Existence of solutions for a class of singular elliptic systems with convection term

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Abstract

We show the existence of positive solutions for a class of singular elliptic systems with convection term. The approach combines sub and supersolution method with the pseudomonotone operator theory and perturbation arguments involving singular terms.

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1 Introduction

In this work, we focus our attention on the existence of solutions for the following class of elliptic system with convection term

\[
(S)_\pm \begin{cases}
-\Delta u = \frac{1}{u^\alpha} \pm v^\beta_1 + g_1(\nabla u, \nabla v) & \text{in } \Omega, \\
-\Delta v = \frac{1}{v^\alpha} \pm u^\beta_2 + g_2(\nabla u, \nabla v) & \text{in } \Omega, \\
u, v > 0 & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain with smooth boundary and \( g_i : \mathbb{R}^{2N} \to [0, +\infty), i = 1, 2, \) are positive continuous functions belonging to \( L^\infty(\mathbb{R}^{2N}) \). We consider the system \((S)_\pm\) in a singular case assuming \( \alpha_i, \beta_i, \in [0, 1) \) for \( i = 1, 2 \).

Hereafter \((u, v)\) is a solution to \((S)_\pm\) if \( u, v \in C^2(\Omega) \cap H^1_0(\Omega) \) are both positive in \( \Omega \) and satisfy the equations of \((S)_\pm\) in the classical sense.

Nonlinear singular boundary value problems are mathematically challenging and important for applications. They arise in several physical situations such as fluid mechanics, pseudoplastics flow, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation, for more details about this subject, we cite the papers of Fulks & Maybe [17], Callegari & Nashman [6, 7] and the references therein.

Systems \((S)_\pm\) can be seen as a version of the singular scalar equations

\[
(P)_\pm \begin{cases}
-\Delta u = \frac{1}{u^\alpha} \pm u^\beta + g(\nabla u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

with \( \alpha, \beta > 0 \) and \( g : \mathbb{R}^N \to \mathbb{R} \) be a continuous function verifying some technical conditions. Several works are devoted to classes of problems covering \((P)_\pm\). For instance, see the papers of Aranda [5], Ghergu & Radulescu [19, 20], Giarrusso & Porru [18], Lair & Wood [22], Zhang [26] and references therein. Problem \((P)_\pm\) without a convection term, that is \( g = 0 \) was also investigated. Relevant contributions regarding this situation can be
found in Crandall, Rabinowitz & Tartar \cite{[9]}, Choi & McKenna \cite{[8]}, Coclite & Palmieri \cite{[10]}, Cîrstea, Ghergu & Radulescu \cite{[11]}, Dávila & Montenegro \cite{[12]} and Diaz, Morel & Oswald \cite{[14]}. The main tools used in the aforementioned works are Sub and Supersolution, Fixed Point Theorems, Bifurcation Theory and Galerkin Method. On the other hand, using variational technique, more precisely mountains pass theorem, de Figueiredo, Girardi & Matzeu \cite{[13]} studied a class of elliptic problems without singularity, where the nonlinearity depends of the gradient of the solution.

Related to systems $(S)_\pm$, to date, the only case considered in the literature, known to authors for $g_i \neq 0$ is the paper due to Alves, Carrião & Faria \cite{[4]}. For the case where $g_i = 0$, we refer the reader to the survey paper by Alves & Corrêa \cite{[1]}, Alves, Corrêa & Gonçalves \cite{[2]}, El Manouni, Perera & Shivaji \cite{[15]} and Motreanu & Moussaoui \cite{[24]}. From the above commentaries, we observe that in recent years singular elliptic problems with convection term has received few attention. Motivated by this fact, our aim is to show the existence of solutions for a class of elliptic systems where the nonlinearity besides a singular term has a convection term. The proof combines results involving pseudomonotone operators, sub and supersolution method and perturbation arguments involving singular terms. We emphasize that our study complete those made in \cite{[1]}, \cite{[2]} and \cite{[24]}, in the sense that in those papers the authors did not consider the case where the nonlinearity has a convection term, and also \cite{[4]}, because a different type of singular term was considered. The method used in the present work is different from those applied in the aforementioned papers.

