Digraphs and variable degeneracy

Jørgen Bang-Jensen∗
IMADA
University of Southern Denmark
Campusvej 55, DK-5320 Odense M, Denmark
jbj@imada.sdu.dk

Thomas Schweser∗  Michael Stiebitz∗
Institute of Mathematics
Technische Universität Ilmenau
D-98684 Ilmenau, Germany
{thomas.schweser, michael.stiebitz}@tu-ilmenau.de

Abstract

Let $D$ be a digraph, let $p \geq 1$ be an integer, and let $f : V(D) \to \mathbb{N}_0^p$ be a vector function with $f = (f_1, f_2, \ldots, f_p)$. We say that $D$ has an $f$-partition if there is a partition $(V_1, V_2, \ldots, V_p)$ of the vertex set of $D$ such that, for all $i \in [1, p]$, the digraph $D_i = D[V_i]$ is weakly $f_i$-degenerate, that is, in every non-empty subdigraph $D'$ of $D_i$ there is a vertex $v$ such that $\min\{d_{D'}^+(v), d_{D'}^-(v)\} < f_i(v)$. In this paper, we prove that the condition $f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d_{D'}^+(v), d_{D'}^-(v)\}$ for all $v \in V(D)$ is almost sufficient for the existence of an $f$-partition and give a full characterization of the bad pairs $(D, f)$. Among other applications, this leads to a generalization of Brooks’ Theorem as well as the list-version of Brooks’ Theorem for digraphs, where a coloring of a digraph is a partition of the digraph into acyclic induced subdigraphs. We furthermore obtain a result bounding the $s$-degenerate chromatic number of a digraph in terms of the maximum of maximum in-degree and maximum out-degree.

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Most of our terminology is defined as in [6] and similar to the papers [4, 5] (see also Section 2). Throughout this paper, for $0 \leq \ell \leq k$, let $[\ell, k] = \{i \in \mathbb{N}_0 \mid \ell \leq i \leq k\}$.

1 Introduction

Most people would define a $k$-coloring of an (undirected) graph $G$ as a function that assigns colors from a color set of cardinality $k$ to the vertices of $G$ such that each set of same-colored vertices induces an edgeless subgraph of $G$. But it is also nothing more than a partition of $G$ into vertex disjoint induced subgraphs $(G_1, G_2, \ldots, G_k)$ such that $G_i$ is edgeless for all $i \in [1, k]$. Naturally, each view of a coloring has its benefits. In this paper, we examine a more general approach regarding the latter definition that will allow us to

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obtain various well-known coloring results. But first of all, we need to clarify what digraph coloring refers to.

A coloring and \(k\)-coloring of a digraph \(D\) is a function \(\varphi : V(D) \to [1,k]\) such that each color class \(\varphi^{-1}(i) = \{v \in V(D) \mid \varphi(v) = i\}\) induces an acyclic subdigraph of \(D\), that is, a subdigraph that does not contain any directed cycle. The dichromatic number \(\chi(D)\) of a digraph \(D\) is the least integer \(k\) such that \(D\) admits a \(k\)-coloring.

This digraph coloring concept was originally introduced by Neumann-Lara [31] in 1982; however, it took over twenty years until it was rediscovered by Mohar [28] in 2003. Ever since, it has attracted much attention amongst graph theorists (see, e.g., [2, 3, 4, 5, 17, 18, 19, 20, 21, 23, 29, 30]). Although this coloring concept might not seem intuitive at first sight, there are various factors stressing why it is especially reasonable. First of all, the dichromatic number of a bidirected graph (replace each edge with a pair of opposite arcs) and the chromatic number of its underlying (undirected) graph coincide. Consequently, many theorems on digraph coloring are generalizations of theorems on coloring of undirected graphs. Moreover, it has been shown that plenty of well-known theorems in graph coloring indeed have digraph counterparts. For example, Harutyunyan and Mohar [20, 21] proved that there exist digraphs \(D\) of maximum total degree \(\Delta\) and arbitrary large directed girth such that \(\chi(D) \geq \frac{\Delta^2}{\log \Delta}\) for some constant \(c\), thereby obtaining the analogue of a famous result of Bollobás [10], respectively Kostochka and Mazurova [24]. Moreover, Andres and Hochstättler [1] obtained the digraph analogue of Chudnovsky, Robertson, Seymour, and Thomas’ celebrated Strong Perfect Graph Theorem [13]. In this paper, we will mainly focus on the analogue to Brooks’ famous theorem [11], which was discovered by Mohar [30] in 2010. Note that, given a digraph \(D\), its maximum out-degree (respectively in-degree) is denoted by \(\Delta^+(D)\) (respectively \(\Delta^-(D)\)).

**Theorem 1 (Mohar, 2010)** Let \(D\) be a connected digraph. Then \(D\) satisfies \(\chi(D) \leq \max\{\Delta^-(D), \Delta^+(D)\} + 1\) and equality holds if and only if \(D\) is

(a) a directed cycle of length \(\geq 2\), or

(b) a bidirected cycle of odd length \(\geq 3\), or

(c) a bidirected complete graph.

In fact, it turns out that it is possible to obtain a choosability version of Brooks’ Theorem for digraphs, i.e., a version regarding list-colorings. Given a digraph \(D\), a color set \(\Gamma\), and a function \(L : V(D) \to 2^\Gamma\) (a so-called list-assignment), an \(L\)-coloring of \(D\) is a function \(\varphi : V(D) \to \Gamma\) such that \(\varphi(v) \in L(v)\) for all \(v \in V(D)\) and \(D[\varphi^{-1}(\alpha)]\) contains no directed cycle for each \(\alpha \in \Gamma\) (if such a coloring exists, we say that \(D\) is \(L\)-colorable). Harutyunyan and Mohar [20] proved the following, thereby extending a well-known theorem of Erdős, Rubin and Taylor [15] for undirected graphs. Note that a block \(B\) of a digraph \(D\) is a maximal connected subdigraph of \(D\) that does not contain a separating vertex, i.e., the underlying graph \(G(B)\) of \(B\) is a block of the underlying graph \(G(D)\) of \(D\). By \(\mathcal{B}(D)\) we denote the set of blocks of a digraph \(D\). For \(v \in V(D)\), \(\mathcal{B}_v(D)\) denotes the set of blocks of \(D\) containing \(v\).

**Theorem 2 (Harutyunyan and Mohar, 2011)** Let \(D\) be a connected digraph, and let \(L\) be a list-assignment such that \(|L(v)| \geq \max\{d^+_{\Delta}(v), d^-_{\Delta}(v)\}\) for all \(v \in V(D)\). Suppose that \(D\) is not \(L\)-colorable. Then the following statements hold:
(a) $D$ is Eulerian and $|L(v)| = d^+_D(v) = d^-_D(v)$ for all $v \in V(D)$.

(b) If $B \in \mathcal{B}(D)$, then $B$ is either a directed cycle of length $\geq 2$, a bidirected complete graph, or a bidirected cycle of odd length $\geq 5$.

(c) For each $B \in \mathcal{B}(D)$ there is a set $\Gamma_B$ of $\Delta^+(B)$ colors such that, for every $v \in V(D)$, the sets $\Gamma_B$ with $B \in \mathcal{B}_v(D)$ are pairwise disjoint and

$$L(v) = \bigcup_{B \in \mathcal{B}_v(D)} \Gamma_B.$$  

In the present paper, we will obtain a generalization of the two previously mentioned results by examining degenerate digraphs. The concept of digraph degeneracy was introduced by Bokal et al. [8] in 2004. Given a positive integer $k$, a digraph $D$ is weakly $k$-degenerate if every non-empty subdigraph $D'$ of $D$ contains a vertex $v$ with

$$\min\{d^+_D(v), d^-_D(v)\} < k.$$  

As a consequence, a digraph is acyclic if and only if it is weakly 1-degenerate and so a coloring of a digraph coincides with a partition of the digraph into induced subdigraphs which are weakly 1-degenerate. We shall extend this definition to the case of variable degeneracy, based on the model of Borodin, Kostochka, and Toft [9] for undirected graphs.

Let $D$ be a digraph and let $h : V(D) \to \mathbb{N}_0$ be a function. Then, $D$ is weakly $h$-degenerate if every non-empty subdigraph $D'$ of $D$ contains a vertex $v$ with

$$\min\{d^+_D(v), d^-_D(v)\} < h(v).$$  

Clearly, if $h \equiv k$ is the constant function, then $D$ is weakly $h$-degenerate if and only if $D$ is weakly $k$-degenerate.

