How do we let students work as 'young mathematicians' in the classroom?

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Abstract. We must encourage students to solve mathematical problems and—in my opinion, more importantly—to develop and justify mathematical theories. If that succeeds, mathematics becomes more than finding the right answer or repeating a teacher’s argument. But how do you do that in the class room? The problems we pose to students and the questions we ask largely determine the image of mathematics that students develop. If we constantly ask for the 'right' answer, students will think that mathematics is primarily a mathematical activity and that mathematical thinking stops when they have found the 'teacher's answer'. However, when we ask students to defend their approach and encourage students to generalize the strategy used, a different, richer picture of mathematics arises.

1. What is the mathematics in the investigation?
There are basically two ways to look at a mathematical investigation that teachers ask children to work on. On the one hand, we can ask ourselves what we need to teach our children so that they can get the right answer to that problem. On the other hand, we can analyze what mathematics our children can develop while working on this problem, and we can try to support our children to develop and understand this mathematics?

Teachers who use a textbook that contains series of small calculation tasks without any internal connection, often follow the first approach. They want to help the children by teaching them how to get the answer. As the problems in those textbook series are unrelated (see fig 1), children will not develop any new insights in mathematics; they might, however, practice fluency in one or more mathematical operations.

Teachers who focus more on the development and understanding of mathematical theory will be more inclined to see investigations as an opportunity for the children to develop mathematical theory. They will look for entry points to discuss the mathematics behind the investigation and therefore go beyond the answer to the problem. They will support children by asking questions that are at the edge of their understanding and

Children will also behave differently. When they focus on getting a correct answer, children will play safe. They will not feel challenged to experiment with strategies they do not fully understand, however they will use lower order strategies that will give them the answer. When children get space to experiment, they will not always find the correct answer. They will be extremely interested in discussing their approach and-if needed-adjust it.

Consider children who are still developing subtraction strategies, trying solve 36 - 19. If pressed for the correct answer they will probably count three times: they tally 36, they strike through 19 tallies and
count the remaining ones. However, if they are giving space to explore this and other problems, some might try subtracting 20 in two jumps of 10: 36, 26, 16. They realize they have to compensate 1, but they might not be sure if it is 15 or 17.

Peter’s method, as describe by Max Stephens [2], is a great example that shows how children’s inventions can be a starting point for vertical mathematizing - for the further development of mathematical. Peter says, “I do these by first adding 5 and then subtracting 10, like $32 - 5 = 32 + 5 - 10$. Working it out this way is easier.” Peter is using a strategy that not many children will use initially. So, the first question might be whether Peter’s method even works: “will it always give the correct answer?” Tim (9 years, 1 month) shows how powerful it can be to discuss Peter’s method with children: “Here is an explanation for all numbers. Whatever number he (Peter) is taking away (assumed to be less than 10), you plus the number that would make a ten, and you take away ten. The bigger the number you are subtracting, the smaller the number you are plussing. They all make a ten together” [2]. Tim generalizes over all minuends and all subtrahends. In algebraic terms, he says: ‘$a - b$ is the same as $a + (10 - b) - 10$.’

In the next paragraph, we will look at an example of an investigation that will allow children to develop mathematical theory. We will not only look at the investigation, but we will also focus on the mathematics the students can invent.

2. One investigation: many problems at many levels
SEAMEO Regional Centre for QITEP in Mathematics (SEAQiM) in Yogyakarta (Ind), organizes yearly a Southeast Asia Realistic Mathematics Education (SEA-RME) course which is based on the regional culture, nature, and characteristics of Southeast Asia. This course gives teachers from SEAMEO Member Countries the opportunity to work collaboratively, celebrate diversity, and bring these experiences to their mathematics class rooms [3]. During the RME course in 2017, a group of 30 teachers worked on a number of mathematical investigations in number explorations, algebra, geometry, and data handling; they observed and discussed video-examples of RME investigations in primary schools; they read and discussed articles about RME in Indonesia; and they designed and implemented an investigation for primary school students.

In the design of an investigation, the course followed a lessons study approach. Groups of participants, supported by SEAQiM staff members, designed an investigation. Each investigation was first given to all other participants. Given the experiences with this try-out, the investigation was further improved. Next, the group who designed the investigation implemented it in a class in Yogyakarta. Other participants observed and made notes. These experiences were discussed with all
participants. Subsequently, the designers redesigned the investigation. This final design was given to all participants to implement in their school and in their country.

