LIOUVILLE TYPE THEOREMS FOR MINIMAL GRAPHS OVER MANIFOLDS

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Abstract. Let $\Sigma$ be a complete Riemannian manifold with the volume doubling property and the uniform Neumann-Poincaré inequality. We show that any positive minimal graphic function on $\Sigma$ is a constant.

1. Introduction

Let $\Sigma$ denote a smooth complete non-compact Riemannian manifold with Levi-Civita connection $D$. Let $\text{div}_\Sigma$ be the divergence operator in terms of the Riemannian metric of $\Sigma$. In this paper, we study the minimal hypersurface equation on $\Sigma$

\begin{equation}
\text{div}_\Sigma \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0,
\end{equation}

which is a non-linear partial differential equation describing the minimal graph

$M = \{(x, u(x)) \in \Sigma \times \mathbb{R} | x \in \Sigma\}$

over $\Sigma$. The equation (1.1) is equivalent to that $u$ is harmonic on $M$, i.e.,

\begin{equation}
\Delta_M u = 0,
\end{equation}

where $\Delta_M$ is the Laplacian on $M$. The solution $u$ to (1.1) is the height function of the minimal graph $M$ in $\Sigma \times \mathbb{R}$. Therefore we call $u$ a minimal graphic function on $\Sigma$.

When $\Sigma$ is a Euclidean space $\mathbb{R}^n$, (1.1) is exactly the famous minimal surface equation on $\mathbb{R}^n$. In 1961, J. Moser [18] derived Harnack’s inequalities for uniformly elliptic equations, which imply Bernstein theorem for minimal graphs of bounded slope in all dimensions. In 1969, Bombieri-De Giorgi-Miranda [2] (see also [15]) showed interior gradient estimates for solutions to the minimal surface equation on $\mathbb{R}^n$, where the 2-dimensional case had already been obtained by Finn [12]. Using the gradient estimates, they get a Liouville type theorem in [2] as follows.

Theorem 1.1. Any positive minimal graphic function on $\mathbb{R}^n$ is a constant.

Without the ‘positive’ condition in Theorem 1.1, it is exact Bernstein theorem (see [1, 9, 13, 22] and the counter-example in [4]). Specially, any minimal graphic function on $\mathbb{R}^n$ is affine for $n \leq 7$.

As the linear analogue of (1.1) or (1.2), harmonic functions have been studied successfully on manifolds of nonnegative Ricci curvature. Yau [23] showed a Liouville theorem for harmonic functions:

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Every positive harmonic function on a complete manifold of nonnegative Ricci curvature is a constant.

Compared with this, it is natural to study Liouville type theorems for the solutions to (1.1) on manifolds of nonnegative Ricci curvature. Since minimal graphs in $\Sigma \times \mathbb{R}$ are area-minimizing, any positive minimal graphic function on a Riemann surface $\Sigma$ of nonnegative curvature is a constant from Fischer-Colbrie and Schoen [14]. For the general dimension $n$, Rosenberg-Schulze-Spruck [20] generalized Theorem 1.1. Specifically, they showed that any positive minimal graphical function on an $n$-dimensional complete manifold $\Sigma$ is a constant provided $\Sigma$ has nonnegative Ricci curvature and sectional curvature uniformly bounded from below. Besides the minimal graphs of dimension $> 7$ constructed by Bombieri-De Giorgi-Giusti [4], for all $n \geq 4$ there are non totally geodesic minimal graphs over $n$-dimensional complete manifolds of positive sectional curvature [10]. In the present paper, we obtain the following Liouville type theorem.

**Theorem 1.2.** Any positive minimal graphic function on a complete manifold of nonnegative Ricci curvature is a constant.

In fact, Theorem 1.2 is a consequence of a much more general result. In order to state this, let us recall the definition of some basic analytic inequalities on complete Riemannian manifold $\Sigma$. Let $B_r(p)$ denote the geodesic ball in $\Sigma$ of the radius $r$ and centered at $p \in \Sigma$. We call that $\Sigma$ has the volume doubling property, if there exists a positive constant $C_D > 1$ such that for all $p \in \Sigma$ and $r > 0$

$$H^n(B_{2r}(p)) \leq C_D H^n(B_r(p)),$$

where $H^n(\cdot)$ denotes the $n$-dimensional Hausdorff measure. We call that $\Sigma$ satisfies a uniform Neumann-Poincaré inequality, if there exists a positive constant $C_N \geq 1$ such that for all $p \in \Sigma$, $r > 0$ and $f \in W^{1,1}(B_r(p))$

$$\int_{B_r(p)} |f - \bar{f}_{p,r}| \leq C_N r \int_{B_r(p)} |Df|,$$

where $\bar{f}_{p,r} = \frac{1}{H^n(B_r(p))} \int_{B_r(p)} f$.

If $\Sigma$ has nonnegative Ricci curvature, then $\Sigma$ automatically satisfies the volume doubling property with doubling constant $C_D = 2^n$ by Bishop-Gromov volume comparison theorem. From P. Buser [5] or Cheeger-Colding [6], $\Sigma$ satisfies a uniform Neumann-Poincaré inequality with $C_N = C_N(n) < \infty$. Now we can state our more general result compared with Theorem 1.2 as follows.

**Theorem 1.3.** Let $\Sigma$ be an $n$-dimensional complete Riemannian manifold with (1.3)(1.4). If $u$ is a positive minimal graphic function on $\Sigma$, then $u$ is a constant.
Let $\Sigma$ be an $n$-dimensional complete manifold with Riemannian metric $\sigma$ and the Levi-Civita connection $D$. Suppose that $\Sigma$ satisfies the volume doubling property \((1.3)\) and the uniform Neumann-Poincaré inequality \((1.4)\). From \((1.3)\), one has
\[
\mathcal{H}^n(B_R(p)) \leq C_D \mathcal{H}^n(B_{\frac{R}{2}}(p)) \leq \cdots \leq C_D^k \mathcal{H}^n(B_{\frac{R}{2^k}}(p)) = 2^{k \log_2 C_D} \mathcal{H}^n(B_{\frac{R}{2^k}}(p))
\]
for any $R > 0$ and $k \in \mathbb{Z}^+$. Hence, there is a constant $\alpha = \log_2 C_D > 0$ such that
\[
\mathcal{H}^n(B_r(p)) \geq \frac{1}{C_D} \mathcal{H}^n(B_R(p)) \left( \frac{r}{R} \right)^\alpha
\]
for all $r \in (0, R)$. Without loss of generality, we assume that $C_D \geq 4$, then
\[
\alpha = \log_2 C_D \geq 2.
\]
From \((1.3)\) and 5-lemma, there is a constant $\Lambda_D \geq 1$ depending on $C_D$ such that
\[
\mathcal{H}^n(B_R(p) \setminus B_{R-r}(p)) \leq \Lambda_D \mathcal{H}^n(B_{R-r}(p) \setminus B_{R-2r}(p))
\]
for all $0 < r < R/2$, which implies
\[
\mathcal{H}^n(B_R(p) \setminus B_{R-r}(p)) \leq \frac{\Lambda_D}{\Lambda_D + 1} \mathcal{H}^n(B_R(p) \setminus B_{R-2r}(p)).
\]
For all $0 < r < R$, there is an integer $k \geq 0$ such that $2^{-k-1} < r/R \leq 2^{-k}$. Then from \((2.3)\), we have
\[
\mathcal{H}^n(B_R(p) \setminus B_{R-r}(p)) \leq \left( \frac{\Lambda_D}{\Lambda_D + 1} \right)^k \mathcal{H}^n(B_R(p) \setminus B_{R-2^k r}(p))
\]
\[
\leq 2^{-k \log_2 (1+1/\Lambda_D)} \mathcal{H}^n(B_R(p)) \leq \left( \frac{2r}{R} \right)^{\log_2 (1+1/\Lambda_D)} \mathcal{H}^n(B_R(p)).
\]
From \((1.4)\), we have
\[
\min \{ \mathcal{H}^n(\Omega), \mathcal{H}^n(B_r(p) \setminus \Omega) \} \leq C_N r \mathcal{H}^{n-1}(B_r(p) \cap \partial \Omega)
\]
immediately for any open set $\Omega$ in $B_r(p)$ with rectifiable boundary. Combining \((1.3)\) and \((1.4)\), one can get an isoperimetric inequality on $\Sigma$ (see the appendix for a self-contained proof). Namely, there exists a constant $C_S \geq 1$ depending only on $C_D, C_N$ such that for any $p \in \Sigma, r > 0$, we have
\[
\mathcal{H}^n(\Omega)^{1 - \frac{1}{\alpha}} \leq C_S (\mathcal{H}^n(B_r(p)))^{-\frac{1}{\alpha}} r \mathcal{H}^{n-1}(\partial \Omega)
\]
for any open set $\Omega \subset B_r(p)$ with rectifiable boundary. By a standard argument (see \cite{21} for instance), for $f \in W^{1,1}_0(B_r(p))$ with rectifiable boundary, we have
\[
\left( \int_{B_r(p)} |f|^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \leq C_S \mathcal{H}^n(B_r(p))^{\frac{1}{\alpha}} r \int_{B_r(p)} |Df|.
\]
From \((1.3)\) and \((1.4)\) on $\Sigma$, we can get the Sobolev-Poincaré inequality on $\Sigma$ (see Theorem 1 in \cite{16} for instance). Namely, up to select the constants $\alpha \geq 2$ and $C_S \geq 1$, for any $p \in \Sigma, r > 0, f \in W^{1,1}(B_r(p))$, we have
\[
\left( \int_{B_r(p)} |f - \bar{f}_{p,r}|^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \leq C_S \mathcal{H}^n(B_r(p))^{-\frac{1}{\alpha}} r \int_{B_r(p)} |Df|,
\]
where \( \tilde{f}_{p,r} = \frac{1}{\mathcal{H}^n(B_r(p))} \int_{B_r(p)} f \). The Sobolev-Poincaré inequality \((2.8)\) implies the following isoperimetric type inequality, compared with \((2.5)\).

**Lemma 2.1.** For any open set \( \Omega \) in \( B_r(p) \) with rectifiable boundary, we have

\[
(2.9) 
\mathcal{H}^{n-1}(B_r(p) \cap \partial \Omega) \geq \frac{(\mathcal{H}^n(B_r(p)))^{\frac{1}{n}}}{2\pi C_{SR}} (\min \{ \mathcal{H}^n(\Omega), \mathcal{H}^n(B_r(p) \setminus \Omega) \})^{\frac{\alpha - 1}{\alpha}}.
\]

**Proof.** Denote \( \Omega_\epsilon = B_r(p) \setminus \Omega \). Let \( f \) be a function of bounded variation defined by \( f \equiv 0 \) on \( \Omega \), and \( f \equiv \mathcal{H}^n(B_r(p)) \) on \( \Omega_\epsilon \). For any \( \epsilon > 0 \), let \( f_\epsilon \) be a Lipschitz function defined on \( B_r(p) \) by letting \( f_\epsilon \equiv 0 \) on \( \Omega \), \( f_\epsilon \equiv t \mathcal{H}^n(B_r(p))/\epsilon \) on \( \{ x \in B_r(p) \mid d(x, \Omega) = t \} \) for any \( t \in (0, \epsilon) \) and \( f_\epsilon \equiv \mathcal{H}^n(B_r(p)) \) on \( \{ x \in B_r(p) \mid d(x, \Omega) > \epsilon \} \), where \( d(\cdot, \Omega) \) is the distance function to \( \Omega \) on \( \Sigma \). Then \( f_\epsilon \) satisfies

\[
\lim_{\epsilon \to 0} \int_{B_r(p)} f_\epsilon = \int_{B_r(p)} f = \mathcal{H}^n(B_r(p)) \mathcal{H}^n(\Omega_\epsilon).
\]

