ON THE MGT EQUATION WITH MEMORY OF TYPE II

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Abstract. We consider the Moore-Gibson-Thompson equation with memory of type II
\[ \partial_{ttt}u(t) + \alpha \partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(t-s)A\partial_t u(s)ds = 0 \]
where \( A \) is a strictly positive selfadjoint linear operator (bounded or unbounded) and
\( \alpha, \beta, \gamma > 0 \) satisfy the relation \( \gamma \leq \alpha \beta \). First, we prove a well-posedness result without
requiring any restriction on the total mass \( \hat{\gamma} \) of \( g \). Then we show that it is always possible
to find memory kernels \( g \), complying with the usual mass restriction \( \hat{\gamma} < \beta \), such that
the equation admits solutions with energy growing exponentially fast. In particular, this
provides the answer to a question raised in [2].

1. Introduction

Let \((H, \langle \cdot, \cdot \rangle, \| \cdot \|)\) be a separable real Hilbert space, and let
\[ A : \mathcal{D}(A) \subset H \to H \]
be a strictly positive selfadjoint linear operator (bounded or unbounded). We consider
the Moore-Gibson-Thompson (MGT) equation with memory of type II
\begin{equation}
\partial_{ttt}u(t) + \alpha \partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(t-s)A\partial_t u(s)ds = 0,
\end{equation}
where \( \alpha, \beta, \gamma \) are strictly positive fixed constants subject to the structural constraint
\begin{equation}
\gamma \leq \alpha \beta,
\end{equation}
and the so-called memory kernel \( g : [0, \infty) \to [0, \infty) \) is an absolutely continuous nonin-
creasing function of total mass
\[ g = \int_0^\infty g(s)ds > 0. \]

The MGT equation without memory, i.e.
\begin{equation}
\partial_{ttt}u + \alpha \partial_{tt}u + \beta A\partial_t u + \gamma Au = 0,
\end{equation}
is a model arising in acoustics and accounting for the second sound effects and the associ-
ated thermal relaxations in viscous fluids [5, 13, 16, 17]. The case \( \gamma < \alpha \beta \) is referred to as
subcritical, since in this regime the associated solution semigroup exhibits an exponential
decay in the natural weak energy space
\[ \mathcal{H} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times H. \]

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ially growing solutions.
On the contrary, the case $\gamma = \alpha \beta$ is critical, since stability (even the nonuniform one) is lost [7, 12]. Finally, in the supercritical case $\gamma > \alpha \beta$, there exist trajectories whose energy blows up exponentially [3, 7, 12]. If additional molecular relaxation phenomena are taken into account, integral terms pop up in the MGT equation, leading to (1.1) with a nonnull memory kernel [6, 8, 9, 11, 14]. In more generality, the convolution term appearing in (1.1) can be taken of the form $\int_0^t g(t-s) A w(s) \, ds$, where the variable $w$ is of the following three types:

$$w(s) = \begin{cases} 
  u(s) & \text{(type I)}, \\
  \partial_t u(s) & \text{(type II)}, \\
  ku(s) + \partial_t u(s) & \text{for } k > 0 \text{ (type III)}. 
\end{cases}$$

For the memory of type I, the picture is quite understood. As shown in [8], in the subcritical case and under proper decay assumptions on the kernel $g$ all the solutions converge exponentially to zero. Instead, in the critical case the decay is only strong, with a counterexample to exponential stability if the operator $A$ is unbounded [2]. An interesting question becomes what is the effect of the memory of type II. Here, in the subcritical case and with strong restrictions on the mass of memory kernel including $\varrho < \beta$, one shows the exponential decay of the energy [8]. In the critical case, exponential stability holds, but with a very special choice of memory of type III, namely $w = \gamma \beta^{-1} u + \partial_t u$.

It is worth noting that in all the results mentioned above a structural restriction on $\varrho$ is required. For the case of memory of type II, such a restriction reads

$$\varrho < \beta.$$ 

To better understand this issue, an interesting comparison can be made with the MGT equation without memory (1.3), which is shown to be ill-posed in $H$ if $A$ is unbounded and $\beta = 0$ (but the same is true if $\beta \leq 0$), in the sense that the equation does not generate a strongly continuous semigroup (see [7]). And indeed, equation (1.3) with $(\beta - \varrho) A \partial_t u$ in place of the term $\beta A \partial_t u$ can be considered the limiting case of (1.1) when the kernel $g$ converges to a multiple $\varrho$ of the Dirac mass at 0. This would somehow indicate that some problems might arise when $\varrho \geq \beta$. Quite unexpectedly, as it will be shown in this work, it is instead possible to have existence and uniqueness of solutions in the natural weak energy space $H$, no matter how is the size of $\varrho$. In fact, the same picture occurs for the MGT equation with memory of type I or III.

