Asymptotic normality of an estimator of kernel-based conditional mean dependence measure

Terence Kevin MANFOUMBI DJONGUET and Guy Martial NKIET

URMI, Université des Sciences et Techniques de Masuku, Franceville, Gabon.

E-mail addresses: tkmpro95@gmail.com, guymartial.nkiet@univ-masuku.org.

Abstract. We propose an estimator of the kernel-based conditional mean dependence measure obtained from an appropriate modification of a naive estimator based on usual empirical estimators. We then get asymptotic normality of this estimator both under conditional mean independence hypothesis and under the alternative hypothesis. A new test for conditional mean independence of random variables valued into Hilbert spaces is then introduced.

AMS 1991 subject classifications: 62E20, 46E22.
Key words: Asymptotic normality; Kernel method; Kernel-based conditional dependence; Reproducing kernel Hilbert space; Functional data analysis.

1 Introduction

Conditional mean dependence is a statistical property that is important to evaluate for given random variables. Indeed, many regression analysis problems consist in modeling conditional mean of a response variable $Y$ given a predictor variable $X$ using either linear models or nonparametric models. Such modeling approaches are in fact not relevant in case of conditional mean independence of the involved variables. That is why testing whether the predictor has a contribution to the mean of the response is of a great interest. However, there exists just a few works dealing with the problem of testing for conditional mean independence between random variables. It was investigated in Shao and Zhang (2014) by using the so-called martingale
difference divergence (MDD) for $Y \in \mathbb{R}$ and $X \in \mathbb{R}^q$. Later, a generalization of MDD was introduced in Park et al. (2015) in order to deal with the case of $Y \in \mathbb{R}^p$ and $X \in \mathbb{R}^q$, and Lee et al. (2020) proposed functional martingale difference divergence (FMDD) which extended MDD to the case where $X$ and $Y$ are functional variables. The case of high-dimensional setting was tackled in Zhang et al. (2018). Recently, Lai et al. (2021) introduced the kernel-based conditional mean dependence measure (KCMD) by means of which a test for conditional mean independence was constructed. This test is based on an unbiased estimator of KCMD which has the form of a U-statistic with asymptotic distribution under null hypothesis equal to an infinite sum of distributions. This last property is a drawback that forced Lai et al. (2021) to resort to a wild bootstrap method for performing the test. Faced with a similar problem with a maximal mean discrepancy (MMD) estimator, Magikusa and Naito (2020) adopted an approach permitting to obtain asymptotic normality for a proposed estimator both under the null hypothesis and under the alternative. This approach was also used later in Balogoun et al. (2021) for the case of generalized maximal mean discrepancy (GMMD). In this paper we tackle this approach consisting in making an appropriate modification on a naive estimator of KCMD. We then obtain asymptotic normality for the resulting estimator under the conditional mean independence hypothesis. This allows to propose a test for conditional mean independence of random variables with values into Hilbert spaces and that can, therefore, be used on functional data. The rest of the paper is organized as follows. The KCMD is recalled in Section 2, and Section 3 is devoted to its estimation by a modification of the naive estimator, and to the main results. All the proofs are postponed in Section 4.

2 KCMD and conditional mean independence

Let $X$ and $Y$ be two random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and taking values in separable Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively; it is assumed that $\mathbb{E}(\|Y\|_Z^2) < +\infty$, where $\| \cdot \|_Z$ denotes the norm associated with the inner product $\langle \cdot, \cdot \rangle_Z$ of the Hilbert space $\mathcal{Z}$. In order to test for conditional mean independence, that is testing for the hypothesis

$$\mathcal{H}_0 : \mathbb{E}(Y|X) = \mathbb{E}(Y) \text{ almost surely}$$
versus
\[ H_1 : P(\mathbb{E}(Y|X) = \mathbb{E}(Y)) < 1, \]
where \( \mathbb{E}(Y|X) \) denotes conditional expectation, Lai et al. (2021) introduced the Kernel Conditional Mean Independence measure (KCMD). Let us consider a reproducing kernel Hilbert space \( \mathcal{H} \) of functions from \( X \) to \( \mathbb{R} \) with associated kernel \( K : \mathcal{X}^2 \to \mathbb{R} \) which is a symmetric function such that, for any \( f \in \mathcal{H} \) and any \( x \in \mathcal{X} \), one has \( K(x, \cdot) \in \mathcal{H} \) and \( f(x) = \langle K(x, \cdot), f \rangle_\mathcal{H} \) (see Berlinet and Thomas-Agnan (2004)). Throughout this paper, we assume that \( K \) satisfies the following condition:

\[ (C_1) : \|K\|_\infty := \sup_{(x,y) \in \mathcal{X}^2} K(x, y) < +\infty; \]

then the kernel mean embedding \( m_X := \mathbb{E}(K(X, \cdot)) \) exists. KCMD is the measure given by

\[ \text{KCMD}(Y, X) = \left\| \mathbb{E}(Y \otimes K(X, \cdot)) - \mu \otimes m_X \right\|_{\text{HS}}^2, \quad (1) \]

where \( \mu = \mathbb{E}(Y) \), the tensor product \( \otimes \) is such that, for any \( (y, f) \in \mathcal{Y} \times \mathcal{H} \), \( y \otimes f \) is the linear operator defined by \( (y \otimes f)(t) = \langle y, t \rangle_\mathcal{Y} f \) for any \( t \in \mathcal{Y} \), and \( \|\cdot\|_{\text{HS}} \) denotes the Hilbert-Schmidt norm of operators. As demonstrated in Lai et al. (2021), when the kernel \( K \) is characteristic, then the null hypothesis \( H_0 \) holds if, and only if, KCMD\((Y, X) = 0\). So, a test for conditional mean independence can be achieved by using an estimator of KCMD\((Y, X) = 0\) as test statistic. An unbiased estimator, based on a i.i.d. sample \{\((X_i, Y_i)\)\}_{1 \leq i \leq n} of \((X, Y)\), was defined in Lai et al. (2021) as:

\[ \text{KCMD}_n(Y, X) = \frac{1}{n(n-3)} \sum_{i \neq j} C_{ij} D_{ij}, \]

where

\[
\begin{align*}
    c_{ij} &= \begin{cases} 
        K(X_i, Y_j) & \text{if } i \neq j \\
        0 & \text{if } i = j 
    \end{cases}, \\
    d_{ij} &= \begin{cases} 
        \langle Y_i, Y_j \rangle_\mathcal{Y} & \text{if } i \neq j \\
        0 & \text{if } i = j 
    \end{cases}, \\
    c_i &= \frac{1}{n-2} \sum_{j=1}^{n} c_{ij}, \quad c_j = \frac{1}{n-2} \sum_{i=1}^{n} c_{ij}, \quad c_\cdot = \frac{1}{(n-1)(n-2)} \sum_{i,j=1}^{n} c_{ij},
\end{align*}
\]

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They derived the asymptotic distribution under null hypothesis of this statistic and proved that, under $\mathcal{H}_0$, $n\text{KCMD}_n(Y, X)$ converges in distribution, as $n \to +\infty$, to $\sum_{m=1}^{+\infty} \gamma_m (N_m^2 - 1)$, where $N_m$s are i.i.d. standard normal distributed random variables and $(\gamma)_m \geq 1$ is a sequence of eigenvalues of a suitable positive autoadjoint operator. This limiting distribution cannot be used to compute critical values for performing the test since the $\gamma_m$s are unknown, and since it is an infinite sum of distributions. That is why Lai et al. (2021) proposed a wild bootstrap procedure to approximate the asymptotic null distribution. As one knows, bootstrap procedures have the disadvantage of leading to rather high computation times, that is why it is preferable to obtain asymptotic normality of the test statistic. Following an approach introduced in Magikusa and Naito (2020) and also tackled in Balogoun et al. (2021), we propose in this paper to modify a naive estimator of KCMD($Y, X$) in order to get asymptotic normality under $\mathcal{H}_0$ and to use this result for performing the test.

