3-Point Off-shell Vertex in Scalar QED in Arbitrary Gauge and Dimension

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We calculate the complete one-loop off-shell three-point scalar-photon vertex in arbitrary gauge and dimension for Scalar Quantum Electrodynamics. Explicit results are presented for the particular cases of dimensions 3 and 4 both for massive and massless scalars. We then propose non-perturbative forms of this vertex that coincide with the perturbative answer to order $e^2$.

PACS numbers: 11.15.Tk, 12.20-m, 11.30.Rd

I. INTRODUCTION

The non-perturbative structure of Green functions in gauge field theories has turned out to be a challenging problem. Aside from the complicated non-abelian scenario of Quantum Chromodynamics (QCD), even simpler examples such as Quantum Electrodynamics (QED) have proved a hard nut to crack in the non-perturbative regime. Nevertheless, gauge covariance relations, such as the Ward-Fradkin-Green-Takahashi Identity (WFGTI) [1], and the Landau-Khalatnikov-Fradkin transformations (LKFT) [2] contain vital clues about the Green functions. Guided by such relations, extensive work has been carried out to construct non-perturbative Green functions [3, 4, 5]. As well, perturbation theory is a reliable guide when constraining acceptable structures in the weak coupling limit [6, 9, 10, 11, 12, 13].

In the context of perturbation theory, a systematic study of spinor QED was initiated by Ball and Chiu [6]. They decomposed the vertex into a ‘longitudinal’ part which ensures that the WFGTI identity is satisfied, and a ‘transverse’ part. In a basis where kinematic singularities are avoided, they gave off-shell results for the one-loop transverse vertex in 4 dimensions in the Fermi-Feynman gauge. Later on, Kızılersi, Reenders and Pennington extended this result to an arbitrary covariant gauge [7]. Results for massless and massive QED3 were obtained afterwards [11, 12, 13, 14]. These results were then generalized to arbitrary dimension by Davydychev, Osland and Saks [15] in the realm of QCD (from which all QED results can be inferred). Whereas the bare fermion-boson vertex in a minimal coupling gauge theory is merely $\gamma^\mu$, in general the vertex can be expanded out in terms of 12 spin amplitudes constructed from $\gamma^\mu$ and two independent four-momenta [16]. The WFGTI fixes four coefficients of the twelve spin amplitudes in terms of the fermion functions comprising the longitudinal component. The transverse part thus involves eight vectors with eight unconstrained scalar coefficients that depend on the gauge parameter $\xi$, the space-time dimension $d = 2\ell$, fermion masses and three kinematical invariants ($k^2, p^2, q^2$); so this is a complicated problem even at one-loop order.

One might hope that, in the absence of spinorial matrices, Scalar Quantum Electrodynamics (SQED) can offer a simpler platform to study non-perturbative solutions [17]. In this scenario, the 3-point vertex can be written in terms of just two independent four-momenta. The WFGTI fixes the coefficient of one of these. Therefore, there is only one unconstrained function which defines the transverse vertex—representing an 8-fold simplification of spinor QED/QCD! The trade-off is that additional four-point interactions occur in SQED. Thus the 1-loop scalar-photon vertex involves two additional Feynman diagrams. Ball and Chiu [6] carried out this calculation for massive scalars in the Fermi-Feynman gauge ($\xi = 1$) for $d = 4$. In this article, we extend their work to arbitrary dimension $d$ and gauge $\xi$ involving the one-loop scalar propagator along the way.

There are several reasons why this calculation is helpful: (i) it keeps track of the correct gauge covariance properties of the Green functions; (ii) one can take on-shell limits to check the gauge invariance of physical observables—this would not be not possible [12] if one only has results near four dimensions; (iii) SQED anyway has interest in lower dimensions, for example non-perturbative SQED in 2+1- and 0+1-dimensions has been examined by [18] and [19] respectively; (iv) three dimensional field theories contain several features of corresponding four dimensional field theories at high temperatures [20].

We have organized the article as follows. In Section II we introduce the notation to calculate the three-point vertex, discuss its decomposition in the light of WFGTI and give the expressions for one-loop scalar propagator and the longitudinal component of the three-point vertex. In Section III we evaluate the complete one-loop vertex in arbitrary gauge and dimensions and hence deduce an expression for its transverse component. We suggest three simple and natural constructions for its non-perturbative counterpart in Section IV and finish by discussing the so-called transverse Takahashi identities in Section V. An appendix serves to summarise many useful expressions arising from the Feynman integrals.
II. PRELIMINARIES

We shall start by setting out the notation, discussing the WF GTI and decomposing the 3-point vertex into longitudinal and transverse components. We then make use of the scalar propagator to present the longitudinal part fully at 1-loop order. Constant reference is made here and in the next section to various (mainly hypergeometric) functions which are listed in an appendix.

A. Notation

We define the bare quantities in the usual form: the scalar propagator $S^0(p) = 1/(p^2 - m^2)$, the photon propagator $\Delta^\mu_\nu = -[g_{\mu\nu}p^2 - (1 - \xi)p_\mu p_\nu]/p^4$, the 3-point vertex $\Gamma^0_\mu = (k + p)_\mu$, and the 4-point double photon vertex $e^2\Gamma^0_\mu_\nu = e^2g_{\mu\nu}$, where $\xi$ is the general covariant gauge parameter (such that $\xi = 0$ corresponds to Landau gauge) and $e$ is the usual QED coupling constant. The 3-point vertex up to one-loop order is diagrammatically represented in Figure 1; it can be written in terms of two vectors alone, namely, $k^\mu$ and $p^\mu$ or, if preferred, $P^\mu \equiv (p + k)^\mu$ and $q^\mu \equiv (k - p)^\mu$.

Due to the presence of the four-point vertex, there are two additional diagrams to be calculated in addition to the usual one required for the spinor QED.

![Figure 1: One loop 3-point vertex in SQED.](image)

The full 3-point vertex satisfies the usual WFGTI:

$$q_\mu \Gamma^\mu(k, p) = S^{-1}(k) - S^{-1}(p)$$

and has the non-singular limit

$$\Gamma^\mu(p, p) = \partial S^{-1}(p)/\partial p_\mu$$

when $k \rightarrow p$. We can use (1) to construct the 'longitudinal' part of the vertex:

$$\Gamma^\mu_L(k, p) = \frac{S^{-1}(k) - S^{-1}(p)}{k^2 - p^2} (k + p)^\mu.$$  

The full vertex can then be written as

$$\Gamma^\mu(k, p) = \Gamma^\mu_L(k, p) + \Gamma^\mu_T(k, p),$$

where the ‘transverse’ part satisfies

$$q_\mu \Gamma^\mu_T(k, p) = 0, \quad \Gamma^\mu_T(p, p) = 0,$$

and can be expanded out only in terms of the basis vector

$$T^\mu(k, p) = k \cdot q + p \cdot k^{\mu} = [q^\mu(k^2 - p^2) - (k + p)^\mu q^2]/2,$$

Thus the full vertex is

$$\Gamma^\mu(k, p) = \frac{S^{-1}(k) - S^{-1}(p)}{k^2 - p^2} (p + k)^\mu + \tau(k^2, p^2, q^2) T^\mu(k, p).$$

