A new parabolic flow in Kähler manifolds

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0.1 Introduction and main theorem

This is a continuation of my earlier work [9], where we study the lower bound of the Mabuchi energy on Kähler manifolds when the first Chern class is negative. The problem of finding a lower bound for the Mabuchi energy is very important in Kähler geometry since the existence of a lower bound is the pre-condition for the existence of constant scalar curvature metric in a Kähler class (cf. [2] and [8]). The Mabuchi energy is first defined by Mabuchi through its derivative. The first explicit formula was given in [23]. In [9], we re-group the right hand side of this formula of Tian in a slightly different way. According to this decomposition formula (cf. [9]), the problem is reduced to the solution of the existence problem for critical metrics of a new functional $J_{\chi}$ (cf. [9] [12]). For convenience, we include the definition of the functional $J_{\chi}$ below. The existence problem is completely solved in any Kähler surface. However, it is still widely open in higher dimensional Kähler manifolds. In this paper, we try to understand the general existence problem via the flow method. Let $(V^n, \omega_0)$ be an $n$ dimensional Kähler manifold and $\omega_0$ be a fixed Kähler form in $V$. Consider the space of Kähler potentials

$$\mathcal{H} = \{ \varphi \in C^\infty(V) \mid \omega_\varphi = \omega_0 + i\partial \bar{\partial} \varphi > 0 \text{ on } V \}.$$ 

For any positive (1,1) form $\chi$, consider the parabolic equation

$$\frac{\partial \varphi}{\partial t} = \frac{n [\chi] \cdot [\omega_0]^{n-1}}{[\omega_0]^n} - \frac{\chi \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n}. \quad (0.1)$$

This is the gradient flow for the functional $J_{\chi}$ which will be defined in Section 0.2. Similar to the case of the Calabi flow [8], the main result of this paper is

Theorem 0.1. The following statements are true:

1. This gradient flow of $J_{\chi}$ always exists for all the time for any smooth initial data. Moreover, the length of any smooth curve in $\mathcal{H}$ and the distance between any two metrics decreases under this flow.\footnote{The definition of the length of a smooth curve in $\mathcal{H}$ is given in equation (1.1). On the other hand, the distance function in $\mathcal{H}$ is given by Definition 1.3.}
2. If the bisectional curvature of $\chi$ is semi-positive, then the gradient flow exists for all time and converges to a smooth critical metric.

In this paper, we will call the gradient flow of the functional $J_\chi$ a $J$–flow.

0.2 Setup of notations

Let $g_0 = \sum_{\alpha,\beta=1}^{n} g_{\alpha,\beta} d\alpha \bar{d}\beta$ be the Kähler metric corresponding to the Kähler form $\omega_0$ and $g = \sum_{\alpha,\beta=1}^{n} g_{\alpha,\beta} d\alpha \bar{d}\beta$ the Kähler metric corresponds to the Kähler form $\omega_\varphi$ for some $\varphi \in \mathcal{H}$. For any fixed positive (1,1) form $\chi$, one introduces a new functional $J_\chi$ with respect to this form $\chi$.

**Definition 0.2.** Suppose $\chi$ is a positive closed (1,1) form in $V$. For any $\varphi(t) \in \mathcal{H}$, the functional $J_\chi$ is defined through its derivative:

$$\frac{dJ_\chi}{dt} = \int_V \frac{\partial \varphi}{\partial t} \chi \wedge \omega_\varphi^{n-1}/(n-1)!.$$ 

It is straightforward to show that this is well defined.

**Remark 0.3.** This version of definition is given by [12]. In [9], we give an independent definition for the functional $J_\chi$ where we assume $\chi$ is a Ricci form (not necessary positive).

In [12], Donaldson pointed out the significance of this functional in its own right: a) $J_\chi$ is a convex functional in the space of Kähler potentials (See proposition 3.1 below); b) $J_\chi$ is a moment map from the space of Kähler potentials to the dual space of the Lie algebra of some symplectic automorphism group.

For any $\varphi \in \mathcal{H}$, there exists a unique Kähler form $\omega_\varphi$. Conversely, for any Kähler form $\omega'$ in the cohomology class of $\omega_0$, there exists a unique Kähler potential $\varphi \in \mathcal{H}$ up to addition of some constants such that

$$\omega' = \omega_\varphi.$$

In order to remove the ambiguity of the correspondence between Kähler forms and Kähler potential, we introduce a well-known functional here.

