Phase Covariant Channel: Quantum Speed Limit of Evolution

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The quantum speed of evolution for the phase covariant map is investigated. This involves absorption, emission, and dephasing processes. The maps under various combinations of the above processes are considered to investigate the effect of phase covariant maps on quantum speed limit time. For absorption-free phase covariant maps, combinations of dissipative and CP-(in)divisible (non)-Markovian dephasing noises are considered. The role of coherence-mixedness balance on the speed limit time is checked in the presence of both vacuum and finite temperature effects. The rate at which Holevo’s information changes and the action quantum speed of evolution for specific cases of the phase covariant map are also investigated.

1. Introduction

Nowadays, it is an established fact that one can manipulate the impact of quantum noise on quantum systems productively.[1] A good amount of work has been devoted to investigating the memory effects of noise on the dynamics of quantum systems.[2–7] In general, (non-)Markovianity discusses the nature of the system’s correlations with the environment. Quantum coherence and correlations are significant resources for quantum technology.[8–12] Non-Markovianity influences the quantum resources in both beneficial and unfavorable ways; consequently, the investigation of quantum correlation and coherence becomes highly significant.[13–15] Along with the memory effects on quantum resources, it is pertinent to discuss the evolution speed of quantum systems. It has been shown that energy-time uncertainty reveals the bound on the speed of the evolution of quantum states.[16] Initially, speed limit time was derived for the dynamics between the orthogonal states for isolated systems.[17] Later, quantum speed limit (QSL) time for the time-independent systems was extended to the arbitrary quantum states.[18] Further, the speed limit for the evolution between the states with the arbitrary angle for a driven quantum system has been determined.[19] Recently, the speed of evolution between arbitrary states for open quantum systems[20–22] has become a lively research topic and is the central theme of the present work.

Not only from the dynamical perspective but also the system’s characteristics revealed by the limit on the speed of evolution display its distinguishable role in quantum communication and technology. To list a few, the bound on speed limit time reveals how fast the quantum information can be communicated, the maximum rate at which information can be processed, and the precision limit in quantum metrology[23–25] among others. Even though there exists no direct connection between non-Markovianity, a class of which is identified by information backflow and quantum speed limit time ($\tau_{\text{QSL}}$),[26] it has been shown that $\tau_{\text{QSL}}$ could be realized as a witness of the decay-revival mechanism of quantum correlations[27] for a certain class of quantum noises. In ref. [21, 28–30], it has been seen that quantum non-Markovianity may speed up the evolution of quantum states. It is also known that non-Markovianity is not always required to speed up quantum evolution.[31–33] From the practical point of view, $\tau_{\text{QSL}}$ finds many applications in a wide range of fields.[14] The phase-covariant map describes the physical processes involving absorption, emission, and pure dephasing. This provides a convenient platform to study both (non-)unital processes from a common perspective. In the present work, we estimate the $\tau_{\text{QSL}}$ for single-qubit states evolving under the phase covariant channel.[15] This channel can be thought of as an approximation of the general spin-Boson problem.[36] Work on similar lines was initiated recently in ref. [37]. Here, in addition to the impact of various processes like heating, dissipation, and dephasing on $\tau_{\text{QSL}}$, we also consider the role of coherence and mixing, as well as the purity of initial states. Coherence is one of the central features of quantum physics.[38] Further, an open system evolution generally makes a system’s states mixed. Hence, it is meaningful to ask how the balance between these two processes, namely, coherence and mixing, impacts the dynamics.[15,39] We further consider different combinations of CP-(in)divisible (non)-Markovian quantum channels and a phenomenological model and show how these combinations influence the speed of quantum evolution for both pure and mixed initial states. The influence of thermal bath on $\tau_{\text{QSL}}$, along with the rate at which the upper bound for Holevo’s information changes, are also checked. The present work is structured as follows. Section 2 discusses the prerequisites for the current work, which contains the details of the phase covariant channel, quantum speed limit, and the measure of non-Markovianity used here. In Section 3, we investigate $\tau_{\text{QSL}}$ and the impact of coherence-mixedness trade-off on $\tau_{\text{QSL}}$ for...
different quantum phase covariant noises for an initial pure state. Quantum speed limit time for an initially mixed state is discussed in Section 4. In Section 5, we briefly discuss the recently introduced action quantum speed limit for the phase covariant map, followed by the concluding remarks in Section 6.