We will connect the concept of degeneracy with partitions of digraphs. A partition and $p$-partition of a digraph $D$ is a sequence $(D_1, D_2, \ldots, D_p)$ of pairwise disjoint induced subdigraphs of $D$ such that

$$V(D) = V(D_1) \cup V(D_2) \cup \ldots \cup V(D_p).$$

Note that if $(D_1, D_2, \ldots, D_p)$ is a partition of a digraph $D$, it might happen that $D_i$ is empty for some $i \in [1, p]$. Now let $f : V(D) \to \mathbb{N}_0^p$ be a vector function. Then we denote by $f_i$ the $i$-th coordinate of $f$, i.e. $f = (f_1, f_2, \ldots, f_p)$. Then, an $f$-partition of $D$ is a $p$-partition $(D_1, D_2, \ldots, D_p)$ of $D$ such that $D_i$ is weakly $f_i$-degenerate for every $i \in [1, p]$.

If $D$ admits an $f$-partition, then we also say that $D$ is $f$-partitionable.

The main aim of this paper is to determine under which degree conditions $D$ admits an $f$-partition. First of all, let us motivate why this is worthwhile considering. To this end, let $D$ be a digraph and let $L$ be a list-assignment for $D$. Moreover, let

$$\Gamma = \bigcup_{v \in V(D)} L(v)$$

be the set of all colors appearing in some list. By renaming the colors if necessary we may assume $\Gamma = [1, p]$. Let $f : V(D) \to \mathbb{N}_0^p$ be the vector function with

$$f_i(v) = \begin{cases} 1 & \text{if } i \in L(v), \\ 0 & \text{if } i \notin L(v) \end{cases}$$

3
for \( i \in [1, p] \) (see also Figure 1).

\[
\begin{align*}
\{1, 3, 4\} & \quad \{1, 2, 4\} \\
\{1, 2, 3\} & \quad \{1, 2, 4\} \\
\{2, 3\} & \quad \{1, 3, 4\}
\end{align*}
\]

\( (D, L) \)

\[
\begin{align*}
\{0, 1, 0, 1\} & \quad \{0, 1, 1, 0\} \\
\{0, 1, 1, 0\} & \quad \{0, 1, 1, 1\} \\
\{0, 1, 0, 1\} & \quad \{0, 1, 1, 1\}
\end{align*}
\]

\( (D, f) \)

Figure 1: Transforming a list-assignment into a vector function

If \( D \) is \( L \)-colorable, then it easy to confirm that the partition \((D_1, D_2, \ldots, D_p)\) of \( D \) where \( D_i \) is the subdigraph of \( D \) induced by vertices of color \( i \) is indeed an \( f \)-partition of \( D \). Conversely, if \( D \) has an \( f \)-partition \((D_1, D_2, \ldots, D_p)\), then setting \( \varphi(v) = i \) if \( v \in V(D_i) \) leads to an \( L \)-coloring of \( D \). Hence, \( D \) is \( L \)-colorable if and only if \( D \) has an \( f \)-partition and so the task of finding an \( f \)-partition generalizes the usual list-coloring problem (and, of course, the usual coloring problem, too). However, since colorings and list-colorings only correspond to very simple functions \( f_1, f_2, \ldots, f_p \), by examining conditions for the existence of an \( f \)-partition we get significantly more general results than for the usual coloring problem.

Since deciding whether the dichromatic number is at most two is already \textbf{NP}-hard (see [8]), it is pointless to try to determine whether a digraph \( D \) is \( f \)-partitionable in general. Instead, we need to find a reasonable condition for the function \( f \) that would allow a digraph satisfying this condition to be \( f \)-partitionable. Note that in the above transformation, it follows from the definition of \( f \) that

\[
|L(v)| = f_1(v) + f_2(v) + \ldots + f_p(v)
\]

for all \( v \in V(D) \). Thus, Theorem 2 suggests that

\[
f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d^+_D(v), d^-_D(v)\}
\]

for all \( v \in V(D) \) might be the right condition to investigate. Indeed, we shall prove that this condition is always sufficient for the existence of an \( f \)-partition, unless \((D, f)\) belongs to the following, recursively defined class of configurations. Clearly, \( D \) admits an \( f \)-partition if and only if each connected component of \( D \) has one and, hence, it suffices to examine connected digraphs.

Let \( D \) be a connected digraph, let \( p \geq 1 \) be an integer, and let \( f : V(D) \to \mathbb{N}_0^p \) be a vector function. We say that \((D, f)\) is a \textbf{hard pair} and that \( D \) is \textbf{\( f \)-hard} if one of the following four conditions hold.

(H1) \( D \) is a block, \( D \) is Eulerian, and there exists an index \( j \in [1, p] \) such that, for all \( i \in [1, p] \) and for each \( v \in V(D) \), we have

\[
f_i(v) = \begin{cases} d^+_D(v) = d^-_D(v) & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]
In this case, we say that \((D,f)\) is a hard pair of type \((M)\).

(H2) \(D\) is a bidirected complete graph and there are integers \(n_1, n_2, \ldots, n_p \geq 0\) with at least two \(n_i\) different from zero such that \(n_1 + n_2 + \ldots + n_p = |D| - 1\) and
\[
f(v) = (n_1, n_2, \ldots, n_p) \quad \text{for all } v \in V(D).
\]
In this case, we say that \((D,f)\) is a hard pair of type \((K)\).

(H3) \(D\) is a bidirected cycle of odd length at least 5 and there are two indices \(k \neq \ell\) from the set \([1,p]\) such that, for all \(i \in [1,p]\) and for each \(v \in V(D)\), we have
\[
f_i(v) = \begin{cases} 
1 & \text{if } i \in \{k, \ell\}, \\
0 & \text{otherwise}.
\end{cases}
\]
In this case, we say that \((D,f)\) is a hard pair of type \((C)\).

(H4) There are two hard pairs \((D^1, f^1)\) and \((D^2, f^2)\) with \(f^j : V(D^j) \to \mathbb{N}_0^p\) for \(j \in \{1, 2\}\) such that \(D\) is obtained from \(D^1\) and \(D^2\) by identifying two vertices \(v^1 \in V(D^1)\) and \(v^2 \in V(D^2)\) to a new vertex \(v\). Furthermore, for \(w \in V(D)\), it holds that
\[
f(w) = \begin{cases} 
f^1(w) & \text{if } w \in V(D^1) \setminus \{v^1\}, \\
f^2(w) & \text{if } w \in V(D^2) \setminus \{v^2\}, \\
f^1(v^1) + f^2(v^2) & \text{if } w = v.
\end{cases}
\]
In this case we say that \((D,f)\) is obtained from \((D^1, f^1)\) and \((D^2, f^2)\) by merging \(v^1\) and \(v^2\) to \(v\).

In order to develop a better feeling of what hard pairs may look like, we refer the reader to Figure 2. The main result of this paper is the following.

**Theorem 3** Let \(D\) be a connected digraph, let \(p \geq 1\) be an integer, and let \(f : V(D) \to \mathbb{N}_0^p\) be a vector function such that \(f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d^+_D(v), d^-_D(v)\}\) for all \(v \in V(D)\). Then \(D\) is not \(f\)-partitionable if and only if \((D,f)\) is a hard pair.

\[\diamond\]

**Figure 2:** Examples of hard pairs
2 Basic Terminology

Let $D = (V(D), A(D))$ be a digraph, where $V(D)$ is the set of vertices of $D$ and $A(D)$ is the set of arcs of $D$. The order $|D|$ of $D$ is the size of $V(D)$. The digraphs in this paper are finite and do not have loops nor parallel arcs; however, there may be two arcs in opposite directions between two vertices (in this case we say that the arcs are opposite). We denote by $uv$ the arc whose initial vertex is $u$ and whose terminal vertex is $v$. Two vertices $u,v$ are adjacent if at least one of $uv$ and $vu$ belongs to $A(D)$. If $u$ and $v$ are adjacent, we also say that $u$ is a neighbor of $v$ and vice versa. If $uv \in A(D)$, then $v$ is called an out-neighbor of $u$, and $u$ is called an in-neighbor of $v$. Given a digraph $D$ and a vertex set $X$, we denote by $D[X]$ the subdigraph of $D$ induced by the vertex set $X$, that is, $V(D[X]) = X$ and $A(D[X]) = \{uv \in A(D) \mid u,v \in X\}$. A digraph $D'$ is said to be an induced subdigraph of $D$ if $D' = D[V(D')]$. As usual, if $X$ is a subset of $V(D)$, we define $D - X = D[V(D) \setminus X]$. If $X = \{v\}$ is a singleton, we use $D - v$ rather than $D - \{v\}$. The out-degree of a vertex $v \in V(D)$, denoted $d^+_D(v)$, is the number of arcs whose initial vertex is $v$. Similarly, the in-degree of $v$, denoted $d^-_D(v)$ is number of arcs whose terminal vertex is $v$. A vertex $v \in V(D)$ is Eulerian if $d^+_D(v) = d^-_D(v)$. Moreover, the digraph $D$ is Eulerian if every vertex of $D$ is Eulerian.