The coefficient $\tau$ is a Lorentz scalar function of $k$ and $p$, and can be expressed in terms of the 3 invariants, $k^2, p^2$ and $q^2$. Thus knowing only one unknown function $\tau$ is sufficient to fix the full 3-point vertex completely in SQED, the rest being tied to the scalar propagator $S$. 
B. Longitudinal Vertex

At one-loop the scalar propagator is given by two diagrams but because massless tadpole type diagrams are zero in dimensional regularization (which we are adopting), only the first diagram contributes.

\[ \Gamma_{\mu}(k, p) = \frac{-i e^2}{(2\pi)^{2\ell}} \left\{ 4(k \cdot p)(k + p)_\mu J^{(0)} + \left[ -8(k \cdot p)g_{\mu\nu} - 2(k + p)_\mu (k + p)_\nu \right] J^{(1)} + 4(k + p)^\nu J^{(2)}_{\mu\nu} + (k + p)_\mu K^{(0)} - 2K^{(1)}_{\mu} \right\} \]

and readily yields the longitudinal part of the 3-point vertex at one-loop:

\[ \Gamma_{\mu}^L(k, p) = \frac{e^2 \pi^2 (k + p)^\mu}{(2\pi)^{2\ell} (k^2 - p^2)} \left\{ 2Q_1(k)(m^2 + k^2) - 2Q_1(p)(m^2 + p^2) + (1 - \xi)(m^2 - p^2)^2 Q_3(p) - (m^2 - k^2)^2 Q_3(k) \right\} \]

where the functions \( Q_i(p) \) are tabulated in the appendix.

III. ONE-LOOP VERTEX

In this section, we shall evaluate the complete off-shell one-loop vertex for all \( \xi \) and \( \ell \). Subtracting the longitudinal part from that produces the remaining transverse part.

A. The Full Vertex

The complete one-loop correction to the vertex is the sum of the three contributions that correspond to the last three graphs of Figure 2:

\[ \Lambda^\mu = \Lambda^\mu_1(k, p) + \Lambda^\mu_2(p) + \Lambda^\mu_3(k) \]

The first contribution involving only 3-point vertices is given by:

\[ \Lambda^\mu_1 = \frac{-i e^2}{(2\pi)^{2\ell}} \left\{ 4(k \cdot p)(k + p)_\mu J^{(0)} + \left[ -8(k \cdot p)g_{\mu\nu} - 2(k + p)_\mu (k + p)_\nu \right] J^{(1)} + 4(k + p)^\nu J^{(2)}_{\mu\nu} + (k + p)_\mu K^{(0)} - 2K^{(1)}_{\mu} \right\} + (\xi - 1) \left\{ (k + p)_\mu K^{(0)} + 4(k + p)_\mu p^\alpha k^\beta I^{(2)}_{\alpha\beta} - 8p^\alpha k^\beta I^{(3)}_{\alpha\beta} - 2(k + p)_\mu (k + p)^\alpha J^{(1)}_{\alpha} + 4(k + p)^\alpha J^{(2)}_{\alpha} - 2K^{(1)}_{\mu} \right\} \]

where

\[ K^{(0)} = \int d^d w \frac{1}{[(p - w)^2 - m^2][(k - w)^2 - m^2]} \]

\[ K^{(1)}_{\mu} = \int d^d w \frac{w_{\mu}}{[(p - w)^2 - m^2][(k - w)^2 - m^2]} \]
The final result is to evaluate as they contain only propagator type loops. Thus we only quote the coefficient

\[ \lambda = 2 \] using cut-off regularization. The explicit answers for the integrals (12) are stated in the appendix; as a generalization to arbitrary dimension and gauge of the work of Ball and Chiu [6] who only examined the case \( \xi = 2 \), using cut-off regularization. The explicit answers for the integrals (12) are stated in the appendix; as a general abbreviation we will write \( X^{(0)} = i \pi^2 \lambda_0 / 2 \) in what follows and use \( (d, m) \) as a superscript (or subscript) to signify dimension and mass, as and when needed.

### B. The Transverse Vertex

The transverse vertex is obtained by subtracting the longitudinal vertex, Eq. (3), from the full one, Eqs. (10–13), at one-loop. Carrying out this exercise, we arrive at the following coefficient \( \tau \) of the transverse vector \( T^\mu \) for massive scalars:

\[
\tau_{d,m}(k^2, p^2, q^2) = \frac{e^2 \pi^2}{2(2\pi)^2 \Delta^2} \left\{ \begin{array}{l}
\left[ (k^2 - 2m^2 + p^2 - 4k \cdot p) [-K_0 + (m^2 + k \cdot p)J_0] \right. \\
\left. + \frac{2Q_1(p)}{k^2 - p^2} [p^2(p^2 - 3k \cdot p) + k^2(k \cdot p - 3p^2) - 2m^2(p^2 + k \cdot p)] \\
- \frac{2Q_1(k)}{k^2 - p^2} [k^2(k^2 - 3k \cdot p) + p^2(k \cdot p - 3k^2) - 2m^2(k^2 + k \cdot p)] \\
+ (\xi - 1)(m^2 - k^2)(m^2 - p^2) [J_0 - (k \cdot p + m^2)I_0 - \\
\frac{2Q_3(p)}{k^2 - p^2}(k \cdot p + p^2) + \frac{2Q_3(k)}{k^2 - p^2}(k \cdot p + k^2)] \end{array} \right\}, \tag{14}
\]

where \( I_0, J_0, K_0, Q_1-6 \) are explicitly stated in the appendix. \( \Delta^2 \equiv (k \cdot p)^2 - k^2p^2 = (k \cdot q)^2 - k^2q^2 \). Thus \( 4\Delta^2 = \lambda(k^2, p^2, q^2) = |p| + k + q^2 - 2p^2k^2 - 2p^2q^2 - 2k^2q^2 | \), the Källén function and is related to the \( (2\times\text{area})^2 \) of a triangle with sides \( \sqrt{k^2}, \sqrt{p^2}, \sqrt{q^2} \). In the massless limit, the above expression reduces to

\[
\tau_{d,0} = \frac{e^2 \pi^2}{2(2\pi)^2 \Delta^2} \left\{ (k^2 + p^2 - 4k \cdot p)[(k \cdot p)J^{d,0}_0 - K^{d,0}_0] \right\}
\]
\[ \begin{align*}
\frac{2Q_{4,0}^4(p)}{k^2 - p^2} [p^4 - 3(k^2 + k \cdot p)p^2 + k^2 k \cdot p] - \frac{2Q_{4,0}^4(k)}{k^2 - p^2} [k^4 - 3(p^2 + k \cdot p)k^2 + p^2 k \cdot p] \\
+ (\xi - 1)k^2 p^2 \left[ j_{0,0} - k \cdot p j_{0,0} - 2Q_{4,0}^4(p) \left( k \cdot p + p^2 \right) + 2Q_{4,0}^4(k) \left( k \cdot p + k^2 \right) \right].
\end{align*} \] (15)

