LAMBERT-W SOLVES THE NONCOMMUTATIVE \( \Phi^4 \)-MODEL

ERIK PANZER AND RAIMAR WULKENHAAR

Abstract. We show that the closed Dyson-Schwinger equation for the 2-point function of the noncommutative \( \lambda \phi^4 \)-model can be rearranged into the boundary value problem

\[ \Psi(a+, b+) \Psi(a-, b-) = \Psi(a+, b-) \Psi(a-, b+) \]

for a sectionally holomorphic function \( \Psi \). This expresses the 2-point function as Hilbert transform of an angle function which itself satisfies a highly non-linear integral equation. A solution of that equation as formal power series in \( \lambda \) shows a surprisingly simple structure. The solution to 10th order is matched by Stirling numbers of the first kind. Its extrapolation to all orders is resummed with the Lagrange-Bürmann formula to Lambert-W. This leads to an explicit exact formula of the 2-point function, real-analytic at any coupling constant \( \lambda > -1/(2 \log 2) \), in terms of Lambert-W.

1. Main result

In this paper we give strong evidence for

Conjecture 1. The non-linear integral equation for a function \( G_{\lambda}: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \),

\[
(1 + a + b) G_{\lambda}(a, b) = 1 + \lambda \int_0^\infty dp \frac{G_{\lambda}(p, b) - G_{\lambda}(a, b)}{p - a} + \frac{G_{\lambda}(a, b)}{1 + p} \\
+ \lambda \int_0^\infty dq \frac{G_{\lambda}(a, q) - G_{\lambda}(a, b)}{q - b} + \frac{G_{\lambda}(a, b)}{1 + q} \\
- \lambda^2 \int_0^\infty dp \int_0^\infty dq \frac{G_{\lambda}(a, b) G_{\lambda}(p, q) - G_{\lambda}(a, q) G_{\lambda}(p, b)}{(p - a)(q - b)},
\]

which describes a scaling limit of the connected 2-point function of the \( \lambda \phi^4 \)-model with harmonic propagation on 2-dimensional noncommutative Moyal space, is for any real coupling constant \( \lambda > -1/(2 \log 2) \approx -0.721348 \) solved by

\[
G_{\lambda}(a, b) = G_{\lambda}(b, a) = \frac{(1 + a + b) \exp(N_{\lambda}(a, b))}{(b + \lambda W \left( \frac{1}{\lambda} e^{(1+a)/\lambda} \right) (a + \lambda W \left( \frac{1}{\lambda} e^{(1+b)/\lambda} \right))}, \quad \text{where} \quad \lambda > 0
\]

\[
N_{\lambda}(a, b) := \frac{1}{2\pi i} \int_{-\infty}^\infty dt \log \left( 1 - \frac{\lambda \log(\frac{t}{2} - it)}{b + \frac{\lambda}{2} + it} \right) d\log \left( 1 - \frac{\lambda \log(\frac{1}{2} + it)}{a + \frac{\lambda}{2} - it} \right).
\]

Here, \( W \) denotes the Lambert function [7, 20], more precisely its principal branch \( W_0 \) for \( \lambda > 0 \) and the other real branch \( W_{-1} \) for \(-1 < \lambda < 0 \) of the solution of \( W(z) e^{W(z)} = z \). The function \( N_{\lambda}(a, b) \) defined for \( \lambda > -1/(2 \log 2) \) has a perturbative expansion into Nielsen polylogarithms [22].
As by-product we establish identities involving the Lambert function:

\[
\int_0^\lambda \frac{dt}{t} \frac{1}{1 + \frac{e^{1/t + \alpha/\lambda}}{W}}} = \log a - \log \left( \frac{e^{(1+a)/\lambda}}{\lambda W} \right) - 1, \tag{4}
\]

\[
\frac{1}{\pi} \int_0^\infty dp \arctan \left( \frac{\lambda \pi}{1 + p - \lambda \log p} \right) = \log \left( \sqrt{1 + a + b - \lambda \log a} + (\lambda \pi)^2 \right)^\lambda, \tag{5}
\]

where \( \int \) denotes a principal value integral and the arctan ranges in \([0, \pi]\).

We explain in sections 2 and 3 how the integral equation (1) arises from a quantum field theory model on a noncommutative geometry. In sec. 4 we rewrite (1) as a boundary value problem for a sectionally holomorphic function in two variables which can partially be integrated to a function \( \tau(a) \) of a single variable. In sec. 5 we determine the first terms of a formal power series for \( \tau(a) \). These terms are surprisingly simple and allow to guess the whole formal power series in \( \lambda \). We resum the series in sec. 6 and convert the result into a manifestly symmetric form in sec. 7. Further treatment of the final integral (3) will be postponed to subsequent work. Some initial thoughts are given in sec. 8. We finish by a discussion (sec. 9).

2. The setup

The \( \lambda \phi^4 \)-model with harmonic propagation on the 2-dimensional Moyal plane is defined by the action functional [13]

\[
S(\phi) = \int_{\mathbb{R}^2} d^2x \left( \frac{1}{2} \left( -\Delta + 4 \Omega^2 |\Theta^{-1}x|^2 + \mu^2 \right) \phi + \frac{\lambda}{4} \phi \phi \phi \phi \right)(x), \tag{6}
\]

where \( \mu^2, \lambda, \Omega \) are real numbers (mass\(^2\), coupling constant, oscillator frequency) and \( \star \) denotes the Moyal product with deformation matrix \( \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \), \( \theta > 0 \).

The Moyal product is defined by the oscillatory integral

\[
(\phi \star \psi)(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{dydk}{(2\pi)^2} \phi(x + \frac{1}{2}\Theta k)\psi(x) e^{i(y,k)}.
\]

There exists a family \( \{f_{mn}\}_{m,n \in \mathbb{N}} \) of “matrix basis functions” on \( \mathbb{R}^2 \) which satisfy \((f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x), \langle f_{mn}(x) \rangle = f_{nm}(x) \) and \( \int_{\mathbb{R}^2} dx f_{mn}(x) = 2\pi \theta \delta_{mn} \). See e.g. [10]. The resulting correspondence

\[
\phi(x) = \sum_{m,n=0}^\infty \Phi_{mn} f_{mn}(x) \quad \Leftrightarrow \quad \Phi_{kl} = \frac{1}{2\pi \theta} \int_{\mathbb{R}^2} dx \phi(x) f_{lk}(x) \quad (7)
\]

defines an isomorphism of Fréchet algebras between Schwartz functions with Moyal product and infinite matrices with rapidly decaying entries. This isomorphism extends to Moyal products between other classes of functions. Real functions \( \phi \) are represented by self-adjoint matrices \( \Phi \).
At critical frequency $\Omega = 1$, the matrix basis functions satisfy

$$(-\Delta + 4 |\Theta^{-1} x|^2) f_{mn} = \frac{4}{\Omega} (m + n + 1) f_{mn}.$$ 

Therefore, at $\Omega = 1$ the isomorphism (7) leads to

$$S(\phi) \equiv S(\Phi) \overset{\Omega=1}{=} V \text{ tr}(E\Phi^2 + \frac{4}{3}\Phi^4),$$

where $V = \frac{4}{\Omega}$ and $E = (E_m \delta_{mn})$ with $E_m = \frac{4}{\Omega^2} + \frac{m}{\Omega}$. For the next steps matrix sizes are restricted to $m, n \leq N$. Now the (Fourier transform of the) partition function is well-defined:

$$Z(J) := \int D\Phi \exp(-S(\Phi) + iV \text{ tr}(J\Phi)),$$

where $D\Phi$ is the Lebesgue measure on $\mathbb{R}^{(N+1)^2}$ and $J$ another matrix with (for $N \to \infty$) rapidly decaying entries. As usual for matrix models, the logarithm of $Z(J)$ has an expansion into boundary cycles ($p^\beta_{N+1} \equiv p^\beta_1$)

$$\log \frac{Z(J)}{Z(0)} = \sum_{B=1}^{\infty} \sum_{N_B \geq N_1 \geq \cdots \geq N_B} \frac{V^{2-B}}{S_{N_1 \ldots N_B}} G_{[p_1 \ldots p_{N_1} | \ldots | p_B \ldots p_{N_B}]} \prod_{\beta=1}^{B} \left( \prod_{j=1}^{N_\beta} \frac{iJ_{j\beta} p^\beta_{j+1}}{p^\beta_j} \right).$$

3. Dyson-Schwinger equation for the 2-point function

The following Dyson-Schwinger equations for the 2- and 4-point functions were derived in [14], here with $O(\frac{1}{N})$-terms suppressed$^1$:

$$G_{[ab]} = \frac{1}{E_a + E_b} + \frac{(-\lambda)}{E_a + E_b} \left( \frac{1}{V} \sum_{m,n=0}^{N} G_{[ab]G_{[am]} - \frac{1}{V} \sum_{a\neq n=0}^{N} \frac{G_{[ab]} - G_{[nb]}}{E_a - E_n} \right),$$

$$G_{[abcd]} = (-\lambda) \frac{G_{[ab]}G_{[cd]} - G_{[ad]}G_{[cb]}}{(E_a - E_c)(E_b - E_d)}. $$

These equations rely on a Ward identity discovered in [8]. By the same techniques one can derive another Dyson-Schwinger equation for the 2-point function (again with $O(\frac{1}{N})$-terms suppressed):

$$G_{[ab]} = \frac{1}{E_a + E_b} + \frac{(-\lambda)}{E_a + E_b} \left( \frac{1}{V^2} \sum_{m,n=0}^{N} G_{[bamn]} + \frac{1}{V} \sum_{n=0}^{N} G_{[ab]}(G_{[an]} + G_{[nb]}) \right).$$

This Dyson-Schwinger equation has an obvious graphical interpretation. The proof combines [14, equations (3.2) and (3.3)] in our conventions and for $a \neq b$ to

$$G_{[ab]} = \frac{1}{E_a + E_b} + \frac{(-\lambda)}{(E_a + E_b)V^3} \sum_{m,n=0}^{N} \frac{\partial^4(\exp(\log \frac{Z(J)}{Z(0)}))}{\partial J_{ba}\partial J_{am}\partial J_{mn}\partial J_{nb}} \bigg|_{J=0}.$$

$^1$A behaviour $\sum_n \sim V$ is assumed so that the sums in (10) are kept.
Generically the \( J \)-differentiations yield the 4-point function \( G_{[barn]} \) to be summed over \( m, n \). But there are also the cases \( m = b \) or \( n = a \) where a disconnected product of 2-point functions contributes in \( \exp(\log(Z(J))) \), producing the last terms in (12). Other contributions such as \( m = n = a \) and \( m = n = b \) are \( \mathcal{O}(\frac{1}{N}) \)-suppressed.

