Non-semimartingale solutions of reflected BSDEs and applications to Dynkin games and variational inequalities

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Abstract

We introduce a new class of reflected backward stochastic differential equations with two càdlàg barriers, which need not satisfy any separation conditions. For that reason, in general, the solutions are not semimartingales. We prove existence, uniqueness and approximation results for solutions of equations defined on general filtered probability spaces. Applications to Dynkin games and variational inequalities, both stationary and evolutionary, are given.

Keywords: Reflected backward stochastic differential equation, Dynkin game, variational inequality.

Mathematics Subject Classification (2010): 60H20, 60G40

1 Introduction

In this paper we consider backward stochastic differential equations (RBSDEs) with two reflecting barriers \( L \) and \( U \). We assume merely that \( L, U \) are adapted càdlàg processes such that \( L^+, U^- \) are of class (D) and \( L_t \leq U_t, t \geq 0 \). Because, in general, the barriers \( L, U \) are not semimartingales and we do not assume any separation condition, to treat such equations requires extending the notion of a solution to encompass the case where the first component of the solution is a more general process than a semimartingale. One of the main novelties of the paper is that we provide such an extension. It is right in the sense that it coincides with the “classical” definition (semimartingale solutions) if there exists a special semimartingale between the barriers. Furthermore, under reasonable assumptions on the terminal condition and the generator of the equation, one can show the existence and uniqueness of solutions, as well as useful approximation and stability results. Let us also stress that we consider equations on probability spaces equipped with general filtration satisfying only the usual conditions. Our motivation for studying such general setting comes from applications to Dynkin games and variational inequalities.

We now describe the content of the paper and give more information about our motivations. We start with a brief account of the literature on reflected BSDEs.

Reflected BSDEs with two separated (by a special semimartingale) barriers were introduced by Karatzas and Shreve [11] in the case where the barriers \( L, U \) are continuous and their supremums are square-integrable, the terminal value \( \xi \) is square-integrable, the terminal time \( T \) is constant and finite, the generator \( f \) is Lipschitz

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continuous and the underlying filtration $\mathbb{F}$ is Brownian. Since then the notion of reflected BSDEs was recognized as a very useful and important tool in application to stochastic control, mathematical finance and the variational inequalities theory (see e.g. [8, 23, 27, 32, 33, 39] and reference therein). Subsequently, in many papers the assumptions adopted in [11] were weakened but the separation condition (called Mokobodzki’s condition) was always assumed (see Remark 2.8). The case of less regular barriers is considered in [19, 22, 24, 36, 47]. Equations with $L^p$-data are studied in [26, 28, 51], and with less regular $f$ in [26, 28, 37, 51]. In [1, 3, 31, 52] equations with random (possibly infinite) terminal time are studied, and in [21, 24, 25] equations with a more general Brownian-Poisson filtration. Up to now, the most general setting was adopted in the paper by Klimsiak [29] in which the underlying filtration $\mathbb{F}$ is a general filtration satisfying only the usual conditions and equations with $L^1$-data and càdlàg barriers of class (D) separated by a special semimartingales are considered.

In [29] it is assumed that the terminal time $T$ is bounded. For the purposes of the present paper, we extend the notion of a reflected BSDE introduced in [29] to arbitrary stopping time $T$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a right-continuous complete filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, and let $T$ be a (possibly infinite) $\mathbb{F}$-stopping time. We assume that we are given an $\mathcal{F}_t$-measurable integrable random variable $\xi$, a function $\Omega \times \mathbb{R}_+ \times \mathbb{R} \ni (\omega, t, y) \mapsto f(\omega, t, y) \in \mathbb{R}$ which is progressively measurable with respect to $(\omega, t)$, and two $\mathbb{F}$-adapted càdlàg processes $L, U$ such that $L^+, U^-$ are of class (D), and moreover, $L_t \leq U_t$ for all $t \in [0, T \wedge a]$, $a \geq 0$, and $\limsup_{a \to \infty} L_{T \wedge a} \leq \xi \leq \liminf_{a \to \infty} U_{T \wedge a}$. In this paper, by a (semimartingale) solution of the reflected BSDE with terminal value $\xi$, generator $f$ and barriers $L$ and $U$ (RBSDE$^T (\xi, f, L, U)$ for short) we mean a triple $(Y, M, R)$ of $\mathbb{F}$-adapted càdlàg processes such that $Y$ is of class (D), $M$ is a local martingale with $M_0 = 0$, $R$ predictable of finite variation, $R_0 = 0$ and

\[
\begin{align*}
Y_t & = Y_{T \wedge a} + \int_t^{T \wedge a} f(r, Y_r) \, dr + \int_t^{T \wedge a} dR_r \\
& \quad - \int_t^{T \wedge a} dM_r, \quad t \in [0, T \wedge a], \quad a \geq 0, \\
L_t & \leq Y_t \leq U_t, \quad t \in [0, T \wedge a], \quad a \geq 0, \\
\int_0^{T \wedge a} (Y_{t-} - L_{t-}) \, dR^+_t & = \int_0^{T \wedge a} (U_{t-} - Y_{t-}) \, dR^-_t = 0, \quad a \geq 0, \\
Y_{T \wedge a} & \to \xi \quad \text{a.s. as } a \to \infty.
\end{align*}
\]

In Section 2 we show the existence, uniqueness and approximation results for solutions of (1.1) under the assumption that there is a special semimartingale between $L$ and $U$, and the generator $f$ is continuous and nonincreasing with respect to $y$.

Note that linear equations of the above form were considered in the financial context in [5, 10]. Note also that the processes $M, R$ are determined uniquely by the process $Y$ through the Doob-Meyer decomposition of the special semimartingale $Y + \int_0^t f(r, Y_r) \, dr$. Therefore, without ambiguity, we may say that $Y$ is a solution of RBSDE$^T (\xi, f, L, U)$.

One of the most important result proved in [11] concerns the connection between solutions of RBSDEs and so-called Dynkin games introduced in [15] and studied extensively by many authors (see, e.g., [2, 6, 14, 15, 34, 38, 41, 54, 60]). In [11] (see [29] in the case of general setting) it is proved that if $Y$ is the first component of the solution of the RBSDE with terminal time $T$, terminal value $\xi$ and generator $f$, then for any...
stopping time $\alpha \leq T$,

$$Y_\alpha = \esssup_{\sigma \geq \alpha} \essinf_{\tau \geq \alpha} E \left( \int_{\alpha}^{\tau \wedge \sigma} f(r, Y_r) \, dr + L_\sigma 1_{\sigma < \tau} + U_\tau 1_{\tau \leq \sigma < T} + \xi 1_{\sigma = \tau = T} \middle| \mathcal{F}_\alpha \right). \tag{1.2}$$

Recently it was proved in [13] (see also [4]) that under some conditions on $f$ the above equality may be equivalently stated as

$$Y_\alpha = \esssup_{\sigma \geq \alpha} \essinf_{\tau \geq \alpha} \mathcal{E}^f_{\sigma, \tau \wedge \alpha} (L_\sigma 1_{\sigma < \tau} + U_\tau 1_{\tau \leq \sigma < T} + \xi 1_{\sigma = \tau = T}), \tag{1.3}$$

where $\mathcal{E}^f$ is the nonlinear $f$-expectation introduced by Peng [45] (see also [46]). In [17] it was shown that the theory of nonlinear pricing systems has a wide application in mathematical finance. When $f = 0$, (1.3) reduces to the classical Dynkin game, and when $f \neq 0$, it is called a generalized Dynkin game (see [13]).

Assume that $Y$ is a solution to (1.2) or (1.3). Here arises a natural question whether $Y$ is the first component of a solution to some reflected BSDE. In general, the answer is \textquotedblleft no\textquotedblright, because if $f = 0$ and $L = U$, then from (1.3) it follows that $Y = L$. Hence, since we only assume that $L^+$ is a càdlàg process of class (D), the process $Y$ need not be a semimartingale. On the other hand, by the very definition (in the existing definitions in the literature) of a solution to RBSDE, $Y$ is a special semimartingale. We see that to obtain a one-to-one correspondence between solutions of RBSDEs and solutions to Dynkin games requires an extension of the notion of a solution to RBSDE.

A similar problem appears in applications of RBSDEs to variational inequalities. Let $(\mathcal{E}, D[\mathcal{E}])$ be a symmetric transient regular Dirichlet form and let $X = (X_t, P_x)$ be a Hunt process with life time $\zeta$ associated with $(\mathcal{E}, D[\mathcal{E}])$. Suppose that $L, U$ and $f$ are of Markov-type, i.e.

$$L_t = h_1(X_t), \quad U_t = h_2(X_t), \quad f(t, y) = \hat{f}(X_t, y), \quad t \geq 0, \quad y \in \mathbb{R}, \tag{1.4}$$

for some $h_1, h_2 \in D[\mathcal{E}]$ such that $h_1 \leq h_2$ and some $\hat{f} : E \times \mathbb{R} \to \mathbb{R}$. It is well known (see [60] for the linear case and [31] for the nonlinear case) that if $(Y^x, M^x, R^x)$ is, under the measure $P_x$, a solution of the Markov-type RBSDE\textsuperscript{T}(ξ, f, L, U), then $u : E \to \mathbb{R}$ defined as $u(x) = E_x Y_0^x$ is a solution of the following variational inequality:

$$u \in D[\mathcal{E}], \quad h_1 \leq u \leq h_2, \tag{1.5}$$

and

$$\mathcal{E}(u, v - u) \geq (\hat{f}(\cdot, u), v - u), \quad v \in D[\mathcal{E}] \text{ such that } h_1 \leq v \leq h_2. \tag{1.6}$$

Moreover, $Y^x = u(X)$ under the measure $P_x$ for q.e. $x \in E$. In general, however, if $u$ is a solution to (1.5) and (1.6), then $u(X)$ need not be a solution to some Markov-type RBSDE. The reason is that $u \in D[\mathcal{E}]$ need not be a difference of potentials, i.e. need not satisfy the condition which is known to be necessary for $u(X)$ to be a semimartingale (see [9]). We see that we may apply RBSDEs methods to optimization problems and variational inequalities as long as the value function is a semimartingale. This is very restricting in practice.

The need of extending the notion of reflected BSDEs also arises in the problems of approximation of the value process in Dynkin games. Recall that there are basically two methods of solving RBSDEs with two reflecting barriers (or solving the related
Dynkin game problem. The first one consists in solving the following system of optimal stopping problems introduced in [6, 7]:
\[
\begin{align*}
Y_1^1 &= \text{ess sup}_{t \leq \tau \leq T} E(Y_2^2 + \int_t^\tau f(r) \, dr + L_\tau \mathbf{1}_{\tau < T} + \xi \mathbf{1}_{\tau = T} | \mathcal{F}_t), \\
Y_2^2 &= \text{ess sup}_{t \leq \tau \leq T} E(Y_1^1 \mathbf{1}_{\tau < T} - U_\tau \mathbf{1}_{\tau < T} | \mathcal{F}_t).
\end{align*}
\]
Putting \( Y = Y^1 - Y^2 \), we obtain a solution of the linear RBSDE\(^T\) \((\xi, f, L, U)\). Next, by a fixed point argument, one can obtain the existence of a solution in the in nonlinear case. Note that the above methods always leads to a semimartingale solution, i.e. \( Y \) is a semimartingale. The second method is the so-called penalty method. It is known (see, e.g., [29]) that if there exists a special semimartingale between the barriers, then under some assumptions on the data, the first component \( Y^n \) of the solution \((Y^n, M^n)\) of the BSDE
\[
Y^n_t = \xi + \int_t^T f(r, Y^n_r) \, dr + n \int_t^T (Y^n_r - L_r)^- \, dr
- n \int_t^T (Y^n_r - U_r)^+ \, dr - \int_t^T dM^n_r, \quad t \in [0, T]
\]
converges as \( n \to \infty \) to a process \( Y \) being the first component of a semimartingale solution of RBSDE\(^T\) \((\xi, f, L, U)\). The question arises whether \( \{Y^n\} \) converges if we omit the assumption of existence of a special semimartingale between the barriers. Secondly, if the answer is “yes”, what kind of equation solves the limit process? The problem is rather subtle. It is worth noting here that the penalty method had been applied to Dynkin games problems much before the notion of BSDEs was introduced (see [50, 54, 55, 57]). From the results of Stettner [55] (see also [54, 56] for the Markovian case) it follows (see Remark 5.5 for details) that in the linear case under some additional assumptions on the barriers the solutions of (1.7) can converge to a solution of (1.2) without the assumption that there is a special semimartingale between the barriers. Part of our results may be viewed as far reaching generalization of Stettner’s results on approximation of the value process in Dynkin games.

As explained above, to show the one-to-one correspondence between solutions to RBSDEs and solutions of the generalized Dynkin problem (1.2) or (1.3), or give the one-to-one correspondence between solutions of Markov-type RBSDEs and solutions of variational inequalities of type (1.5), we find ourselves forced to introduce a new definition of a solution in which we do not require that the process \( Y \) is a semimartingale. From now on, solutions of RBSDEs in the sense of (1.1) will be called semimartingale solutions, and the solutions in the generalized sense will be called non-semimartingale solutions or simply solutions.

To give an idea what we mean by a solution of RBSDE\(^T\) \((\xi, f, L, U)\) for general \( L, U \) such that \( L \leq U \), suppose that \( L, U \) are not semimartingales. Of course, since we require that the first component \( Y \) of a solution lies between the barriers, \( Y \) need not be a semimartingale, at least on the set \( \{L = U\} \). Note, however, that \( Y \) is uniquely determined on \( \{L = U\} \), because we have \( L = Y = U \) on this set. We see that to guarantee uniqueness of solutions, we need to define properly \( Y \) outside the set \( \{L = U\} \). The first idea which appears is to require that \( Y \) is a special semimartingale locally outside \( \{L = U\} \) (here locally means that it is a special semimartingale on each random interval \([\alpha, \beta] \subset \{L \neq U\}\)). With this idea in mind, we would like to call a càdlàg process \( Y \) of class (D) a solution of RBSDE\(^T\) \((\xi, f, L, U)\) if \( Y \) solves RBSDE\(^T\) \((\xi, f, L, U)\)
in the classical way locally outside \( \{ L = U \} \) and \( Y_{T \wedge a} \to \xi \) as \( a \to \infty \). However, we show by examples that in general \( Y \) need not be a special semimartingale outside \( \{ L = U \} \) (see Example 3.1), and moreover, that the above requirement imposed on \( Y \) does not ensure uniqueness (see Example 3.2). The reason for non-uniqueness is that solutions can have jumps on the boundary of the set \( \{ L = U \} \). They are produced by jumps of \( L \) or \( U \). Without control of these jumps, we get multiple solutions. We see that we are forced to modify the initial idea. We make two crucial observations in the paper. The first one is that only some kinds of jumps of \( L, U \) on the boundary of \( \{ L = U \} \) may produce multiple solutions, and the second one is that \( Y \) will be always locally a special semimartingale outside the set \( \{ L = U \} \cup \{ L_\neq U_- \} \). Based on these observations one could try to find some progressively measurable extension of the set \( \{ L_\neq U \} \cap \{ L_\neq U_- \} \) such that it covers all the jumps of the barriers that may produce multiple solutions, and moreover, \( Y \) remains a special semimartingale locally on this extension. Unfortunately, it appears that there is no extension (depending only on \( L, U \)) having these properties (see Example 3.3). Let \( T \) denote the family of all \( \mathbb{F} \)-stopping times \( \tau \) such that \( \tau \leq T \). One of the most important ingredient of our concept of a solution of RBSDE\(^T\)(\( \xi, f, L, U \)) is that we are able to find a family \( \{ C_\tau, \tau \in T \} \) of progressively measurable sets such that \( U_{\tau \in T} \) extends the set \( \{ L \neq U \} \cap \{ L_\neq U_- \} \), it covers all jumps of the barriers on the boundary of \( \{ L = U \} \) responsible for non-uniqueness and \( Y \) is a special semimartingale locally on each \( C_\tau \) (and not on the whole set \( U_{\tau \in T} \) in general).

To make the above idea precise, we first define the family \( \ell := \{ (\gamma_\tau, \Lambda_\tau), \tau \in T \} \) by

\[
\gamma_\tau = \inf \{ \tau < t \leq T : L_{t-} = U_{t-} \} \wedge \inf \{ \tau \leq t \leq T : L_t = U_t \} \wedge T
\]

and

\[
\Lambda_\tau = \{ L_{\gamma_\tau} = U_{\gamma_\tau} \} \cap \{ \tau < \gamma_\tau < \infty \}.
\]

We call it an \( \ell \)-system associated with \( L, U \). We then say that a càdlàg process \( X \) is a special semimartingale with respect to \( \ell \), or simply an \( \ell \)-semimartingale, if for every stopping time \( \tau \leq T \), \( X \) is a special semimartingale on the random interval \([\tau, \gamma_\tau] \) defined by

\[
[\tau(\omega), \gamma_\tau(\omega)) = \begin{cases} 
(\tau(\omega), \gamma_\tau(\omega)), & \omega \notin \Lambda_\tau, \\
(\tau(\omega), \gamma_\tau(\omega)), & \omega \in \Lambda_\tau.
\end{cases}
\]

The intervals \([\tau, \gamma_\tau] \) play the role of the above-mentioned sets \( C_\tau \). Finally, by a solution of RBSDE\(^T\)(\( \xi, f, L, U \)) we mean a pair \((Y, \Gamma)\) of \( \mathbb{F} \)-adapted càdlàg processes such that \( Y, \Gamma \) are special \( \ell \)-semimartingales such that

\[
\begin{align*}
Y_t &= Y_{T \wedge a} + \int_t^{T \wedge a} f(r, Y_r) \, dr + \Gamma_{T \wedge a} - \Gamma_t, \quad t \in [0, T \wedge a], \quad a \geq 0, \\
L_t &\leq Y_t \leq U_t, \quad t \in [0, T \wedge a], \quad a \geq 0, \\
\int_\tau^\tau (Y_{r-} - L_{r-}) \, d\Gamma_{r+}^v = \int_\tau^\tau (U_{r-} - Y_{r-}) \, d\Gamma_{r-}^{v'}(\tau) = 0,
\end{align*}
\]

(1.8)

In (1.8), \( \Gamma^v(\tau) \) is the predictable finite variation part from the Doob-Meyer decomposition of \( \Gamma \) on \([\tau, \gamma_\tau] \). In Section 4 we show that if \((Y, \Gamma)\) satisfies (1.8) and there exists a special semimartingale between the barriers \( L \) and \( U \), then \( Y, \Gamma \) are special semimartingales and the triple \((Y, \Gamma^v, \Gamma^m)\), where \( \Gamma^v \) (resp. \( \Gamma^m \)) is the predictable finite variation...
part (resp. martingale part) from the Doob-Meyer decomposition of $\Gamma$ on $[0, T]$, is a semimartingale solution of $\text{RBSDE}^T(\xi, f, L, U)$, i.e. solution in the sense of (1.1). At first glance the proposed definition of a solution of RBSDE with general càdlàg barriers seems to be quite complicated, but at the matter of fact is very handy in practice.

We now describe the content of the paper. In Section 2 devoted to semimartingale solutions of RBSDE, we extend the results of [29] to the case of arbitrary, possibly unbounded terminal time $T$. Then, in Sections 3 and 4, we prove our main results on existence, uniqueness and approximation of non-semimartingale solutions of general, non-Markov-type RBSDEs. First we show that under the assumption that $y \mapsto f(t, y)$ is nonincreasing a comparison theorem for solutions to RBSDEs holds true. It implies uniqueness of solutions. Moreover, we prove stability of solutions, i.e. we show that if $(Y^i, \Gamma^i), i = 1, 2$, are a solution of $\text{RBSDE}^T(\xi^i, f^i, L^i, U^i)$, then

$$
\|Y^1 - Y^2\|_{1, T} \leq E|\xi^1 - \xi^2| + E\int_0^T |f_1(t, Y^2_t) - f_2(t, Y^2_t)| \, dt
\]

+ $\|L^1 - L^2\|_{1, T} + \|U^1 - U^2\|_{1, T},$

where $\|Y\|_{1, T} = \sup_{\tau \leq T, \tau < \infty} E|Y_\tau|$. To show the existence of a solution, we additionally impose some integrability conditions on $f$. In the paper we assume that

$$
\int_0^T |f(t, y)| \, dt < \infty \quad \text{for every } y \in \mathbb{R}
$$

and there exists a càdlàg process $S$ being a difference of supermartingales of class (D) such that

$$
E\int_0^T |f(t, S_t)| \, dt < \infty.
$$

The second condition is commonly used in the literature with $S = 0$. Both conditions are minimal known conditions ensuring the existence of solutions of BSDEs with no reflection (condition (1.11) is necessary when $f$ is positive). We prove that if the function $y \mapsto f(t, y)$ is continuous and nonincreasing, and moreover, $f$ satisfies (1.10) and (1.11), then there exists a unique solution $(Y, \Gamma)$ of $\text{RBSDE}^T(\xi, f, L, U)$. We also show that under these assumptions for every strictly positive bounded $\mathcal{F}$-progressively measurable process $\eta$ such that

$$
E\int_0^T \eta(t)(S_t - L_t)^- \, dt + E\int_0^T \eta(t)(S_t - U_t)^+ \, dt < \infty
$$

there exists a unique solution to the following penalized BSDE

$$
Y^n_t = \xi + \int_t^T f(r, Y^n_r) \, dr + n\int_t^T \eta_r(Y^n_r - L_r)^- \, dr
\]

- $n\int_t^T \eta_r(Y^n_r - U_r)^+ \, dr - \int_t^T dM^n_r,$

and for every $a \geq 0$, $Y^n_t \to Y_t$, $\int_0^t n(Y^n_r - L_r)^- \, dr - \int_0^t n(Y^n_r - U_r)^+ - M^n_r \to \Gamma_t$, $t \in [0, T \wedge a]$. 