Our main result is the following:

**Theorem 1.1** Assume that $g_i : \mathbb{R}^N \to \mathbb{R}$ are continuous functions and $\alpha_i, \beta_i \in [0, 1)$ for $i = 1, 2$. Then, the systems $(S)_\pm$ has a solution.

The proof of Theorem 1.1 is done in Sections 3 and 4. The first main technical difficulty is that the nonlinearities of $(S)_\pm$ depend of gradient of the solution, which is more a complicating factor. Indeed, for the scalar case, an interesting result is proved by Kazdan & Kramer \cite{[21]} and Leon \cite{[23]}, where the authors develop a sub and supersolution method for scalar problem where the nonlinearity depends of the gradient. Instead, the counter-part of this result for systems with gradient terms is not known in the literature. Thus, we do not know a result involving sub and supersolution that could be use to establishes the existence of solution for this class of system. To overcome this
difficulty, we show in Section 2 a result, see Theorem 2.1, which can be see as a sub and supersolution method for systems whose nonlinearity depends of gradient.

The second main difficulty in the proof of Theorem 1.1 is associated with the fact that the sub and supersolution method in its version involving maximum principle cannot be used directly for systems involving the gradient of the solution. Moreover, the way as the singularities appear in the system \((S)_{\pm}\) is a difficult point to work with maximum principle. In order to overtake the stated problem we first introduce a parameter \(\varepsilon > 0\) in \((S)_{\pm}\), giving rise to regularized systems for \((S)_{\pm}\) whose study is relevant for our initial problem. Then, for the regularized systems, we combine variational methods with the sub-supersolution one to prove the existence of a solution \((u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \cap L^\infty(\Omega) \times H^1(\Omega) \cap L^\infty(\Omega)\). This solution \((u_\varepsilon, v_\varepsilon)\) is located in some rectangle formed by the sub and supersolution, independent for \(\varepsilon > 0\), which does not contains zero for all \(\varepsilon > 0\). Then, a positive solution of \((S)_{\pm}\) is obtained by passing to the limit as \(\varepsilon \to 0\). This is based on a priori estimates and Hardy-Sobolev inequality. The positivity of the solution is derived from the independence of the subsolution of the regularized systems on \(\varepsilon\).

The rest of this article is organized as follows: In Section 2 we state and prove a general theorem about sub and supersolution method for systems with convection term. Sections 3 and 4 contain the proof of Theorem 1.1.

2 An auxiliary result

The main goal in this section is to prove the Theorem 2.1 below, which is a key point in the proof of Theorem 1.1. An interesting point related to Theorem 2.1 is the fact that it is a result of sub and supersolution whose the proof is made using pseudomonotone operator theory.

Theorem 2.1 Let \(H, G : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}\) continuous functions function verifying the following conditions: Given \(T, S > 0\), there exist \(C > 0\) and \(\alpha, \beta \in (0, 1)\), such that

\[
|H(x, s, t, \eta, \xi)|, |G(x, s, t, \eta, \xi)| \leq C(1 + |\eta|^\alpha + |\xi|^\beta)
\]

for all \((x, s, t, \eta, \xi) \in \Omega \times [0, T] \times [0, S] \times \mathbb{R}^N \times \mathbb{R}^N\). Let \(\tilde{g}, \bar{g} \in C^2(\Omega)\) and \(u, \bar{u}, v, \bar{v} \in W^{1,\infty}(\Omega)\) with

\[
u(x) \leq \tilde{g}(x) \leq \bar{u}(x)\]
on \(\partial\Omega\)
and
\[ \underline{v}(x) \leq \hat{g}(x) \leq \overline{v}(x) \] on \( \partial \Omega \).

