The conjectures of Artin–Tate and Birch–Swinnerton-Dyer

Stephen Lichtenbaum, Niranjan Ramachandran, and Takashi Suzuki

Abstract. We provide two proofs that the conjecture of Artin–Tate for a fibered surface is equivalent to the conjecture of Birch–Swinnerton-Dyer for the Jacobian of the generic fibre. As a byproduct, we obtain a new proof of a theorem of Geisser relating the orders of the Brauer group and the Tate–Shafarevich group.

Keywords. Birch–Swinnerton-Dyer conjecture; finite fields; zeta functions; Tate conjecture

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Stephen Lichtenbaum
Department of Mathematics, Brown University, Providence, RI 02912
e-mail: stephen_lichtenbaum@brown.edu

Niranjan Ramachandran
Department of Mathematics, University of Maryland, College Park, MD 20742 USA.
e-mail: atma@math.umd.edu

Takashi Suzuki
Department of Mathematics, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
e-mail: tsuzuki@gug.math.chuo-u.ac.jp

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1. Introduction and statement of results

Let \( k = \mathbb{F}_q \) be a finite field of characteristic \( p \) and let \( S \) be a smooth projective (geometrically connected) curve over \( T = \text{Spec} \ k \) and let \( F = k(S) = \mathbb{F}_q(S) \) be the function field of \( S \). Let \( X \) be a smooth proper surface over \( T \) with a flat proper morphism \( \pi : X \to S \) with smooth geometrically connected generic fiber \( X_0 \) over \( \text{Spec} \ F \). The Jacobian \( J \) of \( X_0 \) is an Abelian variety over \( F \).

Our first main result is a proof of the following statement conjectured by Artin and Tate [Tat66, Conjecture (d)]:

**Theorem 1.1.** The Artin–Tate conjecture for \( X \) is equivalent to the Birch–Swinnerton-Dyer conjecture for \( J \).

Recall that these conjectures concern two (conjecturally finite) groups: the Tate–Shafarevich group \( \text{III}(J/F) \) of \( J \) and the Brauer group \( \text{Br}(X) \) of \( X \). A result of Artin–Grothendieck [Gor79, Theorem 2.3] [Gro68, §4] is that \( \text{III}(J/F) \) is finite if and only if \( \text{Br}(X) \) is finite.

Our second main result is a new proof of a beautiful result (2.18) of Geisser [Gei20, Theorem 1.1] that relates the conjectural finite orders of \( \text{III}(J/F) \) and \( \text{Br}(X) \); special cases of (2.18) are due to Milne–Gonzales-Aviles [Mil81, GA03].

We actually provide two proofs of Theorem 1.1; while our first proof uses Geisser’s result (2.18), the second (and very short) proof in §4, completely due to the third-named author, does not.

1.1. History

Artin and Tate regarded Theorem 1.1 as easier to prove as opposed to the other conjectures in [Tat66]. They proved Theorem 1.1 when \( \pi \) is smooth and has a section ([Tat66, p.427]) using the equality

\[
\text{III}(J/F) = [\text{Br}(X)]
\]

between the orders of the groups \( \text{III}(J/F) \) and \( \text{Br}(X) \) which follows from Artin’s theorem [Tat66, Theorem 3.1], [Gor79, Theorem 2.3]: if \( \pi \) is generically smooth with connected fibers and admits a section, then \( \text{III}(J/F) \equiv \text{Br}(X) \). Gordon [Gor79, Theorem 6.1] used (1.1) to prove Theorem 1.1 when \( \pi \) is cohomologically flat with a section (see [Gor79, Theorem 2.3]). Building on Gordon [Gor79], Liu–Lorenzini–Raynaud [LLR04] proved several new cases of Theorem 1.1 by eliminating the condition of cohomological flatness of \( \pi \); their proof [LLR04, Theorem 4.3] proceeds by proving that Theorem 1.1 is equivalent to a precise relation generalizing (1.1) between \([\text{Br}(X)]\) and \([\text{III}(J/F)]\) which in their case had been proved by Milne and Gonzales-Aviles [Mil81, GA03].

\[\text{There is another proof (up to } p\text{-torsion) in this case due to Z. Yun [Yun15].}\]
As Liu–Lorenzini–Raynaud (and Milne) point out [LLR05, Theorem 2], Theorem 1.1 follows by combining [Tat66, Gro68, Mil75, KT03]:

\[ \text{AT}(X) \xleftrightarrow{\text{Artin–Tate–Milne}} \text{Br}(X) \text{ finite} \xleftrightarrow{\text{Artin–Grothendieck}} \text{III}(J/F) \text{ finite} \xleftrightarrow{\text{Kato–Trihan}} \text{BSD}(J). \]

In 2018, Geisser pointed out that a slight correction is necessary in the relation [LLR04, Theorem 4.3] between [\text{Br}(X)] and [\text{III}(J/F)]; Liu–Lorenzini–Raynaud [LLR18, Corrected Theorem 4.3] showed that Theorem 1.1 holds if and only if this slightly corrected version holds. This precise relation (Theorem 2.11) was then proved by Geisser [Gei20, Theorem 1.1] without using Theorem 1.1. Thus, combining [LLR18, Corrected Theorem 4.3] and [Gei20, Theorem 1.1] gives the second known proof of Theorem 1.1. But this proof relies heavily on the work of Gordon\(^2\) [Gor79] as can be seen from [LLR18, §3, (3.9)].

### 1.2. Our approach

Our first proof depends on [Gor79] only for the elementary result (2.9). As in [Gor79, LLR04, LLR18], this proof also follows the strategy in [Tat66, §4]. We use the localization sequence to record a short proof\(^3\) of the Tate–Shioda relation (Corollary 2.2). In turn, this gives a quick calculation (2.17) of the height pairing \(\Delta_{\text{ar}}(\text{NS}(X))\) on the Néron–Severi group of \(X\). The same calculation in [Gor79, LLR18] requires a detailed analysis of various subgroups of \(\text{NS}(X)\). A beautiful introduction to these results is [Ulm14]; see [Lic83, Lic05, GS20] for Weil-étale analogues.

The second proof (§4) of Theorem 1.1 uses only (2.5) and the Weil-étale formulations of the two conjectures. In this proof, we do not compare each term of the two special value formulas and entirely work in derived categories.

### Notations

Throughout, \(k = \mathbb{F}_q\) is a finite field of characteristic \(p\) and \(T = \text{Spec } k\); if \(\bar{k}\) is an algebraic closure of \(k\), let \(\bar{T} = \text{Spec } \bar{k}\). The function field of \(S\) is \(F = k(S)\). Let \(X\) be a smooth proper surface over \(T\) with a flat proper morphism \(\pi : X \to S\) with smooth geometrically connected generic fiber \(X_0\) over Spec \(F\). The Jacobian \(J\) of \(X_0\) is a finite étale cover of \(F\).

### 1.3. The Artin–Tate conjecture

Let \(k = \mathbb{F}_q\) and \(F = k(S)\). For any scheme \(V\) of finite type over \(T\), the zeta function \(\zeta(V, s)\) is defined as

\[ \zeta(V, s) = \prod_{v \in V} \frac{1}{(1-q_v^{-s})}, \]

the product is over all closed points \(v\) of \(V\) and \(q_v\) is the size of the finite residue field \(k(v)\) of \(v\). If \(V\) is smooth proper (geometrically connected) of dimension \(d\), then the zeta function \(\zeta(V, s)\) factorizes as

\[ \zeta(V, s) = \frac{P_1(V, q^{-s}) \cdots P_{2d-1}(V, q^{-s})}{P_0(V, q^{-s}) \cdots P_{2d}(V, q^{-s})}, \quad P_0 = (1-q^{-s}), \quad P_{2d} = (1-q^{d-s}), \]

where \(P_i(V, t) \in \mathbb{Z}[t]\) is the characteristic polynomial of Frobenius acting on the \(\ell\)-adic étale cohomology \(H^i(V \times T, \mathbb{Q}_\ell)\) for any prime \(\ell\) not dividing \(q\); by Grothendieck and Deligne, \(P_j(V, t)\) is independent of \(\ell\). One has the factorization [Tat66, (4.1)] (the second equality uses Poincaré duality)

\[ \zeta(X, s) = \frac{P_1(X, q^{-s}) \cdot P_3(X, q^{-s})}{(1-q^{-s}) \cdot P_2(X, q^{-s}) \cdot (1-q^{2-s})} = \frac{P_1(X, q^{-s}) \cdot P_1(X, q^{1-s})}{(1-q^{-s}) \cdot P_2(X, q^{-s}) \cdot (1-q^{2-s})}. \]

\(^2\)Known to have several inaccuracies; see [LLR18, §3.3].