In the massive case for small \( \epsilon = 2 - \ell \) one gets
\[ \tau_{4-2\epsilon,m} = \frac{\alpha}{8\pi\Delta^2} \left\{ (k^2 - 2m^2 + p^2 - 4k \cdot p)(m^2 + k \cdot p)J_{0}^{4-2\epsilon,m} + 2S \right\} \]
\[ + \frac{2L(p)}{k^2 - p^2} \left( p^2(p^2 - 3k \cdot p) + k^2(k \cdot p - 3p^2) - 2m^2(p^2 + k \cdot p) \right) \]
\[ - \frac{2L(k)}{k^2 - p^2} \left( k^2(k^2 - 3k \cdot p) + p^2(k \cdot p - 3k^2) - 2m^2(k^2 + k \cdot p) \right) \]
\[ + (\xi - 1)(m^2 - k^2)(m^2 - p^2) \left[ j_{0}^{4-2\epsilon,m} - \frac{2}{\chi}(k \cdot p + m^2) \left( -q^2 S \right) \right. \]
\[ + \frac{p^2[(p^2 - m^2)q^2 + 2m^2(k^2 - p^2)]L(p)}{(p^2 - m^2)^2} + \frac{k^2[(k^2 - m^2)q^2 - 2m^2(k^2 - p^2)]L(k)}{(k^2 - m^2)^2} \left. \right] \]
\[ + \frac{2}{(k^2 - p^2)(p^2 - m^2)^2} \left[ L(p) - 2(p^2 + k^2)(k^2 + m^2) - (k^2 - p^2)(k^2 - m^2)^2 \right]. \] (16)

The quantities \( S, L \) can be found early on in the appendix. In the massless limit for small \( \epsilon \) one reads off
\[ \tau_{4-2\epsilon,0} = \frac{\alpha}{8\pi\Delta^2} \left\{ (k^2 + p^2 - 4k \cdot p)(k \cdot p)J_{0}^{4-2\epsilon,0} + \ln \frac{q^2}{p^2 k^2} \right\} \]
\[ + \frac{(k^2 + p^2)q^2 - 8p^2k^2}{p^2 - k^2} \ln \frac{k^2}{p^2} \]
\[ + (\xi - 1)k^2 p^2 \left[ j_{0}^{4-2\epsilon,0} + 2p^2(k^2 + k \cdot p)\ln(-p^2) - \frac{2}{\chi} \frac{k^2(p^2 + k \cdot p)}{k^2 - p^2} \ln(-k^2) + 2k \cdot p \ln(-q^2) \right]. \] (17)

Note that for \( \xi = 1 \), it agrees with Eq. (2.9) of \( \text{[1]} \). In the massive case with \( d = 3 \) we end up with
\[ \tau_{3,m} = \frac{e^2}{16\pi\Delta^2} \left\{ (k^2 - 2m^2 + p^2 - 4k \cdot p)(m^2 + k \cdot p)J_{0}^{3,m} - 2I \left( \frac{q^2}{4} \right) \right\} \]
\[ + 4(k^2(k^2 - 3k \cdot p) - 2(k^2 + k \cdot p)m^2 + (k \cdot p - 3k^2)p^2) \frac{I(k^2)}{(k^2 - p^2)} \]
\[ - 4(k^2(k \cdot p - 3p^2) + p^2(p^2 - 3k \cdot p) - 2m^2(k \cdot p + p^2)) \frac{I(p^2)}{(k^2 - p^2)} \]
\[ + (\xi - 1)(m^2 - k^2)(m^2 - p^2) \left[ j_{0}^{3,m} - \frac{4m(k^2 + k \cdot p)}{(k^2 - m^2)^2(k^2 - p^2)} + \frac{4m(p^2 + k \cdot p)}{(m^2 - p^2)^2(k^2 - p^2)} \right] \]
\[ - \frac{1}{\chi} \left[ 2(m^2 + k \cdot p) \left( J_{0} + \frac{m}{2} \right) q^2 + m \left( k^2 - m^2 + (k^2 + m^2)(k^2 - p^2)) \right. \right. \]
\[ \left. \left. - \frac{k^2 - p^2 + q^2}{(k^2 - m^2)^2} \right] \right\}. \] (18)

where \( \chi \) is a geometrical quantity listed in the appendix. The result (18) simplifies remarkably in the massless limit:
\[ \tau_{3,0} = \frac{e^2}{2k\mathcal{P}\mathcal{Q}} \left[ \frac{\mathcal{K}^2 + 2\mathcal{Q}\mathcal{P} + \mathcal{P}^2 + 2\mathcal{P}\mathcal{Q}}{(\mathcal{K} + \mathcal{P})(\mathcal{K} + \mathcal{P} + \mathcal{Q})} + (\xi - 1) \frac{1}{4} \right], \] (19)
where we have adopted the Euclidian notation \( \sqrt{-k^2} = \mathcal{K}, \sqrt{-p^2} = \mathcal{P}, \sqrt{-q^2} = \mathcal{Q} \).

IV. ON THE NON-PERTURBATIVE VERTEX

The one-loop expression for \( \tau(l^2, p^2, q^2) \) provides a guide as to its possible form in the strong coupling regime. Any non-perturbative \textit{ansatz} for the transverse vertex should reduce to the perturbative result evaluated above. Eqs. \( \text{[6]} \)
and (14) suggest what \( \tau \) might resemble for general \( \epsilon^2 \):

\[
\tau_{d,m}(k^2, p^2, q^2) = \frac{1}{4\Delta^2} \left[ S^{-1}(k, \xi = 1) - S^{-1}(p, \xi = 1) \right] \left( \frac{m^2 + k^2}{(m^2 + k^2)Q_1(k) - (m^2 + p^2)Q_1(p)} \right) \times \\
\left\{ (k^2 - 2m^2 + p^2 - 4k \cdot p) \left[ -K_0 + (m^2 + k \cdot p)J_0 \right] + \frac{2Q_1(p)}{k^2 - p^2} \left[ p^2(p^2 - 3k \cdot p) + k^2(k \cdot p - 3p^2) - 2m^2(p^2 + k \cdot p) \right] - \frac{2Q_1(k)}{k^2 - p^2} \left[ k^2(k^2 - 3k \cdot p) + p^2(k \cdot p - 3k^2) - 2m^2(k^2 + k \cdot p) \right] \right\} \\
+ \frac{1}{2\Delta^2} \left[ S^{-1}(k, \xi = 1) - S^{-1}(p, \xi = 1) \right] \frac{(m^2 - k^2)(m^2 - p^2)}{(m^2 - k^2)^2Q_3(k) - (m^2 - p^2)^2Q_3(p)} \times \\
\left\{ J_0 - (k \cdot p + m^2)J_0 - \frac{2Q_3(p)}{k^2 - p^2}(k \cdot p + p^2) + \frac{2Q_3(k)}{k^2 - p^2}(k \cdot p + k^2) \right\}.
\]  

(20)

The notation here means that \( S(p, \xi = 1) \) is the scalar propagator in the Fermi-Feynman gauge, whereas, \( S(p, \xi = 1) \) is the coefficient of the scalar propagator proportional to \((\xi - 1)\). By construction, this expression reproduces the one-loop transverse vertex in the weak coupling regime. In specific dimensions and for the massless case, it simplifies.