Assume that
\[
\int_{\Omega} \nabla \underline{u} \nabla \phi dx \leq \int_{\Omega} H(x, u, v, \nabla u, \nabla v) \phi dx,
\]
\[
\int_{\Omega} \nabla \overline{v} \nabla \psi dx \leq \int_{\Omega} G(x, u, v, \nabla u, \nabla v) \psi dx,
\]
\[
\int_{\Omega} \nabla \pi \nabla \phi dx \geq \int_{\Omega} H(x, \pi, v, \nabla \pi, \nabla v) \phi dx
\]
and
\[
\int_{\Omega} \nabla \pi \nabla \psi dx \geq \int_{\Omega} G(x, \pi, v, \nabla \pi, \nabla v) \psi dx,
\]
for all nonnegative functions \( \phi, \psi \in H^{1}(\Omega) \). Then, there is \((u, v) \in (H^{1}(\Omega) \cap L^{\infty}(\Omega)) \times (H^{1}(\Omega) \cap L^{\infty}(\Omega))\) verifying
\[
\underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ and } \underline{v}(x) \leq v(x) \leq \overline{v}(x) \forall x \in \Omega,
\]
\[ u - \hat{g}, v - \hat{g} \in H^{1}_{0}(\Omega) \]
and
\[
\int_{\Omega} \nabla u \nabla \phi dx = \int_{\Omega} H(x, u, v, \nabla u, \nabla v) \phi dx \ \forall \phi \in H^{1}_{0}(\Omega),
\]
\[
\int_{\Omega} \nabla v \nabla \psi dx = \int_{\Omega} G(x, u, v, \nabla u, \nabla v) \psi dx \ \forall \psi \in H^{1}_{0}(\Omega),
\]
that is, \((u, v)\) is a solution of the system
\[
(AS) \quad \begin{cases} 
-\Delta u = H(x, u, v, \nabla u, \nabla v) \text{ in } \Omega, \\
-\Delta v = G(x, u, v, \nabla u, \nabla v) \text{ in } \Omega,
\end{cases}
\]

**Proof.** Here, we will adapt some arguments found in Leon [23]. Firstly, we introduce two new functions
\[
H_{1}(x, s, t, \eta, \xi) = \begin{cases} 
H(x, \underline{u}(x), \underline{v}(x), \nabla \underline{u}(x), \nabla \underline{v}(x)), s \leq \underline{u}(x) \\
H(x, \underline{u}(x), \eta, \nabla \underline{u}(x)), \underline{u}(x) \leq s \leq \underline{\pi}(x) \text{ and } t \leq \underline{v}(x) \\
H(x, s, \underline{v}(x), \eta, \nabla \underline{u}(x)), \underline{v}(x) \leq s \leq \underline{\pi}(x) \text{ and } t \leq \underline{v}(x) \\
H(x, s, t, \eta, \xi), \underline{u}(x) \leq s \leq \underline{\pi}(x) \text{ and } t \leq \underline{v}(x) \\
H(x, \underline{u}(x), \underline{v}(x), \nabla \underline{u}(x), \nabla \underline{v}(x)), s \geq \underline{u}(x).
\end{cases}
\]
and

\[
G_1(x, s, t, \eta, \xi) = \begin{cases} 
G(x, \underline{u}(x), \underline{v}(x), \nabla \underline{u}(x), \nabla \underline{v}(x)), & t \leq \underline{v}(x) \\
G(x, \underline{u}(x), t, \nabla \underline{u}(x), \xi), & \underline{v}(x) \leq t \leq \overline{v}(x) \text{ and } s \leq \underline{u}(x) \\
G(x, s, t, \eta, \xi), & \underline{v}(x) \leq t \leq \overline{v}(x) \text{ and } u(x) \leq \underline{u}(x) \\
G(x, s, t, \eta, \xi), & \underline{v}(x) \leq t \leq \overline{v}(x) \text{ and } u(x) \geq \underline{u}(x) \\
G(x, \overline{u}(x), \overline{v}(x), \nabla \overline{u}(x), \nabla \overline{v}(x)), & t \geq \overline{v}(x).
\end{cases}
\]

