = \frac{1}{8} \left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 - \frac{1}{4} \frac{\delta^2 S_{cl}}{\delta \phi^2}. \quad (2.11)$$

The zero energy ground state for this Hamiltonian is

$$\psi_0[\phi] = e^{-S_{cl}[\phi]/2}. \quad (2.12)$$

The corresponding Lagrangian can be obtained as usual by a Legendre transform:

$$\mathcal{L}[\phi, \dot{\phi}] = \frac{1}{2} \dot{\phi}^2 + U[\phi]. \quad (2.13)$$

An alternative and more general way to find the quantum action consists in considering the partition function

$$Z = \langle 1 \rangle_{\eta} = \int D\eta \ e^{-\frac{1}{2} \int \eta^2(x,t) \ dx dt} \quad (2.14)$$

Changing variables from $\eta$ to $\phi$, this becomes

$$Z = \int D\phi \ \det \left( \frac{\delta \eta}{\delta \phi} \right) e^{-\frac{1}{2} \int \left( \frac{\delta S_{cl}[\phi]}{\delta \phi} + \frac{1}{2} \frac{\delta^2 S_{cl}[\phi]}{\delta \phi^2} \right)^2 \ dx dt}. \quad (2.15)$$

The evaluation of the Jacobian requires some care. In particular, one has to choose a direction for the time evolution, make use of the Stratanovich stochastic calculus (see [11]), and drop an infinite constant. All these issues are neatly resolved in the supersymmetric formulation presented in next section. By choosing propagation in the *positive* time direction, one formally recovers

$$\det \left( \frac{\delta \eta}{\delta \phi} \right) = e^{\frac{1}{2} \int \frac{\delta^2 S_{cl}[\phi]}{\delta \phi^2} \ dx dt}. \quad (2.16)$$

Plugging the Jacobian back into (2.15) and neglecting a total derivative term $\phi \delta S = \dot{S}$, one
finds
\[ Z = \int D\phi \, e^{-\int L[\phi, \phi] \, dx \, dt}, \tag{2.17} \]
where
\[ L[\phi, \phi] = \frac{1}{2} \dot{\phi}^2 + \frac{1}{8} \left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 - \frac{1}{4} \frac{\delta^2 S_{cl}}{\delta \phi^2}, \tag{2.18} \]
as in Eq. (2.13).

2.3 Supersymmetric formulation

It was shown in [9] that any theory obtained by stochastic quantization admits supersymmetry with respect to the fictitious time direction \( t \). In fact, already the classical Brownian motion in one dimension corresponds to supersymmetric quantum mechanics. This supersymmetry is linked to the existence of the so-called Nicolai map. One of the useful consequences of this supersymmetry which is already encoded in the Langevin equation, are Ward identities for the Green’s functions.

The fermionic superpartners of \( \phi \) can be introduced as a calculational device to expand the Jacobian determinant (2.16) as a fermionic path integral:
\[ \det \left( \frac{\delta \eta}{\delta \phi} \right) = \det \left( \frac{\partial}{\partial t} + \frac{\delta S_{cl}}{2 \delta \phi^2} \right) = \int D\psi D\bar{\psi} \, e^{\int dx \, dt \left( \frac{\bar{\psi}}{\bar{\psi}} + \frac{1}{2} \frac{\delta^2 \bar{\psi}}{\delta \phi^2} \right) \psi}. \tag{2.19} \]
The FP Lagrangian becomes
\[ L_{FP}[\phi, \phi, \psi, \bar{\psi}] = \frac{1}{2} \dot{\phi}^2 + \frac{1}{8} \left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 - \bar{\psi} \left( \frac{\partial}{\partial t} + \frac{1}{2} \frac{\delta^2 S_{cl}}{\delta \phi^2} \right) \psi. \tag{2.20} \]
In the scalar case, this can be rephrased in terms of superspace language. We introduce the superfield
\[ \Phi = \phi + \bar{\theta} \psi + \psi \theta + \bar{\theta} \theta F, \tag{2.21} \]
where \( F \) is an auxiliary field needed to achieve closure of the supersymmetry algebra. The super–covariant derivatives are given by
\[ D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \frac{\partial}{\partial \bar{t}}. \tag{2.22} \]
With this, the action becomes
\[ L_{FP}[\Phi] = S_{cl}[\Phi] + \Phi \bar{D}D\Phi, \tag{2.23} \]
where \( S_{cl}[\Phi] \) is the integral over the classical Lagrangian in terms of the superfield \( \Phi \). Note in particular that \( S_{cl}[\Phi] \) now takes the role of a superpotential which is a function of the su-
perfield $\Phi$. A word should be spent on the interpretation of the supersymmetry in this type of problems. When passing from the Langevin to the FP description, there are two possible choices for the direction of the time propagation. They correspond to two Hamiltonians, $\mathcal{H}^-$ and $\mathcal{H}^+$. $\mathcal{H}^-$ is the one we used; $\mathcal{H}^+$ has by construction only strictly positive eigenvalues and hence does not contribute to the $t \to \infty$ dynamics where the classical system is recovered. The partition function can be expressed as the difference $Z = Z^+ - Z^-$ using the corresponding Lagrangians. For some functions $F[\phi]$ of the fields, the expectation values on the two partition functions coincide and we find identities of the type

$$\langle F \rangle_Z = \langle F \rangle_{Z^+} - \langle F \rangle_{Z^-} = 0, \quad (2.24)$$

which can be interpreted as Ward identities in the supersymmetric formalism.

Consider now the special case of the action for a free field $\phi$ which can be written in the form

$$S_{\text{cl}}[\phi] = \langle \phi | L \phi \rangle, \quad (2.25)$$

where $\phi \in \mathcal{H}$, with $\mathcal{H}$ being a Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ and $L$ a self–adjoint operator. The Langevin equation for $\phi_t$ reads

$$\dot{\phi}_t = -L \phi_t + \eta_t, \quad (2.26)$$

where $\eta_t \in \mathcal{H}$ is a white Gaussian noise. The corresponding partition function is

$$Z = \int D\eta_t \exp \left[ \frac{1}{2} \int dt \langle \eta_t | \eta_t \rangle \right]. \quad (2.27)$$

Changing the integration variable,

$$Z = \int D\phi_t \det \left( \frac{\delta \eta_t}{\delta \phi_t} \right) \exp \left[ \frac{1}{2} \int dt \| \eta_t \|^2 \right]. \quad (2.28)$$

