Dependence of river network scaling on initial conditions

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Abstract

We investigate the dependence of river network scaling on the relative dominance of slope vs. noise in initial conditions, using an erosion model. Increasing slope causes network patterns to transition from dendritic to parallel and results in a breakdown of simple power-law scaling. This provides an example of how natural deviations from scaling might originate. Similar network patterns in leaves suggest such deviations may be widespread. Simple power-law scaling in river networks may require a combination of dynamics, initial conditions, and perturbations over time.

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Branching network patterns are ubiquitous in nature, from river networks to lightning, from plant vascular patterns to animal circulatory systems. It is generally accepted that these networks exhibit power-law scaling. While it has been suggested that the scaling is topologically inevitable [1], topology may have limited relevance for networks embedded in physical space [2], and in any case allows a range scaling exponents [1].

The origin of specific exponent values remains a topic of active research. In river networks, three major approaches have been taken: growth models that create static networks through a growth mechanism [3], optimal models that create equilibrium networks through an optimization process [4], and dynamic models based on local erosion rules iterated over time [5, 6, 7]. We are interested in how the scaling of dynamic models is affected by anisotropy in initial conditions. In a different type of network, anisotropic diffusion-limited aggregation, scaling exponents depend on the strength of anisotropy [8]. In river networks, landscape slope is known to affect network appearance [9, 10, 11], but possible effects on scaling have been almost entirely ignored [12].

The study of scaling laws in river networks goes back to Horton’s work in the 1940s [13]. Although a number of scaling laws have been discovered, only two scaling exponents are independent for most networks [2, 14]: the Hack exponent \( h \) and the sinuosity exponent \( d \).

The Hack exponent is defined by \( \langle l \rangle \sim A^h \) and the sinuosity exponent by \( \langle l \rangle \sim L_{\parallel}^d \), where \( \langle l \rangle \) is average main stream length, \( A \) is drainage area (land area draining to a point), and \( L_{\parallel} \) is longitudinal (Euclidean) main stream length. The natural range for \( h \) is 0.5–0.7, with 0.57 a commonly cited average; \( d \) ranges from 1.0–1.2, with 1.1 a common average [2, 15, 16, 17]. The basin shape exponent \( D = d/h \) is also useful in network studies [14]. Since \( A \sim L_{\parallel}^D \), \( D \) indicates whether the scaling of basin shape is self-similar (\( D = 2 \)) or self-affine (\( D \neq 2 \)) [16].

In this Letter, we investigate the dependence of river network scaling on initial conditions (ICs), using a dynamic model. We consider the effect of the relative dominance of slope vs. noise in the ICs, or equivalently, the degree to which ICs are anisotropic vs. isotropic. Anisotropy of ICs is quantified, and related to initial scaling. We find that anisotropy produces deviations from simple power-law scaling in steady-state networks, because the sensitivity of network structure to anisotropy is scale-dependent. This provides a simple example of how scaling deviations in natural rivers may originate. As ICs become more anisotropic, network patterns transition from dendritic to parallel. Similar patterns in leaf
FIG. 1: Four steady-state river networks produced by simulation of Eq. 1. Initial slope-to-noise ratios $\lambda$, from (a)–(d), are 0, 1, 2, and 3. Noise dominated the initial condition of (a). Noise and slope were initially balanced in (b). Initial slope dominated (c), and by (d), slope was so dominant that the final drainage pattern is unnaturally regular. The drainage pattern of (a) is dendritic, while (b)–(d) have parallel patterns [9, 10, 11]. Only rivers with drainage area $A \geq 10$ nodes are shown; line widths are proportional to $\sqrt{A}$.

Veins suggest that deviations from scaling exist in other networks. Since model dynamics do not guarantee simple power-law scaling, power laws in river networks may require a combination of dynamics, ICs, and perturbations that smooth scaling deviations.

We simulated an erosion law on a square lattice. All nodes of the lattice receive uniform precipitation. Water from each node flows to the neighboring node with the steepest downslope gradient; diagonal flow is allowed. Since it is possible to have nodes with no lower neighbors (pits), lakes can form and a lake algorithm is needed to route pit flow. Our lake algorithm fills depressions with water and finds the lowest outlet, where the lake overflows. The algorithm also prevents erosion of nodes below the lake surface. The system evolves from a random initial surface to a static steady-state network.
The simulated erosion law is of the form

$$\frac{dz}{dt} = -KA^m |\nabla z|^n + U$$

where $z$ is elevation, $K$ is an erosivity coefficient (related to rock hardness, rainfall rate, etc.), $m$ and $n$ are constant exponents, and $U$ is tectonic uplift rate. This erosion law may be derived from the assumption that erosion is proportional to a power of shear stress or of unit stream power, combined with hydrodynamic principles [5]. The same form has also been suggested as a modified version of the Kardar-Parisi-Zhang (KPZ) equation [7].

