Extremal Kähler metrics on projectivised vector bundles

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Abstract

We prove the existence of extremal, non-csc, Kähler metrics on certain unstable projectivised vector bundles $\mathbb{P}(E) \to M$ over a cscK-manifold $M$ with discrete holomorphic automorphism group, in certain adiabatic Kähler classes. In particular, the vector bundles $E \to M$ under consideration are assumed to split as a direct sum of stable subbundles $E = E_1 \oplus \cdots \oplus E_s$ all having different Mumford-Takemoto-slope, e.g. $\mu(E_1) > \cdots > \mu(E_s)$.

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1 Introduction

In this first section we shall give an overview of the problem we are considering, including an overview of related previous work, and introduce some notation.

1.1 Previous work

Constant scalar curvature Kähler metrics (cscK in the sequel) on projectivised vector bundles in so-called adiabatic Kähler classes were first constructed by Y.-J. Hong. In his first paper [Ho1], Hong considered the case of a cscK base-manifold $(M, J_M, g_M, \omega_M)$ with discrete holomorphic automorphism group; and a Mumford-Takemoto-slope-stable (with respect to $\omega_M$) Hermitian holomorphic vector bundle $E \to M$ endowed with a Hermitian-Einstein-connection—i.e. the Chern connection corresponding to a Hermitian-Einstein-metric—over it. We denote by $L^* \to \mathbb{P}(E)$ the fibrewise hyperplane bundle $O_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$. The Hermitian-Einstein-connection $\nabla$ on $E$ induces a Hermitian connection $\nabla_{L^*}$ on the line bundle $L^*$; and we denote its curvature form by $F_{\nabla_{L^*}}$. Hong then used an adiabatic limit technique to construct a cscK-metric on $\pi: \mathbb{P}(E) \to M$ in the Kähler class $[\omega_k] = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + k\pi^* [\omega_M]$, for sufficiently large $k$. One of the crucial points of Hong’s technique is, that the Kähler metric

$$\omega_k = \left(\frac{i}{2\pi} F_{\nabla_{L^*}}\right) + k\pi^* \omega_M$$

gives an asymptotic approximation to a cscK-metric on $\mathbb{P}(E)$. It is because of this property, that Hong can proceed by finding a formal power series solution to the cscK-equation on $\mathbb{P}(E)$, which is $O(k^{-s})$-close (in a suitable norm) to a genuine solution, for an integer $s > 0$ arbitrarily large. Obtaining suitable estimates for the scalar curvature map acting on Kähler potentials on $\mathbb{P}(E)$ and applying standard elliptic-PDE-theory, Hong is able to deduce the existence of a genuine cscK-metric on $\mathbb{P}(E)$ for $k \gg 0$ by using an implicit function theorem argument.

Hong’s analysis relies essentially on the bundle $E$ being slope-stable and therefore also simple (i.e. it only has endomorphisms of the form $\lambda \cdot \text{Id}_E$, with $\lambda \in \mathbb{C}^*$ and $\text{Id}_E$ the identity endomorphism). The simplicity of the vector bundle $E$ is reflected in the linearisation of the scalar curvature map on Kähler potentials on $\mathbb{P}(E)$ having trivial co-kernel.

In a second paper on this topic [Ho2], Hong considered the situation of a polystable, non-simple Hermitian holomorphic vectorbundle $E = E_1 \oplus \cdots \oplus E_s$ being projectivised over a cscK base manifold $M$ with a non-trivial Lie algebra $\text{ham}(M, J_M, \omega_M)$ of Hamiltonian Killing vector
fields. The main difference of this situation to the above one is, that the lifting of the action of $\mathfrak{ham}(M,J_M,\omega_M)$ will induce non-trivial Hamiltonian Killing vector fields on $\mathbb{P}(E)$. Moreover, since $E$ is not simple anymore, the Lie-algebra $\mathfrak{g}_E$ of the projectivisation of the automorphism group of $E$ will induce a non-trivial action as well. Hong assumes in [Ho2], that the Futaki invariant with respect to the Kähler class $[\omega_k]$ and $\mathfrak{g}_E + \text{(the lift of)} \mathfrak{ham}(M,J_M,\omega_M)$ on $\mathbb{P}(E)$ is zero. This assumption enables him to solve the cscK-equation on $\mathbb{P}(E)$, without having to deal with any obstruction coming from a non-trivial co-kernel of its linearisation.

Another situation similar to the above ones was considered by J. Fine [F]. He treated the problem of finding a cscK-metric in adiabatic Kähler classes on the total space of a Kodaira fibration $X \to \Sigma$. Here the base is a complex curve of high genus, and the fibres have genus at least two. The fibres and the base admit no non-trivial holomorphic vector fields. From this one can conclude, using the projection formula in cohomology, that the total space $X$ admits no non-trivial holomorphic vector fields either. Therefore, the cscK equation on $X \to \Sigma$ is solvable without any further obstructions (the co-kernel of its linearisation consists of constant functions).

The main difference in Fine’s work is, that the fibres of the Kodaira fibration have non-trivial moduli, which leads to other difficulties in his case.

**Remark 1.** The theorem, that a Hermitian holomorphic vector bundle over a compact Kähler manifold admits a Hermitian-Einstein-metric (and thus a corresponding Hermitian-Einstein-connection) if and only if it is polystable was proven by Narasimhan-Seshadri, Donaldson and Uhlenbeck-Yau (see [NS, UY, D1]). Usually, this result is referred to as the Hitchin-Kobayashi correspondence.

### 1.2 Introduction to the main problem

The situation we are considering differs from the above ones by the fact that we will be searching for an extremal, non-cscK-metric on a projectivised Hermitian holomorphic vector bundle

$$\pi : (\mathbb{P}(E), \omega_k) \to (M, \omega_M)$$

in the Kähler class $[\omega_k] = 2\pi c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + k\pi^* [\omega_M]$ for $k \gg 0$, where $\mathcal{O}_{\mathbb{P}(E)}(1)$ is again the fibrewise hyperplane bundle over $\mathbb{P}(E)$. The crucial difference is, that our vector bundle $E$ will be slope-unstable. However, we will assume a certain special structure and look at a bundle $E$ which splits as a direct sum of slope-stable subbundles (again, slope-stable with respect to $[\omega_M]$)

$$E = E_1 \oplus \cdots \oplus E_s,$$

all having different slopes.

**Remark 2.** For convenience, we shall assume from now on that the slopes $\mu(E_i)$ satisfy

$$\mu(E_1) > \cdots > \mu(E_s).$$
Since the bundles $E_i \to (M, \omega_M)$ are all stable, we can endow each of them with a HE-connection $\nabla_i$, i.e. the Chern-connection corresponding to a Hermitian-Einstein-metric, satisfying
\[
i \Lambda_{\omega_M} F^{\nabla_i} = \lambda_i I d_{E_i}, \quad \lambda_i = \text{const.} \in \mathbb{R}.
\]
The direct sum of these connections will give us a (Chern) connection $\nabla = \bigoplus \nabla_i$ on $E$. As above, this induces a (Chern) connection $\nabla^{L^*}$ on $L^* = \mathcal{O}_{\mathbb{P}(E)}(1)$, the curvature form of which we denote again by $F^{\nabla^{L^*}}$. Similar to Hong, we will start with the Kähler metric
\[
\omega_k = i F^{\nabla^{L^*}} + k \pi^* \omega_M
\]
and see that it gives us an asymptotic approximation—in a sense to be made precise later—to an extremal, non-csc Kähler metric on $\mathbb{P}(E)$. Our main result is.

**Theorem 3.** Given a cscK manifold $(M, \omega_M)$ with no non-trivial holomorphic automorphisms and a Hermitian holomorphic vector bundle $E \to M$ splitting as a direct sum of stable sub-bundles $E = E_1 \oplus \cdots \oplus E_s$, each of them endowed with a Hermitian-Einstein-connection $\nabla_i$ and all of them having different Mumford-Takemoto-slope; then for $k \gg 0$ the projectivised vector bundle $\mathbb{P}(E) \to (M, \omega_M)$ has an extremal, non-csc Kähler-metric in the Kähler class $[\omega_k] = 2 \pi c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + k \pi^* [\omega_M]$.

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## 2 Preliminaries and background material

We shall collect here some background material which we will need in the sequel.

### 2.1 Background on extremal Kähler metrics

The notion of an extremal Kähler metric on a (compact) Kähler manifold $(M, J, g, \omega)$ was first introduced by Calabi in [C1]. They are defined to be the critical points of the so-called Calabi functional
\[
C(\omega) = \int_M (\text{Scal}(\omega) - \overline{S}) g^{\omega_n} \frac{\omega^n}{n!},
\]
in some Kähler-class $[\omega]$, where $\text{Scal}(\omega)$ denotes the scalar curvature of the metric $g$ corresponding to $\omega$, and $\overline{S}$ its average. Of course, cscK-metrics are automatically extremal Kähler metrics. The converse is not always true, the first examples of extremal, non-cscK-metrics were constructed by Calabi on Hirzebruch surfaces in [C1 Section 3].

**Remark 4.** In the sequel, we will often use the Kähler metric $g$ on $(M, J, g, \omega)$ and its associated Kähler form $\omega(\cdot, \cdot) = g(J \cdot, \cdot)$ interchangeably.
**Definition 5** (Extremal Kähler metric). A Kähler metric \( \omega \in [\omega] \) on a compact complex manifold \( (M, J) \) is called extremal (non-cscK) if it is a non-minimal critical point of the Calabi-functional \( I \).

**Definition 6** (Reduced Automorphism Group). For a Kähler manifold \( (M, J, g, \omega) \), we define the (identity component of the) reduced automorphism group \( Aut^0_{\text{red}}(M, J) \) to be the subgroup of \( Aut^0(M, J) \), i.e. the identity component of the holomorphic automorphism group of \( (M, J) \), generated by (real) holomorphic vector fields with non-trivial zero-set on \( (M, J) \).

One can show that \( Aut^0_{\text{red}}(M, J) \) is the unique linear algebraic subgroup of \( Aut^0(M, J) \) such that the quotient \( Aut^0(M, J)/Aut^0_{\text{red}}(M, J) \) is the Albanese torus of \( (M, J) \).

Suppose we are given a Kähler manifold \( (M, J, g, \omega) \). We shall now choose a connected maximal compact subgroup \( G_{\text{max}} \) of the reduced automorphism group \( Aut^0_{\text{red}}(M, J) \).

Then, for any \( G_{\text{max}} \)-invariant Kähler metric \( \omega \in [\omega]^{G_{\text{max}}} \), where \([\omega]^{G_{\text{max}}} \) denotes the set of \( G_{\text{max}} \)-invariant Kähler metrics (forms) in \([\omega]\)—the Lie-algebra \( g_{\text{max}} \) of \( G_{\text{max}} \) is the space of Hamiltonian Killing vector fields (cf. [FM, Introduction and Section 1]). The key point is that the Hamiltonian Killing vector fields in \( g_{\text{max}} \) remain Hamiltonian Killing vector fields as we vary \( \omega \) in \([\omega]^{G_{\text{max}}} \).

**Definition 7** (Extremal vector field). For all \( V \in g_{\text{max}} \) we define the extremal vector field \( X_{[\omega]}^{G_{\text{max}}} \in g_{\text{max}} \), as the vector field satisfying

\[
\mathfrak{F}(V, [\omega]) = \langle X_{[\omega]}^{G_{\text{max}}}, V \rangle;
\]

where \( \mathfrak{F}(V, [\omega]) \) denotes the Futaki-invariant, and \( \langle \cdot, \cdot \rangle \) is the Futaki-Mabuchi inner product restricted to \( g_{\text{max}} \). This inner product is positive definite on \( g_{\text{max}} \) (cf. [FM Theorems A and C]), which is why by duality we can define \( X_{[\omega]}^{G_{\text{max}}} \) as above. The extremal vector field \( X_{[\omega]}^{G_{\text{max}}} \) depends only on the Kähler-class and the choice of \( G_{\text{max}} \), in particular it is independent of the choice of a \( G_{\text{max}} \)-invariant Kähler metric \( \omega \in [\omega]^{G_{\text{max}}} \) (cf. [FM Corollary D]).

**Remark 8.** It was also shown by Futaki-Mabuchi that \( X_{[\omega]}^{G_{\text{max}}} \) lies in the centre of \( g_{\text{max}} \) and generates a torus action (cf. [FM, Theorem F]).

Calabi computed the Euler-Lagrange equation to his functional \( I \) on a compact Kähler manifold \( (M, J, g, \omega) \) in [C1], it is given by (again, \( g \) is the metric corresponding to \( \omega \))

\[
\mathcal{L}_{\nabla_p \text{Scal}(g)} J = 0,
\]

i.e. \( \nabla_p \text{Scal}(g) \) is the real part of a holomorphic section of \( T^{1,0}M \) (where \( \mathcal{L} \) denotes the Lie-derivative). Restricting to Kähler metrics invariant under a chosen maximal connected compact subgroup of the reduced automorphism group, one can reduce the order of this equation as follows: According to [C1], a Kähler metric \( \omega \in [\omega]^{G_{\text{max}}} \) is extremal, if

\[
\text{Scal}(\omega) - H(\omega) - \mathcal{F} = 0,
\]

\[1\]For the definition of the Futaki-Mabuchi inner product, see [FM].
where $H(\omega)$ is a (mean-value zero) Hamiltonian for a Hamiltonian Killing vector field (in the Lie-algebra $\mathfrak{g}_{\text{max}}$) with respect to $\omega \in [\omega]^{\mathfrak{g}_{\text{max}}}$. In fact, if equation (4) is satisfied for a Kähler metric $\omega \in [\omega]^{\mathfrak{g}_{\text{max}}}$, then $H(\omega)$ is the (mean-value zero) Hamiltonian of the extremal vector field $X_{[\omega]}^{\mathfrak{g}_{\text{max}}}$ defined as in Definition 7. This follows from the definition of the Futaki-invariant and the Futaki-Mabuchi inner product, and the calculation

$$\int_M H(\omega) H_V \omega^n = \int_M (\text{Scal}(\omega) - 3) H_V \omega^n = \delta(V, [\omega]) = \langle X_{[\omega]}^{\mathfrak{g}_{\text{max}}}, V \rangle,$$

where $H_V$ denotes the (mean-value zero) Hamiltonian of any Hamiltonian Killing vector field $V \in \mathfrak{g}_{\text{max}}$.

2.2 Preparatory material

Suppose we are given a rank $r := \text{rk}(E)$ complex holomorphic vector bundle $(E, h, \nabla) \to (M, \omega_M)$, with Hermitian metric $h$ and Chern connection $\nabla$, over a (complex) $n$-dimensional Kähler manifold $M$. The Chern connection $\nabla$ defines a splitting of the tangent bundle of $M$.

In the sequel, we consider the natural action of $\text{End}(E)$ of $\mathbb{C}$-endomorphisms of $E$, denote by $\hat{A}$ the vertical vector field defined as follows. Recall that for any $x \in \mathbb{P}(E)$ with projection $\pi(x) = y$, we can identify the vertical (real) tangent space $T^v_x \mathbb{P}(E)$ at $x$ naturally with the space $\mathcal{H}^*$. Taking the tensor product with the pull-back map on $\pi$-forms $\mu^* : \Omega^p_M(\mathfrak{su}(E)) \to \Omega^p_{\mathbb{P}(E)}$, and by complex linearity to $\text{End}(E)$-valued (complex) $p$-forms. Using this notation, we get the precise relationship between $F^\nabla$ and $F^{\nabla_v}$. (The following result and its proof can be found in [FP].)

**Proposition 9 (cf. Proposition 2.1 in [FP]).** With respect to the vertical-horizontal decomposition of two-forms on $\mathbb{P}(E)$: $\Lambda^2 T \mathbb{P}(E)^* \cong \Lambda^2 \mathcal{Y}^* \oplus (\mathcal{Y}^* \otimes \mathcal{H}^*) \oplus \Lambda^2 \mathcal{H}^*$, we get

$$iF^{\nabla_v} = \omega_{FS} \oplus 0 \oplus \mu^*(F^\nabla),$$

where $\omega_{FS}$ restricts to the Fubini-Study metric on the fibres. Moreover, $iF^{\nabla_v}$ is a symplectic form if and only if $\mu^*(F^\nabla)^n$ is nowhere zero.

In the sequel, we consider the natural action of $\text{End}(E)$ on $\mathbb{P}(E)$, and shall now describe the associated infinitesimal action.

For any section $A$ of the vector bundle $\text{End}(E)$ of $\mathbb{C}$-endomorphisms of $E$, denote by $\hat{A}$ the vertical vector field defined as follows. Recall that for any $x \in \mathbb{P}(E)$ with projection $\pi(x) = y$, we can identify the vertical (real) tangent space $T^v_x \mathbb{P}(E)$ at $x$ naturally with the space $\mathcal{H}^*$. Taking the tensor product with the pull-back map on $\pi$-forms $\mu^* : \Omega^p_M(\mathfrak{su}(E)) \to \Omega^p_{\mathbb{P}(E)}$, and by complex linearity to $\text{End}(E)$-valued (complex) $p$-forms. Using this notation, we get the precise relationship between $F^\nabla$ and $F^{\nabla_v}$. (The following result and its proof can be found in [FP].)

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where $\omega_{FS}$ restricts to the Fubini-Study metric on the fibres. Moreover, $iF^{\nabla_v}$ is a symplectic form if and only if $\mu^*(F^\nabla)^n$ is nowhere zero.
\[ \text{Hom}(x, E_y/x) \cong \text{Hom}(x, x^\perp) \] of $C$-linear homomorphisms from the complex line $x$ to the orthogonal subspace $x^\perp$ to $x$ in $E_y$, where we identified $x^\perp \cong E_y/x$ (with the space on the right hand side having a holomorphic structure).

**Remark 10.** Since we identified $x^\perp \cong E_y/x$, we in fact defined a holomorphic structure on $\text{Hom}(x, x^\perp)$ ($\cong \text{Hom}(x, E_y/x)$).

