The maximum turnaround radius for axisymmetric cosmic structures

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Abstract

We compute the leading order effect of non-sphericity on the maximum size $R_{TA,max}$ of realistic large scale bound cosmic structures in the framework of ΛCDM. As a first step, we focus on static or stationary axisymmetric cases, in which the departure from spherical symmetry is due to the mass density distribution or to the angular momentum of the structure. Modeled by a Kerr-de Sitter spacetime, the fractional change $\delta R_{TA,max}(\theta)/R_{TA,max}^{(0)}$ of $R_{TA,max}$ of a given rotating cosmic structure, compared to a spherical one with the same mass, is negative for all values of the polar angle $\theta$, and its average over the angles is $\langle \delta R_{TA,max}(\theta)/R_{TA,max}^{(0)} \rangle \approx -a^2/(3R_{TA,max}^{(0)})^2 \approx -O(v_{\text{out}}^2/c^2)$, with $a = J/M$ the parameter angular momentum per unit mass of the Kerr-de Sitter background and $v_{\text{out}}$ the azimuthal speed of the outmost members of the structure. In contrast, in the case of a homogeneous static spheroidal distribution the leading order effect of its eccentricity on $\langle \delta R_{TA,max}(\theta) \rangle$ vanishes for all values of the eccentricity parameter.

keywords : Large scale structures, non-sphericity, maximum turnaround radius

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1 Introduction

The ΛCDM model is simple and very successful in explaining the observational data up to now [1, 2]. Nevertheless, it is not entirely satisfactory, because apart from issues related to the statistical significance of the supernovae Ia data [3], it requires severe fine tuning of the cosmological constant Λ, interpreted as the vacuum energy density of the Standard Model matter fields [4], and, furthermore, it does not provide any insight towards an explanation of the “cosmic coincidence problem”, i.e., the fact that the current observed values of the dark energy and the cold dark matter energy densities are so close to each other. Despite all efforts of a large part of the physics community no completely satisfactory answer is yet available.

Motivated by these two issues and by the so far lack of any observational evidence of a dark matter particle candidate, the community in recent years has plunged into intense research in looking into alternatives of the ΛCDM model or Einstein’s theory of gravitation, see [5, 6, 7, 8] for exhaustive reviews (also references therein). To distinguish all these alternative gravity models from each other and from the ΛCDM, we require suitable cosmological or astrophysical observable quantities. Owing to the tiny observed current value of Λ (∼ O(10^{-52} m^{-2})), it is natural to expect that its effect would be significant only when one probes astrophysical or cosmological observables comparable to the Hubble horizon length. Redshift of light from the type Ia supernovae or the microwave background are two such well known examples.

However, a novel local check of the viable dark energy models was proposed recently [9, 10, 11], based on the stability of certain low redshift large scale cosmic structures. Precisely, since the dark energy has a repulsive effect growing monotonically with the radial distance from a given centre while the Newtonian attraction force diminishes, one expects these two forces to balance at some radius. Beyond this, the spacetime is expanding with acceleration and there can be no bound structure formation. A stable structure can only exist within this radius, called the maximum turnaround radius, R_{TA,max}. Thus the theoretical prediction of R_{TA,max} of a given stable structure for a certain model of gravity must be greater than or equal to its actual observed size. The next step is then to use R_{TA,max} as an observable to constrain the parameters of a cosmological model by comparing its theoretical prediction for R_{TA,max} with the observed data.

For ΛCDM for instance, the predicted value of R^{(0)}_{TA,max} for a spherical structure equals (3MG/Λc^2)^{1/3} [9]. For superclusters as massive as M ≳ 10^{15} M⊙, this theoretical prediction lies very close from above, with the departure from the observation being only about roughly 10%. This means that the ΛCDM is consistent with the observed bound cosmic structures and secondly, we may indeed use R_{TA,max} to constrain alternative gravity models!

Now, after a structure gets formed via gravitational collapse, it goes through a virialization process and its size gets reduced. It was shown in [11, 13] that structures with M ≳ 10^{13} M⊙ are not virialized today. Thus, R_{TA,max} is a relevant and useful quantity for those non-virialized ‘large’ structures only. It also turns out that they are relatively young and lie close to us, z ∼ 0.01. Hence probing the turnaround radius is basically a local check of the dark energy models.

We refer our reader to [14]-[34] for various computations, data and parameter space analysis and other applications pertaining the R_{TA,max}. The computations yield the same result no matter whether one uses a static or time dependent cosmological geometry. We further refer our reader to [27] for a more detailed review on this.