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In the case where $T$ is bounded, one can take $\eta \equiv 1$, so (1.12) reduces to the usual penalization scheme (1.7). Moreover, we show that if (1.13) is satisfied, then the convergence of $\{Y^n\}$ is uniform in probability on compact subsets of $\mathbb{R}_+$ (the so-called ucp convergence). Finally, let us note that in Section 5 we show that if $L, U$ are of class (D) (and not merely $L^+$ and $U^-$), then there is a solution of $\text{RBSDE}^T(\xi, f, L, U)$ even if we drop condition (1.11). Unfortunately, we do not now whether it is a limit of some penalization scheme. Nevertheless, this result is still interesting because it is known that in general, without condition (1.11), there is no solution of $\text{BSDE}^T(\xi, f)$.

In Section 5, we study connections of RBSDEs with Dynkin games and nonlinear expectation. We show that if $Y$ is a solution of (1.2), then $Y$ is the first component of a solution of $RBSDE^T(\xi, f, L, U)$, and conversely, if $(Y, \Gamma)$ is a solution of $\text{RBSDE}^T(\xi, f, L, U)$ and $E \int_0^T |f(r, Y_r)| \, dr < \infty$, then $Y$ is a solution to (1.2). We also prove that if
\begin{equation}
\rho_L \geq L_-, \quad \rho_U \leq U_-,
\end{equation}
where $\rho_L$ (resp. $\rho_U$) is the predictable projection of $L$ (resp. $U$), then $(\sigma^*_\alpha, \tau^*_\alpha)$ defined by
\begin{equation}
\sigma^*_\alpha = \inf\{t \geq \alpha : Y_t = L_t\} \land T, \quad \tau^*_\alpha = \inf\{t \geq \alpha : Y_t = U_t\} \land T
\end{equation}
is a saddle point for (1.2). Moreover, the process $Y + \int_\alpha^\tau f(r, Y_r) \, dr$ is a uniformly integrable martingale on the closed interval $[\alpha, \sigma^*_\alpha \land \tau^*_\alpha]$.

We next generalize the notion of the nonlinear $f$-expectation introduced in [17] for Brownian filtration and square integrable data, and then extended in [49] to the case of filtration generated by Brownian motion and an independent Poisson random measure, and we show that
\begin{equation}
(Y, \Gamma) \text{ is a solution of } \text{RBSDE}^T(\xi, f, L, U) \quad \text{iff } Y \text{ satisfies } (1.3).
\end{equation}
Let us stress here that (1.15) holds true although in general the integral $E \int_0^T |f(r, Y_r)| \, dr$ may be infinite. Furthermore, we show that under (1.13) the pair (1.14) is a saddle point for the generalized Dynkin game (1.3).

In Section 6 we deal with Markov-type RBSDEs. In the first part of this section, we assume that we are given a Borel right Markov process $X = \{(X, P_x) : x \in E\}$ on $E$, the generator and barriers are of the form (1.4), and $\xi = \psi(X_{\tau_D})$, for some $\psi : E \setminus D \to \mathbb{R}$, where $\tau_D$ is first exit time from a finely open set $D \subset E$. We first show that there exists an $m$-inessential set $N \subset E$ such that for every $x \in E \setminus N$ there exists a unique solution $(Y^x, \Gamma^x)$ of $\text{RBSDE}(\xi, f, L, U)$ under the measure $P_x$, and there exists a nearly Borel function $u$ on $E$ such that
\begin{equation}
Y^x_t = u(X_t), \quad t \in [0, \tau_D], \quad P_x\text{-a.s.}
\end{equation}
As a corollary, we get Stettner’s results on the penalty method and saddle points for Markovian Dynkin games, but in the much more general setting. Then we show the connection of the function $u$ with the stationary variational inequality of the form (1.5) and (1.6) in case $X$ is a Hunt process associated with some semi-Dirichlet form. In particular, we show that under some natural assumptions, if $u$ is a solution of (1.5), (1.6), then for q.e. $x \in E$ the solution of $\text{RBSDE}(\xi, f, L, U)$ under the measure $P_x$ has the form $(u(X), \Gamma)$ with $\Gamma$ defined by
\begin{equation}
-\Gamma_t = A^{[u]}_t + M^{[u]}_t + \int_0^t f(X_r, u(X_r)) \, dr, \quad t \in [0, \tau_D], \quad P_x\text{-a.s.}
\end{equation}
In (1.16), \(A^{[u]}, M^{[u]}\) are additive functionals of \(X\) appearing in Fukushima’s decomposition of \(u(X)\) (see [18]). From (1.16) it follows in particular that

\[
Y_t^x = \psi(X_{\tau_D}) + \int_t^{\tau_D} \hat{f}(X_r, u(X_r)) \, dr + A^{[u]}_t - A^{[u]}_{\tau_D} - \int_t^{\tau_D} dM^{[u]}_r, \quad t \in [0, \tau_D], \quad P_x\text{-a.s.}
\]

Comparing this formula with the first equation in (1.1), we see that the zero energy functional \(A^{[u]}\) plays the role of the reflection process \(R\).

In the second part of Section 6, we give some analogues of the results of the first part for evolutionary variational inequalities.

## 2 Semimartingale solutions to reflected BSDEs

In Sections 2–5, \((\Omega, \mathcal{F}, P)\) is a complete probability space equipped with a right-continuous complete filtration \(\mathcal{F} = \{\mathcal{F}_t, t \in [0, \infty]\}\) with \(\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t\). We assume that we are given a function \(\Omega \times \mathbb{R}_+ \times \mathbb{R} \ni (\omega, t, y) \mapsto f(\omega, t, y) \in \mathbb{R}\), which is \(\mathcal{F}\)-progressively measurable with respect to \((\omega, t)\) for every \(y \in \mathbb{R}\).

Let \(\alpha, \beta\) be two stopping times such that \(\alpha \leq \beta\). We say that an \(\mathcal{F}\)-progressively measurable process \(Y\) is of class (D) on \([\alpha, \beta]\) if the family \(\{Y_\tau, \alpha \leq \tau \leq \beta, \tau < \infty\}\) is uniformly integrable. We set

\[
\|Y\|_{1,\alpha,\beta} = \sup_{\alpha \leq \tau \leq \beta, \tau < \infty} E|Y_\tau|, \quad \|Y\|_{1,\beta} = \|Y\|_{1,0,\beta}. \tag{2.1}
\]

We say that an increasing sequence of stopping times \(\{\tau_k\}\) is a chain on \([\alpha, \beta]\) if \(\alpha \leq \tau_k \leq \beta, k \geq 1\), and the set \(\{k \geq 1 : \tau_k < \beta\}\) is finite a.s. In the rest of this section we assume that \(\alpha, \beta\) are finite a.s.

**Definition 2.1.** We say that a pair \((Y, M)\) of \(\mathcal{F}\)-adapted càdlàg processes is a solution of the backward stochastic differential equation on \([\alpha, \beta]\) with \(\mathcal{F}_\beta\)-measurable terminal condition \(\hat{\xi}\), generator \(f\) (BSDE\(^{a,\beta}(\hat{\xi}, f)\) for short) if

(i) \(Y\) is of class (D), \(M\) is a local martingale with \(M_\alpha = 0\),

(ii) \(\int_\alpha^\beta |f(r, Y_r)| \, dr < \infty\) and

\[
Y_t = \hat{\xi} + \int_t^\beta f(r, Y_r) \, dr - \int_t^\beta dM_r, \quad t \in [\alpha, \beta].
\]

When considering reflected BSDEs we will also assume that we are given two \(\bar{\mathcal{F}}\)-adapted càdlàg processes \(L\) (lower barrier) and \(U\) (upper barrier) such that \(L_t \leq U_t, \ t \geq 0\).

**Definition 2.2.** We say that a triple \((Y, M, K)\) of \(\mathcal{F}\)-adapted càdlàg processes is a solution of the reflected BSDE on \([\alpha, \beta]\) with \(\mathcal{F}_\beta\)-measurable terminal condition \(\hat{\xi}\), generator \(f\) and lower barrier \(L\) (RBSDE\(^{a,\beta}(\hat{\xi}, f, L)\) for short) if

(i) \(Y\) is of class (D), \(K\) is an increasing predictable process with \(K_\alpha = 0\), \(M\) is a local martingale with \(M_\alpha = 0\),

(ii) \(\int_\alpha^\beta |f(r, Y_r)| \, dr < \infty\) and

\[
Y_t = \hat{\xi} + \int_t^\beta f(r, Y_r) \, dr + \int_t^\beta dK_r - \int_t^\beta dM_r, \quad t \in [\alpha, \beta],
\]
(iii) \( L_t \leq Y_t, \ t \in [\alpha, \beta], \) and
\[
\int_\alpha^\beta (Y_r^- - L_r^-) dK_r = 0.
\]

**Definition 2.3.** We say that a triple \((Y, M, A)\) of \(\mathbb{F}\)-adapted càdlàg processes is a solution of the reflected BSDE on \([\alpha, \beta]\) with \(\mathcal{F}_\beta\)-measurable terminal condition \(\xi\), generator \(f\) and upper barrier \(U\) (\(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, U)\) for short) if \((-Y, A, -M)\) is a solution to \(\text{RB}^{\alpha,\beta}(-\hat{\xi}, f, -U)\), where \(\hat{f}(t, y) = -f(t, -y)\).

Let us consider the following hypotheses:

(A1) \(\hat{\xi}\) is \(\mathcal{F}_\beta\)-measurable, \(E|\hat{\xi}| < \infty\) and there exists a càdlàg process \(S\), which is a difference of supermartingales of class \((D)\), such that \(E\int_\alpha^\beta |f(r, S_r)| \, dr < \infty\).

(A2) there exists \(\mu \in \mathbb{R}\) such that for a.e. \(t \in [\alpha, \beta]\) the function \(y \mapsto f(t, y) - \mu y\) is nonincreasing.

(A3) for a.e. \(t \in [\alpha, \beta]\) the function \(y \mapsto f(t, y)\) is continuous.

(A4) \(\int_\alpha^\beta |f(r, y)| \, dr < \infty\) for every \(y \in \mathbb{R}\).

**Theorem 2.4.** Assume that \(\hat{\xi}, f\) satisfy (A1)-(A4), \(L_\beta \leq \hat{\xi}\) and \(L^+\) is of class \((D)\) on \([\alpha, \beta]\).

(i) There exists a unique solution \((Y, M, K)\) of \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L)\).

(ii) Let \(\{\hat{\xi}_n\}\) be a sequence of integrable \(\mathcal{F}_\beta\)-measurable random variables such that \(\hat{\xi}_n \nearrow \hat{\xi}\), and let \((Y^n, M^n)\), \(n \geq 1\), be a solution of \(\text{RB}^{\alpha,\beta}(\hat{\xi}_n, f_n)\) with \(f_n(t, y) = f(t, y) + n(y - L_t^-)\). Then \(Y^n_t \nearrow Y_t, \ t \in [\alpha, \beta]\).

**Proof.** By [32, Theorem 2.7], there exists a unique solution \((\hat{Y}, \hat{M})\) of \(\text{BS}^{\alpha,\beta}(\hat{\xi}, f)\). Since \(\hat{Y} \vee L\) is of class \((D)\) on \([\alpha, \beta]\), by [29, Theorem 4.1] there exists a unique solution \((Y, M, K)\) of \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L \vee \hat{Y})\). Let \((Y^n, M^n)\) be a solution of \(\text{BS}^{\alpha,\beta}(\hat{\xi}_n, f^n)\) with \(\hat{\xi}_n = \hat{\xi} \wedge (-n)\) and \(f^n = f \wedge (-n)\). By [29, Theorem 4.1], there exists a unique solution \((Y^n, M^n, K^n)\) of \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L \vee \hat{Y})\). Furthermore, by [29, Proposition 2.1], \(Y^n \leq \hat{Y} \leq \hat{Y^n}\), which implies that \(\hat{Y^n} \vee L \leq \hat{Y} \vee L \leq Y^n\). Thus \((Y^n, M^n, K^n)\) is a solution of \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L \vee \hat{Y})\). Consequently, by uniqueness (see [29, Corollary 2.2]), \((Y^n, M^n, K^n) = (Y, M, K), n \geq 1\). In particular, for any \(n \geq 1\),
\[
\int_\alpha^\beta (Y_{r^-} - L_{r^-} \vee \hat{Y^n}_{r^-}) \, dK_r = 0.
\]

Letting \(n \to \infty\) we get \(\int_\alpha^\beta (Y_{r^-} - L_{r^-}) \, dK_r = 0\). Since \(Y \geq L\), we see that in fact \((Y, M, K)\) is a solution to \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L)\). We may now repeat step by step the proof of [29, Theorem 4.1], with obvious changes, to show the convergence of \(\{Y^n\}\). \(\square\)

**Remark 2.5.** In the proof of Theorem 2.4 we have showed that under (A1)-(A4) a triple \((Y, M, K)\) is a solution of \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L)\) if and only if it is a solution of \(\text{RB}^{\alpha,\beta}(\hat{\xi}, f, L \vee \hat{Y})\), where \((\hat{Y}, \hat{M})\) is a solution of \(\text{BS}^{\alpha,\beta}(\hat{\xi}, f)\). Therefore, without lost of generality, one can assume that \(L\) is of class \((D)\) (and not merely that \(L^+\) is of class \((D)\)).
Definition 2.6. We say that a triple \((Y, M, R)\) of \(\mathbb{F}\)-adapted càdlàg processes is a solution to reflected BSDE on \([\alpha, \beta]\) with an \(\mathcal{F}_\beta\)-measurable terminal condition \(\hat{\xi}\), generator \(f\), lower barrier \(L\) and upper barrier \(U\) (RBSDE\(^{\alpha,\beta}(\hat{\xi}, f, L, U)\) for short) if

(a) \(Y\) is of class (D), \(R\) is a finite variation predictable process with \(R_\alpha = 0\), \(M\) is a local martingale with \(M_\alpha = 0\),

(b) \(\int_\alpha^\beta |f(r, Y_r)| \, dr < \infty\) and

\[
Y_t = \hat{\xi} + \int_t^\beta f(r, Y_r) \, dr + \int_t^\beta dK_r - \int_t^\beta dM_r, \quad t \in [\alpha, \beta],
\]

(c) \(L_t \leq Y_t \leq U_t, \ t \in [\alpha, \beta]\), and

\[
\int_\alpha^\beta (Y_{r-} - L_{r-}) \, dR^+_r = \int_\alpha^\beta (U_{r-} - Y_{r-}) \, dR^-_r = 0.
\]

If \(\alpha = 0\), we write \(\text{RBSDE}^\beta\) instead of \(\text{RBSDE}^{0,\beta}\).

Theorem 2.7. Assume that \(\hat{\xi}, f\) satisfy (A1)–(A4), \(L_\beta \leq \hat{\xi} \leq U_\beta\) and \(L^+, U^-\) are of class (D) on \([\alpha, \beta]\).

(i) There exists a solution \((Y, M, R)\) of \(\text{RBSDE}^{\alpha,\beta}(\hat{\xi}, f, L, U)\) if and only if there exists a special semimartingale \(X\) such that \(L_t \leq X_t \leq U_t, \ t \in [\alpha, \beta]\).

(ii) Let \(\{\hat{\xi}_n\}\) be a sequence of \(\mathcal{F}_\beta\)-measurable integrable random variables such that \(\hat{\xi}_n \nearrow \hat{\xi}\), and let \((\hat{Y}^n, \hat{A}^n, \hat{M}^n)\) be a solution of \(\text{RBSDE}^{\alpha,\beta}(\hat{\xi}_n, f_n, U)\) with

\[
f_n(t, y) = f(t, y) + n(y - L_t^-).
\]

Then \(\hat{Y}^n_t \nearrow Y_t, \ t \in [\alpha, \beta]\).

Proof. Of course, if there exists a solution \((Y, M, R)\) of \(\text{RBSDE}^{\alpha,\beta}(\hat{\xi}, f, L, U)\), then \(Y\) is a special semimartingale which lies between the barriers. Suppose now that there exists a special semimartingale \(X\) such that \(L_t \leq X_t \leq U_t, \ t \in [\alpha, \beta]\). To show the existence of a solution it suffices to modify slightly the proof of [29, Theorem 4.2]. Indeed, in [29] the existence of a solution of \(\text{RBSDE}^{\alpha,\beta}(\hat{\xi}, f, L, U)\) is proved under the additional assumption that \(E \int_\alpha^\beta d|V|_r < \infty\), where \(V\) is the finite variation part from the Doob-Meyer decomposition of \(X\), and \(L, U\) are of class (D). However, the proof of [29, Theorem 4.2] applies also to our case. The only difference is that in the present situation the sequence \(\{\delta_k\}\) appearing in the proof of [29, Theorem 4.2] should be defined as follows:

\[
\delta_k = \inf \{t \geq \beta : \int_\alpha^t |f(r, X_r)| \, dr \geq k\} \land \sigma_k,
\]

where \(\{\sigma_k\}\) is a chain on \([\alpha, \beta]\) such that \(E \int_\alpha^{\sigma_k} d|V|_r < \infty\). Such a chain exists since \(V\) is predictable (and \(X_\alpha\) is integrable). The fact that \(L, U\) are of class (D) was used in the proof of [29, Theorem 4.2] only to apply [29, Theorem 2.13] to some reflected BSDE with upper barrier \(U\). However, we have shown in Theorem 2.4 that [29, Theorem 2.13] is still true when we only assume that \(U^-\) is of class (D). The proof of part (ii) runs as the proof of [29, Theorem 4.2] with obvious changes (in [29, Theorem 4.2] the case of terminal conditions not depending on \(n\) is considered).

\(\square\)

Remark 2.8. If \(L_t < U_t, \ t \in [\alpha, \beta]\), and \(L_t^- < U_t^-, \ t \in (\alpha, \beta]\), then one can show that there exists a special semimartingale \(X\) such that \(L_t \leq X_t \leq U_t, \ t \in [\alpha, \beta]\) (see [58]).
3 Reflected BSDEs with bounded terminal time

In this section, we assume that $T$ is a bounded $\mathbb{F}$-stopping time. In the sequel, for a given progressively measurable set $A$, we say that some property holds locally on $A$ if it holds on $[\alpha, \beta] \subset A$ for every $\alpha, \beta \in \mathcal{T}$ such that $\alpha \leq \beta$. In particular, we say that a càdlàg progressively measurable process $Y$ is a solution of RBSDE$^T(\xi, f, L, U)$ locally on $A$ if for all $\alpha, \beta \in \mathcal{T}$ such that $\alpha \leq \beta$ and $[\alpha, \beta] \subset A$ it is a semimartingale solution of RBSDE$^{\alpha, \beta}(Y_{\beta}, f, L, U)$.

We start with an example showing that in general a solution of the reflected equation is not a semimartingale locally outside $\{L = U\}$.

**Example 3.1.** Let $\Omega = \mathbb{R}$, $T = 2$ and $\mathcal{F}_t = \{\emptyset, \Omega\}$, $\mathcal{F} = \{\mathcal{F}_t, t \in [0, T]\}$. We set $f \equiv 0$, $\xi \equiv 0$, and

$$L_t = (1 - t) \cos\left(\frac{\pi}{1 - t}\right) 1_{[0, 1]}(t), \quad U_t = (L_t + \frac{1}{2}(1 - t)) 1_{[0, 1]}(t) + 1_{[1, 2]}(t), \quad t \in [0, T].$$

Since the filtration $\mathcal{F}$ is trivial, a process $Y$ is an $\mathcal{F}$-semimartingale if and only if it is a process of finite variation. Of course, any solution of RBSDE$^T(\xi, f, L, U)$ has to satisfy $L \leq Y \leq U$. In particular, putting $t_n = (n - 1)/n$, we have

$$\frac{1}{2n} = L_{t_{2n}} \leq Y_{t_{2n}}, \quad Y_{t_{2n+1}} \leq U_{t_{2n+1}} = \frac{-1}{2n + 1} + \frac{1}{4n + 2}.$$ 

Observe that $\{L = U\} = \emptyset$ and

$$\text{Var}_{[0, 1]}(Y) \geq \sum_{n=2}^{\infty} |Y_{t_{2n}} - Y_{t_{2n+1}}| \geq \sum_{n=2}^{\infty} \frac{1}{2n + 1} = \infty,$$

so $Y$ is not a semimartingale.