Using \((2.5)\) for \( f_\epsilon \), and then letting \( \epsilon \to 0 \) implies

\[
(2.10) 
\left( \left( \mathcal{H}^n(\Omega) \right)^{\frac{\alpha}{\alpha - 1}} \mathcal{H}^n(\Omega_\epsilon) + \left( \mathcal{H}^n(\Omega_\epsilon) \right)^{\frac{\alpha}{\alpha - 1}} \mathcal{H}^n(\Omega) \right)^{\frac{\alpha - 1}{\alpha}} 
\leq C_{SR} \left( \mathcal{H}^n(B_r(p)) \right)^{\frac{1}{\alpha}} r \mathcal{H}^{n-1}(B_r(p) \cap \partial \Omega) \mathcal{H}^n(B_r(p)).
\]

Without loss of generality, we assume \( \mathcal{H}^n(\Omega_\epsilon) \leq \mathcal{H}^n(\Omega) \). Then

\[
C_{SR} \left( \mathcal{H}^n(B_r(p)) \right)^{1 - \frac{1}{\alpha}} r \mathcal{H}^{n-1}(B_r(p) \cap \partial \Omega)
\geq \left( \mathcal{H}^n(\Omega_\epsilon) \right)^{\frac{\alpha - 1}{\alpha}} \mathcal{H}^n(\Omega) \left( 1 + \left( \frac{\mathcal{H}^n(\Omega_\epsilon)}{\mathcal{H}^n(\Omega)} \right)^{\frac{\alpha - 1}{\alpha}} \right)^{\frac{\alpha - 1}{\alpha}}
\geq \left( \mathcal{H}^n(\Omega_\epsilon) \right)^{\frac{\alpha - 1}{\alpha}} \mathcal{H}^n(\Omega) 2^{\frac{1}{\alpha}} \left( 1 + \frac{\mathcal{H}^n(\Omega_\epsilon)}{\mathcal{H}^n(\Omega)} \right)
\geq 2^{-\frac{1}{\alpha}} \left( \mathcal{H}^n(\Omega_\epsilon) \right)^{\frac{\alpha - 1}{\alpha}} \mathcal{H}^n(B_r(p)),
\]

where in the third inequality we have used \( 2^{\frac{1}{\alpha}} (1 + t)^{\frac{\alpha - 1}{\alpha}} \geq 1 + t \geq 1 + t^{\alpha - 1} \) for any \( t \in [0, 1] \) and \( \alpha \geq 2 \). This completes the proof.

Let \( \Sigma \times \mathbb{R} \) be the product manifold with the flat product metric \( \sigma + ds^2 \). For any \( \bar{p} = (p, t_p) \in \Sigma \times \mathbb{R} \), let \( B_r(\bar{p}) \) denote the geodesic ball in \( \Sigma \times \mathbb{R} \) of the radius \( r \) and centered at \( \bar{p} \). Then the volume of the geodesic ball \( B_r(\bar{p}) \) satisfies

\[
(2.12) 
\mathcal{H}^{n+1}(B_r(\bar{p})) = 2 \int_{x \in B_r(p)} \sqrt{r^2 - d(x, p)^2},
\]

where \( d(\cdot, p) \) is the distance function on \( \Sigma \) from \( p \). Combining \((2.1)\), we have

\[
(2.13) 
2r \mathcal{H}^n(B_r(p)) \geq \mathcal{H}^{n+1}(B_r(\bar{p})) \geq 2 \int_{B_{2r}(p)} \frac{\sqrt{3}}{2} = \sqrt{3} r \mathcal{H}^n \left( B_{\frac{3r}{2}}(p) \right) \geq \frac{\sqrt{3} r}{2^\alpha C_D} \mathcal{H}^n \left( B_r(p) \right).
\]

Then combining \((2.1)\) we have

\[
(2.14) 
\mathcal{H}^{n+1}(B_R(\bar{p})) \leq 2R \mathcal{H}^n(B_R(p)) \leq 2RC_D \left( \frac{R}{r} \right)^\alpha \mathcal{H}^n(B_r(p)) \leq 2RC_D \left( \frac{R}{r} \right)^\alpha \mathcal{H}^{n+1}(B_r(\bar{p})) = \frac{2^\alpha C_D}{\sqrt{3}} \left( \frac{R}{r} \right)^{\alpha + 1} \mathcal{H}^{n+1}(B_r(\bar{p})).
\]
for all \( r \in (0, R) \).

Let \( \pi \) denote the projection from \( \Sigma \times \mathbb{R} \) onto \( \Sigma \). For any \( \bar{x} = (x, t_x) \in \Sigma \times \mathbb{R} \), let \( C_{\bar{x}, t} \) be a cylinder in \( \Sigma \times \mathbb{R} \) defined by
\[
(2.15) \quad C_{\bar{x}, t} = B_r(x) \times (t_x - t, t_x + t).
\]
Let \( C_{\bar{x}, r} = C_{\bar{x}, r} \) for convenience.

**Lemma 2.2.** For any open set \( V \) in \( C_{\bar{p}, r} \) with rectifiable boundary, there is an absolute constant \( c > 0 \) such that
\[
(2.16) \quad C_n r \mathcal{H}^n(C_{\bar{p}, r} \cap \partial V) \geq \frac{1}{c} \min \{ \mathcal{H}^{n+1}(V), \mathcal{H}^{n+1}(C_{\bar{p}, r} \setminus V) \}.
\]

**Proof.** Let \( \Gamma = \partial V \cap C_{\bar{p}, r} \). Let \( \delta \) be a positive constant with
\[
(2.17) \quad \mathcal{H}^n(\pi(\Gamma)) = \delta \mathcal{H}^n(B_r(p)).
\]
Suppose that \( \delta \leq \frac{1}{8} \), or else
\[
\mathcal{H}^n(\Gamma) \geq \mathcal{H}^n(\pi(\Gamma)) \geq \frac{1}{8} \mathcal{H}^n(B_r(p)),
\]
which completes the proof. If
\[
(2.18) \quad \min \{ \mathcal{H}^n(\pi(V) \setminus \pi(\Gamma)), \mathcal{H}^n(\pi(C_{\bar{p}, r} \setminus V) \setminus \pi(\Gamma)) \} \leq \frac{1}{8} \mathcal{H}^n(B_r(p)),
\]
then using (2.17) we have
\[
\min \{ \mathcal{H}^n(\pi(V)), \mathcal{H}^n(\pi(C_{\bar{p}, r} \setminus V)) \} \leq \frac{1}{4} \mathcal{H}^n(B_r(p)).
\]
Now we assume
\[
(2.19) \quad \mathcal{H}^n(\pi(V)) \leq \frac{1}{4} \mathcal{H}^n(B_r(p)).
\]
Let \( V_t = V \cap (\Sigma \times \{t\}) \) for all \( |t - t_p| < r \). Clearly, \( \mathcal{H}^n(V_t) \leq \mathcal{H}^n(\pi(V)) \leq \frac{1}{4} \mathcal{H}^n(B_r(p)) \).

From the uniform Neumann-Poincaré inequality (1.4), we get
\[
(2.20) \quad \mathcal{H}^n(V_t) \leq C_n r \mathcal{H}^{n-1}(\partial V \cap (B_r(p) \times \{t\})).
\]
Combining the co-area formula we have
\[
(2.21) \quad \mathcal{H}^{n+1}(V) = \int_{t_p - r}^{t_p + r} \mathcal{H}^n(V_t)dt \leq C_n r \int_{t_p - r}^{t_p + r} \mathcal{H}^{n-1}(\partial V \cap (B_r(p) \times \{t\}))dt \leq C_n r \mathcal{H}^n(\partial V \cap C_{\bar{p}, r}).
\]
Clearly, the above inequality is true provided the assumption (2.19) is replaced by \( \mathcal{H}^n(\pi(C_{\bar{p}, r} \setminus V)) \leq \frac{1}{4} \mathcal{H}^n(B_r(p)) \). Hence, we always have
\[
(2.22) \quad \min \{ \mathcal{H}^{n+1}(V), \mathcal{H}^{n+1}(C_{\bar{p}, r} \setminus V) \} \leq C_n r \mathcal{H}^n(\partial V \cap C_{\bar{p}, r}).
\]

If (2.18) fails, we have
\[
(2.23) \quad \min \{ \mathcal{H}^n(\pi(V) \setminus \pi(\Gamma)), \mathcal{H}^n(\pi(C_{\bar{p}, r} \setminus V) \setminus \pi(\Gamma)) \} \geq \frac{1}{8} \mathcal{H}^n(B_r(p)),
\]
which implies
\[
(2.24) \quad \min \{ \mathcal{H}^n(V_t), \mathcal{H}^n(B_r(p) \times \{t\} \setminus V_t) \} \geq \frac{1}{8} \mathcal{H}^n(B_r(p))
\]
for all $|t - t_p| < r$. With (2.25), we have
\begin{equation}
H^n (\partial V \cap (B_r(p) \times \{t\})) \geq \frac{1}{C_{NT}} \min \{H^n(V_t), H^n(B_r(p) \times \{t\}) \} \geq \frac{H^n(B_r(p))}{8C_{NT}}.
\end{equation}
Combining the co-area formula, we get
\begin{equation}
C_{NT} H^n (\partial V \cap C_{p,r}) \geq C_{NT} \int_{t_p-r}^{t_p+r} H^n (\partial V \cap (B_r(p) \times \{t\})) dt \geq \frac{r}{4} H^n (B_r(p)),
\end{equation}
which implies (2.16). We complete the proof. \hfill \Box

From Lemma 2.2, we have a Neumann-Poincaré inequality in $\Sigma \times \mathbb{R}$ as follows.

**Lemma 2.3.**
\begin{equation}
\int_{B_r(\bar{p})} |f - \tilde{f}_{\bar{p},r}| \leq 2c_{NT} \int_{B_r(\bar{p})} |\nabla f|
\end{equation}
for all function $f \in W_{loc}^{1,1}(\Sigma \times \mathbb{R})$, where $\tilde{f}_{\bar{p},r} = \frac{1}{H^n(V_{r,0})} \int_{B_r(\bar{p})} f$.

