Abstract

It is shown that curvature-dimension bounds \( CD(N, k) \) for a metric measure space \((X, d, m)\) in the sense of Sturm imply a weak \( L^1 \)-Poincaré-inequality provided \((X, d)\) has \( m \)-almost surely no branching points.

1. Introduction. From analysis on manifolds and metric measure spaces \((X, d, m)\) the fundamental importance of Poincaré-type inequalities for the regularity of harmonic, Lipschitz or Sobolev functions is known (cf. \[Stu96\], \[Che99\], \[HK00\] and \[Hei01\], \[SC02\], \[AT04\]). In this note we show that metric measure spaces \((X, d, m)\) with upper dimension-lower Ricci curvature \( CD(N, k) \) bounds in the generalized sense of Sturm \[Stu05\] (cf. \[LV04\] for \( CD(N, 0) \)) support a weak local \( L^1 \)-Poincaré inequality provided the choice of geodesics by which mass travels between cut points in \((X, d)\) can be made symmetric. By some additional argument on the support of the measure \( m \) we show that this will be the case if we assume moreover that the underlying metric admits \( m \)-almost surely no branching points.

The main step is to establish a version of the segment inequality of Cheeger-Colding \[CC96\] which in particular implies the Poincaré inequality in the sense of upper gradients. What is needed for the proof is only a quantitative version of the Bishop-Gromov volume comparison theorem which is called \((N, k)\)-measure contraction property in \[Oht05\] and which is implied by the Lott-Villani-Sturm dimension-curvature bounds.

The assumption on the \( m \)-almost sure absence of branching points of \((X, d)\) may be quite restrictive. However, as a first step in understanding the full meaning of curvature-dimension bounds for metric measure spaces it may be a useful task to study the regularity of admissible spaces \((X, d, m)\) without branching points and study their relation with Alexandrov spaces, for instance.

\* Currently visiting Courant Institute with the support of the Alexander von Humboldt Foundation (AvH).
2. Preliminary study: Proof of the segment inequality on smooth manifolds by mass transportation.

For illustration let us derive the segment inequality in the smooth case using the language of mass transportation. This approach proved recently very useful for the generalization of certain concepts in smooth Riemannian analysis to general metric measure spaces [vRS04, Stu05a, LV04, Stu05b].

We review some standard terminology first. Throughout this note we call a curve \( \gamma : [0, 1] \to (X, d) \) in a metric space a geodesic (segment) if \( d(\gamma(s), \gamma(t)) = d(\gamma(0), \gamma(1)) |s - t| \) for all \( s, t \in [0, 1] \). For \( A, B \subset X \) we define the set \( \Gamma(A, B) \) as the collection of all geodesics \( \gamma \) with \( \gamma(0) \in A \) and \( \gamma(1) \in B \). For \( x, y \in X \) any \( \gamma \in \Gamma(x, y) \) will be denoted \( \gamma_{xy} \) which may not be unique. If \( X = M \) is a Riemannian manifold \((M^n, g)\) then \( d \) is the intrinsic metric induced by \( g \).

**Proposition.** ([CC96]) Let \((M^n, g)\) be a smooth Riemannian manifold with Ricci curvature \( \text{Ricc}_M \geq (n-1)\kappa \), \( \kappa \in \mathbb{R} \). Let \( A_1, A_2 \subset B_R \) be measurable subsets contained in a geodesic \( R \)-Ball and let \( g : B_{2R} \to \mathbb{R} \) be a nonnegative measurable function, then

\[
\int_{A_1} \int_{A_2} \int_0^1 g(\gamma_{xy}(s)) dtdxdy \leq C_n(n, D)(|A_1| + |A_2|) \int_{B_{2R}} g(z) dz,
\]

with

\[
C_n(n, D) = \sup_{t \in [\frac{1}{2}, 1]} t \left( \frac{s_k(kD)}{s_k(tD)} \right)^{n-1}
\]

and \( D = D(A_1, A_2) = \sup_{(x,y) \in A_1 \times A_2} d(x,y) \leq \text{Diam}(A_1 \cup A_2) \).