In the sequel, we will show that a solution of RBSDE$^T(\xi, f, L, U)$ is always a special semimartingale locally outside $\{L = U\} \cup \{L_- = U_-\}$. Unfortunately, the requirement that the solution of RBSDE$^T(\xi, f, L, U)$ is a càdlàg process $Y$ of class (D) with $Y_T = \xi$ solving RBSDE$^T(\xi, f, L, U)$ locally on $\{L \neq U\} \cap \{L_- \neq U_-\}$ is to weak to guarantee uniqueness. The following example shows that actually there can be many càdlàg processes $Y$ of class (D) with $Y_T = \xi$, which are special semimartingales solving RBSDE$^T(\xi, f, L, U)$ locally on $\{L \neq U\}$.

**Example 3.2.** We define $\Omega, T$ and $\mathcal{F}$ as in Example 3.1. Let

$$L_t = -t 1_{[0, 1]}(t), \quad U_t = t 1_{[0, 1]}(t), \quad t \in [0, 2].$$

Observe that $\{L \neq U\} \cap \{L_- \neq U_-\} = \{L \neq U\} = [0, 1)$. Of course, the process $Y \equiv 0$ is of class (D) with $Y_T = 0$, and it is a solution of RBSDE$^{a,b}(Y_0, 0, L, U)$ for every $a, b \in [0, 1)$ such that $a \leq b$. Let $r \in (0, 1)$ and

$$Y^r_t = t 1_{[0,r]}(t) + r 1_{[r, 1]}(t), \quad t \in [0, 2].$$

It is easy to verify that for every $r \in (0, 1)$ the process $Y^r$ is a special semimartingale of class (D) with $Y^r_T = 0$, and that $Y^r$ is a solution of RBSDE$^{a,b}(Y_0, 0, L, U)$ for every $a, b \in [0, 1)$ with $a \leq b$. 

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In this section we will show that there is a family \( \{ \mathcal{C}_\tau, \tau \in T \} \) of progressively measurable sets having the property that if \( Y \) is a càdlàg process of class (D) with \( Y_T = \xi \) and \( Y \) solves RBSDE\(^T\)\((\xi, f, L, U)\) locally on \( \mathcal{C}_\tau \) for every \( \tau \in T \), then \( Y \) is uniquely determined. Sum of this family covers the set \( \{L \neq U\} \cap \{L_- \neq U_-\} \) and some points from the set \( \{L = U\} \). The following example shows that in general there is no extension of \( \{L \neq U\} \cap \{L_- \neq U_-\} \) by a single progressively measurable set having the same property as \( \{\mathcal{C}_\tau, \tau \in T\} \).

**Example 3.3.** We define \( \Omega, T \) and \( \mathbb{F} \) as in Example 3.1. We set

\[
L^0_t = 1 - \sum_{n=1}^{\infty} (t - \frac{1}{n+1}) 1_{\{\tau_n \leq t\}}(t), \quad U^0_t = 1 + \sum_{n=1}^{\infty} (t - \frac{1}{n+1}) 1_{\{\tau_n \leq t\}}(t), \quad t \geq 0,
\]

and then, for \( t \in [0,2] \) we set

\[
L_t = (1 + (t + 1) \cos \frac{\pi}{1+t}) L^0_{t+1} - 1_{[0,1]}(t), \quad U_t = (1 + (t + 1) \cos \frac{\pi}{1+t}) U^0_{t+1} + 1_{[0,1]}(t).
\]

Observe that \( \{L \neq U\} \cap \{L_- \neq U_-\} = [0,2] \setminus N \), where \( N = \{1\} \cup \{1 + \frac{1}{n}, n \geq 2\} \). From Example 3.2 it follows that for every \( a \in N \), if there exists a càdlàg progressively measurable process \( Y \) of class (D) with \( Y_T = \xi \) such that \( Y \) solves RBSDE\(^T\)\((0,0,L,U)\) locally on \([0,T] \setminus a\), then there are infinitely many processes with these properties. Therefore the only extension of \( \{L \neq U\} \cap \{L_- \neq U_-\} \) ensuring uniqueness of \( Y \) is the whole interval \([0,2]\). However, from the construction of \( L,U \) it follows that each process \( Y \) lying between \( L \) and \( U \) is of infinite variation on \([0,2]\), so it is not a special semimartingale.

### 3.1 Definition of a solution

We denote by \( T \) the set of all \( \mathbb{F} \)-stopping times \( \tau \) such that \( \tau \leq T \). For a stopping time \( \sigma \) and \( \Lambda \in \mathcal{F}_\sigma \), we set

\[
\sigma_\Lambda(\omega) = \begin{cases} 
\sigma(\omega), & \omega \in \Lambda, \\
\infty, & \omega \notin \Lambda.
\end{cases}
\]

It is well known that \( \sigma_\Lambda \) is a stopping time. For a given stopping time \( \sigma \), denote by \( \sigma^{[a]}, \sigma^{[i]} \) its accessible and totally inaccessible part, respectively. Let us recall (see [12, Chapter III, T41]) that there exist unique disjoint sets \( \Lambda^{[a]}(\sigma), \Lambda^{[i]}(\sigma) \in \mathcal{F}_{\sigma^-} \) such that

\[
\Lambda^{[a]}(\sigma) \cup \Lambda^{[i]}(\sigma) = \Omega, \quad \sigma^{[a]} = \sigma_{\Lambda^{[a]}(\sigma)}, \quad \sigma^{[i]} = \sigma_{\Lambda^{[i]}(\sigma)}.
\]

**Definition 3.4.** We say that a family \( \{(\gamma_\tau, \Lambda_\tau), \tau \in T\} \) is an \( \ell \)-system if \( \gamma_\tau \in T \), \( \tau \leq \gamma_\tau \) and \( \Lambda_\tau \in \mathcal{F}_{\gamma_\tau^-}, \Lambda_\tau \subseteq \Lambda^{[a]}(\gamma_\tau) \).

Let us fix an \( \ell \)-system \( \{(\gamma_\tau, \Lambda_\tau), \tau \in T\} \). Since \( \gamma_\tau \) is accessible, there exists a sequence of predictable stopping times \( \{S_l\} \) such that \( P(\bigcup_{l \geq 1} \Lambda^l_T) = 1 \), where \( \Lambda^l_T = \{(\gamma_\tau)_{\Lambda_T} = S_l\} \). Let \( \{\delta^{k,l}_{r}\} \) \( (\delta^{k,l}_{r} \geq \tau) \) be an announcing system for \( (\gamma_\tau)_{\Lambda_T} \), i.e. for fixed \( l \geq 1 \) the sequence \( \{\delta^{k,l}_{r}\} \) announces \( (\gamma_\tau)_{\Lambda^l_T} \), and let \( \delta^{k,l}_{r} = \delta^{k,l}_{r} \wedge T \). We put

\[
\gamma^{k,l}_{\tau} = \delta^{k,l}_{\tau} \wedge \gamma_\tau.
\]

In the whole paper we use the following notation

\[
[\tau(\omega), \gamma_\tau(\omega)] = \begin{cases} 
[\tau(\omega), \gamma_\tau(\omega)], & \omega \notin \Lambda_\tau, \\
[\tau(\omega), \gamma_\tau(\omega)], & \omega \in \Lambda_\tau.
\end{cases}
\]
Observe that 

\[ [\tau, \gamma_\tau] = \bigcup_{k,l \geq 1} [\tau, \gamma_{\tau}^{k,l}]. \]

In what follows we also adopt the convention that \([a, a] = [a, a] = \{a\} \). We say that an \( \mathbb{F} \)-adapted process \( \Gamma \) is a (local) martingale (resp. (predictable) increasing process) on \([\tau, \gamma_\tau]\) if \( \Gamma_{\tau} = \tau, \gamma_\tau \) if it is a (local) martingale (resp. (predictable) increasing process) on \([\tau, \gamma_{\tau}^{k,l}]\) for \( k, l \geq 1 \). We say that an \( \mathbb{F} \)-adapted process \( \Gamma \) is a (local) martingale (resp. (predictable) increasing process) on \([\alpha, \beta] \) for \( \alpha, \beta \in \mathcal{T} \) \( \alpha \leq \beta \) if there exist a (local) \( \mathbb{F} \)-martingale \( M \) (resp. a (predictable) increasing \( \mathbb{F} \)-adapted process \( A \) ) such that \( \Gamma_t = M_t, t \in [\alpha, \beta] \) (resp. \( \Gamma_t = A_t, t \in [\alpha, \beta] \)).

**Definition 3.5.** We say that an \( \mathbb{F} \)-adapted càdlàg process \( \Gamma \) is an \( \ell \)-martingale (resp. local \( \ell \)-martingale) if it is a martingale (resp. local martingale) on \([\tau, \gamma_\tau]\) for every \( \tau \in \mathcal{T} \). We say that \( \gamma \) is an \( \ell \)-semimartingale (resp. special \( \ell \)-semimartingale) if \( \Gamma \) is a semimartingale (resp. special semimartingale) on \([\tau, \gamma_\tau]\) for every \( \tau \in \mathcal{T} \).

The barriers \( L, U \) determine some special \( \ell \)-system defined as follows. For \( \tau \in \mathcal{T} \) we define the stopping time \( \hat{\gamma}_\tau \) by

\[
\hat{\gamma}_\tau = \inf \{ \tau < t \leq T : L_{t-} = U_{t-} \} \wedge \inf \{ \tau \leq t \leq T : L_t = U_t \},
\]

and then we set

\[
\gamma_\tau = \hat{\gamma}_\tau \wedge T, \quad \Lambda_\tau = \{ L_{\gamma_\tau} = U_{\gamma_\tau} \} \cap \{ \tau < \gamma_\tau \}. \tag{3.2}
\]

Observe that \( \Lambda_\tau \in \mathcal{F}_{\gamma_\tau} \) and the stopping time \( (\gamma_\tau)_{\Lambda_\tau} \) is predictable since the sequence \( \{ \alpha_n := \inf \{ t > \tau : |L_t - U_t| \leq \frac{1}{n} \} \wedge n \) announces it. Therefore \( \{ (\gamma_\tau, \Lambda_\tau), \tau \in \mathcal{T} \} \) with \( \gamma_\tau, \Lambda_\tau \) defined by (3.2) is an \( \ell \)-system in the sense of Definition 3.4. We call it the \( \ell \)-system associated with \( L \) and \( U \). We shall see that the family \( \{ \mathcal{C}_\tau, \tau \in \mathcal{T} \} \), where \( \mathcal{C}_\tau = [\tau, \gamma_\tau] \) and \([\tau, \gamma_\tau]\) is determined by this system has the crucial property formulated right after Example 3.2.

In what follows we consider the \( \ell \)-system associated with \( L \) and \( U \). Observe that in the case of that system,

\[
\Lambda_\tau \subset \Lambda^1_\tau,
\]

where \( S_1 = (\gamma_\tau)_{\Lambda_\tau} \). We will also need the following notation: \( \hat{\delta}^k_\tau = \hat{\delta}^{k,1}_\tau \), \( \delta^k_\tau = \delta^{k,1}_\tau \) and

\[
\gamma^{k,1}_\tau = \gamma^{k,1}_\tau. \tag{3.3}
\]

For a given special \( \ell \)-semimartingale \( \Gamma \), we denote by \( \Gamma^u(\tau) \) (resp. \( \Gamma^m(\tau) \)) its predictable finite variation part (resp. local martingale part) from the Doob-Meyer decomposition on \([\tau, \gamma_\tau]\). For a process \( \Gamma \) and finite \( \alpha, \beta \in \mathcal{T} \) such that \( \alpha \leq \beta \) we denote by \( \int_{\alpha}^{\beta} d\Gamma_r \) the difference \( \Gamma_\beta - \Gamma_\alpha \).

**Definition 3.6.** We say that a pair \((Y, \Gamma)\) of \( \mathbb{F} \)-adapted càdlàg process is a solution of the reflected backward stochastic differential equation on the interval \([0, T]\) with terminal time \( \xi \), generator \( f \), lower barrier \( L \) and upper barrier \( U \) (RBSDE\(^T\)\((\xi, f, L, U)\) for short) if

(a) \( Y \) is of class (D), \( \Gamma \) is a special \( \ell \)-semimartingale,
\( \int_0^T |f(r,Y_r)| \, dr < \infty \) and
\[
Y_t = \xi + \int_t^T f(r,Y_r) \, dr + \int_t^T d\Gamma_r, \quad t \in [0,T],
\]
(c) \( L_t \leq Y_t \leq U_t, \quad t \in [0,T] \),
(d) for every \( \tau \in T \),
\[
\int_{\tau}^{\gamma_{\tau}} (Y_r - L_r) \, d\Gamma^{u,+}_r(\tau) = \int_{\tau}^{\gamma_{\tau}} (U_r - Y_r) \, d\Gamma^{u,-}_r(\tau) = 0.
\]

**Remark 3.7.** Of course in the above definition process \( \Gamma \) is determined by \( Y \) through the formula
\[
\Gamma_t = Y_0 - Y_t - \int_0^t f(r,Y_r) \, dr, \quad t \in [0,T].
\]
That is why in the whole paper we shall write that a solution of RBSDE is \( Y \) and \((Y,\Gamma)\) interchangeably.

**Remark 3.8.** Consider the very special case where \( L = U \). If \((Y,\Gamma)\) is a solution of RBSDE\(^{(T)}\)(\(\xi,f,L,U\)), then of course
\[
Y_t = L_t, \quad \Gamma_t = -\int_0^t f(r,L_r) \, dr - L_t + L_0, \quad t \in [0,T],
\]
and \( \gamma_{\tau} = \tau \) for every \( \tau \in T \). When there is a semimartingale solution \((Y,M,R)\) of RBSDE\(^{(T)}\)(\(\xi,f,L,U\)), then
\[
\Gamma_t = R_t - M_t, \quad t \in [0,T].
\]

### 3.2 Existence, uniqueness and approximation of solutions

Let us consider the following hypotheses:

(H1) \( E|\xi| < \infty \) and there exists a càdlàg process \( S \), which is a difference of supermartingales of class (D), such that \( E\int_0^T |f(r,S_r)| \, dr < \infty \).

(H2) there exists \( \mu \in \mathbb{R} \) such that for a.e. \( t \in [0,T] \) the function \( y \mapsto f(t,y) - \mu y \) is nonincreasing.

(H3) for a.e. \( t \in [0,T] \) the function \( y \mapsto f(t,y) \) is continuous.

(H4) \( \int_0^T |f(r,y)| \, dr < \infty \) for every \( y \in \mathbb{R} \).

We start with a comparison result.

**Theorem 3.9.** Assume that \( \xi_1 \leq \xi_2, \quad L^1_t \leq L^2_t, \quad U^1_t \leq U^2_t, \quad t \in [0,T] \), and for a.e. \( t \in [0,T] \) we have \( f^1(t,y) \leq f^2(t,y) \) for all \( y \in \mathbb{R} \). Assume also that \( f^1 \) satisfies (H2). Let \((Y^i,\Gamma^i), \ i = 1,2, \) be a solution to RBSDE\(^{(T)}\)(\(\xi^i,f^i,L^i,U^i\)). Then
\[
Y^1_t \leq Y^2_t, \quad t \in [0,T].
\]
Proof. Let \( \tau \in \mathcal{T} \) and \((\gamma_1^\tau, \Lambda_1^\tau), (\gamma_2^\tau, \Lambda_2^\tau)\) be defined by (3.2) but with \( L, U \) replaced by \( L^1, U^1 \) and \( L^2, U^2 \), respectively. Let \((\gamma_1^1, k), \{\gamma_2^2\}\) be the sequences constructed as in (3.3) but for \( \gamma_1^\tau \) replaced by \( \gamma_1^1 \) and \( \gamma_2^2 \), respectively. By the definition, \( Y^\tau \) is a special semimartingale on \([\tau, \gamma_1^\tau] \), \( i = 1, 2 \). In particular, \( Y^1, Y^2 \) are special semimartingales on \([\tau, \gamma_1^1 \wedge \gamma_2^2] \). By the Tanaka-Meyer formula and (H2),

\[
E(Y_\tau^1 - Y_\tau^2)^+ \leq E \left( Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k} \right)^+ + E \int_\tau^{\gamma_1^1 \wedge \gamma_2^2} \text{sgn}(Y_\tau^1 - Y_\tau^2) \, d(\Gamma_\tau^1, \nu(v) - \Gamma_\tau^2, \nu(v)) \\
+ \mu^+ E \int_\tau^{\gamma_1^1 \wedge \gamma_2^2} (Y_\tau^1 - Y_\tau^2)^+ \, dr.
\]

Hence, by condition (d) of Definition 3.6,

\[
E(Y_\tau^1 - Y_\tau^2)^+ \leq E \left( Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k} \right)^+ + \mu^+ E \int_\tau^{\gamma_1^1 \wedge \gamma_2^2} |Y_\tau^1 - Y_\tau^2| \, dr. \tag{3.4}
\]

We will show that

\[
\lim_{k \to \infty} (Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = 0. \tag{3.5}
\]

The reasoning below is for fixed \( \omega \in \Omega \). We consider several cases.

Case I: \( \gamma_1^\tau = \tau \) or \( \gamma_2^\tau = \tau \). Then \( \gamma_1^1 \wedge \gamma_2^2 = \tau \). If \( \tau < T \), then \( Y_\tau^1 = L_\tau^1 \) or \( Y_\tau^2 = U_\tau^2 \).

In both cases (3.5) is satisfied. If \( \tau = T \), then the limit in (3.5) equals \( (\xi_1 - \xi_2)^+ \), so (3.5) is satisfied by the assumptions.

Case II: \( \gamma_1^1 > \tau \) and \( \gamma_2^2 > \tau \). We divide the proof into several subcases.

Case II(a): \( \gamma_1^1 < \gamma_2^2 \). First suppose that there exists \( k_0 \) such that \( \gamma_1^1 \wedge \gamma_2^2 = \gamma_1^1, k \geq k_0 \). Then \( \gamma_1^1 = \gamma_1^1, k \geq k_0 \). Hence \( \omega \notin \Lambda_1^1 \), which implies that \( L_\tau^1 = U_\tau^1 \). Hence we get easily (3.5). Suppose now that \( \gamma_1^1 \wedge \gamma_2^2 < \gamma_1^1, k \geq 1 \). Then \( \gamma_1^1 < \gamma_1^1, k \geq 1 \), which implies that \( \omega \in \Lambda_1^1 \). Thus \( L_\tau^1 = U_\tau^1 \). Therefore

\[
(Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ \to (Y_\tau^{1,1} - Y_\tau^{2,1})^+ = (L_\tau^1 - U_\tau^2)^+ = 0.
\]

Case II(b): \( \gamma_1^1 > \gamma_2^2 \). The proof is analogous to that in Case II(a).

Case II(c): \( \gamma_1^1 = \gamma_2^2 < T \). First suppose that \( \gamma_1^1 \wedge \gamma_2^2 < \gamma_1^1, k \geq 1 \). Then \( \gamma_1^1 < \gamma_1^1, k \geq 1 \), which implies that \( \omega \in \Lambda_1^1 \cup \Lambda_2^1 \) or equivalently \( L_\tau^1 = U_\tau^1 \) or \( L_\tau^2 = U_\tau^2 \). Therefore

\[
(Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ \to (Y_\tau^{1,1} - Y_\tau^{2,1})^+ = (Y_\tau^{1,1} - Y_\tau^{2,1})^+ = 0.
\]

If \( \omega \in \Lambda_1^1 \), then \( (Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = (L_\tau^1 - U_\tau^2)^+ = 0 \). If \( \omega \in \Lambda_2^1 \), then \( (Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = (Y_\tau^{1,1} - U_\tau^2)^+ = 0 \).

Case II(d): \( \gamma_1^1 = \gamma_2^2 = T \). If there exists \( k_0 \in \mathbb{N} \) such that \( \gamma_1^1 \wedge \gamma_2^2 = T \), then \( (Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = (\xi_1 - \xi_2)^+ = 0, k \geq k_0 \). If \( \gamma_1^1 \wedge \gamma_2^2 < T, k \geq 1 \), then \( \omega \in \Lambda_1^1 \cup \Lambda_2^1 \) or equivalently \( L_\tau^1 = U_\tau^1 \) or \( L_\tau^2 = U_\tau^2 \). If \( \omega \in \Lambda_1^1 \), then \( (Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = (L_\tau^1 - U_\tau^2)^+ = 0 \). If \( \omega \in \Lambda_2^1 \), then \( (Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = (Y_\tau^{1,1} - U_\tau^2)^+ = 0 \).