A second intriguing problem is to fully understand the effects of the memory of type II on the longtime dynamics, within the restriction $\varrho < \beta$ (indeed, if $\varrho \geq \beta$, blow up at infinity is the general rule). In particular, whether this damping alone is able to stabilize the equation in the critical case. As we shall see in this paper, the answer is negative. Even more is true: a subcritical MGT equation can be exponentially destabilized by “large” effects of the memory of type II. Hence, a posteriori, for such an equation we may say that no critical value changing the asymptotic dynamics exists, in the sense that blow up of solutions appears, both in the subcritical and in the critical regimes. In particular, this provides an answer to a question raised in [2]. Actually, in the same paper [2] a heuristic explanation was given, by noting that the action of the memory of type II can be interpreted as an addition of a “stabilizer” and of an “antidamper” to the MGT
equation. To wit, observe that

\[-\int_0^t g(t - s)A\partial_t u(s)ds = -g(0)Au(t) + g(t)Au(0) - \int_0^t g'(t - s)Au(s)ds.\]

The above formula indicates that the memory of type II provides two opposite effects. The static damping term \(g(0)Au\) moves the original critical value \(\gamma = \alpha\beta\) to the noncritical region \(\gamma = \alpha\beta - g(0)\). On the other hand, the viscoelastic term \(-\int_0^t g'(t - s)Au(s)ds\) acts as an “antidamper”, due to the negative sign of \(g'\). This renders the issue quite interesting, as it is not clear which “damping” wins the game. Of course, the value \(g(0)\) and \(g'(t)\) will play a crucial role.

**Comparison with the previous literature.** The recent paper [1] is concerned with the existence, uniqueness and stability of the MGT equation with infinite memory, i.e. with a more general convolution term of the form \(\int_{-\infty}^t g(t - s)Aw(s)ds\). In the case of memory of type I, Theorem 3.7 therein proves the exponential decay of solutions in the subcritical case. This result has been already shown for finite memory of type I in [8] (see also [9] for more general relaxation kernels leading to uniform but not exponential decays). In the case of memory of type II, the same [1, Theorem 3.7] establishes the exponential decay of the energy in the subcritical case, but under strong “smallness” type restrictions imposed on the mass of kernel \(\varrho\). Here, again, this is an extension to infinite memory of the results obtained in [8]. In short, this “smallness” condition requires a rather fast decay of the kernel \(g\) with respect to the strictly positive value \(\alpha\beta - \gamma\). For exponentially decaying kernels of the form \(g(t) = \varrho\delta e^{-\delta t}\) with \(\delta > 0\), this condition translates into \(\varrho < \beta - \gamma\alpha^{-1}\).

Regarding the negative result in the case of memory of type II (conjectured in [2]), the paper [1] evokes the lack of dissipativity of the generator for larger values of \(\varrho\). One should note that dissipativity is a property of the considered inner product and, alone, cannot prove the conjecture stated in [2], i.e. to disprove exponential stability. In summary, the analysis carried out in [1] is inconclusive with respect to the open question under consideration.

Coming instead to the MGT equation with memory of type II in the critical case, it is still unknown whether exponential stability could be achieved with suitably calibrated relaxation kernel (fast decay and small mass). Positive results are available in the literature (see [1, 8]), but only in the subcritical regime.

**Notation.** We define the family of nested Hilbert spaces depending on a parameter \(r \in \mathbb{R}\)

\[H^r = \mathcal{D}(A^r), \quad \langle u, v \rangle_r = \langle A^ru, A^rv \rangle, \quad \|u\|_r = \|A^ru\|.\]

The index \(r\) will be always omitted whenever zero. Along the paper, the Hölder, Young and Poincaré inequalities will be tacitly used in several occasions. The phase space of our problem is

\[\mathcal{H} = H^1 \times H^1 \times H,\]

endowed with the (Hilbert) product norm

\[\|(u, v, w)\|_{\mathcal{H}}^2 = \|u\|_1^2 + \|v\|_1^2 + \|w\|^2.\]
2. Well-Posedness

The existence and uniqueness result for (1.1) is ensured by the following theorems.

**Theorem 2.1.** If the derivative \( g' \) is bounded on bounded intervals, then for every initial datum \( U_0 \in \mathcal{H} \), equation (1.1) admits a unique weak solution

\[
U = (u, \partial_t u, \partial_{tt} u) \in \mathcal{C}([0, T], \mathcal{H})
\]

on the interval \([0, T]\), for any \( T > 0 \).

