Unbounded $p$-Convergence in Lattice-Normed Vector Lattices

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Abstract—A net $x_\alpha$ in a lattice-normed vector lattice $(X, p, E)$ is unbounded $p$-convergent to $x \in X$ if $p(|x_\alpha - x| \wedge u) \xrightarrow{uo} 0$ for every $u \in X_+$. This convergence has been investigated recently for $(X, p, E) = (X, |\cdot|, X)$ under the name of $uo$-convergence, for $(X, p, E) = (X, \|\cdot\|, \mathbb{R})$ under the name of un-convergence, and also for $(X, p, \mathbb{R}^X)$, where $p(x)[f] := |f(|x|)|$, under the name $uw$-convergence. In this paper we study general properties of the unbounded $p$-convergence.

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1. INTRODUCTION AND PRELIMINARIES

Lattice-valued norms on vector lattices provide natural and efficient tools in the theory of vector lattices. It is enough to mention the theory of lattice-normed vector lattices (see, for example, [13, 20, 21]). The main aim of the present paper is to illustrate usefulness of lattice-valued norms for investigation of different types of unbounded convergences in vector lattices, which attracted attention of several authors in series of recent papers [4–8, 10–12, 14–18, 22, 26, 27, 32].

The $uo$-convergence was introduced in [25] under the name individual convergence, and the un-convergence was introduced in [28] under the name $d$-convergence. We refer the reader for an exposition on $uo$-convergence to [16, 17] and on un-convergence to [10] (see also [18]). For applications of $uo$-convergence, we refer to [11, 12, 15, 15–17, 22, 24]. Throughout the paper, all vector lattices are assumed to be real, Archimedean, and nonzero. We write $X^\delta$ for the order (or Dedekind) completion of a vector lattice $X$ (cf. [3, p.19]).

Recall that a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice $X$ is order convergent (or $o$-convergent, for short) to $x \in X$, if there exists another net $(y_\beta)_{\beta \in B}$ satisfying $y_\beta \not\rightarrow 0$ and, for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{o} x$. In a vector lattice $X$, a net $x_\alpha$ is unbounded order convergent (or $uo$-convergent, for short) to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{uo} 0$ for every $u \in X_+$. In this case we, write $x_\alpha \xrightarrow{uo} x$. The $uo$-convergence is an abstraction of a.e.-convergence in $L_p$-spaces for $1 \leq p < \infty$, [16, 17]. In a normed lattice $(X, \|\cdot\|)$, a net $x_\alpha$ is unbounded norm convergent (un-convergent) to $x \in X$, written as $x_\alpha \xrightarrow{un} x$, if $\|x_\alpha - x| \wedge u\| \rightarrow 0$ for every $u \in X_+$. Clearly, if the

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norm is order continuous, then \( \text{uo-}\) convergence implies \( \text{un-}\) convergence. For a finite measure \( \mu \), \( \text{un-}\) convergence of sequences in \( L_p(\mu), 1 \leq p < \infty \), is equivalent to convergence in measure (see [10, 28]).

Let \( X \) be a vector space, \( E \) be a vector lattice, and \( p : X \to E^+ \) be a vector norm (i.e. \( p(x) = 0 \iff x = 0 \), \( p(\lambda x) = |\lambda| p(x) \) for all \( \lambda \in \mathbb{R}, x \in X \), and \( p(x + y) \leq p(x) + p(y) \) for all \( x, y \in X \), then the triple \((X, p, E)\) is called a lattice-normed space, abbreviated as LNS. We say that elements \( x \) and \( y \) of an LNS \( X \) are \( p \)-disjoint if their lattice norms are disjoint, and abbreviate this by \( x \perp_p y \). The lattice norm \( p \) in an LNS \((X, p, E)\) is said to be decomposable if, for all \( x \in X \) and \( e_1, e_2 \in E^+ \), from \( p(x) = e_1 + e_2 \) it follows that there exist \( x_1, x_2 \in X \) such that \( x = x_1 + x_2 \) and \( p(x_k) = e_k \) for \( k = 1, 2 \). We abbreviate the convergence \( p(x_\alpha - x) \overset{\alpha}{\to} 0 \) as \( x_\alpha \overset{p}{\to} x \) and say in this case that \( x_\alpha \) \( p \)-converges to \( x \). We refer the reader for more information on LNSs to [20, 21].

If, in addition, \( X \) is a vector lattice and the vector norm \( p \) is monotone (i.e. \( |x| \leq |y| \Rightarrow p(x) \leq p(y) \)), then the triple \((X, p, E)\) is called a lattice-normed vector lattice, abbreviated as LNL. In an LNL \((X, p, E)\), \( p \)-disjointness implies disjointness. Indeed, let \( x \perp_p y \). Then \( p(|x| \wedge |y|) \leq p(x) \wedge p(y) = 0 \) and hence \( x \perp y \). We shall make difference between two notions of bands in an LNL \( X = (X, p, E) \). More precisely, a subset \( B \) of \( X \) is called a band if it is a band in the vector lattice \( X \) in the usual sense. Following to [21, 2.1.2.], we say that a subset \( B \) of \( X \) is a \( p \)-band if

\[
B = M^{\perp_p} = \{ x \in X : (\forall m \in M) x \perp_p m \}
\]

for some \( M \subseteq X \). In general, there are many bands which are not \( p \)-bands. To see this, consider the normed lattice \((\mathbb{R}^2, \| \cdot \|, \mathbb{R}) \). It has four bands, but only two of them are \( p \)-bands. It is easy to see that any \( p \)-band is an order ideal. The following example shows that a \( p \)-band may not be a band in general.

**Example 1.1.** Consider the LNL \( (c, p, c) \) with

\[
p(x) := |x| + \left( \lim_{n \to \infty} |x_n| \right) \cdot 1 \quad (x = (x_n)_n \in c),
\]

where \( I \) denotes the sequence identically equal to 1. Take \( M = \{ e_1 \} \). Then the \( p \)-band \( M^{\perp_p} = \{ x \in c_0 : x_1 = 0 \} \) is not a band.

In Proposition 2.7, we show that, under some mild conditions, every \( p \)-band is a band. Unless otherwise stated, we do not assume lattice norms to be decomposable. While dealing with LNLs, we shall keep in mind also the following examples.

**Example 1.2.** Let \( X \) be a normed lattice with a norm \( \| \cdot \| \). Then \( X \) is the LNL \( (X, \| \cdot \|, \mathbb{R}) \).

**Example 1.3.** Let \( X \) be a vector lattice. Then \( X \) is the LNL \( (X, \| \cdot \|, X) \).

**Example 1.4.** Let \( X = (X, \| \cdot \|) \) be a normed lattice. Consider the closed unit ball \( B_X \) of the dual Banach lattice \( X' \). Let \( E = \ell_\infty(B_X) \) be the vector lattice of all bounded real-valued functions on \( B_X \). Define an \( E \)-valued norm \( p \) on \( X \) by

\[
p(x)[f] := |f|(|x|) \quad (f \in B_X')
\]

for any \( x \in X \). The Hahn-Banach theorem ensures that \( p(x) = 0 \iff x = 0 \). All other properties of lattice norm are obvious for \( p \). Thus \( (X, p, E) \) is an LNL. Notice also that the lattice norm \( p \) takes values in the space \( C(B_X') \) of all continuous functions on the \( w^* \)-compact ball \( B_{X'} \) of \( X' \). Hence, instead of \( (X, p, \ell_\infty(B_X)) \), one may also consider the LNL \( (X, p, C(B_X')) \).

**Example 1.5.** Let \( X \) be a vector lattice, \( X \) be the order dual of \( X \), and \( Y \) be a sublattice of \( X \) such that \( (X, Y) \) is a dual system. Define \( p : X \to \mathbb{R}^Y \) by \( p(x)[y] := |y|(|x|) \). Then \( (X, p, \mathbb{R}^Y) \) is an LNL.

The LNLs in Examples 1.1, 1.2, and 1.3 have decomposable norms. It can be shown easily that, in Examples 1.4 and 1.5, the lattice norms are decomposable if \( \text{dim}(X) = 1 \).

We refer the reader for further examples of LNSs to [21]. It should be noticed that the theory of lattice-normed spaces is well developed in the case of decomposable lattice norms (cf. [20, 21]).

The structure of the paper is as follows. In Section 2, we study several notions related to LNLs in parallel to the theory of Banach lattices. In particular, an LNL \((X, p, E)\) is said to be \textit{op-continuous} if \( X \ni x_\alpha \overset{\alpha}{\to} 0 \) implies \( x_\alpha \overset{p}{\to} 0 \); a \textit{\( p \)-KB-space} if, for any \( 0 \leq x_\alpha \uparrow \) with \( p(x_\alpha) \leq e \in E \), there exists
\[ x \in X \text{ satisfying } x_\alpha \xrightarrow{p} x. \] We give a characterization of op-continuity in Theorem 2.1, and study some properties of \( p \)-KB-spaces, e.g. in Proposition 2.4 and in Proposition 2.5. A vector \( e \in X \) is called a \( p \)-unit if, for any \( x \in X_+ \), \( p(x - ne \wedge x) \xrightarrow{\omega} 0 \). Any \( p \)-unit is a weak unit, whereas strong units are \( p \)-units.

For a normed lattice \((X, \|\cdot\|)\), a vector in \( X \) is a \( p \)-unit in \((X, \|\cdot\|, \mathbb{R})\) if it is a quasi-interior point of the normed lattice \((X, \|\cdot\|)\).

In Section 3, some basic theory of unbounded \( p \)-convergence in LNLs is developed in parallel to \( \omega \)- and \( \omega \)-convergences. For example, it is enough to check out the \( \omega \)-convergence at a weak unit, while the \( \omega \)-convergence needs to be checked only at a quasi-interior point. Similarly, in LNLs, \( \omega \)-continuity needs to be examined at a quasi-interior point. Similarly, in LNLs, \( \omega \)-convergence needs to be checked only at a quasi-interior point. Similarly, in LNLs, \( \omega \)-continuity needs to be checked only at a quasi-interior point.

In Section 4, we introduce and study \( \uparrow \)-regular sublattices. Majorizing sublattices and projection bands are examples of \( \uparrow \)-regular sublattices, by Theorem 4.1. Also some further investigation of \( \uparrow \)-regular sublattices is carried out in certain LNLs in this section.

In the last section, we study properties of mixed-normed LNLs in Proposition 5.1, in Theorem 5.1, and in Theorem 5.2. We also prove that, in a certain LNL, the \( \uparrow \)-null nets are “\( p \)-almost disjoint” (see Theorem 5.3). Those results generalize the correspondent results from [10, 16].

We refer the reader for unexplained notions and terminology to [2, 3, 21, 23].

2. \( p \)-NOTIONS IN LATTICE-NORMED VECTOR LATTICES

Most of notions and results of this preliminary section are direct analogies of well-known facts of the theory of normed lattices. We include them for convenience of the reader. They are also of certain proper interest and some of them will be used in further sections. In the present section, we define and study certain necessary notions such as: \( \text{op} \)-continuity of LNLs, \( p \)-KB-spaces, \( p \)-Fatou spaces, \( p \)-units, etc. In particular, we characterize the \( \text{op} \)-continuity, prove some properties of \( p \)-KB-spaces, discuss \( p \)-dense subsets, and study \( p \)-units in LNLs.

2.1. \( p \)-Continuity of lattice operations in LNLs. The lattice operations in an LNL \( X \) are \( p \)-continuous in the following sense.

**Lemma 2.1.** Let \((x_\alpha)_{\alpha \in A} \) and \((y_\beta)_{\beta \in B} \) be two nets in an LNL \((X, p, E)\). If \( x_\alpha \xrightarrow{p} x \) and \( y_\beta \xrightarrow{p} y \), then \( (x_\alpha \vee y_\beta)_{(\alpha, \beta) \in A \times B} \xrightarrow{p} x \vee y \). In particular, \( x_\alpha \xrightarrow{p} x \) implies that \( x_\alpha^- \xrightarrow{p} x^- \).

Although this result seems to be well-known, we did not find appropriate references for it and therefore, we include its elementary proof for convenience.