All the above computations were performed assuming spherical symmetry of the background spacetime. Since the R_{TA,max} for the large scale structures we are looking into is much bigger than their Schwarzschild

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1 Notably, the same idea was used in [12] to derive an upper bound on Λ on the basis of the existence of stable galaxies.
radii, and as well as non-virialization implies near spherical symmetry, such approximations are expected to work well. Nevertheless, an obvious question to ask is: what is the leading effect of the spacetime non-sphericity on $R_{TA, \text{max}}$? In particular, this question seems very appropriate given the generic non-spherical form of the structures in the Cosmos and in numerical simulations.

To the best of our knowledge, this issue was addressed very recently in [35, 36], using the cosmological quasi-local mass function 2 and a related publication is currently in preparation [37] based on a statistical analysis of structures obtained by extensive computer simulations. In this work we wish to study analytically the maximum turnaround radius in axisymmetric spacetimes as a first departure from spherical symmetry. This, in particular, may be thought of as a reasonable model to study the weak gravitational field of our nearby Corona-Borealis supercluster, which seems to be a binary connected via a filament, e.g. [38]. The plan of the present paper is the following: For orientation, we start in Section 2 with the Newtonian analysis of a homogeneous spheroidal structure [39]. In Section 3 and Section 4 we extend the analysis to the general relativistic backgrounds of a static axisymmetric [40, 41, 42] as well as to the stationary axisymmetric Kerr-de Sitter spacetimes. We end with a summary of our results and a few comments in the Conclusion Section.

We shall use mostly positive signature of the metric and henceforth will set $c = 1 = G$.

## 2 Turnaround radius of a spheroidal structure in de Sitter

We start with the Newtonian approximate treatment of the maximum turnaround radius in the gravitational field of a homogeneous oblate spheroid with semi-axes $\alpha$ and $\beta$ ($\alpha \geq \beta$) and total mass $M$ in a de Sitter background with cosmological constant $\Lambda = 3H_0^2$, where $H_0^{-1} \sim 1.3 \times 10^{10}$ ly is approximately the inverse of the Hubble parameter today.

In the $(v, \xi, \psi)$ ellipsoid coordinate system of the oblate spheroid, the Newtonian potential outside it (i.e. for $v > 1$) is the harmonic function [39]

$$V_N = -\frac{M}{\alpha \epsilon} \left( \cot^{-1} \sigma + \frac{1}{2} ((3\sigma^2 + 1) \cot^{-1} \sigma - 3\sigma) P_2(\cos \xi) \right)$$

(1)

with $\epsilon = \sqrt{1 - \beta^2/\alpha^2}$ the “eccentricity” of the ellipsoid, $\sigma = \sigma(v) = \sqrt{1 - \epsilon^2} v/\epsilon$ and $P_2(x) = (3x^2 - 1)/2$ is the Legendre polynomial.

The position vector in the coordinate system $(v, \xi, \psi)$ is

$$\mathbf{r} = \alpha \left( \sqrt{(1 - \epsilon^2)} v^2 + \epsilon^2 \sin \xi \left( i \cos \psi + j \sin \psi \right) + k \sqrt{1 - \epsilon^2} v \cos \xi \right)$$

and, consequently, the relation of the ellipsoidal coordinates to the spherical polar ones $(r, \theta, \phi)$ is

$$v = \frac{s(r, \theta)}{\alpha \sqrt{1 - \epsilon^2}}, \quad \cos \xi = \frac{r \cos \theta}{s(r, \theta)}, \quad \psi = \phi$$

(2)

with

$$s^2 = \frac{1}{2} \left( r^2 - \alpha^2 \epsilon^2 + \sqrt{(r^2 - \alpha^2 \epsilon^2)^2 + 4\alpha^2 \epsilon^2 r^2 \cos^2 \theta} \right)$$

(3)

