Holographic Phase Transitions with Fundamental Matter

David Mateos,1 Robert C. Myers,2,3 and Rowan M. Thomson,2,3
1 Department of Physics, University of California, Santa Barbara, CA 93106-9530, USA
2 Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J 2Y5, Canada
3 Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

The holographic dual of a finite-temperature gauge theory with a small number of flavours typically contains D-brane probes in a black hole background. At low temperature the branes sit outside the black hole and the meson spectrum is discrete and possesses a mass gap. As the temperature increases the branes approach a critical solution. Eventually they fall into the horizon and a phase transition occurs. In the new phase the meson spectrum is continuous and gapless. At large \( N_f \) and large ’t Hooft coupling, this phase transition is always of first order, and in confining theories with heavy quarks it occurs at a temperature higher than the deconfinement temperature for the glue.

**Introduction:** The gauge/gravity correspondence is a powerful tool to study non-perturbative physics of gauge theories in diverse dimensions. The classical supergravity regime corresponds to the large-\( N_f \), strong ’t Hooft coupling limit of the gauge theory. This allows the study of a large class of theories that share some of the important features of four-dimensional QCD, such as confinement, chiral symmetry breaking, thermal phase transitions, etc. In principle, because of its asymptotic freedom, QCD itself is not in this class. This means that calculations of certain quantitative properties of QCD, such as the detailed mass spectrum, for example, will require going beyond the supergravity approximation. However, this does not exclude the possibility that some aspects of QCD can be studied in this approximation: Some predictions of the gauge/gravity correspondence may be universal enough as to apply to QCD, at least in certain regimes. A suggestive recent example is the gauge/gravity calculation of the shear viscosity in the hydrodynamic regime of strongly coupled finite-temperature gauge theories. The viscosity/entropy ratio is universal for a large class of gauge theories in the regime described by their gravity duals which, for high enough a temperature, generically contain a black hole horizon. Moreover, this ratio appears to be surprisingly close to that inferred from experiments at the Relativistic Heavy Ion Collider (RHIC). It is therefore important to establish as many universal features of the gauge/gravity correspondence as possible.

The quarks in QCD transform in the fundamental representation. For a large class of gauge theories, a small number of flavours of fundamental matter, \( N_f \ll N_c \), may be described by probe D-branes in the appropriate gravitational background. At sufficiently high temperature \( T \), this contains a black hole. The purpose of this paper is to exhibit some universal features of this system that only depend on these two facts. In particular, we demonstrate the existence of a first order phase transition for the fundamental matter, as follows:

At sufficiently small \( T/M_q \) (where \( M_q \) is the quark mass), the brane tension is sufficient to overcome the attraction of the black hole and hence the branes lie outside the horizon in a ‘Minkowski’ embedding — see fig. 1 and below. In this phase the meson spectrum (i.e., the spectrum of quark-antiquark bound states) is discrete and possesses a mass gap. At sufficiently large \( T/M_q \), the gravitational attraction overcomes the brane tension and the branes fall into the horizon yielding a ‘black hole’ embedding. In this case, the meson spectrum is continuous and gapless. In between, a limiting, critical solution exists. We will show that the phase diagram in the vicinity of this solution exhibits a self-similar structure. While this structure went unnoticed, the phase transition that occurs as \( T/M_q \) increases from small to large values was observed in two specific models. In fact, as was first noted in 2 for a D6-brane in a thermal D4 background, the transition is of first order (in the approximations stated above). Rather than dropping continuously through the critical solution, the probe brane jumps discontinuously from a Minkowski to a black hole embedding at some \( T = T_{\text{fund}} \). This leads to a discontinuity in several field theory quantities, such as, for example, the quark condensate \( \langle \bar{\psi} \psi \rangle \) or the entropy density. In the following, we will see that the critical behavior and, as a result, the first order transition are essentially universal to all Dp/Dq systems.