We eliminate \( \sum_n G_{[an]} \), \( \sum_n G_{[nb]} \) in (12) via (10) and express \( G_{[barn]} \) in (12) via (11). The sums can safely exclude \( m = b \) and \( n = a \) because these contribute with an exceeding \( \frac{1}{N} \)-factor which is anyway ignored. We have thus proved:

**Lemma 2.** The 2-point function of the \( \lambda \phi^4 \)-model with harmonic propagation

on 2-dimensional Moyal space satisfies in matrix representation (with cut-off \( N \), up to \( \mathcal{O}(\frac{1}{N}) \)-corrections) the following closed Dyson-Schwinger equation:

\[
G_{[ab]} = \frac{1}{E_a + E_b} \left\{ 1 - \frac{(-\lambda)^2}{V^2} \sum_{a \neq n = 0}^{N} \sum_{b \neq m = 0}^{N} \frac{G_{[ab]} G_{[am]} - G_{[an]} G_{[nb]}}{(E_a - E_n)(E_b - E_m)} \right. \\
- \left. \frac{(-\lambda)}{V} \sum_{a \neq n = 0}^{N} \frac{G_{[ab]} G_{[nb]} - G_{[an]} G_{[nm]}}{E_a - E_n} - \frac{(-\lambda)}{V} \sum_{b \neq m = 0}^{N} \frac{G_{[ab]} - G_{[an]}}{E_b - E_m} \right\}.
\]

Compared with (10) this equation in manifestly symmetric in \( a, b \) and contains the \( G \)-quadratic terms in a more regular way.

As in [14] we take a combined limit \( N, V \to \infty \) with \( \frac{2}{N} = \Lambda^2 \) fixed. In this limit, \( E_n \mapsto \frac{\mu^2}{2} + p \) with \( p = \frac{\mu}{N} \), \( \frac{N}{\Lambda^2} \in [0, \Lambda^2] \). The 2-point function becomes a function \( G(a, b) \) of real arguments \( a, b \in [0, \Lambda^2] \), and the densitised sums converge to principal value Riemann integrals\(^2\) over \([0, \Lambda^2] \):

\[
(a + b + \mu^2)G(a, b) = 1 + \lambda \int_0^{\Lambda^2} dp \frac{G(p, b) - G(a, b)}{p - a} + \lambda \int_0^{\Lambda^2} dq \frac{G(a, q) - G(a, b)}{q - b} \\
- \lambda^2 \int_0^{\Lambda^2} dp \int_0^{\Lambda^2} dq \frac{G(a, b)G(p, q) - G(a, q)G(p, b)}{(p - a)(q - b)}. \tag{13}
\]

This equation is exact: the previously ignored \( \mathcal{O}(\frac{1}{N}) \)-terms are strictly absent.

4. A BOUNDARY VALUE PROBLEM À LA GAKHOV

We employ a method described in Gakhov’s book [9] on boundary value problems.\(^3\) Consider for \( w, z \in \mathbb{C} \setminus [0, \Lambda^2] \) the integral

\[
Q(z, w) = \frac{1}{\pi^2} \int_0^{\Lambda^2} dp \int_0^{\Lambda^2} dq \frac{G(p, q)}{(p - z)(q - w)}. \tag{14}
\]

\(^2\)In fact for \( G(a, b) \) Hölder-continuous the ordinary integral arises. Since the next step takes the numerators apart we write principal values already here.

\(^3\)RW would like to thank Alexander Hock for pointing out this reference.
The Plemelj formulae give for $\sigma_1, \sigma_2 = \pm 1$

$$Q(a + \ii \sigma_1 \epsilon, b + \ii \sigma_2 \epsilon) = \frac{1}{\pi^2} \int_0^{\Lambda^2} \frac{dp}{p - a} \int_0^{\Lambda^2} \frac{dq}{q - b} G(p, q) - \sigma_1 \sigma_2 G(a, b)$$

Equation (15) is reverted to (with$^4 a \pm \equiv a \pm \ii \epsilon$)

$$\frac{1}{\pi^2} \int_0^{\Lambda^2} \frac{dp}{p - a} \int_0^{\Lambda^2} \frac{dq}{q - b} G(p, q)$$

These identities are inserted into (13), where also $\int_0^{\Lambda^2} \frac{dp}{p - a} = \log \frac{\Lambda^2 - a}{a}$ is used:

$$-\frac{\lambda^2 \pi^2}{4} \left( Q(a+, b+)Q(a-, b-) - Q(a+, b-)Q(a-, b+) \right)$$

$$= 1 + \frac{1}{4} \left( a + b + \mu^2 + \lambda \log \frac{\Lambda^2 - a}{a} + \lambda \log \frac{\Lambda^2 - b}{b} \right) Q(a+, b+) + \frac{1}{4} \left( a + b + \mu^2 + \lambda \log \frac{\Lambda^2 - a}{a} + \lambda \log \frac{\Lambda^2 - b}{b} \right) Q(a-, b-)$$

This suggests the renormalisation $\mu^2 = 1 - 2\lambda \log(1 + \Lambda^2)$. Now (16) can easily be rearranged as follows:

**Theorem 3.** The closed integral equation (13) for the 2-point function of the $\lambda \phi^4$-model with harmonic propagation on 2-dimensional Moyal space is in the scaling limit $N \to \infty$ subject to $\frac{N}{2} = \Lambda^2$ fixed equivalent to the following boundary value problem: Define by

$$\Psi(z, w) := 1 + z + w + \lambda \log \frac{\Lambda^2 - z}{(-z)(1 + \Lambda^2)} + \lambda \log \frac{\Lambda^2 - w}{(-w)(1 + \Lambda^2)}$$

$$+ \lambda^2 \int_0^{\Lambda^2} \frac{dp}{p - z} \int_0^{\Lambda^2} \frac{dq}{q - w} G(p, q)$$

$^4$Here and throughout this paper a limit $\epsilon \to 0$ in all $i\epsilon$-descriptions is self-understood.
a function holomorphic on \((\mathbb{C} \setminus [0, \Lambda^2])^2\). Then
\[
\Psi(a+, b+)\Psi(a-, b-) = \Psi(a+, b-)\Psi(a-, b+) .
\] (17)

Note that \(\Psi\) has a well-defined limit \(\Lambda \to \infty\). We find it remarkable that a complicated interacting quantum field theory which when expanded into Feynman graphs evaluates to a huge number of Nielsen polylogarithms (see later) admits such simple presentation. Unfortunately we are not aware of a solution theory for such boundary value problems, and therefore develop an ad hoc approach in the sequel.

Observe that (17) can be written as \(|\Psi(a+, b+)\) = \(|\Psi(a+, b-)|\). Hence there is a real function \(\tau_a(b)\), not symmetric in \(a, b\), with
\[
\Psi(a+, b+)e^{-i\tau_a(b)} = \Psi(a+, b-)e^{i\tau_a(b)}. \tag{18}
\]

Inserting (15) with gives with the finite Hilbert transform
\[
\mathcal{H}_\Lambda^a[f(\bullet)] := \frac{1}{\pi} \int_0^{\Lambda^2} \frac{dp}{p-a} f(p) \quad \text{in one variable and}
\]
\[
\mathcal{H}_{a,b}^\Lambda[f(\bullet, \bullet)] := \frac{1}{\pi^2} \int_0^{\Lambda^2} \frac{dp}{p-a} \int_0^{\Lambda^2} \frac{dq}{q-b} f(p, q)
\]
in two variables:
\[
(1 + a + b + \lambda \log \frac{\Lambda^2-a}{(1+\Lambda^2)q} + \lambda \log \frac{\Lambda^2-b}{(1+\Lambda^2)p} + 2\pi i
+ \lambda^2 \pi^2 (\mathcal{H}_{a,b}^\Lambda[G(\bullet, \bullet)] - G(a, b) + i\mathcal{H}_a^\Lambda[G(a, \bullet)] + i\mathcal{H}_b^\Lambda[G(\bullet, b)])e^{-i\tau_a(b)}
= (1 + a + b + \lambda \log \frac{\Lambda^2-a}{(1+\Lambda^2)q} + \lambda \log \frac{\Lambda^2-b}{(1+\Lambda^2)p}
+ \lambda^2 \pi^2 (\mathcal{H}_{a,b}^\Lambda[G(\bullet, \bullet)] + G(a, b) + i\mathcal{H}_a^\Lambda[G(a, \bullet)] - i\mathcal{H}_b^\Lambda[G(\bullet, b)])e^{i\tau(a,b)}
\]
or
\[
G(a, b) \cot(\tau_a(b)) - \mathcal{H}_b^\Lambda[G(a, \bullet)] = \frac{1}{\lambda \pi}, \tag{19}
\]
\[
G(a, b) \cot(\tau_a(b)) - \mathcal{H}_b^\Lambda[G(a, \bullet)] = \frac{1 + a + b + \lambda \log \frac{\Lambda^2-a}{(1+\Lambda^2)q} - \lambda \log(1+\Lambda^2)}{\lambda^2 \pi^2},
\]
where \(G(a, b) := \frac{1}{\Lambda^2} + \mathcal{H}_a^\Lambda[G(\bullet, b)]\). These equations are Carleman-type singular integral equations for which a solution theory is developed e.g. in [28, §4.4]. It turns out that the cases \(\lambda > 0\) and \(\lambda < 0\) must be carefully distinguished and are not both compatible with the conventions in [28]. We therefore repeat and adapt the algebraic solution strategy of Carleman equations.

