Affinities and disagreements between Mean-field and short-range critical dynamics

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In this work, we explore some interesting details of the time-dependent regime of the long-range systems under mean-field approximation in comparison with the critical dynamics of the short-range systems. First, we discuss some mechanisms of the initial anomalous behavior of the magnetization via two-dimensional Monte Carlo simulations to after compare with results from Mean-field simulations in both: spin 1/2 (Ising) and spin 1 (Blume Capel model) Ising models. The distinction between critical and tricritical points is also investigated. For a complete analysis, we performed short-time simulations in the mean-field regime to determine the critical temperatures optimizing power laws and the critical exponents of the different points, which were independently calculated, i.e. without using previous critical exponents estimates from literature/theory. Our investigations corroborate analytical results here also developed.

Keywords: Long-range systems, Mean-field regime, Time-dependent Monte Carlo simulations

I. INTRODUCTION

The studies in statistical mechanics can be essentially divided into two parts: equilibrium (or stationary regime for non-Hamiltonian systems) and nonequilibrium one \[1\]. A huge number of studies were dedicated to the equilibrium branch, and in this context, the long-range (LR) mean-field (MF) regime approximation for such systems was/is explored in several models, however, we believe that several points in this regime deserve better attention when explored in nonequilibrium regime since some points relatively well established in short-range (SR) systems as the relaxation of the magnetization in MF, did not have suitable exploration in the literature to the best of our knowledge.

Considering for example a description with time-dependent (TD) Monte Carlo (MC) simulations (TDMC) in the MF regime, it could be interesting to investigate the dynamic behavior of systems in MF approximation as well as, investigating the possible similarities/differences in this study in comparison with regular TDMC simulations of spin systems with SR interactions, for instance, in two or three-dimensional lattices.

Ising-like Hamiltonians can be simply generalized as written in the following:

\[ H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + D \sum_{i=1}^{N} \sigma_i^2 - H \sum_{i=1}^{N} \sigma_j \]  

where if \( D = 0 \) and \( \sigma_j = \pm 1 \) (spin 1/2) one has the standard Ising model, while for \( D \geq 0 \) (anisotropy term) and \( \sigma_j = \pm 1 \) (spin 1) one has the known Blume-Capel model. Here \( H \) is the external field that couples with each spin and \( \langle i,j \rangle \) denotes that sum is taken only over the nearest neighbors in a \( d \)-dimensional lattice.

A mean-field approximation considers that each \( i \)-th spin \( \sigma_i \) interacts with a magnetic “cloud” represented by the average magnetization \( \xi_i = \frac{1}{N} \sum_{j=1, j \neq i}^{N} \sigma_j \). Each spin, in the supposed original lattice where it is inserted, is linked to other \( z = 2^d \) neighbors, the number of links in the lattice is \( \frac{Nz}{2} \) since one has to count \( \frac{z}{2} \) links for each read spin by avoiding repeated links. Thus in this approximation, the interacting term \( H_{int} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \) must be replaced in the mean-field approximation by

\[ H_{int}^{(MF)} = -\frac{Jz}{2} \sum_{i=1}^{N} \xi_i \]

\[ = -\frac{Jz}{2N} \sum_{i=1}^{N} \sigma_i \sum_{j=1, j \neq i}^{N} \sigma_j \]

\[ \approx -\frac{Jz}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \]

Finally, one has that mean-field Hamiltonian is given by

\[ H^{(MF)} = -\frac{Jz}{2N} M^2 - HM + D \sum_{i=1}^{N} \sigma_i^2 \]  

\[ (2) \]

where \( M = \sum_{i=1}^{N} \sigma_i \) is the magnetization of the system.

For our aims \( H = 0 \) from here.

At this point, we raise an important question in this manuscript: What are the similarities and differences between the relaxation of SR spin systems (hamiltonian from Eq. \[1\]) and LR-MF spin systems (that one from Eq \[2\])? For that, we must remember some points about the relaxation of spin systems with SR interactions. Such systems, initially at high temperature, and therefore highly disordered, (with very small initial magnetization \( m_0 << 1 \), when suddenly placed in contact with a thermal reservoir at critical temperature \( T_c \), tend to initially mimic the MF characteristic correlations, and the correlations must present a kind of “inertia” until reaching the regime of SR correlations.

Such tendency leads to an initial anomalous behavior which is not exactly the same for critical (see for example \[2\] \[3\]) and tricritical points \[4\] \[5\]. At this same
initial condition, for critical points, the theory of Jansen, Schaub, and Schmittmann [6] and the Monte Carlo (MC) simulations from Zheng [2] predict a crossover between two power laws: the first one is exactly an anomalous increase of the magnetization from the initially disordered state characterized for an exponent \( \theta > 0 \), followed by a decrease of the magnetization which occurs when the system reaches a reasonable ordering state and in this case with an exponent \( \lambda < 0 \). Finally, the system remains decaying but in this case exponentially after reaching the thermodynamic equilibrium.