**Definition 11** (Infinitesimal action induced on $P(E)$ by $\text{End}(E)$). We define the vertical vector field $\hat{A}(x): v \mapsto Av - N_A(x)v$, for any $v$ in $x \subset E_y$, by setting

\[ N_A(x) = \frac{(Au, u)_h}{|u|^2_h}; \] (6)

where $u$ stands for any generator of $x$ in $E_y$ and $(\cdot, \cdot)_h$ denotes the Hermitian inner product with respect to the bundle metric $h$.

**Remark 12.** If $A$ is a constant multiple of the identity, then $\hat{A}$ is indeed zero, as it should be, and $N_A$ is constant on each fibre.

If $A$ is skew-hermitian, the restriction of the vertical vector field $\hat{A}$ to a fibre $\mathbb{P}(E_y)$ is a Hamiltonian Killing and real holomorphic vector field with respect to the Fubini-Study metric on $\mathbb{P}(E_y)$, induced by the Hermitian metric $h$ on $E$. The fibrewise Hamiltonian of this vector field with respect to this (Fubini-Study) metric is just $-iN_A$.

**Remark 13.** Proposition 9 is also true for the more general situation of the fibre being a general co-adjoint orbit $G/H$ (see Proposition 2.1 and Remark 2.3 in [FP]).

### 2.3 Future extensions

It would be interesting to extend the results stated in Section 1 to more general Kähler fibrations. Indeed, the adiabatic limit technique used in the proof of our existence theorem is not limited to projectivised bundles, and could be applied to more general fibrations.

Suppose we are given a principal $G$-bundle $\pi : P \to (M, \omega_M)$, with connection $\nabla$, over a cscK manifold $(M, \omega_M)$ without holomorphic automorphisms. We suppose the fibres of the associated bundle $X \to M$ to be of the form $(G/H, \omega_{G/H})$, while the Kähler metric $\omega_{G/H}$ is supposed to be Kähler-Einstein. Moreover, we assume the existence of a moment map $\mu : G/H \to g^*$, embedding $G/H$ as an integral co-adjoint orbit. (For a detailed discussion of the theory of (co-)adjoint orbits and existence of Kähler-Einstein metrics on them, see [Bes].)

In addition to the existence of the moment map $\mu : G/H \to g^*$, we stipulate that the symplectic form $\omega_{G/H}$ is the curvature form of a (Chern) connection on a Hermitian holomorphic line bundle $L \to G/H$, such that the action of $G$ on $G/H$ lifts to a unitary action on $L$ preserving the connection. Let $\mathcal{L} = P \times_G (G/H, L) \to X$ be the Hermitian holomorphic line bundle, whose fibrewise restriction is $L \to G/H$. The connection $\nabla$ enables us to combine the fibrewise connections in $\mathcal{L}$ to give a (Chern) connection $\nabla^{\mathcal{L}}$. Using the horizontal-vertical decomposition defined by $\nabla$, we obtain for the curvature of $\nabla^{\mathcal{L}}$ (cf. [FP, Remark 2.3])

\[ iF^{\nabla^{\mathcal{L}}} = \omega_{G/H} \oplus 0 \oplus \mu^*(F^\nabla), \]
in which $F^\nabla$ is the curvature form of $\nabla$, and $\mu^i(F^\nabla)$ is defined similarly as before in Proposition 9. Using the theory of stability and Hermitian-Einstein connections on principal bundles of Ramanathan and Subramanian [RS], it should be possible to formulate a criterion similar to the decomposition of the vector bundle $E \to M$ into stable direct summands used before, for the principal fibre bundle $P \to M$. It should be possible to extend the main existence result for extremal metrics on projectivised bundles, Theorem 3 to this more general situation using again an adiabatic limit technique. At the time of writing this paper the author was not able to work everything out in detail, but these question shall be addressed in a sequel to the current paper.

3 The formal solutions

We are now going to construct a pointwise formal power series solution of the extremal metric equation (4) by adding Kähler potentials, found by an inductive scheme, to the metric $\omega_k$. Our induction scheme will be different from the ones of Fine [F] and Hong [Ho1, Ho2], since a non-trivial co-kernel will be present in some of the linear equations we have to solve.

However, since our induction scheme is similar in nature to the one in [F, Section 3], we will loosely follow the structure of the exposition there. All results obtained for the formal solutions in this section are only valid pointwise. Only later we will show how to establish convergence of the formal power series solutions in suitable Banach spaces.

In summary, the purpose of Section 3 is to produce a Kähler metric $\omega_{k,n}$, $n \geq 1$ with $\omega_{k,0} = \omega_k$ on $P(E) \to (M, \omega_M)$, obtained by adding Kähler potentials $\psi_i$ to $\omega_k$—with $\psi_i \in C^\infty_\mathbb{T}(P(E), \mathbb{R})$, where $C^\infty_\mathbb{T}(P(E), \mathbb{R})$ denotes the space of smooth real valued functions on $P(E)$ invariant under the $\mathbb{T}^s$-action induced on $P(E)$ by $Id_{E_1}, \ldots, Id_{E_r} \in \text{End}(E)$ (cf. Definition 11)—such that $\omega_{k,n}$ is an approximate solution to the extremal metric equation (4) in the sense that for certain constants $\mathcal{C}, c_1, \ldots, c_{n+1} \in \mathbb{R}$,

$$\text{Scal}(\omega_{k,n}) - Q(\omega_{k,n}) - \mathcal{C} = \sum_{i=1}^{n+1} c_i k^{-i} + O(k^{-n-2}),$$

(7)

where $Q(\omega_{k,n})$ is a Hamiltonian with respect to the purely vertical part $(\omega_{k,n})_r$ of $\omega_{k,n}$ for a (Hamiltonian Killing) vector field in the Lie algebra $\mathfrak{t}^r$ of the torus $\mathbb{T}^s$ (generated by the vector fields induced by $Id_{E_1}, \ldots, Id_{E_r} \in \text{End}(E)$ on $P(E)$).

In order to produce this approximate solution $\omega_{k,n}$, we have to solve three linear PDEs at each step in our induction scheme. As explained in Subsection 3.1.1 below, the errors we have to correct in order to successively adjust a given approximate solution to a higher order approximate solution live in three function spaces $N_{\mathfrak{sl}(r)}$, $R$, $C^\infty(M)$. The $N_{\mathfrak{sl}(r)}$-parts of the errors are corrected by perturbing the hermitian bundle metric $h$ on $E \to M$ and the (Hamiltonian Killing) vector field which corresponds, with respect to the purely vertical part $(\omega_{k,n})_r$ of $\omega_{k,n}$, to the Hamiltonian $Q(\omega_{k,n})$ at each step (for the details, see Subsection 3.2.2). The $R$-parts of the errors are corrected by adjusting $\omega_{k,n}$ at a certain step in the induction scheme by a $\mathbb{T}^s$-invariant Kähler potential which is $L^2$-orthogonal to the function space $N_{\mathfrak{sl}(r)}$ (for the details, see Subsection 3.2.3). Finally, the $C^\infty(M)$-parts of the errors are corrected by adjusting the Kähler form $\omega_M$ on the base manifold $M$ by suitable Kähler potentials (for the details, see Subsection 3.2.4).
3.1 The first order approximate solution

We shall now compute the scalar curvature of $\omega_{k,0} = \omega_k = iF^{V_+^*} + k \cdot \pi^\ast \omega_M$. But first, we will need some more terminology.

Splitting the trace $\Lambda_{\omega_k}$ with respect to $\omega_k$ up into vertical and horizontal parts motivates the following definitions.

**Definition 14.** The vertical trace is defined by

$$\Lambda_{\omega_{FS}} \alpha = \frac{(r-1) \alpha \wedge \omega_{FS}^{n-2}}{\omega_{FS}^{n-1}},$$

for $\alpha \in \Lambda^2 \mathcal{V}^*$, where the quotient is taken in the line $\det \mathcal{V}$ (as $\omega_{FS} \in \Lambda^2 \mathcal{V}^*$ and $\text{rk}(\mathcal{V}) = r - 1$, $r = \text{rk}(E)$, this is well-defined). The horizontal trace is defined by

$$\Lambda_{\omega_M} \alpha = \frac{n \alpha \wedge \omega_M^{n-1}}{\omega_M^n},$$

for $\alpha \in \Lambda^2 \mathcal{H}^*$, where the quotient is taken in the line $\det \mathcal{H}$ (as $\omega_M \in \Lambda^2 \mathcal{H}^*$ and $\text{rk}(\mathcal{H}) = \dim(M) = n$, this is also well-defined).

**Lemma 15.** Let $\alpha \in \Lambda^2 T^\ast P(E)^*$, then

$$\Lambda_{\omega_k} \alpha = \Lambda_{\omega_{FS}} (\alpha)_\mathcal{V} + k^{-1} \Lambda_{\omega_M} (\alpha)_{\mathcal{H}} + \mathcal{O}(k^{-2}),$$

where $(\alpha)_{\mathcal{H}}$ and $(\alpha)_\mathcal{V}$ denote the purely horizontal and purely vertical components of the form $\alpha$.

**Proof.** The result is obtained by computing:

$$\Lambda_{\omega_k} \alpha = \frac{(n + r - 1) \alpha \wedge \omega_k^{n+r-2}}{\omega_k^{n+r-1}}$$

$$= \frac{(r - 1) (\alpha)_\mathcal{V} \wedge \omega_{FS}^{n-2} \wedge (\mu^\ast(F^\mathcal{V}) + k \omega_M)^n}{\omega_{FS}^{n-1} \wedge (\mu^\ast(F^\mathcal{V}) + k \omega_M)^n}$$

$$+ \frac{n (\alpha)_{\mathcal{H}} \wedge \omega_{FS}^{n-1} \wedge (\mu^\ast(F^\mathcal{V}) + k \omega_M)^{n-1}}{\omega_{FS}^{n-1} \wedge (\mu^\ast(F^\mathcal{V}) + k \omega_M)^n}$$

$$= \Lambda_{\omega_{FS}} (\alpha)_{\mathcal{V}} + k^{-1} \Lambda_{\omega_M} (\alpha)_{\mathcal{H}} + \mathcal{O}(k^{-2});$$

where in the last equality we expanded the second fraction in a power series in terms of $k^{-1}$, and absorbed the terms containing $\mu^\ast(F^\mathcal{V})$ into the $\mathcal{O}(k^{-2})$-terms.

**Definition 16.** The vertical and horizontal Laplacians (on functions) are defined by

$$\Delta_{\mathcal{V}} f = \Lambda_{\omega_{FS}} \left( \overline{\partial \partial} f \right)_{\mathcal{V}},$$

$$\Delta_{\mathcal{H}} f = \Lambda_{\omega_M} (\overline{\partial \partial} f)_{\mathcal{H}}.$$
and
\[ \Delta_{\mathcal{H}} f = \Lambda_{\omega_M} \left( \overline{\partial} \partial f \right)_{\mathcal{H}}. \]
The fibrewise restriction of \( \Delta_{\mathcal{V}} \) is the Laplacian on a fibre determined by \( \omega_{FS} \). Whereas on functions pulled back from the base, \( \Delta_{\mathcal{H}} \) is the Laplacian defined by \( \omega_M \).

**Lemma 17.** The \( \omega_k \)-Laplacian on functions, denoted by \( \Delta_k \), satisfies
\[ \Delta_k f = \Delta_{\mathcal{V}} f + k^{-1} \Delta_{\mathcal{H}} f + O(k^{-2}). \]

**Proof.** This follows immediately from the decomposition of \( \Lambda_{\omega_k} \) obtained in Lemma 15. □

**Lemma 18.** For the first order approximate solution \( \omega_k \) we get
\[ \text{Scal}(\omega_k) = \overline{C} + k^{-1} \left( \text{Scal}(\omega_M) + b \mu^*(\Lambda_{\omega_M} F^\n) \right) + O(k^{-2}), \tag{8} \]
for some constants \( \overline{C}, b \) depending only on \( r \); and \( \mu^* \) is again the map defined at the beginning of Section 2.2.

**Proof.** We have the short exact sequence of vector bundles on \( \mathbb{P}(E) \)
\[ 0 \to \mathcal{V} \to T \mathbb{P}(E) \to \mathcal{H} \to 0. \]
Therefore, we have the \( C^\infty \)-splitting \( T \mathbb{P}(E) = \mathcal{V} \oplus \mathcal{H} \) (as already mentioned in Section 2.2 above). This is not a holomorphic splitting and in general \( \mathcal{H} \) defined via this splitting won’t be a holomorphic subbundle of \( T \mathbb{P}(E) \). However, as the vertical tangent bundle \( \mathcal{V} \) is a holomorphic subbundle of \( T \mathbb{P}(E) \), the quotient bundle \( T \mathbb{P}(E)/\mathcal{V} \) is also a holomorphic vector bundle. Moreover, we have the \( C^\infty \)-isomorphism \( \mathcal{H} \cong T \mathbb{P}(E)/\mathcal{V} \), and for the calculation below we shall use this identification and consider \( \mathcal{H} \) as a holomorphic vector bundle.

Thus we have the isomorphism \( K_{\mathcal{P}(E)} \cong \Lambda^{r-1} \mathcal{V}^* \otimes \Lambda^n \mathcal{H}^* \) of holomorphic line-bundles. Hence the Ricci form
\[ \rho_k = i F^{\Lambda^{r-1} \mathcal{V}^*} + i F^{\Lambda^n \mathcal{H}^*}, \]
where \( F^{\Lambda^{r-1} \mathcal{V}^*}, F^{\Lambda^n \mathcal{H}^*} \) are the curvature forms of \( \Lambda^{r-1} \mathcal{V}^*, \Lambda^n \mathcal{H}^* \).

With \( \Lambda^{r-1} \mathcal{V}^* \cong \mathcal{O}_{\mathbb{P}(E)}(-r) \otimes (\det E)^{-1} \), we see that \( \omega_k = \omega_{FS} \oplus \mu^*(F^\n) + k \omega_M \) induces a metric \( h_{\mathcal{V}} \) on \( \Lambda^{r-1} \mathcal{V}^* \) which is determined by the fibrewise Fubini-Study metrics. So, \( h_{\mathcal{V}} \) is the \( r \)-th power of the metric \( \mathcal{O}_{\mathbb{P}(E)}(1) \) (which is induced by the metric \( h \) on \( E \)), hence its curvature is just \( r F \n \).

The curvature \( F^{\Lambda^n \mathcal{H}^*} \) of \( \Lambda^n \mathcal{H}^* \) depends on \( k \), as the metric on \( \mathcal{H} \) corresponds to the Kähler-form \( \mu^*(F^\n) + k \omega_M \). Denote by \( \rho_M \) the Ricci form (pulled back to \( \mathcal{P}(E) \)), i.e. the curvature form of the Chern connection on \( K^*_M \), the anti-canonical line bundle of \( M \), determined by \( \omega_M \). Since the horizontal tangent bundle \( \mathcal{H} \) projects to the tangent bundle \( TM \) of the base manifold \( M \), we will identify \( \Lambda^n \mathcal{H}^* \cong \pi^* K_M^* \) as holomorphic line-bundles.

1We won’t denote the pullback of functions, forms, etc. explicitly.
The ratio of the top exterior powers of the two Kähler forms \( \mu^*(F^V) + k \omega_M \) and \( \omega_M \) gives us the ratio of the corresponding metrics on the (holomorphic) line bundle \( \Lambda^n, \mathcal{H}^* \). By general theory, we then know that \( iF^{A, \mathcal{H}^*} \) and \( \rho_M \) are related by

\[
iF^{A, \mathcal{H}^*} - \rho_M = i\overline{\partial} \partial \log \left( \frac{\mu^*(F^V) + k \omega_M}{\omega_M^k} \right)
= i\overline{\partial} \partial \log \left( k^n + \mu^*(\Lambda_{\omega_M}F^V)k^{n-1} + O(k^{n-2}) \right).
\]

Thus, the Ricci form of \( \omega_k \) is given by

\[
\rho_k = iF^{\Lambda^n, \mathcal{H}^*} + iF^A, \mathcal{H}^*
= riF^{\Lambda^n} + iF^A, \mathcal{H}^*
= riF^{\Lambda^n} + \rho_M + i\overline{\partial} \partial \log \left( k^n + \mu^*(\Lambda_{\omega_M}F^V)k^{n-1} + O(k^{n-2}) \right)
= riF^{\Lambda^n} + \rho_M + \overline{\partial} \partial \log (1 + \mu^*(\Lambda_{\omega_M}F^V)k^{-1} + O(k^{-2})).
\]

Using the power series expansion \( \log(1 + x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i} \), \( |x| < 1 \) (which is possible since \( k \gg 0 \)), we obtain

\[
\rho_k = riF^{\Lambda^n} + \rho_M + k^{-1} \overline{\partial} \partial (\mu^*(\Lambda_{\omega_M}F^V)) + O(k^{-2})
= r \omega_{FS} + r \mu^*(F^V) + \rho_M + k^{-1} i\overline{\partial} \partial (\mu^*(\Lambda_{\omega_M}F^V)) + O(k^{-2})
\]

Using Lemmas 15-17 and the fact that the Ricci-form of the Fubini-Study metric induced on the fibres by \( \Theta_{P(E)}(1) \) is \( \rho_{FS} = r \omega_{FS} \), we get by taking the trace of \( \rho_k \) with \( \omega_k \)

\[
Scal(\omega_k) = Scal(\omega_{FS}) + k^{-1} \left( r \mu^*(\Lambda_{\omega_M}F^V) + Scal(\omega_M) + \Delta_F (\mu^*(\Lambda_{\omega_M}F^V)) \right) + O(k^{-2}).
\]

Moreover, using that \( \mu^*(\Lambda_{\omega_M}F^V) \) is in the first eigenspace of \( \Delta_F \)—with first eigenvalue \( \nu_1 = 2r \)—we get

\[
Scal(\omega_k) = Scal(\omega_{FS}) + k^{-1} \left( Scal(\omega_M) + b \mu^*(\Lambda_{\omega_M}F^V) \right) + O(k^{-2}),
\]

with some constant \( b \) depending only on \( r \). Setting \( C := Scal(\omega_{FS}) = 2r(r - 1) \) gives us equation (8).