One of the investigations was inspired by work of Wahid Yunianto, head of the ‘Research and Development, Capacity Building and Training’ division [4,5]. Let me summarize the investigation in my words: a farmer wants to fence off a small coral. He has 25 fence pieces; each fence piece was one-meter long. He wants to know what shape would give him the largest enclosed area. As support, the students were given 25 yellow squares (the fence) and a larger set of green squares (the field).

During the try-out in a class in Yogyakarta, the children started to create rectangles with the green squares and a fence with the yellow squares. All groups started with a rectangle, most groups started with a square, a few used a non-square rectangle. Once all yellow squares were used, the children counted the number of enclosed greens squares and gave that total as their answer. If we only focus on the answer, this would have been the endpoint of the investigation. The students would say “25” and a happy teacher would say “great, next problem.” However, so much more could be at stake. I will discuss five issues that are possible discussion points in this investigation: the structure of the array model, the shape of the largest enclosed area, the relationship between perimeter and area, the transition from counting to skip-counting, and the transition from skip-counting to multiplication.

2.1. Understanding the array model
To go beyond counting, children need to understand the structure of an array as Figure 2 below. They need to see an array as a series of connected rows or columns to start using skip counting or multiplication. Seeing an array as a combination of rows or columns is not something children will automatically do; they need to construct it [6,7]. Therefore, it is not supporting when a teacher tells children that they need to count the number of tiles in one row and count the number of rows. We can assess if children have constructed this idea by asking them — the next day when the green and yellow tiles are not around — to draw the field the farmer has built.

![Figure 2. Children are investigating what the size is of the largest field enclosed by 25 (yellow) fence pieces.](image)

2.2. What border shape will create the largest enclosed area?
Starting with a square does not mean that the children are convinced that a square will give the largest area, nor that they will that be able to justify why the square will lead to the largest number of enclosed tiles. It is not too hard to challenge them to explore if another rectangular shape, say 4 by 6, would create a larger area. Figure 2 shows a group that also explores a concave, non-rectangular shape. They might even explore octagons or other more circular shapes; however, we need to keep in mind that the green squares will hinder the children to go in this direction. Once the children are convinced that the square will give a larger area compared to non-square rectangles, they will need to prove that statement.
2.3. **Rectangles with the same perimeter do not need to have the same area**

While they are exploring what shape will enclose the largest area, they will find that different rectangles with a perimeter of 25 will not always have the same area. So, not only will they start to discuss why the square has the largest area, in the end, they might also be able to discuss a big idea: “when the perimeter of two rectangles are the same, the area does not have to be the same”, and also: “when the area of two rectangles are the same, the perimeter does not have to be the same”.

2.4. **From counting-all to skip counting.**

When children see the array as a series of connected rows or columns, they will understand that counting the number of tiles in one row and the number of rows will help to find the total number of tiles. However, something odd happens. While counting the children know that you have to count each tile once. Yet, when children count the number of tiles in a row and count the number of rows, it appears as if they count at one tile twice. They point at that particular tile counting the number of tiles in the row, and they point at that same tile again when they count the number of rows. Being able to justify that they do not count that tile double, requires a deep understanding of the structure of the array.

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1 2 3 4 5
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Figure 3. Counting the number of tiles in one row (left) and counting the number of rows (right). Did we count the top-left tile twice?

The size of the array matters. For children, there is not much difference in skip counting 5 fives or counting 25 tiles. Larger numbers will make it more rewarding to skip count. Imagine that the field is 10 by 16. Counting 160 tiles is a lot of work, skip counting 10, 20, …, 160 will be rewarding and intriguing. In designing an investigation, we have to decide between using numbers that make the investigation doable or choosing numbers that will entice children to look for more advanced approaches.

2.5. **From skip counting to multiplication**

A deeper understanding will be developed when students start to talk about multiplication. When students skip count, they count the number of tiles. However, what does it mean when they say the area is 5 \( \times \) 5. Clearly, they do not mean that the area is 5 tiles by 5 tiles or 25 square tiles. In the multiplication, the 5 does not refer the number of tiles, but to the length of one side of the rectangle.