2. Preliminaries

Here, we present the preliminary information required for this work. This begins with a brief overview of the phase covariant map, followed by a discussion of the QSL time using the geometric approach. Since the interplay of coherence and mixing, along with the non-Markovian behavior of the combination of different quantum channels, is central to the present work, these notions are briefly introduced.

2.1. The Phase Covariant Map

The master equation for a single qubit phase covariant dynamics has the form [35]:

\[
\frac{d\rho(t)}{dt} = -i\frac{\alpha(t)}{2}\sigma_+\rho(t)\sigma_- + \frac{\gamma_1(t)}{2}\mathcal{L}_1(\rho(t)) + \frac{\gamma_2(t)}{2}\mathcal{L}_2(\rho(t))
\]

(1)

where

\[
\mathcal{L}_1(\rho(t)) = \sigma_+\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}
\]

(2)

\[
\mathcal{L}_2(\rho(t)) = \sigma_-\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}
\]

(3)

\[
\mathcal{L}_3(\rho(t)) = \sigma_\tau\rho(t)\sigma_\tau - \rho(t),
\]

(4)

and \(\sigma_\tau = \frac{1}{2}(\sigma_+ \pm i\sigma_\tau)/2\). The action of the phase covariant map on an arbitrary single-qubit density matrix \(\rho(0)\) is:

\[
\Phi_\tau(\rho(0)) = \rho(t) = \left(1 - p_1(t)\right)\frac{a^*(t)}{a(t)}\frac{p_1(t)}{1 - p_1(t)}
\]

(5)

where

\[
p_1(t) = e^{-i\gamma_1(t)}[G(t) + p_1(0)]
\]

(6)

\[
a(t) = a(0)e^{aG(t)} = a(t)\frac{2 - G(t)}{2 - G(0)}
\]

(7)

and

\[
\Gamma(t) = \int_0^t \gamma_1(\tau) + \gamma_3(\tau)\,d\tau
\]

(8)

\[
G(t) = \int_0^t e^{a(\tau)}\gamma_1(\tau)\,d\tau
\]

(9)

\[
\Omega(t) = \int_0^t 2\omega(\tau)d\tau
\]

(10)

\[
\tilde{\Gamma}(t) = \int_0^t \gamma_3(\tau)d\tau
\]

(11)

Here, \(\gamma_1(t), \gamma_2(t), \gamma_3(t)\) correspond to energy gain, energy loss, and pure dephasing rates, respectively. Also, \(\Omega(t)\) corresponds to rotations around the z-axis of the Bloch ball. Phase covariant dynamics for a single qubit satisfies the relation \(\exp(-i\sigma_\tau\phi)\Phi_\tau[\rho]\exp(i\sigma_\tau\phi) = \Phi_\tau[\rho]\exp(-i\sigma_\tau\phi)\exp(i\sigma_\tau\phi)\) for all real \(\phi\)[41]. It can be shown that phase covariant dynamics implies uniform deformation of the x and y Bloch vectors and permits a deformation as well as translation of the z Bloch vector.

