A SHORT PROOF OF A CONJECTURE OF AOUGAB-HUANG

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Abstract. In response to Sanki-Vadnere [SV19], we present a short proof of the following theorem: a pair of simple curves on a hyperbolic surface whose complementary regions are disks has length at least half the perimeter of the regular right-angled \((8g - 4)\)-gon.

1. INTRODUCTION

Let \(S = S_g\) be an oriented closed surface of genus \(g\), and let \(P = P_g\) be the hyperbolic regular right-angled \((8g - 4)\)-gon. A set of curves on \(S\) is filling if the complementary components are disks.

Theorem. A filling pair of simple geodesics on a hyperbolic surface homeomorphic to \(S\) has length at least \(\frac{1}{2} \text{perim}(P)\).

This theorem was conjectured by Aougab-Huang [AH15] in the context of their study of minimal filling pairs, i.e. those for which the complement has one component. For minimal filling pairs, the above theorem follows directly from an isoperimetric inequality in the hyperbolic plane, due to Bezdek [Bez84]. When there are two complementary polygons, one may glue them together along a common side. After erasing two superfluous vertices, the result is an \((8g - 4)\)-gon, and the same isoperimetric inequality holds [AH15, Cor. 4.5].

The purpose of this note is to demonstrate that Aougab-Huang’s approach generalizes to arbitrarily many components: the complementary pieces can be glued together so that Bezdek’s isoperimetric becomes available. Of course, a difficulty arises, in that the number of sides of the polygon so obtained may have become unwieldy. Here one should glue with a bit more care, avoiding the possibility of corners with angle greater than \(\pi\).

An alternative technical approach to the above theorem was developed prior to the present paper by Sanki-Vadnere [SV19]. There, the surface \(S\) plays a lesser role, and one compares perimeters of the complementary pieces to that of a single regular polygon directly. Sanki-Vadnere show: for \(i = 1, \ldots, r\), let \(P_i\) be a polygon with \(2n_i\) sides, and suppose that \(P\) is the regular hyperbolic polygon with \(\text{area}(P) = \sum \text{area}(P_i)\) and \(2m\) sides, where \(m + 2r = 2 + \sum n_i\). Then, provided \(P\) is not acute, we have \(\sum \text{perim}(P_i) \geq \text{perim}(P)\). This somewhat complicated statement implies the Aougab-Huang conjecture: The sum of the lengths of the geodesics is at least half of the sum of the perimeters of their complementary components, and by Gauss-Bonnet the polygon obtained above is \(P \approx P\).

The Sanki-Vadnere result is more general than the above theorem, as it applies to polygons that do not tile a closed surface. On the other hand, the approach contained here demonstrates slightly more: if one obtains equality in the above Theorem, then the complement of the geodesics is isometric to \(P\) (see the Corollary below).

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2. A lemma about spanning trees

Let $G$ be a graph embedded on $S$. For each vertex $p$ of $G$, the orientation of the tangent space $T_pS$ endows the edges incident to $p$ with a cyclic order.

**Definition.** A subgraph $H \subset G$ is spread if: for every vertex $p \in H$ and edges $e, e'$ of $H$ at $p$, in the cyclic order at $p$ the edges $e$ and $e'$ are not consecutive.

**Lemma.** Let $\alpha, \beta$ be a filling pair of simple closed curves in minimal position on $S$. If $\alpha$ is nonseparating, then the dual graph to $\alpha \cup \beta$ admits a spread spanning tree. If $\alpha$ is separating, the dual graph admits a spread spanning forest with two components.

**Proof.** Observe that by the assumptions $S$ is homeomorphic to $A/\sim$, where $A$ is a Euclidean annulus formed by $|\alpha \cap \beta|$ unit squares in a ring, and where $\sim$ is a side-pairing of $A$, so that the core curve of $A$ projects to the homotopy class of $\alpha$ under $A \to A/\sim \approx S$. Let $G$ be the 1-skeleton of $A/\sim$, or, equivalently, the graph dual to $\alpha \cup \beta$.

The square complex $A/\sim$ partitions the edges of $G$ into horizontal and vertical. We suppose that $\alpha$ is horizontal, and let $\Gamma_0$ be the subgraph of $G$ spanned by the horizontal edges. Evidently, $\Gamma_0$ spans $G$, since every vertex is incident to a horizontal edge. Moreover, $\Gamma_0$ is spread, since edges alternate between horizontal and vertical at each vertex of $G$.