2.6. **An accessible investigation**

The children realize what they can do in this context and with the given materials. They start laying the tiles down, building the fences, and find the enclosed area. They can work on this investigation at a very concrete level, therefore this investigation is accessible for all. It clearly allows for natural differentiation, as it also offers many challenging problems at many levels. Children can be challenged to continue working on an issue that is at the edge of their understanding. Here, the support of the teacher is very important. The teacher needs to be able to challenge each group of children to work as young mathematicians.
3. How do we change classrooms into communities of mathematicians?

Changing classroom practice is not easy. This is not so much a problem for the children, it is foremost a problem for the teachers. They have to change their habits, teaching styles, believes, and split-second behavior. They have to provide children with interesting and challenging investigations. A good start is to:

- Present an investigation in a clear context. It can be a real-world context, it can be a fantasy context, it might even be the world of mathematics. Presenting a context means more than handing out a worksheet. Tell a story that connects to the children’s experience. It should allow all children to work at their own level: they can work at a very concrete level, or they work at a more abstract level.
- Let the children work on the investigation in small groups. I would suggest creating an optimal mismatch when you group the children. When you create a group with a child who works concrete and a student who works abstract, they might not understand what the other is talking about. When all children are working in an identical way, there is nothing to discuss. It is better to create groups of children who are close but not too closely aligned in their approach to the investigation.
- Support the children during their investigation. Do not tell them how they can solve their problems but support them to work on the mathematics themselves. Cathy Fosnot [8] talks about three stages of a conferral: clarification, celebration, and challenge. When you start a conversation or a conferral with children, you might start by listening to the children talking and clarifying what you have heard and seen the children do. This gives them a chance to explain their understanding and their wonderments. Next, you celebrate their findings. And you end by posing some further investigations at the edge of their understanding.
- When most groups have solved the problem, ask each group to create a poster that shows and justifies their thinking. Creating a poster that only shows a solution is not informative as all groups will have solved the investigation. All children will know the answer; however, they might not understand the approach of other groups. A poster should contain a justification of their solution, a viable mathematical argument [9], open questions, wonderments… This will force the children to reflect on their own work and thinking, to develop mathematical argumentation, and to be ready to defend their thinking.
- Ask a few of the groups to explain their poster to the other children in the class. Let other children ask questions and clarifications. It is not a share, where all groups have to tell their story, it is a mathematical congress where the community of learners defends, questions, develops, and explores mathematical understanding. You could ask questions that might lead to vertical mathematizing, to the further development of mathematical theory. This might happen when you contrast two approaches, or when you build on an invention by one of the groups, or when you explore one of their wonderments.

References
[1] Malmberg 2010 De Wereld in Getallen: voorproefje [Online] Available: http://www2.malmberg.nl/BAO/WIG/wig-8-lb-lessen/files/513830-01_lessen_180dpi1.pdf. [Accessed 9 5 2018]
[2] Stephens M 2007 Using number sentences to introduce the idea of variable [Online] Available: https://nzmaths.co.nz/2007-national-numeracy-facilitators-conference [Accessed 9 5 2018]
[3] SEAQiM Southeast Asia Realistic Mathematics Education (SEA-RME) [Online] Available: http://www.qitepinmath.org/en/programmes/regular-courses/southeast-asia-realistic-mathematics-education-sea-rme/. [Accessed 18 5 2018]
[4] Yunianto W Supporting 7th grade students’ understanding of the area measurement of quadrilaterals and triangles through reallocation activities [THESIS]. Palembang, Sriwijaya University, 2014
[5] W Yunianto "Developing understanding of area through reallocation activities," in Examples of Mathematics Investigations for Primary Schools in Indonesia (working title) ed F van Galen, D van Eeerde (Utrecht, Freudenthal Institute) (Under development) Available: http://www.fisme.science.uu.nl/en/impome/investigations

[6] Battista M T, Clements D H, Arnoff J and Van Auken Borrow C 1998 Students’ Spatial Structuring of 2D Arrays of Squares Journal for Research in Mathematics Education 29 503

[7] Outhred L and Mitchelmore M 2004 Students’ Structuring of Rectangular Arrays Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (Bergen, Norway)

[8] Fosnot c 2016 Conferring with Young Mathematicians at Work (New London: New Perspectives on Learning LLC)

[9] Stylianou D and Blanton M 2018 Teaching with Mathematical Argument (Portsmouth, NH: Heinemann)

[10] Gravemeijer K, Stephan M, Julie C, Lin F L and Ohtani M 2017 What Mathematics Education May Prepare Students for the Society of the Future? Int J of Sci and Math Educ 15 115