3 Modification of KCMD and asymptotic normality

Replacing each expectation in (1) by its empirical counterpart leads to the simple estimator of KCMD given by

$$\hat{\text{KCMD}}_n = \left\| \frac{1}{n} \sum_{i=1}^{n} Y_i \otimes K(X_i, \cdot) - \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right) \otimes \left( \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot) \right) \right\|_{\text{HS}}^2$$
and which can be expanded as

\[
\widehat{\text{KCMD}}_n = \frac{1}{n^2} \sum_{i,j=1}^{n} \langle Y_i, Y_j \rangle_y K(X_i, X_j) + \frac{1}{n^4} \sum_{i,j,q,r}^{n} \langle Y_q, Y_r \rangle_y K(X_i, X_j) \\
- \frac{2}{n^3} \sum_{i,j,q=1}^{n} \langle Y_i, Y_q \rangle_y K(X_i, X_j) + 1
\]

by using properties of \( \otimes \) and reproducing property of \( K \). We propose another estimator of \( \text{KCMD}(Y, X) \) obtained from a modification of \( \widehat{\text{KCMD}}_n \). This modification just consists to introduce a weight in the cross-product term of (2). Let \( \{w_{i,n}(\gamma)\}_{1 \leq i \leq n} \) be a triangular array of positive real numbers depending on a parameter \( \gamma \in ]0, 1[ \). We consider the estimator \( \widehat{K}_{n,\gamma} \) of \( \text{KCMD}(Y, X) \) given by:

\[
\widehat{K}_{n,\gamma} = \frac{1}{n^2} \sum_{i,j=1}^{n} \langle Y_i, Y_j \rangle_y K(X_i, X_j) + \frac{1}{n^4} \sum_{i,j,q,r}^{n} \langle Y_q, Y_r \rangle_y K(X_i, X_j) \\
- \frac{2}{n^3} \sum_{i,j,q=1}^{n} w_{i,n}(\gamma) \langle Y_i, Y_q \rangle_y K(X_i, X_j),
\]

and we take it as test statistic. For obtaining its asymptotic normality, we suppose that the sequence of weights that is used satisfy the following conditions:

\((C_2)\) : There exists a strictly positive real number \( \tau \) and an integer \( n_0 \) such that for all \( n > n_0 \):

\[
n \left| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) - 1 \right| \leq \tau.
\]

\((C_3)\) : There exists \( C > 0 \) such that \( \max_{1 \leq i \leq n} w_{i,n}(\gamma) < C \) for all \( n \in \mathbb{N}^* \) and \( \gamma \in ]0, 1[ \).

\((C_4)\) : For any \( \gamma \in ]0, 1[ \), \( \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} w_{i,n}^2(\gamma) = w^2(\gamma) > 1 \).

Such sequence was first introduced in [1] where an example defined as \( w_{i,n}(\gamma) = 1 + (-1)^i \gamma \) was given. For this example, one has \( C = 2, w^2(\gamma) = 1 + \gamma^2 \) and \( \tau \) is any positive real number. Another example is \( w_{i,n}(\gamma) = 1 + \sin(i\pi\gamma) \) which corresponds to \( \tau = 1/|\sin(\pi\gamma/2)| \), \( C = 2 \) and \( w^2(\gamma) = 3/2 \). Putting
\[ \eta = \mathbb{E}(Y \otimes K(X, \cdot)) \] and \( \nu = \mu \otimes m_X \), and considering the functions \( \mathcal{U} \) and \( \mathcal{V} \) from \( \mathcal{X} \times \mathcal{Y} \) to \( \mathbb{R} \) defined as
\[
\mathcal{U}(x, y) = \langle y \otimes K(x, \cdot) - \eta, \eta \rangle_{\text{HS}} + \langle y \otimes m_X + \mu \otimes K(x, \cdot) - 2\nu, \nu - \eta \rangle_{\text{HS}},
\]
\[
\mathcal{V}(x, y) = \langle y \otimes K(x, \cdot) - \eta, \nu \rangle_{\text{HS}},
\]
where \( \langle \cdot, \cdot \rangle_{\text{HS}} \) denotes the Hilbert-Smidt inner product, we have:

**Theorem 1** Assume conditions (C1) to (C4). Then
\[
\sqrt{n} \left( \hat{K}_{n, \gamma} - \text{KCMD}(Y, X) \right) \overset{p}{\longrightarrow} \mathcal{N}(0, \sigma^2_\gamma),
\]
as \( n \to +\infty \), where
\[
\sigma^2_\gamma = 4 \text{Var} (\mathcal{U}(X_1, Y_1)) + 4w(\gamma) \text{Var} (\mathcal{V}(X_1, Y_1)) - 8 \text{Cov} (\mathcal{U}(X_1, Y_1), \mathcal{V}(X_1, Y_1)).
\]

