The Podleš quantum sphere $S^2_q$ admits a natural commutative $C^*$-subalgebra $I_q$ with spectrum $\{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\}$, which may therefore be considered as a quantized version of a classical interval. We study here the compact quantum metric space structure on $I_q$ inherited from the corresponding structure on $S^2_q$, and provide an explicit formula for the metric induced on the spectrum. Moreover, we show that the resulting metric spaces vary continuously in the deformation parameter $q$ with respect to the Gromov-Hausdorff distance, and that they converge to a classical interval of length $\pi$ as $q$ tends to 1.

1. Introduction

The study of compact quantum metric spaces dates back to the work of Connes [Co89], in which he studied metrics on state spaces of spectral triples. This notion was later formalised in the works of Rieffel [Ri98, Ri99, Ri05], in which the weak $*$-topology on the state space is metrised by the Monge-Kantorovich metric coming from a so-called Lip-norm on a $C^*$-algebra (see Section 2 for details). As shown by Rieffel, the classical Gromov-Hausdorff distance admits an analogue, known as quantum Gromov-Hausdorff distance, for compact quantum metric spaces, and this notion was later refined by Latrémolière through his notion of propinquity [La16]. Although examples of compact quantum metric spaces are abundant, some of the most basic examples from non-commutative geometry are not well understood from this point of view, and only very recently, Aguilar and Kaad [AK18] showed that the Podleš standard sphere $S^2_q$, introduced as a homogeneous space of Woronowicz’ $q$-deformed $SU(2)$ [Po87, Wo87], admits a natural compact quantum metric space structure stemming from its non-commutative geometry. More precisely, Aguilar and Kaad show that the Lip-norm arising from the Dirac operator $D_q$ of the Dąbrowski-Sitarz spectral triple [DS03], does indeed provide a quantum metric structure on $S^2_q$. The main question left open in [AK18] is that of quantum Gromov-Hausdorff convergence of $S^2_q$ to the classical 2-sphere $S^2$ as the deformation parameter tends to 1. This question seems rather difficult to settle\footnote{We are currently working on this.}, and the aim of the present paper is to show that the Podleš sphere $S^2_q$ contains a natural commutative $C^*$-algebra $I_q$ for which the corresponding convergence question can be settled, and that the answer supports the more general conjecture that $S^2_q$ converges to $S^2$ as $q$ tends to 1. The Podleš sphere is generated by a self-adjoint operator $A$ and a non-normal operator $B$ (see Section 2 for precise definitions), and the $C^*$-algebra $I_q$ is simply the unital $C^*$-algebra generated by $A$ inside $S^2_q$. Since $S^2_q$ admits a rather accessible representation on $B(ℓ^2(\mathbb{N}_0))$ [Po87, Proposition 4], the spectrum of the self-adjoint generator $A \in S^2_q$ is easily derivable, and one finds that for $q \in (0, 1)$ this is exactly the set \[ 2\pi \mathbb{Z} / q^2 \mathbb{Z} \]
which can therefore be viewed as a quantised version of a classical interval. The Lip-norm $L_{D_q}$ coming from the Dirac operator on $S^2_q$ therefore, in particular, provides a metric on the state space of $I_q \cong C(X_q)$ and embedding $X_q$ into the state space of $C(X_q)$ as point-evaluations, we obtain a metric $d_q$ on $X_q$. Our first main result determines an explicit formula for this metric.

**Theorem A.** For $q \in (0,1)$, the metric $d_q$ on $X_q$ is given by the following formula:

\[
d_q(x, y) := \begin{cases}
0 & \text{if } x = y \\
\sum_{k=\min(m,n)}^{\max(m,n)-1} \frac{(1 - q^2)q^k}{\sqrt{1 - q^{2(k+1)}}} & \text{if } x = q^{2n} \text{ and } y = q^{2m} \text{ with } n \neq m \\
\sum_{k=n}^{\infty} \frac{(1 - q^2)q^k}{\sqrt{1 - q^{2(k+1)}}} & \text{if } x = q^{2n} \text{ and } y = 0 \text{ or } x = 0 \text{ and } y = q^{2n}.
\end{cases}
\]

When $q = 1$, the spectrum of the operator $A$ becomes $X_1 := [0,1]$ and in Section 3.1 we will show that when $X_1$ is equipped with the metric $d_1$ inherited from the classical 2-sphere $S^2$, then the space $(X_1, d_1)$ becomes isometrically isomorphic to $[-\pi/2, \pi/2]$ with its standard Euclidian metric. Our second main theorem therefore confirms that the quantised intervals do indeed converge to the appropriate classical interval as the deformation parameter tends to 1:

**Theorem B.** The metric spaces $(X_q, d_q)$ vary continuously with respect to the Gromov-Hausdorff distance in the deformation parameter $q \in (0,1)$ and converge to the interval $[-\pi/2, \pi/2]$ with its standard metric as $q$ tends to 1.

On the class of commutative compact quantum metric spaces, convergence in both Latrémolière’s propinquity [La16] and Rieffel’s quantum Gromov-Hausdorff distance [Ri04] is implied by convergence in classical Gromov-Hausdorff distance (see Remark 3.7) and Theorem B therefore settles all the natural convergence question for the algebras $I_q \cong C(X_q)$.

The paper is structured as follows: The first part will introduce the basic definitions concerning quantum metric spaces, Gromov-Hausdorff distance, $SU_q(2)$ and the standard Podleś sphere and the associated Dąbrowski-Sitarz spectral triple. In the second part we first give a description of $I_q$ in the continuum? case, i.e. when $q = 1$, followed by a thorough treatment of the quantised? case, where $SU(2)$ is deformed by a parameter $q \in (0,1)$. For this we provide a thorough treatment of the metric $d_q$, on $X_q$ and its Lipschitz semi-norm from which we can prove Theorem A and finally we use this to prove Theorem B.