Note however, that the formula for the maximum turnaround radius in [35] was obtained by equating the mass of the structure to the mass function due to the positive cosmological constant. Our approach on the other hand, utilises the equality of the attractive and repulsive forces, instead of the energy, due to them. Accordingly, our formula differs from that of [35] by a numerical factor.
We are interested in the behavior of the Newtonian potential at large distances \( r \). We expand \( s(r, \theta) \) in Eq. (3) for large \( r \) and use the first two of Eq. (2) and the definition of \( \sigma \), to obtain
\[
\sigma \simeq \frac{r}{\alpha e} \left( 1 - \frac{\alpha^2 e^2 \sin^2 \theta}{2 r^2} + \mathcal{O}(r^{-4}) \right), \quad \cos \xi \simeq \cos \theta + \mathcal{O}(r^{-2})
\]
and from these
\[
\cot^{-1} \sigma \simeq \frac{1}{\sigma} - \frac{1}{3 \sigma^3} + \mathcal{O}(\sigma^{-5}) \simeq \frac{\alpha \varepsilon}{r} + \frac{\alpha^3 e^3}{2 r^3} \left( \sin^2 \theta - \frac{2}{3} \right) + \mathcal{O}(r^{-5}) \quad \text{and} \quad P_2(\cos \xi) \simeq P_2(\cos \theta) + \mathcal{O}(r^{-2})
\]
Substituting, finally, the above into the Newtonian potential Eq. (1), we get
\[
V_N \simeq -\frac{M}{r} \left( 1 - \frac{\alpha^2 e^2}{5 r^2} P_2(\cos \theta) \right) + \mathcal{O}(r^{-5})
\]
Adding to \( 2V_N \) the repulsive potential \(-H_0^2 r^2\) representing the effect of the cosmological constant at large distances from the oblate ellipsoidal body, we obtain in the Newtonian approximation of General Relativity the effective gravitational potential at \((r, \theta, \phi)\)
\[
U_{\text{eff}} \simeq -\frac{2M}{r} \left( 1 - \frac{\alpha^2 e^2}{5 r^2} P_2(\cos \theta) \right) - H_0^2 r^2
\]
For the spherical body, \( \varepsilon = 0 \), we obtain the well known result \( R_{\text{TA,max}}^{(0)} = (M/H_0^3)^{1/3} \) [9].

The fractional deviation of the effective potential from the one for a spherical structure is of the order of \( \mathcal{O}(\alpha^2/r^2) \). Correspondingly, the change in the maximum turnaround radius, \( \delta R_{\text{TA,max}}(\theta) = R_{\text{TA,max}}^{(\epsilon)}(\theta) - R_{\text{TA,max}}^{(0)} \), for fixed \( \theta = \theta_0 \), as obtained by the condition \( U' = 0 \), is
\[
\frac{\delta R_{\text{TA,max}}(\theta_0)}{R_{\text{TA,max}}^{(0)}} \simeq -\frac{e^2}{5} \left( \frac{\alpha}{R_{\text{TA,max}}^{(0)}} \right)^2 P_2(\cos \theta_0)
\]
whose average over the solid angle \((\theta_0, \phi)\) is zero,
\[
\left\langle \frac{\delta R_{\text{TA,max}}(\theta_0)}{R_{\text{TA,max}}^{(0)}} \right\rangle \simeq 0.
\]
For \( \theta_0 = 0, \pi \) and \( \theta_0 = \pi/2 \), in particular, we obtain
\[
\frac{\delta R_{\text{TA,max}}(0, \pi)}{R_{\text{TA,max}}^{(0)}} \simeq -\frac{e^2}{5} \left( \frac{\alpha}{R_{\text{TA,max}}^{(0)}} \right)^2 \quad \text{and} \quad \frac{\delta R_{\text{TA,max}}(\pi/2)}{R_{\text{TA,max}}^{(0)}} \simeq \frac{e^2}{10} \left( \frac{\alpha}{R_{\text{TA,max}}^{(0)}} \right)^2
\]
respectively. The result is as expected on the basis of Newtonian gravity. The \( R_{\text{TA,max}} \) becomes smaller (larger) when the attraction to the origin diminishes (increases). In the case of a pancake-like axisymmetric structure the gravitational attraction on the symmetry axis, \( \theta_0 = 0 \) (the equatorial plane, \( \theta_0 = \pi/2 \)) is weaker (stronger) than it would be if all its mass were at its center. Note that even for \( \varepsilon \sim \mathcal{O}(1) \), away from spherical symmetry, these are very small for realistic structures. Also, on the basis of Newtonian gravity the reader can easily convince her/himself that in the case of a prolate (cigar-shaped) spheroidal structure the behaviour of \( \delta R_{\text{TA,max}} \) as a function of \( \theta_0 \) is opposite to the one in formulae Eq. (8) and Eq. (9). In particular, it is positive on the symmetry axis \( \theta_0 = 0, \pi \) and negative on the plane \( \theta_0 = \pi/2 \).

We shall extend the above analysis below to general relativistic scenarios, for both static axisymmetric and stationary axisymmetric spacetimes.
3 A static axisymmetric metric