Compatibility with perturbation theory requires \(G(a, b) > 0\) for all \(a, b\) at least in a neighbourhood of \(\lambda = 0\). Then (19) together with continuity of the angle function – necessary for its Hilbert transform – imply the convention
\[
\tau_b(a) \in [0, \pi] \quad \text{for } \lambda > 0, \quad \tau_b(a) \in [-\pi, 0] \quad \text{for } \lambda < 0. \tag{20}
\]
Consider the function \( \varphi^\pm_a(z) := \exp \left( \frac{\mp i}{\pi} \int_0^{\Lambda^2} dp \, \frac{\tau_a(p)}{p-z} \right) - 1 \) which (for \( \tau_a \) continuous) is holomorphic on \( \mathbb{C} \setminus [0, \Lambda^2] \) and vanishes for \( z \to \infty \). The general properties (Plemelj formulae or dispersion relations) of the Hilbert transform imply that the Hilbert transform – here understood as integral over the whole real line – of the imaginary part of \( \varphi^+_a(\bullet+) \) is the real part of \( \varphi^-_a(b+) \). Taking \( \frac{1}{\pi} \int_0^{\Lambda^2} dp \, \frac{\tau_a(p)}{p-b} = H^\Lambda_b[\tau_a(\bullet)] + i\tau_a(b) \) into account, we arrive at

\[
\pm \mathcal{H}^\Lambda_b[e^{\pm \mathcal{H}^\Lambda_b[\tau_a]} \sin \tau_a(\bullet)] = e^{\pm \mathcal{H}^\Lambda_b[\tau_a]} \cos \tau_a(b) - 1, \tag{21}
\]

which is essentially \([28, \S 4.4, \text{eqs.} \ (18)+(28)]\). Chosing the upper sign we can immediately solve or partially solve (19) to

\[
G(a, b) = e^{\mathcal{H}^\Lambda_b[\tau_a(\bullet)]} \sin \tau_a(b) \frac{1}{\lambda \pi},
\]

\[
G(a, b) = \frac{1 + a + \lambda \log \left( \frac{\Lambda^2 - a}{1 + \Lambda^2} \right) - \lambda \log(1 + \Lambda^2)}{\lambda \pi} G(a, b) + \gamma_a(b), \tag{22}
\]

where \( \gamma_a \) solves \( \gamma(b) \cot(\tau_a(b)) = -\mathcal{H}^\Lambda_b[\gamma(\bullet)] \). Let \( \beta_a(z) := \frac{1}{\pi} \int_0^{\Lambda^2} dt \, \frac{\tau_a(t)}{t-z} \) for \( z \in \mathbb{C} \setminus [0, \Lambda^2] \). The dispersion relations \( \gamma_a(b) = \frac{\beta_a(b+) - \beta_a(b-)}{2} \) and \( \mathcal{H}^\Lambda_b[\gamma_a(\bullet)] = \frac{\beta_a(b+)}{\Lambda^2} \sin \tau_a(b) \) give

\[
\beta_a(b+)e^{-i\tau_a(b)} - \beta_a(b-)e^{i\tau_a(b)} = \frac{2ib}{\lambda \pi^2} \sin \tau_a(b). \tag{23}
\]

The standard ansatz is \( \beta_a(z) = F_a(z) e^{T_a(z)} \) for sectionally holomorphic functions \( F_a, T_a \) which are chosen according to \( e^{T_a(b+)} e^{-i\tau_a(b)} = e^{T_a(b-)} e^{i\tau_a(b)} \). That equation is solved by \( T_a(z) = \frac{1}{\pi} \int_0^{\Lambda^2} dp \, \frac{\tau_a(p)}{p-z} \), which turns (23) into \( F_a(b+) - F_a(b-) = e^{-\mathcal{H}^\Lambda_b[\tau_a]} \frac{b}{\lambda \pi^2} \sin \tau_a(b) \) with solution

\[
F_a(z) = \frac{1}{\pi} \int_0^{\Lambda^2} dp \, \frac{dp}{p-z} e^{-\mathcal{H}^\Lambda_b[\tau_a]} \frac{p}{\lambda \pi^2} \sin \tau_a(p), \quad z \notin [0, \Lambda^2].
\]

We easily extract \( F_a(b\pm) \) and obtain together with \( T_a(b\pm) \) derived before the solution

\[
\gamma_a(b) = \frac{e^{\mathcal{H}^\Lambda_b[\tau_a]} \sin(\tau_a(b))}{\lambda \pi^2} \left( be^{-\mathcal{H}^\Lambda_b[\tau_a]} \cos(\tau_a(b)) + \mathcal{H}^\Lambda_b[e^{-\mathcal{H}^\Lambda_b[\tau_a]} \bullet \sin(\tau_a(\bullet))] \right)
\]

\[
= \frac{G(a, b)}{\lambda \pi} \left( b + \frac{1}{\pi} \int_0^{\Lambda^2} dp \, e^{-\mathcal{H}^\Lambda_b[\tau_a]} \sin(\tau_a(p)) \right), \tag{24}
\]

where the 2nd line follows from \( \bullet = (\bullet - b) + b \) together with the lower sign in (21). Inserting (24) together with \( G(a, b) = \frac{1}{\lambda \pi} + \mathcal{H}^\Lambda_a[G(\bullet, b)] \) back into the 2nd equation (22) we obtain exactly the symmetric partner \( a \leftrightarrow b \) of the 1st equation.
provides the unique (partial) solution. Note that Carleman-type singular integral equations permit solutions of the homogeneous case because of $\mathcal{H}_b^\lambda \frac{\sin \tau_b(p)}{\Lambda^2 - b} = \frac{\sin \tau_b(0)}{\Lambda^2 - b}$.

In particular, $\cot \tau_0(a) = \frac{a}{\Lambda^2} + \cot \tau_0(a)$. This equation and (22) are analogous to identities for the $\lambda \phi^4_1$-model [14] derived by a completely different strategy. We summarise the results obtained so far for the limit $\Lambda^2 \to \infty$:

**Proposition 4.** Let $\mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \int_0^\infty \frac{dp}{p^2 - a} f(p)$ denote the one-sided Hilbert transform. The boundary value problem of Theorem 3 is partially solved by $G_\lambda(a, b) = \frac{\sin \tau_0(a)}{\Lambda^2 - a} e^{\mathcal{H}_a[\tau_0(\bullet)]}$, where the angle function obeys the non-linear integral equation

\[
\lambda \pi \cot \tau_b(a) = 1 + a + b - \lambda \log a + I_\lambda(a), \quad \text{where}
\]

\[
I_\lambda(a) := \frac{1}{\pi} \int_0^\infty dp \left( e^{-\mathcal{H}_a[\tau_0(\bullet)]} \sin \tau_a(p) - \frac{\lambda \pi}{1 + p} \right)
\]  

and the branch $\tau_0(a) \in [0, \pi]$ for $\lambda \geq 0$ and $\tau_0(a) \in [-\pi, 0]$ for $\lambda \leq 0$ is chosen.

We do not claim that Proposition 4 provides the unique (partial) solution. Note that Carleman-type singular integral equations permit solutions of the homogeneous case because of $\mathcal{H}_b^\lambda \frac{\sin \tau_b(0)}{\Lambda^2 - b} = \frac{\sin \tau_b(0)}{\Lambda^2 - b}$. We discuss this freedom in sec. 9.

5. Perturbative solution

We try to solve (26) as a formal power series in $\lambda$. This strategy leads surprisingly far. The solution clearly starts with $\tan \tau_0(a) = \frac{\lambda \pi}{1 + a + b} + \mathcal{O}(\lambda^2)$ which gives $G_\lambda(a, b) = \frac{\lambda \pi}{1 + a + b} + \mathcal{O}(\lambda)$.

5.1. 1st order. From $\tau_0(a) = \frac{\lambda \pi}{1 + a + b} + \mathcal{O}(\lambda^2)$ we conclude

\[
I_\lambda(a) = \lambda \int_0^\infty dp \left( \frac{1}{1 + p + a} - \frac{1}{1 + p} \right) + \mathcal{O}(\lambda^2) = -\lambda \log(1 + a) + \mathcal{O}(\lambda^2).
\]

Inserted into (26) gives

\[
\tau_b(a) = \frac{\lambda \pi}{1 + a + b} + \frac{\lambda^2 \pi}{(1 + a + b)^2} \left( \log a + \log(1 + a) \right) + \mathcal{O}(\lambda^3).
\]

This produces $\frac{\sin \tau_0(a)}{\Lambda^2}$ necessary for the 1st order of (22). For the same order we need the Hilbert transform of (27) also to 1st order. This is an elementary exercise; nonetheless we describe our strategy which permitted to compute $\mathcal{H}_a[\tau_b(\bullet)]$ up to $\mathcal{O}(\lambda^4)$ by hand: One should identify $\tau_b(p)$ as imaginary part of a sectionally
holomorphic function with branch cut exactly \( \mathbb{R}_+ \). The Hilbert transform is then the real part of that function:

\[
\mathcal{H}_a \left[ \frac{\pi}{1 + \bullet + b} \right] = \mathcal{H}_a \left[ \text{Im} \left( \frac{-\log(-(\bullet + i\epsilon)) + \log(1+b)}{1 + \bullet + i\epsilon + b} \right) \right] = \text{Re} \left( \frac{-\log(-(a + i\epsilon)) + \log(1+b)}{1 + a + i\epsilon + b} \right) = -\frac{\log(a) + \log(1+b)}{1 + a + b}.
\]

In the 1st line the term \( \log(1+b) \) has vanishing imaginary part but must be added to kill the pole at \( \bullet = -(1+b) \) from the denominator. The above Hilbert transform gives \( e^{\mathcal{H}_a[\tau_0(\bullet)]} = 1 + \frac{\lambda(-\log a + \log(1+b))}{1+a+b} + O(\lambda^2) \).

5.2. 2nd to 4th order. By elementary techniques we pushed the integrations \( I_\lambda(a) \) and \( \mathcal{H}_a[\tau_0(\bullet)] \) up to \( O(\lambda^5) \). This involved polylogarithms up to \( \text{Li}_4 \) which arise via \( \text{Im}(\text{Li}_n(1 + a + i\epsilon)) = \frac{\pi}{(n-1)!} \log(1+a)^n \). We obtained:

\[
I_\lambda(a) = (-\lambda) \log(1+a) + \frac{(-\lambda)^2}{1+a} ((1+a) + a) \log(1+a) - \frac{(-\lambda)^3}{(1+a)^2} \left( (1+a)^2 + a^2 \right) \log(1+a) - \frac{(-\lambda)^4}{(1+a)^3} \left( (1+a)^3 + a^3 \right) \log(1+a) + O(\lambda^5).
\]

This is astonishing because the very many individual contributions to that integral evaluate to combinations of \( \text{Li}_3(a), \text{Li}_3\left(\frac{1}{1+a}\right), \text{Li}_2(-a), \log a, \log(1+a), \zeta(2), \zeta(3) \) of weight \( \leq 3 \), but in the end everything collapses. So the following conjecture arises:

**Conjecture 5.** In the class of formal power series, the integral in (26) evaluates to \( O(\lambda^n) \) into a polynomial in \( \log(1+a) \) of degree \( n-1 \) with coefficients in rational functions of \( a \).