We introduce the Grassmann fields $\psi, \bar{\psi} \in \mathcal{H}$ to express the determinant in a fermionic representation,

$$\det \left( \frac{\delta \eta_t}{\delta \phi_t} \right) = \int D\bar{\psi}_t D\psi_t \exp \left[ -\int dt \langle \bar{\psi}_t | \frac{\delta \eta_t}{\delta \phi_t} | \psi_t \rangle \right]. \quad (2.29)$$

Now the final action takes the form

$$S_q[\phi_t, \dot{\phi}_t, \psi_t, \dot{\psi}_t] = \frac{1}{2} \| \phi + L \phi \|^2 - \langle \bar{\psi} | \psi + L \psi \rangle. \quad (2.30)$$

This same action can be written in a manifestly supersymmetric form as

$$S_q[\Phi] = \int d\bar{\theta} d\theta S_{\text{susy}} = \frac{1}{2} \int d\bar{\theta} d\theta \langle D\Phi | D\Phi \rangle + \langle \Phi | L \Phi \rangle. \quad (2.31)$$
2.4 Discrete analog

It is possible to quantize discrete stochastic models in a manner analogous to stochastic quantization as discussed above. The dynamics of any discrete stochastic model is described by the master equation

\[
\frac{d}{dt} P_\alpha(t) = \sum_{\beta \neq \alpha} \left( W_{\alpha\beta} P_\beta(t) - W_{\beta\alpha} P_\alpha(t) \right),
\]

where \( P_\alpha(t) \) is the probability to be in configuration \( \alpha \) at time \( t \), and \( W_{\alpha\beta} \) is the transition rate to state \( \alpha \) if the system is in state \( \beta \). We choose the following transition rates:

\[
W_{\alpha\beta} = C_{\alpha\beta} e^{-g(\overline{H}(\alpha) - \overline{H}(\beta)) / 2},
\]

where \( g \) is a coupling constant, \( \overline{H}(\alpha) \) is the classical Hamiltonian evaluated on the configuration \( \alpha \) and \( C_{\alpha\beta} \) is the adjacency matrix of the state graph. One can verify that the (unique) stationary distribution for \( P_\alpha \) is

\[
P_\alpha^{(0)} = \frac{1}{Z} e^{-g\overline{H}(\alpha)},
\]

where

\[
Z = \sum_\alpha e^{-g\overline{H}(\alpha)}.
\]

This is equivalent to saying that the system is in a Boltzmann distribution with classical energy \( \overline{H} \). One can easily verify not only that \( P_\alpha^{(0)} \) describes an equilibrium state, but that it also satisfies the detailed balance condition

\[
W_{\beta\alpha} P_\alpha^{(0)} = W_{\alpha\beta} P_\beta^{(0)}.
\]

Detailed balance implies the Markov property for a stochastic process.

It is customary to define the exit rate from state \( \alpha \) as

\[
W_{\alpha\alpha} = -\sum_{\beta \neq \alpha} W_{\beta\alpha} = -\sum_{\beta \neq \alpha} C_{\alpha\beta} e^{-g(\overline{H}(\beta) - \overline{H}(\alpha)) / 2}.
\]

In this way, the evolution can be described by the vector equation

\[
\frac{d}{dt} P(t) = WP(t),
\]

where \( W \) is the matrix with entries \( W_{\alpha\beta} \). The stationarity condition becomes

\[
WP^{(0)} = 0.
\]
Instead of using the matrix $W$, one can define the symmetrized version with entries

$$\tilde{W}_{\alpha\beta} = \left( P_{\alpha}^{(0)} \right)^{-1/2} W_{\alpha\beta} \left( P_{\beta}^{(0)} \right)^{-1/2} \quad (2.40)$$

(no summation implied), which is now interpreted as the Hamiltonian of the dynamical system. Explicitly, one finds

$$\begin{aligned}
\tilde{W}_{\alpha\beta} &= C_{\alpha\beta} \quad \text{if } \alpha \neq \beta \\
\tilde{W}_{\alpha\alpha} &= W_{\alpha\alpha} .
\end{aligned} \quad (2.41)$$

Note that this expression coincides with the definition for the Laplacian on a directed graph given in [12].

Given the characteristic equation $\tilde{W}_{\alpha\beta} \tilde{\phi}^{(\lambda)}_{\beta} = \lambda \tilde{\phi}^{(\lambda)}_{\alpha}$ one can verify that the ground state is now represented by the vector with components

$$\tilde{\phi}^{(0)}_{\alpha} = e^{-g\widetilde{F}(\alpha)/2} . \quad (2.42)$$

This whole construction can now be interpreted as a quantization procedure by introducing a Hilbert space $\mathcal{H}$ generated by vectors labelled by the states $\alpha$. Now $\tilde{W}$ is interpreted as a Hamiltonian operator, and the evolution equation for $\tilde{\phi} \in \mathcal{H}$ as a real time Schrödinger equation.

The analogy to the stochastic quantization for a continuous theory becomes obvious once one considers the following points.

- The Markov property lies at the root of both quantization schemes.
- Just as Eq. (2.32), also the Fokker–Planck equation (2.6) is a master equation.
- The stationary distribution (2.34) has the same form as the equilibrium probability in Eq. (2.8).
- The procedure of symmetrizing the matrix $W$ is analogous to the one of bringing the FP equation into a Schrödinger–like form leading to the Hamiltonian (2.11).

Last but not least,

- the ground states (2.12) and (2.42) are the same.

### 3 Examples

This section forms the main body of this article and gives several examples of models which are related by the framework of stochastic quantization. We cite some illustrative examples from the literature along with examples which we present here for the first time.
We start by considering the prototypical example of a zero dimensional field theory which becomes supersymmetric quantum mechanics after quantization. We next discuss the simple but rich example of the stochastic quantization of a bosonic field. Our treatment offers a reinterpretation of the discussion of the quantum Lifshitz model in [2] and allows the generalization beyond the free case. We next discuss the relation of the gauged WZW model to the strong coupling limit of topologically massive gauge theory, which belongs to the same class of models even though stochastic quantization cannot be applied directly. The resulting limit of topologically massive gauge theory is, unlike the pure Chern–Simons theory, dynamical.

Then we move on to the discrete examples of the quantum dimer model and the quantum crystal. Here we stress the point that this discrete quantization scheme is the analog of stochastic quantization and explore possible interpretations of the results in terms of string theory. Finally, we sketch the quantization of the six–dimensional Kähler gravity action.

Recently, theories with anisotropic scaling which can be thought of as stemming from a stochastic quantization process have appeared in the literature in the context of gauge theory, membrane actions, and non—Lorentz invariant gravity [13, 14, 15].

Another example which we are not discussing here deserves being mentioned, namely the connection between $d = 3$ Chern–Simons theory and $d = 4$ topological Yang—Mills theory via stochastic quantization, as given in [16].

