Geometrization of symplecticity conditions for implicit schemes

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Abstract

In this note we give simple symplecticity conditions for implicit schemes in symplectic vector spaces. We consider implicit maps on generic symplectic manifold and we introduce the concept of consistent implicit maps to generalize the symplecticity conditions on symplectic manifolds. Additionally, we give a preliminary geometrical interpretation of those conditions.

1 Introduction

The most widely numerical methods for simulating Hamiltonian dynamics are symplectic integrators. Nowadays, there are a multitude of techniques and types of simplectic integrators, both explicit and implicit. Implicit methods, in general, are approximations which minimize the error but, most geometrical information is lost. The geometry of explicit schemes has been widely studied and it is well understood at present, which is not the case for implicit schemes. The main conceptual difficulty for the implicit case, arrives from the fact that symplectic maps are natural operators on cotangent bundles to smooth manifolds: every diffeomorphism on the base manifold lifts to a symplectomorphism on the cotangent bundle; this mapping is called the cotangent lift \cite{1, 5}. Such a symplectomorphism consists on a covariant and a contravariant part acting in opposite directions, specifically, the mapping on the fibers is defined by the inverse of the pullback. This fact is connected with the idea that generating functions of type II and III are well adapted for constructing symplectic integrators, but not the generating functions of type I nor IV.

To be more specific, suppose that two symplectic manifolds \((M_1, \omega_1)\) and \((M_2, \omega_2)\) are given, with canonical coordinates \((q, p) \in M_1\) and \((Q, P) \in M_2\), and a symplectomorphism \(\phi : M_1 \to M_2\) between them. For generating functions of type II and III the interchange of geometrical information goes from \((q, P) \rightarrow (Q, p)\) and \((Q, p) \rightarrow (q, P)\) respectively. The image coordinates are obtained by solving the Hamilton-Jacobi equation. Looking for intermediate
points preserving the geometrical structure, we must keep this balance, generalized by the rule

\[(\alpha q + (1 - \alpha)Q, (1 - \alpha)p + \alpha P) \mapsto ((1 - \alpha)q + \alpha Q, \alpha p + (1 - \alpha)P)\] (1)

which is encoded in the Liouvillian form associate to the Hamiltonian system [4]. In this generalization, generating functions of type II and III correspond to the expression (1) with values \(\alpha = 1\) and \(\alpha = 0\) respectively. The other well-known case, corresponding to \(\alpha = 1/2\), is the mid-point rule. Other possibilities are not taken into account since for \(\alpha \neq 1/2\) the methods obtained are of first order and for \(\alpha = 1/2\) is second order (the symmetric case).

With this paper we start a systematic study on implicit symplectic integrators from the geometrical point of view. We outline its content. In Section 2, we state the definitions of explicit and implicit symplectic integrators and we give simple symplecticity conditions for the case of linear symplectic spaces, modeled by \((\mathbb{R}^{2n}, \omega_0)\). They are already known results. In Section 3, we state some preliminary definitions in order to restrict the type of implicit schemes where the conditions apply. We are interested on implicit maps which can be given by the composition of two explicit maps. The main idea is to find an intermediary point given in terms of two consecutive points of the discretized flow, such that we can construct a flow line passing by three successive points on the manifold. Then, we generalize and give an interpretation of the symplecticity conditions stated in Section 2 to the case of a Hamiltonian system on a generic symplectic manifold.

2 Explicit and implicit symplectic integrators

In a single phrase, we can state the definition of a symplectic integrator as follows: A symplectic integrator for a Hamiltonian system is a numerical method which preserves the structure of the Hamiltonian vector field. However, the symplecticity of an integrating method only constrains the numerical scheme to preserve the form of the vector field. Moreover, if the Hamilton equations are independent of the time (i.e. if the Hamiltonian function \(H\) is autonomous), a symplectic integrator preserves also the energy integral. Energy preservation restricts the numerical solution to be “close” to a submanifold \(\Sigma_h = H^{-1}(h)\) of codimension \(\text{codim}(\Sigma_h) = 1\), which is advantageous for a lower-dimensional Hamiltonian system, but weak for a higher-dimensional system. Preserving the form of the vector field says that we can apply the method to the image point without any additional analysis but no additional constraints are given. The reason of this, rise as a consequence of Darboux’s theorem which states that the symplectic structure does not recognizes the local structure of the Hamiltonian flow. Unfortunately, numerical integration is intrinsically a local procedure and a new point of view is needed to go beyond in the subject of numerical symplectic integration.