2.2. Quantum Speed Limit Time

Mandelstam and Tamm (MT) and Margolus and Levitin (ML)-type bounds on speed limit time are estimated using the geometric approach to quantify the closeness between the initial and final states. Here, the Bures angle measures the distance between two quantum states. In ref. [21], for the initial pure state \(\rho_0 = |\psi_0\rangle\langle\psi_0|\), a bound on the speed limit time based on Bures angle \(B(\rho_0, \rho_f)\) is

\[
\tau_{QSL} = \max\left\{\frac{1}{\Lambda_{\text{op}}}, \frac{1}{\Lambda_{\text{tr}}}, \frac{1}{\Lambda_{\text{hs}}}\right\} \sin^2\langle B \rangle
\]

(12)

where \(B(\rho_0, \rho_f) = \arccos(\sqrt{|\langle \psi_0 | \rho_f \rangle|})\).

\[
\Lambda_{\text{op, tr, hs}} = \frac{1}{\tau} \int_0^\tau |\mathcal{L}[\rho(t)]|_{\text{op, tr, hs}}
\]

(13)

where \(\Lambda_{\text{op}}, \Lambda_{\text{tr}}, \Lambda_{\text{hs}}\) are the operator, Hilbert-Schmidt, and trace norms, respectively. Operators satisfy the von Neumann trace inequality \(||A||_{\text{op}} \leq ||A||_{\text{hs}} \leq ||A||_{\text{tr}},\) which gives, \(1/\Lambda_{\text{op}} \geq 1/\Lambda_{\text{tr}} \geq 1/\Lambda_{\text{hs}}\). The tighter bound on the quantum speed limit time is achieved by using operator norm of the generator. An upper bound on fidelity for any density matrices \(\rho_1 \) and \(\rho_2\) shows that\([42] P(\rho_1, \rho_2) \leq tr(\rho_1\rho_2 + \sqrt{(1-tr\rho_1^2)(1-tr\rho_2^2)}).\) Making use of this super-fidelity, a bound on quantum speed limit time for both pure and mixed initial states can be written by multiplying the RHS in (Equation 13) by a factor \((1 + \sqrt{1-tr\rho_1^2})\).[43]

2.3. Coherence-Mixing Balance and the Upper Limit of the Holevo Bound

The mixedness of a quantum system imposes a limit on the amount of quantum coherence it can possess.[15,44] For a d-level system, this trade-off can be expressed as an inequality:

\[
M_d = \frac{C_d^2}{(d-1)^2} + M_\tau(\rho) \leq 1
\]

(14)

For a two-level system, using the density matrix equation Equation (5), the trade-off equation can be shown to be in the following helpful form

\[
4p_1(t)(1 - p_1(t)) \leq 1
\]

(15)

See the Appendix for the details of the derivation. It was recently shown that the rate at which accessible information quantified by
the Holevo quantity $\chi$ changes is upper bounded by the quantum speed limit time as $^{34,45}$

$$\dot{\chi} \leq \frac{\Delta \chi}{\tau_{QSL}}$$

(16)

Here, $\Delta \chi$ represents the change of $\chi$. This suggests that $\tau_{QSL}$, in particular, $1/\tau_{QSL}$, upper bounds the rate with which the accessible information changes.

2.4. Self Similarity Measure of Non-Markovianity

Recently, a measure, which we call the SSS measure, was defined that approaches non-Markovian behavior from the perspective of temporal self-similarity, the property of a system dynamics wherein the propagator between two intermediate states is independent of the initial time.$^{46}$ In particular, it quantifies non-Markovian behavior in terms of deviation $\zeta$ from the temporal self-similarity

$$\zeta = \frac{1}{T} \int_0^T ||L(t) - L^*||_1 dt$$

(17)

Here, $||A||_1 = tr\sqrt{A^*A}$ is the trace norm of matrix $A$, $L(t)$ is the Lindbladian corresponding to the time-homogeneous master equation, and $L^*$ is a time-independent Lindblad generator.

3. Quantum Speed Limit Time: Analysis of Various Phase Covariant Maps

The master equation for single qubit phase covariant dynamics (Equation (1)) discusses the evolution of quantum states under various physical processes such as absorption, emission, and dephasing, which are characterized by the rate constants $\gamma_1(t), \gamma_2(t)$, and $\gamma_3(t)$, respectively, from the framework of phase covariant map. Here, we investigate the quantum speed limit time for pure and mixed initial qubit states under the combination of different (non)-unital (non)-Markovian quantum channels and analyse the impact of coherence-mixing on $\tau_{QSL}$.