**Definition 0.4.** For any smooth curve $\varphi(t) \in \mathcal{H}$, we define the functional $I$ such as

$$\frac{d I(\varphi(t))}{dt} = \int_V \frac{\partial \varphi}{\partial t} \omega_\varphi^n/n!.$$

Clearly, $I$ is well defined.
Obviously, $I$ defines a function from $\mathcal{H}$ to the real line. Consider the 0-level surface $\mathcal{H}_0$ of the functional $I$ in $\mathcal{H}$. This level surface can be defined as

$$\mathcal{H}_0 = \{ \phi \in \mathcal{H} \mid I(\phi) = 0 \}.$$

Set the variational space of $J_\chi$ as $\mathcal{H}_0$. In local coordinates, suppose that $\chi$ can be expressed as the following:

$$\chi = \sum_{\alpha, \beta=1}^{n} \chi_{\alpha\beta} \, dz_\alpha \, d\bar{z}_\beta.$$

Then the Euler-Lagrange equation for $J_\chi$ is:

$$\sum_{\alpha, \beta=1}^{n} g^{\alpha\bar{\beta}} \chi_{\alpha\bar{\beta}} = \text{tr}_g \chi = c \tag{0.2}$$

where

$$g_{\alpha\bar{\beta}} = g_{0\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}, \tag{0.3}$$

Here $g = \sum_{\alpha, \beta=1}^{n} g_{\alpha\bar{\beta}} \, dz_\alpha \, d\bar{z}_\beta$ is the critical metric of $J_\chi$ in $\mathcal{H}_0$ and $c$ is a constant which depends only on the Kähler class of $[\chi]$ and $[\omega_0]$, i.e.,

$$c = \frac{\int_V \chi \wedge \frac{1}{(n-1)!} \omega^{(n-1)}}{\int_V \frac{1}{n!} \omega^n} = \frac{n [\chi] \cdot [\omega_0]^{n-1}}{[\omega_0]^n}. \tag{0.4}$$

From equation (0.2), it is easy to see that a necessary condition for a solution to exist is (also see [12])

$$c \cdot \omega_g - \chi > 0,$$

where $\omega_g$ is the Kähler form associated to $g$. In other words, there exists at least one Kähler form $\omega$ in the Kähler class of $[\omega_0]$ such that the following holds:

$$c \cdot \omega - \chi > 0. \tag{0.5}$$

**Conjecture 0.5.** (Donaldson [12]) *If the aforementioned necessary condition is satisfied, then there exists a critical point for $J_\chi$ in that Kähler class.*

**Historic remarks:** Using the heat flow method to study the nonlinear PDE is a well known method in differential geometry. In recent years, it has been the source of active researches since the famous work of J. Eells and J. Sampson [13]. Interested readers are refereed to important works by R. Hamilton [16].
G. Huisken \[18\] and a survey paper by H. D. Cao and B. Chow \[7\] and the references therein.

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1 **Summary of recent developments in the Riemannian metric in space of Kähler potentials**

Mabuchi (\[19\]) in 1987 defined a Riemannian metric on the space of Kähler metrics, under which it becomes (formally) a non-positive curved infinite dimensional symmetric space. Apparently unaware of Mabuchi’s work, Semmes \[21\] and Donaldson \[11\] re-discovered this same metric from different angles. For any vector $\psi$ in the tangential space $T_\phi \mathcal{H}$, we define the length of this vector as

$$\|\psi\|_\phi^2 = \int_V \psi^2 \omega^n_\phi.$$  

For any smooth curve $\varphi(t) : [0, 1] \to \mathcal{H}$, the length of this curve is given by

$$\int_0^1 \left( \int_V \left( \frac{\partial \varphi(t)}{\partial t} \right)^2 \omega^n_\varphi \right)^{\frac{1}{2}} dt.$$  

(1.1)

The geodesic equation is

$$\frac{\partial^2 \varphi(t)}{\partial t^2} - \frac{1}{2} \nabla \frac{\partial \varphi(t)}{\partial t} \frac{\partial \varphi(t)}{\partial t} = 0,$$  

(1.2)

where the derivative and norm in the 2nd term of the left hand side are taken with respect to the metric $\omega_\varphi(t)$.