We are convinced that implicit schemes can be very accurate methods for simulating the Hamiltonian flow, giving additionally the right direction of the
numerical flow. To have suitable numerical integrators we must return to the roots of the geometrical problem. We start by some basic definitions and we state new definitions formalizing the implicit schemes of our interest.

2.1 Some basic definitions

A **symplectic manifold** is a $2n$-dimensional manifold $M$ equipped with a non-degenerated, skew-symmetric, closed 2-form $\omega$, such that at every point $m \in M$, the tangent space $T_m M$, has the structure of a symplectic vector space.

A Hamiltonian system $(M, \omega, X_H)$ is a vector field $X = X_H$ on a symplectic manifold $(M, \omega)$ such that

$$i_{X_H}\omega = -dH,$$

for a differentiable function $H : M \rightarrow \mathbb{R}$ known as the total energy or the Hamiltonian function. There is an alternative definition in vector field form as

$$X_H = J \nabla_H,$$

with evolution equations

$$\dot{z} = J \nabla_H(z), \quad z \in M. \quad (3)$$

where $J$ is the canonical complex structure on $T_z M$ given by

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}, \quad 0_n, I_n \in \mathbb{M}_{n \times n}(\mathbb{R}). \quad (4)$$

and $\nabla$ is the standard gradient operator on $T_z M \cong \mathbb{R}^{2n}$.

Denoting by $\phi^h_H$ the Hamiltonian flow it is well-known that for each fixed $h \in \mathbb{R}$, the map $\phi^h_H$ is a symplectic map. Let $z_0 \in M$ be a point on the symplectic manifold and $z(t)$ the integral curve to $X_H$ such that $z_0 = z(0)$. By definition of the flow, the mapping

$$z(t + h) = \phi^h_H(z(t))$$

will propagates the solution from the time $t$ to time $t + h$. A **symplectic algorithm** with stepsizes $h$, is the numerical approximation $\psi_h$ of the flow $\phi^h_H : M \rightarrow M$, which is an isometry of the symplectic form $\omega$. Specifically, consider the exact solution $z(t)$ of a Hamiltonian system for the time $t \in [0, T]$, and a discretization of the time interval $\{t_i\}_{i=0}^N$ such that $t_0 = 0$, $t_N = T$, $h = T/N = t_{k+1} - t_k$, and $z_k = z(t_k)$ for $0 \leq k < N$.

We define an **explicit symplectic integrator** as a map

$$\psi_h : U \subset M \rightarrow U \quad z_k \mapsto z_{k+1} = \psi_h(z_k)$$

smooth on $h$ an $H$, such that $(\psi_h)^* \omega = \omega$, where $(\psi_h)^*$ is the pullback of $\psi_h$.

In an analogous way, we define an **implicit symplectic integrator** as a map

$$\varphi_h : U \times U \rightarrow U \quad (z_k, z_{k+1}) \mapsto z_{k+1} = \varphi_h(z_k, z_{k+1})$$
smooth with respect to \( h \) and \( H \), and such that \( \varphi_h^* \omega = \omega \). However, we have a problem to state what means \( \varphi_h^* \omega = \omega \) in the implicit case. Some authors consider maps of type

\[
\varphi_h(z_k, z_{k+1}) = z_k + \phi_h(z_k, z_{k+1}),
\]

or the general implicit rule

\[
\Psi(z_{k+1}, z_k) = z_{k+1} - \varphi_h(z_{k+1}, z_k) = 0
\]

The test of symplecticity is obtained by implicit differentiation of expression (10), obtaining the linearized algorithm \( \delta z_{k+1} = A \delta z_k \), where \( A \) is called by some authors the linearized amplification matrix of the scheme [A, Appx II]. The matrix \( A \) is given by

\[
A_1 = \frac{\partial \Psi(z_{k+1}, z_k)}{\partial z_{k+1}} \quad \text{and} \quad A_2 = -\frac{\partial \Psi(z_{k+1}, z_k)}{\partial z_k},
\]

and the implicit mapping \( \Psi \) is simplectic if \( A_2^T A_1^{-1} J A_1^{-1} A_2 = J \). With this information we can state the following result on \( \mathbb{R}^{2n} \), equipped with the canonical symplectic form \( \omega_0 \), considering \((\mathbb{R}^{2n}, \omega_0)\) as a symplectic manifold.

