Instanton operators and symmetry enhancement in 5D supersymmetric gauge theories

Yuji Tachikawa$^{1,2,*}$

$^1$Department of Physics, Faculty of Science, University of Tokyo, Bunkyo-ku, Tokyo 133-0033, Japan
$^2$Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8583, Japan
$^*$E-mail: yuji.tachikawa@ipmu.jp

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Supersymmetric gauge theories in five dimensions often exhibit less symmetry than the ultraviolet fixed points from which they flow. The fixed points might have larger flavor symmetry or they might even be secretly 6D theories on $S^1$. Here we provide a simple criterion when such symmetry enhancement in the ultraviolet should occur, by a direct study of the fermionic zero modes around one-instanton operators.

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1. Introduction

In four dimensions and in lower dimensions, we often start from a Lagrangian gauge theory in the ultraviolet (UV) and study the behavior of the system in the infrared (IR). In five dimensions (5D), Lagrangian gauge theories are always non-renormalizable, and instead one needs to look for nontrivial ultraviolet superconformal field theories (SCFTs) from which they flow out. With supersymmetry, such a study is indeed possible, as demonstrated in Refs. [1–4], by combining field-theoretical analyses and embedding into string theory.

There and in other works, it was found that the UV fixed point can have enhanced symmetry: the instanton number symmetry sometimes enhances to a non-Abelian flavor symmetry, and sometimes enhances to the Kaluza–Klein mode number of a 6D theory on $S^1$. For example, the UV SCFT for $\mathcal{N} = 1$ SU(2) theory with $N_f \leq 7$ flavors has $E_{N_f+1}$ flavor symmetry; with $N_f = 8$ flavors, the UV fixed point is instead a 6D $\mathcal{N} = (1, 0)$ theory with $E_8$ symmetry compactified on $S^1$ [5]. For SU($N$) theory, the flavor symmetry enhancement only occurs for some specific choice of the number of the flavors and of the Chern–Simons levels. $\mathcal{N} = 2$ theory with gauge group $G$ in 5D, in contrast, comes from some 6D $\mathcal{N} = (2, 0)$ theory compactified on $S^1$, as originally found in the context of string duality [6,7].

These results were soon extended to include more models, using webs of five-branes, in, e.g., Refs. [8–10]. More recently, various sophisticated techniques such as supersymmetric localizations, Nekrasov partition functions, and refined topological strings have been applied to the analysis of these 5D systems. The symmetry enhancement of the models mentioned above has been successfully confirmed by these methods, and even more diverse models are being explored; see, e.g., Refs. [11–27].
These results are impressive, but the techniques used are rather unwieldy.\(^1\) In this paper, we describe a simpler method to identify the symmetry enhancement of a given 5D gauge theory, assuming that it has an ultraviolet completion either in 5D or 6D. Although heuristic, this method tells us what the enhanced flavor symmetry will be and whether the ultraviolet completion is a 5D SCFT or a 6D SCFT on $S^4$.

We do this by identifying the supermultiplet of broken symmetry currents\(^2\) by studying instanton operators that introduce non-zero instanton number on a small $S^4$ surrounding a point.\(^3\) When the instanton number is one, the structure of the moduli space and the zero modes is particularly simple, allowing us to find the broken symmetry currents rather directly. The spirit of the analysis will be close to the analysis of monopole operators of 3D supersymmetric gauge theories in Ref. [30], except that we need to identify an operator in the IR that would come from a UV current operator, rather than vice versa.

The rest of the paper is organized as follows. In Sect. 2, we collect the basic facts on the supersymmetry and on instantons on $S^4$ that we use. In Sect. 3, we analyze $\mathcal{N} = 1$ SU(2) gauge theories with $N_f \leq 8$ flavors and $\mathcal{N} = 2$ SU(2) theory. The effect of the discrete theta angle will also be discussed. In Sect. 4, we extend the discussion to $\mathcal{N} = 1$ SU(N) gauge theories with fundamentals and Chern–Simons terms, and to $\mathcal{N} = 2$ SU(N) theory. We use the results that will be obtained up to this point in Sect. 5 to study symmetry enhancement in quiver gauge theories made of SU gauge groups and bifundamentals. We assume that the effective number of flavors of each SU(N) node is $2N$ and that the Chern–Simons levels are all zero. We conclude with a discussion in Sect. 6.

2. Preliminaries

2.1. Supermultiplet of broken currents

Suppose that the 5D supersymmetric gauge theory is obtained by a mass deformation of a 5D superconformal theory. In this setup, supersymmetric mass deformations are always associated with the Cartan part of a possibly non-Abelian conserved current supermultiplet\(^4\) that contains the conformal primaries

\[ \mu_{(ij)}^a, \quad \psi_{i\alpha}^a, \quad J_{\mu}^a, \quad M^a \]  

with scaling dimensions 3, 3.5, 4, 4, respectively. Here $a$ is the adjoint index of the flavor symmetry, $i = 1, 2$ is the index of Sp(1)$_R$, and $\alpha$ is the spinor index of SO(5); the symplectic Majorana condition is imposed on $\psi_{i\alpha}^a$.

\(^1\) One of the trickiest aspects is the need to remove spurious contributions from the so-called “U(1) parts” in the instanton counting method and from the parallel legs in the toric diagram in the topological string method.

\(^2\) The importance and the special property of the current supermultiplets in 5D SCFTs were emphasized in Ref. [13]. The analysis presented here is strongly influenced by that paper.

\(^3\) Recently, a paper [28] appeared in which instanton operators of 5D $\mathcal{N} = 2$ gauge theories were also studied. There the emphasis was on the multi-point functions of instanton operators. In this paper we just consider a single instanton operator, and the multiplet it forms under the flavor symmetry and the supersymmetry. In another recent paper [29], a different type of instanton operator was studied, where an external Sp(1)$_R$ background was introduced on $S^4$ to have manifest supersymmetry at the classical level. Here we do not introduce such additional backgrounds.