**Proof.** Let
\begin{equation}
V_{s,\bar{p}}^+ = \{x \in B_s(\bar{p}) | f(x) > \tilde{f}_{\bar{p},r} + t\},
\end{equation}
and
\begin{equation}
V_{s,\bar{p}}^- = \{x \in B_s(\bar{p}) | f(x) < \tilde{f}_{\bar{p},r} + t\}
\end{equation}
for all $s \in (0, 2r]$ and $t \in \mathbb{R}$. Without loss of generality, we assume $H^{n+1}(V_{r,0}^+) \leq H^{n+1}(V_{r,0}^-)$. Then
\begin{equation}
H^{n+1}(V_{r,0}^+) \leq \frac{1}{2} H^{n+1}(B_r(\bar{p}))
\end{equation}
for any $t \geq 0$. From Lemma 2.2 there holds
\begin{equation}
H^n (\partial V_{r,0}^+ \cap B_{\sqrt{n}}(\bar{p})) \geq \frac{1}{c_{NT}} H^{n+1}(V_{r,0}^+).
\end{equation}
From co-area formula, we have
\begin{equation}
\int_{V_{r,0}^+} (f - \tilde{f}_{\bar{p},r}) = \int_0^\infty H^{n+1}(V_{r,0}^+) dt \leq c_{NT} \int_0^\infty H^n (\partial V_{\sqrt{n},0}^+ \cap B_{\sqrt{n}}(\bar{p})) dt
\end{equation}
\begin{equation}
\leq c_{NT} \int_{B_{\sqrt{n}}(\bar{p})} |\nabla f|.
\end{equation}
Hence
\begin{equation}
\int_{B_r(\bar{p})} |f - \tilde{f}_{\bar{p},r}| = \int_{V_{r,0}^+} (f - \tilde{f}_{\bar{p},r}) - \int_{V_{r,0}^-} (f - \tilde{f}_{\bar{p},r})
\end{equation}
\begin{equation}
= 2 \int_{V_{r,0}^+} (f - \tilde{f}_{\bar{p},r}) \leq 2c_{NT} \int_{B_{\sqrt{n}}(\bar{p})} |\nabla f|.
\end{equation}
This completes the proof. \hfill \Box

Combining (2.14), Lemma 2.3 and Theorem 1 in 16, up to select the constants $\alpha \geq 2$ and $C_S \geq 1$, for any $\bar{p} \in \Sigma \times \mathbb{R}$, $r > 0$, and $f \in W^{1,1}(B_r(\bar{p}))$, we have
\begin{equation}
\left( \int_{B_r(\bar{p})} |f - \tilde{f}_{\bar{p},r}|^{\alpha+1} \right)^{\frac{1}{\alpha+1}} \leq C_S \left( H^{n+1}(B_r(\bar{p})) \right)^{-\frac{1}{\alpha+1}} r \int_{B_r(\bar{p})} |D f|,
\end{equation}
where \( \tilde{f}_{\beta,r} = \frac{1}{H^{n+1}(B_r(\bar{p}))} \int_{B_r(\bar{p})} f \). From the argument of Lemma 2.1, we have

\[
(2.33) \quad \mathcal{H}^n(B_r(\bar{p}) \cap \partial V) \geq \frac{(H^{n+1}(B_r(\bar{p})))^{\frac{1}{n+1}}}{2^{\frac{n+1}{n+1}}CSR} \left( \min \{ \mathcal{H}^{n+1}(V), \mathcal{H}^{n+1}(B_r(\bar{p}) \setminus V) \} \right)^{\frac{\alpha}{\alpha+1}}
\]

for any open set \( V \) in \( B_r(\bar{p}) \) with rectifiable boundary. In particular,

\[
(2.34) \quad \mathcal{H}^n(B_r(\bar{p}) \cap \partial V) \geq \frac{(H^{n+1}(B_r(\bar{p})))^{\frac{1}{n+1}}}{2^{\frac{n+1}{n+1}}CSR} \min \{ \mathcal{H}^{n+1}(V), \mathcal{H}^{n+1}(B_r(\bar{p}) \setminus V) \} \left( \frac{\mathcal{H}^{n+1}(B_r(\bar{p}))}{2} \right)^{-\frac{\alpha}{\alpha+1}}
\]

\[
= \frac{1}{CSR} \min \{ \mathcal{H}^{n+1}(V), \mathcal{H}^{n+1}(B_r(\bar{p}) \setminus V) \}.
\]

From the argument in the appendix, there holds the isoperimetric inequality and the Sobolev inequality on \( \Sigma \times \mathbb{R} \) from the doubling property (2.13) and (2.34). Namely, up to choose the constant \( C_S \geq 1 \) depending only on \( C_D, C_N \), we have

\[
(2.35) \quad \left( \mathcal{H}^{n+1}(V) \right)^{\frac{\alpha}{n+1}} \leq C_S \left( \mathcal{H}^{n+1}(B_r(\bar{p})) \right)^{-\frac{1}{n+1}} r \mathcal{H}^{n+1}(\partial V)
\]

for any open set \( V \) in \( B_r(\bar{p}) \) with rectifiable boundary.

3. SOBOLEV AND NEUMANN-POINCARÉ INEQUALITIES ON MINIMAL GRAPHS

Let \( M \) be a minimal graph over the geodesic ball \( B_R(p) \subset \Sigma \) with the graphic function \( u \), i.e., \( u \) satisfies (3.1) on \( B_R(p) \). Denote \( \bar{p} = (p, u(p)) \in M \). Similar to the Euclidean case, \( M \) is an area-minimizing hypersurface in \( B_R(p) \times \mathbb{R} \) (see [19] or Lemma 2.1 in [11]). Namely, for any rectifiable hypersurface \( S \) in \( B_R(p) \times \mathbb{R} \) with boundary \( \partial S = \partial M \), we have

\[
(3.1) \quad \mathcal{H}^n(M) \leq \mathcal{H}^n(S).
\]

For the fixed \( R > 0 \), let

\[
(3.2) \quad \Omega_\pm = \{(x, t) \in B_R(p) \times \mathbb{R} \mid t > (<) u(x)\}.
\]

**Lemma 3.1.** There is a constant \( \beta \in (0, \frac{1}{2}] \) depending only on \( C_D, C_N \) such that for any \((n+1)\)-dimensional ball \( B_t(z) \subset B_R(p) \times \mathbb{R} \) with \( z \in M \) we have

\[
\mathcal{H}^{n+1}(\Omega_+ \cap B_t(z)) \geq \beta \mathcal{H}^{n+1}(B_t(z)).
\]

**Proof.** Let \( \Omega_{s,z} = \Omega_+ \cap \partial B_s(z) \) for any \( 0 < s < r \). Then

\[
\partial \Omega_{s,z} = \partial \Omega_+ \cap \partial B_s(z) = \partial (B_s(z) \cap \partial \Omega_+).
\]

Since \( \partial \Omega_+ \cap B_s(z) \) is area-minimizing in \( B_R(p) \times \mathbb{R} \), then

\[
(3.3) \quad \mathcal{H}^n(\Omega_{s,z}) \geq \mathcal{H}^n(B_s(z) \cap \partial \Omega_+).
\]

By the isoperimetric inequality (2.35), for any \( 0 < t < r \) we have

\[
(3.4) \quad \left( \mathcal{H}^{n+1}(\Omega_+ \cap B_t(z)) \right)^{\frac{\alpha}{n+1}} \leq C_S \left( \mathcal{H}^{n+1}(B_t(z)) \right)^{-\frac{1}{n+1}} \mathcal{H}^n(\partial(\Omega_+ \cap B_t(z)) \).
\]

Since

\[
\mathcal{H}^n(\partial(\Omega_+ \cap B_t(z))) = \mathcal{H}^n(\Omega_{t,z}) + \mathcal{H}^n(B_t(z) \cap \partial \Omega_+) \leq 2 \mathcal{H}^n(\Omega_{t,z}),
\]

then

\[
(3.5) \quad \left( \mathcal{H}^{n+1}(B_t(z)) \right)^{\frac{1}{n+1}} \left( \mathcal{H}^{n+1}(\Omega_+ \cap B_t(z)) \right)^{\frac{\alpha}{n+1}} \leq 2C_S t \mathcal{H}^n(\Omega_{t,z})
\]

for any $t \in (0, r)$, which means

\[
(3.6) \quad (\mathcal{H}^{n+1}(B_t(z)))^{\alpha+1} \left( \int_0^t \mathcal{H}^n(\Omega_{s,z}) \, ds \right)^{\alpha+1} \leq 2Cst\mathcal{H}^n(\Omega_{t,z}).
\]

As $z \in M$, one has $\int_0^t \mathcal{H}^n(\Omega_{s,z}) \, ds > 0$ for each $t > 0$. From (3.6), we get

\[
(3.7) \quad \frac{\partial}{\partial t} \left( \int_0^t \mathcal{H}^n(\Omega_{s,z}) \, ds \right)^{\alpha+1} \geq \frac{1}{2(\alpha+1)Cs} \left( \mathcal{H}^{n+1}(B_t(z)) \right)^{\alpha+1}
\]
on $(0, r)$, which gives

\[
\left( \int_0^r \mathcal{H}^n(\Omega_{s,z}) \, ds \right)^{\alpha+1} \geq \frac{1}{2(\alpha+1)Cs} \left( \mathcal{H}^{n+1}(B_{\tilde{z}}(z)) \right)^{\alpha+1} \int_0^r \frac{1}{t} \, dt
\]

\[
= \frac{\log 2}{2(\alpha+1)Cs} \left( \mathcal{H}^{n+1}(B_{\tilde{z}}(z)) \right)^{\alpha+1}.
\]

In particular,

\[
(3.8) \quad \mathcal{H}^{n+1}(\Omega_+ \cap B_r(z)) = \int_0^r \mathcal{H}^n(\Omega_{s,z}) \, ds \geq \left( \frac{\log 2}{2(\alpha+1)Cs} \right)^{\alpha+1} \mathcal{H}^{n+1}(B_{\tilde{z}}(z)),
\]

and this completes the proof. \hfill \Box

From the above lemma, we also have

\[
\mathcal{H}^{n+1}(\Omega_- \cap B_r(z)) \geq \beta \mathcal{H}^{n+1}(B_r(z))
\]

for any $B_r(z) \subset B_R(p) \times \mathbb{R}$ with $z \in M$. Combining (2.34), we can give a lower bound of the volume of minimal graphs as follows.

\[
(3.9) \quad \mathcal{H}^n(M \cap B_r(z)) \geq \frac{\beta}{Csr} \mathcal{H}^{n+1}(B_r(z)).
\]

Recall $C^r_{x,r} = B_r(x) \times (t_x - r, t_x + r)$ for $\bar{x} = (x, t_x) \in M$, and $\pi$ denotes the projection from $\Sigma \times \mathbb{R}$ onto $\Sigma$. From Lemma 3.1 we have

\[
(3.10) \quad \mathcal{H}^{n+1}(\Omega_+ \cap C^r_{x,r}) \geq \mathcal{H}^{n+1}(\Omega_+ \cap B_r(z)) \geq \beta \mathcal{H}^{n+1}(B_r(z)).
\]

Since

\[
\Omega_+ \cap (B_r(\pi(z)) \times \{t_1\}) \subset \Omega_+ \cap (B_r(\pi(z)) \times \{t_2\})
\]

for all $t_1 < t_2$, then from co-area formula and (3.10), we have

\[
\mathcal{H}^n(\Omega_+ \cap (B_r(\pi(z)) \times \{t\})) \geq \mathcal{H}^n(\Omega_+ \cap (B_r(\pi(z)) \times \{r\}))
\]

\[
\geq \frac{1}{2r} \mathcal{H}^{n+1}(\Omega_+ \cap C^r_{x,r}) \geq \frac{\beta}{2r} \mathcal{H}^{n+1}(B_r(z))
\]

for any $t \geq r$.

Now let us define several open sets in $\Sigma \times \mathbb{R}$ as follows. For any $r > 0$, $t \in \mathbb{R}$ and $\bar{x} = (x, t_x) \in \Sigma \times \mathbb{R}$, we define

\[
\mathcal{D}^r_{x,r} = \{(y, s) \in \Sigma \times \mathbb{R} | d(x, y) + |s - t_x| < r\},
\]

\[
\mathcal{D}^\perp_x = \{(y, s) \in \Sigma \times \mathbb{R} | d(x, y) + |s - t_x| < r, s > t + t_x\},
\]

and

\[
\mathcal{D}^\perp_x = \{(y, s) \in \Sigma \times \mathbb{R} | d(x, y) + |s - t_x| < r, s > (\leq) t_x\}.
\]

Then

\[
B_r(\bar{x}) \subset \mathcal{D}^r_{x,\sqrt{2}r} \subset B_{\sqrt{2}r}(\bar{x}).
\]
For any $n$-rectifiable set $\Omega \subset \partial \mathcal{D}^+_{x,r} \setminus (B_r(x) \times \{t\})$, one has
\begin{equation}
\mathcal{H}^n(\Omega) = \sqrt{2}\mathcal{H}^n(\pi(\Omega)).
\end{equation}

From now on, we assume that the graphic function $u$ of the minimal graph $M$ is not a constant in this section. Since $M$ is area-minimizing in $B_r(p) \times \mathbb{R}$, then
\begin{equation}
\mathcal{H}^n(M \cap \mathcal{D}_{x,r}) \leq \frac{1}{2}\mathcal{H}^n(\partial \mathcal{D}_{x,r}) = \sqrt{2}\mathcal{H}^n(B_r(x)).
\end{equation}
Combining (2.13) and (3.9), there is a constant $\beta_* \in (0, \beta]$ depending only on $C_D, C_N$ such that
\begin{equation}
\beta_*\mathcal{H}^n(B_r(x)) \leq \mathcal{H}^n(M \cap \mathcal{D}_{x,r}) \leq \sqrt{2}\mathcal{H}^n(B_r(x))
\end{equation}
for all $B_r(x) \subset B_R(p)$. From (2.1) (2.13) and Lemma 3.1, there is a constant $\hat{\beta} > 0$ depending only on $C_D, C_N$ such that
\begin{equation}
\mathcal{H}^{n+1}(\Omega^+ \cap \mathcal{D}_{x,r}) \geq \hat{\beta} \mathcal{H}^n(B_r(x)).
\end{equation}

Let us prove an isoperimetric type inequality on $M$.