In [40, 41] (see also [42] and references therein), a rotating axisymmetric metric (with $H_0 = 0$) was introduced to study the uniqueness properties of a stationary black hole,

\begin{align*}
g_{tt} &= -\left(\frac{\Delta_r - a^2 \sin^2 \theta}{\rho^2}\right) + \frac{10 \, \dot{\epsilon} P_2(\cos \theta)}{16 M^2 r^2} \left[2M \left(3r^3 - 9Mr^2 + 4M^2 r + 2M^3\right) - 3r^2(r - 2M)^2 \ln \left(\frac{r}{r - 2M}\right)\right], \\
g_{\phi\phi} &= -\frac{2Mar \sin^2 \theta}{\rho^2}, \\
g_{rr} &= \rho^2 \frac{\Delta_r}{\Delta} + \frac{10 \, \dot{\epsilon} P_2(\cos \theta)}{16 M^2 (r - 2M)^2} \left[2M \left(3r^3 - 9Mr^2 + 4M^2 r + 2M^3\right) - 3r^2(r - 2M)^2 \ln \left(\frac{r}{r - 2M}\right)\right],
\end{align*}

\begin{align*}
g_{\theta\theta} &= \rho^2 + \frac{10 \, \dot{\epsilon} r P_2(\cos \theta)}{16 M^2} \left[2M \left(3r^2 + 3Mr - 2M^2\right) - 3r \left(r^2 - 2M^2\right) \ln \left(\frac{r}{r - 2M}\right)\right], \\
g_{\phi\phi} &= \left[r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2} + \frac{10 \, \dot{\epsilon} r P_2(\cos \theta)}{16 M^2} \right] \left[2M \left(3r^2 + 3Mr - 2M^2\right) - 3r \left(r^2 - 2M^2\right) \ln \left(\frac{r}{r - 2M}\right)\right] \sin^2 \theta,
\end{align*}

where $\dot{\epsilon}$ is a dimensionless parameter and

\[\tilde{\Delta}_r = r^2 - 2Mr + a^2 \quad \text{and} \quad \rho^2 = r^2 + a^2 \cos^2 \theta.\]  \hspace{1cm} (11)

Setting $\dot{\epsilon} \to 0$ one recovers the Kerr spacetime written in the Boyer-Lindquist coordinates. The above metric usually possesses a naked curvature singularity rendering it unphysical globally. One plausible remedy of this seems to impose a cut off radius inside which the metric gets replaced with a suitable interior one.

We shall be concerned about the \textit{static} (i.e., non-rotating) limit of the above metric, $a = 0$, and use it to model the gravitational field of a large scale structure, which is essentially not an isolated stationary black hole. The parameter $\dot{\epsilon}$ thus represents some intrinsic non-sphericity in the shape of the structure, analogous to the eccentricity parameter in the preceding section. Permissible solutions for $a = 0$ requires $\dot{\epsilon} \geq -0.8$ [42]. Setting $\dot{\epsilon} = 0$ further reduces the metric to the Schwarzschild one.

We are interested in structures with $M \ll R_{\text{TA, max}} \ll H_0^{-1}$, i.e. much bigger than their Schwarzschild radius $2M$ and much smaller than the size of the observed Universe, so that they fit inside it. Accordingly, we shall work with the weak field regime of Eq. (10), with the leading modification due to a positive $\Lambda$.

Expanding Eq. (10) (with $a = 0$) up to $O(M^3/r^3)$, appropriate to study the weak gravity regime, we obtain

\begin{align*}
g_{tt} &\approx -\left(1 - \frac{2M}{r} + 2\dot{\epsilon} P_2(\cos \theta) \frac{M^3}{r^3}\right), \quad g_{rr} \approx \left(1 - \frac{2M}{r}\right)^{-1} - 2\dot{\epsilon} P_2(\cos \theta) \frac{M^3}{r^3} \\
g_{\theta\theta} &\approx r^2 \left(1 - 10\dot{\epsilon} P_2(\cos \theta) \frac{M}{r} + 16\dot{\epsilon} P_2(\cos \theta) \frac{M^3}{r^3}\right), \quad g_{\phi\phi} = g_{\theta\theta} \sin^2 \theta
\end{align*}

(12)

The leading modification of the above metric functions in the presence of a positive $\Lambda$ will be to make the replacement,

\[1 - \frac{2M}{r} \quad \longrightarrow \quad 1 - \frac{2M}{r} - H_0^2 r^2\]
Expanding now the dispersion relation for a test particle following a timelike geodesic, \( u_\alpha u^\alpha = -1 \), in the above background we obtain,

\[
\left( \frac{dr}{d\tau} \right)^2 = E^2 - \left( 1 - \frac{2M}{r} - H_0^2 r^2 + 2\tilde{\epsilon} P_2(\cos \theta) \frac{M^3}{r^4} \right) \left[ \frac{L^2}{g_{\phi\phi}} + g_{\theta\theta} \left( \frac{d\theta}{d\tau} \right)^2 + 1 \right]
\]

where \( \tau \) is the proper time along the trajectory. We have also defined, owing to the time translation and azimuthal symmetries of the spacetime, the conserved energy and the orbital angular momentum of the test particle,

\[
E = -g_{ab}(\partial_t)^a u^b = -g_{tt} \frac{dt}{d\tau}, \quad L = g_{ab}(\partial_\phi)^a u^b = g_{\phi\phi} \frac{d\phi}{d\tau}
\]

The maximum turnaround condition is obtained by setting \( d^2r/d\tau^2 = 0 \) in Eq. (13). Note that there can be analogous turnaround condition in the polar direction \( \theta \), as well. However, the surface of the compact axisymmetric structure we are looking into is spanned by \( \theta \) and \( \phi \). Hence any turnaround condition in the polar direction will not carry any information about the maximum size of the structure. Accordingly, for our current purpose we shall only be concerned about the turnaround condition along the radial direction.