**Proof.** There exist two nets \((z_\gamma)_{\gamma \in \Gamma} \) and \((w_\lambda)_{\lambda \in \Lambda} \) in \( E \) satisfying \( z_\gamma \downarrow 0 \), and \( w_\lambda \downarrow 0 \), and, for all \((\gamma, \lambda) \in \Gamma \times \Lambda \), there are \( \alpha_\gamma \in A \) and \( \beta_\lambda \in B \) such that \( p(x_{\alpha_\gamma} - x) \leq z_\gamma \) and \( p(y_{\beta_\lambda} - y) \leq w_\lambda \) for all \((\alpha, \beta) \geq (\alpha_\gamma, \beta_\lambda) \). It follows from the inequality \( |a \vee b - a \vee c| \leq |b - c| \) that

\[
\begin{align*}
p(x_\alpha \vee y_\beta - x \vee y) &= p(|x_\alpha \vee y_\beta - x_\alpha \vee y + x_\alpha \vee y - x \vee y|) \\
&\leq p(|x_\alpha \vee y_\beta - x_\alpha \vee y|) + p(|x_\alpha \vee y - x \vee y|) \\
&\leq p(|y_\beta - y|) + p(|x_\alpha - x|) \leq w_\lambda + z_\gamma
\end{align*}
\]

for all \( \alpha \geq \alpha_\gamma \) and \( \beta \geq \beta_\lambda \). Since \( (w_\lambda + z_\gamma) \downarrow 0 \), then \( p(x_\alpha \vee y_\beta - x \vee y) \xrightarrow{\omega} 0 \). \( \square \)

**Definition 2.1.** Let \((X, p, E)\) be an LNL and \( Y \subseteq X \). Then \( Y \) is called \( p \)-closed in \( X \) if, for any net \( y_\alpha \) in \( Y \) that is \( p \)-convergent to \( x \in X \), there holds that \( x \in Y \).

**Remark 2.1.**

1. Every band is \( p \)-closed. Indeed, let \( B \) be a band in an LNL \((X, p, E)\). If \( B \ni x_\alpha \xrightarrow{p} x \), then, by Lemma 2.1, \( |x_\alpha| \wedge |y| \xrightarrow{p} |x| \wedge |y| \) for any \( y \in B^\perp \). Since \( |x_\alpha| \wedge |y| = 0 \) for all \( \alpha \), then \( |x| \wedge |y| = 0 \), and so \( x \in B^\perp = B \).
2. Every $p$-band is $p$-closed. Indeed, let $B = M^+p$ for some $M \subseteq X$ and $B \ni x_\alpha \xrightarrow{p} x_0 \in X$. Take any $m \in M$. It follows from

$$p(x_0) \wedge p(m) \leq (p(x_0 - x_\alpha) + p(x_\alpha)) \wedge p(m) \leq p(x_0 - x_\alpha) \wedge p(m) + p(x_\alpha) \wedge p(m) = p(x_0 - x_\alpha) \wedge p(m) \xrightarrow{\alpha} 0,$$

that $p(x_0) \wedge p(m) = 0$. Since $m \in M$ is arbitrary, then $x_0 \in B$.

The following well-known property is a direct consequence of Lemma 2.1.

**Proposition 2.1.** The positive cone $X_+$ in any LNL $X$ is $p$-closed.

Proposition 2.1 implies the following well-known fact.

**Proposition 2.2.** Any monotone $p$-convergent net in an LNL $o$-converges to its $p$-limit.

**Proof.** It is enough to show that if $(X, p, E) \ni x_\alpha \uparrow$ and $x_\alpha \xrightarrow{p} x$ then $x_\alpha \uparrow x$.

Fix an arbitrary $\alpha$. Then $x_\beta - x_\alpha \in X_+$ for $\beta \geq \alpha$. By Proposition 2.1, $x_\beta - x_\alpha \xrightarrow{p} x - x_\alpha \in X_+$. Therefore, $x \geq x_\alpha$ for any $\alpha$. Since $\alpha$ is arbitrary, then $x$ is an upper bound of $x_\alpha$.

If $y \geq x_\alpha$ for all $\alpha$, then, by Proposition 2.1 again, $y - x_\alpha \xrightarrow{p} y - x \in X_+$, or $y \geq x$. Thus $x_\alpha \uparrow x$. □

2.2. Several basic $p$-notions in LNLs. We continue with several basic notions in LNLs, which are motivated by their analogies from the vector lattice theory.

**Definition 2.2.** Let $X = (X, p, E)$ be an LNL. Then

(i) a net $(x_\alpha)_{\alpha \in A}$ in $X$ is said to be $p$-Cauchy if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A}$ $p$-converges to 0;

(ii) $X$ is called $p$-complete if every $p$-Cauchy net in $X$ is $p$-convergent;

(iii) a subset $Y \subseteq X$ is called $p$-bounded if there exists $e \in E$ such that $p(y) \leq e$ for all $y \in Y$;

(iv) $X$ is called op-continuous if $x_\alpha \xrightarrow{\alpha} 0$ implies that $p(x_\alpha) \xrightarrow{\alpha} 0$;

(v) $X$ is called a $p$-KB-space if every $p$-bounded increasing net in $X_+$ is $p$-convergent;

(vi) $p$ is said to be additive on $X_+$ if $p(x + y) = p(x) + p(y)$ for all $x, y \in X_+$.

**Remark 2.2.** 1. $p$-convergence, a $p$-Cauchy net, $p$-completeness, and $p$-boundedness in LNLs are also known as $bo$-convergence, a $bo$-fundamental net, $bo$-completeness, and $bo$-boundedness respectively (e.g., see [21, p.48])

2. Clearly, any LNL $(X, |\cdot|, X)$ is $op$-continuous.

3. In Definition 2.2(v), we do not require $p$-completeness of $X$.

4. It is easy to see that a $p$-KB-space $(X, |\cdot|, E)$ is always $p$-complete (see, e.g. [31, Ex.95.4]). Therefore, in the case of a normed lattice, the notion of $p$-KB-space coincides with the notion of $KB$-space.

5. Consider an LNL $(X, p, E)$. The vector lattice $E$ is regular in its order completion $E^\delta$ [16, p.6]. Since $p(X) \subseteq E$, then $(X, p, E)$ is $p$-complete if $(X, p, E^\delta)$ is $p$-complete.

6. Clearly, an LNL $X = (X, |\cdot|, X)$ is a $p$-KB-space iff $X$ is order complete.

7. Notice that, for a $p$-KB-space $X = (X, p, E)$, the vector lattice $p(X)^{\perp\perp}$ does not need to be order complete. To see this, take a $KB$-space $(X, |\cdot|)$ and $E = C[0, 1]$. Then the LNL $X = (X, p, E)$ with $p(x) := ||x|| \cdot 1_{[0,1]}$ is, clearly, a $p$-KB-space, yet $p(X)^{\perp\perp} = E$ is not order complete.

**Lemma 2.2.** For an LNL $(X, p, E)$, the following statements are equivalent.

(i) $X$ is $op$-continuous;

(ii) $x_\alpha \downarrow 0$ in $X$ implies $p(x_\alpha) \downarrow 0$.

**Proof.** The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): Let $x_\alpha \xrightarrow{\alpha} 0$, then there exists a net $z_\beta \downarrow 0$ in $X$ such that, for any $\beta$, there exists $\alpha_\beta$ so that $|x_\alpha| \leq z_\beta$ for all $\alpha \geq \alpha_\beta$. Hence $p(x_\alpha) \leq p(z_\beta)$ for all $\alpha \geq \alpha_\beta$. By (ii), $p(z_\beta) \downarrow 0$. Therefore, $p(x_\alpha) \xrightarrow{\alpha} 0$ or $x_\alpha \xrightarrow{p} 0$. □
It follows from Lemma 2.2, that the \( op \)-continuity in LNLs is equivalent to the order continuity in the sense of [21, 2.1.4, p.48]. In the case of a \( p \)-complete LNL, we have further conditions for \( op \)-continuity.

**Theorem 2.1.** For a \( p \)-complete LNL \((X, p, E)\), the following statements are equivalent:

(i) \( X \) is \( op \)-continuous;

(ii) if \( 0 \leq x_{\alpha} \uparrow \leq x \) holds in \( X \), then \( x_{\alpha} \) is a \( p \)-Cauchy net;

(iii) \( x_{\alpha} \downarrow 0 \) in \( X \) implies \( p(x_{\alpha}) \downarrow 0 \).

**Proof.** (i) \( \Rightarrow \) (ii): Let \( 0 \leq x_{\alpha} \uparrow \leq x \) in \( X \). By [2, Lm.4.8], there exists a net \( y_{\beta} \) in \( X \) such that \( (y_{\beta} - x_{\alpha})_{\alpha, \beta} \downarrow 0 \). So \( p(y_{\beta} - x_{\alpha}) \downarrow 0 \), and hence the net \( x_{\alpha} \) is \( p \)-Cauchy.

(ii) \( \Rightarrow \) (iii): Assume that \( x_{\alpha} \downarrow 0 \) in \( X \). Fix an arbitrary \( \alpha_{0} \), then, for \( \alpha \geq \alpha_{0} \), \( x_{\alpha} \leq x_{\alpha_{0}} \), and so \( 0 \leq (x_{\alpha_{0}} - x_{\alpha})_{\alpha \geq \alpha_{0}} \downarrow \leq x_{\alpha_{0}} \). By (ii), the net \( (x_{\alpha_{0}} - x_{\alpha})_{\alpha \geq \alpha_{0}} \) is \( p \)-Cauchy. Since \( X \) is \( p \)-complete, then there exists \( x \in X \) satisfying \( p(x_{\alpha} - x) \downarrow 0 \). By Proposition 2.2, \( x_{\alpha} \downarrow x \) and hence \( x = 0 \). As a result, \( x_{\alpha} \Rightarrow 0 \) and the monotonicity of \( p \) implies \( p(x_{\alpha}) \downarrow 0 \).

(iii) \( \Rightarrow \) (i): It is just the implication (ii) \( \Rightarrow \) (i) of Lemma 2.2. \( \square \)

**Corollary 2.1.** Let \((X, p, E)\) be an \( op \)-continuous and \( p \)-complete LNL, then \( X \) is order complete.

**Proof.** Assume \( 0 \leq x_{\alpha} \uparrow \leq u \), then by Theorem 2.1 (ii), the net \( x_{\alpha} \) is \( p \)-Cauchy and, since \( X \) is \( p \)-complete, \( x_{\alpha} \Rightarrow x \) for some \( x \in X \). By Proposition 2.2, \( x_{\alpha} \uparrow x \), and so \( X \) is order complete. \( \square \)

**Corollary 2.2.** Any \( p \)-KB-space is \( op \)-continuous.

**Proof.** Let \( x_{\alpha} \downarrow 0 \). Take any \( \alpha_{0} \) and let \( y_{\alpha} := x_{\alpha_{0}} - x_{\alpha} \) for \( \alpha \geq \alpha_{0} \). Clearly, \( y_{\alpha} \uparrow \leq x_{\alpha_{0}} \). Hence \( p(y_{\alpha}) \uparrow \leq p(x_{\alpha_{0}}) \) for \( \alpha \geq \alpha_{0} \). Since \( X \) is a \( p \)-KB-space, there exists \( y \in X \) such that \( p(y_{\alpha} - y) \downarrow 0 \).

Since \( y_{\alpha} \uparrow \) and \( y_{\alpha} \Rightarrow y \), Proposition 2.2 ensures that

\[
y = \sup_{\alpha \geq \alpha_{0}} y_{\alpha} = \sup_{\alpha \geq \alpha_{0}} (x_{\alpha_{0}} - x_{\alpha}) = x_{\alpha_{0}},
\]

and hence \( y_{\alpha} = x_{\alpha_{0}} - x_{\alpha} \Rightarrow x_{\alpha_{0}} \) or \( x_{\alpha} \Rightarrow 0 \). By monotonicity of \( p \), we get \( p(x_{\alpha}) \downarrow 0 \). So, \( X \) is \( op \)-continuous in view of Lemma 2.2. \( \square \)

**Proposition 2.3.** Any \( p \)-KB-space is order complete.

**Proof.** Let \( X \) be a \( p \)-KB-space and \( 0 \leq x_{\alpha} \uparrow \leq z \) in \( X \). Then \( p(x_{\alpha}) \leq p(z) \). Hence the net \( x_{\alpha} \) is \( p \)-bounded and therefore, \( x_{\alpha} \Rightarrow x \) for some \( x \in X \). By Proposition 2.2, \( x_{\alpha} \Rightarrow x \).