### 3.1 From zero dimensions to supersymmetric quantum mechanics

The simplest example of stochastic quantization is obtained by considering zero dimensional quantum field theory. In this case, the field is a map from a point $P$ to the real line, $x : P \to \mathbb{R}$, i.e. a variable. The action is a function of this variable, $S_{\text{cl}} = S_{\text{cl}}(x)$, and the classical partition function is given by the integral

$$Z_{\text{cl}} = \int_{\mathbb{R}} dx \ e^{-S_{\text{cl}}(x)}. \quad (3.1)$$

The stochastic quantization is performed by adding a time direction and promoting $x$ to a function of time, $x : \mathbb{R} \to \mathbb{R}$, $t \mapsto x(t)$. We can now impose the Langevin equation given in Eq. (2.2):

$$\frac{d}{dt} x(t) = -\frac{1}{2} \frac{d}{dx} S_{\text{cl}}[x(t)] + \eta(t), \quad (3.2)$$

where $\eta(t)$ is a white Gaussian noise. This system corresponds to a one dimensional Brownian motion. Using the procedure outlined above, we can write the quantum partition function as:

$$Z = \int Dx(t) \ e^{-\frac{1}{2} \int dt \eta(t)^2} = \int Dx(t) \ det \left[ \frac{\partial \eta(t)}{\partial x(t)} \right] \ e^{-\frac{1}{2} \int dt \left( x^2 + \frac{1}{2} S_{\text{cl}}[x(t)] \right)^2}, \quad (3.3)$$
whence we can directly read the quantum action, which is given precisely by the usual supersymmetric quantum mechanics,

\[ S_q[x, \psi, \bar{\psi}] = \int dt \left[ \frac{1}{2} \dot{x}(t)^2 + \frac{1}{8} (S'_cl[x(t)])^2 - \dot{\psi}(t) \psi(t) - \frac{1}{2} S''_cl[x(t)] \bar{\psi}(t) \psi(t) \right], \tag{3.4} \]

where \( \psi \) and \( \bar{\psi} \) describe a complex fermion. The corresponding Hamiltonian density is given by

\[ H[\pi, x, \psi, \bar{\psi}] = \frac{1}{2} \pi^2 + \frac{1}{8} (S'_cl[x])^2 + \frac{1}{2} S''_cl[x] [\bar{\psi}, \psi], \tag{3.5} \]

where \( \pi \) is the conjugate momentum to \( x(t) \). Being a supersymmetric theory, it makes sense to introduce the charges

\[ Q = \bar{\psi} \left( \frac{\partial}{\partial x} + \frac{1}{2} S'_cl[x] \right), \tag{3.6} \]

\[ \bar{Q} = \psi \left( -\frac{\partial}{\partial x} + \frac{1}{2} S'_cl[x] \right), \tag{3.7} \]

such that the Hamiltonian is obtained as the anticommutator

\[ \mathcal{H} = \frac{1}{2} \{ Q, \bar{Q} \}. \tag{3.8} \]

The ground state of the theory is annihilated by both charges and can be written as:

\[ \psi_0[x] = e^{-S_{cl}[x]/2}. \tag{3.9} \]

As expected, the classical partition function factorizes,

\[ Z_{cl} = \int dx \ e^{-S_{cl}(x)} = \int dx \ |\psi_0[x]|^2 = \langle \psi_0 | \psi_0 \rangle_{L^2}. \tag{3.10} \]

### 3.2 Stochastic quantization of a bosonic field

As a slightly more complicated example, we treat next the stochastic quantization of a free boson. Consider the action for a free boson in \( d \) Euclidean dimensions,

\[ S'_cl[\phi] = \frac{\kappa}{2} \int d^d x \left[ \partial_i \phi(x^i) \partial^i \phi(x^i) \right] \quad i = 1, 2, \ldots, d. \tag{3.11} \]

The Langevin equation, describing the evolution in the fictitious time \( t \) is given by

\[ \partial_t \phi(t, x^i) = \frac{\kappa}{2} \partial_i \partial^i \phi(t, x^i) + \eta(t, x^i). \tag{3.12} \]

where \( \eta(t, x^i) \) is a white Gaussian noise, \textit{i.e.} a stochastic variable whose second momentum is the only one that is non–vanishing, see (2.3). In this way, \( \varphi(t, x^i) \) itself becomes a stochastic
variable and the expectation value of any functional $F[\phi]$ is obtained by averaging over the noise:

$$\langle F[\phi] \rangle_\eta = \frac{1}{Z} \int \mathcal{D}\eta \ F[\phi] e^{-\frac{1}{2} \int dt \ dx \ \eta(t,x)^2}, \quad (3.13)$$

where the partition function is defined by

$$Z = \int \mathcal{D}\eta \ e^{-\frac{1}{2} \int dt \ dx \ \eta(t,x)^2}. \quad (3.14)$$

It is convenient to change the integration variable from $\eta$ to $\phi$. The expression becomes:

$$Z = \int \mathcal{D}\phi \ \det \left[ \frac{\delta \eta}{\delta \phi} \right]_{\eta = \partial_t \phi(t,x_i) + \kappa \partial_i \partial_i \phi} \ e^{-\frac{1}{2} \int dt \ dx \ \phi(t,x_i)^2} \ . \quad (3.15)$$

The Jacobian is easily expressed by introducing two fermionic fields $\psi(t,x^i)$ and $\bar{\psi}(t,x^i)$ such that

$$\det \left[ \frac{\delta \eta}{\delta \phi} \right]_{\eta = \partial_t \phi(t,x_i) + \kappa \partial_i \partial_i \phi} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int dt \ dx \ \bar{\psi}(t,x_i)(\partial_t - \frac{\kappa}{2} \partial_i \partial_i) \psi(t,x_i)}. \quad (3.16)$$

In this way we can directly read off the $(d + 1)$–dimensional action:

$$S_{d+1}^{q}[\phi, \psi, \bar{\psi}] = -\int dt \ dx \left[ \frac{1}{2} \left( \partial_t \phi \right)^2 + \frac{\kappa^2}{8} \left( \partial_i \partial_i \phi \right)^2 - \bar{\psi} \left( \partial_t - \frac{\kappa}{2} \partial_i \partial_i \right) \psi \right]. \quad (3.17)$$