This theorem gives asymptotic normality both under \( \mathcal{H}_0 \) and under \( \mathcal{H}_1 \). Under \( \mathcal{H}_0 \), we have \( \text{KCMD}(Y, X) = 0 \), which is equivalent to \( \eta = \nu \) and implies that \( \mathcal{U} = \mathcal{V} \); then, \( \sqrt{n} \hat{K}_{n, \gamma} \overset{p}{\longrightarrow} \mathcal{N}(0, \sigma^2_\gamma) \), as \( n \to +\infty \), with \( \sigma^2_\gamma = 4 (w(\gamma)^2 - 1) \text{Var} (\langle Y_1 \otimes K(X_1, \cdot), \eta \rangle_{\text{HS}}) \). This variance is unknown since its depends on \( \nu \). So, for performing the test we have to estimate it. We consider the estimator
\[
\hat{\sigma}^2_\gamma = 4(w(\gamma)^2 - 1)\hat{\alpha}^2,
\]
where
\[
\hat{\alpha}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \langle Y_i, Y_j \rangle_\gamma K(X_i, X_j) - \frac{1}{n^2} \sum_{m=1}^{n} \sum_{p=1}^{n} \langle Y_m, Y_p \rangle_\gamma K(X_m, X_p) \right)^2,
\]
and we have:

**Theorem 2** Assume conditions (C1) to (C4). Then, under \( \mathcal{H}_0 \),
\[
\sqrt{n} \frac{\hat{K}_{n, \gamma}}{\hat{\sigma}_\gamma} \overset{p}{\longrightarrow} \mathcal{N}(0, 1),
\]
as \( n \to +\infty \).
This theorem allows to achieve the test in practice. The null hypothesis $H_0$ is to be rejected when \( \sqrt{n\widehat{K}_{n,\gamma}} > \widehat{\sigma}_n \Phi^{-1}(1 - \alpha) \), where $\alpha$ is the chosen significance level and $\Phi$ is the cumulative distribution function of the standard normal distribution.

**Remark 1.** This test can be applied on functional data corresponding, for instance, to the case where the $X_i$s and the $Y_i$s are random functions belonging in $L^2([0,1])$ and observed on points $t_1, \cdots, t_r$ and $s_1, \cdots, s_q$, respectively, of fine grids in $[0,1]$ such that $t_1 = s_1 = 0$ and $t_r = s_q = 1$. In this case, one has

\[
\langle Y_i, Y_j \rangle_{\mathcal{Y}} = \int_0^1 Y_i(s) Y_j(s) \, ds,
\]

what can be approximated by using trapezoidal rule so as to obtain

\[
\langle Y_i, Y_j \rangle_{\mathcal{Y}} \simeq \sum_{m=1}^{q-1} \frac{s_{m+1} - s_m}{2} (Y_i(s_m) Y_j(s_m) + Y_i(s_m+1) Y_j(s_m+1)).
\]  

If the gaussian kernel is used, one has

\[
K(X_i, X_j) = \exp \left( -\omega^2 \| X_i - X_j \|^2_X \right) = \exp \left( -\omega^2 \int_0^1 (X_i(t) - X_j(t))^2 \, dt \right),
\]

where $\omega > 0$, and this term can also be approximated by using trapezoidal rule:

\[
K(X_i, X_j) \simeq \exp \left( -\omega^2 \sum_{m=1}^{r-1} \frac{t_{m+1} - t_m}{2} \left( (X_i(t_m) - X_j(t_m))^2 + (X_i(t_{m+1}) - X_j(t_{m+1}))^2 \right) \right).
\]  

Then, $\widehat{K}_{n,\gamma}$ and $\widehat{\alpha}^2$ are to be computed by using (3) and (4).
4 Proofs

