PATTERN RECOGNITION ON ORIENTED MATROIDS:
LAYERS OF TOPE COMMITTEES

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Abstract. A tope committee $K^*$ for a simple oriented matroid $M$ is a subset of its maximal covectors such that every positive halfspace of $M$ contains more than half of the covectors from $K^*$. The structures of the family of all committees for $M$, and of the family of its committees that contain no pairs of opposites, are described. A Farey subsequence associated with the elements of the $m$th layer of the Boolean lattice of rank $2m$ is explored.

Contents

1. Introduction 1
2. Tope Committees and Relative Blocking 4
3. The Farey Subsequence $\mathcal{F}(B(2^m), m)$ 5
4. Layers of Tope Committees 10
5. Layers of Tope Committees Containing no Pairs of Opposites 11
References 14

1. Introduction

Let $\mathcal{H} := \{H_1, \ldots, H_t\}$ be a finite arrangement of oriented affine hyperplanes $H_i := \{x \in \mathbb{R}^n : \langle e_i, x \rangle = a_i\}$ in $\mathbb{R}^n$, where any two distinct vectors $e_i$ and $e_j$ are linearly independent, $a_i \in \mathbb{R}$, and $\langle e_i, x \rangle := \sum_{j=1}^{n} e_{ij} x_j$. See, e.g., [25] [29] on hyperplane arrangements. The regions (or chambers) of $\mathcal{H}$ are the connected components of $\mathbb{R}^n - \bigcup_{H \in \mathcal{H}} H$. A region $R$ lies on the positive side of the hyperplane $H_i$ if $\langle e_i, v \rangle > a_i$, for a vector $v \in R$. Let $T_i^+$ denote the set of all regions of the arrangement $\mathcal{H}$ that lie on the positive side of the hyperplane $H_i$. A majority committee of regions (or a committee, for short) for the arrangement $\mathcal{H}$ is a subset of regions $\mathcal{K}^* := \{R_1, \ldots, R_t\}$ such that it holds $|\mathcal{K}^* \cap T_i^+| > \frac{|\mathcal{K}^*|}{2}$, $1 \leq i \leq t$. A representative system $\{w_k \in R_k : 1 \leq k \leq t\}$ is called a committee for the system of strict linear inequalities $\{\langle e_i, x \rangle > a_i : 1 \leq i \leq t\}$, see, e.g., [1, 2, 14, 22, 23].

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Oriented matroids are defined by various equivalent axiom systems, and they can be thought of as a combinatorial abstraction of point configurations over the reals, of real hyperplane arrangements, of convex polytopes, and of directed graphs. Oriented matroids are thoroughly studied, e.g., in [3, 7, 8, 27, 28, 31].

An application of constructions, that generalize committees for arrangements of linear hyperplanes, to the pattern recognition problem in its abstract setting is as follows.

For a positive integer $t$, let $E_t$ and $[1, t]$ denote the set $\{1, 2, ..., t\}$. Let $M := (E_t, T)$ be a simple oriented matroid (throughout the paper, ‘simple’ means, in a partly nonstandard way, that $M$ has no loops, parallel or antiparallel elements) of rank $r(M) \geq 2$, on the ground set $E_t$, with set of topes $T \subset \{-, +\}^{E_t}$.

The number of topes $|T|$ can be computed with the help of the technique which was independently proposed by Las Vergnas and Zaslavsky, see [7, §4.6]. The positive halfspace associated with an element $e \in E_t$ is the set $T^+_e := \{T \in T : T(e) = +\}$. A subset $K^* \subset T$ is called a tope committee for $M$ if it holds $|K^* \cap T^+_e| > \frac{|K^*|}{2}$, for all $e \in E_t$. This paper is a sequel to [19], where it was shown that the family of tope committees for $M$, denoted by $K^*(M)$, is nonempty.

Let $M$ denote the nontrivial extension of $M$ by a new element $g$ which is not a loop, and which is parallel or antiparallel to neither of the elements of $E_t$; let $\sigma$ be the corresponding localization. Fix a tope committee $K^*$ for $M$. Let $C^*$ denote the set of cocircuits of $M$, and suppose that the sets $\{(X, \Sigma_K := \sigma(X)) : X \in C^*, X \text{ restriction of } K\}$ are conformal, for all topes $K \in K^*$. The committee decision rule, corresponding to $K^*$, relates the element $g$ to a class $A$ if $|\{K \in K^* : \Sigma_K = +\}| < \frac{|K^*|}{2}$; on the contrary, $g$ is recognized as an element of the other class $B$ if $|\{K \in K^* : \Sigma_K = +\}| > \frac{|K^*|}{2}$, see [19] for more on the two-class problem of pattern recognition on oriented matroids.

In this paper, we present the structural description of the family $K^*(M)$ of all tope committees for a simple oriented matroid $M$ which is not acyclic. The description involves certain specific subsequences of the Farey sequences. See, e.g., [13, Chapter 4] on the standard Farey sequences $F_n$ of order $n$, which are defined to be the ascending sequences of irreducible fractions $\frac{h}{k}$ such that $0 \leq \frac{h}{k} \leq 1$ and $k \leq n$.

Among interesting subsequences of $F_n$ there are some sequences which have a neat set-theoretic and combinatorial meaning: Let $A$ be a proper $m$-subset of a nonempty finite set $C$ of cardinality $n$. For all nonempty subsets $B \subseteq C$, arrange the fractions $\frac{|B \cap A|}{|B|}$, reduced to their lowest terms, in ascending order, without repetition. The resulting sequence $F(B(n), m)$
dowed with the order-reversing bijection \( h \mapsto \frac{k-h}{h} \), with \( h \) making reference to the poset rank function on the Boolean lattice \( \mathbb{B}(n) \) of rank \( n \) inherits many properties of \( F_n \).