We can rewrite this result by introducing the superfield $\Phi$ defined by

$$\Phi(t,x^i, \theta, \bar{\theta}) = \phi(t,x^i) + \bar{\theta} \psi(t,x^i) + \bar{\psi}(t,x^i) \theta + \theta \theta F(t,x^i), \quad (3.18)$$

where $\theta$ and $\bar{\theta}$ are Grassmann variables and $F(t,x^i)$ is an auxiliary bosonic field. One finds that

$$S_{d+1}^{q}[\phi, \psi, \bar{\psi}] = \int d\theta d\bar{\theta} \ dt \ dx \left[ D\Phi \overline{D}\Phi + \mathcal{L}_{cl}[\Phi] \right] = \int d\theta d\bar{\theta} \ dt \ dx \left[ D\Phi \overline{D}\Phi + \frac{\kappa}{2} \partial_i \Phi \partial_i \Phi \right], \quad (3.19)$$

where $D$ and $\overline{D}$ are the super–covariant derivatives (2.22). Note that here, as remarked before, $\mathcal{L}_{cl}$ takes the role of a superpotential. In this form the action is manifestly covariant under the supersymmetric variations

$$\delta_c \phi = \bar{\epsilon} \psi - \bar{\psi} \epsilon, \quad \delta_c F = \bar{\epsilon} \dot{\psi} + \dot{\bar{\psi}} \epsilon, \quad (3.20)$$

$$\delta_c \psi = \epsilon (\phi + F), \quad \delta_c \bar{\psi} = \bar{\epsilon} (\phi - F). \quad (3.21)$$

There are different possible approaches to obtain the Hamiltonian description. One possibility consists in starting with the Fokker–Planck equation. Alternatively, one can use the
supersymmetric structure and define the two supercharges

\[ Q = \bar{\psi} \left( i \Pi_{\varphi} - \frac{\kappa}{2} \partial_i \partial^i \varphi \right), \quad \bar{Q} = \psi \left( -i \Pi_{\varphi} - \frac{\kappa}{2} \partial_i \partial^i \varphi \right), \tag{3.22} \]

where \( \Pi_{\varphi} = -i \frac{\delta}{\delta \varphi} \) is the conjugate momentum to \( \varphi \). One can easily see that \( \{ Q, Q \} = \{ Q, \bar{Q} \} = 0 \) and

\[ \{ Q, \bar{Q} \} = 2 \mathcal{H}, \tag{3.23} \]

where \( \mathcal{H} \) is the Hamiltonian operator obtained by a Legendre transformation of the action \( S^d_{d+1} \) in Eq. (3.17). Using the expressions for \( Q \) and \( \bar{Q} \), one can verify that the (bosonic) ground state,

\[ |\Psi_0\rangle = e^{-\frac{\kappa}{4} \int d^d x \partial_i \partial^i \varphi} |0\rangle, \tag{3.24} \]

where \( |0\rangle \) is the vacuum annihilated by \( \psi \), is in fact a supersymmetric ground state, since

\[ Q |\Psi_0\rangle = \bar{Q} |\Psi_0\rangle = 0. \tag{3.25} \]

The bosonic part of the action in Eq. (3.17) has already been conjectured in [17] and studied in [2]. The case \( d = 2 \) is particularly interesting since the ground state expectation values of the \( d = 3 \) quantum theory can be evaluated as correlators of a conformal field theory.

The supersymmetric formulation presented here has the advantage that due to the presence of the fermionic part, both \( Q \) and \( \bar{Q} \) annihilate the bosonic ground state, which was not the case for the quantum Lifshitz model discussed in [2]. In their case, the contribution to the Hamiltonian coming from \( Q \bar{Q} \) was interpreted as a UV divergent zero–point energy and had to be subtracted. Using the stochastic quantization formalism thus allows us to deal with more general problems where the boson evolves in a potential and is described by an action of the type

\[ S^d_{cl}[\varphi] = \int d^d x \left[ \frac{\kappa}{2} \partial_i \varphi \partial^i \varphi + V[\varphi] \right]. \tag{3.26} \]

Take for example a massive theory,

\[ V[\varphi] = \frac{m^2}{2} \varphi^2. \tag{3.27} \]

The \((d + 1)\)–dimensional action is found to be

\[ S^d_{q}[\varphi, \psi, \bar{\psi}] = \int d^{d+1} x \left[ \frac{1}{2} \left( \partial_t \varphi \right)^2 + \frac{\kappa^2}{8} \left( \partial_i \partial^i \varphi \right)^2 + \frac{\kappa m^2}{4} \partial_\xi \varphi \partial^\xi \varphi + \frac{m^4}{8} \varphi^2 - \bar{\psi} \left( \partial_t + \frac{m^2}{2} - \frac{\kappa}{2} \partial_i \partial^i \right) \psi \right]. \tag{3.28} \]

This action provides a supersymmetric generalization of the Landau free energy expression used to describe the Lifshitz points in the study of liquid crystals [18] (a Lifshitz point oc-
curs where a high-temperature disordered phase, a spatially uniform ordered phase and a
spatially modulated phase meet). The supersymmetric charges are found to be

\[ Q = \bar{\psi} \left( \frac{\delta}{\delta \varphi} - \frac{\kappa}{2} \partial_i \partial^i \varphi + \frac{m^2}{2} \varphi \right), \quad \bar{Q} = \psi \left( -\frac{\delta}{\delta \varphi} - \frac{\kappa}{2} \partial_i \partial^i \varphi + \frac{m^2}{2} \varphi \right). \] (3.29)

Another interesting example is provided by starting with a sine–Gordon type action.

\[ S_{cl}^d[\varphi] = \int d^d x \left[ \frac{\kappa}{2} \partial_i \varphi \partial^i \varphi + \frac{\lambda}{2\pi} \cos(2\pi \varphi) \right]. \] (3.30)

In this case, the quantization yields

\[ S_{q}^{d+1}[\varphi, \psi, \bar{\psi}] = \int d^{d+1} x \left[ \frac{1}{2} (\partial_i \varphi)^2 + \frac{1}{2} \left( \kappa \partial_i \varphi \varphi + \lambda \sin(2\pi \varphi) \right)^2 - \bar{\psi} \left( \partial_t - \frac{\kappa}{2} \partial_i \partial^i - \pi \lambda \cos(2\pi \varphi) \right) \psi \right], \] (3.31)

and the \( Q \) and \( \bar{Q} \) operators read:

\[ Q = \bar{\psi} \left( \frac{\delta}{\delta \varphi} - \frac{\kappa}{2} \partial_i \partial^i \varphi - \frac{\lambda}{2} \sin(2\pi \varphi) \right), \quad \bar{Q} = \psi \left( -\frac{\delta}{\delta \varphi} - \frac{\kappa}{2} \partial_i \partial^i \varphi - \frac{\lambda}{2} \sin(2\pi \varphi) \right). \] (3.32)

Since for a precise choice of \( \lambda \) and \( \kappa \), the sine–Gordon action corresponds to a massive Dirac fermion, we expect this action to be related to the continuum limit of the quantum crystal. It was argued in [2] that this action should give rise to a mass gap which is in agreement with the results in [3] concerning the discrete model. It is likely that the supersymmetric structure we identified here will help in gaining a better understanding of the properties of the system away from the Lifshitz point represented by the free boson (quantum dimer).

