EMBEDDING THEOREMS FOR THE DUNKL HARMONIC OSCILLATOR ON THE LINE

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ABSTRACT. Embedding results of Sobolev type are proved for the Dunkl harmonic oscillator on the line.

1. INTRODUCTION

The subindex ev/odd is added to any space of functions on \( \mathbb{R} \) to indicate its subspace of even/odd functions; in particular, \( C^\infty = C^\infty_{\text{ev}} \oplus C^\infty_{\text{odd}} \) for \( C^\infty := C^\infty(\mathbb{R}) \).

The Dunkl operator \( T_\sigma \) (\( \sigma > -1/2 \)) on \( C^\infty \) is the perturbation of \( \frac{d}{dx} \) defined by \( T_\sigma = \frac{d}{dx} \) on \( C^\infty_{\text{ev}} \) and \( T_\sigma = \frac{d}{dx} + 2\sigma\frac{x}{2} \) on \( C^\infty_{\text{odd}} \). The corresponding Dunkl harmonic oscillator is the perturbation \( L_\sigma = -T_\sigma^2 + s^2x^2 \) of the harmonic oscillator \( H = -\frac{d^2}{dx^2} + s^2x^2 \) (\( s > 0 \)). The conjugation \( E_\sigma = |x|^{\sigma}T_\sigma|x|^{-\sigma} \) on \( |x|^\sigma C^\infty \) is equal to \( \frac{d}{dx} - \sigma x^{-1} \) on \( |x|^\sigma C^\infty_{\text{ev}} \) and \( \frac{d}{dx} + \sigma x^{-1} \) on \( |x|^\sigma C^\infty_{\text{odd}} \); note that \( |x|^\sigma C^\infty_{\text{ev/odd}} \) consists of even/odd functions, possibly not smooth or not even defined at 0. Up to the product by a constant, \( E_\sigma \) was introduced by Yang [39]. In the form \( T_\sigma \), this operator was generalized to \( \mathbb{R}^n \) by Dunkl [12, 13, 14], giving rise to what is now called Dunkl theory (see the survey [31]); in particular, the Dunkl harmonic oscillator on \( \mathbb{R}^n \) was studied in [29, 15, 26, 25]. See [27] for further generalizations on \( \mathbb{R} \). Sometimes the terms Yang-Dunkl operator and Yang-Dunkl harmonic oscillator are used in the case of \( \mathbb{R}^2 \).

Let \( p_k \) be the sequence of orthogonal polynomials for the measure \( e^{-sx^2}|x|^{2\sigma} \, dx \), taken with norm one and positive leading coefficient. Up to normalization, these are the generalized Hermite polynomials [32, p. 380, Problem 25]; see also [9, 11, 14, 10, 29, 30]. The corresponding generalized Hermite functions are \( \phi_k := p_k e^{-sx^2}/2 \).

For each \( m \in \mathbb{N} \), let \( S^m \) be the Banach space of functions \( \phi \in C^m(\mathbb{R}) \) with \( \sup_x |x^i\phi^{(j)}(x)| < \infty \) for \( i + j \leq m \); the corresponding Fréchet space \( S = \bigcap_m S^m \) is the Schwartz space on \( \mathbb{R} \). With domain \( S \), \( L_\sigma \) is essentially self-adjoint in \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \), and the spectrum of its self-adjoint extension, \( L_\sigma \), consists of the eigenvalues \( (2k + 1 + 2\sigma)\lambda \) (\( k \in \mathbb{N} \)), with corresponding eigenfunctions \( \phi_k \) [29]. For each real \( m \geq 0 \), let \( W^m_{\sigma} \) be the Hilbert space completion of \( S \) with respect to the scalar product \( \langle \phi, \psi \rangle_{W^m_{\sigma}} := \langle (1 + L_\sigma)^m \phi, \psi \rangle_\sigma \), where \( \langle \ , \ \rangle_\sigma \) denotes the scalar product of \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \), obtaining a Fréchet space \( W^\infty_{\sigma} = \bigcap_m W^m_{\sigma} \). We show the following embedding theorems; the second one is of Sobolev type.

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Theorem 1.1. For each \( m \in \mathbb{N} \), \( S_{m, \text{ev/odd}}^{m, \text{ev/odd}} \subset W_{\sigma, \text{ev/odd}}^{m} \) continuously, where

\[
M_{m, \text{ev/odd}} = \begin{cases} \frac{3m+3}{2} + \frac{m+1}{4} [\sigma] [\sigma] + 3 + [\sigma] & \text{if } \sigma \geq 0 \text{ and } m \text{ is odd} \\ 2m + 3 & \text{if } \sigma < 0 \text{ and } m \text{ is odd} \\ \end{cases}
\]

\[
M_{m, \text{ev}} = \begin{cases} \frac{3m+2}{2} + \frac{m}{4} [\sigma] [\sigma] + 3 + [\sigma] & \text{if } \sigma \geq 0 \text{ and } m \text{ is even} \\ 2m + 2 & \text{if } \sigma < 0 \text{ and } m \text{ is even} \\ \end{cases}
\]

\[
M_{m, \text{odd}} = \begin{cases} \frac{3m+4}{2} + \frac{m+2}{4} [\sigma] [\sigma] + 3 + [\sigma] & \text{if } \sigma \geq 0 \text{ and } m \text{ is even} \\ 2m + 4 & \text{if } \sigma < 0 \text{ and } m \text{ is even} \\ \end{cases}
\]

Theorem 1.2. For all \( m \in \mathbb{N} \) and \( m_\sigma = m + 1 + \frac{1}{2} [\sigma] [\sigma] + 1 \), \( W_{\sigma, \text{ev/odd}}^{m', \text{ev/odd}} \subset S_{\sigma, \text{ev/odd}}^{m} \) continuously if \( m' > N_{\text{ev/odd}}^{m} \), where \( N_{\text{ev}} = 2 \text{ and } N_{\text{odd}} = 5 \) if \( m_\sigma = 1 \), \( N_{\text{ev}} = 6 \) and \( N_{\text{odd}} = 5 \) if \( m_\sigma = 2 \), \( N_{\text{ev}} = 6 \) and \( N_{\text{odd}} = 7 \) if \( m_\sigma = 3 \), and \( N_{\text{ev/odd}} = m_\sigma + 3 \) for \( m_\sigma \geq 4 \).

Corollary 1.3. \( S = W_{\sigma}^{\infty} \) as Fréchet spaces.

In other words, Corollary 1.3 states that an element \( \phi \in L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \) is in \( S \) if and only if the “Fourier coefficients” \( \langle \phi, \phi_k \rangle_\sigma \) are rapidly decreasing on \( k \). This also means that \( S = \bigcap_m \mathcal{P}(\mathcal{L}_\sigma^m) \) (\( S \) is the smooth core of \( \mathcal{L}_\sigma \) with the terminology of \( [B] \)) because the sequence of eigenvalues of \( \mathcal{L}_\sigma \) is in \( O(k) \) as \( k \to \infty \).

We introduce a version \( S_\sigma^m \) of every \( S^m \), whose definition involves \( T_\sigma \) instead of \( \frac{d}{dx} \). They satisfy much simpler embeddings: \( S_\sigma^m \subset W_{\sigma}^m \), and \( W_{\sigma}^m \subset S_\sigma^m \) if \( m' - m > 1 \). Even though \( S = \bigcap_m S_\sigma^m \), the inclusion relations between the spaces \( S_\sigma^m \) and \( S_\sigma^{m'} \) are complicated, giving rise to the complexity of Theorems 1.1 and 1.2.

Other Sobolev type embedding theorems, for different operators and with different techniques, were recently proved in \([33, 36, 37]\).

Next, we consider other perturbations of \( H \) on \( \mathbb{R}^+ \). Let \( S_{\text{ev/odd}} \) denote the space of restrictions of even Schwartz functions to some open set \( U \), and set \( \phi_{k,U} = \phi_k |_U \).