The following question arises naturally.

**Problem 2.1.** Let \((X, p, E)\) be a \( p \)-KB-space. Is \((X, p, E)\) \( p \)-complete?

We do not know the answer to Problem 2.1 even under the assumption that \( E \) is order complete.

**Proposition 2.4.** Let \((X, p, E)\) be a \( p \)-KB-space, and \( Y \subseteq X \) be an order closed sublattice. Then \((Y, p, E)\) is also a \( p \)-KB-space.

**Proof.** Let \( Y_{+} \ni y_{\alpha} \uparrow \) and \( p(y_{\alpha}) \leq e \in E_{+} \) for all \( \alpha \). Since \( X \) is a \( p \)-KB-space, there exists \( x \in X_{+} \) such that \( y_{\alpha} \Rightarrow x \). By Proposition 2.2, we have \( y_{\alpha} \Rightarrow x \), and so \( x \in Y \), because \( Y \) is order closed. Thus \((Y, p, E)\) is a \( p \)-KB-space. \( \square \)

It is clear from the proof of Proposition 2.4, that every \( p \)-closed sublattice \( Y \) of a \( p \)-KB-space \( X \) is also a \( p \)-KB-space.

**Proposition 2.5.** Let \((X, p, E)\) be a \( p \)-complete LNL such that \( p \) is additive on \( X_{+} \). Then \( X \) is a \( p \)-KB-space.

**Proof.** Let \( x_{\alpha} \) be an increasing and \( p \)-bounded nonnegative net in \((X, p, E)\). Then \( x_{\alpha} \) is also increasing and \( p \)-bounded in \((X, p, E^{\oplus})\).

By additivity of \( p \) on \( X_{+} \), \( p(x_{\alpha_{2}}) - p(x_{\alpha_{1}}) = p(x_{\alpha_{2}} - x_{\alpha_{1}}) \) for all \( \alpha_{2} \geq \alpha_{1} \). Since \( E^{\oplus} \) is Dedekind complete, then \( e_{\alpha} := \sup_{\gamma} p(x_{\gamma} - x_{\alpha}) \) exists in \( E^{\oplus} \) for any \( \alpha \). Clearly, \( e_{\alpha} \downarrow 0 \). We claim \( e_{\alpha} \downarrow 0 \).
Suppose, in contrary, $e_\alpha \geq z$ for some nonzero $z \in E^d_+$ and for all $\alpha$. The increasing net $p(x_\alpha)$ is order bounded in $E^d$. Then $p(x_\alpha)$ is order convergent in $E^d$. So, $p(x_\alpha)$ is $(o)$-Cauchy in $E^d$ and hence in $E$. Thus, there is a net $g_\psi \downarrow 0$ in $E$ such that, for any $\psi$, there exists an $\alpha_\psi$ satisfying

$$|p(x_{\alpha_2}) - p(x_{\alpha_1})| \leq g_\psi \quad (\forall \alpha_2, \alpha_1 \geq \alpha_\psi).$$

Since $E$ is regular in $E^d$, $g_\psi \downarrow 0$ in $E^d$. Take an arbitrary $\psi$. Then

$$0 < z \leq e_{\alpha_1} = \sup_{\alpha_2 \geq \alpha_1} (p(x_{\alpha_2}) - p(x_{\alpha_1})) = \sup_{\alpha_2 \geq \alpha_1} |p(x_{\alpha_2}) - p(x_{\alpha_1})| \leq g_\psi \quad (\forall \alpha_1 \geq \alpha_\psi),$$

violating $g_\psi \downarrow 0$ in $E^d$, because $\psi$ was taken arbitrary. The obtained contradiction proves $e_\alpha \downarrow 0$.

It follows from

$$p(x_\alpha - x_\beta) \leq p(x_\gamma - x_\alpha) + p(x_\gamma - x_\beta) \leq (e_\alpha + e_\beta) \downarrow 0,$$

that $x_\alpha$ is $p$-Cauchy in the $p$-complete LNL $(X, p, E)$, and hence $x_\alpha$ $p$-converges to some $x \in X$. Therefore, $(X, p, E)$ is a $p$-KB-space. \qed

Remark that the LNL $(c_0, | \cdot |, \ell_{\infty})$ is not $p$-complete, yet its lattice norm is additive on $(c_0)_\alpha$. Consider the sequence $x_n = \sum_{i=1}^{n} e_i$, where $e_i$'s are the standard unit vectors of $c_0$. Then $0 \leq x_n \uparrow$ and $x_n$'s are $p$-bounded by $1 = (1, 1, \cdots) \in \ell_{\infty}$. Clearly, it is not $p$-convergent, so the LNL $(c_0, | \cdot |, \ell_{\infty})$ is not $p$-KB-space. Notice also that $(c_0, | \cdot |, \ell_{\infty})$ is op-continuous.

2.3. Let us discuss Example 1.4 in more details.

(i) If $X$ is an order continuous Banach lattice, then $(X, p, \ell_{\infty}(B_X))$ is op-continuous.

Proof. Assume $x_\alpha \downarrow 0$, we show $p(x_\alpha) \downarrow 0$. Our claim is the following: $p(x_\alpha) \downarrow 0$ iff $p(x_\alpha)[f] \downarrow 0$ for all $f \in B_{X'}$.

For the necessity, let $p(x_\alpha) \downarrow 0$ and $f \in B_{X'}$. Trivially, $|f|(x_\alpha) \downarrow$. Let $z_f \in \mathbb{R}$ be such that $0 \leq z_f \leq |f|(x_\alpha)$ for all $x_\alpha$, then

$$0 \leq z_f \leq |f|(x_\alpha) \leq \|f\| \|x_\alpha\| \downarrow 0.$$

Hence $z_f = 0$ and $p(x_\alpha)[f] = |f|(x_\alpha) \downarrow 0$.

For the sufficiency, let $p(x_\alpha)[f] \downarrow 0$ for every $f \in B_{X'}$. Since $p$ is monotone and $x_\alpha \downarrow$, then $p(x_\alpha) \downarrow$. If $0 \leq \varphi \leq p(x_\alpha)$ for all $x_\alpha$, then

$$0 \leq \varphi(f) \leq p(x_\alpha)[f] \quad (\forall f \in B_{X'}).$$

So, by the assumption, we get $\varphi(f) = 0$ for all $f \in B_{X'}$, and hence $\varphi = 0$. Therefore, $p(x_\alpha) \downarrow 0$. \Box

(ii) If $(X, \| \cdot \|)$ is a KB-space, then $(X, p, \ell_{\infty}(B_X))$ is a $p$-KB-space.

Proof. Suppose that $0 \leq x_\alpha \uparrow$ and $p(x_\alpha) \leq \varphi \in \ell_{\infty}(B_{X'})$. As

$$\|x_\alpha\| = \sup_{f \in B_{X'}} |f(x_\alpha)| \leq \sup_{f \in B_{X'}} |f|(x_\alpha) \leq \sup_{f \in B_{X'}} p(x_\alpha)[f] \leq \sup_{f \in B_{X'}} \varphi[f] \leq \|\varphi\|_{\infty} < \infty \quad (\forall x_\alpha),$$

and since $X$ is a KB-space, we get $\|x_\alpha - x\| \to 0$ for some $x \in X_+$. So, for any $f \in B_{X'}$, we have $|f|(x_\alpha - x) \to 0$ or $p(x_\alpha - x)[f] \to 0$. So $p(x_\alpha - x) \overset{\alpha}{\Rightarrow} 0$ in $\ell_{\infty}(B_{X'})$ and hence $x_\alpha \overset{p}{\Rightarrow} x$. \Box

Recall that a vector lattice $X$ is called perfect if the natural embedding from $X$ into $(X^*_n)_n$ is one-to-one and onto, where $X^*_n$ denotes the order continuous dual of $X$ [2, p.63]. If $X$ is a perfect vector lattice, then $X^*_n$ separates the points of $X$ [2, Thm.1.71(1)].
Proposition 2.6. Let $X$ be a perfect vector lattice, $Y = X^\sim_\vee$ and $p : X \to \mathbb{R}^Y$ be defined as $p(x)[f] := |f|(x)$, where $f \in Y$. Then the LNL $(X, p, \mathbb{R}^Y)$ is a $p$-KB-space.

Proof. Assume $0 \leq x_\alpha \uparrow x$ in $X$ and $p(x_\alpha) \leq \varphi \in \mathbb{R}^Y$. Then, for all $f \in Y$, we have $p(x_\alpha)[f] \leq \varphi(f)$ or $|f|(x_\alpha) \leq \varphi(f)$. So, for all $f \in Y$, sup $|f|(x_\alpha) < \infty$, and hence, by [2, Thm.1.71(2)], there is $x \in X$ with $x_\alpha \uparrow x$. An argument similar to (i) above shows that $X$ is op-continuous. Therefore, $x_\alpha \overset{p}{\to} x$. □

2.4. Fatou space. In this subsection, we introduce and discuss $p$-Fatou spaces.

Definition 2.3. An LNL $(X, p, E)$ is called a $p$-Fatou space if $0 \leq x_\alpha \uparrow x$ in $X$ implies $p(x_\alpha) \uparrow p(x)$.

Note that $(X, p, E)$ is a $p$-Fatou space if $p$ is order semicontinuous [21, 2.1.4, p.48]. Clearly any op-continuous LNL $(X, p, E)$ is a $p$-Fatou space. It is easy to see that the LNL $(c, p, c)$ in Example 1.1 is not a $p$-Fatou space. Moreover the $p$-Fatou property ensures that $p$-bands are bands.

Proposition 2.7. Let $B$ be a $p$-band in a $p$-Fatou space $(X, p, E)$. Then $B$ is a band in $X$.

Proof. Let $B = M^\perp_p = \{x \in X : (\forall m \in M) p(x) \perp p(m)\}$ for some $M \subseteq X$. Since $B$ is an ideal in $X$, in order to show that $B$ is a band, it is enough to prove that $B_+ \ni b_+ \uparrow x \in X$ implies $x \in B$. That is easy, since $p(b_+) \uparrow p(x)$ as $X$ is a $p$-Fatou space. By $\alpha$-continuity of lattice operations in $E$, we obtain

\[0 = p(b_+) \land p(m) \overset{\alpha}{\rightarrow} p(x) \land p(m) \quad (\forall m \in M).\]

Therefore, $p(x) \land p(m) = 0$ for all $m \in M$, and hence $x \in B$. □

In connection with Proposition 2.7 and Example 1.1, the following open problem arises.

Problem 2.2. Let $(X, p, E)$ be a decomposable LNL in which every $p$-band is a band. Is $X$ a $p$-Fatou space?

2.5. $p$-Denseness and $p$-units. In the present paper, we use the following definition of a $p$-dense subset in an LNS, which is motivated by the notion of a dense subset in a normed space.

Definition 2.4. Given an LNS $(X, p, E)$ and $A \subseteq X$. A subset $B \subseteq A$ is said to be $p$-dense in $A$ if, for any $a \in A$ and for any $0 \neq u \in p(X)$, there is $b \in B$ such that $p(a - b) \leq u$.

Remark 2.3.

1. Consider the LNL $(X, p, E)$ with $p = |\cdot|$, $E = \mathbb{X}$, and let $Y$ be a sublattice in $X$. If $Y$ is $p$-dense in $X$, then $Y$ is order dense. Indeed, let $0 \neq x \in X_\vee$, then there is $y \in Y$ such that $|y - \frac{1}{2} x| \leq \frac{1}{2} x$, which implies $0 < \frac{1}{6} x \leq y \leq \frac{1}{3} x$, and so $0 < y \leq x$.