Another approach is to tie in the asymptotic behaviour with the anomalous dimension of the scalar field in 4-d. To see how this is done, return to the 1-loop self energy as obtained previously in eq. (8); by using contiguity relations of hypergeometric functions, this can be cast in the simpler form

\[
\Sigma(p) = -\frac{e^2}{m^2} \left( \frac{m^2}{4\pi} \right)^\ell \Gamma(1-\ell) \left[ 1 + 2(\ell - 1)(1 - \xi) + ((\ell - 2) - (3 - 2\ell)\xi) \right] \left( 1 + \frac{p^2}{m^2} \right) _2F_1 \left( 2 - \ell, 1; \ell; \frac{p^2}{m^2} \right).
\]

According to the procedure for self-consistent regularization by higher-order corrections \(^7\), we then make the substitution \( \ell \to 2 + \gamma \) in eq. (21) and renormalize by ensuring that the propagator behaves as \((-p^2)^{1+\gamma}\) as \( p^2 \to \infty \). This gives the self-consistent asymptotic equation,

\[
\left( -\frac{p^2}{m^2} \right)^{1+\gamma} \sim \frac{e^2}{16\pi^2} \frac{\Gamma^2(1+\gamma)}{\Gamma(2\gamma+2)} \left[ (\xi - 3)\Gamma(-\gamma) + 2(1 - \xi)\Gamma(1 - \gamma) \right] \left( 1 + \frac{p^2}{m^2} \right) ^\gamma \left( 1 - \frac{p^2}{m^2} \right),
\]

which fixes \( \gamma = (3 - \xi)e^2/16\pi^2 + O(e^4) \). But anyway it produces a non-perturbative form of the 4-d propagator

\[
S^{-1}(p) \sim \frac{e^2}{16(\gamma + 1)\pi^2} \left[ (\xi - 3)\Gamma(-\gamma) + 2(1 - \xi)\Gamma(1 - \gamma) \right] \left( 1 + \frac{p^2}{m^2} \right) _2F_1 \left( -\gamma, 1; 1 + \gamma; 2; p^2/m^2 \right).
\]

(22)

This non-perturbative method succeeds in the ultraviolet but fails in the infrared limit \( p^2 \to m^2 \), when he propagator \( S \sim 1/(p^2 - m^2)^{1+(3-\xi)e^2/8\pi^2} \). For infrared exponentiation it is much easier to resort to the gauge technique \(^8\).

Exactly the same procedure can be applied to the transverse vertex. If \( \tau \) is expressed in Feynman parametric form,

\[
(p^2 - k^2)\tau(p, k, q) = \frac{4e^2\Gamma(2 - \ell)}{4\pi^\ell} \int_0^1 ds \sigma(1 - \sigma)^{\ell - 2} \left( (m^2 - p^2\sigma)^{\ell - 2} - (p \to k) \right) \\
+ \frac{e^2(2m^2 + k^2 + p^2 - 2q^2)\Gamma(3 - \ell)}{4\pi^\ell} \int_0^1 ds \sigma(1 - \sigma)^{\ell - 1} \int_0^1 du uD^{\ell - 3} \\
+ \frac{e^2(\xi - 1)(k^2 - m^2)(p^2 - m^2)\Gamma(4 - \ell)}{4\pi^\ell} \int_0^1 ds \sigma(1 - \sigma)^{\ell - 2} \int_0^1 du uD^{\ell - 4},
\]

(23)

with \( D \equiv m^2 - q^2(1 - \sigma)(1 - u^2)/4 - p^2\sigma(1 + u)/2 - k^2\sigma(1 - u) \), then with the above substitution, an ansatz for the non-perturbative transverse vertex in 4-d emerges:

\[
(p^2 - k^2)\tau(p, k, q) = \frac{4e^2\Gamma(\gamma)}{4\pi^2} \int_0^1 ds \sigma(1 - \sigma)^\gamma \left( (m^2 - p^2\sigma)^\gamma - (p \to k) \right)
\]
That the anomalous dimension of the scalar field makes an appearance should come as no surprise: the WFGTI identity is at work. What is rather interesting about the form (24) is that for $q^2 = 0$ it takes the form $[F(p^2) - F(k^2)]/(p^2 - k^2)$ even though it is associated with the transverse piece; but for $q^2 \neq 0$ this particular structure disappears as one can see from the form of $D$. Note anyway that (23) and (24) both have intrinsic dependences on all three variables $p^2$, $k^2$ and $q^2$ (or $p \cdot k$).

A third way of going non-perturbative relies upon dispersion relations; while these are well-established for the two-point function, in the form of the Lehmann-Källén representation, they are trickier for the vertex function but can nevertheless be found as follows for graphs with triangular topology. Make the change of variable $\rho \equiv (1 - \sigma)/\sqrt{2}$. Then, $W$ takes the form $\rho^2 W^2 + W + \xi^2$. Note anyway that (23) and (24) both have intrinsic dependences on all three variables.

$$S^{-1}(p) = \int_{m^2}^{\infty} dW^2 \frac{\rho(W^2)}{[p^2 - W^2 + i\epsilon]}$$

$$\tau(p, k, q) = \int_{m^2}^{\infty} dW^2 \int_{-1}^{1} du \frac{\rho(W^2, u)}{[p^2(1 + u)/2 + k^2(1 - u)/2 + q^2(1 - u^2)(W^2 - m^2)/4 - W^2 + i\epsilon]}.$$

The idea is then to determine $\rho$ and $\mathcal{P}$ self-consistently through the Schwinger-Dyson equations for the propagator and the vertex: the latter inevitably brings in the 4-point function, but we can use its own WFGTI to approximate it by connected 3-point graphs. While we have not solved this problem for $\mathcal{P}$, the idea has been taken to fruition [8] for $\rho$ in SQED and QED, giving results that coincide with perturbation theory up to order $e^4$ for the charged field propagators. There is much more work involved in obtaining the spectral functions $\mathcal{P}$ accurately to order $e^4$ and higher and this has not yet been done. These ansätze are unlikely to be the whole story. However, one may ask how close these are to the real vertex and how these can be compared to each other. By construction, our ansätze agree with perturbation theory order by order depending upon the exact knowledge of the propagators. There is much more work involved in obtaining the spectral functions $\mathcal{P}$ accurately to order $e^4$ and higher and this has not yet been done. These ansätze are unlikely to be the whole story. However, one may ask how close these are to the real vertex and how these can be compared to each other. By construction, our ansätze agree with perturbation theory at the one loop order.

- **Ansatz (20)** agrees with perturbation theory to $O(e^2)$ in all momentum regimes, dimensions and gauges. To see whether this relation between the vertex and the propagator survives at $O(e^4)$, one needs to know these Green functions at that order.

- **Ansatz (24)** would be in accord with the real vertex in 4 dimensions for large $k^2$ and $p^2$ but would fail for $k^2$ and $p^2 \approx m^2$. As one knows $\gamma$ to $O(e^2)$, it amounts to knowing the vertex to all orders in the leading logarithm approximation for asymptotically large values of momenta.

- **Ansatz (25)** would agree with perturbation theory order by order depending upon the exact knowledge of the $\rho$ and $\mathcal{P}$ functions to a given order. In principle, one could evaluate these functions non-perturbatively through SDEs. However, this exercise for $\mathcal{P}$ is a hard nut to crack.

