The effect of Coulomb interaction upon superconductive proximity effect in disordered metals is studied, employing newly developed Keldysh functional approach. We have calculated subgap Andreev conductance between superconductor and 2D dirty film, as well as Josephson coupling via such a film. Both two qualitatively different Coulomb effects - suppression of the tunneling density of states and disorder-enhanced repulsion in the Cooper channel - are shown to be important at sufficiently low temperatures, \( \ln^2 T \tau \geq \frac{1}{g} \), where \( g = \hbar \sigma / e^2 \) is the dimensionless sheet conductance of the film.

1 Introduction

Electron transport in hybrid superconductive-normal (S-N) systems at low temperatures is governed by the Andreev reflection. Both finite-voltage conductance \( G_A \) between superconductive and normal electrodes, and Josephson critical current \( I_c \) between two superconductive banks, separated by normal region, are determined by the Cooper pair propagation in the normal metal. The theory of Andreev conductance (without Coulomb effects) was developed in e. g. [1], whereas Josephson coupling was calculated (with the account of short-range electron interaction) in [2]. When normal conducting region is made of a dirty metal film, or two-dimensional electron gas with low density of electrons, Coulomb interaction in the normal region may lead to strong quantum fluctuations which suppress both Andreev conductance and Josephson proximity effect.

There are two different quantum effects which play major role in dirty superconductive systems. First of all, critical temperature of a uniformly disordered superconductive film is suppressed by disorder, and eventually vanishes \( T_c = (2\pi)^{-2} \ln^2 \frac{1}{T_0 \tau} \); here \( \tau \) is the elastic scattering time and \( T_0 \) is the BCS transition temperature. In this range of parameters weak-localization and interaction-induced corrections to conductivity of 2D metal are of the relative order of \( g^{-1} \ln \frac{1}{T_0 \tau} \ll 1 \). This fact brings about the issue of a quantum (i.e. \( T = 0 \)) superconductor-metal transition (cf. [3, 4, 5]) as opposed to the usually assumed direct S-I transition. The second effect that is important for hybrid superconductor-normal systems with tunnel barriers at the S/N interface, is the effect of tunneling conductance suppression ("zero-bias anomaly") due to slow relaxation of an extra charge after the tunneling event; as noted in [6], this is physically the same effect as Coulomb blockade of tunneling into a finite system. In a 2D system with a long-range Coulomb interaction this effect scales as \( g^{-1} \ln \frac{1}{T_0 \tau} \). Whereas the first discussed effect is insensitive to the form of long-range Coulomb asymptotics, the zero-bias anomaly does crucially depend on it, and may be suppressed by screening of the Coulomb interaction by nearby external electrodes.

In the present paper we discuss behaviour of the subgap conductance between superconductor and thin dirty metal film, as well as Josephson coupling between two superconductors via such a film, assuming that relevant energy scale \( \hbar \omega = \max(eV, T) \) is such that \( \sqrt{g} \leq \ln \omega \tau \ll g \). The second inequality allows us to neglect weak localization and interaction-induced corrections to the film conductance, whereas the first one calls for a nonperturbative treatment of the zero-bias anomaly (ZBA) and Finkelstein effects. The first of these effects was recently considered in [4], where renormalization of the tunneling DoS on the N side of N/S sandwich was calculated within replica functional method (cf. [5]). We prefer to avoid the use of replicas; instead, we have developed the functional integral approach based on the Keldysh representation of Green functions for dirty superconductors. An advantage of this method is that it allows to calculate nonequilibrium quantities, and does not contain any analytic continuation procedures. Recently
2 Keldysh $\sigma$-model

In the simplest case of spin-independent interactions, and neglecting relativistic effects, we can write down an effective low-energy action for a dirty film of superconductor as:

$$S = \frac{i\pi\nu}{4} \text{Tr} \left[D(\nabla Q)^2 + 4i(\tau_z \partial_\tau + \Phi + \Delta)Q \right] + 2\nu \text{Tr} \bar{\Phi}^T \sigma_x \Phi + \text{Tr} \phi^T \sigma_x \bar{V}_0^{-1} \bar{\phi} + \frac{2\nu}{\lambda} \text{Tr} \Delta^\dagger \sigma_x \Delta$$

$$+ \frac{i\nu}{4} \gamma \text{Tr}_T e^{i(K(t) - K_S(t))\tau_z} Q e^{-i(K(t) - K_S(t))\tau_z} Q_S.$$  