Since the cardinality of every positive halfspace of a simple oriented matroid \( M := (E, T) \) equals \( \frac{|T|}{2} \), the Farey subsequences of special interest are those analogous to \( F(\mathbb{B}(|T|), \frac{|T|}{2}) \).

Although, for any positive integer \( m \), the sequence \( F(\mathbb{B}(2m), m) \) is endowed with the order-reversing bijection \( h \mapsto \frac{k-h}{k} \), the entries within the left halfsequence \( F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m) := \{ f \in F(\mathbb{B}(2m), m) : f \leq \frac{1}{2} \} \) and the right halfsequence \( F^{\geq \frac{1}{2}}(\mathbb{B}(2m), m) := \{ f \in F(\mathbb{B}(2m), m) : f \geq \frac{1}{2} \} \) exhibit different behavior. For example, if \( m > 1 \) then the numerators \( h \) of the fractions \( \frac{h}{k} \geq \frac{1}{2} \) are symmetrically distributed with respect to the numerator 2 of the fraction \( \frac{2}{k} \) which occupies the central position in the subsequence \( F^{\geq \frac{1}{2}}(\mathbb{B}(2m), m) \); the subsequence \( F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m) \) does not have such a property. It is shown in \( \cite{16} \) \( \S \S 3, 4 \) relying on results from the present paper that in fact the sequence of numerators from \( F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m) \) is exactly the sequence of numerators from \( F_m \), and the sequence of numerators from \( F^{\geq \frac{1}{2}}(\mathbb{B}(2m), m) \) is the sequence of denominators from \( F_m \):

Let \( m > 1 \). The maps

\[
\begin{align*}
F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m) & \to F_m, \\
F_m & \to F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m),
\end{align*}
\]

and

\[
F_m \to F^{\geq \frac{1}{2}}(\mathbb{B}(2m), m), \quad \frac{h}{k} \mapsto \frac{k}{2k-h},
\]

are order-preserving and bijective.

The maps

\[
\begin{align*}
F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m) & \to F_m, \\
F_m & \to F^{\leq \frac{1}{2}}(\mathbb{B}(2m), m),
\end{align*}
\]

and

\[
F_m \to F^{\geq \frac{1}{2}}(\mathbb{B}(2m), m), \quad \frac{h}{k} \mapsto \frac{k}{k+h},
\]

are order-reversing and bijective.

If \( K^* \) is a tope committee for an oriented matroid \( M := (E, T) \) then the disjoint union \( K^* \cup \{T, -T\} \) of \( K^* \) with a pair of opposites \( \{T, -T\} \subset T \) is also a committee for \( M \). In a similar way, if \( K^{**} \) and \( K^{***} \) are disjoint committees, then their union is a committee as well. Such redundant committees are not of applied importance because in practice one searches in
reverse direction, for inclusion-minimal committees: if $K^r$ and $K'^r$ are tope committees for $M$, with $K^r \subsetneq K'^r$, then the committee $K^r$ is preferred. A committee $K^r$ is called minimal if any its proper subset is not a committee. Minimal committees do not contain opposites. Among the minimal committees the most interesting are the committees of minimal cardinality, so-called minimum committees.

In Section 2 of the paper, we interpret tope committees for oriented matroids in terms of relatively blocking elements in Boolean lattices. The Farey subsequences $F(\mathbb{B}(2m), m)$ are explored in Section 3. Subfamilies of the family $K^r(M)$ of tope committees for an oriented matroid $M := (E_t, T)$, lying on layers of the Boolean lattice $\mathbb{B}(T)$ of subsets of the tope set $T$, are described in Section 4. In Section 5 we consider the subfamily $\hat{K}^r(M) \subset K^r(M)$ of tope committees for $M$, that contain no pairs of opposites, and we describe its structure.

2. Tope Committees and Relative Blocking

Let $M = (E_t, T)$ be a simple oriented matroid on the ground set $E_t$, with set of topes $T$. The Boolean lattice of all subsets of $T$ is denoted by $\mathbb{B}(T)$. Let $\Upsilon$ denote the antichain $\{\upsilon_1, \ldots, \upsilon_t\}$ corresponding to the family of positive halfspaces $\{T_1^+, \ldots, T_t^+\}$ which is thought of as a subset of $\mathbb{B}(T)$. The poset rank $\rho(\upsilon_e)$ of any element $\upsilon_e$ in $\mathbb{B}(T)$ is $\frac{|T_e|}{T_e}$, therefore the antichain $\Upsilon$ is pure in the sense that it lies entirely on a layer of the graded lattice $\mathbb{B}(T)$.

Interpret a tope committee for $M$ as an element $b \in \mathbb{B}(T)$. Then the family

$$
K^r(M) := \{K^r \subset T : |K^r \cap T_e^+| > \frac{1}{2}|K^r|, \ \forall e \in E_t\}
$$

of all tope committees for $M$ can be viewed as the subposet

$$
I_{\frac{1}{2}}(\mathbb{B}(T), \Upsilon) := \left\{ b \in \mathbb{B}(T) : \rho(b) > 0, \ \frac{\rho(b \wedge \upsilon_e)}{\rho(b)} > \frac{1}{2}, \ \forall e \in E_t\right\}
$$

of all relatively $\frac{1}{2}$-blocking elements for the antichain $\Upsilon$ in $\mathbb{B}(T)$. Relative blocking in posets is discussed in [21]. The antichain $\eta_{\frac{1}{2}}(\mathbb{B}(T), \Upsilon)$, called in [21] the relative $\frac{1}{2}$-blocker of $\Upsilon$ in $\mathbb{B}(T)$, is the family of all minimal tope committees for $M$; throughout the paper, $\text{min}$ denotes the set of all minimal elements of a subposet.