Interestingly, this simple angle function produces for the 2-point function a rich number-theoretic structure. To 2nd order one finds

\[
G_\lambda(a, b) = \frac{1}{1 + a + b} + \frac{\lambda}{(1 + a + b)^2} \left( \log(1+a) + \log(1+b) \right)
\]

\[
- \frac{\lambda^2}{(1 + a + b)^2} \left( \frac{1 + 2a}{a(1+a)} \log(1+a) + \frac{1 + 2b}{b(1+b)} \log(1+b) \right)
\]

\[
+ \frac{\lambda^2}{(1 + a + b)^3} \left( \zeta(2) + (\log(1+a))^2 + (\log(1+b))^2 \right) + \log(1+a) \log(1+b) - \text{Li}_2(-a) - \text{Li}_2(-b) + O(\lambda^3).
\]

(28)
5.3. Higher orders. As illustrated above, the perturbative calculation leads to expressions that are rational linear combinations of polylogarithmic functions. Furthermore, they have only very simple singularities, confined to the hyperplanes
\[ a = 0, \quad b = 0, \quad a + 1 = 0, \quad b + 1 = 0 \quad \text{and} \quad 1 + a + b = 0. \]
The integration theory on such hyperplane complements\(^5\) is completely understood [5, 25] in terms of iterated integrals, and computer implementations are available [3, 23]. We note that there is also an alternative approach based on the toolbox of holonomic recurrences [2, 26].

Using HyperInt [23], we confirmed Conjecture 5 up to corrections in \( \mathcal{O}(\lambda^{11}) \). As an illustration, let us consider the Hilbert transform of (27). With
\[
> \text{read } "\text{HyperInt.mpl}";
> \text{tau := } \text{Pi}/(1+a+b)^2*(\log(a)+\log(1+a));
> \text{H := hyperInt(eval(tau, a=p)/(p-a)/Pi, p=0..infinity);} \]
one computes the coefficient \( H \) of \( \lambda^2 \) of the integral
\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{d}{p-a} \tau(p).
\]
The program HyperInt assumes that \( a \) has a non-zero imaginary part and computes the integral over \( p \) along positive reals \( p \in (0, \infty) \). The result is revealed with the command
\[
> \text{fibrationBasis(H, [a,b]);}
\]
\[
\text{Hlog (b, [0, -1]) + Hlog (b, [-1, -1]) - Hlog (a, [-1, 0]) - Hlog (a, [0, 0]) + 4} \zeta(2)
\]
\[
+ \frac{\text{Hlog (b, [-1])}}{1 + a + b} \left( \frac{1}{b} + \frac{1}{1 + b} \right) + I\pi \delta_a \frac{\text{Hlog (a, [0])} + \text{Hlog (a, [-1])}}{(1 + a + b)^2}
\]
which brings the expression into a normal form where \( \text{Hlog (a, [-1])} = \log(1 + a) \) and \( \text{Hlog (a, [\sigma, \tau])} = \int_{0}^{a} \frac{dz}{z-\sigma} \int_{0}^{z} \frac{dw}{w-\tau} \) denote hyperlogarithms. Some of these can be expressed as polylogarithms, e.g. \( \text{Hlog (b, [0, -1])} = -\text{Li}_2(-b) \). But there is no need to do so, since in the final result for \( I_\lambda \), all multiple polylogarithms cancel and collapse to barely powers of the logarithm \( \log(1 + a) \).

The sign of the imaginary part of \( a \) is denoted by \( \delta_a \). The Hilbert transform \( \mathcal{H} \) is obtained by taking the real part of \( H \), i.e. dropping the term with \( \delta_a \) altogether. The subsequent integration over \( p \) in (26) is even simpler, as no imaginary parts have to be taken care of.

Hence it is straightforward to compute moderately high orders, and we obtained the coefficients of \( \lambda^{\leq 10} \) in this way. The results are of such striking simplicity and structure that we could obtain an explicit formula. Concretely,
\[
I_\lambda(a) = -\lambda \log(1 + a) + \sum_{n=1}^{\infty} \lambda^{n+1} \left( \frac{(\log(1 + a))^n}{n a^n} + \frac{(\log(1 + a))^n}{n(1 + a)^n} \right)
\]
\[
+ \sum_{n=1}^{\infty} \frac{(n-1)! \lambda^{n+1}}{(1+a)^n} \sum_{j=1}^{n-1} \sum_{k=0}^{j} (-1)^j s_{j,n-k} \left( \left( \frac{1+a}{a} \right)^{n-j} + 1 \right) \left( \log(1+a) \right)^k \tag{29}
\]
\(^5\)We note that our case is isomorphic to the moduli space \( \mathfrak{M}_{0,5} \).
correctly reproduces the first 10 terms of the expansion in $\lambda$. We conjecture that it holds true to all orders. By $s_{n,k}$ we denote the Stirling numbers of the first kind, with sign $(-1)^{n-k}$.

We remark that HyperInt can easily find (28) by direct integration of the perturbative expansion of (1).

6. Resummation

The Stirling numbers of first kind have generating function

$$(1 + z)^u = \sum_{n=0}^{\infty} \frac{n^u}{n!} s_{n,k} , \quad (-1)^j s_{j,n-k} = \frac{1}{(n-k)!} \frac{d^{n-k}}{du^{n-k}} \frac{\Gamma(j-u)}{\Gamma(-u)} \bigg|_{u=0}. $$

Let $\left(29\right)_2$ be the 2nd line of (29). Writing also $(\log(1 + a))^k = \frac{d^k}{da^k} (1 + a)^u |_{u=0}$, this line takes the from

$$\left(29\right)_2 = \sum_{n=1}^{\infty} \frac{\lambda^{n+1}}{n(1 + a)^n} \sum_{j=1}^{n-1} \left( \frac{1 + a}{a} \right)^{n-j} + 1 \\
\times \sum_{k=0}^{n} \frac{n^k}{k!} \left( \frac{d^{n-k}}{du^{n-k}} \frac{n!}{j! \Gamma(-u)} \right) \left( \frac{d^k}{du^k} (1 + a)^u \right) \bigg|_{u=0} \quad (30) \tag{30}$$

The summation over $k$ gives for the 2nd line of (30)

$$\frac{d^n}{du^n} \left( \frac{\Gamma(j-u)}{j! \Gamma(-u)} (1 + a)^u \right) \bigg|_{u=0} = \frac{d^n}{du^n} \left( (-1)^j (1 + a)^j \frac{d^j}{da^j} (1 + a)^u \right) \bigg|_{u=0} = (-1)^j (1 + a)^j \frac{d^j}{da^j} (\log(1 + a))^n. $$

This is inserted back into (30) and the 2nd line of (29). Now the first line of (29) is the missing case $j = 0$ to extend $I_\lambda(a)$ to

$$I_\lambda(a) = -\lambda \log(1 + a) + \sum_{n=1}^{\infty} \frac{\lambda^{n+1}}{n} \sum_{j=0}^{n-1} \left( \frac{1}{(1 + a)^{n-j}} + \frac{1}{a^{n-j}} \right) (-1)^j \frac{d^j}{da^j} (\log(1 + a))^n. $$

Writing $\frac{1}{a^{n-j}} = (-1)^{n-1-j} \frac{d^{n-1-j}}{da^{n-1-j}} a$ and similarly for $\frac{1}{(1+a)^{n-j}}$, we thus arrive at

$$I_\lambda(a) = -\lambda \log(1 + a) + \sum_{n=1}^{\infty} \frac{(-\lambda)^{n+1}}{n!} \frac{d^{n-1}}{da^{n-1}} \left( \frac{(\log(1 + a))^n}{1 + a} + \frac{(\log(1 + a))^n}{a} \right) \\
= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} (-\log(1 + a))^n - \lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} (-\log(1 + a))^n \quad (31) \tag{31}$$

There are several ways to sum these series. The most efficient approach seems to be the Lagrange-Bürmann inversion formula [6, 19]:
Theorem 6. Let $\phi(w)$ be analytic at $w = 0$ with $\phi(0) \neq 0$ and $f(w) := \frac{w}{\phi(w)}$. Then the inverse $g(z)$ of $f(w)$, such that $z = f(g(z))$, is analytic at $z = 0$ and given by

$$g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \bigg|_{w=0} \phi(w)^n.$$ (32)

More generally, if $H(z)$ is an arbitrary analytic function with $H(0) = 0$, then

$$H(g(z)) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \bigg|_{w=0} (H'(w)\phi(w)^n).$$ (33)

By virtue of (32), we see upon setting $z = \lambda$ and $\phi(w) = -\log(1 + a + w)$ that the first summand in (31),

$$K(a, \lambda) := \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} (-\log(1 + a))^n,$$ (34)

is the inverse of the function $\lambda(w) = -\frac{w}{\log(1+a+w)}$:

$$K(a, \lambda) = -\lambda \log(1 + a + K(a, \lambda)).$$ (35)

This functional equation is easily solved in terms of Lambert-W [7],

$$K(a, \lambda) = \lambda W \left( \frac{1}{\lambda} e^{\frac{1+a}{\lambda}} \right) - 1 - a.$$ (36)

Let us now turn to the second summand (up to a factor $-\lambda$) in (31),

$$L(a, \lambda) := \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} \frac{(-\log(1 + a))^n}{a}.$$ (37)

This can directly be recognised as (33) with $H(w) = \log(1 + w/a)$, such that

$$L(a, \lambda) = \log \left( 1 + \frac{K(a, \lambda)}{a} \right) = \log \frac{\lambda W \left( \frac{1}{\lambda} e^{(1+a)/\lambda} \right) - 1}{a}.$$ (38)