The second term on the right hand side of Eq. (13) can be interpreted as the effective potential \((r, \theta \text{ and velocity dependent})\), the test particle experiences in the static and axisymmetric gravitational field of the structure. It is a positive definite quantity in our region of interest. Let us now imagine a test particle approaching the maximum turnaround point, where the effective potential has a maximum and the radial speed becomes, by definition, zero or vanishingly small. Sufficiently close to that point, the potential must be monotonically increasing. Now since the quantity appearing in the square bracket is greater than or equal to unity, it is clear that the maximum upper bound of all the turnaround radii, i.e. \( R_{\text{TA, max}} \), will simply correspond to \( L = 0 = d\theta/d\tau \) in Eq. (13). Equivalently, any motion along the angular directions will always create centrifugal force on the test particle at least at the leading order, which will reduce the turnaround radius.

Thus \( R_{\text{TA, max}} \) is found by setting the first radial derivative of the resulting effective potential to zero, keeping the angle \( \theta = \theta_0 \) as an input parameter. We find the leading correction \( \delta R_{\text{TA, max}} \) over the spherically symmetric case due to the non-sphericity parameter \( \tilde{\epsilon} \),

\[
\frac{\delta R_{\text{TA, max}}(\theta_0)}{R_{\text{TA, max}}^{(0)}} = -\tilde{\epsilon} P_2(\cos \theta_0) \left( \frac{M}{R_{\text{TA, max}}^{(0)}} \right)^2 \tag{14}
\]

whose average over all directions \((\theta_0, \phi)\) vanishes. Note in particular that if we take \( \tilde{\epsilon} \) to be positive, the above result is in perfect qualitative agreement with that of the previous section, thereby describing a pancake shaped structure. On the other hand, if we take \( \tilde{\epsilon} < 0 \), we obtain instead an ellipsoidal structure. On the other hand, note also that Eq. (9) contains the actual length scale of the structure, \( \alpha \), whereas the above formula contains the Schwarzschild radius. Accordingly, we expect Eq. (9) would be large compared to Eq. (14), for a given structure.

4 The case of the Kerr-de Sitter spacetime

We shall next study the example of the Kerr-de Sitter spacetime, which is stationary and axisymmetric. It does not represent a structure which is inherently non-spherical — but the departure from the spherical
symmetry is obtained via the rotation of the structure. Accordingly, we expect to find qualitatively different features in $R_{TA, max}$, compared to what we have seen so far.

The Kerr-de Sitter spacetime represents an axisymmetric structure, spinning along an azimuthal direction with respect to a given axis, embedded in the de Sitter universe. The spacetime is axisymmetric and stationary. It asymptotically approaches the de Sitter spacetime at large ‘distances’ from the structure. Since the de Sitter spacetime is spherically symmetric and also static inside the cosmological event horizon, it is natural to choose a coordinate system of the Kerr-de Sitter spacetime such that (a) it is manifestly stationary and axisymmetric and (b) it asymptotically coincides with that of the de Sitter. Such description is achieved by the so called Boyer-Lindquist coordinates [43],

$$ds^2 = -\frac{\Delta_r - a^2 \sin^2 \theta \Delta_\theta}{\rho^2} dt^2 - \frac{2a \sin^2 \theta}{\rho^2 \Xi} ((r^2 + a^2) \Delta_\theta - \Delta_r) dtd\phi$$

$$+ \frac{\sin^2 \theta}{\rho^2 \Xi^2} ((r^2 + a^2)^2 \Delta_\theta - \Delta_r a^2 \sin^2 \theta) d\phi^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2$$

(15)

where,

$$\Delta_r = (r^2 + a^2) (1 - H_0^2 r^2) - 2M r, \quad \Delta_\theta = 1 + H_0^2 a^2 \cos^2 \theta, \quad \Xi = 1 + H_0^2 a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, (16)$$

$M$ is the mass parameter of the structure and $a = J/M$ is its angular momentum per unit mass. For $a = 0$, in particular, we recover the Schwarzschild-de Sitter spacetime, whereas setting in addition $M = 0$ it reduces to the de Sitter spacetime written in the static patch.

