Islands of Conformational Stability for Filopodia

D. Robert Daniels¹*, Matthew S. Turner²

1 Multidisciplinary Nanotechnology Centre, College of Engineering, Swansea University, Swansea, United Kingdom, 2 Department of Physics and Centre for Complexity Science, University of Warwick, Coventry, United Kingdom

Abstract

Filopodia are long, thin protrusions formed when bundles of fibers grow outwardly from a cell surface while remaining closed in a membrane tube. We study the subtle issue of the mechanical stability of such filopodia and how this depends on the deformation of the membrane that arises when the fiber bundle adopts a helical configuration. We calculate the ground state conformation of such filopodia, taking into account the steric interaction between the membrane and the enclosed semiflexible fiber bundle. For typical filopodia we find that a minimum number of fibers is required for filopodium stability. Our calculation elucidates how experimentally observed filopodia can obviate the classical Euler buckling condition and remain stable up to several tens of μm. We briefly discuss how experimental observation of the results obtained in this work for the helical-like deformations of enclosing membrane tubes in filopodia could possibly be observed in the acrosomal reactions of the sea cucumber Thyone, and the horseshoe crab Limulus. Any realistic future theories for filopodium stability are likely to rely on an accurate treatment of such steric effects, as analysed in this work.

Introduction

Filopodia are formed by the growth of bundles of biological fibers outwards from a biological cell surface that remain enclosed in a membrane tube. They are implicated in many processes vital to life, including sensing and motility [1] [2] [3]. There has therefore been much interest recently in the formation and growth of long, thin cellular protrusions due to the polymerization of bundles of fibers, including actin [4]. Such structures appear on cell membranes as familiar filopodia [1] [4], but can also appear on neural growth cones [5], sickled red blood cells [6] [7], the acrosomal reaction of the sea cucumber Thyone, and the horseshoe crab Limulus. Any realistic future theories for filopodium stability are likely to rely on an accurate treatment of such steric effects, as analysed in this work.

A naive Euler buckling type estimate for the stability of filopodia [11] [12] suggests a limiting length of $L = 2\mu m$. Additionally, the presence of cross-linking in the fiber bundle, and hence increased stiffness, further suggest a limiting length of $L = 20\mu m$ for stable filopodia [11] [12]. However, filopodia many tens of μm have been observed experimentally [8] [13].

In [11], a helical ansatz was employed for the conformation of the polymer bundle. However, for analytical calculational purposes this was assumed to reside inside an enclosing membrane tube that remained perfectly cylindrical, despite simulation snapshot evidence to the contrary [11]. Energetically stable ground state configurations were calculated for filopodia within the range of physiologically relevant parameters. However, due to the presence of very soft modes [14] for membrane tube deformations, it is unrealistic to analytically assume that the enclosing membrane tube will remain perfectly cylindrical. It would cost the membrane tube very little energy to deform in order to accommodate the enclosed helical fiber bundle (see Fig. 1). In order to calculate analytically the ground state configurations of realistic filopodia, and their corresponding energetic stability, we find that it is necessary to explicitly compute the conformation of the enclosing membrane tube. This is achieved by minimising a rigorously derived energy functional (defined below) that includes the elastic response of both the membrane and the fiber bundle while respecting the constraint that the helical polymer bundle must remain enclosed by the membrane tube.

Results

Typical experimental parameter values for biological membranes range from 

$$\sigma \approx 0.013 - 0.025 k_B T \text{nm}^2.$$ 

In order to compare the results of this work with that of [11], we take $\kappa = 40 k_B T$ and
the minimum number of fibers required for filopodium stability is given by \( N = 4 \), as shown in Fig. 3, which gives rise to a local energy minimum at: \( r_P = 122 \text{nm} \) and \( \gamma = 0.77 \), corresponding to one helical winding per 2 \( \mu \text{m} \) of contour length. From Fig. 4 we can see that for \( N = 20 \) we have a local energy minimum at: \( r_P = 99 \text{nm} \) and \( \gamma = 0.98 \), corresponding to one helical winding per 4 \( \mu \text{m} \) of contour length. Moreover, from Fig. 5 we can see that as the number of fibers \( N \) in a bundle increases, our filopodium remains stable, with the \( \gamma \) extension factor rapidly approaching the maximum allowed value of 1. Furthermore, as \( N \) increases, we can see from Fig. 6 that \( r_P \) decreases, tending towards the limiting value of \( r_0 - b \), as the number of fibers becomes large. Additionally, we can see from Fig. 7 that the amount of helical winding required for stability reduces concomitantly also, as the number of fibers in a bundle \( N \) increases.