**Proposition 2.1** Let \( U \subset \mathbb{R}^{2n} \) be a convex open set and \( z_k, z_{k+1} \in U \) two interior points. Consider a point \( \bar{z} \in U \) and a differentiable map \( f : U \times U \to U \) such that \( \bar{z} = f(z_k, z_{k+1}) \). Denote the matrices of partial derivatives by

\[
B = \frac{\partial f(z_k, z_{k+1})}{\partial z_k} \quad \text{and} \quad C = \frac{\partial f(z_k, z_{k+1})}{\partial z_{k+1}},
\]

and suppose that \( B \) and \( C \) fulfills the following conditions

\[
(i) \quad B + C = I_{2n} \quad \text{and} \quad (ii) \quad BJ = JC^T,
\]

where \( I_{2n} \in M_{2n \times 2n}(\mathbb{R}) \) is the identity matrix and \( J \) is the almost complex structure defined in (4).

Then, the map

\[
z_{k+1} = z_k + hX_H(\bar{z})
\]

defines an implicit symplectic integrator with stepsize \( h \).

**Proof.** Consider the implicit mapping \( \Psi \) given by

\[
\Psi(z_k, z_{k+1}) = z_{k+1} - z_k - hX_H(f(z_k, z_{k+1})) = 0.
\]

Implicit differentiation of (10) using the chain rule and expressions (7) gives

\[
\frac{\partial \Psi(z_k, z_{k+1})}{\partial z_k} = I - hJH_{zz}B
\]

and

\[
\frac{\partial \Psi(z_k, z_{k+1})}{\partial z_{k+1}} = -I - hJH_{zz}C
\]
where $H_{zz}$ is the Hessian matrix of $H$. Denote the partial derivatives of $\Psi$ by

$$A_1 = \frac{\partial \Psi(z_k, z_{k+1})}{\partial z_k}, \quad \text{and} \quad A_2 = -\frac{\partial \Psi(z_k, z_{k+1})}{\partial z_{k+1}}.$$  

The amplification matrix of the linearized system is $A = A_2^{-1} \circ A_1$, and $\Psi$ is symplectic if the matrix $A$ of the linearized system is symplectic. We recall that $A$ is symplectic if and only if $A_T = A_1^T \circ A_2^{-T}$ is symplectic, i.e. if the equality $A_2^{-1} \circ A_1 JA_1^T \circ A_2^{-T} = J$ holds, or equivalently if $A_1 JA_1^T - A_2 JA_2^T = 0$.

Using the last expression, symplecticity condition becomes

$$(I - hJH_{zz}B)(I - hJH_{zz}B)^T - (I + hJH_{zz}C)(I + hJH_{zz}C)^T = 0.$$  

Developing and simplifying we have

$$h \left( JH_{zz}BJ + JB^T H_{zz}^T J^T + JH_{zz}CJ + JC^T H_{zz}^T J^T \right) - h^2 \left( JH_{zz}BJB^T H_{zz}^T J^T - JH_{zz}CJC^T H_{zz}^T J^T \right) = 0.$$  

Using the facts that $H_{zz} = H_{zz}^T$, $J^T = -J$ and $h \neq 0$, factoring and reordering we obtain the system of equations

$$0 = H_{zz}(B + C) - (B^T + C^T)H_{zz}$$

$$0 = BJB^T - CJC^T.$$  

First equation is satisfied applying hypothesis (i). For the second equation we substitute $B^T = I - C^T$ in $BJB^T - CJC^T$ to obtain successively

$$BJB^T - CJC^T = BJ(I - C^T) - CJC^T = BJ - (B + C)JC^T \overset{(i)}{=} BJ - JC^T.$$  

By hypothesis (ii) the second equation is satisfied. Consequently, the implicit method [ii] is symplectic as we want to prove. \qed

**Lemma 2.2** Let $B, C \in GL(2n)$ be two matrices defined on a symplectic vector space $(V, \omega_0)$ and $J \in GL(2n)$ the complex matrix associated to $\omega_0$. If $B + C = I$ is the identity matrix, then the following statements are equivalents:

1. $BJ = JC^T$,  

2. $(B - C)$ is a Hamiltonian matrix.