**Theorem 2.2.** Assume the mass restriction \( \varrho < \beta \). Then, for every initial datum \( U_0 \in \mathcal{H} \), equation (1.1) admits a unique weak solution

\[
U = (u, \partial_t u, \partial_{tt} u) \in \mathcal{C}([0, \infty), \mathcal{H})
\]

whose corresponding energy

\[
F(t) = \|u(t)\|_1^2 + \|\partial_t u(t)\|_1^2 + \|\partial_{tt} u(t)\|_1^2 + \int_0^t g(t-s)\|\partial_t u(t) - \partial_t u(s)\|_1^2\, ds
\]

satisfies the energy inequality

\[
F(t) \leq K F(0)e^{\omega t},
\]

for some structural constants \( K, \omega > 0 \) and for all \( t \geq 0 \).

As already mentioned in the Introduction, the first Theorem 2.1 above, within a very mild assumption on the derivative \( g' \) of the memory kernel, provides existence and uniqueness of solutions in the space \( \mathcal{H} \) without imposing the usual restriction \( \varrho < \beta \) on the size of the mass of \( g \). In which case, the solutions will be typically unbounded in time and exhibit a “rough” asymptotic behavior as \( t \to \infty \). To the best of our knowledge, this is the first well-posedness result obtained for the MGT equation with memory without assuming “smallness” restrictions imposed on the relaxation kernel.

If instead we assume \( \varrho < \beta \), then the second Theorem 2.2 provides existence and uniqueness of solutions in \( \mathcal{H} \) which enjoy an exponential-type growth at infinity.

In order to prove the theorems, we first show a well-posedness result in the more regular space

\[
\dot{\mathcal{H}} = H^2 \times H^1 \times H,
\]

by constructing solutions to a memoryless nonhomogeneous MGT equation exploiting the so-called MacCamy trick (see e.g. [15]).

**Lemma 2.3.** For every initial datum \( U_0 \in \dot{\mathcal{H}} \) (and every \( \varrho > 0 \)), equation (1.1) admits a unique weak solution

\[
U = (u, \partial_t u, \partial_{tt} u) \in \mathcal{C}([0, T], \dot{\mathcal{H}})
\]

on the interval \([0, T]\), for any \( T > 0 \).

**Proof.** Let \( T > 0 \) be arbitrarily fixed, and let \( R_u(t) \) denote the resolvent operator associated with the kernel

\[
\mu(s) = -\frac{1}{\beta} g(s).
\]
This means that $R_\mu$ solves the equation

$$R_\mu(t) + \int_0^t \mu(t-s)R_\mu(s) \, ds = \mu(t), \quad \forall t \geq 0.$$  

We now rewrite (1.1) in the form

$$A^{\frac{1}{2}}\partial_t u(t) + \int_0^t \mu(t-s)A^{\frac{1}{2}}\partial_t u(s) \, ds = Y(t),$$

where

$$Y(t) = -\frac{1}{\beta} [A^{-\frac{1}{2}}\partial_{tt} u(t) + \alpha A^{-\frac{1}{2}}\partial_{tt} u(t) + \gamma A^{\frac{1}{2}} u(t)].$$

If we knew in advance that $Y \in C([0,T], H)$, then the function $X(t) = A^{\frac{1}{2}}\partial_t u(t)$, being the solution to the Volterra equation on $[0,T]$

$$X(t) + \int_0^t \mu(t-s)X(s) \, ds = Y(t),$$

has the explicit representation

$$X(t) = Y(t) - \int_0^t R_\mu(t-s)Y(s) \, ds.$$

Applying $\beta A^{\frac{1}{2}}$ to both sides, we conclude that the function $U = (u, \partial_t u, \partial_{tt} u)$ satisfies the equation

(2.2) $$\partial_{ttt} u + \alpha \partial_{tt} u + \beta A \partial_t u + \gamma Au = Q_U,$$

having set

$$Q_U(t) = \int_0^t R_\mu(t-s)[\partial_{tt} u(s) + \alpha \partial_{tt} u(s) + \gamma Au(s)] \, ds.$$  

Integrating by parts the first term above, we obtain

$$Q_U(t) = \int_0^t R_\mu(t-s)\partial_{tt} u(s) \, ds - R_\mu(t)\partial_{tt} u(0) + R_\mu(0)\partial_{tt} u(t)$$

$$+ \int_0^t R_\mu(t-s)[\alpha \partial_{tt} u(s) + \gamma Au(s)] \, ds.$$  

At this point, we observe that $R_\mu(0) = \mu(0) < 0$. Thus, calling $\hat{\alpha} = \alpha - R_\mu(0) > 0$ and