4.1 Proof of Theorem 1

Putting $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$, $\bar{K}_n = \frac{1}{n} \sum_{i=1}^n K(X_i, \cdot)$ and $\overline{Y \otimes K^n} = \frac{1}{n} \sum_{i=1}^n Y_i \otimes K(X_i, \cdot)$, we have

$$\sqrt{n} \left( \bar{K}_{n, \gamma} - \text{CMD}(Y, X) \right)$$

$$= \sqrt{n} \left[ \left\| \overline{Y \otimes K^n} - \eta \right\|_{\text{HS}}^2 + 2 \left\langle \overline{Y \otimes K^n}, \eta \right\rangle_{\text{HS}} - \left\| \eta \right\|_{\text{HS}}^2 + \left\| \bar{Y}_n \otimes \bar{K}_n - \nu \right\|_{\text{HS}}^2 + 2 \left\langle \bar{Y}_n \otimes \bar{K}_n, \nu \right\rangle_{\text{HS}} ight. - \| \nu \|_{\text{HS}}^2 - \frac{2}{n} \sum_{i=1}^n (w_{i,n}(\gamma) - 1) \left\langle Y_i \otimes K(X_i, \cdot), \bar{Y}_n \otimes \bar{K}_n - \nu \right\rangle_{\text{HS}} \right.$$

$$- \frac{2}{n} \sum_{i=1}^n w_{i,n}(\gamma) \left\langle Y_i \otimes K(X_i, \cdot), \nu \right\rangle_{\text{HS}}$$

$$- \| \eta - \nu \|_{\text{HS}}^2 - 2 \left\langle \overline{Y \otimes K^n} - \eta; \overline{Y}_n \otimes \overline{K}_n - \nu \right\rangle_{\text{HS}} - 2 \left\langle \bar{Y}_n \otimes \bar{K}_n, \eta \right\rangle_{\text{HS}}$$

$$- \frac{2}{n} \sum_{i=1}^n (w_{i,n}(\gamma) - 1) \left\langle \eta, \nu \right\rangle_{\text{HS}} + \frac{2}{n} \sum_{i=1}^n w_{i,n}(\gamma) \left\langle \eta, \nu \right\rangle_{\text{HS}} \right]$$

$$= A_n + B_n + C_n + D_n,$$

where

$$A_n = n^{-1/2} \left( \left\| \sqrt{n}(Y \otimes K^n) - \eta \right\|_{\text{HS}}^2 + \left\| \sqrt{n}(\bar{Y}_n \otimes \bar{K}_n - \nu) \right\|_{\text{HS}}^2 \right)$$

$$= n^{-1/2} \left( \left\| \sqrt{n}(Y \otimes K^n) - \eta \right\|_{\text{HS}}^2 + \left\| \sqrt{n}(\bar{Y}_n \otimes \bar{K}_n - \nu) \right\|_{\text{HS}}^2 \right)$$

$$+ \left( \sqrt{n}(\bar{Y}_n - \mu) \otimes m_X + \mu \otimes \left( \sqrt{n}(\bar{K}_n - m_X) \right) \right)$$

$$B_n = -2 \left( \left\langle \frac{1}{n} \sum_{i=1}^n (w_{i,n}(\gamma) - 1) Y_i \otimes K(X_i, \cdot), \sqrt{n}(\bar{Y}_n \otimes \bar{K}_n - \nu) \right\rangle_{\text{HS}} \right.$$

$$+ \left. \frac{1}{n} \sum_{i=1}^n (w_{i,n}(\gamma) - 1) \left\langle \eta, \nu \right\rangle_{\text{HS}} \right),$$

$$C_n = -2 \left\langle \sqrt{n}(Y \otimes K^n) - \eta), \bar{Y}_n \otimes \bar{K}_n - \nu \right\rangle_{\text{HS}} + 2 \left\langle \sqrt{n}(\bar{Y}_n - \mu) \otimes (\bar{K}_n - m_X), \nu - \eta \right\rangle_{\text{HS}},$$

$$D_n = \frac{1}{n} \sum_{i=1}^n (w_{i,n}(\gamma) - 1) \left\langle \eta, \nu \right\rangle_{\text{HS}}.$$
\[ D_n = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \mathcal{U}(X_i, Y_i) - w_{i,n}(\gamma) \mathcal{V}(X_i, Y_i). \]