This geodesic equation shows us how to define a connection on the tangent bundle of $\mathcal{H}$. If $\phi(t)$ is any path in $\mathcal{H}$ and $\psi(t)$ is a field of tangent vectors along the path (that is, a function on $V \times [0, 1]$), we define the covariant derivative along the path to be

$$D_t \psi = \frac{\partial \psi}{\partial t} - \frac{1}{2} (\nabla \psi, \nabla \frac{\partial \phi(t)}{\partial t})_\phi.$$  

The main theorem formally proved in \[19\] (and later reproved in \[21\] and \[11\]) is:
Theorem A  The Riemannian manifold $\mathcal{H}$ is an infinite dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\phi \in \mathcal{H}$ the curvature is given by

$$R_{\phi}(\delta_1 \phi, \delta_2 \phi)\delta_3 \phi = -\frac{1}{4}\{\{\delta_1 \phi, \delta_2 \phi\}_\phi, \delta_3 \phi\}_\phi,$$

where $\{\ , \ \}_\phi$ is the Poisson bracket on $C^\infty(V)$ of the symplectic form $\omega_\phi$; and $\delta_1 \phi, \delta_2 \phi \in T^*_\phi \mathcal{H}$. Then the sectional curvature is non-positive, given by

$$K_{\phi}(\delta_1 \phi, \delta_2 \phi) = -\frac{1}{4}\|\{\delta_1 \phi, \delta_2 \phi\}_\phi\|^2_\phi.$$

We will skip the proof here. Interested readers are referred to paper of Mabuchi [19] or [21] and [11] for the proof.

In [11], Donaldson outlined a connection between this Riemannian metric in the infinite dimensional space $\mathcal{H}$ and the traditional Kähler geometry through a series important conjectures and theorems. In 1997, following his program, the author proved some of his conjectures:

Theorem B [8] The following statements are true:

1. The space of Kähler potentials $\mathcal{H}$ is convex by $C^{1,1}$ geodesic (cf. Definition 1.1 below). More specifically, if $\varphi_0, \varphi_1 \in \mathcal{H}$ and $\varphi(t)$ ($0 \leq t \leq 1$) is a geodesic connecting these two points in $\mathcal{H}$, then the mixed covariant derivative of $\varphi(t)$ is uniformly bounded from above.

2. $\mathcal{H}$ is a metric space. In other words, the infimum of the lengths of all possible curves between any two points in $\mathcal{H}$ is strictly positive.

Definition 1.1. For any $\epsilon > 0$, a smooth path $\varphi(t)$ in $\mathcal{H}$ is called $\epsilon$-approximate geodesic if the following holds:

$$\left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \nabla \frac{\partial \varphi}{\partial t} \right)^2 \omega^n_\varphi = \epsilon \cdot \omega^n_0. \quad (1.3)$$

Remark 1.2. Note that for $\epsilon \to 0$, the $\epsilon$-approximated geodesic converges to the unique $C^{1,1}$ geodesic between the two end points. Theorem B implies that the second mix derivatives of $\varphi(t)$ stay uniformly bounded as $\epsilon \to 0$.

Following Theorem B, we can define a distance function in $\mathcal{H}$.

Definition 1.3. For any two points $\varphi_1, \varphi_2 \in \mathcal{H}$, the distance $d(\varphi_1, \varphi_2)$ is defined to be the length of the unique $C^{1,1}$ geodesic which connects these two points.

In [4], E. Calabi and the author proved the following:

Theorem C [4] The following statements are true:
1. $\mathcal{H}$ is a non-positive curved space in the sense of Alexandrov.

2. The length of any curve in $\mathcal{H}$ is decreased under the Calabi flow unless it is represented by a holomorphic transformation. The distance in $\mathcal{H}$ is also decreasing if the Calabi flow exists for all the time for any initial smooth data.

2 General preparation

For convenience, from now on, we drop the dependence of $\chi$ in $J_\chi$. Then

**Proposition 2.1.** $J$ is a strictly convex functional on any $C^{1,1}$ geodesic. In particular, $J$ has at most one critical point in $\mathcal{H}_0$.

**Remark 2.2.** This proposition was pointed out to the author by Donaldson in spring 1999. The proof here is somewhat different from his original idea.