3.1.1 Splitting of function spaces on \( P(E) \)

The space of smooth functions \( C^\infty(P(E)) \) on \( P(E) \to M \) splits as follows

\[
C^\infty(P(E)) = C^\infty_0(P(E)) \oplus C^\infty(M),
\]

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where $C^\infty(M)$ are the smooth functions pulled back from the base; and the space $C^\infty_0(\mathbb{P}(E))$ of smooth functions of fibrewise mean-value zero splits further into

$$C^\infty_0(\mathbb{P}(E)) = N_{su(r)} \oplus R,$$

where functions in $N_{su(r)}$ restrict to mean-value zero Hamiltonians for an isometry of a fibre with respect to the Fubini-Study metric, while the functions in $R$ are $L^2$-orthogonal to $N_{su(r)}$ and the constant functions. In total we get a splitting into three function spaces

$$C^\infty(\mathbb{P}(E)) = N_{su(r)} \oplus R \oplus C^\infty(M), \quad (10)$$

which depends on the Fubini-Study metric induced on the fibres of $\mathbb{P}(E) \to M$, and thus on the Hermitian bundle metric $h$ and the corresponding Chern connection $\nabla_h$ on $E \to M$.

In order to perturb $\omega_k$ to a higher order approximation of an extremal Kähler metric, we will have to deal with errors living in these three function spaces. As already mentioned above, these errors will be corrected by solving linear PDEs.

### 3.2 The second order approximate solution

#### 3.2.1 Linearisation formulas

The next lemma is the same as [F, Lemma 2.1], about the linearisation of the scalar curvature map on Kähler potentials on a Kähler manifold $(M,J,g,\omega)$; similar formulas can also be found in [LS2, Section 2]. We are considering the map $\text{Scal}: \phi \mapsto \text{Scal}(\omega_\phi)$, with $\omega_\phi := \omega + i\theta \partial \bar{\partial} \phi$; which is defined on some open set $U \subset C^\infty(M)$.

**Lemma 19** (cf. Lemma 2.1 in [F]). On a Kähler manifold $(M,J,g,\omega)$, let $V$ denote the $L^p_{m+4}$-Sobolev completion of $U \subset C^\infty(M)$. The scalar curvature map on Kähler potentials, $\text{Scal}$, extends to a smooth map $\text{Scal} : V \to L^p_{m}$ whenever $(m+2)p - 2n > 0$, where $n = \dim \mathbb{C} M$ is the dimension of the underlying manifold $M$. Its linearisation at $0 \in V$ is given by

$$L_{\text{Scal},\omega}(\phi) = \left( \Delta^2 - \text{Scal}(\omega_0)\Delta \right) \phi + n(n-1) \frac{i\theta \partial \bar{\partial} \phi \wedge \rho \wedge \omega^{n-2}}{\omega^n}, \quad (11)$$

where $\rho$ denotes the Ricci-form of $\omega$.

Frequently, we will have to use another form of the linearisation of the scalar curvature map on Kähler potentials. Using a Weitzenböck-type formula for the Lichnerowicz-operator $\mathcal{D}^* \mathcal{D}$, equation (11) can also be (re-)written as in the following Lemma. (Rigorous proofs of the two lemmas stated below, with a slightly different convention for the Laplacian and scalar curvature, can be found in [LS2, Section 2].)

**Lemma 20.** On a Kähler manifold $(M,J,g,\omega)$, the linearisation $L_{\text{Scal},\omega}$ of the scalar curvature map on Kähler potentials is given by

$$L_{\text{Scal},\omega}(\phi) = \mathcal{D}^* \mathcal{D} \phi + \frac{1}{2} \nabla \text{Scal} \cdot \nabla \phi, \quad (12)$$

where the gradient and inner product in the last summand are taken with respect to the metric $g$ corresponding to $\omega$. 

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In the same vein, we obtain the analogous result for the linearisation of the extremal metric operator $Scal(\omega) - H(\omega) - \mathfrak{F}$.

**Lemma 21.** On a Kähler manifold $(M, J, g, \omega)$, the linearisation $L_{E.str}\omega$ of the extremal metric operator $Scal(\omega) - H(\omega) - \mathfrak{F}$ on Kähler potentials invariant under the chosen maximal connected compact subgroup $G_{\max}$ of the reduced automorphism group $Aut_{\red}^0(M, J)$, is given by

$$L_{E.str}\omega(\phi) = \mathcal{D}^*\mathcal{D}(\phi) + \frac{1}{2} \nabla Scal(\omega) \cdot \nabla \phi - \frac{1}{2} \nabla H(\omega) \cdot \nabla \phi,$$

(13)

where the gradients and inner products are taken with respect to the metric $g$ corresponding to $\omega$. Here, $H(\omega)$ is the Hamiltonian with respect to $\omega$ of the extremal vector field determined by $G_{\max}$ and $[\omega]$ (cf. Definition [7]).

Hence if we linearise the extremal metric operator $Scal(\omega) - H(\omega) - \mathfrak{F}$, at an extremal metric, the last two summands in equation (13) drop out as the metrical already satisfies equation (4), and we get the Lichnerowicz-operator $\mathcal{D}^*\mathcal{D}(\phi)$.

### 3.2.2 Correcting the $N_{\mathrm{su}(r)}$-part

The $O(k^{-2})$-error in equation (8), which we will denote by $\eta_{O(k^{-2})}$, splits according to the splitting (10) of the function space $C_0^0(\mathbb{P}(E))$,

$$\eta_{O(k^{-2})} = \eta_{O(k^{-2}), N_{\mathrm{su}(r)}} + \eta_{O(k^{-2}), R} + \eta_{O(k^{-2}), C^\infty(M)}.$$

In order to get rid of the $\eta_{O(k^{-2}), N_{\mathrm{su}(r)}}$-part of the $O(k^{-2})$-error $\eta_{O(k^{-2})}$, we will employ a technique which involves perturbing the Hermitian metric $h$ on $E \to M$ by a suitable Hermitian bundle endomorphism. In the current section, it becomes important that $\mu^*(\Lambda_{\mathrm{adj}} F^V)$ depends on the (Hermitian) bundle metric $h$. For this reason, we shall write $\mu^*(\Lambda_{\mathrm{adj}} F^V) = \mu^*(h, \Lambda_{\mathrm{adj}} F^V_h)$—emphasising on the $h$-dependence of the map $\mu^*$ and the Chern connection $\nabla = \nabla_h$ on $E \to M$—from now on.

Remember equation (8) which says that the scalar curvature of $\omega_h$ is given by

$$Scal(\omega_h) = \mathcal{C} + k^{-1} \left( Scal(\omega_M) + b \mu^*(h, \Lambda_{\mathrm{adj}} F^V_h) \right) + k^{-2} \left( \eta_{O(k^{-2}), N_{\mathrm{su}(r)}} + \eta_{O(k^{-2}), R} + \eta_{O(k^{-2}), C^\infty(M)} \right) + O(k^{-3}),$$

(14)

where we explicitly wrote out the $O(k^{-2})$-error.

**Step 1.** We are going to change $h$ to a new bundle metric $h' := h \left( 1 + k^{-1} V \right)$, where $V$ is a Hermitian bundle endomorphism, i.e. the two metrics $h, h'$ are related via

$$\left( (1 + k^{-1} V)(\cdot, \cdot) \right)_h = (\cdot, \cdot)_{h'}.$$

This change of the metric $h$ will cause two types of changes in $\mu^*(h, \Lambda_{\mathrm{adj}} F^V_h)$. Namely, the one caused by the $h$-dependence of $\mu^*$ itself—indicated by the first argument of $\mu^*(\cdot, \cdot)$; and
the other comes from varying $\Lambda_{\omega(h)} F^{V_h}$—the second argument of $\mu^*(\cdot, \cdot)$ in which it is actually linear. We write the total variation $\delta \mu^*(h, \Lambda_{\omega(h)} F^{V_h})$ as the sum of these two variations

$$
\delta \mu^*(h, \Lambda_{\omega(h)} F^{V_h}) = \delta_h \mu^*(h, \Lambda_{\omega(h)} F^{V_h}) + \delta_{\Lambda_{\omega(h)} F^{V_h}} \mu^*(h, \Lambda_{\omega(h)} F^{V_h}).
$$

In order to correct the $\eta_{\partial(k^{-2} N_{su(r)})}$-part of the $\mathcal{O}(k^{-2})$-error, we set

$$
\delta_{\Lambda_{\omega(h)} F^{V_h}} \mu^*(h, \Lambda_{\omega(h)} F^{V_h}) = -\eta_{\partial(k^{-2} N_{su(r)})},
$$

which will give us an equation for $V$.

For the Hamiltonian of the (real holomorphic) Hamiltonian Killing vector field $\Lambda_{\omega(h)} F^{V_h} = -i \sum_{p=1}^2 \lambda_p L dE_p$ (defined as in Definition [11]) with respect to the metric $\omega_{FS}(h')$—which is the purely vertical part of $\omega_0(h')$ with respect to the perturbed bundle metric $h'$—we will use the abbreviation $\mu^*(h', \Lambda_{\omega(h)} F^{V_h}) = \mu^*(h, \Lambda_{\omega(h)} F^{V_h}) + \delta_t \mu^*(h, \Lambda_{\omega(h)} F^{V_h})$.

**Step 2.** Using the formula

$$
h'^{-1} = (1 - k^{-1} V) h^{-1} + \mathcal{O}(k^{-2}),
$$

where $h^{-1}, h'^{-1}$ denote the (local) inverses of the metrics $h, h'$, we are ready to compute the change $\delta_{\Lambda_{\omega(h)} F^{V_h}} \mu^*(h, \Lambda_{\omega(h)} F^{V_h})$ of $\mu^*(h, \Lambda_{\omega(h)} F^{V_h})$. Since locally the curvature of the Chern connection $\nabla_h$ is given by $F^{V_h} = \mathcal{F}(h'^{-1} \partial h')$,

$$
\nabla_h = \nabla_{\Lambda_{\omega(h)} F^{V_h}} + k^{-1} i \Delta_{\omega(h)} \nabla_h + \mathcal{O}(k^{-2}),
$$

where $\partial h$ is the $(1, 0)$-part of the Chern connection of the bundle metric $h$ (for the $(0, 1)$-part we have $\overline{\nabla}_h = \overline{\partial}$, thus we dropped the index). Contracting, using the Kähler identity $\partial\overline{\partial} = i [\Lambda, \overline{\partial}]$, gives

$$
\Lambda_{\omega(h)} F^{V_h} = \Lambda_{\omega(h)} F^{V_h} - k^{-1} i \Delta_{\omega(h)} V + \mathcal{O}(k^{-2}),
$$

where $\Delta_{\omega(h)}$ denotes the $\partial\overline{\partial}$-Laplacian acting on endomorphisms (determined by $h$).

Hence for $\mu^*(h', \Lambda_{\omega(h)} F^{V_h})$ we get

$$
\mu^*(h', \Lambda_{\omega(h)} F^{V_h}) = \mu^*(h, \Lambda_{\omega(h)} F^{V_h}) + \delta_h \mu^*(h, \Lambda_{\omega(h)} F^{V_h}) - k^{-1} \mu^*(h, i \Delta_{\omega(h)} V) + \mathcal{O}(k^{-2})
$$

$$
= \mu^*(h', \Lambda_{\omega(h)} F^{V_h}) - k^{-1} \mu^*(h, i \Delta_{\omega(h)} V) + \mathcal{O}(k^{-2}).
$$

Therefore, after changing $h$ to $h' = h (1 + k^{-1} V)$, the scalar curvature of $\omega_h(h')$ is

$$
Scal(\omega_h(h')) = \overline{\mathcal{C}} + k^{-1} \left( Scal(\omega_M) + b \mu^*(h', \Lambda_{\omega(h)} F^{V_h}) \right) + k^{-2} \left( -2 b \mu^*(h, i \Delta_{\omega(h)} V) + \eta_{\partial(k^{-2} N_{su(r)})} + \eta_{\partial(k^{-2} N_{su(r)})} + \eta_{\partial(k^{-2} N_{su(r)})} \right) + \mathcal{O}(k^{-2}).
$$

Hence equation (15) becomes

$$
 b \mu^*(h, i \Delta_{\omega(h)} V) = \eta_{\partial(k^{-2} N_{su(r)})}.
$$
Writing $\mu^*(h,U) := \eta_{\sigma(k^{-2}),N_{su(r)}}$ for some skew-hermitian endomorphism $U$, which is possible since $\eta_{\sigma(k^{-2}),N_{su(r)}} \in N_{su(r)}$—the space of mean-value zero Hamiltonians for isometries on the fibres of $\mathbb{P}(E) \to M$, gives
\begin{equation}
bi\Delta_{h_b} V = U. 
\end{equation}

**Step 3.** In this last step, we solve equation (17).

The Laplacian $\Delta_{h_b}$ has a non-trivial (co-)kernel in $\text{End}(E)$. Since the vector bundle we consider splits as a direct sum of stable subbundles of different slopes $E = E_1 \oplus \cdots \oplus E_n$, this (co-)kernel is generated by the identity endomorphisms $Id_{E_1}, \ldots, Id_{E_n}$. Therefore, the projection of $U$ to $\text{coker}_{\text{End}(E)} \Delta_{h_b}$ can be written as

$$
\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U) = i(\gamma_1 Id_{E_1} + \cdots + \gamma_n Id_{E_n}),
$$

for suitably chosen $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$. Subtracting $\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U)$ from the right hand side of equation (17), we can now solve (using standard elliptic PDE-theory)

$$
b_i \Delta_{h_b} V = U - \text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U) = U - i(\gamma_1 Id_{E_1} + \cdots + \gamma_n Id_{E_n})
$$

for $V$. Thus, we have found the desired bundle endomorphism $V$ and can therefore correct the $\eta_{\sigma(k^{-2}),N_{su(r)}}$-error by setting $h' = h(1 + k^{-1}V)$.

However, subtracting $\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U)$ from the right hand side of equation (17), we have to add it back on to the right hand side of equation (14). In fact, with $U$ given by $\mu^*(h,U) := \eta_{\sigma(k^{-2}),N_{su(r)}}$, re-writing equation (14) as

$$
\text{Scal}(\omega_k) = \bar{c} + k^{-1}\left(\text{Scal}(\omega_M) + b\mu^*(h,\Lambda_{\omega_M} F^\nabla)\right) + k^{-2}\mu^*(h,\left(\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U) - \text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U)\right)) + k^{-2}\left(\eta_{\sigma(k^{-2}),N_{su(r)}} + \eta_{\sigma(k^{-2}),R} + \eta_{\sigma(k^{-2}),L^\infty(M)}\right) + O(k^{-3})
$$

leaves it unchanged (because the terms in the second line add to zero). Since $\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U) = i(\gamma_1 Id_{E_1} + \cdots + \gamma_n Id_{E_n})$, using that $\mu^*(\cdot,\cdot)$ is linear in its second argument, one can further re-write this as

$$
\text{Scal}(\omega_k) = \bar{c} + k^{-1}\left(\text{Scal}(\omega_M) + \mu^*(h,h\Lambda_{\omega_M} F^\nabla) + k^{-1}i(\gamma_1 Id_{E_1} + \cdots + \gamma_n Id_{E_n})\right) + k^{-2}\mu^*(h,\left(\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U)\right)) + k^{-2}\left(\eta_{\sigma(k^{-2}),N_{su(r)}} + \eta_{\sigma(k^{-2}),R} + \eta_{\sigma(k^{-2}),L^\infty(M)}\right) + O(k^{-3}).
$$

Therefore, using $b\Lambda_{\omega_M} F^\nabla = -bi\sum_{p=1}^n \lambda_p Id_{E_p}$, the “trick” we used to solve equation (17)—i.e. adding and subtracting $\text{proj}_{\text{coker}_{\text{End}(E)} \Delta_{h_b}}(U)$—can be interpreted as changing the weights of the Hamiltonian $\mathbb{T}^n$-action, induced by $Id_{E_1}, \ldots, Id_{E_n} \in \text{End}(E)$ on $\mathbb{P}(E)$ (as in Definition III), since $b\mu^*(h,\Lambda_{\omega_M} F^\nabla)$ becomes

$$
\mu^*(h,i \sum_{p=1}^n (-b\lambda_p + k^{-1}\gamma_p) Id_{E_p})
$$

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Using the re-written version of equation (14), equation (19), we can go through the steps 1–3 explained above again via setting \( h' = h(1 + k^{-1}V) \) and solving equation (18) for \( V \), which gives us

\[
Scal(\omega_k(h')) = C + k^{-1}
\begin{align*}
&Scal(\omega_M) + \mu^* \left( h', i \sum_{p=1}^{s} (-b_\lambda + k^{-1} \gamma_p) Id_{E_p} \right)
\end{align*}
\]

+ \( k^{-2} \left( \eta_{\mathcal{O}(k^{-2}), R} + \eta_{\mathcal{O}(k^{-2}), \mathcal{C}^r(M)} \right) + O(k^{-3}).
\] (20)

By Proposition 9 and our definition of \( \omega_k \),

\[
\omega_k(h) = \omega_{FS}(h) + \mu^*(h, F^\omega) + k\omega_M,
\]

where we emphasised on the \( h \)-dependence of the first two summands. These first two summands are representatives of the class \( c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \), and therefore for any two metrics \( h, h' = h(1 + k^{-1}V) \) on \( E \) they are cohomologous. By general theory, the two metrics \( \omega_k(h), \omega_k(h') \) are related by

\[
\omega_k(h') - \omega_k(h) = k^{-1}i\partial\bar{\partial}\left( \sum_{d=0}^{\infty} k^{-d} \sum_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}} \right) = k^{-1}i\partial\bar{\partial}\phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}},
\]

where it is crucial (in particular for the analysis done later in Section 4) to observe that

\[
\omega_k(h') - \omega_k(h) = O(k^{-1}).
\]

Therefore, the same effect as varying the metric \( h \) on the bundle \( E \) can also be achieved by adding \( k^{-1}i\partial\bar{\partial}\phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}} \)—where the Kähler potential \( \phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}} \) depends on powers of \( k^{-1} \)—to \( \omega_k = \omega_k(h) \). Clearly the Kähler potential \( \phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}} \in C^\infty(\mathbb{P}(E), \mathbb{R}) \) is \( T^r \)-invariant, i.e. \( \phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}} \in C^\infty(T^r(\mathbb{P}(E), \mathbb{R}), \mathbb{R}) \), which follows directly from the fact that the metrics \( \omega_k(h), \omega_k(h') \) and also their difference are \( T^r \)-invariant. Using \( \omega_k(h') = \omega_k(h) + k^{-1}i\partial\bar{\partial}\phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}} \), we write as the conclusion of this section

\[
Scal(\omega_k + k^{-1}i\partial\bar{\partial}\phi_{\mathcal{O}(k^{-2}), N_{\text{au(r)}}}) = C + k^{-1}
\begin{align*}
&Scal(\omega_M) + \mu^* \left( h', i \sum_{p=1}^{s} (-b_\lambda + k^{-1} \gamma_p) Id_{E_p} \right)
\end{align*}
\]

+ \( k^{-2} \left( \eta_{\mathcal{O}(k^{-2}), R} + \eta_{\mathcal{O}(k^{-2}), \mathcal{C}^r(M)} \right) + O(k^{-3}).
\] (21)

### 3.2.3 Correcting the \( R \)-part

Using the results in Section 3.2.1, we get.