We have shown that (3.5) is satisfied. Since \( Y^1, Y^2 \) are of class (D), it follows from (3.5) that

\[
\lim_{k \to \infty} E(Y_\tau^{1,k,\gamma_1^2,k} - Y_\tau^{2,k,\gamma_2^2,k})^+ = 0. \tag{3.6}
\]
Assume that Theorem 3.12. By Theorem 2.4, \( \bar{\tau} \) then the triple \((\bar{Y}, M, R)\) of \(Y\) we have \(\gamma_1^1 \wedge \gamma_2^2 = \gamma_\sigma^1 \wedge \gamma_\sigma^2\). Therefore from (3.7) it follows that

\[
E(Y_{\sigma}^1 - Y_{\sigma}^2)^+ \leq \mu^+ E \int_{\sigma}^{\gamma_\sigma^1 \wedge \gamma_\sigma^2} (Y_{r}^1 - Y_{r}^2)^+ dr.
\]  

(3.7)

Observe that for every \(\sigma \in \mathcal{T}\) such that

\[
\tau \leq \sigma \leq \gamma_\tau^1 \wedge \gamma_\tau^2 \quad \text{and} \quad \tau \leq \sigma < (\gamma_\tau^1)_\Lambda^1 \wedge (\gamma_\tau^2)_\Lambda^2
\]

(3.8)

we have \(\gamma_\tau^1 \wedge \gamma_\tau^2 = \gamma_\sigma^1 \wedge \gamma_\sigma^2\). Then there exists at most one solution \((\bar{Y}, \bar{M}, \bar{R})\) of RBSDE \(\Gamma\) is a solution of RBSDE \(\alpha, \beta\). By [29, Proposition 2.1], there exists a unique solution \((\bar{Y}_\tau, \bar{M}_\tau, \bar{R}_\tau)\) of the equation \(\bar{Y}_\tau \wedge \gamma_{\bar{Y}} = \gamma_\tau^1 \wedge \gamma_{\bar{Y}}\). Thus \(Y^1_{\tau} \leq Y^2_{\tau}\). Since \(\tau \in \mathcal{T}\) was arbitrary, we get \(Y^1_{\tau} \leq Y^2_{\tau}, t \in [0, T]\), by the Section Theorem. □

**Corollary 3.10.** Assume that (H2) is satisfied. Then there exists at most one solution of RBSDE\(^T\)\((\xi, f, L, U)\).

**Remark 3.11.** If \((Y, M, R)\) is a solution of RBSDE\(^T\)\((\xi, f, L, U)\) and \(\alpha, \beta \in \mathcal{T}, \alpha \leq \beta\), then the triple \((\bar{Y}, M - M_{\alpha}, R - R_{\alpha})\) is a solution of RBSDE\(^{\alpha, \beta}\)\((Y, f, L, U)\).

**Theorem 3.12.** Assume that (H1)–(H4) are satisfied.

(i) There exists a unique solution \((Y, \Gamma)\) of RBSDE\(^T\)\((\xi, f, L, U)\).

(ii) Let \(\{\xi_n\}\) be a sequence of integrable \(\mathcal{F}_T\)-measurable random variables such that \(\xi_n \nearrow \xi\), and let

\[
f_n(t, y) = f(t, y) + n(y - L_t)^-.
\]

Then for each \(n \in \mathbb{N}\) there exists a unique solution \((Y^n, M^n, A^n)\) of the equation \(\tilde{\text{RBSDE}}^T(\xi_n, f_n, U)\), and moreover, \(Y^n_t \nearrow Y_t, \Gamma^n_t \rightarrow \Gamma_t, t \in [0, T]\), where \(\Gamma^n_t = \int_0^t n(Y^n_r - L_r)^- dr - A^n_t - M^n_t, t \in [0, T]\).

**Proof.** By Theorem 2.4, for every \(n \geq 1\) there exists a unique solution \((Y^n, M^n, A^n)\) of \(\tilde{\text{RBSDE}}^T(\xi_n, f_n, U)\). By [29, Proposition 2.1], \(Y^n \leq Y^{n+1}\). Set

\[
Y_t = \lim_{n \to \infty} Y^n_t, \quad t \in [0, T] .
\]

By [29, Proposition 2.1] \(Y^n \leq \tilde{Y}^n\), where \((\tilde{Y}^n, \tilde{M}^n)\) is a solution of BSDE\(^T\)\((\xi, f_n)\). By Theorem 2.4, \(\tilde{Y}^n \nearrow \tilde{Y}\), where \((\tilde{Y}, \tilde{M}, \tilde{K})\) is a solution of \(\tilde{\text{RBSDE}}^T(\xi, f, L)\). Hence \(Y^1 \leq Y^n \leq \tilde{Y}, n \geq 1\), so \(Y\) is of class (D). By Theorem 2.4, for all \(\varepsilon > 0\) and \(n \geq 1\) there exists a unique solution \((Y^{n, \varepsilon}, M^{n, \varepsilon}, A^{n, \varepsilon})\) of \(\tilde{\text{RBSDE}}^T(\xi_n, f_n, \varepsilon, U)\) with

\[
f_{n, \varepsilon}(t, y) = f(t, y) + n(y - L^n_t)^-, \quad L^n = L - \varepsilon.
\]
By [29, Proposition 2.1], $Y^{n,\varepsilon} \leq Y^n$, while by Theorem 2.7 and Remark 2.8, $Y_t^{n,\varepsilon} \not
earrow Y_t^n$, $t \in [0,T]$, where $(Y_t^n, M_t^n, R_t^n)$ is a solution to $\text{RBSD}^T(\xi, f, L^n, U)$. Therefore $L^n \leq Y^n$, and since $\varepsilon > 0$ was arbitrary, $L \leq Y$. Of course $Y \leq U$. Now we will show that $Y$ is càdlàg. Let $\tau \in T$. Applying Theorem 2.7 (see also Remark 2.8, Remark 3.11) on $[\tau, \gamma_k^n]$ (see (3.2), (3.3) for the definition of $\gamma_\tau, \gamma_k^n$) with $\hat{\xi}^n = Y_{\tau_k}^n$ we see that $Y$ is càdlàg on $[\tau, \gamma_k^n]$. If $\tau < \gamma$ then we get that $Y$ is right-continuous in $\tau$ if $\tau = \gamma$, then $L = U = Y$, so $Y$ is right-continuous in $\tau$ by the right-continuity of $L, U$ and the fact that $L \leq Y \leq U$. Hence, by [12, IV.T28], $Y$ is right-continuous on $[0, T]$. Now let $\{\tau_n\} \subset T$ be an increasing sequence and $\tau := \sup_{n \geq 1} \tau_n$. It is clear that on the set $\{\omega \in \Omega; \tau_n(\omega) = \tau(\omega), n \geq n_\omega\} \cup \{L_{\tau^-} = U_{\tau^-}\}$ the limit $\lim_{n \to \infty} Y_{\tau_n}$ exists. Now we will show that this limit exists on the set

$$A = \{\tau_n < \tau, n \geq 1\} \cap \{L_{\tau^-} < U_{\tau^-}\}.$$ 

Applying Theorem 2.7 (see also Remark 2.8, Remark 3.11) on the interval $[\tau_n, \gamma_k^n]$ with $\tilde{\xi}^n = Y_{\gamma_k^n}$ for every $k \geq 1$ we see that $Y$ is càdlàg on $[\tau_n, \gamma_k^n]$. Since $A \subset \{L_{\tau^-} < U_{\tau^-}\}$, for every $\omega \in A$ there exists $n_\omega$ such that

$$[\tau_{n_\omega}(\omega), (\tau(\omega))] \subset [\tau_{n_\omega}(\omega), \gamma_{n_\omega}(\omega)].$$

Therefore $\lim_{n \to \infty} Y_{\tau_n}$ exists on $A$. Summing up we have that $\lim_{n \to \infty} Y_{\tau_n}$ exists a.s., so again by [12, IV.T28], $Y$ has left limits on $[0, T]$. Set

$$\Gamma^n_t = \int_0^t n(Y^n_r - L^n_r) - A^n_t - M^n_t, \quad t \in [0, T].$$

It is clear that

$$Y^n_t = \xi_n + \int_t^T f(r, Y^n_r) dr + \int_t^T d\Gamma^n_r, \quad t \in [0, T].$$

By (H2) and (H4) we may pass to the limit in the above equation getting condition (c) of the definition of RBSD$^T(\xi, f, L, U)$ with $\Gamma_t = Y_t + Y_0 - \int_0^t f(r, Y_r) dr$. Let $\tau \in T$. Applying Theorem 2.7 (see also Remark 3.11) on $[\tau, \gamma_k^n]$, $k \geq 1$ with $\tilde{\xi}^n = Y_{\gamma_k^n}$ (see also Remark 2.8) we see that $\Gamma$ is a special semimartingale on $[\tau, \gamma]$ and

$$\int_\tau^{\gamma} (Y^n_{\tau^-} - L^n_{\tau^-}) d\Gamma^n_{\tau^+}(\tau) = \int_\tau^{\gamma} (U^n_{\tau^-} - Y^n_{\tau^-}) d\Gamma^n_{\tau^-}(\tau) = 0,$$

which completes the proof. \hfill $\square$

**Corollary 3.13.** Assume that (H1)–(H4) are satisfied. Let $(Y, \Gamma)$ be a solution of $\text{RBSD}^T(\xi, f, L, U)$ and $(Y^n, M^n)$ be a solution of $\text{BSDE}^T(\xi, f_n)$ with

$$f_n(t, y) = f(t, y) + n(y - L_t)^- - n(y - U_t)^+.$$ 

Then $Y^n_t \to Y_t$, $t \in [0, T]$.

**Proof.** By Theorem 3.12, $Y^n \not\nearrow Y$, where $(\bar{Y}^n, \bar{M}^n, \bar{A}^n)$ is a solution of the equation $\overline{\text{RBSD}}^T(\xi, f_n, U)$ with $\bar{f}_n(t, y) = f(t, y) + n(y - L_t)^-$. In much the same manner one can show that $Y^n \not\nearrow Y$, where $(\bar{Y}^n, \bar{M}^n, \bar{K}^n)$ is a solution of $\overline{\text{RBSD}}^T(\xi, f_n, U)$ with $\bar{f}_n(t, y) = f(t, y) - n(y - U_t)^+$. By [29, Proposition 2.1], $\bar{Y}^n \leq Y^n \leq \bar{Y}^n$, which implies the desired result. \hfill $\square$
Corollary 3.14. Let $\alpha \in \mathbb{R}$ and $(Y, \Gamma)$ be a solution of $\text{RBSDE}^T(\xi, f, L)$. Then $(Y^\alpha, \Gamma^\alpha)$ is a solution of $\text{RBSDE}^T(\xi^\alpha, f^\alpha, L^\alpha, U^\alpha)$ with

$$\xi^\alpha = e^{\alpha T} \xi, \quad f^\alpha(t, y) = e^{\alpha t} f(t, e^{-\alpha t} y) - \alpha y, \quad L^\alpha_t = e^{\alpha t} L_t, \quad U^\alpha_t = e^{\alpha t} U_t,$$

where

$$Y^\alpha_t = e^{\alpha t} Y_t, \quad \Gamma^\alpha_t = e^{\alpha t} \Gamma_t - \int_0^t \alpha e^{\alpha r} \Gamma_r dr.$$

Proof. We first assume that $E \int_0^T |f(r, Y_r)| \, dr < \infty$. By Theorem 3.12, $Y^\alpha_t \wedge Y_t, t \in [0, T]$, and $\Gamma^\alpha_t \to \Gamma_t, t \in [0, T]$, where $(Y^n, M^n, A^n)$ is a solution of $\text{RBSDE}^T(\xi_n, f_n, U)$ with $f_n(t, y) = f(t, Y_t) + n(y - L_t)^-$ and

$$\Gamma^n_t = \int_0^t n(Y^n_r - L_r)^- \, dr - A^n_t - M^n_t, \quad t \in [0, T].$$

It is clear that

$$Y^n_t = \xi + \int_t^T f(r, Y_r) \, dr + \int_t^T d\Gamma^n_r, \quad t \in [0, T].$$

Integrating by parts we obtain

$$e^{\alpha t} Y^n_t = \xi^\alpha + \int_t^T e^{\alpha r} f(r, Y_r) \, dr - \int_t^T \alpha e^{\alpha r} Y^n_r \, dr + \int_t^T d\Gamma^n_{\tau}, \quad t \in [0, T],$$

with

$$\Gamma^n_{\tau} = e^{\alpha \tau} \Gamma^n_\tau - \int_0^\tau \alpha e^{\alpha r} \Gamma^n_r \, dr.$$

Therefore letting $n \to \infty$ in (3.10) we get

$$Y^\alpha_t = \xi^\alpha + \int_t^T f^\alpha(r, Y^\alpha_r) \, dr + \int_t^T d\Gamma^\alpha_r, \quad t \in [0, T].$$

It is clear that $Y^\alpha$ is of class (D) and $L^\alpha \leq Y^\alpha \leq U^\alpha$. What is left is to show that condition (d) of Definition 3.6 is satisfied. However, this condition easily follows from the fact that on the interval $[\tau, \gamma_r]$ we have

$$\Gamma^\alpha_{\tau} - \Gamma^\alpha_{\gamma_r} = \int_\tau^{\gamma_r} e^{\alpha r} \, d\Gamma^\alpha_r.$$

Let $\{\tau_k\}$ be a chain on $[0, T]$ such that $E \int_0^{\tau_k} |f(r, Y_r)| \, dr < \infty, k \geq 1$. By what has been already proved $(Y^\alpha, \Gamma^\alpha)$ is a solution to $\text{RBSDE}^{\tau_k}(Y^\alpha_{\tau_k}, f^\alpha, L^\alpha, U^\alpha)$. Since $\{\tau_k\}$ is a chain we get the result.

The following theorem shows that the solutions of reflected equation BSDE are stable with respect to the norm. $\| \cdot \|_{1, T}$ defined by (2.1).

Theorem 3.15. Let $(Y^i, \Gamma^i), i = 1, 2,$ be a solution of $\text{RBSDE}^T(\xi^i, f^i, L^i, U^i)$ and $f^i$ satisfy (H2). Then for all $\tau \in T$ and $\varepsilon > 0$,

$$(Y^1_\tau - Y^2_\tau)^+ \leq E \left( e^{(T-\tau)\mu^+}(\xi^1_\tau - \xi^2_\tau) + \int_\tau^{\hat{\beta}^1_{\tau}} e^{(r-\tau)\mu^+}(f^1(r, Y^2_r) - f^2(r, Y^2_r)) \, dr + e^{(\hat{\alpha}^1_{\tau} - \tau)\mu^+}1_{\hat{\beta}^1_{\tau} < T}(L^1_{\hat{\beta}^1_{\tau}} - L^2_{\hat{\beta}^1_{\tau}})^+ + e^{(\hat{\beta}^2_{\tau} - \tau)\mu^+}1_{\beta^2_{\tau} < T}(U^1_{\beta^2_{\tau}} - U^2_{\beta^2_{\tau}})^+ \right) + \varepsilon,$$

where $\hat{\beta}^1_{\tau} = \beta^1_{\tau} \land \beta^2_{\tau}$ and

$$\beta^1_{\tau} = \inf\{t \geq \tau : Y^1_t \leq L^1_t + \varepsilon\} \land T, \quad \beta^2_{\tau} = \inf\{t \geq \tau : Y^2_t \geq U^2_t - \varepsilon\} \land T.$$
For stopping times $\alpha, \tau$ for the barriers, we assume that

In this section we assume that

We define $\hat{\gamma}_\tau = \gamma_\tau^{1,k} \wedge \gamma_\tau^{2,k}$, $\hat{\gamma}_\tau = \gamma_\tau^{1} \wedge \gamma_\tau^{2}$ and $\sigma^{k}_\tau = \hat{\beta}_\tau \wedge \hat{\gamma}_\tau$. Observe that $\hat{\beta}_\tau \leq \hat{\gamma}_\tau$. By the minimality condition (d) in Definition 3.6 and the definition of $\hat{\beta}_\tau$, we have

$$\int_\tau^{\sigma^{k}_\tau} d\Gamma^{1,v,+}(\tau) + \int_\tau^{\sigma^{k}_\tau} d\Gamma^{2,v,-}(\tau) = 0.$$ 

Therefore applying the Tanaka-Meyer formula on $[\tau, \sigma^{k}_\tau]$ and using (H2) we get

$$\left( Y^1_\tau - Y^2_\tau \right)^+ \leq E\left( \left( Y^1_{\sigma^{k}_\tau} - Y^2_{\sigma^{k}_\tau} \right)^+ + \int_\tau^{\sigma^{k}_\tau} (f_1(r, Y^2_r) - f_2(r, Y^2_r)) + dr \right| F_\tau).$$

The following calculations are made for fixed $\omega \in \Omega$. We consider two cases.

Case I: $\hat{\gamma}_\tau = \tau$. If $\tau < T$, then $L^{1}_{\sigma^{k}_\tau} = U^{1}_{\sigma^{k}_\tau} = Y^1_{\sigma^{k}_\tau}$, $k \geq 1$ or $L^{2}_{\sigma^{k}_\tau} = U^{2}_{\sigma^{k}_\tau} = Y^2_{\sigma^{k}_\tau}$, $k \geq 1$. In both cases we have

$$\left( Y^1_{\sigma^{k}_\tau} - Y^2_{\sigma^{k}_\tau} \right)^+ \leq \max\left\{ (L^1_{\sigma^{k}_\tau} - L^2_{\sigma^{k}_\tau})^+, (U^1_{\sigma^{k}_\tau} - U^2_{\sigma^{k}_\tau})^+ \right\}$$

$$= \max\left\{ (L^1_{\gamma_\tau} - L^2_{\gamma_\tau})^+, (U^1_{\gamma_\tau} - U^2_{\gamma_\tau})^+ \right\}.$$ 

If $\tau = T$, then $(Y^1_{\sigma^{k}_\tau} - Y^2_{\sigma^{k}_\tau})^+ = (\xi^1 - \xi^2)^+.$

Case II: $\hat{\gamma}_\tau > \tau$. We consider the following three subcases.

Case II(a): $\hat{\beta}_\tau \in [\tau, \hat{\gamma}_\tau)$. Then $\hat{\beta}_\tau < \hat{\gamma}_\tau$, $k \geq k_0$. Moreover, $Y^1_{\sigma^{k}_\tau} \leq L^1_{\sigma^{k}_\tau} + \varepsilon$ or $Y^2_{\sigma^{k}_\tau} \geq U^2_{\sigma^{k}_\tau} - \varepsilon$, $k \geq k_0$. Therefore

$$\left( Y^1_{\sigma^{k}_\tau} - Y^2_{\sigma^{k}_\tau} \right)^+ \leq \max\left\{ (L^1_{\sigma^{k}_\tau} - L^2_{\sigma^{k}_\tau})^+, (U^1_{\sigma^{k}_\tau} - U^2_{\sigma^{k}_\tau})^+ \right\} + \varepsilon$$

$$= \max\left\{ (L^1_{\beta_\tau} - L^2_{\beta_\tau})^+, (U^1_{\beta_\tau} - U^2_{\beta_\tau})^+ \right\} + \varepsilon, \quad k \geq k_0.$$ 

Case II(b): $\hat{\gamma}_\tau = \hat{\beta}_\tau < T$. Then $\omega \notin \Lambda^1_\tau \cup \Lambda^2_\tau$. Hence $L^{1}_{\sigma^{k}_\tau} = U^{1}_{\sigma^{k}_\tau}$ or $L^{2}_{\sigma^{k}_\tau} = U^{2}_{\sigma^{k}_\tau}$, $k \geq k_0$. In both cases (3.12) is satisfied.

Case II(c): $\hat{\gamma}_\tau = \hat{\beta}_\tau = T$. Then $\omega \notin \Lambda^1_\tau \cup \Lambda^2_\tau$, so $(Y^1_{\sigma^{k}_\tau} - Y^2_{\sigma^{k}_\tau})^+ = (\xi^1 - \xi^2)^+ + k \geq k_0$. Combining Case I with Case II and (3.11) we get the desired result. \hfill \Box

Note that if the $f^1$ is nonincreasing with respect to $y$, i.e. (H2) is satisfied with $\mu \leq 0$, then Theorem 3.15 implies (1.9).