Moreover, for each \(l \in (0, 1)\), we consider

\[
\gamma_1(x, s) = -((\underline{u}(x) - s)_+)^l + ((s - \overline{u}(x))_+)^l
\]

and

\[
\gamma_2(x, t) = -((\underline{v}(x) - t)_+)^l + ((t - \overline{v}(x))_+)^l.
\]

Using the above functions, we will work with the ensuing auxiliary system

\[
\begin{cases} 
-\Delta u = H_1(x, u, v, \nabla u, \nabla v) - \gamma_1(x, u) \text{ in } \Omega, \\
-\Delta v = G_1(x, u, v, \nabla u, \nabla v) - \gamma_2(x, v) \text{ in } \Omega, \\
u(x) = \tilde{g}, v(x) = \hat{g} \text{ on } \partial \Omega.
\end{cases}
\]

Setting the functions

\[
H_2(x, s, t, \eta, \xi) = H_1(x, s, t, \eta, \xi) - \gamma_1(x, s)
\]

and

\[
G_2(x, s, t, \eta, \xi) = G_1(x, s, t, \eta, \xi) - \gamma_2(x, t),
\]

we define the operator \(B : E \rightarrow E'\) given by

\[
\langle B(u, v), (\phi, \psi) \rangle = \int_{\Omega} (\nabla u \nabla \phi + \nabla v \nabla \psi) \, dx - \int_{\Omega} H_2(x, u, v, \nabla u, \nabla v) \phi \, dx
\]

\[-\int_{\Omega} G_2(x, u, v, \nabla u, \nabla v) \psi \, dx,
\]

where \(E = H^1_0(\Omega) \times H^1_0(\Omega)\) is endowed of the norm

\[
\| (u, v) \| = (\| u \|^2 + \| v \|)^{\frac{1}{2}}.
\]
with $\|\cdot\|$ being the usual norm in $H^1_0(\Omega)$.

Using the hypotheses on $H$ and $G$ together with the definition of $H_1, H_2, G_1, G_2, \gamma_1$ and $\gamma_2$, we can prove the ensuing properties for operator $B$:

I) $B$ is continuous:

The proof of this property follows by using the fact that $H_1, G_1$ belong to $L^\infty$.

II) $B$ is bounded:

Here, the boundedness of $B$ is understood in the sense that if $U \subset E$ is a bounded set, then $B(U) \subset E'$ is also bounded. This property also follows using the boundedness of $H_1$ and $G_1$.

III) $B$ is coercive:

Here, it is enough to prove that

$$\frac{\langle B(u, v), (u, v) \rangle}{\|(u, v)\|} \to +\infty \quad \text{as} \quad \|(u, v)\| \to +\infty.$$

Using again the boundedness of $H_1$ and $G_1$, we derive

$$\langle B(u, v), (u, v) \rangle \geq \|(u, v)\|^2 - C_1 \|(u, v)\| - C_2 \|(u, v)\|^{l+1}.$$

Thus,

$$\frac{\langle B(u, v), (u, v) \rangle}{\|(u, v)\|} \geq \|(u, v)\| - C_1 - C_2 \|(u, v)\|^{l},$$

showing that

$$\frac{\langle B(u, v), (u, v) \rangle}{\|(u, v)\|} \to +\infty \quad \text{as} \quad \|(u, v)\| \to +\infty.$$

IV) $B$ is pseudomonotone:

First of all, we recall that $B$ is a pseudomonotone operator if $(u_n, v_n) \rightharpoonup (u, v)$ in $E$ and verifies

$$\limsup_{n \to +\infty} \langle B(u_n, v_n), (u_n, v_n) - (u, v) \rangle \leq 0, \quad (2.1)$$
then
\[
\liminf_{n \to +\infty} \langle B(u_n, v_n), (u_n, v_n) - (\phi, \psi) \rangle \geq \langle B(u, v), (u, v) - (\phi, \psi) \rangle \quad \forall (\phi, \psi) \in E.
\]
(2.2)