3.3 From the gauged WZW model to topologically massive gauge theory

As the next easiest model to study, let us now consider the gauged WZW model. This example differs a little from the others since we are not using stochastic quantization directly. But it belongs to this collection because the two theories (in \( d \), resp. \( d + 1 \) dimensions) are related by the classical partition function of one being the norm square of the ground state wave function of the other. This is the common property of all the examples discussed here.

We first summarize the WZW model very briefly and refer the reader to the literature for details. The basic field \( g \) of the WZW model is a map from a 2d Riemann surface \( \Sigma \) to a compact Lie group \( G \). Its basic functional is the action

\[ I[\mathbf{g}] = -\frac{1}{8\pi} \int_{\Sigma} d^2 \sigma \sqrt{g}^{ij} \text{Tr}(g^{-1} \partial_i g \cdot g^{-1} \partial_j g) - i \Gamma[\mathbf{g}], \] (3.33)
where $\rho$ is a metric on $\Sigma$ and $\Gamma$ is the Wess–Zumino term,

$$\Gamma[g] = \frac{1}{12\pi} \int_B d^3\sigma \epsilon^{ijk} \text{Tr} \, g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g,$$

(3.34)

where $B$ is a 3–manifold such that $\partial B = \Sigma$. The main property of the WZW model that we will make use of in the following is the existence of a *holomorphic factorization*. This means that the partition function $Z$ can be expressed as a square, $Z = \langle f | f \rangle$, of a holomorphic section $f$ of a flat vector bundle over moduli space.

As it was already noted in [2], a direct stochastic quantization of the WZW action is not possible because of the presence of the $\text{WZ}$ term. Instead we follow Witten’s treatment [19], where the WZW model is gauged in order to derive the existence of the holomorphic factorization property. In the gauged model, $g$ becomes a section of a bundle $X \to \Sigma$ with fiber $G$ and structure group $G_L \times G_R$. We take $A$ to be a connection on such a bundle. Let us now define

$$I[g, A] = I[g] + \frac{1}{2\pi} \int_{\Sigma} d^2z \text{Tr} A_z g^{-1} \partial_z g - \frac{1}{4\pi} \int_{\Sigma} d^2z \text{Tr} A_z A_z.$$

(3.35)

Now we can formally define a functional of $A$,

$$\Psi[A] = \int Dg \, e^{-k I[g, A]}.$$

(3.36)

$\Psi[A]$ obeys two key equations,

$$\left( \frac{\delta}{\delta A_z} - \frac{k}{4\pi} A_z \right) \Psi[A] = 0,$$

(3.37)

$$\left( D_z \frac{\delta}{\delta A_z} + \frac{k}{4\pi} D_z A_z - \frac{k}{2\pi} F_{zz} \right) \Psi[A] = 0,$$

(3.38)

where the covariant derivative is defined by

$$D_i u = \partial_i u + [A_i, u].$$

(3.39)

These two conditions mean that $\Psi[A]$ is a *holomorphic section* and *gauge invariant*. This can be interpreted to mean that $\Psi[A]$ is a physical state, i.e. a wave function, of 2 + 1 dimensional Chern–Simons theory, and thus relates the WZW model to Chern–Simons theory. In [19] it is shown that the $\Psi[A]$ we constructed in Eq. (3.36) is indeed the holomorphic section that squares to the partition function,

$$Z(\Sigma) = \frac{1}{\text{vol}(\hat{G})} \int DA \, \overline{\Psi[A]} \Psi[A] = \|\Psi\|^2.$$

(3.40)

\footnote{The $\text{WZ}$ term, being imaginary, would not contribute to the modulus square of the action. As a result, one would find at thermal equilibrium, $t \to \infty$, that the system converges to the principal chiral model instead of the expected WZW model.}
We recognize here the by now familiar structure of a theory resulting from stochastic quantization: the partition function of a classical theory in \(d\) dimensions (in this case WZW and \(d=2\)) is expressed as the square of the ground state wave function of a quantum theory in \(d+1\) dimensions. All we need to know now is the precise theory whose Hamiltonian annihilates \(\Psi[A]\). The corresponding theory was described in [20] and is a strong coupling limit of topologically massive gauge theory, whose action is given by

\[
S = -\int d^3x \, \text{Tr}[\frac{1}{2} F_0 E^0] + \frac{k}{4\pi} \int dx \, \text{Tr}[AdA - \frac{2}{3} A^3].
\]  

(3.41)

We would now like to point out another similarity with stochastic quantization. Interestingly enough, the Chern–Simons term is reminiscent of the total derivative term \(\frac{dS}{dt}\) that we dropped in the derivation of the action in Eq. (2.18). When considering a 3–manifold \(Y = \Sigma \times [0, T]\), on the one hand the contribution of \(\int_Y dx dt \frac{dS}{dt}\) is \(\int_{\Sigma = \partial Y} (S(T) - S(0))\), and on the other, it is a known fact that on such a manifold, the CS action is equivalent to the (chiral) WZW model on the boundary \(\partial Y\) [21, 22].

Note moreover that in the strong coupling limit we consider, the magnetic component of the Yang–Mills term drops out. As a result, the Lagrangian, as it is usually the case for stochastic quantization, is not Poincaré invariant. Canonically quantizing this model on \(\Sigma \times \mathbb{R}\) (where \(\Sigma\) is the Riemann sphere) in the Weyl gauge \(A_0 = 0\), one obtains the Hamiltonian\(^2\)

\[
\mathcal{H} = \int d^2x \, \text{Tr}[E(x'^i)^2], \quad i = 1, 2,
\]  

(3.43)

where

\[
E^{a,i}(x) = \Pi^{a,i}(x) - \frac{k}{8\pi} \varepsilon^{ij} A^a_j(x),
\]  

(3.44)

and \(\Pi^{a,i}(x)\) is the conjugate momentum of \(A^a_i(x')\):

\[
[A^a_i(x), \Pi^b_j(y)] = i \delta_{ij} \delta^{ab} \delta(x-y).
\]  

(3.45)

In this formalism, we can now understand the two conditions in Eq. (3.37) and Eq. (3.38):

- Since \(A_0\) appears in the action as a Lagrange multiplier, the gauge choice \(A_0 = 0\) is implemented by imposing Eq. (3.38) as the constraint that the state \(|\Psi\rangle\) be physical.

