Teleportation of arbitrary $n$-qudit state with multipartite entanglement

Zhan-jun Zhang$^{a,b}$

$^a$ Department of Physics and Center for Quantum Information Science, National Cheng Kung University, Tainan 70101, Taiwan
$^b$ School of Physics & Material Science, Anhui University, Hefei 230039, China

We propose a protocol $D_n$ for faithfully teleporting an arbitrary $n$-qudit state with the tensor product state (TPS) of $n$ generalized Bell states (GBSs) as the quantum channel. We also put forward explicit protocol $D'_n$ and $D''_n$ for faithfully teleporting an arbitrary $n$-qudit state with two classes of $2n$-qudit GESs as the quantum channel, where the GESs are a kind of genuine entangled states we construct and can not be reducible to the TPS of $n$ GBSs.

I. Introduction

No-cloning theorem forbids a perfect copy of an arbitrary unknown quantum state. How to interchange different resources has ever been a question in quantum computation and quantum information. In 1993, Bennett et al.[1] first presented a quantum teleportation scheme $T_0$. In the scheme, an arbitrary unknown quantum state in Alice’s qubit can be teleported to a distant qubit $B$ with the aid of Einstein-Podolsky-Rosen (EPR) pair. Suppose Alice has a qubit $x$ in an arbitrary unknown normalized state

$$|\Lambda\rangle_x = \alpha |0\rangle_x + \beta |1\rangle_x,$$

where $\alpha$ and $\beta$ are complex. Alice and a remote Bob share an EPR pair $(a, b)$, say, in the state

$$|\Psi_0\rangle_{ab} = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} (|j\rangle|j\rangle)_{ab}.$$ (2)

This teleportation between Alice and Bob can be seen intuitively from the following equation,

$$|\Lambda\rangle_x |\Psi_0\rangle_{ab} = \sum_{i=0}^{3} |\Psi_i\rangle_{ax} \sigma^{(i)} |\Lambda\rangle_b,$$ (3)

where $|\Psi_i\rangle_{ab} = \sigma^{(i)} |\Psi_0\rangle_{ab}$, $\sigma^{(0)} = |1\rangle|1\rangle + |0\rangle|0\rangle$, $\sigma^{(1)} = |1\rangle|1\rangle + |1\rangle|0\rangle$, $\sigma^{(2)} = |0\rangle|1\rangle - |1\rangle|0\rangle$ and $\sigma^{(3)} = |1\rangle(1) - |0\rangle|0\rangle$. Bennett et al’s work showed in essence the interchangeability of different quantum resources[2].

The teleportation of multi-qubit teleportation has been studied by Lee et al[3] and Yang et al[4]. Suppose that the arbitrary $n(n \geq 2)$-qubit state Alice wants to teleport to Bob is written as

$$|\Lambda\rangle_{x_1x_2...x_n} = \sum_{m_N=0}^{1} \cdots \sum_{m_2=0}^{1} \sum_{m_1=0}^{1} C_{m_1m_2...m_N} |m_1\rangle_{x_1} |m_2\rangle_{x_2} \cdots |m_n\rangle_{x_n},$$ (4)

where $C$’s are complex coefficients and $|\Lambda\rangle_{x_1x_2...x_n}$ is assumed to be normalized. Alice and Bob share in advance $N$ same Bell states, say, $|\Psi_0\rangle_{a_0b_0} \otimes \cdots \otimes |\Psi_0\rangle_{a_2b_2} \otimes |\Psi_0\rangle_{a_1b_1}$. The $n$ qubits $a_1, a_2, \cdots, a_{n-1}$ and $a_n$ is in Alice’s site. The $n$ qubits $b_1, b_2, \cdots, b_{n-1}$ and $b_n$ in Bob’s site are used to “receive” the teleported state from Alice. Hence, the initial joint state is

$$|\Lambda\rangle_{x_1x_2...x_n} \otimes |\Psi_0\rangle_{a_0b_0} \otimes \cdots \otimes |\Psi_0\rangle_{a_2b_2} \otimes |\Psi_0\rangle_{a_1b_1}.$$ (5)
It can be rewritten as[5]

\[
\frac{1}{2^n} \sum_{i=1}^{2^n} |\psi_{i_n}\rangle_a x_n |\psi_{i_{n-1}}\rangle_a x_{n-1} \cdots |\psi_{i_1}\rangle_a x_1 U_{i_n i_{n-1} \cdots i_1} b_n b_{n-1} \cdots b_1 |\Lambda\rangle b_1 b_2 \cdots b_{n-1} b_n. \tag{6}
\]