In our case, the weak limit \((u_n, v_n) \rightharpoonup (u, v)\) in \(E\) yields
\[
\int_{\Omega} H_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u) \to 0
\]
and
\[
\int_{\Omega} G_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(v_n - v) \to 0.
\]
Thereby, the above limits combined with (2.1) load to
\[
\limsup_{n \to +\infty} \int_{\Omega} \left( \nabla u_n \nabla (u_n - u) + \nabla v_n \nabla (v_n - v) \right) \leq 0,
\]
from where it follows that
\[
(u_n, v_n) \to (u, v) \quad \text{in} \quad E.
\]

The properties I) - IV) allow us to use [16, Theorem 3.3.6] to conclude that \(B\) is surjective. Therefore, there exists \((u, v) \in E\) such that
\[
\langle B(u, v), (\phi, \psi) \rangle = 0 \quad \forall (\phi, \psi) \in E,
\]
implying that \((u, v)\) is a solution of \((S_1)\). Now, our goal is showing that
\[
(i) \quad u \leq u \leq \overline{u} \quad \text{and} \quad (ii) \quad v \leq v \leq \overline{v}.
\]
(2.3)
We will show only \((i)\), because the same arguments can be used to prove \((ii)\).
Choosing \((\phi, \psi) = ((u - \overline{u})_+, 0)\) as a test function, we have
\[
\int_{\Omega} \nabla u \nabla (u - u)_+ = \int_{\Omega} H_2(x, u, v, \nabla u, \nabla v)(u - \overline{u})_+ dx.
\]
From definition of \(H_2\),
\[
\int_{\Omega} \nabla u \nabla (u - \overline{u})_+ = \int_{\Omega} H_1(x, u, v, \nabla u, \nabla v)(u - \overline{u})_+ dx - \int_{\Omega} \gamma_1(x, u)(u - \overline{u})_+ dx
\]
and so,
\[ \int_{\Omega} \nabla u \nabla (u - \overline{u})_+ \, dx = \int_{\Omega} H(x, \overline{u}, \overline{v}, \nabla \overline{u}, \nabla \overline{v})(u - \overline{u})_+ \, dx - \int_{\Omega} (u - \overline{u})^{t+1}_+ \, dx. \]

Since \((\overline{u}, \overline{v})\) is a supersolution, it follows that
\[ \int_{\Omega} \nabla u \nabla (u - \overline{u})_+ \, dx \leq \int_{\Omega} \nabla \overline{u} \nabla (u - \overline{u})_+ \, dx - \int_{\Omega} (u - \overline{u})^{t+1}_+ \, dx, \]
or equivalently,
\[ \int_{\Omega} |\nabla (u - \overline{u})_+|^2 \, dx \leq - \int_{\Omega} (u - \overline{u})^{t+1}_+ \, dx \leq 0, \]
showing that \((u - \overline{u})_+ = 0\), from where it follows that \(u \leq \overline{u}\). To prove that \(u \leq u\), we choose \((\phi, \psi) = ((u - u)_+, 0)\) as a test function. Repeating the above arguments, we get
\[ \int_{\Omega} \nabla u \nabla (u - u)_+ \, dx = \int_{\Omega} H(x, u, v, \nabla u, \nabla v)(u - u)_+ \, dx + \int_{\Omega} (u - u)^{t+1}_+ \, dx. \]