However tricritical points present peculiar aspects when initially prepared with a small magnetization as predicted by Oerding and Jansen [7]. The anomalous initial behavior in two dimensions is characterized by a power-law with \( \theta < 0 \). Here, only for illustration, we performed standard TDMC simulations according to Metropolis dynamics, the magnetization \( m(t) \) averaged over different \( N_{\text{run}} \) = 15 000 runs, for the two-dimensional Blume-Capel model, for one critical point (anisotropy \( D/J = 0 \) and \( k_B T_c/J = 1.6950 \)) and the tricritical one (\( D/J = 1.9655 \) \( k_B T_c/J = 0.610 \)), for \( L = 160 \) for several values of initial magnetization (fixed but randomly established in the beginning of each run). Please, for details of how to perform these simulations, see for example [2, 4].

\[
\frac{1}{N} \left( \sum_{i=1}^{N} \sigma_i \right) \sim \frac{1}{\sqrt{N}} \quad \text{and} \quad \frac{1}{N} \left( \sum_{i=1}^{N} \left( \Delta_i \right)^2 \right) \sim \frac{1}{N} \\
\text{for} \quad t \rightarrow 0
\]

Figure 1. Time evolutions of magnetization for the two-dimensional BC model in (a) Critical point (b) Tricritical point. These plots exactly illustrate which is theoretically predicted in [6] and [2] respectively.

\[ m_{m_0=1}(t) \sim t^{-\lambda}. \] (3)

where \( \lambda = \frac{\beta}{\nu} \). Here \( \beta \) is related to magnetization, and \( \nu \) related to correlation length \( \xi \), and a dynamic exponent \( z \) which links the length correlation with time correlation \( \Delta \) such that \( z \sim \frac{m_0^2}{\Delta} \), is the dynamic exponent.

The exponent \( \lambda \) can be extracted, in an independent way, from the ratio [3]:

\[
F_2(t) = \frac{m_{m_0=0}^{(2)}(t)}{m_{m_0=1}^{(2)}(t)} \sim t^\lambda
\]

where \( \xi = d/2 \), since \( m_{m_0=0}^{(2)}(t) = \frac{1}{N^2} \left( \sum_{i=1}^{N} \sigma_i(t) \right)^2 \sim t^{(d-2\beta/\nu)/z} \), where \( m_0 = 0 \), but with spins randomly distributed. With exponent \( \lambda \), one estimates \( \nu \) from a power law expected by the derivative of log-magnetization [2]:

\[
D(t) = \frac{1}{2\delta} \ln \left[ \frac{m_{m_0=1}(T_c+\delta)}{m_{m_0=1}(T_c-\delta)} \right] \sim t^\theta
\]

with \( \theta = \frac{1}{\nu} \), which is valid when \( \delta << 1 \), and finally with \( \nu \) and \( z \) in hands, we return to [3] and calculate \( \beta \). These three power laws can be used to determine \( \beta \), \( \nu \) and \( z \) for the critical (see for example [8]) and tricritical [4, 5] in both two and three-dimensional nonequilibrium simulations which is known as short-time dynamics, but the question is: could they be observed in mean-field regime? Moreover, how works the initial anomalous behavior of the magnetization for \( m_0 << 1 \)!

In this paper, we will show the peculiarities of the nonequilibrium critical dynamics of mean-field Ising-like systems for both: spin 1/2 (Ising) and spin 1 (BC). We will localize the critical temperature with a method previously used for systems with SR interactions [9] but that we here show to be properly interesting to estimate the critical parameters in MF systems. After that, we show that there is a notorious difference between the relaxation of the nonequilibrium critical dynamics from disordered initial systems for two-dimensional and MF systems, via MC simulations and some analytical results. However, we show that power laws are described by Eqs. [3, 4] and [5] are valid in the MF regime and the exponents \( \beta \), \( \nu \) and \( z \) were calculated corroborating the classical exponents in both: critical and tricritical points. Finally, the persistence phenomenon is also investigated in the MF regime. In section [IV] we develop the equations that describe the time evolution of the magnetization of Ising and BC models, valid for critical and tricritical points as a function of the initial magnetization. Our results are presented in section [III]. Our main conclusions are resumed in section [IV].