**Lemma 22.** Denote again by \( L\text{Scal,} \omega_k \) the formal linearisation of the scalar curvature map on Kähler potentials defined by \( \omega_k \). Then

\[
L\text{Scal,} \omega_k = L\text{Scal,} F + O(k^{-1}),
\]

where \( L\text{Scal,} F \) is the fibrewise linearisation of the scalar curvature map (on Kähler potentials), i.e. \( L\text{Scal,} F(\phi) \) is defined as the change in scalar curvature determined by adding \( i\partial\bar{\partial}(\phi|_{\text{fibre}}) \) on the Fubini-Study metrics induced on the fibres of \( \mathbb{P}(E) \rightarrow M \).
Proof. For the linearisation $L_{\text{Scal}, \omega}$ of the scalar curvature map on Kähler potentials on a Kähler manifold $(M, J, g, \omega)$, \( \text{Scal} : \phi \in C^\infty \to \text{Scal}(\omega_\phi) \), \( \omega_\phi = \omega + i\partial \bar{\partial} \phi \), we have by equation (11)

$$L_{\text{Scal}, \omega}(\phi) = (\Delta^2 - \text{Scal}(\omega_\phi) \Delta) \phi + \frac{(n(n-1))^{\frac{1}{2}}(\partial^2 \phi \wedge \rho \wedge \omega^{n-2})}{\omega^n},$$

(22)

where $\rho$ is the Ricci-from of the Kähler metric induced by $\omega$. Applying this to the scalar curvature map on Kähler potentials on $(\mathbb{P}(E), \omega_k)$ gives us

$$L_{\text{Scal}, \omega_k}(\phi) = (\Delta_k^2 - \text{Scal}(\omega_k) \Delta_k) \phi + (n + r - 1)(n + r - 2)\frac{\partial^2 \phi \wedge \rho_k \wedge \omega_k^{n+r-3}}{\omega_k^{n+r-1}},$$

where as above, $\Delta_k$ is the Laplacian defined by $\omega_k$. Using equation (8), and equation (9) for $\rho_k$ together with Lemma [17] gives

$$L_{\text{Scal}, \omega_k}(\phi) = (\Delta_k^2 - \text{Scal}(\omega_{FS}) \Delta_y) \phi + (r - 1)(r - 2)\frac{\partial^2 \phi \wedge (\rho_k) \wedge \omega_{FS}^{r-3} \wedge (\mu^* (F \overline{V}) + k \omega_M)^n}{\omega_{FS}^{r-1} \wedge (\mu^* (F \overline{V}) + k \omega_M)^n} + O(k^{-1})$$

$$= (\Delta_y^2 - \text{Scal}(\omega_{FS}) \Delta_y) \phi + (r - 1)(r - 2)\frac{\partial^2 \phi \wedge (\rho_k) \wedge \omega_{FS}^{r-3}}{\omega_{FS}^{r-1}} + O(k^{-1})$$

$$= L_{\text{Scal}, F} + O(k^{-1}).$$

(Essentially, this computation is the same as the one in the proof of Lemma [15].)

From equation (12), we know that since the Fubini-Study metrics induced on the fibres of $\mathbb{P}(E) \to M$ have constant scalar curvature,

$$L_{\text{Scal}, F}(\phi) = D^+ \mathcal{D}_F(\phi),$$

where $D^+ \mathcal{D}_F$ is the Lichnerowicz operator on the fibres.

Remark 23.

1. On a Kähler manifold $(M, J, g, \omega)$ endowed with a $G_{\text{max}}$-invariant Kähler metric—where again $G_{\text{max}}$ is some chosen maximal connected compact subgroup of $\text{Aut}_{\text{red}}^0(M, J)$—the Lichnerowicz operator $D^+ \mathcal{D}$ is a self-adjoint, fourth order linear elliptic differential operator which is moreover $G_{\text{max}}$-invariant. Naturally, $D^+ \mathcal{D}$ acts on the space of smooth, real-valued, $G_{\text{max}}$-invariant functions $C^\infty_{G_{\text{max}}}(M, \mathbb{R})$; and has a continuous linear extension—also denoted by $D^+ \mathcal{D}$—mapping between the Sobolev-completions $L^2_{m, G_{\text{max}}}(M, \mathbb{R})$ of $C^\infty_{G_{\text{max}}}(M, \mathbb{R})$ in $L^2_{G_{\text{max}}}(M, \mathbb{R})$.

2. Moreover on the Kähler manifold $(M, J, g, \omega)$, the space of Hamiltonian Killing vector fields $\mathfrak{h}(M, J, \omega) = \text{iso}^0(M, g) \cap \text{aut}^0_{\text{red}}(M, J)$—where $\text{iso}^0(M, g)$ is the Lie algebra of the isometry group $\text{Isom}^0(M, g)$ of $(M, g)$—can be identified via the Hamiltonian construction for $\omega$ with the (co-)kernel of $D^+ \mathcal{D}$ in $C^\infty(M, \mathbb{R})$, since a vector field $V \in \mathfrak{h}(M, J, \omega)$ if and only if it is of the form $V = J \text{grad}_g f = \text{grad}_\omega f$ for a real function $f \in \ker D^+ \mathcal{D}$ (For a proof of this result, cf. [LS2] Theorem 1 and Proposition 1).
3. In particular, the Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$ on $\mathbb{P}(E) \to M$ is invariant under the (Hamiltonian) $\mathbb{T}^\nu$-action induced on $\mathbb{P}(E)$ by the bundle-endomorphisms $Id_{E_1}, \ldots, Id_{E_s}$—remember, the vector bundle $E \to M$ is supposed to split as a direct sum $E = E_1 \oplus \cdots \oplus E_s$ of stable, hence simple, sub-bundles of different slope—via the (infinitesimal) action described in Definition \ref{def:infinitesimal}. This is relevant, for example, since we perturb the Kähler metric $\omega_k$ on $\mathbb{P}(E) \to M$ by adding $\mathbb{T}^\nu$-invariant Kähler potentials $\phi \in \mathcal{C}^\infty_c(\mathbb{P}(E), \mathbb{R})$.

By point 2. of Remark \ref{rem:vectorbundle} we know, since $\mathcal{D}^*\mathcal{D}_F \cong \ker \mathcal{D}^*\mathcal{D}_F$ can be identified via the Fubini-Study metric induced on the fibres of $\mathbb{P}(E) \to M$ with the function space $N_{su(r)}$ in the splitting \ref{eq:splitting} of $\mathcal{C}^\infty(\mathbb{P}(E))$. Therefore, we can invert $L_{\text{Scal}, F} = \mathcal{D}^*\mathcal{D}_F$ only in the function space $R$—which consists of the functions which are $L^2$-orthogonal to $N_{su(r)}$ and the constant functions.

The $R$-component of $\eta_{\phi(k^{-2})}$ will be corrected by adding a suitably chosen Kähler potential $k^{-2}\phi_{(k^{-2}), R}$ to $\omega_k$. Applying Lemma \ref{lem:kahlersolutions} gives

\begin{align} 
\text{Scal}(\omega_k + k^{-1}i\partial \bar{\partial} \phi_{(k^{-2}), N_{su(r)}} + k^{-2}i\partial \bar{\partial} \phi_{(k^{-2}), R}) = & \\
\sqrt{\gamma} + k^{-1} \left( \text{Scal}(\omega_M) + \mu^* \left( h', i \sum_{p=1}^{s} \left( -b \lambda_p + k^{-1} \gamma_p \right) Id_{E_p} \right) \right) \\
+ & k^{-2} \left( L_{\text{Scal}, F}(\phi_{(k^{-2}), R}) + \eta_{\phi(k^{-2}), R} + \eta_{\phi(k^{-2}), C^\infty(M)} \right) + O(k^{-3}).
\end{align}

Therefore, the $\eta_{R}$-part of the $O(k^{-2})$-error can be corrected by solving

\begin{equation}
L_{\text{Scal}, F}(\phi_{(k^{-2}), R}) = -\eta_{\phi(k^{-2}), R},
\end{equation}

for the Kähler potential $\phi_{(k^{-2}), R}$. Indeed, $\phi_{(k^{-2}), R}$ can be chosen to be invariant under the $\mathbb{T}^\nu$-action induced on $\mathbb{P}(E) \to M$, since the differential operator $L_{\text{Scal}, F} = \mathcal{D}^*\mathcal{D}_F$ itself is invariant under this action (See point 3. of Remark \ref{rem:vectorbundle}).

**Lemma 24.** For $\theta \in R$, there exists a unique $\rho \in R$ such that

$L_{\text{Scal}, F}(\rho) = \theta$.

**Proof.** (Modified from the analogous result for Kodaira fibrations, \cite[Lemma 3.6]{F}).

Given the function $\rho \in R$, denote by $\rho_\sigma$ the restriction of $\rho$ to the fibre of $\mathbb{P}(E) \to M$ over $\sigma \in M$. The operator $L_{\text{Scal}, F}$ is just the linearisation of the scalar curvature map on Kähler potentials determined by the induced Fubini-Study metric on that fibre. By point 2. of Remark \ref{rem:vectorbundle} this operator is linear elliptic, self-adjoint and also an isomorphism for functions in $R$. Since functions in $R$ are $(L^2)$-orthogonal to $N_{su(r)}$ and also to the constant functions, we can certainly solve the fibrewise equation $(L_{\text{Scal}, F})_\sigma \rho_\sigma = \theta_\sigma$, uniquely. Patching together, using the uniqueness of the fibrewise solutions $\rho_\sigma$, gives a solution to $L_{\text{Scal}, F}(\rho) = \theta$. Because the operator $L_{\text{Scal}, F}$ is only elliptic in the vertical directions, we have to check that the function $\rho$ is also smooth transverse to the fibres. However, since $\rho_\sigma = (L_{\text{Scal}, F})_\sigma^{-1} \theta_\sigma$, and the fact that $(L_{\text{Scal}, F})_\sigma$ is a smooth family of differential operators, the required regularity properties follow.
Applying Lemma 24 and using point 3. of Remark 23 gives us the existence of a $\mathbb{T}^s$-invariant solution $\phi_{\theta(k-z),R} \in R \cap C^\infty(T(E),\mathbb{R})$ of equation (24).

Adding the $\mathbb{T}^s$-invariant potential $\bar{\partial} \partial k^{-2} \phi_{\theta(k-z),R}$ with $L_{\text{Scal}_F}(\phi_{\theta(k-z),R}) = -\eta_{\theta(k-z),R}$ to $\omega_k$ can be considered as changing

$$\omega_{FS} \mapsto \omega_{FS} + k^{-2} \bar{\partial} \partial \phi_{\theta(k-z),R}.$$ 

So for the term

$$\mu^* \left( h', i \sum_{p=1}^s (-b \lambda_p + k^{-1} \gamma_p) Id_{E_p} \right)$$

in equation (23) to remain a Hamiltonian for the vector field

$$i \sum_{p=1}^s (-b \lambda_p + k^{-1} \gamma_p) Id_{E_p}$$

(again defined as in Definition 11) with respect to the perturbed metric $\omega_{FS} + k^{-2} \bar{\partial} \partial \phi_{\theta(k-z),R}$, it will change according to the following Lemma.

**Lemma 25.** Given a Hamiltonian $F$ for some vector field $V$ in the Lie-algebra $\mathfrak{ham}(M,J,\omega)$ of the Hamiltonian isometry group $\text{Ham}(M,J,\omega)$ of a (compact) Kähler manifold $(M,J,\omega,g)$. Varying $\omega$ by adding a $V$-invariant Kähler potential $\psi \in C^\infty(M)$, i.e. $\mathcal{L}_V \psi = 0$, such that $\omega' = \omega + i \bar{\partial} \partial \psi$, varies $F$ according to the rule

$$F' = F + \frac{1}{2} d\psi(JV) = F - \frac{1}{2} \nabla F \cdot \nabla \psi, \quad (25)$$

up to the addition of a constant. The gradient and inner product are both taken with respect to the metric $g$ corresponding to $\omega$.

**Proof.** The vector field $V$ and the Hamiltonian $F$ are related via

$$t_V \omega = -dF.$$ 

If $\omega' = \omega + i \bar{\partial} \partial \psi = \omega - \frac{1}{2} dd^c \psi$, then

$$t_V \omega' = t_V \omega - \frac{1}{2} t_V dd^c \psi$$

$$= -dF - \frac{1}{2} \mathcal{L}_V(d^c \psi) + \frac{1}{2} d(t_V d^c \psi) \quad \text{(by Cartan’s formula)}$$

$$= -dF - \frac{1}{2} d^c(\mathcal{L}_V \psi) + \frac{1}{2} d(t_V d^c \psi) \quad \text{(as $V$ is real holomorphic)}$$

$$= -d(F - \frac{1}{2} t_V d^c \psi) \quad \text{(since $\mathcal{L}_V \psi = 0$)}$$

$$= -d(F + \frac{1}{2} d\psi(JV)).$$

This computation shows that $V$ is a Hamiltonian vector field with respect to $\omega'$, and the corresponding Hamiltonian function is $F' = F + \frac{1}{2} d\psi(JV) = F - \frac{1}{2} \nabla F \cdot \nabla \psi$. 

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Applying Lemma 25 to $\mu^* \left( h', i \sum_{p=1}^s (-b\lambda_p + k^{-1}\gamma_p)Id_{E_p} \right)$ and $\omega_F + k^{-2}i\overline{\partial}\partial \phi_{O(k^{-2}),R}$ shows that $\mu^* \left( h', i \sum_{p=1}^s (-b\lambda_p + k^{-1}\gamma_p)Id_{E_p} \right)$ transforms via

$$
\mu^* \left( h', i \sum_{p=1}^s (-b\lambda_p + k^{-1}\gamma_p)Id_{E_p} \right) \mapsto \mu^* \left( h', i \sum_{p=1}^s (-b\lambda_p + k^{-1}\gamma_p)Id_{E_p} \right) - \frac{k^{-2}}{2} \nabla \phi_R \cdot \nabla \mu^* \left( h', i \sum_{p=1}^s (-b\lambda_p + k^{-1}\gamma_p)Id_{E_p} \right).
$$

This completes the task of correcting the $\mathcal{O}(k^{-2})$-term of the $\mathcal{O}(k^{-2})$-error $\eta_{\mathcal{O}(k^{-2})}$.

### 3.2.4 Correcting the $C^\infty(M)$-part

In order to correct the $C^\infty(M)$-component $\eta_{\mathcal{O}(k^{-2}),C^\infty(M)}$ of the $\mathcal{O}(k^{-2})$-error $\eta_{\mathcal{O}(k^{-2})}$, we will perturb the metric $\omega_M$, pulled back from the base, with a Kähler potential $\phi_{\mathcal{O}(k^{-2}),C^\infty(M)} \in C^\infty(M)$ in a suitable way.

From equation (8) we know that the scalar curvature $\text{Scal}(\omega_M)$ of $\omega_M$ (the pulled back metric from the base) appears at order $\mathcal{O}(k^{-1})$ in $\text{Scal}(\omega_k)$—it is the $C^\infty(M)$-part of the $\mathcal{O}(k^{-1})$-term of $\text{Scal}(\omega_k)$. Using the Kähler potential $\phi_{\mathcal{O}(k^{-2}),C^\infty(M)} \in C^\infty(M)$ to perturb $\omega_k$ can be thought of as changing the metric $\omega_M$, scaled by the factor of $k$ in the definition of $\omega_k$. Because of this scaling, the effect of adding $i\overline{\partial}\partial \phi_{\mathcal{O}(k^{-2}),C^\infty(M)}$ to $\omega_k$ is the same as adding $k^{-1}i\overline{\partial}\partial \phi_{\mathcal{O}(k^{-2}),C^\infty(M)}$ to $\omega_M$.