3.1. Non-Markovian Amplitude Damping and Random Telegraph Noise

Here, we consider the case of absorption free phase covariant dynamics. A combination of the non-unital non-Markovian amplitude damping (nMAD) channel and the unital random telegraph noise (RTN), a pure dephasing channel$^{46}$ is taken into consideration. Amplitude damping channel models the physical processes like spontaneous emission. For (non)-Markovian RTN dephasing channels, decoherence processes are induced by low-frequency noise modeled through stochastic processes. Since it is an absorption-free process, we take $\gamma_3(t) = 0$ in Equation (1). For the nMAD channel, $\gamma_1(t) = -\frac{\Lambda(t)}{\Lambda(0)} \frac{dz}{dt}$, where $\Lambda(t) = e^{-lt/2} (\cosh(zt/2) + \frac{i}{z} \sinh(zt/2))^2$ is the decoherence function. The decoherence rate then becomes

$$\gamma_1(t) = \frac{-4z \sqrt{l} \sinh(zt/2)}{z \cosh(zt/2) + l \sinh(zt/2)}$$

(18)

where $z = \sqrt{I - 2kI}$. $k$ describes the qubit-environment coupling strength, and $l$ is the spectral width related to the reservoir correlation time. The dynamics are Markovian in the region $l > 2k$, whereas it is non-Markovian in the region $l < 2k$. The dephasing rate for RTN is $\gamma_1(t) = \frac{1}{\Lambda(0)} \frac{\Lambda(t)}{\Lambda(0)}$, where $\Lambda(t) = e^{-lt} (\cos(\mu t \Lambda(t)) + \frac{i}{\mu} \sin(\mu t \Lambda(t))$ and is

$$\gamma_1(t) = \frac{\eta(\mu^2 + 1) \sin(\mu t \Lambda(t))}{\mu \cos(\mu t \Lambda(t)) + \sin(\mu t \Lambda(t))}$$

(19)

where $\mu = \sqrt{(\frac{2z}{l})^2 - 1}$. Here, $\eta$ is the spectral bandwidth and $a$ is the coupling strength between the qubit and the reservoir. Depending on whether $(\frac{2z}{l})^2 > 1$ or $(\frac{2z}{l})^2 < 1$, the dynamics is non-Markovian or Markovian, respectively. Using Equation (17), we calculate the channel’s memory $\zeta$; $\zeta > 0$ indicates the presence of the memory.

Figure 1a depicts the behavior of $\tau_{QSL}$ with respect to the dimensionless time $(\kappa \tau)$ for the maximally coherent initial state $\frac{1}{\sqrt{2}} |0\rangle + |1\rangle$. We have QSL time and $1/\tau_{QSL}$ versus $\zeta$ in Figures 1b1 and 1b2, respectively, $\tau_{QSL}$ versus QSL time is shown in Figure 1c. We find that a more non-Markovian combination of channels does not always lead to the speed-up of quantum evolution. It generally depends on the strength of the coupling and the initial state chosen. The wiggling nature of the QSL curve could be
attributed to the RTN noise in the non-Markovian regime due to the highly oscillatory nature of the noise in this regime. The $\tau_{QSL}$ versus $M_{cl}$ plot brings out the influence of coherence and mixing on the speed of evolution.

### 3.2. Non-Markovian Amplitude Damping and Modified Ornstein–Uhlenbeck noise

Now, RTN noise in the previous case (Section 3.1) is replaced by the modified Ornstein–Uhlenbeck noise (OUN),[47] a stationary Gaussian random process. The spin of an electron interacting with a magnetic field subject to stochastic fluctuations is a physical scenario that leads to the occurrence of OUN. OUN is CP visible, but it is still non-Markovian as it exhibits memory effects, as captured by the SSS measure.[46] This noise has a well-defined Markovian limit, referred to here as MOUN. The OUN dephasing rate is $\gamma_3(t) = -\frac{1}{\Lambda(t)} \frac{d\Lambda(t)}{dt}$, where $\Lambda(t) = \exp\left(\frac{-1}{2} (t + \frac{1}{m}(e^{-mt} - 1))\right)$ which is calculated as

$$\gamma_3(t) = \frac{p}{2}(1 - e^{-m\tau})$$

(20)

Here, $p$ is the inverse of the effective relaxation time, and $m$ is related to the noise bandwidth. This channel is Markovian in the limit $m \to \infty$ for which case $\Lambda(t) = e^{-p/2}$ and $\gamma_1(t) = p/2$. We use Equation (18) for $\gamma_3(t)$ and $\gamma_1(t) = 0$.

