Real Mutually Unbiased Bases

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Abstract

We tabulate bounds on the optimal number of mutually unbiased bases in $\mathbb{R}^d$. For most dimensions $d$, it can be shown with relatively simple methods that either there are no real orthonormal bases that are mutually unbiased or the optimal number is at most either 2 or 3. We discuss the limitations of these methods when applied to all dimensions, shedding some light on the difficulty of obtaining tight bounds for the remaining dimensions that have the form $d = 16n^2$, where $n$ can be any number. We additionally give a simpler, alternative proof that there can be at most $d/2 + 1$ real mutually unbiased bases in dimension $d$ instead of invoking the known results on extremal Euclidean line sets by Cameron and Seidel, Delsarte, and Calderbank et al.

Keywords: Quantum Information Processing, Quantum Computing, Hadamard Matrices, Euclidean Line Sets, Hadamard Conjecture.

1 Introduction

Two orthonormal bases $B$ and $B'$ of the Hilbert space $\mathbb{C}^d$ are called mutually unbiased if and only if the modulus of the standard complex inner product is

$$\langle \phi | \psi \rangle = 1/\sqrt{d}$$

for all $|\phi\rangle \in B$ and all $|\psi\rangle \in B'$. The problem of determining bounds on the maximum number $M_{\mathbb{C}^d}$ of bases over $\mathbb{C}^d$ that are mutually unbiased is an important open problem [15] which has received much attention [13] [18] [2] [14] [11] [17]. We refer the reader to e.g. [1] for an overview of known bounds.

The most common application of MUBs is found in quantum cryptography where MUBs are the quantum states used in most QKD protocols [5] [3] [7] [4].

The MUB problem has remained open for over 20 years. For example, it is not even known whether:
- for all $M$ there is a $d'$ such that $M_{\mathbb{C}^d} \geq M$ for all $d > d'$? In this manuscript, we consider the problem of determining $M_{\mathbb{R}^d}$ the maximum number of mutually unbiased bases over $\mathbb{R}^d$ (real MUBs), defined analogous to the above. In the real case, the problem appears to have a somewhat different character: elementary arguments give tight bounds on almost all $d$; and the answer to the question is easily seen to be negative. Additionally, we are able to directly construct real MUBs which cannot be extended to an optimal set even using complex bases; which means in general, maximal sets of MUBs are not necessarily optimal.

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Table 1: Here \( s \) is any positive odd integer, and \( i \) is any non-negative integer; \( MOLS(r) \) denotes the maximum number of mutually orthogonal Latin squares of order \( r \). Recall that a Hadamard matrix (abbreviated by HM) is a square matrix of size \( d \) with entries in \( \{-1, 1\} \) with pairwise orthogonal columns.

Obtaining real MUBs is a special case of the geometric problem of obtaining “extremally or uniformly distributed” collections of Euclidean line sets \([10]\) with arbitrary, prescribed angles. Observe that a collection of MUBs is exactly a collection of orthogonal sets of lines passing through the origin such that any one of the following equivalent statements holds: (i) the angle between any pair of lines taken from different sets is the same; (ii) the minimum angle over any pair of lines taken from different sets is maximized; (iii) the maximum angle over any pair of lines taken from different sets is minimized. (This formulation of MUBs applies to any Hilbert space when “angle” is replaced by “magnitude of inner product.”)

In this paper, we show upper and lower bounds on the number of real MUBs in various dimensions using elementary methods or known results \([17, 10]\). These results are summarized in Table 1. The paper is organized as follows: Section 2 introduces a simple proof technique which we use throughout this paper. We use a simple technique to find the number of real MUBs in dimensions which are not divisible by four (\( M_{d/2} = 1 \)) and non-square dimensions (\( M_{d/2} \leq 2 \)). The remaining dimensions can be written as \( d = 4^i s^2 \) with \( s \) odd and \( i \geq 1 \). We consider such square dimensions \( 4^i s^2 \) in Section 3 where we find an upper bound (\( M_{d/2} \leq 3 \)) in the case of \( i = 1 \), and we use a previous result \([17]\) to show a lower bound in the case of \( i > 1 \). Section 4 gives a simpler, alternate proof for a previously known upper bound (\( M_{d/2} \leq d/2 + 1 \)).

Finally, in Section 5 we consider the question of whether all sets of real MUBs can be extended into an optimal set of (complex or real) MUBs and find that the answer is negative.

### 2 Most \( d \)'s admit at most 1 or 2 real MUBs

In this section, we consider non-square dimensions and dimensions not divisible by four. The proofs use an elementary technique which we will make use of in later sections. We will repeatedly use the following standard canonical form for real MUBs.