Theorem 1.4. Let \( P = H - 2f_1 \frac{d^2}{dx^2} + f_2 \), where \( f_1 \in C^1(U) \) and \( f_2 \in C(U) \) for some open \( U \subset \mathbb{R}^+ \) of full Lebesgue measure. Assume that \( f_2 = \sigma(x-1)x^{-2} - f_1' \) for some \( \sigma > -1/2 \). Let \( h = x^r e^{F_1} \), where \( F_1 \in C^2(U) \) is a primitive of \( f_1 \). Then the following properties hold:

(i) \( P \), with domain \( h S_{\text{ev/odd}} \), is essentially self-adjoint in \( L^2(\mathbb{R}^+, e^{2F_1} \, dx) \);

(ii) the spectrum of its self-adjoint extension, \( \mathcal{P} \), consists of the eigenvalues \( \sqrt{2} h \phi_{k,U} \); and

(iii) the smooth core of \( \mathcal{P} \) is \( h S_{\text{ev/odd}} \).

This theorem follows by showing that the stated condition on \( f_1 \) and \( f_2 \) characterizes the cases where \( P \) can be obtained by the following process: first, restricting \( L_\sigma \) to even functions, then restricting to \( U \), and finally conjugating by \( h \). The term of \( P \) with \( \frac{d}{dx} \) can be removed by conjugation with the product of a positive function, obtaining the operator \( H + \sigma(x-1)x^{-2} \); in this way, we get all operators of the form \( H + cx^{-2} \) with \( c > -1/4 \).

The conditions of Theorem 1.4 are satisfied by \( P = H - 2c_1 x^{-1} \frac{d}{dx} + c_2 x^{-2} \) \( (c_1, c_2 \in \mathbb{R}) \) on \( \mathbb{R}^+ \) if and only if there is some \( a \in \mathbb{R} \) such that \( a^2 + (2c_1 - 1)a - c_2 = 0 \) and \( a + c_1 > -1/2 \); in this case, \( h = x^a \) and \( e^{2F_1} = x^{2c_1} \). For some \( c_1, c_2 \in \mathbb{R} \), there
are two values of $a$ satisfying these conditions, obtaining two different self-adjoint operators defined by $P$ in different Hilbert spaces. For instance, $L_\sigma$ may define a self-adjoint operator when $\sigma < -1/2$.

This example is applied in [2] to prove a new type of Morse inequalities on strata of compact stratifications [33, 20, 34] with adapted metrics [23, 24, 5], where Witten’s perturbation [38] is used for the minimal/maximal ideal boundary conditions of de Rham complex [7, 8, 6]. The version of Morse functions used in [2] is different from the version of Goresky-MacPherson [18]. More precisely, the operator $P$ describes the radial direction of Witten’s perturbed Laplacian in the local conic model of a stratification around each critical point. The two possible choices of $a$ give rise to the minimal/maximal ideal boundary conditions.

2. Preliminaries

2.1. Dunkl operator on the line. For any $\phi \in C^\infty := C^\infty(\mathbb{R})$, there exists some $\psi \in C^\infty$ so that $\phi(x) - \phi(0) = x\psi(x)$; moreover

$$\psi^{(m)}(x) = \int_0^1 t^m \phi^{(m+1)}(tx) \, dt$$

(1)

for all $m \in \mathbb{N}$ (see e.g. [19, Theorem 1.1.9]). Let us use the notation $\psi = x^{-1}\phi$. The Dunkl operator on $T_\sigma$ ($\sigma \in \mathbb{R}$) on $C^\infty$ is the perturbation of $d/dx$ defined by

$$(T_\sigma \phi)(x) = \phi'(x) + 2\sigma \frac{\phi(x) - \phi(-x)}{x}.$$ 

Consider matrix expressions of operators on $C^\infty$ with respect to the decomposition $C^\infty = C^\infty \oplus C^\infty_{\text{odd}}$, as direct sum of subspaces of even and odd functions. For each function $h$, the notation $h$ is also used for the operator of multiplication by $h$. Then

$$\frac{d}{dx} = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$$

and

$$T_\sigma = \begin{pmatrix} 0 & \frac{d}{dx} + 2\sigma x^{-1} \\ \frac{d}{dx} + 2\sigma x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} d/dx + 2\sigma & 0 \\ 0 & 0 \end{pmatrix}$$

on $C^\infty$. With $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$, we get

$$[T_\sigma, x] = 1 + 2\Sigma, \quad T_\sigma \Sigma + \Sigma T_\sigma = x \Sigma + \Sigma x = 0.$$ 

(2)

The perturbed factorial $m!_\sigma$ ($m \in \mathbb{N}$) is inductively defined by $0!_\sigma = 1$, and

$$m!_\sigma = \begin{cases} (m - 1)!_\sigma m & \text{if } m \text{ is even} \\ (m - 1)!_\sigma (m + 2\sigma) & \text{if } m \text{ is odd} \end{cases} \quad (m \geq 1).$$

Notice that $m!_\sigma > 0$ if $\sigma > -1/2$. For $k \leq m$, even when $k!_\sigma = 0$, the quotient $m!_\sigma/k!_\sigma$ can be understood as the product of the factors used in the definition of $m!_\sigma$ and not used in the definition of $k!_\sigma$. For $\phi \in C^\infty$ and $m \in \mathbb{N}$, by (1) and induction on $m$, we get

$$(T_\sigma^m \phi)(0) = \frac{m!_\sigma}{m!} \phi^{(m)}(0).$$

(3)

\footnote{We adopt the convention $0 \in \mathbb{N}$.}
2.2. Dunkl harmonic oscillator on the line. On $C^\infty$, the harmonic oscillator, and the annihilation and creation operators are $H = -\frac{d^2}{dx^2} + s^2x^2$, $A = sx + \frac{d}{dx}$ and $A' = sx - \frac{d}{dx}$ ($s > 0$). Their perturbations $L = -T_0^2 + s^2x^2$, $B = sx + T_0$ and $B' = sx - T_0$ are called Dunkl harmonic oscillator, and Dunkl annihilation and creation operators. By (2),

$$L = BB' - (1 + 2\Sigma)s = B'B + (1 + 2\Sigma)s = \frac{1}{2}(BB' + B'B') ,$$

(4)

$$[L, B] = -2sB , \quad [L, B'] = 2sB' ,$$

(5)

$$[B, B'] = 2s(1 + 2\Sigma) ,$$

(6)

$$[L, \Sigma] = B\Sigma + \Sigma B = B'\Sigma + \Sigma B' = 0 .$$

(7)

For each $m \in \mathbb{N}$, let $S^m$ be the space of functions $\phi \in C^\infty$ such that

$$\|\phi\|_{S^m} = \sum_{i+j \leq m} \sup_{x} |x^i\phi^{(j)}(x)| < \infty .$$

This expression defines a norm $\|\cdot\|_{S^m}$ on $S^m$, which becomes a Banach space. We have $S^{m+1} \subset S^m$ continuously and $S = \bigcap_m S^m$, with the induced Fréchet topology, is the Schwartz space on $\mathbb{R}$. Note that $\|\phi\|_{S^m} \leq \|\phi\|_{S^{m+1}}$ for all $m$.

We can restrict the above decomposition of $C^\infty$ to every $S^m$ and $S$, obtaining $S^m = S^m_{ev} \oplus S^m_{odd}$ and $S = S_{ev} \oplus S_{odd}$. The matrix expressions of operators on $S$ are taken with respect to this decomposition. For $\phi \in C^\infty_{ev}$, $\psi = x^{-1}\phi$ and $i, j \in \mathbb{N}$, we get from (11) that

$$|x^i\psi^{(j)}(x)| \leq \sup_{y \in \mathbb{R}} |t^{j-1}t^{i}(t^{j+1})(t^{i})| dt \leq \sup_{y \in \mathbb{R}} |y^{i} \psi^{(j+1)}(y)|$$

for all $x \in \mathbb{R}$. So $\|\psi\|_{S^m} \leq \|\phi\|_{S^{m+1}}$ for all $m \in \mathbb{N}$, obtaining that $S_{odd} = xS_{ev}$ and $x^{-1} : C^\infty_{ev} \rightarrow C^\infty_{odd}$ restricts to a continuous operator $x^{-1} : S_{odd} \rightarrow S_{ev}$. Hence $x : S_{ev} \rightarrow S_{odd}$ is an isomorphism of Fréchet spaces, and $T_0$, $B$, $B'$ and $L$ define continuous operators on $S$.