A full 2-loop calculation of the vertex should narrow down possible forms of any ansatz. Techniques for the 2-loop vertex calculation, especially for the massless particles, have been developed in [21, 22, 23]. All the master integrals for massless 2-loop vertex diagram with three off-shell legs have been calculated in [24]. These advances indicate that the calculation of the two-loop transverse vertex should not be too difficult, at least for the massless case. This work is under progress.

V. ON THE TRANSVERSE TAKAHASHI IDENTITIES

Eq. (20) is effectively a Ward-identity type relation linking the transverse vertex to the scalar propagator. There have been attempts to look for formal relations of this kind. Takahashi [23] discovered what are called transverse identities whose implications for the vertex have been examined for spinor QED [24, 25, 26, 27, 28]. In case of SQED, as there is just one unknown which remains undetermined by the conventional WFGTI, it is tempting to look for a transverse Takahashi identity, hoping one might be able to determine the three-point vertex more realistically.

It should be noted that the general form of the vertex (7) shows that the transverse coefficient contributes to both basis vectors $P_\mu = (p + k)_\mu$ and $q_\mu = (k - p)_\mu$. The curl of the vertex $q_\mu \Gamma_\nu - q_\nu \Gamma_\mu$ will eliminate the component of $\Gamma$ proportional to $q$, leaving us with the kinematic mixture $q_\mu P_\nu - q_\nu P_\mu$, multiplying the coefficient

$$\frac{S^{-1}(p) - S^{-1}(k)}{p^2 - k^2} + q^2 \tau(p^2, k^2, q^2)/2,$$
which is unwieldy. However the same kinematic combination can also be obtained by forming the ‘modified curl’ 
\( P_\mu \Gamma_\nu - P_\nu \Gamma_\mu \); this has the advantage of killing off the longitudinal part of \( \Gamma \) and only bringing in \( \tau \). We suggest that 
new identities involving the modified curl are more appropriate and will prove more promising for SQED.

ACKNOWLEDGMENTS

We thank A. Raya and M.E. Tejeda for valuable discussions. We acknowledge CIC and CONACyT for their financial 
support under grants number 4.10 and 46614-I.

APPENDIX

In this appendix we summarise the results for various integrals involved in the calculation of the 3-point vertex for 
quick reference. We write down the results for arbitrary \( d \) as well as small \( \epsilon = 2 - \ell \) and \( d = 3 \). This way we aim to 
present all the integrals including the ones which have not been considered in the articles [6, 9, 11, 12, 13]. Wherever 
possible, we compare the results of specific cases with the ones in the above-mentioned articles.

Subsidiary quantities :

There are a number of quantities which arise in various integrations that constantly appear later on, so we shall 
summarise them first and invoke them as they turn up. Unless specified we work in general dimension \( 2\ell \):

\[
S = \sqrt{1 - \frac{4m^2}{q^2}} \ln \left( \frac{\sqrt{1 - \frac{4m^2}{q^2} + 1}}{\sqrt{1 - \frac{4m^2}{q^2} - 1}} \right) = 2\sqrt{4m^2/q^2 - 1} \arctan \left( \frac{1}{\sqrt{4m^2/q^2 - 1}} \right)
\]

\[I(p^2) \equiv (1/\sqrt{-p^2}) \arctan \sqrt{-p^2/m^2},\]

\[L(p^2) \equiv (1 - m^2/p^2) \ln(1 - p^2/m^2),\]

\[Q_1(k) \equiv (\pi m^2)^\ell^2 \Gamma(1 - \ell) \, _2F_1(2 - \ell, 1; \ell; k^2/m^2),\]

\[\ell Q_2(k) \equiv (\pi m^2)^{\ell-2} \Gamma(2 - \ell) \, _2F_1(2 - \ell, 1; \ell + 1, k^2/m^2),\]

\[m^2 Q_3(k) \equiv (\pi m^2)^{\ell-2} \Gamma(1 - \ell) \, _2F_1(3 - \ell, 2; \ell, k^2/m^2),\]

\[m^2 Q_4(k) \equiv (\pi m^2)^{\ell-2} (2 - \ell) \Gamma(-\ell) \, _2F_1(3 - \ell, 2; \ell + 1; k^2/m^2),\]

\[Q_5(k) \equiv -2\pi^2 (\pi m^2)^{\ell-2} \Gamma(2 - \ell) \, _2F_1(1/2, 1; \ell - 1; 4m^2/q^2),\]

\[Q_6(p) \equiv i\pi^\ell (\ell - 3)(m^2)^{\ell-3} \Gamma(1 - \ell) \, _2F_1(1, 4 - \ell; \ell; p^2/m^2)\]

\[\chi \equiv m^2 (k^2 - p^2)^2 + (m^2 - k^2)(m^2 - p^2)q^2.\]

\( \chi \) represents \((12 \times \text{volume})^2\) of a tetrahedron constructed with base triangle lengths \(\sqrt{-k^2}, \sqrt{-p^2}, \sqrt{-q^2}\) and lateral 
sides to the apex of lengths \(m, m, 0\); thus it has geometrical significance. It is worthwhile noting the zero mass limits 
of the \( Q_i \) as we will consider such situations later:

\[Q_1^{d,0}(k) = -(-\pi k^2)^{\ell-2} \Gamma(\ell - 1) \Gamma(2 - \ell) / \Gamma(2\ell - 2) = -Q_2^{d,0}(k),\]

\[2\pi Q_3^{d,0}(k) = (-\pi k^2)^{\ell-3} \Gamma(\ell - 2) \Gamma(3 - \ell) / \Gamma(2\ell - 4) = \pi Q_4^{d,0}(k)\]
\[(m^2)^{2-\ell} Q^{d,0}_6(k) \to -2i\pi^\ell \Gamma(2-\ell), \quad Q^{d,0}_6(k)/m^2 \to i\pi^\ell (-k^2)^{\ell-4} \Gamma(\ell)\Gamma(\ell-2)/\Gamma(2\ell-4) \]

\[\chi^{d,0} = k^2 p^2 q^2 \text{ very simply.}\]

The $K$ Integrals:

$K^{(0)}$ in the list \[12\] for arbitrary $d = 2\ell$ equals

\[i\pi^2 K_0/2 = K^{(0)} = i\pi^\ell \Gamma(2-\ell) (m^2)^{\ell-2} _2F_1 \left(1, 2-\ell; \frac{3}{2}; \frac{q^2}{4m^2} \right), \quad (26)\]

\[K^{(0)}_{d,0} = i\pi^\ell (-q^2)^{\ell-2} \frac{\Gamma^2(\ell-1)\Gamma(2-\ell)}{\Gamma(2\ell-2)}. \quad (27)\]

Corresponding massive and massless expressions in the neighborhood of 4-d ($\epsilon = 2 - \ell$) are:

\[K^{(0)}_{4-2\epsilon,m} = i\pi^2 [C - S], \quad K^{(0)}_{4-2\epsilon,0} = i\pi^2 [C - \ln(-q^2/m^2)], \quad (28)\]

where

\[C = \frac{1}{\epsilon} - \gamma - \ln(\pi m^2) + 2, \quad (29)\]

The first of the results \[28\] agrees with Eqs. (44, 46-48) of \[9\]. When $d = 3$, this integral simplifies even more:

\[K^{(0)}_{3,m} = i\pi^2 I(q^2/4), \quad K^{(0)}_{3,0} = i\pi^3/\sqrt{-q^2} \quad (30)\]

Expressions \[30\] coincide with (A1) of \[12\] and (A3) of \[13\] respectively. For the $K^{(1)}_{\mu}$ integral in (12), it is easy to show that $K^{(1)}_{\mu} = (p + k)_{\mu} K^{(0)}/2$.