With the following formal linearisation formula—derived exactly the same way as equation (11)—giving the variation of $\text{Scal}(\omega_M)$

$$
\text{Scal} \left( \omega_M + k^{-1}i\overline{\partial}\partial \phi_{\mathcal{O}(k^{-2}),C^\infty(M)} \right) = \text{Scal}(\omega_M) + k^{-1}L_{\text{Scal},M}(\phi_{\mathcal{O}(k^{-2}),C^\infty(M)}) + \mathcal{O}(k^{-2}),
$$

(in which $L_{\text{Scal},M}$ denotes the formal linearisation of the scalar curvature map on Kähler potentials on the base) we obtain by adding $i\overline{\partial}\partial \phi_{\mathcal{O}(k^{-2}),C^\infty(M)}$ to the perturbed metric $\omega_k + k^{-1}i\overline{\partial}\partial \phi_{\mathcal{O}(k^{-2}),C^\infty(M)}$, considering it as a change in $\omega_M$, using equations (23) and (26)

$$
\text{Scal} \left( \omega_k + k^{-1}i\overline{\partial}\partial \phi_{\mathcal{O}(k^{-2}),C^\infty(M)} \right)
= \nabla + k^{-1} \left( \text{Scal}(\omega_M) + \mu^* \left( h', i \sum_{p=1}^s (-b\lambda_p + k^{-1}\gamma_p)Id_{E_p} \right) \right)
+ k^{-2} \left( \eta_{\mathcal{O}(k^{-2}),C^\infty(M)} + L_{\text{Scal},M}(\phi_{\mathcal{O}(k^{-2}),C^\infty(M)}) \right) + \mathcal{O}(k^{-3}).
$$

Since the base metric $\omega_M$ is cscK, using equation (12) gives

$$
L_{\text{Scal},M} = D^*D_M,
$$

where $D^*D_M$ is the (self-adjoint, fourth-order linear elliptic) Lichnerowicz operator on the base. Analogous to Lemma 24 we have.
Lemma 26. For $\beta \in C^0_0(M)$, there exists a unique $\alpha \in C^0_0(M)$ such that

$$L_{\text{Scal,}M}(\alpha) = \beta.$$ 

Proof. The cscK base manifold $(M,\omega_M)$ is assumed to have no holomorphic automorphisms. By the Matsushima-Lichnerowicz theorem, holomorphic automorphisms complexify Hamiltonian isometries modulo trivial isometries on a cscK manifold; hence the base has no non-trivial Hamiltonian isometries, and thus by point 2. of Remark 23, $\ker L_{\text{Scal,}M} = \ker \mathcal{D}^* \mathcal{D}_M \cong \mathbb{R}$. Since $L_{\text{Scal,}M} = \mathcal{D}^* \mathcal{D}_M$ is a self-adjoint, fourth-order linear elliptic differential operator on the compact manifold $(M,\omega_M)$, standard elliptic PDE-theory immediately gives the (unique) invertibility of $L_{\text{Scal,}M} : C^\infty_0(M) \to C^\infty_0(M)$ as $C^\infty_0(M) = C^\infty_0(M)/\mathbb{R}$ on the compact manifold $M$. 

Hence up to the addition of constants we can solve

$$L_{\text{Scal,}M}(\phi_{\theta(k^{-2}),C^\infty_0(M)}) = -\eta_{\theta(k^{-2}),C^\infty_0(M)};$$

in case the right hand side has mean-value zero. Denoting by $c_2 := \eta_{\theta(k^{-2}),C^\infty_0(M)}$ the mean value of $\eta_{\theta(k^{-2}),C^\infty_0(M)}$ (with respect to the Kähler metric corresponding to $\omega_M$), we define $\phi_{\theta(k^{-2}),C^\infty_0(M)}$ to be the solution of

$$L_{\text{Scal,}M}(\phi_{\theta(k^{-2}),C^\infty_0(M)}) = \mathcal{D}^* \mathcal{D}_M(\phi_{\theta(k^{-2}),C^\infty_0(M)}) = c_2 - \eta_{\theta(k^{-2}),C^\infty_0(M)}. \quad (27)$$

By Lemma 26 this equation can be solved in $C^\infty_0(M)$ since its right hand side has mean-value zero. Moreover, since the Kähler potential $\phi_{\theta(k^{-2}),C^\infty_0(M)}$ is pulled back from the base, it is automatically invariant under the (Hamiltonian) $\mathbb{T}^s$-action induced by $\text{Id}_E_1,\ldots,\text{Id}_E_s \in \text{End}(E)$ on $\mathbb{P}(E)$.

Therefore, the $C^\infty_0(M)$-part $\eta_{\theta(k^{-2}),C^\infty_0(M)}$ of the $\mathcal{O}(k^{-2})$-error $\eta_{\theta(k^{-2})}$ is corrected modulo the constant $c_2$, i.e.

$$\text{Scal} \left( \omega_k + k^{-1}i\theta \partial \phi_{\theta(k^{-2}),N_{\text{null}}} + k^{-2}i\theta \partial \phi_{\theta(k^{-2}),R} + i\theta \partial \phi_{\theta(k^{-2}),C^\infty_0(M)} \right)$$

$$= \mathcal{C} + k^{-1} \left( \text{Scal}(\omega_M) + \mu^* \left( h', i \sum_{p=1}^{s} (-b\lambda_p + k^{-1} \gamma_p) Id_{E_p} \right) \right) + c_2 k^{-2} + \mathcal{O}(k^{-3}). \quad (28)$$

Thus we completely corrected the $\mathcal{O}(k^{-2})$-error $\eta_{\theta(k^{-2})}$.

3.3 The higher order approximate solutions

In this section we will complete our approximation scheme. This enables us to find—in the sense of equation (7)—an approximate formal power series solution to the extremal metric equation (4), pointwise arbitrarily close to a genuine solution.

Remark 27. From now on, in order to save on notation, we will denote the Hamiltonian constructed while perturbing the map $\mu^*$ in our induction scheme by $Q$.

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Theorem 28 (Formal solutions to the extremal metric equation). Given an integer \( n \geq 1 \) we can find Kähler potentials, invariant under the \( T^s \)-action induced by \( \text{Id}_{E_1}, \ldots, \text{Id}_{E_s} \in \text{End}(E) \) on \( \mathbb{P}(E) \),

\[
\phi_{i,N^{u(i)}} \in \mathcal{C}^\infty_T(\mathbb{P}(E), \mathbb{R}), \quad \phi_i,R \in \mathbb{R} \cap \mathbb{C}^\infty_T(\mathbb{P}(E), \mathbb{R}), \quad \phi_i,C^{-}(M) \in \mathcal{C}^\infty_T(M) \cap \mathcal{C}^\infty_T(\mathbb{P}(E), \mathbb{R}),
\]

for \( i = 1, \ldots, n \) such that the metric

\[
\omega_{k,n} = \omega_k + i\partial \bar{\partial} \sum_{i=1}^{n} \left( k^{-i}\phi_{i,N^{u(i)}} + k^{-i-1}\phi_i,R + k^{-i+1}\phi_i,C^{-}(M) \right)
\]

is an \((n + 1)\)-th order approximate solution to the extremal metric equation (4), by which we mean, as in equation (7), that pointwise on \( \mathbb{P}(E) \)

\[
\text{Scal}(\omega_{k,n}) - Q(\omega_{k,n}) - \overline{C} = \sum_{i=1}^{n+1} c_i k^{-i} + \mathcal{O}(k^{-n-2}), \quad \overline{C}, c_1, \ldots, c_{n+1} \in \mathbb{R}, \quad (29)
\]

where \( Q(\omega_{k,n}) \) is a Hamiltonian with respect to the purely vertical part \( (\omega_{k,n})_V \) of \( \omega_{k,n} \) for a (Hamiltonian Killing) vector field in the Lie algebra \( \mathfrak{t}^s \) of the torus \( T^s \) (generated by the vector fields induced by \( \text{Id}_{E_1}, \ldots, \text{Id}_{E_s} \in \text{End}(E) \) on \( \mathbb{P}(E) \)).

Proof. The proof follows by induction using the steps carried out in order to find the second order approximate solution in Sections 3.2.2, 3.2.3, 3.2.4 as the inductive steps. \( \square \)

4 Analytic aspects

The whole Section 4 bears many similarities with [F, Sections 4–7], and in fact many results and ideas of J. Fine were adapted for our case and are variations of his results.

4.1 The Implicit Function Theorem

We are going to use a parameter-dependent implicit function theorem (IFT), the parameter being the adiabatic parameter \( k \), in order to show the existence of a genuine solution of the extremal metric equation, lying nearby the approximate solution found in Theorem 28.

Theorem 29 (Implicit function theorem).

- Let \( F : B_1 \to B_2 \) be a differentiable map of Banach spaces, whose derivative at 0, \( DF|_0 \), is an epimorphism of Banach spaces, with right-inverse \( P \).

- Let \( \delta' \) be the radius of the closed ball in \( B_1 \), centred at 0, on which \( F - DF|_0 \) is Lipschitz, with constant \( 1/(2\|P\|) \).

- Let \( \delta = \delta'/|P| \).

Whenever \( y \in B_2 \) satisfies \( \|y - F(0)\| < \delta \), there exists \( x \in B_1 \) with \( F(x) = y \).
In fact, this statement of the IFT is the same as [F, Theorem 4.1], except that we assume $DF|_0$ to be an epimorphism of Banach spaces having only a right-inverse $P$, instead of a “two-sided” inverse. The reason is that unlike Fine [F] or Hong [Ho1], we actually have non-trivial Hamiltonian Killing vector fields on $\mathbb{P}(E)$, induced by the non-trivial automorphism group of the vector bundle $E \to M$ (remember: $E$ is unstable, and not simple since $\text{Aut}(E) \cong U(1)^s$).

Therefore, the leading order part of the linearisation at $\omega_{k,n}$ of the “approximate extremal metric operator”\footnote{We shall use “AEMO” as abbreviation for “approximate extremal metric operator”.
}

$$\text{AEMO}(\phi) := \text{Scal}(\omega_{k,n} + i\overline{\partial}\partial \phi) - Q(\omega_{k,n} + i\overline{\partial}\partial \phi) - \overline{S}$$  \tag{30}

(i.e. the left hand side of equation (29)) on $\mathbb{T}^s$-invariant Kähler potentials will have a non-trivial co-kernel in the function spaces $C^m_{\mathbb{T}^s}(\mathbb{P}(E), \mathbb{R})$ and $L^2_{m,\mathbb{T}^s}(\mathbb{P}(E), \mathbb{R})$; where we use the standard notation and denote by $L^2_m$ the Sobolev space of functions whose derivatives up to order $m$ are in $L^2$.

**Remark 30.** The Sobolev space $L^2_m(g_{k,n})$, defined via the metric $g_{k,n}$ corresponding to $\omega_{k,n}$, contains the same functions for different values of the adiabatic parameter $k$, since the corresponding Sobolev norms $\| \cdot \|_{L^2_m(g_{k,n})}$ are equivalent for different values of $k$. (Although the constants of equivalence depend on $k$.)

**The parametrised equation**

Our goal is to solve the extremal metric equation (4), for $k \gg 0$ and fixed $n$,

$$\text{Scal}(\omega_{k,n} + i\overline{\partial}\partial \phi) - H(\omega_{k,n} + i\overline{\partial}\partial \phi) - \overline{S} = 0,$$  \tag{31}

where $\overline{S}$ is the average scalar curvature and $\phi$ is a $\mathbb{T}^s$-invariant Kähler potential. So it is reasonable to try to solve $\text{AEMO}(\phi) = 0$, with AEMO as in (30), which we want to do without having to worry about the obstructions coming from the non-trivial co-kernel of the leading order part of its linearisation. In order to handle this non-trivial co-kernel, we will employ essentially the same trick as already used in Section 3.2.2 above. More precisely, denote the linearisation of AEMO at $\omega_{k,n}$ by $L_{\text{AEMO},\omega_{k,n}}(\phi)$. Using Lemma 20 to linearise the $\text{Scal}(\omega_{k,n})$-part in (30), and Lemma 25 to linearise the $Q(\omega_{k,n})$-part, on $(\mathbb{T}^s$-invariant) Kähler potentials, we obtain

$$L_{\text{AEMO},\omega_{k,n}}(\phi) = \mathbb{D}^* \mathbb{D}(\phi) + \frac{1}{2} \nabla \text{Scal}(\omega_{k,n}) \cdot \nabla \phi - \frac{1}{2} \nabla Q(\omega_{k,n}) \cdot \nabla \phi = \mathbb{D}^* \mathbb{D}(\phi) + O(k^{-n-2}) \quad \text{(by using equation (29))},$$  \tag{32}

where the gradient and inner product in the first line, and $\mathbb{D}^* \mathbb{D}$ in both lines, are taken with respect to the metric corresponding to $\omega_{k,n}$.

The (co-)kernel of the self-adjoint operator $\mathbb{D}^* \mathbb{D}$ in $C^m_{\mathbb{T}^s}(\mathbb{P}(E), \mathbb{R})$ is isomorphic, via $\omega_{k,n}$, to the space of Hamiltonian Killing vector fields induced—as in Definition 11—on $\mathbb{P}(E)$ by linear combinations of $\text{Id}_{E_1}, \ldots, \text{Id}_{E_s} \in \text{End}(E)$. Also, $Q(\omega_{k,n})$ in equation (29) is the Hamiltonian,
with respect to the purely vertical part \((\omega_k)_y\) of \(\omega_k\), for the vector field

\[
B := i \sum_{p=1}^s \left( -b k^{-1} \lambda_p + \sum_{l=1}^n k^{-l-1} \gamma_{p,l} \right) \text{Id}_{E_p}, \quad \lambda_p, \gamma_{p,l} \in \mathbb{R} \text{ for } p = 1, \ldots, s, \ l = 1, \ldots, n;
\]

constructed by iterating the procedure in Section 3.2.2 in order to find \(\omega_k\). (The additional factor of \(k^{-1}\) in (33) is due to \(Q(\omega_k)_n\) not being multiplied by \(k^{-1}\) in equation (29); in contrast to \(\mu^*(\cdots)\) in equation (28).) Therefore, we introduce an \(s\)-tuple of parameters \(\Theta := (\theta_1, \ldots, \theta_s)\) with \(\theta_1, \ldots, \theta_s \in \mathbb{R}\), in the vector field \(B\) in (33) and define

\[
B' := B + B_\Theta := i \sum_{p=1}^s \left( -b k^{-1} \lambda_p + \sum_{l=1}^n k^{-l-1} \gamma_{p,l} \right) (1 + \theta_p) \text{Id}_{E_p},
\]

with

\[
B_\Theta := i \sum_{p=1}^s \left( -b k^{-1} \lambda_p + \sum_{l=1}^n k^{-l-1} \gamma_{p,l} \right) \theta_p \text{Id}_{E_p};
\]

which can be interpreted as varying the (infinitesimal) action of \(B\) on \(\mathbb{P}(E)\). The parameter-dependent vector fields \(B', B_\Theta\) are again Hamiltonian Killing vector fields on \(\mathbb{P}(E)\) with \(B', B_\Theta \in \mathfrak{t}'\) — the Lie algebra of \(\mathbb{T}^s\) (which is generated by the vector fields induced by \(\text{Id}_{E_1}, \ldots, \text{Id}_{E_s} \in \text{End}(E)\) on \(\mathbb{P}(E)\)). Of course, the introduction of the \(s\)-tuple of parameters \(\Theta\) makes the Hamiltonian for \(B\) with respect to \((\omega_k)_y\) — which we denote by \(Q(\omega_k, B)\) — parameter-dependent, as well. Thus,

\[
Q(\omega_k, B') = Q(\omega_k, B + B_\Theta) = Q(\omega_k, B) + Q(\omega_k, B_\Theta),
\]

since \(Q\) is linear in the second argument. So, instead of solving \(\text{AEMO}(\phi) = 0\) directly for \(\phi \in C^\infty_{\mathbb{P}(E), \mathbb{R}}\), we will solve a “parametrised version”. Therefore, we shall also consider the constant \(C \in \mathbb{R}\) in (30) as a parameter, which we write as \(C + k\bar{R}\), and solve

\[
\text{Scal}(\omega_k + i\bar{R} \partial \phi) - Q(\omega_k + i\bar{R} \partial \phi, B) - Q(\omega_k, B_\Theta) - C - k\bar{R} = 0, \quad \Theta \in \mathbb{R}^s, \ \bar{R} \in \mathbb{R},
\]

for \(\phi \in C^\infty_{\mathbb{P}(E), \mathbb{R}}\) and \(\Theta \in \mathbb{R}^s, \ \bar{R} \in \mathbb{R}\). We define the corresponding “parametrised extremal metric operator” to be

\[
\text{AEMO}^{\Theta, \bar{R}}(\phi) := \text{Scal}(\omega_k + i\bar{R} \partial \phi) - Q(\omega_k + i\bar{R} \partial \phi, B) - Q(\omega_k, B_\Theta) - C - k\bar{R};
\]

and will denote its linearisation at \(\omega_k\) by \(L_{\text{AEMO}, \omega_k}^{\Theta, \bar{R}}\). Hence we get, as the operator is linear in the parameters \((\Theta, \bar{R})\),

\[
L_{\text{AEMO}, \omega_k}^{\Theta, \bar{R}}(\phi) = \mathcal{D}^* \mathcal{D}(\phi) + \frac{1}{2} \nabla \text{Scal}(\omega_k) \cdot \nabla \phi - \frac{1}{2} \nabla Q(\omega_k, B) \cdot \nabla \phi - Q(\omega_k, B_\Theta) - k\bar{R} + \mathcal{O}(k^{-n-2}).
\]

**Lemma 31.** For the linearisation of \(\text{AEMO}^{\Theta, \bar{R}}(\phi)\) (defined in (37)) at \(\omega_k\), we get

\[
L_{\text{AEMO}, \omega_k}^{\Theta, \bar{R}}(\phi) = \mathcal{D}^* \mathcal{D}(\phi) - Q(\omega_k, B_\Theta) - k\bar{R} + \mathcal{O}(k^{-n-2}).
\]
Later, we will show that the map
\[ L^\Theta \mathcal{P}_{\text{AEMO}, \omega_k} : L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \to L^2_{m, T^s}, \]
is a Banach space epimorphism for which we can construct a right-inverse with suitable bounds.

**Remark 32.** Because there is only one \( T^s \)-action on \( \mathbb{P}(E) \), we know by the theory outlined in Section 2.1 that the \( s \)-tuple of parameters \( \Theta \) is determined by the Kähler-class \([\omega_k]\) and the \( T^s \)-action. In particular the extremal vector field—defined in Definition 7—is determined by this data, and the variation of \( \Theta \) will perturb \( Q(\omega_k, B) + Q(\omega_k, B_\Theta) \) to the Hamiltonian of the extremal vector field, as we apply the IFT.

If equation (36) is satisfied for the \( T^s \)-invariant Kähler metric \( \omega_k + i\partial \bar{\partial} \phi, \Theta, \mathcal{R} \), it follows from the calculation in equation (5) that \( \mathcal{C} + \mathcal{R} \) is the average of \( \text{Scal}(\omega_k + i\partial \bar{\partial} \phi) \) and \( Q(\omega_k, B) + Q(\omega_k, B_\Theta) \) is the (mean-value zero) Hamiltonian of the extremal vector field.