Let $\langle , , \rangle_{\sigma}$ and $\|\cdot\|_{\sigma}$ be the scalar product and norm of $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$. Suppose from now on that $\sigma > -1/2$, obtaining that $S$ is dense in $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$. The following properties hold considering these operators in $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$ with domain $S$:

$-T_0$ is adjoint of $T_0$, $B'$ is adjoint of $B$, and $L$ is essentially self-adjoint. Let $L$, or $L_0$, denote the self-adjoint extension of $L$ (with domain $S$). Its spectrum consists of the eigenvalues $(2k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$). The corresponding normalized eigenfunctions $\phi_k$ are inductively defined by

$$\phi_0 = s^{(2\sigma+1)/4}\Gamma(\sigma + 1/2)^{-1/2}e^{-sx^2/2} ,$$

(8)

$$\phi_k = \begin{cases} (2ks)^{1/2}B'\phi_{k-1} & \text{if } k \text{ is even} \\
(2(k+2\sigma)s)^{1/2}B'\phi_{k-1} & \text{if } k \text{ is odd} \end{cases} \quad (k \geq 1) .$$

(9)

Furthermore

$$B\phi_0 = 0 ,$$

(10)

$$B\phi_k = \begin{cases} (2ks)^{1/2}\phi_{k-1} & \text{if } k \text{ is even} \\
(2(k+2\sigma)s)^{1/2}\phi_{k-1} & \text{if } k \text{ is odd} \end{cases} \quad (k \geq 1) .$$

(11)

\footnote{For topological vector spaces $X$ and $Y$, it is said that $X \subset Y$ continuously if $X$ is a linear subspace of $Y$ and the inclusion map $X \hookrightarrow Y$ is continuous.}
These properties follow from 1–4, like in the case of $H$.

2.3. Generalized Hermite polynomials. By 8, 9 and the definition of $B'$, we get $\phi_k = p_k e^{-sx^2/2}$, where $p_k$ is the sequence of polynomials inductively given by $p_0 = s^{2(\sigma+1)/4} \Gamma(\sigma + 1/2)^{-1/2}$ and

$$p_k = \begin{cases} (2ks)^{-1/2} (2sx_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is even} \quad (k \geq 1) \\ (2(k+2\sigma)s)^{-1/2} (2sx_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is odd} \end{cases}$$

(12)

Up to normalization, $p_k$ and $\phi_k$ are the generalized Hermite polynomials and functions 8, p. 380, Problem 25. Each $p_k$ is of degree $k$, even/odd if $k$ is even/odd, and with positive leading coefficient. Moreover $T_\sigma p_0 = 0$ and

$$T_\sigma p_k = \begin{cases} (2ks)^{1/2} p_{k-1} & \text{if } k \text{ is even} \quad (k \geq 1) \\ (2(k+2\sigma)s)^{1/2} p_{k-1} & \text{if } k \text{ is odd} \end{cases}$$

(13)

From 12 and 13, we obtain the recursion formula

$$p_k = \begin{cases} k^{-1/2} (2s)^{1/2} x_{k-1} - (k - 1 + 2\sigma)^{1/2} p_{k-2} & \text{if } k \text{ is even} \\ (k+2\sigma)^{-1/2} (2s)^{1/2} x_{k-1} - (k - 1)^{1/2} p_{k-2} & \text{if } k \text{ is odd} \end{cases}$$

(14)

By 14 and induction on $k$, we easily get the following when $k$ is odd:

$$x^{-1} p_k = \sum_{\ell \in \{0, 1, \ldots, k-1\}} (-1)^{k-\ell-1} \sqrt{\frac{(k-1)(k-3) \cdots (\ell+2)2s}{(k+2\sigma)(k-2+2\sigma) \cdots (\ell+1+2\sigma)}} p_\ell.$$  

(15)

The following theorem contains a simplified version of the asymptotic estimates satisfied by $\phi_k$ and $\xi_k = |x|^r \phi_k$. They can be obtained by expressing $p_n$ in terms of the Laguerre polynomials 30, 31, whose the asymptotic estimates are studied in 17, 3, 22, 21. The method of Bonan-Clark 4 can be also applied 11.

Theorem 2.1. There exist $C, C', C'' > 0$, depending on $\sigma$ and $s$, such that:

(i) if $k$ is odd or $\sigma \geq 0$, then $\xi_k^2(x) \leq C' k^{-1/6}$ for all $x \in \mathbb{R}$;

(ii) if $k$ is even and positive, and $\sigma < 0$, then $\xi_k^2(x) \leq C'' k^{-1/6}$ for $|x| \geq 1$; and

(iii) if $\sigma < 0$, then $\phi_k^2(x) \leq C''$ for all $k$ and $|x| \leq 1$.

3. Perturbed Schwartz space

We introduce a perturbed version $S^m_\sigma$ of each $S^m$ that will be appropriate to show our embedding results. Since $S^m_\sigma$ must contain the functions $\phi_k$, Theorem 2.1 indicates that different definitions must be given for $\sigma \geq 0$ and $\sigma < 0$.

When $\sigma \geq 0$, for any $\phi \in C^\infty$ and $m \in \mathbb{N}$, let

$$\|\phi\|_{S^m_\sigma} = \sum_{i+j \leq m} \sup_x |x|^\sigma |x^iT_j^s \phi(x)|.$$  

(16)

This defines a norm $\| \cdot \|_{S^m_\sigma}$ on the linear space of functions $\phi \in C^\infty$ with $\|\phi\|_{S^m_\sigma} < \infty$, and let $S^{m}_{\sigma}$ denote the corresponding Banach space completion. There are direct sum decompositions into subspaces of even and odd functions, $S^m_\sigma = S^m_{\sigma,\text{ev}} \oplus S^m_{\sigma,\text{odd}}$.

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3As a convention, the product of an empty set of factors is 1. Thus $(k-1)(k-3) \cdots (\ell+2) = 1$ for $\ell = k-1$ in 15.
When $\sigma < 0$, the even and odd functions are considered separately: let
\[
\|\phi\|_{S^m_\sigma} = \sum_{i+j \leq m, \; i, j \text{ even}} \left( \sup_{|x| \leq 1} |x^i(T^j_\sigma \phi)(x)| + \sup_{|x| \geq 1} |x^i| |x^j(T^j_\sigma \phi)(x)| \right) + \sum_{i+j \leq m, \; i, j \; \text{odd}} \sup_{x \neq 0} |x^i| |x^j(T^j_\sigma \phi)(x)|
\]
(17)
for $\phi \in C^\infty$, and
\[
\|\phi\|_{S^m_\sigma} = \sum_{i+j \leq m, \; i, j \text{ even}} \sup_{|x| \leq 1} |x^i| |x^j(T^j_\sigma \phi)(x)|
\]
\[
\quad + \sum_{i+j \leq m, \; i, j \; \text{odd}} \left( \sup_{|x| \leq 1} |x^i(T^j_\sigma \phi)(x)| + \sup_{|x| \geq 1} |x^i| |x^j(T^j_\sigma \phi)(x)| \right)
\]
(18)
for $\phi \in C^\infty_{\text{odd}}$. These expressions define a norm $\| \cdot \|_{S^m_\sigma}$ on the linear spaces of functions $\phi \in C^\infty_{\text{even/odd}}$ with $\|\phi\|_{S^m_\sigma} < \infty$. The corresponding Banach space completions will be denoted by $S^m_{\sigma,\text{even/odd}}$ and let $S^m_{\sigma} = S^m_{\sigma,\text{even}} \oplus S^m_{\sigma,\text{odd}}$.

In any case, there are continuous inclusions $S^m_{\sigma+1} \subset S^m_{\sigma}$, and a perturbed Schwartz space is defined as $S_{\sigma} = \bigcap_m S^m_{\sigma}$, with the corresponding Fréchet topology, which decomposes as direct sum of the subspaces of even and odd functions, $S_{\sigma} = S_{\sigma,\text{even}} \oplus S_{\sigma,\text{odd}}$; in particular, $S_0 = S$. It easily follows that $S_{\sigma}$ consists of functions that are $C^\infty$ on $\mathbb{R} \setminus \{0\}$ but a priori possibly not even defined at zero, and $S^m_{\sigma} \cap C^\infty$ is dense in $S^m_{\sigma}$ for all $m$; thus $S_{\sigma} \cap C^\infty$ is dense in $S_{\sigma}$.