\(^3\)This is similar to the ideas of Hindry–Pacheco and Kahn in [Kah09, §§3.2-3.3].
Let \( \rho(X) \) be the rank of the finitely generated Néron–Severi group \( \text{NS}(X) \). The intersection \( D \cdot E \) of divisors \( D \) and \( E \) provides a symmetric non-degenerate bilinear pairing on \( \text{NS}(X) \); the height pairing \( (D,E)_{\text{ar}} \) \cite[Remark 3.11]{LLR18} on \( \text{NS}(X) \) is related to the intersection pairing as follows:

\[
\text{NS}(X) \times \text{NS}(X) \to \mathbb{Q}((\log q)), \quad D,E \mapsto (D \cdot E)\log q.
\]

Let \( A \) be the reduced identity component \( \text{Pic}^\text{red,0}_{X/k} \) of the Picard scheme \( \text{Pic}_{X/k} \) of \( X \). Let 

\[
\alpha(X) = \chi(X,\mathcal{O}_X) - 1 + \dim(A).
\]

We write \([G]\) for the order of a finite group \( G \).

**Conjecture 1.2** \cite[Conjecture (C)]{Artin-Tate}[Tat66, Conjecture (C)]. The Brauer group \( \text{Br}(X) \) is finite, \( \text{ord}_{s=1} P_2(X, q^{-s}) = \rho(X) \), and the special value

\[
P_2^s(X, q^{-1}) := \lim_{s \to 1} \frac{P_2(X, q^{-s})}{(s-1)^{\rho(X)}}
\]

of \( P_2(X, t) \) at \( t = 1/q \) \((\text{this corresponds to } s = 1)\) satisfies

\[
P_2^s(X, q^{-1}) = [\text{Br}(X)] \cdot \Delta_{\text{ar}}(\text{NS}(X)) \cdot q^{-\alpha(X)}.
\]

Here \( \Delta_{\text{ar}}(\text{NS}(X)) \) is the discriminant \((\text{see } \S 1.4)\) of the height pairing on \( \text{NS}(X) \).

**Remark.** The discriminant \( \Delta_{\text{ar}}(\text{NS}(X)) \) of the height pairing on \( \text{NS}(X) \) is related to the discriminant \( \Delta(\text{NS}(X)) \) of the intersection pairing as follows: \( \Delta_{\text{ar}}(\text{NS}(X)) = \Delta(\text{NS}(X)) \cdot (\log q)^{\rho(X)} \).

### 1.4. Discriminants

For more details on the basic notions recalled next, see \cite[§2.8]{Yun15} and \cite{Blo87}. Let \( N \) be a finitely generated Abelian group \( N \) and let \( \psi : N \times N \to K \) be a symmetric bilinear form with values in any field \( K \) of characteristic zero. If \( \psi : N/\text{tor} \times N/\text{tor} \to K \) is non-degenerate, the discriminant \( \Delta(N) \) is defined as the determinant of the matrix \( \psi(b_i, b_j) \) divided by \( (N : N')^2 \) where \( N' \) is the subgroup of finite index generated by a maximal linearly independent subset \( \{b_i\} \) of \( N \). Note that \( \Delta(N) \) is independent of the choice of the subset \( \{b_i\} \) and the subgroup \( N' \) and incorporates the order of the torsion subgroup of \( N \). For us, \( K = \mathbb{Q} \) or \( \mathbb{Q}(\log q) \).

Given a short exact sequence \( 0 \to N' \to N \to N'' \to 0 \) which splits over \( \mathbb{Q} \) as an orthogonal direct sum \( N_\mathbb{Q} \cong N'_\mathbb{Q} \oplus N''_\mathbb{Q} \) with respect to a definite pairing \( \psi \) on \( N \), one has the following standard relation

\[
\Delta(N) = \Delta(N') \cdot \Delta(N'').
\]

Given a map \( f : C \to C' \) of Abelian groups with finite kernel and cokernel, the invariant \( z(f) = \frac{\left[\text{Ker}(f)\right]}{\left[\text{Coker}(f)\right]} \) \cite{Tat66} extends to the derived category \( D \) of complexes in Abelian groups with bounded and finite homology: given any such complex \( C_* \), the invariant

\[
z(C_*) = \prod_i \left[H_i(C_*)\right]^{(-1)^i}
\]

is an Euler characteristic; for any triangle \( K \to L \to M \to K[1] \) in \( D \), the following relation holds

\[
z(K) \cdot z(M) = z(L).
\]

One recovers \( z(f) \) viewing \( f : C \to C' \) as a complex in degrees zero and one. For any pairing \( \psi : N \times N \to \mathbb{Z} \), the induced map \( N \to \text{RHom}(N, \mathbb{Z}) \) recovers \( \Delta(N) \) above:

\[
\Delta(N) = z(N \to \text{RHom}(N, \mathbb{Z}))^{-1}.
\]

\(\square\)
1.5. The Birch–Swinnerton-Dyer conjecture

For more details on the basic notions recalled next, see [GS20]. Let $J$ be the Jacobian of $X_0$. Recall that the complete $L$-function [Ser70, Mil72], [GS20, §4] of $J$ is defined as a product of local factors

\[
L(J, s) = \prod_{v \in S} L_v(J, q_v^{-s}).
\]

For any closed point $v$ of $S$, the local factor $L_v(J, t)$ is the characteristic polynomial of Frobenius on

\[
H^1_{\text{ét}}(J \times F_v \text{sep}, \mathbb{Q}_\ell)^I_v,
\]

where $F_v$ is the complete local field corresponding to $v$ and $I_v$ is the inertia group at $v$. By [GS20, Proposition 4.1], $L_v(J, t)$ has coefficients in $\mathbb{Z}$ and is independent of $\ell$, for any prime $\ell$ distinct from the characteristic of $k$. Let $\Pi(J/F)$ be the Tate–Shafarevich group of $J$ over $F$ and let $r$ be the rank of the finitely generated group $J(F)$. Let $\Delta_{NT}(J(F))$ be the discriminant of the Néron–Tate pairing [Tat66, p. 419], [KT03, §1.5] on $J(F)$:

\[
J(F) \times J(F) \to \mathbb{Q}(\log q), \quad (\gamma, \kappa) \mapsto \langle \gamma, \kappa \rangle_{NT}.
\]

Let $\mathcal{J}$ be the Néron model of $J$; for any closed point $v \in S$, define $c_v = [\Phi_v(k_v)]$ where $\Phi_v$ is the group of connected components of $\mathcal{J}_v$ and put $c(J) = \prod_{v \in S} c_v$; this is a finite product as $c_v = 1$ for all but finitely many $v$. Let $\text{Lie} \mathcal{J}$ be the locally free sheaf on $S$ defined by the Lie algebra of $\mathcal{J}$. Recall the $^4$

Conjecture 1.3 (Birch–Swinnerton-Dyer). The group $\Pi(J/F)$ is finite, $\text{ord}_{s=1} L(J, s) = r$, and the special value

\[
L^*(J, 1) := \lim_{s \to 1} \frac{L(J, s)}{(s-1)^r}
\]

satisfies

\[
L^*(J, 1) = [\Pi(J/F)] \cdot \Delta_{NT}(J(F)) \cdot c(J) \cdot q^{\chi(S, \text{Lie} \mathcal{J})}.
\]

The proof of Theorem 1.1, i.e. the equivalence of Conjectures 1.2 and 1.3, naturally divides into four parts:

- $\text{Br}(X)$ is finite if and only if $\Pi(J/F)$ is finite. This is known [Gro68, (4.4), Corollaire (4.4)].
- Comparison of $\chi(S, \text{Lie} \mathcal{J})$ and $\alpha(X)$ given in (2.5). This is known [LLR04, p. 483]. For the convenience of the reader, we recall it in §2.2.
- (Proposition 2.4) $\text{ord}_{s=1} P_2(X, q^{-s}) = \rho(X)$ if and only if $\text{ord}_{s=1} L(J, s) = r$.
- ($\S$3) $P_2^*(X, 1)$ satisfies (1.4) if and only if $L^*(J, 1)$ satisfies (1.10).

The first two parts are not difficult and we provide elementary proofs of the last two parts.