**Applying the parameter-dependent implicit function theorem**

Once we showed that the map \( L^\Theta \mathcal{P}_{\text{AEMO}, \omega_k} : L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \to L^2_{m, T^s} \) is a Banach space epimorphism with bounded right-inverse, applying the implicit function Theorem 29 to the map
\[ L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \ni (\phi, \Theta, \mathcal{R}) \mapsto \text{Scal}(\omega_k + i\partial \bar{\partial} \phi) - Q(\omega_k + i\partial \bar{\partial} \phi, B) - Q(\omega_k, B_\Theta) - \mathcal{C} - \mathcal{R} \in L^2_{m, T^s}; \]

we see that if the evaluation of this map at \((0, 0, 0) \in L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \) for small \( \delta \) satisfies
\[ \| \text{Scal}(\omega_k, B) - Q(\omega_k, B) - \mathcal{C} \|_{L^2_{m, T^s}(0, \delta)} < \delta, \]

then there exist \((\phi, \Theta, \mathcal{R}) \in L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \) satisfying equation (36). Hence we can conclude the proof once we have shown that \( \text{Scal}(\omega_k, B) - Q(\omega_k, B) - \mathcal{C} \) converges to zero quicker than \( \delta \), for suitably chosen \( n \).

### 4.2 Local analysis

In this section we will establish Sobolev inequalities, and elliptic estimates for \( L^\Theta \mathcal{P}_{\text{AEMO}, \omega_k} \), uniformly in the adiabatic parameter \( k \). Most results in this section were already proven in [F, Section 5], to which we will often refer.

#### 4.2.1 The local model

The most important result of this subsection, Proposition 34, states that the geometry of the fibres dominates the local geometry of the total space \( \mathbb{P}(E) \) in an adiabatic limit for \( k \gg 0 \). The local model we use in this section was first constructed in [F, Section 5.1], and our construction is an adaptation of it.

Let \( B_{\text{flat}} \subset M \) be a ball in the base manifold \( M \) with centre \( p_0 \in M \), endowed with the flat Kähler metric. Since this ball is contractible, \( \mathbb{P}(E)|_{B_{\text{flat}}} \) (the part of \( \mathbb{P}(E) \) over \( B_{\text{flat}} \)) is diffeomorphic to \( \mathbb{P}^{r-1} \times B_{\text{flat}} \). The identification \( \mathbb{P}(E)|_{B_{\text{flat}}} \cong \mathbb{P}^{r-1} \times B_{\text{flat}} \) can be arranged, such that
the horizontal distribution on the (central) fibre \(\mathbb{P}^{r-1}_{(p_0)}\) coincides with the restriction of the \(TB_{\text{flat}}\)-summand to \(\mathbb{P}^{r-1}_{(p_0)}\) in the splitting

\[
T(\mathbb{P}(E)|_{B_{\text{flat}}}) \cong T(\mathbb{P}^{r-1} \times B_{\text{flat}}) \cong T\mathbb{P}^{r-1} \oplus TB_{\text{flat}}.
\]  

(41)

For every \(k\), two Kähler structures on \(\mathbb{P}^{r-1} \times B_{\text{flat}}\) will be of interest: the first one is simply the restriction of the Kähler structure \((\mathbb{P}(E), J, \omega_{k,n})\) to \(\mathbb{P}(E)|_{B_{\text{flat}}}\).

The second Kähler structure of interest is the product structure \((J', \omega_k')\), scaled by \(k\) in the \(B_{\text{flat}}\)-direction. With respect to the splitting \(\{1\}\), we have

\[
J' = J_{\mathbb{P}^{r-1}} \oplus J_{B_{\text{flat}}},
\]

\[
\omega_k' = \omega_{\mathbb{P}^{r-1}} \oplus k\omega_{B_{\text{flat}}},
\]

where \(\omega_{B_{\text{flat}}}\) is the flat Kähler form on \(B_{\text{flat}}\) agreeing with \(\omega_M\) at \(p_0 \in M\). \(J_{B_{\text{flat}}}\) is the complex structure on \(B_{\text{flat}}\), and \(J_{\mathbb{P}^{r-1}}\) are the complex structure and (Fubini-Study) Kähler form on the (central) fibre \(\mathbb{P}^{r-1}_{(p_0)}\). The corresponding product metric induced by \((J', \omega_k')\) on \(\mathbb{P}^{r-1} \times B_{\text{flat}}\) will be denoted by \(g_k'\). Observe, since the fibration \(\mathbb{P}(E) \to M\) is locally holomorphically trivial, the complex structure \(J\) induced on \(\mathbb{P}(E)|_{B_{\text{flat}}}\) by restricting the Kähler structure \((\mathbb{P}(E), J, \omega_{k,n})\) to \(\mathbb{P}(E)|_{B_{\text{flat}}}\), and the complex structure \(J'\) induced by the product Kähler structure coincide over \(B_{\text{flat}}\), i.e. \(J'|_{B_{\text{flat}}} = J|_{B_{\text{flat}}}\).

Later on, we will need the following lemma.

**Lemma 33 (cf. Lemma 5.1 in [F])**. Let \(\alpha \in C^m(T^*\mathbb{P}(E)^{\oplus i})\). Over \(\mathbb{P}(E)|_{B_{\text{flat}}}, \|\alpha\|_{C^m(g_k')} = \mathcal{O}(1)\). If \(\alpha\) is pulled up from the base, we have \(\|\alpha\|_{C^m(g_k')} = \mathcal{O}(k^{-i/2})\).

The proof of the lemma is the same as the one of [F, Lemma 5.1]. The main result of this subsection, which is the analogue of [F, Theorem 5.2], is

**Proposition 34**. For all \(\varepsilon > 0\), \(p_0 \in M\), there exists a ball \(B_{\text{flat}} \subset M\), centred at \(p_0\), such that for all sufficiently large \(k\), over \(\mathbb{P}(E)|_{B_{\text{flat}}}\) we have

\[
\|(J', \omega_k') - (J, \omega_{k,n})\|_{C^m(g_k')} < \varepsilon.
\]

The proof of the proposition is similar to the proof of [F, Theorem 5.2], and we refer to this reference for the details; in fact, the proof in our case is easier since we just have to deal with a holomorphically trivial fibration \(\mathbb{P}(E) \to M\), so \(J'|_{B_{\text{flat}}} = J|_{B_{\text{flat}}}\), whereas [F] considers Kodaira fibrations, the fibres of which have non-trivial moduli.

### 4.2.2 Analysis in the local model

This section contains analytic results on the product model \((\mathbb{P}^{r-1} \times B_{\text{flat}}, J', \omega_k')\), needed in the sequel.

The proofs of the following results won’t be reproduced, since they can be taken over (almost) verbatim from the book [D3, Chapter 3], or from [F, Section 5.2].

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Lemma 35 (cf. Lemma 5.3 in [F]). For indices \(m, l, q \geq p\) satisfying \(m - \dim_{\mathbb{R}}(\mathbb{P}(E))/p \geq l - \dim_{\mathbb{R}}(\mathbb{P}(E))/q\), there is a constant \(c\) (depending only on \(m, l, q, p\)) such that for all \(\phi \in L^p_{m, T^s}(\mathbb{P}^{r-1} \times B_{\text{flat}})\),
\[
\|\phi\|_{L^p_{m, T^s}(g'_k)} \leq c\|\phi\|_{L^p_{m, T^s}(g'_k)}.
\]
For indices \(p, m\) satisfying \(m - \dim_{\mathbb{R}}(\mathbb{P}(E))/p > 0\), there exists a constant \(c\) (depending only on \(p, m\)), such that for all \(\phi \in L^p_{m, T^s}(\mathbb{P}^{r-1} \times B_{\text{flat}})\),
\[
\|\phi\|_{C^2_{T^s}} \leq c\|\phi\|_{L^p_{m, T^s}(g'_k)},
\]
where \(g'_k\) is the scaled product metric, from Proposition \([34]\) on \(\mathbb{P}^{r-1} \times B_{\text{flat}}\).

Remark 36. Even though the result above, and also several results below (Lemmas \([37–42]\)), are proven in [D3, F] for general Sobolev spaces, restricting to \(\mathbb{T}^s\)-invariant functions in these spaces doesn’t cause problems and the proofs are the same.

The product Kähler structure \((J', \omega'_k)\) on \(\mathbb{P}^{r-1} \times B_{\text{flat}}\), as defined in Proposition \([34]\) determines a “product extremal metric operator” on \((\mathbb{T}^s\)-invariant) Kähler potentials
\[
\phi \mapsto \text{Scal}(\omega'_k + i\overline{\theta} \partial \phi) - Q(\omega'_k + i\overline{\theta} \partial \phi, B) - \overline{c},
\]
with the Hamiltonian
\[
Q(\omega'_k + i\overline{\theta} \partial \phi, B),
\]
which is the analogue of \(Q(\omega_k, \omega'_k)\) for the product structure \((J', \omega'_k)\); i.e. \(Q(\omega'_k, B)\) is the Hamiltonian for \(B\) with respect to \(\omega_{\mathbb{P}^{r-1}}\) — the metric on the first factor of the product (where the vector field \(B\) is induced on \(\mathbb{P}^{r-1}\) as in Definition \([11]\)). We denote the linearisation of the map \([12]\) at \(\omega'_k\) by \(L_{\text{AEMO}, \omega'_k}(\phi) : L^2_{m+4, \mathbb{T}^s}(g'_k) \to L^2_{m, \mathbb{T}^s}(g'_k)\). Using the results from Chapter 3 of [D3], or [F] Section 5.2, gives the following elliptic estimate for \(L_{\text{AEMO}, \omega'_k}(\phi)\). (Indeed, the estimates presented in Chapter 3 of [D3] are valid for any elliptic operator determined by the local geometry of the underlying manifold.)

Lemma 37 (cf. Lemma 5.4 in [F]). There exists a constant \(C\) such that for all \(\phi \in L^2_{m+4, \mathbb{T}^s}(\mathbb{P}^{r-1} \times B_{\text{flat}})\),
\[
\|\phi\|_{L^2_{m+4, \mathbb{T}^s}(g'_k)} \leq C \left( \|\phi\|_{L^2_{m, \mathbb{T}^s}(g'_k)} + \|L_{\text{AEMO}, \omega'_k}(\phi)\|_{L^2_{m, \mathbb{T}^s}(g'_k)} \right).
\]

Later on when carrying out the patching arguments to transform those results from the product to the total space of \(\mathbb{P}(E) \to M\), we will also need

Lemma 38 (cf. Lemma 5.5 in [F]). There exists a constant \(P\), such that for all \(\phi \in C^{m+4}_{\mathbb{T}^s}(B_{\text{flat}})\), and all \(\phi \in L^p_{m+4, \mathbb{T}^s}(\mathbb{P}^{r-1} \times B_{\text{flat}})\),
\[
\|L_{\text{AEMO}, \omega'_k}(u \phi) - u L_{\text{AEMO}, \omega'_k}(\phi)\|_{L^p_{m, \mathbb{T}^s}(g'_k)} \leq P \sum_{j=1}^{m+4} \|\nabla^j u\|_{C^0_{T^s}(g'_k)} \|\phi\|_{L^p_{m+4, \mathbb{T}^s}(g'_k)}.
\]
4.2.3 Local analysis and patching arguments

This section will show how to convert results from the product \((\mathbb{P}^r \times \mathbb{R}, J', \omega_k')\) to uniform results over \((\mathbb{P}(E), J, \omega_{k,n})\), and corresponds to [F94] Section 5.3. Applying Proposition 5.4 with \(\varepsilon < 1\), we obtain that over \(\mathbb{P}(E)|_{\mathbb{R}^{\mathbb{R}}}\), the difference \(g_k - g_k'\) is uniformly bounded in the space \(C^m(g_k')\). This choice of \(\varepsilon\) ensures that the metrics are sufficiently close, so that the difference \(g^{T^{\flat}\mathbb{P}(E)} - g^{T^{\flat}\mathbb{P}(E)}\) of the induced metrics on the cotangent bundle is also uniformly bounded.

Hence the Banach space norms on tensors determined by \(g_k\) and \(g_k'\) are uniformly equivalent, i.e.

\[
L\|t\|_{C^m(g_k)} \leq \|t\|_{C^m(g_k')} \leq L\|t\|_{C^m(g_k)},
\]

for some tensor \(t\) and fixed, positive constants \(L, L\). From this we get

**Lemma 39** (cf. Lemma 5.6 in [F94]). For a tensor \(\alpha \in C^m(T^\flat\mathbb{P}(E)_{\mathbb{R}})\), \(\|\alpha\|_{C^m(g_k)} = O(1)\). Additionally, if \(\alpha\) is pulled up from the base, we have \(\|\alpha\|_{C^m(g_k)} = O(k^{-i/2})\).

**Proof.** The same argument as in the proof of [F94] Lemma 5.6] applies, which we shall repeat for the reader’s convenience. By Lemma 33 the statement is true for the local product model. Let again \(B_{\mathbb{R}^{\mathbb{R}}} \subset M\) be a ball over which Proposition 33 holds with \(\varepsilon = 1/2\), for example.

Since the two norms \(\|\cdot\|_{C^m(g_k)}\) and \(\|\cdot\|_{C^m(g_k')}\) are uniformly equivalent over \(\mathbb{P}(E)|_{B_{\mathbb{R}^{\mathbb{R}}}}\), the result holds in the function space \(C^m(g_k)\) over \(\mathbb{P}(E)|_{B_{\mathbb{R}^{\mathbb{R}}}}\). Cover \(M\) with finitely many such balls \(B_{\mathbb{R}^{\mathbb{R}}}\). The result holds in \(C^m(g_k)\) over each \(\mathbb{P}(E)|_{B_{\mathbb{R}^{\mathbb{R}}}}\), and so over all of \(\mathbb{P}(E)\) by adding.

The next lemma gives us a convergence result in the function spaces \(C^m_{\text{flat}}(g_k), L^2_{\text{flat}}(g_k)\) needed later in order to apply the implicit function theorem. N.B.: up to now we only established pointwise convergence for the formal solution constructed in Section 3.

**Lemma 40** (cf. Lemma 5.7 in [F94]). We have

\[
\text{Scal}(\omega_{k,n}) - Q(\omega_{k,n}) - \overline{C} = O(k^{-n-2}) \quad \text{in} \quad C^m_{\text{flat}}(g_k) \quad \text{as} \quad k \to \infty,
\]

\[
\text{Scal}(\omega_{k,n}) - Q(\omega_{k,n}) - \overline{C} = O(k^{-n-2+\text{dim}(M)/2}) \quad \text{in} \quad L^2_{\text{flat}}(g_k) \quad \text{as} \quad k \to \infty.
\]

**Proof.** The proof given here is similar to the proof of [F94] Lemma 5.7]; in fact the proof of convergence in \(C^m_{\text{flat}}(g_k)\) is more or less the same as the one given there, adapted for our purposes. The proof of \(L^2_{\text{flat}}(g_k)\)-convergence is different and brings in dimension considerations.

Since we established \(\text{Scal}(\omega_{k,n}) - Q(\omega_{k,n}) - \overline{C} = O(k^{-n-2})\) pointwise in Theorem 28 we shall first deduce that with respect to some fixed metric \(g\), we have

\[
\text{Scal}(\omega_{k,n}) - Q(\omega_{k,n}) - \overline{C} = O(k^{-n-2}) \quad \text{in} \quad C^m_{\text{flat}}(g) \quad \text{as} \quad k \to \infty.
\]

In order to see this, we argue as follows.

All the calculations done in Section 3 involve absolutely convergent power series and algebraic manipulations of them.

Concerning the \(Q(\omega_{k,n})\)-term, observe that the right hand side of equation 28 is obtained by manipulations such as: expansions of terms in (absolutely convergent) power series, involving
negative powers of \( k \); or the power-series-expansion of \( \log(1 + x) \). I.e. concerning the computations done in the proof of Lemma \([18]\) we can argue that for \( \mu^*(\Lambda_{0\text{vol}} F^V) \), \( \log(1 + k^{-1} \mu^*(\Lambda_{0\text{vol}} F^V)) \) is \( O(k^{-1}) \) in \( C_{m\text{tr}}^m(g) \) since

\[
\| \log(1 + k^{-1} \mu^*(\Lambda_{0\text{vol}} F^V)) \|_{C_{m\text{tr}}^m(g)} \leq \sum_{i \geq 1} k^{-(i+1)} C_i \| \mu^*(\Lambda_{0\text{vol}} F^V) \|_{C_{m\text{tr}}^m(g)}
\]

with a constant \( C \) such that \( \| \rho \sigma \|_{C_{m\text{tr}}^m(g)} \leq C \| \rho \|_{C_{m\text{tr}}^m(g)} \| \sigma \|_{C_{m\text{tr}}^m(g)} \). Since the Hamiltonian function \( Q(\omega_{k,n}) \) from Theorem \([28]\) constructed in our induction scheme, is an \( O(k^{-1}) \)-perturbation of the \( \mu^*(\Lambda_{0\text{vol}} F^V) \)-term from equation (8), it follows that it is also \( O(k^{-1}) \) in \( C_{m\text{tr}}^m(g) \).

Hence, for the statement to be true in the \( C_{m\text{tr}}^m(g_k) \)-norm, a fixed function has to be bounded in this norm as \( k \to \infty \) (the constant \( C \) in the last two inequalities does not depend on \( g \)). Therefore, we can deduce the \( C_{m\text{tr}}^m(g_k) \)-result from Lemma \([39]\).

In order to establish the \( L_{m\text{tr}}^2 \)-result, we observe that the \( g_k \)-volume is \( k^{\dim M} \) times a fixed volume form. So, over a ball \( B_{\text{flat}} \subset M \) where Proposition \([34]\) holds with \( \varepsilon = 1/2 \), the \( g_k \)-volume is \( O(k^{\dim M}) \) times a fixed volume form. Hence, with respect to \( g_k \), the volume of \( \mathcal{P}(E)|_{B_{\text{flat}}} \) is \( O(k^{\dim M}) \). Cover \( M \) with finitely many such balls, \( B_{\text{flat},i} \). Then, the volume \( \text{vol}_k \) of \( \mathcal{P}(E) \), with respect to \( g_k \), satisfies

\[
\text{vol}_k \leq \sum_i \text{vol}(\mathcal{P}(E)|_{B_{\text{flat},i}}) = O(k^{\dim M}).
\]

With all that in our hands, the result follows from the \( C_{m\text{tr}}^m \)-result and the fact that \( \| \phi \|_{L_{m\text{tr}}^2(g_k)} \leq \text{vol}^{1/2}_k \| \phi \|_{C_{m\text{tr}}^m(g_k)} \).