Here and in the sequel \( s_k \) denotes the usual Sturm-Liouville function

\[
s_k(t) = \begin{cases} 
sin(\sqrt{k}t) & \text{if } k > 0 \\
t & \text{if } k = 0 \\
sinh(\sqrt{-k}t) & \text{if } k < 0.
\end{cases}
\]

Let us recall the necessary basic facts from optimal mass transportation theory we need (cf. [CEMS01] and [Vil03] as general reference).

Let \( \mu_0 \) and \( \mu_1 \) be two probability measures on a Riemannian manifold \( M^n \) with \( \text{Ricc}(M) \geq (n-1)\kappa \) and let \( \tau_t : M \to M \), \( t \in [0, 1] \) the optimal transportation map associated to the \( L^2 \)-Wasserstein metric with the squared distance as cost function. General theory says that for \( \mu_0 \ll d\text{vol}_g \) this map is of the form

\[
\tau_t(x) = \exp_x(-t\nabla \psi(x))
\]

where \( \psi : M \to \mathbb{R} \) is a \( d^2/2 \)-concave function (i.e \( \psi \) it is the inf-convolution of another function \( \phi \) with respect to the potential \( d^2/2 \)).

The main ingredient of the proof given below is the concavity estimate for the \( n \)-th root of the Jacobian of \( \tau_t \) with respect to \( t \) which is one
of the main results in the fundamental paper \cite{CEMS01}. It reads as follows. If \(J_t(x) := \det d\tau_t(x)\) denotes the Jacobian determinant of the map \(\tau_t\) in \(x\) then
\[
J_t^{1/n}(x) \geq \tau^{(1-t)}_k(x)J_0^{1/n}(x) + \tau^{(1-t)}_k(x)J_1^{1/n}(x)
\]
with
\[
\tau^{(t)}_k(x) := t^{1/n} \left( \frac{s_k(t \delta x(x), \tau_1(x))}{s_k(d(x, \tau_1(x)))} \right)^{1-1/n} =: \tau^{t,n}_k(t \delta x(x, \tau_1(x))).
\]
Here \(d(\tau_1(x), x)\) is the distance between a point \(x\) and its target point \(\tau_1(x)\). It is obtained using the structure of \(\tau_t\) and Jacobi-field estimates together with the central arithmetic-geometric mean inequality on nonnegative matrices (cf. \cite{Stu05b} for a nice presentation).