If Alice performs \( n \) Bell-state measurements on the qubit pairs \((a_n, x_n), \ldots, (a_1, x_1)\) and publishes a \( 2^n \)-bit classical message corresponding to her measurement outcomes on the qubit pairs, then conditioned on Alice’s information, Bob can recover the arbitrary state \(|\Lambda\rangle\) by performing at most \( 2n \) single-qubit operations. To our knowledge, as far as the multipartite quantum state teleportation is concerned, only protocols for \( n \)-qubit state teleportation are proposed[3-9], and so far there does not exist any protocol for teleporting an arbitrary \( n \)-qubit state though teleportation for one-qudit state has ever been studied by Zubairy[10], Stenholm and Bardroff[11], and Roa et al[12]. In this paper we will extend such studies.

We will directly consider the general case of teleporting an arbitrary \( n \)-qubit state with the tensor product state (TPS) of \( n \) generalized Bell states (GBSs) as the quantum channel.

On the other hand, recently Rigolin[6] has proposed a protocol for teleporting an arbitrary two-qubit state with a four-particle generalized Bell state as a genuine quantum teleportation channel and a four-particle joint measurement. However, the multipartite state in the Rigolin’s protocol is just a tensor product state of two Bell states in essence, not a genuine multipartite entangled state[7]. As a consequence, the Rigolins protocol[6] is equivalent to the Yang-Guo protocol[4] for teleporting an arbitrary multipartite state in principle. Very recently, Yeo and Chua[8] have presented an explicit protocol for faithfully teleporting an arbitrary two-qubit state via a genuine 4-qubit entangled state they constructed. They think it is an important consideration because the four-qubit entangled state, in addition to two Bell states, could be a likely candidate for the genuine four-particle analogue to a Bell state. Soon later, Cheng, Zhu and Guo[9] presented a general form of genuine multipartite entangled quantum channels for arbitrary qubit-state teleportation. In this paper we will present an explicit protocol for faithfully teleporting an arbitrary \( n \)-qubit state with two classes of \( 2n \)-qudit GBSs, where GES is referred to as a kind of genuine entangled states we construct and can not be reducible to the TPS of \( n \) GBSs.

This paper is organized as follows: In section II we will propose a faithful teleportation protocol \( \mathcal{D}_n \) of multipartite \( n \)-qudit state with the TPS of \( n \) GBSs as the quantum channel. In section III we will present an explicit protocols \( \mathcal{D}_n’ \) and \( \mathcal{D}_n'' \) for faithfully teleporting an arbitrary multipartite \( n \)-qudit state with two classes of GESs. A brief summary is given in section IV.

II. Protocol \( \mathcal{D}_n \) for teleporting arbitrary \( n \)-qudit state using TPS of \( n \) GBSs

Teleportation for one-qudit state has ever been studied by Zubairy[10], Stenholm and Bardroff[11], and Roa et al[12]. However, so far there does not exist any protocol concerning the teleportation of an arbitrary \( n \)-qudit state. In this section we will focus on this issue and propose a faithful teleportation protocol \( \mathcal{D}_n \) of multipartite \( n \)-qudit state with the TPS of \( n \) GBSs as the quantum channel.

Suppose Alice has \( n \) qudits \( \{X_1, X_2, \ldots, X_n\} \) in the state of

\[
|\Lambda\rangle_{X_1 X_2 \cdots X_n} = \sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \cdots \sum_{j_n=0}^{d-1} C_{j_1 j_2 \cdots j_n} |j_1 j_2 \cdots j_n\rangle_{X_1 X_2 \cdots X_n}, \tag{7}
\]

where \( C \)'s are complex coefficients and \(|\Lambda\rangle_{X_1 X_2 \cdots X_n}\) is assumed to be normalized. Moreover, Alice and Bob share in advance \( n \) generalized Bell states (GBSs) in the form

\[
|\Theta_{0000 \cdots 00}\rangle_{A_1 B_1 A_2 B_2 \cdots A_n B_n} = |\Phi_{00}\rangle_{A_1 B_1} |\Phi_{00}\rangle_{A_2 B_2} \cdots |\Phi_{00}\rangle_{A_n B_n}, \tag{8}
\]
where

$$|\Phi_{00}\rangle = \sum_{j=0}^{d-1} |jj\rangle/\sqrt{d}. \quad (9)$$

Alice has the \( n \) qudits \( \{A_1, A_2, \ldots, A_n\} \) while Bob the \( n \) qudits \( \{B_1, B_2, \ldots, B_n\} \). Hence the state of the \( 3n \)-qudit system is

$$|\Gamma\rangle x_1 x_2 \cdots x_n A_1 B_1 A_2 B_2 \cdots A_n B_n = |\Lambda\rangle x_1 x_2 \cdots x_n |\Phi_{00}\rangle A_1 B_1 |\Phi_{00}\rangle A_2 B_2 \cdots |\Phi_{00}\rangle A_n B_n. \quad (10)$$