**Discussion**

We have calculated theoretically the ground state configurations of filopodia, and found ‘islands of stability’ for typical filopodia within physiologically relevant parameters. Our calculation elucidates how experimentally observed filopodia can obviate the classical Euler buckling condition and remain stable up to several tens of \( \mu \text{m} \) [1] [8] [13]. We find, as in [11] that the enclosing membrane tube tends to stabilise filopodia, rather than de-stabilise as a naive Euler buckling estimate might suggest.

The work presented here differs from that presented in [11] in the following, important ways. Firstly, we correctly incorporate the effects of a finite fiber bundle radial size \( b \), in this work, which was absent in [11]. Secondly, the total energy \( \frac{E_{\text{tot}}}{L} \) given by Eq. (12) of this work is calculated rigorously and analytically, taking into proper account the steric constraint of membrane tube enclosure of our semiflexible fiber bundle. The presence of soft modes for membrane tube deformations, implies that the membrane tube typically deforms in order to accommodate the enclosed helical fiber bundle, and does not remain perfectly straight, as analytically assumed in [11]. Thirdly, as we have found, there exists a delicate interplay between a fluid membrane tube and an enclosed semiflexible fiber bundle in filopodia. It is therefore imperative that the most accurate and correct total energy function for filopodia be calculated, as achieved in this work. Only then does it become possible to realistically describe the rather subtle issue of whether a given filopodium exists in a stable or a collapsed state. For example, we find in this work that the minimum number of fibers required for stability is given by \( N = 4 \), whereas in [11] all fibers with \( N < 6 \) are deemed unstable.

Experimental observation of the results obtained in this work for the helical-like deformations of enclosing membrane tubes in filopodia would presumably be difficult. However, such helical membrane conformations are qualitatively supported by the snapshot pictures of simulation work carried out in [11], and could possibly be observed in the acrosomal reactions of the sea cucumber *T. tridentata* [8], and the horseshoe crab *L. polyphemus* [16].

We adopt a ground state approximation in which thermal fluctuations are assumed to be small. Since the amplitude of these fluctuations is small at the high tensions of interest to us here, perhaps a few \( \text{nm} \) or less [17], this is a reasonable approximation. Analogous steric constraints to those considered here are likely to be of relevance in other similar and important biological contexts, such as the packaging of semiflexible DNA in viral capsids, for example [18] [19] [20]. The interesting issue of mechanical stability in biological cellular tubes without an enclosed stiff polymer has also recently been considered in [21].

---

**Figure 1. Sketch of a helically deformed membrane enclosing a helical fiber bundle.** The membrane radius is given by \( r_M \), the helical polymer radius is given by \( r_P \), and \( b \) is the radial size of the enclosed polymer filament bundle. doi:10.1371/journal.pone.0059010.g001
Figure 2. Contour plot of the total energy per unit contour length $E_{\text{tot}}/L$ from Eq.(12). The energy is plotted as a function of the enclosed filament helical radius $r_p$, and the z extension factor $\gamma$. The membrane bending modulus is $\kappa = 40k_BT$ and the surface tension is $\sigma = 0.0025k_BT/\text{nm}^2$. The same values of $\kappa$ and $\sigma$ are used in all subsequent figures. The number of filaments in this case is given by $N = 1$. These parameters do not give rise to a local energy minimum. The contours near the top of the plot have values around $\sim 11\mu N$, those contours near the middle $\sim 7\mu N$, and the nearest to bottom contours on the plot $\sim 2\mu N$, (at room temperature).