2. $c$ is order dense, but is not $p$-dense in both of the following LNLs: $(\ell_\infty, |\cdot|, \mathbb{R})$ and $(\ell_\infty, |\cdot|, \ell_\infty)$.

3. If $X = (X, |\cdot|)$ is a normed lattice, $p = |\cdot|$, and $E = \mathbb{R}$, then clearly any subset $Y$ of $X$ is $p$-dense iff $Y$ is norm dense.

The following notion is motivated by the notion of a weak order unit in a vector lattice $X = (X, |\cdot|, X)$ and by the notion of a quasi-interior point in a normed lattice $X = (X, |\cdot|, \mathbb{R})$.

Definition 2.5. Let $(X, p, E)$ be an LNL. A vector $e \in X$ is called a $p$-unit if, for any $x \in X_\vee$, we have $p(x - x \land ne) \overset{\alpha}{\rightarrow} 0$.

Remark 2.4. Let $(X, p, E)$ be an LNL.

1. If $X \neq \{0\}$, then, for any $p$-unit $e$ in $X$, there holds $e > 0$. Indeed, let $e$ be a $p$-unit in $X \neq \{0\}$. Trivially, $e \neq 0$. Suppose $e^- > 0$. Then, for $x := e^-$, we obtain

\[p(x - x \land ne) = p(e^- - (e^- \land n(e^- + e^-))) =\]

\[p(e^- - (e^- \land n(-e^-))) = p(e^- - (-ne^-)) = p((n + 1)e^-) =\]

\[(n + 1)p(e^-) \overset{\alpha}{\rightarrow} 0\]

as $n \to \infty$. This is impossible, because $e$ is a $p$-unit. Therefore, $e^- = 0$ and $e > 0$.\]
2. Let \( e \in X \) be a \( p \)-unit. Given \( 0 < \alpha \in \mathbb{R}_+ \) and \( z \in X_+ \). Observe that, for \( x \in X_+ \), \( p(x - n\alpha e \land x) = ap \left( \frac{x}{\alpha} - ne \land \frac{x}{\alpha} \right) \) and \( p(x - n(e + z) \land x) \leq p(x - x \land ne) \), from which it follows easily that \( \alpha e \) and \( e + z \) are both \( p \)-units.

3. If \( e \in X \) is a strong unit, then \( e \) is a \( p \)-unit. Indeed, let \( x \in X_+ \), then there is \( k \in \mathbb{N} \) such that \( x \leq ke \), so \( x - x \land ne = 0 \) for any \( n \geq k \).

4. If \( e \in X \) is a \( p \)-unit, then \( e \) is a weak unit. Assume \( x \land e = 0 \), then \( x \land ne = 0 \) for any \( n \in \mathbb{N} \). Since \( e \) is a \( p \)-unit, then \( p(x) = 0 \) and hence \( x = 0 \).

5. If \( X \) is \( op \)-continuous, then clearly every weak unit of \( X \) is a \( p \)-unit.

6. In \( X = (X, |\cdot|, X) \), the lattice norm \( p(x) = |x| \) is always order continuous. Therefore, the notions of \( p \)-unit and of weak unit coincide in \( X \).

7. If \( X = (X, \|\cdot\|) \) is a normed lattice, \( p = \|\cdot\| \), \( E = \mathbb{R} \), and \( e \in X \), then \( e \) is a \( p \)-unit if \( e \) is a quasi-interior point of \( X \).

In the proof of the following proposition, we use the same technique as in the proof of \([1, \text{Lm.4.15}]\).

**Proposition 2.8.** Let \((X, p, E)\) be an LNL, \( e \in X_+ \), and \( I_e \) be the order ideal generated by \( e \) in \( X \). If \( I_e \) is \( p \)-dense in \( X \), then \( e \) is a \( p \)-unit.

**Proof.** Let \( 0 \neq u \in p(X) \). Let \( x \in X_+ \), then there exists \( y \in I_e \) such that \( p(x - y) \leq u \). Since \( |y^+ \land x - x| \leq |y^+ - x| = |y^+ - x^+| \leq |y - x| \), then, by replacing \( y \) by \( y^+ \land x \), we may assume without loss of generality that there is \( y \in I_e \) such that \( 0 \leq y \leq x \) and \( p(x - y) \leq u \). Thus, for any \( m \in \mathbb{N} \), there is \( y_m \in I_e \) such that \( 0 \leq y_m \leq x \) and

\[ p(x - y_m) \leq \frac{1}{m} u. \]

Since \( y_m \in I_e \), then there exists \( k = k(m) \in \mathbb{N} \) such that \( 0 \leq y_m \leq ke \), and so \( 0 \leq y_m \leq ke \land x \).

For \( n \geq k \), \( x - x \land ne \leq x - x \land ke \leq x - y_m \), and so \( p(x - x \land ne) \leq p(x - y_m) \leq \frac{1}{m} u \). Hence \( p(x - x \land ne) \overset{\alpha}{\rightarrow} 0 \). Thus \( e \) is a \( p \)-unit. \( \square \)

### 3. UNBOUNDED \( p \)-CONVERGENCE

The \( up \)-convergence in LNLs generalizes the \( uo \)-convergence in vector lattices \([14, 16, 17] \), the \( un \)-convergence \([10, 18] \) and the \( uaw \)-convergence \([32, \text{p.501}] \) in Banach lattices. We study basic properties of the \( up \)-convergence and characterize the \( up \)-convergence in certain LNLs.

**3.1. The main definition and its motivation.** Let \((X, p, E)\) be an LNL. The following definition is motivated by its special case when it is reduced to the \( un \)-convergence for a normed lattice \((X, p, E) = (X, \|\cdot\|, \mathbb{R}) = (X, \|\cdot\|)\).

**Definition 3.1.** A net \( x_\alpha \subseteq X \) is said to be unbounded \( p \)-convergent to \( x \in X \) (shortly, \( x_\alpha \overset{up}{\rightarrow} x \)), if

\[ p(|x_\alpha - x| \land u) \overset{\alpha}{\rightarrow} 0 \quad (\forall u \in X_+). \]

It is immediate to see that \( up \)-convergence coincides with \( un \)-convergence in the case when \( p \) is the norm in a normed lattice, and with the \( wo \)-convergence in the case when \( X = E \) and \( p(x) = |x| \). It is clear that \( x_\alpha \overset{p}{\rightarrow} x \) implies \( x_\alpha \overset{up}{\rightarrow} x \), and, for ordered bounded nets, \( up \)-convergence and \( p \)-convergence agree. It should be also clear that, if an LNL \( X \) is \( op \)-continuous, then \( wo \)-convergence in \( X \) implies \( up \)-convergence.

**Proposition 3.1.** Let \( X \) be a vector lattice, and let \( Y \) be a sublattice of the order dual \( X \) of \( X \) which separates points of \( X \). Define \( p : X \rightarrow \mathbb{R}^Y \) by \( p(x)[y] := |y||x| \) (cf. Example 1.5).

Then \( x_\alpha \overset{up}{\rightarrow} x \) in \((X, p, \mathbb{R}^Y)\) iff \( y(|x_\alpha - x| \land u) \rightarrow 0 \) for all \( u \in X_+ \) and for all \( y \in Y \).
Proof. One has \( x_{\alpha} \xrightarrow{up} x \) in \( X \) iff \( p(|x_{\alpha} - x| \wedge u) \xrightarrow{\alpha} 0 \) in \( \mathbb{R}^Y \) for all \( u \in X_+ \) iff
\[
p(|x_{\alpha} - x| \wedge u)[y] = |y|(|x_{\alpha} - x| \wedge u) \to 0 \quad (\forall y \in Y, u \in X_+)
\]
iff
\[
y(|x_{\alpha} - x| \wedge u) \to 0 \quad (\forall y \in Y, u \in X_+).
\]

Recently, in [32, p.501], the following convergence in a Banach lattices was introduced. A net \( x_{\alpha} \) in a Banach lattice \( X \) is uaw-convergent to \( x \in X \) if, for each \( u \in X_+ \), \( |x_{\alpha} - x| \wedge u \to 0 \) weakly. By the Nakano-Roberts theorem, the topological dual \( X' \) of the Banach lattice \( X \) is an ideal in its order dual \( X \) (cf. [3, Thm.2.22]). Proposition 3.1 says that the uaw-convergence in \( X \) is exactly the up-convergence in the LNL \( (X, p, \mathbb{R}^X) \).

3.2. Basic results on up-convergence. We begin with the next list of properties of up-convergence, which follows directly from Lemma 2.1.

**Lemma 3.1.** Let \( x_{\alpha} \xrightarrow{up} x \) and \( y_{\alpha} \xrightarrow{up} y \) in an LNL \( (X, p, E) \), then:

(i) \( ax_{\alpha} + by_{\alpha} \xrightarrow{up} ax + by \) for any \( a, b \in \mathbb{R} \);

(ii) \( x_{\alpha \beta} \xrightarrow{up} x \) for any subnet \( x_{\alpha \beta} \) of \( x_{\alpha} \);

(iii) \( |x_{\alpha}| \xrightarrow{up} |x| \);

(iv) if \( x_{\alpha} \geq y_{\alpha} \) for all \( \alpha \) then \( x \geq y \), in particular, if \( x_{\alpha} = y_{\alpha} \) for all \( \alpha \) then \( x = y \).

**Lemma 3.2.** Let \( x_{\alpha} \) be a monotone net in an LNL \( (X, p, E) \) such that \( x_{\alpha} \xrightarrow{up} x \), then \( x_{\alpha} \xrightarrow{\alpha} x \).

**Proof.** The proof of Proposition 2.2 is applicable here as well. \( \square \)

The following result is a \( p \)-generalization of [18, Lm.1.2(ii)].

**Theorem 3.1.** Let \( x_{\alpha} \) be a monotone net in an LNL \( (X, p, E) \) which up-converges to \( x \). Then \( x_{\alpha} \xrightarrow{p} x \).

**Proof.** Without loss of generality, we may assume that \( 0 \leq x_{\alpha} \uparrow \). From Lemma 3.2 it follows that \( 0 \leq x_{\alpha} \uparrow x \). So \( 0 \leq x - x_{\alpha} \leq x \) for all \( \alpha \). Since, for each \( u \in X_+ \), we know that
\[
p((x - x_{\alpha}) \wedge u) \xrightarrow{\alpha} 0,
\]
in particular, for \( u = x \), we obtain that
\[
p(x - x_{\alpha}) = p((x_{\alpha} - x) \wedge x) \xrightarrow{\alpha} 0.
\]

Similar to [14, Lm.1.2(1)] we have that if \( x_{\alpha} \xrightarrow{up} 0 \) in an LNL \( (X, p, E) \), then \( \inf_{\beta} |y_{\beta}| = 0 \) for any subnet \( y_{\beta} \) of the net \( x_{\alpha} \). Indeed, let \( y_{\beta} \) be a subnet of \( x_{\alpha} \). Clearly, \( y_{\beta} \xrightarrow{up} 0 \). If \( 0 \leq z \leq |y_{\beta}| \) for all \( \beta \), then \( p(z) = p(z \wedge |y_{\beta}|) \xrightarrow{\beta} 0 \), and so \( z = 0 \). Hence \( \inf_{\beta} |y_{\beta}| = 0 \).

The following two results, which are analogies of Lemma 2.8 in [10] and of Lemma 3.6 in [17], we have respectively.

**Lemma 3.3.** Let \( (X, p, E) \) be an LNL. Assume that \( E \) is order complete and \( x_{\alpha} \xrightarrow{up} x \), then \( p(|x| - |x| \wedge |x_{\alpha}|) \xrightarrow{\alpha} 0 \) and \( p(x) = \lim \inf_{\alpha} p(|x| \wedge |x_{\alpha}|) \). Moreover, if \( x_{\alpha} \) is \( p \)-bounded, then \( p(x) \leq \lim \inf_{\alpha} p(x_{\alpha}) \).