\(^4\) For the details of the superconformal multiplets in 5D and in 6D, see, e.g., Ref. [31] and references therein.
The mass deformation is done by adding $\delta L = h_\mu M^\mu$ to the Lagrangian, thus breaking some of the flavor symmetry:

$$\partial_\mu J^\mu_\mu \propto f^{ab\ell} h_\mu M^\mu,$$

(2.2)

where $f^{ab\ell}$ is the structure constant. The theory is no longer superconformal, but we can still consider the supermultiplets of operators under the 5D $\mathcal{N} = 1$ supersymmetry, modulo spacetime derivatives $\partial_\mu$. No comprehensive analysis of the structure of such supermultiplets in 5D is available in the literature, to the knowledge of the author. We can still see that the supermultiplet containing $\mu_{(ij)}^a$ of a broken generator is short, because the superderivative of $\mu_{(ij)}^a$ only contains $\psi^a_{i\alpha}$, which is a doublet under $\text{Sp}(1)_R$. This fact allows us to identify the broken current supermultiplets clearly in the infrared gauge theory: they are supermultiplets containing the following:

$$\mu_{(ij)}, \, \psi_{i\alpha}, \, J_\mu$$

(2.3)

where $\mu_{(ij)}$ is an $\text{Sp}(1)_R$ triplet scalar $\mu_{(ij)}$, $\psi_{i\alpha}$ is a symplectic Majorana $\text{Sp}(1)_R$ doublet fermion $\psi_{i\alpha}$, and $J_\mu$ is a vector. The component $M$ in (2.1) is a divergence of $J_\mu$, and therefore is a descendant.

Suppose instead that the 5D theory is obtained by putting an $\mathcal{N} = (1, 0)$ 6D SCFT on $S^1$, possibly with a nontrivial holonomy for the flavor symmetry; we consider $\mathcal{N} = (2, 0)$ theories as special cases of $\mathcal{N} = (1, 0)$ theories. The conserved current supermultiplet in 6D contains

$$\mu_{(ij)}, \, \psi_{i\alpha}, \, J_A, \tag{2.4}$$

where $A$ is the 6D vector index and we have omitted the adjoint index for brevity. On $S^1$, we have Kaluza–Klein (KK) modes $J^{(n)}_\mu$ and $M^{(n)} := J^{(n)}_6$ where $n$ is the KK mode number. Then the 6D conservation gives

$$\partial_\mu J^{(n)}_\mu \propto \frac{n}{L} M^{(n)}$$

(2.5)

in 5D, where $L$ is the circumference of $S^1$. Therefore, we again find the same supermultiplet of broken currents.

### 2.2. Instanton operators

Suppose now that we are given a 5D gauge theory, and further assume that it is a mass deformation of a 5D SCFT or a 6D SCFT on $S^1$. Pick a point $p$ in $\mathbb{R}^5$ and insert there a current operator that is broken by the mass deformation in the former case, and a current operator with non-zero KK mode number in the latter case. Surround $p$ by an $S^4$, on which we have a certain state. Let the size of $S^4$ be sufficiently larger than the characteristic length scale of the system set by the mass deformation or the inverse radius of $S^1$. Then the state on $S^4$ can be analyzed using the gauge theory. When the instanton number of the gauge configuration on $S^4$ is non-zero, we call the original operator inserted at $p$ an instanton operator.

It should be possible to study the supermultiplet structure of instanton operators in detail using supersymmetric 5D Lagrangians on $S^4$ times $\mathbb{R}$ or $S^1$, using the results in, e.g., Refs. [11,32–34]. Here we only provide a rather impressionistic analysis of one-instanton operators, i.e., the instanton operators when the instanton number on $S^4$ is one.

In the rest of this section we gather known facts on one-instanton moduli spaces and fermion zero modes. We will be brief; a comprehensive account of $\text{SU}(N)$ instantons can be found in, e.g., Ref. [35]. For instantons of general gauge groups, see, e.g., Refs. [36,37].
When the gauge group is SU(2). Any instanton configuration on $S^4$ can be obtained by conformal transformations from one on a flat $\mathbb{R}^4$. When the gauge group is SU(2), the one-instanton moduli space on $\mathbb{R}^4$ has the form

$$\mathbb{R}^4 \times \mathbb{R}_{>0} \times S^3/\mathbb{Z}_2,$$

where $\mathbb{R}^4$ parametrizes the position, $\mathbb{R}_{>0}$ the size, and $S^3/\mathbb{Z}_2$ the gauge rotation at infinity. When the instanton is mapped to $S^4$, the first two factors $\mathbb{R}^4$ and $\mathbb{R}_{>0}$ combine to form the ball $B^5$ with a standard hyperbolic metric. The asymptotic infinity of $\mathbb{R}^4$ is mapped to a point on $S^4$, and therefore the gauge symmetry there should really be gauged. Therefore we lose the last factor $S^3/\mathbb{Z}_2$, but we should remember that the gauge group SU(2) is broken to $\mathbb{Z}_2$.

A point in $B^5$ very close to a point $x$ in $\mathbb{R}^4$ describes an almost point-like instanton localized at $x$. A point at the center of $B^5$ corresponds to the largest possible instanton configuration, which is in fact SO(5) invariant. This invariant configuration can be identified with the positive-chirality spinor bundle on the round $S^4$. This configuration is also known as Yang’s monopole [42].

A Weyl fermion of the correct chirality in the doublet of the SU(2) gauge group has just one zero mode on a one-instanton configuration. On $S^4$, this is a singlet of SO(5) rotational symmetry.

A Weyl fermion of the correct chirality in the adjoint of the SU(2) gauge group has four zero modes. Two are obtained by applying supertranslations and the other two by applying special superconformal transformations to the original bosonic configuration. When conformally mapped to $S^4$, these four modes transform in the spinor representation of SO(5). They are exactly the modes obtained by applying the 5D supersymmetry to the bosonic instanton configuration. In this sense the instanton configuration breaks all the supersymmetry classically, but this does not mean that the instanton operator is in a generic, long multiplet of supersymmetry, as we will soon see.

When the gauge group is general. When the gauge group is a general simple group $G$, any one-instanton configuration is obtained by embedding an SU(2) one-instanton configuration by a homomorphism $\varphi : SU(2) \to G$ determined by a long root. Therefore, the one-instanton moduli space on $\mathbb{R}^4$ is of the form

$$\mathbb{R}^4 \times \mathbb{R}_{>0} \times G/H$$

where $H$ is the part of $G$ that is unbroken by the SU(2) embedded by $\varphi$. A further quotient $G/(SU(2) \times H)$ is known as the Wolf space of type $G$. The form of $H$ is well known; here we only note that for $G = SU(N)$ we have $H = U(1) \times SU(N - 2)$.