(2.3) $$\hat{Q}_U(t) = \int_0^t R_\mu'(t-s)\partial_{tt} u(s) \, ds - R_\mu(t)\partial_{tt} u(0)$$

$$+ \int_0^t R_\mu(t-s)[\alpha \partial_{tt} u(s) + \gamma Au(s)] \, ds,$$

equation (2.2) reads

(2.4) $$\partial_{ttt} u + \hat{\alpha} \partial_{tt} u + \beta A \partial_t u + \gamma Au = \hat{Q}_U.$$  

Introducing the three-component vector

$$Q_U = (0,0,\hat{Q}_U),$$
we can write (2.4) in the abstract form
\[ \frac{d}{dt} U = \mathcal{A}U + \mathcal{Q}U, \]
where
\[ \mathcal{A}U = (\partial_t u, \partial_{tt} u, -\hat{\alpha} \partial_{tt} u - \beta A \partial_t u - \gamma Au). \]
It is known from [7] that the MGT equation without memory generates a strongly continuous semigroup \( S(t) = e^{\mathcal{A}t} \) on \( \hat{\mathcal{H}} \). Hence, its nonhomogeneous version (2.5) driven by a generic forcing term \( \mathcal{Q} \in L^1(0, T; \hat{\mathcal{H}}) \) admits a unique solution
\[ U = (u, \partial_t u, \partial_{tt} u) \in C([0, T], \hat{\mathcal{H}}). \]
Accordingly, if \( \mathcal{Q}U \in L^1(0, T; \hat{\mathcal{H}}) \), from the variation-of-constant formula we end up with
\[ U(t) = S(t)U_0 + \int_0^t S(t-s) \mathcal{Q}_U(s)ds. \]
In order to finish the proof, we merely apply the Banach contraction principle, first on the space
\[ \mathcal{X} = \{ U \in C([0, T_0], \hat{\mathcal{H}}) : U(0) = U_0 \}, \]
with \( T_0 \) sufficiently small, and then reiterated (due to the linearity) to the intervals \([nT_0, (n+1)T_0]\) until \( T \) is reached. □

The next step is extending the result of Lemma 2.3 to the whole space \( \mathcal{H} \). This will be easily accomplished by standard density arguments, once a priori estimates involving initial data belonging to \( \mathcal{H} \) are established. Here, different arguments are needed depending whether we are in the framework of Theorem 2.1 or of Theorem 2.2.