**Proof.** Note that
\[
|x| - |x| \wedge |x_{\alpha}| = |x_{\alpha}| \wedge |x| - |x| \wedge |x| \leq |x_{\alpha}| - |x| \wedge |x| \leq |x_{\alpha} - x| \wedge |x|.
\]
Since \( x_{\alpha} \xrightarrow{up} x \), we get \( p(|x| - |x| \wedge |x_{\alpha}|) \xrightarrow{\alpha} 0 \). Thus
\[
p(x) = p(|x|) \leq p(|x| - |x| \wedge |x_{\alpha}|) + p(|x| \wedge |x_{\alpha}|).
\]
So \( p(x) \leq \lim \inf_{\alpha} p(|x| \wedge |x_{\alpha}|) \). Hence \( p(x) = \lim \inf_{\alpha} p(|x| \wedge |x_{\alpha}|) \). \( \square \)
**Lemma 3.4.** Let \((X, p, E)\) be an \(op\)-continuous LNL. Assume that \(E\) is order complete and \(x_\alpha \xrightarrow{uo} x\), then \(p(|x| - |x| \wedge |x_\alpha|) \rightarrow 0\) and \(p(x) = \lim \inf \alpha \ p(|x| \wedge |x_\alpha|)\). Moreover, if \(x_\alpha\) is \(p\)-bounded, then \(p(x) \leq \lim \inf \alpha \ p(x_\alpha)\).

We finish this subsection with the following technical lemma. Recall that a net \(x_\alpha\) in a vector lattice \(X\) is \(uo\)-Cauchy if the net \((x_\alpha - x_{\alpha'})_{(\alpha, \alpha')}\) \(uo\)-converges to zero.

**Lemma 3.5.** Given an LNL \((X, p, E)\). If \(x_\alpha \xrightarrow{p} x\) and \(x_\alpha\) is an \(o\)-Cauchy net, then \(x_\alpha \xrightarrow{o} x\). Moreover, if \(x_\alpha \xrightarrow{p} x\) and \(x_\alpha\) is \(uo\)-Cauchy, then \(x_\alpha \xrightarrow{uo} x\).

**Proof.** Since \(x_\alpha\) is \(o\)-Cauchy, then \(x_\alpha - x_\beta \xrightarrow{o} 0\) as \(\alpha, \beta \rightarrow \infty\). So there exists \(z_\gamma \downarrow \infty\) such that, for every \(\gamma\), there exists \(\alpha_\gamma\) satisfying

\[
|x_\alpha - x_\beta| \leq z_\gamma, \quad \forall \alpha, \beta \geq \alpha_\gamma.
\]

By taking \(p\)-limit over \(\beta\) in (1) and applying Lemma 2.1, we get \(|x_\alpha - x| \leq z_\gamma\) for all \(\alpha \geq \alpha_\gamma\). Thus \(x_\alpha \xrightarrow{o} x\).

For the \(uo\)-convergence, the similar argument is used, so the proof is omitted.

**3.3. up-Convergence and p-units.** The following result is a generalization of [10, Lm. 2.11] and of [16, Cor. 3.5] in LNLs.

First of all, we recall some useful characterizations of order convergence. For any order bounded net \(x_\alpha\) in an order complete vector lattice \(E\), \(x_\alpha \xrightarrow{o} x\) iff \(\lim \sup \alpha \ |x_\alpha - x| = \inf \alpha \sup_{\beta \geq \alpha} |x_\beta - x| = 0\).

Moreover, for any net \(x_\alpha\) in a vector lattice \(E\), \(x_\alpha \xrightarrow{o} 0\) in \(E\) iff \(x_\alpha \xrightarrow{o} 0\) in \(E^6\) (e.g., see [16, Cor.2.9]).

**Theorem 3.2.** Let \((X, p, E)\) be an LNL and \(e \in X_+\) be a \(p\)-unit. Then \(x_\alpha \xrightarrow{up} 0\) iff \(p(|x_\alpha| \wedge e) \xrightarrow{o} 0\) in \(E\).

**Proof.** The “only if” part is trivial. For the “if” part, let \(u \in X_+\), then

\[
|x_\alpha| \wedge u \leq |x_\alpha| \wedge (u - u \wedge ne) + |x_\alpha| \wedge (u \wedge ne) \leq (u - u \wedge ne) + n(|x_\alpha| \wedge e),
\]

and so

\[
p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne) + np(|x_\alpha| \wedge e)
\]

holds in \(E^6\) for any \(\alpha\) and any \(n \in \mathbb{N}\). Hence

\[
\lim \sup \alpha \ p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne) + n \lim \sup \alpha \ p(|x_\alpha| \wedge e)
\]

holds in \(E^6\) for all \(n \in \mathbb{N}\). Since \(p(|x_\alpha| \wedge e) \xrightarrow{o} 0\) in \(E\), then \(p(|x_\alpha| \wedge e) \xrightarrow{o} 0\) in \(E^6\), and so \(\lim \sup \alpha \ p(|x_\alpha| \wedge e) = 0\) in \(E^6\). Thus

\[
\lim \sup \alpha \ p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne)
\]

holds in \(E^6\) for all \(n \in \mathbb{N}\). Since \(e\) is a \(p\)-unit, we have that \(\lim \sup \alpha \ p(|x_\alpha| \wedge u) = 0\) in \(E^6\) or \(p(|x_\alpha| \wedge u) \xrightarrow{o} 0\) in \(E^6\). It follows that \(p(|x_\alpha| \wedge u) \xrightarrow{o} 0\) in \(E\) and hence \(x_\alpha \xrightarrow{up} 0\).

**3.4. up-Convergence and sublattices.** Given an LNL \((X, p, E)\), a sublattice \(Y\) of \(X\), and a net \((y_\alpha)_\alpha \subseteq Y\). Then \(y_\alpha \xrightarrow{up} 0\) in \(Y\) has the meaning that

\[
p(|y_\alpha| \wedge u) \xrightarrow{o} 0 \quad (\forall u \in Y_+).
\]

The following lemma is a \(p\)-version of [17, Lm.3.3].

**Lemma 3.6.** Let \((X, p, E)\) be an LNL, \(B\) be a projection band of \(X\), and \(P_B\) be the corresponding band projection. If \(x_\alpha \xrightarrow{up} x\) in \(X\), then \(P_B(x_\alpha) \xrightarrow{up} P_B(x)\) in both \(X\) and \(B\).
Proof. It is known that $P_B$ is a lattice homomorphism and $0 \leq P_B \leq I$. Since $|P_B(x_\alpha) - P_B(x)| = P_B|x_\alpha - x| \leq |x_\alpha - x|$, then it follows easily that $P_B(x_\alpha) \xrightarrow{up} P_B(x)$ in both $X$ and $B$.

Let $(X, p, E)$ be an LNL and $Y \subseteq X$. Then $Y$ is called up-closed in $X$ if, for any net $y_\alpha$ in $Y$ that is $up$-convergent to $x \in X$, we have $x \in Y$. Clearly, every band is up-closed.

We present a $p$-version of [16, Prop. 3.15] with a similar proof.

**Proposition 3.2.** Let $X$ be an LNL and $Y$ be a sublattice of $X$. Suppose that either $X$ is op-continuous or $Y$ is a $p$-KB-space in its own right. Then $Y$ is up-closed in $X$ iff it is $p$-closed in $X$.

Proof. Only the sufficiency requires a proof. Let $Y$ be $p$-closed in $X$ and $y_\alpha$ be a net in $Y$ with $y_\alpha \xrightarrow{up} x \in X$. Without loss of generality, we assume $(y_\alpha)_\alpha \subseteq Y_\alpha$, because the lattice operations in $X$ are $p$-continuous. Note that, for every $z \in X_+$, $|y_\alpha \land x \land z| \leq |y_\alpha - x| \land z$. So $p(y_\alpha \land x \land z) \leq p(|y_\alpha - x| \land z) \xrightarrow{p} 0$. In particular, $Y \ni y_\alpha \land y \xrightarrow{p} x \land y$ in $X$ for any $y \in Y_\alpha$. Since $X$ is $p$-closed, $x \land y \in Y$ for any $y \in Y_\alpha$. So for any $0 \leq z \in Y_\perp$ and for any $\alpha$, we have $y_\alpha \land z = 0$, then

$$|x \land z| = |y_\alpha \land z - x \land z| \leq |y_\alpha - x| \land z \xrightarrow{p} 0.$$ 

Therefore, $x \land z = 0$, and hence $x \in Y_\perp$. Since $Y_\perp$ is the band generated by $Y$ in $X$, there is a net $(z_\beta)_\beta \in \mathcal{B}$ in the ideal $I_Y$ generated by $Y$ such that $0 \leq z_\beta \uparrow x$ in $X$. Take, for every $\beta$, an element $w_\beta \in Y$ with $z_\beta \leq w_\beta$. Then $x \geq w_\beta \land x \geq z_\beta \land x = z_\beta \uparrow x$ in $X$, and so $w_\beta \land x \xrightarrow{o} x$ in $X$.

Case 1: If $X$ is op-continuous, then $w_\beta \land x \xrightarrow{p} x$. Since $w_\beta \land x \in Y$ and $Y$ is $p$-closed, we get $x \in Y$.

Case 2: Suppose $Y$ is a $p$-KB-space in its own right. Let $\Delta$ be the collection of all finite subsets of the index set $B$. For each $\delta = \{\beta_1, \ldots, \beta_n\} \in \Delta$, let $y_\delta := (w_{\beta_1} \lor \ldots \lor w_{\beta_n}) \land x$. Clearly, $y_\delta \in Y$, $0 \leq y_\delta \uparrow$, and the net $(y_\delta)_\delta$ is $p$-bounded in $Y$. Since $X$ is a $p$-KB-space, then there is $y_0 \in Y$ such that $y_\delta \xrightarrow{p} y_0$ in $Y$ and trivially in $X$. Since $(y_\delta)$ is monotone, then it follows from Proposition 2.2, that $y_\delta \uparrow y_0$ in $X$. Also, we have $y_\delta \xrightarrow{o} x$ in $X$. Thus, $x = y_0 \in Y$.

3.5. $p$-Almost order bounded sets. Recall that a subset $A$ in a normed lattice $(X, \| \cdot \|)$ is said to be almost order bounded if, for any $\epsilon > 0$, there is $u_\epsilon \in X_+$ such that $\|\|x - u_\epsilon\|\| = \|\|x - u_\epsilon \land x\|\| \leq \epsilon$ for any $x \in A$. Similarly we have:

**Definition 3.2.** Given an LNL $(X, p, E)$. A subset $A$ of $X$ is called a $p$-almost order bounded if, for any $w \in E_+$, there is $x_\alpha \in X_+$ such that $p((|x_\alpha - x_\alpha|)\alpha) = \|\|x - u_\epsilon \land x\|\| \leq \epsilon$ for any $x \in A$.

It is clear that $p$-almost order boundedness notion in LNLs is a generalization of almost order boundedness in normed lattices. In the LNL $(X, | \cdot |, X)$, a subset of $X$ is $p$-almost order bounded iff it is order bounded in $X$.

The following result is a $p$-version of [10, Lm. 2.9], and it is also similar to [17, Prop. 3.7].

**Proposition 3.3.** If $(X, p, E)$ is an LNL, $x_\alpha$ is $p$-almost order bounded, and $x_\alpha \xrightarrow{up} x$, then $x_\alpha \xrightarrow{p} x$.

Proof. Since $x_\alpha$ is $p$-almost order bounded, then it is easy to see that the net $(|x_\alpha - x|)\alpha$ is also $p$-almost order bounded. So given $w \in E_+$. Then there exists $x_w \in X_+$ with

$$p(|x_\alpha - x| - |x_\alpha - x| \land x_w) \leq w.$$ 

But $x_\alpha \xrightarrow{up} x$, so $\limsup\alpha p(|x_\alpha - x| \land x_w) = 0$ in $E^\delta$. Thus, for any $\alpha$,

$$p(x_\alpha - x) = p(|x_\alpha - x|) \leq p(|x_\alpha - x| - |x_\alpha - x| \land x_w) + p(|x_\alpha - x| \land x_w) \leq w + p(|x_\alpha - x| \land x_w).$$

Hence

$$\limsup\alpha p(x_\alpha - x) \leq w + \limsup\alpha p(|x_\alpha - x| \land x_w) \leq w.$$
holds in $E^\delta$. But $w \in E_+$ is arbitrary, so $\limsup_{\alpha} p(x_\alpha - x) = 0$ in $E^\delta$. Thus $p(x_\alpha - x) \xrightarrow{\alpha} 0$ in $E^\delta$, and so in $E$.