![FIG. 1: Profiles of D7-brane embeddings in a D3-brane background. The thick black circle is the horizon (\( \rho = 1 \)).](image-url)
The case of confining gauge theories is particularly interesting because, for sufficiently heavy quarks, two distinct phase transitions occur. At $T = T_{\text{decou}}$, the gravitational background changes from a horizon-free background to a black hole background. At this temperature the gluons and the adjoint matter become deconfined, but the branes remain outside the horizon and hence stable quark-antiquark bound states still exist in a range $T_{\text{decou}} < T < T_{\text{fund}}$. At $T = T_{\text{fund}}$, the phase transition described above for the fundamental matter occurs.

**The Setup:** The black D$p$-brane metric in the string-frame takes the form

$$ds^2 = H^{-\frac{d}{2}} - f dt^2 + dx_p^2 + H^{\frac{d}{2}} \left( \frac{du^2}{f} + u^2 d\Omega_{8-p}^2 \right),$$

where $H(u) = (L/u)^{7-p}$, $f(u) = 1 - (u_0/u)^{7-p}$ and $L$ is a length scale (the AdS radius in the case $p = 3$). There is also a non-trivial dilaton $e^\phi = g_s H^{(3-p)/4}$ and a Ramond-Ramond field $C_{01...p} = H^{-1}$. The horizon lies at $u = u_0$. As usual, regularity of the Euclidean section, obtained through $t \to it$, requires that $t_\text{E}$ be identified with period

$$\frac{1}{T} = \frac{4\pi L}{7-p} \left( \frac{L}{u_0} \right)^{\frac{2}{7-p}}.$$

According to the gauge/gravity correspondence, string theory on the background above is dual to a $(p+1)$-dimensional supersymmetric gauge theory at temperature $T$. In some cases one periodically identifies some of the ‘Poincaré’ directions $x_p$ in order to render the theory effectively lower-dimensional at low energies; a prototypical example is that of a D4-brane with one compact space direction. Under these circumstances a different background with no black hole may describe the low-energy physics, and a phase transition may occur as $T$ increases.

In the gauge theory this is typically a confinement/deconfinement phase transition for the gluonic (or adjoint) degrees of freedom. Throughout this paper we assume that $T$ is high enough, in which case the appropriate gravitational background is always.

Consider now a D$q$-brane probe that shares $d$ spacelike ‘Poincaré’ directions with the background D$p$-branes and wraps an $S^n$ inside the $S^{8-p}$. We will assume that the D$q$-brane also extends along the radial direction, so that $q = d + n + 1$. In the gauge theory this corresponds to introducing fundamental matter that propagates along a $(d+1)$-dimensional defect. To ensure stability, we will assume that the D$p$/D$q$ intersection under consideration is supersymmetric at zero temperature. Under these conditions the RR field sourced by the D$p$-branes does not couple to the D$q$-branes.

Two cases of special interest here are the D3/D7 ($n = 3$) and the D4/D6 ($n = 2$) systems. If one of the D4 directions is compact, then both cases can effectively be thought of as describing the dynamics of a four-dimensional gauge theory with fundamental matter.

**Criticality and Scaling:** In this section we follow closely. We begin by studying the behaviour of the brane probe near the horizon. In order to do so we write

$$d\Omega_7^2 = d\theta^2 + \sin^2 \theta d\Omega_5^2 + \cos^2 \theta d\Omega_{7-p-n}^2,$$

$$u = u_0 + \pi T z^2, \quad \theta = \frac{L}{u_0} \left( \frac{u_0}{L} \right)^{\frac{p-3}{4}}, \quad x = \left( \frac{u_0}{L} \right)^{\frac{p-3}{4}} x.$$ 

with $T$ the temperature above. Expanding the metric to lowest order in $z, y$ gives Rindler space together with some spectator directions:

$$ds^2 = -(2\pi T)^2 z^2 dt^2 + dz^2 + dy^2 + y^2 d\Omega_5^2 + d\tilde{x}_n^2 + \cdots.$$ 