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2. Preparations

2.1. Elementary identities and known results

The Néron–Severi group $\text{NS}(X)$ is the group of $k$-points of the group scheme $\text{NS}_{X/k} = \pi_0(\text{Pic}_{X/k})$ of connected components of the Picard scheme $\text{Pic}_{X/k}$ of $X$. Let $A = \text{Pic}_{X/k}^{\text{red}, 0}$. The Leray spectral sequence for

$^{4}$By [GS20, Corollary 4.5], this is equivalent to the formulation in [Tat66].
the morphism $X \to \text{Spec } k$ and the étale sheaf $\mathbb{G}_m$ provides the first exact sequences \cite[Proposition 4, p. 204]{BLR90} below:

$$0 \to \text{Pic}(k) \to \text{Pic}(X) \to \text{Pic}_{X/k}(k) \to \text{Br}(k) \quad \text{and} \quad 0 \to \text{Pic}_{X/k}^0 \to \text{Pic}_{X/k} \to \pi_0(\text{Pic}_{X/k}) \to 0.$$ 

Since $\text{Br}(k) = 0$, $H^1_{\text{et}}(\text{Spec } k, \text{Pic}_{X/k}^0) = H^1_{\text{et}}(\text{Spec } k, \text{Pic}_{X/k}^{\text{red},0})$ and $H^1_{\text{et}}(\text{Spec } k, A) = 0$ (Lang’s theorem \cite[p. 209]{Tat66}), this provides

\begin{equation}
\text{Pic}_{X/k}(k) = \text{Pic}(X) \quad \text{and} \quad NS(X) = NS_{X/k} = \frac{\text{Pic}(X)}{\mathcal{A}(k)}.
\end{equation}

Let $P$ be the identity component of the Picard scheme $\text{Pic}_{S/k}$ of $S$. Let $B$ be the cokernel of the natural injective map $\pi^*: P \to A$. So one has short exact sequences (using Lang’s theorem \cite[p. 209]{Tat66} for the last sequence)

\begin{equation}
A = \text{Pic}_{X/k}^{\text{red},0}, \quad P = \text{Pic}_{S/k}^0, \quad 0 \to P \to A \to B \to 0, \quad \text{and} \quad 0 \to P(k) \to A(k) \to B(k) \to 0.
\end{equation}

It is known \cite[p. 428]{Tat66} that

\begin{equation}
P_1(S, q^{-s}) = P_1(P, q^{-s}), \quad P_1(X, q^{-s}) = P_1(A, q^{-s}), \quad \text{and} \quad P_1(A, q^{-s}) = P_1(P, q^{-s}) \cdot P_1(B, q^{-s}).
\end{equation}

For any Abelian variety $G$ of dimension $d$ over $k = \mathbb{F}_q$, it is well known that \cite[p. 429, top line]{Tat66} (or \cite[61.3]{Gor79})

\begin{equation}
P_1(G, 1) = [G(k)] \quad \text{and} \quad P_1(G, q^{-1}) = [G(k)]q^{-d}.
\end{equation}

### 2.2. Comparison of $\chi(S, \text{Lie } J)$ and $\alpha(X)$

It is known \cite[p. 483]{LLR04} that

\begin{equation}
\chi(S, \text{Lie } J) - \dim(B) = -\alpha(X).
\end{equation}

We include their proof here for the convenience of the reader. A special case of this is due to Gordon \cite[Proposition 6.5]{Gor79}. The Leray spectral sequence for $\pi$ and $\mathcal{O}_X$ provides $H^0(\mathcal{O}_S) \cong H^0(X, \mathcal{O}_X)$,

$$0 \to H^1(S, \mathcal{O}_S) \to H^1(X, \mathcal{O}_X) \to H^0(S, R^1\pi_*\mathcal{O}_X) \to 0, \quad H^2(X, \mathcal{O}_X) \cong H^1(S, R^1\pi_*\mathcal{O}_X).$$

This proves $\chi(X, \mathcal{O}_X) = \chi(S, \mathcal{O}_S) - \chi(S, R^1\pi_*\mathcal{O}_X)$. Recall that $J$ is the Néron model of the Jacobian $J$ of $X_0$. As the kernel and cokernel of the natural map\footnote{The map $\phi$ is obtained by the composition of the maps $R^1\pi_*\mathcal{O}_X \to \text{Lie } P$ \cite[Proposition 1.3 (b)]{LLR04} and $\text{Lie } P \to \text{Lie } Q$ \cite[Theorem 3.1]{LLR04} with $Q \to J$ \cite[Facts 3.7 (a)]{LLR04}; it uses the fact that $X$ is regular, $\pi: X \to S$ is proper flat, and $\pi_*\mathcal{O}_X = \mathcal{O}_S$.} $\phi: R^1\pi_*\mathcal{O}_X \to \text{Lie } J$ are torsion sheaves on $S$ of the same length \cite[Theorem 4.2]{LLR04}, we have \cite[p. 483]{LLR04}

\begin{equation}
\chi(S, R^1\pi_*\mathcal{O}_X) = \chi(S, \text{Lie } J).
\end{equation}

Thus,

$$\alpha(X) = \chi(X, \mathcal{O}_X) + 1 + \dim(A) = \chi(S, \mathcal{O}_S) - \chi(S, R^1\pi_*\mathcal{O}_X) - 1 + \dim(A)$$

$$= 1 - \dim(P) - \chi(S, \text{Lie } J) - 1 + \dim(A) = -\chi(S, \text{Lie } J) + \dim(A) - \dim(P)$$

\begin{equation}
= -\chi(S, \text{Lie } J) + \dim(B).
\end{equation}

### 2.3. The Tate–Shioda relation about the Néron–Severi group

The structure of $\text{NS}(X)$ depends on the singular fibers of the morphism $\pi: X \to S$. 
2.3.1. Singular fibers.— Let $Z = \{ v \in S \mid \pi^{-1}(v) = X_v \text{ is not smooth}\}$. For any $v \in S$, let $G_v$ be the set of irreducible components $\Gamma_i$ of $X_v$, let $m_v$ be the cardinality of $G_v$, and $m := \sum_{v \in Z}(m_v - 1)$; for any $i \in G_v$, let $r_i$ be the number of irreducible components of $\Gamma_i \times \kappa(v)$. Let $R_v$ be the quotient

$$R_v = \frac{Z^{G_v}}{Z}$$

of the free Abelian group generated by the irreducible components of $X_v$ by the subgroup generated by the cycle associated with $X_v = \pi^{-1}(v)$. If $v \notin Z$, then $R_v$ is trivial.

Let $U = S - Z$; the map $X_{U} = \pi^{-1}(U) \to U$ is smooth. For any finite $Z' \subseteq S$ with $Z \subseteq Z'$, we consider $U' = S - Z'$ and $X_{U'} = X - \pi^{-1}(U')$. The following proposition provides a description of $\text{NS}(X) \cong \text{Pic}(X)/A(k)$.

**Proposition 2.1.**

(i) The natural maps $\pi^* : \text{Pic}(S) \to \text{Pic}(X)$ and $\pi^*: \text{Pic}(U') \to \text{Pic}(X_{U'})$ are injective.

(ii) There is an exact sequence

$$0 \to \bigoplus_{v \in Z} R_v \to \frac{\text{Pic}(X)}{\pi^* \text{Pic}(S)} \to \text{Pic}(X_0) \to 0. \tag{2.7}$$

**Proof.** (i) From the Leray spectral sequence for $\pi : X \to S$ and the étale sheaf $\mathbb{G}_m$ on $X$, we get the exact sequence

$$0 \to H^1_\text{ét}(S, \mathbb{G}_m) \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^0(S, R^1 \pi_* \mathbb{G}_m) \to \text{Br}(S).$$

Now $X_0$ being geometrically connected and smooth over $F$ implies [Mil81, Remark 1.7a] that $\pi_* \mathbb{G}_m$ is the sheaf $\mathbb{G}_m$ on $S$. This provides the injectivity of the first map. The same argument with $U'$ in place of $S$ provides the injectivity of the second.

(ii) The class group $\text{Cl}(Y)$ and the Picard group $\text{Pic}(Y)$ are isomorphic for regular schemes $Y$ such as $S$ and $X$. The localization sequences for $X_{U'} \subset X$ and $U' \subset S$ can be combined as

$$0 \to \Gamma(S, \mathbb{G}_m) \longrightarrow \Gamma(U', \mathbb{G}_m) \longrightarrow \bigoplus_{v \in Z'} \mathbb{Z} \longrightarrow \text{Pic}(S) \longrightarrow \text{Pic}(U') \longrightarrow 0$$

$$0 \to \Gamma(X, \mathbb{G}_m) \longrightarrow \Gamma(X_{U'}, \mathbb{G}_m) \longrightarrow \bigoplus_{v \in Z'} Z^{G_v} \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(X_{U'}) \longrightarrow 0.$$

Here $\Gamma(X, \mathbb{G}_m) = H^0_{\text{ét}}(X, \mathbb{G}_m) = H^0_{\text{Zar}}(X, \mathbb{G}_m)$. The induced exact sequence on the cokernels of the vertical maps is

$$0 \to \bigoplus_{v \in Z'} R_v \to \frac{\text{Pic}(X)}{\pi^* \text{Pic}(S)} \to \frac{\text{Pic}(X_{U'})}{\pi^* \text{Pic}(U')} \to 0.$$

In particular, we get this sequence for $Z$ and $U$. By assumption, $X_v$ is geometrically irreducible for any $v \notin Z$; so $R_v = 0$ for any $v \notin Z$. So this means that, for any $U' = S - Z'$ contained in $U$, the induced maps

$$\frac{\text{Pic}(X_{U'})}{\pi^* \text{Pic}(U')} \to \frac{\text{Pic}(X_{U'})}{\pi^* \text{Pic}(U')}$$

are isomorphisms. Taking the limit over $Z'$ gives us the exact sequence in the proposition. \hfill \square

**Corollary 2.2.**

(i) The Tate–Shioda relation [Tat66, (4.5)] $p(X) = 2 + r + m$ holds.