Obviously, $\Sigma$ defines a bounded operator on each $S^m_{\sigma}$. It is also easy to see that $T_{\sigma}$ defines a bounded operator $S^m_{\sigma+1} \rightarrow S^m_{\sigma}$ for any $m$; notice that, when $\sigma < 0$, the role played by the parity of $i + j$ fits well to prove this property. Similarly, $x$ defines a bounded operator $S^{m+1}_{\sigma} \rightarrow S^m_{\sigma}$ for any $m$ because, by (2),
\[
[T_{\sigma}^j, x] = \begin{cases} jT_{\sigma}^{j-1} & \text{if } j \text{ is even} \\ (j + 2\Sigma)T_{\sigma}^{j-1} & \text{if } j \text{ is odd} \end{cases}
\]
So $B$ and $B'$ define bounded operators $S^m_{\sigma+1} \rightarrow S^m_{\sigma}$, and $L$ a bounded operator $S^{m+2}_{\sigma} \rightarrow S^m_{\sigma}$. Thus $T_{\sigma}$, $x$, $\Sigma$, $B$, $B'$ and $L$ define continuous operators on $S_{\sigma}$.

In order to prove Theorems 1.1 and 1.2 we introduce an intermediate weakly perturbed Schwartz space $S_{w,\sigma}$. Like $S_{\sigma}$, it is a Fréchet space of the form $S_{w,\sigma} = \bigcap_m S^m_{w,\sigma}$, where each $S^m_{w,\sigma}$ is the Banach space defined like $S^m_{\sigma}$ but a priori possibly not even defined at zero, $S_{w,\sigma} \cap C^\infty$ is dense in $S_{w,\sigma}$, there is a canonical decomposition $S_{w,\sigma} = S_{w,\sigma,\text{even}} \oplus S_{w,\sigma,\text{odd}}$, and $x$ define bounded operators on $S^{m+1}_{w,\sigma} \rightarrow S^m_{w,\sigma}$. Thus $\frac{d}{dx}$ and $x$ define continuous operators on $S_{w,\sigma}$.

**Lemma 3.1.** If $\sigma \geq 0$, then $S^{m+\lceil \sigma \rceil} \subset S^m_{w,\sigma}$ continuously.

**Proof.** Let $\phi \in S$. For all $i$ and $j$, we have $|x|^i |x^j(\phi)(x)| \leq |x^{i+\lceil \sigma \rceil} \phi(x)|$ for $|x| \geq 1$, and $|x|^i |x^j(\phi)(x)| \leq |x^i \phi(x)|$ for $|x| \leq 1$. So $\|\phi\|_{S^{m+\lceil \sigma \rceil}_{w,\sigma}} \leq \|\phi\|_{S^m_{w,\sigma}}$. □

**Lemma 3.2.** If $\sigma \geq 0$, $S^m_{w,\sigma} \subset S^m$ continuously, where $m_{\sigma} = m + 1 + \frac{1}{2} \lceil \sigma \rceil (\lceil \sigma \rceil + 1)$.
Proof. Let \( \phi \in S_{w, \sigma} \). For all \( i \) and \( j \),

\[
|x^i \phi^{(j)}(x)| \leq |x|^\sigma |x^i \phi^{(j)}(x)|
\]

(19)

for \( |x| \geq 1 \). It remains to prove an inequality of this type for \( |x| \leq 1 \), which will be a consequence of the following assertion.

**Claim 1.** For each \( n \in \mathbb{N} \), there are finite families of real numbers, \( c_{a,b}^n \), \( d_{k,l}^n \) and \( e_{u,v}^n \), where the indices \( a, b, k, \ell, u \) and \( v \) run in finite subsets of \( \mathbb{N} \) with \( b, \ell, v \leq M_n = 1 + \frac{n(n+1)}{2} \) and \( k \geq n \), such that, for all \( \phi \in C^\infty \),

\[
\phi(x) = \sum_{a,b} c_{a,b}^n x^a \phi^{(b)}(1) + \sum_{k,\ell} d_{k,\ell}^n x^k \phi^{(\ell)}(x) + \sum_{u,v} e_{u,v}^n x^u \int_x^1 t^n \phi^{(v)}(t) \, dt .
\]

Assuming that Claim 1 is true, the proof can be completed as follows. Let \( \phi \in S_{w, \sigma} \) and set \( n = |\sigma| \). For \( |x| \leq 1 \), according to Claim 1 \( |\phi(x)| \) is bounded by

\[
\sum_{a,b} |c_{a,b}^n| |\phi^{(b)}(1)| + \sum_{k,\ell} |d_{k,\ell}^n| |x^k \phi^{(\ell)}(x)| + \sum_{u,v} |e_{u,v}^n| 2 \max_{|t| \leq 1} |t^n \phi^{(v)}(t)|
\]

\[
\leq \sum_{i,j} |c_{a,b}^n| |\phi^{(b)}(1)| + \sum_{k,\ell} |d_{k,\ell}^n| |x|^\sigma |\phi^{(\ell)}(x)| + \sum_{u,v} |e_{u,v}^n| 2 \max_{|t| \leq 1} |t|^\sigma |\phi^{(v)}(t)| .
\]

Let \( m, i, j \in \mathbb{N} \) with \( i + j \leq m \). By applying the above inequality to the function \( x^i \phi^{(j)} \), and expressing each derivative \( (x^i \phi^{(j)})(r) \) as a linear combination of functions of the form \( x^{p+q} \) with \( p+q \leq i+j+r \), it follows that there is some \( C \geq 1 \), depending only on \( \sigma \) and \( m \), such that

\[
|x^i \phi^{(j)}(x)| \leq C \|\phi\|_{S_{w, \sigma}^m}^m
\]

(20)

for \( |x| \leq 1 \). By (19) and (20), \( \|\phi\|_{S^m} \leq C \|\phi\|_{S_{w, \sigma}^m} \) because \( m = m + M_n \).

Now, let us prove Claim 1. By induction on \( n \) and using integration by parts, it is easy to prove that

\[
\int_x^1 t^n \phi^{(n+1)}(t) \, dt = \sum_{r=0}^n (-1)^{n-r} \frac{n!}{r!} \left( \phi^{(r)}(1) - x^r \phi^{(r)}(x) \right) .
\]

(21)

This shows directly Claim 1 for \( n \in \{0, 1\} \). Proceeding by induction, let \( n \geq 2 \) and assume that Claim 1 holds for \( n-1 \). By (21), it is enough to find appropriate expressions of \( x^r \phi^{(r)}(x) \) for \( 0 < r < n \). For that purpose, apply Claim 1 for \( n-1 \) to each function \( \phi^{(r)} \), and multiply the resulting equality by \( x^r \) to get

\[
x^r \phi^{(r)}(x) = \sum_{a,b} c_{a,b}^{n-1} x^{r+a} \phi^{(r+b)}(1) + \sum_{k,\ell} d_{k,\ell}^{n-1} x^{r+k} \phi^{(r+\ell)}(x)
\]

\[
+ \sum_{u,v} e_{u,v}^{n-1} x^{r+u} \int_x^1 t^{n-1} \phi^{(r+v)}(t) \, dt ,
\]

where \( a, b, k, \ell, u \) and \( v \) run in finite subsets of \( \mathbb{N} \) with \( b, \ell, v \leq M_{n-1} \) and \( k \geq n-1 \); thus \( r + k \geq n \) and \( r + b, r + \ell, r + v \leq n - 1 + M_{n-1} = M_n - 1 \). Therefore it only remains to rise the exponent of \( t \) by a unit in the integrals of the last sum. Once more, by integration by parts makes the job:

\[
\int_x^1 t^n \phi^{(r+v+1)}(t) \, dt = \phi^{(r+v)}(1) - x^n \phi^{(r+v)}(x) - n \int_x^1 t^{n-1} \phi^{(r+v)} \, dt .
\]
Lemma 3.3. If $\sigma < 0$, then $S^{m+1}_{w,\sigma} \subset S^m_{w,\sigma}$ continuously.