**Lemma 3.2.** There is a constant $\theta$ depending only on $C_D, C_N$ such that for any $\bar{x} = (x, u(x))$ with $B_{2r}(x) \subset B_R(p)$ and $\tau \in (0, r)$, we have
\begin{equation}
(\mathcal{H}^n(M \cap \mathcal{D}_{x,r}))^{\frac{1}{n}} (\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}))^{\frac{n-1}{n}} \leq \theta \left(\frac{r}{t} \mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{x,r}) + \frac{r}{r} \mathcal{H}^n(E_t \cap \mathcal{D}_{x,r})\right),
\end{equation}
where $E_t = M \cap \mathcal{D}_{x,2r}$ or $E_t = M \cap (\mathcal{D}_{x,2r} \setminus \mathcal{D}_{x,2r})$ for any $t \in (-r, r)$.

**Proof.** Using $u - u(x)$ instead of $u$, we can assume $u(x) = 0$. By considering $-u, E_{-t}$ instead of $u, E_t$, respectively, we only need to show the case $E_t = M \cap \mathcal{D}_{x,2r}$ for any $t \in (-r, r)$.

From (2.1), there exists a constant $\beta' \in (0, 1)$ depending only on $C_D, C_N$ such that
\begin{equation}
\mathcal{H}^n(B_r(x) \setminus B_{(1-\beta')r}(x)) \leq \frac{\hat{\beta}}{16} \mathcal{H}^n(B_r(x)),
\end{equation}
where $\hat{\beta}$ is the constant in (3.15). From (2.1) and (3.14), for $|t| > 0$ we have
\begin{equation}
\mathcal{H}^n(M \cap \mathcal{D}_{x,|t|}) \geq \beta_* \mathcal{H}^n(B_{|t|}(x)) \geq \frac{\beta_*}{C_D} \mathcal{H}^n(B_r(x)) \left(\frac{|t|}{r}\right)^{\alpha} \geq \frac{\sqrt{2} \beta_*}{2C_D} \mathcal{H}^n(M \cap \mathcal{D}_{x,r}) \frac{|t|^\alpha}{r^\alpha}.
\end{equation}
Hence, we only need to prove (3.16) for $t > r \geq -\min\{\frac{1}{2}, \frac{1}{8} \beta' r, \beta' r\}$. Let $\delta$ be a positive constant satisfying
\begin{equation}
\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}) = \delta \left(\frac{t}{r}\right)^{\alpha} \mathcal{H}^n(B_r(x)).
\end{equation}
We can assume $\delta < \frac{\sqrt{2}}{8} \hat{\beta}$, or else we have complete the proof combining (3.14).

Let $U^-_t$ be a subset in $B_{r-t}(x) \times \{t\}$ defined by
\begin{equation}
U^-_t = \Omega^- \cap (B_{r-t}(x) \times \{t\}),
\end{equation}
and
\begin{equation}
U^+_t = \Omega^+ \cap (B_{r-t}(x) \times \{t\}) = B_{r-t}(x) \times \{t\} \setminus U^-_t.
\end{equation}
We claim
\begin{equation}
\mathcal{H}^n(U^+_t) \geq \delta_* \mathcal{H}^n(B_{r-t}(x))
\end{equation}
for some positive constant $\delta_*$ depending only on $C_D, C_N$. The proof of the claim is divided into 2 cases as follows.

- Case 1: $\mathcal{H}^n(\pi(M \cap \mathcal{D}_{x,r})) \leq \frac{1}{4} \mathcal{H}^n(B_r(x))$. Let $\Omega^+_x = \{(x,s) \in \Omega^+ \cap \mathcal{D}_{x,r} \mid x \in \pi(M \cap \mathcal{D}_{x,r})$, $s \in \mathbb{R}\}$. Then $\Omega^+_x$ is symmetric with respect to $\Sigma \times \{0\}$, and

$$\mathcal{H}^{n+1}(\Omega^+_x) \leq 2r\mathcal{H}^n(\pi(M \cap \mathcal{D}_{x,r})) \leq \frac{1}{2} \mathcal{H}^n(B_r(x)).$$

Combining (3.15), we have

$$\mathcal{H}^{n+1}(\Omega^+ \cap \mathcal{D}_{x,r} \setminus \Omega^+_x) \geq \frac{1}{2} \mathcal{H}^n(B_r(x)).$$

(3.21)

For any $s \in \mathbb{R}$, we define a family of subsets $W_{x,r}(s) \subset B_r(x)$ by

$$\Omega^+ \cap \mathcal{D}_{x,r} \setminus \Omega^+_x = \bigcup_{s \in \mathbb{R}} W_{x,r}(s) \times \{s\}.$$

Since $\Omega^+ \cap \mathcal{D}_{x,r} \setminus \Omega^+_x$ is symmetric with respect to $\Sigma \times \{0\}$, then $W_{x,r}(s) = W_{x,r}(-s)$. By the definition of $\Omega^+_x$, we have

$$\mathcal{H}^{n+1}(\Omega^+ \cap \mathcal{D}_{x,r} \setminus \Omega^+_x) \leq \frac{1}{4} \mathcal{H}^n(W_{x,r}(0)) + 2r\mathcal{H}^n(W_{x,r}(\hat{\beta}/8)) \leq \frac{1}{4} \mathcal{H}^n(B_r(x)) + 2\mathcal{H}^n(W_{x,r}(\hat{\beta}/8)).$$

(3.22)

Combining (3.21), we get

$$\mathcal{H}^n(W_{x,r}(-\hat{\beta}/8)) = \mathcal{H}^n(W_{x,r}((\hat{\beta}/8)) \geq \frac{1}{2} \mathcal{H}^n(B_r(x)).$$

(3.23)

Uniting with (3.21) for $t \geq -\min\{\frac{1}{2} \hat{\tau}, \frac{1}{2} \hat{\beta}r, \beta' r\}$, we complete the proof of the claim (3.20).

- Case 2: $\mathcal{H}^n(\pi(M \cap \mathcal{D}_{x,r})) > \frac{1}{4} \mathcal{H}^n(B_r(x))$. From (3.18) and $\delta < \frac{1}{8} \hat{\beta}$, we have

$$\mathcal{H}^n(E_t) \leq \delta\mathcal{H}^n(B_r(x)) \leq \frac{1}{8} \mathcal{H}^n(B_r(x)).$$

For $|t| \leq \beta' r$, we have

$$\mathcal{H}^n(U^+_t) + \mathcal{H}^n(B_r(x) \setminus B_{\sqrt{1-(\beta')^2}r}(x)) \geq \mathcal{H}^n(\pi(M \cap \mathcal{D}_{x,r} \setminus E_t)) \geq \mathcal{H}^n(\pi(M \cap \mathcal{D}_{x,r})) - \mathcal{H}^n(E_t) \geq \frac{\hat{\beta}}{4} \mathcal{H}^n(B_r(x)) - \frac{\hat{\beta}}{8} \mathcal{H}^n(B_r(x)) = \frac{\hat{\beta}}{8} \mathcal{H}^n(B_r(x)),$$

(3.24)

then combining (3.17), we have

$$\mathcal{H}^n(U^+_t) \geq \frac{\hat{\beta}}{8} \mathcal{H}^n(B_r(x)) - \mathcal{H}^n(B_r(x) \setminus B_{(1-\beta')r}(x)) \geq \frac{\hat{\beta}}{8} \mathcal{H}^n(B_r(x)) - \frac{\hat{\beta}}{16} \mathcal{H}^n(B_r(x)) = \frac{\hat{\beta}}{16} \mathcal{H}^n(B_r(x)).$$

(3.25)

Uniting with (3.21) for $t \geq -\min\{\frac{1}{2} \hat{\tau}, \frac{1}{8} \hat{\beta}r, \beta' r\}$, we complete the proof of the claim (3.20).

Combining Lemma 2.1 and (3.20), for $t \geq -\min\{\frac{1}{2} \hat{\tau}, \frac{1}{8} \hat{\beta}r, \beta' r\}$ we have

$$\mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{x+r,t}) \geq \mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{(x,t),r-t}) \geq \frac{\delta^*}{r} (\mathcal{H}^n(B_r(x)))^{\frac{1}{n}} (\mathcal{H}^n(U^-_t))^{\frac{n-1}{n}}$$

(3.26)
for some positive constant $\delta^*$ depending only on $C_D, C_N$. Let $V_{x,t}$ be a domain enclosed by $\partial \mathcal{D}_{(x,t), r-t}$ and $M$ in $\mathcal{D}_{(x,t), r-t}$ such that $\partial V_{x,t} \cap \mathcal{D}_{(x,t), r-t} \subset E_t \cup U_t^-$. Since $M$ is area-minimizing in $B_R(p) \times \mathbb{R}$, then

\begin{equation}
\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}) = \mathcal{H}^n(\partial V_{x,t} \cap M) \leq \mathcal{H}^n(\partial V_{x,t} \cap \partial \mathcal{D}_{(x,t), r-t}) + \mathcal{H}^n(U_t^-) \leq \left(\sqrt{2} + 1\right) \mathcal{H}^n(U_t^-).
\end{equation}

Combining (3.26), we have

\begin{equation}
\mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{x,r} + r) \geq \left(\sqrt{2} + 1\right) \frac{1-n}{r} \mathcal{H}^n(B_r(x)) \mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}) \leq \mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{x,r}) \leq \mathcal{H}^n(M \cap \mathcal{D}_{x,r}).
\end{equation}

provided $\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}) \leq \frac{1}{2} \mathcal{H}^n(M \cap \mathcal{D}_{x,r})$.