(ii) One has an exact sequence

$$0 \to B(k) \to \frac{\text{Pic}(X)}{\pi^* \text{Pic}(S)} \to \frac{\text{NS}(X)}{\pi^* \text{NS}(S)} \to 0.$$
Proof. (i) Since \( r \) is the rank of \( J(F) \), the rank of \( \text{Pic}(X_0) \) is \( r + 1 \). Since \( \text{Pic}(S) \) has rank one, \( A(k) \) is finite and \( m = \sum_{v \in \mathbb{Z}} (m_v - 1) \), this follows from (2.1) and (2.8).

(ii) This follows from the diagram

\[
\begin{array}{cccccc}
0 & \to & P(k) & \xrightarrow{\pi^*} & A(k) & \to & B(k) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Pic}(S) & \xrightarrow{\pi^*} & \text{Pic}(X) & \to & \text{Pic}(X) / \pi^* \text{Pic}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{NS}(S) & \xrightarrow{\pi^*} & \text{NS}(X) & \to & \text{NS}(X) / \pi^* \text{NS}(S) & \to & 0.
\end{array}
\]

\[\blacksquare\]

2.4. Relating the order of vanishing at \( s = 1 \) of \( P_2(X, q^{-s}) \) and \( L(J, s) \)

By \(^6\) [Gor79, Proposition 3.3], one has

\[
(2.9) \quad \zeta(X_v, s) = \frac{P_1(X_v, q_v^{-s})}{(1 - q_v^{-s})}, \quad \text{and} \quad P_2(X_v, q_v^{-s}) = \left\{ \begin{array}{ll} (1 - q_v^{1-s}), & \text{for } v \notin \mathbb{Z} \\ \prod_{i \in G_v} (1 - (q_v)^r(1-s)), & \text{for } v \in \mathbb{Z} \end{array} \right.,
\]

see §2.3.1 for notation. Using

\[
Q_2(s) = \prod_{v \in \mathbb{Z}} P_2(X_v, q_v^{-s}) = \frac{P_1(S, q^{-s})}{(1 - q^{-s})} \cdot (1 - q^{1-s}), \quad \zeta(S, s) = \frac{P_1(S, q^{-s})}{(1 - q^{-s})} \cdot (1 - q^{1-s}), \quad \text{and} \quad Q_1(s) = \prod_{v \in S} P_1(X_v, q_v^{-s}),
\]

we can rewrite

\[
\zeta(X, s) = \prod_{v \in S} \zeta(X_v, s) = \frac{1}{Q_2(s)} \cdot \prod_{v \in \mathbb{Z}} P_1(X_v, q_v^{-s}) = \frac{\zeta(S, s) \cdot \zeta(S, s-1) \cdot Q_1(s)}{Q_2(s)}.
\]

The precise relation between \( P_2(X, q^{-s}) \) and \( L(J, s) \) is given by (2.11).

**Proposition 2.3.** One has \( \text{ord}_{s=1} Q_2(s) = m \) and

\[
(2.10) \quad Q_2(1) = \lim_{s \to 1} \frac{Q_2(s)}{(s-1)^m} = \prod_{v \in \mathbb{Z}} \left( \log q_v \right)^{(m_v - 1)} \prod_{i \in G_v} r_i,
\]

\[
(2.11) \quad P_2(X, q^{-s}) = \prod_{v \in \mathbb{Z}} \frac{P_1(B, q^{-s})}{(1 - q_v^{-s})^2} = P_1(B, q^{-s}) \cdot P_1(B, q^{1-s}) \cdot L(J, s) \cdot Q_2(s).
\]

**Proof.** Observe that (2.10) is elementary: for any positive integer \( r \), one has

\[
\lim_{s \to 1} \frac{(1 - q_v^{r(1-s)})}{(s-1)} = \lim_{s \to 1} \frac{(1 - q_v^{r(1-s)})}{(1 - q_v^{1-s})} \cdot \frac{(1 - q_v^{1-s})}{(s-1)} = \lim_{s \to 1} (1 + q_v^{1-s} + \cdots + q_v^{r(1-s)}) \cdot \log q_v = r \cdot \log q_v.
\]

For each \( v \in \mathbb{Z} \), this shows that

\[
\lim_{s \to 1} \frac{P_2(X_v, q^{-s})}{(s-1)^m_v} = (\log q_v)^{m_v} \prod_{i \in G_v} r_i.
\]

Therefore, we obtain that

\[
\lim_{s \to 1} \frac{Q_2(s)}{(s-1)^m} = \prod_{v \in \mathbb{Z}} \lim_{s \to 1} \frac{P_2(X_v, q^{-s})}{(1 - q_v^{-s})^{m_v - 1}} = \prod_{v \in \mathbb{Z}} \lim_{s \to 1} \frac{P_2(X_v, q^{-s})}{(s-1)^{m_v}} = \prod_{v \in \mathbb{Z}} \left( (\log q_v)^{m_v} \prod_{i \in G_v} r_i \right).
\]

\(^6\)This proposition, first stated on Page 176 of [Gor79], has a typo in the formula for \( P_2 \) which is corrected in its restatement on Page 193. We only need the part about \( P_2 \) (and this is elementary).
We now prove (2.11). Simplifying the identity
\[
\frac{P_1(X,q^{-s}) \cdot P_1(X,q^{1-s})}{(1-q^{-s}) \cdot (1-q^{1-s})} = \frac{P_1(S,q^{-s}) \cdot P_1(S,q^{1-s})}{(1-q^{-s}) \cdot (1-q^{1-s})} \cdot \frac{Q_1(s)}{Q_2(s)}
\]
from (1.2) using (2.3), one obtains
\[
\frac{P_1(B,q^{-s}) \cdot P_1(B,q^{1-s})}{P_2(X,q^{-s})} = \frac{1}{(1-q^{1-s})} \cdot \frac{Q_1(s)}{Q_2(s)}.
\]
On reordering, this becomes
\[
\frac{P_2(X,q^{-s})}{(1-q^{1-s})^2} = \frac{P_1(B,q^{-s}) \cdot P_1(B,q^{1-s}) \cdot Q_2(s)}{Q_1(s)}.
\]
Let $T/J$ be the $\ell$-adic Tate module of the Jacobian $J$ of $X$. For any $v \in S$, the Kummer sequence on $X$ and $J$ provides a $\text{Gal}(F_v^e/F)$-equivariant isomorphism
\[
H^1_{\text{ét}}(X \times_S F_v^e, Z_\ell(1)) \xrightarrow{\sim} T/J \xleftarrow{\sim} H^1_{\text{ét}}(J \times_S F_v^e, Z_\ell(1)),
\]
as $J$ is a self-dual Abelian variety: this provides the isomorphisms
\[
H^1_{\text{ét}}(J \times_S F_v^e, Q_\ell) \equiv H^1_{\text{ét}}(X \times_S F_v^e, Q_\ell), \quad H^1_{\text{ét}}(J \times_S F_v^e, Q_\ell)^L \equiv H^1_{\text{ét}}(X \times_S F_v^e, Q_\ell)^L.
\]
From [Del80, Théorème 3.6.1, pp.213–214] (the arithmetic case is in [Blo87, Lemma 1.2]), we obtain an isomorphism
\[
H^1_{\text{ét}}(X \times_S F_v^e, Q_\ell) \xrightarrow{\sim} H^1_{\text{ét}}(X \times_S F_v^e, Q_\ell)^L.
\]
The definition of $L_v(J,t)$ in (1.8) now implies that $P_1(X_v,q_v^{-s}) = L_v(J,q_v^{-s})$ and hence $Q_1(s) \cdot L(J,s) = 1$. \hfill \qed

**Proposition 2.4.**

(i) $\text{ord}_{s=1} P_2(X,q^{-s}) = \rho(X)$ if and only if $\text{ord}_{s=1} L(J,s) = r$.

(ii) One has
\[
P_2^*(X, \frac{1}{q}) = \frac{P_1(B,q^{-1}) \cdot P_1(B,1) \cdot L^*(J,1) \cdot Q_2^*(1) \cdot (\log q)^2}{(2.4) \cdot [B(k)]^2 \cdot q^{\dim(B)} \cdot L^*(J,1) \cdot Q_2^*(1) \cdot (\log q)^2}.
\]

**Proof.** As $P_1(B,q^{-s}) \cdot P_1(B,q^{1-s})$ does not vanish at $s = 1$ by (2.4), it follows from (2.11) that
\[
\text{ord}_{s=1} P_2(X,q^{-s}) - 2 = \text{ord}_{s=1} L(J,s) + \text{ord}_{s=1} Q_2(s).
\]
Corollary 2.2 says $\rho(X) = r + m + 2$; (i) follows as $\text{ord}_{s=1} Q_2(s) = m$.