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**Standing conventions.** The semi-norms appearing in this text are defined everywhere on unital $C^*$-algebras and may take the value infinity.
2. Preliminaries

2.1. Quantum metric spaces. We begin this section by recalling some basic facts about metric spaces. Let \((X, d)\) be a compact metric space. The Lipschitz semi-norm, \(L_d: C(X) \to [0, \infty]\), on \(C(X)\) is defined by the formula

\[
L_d(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \neq y \right\}; \quad f \in C(X).
\]

A continuous function \(f : X \to \mathbb{C}\) is then said to be a Lipschitz function when \(L_d(f) < \infty\) and in this case \(L_d(f)\) agrees with the Lipschitz constant. The Lipschitz functions on \(X\) form a *-subalgebra which we denote by \(C_{\text{Lip}}(X) \subset C(X)\). Given subsets \(A, B \subset X\), their Hausdorff-distance is defined as

\[
\text{dist}^d_H(A, B) := \inf \{ r > 0 | A \subset \mathcal{B}(B, r) \text{ and } B \subset \mathcal{B}(A, r) \},
\]

where \(\mathcal{B}(A, r)\) denotes the set \(\{ x \in X : \exists a \in A : d(x, a) < r \}\). For two metric spaces \((X, d_X), (Y, d_Y)\), their Gromov-Hausdorff distance is defined as

\[
\text{dist}_{GH}(X, Y) = \inf \{ \text{dist}^d_H(\iota_X(X), \iota_Y(Y)) \},
\]

where the infimum ranges over all metric spaces \((Z, d_Z)\) and all isometric embeddings \(\iota_X : X \to Z\) and \(\iota_Y : Y \to Z\). Next, we will recall the relevant definitions for quantum metric spaces.

**Definition 2.1** ([1098] [1099] [1105]). Let \(A\) be a unital \(C^*\)-algebra, and let \(L : A \to [0, \infty]\) be a semi-norm. We say that \((A, L)\) is a compact quantum metric space, and that \(L\) is a Lip-norm, if the following conditions are satisfied:

1. \(\text{Dom}(L) := \{ a \in A : L(a) < \infty \}\) is dense in \(A\);
2. \(L\) is *-invariant and lower semi-continuous on \(A\);
3. \(\ker(L) := \{ a \in A : L(a) = 0 \} = \mathbb{C}1_A\);
4. The Monge-Kantorovich metric on the state space \(S(A)\) of \(A\), given by

\[
\text{mk}_L(\mu, \nu) := \sup \{ |\mu(a) - \nu(a)| : a \in A, L(a) \leq 1 \}, \quad \text{for } \mu, \nu \in S(A)
\]

metrises the weak *-topology.

The model example for a compact quantum metric space is, unsurprisingly, \((C(X), L_d)\) where \((X, d)\) is a compact metric space. In this case it is a well-known fact that the Monge-Kantorovich metric recaptures the metric \(d\) on \(X\) when the latter is viewed as a subset of the state space of \(C(X)\):

\[
d(x, y) = \sup \{ |f(x) - f(y)| : f \in C(X), L_d(f) \leq 1 \}.
\]

Another interesting class of examples, which dates back to the work of Connes [1089], comes from certain spectral triples: the setting is thus that of a separable Hilbert space \(H\) with a self-adjoint densely defined operator \(D : \text{Dom}(D) \to H\), and a unital \(C^*\)-algebra \(A\) represented on \(H\) via a *-homomorphism \(\rho : A \to B(H)\). Then one can define the Lipschitz algebra \(\text{Lip}_D(A)\), to consist of all elements \(x \in A\) which preserve \(\text{Dom}(D)\), and for which \([D, \rho(x)] : \text{Dom}(D) \to H\) admits a bounded extension to \(H\), which will be denoted by \(\partial(x) \in B(H)\). Clearly, \(\text{Lip}_D(A) \subset A\) is a *-subalgebra and it follows from the definition of a spectral triple that \(\text{Lip}_D(A) \subset A\) is norm-dense. From the spectral triple \((A, H, D)\), we also obtain a semi-norm as follows:

**Definition 2.2.** Define \(L_D : A \to [0, \infty]\) by the formula

\[
L_D(x) := \sup \{ |\langle \xi, \rho(x^*)D\eta \rangle - \langle \rho(x)D\xi, \eta \rangle| : \xi, \eta \in \text{Dom}(D), \|\xi\| = \|\eta\| = 1 \}.
\]
A first result says that \( x \in \text{Lip}_D(A) \) exactly when \( L_D(x) \) is finite, and in this case \( L_D(x) = \| \partial(x) \| \), see e.g. [AK18, Lemma 2.3]. Moreover, \( L_D : A \to [0, \infty] \) is lower semi-continuous and \(*\)-invariant, see [R99, Proposition 3.7]. The above construction does in general not yield a quantum metric space, but due to the work of Rieffel, there are tools available for verifying whether or not this is the case (see for instance [R98, Theorem 1.8]).

Quantum analogues of the Gromov-Hausdorff distance have been defined by Rieffel and La-trémolière, and we refer the reader to [Ri04, La16] for concrete definitions. For our purposes, it suffices to know that when the compact quantum metric spaces in question are of the form \((C(X), L_d)\), then both analogues are dominated by the classical Gromov-Hausdorff distance, see Remark 3.7.