**Proof of the Riemannian segment inequality.** Let be \(A_1\) and \(A_2\) be two sets which are both embedded in a larger ball \(B_R\) and let \(y \in A_2\) by a point. Let \(\mu_t = \tau_t \cdot \delta_y\) the Wasserstein geodesic connecting \(\mu_0 = \frac{1}{|A_1|} \delta x_{1,A_1}\) on \(A_1\) with \(\delta_y\), the Dirac measure in \(y\). From the structure of it is clear that each \(x \in A_1\) travels along a geodesic connecting \(y\) with \(x\) (i.e. \(\psi(x) = (d\mu_1^y)/(1 \cdot n)\) in \(\tau_t\)). Since the cut locus of \(y\) has measure zero we can assume below that there is only one such geodesic for each pair \((x, y) \in M \times M\). Let \(g : B_{2R} \to \mathbb{R}\) be nonnegative, then
\[
\frac{1}{|A_1|} \int_{A_1} \int_0^{1/2} g(\gamma_{xy}(s))dtdx = \frac{1}{|A_1|} \int_{A_1} \int_0^{1/2} g(\tau^{(t)}_y(x))dtdx
\]
which by the general integral transformation formula equals
\[
= \int_0^{1/2} \int_{A_1} g(z)\mu_z(z)dtdz = \int_0^{1/2} \int_{A_1} g(z)\frac{m_0(x(t, z))}{\det d\tau_t(x(t, z))}dtdz
\]
with \(A_{1_t} := \{ \gamma_{xy}(t) \mid y \in A_1\}\) and where we used the Jacobi identity for \(m_t = \frac{d\mu_t}{dx}\)
\[
m_t(\tau_t(.))\det d\tau_t(.) = m_0(.
\]
with \(x(t, z) = \tau_t^{-1}(z)\) being the origin of the transport ray which hits \(z\) at time \(t\). By the Jacobian concavity
\[
\frac{1}{\det d\tau_t(x(t, z))} \leq \left( \tau^{(1-t)}_k(x(t, z)) + \tau^{(t)}_k(x(t, z)) \det d\tau_1^{1/n}(x(t, z)) \right)^{-n}
\]
where in the present case \(\det d\tau_1(x(t, z)) = 0\) since \(\mu_1 = \delta_y\). (To see this approximate \(\mu_1 = \delta_y\) by the family \(\mu_1^t = \frac{1}{m_0(y)} d\mu_{B_1(y)}\) and use Jacobian identity for the density \(m_1^t\) of \(\mu_1^t\)
\[
m_1^t(z) = \frac{1}{|B_1(y)|} \frac{\mu_{B_1(y)}(z)}{d\mu_{B_1(y)}(z)} = \frac{m_0(x(z))}{\det d\tau_1^t(x(z))} = \frac{1/|A_1| \frac{\mu_{A_1}(x(z))}{d\mu_{A_1}(x(z))}}{\det d\tau_1^t(x(z))}
\]
which implies \(\det \det d\tau_1^t(x(z)) = \frac{|B_1(y)|}{|A_1|}\) for all \(z \in B_1(y)\), equivalently \(\det d\tau_1(x) = \frac{|B_1(y)|}{|A_1|}\) for all \(x \in A_1\) or \(\det d\tau_1^t(x(t, z)) = \frac{|B_1(y)|}{|A_1|}\) for all
\[ z \in A_1, \text{ and let } \epsilon \to 0. \text{ Since } m_0(x(t, z)) = \mathbb{1}_{A_1}(z)/|A_1| \text{ we arrive at the estimate} \]
\[
\frac{1}{|A_1|} \int_{A_1} \int_0^1 g(\gamma_{xy}(s))dtdx \leq \int_0^1 \int_{A_1} g(z)m_0(x(t, z)) \left( \tau^t(1-t)(x(t, z)) \right)^{-n} dzdt \leq \frac{1}{2|A_1|} \sup_{t \in [0, 1]} (\tau^t(x))^{-n} \int_{B_{2R}} g(z)dz,
\]

where we used \( A_1, A_2 \subset B_R \). Integration respect to \( y \in A_2 \) yields
\[
\int_{A_1} \int_{A_2} \int_0^1 g(\gamma_{xy}(s))dtdxdy \leq C_\kappa(n, D) \int_{B_{2R}} g(z)dz
\]
where \( C_\kappa(n, D) \) is the constant as defined in the statement of the proposition. This is because the expression for \( \tau^t(x) \) is monotone increasing in \( d(x, y) \). - Using the symmetry of the integral estimate we can bound the expression
\[
\int_{A_1} \int_{A_2} \int_0^1 g(\gamma_{xy}(s))dtdxdy \leq C_\kappa(n, D) \frac{|A_2|}{2} \int_{B_{2R}} g(z)dz
\]
by repeating the previous arguments to the corresponding integral over the time interval \([0, \frac{1}{2}]\) when \( A_1 \) and \( A_2 \) are interchanged (see also section 3.) which by adding the two estimates concludes the proof. \( \square \)

In the estimate above we defined the geodesic to be parameterized on the unit interval. Using unit speed parameterization it reads
\[
\int_{A_1} \int_{A_2} \int_0^d(x, y) \frac{g(\gamma_{xy}(t))}{d(x, y)}dtdxdy \leq \frac{1}{2} C_\kappa(n, D)(|A_1| + |A_2|) \int_{B_{2R}} g(z)dz.
\]

3. Segment Inequality on metric measure spaces with transportation lower Ricci bounds.

3.1. Measure contraction.

We start out from the following definition which is a quantified version of the measure contraction property formulated in \cite{Stu98}.