This phase covariant channel exhibits memory effects and is quantified using Equation (17). The variation of $\tau_{QSL}$ for OUN in combination with nMAD to coupling strength is depicted in Figure 2a for the initial maximally coherent state. From Figure 2a, it is clear that the combination of only non-Markovian channels does not always contribute to the speed-up of quantum evolution. It depends on the nature of the coupling strength and initial states. The behavior of QSL time with respect to the memory is depicted in Figure 2b, and the coherence-mixedness impact on quantum speed limit time is shown in Figure 2c. In Figure 3, $\tau_{QSL}$ versus $\kappa \tau$ is plotted for the dynamics which involve only emission processes (nMAD) as well as the maps with both the emission and dephasing. In these cases, the emission process is always non-Markovian in nature, whereas the dephasing occurs in both (non)-Markovian regimes. From Figure 3, it is seen that considering dephasing noises (RTN and OUN) along with nMAD can decrease the speed of evolution. The decoherence process due to the spontaneous emission and pure dephasing processes cannot always increase the speed of evolution compared with the decoherence process due to only spontaneous emission. In fact, for all range of parameter values, except for a small interval (Figure 3), adding dephasing channels decreases the speed of evolution for the initial states considered.

### 3.3. A Phenomenological Model

We next consider a phenomenological model which discusses the decoherence of states due to the physical processes such as absorption, emission, and pure dephasing. All the three rates $\gamma_1(t)$, $\gamma_2(t)$, and $\gamma_3(t)$ take nonzero values (Equation (1)) and reproduce the Markovian master equation in the appropriate limit.[48] The decay rates are given as

$$\gamma_1(t) = 2\mathcal{N}f(t)$$

(21)

$$\gamma_2(t) = 2(\mathcal{N} + 1)f(t)$$

(22)
where

\[ f(t) = -2\text{Re}\left\{ \frac{1}{c(t)} \frac{dc(t)}{dt} \right\} \]  

(23)

\[ c(t) = c(0)e^{-\gamma t / \omega_c^2} \left[ \cosh\left(\sqrt{1 - 2Rt/\omega_c^2}\right) + \frac{1}{\sqrt{1 - 2R}} \right] \]  

(24)

Here \( R \) is a dimensionless constant greater than zero, which depicts the coupling between the system and the environment as well as the environmental spectral properties. \( \mathcal{N}'(T) = [\exp(v_b/T) - 1]^{-1} \) is the mean number of excitation in modes of the thermal environment, where \( T \) is the temperature and \( v_b \) is the Bohr frequency. When \( R < 1/2 \), \( \gamma_1(t) \) and \( \gamma_2(t) \) are always positive. They become negative for certain time intervals when \( R > 1/2 \) and the dynamics become non-Markovian. At temperature \( T = 0 \), \( \mathcal{N}'(T) = 0 \), and it reduces to non-Markovian amplitude damping. This matches with \( \gamma_3(t) \) in Equation (18) with \( l \) equal to one. The pure dephasing rate is given by

\[ \gamma_3(t) = 2 \int d\omega J(\omega)\coth(\omega/k_B T)\sin(\omega t) \]  

(25)

where the spectral density \( J(\omega) \) is

\[ J(\omega) = \frac{\nu\omega^2}{\omega_c^2} e^{-\omega/\omega_c} \]  