II. MEAN-FIELD MC DYNAMICS FOR ISING-LIKE SYSTEMS AND EFFECTS OF INITIAL MAGNETIZATION

In discrete systems, as Ising-like ones, if one denotes \( \Pr(\sigma, t) \) is the probability of the occurrence of configuration \( \sigma = (\sigma_1, ..., \sigma_N) \) at time \( t \). Thus, the expected value
of a spin at site $j = 1, \ldots, N$, is given by

$$
\langle \sigma_j \rangle = \sum_\sigma \sigma_j \Pr(\sigma, t)
$$

(6)

By using the master equation, and for systems with spin 1/2 (the usual Ising model), one can show as direct consequence of the master equation (see for example [12]) that:

$$
\frac{d \langle \sigma_j \rangle}{dt} = -2 \sum_\sigma \sigma_j w(\sigma \rightarrow \sigma^{(j)}) \Pr(\sigma, t),
$$

where $w(\sigma \rightarrow \sigma^{(j)})$ is the transition rate to from $\sigma$ to $\sigma^{(j)}$, where this latter denotes $\sigma^{(i)} = (\sigma_1, \ldots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \sigma_N)$ which only difference is that $\sigma_i$ is substituted by $-\sigma_i$, which represents a local transition in the regular representation.

We can choose any prescription that satisfies the detailed balance, for example the usual Glauber dynamics:

$$
w(\sigma \rightarrow \sigma^{(j)}) = \frac{1}{\tau} \left[ 1 - \tanh \left( \frac{\Delta H_{MF}}{2} \right) \right]
$$

In the case of the mean-field Ising model, one has

$$
\Delta H_{MF} = -\frac{J}{2N} \left[ \left( \sum_{i=1}^N, i \neq j \sigma_i - \sigma_j \right)^2 - \left( \sum_{i=1}^N, i \neq j \sigma_i + \sigma_j \right)^2 \right] = \frac{2J}{N} \sigma_j \sum_{i=1, i \neq j}^N \sigma_i = \frac{2J}{N} \sigma_j (M - \sigma_j) \approx \frac{2J}{N} \sigma_j M
$$

Thus, $w(\sigma \rightarrow \sigma^{(j)}) = \frac{1}{2\tau} \left[ 1 - \tanh \left( \frac{\beta J z M}{2} \right) \right]$, and since $\tanh(x)$ is an odd function, $w(\sigma \rightarrow \sigma^{(j)}) = \frac{1}{2\tau} \left[ 1 - \sigma_j \tanh \left( \frac{\beta J z M}{2} \right) \right]$. Thus, since $\sigma_j^2 = 1$

$$
\frac{d \langle \sigma_j \rangle}{dt} = -\frac{1}{\tau} \left[ \sum_\sigma \sigma_j \Pr(\sigma, t) - \sum_\sigma \sigma_j^2 \tanh \left( \frac{\beta J z M}{2} \right) \Pr(\sigma, t) \right] = -\frac{1}{\tau} \left[ \langle \sigma_j \rangle - \left\langle \tanh \left( \frac{\beta J z M}{2} \right) \right\rangle \right]
$$

Since $\langle \sigma_j \rangle = m$, we have and considering that in the mean-field approximation one has $\left\langle \tanh \left( \frac{\beta J z M}{2} \right) \right\rangle = \tanh \left( \frac{\beta J z M}{2} \right)$, we have

$$
\tau \frac{dm}{dt} = -m + \tanh \beta J z m
$$

(7)

where $m = \lim_{N \rightarrow \infty} \frac{\langle M \rangle}{N}$ since one has $N \gg 1$.

It is important to observe that the right side of this equation is exactly the negative of free energy of the Ising model:

$$
\frac{dm}{dt} = \frac{\partial f}{\partial y} \bigg|_{y=m}
$$

where $f(y) = \Phi(y, h = 0) = \frac{y^2}{2} - \frac{1}{\beta D} \ln(2 \cosh(\beta J z y))$. Sure for $\beta J z = 1$ (critical point), $m \ll 1$, and thus $\tau \frac{dm}{dt} = -m + \tanh m \approx -\frac{1}{3} m^3$. And then

$$
m(t) = m_0 \sqrt{\frac{3}{3 + 2m_0^2 t / \tau}} \sim t^{-1/2}
$$

(8)

for $t \rightarrow \infty$.