**Proof.** Suppose $\varphi(t)$ is a $C^{1,1}$ geodesic. In other words, $\varphi(t)$ is a weak limit of the following continuous equation as $\epsilon \to 0$ (with uniform bounds on the second mixed derivatives of Kähler potentials):

$$
\left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \right) \omega_{\varphi(t)}^n = \epsilon \cdot \omega_0^n.
$$

Denote $g(t)$ is the corresponding Kähler metric corresponds to the Kähler form $\omega_{\varphi(t)}$. Again, we drop the dependence of $t$ from $g(t)$ for convenience from now on. Recall the definition of $J$, we have

$$
\frac{dJ}{dt} = \int_V \frac{\partial \varphi}{\partial t} (g^{\alpha\overline{\beta}} \chi^\alpha_{\alpha\overline{\beta}}) \frac{\omega^n}{n!}.
$$

Then (denote $\sigma = g^{\alpha\overline{\beta}} \chi^\alpha_{\alpha\overline{\beta}}$ in the following calculation):

$$
\frac{d^2 J}{dt^2} = \int_V \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \right) \sigma_r g^{-1} \chi^\alpha_{\alpha\overline{\beta}} + \frac{\partial \varphi}{\partial t} \sigma \Delta_g \frac{\partial \varphi}{\partial t} \omega^n
$$

$$
= \int_V \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \right) \sigma_r g^{-1} \chi^\alpha_{\alpha\overline{\beta}} - \frac{\partial \varphi}{\partial t} \sigma \Delta_g \frac{\partial \varphi}{\partial t} \omega^n
$$

$$
= \int_V \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \right) \sigma r \nabla \varphi \nabla \varphi - \frac{\partial \varphi}{\partial t} \sigma \nabla \varphi \nabla \varphi - \frac{\partial \varphi}{\partial t} \sigma \nabla \varphi \nabla \varphi
$$

$$
\geq \int_V \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 \right) \sigma \frac{\omega^n}{n!} \geq 0.
$$

The last equality holds along any $C^{1,1}$ geodesic. $\square$
Definition 2.3. According to [12], the functional $J$ can be regarded as a moment map. Therefore, we can define its "norm" as a new energy functional $E$. In other words, we have,

$$E = \int_V (\text{tr}_g(\chi))^2 \frac{\omega^n}{n!}.$$ 

Proposition 2.4. $E$ has similar critical points as $J$. Moreover, $E$ is decreasing along the gradient flow of $J$.

Proof. Set $\sigma = \text{tr}_g \chi$. All of the derivatives, norm and integration are taken with respect to $g$ in the following calculation:

$$\delta_v E(g) = \int_V \left( 2\sigma (- \sum_{\alpha, \beta, r, \delta} g^{\alpha \beta} v_{,\sigma} \bar{g}^\delta \chi_{\alpha \beta} + \sigma^2 \Delta_g v) \right) \frac{\omega^n}{n!}$$

$$= \int_V \left( 2 \sum_{\alpha, \beta, r, \delta} (g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta} + g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta}) - 2\sigma \sigma v_{,r} \bar{g}^\delta \right) \frac{\omega^n}{n!}$$

$$= \int_V \left( 2 \sum_{\alpha, \beta, r, \delta} (g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta} + g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta}) - 2\sigma \sigma v_{,r} \bar{g}^\delta \right) \frac{\omega^n}{n!}$$

$$= 2 \int_V \left( \sum_{\alpha, \beta, r, \delta} g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta} \right) \frac{\omega^n}{n!}$$

$$= -2 \int_V \sum_{\alpha, \beta, r, \delta} (g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta}) \frac{\omega^n}{n!}.$$

The Euler-Lagrange equation for functional $E$ is:

$$\sum_{\alpha, \beta, r, \delta} (g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta}) \frac{\omega^n}{n!} = 0. \quad (2.1)$$

The left hand side is a divergence form and the equation holds on the manifold without boundary. Multiple $\sigma$ in both sides of the equation (2.1) and integrating over the entire manifold, we have

$$0 = \int_V \left( g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta} \frac{\omega^n}{n!} \right) \sigma \frac{\omega^n}{n!}$$

$$= - \int_V \sum_{\alpha, \beta, r, \delta} \sum_{\alpha, \beta, r, \delta} (g^{\alpha \beta} \sigma v_{,r} \bar{g}^\delta \chi_{\alpha \beta}) \frac{\omega^n}{n!}$$

Therefore, $\sigma = \text{const}$ in the manifold $V$ since $\chi$ is a strictly positive (1,1) form.
Now, along the $J$–flow, we have $\frac{\partial \phi}{\partial t} = c - tr_g \chi = c - \sigma$. Thus,

$$\frac{d E}{dt} = 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$= -2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$\leq 0. \quad (2.2)$$