The following proposition is a $p$-version of [17, Prop.4.2].

**Proposition 3.4.** Given an $op$-continuous and $p$-complete LNL $(X,p,E)$. Then every $p$-almost order bounded $uo$-Cauchy net is $uo$- and $p$-convergent to the same limit.

**Proof.** Let $x_\alpha$ be a $p$-almost order bounded $uo$-Cauchy net. Then the net $(x_\alpha - x_{\alpha'})$ is $p$-almost order bounded and $uo$-converges to $0$. Since $X$ is $op$-continuous, then $x_\alpha - x_{\alpha'} \xrightarrow{up} 0$ and, by Proposition 3.3, we get $x_\alpha - x_{\alpha'} \xrightarrow{\alpha} 0$. Thus $x_\alpha$ is $p$-Cauchy, and so is $p$-convergent. By Lemma 3.5, we get that $x_\alpha$ is also $uo$-convergent to its $p$-limit.

**3.6. rup-Convergence.** Recall that a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice $E$ is relatively uniform convergent (or $r$-convergent, for short) to $x \in E$ if there is $y \in E_+$, such that, for any $\varepsilon > 0$, there exists $\alpha_0 \in A$ satisfying $|x_\alpha - x| \leq \varepsilon y$ for all $\alpha \geq \alpha_0$, \cite[Thm.16.2]{23}. In this case, we write $x_\alpha \xrightarrow{rup} x$.

**Definition 3.3.** A net $x_\alpha$ in an LNL $(X,p,E)$ is said to be relatively unbounded $p$-convergent (rup-convergent) to $x \in X$ if

$$p(|x_\alpha - x| \wedge u) \xrightarrow{\alpha} 0 \quad (\forall u \in X_+).$$

In this case we write $x_\alpha \xrightarrow{rup} x$.

Clearly, rup-convergence implies up-convergence, but the converse need not be true.

**Definition 3.4.** Given an LNL $(X,p,E)$. A vector $e \in X$ is called an $rp$-unit if, for any $x \in X_+$, we have $p(x - x \wedge ne) \xrightarrow{r} 0$ as $n \to \infty$.

Obviously, every $rp$-unit is a $p$-unit. So, by Remark 2.4 (1) after Definition 2.5, if $e \in X \neq \{0\}$ is an $rp$-unit then $e > 0$. Not every $p$-unit is an $rp$-unit. To see this, take $X = (C_b(\mathbb{R}), |\cdot|, C_b(\mathbb{R}))$ and $e = \exp(-|t|)$. Then $e$ is a $p$-unit. However, $e$ is not an $rp$-unit since $p(1 - 1 \wedge ne)$ does not $r$-converge to $0$, where $1(t) \equiv 1$.

The following result generalizes \cite[Cor.3.5]{16} and \cite[Lm.2.11]{10}.

**Proposition 3.5.** Let $(X,p,E)$ be an LNL with an $rp$-unit $e$. Then $x_\alpha \xrightarrow{rup} 0$ iff $p(|x_\alpha| \wedge e) \xrightarrow{r} 0$.

**Proof.** The “only if” part is trivial. For the “if” part, let $u \in X_+$, then

$$|x_\alpha| \wedge u \leq |x_\alpha| \wedge (u - u \wedge ne) + |x_\alpha| \wedge (u \wedge ne) \leq (u - u \wedge ne) + n(|x_\alpha| \wedge e),$$

and so

$$p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne) + np(|x_\alpha| \wedge e)$$

holds for all $\alpha$ and for all $n \in \mathbb{N}$.

Since $e$ is an $rp$-unit, then $p(u - u \wedge ne) \xrightarrow{r} 0$. Then, there exists $y \in E_+$ such that, for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that

$$p(u - u \wedge ne) \leq \frac{\varepsilon}{2} y \quad (\forall n \geq n_\varepsilon).$$

Take an arbitrary $\varepsilon > 0$. It follows from (1) and (2) that

$$p(|x_\alpha| \wedge u) \leq \frac{\varepsilon}{2} y + n_\varepsilon p(|x_\alpha| \wedge e) \quad (\forall \alpha)$$

Since $p(|x_\alpha| \wedge e) \xrightarrow{r} 0$, there exist $z \in E_+$ and $\alpha_\varepsilon$ such that

$$p(|x_\alpha| \wedge e) \leq \frac{\varepsilon}{2n_\varepsilon} z \quad (\forall \alpha \geq \alpha_\varepsilon).$$

Taking $w := y \vee z$ and substituting (4) into (3) gives

$$p(|x_\alpha| \wedge u) \leq \frac{\varepsilon}{2} y + n_\varepsilon \frac{\varepsilon}{2n_\varepsilon} z \leq \varepsilon w \quad (\forall \alpha \geq \alpha_\varepsilon).$$

Since $\varepsilon > 0$ was taken arbitrary, (5) implies that $p(|x_\alpha| \wedge u) \xrightarrow{r} 0$.

\[\square\]
4. up-REGULAR SUBLATTICES

The up-convergence passes obviously to any sublattice of \( X \). As it was remarked in [10, p.3], in opposite to \( wo \)-convergence [16, Thm.3.2], the \( un \)-convergence does not pass even from regular sublattices. These two facts motivate the following notion in LNLs.

**Definition 4.1.** Let \((X, p, E)\) be an LNL and \( Y \) be a sublattice of \( X \). Then \( Y \) is called up-regular if, for any net \( y_\alpha \) in \( Y \), \( y_\alpha \uparrow \) in \( Y \) implies \( y_\alpha \uparrow \) in \( X \). Equivalently, \( Y \) is up-regular in \( X \) when \( y_\alpha \uparrow \) in \( Y \) if and only if \( y_\alpha \uparrow \) in \( X \) for any net \( y_\alpha \) in \( Y \).

It is clear that, if \( Y \) is a regular sublattice of a vector lattice \( X \), then \( Y \) is up-regular in the LNL \((X, \cdot|\cdot, X)\) (see [16, Thm.3.2]). The converse does not hold in general.

**Example 4.1.** Let \( X = \ell_\infty([0,1]) \) be the Banach lattice of all real-valued bounded functions on \([0,1]\) equipped with the sup-norm, and let \( Y \) be its sublattice \( C[0,1] \).

We claim that \( Y = (Y, \cdot|\cdot, X) \) is a up-regular sublattice of \( X = (X, \cdot|\cdot, X) \). Let \( f_\alpha \) be a net in \( Y \) such that \( f_\alpha \uparrow \) in \( Y \). That is \( |f_\alpha| \wedge \alpha \to \infty \) in \( X \) for any \( \alpha \in \mathbb{Y} \). In particular, we have \( |f_\alpha| \wedge |1| \wto \infty \) in \( X \), where \( |1| \) denotes the constant function one. Since \( I \) is a strong unit in \( X \), then it is a p-unit for the LNL \((X, \cdot|\cdot, X)\). It follows from Theorem 3.2, that \( f_\alpha \uparrow \) in \( X \). However, the sublattice \( Y \) is not regular in \( X \). Indeed, for each \( n \in \mathbb{N} \), let \( f_n(t) = t^n \). Then \( f_n \downarrow 0 \) in \( C[0,1] \), but \( f_n \downarrow |x_0| \) in \( \ell_\infty([0,1]) \). So, by Lemma 2.5 in [16], we have that \( Y \) is not regular in \( X \).

Consider the sublattice \( c_0 \) of \( \ell_\infty \). Then \((c_0, \cdot|\cdot, \ell_\infty, \mathbb{R})\) is not up-regular in the LNL \((\ell_\infty, \cdot|\cdot, \ell_\infty, \mathbb{R})\). Indeed, \((e_n)\) is un-convergent in \( c_0 \) but not in \( \ell_\infty \). However, \((c_0, \cdot|\cdot, \ell_\infty)\) is up-regular in the LNL \((\ell_\infty, \cdot|\cdot, \ell_\infty)\).

4.1. Several basic results. We begin with the following result which is a \( p \)-version of [18, Thm.4.3].

**Theorem 4.1.** Let \( Y \) be a sublattice of an LNL \( X = (X, p, E) \). Then \( Y \) is up-regular in each of the following cases:

(i) \( Y \) is majorizing in \( X \); 
(ii) \( Y \) is \( p \)-dense in \( X \); 
(iii) \( Y \) is a projection band in \( X \).

**Proof.** Let \((y_\alpha) \subseteq Y \) be such that \( y_\alpha \uparrow \) in \( Y \). Let \( 0 \neq x \in X_+ \).

(i) There exists \( y \in Y \) such that \( x \leq y \). It follows from 
\[ 0 \leq |y_\alpha| \wedge x \leq |y_\alpha| \wedge y \overset{p}{\to} 0, \]
that \( y_\alpha \uparrow \) in \( X \).

(ii) Choose an arbitrary \( 0 \neq u \in p(X) \). Then there exists \( y \in Y \) with \( p(x - y) \leq u \). Since 
\[ |y_\alpha| \wedge x \leq |y_\alpha| \wedge (x - y) + |y_\alpha| \wedge |y|, \]
then 
\[ p(|y_\alpha| \wedge x) \leq p(|y_\alpha| \wedge (x - y)) + p(|y_\alpha| \wedge |y|) \leq u + p(|y_\alpha| \wedge |y|). \]
Since \( 0 \neq u \in p(X) \) is arbitrary and \( |y_\alpha| \wedge |y| \overset{p}{\to} 0 \), then \( |y_\alpha| \wedge x \overset{p}{\to} 0 \). Hence \( y_\alpha \uparrow \) in \( X \).

(iii) \( Y = Y^\perp \perp \) implies that \( X = Y \oplus Y^\perp \). Hence \( x = x_1 + x_2 \) with \( x_1 \in Y \) and \( x_2 \in Y^\perp \). Since \( y_\alpha \wedge x_2 = 0 \), we have 
\[ p(y_\alpha \wedge x) = p(y_\alpha \wedge (x_1 + x_2)) = p(y_\alpha \wedge x_1) \overset{p}{\to} 0. \]
Hence \( y_\alpha \uparrow \) in \( X \).

The following result deals with a particular case of Example 1.5.

**Theorem 4.2.** Let \( X \) be a vector lattice and \( Y = X_n^\twoheadrightarrow \). Assume \( X_n^\twoheadrightarrow \) separates points of \( X \). Define \( p : X \to \mathbb{R}^Y \) by \( p(x)[y] = |y|(|x|) \). Then any ideal of \( X \) is up-regular in \((X, p, \mathbb{R}^Y)\).
Proof. Let $I$ be an ideal of $X$ and $x_\alpha$ be a net in $I$ such that $x_\alpha \uparrow 0$ in $I$. We show $x_\alpha \uparrow 0$ in $X$. By Proposition 3.1, it is enough to show that $|x_\alpha| \cap u \xrightarrow{\sigma(X,Y)} 0$ for any $u \in X_+$. First note that, if $v \in I^\bot$, then $|x_\alpha| \cap |v| = 0$, and so, for any $w \in (I \oplus I^\bot)_+$, we have $|x_\alpha| \cap w \xrightarrow{\sigma(X,Y)} 0$. Note also that $I \oplus I^\bot$ is order dense (e.g., see [3, Thm.1.12]). Let $u \in X_+$ and $y \in Y$, then there is a net $w_\beta$ in $(I \oplus I^\bot)_+$ such that $w_\beta \uparrow u$, and so $|y|(w_\beta \wedge u) \uparrow |y|(u)$. Given $\varepsilon > 0$. There is $\beta_0$ such that

$$|y|(u) - |y|(w_\beta \wedge u) < \frac{\varepsilon}{2}.$$ 

Also there is $\alpha_0$ such that

$$|y|(x_\alpha \cap w_{\beta_0}) < \frac{\varepsilon}{2}$$

for all $\alpha \geq \alpha_0$. Taking into account the inequality $|a \wedge c - b \wedge c| \leq |a - b|$, we have, for any $\alpha \geq \alpha_0$,

$$|y|(x_\alpha \cap u) = |y|(x_\alpha \cap u) - |y|(x_\alpha \cap u \wedge w_{\beta_0}) + |y|(x_\alpha \cap u \wedge w_{\beta_0}) \leq |y|(u) - |y|(w_{\beta_0} \wedge u) + |y|(x_\alpha \cap w_{\beta_0}) \leq \varepsilon.$$

Since $u \in X_+$ and $y \in Y$ are arbitrary, we get $|x_\alpha| \cap u \xrightarrow{\sigma(X,Y)} 0$ for any $u \in X_+$, and this completes the proof.