The conformal transformation to $S^4$ again combines $\mathbb{R}^4$ and $\mathbb{R}_{>0}$ to the ball $B^5$, and we lose $G/H$ as before. The remaining effect is that we have $H$ as the unbroken gauge symmetry. Analyzing the fermionic zero modes of an arbitrary representation $R$ of $G$ around this configuration is not any harder than for SU(2), since the actual gauge configuration is still essentially that of SU(2). We only have to decompose $R$ under SU(2) × $H$, and to utilize our knowledge for SU(2).
3. SU(2)

3.1. Pure $\mathcal{N} = 1$ theory

After these preparations, let us first consider one-instanton operators of pure $\mathcal{N} = 1$ SU(2) gauge theory. As recalled in the previous section, the one-instanton moduli space on $S^4$ is just the ball $B^5$. We consider a state corresponding to the lowest SO(5)-invariant wavefunction.

Now we need to take fermionic zero modes into account. The gaugino is a spinor field in five dimensions, which gives two Weyl fermions with the correct chirality in the adjoint of the SU(2) gauge group and in the doublet of Sp$(1)_R$ when restricted on $S^4$. As recalled in the previous section, the zero modes are in the doublet of Sp$(1)_R$ and in the spinor of SO(5) rotational symmetry. Let us denote them by $\lambda_{i\alpha}$, where $i = 1, 2$ is for Sp$(1)_R$ and $\alpha = 1, 2, 3, 4$ is for SO(5). The symplectic Majorana condition in 5D means that we need to quantize these zero modes into “gamma matrices” satisfying

$$\{\lambda_{i\alpha}, \lambda_{j\beta}\} = \epsilon_{ij}J_{\alpha\beta}. \quad (3.1)$$

The states on which these zero modes act, then, can be found by decomposing the Dirac spinor representation of SO(8) in terms of its subgroup Sp$(1)_R \times$ SO(5) such that the vector representation of SO(8) becomes the doublet times the quartet. We find the following sixteen states:

$$|\mu^+_{(ij)}, \psi_{i\alpha}^+, J^+_\mu\rangle, \quad (3.2)$$

which form exactly the broken current supermultiplet (2.3) recalled in the last section. We put the plus signs as superscripts to remind us that they are one-instanton operators.

Let us assume that this gauge theory is a mass deformation of a UV 5D SCFT. Then the UV SCFT should simultaneously have both the multiplet that contains $J^+\mu$ in the gauge theory and the multiplet that becomes the instanton number current

$$J^0_\mu \propto \epsilon_{\mu\nu\rho\sigma\upsilon} \text{tr} F_{\nu\rho} F_{\sigma\upsilon}. \quad (3.3)$$

Now, $J^+\mu$ has charge 1 under $J^0_\mu$, because $J^+\mu$ is a one-instanton operator. Therefore, they should form an SU(2) flavor symmetry current. This conclusion agrees with the original stringy analysis [1].

Effect of the discrete theta angle. In the analysis so far, we have neglected the effect of the discrete theta angle of the SU(2) theory. The theta angle is associated with $\pi_4$(SU(2)) = $\mathbb{Z}_2$, and therefore only takes the values $\theta = 0$ or $\pi$. On a five-manifold of the form $M \times S^1$ such that the SU(2) configuration on $M$ has instanton number 1 and there is a nontrivial $\mathbb{Z}_2$ holonomy around $S^1$, the theta angle $\theta = \pi$ assigns an additional sign factor $-1$ in the path integral. On $\mathbb{R}^4 \times S^1$, this has an effect that makes the wavefunctions on the one-instanton moduli space sections of a nontrivial line bundle with holonomy $-1$ on $S^3/\mathbb{Z}_2$.

In our setup, the supermultiplet (3.2) is kept when $\theta = 0$, but is projected out when $\theta = \pi$. Therefore, there is an enhancement of the instanton number symmetry to SU(2) when $\theta = 0$, but we see no enhancement when $\theta = \pi$. This effect of the discrete theta angle matches what was found originally in Ref. [2] using stringy analysis.

A caveat here is that, in our crude analysis, we can only say that the states considered so far do not give any broken current supermultiplet. It is logically possible that exciting non-zero modes in the one-instanton sector or considering operators with instanton number 2 or larger gives rise to a

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Footnote 6: For a derivation, see, e.g., Appendix A of Ref. [43].
broken current supermultiplet, enhancing the instanton number symmetry to SU(2) even with θ = π. A careful study on this point is definitely worthwhile, but is outside the scope of this paper. The same caveat is also applicable to the rest of the article, but we will not repeat it.

3.2. With fundamental flavors

Next, let us consider \( N = 1 \) SU(2) theory with \( N_f \) flavors in the doublet. Stated differently, we add \( 2N_f \) half-hypermultiplets in the doublet of SU(2). At the classical Lagrangian level, they transform under SO(2\(N_f\)) symmetry.

In a one-instanton background, they give \( 2N_f \) fermionic zero modes, which need to be quantized as gamma matrices:

\[
\{ \Gamma^a, \Gamma^b \} = \delta^{ab} \tag{3.4}
\]

where \( a = 1, \ldots, 2N_f \) is the index of the vector representation of SO(2\(N_f\)). They act on the Dirac spinor representation \( S_+ \oplus S_- \) of SO(2\(N_f\)). Tensoring with the result (3.2) of the quantization of the adjoint zero modes, we find that the one-instanton operator is a broken current supermultiplet in the Dirac spinor of SO(2\(N_f\)).

Now we need to take the unbroken \( Z_2 \) gauge symmetry into account. The generator of this \( Z_2 \) symmetry acts by \(-1\) on the doublet representation, and therefore by \(-1\) on the gamma matrices in (3.4). Therefore, this \( Z_2 \) acts as the chirality operator on \( S_+ \oplus S_- \). Therefore, depending on the value of the discrete theta angle being 0 or \( \pi \), we keep only one-instanton operators in \( S_+ \) or \( S_- \). These two choices are related by the parity operation of flavor O(2\(N_f\)) and are therefore equivalent.