**Observation 2.1.** If \( B_1, B_2, \ldots, B_m \) are a collection of real MUBs, without loss of generality, we can always choose \( B_1 \) to be the standard basis (simply multiply all matrices by \( B_1^{-1} \) from the left). Putting the elements of \( B_1 \) as columns, we get the identity matrix \( 1 \). By Equation (1), all the other \( B_i \) are orthogonal and unbiased to \( B_1 \) if and only if their basis elements are the columns of a Hadamard matrix scaled by \( \frac{1}{\sqrt{d}} \); and \( B_i, B_j \) are pairwise unbiased if and only if \( B_i^{-1} B_j = B_i^t B_j \) is a Hadamard matrix scaled by \( \frac{1}{\sqrt{d}} \).
Proposition 2.2. If $4 \nmid d$, then $M_{\mathbb{R}^d} = 1$.

Proof. Let $B_1$, $B_2$ be two real MUBs in $\mathbb{R}^d$. Using the canonical form of Observation 2.1 for $B_1$ and $B_2$, it is a simple folklore fact that “$d$ must therefore be 2 or a multiple of 4 since otherwise Hadamard matrix of size $d$ cannot exist.” (The famous Hadamard conjecture [12] is that for every multiple of 4, there exists in fact a Hadamard).

We prove this simple folklore fact here, since many of the arguments presented in this note are elementary variants of this proof. This proof is similar to the one found in [16, page 74]. We show the stronger statement that no more than 3 orthogonal vectors in $\{-1,1\}^d$ can exist unless $d$ is a multiple of 4.

Assume there are three vectors $v_1, v_2, v_3 \in \{-1,1\}^d$ that are orthogonal to each other. Observe that permuting the entries of the three vectors according to any permutation does not change the inner product of any pair. Also note that changing the sign of the entries by multiplying the three vectors entry-wise with any column vector does not change the inner product of any pair, nor does it change the magnitude of any entry.

We use these two facts to assume without loss of generality, that the first vector $v_1$ is the all ones vector $(+,+,\ldots,+)$. The second vector $v_2$ should be orthogonal to the first, so again without loss of generality it can be chosen as the vector $(+,+\ldots,+,−\ldots,−)$, with the first half $+$’s and the second half $-$’s. This requires $d$ to be even and completes the Hadamard conjecture for $d = 2$.

The third vector $v_3$ should be orthogonal to the first and the second, hence without loss of generality, it can be chosen to be of the form $(+,\ldots,+−\ldots,−\ldots,+\ldots,−\ldots)$, with 4 blocks of $+$’s and $-$’s with alternating signs. This is due to the ability to permute the entries in any way we like. Any permutation that fixes $v_1$ and $v_2$ is allowed, so we permute $v_3$ such that there are $m_{++}$ values of $+$ followed by $m_{+-}$ values of $-$ in the first half of the vector. In the second half of the vector there are $m_{-+}$ values of $+$ followed by $m_{--}$ values of $−$. This gives rise to 4 independent equations:

\[
\begin{align*}
m_{++} + m_{+-} &= \frac{d}{2} \\
m_{-+} + m_{--} &= \frac{d}{2} \\
\langle v_1, v_3 \rangle &= m_{++} - m_{+-} + m_{-+} - m_{--} = 0 \\
\langle v_2, v_3 \rangle &= m_{++} - m_{+-} - m_{-+} + m_{--} = 0
\end{align*}
\]

The above equations give rise to the unique solution $m_{\pm\pm} = d/4$ and consequently $4 \mid d$. \(\square\)

This shows that there are two real orthonormal bases are that are mutually unbiased whenever $4 \nmid d$. Next we consider non-square dimensions of the form $d = 4n$.

Proposition 2.3. If $4 \mid d$, but $d$ is not a square, then $M_{\mathbb{R}^d} \leq 2$. In this case, $M_{\mathbb{R}^d} = 2$ if and only if a Hadamard of order $d$ exists.

Proof. Assume the contrary and let $B_1$, $B_2$ and $B_3$ three such orthonormal bases that are mutually unbiased. We also use $B_1, B_2$ and $B_3$ to denote the corresponding real matrices in the standard canonical form of Observation 2.1.

As in the proof of the simple folklore fact and Proposition 2.2 given above, we can always achieve without loss of generality that the first column of $B_2$ is $\frac{1}{\sqrt{d}}(1,1,\ldots,1)$. This is done as follows. If the $i$th entry of $B_2$ is $\frac{1}{\sqrt{d}}$ then multiply the $i$th row of $B_2$ and the $i$th row of $B_3$ by $-1$. Clearly, this does not change the absolute values of the inner products between the columns vectors of $B_1$, $B_2$ and $B_3$, nor does it change the orthogonality of any of them.
Now, let $\frac{1}{\sqrt{d}}(s_1, s_2, \ldots, s_d)$ (with $s_i = \pm 1$) be the first column of $B_3$. Eq. (1) applied to the first column of $B_2$ and the first column of $B_3$ implies that

$$|\sum_{i=1}^{d} s_i| = \sqrt{d}. \quad (2)$$

The absolute value of the sum is clearly a natural number, whereas $\sqrt{d}$ is irrational if $d$ is not a square. Therefore, there cannot exist a third orthonormal real basis $B_3$ that is mutually unbiased to $B_1$ and $B_2$.