**Definition.** \cite{Oht05} A metric measure space \((X, d, m)\) is having the \((N, k)\) measure contraction property if for each pair \((x, M) \in X \times 2^X\) there exists a probability measure \(\Pi\) on the set of geodesics \(\Gamma(x, M) = \{ \gamma_{xy} | y \in A \}\) with \(e_0, \Pi = m_{\mid M} := \frac{1}{|M|} m_{\mid M}\) and \(e_1, \Pi = \delta_x\) such that
\[
dm \geq e_{t*} \left( (1 - t) \frac{s_k((1 - t)\ell(\gamma))}{s_k(\ell(\gamma))} \right)^{N-1} \mu(M)d\Pi,
\]
where \(e_t : \Gamma \to X\) with \(e_t \gamma := \gamma(t), t \in [0, 1]\) is the evaluation map.
Let us abbreviate this property by $\text{MCP}(N,k)$. Its meaning is the following. Disintegrating the measure $\Pi$ with respect to the evaluation map $e_0$ and using the condition that $e_0_*\Pi = m_M$ we obtain a mixing representation $\Pi(d\gamma) = \lambda_y(d\gamma)m_M(y)$ where the measures $\lambda_y$ are supported on $\Gamma(y,x)$. Moreover, the measures $\lambda_{yx}$ are determined uniquely by $\Pi$ for $m$-almost $y$ and vice versa.

Let now $M_t := e_t\Gamma(M,x)$ be the set hit by all geodesics from $M$ to $x$ then the statement above is equivalent to

$$m(Z) \geq (1-t) \int_M \int_{\Gamma(y,x)} \mathbb{1}_Z(\gamma) \left\{ \frac{s_k((1-t)d(x,y))}{s_k(d(x,y))} \right\}^{N-1} \lambda_y(d\gamma)m(dy)$$

(MCP)

for all measurable $Z$ with w.l.o.g $Z \subset M_t$, since for $Z \subset X \setminus M_t$ the right hand side is zero. - Written in this form the MCP-condition gives a lower bound for the concentration of $m$ under the generalized homothetic map defined by the Markov kernels $\Lambda^y_t(dx) = e_t^\ast(\lambda_y)(dx)$ (cf. next section). It may also be seen as a requirement on the minimal 'mean spreading' of all geodesics to $x$, where the mean is taken with respect to the collection of weights $(\lambda_y)$ and $m$.

The relevance of the $(N,k)$-measure contraction property is its robustness with respect to the measured Gromov-Hausdorff convergence (cf. [Oht05]). Moreover, it is implied by the generalized dimension-curvature bounds defined recently by Sturm.

**Definition.** ([Stu05b]) A metric measure space $(X,d,m)$ satisfies the curvature dimension condition $\text{CD}(N,k)$ if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M,d,m)$ there exists an optimal $d^2$-coupling $q \in \mathcal{P}(X \times X)$ and a geodesic $\Gamma : [0,1] \to P_2(X,d,m)$ connecting $\nu_0$ and $\nu_1$ such that

$$S^N_{N'}(\Gamma(t)|m) \leq -\int_{X \times X} \left[ r^{(1-t)}_{K,N'}(d(x_0,x_1) \cdot \rho_0^{-1/N'}(x_0)} + r^{t}_{K,N'}(d(x_0,x_1) \cdot \rho_1^{-1/N'}(x_1)} \right]dq(x_0,x_1)$$

where $\rho_i = d\mu_i/dm$ are the respective densities of $\mu_i$, $S^N_{N'}(\mu|m) = -\int_X \rho^{-1/N'}dm$ is the Renyi entropy of a measure $d\mu = \rho dm$ with respect to $m$ and the functions $r_{K,N'}$ are defined as above.

The following is easily verified (see [Stu05b], proposition 1.7.iv.]).

**Proposition.** $\text{CD}(N,k)$ implies $\text{MCP}(N,k)$ for a metric measure space $(X,d,m)$.

From the results in [CEMS01] one deduces that smooth Riemannian manifolds $(M^n,g)$ with $\text{Ricc}(M^n,g) \geq (n-1)k$ satisfy the $\text{CD}(n,k)$ condition. - One of the main results in Sturm’s theory is the stability of the $\text{CD}(N,k)$ bounds with respect to convergence.

**Theorem.** ([Stu05b]) For fixed $N,k \in \mathbb{R}$ the set of metric measure spaces satisfying the $\text{CD}(N,k)$ is closed with respect to measured Gromov-Hausdorff convergence.
In particular, measured Gromov-Hausdorff limits of \( CD(N, k) \)-spaces will satisfy the \((N, k)\)-measure contraction property. (The corresponding stability result of the \((N, K)\)-MCP condition alone is a little stronger and is obtained in [Oht05] based on [LV04]).