The central limit theorem ensures that \( \sqrt{n}(\overline{Y} \otimes \overline{K}^n - \eta), \sqrt{n}(\overline{Y}_n - \mu) \) and \( \sqrt{n}(\overline{K}_n - m_X) \) converge in distribution to random variables having normal distributions as \( n \to +\infty \). Moreover, by the law of large numbers \( \overline{K}_n - m_X \) converges in probability to 0 as \( n \to +\infty \). Then, by the continuous mapping theorem we deduce that \( A_n = o_p(1) \). Concerning \( B_n \), we get by the Cauchy-Schwarz inequality

\[
|B_n| \leq 2 \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) Y_i \otimes K(X_i, \cdot) \right\|_{\text{HS}} \left( \left\| \sqrt{n}(\overline{Y}_n \otimes \overline{K}_n - \nu) \right\|_{\text{HS}} \right)
+ \sqrt{n} \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \right\|_{\text{HS}} \left\| \nu \right\|_{\text{HS}} \right];
\]

and since \( \sqrt{n}(\overline{Y}_n \otimes \overline{K}_n - \nu) = \sqrt{n}(\overline{Y}_n - \mu) \otimes \overline{K}_n + \mu \otimes \sqrt{n}(\overline{K}_n - m_X) \) and, under \((\mathcal{C}2)\) for \( n \) large enough, \( n \left| \sum_{i=1}^{n} w_{i,n}(\gamma) - 1 \right| \leq \tau \), it follows

\[
|B_n| \leq 2 \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) Y_i \otimes K(X_i, \cdot) \right\|_{\text{HS}} \left( \left\| \sqrt{n}(\overline{Y}_n - \mu) \otimes \overline{K}_n \right\|_{\text{HS}} \right)
+ \left\| \mu \otimes \sqrt{n}(\overline{K}_n - m_X) \right\|_{\text{HS}} \right] + \frac{\tau}{\sqrt{n}} \left\| \eta \right\|_{\text{HS}} \left\| \nu \right\|_{\text{HS}} \right];
\]

\[
= 2 \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) Y_i \otimes K(X_i, \cdot) \right\|_{\text{HS}} \left( \left\| \sqrt{n}(\overline{Y}_n - \mu) \right\|_{\text{HS}} \left\| \overline{K}_n \right\|_{\text{H}} \right)
+ \left\| \mu \right\|_{\text{Y}} \left\| \sqrt{n}(\overline{K}_n - m_X) \right\|_{\text{H}} \right] + \frac{\tau}{\sqrt{n}} \left\| \eta \right\|_{\text{HS}} \left\| \nu \right\|_{\text{HS}} \right].
\]

By the reproducing property we obtain

\[
\left\| \overline{K}_n \right\|_{\text{H}} \leq \frac{1}{n} \sum_{i=1}^{n} \left\| K(X_i, \cdot) \right\|_{\text{H}_X} \leq \left\| K \right\|_{\text{H}}^{1/2};
\]
hence

$$|B_n| \leq 2 \left[ \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) Y_i \otimes K(X_i, \cdot) \right] \left[ \left\| \sqrt{n}(\bar{Y}_n - \mu) \right\|_{HS} \left\| \sqrt{n}(\bar{K}_n - m_X) \right\|_{HS} + \left\| \mu \right\|_Y \left\| \sqrt{n}(\bar{K}_n - m_X) \right\|_{HS} \right] + \frac{\tau}{\sqrt{n}} \left\| \nu \right\|_{HS} \left\| \eta \right\|_{HS}. $$

From Lemma 1 in Manfoumbi Djonguet et al. (2022) we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) Y_i \otimes K(X_i, \cdot) \right\|_{HS} = o_p(1),$$

and from the central limit theorem $\sqrt{n}(\bar{K}_n - m_X)$ and $\sqrt{n}(\bar{Y}_n - \mu)$ converge in distribution as $n \to +\infty$. We then deduce from the preceding inequality that $B_n = o_p(1)$. Another use of the Cauchy-Schwartz inequality yields:

$$|C_n| \leq 2 \left\| \sqrt{n}(\bar{Y} \otimes \bar{k}^\gamma - \eta) \right\|_{HS} \left\| \bar{Y}_n \otimes \bar{K}_n - \nu \right\|_{HS} + 2 \left\| \sqrt{n}(\bar{Y}_n - \mu) \otimes (\bar{K}_n - m_X) \right\|_{HS} \left\| \nu - \eta \right\|_{HS}$$

$$= 2 \left\| \sqrt{n}(\bar{Y} \otimes \bar{k}^\gamma - \eta) \right\|_{HS} \left\| \bar{Y}_n \otimes \bar{K}_n - \nu \right\|_{HS} + 2 \left\| \sqrt{n}(\bar{Y}_n - \mu) \right\|_Y \left\| \bar{K}_n - m_X \right\|_H \left\| \nu - \eta \right\|_{HS}. $$