Proof of Theorem 2.1. Let \( T > 0 \) be arbitrarily fixed. We start from equation (2.4) of the previous proof, that is,
\[ \partial_{tt} u + \hat{\alpha} \partial_t u + \beta A \partial_t u + \gamma Au = \hat{Q}_U, \]
whose solution \( U \) (which is in fact the solution to the original equation) exists in \( \hat{\mathcal{H}} \) for initial data \( U_0 \in \mathcal{H} \). Then, we multiply by \( 2 \partial_t u \) in \( H \), and we add to both sides the term \( 2m\langle u, \partial_t u \rangle_1 \) for \( m > 0 \) to be fixed later. We obtain the differential equality
\[ \frac{d}{dt} V_m + 2\hat{\alpha}\| \partial_t u \|^2 + 2\gamma\| \partial_t u \|_1^2 + 2m\langle u, \partial_t u \rangle_1 + 2\langle \hat{Q}_U, \partial_t u \rangle, \]
where we set
\[ V_m(t) = \| \partial_t u(t) \|^2 + \beta\| \partial_t u(t) \|_1^2 + 2\gamma\langle u(t), \partial_t u(t) \rangle_1 + m\| u(t) \|_1^2. \]
It is readily seen that, up to choosing \( m > 0 \) sufficiently large, there exist \( \kappa_2 > \kappa_1 > 0 \) such that
\[ \kappa_1\| U(t) \|^2_{\mathcal{H}} \leq V_m(t) \leq \kappa_2\| U(t) \|^2_{\mathcal{H}}. \]
Accordingly,
\[ \frac{d}{dt} V_m \leq CV_m + 2\langle \hat{Q}_U, \partial_t u \rangle, \]
for some $C > 0$. We now claim that the inequality
\begin{equation}
\int^t_0 \langle \dot{Q}U(s), \partial_t u(s) \rangle ds \leq \frac{1}{4} V_m(t) + C_T V_m(0) + C_T \int^t_0 V_m(s) ds
\end{equation}
holds for every $t \in [0, T]$. Here and till the end of the proof, $C_T > 0$ denotes a generic constant, independent of the initial data, but depending on $T$. Then, integrating (2.7) on $[0, t]$, we end up with
\[ V_m(t) \leq C_T V_m(0) + C_T \int^t_0 V_m(s) ds, \quad \forall t \in [0, T], \]
and the standard Gronwall lemma together with (2.6) entail
\[ \|U(t)\|_H \leq C_T \|U(0)\|_H, \quad \forall t \in [0, T]. \]
We are left to prove (2.8). Recalling (2.3), we limit ourselves to show the more difficult estimate of the higher-order term, namely,
\[ \mathcal{J} := \int^t_0 \int^s_0 R_\mu(s - y) \langle u(y), \partial_t u(s) \rangle_1 dy ds. \]
We write
\[ \int^s_0 R_\mu(s - y) \langle u(y), \partial_t u(s) \rangle_1 dy = \frac{d}{ds} \left[ \int^s_0 R_\mu(s - y) \langle u(y), \partial_t u(s) \rangle_1 dy \right] \]
\[ - R_\mu(0) \langle u(s), \partial_t u(s) \rangle_1 - \int^s_0 R'_\mu(s - y) \langle u(y), \partial_t u(s) \rangle_1 dy. \]
Note that, within the boundedness assumption on $g'$, we have that $R'_\mu$ (as well as $R_\mu$) is bounded on $[0, T]$. Then, integrating on $[0, t]$ the identity above, we are led to
\[ \mathcal{J} = \int^t_0 R_\mu(t - s) \langle u(s), \partial_t u(t) \rangle_1 ds - R_\mu(0) \int^t_0 \langle u(s), \partial_t u(s) \rangle_1 ds \]
\[ - \int^t_0 \int^s_0 R'_\mu(s - y) \langle u(y), \partial_t u(s) \rangle_1 dy ds. \]
We finally estimate the three terms in the right-hand side as follows:
\[ \int^t_0 R_\mu(t - s) \langle u(s), \partial_t u(t) \rangle_1 ds - R_\mu(0) \int^t_0 \langle u(s), \partial_t u(s) \rangle_1 ds \]
\[ \leq \varepsilon \|U(t)\|^2_H + \frac{C_T}{\varepsilon} \int^t_0 \|U(s)\|^2_H ds, \]
for any $\varepsilon > 0$ small, and
\[ - \int^t_0 \int^s_0 R'_\mu(s - y) \langle u(y), \partial_t u(s) \rangle_1 dy ds \leq C_T \int^t_0 \|U(s)\|_H \int^s_0 \|U(y)\|_H dy ds \]
\[ \leq C_T \left( \int^t_0 \|U(s)\|_H ds \right)^2 \]
\[ \leq C_T \int^t_0 \|U(s)\|^2_H ds. \]
Therefore,

\[ J \leq \varepsilon \|U(t)\|_{H_t}^2 + \frac{C_T}{\varepsilon} \int_0^t \|U(s)\|_{H_t}^2 \, ds. \]

The remaining terms of \( \int_0^t \langle \dot{Q}_U(s), \partial_t u(s) \rangle \, ds \), as we said, are controlled in a similar (in fact easier) way, and at the end one has to use (2.6). Only at that point, one fixes \( \varepsilon \) in order to get the desired coefficient 1/4 (or smaller) in front of \( V_m(t) \). This finishes the proof.

\[ \square \]

**Proof of Theorem 2.2.** We only need to show the energy inequality (2.1). To this aim, similarly to the proof of Theorem 2.1, we take the product in \( H \) of (1.1) and \( 2 \partial_t u \), and we add to both sides the term \( 2m \langle u, \partial_t u \rangle_1 \) for \( m > 0 \) to be fixed later. This yields

\[
\frac{d}{dt} \left[ \|\partial_t u(t)\|^2 + \beta \|\partial_t u(t)\|_1^2 + m\|u(t)\|^2 + 2\gamma \langle u(t), \partial_t u(t) \rangle_1 \right] \\
= -2 \int_0^t g(t-s) \langle \partial_t u(s), \partial_t u(t) \rangle_1 \, ds \\
= 2\gamma \|\partial_t u(t)\|_1^2 - 2\alpha \|\partial_t u(t)\|^2 + 2m \langle u(t), \partial_t u(t) \rangle_1 \\
\leq 2(\gamma + m)F(t).
\]