Proof. This is proved by induction on $m$. For $\phi \in C^\infty_{ev}$ and $|x| \geq 1$, we have $|x|\sigma |\phi(x)| \leq |\phi(x)|$, obtaining $\|\phi\|_{S^m_{w,\sigma}} \leq \|\phi\|_{S^0_{w,\sigma}}$ on $C^\infty_{ev}$. On the other hand, for $\phi \in C^\infty_{odd}$ and $\psi = x^{-1}\phi \in C^\infty_{ev}$, we get

$$|x|\sigma |\phi(x)| \leq \begin{cases} |\psi(x)| & \text{if } 0 < |x| \leq 1 \\ |\phi(x)| & \text{if } |x| \geq 1. \end{cases}$$

So $\|\phi\|_{S^m_{w,\sigma}} \leq \max\{\|\phi\|_{S^0}, \|\psi\|_{S^0}\} \leq \|\phi\|_{S^1}$ by (1).

Now, assume that $m \geq 1$ and the result holds for $m - 1$. Let $i, j \in \mathbb{N}$ such that $i + j \leq m$, and let $\psi \in S^m_{ev} \cup S^m_{odd}$. Independently of the parity of $\phi$ and $i + j$, we have $|x|\sigma |x^i\phi(j)(x)| \leq |x^i\phi(j)(x)|$ for $|x| \geq 1$.

Suppose that $\phi \in S^m_{ev}$. If $i = 0$ and $j$ is odd, then $\phi(j) \in S^m_{odd}$. Thus there is some $\psi \in S^m_{ev}$ such that $\phi = x^{-1}\psi$, obtaining $|x|\sigma |\phi(j)(x)| \leq |\psi(x)|$ for $0 < |x| \leq 1$. If $i + j$ is odd and $i > 0$, then $|x|\sigma |x^i\phi(j)(x)| \leq |x^i\phi(j)(x)|$ for $0 < |x| \leq 1$. Hence, by (1), there is some $C > 0$, independent of $\phi$, such that

$$\|\phi\|_{S^m_{w,\sigma}} \leq C \max\{\|\phi\|_{S^m_{ev}}, \|\psi\|_{S^0_{w,\sigma}}\} \leq C \max\{\|\phi\|_{S^m_{ev}}, \|\phi(j)\|_{S^1_{w,\sigma}}\} \leq C \|\phi\|_{S^{m+1}_{w,\sigma}}.$$ 

Finally, assume $\phi \in S^m_{odd}$, and let $\psi = x^{-1}\phi \in S^m_{ev}$. If $i$ is even and $j = 0$, then $|x|\sigma |x^i\psi(x)| \leq |x^i\psi(x)|$ for $0 < |x| \leq 1$. If $i + j$ is even and $j > 0$, then

$$|x|\sigma |x^i\phi(j)(x)| \leq |x^i\phi(j)(x)| + j |x|\sigma |x^i\phi(j-1)(x)|$$

for $0 < |x| \leq 1$ because $\frac{d^{j-1}}{dx^{j-1}} |x| = j \frac{d^{j-1}}{dx^{j-1}}$. Therefore, by (1) and the induction hypothesis, there are some $C', C'' > 0$, independent of $\phi$, such that

$$\|\phi\|_{S^m_{w,\sigma}} \leq C' \max\{\|\phi\|_{S^m_{ev}}, \|\psi\|_{S^m_{odd}} + \|\psi\|_{S^{m+1}_{w,\sigma}}\} \leq C'' \|\phi\|_{S^{m+1}_{w,\sigma}}. \quad \Box$$

Lemma 3.4. If $\sigma < 0$, then $S^{m+1}_{w,\sigma} \subset S^m_{w,\sigma}$ continuously.

Proof. Let $i, j \in \mathbb{N}$ such that $i + j \leq m$. Since

$$|x^i\phi(j)(x)| \leq \begin{cases} |x|\sigma |x^i\phi(j)(x)| & \text{if } 0 < |x| \leq 1 \\ |x|\sigma |x^{i+1}\phi(j)(x)| & \text{if } |x| \geq 1. \end{cases}$$

for any $\phi \in C^\infty$, we get $\|\phi\|_{S^m_{w,\sigma}} \leq \|\phi\|_{S^{m+1}_{w,\sigma}}$. \quad \Box

Corollary 3.5. $S = S^m_{w,\sigma}$ as Fréchet spaces.

Corollary 3.6. $x^{-1}$ defines a bounded operator $S^{m'}_{w,\sigma,odd} \to S^m_{w,\sigma,even}$, where

$$m' = \begin{cases} m + 2 + \frac{1}{2}[\sigma][\sigma] + 3 & \text{if } \sigma \geq 0 \\ m + 3 & \text{if } \sigma < 0. \end{cases}$$

Proof. If $\sigma \geq 0$, the composite

$$S^{m+2+\frac{1}{2}[\sigma][\sigma]+3}_{w,\sigma,odd} \hookrightarrow S^{m+[\sigma]+1}_{odd} \xrightarrow{x^{-1}} S^{m+[\sigma]}_{ev} \hookrightarrow S^m_{w,\sigma,even}$$

is bounded by Lemmas 3.1 and 3.2. If $\sigma < 0$, the composite

$$S^{m+3}_{w,\sigma,odd} \hookrightarrow S^{m+2}_{odd} \xrightarrow{x^{-1}} S^{m+1}_{ev} \hookrightarrow S^m_{w,\sigma,even},$$

is bounded by Lemmas 3.3 and 3.4. \quad \Box

Corollary 3.7. $x^{-1}$ defines a continuous operator $S^{m}_{w,\sigma,odd} \to S^m_{w,\sigma,even}$. 

Lemma 3.8. \( S_{w,σ, ev/odd}^{M_{m, σ, ev/odd}} \subset S_{σ, ev/odd}^m \) continuously, where

\[
M_{m, ev/odd} = \begin{cases} 3m & \text{if } σ \geq 0 \text{ and } m \text{ is even} \\ \frac{3m - 1}{2} & \text{if } σ < 0 \text{ and } m \text{ is even} \\ 3m + 1 & \text{if } σ \geq 0 \text{ and } m \text{ is odd} \\ \frac{3m + 1}{2} & \text{if } σ < 0 \text{ and } m \text{ is odd} \end{cases}
\]

\[
M_{m, ev} = \begin{cases} 3m - 1 & \text{if } σ \geq 0 \text{ and } m \text{ is odd} \\ \frac{3m - 2}{2} & \text{if } σ < 0 \text{ and } m \text{ is odd} \end{cases}
\]

\[
M_{m, odd} = \begin{cases} 3m + 1 & \text{if } σ \geq 0 \text{ and } m \text{ is odd} \\ \frac{3m + 2}{2} & \text{if } σ < 0 \text{ and } m \text{ is odd} \end{cases}
\]

Proof. This follows by induction on \( m \). It is true for \( m = 0 \) because \( S_{w,σ}^0 = S_{σ}^0 \) as Banach spaces. Now, let \( m \geq 1 \), and assume that the result holds for \( m - 1 \).

For \( φ \in C_{ev}^∞ \), \( i + j \leq m \) with \( j > 0 \), and \( x \in \mathbb{R} \), we have \( |x^i(T_d^j φ)(x)| = |x^i(T_d^j φ')(x)| \) with \( φ' \in C_{odd}^∞ \), obtaining \( \|φ\|_{S^m_w} ≤ \|φ\|_{S^m_{σ}} \). But, by the induction hypothesis and since \( M_{m, ev} = M_{m-1, odd} + 1 \), there are some \( C, C' > 0 \), independent of \( φ \), such that

\[
\|φ'\|_{S^m_{σ}} ≤ C \|φ\|_{S^m_{w,σ}} ≤ C' \|φ\|_{S^m_{w, σ, ev/odd}} .
\]

Corollary 3.9. \( S_{w,σ} \subset S_{σ} \) continuously.