The D$q$-brane lies at constant values of the omitted coordinates, so these play no role in the following. The horizon is of course at $z = 0$. The D$q$-brane embedding is specified by a curve $(z(\sigma), y(\sigma))$ in the $(z, y)$-plane. Since the dilaton approaches a constant near the horizon, up to an overall constant the D$q$-brane (Euclidean) action is simply the volume of the brane, namely

$$I \propto \int d\sigma \sqrt{\dot{z}^2 + \dot{y}^2} z y^n,$$

where the dot denotes differentiation with respect to $\sigma$. This is precisely the action considered in ref.\cite{7}. In the gauge $z = \sigma$ the equation of motion takes the form

$$z y \dot{y} + (y \dot{y} - n z)(1 + \dot{y}^2) = 0.$$ 

Solutions fall into two classes that we call ‘black hole’ embeddings and ‘Minkowski’ embeddings; see fig.\cite{1}. Black hole embeddings are those for which the brane falls into the horizon, and may be characterised by the size of the induced horizon, $y_0$. The appropriate boundary condition in this case is $\dot{y} = 0, y = y_0$ at $z = 0$. Minkowski embeddings are those for which the brane closes off smoothly above the horizon. These are characterised by the distance to the horizon, $z_0$, and satisfy the boundary condition $\dot{z} = 0, z = z_0$ at $y = 0$. The limiting solution is the critical solution $y = \sqrt{n} z$, which touches the horizon at the point $y = z = 0$.

The equation of motion enjoys a scaling symmetry: If $y = f(z)$ is a solution, then so is $y = f(\mu z)/\mu$ for any real positive $\mu$. This transformation rescales $z_0 \to z_0/\mu$ for Minkowski embeddings, or $y_0 \to y_0/\mu$ for black hole embeddings, which implies that all solutions of a given type can be generated from any other one by this transformation.

Consider now a solution very close to the critical one, $y(z) = \sqrt{n} z + \xi(z)$. Linearising the equation of motion, one finds that for large $z$ the solutions are of the form $\xi(z) = z^{n/2} + \epsilon_n z^{n/2}$, with $\nu_n = n/2 \pm \sqrt{n^2 - 4(n+1)/2}$. If $n \geq 5$ these exponents are real, whereas if $n \leq 4$ they have non-vanishing imaginary parts that lead to oscillatory behaviour. Under the assumptions stated above $n \geq 5$ implies $d \leq 1$ (an example of this is a D3/D7 intersection over a string). Since we are mostly interested
in higher dimensional defects, we will henceforth assume that \( n \leq 4 \). In this case it is convenient to write the general solution as

\[
y = \sqrt{n} z + \frac{T^{-1}}{(TZ)^2} \left[ a \sin(\alpha \log T z) + b \cos(\alpha \log T z) \right],
\]

where \( \alpha = \sqrt{4(n+1) - n^2}/2 \) and \( a, b \) are dimensionless constants determined by \( z_0 \) or \( y_0 \). It is easy to show that under the rescaling discussed above, these constants transform as

\[
\left( \frac{a}{b} \right) \rightarrow \frac{1}{\mu^{n+1}} \left( \frac{\cos(\alpha \log \mu)}{\sin(\alpha \log \mu)} \sin(a \log \mu) \cos(b \log \mu) \right) \left( \frac{a}{b} \right). \tag{8}
\]

This transformation law implies that the solutions exhibit discrete self-similarity and can be used to derive critical exponents that characterise the near-critical behaviour. We refer the reader to [7, 8] for details but we emphasise that this behaviour depends only on the dimension of the sphere, and is therefore universal for all Dp/Dq systems with \( n \leq 4 \).