Proof. Without loss of generality, we can assume $u(x) = 0$. By considering $-u, E_{-t}$ instead of $u, E_t$, respectively, we only need to show the case $E_t = M \cap \mathcal{D}_{x,r}$ for any $t \in (-r, r)$. Let $U_t^-$ be a subset of $B_{r+|t|} \times \{t\}$ defined by

\[ U_t^- = \Omega^- \cap (B_{r+|t|} \times \{t\}), \]

and $U_t^+ = B_{r+|t|} \times \{t\} \setminus U_t^-$. Let $V_{x,t}$ be a domain enclosed by $\partial \left(\mathcal{D}_{(x,t), r+|t|} \setminus \mathcal{D}_{(x,t), r+|t|}\right)$ and $M$ in $\mathcal{D}_{(x,t), r+|t|}$ such that $\partial V_{x,t} \cap \mathcal{D}_{(x,t), r+|t|} \subset (M \setminus E_t) \cup U_t^+$. Since $M$ is area-minimizing in $B_R(p) \times \mathbb{R}$, then

\begin{equation}
\mathcal{H}^n(\partial V_{x,t} \cap M) \leq \mathcal{H}^n(\partial V_{x,t} \cap \partial \mathcal{D}_{(x,t), r+|t|}) + \mathcal{H}^n(U_t^+) \leq \left(\sqrt{2} + 1\right) \mathcal{H}^n(U_t^+).
\end{equation}

Combining $\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}) \leq \frac{1}{2} \mathcal{H}^n(M \cap \mathcal{D}_{x,r})$, we have

\begin{equation}
\mathcal{H}^n(M \cap \mathcal{D}_{x,r}) \leq \mathcal{H}^n(\mathcal{D}_{x,r} \cap M \setminus E_t) \leq \mathcal{H}^n(\partial V_{x,t} \cap M) \leq \left(\sqrt{2} + 1\right) \mathcal{H}^n(U_t^+).
\end{equation}

With (3.14), we have

\begin{equation}
\mathcal{H}^n(U_t^+) \geq \frac{\sqrt{2} - 1}{2} \mathcal{H}^n(B_r(x)).
\end{equation}

Combining Lemma 2.1, we have

\begin{equation}
\mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{x,3r}) \geq \mathcal{H}^{n-1}(\partial E_t \cap \mathcal{D}_{(x,t), r+|t|}) \geq \frac{\epsilon^*}{r} \left(\mathcal{H}^n(B_r(x)) \mathcal{H}^n(U_t^-)\right)^{\frac{\alpha - 1}{\alpha}}
\end{equation}

for some positive constant $\epsilon^*$ depending only on $C_D, C_N$. Let $W_{x,t}$ be a domain enclosed by $\partial \mathcal{D}_{(x,t), r+|t|}$ and $M$ in $\mathcal{D}_{(x,t), r+|t|}$ such that $\partial W_{x,t} \cap \mathcal{D}_{(x,t), r+|t|} \subset E_t \cup U_t^-$. Area-minimizing $M$ in $B_R(p) \times \mathbb{R}$ implies

\begin{equation}
\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}) \leq \mathcal{H}^n(\partial W_{x,t} \cap \partial \mathcal{D}_{(x,t), r+|t|}) + \mathcal{H}^n(U_t^-) \leq \left(\sqrt{2} + 1\right) \mathcal{H}^n(U_t^-).
\end{equation}
From Lemma 3.2, we have
\[
(\mathcal{H}^n(\partial E_t \cap \mathcal{D}_{x,3r}) \geq (\sqrt{2} + 1)^{\frac{1}{1 - \alpha}} \frac{\epsilon_s}{r}(\mathcal{H}^n(B_r(x)))^{\frac{1}{1 - \alpha}}(\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}))^{\frac{\alpha - 1}{\alpha}}.
\]

With (3.14), we complete the proof. \(\square\)

Assume \(u > 0\) on \(B_R(p)\). Let \(\phi\) be a monotonic increasing or monotonic decreasing \(C^1\)-function on \(\mathbb{R}^+\), and \(\Phi(x) = \phi(u(x))\) on \(B_R(p) \subset \Sigma\). For any \(\bar{x} \in \Sigma \times \mathbb{R}\), and \(r > 0\), we put
\[
\mathcal{B}_r(\bar{x}) = M \cap \mathcal{D}_{x,r}.
\]
Let \(\nabla\) be the Levi-Civita connection of \(M\) with respect to its induced metric from \(\Sigma \times \mathbb{R}\). Let us prove Sobolev inequalities for \(\Phi\) on \(M\) using Lemma 3.2.

**Lemma 3.4.** Suppose \(\Phi > 0\) on \(B_R(p)\). There are
\[
(\mathcal{H}^n(\mathcal{B}_r(\bar{x})))^{\frac{1}{2}} \left(\int_{\mathcal{B}_r(\bar{x})} \Phi^{\frac{\alpha}{\alpha - 1}}\right)^{\frac{\alpha - 1}{\alpha}} \leq \theta r \left(\int_{\mathcal{B}_{r+r}(\bar{x})} |\nabla \Phi| + \frac{1}{r} \int_{\mathcal{B}_r(\bar{x})} \Phi\right),
\]
and
\[
(\mathcal{H}^n(\mathcal{B}_r(\bar{x})))^{\frac{1}{2}} \left(\int_{\mathcal{B}_r(\bar{x})} \Phi^{\frac{2\alpha}{\alpha - 1}}\right)^{\frac{\alpha - 1}{\alpha}} \leq \theta \left(r^2 \int_{\mathcal{B}_{r+r}(\bar{x})} |\nabla \Phi|^2 + \frac{2r}{r} \int_{\mathcal{B}_r(\bar{x})} \Phi^2\right)
\]
for all \(r \geq \tau > 0\) and \(B_{2r}(x) \subset B_R(p)\) with \(\bar{x} = (x, u(x)) \in M\), where \(\theta\) is the constant depending only on \(C_D, C_N\) defined in Lemma 3.2.

**Proof.** We fix \(r > 0\) and \(\tau \in (0, r]\). For any \(t \in \mathbb{R}\), we define an open set in \(M\) by
\[
E_t = \{x \in M \cap \mathcal{D}_{x,r+\tau} | \Phi(x) > t\}.
\]
From the monotonicity of \(\phi\), there exists a number \(s_t \in \mathbb{R}\) with \(\phi(s_t) = t\) such that
\[
E_t = \{(q, u(q)) \in M \cap \mathcal{D}_{x,r+\tau} | u(q) > s_t\}
\]
or
\[
E_t = \{(q, u(q)) \in M \cap \mathcal{D}_{x,r+\tau} | u(q) < s_t\}.
\]
From Lemma 3.2
\[
(\mathcal{H}^n(\mathcal{B}_r(x)))^{\frac{1}{2}} (\mathcal{H}^n(E_t \cap \mathcal{D}_{x,r}))^{\frac{\alpha - 1}{\alpha}} \leq \theta \left(r^{\alpha - 1}(\partial E_t \cap \mathcal{D}_{x,r+\tau}) + \frac{r}{\tau} \mathcal{H}^n(E_t \cap \mathcal{D}_{x,r})\right).
\]
for the constant \(\theta\) depending only on \(C_D, C_N\).

By co-area formula,
\[
\int_{M \cap \mathcal{D}_{x,r+\tau}} |\nabla \Phi| = \int_0^\infty \mathcal{H}^n(\partial E_t \cap \mathcal{D}_{x,r+\tau})dt.
\]
By Fubini’s theorem, the \((n + 1)\)-dimensional Hausdorff measure of \(\{(x, t) \in M \cap \mathcal{D}_{x,r} | 0 < t < \Phi(x)\}\) is equal to
\[
\int_{M \cap \mathcal{D}_{x,r}} \Phi = \int_0^\infty \mathcal{H}^n(E_t \cap \mathcal{D}_{x,r})dt.
\]
Then from (3.38) we have
\[
\int_{M \cap \mathcal{D}_{x,r+\tau}} |\nabla \Phi| + \frac{1}{\tau} \int_{M \cap \mathcal{D}_{x,r}} \Phi = \int_0^{\infty} H^{n-1}(\partial E_t \cap \mathcal{D}_{x,r+\tau}) dt + \frac{1}{\tau} \int_0^{\infty} H^n(E_t \cap \mathcal{D}_{x,r}) dt \\
\geq \left( \frac{H^n(\mathcal{B}_r(\bar{x}))}{\theta} \right)^{\frac{1}{\alpha}} H^n(E_t \cap \mathcal{D}_{x,r}) dt.
\]

From a result of Hardy-Littlewood-Pólya (see also the proof of co-area formula in [21]), one gets
\[
\int_{M \cap \mathcal{D}_{x,r+\tau}} |\nabla \Phi| + \frac{1}{\tau} \int_{M \cap \mathcal{D}_{x,r}} \Phi \geq \left( \frac{H^n(\mathcal{B}_r(\bar{x}))}{\theta} \right)^{\frac{1}{\alpha}} \left( \int_0^{\infty} H^n(E_{s \alpha^{-1}} \cap \mathcal{D}_{x,r}) ds \right)^{\frac{\alpha-1}{\alpha}}
\]
\[
= \left( \frac{H^n(\mathcal{B}_r(\bar{x}))}{\theta} \right)^{\frac{1}{\alpha}} \left( \int_{M \cap \mathcal{D}_{x,r}} \Phi^{\frac{\alpha}{\alpha-1}} d\mu \right)^{\frac{\alpha-1}{\alpha}}.
\]

This completes the proof of (3.36).

Note that $\phi^2$ is still a monotonic increasing or monotonic decreasing $C^1$-function on $\mathbb{R}^+$. For any $r \geq \tau$ and $B_{r+\tau}(x) \subset B_R(p)$ with $(x,u(x)) \in M$, from (3.36) and Cauchy inequality we have
\[
(H^n(\mathcal{B}_r(\bar{x})))^{\frac{1}{\alpha}} \left( \int_{\mathcal{B}_r(\bar{x})} \Phi^{\frac{2\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \leq \theta r \left( 2 \int_{\mathcal{B}_{r+\tau}(\bar{x})} \Phi |\nabla \Phi| + \frac{1}{\tau} \int_{\mathcal{B}_r(\bar{x})} \Phi^2 \right)
\]
\[
\leq \theta \left( r^2 \int_{\mathcal{B}_{r+\tau}(\bar{x})} |\nabla \Phi|^2 + \frac{2r}{\tau} \int_{\mathcal{B}_r(\bar{x})} \Phi^2 \right).
\]

This completes the proof of (3.37). \qed

Let us show the following Neumann-Poincaré inequality using Lemma 3.3.

**Lemma 3.5.** Let $\theta^*$ be the constant in Lemma 3.3. Then
\[
\int_{\mathcal{B}_r(\bar{x})} |\Phi - \Phi_{x,r}| \leq 2\theta^* r \int_{\mathcal{B}_{3r}(\bar{x})} |\nabla \Phi|
\]
for all $B_{3r}(x) \subset B_R(p)$ with $(x,u(x)) \in M$, where $\Phi_{x,r}$ is the mean of $\Phi$ on $\mathcal{B}_r(\bar{x})$, i.e.,
\[
\Phi_{x,r} = \left( \int_{\mathcal{B}_r(\bar{x})} \Phi \cdot \frac{1}{H^n(\mathcal{B}_r(\bar{x}))} \right) \Phi.
\]

**Proof.** Let $\Phi_{x,r}$ be the mean of $\Phi$ on $\mathcal{B}_r(\bar{x})$, i.e.,
\[
\Phi_{x,r} = \left( \int_{\mathcal{B}_r(\bar{x})} \Phi \cdot \frac{1}{H^n(\mathcal{B}_r(\bar{x}))} \right) \Phi.
\]
For any fixed $\bar{x} = (x,u(x)) \in M$, $r > 0$ with $B_{3r}(x) \subset B_R(p)$, let
\[
U_{s,t}^+ = \{ y \in \mathcal{B}_s(\bar{x}) \mid \Phi(y) > \Phi_{x,r} + t \},
\]
and
\[
U_{s,t}^- = \{ y \in \mathcal{B}_s(\bar{x}) \mid \Phi(y) < \Phi_{x,r} + t \}.
\]
for all \( s \in (0, 3r] \) and \( t \in \mathbb{R} \). Without loss of generality, we assume \( \mathcal{H}^n(U_{r,t}^+) \leq \mathcal{H}^n(U_{r,0}^-) \). Then

\[
\mathcal{H}^n(U_{r,t}^+) \leq \mathcal{H}^n(U_{r,0}^+) \leq \mathcal{H}^n(U_{r,0}^-) \leq \mathcal{H}^n(U_{r,t})
\]

for any \( t \geq 0 \). In particular,

\[
\mathcal{H}^n(U_{r,t}) \geq \frac{1}{2} \mathcal{H}^n(\mathcal{B}_r(\bar{x})).
\]

From Lemma 3.3 there holds

\[
\mathcal{H}^{n-1}(\partial U_{3r,t}^+ \cap \mathcal{B}_{3r}(\bar{x})) \geq \frac{1}{\theta^*} (\mathcal{H}^n(\mathcal{B}_r(\bar{x})))^{\frac{1}{n}} (\mathcal{H}^n(U_{r,t}^+))^{\frac{n-1}{n}} \geq \frac{1}{\theta^*} \mathcal{H}^n(U_{r,t}^+),
\]

where \( \theta^* \) is the constant in Lemma 3.3. Combining co-area formula, we have

\[
\int_{U_{r,0}^+} (\Phi - \Phi_{x,r}) = \int_0^\infty \mathcal{H}^n(U_{r,t}^+) dt \leq \theta^* r \int_{\mathcal{B}_{3r}(\bar{x})} |\nabla \Phi|.
\]

Combining the definition of \( \Phi_{x,r} \), we get

\[
\int_{\mathcal{B}_{3r}(\bar{x})} |\Phi - \Phi_{x,r}| = \int_{U_{r,0}^+} (\Phi - \Phi_{x,r}) - \int_{U_{r,0}^-} (\Phi - \Phi_{x,r}) = 2 \int_{U_{r,0}^+} (\Phi - \Phi_{x,r}) \leq 2\theta^* r \int_{\mathcal{B}_{3r}(\bar{x})} |\nabla \Phi|.
\]

This completes the proof. \( \square \)

Remark. From the proof of Lemma 3.5 clearly the inequality (3.44) holds without the monotonicity of \( \phi \).