(26)

with \( \omega_c \) being the cut-off frequency and \( \nu \) being a dimensionless constant. In the pure dephasing case, that is, when \( \gamma_1(t) = \gamma_2(t) = 0 \), the Ohmic parameter \( s \) as a function of temperature determines the Markovianity of the model. The system is non-Markovian when \( s > s_{\text{crit}}(T) \). The critical value of \( s \) has a minimum of \( s_{\text{crit}}(T = 0) = 2 \) at \( T = 0 \) and a maximum of \( s_{\text{crit}}(T \rightarrow \infty) = 3 \) in the high temperature limit. Using the time-independent and dependent generators in Equation (17), we calculate the channel’s memory; \( \zeta > 0 \) implies the presence of the quantum memory.

With the maximally coherent state taken as the initial state, Figure 4a shows the behavior of \( \tau_{\text{QSL}} \) with respect to \( R \). We find that \( \tau_{\text{QSL}} \) is higher at finite temperatures and continues to increase with the increase in temperature. The increase in temperature tends to slow down the process. The change of QSL time for the memory \( \zeta \) is depicted in Figure 4b. Figure 4c shows the trajectories taken by the \( \tau_{\text{QSL}} \) with respect to \( M_{\text{Cl}} \) at different temperatures. The system never reaches a pure state at finite temperatures corresponding to \( M_{\text{Cl}} = 0 \).

### 3.4. Eternally CP-Indivisible Dynamics

A family of non-unital eternal CP-indivisible dynamical maps was introduced in ref. [41] with the decay rates of absorption, emission, and dephasing

\[ \gamma_1(t) = 2\nu(1 + b) \]  

(27)

\[ \gamma_2(t) = 2\nu(1 - b) \]  

(28)

\[ \gamma_3(t) = -\frac{\nu(1 - b^2)\sinh(2\nu t)}{[1 + b^2 + (1 - b^2)\cosh(2\nu t)]} \]  

(29)

Here, \( \nu > 0 \). When \( |b| < 1, \gamma_1(t) < 0 \) for all \( t > 0 \), that is, eternally indivisible dynamics which is neither unital nor commutative.

The SSS measure for this channel is \( \zeta = \frac{1}{2} \ln\left(\frac{1 + \nu T}{2(2 - \nu)}\right) \). As \( \zeta > 0 \) for \( T > 0 \), we see that this channel possesses memory. In Figure 5a, we plot \( \tau_{\text{QSL}} \) as a function of \( r \), the driving time and \( r \), where \( r \) is used to parameterize the single qubit wave function, \( |\psi(\tau)| = \sqrt{\tau}|0\rangle + \sqrt{1 - \tau}|1\rangle \).

Figure 4. \( \tau_{\text{QSL}} \) versus \( R \), \( \tau_{\text{QSL}} \) versus \( \zeta \), and \( \tau_{\text{QSL}} \) versus \( M_{\text{Cl}} \) are plotted in subplots (a), (b), and (c), respectively, at zero and finite temperatures. QSL time for the pure state (\( 1/2 |0\rangle + |1\rangle \)) is estimated for temperature-dependent decay and dephasing processes at temperatures \( T = 0, 0.5 \) for super Ohmic spectral density \( s = 4 \). We have \( c(0) = 1, \omega_c = 1, \nu = 1 \), and the actual driving time \( r = 1 \).

Figure 5. a) \( \tau_{\text{QSL}} / \tau \) and b) \( 1/\tau_{\text{QSL}} \) with respect to \( r \) and \( r \), c) \( \tau_{\text{QSL}} \) plotted as a function of \( M_{\text{Cl}} \) with the maximally coherent state taken as the initial state for eternal CP-indivisible dynamics. Here \( b = 0.5, \nu = 1 \).
Depending on the initial state chosen, the behavior of $\tau_{\text{QSL}}$ varies. We find $\tau_{\text{QSL}}/\tau = 1$ for two different states, when $\tau = 0$ and when $\tau = 1$, while for all other states, it remains below 1. The upper bound to the Holevo rate decreases rapidly with the increase in driving time for all states, as can be seen in Figure 5b. The $\tau_{\text{QSL}}$ is plotted as a function of $M_d$ in Figure 5c. With the maximally coherent state taken as the initial state, $M_d$ first decreases from 1 as $\tau_{\text{QSL}}$ increases and then remains steady at 0.75 in Figure 5c. This can be explained using Equation (15). In the limit $\tau \to \infty$, we find $M_d = (1 - b^2)$, which is equal to 0.75 for our chosen $b = 0.5$.