This shows that the maximum number of real MUBs is at most 2 for dimensions $d$ that are not squares. However, if there is a Hadamard matrix of order $d$, then in fact, $B_2$ exists, and we have exactly 2 real MUBs.

In summary, we have handled the case of all dimensions not divisible by 4 by showing that there are not even two MUBs. In non-square dimensions which are divisible by four, there are at most two MUBs. In the next section, we will consider all other dimensions.

### 3 Square dimensions divisible by 4

We have so far shown that unless $d = 4m^2$ for some positive integer $m$, there are at most 2 real MUBs, such dimensions can be written as $d = 4^i s^2$ with $s$ odd. First we consider the case of $i = 1$ and then consider the case of $i > 1$.

**Proposition 3.1.** If $d = 4s^2$ for an odd positive integer $s$, the $M_{\mathbb{R}^d} \leq 3$; furthermore $M_{\mathbb{R}^d} \geq 2$ provided that a Hadamard matrix of order $d$ exists.

**Proof.** We show a stronger Lemma 3.3 that involves mutually unbiased lattice lines (as opposed to entire bases). Using this lemma and the canonical form of Observation 2.1 it follows that only 2 further MUBs exist that are unbiased to the standard basis.

**Definition 3.2.** A lattice line in $\mathbb{R}^d$ is a line passing through the origin and some point $v$ in $\{-1, 1\}^d$. We denote the line by either of the vectors $v$ or $-v$. Two lattice lines $v$ and $w$ are mutually unbiased if $|\langle v, w \rangle| = \frac{1}{\sqrt{d}}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. We denote by $L_{\mathbb{R}^d}$ the maximum number of mutually unbiased lattice lines that can be found in $\mathbb{R}^d$.

**Lemma 3.3.** If $d = 4s^2$ for an odd positive integer $s$, then $L_{\mathbb{R}^d} \leq 2$.

**Proof.** Assume to the contrary that there are 3 such lines represented by the vectors $v_1, v_2, v_3 \in \{-1, 1\}^d$. Consider the vectors corresponding to those lines. As in the proof of the simple folklore fact and Proposition 2.2 we assume that $v_1 = (+, +, \ldots, +)$ is the all-one-vector and we consider a partition of these vectors into 4 blocks with lengths $m_{++}, m_{+-}, m_{-+},$ and $m_{--}$. We obtain the following equations:

$$\langle v_i, v_i \rangle = m_{++} + m_{+-} + m_{-+} + m_{--} = 4s^2$$
$$\langle v_1, v_2 \rangle = m_{++} - m_{+-} + m_{-+} - m_{--} = \pm 2s$$
$$\langle v_2, v_2 \rangle = m_{++} + m_{+-} - m_{-+} - m_{--} = \pm 2s$$
$$\langle v_2, v_3 \rangle = m_{++} - m_{+-} - m_{-+} + m_{--} = \pm 2s$$
One can solve for $m_{++}$ by adding all the equations together to get:

$$m_{++} = s^2 + \frac{s(\pm1 \pm 1 \pm 1)}{2}$$

Since the value in the parenthesis can only add up to an odd number $(-3, -1, 1, 3)$, and due to the fact that $s$ is odd and the product of two odds is odd, $\frac{s(\pm1 \pm 1 \pm 1)}{2}$ cannot be an integer. However, if $m_{++}$ is not an integer, we have a contradiction, proving the result. \[\Box\]

Hence we see that if $d = 4^i s^2$ with $s$ odd and $i = 1$, there are at most 3 real MUBs. Now we consider the other half of the remaining cases, i.e., where $i > 1$. In this case, an earlier construction of MUBs using Latin squares immediately gives a lower bound.

**Proposition 3.4.** \cite{17} If $d = 4^i s^2$, where $s$ is any positive integer, then $M_{R,d} \geq MOLS(2^i s) + 2$, provided that there exists a Hadamard matrix of order $2^i s$, where $MOLS(m)$ denotes the maximum number of mutually orthogonal Latin squares of order $m$.

This construction is contained in Appendix A. Since there is a Hadamard of order 2, the above construction also works in dimension 4 and is optimal in that case. We obtain the following three mutually unbiased bases:

$$B_1 := \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\},$$

$$B_2 := \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\},$$

$$B_3 := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

So far, we have seen that the upper bound on the number of real MUBs in dimension $d$ is constant (either 1, 2 or 3) unless $d = 4^i s^2$ with $s$ odd and $i > 1$. In the latter case, we found a lower bound on the number of real MUBs following from the construction based on mutually orthogonal Latin squares. In the next section we give an upper bound on the number of real MUBs and see that the upper bound is tight in dimension $d = 4^i$.