(1)

Here $Q = Q_{t,t'}$ is a $4 \times 4$ matrix in the $K \otimes N$ space, which depends upon two time arguments $t, t'$. Pauli matrices in the Keldysh ($K$) $2 \times 2$ space are denoted by $\sigma_{x,y,z}$, where $\tau_{x,y,z}$ stand for the Nambu ($N$) space. Matrix $Q$ is subject to the usual $\sigma$-model constraint $Q^2 = 1$. Operation $\text{Tr}$ means taking trace over $K \otimes N$ matrix spaces, as well as over time and real spaces. $\nu$ is the bare DoS per one spin projection at the Fermi level, $D$ is the diffusion coefficient, $\phi$ is the (fluctuating) electric potential, $\Delta$ is the BCS order parameter field, $\lambda$ is the interaction constant in the Cooper channel (its negative sign corresponds to attraction). $V_0(q) = 2\pi e^2/q$ is the Fourier component of Coulomb interaction in 2D. We use both vector and tensor (boldfaced) notations for bosonic fields in the Keldysh space, e.g. $\vec{\Delta} = (\Delta_1, \Delta_2)^T$ and $\Delta = \Delta_1 \sigma_0 + \Delta_2 \sigma_1$, where $\Delta_{1,2}$ are certain linear combinations of the $\Delta$ field on the forward and backward branches of the Keldysh time integration path. The field $\vec{\Phi} = \vec{\phi} - \partial_\tau \vec{K}$, where $\vec{K}$ is the phase of the gauge transformation similar to the one proposed in.\cite{20}, in the superconductive case the original matrix field $\dot{Q}$ changes under this gauge transformation according to $\dot{Q}_{t,t'} = e^{iK(t)\tau_z} Q_{t,t'} e^{-iK(t')\tau_z}$. Covariant space derivative $\nabla X \equiv \partial_\tau X + i[\tau_z \partial_\tau, \vec{K}, X]$. The last term in (1), denoted below as $S_S$, describes electron tunneling across the barrier (cf.21). Here $Q_S$ and $K_S$ refer to the S side of the interface boundary $\Gamma$; the notation $\text{Tr}_T$ means that the space integral is taken over the interface surface, $\gamma$ is the (dimensionless) normal-state tunneling conductance per unit area of the boundary. Variation of the total action with respect to $Q$ and $Q_S$ with the constraints $Q^2_S = Q^2_S = 1$ leads (at $K = 0$) to the dynamic Usadel equation,\cite{22} together with the standard\cite{23} boundary conditions. In the absence of the boundary term, the equilibrium saddle-point solution of the Usadel equation is given by $Q = u \cdot \text{diag}(Q^R, Q^A) \cdot u$, where $Q^R(A)$ are retarded (advanced) Green functions, $u = u^{-1} = \sigma_z + \sigma_+ \hat{F}$, and $\hat{F} = f + f_1 \tau_z$ is the generalized fermion distribution function, $\sigma_+ = (\sigma_x + i\sigma_y)/2$. This representation suggests the use of a new variable, $Q = u Qu$. Below we consider low-energy limit $\epsilon \ll |\Delta|$, so the superconductive matrix reduces to purely phase rotations: $Q_S = -i\Delta/|\Delta|$. We will also assume weak tunneling across S/N interface, which makes it possible to start (on the normal side of interface) from purely metallic saddle-point $Q_0 = \sigma_+ \tau_z$. Slow rotations of this trivial solution can be parametrized as $Q = e^{-W/2} \sigma_z \tau_z e^{W/2}$ by the matrix $W$ subject to the constraint $\{W, \sigma_+ \tau_z\} = 0$, which is resolved as $W = \begin{pmatrix} w_x \tau_x + w_y \tau_y & w_0 + w_z \tau_z \\ \bar{w}_0 + \bar{w}_z \tau_z & \bar{w}_x \tau_x + \bar{w}_y \tau_y \end{pmatrix}_K$. The diagonal (off-diagonal) in the Nambu space excitations, $w_i$ and $\bar{w}_i$ with $i = 0, z (i = x, y)$, correspond to diffusion (Cooper) modes in the metal. Following\cite{23} we choose $K$ to be a linear functional of $\phi$ and require the vanishing of the term bilinear in $W$ and $\vec{\Phi}$ in the $\sigma$-model action (1).