Now, we have everything we need in order to transfer the “product results” from Section \([4, 2.2]\) to \( (\mathcal{P}(E), J, \omega_{k,n}) \). The next lemma is exactly the same as \([\mathbf{f}, \text{Lemma 5.8}]\), thus we shall omit its proof since restricting to the \( T^\nu \)-invariant functions in the respective Sobolev spaces doesn’t change it (cf. Remark \([36]\)).

**Lemma 41** (cf. Lemma 5.8 in \([\mathbf{f}]\)). For indices \( m, l \) and \( q \geq p \) satisfying \( m - \dim_{\mathbb{R}}(\mathcal{P}(E))/p \geq l - \dim_{\mathbb{R}}(\mathcal{P}(E))/q \), there is a constant \( c \) (depending only on \( m, l, q \) and \( p \), but not on \( k \)) such that for all \( \phi \in L_{m \text{tr}}^p(\mathcal{P}(E)) \) and all sufficiently large \( k \),

\[
\| \phi \|_{L_{m \text{tr}}^p(g_k)} \leq c \| \phi \|_{L_{m \text{tr}}^p(g_k)}.
\]

For indices \( p, m \) satisfying \( m - \dim_{\mathbb{R}}(\mathcal{P}(E))/p > 0 \), there exists a constant \( c \) (depending only on \( p, m \) and not on \( k \)), such that for all \( \phi \in L_{m \text{tr}}^p(\mathcal{P}(E)) \) and all sufficiently large \( k \),

\[
\| \phi \|_{C_{m\text{tr}}^m(g_k)} \leq c \| \phi \|_{L_{m \text{tr}}^p(g_k)}.
\]

We are now in a position to prove a uniform elliptic estimate for \( L_{AEMO, \omega_{k,n}}^0(\text{N.B. } (\Theta, \mathcal{R}) = 0 \) here).
Lemma 42 (cf. Lemma 5.9 in [F]). There exists a constant $C$, depending only on $m$, such that for all $\phi \in L^2_{m+4, T^0}(\mathbb{P}(E))$ and all sufficiently large $k$, 

$$
\|\phi\|_{L^2_{m+4, T^0}(\mathbb{R}_k)} \leq C \left( \|\phi\|_{L^2_{m, T^0}(\mathbb{R}_k)} + \|L_{\text{AEMO}, \omega_{k,n}}^{0,0}(\phi)\|_{L^2_{m, T^0}(\mathbb{R}_k)} \right),
$$

where as in Lemma 37 above, $L_{\text{AEMO}, \omega_{k,n}}^{0,0}$ is the linearisation, for $(\Theta, \mathcal{R}) = 0$, of the “parametrised extremal metric operator” on $(\mathbb{T}^4, \text{-invariant})$ Kähler potentials determined by $\omega_{k,n}$.

**Proof.** Even though the elliptic operators under consideration are different, the proof is similar to the one of [F, Lemma 5.9].

Following the strategy of proof of [F, Lemma 5.9], one makes two observations:

- Applying Lemma 25 to the parts of $L_{\text{AEMO}, \omega_{k,n}}^{0,0}, L_{\text{AEMO}, \omega'_{k,n}}$ corresponding to the linearisations of $Q(\omega_{k,n} + i\partial \bar{\partial} \phi, B)$, $Q(\omega'_{k,n} + i\partial \bar{\partial} \phi, B)$ shows that—since both Hamiltonians are formed with respect to the same vector field $B$ and varied by the same invariant Kähler potential—the difference of their variations (linearisations) is zero by using the first equality in equation (25) (recall that $J'[B_{flat}] = J[B_{flat}]$, so we don’t have to worry about $J$ in the first equality of equation (25)).

- For the parts of $L_{\text{AEMO}, \omega_{k,n}}^{0,0}, L_{\text{AEMO}, \omega'_{k,n}}$ corresponding to the linearisations of the scalar curvature maps $\text{Scal}(\omega_{k,n} + i\partial \bar{\partial} \phi), \text{Scal}(\omega'_{k,n} + i\partial \bar{\partial} \phi)$ on invariant Kähler potentials $\phi$, one can argue exactly as in the proof of [F, Lemma 5.9].

These two observations enable us to replace $L_{\text{AEMO}, \omega'_{k,n}}$ with $L_{\text{AEMO}, \omega_{k,n}}^{0,0}$ in Lemma 37 just as in the case treated in [F, Lemma 5.9], and hence we conclude. 

\[\square\]

### 4.3 Global Analysis

In this section we will derive the global estimates, in order to find a lower bound for the first non-zero eigenvalue of the operator $L_{\text{AEMO}, \omega_{k,n}}^{\Theta, \mathcal{R}}$. Following [F, Section 6] we will construct a global model, which has the crucial property of being a Riemannian submersion for $\mathbb{P}(E) \rightarrow (M, k\omega_M)$.

The current section is similar in nature to [F, Section 6], and many of the results presented here are a variation of Fine’s results. In particular, the construction of the global model used below is due to Fine—our analysis is slightly different however, since we work with an operator involving parameters and have to deal with a non-trivial co-kernel.

In fact, the parameters $\Theta, \mathcal{R}$ will play a crucial role to obtain the results below. As main result of this section, we are going to prove:

**Theorem 43.** For all large $k$ and suitable $n$, the operator $L_{\text{AEMO}, \omega_{k,n}}^{\Theta, \mathcal{R}} : L^2_{m+4, T^0} \times \mathbb{R}^n \rightarrow L^2_{m, T^0}$ is a Banach space epimorphism. There exist a constant $C$ and parameters $(\Theta, \mathcal{R}) \in \mathbb{R}^n$, such that for all large $k$ and all functions $\phi \in L^2_{m, T^0}$, the right-inverse operator $L_{\text{AEMO}, \omega_{k,n}}^{\Theta, \mathcal{R}}$ satisfies the estimate

$$
\|L_{\text{AEMO}, \omega_{k,n}}^{\Theta, \mathcal{R}}(\phi)\|_{L^2_{m+4, T^0}(\mathbb{R}_k)} \leq C \|\phi\|_{L^2_{m, T^0}(\mathbb{R}_k)}. \quad (43)
$$

Proving such an estimate is a genuine global issue. Therefore we are now going to describe the global model, first constructed in [F, Section 6.1].

\[\square\]
4.3.1 The global model

We define a Riemannian metric \( h_k \) on \( \mathbb{P}(E) \) by using the fibrewise metrics determined by the purely vertical part of \( iF^\nabla L^* \) (for the definition of \( iF^\nabla L^* \), see Proposition \[9\], i.e. \( \omega_F \), and adding the metric \( k\omega_M \) (in horizontal directions). In this setup, \( (\mathbb{P}(E), h_k) \rightarrow (M, k\omega_M) \) is a Riemannian submersion.

With this construction, \( g_{k,0} = h_k + a \), for some purely horizontal tensor \( a \in s^2(T^*\mathbb{P}(E)) \), independent of \( k \) (it is given by the horizontal components of \( iF^\nabla L^* \)). Horizontal 1-forms scale by \( k^{-1/2} \) in the metric \( h_k \), so we have for \( k \) sufficiently large

\[
\|g_{k,0} - h_k\|_{C^0(h_k)} \leq \frac{1}{2}. \tag{44}
\]

Also since \( \|g_k - g_{k,0}\|_{C^0(h_k)} = \mathcal{O}(k^{-1}) \), the inequality \( (44) \) holds with \( g_{k,0} \) replaced by \( g_k \). From all this one infers that the difference in the induced metrics on \( T^*\mathbb{P}(E) \) is uniformly bounded and hence the \( L^2 \)-norms on tensors determined by \( h_k \) and \( g_k \) are uniformly equivalent (this will be crucial in the sequel).

**Lemma 44** (cf. Lemma 6.2 in [F]). Let \( T \rightarrow \mathbb{P}(E) \) be any bundle of tensors. Then there exist positive constants \( s, S \), such that \( \forall t \in \Gamma(T) \) and sufficiently large \( k \) we have the equivalence of norms

\[
s\|t\|_{L^2(h_k)} \leq \|t\|_{L^2(g_k)} \leq S\|t\|_{L^2(h_k)}.
\]

4.3.2 Controlling the lowest eigenvalue of the parametrised Lichnerowicz operator

As shown in Lemma [31], we have \( L_{\mathbb{R}EM, \omega_k, n}^\phi = \mathcal{D}^* \mathcal{D}(\phi) - Q(\omega_k, B_\phi) - \mathcal{R} + \mathcal{O}(k^{-n-2}) \); with \( \mathcal{D} = \nabla \) the \( \nabla \)-operator on the holomorphic tangent bundle of \( \mathbb{P}(E) \), \( \nabla \) the gradient, and \( \mathcal{D}^* \) is the \( L^2 \)-adjoint of \( \mathcal{D} \). \( \mathcal{D}^* \mathcal{D} \) depends on the metric corresponding to \( \omega_k, n \) and hence also on \( k \). Since it is notationally more convenient, we shall just write \( \nabla \) for \( \nabla_{g_k} \), \( \nabla \) for \( \nabla_{g_k} \) and \( \mathcal{D} \) for \( \mathcal{D}_{g_k} \).

The bound for the lowest non-zero eigenvalue of the “parametrised Lichnerowicz operator” \( \mathcal{D}^* \mathcal{D}(\phi) - Q(\omega_k, B_\phi) - \mathcal{R} \) will be found by linking together two eigenvalue estimates: the first being the one for the ordinary Hodge Laplacian (Lemma 45), and the second being the one for the \( \nabla \)-Laplacian acting on sections of the holomorphic tangent bundle (Lemma 46).

**Lemma 45** (cf. Lemma 6.5 in [F]). There exists a constant \( C_1 > 0 \) such that for all functions \( \phi \) with \( g_k \)-mean value zero and all sufficiently large \( k \),

\[
\|d\phi\|^2_{L^2(g_k)} \geq C_1 k^{-1} \|d\phi\|^2_{L^2(h_k)}. \tag{45}
\]

**Proof.** The proof of this Lemma is, up to dimension considerations, the same as the proof of [F] Lemma 6.5; however, for the reader’s convenience we will provide the details. One can find a constant \( w \) such that \( \phi - w \) has \( h_1 \)-mean value zero. Since \( d\phi = d(\phi - w) \), using Lemma 44 gives \( \|d\phi\|_{L^2(h_k)} \geq \text{const} \|d(\phi - w)\|_{L^2(h_k)} \). Let \( |\cdot|_{h_k} \) denote the norm induced by the pointwise
inner product defined by $h_k$. By definition of $h_k$, we have $|d(\phi - w)|^2_{h_k} \geq k^{-1}|d(\phi - w)|^2_{h_1}$; and since the volume form satisfies $d\text{vol}(h_k) \geq k^{\dim M}d\text{vol}(h_1)$ we get

$$\|d(\phi - w)\|^2_{L^2(h_1)} \geq k^{(\dim M)-1}\|d(\phi - w)\|^2_{L^2(h_k)}.$$ 

Let $\mu_1$ be the first (non-zero) eigenvalue of the $h_1$-Laplacian. Using that $\phi - w$ has mean value zero with respect to $h_1$ gives

$$\|d(\phi - w)\|^2_{L^2(h_1)} \geq \mu_1\|\phi - w\|^2_{L^2(h_1)} \geq \mu_1 k^{-\dim M}\|\phi - w\|^2_{L^2(h_k)}.$$ 

A further application of Lemma 44 renders

$$\|\phi - w\|^2_{L^2(h_k)} \geq \text{const}\|\phi - w\|^2_{L^2(h_k)} \geq \text{const}\|\phi\|^2_{L^2(h_k)},$$

whereas the second inequality follows from the assumption that $\phi$ has $g_k$-mean value zero. Putting the inequalities together completes the proof. 

**Lemma 46** (cf. Lemma 6.6 in [F]). There exists a positive constant $C_2$ such that for all $\zeta = \nabla f$, with $\zeta \perp \ker \partial$, and sufficiently large $k$ we have

$$\|\partial \zeta\|^2_{L^2(g_k)} \geq C_2 k^{-2}\|\zeta\|^2_{L^2(g_k)}.$$  

**Proof.** The proof is the same as the proof of [F] Lemma 6.6], modified for our purposes as the proof of Lemma 45 above. In fact, up to dimension considerations, the proof is the same as in Fine’s case since we assume $\zeta \perp \ker \partial$. 

Linking the two estimates just proved gives us an estimate for $\mathcal{D}$. 

**Lemma 47** (cf. Lemma 6.7 in [F]). There exists a constant $C$ such that for all $\phi \perp \ker \mathcal{D}$ and sufficiently large $k$,

$$\|\mathcal{D}\phi\|^2_{L^2(g_k)} \geq C k^{-3}\|\phi\|^2_{L^2(g_k)}.$$  

**Proof.** The same proof as in [F] Lemma 6.7] works here as well: Combining Lemmas 45 and 46 shows that when $\phi \perp \ker \mathcal{D},$

$$\|\partial \nabla \phi\|^2_{L^2(g_k)} \geq C_2 k^{-2}\|\nabla \phi\|^2_{L^2(g_k)} = C_2 k^{-2}\|d\phi\|^2_{L^2(g_k)} \geq C_1 C_2 k^{-3}\|\phi\|^2_{L^2(g_k)}.$$ 

From this Lemma, it follows that for $\phi \perp \ker \mathcal{D}^*\mathcal{D},$

$$\|\mathcal{D}^*\mathcal{D}\phi\|^2_{L^2(g_k)} \geq C k^{-3}\|\phi\|^2_{L^2(g_k)}.$$ 

**Remark 48.** The elements $f \in \ker \mathcal{D}^*\mathcal{D} \cong \text{coker} \mathcal{D}^*\mathcal{D}$ can be identified with the (real holomorphic) Hamiltonian Killing vector fields on the underlying (compact) Kähler manifold via the Hamiltonian construction, cf. Remark 23. In our situation all Hamiltonian Killing vector fields on $\mathcal{P}(E)$ are induced by the bundle endomorphisms $Id_{E_1}, \ldots, Id_{E_r}$ as in Definition 11 since the bundle $E$ splits as a direct sum of stable subbundles all having different slope and the base admits no holomorphic automorphisms. Therefore, the parameters $(\Theta, \vec{R}) \in \mathbb{R}^{r+1}$ can be chosen such that the projection $\text{proj}_{\ker \mathcal{D}^*\mathcal{D}} \phi$ of any $\phi$ to $\ker \mathcal{D}^*\mathcal{D} \cong \text{coker} \mathcal{D}^*\mathcal{D}$ can be written as $\text{proj}_{\ker \mathcal{D}^*\mathcal{D}} \phi = -Q(\omega_k, B_\Theta) - \vec{R}$.
Thus, the estimate (48) can be extended, for suitably chosen \((\Theta, \mathcal{R}) \in \mathbb{R}^{s+1}\), to all \(\phi\) as

\[
\| \mathcal{D}^* \mathcal{D} \phi - Q(\omega_{k,n}, B_{\Theta}) - \mathcal{R} \|_{L^2(g_\mathcal{R})} \geq C k^{-3} \| \phi \|_{L^2(g_\mathcal{R})}.
\]

We formulate this observation as a Lemma.

**Lemma 49.** There exist a constant \(C\) and parameters \((\Theta, \mathcal{R}) \in \mathbb{R}^{s+1}\) such that for all \(\phi\) and sufficiently large \(k\),

\[
\| \mathcal{D}^* \mathcal{D} \phi - Q(\omega_{k,n}, B_{\Theta}) - \mathcal{R} \|_{L^2(g_\mathcal{R})} \geq C k^{-3} \| \phi \|_{L^2(g_\mathcal{R})}.
\] (49)

**Remark 50.** Lemmas 45–49 were proved for functions \(\phi\) not necessarily invariant under the \(T^s\)-action induced by \(I_{D_k} \ldots, I_{D_E}\) on \(\mathcal{P}(E)\). However, restricting to \(T^s\)-invariant functions does not affect the proofs and the results are valid for such functions as well (cf. also Remark 36).