doi:10.1371/journal.pone.0059010.g002

Figure 3. Contour plot of the total energy per unit contour length $E_{\text{tot}}/L$ from Eq.(12). The energy is plotted as a function of the enclosed filament helical radius $r_p$, and the z extension factor $\gamma$. The number of filaments in this case is given by $N = 4$. These parameters give rise to a local energy minimum at $r_p = 122\mu m$ and $\gamma = 0.77$, corresponding to one helical winding per $2\mu m$ of contour length. Both the contours near the top and bottom of the plot have values around $\sim 12\mu N$, while the closed contour near the middle has a value of $\sim 10\mu N$.

doi:10.1371/journal.pone.0059010.g003
Models

Polymer Energy

In order to describe the filament bundle, inside filopodia, we study the semi-flexible polymer Hamiltonian $H_p$ (where we chose energy units such that $k_B T \sim 1$ throughout):

$$H_p = \frac{l_3}{2} \int_0^L ds \left( \frac{\dot{R}_P(s)}{\dot{s}} \right)^2$$

(1)

with $l_3 = \frac{\varepsilon R_P(s)}{\varepsilon s}$. $L$ is the contour length of the fiber bundle, and we take the persistence length for un-crosslinked bundles of $N$ fibers to be $l_p = \kappa_p N$ [12], where $\kappa_p$ is the bending modulus of a single fiber ($\kappa_p \sim 15 \mu m$ for actin [2]).

Any realistic deformation of the polymer must be able to pack a given contour length $L$ within a given radius and extension along the $z$ axis, as prescribed by the enclosing membrane tube. We therefore assume the most plausible conformation for the polymer as being that of a helix, as also outlined in [11].

$$R_P(z) = R_P \left( \cos (\Omega z) \hat{i} + \sin (\Omega z) \hat{j} + z \hat{k} \right)$$

(2)

Figure 4. Contour plot of the total energy per unit contour length $\frac{E_{tot}}{L}$ from Eq.(12). The energy is plotted as a function of the enclosed filament helical radius $r_P$, and the $z$ extension factor $\gamma$. The number of filaments in this case is given by $N = 20$. These parameters give rise to a local energy minimum at $r_P = 99$ nm and $\gamma = 0.98$, corresponding to one helical winding per $4 \mu m$ of contour length. The closed contour near the top of the plot has a value of $\sim 11 pN$, while the contours close to the bottom of the plot have values $\sim 13 pN$.

doi:10.1371/journal.pone.0059010.g004

Figure 5. Plot of the extension factor $\gamma$ along the $z$ axis versus the number of filaments $N$. The $\gamma$ values plotted correspond to the energetic minima of the total energy per unit contour length $\frac{E_{tot}}{L}$ from Eq.(12), for a given number of filaments $N$.

doi:10.1371/journal.pone.0059010.g005
We have chosen to parameterise the polymer in terms of the $z$ coordinate, as opposed to the arc-length $s$, in order to simplify consideration of the required steric constraint between the polymer and the membrane as outlined below. Inextensibility for the polymer is maintained by requiring that:

$$\gamma = \frac{dz}{ds} = \frac{1}{[c_R R_p]} = \frac{1}{\sqrt{1 + \Omega^2 r_p^2}}$$

In this way we can easily translate between the arc-length $s$, and $z$ extension representations, by defining: $L = \gamma$ and $\omega = \gamma \Omega$, such that: $\gamma^2 + \omega^2 r_p^2 = 1$.

The polymer part $H_P$ is thus straightforwardly calculated to be:

$$H_P = L \beta \frac{(1 - \gamma^2)^2}{r_p}$$

Membrane Energy

In order to describe deformations of our membrane tube, we use:

$$H_M = \left[\sigma + \frac{K}{2} \gamma^2\right] \sqrt{\partial s}\phi dz$$

where $H_M$ is the usual Hamiltonian for membrane elasticity [22] [23], containing both surface tension ($\sigma$) and rigidity ($K$) controlled terms.