For a graded poset $\mathcal{P}$ with rank function $\rho$, and for a nonnegative integer $k$, we let $\mathcal{P}^{(k)}$ denote the $k$th layer of $\mathcal{P}$, that is the antichain $\{p \in \mathcal{P} : \rho(p) = k\}$. If $A$ is an antichain in $\mathcal{P}$, then $\mathcal{I}(A)$ and $\mathcal{F}(A)$ denote the order ideal and filter in $\mathcal{P}$, generated by $A$, respectively.

For $k \in [1, |T| - 1]$, the subposet

$$
I_{\frac{1}{2}, k}(\mathbb{B}(T), \Upsilon) := \mathbb{B}(T)^{(k)} \cap I_{\frac{1}{2}}(\mathbb{B}(T), \Upsilon)
$$

(2.1)
of elements of rank $k$ from $I_{\frac{1}{2}}(B(T), Y)$ is the antichain
\[ B(T)^{(k)} \cap \bigcap_{e \in E_i} \mathcal{F}(v_e) \cap B(T)^{([k+1]/2])}, \]
see [21 Proposition 5.1(ii)].

3. The Farey Subsequence $\mathcal{F}(B(2m), m)$

Let $C$ be a finite nonempty set of even cardinality $2m$, and $A$ an $m$-subset of $C$. Arrange in ascending order (without repetition) the fractions reduced to their lowest terms, for all nonempty subsets $B \subseteq C$; the resulting Farey subsequence is
\[ \mathcal{F}(B(2m), m) := \left\{ \frac{h}{k} \in \mathcal{F}_{2m} : h \leq m, k - h \leq m \right\}. \]

Example 3.1.
\[ \mathcal{F}(B(8), 4) = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{2}{7}, \frac{3}{7}, \frac{3}{5}, \frac{4}{7}, \frac{5}{7}, \frac{5}{6}, \frac{4}{5}, \frac{7}{8}, \frac{7}{9}, \frac{6}{5}, \frac{5}{4}, \frac{9}{8} \right\}; \]
\[ \mathcal{F}(B(10), 5) = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{2}{7}, \frac{3}{7}, \frac{3}{5}, \frac{2}{5}, \frac{4}{7}, \frac{5}{7}, \frac{5}{6}, \frac{4}{5}, \frac{7}{8}, \frac{7}{9}, \frac{6}{5}, \frac{5}{4}, \frac{9}{8} \right\}. \]

In this section, we explore such sequences, and we start by recalling some basic properties of general sequences
\[ \mathcal{F}(B(n), m) := \left\{ \frac{h}{k} \in \mathcal{F}_n : h \leq m, k - h \leq n - m \right\}, \]
see also [17].

Lemma 3.2. [21 Proposition 7.5] Let $\frac{h}{k} \in \mathcal{F}(B(n), m) \setminus \left\{ \frac{0}{1}, \frac{1}{1} \right\}$, where $0 < m < n$.

(i) Let $x_0$ be the integer such that $kx_0 \equiv -1 \pmod{h}$ and $m - h + 1 \leq x_0 \leq m$. Define integers $y_0$ and $t^*$ by $y_0 := \frac{kx_0 + 1}{h}$ and $t^* := \left\lfloor \min\left\{ \frac{m-x_0}{h}, \frac{n-y_0}{k}, \frac{n-m+x_0-y_0}{k-h} \right\} \right\rfloor$.

The fraction $\frac{y_0 + t^* h}{y_0 + t^* k}$ precedes the fraction $\frac{h}{k}$ in $\mathcal{F}(B(n), m)$.

(ii) Let $x_0$ be the integer such that $kx_0 \equiv 1 \pmod{h}$ and $m - h + 1 \leq x_0 \leq m$. Define integers $y_0$ and $t^*$ by $y_0 := \frac{kx_0 - 1}{h}$ and $t^* := \left\lfloor \min\left\{ \frac{m-x_0}{h}, \frac{n-y_0}{k}, \frac{n-m+x_0-y_0}{k-h} \right\} \right\rfloor$.

The fraction $\frac{y_0 + t^* h}{y_0 + t^* k}$ succeeds the fraction $\frac{h}{k}$ in $\mathcal{F}(B(n), m)$.
Lemma 3.3. [21 Proposition 7.8] Let \( \frac{h}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}} \) be three successive fractions of \( \mathcal{F}(\mathbb{B}(n), m) \), where \( 0 < m < n \).

(i) The integers \( h_j \) and \( k_j \) are computed by
\[
\begin{align*}
    h_j &= \left\lfloor \frac{h_{j+2} + m}{h_{j+1}} \right\rfloor \left( h_{j+1} - h_j + n - m \right) - h_j + 2, \\
    k_j &= \left\lfloor \frac{h_{j+2} + m}{k_{j+1}} \right\rfloor \left( k_{j+1} - h_j + n - m \right) - k_j + 2.
\end{align*}
\]

(ii) The integers \( h_{j+2} \) and \( k_{j+2} \) are computed by
\[
\begin{align*}
    h_{j+2} &= \left\lfloor \frac{h_j + m}{h_{j+1}} \right\rfloor \left( h_{j+1} - h_j + n - m \right) - h_j, \\
    k_{j+2} &= \left\lfloor \frac{h_j + m}{k_{j+1}} \right\rfloor \left( k_{j+1} - h_j + n - m \right) - k_j.
\end{align*}
\]

The first observation is as follows:

Remark 3.4. The map
\[
\mathcal{F}(\mathbb{B}(2m), m) \to \mathcal{F}(\mathbb{B}(2m), m), \quad \frac{h}{k} \mapsto \frac{k-h}{h}
\]
is order-reversing and bijective.