3.2. Segment inequality for \((X, d, m)\).

For the precise formulation of the subsequent results we need a little more notation. Let \( B \subset X \) be a set and \( t \in [0, 1] \). Then we define the following set valued geodesic contraction map in direction \( B \)

\[
\Gamma_t(\cdot, B) : 2^X \to 2^X; \quad \Gamma_t(A, B) := e_t(\Gamma(A, B)).
\]

By abuse of notation for \( t \in [0, 1] \) we define also \( \Phi_{-t}(A, B) = \{ z \in X | \exists b \in B, \gamma_{zb}(t) \in A \} \) the inverse \( A \) with respect to geodesic contraction in direction \( B \). When \( B = \{ b \} \) we write \( A_t(b) := \Gamma_t(A, \{ b \}) \) and \( A_t^{-1}(b) := \Gamma_{-t}(A, \{ b \}) \).

Note that in the \((N, k)\)-MCP statement above the transference plan \( \Pi \) and thus the measures \( (\lambda_{yx})_{y \in M} = (\Lambda_{yx}^M)_{y \in M} \) may depend on \( M \). Let us say that the family of measures \( (\lambda_{yx}^M)_{x,y \in X} \) is symmetric in \( (x, y) \) if

\[
\lambda_{xy}^M(\gamma) = \lambda_{yx}^M(\gamma),
\]

where \( \gamma \to \gamma \) is the inversion map \( \gamma(t) = \gamma(1-t), t \in [0, 1] \).

**Proposition.** Let \((X, d, m)\) satisfy the \((N, k)\)-measure contraction property and assume that the map \((x, y) \to \lambda_{xy}^M \in \mathcal{P}(\Gamma_{xy})\) can be chosen to be symmetric for \( m \times m \) almost every pair \((x, y)\) and for some set \( M \subset X \). Then for two measurable subset \( A_1, A_2 \subset B_R \) contained in a geodesic ball \( B_R \subset M \) and \( g : B_{2R} \to \mathbb{R} \) nonnegative

\[
\int \int \int_{A_1 \times A_2 \times \Gamma_{(x,y)}} g(\gamma_{xy}(t)) dt \lambda_{xy}^M(\gamma) m(dx)m(dy) \\
\leq \frac{1}{2} C_n(n, D)(|A_1| + |A_2|) \int_{B_{2R}} g(z) m(dz).
\]

**Proof.** We write the \((N, k)\)-measure contraction inequality relative to the ambient set \( M \) yet in another form, namely for all \((x, t) \in X \times [0, 1] \)

\[
m \geq (\tau_t^x \cdot \Lambda_t^x) \ast m,
\]

where the sign \( \ast \) means convolution with the transition kernel

\[
(\tau_t^x \cdot \Lambda_t^x)(y, dz) = \tau(x, y) \Lambda_t^x(y, dz)
\]

with the symmetric function \( \tau(x, y) = t \left\{ \frac{A_t((1-t)d(x,y))}{A_t(d(x,y))} \right\}^{N-1} \) and the Markov kernel

\[
(y, Z) \to \Lambda_t^x(y, Z) = \int_{\Gamma(y, x)} \mathbb{1}_Z(\gamma_t) \lambda_{gz}(d\gamma), \quad y \in X, Z \subset X.
\]
Here we omitted the upper index for $\lambda_{yx} = \lambda_{yx}^M$ as we shall in the rest of the proof because $M$ is fixed.

With this notation the $(N, k)$-measure contraction inequality is written

$$\int_Z g(z)m(dz) \geq \int_X \int_Z g(y)t(x, z)\Lambda^x_t(z, dy)m(dz)$$

for all measurable $Z \subset X$ and nonnegative measurable $f : X \to \mathbb{R}$. Note that it suffices to take the outer integral on the set $Z_t^{-1}(x)$. Since $d(z, x) = d(\gamma_{zx}(t), x)/(1 - t)$ for all $z \in X$ we have

$$\inf_{z \in Z_t^{-1}(x)} [\tau_t(x, z)] = \inf_{z \in Z} [\tau_t(d(x, z)/(1 - t))]$$

such that the estimate

$$\int_Z g(z)m(dz) \geq \inf_{z \in Z} [\tau_t(d(x, z)/(1 - t))] \int_X \int_Z g(y)\Lambda^x_t(z, dy)m(dz).$$

is obtained. For $A \subset X$ set $Z = A_1(x)$, then under the assumption $A \subset B_\infty$, $x \in B_1$ it follows $A_1(x) \subset B_2$, thus

$$\int_{B_2} g(z)m(dz) \geq \inf_{z \in A} [\tau_t(z, x)] \int_X \int_{A_1(x)} g(y)\Lambda^x_t(z, dy)m(dz).$$