2.2. The Standard Podleś Sphere. The central object of interest in this paper is the standard Podleś quantum sphere, which is defined as a particular \(*\)-subalgebra of Woronowicz’ quantum group \( SU_q(2) \) as given below. Fix \( q \in (0, 1] \), and let \( SU_q(2) \) denote the universal unital \(*\)-algebra with generators \( a \) and \( b \) defined such that the following relations are satisfied:

\[
ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b
\]

We denote the unital \(*\)-subalgebra generated by \( a \) and \( b \) by \( \mathcal{O}(SU_q(2)) \), and by \( \mathcal{O}(S^2_q) \) the unital \(*\)-subalgebra of \( \mathcal{O}(SU_q(2)) \) generated by the elements \( A := b^*b \) and \( B := ab^* \).

The standard Podleś quantum sphere, \( S^2_q \), is defined as the norm-closure of \( \mathcal{O}(S^2_q) \subset SU_q(2) \) [Po87]. We remark that from the defining relations of \( SU_q(2) \) we obtain a similar set of relations for \( A \) and \( B \):

\[
AB = q^2BA, \quad A = A^*
\]

\[
BB^* = q^{-2}A(1 - A), \quad B^*B = A(1 - q^2A).
\]

The \(*\)-algebra \( SU_q(2) \) comes equipped with a natural faithful state, called the Haar state, which we denote by \( h : SU_q(2) \to \mathbb{C} \), see e.g. [KS97, Section 11.3.2]. We let \( L^2(SU_q(2)) \) denote the separable Hilbert space obtained by applying the GNS-construction to the \(*\)-algebra \( SU_q(2) \) equipped with the Haar state.

From now on, we assume that \( q \neq 1 \). Define an automorphism \( \partial_k \) on \( \mathcal{O}(SU_q(2)) \) by \( \partial_k(x) = q^{1/2}x \) if \( x \in \{a, b\} \), and \( \partial_k(x) = q^{-1/2}x \) if \( x \in \{a^*, b^*\} \), and for each \( n \in \mathbb{Z} \), define the vector subspaces

\[
A_n := \{ x \in \mathcal{O}(SU_q(2)) : \partial_k(x) = q^{n/2}x \} \subset \mathcal{O}(SU_q(2)).
\]

It turns out that \( A_0 = \mathcal{O}(S^2_q) \) and that the algebra structure on \( \mathcal{O}(SU_q(2)) \) allows us to consider each \( A_n \) as a left module over \( \mathcal{O}(S^2_q) \). We let \( H_+ \) and \( H_- \) denote the separable Hilbert spaces obtained by taking the Hilbert space closures of \( A_1 \) and \( A_{-1} \) (respectively) when considered as subspaces of \( L^2(SU_q(2)) \). The GNS-representation of \( SU_q(2) \) on \( L^2(SU_q(2)) \) (when properly restricted) then provides us with two unital \(*\)-homomorphisms \( \rho_+ : S^2_q \to B(H_+) \) and \( \rho_- : S^2_q \to B(H_-) \).
By [DS03] there exists an even spectral triple, \((S_q^2, H_+ \oplus H_-, D_q)\), where the representation in question is given by the direct sum \(\rho : \rho_+ \oplus \rho_- : S_q^2 \to B(H_+ \oplus H_-)\). For an explicit construction of the Dirac operator \(D_q : \text{Dom}(D_q) \to H_+ \oplus H_-\), we refer to [DS03, NeTu05] or [AK18].

For \(x \in \text{Lip}_{D_q}(S_q^2)\), the associated operator \(\partial(x)\) (obtained as the closure of \([D_q, \rho(x)]\)) takes the form

\[
\begin{pmatrix}
0 & \partial_2(x) \\
\partial_1(x) & 0
\end{pmatrix} : H_+ \oplus H_- \to H_+ \oplus H_-,
\]

where \(\partial_1 : \text{Lip}_{D_q}(S_q^2) \to B(H_+, H_-)\) and \(\partial_2 : \text{Lip}_{D_q}(S_q^2) \to B(H_-, H_+)\) are derivations satisfying \(\partial_2(x^*) = -\partial_1(x)^*\) (remark in this respect that \(B(H_+, H_-)\) and \(B(H_-, H_+)\) can be considered as bimodules over \(S_q^2\) via the representations \(\rho_+\) and \(\rho_-\)). Consequently the Lip-norm is, for \(x \in \text{Lip}_{D_q}(S_q^2)\), given by

\[
L_{D_q}(x) = \max \{\|\partial_1(x)\|, \|\partial_1(x^*)\|\}.
\]

By [Po87] Proposition 4], \(S_q^2\) admits a faithful representation, \(\pi : S_q^2 \to B(\ell^2(\mathbb{N}_0))\), defined by

\[
\pi(A)(e_k) := q^{2k}e_k, \quad \pi(B)(e_k) = q^k\sqrt{1 - q^{2(k+1)}}e_{k+1},
\]

where \(e_k\) denotes the characteristic function on the point-set \(\{k\} \subset \mathbb{N}_0\). In fact, this representation even provides a \(*\)-isomorphism to the unitisation of the compact operators on \(\ell^2(\mathbb{N}_0)\). Using this representation it is easy to see that the spectrum of the operator \(A\) for a specific \(q \in (0, 1)\) is given by

\[
X_q := \{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\}.
\]