In conclusion, we found the following current multiplets:

- the conserved SO(2\(N_f\)) currents \( J_\mu^{[ab]} \),
- the conserved instanton number current \( J_\mu^0 \),
- the broken currents coming from one-instanton operators \( J_\mu^{+,A} \) where \( A \) is the chiral spinor index of SO(2\(N_f\)).

Take \( 1 \leq N_f \leq 7 \). If we assume that the gauge theory is a mass deformation of a UV fixed point, we see that the currents listed above need to combine to give the flavor symmetry \( E_{N_f+1} \). This can be seen by attaching an additional node, representing a Cartan element for \( J_\mu^0 \), to the node of the Dynkin diagram of SO(2\(N_f\)) that gives the chiral spinor representation. For example, we have

\[
\text{(3.5)}
\]

when \( N_f = 7 \), where the black node is for the instanton number current.\(^7\) This enhancement pattern agrees with what was found originally in Ref. [1].

Let us boldly take \( N_f = 8 \). We now have the combined Dynkin diagram

\[
\text{(3.6)}
\]

which is known as \( E_9 \) or \( \hat{E}_8 \). If we assume that the gauge theory is an outcome of a massive deformation of some UV completion, the UV completion needs to have an affine \( E_8 \) flavor symmetry as

\[^7\]Strictly speaking, this procedure is not unique when a theory with very small \( N_f \) is considered alone. As a simpler example of this issue, suppose we know that we have an SU(2) current, an instanton number current, and a broken current coming from one-instanton operators in the doublet of SU(2). Then we have two choices, either an SU(3) corresponding to •\(-\)○ or an Sp(2) corresponding to ○\(-\)○•. They can only be distinguished by studying the two-instanton operators. In our case, however, we can just study the \( N_f = 7 \) case and then apply the mass deformation to make \( N_f \) smaller, to conclude that the flavor symmetry is always \( E_{N_f+1} \).
a 5D theory. Stated differently, this means that the UV completion needs to be a 6D theory with $E_8$ flavor symmetry. This conclusion matches what was found in Ref. [5].

We see a problem when $N_f > 8$: the combined Dynkin diagram defines a hyperbolic Kac–Moody algebra, whose number of roots grows exponentially. It is therefore unlikely that there is a UV completion when $N_f > 8$. Again, this matches the outcome of a different analysis in Ref. [1].

Before proceeding, we note here that it is well known that the instanton particles in this gauge theory give rise to spinors of $SO(2N_f)$ flavor symmetry. The only slightly new point in this section is that, when studied in the context of instanton operators, they are indeed part of the broken current multiplets given by (3.2).

### 3.3. $\mathcal{N} = 2$ theory

As a final example of SU(2) gauge theory, let us consider $\mathcal{N} = 2$ gauge theory. On the one-instanton background on $S^4$, we have four Weyl fermions in the adjoint of SU(2), and the zero modes can be denoted by $\lambda_{i\alpha}$ where $i = 1, 2, 3, 4$ is now for Sp(2)$_R$ and $\alpha = 1, 2, 3, 4$ is for the spacetime SO(5). Again, the symplectic Majorana condition in 5D means that they become “gamma matrices” with the commutation relation

$$\{\lambda_{i\alpha}, \lambda_{j\beta}\} = J_{ij} J_{\alpha\beta}. \quad (3.7)$$

These gamma matrices act on the following one-instanton states on $S^4$:

$$|\mu^+_{(ab)}\rangle, \quad |\psi^+_{a\mu}\rangle, \quad |J^+_{[ab] \mu}\rangle, \quad |Q^+_{i\mu\alpha}\rangle, \quad |T^+_{(\mu\nu)}\rangle, \quad |X^+_{a[\mu\nu]}\rangle, \quad (3.8)$$

where $a = 1, 2, 3, 4, 5$ is the vector index for Sp(2)$_R = SO(5)_R$. The gamma-tracelessness condition on $\psi^+, Q^+$ and the tracelessness condition on $\mu^+, T^+$ need to be imposed.

When the discrete theta angle is zero, these operators are all kept, and the structure of the operators is exactly that of KK modes of the 6D $\mathcal{N} = 2$ energy–momentum supermultiplet. For example, the currents $J^+_{[ab] \mu}$ in the adjoint of Sp(2)$_R$ suggest that the Sp(2)$_R$ symmetry of the 5D gauge theory enhances to the affine Sp(2)$_R$, and the symmetric traceless $T^+_{(\mu\nu)}$ is the KK mode of the 6D energy–momentum tensor. This is as it should be, since the $S^4$ compactification of $\mathcal{N} = (2, 0)$ theory of type SU(2) on $S^4$ is described by $\mathcal{N} = 2$ SU(2) gauge theory in 5D. The relation between the 5D $\mathcal{N} = 2$ theory and the 6D $\mathcal{N} = (2, 0)$ theory on $S^4$ has been extensively studied; see, e.g., Refs. [44,45].

When the discrete theta angle is $\pi$, these operators are all projected out. In this case too, the gauge theory is the IR description of $\mathcal{N} = (2, 0)$ theory of type SU(3) on $S^4$ with an outer-automorphism $\mathbb{Z}_2$ twist around it [43]. We expect operators with the same structure to arise in a sector with higher instanton number, but to check this is outside the scope of this paper.

### 4. SU($N$)

#### 4.1. Pure $\mathcal{N} = 1$ theory

Now let us move on to SU($N$) gauge theories. Our first example is the pure SU($N$) theory. As recalled in Sect. 2, in one-instanton configurations on $S^4$, the gauge fields take values in SU($2$) $\subset$ SU($N$). The unbroken subgroup is U(1) $\times$ SU($N - 2$). We take the generator of the U(1) part to be

$$\text{diag}(N - 2, N - 2, -2, -2, \ldots, -2). \quad (4.1)$$

In this normalization, when the Chern–Simons level is $\kappa$, the one-instanton configuration has the U(1) charge $(N - 2)\kappa$.

An adjoint Weyl fermion of SU($N$) on this background decomposes into an SU(2) adjoint Weyl fermion, $N - 2$ Weyl fermions in the doublet of SU(2) in the fundamental of SU($N - 2$) and with
U(1) charge $N_f$ and Weyl fermions that are neutral. The gaugino in 5D gives two adjoint Weyl fermions of SU($N$) on $S^4$. The SU(2) adjoint part gives the same broken current supermultiplet (3.2), and we need to take additional $2 \times (N - 2)$ zero modes coming from the SU(2) doublet into account.