4 Reflected BSDEs with unbounded terminal time

In this section we assume that $T$ is a general (possibly infinite) $\mathbb{F}$-stopping time. As for the barriers, we assume that

$$\limsup_{a \to \infty} L_{T \wedge a} \leq \xi, \quad \liminf_{a \to \infty} U_{T \wedge a} \geq \xi.$$ 

We also modify the definition of the set $\Lambda_\tau$ introduced in Section 3. Now we set

$$\Lambda_\tau = \{ L_{\gamma_\tau} = U_{\gamma_\tau} \} \cap \{ \tau < \gamma_\tau < \infty \}. $$

For stopping times $\alpha \leq \beta$ we denote by $[[\alpha, \beta]]$ the random interval defined as

$$[[\alpha, \beta]] = \{ (t, \omega) \in [0, \infty) \times \Omega : \alpha(\omega) \leq t \leq \beta(\omega) \}. $$

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We say that $Y^1 = Y^2$ on $[[\alpha, \beta]]$ if for a.e. $\omega \in \Omega$ we have $Y^1_t(\omega) = Y^2_t(\omega)$ for every $t \geq 0$ such that $(t, \omega) \in [[\alpha, \beta]]$. We write that $Y^1 \geq Y^2$ on $[[\alpha, \beta]]$ if $(Y^1 - Y^2)^- = 0$ on $[[\alpha, \beta]]$. If the interval is clear from the context, we omit it in the notation. We also put $[\alpha(\omega), \beta(\omega)] = [\alpha(\omega), \infty)$ if $\beta(\omega) = \infty$.

The main difference between reflected BSDEs with bounded and unbounded terminal times lies in the definition of a solution, especially in condition (4.1) formulated below. Moreover, in case of unbounded terminal times we assume additionally that $\mu \leq 0$ in hypothesis (H2). One another difficulty which appears in the case of unbounded terminal time concerns the integrability of $f$. Recall that one of the standard assumptions when considering BSDEs with generator $f$ is the integrability of $f(\cdot, 0)$. In this paper we consider a slightly more general condition (H1). Of course, the same condition should be required for reflected BSDEs. For bounded terminal time, $f_n(\cdot, S)$ is integrable if and only if $f(\cdot, S)$ is integrable for $S$ appearing in (H1), because $L^+, S$ are of class (D). This is no longer true for unbounded terminal time. This forces some additional assumptions when considering the penalization scheme or some its modifications.

**Remark 4.1.** Let (H2) be satisfied with $\mu \leq 0$. Then Theorem 3.9 and Theorem 3.15 hold true for unbounded $T$. The proofs of these results run, without any changes, as the proofs of Theorems 3.9 and 3.15 (the proof of Theorem 3.9 is even simpler since the right-hand side of (3.7) equals zero).

### 4.1 Semimartingale solutions

**Definition 4.2.** We say that a pair $(Y, M)$ of $\mathbb{F}$-adapted càdlàg processes is a solution of the backward stochastic differential equation on $[\alpha, \beta]$ with an $\mathcal{F}_\beta$-measurable terminal condition $\xi$, generator $f$ (BSDE$^{\alpha, \beta}_t(\xi, f)$ for short) if for every $a \geq 0$ it is a solution of BSDE$^{\alpha, (\beta \land a) \lor a}_t(Y_{(\beta \land a) \lor a}, f)$ and

$$Y_{(\beta \land a) \lor a} \to \hat{\xi} \quad \text{a.s. as } a \to \infty. \quad (4.1)$$

**Definition 4.3.** We say that a triple $(Y, M, K)$ of $\mathbb{F}$-adapted càdlàg processes is a solution of the reflected BSDE on $[\alpha, \beta]$ with an $\mathcal{F}_\beta$-measurable terminal condition $\hat{\xi}$, generator $f$ and lower barrier $L$ (RBSDE$^{\alpha, \beta}_t(\hat{\xi}, f, L)$ for short) if for every $a \geq 0$ it is a solution of RBSDE$^{\alpha, (\beta \land a) \lor a}_t(\hat{\xi}, f, L)$ and (4.1) is satisfied.

**Definition 4.4.** We say that a triple $(Y, M, A)$ of $\mathbb{F}$-adapted càdlàg processes is a solution of the reflected BSDE on $[\alpha, \beta]$ with an $\mathcal{F}_\beta$-measurable terminal condition $\hat{\xi}$, generator $f$ and upper barrier $U$ (RBSDE$^{\alpha, \beta}_t(\hat{\xi}, f, U)$ for short) if $(-Y, A, -M)$ is a solution of RBSDE$^{\alpha, \beta}_t(-\hat{\xi}, \hat{f}, -U)$ with $\hat{f}(t, y) = -f(t, -y)$.

**Definition 4.5.** We say that a triple $(Y, M, R)$ of $\mathbb{F}$-adapted càdlàg processes is a solution of reflected BSDE on $[\alpha, \beta]$ with an $\mathcal{F}_\beta$-measurable terminal condition $\hat{\xi}$, generator $f$ and barriers $L$ and $U$ (RBSDE$^{\alpha, \beta}_t(\hat{\xi}, f, L, U)$ for short) if for every $a \geq 0$ it is a solution of RBSDE$^{\alpha, (\beta \land a) \lor a}_t(\hat{\xi}, f, L, U)$ and (4.1) is satisfied.

**Remark 4.6.** If $\beta < \infty$, then the above definitions are equivalent to the corresponding definitions of Section 2.

**Remark 4.7.** A brief inspection of the proofs reveals that all the results of Sections 2 and 3 concerning the convergence of the penalization schemes, i.e. schemes including
From now on, \( \eta \) is a strictly positive bounded \( \mathbb{F} \)-progressively measurable process such that \( E \int_0^T \eta_t (S_t - L_t)^- \, dt + E \int_0^T \eta_t (S_t - U_t)^+ \, dt < \infty \). Such a process always exists. For instance, the process defined as

\[
\eta_t = \frac{2}{\pi} \frac{1}{1 + t^2}, \quad t \geq 0,
\]

has the desired property because

\[
E \int_0^T \eta_t (S_t - L_t)^- \, dt + E \int_0^T \eta_t (S_t - U_t)^+ \, dt \leq \|S\|_1 + \|L^+\|_1 + \|U^-\|_1.
\]

**Theorem 4.8.** Assume that \( f \) satisfies (H1)–(H4) on \([\alpha, \beta]\) with \( \mu \leq 0 \), \( \hat{\xi} \) is an \( \mathcal{F}_\beta \)-measurable integrable random variable such that \( \lim \sup_{u \to \infty} L_{(\beta \wedge a) \vee a} \leq \hat{\xi} \) and \( L^+ \) is of class (D) on \([\alpha, \beta]\). Then there exists a unique solution \((Y, M, K)\) of RBSDE\(^{\alpha, \beta} \)(\( \hat{\xi}, f, L \)). Moreover, \( Y^n_t \nrightarrow Y_t \), \( t \in [\alpha, \beta] \), where \((Y^n, M^n)\) is a solution to BSDE\(^{\alpha, \beta} \)(\( \xi, f_n \)) with

\[
f_n(t, y) = f(t, y) + n\eta(t - L_t^-).
\]

**Proof.** Without lost of generality we can assume that \( L \) is of class (D) (see Remark 2.5). By [31, Theorem 2.8], for every \( n \geq 1 \) there exists a unique solution \((Y^n, M^n)\) of BSDE\(^{\alpha, \beta} \)(\( \xi, f_n \)). By [32, Proposition 3.1], \( Y^n_t \leq Y^n_{t+1}, \ t \in [\alpha, \beta] \). Define \( Y \) as \( Y_t = \lim_{n \to \infty} Y^n_t \), \( t \in [\alpha, \beta] \). Observe that \((Y^n, M^n, K^n)\) with \( K^n_t = \int_0^t n(Y^n_r - L_r)^- \, dr \) is a solution of RBSDE\(^{\alpha, \beta} \)(\( \hat{\xi}, f, L^n \)) with \( L^n = L - (Y^n - L)^- \). Let

\[
S_t = S_0 + V_t + N_t, \quad t \in [\alpha, \beta],
\]

be the Doob-Meyer decomposition of \( S \). Let \( \tau \) be a stopping times such that \( \alpha \leq \tau \leq \beta \) and let

\[
\sigma_n = \inf \{ t \geq \tau; Y^n_t \leq L^n_t + \varepsilon \} \wedge \beta.
\]

By the Tanaka-Meyer formula, (H2) and the minimality condition (see (iii) of Definition 2.2)

\[
(Y^n_\tau - S_\tau)^+ \leq E \left( (Y^n_{\tau^-} - S_{\tau^-})^+ + \int_{\tau^-}^{\tau} 1_{\{Y^n_r > S_r^-\}} f(r, Y^n_r) \, dr \right.
\]

\[
+ \int_{\tau^-}^{\sigma_n} 1_{\{Y^n_r > S_r^-\}} \, dV_r + \int_{\tau^-}^{\sigma_n} 1_{\{Y^n_r > S_r^-\}} \, dK^n_r |\mathcal{F}_\tau \bigg)
\]

\[
\leq E \left( (L_{\sigma_n} - S_{\sigma_n})^+ 1_{\{\sigma_n < \beta\}} + (\xi - S_{\beta})^+ 1_{\{\sigma_n = \beta\}} \right)
\]

\[
+ \int_{\tau^-}^{\sigma_n} 1_{\{Y^n_r > S_r^-\}} f(r, S_r) \, dr + \int_{\tau^-}^{\beta} d|V_r| |\mathcal{F}_\tau \bigg) + \varepsilon.
\]

From this inequality, the fact that \( L^+, S \) are of class (D), \( Y^n \nrightarrow Y \) and \( E \int_{\alpha}^{\beta} |f(r, S_r)| \, dr + E \int_{\alpha}^{\beta} d|V_r| < \infty \) we get that \( Y^+ \) is of class (D). Since \( Y^1 \leq Y \) we have that \( Y \) is of class (D). Write \( \beta_a = (\beta \wedge a) \vee \alpha \). By Theorem 2.4 and Remark 4.7 applied on the interval
Let $\alpha, \beta \in \mathbb{R}$, $Y^n_t \Rightarrow Y^n_t$, $t \in [\alpha, \beta]$, where $(Y^a, M^n)$ is a solution of $RBSDE^{a,\beta}(Y^a, M, f, L)$. Let $M = M^n_t$, $t \in [\alpha, \beta]$. By uniqueness, $M$ is well defined. We see that $(Y, M)$ is a solution to $RBSDE^{a,\beta}(Y^a, M, f, L)$ for every $a \geq 0$. What is left is to show that (4.1) is satisfied. Since $Y$ is of class (D), $\sup_{t \in [\alpha, \beta]} |Y_t|$ is finite a.s. Hence, by (H2) and (H4), there exists a chain $\{\tau_k\}$ on $[\alpha, \beta]$ such that

$$E \int_0^{\tau_k} |f(r, Y_t)| dr < \infty, \quad k \geq 1.$$  

Applying now [31, Lemma 3.8] on the interval $[\alpha, \tau_k]$, we get

$$Y_{(\tau_k \wedge a)\cap \alpha} \to Y_{\tau_k}$$

as $a \to \infty$. Since $\{\tau_k\}$ is a chain on $[\alpha, \beta]$, $P(\tau_k < \beta) \to 0$ as $k \to \infty$. Consequently, (4.1) is satisfied.

### 4.2 Non-semimartingale solutions

**Definition 4.9.** We say that a pair $(Y, \Gamma)$ of $\mathbb{F}$-adapted càdlàg process is a solution of the reflected backward stochastic differential equation on the interval $[0, T]$ with terminal time $\xi$, generator $f$, lower barrier $L$ and upper barrier $U$ ($RBSDE^T(\xi, f, L, U)$ for short) if for every $a \geq 0$ it is a solution of $RBSDE^{T\wedge a}(Y_{T\wedge a}, f, L, U)$ and (4.1) is satisfied with $\alpha = 0, \beta = T$.

**Theorem 4.10.** Assume that (H1)–(H4) are satisfied with $\mu \leq 0$.

(i) There exists a unique solution $(Y, \Gamma)$ of $RBSDE^T(\xi, f, L, U)$.

(ii) Let $\{\xi_n\}$ be an increasing sequence of integrable $\mathcal{F}_T$-measurable random variables such that $\xi_n \Rightarrow \xi$, and let

$$f_n(t, y) = f(t, y) + n\eta(y - L_t)^-.$$  

Then for each $n \in \mathbb{N}$ there exists a unique solution $(Y^n, M^n, A^n)$ to the equation $\underline{RBSDE}^T(\xi_n, f_n, U)$. Moreover, $Y^n_t \Rightarrow Y_t$ and $\Gamma^n_t \to \Gamma_t$, $t \in [0, T \wedge a]$, where $\Gamma^n_t = \int_0^t n(Y^n_r - L_r)^- dr - A^n_t - M^n_t$, $t \in [0, T \wedge a]$, $a \geq 0$.

**Proof.** By Theorem 4.8, for each $n \in \mathbb{N}$ there exists a unique solution $(Y^n, M^n, A^n)$ of $\underline{RBSDE}^T(\xi_n, f_n, U)$. By Theorem 3.9, $Y^n \leq Y^{n+1}$. Set

$$Y_t = \lim_{n \to \infty} Y^n_t, \quad t \in [0, T].$$

Observe that $Y^n \leq Y^n$, where $(\hat{Y}^n, \hat{M}^n)$ is a solution of $BSDE^T(\xi_n, f_n)$. By Theorem 4.8, $\hat{Y}^n \Rightarrow \hat{Y}$, where $(\hat{Y}, \hat{M}, \hat{K})$ is a solution of $\underline{RBSDE}^T(\xi, f, L)$. Hence $Y^1 \leq Y^n \leq Y^n$, $n \geq 1$, so $Y$ is of class (D). From Theorem 4.8 and Remark 4.7 applied on the interval $[0, T \wedge a]$ it follows that $Y^n \Rightarrow Y^a$, where $(Y^a, \Gamma^a)$ is a solution of $RBSDE^{T\wedge a}(Y_{T\wedge a}, f, L, U)$. Set $\Gamma_t = \Gamma^a_t$, $t \in [0, T \wedge a]$. It is clear that $\Gamma$ is well defined. What is left is to show that (4.1) is satisfied with $\alpha = 0, \beta = T$. But this is a consequence of the inequality $Y^1 \leq Y \leq \hat{Y}$.

In Proposition 5.9 we will prove that in part (i) of Theorem 4.10 hypotheses (H1) can be omitted if we assume that $L, U$ are of class (D).

The following theorem says that under Mokobodzki’s condition a solution in the sense of Definition 4.9 becomes semimartingale solution.

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Theorem 4.11. Assume that \((Y, \Gamma)\) is a solution to RBSDE\(^T\)\((\xi, f, L, U)\). If there exists a special semimartingale between the barriers \(L, U\), then \(Y, \Gamma\) are special semimartingales and the triple \((Y, \Gamma^u, \Gamma^m)\) is a semimartingale solution of RBSDE\(^T\)\((\xi, f, L, U)\), i.e. in the sense of Definition 2.6, where \(\Gamma^u\) (resp. \(\Gamma^m\)) is a predictable finite variation part (resp. martingale part) from the Doob-Meyer decomposition of the special semimartingale \(\Gamma\).

**Proof.** Let \(\{\theta_k\}\) be a chain on \([0, T]\) such that \(E \int_0^{\tau_k} |f(r, Y_r)| \, dr < \infty\) for every \(k \geq 1\), and let \(\tau_k = \theta_k \wedge k\). Write \(f_Y(t) = f(t, Y_t)\). It is clear that \((Y, \Gamma)\) is a solution of RBSDE\(^T\)\((Y_{\tau_k}, f_Y, L, U)\) for every \(k \geq 1\). By Theorem 4.10, \(Y_{t,k}^{k,n} \nearrow Y_t, t \in [0, \tau_k]\), where \((Y^{k,n}, A^{k,n}, M^{k,n})\) is a solution of RBSDE\(^T\)\((Y_{\tau_k}, f_Y, U)\) with

\[
f_Y^n(t, y) = f_Y(t) + n \eta \gamma(y - L_t)^-.
\]

On the other hand, by Theorem 2.7, \(Y_t^{k,n} \nearrow \bar{Y}_t^k, t \in [0, \tau_k]\), where \((\bar{Y}_t^k, \bar{R}_t^k, \bar{M}_t^k)\) is a semimartingale solution of RBSDE\(^T\)\((Y_{\tau_k}, f_Y, L, U)\). By Theorem 3.9, \(Y = \bar{Y}_k\) on \([0, \tau_k]\), \(k \geq 1\). From this the result follows. \(\square\)

**Remark 4.12.** Under the assumptions of Theorem 4.10, \(Y^n \to Y\), where \((Y^n, M^n)\) is a solution of BSDE\(^T\)\((\xi, f_n)\) and

\[
f_n(t, y) = f(t, y) + n \eta \gamma(y - L_t)^- - n \eta \gamma(y - U_t)^+.
\]

To see this, we denote by \((\bar{Y}^n, \bar{M}^n, \bar{A}^n)\) a solution of RBSDE\(^T\)\((\xi, f_n, U)\) with

\[
\bar{f}_n(t, y) = f(t, y) + n \eta \gamma(y - L_t)^-,
\]

and by \((\bar{Y}^n, \bar{M}^n, \bar{K}^n)\) a solution of RBSDE\(^T\)\((\xi, \bar{f}_n, L)\) with

\[
\bar{f}_n(t, y) = f(t, y) - n \eta \gamma(y - U_t)^+.
\]

By Theorem 4.10, \(\bar{Y}^n \nearrow Y\) and \(\bar{Y}^n \searrow Y\), whereas by Theorem 3.9, \(\bar{Y}^n \leq Y^n \leq \bar{Y}^n\), from which the desired result follows.

**Theorem 4.13.** Assume that (H1)–(H4) with \(\mu \leq 0\) are satisfied and \(pL \geq L_-,\)

\(pU \leq U_-\). Then

\[Y^n \to Y\] in ucp,

where \((\bar{Y}^n, \bar{M}^n, \bar{A}^n)\), \((Y, \Gamma)\) are processes defined in Theorem 4.10.

**Proof.** By Theorem 4.10, \(\bar{Y}^n \nearrow Y\), so \((\bar{Y}^n - L)^- \searrow 0\), and hence \(p(\bar{Y}^n - L)^- \searrow 0\).

By the assumption on \(U\) and \([29,\text{Proposition 4.3}]\), \(\bar{A}^n\) is continuous, so \(p\bar{Y}^n = Y^n\).

Therefore by the assumption on \(L\),

\[
p(\bar{Y}^n - L)^- = (\bar{Y}^n - pL)^- \geq (\bar{Y}^- - L_-)^-.
\]

Consequently, \((\bar{Y}^n - L)^- \searrow 0\) and \((\bar{Y}^- - L_-)^- \searrow 0\), which by Dini’s theorem implies that \((\bar{Y}^n - L)^- \to 0\) in ucp. Since \(0 \leq (\bar{Y}^n - L)^- \leq |\bar{Y}^1| + L^+\) and \(\bar{Y}^1, L^+\) are of class (D), it follows that for every \(a \geq 0\), \(\|(\bar{Y}^n - L)^-\|_{1,T \wedge a} \to 0\) as \(n \to \infty\). Observe that the triple \((\bar{Y}^n, \bar{M}^n, \bar{R}^n)\) is a solution of RBSDE\(^T\)\((\xi_n, f, L_n, U)\) with

\[
L_n = L - (\bar{Y}^n - L)^-, \quad \bar{R}^n = n(\bar{Y}^n - L)^- - \bar{A}^n.
\]

By Theorem 3.15, \(\|\bar{Y}^n - Y\|_{1,T \wedge a} \leq E|\bar{Y}^n_{T \wedge a} - Y_{T \wedge a}| + \|(\bar{Y}^n - L)^-\|_{1,T \wedge a}\). Combining the above arguments, we easily obtain the desired result. \(\square\)
Corollary 4.14. Under the assumptions of Theorem 4.13,

\[ Y_n \to Y \quad \text{in ucp,} \]

where \((Y^n, M^n)\) is defined in Remark 4.12.

Proof. See the reasoning in Remark 4.12. \qed

5 Dynkin games, RBSDEs and nonlinear \(f\)-expectation

In this section we maintain the notation and general assumptions on \(T\) and \(L, U\) from Section 4.