Taking this idea and transposing it for the Blume Capel Model:

$$
f(y) = \frac{y^2}{2} - \frac{1}{\beta D} \ln(2e^{-\beta D \sinh(\beta J z m)} + 1)
$$

(9)

one has

$$
\tau \frac{dm}{dt} = -m + \frac{2e^{-\beta D \sinh(\beta J z m)}}{2e^{-\beta D \cosh(\beta J z m)} + 1}
$$

(10)

The critical line in mean field regime is given by:

$$
\frac{D}{J_z} = K_{BT} \ln \left[ \frac{2(\alpha - K_{BT})}{K_{BT}} \right], \quad \text{thus} \quad \frac{D}{K_{BT}} = \beta D = \ln \left[ \frac{2(\alpha - K_{BT})}{K_{BT}} \right] = \ln(2(\alpha - 1)), \quad \text{where} \quad \alpha = \beta J z. \quad \text{A special point is the tricritical one:} \quad \frac{D}{K_{BT}} = 2 \ln 2, \quad \text{which leads to} \quad \alpha_t = \frac{J_z}{k_B T} = 3. \quad \text{In this paper one considers only the critical line: from} \quad \frac{D}{J_z} = 0 \quad \text{until the critical point} \quad \frac{D}{J_z} = \frac{2}{3} \ln 2. \quad \text{For} \quad \frac{D}{J_z} > \frac{2}{3} \ln 2 \quad \text{one has first order transition points.}$$

Thus one has:

$$
\tau \frac{dm}{dt} = -m + \frac{(\alpha - 1)^{-1} \sinh(\alpha m)}{(\alpha - 1)^{-1} \cosh(\alpha m)} \tau, \quad \text{and considering the approximations} \quad \sinh(\alpha m) \approx \alpha m + \frac{1}{6} m^3 \alpha^3 + \frac{1}{120} m^5 \alpha^5 + O(m^7) \quad \text{and} \quad \cosh(\alpha m) = 1 + \frac{1}{2} m^2 \alpha^2 + \frac{1}{24} m^4 \alpha^4 + O(m^6), \quad \text{one has executing few steps of algebra}$$
that:

\[ \frac{\tau_m}{\alpha t} = -m + (\kappa_1 - 1) \left[ m^0 + m^2 \alpha^2 + O(m^3) \right] \]

\[ = -m + m \left[ \frac{1}{2} m^2 \alpha^2 + \frac{1}{2} m^4 \alpha^4 + O(m^6) \right] \]

\[ = -m^3 \left( \frac{1}{2} \alpha - \frac{1}{6} \alpha^2 \right) + m^5 \left( \frac{1}{120} \alpha^4 - \frac{1}{8} \alpha^3 + \frac{1}{4} \alpha^2 \right) + O(m^7) \]

\[ \approx c_3 m^3 + c_5 m^5 \]

where

\[ c_3 = \frac{1}{6} \alpha^2 - \frac{1}{2} \alpha \]

and

\[ c_5 = \frac{1}{120} \alpha^4 - \frac{1}{8} \alpha^3 + \frac{4}{4} \alpha^2 \]

since at the critical (or tricritical) points \( m \approx 0 \). The general solution is then given by:

\[ k + \frac{t}{\tau} = -c_5 \frac{\ln m(t)}{c_3} + \frac{c_5}{c_3} \ln (c_3 + c_5 m^2(t)) - \frac{1}{2c_3 m^2(t)} \]

where \( k = \frac{c_5}{c_3} \ln \left( \frac{c_3 + c_5 m^2}{m_0} \right) - \frac{1}{2c_3 m^2} \), which is a transcendental equation. More details of this solution will be discussed in the next section.

### III. RESULTS

We perform TD-MC simulations in the MF regime. We start with the standard Ising model (spin 1/2). We prepare \( N \) spins such that the total initial magnetization is \( M_0 = \sum_{i=1}^{N} \sigma_i(0) \) (in our notation \( m_0 = M_0 / N \) is the initial magnetization per spin). Thus, at temperature \( T \), we evolve the system by \( N_{\text{steps}} \) MC steps, under \( N_{\text{run}} \) different runs. Each \( t \) th MC step considers that \( N \) spins are drawn and each drawn spin \( \sigma_i \) is flipped, for example, with MF-Metropolis probability:

\[ p(\sigma_i \rightarrow -\sigma_i) = \min \left\{ 1, \exp (\beta \Delta M_{MF}) \right\} \]

\[ = \min \left\{ 1, \exp \left[ \frac{2Jz}{k_B T} (1 - \sigma_i M(t)) \right] \right\} \]

where \( M(t) = \sum_{i=1}^{N} \sigma_i(t) \) is the magnetization at time \( t \). The simulated moments are calculated by:

\[ m_{m_0}^{(k)}(t) = \frac{1}{N^x} \left\langle \left( \sum_{i=1}^{N} \sigma_i(t) \right)^k \right\rangle_{\text{MC}} \]

\[ = \frac{1}{N_{\text{run}} N_{\text{run}}} \sum_{N=1}^{N_{\text{run}}} \left( \sum_{i=1}^{N} \sigma_i(t) \right)^k, \]

where \( \sigma_i(t) \) denotes the \( i \)-th spin, at \( j \)-th run, at \( t \)-th time step.