Integration of this inequality with respect to time yields

$$\int_0^1 \int_X \int_{A_1(x)} g(y)\Lambda^x_t(z, dy)m(dz)dt \leq \frac{1}{2} \sup_{t \in [0, 1]} [\tau_t^{-1}(x, z)] \int_{A_1(x)} g(z)m(dz).$$

Since

$$\int_X \int_{A_1(x)} g(y)\Lambda^x_t(z, dy)m(dz) = \int_{\Gamma_0(A_1(x), x)} \int_{A_1(x)} g(y)\Lambda^x_t(z, dy)m(dz)$$

$$\geq \int_A \int_{\Gamma(X, X)} g(\gamma(t))\lambda_{zx}(d\gamma)m(d\gamma)$$

the estimate

$$\int_A \int_0^1 \int_{\Gamma(X, X)} g(\gamma(t))\lambda_{zx}(d\gamma)m(d\gamma)dt \leq \frac{1}{2} \sup_{t \in [0, 1]} [\tau_t^{-1}(x, z)] \int_A g(z)m(dz).$$

is obtained. Now put $A = A_1$ and integrate the last inequality with respect to $x \in A_2$. This leads to

$$\int_{A_2} \int_A \int_0^1 \int_{\Gamma(X, X)} g(\gamma(t))\lambda_{zx}(d\gamma)m(d\gamma)$$

$$\leq \frac{1}{2} |A_2| \sup_{t \in [0, 1]} [\tau_t^{-1}(x, z)] \int_A g(z)m(dz)$$

$$\leq \frac{1}{2} |A_2| C_k(N, D) \int_{A_1 \cup A_2} g(z)m(dz),$$

7
with \( C_k(N,D) \) and \( D = D(A_1,A_2) \) defined as in the smooth case above.

Finally, interchanging the roles of \( A_1 \) and \( A_2 \) and using the symmetry of the measures \((\lambda_{xy})\) we obtain

\[
\frac{1}{2} |A_2| C_k(N,D) \int_{B_{2R}} g(z)m(dz)
\geq \int_{A_1} \int_{A_2} \int_{0}^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t)) \lambda_{xz}(d\gamma)m(dx)m(dz)
\]

\[
= \int_{A_1} \int_{A_2} \int_{0}^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t)) \lambda_{xz}(d\gamma)m(dx)m(dz)
\]

\[
= \int_{A_1} \int_{A_2} \int_{0}^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t)) \lambda_{xz}(d\gamma)m(dx)m(dz)
\]

\[
= \int_{A_1} \int_{A_2} \int_{0}^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t)) \lambda_{xz}(d\gamma)m(dx)m(dz)
\]

\[
= \int_{A_1} \int_{A_2} \int_{0}^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t)) \lambda_{xz}(d\gamma)m(dx)m(dz)
\]

Adding this inequality to the first one the claim is established. \(\square\)

**Corollary.** The assertion of the proposition above is true in particular when the cut-locus \( C_x := \{ y \in X \mid \#\Gamma(x,y) \geq 2 \} \) satisfies \( m(C_x) = 0 \) for \( m \)-a.e. \( x \in X \).

**Proof.** If \( x \notin C_y \) then \( \lambda_{xy} = \delta_{\gamma_{xy}} \) for the unique \( \gamma_{xy} \in \Gamma(x,y) \). This forces \( \lambda_{xy} \) to be symmetric \( m \times m \)-almost surely. \(\square\)

For the following version of the Poincaré inequality recall that for \( f : X \to \mathbb{R} \) the function \( g : X \to \mathbb{R}_+ \) is called an upper gradient if

\[
|f(x) - f(x)| \leq \int_{0}^{d(x,y)} h(\gamma_s)ds
\]

for any unit speed geodesic connecting \( x \) and \( y \).