The next Corollary might be compared with [18, Cor.4.6].

**Corollary 4.1.** Let $X$ be a vector lattice and $Y = X^\bot_\alpha$ be the order continuous dual. Assume that $X_\alpha^\bot$ separates points of $X$. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] = |y|(x)$. Then any sublattice of $X$ is up-regular in the LNL $(X,p,\mathbb{R}^Y)$.

**Proof.** Let $X_0$ be a sublattice of $X$ and $x_\alpha$ be a net in $X_0$ such that $x_\alpha \uparrow 0$ in $X_0$. Let $I_{X_0}$ be the ideal generated by $X_0$ in $X$. Then $X_0$ is majorizing in $I_{X_0}$, and, by Theorem 4.1(i), we get $x_\alpha \uparrow 0$ in $I_{X_0}$. Now, Theorem 4.2 implies $x_\alpha \uparrow 0$ in $X$.

4.2. The order completion. We remind that the order completion of a vector lattice $X$ is denoted by the symbol $X^\delta$.

**Lemma 4.1.** Let $(X^\delta,p,E)$ be an LNL. For any sublattice $Y \subseteq X$, if $Y^\delta$ is up-regular in $X^\delta$, then $Y$ is up-regular in $X = (X,p,X,E)$.

**Proof.** Take a net $(y_\alpha)_\alpha \subseteq Y$ such that $y_\alpha \uparrow 0$ in $Y$. Then $p(|y_\alpha| \wedge u) \xrightarrow{\alpha} 0$ for all $u \in Y_+$. Let $w \in Y^\delta$ and, since $Y$ is majorizing in $Y^\delta$, there exists $y \in Y$ such that $w \leq y$. Therefore, we obtain $y_\alpha \uparrow 0$ in $Y^\delta$. Since $Y^\delta$ is up-regular in $X^\delta$, the net $y_\alpha$ is up-convergent to $0$ in $X^\delta$, and so in $X$.

**Lemma 4.2.** Let $(X^\delta,p,E)$ be an LNL. For any sublattice $Y \subseteq X$, if $Y$ is up-regular in $X$, then $Y$ is up-regular in $X^\delta$.

**Proof.** Let $(y_\alpha)_\alpha \subseteq Y$ and $y_\alpha \uparrow 0$ in $Y$. By the assumption, $y_\alpha \uparrow 0$ in $X$. Let $u \in X^\delta_+$, then there exists $x \in X$ such that $u \leq x$. Therefore, we obtain $p(|y_\alpha| \wedge x) \leq p(|y_\alpha| \wedge x) \xrightarrow{\alpha} 0$, i.e. $y_\alpha \uparrow 0$ in $X^\delta$.

In connection with Lemma 4.2, the following question arises.

**Problem 4.1.** Is it true that $I^\delta$ is up-regular in $X^\delta$, whenever $I$ is a up-regular ideal in $X$?

**Proposition 4.1.** Let $(X,p,E)$ be an LNL. As $X$ majorizes $X^\delta$ and $X^\delta$ is Dedekind complete, the mappings $p^\delta_L,p^\delta_U : X^\delta \to E^\delta$:

$$p^\delta_L(z) := \sup_{0 \leq x \leq |z|} p(x) \quad p^\delta_U(z) := \inf_{|z| \leq x} p(x) \quad (z \in X^\delta)$$

are well defined (clearly, both $p^\delta_L$ and $p^\delta_U$ are extensions of $p$). Furthermore:

(i) $p^\delta_L$ is a monotone $E^\delta$-valued norm.
(ii) $p^\delta_U$ is a monotone $E^\delta$-valued seminorm;  
(iii) if $X$ is op-continuous, then $p^\delta_U$ is o-continuous (i.e., $z_\gamma \downarrow 0$ in $X^\delta$ implies $p^\delta_U(z_\gamma) \downarrow 0$ in $E^\delta$);  
(iv) if $X$ is op-continuous, then $p^\delta_U = p^\delta_L$.

Proof. (i) Let $X^\delta \ni z \neq 0$. Since $X$ is order dense in $X^\delta$, there is $x \in X$ such that $0 < x \leq |z|$, and so $p^\delta_L(z) \geq p(x) > 0$.

Let $0 \neq \lambda \in \mathbb{R}$, then

$$p^\delta_L(\lambda z) = \sup_{0 \leq x \leq |\lambda z|} p(x) = \sup_{0 \leq \frac{|\lambda|}{|\lambda|} x \leq |z|} p(|\lambda|^{-1} x) = |\lambda| p^\delta_L(z).$$

Let $z, w \in X^\delta$, we show $p^\delta_L(z + w) \leq p^\delta_L(z) + p^\delta_L(w)$. Suppose $0 \leq x \leq |z + w|$, then $0 \leq x \leq |z| + |w|$. By the Riesz Decomposition Property, there exist $x_1, x_2 \in X$ such that $0 \leq x_1 \leq |z|, 0 \leq x_2 \leq |w|, \text{ and } x = x_1 + x_2$. So

$$p(x) = p(x_1 + x_2) \leq p(x_1) + p(x_2) \leq p^\delta_L(z) + p^\delta_L(w).$$

Thus $p^\delta_L(z + w) = \sup_{0 \leq x \leq |z + w|} p(x) \leq p^\delta_L(z) + p^\delta_L(w)$.

Now, we prove the monotonicity of the lattice norm $p^\delta_L$. If $|z| \leq |w|$ then, for any $x \in X$ with $0 \leq x \leq |z|$, we get $0 \leq x \leq |w|$. So

$$p^\delta_L(z) = \sup_{0 \leq x \leq |z|} p(x) \leq \sup_{0 \leq x \leq |w|} p(x) = p^\delta_L(w).$$

(ii) We show firstly the triangle inequality. Let $z, w \in X^\delta$ and $x_1, x_2 \in X$ be such that $|z| \leq x_1$ and $|w| \leq x_2$, then $|z + w| \leq |z| + |w| \leq x_1 + x_2$. So

$$p^\delta_U(z + w) = \inf_{|z + w| \leq x} p(x) \leq p(x_1 + x_2) \leq p(x_1) + p(x_2).$$

Thus $p^\delta_U(z + w) - p(x_1) \leq p(x_2)$ for any $x_2 \in X$ with $|w| \leq x_2$. Hence $p^\delta_U(z + w) - p(x_1) \leq p^\delta_U(w)$ or $p^\delta_U(z + w) - p^\delta_U(w) \leq p(x_1)$, which holds for all $x_1 \in X$ with $|z| \leq x_1$. Therefore, $p^\delta_U(z + w) - p^\delta_U(w) \leq p^\delta_U(z)$ or $p^\delta_U(z + w) \leq p^\delta_U(w) + p^\delta_U(z)$.

Let $0 \neq \lambda \in \mathbb{R}$, then

$$p^\delta_U(\lambda z) = \inf_{|\lambda z| \leq x} p(x) = \inf_{|z| \leq \frac{1}{|\lambda|} x} p(|\lambda|^{-1} x) = |\lambda| p^\delta_U(z).$$

Finally, we check monotonicity of $p^\delta_U$. Let $|z| \leq |w|$, then, for any $x \in X$ with $|w| \leq x$, we have $|z| \leq x$, and hence

$$p^\delta_U(z) = \inf_{|z| \leq x} p(x) \leq \inf_{|w| \leq x} p(x) = p^\delta_U(w).$$

(iii) Assume $z_\gamma \downarrow 0$ in $X^\delta$. Let $A = \{a \in X : z_\gamma \leq a \text{ for some } \gamma\}$. Then $\inf A = 0$. Indeed, if $0 \leq x \leq a$ for all $a \in A$, then $0 \leq x \leq A_\gamma$ for all $\gamma$, where $A_\gamma = \{a \in X : z_\gamma \leq a\}$. So, by [16, Lm.2.7], we have $x \leq z_\gamma$. Thus $x = 0$.

Clearly, $A$ is directed downward and dominates the net $(z_\gamma)_\gamma$. Since $X$ is op-continuous, then $p(A) \downarrow 0$ and, by the definition of $p^\delta_U$, we get that $p(A)$ dominates the net $(p^\delta_U(z_\gamma))$. Therefore, $p^\delta_U(z_\gamma) \downarrow 0$.

(iv) Let $z \in X^\delta$, then $|z| = \sup\{x \in X : 0 \leq x \leq |z|\}$. By (iii), we have

$$p^\delta_U(z) = p^\delta_U(|z|) = \sup\{p^\delta_U(x) : x \in X, 0 \leq x \leq |z|\} = \sup\{p(x) : x \in X, 0 \leq x \leq |z|\} = p^\delta_L(z).$$
In connection with Proposition 4.1(iv), the following question arises.

**Problem 4.2.** Does the equality $p^\delta_U = p^\delta_L$ imply op-continuity of $X$?

**Proposition 4.2.** Let $(X, p, E)$ be an LNL. Then, for every net $x_\alpha$ in $X$,

$$x_\alpha \xrightarrow{up} 0 \text{ in } (X, p, E) \iff x_\alpha \xrightarrow{up} 0 \text{ in } (X^\delta, p^\delta, E^\delta),$$

where $p^\delta = p^\delta_L$.

**Proof.** Assume $x_\alpha \xrightarrow{up} 0$ in $(X, p, E)$. Then $p(|x_\alpha| \wedge x) \xrightarrow{a} 0$ in $E$ for all $x \in X_+$, and so $p(|x_\alpha| \wedge x) \xrightarrow{a} 0$ in $E^\delta$ for all $x \in X_+$, by [16, Cor.2.9]. Hence

$$p^\delta(|x_\alpha| \wedge x) \xrightarrow{a} 0$$

in $E^\delta$ for all $x \in X_+$. Let $u \in X_+^\delta$, then there exists $x_u \in X_+$ such that $u \leq x_u$, since $X$ majorizes $X^\delta$. From (2) it follows that $p^\delta(|x_\alpha| \wedge u) \xrightarrow{a} 0$ in $E^\delta$. Since $u \in X_+^\delta$ is arbitrary, then $x_\alpha \xrightarrow{up} 0$ in $(X^\delta, p^\delta, E^\delta)$.

Conversely, assume $x_\alpha \xrightarrow{up} 0$ in $(X^\delta, p^\delta, E^\delta)$ then, for all $u \in X_+^\delta$, $p^\delta(|x_\alpha| \wedge u) \xrightarrow{a} 0$ in $E^\delta$. In particular, for all $x \in X_+$, $p(|x_\alpha| \wedge x) = p^\delta(|x_\alpha| \wedge x) \xrightarrow{a} 0$ in $E^\delta$. By [16, Cor.2.9], $p(|x_\alpha| \wedge x) \xrightarrow{a} 0$ in $E$ for all $x \in X_+$. Hence $x_\alpha \xrightarrow{up} 0$ in $(X, p, E)$.

5. MIXED-NORMED SPACES

In this section, we study LNLs with mixed lattice norms.

**5.1. Mixed norms.** Let $(X, p, E)$ be an LNS and $(E, \| \cdot \|)$ be a normed lattice. The mixed norm on $X$ is defined by

$$p^-(\|x\|) = \|p(x)\| \quad (\forall x \in X).$$

In this case, the normed space $(X, p^-(\| \cdot \|))$ is called a mixed-normed space (e.g., see [21, 7.1.1, p.292]).