4. Initial Mixed States

So far, we have considered only initial pure states. This section calculates the quantum speed limit time for initial mixed states under the eternal CP-indivisible dynamics. Different approaches are used to estimate the QSL time according to the purity of initial states\textsuperscript{[49–51]} The single qubit density matrix is

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + r_x & r_y - ir_z \\ r_y + ir_x & 1 - r_x \end{pmatrix}$$ (30)

where $|r| \leq 1$ with the inequality for mixed states. The time-evolved density matrix for the eternally CP-indivisible channel then becomes,

$$\rho(t) = \frac{1}{2} \begin{pmatrix} 1 + e^{-2it}(r_x + b(e^{2it} - 1)) & (r_x - ir_y)e^{-2it}\sqrt{1 + b^2 + (1 - b^2)\cosh(2vt)} \\ (r_y + ir_x)e^{-2it}\sqrt{1 + b^2 + (1 - b^2)\cosh(2vt)} & 1 - e^{-2it}(r_x + b(e^{2it} - 1)) \end{pmatrix}$$ (31)

We use the modified $\tau_{\text{QSL}}$ with the factor $(1 + \sqrt{1 - \rho_{12}^2})$ multiplied to Equation (13). We look for the minimal time required for the system to evolve from a mixed initial state $\rho_c$ to the final state $\rho_{c+\epsilon}$. As the open system quantum evolution is non-unitary, $\rho_{c+\epsilon}$ is also a mixed state. The modified $\tau_{\text{QSL}}$\textsuperscript{[51]} is

$$\tau_{\text{QSL}} = \frac{\sin^2[B_{c,c+\epsilon}]}{\frac{1}{2} \int_{\epsilon}^{\epsilon+\epsilon/2} dt ||\mathcal{L}(\rho_t)||_{\text{op}} \left(1 + \sqrt{1 - \rho_{12}^2} \right)}$$ (32)

where $B_{c,c+\epsilon} = \arccos(F(\rho_c, \rho_{c+\epsilon}))$ and $F$ is the super fidelity. Using this definition, $\tau_{\text{QSL}}$ is plotted with respect to $\tau$ and memory $\zeta$ in Figures 6a and 6b, respectively. They indicate that the minimum time required to achieve a particular overlap between two mixed states decreases with evolution.

5. Action Quantum Speed Limit

Until now, we have used quantum speed limit time based on a geometrical approach. The geometrical approach for quantum speed limits relies on the idea that the geodesic distance between any two points is the shortest possible length connecting them. The speed of evolution depends on the path of evolution. Geometric quantum speed limits are not sensitive to instantaneous speed. This problem is resolved by calculating the action $\tau_{\text{QSL,0}}$\textsuperscript{[52]}

Here, we optimize our action $\tau_{\text{QSL}}$. We implement this on the phase covariant map with $\gamma_1(t) = \gamma, \gamma_2(t) = \Gamma, \gamma_3(t) = 0$. This is the generalized amplitude damping channel (system in contact with a thermal bath at non-zero temperature). We consider a pure initial state, $|\psi\rangle = \cos(\frac{\pi}{4})|0\rangle + \sin(\frac{\pi}{4})|1\rangle$.