**Proof.** We recall that a square matrix $A \in GL(2n)$ is Hamiltonian if $A^T J + JA = 0$. A direct computation shows that

$$(B - C)J + J(B - C)^T = BJ - CJ + JB^T - JC^T$$

$$= BJ - (I - B)J + J(I - C^T) - JC^T$$

$$= 2(BJ - JC^T)$$

$$= 0.$$
where we used the fact that $A^T$ is Hamiltonian if and only if $A$ does.

□

Using this lemma, we have that the map (9) is symplectic when the matrices $B$ and $C$ satisfy that their addition is the identity matrix and their difference is a Hamiltonian one. From the previous results, they can be rewritten as

$$B := \frac{1}{2}(I + b), \quad C := \frac{1}{2}(I - b),$$

and conditions (i) and (ii) become $b^T J + Jb = 0$.

Remark 1 In a slightly different context, Ge and Dau-liu obtain a similar condition for some matrix $b$ (presumably due to Feng) when looking for examples of generating functions which are invariant under symplectic transformations of Feng’s type $\alpha_0$ (see [2], sec. 6). In particular, the case $b = 0$ corresponds to the symplectic midpoint rule which Feng associated to the Poincaré’s generating function and the study of its invariance under symplectic transformations is due to Weinstein [7]. However, in [3] the author shows that Poincaré’s generating function does not produce a symplectic map profitable for numerical integrators. It looks that this condition is related to a different type of symplectic maps adapted for dealing with periodic orbits.

Note that the matrices $B$ and $C$ are well defined by the natural diffeomorphisms

$$T^*\mathbb{R}^n \cong T\mathbb{R}^n \cong (\mathbb{R}^{2n})^* \cong \mathbb{R}^{2n}. \quad (14)$$

However, for a generic symplectic manifold $(M, \omega)$, $B$ and $C$ will be linear operators acting on different linear spaces (for instance, they act on different fibers of the tangent bundle). In the next section we develop the equivalent conditions for the symplectic generic case.

### 3 Consistent implicit maps

We are looking for a geometrical generalization to the conditions of Proposition (2.1) when the phase space is considered as a generic smooth manifold of dimension $2n$. In this case, tangent vectors on generic curves belong to different tangent spaces and we study what is the generalization of the matrices $B$ and $C$ as objects of the differential geometry of the smooth manifold. In what follows, we consider $M$ as any smooth manifold of arbitrary dimension and $U \subset M$ an open convex set of $M$. Restrictions on the dimension and geometry of $M$ will be stated when necessary. All the results studied here are localized into the open set $U \subset M$.

Let $\phi : U \times U \to U$ an implicit map such that $z_{k+1} = \phi(z_k, z_{k+1})$. We say that $\phi$ is consistent if there exists $\bar{z} \in U$ and two local diffeomorphisms $\psi_1, \psi_2 : U \to U$ with $\bar{z} = \psi_1(z_k)$ and $\bar{z} = \psi_2(z_{k+1})$, such that:
1. It is possible to rewrite $\phi$ in the form
\[
\phi(z_k, z_{k+1}) = \psi^{-1}(\bar{z}) = z_{k+1},
\]
(15)

2. The limit
\[
\lim_{z_{k+1} \to z_k} \psi_i = \text{Id}, \quad i = 1, 2.
\]
(16)

Hold.

We call $\bar{z}$ the consistency point.

There is a natural local diffeomorphism $\psi: U \to U$ given by $\psi = \psi^{-1} \circ \psi_1$ which is the explicit counterpart of the implicit map $\phi$. This is called the consistency map and it enables the construction of solutions passing by the three points $z_k, \bar{z}$ and $z_{k+1}$.