The $J$ Integrals:

The $J^{(0)}$ integral can be found in various sources \[9, 10, 12, 13, 30\] but the most general case (massive mesons and any $d$) is not known; we shall simply cite the known answers. $J_0$ is probably the most difficult one to work out as it brings in dilogarithmic or Spence ($Sp$) functions when $\ell$ is integer. For any $\ell$ the massless case has been given a completely elegant representation by Davydychev \[30\] :

\[J^{(0)} = 2i(-i\pi)^\ell (k^2, p^2, q^2)^{\ell-2} \frac{\Gamma^2(\ell-1)\Gamma(2-\ell)}{\Gamma(2\ell-2)} \left[ \frac{\Gamma(\ell-1)}{\Gamma(2\ell-2)} \right] _2F_1 \left(1, 1/2; \ell - 1/2, -\frac{\Delta}{2(p^2 + q^2 - k^2)^2} \right) + \text{two perms} - \frac{\Gamma(2\ell-2)}{\Gamma^2(\ell-1)} \frac{\Delta^{\ell-3/2}}{(2\Delta)^{\ell-3/2}} \right] \quad (31)\]

For massive mesons, like we have, the result $J^{3,m}_0$ in 4-D is too lengthy (and uninformative) to quote. It is given in eq. ((16) of ref. \[9\]) and involves Spence functions of complicated arguments. In 3-D the result is \[13\] easy to state:

\[J^{3,m}_0 = \eta(k, p)I(\eta^2(k, p)\chi/4) + \eta(p, k)I(\eta^2(p, k)\chi/4); \quad \eta(k, p) = \frac{m^2(k^2 - p^2)(2m^2 - k^2 - p^2) + \chi}{\chi(m^2 - k^2)} \quad (32)\]

In the massless limit one obtains \[10, 12\] from all of these forms,

\[J^{4,0}_0 = \frac{2}{\Delta} \left[ Sp \left( \frac{p \cdot q - \Delta}{-p^2} \right) - Sp \left( \frac{p \cdot q + \Delta}{-p^2} \right) + \frac{1}{2} \left( \frac{p^2 + p \cdot q - \Delta}{p^2} \right) \ln \left( \frac{q^2}{p^2} \right) \right] \quad (33)\]

\[J^{3,0}_0 = \pi(K^2 + p^2)/K \chi; \quad K = \sqrt{-k^2}, \quad p = \sqrt{-p^2}, \quad Q = \sqrt{-q^2} \quad (34)\]
The $J_{\mu}$ Integral:

In its most general form, this can be written as

$$J_{\mu}^{(1)} = \frac{i\pi^2}{2} \left[ k_{\mu} J_A(k, p) + p_{\mu} J_B(k, p) \right] ; \quad J_A(k, p) = J_B(p, k).$$

We find

$$J_A(k, p) = -\frac{1}{2\Delta^2} \left\{ \left[ p^2 - k \cdot p \right] K_0 + \left[ p^2(k^2 - m^2) - k \cdot p(p^2 - m^2) \right] J_0 + 2p^2Q_1(p) - 2k \cdot pQ_1(k) \right\}.$$  \hspace{3cm} (35)

In the massless case,

$$J_{d,0}^A(k, p) = -\frac{1}{2\Delta^2} \left\{ \left[ p^2 - k \cdot p \right] 0_{d,0} + p^2[2k^2 - k \cdot p]J_{d,0} + 2p^2Q_{d,0}(p) - 2k \cdot pQ_{d,0}(k) \right\}.$$  \hspace{3cm} (36)

For small $\epsilon = 2 - \ell$, Eq. (35) reduces to the following expressions (see (A16, A17, 3.9(a,b,c)) of [6] and (39 - 44) of [9])

$$J_{A}^{4-2\epsilon, m} = \frac{1}{2\Delta^2} \left\{ -42\epsilon p \cdot q + p^2k \cdot q \right\} + \left( 2\epsilon + 2p^2 - 4k \cdot p \right) I_0(q^2/4) + 4p^2I(p^2) - 2k \cdot p.$$  \hspace{3cm} (37)

Similar results for $d = 3$ are:

$$J_{A}^{m} = \frac{1}{2\Delta^2} \left\{ [2^2(p^2 - k^2)] + [2(2p^2 - 2k^2)I_0(q^2/4) + 4p^2I(p^2) - 4k \cdot pI(k^2)] \right\},$$  \hspace{3cm} (38)

$$J_{A}^{0} = \frac{1}{2\Delta^2} \left\{ [2^2(p^2 - k^2)] + 2\pi \left( 2\epsilon + 2p^2 - 2k \cdot p \right) \right\}. $$

The first of these expressions coincides with Eq. (A5) of [13] and the second with Eq. (A3) of [12] after appropriate change of notation.

The $J_{\mu\nu}^{(2)}$ Integral:

From symmetry considerations, the integral $J_{\mu\nu}^{(2)}$ of the list [12] can be expanded out as follows:

$$J_{\mu\nu}^{(2)} = \frac{i^2\pi}{2} \left[ g_{\mu\nu} K_0 + 2k_{\mu}k_{\nu} - \frac{g_{\mu\nu}k^2}{2\ell} \right] K_A + \left( 2(k \cdot p - p^2)I_0(q^2/4) + 4p^2I(p^2) - 4k \cdot p \right) J_0 + \left( 2p^2 - 2k \cdot p \right) J_E.$$

The coefficients $J_A, J_B$ and $J_E$ in the above expressions are:

$$J_A(k, p) = \frac{1}{4(\ell - 1)\Delta^2} \left\{ [(2\ell - 2)p^2 - (2\ell - 1)(k^2 - m^2)p^2]J_A - [(2\ell - 2)p^2 - (2\ell - 1)(k^2 - m^2)p^2]J_B + [(2\ell - 2)p^2 + (\ell - 1)k \cdot p]K_0 - 4(\ell - 1)k \cdot pQ_2(k) \right\},$$

$$J_B(k, p) = \frac{1}{8(\ell - 1)\Delta^2} \left\{ [2\ell k \cdot p(k^2 - m^2) + 4(\ell - 1)k \cdot p]K_0 + 4(\ell - 1)[k^2Q_2(k) + p^2Q_2(p)] \right\}. $$

In the massless case we obtain the following expressions:

$$J_{\mu\nu}^{\ell, 0}(k, p) = \frac{1}{4(\ell - 1)\Delta^2} \left\{ [(2\ell - 2)p^2 - (2\ell - 1)k^2p^2] J_{\mu\nu}^{\ell, 0} \right\}.$$
The massive Eq. (43) is in agreement with (A6, A7) of [13] and Eq. (44) with (A4) of [12].