The term graph refers to a finite, undirected simple graph. The underlying graph $G(D)$ of a digraph $D$ is the graph with $V(G(D)) = V(D)$ and $\{u,v\} \in E(G(D))$ if and only if at least one of $uv$ and $vu$ belongs to $A(D)$. A connected component of a digraph $D$ is a subdigraph $D'$ of $D$ such that $G(D')$ is a component of $G(D)$; and $D$ is connected if $G(D)$ is connected. A separating vertex of a connected digraph $D$ is a vertex $v \in V(D)$ such that $D - v$ has at least two connected components. Furthermore, a block of $D$ is a maximal connected subdigraph $D'$ of $D$ such that $D'$ has no separating vertex. By $\mathcal{B}(D)$, we denote the set of blocks of $D$. Moreover, if $v \in V(D)$, we denote by $\mathcal{B}_v(D)$ the set of blocks of $D$ containing $v$. A bidirected graph is a digraph that can be obtained from a graph $G$ by replacing each edge by two opposite arcs, we denote it by $D(G)$.

3 Proof of the main result

In order to prove Theorem 3, we need two classic results for graphs. The first one is an easy consequence of Menger’s Theorem and usually referred to as the Fan Lemma (see, e.g. [14, Corollary 3.3.4]). The second lemma is due to Gallai [16, Satz 1.9]. Recall that a chord of a cycle $C$ in a graph $G$ is an edge of $G$ between vertices of $C$ that does not belong to the edges of $C$.

Lemma 4 (The Fan Lemma) Let $G$ be a $k$-connected graph, let $v \in V(G)$, and let $X \subseteq V(G) \setminus \{v\}$ be a set of cardinality at least $k$. Then, there are $k$ paths from $v$ to vertices of $X$ whose only common vertex is $v$ and whose only intersection with $X$ are the respective end-vertices.  

Lemma 5 (Gallai, 1963) If $G$ is a graph in which each even cycle has at least two chords, then every block of $G$ is a complete graph or an odd cycle.
Proposition 6 Let $D$ be a connected digraph, and let $f : V(D) \to \mathbb{N}_0^p$ be a vector function with $p \geq 1$ such that $(D, f)$ is a hard pair. Then, for each $B \in \mathcal{R}(D)$ there is a uniquely determined function $f_B : V(B) \to \mathbb{N}_0^p$ such that the following statements hold:

(a) $(B, f_B)$ is a hard pair of type (M), (K), or (C).

(b) $f(v) = \sum_{B \in \mathcal{R}(B)} f_B(v)$ for all $v \in V(D)$. In particular, $f_B(v) = f(v)$ for all non-separating vertices $v$ of $D$ belonging to $B$.

Proof: We use Proposition 6 in the proof of the following proposition.

Proposition 7 Let $D$ be a connected digraph, and let $f : V(D) \to \mathbb{N}_0^p$ be a vector function with $p \geq 1$ such that $(D, f)$ is a hard pair. Then the following statements hold:

(a) $f_1(v) + f_2(v) + \ldots + f_p(v) = d_D^+(v) = d_D^-(v)$ for all $v \in V(D)$. As a consequence, $D$ is Eulerian.

(b) $D$ is not $f$-partitionable.

Proof: Statement (a) is an easy consequence of Proposition 6. The proof of statement (b) is by reductio ad absurdum. So let $(D, f)$ be a minimal counter-example, that is,

(1) $(D, f)$ is a hard pair,

(2) $D$ admits an $f$-partition $(D_1, D_2, \ldots, D_p)$, and

(3) $|D|$ is minimum with respect to (1) and (2).

We proceed by induction on $|D|$. Note that the empty digraph is the only weakly 0-degenerate digraph and so if $f_i \equiv 0$ for some $i$, then $D_i = \emptyset$.

First suppose that $(D, f)$ is of type (M). Then, there is an index $j \in [1, p]$ such that, for all $i \in [1, p]$ and for each $v \in V(D)$, we have $f_i(v) = d_D^+(v) = d_D^-(v)$ if $i = j$ and $f_i(v) = 0$ otherwise. Thus, $D_i$ is empty for all $i \in [1, p] \setminus \{j\}$ and so $D = D_j$. However, as $f_j(v) = d_D^+(v) = d_D^-(v)$ for all $v \in V(D)$, the digraph $D_j = D$ is not weakly $f_j$-degenerate, contradicting (2).

Next suppose that $(D, f)$ is of type (C), i.e., $D$ is a bidirected cycle of odd length at least 5 and there are two indices $k \neq \ell$ from $[1, p]$ such that, for all $i \in [1, p]$ and for each $v \in V(D)$, we have $f_i(v) = 1$ if $i \in \{k, \ell\}$ and $f_i(v) = 0$, otherwise. Thus, for $i \in [1, p] \setminus \{k, \ell\}$, $f_i \equiv 0$ and so we conclude that $D_i$ is empty. Moreover, for $j \in \{k, \ell\}$, it is not possible that two adjacent vertices $u, v$ belong to $D_j$ since $D[\{u, v\}]$ is a digon and therefore not weakly $f_j$-degenerate (as $f_j \equiv 1$). Consequently, the vertices of $D$ need to belong alternately to $D_k$ and $D_\ell$, which is impossible since $D$ is of odd length.

Finally suppose that $(D, f)$ is of type (K), i.e., $D = D(K_n)$ and there are integers $n_1, n_2, \ldots, n_p \geq 0$ with at least two $n_i$ different from zero such that $n_1 + n_2 + \ldots + n_p = n - 1$ and $f(v) = (n_1, n_2, \ldots, n_p)$ for all $v \in V(D)$. As $D$ is a bidirected complete graph, so is $D_i$ for $i \in [1, p]$, and hence $D_i$ is weakly $f_i$-degenerate if and only if $|D_i| \leq n_i$. Consequently, we have

\[ n = |D| = |D_1| + |D_2| + \ldots + |D_p| \geq n_1 + n_2 + \ldots + n_p = n - 1, \]

which is impossible. Summarizing, we obtain that if $D$ is a block, then $D$ is not $f$-partitionable.
Now assume that $D$ has at least two blocks. Then it follows from (H4) that there are two hard pairs $(D^1, f^1)$ and $(D^2, f^2)$ with $|D^j| < |D|$ for $j \in \{1, 2\}$ such that $(D, f)$ is obtained from $(D^1, f^1)$ and $(D^2, f^2)$ by merging vertices $v^j \in V(D^j)$ to a new vertex $v$. For the sake of readability, we use $v$ below for $v^1 = v^2 = v$. By (3), the digraph $D^j$ is not $f^j$-partitionable for $j \in \{1, 2\}$. Now we regard the $f$-partition $(D_1, D_2, \ldots, D_p)$ of $D$ and set $D^j_i = D^j \cap D_i$ for $j \in \{1, 2\}$ and $i \in [1, p]$. By symmetry, we may assume that $v \in V(D_1)$. Then, $D^j_i$ is weakly $f^j_i$-degenerate for all $i \in [2, p]$ and $j \in \{1, 2\}$ (as $D^j_i \subseteq D_i$ and $f^j_i(w) = f_i(w)$ for all $w \in V(D^j_i)$). As $D^j$ is not $f^j$-partitionable, it follows that $D^j_i$ is not $f^j_1$-partitionable for $j \in \{1, 2\}$ and so there are non-empty subdigraphs $\tilde{D}^j \subseteq D^j_i$ with $\min\{d_{\tilde{D}^j}^+(w), d_{\tilde{D}^j}^-(w)\} \geq f^j_i(w)$ for all $w \in V(\tilde{D}^j)$. Let $\tilde{D} = D_1 \cup \tilde{D}^2$. If $v \notin \tilde{D}$, then $\tilde{D}$ is the disjoint union of $\tilde{D}^1$ and $\tilde{D}^2$ and we clearly have $\min\{d_{\tilde{D}}^+(w), d_{\tilde{D}}^-(w)\} \geq f_1(w)$ for all $w \in V(\tilde{D})$ and so $D_1$ is not weakly $f_1$-degenerate, contradicting (2).

Thus, by Proposition 7, the “if”-direction of Theorem 3 is proved. The hard part, i.e., the “only if”-direction, is covered in the next theorem.

**Theorem 8** Let $D$ be a connected digraph, let $p \geq 1$ be an integer, and let $f : V(D) \to \mathbb{N}_0^p$ be a vector function such that $f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d_{D}^+(v), d_{D}^-(v)\}$ for all $v \in V(D)$. If $D$ is not $f$-partitionable, then $(D, f)$ is a hard pair.