Alice performs 2-qudit \( \Phi \)-state projective measurements on the qudit pairs \( (X_1, A_1), (X_2, A_2), \ldots, (X_2, A_2) \), respectively. The 2-qudit \( \Phi \)-state set \( \{ |\Phi_{kl}\rangle_{AB} = U^{(kl)}_A |\Phi_{00}\rangle_{AB} = V^{(kl)}_B |\Phi_{00}\rangle_{AB}; \ k, l \in \{0, 1, \ldots, d-1\} \} \) is a complete orthonormal basis set in \( d^2 \) dimensional Hilbert space for two qudits, where

$$U^{(kl)} = \sum_{j=0}^{d-1} e^{-2\pi i j l / d} |j\rangle |j\rangle/\sqrt{d}, \quad (11)$$

$$V^{(kl)} = \sum_{j=0}^{d-1} e^{2\pi i j k / d} |j\rangle |j\rangle/\sqrt{d}, \quad (12)$$

$$j + l = (j + l) \mod d. \quad (13)$$

Incidently, since \( \Phi \)-states can be transformed into each other via the local unitary operations, the quantum channel linking Alice and Bob can also be other TPSs such as \( |\Theta_{k_1 k_2 \ldots k_n \ell n}\rangle_{A_1 B_1 A_2 B_2 \cdots A_n B_n} \) instead of the TPS \( |\Phi_{0000 \cdots 00}\rangle_{A_1 B_1 A_2 B_2 \cdots A_n B_n} \), where

$$|\Theta_{k_1 k_2 \ldots k_n \ell n}\rangle_{A_1 B_1 A_2 B_2 \cdots A_n B_n} = \sum_{j=0}^{d-1} V^{(k_1 j)}_{A_1} U^{(k_2 j)}_{A_2} \cdots U^{(k_n \ell n)}_{A_n} |\Theta_{0000 \cdots 00}\rangle_{A_1 B_1 A_2 B_2 \cdots A_n B_n} \quad (14)$$

After Alice’s measurements, the system’s state collapses to

$$|\Psi_{k_1 k_2 \ldots k_n \ell n}\rangle_{A_1 X_1 A_2 X_2 \cdots A_n X_n} = \sum_{j=0}^{d-1} (U^{(k_1 j)}_{A_1} V^{(k_2 j)}_{A_2} \cdots V^{(k_n \ell n)}_{A_n}) |\Psi_{0000 \cdots 00}\rangle_{A_1 B_1 A_2 B_2 \cdots A_n B_n}. \quad (15)$$

This means that if Alice gets the state \( |\Phi_{k_1 l_1}\rangle_{A_1 X_1} |\Phi_{k_2 l_2}\rangle_{A_2 X_2} \cdots |\Phi_{k_n l_n}\rangle_{A_n X_n} \) via her measurement, then the state of Bob’s qudits \( \{B_1, B_2, \ldots, B_n\} \) collapses to the state \( V^{(k_1 l_1)}_{B_1} V^{(k_2 l_2)}_{B_2} \cdots V^{(k_n l_n)}_{B_n} |\Lambda\rangle_{B_1 B_2 \cdots B_n} \). Further, if Alice tells Bob her results (i.e., \( k_1 l_1 k_2 l_2 \cdots k_n l_n \)) via public channel, then Bob can recover the state \( |\Lambda\rangle \) in his qudits \( \{B_1, B_2, \ldots, B_n\} \) by performing the local unitary operations \( V^{(k_1 l_1)}_{B_1}, V^{(k_2 l_2)}_{B_2}, \ldots, V^{(k_n l_n)}_{B_n} \), respectively. Up to now, we have presented the protocol \( D_n \) for teleporting arbitrary
III. Protocols $D'_n$ and $D''_n$ using two classes of GESs

In the last section we have shown a protocol $D_n$ for teleporting arbitrary $n$-qudit state using TPS of $n$ GBSs. Now we consider the teleportation of the same $n$-qudit state using another entangled quantum channel between Alice and Bob as follows,

$$\left| \Xi_{0000\cdots00} \rightangle_{A_1B_1A_2B_2\cdots A_nB_n} = \Upsilon_{A_1A_2\cdots A_n} \left| \Theta_{0000\cdots00} \rightangle_{A_1B_1A_2B_2\cdots A_nB_n},$$

where $\Upsilon_{A_1A_2\cdots A_n}$ is a global unitary operator acting on the $n$ qudits $A_1, A_2, \cdots, A_n$ and $\Theta_{0000\cdots00}$ can be reducible to the $\Theta$ states. Hence, other $\Xi$ states can not be reducible to the $\Theta$ states. The present protocol becomes the Lee-Min-Oh protocol in Ref.[3].