Then for small $\epsilon = 2 - \ell$, we arrive at (compare Eq. (41) with Eqs. (A18-A20) of [6] and Eqs. (49-51) of [9])

$$J^{4-2\epsilon,m}_C = \frac{1}{4\Delta^2} \left\{[p^2 - m^2]J^{4-2\epsilon,m}_A + [p^2 - m^2]J^{4-2\epsilon,m}_B + \frac{2k \cdot p}{k^2}J^{4-2\epsilon,m}_D\right\}$$

Also for $d = 3$,

$$J^{3,m}_C = \frac{1}{2\Delta^2} \left\{[p^2 - m^2]J^{3,m}_A - 2m \cdot \frac{k \cdot p}{k^2}I(k^2) - (k \cdot p)^2I(q^2/4) + 4m\right\}$$

$$J^{3,m}_D = \frac{1}{4\Delta^2} \left\{3k^2 J^{3,m}_A + [p^2 - m^2]J^{3,m}_B\right\}$$

Eq. (13) is in agreement with (A6, A7) of [13] and Eq. (44) with (A4) of [12].

The $I^{(0)}$ Integral:

The massive $I^{(0)}$ integral in arbitrary dimensions is given by

$$I^{(0)} = \frac{1}{2\chi} \left\{4(2 - \ell)q^2(m^2 + k \cdot p)J^{(0)} + (4m^2 - q^2)Q_5(q)\right\}$$

Hence the massless case reduces to

$$I^{(0)}_{d,0} = \frac{i\pi^2}{k^2p^2} \left\{(2 - \ell)k \cdot p J^{d,0}_0 - k^2Q^d_3(k) - p^2Q^d_3(p) + q^2Q^d_3(q)\right\}.$$
When \( \epsilon = 2 - \ell \) is small, we have

\[
I^{(0)}_{4-2\epsilon, m} = i\pi^2 \left\{ \frac{1}{\chi} \left[ -q^2 S + p^2 \left( \frac{(p^2 - m^2)q^2 + 2m^2(k^2 - p^2)}{(p^2 - m^2)^2} \right) L(p) + k^2 \left( \frac{(k^2 - m^2)q^2 - 2m^2(k^2 - p^2)}{(k^2 - m^2)^2} \right) L(k) \right] \right. \\
- \left. \frac{C - 2}{(p^2 - m)^2(k^2 - m^2)} \right\}
\]

\[
I^{(0)}_{4-2\epsilon, 0} = \frac{i\pi^2}{k^2p^2} \left[ 2 - C \ln \left( \frac{k^2p^2}{q^2m^2} \right) \right].
\]

whereas for \( d = 3 \),

\[
I^{(0)}_{3,m} = \frac{1}{\chi} \left\{ q^2(m^2 + k \cdot p) J^{(0)}_{3,m} + i\pi^2 m \left[ \frac{q^2(k^2 - m^2) - (k^2 - p^2)(k^2 + m^2)}{(k^2 - m^2)^2} + \frac{q^2(p^2 - m^2) + (k^2 - p^2)(p^2 + m^2)}{(p^2 - m^2)^2} \right] \right\}
\]

\[
I^{(0)}_{3,0} = \frac{k \cdot p}{k^2p^2} J^{(0)}_{3,m}.
\]

The first of Eq. (48) agrees with (A8) of [13].

The \( I^{(1)}_{\mu} \) Integral:

In analogy with the integral \( J^{(1)}_{\mu} \), \( I^{(1)}_{\mu} \) can be expanded out as

\[
I^{(1)}_{\mu} = \frac{i\pi^2}{2} [k_\mu I_A(k, p) + p_\mu I_B(k, p)]; \quad I_B(k, p) = I_A(p, k),
\]

where

\[
I_A(k, p) = \frac{1}{2\Delta^2} \left\{ [k \cdot p(p^2 - m^2) - p^2(k^2 - m^2)]I_0 + [k \cdot p - p^2]I_0 + 2k \cdot pQ_3(k) - 2p^2Q_3(p) \right\}.
\]

In the massless case

\[
I^{d,0}_{A}(k, p) = \frac{1}{2\Delta^2} \left\{ p^2[k \cdot p - k^2]J^{d,m}_{0} + [k \cdot p - p^2]J^{d,m}_{0} + 2k \cdot pQ_3^{d,0}(k) - 2p^2Q_3^{d,0}(p) \right\}.
\]

Near 4-dimensions

\[
I^{4-2\epsilon, m}_{A} = \frac{1}{2\Delta^2} \left\{ -k \cdot q J^{4-2\epsilon, m}_{0} - 2q^2(m^2 - k^2)(k^2 - k \cdot p)S/\chi \right. \\
+ \left. \frac{2L(p)}{(m^2 - p^2)}[p^2 - k \cdot p + p^2q^2(k^2 - m^2)(m^2 + k \cdot p)/\chi + 2k^2q^2(m^2 + k \cdot p)L(k)/\chi] \right\},
\]

\[
I^{4-2\epsilon, 0}_{A} = \frac{1}{2\Delta^2} \left\{ [k \cdot p - p^2]J^{4-2\epsilon, 0}_{0} - 2[k \cdot p - k^2] \ln \left( \frac{-q^2}{k^2} \right) + 2\frac{k \cdot p}{k^2} \ln(-p^2) - 2\ln(-k^2) \right\}.
\]

Eq. (51) is in agreement with the expression (53) of [3]. Similar answers for \( d = 3 \) are

\[
I^{3,m}_{A} = \frac{2}{\Delta^2} \left\{ [k \cdot p(p^2 - m^2) - p^2(k^2 - m^2)]J^{3,m}_{0} + [k \cdot p - p^2]J^{3,m}_{0} + \frac{mp^2}{(m^2 - p^2)^2} - \frac{mk \cdot p}{(m^2 - k^2)^2} \right\},
\]

\[
I^{3,0}_{A} = -\frac{\pi}{k^2\sqrt{-q^2k^2p^2}}.
\]

Eq. (53) agrees with (A11) of [13].

The \( I^{(2)}_{\mu\nu} \) Integral:

The integral \( I^{(2)}_{\mu\nu} \) of the list [12] may be decomposed as follows:

\[
I^{(2)}_{\mu\nu} = \frac{i\pi^2}{2} \left[ \frac{g_{\mu\nu}J_0 + (k_\mu k_\nu - g_{\mu\nu}k^2/2\ell)I_C + (p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu}k \cdot p/\ell)I_D + (p_\mu p_\nu - g_{\mu\nu}p^2/2\ell)I_E}{2\ell} \right],
\]

\[
I_E(k, p) = I_C(p, k).
\]
where, in arbitrary dimensions,

\[ I_C(k, p) = \frac{1}{4(\ell - 1)\Delta^2} \left\{ \left( (2\ell - 2)(p^2 - m^2)k \cdot p - (2\ell - 1)(k^2 - m^2)p^2 \right) I_A - (p^2 - m^2)p^2 I_B \\
+((2\ell - 2)k \cdot p - (2\ell - 1)p^2)J_A - p^2 J_B + 2p^2 J_0 + 4(\ell - 1)k \cdot pQ_4(k) \right\}. \]  