In conclusion, we have resummed (31) in the form

$$I_\lambda(a) = \lambda W \left( \frac{1}{\lambda} e^{\frac{1+a}{\lambda}} \right) - \lambda \log \left( \lambda W \left( \frac{1}{\lambda} e^{\frac{1+a}{\lambda}} \right) - 1 \right) - 1 - a + \lambda \log a.$$ (39)

Remark 7 (Negative coupling). In the above formulae, we assumed $\lambda > 0$ and $W$ denotes the principal branch $W_0$ of the Lambert W function. However, by Theorem 6, the power series (31) define analytic functions of $\lambda$, i.e. with a non-zero radius of convergence. Hence, $I_\lambda(a)$ extends into the negative domain of $\lambda$. In the standard notation [7] for the Lambert W function, the correct branch in
that negative domain is denoted by $W_{-1}$. That is, (39) remains valid for negative $\lambda$ when we define

$$W(z) := \begin{cases} W_0(z) & \text{for } z \geq 0 \text{ and} \\ W_{-1}(z) & \text{for } z < 0. \end{cases}$$ (40)

With this definition, $\lambda W(z)$ with $z = \frac{1}{\lambda} e^{(1+a)/\lambda}$ extends smoothly from positive $\lambda$ through $\lambda = 0$ to an analytic function for all $\lambda > -1$. Although $W_{-1}(z)$ is well-defined and real for all negative $\lambda$, due to

$$\frac{1}{\lambda} e^{(1+a)/\lambda} \geq -\frac{1}{e} \frac{1}{1+a} \geq -\frac{1}{e} \text{ for all } \lambda < 0 \leq a,$$

beware that the value $K(0, \lambda) = \lambda W_{-1} \left( \frac{1}{\lambda} e^{1/\lambda} \right) - 1$ at $a = 0$ becomes non-zero for $\lambda < -1$, which is incompatible with the perturbation series. This shows that we must restrict to $\lambda > -1$. The non-analyticity keeps in at the critical value $\lambda = -1$, where $K(a, -1) = \sqrt{2a} - \frac{a}{\pi} + O(a^{3/2})$ develops a square root singularity at $a = 0$, corresponding to the branch point of $W_{-1}(z)$ at $z = -e^{-1}$. We also see that $L(0, \lambda) = -\log(1 + \lambda)$ from (38) has a branch cut on $\lambda \in (-\infty, -1]$.

Hence, $I_\lambda(a)$ can only be extended to $\lambda > -1$, if it is to be defined for all $a \geq 0$.

**Remark 8** (Complex coupling). To obtain the maximal domain of analytic continuation of $I_\lambda(a)$ into the complex plane, let us assume a to be fixed. To keep the symmetry $K(a, \lambda) = K(a, \bar{\lambda})$, we consider $K$ in the slit plane $\mathbb{C} \setminus (-\infty, -1 - a]$ and choose the principal branch for the logarithm in the functional equation (35). Note that a branch point of $K(a, \lambda)$ at $\lambda = \lambda^*$ corresponds to a zero at $\lambda = K^* := K(a, \lambda^*)$ of the derivative

$$0 = \frac{d \lambda}{d K} = \frac{\frac{\partial K}{\partial K} \log(1 + a + K)}{(1 + a + K)^2} = \frac{K - (1 + a + K) \log(1 + a + K)}{(1 + a + K)^2},$$

which are given by the complex conjugate pair $K_j^*(a) = e^{1+W_j(-\frac{1+a}{e})} - 1 - a$ where $j \in \{0, -1\}$. Inserting this into (35) gives

$$\lambda_j^*(a) = \frac{1 + a}{W_j(-\frac{1+a}{e})} \text{ where } \lambda_{-1}^* = \overline{\lambda}_0^.$$. (41)

These branch points bound the maximal domain in $\lambda$ for which $K(a, \lambda)$ can be defined analytically (for all $a > 0$ simultaneously). This boundary curve can be parametrised with $\xi + i \eta = W_0(-\frac{1+a}{e})$ as $-e^{1-\eta \cot \eta + i \eta}$ (one finds $\xi = -\eta \cot \eta$ for $\eta \in [0, \infty)$. One sees that $|\lambda_j^*(a)|$ increases with $a$ (and $\eta$), such that the radius of convergence of $I_\lambda(a)$ is given by

$$\frac{1 + a}{|W_j(-\frac{1+a}{e})|}.$$ (42)

Its infimum is unity, taken at $a = 0$. Also note that $\arg \lambda_j^*(a) \to 0$ for $a \to \infty$, such that $\Re_j$ is the only ray in the $\lambda$ plane along which $K(a, \lambda)$ may be analytically continued for all values of $a > 0$. 

LAMBERT-W SOLVES THE NONCOMMUTATIVE $\Phi^4$-MODEL
Remark 9 (Strong coupling). When $\lambda \to \infty$, eventually $z = \frac{1}{\lambda} e^{(1+a)/\lambda}$ fulfils $|z| < 1/e$, such that the power series $W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n$ yields a convergent strong-coupling expansion of $I_{\lambda}(a)$. For $\lambda > 0$, the condition $|z| < 1/e$ is equivalent to

$$\lambda > \frac{1 + a}{W_0 \left( \frac{1 + a}{e} \right)}.$$ 

Remark 10 (Alternative solution). One could express the multiple derivatives via Cauchy’s formula, which after inserting $\frac{1}{n} = \int_0^1 \frac{dt}{t^n}$ gives rise to a geometric series. It results a single residue located at the solution of $1 + a = 1 + z + \lambda t \log(1+z)$.

Remark 11 (Alternative solution II). Starting from the series expansion

$$\log(1 + a) = \sum_{n=1}^{\infty} \frac{(-1)^n a^n}{n!} \text{valid for } |a| < 1,$$

where $s_{m,n}$ are again the Stirling numbers of first kind, we can expand (34) and (37) for $|a| < 1$ as

$$K(a, \lambda) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} s_{m+n-1,n} (-\lambda)^n \frac{a^m}{m!}, \quad L(a, \lambda) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} s_{m+n,n} (-\lambda)^n \frac{a^m}{m!}.$$

The recursion relation $n s_{n,k} = s_{n,k-1} - s_{n+1,k}$ of the Stirling numbers translates into the following PDE:

$$\left[ (1 + a + \lambda) \frac{\partial}{\partial a} + \lambda \frac{\partial}{\partial \lambda} \right] K(a, \lambda) = K(a, \lambda) - \lambda,$$

$$\left[ a \frac{\partial}{\partial a} + \lambda \frac{\partial}{\partial \lambda} \right] L(a, \lambda) = \frac{\partial K(a, \lambda)}{\partial a}.$$ 

(43)

It is straightforward to check that the solutions (36) and (38) solve these equations. In fact, the solution (36) of the first equation (for $K$) is found by Maple and fixed through the boundary conditions $K(0, \lambda) = 0$ and $K(a, 0) = 0$. Remark 7 gives the additional information to select for $\lambda > 0$ the principal branch $W_0$ and for $\lambda < 0$ the branch $W_{-1}$.

Changing variables $(a, \lambda) \mapsto (v = \frac{a}{\lambda}, \lambda)$, the 2nd equation in (43) becomes the ODE

$$\lambda \frac{d}{d\lambda} L(v, \lambda) = -1 + W'(\frac{1}{\lambda} e^{v+\frac{1}{\lambda}}) \frac{1}{\lambda} e^{v+\frac{1}{\lambda}} = -\frac{1}{1 + W(\frac{1}{\lambda} e^{v+\frac{1}{\lambda}})}$$

in which $v$ is merely a parameter. Again, it is easily checked that (38) solves this equation. However, the integration starting from the boundary value $L(v, 0) = 0$

---

6It is well-known that the expansion of Lambert-W at infinity is related to Stirling numbers, see [7]. However, we did not find the precise form we obtain here in the literature.
provides the solution in a different form,

\[ L(a, \lambda) = - \int_0^\lambda \frac{1}{t} \frac{1}{1 + W(\frac{1}{\lambda} e^{1/1+a/\lambda})}. \]  \hfill (44)

We conclude the non-trivial identity (4) given in the beginning.

With \( I(\lambda) \) established in (39) we can resolve (26) for \( \tau_b(a) \). The result is inserted into the 1st equation (22) and gives

\[ G(\lambda, a, b) \equiv G(\lambda, b, a) \]

\[ \exp \left( \frac{1}{\pi} \int_0^\infty \frac{dp}{p-a} \arctan \left( \frac{\lambda \pi}{b + \lambda W(e^{(1+p)/\lambda}) - \lambda \log \left( \lambda W(e^{(1+p)/\lambda}) - 1 \right)} \right) \right) \]

\[ \sqrt{(\lambda \pi)^2 + \left( b + \lambda W(e^{(1+p)/\lambda}) - \lambda \log \left( \lambda W(e^{(1+p)/\lambda}) - 1 \right) \right)^2} \]

For \( \lambda \geq 0 \), \( W \) denotes the principal branch \( W_0 \) of the Lambert function \([7, 20]\), and the branch \([0, \pi]\) of the \( \arctan \) is chosen. For \( -1 < \lambda < 0 \), \( W \) denotes the other real branch \( W_{-1} \) of the Lambert function, and the branch \([-\pi, 0]\) of the \( \arctan \) is chosen.

7. Simplification of the Hilbert transform

For a direct verification that (26) is solved by (39) more information about the Hilbert transform of \( \tau_b(a) \) is necessary. In the following we let \( \gamma_\epsilon \) be the curve in the complex plane which encircles the positive real axis clockwise at distance \( \epsilon \).