Theorem 5.1. Let \((Y, \Gamma)\) be a solution of RBSDE\(^T\)\((\xi, f, L, U)\). Assume additionally that \(E \int_0^T |f(r, Y_r)| \, dr < \infty\). Then for every \(\alpha \in T\),

\[ Y_\alpha = \text{ess sup}_{\sigma \geq \alpha} \text{ess inf}_{\tau \geq \alpha} J_\alpha(\tau, \sigma) = \text{ess inf}_{\tau \geq \alpha} \text{ess sup}_{\sigma \geq \alpha} J_\alpha(\tau, \sigma), \quad (5.1) \]

where

\[ J_\alpha(\tau, \sigma) = E \left( \int_\tau^{\sigma \land \sigma} f(r, Y_r) \, dr + L_\sigma 1_{\sigma < \tau} + U_\tau 1_{\tau \leq \sigma < T} + \xi 1_{\sigma = \tau = T} | F_\alpha \right). \quad (5.2) \]

Moreover, for all \(\sigma, \tau \in T_\alpha\),

\[ J_\alpha(\tau_\epsilon, \sigma) - \epsilon \leq Y_\alpha \leq J_\alpha(\tau, \sigma_\epsilon) + \epsilon, \quad (5.3) \]

where

\[ \tau_\epsilon = \inf\{ t \geq \alpha : Y_t \geq U_t - \epsilon \} \land T, \quad \sigma_\epsilon = \inf\{ t \geq \alpha : Y_t \leq L_t + \epsilon \} \land T. \quad (5.4) \]

Proof. Let \(\tau, \sigma \in T_\alpha\). It is clear that \(\sigma_\epsilon, \tau_\epsilon \leq \gamma_\alpha\). Let \(\{\delta_n\}\) be a fundamental sequence for the local martingale \(\Gamma^m(\alpha)\) on \([\alpha, \gamma_\alpha]\), and let

\[ \theta_k = \tau_\epsilon \land \sigma \land \gamma^n_\alpha. \]

By the minimality condition on \(U\) (see Definition 2.6(c)),

\[ Y_\alpha = Y_{\theta_k \land \delta_n} + \int_{\alpha}^{\theta_k \land \delta_n} f(r, Y_r) \, dr + \int_{\alpha}^{\theta_k \land \delta_n} \mu(r) \, d\mu_r + \int_{\alpha}^{\theta_k \land \delta_n} d\Gamma^w_r(\alpha) - \int_{\alpha}^{\theta_k \land \delta_n} dM_r. \quad (5.5) \]

Since \(Y\) is of class (D), taking the conditional expectation with respect to \(F_\alpha\) of both sides of the above equality and then letting \(n \to \infty\) we get (observe that \(\theta_k \land \delta_n(\omega) = \theta_k(\omega), \, n \geq n_0(\omega)\))

\[ Y_\alpha = E \left( Y_{\theta_k} + \int_{\alpha}^{\theta_k} f(r, Y_r) \, dr + \int_{\alpha}^{\theta_k} \mu(r) \, d\mu_r + \int_{\alpha}^{\theta_k} d\Gamma^w_r(\alpha) | F_\alpha \right). \quad (5.6) \]

As \(k \to \infty\), we have

\[ Y_{\theta_k} \to Y_{\tau_\epsilon \land \sigma}. \quad (5.7) \]

To see this, let us consider two cases: (a) \(\theta_k = \tau_\epsilon \land \sigma\) for some \(k \geq k_0\) (\(k_0\) depends on \(\omega\)), and (b) \(\theta_k < \tau_\epsilon \land \sigma\), \(k \geq 1\). It is clear that (5.7) is satisfied in case (a). In case (b),

\[ Y_{\theta_k} \to Y_{\tau_\epsilon \land \sigma}. \quad (5.7) \]
Theorem 4.11, the triple $(Y,\omega \notin \Lambda_\alpha)$, for otherwise we would have $\tau_\varepsilon < \gamma_\alpha$ (since $L_{\gamma_\alpha} = U_{\gamma_\alpha}$ if $\omega \in \Lambda_\alpha$), which in turn implies (a) (since $\gamma_k^{\alpha} \nleq \gamma_\alpha$). Hence, in case (b), $\gamma_k^{\alpha} < \gamma_\alpha$, $k \geq 1$. This is possible only if $\gamma_\alpha = \infty$, so $\gamma_k^{\alpha} \to \infty$. Since $\theta_k = \gamma_k^{\alpha}$ in case (b), it follows that $\theta_k \to \infty$ and $\tau_\varepsilon \wedge \sigma = \infty$, which implies that $Y_{\theta_k} \to \xi = Y_{\tau_\varepsilon \wedge \sigma}$, i.e. (5.7) is satisfied. Letting $k \to \infty$ in (5.6) and using (5.7) and the definition of $\tau_\varepsilon$, we get

$$J_\alpha(\tau_\varepsilon, \sigma) - \varepsilon \leq Y_\alpha.$$

A similar argument applied to the pair $\tau,\sigma_\varepsilon$ gives the second inequality in (5.3). From (5.3) we easily deduce (5.1).

**Corollary 5.2.** Assume that $Y$ is a progressively measurable process such that we have $E \int_0^T |f(r,Y_r)| \, dr < \infty$ and (5.1) holds for every $\alpha \in \mathcal{T}$. Then $Y$ is a solution of RBSDE$_T^\alpha(\xi,f,Y,L,U)$.

**Proof.** By Theorem 4.10, there exists a unique solution $Y$ of RBSDE$_T^\alpha(\xi,f,Y,L,U)$ with $f(t) = f(t,Y_t)$. By Theorem 5.1, for every $\alpha \in \mathcal{T}$,

$$\hat{Y}_\alpha = \text{ess sup}_{\sigma \geq \alpha} \text{ess inf}_{\tau \geq \alpha} J_\alpha(\tau,\sigma) = \text{ess inf}_{\tau \geq \alpha} \text{ess sup}_{\sigma \geq \alpha} J_\alpha(\tau,\sigma),$$

where $J_\alpha(\tau,\sigma)$ is given by (5.2). Thus $Y = \hat{Y}$, so $Y$ is a solution of RBSDE$_T^\alpha(\xi,f,Y,L,U)$.

**Theorem 5.3.** Assume that

$$pL \geq L_-, \quad pU \leq U_-.$$  \hfill (5.8)

Let $(Y,\Gamma)$ be a solution of RBSDE$_T^\alpha(\xi,f,Y,L,U)$ such that $E \int_0^T |f(r,Y_r)| \, dr < \infty$. Then for every $\alpha \in \mathcal{T}$,

$$Y_\alpha = J_\alpha(\sigma_\alpha^*, \tau_\alpha^*),$$

where $J_\alpha$ is given by (5.2) and

$$\sigma_\alpha^* = \inf\{t \geq \alpha : Y_t = L_t\} \wedge T, \quad \tau_\alpha^* = \inf\{t \geq \alpha : Y_t = U_t\} \wedge T.$$  \hfill (5.10)

**Proof.** Step 1. We assume additionally that $Y$ (or, equivalently, $\Gamma$) is a special semimartingale. Under this additional condition we will show that

$$\int_0^{\sigma_\alpha^*} d\Gamma_r^{\nu^+} = \int_0^{\tau_\alpha^*} d\Gamma_r^{\nu^-} = 0.$$ \hfill (5.11)

By Theorem 4.11, the triple $(Y,\Gamma^\nu,\Gamma^m)$ is a semimartingale solution of the problem RBSDE$_T^\alpha(\xi,f,Y,L,U)$. Let $\{\tau_k\}$ be a fundamental sequence for the local martingale $\Gamma^m - \Gamma^\alpha_\alpha$ on $[\alpha,T]$. We set

$$\theta_k = \tau_\alpha^* \wedge \sigma_\alpha^* \wedge \tau_k,$$

and then

$$A_k = \{(t,\omega) \in [\alpha,\theta_k] : Y_{t-}(\omega) = L_{t-}(\omega), \Delta \Gamma_t^{\nu^+}(\alpha)(\omega) > 0\},$$

$$B_k = \{(t,\omega) \in [\alpha,\theta_k] : Y_{t-}(\omega) = U_{t-}(\omega), \Delta \Gamma_t^{\nu^-}(\alpha)(\omega) > 0\}.$$
We will show that $P(\Pi(A_k)) = P(\Pi(B_k)) = 0$. Assume that $P(\Pi(A_k)) > 0$. Since $A_k$ is predictable, by the Section Theorem, for every $\varepsilon > 0$ there exists a predictable stopping time $\tau$ (depending on $k, \varepsilon$) such that

$$[[\tau]] \subset A_k, \quad P(\Pi(A_k)) \leq P(\tau < \infty) + \varepsilon. \quad (5.12)$$

Observe that on the set $\{\tau < \infty\}$ we have

$$Y_\tau - L_{\tau-} + \Delta \Gamma^\varepsilon_{\tau-} = \Delta \Gamma^m_{\tau-}. \quad (5.13)$$

Since $\tau$ is predictable and $L_\tau \leq Y_\tau$, we have $E1_{\{\tau < \infty\}}(Y_\tau - L_{\tau-}) \geq 0$ by (5.8). By predictability of $\tau$, we also have $E1_{\{\tau < \infty\}}\Delta \Gamma^m_{\tau-} = 0$. Hence, by (5.13), $E1_{\{\tau < \infty\}}\Delta \Gamma^\varepsilon_{\tau-} = 0$. Therefore $P(\Pi(A_k)) = 0$ by (5.13). In much the same way one can show that $P(\Pi(B_k)) = 0$. From this and Definition 2.6(c), we get (5.11).

Step 2. The general case. Let $(Y^\varepsilon, \Gamma^\varepsilon)$ be a solution of RBSDE$^T(\xi, f, L, U + \varepsilon)$, and $(Y^\varepsilon, \Gamma^\varepsilon)$ be a solution of RBSDE$^T(\xi, f, L - \varepsilon, U)$. By Remark 2.8 and Theorem 4.11, $Y^\varepsilon, Y^\varepsilon$ are special semimartingales and $(Y^\varepsilon, \Gamma^\varepsilon, \Gamma^\varepsilon, m)$, $(Y^\varepsilon, \Gamma^\varepsilon, \Gamma^\varepsilon, m)$ are usual semimartingale solutions. Moreover, by Theorem 3.9, $Y^\varepsilon \leq Y \leq Y^\varepsilon$. Hence $\tau^*_{\varepsilon} \geq \tau^*_k$ and $\sigma^*_{\varepsilon} \geq \sigma^*_k$, where

$$\tau^*_{\varepsilon} = \inf\{t \geq \alpha : Y^\varepsilon_t = U_t\} \wedge T, \quad \sigma^*_{\varepsilon} = \inf\{t \geq \alpha : Y^\varepsilon_t = L_t\} \wedge T.$$

By (5.11),

$$Y^\varepsilon_t = Y^\varepsilon_\alpha - \int^t_\alpha f(r, Y^\varepsilon_r) dr + \int^t_\alpha d\Gamma^\varepsilon_{\alpha \rightarrow +} + \int^t_\alpha d\Gamma^\varepsilon_{\alpha \rightarrow -}, \quad t \in [\alpha, \tau^*_{\varepsilon} \wedge \sigma^*_{\varepsilon}]$$

and

$$Y^\varepsilon_t = Y^\varepsilon_\alpha - \int^t_\alpha f(r, Y^\varepsilon_r) dr - \int^t_\alpha d\Gamma^\varepsilon_{\alpha \rightarrow +} + \int^t_\alpha d\Gamma^\varepsilon_{\alpha \rightarrow -}, \quad t \in [\alpha, \tau^*_{\varepsilon} \wedge \sigma^*_{\varepsilon}].$$

Therefore $Y^\varepsilon + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ is a submartingale of class (D) on $[\alpha, \tau^*_{\varepsilon} \wedge \sigma^*_{\varepsilon}]$ and $Y^\varepsilon + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ is a supermartingale of class (D) on $[\alpha, \tau^*_{\varepsilon} \wedge \sigma^*_{\varepsilon}]$. By Theorem 3.15, $Y^\varepsilon + \int^t_\alpha f(r, Y^\varepsilon_r) dr \rightarrow Y + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ and $Y^\varepsilon + \int^t_\alpha f(r, Y^\varepsilon_r) dr \rightarrow Y + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ in the norm $\|\cdot\|_{1, \alpha, \tau^*_{\varepsilon} \wedge \sigma^*_{\varepsilon}}$. It follows that $Y + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ is a uniformly integrable martingale on $[\alpha, \tau^*_{\varepsilon} \wedge \sigma^*_{\varepsilon}]$. From this one can deduce (5.9). □

**Remark 5.4.** Assume that (H1), (H2), (5.8) are satisfied. Let $(Y, \Gamma)$ be a solution to RBSDE$^T(\xi, f, L, U)$. Then for every $\alpha \in T,$

$$E \int^\alpha_{\tau_k \wedge \sigma^*_{\varepsilon}} |f(r, Y_r)| dr < \infty, \quad (5.14)$$

where $\sigma^*_{\varepsilon}, \tau^*_{\varepsilon}$ are defined by (5.10). Moreover, the process $Y + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ is a uniformly integrable martingale on the closed interval $[\alpha, \sigma^*_{\varepsilon} \wedge \tau^*_{\varepsilon}].$ To see this, we set

$$\tau_k = \inf\{t \geq \alpha : \int^t_\alpha |f(r, Y^\varepsilon_r)| dr \geq k\} \wedge T, \quad \theta_k = \sigma^*_{\varepsilon} \wedge \tau^*_{\varepsilon} \wedge \tau_k.$$

By Theorem 5.3, the process $Y + \int^t_\alpha f(r, Y^\varepsilon_r) dr$ is a uniformly integrable martingale on $[\alpha, \theta_k].$ Therefore $Y$ is the first component of the solution of BSDE$^T(\alpha, \theta_k, Y_{\theta_k}, f).$ By (H1), (H2) and [31, Theorem 2.8],

$$E \int^\theta_k_\alpha |f(r, Y^\varepsilon_r)| dr \leq E|Y_{\theta_k}| + E|S_{\theta_k}| + E \int^\theta_k_\alpha |f(r, S_r)| dr + E \int^\theta_k_\alpha d[S^u]|_{r},$$

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where $S^v$ is the predictable finite variation part of the Doob-Meyer decomposition of $S$. Letting $k \to \infty$ and using (H1), (H2) and the fact that $Y$ is of class (D) yields (5.14). From this we easily conclude that the process $Y + \int_0^\tau f(r, Y_r) \, dr$ is a uniformly integrable martingale on $[\alpha, \sigma^\ast \land \tau^\ast]_\alpha$.

**Remark 5.5.** By Theorem 5.1 and Remark 4.12, the value process $Y$ in the Dynkin game (1.2) can be approximated by solutions $Y^n$ of the penalized equation (1.7). This kind of results had appeared in the literature much before the notion of reflected BSDEs was introduced. In [55] (see also [54, 56] for Markovian case) Stettner proved that $Y$ given by (1.2), but with $f \equiv 0$, $T = \infty$ and barriers of the following special form

$$L_t = e^{-at} \hat{L}_t, \quad U_t = e^{-at} \hat{U}_t, \quad t \geq 0,$$

where $a > 0$ and $\hat{L}, \hat{U}$ are bounded right-continuous adapted processes, can by approximated by solutions of the following equation

$$Y_t^n = nE \left( \int_t^\infty e^{-ar}(Y_r^n - L_r)^- \, dr - \int_t^\infty e^{-ar}(Y_r^n - U_r)^+ \, dr \bigg| \mathcal{F}_t \right).$$

Observe that if we define $M^n$ as

$$M_t^n = nE \left( \int_0^\infty e^{-ar}(Y_r^n - L_r)^- \, dr - \int_0^\infty e^{-ar}(Y_r^n - U_r)^+ \, dr \bigg| \mathcal{F}_t \right) - Y_0^n,$$

then the pair $(Y^n, M^n)$ is a solution of the penalized BSDE (1.7) with $f(r, y) = -\alpha y$, $\xi = 0$ and $T = \infty$.

We now introduce the notion of the nonlinear expectation

$$\mathcal{E}_{\alpha, \beta}^f : L^1(\Omega, \mathcal{F}_\beta; P) \to L^1(\Omega, \mathcal{F}_\alpha; P)$$

for $\alpha, \beta \in \mathcal{T}$ such that $\alpha \leq \beta$ and for $f$ satisfying (H1)-(H4) with $\mu \leq 0$. For $\xi \in L^1(\Omega, \mathcal{F}_\beta; P)$ we put

$$\mathcal{E}_{\alpha, \beta}^f(\xi) = Y_\alpha,$$

where $(Y, M)$ be the unique solution of BSDE$^\beta(\xi, f)$.

We say that a càdlàg process $X$ of class (D) is an $\mathcal{E}^f$-supermartingale (resp. $\mathcal{E}^f$-submartingale) on $[\alpha, \beta]$ if $\mathcal{E}_{\alpha, \tau}^f(X_\tau) \leq X_\sigma$ (resp. $\mathcal{E}_{\sigma, \tau}^f(X_\tau) \geq X_\sigma$) for all $\tau, \sigma \in \mathcal{T}$ such that $\alpha \leq \sigma \leq \tau \leq \beta$. Of course, $X$ is called an $\mathcal{E}^f$-martingale on $[\alpha, \beta]$ if it is both $\mathcal{E}^f$-supermartingale and $\mathcal{E}^f$-submartingale on $[\alpha, \beta]$.

**Proposition 5.6.** Assume that $f$ satisfies (H1)-(H4) with $\mu \leq 0$ and $\alpha, \beta \in \mathcal{T}$, $\alpha \leq \beta$.

(i) Let $\xi \in L^1(\Omega, \mathcal{F}_\beta; P)$ and $V$ be a càdlàg $\mathcal{F}$-adapted finite variation process such that $V_\alpha = 0$ and $E|V|_\beta < \infty$. Let $(X, N)$ denote a solution of BSDE$^\alpha, \beta(\xi, f + dV)$. If $V$ (resp. $-V$) is an increasing process, then $X$ is an $\mathcal{E}^f$-supermartingale (resp. $\mathcal{E}^f$-submartingale) on $[\alpha, \beta]$.

(ii) If $\xi_1, \xi_2 \in L^1(\Omega, \mathcal{F}_\beta; P)$ and $\xi_1 \leq \xi_2$, then $\mathcal{E}_{\alpha, \beta}^f(\xi_1) \leq \mathcal{E}_{\alpha, \beta}^f(\xi_2)$.
(iii) If \( f_1, f_2 \) satisfy (H1)–(H4) with \( \mu \leq 0, \alpha, \beta_1, \beta_2 \in \mathcal{T}, \alpha \leq \beta_1 \leq \beta_2, \xi_1 \in L^1(\Omega, \mathcal{F}_{\beta_1}; P), \xi_2 \in L^1(\Omega, \mathcal{F}_{\beta_2}; P) \) then

\[
|\mathcal{E}_{\alpha,\beta_1}^f(\xi_1) - \mathcal{E}_{\alpha,\beta_2}^f(\xi_2)| \leq E \left[ |\xi_1 - \xi_2| + \int_{\alpha}^{\beta_1} |f_1(r, Y^1_r) - f_2(r, Y^1_r)| \, dr \right. \\
+ \left. \int_{\beta_1}^{\beta_2} |f_2(r, Y^2_r)| \, dr |\mathcal{F}_\alpha \right),
\]

where \( Y^1_t = \mathcal{E}_{\tau \wedge \beta_1, \beta_1}^f(\xi_1), Y^2_t = \mathcal{E}_{\tau \wedge \beta_2, \beta_2}^f(\xi_2) \).

**Proof.** Assertion (iii) follows from Theorem 3.15 and (ii) follows from Theorem 3.9. Now assume that \( X \) is as in (i) and \( V \) is an increasing process. Let \( \sigma, \tau \in \mathcal{T} \) be such that \( \alpha \leq \sigma \leq \tau \leq \beta_1 \) and let \( (X^\tau, N^\tau) \) be a solution of BSDE\(^{\alpha,\tau}(X^\tau, f) \). It is clear that \( (X, N) \) is a solution of BSDE\(^{\alpha,\tau}(X^\tau, f + dV) \). Therefore, by Theorem 3.9, \( X \geq X^\tau \) on \([\alpha, \tau]\). In particular, \( X_\sigma \geq X^\tau_\sigma \). By the definition of the nonlinear expectation, \( \mathcal{E}_{\tau}^f(X^\tau_\sigma) = X^\tau_\sigma \), so \( \mathcal{E}_{\tau}^f(X_\tau) \leq X_\sigma \). A similar reasoning in the case where \( -V \) is increasing gives the result. \( \square \)

**Theorem 5.7.** Assume that (H1)–(H4) are satisfied with \( \mu \leq 0 \) and \( L, U \) are of class (D).

(i) \((Y, \Gamma)\) is a solution of RBSDE\(^T\)(\(\xi, f, L, U\)) if and only if for every \( \alpha \in \mathcal{T}, \)

\[
Y_\alpha = \mathrm{ess} \sup_{\sigma \geq \alpha} \mathrm{ess} \inf_{\tau \geq \alpha} J^f_\alpha(\tau, \sigma) = \mathrm{ess} \inf_{\tau \geq \alpha} \mathrm{ess} \sup_{\sigma \geq \alpha} J^f_\alpha(\tau, \sigma),
\]

where

\[
J^f_\alpha(\tau, \sigma) = \mathcal{E}_{\alpha, \tau \wedge \sigma}^f(L_\alpha 1_{\sigma < \tau} + U_\tau 1_{\tau \leq \sigma < T} + \xi 1_{\sigma = \tau = T}).
\]

(ii) Let \((Y, \Gamma)\) be a solution of RBSDE\(^T\)(\(\xi, f, L, U\)). Then for all \( \sigma, \tau \in \mathcal{T}_\alpha \) we have

\[
J^f_\alpha(\tau_\varepsilon, \sigma) - \varepsilon \leq Y_\alpha \leq J^f_\alpha(\tau, \sigma_\varepsilon) + \varepsilon,
\]

where \( \tau_\varepsilon, \sigma_\varepsilon \) are defined by (5.4).