### 4 A general upper bound which is tight for $d = 4^i$

In this section we consider general dimensions and give a new proof that $M_{R,d} \leq d/2 + 1$. Then we cite a construction for $d = 4^i$ which obtains the upper bound, and thus we see that the bound is tight in the case of $d = 4^i$.

**Proposition 4.1.** \cite{18} \cite{19} We have the upper bound $M_{R,d} \leq d/2 + 1$ for any $d$. 


Proposition 4.1 is a special case of the type of bound proved by [9] (using Jacobi polynomials) and [10] (using tensor algebra first principles), where they consider the maximum number of lines that can be packed into \( \mathbb{R}^d \) such that the angle between any pair of lines is one of two specified angles.

Next, we give an alternative, simpler proof of this upper bound. This is based on an adaptation to \( \mathbb{R}^d \) of a result of [2] which equates the MUBs over \( \mathbb{C}^d \) to so-called commuting classes of unitary matrices of size \( d \times d \).

**Theorem 4.2.** [2] There exists a set of \( m \) MUBs in \( \mathbb{C}^d \) if and only if there are \( m \) classes \( C_1, \ldots, C_m \) of the following properties:

- each class \( C_j \) consists of \( d \) commuting matrices,
- any two classes have only the identity matrix in common, and
- all matrices in \( C_1, \ldots, C_m \) are pairwise orthogonal with respect to the trace inner product.

The corresponding MUBs \( B_1, \ldots, B_m \) are the common eigenvectors of the matrices of the commuting classes \( C_1, \ldots, C_m \), respectively.

**Corollary 4.3.** There exists a set of \( m \) MUBs in \( \mathbb{R}^d \) if and only if there are \( m \) classes \( C_1, \ldots, C_m \) with properties as in Theorem 4.2 with the additional property that all matrices are real symmetric.

**Proof.** Since the eigenvectors of real symmetric matrices are real, it follows from Theorem 4.2 that there are \( m \) real MUBs under the assumptions of the corollary. This proves one direction of the equivalence.

For the other direction, we show how to construct \( m \) commuting classes \( C_1, \ldots, C_m \) of real symmetric matrices satisfying the properties of Theorem 4.2 starting from \( m \) MUBs.

Let the \( j \)th MUB be given by

\[
B_j = \{ |\psi_{j1}\rangle, \ldots, |\psi_{jd}\rangle \}.
\]

Assuming that \( m \geq 2 \), we can construct at least one real Hadamard

\[
H := \sqrt{d}B_1^\dagger B_2,
\]

where we use \( B_1 \) and \( B_2 \) to also denote the unitary matrices whose column vectors are given by basis vectors of \( B_1 \) and \( B_2 \), respectively. We may assume without loss of generality that the \( B_1 \) is the standard basis and that the first vector of \( B_2 \) is given by the normalized. Then the first row of \( H \) is the all-one-vector. Denote the entries of \( H \) by \( h_{tk} \).

Then we define the \( j \)th commuting classes \( C_j := \{ U_{j,1}, U_{j,2}, \ldots, U_{j,d} \} \), where the matrices within the class are given by

\[
U_{j,t} := \sum_{k=1}^{d} h_{t,k} |\psi_{kj}\rangle \langle \psi_{kj}|
\]

for \( t = 1, \ldots, d \) and \( j = 1, \ldots, m \).

Observe that the matrix \( U_{j,t} \) is diagonal with respect to the basis \( B_j \) and that its eigenvalues are given by the entries of the \( t \)th row of the real Hadamard matrix \( H \). Therefore, theses matrices are not only unitary but also real symmetric. Clearly the matrices within each commuting class commute because they are diagonal with respect to the same basis. The only matrix that is contained in all commuting classes is \( I = U_{j,1} \). It remains to show that they are all orthogonal with respect to the trace inner product.

Observe that all matrices within a commuting class are orthogonal because the rows of \( H \) are orthogonal. This implies that all matrices not equal to the identity matrix are traceless.
We can compute the inner product directly:

\[ \text{tr}(U_{i,r}^\dagger U_{j,s}) = \text{tr}(U_{i,r} U_{j,s}) = \sum_{x=1}^{d} \sum_{y=1}^{d} h_{r,x} h_{s,y} |\langle \psi^j_x | \psi^j_x \rangle|^2 \]

\[ = \frac{1}{d} \sum_{x=1}^{d} \sum_{y=1}^{d} h_{r,x} h_{s,y} \]

\[ = \frac{1}{d} \text{tr}(U_{i,r}) \text{tr}(U_{j,s}) = 0 \]

because one of the matrices is not the identity matrix. This completes the proof.

Now we are ready to give the alternative proof of Proposition 4.1.