Now $\Gamma_0$ has either one or two components, according to whether $\alpha$ is nonseparating or separating. Indeed, $\Gamma_0$ is the image of $\partial A \to \partial A/\sim$, and the two components of $\partial A$ are connected in the image exactly when $\alpha$ is nonseparating. We now apply the while-loop:

\[
(*) \quad \text{While } \Gamma_i \text{ has an embedded loop, let } \Gamma_{i+1} \text{ be the graph obtained by deleting an edge that lies in an embedded loop from } \Gamma_i.
\]

This algorithm terminates in a spread spanning tree if $\alpha$ is nonseparating, and it terminates in a spread spanning forest with two components if $\alpha$ is separating. \qed

![Figure 1. A filling pair whose dual graph contains no spread spanning trees.](image)

**Remark.** The separating / nonseparating dichotomy in this Lemma leads to a dichotomy in the proof of the Main Theorem. If one of the two curves is nonseparating, the Aougab-Huang approach goes through unmolested. When both curves are separating, more care must be taken. This dichotomy is not artificial: Figure 1 shows a filling pair of separating curves on $S_2$ whose complementary components consist of two octagons and eight squares. One can check that there does not exist a spread path between the two octagons.

**Question.** Which filling graphs embedded in $S$ admit spread spanning trees?
3. The proof of the main theorem

We now mimic the proof of [AH15, Cor. 4.5], gluing together the complementary polygons to a filling pair of simple geodesics using the above Lemma. We indicate the perimeter of a polygon $Q$ by $\text{perim}(Q)$ and its number of sides (or, equally, vertices) by $n(Q)$.

**Proof of Main Theorem.** Let $\alpha, \beta$ be simple geodesics on $X \approx S$, and let $\mathcal{G} \subset X$ be the graph induced by $\alpha \cup \beta \subset X$. The complementary components of $\mathcal{G}$ determine hyperbolic polygons $P_1, \ldots, P_r$, and the length of $\mathcal{G}$ is equal to $\frac{1}{2} \sum \text{perim}(P_k)$.

Observe that the sum $\sum n(P_k)$ is two times the number of edges of $\mathcal{G}$, which is equal to four times the number of vertices. The number of faces is $r$, so by Gauss-Bonnet we find

$$\sum_{k=1}^{r} n(P_k) = 8g - 8 + 4r.$$  

Suppose first that $\alpha$ is nonseparating. By the lemma, the dual graph to $\mathcal{G}$ admits a spread spanning tree $T$, which we may regard as embedded in $X$ dual to $\mathcal{G}$. Let $\hat{Q} = \sqcup_k P_k / \sim_T$ be obtained as follows: for each edge $e$ of $T$ whose endpoints are polygons $P_i$ and $P_j$, we identify the sides of $P_i$ and $P_j$ along their shared side dual to $e$. As $T$ is a tree, $\hat{Q}$ is again a polygon.

Moreover, the vertices of $\hat{Q}$ can be partitioned into old vertices, whose $\sim_T$–equivalence class is a singleton, and the complementary new vertices.

Choose a vertex $q \in \mathcal{G}$.Because $T$ is spread, the edges of $\mathcal{G}$ incident to $q$ and dual to edges of $T$ are non-consecutive in the cyclic order of $\mathcal{G}$ at $q$. Each $\sim_T$–equivalence class of vertices of $\sqcup_k P_k$ therefore has either one or two elements, and the number of new vertices is exactly $2e(T)$, where $e(T)$ is the number of edges of $T$. Moreover, any new vertex of $\hat{Q}$ must have angle $\pi$, so we may construct a polygon $Q$ by erasing the new vertices of $\hat{Q}$.

Now it is evident that the number of vertices of $Q$ is equal to the number of old vertices of $\hat{Q}$, so $n(Q) = n(\hat{Q}) - 2e(T)$. Because $\sim_T$ erases two edges of $\sqcup_k P_k$ for each edge of $T$,

$$n(\hat{Q}) = -2e(T) + \sum_{k=1}^{r} n(P_i).$$

Together with [1], this implies that $n(Q) = -4e(T) + 8g - 8 + 4r$.

As $T$ is spanning, its number of vertices is $r$, and as $T$ is a tree we find $e(T) = r - 1$. Hence

$$n(Q) = -4(r - 1) + 8g - 8 + 4r = 8g - 4,$$

and by [Bez84] we find $\text{perim}(Q) \geq \text{perim}(P)$. Of course, $\sum \text{perim}(P_k) \geq \text{perim}(Q)$.

Now suppose that $\alpha$ separates $X$ into totally geodesic subsurfaces $X_1$ and $X_2$, of genus $g_1$ and $g_2$ respectively. In that case, the Lemma provides the spanning forest $T_1 \sqcup T_2 \subset \mathcal{G}$, where $T_1$ and $T_2$ are spread trees. The same construction above yields polygons $Q_1$ and $Q_2$ with $\sum \text{perim}(P_k) \geq \text{perim}(Q_1) + \text{perim}(Q_2)$. Moreover, $X_i$ is isometric to a gluing of $Q_i$.