Under the metrics chosen in this work, the action $\tau^a_{\text{QSL}}$ can be calculated as

$$\tau^a_{\text{QSL}} = \frac{\left(\sin^2[B_{\rho_0, \rho_t}]\right)^2}{a^2}$$ (33)

Here, $a^2$ is the action which needs to be minimized along the path and is expressed as

$$a^2 = \int_0^t dt \mathcal{L}(\rho_t, \dot{\rho}_t, \dot{\gamma}(t))$$

$$= \int_0^t dt \mathcal{L}(\rho_t) ||_{\text{op}}$$

$$= \int_0^t dt \dot{\rho}_t^2 \left(\frac{\sin^2 2\theta}{16(1 - q(t))} + (\sin^2 \theta - \eta)^2\right)$$ (34)
consideration. Its appearance in the Kraus operators of the channel under the Legendre transform. Modified gradient descent algorithm [53] trade-off on feature of this study is the impact of the coherence-mixedness the increase in temperature for the chosen initial state. Another tion of the path traversed, and we have $\tau = 1$.

where $q(t)$ describes the path traversed by the channel and makes its appearance in the Kraus operators of the channel under consideration. $L(q, \rho, \dot{q}(t))$ is the control Lagrangian. Also, $\eta = \frac{1}{2}(1 + \tanh \beta)$, with $\beta = \frac{1}{2} \ln 2$. The Hamiltonian is obtained using the Legendre transform. Modified gradient descent algorithm is used to obtain the consecutive $\dot{q}(t)$’s.

From Figure 7, we see that optimization of the action leads to saturation of the Cauchy Schwartz inequality. Thus, the geometric $\tau_{QSL}$ is seen to be an upper bound of the action $\tau_{QSL}^e$.

6. Conclusion

As is known, the phase-covariant map describes the physical process involving absorption, emission, and pure dephasing. This provides a convenient platform to study both (non)-unitary processes from a common perspective. We made use of this to make an exhaustive study of QSL time for different sets of physical processes ((non)-unitary, (non)-Markovian) within the framework of the phase covariant channel. For this, we considered maps that do not involve the absorption process and maps containing all three processes. For absorption-free phase covariant maps, we considered the cases where both CP-(in)divisible (non)-Markovian dephasing are involved. For the initial maximal coherent state, it was found that the presence of a CP-indivisible non-Markovian dephasing map does not always speed up the quantum evolution.

In the phenomenological model, where absorption, emission, and dephasing rates are considered together, the temperature was found to affect $\tau_{QSL}$ significantly, with $\tau_{QSL}$ increasing with the increase in temperature for the chosen initial state. Another feature of this study is the impact of the coherence-mixedness trade-off on $\tau_{QSL}$ for evolution generated by the phase-covariant map. Various features of the dynamics, such as information backflow and temperature effects, could be ascertained from this. In general, the speed of evolution depends on the initial state and the channel parameters. A newly developed formulation for quantum speed limit called action speed limit, which takes into account the path of the evolution, was also considered and compared with the geometrical QSL time. The generality of the present study implies that special cases of the phase-covariant map, such as the amplitude damping and the pure dephasing channels, can be easily obtained as simplifications of the models studied.

Appendix A

The $l_1$ norm of coherence is given by:

$$C_{i_1} = \sum_{i,j} |\rho_{ij}|$$  \hspace{1cm} (A1)

For a qubit, this is the sum of the absolute value of the off-diagonal elements. The mixedness, based on normalized linear entropy for a single qubit, is:

$$M_{i_1}(\rho) = 2(1 - \text{Tr}(\rho^2))$$  \hspace{1cm} (A2)

Using (5), (A1), and (A2), we find for the phase covariant noise:

$$C_{i_1} = 2|\alpha(t)|.$$  \hspace{1cm} (A3)

$$M_{i_1}(\rho) = 4(\rho_{11}(t) - \rho_{11}(t)^2 - |\alpha(t)|^2)$$  \hspace{1cm} (A4)

The trade-off between mixedness and coherence $(C_{i_1}^2 + M_{i_1}(\rho) := M_{i_1})$ is then calculated as:

$$M_{i_1} = 4\rho_{11}(t)(1 - \rho_{11}(t)) \leq 1$$  \hspace{1cm} (A5)

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

open quantum systems, phase covariant channels, quantum speed limit

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