For (ii), use (2.4) and (2.11). \hfill \qed

### 2.5. Pairings on $\text{NS}(X)$

Our next task is to compute $\Delta(\text{NS}(X))$.

**Definition 2.5.**

(i) Let $\text{Pic}^0(X_0)$ be the kernel of the degree map $\text{deg} : \text{Pic}(X_0) \rightarrow \mathbb{Z}$; the order $\delta$ of its cokernel is, by definition, the index of $X_0$ over $F$.

(ii) Let $\alpha$ be the order of the cokernel of the natural map $\text{Pic}^0(X_0) \hookrightarrow \text{Jac}(F)$.

(iii) Let $H$ (horizontal divisor on $X$) be the Zariski closure in $X$ of a divisor $d$ on $X_0$, rational over $F$, of degree $\delta$.

(iv) The (vertical) divisor $V$ on $X$ is $\pi^{-1}(s)$ for a divisor $s$ of degree one on $S$. Such a divisor $s$ exists as $k$ is a finite field and so the index of the curve $S$ over $k$ is one. Writing $s = \sum a_i v_i$ as a sum of closed points $v_i$ on $S$ gives $V = \sum a_i \pi^{-1}(v_i)$. Note that $V$ generates $\pi^*\text{NS}(S) \subset \text{NS}(X)$. 


Remark. The definitions show that the intersections of the divisor classes \( H \) and \( V \) in \( \text{NS}(X) \) are given by

\[
H \cdot V = \delta = V \cdot H \quad \text{and} \quad V \cdot V = 0.
\]

Also, since \( \pi : X \to S \) is a flat map between smooth schemes, the map \( \pi^* : \text{CH}(S) \to \text{CH}(X) \) on Chow groups is compatible with intersection of cycles. Since \( V = \pi^*(s) \) and the intersection \( s \cdot s = 0 \) in \( \text{CH}(S) \), one has \( V \cdot V = 0 \).

Let \( \text{NS}(X)_0 = (\pi^* \text{NS}(S))^{1/2} \); as \( V \) generates \( \pi^* \text{NS}(S) \), we see that \( \text{NS}(X)_0 \) is the subgroup of divisor classes \( Y \) such that \( Y \cdot X_v = 0 \) for any fiber \( \pi^{-1}(v) = X_v \) of \( \pi \); let \( \text{Pic}(X)_0 \) be the inverse image of \( \text{NS}(X)_0 \) under the projection \( \text{Pic}(X) \to \text{NS}(X) \equiv \frac{\text{Pic}(X)}{\text{A}(k)} \).

**Lemma 2.6.** \( \text{NS}(X)_0 \) is the subgroup of \( \text{NS}(X) \) generated by divisor classes whose restriction to \( X_0 \) is trivial.

**Proof.** We need to show that \( \text{NS}(X)_0 \) is equal to \( K : = \text{Ker}(\text{NS}(X) \to \text{NS}(X_0)) \). If \( D \) is a vertical divisor \( (\pi(D) \subset S) \) is finite), then \( D \) is clearly in \( K \); by [Liu02, §9.1, Proposition 1.21], \( D \) is in \( \text{NS}(X)_0 \).

If \( D \) has no vertical components, then \( D \cdot V = \deg(D_0) \). To see this, clearly we may assume \( D \) is reduced and irreducible (integral) and so flat over \( S \). So \( \mathcal{O}_D \) is locally free over \( \mathcal{O}_S \) of constant degree \( n \) since \( S \) is connected. But then \( \deg(D_0) \) is equal to \( n \) as is the integer \( D \cdot V \).

**Lemma 2.7.** Let us denote

\[
R = \bigoplus_{v \in Z} R_v \quad \text{and} \quad E = B(k) \cap R \subset \frac{\text{Pic}(X)_0}{\pi^* \text{Pic}(S)}.
\]

One has the exact sequences

\[
0 \to R \to \frac{\text{Pic}(X)_0}{\pi^* \text{Pic}(S)} \to \frac{\text{Pic}^0(X_0)}{\pi^* \text{Pic}(S)} \to 0, \quad \text{and}
\]

\[
0 \to R/E \to \frac{\text{NS}(X)_0}{\pi^* \text{NS}(S)} \to \frac{\text{Pic}^0(X_0)}{B(k)/E} \to 0.
\]

**Proof.** Lemma 2.6 shows that \( R \subset \frac{\text{Pic}(X)_0}{\pi^* \text{Pic}(S)} \). As \( A(k) \) is the kernel of the map \( \text{Pic}(X) \to \text{NS}(X) \), it follows that \( A(k) \subset \text{Pic}(X)_0 \). Thus, \( B(k) \) is a subgroup of \( \frac{\text{Pic}(X)_0}{\pi^* \text{Pic}(S)} \).

The first exact sequence follows from Lemma 2.6; the second one follows from Corollary 2.2 (ii).

**Lemma 2.8.** One has the equality

\[
\Delta_{\text{ar}} \left( \frac{\text{NS}(X)_0}{\pi^* \text{NS}(S)} \right) = [B(k)]^2 \cdot \alpha^2 \cdot \Delta_{\text{NT}}(\text{I}(F)) \cdot \prod_{v \in Z} \Delta_{\text{ar}}(R_v).
\]

**Proof.** The exact sequence (2.14) splits orthogonally over \( \mathbb{Q} \); for any divisor \( \gamma \) representing an element of \( \text{Pic}(X_0) \), consider its Zariski closure \( \tilde{\gamma} \) in \( X \). Since the intersection pairing on \( R_v \) is negative-definite [Liu02, §9.1, Theorem 1.23], the linear map \( R_v \to \mathbb{Z} \) defined by \( \beta \mapsto \beta \cdot \tilde{\gamma} \) is represented by a unique element

\[
\psi_v(\gamma) \in R_v \otimes \mathbb{Q} \subset \frac{\text{NS}(X)_0}{\pi^* \text{NS}(S)} \otimes \mathbb{Q}.
\]

Thus, the element

\[
\tilde{\gamma} := \tilde{\gamma} - \sum_{v \in Z} \psi_v(\gamma)
\]

is good in the sense of [Gor79, §5, p. 185]; by construction, the divisor \( \tilde{\gamma} \) on \( X \) intersects every irreducible component of every fiber of \( \pi \) with multiplicity zero. Fix \( \gamma, \kappa \in \text{Pic}^0(X_0) \); viewing them as elements of \( \text{I}(F) \), one computes their Neron–Tate pairing (1.9); also, one can compute the height pairing of \( \tilde{\gamma} \) and \( \tilde{\kappa} \) in \( \text{NS}(X) \). These two are related by the identity [Tat66, p. 429] [LLR18, Remark 3.11]

\[
\langle \gamma, \kappa \rangle_{\text{NT}} = -\langle \tilde{\gamma}, \tilde{\kappa} \rangle_{\text{ar}} = -\langle \tilde{\gamma} \cdot \tilde{\kappa} \rangle \cdot \log q.
\]
This says that
\begin{equation}
\Delta_{\text{ar}}(\text{Pic}^0(X_0)) = \Delta_{\text{NT}}(\text{Pic}^0(X_0)).
\end{equation}

The map
\[
\text{Pic}^0(X_0) \otimes \mathbb{Q} \to \frac{\text{NS}(X_0)}{\pi^* \text{NS}(S)} \otimes \mathbb{Q}, \quad \gamma \mapsto \tilde{\gamma}
\]
provides an orthogonal splitting of (2.14) (over \(\mathbb{Q}\)). So
\[
\Delta_{\text{ar}}\left(\frac{\text{NS}(X_0)}{\pi^* \text{NS}(S)}\right) = \Delta_{\text{ar}}\left(\text{Pic}^0(X_0)\right) \cdot \Delta_{\text{ar}}(R) = \left[\frac{[B(k)]^2}{e^2}\right] \Delta_{\text{ar}}\left(\text{Pic}^0(X_0)\right) \cdot \Delta_{\text{ar}}(R)
\]
where \(e = |E|\) as the size of \(E\). As
\begin{equation}
\Delta_{\text{NT}}(\text{Pic}^0(X_0)) = \alpha^2 \cdot \Delta_{\text{NT}}(J(F)) \quad \text{and} \quad \Delta_{\text{ar}}(R) = \prod_{v \in Z} \Delta_{\text{ar}}(R_v),
\end{equation}
this proves the lemma.

With Lemma 2.8 at hand we are almost ready to compute \(\Delta_{\text{ar}}(\text{NS}(X))\). As the intersection pairing on \(\text{NS}(X)\) is not definite (Hodge index theorem), we cannot apply (1.5). Instead, we use a variant of a lemma of Z. Yun [Yun15].