**Proof.** The proof of (ii) and the necessity part of (i) is similar to the proof of Theorem 5.1. The only difference is that the sequence \( \{\delta_n\} \) defined in that proof should now satisfy the additional condition \( E \int_{\alpha}^{\delta_n} |f(r, Y_r) - f_\alpha(r, Y_r)| \, dr + E \int_{\alpha}^{\delta_n} d\Gamma^{\alpha+}_n(\sigma) < \infty \). Now by (5.6) and Proposition 5.6(i) we get

\[
\mathcal{E}_{\alpha, \delta_n}^f(Y_{\delta_n}) \leq Y_\alpha.
\]

By the reasoning following (5.6) we know that \( Y_{\delta_n} \rightarrow Y_{\tau_\varepsilon \wedge \sigma} \) as \( k \rightarrow \infty \). So, by Proposition 5.6(iii) and the above inequality

\[
\mathcal{E}_{\alpha, \tau_\varepsilon \wedge \sigma}^f(Y_{\tau_\varepsilon \wedge \sigma}) \leq Y_\alpha.
\]

By the definition of \( \tau_\varepsilon \) and Proposition 5.6(ii) we conclude from the above inequality

\[
\mathcal{E}_{\alpha, \tau_\varepsilon \wedge \sigma}^f(L_\alpha 1_{\sigma < \tau_\varepsilon} + U_{\tau_\varepsilon} 1_{\tau_\varepsilon \leq \sigma < T} - \varepsilon + \xi 1_{\sigma = \tau_\varepsilon = T}) \leq Y_\alpha.
\]

From this and Proposition 5.6(iii) we get the left-hand side inequality in (5.17). An analogous reasoning applied to the pair \((\sigma_\varepsilon, \tau)\) gives the right-hand side inequality in (5.17). From (5.17) we easily get (5.15). The sufficiency of (i) is obvious. \( \square \)
Theorem 5.8. Assume that (H1)–(H4) with \( \mu \leq 0 \) hold true and (5.8) is satisfied for every finite predictable stopping time \( \tau \in \mathcal{T} \). Let \((Y, \Gamma)\) be a solution of RBSDE\(^T\)(\(\xi, f, L, U\)). Then for every \( \alpha \in \mathcal{T} \),
\[
Y_\alpha = J^f_\alpha (\sigma^{\alpha}_\alpha, \tau^{\alpha}_\alpha),
\]
where \( J^f_\alpha \) is defined by (5.16) and \( \sigma^{\alpha}_\alpha, \tau^{\alpha}_\alpha \) are defined by (5.10).

Proof. By Remark 5.4 and Proposition 5.6(i), \( E^{f}_{\alpha,\sigma^{\alpha}_\alpha,\tau^{\alpha}_\alpha}(Y^{\alpha}_{\sigma^{\alpha}_\alpha,\tau^{\alpha}_\alpha}) = Y_\alpha \). From this we get (5.18).

In Theorem 4.10 we have assumed that \( L^+, U^- \) are of class (D). Under the stronger assumption that \( L, U \) are of class (D), in its first part we can drop hypothesis (H1).

Proposition 5.9. Assume that (H1)–(H4) are satisfied with \( \mu \leq 0 \) and \( L, U \) are of class (D). Then there exists a unique solution of RBSDE\(^T\)(\(\xi, f, L, U\)).

Proof. We only need to prove the existence of a solution. To this end, we write
\[
f_{n,m}(t, y) = (f(t, y) \wedge m \eta_t) \vee (-m \eta_t).
\]

Then by Theorem 4.10 there exists a unique solution of RBSDE\(^T\)(\(\xi, f_{n,m}, L, U\)). By Theorem 3.9, \( Y_{n,m} \leq Y_{n+1,m} \) and \( Y_{n,m} \geq Y_{n,m+1} \). We put \( Y^m = \lim_{n \to \infty} Y_{n,m} \). Of course, \( L \leq Y^m \leq U \), so \( Y^m \) is of class (D). Next, we observe that \( Y^m \geq Y^{m+1} \) and we put \( Y = \lim_{m \to \infty} Y^m \). Of course, \( L \leq Y \leq U \), so \( Y \) is of class (D). By the definition,
\[
Y_{t}^{n,m} = Y_{T \wedge a}^{n,m} + \int_t^{T \wedge a} f_{n,m}(r, Y_r^{n,m}) \, dr + \int_t^{T \wedge a} d\Gamma_r^{n,m}, \quad t \in [0, T \wedge a].
\]

Since \( L \leq Y_{n,m} \leq U \), letting \( n \to \infty \) and then \( m \to \infty \) in the above equation and using (H2)–(H4) we obtain
\[
Y_t = Y_{T \wedge a} + \int_t^{T \wedge a} f(r, Y_r) \, dr + \int_t^{T \wedge a} d\Gamma_r, \quad t \in [0, T \wedge a].
\]

Since \( L, U \) are of class (D), by (H4) there exists a chain \( \{\tau_k\} \) on \([0, T]\) such that
\[
E \int_0^{\tau_k} |f(r, L_r)| \, dr + E \int_0^{\tau_k} |f(r, U_r)| \, dr < \infty.
\]

From what has already been proved and (H2)–(H4) it follows that
\[
\lim_{m \to \infty} \lim_{n \to \infty} E \int_0^{\tau_k} |f_{n,m}(r, Y_r^{n,m}) - f(r, Y_r)| \, dr = 0.
\]

Hence, by Theorem 3.15, \( \lim_{m \to \infty} \lim_{n \to \infty} \|Y_{n,m} - Y\|_{1,\tau_k} = 0 \). Therefore \( Y \) is a c\`adl\`ag process and \( Y_{\tau_k \wedge a} \to Y_{\tau_k} \) as \( a \to \infty \). Since \( \{\tau_k\} \) is a chain on \([0, T]\), we get (4.1). By Theorem 5.1,
\[
Y_{\alpha}^{n,m} = \text{ess sup}_{\tau_k \wedge a \geq \alpha} \text{ess inf}_{\tau_k \wedge a \geq \tau \geq \alpha} E \left( \int_\alpha^{\tau \wedge a} f_{n,m}(r, Y_r^{n,m}) \, dr \right.
\]
\[
+ L_r 1_{\sigma < \tau} + U_r 1_{\tau < \sigma < \tau \wedge a} + Y_{\tau \wedge a}^{n,m} 1_{\sigma = \tau = \tau \wedge a} \big| \mathcal{F}_\alpha \big)
\]
Letting $n \to \infty$ and then $m \to \infty$ and using (5.19) we obtain
\[
Y_\alpha = \text{ess sup}_{\tau_k \land \alpha \geq \alpha} \text{ess inf}_{\tau_k \land \alpha \geq \tau \geq \alpha} E \left( \int_0^{\tau \wedge \sigma} f(r, Y_r) \, dr + L_\sigma 1_{\sigma < \tau} + U_\tau 1_{\sigma \leq \tau < \tau_k \wedge \alpha} + Y_{\tau_k \wedge \alpha} 1_{\sigma = \tau_k \wedge \alpha} \, d\mathcal{F}_\alpha \right).
\]
By Corollary 5.2 we get that the pair $(Y, \Gamma)$ is a solution of RBSDE_{\tau_k \wedge \alpha}^a(Y_{\tau_k \wedge \alpha}, f, L, U)$. Since $\{\tau_k\}$ is a chain, we conclude that $(Y, \Gamma)$ is a solution of RBSDE_T^\xi(\xi, f, L, U). \qed

6 Markov-type RBSDEs

In this section we show that the value process for Markov-type Dynkin games has the Markovian structure. As a corollary we get formulas for saddle points and we show that the value function can be approximated by the penalty method. This generalizes the results of [16, 44, 54, 56] to general Markov processes and data (besides continuity of value function which depends on the special structure of the problem). We also show that in the important special case where the underlying Markov process $X$ is associated with some semi-Dirichlet (resp. generalized semi-Dirichlet form), the value function solves some stationary (resp. evolutionary) variational inequality. This generalizes the results of [42, 59, 60].

In what follows $E$ is a Lusin space and $X = (\{X_t, t \geq 0\}, \{P_x, x \in E \cup \{\Delta\}\}, \mathcal{F} = \{\mathcal{F}_t, t \geq 0\}, \{\theta_t, t \geq 0\}, \zeta)$ is a Borel right process on $E$. Here $\Delta$ is an isolated point adjacent to $E$ and $\zeta$ is the life time of $X$. Let $m$ be a $\sigma$-finite excessive measure for $X$. Recall that a nearly Borel set $B \subset E$ is called $m$-polar if
\[
P_m(\sigma_B < \infty) = 0,
\]
where $P_m(\cdot) = \int_E P_x(\cdot) m(dx)$ and $\sigma_B = \inf\{t > 0 : X_t \in B\}$. We say that some property holds q.e. if it holds outside some $m$-polar set.

Let $D$ be a nonempty finely open subset of $E$. We set
\[
\tau_D = \inf\{t > 0 : X_t \notin D\}.
\]
It is well known that
\[
P_x(\tau_D > 0) = 0 \quad \text{for q.e. } x \in E \setminus D. \quad (6.1)
\]
Let $X^D$ denote the process $X$ killed upon leaving $D$. Assume that $X^D$ is transient, i.e. there exists a bounded nearly Borel function $\rho : D \to \mathbb{R}^+$ such that $\rho > 0$ q.e. and $E_x \int_0^{\tau_D} \rho(X_r) \, dr < \infty$ for q.e. $x \in E$. We set $\rho_t = \rho(X_t)$.

Let $h_1, h_2 : E \to \mathbb{R}$ be functions such that $h_1 \leq h_2$ q.e. We also assume that $h_1$ and $h_2$ are quasi-càdlàg, i.e. the processes $h_1(X), h_2(X)$ are càdlàg under the measure $P_x$ for q.e. $x \in E$, and that $h_1^+(X), h_2^-(X)$ are of class (D) under the measure $P_x$ for q.e. $x \in E$. Note that each quasi-continuous function is quasi-càdlàg, and each excessive function is quasi-càdlàg. Let $\psi : D^c \to \mathbb{R}$ be a nearly Borel function such that $E_x|\psi(X_{\tau_D})| < \infty$ for q.e. $x \in E$. We assume that for q.e. $x \in E$ we have
\[
\limsup_{a \to \infty} h_1(X_{\tau_D \wedge a}) \leq \psi(X_{\tau_D}), \quad \liminf_{a \to \infty} h_2(X_{\tau_D \wedge a}) \geq \psi(X_{\tau_D}), \quad P_x\text{-a.s.}
\]
Let $\hat{f} : E \times \mathbb{R} \to \mathbb{R}$, and let $g : D \to \mathbb{R}$ be a nearly Borel function such that $E_x \int_0^{\tau_D} |g(X_t)| \, dt < \infty$ for q.e. $x \in E$. Recall that a nearly Borel set is called $m$-inessential if it is $m$-polar and $E \setminus N$ is absorbing for $X$. It is well known that each
$m$-polar set is contained in an $m$-inessential set. In what follows by $N$ we denote an $m$-inessential set such that all the above property (holding q.e) holds outside $N$. By $\|\cdot\|_{1;\tau_D}$ we denote norm (2.1) with $\alpha = 0, \beta = \tau_D$ under measure $P_x$.

6.1 Structure theorems

**Lemma 6.1.** Let $v(x) = E_x\psi(X_{\tau_D})$, $x \in E \setminus N$. Then for every stopping time $\alpha$ such that $\alpha \leq \tau_D$ we have

$$v(X_\alpha) = E_x(\psi(X_{\tau_D})|\mathcal{F}_\alpha), \quad P_x\text{-a.s.}, \quad x \in E \setminus N.$$ 

**Proof.** Let $A = \{\alpha < \tau_D\}$, $B = \{\alpha = \tau_D\}$. All the following equations hold $P_x$-a.s. for $x \in E \setminus N$. By the strong Markov property,

$$v(X_\alpha) = E_{X_\alpha}\psi(X_{\tau_D}) = E_x(\psi(X_{\tau_D} \circ \theta_\alpha)|\mathcal{F}_\alpha) = E_x(1_A\psi(X_{\tau_D} \circ \theta_\alpha)|\mathcal{F}_\alpha) + E_x(1_B\psi(X_{\tau_D} \circ \theta_\alpha)|\mathcal{F}_\alpha).$$

On the set $A$ we have $\tau_D \circ \theta_\alpha = \tau_D - \alpha$, so $1_A\psi(X_{\tau_D} \circ \theta_\alpha) = 1_A\psi(X_{\tau_D})$. By (6.1), $1_B\psi(X_{\tau_D} \circ \theta_\alpha) = 1_B\psi(X_{\tau_D})$, which proves the lemma.

**Corollary 6.2.** Let $\hat{f} : E \times \mathbb{R} \to \mathbb{R}$ be a function such that $f$ defined as

$$f(t, y) = \hat{f}(X_t, y), \quad t \geq 0, \quad y \in \mathbb{R},$$

satisfies hypotheses (H2)–(H4) under the measure $P_x$ for $x \in E \setminus N$. Assume that for every $x \in E \setminus N$ there exists a unique solution $(Y^x, M^x)$ of BSDE$^{\tau_D}(\psi(X_{\tau_D}), f)$ under the measure $P_x$ such that $E_x\int_0^{\tau_D} |f(r, Y^x_r)| \, dr < \infty$. Then there exists a nearly Borel function $w$ such that

$$Y^x_t = w(X_t), \quad t \leq \tau_D, \quad x \in E \setminus N.$$  \hspace{1cm} (6.2)

**Proof.** We first assume additionally that $E_x\int_0^{\tau_D} |f(t, v(X_t))| \, dt < \infty$, where $v$ is defined as in Lemma 6.1. By [31, Remark 2.3] and Lemma 6.1, $Y^x = \bar{Y}^x + v(X)$, where $(\bar{Y}^x, M^x)$ is a solution of BSDE$^{\tau_D}(0, f_v)$ with

$$f_v(t, y) = f(t, y + v(X_t)).$$

By [32, Theorem 4.7], there exists a nearly Borel function $\tilde{w}$ such that $\bar{Y}^x = \tilde{w}(X)$, $x \in E \setminus N$. Thus we have (6.2) with $w = \tilde{w} + v$. To prove the general case, we set

$$f_n(t, y) = \frac{n\theta_t}{1 + n \theta_t}[(f \land n) \lor (-n)](t, y). \hspace{1cm} (6.3)$$

By [31, Theorem 2.8], for every $x \in E \setminus N$ there exists a unique solution $(Y^{x,n}, M^{x,n})$ of BSDE$^{\tau_D}(\psi(X_{\tau_D}), f_n)$ under the measure $P_x$. By what has already been proved, for every $n \geq 1$ there exists a nearly Borel function $w_n$ such that $Y^{x,n} = w_n(X)$, $x \in E \setminus N$. By Theorem 3.15,

$$\|Y^{x,n} - Y^x\|_{1;\tau_D} \leq \int_0^{\tau_D} |f_n(r, Y^x_r) - f(r, Y^x_r)| \, dr.$$ 

Therefore the function $w := \lim_{n \to \infty} w_n$ is well defined on $E \setminus N$ and possesses the desired property. \hfill \Box
In the sequel \( \eta : E \to \mathbb{R} \) stands for a strictly positive bounded nearly Borel function such that

\[
E_x \int_0^{	au_D} \eta(X_r) h_1^+(X_r) \, dr + E_x \int_0^{	au_D} \eta(X_r) h_2^-(X_r) \, dr < \infty, \quad x \in E \setminus N.
\]

Such a function always exists. For instance, one can consider \( \eta = \eta^1 \land \eta^2 \), where

\[
\eta_1 = \sum_{n \geq 1} n^{-1} 1_{\{n-1 \leq h_1^- < n\}} \mathbb{Q}, \quad \eta_2 = \sum_{n \geq 1} n^{-1} 1_{\{n-1 \leq h_2^- < n\}} \mathbb{Q}.
\]

In what follows, we set \( \eta_t = \eta(X_t), \ t \geq 0 \).

**Theorem 6.3.** Assume that for \( x \in E \setminus N \) the function \( f \) defined as \( f(t,y) = \tilde{f}(X_t,y) \) satisfies hypotheses (H2)–(H4), and for \( x \in E \setminus N \) let \((Y^x, \Gamma^x)\) be a solution, under the measure \( P_x \), of RBSDE\(^D\)\((\psi(X_{\tau_D}), f, h_1(X), h_2(X))\) such that \( E_x \int_0^{	au_D} |f(t,Y^x_r)| \, dt < \infty \). Then there exists a nearly Borel function \( u \) such that

\[
Y^x_t = u(X_t), \quad t \in [0, \tau_D), \quad P_x\text{-a.s.,} \quad x \in E \setminus N.
\]

**Proof.** We first assume additionally that \( E_x \int_0^{	au_D} |f(t,0)| \, dt < \infty \) for \( x \in E \setminus N \). Let \((Y^{x,n}, M^{x,n})\) be a solution of BSDE\(^D\)\((\psi(X_{\tau_D}), f_n)\) under measure \( P_x \) with

\[
f_n(t,y) = f(t,y) + n \eta_t(y - h_1(X_t)) - n \eta_t(y - h_2(X_t))^+.
\]  

By Remark 4.12, \( Y^{x,n} \to Y^x \), \( x \in E \setminus N \). Furthermore, by [31, Theorem 2.8], \( E_x \int_0^{	au_D} |f_n(r,Y^{x,n}_r)| \, dr < \infty \), \( x \in E \setminus N \), so by Corollary 6.2 there exists a nearly Borel function \( u_n \) such that \( Y^{x,n} = u_n(X), \ x \in E \setminus N \). From the convergence of \( \{Y^{x,n}\} \) it follows that \( u := \lim_{n \to \infty} u_n \) is well defined on \( E \setminus N \). It is clear that \( u \) is nearly Borel and \( Y^x = u(X), \ x \in E \setminus N \). Consider now the general case. Let \( f_n \) be given by (6.3). By Theorem 4.10, for all \( n \geq 1 \) and \( x \in E \setminus N \) there exists a unique solution \((Y^{x,n}, \Gamma^{x,n})\) of RBSDE\(^D\)\((\psi(X_{\tau_D}), f_n, h_1(X), h_2(X))\) under the measure \( P_x \), and by the first part of the proof, for each \( n \geq 1 \) there exists a nearly Borel function \( u_n \) such that \( Y^{x,n} = u_n(X), \ x \in E \setminus N \). By Theorem 3.15,

\[
\|Y^{x,n} - Y^x\|_{1;x,\tau_D} \leq E_x \int_0^{	au_D} |f_n(r,Y^x_r) - f(r,Y^x_r)| \, dr, \quad x \in E \setminus N.
\]

Therefore \( u \) defined as \( u := \lim_{n \to \infty} u_n \) has the desired properties.

\[ \square \]

### 6.2 Dynkin games

For \( x \in E \setminus N \) and stopping times \( \alpha \leq \sigma, \tau \leq \tau_D \), we set

\[
J_\alpha(x; \sigma, \tau) = E_x \left( \int_{\alpha}^{\tau \wedge \sigma} g(X_r) \, dr + h_1(X_\sigma) 1_{\sigma < \tau} 
+ h_2(X_\tau) 1_{\tau \leq \sigma < \tau_D} + \psi(X_{\tau_D}) 1_{\sigma = \tau = \tau_D} |\mathcal{F}_\alpha| \right).
\]

and then

\[
u(x) = \sup_{\sigma \leq \tau_D} \inf_{\tau \leq \tau_D} J_0(x; \sigma, \tau).
\]  

**Theorem 6.4.** Let \( u \) be defined by (6.5).
that by the Lax-Milgram theorem, for every $c$

\[ u(X_\alpha) = \text{ess sup}_{\alpha \leq \tau \leq \tau_D} \text{ess inf}_{\alpha \leq \tau \leq \tau_D} J_\alpha(x;\sigma,\tau) = \text{ess inf}_{\alpha \leq \tau \leq \tau_D} \text{ess sup}_{\alpha \leq \tau \leq \tau_D} J_\alpha(x;\sigma,\tau) \quad P_x\text{-a.s.} \quad (6.6) \]

(ii) Let $f_n$ be defined by (6.4) with $f(t,y) = g(X_t)$, and let $(u_n(X), M^{x,n})$ be a solution of $BSDE^D(\psi(X_{\tau_D}), f_n)$ under the measure $P_x$. Then $u_n \to u$ on $E \setminus N$.

(iii) For every $\varepsilon > 0$,

\[ J_\alpha(x;\tau_\varepsilon,\sigma) - \varepsilon \leq u(X_\alpha) \leq J_\alpha(x;\tau,\sigma) + \varepsilon, \quad P_x\text{-a.s.,} \quad x \in E \setminus N, \]

where

\[ \tau_\varepsilon = \inf\{t \geq \alpha : u(X_t) \geq h_2(X_t) - \varepsilon\} \land \tau_D, \]
\[ \sigma_\varepsilon = \inf\{t \geq \alpha : u(X_t) \leq h_1(X_t) + \varepsilon\} \land \tau_D. \]

Proof. Follows from Theorem 5.1 and Theorem 6.3.