Corollary 3.10. \( S_{ev/odd}^{M_{m, ev/odd}} \subset S_{σ, ev/odd}^m \) continuously, where, with the notation of Lemma 3.8,

\[
M'_{m, ev/odd} = \begin{cases} M_{m, ev/odd} + [σ] & \text{if } σ \geq 0 \\ M_{m, ev/odd} + 1 & \text{if } σ < 0 \end{cases}
\]

Proof. This follows from Lemmas 3.1, 3.3 and 3.8

4. Perturbed Sobolev spaces

Observe that \( S_{σ} \subset L^2(\mathbb{R}, |x|^{2σ} \, dx) \). Like in the case where \( S \) is considered as domain, it is easy to check that, in \( L^2(\mathbb{R}, |x|^{2σ} \, dx) \), with domain \( S_{σ} \), \( B \) is adjoint of \( B' \) and \( L \) is symmetric.

Lemma 4.1. \( S_{σ} \) is a core of \( L \).

\(^4\)Recall that a core of a closed densely defined operator \( T \) between Hilbert spaces is any subspace of its domain \( D(T) \) which is dense with the graph norm.
Proof. Let $R$ denote the restriction of $\mathcal{L}$ to $\mathcal{S}_\sigma$. Then $\mathcal{L} \subset \overline{\mathcal{R}} \subset \mathbb{R}^* \subset \mathcal{L} = \mathcal{L}$ in $L^2(\mathbb{R}, |x|^{2\sigma} dx)$ because $\mathcal{S} \subset \mathcal{S}_\sigma$ by Corollaries 3.5 and 3.9.

For each $m \in \mathbb{R}$, let $W^m_\sigma$ be the Hilbert space completion of $\mathcal{S}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{W^m_\sigma}$ defined by $\langle \phi, \psi \rangle_{W^m_\sigma} = \langle (1 + \mathcal{L})^m \phi, \psi \rangle_\sigma$. The corresponding norm will be denoted by $\| \cdot \|_{W^m_\sigma}$, whose equivalence class is independent of the parameter $s$ used to define $L$. In particular, $W^m_\sigma = L^2(\mathbb{R}, |x|^{2\sigma} dx)$. As usual, $W^m_\sigma \subset W^m_\sigma$ when $m' > m$, and let $W^\infty_\sigma = \bigcap_m W^m_\sigma$ with the induced Fréchet topology. Once more, there are direct sum decompositions into subspaces of even and odd (generalized) functions, $W^m_\sigma = W^{m}_{\sigma,\text{ev}} \oplus W^m_{\sigma,\text{odd}} (m \in [0, \infty))$. By Lemma 4.1 $\mathcal{S}_\sigma$ can be used instead of $\mathcal{S}$ in the definition of $W^m_\sigma$.

Obviously, $L$ defines a bounded operator $W^{m+2}_\sigma \rightarrow W^m_\sigma$ for each $m$, and therefore a continuous operator on $W^\infty_\sigma$. By (7), $\Sigma$ defines a bounded operator on each $W^m_\sigma$, and therefore a continuous operator on $W^\infty_\sigma$. Moreover $B$ and $B'$ define bounded operators $W^{m+1}_\sigma \rightarrow W^m_\sigma$ for each $m \in \mathbb{N}$; this follows easily by induction on $m$, using (4) and (5) (the details of the proof are omitted because this observation will not be used). Thus $L$, $\Sigma$, $B$ and $B'$ define bounded operators on $W^\infty_\sigma$. Also on the spaces $W^m_\sigma$, the parity of (generalized) functions is preserved by $L$ and $\Sigma$, and reversed by $B$ and $B'$. Observe that $B'$ is not adjoint of $B$ in $W^m_\sigma$ for $m \neq 0$.

The motivation of our tour through perturbed Schwartz spaces is the following embedding results; the second one is a version of the Sobolev embedding theorem.

Proposition 4.2. $S^{m+1}_\sigma \subset W^m_\sigma$ continuously for all $m \in \mathbb{N}$.

Proposition 4.3. For all $m \in \mathbb{N}$, $W^{m'}_\sigma \subset S^m_\sigma$ continuously if $m' - m > 1$.

Corollary 4.4. $\mathcal{S}_\sigma = W^\infty_\sigma$ as Fréchet spaces.

For each non-commutative polynomial $p$ (of two variables, $X$ and $Y$), let $p'$ denote the non-commutative polynomial obtained by reversing the order of the variables in $p$; e.g., if $p(X, Y) = X^2Y^3X$, then $p'(X, Y) = XY^3X^2$. It will be said that $p$ is symmetric if $p(X, Y) = p'(Y, X)$. Note that $p'(Y, X)p(X, Y)$ is symmetric for any $p$. Given any non-commutative polynomial $p$, the continuous operators $p(B, B')$ and $p'(B', B)$ on $\mathcal{S}_\sigma$ are adjoint of each other in $L^2(\mathbb{R}, |x|^{2\sigma} dx)$; thus $p(B, B')$ is a symmetric operator if $p$ is symmetric.

Lemma 4.5. For $m \in \mathbb{N}$, we have $(1 + L)^m = \sum a q_a(B', B)q_a(B, B')$ for some finite family of homogeneous non-commutative polynomials $q_a$ of degree $\leq m$.

Proof. The result follows easily from the following assertions.

Claim 2. If $m$ is even, then $L^m = g_m(B, B')^2$ for some symmetric homogeneous non-commutative polynomial $g_m$ of degree $m$.

Claim 3. If $m$ is odd, then $L^m = g_{m, 1}(B', B)g_{m, 1}(B, B') + g_{m, 2}(B', B)g_{m, 2}(B, B')$ for some homogeneous non-commutative polynomials $g_{m, 1}$ and $g_{m, 2}$ of degree $m$.

If $m$ is even, then $L^{m/2} = g_m(B, B')$ for some symmetric homogeneous non-commutative polynomial $g_m$ of degree $m$ by (4). So $L^m = g_m(B, B')^2$, showing Claim 2.
If \( m \) is odd, write \( L^{\lfloor m/2 \rfloor} = f_m(B, B') \) as above for some symmetric homogeneous non-commutative polynomial \( f_m \) of degree \( m \). Then, by (1),
\[
L^m = \frac{1}{2} f_m(B, B')(BB' + B'B)f_m(B, B')
\]
Thus Claim 3 follows with
\[
g_{m,1}(B, B') = \frac{1}{\sqrt{2}} B' f_m(B, B') \, ; \quad g_{m,2}(B, B') = \frac{1}{\sqrt{2}} B f_m(B, B') \, .
\]

Proof of Proposition 4.2. By the definitions of \( B \) and \( B' \), and (10)–(18), for each homogeneous non-commutative polynomial \( p \) of three variables with degree \( d \leq m + 1 \), there exists some \( C_p > 0 \) such that, for all \( \phi \in S_\sigma \); if \( \sigma < 0 \), \( |x| \leq 1 \), and \( \phi \) and \( d \) have the same parity, then \( |(p(x, B, B')\phi)(x)| \leq C_p \| \phi \|_{S^{m+1}_\sigma} \); and, otherwise, \( |x|^\sigma |(p(x, B, B')\phi)(x)| \leq C_p \| \phi \|_{S^{m+1}_\sigma} \).