Lemma 12. In the previous convention for branches of \( \arctan \) and the Lambert function one has for all \( \lambda > -1 \)

\[ \mathcal{H}_a \left[ \arctan \left( \frac{\lambda \pi}{1+b+p-\lambda \log(p)} \right) \right] = \log \left( \frac{\sqrt{(1+a+b-\lambda \log a)^2 + (\lambda \pi)^2}}{a + \lambda W(e^{(1+b)/\lambda})} \right) \]

\[ \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{dz}{z-a} \log \left( \frac{1 - \lambda \log(-z)}{1+b+z} \right) = \log \left( \frac{1 + a + b}{a + \lambda W(e^{(1+b)/\lambda})} \right). \]  \hfill (46)

Proof. The branch conventions give for any \( p, b \geq 0 \) and \( \lambda > -1 \)

\[ \arctan \left( \frac{\lambda \pi}{1+b+p-\lambda \log(p)} \right) = \text{Im} \left( \log \left( 1+b+p - \lambda \log(-p-i\epsilon) \right) \right) \]

\[ = \text{Im} \left( \log \left( 1 - \frac{\lambda \log(-p-i\epsilon)}{1+b+p} \right) \right). \]  \hfill (47)

We divide the integral of the Hilbert transform into \([0, \delta]\) and \([\delta, \infty)\). The contribution of \([0, \delta]\) can be bounded by \( O(-\delta \log \delta) \). For any \( p \in [\delta, \infty) \) the logarithm
in (47) has finite radius of convergence in $\lambda$. Denoting by $\delta$ equality up to $\mathcal{O}(-\delta \log \delta)$ we have for $a > \delta$

$$
\mathcal{H}_a \left[ \arctan \left( \frac{\lambda \pi}{1+b+\bullet - \lambda \log(\bullet)} \right) \right]
\begin{equation}
\delta = \sum_{n=1}^{\infty} \frac{\lambda^n}{\pi n} \int_0^\infty \frac{dp}{p - (a-\delta)} \frac{\text{Im} \left( (\log(-p - \delta - ie))^n \right)}{(1+b+p+\delta)^n}
\end{equation}
\begin{equation}
= \sum_{n=1}^{\infty} \frac{\lambda^n}{\pi n! \partial^n \partial_1^{n-1}} \int_0^\infty \frac{dp}{p - (a-\delta)} \text{Im} \left( \frac{(-\log(-p - \delta - ie))^n - (-\log(1+b-\delta - ie))^n}{(1+b+p+\delta)^n} \right)
\end{equation}
\begin{equation}
\delta = \sum_{n=1}^{\infty} \frac{\lambda^n}{n! \partial^n \partial_1^{n-1}} \text{Re} \left( \frac{(-\log(-a - ie))^n - (-\log(1+b-\delta - ie))^n}{(1+b+a)} \right).
\end{equation}
(48)

From the 2nd to 3rd line we wrote $(1+b+p+\delta)^{-n}$ as $(n-1)$-fold derivative and subtracted $(-\log(1+b-\delta - ie))^n$ which has vanishing imaginary part for $\epsilon \to 0$. This term cancels the pole at $p + \delta = -(1+b)$. The extension to $p \in (-\infty,-\delta]$ has vanishing imaginary part, and the missing $[-\delta,0]$ can be estimated by $\mathcal{O}(-\delta \log \delta)$. The Hilbert transforms thus extends to $\mathbb{R}$ and produces via Plemelj formulae the real part of the sectionally holomorphic integrand at $p = a - \delta$. The contributions of $(-\log(-a - ie))^n$ are straightforward to differentiate and resum to $\log \left( \sqrt{1 + a + b - \lambda \log a + \frac{(\lambda \pi)^2}{1+b+a}} \right)$. The other term is again a Lagrange-Bürmann problem (33) for $\phi(w) = -\log(1+b+w)$ and $H(w) = \log(1 + \frac{w}{1+a+b})$.

Therefore, the result (46) follows first up to $\mathcal{O}(-\delta \log \delta)$ and $|\lambda|$ small enough, but then extends to all $\lambda > -1$ because of analyticity of both sides. The limit $\delta \to 0$ is now safe.

Alternatively, the Cauchy integral of (47) becomes

$$
\mathcal{H}_a \left[ \arctan \left( \frac{\lambda \pi}{1+b+\bullet - \lambda \log(\bullet)} \right) \right] + i \arctan \left( \frac{\lambda \pi}{1+b+a - \lambda \log a} \right)
\begin{equation}
= \frac{1}{\pi} \int_0^\infty \frac{dp}{p - (a + ie)} \text{Im} \left( \log \left( 1 - \frac{\lambda \log(-p - ie)}{1+b+p} \right) \right)
\end{equation}
\begin{equation}
= \frac{1}{2\pi i} \int_{\gamma_{ie}} \frac{dz}{z - (a + ie)} \log \left( 1 - \frac{\lambda \log(-z)}{1+b+z} \right).
\end{equation}
(49)

We subtract and add the same integral but with curve $\gamma_{2\epsilon}$. The difference retracts to the residue at $z = a$. Its imaginary part is, unsurprisingly, the imaginary part in the 1st line of (48). Spelling out the real part and shifting $a + ie \mapsto a$, now safe, we arrive at the 2nd equation (46).

We are now able to split $\mathcal{H}_a[\tau_0(\bullet)]$ into an explicit formula and an integral which is symmetric in $a,b$:
Proposition 13. Let \( \tau_\lambda(a) = \arctan \left( \frac{\lambda\pi}{1 + a + b - \lambda \log a + I_\lambda(a)} \right) \) where \( I_\lambda(a) \) is given in (39). For any \( \lambda > -\frac{1}{2\log 2} \) one has
\[
\mathcal{H}_a[\tau_\lambda(\bullet)] = \log \sqrt{(1 + \lambda W(\frac{1}{\lambda} e^{(1+a)/\lambda}) - \lambda \log(\lambda W(\frac{1}{\lambda} e^{(1+a)/\lambda}) - 1)) \lambda \log(1 + \lambda W(\frac{1}{\lambda} e^{(1+a)/\lambda})))} + (\lambda\pi)^2
\]
\[
+ \log \left( \frac{(1 + a + b) \exp(N_\lambda(a, b))}{(b + \lambda W(\frac{1}{\lambda} e^{(1+a)/\lambda}))(a + \lambda W(\frac{1}{\lambda} e^{(1+b)/\lambda}))} \right),
\]
where \( N_\lambda(a, b) = N_\lambda(b, a) \) is given by
\[
N_\lambda(a, b) = \frac{1}{2\pi i} \int_{\gamma_e} dz \log \left( 1 - \frac{\lambda \log(-z)}{1 + b + z} \right) \frac{d}{dw} \log \left( 1 - \frac{\lambda \log(1 + z + w)}{1 + a - (1 + z + w)} \right) \bigg|_{w=0}
\]
or equivalently by (3). In particular, formula (2) follows.

Proof. A substitution \( p = u + \lambda \log(1 + u) \) with inverse \( u = \lambda W(\frac{1}{\lambda} e^{(1+p)/\lambda}) - 1 \) in the integration variable leads to the alternative formula
\[
\mathcal{H}_a[\tau_\lambda(\bullet)] + i\tau_\lambda(a) = \frac{1}{\pi} \int_0^\infty du \left( \frac{1 + u + \lambda}{1 + u} \right) \arctan \left( \frac{\lambda\pi}{1 + b + u - \lambda \log u} \right).
\]
Similar as in in (49) this integral is rewritten as a contour integral of a logarithm over \( \gamma_{e/2} \). We subtract and add the same integral over the curve \( \gamma_{2\epsilon} \). The difference produces the residue at the solution of \( z + \lambda \log(1 + z) = a \) which makes it convenient to transform back to the Lambert function. The imaginary part of the residue is again \( i\tau_\lambda(a) \), and it remains
\[
\mathcal{H}_a[\tau_\lambda(\bullet)] = \frac{1}{2\pi i} \int_{\gamma_e} dz \frac{(1 + z + \lambda) \log \left( 1 - \frac{\lambda \log(-z)}{b + 1 + z} \right)}{(1 + z)(z + \lambda \log(1 + z) - a)}
\]
\[
+ \Re \left( \log \left( 1 - \frac{\lambda \log(-\lambda W(\frac{1}{\lambda} e^{(1+p)/\lambda}) + 1 - \lambda\pi) + 1 + \lambda\pi)}{b + \lambda W(\frac{1}{\lambda} e^{(1+p)/\lambda})} \right) \right).
\]
Adding \( 0 = \text{rhs} - \text{lhs} \) of the 2nd identity (46) identifies \( N_\lambda(a, b) \) in (51).

It remains to prove the symmetry. This would be achieved via deformation of \( \gamma_\epsilon \) into a curve encircling \((-\infty, -1] \). However, this is not directly possible because of the infinite number of branch cuts in the complex plane traced back to the branches of the Lambert function. But we can use the same trick as in (48). Outside a ball of radius \( \delta \) which contributes at most \( O(-\delta \log \delta) \) we can expand the logarithms into power series in \( \lambda \) with finite radius of convergence. All terms are holomorphic outside \((-\infty, -1] \cup [0, \infty) \) so that we can deform the contour \( \gamma_\epsilon \) into the straight line \(-\frac{1}{2} + i\mathbb{R} \) which (for small \(|\lambda|\)) does not intersect any branch cuts. Therefore, the series can be resummed and yields equation (3) given in the
beginning. This formula has a maximal real-analytic extension to $\lambda > -\frac{1}{2\log 2}$ and is – by integration by parts – manifestly symmetric, $N_{\lambda}(a, b) = N_{\lambda}(b, a)$.

The argument of the logarithms in (3) is real at the solutions of $t = -\lambda \arctan 2t$. For $\lambda \geq -\frac{1}{2}$ only the trivial solution $t = 0$ arises. For any $\lambda \leq -\frac{1}{2\log 2}$ both logarithms in (3) are singular at $t = 0$ for some $a, b \geq 0$. Since the argument of the logarithms is strictly positive at the other two solutions arising for $-\frac{1}{2\log 2} < \lambda < \frac{1}{2}$, no other singularity occurs, and (3) is real-analytic for any $\lambda \geq -\frac{1}{2\log 2}$ and $a, b \geq 0$.