Each near-horizon solution gives rise to a global solution when extended over the full spacetime. Each of these solutions is characterised by a quark mass \( M_\lambda \) and a quark condensate \( \langle \bar{\psi} \psi \rangle \) that can be read off from its asymptotic behaviour. Both of these quantities are fixed by \( z_0 \) or \( y_0 \). As we will see, the values corresponding to the critical solution, \( M_\lambda^* \) and \( \langle \bar{\psi} \psi \rangle^* \), give a rough estimate of the point at which the phase transition occurs.

**Fundamental Phase Transitions:** In order to study the global solutions, it is convenient to introduce an isotropic, dimensionless radial coordinate \( \rho \) through

\[
(u_0 \rho)^{\frac{7-p}{5}} = u_0^{\frac{7-p}{5}} + \sqrt{\frac{\rho^7-p - u_0^{7-p}}{u_0^{7-p}}}. \tag{9}
\]

Note that the horizon is at \( \rho = 1 \). Just for concreteness, we will now assume that the Dp/Dq system under consideration is T-dual to the D3/D7 one, in which case \( (p-d) + (n+1) = 4 \). Under these circumstances, the Euclidean Dq-brane action density in the bulk Dp-brane background is

\[
\frac{I_{dp}}{\mathcal{N}_p} = \int_{\rho_{min}}^\infty d\rho \left( \frac{u}{u_0 \rho} \right)^{d-3} \left( 1 - \frac{1}{\rho^{2(7-p)}} \right) \rho^n \times (1 - \chi^2)^{\frac{n+1}{2}} \sqrt{1 - \chi^2 + \rho^2 \chi^2}, \tag{10}
\]

where \( \mathcal{N} \) is a normalisation constant:

\[
\mathcal{N}_p = \frac{N I_{Dp} u_0^{n+1} \Omega_n}{4T}. \tag{11}
\]

\( T_{Dp} = 1/(2\pi \ell_s)^q g_s \ell_s \) is the Dq-brane tension, \( \Omega_n \) is the volume of a unit \( n \)-sphere, \( \chi = \cos \theta \), and \( \chi = d\chi/d\rho \). Multiple flavours arise by introducing \( N_i \) coincident probe branes. Up to a numerical constant the normalisation factor is found to be

\[
\mathcal{N} \sim N_i N_c T^d g_{eff}(T)^{\frac{4d-1}{3-d}} \tag{12},
\]

where \( g_{eff}(T) = \lambda T^{p-3} \) is the dimensionless effective 't Hooft coupling for a \((p+1)\)-dimensional theory at temperature \( T \). \( \lambda = g_s^2 N_c \) and we have used the standard gauge/gravity relations \( g_{eff}^2 \sim g_s^2 \ell_s^{p-3} \) and \( L^7-p \sim g_s N_c \ell_s^{p-3} \) with \( \ell_s \) the fundamental string length.

The equation of motion that follows from (10) leads to the large-\( \rho \) behaviour:

\[
\chi = \frac{m}{\rho} + \frac{c}{\rho^p} + \cdots. \tag{13}
\]

The constants \( m, c \) are related to the quark mass and condensate through \( [5, 6] \)

\[
M_n = \frac{u_0 m}{2^{\frac{3-p}{2}} \pi \ell_s^2}, \quad \langle \bar{\psi} \psi \rangle = -\frac{2\pi^2 \ell_s(n-1)\Omega_n T_{Dn} u_0^q c}{4L}. \tag{14}
\]

This implies the relation \( m^{(5-p)/2} = M/T \) between the dimensionless quantity \( m \), the temperature \( T \) and the mass scale

\[
M = \frac{7-p}{2^{\frac{3-p}{2}} \pi L} \left( \frac{2\pi^2 \ell_s M_n}{L} \right)^{\frac{4}{7-p}} \sim g_{eff}(M_n), \tag{15}
\]

This is precisely the scale of the mass gap in the discrete meson spectrum at temperatures well below the phase transition \( [5, 6] \). We see here that it is also the scale of the transition temperature for the fundamental degrees of freedom, \( T_{fund} \sim M \), since this takes place at \( m \sim 1 \).