In the case where \( n := 2m \), Lemmas 3.2 and 3.3 can be refined.

Corollary 3.5. (i) Let \( \frac{h}{k} \in \mathcal{F}(\mathbb{B}(2m), m) \). Suppose that \( \frac{h}{k} > \frac{1}{2} \).

Let \( x_0 \) be the integer such that \( kx_0 \equiv -1 \) (mod \( h \)) and \( m - h + 1 \leq x_0 \leq m \). The fraction
\[
x_0/\frac{kx_0+1}{h}
\]
precedes the fraction \( \frac{h}{k} \) in \( \mathcal{F}(\mathbb{B}(2m), m) \).

(ii) Let \( \frac{h}{k} \in \mathcal{F}(\mathbb{B}(2m), m) \). Suppose that \( \frac{1}{2} \leq \frac{h}{k} < \frac{1}{2} \).

Let \( x_0 \) be the integer such that \( kx_0 \equiv 1 \) (mod \( h \)) and \( m - h + 1 \leq x_0 \leq m \). The fraction
\[
x_0/\frac{kx_0-1}{h}
\]
succeeds the fraction \( \frac{h}{k} \) in \( \mathcal{F}(\mathbb{B}(2m), m) \). In particular, the fraction \( \frac{m}{2m-1} \) succeeds \( \frac{1}{2} \).

(iii) Let \( \frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}} \) be three successive fractions in \( \mathcal{F}(\mathbb{B}(2m), m) \), where \( m > 1 \), with \( \frac{h_i}{k_i} \geq \frac{1}{2} \).

The integers \( h_j, k_j, h_{j+2} \) and \( k_{j+2} \) are computed by
\[
\begin{align*}
    h_j &= \left\lfloor \frac{h_{j+2} + m}{h_{j+1}} \right\rfloor \left( h_{j+1} - h_j + 2 \right), \\
    k_j &= \left\lfloor \frac{h_{j+2} + m}{k_{j+1}} \right\rfloor \left( k_{j+1} - k_j + 2 \right), \\
    h_{j+2} &= \left\lfloor \frac{h_j + m}{h_{j+1}} \right\rfloor \left( h_{j+1} - h_j \right), \\
    k_{j+2} &= \left\lfloor \frac{h_j + m}{k_{j+1}} \right\rfloor \left( k_{j+1} - k_j \right).
\end{align*}
\]
Proof. We prove (ii); assertion (i) is proved in a similar way.

(ii) In terms of Lemma 3.2(ii), \( t^* := \left\lfloor \min\left\{ \frac{m-x_0}{k}, \frac{2m-y_0}{k}, \frac{m+x_0+y_0}{k} \right\} \right\rfloor \). Since \( \frac{h}{k} \geq \frac{1}{2} \), we have \( \min\{ \frac{m-x_0}{k}, \frac{2m-y_0}{k}, \frac{m+x_0+y_0}{k} \} = \frac{m-x_0}{k} \). Therefore \( t^* = \left\lfloor \frac{m-x_0}{h} \right\rfloor \). But the constraint \( 0 \leq m - x_0 \leq h - 1 \) implies \( t^* = 0 \), and the assertion follows from Lemma 3.2(ii).

(iii) We prove (3.3). Lemma 3.3(ii) implies that

\[
\begin{align*}
h_{j+2} &= \left\lfloor \min \left\{ \frac{h_j + m}{h_{j+1}}, \frac{k_j + 2m}{k_{j+1}}, \frac{k_j - h_j + m}{k_{j+1} - h_{j+1}} \right\} \right\rfloor \cdot h_{j+1} - h_j , \\
k_{j+2} &= \left\lfloor \min \left\{ \frac{h_j + m}{h_{j+1}}, \frac{k_j + 2m}{k_{j+1}}, \frac{k_j - h_j + m}{k_{j+1} - h_{j+1}} \right\} \right\rfloor \cdot k_{j+1} - k_j .
\end{align*}
\]

Since \( \frac{h_j}{k_j} \geq \frac{1}{2} \), we have \( \min\{ \frac{h_j + m}{h_{j+1}}, \frac{k_j + 2m}{k_{j+1}}, \frac{k_j - h_j + m}{k_{j+1} - h_{j+1}} \} = \frac{h_j + m}{h_{j+1}} \); hence (3.3). \( \square \)

Analogous properties of the left halfsequence of \( \mathcal{F}(\mathbb{B}(2m), m) \) are presented in Proposition 3.8 below.

For a fraction \( f := \frac{h}{k} \), we denote by \( \underline{f} := h \) and \( \overline{f} := k \) its numerator and denominator, respectively.

The fractions \( f_s \in \mathcal{F}(\mathbb{B}(n), m) \) are always indexed starting with zero, thus \( f_0 := \frac{0}{1} \).

**Proposition 3.6.** Let \( f_s, f_t \in \mathcal{F}(\mathbb{B}(2m), m) \), where \( m > 1 \), \( f_s := \frac{1}{2} \) and \( f_t := \frac{2}{3} \). Fractions from \( \mathcal{F}(\mathbb{B}(2m), m) \) satisfy the equalities

\[
\begin{align*}
f_{t+v} &= f_{t-v} , \\
\left( \frac{f_{t+v}}{f_{t-v}} + \frac{f_{t-v}}{f_{t+v}} \right) \cdot 
\end{align*}
\]

for all \( v \), \( 0 \leq v \leq t - s \).

