Cosmological dynamics of $f(R)$ gravity scalar degree of freedom in Einstein frame

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$f(R)$ gravity models belong to an important class of modified gravity models where the late time cosmic accelerated expansion is considered as the manifestation of the large scale modification of the force of gravity. $f(R)$ gravity models can be expressed in terms of a scalar degree of freedom by redefinition of model’s variable. The conformal transformation of the action from Jordan frame to Einstein frame makes the scalar degree of freedom more explicit and can be studied conveniently. We have investigated the features of the scalar degree of freedoms and the consequent cosmological implications of the power-law ($\xi R^n$) and the Starobinsky (disappearing cosmological constant) $f(R)$ gravity models numerically in the Einstein frame. Both the models show interesting behaviour of their scalar degree of freedom and could produce the accelerated expansion of the Universe in the Einstein frame with the negative equation of state of the scalar field. However, the scalar field potential for the power-law model is the well behaved function of the field, whereas the potential becomes flat for higher value of field in the case of the Starobinsky model. Moreover, the equation of state of the scalar field for the power-law model is always negative and less than $-1/3$, which corresponds to the behaviour of the dark energy, that produces the accelerated expansion of the Universe. This is not always the case for the Starobinsky model. At late times, the Starobinsky model behaves as cosmological constant $\Lambda$ as behaves by power-law model for the values of $n \to 2$ at all times.

PACS numbers: 96.10.+i, 04.50.Kd, 03.50.-w, 02.60.-x
Keywords: $f(R)$ gravity, scalar degree of freedom, numerical analysis

I. INTRODUCTION

The discovery of the late time accelerated expansion of the Universe [1, 2] demands a new theory as the General Theory of Relativity (GTR) could not provide any explanation to this phenomenon within its ambit. However, because of all the successful test performed so far on GTR for the low scales, attempts have been made to modify GTR in order to account the late time behaviour of the cosmic expansion. There are two main conceptual approaches that have been led to the modification of GTR. In one approach, a new scalar degree of freedom is incorporated in the energy-momentum tensor on the right hand side of the Einstein’s equation, which is dubbed as dark energy for its exotic behaviour with large negative pressure [3–5]. Initially cosmological constant was considered as the source of dark energy which was faced with fine tuning problem of an acceptable level. In recent years, as an alternative to the cosmological constant, a variety of scalar field models is proposed to provide a viable explanation for the phenomenon of late time cosmic acceleration [6–8]. In the other approach, the left hand side of the Einstein’s equation can be modified by considering the late time cosmic accelerated expansion is due to the large scale modification of the force of gravity, which leads to the modified gravity theories [29, 32]. It needs to be mentioned that, there are many other alternative attempts to understand the physical mechanism of this late time behavior of the universe (e.g. see [9–12]).

The simplest class of modified gravity theories is the $f(R)$ gravity theories in which Einstein gravity is modified by replacing the Ricci curvature scalar $R$ by an arbitrary curvature function $f(R)$. In the special case when $f(R) \to R$, $f(R)$ gravity converge to GTR without a cosmological constant [24, 32]. In last few years, the $f(R)$ gravity models have been studied extensively in the cosmological and astrophysical aspects because of a number of very interesting results (see [13] and see references there in). There are many $f(R)$ gravity models which could produce the late time cosmic acceleration, but all of them are not cosmologically viable and many of them also suffered from the singularity and stability problem [14, 15]. So certain restrictions have to be imposed on $f(R)$ gravity models to be linearly stable and cosmologically viable. Some of such models can be found in [16, 17].