4. HARNACK’S INEQUALITY FOR MINIMAL GRAPHIC FUNCTIONS

For \( R > 0 \), let \( M \) be a minimal graph over \( B_{4R}(p) \subset \Sigma \) with the graphic function \( u \). We always assume that the minimal graphic function \( u \) is not a constant and \( u > 0 \) on \( B_{4R}(p) \). Let \( \tilde{u}(z) = u(x) \) for any \( z = (x, u(x)) \in M \), and we usually denote \( \tilde{u} \) by \( u \), which will not cause confusion from the context in general. Let \( \Delta \) be the Laplacian on \( M \) with respect to its induced metric from \( \Sigma \times \mathbb{R} \). Then \( u \) is harmonic on \( M \) (see also (2.2) in [10]), i.e.,

\[
\Delta u = 0.
\]

Denote \( \tilde{p} = (p, u(p)) \in \Sigma \times \mathbb{R} \). Recall \( \mathcal{B}_r(\tilde{p}) = M \cap \mathcal{D}_{\tilde{p},r} \) and

\[
\mathcal{D}_{\tilde{p},r} = \{(y, s) \in \Sigma \times \mathbb{R} | d(p, y) + |s - u(p)| < r\}.
\]

For any function \( \psi \in L^k(\mathcal{B}_r(\tilde{p})) \) with each \( k > 0 \) and \( r \in (0, 4R) \), we set

\[
||\psi||_{k,r} = \left( \frac{1}{\mathcal{H}^n(\mathcal{B}_r(\tilde{p}))} \int_{\mathcal{B}_r(\tilde{p})} |\psi|^k d\mu \right)^{1/k},
\]

where \( d\mu \) is the volume element of \( M \). Let \( \rho_\tilde{p} \) be a Lipschitz function on \( \Sigma \times \mathbb{R} \) defined by

\[
\rho_\tilde{p}(\bar{x}) = d(x, p) + |t - u(p)|
\]
for any $\bar{x} = (x,t)$. Then $\mathcal{D}_{p,r} = \{ z \in \Sigma \times \mathbb{R} | \rho_{\bar{p}}(z) < r \}$. Let $\nabla$ and $\nabla$ be the Levi-Civita connections of $M$ and $\Sigma \times \mathbb{R}$, respectively. Then

$$|\nabla \rho_{\bar{p}}| \leq |\nabla \rho_{p}| \leq \sqrt{2}.$$  

Let $\phi$ be a monotonic increasing or monotonic decreasing positive $C^1$-function on $\mathbb{R}^+$, and $\Phi(x) = \phi(\eta(x))$ on $B_R(p) \subset \Sigma$. Now, let us carry out De Giorgi-Nash-Moser iteration for getting the Harnack' inequality of $u$ with the help of the Sobolev inequality and the Neumann-Poincaré inequality for the function $\Phi$.

**Lemma 4.1.** Suppose $\Delta \Phi \geq 0$ on $M$. For any $r \in (0,2R]$, there is a constant $c_0$ depending only on $C_D,C_N$ such that

$$||\Phi||_{\infty,\delta r} \leq c_0(1 - \delta)^{\frac{-2}{a+1}}||\Phi||_{k,r}$$

for any $k \geq 2$ and $\delta \in (0,1)$.

**Proof.** For any constant $\ell \geq 1$ and any Lipschitz function $\eta$ with $\text{supp}\eta \subset B_{4R}(p)$, from $\Delta \Phi \geq 0$ we have

$$0 \geq -\int \Phi^{2\ell-1}\eta^2 \Delta \Phi = (2\ell - 1) \int \Phi^{2\ell-2}\eta^2 |\nabla \Phi|^2 + 2 \int \Phi^{2\ell-1}\eta \nabla \Phi \cdot \nabla \eta$$

$$\geq (2\ell - 1) \int \Phi^{2\ell-2}\eta^2 |\nabla \Phi|^2 - \frac{\ell}{2} \int \Phi^{2\ell-2}\eta^2 - \frac{2}{\ell} \int \Phi^{2\ell}|\nabla \eta|^2$$

$$= \left(\frac{3}{2} - 1\right) \int \Phi^{2\ell-2}\eta^2 |\nabla \Phi|^2 - \frac{2}{\ell} \int \Phi^{2\ell}|\nabla \eta|^2,$$

which infers

$$\int |\nabla \Phi|^2 \eta^2 \leq \frac{4\ell}{3\ell - 2} \int \Phi^{2\ell}|\nabla \eta|^2 \leq 4 \int \Phi^{2\ell}|\nabla \eta|^2.$$  

For each $r,\tau > 0$ with $\tau \leq r$, let $\eta$ be a Lipschitz function defined by $\eta = 1$ on $B_{r+\frac{\tau}{2}}(p)$, $\eta = \frac{2}{\tau}(r+\tau-\rho_{\bar{p}})$ on $B_{r+\tau}(p) \setminus B_{r+\frac{\tau}{2}}(p)$, $\eta = 0$ outside $B_{r+\tau}(p)$. Then $|\nabla \eta| \leq \frac{2}{\sqrt{2}}/\tau$ from (4.2). Combining Lemma 3.4 and (4.5), we have

$$\|\Phi^{2\ell}\|_{\frac{a}{a-1},r} \leq \theta \left( r^2 \|\nabla \Phi^{\ell}\|^2 |_{2,r+\frac{\tau}{2}} + \frac{4r}{\tau} \|\Phi^{2\ell}\| |_{1,r+\frac{\tau}{2}} \right) \leq \theta \left( 4r^2 \|\Phi^{2\ell}|\nabla \eta|^2 + \frac{4r}{\tau} \|\Phi^{2\ell}\| |_{1,r+\tau} \right)$$

$$\leq \theta \left( 32 \frac{r^2}{\tau^2} \|\Phi^{2\ell}\| |_{1,r+\tau} + \frac{4r}{\tau} \|\Phi^{2\ell}\| |_{1,r+\tau} \right) \leq c_0 \frac{r^2}{\tau^2} \|\Phi^{2\ell}\| |_{1,r+\tau}$$

with $c_0 = 36\theta$, where $\theta$ is the constant depending only on $C_D,C_N$ defined in Lemma 3.2.

Then one has

$$\|\Phi\|_{\frac{a}{a-1},r} \leq c_0 \frac{1}{r^{\frac{1}{a-1}}} \left( \tau^{-\frac{1}{a-1}} \right)^{\frac{1}{a-1}} \|\Phi\|_{2\ell,r+\tau}.$$  

For any $\delta \in (0,1)$, $k \geq 2$ and any integer $i \geq -1$, set $\ell_i = \left( \frac{2}{a-1} \right)^i$, $\tau_i = 2^{-(1+i)}(1-\delta)r$ and $r_i = r - \sum_{j=0}^{i} \tau_j = \delta r + \tau_i \leq r$. By iterating (4.6), for $i \geq 0$ we have

$$\|\Phi\|_{\frac{a}{a-1},\ell_i,r_i} \leq c_\theta \tau_{-1}^{\frac{1}{a-1}} \|\Phi\|_{\ell_i-1,r_i-1} \leq \prod_{j=0}^{i} c_\theta \tau_{j}^{\frac{1}{a-1}} \|\Phi\|_{k,r}.$$
Note that \( \tau_j/r_j \geq 2^{-(1+j)(1-\delta)} \) for every \( j \geq 0 \). Letting \( i \to \infty \), then
\[
||\Phi||_{\infty,\delta r} \leq \prod_{j=0}^{\infty} c_g^{\frac{1}{\alpha}} \left( \frac{2^{1+j}}{(1-\delta)} \right)^{\frac{1}{\alpha}} \sum_{j=0}^{\infty} \frac{2^{j+i}}{(1-\delta)^{j+i}} \frac{1}{\alpha} ||\Phi||_{k, r} = c_g \sum_{j=0}^{\infty} 2^{j+i} \sum_{j=0}^{\infty} \frac{1}{\alpha} (1-\delta)^{j} - \sum_{j=0}^{\infty} \frac{1}{\alpha} ||\Phi||_{k, r}.
\]

By the definition of \( \ell_j \), it follows that
\[
\sum_{j=0}^{\infty} \frac{1}{\ell_j} = \frac{1}{k} \sum_{j=0}^{\infty} \left( \frac{\alpha - 1}{\alpha} \right)^j = \frac{\alpha}{k}.
\]

Hence from (4.8) we get
\[
||\Phi||_{\infty,\delta r} \leq 2^{\frac{2\alpha}{\alpha}} c_g \left( 1 - \delta \right)^{-\frac{2\alpha}{\alpha}} ||\Phi||_{k, r}.
\]

This completes the proof. \( \square \)

**Theorem 4.2.** Suppose \( \Delta \Phi \geq 0 \) on \( M \). For any \( k > 0 \), there is a constant \( c_k \) depending only on \( k, C_D, C_N \) such that for any \( r \in (0, 2R) \) one has
\[
\sup_{B_r(\bar{p})} \Phi \leq c_k (1 - \delta)^{-\frac{2\alpha}{\alpha}} ||\Phi||_{k, r}.
\]

**Proof.** From (4.3), we only need to show (4.11) for \( 0 < k < 2 \). From (4.3), we have
\[
||\Phi||_{\infty,\delta r} \leq c_0 (1 - \delta)^{-\alpha} ||\Phi||_{2, r}.
\]

Put \( r_0 = \delta r, r_i = \delta r + \sum_{j=1}^{i} 2^{-j}(1-\delta)r = r - 2^{-i}(1-\delta)r \), and \( \delta_i = r_{i-1}/r_i \). Then
\[
1 - \delta_i = \frac{r_{i-1}}{r_i} = \frac{2^{-i}(1-\delta)r}{r - 2^{-i}(1-\delta)r} \geq 2^{-i}(1-\delta).
\]

Note that \( \frac{1+\delta}{2} \leq r_i \leq r \) for all \( i \geq 1 \). Then from (2.11) (3.14) we have
\[
\mathcal{H}^n(B_r(\bar{p})) \leq 2^{\alpha+\frac{1}{2}} C_D \mathcal{H}^n \left( B_{\frac{r}{2}}(p) \right) \leq 2^{\alpha+\frac{1}{2}} C_D \delta^{-1}\mathcal{H}^n(B_r(\bar{p})).
\]