The term in the action, which is responsible for Andreev subgap conduction, comes from the averaging of $S_A^2$ over Cooperon modes: $\bar{S}_A = \frac{i\epsilon}{\pi} G_A \text{Tr} (Q_S \sigma_+ \tau_z)^2$, where $G_A$ is the (dimensionless) Andreev conductance. At low energies ($\omega, T, eV \ll E_{th} = \hbar D / L^2$, $L$ being the relevant length scale of the N region, and in the absence of quantum corrections, the known result:\cite{11} $G_A^{(0)} = G_T^2 R_D$.
is recovered, where \( R_D \) is the total resistance of the diffusive metal region and \( G_T = \gamma A \) is the total normal-state tunneling conductance, \( A \) being the area of the junction. In the geometry of rectangular N/S contact (normal metal film with the length \( L_y \) along the boundary with SC, and the length \( L_x \) between SC and normal reservoir), \( G_A^{(0)} = \frac{G^2}{y} \frac{L_x}{L_y} \). At higher \( \omega \) and/or \( T \), the length \( L_x \) should be replaced by \( L_{\text{eff}}(\omega,T) \), which is equal to \( L_{\text{eff}}(\omega) = \sqrt{2D/|\omega|} \) at \( \hbar \omega \gg T \), and to \( L_{\text{eff}}(T) = 0.95\sqrt{D/2T} \) in the opposite limit.

### 3 Renormalization group

Quantum fluctuations lead to renormalization of the Cooper-channel interaction constant \( \lambda \) in the N metal and of the barrier transparency \( \gamma \); note that the sheet conductance \( g = 2\nu D \) is constant within our approximation. Renormalization of \( \lambda \) comes about in the second order over the interaction term \( S_{\phi,Q} = -\pi \nu \text{Tr} [u \Phi u Q] \), which should be averaged over diffusion/Cooperon modes of the \( Q \) matrix field and electric potential fluctuations \( \phi \). This calculation can be done in the standard gauge with \( K = 0 \) (cf.\cite{2} for detailed discussion), the result coincides with the one known from\cite{2}: \[
\frac{d\lambda}{d\zeta} = \frac{1}{4\pi^2 g} - \lambda^2;
\zeta = \ln \frac{1}{\omega \tau},
\tag{2}
\]
where \( \omega \) is the running low-frequency cutoff of RG procedure, which stops eventually at \( \omega = \max(T, eV, E_{\text{th}}) \). The last term in \( (2) \) is the usual BCS ladder contribution, which can also be considered within RG approach, as coming from the second order term over \( S_{\Delta,Q} = -\pi \nu \text{Tr} [u \Delta u Q] \). Eq. \( (2) \) has a locally stable infrared fixed point \( \lambda_0 = 1/2\pi \sqrt{g} \). Integration of Eq. \( (2) \) with initial condition \( \lambda_0 < 0 \) leads to Finkelstein’s result for the \( T_c \) suppression: \( T_c \tau = (1 - \lambda_0 \ln(1/T_c \tau))^{1/2 \lambda_0} \).