Fig. 2 shows the time evolution of the magnetization for different values of \( m_0 \) for \( k_B T = 1 \). Points correspond to MC simulations while curves correspond to fit with the function from Eq. 8.

For \( m_0 = 0.8 \), for \( t > t_{\text{min}} \) \( \approx 10 \) MC steps, one observes the expected power-law \( m(t) \sim t^{-\lambda} \), where \( \lambda \) is expected to be equal to 1/2 from theory. In order to check if the critical temperature is correct one can use the method developed in [9] and highly used in several different models with [10] and without [11] defined Hamiltonian, which find the optimal \( K = k_B T / J_z \), denoted by \( K^{(\text{opt})} \), for which \( m(t) \sim t^{-\lambda} \) leads to the best power law. The idea is simple since at criticality it is expected that the order parameter obeys the power-law behavior given by \( m(t) \sim t^{-\lambda} \), we performed MC simulations for each value \( K = K^{(\text{min})} + i \Delta K \), with \( i = 1, \ldots, n \), where \( n = [(K^{(\text{max})} - K^{(\text{min})}) / \Delta K] \), and calculated the coefficient of determination \( r \), which is given by

\[ r = \frac{\sum_{t=t_{\text{min}}}^{t_{\text{max}}} (\ln m - a - b \ln t)^2}{\sum_{t=t_{\text{min}}}^{t_{\text{max}}} (\ln m - \ln (m(t)))^2}, \]

with \( \ln m = \frac{1}{(t_{\text{max}} - t_{\text{min}})} \sum_{t=t_{\text{min}}}^{t_{\text{max}}} \ln m(t) \). The critical value \( K_c = k_B T / J_z \) corresponds to \( K^{(\text{opt})} \).
arg max_{K \epsilon [K_{\text{min}},K_{\text{max}}]} \{ r \} and, a and b are, respectively, the slope and intercept obtained from the linearization. Here, \( t_{\text{min}} \) is the number of discarded MC steps and \( t_{\text{max}} \) the maximum number of MC steps used in our simulations.

Fig. 3(a) shows the coefficient of determination as function of \( K \). We obtain curves for both Metropolis (Eq. 16) and we also added results for Glauber dynamics: \( p(\sigma_j \rightarrow -\sigma_j) = \frac{1}{2} \left[ 1 - \tanh \left( \frac{Jz}{Nk_B T} \sigma_j M(t) \right) \right] \). We observe that regardless of the dynamics, the maximum value of \( r \) is obtained for \( K_c \approx 1 \), corroborating that TDMC in the MF regime can determine the critical temperature of the spin-1/2 Ising model.

To avoid doubts about the required size of systems to be used, fig. 3(b) shows a finite-size study observing the magnetization decay for different sizes. We can observe that a robust power law is obtained for \( L = 10^5 \). In this paper, we used \( L = 10^5 \) spins to obtain all results.

Now, it is interesting to observe if the critical exponents of the MF Ising model are numerically corroborated. In mean-field is expected \( \nu = 1/2, \beta = 1/2, \) and \( z = 2 \) \cite{12}, which means \( \lambda = 1/2, \vartheta = 2 \). For \( c \) that for SR system corresponds to \( d/z \), in MF regime it is a little more confused. There is a rule which says that above a certain critical dimension (upper critical dimension) the exponents are given by the MF theory (classical exponents). Therefore, when we have a scaling relation where \( d \) is considered, we must use the critical dimension that in the case of Ising model is \( d_c = 4 \) \cite{12, 13}. Therefore we must do \( c = 4/z \) in the MF regime, which results in \( c = 2 \) for the Ising model. However for the tricritical point \( d_c = 3 \) \cite{14}, and then in this case: \( c = 3/z \).

Fig. 4 shows the simulated power laws corroborating the ones from Eqs. 3, 4, and 5 for the MF Ising model.

Since these power laws were obtained for \( m_0 = 1, N_{\text{run}} = 2000 \) runs which are more than enough in this condition. The uncertainty bars are obtained performing \( N_{\text{bin}} = 5 \) five different sets corresponding to different seeds in our simulations. The results lead to: \( \lambda = 0.5484(6), \varsigma = 2.033(3), \) and \( \vartheta = 1.078(2) \). This leads to \( \beta = 0.509(1), z = 1.967(3), \) and \( \nu = 0.472(1) \) that corroborate the critical classical exponents.