**Proof:** The proof is by reductio ad absurdum. So let $(D, f)$ be a smallest counterexample, that is, 

(C1) $f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d_{D}^+(v), d_{D}^-(v)\}$ for all $v \in V(D)$,

(C2) $D$ is not $f$-partitionable,

(C3) $(D, f)$ is not a hard pair, and

(C4) $|D|$ is minimum subject to (C1), (C2), and (C3).

Clearly, $D$ is a connected digraph. In order to derive a contradiction, we establish a sequence of eight claims.

**Claim 1** $D - v$ is $f$-partitionable for every $v \in V(D)$.

**Proof:** Suppose that $D - v$ is not $f$-partitionable for some $v \in V(D)$. Then there is a connected component $D'$ of $D - v$ such that $D'$ is not $f$-partitionable and so $(D', f)$ is a hard pair (by (C4)). As $D$ is connected, $v$ has in $D$ a neighbor $u$ belonging to $D'$. Since $(D', f)$ is a hard pair, Proposition 7(a) implies that $f_1(u) + f_2(u) + \ldots + f_p(u) = d_{D'}^+(u) = d_{D'}^-(u)$. As $D$ contains at least one of the arcs $vu$ and $uv$, we conclude that

$$f_1(u) + f_2(u) + \ldots + f_p(u) = d_{D'}^+(u) = d_{D'}^-(u) < \max\{d_{D}^+(u), d_{D}^-(u)\},$$

contradicting (C1).
The next claim is central for the proof of Theorem 8. Casually speaking, it says that not only is $D$ Eulerian and in- and out-degree of each vertex coincides with the sum of its $f$-values, but also, given a fixed vertex $v$ and an $f$-partition $(D_1, D_2, \ldots, D_p)$ of $D - v$, the connected component of the digraph $D_i + v = D[V(D_i) \cup \{v\}]$ containing $v$ is Eulerian, too, and the respective degrees coincide with the $f_i$-values.

**Claim 2** Let $v$ be an arbitrary vertex of $D$ and let $(D_1, D_2, \ldots, D_p)$ be an $f$-partition of $D - v$. Then the following statements hold:

(a) $f_1(v) + f_2(v) + \ldots + f_p(v) = d^{+}_D(v) = d^{-}_D(v)$. As a consequence, $D$ is Eulerian.

(b) $d^{+}_{D_i+v}(v) = d^{-}_{D_i+v}(v) = f_i(v)$ for all $i \in [1, p]$.

(c) For $i \in [1, p]$, if $u$ is a neighbor of $v$ in $D_i$, then the sequence $(D'_1, D'_2, \ldots, D'_p)$ with $D'_i = (D_i + v) - u$ and $D'_j = D_j$ for $j \neq i$ is an $f$-partition of $D - u$.

(d) For $i \in [1, p]$, the connected component $D'$ of $D_i + v$ that contains $v$ is Eulerian and $d^{+}_{D'}(w) = d^{-}_{D'}(w) = f_i(w)$ for all $w \in V(D')$.

**Proof:** Let $i \in [1, p]$ be an arbitrary index. As $D$ is not $f$-partitionable, the digraph $D_i + v$ is not weakly $f_i$-degenerate. Thus, there is a non-empty subdigraph $D^*$ of $D_i + v$ such that $\min\{d^{+}_{D'}(w), d^{-}_{D'}(w)\} \geq f_i(w)$ for all $w \in V(D^*)$. Since $D_i$ is weakly $f_i$-degenerate, $D^*$ contains $v$ and so

$$f_i(v) \leq \min\{d^{+}_{D'}(v), d^{-}_{D'}(v)\} \leq \min\{d^{+}_{D_i+v}(v), d^{-}_{D_i+v}(v)\}.$$  \hspace{1cm} (3.1)

Since $(D_1, D_2, \ldots, D_p)$ is a partition of $D - v$, we have

$$\sum_{i=1}^{p} d^{+}_{D_i+v}(v) = d^{+}_D(v) \quad \text{and} \quad \sum_{i=1}^{p} d^{-}_{D_i+v}(v) = d^{-}_D(v)$$  \hspace{1cm} (3.2)

Since $i$ was chosen arbitrarily, we conclude from (3.1), (3.2), and (C1) that

$$\sum_{i=1}^{p} f_i(v) \leq \sum_{i \in [1, p]} \min\{d^{+}_{D_i+v}(v), d^{-}_{D_i+v}(v)\} \leq \min\{d^{+}_D(v), d^{-}_D(v)\}$$ \hspace{1cm} (3.3)

and so we have equality everywhere in (3.1) and (3.3). Hence, statement (a) holds and, by (3.2), also (b) holds. Furthermore, we obtain that

$$f_i(v) = d^{+}_{D_i+v}(v) = d^{-}_{D_i+v}(v) \geq \max\{d^{+}_D(v), d^{-}_D(v)\} \geq f_i(v),$$

implying that $D^*$ contains all neighbors of $v$ in $D_i$. Let $u$ be such a neighbor of $v$ in $D_i$. Since $D_i$ is weakly $f_i$-degenerate, it then follows from (b) that $(D_i + v) - u$ is weakly $f_i$-degenerate and hence $(D'_1, D'_2, \ldots, D'_p)$ with $D'_i = (D_i + v) - u$ and $D'_j = D_j$ for $j \neq i$ is an $f$-partition of $D - u$, which proves (c). For the proof of (d), let $D'$ be the connected component of $D_i + v$ that contains $v$ and let $w$ be an arbitrary vertex of $D'$. Suppose that
$d$ is the distance between $v$ and $w$ in the underlying graph $G(D')$. We prove by induction on $d$ that

$$
\oplus d^+_{D'}(w) = d^-_{D'}(w) = f_i(w).
$$

If $d = 0$, then $w = v$ and $\oplus$ holds by (b). If $d \geq 1$, then there is a neighbor $u$ of $v$ in $D_i$ such that the distance between $u$ and $w$ in $G(D')$ is $d - 1$. By (c), the sequence $(D'_1, D'_2, \ldots, D'_p)$ with $D'_i = (D_i + v) - u$ and $D'_j = D_j$ for $j \neq i$ is an $f$-partition of $D - u$. Clearly, $D'$ is the connected component of $D'_1 + u$ that contains $u$. Then the induction hypothesis implies that $\oplus$ holds. This proves (d). \hfill \Box

**Claim 3** $D$ is a block.

**Proof:** The proof is by reductio ad absurdum. So assume that $D$ is the union of two connected induced subdigraphs $D^1$ and $D^2$ with $V(D^1) \cap V(D^2) = \{v\}$ and $|D^i| < |D|$ for $j \in \{1, 2\}$. Then there are no arcs in $D$ between $D^1 - v$ and $D^2 - v$. We will model two functions $f^1$ and $f^2$ such that $(D^1, f^1)$ and $(D^2, f^2)$ are hard pairs and $(D, f)$ is obtained from the two hard pairs via the merging operation, thereby giving us the desired contradiction. By Claim 1, $D - v$ has an $f$-partition, say $(D_1, D_2, \ldots, D_p)$. For $i \in [1, p]$ and $j \in \{1, 2\}$, let $D'_i = D_i \cap D^j$. Clearly, for all $i \in [1, p]$, the digraphs $D'_i$ and $D^j_i$ are disjoint and there are no arcs in $D$ between these two digraphs. Next we claim that

$$
d^+_{D'_i + v}(v) = d^-_{D'_i + v}(v) \quad \text{for all } i \in [1, p] \text{ and } j \in \{1, 2\}. \tag{3.4}
$$

Suppose this is false. Then there is a pair $(i, j)$ with $i \in [1, p]$ and $j \in \{1, 2\}$ such that $v$ is not Eulerian in $D'_i + v$. Let $D'$ be the connected component of $D_i + v$ containing $v$. Then, we obtain that

$$
D' = (D' \cap (D'_i + v)) \cup (D' \cap (D'_2 + v)),
$$

$(D' \cap (D'_1 + v))$ and $(D' \cap (D'_2 + v))$ have only $v$ in common, and there are no arcs in $D$ between vertices of $D'_1$ and $D'_2$. Thus vertex $v$ is not Eulerian in $D' = (D' \cap (D'_i + v))$

Since in every digraph, the sum of out-degrees over all vertices equals the sum of in-degrees over all vertices, this implies that $D'$ contains a vertex $w \neq v$ that is not Eulerian in $D'$. But then $D'$ is not Eulerian, a contradiction to Claim 2(d). This proves the claim that (3.4) holds.