The key observation [5, 6] is that the values \( (m, c) \) of a near-critical solution are linearly related to the corresponding values in the near-horizon region. Combining this with the transformation rule [5, 6] for the near-horizon constants \( (a, b) \) and eliminating \( \mu \), we deduce that \((m - m^*)/z_0^{7-p} + (c - c^*)/z_0^{7-p} \) are periodic functions of \((\alpha/2\pi) \log z_0 \) with unit period for Minkowski embeddings, and similarly with \( z_0 \) replaced by \( y_0 \) for black hole embeddings. This is confirmed by our numerical results, as illustrated in fig. 2.

The oscillatory behaviour of \( m \) and \( c \) as functions of \( z_0 \) or \( y_0 \) implies that the quark condensate is not a single-valued function of the quark mass. The preferred solution will of course be the one that minimises the free energy density of the Dq-brane, \( F = TI_{Dp} \). This quantity contains a volume divergence, as can be seen by using the asymptotic behaviour [5, 6]. It therefore needs to be regularised and renormalised. We achieve the former by replacing the upper limit of integration by a finite ultraviolet cut-off \( \rho_{max} \). The latter can be done by subtracting the free energy of a fiducial embedding, as was done for the D4/D6 system in ref. [5, 6]. The asymptotic geometry of a D7-brane in a D3 background is \( AdS_5 \times S^3 \), so in this case the more elegant method of holographic renormalisation [12] for brane probes [13] is available. This consists of adding to the brane action the boundary `counter-term’

\[
\frac{I_{bound}}{\mathcal{N}} = -\frac{1}{4} \left( \rho_{max}^{4} - m^{4} \right)^2 - 4mc. \tag{16}
\]
The renormalised brane energy \( I = I_0 + I_{\text{bound}} \) is then finite as the cut-off is removed, \( \rho_{\text{max}} \to \infty \). The results for the D3/D7 case are shown in fig. 3. We see that the transition occurs discontinuously by jumping from a Minkowski embedding (point A) to a black hole embedding (point B). We emphasize again that this first order transition is a direct consequence of the multi-valued nature of the physical quantities brought on by the critical behaviour described in the previous section. It may be possible to access this self-similar region by super-cooling the system.

It is interesting to ask if the strong coupling results obtained here could in principle be compared with a weak coupling calculation. It follows from our analysis that the free energy density takes the form \( F = NTf(m^2) \), where the function \( f \) can only depend on even powers of \( m \) because of the reflection symmetry \( \chi \to -\chi \). The strong coupling limit \( g_{\text{eff}} \to \infty \) corresponds to \( m \to 0 \), which may be equivalently regarded as a zero quark mass limit or as a high-temperature limit. In this limit the brane lies on the equatorial embedding \( \chi = 0 \) and slices the horizon in two equal parts. In general \( f(0) \) is a non-zero numerical constant; in the D3/D7 case, for example, a straightforward calculation yields \( f(0) = -1/2 \). Thus at strong coupling the free energy density scales as

\[
F \sim N_c N_* T^{d+1} g_{\text{eff}}(T)^{2(p-3)/(p-1)}.
\]  

(17)

The temperature dependence is that expected on dimensional grounds for a \( d \)-dimensional defect, and the \( N_c N_* \) dependence follows from large-\( N_c \) counting rules. However, the dependence on the effective 't Hooft coupling indicates that this contribution comes as a strong coupling effect, without direct comparison to any weak coupling result. The same is true for other thermodynamic quantities such as, for example, the entropy density \( S = -\partial F/\partial T \). We remind the reader that the background geometry makes the leading contribution to the free energy density \( 14 \)

\[
F \sim N_c^2 T^{d+1} g_{\text{eff}}(T)^{2(p-3)/(p-1)},
\]  

(18)

which corresponds to that coming from the gluons and adjoint matter. Of course, for \( p = 3 \), the effective coupling factor is absent and this bulk contribution differs from the weak coupling result by only a factor of 3/4 15.