**Proof.** (of Proposition 4.1). If there are \( m \) real MUBs, then according to Corollary 4.3 we can construct \( m \) classes of commuting real symmetric unitaries. Each class contains \( d - 1 \) traceless matrices. Additionally, we can consider the identity matrix, which is also a real symmetric unitary. All of these \( m(d - 1) + 1 \) matrices are orthogonal. The space of real symmetric matrices is a vector space of dimension \( d(d + 1)/2 \). The span of our \( m(d - 1) + 1 \) matrices is a subspace of all real symmetric matrices, thus:

\[ m(d - 1) + 1 \leq \frac{d(d + 1)}{2} \]

\[ m \leq \frac{d}{2} + 1 \]

This constitutes a new proof for the upper bound of real MUBs in dimension \( d \).

A construction of \( d/2 + 1 \) line sets that correspond to real MUBs is given in [6, 10] for the special case of \( d \) being a power of 4. The general upper bound of Proposition 4.1 shows that construction is optimal.

In the next section we consider the question of extending sets of MUBs to try to reach the upper bounds of this section. Unfortunately, we will see that such extensions in general are not optimal.

## 5 Greedy methods do not work

In this section, we consider the efficacy of greedy methods of constructing real MUBs. We ask the question: **Starting with an arbitrary set of MUBs or mutually unbiased lattice lines, can one extend the set to find an optimal number of MUBs or mutually unbiased lattice lines?** Put another way: can we take any MUB that comes our way, or must we be careful to choose sets that fit well together?

We consider the question: can the method of Proposition 3.1 and Lemma 3.3 be extended to show an upper bound on the number of real MUBs for \( d = 4^i s^2 \) (to complement the lower bound of Proposition 3.4)?

We could continue, without loss of generality, to partition \( j \) vectors into \( 2^j \) blocks and write down equations that they must satisfy in order to represent mutually unbiased lattice lines. Showing the conditions on \( j \) and \( d \) under which such systems do not have solutions would then give us an upper bound on \( L_{\mathbb{R}^d} \) (and \( M_{\mathbb{R}^d} \)) for specific dimensions \( d \). However, in this approach, we end up with a family of linear Diophantine systems one for each \( j \) and \( d \), with inequalities and equalities, since the
solutions \( m_{ij} \) (to the sizes of the blocks in the partition) should be non-negative integers. Showing the conditions on \( j \) and \( d \) for which such a system does not have a solution does not appear to be tractable.

Our approach is instead the following. We construct a particular set of \( 2^i \) lattice lines in dimension \( d = 4i s^2 \) using the \( 2^i \times 2^i \) Sylvester Hadamard \([12]\). We then show that this particular set cannot be extended by even one lattice line. If any set can be extended into an optimal set, then \( L_{Rd} = 2^i \) for \( d = 4i s^2 \). However, this result is not true, because we later give a construction that gives \( L_{Rd} \geq 2^i (2^i s^2 - s) \) for some cases of \( i \), which shows that not every set can be extended into an optimal set.

Let \( H \) be the Sylvester Hadamard matrix of order \( 2^i \) \([12]\) and denote its entries by \( s_{kl} \) for \( k, l = 0, \ldots, N - 1 \), where \( N = 2^i \). Let \( w_0, w_1, \ldots, w_{N-1} \) be all-one-vectors of lengths \( b_0, b_1, \ldots, b_{N-1} \), respectively. The lengths will be determined later. The vectors \( v_0, v_1, \ldots, v_{N-1} \) have the following block structure:

\[
v_k := (s_{k,0} w_0 \mid s_{k,1} w_1 \mid \ldots \mid s_{k,N-1} w_{N-1})^t,
\]

where \( \mid \) means that we concatenate the sub-vectors \( s_{ij} v_j \) and \( t \) denotes the transposition. The conditions for the vectors \( v_k \) to for a collections of lattice lines are given by the following equations

\[
\langle v_k, v_l \rangle = 4^i s^2 \quad \langle v_k, v_l \rangle = 2^i s
\]

for \( 0 \leq i < j \leq N - 1 \). It may appear that there are \( 2^i + 2^i (2^i - 1)/2 \) equations. But that is not so. Many of the equations are equivalent to others. This is because the columns of the Sylvester Hadamard matrix form a group under point-wise multiplication. So there are only \( 2^i \) equations, rather than the much larger number that appears above. More precisely, if \( \oplus \) denotes the group operation, then we have

\[
\langle v_k, v_l \rangle = \langle v_0, v_{k \oplus l} \rangle.
\]

So, in fact we see the only equations we have to consider are of the form

\[
\langle v_0, v_0 \rangle = \sum_{j=0}^{N-1} b_j = 4^i s^2, \quad \langle v_0, v_k \rangle = \sum_{j=0}^{N-1} s_{k,j} b_j = \pm 2^i s,
\]

for \( 1 \leq k \leq N - 1 \). In matrix notation, this means

\[
H \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{pmatrix} = \begin{pmatrix} 4^i s^2 \\ \pm 2^i s \\ \vdots \\ \pm 2^i s \end{pmatrix}.
\]