**Lemma 3.1** For every consistent implicit map $\phi: U \times U \to U$ it is possible to generate an implicit map $\rho: U \times U \to U$ such that
\[
\bar{z} = \rho(z_k, z_{k+1}).
\]
(17)

**Proof.** Since $\phi$ is a consistent implicit map, there exist local diffeomorphisms $\psi_1, \psi_2 \in \text{Diff}_0(U)$ and $\bar{z} \in U$ such that $\bar{z} = \psi_1(z_n)$ and $\bar{z} = \psi_1(z_{n+1})$. Consider a convex combination
\[
\rho(z_k, z_{k+1}) := a\psi_1(z_k) + (1-a)\psi_2(z_{k+1}), \quad a \in \mathbb{R}.
\]
(18)

Then we have successively
\[
\rho(z_k, z_{k+1}) = a\psi_1(z_k) + (1-a)\psi_2(z_{k+1})
\]
\[
= a\bar{z} + (1-a)\bar{z}
\]
\[
= \bar{z}.
\]
\[\square\]

Note that the general case $a \in \mathbb{R}$ is well defined, however, we are looking for localized maps in the open $U$. Moreover, we want to constrain the point $\bar{z}$ to be an intermediate point on the same flow line of $z_k$ and $z_{k+1}$ then we restrict its domain to $a \in [0, 1]$. From now on, we will see the map (18) as a partition of the unity.

**Lemma 3.2** If $\phi: U \times U \to U$ is a consistent implicit map and $\rho: U \times U \to U$ the map given by (18) with image on the consistency point $\bar{z} = \rho(z_k, z_{k+1})$. Then its tangent map $T_\rho: T_U \times T_U \to T_U$ corresponds to the identity map on $T_{\bar{z}}U$.

**Proof.** By the consistency hypothesis, we have a point $\bar{z} \in U$ and two local diffeomorphisms $\psi_1, \psi_2 \in \text{Diff}_0(U)$ satisfying the hypothesis of Lemma 3.1.

For every vector $v \in T_{\bar{z}}U$, there exist vectors $v_1 \in T_{z_k}U$ and $v_2 \in T_{z_{k+1}}U$ such
that \( v = T\psi_1(v_1) \) and \( v = T\psi_2(v_2) \), where \( T\psi_i : TU \to TU \) are the tangent maps to \( \psi_i \) for \( i = 1, 2 \).

To be more specific, we have

\[
T\psi_1|_{z_k} : Tz_k U \to T\bar{z}_k U \quad \text{and} \quad T\psi_2|_{z_{k+1}} : Tz_{k+1} U \to T\bar{z}_k U.
\]  

(19)

If \( \rho : U \times U \to U \) is of the form (18), its tangent map \( T\rho : Tz_k U \times Tz_{k+1} U \to T\bar{z}_k U \) takes \((v_1, v_2)^T \mapsto v = aT\psi_1(v_1) + (1 - a)T\psi_2(v_2)\).

Finally note that

\[
T\rho \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) = (aT\psi_1, (1 - a)T\psi_2) \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) = v
\]  

(20)

where we used the identity map \( v = \text{Id}(v) \) on \( T\bar{z}_k U \) on the right hand side.

\[\square\]

In local coordinates we obtain the expression

\[
\frac{\partial \rho(z_k, z_{k+1})}{\partial z_k}(v_1) + \frac{\partial \rho(z_k, z_{k+1})}{\partial z_{k+1}}(v_2) = I_{2n}(v),
\]  

(21)

which corresponds to the generalization of \( B + C = I_{2n} \).

3.1 Symplectic constraints for consistent implicit maps

From now on, the analysis concerns the symplectic case and the manifold of interest are the generic symplectic manifold \((M, \omega)\) of dimension \( \dim(M) = 2n \).

We say that a consistent implicit map \( \phi : U \times U \to U \) in an open convex set of a symplectic manifold \((M, \omega)\) interleaves a symplectic map if its consistency map \( \psi \) is symplectic.

Lemma 3.3 Let \( \phi : U \times U \to U \) be a consistent implicit map and \( \psi = \psi_2^{-1} \circ \psi_1 \) its consistency map. Then \( \phi : U \times U \to U \) interleaves a symplectic map if

\[
(T\psi_1)^{-T} J(T\psi_1)^{-1} = (T\psi_2)^{-T} J(T\psi_2)^{-1}.
\]  

(22)

where \( T\psi_1 \) and \( T\psi_2 \) are the tangent maps and \((\cdot)^{-T} = (\cdot)^{-1}\) is the transpose of the inverse map. Moreover, if \( \psi_1 \) and \( \psi_2 \) are symplectic then

\[
(T\psi_1)^T JT\psi_1 = (T\psi_2)^T JT\psi_2.
\]  

(23)