- Equation (3.37) is now understood as \(\Psi[A]\) being the ground state annihilated by the

\[^2\]If instead of taking the strong coupling limit we had considered the complete Yang–Mills term, this would have resulted in an extra term in the Hamiltonian,

\[
\mathcal{H} = \int d^2x \, \text{Tr}[E(x'^i)^2 + B(x_i)^2],
\]  

(3.42)

which does not lead to the ground state we expect.
operator $E^a$: 

$$E^a \Psi[A] = \left( \frac{2}{i} \frac{\delta}{\delta A^a_z} - \frac{ik}{8\pi} A^a_z \right) \Psi[A] = 0 . \quad (3.46)$$

(Here, we passed to complex coordinates where $A_z = A_1 + i A_2$ and $A_{\bar{z}} = A_1 - i A_2$).

Being a first order equation, we would like to argue that the last relation can be understood as the annihilation of the ground state in a supersymmetric theory. To do so, we introduce the following operators:

$$Q^a(x) = \bar{\chi}(x) \left( \frac{2}{i} \frac{\delta}{\delta A^a_z} - \frac{ik}{8\pi} A^a_z \right) \chi(x) = \bar{\chi}(x) \chi(x) , \quad (3.47)$$

$$\overline{Q}^a(x) = \left( 2 \frac{\delta}{\delta A^a_{\bar{z}}} + \frac{ik}{8\pi} A^a_{\bar{z}} \right) \chi(x) = \bar{E}^a(x) \chi(x) , \quad (3.48)$$

where $\chi$ is a complex fermion satisfying

$$\{ \chi(x), \bar{\chi}(y) \} = \delta(x - y) . \quad (3.49)$$

The anticommutator between $Q^a$ and $\overline{Q}^a$ yields

$$\{ Q^a(x), \overline{Q}^a(y) \} = \left( \bar{E}^a(x) E^a(x) + \frac{k}{2\pi} \bar{\chi}(x) \chi(x) \right) \delta(x - y) , \quad (3.50)$$

where we used the commutation relation

$$[E^a(x), \bar{E}^b(y)] = \frac{k}{2\pi} \delta^{ab} \delta(x - y) . \quad (3.51)$$

The anticommutator is a supersymmetric extension of the Hamiltonian of topologically massive gauge theory, see Eq. (3.43), which only contains the bosonic part. Using the charges $Q^a(x)$, $\overline{Q}^a(x)$, one can directly derive the supersymmetry transformations for the fields $A^a$ and $\chi$, which are given by

$$\delta_\epsilon A^a = [\epsilon Q^b + \bar{\epsilon} \overline{Q}^b, A^a] , \quad (3.52)$$

$$\delta_\epsilon \chi = [\epsilon Q^b + \bar{\epsilon} \overline{Q}^b, \chi] . \quad (3.53)$$

Going to the Lagrangian formalism, the system is described by the action

$$S = - \int d^3x \ Tr \left[ \frac{1}{2} F_0 F_0^{[1]} + \bar{\chi} \chi + \frac{k}{2\pi} \bar{\chi} \chi + \frac{k}{4\pi} \int dx \ Tr[A d A - \frac{2}{3} i A^3] \right] . \quad (3.54)$$

This can be seen as a strong coupling limit of the supersymmetric topologically massive gauge theory studied in [23].
3.4 The quantum dimer model

This example comes from the world of solid state physics. The quantum dimer model is a so-called resonating valence bond model and was first introduced by Rokhsar and Kivelson \[4\] as a candidate model for high temperature superconductivity. It starts from the dimer model\[3\], a lattice model living on a bipartite two-dimensional graph. Dimers live on the edges of the graph and each vertex can only be touched by one dimer. In the dimer model, the number of so-called perfect matchings is counted. The basic move in the dimer model is a plaquette flip, in which the dimers of a fully occupied plaquette of the lattice are each turned by one position. The quantization of the dimer model is an example of a discrete analog of stochastic quantization as discussed in Section 2.4. Its Hilbert space is spanned by vectors in one-to-one correspondence with the perfect matchings and its quantum Hamiltonian can be expressed as

\[ H = -J \left( \sum |\Box\rangle \langle \Box| + |\square\rangle \langle \square| + \sum |\Box\rangle \langle \Box| + |\square\rangle \langle \square| \right), \tag{3.55} \]

where the black and white squares correspond to the two states in which a fully occupied plaquette can be. The sum runs over all flippable plaquettes of a given perfect matching. The first two terms are kinetic and flip a plaquette either to the left or the right, the last two terms are potential terms and count the number of plaquette moves. The ground state corresponds to the equally weighted sum over all perfect matchings.

In the case of the dimer model on the hexagonal lattice, there exists a one-to-one map between perfect matchings and the configurations of an idealized three-dimensional crystal corner. The Hamiltonian in Eq. (3.55) can be generalized to describe a (Markov) growth process as detailed in the next section.

3.5 Quantum crystal melting

Let us now discuss the example of quantizing the melting crystal corner [3], which is interesting from the point of view of string theory. This is as well an example of a discrete analog of stochastic quantization as discussed in Section 2.4.

The melting crystal corner is often mapped to the problem of stacking cubes in an empty corner of 3D space. The growth rules are the following. A cube can be added to a configuration if three of its sides will touch either the wall or other cubes. This leads to a minimum energy configuration without any free-standing cubes. The partition function of the melting crystal corner takes the following form:

\[ Z = \sum_{3d \text{ partitions}} q^{\# \text{ boxes}} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n}, \tag{3.56} \]

For the detailed definitions see for example [24].
where the rightmost expression is the so-called MacMahon function.