Since \((u, v)\) is a subsolution, it follows that
\[ \int_{\Omega} \nabla u \nabla (u - u)_+ \, dx \geq \int_{\Omega} \nabla u \nabla (u - u)_+ \, dx + \int_{\Omega} (u - u)^{t+1}_+ \, dx, \]
or equivalently,
\[ \int_{\Omega} |\nabla (u - u)_+|^2 \, dx \leq - \int_{\Omega} (u - u)^{t+1}_+ \, dx \leq 0, \]
showing that \((u - u)_+ = 0\), and so, \(u \leq u\). Combining (2.3) with the definition of \(H_2\) and \(G_2\), it follows that
\[ H_2(x, u, v, \nabla u, \nabla v) = H(x, u, v, \nabla u, \nabla v) \]
and
\[ G_2(x, u, v, \nabla u, \nabla v) = G(x, u, v, \nabla u, \nabla v), \]
showing that \((u, v)\) is a solution for system \((AS)\).
3 Existence of solution for system \((S)_-\)

In this section, we will study the existence of solution for the following singular elliptic system

\[
\begin{aligned}
-\Delta u &= \frac{1}{|v|^{2\alpha_1}} - v^{\beta_1} + g_1(\nabla u, \nabla v) \quad \text{in } \Omega, \\
-\Delta v &= \frac{1}{|u|^{2\alpha_2}} - u^{\beta_2} + g_2(\nabla u, \nabla v) \quad \text{in } \Omega, \\
u, v > 0 &\quad \text{in } \Omega, \\
u = v = 0 &\quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary, \(g_i : \mathbb{R}^{2N} \to [0, +\infty)\) are positive continuous functions belong to \(L^\infty(\mathbb{R}^{2N})\) and \(\alpha_i, \beta_i \in [0, 1)\).

Our approach consists in considering for \(\epsilon > 0\) the approximated system

\[
\begin{aligned}
-\Delta u &= \frac{1}{(|v|^{2+\epsilon})^{\frac{2\alpha_1}{2}}} - v^{\beta_1} + g_1(\nabla u, \nabla v) \quad \text{in } \Omega, \\
-\Delta v &= \frac{1}{(|u|^{2+\epsilon})^{\frac{2\alpha_2}{2}}} - u^{\beta_2} + g_2(\nabla u, \nabla v) \quad \text{in } \Omega, \\
u = v = 0 &\quad \text{on } \partial \Omega.
\end{aligned}
\]

For this class of system it is possible to find sub and supersolution which do not depend of \(\epsilon\). For example, by using the positive eigenfunction associated with the first eigenvalue, we can find easily a subsolution \((u, u)\).

To get the supersolution, we observe that any large constant \(M > 0\) can be used to get a supersolution \((u, v) = (M, M)\) with \(M > \|u\|_\infty\).

Now, let us consider the functions

\[
H_\epsilon(x, s, t, \eta, \xi) = \frac{1}{(|t|^{2+\epsilon})^{\frac{2\alpha_1}{2}}} - t^{\beta_1} + g_1(\eta, \xi)
\]

and

\[
G_\epsilon(x, s, t, \eta, \xi) = \frac{1}{(|s|^{2+\epsilon})^{\frac{2\alpha_2}{2}}} - s^{\beta_2} + g_2(\eta, \xi),
\]

which are well defined in \(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N\).
By using Theorem 2.1, there exists a positive \((u_\epsilon, v_\epsilon)\) verifying system \((S_\epsilon)_-\). Set \(\epsilon = \frac{1}{n}\) with any integer \(n \geq 1\). From now on, we denote by \((u_n, v_n)\) the solution \((u_n, v_n)\). Hence,

\[
(S_n)_- \quad \left\{
\begin{array}{l}
-\Delta u_n = \frac{1}{(|u_n|^2 + \frac{1}{n})^{\frac{3}{2}}} - v_n^\beta_1 + g_1(\nabla u_n, \nabla v_n) \quad \text{in } \Omega, \\
-\Delta v_n = \frac{1}{(|u_n|^2 + \frac{1}{n})^{\frac{3}{2}}} - u_n^\beta_2 + g_2(\nabla u_n, \nabla v_n) \quad \text{in } \Omega, \\
u_n = v_n = 0 \text{ on } \partial \Omega.
\end{array}
\right.
\]