Next, calling \( G(t) = \int_0^t g(s) \, ds \), we compute the integral in the left-hand side as

\[
-2 \int_0^t g(t-s) \langle \partial_t u(s), \partial_t u(t) \rangle_1 \, ds \\
= \frac{d}{dt} \left[ \int_0^t g(t-s) \|\partial_t u(t) - \partial_t u(s)\|_1^2 \, ds - G(t) \|\partial_t u(t)\|_1^2 \right] \\
+ g(t) \|\partial_t u(t)\|_1^2 - \int_0^t g'(t-s) \|\partial_t u(t) - \partial_t u(s)\|_1^2 \, ds.
\]

Setting

\[
E_m(t) = \|\partial_t u(t)\|^2 + (\beta - G(t)) \|\partial_t u(t)\|_1^2 + m\|u(t)\|^2 + 2\gamma \langle u(t), \partial_t u(t) \rangle_1 \\
+ \int_0^t g(t-s) \|\partial_t u(t) - \partial_t u(s)\|_1^2 \, ds,
\]

since \( g \) is nonnegative and nonincreasing we arrive at the inequality

\[
\frac{d}{dt} E_m \leq 2(\gamma + m)F.
\]

Recalling that \( G(t) \leq \varrho < \beta \), is then clear that, up to choosing \( m \) large enough,

\[
\kappa_1 F(t) \leq E_m(t) \leq \kappa_2 F(t),
\]

for some \( \kappa_2 > \kappa_1 > 0 \). The desired conclusion follows by an application of the Gronwall lemma.

\[ \square \]

**Remark 2.4.** If \( A \) is a bounded operator, the conclusions of Theorem 2.2 are easily attained removing the restriction \( \varrho < \beta \).
Remark 2.5. As a final comment, it is interesting to observe that the trick of multiplying both sides of the equation by \( \langle u, \partial_t u \rangle \), employed in the proofs above, allows to provide a two-line proof of the well-posedness of the strongly damped wave equation with the “wrong” sign of \( Au \), namely,

\[
\partial_t u + A\partial_t u - Au = 0,
\]

which highlights the essential parabolicity of the original equation. This is not the case if one has a lower-order dissipation. Indeed, the equation

\[
\partial_t u + A^{\vartheta} \partial_t u - Au = 0
\]

is ill-posed for \( \vartheta < 1 \), as the real part of the spectrum of the associated linear operator is not bounded above.

3. The Case of the Exponential Kernel

We now dwell on the particular case of the exponential kernel

\[
g(s) = \varrho \delta e^{-\delta s},
\]

with

\[
\varrho \in (0, \beta) \quad \text{and} \quad \delta > 0.
\]

For this choice, equation (1.1) reads

\[
(3.1) \quad \partial_{tt} u(t) + \alpha \partial_t u(t) + \beta A \partial_t u(t) + \gamma Au(t) - \varrho \delta \int_0^t e^{-\delta (t-s)} A \partial_t u(s) \, ds = 0.
\]

In the same spirit of [2], taking the sum \( \partial_t (3.1) + \delta (3.1) \) we obtain the fourth-order equation

\[
(3.2) \quad \partial_{ttt} u + (\alpha + \delta) \partial_{tt} u + \alpha \partial_t u + \beta A \partial_t u + (\gamma + \delta \beta - \varrho \delta) A \partial_t u + \gamma \delta Au = 0.
\]

Note that

\[
\gamma + \delta \beta - \varrho \delta > 0,
\]

as \( \varrho < \beta \). Introducing the 4-component space

\[
\mathcal{V} = H^1 \times H^1 \times H^1 \times H,
\]

it is known from [4] that (3.2) admits a unique (weak) solution

\[
\hat{U} = (u, \partial_t u, \partial_{tt} u, \partial_{ttt} u) \in C([0, \infty), \mathcal{V}),
\]

for every initial datum \( \hat{U}_0 \in \mathcal{V} \). Besides, the analysis in [4] provides necessary and sufficient conditions in order for (3.2) to be (exponentially) stable, depending on two stability numbers \( \kappa \) and \( \varpi \), which in turn depend only on the (positive) structural constants of the equation. For this particular case, the two stability numbers read

\[
\kappa = \frac{\alpha \beta - \gamma + \varrho \delta}{\alpha + \delta} > 0 \quad \text{and} \quad \varpi = \frac{\alpha \beta \delta^2 - \gamma \delta^2 - \alpha \varrho \delta^2}{\gamma + \delta \beta - \varrho \delta}.
\]

In particular, if \( \varrho \in (\beta - \frac{\varpi}{\alpha}, \beta) \) and \( \delta \) is large enough, then

\[
\varpi < -\lambda_1 \kappa
\]

where \( \lambda_1 > 0 \) is the smallest element of the spectrum \( \sigma(A) \) of the operator \( A \). In this regime, the results of [4] predict the existence of solutions growing exponentially fast, which gives a clear indication that our energy \( F \) might blow up exponentially for some
initial data. At the same time, the equivalence between (3.1) and (3.2) is, at this stage, only formal. The next proposition establishes such an equivalence in a rigorous way.