We parameterise our membrane given by $R_M(z, \phi)$ in the usual way as:

$$R_M(z, \phi) = r_0(z, \phi)(\cos(\phi) \hat{i} + \sin(\phi) \hat{j}) + z \hat{k}$$

The membrane contribution $H_M$ is calculated as follows. We proceed by writing perturbatively: $r_M(z, \phi) = r_0 + \delta r_M(z, \phi)$, which involves the radial length scale $r_0 = \sqrt{\frac{K}{2\sigma}}$. In this way we obtain:

$$\delta r_M(z, \phi) = r_0 + \delta r_M(z, \phi)$$

Figure 6. Plot of the polymer helical radius $r_P$ versus the number of filaments $N$. The $r_P$ values plotted correspond to the energetic minima of the total energy per unit contour length $E_{tot}/L$ from Eq.(12), for a given number of filaments $N$. For comparison, note that $r_0 = 89$ nm.

doi:10.1371/journal.pone.0059010.g006

Figure 7. Plot of the polymer helical winding length versus the number of filaments $N$. The winding length values plotted correspond to the polymer contour length required for one complete helical turn in order to maintain stability of the filopodium.

doi:10.1371/journal.pone.0059010.g007
\[ H_M = L \frac{2\pi k}{r_0} + \frac{\kappa}{2r_0} \int d\phi d\zeta \delta r_M(\phi,\zeta) K_M \delta r_M(\phi,\zeta) \]  

where the kernel \( K_M \) is given by: 
\[ K_M = (1 + 2\epsilon^2 + (r_0^2 \epsilon^2 + \epsilon^4)^2). \]

Steric Constraint

By inspection of Eqs. (2) and (6), we can see that the steric condition we need to apply to the membrane in order to guarantee polymer enclosure is given by:
\[ r_M(\phi,\zeta) = r_p + b \]  

where \( b \) is the radial size of the polymer filament bundle. By writing perturbatively:
\[ r_M(\phi,\zeta) = r_0 + \delta r_M(\phi,\zeta), \]
the steric constraint of Eq. (3) now implies:
\[ \delta r_M(\phi,\zeta) = r_0 + b - r_0. \]

By introducing the following Hamiltonian \( H_C \):
\[ H_C = \int d\phi d\zeta (\delta r_M(\phi,\zeta) - (r_0 + b - r_0)) \]

which includes a Lagrange multiplier \( \lambda(\zeta) \) that ensures membrane tube enclosure of the confined polymer helix. While the steric relationship is strictly an inequality, on physical grounds the ground state polymer configuration always tends to contact the membrane because the longer the polymer the smaller the compressive load it can support before it buckles, becoming helical. Thus a long polymer will always tend to adopt a helical configuration, stabilised by the inward-pointing membrane force, at the maximum radius allowed by the steric constraint.

Total Energy

In order to find the ground-state configuration of our filopodium, we need to find the conformation which minimises the total energy \( E_{tot} \) given by: 
\[ E_{tot} = H_M + H_C \]  

where \( H_M + H_C \) w.r.t. \( \delta r_M(\phi,\zeta) \) and \( \lambda(\zeta) \), and by using the relevant Green functions, we obtain:
\[ \delta r_M(\phi,\zeta) = (r_0 + b - r_0) \sum m a_m \cos (m(\phi - \zeta)) / \sum m a_m \]  

along with \( \lambda(\zeta) = -\frac{2\pi k}{r_0} (r_0 + b - r_0) / \sum m a_m \), and where the Fourier coefficients \( a_m \) are given by:
\[ a_m = \frac{1}{(1 - 2m^2 + m^4(1 + \Omega^2 r_0^2))}. \]

Note that an ansatz loosely similar to Eq. (10) was also used in [24] to minimise the energy for a stack of cylinder membranes in order to describe the helical coiling behaviour of myelin tubes. Indeed, the filopodia described in this work, consisting of a fiber bundle of radius \( b \) enclosed by a membrane tube, can analogously be thought of as an ‘\( n=2 \)’ cylindrical membrane stack. In terms of the Fourier coefficients \( a_m \), the membrane radius solution of Eq. (10) can additionally be seen to automatically satisfy the steric constraint:
\[ \delta r_M(\phi,\zeta) = r_0 + b - r_0. \]