The next proposition follows directly from the basic definitions and results, so its proof is omitted.

**Proposition 5.1.** Let $(X, p, E)$ be an LNL, $(E, \| \cdot \|)$ be a Banach lattice, and $(X, p^-(\| \cdot \|))$ be a mixed-normed space. The following statements hold:

(i) if $(X, p, E)$ is op-continuous and $E$ is order continuous, then $(X, p^-(\| \cdot \|))$ is an order continuous normed lattice;

(ii) if a subset $Y$ of $X$ is $p$-bounded (respectively, $p$-dense) in $(X, p, E)$, then $Y$ is norm bounded (respectively, norm dense) in $(X, p^-(\| \cdot \|))$;

(iii) if $e \in X$ is a $p$-unit and $E$ is order continuous, then $e$ is a quasi-interior point of $(X, p^-(\| \cdot \|))$;

(iv) if $(X, p, E)$ is a $p$-Fatou space and $E$ is order continuous, then $p^-(\| \cdot \|)$ is a Fatou norm, [23, p.42];

(v) if $Y$ is a $p$-almost order bounded subset of $X$, then $Y$ is almost order bounded set in $(X, p^-(\| \cdot \|))$.

**Theorem 5.1.** Let $(X, p, E)$ and $(E, m, F)$ be two $p$-KB-spaces. Then the LNL $(X, m \circ p, F)$ is also a $p$-KB-space.

**Proof.** Let $0 \leq x_\alpha \uparrow$ and $m(p(x_\alpha)) \leq g \in F$. Since $0 \leq p(x_\alpha) \uparrow$ and since $(E, m, F)$ is a $p$-KB-space, then there exists $y \in E$ such that $m(p(x_\alpha) - y) \rightarrow 0$. By Proposition 2.2, $p(x_\alpha) \uparrow y$. Thus the net $x_\alpha$ is increasing and $p$-bounded. Since $X$ is $p$-KB-space, then there exists $x \in X$ such that $p(x_\alpha - x) \rightarrow 0$. As $(E, m, F)$ is $op$-continuous by Corollary 2.2, then $m(p(x_\alpha - x)) \xrightarrow{a} 0$ i.e. $(m \circ p)(x_\alpha - x) \xrightarrow{a} 0$. Thus $(X, m \circ p, F)$ is a $p$-KB-space.

**Corollary 5.1.** Let $(X, p, E)$ be a $p$-KB-space and $(E, \| \cdot \|)$ be a KB-space. Then $(X, p^-(\| \cdot \|))$ is a KB-space.
5.2. up-Completeness. A net \( x_\alpha \) in an LNL \((X, p, E)\) is said to be up-Cauchy if, for every \( u \in X_+ \), \( p(|x_\alpha - x_{\alpha'}| \wedge u) \to 0 \). A LNL \((X, p, E)\) is said to be up-complete if every up-Cauchy net in \((X, p, E)\) is \( p \)-convergent.

We remind the following well-known technical lemma which is a particular case of Lemma 3.5.

**Lemma 5.1.** Given a Banach lattice \((X, \| \cdot \|)\). If \( x_\alpha \xrightarrow{p} x \) and \( x_\alpha \) is \( o \)-Cauchy, then \( x_\alpha \xrightarrow{o} x \).

Recall also that a Banach lattice is called \( un \)-complete if every \( un \)-Cauchy net is \( un \)-convergent, [18].

**Theorem 5.2.** Let \((X, p, E)\) be an LNL and \((E, \| \cdot \|)\) be an order continuous Banach lattice. If \((X, p\| \cdot \|)\) is a up-complete Banach lattice, then \( X \) is up-complete.

**Proof.** Let \( x_\alpha \) be a up-Cauchy net in \( X \). So, for every \( u \in X_+ \), \( p(|x_\alpha - x_{\beta}| \wedge u) \to 0 \). Since \( E \) is order continuous, then, for every \( u \in X_+ \), \( p(|x_\alpha - x_{\beta}| \wedge u) \to 0 \) or for every \( u \in X_+ \), \( p-\|x_\alpha - x_{\beta}\| \wedge u) \to 0 \), i.e. \( x_\alpha \) is \( un \)-Cauchy in \((X, p\| \cdot \|)\). Since \((X, p\| \cdot \|)\) is up-complete, then there exists \( x \in X \) such that \( x_\alpha \xrightarrow{up} x \) in \((X, p\| \cdot \|)\). That is, for every \( u \in X_+ \), \( p(|x_\alpha - x| \wedge u) \to 0 \). Next we show the net \((p(|x_\alpha - x| \wedge u))\) is order Cauchy in \( E \). Indeed,

\[
p(|x_\alpha - x| \wedge u) - p(|x_{\beta} - x| \wedge u) \leq p(|x_\alpha - x| \wedge u) - p(|x_{\beta} - x| \wedge u) \xrightarrow{o} 0.
\]

Now, Lemma 5.1 above, implies that \( p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0. \)

5.3. up-Null nets and up-null sequences in mixed-normed spaces. The following theorem is a \( p \)-version of [10, Thm.3.2] and a generalization of [16, Lm.6.7], as we take \((X, p, E) = (X, \| \cdot \|, \mathbb{R})\).

**Theorem 5.3.** Let \((X, p, E)\) be an op-continuous and \( p \)-complete LNL, \( E \) an order continuous Banach lattice, and \( X \ni x_\alpha \overset{up}{\to} 0 \). Then there exists an increasing sequence \( \alpha_k \) of indices and a disjoint sequence \( d_k \in X \) such that \( (x_{\alpha_k} - d_k) \xrightarrow{p} 0 \) as \( k \to \infty \).

**Proof.** Consider the mixed norm \( p-\|x\| = \|p(x)\| \). Since \( p(|x_\alpha \wedge u) \xrightarrow{o} 0 \) for all \( u \in X_+ \), then \( p-\|x_\alpha \wedge u\| = \|p(x_\alpha \wedge u)\| \xrightarrow{o} 0 \), that means \( x_\alpha \xrightarrow{op} 0 \) in \((X, p\| \cdot \|)\), by \( o \)-continuity of \((E, \| \cdot \|)\).

By [10, Thm.3.2], there exist an increasing sequence \( \alpha_n \) of indices and a disjoint sequence \( d_n \) in \( X \) such that \( x_{\alpha_n} - d_n \xrightarrow{p\| \cdot \|} 0 \). Now \((X, p\| \cdot \|)\) is a Banach lattice, by [21, 7.1.3 (1), p.294]. So, by [29, Thm.VII.2.1], there is a further subsequence \( (\alpha_{n_k}) \) such that \( |x_{\alpha_{n_k}} - d_{n_k}| \xrightarrow{o} 0 \) in \( X \). By \( op \)-continuity of \( X \), \( p(x_{\alpha_{n_k}} - d_{n_k}) \xrightarrow{o} 0 \).

The next corollary is a \( p \)-version of [10, Cor.3.5].

**Corollary 5.2.** Let \((X, p, E)\) be an op-continuous LNL, \( E \) be an order continuous Banach lattice, and \( X \ni x_\alpha \overset{up}{\to} 0 \). Then there exists an increasing sequence \( \alpha_k \) of indices such that \( x_{\alpha_k} \overset{up}{\to} 0 \).

**Proof.** Let \( \alpha_k \) and \( d_k \) be as in Theorem 5.3. Since the sequence \( d_k \) is disjoint, then \( d_k \xrightarrow{up} 0 \), by [16, Cor.3.6.]. Since \( X \) is \( op \)-continuous, then \( d_k \xrightarrow{p} 0 \). Since

\[
p(|x_{\alpha_k} - d_k| \wedge u) \leq p(x_{\alpha_k} - d_k) \xrightarrow{o} 0 \quad (\forall u \in X_+),
\]

then \( x_{\alpha_k} - d_k \xrightarrow{up} 0 \). Since \( d_k \xrightarrow{up} 0 \), then \( x_{\alpha_k} \xrightarrow{up} 0 \).

The following proposition extends [10, Prop.4.1] to LNLs.

**Proposition 5.2.** Let \((X, p, E)\) be a \( p \)-complete LNL, \((E, \| \cdot \|)\) be an order continuous Banach lattice, and \( X \ni x_n \overset{up}{\to} 0 \). Then there exists a subsequence \( x_{n_k} \) of \( x_n \) such that \( x_{n_k} \xrightarrow{op} 0 \) as \( k \to \infty \).
Proof. Suppose \( x_n \xrightarrow{up} 0 \), then, for all \( u \in X_+ \), \( p(|x_n| \wedge u) \xrightarrow{\omega} 0 \), and so \( \|p(|x_n| \wedge u)\| \to 0 \), since \( E \) is order continuous. Thus \( |x_n| \wedge u \xrightarrow{p-\|\|} 0 \), i.e. \( x_n \xrightarrow{un} 0 \) in \( (X, p-\|\|) \). It follows from [21, 7.1.3(1), p.294] that the mixed-normed space \( (X, p-\|\|) \) is a Banach lattice, and so, by [10, Prop.4.1], there is a subsequence \( x_{nk} \) of \( x_n \) such that \( x_{nk} \xrightarrow{uo} 0 \) as \( k \to \infty \).

The next result is a \( p \)-version of [10, Thm.4.4].

**Proposition 5.3.** Let \( (X, p, E) \) be an op-continuous and \( p \)-complete LNL such that \( (E, \|\|) \) is an order continuous atomic Banach lattice. Then a sequence in \( X \) is \( up \)-null iff every its subsequence has a further subsequence which \( uo \)-converges to zero.

**Proof.** The necessity follows from Proposition 5.2.

For the sufficiency, let \( x_n \) be a sequence in \( X \) such that, every subsequence of \( x_n \) has a further subsequence, which is \( uo \)-null, yet \( x_n \xrightarrow{up} 0 \). Then, for some \( u \in X_+ \), \( p(|x_n| \wedge u) \xrightarrow{\omega} 0 \). Since the sequence \( p(|x_n| \wedge u) \) is order bounded in \( E \), [10, Lm.5.1] implies \( \|p(|x_n| \wedge u)\| \neq 0 \). Thus there exists a subsequence \( x_{nk} \) such that \( \|p(|x_{nk}| \wedge u)\| \geq r \) for some real \( r > 0 \) and for all \( k \).

By the compactness of the order intervals in the order continuous atomic Banach lattices, \( x_{nk} \) has further subsequence \( x_{nk_{k}} \) such that \( p(|x_{nk_{k}}| \wedge u) \xrightarrow{\|\|} y \in [0, p(u)] \). By the assumption, \( x_{nk_{k}} \) has a further subsequence \( x_{nk_{k_{k}}} \) such that \( x_{nk_{k_{k}}} \xrightarrow{uo} 0 \) and hence \( |x_{nk_{k_{k}}}| \wedge u \xrightarrow{\omega} 0 \). Since \( (X, p, E) \) is op-continuous, \( p(|x_{nk_{k_{k}}}| \wedge u) \xrightarrow{\omega} 0 \). Order continuity of \( E \) implies \( \|p(|x_{nk_{k_{k}}}| \wedge u)\| \xrightarrow{\omega} 0 \) violating \( \|p(|x_{nk_{k_{k}}}| \wedge u)\| \geq r > 0 \). The obtained contradiction completes the proof. 

Our last result is a \( p \)-version of [10, Lm.5.1].

**Proposition 5.4.** Let \( (X, p, E) \) be an op-continuous \( p \)-complete LNL and \( (E, \|\|) \) be an order continuous Banach lattice. If \( X \) is atomic and \( x_n \) is an order bounded sequence such that \( x_n \xrightarrow{p} 0 \) in \( X \), then \( x_n \xrightarrow{\omega} 0 \).

**Proof.** The mixed-normed space \( (X, p-\|\|) \) is an atomic order continuous Banach lattice such that \( x_n \xrightarrow{p-\|\|} 0 \), and so \( x_n \xrightarrow{\omega} 0 \), by [10, Lm.5.1].

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