By quantizing them, we have fermionic creation operators $B_{ia}$ where $i = 1, 2$ is now for Sp($1$)$_R$ and $a = 1, \ldots, (N - 2)$ is for SU($N - 2$). The U(1) charge of $B_{ia}$ is $N_f$. The SU($N - 2$) invariant states are then

$$|0\rangle, \quad e^{a_1 \cdots a_{N-2}} B_{i_1a_1} B_{i_2a_2} \cdots B_{i_{N-2}a_{N-2}} |0\rangle, \quad (B_{ia})^{2(N-2)} |0\rangle,$$

with U(1) charge $-(N - 2)N, 0, +(N - 2)N$, respectively.

When $\kappa$ is neither 0 nor $\pm N$, all states are projected out, due to the non-zero U(1) gauge charge. When $\kappa = \pm N$, one singlet state is kept, and we have a broken current supermultiplet (3.2). Therefore we expect the enhancement of the instanton number symmetry to SU(2). This enhancement was recently discussed in Ref. [19].

When $\kappa$ is 0, the one-instanton operators are the tensor product of the broken current supermultiplet (3.2) times $e^{a_1 \cdots a_{N-2}} B_{i_1a_1} B_{i_2a_2} \cdots B_{i_{N-2}a_{N-2}} |0\rangle$, which transforms in the $N - 1$-dimensional irreducible representation of Sp($1$)$_R$. This is a short supermultiplet, but does not correspond to a broken flavor symmetry.

4.2. With fundamental flavors

Our next example is SU($N$) theory with $N_f$ hypermultiplets in the fundamental representation. On one-instanton configurations on $S^4$, each flavor decomposes into a pair of SU(2) doublets and a number of neutrals; they all have U(1) charge $N - 2$. Therefore, we have additional fermionic creation operators $C_s$, $s = 1, \ldots, N_f$, of U(1) charge $N - 2$. They act on the states of the form

$$C_{s_1} C_{s_2} \cdots C_{s_k} |0\rangle$$

for $k = 0, \ldots, N_f$, with U(1) charge $(N - 2)(k - N_f/2)$.

Tensoring (3.2), (4.2), and (4.3) and imposing the U(1) gauge neutrality condition, we see that a broken symmetry supermultiplet survives when we have

$$\kappa \pm N + (k - N_f/2) = 0. \quad (4.4)$$

Now, in Ref. [4] it was shown that we need $|\kappa| \leq N - N_f/2$ to have a 5D UV SCFT behind the gauge theory. Therefore $|\kappa \pm N| \geq N_f/2$. We also trivially have $|k - N_f/2| \leq N_f/2$. Therefore, the equality (4.4) can only be satisfied when $\pm \kappa = N - N_f/2$.

When $\pm \kappa = N - N_f/2$, the surviving broken current supermultiplet comes from $k = 0$ or $k = N_f$ in (4.3). They have instanton number one and baryonic charge $\pm N_f/2$. Therefore, when $\kappa = N - N_f/2$, one U(1) enhances to SU(2), and, when $-\kappa = N - N_f/2$, another U(1) enhances to SU(2). When $\kappa = N - N_f/2 = 0$, the combination $I \pm B/N$ of the instanton charge $I$ and the baryonic charge $B$ both enhance to SU(2)$_\pm$, making the UV flavor symmetry SU($N_f$) $\times$ SU(2)$_+ \times$ SU(2)$_-$. This enhancement pattern was found in Refs. [19,27].

4.3. $\mathcal{N} = 2$ theory

Let us next consider $\mathcal{N} = 2$ SU($N$) theory. The Chern–Simons level $\kappa$ is automatically zero. Again, the adjoint Weyl fermions of SU($N$) decompose into those that are adjoint of SU(2) and those that are doublets of SU(2). The zero modes of the former generate the states (3.8).
The zero modes of the latter give us fermionic creation operators \( B_{ia} \), where \( i = 1, 2, 3, 4 \) is for \( \text{Sp}(2)_R \) and \( a = 1, \ldots, (N - 2) \) is for gauge \( \text{SU}(N - 2) \). The \( \text{U}(1) \times \text{SU}(N - 2) \) neutral states then have the form

\[
\epsilon^{a_1 \cdots a_{N-2}} B_{1a_1} B_{2a_2} \cdots B_{i_{N-2} a_{N-2}} \epsilon^{b_1 \cdots b_{N-2}} B_{j_1 b_1} B_{j_2 b_2} \cdots B_{j_{N-2} b_{N-2}} |0 \rangle
\]

\[
= B_{i_1} B_{j_1} B_{i_2} B_{j_2} \cdots B_{i_{N-2}} B_{j_{N-2}} |0 \rangle.
\]

(4.5)

We want to decompose them under the action of \( \text{Sp}(4)_R \). Let us think that \( \text{SU}(4) \) acts on the indices \( i \) and \( j \). The indices \( i_n \) and \( j_n \) are antisymmetrized. Combined, \( [i_n,j_n] \) is a vector of \( \text{SO}(6) \simeq \text{SU}(4) \). The indices are symmetrized under the combined exchange of \( [i_n,j_n] \) and \( [i_m,j_m] \). Therefore, the states (4.5) transform under the \((N-2)\)nd symmetric power of the vector of \( \text{SO}(6) \). Decomposing it under \( \text{Sp}(4)_R \simeq \text{SO}(5)_R \subset \text{SO}(6) \), we see that the states (4.5) are in the representation

\[
V_0 \oplus V_1 \oplus \cdots V_{N-2},
\]

(4.6)

where \( V_k \) is the \( k \)th symmetric traceless representation of \( \text{SO}(5)_R \).

This structure is precisely what we would expect for the KK modes of \( \mathcal{N} = (2, 0) \) theory of type \( \text{SU}(N) \) put on \( S^1 \). In general, \( \mathcal{N} = (2, 0) \) theory of type \( G \) has short multiplets containing a spacetime symmetric traceless tensor that is in \( V_{N-2} \) of \( \text{SO}(5)_R \), for each \( n \) that gives a generator of invariant polynomials of \( G \). For \( G = \text{SU}(N), \) \( n \) runs from 2 to \( N \), thus giving (4.6) tensored with (3.8).