Hence the indicator functions \(\chi_{\{q^{2k}\}} : X_q \to \{0, 1\}\) are continuous for all \(k\). In fact, the continuous indicator functions generate \(C(X_q)\), since any continuous function, \(f : X_q \to \mathbb{C}\), can be written as \(\sum_{k=0}^{\infty} f(q^{2k})\chi_{\{q^{2k}\}}\), where \(\lim_{k \to \infty} f(q^{2k}) = f(0)\). By [AK18] Theorem 8.3], \((S_q^2, L_{D_q})\) is a compact quantum metric space, and consequently so is \(I_q := C^*(A, 1) \cong C(X_q)\) with the restricted Lip-norm. The compact quantum metric space \((I_q, L_{D_q})\) is our main object of interest in the present paper. As \(I_q\) is commutative, the Lip-norm \(L_{D_q}\) defines a genuine metric \(d_q\) on \(X_q\) when the latter is considered as a subset of the state space \(S(S_q^2)\). In order to describe \(d_q\) explicitly, the following lemma will be key:

**Lemma 2.3** ([AK18] Lemma 5.3]). Let \(k \in \mathbb{N}_0\) and let \(q \in (0, 1)\). We have that \(\chi_{\{q^{2k}\}}(A) \in \text{Lip}_{D_q}(S_q^2)\) and the derivative is given by

\[
\partial_1(\chi_{\{q^{2k}\}}(A)) = \frac{1}{q^{2k}(1 - q^2)}\chi_{\{q^{2k}\}}(A) \cdot b^*a^* - \frac{1}{q^{2(k-1)}(1 - q^2)}\chi_{\{q^{2(k-1)}\}}(A) \cdot b^*a^*.
\]

In particular, we obtain that

\[
\partial_1(f(A)) = \sum_{k=0}^{\infty} \frac{f(q^{2k}) - f(q^{2(k+1)})}{q^{2k}(1 - q^2)}\chi_{\{q^{2k}\}}(A) \cdot b^*a^* \tag{2}
\]

for every \(f \in \text{span}_\mathbb{C}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\}\).
Remark 2.4. The formula in (2) for \( \partial_t(f(A)) \) is related to the notion of \( q \)-differentiation from \( q \)-calculus. Indeed, the \( q^2 \)-differentiation of \( f \in \text{span}_\mathbb{C}\{\chi_{q^k} : k \in \mathbb{N}_0\} \) would be given by
\[
\mathcal{D}_{q^2}(f) = \sum_{k=0}^{\infty} \frac{f(q^{2k}) - f(q^{2(k+1)})}{q^{2k}(1 - q^2)} \chi_{q^{2k}},
\]
see for example [KS97, Chapter 2.2]. The extra term \( b^*a^* \) appearing in (2) comes from the geometry of the quantised 2-sphere as it operates between the Hilbert space completions \( H_+ \) and \( H_- \) of the quantised spinor bundles \( A_1 \) and \( A_{-1} \).

3. Metric Properties of the Quantised Interval

In this section we first provide the explicit descriptions of the compact metric spaces \((X_q, d_q)\) which encode the compact quantum metric space structure of \((I_q, L_{D_q})\). More precisely, the algebra of Lipschitz functions of the metric space \((X_q, d_q)\) must agree with the Lipschitz algebra \( \text{Lip}_{D_q}(C^0_q) \cap I_q \) and the two semi-norms must agree, in the sense that \( L_{D_q}(f(A)) = L_{d_q}(f) \) whenever \( f \) is a Lipschitz function on \((X_q, d_q)\). This analysis is separated into the case \( q = 1 \), referred to as the continuum case, and the case \( q < 1 \), referred to as the quantised case.

3.1. The continuum case. We consider the 2-sphere \( S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \) whereas \( S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \) both equipped with the subspace topology coming from the usual topology on \( \mathbb{R}^3 \) and \( \mathbb{C}^2 \).

In the situation where \( q = 1 \) we have a homeomorphism between the characters of \( SU_q(2) \) and the 3-sphere \( S^3 \), which sends \((z, w) \in S^3 \subset \mathbb{C}^2 \) to the unique character \( \chi_{z,w} \) satisfying that \( \chi_{z,w}(a) = z \) and \( \chi_{z,w}(b) = w \) (see [Wo87]). Consequently, we can identify \( SU_q(2) \) with \( C(S^3) \) such that \( a(z, w) = z \) and \( b(z, w) = w \). We may moreover view the 2-sphere \( S^2 \) as the quotient space of \( S^3 \) under the circle action \( \lambda \cdot (z, w) := (\lambda \cdot z, \lambda \cdot w) \) and this identification happens via the Hopf-fibration
\[
S^3 \ni (z, w) \mapsto (2\text{Re}(z\bar{w}), 2\text{Im}(z\bar{w}), |z|^2 - |w|^2) \in S^2.
\]
Since both \( A(z, w) = (b^*b)(w) = |w|^2 \) and \( B(z, w) = z\bar{w} \) are invariant under the circle action we may consider them as continuous function on \( S^2 \) and as such they are given by
\[
A(x_1, x_2, x_3) = \frac{1 - x_3}{2} \quad \text{and} \quad B(x_1, x_2, x_3) = \frac{x_1 + ix_2}{2}.
\]
It is now clear that \( A \) has range \([0, 1]\) and so we have a \( * \)-isomorphism \( C([0, 1]) \cong I_1 \). Let \( d_1 \) be the metric on \([0, 1]\) obtained from the standard round metric on \( S^2 \) so that
\[
d_1(s, t) := \inf \{d_{S^2}((x_1, x_2, 1 - 2s), (y_1, y_2, 1 - 2t)) : x_1^2 + x_2^2 + (1 - 2s)^2 = 1 = y_1^2 + y_2^2 + (1 - 2t)^2\}
\]
for all \( s, t \in [0, 1] \). We record the following elementary result:

Proposition 3.1. The map \( \phi : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [0, 1] \) given by \( \phi(t) = \frac{1}{2} + \frac{1}{4} \sin(t) \) is an isometric isomorphism when \([-\frac{\pi}{2}, \frac{\pi}{2}]\) is equipped with the standard Euclidean metric \( d \) and \([0, 1]\) is equipped with the metric \( d_1 \). In particular, we have a \( * \)-isomorphism \( \beta : C([-\frac{\pi}{2}, \frac{\pi}{2}]) \rightarrow I_1 \), \( \beta(f) = (f \circ \phi^{-1})(A) \), which maps \( C_{\text{Lip}}([-\frac{\pi}{2}, \frac{\pi}{2}]) \) onto \( I_1 \cap C_{\text{Lip}}(S^2) \) and satisfies \( L_{d_{S^2}}(\beta(f)) = L_d(f) \).

Remark 3.2. For completeness, we note that when \( q = 1 \), the standard Podleś sphere is of course isomorphic to \( C(S^2) \). Indeed, the continuous maps corresponding to \( A \) and \( B \) separate points in \( S^2 \) and the Stone-Weierstrass Theorem then shows that \( S^1_q = C^*(1, A, B) \cong C(S^2) \).
3.2. The quantised case. We will now address the case of a fixed \( q \in (0,1) \). We let \( X_q \) denote the spectrum of \( A \in S_q^2 \), and, as we already saw, \( X_q = \{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\} \). As explained in the introduction, the Lip-norm \( L_{D_q} \) gives rise to a metric on the state space of \( C^*(A,1) \cong C(X_q) \), which therefore, in particular, determines a metric \( d_q \) on \( X_q \) when the latter is viewed as a subset of the state space via point evaluations. The aim of the current section is to find an explicit formula for this metric, and show that the metric spaces \((X_q,d_q)\) converge in the Gromov-Hausdorff distance to the Euclidean interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) as \( q \) tends to 1.

Define the function \( \rho_q : [-1, \infty) \to \mathbb{R} \) by

\[
\rho_q(x) := \frac{\sqrt{1 - q^{2(x+1)}}}{(1 - q^2)q^x}.
\]

**Definition 3.3.** Define the metric \( d_q : X_q \times X_q \to [0, \infty) \) by

\[
d_q(x,y) := \begin{cases} 
0 & \text{if } x = y \\
\max\{m,n\} - 1 & \text{if } x = q^{2m} \text{ and } y = q^{2n} \text{ with } n \neq m \\
\sum_{k=n}^{\infty} \frac{1}{\rho_q(k)} & \text{if } x = q^{2n} \text{ and } y = 0 \text{ or } x = 0 \text{ and } y = q^{2n}.
\end{cases}
\]

Remark that the series \( \sum_{k=0}^{\infty} \frac{1}{\rho_q(k)} \) is convergent as can be seen from the estimate

\[
\frac{1}{\rho_q(k)} = \frac{q^k (1-q^2)}{\sqrt{1-q^{2(k+1)}}} \leq q^k \quad \text{for all } k \in \mathbb{N}_0.
\] (3)

In order to prove Theorem A we need several lemmas, the first of which shows that the Lipschitz semi-norm on \( C(X_q) \) defined by the metric \( d_q \) and the Lip-norm \( L_{D_q} \) on \( I_q \) agree on all finite linear combinations of characteristic functions on \( X_q \):

**Lemma 3.4.** For any \( f \in \operatorname{span}_{\mathbb{C}}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\} \subset C(X_q) \), it holds that \( f(A) \in \operatorname{Lip}_{D_q}(S_q^2) \cap I_q \). Moreover, we have the identities

\[
L_{D_q}(f(A)) = \max\{\rho_q(k) : |f(q^{2k}) - f(q^{2(k+1)})| : k \in \mathbb{N}_0\} = L_{d_q}(f).
\]

In particular, \( f \) is also Lipschitz with respect to the metric \( d_q \).

Note that the maximum is indeed well-defined, since \( f \) is non-zero at at most finitely many elements from \( X_q \).

**Proof.** Let \( f \in \operatorname{span}_{\mathbb{C}}\{\chi_{\{q^{2k}\}} : k \in \mathbb{N}_0\} \) be given. The fact that \( f(A) \in \operatorname{Lip}_{D_q}(S_q^2) \cap I_q \) is a consequence of Lemma 2.3. Moreover, from Lemma 2.3 and the defining identities for \( SU_q(2) \) we obtain that

\[
\partial_t(f(A))\partial_t(f(A)^*) = A(1 - q^2 A) \sum_{k=0}^{\infty} \frac{|f(q^{2k}) - f(q^{2(k+1)})|^2}{q^{4k}(1 - q^2)^2} \chi_{\{q^{2k}\}}(A)
\]

\[
= \sum_{k=0}^{\infty} \rho_q(k)^2 \cdot |f(q^{2k}) - f(q^{2(k+1)})|^2 \chi_{\{q^{2k}\}}(A).
\]

The continuous functional calculus applied to \( A \in I_q \) then implies that

\[
\|\partial_t(f(A))\|^2 = \max\{\rho_q(k) : |f(q^{2k}) - f(q^{2(k+1)})| : k \in \mathbb{N}_0\}.
\] (4)
The identity

\[ L_{D_q}(f(A)) = \max\{\rho_q(k) \cdot |f(q^{2k}) - f(q^{2(k+1)})| : k \in \mathbb{N}_0\} \]

now follows since the formula in (1) implies that \( \|\varphi_\partial(f(A))\| = \|\varphi_\partial(f(A))\| = \|\varphi_\partial(f(A))\| \).