**Proof.** Notice that (3.5) holds for \( v := 0 \).

Let \( v := 1 \). By Corollary 3.5(ii), the fraction \( \frac{h_t}{k_t} := f_t \) precedes the fraction \( \frac{h_{t+v}}{k_{t+v}} := f_{t+v} \) with \( h_{t+v} = x_0 \) and \( k_{t+v} = \frac{k_t x_0 - 1}{h_t} \), where \( 3x_0 \equiv 1 \) (mod 2) and \( m - 1 \leq x_0 \leq m \); therefore the numerator and denominator of \( f_{t+v} \) are

\[
\begin{align*}
h_{t+v} := f_{t+v} &= x_0 = \begin{cases} m - 1, & \text{if } m \text{ is even} , \\
m, & \text{if } m \text{ is odd} ; \end{cases} \\
k_{t+v} := f_{t+v} &= \frac{k_t x_0 - 1}{h_t} = \begin{cases} \frac{3m-4}{2}, & \text{if } m \text{ is even} , \\
\frac{3m-1}{2}, & \text{if } m \text{ is odd} . \end{cases}
\end{align*}
\]
Corollary \[3.5\] implies that the numerator and denominator of the fraction \(h_{t-v}/k_{t-v} := f_{t-v}\) are

\[
h_{t-v} := f_{t-v} = \begin{cases} m-1, & \text{if } m \text{ is even,} \\ m, & \text{if } m \text{ is odd;} \end{cases}
\]
\[
k_{t-v} := f_{t-v} = \begin{cases} \frac{3m-2}{2}, & \text{if } m \text{ is even,} \\ \frac{3m+1}{2}, & \text{if } m \text{ is odd.} \end{cases}
\]

We see that equalities (3.4) and (3.5) hold for \(v := 1\).

Let \(v := 2\). Equalities (3.3) and (3.2) yield

\[
h_{t+v} := f_{t+v} = \left[ \frac{h_{t+v-2} + m}{h_{t+v-1}} \right] h_{t+(v-1)} - h_{t+(v-2)},
\]
\[
h_{t-v} := f_{t-v} = \left[ \frac{h_{t-v-2} + m}{h_{t-v-1}} \right] h_{t-(v-1)} - h_{t-(v-2)},
\]

respectively. It has been shown that \(h_{t+v-1} = h_{t-(v-1)}\) and, by convention, we have \(h_{t+(v-2)} = h_{t-(v-2)} = 2\). Thus, (3.4) holds for \(v := 2\).

Let \(k_{t+v} := f_{t+v}\) and \(k_{t-v} := f_{t-v}\). Equalities (3.3) and (3.2) yield

\[
\left( \frac{f_{t+v}}{f_{t-v}} + \frac{f_{t-v}}{f_{t+v}} \right) / f_{t+v} = \frac{k_{t+v} + k_{t-v}}{h_{t+v}}
\]
\[
= \left[ \frac{h_{t+(v-2)} + m}{h_{t+(v-1)}} \right] k_{t+(v-1)} - k_{t+(v-2)} + \left[ \frac{h_{t-(v-2)} + m}{h_{t-(v-1)}} \right] k_{t-(v-1)} - k_{t-(v-2)}
\]
\[
= \left[ \frac{h_{t+(v-2)} + m}{h_{t+(v-1)}} \right] (k_{t+(v-1)} + k_{t-(v-1)}) - (k_{t+(v-2)} + k_{t-(v-2)})
\]
\[
= \left[ \frac{h_{t+(v-2)} + m}{h_{t+(v-1)}} \right] h_{t+(v-1)} - h_{t+(v-2)}
\]
\[
= \left[ \frac{h_{t+(v-2)} + m}{h_{t+(v-1)}} \right] h_{t+(v-1)} - 3h_{t+(v-2)}
\]
\[
= 3 \left[ \frac{h_{t+(v-2)} + m}{h_{t+(v-1)}} \right] h_{t+(v-1)} - h_{t+(v-2)}
\]

that is, we have obtained (3.5) for \(v := 2\).

Let \(v\) be any integer \(\geq 3\) such that the fraction \(f_{t-v}\) from \(\mathcal{F}(B(2m), m)\) is greater than or equal to \(\frac{1}{2}\). Equalities (3.4) and (3.5) are proved by induction.

Recall that for \(v := t - s\), by convention, we have \(f_{t-s} = 1\) and \(f_{t-s} = 2\).

We conclude from (3.4) and (3.5) that \(f_{t-v} = \frac{1}{3}\).

Along with Remark (3.4), Proposition (3.6) leads to the following observation:
Proposition 3.8. (i) The maps
\[ \mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(2m), m) \rightarrow \mathcal{F}^{\leq \frac{1}{2}}(\mathbb{B}(2m), m), \quad \frac{h}{k} \mapsto \frac{h}{3m-k}, \]
and
\[ \mathcal{F}^{\leq \frac{1}{2}}(\mathbb{B}(2m), m) \rightarrow \mathcal{F}^{\geq \frac{1}{2}}(\mathbb{B}(2m), m), \quad \frac{h}{k} \mapsto \frac{k-2h}{2k-3h}, \]
are order-reversing and bijective.

(ii) Let \( f_s, f_t \in \mathcal{F}(\mathbb{B}(2m), m) \), where \( m > 1 \), \( f_s := \frac{1}{3} \) and \( f_t := \frac{1}{2} \). Fractions from \( \mathcal{F}(\mathbb{B}(2m), m) \) satisfy the equalities
\[ \left( \frac{f_{s+v} + f_{s-v}}{f_{s+v} - f_{s-v}} \right) = 3, \]
for all \( v, 0 \leq v \leq t-s \).