The proof of our initial Conjecture 1 is thus reduced to the verification that two formulae for $I_{\lambda}(a)$ coincide:

$$
\lambda \int_{0}^{\infty} dp \left( \frac{(a + \lambda W(e^{(1+p)/\lambda}))(p + \lambda W(e^{(1+a)/\lambda})) \exp(-N_{\lambda}(a, p))}{(\lambda \pi)^2 + (a + \lambda W(e^{(1+p)/\lambda}) - \lambda \log(\lambda W(e^{(1+p)/\lambda}) - 1))^2} - \frac{1}{1 + p} \right)
$$

$$= \lambda W\left(\frac{1}{\lambda e^{1/a}}\right) - \lambda \log \left(\lambda W\left(\frac{1}{\lambda e^{1/a}}\right) - 1\right) - 1 - a + \lambda \log a.
$$

(54)

8. Perturbative Expansion of the Final Integral

The $\lambda$-expansion of $N_{\lambda}(a, b)$ from (51) is computable symbolically with HyperInt:

```maple
> ln(1-lambda*ln(-z)/(1+b+z))*ln(1-lambda*ln(1+z+w)/(a-z-w));
> eval(diff(%,w),w=0);
> series(%,lambda) :
> coeff(%,lambda,2);
> hyperInt(%,2*I*Pi),z);
> X := fibrationBasis(%,[a,b]);
```

The integral $X$ over $z$ from 0 to $\infty$ depends on the signs $\delta_z$ and $\delta_a$ of the imaginary parts of $z$ and $a$. To get the half of $\gamma_{\epsilon}$ above $\mathbb{R}_+$, we have to set $\delta_z = 1$ and $\delta_a = -1$ (since $a$ lies below $z$). The first half of $\gamma_{\epsilon}$ subtracts the conjugate, so

```maple
> Above := eval(X,[delta[z]=1,delta[a]=-1]);
> Below := eval(X,[delta[z]=-1,delta[a]=1]);
> N[2] := collect(Above-Below,Hlog,factor);
```

gives $[\lambda^2]N_{\lambda}(a, b) = \frac{\zeta(2) - \log(1+a) - \log(1+b) - \log(1+a+b)}{(1+a+b)^2}$. These terms show up in (28). In fact, we can characterize the emerging polylogarithms very precisely: They belong to the family of multiple polylogarithms studied by Nielsen, [16, 22],

$$S_{n,p}(z) = \frac{(-1)^n}{(n-1)!p!} \int_{0}^{1} \frac{\log^{n-1}(t) \log^p(1 - zt)}{t} \ dt.
$$

(55)
To make this clear, we expand (51) in the form

$$N_\lambda(a, b) = \frac{\partial}{\partial w} \sum_{m,n=1}^{\infty} \frac{(-\lambda)^{m+n}}{m!n!} \frac{\partial^{m-1}}{\partial a^{m-1}} \frac{\partial^{n-1}}{\partial b^{n-1}} \int_\gamma \frac{d_z (\log(z+w))^n (\log(-z))^n}{2\pi i} (1+b+z)(a-z-w).$$

(56)
taken at $w = 0$. Pulling out the prefactor $(1+a+b-w)^{-1}$, the decomposition

$$\frac{1+a+b-w}{(1+b+z)(a-z-w)} = \frac{1+a}{(a-z-w)(1+w+z)} - \frac{b-w}{(1+b+z)(1+w+z)}$$
of the integration kernel completely separates the $a$- and $b$-dependence of the integral (up to the prefactor). The remaining integrals can be transformed into the form (55). In fact, we can compute the generating function

$$R_{\alpha,\beta}(a, b; w) := \frac{1}{\pi} \int_0^\infty d_z \text{Im} \left( \frac{(-z-i\epsilon)^\beta (1+z+w+i\epsilon)^\alpha}{(1+b+z+i\epsilon)(a-z-w-i\epsilon)} \right)$$
defined so that the coefficient of $\alpha^m \beta^n$ is $m!n!$ times the contour integral in (56):

$$N_\lambda(a, b) = \sum_{m,n=1}^{\infty} \frac{\partial}{\partial a^{m-1}} \frac{\partial}{\partial b^{n-1}} [\alpha^m \beta^n] \frac{\partial}{\partial w} \bigg|_{w=0} R_{-\lambda a,-\lambda \beta}(a, b; w).$$

(58)

Resolving the $i\epsilon$-descriptions in (57) and the substitution $z = (1+w)p$ give

$$R_{\alpha,\beta}(a, b; w) = \cos(\beta \pi) \frac{(a-w)^\beta (1+a)^\alpha}{(1+a+b-w)} + \frac{(1+w)^{\alpha+\beta-1} \sin(\pi \beta)}{\pi (1+a+b-w)} (b-w) \int_0^\infty dp \frac{p^\beta (1+p)^{\alpha-1}}{(p+\frac{a-w}{1+w})}$$

$$+ \frac{(1+w)^{\alpha+\beta-1} \sin(\pi \beta)}{\pi (1+a+b-w)} (1+a) \int_0^\infty dp \frac{p^\beta (1+p)^{\alpha-1}}{(p-\frac{a-w}{1+w})}.$$  

(59)

The integral in the 2nd line is a standard hypergeometric integral. Writing $p^\beta (1+p)^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} G_{1,1}^{1,0}(p; \alpha+\beta)$, the last line of (59) is the Hilbert transform at $\frac{-w}{1+w}$ of a Meijer-G function. This Hilbert transform is simply obtained by adding a leading 0 and a terminating $\frac{1}{2}$ to both rows of arguments:

$$\frac{1}{\pi} \int_0^\infty dp \frac{p^\beta (1+p)^{\alpha-1}}{(p-\frac{a-w}{1+w})} = \frac{1}{\Gamma(1-\alpha)} G_{3,3}^{2,2}(a-w \mid 0, 0, \alpha+\beta; 0, 0, \frac{1}{2});$$

The Meijer-G function on the rhs is expanded into a $\,_2F_1$ function and a $\,_1F_0$ function. The latter one cancels the first term on the rhs of (59), giving

$$R_{\alpha,\beta}(a, b; w) = \frac{(1+w)^{\alpha+\beta}}{(1+a+b-w)} \frac{(b-w) \beta \Gamma(1-\alpha-\beta)}{(1+b) \Gamma(2-\alpha) \Gamma(1-\beta)} \,_2F_1 \left( 1+\beta \mid b-w \mid 1+b \right)$$

$$+ \frac{(1+w)^{\alpha+\beta}}{(1+a+b-w)} \frac{\Gamma(1-\alpha-\beta) \, \beta}{\Gamma(1-\alpha) \Gamma(1-\beta)} \,_2F_1 \left( 1, 1-\alpha-\beta \mid w-a \mid 1+w \right).$$
A contiguous relation in the first line together with fractional transformations of both lines bring this formula into the following manifestly symmetric form:

\[
R_{\alpha,\beta}(a, b; w) = \frac{1}{(1 + a + b - w) \Gamma(1 - \alpha - \beta)} \left\{ - (1 + w)^{\alpha + \beta} 
+ (1 + w)^\alpha (1 + b)^{\beta} \binom{-\alpha, \beta}{1 - \alpha} w - b \binom{1}{1 + w} 
+ (1 + w)^\beta (1 + a)^{\alpha} \binom{-\beta, \alpha}{1 - \beta} w - a \binom{1}{1 + w} \right\}.
\]

This hypergeometric function generates the Nielsen polylogarithms, as observed in [17, Equation (2.12)] and [4, Theorem 6.6]:

\[
\binom{-x, y}{1 - x} = 1 - \sum_{n,p \geq 1} S_{n,p}(z)x^n y^p.
\]

We note that the Gamma functions in (60) expand into Riemann zeta values,

\[
\frac{\Gamma(1 - \alpha - \beta)}{\Gamma(1 - \alpha) \Gamma(1 - \beta)} = \exp \left( \sum_{k=2}^{\infty} \left( (\alpha + \beta)^k - \alpha^k - \beta^k \right) \frac{\zeta(k)}{k} \right).
\]

For the derivative with respect to \( w \), note that for \( z = (w - a)/(1 + w) \), we have

\[
\frac{\partial}{\partial w} \bigg|_{w=0} S_{n,p}(z) = S'_{n,p}(-a) \frac{\partial z}{\partial w} \bigg|_{w=0} = (1 + a) S'_{n,p}(-a) = -(1 + a) \partial_a S_{n,p}(-a).
\]

The contribution to \( \partial_w |_{w=0} R \) from the third line in (60) is then

\[
\frac{(1 + a)^\alpha}{1 + a + b} \left\{ \frac{1}{1 + a + b} + \beta - (1 + a) \partial_a \right\} \left( 1 - \sum_{n,p=1}^{\infty} S_{n,p}(-a) \beta^n \alpha^p \right),
\]

up to the Gamma prefactor. Hence, we can compute the expansion of \( \partial_w |_{w=0} R \) in terms of zeta values, logarithms \( \log(1 + a) \) and polylogarithms \( S_{n,p}(-a) \) (and those with \( a \) replaced by \( b \)), with rational functions of \( a \) and \( b \) as coefficients.

9. Discussion

9.1. Conjecture 1 is true if one accepts the following hypothesis: A sequence \( (a_n) \) of rational numbers arising from a seemingly simple mathematical problem such as (17), for which \( a_1, \ldots, a_{36} \) are integers of combinatorial significance, will continue as the same integer sequence. The integer sequence in question is \( (s_{j,k}) \) in (29) for \( 1 \leq j \leq n - 2 \) and \( 1 \leq k \leq j \) which was verified by HyperInt for \( n \leq 10 \). In principle, Conjecture 1 is verifiable by induction. For that one would restrict (29) to \( n \leq n_0 \), insert into (26) and reexpand to \( \tau_a(p) \) up to order \( n_0 + 1 \). The resulting polynomials in \( \log p \) and \( \log(1 + p) \) of degree \( \leq n_0 \) must be reshuffled into the basis of Nielsen polylogarithms. Evaluate the Hilbert transform, exponentiate, integrate the product with \( \sin \tau_a(p) \). Comparison with (29) for \( n \leq n_0 + 1 \) gives rise to a tremendous number of combinatorial identities.
involving polynomials in Stirling numbers of degree \( \leq n_0 \). Proving Conjecture 1 could be a treasure for combinatorics.

9.2. In the derivation of (26) we ignored possible non-trivial solutions of the homogeneous Carleman equation which we briefly mentioned after Proposition 4. Such solutions are constant in the Hilbert transform variable but might depend on the parameter \( a \) of the angle function \( \tau_a(\cdot) \). Expanding these functions \( C_a \) in \( \lambda \) and comparing with the Feynman graph expansion of the original partition function shows that at least the first orders are absent. In the spirit of the above hypothesis – that a sequence starting with many zeros is zero – we have \( C_a = 0 \) to all orders in \( \lambda \). This does not rule out flat functions such as \( e^{-\frac{1}{\lambda}} \). However, their presence is hard to make compatible with the symmetry \( G_\lambda(a, b) = G_\lambda(b, a) \) at large \( \lambda \). With this numerical reasoning we are convinced that Conjecture 1 is not only true as a whole, it also gives the unique symmetric solution of (1).