The equality holds unless $\sigma$ is a constant. Thus $E$ is strictly decreasing under the $J$–flow. □

**Proposition 2.5.** Any critical point of $E$ is a local minimizer.

**Proof.** Let $u, v$ be the tangential vectors in $V$. Recall the first variation of $E$ as:

$$\delta_v E(g) = 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

Next we want to compute the second variation of $E$:

$$\delta_u \delta_v E(g)$$

$$= -2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$= 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$+ 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$= 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$+ 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!}$$

$$= 2 \int_V \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} \sigma g^\delta \chi^\alpha \varepsilon \right) \frac{\omega_n^2}{n!} > 0.$$

The last equality holds since $\sigma = \text{const}$ at the critical point. □

**Proposition 2.6.** Along the $J$–flow, any smooth curve in $\mathcal{H}$ strictly decreases. Moreover, the distance between any two points decreases as well.
Proof. Suppose \( \varphi(s) : [0,1] \to \mathcal{H} \) is a smooth curve in the space of Kähler potentials. Now consider the energy of this curve as

\[
En = \int_{s=0}^{1} \int_{V} \left( \frac{\partial \varphi}{\partial s} \right)^2 \frac{\omega^n}{n!} \, ds.
\]

Under the \( J \) flow, suppose that the energy of the evolved curve at time \( t > 0 \) is \( En(t) \). We want to show that the energy is strictly decreasing under this flow (All of the derivatives, norm and integration are taken with respect to \( g \) in the following calculation):

\[
\frac{d En(t)}{dt} = 2 \int_{s=0}^{1} \int_{V} \left( \frac{\partial \varphi}{\partial s} \cdot \frac{\partial^2 \varphi}{\partial s \partial t} + \left( \frac{\partial \varphi}{\partial s} \right)^2 \Delta_g \frac{\partial \varphi}{\partial t} \right) \frac{\omega^n}{n!} \, ds
\]

\[
= -2 \int_{s=0}^{1} \int_{V} \left( \frac{\partial \varphi}{\partial s} \cdot \frac{\partial \varphi}{\partial s} + \left( \frac{\partial \varphi}{\partial s} \right)^2 \Delta_g \frac{\partial \varphi}{\partial t} \right) \frac{\omega^n}{n!} \, ds
\]

\[
= 2 \int_{s=0}^{1} \int_{V} \frac{\partial \varphi}{\partial s} \left( \sum_{\alpha,\beta,r,\delta=1} g^{\alpha\beta} (\partial \varphi/\partial s)_{\alpha} g^{\rho\sigma} (\partial \varphi/\partial s)_{\rho} g^{\sigma\delta} \chi_{\alpha} \chi_{\delta} + (\partial \varphi/\partial s)_{\alpha} \sigma g^{\rho\sigma} \omega^n \right) \frac{\omega^n}{n!} \, ds
\]

\[
= -2 \int_{s=0}^{1} \int_{V} \sum_{\alpha,\beta,r,\delta=1} \left( g^{\alpha\beta} (\partial \varphi/\partial s)_{\alpha} g^{\rho\sigma} (\partial \varphi/\partial s)_{\rho} g^{\sigma\delta} \chi_{\alpha} \chi_{\delta} \right) \frac{\omega^n}{n!} \, ds + 2 \int_{s=0}^{1} \int_{V} \frac{\partial \varphi}{\partial s} \left( \frac{\partial \varphi}{\partial s} \right)_{\sigma} \sigma g^{\alpha\sigma} \omega^n \frac{\omega^n}{n!} \, ds
\]

\[
= -2 \int_{s=0}^{1} \int_{V} \sum_{\alpha,\beta,r,\delta=1} \left( g^{\alpha\beta} (\partial \varphi/\partial s)_{\alpha} g^{\rho\sigma} (\partial \varphi/\partial s)_{\rho} g^{\sigma\delta} \chi_{\alpha} \chi_{\delta} \right) \frac{\omega^n}{n!} \, ds + 2 \int_{s=0}^{1} \int_{V} \frac{\partial \varphi}{\partial s} \left( \frac{\partial \varphi}{\partial s} \right)_{\sigma} \sigma g^{\alpha\sigma} \omega^n \frac{\omega^n}{n!} \, ds
\]

The equality holds unless \( \frac{\partial \varphi}{\partial s} \) is a constant in \( V \times [0,1] \) or the geodesic is trivial.