Thus, fragments of \( \mathcal{F}(\mathbb{B}(2m), m) \), for large \( m \), look like
\[ \mathcal{F}(\mathbb{B}(2m), m) \approx \left( \frac{0}{1} < \frac{1}{m+1} < \frac{1}{m} < \frac{1}{m-1} < \cdots < \frac{1}{3} < \cdots \right) \]
\[ < \frac{m-3}{2m-5} < \frac{m-2}{2m-3} < \frac{m-1}{2m-1} < \frac{1}{2} < \frac{m}{2m-3} < \frac{m-1}{2m-5} < \cdots \]
\[ < \cdots < \frac{2}{3} < \cdots < \frac{m-2}{m-1} < \frac{m-1}{m} < \frac{m}{m+1} < \frac{1}{2} \); see Example 3.3.1 this observation is discussed in more detail in [18] §4. In particular, the fraction \( \frac{h}{k} \), with
\[ h = \begin{cases} \frac{m-2}{2} & \text{if } m \text{ is even} \\ \frac{m-1}{2} & \text{if } m \text{ is odd} \end{cases}, \quad k = \begin{cases} \frac{3m-4}{2} & \text{if } m \text{ is even} \\ \frac{3m-1}{2} & \text{if } m \text{ is odd} \end{cases}, \]
precedes \( \frac{1}{2} \) in \( \mathcal{F}(\mathbb{B}(2m), m) \), and the fraction \( \frac{h}{k} \), with
\[ h = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}, \quad k = \begin{cases} \frac{3m-2}{2} & \text{if } m \text{ is even} \\ \frac{3m+1}{2} & \text{if } m \text{ is odd} \end{cases}, \]
succeeds \( \frac{1}{2} \) in \( \mathcal{F}(\mathbb{B}(2m), m) \); see [18] for more on neighboring fractions in \( \mathcal{F}(\mathbb{B}(2m), m) \).

We conclude this section by presenting an analogue of Corollary 3.5. It describes some properties (see [16] §4) of the left halfsequence of \( \mathcal{F}(\mathbb{B}(2m), m) \):

Proposition 3.8. (i) Let \( \frac{h}{k} \in \mathcal{F}(\mathbb{B}(2m), m) \). Suppose that \( \frac{1}{3} < \frac{h}{k} \leq \frac{1}{2} \). Let \( x_0 \) be the integer such that \( hx_0 \equiv 1 \pmod{(k-h)} \) and \( m-k+h+1 \leq x_0 \leq m \). The fraction
\[ \frac{hx_0-1}{k-h} / \frac{kx_0-1}{k-h} \]
precedes \( \frac{h}{k} \) in \( \mathcal{F}(\mathbb{B}(2m), m) \). In particular, the fraction \( \frac{m-1}{2m-1} \) precedes \( \frac{1}{2} \).
(ii) Let \( \frac{h}{k} \in \mathcal{F}(B(2m), m) \). Suppose that \( 0 \leq \frac{h}{k} < \frac{1}{2} \). Let \( x_0 \) be the integer such that \( hx_0 \equiv -1 \pmod{(k-h)} \) and \( m-k+h+1 \leq x_0 \leq m \). The fraction

\[
\frac{hx_0+1}{k-h} / \frac{hx_0+1}{k-h}
\]

succeeds \( \frac{h}{k} \) in \( \mathcal{F}(B(2m), m) \).

(iii) Let \( \frac{h_1}{k_1} < \frac{h_{i+1}}{k_{i+1}} < \frac{h_{i+2}}{k_{i+2}} \) be three successive fractions in \( \mathcal{F}(B(2m), m) \), where \( m > 1 \), with \( \frac{h_{i+2}}{k_{i+2}} \leq \frac{1}{2} \).

The integers \( h_j, k_j, h_{j+2} \) and \( k_{j+2} \) are computed by

\[
\begin{align*}
    h_j &= \left\lfloor \frac{k_{j+2} - h_{j+2} + m}{k_{j+1} - h_{j+1}} \right\rfloor h_{j+1} - h_{j+2}, & k_j &= \left\lfloor \frac{k_{j+2} - h_{j+2} + m}{k_{j+1} - h_{j+1}} \right\rfloor k_{j+1} - k_{j+2}, \\
    h_{j+2} &= \left\lfloor \frac{k_j - h_j + m}{k_{j+1} - h_{j+1}} \right\rfloor h_{j+1} - h_j, & k_{j+2} &= \left\lfloor \frac{k_j - h_j + m}{k_{j+1} - h_{j+1}} \right\rfloor k_{j+1} - k_j.
\end{align*}
\]

4. LAYERS OF TOPE COMMITTEES

We now describe the structure of the family \( K^*(M) \) of all tope committees for an oriented matroid \( M := (E_t, T) \) which is not acyclic.

The following assertion is a consequence of [21, Theorem 8.4].

**Proposition 4.1.** Let \( M \) be a simple oriented matroid, which is not acyclic, on the ground set \( E_t \), with set of topes \( T \).