As $n \to +\infty$, $\sqrt{n}(\bar{Y} \otimes \bar{k}^\gamma - \eta)$ and $\sqrt{n}(\bar{Y}_n - \mu)$ converge in distribution to normal random variables, $\bar{K}_n$ and $\bar{Y}_n$ converge in probability to $m_X$ and $\mu$ respectively. Thus, by the continuous mapping theorem, $\bar{Y}_n \otimes \bar{K}_n$ converge in probability to $\nu$ as $n \to +\infty$, and the preceding inequality implies that $C_n = o_p(1)$. Finally, we got

$$\sqrt{n} \left( \hat{K}_{n,\gamma} - \text{KCMD}(Y, X) \right) = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} U(X_i, Y_i) - w_{i,n}(\gamma) V(X_i, Y_i) + o_p(1).$$

From Slutsky’s theorem, $\sqrt{n} \left( \hat{K}_{n,\gamma} - \text{KCMD}(Y, X) \right)$ has the same limiting distribution than $D_n$. Let us set

$$s_{n,\gamma}^2 = \sum_{i=1}^{n} \text{Var} \left( U(X_i, Y_i) - w_{i,n}(\gamma) V(X_i, Y_i) \right).$$
By similar arguments as in the proof of Theorem 1 in Magikusa and Naito (2020) we obtain that, for any $\varepsilon > 0$,

$$s_{n,\gamma}^{-2} \sum_{i=1}^{n} \int_{\{(x,y) : |U(x,y) - w_{i,n}(\gamma) V(x,y)| > \varepsilon s_{n,\gamma}\}} \left( U(x,y) - w_{i,n}(\gamma) V(x,y) \right)^2 dP_{XY}(x,y)$$

converges to 0 as $n \to +\infty$. Therefore, by Section 1.9.3 in [?] we obtain that

$$\sqrt{ns_{n,\gamma}} \frac{D_n}{2} \xrightarrow{p} N(0, 1).$$

However,

$$\left( \frac{s_{n,\gamma}}{\sqrt{n}} \right)^2 = Var(U(X_1, Y_1)) + \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) Var(V(X_1, Y_1))$$

$$- 2 \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) Cov(U(X_1, Y_1), V(X_1, Y_1)),$$

then, using ($C_2$) and ($C_4$), we get

$$\lim_{n \to +\infty} \left( n^{-1}s_{n,\gamma}^2 \right) = Var(U(X_1, Y_1)) + w^2(\gamma) Var(V(X_1, Y_1)) - 2 Cov(U(X_1, Y_1), V(X_1, Y_1))$$

and, therefore, $D_n \xrightarrow{p} N(0, \sigma^2)$. 

### 4.2 Proof of Theorem 2

It suffices to prove that $\hat{\sigma}^2$ converges in probability to $\sigma^2$ as $n \to +\infty$, what is obtained from the convergence in probability of $\hat{\alpha}^2$ to $Var \left( \left\langle Y_1 \otimes K(X_1, \cdot), \eta \right\rangle_{HS} \right)$. From the definition of the Hilbert-Schmidt inner product and the reproducing property of $K$ one can easily see that

$$\left\langle Y_i \otimes K(X_i, \cdot), Y \otimes K^n \right\rangle_{HS} = n^{-1} \sum_{i=1}^{n} \left\langle Y_i, Y_j \right\rangle_{\gamma} K(X_i, X_j)$$

and, therefore, that

$$\hat{\alpha}^2 = \frac{1}{n} \sum_{i=1}^{n} \left\langle Y_i \otimes K(X_i, \cdot), Y \otimes K^n \right\rangle_{HS}^2 - \left( \frac{1}{n} \sum_{i=1}^{n} \left\langle Y_i \otimes K(X_i, \cdot), Y \otimes K^n \right\rangle_{HS} \right)^2.$$