With the notation of Lemma 4.5, let \( d_a \) denote the degree of each \( q_a \), and let \( \bar{q}_a(x, B, B') = x q_a(B, B') \). If \( \sigma \geq 0 \), then
\[
\| \phi \|^2_{W_{m,\sigma}} = \sum_a \| q_a(B, B')\phi \|^2_\sigma = \sum_a \int_{-\infty}^{\infty} |(q_a(B, B')\phi)(x)|^2 |x|^{2\sigma} \, dx \\
\leq 2 \sum_a \left( C_{q_a}^2 + C_{q_a}^2 \int_1^\infty x^{-2} \, dx \right) \| \phi \|^2_{S^{m+1}_\sigma}
\]
for \( \phi \in S_\sigma \). Similarly, if \( \sigma < 0 \), then \( \| \phi \|^2_{W_{m,\sigma}} \) is bounded by
\[
2 \left( \sum_{d_a \text{ even}} C_{q_a}^2 \int_0^1 x^{2\sigma} \, dx + \sum_{d_a \text{ odd}} C_{q_a}^2 + \sum_a C_{q_a}^2 \int_{1}^{\infty} x^{-2} \, dx \right) \| \phi \|^2_{S^{m+1}_\sigma}
\]
for \( \phi \in S_{\sigma, ev} \), and by
\[
2 \left( \sum_{d_a \text{ even}} C_{q_a}^2 + \sum_{d_a \text{ odd}} C_{q_a}^2 \int_0^1 x^{2\sigma} \, dx + \sum_a C_{q_a}^2 \int_{1}^{\infty} x^{-2} \, dx \right) \| \phi \|^2_{S^{m+1}_\sigma}
\]
for \( \phi \in S_{\sigma, odd} \).
According to Section 2.2, the “Fourier coefficients” mapping $\phi \mapsto (\langle \phi_k, \phi \rangle_\sigma)$ defines a quasi-isometry $W^m_\sigma \to \ell^2_m$ for all finite $m$, and therefore an isomorphism $W^\infty_\sigma \to C_\infty$, of Fréchet spaces. This map is compatible with the decompositions into even and odd subspaces.

**Corollary 4.8.** Any $\phi \in L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$ is in $S_\sigma$ if and only if its “Fourier coefficients” $(\langle \phi, \phi \rangle_\sigma)$ are rapidly decaying on $k$.

**Proof.** By Corollary 4.4, the “Fourier coefficients” mapping defines an isomorphism $S_\sigma \to C_\infty$, of Fréchet spaces. □

The operator $\ell^2_{m'} \to \ell^2_m$ is compact if $m' > m$ (see e.g. [28, Theorem 5.8]). So, by using the “Fourier coefficients” mapping, the operator $W^m_{\sigma'} \to W^m_\sigma$ is compact if $m' > m$ (a version of the Rellich theorem). 

**Proof of Proposition 4.3.** For $\phi \in S_\sigma$, its “Fourier coefficients” $c_k = (\langle \phi_k, \phi \rangle_\sigma)$ form a sequence $c = (c_k)$ in $C_\infty$, and $\sum_k |c_k| (1 + k)^{m/2} \leq \|c\|_{\ell^2_{m'}} \sum (1 + k)^{m-m'}$ by Cauchy-Schwarz inequality, where the last series is convergent since $m - m' < -1$. Therefore there is some $C > 0$, independent of $\phi$, such that

$$\sum_k |c_k| (1 + k)^{m/2} \leq C \|\phi\|_{W^m_{\sigma'}}.$$  \hfill (22)

On the other hand, for all $i, j \in \mathbb{N}$ with $i + j \leq m$, there is some homogeneous non-commutative polynomial $p_{ij}$ of degree $i + j$ such that $x^i T^j_\sigma = p_{ij}(B, B')$. Then, by (9)–(11), there is some $C_{ij} > 0$, independent of $\phi$, such that

$$|\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| \leq C_{ij} (1 + k)^{m/2} \sum_{|\ell-k| \leq m} |c_{\ell}|.$$  \hfill (23)

Assume that $\sigma \geq 0$. By (22), (23) and Theorem 2.1(i), there is some $C_{ij} > 0$, independent of $\phi$ and $x_0$, so that

$$|x_0|^{\sigma} |\langle x^i T^j_\sigma \phi \rangle_\sigma(x_0)| \leq |x_0|^{\sigma} \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| |\phi_k(x_0)|$$

$$= \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| |\phi_k(x_0)|$$

$$\leq C_{ij} \|\phi\|_{W^m_{\sigma'}}.$$  \hfill (24)

for all $x_0 \in \mathbb{R}$. Hence $\|\phi\|_{S^\tau_{\sigma'}} \leq C' \|\phi\|_{W^m_{\sigma'}}$ for some $C' > 0$, independent of $\phi$.

Now, suppose that $\sigma < 0$. From (22), (23) and Theorem 2.1(ii),(iii), it follows that there is some $C_{ij} > 0$, independent of $\phi$ and $x_0$, such that

$$|\langle x^i T^j_\sigma \phi \rangle_\sigma(x_0)| \leq \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| |\phi_k(x_0)| \leq C_{ij} \|\phi\|_{W^m_{\sigma'}}$$

if $\phi$ and $i + j$ have the same parity, and $|x_0| < 1$; otherwise, $|x_0|^\sigma |\langle x^i T^j_\sigma \phi \rangle_\sigma(x_0)| \leq C_{ij} \|\phi\|_{W^m_{\sigma'}}$ like in (24). So $\|\phi\|_{S^\tau_{\sigma'}} \leq C' \|\phi\|_{W^m_{\sigma'}}$ with $C' > 0$, independent of $\phi$. □

As suggested by (15), consider the mapping $c = (c_k) \mapsto \Xi(c) = (d_k)$, where $c$ is odd and $\Xi(c)$ is even, with

$$d_\ell = \sum_{k \in \{\ell+1,\ell+3,\ldots\}} (-1)^{k-\ell-1} \sqrt{\frac{(k-1)(k-3) \cdots (\ell+2)2s}{(k+2\sigma)(k-2+2\sigma) \cdots (\ell+1+2\sigma)}} c_k$$

for $\ell$ even, assuming that this series is convergent.
Lemma 4.9. \( \Xi \) defines a bounded map \( \ell^2_{m', \text{odd}} \rightarrow C_{m, \text{ev}} \) if \( m' - m > 1 \).

Proof. By the Cauchy-Schwartz inequality,

\[
\|d\|_{C_{m}} = \sup_{\ell} \sum_{k \in \{\ell + 1, \ell + 3, \ldots\}} \sqrt{\frac{(k - 1)(k - 3) \cdots (\ell + 2)2s}{(k + 2\sigma)(k + 2 + 2\sigma) \cdots (\ell + 1 + 2\sigma)}} |c_k|(1 + \ell)^m
\]

\[
\leq \sqrt{2s} \sup_{\ell} \sum_{k \in \{\ell + 1, \ell + 3, \ldots\}} |c_k|(1 + \ell)^m
\]

\[
\leq \sqrt{2s} \|c\|_{\ell^2_{m'}} \sup_{\ell} \left( \sum_{k \in \{\ell + 1, \ell + 3, \ldots\}} (1 + k)^{-m'}(1 + \ell)^m \right)^{1/2}
\]

\[
\leq \sqrt{2s} \|c\|_{\ell^2_{m'}} \left( \sum_{k}(1 + k)^{m'-m} \right)^{1/2},
\]

where the last series is convergent because \( m - m' < -1 \).

\[
\square
\]

Corollary 4.10. \( x^{-1} \) defines a bounded operator \( S^m_{\sigma, \text{odd}} \rightarrow S^m_{\sigma, \text{ev}} \) if \( 2m' > m + 6 \).

Proof. Since \( 2m' > m + 6 \), there are \( m_1, m_2 \in \mathbb{R} \) so that \( m' - m_2 > 2, 2m_1 - m_1 > 1 \) and \( m_1 - m > 1 \). Then, by Propositions 4.2 and 4.3, Lemmas 4.6 and 4.9 and using the “Fourier coefficients” mapping, we get the composition of bounded maps,

\[
S^m_{\sigma, \text{odd}} \hookrightarrow W^m_{\sigma, \text{odd}} \rightarrow \ell^2_{m'-1, \text{odd}} \overset{\Xi}{\rightarrow} C_{m_2, \text{ev}} \hookrightarrow \ell^2_{m_1, \text{ev}} \rightarrow W^m_{\sigma, \text{ev}} \hookrightarrow S^m_{\sigma, \text{ev}},
\]

which extends \( x^{-1} : S_{\sigma, \text{odd}} \rightarrow S_{\sigma, \text{ev}} \) by (15).

\[
\square
\]

Question 4.11. Is it possible to prove Corollary 4.10 without using (15)?

Corollary 4.12. \( x^{-1} \) defines a continuous operator \( S^m_{\sigma, \text{odd}} \rightarrow S^m_{\sigma, \text{ev}} \).