For the second equality, choose \( l \in \mathbb{N}_0 \) such that

\[ \rho_q(l) \cdot |f(q^{2l}) - f(q^{2(l+1)})| = \max\{\rho_q(k) \cdot |f(q^{2k}) - f(q^{2(k+1)})| : k \in \mathbb{N}_0\}. \]

For \( m < n \) we estimate that

\[ |f(q^{2m}) - f(q^{2n})| \leq \sum_{k=m}^{n-1} |f(q^{2k}) - f(q^{2(k+1)})| \leq \sum_{k=m}^{n-1} |f(q^{2l}) - f(q^{2(l+1)})| \cdot \frac{\rho_q(l)}{\rho_q(k)} \]

(5)

This shows that \( f : X_q \to \mathbb{C} \) is Lipschitz with \( L_{d_q}(f) \leq L_{D_q}(f(A)) \). The fact that equality is achieved is a consequence of the identity

\[ |f(q^{2l}) - f(q^{2(l+1)})| = |f(q^{2l}) - f(q^{2(l+1)})| \cdot \rho_q(l) \cdot d_q(q^{2l}, q^{2(l+1)}). \]

The next lemma computes the Lipschitz semi-norms of general continuous functions on \( X_q \) and provides information on the behaviour of the Lipschitz constants of a particularly interesting approximation.

**Lemma 3.5.** For any \( f \in C(X_q) \) one has

\[ L_{d_q}(f) = \sup\{|f(q^{2k}) - f(q^{2(k+1)})| \cdot \rho_q(k) : k \in \mathbb{N}_0\}. \]

Moreover, if \( f(0) = 0 \) and \( f \) is Lipschitz with respect to the metric \( d_q \), then the sequence \( \{L_{d_q}(f \cdot \chi_{\{q^{2k} : k \leq n\}})\}_{n=0}^\infty \) is bounded.

**Proof.** It follows from Lemma 3.4 that

\[ \frac{|f(x) - f(y)|}{d_q(x, y)} \leq \sup\{|f(q^{2k}) - f(q^{2(k+1)})| \cdot \rho_q(k) : k \in \mathbb{N}_0\} \]

(6)

whenever \( x, y \in X_q \setminus \{0\} \) satisfy \( x \neq y \). Moreover, each value \( |f(q^{2n}) - f(q^{2(n+1)})| \cdot \rho_q(n) \) may be attained by choosing \( x = q^{2n} \) and \( y = q^{2(n+1)} \). Thus, to establish the claimed identity, it only remains to be shown that the supremum in (6) is still an upper bound for some \( n \in \mathbb{N}_0 \) and \( y = 0 \). However, this follows immediately from the estimate in (6) together with continuity of the function \( f \) and the metric \( d_q \).

For the second part we assume that \( f \) is Lipschitz and that \( f(0) = 0 \). By Lemma 3.4 it suffices to show that the sequence \( \{|f(q^{2n})| : \rho_q(n)\}_{n=0}^\infty \) is bounded. To this end, we first note that since \( f \) is Lipschitz we may find a constant \( C \) such that \( |f(q^{2n})| \leq C \cdot d_q(q^{2n}, 0) \) for all \( n \in \mathbb{N}_0 \). It follows that

\[ \rho_q(n) \cdot |f(q^{2n})| \leq C \cdot \sum_{k=0}^\infty \rho_q(n) \cdot \rho_q(k) = C \cdot \sum_{k=0}^\infty q^k \sqrt{1 - q^{2(n+1)}} \leq C \cdot \sum_{k=0}^\infty q^k = \frac{C}{1 - q} \]

for all \( n \in \mathbb{N}_0 \). This ends the proof of the lemma. \( \square \)

**Theorem 3.1.** The Lip-algebra of \( I_q \) associated with the Dąbrowski-Sitarz spectral triple \((S^2_q, H_+ \oplus H_-, D_q)\) agrees with \( \{f(A) : f \in C_{Lip}(X_q)\} \), and for \( f \in C_{Lip}(X_q) \), we have \( L_{D_q}(f(A)) = L_{d_q}(f) \).
Proof. Let \( f \in C(X_q) \) be given.

Suppose first that \( L_{D_q}(f(A)) < \infty \). For each \( n \in \mathbb{N}_0 \) we define the projection \( Q_n := \sum_{k=0}^n \chi_{\{q^{2k}\}}(A) \). Since \( \partial_1 \) is a derivation, we obtain from Lemma \ref{lem:derivation}
that\
\[
\partial_1(f(A))Q_n = \partial_1(f(A)Q_n) - f(A)\partial_1(Q_n) = \sum_{k=0}^{n-1} (f(q^{2k}) - f(q^{2(k+1)})) \frac{1}{q^{2k}(1 - q^2)} \chi_{\{q^{2k}\}}(A) \cdot b^*a^*.
\]

Following the proof of Lemma \ref{lem:derivation} we then get that
\[
\| \partial_1(f(A))Q_n \| = \max\{|f(q^{2k}) - f(q^{2(k+1)})| \cdot \rho_q(k) : k \in \{0, 1, \ldots, n - 1\}\}
\] (7)

and hence (using that \( Q_n \) is an orthogonal projection) we obtain the estimate
\[
\sup\{|f(q^{2k}) - f(q^{2(k+1)})| \cdot \rho_q(k) : k \in \mathbb{N}_0\} = \sup\{\|\partial_1(f(A))Q_n\| : n \in \mathbb{N}_0\}
\] (8)