It would be interesting to perform similar computations for \( \mathcal{N} = 2 \) theories for other \( G \). For example, when \( G = E_6 \), we should have \( V_0 \oplus V_3 \oplus V_4 \oplus V_6 \oplus V_7 \oplus V_{10} \), since the generators of invariant polynomials of \( E_6 \) have degrees 2, 5, 6, 8, 9, and 12. When \( G \) is non-simply laced, the UV completion is \( \mathcal{N} = (2, 0) \) theory of some simply laced type, with an outer-automorphism twist around \( S^1 \). This again predicts which \( V_n \) should appear among the one-instanton operators.

5. Quivers

In this section, we study the symmetry enhancement in the quiver gauge theory with SU gauge groups and bifundamental hypermultiplets. In this note we do not aim at comprehensiveness; instead we only treat the case where the effective number of flavors at each node \( \text{SU}(N_i) \) is \( 2N_i \) and the Chern–Simons levels are all zero.

We use the by-now standard notation where \( \bullet \circ \) stands for an \( \text{SU}(N_1) \) flavor symmetry node and an \( \text{SU}(N_2) \) gauge symmetry node connected by a bifundamental hypermultiplet, etc. We also use a special convention that two flavors of “\( \text{SU}(1) \)”, \( \bullet \circ \), stand for \( \text{SU}(2) \). The rationale behind this convention will be explained later in Sect. 5.3.

5.1. \( \text{SU}(2)^2 \) theory

Let us begin our analysis by considering the theory

\[
\begin{array}{c}
\text{2} \quad Q_0 \\
\text{2} \quad Q_1 \\
\text{2} \quad Q_2 \\
\end{array}
\]

(5.1)

We denote the gauge groups as \( \text{SU}(2)_1 \times \text{SU}(2)_2 \). The hypermultiplets \( Q_0, Q_1, Q_2 \) have flavor symmetries \( \text{SO}(4)_F, \text{SU}(2)_F, \text{SO}(4)_F, \) respectively. The gauge group \( \text{SU}(2)_1 \) effectively has \( N_f = 4 \) flavors, and thus it has \( E_{4+1} = \text{SO}(10) \) flavor symmetry when \( \text{SU}(2)_2 \) is not gauged. After gauging, the remaining flavor symmetry is the commutant of \( \text{SU}(2)_2 \), which is \( \text{SU}(4) \times \text{SU}(2) \). We can
summarize this enhancement pattern as

\[
\begin{array}{c}
\circ \longrightarrow \circ \\
(5.2)
\end{array}
\]

where the two white nodes on the left are \( \text{SO}(4)_{F0} \), the white node on the right is \( \text{SU}(2)_F \), and the black node is the contribution from the instanton operator of \( \text{SU}(2)_1 \).

The same argument can be applied to the \( \text{SU}(2)_2 \) side, and we conclude that the full flavor symmetry is

\[
\begin{array}{c}
\circ \longrightarrow \circ \\
(5.3)
\end{array}
\]
i.e., \( \text{SU}(2) \times \text{SU}(6) \times \text{SU}(2) \).

Let us study one implication of this enhancement. The adjoint of \( \text{SU}(6) \) can be decomposed as follows:

\[
\begin{pmatrix}
A & B_1 & C \\
B_1^\dagger & A' & B_2 \\
C^\dagger & B_2^\dagger & A''
\end{pmatrix},
\]

(5.4)

where each symbol stands for a \( 2 \times 2 \) block. The blocks \( A, A', \) and \( A'' \) are three \( \text{SU}(2) \) flavor symmetries that can be seen in the Lagrangian; \( B_i \) comes from one-instanton operators of the \( \text{SU}(2)_i \) gauge group; and \( C \) comes from instanton operators that have instanton number one for both gauge groups \( \text{SU}(2)_{1,2} \). Let us call these last ones \((1, 1)\)-instanton operators. They transform as chiral spinors under \( \text{SO}(4)_{F0, F2} \) and are neutral under \( \text{SU}(2)_F \).

Let us try to study the \((1, 1)\)-instanton operators directly. The zero modes of gauginos of \( \text{SU}(2)_{1,2} \) give two copies of the broken current multiplets \((3.2)\); those of hypermultiplets \( Q_0 \) and \( Q_2 \) give the spinors of \( \text{SO}(4)_{F0} \) and \( \text{SO}(4)_{F2} \).

Finally, the hypermultiplet \( Q_1 \) couples to one-instanton configurations of both \( \text{SU}(2)_{1,2} \). Therefore this is effectively a triplet coupled to an \( \text{SU}(2) \) one-instanton configuration. It is also a doublet of \( \text{SU}(2)_F \). Therefore they give rise to the states

\[
|\mu_{(ab)}^+, \psi_{aa}^+, J_\mu^+\rangle,
\]

(5.5)

where \( a, b = 1, 2 \) is now the index of the doublet of \( \text{SU}(2)_F \).

We therefore need to take the tensor product of two copies of the broken current multiplet \((3.2)\), the spinors of \( \text{SO}(4)_{F0, F2} \), and the multiplet \((5.5)\). We then need to impose the \( \mathbb{Z}_2 \) projections for \( \text{SU}(2)_{1,2} \).

At present we do not know enough about the behavior of the tensor product of the supersymmetry multiplets in 5D. Instead, we learn the following by using the knowledge of the flavor symmetry properties of \((1, 1)\)-instanton operators deduced from the block decomposition \((5.4)\): the tensor product of two copies of the broken current multiplet \((3.2)\) and the multiplet \((5.5)\) contains again a unique broken current multiplet. Furthermore, it is \( \text{SU}(2)_F \) neutral, and therefore it comes from the factor \( J_\mu^+ \) in \((5.5)\).