There are two sources of renormalization corrections to \( \gamma \). The first one is due to Cooper-channel interaction; to find it one should average over Cooperon modes the product \( S_{\gamma} S_{\lambda} \), where \( S_{\lambda} = (1/4)\pi^2 \nu \lambda \int dtd^2r \text{Tr} \sigma_z [Q^2 - (\tau_z Q_{\mu})^2] \) is the result of Gaussian integration over \( \Delta \) field in the action \( (1) \). The second contribution to \( \gamma \) is due to fluctuations of the Coulomb-induced phase \( K(t) \), describing the ZBA effect; here we neglect similar fluctuations of the phase \( K_S(t) \), assuming that voltage at the superconductor side is fixed by external circuit, and SC diffusion constant \( D_S \gg D \) (for the opposite situation cf.\cite{1}). Mean-squared fluctuations of the field \( K(t) \) can be expressed via the function \( \rho(\omega) \) which generalizes a notion of "environmental impedance" introduced in\cite{3} within phenomenological approach to the Coulomb suppression of Andreev conductance. We calculate \( \rho(\omega) \) microscopically\cite{2} for all considered geometries, see below. The resulting RG equation for \( \gamma \) reads \[
\frac{d\gamma}{d\zeta} = -\gamma \left( \lambda + \frac{2}{\pi} \rho \right) \quad \text{at} \quad \zeta > \zeta_\Delta = \ln \frac{1}{\Delta \tau}.
\tag{3}
\]
At high frequencies \( \omega \geq \Delta \) one has instead \( d\gamma/d\zeta = -\gamma \rho / \pi \), as it should be for the tunneling in the absence of superconductivity.\cite{1} Eqs. \( (2), (3) \) are sufficient to study low-energy behaviour of the Andreev conductance in a rectangular geometry. Another interesting problem is the subgap conductance between small SC island of size \( d \) and large metal film (cf.\cite{3}), when the "diffusive" resistance \( R_D \) logarithmically depends on the relevant space scale. As a result, the Andreev conductance becomes itself subject to the RG transformation: \[
\frac{dG_A}{d\zeta} = A^2 \frac{\gamma^2}{4\pi g} - \frac{4}{\pi} \rho G_A \quad \text{at} \quad \zeta > \zeta_\Delta,
\tag{4}
\]
where we assume, for simplicity, \( \zeta_\Delta \approx \zeta_d = \ln \frac{\rho^2}{D_\tau} \).
4 Andreev conductance

We start from the case of rectangular geometry of the contact, when it suffices to find renormalized value $\gamma_R$ by integration of Eqs. (3), (4), so that $G_A = G_A^{(0)}(\gamma_R/\gamma_0)^2$. In the practically interesting (at $\omega \lesssim \Delta$) limit $\omega \gg \Delta$ we find $\rho(\omega) = \frac{1}{\pi g} \ln \frac{L}{L_{\text{eff}}(\omega)}$ (we assumed here $L = L_x \sim L_y$). The result for the Andreev conductance reads

$$G_A(\omega) = \frac{G_{T\Delta}^2}{gL_y} \frac{L_{\text{eff}}(\omega, T)}{L_y} \frac{4(\omega/\Delta)^{2\lambda_n}}{[1 + \lambda_n/\lambda_g + (1 - \lambda_n/\lambda_g)(\omega/\Delta)^{2\lambda_g}]^2} \cdot \exp \left( -\frac{1}{\pi^2 g} \ln \frac{\Delta}{\omega} \cdot \ln \frac{\omega\Delta}{E_{\text{th}}^2} \right) \quad (5)$$

where $\lambda_n$ is the Cooper-channel repulsion constant at the energy scale $\hbar \tau^{-1}$, $G_{T\Delta}$ is the normal-state tunneling conductance at $eV \approx \Delta$. The last multiplier in the r.h.s. of Eq. (4) describes the ZBA effect, with the doubled coefficient in front of the log accounting for the fact that in this geometry extra charge spreads (after tunneling) over half-plane. Eq. (5) is valid at the ZBA effect, with the doubled coefficient in front of the log accounting for the fact that in this geometry extra charge spreads (after tunneling) over half-plane. Eq. (5) is valid at $\omega \geq E_{\text{th}} = \frac{\hbar D}{2}$. In the opposite limit the result (4) can be used after replacements $\omega \rightarrow \frac{D}{\pi^2}$, $L_{\text{eff}}(\omega) \rightarrow L_x$.

Next we consider geometry of small SC island siting in the middle of the N thin film of characteristic size $L$. Now we need to integrate the whole set of Eqs. (2)–(4). We assume, for simplicity, that the island size $d \sim \xi_\Delta = \sqrt{D/\Delta}$. Thus at the energy scale $\omega \ll \Delta$ extra charge spreads symmetrically over 2D plane and $\rho(\omega) = \frac{1}{\pi g} \ln L_{\text{eff}}(\omega)$. The result reads (assuming again $\omega \gg \Delta$)

$$G_A(\omega) = \frac{G_{T\Delta}^2}{4\pi g} \frac{1 - (\omega/\Delta)^{2\lambda_g}}{(\lambda_g + \lambda_n) + (\lambda_g - \lambda_n)(\omega/\Delta)^{2\lambda_g}} \cdot \exp \left( -\frac{1}{2\pi^2 g} \ln \frac{\Delta}{\omega} \cdot \ln \frac{\omega\Delta}{E_{\text{th}}^2} \right) \quad (6)$$

At lowest energies $\omega$ should be replaced by $E_{\text{th}}$ in Eq. (5). The whole effect of quantum fluctuations in this geometry is that initial “semiclassical” logarithmic growth of $G_A$ with $\omega$ decrease, $G_A \propto \ln \frac{D}{\omega^2}$, crosses over to the “log-normal” law.