By multiplying both sides by \( H^t/2^i \) we obtain the solution

\[
\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{pmatrix} = H^t \begin{pmatrix} 2^i s^2 \\ \pm s \\ \vdots \\ \pm s \end{pmatrix}.
\]

\[ ^1 \text{This group is isomorphic to } \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \text{ (the } i \text{-fold direct product of the group } \mathbb{Z}_2). \text{ The multiplication of the } k \text{th and } l \text{th column gives } m \text{th column, where } m \text{ is obtained by XORING the binary numbers corresponding to } k \text{ and } l \text{ and converting the resulting binary number to decimal notation. We denote this operation by } m = k \oplus l. \]
and it is clear that the resulting lengths \( b_0, b_1, \ldots, b_{N-1} \) are integers. This completes the proof that there are at least \( 2^i \) lattice lines in dimension \( d = 4s^2 \).

Next, we show that such a collection of Sylvester lattice lines cannot be extended. Let \( v_0, \ldots, v_{N-1} \) be a collection of \( N = 2^i \) such lines in dimension \( d = 4s^2 \). Assume there is vector \( w \) that is mutually unbiased to the vectors \( v_0, \ldots, v_{N-1} \). We may assume without loss of generality that in the \( b_j \) block of \( w \) consists of a sub-block of \(+1\)'s of length \( b'_j \) and a sub-block of \(-1\)'s, where \( b_j = b'_j + b''_j \). This follows from the fact that permuting the entries of the vectors within the \( j \)th block according to the same permutation does not change the vectors \( v_0, \ldots, v_{N-1} \) and does not change the inner product between \( w \) and \( v_0, \ldots, v_{N-1} \). This leads to the following equation system:

\[
(I_2 \otimes H_2) \begin{pmatrix}
  b'_0 + b''_0 \\
  \vdots \\
  b'_{N-1} + b''_{N-1} \\
  b'_0 - b''_0 \\
  \vdots \\
  b'_{N-1} - b''_{N-1}
\end{pmatrix} = (H_{2^i} \otimes (1, 1)) \begin{pmatrix}
  b'_0 \\
  \vdots \\
  b'_{N-1} \\
  b''_0 \\
  \vdots \\
  b''_{N-1}
\end{pmatrix}
\]

\[
= H_{2^{i+1}} (b'_0, b''_0, \ldots, b'_{N-1}, b''_{N-1})^t
\]

\[
= P (4s^2, \pm 2s, \ldots, \pm 2s)^t,
\]

where \( P \) is a permutation matrix. Multiplying by \( H_{2^{i+1}/2^{i+1}} \) on both sides we obtain

\[
b'_0 = \pm 2^{i-1}s^2 \pm \frac{s(\pm 1 \pm 1 \ldots \pm 1)}{2} = 2^{i-1}s^2 \pm \frac{sk}{2}
\]

for some odd number \( k \) corresponding to the sum in the parenthesis. But since \( s \) is also odd the resulting number \( b'_0 \) is not an integer. This is a contradiction completing the proof that the collection of \( 2^i \) Sylvester lattice lines cannot be extended.

Unfortunately, it turns out that for \( d = 4s^2 \), our greedy construction gave \( 2^i \) unbiased lattice lines, but we can do a lot better for \( i > 1 \) as shown below.

Specifically, in the (generalized) construction of Lemma 5.3 given above, while the partitioning and choices of the first three vectors (needed for \( i = 1 \)) are without loss of generality, the choice of the \( 4 \)th and further vectors (needed for \( i \geq 2 \)) is specific and loses generality. Therefore, non-extendibility does not imply optimality, except for \( i = 1 \).

**Proposition 5.1.** We have \( L_{\mathbb{R}^d} \geq d - \sqrt{d} \) provided that there is a Hadamard matrix of order \( d - \sqrt{d} \).

**Proof.** Assume that there is a Hadamard matrix \( H \) of order \( d - \sqrt{d} \). We obtain \( d - \sqrt{d} \) lattice lines in dimension \( d \) by appending to the row vectors of \( H \) the all-one-vector of length \( \sqrt{d} \).

In other words, if the \( h_k \) is the \( k \)th row vector of \( H \), then the \( k \) lattice vector has the form

\[
\vec{v}_k = (h_{k,1}, h_{k,2}, \ldots, h_{k,(d-\sqrt{d})}, 1, 1, \ldots, 1),
\]

It is obvious that these vectors are lattice lines. \( \square \)

The above proposition along with many examples gives counter-examples to the greedy conjecture. For instance consider \( d = 4^2 \). If any maximal set were optimal, then \( L_{\mathbb{R}^d} \leq 2^2 \). But since there is a Hadamard of dimension 12 \( (d - \sqrt{d}) \), we see that \( L_{\mathbb{R}^d} \geq 12 \).