We used the following typical literature values for the constants in Eq. (1): $K = 10^{-5}$ yr$^{-1}$; $U = 10^{-3}$ m/yr; and $m = 1/2$ and $n = 1$ (consistent with the empirical observation that often $m/n \approx 1/2$) [5].

Simulations were performed on lattices with a 2:1 width:height aspect ratio. Left-right boundary conditions were periodic; top-bottom were Dirichlet $z = 0$. Boundary conditions were chosen to minimize edge effects; basin boundaries form spontaneously.

Initial conditions consist of two sloping surfaces that meet in the center. Random noise in elevation from a uniform distribution is added to these sloping surfaces. To specify the relative dominance of slope vs. noise (anisotropy vs. isotropy) in the initial conditions, we introduce the initial slope-to-noise ratio $\lambda = |s|\Delta x/N$, where $s$ is the slope of the initial condition in the absence of noise, $\Delta x$ is the lattice spacing, and $N$ is the maximum magnitude of the noise in elevation (1 m for our simulations). $\lambda$ is the ratio of elevation change over one node from slope to the maximum possible change from noise. $\lambda = 0$ indicates a flat surface with noise, or alternatively the unphysical limit where $N \to \infty$ for a finite slope. While the initial network for $\lambda = 0$ will depend on how flow through lakes is defined, since no direction is favored the overall scaling should be similar to that of a random spanning tree, which has $(h, d) = (5/8, 5/4)$ [2]. For $\lambda \geq 1$ the slope is strong enough to prevent lakes, and initial scaling will be that of a Scheidegger network with $(h, d) = (2/3, 1)$, until $\lambda$ is large enough to produce non-convergent flow with the trivial exponents $(h, d) = (1, 1)$ [2]. For $0 < \lambda < 1$, some lakes will be present, and initial scaling will exist in a crossover between $\lambda = 0$ scaling and the Scheidegger network.

We will now show that the combination of a square lattice and sloped initial conditions limits the meaningful range of $\lambda$ to $0 \leq \lambda \leq \sqrt{2}/(\sqrt{2} - 1) \approx 3.4$. Assume that the initial slope $s$, in the absence of noise, is parallel to the square lattice. Then the slope from a
FIG. 2: (Color online) The local Hack exponent $h(A)$ as a function of drainage area $A$ for four initial slope-to-noise ratios $\lambda$. Simple power-law scaling is observed only for $\lambda = 0$. For this case, Hack’s exponent $h$ was estimated by taking the mean and standard deviation of $h(A)$ over the region designated in the figure.

A node to a diagonal neighbor will be $s/\sqrt{2}$. If flow is ever to be diagonal, then $|s|/\sqrt{2} > |s| - N/\Delta x$. That is, if noise is maximized ($N$) in the adjacent direction and minimized (0) in the diagonal direction, it must be sufficient to cause diagonal flow. Rearranging and substituting $\lambda = |s|\Delta x/N$ yields $\lambda < \sqrt{2}/(\sqrt{2} - 1) \approx 3.4$ as the condition for diagonal flow. If $\lambda > 3.4$, and the system does not experience perturbations, flow will be non-convergent for the entire system evolution. Somewhat lower values of $\lambda$ are also problematic, because the noise is small enough that flow is difficult to divert, resulting in drainage patterns that are unnaturally regular (Fig. 1).

Fifty simulations were run for each integer and half-integer value of $\lambda \in [0, 3]$, with lattice dimensions 800x400. All simulations were run to steady state, defined as no change from one timestep to the next within the limits of numeric precision.

The simulated networks show a transition from dendritic to parallel drainage patterns with increasing $\lambda$. The $\lambda = 0$ networks are dendritic, with a tree-like pattern composed of shorter, more branching streams (Fig. 1a). The patterns become more parallel with increasing $\lambda$, with longer, straighter streams that more rarely intersect (Fig. 1b–d). In nature, the key distinction between dendritic and parallel networks is the extent to which flow is controlled by gradient [9, 10, 11], consistent with our model.
The simulations with $\lambda > 0$ do not exhibit simple power-law scaling; log-log plots of $\langle l \rangle$ vs. $A$ and $\langle l \rangle$ vs. $L_{\parallel}$ do not exhibit linear behavior. To quantify the scaling, we introduce the local exponents

$$h(A) = \frac{d \log \langle l(A) \rangle}{d \log A} \quad \text{and} \quad d(L_{\parallel}) = \frac{d \log \langle l(L_{\parallel}) \rangle}{d \log L_{\parallel}}.$$  

The local Hack exponent $h(A)$ has been used previously to quantify deviations from scaling [18]; local exponents have also been used in other cases where fractal properties are scale-dependent [19, 20]. For each value of $\lambda$, local exponents were measured by calculating the averages $\langle l(A) \rangle$ and $\langle l(L_{\parallel}) \rangle$ over all runs, then performing linear regression in log-log space over a small moving window (0.15 orders of magnitude for $d(A)$ and 0.2 for $h(A)$). For the figures, local exponents were smoothed by taking running averages over 0.05 orders of magnitude.