Crystal melting is an extension of the dimer model discussed in the last section in the following sense. Via a rhombus tiling, the configurations of the dimer model on the hexagonal lattice are in one-to-one correspondence with the configurations of the melting crystal corner. While in the dimer model, only the overall number of configurations are counted, the model of the melting crystal corner contains more information, it also keeps track of the number of cubes of each configuration. The quantum Hamiltonian of the melting crystal can be expressed as

\[
H = -J \left( \sum |\blacksquare\rangle\langle\square| + |\square\rangle\langle\blacksquare| + \sum \sqrt{q} |\square\rangle\langle\square| + \frac{1}{\sqrt{q}} |\blacksquare\rangle\langle\blacksquare| \right),
\]

which acts on the Hilbert space generated by the orthonormal basis of the configurations \( |\alpha\rangle \). The sum runs over all places in a given configuration where a cube can be added or removed. The ket \( |\blacksquare\rangle \) represents a place where a cube can be removed, while \( |\square\rangle \) represents a place where a cube can be added. Once one takes the limit \( q \rightarrow 1 \) in (3.57), one recovers the Hamiltonian of the quantum dimer model. The ground state is in fact the unique zero energy ground state, fulfilling

\[
H |\text{ground}\rangle = 0,
\]

and has the form

\[
|\text{ground}\rangle = \sum_{\alpha} q^{N(\alpha)/2} |\alpha\rangle.
\]

Note the normalization for the wave function,

\[
\langle\text{ground}|\text{ground}\rangle = \sum_{\alpha} q^{\#\text{boxes}} = Z.
\]

Once we identify

\[
q = e^{-g_s},
\]

it turns out that the closed string partition function of the A–model with the Calabi–Yau \( X = \mathbb{C}^3 \) as target space corresponds exactly to the partition function of the melting crystal corner (3.56) \[5, 6\]. For the crystal, the above identification (3.61) translates into a lattice spacing of \( g_s \). The crystal melting configurations can be roughly mapped to (multiple) blow–ups \( \hat{X} \) of the original Calabi–Yau \( X \). Moreover, in \[25\] it was shown that a crystalline structure similar to the one corresponding to \( \mathbb{C}^3 \) which we used, can be constructed for any toric Calabi–Yau manifold, based on the corresponding quiver gauge theory. The partition function of this crystalline model will then correspond to the partition function of the topological A–model on this toric Calabi–Yau modulo possible wall crossings (see e.g. \[26\]). This means that also our quantization scheme can be generalized to any toric Calabi–Yau. All that is needed is the state graph whose nodes are crystal configurations. Its arrows mark which
configurations are related by adding or removing an atom. The quantization procedure does not depend on the details of the crystal. Consider the state graph for the melting of a crystal made of $A$ different atoms. To each configuration $\alpha$ we associate a weight

$$w(\alpha) = \prod_{i=1}^{A} q_i^{n_i(\alpha)},$$

(3.62)

where $0 < q_i < 1$, $i = 1, \ldots, A$ are parameters and $n_i(\alpha)$ is the number of atoms of species $i$ in the configuration $\alpha$. Then the Laplacian $\tilde{W}$ in Eq. (2.41) becomes the matrix with entries

$$\tilde{W}_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ -\sum_{\gamma \text{ neighbour of } \alpha} \sqrt{\frac{w(\gamma)}{w(\alpha)}} & \text{if } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

(3.63)

One can verify that the vector

$$|\text{ground}\rangle = \sum_{\alpha} \left( \prod_{i=1}^{A} q_i^{n_i(\alpha)/2} \right) |\alpha\rangle$$

(3.64)

is a zero energy ground state for $\tilde{W}$ and its norm square reproduces the classical crystal melting partition function

$$\langle \text{ground}|\text{ground}\rangle = \sum_{\alpha} w(\alpha) = Z \sim Z^{\text{top}} \quad \text{(modulo wall crossings)}. \quad (3.65)$$

Let us now consider possible interpretations of the quantized crystal in terms of string theory. By adding a time evolution to the statistical system, we created a quantum theory with one dimension more than the system we started from.

From the string theory point of view, this means that instead of the six dimensional topological A–model, we should be looking at a seven dimensional theory. An immediate candidate would be topological M–theory, or an effective version thereof. Given that the new time dimension is not treated on the same footing as the original six dimensions, we cannot be looking at M–theory itself, but rather at a non–Lorentz invariant limit. Unfortunately, topological M–theory has not yet been clearly defined. Its status is best discussed in [8], where its connection to Hitchin’s functionals is described.

Given that the classical configurations of the melting crystal are related to geometries, it would be interesting to try to find a completely geometric interpretation of the Hamiltonian (3.57), which is the Laplacian on the graph of states (geometries). Its kinetic term creates ”neighboring” geometries from a given configuration. It would be interesting to see if also the potential term has a precise geometric meaning.
Concerning the interpretation of the quantum crystal, another observation can be made that we consider to be one of the central conceptual points of this paper. If we identify the three–dimensional crystal configurations with multiple blow–ups of $\mathbb{C}^3$, we find that the Hilbert space generated by the configurations has a natural interpretation as mini–superspace and the ground state can be seen as a wavefunction of the Universe.

Looking at formula (3.58),

$$H \Psi_0 = 0,$$

one is strongly reminded of the Wheeler–De Witt equation. The WDW equation also has the form $H_G \Psi = 0$, where $H_G$ is the Hamiltonian associated to the Einstein–Hilbert action, and $\Psi$ is the wave function of the universe. The WDW equation is in fact nothing else but a zero energy Schrödinger equation and the crucial information needed is the initial quantum state which is then evolved. The wave function $\Psi$ is defined on superspace, an infinite dimensional space of all possible geometries and matter configurations. In practice, one usually works in the finite dimensional mini–superspace, in which all but a few degrees of freedom are frozen out.

The sum over classical crystal melting configurations corresponds on the A–model side to a sum over "quantum Kähler geometries", or to be more precise, a sum over ideal sheaves, which are torsion–free sheaves with vanishing first Chern class [6]. In this framework, the mini–superspace we are looking at is thus the moduli space of all ideal sheaves $M^{\text{sheaf}}$ over the original manifold $X$.

In our case however, it is not obvious how to find the natural classical gravity theory whose quantization leads to $H$ (i.e. the analog of the Einstein–Hilbert action the usual WDW equation descends from). Let us therefore concentrate on the meaning of $H$. As we have stressed already, $H$ is the Laplacian on the space of configurations. The condition in Eq. (3.66) can be understood as requiring $\Psi_0$ to be a harmonic function in mini–superspace. This is strongly reminiscent of the results in [27] concerning quantum gravity in $2 + 1$ dimensions. In this case, the Einstein–Hilbert gravity turns into a Hamiltonian system on Teichmüller space after the ADM reduction. In particular, for genus $g = 1$, the WDW equation takes the form

$$\sqrt{\Delta} \Psi = 0,$$

where $\Delta$ is the Laplacian on the torus moduli space. Two observations can be made about this:

• (quantum) gravity in $2 + 1$ dimensions is special because there are no propagating degrees of freedom. It is therefore not surprising that this case is very close to our construction which is related to topological strings.

• The square root in (3.67) is reminiscent of the fact that in supersymmetric systems the ground state is found by imposing $Q \Psi_0 = 0$. The operator $Q$ is in some sense close
to being the “square root” of the Hamiltonian, since $\mathcal{H} = \frac{1}{2}\{Q, \overline{Q}\}$. Also the discrete quantum crystal can be supersymmetrized (by enlarging the space of states) and the ground state then satisfies a first order difference equation as opposed to the second order equation (3.66). Apart from this, Eq. (3.67) results in the same ground state as Eq. (3.66).