Once that \(\alpha_i, \beta_i \in [0, 1), i = 1, 2\), and \(g_i\) belongs to \(L^\infty(\mathbb{R}^{2N})\), it follows that \((u_n, v_n)\) is bounded in \(H^1_0(\Omega) \times H^1_0(\Omega)\). Indeed, since

\(u_n \geq u > 0\) and \(v_n \geq v > 0\) in \(\Omega\),

we have

\[
\|u_n\|^2 \leq \int_{\Omega} \frac{u_n}{v_n^{\alpha_1}} dx + \int_{\Omega} g_1(\nabla u_n, \nabla v_n) u_n dx
\]

and so,

\[
\|u_n\|^2 \leq \int_{\Omega} \frac{u_n}{v_n^{\alpha_1}} dx + \|g_1\|_\infty \int_{\Omega} |u_n| dx. \quad (3.4)
\]

Similarly we derive

\[
\|v_n\|^2 \leq \int_{\Omega} \frac{v_n}{u_n^{\alpha_2}} dx + \|g_2\|_\infty \int_{\Omega} |v_n| dx. \quad (3.5)
\]

On the other hand, for \(\alpha_i \in [0, 1), i = 1, 2\), we may invoke the Hardy-Sobolev inequality in the form stated in [11, Lemma 2.3] to infer that

\[
\int_{\Omega} \frac{u_n}{x^{\alpha_1}} dx \leq C\|u_n\| \quad \text{and} \quad \int_{\Omega} \frac{v_n}{x^{\alpha_2}} dx \leq C\|v_n\|. \quad (3.6)
\]

Combining (3.4)-(3.6) with Sobolev embedding, it follows that \((u_n, v_n)\) is bounded in \(H^1_0(\Omega) \times H^1_0(\Omega)\). Consequently, we can assume that there is \((u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)\) and \(G_i \in L^2(\Omega)\) verifying

\[
u_n \to u, v_n \to v \text{ in } H^1_0(\Omega),
\]

\[
u_n \to u, v_n \to v \text{ in } L^p(\Omega) \text{ for all } p \in [1, +\infty),
\]
\[ u_n(x) \to u(x), \ v_n(x) \to v(x) \text{ a.e. in } \Omega \quad (3.9) \]

and

\[ g_i(\nabla u_n, \nabla v_n) \to G_i \text{ in } L^2(\Omega). \quad (3.10) \]

Once that

\[ u \leq u_n \leq M \text{ and } v \leq v_n \leq M \text{ a.e in } \Omega, \text{ for all } n \in \mathbb{N}, \]

the limit \((3.9)\) gives

\[ u \leq u \leq M \text{ and } v \leq v \leq M \text{ a.e in } \Omega. \]

Recall that \((S_n)_-\) entails

\[
\begin{align*}
\int \nabla u_n \nabla \varphi \, dx &= \int \left( \frac{1}{(|v_n|^2 + \frac{1}{n})^{n/2}} - v_n^{2^*} + g_1(\nabla u_n, \nabla v_n) \right) \varphi \, dx \\
\int \nabla u_n \nabla \psi \, dx &= \int \left( \frac{1}{(|u_n|^2 + \frac{1}{n})^{n/2}} - u_n^{2^*} + g_2(\nabla u_n, \nabla v_n) \right) \psi \, dx
\end{align*}
\]

\[(3.11)\]

for all \((\varphi, \psi) \in H^1_0(\Omega) \times H^1_0(\Omega)\). Setting \((\varphi, \psi) = (u_n - u, v_n - v)\) in \((3.11)\) yields

\[
\begin{align*}
\int \nabla u_n \nabla (u_n - u) \, dx &= \int \left( \frac{1}{(|v_n|^2 + \frac{1}{n})^{n/2}} - v_n^{2^*} + g_1(\nabla u_n, \nabla v_n) \right) (u_n - u) \, dx \\
\int \nabla v_n \nabla (v_n - v) \, dx &= \int \left( \frac{1}{(|u_n|^2 + \frac{1}{n})^{n/2}} - u_n^{2^*} + g_2(\nabla u_n, \nabla v_n) \right) (v_n - v) \, dx.
\end{align*}
\]