Proof. By definition, \( \phi \) interleaves a symplectic map if its consistency map \( \psi = \psi_2^{-1} \circ \psi_1 \) is symplectic, it means if \( \psi^*\omega = \omega \). This holds if and only if we have successively the following equalities

\[
\psi^*\omega = (\psi_2^{-1} \circ \psi_1)^*\omega = \psi_1^* \circ ((\psi_2^{-1})^*\omega) = \omega
\]
from where we obtain $\left(\psi_2^{-1}\right)^* \omega = \left(\psi_1^{-1}\right)^* \omega$. This condition is equivalent to the equation

$$
(T\psi_1^{-1})^T J (T\psi_1^{-1}) = (T\psi_2^{-1})^T J (T\psi_2^{-1}).
$$

(24)
on $T\bar{z}U$ for $\bar{z} \in U$ the consistency point of $\phi$. This proves the first result. For the second part, the symplectic hypothesis on $\psi_1$ and $\psi_2$ implies that $\psi_1^{-1}$ and $\psi_2^{-1}$ are also symplectic (local) diffeomorphisms by the group property of $Sp(U,\omega)$.

Remark 2 Condition (22) says nothing about the symplecticity of the mappings $\psi_1$ and $\psi_2$, but only that the composition is symplectic. For instance, suppose that $\psi_1 = f \circ g$ and $\psi_2 = f \circ h$, (25) for $h,g \in Sp(U,\omega)$, $h \neq g$ and $f \in Diff_0(U)$, but $f \notin Sp(U,\omega)$. Then, $\psi_1, \psi_2 \notin Sp(U,\omega)$ however $\psi_2^{-1} \circ \psi_1 = h \circ g \in Sp(U,\omega)$.

In other words, condition (22) says that the complex structures $J$ on $Tz_k U$ and $Tz_{k+1} U$ are equivalent, but it might not be well defined on $T\bar{z} U$ when $\psi_1, \psi_2 \notin Sp(U,\omega)$. Expression (23), on the other hand, says that there exists a structure on $T\bar{z} U$ which is equivalent to those $J$’s on $Tz_k U$ and $Tz_{k+1} U$ and it defines a complex structure on $T\bar{z} U$.

Since we are interested in simulating Hamiltonian flows, we impose the condition that the components of the consistency map $\psi_1, \psi_2 \in Sp(U,\omega)$ are symplectic in the rest of this work.

Lemma 3.4 Let $\phi : U \times U \to U$ be a consistent implicit map interleaving a symplectic map. If the components of the consistency map $\psi = \psi_2^{-1} \circ \psi_1$ are symplectic, then

$$
T\psi_1 - T\psi_2 : TU \times TU \to TU
$$

(26)
is a Hamiltonian operator on $T\bar{z} U$.

Proof.- 1) Consider the map $\bar{z} = \rho(z_n, z_{n+1}) = a\psi_1(z_n) + (1 - a)\psi_2(z_{n+1})$ as a partition of the unity of two local charts associated to $\psi_1$ and $\psi_2$. Since $\psi_1, \psi_2 \in Sp(U,\omega)$ we can consider the curve

$$
\gamma_\tau = \tau \psi_1 + (1 - \tau)\psi_2
$$

(27)
as a one parameter family of symplectic diffeomorphisms with image $\bar{z}$. Now, we consider the properties of $T\bar{z}u$ as a vector space.

For every $v \in T\bar{z} U$, and for every $\tau \in [0,1]$, there exist $\hat{v}_1, \hat{v}_2 \in T\bar{z} U$ such that $v = \tau \hat{v}_1 + (1 - \tau)\hat{v}_2$. This defines a $(2n - 1)$-dimensional subspace of $T\bar{z} U$ (the hyper-plane perpendicular to $v$). We consider the space of smooth curves joining the origin in $T\bar{z} U$ with the end point of $v$, and it is possible to write

$$
\gamma_\tau = \tau \hat{v}_1 + (1 - \tau)\hat{v}_2
$$

(28)
We look for vectors \( v_1 \in T_{z_0} U \) and \( v_2 \in T_{z_0+1} U \) such that \( \hat{v}_1 = T\psi_1(v_1) \) and \( \hat{v}_2 = T\psi_2(v_2) \), and the curve \( \gamma \) can be rewritten as

\[
\gamma_\tau = \tau T\psi_1(v_1) + (1-\tau)T\psi_2(v_2)
\]  

(29)

Since \( \psi_1, \psi_2 \in Sp(U, \omega) \) we can consider the curve \( \gamma \) as a one parameter family of symplectic diffeomorphisms with image \( \bar{z} \), joining the symplectomorphisms \( \psi_1 \) and \( \psi_2 \).