By the way, if $\Upsilon_{A_1A_2\cdots A_n}$ is a global unitary operator acting on the $n$ qudits $A_1, A_2, \cdots, A_n$, and can not be reducible to $n$ local operators acting the $n$ qudits. The state of the $3n$ qudits $X_1, X_2, \cdots, X_n, A_1, B_1, A_2, B_2, \cdots, A_n, B_n$ is

$$\left| \Gamma' \rightangle_{X_1X_2\cdots X_n A_1B_1A_2B_2\cdots A_nB_n} = \left| \Lambda \rightangle_{X_1X_2\cdots X_n} \left| \Xi_{0000\cdots00} \rightangle_{A_1B_1A_2B_2\cdots A_nB_n}.$$  

The $2n$-qudit state set \{\$|\Xi_{k_1,k_2\cdots k_n}\$\}_{A_1B_1A_2B_2\cdots A_nB_n} = \Upsilon_{A_1A_2\cdots A_n} |\Theta_{k_1,k_2\cdots k_n}\rangle_{A_1B_1A_2B_2\cdots A_nB_n}, \text{ where } k_x, l_x \in \{0,1,\cdots ,d\}\} is another complete orthonormal basis set for $2n$ qudits. Different $\Xi$ states can be transformed into each other via local unitary operations. Hence, other $\Xi$ states can also be used as the quantum channel instead of the state $|\Xi_{0000\cdots00}\rangle_{A_1B_1A_2B_2\cdots A_nB_n}$. Alice performs the $\Xi$-state projective measurement on the qubits $X_1, X_2, \cdots, X_n, A_1, A_2, \cdots, A_n$ in her site,

$$\left| \Xi_{k_1,k_2\cdots k_n} \rightangle_{A_1X_1A_2X_2\cdots X_n A_1A_2} \left| \Xi_{k_1,k_2\cdots k_n} \rightangle_{X_1X_2\cdots X_n A_1B_1A_2B_2\cdots A_nB_n} \left| \Gamma' \rightangle_{X_1X_2\cdots X_n A_1B_1A_2B_2\cdots A_nB_n} \left| \Lambda \rightangle_{X_1X_2\cdots X_n} \Upsilon_{A_1A_2\cdots A_n} |\Theta_{k_1,k_2\cdots k_n}\rangle_{A_1B_1A_2B_2\cdots A_nB_n}$$

This indicates that if Alice obtains the state $\left| \Xi_{k_1,k_2\cdots k_n} \rightangle_{A_1X_1A_2X_2\cdots X_n A_1} \text{ via her measurement, then the state of Bob’s } n \text{ qudits } B_1, B_2, \cdots, B_n \text{ collapses to } V^{(k_1)}_{B_1} V^{(k_2)}_{B_2} \cdots V^{(k_n)}_{B_n} |\Lambda \rangle_{B_1B_2\cdots B_n}.$ Further, if Alice informs Bob of her results (i.e., $(k_1,k_2,\cdots ,k_n)$) via public channel, then Bob can recover the state $\left| \Lambda \rightangle$ in his $n$ qudits $B_1, B_2, \cdots, B_n$, by performing the local unitary operations $V^{(k_1)}_{B_1}, V^{(k_2)}_{B_2}, \cdots, \text{ and } V^{(k_n)}_{B_n},$ respectively. Since $\Upsilon_{A_1A_2\cdots A_n}$ is a global unitary operator and can not be reducible to $n$ local operators acting on the $n$ qudits $A_1, A_2, \cdots, A_n$, the $\Xi$ states can not be reducible to the $\Theta$ states. Hence, $\Xi$ states is different from the TPSs of $n$ GBSs and is referred to as a kind of genuine entangled states for it is also a candidates for teleporting an arbitrary $n$-qudit state.