5.2. \( \text{SU}(N_1) \times \text{SU}(N_2) \) theory

Next let us discuss more general two-node quivers given by

\[
\begin{array}{c}
\text{[N_1]} \arrrow Q_k \arrrow \text{[N_2]} \arrrow Q_{k+1} \arrrow \text{[N_3]} \arrrow \text{[N_4]}
\end{array}
\]

(5.6)

We assume \( N_1 > 2, N_2 > 2, N_0 + N_2 = 2N_1, \) and \( N_1 + N_3 = 2N_2 \). We also set both the Chern–Simons levels to zero. Let us denote by \( \text{U}(1)_{B1} \) and \( \text{U}(1)_{B2} \) the baryonic flavor symmetries that
assign charge 1 to a field in the fundamental of SU($N_1$) and SU($N_2$), respectively. Let us also denote by $U(1)_{I_1}$ and $U(1)_{I_2}$ the instanton number charge of SU($N_1$) and SU($N_2$), respectively.

We already saw that the combinations $I_{1\pm} := I_1 \pm B_1/N_1$ and $I_{2\pm} := I_2 \pm B_2/N_2$ are each enhanced to an SU(2), giving SU(2)$^4$ flavor symmetry. Let us show that $I_{1+}$ and $I_{2+}$ combine to form an SU(3)$_+$ and similarly that $I_{1-}$ and $I_{2-}$ combine to form an SU(3)$_-$. To see this, we need to analyze separately the cases when one of the gauge groups is SU($N_i$) is broken to $U(1)_{I_j} \times SU(N_i - 2)$. The gaugino zero modes can be analyzed as before. The hypermultiplets $Q_0$ and $Q_2$ give doublets of SU(2) one-instanton configuration; the hypermultiplet $Q_1$ similarly gives a lot of doublets and just one triplet of SU(2) one-instanton configuration. Then, we need to find states that are neutral under the unbroken gauge group from the tensor product of the following contributions:

1. From fields that are doublets of the SU(2) one-instanton configuration, we have
   1a. contributions (4.2) from gauginos of SU($N_1$) and SU($N_2$),
   1b. and contributions (4.3) from $Q_0$, $Q_1$, and $Q_2$.

2. From fields that are triplets of the SU(2) one-instanton configuration, we have
   2a. a contribution (5.5) from $Q_1$,
   2b. and two copies of the broken current multiplet (3.2) from SU($N_{1,2}$).

From the contributions 1, we find two states that are neutral under the unbroken gauge group $U(1)_1 \times SU(N_1 - 2) \times U(1)_2 \times SU(N_2 - 2)$, by tensoring the ground state or the top state of (4.3) from the contributions 1a by the ground state or the top state of (4.3) from the contributions 1b.

From the contributions 2a, we note that the only $U(1)_{I_1}$- and $U(1)_{I_2}$-neutral state is the component $J_{I_2 \mu}$ in (5.5). Tensoring with the contributions 2b, we find a broken current multiplet, as we found at the end of the last subsection.

In total, we find at least two gauge-invariant broken current multiplets. The charges under the Lagrangian flavor symmetries can be easily found: both are neutral under SU($N_0$) and SU($N_3$), and the charges under $U(1)_{Q_0}$, $U(1)_{Q_1}$, $U(1)_{Q_2}$ are

$$\pm \left( \frac{N_0}{2}, \frac{N_1 - N_2}{2}, \frac{-N_3}{2} \right).$$

Therefore we have found two broken current multiplets with charges under $(I_{1+}, I_{2+}; I_{1-}, I_{2-})$ given by $(1, 1; 0, 0)$ and $(0, 0; 1, 1)$, respectively. We already know that one-instanton operators of SU($N_1$) give broken current multiplets with charges $(1, 0; 0, 0)$ and $(0, 0; 1, 0)$, and similarly those of SU($N_2$) give multiplets with charges $(0, 1; 0, 0)$ and $(0, 0; 0, 1)$. Therefore we see that the instanton number currents $I_{1+}$, $I_{2+}$ combine to form SU(3)$_+$, and the currents $I_{1-}$ and $I_{2-}$ combine to form SU(3)$_-$. The total flavor symmetry is therefore at least

$$SU(3)_+ \times SU(3)_- \times SU(N_0) \times SU(N_3) \times U(1).$$

The last U(1) is absent when $N_0$ or $N_3$ is zero.

5.3. Some special two-node quivers

We need to analyze separately the cases when one of the gauge groups is SU(2) or “SU(1)”. As already mentioned, we use the convention where “two flavors of SU(1)”

$$SU(1) \times SU(1) \times SU(1)$$

(5.9)
SU quiver: “instantons” that naturally give rise to the setup (5.10). This is another rationale for our convention.

Consider as an example the quiver

\[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\]

(5.10)

We also apply the same convention where the SU(2) on the left is gauged.

From a purely 5D field theory point of view, this is really just a convention, but it is useful because “SU(1) with two flavors” shows an enhanced symmetry of SU(2) \( \times \) SU(2) \( \times \) SU(2), just as a special case of SU(N) with 2N flavors with the symmetry enhancement SU(2) \( \times \) SU(2) \( \times \) SU(2N).

From a string/M theory point of view, when “SU(1) with two flavors” is engineered, say, in the brane web construction, one in fact finds additional hypermultiplets coming from “point-like SU(1) instantons” that naturally give rise to the setup (5.10). This is another rationale for our convention.

Now, let us couple this “SU(1) with two flavors” to an SU(2) gauge group to form a two-node quiver:

\[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\]

(5.11)

Using our convention, this is just SU(2) with five flavors that show an enhancement to \( E_6 \). This contains SU(3) \( _+ \) \( \times \) SU(3) \( _- \) \( \times \) SU(3), showing the general pattern that we found in (5.8).

We already treated

\[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\]

(5.12)

and saw that the symmetry is SU(2) \( \times \) SU(6) \( \times \) SU(2). As SU(6) \( \supset \) SU(3) \( _+ \) \( \times \) SU(3) \( _- \) \( \times \) U(1), it again shows the general pattern (5.8).

Finally, let us consider

\[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\]

(5.13)

The SU(2) theory before coupling to SU(4) has an enhanced symmetry \( SO(10) \). After coupling to SU(4) the remaining part is SU(2) \( _+ \) \( \times \) SU(2) \( _- \), and they are enhanced by the dynamical SU(4) to SU(3) \( _+ \) \( \times \) SU(3) \( _- \), again following the general pattern (5.8).

5.4. Multi-node quivers

After our preparation on the two-node quivers, it is easy to analyze general multi-node quivers, again with the restriction that each SU(N) node has effectively 2N flavors and zero Chern–Simons terms.