If the \( J \) flow exists for all the time for any initially smooth metric, then this will imply that \( J \) flow decreases distance between any two points in \( \mathcal{H} \). We will prove the long term existence of the \( J \) flow in the next section. \( \square \)

There are surprising similarities between these two group of functionals: the first group is the Mabuchi energy and the Calabi energy, both are well known, but perhaps not well understood. The second group is our \( J \) functional and its norm \( E \). Here \( J \) plays the "role" of the Mabuchi energy and \( E \) plays the role of the Calabi energy. Calabi first proved in [3] that the critical point of the Calabi energy is a local minimizer. We shall compare this to Proposition 2.4 aforementioned. In [4], we showed that the Calabi flow (gradient flow of The Mabuchi energy ) decreases the length of any smooth curves in the space of Kähler metrics. We shall compare this to Proposition 2.5 aforementioned.
In [8], we prove that the critical point of the Calabi energy is unique if the first Chern class is negative; and in general it is conjectured that critical point of the Calabi energy is unique in each Kähler class. We shall compare this to Proposition 2.1 aforementioned. The list of similarities can go on and on. The critical point of the Calabi energy is well known. The critical point of $E$ is not so well-known and is also not clear about its significance in geometry. But the similarity between $E$ and the Calabi energy makes one thing clear: to study the critical point of $E$ or its gradient flow, is amount to study a junior version of extremal metrics or the Calabi flow. The insight we will learn and the technique we will develop from the studying of the Euler-Lagrange equation for the functional $J$ must be helpful to the studying of the extremal metric and the Calabi flow.

3 $C^2$ estimate of heat flow depending time $t$

**Lemma 3.1.** $\sigma > 0$ is bounded from above and below along the gradient flow of $J$.

**Proof.** Taking second derivatives with respect time, we have

$$\frac{\partial^2 \varphi}{\partial t^2} = \sum_{\alpha,\beta,r,s=1}^n g^{\alpha\beta} \left( \frac{\partial \varphi}{\partial t} \right)_{r\beta} g^r_s \chi_{\alpha s}.$$

By the ordinary maximum principle for parabolic equation, we have $\max_V \frac{\partial \varphi}{\partial t}$ is decreasing while $\min_V \frac{\partial \varphi}{\partial t}$ is increasing as $t$ increases. Since $\frac{\partial \varphi}{\partial t} = c - \sigma$, we then prove this lemma. $\square$

**Corollary 3.2.** Under the $J$–flow the evolved metric is strictly bounded from below.

**Theorem 3.3.** There exists a const $C(t)$ depends on $t$ only, such that $\omega_0 + \partial \tilde{\varphi} \leq C \cdot \omega_0$.

**Proof.** Consider the heat flow equation:

$$\frac{\partial \varphi}{\partial t} = c - \sum_{\alpha,\beta=1}^n g^{\alpha\beta} \chi_{\alpha\beta}.$$

In the following calculation, all indices are running from 1 to $n$. We adopt Einstein’s convention that the repeated indices mean summation from 1 to $n$. Taking first derivatives on both side of this equation.

$$\frac{\partial \varphi_{,i}}{\partial t} = g^{\alpha\beta} g_{\alpha\beta, i} g^{\alpha\beta} \chi_{\alpha\beta} - g^{\alpha\beta} \chi_{\alpha\beta, i}.$$
Next we take second derivatives on both sides of the above equation. We have
\[
\frac{\partial F}{\partial t} = g^{\alpha \tau} g_{\beta \gamma} g_{\delta \iota} \chi_{\alpha \beta} \chi_{\gamma \delta} - g^{\alpha \tau} g_{\mu \nu} g^{\rho \sigma} g_{\delta \iota} g^{\beta \gamma} \chi_{\chi} - g^{\alpha \tau} g_{\mu \nu} g^{\rho \sigma} g_{\delta \iota} g^{\beta \gamma} \chi_{\chi}.
\]
Indices after comma represent partial derivatives in above two equations. Next we multiple the equation (3) with \(\chi^\jmath\) and summarize everything. Set
\[
F = \chi^\jmath g^\jmath = \chi^\jmath (g_0^\jmath + \varphi^\jmath).
\]