Now we are ready to define the vector functions $f^1$ and $f^2$. For $i \in [1, p]$ and $j \in \{1, 2\}$ let

$$
f^j_i(w) = \begin{cases} f_i(w) & \text{if } w \in V(D^j - v), \\ d^+_{D'_i + v}(v) & \text{if } w = v. \end{cases} \tag{3.5}
$$

By Claim 2(a), we obtain that $f_1(w) + f_2(w) + \ldots + f_p(w) = d^+_{D}(w) = d^-_{D}(w)$ for all $w \in V(D)$. Consequently, for $j \in \{1, 2\}$ and all $w \in V(D') \setminus \{v\}$, we obtain that

$$
f^j_1(w) + f^j_2(w) + \ldots + f^j_p(w) = f_1(w) + f_2(w) + \ldots + f_p(w) = \max\{d^+_{D}(w), d^-_{D}(w)\} = \max\{d^+_{D}(w), d^-_{D}(w)\},
$$
Thus, by (3.4), and since $(D_1', D_2', \ldots, D_p')$ is a partition of $D^j - v$ (for $j \in \{1, 2\}$), we obtain for $j \in \{1, 2\}$ that

$$d_{D_1'}^+(v) = \sum_{i=1}^{p} d_{D_1'+v}^+(v) = \sum_{i=1}^{p} d_{D_1'+v}^-(v) = d_{D_1'}^-(v),$$  \hspace{1cm} (3.6)$$

and it follows from (3.5) and (3.6) that

$$f_1^j(v) + f_2^j(v) + \ldots + f_p^j(v) = \sum_{i \in [1, p]} d_{D_1'+v}^+(v) = d_{D_1'}^+(v) = d_{D_1'}^-(v).$$  \hspace{1cm} (3.7)$$

As a consequence, both $(D^1, f^1)$ and $(D^2, f^2)$ fulfill the hypothesis of Theorem 8.

First, assume that, for some $j \in \{1, 2\}$, the digraph $D^j$ is $f^j$-partitionable. By symmetry, we may assume $j = 1$. Then, $D^1$ admits an $f^1$-partition, say $(D_1', D_2', \ldots, D_p')$. By symmetry, we may assume $v \in V(D_1')$. Let $D_i' = D_1' \cup (D_i^1 + v)$ and $D_i'' = D_i' \cup D_i^2$ for $i \in [2, p]$. Then $(D_1', D_2', \ldots, D_p')$ is a partition of $D$. As $D_1'$ and $D_i''$ are disjoint and as $(D_1^1, D_2^1, \ldots, D_p^1)$ is an $f$-partition of $D - v$, $D_i''$ is weakly $f_i$-degenerate for $i \in [2, p]$. We claim that $D_1'$ is weakly $f_1$-degenerate. Suppose this is false. Then there is a non-empty subdigraph of $D_1' = D_1' \cup (D_2^1 + v)$, say $\tilde{D}$, such that

$$\min\{d_{\tilde{D}}^+(w), d_{\tilde{D}}^-(w)\} \geq f_1(w)$$

for all $w \in V(\tilde{D})$. Since $D_1^1$ is weakly $f_1$-degenerate, the digraph $\tilde{D}^1 = \tilde{D} \cap D^1$ is non-empty. Since $D^1$ is a subdigraph of $D_1'$ and therefore weakly $f_1$-degenerate, there is a vertex $w \in V(\tilde{D}^1)$ such that $\min\{d_{\tilde{D}^1}^+(w), d_{\tilde{D}^1}^-(w)\} < f_1(w)$. If $w \neq v$, then

$$\min\{d_{\tilde{D}^1}^+(w), d_{\tilde{D}^1}^-(w)\} = \min\{d_{D_1'}^+(w), d_{D_1'}^-(w)\} < f_1(w) = f_1(w),$$  \hspace{1cm} (3.8)$$

a contradiction. It remains to consider the case that $w = v$. Since

$$d_{D_2'+v}^+(v) = d_{D_2'+v}^-(v) = f_1^2(v)$$

(by (3.4) and (3.5)), this implies that

$$\min\{d_{\tilde{D}}^+(v), d_{\tilde{D}}^-(v)\} \leq \min\{d_{D_1'}^+(v), d_{D_1'}^-(v)\} + d_{D_2'+v}^+(v) + d_{D_2'+v}^-(v) < f_1(v) + f_1^2(v) = f_1(v),$$

a contradiction, too. Consequently, $D_1'$ is weakly $f_1$-degenerate, as claimed, and so $(D_1', D_2', \ldots, D_p')$ is an $f$-partition of $D$, a contradiction to (C2).

Now, assume that $D^j$ is not $f^j$-partitionable for $j \in \{1, 2\}$. Then by the minimality of $D$, both $(D^1, f^2)$ and $(D^2, f^2)$ are hard pairs. From Claim 2(b) and (3.4) it follows that

$$f_1^j(v) + f_2^j(v) = d_{D_1'+v}^+(v) + d_{D_2'+v}^+(v) = d_{D_1'+v}^+(v) = f_1(v).$$

Thus, $(D, f)$ is obtained from $(D^1, f^2)$ and $(D^2, f^2)$ via the merging operation. As a consequence, $(D, f)$ is a hard pair, contradicting (C3). This completes the proof of the claim.

By the above claim, $D$ is a block. Hence it remains to show that $(D, f)$ is a hard pair of type (M), (K), or (C), giving us the desired contradiction. The next claim eliminates the pairs of type (M).
Claim 4 For every \(v \in V(D)\) and each \(f\)-partition \((D_1, D_2, \ldots, D_p)\) of \(D - v\), there are two indices \(i, j \in [1, p]\) with \(i \neq j\) such that \(D_i\) and \(D_j\) are non-empty.

Proof: Suppose that there is a vertex \(v \in V(D)\) and a partition \((D_1, D_2, \ldots, D_p)\) of \(D - v\) such that exactly one part of the partition is non-empty, say \(D_1\). Then \(D_1 + v = D\) and so it follows from Claim 2(d), and the fact that \(D\) is connected, that \(d^+(w) = d^-(w) = f_1(w)\) for all \(w \in V(D)\). Thus it follows from Claim 2(b) that \(f_j(w) = 0\) for all \(j \in [2, p]\) and for all \(w \in V(D)\) and \((D, f)\) is a hard pair of type \((M)\), contradicting \((C3)\).

Actually, Claim 2(c) provides us with a powerful tool that we shall use in the following. Let \(v \in V(D)\) be an arbitrary vertex and let \((D_1, D_2, \ldots, D_p)\) be an \(f\)-partition of \(D - v\). Moreover, let \(u \in V(D)\) be a neighbor of \(v\). Then, \(u \in V(D_i)\) for some \(i \in [1, p]\) and Claim 2(c) implies that replacing \(D_i\) with \((D_i + v) - u\) leads to an \(f\)-partition of \(D - u\) in which \(v\) is contained in what was previously \(D_i\). Thus, we can swap \(v\) and any neighbor \(u\) of \(v\) and obtain a new \(f\)-partition of \(D - u\). In order to make this observation a bit more graphical, we introduce the following terms. Given a vertex \(v \in V(D)\), we call an \(f\)-partition \((D_1, D_2, \ldots, D_p)\) of \(D - v\) an \(f\)-coloring \(\varphi\) of \(D - v\). Moreover, for \(w \in V(D)\backslash\{v\}\), we set \(\varphi(w) = i\) if \(w \in V(D_i)\) and say that \(w\) has color \(i\). Finally, we say that the vertex \(v\) is uncolored. Statements (a) and (b) of the following claim are just a reformulation of Claim 4 and Claim 2(b) to fit the new terminology. Statement (c) reformulates the above introduced "swapping"-technique and follows immediately from Claim 2(c).

Claim 5 Let \(v \in V(D)\) be an arbitrary vertex and let \(\varphi\) be an \(f\)-coloring of \(D - v\). Then, the following statements hold:

(a) At least two color-classes of \(\varphi\) are non-empty.

(b) \(d^+_{D\left[\varphi^{-1}(i)\right]}(v) = d^-_{D\left[\varphi^{-1}(i)\right]}(v) = f_i(v)\) for all \(i \in [1, p]\).

(c) Let \(u\) be a neighbor of \(v\). Then, uncoloring \(u\) and coloring \(v\) with color \(\varphi(u)\) leads to an \(f\)-coloring of \(D - u\).

We will call the process that is described in statement (c) of the above claim shifting the color from \(u\) to \(v\). The original idea of shifting goes back to Gallai [16].

Now let \(C\) be a cycle in \(G(D)\), let \(v \in V(C)\) be an arbitrary vertex, and let \(\varphi\) be an \(f\)-coloring of \(D - v\). Moreover, let \(u\) and \(w\) be the vertices such that \(uw\) and \(vw\) are in \(E(C)\). Then, we can shift the color from \(u\) to \(v\) and obtain an \(f\)-coloring of \(G - u\) by Claim 5(c). Afterwards, we shift the color from the other neighbor of \(u\) on \(C\) to \(u\). Continuing like this, we can shift the color of each vertex of \(C\), one after another, clockwise, until eventually we shift the color from \(v\) to \(w\). This gives us a new \(f\)-coloring \(\varphi'\) of \(D - v\) (see Figure 3). In particular, \(\varphi'(w) = \varphi(u)\).
Similarly, starting from $\varphi$ with shifting the color from $w$ to $v$, we can shift the color of each vertex counter-clockwise on the cycle and obtain a third $f$-coloring of $D - v$. The next claim uses this observation.