So far, we have shown that greedy methods for constructing mutually unbiased lattice lines do not work. Next, we consider the efficacy of greedy methods of constructing general (complex) MUBs.
Specifically, we give a negative answer to the question: is every maximal, or non-extendible set of MUBs necessarily optimal?

We consider the general, complex version of the MUBs constructed in Proposition 3.4 [17] based on Latin squares, we call these the Latin MUBs: note that this construction yields at most \( \sqrt{d} + 1 \) MUBs for dimensions of the form \( d = p^2 \), where \( p \) is a prime and \( e \) is an integer, whereas in prime power dimensions it is known that there are in fact \( d + 1 \) MUBs [2]. Nets and the Latin MUB construction are reviewed in Appendix A. We obtain restrictions on possible extensions to latin MUBs in general and show that the set of latin MUBs in \( d = 4 \) is maximal, i.e., not extendible, thus answering the above question in the negative.

Observation 5.2. The \( s(s+1) \) incidence vectors of a \((s+1,s)\)-net span \( \mathbb{R}^{s^2} \).

Proof. For any \( p \in \{1, \ldots, s^2\} \), there is exactly one incidence vector in each parallel class that has 1 as entry at position \( p \). Sum all these vectors. This gives the vector \( v_p = (1, \ldots, 1, (s+1), 1, \ldots, 1) \), where the value \( (s+1) \) appears at position \( p \). Now the \( s^2 \) vectors \( v_p \) span \( \mathbb{R}^{s^2} \) because the \( p \)th standard basis vector \( e_p \) is simply: \( (1/(s+1))^2((s^2+s-1)v_p - \sum_{q \neq p} v_q) \).

Lemma 5.3. In dimension \( d = s^2 \), any new basis unbiased to a \((s+1)\) collection of Latin MUBs corresponds to a Hadamard matrix. (Note the standard basis is not unbiased to any Latin MUB).

Proof. Assume there is a \((s+1,s)\)-net and let us consider the \((s+1)\) Latin MUBs that are constructed based on this net. We show that any orthonormal basis of \( \mathbb{C}^d \) that is mutually unbiased to the Latin MUBs defines a generalized Hadamard matrix of order \( d \), that is, the absolute values of all entries is \( 1/\sqrt{d} \).

Let us consider the \( i \)th incidence vector \( m_{bi} \) of the block \( b \). This incidence vector gives rise to \( s \) basis vectors of the \( b \)th Latin MUB. Denote the support of \( m_{bi} \) by \( S_i \). By construction the \( s \) basis vectors contain a Hadamard matrix \( H \) of order \( s \) as a sub-matrix if we look at the entries at positions contained in \( S_i \).

Let \( y = (y_1, \ldots, y_d) \) be any vector of the new ONB and denote by \( \tilde{y}_i = (y_{i_1}, \ldots, y_{i_s}) \) the sub-vector of length \( s \) of \( y \) defined by the support \( S_i = \{i_1, \ldots, i_s\} \subset \{1, \ldots, d\} \). Since \( y \) is mutually unbiased to the \( s \) basis vectors obtained from \( m_{bi} \), we have

\[
\frac{1}{\sqrt{s}} H \tilde{y}_i = \tilde{u}
\]

where \( u = (u_1, \ldots, u_s) \) is some vector in \( \mathbb{C}^s \) with \( |u_j|^2 = 1/d \) for \( j = 1, \ldots, s \). Since the matrix \( \frac{1}{\sqrt{s}} H \) is unitary, it follows that \( ||\tilde{y}_i||^2 = ||u||^2 \). We have \( ||u||^2 = s/d = 1/s \) and consequently

\[
\sum_{j \in S_i} |y_j|^2 = \langle m_{bi}, (|y_1|^2, \ldots, |y_d|^2) \rangle = \frac{1}{s}.
\]

The above equation is true for all incidence vectors \( m_{bi} \) and gives a system of linear equations for the squares of the absolute values \( |y_j|^2 \).

Since the support of each incidence vector has cardinality \( s \), one solution to the above system is by choosing all entries of \( y \) to have the same magnitude, that is, \( |y_j| = 1/\sqrt{s} \). Due to Observation 5.2 the system has full rank and thus this solution is always a unique solution. This complete the proof that the entries of all basis vectors of the new ONB have the same magnitude and consequently that corresponds to a generalized Hadamard matrix.

Proposition 5.4. The \( 3 \) standard Latin MUBs in dimension \( d = 4 \) are unextendible.
Proof. Assume there is a vector $y$ that is mutually unbiased to all vectors of the three Latin MUBs. It follows from Lemma 5.3 that all its entries have the same modulus. Without loss of generality we may assume that $y = (1, a, b, c)/2$ after factoring out a common phase factor.