Consider all differentials as covariant differentials with respect to metric \(\chi\). Denote the bisectional curvature of \(\chi\) as \(R(\chi)\). We then have
\[
\frac{\partial F}{\partial t} = g^{\alpha \tau} g_{\beta \gamma} g_{\delta \iota} \chi_{\alpha \beta} \chi_{\gamma \delta} - 2g^{\alpha \tau} g_{\mu \nu} g^{\rho \sigma} g_{\delta \iota} g^{\beta \gamma} \chi_{\chi} - 2g^{\alpha \tau} g_{\mu \nu} g^{\rho \sigma} g_{\delta \iota} g^{\beta \gamma} \chi_{\chi}.
\]

Since \(\chi\) is a fixed Kähler metric, then there exists a constant \(C\) such that
\[
| R(\chi) \| \leq C \cdot (\chi^\jmath g^\jmath + \chi^\jmath g^\jmath).
\]

Also, we can choose a coordinate such that \(\chi^\jmath = \delta^\jmath\). Thus,
\[
g^{\alpha \tau} (R(\chi) \| \chi^\jmath - R(\chi) \| \chi^\jmath) \leq C \cdot g^{\alpha \tau} \left( \delta^\jmath \delta^\jmath + \delta^\jmath \delta^\jmath \right) g_{\chi} + (\delta^\jmath \delta^\jmath + \delta^\jmath \delta^\jmath) g_{\chi} \leq C \cdot g^{\alpha \tau} = C \cdot g^{\alpha \tau}.
\]

Thus, we have
\[
\frac{\partial F}{\partial t} = \hat{\Delta} F - 2g^{\alpha \tau} g_{\mu \nu} g^{\rho \sigma} g_{\delta \iota} g^{\beta \gamma} \chi_{\chi} + C \cdot g^{\alpha \tau}.
\]

Where \(\hat{\Delta} f = g^{\alpha \tau} f_{\alpha \beta} g^{\beta \gamma} \chi_{\chi} \) for any smooth function \(f\). From here, we quickly deduce that \(F\) is bounded from above since \(\sigma\) is uniformly bounded from above.

Recall the estimate of equation (2.2), we have
\[
\int_0^\infty \frac{dE}{dt} dt = -2 \int_0^\infty \int_V \left( \sum_{\alpha, \beta, r, s = 1}^n g^{\alpha \tau} g_{\chi} g^{\beta \gamma} \chi_{\chi} \right) \frac{\omega^n}{n!} dt \geq C.
\]
In other words,
\[
\int_0^\infty \int_V \left( \sum_{\alpha, \beta, r, \delta = 1}^n g^{\alpha\beta} g^{r\delta} \chi_{\alpha\beta\delta} \right) \frac{\omega^n}{n!} d t \leq C.
\]

There exists a subsequence of \( t_i \to \infty \) such that
\[
\int_V \left( \sum_{\alpha, \beta, r, \delta = 1}^n g^{\alpha\beta} g^{r\delta} \chi_{\alpha\beta\delta} \right) \frac{\omega^n}{n!} \bigg|_{t = t_i} \to 0.
\]

This last expression, shall suggest that \( \chi_{\alpha\beta\delta} \big|_{t \to c} \) in some sense for some constant \( c \).

Next we return to prove the first part of Theorem 0.1.

**Proof.** Following from the interior estimate by Evans and Krylov, we can imply \( C^{2,0} \) estimate for any finite time \( t \). Then standard theory of the parabolic equation will imply that \( g \) is \( C^\infty \) at any finite time. Thus the flow exists for long time. Then proposition 2.5 implies that \( J \) flow decreases distance between any two points in \( H \).

### 4 Uniform \( C^2 \) estimate for heat flow for manifolds with semi-positive definite curvature tensors.

In this section, we assume that the bisectional curvature of \( \chi \) is non-negative, we want to show that there exists a uniform bound on the second derivatives of \( \varphi \).

**Theorem 4.1.** If the bisectional curvature of \( \chi \) is non-negative, then there exists a uniform bound on the second derivatives of \( \varphi \).