**Claim 6** Let $C$ be a cycle in $G(D)$, let $v \in V(C)$ be an arbitrary vertex, and let $\varphi$ be an $f$-coloring of $D - v$. Moreover, let $u$ and $w$ be the neighbors of $v$ on $C$, that is, \{w, uv, vw\} $\subseteq$ $E(C)$. Then the following statements hold:

(a) For any edge $u'w' \in E(C)$ such that $v \notin \{u', w'\}$, there exists an $f$-coloring $\varphi'$ of $D - v$ such that $\{\varphi'(u), \varphi'(w)\} = \{\varphi(u'), \varphi(w')\}$ and $\varphi(v') = \varphi(v')$ for all $v' \in V(D) \setminus V(C)$.

(b) For any vertex $z \in V(C) \setminus \{v\}$, there is an $f$-coloring $\varphi'$ of $D - v$ such that $\varphi'(u) = \varphi(z)$ and $\varphi'(v') = \varphi(v')$ for all $v' \in V(D) \setminus V(C)$.

**Proof:** Let $v, v_0, v_1, \ldots, v_r, v$ be a consecutive ordering of the vertices of the cycle $C$, and let $\varphi$ be an $f$-coloring of $D - v$. By symmetry, we may assume that $u = v_0, w = v_r, u' = v_i$ and $u' = v_{i+1}$ for some $i \in [0, r - 1]$ (where all indices are calculated modulo $r+1$, i.e. $v_{r+1} = v_0$ etc.).

We prove (a) and (b) simultaneously. In particular, we show that there is an $f$-coloring $\varphi'$ of $D - v$ such that $\varphi'(u) = \varphi(u'), \varphi'(w) = \varphi(w')$ and $\varphi'(v') = \varphi(v')$ for all $v' \in V(D) \setminus V(C)$. For the sake of readability, shifting the color of a vertex $x$ to an uncolored vertex $y$ is denoted by $x \rightarrow y$.

We begin with $v_0 \rightarrow v$. Now, $v_0$ is uncolored while $v$ has color $\varphi(v_0)$. Afterwards, we shift $v_1 \rightarrow v_0$, $v_2 \rightarrow v_1, \ldots, v_r \rightarrow v_{r-1}$, and finally $v \rightarrow v_r$. This leads to a new coloring $\varphi_1$ of $D - v$ such that $\varphi_1(v_j) = \varphi(v_{j+1})$ for $j \in [0, r]$ and $\varphi_1(v_r) = \varphi(v')$ for all $v' \in V(D) \setminus V(C)$. By repeating this procedure we obtain a coloring $\varphi_2$ of $D - v$ satisfying $\varphi_2(v_j) = \varphi_2(v_{j+2})$ for $j \in [0, r]$ and $\varphi_2(v') = \varphi_2(v')$ for all $v' \in V(D) \setminus V(C)$. Thus, after $i + 1$ steps, we get a coloring $\varphi_{i+1}$ of $D - v$ with $\varphi_{i+1}(v_j) = \varphi(v_{j+i+1})$ for $j \in [0, r]$ and $\varphi_{i+1}(v') = \varphi(v')$ for all $v' \in V(D) \setminus V(C)$. In particular,

$$\varphi_{i+1}(w) = \varphi_{i+1}(v_r) = \varphi(v_{r+i+1}) = \varphi(v_i) = \varphi(w')$$

Figure 3: Clockwise shifting of colors around a cycle in $G(D)$. 
and
\[ \varphi_{i+1}(u) = \varphi_{i+1}(v_0) = \varphi(v_{i+1}) = \varphi(u') \]

and so \( \varphi' = \varphi_{i+1} \) is the desired coloring for statement (a). Moreover, \( \varphi_{i+1}(w) = \varphi(v_i) \) and \( \varphi_i(u) = \varphi(v_i) \), which proves (b).

Using Claims 5 and 6, we are able to prove the next claim.

**Claim 7** Let \( C \) be a cycle in \( G(D) \) and let \( v \in V(C) \) be a vertex of \( C \) that is not contained in a chord of \( C \) in \( G(D) \). Then, \( C \) is an odd cycle and there is an \( f \)-coloring \( \varphi^* \) of \( D - v \) and indices \( k \neq \ell \) from \([1, p]\) such that the vertices of \( C - v \) are colored alternately with \( k \) and \( \ell \).

**Proof:** Let \( u \) and \( w \) be the vertices with \( \{uw, vw\} \subseteq E(C) \). As \( v \) is not contained in a chord, \( u \) and \( w \) are the only neighbors of \( v \) from \( V(C) \) in \( G(D) \).

We first claim that there is an \( f \)-coloring \( \varphi^* \) of \( D - v \) with \( \varphi^*(u) \neq \varphi^*(w) \). To this end, let \( \varphi \) be an \( f \)-coloring of \( D - v \) and assume that \( \varphi(u) = \varphi(w) \). From Claim 5(a) we know that at least two color classes of \( \varphi \) are non-empty. Thus, in \( D - v \) there is a vertex \( z \) with \( \varphi(z) \neq \varphi(u) \). As \( C \) is a cycle contained in the block \( G(D) \), we have \( |G(D)| \geq 3 \) and so \( G(D) \) is 2-connected. Then, it follows from Lemma 4 that in \( G(D) \) there are two paths \( P, P' \) from \( z \) into the set \( \{u, v, w\} \) whose only common vertex is \( z \) and whose internal vertices are not in \( \{u, v, w\} \). Note that the union \( P \cup P' \) together with either one or both of the edges \( \{uv, vw\} \) forms a cycle \( C' \) in \( G(D) \), which contains \( v \). Since \( \varphi(z) \neq \varphi(u) = \varphi(w) \), \( C' \) contains two consecutive vertices of different colors, say \( v_1 \) and \( v_2 \). If \( C' \) contains both \( u \) and \( w \), it follows from Claim 6(a) that there is a coloring \( \varphi^* \) of \( D - v \) with

\[ \{\varphi^*(u), \varphi^*(w)\} = \{\varphi(v_1), \varphi(v_2)\}, \]

and we are done. In order to complete the first part of the claim, it remains to consider the case that \( C' \) contains only one of \( u, v \), say \( u \). By symmetry, we may assume \( \varphi(v_1) \neq \varphi(w) \).

Then, it follows from Claim 6(b) that there is a coloring \( \varphi^* \) of \( D - v \) with

\[ \varphi^*(u) = \varphi(v_1) \neq \varphi(w) = \varphi^*(w) \]

and so we are done.

Now let \( \varphi^* \) be an \( f \)-coloring of \( D - v \) with \( \varphi^*(u) = k \) and \( \varphi^*(w) = \ell \), where \( k \neq \ell \). We claim that the vertices of \( C - v \) are colored alternately with \( k \) and \( \ell \). Otherwise, there is an edge \( u_1u_2 \in E(C) \) with \( v \not\in \{u_1, u_2\} \) such that

\[ \{\varphi^*(u_1), \varphi^*(u_2)\} \neq \{k, \ell\}. \]

By symmetry, we may assume that \( \ell \not\in \{\varphi^*(u_1), \varphi^*(u_2)\} \). Then, by Claim 6(a), there is a coloring \( \varphi' \) of \( D - v \) such that \( \{\varphi'(u), \varphi'(w)\} = \{\varphi^*(u_1), \varphi^*(u_2)\} \) and \( \varphi'(v') = \varphi^*(v') \) for all \( v' \in V(D) \setminus V(C) \). Again by symmetry, assume that \( \varphi'(u) = \varphi^*(u_1) \) and \( \varphi'(w) = \varphi^*(u_2) \).

As \( \ell \not\in \{\varphi'(u), \varphi'(w)\} \), but \( \ell \in \{\varphi^*(u), \varphi^*(w)\} \), and as \( u \) and \( w \) are the only neighbors of \( v \) from \( V(C) \) in \( G(D) \), we obtain that either

\[ d_{D[[\varphi^*]^{-1}(\ell)]+u}^+(v) \neq d_{D[[\varphi^*]^{-1}(\ell)]+v}^+(v) = f_\ell(v) \]

or

\[ d_{D[[\varphi^*]^{-1}(\ell)]+v}^-(v) \neq d_{D[[\varphi^*]^{-1}(\ell)]+u}^-(v) = f_\ell(v), \]

14
(where the equalities follow from Claim 5(b)), and so \( \varphi' \) is in contradiction to Claim 5(b). This proves the claim that the vertices of \( C - v \) are colored alternately with \( k \) and \( \ell \). As \( \varphi'(u) \neq \varphi^*(w) \), this implies that \( C \) is an odd cycle and so the proof of the claim is complete. \( \square \)

As a consequence of the above claim, we obtain that every even cycle in \( G(D) \) has at least two chords. Thus, it follows from Lemma 5 that every block of \( G(D) \) is an odd cycle or a complete graph. As \( D \) and therefore \( G(D) \) itself is a block, we conclude that \( G(D) \) is either a cycle of odd length or a complete graph. To complete the proof of Theorem 3, we show that both cases are impossible.