The metric Eq. (15) is $t-$ and $\phi-$ independent. The corresponding conserved energy and angular momentum of the test particle with velocity $u^a = dx^a/d\tau$, where $\tau$ is the proper time along the trajectory, are $E = -g_{\tau u}^\tau$ and $L = g_{\phi u}^\phi$. These, along with the expansion of the dispersion relation ($u_\alpha u^\alpha = -1$) for a timelike geodesic in the background of Eq. (15) gives [43],

$$\frac{dt}{d\tau} = \frac{a \Xi}{\Delta_r \Delta_\theta \rho^2} \left[ \Delta_r \left( L - \frac{E a \sin^2 \theta}{\Xi} \right) - \Delta_\theta (r^2 + a^2) \left( L - \frac{E (r^2 + a^2)}{a \Xi} \right) \right]$$

$$\frac{d\phi}{d\tau} = \frac{\Xi^2}{\Delta_r \Delta_\theta \rho^2 \sin^2 \theta} \left[ \Delta_r \left( L - \frac{E a \sin^2 \theta}{\Xi} \right) - a^2 \sin^2 \theta \Delta_\theta \left( L - \frac{E (r^2 + a^2)}{a \Xi} \right) \right]$$

$$\rho^2 \left( \frac{dr}{d\tau} \right)^2 = a^2 \Xi^2 \left( L - \frac{E (r^2 + a^2)}{a \Xi} \right)^2 - \Delta_r \left( K_C + r^2 \right),$$

$$\rho^2 \left( \frac{d\theta}{d\tau} \right)^2 = -\frac{\Xi^2}{\sin^2 \theta} \left( L - \frac{E a \sin^2 \theta}{\Xi} \right)^2 + \Delta_\theta \left( K_C - a^2 \cos^2 \theta \right) \equiv \lambda(\theta)$$

(17)

where $K_C$ is Carter’s constant of variable separation and $\lambda(\theta)$ is an abbreviation of the right hand side of Eq. (17). The general expression for the maximum turnaround radius or zero acceleration condition in the radial direction can be found from the third and fourth of Eq. (17), by setting as earlier, $d^2 r/d\tau^2 = 0$, and treating $E$, $L$, $M$, $a$, $K_C$ and $\theta$ as numerical inputs. Note from Eq. (17) that there will be terms linear in $L$, showing that unlike the static and spherically symmetric case ($a = 0$), the direction of rotation of orbits will be distinguished here. In particular, using $K_C \geq 0$ e.g. [44], we can show that for the retrograde ($L < 0$) orbits, the turnaround radius will be higher than that of the prograde ($L > 0$) ones.

Note also that just like the case of the static axisymmetric spacetime discussed in the previous section, we shall not consider any turnaround condition along the $\theta$ direction.
As earlier, we shall focus on cosmic structures satisfying the conditions $M \ll R_{\text{TA, max}} \ll H_0^{-1}$. The constraint $a \lesssim O(M)$, known in the case of an isolated black hole, does not apply here. Any potential naked singularity is hidden inside the body of the structure, where the above metric is not valid. However, the analysis of simulations [37] shows that the condition $a \lesssim O(M)$ is comfortably satisfied by the large scale structures studied here.\footnote{Incidentally, we refer our reader also to [46] and references therein, for the so called super-spinning Kerr solution, where $a$ can exceed $M$, but at the expense of introducing additional terms in the metric.}

Now, keeping in mind that we shall work essentially in a weak gravity regime, it will be more convenient for us, instead of using Eq. (17), to use a simple alternative derivation of the $R_{\text{TA, max}}$ for Eq. (15) below, without introducing Carter’s constant. We introduce the timelike vector field $\chi^a$,

$$\chi^a = (\partial_t)^a - \frac{(\partial_t \cdot \partial_\phi) (\partial_\phi)}{(\partial_\phi \cdot \partial_\phi)} (\partial_\phi)^a = (\partial_t)^a - \frac{g_{t\phi}}{g_{\phi\phi}} (\partial_\phi)^a$$