**Proof.** Following the calculation in Section 3, we have (equation (3)):
\[
\frac{\partial \varphi, i^j}{\partial t} = g^{\alpha\beta} g_{\alpha\beta, i^j} \chi_{\alpha\beta} - g^{\alpha\beta} g_{\alpha\beta, j^i} \chi_{\alpha\beta} - g^{\alpha\beta} \chi_{\alpha\beta, i^j},
\]
Simplifying this equation by using covariant derivatives in terms of the metric \( \chi \), we have
\[
\frac{\partial \varphi, i^j}{\partial t} = g^{\alpha\beta} g_{\alpha\beta, i^j} \chi_{\alpha\beta} - g^{\alpha\beta} g_{\alpha\beta, j^i} \chi_{\alpha\beta} - g^{\alpha\beta} \chi_{\alpha\beta, i^j}.
\]
Define an auxiliary tensor \( T_{i^j} \) as
\[
T_{i^j} = g^{\alpha\beta} g_{\alpha\beta, i^j} \chi_{\alpha\beta} - g^{\alpha\beta} g_{\alpha\beta, j^i} \chi_{\alpha\beta} - g^{\alpha\beta} \chi_{\alpha\beta, i^j}. \quad (4.1)
\]
Choose $C_0$ big enough so that $T_{ij} < 0$ as a tensor at time $t = 0$.

**Claim:** $T_{ij} < 0$ is preserved under this gradient flow.

From equation (4.1), we have

\[
\frac{\partial T_{ij}}{\partial t} = g^{\alpha\beta} T_{\kappa\mu} g_{\alpha\beta} T_{\kappa\mu} - g^{\alpha\beta} g_{\alpha\beta} T_{\kappa\mu} = g^{\alpha\beta} \left( T_{\kappa\mu} + R(\chi)_{\kappa\mu} T_{\rho\sigma} - R(\chi)_{\rho\sigma} T_{\kappa\mu} \right) g_{\alpha\beta}
\]

Next we want to apply Hamilton’s maximal principle for tensors. Since $T_{ij} < 0$ at $t = 0$, that there is a first time $t = t_0 > 0$ and a point $O$, where $T$ has a degenerate direction in $T_O V$. We assume that this direction is $\xi(O) = (\xi^1, \xi^2, \cdots, \xi^n)$. Parallel transport this vector along a small neighborhood $O$ of $O$ by metric $\chi$. By definition, we have

\[
T_{ij} \xi^i \xi^j < 0, \ \forall \ t < t_0,
\]

and at $t = t_0$ we have

\[
T_{ij} \xi^i \xi^j (O) = 0, \ \text{and} \ \ T_{ij} \leq 0 \ \text{in} \ O.
\]

In particular, at $t = t_0$ and at point $O$, we have $T_{ij} \xi^i = T_{ij} \xi^j = 0$. Now plugging everything into the equation (4), we have

\[
\frac{\partial T_{ij} \xi^i \xi^j}{\partial t} = g^{\alpha\beta} \left( T_{\kappa\mu} v_{\alpha\beta} T_{\kappa\mu} - R(\chi)_{\kappa\mu} T_{\rho\sigma} - R(\chi)_{\rho\sigma} T_{\kappa\mu} \right) g_{\alpha\beta} \xi^i \xi^j
\]

Now at the point $O$ and at time $t = t_0$, by the standard maximum principle, we have $\Delta(T_{ij} \xi^i \xi^j) \leq 0$. Moreover,

\[
g^{\alpha\beta} R(\chi)_{\kappa\mu} T_{\rho\sigma} \xi^i g_{\alpha\beta} \chi_{\alpha\beta} \xi^j (O) = 0
\]
and
\[ g^{\alpha\beta} R(\chi)_{\alpha\beta} - \xi^i T_i \delta^j \chi_{\alpha\beta} \leq 0. \]

The last inequality holds since \( R(\chi) \) is a non-negative tensor while \( T \) is a non-positive tensor. Thus
\[ \frac{\partial(T_{ij} \xi^i \xi^j)}{\partial t} (O) \leq 0. \]

This implies that \( T \) will remain non-positive. In other words,
\[ g_{ij} \leq C_0 \cdot \chi_{ij} \]

holds for all \( t \) where the flow exists. Thus, all of the second derivatives of \( \varphi \) is bounded from above.

Finally, we want to prove the second part of Theorem 0.1.

**Proof.** Again, following from the interior estimate by Evans and Krylov, we can obtain a uniform \( C^{2,\alpha} \) estimate for any finite time \( t \) from the Theorem 4.1 above. Then standard elliptic regularity theorem would imply that \( g \) is \( C^{\infty} \) at any finite time. Thus the flow exists for long time. Since the \( C^{2,\alpha} \) estimate is uniform (independent of time \( t \)), thus the flow converges to a critical point of \( J \), at least by sequence. The uniqueness of sequential limit is provided by the fact that \( J \) is strictly convex.

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