From (4.12), for \( i \geq 1 \) we have
\[
||\Phi||_{\infty, r_{i-1}} \leq c_0 (1 - \delta_i)^{-\alpha} ||\Phi||_{2, r_i} \leq c_0 (1 - \delta_i)^{-\alpha} ||\Phi||_{\frac{1}{k, r_i}} ||\Phi||_{\frac{1}{\infty, r_i}} \leq c_0 2^{\alpha}(1 - \delta)^{-\alpha} \left( 2^{\alpha+\frac{1}{2}} C_D \delta^{-1} \right)^{\frac{1}{2}} ||\Phi||_{\frac{1}{k, r}} ||\Phi||_{\frac{1}{\infty, r_i}}.
\]

Set \( \tilde{c}_0 = c_0 \left( 2^{\alpha+\frac{1}{2}} C_D \delta^{-1} \right)^{\frac{1}{2}} \), which is a constant depending only on \( C_D, C_N \). Iterating the above inequality implies
\[
||\Phi||_{\infty, r_0} \leq \tilde{c}_0 2^{\alpha}(1 - \delta)^{-\alpha} ||\Phi||_{\frac{1}{k, r}} ||\Phi||_{\frac{1}{\infty, r_1}}^{1-\frac{\alpha}{2}}
\]
\[
\leq \prod_{j=0}^{i} \left( \tilde{c}_0 2^{(j+1)\alpha}(1 - \delta)^{-\alpha} ||\Phi||_{\frac{1}{k, r}} ||\Phi||_{\frac{1}{\infty, r_j}} \right)^{(1-k/2)^j} ||\Phi||_{\frac{1}{\infty, r_{i+1}^{i+1}}}.}
\]
By a direct computation, one has
\[ \sum_{j=0}^{\infty} \left( 1 - \frac{k}{2} \right)^j = \frac{2}{k}, \]
(4.15)
\[ \sum_{j=0}^{\infty} (j+1) \left( 1 - \frac{k}{2} \right)^j = \frac{4}{k^2}. \]

Letting \( i \to \infty \) in (4.14) infers
\[ ||\Phi||_{\infty, \delta r} \leq \left( \prod_{j=0}^{\infty} \left( \tilde{c}_0 (1 - \delta)^\alpha (1 - k/2)^j \right) \right)^{2}\]
(4.16)
\[ = \left( \prod_{j=0}^{\infty} \left( \tilde{c}_0 (1 - \delta)^\alpha (1 - k/2)^j \right) \right) 2^\alpha \sum_{j=0}^{\infty} (j+1)(1-k/2)^j ||\Phi||_{k,r} \]
\[ = 2^{\frac{4\alpha}{k}} \tilde{c}_0 \frac{2}{k} (1 - \delta) - \frac{2\alpha}{k} ||\Phi||_{k,r}. \]

This completes the proof. \( \square \)

**Theorem 4.3.** Suppose \( u > 0 \) on \( B_{4R}(p) \). Then \( u \) satisfies Harnack’s inequality on \( B_{2R}(\bar{p}) \):
\[ \sup_{B_{2R}(\bar{p})} u \leq \vartheta \inf_{B_{2R}(\bar{p})} u \]
(4.17)
for some constant \( \vartheta \) depending only on \( C_D, C_N \).

**Proof.** Let
\[ w = \log u - \int_{S_r(p)} \log u, \]
then from (4.11) one has
\[ \Delta w = -|\nabla w|^2. \]

Let \( \eta \) be a Lipschitz function with compact support in \( M \cap B_{4R}(p) \). From (4.18), for any \( q \geq 0 \) integrating by parts implies
\[ \int |\nabla w|^2 |\eta|^2 |w|^q = - \int \eta^2 |w|^q \Delta w = 2 \int \eta |w|^q \nabla \eta \cdot \nabla w + q \int \eta^2 |w|^{q-2} w |\nabla w|^2 \]
(4.19)
\[ \leq \frac{1}{2} \int |\nabla w|^2 \eta^2 |w|^q + 2 \int |\nabla \eta|^2 |w|^q + q \int \eta^2 |w|^{q-1} |\nabla w|^2. \]

Then
\[ \int \eta^2 |w|^q |\nabla w|^2 \leq 4 \int |\nabla \eta|^2 |w|^q + 2q \int \eta^2 |w|^{q-1} |\nabla w|^2. \]
(4.20)

For any \( r \in (0, R) \), if we choose \( \eta_0 = 1 \) on \( B_{3r}(\bar{p}) \), \( \eta_0 = \frac{4r - \rho_0}{r} \) on \( B_{4r}(\bar{p}) \setminus B_{3r}(\bar{p}) \), \( \eta_0 = 0 \) on \( M \setminus B_{4r}(\bar{p}) \). Then combining (4.12), we get \( |\nabla \eta_0| \leq \sqrt{2}/r \). Choosing \( q = 0 \) in (4.20), we have
\[ \int_{B_{3r}(\bar{p})} |\nabla w|^2 \leq 4 \int |\nabla \eta_0|^2 \leq \frac{8}{r^2} \hat{H}^n(B_{4r}(\bar{p})). \]
(4.21)
Combining the Neumann-Poincaré inequality \((4.24)\) for \(w\), we have
\[
\int_{B_r(\bar{\rho})} |w| \leq 2\theta^* r \int_{B_{3r}(\bar{\rho})} |\nabla w|
\]
\[
(4.22)
\]
\[
\leq 2\theta^* r (\mathcal{H}^n(B_{3r}(\bar{\rho})))^{\frac{1}{2}} \left( \int_{B_r(\bar{\rho})} |\nabla w|^2 \right)^{\frac{1}{2}} \leq 4\sqrt{2}\theta^* \mathcal{H}^n(B_{4r}(\bar{\rho})).
\]

We choose \(\eta_1 = 1\) on \(B_{\frac{4}{3}r}(\bar{\rho})\), \(\eta_1 = \frac{4-r}{r} \eta_2 + \frac{r}{4-r} \eta_1\) on \(B_r(\bar{\rho}) \setminus B_{\frac{4}{3}r}(\bar{\rho})\), \(\eta_1 = 0\) on \(M \setminus B_r(\bar{\rho})\). Then we get \(|\nabla \eta_1| \leq 4\sqrt{2}/r\) from \((4.2)\). Without loss of generality, we may assume \(\theta^* \geq 1\).

Choosing \(q = 1\) in \((4.20)\), combining \((4.21)\) \((4.22)\) we have
\[
(4.23)
\]
\[
\int_{B_{\frac{4}{3}r}(\bar{\rho})} |w| |\nabla w|^2 \leq \frac{2^7}{r^2} \int_{B_r(\bar{\rho})} |w| + 2 \int_{B_r(\bar{\rho})} |\nabla w|^2 \leq \frac{2^{10}}{r^2} \theta^* \mathcal{H}^n(B_{4r}(\bar{\rho})).
\]

Let \(r_j = \frac{1}{2}(1 + 2^{-j})r\) for each integer \(j \geq 0\). Let \(\eta\) be the cut-off function on \(B_{r_j}(\bar{\rho})\) such that \(\eta = 1\) on \(B_{r_j}(\bar{\rho})\), \(\eta = \frac{r_j-r_{j+1}}{r_j-r_{j+1}}\) on \(B_{r_j}(\bar{\rho}) \setminus B_{r_{j+1}}(\bar{\rho})\), \(\eta = 0\) on \(M \setminus B_{r_j}(\bar{\rho})\). Then combining \((4.2)\) we get \(|\nabla \eta| \leq 2^{j+\frac{5}{2}}/r\). From \((4.20)\), for any number \(q \geq 0\) and any integer \(j \geq 0\) we have
\[
(4.24)
\]
\[
\int_{B_{r_j}(\bar{\rho})} \eta^2 |w|^q |\nabla w|^2 \leq \frac{2^{j+7}}{r^2} \int_{B_{r_j}(\bar{\rho})} |w|^q + 2q \int_{B_{r_j}(\bar{\rho})} \eta^2 |w|^{q-1} |\nabla w|^2.
\]

From Young’s inequality, one has
\[
(4.25)
\]
\[
2q |w|^{q-1} \leq \frac{1}{2} |w|^q + 2^{q-1}(q - 1)^{q-1} \quad \text{for } q \geq 1,
\]
\[
|w|^q \leq q |w| + (1-q) \quad \text{for } q \in [0,1).
\]

Here, we let \(0^0 = 1\) for the case \(q = 1\). Then for \(j \geq 0\) and \(q \geq 1\), substituting \((4.25)\) into \((4.24)\) infers
\[
(4.26)
\]
\[
\frac{1}{2} \int_{B_{r_{j+1}}(\bar{\rho})} |w|^q |\nabla w|^2 \leq \frac{2^{j+7}}{r^2} \int_{B_{r_j}(\bar{\rho})} |w|^q + 2^{q-1}(q - 1)^{q-1} \int_{B_{r_j}(\bar{\rho})} |\nabla w|^2 \leq \frac{2^{j+7}}{r^2} \int_{B_{r_{j+1}}(\bar{\rho})} |w|^q + 2^{2q+2}(q - 1)^{q-1} - 2\mathcal{H}^n(B_{4r}(\bar{\rho})),
\]

where we have used \((4.21)\) in the above inequality. Combining Cauchy inequality, we have
\[
(4.27)
\]
\[
\int_{B_{r_{j+1}}(\bar{\rho})} |w|^q |\nabla w| \leq \frac{r}{2^{j+5}} \int_{B_{r_{j+1}}(\bar{\rho})} |w|^q |\nabla w|^2 + \frac{2^{j+3}}{r} \int_{B_{r_{j+1}}(\bar{\rho})} |w|^q \leq \frac{2^{j+4}}{r} \int_{B_{r_j}(\bar{\rho})} |w|^q + 2^{2q-j-2}(q - 1)^{q-1} - 2\mathcal{H}^n(B_{4r}(\bar{\rho})),
\]

for \(q \geq 1\) and \(j \geq 0\). Moreover, for \(j \geq 0\) and \(0 \leq q < 1\), combining \((4.24)\) \((4.26)\) \((4.25)\) one has
\[
(4.28)
\]
\[
\int_{B_{r_{j+1}}(\bar{\rho})} |w|^q |\nabla w|^2 \leq q \int_{B_{r_j}(\bar{\rho})} |w||\nabla w|^2 + (1-q) \int_{B_{r_j}(\bar{\rho})} |\nabla w|^2 \leq \frac{2^{10}}{r^2} \theta^* \mathcal{H}^n(B_{4r}(\bar{\rho})) + (1-q) \frac{8}{r^2} \mathcal{H}^n(B_{4r}(\bar{\rho})) \leq \frac{2^{10}}{r^2} \theta^* \mathcal{H}^n(B_{4r}(\bar{\rho})).
\]
Then with Cauchy inequality, we have

\[
\int_{B_{rj+1}(\bar{\rho})} |w|^q |\nabla w| \leq \frac{r}{2j+6} \int_{B_{rj+1}(\bar{\rho})} |w|^q |\nabla w|^2 + \frac{2j+4}{r} \int_{B_{rj+1}(\bar{\rho})} |w|^q
\]

(4.29)

\[
\leq \frac{2j+4}{r} \int_{B_{rj+1}(\bar{\rho})} |w|^q + 2^4 \theta^q r^{-1} H^n(B_{4r}(\bar{\rho}))
\]

for \(0 \leq q < 1\) and \(j \geq 0\). Combining (4.27) and (4.29), we get

\[
\int_{B_{rj+1}(\bar{\rho})} |w|^q |\nabla w| \leq \frac{2j+4}{r} \int_{B_{rj}(\bar{\rho})} |w|^q + 2^{2q+4-j}(q+1)^q \theta^q r^{-1} H^n(B_{4r}(\bar{\rho}))
\]

(4.30)

for \(q \geq 0\) and \(j \geq 0\).