2.5.1. A lemma of Yun. — Given a non-degenerate symmetric bilinear pairing \(\Lambda \times \Lambda \to \mathbb{Z}\) on a finitely generated Abelian group \(\Lambda\), an isotropic subgroup \(\Gamma\), a subgroup \(\Gamma'\) containing \(\Gamma\) and with finite index in \(\Gamma',\) let \(\Lambda_0 = \frac{\Lambda}{\Gamma}\). We recall from §1.4 that \(\Delta(\Lambda) = z(D)^{-1}\) where \(D := \Lambda \to \text{RHom}(\Lambda, \mathbb{Z})\) and \(\Delta(\Lambda_0) = z(D_0)^{-1}\) where \(D_0 := \Lambda_0 \to \text{RHom}(\Lambda_0, \mathbb{Z})\). Let \(\Delta\) be the discriminant of the induced non-degenerate pairing \(\Gamma' \times \frac{\Lambda}{\Gamma'} \to \mathbb{Z}\):

\[
\Delta = \frac{1}{z(C)} = \frac{1}{z(C')}, \quad C := \Gamma \to \text{RHom}\left(\frac{\Lambda}{\Gamma'}, \mathbb{Z}\right), \quad \text{and} \quad C' := \frac{\Lambda}{\Gamma'} \to \text{RHom}(\Gamma, \mathbb{Z}).
\]

Lemma 2.9 (cf. [Yun15, Lemma 2.12]). One has \(\Delta(\Lambda) = \Delta(\Lambda_0) \cdot \Delta^2\).

Proof. Applying (1.6) to the maps of triangles
\[
\begin{array}{ccccccc}
\Gamma & \longrightarrow & \Lambda & \longrightarrow & \frac{\Lambda}{\Gamma} & \longrightarrow & \Gamma[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{RHom}\left(\frac{\Lambda}{\Gamma}, \mathbb{Z}\right) & \longrightarrow & \text{RHom}(\Lambda, \mathbb{Z}) & \longrightarrow & \text{RHom}(\Gamma, \mathbb{Z}) & \longrightarrow & \text{RHom}\left(\frac{\Lambda}{\Gamma}, \mathbb{Z}\right)[1]
\end{array}
\]
and
\[
\begin{array}{ccccccc}
\Gamma' & \longrightarrow & \frac{\Lambda}{\Gamma'} & \longrightarrow & \frac{\Lambda}{\Gamma'} & \longrightarrow & \frac{\Gamma'}{\Gamma'}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{RHom}\left(\frac{\Gamma'}{\Gamma'}, \mathbb{Z}\right) & \longrightarrow & \text{RHom}(\Gamma', \mathbb{Z}) & \longrightarrow & \text{RHom}(\Gamma, \mathbb{Z}) & \longrightarrow & \text{RHom}\left(\frac{\Gamma'}{\Gamma'}, \mathbb{Z}\right)[1]
\end{array}
\]
shows that \(z(D) \cdot z(C)^{-1} = z(D_0) \cdot z(C')\).

We can finally compute \(\Delta_{\text{ar}}(\text{NS}(X))\).

Proposition 2.10. The following relations hold
\[
\Delta_{\text{ar}}(\text{NS}(X)) = \delta^2 \cdot \Delta_{\text{ar}}\left(\frac{\text{NS}(X_0)}{\pi^* \text{NS}(S)}\right) \cdot (\log q)^2 \quad \text{and} \quad \Delta(\text{NS}(X)) = \delta^2 \cdot \Delta\left(\frac{\text{NS}(X_0)}{\pi^* \text{NS}(S)}\right).
\]
**Proof.** Let \( \Lambda \cong \pi^* \text{NS}(S) \subset \text{NS}(X) = \Lambda \) with \( \Gamma = \text{NS}(X)_{0} \) and \( \Lambda_0 = \frac{\text{NS}(X)_0}{\pi^* \text{NS}(S)} \). Lemma 2.6 implies that

\[
\Delta_{\text{tor}} = \frac{\text{NS}(X)}{\text{NS}(X)_0} \cong \mathbb{Z} \quad \text{and} \quad C = \Gamma \to \text{Hom}\left( \frac{\text{NS}(X)}{\text{NS}(X)_0}, \mathbb{Z} \right),
\]

with \( C \) as in Lemma 2.9. Now (2.13) shows that \( \pi^* \text{NS}(S) \) is isotropic and \( \Delta = \delta \). The result follows from Lemma 2.9.

Combining the previous proposition with Lemma 2.8 provides the identity

\[
\Delta_{\text{tor}}(\text{NS}(X)) = \delta^2 \cdot [B(k)]^2 \cdot \alpha^2 \cdot \Delta_{\text{NT}}(J(F)) \cdot \prod_{v \in \mathbb{Z}} \Delta_{\text{tor}}(R_v) \cdot (\log q)^2.
\]

For \( v \in S \), we put \( \delta_v \) and \( \delta_v' \) for the (local) index and period of \( X \times F_v \) over the local field \( F_v \).

**Theorem 2.11.** [Gei20, Theorem 1.1] Assume that \( \text{Br}(X) \) is finite. The following equality holds:

\[
[\text{Br}(X)] \alpha^2 \delta^2 = [\text{III}(J/F)] \prod_{v \in S} \delta_v' \delta_v.
\]

**Remark 2.12.** Note that for \( v \in U \), one has \( \delta_v = 1 = \delta_v' \) [LLR18, p. 603], [FS21, (74)] (for \( \delta_v = 1 \), [Gro68, Proposition (4.1) (a)] \( \delta_v' \) divides \( \delta_v \); the basic reason is that if \( v \in U \), then \( X_v \) has a rational divisor of degree one as \( k(v) \) is finite; this divisor lifts to a rational divisor of degree one on \( X \times F_v \) by smoothness of \( X_v \). Also, \( c_v = 1 \) [BLR90, Theorem 1, §9.5 p. 264]. So \( c(J) := \prod_{v \in S} c_v \) satisfies

\[
c(J) = \prod_{v \in \mathbb{Z}} c_v.
\]

**Lemma 2.13.** One has

\[
c(J) \cdot Q_2^*(1) = \prod_{v \in \mathbb{Z}} \delta_v' \delta_v \cdot \Delta_{\text{tor}}(R_v).
\]

**Proof.** By a result of Flach and Siebel [FS21, Lemma 17] (using Raynaud’s theorem [Gor79, Theorem 5.2] in [BL99]), one has

\[
\Delta_{\text{tor}}(R_v) = \frac{c_v}{\delta_v \delta_v'} \cdot (\log q_v)^{-1} \cdot \prod_{i \in G_v} r_i.
\]

So we find that

\[
\prod_{v \in \mathbb{Z}} \delta_v' \delta_v \cdot \Delta_{\text{tor}}(R_v) = \prod_{v \in \mathbb{Z}} \left( \frac{c_v}{\delta_v \delta_v'} \cdot (\log q_v)^{-1} \cdot \prod_{i \in G_v} r_i \right)
\]

\[
= \prod_{v \in \mathbb{Z}} c_v \cdot (\log q_v)^{-1} \cdot \prod_{i \in G_v} r_i
\]

\[
= c(J) \cdot Q_2^*(1).
\]

\[
(\text{2.19})
\]

**3. First proof of Theorem 1.1**

**Proof of Theorem 1.1.** By (2.17) and (2.20), we have

\[
\Delta_{\text{tor}}(\text{NS}(X)) = \frac{\alpha^2 \delta^2}{\prod_{v \in \mathbb{Z}} \delta_v' \delta_v} \cdot \Delta_{\text{NT}}(J(F)) \cdot c(J) \cdot [B(k)]^2 \cdot Q_2^*(1) \cdot (\log q)^2.
\]

From Theorem 2.11, we have

\[
[\text{Br}(X)] \cdot \Delta_{\text{tor}}(\text{NS}(X)) = [\text{III}(J/F)] \cdot \Delta_{\text{NT}}(J(F)) \cdot c(J) \cdot [B(k)]^2 \cdot Q_2^*(1) \cdot (\log q)^2.
\]
Further with (2.5), we obtain
\[
[\text{Br}(X)] \cdot \Delta_{\text{Ar}}(\text{NS}(X)) \cdot q^{-\alpha(X)} = [\text{III}(J/F)] \cdot \Delta_{\text{NT}}(J(F)) \cdot c(J) \cdot q^{\chi(S,\text{Lie } J)} \cdot [B(k)]^2 \cdot Q^*_2(1) \cdot q^{-\dim(B)} \cdot (\log q)^2.
\]
On the other hand, recall (2.12)
\[
P^*_2(X, \frac{1}{q}) = L^*(J, 1) \cdot [B(k)]^2 \cdot Q^*_2(1) \cdot q^{-\dim(B)} \cdot (\log q)^2.
\]
The ratio of the previous two equalities gives
\[
\frac{P^*_2(X, \frac{1}{q})}{[\text{Br}(X)] \cdot \Delta_{\text{Ar}}(\text{NS}(X)) \cdot q^{-\alpha(X)}} = \frac{L^*(J, 1)}{[\text{III}(J/F)] \cdot \Delta_{\text{NT}}(J(F)) \cdot c(J) \cdot q^{\chi(S,\text{Lie } J)}}.
\]
This equality implies Theorem 1.1. □

4. Second proof of Theorem 1.1

We will give another more direct proof of Theorem 1.1 using Weil-étale cohomology. We refer the reader to [Lic05, Gei04, GS20] for basics about Weil-étale cohomology over finite fields. Throughout this section, we assume that Br(X) (and hence III(J/F)) is finite.