The local Hack exponent $h(A)$ shows simple power-law scaling for $\lambda = 0$ (Fig. 2). The region with $h \approx \text{const}$ gives $h = 0.53 \pm 0.02$, where the error is the standard deviation of $h(A)$ within the region and the values are calculated from unsmoothed data. There are deviations in $h(A)$ below $A \approx 10$ nodes due to grid effects [21]; deviations due to finite size effects are observed at large $A$. For $\lambda > 0$, no significant regions of simple scaling are observed. After increasing for small $A \gtrsim 10$, $h(A)$ decreases for intermediate $A$, increases again for large $A$, and finally decreases for the largest $A$ due to finite size effects. The magnitude of variation in $h(A)$ increases with increasing $\lambda$.

The behavior of $d(L_{\parallel})$ is similar (Fig. 3). For $\lambda = 0$, there is a region of approximate simple scaling that gives $d = 1.07 \pm 0.02$. For larger $\lambda$, no significant regions of simple scaling are observed at intermediate scales.

The breakdown of simple scaling for $\lambda > 0$ may be traced to a scale-dependent effect of initial slope on network structure. The largest rivers form on the initial slope and evolve under its full effect. Their tributaries also initially flow down this slope, but as the major rivers carve out valleys, their tributaries redirect to flow down the valley sides. This rearrangement introduces additional noise into smaller-scale networks. Thus, the effect of initial anisotropy decays with decreasing scale, so that networks become more irregular and more dendritic as scale decreases. The resulting deviations from scaling are exacerbated by the structure of parallel networks. Between major junctions, length and area will increase roughly linearly [18]. Since streams intersect more rarely in networks that are more parallel,
such networks will exhibit larger fluctuations in their Hack distributions.

Deviations from simple power-law scaling for Hack’s law have been observed in continent-scale river networks [18]. There are deviations at small scales due to long, thin basins, and at large scales due to statistical fluctuations and boundaries. At intermediate scales, where simple scaling would be anticipated, $h(A)$ exhibits gradual drift rather than fluctuating about an average value. These deviations cannot be attributed solely to ICs, since continent-scale networks have complex histories and are shaped by numerous processes, but the conditions under which such networks form are surely responsible for some deviations. More importantly, our results provide an example of how such deviations in river network scaling may be produced.

The Hack and sinuosity exponents have not been measured for different drainage patterns as such, but recently Mejía and Niemann [12] have shown that the scaling of basin shape is self-similar for dendritic networks and self-affine for parallel networks. For our simulations, the basin shape exponent $D \approx 2$ (self-similar) for the dendritic network ($\lambda = 0$). This suggests that while $h$ and $d$ evolve from the initial values, self-similarity is inherited from the ICs. For larger $\lambda$ (more parallel), a local shape exponent $D(A)$ shows similar behavior to $1/h(A)$. Self-affinity is not inherited from the ICs since simple scaling breaks down, but a narrow aspect ratio is inherited. Since the scaling of parallel networks reported by Mejía...
and Niemann shows larger fluctuations than that of dendritic networks, it is possible that they actually measured an average over local scaling rather than simple scaling. For our simulations, an averaged $D(A)$ would be less than 2 and thus self-affine.

It is also possible that at least some natural parallel networks do exhibit simple scaling. Such networks might result from ICs with $\lambda$ very close to zero, but they might also indicate shortcomings in our model. It has been shown that dynamic models give cleaner power laws when there are perturbations from variable boundary conditions. Steady-state solution methods for Eq. 1 that involve perturbing the network also yield cleaner power laws [22]. Our results show that Eq. 1 does not guarantee even approximate power laws. Perhaps simple scaling in river networks results from a combination of ICs that set drainage patterns, dynamics that govern network evolution, and perturbations that smooth deviations from scaling.

We have shown that as initial slope becomes more dominant, drainage patterns transition from dendritic to parallel, and simple power-law scaling is replaced by scale-dependent exponents. Initial conditions deserve greater consideration in river network models, especially given the range of initial conditions in use [6, 7, 23, 24, 25]. While our results challenge attempts to explain river network scaling in terms of a single principle or universality class, they also open a number of avenues for future research, such as the effect of other initial geometries and other sources of anisotropy and noise. In particular, perturbations over time may be important in producing simple scaling. Our results may be relevant to other branching patterns as well. For example, the vein structure of some leaves [26] resembles our parallel river networks, with a more ordered structure at large scales transitioning to a more disordered structure at small scales, hinting that other networks may exhibit complex scaling. More broadly, Chen and Bak have suggested that length-scale-dependent scaling “may represent a quite general geometrical form for nonequilibrium dissipative systems” [19].

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