Note that in both cases (4), (5) the ZBA effect is represented as a separate multiplicative factor; the reason is that fluctuations responsible for the ZBA have frequencies $\omega \gg Dq^2$, and thus they are not mixed with the low-$\omega$ fluctuations responsible for the $\lambda$ renormalization. Detailed dependences of the ZBA factor on $V$ and $T$ can be found using formulae from (2) with $\rho(\omega)$ employed as the effective impedance. The power-law factors due to Finkelstein’s renormalizations are determined always by the largest of the scales $eV, T, E_{\text{th}}$.

5 Josephson proximity coupling

The term in the effective action, which is responsible for the Josephson proximity coupling, can be written in the form $S_J = \frac{1}{2} E_J \text{Tr}[Q_S^{(1)} Q_S^{(2)} \sigma_z]$, where superscripts (1) and (2) refer to two superconductive banks or islands. Using low-energy representation for $Q_S$, and neglecting phase factors $\exp(iK(t)\tau_z)$, one finds that calculation of the Josephson current with the use of $S_J$ produces standard expression $I_J = \frac{2e}{h} E_J \sin(\theta_1 - \theta_2)$. The contribution of $K(t)$ fluctuations factorizes as above, and is taken care of by the same multiplicative factor as in Eqs. (5), (6), with $\omega$ replaced by an appropriate inverse diffusion time. We start from the calculation of $E_J$ in the absence of the ZBA. For both geometries discussed above, $E_J$ can be expressed via the same function defined in the Fourier space, the zero-frequency Cooperon amplitude $J(q)$, which obeys the RG equation $\frac{\partial J(q)}{\partial q} = A^2 \frac{\partial J(q)}{\partial q}$, where $\zeta_q = \ln(\Delta/Dq^2)$. First we consider an example of two small SC islands of radius $d$, separated by the distance $R \gg d$. In this case $E_J$, which is given by the inverse Fourier transform of $J(q)$, can be represented as $E_J(R) = \frac{1}{\pi R^2} \frac{\partial J(\zeta_q)}{\partial \zeta_q} \big|_{\zeta_q = \ln(Dq^2)}$.
Using renormalized $\gamma(\zeta_d)$ according to Eqs. (2,3), and adding the ZBA factor from (3) estimated at $\omega = D/R^2$, we find (with the total size $L$ of the film being in the range $R \ll L \ll 4 \pi \nu e^2 R^2$):

$$E_J(R) = \frac{\lambda_g^2 G^2_{T\Delta}}{2\pi \nu(\lambda_n + \lambda_g)^2} \frac{1}{R^2} \left( \frac{\xi_\Delta}{R} \right)^{4\lambda_g} \frac{1}{[1 + \beta(\xi_\Delta/R)^{4\lambda_g}]} \exp \left( -\frac{2}{\pi^2 g} \ln \frac{R}{\xi_\Delta} \ln \frac{L^2}{\xi_\Delta} \right)$$

(7)

where $\beta = \frac{\lambda_n - \lambda_g}{\lambda_n + \lambda_g}$. The increase of the power-law exponent up to $x_J = 2 + 4 \lambda_g$ is due to Finkelstein’s corrections, whereas additional log-normal decay factor is due to the ZBA.

In the case of the rectangular geometry the discrete nature of diffusion modes in the N region should be taken into account, while calculating $E_J$. The result (for $L_x \sim L_y$) is:

$$E_J(L_x, L_y) = \frac{\lambda_g G^2_{T\Delta}}{4\nu(\lambda_n + \lambda_g)^2} \frac{1}{L_x L_y} \left( \frac{\xi_\Delta}{L_x} \right)^{4\lambda_g} \frac{1}{1 + \beta(\xi_\Delta/L_x)^{4\lambda_g}} \exp \left( -\frac{4}{\pi^2 g} \ln^2 \frac{L_x}{\xi_\Delta} \right)$$

(8)

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