By Lemma \ref{lem:derivation} this shows that \( f \) is Lipschitz with respect to the metric \( d_q \) and that
\[
L_{D_q}(f) \leq \| \partial_1(f(A)) \|.
\]

To prove that equality holds, we observe that by \cite{T08} Theorem 6.2.17,
\[
h(Q_n) = (1 - q^2) \sum_{k=0}^{n} q^{2k} \to 1,
\]

where \( h \) denotes the Haar state on \( SU_q(2) \). Since \( h \) is faithful and \( \{Q_n\}_{n=0}^{\infty} \) is an increasing sequence of projections, \( Q_n \) converges to the identity in the strong operator topology on \( B(L^2(SU_q(2))) \), and hence also on \( B(H_+) \). It now follows from \cite{S} and \ref{lem:derivation} that for any \( \xi \) in the unit ball of \( H_+ \), we have
\[
\| \partial_1(f(A))\xi\| = \lim_{n \to \infty} \| \partial_1(f(A))Q_n\xi\| \leq \sup\{\|\partial_1(f(A))Q_n\| : n \in \mathbb{N}_0\}
\]
\[
= L_{D_q}(f).
\]

and hence that \( \| \partial_1(f(A))\| = L_{D_q}(f) \). Since we moreover have the identities
\[
\| \partial_2(f(A))\| = \| \partial_1(f(A))\| = L_{D_q}(f)
\]
we may conclude that \( L_{D_q}(f(A)) = L_{D_q}(f) \).

Suppose next that \( f \in C(X_q) \) is Lipschitz with respect to the metric \( d_q \). Since subtracting a constant changes neither the Lipschitz constant of \( f \) nor \( L_{D_q}(f(A)) \), we may, without loss of generality, assume that \( f(0) = 0 \). For each \( n \in \mathbb{N}_0 \) define the function \( f_n := f \cdot \chi_{\{q^{2k} : k \leq n\}} \).

By Lemma \ref{lem:derivation} the sequence \( \{L_{D_q}(f_n)\}_{n=0}^{\infty} \) is then bounded and moreover \( f_n(A) \) converges to \( f(A) \) in operator norm.

Consequently, since \( L_{D_q}(f_n(A)) = L_{D_q}(f_n) \) by Lemma \ref{lem:derivation} we obtain by lower semi-continuity of \( L_{D_q} : I_q \to [0, \infty] \) that
\[
L_{D_q}(f(A)) \leq \sup\{L_{D_q}(f_n(A)) : n \in \mathbb{N}_0\} < \infty.
\]

This shows that \( f(A) \in \operatorname{Lip}_{D_q}(I_q) \) and this ends the proof of the theorem. \( \square \)

Theorem \ref{thm:main} now follows easily:
Proof of Theorem A. The metric $d'_q$ on $X_q$ induced by $L_{D_q}$ is by definition given by
\[ d'_q(x,y) := \sup\{|f(x) - f(y)| : f \in C(X_q), L_{D_q}(f) \leq 1\}. \]
However, by Theorem 3.1 we have
\[ d_q(x,y) = \sup\{|f(x) - f(y)| : f \in C(X_q), L_{D_q}(f) \leq 1\} \]
and hence the two metrics agree.

In the following, we will consider the behaviour of $(X_q, d_q)$ with respect to the Gromov-Hausdorff metric, and provide a proof of Theorem B. To this end, we first establish a preliminary result about the diameter of $X_q$:

Lemma 3.6. It holds that $\lim_{q \to 1} d_q(0,1) = \pi$.

Proof. Observe that the function $\frac{1}{\rho_q}: x \mapsto (1-q^2)\frac{q^x}{\sqrt{1-q^{2x+1}}}$ is positive and decreasing function on $(-1, \infty)$. This yields the estimates
\[ \int_1^\infty \frac{1}{\rho_q(x)} \, dx \leq \sum_{k=0}^\infty \frac{1}{\rho_q(k)} \leq \int_0^\infty \frac{1}{\rho_q(x)} \, dx \tag{9} \]
Furthermore, one has that $F(x) := \frac{1}{\rho_q^2(q^x)} \ln(q)\frac{q^x}{\sqrt{1-q^{2x+1}}} \ln(q)$ is an antiderivative of $\frac{1}{\rho_q(x)}$ and $\lim_{x \to \infty} F(x) = 0$. We therefore obtain the inequalities
\[ -\frac{1-q^2}{q \ln(q)} \ln(q) \leq d_q(0,1) \leq -\frac{1-q^2}{q \ln(q)} \ln(q). \]
Since $\lim_{q \to 1} \frac{1-q^2}{q \ln(q)} = -2$ and $\ln(1) = \frac{\pi}{2}$ we may conclude that $\lim_{q \to 1} d_q(0,1) = \pi$. \(\square\)

Proof of Theorem B. For each $q \in (0,1)$, we consider the isometric embedding $\iota_q: X_q \to \mathbb{R}$ given by $\iota_q(x) = d_q(1,x) - \frac{\pi}{2}$.