Since we have performed a preliminary study with the Ising model, now we can explore the critical-tricritical behavior of the MF Blume Capel model. In this case, for the sake of simplicity, here one uses heat-bath dynamics. According to Eq. 2 one has that energy in a BC model (\( H = 0 \)) with current spin \( \sigma_i \) replaced by spin \( \sigma_i' \) is given by:

\[
E(\sigma_i') = -\frac{Jz}{N}(M - \sigma_i + \sigma_i')^2 + D \sum_{k=1}^{N} \sigma_k^2 + D\sigma_i'^2
\]
thus the transition probability to spin $\sigma_i'$ is:

$$p(\sigma_i') = \frac{e^{-\beta E(\sigma_i')}}{e^{-\beta E(0)} + e^{-\beta E(+)} + e^{-\beta E(-)}}$$

and thus: $p(0) = \left[1 + e^{-\beta(E(+)-E(0))} + e^{-\beta(E(-)-E(0))}\right]^{-1}$, similarly $p(-) = \left[1 + e^{-\beta(E(+)-E(-))} + e^{-\beta(E(0)-E(-))}\right]^{-1}$, and naturally $p(+) = 1 - p(0) - p(-) = \left[1 + e^{-\beta(E(-)-E(0))} + e^{-\beta(E(0)-E(-))}\right]^{-1}$.

Thus what we have calculate is $\Delta E_{x,y} = E(x) - E(y)$, which is given by:

$$\Delta E_{x,y} = -\frac{J_z}{2N} [(M - \sigma_i + x)^2 - (M - \sigma_i + y)^2] + D(x^2 - y^2)$$

$$= -\frac{J_z}{2N} [2(M - \sigma_i)(x - y)] + \left(D - \frac{J_z}{2N}\right)(x^2 - y^2)$$

$$\approx -J_z \frac{2x}{N}(x - y) + D(x^2 - y^2)$$

(18)

However, to avoid any doubt, one used the exact form of equation (second line in the [18] in our simulations, and not the approximation suggested in the last line of this same equation.

Thus, by fixing the value of $D/k_BT$, we change $K = \frac{k_BT}{J_z}$ to find $K_c$, i.e., the value that maximizes $r$. One uses $\Delta K = 0.005$. The values used are $D/k_BT = 0, 0.28, 0.54, 0.84, 1.12$, and $\ln 4$, this last one corresponding to the tricritical point. Fig 5 (a) shows the coefficient of determination as function of $K = \frac{k_BT}{J_z}$. We can observe that critical temperatures, corresponding to the maximum values of coefficient of determination, are found in all studied cases. The corresponding Fig. 5 (b) is build with values found in Fig 5 (a). We can observe an excellent agreement between the results found in our simulations with theoretical prediction, which shows that TDMC simulations in the MF regime can be performed with theoretical prediction, which shows that TDMC simulations in the MF regime can be performed as occurs in regular time-dependent MC simulations in lattices.

Since one has $\tau_{dm}\tau = c_3m^4 + c_5m^5$, it is interesting to check two extremal cases: $c_3 = 0$, and $c_5 = 0$. For example, when one has $c_3 = 0$, it leads to $\frac{1}{3}\alpha^2 - \frac{1}{2}\alpha = 0$ that has as non-trivial solution $\alpha_t = \frac{J_z}{k_BT} = 3$. This corresponds to $\tau_{dm} = -\frac{9}{2m^3}$, which solution is

$$m_{cri}(t) = \frac{1}{(m_0^{-2} + \frac{9}{8}\tau t)^{1/4}}$$

(19)

which results in

$$m_{cri}(t) \sim t^{-1/4}$$

(20)

for $t \to \infty$. On the other hand, if $c_5 = \frac{1}{4}\alpha^2(\frac{a^2}{m_0^2} - \frac{a}{2} + 1) = 0$, it has as non-trivial solution $\alpha_c = \frac{15}{2} - \frac{1}{2}\sqrt{105}$ and $\alpha_c = \frac{15}{2} + \frac{1}{2}\sqrt{105}$, but $\alpha_c = \frac{15}{2} + \frac{1}{2}\sqrt{105} > 3 = \alpha_{tri}$, which belongs to first order region. Thus we must concentrate our attentions to $c_3m^3 + c_5m^5$, it is interesting to check two extremal cases: $c_3 = 0$, and $c_5 = 0$. For example, when one has $c_3 = 0$, it leads to $\frac{1}{3}\alpha^2 - \frac{1}{2}\alpha = 0$ that has as non-trivial solution $\alpha = \frac{J_z}{k_BT} = 3$. This corresponds to $\tau_{dm} = -\frac{9}{2m^3}$, which solution is

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(19)

which results in

$$m_{cri}(t) \sim t^{-1/4}$$

(20)

Asymptotically one has $m_{cri}(t) \sim t^{-1/2}$ that corresponds to the behavior of the only acceptable critical point: $\alpha_{cri} = \frac{15}{2} - \frac{1}{2}\sqrt{105}$. However, when we perform TDMC simulations in the MF regime, one should observe such similar power law to other critical points but a crossover to the tricritical point is also expected since according to Eq. [20] the exponent changes.