Note that from Young’s inequality (4.25), for \(q \geq 0\) one has

\[
2^q (q+1)|w|^q \leq |w|^{q+1} + 2^{2q+2} q^q.
\]

Note \(r_j \leq r\) for all \(j \geq 0\). Combining Lemma 3.4 and (4.30), for \(j \geq 0\) and \(q \geq 0\), we have

\[
(H^n(B_{rj+2}(\bar{\rho})))^\frac{1}{n} \left( \int_{B_{rj+2}(\bar{\rho})} |w|^{(q+1)\alpha \alpha^{-1}} \right)^\frac{\alpha^{-1}}{\alpha}
\]

\[
\leq \theta r_{j+2} \left( (q+1) \int_{B_{rj+2}(\bar{\rho})} |w|^q |\nabla w| + \frac{2^j+3}{r} \int_{B_{rj+2}(\bar{\rho})} |w|^{q+1} \right)
\]

\[
\leq \theta 2^{j+3} \left( 2(q+1) \int_{B_{rj}(\bar{\rho})} |w|^q + 2^{2q+1}(q+1)^q \theta^q r^{-1} H^n(B_{4r}(\bar{\rho})) + \int_{B_{rj+2}(\bar{\rho})} |w|^{q+1} \right)
\]

\[
\leq \theta 2^{j+4} \left( \int_{B_{rj}(\bar{\rho})} |w|^{q+1} + 2^{2q+2}(q+1)^q \theta^q H^n(B_{4r}(\bar{\rho})) \right).
\]

In other words, there exists a constant \(c_*\) depending only on \(C_D, C_N\) such that

\[
\left\| |w|^q \right\|_{\frac{\alpha}{\alpha-1}, r_{j+2}^j} \leq c_* 2^j \left\| |w|^q + 2^{2q} q^q \right\|_{\alpha, \alpha-1, r_{j+2}^j}
\]

(4.33)

for any \(j \geq 0\), \(q \geq 1\), which implies

\[
\left\| |w| \right\|_{\alpha-1, r_{j+2}^j} \leq (c_* 2^j)^\frac{1}{q} \left( \left\| |w| \right\|_{\alpha, \alpha-1, r_{j+2}^j} + 4q \right).
\]

(4.34)

Let \(q_j = (\frac{\alpha}{\alpha-1})^j\) and \(a_j = \left\| |w| \right\|_{q_j, r_{j+2}} / q_j\) for \(j \geq 0\). Then from (4.34) we have

\[
\left\| |w| \right\|_{q_j, r_{j+2}} \leq c_* 2^j \left( \left\| |w| \right\|_{q_j, r_{j+2}} + 4q_j \right),
\]

(4.35)

and

\[
a_{j+1} \leq \frac{\alpha-1}{\alpha} c_* 2^j \left( a_j + 4 \right)
\]

(4.36)

for every \(j \geq 0\). There is an integer \(j_* > 0\) depending only on \(C_D, C_N\) such that \(\frac{1}{\alpha} 2^{\frac{1}{\alpha}} \leq \frac{\alpha+1}{\alpha} \) for all \(j \geq j_*\). From (4.36), we get

\[
a_{j+1} \leq \frac{\alpha^2 - 1}{\alpha^2} (a_j + 4) \quad \text{for} \quad j \geq j_*,
\]

(4.37)
which implies
\[ a_{j+1} \leq \max\{a_j, 4(\alpha^2 - 1)\} \quad \text{for } j \geq j_*, \]
Namely, \( \max\{a_j, 4(\alpha^2 - 1)\} \) is monotonic nonincreasing for \( j \geq j_* \). Note that \( a_0 \) is bounded by a constant depending only on \( C_D, C_N \) from (4.22). Hence there is a constant \( c^* \) depending only on \( C_D, C_N \) such that
\[ a_j = \|w\|_{q_j, r_2j} \leq c^* \]
for all \( j \geq 0 \). For each integer \( k \geq 1 \), there is an integer \( j_k \geq 0 \) such that \( q_{j_k} \leq k \leq q_{j_k+1} \). With Hölder inequality, we have
\[ \|w\|_{k, r_{\infty}} \leq \|w\|_{k, r_{2j_k+2}} \leq \|w\|_{q_{j_k+1}, r_{2j_k+2}} \leq c^* q_{j_k+1} \leq \frac{c^* \alpha}{\alpha - 1} k, \]
where \( r_{\infty} = \lim_{i \to \infty} r_i = \frac{\alpha}{2} \). Therefore, by Stirling’s formula
\[ \|w\|_{k, r_{\infty}} \leq \left(\frac{c^* \alpha}{\alpha - 1}\right)^k k^{\frac{\alpha}{2}} \leq \left(\frac{c^* \alpha}{\alpha - 1}\right)^k e^{k - \frac{1}{2} k!}, \]
which implies
\[ \frac{1}{k!} \|w\|_{k, r_{\infty}} \leq \left(\frac{c^* \alpha}{\alpha - 1}\right)^k k^{\frac{\alpha}{2}} \leq \frac{1}{k^{\frac{1}{2}}}. \]
Hence, there are constants \( \lambda_* \in (0, \frac{\alpha-1}{c^* \alpha}) \) and \( C_* > 0 \) depending only on \( C_D, C_N \) such that
\[ \int_{\mathcal{B}_2^\alpha} e^{\lambda_* w} \int_{\mathcal{B}_2^\alpha} e^{-\lambda_* w} \leq C_* \]
Namely,
\[ \int_{\mathcal{B}_2^\alpha} u^{\lambda_*} \int_{\mathcal{B}_2^\alpha} u^{-\lambda_*} \leq C_* \]
Combining Theorem 4.2 we have
\[ \sup_{\mathcal{B}_2^\alpha} u^{\lambda_*} \leq \left(\frac{\lambda_*^2}{\alpha} \right)^\lambda_\ast \|u\|_{\lambda_\ast, \mathcal{B}_2^\alpha} \leq 2^{2\alpha} C_\ast C_\lambda^\lambda_\ast \left(\int_{\mathcal{B}_2^\alpha} u^{-\lambda_*}\right)^{-\frac{1}{2}}. \]
Since
\[ \Delta u^{-\lambda_*} = \lambda_\ast (\lambda_\ast + 1) u^{-\lambda_* - 2} |\nabla u|^2 \geq 0, \]
then combining Theorem 4.2 and (4.45) we have
\[ \sup_{\mathcal{B}_2^\alpha} u^{-\lambda_*} \leq C_1 2^{\alpha} \int_{\mathcal{B}_2^\alpha} u^{-\lambda_*} \leq 2^{4\alpha} C_1 C_\ast C_\lambda^\lambda_\ast \left(\sup_{\mathcal{B}_2^\alpha} u^{\lambda_*}\right)^{-\frac{1}{2}}, \]
which implies
\[ \sup_{\mathcal{B}_2^\alpha} u \leq (2^{4\alpha} C_1 C_\ast)^{1/\lambda_\ast} C_\lambda^\lambda_* \inf_{\mathcal{B}_2^\alpha} u. \]
Hence for any \( z \in \mathcal{B}_2^\alpha \) we have
\[ \sup_{\mathcal{B}_2^\alpha(z)} u \leq (2^{4\alpha} C_1 C_\ast)^{1/\lambda_\ast} C_\lambda^\lambda_* \inf_{\mathcal{B}_2^\alpha(z)} u. \]
This completes the proof.

Using Theorem 4.17, we can show the following Liouville type theorem for the minimal graphic function \( u \) by considering \( u - \inf_{\Sigma} u \).

**Theorem 4.4.** Let \( \Sigma \) be an \( n \)-dimensional complete Riemannian manifold with (1.3) and (1.4). If \( u \) is a positive minimal graphic function on \( \Sigma \), then \( u \) is a constant.

As a corollary, if \( \Sigma \) is an open manifold which is quasi isometric to an open manifold with nonnegative Ricci curvature, then any positive minimal graphic function on \( \Sigma \) is a constant.

**Remark 4.5.** The Harnack’s inequality (4.17) implies Hölder continuity of the solution \( u \) (see [18] for instance). Namely, there is a constant \( \delta > 0 \) depending only on \( C_D, C_N \) such that for any entire minimal graphic function \( u \) on \( \Sigma \) satisfies

\[
\sup_{B_r(p)} u - \inf_{B_r(p)} u \leq \frac{1}{\delta} \left( \frac{r}{R} \right)^\delta \left( \sup_{B_R(p)} u - \inf_{B_R(p)} u \right)
\]

for all \( 0 < r < R < \infty \). Therefore, the above Theorem can be improved somewhat to allow

\[
\limsup_{R \to \infty} R^{-\delta} \max \left\{ - \inf_{B_R(p)} u, 0 \right\} = 0.
\]

5. **Appendix**

Let \( \Sigma \) be an \( n \)-dimensional complete manifold with Riemannian metric \( \sigma \) and the Levi-Civita connection \( D \). Suppose that \( \Sigma \) satisfies the volume doubling property (1.3) and the uniform Neumann-Poincaré inequality (1.4). Now let us show an isoperimetric inequality on \( \Sigma \).

Let \( \Omega \) be an open set in \( B_r(p) \) with rectifiable boundary. For any \( x \in \Omega \setminus \partial \Omega \), there is a constant \( r_x > 0 \) such that

\[
\mathcal{H}^n(\Omega \cap B_{r_x}(x)) = \frac{1}{2} \mathcal{H}^n(B_{r_x}(x)).
\]

From (2.1), we have

\[
\mathcal{H}^n(\Omega) \geq \frac{1}{2} \mathcal{H}^n(B_{r_x}(x)) \geq \frac{1}{2C_D} \mathcal{H}^n(B_{2r}(x)) \left( \frac{r_x}{2r} \right)^\alpha \geq \frac{1}{2C_D} \mathcal{H}^n(B_r(p)) \left( \frac{r_x}{2r} \right)^\alpha,
\]

which implies

\[
r_x \leq 2r \left( \frac{2C_D \mathcal{H}^n(\Omega)}{\mathcal{H}^n(B_r(p))} \right)^\frac{1}{\alpha}.
\]
By 5-lemma, there is a sequence of points \( x_i \in \Omega \setminus \partial \Omega \) such that \( \Omega \subset \bigcup_i B_{r_{x_i}}(x_i) \) and \( B_{r_{x_i}}(x_i) \) are mutually disjoint. Then combining (2.1)(2.5)(5.1)(5.3), we get

\[
\mathcal{H}^n(\Omega) \leq \sum_i \mathcal{H}^n(B_{5r_{x_i}}(x_i)) \leq 5^n C_D \sum_i \mathcal{H}^n(B_{r_{x_i}}(x_i)) = 2 \times 5^n C_D \sum_i \mathcal{H}^n(\Omega \cap B_{r_{x_i}}(x_i)) \leq 2 \times 5^n C_D C_N \sum_i r_{x_i} \mathcal{H}^{n-1}(\partial \Omega \cap B_{r_{x_i}}(x_i))
\]

\[
\leq 4 \times 5^n C_D C_N r \left( \frac{2C_D \mathcal{H}^n(\Omega)}{\mathcal{H}^n(B_r(p))} \right)^{\frac{1}{n}} \sum_i \mathcal{H}^{n-1}(\partial \Omega \cap B_{r_{x_i}}(x_i)) \leq 4 \times 5^n C_D C_N r \left( \frac{2C_D \mathcal{H}^n(\Omega)}{\mathcal{H}^n(B_r(p))} \right)^{\frac{1}{n}} \mathcal{H}^{n-1}(\partial \Omega).
\]

Note \( \alpha \geq 2 \) from (2.2). Then the above inequality implies an isoperimetric inequality:

\[
(\mathcal{H}^n(\Omega))^{\frac{1}{n}} \leq 8 \times 5^n C_D C_N \left( \frac{\mathcal{H}^n(B_r(p))}{\mathcal{H}^n(B_r(\partial \Omega))} \right)^{-\frac{1}{n}} r \mathcal{H}^{n-1}(\partial \Omega).
\]

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