**Corollary.** (\( L^1 \)-Poincaré-inequality) Under the conditions above let \( h \) be an upper gradient of \( f \) then

\[
\int_{B_{2R}} \int_{B_{2R}} \frac{|f(x) - f(y)|}{d(x,y)} m(dx)m(dy) \leq |B_R| C_{n,k}(D) \int_{B_{2R}} h(x)m(dx).
\]

**Proof.** In order to prove this inequality for each pair \((x,y)\) let \( \lambda_{xy} \) be the associated measure from proposition above, then the assertion follows from

\[
\frac{|f(x) - f(y)|}{d(x,y)} = \int_{\Gamma_{xy}} |f(\gamma(0)) - f(\gamma(1))| \lambda_{xy}(d\gamma) \leq \int_{\Gamma_{xy}} \int_{0}^{1} h(\gamma_s)ds \lambda_{xy}(d\gamma)
\]

\[8\]
which can be inserted into the segment inequality.

\[ \square \]

**Examples.** Consider the Banach space \( (X, d) = (\mathbb{R}^n, \|\cdot\|_p) \), \( p \in (1, \infty) \) equipped with \( m = \lambda^n \) the \( n \)-dimensional Lebesgue measure, where \( \|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \). The geodesics are straight Euclidean line segments \( \gamma_{xy} = x + t(x - y) \), hence \( C_x = \emptyset \) for all \( x \in X \). Obviously \( (\mathbb{R}^n, \|\cdot\|_p, \lambda^n) \) satisfies \( MCP(0, n) \). - The case \( (\mathbb{R}^n, \|\cdot\|_1, \lambda^n) \) is a little more interesting, since here \( C_x = X \) for all \( x \in X \). However, choosing \( \lambda_{xy}(d\gamma) = \delta_{(x+ty)}(d\gamma) \) for all \( x, y \in X \) the \( MCP(0, n) \) property remains true. Since this choice of \( \lambda \) is symmetric, the segment and Poincaré inequalities hold.

### 3.3. Extendable geodesics and branching points

**Definition.** Let \( (X, d) \) be a metric space and \( x \in X \). Define \( I_p := \{\gamma_{xp}(t) \mid t \in (0, 1], x \in X\} \) as all points \( x \) which are connected to \( p \) by at least one extendable geodesic segment and let \( T_p = X \setminus I_p \).

**Proposition.** Let \( (X, d, m) \) satisfy an \( (N, k) \)-measure contraction property then \( m(T_p) = 0 \) for all \( p \in X \).

**Proof.** The proof is an adaption of the idea behind proposition 3.1. in [OS94]. - Note first that \( I_p = \bigcup_{t \in (0, 1]} X_t(p) \) where the sets \( X_t(p) \) are monotone decreasing for \( t \in [0, 1] \), i.e. \( X_t(p) \subset X_s(p) \subset \emptyset \) for \( s \leq t \). Let \( A \subset X \) be a measurable bounded set and choose the weight functions \( \lambda_{xy} = \lambda_{xy}^M \) for \( M \) large enough such that \( A \subset M_{s-1}(p) \) for all \( t \in [0, 1] \). The measure contraction inequality at time \( t \) applied to the set \( A \cap X_t(p) \) yields

\[
m(A \cap X_t(p)) \geq (1 - t) \int_X \frac{s_k((1-t)d(p,y))}{s_k(d(p,y))} \lambda_{yp}^t(A \cap X_t(p))m(dy)
\]

\[
= (1 - t) \int_X \frac{s_k((1-t)d(p,y))}{s_k(d(p,y))} \lambda_{yp}^t(A)m(dy)
\]

because \( \lambda_{yp}^t(A \setminus X_t(p)) = 0 \) for all \( y \in X \). Hence for \( s \leq t \) we obtain from the monotonicity of the family \( (M_s(p))_{s \in [0, 1]} \) and

\[
m(A \cap X_s(p)) \geq (1 - t) \int_X \frac{s_k((1-t)d(p,y))}{s_k(d(p,y))} \lambda_{yp}^t(A)m(dy).
\]