Lemma 4.13. \( S^m_{\sigma, \text{ev/odd}} \subset S^m_{\sigma, \text{ev/odd}} \) continuously for all \( m \), where \( M_{0, \text{ev/odd}} = 0 \), \( M_{1, \text{ev}} = 1 \), \( M_{1, \text{odd}} = M_{2, \text{odd}} = 4 \), \( M_{2, \text{ev}} = M_{3, \text{ev}} = 5 \), \( M_{3, \text{odd}} = 6 \), and \( M_{m, \text{ev/odd}} = m + 2 \) for \( m \geq 4 \).

Proof. We proceed by induction on \( m \). The case \( m = 0 \) was already indicated in the proof of Lemma 4.8. Now, let \( m \geq 1 \) and assume that the result holds for \( m-1 \).

For \( \phi \in C^\infty_{ev} \), \( i + j \leq m \) with \( j > 0 \) and \( x \in \mathbb{R} \), we have \( |x^i\phi^{(j)}(x)| = |x^i(T_\sigma \phi)^{(j-1)}(x)| \) with \( T_\sigma \phi \in C^\infty_{ev} \), obtaining \( \|\phi\|_{S^m_{\sigma, ev}} \leq \|T_\sigma \phi\|_{S^{m-1}_{\sigma, ev}} \). But, by the induction hypothesis and because \( M_{m, \text{ev}} = M_{m-1, \text{odd}} + 1 \), there are some \( C, C' > 0 \), independent of \( \phi \), such that \( \|T_\sigma \phi\|_{S^m_{\sigma, ev}} \leq C \|\phi\|_{S^m_{\sigma, ev}} \).

For \( \phi \in C^\infty_{ev} \), let \( \psi = x^{-1}\phi \), and take \( i, j \) and \( x \) as above. We have

\[
|x^i\psi^{(j)}(x)| \leq |x^i(T_\sigma \phi)^{(j-1)}(x)| + 2|\sigma| |x^i\psi^{(j-1)}(x)|
\]

with \( T_\sigma \phi, \psi \in C^\infty_{ev} \), obtaining \( \|\phi\|_{S^m_{\sigma, ev}} \leq \|T_\sigma \phi\|_{S^{m-1}_{\sigma, ev}} + 2|\sigma| \|\psi\|_{S^{m-1}_{\sigma, ev}} \). But, by the induction hypothesis, Corollary 4.10 and since \( M_{m, \text{odd}} \leq M_{m-1, \text{ev}} + 1 \) and \( 2M_{m, \text{odd}} > M_{m-1, ev} + 6 \), there are some \( C, C' > 0 \), independent of \( \phi \), such that

\[
\|T_\sigma \phi\|_{S^{m-1}_{\sigma, ev}} \leq C \|\phi\|_{S^m_{\sigma, ev}} \leq C' \|\phi\|_{S^m_{\sigma, ev}}.
\]

\[
\square
Corollary 4.14. $\mathcal{S}_{σ, ev/odd}^m \subset \mathcal{S}_{ev/odd}^m$ continuously, where, with the notation of Lemma 4.13, $M_{σ, ev/odd}^m = M_{m, σ, ev/odd}^m$ for $m_σ = m + 1 + 1/2[σ]([σ] + 1)$.

Proof. This follows from Lemmas 3.2, 3.4, and 4.13.

Corollary 4.15. $\mathcal{S}_σ = \mathcal{S}$ as Fréchet spaces.

Proof. This is a consequence of Corollaries 3.10 and 4.14.

Corollaries 3.10 and 4.14 and Propositions 4.2 and 4.3 give Theorems 1.1 and 1.2.

5. Perturbations of $H$ on $\mathbb{R}_+$

Since the function $|x|^{2σ}$ is even, the decomposition $\mathcal{S} = \mathcal{S}_{ev} \oplus \mathcal{S}_{odd}$ extends to an orthogonal decomposition

$$L^2(\mathbb{R}, |x|^{2σ} \, dx) = L^2_{ev}(\mathbb{R}, |x|^{2σ} \, dx) \oplus L^2_{odd}(\mathbb{R}, |x|^{2σ} \, dx).$$

Let $\mathcal{L}_{ev/odd}$ or $\mathcal{L}_{σ, ev/odd}$ and $\mathcal{L}_{σ, ev/odd}$ denote the corresponding components of $\mathcal{L}$ and $\mathcal{L}$. $\mathcal{L}_{ev/odd}$ is essentially self-adjoint in $L^2_{ev/odd}(\mathbb{R}, |x|^{2σ} \, dx)$, and its self-adjoint extension is $\mathcal{L}_{ev/odd}$, which satisfies an obvious version of Corollary 1.3.

Fix an open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure. Let $\mathcal{S}_{ev/odd, U} \subset C^∞(U)$ denote the linear subspace of restrictions to $U$ of the functions in $\mathcal{S}_{ev/odd}$. The restriction to $U$ defines a linear isomorphism $\mathcal{S}_{ev/odd} \cong \mathcal{S}_{ev/odd, U}$, and a unitary isomorphism $L^2_{ev/odd}(\mathbb{R}, |x|^{2σ} \, dx) \cong L^2(\mathbb{R}_+, |x|^{2σ} \, dx)$. Via these isomorphisms, $L_{ev/odd}$ corresponds to an operator $L_{ev/odd, U}$ on $\mathcal{S}_{ev/odd, U}$, and $L_{ev/odd}$ corresponds to a self-adjoint operator $L_{ev/odd, +}$ in $L^2(\mathbb{R}_+, |x|^{2σ} \, dx)$; the more explicit notation $L_{σ, ev/odd, U}$ and $L_{σ, ev/odd, +}$ may be used. Let $φ_k,U = φ_k|U$, whose norm in $L^2(\mathbb{R}_+, |x|^{2σ} \, dx)$ is $1/\sqrt{2}$.

Going one step further, for any positive function $h \in C^2(U)$, the multiplication by $h$ defines a unitary isomorphism $h : L^2(\mathbb{R}_+, |x|^{2σ} \, dx) \rightarrow L^2(\mathbb{R}_+, h^{-2} \, dx)$. Thus $hL_{ev, U} h^{-1}$, with domain $h\mathcal{S}_{ev, U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, h^{-2} \, dx)$, and its self-adjoint extension is $hL_{ev, U} h^{-1}$. Via these unitary isomorphisms, we get an obvious version of Corollary 1.3 for $hL_{ev, U} h^{-1}$. By using

$$\frac{d}{dx}h = h', \quad \frac{d^2}{dx^2}h = 2h'' \frac{d}{dx} + h''',$$

it easily follows that $hL_{ev, U} h^{-1}$ has the form of $P$ in Theorem 1.4. Then Theorem 1.4 is a consequence of the following.

Lemma 5.1. For $σ > -1/2$, a positive function $h \in C^2(U)$, and $P = H - 2f_1 \frac{d}{dx} + f_2$ with $f_1 \in C^1(U)$ and $f_2 \in C(U)$, we have $P = hL_{σ, ev, U} h^{-1}$ on $h\mathcal{S}_{ev, U}$ if and only if $f_1, f_2$ and $h$ satisfy the conditions of Theorem 1.4.

Proof. By (25),

$$h^{-1}Ph = H - 2(h^{-1}h' + f_1) \frac{d}{dx} - h^{-1}h'' - 2h^{-1}f_1 h' + f_2.$$

So $P = hL_{σ, ev, U} h^{-1}$ if and only if $h^{-1}h' = σx^{-1} - f_1$ and $f_2 = h^{-1}h'' + 2h^{-1}h' f_1$, which are easily seen to be equivalent to the conditions of Theorem 1.4.

Remark 1. By (25), we get an operator of the same type if $h$ and $\frac{d}{dx}$ is interchanged in the operator $P$ of Theorem 1.4.
Remark 2. By using (25) with \( h = x^{-1} \) on \( \mathbb{R}_+ \), it is easy to check that \( \Im_{\sigma, \text{odd}} = xL_{1+\sigma, \text{ev}} \), \( \Re_{\sigma, \text{odd}} = xS_{\sigma, \text{ev}} \). So no new operators are obtained with this process by using \( \Im_{\sigma, \text{odd}} \) instead of \( \Im_{\sigma, \text{ev}} \).

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