It satisfies $\chi \cdot \partial_\phi = 0$, while the square of its norm is

$$\chi^a \chi_a = \frac{g_{tt}g_{\phi\phi} - g_{t\phi}^2}{g_{\phi\phi}} = -\frac{\rho^2 \Delta_r \Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - \Delta_r a^2 \sin^2 \theta}$$

which, with $\Delta_r > 0$, is easily seen to be negative. In other words, $\chi^a$ is a timelike vector field. It is convenient to choose the orthogonal basis for Eq. (15) : $\{\chi^a, (\partial_\phi)^a, (\partial_\theta)^a, (\partial_r)^a\}$. Expanding the dispersion relation, $\mathbf{u} \cdot \mathbf{u} = -1$ in this orthogonal basis we obtain

$$\left( \frac{dr}{d\tau} \right)^2 = \frac{(E - A(r, \theta)L)^2}{\rho^4} \left( (r^2 + a^2)^2 \Delta_\theta - \Delta_r a^2 \sin^2 \theta \right) - \frac{L^2 \Xi^2 \Delta_r}{\sin^2 \theta \left( (r^2 + a^2)^2 \Delta_\theta - \Delta_r a^2 \sin^2 \theta \right)} - \frac{\Delta_r}{\rho^2} \frac{\lambda(\theta) \Delta_r}{\rho^4} \tag{18}$$

where $A(r, \theta)$ is defined by

$$A(r, \theta) = \frac{a \Xi (2Mr + H_0^2 R)}{(r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta}$$

and the positive semi-definite $\lambda(\theta)$ is the function appearing on the right hand side of the last of Eq. (17). It is easy to see then that the leading radial force originating from this term is repulsive, indicating decrease in the maximum turnaround radius. Thus, in order to find $R_{\text{TA, max}}$ we may ignore the kinetic energy of the test particle along the polar angle, compared to that of along the radial direction. We, thus, set $\theta = \theta_0 = \text{constant}$ in Eq. (18) and obtain

$$\left( \frac{dr}{d\tau} \right)^2 = \frac{(E - A(r, \theta_0)L)^2}{\rho_0^4} \left( (r^2 + a^2)^2 \Delta_\theta_0 - \Delta_r a^2 \sin^2 \theta_0 \right) - \frac{L^2 \Xi^2 \Delta_r}{\sin^2 \theta_0 \left( (r^2 + a^2)^2 \Delta_\theta_0 - \Delta_r a^2 \sin^2 \theta_0 \right)} - \frac{\Delta_r}{\rho_0^2} \tag{19}$$

where the subscript 0 indicates that $\theta$ is replaced with $\theta_0$. Notice, although not surprisingly, the above equation indicates that a test particle with $L \neq 0$ cannot be sitting at the poles, $\theta = 0, \pi$. 
Expanding next the right hand side of Eq. (19) up to the third order of the metric functions we obtain

\[
\left( \frac{dr}{d\tau} \right)^2 \approx E^2 \left[ 1 + \frac{a^2 \sin^2 \theta_0}{r^2} \left( 1 + \frac{2M}{r} + H_0^2 r^2 \right) \right] - 2ELa \left( \frac{2M}{r^3} + H_0^2 \right) - \frac{L^2}{r^2 \sin^2 \theta_0} \left[ 1 - H_0^2 r^2 - \frac{2M}{r} - \frac{a^2 \cos^2 \theta_0}{r^2} - \frac{2Ma^2 \sin^2 \theta_0}{r^3} - 2H_0^2 a^2 \right] - \frac{1}{r} - \frac{2M}{r} - H_0^2 r^2 + \frac{a^2 \sin^2 \theta_0}{r^2} + \frac{2Ma^2 \cos^2 \theta_0}{r^3} - H_0^2 a^2 \sin^2 \theta_0 \right] \quad (20)
\]

As a consistency check, we note that as \( a \to 0 \), the largest root of \( d^2 r/d\tau^2 = 0 \) corresponds to \( L = 0 \), recovering the result of the static spherically symmetric case, \( R_{T\Lambda,\text{max}}^{(0)} = (M/H_0^6)^{1/3} \) [9].

The distinction between the \( L > 0 \) and \( L < 0 \) trajectories is now apparent. In particular for \( L < 0 \), the term proportional to \( LaE \) on the right hand side generates an attractive potential. Note also that the leading of the terms containing \( L^2 \) creates repulsion, indicating decrease in the size of the maximum turnaround radius. We would thus like to investigate the effect of the interplay between these two terms on \( R_{T\Lambda,\text{max}} \). However, as we argue next these \( L \)-dependent terms are subleading and can be ignored.

As long as \( E \sim O(1) \), which is the case of interest to us, Eq. (20) implies that \( L \) may have significant contribution to \( R_{T\Lambda,\text{max}} \) only if (a) it is negative generating an attractive force, and (b) it is on the order of \( L \sim O(R_{T\Lambda,\text{max}}) \), for only then the term linear in \( L \), can be expected to become comparable to the other terms. Note also that according to Eq. (20) the attractive terms proportional to \( L^2 \) are subleading compared to the dominant repulsive one.