**Discussion:** We have shown that, in a large class of gauge theories with fundamental matter, quark-antiquark bound states survive the deconfining phase transition for the gluonic degrees of freedom provided \( M \gtrsim T_{\text{deconf}} \), where \( M \) is the typical mesonic scale. This is potentially interesting in connection with QCD, since in QCD heavy quark mesonic bound states with \( M \gg T_{\text{deconf}} \sim 175 \text{ MeV} \) certainly exist. One generic feature of the low-lying mesons in the class of theories discussed here is that they are extremely deeply bound at strong coupling 10,11, as is apparent from eq. (15). It is intriguing that the mesonic states claimed to explain certain features (such as the entropy density) of the strongly coupled quark-gluon plasma formed at RHIC are also deeply bound 16. It would be remarkable if a precise relationship between the two could be established.

Thermal phase transitions in gauge theories with spontaneously broken chiral symmetries at zero temperature are particularly interesting, since they raise the question of whether chiral symmetry is restored at the phase transition. This has been found to be the case 17 in the D4/D8/D8 holographic model of 18. In this model the D8/D8 pair is connected before it falls into the horizon, but splits into two disconnected pieces when it does. At this point the symmetry is enhanced from the diagonal \( U(N_c)_L \) subgroup to the full \( U(N_c)_L \times U(N_c)_R \) group. This contrasts with the cases discussed in this paper, in which the branes remain connected when they fall into the horizon. From the gauge theory viewpoint, this difference in topologies is due to the fact that in the first case the fundamental matter lives on a defect/anti-defect pair, whereas in the second it lives on a single defect.

The detailed results found in this paper rely on the approximation that \( 1/N_c, 1/\lambda \to 0 \) with \( N_c \) fixed. However, the fact that the phase transition is first order implies that it should be stable under small perturbations, and so its order and other qualitative details should hold within a finite radius of the \( 1/N_c, 1/\lambda \) expansions. Of course, finite-\( N_c \) and finite-\( \lambda \) corrections may eventually modify the behaviour uncovered here. For example, at large but finite \( N_c \), the black hole will Hawking-radiate and each bit of the brane probe will experience a thermal bath at a temperature determined by the local acceleration. This effect becomes more and more important as the lower part of a Minkowski brane approaches the horizon, and may potentially blur the self-similar, scaling behaviour.
found here. Note that this effect is of order $1/N^2$, and therefore subleading with respect to the order-$N_f/N_c$ correction to physical quantities from the presence of the brane probes.

Finite 't Hooft coupling corrections correspond to higher-derivative corrections both to the supergravity action and the D-brane action. These may also blur the structure discussed above. For example, higher-derivative corrections to the D-brane equation of motion are likely to spoil the scaling symmetry of eq. (6), and hence the self-similar behaviour. These corrections also become important as the lower part of a Minkowski brane approaches the horizon, since the (intrinsic) curvature of the brane becomes large there.

Yet another type of correction one may consider is due to the backreaction on the background spacetime of the Dq-branes, whose magnitude is controlled by the ratio $N_f/N_c$. These have been considered for the D2/D6 system in ref. [19]. In particular, this ref. finds that the energy density scales as $F \sim N_f^{1/2} N_c^{3/2} T^3$, which obviously differs from [17] with $p = 2, d = 2$. This discrepancy is not at all a contradiction, and has the same origin as the discrepancy found for the meson spectrum [10]. This is the fact that the calculation in [19] applies in the far infrared of the gauge theory, whereas that presented here applies at high temperatures, i.e., at $T \gg g_s^2 T_{YM}$.

We will return to these and other issues in [20].

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