There is yet another way of looking at the classical crystal configurations, as adopted in [25], namely as BPS bound states of $D$–branes ($D0$ and $D2$ branes bound to a single $D6$ brane). Removing an atom of the crystal corresponds here to adding a specific combination of $D0$ and $D2$ brane charges. In this picture, the quantum Hamiltonian (3.57) jumps thus between configurations with different D–brane charges.

In the above paragraphs, we have outlined possible directions to pursue in order to arrive at a meaningfull interpretation of the quantum crystal on the string theory side of the correspondence. We leave a more thorough investigation of these ideas for future research.

### 3.6 Kähler gravity?

In the previous section we proposed a possible interpretation of our quantum crystal construction in terms of a Wheeler–De Witt equation. To make this connection more precise we would need an appropriate theory of gravity to quantize. A possible candidate is a 6–dimensional form theory of gravity, namely Kähler gravity, which describes variations of the Kähler structure on a complex manifold $M$ [7]. Also this example is interesting from the point of view of string theory, since it provides the target space description of the topological A–model. Following the logic in [5, 6], its stochastic quantization should provide a theory that corresponds to (or is at least in the same universality class and thus shares important properties with) the quantum crystal. A relation of this theory to topological M–theory is therefore to be expected. Here we sketch a possible approach leading to the stochastic quantization of this theory.

The action of Kähler gravity is given by

$$S_{\text{Kähler}}[K] = \int_M \frac{1}{2} K \frac{1}{d^c} dK + \frac{1}{3} K \wedge K \wedge K,$$

where $K$ is a variation of the (complexified) Kähler form on $M$, and $d^c = \partial - \bar{\partial}$. The Kähler gravity action is invariant under gauge transformations of the form

$$\delta_\alpha K = d\alpha - d^c (K \wedge \alpha),$$

where $\alpha$ is a 1-form on $M$, such that $d^c \alpha = 0$. The equations of motion for Kähler gravity
theory have the form
\[ dK + d^c (K \wedge K) = 0. \tag{3.70} \]

We can decompose \( K \) into massless and massive modes,

\[ K = x + d^c \gamma, \quad x \in H^{1,1}(M, \mathbb{C}), \tag{3.71} \]

where \( x \in H^{1,1}(M, \mathbb{C}) \) represents the Kähler moduli, which are not integrated over, and \( \gamma \in \Omega^3(M) \) contains the massive modes of \( K \). Using the above decomposition we can write the Kähler gravity action without non–local terms as follows:

\[ S_{\text{Kaehler}}[x, \gamma] = \int_M \frac{1}{2} d\gamma \wedge d^c \gamma + \frac{1}{3} K \wedge K \wedge K. \tag{3.72} \]

Lagrangian D–branes of the A–model are charged under \( \gamma \), implying that these branes are sources for \( K \) and hence modify the integral of \( K \) on 2–cycles which link them. This also implies that the partition function of the A–model depends non–perturbatively on the choice of a cohomology class in \( H^3(M) \) as well as on \( x \in H^2(M) \).

Let us now implement the stochastic quantization procedure in the supersymmetric approach. The seven–dimensional theory can be written as

\[ S_{7d} = \int d\bar{\theta}d\theta dt \int_M \bar{D}\Gamma \wedge *D\Gamma + S_{\text{Kaehler}}[x, \Gamma], \tag{3.73} \]

where \( D \) and \( \bar{D} \) are the super–covariant derivatives, \( * \) is the Hodge star on the 6d manifold, and \( \Gamma \) is the superfield defined by

\[ \Gamma = \gamma + \bar{\theta}\psi + \hat{\psi}\theta + \bar{\theta}\theta F, \tag{3.74} \]

where \( \psi, \hat{\psi}, F \) are all three–forms in space. Note that \( x \) contains the Kähler moduli and is not promoted to a superfield. Just as we have seen above, the classical action now takes the role of a superpotential.

The above derivation is purely formal: the properties of the seven dimensional action have to be understood, and it is not obvious that the stochastic quantization of Kähler gravity is the most direct way towards a description of the seven dimensional topological M–theory. On the other hand we believe that some of the general features that we encountered in the previous examples (quantum theory in \( d + 1 \) dimensions whose ground state reproduces the partition function of the classical theory in \( d \) dimensions, supersymmetry in the extra dimension) are to be expected for such a theory.
4 Conclusions

In this article we have discussed a rather widespread, even though often unrecognized, scheme to relate a classical to a quantum field theory. In condensed matter physics, this method of quantization gave rise to the (discrete) quantum dimer model and its generalization, the quantum crystal. In the continuous case, this approach goes under the name of stochastic quantization. In topological field theories it has appeared for example in relating the WZW model to the Chern–Simons action. Its main features are the following.

1. The classical theory lives on a \( d \)-dimensional manifold \( \Sigma \), while the \((d + 1)\)-dimensional (quantum) theory lives on the manifold \( Y = \Sigma \times [0, T] \).

2. In the Hamiltonian description, the quantum theory admits a zero–energy ground state wavefunction \( \Psi_0 \). Its norm square reproduces the classical partition function \( Z_{\text{cl}} \):

\[
\langle \Psi_0 | \Psi_0 \rangle = Z_{\text{cl}} .
\]  

3. The quantum theory can be interpreted as a Markov process that at equilibrium converges to the minima of the classical action. In this sense, the quantum theory describes the same physics in the \( T \to \infty \) limit as the classical theory.

4. There is a natural supersymmetric structure in the extra dimension.

Based on point number 2 above, one might even venture to say that every theory that admits a holomorphic factorization of its partition function might have a \((d + 1)\)-dimensional partner whose ground state square results in the partition function of the original theory.

In this note, we have supplied a number of examples of models which are as such widely known, while their relation to each other by stochastic quantization on the other hand is largely unrecognized.

We have done two things:

1. Having recognized a common principle in seemingly unrelated models, we are able to overcome some of the problems encountered in previous works, notably by making use of the inherent supersymmetry.

2. Since this quantization scheme underlies the quantum crystal, which is deeply related to topological strings, we would like to propose it as an approach towards topological M–theory.

We have sketched some ideas for interpreting the quantum crystal on the string theory side of the correspondence, exploring interpretations in the WDW framework and in terms of a non–Lorentz invariant limit of topological M–theory. Much work remains to be done in both directions and a more thorough study of these ideas is left for future research.
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