### 4.3.3 Controlling the (right-)inverse

**Lemma 51** (cf. Lemma 6.8 in [E]). There is a constant \(C\), depending only on \(m\), and parameters \((\Theta, \mathcal{R}) \in \mathbb{R}^{s+1}\), such that for all \(\phi \in L^2_{m+4,T^s}\) and sufficiently large \(k\),

\[
\| \phi \|_{L^2_{m+4,T^s}(g_\mathcal{R})} \leq C \left( \| \phi \|_{L^2_{m}(g_\mathcal{R})} + \| \mathcal{D}^* \mathcal{D} (\phi) - Q(\omega_{k,n}, B_{\Theta}) - \mathcal{R} \|_{L^2_{m+4,T^s}(g_\mathcal{R})} \right).
\]

**Proof.** The proof is very similar to the one in [E Lemma 6.8]. Using Lemma 51 with \((\Theta, \mathcal{R}) = (0,0)\), we have

\[
L_{\text{AEMO}, \omega_{k,n}}^{\Theta}(\phi) = \mathcal{D}^* \mathcal{D} (\phi) + O(k^{-n-2}).
\]

Since by equation (38) and Lemma 40 the \(O(k^{-n-2})\)-terms tend to zero in the \(C^m_{T^s}(g_\mathcal{R})\)-norm, \(L_{\text{AEMO}, \omega_{k,n}}^{\Theta} - \mathcal{D}^* \mathcal{D}\) converges to zero in the operator norm induced by the \(L^2_{m+4,T^s}(g_\mathcal{R})\)-Sobolev norm. Hence the estimate follows for \((\Theta, \mathcal{R}) = (0,0)\) from Lemma 42. Choosing the parameters \((\Theta, \mathcal{R})\) as in Remark 36 we obtain the desired estimate. \(\square\)

Now, everything is in place to prove

**Theorem 52.** The operator \(\mathcal{D}^* \mathcal{D} - Q(\omega_{k,n}, B_{\Theta}) - \mathcal{R} : L^2_{m+4,T^s} \times \mathbb{R}^{s+1} \rightarrow L^2_{m,T^s}\) is a Banach space epimorphism. There exist a constant \(S\), depending only on \(m\), and parameters \((\Theta, \mathcal{R})\), such that for all large \(k\) and all \(\rho \in L^2_{m,T^s}\), the right-inverse operator \(W_{\Theta}^{\mathcal{R}, \omega_{k,n}}\) satisfies

\[
\| W_{\Theta}^{\mathcal{R}, \omega_{k,n}} \rho \|_{L^2_{m+4,T^s}(g_\mathcal{R})} \leq S k^3 \| \rho \|_{L^2_{m,T^s}(g_\mathcal{R})}.
\]

**Proof.** Since \(\mathcal{D}^* \mathcal{D}\) is a fourth-order, linear-elliptic and self-adjoint differential operator, the right-inverse \(W_{\Theta}^{\mathcal{R}, \omega_{k,n}}\) of \(\mathcal{D}^* \mathcal{D} - Q(\omega_{k,n}, B_{\Theta}) - \mathcal{R}\) exists since we can vary the parameters \((\Theta, \mathcal{R}) \in \mathbb{R}^{s+1}\) such that we can deal with the (co-)kernel of \(\mathcal{D}^* \mathcal{D}\) (see Remark 45). It follows from Lemma 49 applied to \(\phi = W_{\Theta}^{\mathcal{R}, \omega_{k,n}} \rho\), with the parameters \((\Theta, \mathcal{R})\) chosen such that they kill the projection of \(\rho\) to \(\text{coker}\mathcal{D}^* \mathcal{D}\), that there is a constant \(C\) such that for all \(\rho \in L^2_{m,T^s}\) we get

\[
\| W_{\Theta}^{\mathcal{R}, \omega_{k,n}} \rho \|_{L^2_{m+4,T^s}(g_\mathcal{R})} \leq C k^3 \| \rho \|_{L^2_{m,T^s}(g_\mathcal{R})}.
\]

By applying Lemma 51 to \(\phi = W_{\Theta}^{\mathcal{R}, \omega_{k,n}} \rho\), we obtain the required bound. \(\square\)
The following standard lemma, the proof of which shall be omitted, essentially states the openness of (right) invertibility in the Banach space of bounded linear operators endowed with a suitable operator norm.

**Lemma 53.** Let \( L, D : B_1 \to B_2 \) be linear maps between Banach spaces. If \( D \) is a bounded right-invertible linear map with bounded right-inverse \( W \), such that

\[
\|L - D\| \leq (2\|W\|)^{-1},
\]

then \( L \) is also right-invertible and has a bounded right-inverse \( I \) satisfying \( \|I\| \leq 2\|W\| \).

Now we have all the ingredients for completing the proof of Theorem 43.

**Proof of Theorem 43.** By Lemma 31,

\[
\|m^{\ast} \Theta - D T\| \leq \Theta - \Theta, \quad \|W\| \leq \Theta - \Theta,
\]

so by Lemma 40 there exists a constant \( c \) such that in the operator norm determined by the \( g_k \)-Sobolev norms, we have

\[
\|m^{\ast} \Theta - D T\| \leq \Theta - \Theta, \quad \|W\| \leq \Theta - \Theta.
\]

Therefore, if \( n \) and \( k \) are sufficiently large:

\[
\|L^{\ast} \Theta - D T\| \leq \Theta - \Theta, \quad \|W\| \leq \Theta - \Theta.
\]

Now, Lemma 53 shows that \( L^{\ast} \Theta \) is right-invertible and provides us with a bound for its right-inverse

\[
\|L^{\ast} \Theta - D T\| \leq \Theta - \Theta, \quad \|W\| \leq \Theta - \Theta.
\]

for some constant \( C \).

\[\square\]

### 4.4 Estimating the non-linear terms

What remains in our discussion of the analysis is the issue of estimating the *non-linear terms* of the “parametrised extremal metric operator”

\[\text{AEMO}^{\Theta, \bar{\pi}}(\phi) := \text{Scal}(\omega, \bar{\pi} + \bar{\pi} \partial \phi) - Q(\omega, B_{\Theta}) - \bar{\pi} - \bar{\pi}, \]

defined in (37). This can be done in our case in a similar way as in Lemma 7.1.

The operator corresponding to the non-linear terms of \( \text{AEMO}^{\Theta, \bar{\pi}} \) shall be denoted by

\[N_k^{\Theta, \bar{\pi}}(\phi) := \text{AEMO}^{\Theta, \bar{\pi}}(\phi) - L^{\Theta, \bar{\pi}}(\phi), \]

where the two operators on the right hand side are evaluated on the same \( T^n \)-invariant Kähler potential \( \phi \).

**Proposition 54.** Let \( m > \text{dim}_C \bar{\pi}(E) \). There exist positive constants \( c, K \) such that for all \( \phi, \psi \in L^2_{m+4, T^n} (g_k) \) with \( \|\phi\|_{L^2_{m+4, T^n} (g_k)} \|\psi\|_{L^2_{m+4, T^n} (g_k)} \leq c \) and \( k \) sufficiently large,

\[
\|N_k^{\Theta, \bar{\pi}}(\phi) - N_k^{\Theta, \bar{\pi}}(\psi)\|_{L^2_{m+4, T^n} (g_k)} \leq K \max \left\{ \|\phi\|_{L^2_{m+4, T^n} (g_k)}, \|\psi\|_{L^2_{m+4, T^n} (g_k)} \right\} \|\phi - \psi\|_{L^2_{m+4, T^n} (g_k)}. \tag{50}
\]
Lemma 2.10 holds for "parametrised extremal metric operator", defined in (37), at \( g \) gives

\[
\text{Proof.} \quad \text{The proof is similar to the one given in } \text{[F] Lemma 7.1}. \text{ Using the mean value theorem gives}
\[
\| N_k^R(\phi) - N_k^R(\psi) \|_{L^2_{m+4,T^s}(g)} \leq \sup_{\phi \in [\phi, \psi]} \| (DN_k^R)_\phi \|_{L^2_{m+4,T^s}(g)} \| \phi - \psi \|_{L^2_{m+4,T^s}(g)},
\]

with \( \| (DN_k^R)_\phi \| \) being the operator norm of the derivative of \( N_k^R \) at \( \phi \); and

\[
\vartheta \in [\phi, \psi] := \{ \vartheta \in L^2_{m+4,T^s} \text{ such that } \vartheta = \phi + t(\psi - \phi), \text{ for some } t \in [0,1] \}.
\]

So \( DN_k^R = L_{AEMO,\omega_k}^R + i\bar{\omega}d\vartheta - L_{AEMO,\omega_k}^R \); where \( L_{AEMO,\omega_k}^R \) is the linearisation of the "parametrised extremal metric operator", defined in (37), at \( \omega_k + i\bar{\omega}d\vartheta \). We apply

- [F] Lemma 2.10 \footnote{Since by our assumption \( m > \dim C \mathbb{P}(E) \), the condition on the indices in [F] Lemma 2.10 is fulfilled ([F] Lemma 2.10 holds for \( \mathbb{T}^s \)-invariant functions, as well). This lemma also requires the constants in the \( g_k \)-Sobolev inequalities to be uniformly bounded—which was proven in Lemma 41. Moreover, it is required in order to apply [F] Lemma 2.10, that the \( C^m(g_k) \)-norm of the curvature of \( \omega_k \) is bounded above—which follows from Proposition 34 and [F] Lemma 2.7.} to the parts of \( L_{AEMO,\omega_k}^R + i\bar{\omega}d\vartheta \), \( L_{AEMO,\omega_k}^R \) corresponding to the linearisations of the scalar curvature maps \( Scal(\omega_k + i\bar{\omega}d\vartheta + i\bar{\omega}d\varphi) \), \( Scal(\omega_k + i\bar{\omega}d\varphi) \) on invariant Kähler potentials \( \nu \),

- Lemma 25 \footnote{Requirements on tensors, derived from the Leibniz rule and further explained in [F] Section 2.2.2}. to the parts of \( L_{AEMO,\omega_k}^R + i\bar{\omega}d\vartheta \), \( L_{AEMO,\omega_k}^R \) corresponding to the linearisations of \( Q(\omega_k + i\bar{\omega}d\vartheta + i\bar{\omega}d\varphi,B) \), \( Q(\omega_k + i\bar{\omega}d\varphi,B) \)—since both Hamiltonians are formed with respect to the same vector field \( B \) and varied by the same invariant Kähler potential, the difference of their variations (linearisations) is zero by using the first equality in equation (25).

- Lemma 25 \footnote{Requirements on tensors, derived from the Leibniz rule and further explained in [F] Section 2.2.2}. to the \( Q(\omega_k + i\bar{\omega}d\vartheta,B\vartheta) - Q(\omega_k,B\vartheta) \)-part of \( DN_k^R = L_{AEMO,\omega_k}^R + i\bar{\omega}d\vartheta - L_{AEMO,\omega_k}^R \), (N.B. this parameter-dependent part of the operator is linear in the parameter \( \vartheta \), and not linearised with respect to the invariant Kähler potential). We can estimate using the first equality in equation (25).

\[
\| Q(\omega_k + i\bar{\omega}d\vartheta,B\vartheta) - Q(\omega_k,B\vartheta) \|_{L^2_{m+4,T^s}} \leq C\| d\vartheta(JB\vartheta) \|_{L^2_{m+4,T^s}},
\]

\[
\leq C\| JB\vartheta \|_{L^2_{m+4,T^s}} \| d\vartheta \|_{L^2_{m+4,T^s}} \leq C' \| \omega_k \|_{L^2_{m+4,T^s}}, \quad (51)
\]

where \( C, C', C'' \) are constants. In the third inequality of (51) we used the following inequality on tensors, derived from the Leibniz rule and further explained in [F] Section 2.2.2]. For tensors \( T, T' \in L^p_m \), we have

\[
\| T \ast T' \|_{L^p_m} \leq C \| T \|_{L^p_m} \| T' \|_{L^p_m}, \quad (52)
\]

where "\( \ast \)" stands for any algebraic operation consisting of tensor products and contractions. Here, the constant \( C \) depends only on \( m \), and not on the metric determining the norm; this follows from the uniform bound on the constants in the \( g_k \)-Sobolev inequalities (see [F] Section 2.2.2] for details).
Putting the three points above together gives us the estimate
\[ \| L_{\text{AEMO}, \omega_k}^{\Theta, \mathcal{R}} \varphi - L_{\text{AEMO}, \omega_k}^{\Theta, \mathcal{R}} \varphi \| \leq \text{const} \| \varphi \|_{L^2_{m+4, T^s}(\mathbb{R})}. \]
Since for all \( \varphi \in [\varphi, \psi] \),
\[ \| \varphi \|_{L^2_{m+4, T^s}} \leq \max \left\{ \| \varphi \|_{L^2_{m+4, T^s}}, \| \psi \|_{L^2_{m+4, T^s}} \right\}, \]
the result follows. \( \square \)

4.5 Applying the implicit function theorem

In this section we will complete the proof of our main result, Theorem 3, by using the parameter-dependent Implicit Function Theorem (Theorem 29).

Proof of Theorem 3 For all \( k \gg 0 \) and sufficiently large \( n \), the “parametrised extremal metric operator”
\[ \text{AEMO}^{\Theta, \mathcal{R}} : L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \rightarrow L^2_{m, T^s} \]
satisfies
- \( \text{AEMO}^{\Theta, 0}(0) = \mathcal{O}(k^{-n-2+(\dim \mathcal{M})/2}) \) in \( L^2_{m, T^s}(g_k) \), by Lemma 40.
- Its linearisation at \( \omega_k, \text{AEMO}^{\Theta, \mathcal{R}} \omega_k : L^2_{m+4, T^s} \times \mathbb{R}^{s+1} \rightarrow L^2_{m, T^s} \), is a Banach space epimorphism with right-inverse \( \text{AEMO}^{\Theta, \mathcal{R}} \omega_k \), which is \( \mathcal{O}(k^3) \) in operator norm by Theorem 43.
- There exists a constant \( K \) such that for all sufficiently small \( V \), the non-linear piece \( N_k^{\Theta, \mathcal{R}} \) of \( \text{AEMO}^{\Theta, \mathcal{R}} \) is Lipschitz with constant \( V \) on a ball about 0 of radius \( KV \). This follows from Proposition 54.
- There is only one \( T^s \)-action on \( \mathbb{P}(E) \), generated by \( \text{Id}_{E_1}, \ldots, \text{Id}_{E_s} \in \text{End}(E) \). This allows us to deal with the non-trivial co-kernel of \( \mathcal{D} \mathcal{D} \) by varying the parameters \( (\Theta, \mathcal{R}) \in \mathbb{R}^{s+1} \), see Remark 18. In the end, there is only one choice for the parameters \( (\Theta, \mathcal{R}) \), since \( C + \mathcal{R} \) in equation (36) is a topological constant (the average scalar curvature), and the parameters \( \Theta \) are determined by the Futaki invariant which by Definition 7 is dual—with respect to the Futaki-Mabuchi inner product—to the extremal vector field.

By the Implicit Function Theorem (Theorem 29), the second and third points above imply that the radius \( \delta_k' \) of the ball about the origin on which \( N_k^{\Theta, \mathcal{R}} \) is Lipschitz with constant \( (2\| L_{\text{AEMO}, \omega_k}^{\Theta, \mathcal{R}} \|)^{-1} \),
is bounded below by \( Ck^{-3} \) for some constant \( C > 0 \). Since \( \delta_k = \delta_k'(2\| L_{\text{AEMO}, \omega_k}^{\Theta, \mathcal{R}} \|)^{-1} \), it follows that \( \delta_k \) is bounded by \( Ck^{-6} \) for some constant \( C > 0 \).

Looking at Theorem 29 we see that for \( \rho \in L^2_{m, T^s} \), with \( \| \text{AEMO}^{\Theta, 0}(0) - \rho \|_{L^2_{m, T^s}(\mathbb{R})} \leq Ck^{-6} \), the equation \( \text{AEMO}^{\Theta, \mathcal{R}}(\rho) = \rho \) has a solution. The first of the above properties implies then,
that for sufficiently large \( n \) and \( k \gg 0 \), the equation \( AEMO_{\Theta, R}(\phi) = 0 \) has a solution \((\phi, \Theta, R)\) with \( \phi \in L^2_{m+4, T^*}(g_k) \), where the parameters \( \Theta \) and \( R \) are determined as in the fourth point above.

Provided \( m \) is big enough such that \( L^2_{m+4, T^*} \to C^2_{T^*} \), applying the regularity Lemma 55 from below iteratively shows that \( \phi \) is smooth.

In order to carry out our arguments above, we still need to establish a \textit{regularity result} about extremal Kähler metrics. This will ensure that the \( T^* \)-invariant Kähler potential \( \phi \), found in Section 4.5, is smooth.

As already mentioned in equation (3), a Kähler metric \( g \) on a (compact) Kähler manifold \((M, J, g, \omega)\) is \textit{extremal} if the gradient of its scalar curvature \( \nabla g \text{Scal}(g) \) preserves the complex structure \( J \), i.e. it is the real part of a holomorphic section of \( T^{1,0}M \). So, instead of using equation (4), another condition for a Kähler metric to be extremal is

\[
\mathcal{L}_{\nabla g \text{Scal}(g)} J = 0, \tag{53}
\]

where \( \mathcal{L} \) denotes the Lie-derivative.

The extremal Kähler metric we constructed in Theorem 3 therefore satisfies Equation (53), and we will use this equation to prove the following regularity result (similar results were already proven in [F, Lemma 2.3] and [LS1, Proposition 4]).

**Lemma 55.** If the Kähler metric \( g_\phi \) corresponding to \( \omega_\phi = \omega + i \partial \bar{\partial} \phi \), on a compact Kähler manifold, is extremal with \( \phi \in C^{m, \alpha} \), \( m \geq 2 \), then \( \phi \in C^{m+3, \alpha} \).

**Proof.** We follow the proof of [F, Lemma 2.3]. For an extremal Kähler metric \( g \), the gradient of the scalar curvature \( \nabla g \text{Scal}(g) \) is the real part of a holomorphic vector field, hence it is \textit{real-analytic}. It therefore follows that the metric dual of \( \nabla g \text{Scal}(g_\phi) \), i.e. \( d \text{Scal}(g_\phi) \) is of class \( C^{m-2, \alpha} \) (as the metric \( g_\phi \) corresponding to \( \omega_\phi \) is of class \( C^{m-2, \alpha} \)); so \( \text{Scal}(g_\phi) \) is therefore of class \( C^{m-1, \alpha} \).

Now, \( \text{Scal}(g_\phi) = \Delta g_\phi U \), where \( \Delta g_\phi \) is the \( g_\phi \)-Laplacian and

\[
U = -\log \det(g + \Phi),
\]

where \( \Phi \) is the real symmetric tensor corresponding to the (1,1)-form \( i \partial \bar{\partial} \phi \), and \( g \) is the Kähler metric corresponding to \( \omega \).

Since \( \phi \in C^{m, \alpha} \), \( \Delta g_\phi \) is a linear second order elliptic operator with coefficients in \( C^{m-2, \alpha} \). By standard elliptic regularity results (cf. [Aub] Theorem 3.59) and since \( \text{Scal}(g_\phi) \in C^{m-1, \alpha} \), we get \( U \in C^{m, \alpha} \).

The map \( \phi \mapsto -\log \det(g + \Phi) \) is non-linear, but also second order and elliptic. Therefore, it also satisfies an elliptic regularity result (cf. [Aub] Theorem 3.56), hence \( \phi \in C^{m+2, \alpha} \).

Therefore, \( \Delta g_\phi \) has \( C^{m, \alpha} \)-coefficients; hence \( U \in C^{m+1, \alpha} \) and \( \phi \in C^{m+3, \alpha} \).

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