Taking the derivative with respect to the parameter \( \tau \) we have

\[
\frac{\partial \gamma_\tau}{\partial \tau} = T\psi_1 - T\psi_2
\]  

(30)

Which by definition is a Hamiltonian operator.

\( \square \)

Remark 3 Other parameterizations like \( \bar{\gamma}_\tau = \cos^2(\tau)\hat{v}_1 + \sin^2(\tau)\hat{v}_2 \) with derivative

\[
\frac{\partial \bar{\gamma}_\tau}{\partial \tau} = 2\cos(\tau)\sin(\tau)(T\psi_1 - T\psi_2)
\]  

(31)

or \( \bar{\gamma}_\tau = \text{cn}^2(\tau,k)\hat{v}_1 + \text{sn}^2(\tau,k)\hat{v}_2, \ k \in (0,1) \) with derivative

\[
\frac{\partial \bar{\gamma}_\tau}{\partial \tau} = 2\text{sn}(\tau,k)\text{cn}(\tau,k)\text{dn}(\tau,k)(T\psi_1 - T\psi_2)
\]  

(32)

leads to equivalent results modulo a scalar function on the open set \((0,1)\). All of them are parallel Hamiltonian operators. Expression (32) is given in terms of the elliptic functions of Jacobi \( \text{sn}(\tau,k) \), \( \text{cn}(\tau,k) \) and \( \text{dn}(\tau,k) \), which also satisfy \( \text{cn}^2(\tau,k) + \text{sn}^2(\tau,k) = 1 \).

The conditions \( i) B + C = I, \) and \( ii) BJ = JC^T \) in Proposition 2.1 only constraint the implicit mapping to be consistent and symplectic, but no relationship with any particular Hamiltonian flow is stated. From the geometrical interpretation of condition \( ii) \) the only requirement was that vectors \( \hat{v}_1 \) an \( \hat{v}_2 \) must depend smoothly on the parameter \( \tau \) satisfied by the use of smooth curves. However, we are interested in constrain these vectors such that they were related with the Hamiltonian vector field in the following way: \( \hat{v}_1 = T\psi_1(X_H(z_n)) \) and \( \hat{v}_2 = T\psi_2(X_H(z_{n+1})) \). A good candidate might be

\[
v = \frac{1}{2}(T\psi_1(X_H(z_n)) + T\psi_2(X_H(z_{n+1})))
\]

however, since we know the form of the Hamiltonian vector field, we have that \( v = X_H(\bar{z}) \). The problem is not to find the good vector \( v \), instead we look for a point \( \bar{z} \in U \) on the Hamiltonian flow producing the good value for \( v = X_H(\bar{z}) \).

Remark 4 Conditions stated in Proposition 2.1 give rise to continuous high-dimensional families of implicit symplectic integrators. We study some families in [4] using the classical framework used in the method of generating functions.
The proof of Lemma 3.4 looks a slightly tricky and artificial, at the same time the generalization of the relation $B + C = I_{2n}$ might be unsatisfactory. This arises since the natural definitions for symplectic and Hamiltonian operators are given between manifolds of the same dimension.

This problem is solved if we define all the operators on the product manifold of two copies of the symplectic manifold $(M,\omega)$. For instance, consider the manifold $M = M_1 \times M_2$ with the 2-form $\omega_{\oplus} = \pi_1^*\omega_1 - \pi_2^*\omega_2$, defined by the pullback of the canonical projections $\pi_i : M \to M_i$, $i = 1, 2$; the couple $(M,\omega_{\oplus})$ is a symplectic manifold of dimension $4n$ (see [1]). The procedure consists in create a symplectic path in $\tilde{M}$ from $(M_1 \times \{0\})$ to $(\{0\} \times M_2)$. It means, a continuous family of symplectic subspaces in $M$ of dimension $2n$ joining $M_1$ to $M_2$. For some intermediary element in this family, we project the (mixed) coordinates on the original manifold $(M,\omega)$, and these becomes the coordinates of the point $\tilde{z} \in U \subset M$ that we are looking for. However, there are some subtleties which are worked out in the companion article [4].

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