Performing the calculation [1] for each subsurface, we find $n(Q_i) = 8g_i$. Now let $Q_i$ be a regular $8g_i$-gon with area $\pi(4g_i - 2)$, so that by Bezdek we find $\text{perim}(Q_i) \geq \text{perim}(Q_i)$. Observe that $\hat{Q}_i$ is necessarily right-angled. Indeed, the common angle of $\hat{Q}_i$ is given by \[ \frac{\pi}{8g_i}(8g_i - 2 - (4g_i - 2)) = \frac{\pi}{2} \]

The following comparison now completes the proof:

**Proposition.** Suppose that $R_1$, $R_2$, and $R$ are regular right-angled polygons with $n(R_i) = n_i$, $n(R) = m$, and suppose that $n_1 + n_2 = m + 4$. Then $\text{perim}(R_1) + \text{perim}(R_2) > \text{perim}(R)$.

Observe that we may conclude as well: if $r > 1$, then $\sum \text{perim}(P_k) > \text{perim}(P)$. Therefore,
Corollary. With the setup of the Theorem, if we find equality in the conclusion, then the filling pair is minimal and $X$ is obtained as a gluing of $P$.

It remains to prove the above proposition. We emphasize that the right-angled hypothesis makes this statement far simpler than the involved calculations of [SV19] Thm. 2.4.

Proof of Proposition. As $R_i$ is a right-angled hyperbolic polygon, we have $n_i \geq 5$, so the constraint $n_1 + n_2 = m + 4$ implies that $n_1, n_2 \in \{5, \ldots, m - 1\}$. We first show that, for fixed $m$, the sum $\text{perim}(R_1) + \text{perim}(R_2)$ is minimized for $\{n_1, n_2\} = \{5, m - 1\}$.

One may use hyperbolic trigonometry to calculate the perimeter of a regular polygon (see [RAR94, p. 97]). Using the right-angled assumption we find that $\text{perim}(R_1) = f(n_1)$, $\text{perim}(R_2) = f(n_2)$, and $\text{perim}(R) = f(m)$, where

$$f(x) = 2x \cosh^{-1} \left( \sqrt{2} \cos \left( \frac{\pi}{x} \right) \right).$$

It is straightforward to compute

$$f'(x) = 2 \cosh^{-1} \left( \sqrt{2} \cos \left( \frac{\pi}{x} \right) \right) + \frac{2\pi \sqrt{2} \sin \left( \frac{\pi}{x} \right)}{x \sqrt{\cos \left( \frac{2\pi}{x} \right)}}, \text{ and } f''(x) = -\frac{2\pi^2 \sqrt{2} \cos \left( \frac{\pi}{x} \right)}{x^3 \sqrt{\cos^3 \left( \frac{2\pi}{x} \right)}}.$$

Let $C > 4$. Because $f$ is concave (i.e. $f''(x) < 0$), the function $g : (4, C - 4) \to \mathbb{R}$ given by $g(x) = f(x) + f(C - x)$ is concave as well. Therefore, as a function of $n_1 \in (4, m)$, $\text{perim}(R_1) + \text{perim}(R_2) = f(n_1) + f(m + 4 - n_1)$ is concave. As $n_1$ is an integer, $\text{perim}(R_1) + \text{perim}(R_2) \geq f(5) + f(m - 1)$. Our conclusion will follow from $f(5) + f(m - 1) > f(m)$.

Observe that concavity of $f$ implies that

$$f(m) - f(m - 1) < (m - (m - 1)) \cdot f'(m) = f'(5).$$

It remains to show that $f'(5) \leq f(5)$. As $\cos \left( \frac{\pi}{5} \right) = \frac{\phi}{2}$, where $\phi = \frac{1 + \sqrt{5}}{2}$, we compute

$$f'(5) = 2 \cosh^{-1} \left( \frac{1}{\sqrt{2}} \Phi \right) + \frac{2\pi}{5} \sqrt{\Phi + \frac{1}{\Phi}}, \text{ and } f(5) = 10 \cosh^{-1} \left( \frac{1}{\sqrt{2}} \Phi \right).$$

As $\pi < 4$, we find that $f'(5) < f(5)$ is implied by $\frac{1}{5} \sqrt{\Phi + \frac{1}{\Phi}} \leq \cosh^{-1} \left( \frac{1}{\sqrt{2}} \Phi \right)$.

While one can check that $\frac{1}{5} \sqrt{\Phi + \frac{1}{\Phi}} \approx .299$ and $\cosh^{-1} \left( \frac{1}{\sqrt{2}} \Phi \right) \approx .531$ with a calculator, in fact this can be checked by hand.

Exploiting $\Phi^2 = 1 + \Phi$, $\frac{1}{\Phi} = \Phi - 1$, and $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$, one finds:

$$\cosh^{-1} \left( \frac{1}{\sqrt{2}} \Phi \right) = \frac{1}{2} \log \left( \Phi + \sqrt{\Phi} \right) > \frac{1}{2} > \frac{1}{5} \sqrt{\Phi + \frac{1}{\Phi}},$$

where on the last line we’ve used the elementary estimates $\Phi + \sqrt{\Phi} > e$ and $\Phi + \frac{1}{\Phi} < 3$. \hfill $\square$

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