**Claim 8** \( G(D) \) is not an odd cycle.

**Proof**: For otherwise, let \( v \in V(D) \) be an arbitrary vertex. As \( C = G(D) \) is an odd cycle, \( C \) obviously has no chords and so it follows from Claim 7 that there is a coloring \( \varphi \) of \( D - v \) and indices \( k \neq \ell \) from \([1, p] \) so that the vertices of \( C - v \) are colored alternately with \( k \) and \( \ell \). Then, it follows from Claim 5(b) that

\[
\begin{align*}
  f_k(v) &= f_l(v) = 1 \quad \text{and} \quad f_i(v) = 0 \quad \text{for } i \in [1, p] \setminus \{k, \ell\} \\
  \text{and } \{uv, vu, vw, wv\} &\subseteq A(D) \quad \text{where } u \text{ and } w \text{ are the neighbors of } v \text{ in } C.
\end{align*}
\]

(3.9)

By shifting the color from the neighbor \( u \) of \( v \) to \( v \) we obtain (3.9) for \( u \) instead of \( v \). Repeating this argument proves that \((D, f)\) is a hard pair of type \((C)\), a contradiction. \( \square \)

It remains to consider the case that \( G(D) \) is a complete graph. Claim 5 implies that \( |D| \geq 3 \).

First we claim that \( D \) is bidirected. For otherwise, there are two vertices \( u, v \in V(D) \) with \( uv \in A(D) \) but \( vu \notin A(D) \). Let \( \varphi \) be an \( f \)-coloring of \( D - v \). Then, by Claim 5(a), there is a vertex \( w \in V(D) \setminus \{v\} \) with \( \varphi(w) \neq \varphi(u) \). As \( G(D[\{u, v, w\}]|) \) is a triangle, it follows from Claim 6(b) that swapping the colors of \( D - w \) in \( \varphi \) results in another \( f \)-coloring \( \varphi' \) of \( D - v \). Then, we obtain that \( uv \in A(D) \) but \( vu \notin A(D) \), since otherwise \( \varphi' \) would not satisfy the degree condition in Claim 5(b). By shifting the color from \( w \) to \( v \) we conclude from Claim 5(b) that \( uv \in A(D) \) but \( vu \notin A(D) \). Similarly, by instead shifting the color from \( u \) to \( w \) we conclude from Claim 5(b) that \( wu \in A(D) \), but \( uw \notin A(D) \), which clearly is impossible. This proves the claim that \( D \) is bidirected and so \( D \) is a bidirected complete graph.

Finally, we prove that \((D, f)\) is a hard pair of type \((K)\), leading to the desired contradiction. To this end, let \( v \in V(D) \) be an arbitrary vertex, let \((D_1, D_2, \ldots, D_p)\) be an \( f \)-partition of \( D - v \), and let \( n_i = |D_i| \). As \( D \) is a bidirected complete graph, it then follows from Claim 2(b) that \( f_i(v) = n_i \) for all \( i \in [1, p] \). Moreover, if \( u \in V(D) \setminus \{v\} \), say \( u \in V(D_1) \) (by symmetry), then the sequence \((D_1', D_2', \ldots, D_p')\) with \( D_1' = (D_1 + v) - u \) and \( D_j' = D_j \) for \( j \in [2, p] \) is an \( f \)-partition of \( D - u \) (by Claim 2(c)) with \( |D_i'| = |D_i| \) for all \( i \in [1, p] \) and applying Claim 2(b) to \( u \) leads to \( f_i(u) = n_i \) for all \( i \in [1, p] \). Note that

\[
n_1 + n_2 + \ldots + n_p = |D_1| + |D_2| + \ldots + |D_p| = |V(D)| - 1
\]

and so \((D, f)\) is a hard pair of type \((K)\), contradicting \((C3)\). This contradiction completes the proof of Theorem 3.
Since a bidirected graph $D$ is weakly $h$-degenerate if and only if its underlying graph $G(D)$ is strictly $h$-degenerate (i.e., in each non-empty subgraph $G'$ of $G$ there is a vertex $v$ with $d_{G'}(v) < h(v)$), the restriction of Theorem 3 to bidirected graphs gives us also a result regarding strict degeneracy of graphs. This result for graphs was originally proven by Borodin, Kostochka, and Toft [9]. The proof structure of Claim 3 is similar to that of the first part of their proof. However, apart from this, our proof leads to another proof for the undirected case that uses completely different methods than the original proof. A major benefit of our proof is that—contrary to the original proof in [9]—it generalizes easily to the case of multidigraphs. Of course, the definition of weak degeneracy and of $f$-partitions also works if we allow multiple arcs between vertices going in the same direction. Also, the definition of hard pair needs only to be generalized slightly. To this end, we need the following term. Let $G$ be a graph and let $t \geq 1$ be an integer. Then we denote by $tG$ the multigraph that results from $G$ by replacing each edge by $t$ parallel edges. Now let $D$ be a connected multidigraph, and let $f : V(D) \rightarrow \mathbb{N}^p_0$ be a vector function. We say that $(D, f)$ is a hard pair, if either

- $(D, f)$ is of type (M), or
- $D$ is a bidirected $tK_n$ and there are integers $n_1, n_2, \ldots, n_p \geq 0$ with at least two $n_i$ different from zero such that $n_1 + n_2 + \ldots + n_p = n - 1$ and $f(v) = (tn_1, tn_2, \ldots, tn_p)$ for all $v \in V(D)$, or
- $D$ is a bidirected $tC_n$ with $n$ odd and there are two indices $k \neq \ell$ from the set $[1, p]$ such that, for all $v \in V(D)$, we have $f_i(v) = t$ if $i \in \{k, \ell\}$ and $f_i(v) = 0$ otherwise, or
- $(D, f)$ is obtained from two hard pairs via the merging operation.

For multidigraphs, we obtain the following theorem.

**Theorem 9** Let $D$ be a connected multidigraph, let $p \geq 1$, and let $f : V(D) \rightarrow \mathbb{N}^p_0$ be a vector function such that $f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$. Then $D$ is not $f$-partitionable if and only if $(D, f)$ is a hard pair. \hfill \diamond

By inspecting their proofs, it is easy to check that Claims 1-7 also hold for multidigraphs. Only the remaining part of the proof needs to be changed slightly, but still can easily be done adapting the methods described there. Therefore, we abstain from giving an extra proof.

It is possible to deduce a polynomial time algorithm from our proof that, given a pair $(D, f)$ with $f_1(v) + f_2(v) + \ldots + f_p(v) \geq \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$, either verifies that $(D, f)$ is a hard pair or returns an $f$-partition of $D$. The exact procedure can be found in the extended version [7] of our paper.

### 4 Applications of Theorem 3

#### 4.1 Brooks’ Theorem for list-colorings of digraphs

As mentioned in the introduction, Theorem 3 implies Theorem 2 due to Harutyunyan and Mohar [20]. Let us recall the theorem for the reader’s convenience.
**Theorem 2 (Harutyunyan and Mohar, 2011)** Let \( D \) be a connected digraph, and let \( L \) be a list-assignment such that \(|L(v)| \geq \max\{d^+_D(v), d^-_D(v)\}\) for all \( v \in V(D) \). Suppose that \( D \) is not \( L \)-colorable. Then the following statements hold:

(a) \( D \) is Eulerian and \(|L(v)| = \max\{d^+_D(v), d^-_D(v)\}\) for all \( v \in V(D) \).

(b) If \( B \in \mathcal{B}(D) \), then \( B \) is a directed cycle of length \( \geq 2 \), or \( B \) is a bidirected complete graph, or \( B \) is a bidirected cycle of odd length \( \geq 5 \).