**Proposition 3.1.** Let \( U_0 = (u_0, v_0, w_0) \in \mathcal{H}^1 \times \mathcal{H}^1 \times \mathcal{H}^1 \) be an arbitrarily fixed vector satisfying the further regularity assumption

\[
\beta v_0 + \gamma u_0 \in \mathcal{H}^2.
\]

Then the projection \( U = (u, \partial_t u, \partial_{tt} u) \) onto the first three components of the solution \( \tilde{U} = (u, \partial_t u, \partial_{tt} u, \partial_{ttt} u) \) to (3.2) with initial datum

\[
(u_0, v_0, w_0, -\alpha w_0 - A(\beta v_0 + \gamma u_0)) \in \mathcal{V}
\]

is the unique solution to (3.1) with initial datum \( U_0 \).

**Proof.** We introduce the auxiliary variable

\[
\phi(t) = \partial_{ttt} u(t) + \alpha \partial_{tt} u(t) + \beta A \partial_t u(t) + \gamma A u(t).
\]

Since \( u \) solves (3.2), the function \( \phi \) fulfills the identity

\[
\partial_t \phi + \delta \phi - \varrho \delta A \partial_t u = 0.
\]

A multiplication by \( e^{\delta t} \) yields

\[
\frac{d}{dt}[e^{\delta t} \phi(t)] - \varrho e^{\delta t} A \partial_t u(t) = 0.
\]

Noting that \( \phi(0) = 0 \), an integration on \([0, t]\) leads at once to (3.1). \( \square \)

Still, this is not enough to conclude that \( \mathcal{F} \) can grow exponentially fast, since one has to verify that this occurs for a particular trajectory of (3.2), with initial conditions complying with the assumptions above.

## 4. Exponentially Growing Solutions

In this section, we state and prove the second main result of the paper. Namely, we produce an example of memory kernel \( g \) for which equation (1.1) admits solutions with energy growing exponentially fast. To this end, we consider the exponential kernel \( g(s) = \varrho s e^{-\delta s} \) of the previous section. For simplicity, we also assume that the spectrum of the operator \( A \) contains at least one eigenvalue \( \lambda > 0 \), which is always the case in concrete situations.

**Theorem 4.1.** Let \( \varrho \in (\beta - \frac{\gamma}{\alpha}, \beta) \) be arbitrarily fixed. Then, for every \( \delta > 0 \) sufficiently large, there exist \( \varepsilon > 0 \), an initial datum \( U_0 \in \mathcal{H} \), and a sequence \( t_n \to \infty \) such that the energy \( \mathcal{F}(t) \) associated to the solution to (3.1) originating from \( U_0 \) satisfies the estimate

\[
\mathcal{F}(t_n) \geq \lambda e^{\varepsilon t_n}, \quad \forall n \in \mathbb{N}.
\]

**Remark 4.2.** According to [8], for this particular kernel exponential stability occurs in the subcritical case, within the following assumption: there exist \( k \in \left( \frac{\gamma}{\beta}, \alpha \right) \) and \( \theta > \frac{k}{\delta} \) such that

\[
\varrho \leq \left( \beta - \frac{\gamma}{k} \right) \min \left\{ 1, \frac{2}{k(2 + \theta)} \right\}.
\]

The reader will have no difficulties to check that the condition above implies that \( \varrho < \beta - \frac{\gamma}{\alpha} \), which contradicts \( \varrho \in (\beta - \frac{\gamma}{\alpha}, \beta) \) assumed in Theorem 4.1.
In order to prove the theorem, we introduce the fourth-order polynomial in the complex variable \( \xi \)

\[
\mathcal{P}(\xi) = \xi^4 + (\alpha + \delta)\xi^3 + (\alpha\delta + \beta\delta)\xi^2 + \lambda(\gamma + \delta\beta - \rho\delta)\xi + \lambda\gamma\delta.
\]

Moreover, for all \( x \geq 0 \), we set

\[
q(x) = \sqrt{\frac{4x^3 + 3(\alpha + \delta)x^2 + 2(\alpha\delta + \beta\lambda)x + \lambda(\gamma + \delta\beta - \rho\delta)}{\alpha + \delta + 4x}} > 0.
\]

The next algebraic result will be crucial for our purposes.