So far, we have presented the protocol $D'_n$ for teleporting arbitrary $n$-qudit state using a class of GESs. In the special case of $n = 2, d = 2$ and $\Upsilon = \cos \theta_{12} |00\rangle \langle 00| + \sin \theta_{12} |11\rangle \langle 00| - \sin \theta_{12} |00\rangle \langle 11| + \cos \theta_{12} |11\rangle \langle 11| - \sin \phi_{12} |01\rangle \langle 01| + \cos \phi_{12} |01\rangle \langle 10| + \sin \phi_{12} |10\rangle \langle 01| + \cos \phi_{12} |10\rangle \langle 10|$, the present protocol $D'_n$ is exactly the Yeo-Chua protocol in Ref.[8]. By the way, if $\Upsilon_{A_1A_2\cdots A_n}$ can be reducible to $n$ local operators acting on the $n$ qudits $A_1, A_2, \cdots, A_n$, then the protocol $D'_n$ is transformed into the protocol $D_n$.

Now let us present our protocol $D''_n$ for teleporting an arbitrary $n$-qudit state using another class of GESs as quantum channel. The entangled quantum channel between Alice and Bob is as follows,

$$\left| \Xi'_{0000\cdots00} \rightangle_{A_1B_1A_2B_2\cdots A_nB_n} = \Upsilon_{A_1A_2\cdots A_n} \Omega_{B_1B_2\cdots B_n} |\Theta_{0000\cdots00}\rangle_{A_1B_1A_2B_2\cdots A_nB_n},$$

where $\Omega_{B_1B_2\cdots B_n}$ is a global unitary operator acting on the $n$ qudits $B_1, B_2, \cdots, B_n$ and can not be reducible to $n$ local operators acting the $n$ qudits. Obviously, this entangled quantum channel
\(|\Xi_{0000...00}^\prime\rangle_{A_1B_1A_2B_2...A_nB_n} is different from \(|\Xi_{0000...00}^\prime\rangle_{A_1B_1A_2B_2...A_nB_n} used in the protocol \mathcal{D}'_n\). However, since the \(n\) qudits \(B_1, B_2, \cdots, B_n\) are in Bob’s site, before teleportation he can perform the unitary operation \(\Omega_{B_1B_2...B_n}^\dagger\). After his performance, the quantum channel is transformed into the first class of GESs. Surely, the teleportation can be realized. Hence the protocol \(\mathcal{D}''_n\) is only a slight variation of the protocol \(\mathcal{D}'_n\) but using different quantum channels. One can easily see that, the Chen-Zhu-Guo protocol is only our present protocol \(\mathcal{D}''_n\) in the special case of \(d = 2\). By the way, if \(\Omega_{B_1B_2...B_n}\) can be reducible to \(n\) local operators acting the \(n\) qudits \(B_1, B_2, \cdots, B_n\), then the protocol \(\mathcal{D}''_n\) is transformed into the protocol \(\mathcal{D}'_n\) with other \(\Xi\) state as quantum channel.

4 Summary

To summarize, in this paper we have presented a protocol \(\mathcal{D}_n\) for faithfully teleporting an arbitrary \(n\)-qudit state with the TPS of \(n\) GBSs as the quantum channel. Moreover, we have also put forward explicit protocols \(\mathcal{D}'_n\) and \(\mathcal{D}''_n\) for faithfully teleporting an arbitrary \(n\)-qudit state with two classes of \(2^n\)-qudit GESs as the quantum channel, respectively.

Acknowledgements

This work is supported by the National Natural Science Foundation of China under Grant Nos. 60677001 and 10304022, the science-technology fund of Anhui province for outstanding youth under Grant No.06042087, the general fund of the educational committee of Anhui province under Grant No.2006KJ260B, and the key fund of the ministry of education of China under Grant No.2006063.

References

[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wotters, Phys. Rev. Lett. 70, 1895 (1993).
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[3] J. Lee, H. Min, and S. D. Oh, Phys. Rev. A 66, 052318 (2002).
[4] C. P. Yang and G. C. Guo, Chin. Phys. Lett. 17, 162 (2000).
[5] Z. J. Zhang, Phys. Lett. A 351, 55 (2006).
[6] G. Rigolin, Phys. Rev. A 71, 032303 (2005).
[7] F. G. Deng, Phys. Rev.A 72, 036301 (2005).
[8] Y. Yeo and W. K. Chua, Phys. Rev. Lett. 94, 060502 (2006).
[9] P. X. Chen, S. Y. Zhu and G. C. Guo, Phys. Rev. A 74, 032324 (2006).
[10] M. S. Zubairy, Phys. Rev. A 58, 4368 (1998).
[11] S. Stenholm, P. J. Bardroff, Phys. Rev. A 58, 4373 (1998).
[12] L. Roa, A. Delgado, and I. Fuentes-Guridi, Phys. Rev. A 68, 022310 (2003).