On the one hand,

\[
\mathcal{I}_{\frac{1}{2}}(\mathbb{B}(T), \mathcal{Y}) = \bigcup_{3 \leq k \leq |T| - 3} \left( \mathbb{B}(T)^{(k)} \cap \bigcap_{e \in E_t} \mathcal{F} \left( \mathcal{I}(v_e) \cap \mathbb{B}(T)^{\lceil (k+1)/2 \rceil} \right) \right)
\]

and, in particular,

\[
\mathcal{I}_{\frac{1}{2}}(\mathbb{B}(T), \mathcal{Y}) = \mathbb{B}(T)^{(3)} \cap \bigcap_{e \in E_t} \mathcal{F} \left( \mathcal{I}(v_e) \cap \mathbb{B}(T)^{(2)} \right)
\]

On the other hand,

\[
\mathcal{I}_{\frac{1}{2}}(\mathbb{B}(T), \mathcal{Y}) = \bigcap_{e \in E_t} \left( \bigcup_{f \in \mathcal{F}(\mathbb{B}(|T|/2))^{|T|}} \bigcup_{s \in [1, |T|/2]} \mathbb{B}(T)^{(s, f)} \cap \left( \mathcal{F}(\mathcal{I}(v_e) \cap \mathbb{B}(T)^{(s, f+1)}) \right) \right)
\]

Recall that one structure refining the description of a layer \( \mathbb{B}(n)^{(d)} \) of the Boolean lattice \( \mathbb{B}(n) \) of subsets of an \( n \)-set, \( 1 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor \), is that of the Johnson association scheme, see, e.g., [5, §3.2], [8, §2.7, §9.1], [10, §4.2], [15, §21.6]. The Johnson scheme \( \mathcal{J}(n, d) \) is the pair \((X, \mathcal{R})\), where \( X := \mathbb{B}(n)^{(d)} \) with \( |X| = \binom{n}{d} \), and \( \mathcal{R} := (R_0, R_1, \ldots, R_d) \) is a partition of \( X \times X \), defined by

\[
R_i := \{(x, y) : \partial(x, y) := d - \rho(x \wedge y) = i\}, \quad 0 \leq i \leq d.
\]
For any \( x, y \in X \) with \( \partial(x, y) = k \), and for any integers \( i \) and \( j \), \( 0 \leq i, j \leq d \), the intersection numbers
\[
\begin{align*}
\p_{ij}^k & := |\{ z \in X : \partial(z, x) = i, \partial(z, y) = j \}| \\
& = | \mathcal{B}(n)^d \cap \left( \mathcal{I}(\mathcal{J}(x) \cap \mathcal{B}(n)^{(d-i)}) - \mathcal{I}(\mathcal{J}(y) \cap \mathcal{B}(n)^{(d-j)}) \right) \cap \left( \mathcal{I}(\mathcal{J}(y) \cap \mathcal{B}(n)^{(d-j+1)}) - \mathcal{I}(\mathcal{J}(x) \cap \mathcal{B}(n)^{(d-i+1)}) \right) |
\end{align*}
\]
are the same. We have
\[
\p_{ij}^k = \sum_c \binom{d-k}{c} \binom{k}{d-i-c} \binom{k}{d-j-c} \binom{n-d-k}{i+j+d+c};
\]
the quantity \( n_i := \p_{ij}^0 = |\{ z \in X : \partial(z, x) = i \}| \), for any \( x \in X \), called the valency of \( R_i \), is \( n_i = \binom{n}{i} \binom{n-i}{d-i} \), see, e.g., [24].

The family \( \mathbf{K}^*(M) \) of all tope committees for a simple oriented matroid \( M := (E_t, T) \) can be considered as the poset \( \mathbf{1}_k(\mathcal{B}(T), \mathcal{Y}) \) of relatively \( \frac{1}{2} \)-blocking elements for the subset \( \mathcal{Y} \) of the Johnson scheme \( J(|T|, \frac{|T|}{2}) := (X, \mathcal{R}) \) on the set \( X := \mathcal{B}(T)^{(|T|/2)} \), with the partition \( \mathcal{R} := (R_0, R_1, \ldots, R_{\frac{|T|}{2}}) \) of \( X \times X \), defined by \( R_i := \{ (x, y) : \partial(x, y) := \frac{|T|}{2} - \rho(x \wedge y) = i \} \), for all \( 0 \leq i \leq \frac{|T|}{2} \).

5. Layers of Tope Committees Containing no Pairs of Opposites

For any element \( e \in E_t \), the corresponding positive halfspace \( T_e^+ \) of a simple oriented matroid \( M := (E_t, T) \) contains no pairs of opposites. On the other hand, the tope committees containing pairs of opposites have no applied value. We now describe the structure of the family
\[
\tilde{\mathbf{K}}^*(M) := \{ K^* \in \mathbf{K}^*(M) : T \in K^* \Rightarrow -T \notin K^* \}
\]
of all the committees for \( M \) which include no pairs of opposites.

Denote by \( \mathbf{O}'(T) \) a graded meet-sub-semilattice, of rank \( \frac{|T|}{2} \), of the lattice \( \mathcal{B}(T) \), defined in the following way: the elements of \( \mathbf{O}'(T) \) are the subsets of topes without pairs of opposites, which are ordered by inclusion. This poset is isomorphic to the face poset of the boundary of a \( \frac{|T|}{2} \)-dimensional crosspolytope.

Let \( \gcd(\cdot, \cdot) \) denote the greatest common divisor of a pair of integers, and define a Farey subsequence \( \mathcal{F}(\mathbf{O}'(T), \rho(a)) \), associated with an element \( a \in \mathbf{O}'(T) \), by
\[
\mathcal{F}(\mathbf{O}'(T), \rho(a)) := \left( \frac{\rho(b \wedge a)}{\gcd(\rho(b \wedge a), \rho(b))} \bigg/ \frac{\rho(b)}{\gcd(\rho(b \wedge a), \rho(b))} : b \in \mathbf{O}'(T) - \{0\} \right),
\]
If \( \rho(a) < \frac{|T|}{2} \) then the sequence \( \mathcal{F}(\mathbf{O}'(T), \rho(a)) = \left( \frac{h}{n} \in \mathcal{F}_{\frac{|T|}{2}} : h \leq \rho(a) \right) \) is a member of the family of the Farey subsequences of the form \( \left( \frac{h}{n} \in \mathcal{F}_n : \right) \)
\( h \leq m \), considered, e.g., in [31, 21, §7]. In the case where \( \rho(a) = \frac{|T|}{2} \), relevant to our context, the sequence \( F(O'(T), \rho(a)) \) is nothing else than \( F_{|T|/2} \), the standard Farey sequence of order \( \frac{|T|}{2} \).