Sending \( s \to 0 \), since \( X_s(p) \searrow I_p \) for \( s \searrow 0 \) by monotone convergence

\[
m(A \cap I_p) \geq (1 - t) \int_X \frac{s_k((1-t)d(p,y))}{s_k(d(p,y))} \lambda_{yp}^t(A)m(dy).
\]

Finally, upon sending \( t \to 0 \) and using \( \lambda_{yp}^0(dz) = \delta_y(dz) \) dominated convergence yields

\[
m(A \cap I_p) \geq m(A),
\]

which by the arbitrariness of \( A \) establishes the claim that \( I_p \) has full \( m \)-measure.
A geodesic tripod is a space $(X,d)$ obtained by metric gluing of three segments in one common endpoint which is called the center.

**Definition.** A point $p \in X$ in a metric space $(X,d)$ is called branching point if it is the center an embedded geodesic tripod.

**Corollary.** If $(X,d)$ admits no branching points $m$-almost surely and $(X,d,m)$ satisfies an $\text{MCP}(n,k)$-property then $m(C_x) = 0$ for $x \in X$. Also, the segment and Poincaré inequalities hold for $(X,d,m)$ in this case.

**Proof.** Since for $m$-almost all $y \in C_x$ at least one $\gamma_{xy} \in \Gamma(x,y)$ can be extended beyond $y$ as segment, $y$ must be a branching point. By assumption branching points are $m$-negligible. □

**Remark.** The example $(\mathbb{R}^n, \| \cdot \|_1, \lambda^n)$ shows that the $\text{MCP}(N,k)$-property is not strong enough to prevent a 'large' (with respect to $m$) amount of branching points, even if branching points indicate infinite negative sectional curvature in Alexandrov sense. It is natural to ask which additional assumptions on $(X,d,m)$ inhibit a set of branching points with positive $m$-mass. For example, $(X,d)$ admits no branching if it is a limit of Riemannian manifolds with uniform local lower sectional curvature bounds.

**Acknowledgments.** Thanks to Jeff Cheeger for raising the question and to Karl-Theodor Sturm sending his preprints. Thanks also to Shin-ichi Ohta in particular for bringing the preprint [RM02] to my attention. Ranjbar-Motlagh obtains very similar results independent of the mass transportation approach but assumes a 'strong local doubling' property of $m$ instead.

**References**

[AT04] Luigi Ambrosio and Paolo Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[CC96] Jeff Cheeger and Tobias H. Colding. Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)*, 144(1):189–237, 1996.

[CEMS01] Dario Cordero-Erausquin, Robert J. McCann, and Michael Schmuckenschläger. A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. *Invent. Math.*, 146(2):219–257, 2001.

[Che99] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[Hei01] Juha Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.
[HK00] Piotr Hajlasz and Pekka Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[LV04] John Lott and Cedric Villani. Ricci curvature for metric measure spaces via optimal transportation. 2004. Preprint, http://arxiv.org/abs/math.DG/0412127

[Oht05] Shin-Ichi Ohta. On measure contraction property of metric measure spaces. 2005. Preprint, available at http://www.math.kyoto-u.ac.jp/~sohta/

[OS94] Yukio Otsu and Takashi Shioya. The Riemannian structure of Alexandrov spaces. *J. Differential Geom.*, 39(3):629–658, 1994.

[RM02] Alireza Ranjbar-Motlagh. On the Poincaré inequality for abstract spaces. 2002. Preprint, Sharif University of Technology, Teheran.

[SC02] Laurent Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[Stu96] Karl-Theodor Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl. (9)*, 75(3):273–297, 1996.

[Stu98] Karl-Theodor Sturm. Diffusion processes and heat kernels on metric spaces. *Ann. Probab.*, 26(1):1–55, 1998.

[Stu05a] Karl-Theodor Sturm. Generalized Ricci bounds and convergence of metric measure spaces. *C. R. Math. Acad. Sci. Paris*, 340(3):235–238, 2005.

[Stu05b] Karl-Theodor Sturm. On the geometry of metric measure spaces II. 2005. Preprint.

[Vil03] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[vRS04] Max-K von Renesse and Karl-Theodor Sturm. Transport inequalities, gradient estimates, entropy and Ricci curvature. *Comm. Pure Appl. Math.*, 58(7):923–940, 2004.