4.1. Setup

Let C ∈ D^b(T_{et}) be an object of the bounded derived category of sheaves of Abelian groups on the small étale site T_{et}. Let D ∈ D^b(\text{FDVect}_k) be an object of the bounded derived category of finite-dimensional vector spaces over k. Assume that the Weil-étale cohomology H^i_W(T, C) is finitely generated and the cohomology sheaf H^*(C \otimes^L \mathbb{Z}/l^n\mathbb{Z}) is finite in all degrees for all prime numbers l \mid q. Let ε: H^i_W(T, C) → H^{i+1}_W(T, C) be the map defined by cup product with the arithmetic Frobenius ε \in H^1_W(T, \mathbb{Z}). It defines a complex
\[
\cdots \to H^i_W(T, C) \xrightarrow{\epsilon} H^{i+1}_W(T, C) \xrightarrow{\epsilon} \cdots
\]
with finite cohomology. Set C_{Q_l} = \lim_n (C \otimes^L \mathbb{Z}/l^n\mathbb{Z}) \otimes_{\mathbb{Z}_l} Q_l, whose cohomologies are finite-dimensional vector spaces over Q_l (by the finiteness of H^*(C \otimes^L \mathbb{Z}/l^n\mathbb{Z})) equipped with an action of the geometric Frobenius ε of k. Define
\[
Z(C, t) = \prod_i \det(1 - \varphi t | H^i_W(C_{Q_l}))^{(-1)^{i+1}},
\]
\[
\rho(C) = \sum_j (-1)^{i+1} \cdot j \cdot \text{rank } H^i_W(T, C),
\]
\[
\chi_W(C) = \chi(H^*_W(T, C), \epsilon), \quad \text{and}
\]
\[
\chi(D) = \sum_j (-1)^j \dim H^j(D).
\]
Assume that Z(C, t) ∈ Q(t) and is independent of l. Define Q(C, D) ∈ Q_{>0} × (1 - t)^Z to be the leading term of the (1 - t)-adic expansion of the function
\[
\pm \frac{Z(C, t)(1 - t)^{\rho(C)}}{\chi_W(C)q^{\chi(D)}}
\]
(the sign is the one that makes the coefficient positive). It is the defect of a zeta value formula of the form
\[
\lim_{t \to 1} Z(C, t)(1 - t)^{\rho(C)} = \pm \chi_W(C)q^{\chi(D)}.
\]
We mention $Q(C, D)$ only when $H^i_W(T, C)$ is finitely generated, $H^*(C \otimes Z/IZ)$ is finite and $Z(t) \in \mathbb{Q}(t)$ is independent of $l$. These conditions are satisfied for the cases of interest below. We have
\[ Q(C[1], D[1]) = Q(C, D)^{-1}. \]

If $(C, D)$, $(C', D')$ and $(C'', D'')$ are pairs as above, and $C \to C' \to C'' \to C[1]$ and $D \to D' \to D'' \to D[1]$ are distinguished triangles, then $Q(C', D') = Q(C, D)Q(C'', D'')$.

### 4.2. Special cases

We give two special cases of the above constructions. First, let $\pi_X : X_{\text{et}} \to T_{\text{et}}$ be the structure morphism. Let $P^s_2(X, 1)(1-t)^{\rho(X)}$ be the leading term of the $(1-t)$-adic expansion of $P_2(X, t/q)$.

**Proposition 4.1.** Let $(C, D) = (R\pi_X, G_m[-1], R\Gamma(X, O_X))$. Then $H^*(C \otimes Z/IZ)$ is finite, $H^*_W(T, C)$ is finitely generated, $Z, q^{-s}) = \zeta(X, s + 1)$ and
\[ Q(C, D)^{-1} = \frac{P^s_2(X, 1) \cdot (1-t)^{\rho(X)}}{\left|\text{Br}(X)\right| \cdot \Delta(\text{NS}(X)) \cdot q^{-s}(X)}. \]

In particular, the statement $Q(C, D) = 1$ is equivalent to Conjecture 1.2.

**Proof.** We have $H^*_W(T, C) \cong H^*_W(X, G_m[-1]) \cong H^*_W(X, Z(1))$. The finiteness assumption on $\text{Br}(X)$ implies the Tate conjecture for divisors on $X$ and hence the finite generation of $H^*_W(X, Z(1))$ by [Gei04, Theorems 8.4 and 9.3]. The object $C \otimes Z/IZ \cong R\pi_X, Z/IZ(1) \in D^b(T_{\text{et}})$ is constructible and hence its cohomologies are finite. We have $H^i(C, Q) \cong R^i\pi_X, Q-t(1)$, which is the vector space $H^i_{\text{et}}(X \times \k, Q_t(1))$ equipped with the natural Frobenius action. It follows that $Z, q^{-s}) = \zeta(X, s + 1)$.

We calculate $Q(C, D)^{-1}$. By (1.2), (2.3) and (2.4), the leading term of the $(1-t)$-adic expansion of $Z(C, t)$ is
\[ (4.1) \quad \frac{[A(k)]^2}{P^s_2(X, 1) \cdot (q - 1)^2 \cdot q^{\dim A - 1} \cdot (1-t)^{\rho(X)}}. \]

By [Gei04, Theorems 7.5 and 9.1], we have
\[ \chi_W(C) = \prod_i [H^i_W(X, Z(1))_{\text{tor}}]^{-1} \cdot R^{-1}, \]
where $R$ is the determinant of the pairing
\[ H^2_W(X, Z(1)) \times H^2_W(X, Z(1)) \to H^4_W(X, Z(2)) \to H^4_W(X \times \k, Z(2)) \cong \text{CH}^2(X \times \k) \to Z. \]

We have $H^i_W(X, Z(1)) = 0$ for $n > 5$ by [Gei04, Theorem 7.3] and for $n < 1$ obviously. Also
\[ H^i_W(X, Z(1)) \cong k^*, \quad H^i_W(X, Z(1)) \cong \text{Pic}(X), \quad \text{and} \quad H^i_W(X, Z(1))_{\text{tor}} \cong \text{Br}(X) \]
by [Gei04, Proposition 7.4 (c) and (d)]. By [Gei18, Remark 3.3], the group $H^i_W(X, Z(1))_{\text{tor}}$ is Pontryagin dual to $H^{6-i}_W(X, Z(1))_{\text{tor}}$ for any $i$. The above pairing defining $R$ can be identified with the intersection pairing $\text{Pic}(X) \times \text{Pic}(X) \to Z$. Thus, with (2.1), we have
\[ (4.2) \quad \chi_W(C) = \frac{[A(k)]^2}{\left|\text{Br}(X)\right| \cdot \Delta(\text{NS}(X)) \cdot (q - 1)^2}. \]

Since the rank of $H^i_W(X, Z(1))$ is $\rho(X)$ for $i = 2, 3$ and zero otherwise by [Gei04, Proposition 7.4 (c) and (d)], we have
\[ (4.3) \quad \rho(C) = \rho(X). \]
Combining (1.3), (4.1), (4.2) and (4.3), we get the desired formula for $Q(C, D)^{-1}$. \qed
Next, let $\pi_S : S_{\text{et}} \to T_{\text{et}}$ be the structure morphism. Let $L^\circ(J, 1)(1 - q^{-s})^r$ be the leading term of the $(1 - q^{-s})$-adic expansion of $L(J, s + 1)$. Let $\Delta(J(F))$ be the discriminant of the pairing $\langle \gamma, \kappa \rangle \mapsto \langle \gamma, \kappa \rangle_{NT}/\log q$ on $J(F)$.

**Proposition 4.2.** Let $(C, D) = \left(\varinjlim_{\dagger} \varprojlim_{\dagger} \mathbb{Q}(C, q^{-s})\right)$. Then $H^*(C \otimes \mathbb{Q}(C, q^{-s}))$ is finite, $H^W(T, C)$ is finitely generated, $Z(C, q^{-s}) = L(J, s + 1)$ and

$$Q(C, D) = \frac{L^\circ(J, 1)(1 - t)^r}{[\Pi(J/F)] \cdot \Delta(J(F)) \cdot c(J) \cdot q^g(S, \text{Lie} J)}.$$ 

In particular, the statement $Q(C, D) = 1$ is equivalent to Conjecture 1.3.