(57)

\[ I_D(k, p) = \frac{1}{8(\ell - 1)\Delta^2} \left\{ \left( 2\ell(k^2 - m^2)k \cdot p - (2\ell - 2)(p^2 - m^2)k^2 \right) I_A + \left( 2\ell(p^2 - m^2)k \cdot p - (2\ell - 2)(k^2 - m^2)p^2 \right) I_B \\
+2\ell k \cdot p - (2\ell - 2)k^2)J_A - 4k \cdot pJ_0 + \left( 2\ell k \cdot p - (2\ell - 2)p^2 \right) J_B - 4(\ell - 1)p^2 Q_4(p) + k^2 Q_4(k) \right\}. \]  

(58)

For the massless case, we arrive at the following simplified results

\[ I_{C, d, 0} = \frac{1}{4(\ell - 1)\Delta^2} \left\{ p^2((2\ell - 2)k \cdot p - (2\ell - 1)k^2) I_A^{d, 0} - p^4 I_B^{d, 0} + [(2\ell - 2)k \cdot p - (2\ell - 1)p^2] J_A^{d, 0} - p^2 J_B^{d, 0} \\
+2p^2 J_0^{d, 0} + 4(\ell - 1)k \cdot p Q_4^{d, 0}(k) \right\}, \]

\[ I_{D, d, 0} = \frac{1}{4(\ell - 1)\Delta^2} \left\{ [\ell k \cdot p - (\ell - 1)p^2](J_A^{d, 0} + k^2 I_A^{d, 0}) + [\ell k \cdot p - (\ell - 1)k^2](J_A^{d, 0} + p^2 I_B^{d, 0}) \\
-2k \cdot p J_0^{d, 0} - 2(\ell - 1)p^2 Q_4^{d, 0}(p) + k^2 Q_4^{d, 0}(k) \right\}. \]  

(59)

Near 4-dimensions, these expressions yield

\[ I_{C, 4-2\varepsilon, m} = \frac{1}{4\Delta^2} \left\{ 2p^2 J_0^{4-2\varepsilon, m} - \frac{4k \cdot p}{k^2} \left( 1 + \frac{m^2 L(k)}{(k^2 - m^2)} \right) + \left( 2k \cdot p - 3p^2 \right) J_A^{4-2\varepsilon, m} - p^2 J_B^{4-2\varepsilon, m} + \frac{-2k \cdot p(m^2 - p^2) + 3p^2 (m^2 - k^2)}{(k^2 - m^2)} I_A^{4-2\varepsilon, m} + p^2 (m^2 - p^2) I_B^{4-2\varepsilon, m} \right\}, \]

\[ I_{D, 4-2\varepsilon, m} = \frac{1}{4\Delta^2} \left\{ 2(k \cdot p) J_0^{4-2\varepsilon, m} + 2 \left( 1 + \frac{m^2 L(k)}{(k^2 - m^2)} \right) + 2 \left( 1 + \frac{m^2 L(p)}{(k^2 - m^2)} \right) + (2k \cdot p - k^2) J_A^{4-2\varepsilon, m} \\
+2(2k \cdot p - p^2) J_B^{4-2\varepsilon, m} + 2k \cdot p(m^2 - p^2) - 2k \cdot p(m^2 - k^2) I_A^{4-2\varepsilon, m} \right\}, \]

\[ I_{C, 4-2\varepsilon, 0} = \frac{1}{4\Delta^2} \left\{ 2p^2 J_0^{4-2\varepsilon, 0} - \frac{4k \cdot p}{k^2} + (2k \cdot p - 3p^2) J_A^{4-2\varepsilon, 0} - p^2 J_B^{4-2\varepsilon, 0} + p^2 (2k \cdot p - 3k^2) I_A^{4-2\varepsilon, 0} \right\}, \]

\[ I_{D, 4-2\varepsilon, 0} = \frac{1}{4\Delta^2} \left\{ -2k \cdot p J_0^{4-2\varepsilon, 0} + 4(k \cdot p - k^2) J_A^{4-2\varepsilon, 0} + (2k \cdot p - p^2) J_B^{4-2\varepsilon, 0} - 2k(p^2 - 2k \cdot p) I_A^{4-2\varepsilon, 0} \right\}. \]  

(60)

Finally for \( d = 3 \), we have

\[ I_{C, 3}^{m} = \frac{1}{2\Delta^2} \left\{ 2p^2 J_0^{3, 0} + [p^2(k \cdot p - 2k^2) - m^2(k \cdot p - 2p^2)] J_A^{3, m} - p^2(p^2 - m^2) J_B^{3, m} + (k \cdot p - 2p^2) J_A^{3, m} - p^2 J_B^{3, m} \\
+ \frac{2mk \cdot p}{k^2(k^2 - m^2)} - \frac{2k \cdot p}{k^2 I(k^2)} \right\}, \]

\[ I_{D, 3}^{m} = \frac{1}{4\Delta^2} \left\{ -4k \cdot p J_0^{3, m} + [k^2(3k \cdot p - p^2) - m^2(3k \cdot p - k^2)] J_A^{3, m} + [p^2(3k \cdot p - k^2) - m^2(3k \cdot p - p^2)] J_B^{3, m} \\
+ (3k \cdot p - k^2) J_A^{3, m} + (3k \cdot p - p^2) J_B^{3, m} - \frac{2m}{m^2 - k^2} - \frac{2m}{m^2 - p^2} + 2I(k^2) + 2I(p^2) \right\}, \]

\[ I_{C, 3}^{0} = \frac{1}{2\Delta^2} \left\{ p^2(k \cdot p - 2k^2) J_0^{3, 0} - p^4 J_B^{3, 0} + (k \cdot p - 2p^2) J_A^{3, 0} - p^2 J_B^{3, 0} - \frac{\pi k \cdot p}{k^2 \sqrt{-k^2}} - \frac{4\pi p^2}{\sqrt{-k^2 p^2}} \right\}. \]  

(61)
\[ I_{D}^{3,0} = \frac{1}{4\Delta^2} \left\{ k^2(3k \cdot p - p^2)I_{A}^{3,0} + p^2(3k \cdot p - k^2)I_{B}^{3,0} + (3k \cdot p - k^2)J_{A}^{3,0} + (3k \cdot p - p^2)J_{B}^{3,0} \right\} \]

\[ + \frac{\pi}{\sqrt{-k^2}} + \frac{\pi}{\sqrt{-p^2}} + \frac{8k \cdot p}{\sqrt{-k^2p^2q^2}} \right\} . \]  

(62)

The \( I_{\mu\nu\beta}^{(3)} \) Integral:

The integral \( I_{\mu\nu\beta}^{(3)} \) comes contracted with vectors \( p^\alpha \) and \( k^\beta \) so it is straightforward to show that

\[ -8p^\alpha k^\beta I_{\mu\nu\beta}^{(3)} = -2p^\alpha(k^2 - m^2)I_{\mu\nu\alpha}^{(2)} - 2p^\alpha J_{\mu\nu\alpha}^{(2)} + i\pi^2 p_\mu [Q_2(p) - (p^2 - m^2)Q_4(p)] \]

\[ -2k^\beta(p^2 - m^2)I_{\mu\nu\beta}^{(2)} - 2k^\beta J_{\mu\nu\beta}^{(2)} + i\pi^2 k_\mu [Q_2(k) - (k^2 - m^2)Q_4(k)] . \]  

(63)

Therefore, we have \( p^\alpha k^\beta I_{\mu\nu\beta}^{(3)} \) in terms of integrals we already know.

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