9.3. We recall that (29) has non-zero radius of convergence, in contrast to what is expected from non-integrable quantum field theory.

9.4. More work is necessary to better understand the remaining integral \( N_\lambda(a) \) given in (3). Its perturbative expansion started in (56) gives rise to linear combinations of Nielsen polylogarithms which must reproduce the perturbative solution of (26). We expect that \( N_\lambda(a) \) will also be given as solution of an implicit equation, similar to the occurrence of the Lambert function. This solution should involve the same terms \( \frac{1}{\lambda} e^{(1+\alpha)/\lambda} \) as in (2) because otherwise \( G_\lambda(a, b) \) would have an infinite number of branch cuts in the complexified variables \( a, b \). Experience with the \( \lambda \phi^4 \)-model tells us that \( G_\lambda(a, b) \) is holomorphic at least in \( \text{Re}(a), \text{Re}(b) \geq 0 \).

9.5. The solution method should also extend to the \( \lambda \phi^4 \)-model on 4-dimensional Moyal space. For finite \( \Lambda^2 \) this amounts to change the integration measure in (13) from \( dp \) to \( p dp \). This creates much more severe divergences for \( \Lambda^2 \to \infty \) which require subtle rescaling by a wavefunction renormalisation \( Z(\Lambda^2) \) and a more complicated dependence \( \mu^2(\Lambda^2) \). Whereas (22) already agrees with [14, Thm. 4.7], up to a global factor \( a \) from the changed measure and a global renormalisation constant, an analogue of (26) was missing in [14]. This lack was compensated by a symmetry argument which allowed to prove existence of a solution, but there was no way to obtain an explicit formula. The methods developed here give hope to achieve such a formula.

9.6. Solving a non-linear problem such as (1) by (generalised) radicals can only be expected if some deep algebraic structure is behind. We have no idea what it is\(^7\), but we find it worthwhile to explore that connection. We remark that the initial action (8) is closely related to the action \( S(\Phi) = V \text{ tr}(E \Phi^2 + \frac{1}{3} \Phi^3) \) of

\(^7\)We quote from the bibliographical notes of [7]: “We find it a remarkable coincidence that the curves defining the branch cuts of the Lambert W function (which contain the Quadratrix of Hippias) can be used not only to square the circle—which, by proving \( \pi \) irrational, Lambert went
the Kontsevich model [18]. This model gives rise to solvable $\lambda\Phi^3$-matricial QFT-models in dimension 2, 4 and 6 [11, 12] which, however, are modest from a number-theoretical point of view: In a perturbative expansion of correlation functions only $\log(1 + a)$ arises and only at lowest order, no polylogarithms as in (28). The $\Phi^4$-model is much richer and closer to true QFT-models. The Kontsevich model relates to infinite-dimensional Lie algebras and to the $\tau$-function of the KdV-hierarchy. It generates intersection numbers of stable cohomology classes on the moduli space of complex curves [18]. Something similar should exist for $\Phi^4$ as well.

9.7. Perturbative expansions in realistic quantum field theories like the Standard Model also produce (much more complicated) polylogarithms and other transcendental functions; see for example [1, 15, 21, 24, 27]. It would be exciting if the tremendous apparent complexity of those series could also be produced by an integral transform of a simpler function.

ACKNOWLEDGEMENTS

We are grateful to Spencer Bloch and Dirk Kreimer for invitation to the Les Houches summer school “Structures in local quantum field theories” where the decisive results of this paper were obtained. Discussions during this school in particular with Johannes Blümlein, David Broadhurst and Gerald Dunne contributed valuable ideas. RW would like to thank Alexander Hock for pointing out reference [9] and Harald Grosse for the long-term collaboration which preceded this work.

REFERENCES

[1] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, The three-loop splitting functions $P_{gg}^{(2)}$ and $P_{gg}^{(2,N_F)}$, Nuclear Physics B 922 (Sept., 2017) pp. 1–40, arXiv:1705.01508 [hep-ph].

[2] J. Ablinger, J. Blümlein and C. Schneider, Analytic and algorithmic aspects of generalized harmonic sums and polylogarithms, J. Math. Phys. 54 (Aug., 2013) p. 082301, arXiv:1302.0378 [math-ph].

[3] C. Bogner, MPL—a program for computations with iterated integrals on moduli spaces of curves of genus zero, Comput. Phys. Commun. 203 (2016) pp. 339–353, arXiv:1510.04562 [physics.comp-ph].

[4] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisoněk, Special values of multiple polylogarithms, Trans. Amer. Math. Soc. 353 (Mar., 2001) pp. 907–941, arXiv:math/9910045.

a long way towards proving was impossible by compass and straightedge—but also to trisect a given angle.” Is the same capability of Lambert-W used to solve (1)?
[5] F. C. S. Brown, *Multiple zeta values and periods of moduli spaces $\overline{M}_{0,n}$*, *Ann. Sci. Éc. Norm. Supér.* (4) 42 (2009), no. 3 pp. 371–489, arXiv:math/0606419.

[6] Bürmann, H., *Essai de calcul fonctionnaire aux constantes ad-libitum*, *Mem. Inst. Nat. Sci Arts. Sci. Math. Phys.* 2 (1799) pp. 13–17.

[7] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, *On the Lambert W function*, *Adv. Comput. Math.* 5 (1996) pp. 329–359.

[8] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, *Vanishing of beta function of non commutative $\phi^4_4$ theory to all orders*, *Phys. Lett.* B649 (2007) pp. 95–102, arXiv:hep-th/0612251.

[9] F. D. Gakhov, *Boundary value problems*. Pergamon Press, Oxford, 1966.

[10] J. M. Gracia-Bondía and J. C. Várilly, *Algebras of distributions suitable for phase space quantum mechanics. I.*, *J. Math. Phys.* 29 (1988) pp. 869–879.

[11] H. Grosse, A. Sako and R. Wulkenhaar, *Exact solution of matricial $\Phi^3_3$ quantum field theory*, *Nucl. Phys.* B925 (2017) pp. 319–347, arXiv:1610.00526 [math-ph].

[12] H. Grosse, A. Sako and R. Wulkenhaar, *The $\Phi^3_3$ and $\Phi^3_6$ matricial QFT models have reflection positive two-point function*, *Nucl. Phys.* B926 (2018) pp. 20–48, arXiv:1612.07584 [math-ph].

[13] H. Grosse and R. Wulkenhaar, *Renormalisation of $\phi^4$-theory on noncommutative $\mathbb{R}^4$ in the matrix base*, *Commun. Math. Phys.* 256 (2005) pp. 305–374, arXiv:hep-th/0401128.

[14] H. Grosse and R. Wulkenhaar, *Self-dual noncommutative $\phi^4$-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory*, *Commun. Math. Phys.* 329 (2014) pp. 1069–1130, arXiv:1205.0465 [math-ph].

[15] J. M. Henn and V. A. Smirnov, *Analytic results for two-loop master integrals for Bhabha scattering I*, *JHEP* 1311 (2013) p. 041, arXiv:1307.4083 [hep-th].

[16] K. S. Kölblig, *Nielsen’s generalized polylogarithms*, *SIAM Journal on Mathematical Analysis* 17 (1986), no. 5 pp. 1232–1258.

[17] K. S. Kölblig, J. A. Mignaco and E. Remiddi, *On Nielsen’s generalized polylogarithms and their numerical calculation*, *BIT Numerical Mathematics* 10 (1970), no. 1 pp. 38–73.

[18] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.* 147 (1992) pp. 1–23.

[19] J. L. Lagrange, *Nouvelle méthode pour résoudre des équations littérales par le moyen de séries*, *Mém. Acad. Roy. des Sci. et Belles-Lettres de Berlin* 24 (1770).

[20] J. H. Lambert, *Observationes variae in mathesin puram*, *Acta Helvetica, physico-mathematico-anatomico-botanico-medica* 3 (1758) pp. 128–168.

[21] S. Laporta, *High-precision calculation of the 4-loop contribution to the electron $g − 2$ in QED*, *Physics Letters B* 772 (Sept., 2017) pp. 232–238.
24 ERIK P. ANZER AND RAIMAR WULKENHAAR

arXiv:1704.06996 [hep-ph].

[22] N. Nielsen, Der Eulersche Dilogarithmus und seine Verallgemeinerungen, Nova acta - Kaiserlich Leopoldinisch-Carolinische Deutsche Akademie der Naturforscher 90 (1909), no. 3 pp. 121–212.

[23] E. Panzer, Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals, Computer Physics Communications 188 (Mar., 2015) pp. 148–166, arXiv:1403.3385 [hep-th]. maintained and available at https://bitbucket.org/PanzerErik/hyperint.

[24] E. Panzer and O. Schnetz, The Galois coaction on $\phi^4$ periods, CNTP 11 (2017), no. 3 pp. 657–705, arXiv:1603.04289 [hep-th].

[25] E. Panzer, Feynman integrals and hyperlogarithms. PhD thesis, Humboldt-Universität zu Berlin, 2014. arXiv:1506.07243 [math-ph].

[26] C. Schneider, Modern Summation Methods for Loop Integrals in Quantum Field Theory: The Packages Sigma, EvaluateMultiSums and SumProduction, in Journal of Physics Conference Series, vol. 523 of Journal of Physics Conference Series, p. 012037, June, 2014. arXiv:1310.0160 [cs.SC].

[27] I. Todorov, Polylogarithms and multizeta values in massless Feynman amplitudes, in Lie Theory and Its Applications in Physics (V. Dobrev, ed.), vol. 111 of Springer Proceedings in Mathematics & Statistics, pp. 155–176. Springer Japan, 2014.

[28] F. G. Tricomi, Integral equations. Interscience, New York, 1957.

All Souls College, University of Oxford, OX1 4AL Oxford, United Kingdom
Mathematisches Institut der WWU, Einsteinstr. 62, 48149 Münster, Germany