Thus, we start by studying $m(t)$, starting from $m_0 = 1$, for all points which we localized according to Fig. 5. The idea is to study how $\lambda$ changes as function of $D/(k_BT)$. Fig. 6 shows $m(t)$ as function of $t$ for the different values of $D/(k_BT)$.

One can observe that values of $\lambda$ (second row in table 3) corroborate what we observed with the power-law predicted via MF approximation. For the tricritical
Table I. Mean-field raw exponents from Blume Capel model along the critical line obtained with Monte Carlo simulations.

| $D_{k_B T}/k_B T$ | 0.00 | 0.28 | 0.56 | 0.84 | 1.12 | 2 ln 2 (TP) |
|-------------------|------|------|------|------|------|------------|
| $\lambda$         | 0.5162(4) | 0.5119(3) | 0.5096(3) | 0.5014(3) | 0.4724(5) | 0.2665(1)  |
| $\varsigma$       | 1.951(2)  | 1.954(3)  | 1.950(3)  | 1.952(2)  | 1.897(2)  | 1.529(1)   |
| $\eta$            | 1.0362(6) | 1.0333(5) | 1.0299(6) | 1.0242(3) | 1.0141(3) | 1.0258(2)  |

Figure 6. Decay of the magnetization for different values of $D_{k_B T}/k_B T$: 0, 0.28, 0.56, 0.84, 1.12, and 2 ln 2 (tricritical).

Point: $D_{k_B T}/k_B T = 2 \ln 2$, one obtains $\lambda = 0.2665(1)$ in relation to theoretical value $\lambda = \frac{1}{4}$. For values of $D_{k_B T}/k_B T$ from 0 until 0.84, the exponents agree with Ising universality: $\lambda = \frac{1}{2}$, however, this has no obligation to happen since $\lambda = 0$ for this point in the last point of the critical line: $\lambda = 0.5162(4)$, which resulted in $\lambda = 0.4963(4)$. Backing to the table [4], one observes a smaller value of $\lambda$ for $D_{k_B T}/k_B T = 1.12$ ($\lambda = 0.4724(5)$). Such tendency must continue until the minimal value of $\lambda$ that will happen in the last point of the critical line: TP. For example in another extra point simulated (which also is not presented in table [4]), that one where $c_3 = c_5$ (crossover between curves), resulting in $D_{k_B T}/k_B T = 1.91064$, we find $\lambda = 0.4278(2)$, which corroborates the crossover phenomenon from critical to tricritical point.

In the following, we show in Figs. 7 (a) and (b) the time evolution of $F_2(t)$ and $D(t)$, corresponding to exponents $\varsigma$ and $\eta$, for the same values of $D_{k_B T}/k_B T$ used to plot the time evolution of $m(t)$. For $F_2(t)$ (Fig. 7 (a) ) we can observe a monotonic diminution of $\varsigma$ that leads to the minimal value for the TP: $\varsigma = 1.529(1)$, in addition the values of $\varsigma$ and $\eta$ are shown in table II. On the other hand, as suggested by the Fig. 7 (b) the exponent $\vartheta$ seems to remain approximately equal to 1 along the critical line (see the values in table II).

The critical exponents $\beta$, $\nu$, and $\nu$, obtained from raw exponents, are shown in table II.

One can observe that classical Ising exponents were observed until $D_{k_B T}/k_B T = 0.84$. For $D_{k_B T}/k_B T = 1.12 > 1.0127$ and thus, above the point where $c_5 = 0$, the exponent presents a sensibility with proximity to the tricritical
point (crossover) and a decrease in $\beta$, and an increase in $z$ and $\nu$ is observed. However, exactly in the TP one observes a decrease in $z$, but yet $z \approx 2$ (spin 1/2 Ising) unlike what happens in two dimensions (see [4]), since one observes an increase in $z$ when compared to the critical points. Here, it is important to say that for TP, one used $d_c = 3$, and for the critical points one used $d_c = 4$. The exponents $\beta$ and $\nu$ corroborates the classical estimates for TP ($\beta = 1/4$ and $\nu = 1/2$) corroborating that $d_c$ is indeed 3 for TP.