Consider as an example the quiver

\[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\] \( \longrightarrow \) \[
\begin{array}{c}
\text{SU}(2) \\
1
\end{array}
\]

(5.14)

Each SU(N\( _i \)) node with \( N_i = 4, 3, 2, 1 \) shows an enhancement of the linear combination of the instanton current and the baryonic current, \( I_{i \pm} = I_i \pm B_i / N_i \) to SU(2) \( _{i \pm} \). For each neighboring pair of nodes SU(N\( _j \)) \( \supset \) SU(N\( _j \)) \( \supset \) SU(N\( _j \)) \( \supset \) SU(N\( _j \)) \( \supset \) SU(N\( _j \)) \( \supset \) SU(N\( _j \)). Therefore, in total, we should have SU(5) \( _+ \) \( \times \) SU(5) \( _- \) from the enhancement of the instanton number symmetry and the baryonic symmetry. Combined with the original flavor symmetry SU(5) of the leftmost node, we have

\[
\text{SU}(5) \supset \text{SU}(5) \supset \text{SU}(5)
\]

(5.15)

as the enhanced symmetry. We can easily generalize this analysis to an analogous linear quiver with the gauge group SU(N\( -1 \)) \( \times \) SU(N\( -2 \)) \( \times \cdots \times \) SU(2) \( \times \) “SU(1)”, with bifundamentals between the neighboring gauge nodes and additional \( N \) fundamentals for SU(N\( -1 \)); we see the symmetry SU(N\( _+ \)) \( \times \) SU(N\( _- \)) \( \times \) SU(N). The 5D SCFT is called the 5D \( T_N \) theory, and this linear quiver presentation was recently studied in Refs. [17,20,25,26].
As another example, consider the following quiver:

\[
\begin{array}{cccccccc}
1m & 2m & 3m & 4m & 5m & 6m & 4m & 2m \\
\end{array}
\]  
(5.16)

We have gauge groups SU\((N_i)\) with \(i = 1, 2, \ldots, 9\), with \(N_i = k_i m\). Let us define \(I_{i\pm} = I_i \pm B_i/N_i\). Then, applying exactly the same argument as above, we see that the currents \(I_{i+}\) combine to form \((\hat{E}_8)_{+}\) and the currents \(I_{i-}\) enhance to \((\hat{E}_8)_{-}\). The total flavor symmetry is not quite their product, however. The Cartan generator corresponding to a pure KK momentum of \((\hat{E}_8)_{+}\) is

\[
\sum k_i I_{i+} = \sum k_i I_i + \frac{1}{m} \sum B_i
\]  
(5.17)

but \(\sum B_i\) does not act on the hypermultiplets and therefore is trivial. Thus we have

\[
\sum k_i I_{i+} = \sum k_i I_{i-},
\]  
(5.18)

meaning that the total flavor symmetry is

\[
\hat{E}_8 \times E_8,
\]  
(5.19)

showing that the possible UV completion of this gauge theory is a 6D SCFT with flavor symmetry \(E_8 \times E_8\) on \(S^1\). This is as expected: \(m\) M5-branes on the ALE singularity of type \(E_8\) gives a 6D \(\mathcal{N} = (1, 0)\) SCFT in the infrared, with \(E_8 \times E_8\) flavor symmetry. Compactifying it on \(S^1\) and reducing it to type IIA, we have \(m\) D4-branes probing the ALE singularity of type \(E_8\). Using the standard technique [46], we find the quiver theory given above.

The general statement is now clear. Take a 5D quiver gauge theory, with each SU\((N)\) gauge node having effectively \(2N\) flavors. If the quiver is a finite simply laced Dynkin diagram of type \(G\), the instanton number currents enhance to \(G \times G\); if the quiver is an affine simply laced Dynkin diagram of type \(G\), the instanton number currents enhance to \(\hat{G} \times \hat{G}\).

6. Conclusions

In this paper we have analyzed the one-instanton operators of 5D gauge theories with SU\((N)\) gauge groups with hypermultiplets in the fundamental, adjoint, or bifundamental representations. We saw that a simple exercise in the treatment of fermionic zero modes gives rise to the expected patterns of symmetry enhancements.

There are many areas to be further explored. One is to extend our analysis to include SU\((N)\) gauge theories with other matter representations, such as antisymmetric or symmetric two-index tensor representations, and to consider other gauge groups, both classical and exceptional. There should not be any essential difficulty in performing this generalization, since a one-instanton configuration in any group \(G\) is always just an SU\((2)\) one-instanton configuration embedded into \(G\). Our analysis of the SU quiver theory is by no means exhaustive, and it would be interesting to consider more general cases.

It might be interesting to study instanton operators with higher instanton numbers. This will be significantly harder, however, since the instanton moduli space is much more complicated. Presumably, we will need to use the localization etc. to analyze it, and the method would become equivalent to what has already been done in the literature in the study of the superconformal index of the 5D SCFTs.
Another direction is to study in more detail the structure of the supermultiplets formed by operators in non-conformal 5D supersymmetric theories. In this paper we relied on some heuristics based on the known supermultiplet structures of superconformal theories. The gauge theories in the infrared are, however, non-conformal, and we should analyze them as they deserve. For example, in our analysis of $\text{SU}(N_1) \times \text{SU}(N_2)$ theory, we could not directly analyze the tensor product decomposition of the two copies of (3.2) and the contribution (5.5); instead we needed to import the knowledge gained by the analysis of the special case $\text{SU}(2)^2$. This is not an ideal situation. With a proper understanding of the supermultiplet structures of operators in non-conformal theories, we would be able to analyze this tensor product directly.

We also assumed throughout this paper that we only have to consider fermionic zero modes around the one-instanton configuration, and that the states with excited non-zero modes do not give broken current supermultiplets. This is at least plausible, since non-zero modes would likely produce descendant operators, but this is not at all a rigorous argument. This needs to be better investigated.

Finally, we assumed in this paper that the gauge theory that we analyze is a mass deformation of a UV fixed point, either a 5D one or a 6D one compactified on $S^1$, and then studied what would be the enhanced symmetry in the ultraviolet. It would be desirable to understand the criterion to tell which 5D gauge theory has a UV completion.

The author would like to come back to these questions in the future, but he will not have time in the next few months due to various duties in the university. He hopes that some of the readers get interested and make great progress in the meantime.

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