By inspection of the Fourier coefficients \( a_m \), it can be seen that for small \( \Omega \) winding the leading order contribution to \( H_M \) comes from the \( m=1 \) mode, and is proportional to \( \Omega^2 \). This leads to a relatively weak strength for the quadratic potential in \( r_0 + b - r_0 \), and is due to the fact that the \( m=1 \), \( \Omega \approx 0 \) mode is an extremely soft mode for membrane tubes as shown in [14]. Indeed, the \( m=1 \), \( \Omega = 0 \) mode corresponds precisely to a rigid translation of the entire tube, and cannot therefore make any contribution to the membrane energy \( H_M \). It can also be shown that the modes that contribute to \( H_M \) to next to leading order are the \( a_0 \) mode corresponding to a uniform dilation of the membrane tube, and the \( a_2 \) mode, which corresponds to a small deformation of the cross-section of our tube from a circular shape to that of an ellipse.

Utilising the inextensibility conditions outlined above, we can easily re-write \( H_M \) in terms of the contour length \( L \), and the \( z \) extension factor \( \gamma \). In particular we find for the winding rate:
\[ \Omega = \frac{1}{r_0} \sqrt{1 - 1 - \frac{1}{r_0 - b} \sqrt{1 - 1}}, \]  
to leading order. We thus arrive at our final expression for the total energy per unit length of a filopodium as (valid to quadratic order in \( r_0 + b - r_0 \)):

\[ E_{tot} = L \frac{2\pi k \gamma}{r_0} (1 + \frac{(r_0 + b - r_0)^2}{2r_0 \sum m a_m}) \]  

where the Fourier coefficients \( a_m \) are now functions of \( \gamma \). The ground-state configuration of our filopodium can now be determined by minimising the total energy per unit length \( E_{tot} \) of Eq. (12), with respect to the two parameters \( \gamma \) (the \( z \) extension factor) and \( r_0 \) (the helical radius of the enclosed fiber bundle).

Acknowledgments

We would like to thank the anonymous referees for their helpful comments.

Author Contributions

Analysed the data: DRD MST. Contributed reagents/materials/analysis tools: DRD MST. Wrote the paper: DRD MST.
12. Mogilner A, Rubinstein B (2005) The physics of filopodial protrusion. Biophys J 89: 782–795.
13. Wood W, Martin P (2002) Structures in focus - filopodia. Int J Biochem Cell Biol 34: 726–730.
14. Fournier JB, Gakotola P (2007) Critical fluctuations of tense fluid membrane tubules. Phys Rev Lett 98: 018103.
15. Derenyi I, Julicher F, Prost J (2002) Formation and interaction of membrane tubes. Phys Rev Lett 88: 238101.
16. DeRosier DJ, Tilney L, Flicker P (1980) A change in the twist of the actin containing filaments occurs during the extension of the acrosomal process in limulus sperm. J Mol Biol 137: 375–389.
17. Daniels DR, Turner MS (2005) Spicules and the effect of rigid rods on enclosing membrane tubes. Phys Rev Lett 95: 238101.
18. Marenduzzo D, Micheletti C, Orlandini E (2010) Biopolymer organization upon confinement. J Phys - Cond Mat 22: 283102.
19. Morrison G, Thirumalai D (2009) Semiflexible chains in confined spaces. Phys Rev E 79: 011924.
20. Hu Y, Zandi R, Anavitarte A, Knoehler CM, Gelhart WM (2008) Packaging of a polymer by a viral capsid: The interplay between polymer length and capsid size. Biophys J 94: 1429–1436.
21. Hannezo E, Prost J, Joanny JF (2012) Mechanical instabilities of biological tubes. Phys Rev Lett 108: 018101.
22. Safran SA (1994) Statistical Thermodynamics of Surfaces Interfaces and Membranes. Boston, MA: Addison Wesley Publishing.
23. Nelson D (1989) in Statistical Mechanics of Membranes and Surfaces. Singapore: World Scientific.
24. Santangelo CD, Pincus P (2002) Coiling instabilities of multilamellar tubes. Phys Rev E 66: 061501.