Finally, we would like to revisit the discussion about the initial behavior of the magnetization, in light of TDMC simulations in the MF regime. Both, Ising-like and tricritical points in TDMC in two dimensions present a power-law behavior $m(t) \sim t^\theta$ when the initial magnetization $m_0 << 1$, but with different exponents $\theta > 0$ for the first case and $\theta < 0$ as we previously remembered at the beginning of this work. For example, for Ising-like we found $\theta^{\text{Ising}} \approx 0.2$ while $\theta^{\text{TP}} \approx -0.5$. An explanation of this fact could be related to the global persistence phenomena in both situations. Global persistence $P(t)$, is the probability of the magnetization remains positive until time $t$ initially proposed by Majumdar et al. [13] and that can find applications in systems from game theory until examples in Econophysics [17]. Details in how to numerically calculate $P(t)$ in the context of MC simulations can be found for example [10].

One knows that at the critical temperature, the persistence in two-dimensional spin systems must behave as a power-law $P(t) \sim t^{-\theta_g}$, where $\theta_g$ is the persistence exponent, which is valid for simulations starting from a fixed (but random) $m_0 << 1$. Thus, we performed TDMC simulations in the MF regime to obtain $\theta_g$ comparing it with two-dimensional results.

Fig. 8 (a) shows the time evolution of $P(t)$ for Ising model. We used different values of $m_0$, and a robust power law can be observed for $m_0 = 10^{-4}$. We measure $\theta_g$ in this situation, and one finds $\theta^{\text{Ising-MF}}_g = 0.542(2)$ which is very different from one found for two-dimensional Ising model: $\theta^{\text{Ising-2d}}_g \approx 0.238(3)$ [18]. This result corroborates Majumdar’s result for the $n \to \infty$ limit of the $O(n)$ model: $\theta_g = 1/2$ for $d > 4$. Now, let us consider $\theta_g$ in the MF regime for the BC model. Thus we consider two cases one critical, $D_{k_BT} \to 0$ and the tricritical $D_{k_BT} = 2 \ln 2$. The time evolutions can be observed in Fig. 8 (b). We used in both cases $m_0 = 10^{-4}$. For the critical point one obtained $\theta^{\text{BC-MF-Crit}}_g = 0.5011(7)$, while a similar value is found for TP $\theta^{\text{BC-MF-TP}}_g = 0.488(3)$, differently from occurs for the two dimensional BC $\theta^{\text{BC-2d-Crit}}_g = 0.241(4)$ and $\theta^{\text{BC-2d-TP}}_g = 1.080(4)$. It is suggestive that $\theta^{\text{BC-MF-Crit}}_g > \theta^{\text{BC-2d-Crit}}_g$ and one observes an initial increase of the magnetization characterized by a $\theta > 0$ in the case of critical points while that for the TP point: $\theta^{\text{BC-MF-TP}}_g < \theta^{\text{BC-2d-TP}}_g$ an initial decrease of the magnetization characterized by a $\theta < 0$ is observed. Short-range systems at high temperature that are suddenly placed at $T = T_c$, presents characteristics of a system in the MF regime that has a kind of inertia to behave as a short-range system, therefore this attempt of the system into establishing its new behavior leads to the initial anomalous behavior observed from different ways.

| $\frac{D}{k_B T}$ | 0.00 | 0.28 | 0.56 | 0.84 | 1.12 | $2 \ln 2$ (TP) |
|-------------------|------|------|------|------|------|----------------|
| $\beta$           | 0.4982(5) | 0.4994(4) | 0.4994(5) | 0.4995(5) | 0.49658(5) | 0.2908(1) |
| $z$               | 2.050(2) | 2.047(3) | 2.051(3) | 2.049(2) | 2.109(2) | 1.962(1) |
| $\nu$             | 0.4946(6) | 0.4957(8) | 0.4979(8) | 0.5002(5) | 0.5198(5) | 0.4968(3) |

Table II. Mean-field critical exponents from Blume Capel model along the critical line obtained with TDMC simulations.
for critical and tricritical points.

IV. CONCLUSIONS AND SUMMARIES

In this paper, we establish time-dependent Monte Carlo simulations in the Mean-Field regime to study the relaxation of magnetization and other quantities in Ising-like systems with spin 1/2 and 1. Differently from short-range systems where appear an anomalous initial increase (critical) or decrease (tricritical) of magnetization when properly prepared, such MF systems are described by Eq. 8 for spin 1/2 and by transcendental equation 14 for the Blume-Capel model, this last one which can be analytically solved in some particular cases. Our simulations corroborate such behavior and in addition, we obtained the critical and tricritical parameters considering the optimization of the expected power laws and we obtained the critical mean-field exponents under the context of the crossover between the critical and tricritical behavior. Finally, we explore the global persistence phenomena for both: critical and tricritical points showing that mean-field global persistence exponents, differently as occurs in two-dimensional systems are similar for critical and tricritical points, which in our opinion suggests a possible explanation to the differences between the initial anomalous behavior of magnetization between critical and tricritical points in two dimensions.

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