We point out that \((3.7)-(3.10)\) implies that

\[
\lim_{n \to \infty} \int \left( \frac{1}{(|v_n|^2 + \frac{1}{n})^{n/2}} - v_n^{2^*} + g_1(\nabla u_n, \nabla v_n) \right) (u_n - u) \, dx = 0
\]

and

\[
\lim_{n \to \infty} \int \left( \frac{1}{(|u_n|^2 + \frac{1}{n})^{n/2}} - u_n^{2^*} + g_2(\nabla u_n, \nabla v_n) \right) (v_n - v) \, dx = 0.
\]

Consequently,

\[
\lim_{n \to \infty} \int \nabla u_n \nabla (u_n - u) \, dx = \lim_{n \to \infty} \int \nabla v_n \nabla (v_n - v) \, dx = 0
\]
leading to

$$(u_n, v_n) \to (u, v) \text{ in } H^1_0(\Omega) \times H^1_0(\Omega).$$  \hspace{1cm} (3.12)

This way, passing to relabeled subsequences, we have the limits

$$\nabla u_n(x) \to \nabla u(x) \text{ and } \nabla v_n(x) \to \nabla v(x) \text{ a.e in } \Omega,$$

which imply that

$$g_i(\nabla u_n, \nabla v_n) \rightharpoonup g_i(\nabla u, \nabla v) \text{ in } L^2(\Omega) \ (i = 1, 2).$$  \hspace{1cm} (3.13)

Now, from (3.12) and (3.13), we may pass to the limit in (3.11) to conclude that $(u, v)$ is a solution for $(S)_-$. This completes the proof.

4 Existence of solution for system $(S)_+$

In this section, we will study the existence of solution for the following singular elliptic system

$$(S)_+ \begin{cases} -\Delta u = \frac{1}{v^{\alpha_1}} + v^{\beta_1} + g_1(\nabla u, \nabla v) \text{ in } \Omega, \\ -\Delta v = \frac{1}{u^{\alpha_2}} + u^{\beta_2} + g_2(\nabla u, \nabla v) \text{ in } \Omega \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial \Omega, \end{cases}$$

by supposing the same hypotheses of Section 3.

The existence of solution for $(S)_+$ can be obtained by using the same arguments explored in the previous section. The unique difference is in the construction of the supersolution that we will work. Here, the idea is the following:

Fix $R > 0$ large enough such that $\Omega \subset B_R(0)$ and denote by $e$ the unique solution of the problem

$$\begin{cases} -\Delta e = 1, \text{ in } B_R(0) \\ e = 0, \text{ on } \partial B_R(0). \end{cases}$$
Recalling that \( g \in L^\infty \), for \( M > \|\mathbf{u}\|_\infty \) large enough, a simple computation shows that

\[
(S)_+ \begin{cases} 
-\Delta (Me) = M \geq \frac{1}{(Me)^{\alpha_1}} + (Me)^{\beta_1} + g_1(\nabla w_1, \nabla w_2) \text{ in } \Omega, \\
-\Delta (Me) = M \geq \frac{1}{(Me)^{\alpha_2}} + (Me)^{\beta_2} + g_2(\nabla w_1, \nabla w_2) \text{ in } \Omega, \\
(Me) > 0 \text{ in } \Omega,
\end{cases}
\]

for any \( w_1, w_2 \in H^1_0(\Omega) \). Thereby, the pairs \((u, u)\) and \((\mathbf{u}, \mathbf{v}) = (Me, Me)\) satisfy the hypotheses of Theorem 1.1.  

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