(5)
Noticing that

\[
\frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} - \eta \rangle_{HS}^2 = \frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS}^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), Y \otimes K^n - \eta \rangle_{HS}^2
\]

\[
+ \frac{2}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS} \langle Y_i \otimes K(X_i, \cdot), Y \otimes K^n - \eta \rangle_{HS}
\]

we have to treat each term is this sum. First, using the Cauchy-Schwarz inequality, the reproducing property of \( K \) and condition (C1), we get

\[
\frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} - \eta \rangle_{HS}^2 \leq \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \otimes K(X_i, \cdot) \|_{HS}^2 \right) \| \overline{Y \otimes K^n} - \eta \|_{HS}^2
\]

\[
= \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \|_Y^2 \| K(X_i, \cdot) \|_H^2 \right) \| \overline{Y \otimes K^n} - \eta \|_{HS}^2
\]

\[
= \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \|_Y^2 \| K(X_i, X_i) \|_H \right) \| \overline{Y \otimes K^n} - \eta \|_{HS}^2
\]

\[
\leq \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \|_Y^2 \right) \| K \|_\infty \| \overline{Y \otimes K^n} - \eta \|_{HS}^2.
\]

Since, from the law of large numbers, \( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \|_Y^2 \) and \( \overline{Y \otimes K^n} \) converge in probability, as \( n \to +\infty \), to \( E(\| Y \|_Y^2) \) and \( \eta \) respectively, we deduce from the preceding inequality that

\[
\frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} - \eta \rangle_{HS}^2 = o_p(1).
\]
Secondly, using again the Cauchy-Schwarz inequality, the reproducing property of $K$ and condition $(C_1)$, we obtain the inequality

$$
\left| \frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} - \eta \rangle_{HS} \right|
\leq \left( \frac{1}{n} \sum_{i=1}^{n} \left\| Y_i \otimes K(X_i, \cdot) \right\|_{HS}^2 \right) \left\| \overline{Y \otimes K^n} - \eta \right\|_{HS} \|\eta\|_{HS}
\leq \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \|^2 \right) \| K \|_{\infty} \left\| \overline{Y \otimes K^n} - \eta \right\|_{HS} \|\eta\|_{HS}
$$

from which we conclude that

$$
\frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} - \eta \rangle_{HS} = o_p(1).
$$

Consequently, from (6) it is seen that $\frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} \rangle_{HS}^2$ has the same limit in probability than $\frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS}^2$. From the law of large numbers this latter converges in probability, as $n \to +\infty$ to $E \left( \langle Y_1 \otimes K(X_1, \cdot), \eta \rangle_{HS}^2 \right)$. On the other hand, we have the inequality

$$
\left| \frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} \rangle_{HS} - \frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS} \right|
= \left| \frac{1}{n} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \overline{Y \otimes K^n} - \eta \rangle_{HS} \right|
\leq \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \otimes K(X_i, \cdot) \|_{HS} \right) \left\| \overline{Y \otimes K^n} - \eta \right\|_{HS}
= \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \| \| K(X_i, \cdot) \|_{H} \right) \left\| \overline{Y \otimes K^n} - \eta \right\|_{HS}
= \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \| \sqrt{K(X_i, X_i)} \right) \left\| \overline{Y \otimes K^n} - \eta \right\|_{HS}
\leq \left( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \| \right) \| K \|_{\infty}^{1/2} \left\| \overline{Y \otimes K^n} - \eta \right\|_{HS}
$$
which implies that 
\[ n^{-1} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), Y \otimes K \rangle_{HS} - n^{-1} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS} = o_p(1) \]
since, from the law of large numbers, \( \frac{1}{n} \sum_{i=1}^{n} \| Y_i \| Y \) and 
\( Y \otimes K \) converge in probability, as \( n \to +\infty \), to \( \mathbb{E}(\| Y \| Y) \) and \( \eta \) respectively. Consequently, 
\[ n^{-1} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), Y \otimes K \rangle_{HS} \] converges in probability, as \( n \to +\infty \), to the same limit as 
\[ n^{-1} \sum_{i=1}^{n} \langle Y_i \otimes K(X_i, \cdot), \eta \rangle_{HS} \], 
that is \( \mathbb{E}(\langle Y_1 \otimes K(X_1, \cdot), \eta \rangle_{HS}) \). Finally, from (5), we deduce that \( \hat{\alpha}^2 \) converges in probability, as \( n \to +\infty \), to 
\( \text{Var}(\langle Y_1 \otimes K(X_1, \cdot), \eta \rangle_{HS}) \).

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