We start by proving continuity at a fixed $q_0 \in (0,1)$. Let $\varepsilon > 0$ be given. Choose a $\delta_0 > 0$ such that $J := [q_0 - \delta_0, q_0 + \delta_0] \subset (0,1)$. From the estimate in (3) we obtain that
\[ \sum_{k=0}^\infty \sup_{q \in J} \left\{ \frac{1}{\rho_q(k)} \right\} \leq \sum_{k=0}^\infty \sup_{q \in J} \left\{ q^k \sqrt{1-q^2} \right\} \leq \sum_{k=0}^\infty (q_0 + \delta_0)^k < \infty. \]
We may therefore choose an $n_0 \in \mathbb{N}$ such that
\[ \sum_{k=n_0}^\infty \frac{1}{\rho_q(k)} < \frac{\varepsilon}{3} \tag{10} \]
for all $q \in J = [q_0 - \delta_0, q_0 + \delta_0]$. Now, for each $k \in \mathbb{N}_0$, the function $q \mapsto \sum_{k=0}^{n_0-1} \frac{1}{\rho_q(k)}$ is continuous and we may thus choose a $\delta \in (0,\delta_0)$ such that
\[ \left| \sum_{k=0}^{m-1} \frac{1}{\rho_q(k)} - \sum_{k=0}^{m-1} \frac{1}{\rho_{q_0}(k)} \right| < \frac{\varepsilon}{3} \tag{11} \]
for all $m \in \{1, \ldots, n_0\}$ and all $q \in (q_0 - \delta, q_0 + \delta)$.

Let now $q \in (q_0 - \delta, q_0 + \delta) \subset J$ be given. It then follows immediately from (11) that
\[ |\iota_q(q^{2m}) - \iota_{q_0}(q_0^{2m})| < \frac{\varepsilon}{3} < \varepsilon \]
for all \(m \in \{1, \ldots, n_0\}\). Moreover, for \(m > n_0\) we apply (\[10\]) and (\[11\]) to estimate that
\[
|\iota_q(q^{2m}) - \iota_{q_0}(q_0^{2m})| = \left| \sum_{k=0}^{n_0-1} \frac{1}{\rho_q(k)} + \sum_{k=n_0}^{m-1} \frac{1}{\rho_q(k)} - \left( \sum_{k=0}^{n_0-1} \frac{1}{\rho_{q_0}(k)} - \sum_{k=n_0}^{m-1} \frac{1}{\rho_{q_0}(k)} \right) \right|
\]
\[
\leq |\iota_q(q^{2n_0}) - \iota_{q_0}(q_0^{2n_0})| + \sum_{k=n_0}^{\infty} \frac{1}{\rho_q(k)} + \sum_{k=n_0}^{\infty} \frac{1}{\rho_{q_0}(k)}
\]
\[
< \varepsilon.
\]
A similar argument also shows that \(|\iota_q(0) - \iota_{q_0}(0)| < \varepsilon\). We conclude that
\[
\text{dist}_H(\iota_q(X_q), \iota_{q_0}(X_{q_0})) \leq \varepsilon
\]
and hence that \((0, 1) \ni q \mapsto (X_q, d_q)\) varies continuously in Gromov-Hausdorff distance.

For convergence, it suffices to show that the Hausdorff distance between \(\iota_q(X_q)\) and \([\frac{-\pi}{2}, \frac{\pi}{2}]\) converges to 0 as \(q \to 1\). To this end, let \(\varepsilon > 0\) be arbitrary. By Lemma \([36\]) we may find a \(q_1 \in \{0, 1\}\) such that for any \(q \in (q_1, 1)\), we have \(|\iota_q(0) - \frac{\pi}{2}| < \varepsilon\). Moreover, since \(-\frac{\pi}{2} \leq \iota_q(x) \leq \iota_q(0)\) for all \(x \in X_q\), it follows that for every \(x \in X_q\) there exists a \(y \in [\frac{-\pi}{2}, \frac{\pi}{2}]\) with \(|\iota_q(x) - y| < \varepsilon\). It remains to be shown that we can find a \(q_2 \in (0, 1)\) such that given any \(y \in [\frac{-\pi}{2}, \frac{\pi}{2}]\) and any \(q \in (q_2, 1)\), we can find \(x \in X_q\) such that \(|y - \iota_q(x)| < \varepsilon\). Since \(\frac{1}{\rho_q(0)} = \frac{1}{1 - q^2} \to 0\) as \(q \to 1\) and \(d_q(0, 1) \to \pi\) by Lemma \([36\]) we can find a \(q_2 \in (0, 1)\) such that \(|\iota_q(0) - \frac{\pi}{2}| < \varepsilon\) and \(|\iota_q(0) - \frac{\pi}{2}| < \varepsilon\) for all \(q \in (q_2, 1)\). Let now \(q \in (q_2, 1)\) be given. It follows that \(|y - \iota_q(0)| < \varepsilon\) for \(y \in (\frac{-\pi}{2}, \frac{\pi}{2})\). On the other hand, we may for each \(y \in [\frac{-\pi}{2}, \frac{\pi}{2}]\) find an \(n \in \mathbb{N}\) such that \(y \in [\iota_q(q^{2n}), \iota_q(q^{2(n+1)})]\) and consequently
\[
|\iota_q(q^{2n}) - \iota_q(q^{2(n+1)})| = \frac{1}{\rho_q(n)} \leq \frac{1}{\rho_q(0)} < \varepsilon.
\]

**Remark 3.7.** As stated in the introduction, Theorem \([13\]) also applies if we replace the classical Gromov-Hausdorff distance with respectively the quantum Gromov-Hausdorff distance of Rieffel \([66\]) or Latrémoilère’s propinquity. To see this, note that by [La16, Corollary 6.4] the former is dominated by two the latter and by [La16, Theorem 6.6], propinquity is dominated by the classical Gromov-Hausdorff distance on the class of compact metric spaces, and hence the convergence and continuity are also obtained for these distances.

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