The family \( \mathcal{K}^s(M) \) can be viewed as the subposet

\[
I^s_2(O'(T), \Upsilon) := \left\{ b \in O'(T) : \rho(b) > 0, \frac{\rho(b \wedge v_e)}{\rho(b)} > \frac{1}{2}, \forall e \in E_t \right\}
\]

of all relatively \( \frac{1}{2} \)-blocking elements for the antichain \( \Upsilon \) in \( O'(T) \); for any \( k, 1 \leq k \leq \frac{|T|}{2} \), define the antichain \( I^s_{k,1}(O'(T), \Upsilon) := O'(T)(k) \cap I^s_2(O'(T), \Upsilon) \).

This subposet can be described, in view of [21, Theorem 8.4], in the following way:

**Theorem 5.1.** Let \( M \) be a simple oriented matroid, which is not acyclic, on the ground set \( E_t \), with set of topes \( T \).

On the one hand,

\[
I^s_{1,1}(O'(T), \Upsilon) = \bigcup_{3 \leq k \leq \frac{|T|}{2}} \left( O'(T)(k) \cap \bigcap_{e \in E_t} \mathfrak{F}(v_e) \cap O'(T)(\lfloor \frac{k+1}{2} \rfloor) \right)
\]

and, in particular,

\[
I^s_{2,3}(O'(T), \Upsilon) = O'(T)(3) \cap \bigcap_{e \in E_t} \mathfrak{F}(v_e) \cap O'(T)(2).
\]

On the other hand,

\[
I^s_{1,1}(O'(T), \Upsilon) = \bigcap_{e \in E_t} \bigcup_{f \in F_{|T|/2} : \frac{1}{2} < f} \bigcup_{s \in [1, \lfloor |T|/(2f) \rfloor]} \left( O'(T)(s \cdot 1) \cap \left( \mathfrak{F}(v_e) \cap O'(T)(s \cdot f) - \mathfrak{F}(v_e) \cap O'(T)(s \cdot f+1) \right) \right).
\]

The three-tope committees for \( M \) composing the antichains \( I^s_{2,3}(B(T), \Upsilon) \) and \( I^s_{2,3}(O'(T), \Upsilon) \), described in Proposition 4.1 and Theorem 5.1, are treated and enumerated in [20].

For an integer \( m > 1 \), let \( \pm [1, m] \) denote the \( 2m \)-set \( \{-m, -(m-1), \ldots, -2, -1, 1, 2, \ldots, (m-1), m\} \). Fix some \( d \in [1, m] \), and denote by \( X \) the family of all the \( d \)-subsets \( V \subset \pm [1, m] \) such that

\[
s \in V \implies -s \notin V.
\]  

(5.1)

Let \( O(m) \) denote the family of all the subsets \( V \subset \pm [1, m] \), satisfying (5.1), ordered by inclusion, and augmented by a greatest element 1. The least element of \( O(m) \) is the empty subset of \( \pm [1, m] \). See, e.g., [6, 26] on the structural and combinatorial properties of the face lattice \( O(m) \) of an \( m \)-dimensional crosspolytope (hyperoctahedron, orthoplex), dual to that of an \( m \)-dimensional hypercube. The graded lattice \( O(m) \), of rank \( m + 1 \), is Eulerian. The fundamental properties of Eulerian
the following way: let the pair \((x, y)\) be an association scheme. Indeed, for any \(x, y \in X\) with \(\partial(x, y) := m - \rho(x \wedge y) = k\), and for any integers \(i, j\) and \(k\), \(0 \leq i, j, k \leq m\), the quantities

\[
P_{ij}^k := \left| \{ z \in X : \partial(z, x) = i, \partial(z, y) = j \} \right|
\]

are those of the Hamming association scheme \(H(m, 2)\), see, e.g., [3] §3.2, [9] §2.5, §9.2, [10] §4.1, [11] [12], [13] [21.3]. The \(m\)-cube \(H(m, 2) := (X, R)\) is the family \(X\), of cardinality \(2^m\), of all words \(x \in \{-1, 1\}^m\), together with the partition \(R := (R_0, R_1, \ldots, R_m)\) of \(X \times X\), defined by \(R_i := \{(x, y) : \partial(x, y) := |i| \in [1, m] : x_i \neq y_i\} = i\), \(0 \leq i \leq m\).

Let \(M := (E_t, T)\) be a simple oriented matroid. The family \(K^t(M)\) of its tope committees, containing no pairs of opposites, can be considered as the poset \(I^t \left( O'(T), Y \right) \) of relatively \(\frac{1}{2}\)-blocking elements for the subset \(Y\) of the association scheme \((X, R)\) on the set \(X := O'(T)(\lfloor T \rfloor / 2)\), with the partition \(R := (R_0, R_1, \ldots, R_{\lfloor T \rfloor / 2})\) of \(X \times X\), defined by \(R_i := \{(x, y) : \partial(x, y) := \lfloor T \rfloor / 2 - \rho(x \wedge y) = i\}\), for all \(0 \leq i \leq \lfloor T \rfloor / 2\); the parameters of \((X, R)\) are those of the \(\lfloor T \rfloor / 2\)-cube \(H(\lfloor T \rfloor / 2, 2)\).
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