However, for such high values of \( L \), we have in the turnaround region,

\[
\frac{L^2}{r^2 \sin^2 \theta_0} \gtrsim O(1)
\]

whereas, recalling that \( M \ll R_{T\Lambda,\text{max}} \ll H_0^{-1} \) we have

\[
LaE \left( \frac{2M}{r^3} + H_0^2 \right) \ll 1
\]

Thus, the repulsive term of the orbital angular momentum still dominates over the attractive term linear in \( L \). Accordingly, we conclude that even though a negative \( L \) generates an attractive force in the Kerr-de Sitter geometry, the \( R_{T\Lambda,\text{max}} \) would still correspond to \( L = 0 \). Thus, the leading shift \( \delta R_{T\Lambda,\text{max}}(\theta_0) \) of \( R_{T\Lambda,\text{max}}^{(0)} \) is given by

\[
\frac{\delta R_{T\Lambda,\text{max}}(\theta_0)}{R_{T\Lambda,\text{max}}^{(0)}} \approx \frac{a^2 (E^2 \sin^2 \theta_0 - \cos^2 \theta_0)}{(R_{T\Lambda,\text{max}}^{(0)})^2} + \frac{a^2 \sin^2 \theta_0 (E^2 - 1)}{3MR_{T\Lambda,\text{max}}^{(0)}}
\]

\[
\approx - \left( \frac{a}{R_{T\Lambda,\text{max}}^{(0)}} \right)^2 \left[ (1 - 3H_0^2 R_{T\Lambda,\text{max}}^{(0)} \cos^2 \theta_0 + 3H_0^2 R_{T\Lambda,\text{max}}^{(0)}) \right] \quad (21)
\]

where in the last step we used for the energy \( E \) its value for a particle at rest in the maximum turnaround region

\[
E = -gt \frac{dt}{dr} = \sqrt{-gt} \simeq 1 - \frac{3}{2}H_0^2 \left( R_{T\Lambda,\text{max}}^{(0)} \right)^2
\]
which is consistent with our assumption $E \simeq 1$. The average of Eq. (21) over the solid angle $(\theta_0, \phi)$ is
\[
\left\langle \delta R_{TA,\text{max}}(\theta_0)/R_{TA,\text{max}}^{(0)} \right\rangle \simeq -a^2/(3R_{TA,\text{max}}^{(0)2}).
\]

A very rough estimate of this quantity for the case of a “rotating galaxy cluster” can be obtained if we assume that the fractional change of $R_{TA,\text{max}}$ is small. Then, with $I, R$ and $M$ the moment of inertia, the size and the mass, respectively, of the cluster, we obtain $a = J/M \sim I\omega/M \sim R^2\omega \lesssim R v_{\text{out}}$, with $v_{\text{out}}$ the azimuthal speed of the outmost galaxies in the structure. Thus, we have roughly
\[
\left\langle \delta R_{TA,\text{max}}(\theta_0)/R_{TA,\text{max}}^{(0)} \right\rangle \sim O(v_{\text{out}}^2),
\]
which is much smaller than unity in realistic structures.

Note in particular that Eq. (21) shows decrease in the value of $R_{TA,\text{max}}$ compared to the case of a spherically symmetric structure on the axis ($\theta_0 = 0, \pi$) as well as on the equatorial plane ($\theta_0 = \pi/2$). This is in contrast to the cases discussed in Section 2 and Section 3, making the Kerr-de Sitter qualitatively different. Such decrease could be understood as the repulsive effect originating from the spacetime rotation. Note also that the decrease is minimum at $\theta = \pi/2$ and it monotonically increases to its maximum value at $\theta = 0, \pi$. This could be understood as the flattening of the structure on the equatorial plane due to its rotation.

5 Conclusions

We computed the leading order effect of eccentricity on the maximum size $R_{TA,\text{max}}$ of realistic bound large cosmic structures, whose departure from spherical symmetry is due to either intrinsic non-sphericity or rotation. To that purpose, we studied timelike geodesics in static and stationary axisymmetric backgrounds. In all cases the effect for realistic structures is predicted to be small, for practically any value of the eccentricity parameter, being proportional to the square of the ratio of its characteristic “size” or mass over its zeroth order spherically symmetric turnaround radius $R_{TA,\text{max}}^{(0)}$. Furthermore, we showed that in the Kerr-de Sitter case, analyzed in Section 4, the change $\delta R_{TA,\text{max}}(\theta)$ in the turnaround radius is negative for all values of the polar angle. This is in contrast to the examples studied in Section 2 and Section 3, concerning structures with intrinsic non-sphericity in their shapes, in which $\delta R_{TA,\text{max}}(\theta)$ has positive and negative values, while it vanishes when averaged over the angles.

It is interesting to see to what extent the above conclusions will be confirmed, when confronted by the actual numerical simulation data [37].

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