(c) For each \( B \in \mathcal{B}(D) \) there is a set \( \Gamma_B \) of \( \Delta^+(B) \) colors such that, for every \( v \in V(D) \), the sets \( \Gamma_B \) with \( B \in \mathcal{B}_v(D) \) are pairwise disjoint and

\[
L(v) = \bigcup_{B \in \mathcal{B}_v(D)} \Gamma_B.
\]

**Proof of Theorem 2:** Let \( D \) and \( L \) be as described in the theorem. By using the method from the introduction, we transform the list-coloring problem to that of finding an \( f \)-partition: Let \( \Gamma = \bigcup_{v \in V(D)} L(v) \). By renaming the colors if necessary, we may assume \( \Gamma = [1, p] \). Now let \( f : V(D) \to \mathbb{N}_0^p \) be the vector function with

\[
f_i(v) = \begin{cases} 
1 & \text{if } i \in L(v), \\
0 & \text{if } i \notin L(v)
\end{cases}
\]

for \( i \in [1, p] \). By the definition of \( f \), we have

\[
f_1(v) + f_2(v) + \ldots + f_p(v) = |L(v)| \geq \max\{d^+_D(v), d^-_D(v)\}
\]

for all \( v \in V(D) \) and \( D \) is not \( f \)-partitionable. Thus, it follows from Theorem 3 that \((D, f)\) is a hard pair. Then, statement (a) follows from Proposition 7(a). Moreover, Proposition 6(a) implies that for each block \( B \in \mathcal{B}, (B, f_B) \) is of type (M), (K), or (C), where \( f_B \) is the function as defined in Proposition 6. Note that if \((B, f_B)\) is of type (M), then from the definition of \( f \) it follows that the only non-zero coordinate of \( f_B \) is constant 1 and, hence, \( d^+_B(v) = d^-_B(v) = 1 \) for all \( v \in V(B) \). Consequently, \( B \) is a directed cycle. Thus, (b) holds true. Statement (c) follows easily from Proposition 6(a) and (b).

\[\qed\]

### 4.2 The \( s \)-degenerate dichromatic number

In [17], Golowich introduces a generalization of the dichromatic number as follows. Let \( s \geq 1 \) be an integer. Then, the **\( s \)-degenerate dichromatic number** of a digraph \( D \), denoted by \( \chi_s(D) \), is the least integer \( p \) such that \( D \) admits a \( p \)-partition into weakly \( s \)-degenerate subdigraphs. Clearly, \( \chi_1 \) corresponds to the dichromatic number. Note that the \( s \)-degenerate dichromatic number is the digraph counterpart to the **point partition number** of a graph, which was introduced by Lick and White [26] and which is also known as the **\( s \)-chromatic number**. The point partition number \( \chi_s \) of a graph \( G \) is usually defined as the minimum number \( p \) such that \( G \) admits a \( p \)-partition into strictly \( s \)-degenerate subgraphs. In particular, \( \chi_1 \) is the chromatic number. By setting \( f_i \equiv s \) in Theorem 3, we easily obtain the following theorem, which was proven for graphs by Mitchem [27]. If the reader appreciates a bit of assistance by obtaining a short yet elegant proof, he or she might follow the argumentation in the proof of Theorem 11 (which is the generalization of Theorem 10 to list-colorings regarding the \( s \)-degenerate dichromatic number).
Theorem 10 Let $D$ be a connected digraph and let $m = \max_{v \in V(D)} \{d^+_D(v), d^-_D(v)\}$. Moreover, let $s \geq 1$ be an integer and let $p = \lceil \frac{m}{s} \rceil$. If $D$ is neither a bidirected odd cycle, a bidirected complete graph, nor an Eulerian digraph in which every vertex has in- and out-degree $s$, then $\chi_s(D) \leq p$. 

To conclude the paper, let us demonstrate how to obtain a list-version of Theorem 10. Given a digraph $D$ and a list-assignment $L$ of $D$, we say that $D$ is $(L, s)$-colorable if $D$ admits an $L$-coloring in which every color class induces a weakly $s$-degenerate subdigraph.

Theorem 11 Let $D$ be a connected digraph and let $m = \max_{v \in V(D)} \{d^+_D(v), d^-_D(v)\}$. Moreover, let $s \geq 1$ be an integer and let $L$ be a list-assignment with $|L(v)| \geq \frac{m}{s}$ for all $v \in V(H)$. Then, $D$ is not $(L, s)$-colorable if and only if the following two conditions are fulfilled:

(a) $D$ is a bidirected complete graph with $|D| \equiv 1(\text{mod } s)$, or $D$ is a bidirected odd cycle and $s = 1$, or $D$ is an Eulerian digraph in which every vertex has in-degree and out-degree $s$.

(b) There is a color set $\Gamma$ such that $L(v) = \Gamma$ for all $v \in V(D)$ and $|\Gamma| = m/s$. 

Proof: If $m = 0$, then $D$ consists of just one vertex and the statement is evident. So assume $m \geq 1$. Let $\Gamma = \bigcup_{v \in V(D)} L(v)$. By renaming the colors if necessary, we can assume that $\Gamma = [1, p]$. Now we define a vector function $f : V(D) \to \mathbb{N}_0^p$ as follows. Let 

$$f_i(v) = \begin{cases} 
  s & \text{if } i \in L(v), \\
  0 & \text{otherwise}
\end{cases}$$

for $i \in [1, p]$. Then, $D$ is not $(L, s)$-colorable if and only if $D$ is not $f$-partitionable. By definition, we have $f_1(v) + f_2(v) + \ldots + f_p(v) = s|L(v)| \geq m$ for all $v \in V(D)$ and so (by Theorem 3) $D$ is not $(L, s)$-colorable if and only if $(D, f)$ is a hard pair.

First assume that $D$ and $L$ satisfy (a) and (b). Then it is easy to check that $(D, f)$ is indeed a hard pair and so $D$ is not $(L, s)$-colorable. Here, the only non-obvious case is if $D$ is an Eulerian digraph in which every vertex has in-degree and out-degree $s$. Then, by (b), there is exactly one index $i$, such that for all $v \in V(D)$ we have $f_i(v) = d^+_D(v) = d^-_D(v)$ and $f_j(v) = 0$ for $j \in [1, p] \setminus \{i\}$. By symmetry, we may assume $i = 1$ and so $f(v) = (d^+_D(v), 0, 0, \ldots, 0)$ for all $v \in V(D)$. Now we deduce that $(D, f)$ is a hard pair by induction on the number of blocks of $D$: If $D$ itself is a block, then $(D, f)$ is a hard pair of type $(M)$, and we are done. If $D$ has at least two blocks, let $B$ be an end-block of $D$ and let $v$ be the only separating vertex of $D$ in $B$. As $D$ is Eulerian, so is $B$. For $w \in V(B)$, let $f_B(w) = f(w)$ if $w \neq v$ and $f_B(w) = (d^+_B(v), 0, 0, \ldots, 0)$ if $w = v$. Then $(B, f_B)$ is a hard pair of type $M$. Let $D' = D - (V(B) \setminus \{v\})$ and let $f'$ be the mapping with $f'(w) = f(w)$ for $w \in V(D') \setminus \{v\}$ and $f'(v) = (d^+_D(v) - d^-_D(v), 0, 0, \ldots, 0)$. Then, $D'$ is still Eulerian and $f'(w) = (d^+_D(w), 0, 0, \ldots, 0)$ for all $w \in V(D')$. Now, it follows from the induction hypothesis that $(D', f')$ is a hard pair. Consequently, $(D, f)$ is obtained from the two hard pairs $(D', f')$ and $(B, f_B)$ via the merging operation and, thus, a hard pair, as claimed.

Now assume that $D$ is not $(L, s)$-colorable. Then $(D, f)$ is a hard pair and from Proposition 7(a) it follows that

$$m \leq s|L(v)| = d^+_D(v) = d^-_D(v)$$
for all \( v \in V(D) \) and so \( D \) is Eulerian and each vertex \( v \) satisfies \( d_D^+(v) = d_D^-(v) = m \). If \( m = s \), then \( |L(v)| = 1 \) for all \( v \in V(D) \) and \( D \) is not \((L,s)\)-colorable if and only if there is a color \( \alpha \in \Gamma \) with \( L(v) = \{\alpha\} \) for all \( v \in V(D) \), and we are done. So assume \( m > s \). Then, \( |L(v)| \geq 2 \) and so for each vertex \( v \) there are two indices \( i \neq j \) such that \( f_i(v) \geq 1 \) and \( f_j(v) \geq 1 \). Consequently, by Proposition 6, no end-block \( B \) of \( D \) together with its uniquely determined function \( f_B \) from Proposition 6 is of type (M). Since every vertex of \( D \) has in-degree and out-degree \( m \), this implies that \( D \) itself is a block and so \( D \) is either a bidirected complete graph or a bidirected odd cycle. In the first case, \((D,f)\) is of type (K) and, since every coordinate of \( f \) is either \( s \) or zero we easily conclude from the definition of hard pair of type (K) that \( |D| - 1 \equiv 0 \pmod{s} \). If \((D,f)\) is of type (C), we conclude that \( s = 1 \). Moreover, in both cases the function \( f \) is constant and so the list-assignment \( L \) is constant, too, and \( |L(v)| = m/s \) for all \( v \in V(D) \). This completes the proof. 

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