**Theorem 6.5.** Let $u$ be defined by (6.5). Assume that for every predictable stopping time $\tau \leq \tau_D$ we have

\[ E1_{\tau < \infty} \Delta[h_1(X)]_\tau = 0, \quad E1_{\tau < \infty} \Delta[h_2(X)]_\tau \leq 0. \]

Then for every $x \in E \setminus N$ and every stopping time $\alpha \leq \tau_D$,

\[ u(X_\alpha) = J_\alpha(x;\sigma^*_\alpha,\tau^*_\alpha), \]

where

\[ \sigma^*_\alpha = \inf\{t \geq \alpha : u(X_t) = h_1(X_t)\} \land \tau_D, \quad \tau^*_\alpha = \inf\{t \geq \alpha : u(X_t) = h_2(X_t)\} \land \tau_D. \]

Proof. Follows from Theorem 5.1, Theorem 5.3 and Theorem 6.3.

**Remark 6.6.** Assume that $X$ is a Hunt process and $h_1, h_2$ are quasi-continuous (let us recall that $u$ is quasi-continuous if $u(X)$ is right-continuous and $u(X_\tau)$ is left continuous, see [35]). Then $[\Delta h_i(X)]_\tau = h_i(X_\tau) - h_i(X_{\tau-}), i = 1, 2$. Hence, if $\tau$ is predictable, then $[\Delta h_i(X)]_\tau = 0, i = 1, 2$, since $X$ is quasi-left continuous.

### 6.3 Stationary variational inequalities

Let $(\mathcal{E}, D[\mathcal{E}])$ be a regular semi-Dirichlet form (see [43]) on $L^2(E;m)$ for which there exist $c_1, c_2 > 0$ such that

\[ c_1(u,u) \leq \mathcal{E}(u,u), \quad \mathcal{E}(u,v) \leq c_2 \mathcal{E}^{1/2}(u,u) \mathcal{E}^{1/2}(v,v), \quad u, v \in D[\mathcal{E}] \quad (6.7) \]

(here $(\cdot, \cdot)$ is the standard inner product in $L^2(E;m)$). In this section we assume that $X$ is a Hunt process associated with the form $(\mathcal{E}, D[\mathcal{E}])$, and that $D = E$, $\psi \equiv 0$. Recall that by the Lax-Milgram theorem, for every $g \in L^2(E;m)$ there exists a unique function $Gg \in D[\mathcal{E}]$ such that

\[ \mathcal{E}(Gg,v) = (g,v), \quad v \in D[\mathcal{E}]. \]

Since $X$ is associated with $(\mathcal{E}, D[\mathcal{E}])$, we have

\[ Gg(x) = E_x \int_0^\xi g(X_r) \, dr \quad \text{for} \quad m\text{-a.e.} \ x \in E. \quad (6.8) \]
In fact, by [43, Theorem 3.3.4], the right-hand side of the above equality is a quasi-continuous \( m \)-version of \( Gg \). Hence, in particular, it follows that for every \( g \in L^2(E; m) \),

\[
E_x \int_0^\zeta |g(X_r)| \, dr < \infty \quad \text{for q.e. } x \in E. \tag{6.9}
\]

Let \( \hat{f} : E \times \mathbb{R} \to \mathbb{R} \) and \( h_1, h_2 : E \to \mathbb{R} \). We consider the following conditions:

(S1) the function \( \mathbb{R} \ni y \mapsto \hat{f}(x, y) \) is nonincreasing and continuous for every \( x \in E \), and \( E \ni x \mapsto \hat{f}(x, y) \) is measurable for every \( y \in \mathbb{R} \),

(S2) there exists \( \rho \in L^2(E; m) \) such that \( |\hat{f}(x, y)| \leq \rho(x) + |y| \) for all \( x \in E, y \in \mathbb{R} \),

(S3) \( h_1, h_2 \) are quasi-continuous and there exists \( v \in D[\mathcal{E}] \) such that \( h_1 \leq v \leq h_2 \) q.e.

Let \( K = \{ v \in D[\mathcal{E}] : h_1 \leq v \leq h_2 \} \). We consider the following variational inequality: find \( u \in K \) such that

\[
\mathcal{E}(u, v - u) \geq (\hat{f}(\cdot, u), v - u) \quad \text{for every } v \in K. \tag{6.10}
\]

In what follows our focus is on the relation between solutions of the reflected BSDEs and solutions of the above variational inequality. For the following result see [40, Theorem 5.2, Chapter 3].

**Proposition 6.7.** Assume (S1)–(S3). Then there exists a unique solution \( u \in K \) of (6.10). Moreover, for every \( n \geq 1 \) there exists a unique solution \( u_n \in D[\mathcal{E}] \) of the problem

\[
\mathcal{E}(u_n, v) = (\hat{f}_n(\cdot, u), v), \quad v \in D[\mathcal{E}] \tag{6.11}
\]

with

\[
\hat{f}_n(x, y) = \hat{f}(x, y) + n(y - h_1(x))^+ - n(y - h_2(x))^-, \tag{6.12}
\]

and \( u_n \to u \) weakly in \( (\mathcal{E}, D[\mathcal{E}]) \).

Since for \( v \in D[\mathcal{E}] \) the process \( v(X) \) is of class (D) under the measure \( P_x \) for q.e. \( x \in E \), from (S3) it follows that \( h_1^+ (X), h_2^+ (X) \) are of class (D) under \( P_x \) for q.e. \( x \in E \). Let \( N \) be an \( m \)-inessential nearly Borel set such that for every \( x \in E \setminus N \) the inequality (6.9) holds with \( g \) replaced by \( \rho + h_1^- + h_2^- \), and moreover, \( h_1 \leq h_2 \) on \( E \setminus N \) and \( h_1(X), h_2(X) \) are continuous processes such that \( h_1^-(X), h_2^+(X) \) are of class (D) under the measure \( P_x \) for \( x \in E \setminus N \). Set

\[
f(t, y) = \hat{f}(X_t, y), \quad f_n(t, y) = \hat{f}_n(X_t, y), \tag{6.13}
\]

where \( \hat{f}_n \) is defined by (6.12).

Recall that for each quasi-continuous function \( u \in D[\mathcal{E}] \) the additive functional \( u(X) - u(X_0) \) admits the unique Fukushima’s decomposition

\[
u(X_t) - u(X_0) = A_t^{[u]} + M_t^{[u]}, \quad t \geq 0, \quad P_x\text{-a.s.} \quad \text{q.e. } x \in E \tag{6.14}
\]

into a continuous additive functional \( A^{[u]} \) of \( X \) of zero energy and a martingale additive functional \( M^{[u]} \) of \( X \) of finite energy.
Hence, by the definition of the set $\mathcal{D}$ with $X$ being a Hunt process associated with the semi-Dirichlet form $\mathcal{D}$. In addition, we assume that $\mathcal{D}[B^{(t)}] = D[B^{(0)}]$, $t \in \mathbb{R}$, and there exists $c_3 > 0$ such that

$$\frac{1}{c_3} B^{(t)}(u,u) \leq B^{(0)}(u,u) \leq c_3 B^{(t)}(u,u), \quad u \in D[B^{(0)}], \quad t \in \mathbb{R}.$$
To simplify notation, we set \( F = D[B^{(0)}] \). Let
\[
\mathcal{F} = L^2(\mathbb{R}; F), \quad \mathcal{W} = \{ u \in \mathcal{F} : u_t \in L^2(\mathbb{R}; F') \},
\]
where \( F' \) denotes the dual of \( F \), and for \( T > 0 \) let
\[
\mathcal{F}_{0,T} = L^2(0, T; F), \quad \mathcal{W}_{0,T} = \{ u \in \mathcal{F}_{0,T} : u_t \in \mathcal{F}'_{0,T} \},
\]
\[
\mathcal{W}_{0,T}^0 = \{ u \in \mathcal{W}_{0,T} : u(T) = 0 \}, \quad \mathcal{W}_{0,T}^0 = \{ u \in \mathcal{W}_{0,T} : u(0) = 0 \}.
\]
We also assume that \( \mathbb{R} \ni t \mapsto B(t)(u, v) \) is measurable for all \( u, v \in F \), and we set
\[
\mathcal{B}(u, v) = \int_\mathbb{R} B(t)(u(t), v(t)) \, dt, \quad \mathcal{B}_{0,T}^0(u, v) = \int_0^T B(t)(u(t), v(t)) \, dt, \quad u, v \in \mathcal{F}.
\]
Finally, we set \( E_{0,T} = (0, T) \times E, m_1 = m \otimes dt \) and we define the forms \( \mathcal{E} \) and \( \mathcal{E}^{0,T} \) by
\[
\mathcal{E}(u, v) = \begin{cases} (-\partial_v, v) + \mathcal{B}(u, v), & (u, v) \in \mathcal{W} \times \mathcal{F}, \\ (u, \partial_v) + \mathcal{B}(u, v), & (u, v) \in \mathcal{F} \times \mathcal{W}, \end{cases}
\]
and
\[
\mathcal{E}^{0,T}(u, v) = \begin{cases} (-\partial_v, v)_{0,T} + \mathcal{B}_{0,T}^0(u, v), & (u, v) \in \mathcal{W}_{0,T} \times \mathcal{F}_{0,T}, \\ (u, \partial_v)_{0,T} + \mathcal{B}_{0,T}^0(u, v), & (u, v) \in \mathcal{F}_{0,T} \times \mathcal{W}_{0,T}, \end{cases}
\]
where \((-\cdot, \cdot)_{0,T}\) is the usual inner product in \( L^2(E_{0,T}; m_1) \). It is known that \( \mathcal{E}, \mathcal{E}^{0,T} \) are generalized Dirichlet forms (see [43, 53]).

Assume we are given \( \varphi : E \to \mathbb{R}, \hat{f} : E_{0,T} \times \mathbb{R} \to \mathbb{R} \) and \( h_1, h_2 : \mathbb{R} \times E \to \mathbb{R} \) satisfying the following assumptions:

(E1) the function \( \mathbb{R} \ni y \mapsto \hat{f}(x, y) \) is nonincreasing and continuous for every \( x \in E_{0,T} \), and \( E \ni x \mapsto \hat{f}(x, y) \) is measurable for every \( y \in \mathbb{R} \).

(E2) there exists \( \rho \in L^2(E_{0,T}; m_1) \) such that \( |\hat{f}(x, y)| \leq \rho(x) + |y| \) for all \( x \in E_{0,T}, y \in \mathbb{R} \).

(E3) \( \varphi \in L^2(E; m) \).

(E4) \( h_1, h_2 \) are quasi-continuous functions such that \( h_1(T, \cdot) \leq \varphi \leq h_2(T, \cdot) \) m-a.e. and there exists \( v \in \mathcal{W}_{0,T} \) with the property that \( h_1 \leq v \leq h_2 \) q.e.

We define the convex set \( K \) by
\[
K = \{ v \in \mathcal{F}_{0,T} : h_1 \leq v \leq h_2 \text{ m-a.e.} \}.
\]
We are interested in existence, uniqueness and stochastic representation of a solution of the following variational problem: find \( u \in K \) such that
\[
(u, \partial_v)_{0,T} + \mathcal{B}_{0,T}^0(u, v - u) + \frac{1}{2} \| \varphi \|^2_{L^2(E; m)}
\]
\[
+ (\hat{f}(\cdot, u), v - u)_{0,T} + (\varphi, v(T))_{L^2(E; m)} \geq 0 \tag{6.18}
\]
for all \( v \in K \cap \mathcal{W}_{0,T} \). To state our results, we need some more notation.
By \( \mathbb{X} = (\{X_t, t \geq 0\}, \{P_x, x \in (\mathbb{R} \times E) \cup \{\Delta\}\}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \{\theta_t, t \geq 0\}, \zeta) \) we denote the unique Hunt process associated with \( \mathcal{E} \) (see [43]). It is well known (see [43]) that
\[
X_t = (v(t), X^0_{v(t)}), \quad t \geq 0,
\]
where \( v \) is the uniform motion to the right, i.e. \( v(t) = v(0) + t \) and \( v(0) = s \), \( P_{s,x_0} \)-a.s. for every \((s, x_0) \in \mathbb{R} \times E\). We set
\[
\zeta_v = (T - v(0)) \wedge \zeta.
\]

By \( \{G_{\alpha}^{0,T}, \alpha \geq 0\} \) we denote the resolvent associated with the form \( (\mathcal{E}^{0,T}, D[\mathcal{E}^{0,T}]) \), and we let \( G^{0,T} = G_{\alpha}^{0,T} \). By [43, Section 6.3], for every \( g \in L^2(E_{0,T}; m_1) \) we have
\[
G^{0,T} g(x) = E_x \int_0^{\zeta_v} g(X_t) \, dt \quad \text{for } m\text{-a.e. } x \in E_{0,T}. \tag{6.19}
\]

Moreover, \( G^{0,T} g \in \mathcal{W}_{0,T} \) and the right-hand side of the above equation is a quasi-continuous \( m \)-version of \( G^{0,T} g \). Also note that by [30, Theorem 4.5] the function \( x \mapsto E_x \varphi(X_T^0) \) is quasi-continuous (hence finite q.e.) and belongs to \( \mathcal{W}_{0,T} \).

By [48, Proposition II.4], the set \( \{T \} \times B \) is \( m \)-polar if and only if \( m(B) = 0 \). Therefore, for every nearly Borel set \( B \subset E \) such that \( m(B) = 0 \), we have
\[
P_x(X^0_T \in B) = 0 \quad \text{for q.e. } x \in E_{0,T}. \tag{6.20}
\]

Indeed, since \( m(B) = 0 \), \( \{T\} \times B \) is \( m \)-polar. Hence \( P_x(\exists t > 0 : X_t \in \{T\} \times B) = 0 \) for \( m_1 \)-a.e. \( x \in E_{0,T} \). One can check that \( x \mapsto P_x(\exists t > 0 : X_t \in \{T\} \times B) \) is an excessive function, so it is finely continuous. Consequently, \( P_x(\exists t > 0 : X_t \in \{T\} \times B) = 0 \) for q.e. \( x \in E_{0,T} \). This implies (6.20) since
\[
P_x(\exists t > 0 : X_t \in \{T\} \times B) = P_x(X^0_T \in B, v(0) \leq T).
\]

From (6.20) it follows that if \( h_1(T, \cdot) \leq \varphi \leq h_2(T, \cdot) \) \( m \)-a.e., then
\[
h_1(X_{\zeta_v}) = h_1(T, X^0_T) \leq \varphi(X^0_T) \leq h_2(T, X^0_T) = h_2(X_{\zeta_v}) \quad P_x\text{-a.s.} \tag{6.21}
\]
for q.e. \( x \in E_{0,T} \).

From now on \( N \) is an \( m_1 \)-inessential set such that for every \( x \in E_{0,T} \setminus N \), \( h_1(X) \) and \( h_2(X) \) are càdlàg processes of class (D) under the measure \( P_x \), \( h_1(x) \leq h_2(x) \), \( h_1(X_{\zeta_v}) \leq \varphi(X^0_T) \leq h_2(X_{\zeta_v}) \) \( P_x \)-a.s., and moreover, \( E_x \int_0^{\zeta_v} \rho(X_t) \, dt + E_x |\varphi(X^0_T) < \infty \). We also adopt the notation introduced in (6.13).

We begin with the study of the following problem with no obstacles: find \( u \in \mathcal{W}_{0,T} \) such that
\[
(u, \frac{\partial v}{\partial t})_{0,T} + B^{0,T}(u, v) = (\varphi, v(T))_{L^2(E;m)} + (\hat{f}(\cdot, u), v)_{0,T} \tag{6.22}
\]
for every \( v \in \mathcal{W}_{0,T} \).

**Proposition 6.10.** Assume (E1)–(E3). For every \( x \in E_{0,T} \setminus N \) there exists a unique solution \( (Y^x, M^x) \) of BSDE\(^{\zeta_v}(\varphi(X^0_T), f) \) under the measure \( P_x \). Moreover, there exists a quasi-continuous function \( u : E_{0,T} \to \mathbb{R} \) such that \( Y^x = u(X), x \in E_{0,T} \setminus N \), and \( u \) is a unique solution of problem (6.22).
Proof. By [32, Theorem 2.7], for every \( x \in E_{0,T} \setminus N \) there exists a unique solution \((Y^x, M^x)\) of BSDE\(^c\)(\(\varphi(X^0_T), f\)) under measure \(P_x\), and by [32, Lemma 2.3],

\[
E_x|Y^x_0| + E_x \int_0^\zeta |f(r, Y^x_r)| dr \leq E_x|\varphi(X^0_T)| + E_x \int_0^\zeta \rho(X_r) dr. \tag{6.23}
\]

By Corollary 6.2, there exists a nearly Borel function \( u \) such that \( Y^x = u(X) \), \( x \in E_{0,T} \setminus N \). Since the right-hand side of (6.23), considered as a function of \( x \), belongs to \( L^2(E_{0,T}; m_1) \)(see the comments following (6.19)), it follows that \( u \in L^2(E_{0,T}; m_1) \), and hence that \( \hat{f}(\cdot, u) \in L^2(E_{0,T}; m_1) \). The desired result now follows from [30, Proposition 3.6, Theorem 3.7]. \( \square \)

We now turn to (6.18). We set

\[
\hat{f}_n(x, y) = f(x, y) + n(y - h_1(x)) - n(y - h_2(x))^+, \quad x \in E_{0,T}, y \in \mathbb{R},
\]

and

\[
f_n(t, y) = \hat{f}_n(X_t, y), \quad t \in [0, \zeta], y \in \mathbb{R},
\]

**Theorem 6.11.** Assume (E1)–(E4). For every \( x \in E_{0,T} \setminus N \) there exists a unique solution of the equation RBSDE\(^c\)(\(\varphi(X^0_T), f, h_1(X), h_2(X)\)) under the measure \(P_x\). Moreover, there exists a nearly Borel function \( u \) such that \( u(X) = Y^x \), \( x \in E_{0,T} \setminus N \), and \( u \) is a solution of (6.18).

Proof. By Theorem 4.10, for every \( x \in E_{0,T} \setminus N \) there exists a unique solution \((Y^x, M^x)\) of BSDE\(^c\)(\(\varphi(X^0_T), f, h_1(X), h_2(X)\)) under the measure \(P_x\). Moreover, \( Y^{x,n} \to Y^x \), \( x \in E_{0,T} \setminus N \), where \( Y^{x,n} \) is a solution of BSDE\(^c\)(\(\varphi(X^0_T), f, h_1(X), h_2(X)\)) under \(P_x\). By Proposition 6.10, \( Y^{x,n} = u_n(X) \), \( x \in E_{0,T} \setminus N \), where \( u_n \) is a quasi-continuous \(m_1\)-version of the solution of (6.22) with \( \hat{f} \) replaced by \( \hat{f}_n \). Therefore the function \( u := \lim_{n \to \infty} u_n \) is well defined on \( E_{0,T} \setminus N \) and \( Y^x = u(X) \), \( x \in E_{0,T} \setminus N \). By the definition of a solution, for every \( v \in W_{0,T} \) we have

\[
(u_n, \frac{\partial(v - u_n)}{\partial t})_{0,T} + \mathcal{B}^{0,T}(u_n, v - u_n) = (\varphi, (v - u_n)(T))_{L^2(E;m)} + (\hat{f}_n(\cdot, u_n), v - u_n)_{0,T}.
\]

Hence, for every \( v \in K \cap W_{0,T} \),

\[
(u_n, \frac{\partial v}{\partial t})_{0,T} + \mathcal{B}^{0,T}(u_n, v - u_n) + \frac{1}{2}\|\varphi\|^2_{L^2(E;m)} - (\hat{f}(\cdot, u_n), v - u_n)_{0,T} + (\varphi, v(T))_{L^2(E;m)} \geq 0. \tag{6.24}
\]

Observe that \(-(\hat{f}(\cdot, u_n), v - u_n)_{0,T} \leq -(\hat{f}(\cdot, v), v - u_n)\). From this, (6.7) and (6.24) we get

\[
\mathcal{B}^{0,T}(u_n, u_n) \leq c(\|\varphi\|^2_{L^2(E;m)} + \|v\|^2_{W_{0,T}}).
\]

Hence, up to a subsequence, \( u_n \to u \) weakly in \( \mathcal{F}_{0,T} \), which when combined with (6.24) and monotonicity of \( \hat{f} \) gives (6.18). \( \square \)

**Corollary 6.12.** Define \( u \) by (6.5) with \( T = \zeta, \); \( \psi = \varphi, D = E_{0,T} \); \( g \in L^2(E_{0,T}; m_1) \) and with \( X \) being the Hunt process associated with a generalized semi-Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\). Then \( u \) is a solution of (6.18) with \( \hat{f} \) replaced by \( g \).
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