**Proof.** We have $H^W(T, C) \cong H^W(S, J)$. The finiteness assumption of $\Pi(J/F)$ implies the finite generation of $H^W(S, J)$ by [GS20, Proposition 6.4]. We have $C \otimes \mathbb{Z}/l\mathbb{Z} = \varprojlim_{\dagger} \mathbb{Q}(C, q^{-s})\otimes \mathbb{Z}/l\mathbb{Z}[1]$. By the paragraph before the proof of [GS20, Proposition 9.2] and the first displayed equation in the proof of [GS20, Proposition 9.2], we know that $J \otimes \mathbb{Z}/l\mathbb{Z} \in D^b(S_{\text{et}})$ is constructible. Hence $H^i(C \otimes \mathbb{Z}/l\mathbb{Z})$ is finite. We also have $H^i(C_{\text{et}}) \cong R^i\pi_{S, s}V_i(J)$ (where $V_i$ is the $i$-adic Tate modules tensored with $\mathbb{Q}_l$), which is the vector space $H^i_{\text{et}}(S \times_{\mathbb{Z}} k, V_i(J))$ equipped with the natural Frobenius action. Hence we have $Z(C, q^{-s}) = L(J, s + 1)$ by [Sch82, Satz 1]. We have

$$\chi_W(C) = [\Pi(J/F)] \cdot \Delta(J(F)) \cdot c(J)$$

by [GS20, Proposition 8.3]. By [GS20, Proposition 7.1], the rank of $H^i_{\text{et}}(S, J)$ is $r$ for $i = 0, 1$ and zero otherwise. Hence $\rho(C) = -r$. The formula for $Q(C, D)$ follows. \(\square\)

### 4.3. Comparison

Now Theorem 11 follows from the following

**Proposition 4.3.** One has

$$Q(R\pi_{X, G_m}[1], \Gamma(X, \mathcal{O}_X))^{-1} = Q(R\pi_{S, s}J[-1], \Gamma(S, \text{Lie} J)).$$

**Proof.** We have $R^i\pi_{\text{et}}G_m = 0$ over $S_{\text{et}}$ for all $i \geq 2$ by [Gro68, Corollaire (3.2)]. Hence we have a distinguished triangle

$$R\pi_{S, s}G_m \to R\pi_{X, s}G_m \to R\pi_{S, s}\text{Pic}_{X/S}[1] \to R\pi_{S, s}G_m[1]$$

in $D(T_{\text{et}})$. Similarly, we have a distinguished triangle

$$R\Gamma(S, \mathcal{O}_S) \to R\Gamma(X, \mathcal{O}_X) \to R\Gamma(S, R^1\pi_{\text{et}}\mathcal{O}_X)[-1] \to R\Gamma(S, \mathcal{O}_S)[1].$$

We have $Q(R\pi_{S, s}G_m[-1], R\Gamma(S, \mathcal{O}_S)) = 1$ by the class number formula ([Gei04, Theorems 9.1 and 9.3], or [Liu05, Theorems 5.4 and 7.4] and the functional equation). Therefore

$$Q(R\pi_{S, s}G_m[-1], R\Gamma(X, \mathcal{O}_X))^{-1} = Q(R\pi_{S, s}\text{Pic}_{X/S}[1], R\Gamma(S, R^1\pi_{\text{et}}\mathcal{O}_X)).$$

For a closed point $v \in S$, let $i_v : \text{Spec } k(v) \hookrightarrow S$ be the inclusion. For any $i \in G_v$, let $k(v)_i$ be the algebraic closure of $k(v)$ in the function field of $\Gamma_i$. Let $i_{v,i} : \text{Spec } k(v)_i \to S$ be the natural morphism. Set

$$E = \bigoplus_{v \in \mathcal{Z}} \bigoplus_{i \in G_v} i_{v,i}^*\mathcal{Z}.$$ 

Let $j : \text{Spec } F \hookrightarrow S$ be the inclusion. Then we have a natural exact sequence

$$0 \to E \to \text{Pic}_{X/S} \to j_!\text{Pic}_{X/F} \to 0.$$ 

---

\[\text{Here Pic}_{X/F} = R^1\pi_{\text{et}}G_m \text{ is only an étale sheaf. The fppf sheaf denoted by the same symbol is not an algebraic space in general.}\]
over $S_{\etale}$ by [Gro68, Equations (4.10 bis) and (4.21)] (where the assumption [Gro68, Equation (4.13)] is satisfied since $k(v)$ is finite and hence perfect for all closed $v \in S$). Therefore we have a distinguished triangle

$$R\pi_{S*}E \to R\pi_{S*}j_! \to R\pi_{S*}E[1].$$

Since $E$ is skyscraper, we have $Q(R\pi_{S*}E,0) = 1$ by [GS21, Theorem 3.1] (Step 3 of the proof is sufficient). Therefore

$$(4.5) \quad Q(R\pi_{S*} j_! \Pic_{X/S}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)) = Q(R\pi_{S*}j_! \Pic_{X_0/F}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)).$$

Applying $j_*$ to the exact sequence

$$0 \to J \to \Pic_{X_0/F} \to \mathbb{Z} \to 0$$

over $\text{Spec} F_{\etale}$, we obtain an exact sequence

$$0 \to \mathcal{J} \to j_* \Pic_{X_0/F} \to \mathbb{Z}$$

over $S_{\etale}$. Let $I$ be the image of the last morphism, so that we have an exact sequence

$$0 \to \mathcal{J} \to j_* \Pic_{X_0/F} \to I \to 0.$$

Then we have distinguished triangles

$$R\pi_{S*}\mathcal{J} \to R\pi_{S*}j_* \Pic_{X_0/F} \to R\pi_{S*}I \to R\pi_{S*}\mathcal{J}[1], \text{ and}$$

$$R\pi_{S*}I \to R\pi_{S*}\mathcal{J} \to R\pi_{S*}(\mathcal{J}/I) \to R\pi_{S*}I[1].$$

We have $Q(R\pi_{S*}\mathcal{J},0) = 1$ again by the class number formula ([Gei04, Theorems 9.1 and 9.2] or [Lic05, Theorem 7.4]). Since $\mathbb{Z}/I$ is skyscraper with finite stalks, we have $Q(R\pi_{S*}(\mathcal{J}/I),0) = 1$ by [GS21, Theorem 3.1] (Step 2 of the proof is sufficient). Therefore

$$(4.6) \quad Q(R\pi_{S*}j_* \Pic_{X_0/F}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)) = Q(R\pi_{S*}j_* \Pic_{X_0/F}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)).$$

The complexes $R\Gamma(S, R^1\pi_*\mathcal{O}_X)$ and $R\Gamma(S, \text{Lie } \mathcal{J})$ have the same Euler characteristic by (2.15). Hence

$$(4.7) \quad Q(R\pi_{S*}j_* \Pic_{X_0/F}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)) = Q(R\pi_{S*}j_* \Pic_{X_0/F}[-1], R\Gamma(S, \text{Lie } \mathcal{J})).$$

Combining (4.4)–(4.7), we get the desired equality. \hfill \Box

4.4. A new proof of Geisser’s formula

The above proposition, combined with the results of the previous sections, also gives a new proof of Theorem 2.11 as follows.

**Proof of Theorem 2.11.** By Proposition 4.3, we have

$$P^0_2(X, 1) = \frac{L^0(j, 1)}{[\text{Br}(X)] \cdot \Delta(\text{NS}(X)) \cdot q^{-\alpha(X)}} = \frac{L^0(j, 1)}{\left[\text{Br}(F)\right] \cdot \Delta(j(F)) \cdot c(j) \cdot q^{\text{NS}(S, \text{Lie } \mathcal{J})}}.$$

By (2.12), we have

$$P^0_2(X, 1) = L^0(j, 1) \cdot q^{-\text{dim } B} \cdot [B(k)]^2 \cdot Q_2^0(1),$$

where $Q_2^0(1)$ is the leading coefficient of the $(1 - q^{-s})$-adic expansion of $Q_2(s + 1)$. By (2.17) and (2.20), we have

$$\Delta(\text{NS}(X)) = \frac{\alpha^2 \delta^2}{1 + \prod_{v \in \mathbb{Z}} \delta_v \delta_v} \cdot \Delta(j(F)) \cdot c(j) \cdot [B(k)]^2 \cdot Q_2^0(1).$$

By (2.5), we have

$$q^{-\alpha(X)} = q^{\text{NS}(S, \text{Lie } \mathcal{J})} \cdot q^{-\text{dim } B}.$$

Taking a suitable alternating product of these four equalities, we obtain (2.18). \hfill \Box
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