**Lemma 4.3.** Let \( \rho \in (\beta - \frac{\gamma}{\alpha}, \beta) \) be arbitrarily fixed. Then, for every \( \delta > 0 \) sufficiently large, there exists \( p \) such that the complex number \( \hat{\xi} = p + iq(p) \) solves the equation \( \mathcal{P}(\hat{\xi}) = 0 \).

**Proof.** For all \( x \geq 0 \), by direct calculations we find the equalities

\[
\text{Im} \left( \mathcal{P}(x + iq(x)) \right) = 0 \quad \text{and} \quad \text{Re} \left( \mathcal{P}(x + iq(x)) \right) = f(x),
\]

where

\[
f(x) = x^4 + (\alpha + \delta)x^3 + (\alpha\delta + \beta\lambda)x^2 + (\gamma + \delta\beta - \rho\delta)x + \gamma\delta\lambda + q(x)^4
\]

\[-(6x^2 + 3(\alpha + \delta)x + \alpha\delta + \beta\lambda)q(x)^2.\]

By means of direct computations, with the aid of (4.2) and the assumption \( \rho > \beta - \frac{\gamma}{\alpha} \), it is readily seen that

\[
\lim_{x \to +\infty} f(x) = -\infty
\]

and

\[
f(0) = \gamma\delta\lambda + q(0)^4 - (\alpha\delta + \beta\lambda)q(0)^2 \sim \delta\lambda(\gamma - \alpha\beta + \alpha\rho) > 0, \quad \text{as} \quad \delta \to +\infty.
\]

As a consequence, being \( f \) continuous on \([0, \infty)\), once \( \delta > 0 \) has been fixed sufficiently large there exists \( p > 0 \) such that

\[
0 = f(p) = \text{Re} \left[ \mathcal{P}(p + iq(p)) \right].
\]

The proof is finished. \( \square \)

**Proof of Theorem 4.1.** Denoting by \( w \in H \) the normalized eigenvector of \( A \) corresponding to \( \lambda \), we consider the function

\[
u(t) = e^{pt} [r \sin(qt) + \cos(qt)] w.
\]

Here, \( p > 0 \) is given by Lemma 4.3, \( q = q(p) > 0 \) is given by (4.2) and

\[
r = r(p) = \frac{p^3 - 3pq^2 + \alpha(p^2 - q^2) + \beta\lambda p + \gamma\lambda}{q^3 - 3pq^2 - 2\alpha pq - \beta\lambda q}.
\]

Note that \( r \) is well defined, since (1.2) and (4.2) ensure that

\[
q^3 - 3pq^2 - 2\alpha pq - \beta\lambda q = -\frac{q(8p^3 + 8\alpha p^2 + 2\alpha^2 p + 2\beta\lambda p + \lambda\rho\delta + \lambda(\alpha\beta - \gamma))}{\alpha + \delta + 4p} < 0.
\]
The function \( u \) defined above solves the fourth-order equation (3.2). Indeed, calling for simplicity
\[
\psi(t) = e^{pt} \left[ r \sin(qt) + \cos(qt) \right]
\]
and recalling that due to Lemma 4.3 the complex number \( p + iq \) is a root of the polynomial \( P \) defined in (4.1), we have
\[
\partial_{tttt}u + (\alpha + \delta)\partial_{tt}u + \alpha\delta\partial_{tt}u + \beta A\partial_{tt}u + (\gamma + \delta\beta - \varrho\delta)A\partial_{t}u + \gamma\delta Au = 0.
\]
Moreover, being
\[
\begin{align*}
u(0) &= w, \\
\partial_t u(0) &= (p + rq)w, \\
\partial_{tt} u(0) &= (p^2 + 2rpq - q^2)w, \\
\partial_{ttt} u(0) &= (p^3 + 3rp^2q - 3pq^2 - rq^3)w,
\end{align*}
\]
thanks to the choice of \( r \) it is true that
\[
\partial_{ttt} u(0) = -\alpha(p^2 + 2rpq - q^2)w - \beta\lambda(p + rq)w - \gamma\lambda w
\]
\[
= -\alpha\partial_t u(0) - \beta A\partial_t u(0) - \gamma Au(0).
\]
Invoking Lemma 3.1, the function \( u \) turns out to be the unique solution to (3.1) corresponding to the initial datum
\[
z_0 = (w, (p + rq)w, (p^2 + 2rpq - q^2)w).
\]
Finally, setting
\[
t_n = \frac{2n\pi}{q} \rightarrow +\infty \quad \text{and} \quad \varepsilon = 2p > 0,
\]
we conclude that
\[
F(t_n) \geq \|u(t_n)\|_{1}^2 = \lambda e^{\varepsilon t_n}.
\]
The proof of Theorem 4.1 is finished.

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