Writing the equalities $|1 + a|^2 = |1 + b|^2 = |a + b|^2 = 2$ that follow from eq. 3 it is easily seen that there is no such $a$ and $b$ satisfying these conditions. This shows that we cannot even find a single vector that is unbiased to the Latin MUBs for $d = 4$. This completes the proof that the three Latin MUBs cannot be extended. 

In dimension $d = 4$ if one allows complex MUBs, 5 MUBs may be found. However, if we start with the three real Latin MUBs, we cannot even find a fourth MUB, complex or real, unbiased to the previous three. Hence it is clear that a set of MUBs which cannot be extended is not necessarily optimal.

6 Conclusion

In this paper we have found upper and lower bounds on the number of real MUBs in all dimensions. In most dimensions, the problem is solved. Assuming the Hadamard conjecture is true, the only interesting dimensions left are $d = 4s^2$, where $s$ is odd. In the case of $d = 4s^2$, assuming the Hadamard conjecture is true, the only question is: does $M_{Rd} = 3$ hold always, or only for certain values of $s$. Additionally, what is $M_{Rd}$ for $d = 4s^2$ with $i > 1$ and $s$ odd? Assuming the Hadamard conjecture is true, this paper shows a lower bound valid for all $i > 1$. It would be interesting to find an example that exceeds that lower bound.

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A Nets and Latin MUBs

We quote the definition of a net, which is used in the Latin MUB construction[17]. Note that nets are equivalent to orthogonal arrays and traversal designs[16].

**Definition A.1 (Net).**

Let \( \{ \mathbf{m}_{11}, \ldots, \mathbf{m}_{1s}, \mathbf{m}_{21}, \ldots, \mathbf{m}_{2s}, \ldots, \mathbf{m}_{k1}, \ldots, \mathbf{m}_{ks} \} \) be a collection of \( k s \) incidence vectors of size \( d = s^2 \) that are partitioned into \( k \) blocks where each block contains \( s \) incidence vectors. The incidence vectors are denoted by \( \mathbf{m}_{bi} \), where \( b = 1, \ldots, k \) identifies the block and \( i = 1, \ldots, s \) the vector within a block. If the incidence vectors satisfy the following conditions we say that they form a \( (k, s) \)-net.

1. The supports of all vectors within one block are disjoint, i.e.,
   \[
   \mathbf{m}_{bi}^T \mathbf{m}_{bj} = 0 \tag{4}
   \]
   for all \( 1 \leq b \leq k \) and all \( 1 \leq i \neq j \leq s \).

2. The intersection of any incidence vectors from two different blocks contains exactly one element, i.e.,
   \[
   \mathbf{m}_{bi}^T \mathbf{m}_{cj} = 1 \tag{5}
   \]
   for all \( 1 \leq b \neq c \leq s \) and all \( 1 \leq i, j \leq s \).
Note that our definition of \((k, s)\)-nets is in accordance with the usual definition of nets in design theory \cite[page 172]{8}.

Let \(m \in \{0,1\}^d\) be an incidence vector of Hamming weight \(s\) and \(h \in \mathbb{C}^s\) an arbitrary column vector. Then we define the embedding of \(h\) into \(\mathbb{C}^d\) controlled by \(m\), denoted by \(h \uparrow m\), to be the following vector in \(\mathbb{C}^d\)

\[
h \uparrow m := \sum_{r=1}^{s} h[r] |j_r\rangle,
\]

where \(h[r]\) is the \(r\)th entry of the vector \(h\), \(\{j_1, j_2, \ldots, j_s\}\) the support of \(m\) with the ordering \(j_1 < j_2 < \ldots < j_s\) and \(|j_r\rangle\) the \(j_r\)th standard basis vector of \(\mathbb{C}^d\). A less formal way to define this vector is: the first non-zero entry of \(m\) is replaced by the first entry of \(h\), the second non-zero entry of \(m\) by the second entry of \(h\), etc.

This operation is best illustrated by a simple example:

\[
m := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \{0,1\}^9, \quad h := \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \in \mathbb{C}^3, \quad h \uparrow m := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^9
\]

**Theorem A.2 (Construction of MUBs).**

Let \(\{m_{11}, \ldots, m_{1s}, m_{21}, \ldots, m_{2s}, \ldots, m_{k1}, \ldots, m_{ks}\}\) be a \((k, s)\)-net and \(H\) an arbitrary generalized Hadamard matrix of size \(s\). Then the \(k\) sets for \(b = 1, \ldots, k\)

\[
B_b := \left\{ \frac{1}{\sqrt{s}} (h_l \uparrow m_{bi}) \mid l = 1, \ldots, s, \ i = 1, \ldots, s \right\}
\]

are \(k\) mutually orthogonal bases for the Hilbert space \(\mathbb{C}^d\).