It is interesting that a new scalar degree of freedom (sometimes referred as scalaron) appears in $f(R)$ theories of gravity due to redefinition of model’s variable. The conformal transformation of the metric to the Einstein frame makes it explicit in the action [32, 44, 45, 52, 53, 54]. So it is possible to extend $f(R)$ gravity to generalized Brans-Dicke (BD) theory with a field potential and an arbitrary BD parameter. In $f(R)$ gravity, the scalar field is coupled to non-relativistic matter (dark matter, baryons) with a universal coupling constant $-1/\sqrt{6}$, and hence as far as the scalar field is concerned it is convenient work on $f(R)$ gravity in the Einstein frame, in which a canonical scalar field is coupled to non-relativistic matter. However, conventionally, the Jordan frame, where the baryons obey the standard continuity equation ($\rho_m \propto a^{-3}$), is considered as the physical frame in which physical quantities are compared with observations and experiments. We are treating essentially the same physics in both frames but with

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different time and length scales that give rise to the apparent difference between the observables in two frames. This apparent difference can be overlooked by going back to the Jordan frame when we work on the Einstein frame for some convenience [32, 60, 61].

In this work we have studied the $\xi R^n$ and the Starobinsky disappearing cosmological constant $f(R)$ gravity [62] models using their scalar degree of freedoms in the Einstein frame in the Friedmann-Lemaître-Robertson-Walker (FLRW) background. The basic aim of this study is to see numerically the features of the scalar field coupled to these $f(R)$ gravity models and hence to see the consequent evolution of the cosmological parameters in the Einstein frame. We have considered these two models because: (i) The $f(R) = \xi R^n$ model has exact power-law solutions and as a result the singularity is not appear in it as suffered by other $f(R)$ models. This is manifested in the scalar degree of freedom as the scalar field potential of this model is well behaved function of the field [63]. (ii) The Starobinsky $f(R)$ gravity model with disappearing cosmological constant [62] is a viable cosmological model within the framework of modified gravity which has been studied extensively in last few years in different cosmological and astrophysical context [44, 45, 64, 67]. This will help us to compare the results of $\xi R^n$ gravity model with results of the standard model [62]. We organized this work as follows: In Sec.II we discuss the basics of formalism of the $f(R)$ gravity theory in the Jordan frame and then we transform the formalism to the Einstein frame using the conformal transformations to developed field equations of the scalar degree of freedom. The numerical analysis of the work based on the formalism developed in the Sec.II is done in the Sec.III for our models as mentioned above. Finally in the Sec.IV we concludes the results of our analysis.

II. $f(R)$ GravitY Formalism in Einstein Frame

In this section we will develop the field equations of $f(R)$ gravity in Einstein frame starting from the action of $f(R)$ gravity in the Jordan frame. The $f(R)$ gravity action in the Jordan frame is given by [29, 32],

$$ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \int d^4L_m(g_{\mu\nu}, \Psi_m), \quad (1) $$

where $\kappa^2 = 8\pi G$, $g$ is the determinant of the metric $g_{\mu\nu}$, and $L_m$ is the matter Lagrangian, which is the function of the metric $g_{\mu\nu}$ and the matter fields $\Psi_m$. The variation of the action (1) with respect to $g_{\mu\nu}$ the leads to the equation of motion [32, 45, 52]:

$$ f'R_{\mu\nu} - \nabla_{\mu}f' + \left(\square f' - \frac{1}{2}f\right)g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (2) $$

where $(\square)$ denotes the derivatives with respect to $R$. The term $\square f'$ does not vanish in modified gravity, which means that there is a propagating scalar degree of freedom, defined as $\varphi = f'$, whose dynamics is governed by the trace of the equation (2) given by,

$$ 3\square f' + f'R - 2f = \kappa^2 T. \quad (3) $$

In practice, it is usually difficult to invert the definition of the scalar degree of freedom explicitly for a given $f(R)$ and only convenient to determine effective potential in a parametric form [45]. However, using the conformal transformation to the Einstein frame it is possible to express the scalar degree of freedom of a given $f(R)$ in the explicit form [45]. In view this we now switch to the Einstein frame under the following conformal transformation [32, 68, 69]:

$$ \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (4) $$

where $\Omega^2$ is the conformal factor and the quantities in the Einstein frame are represented by a tilde over them. In these two frames the Ricci scalars have the following relation:

$$ R = \Omega^2 (\tilde{R} + 6\tilde{\Box} \omega - 6\tilde{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega), \quad (5) $$

where

$$ \omega = \ln \Omega, \quad \partial_\mu \omega = \frac{\partial}{\partial x^\mu}, \quad \tilde{\Box} \omega = \frac{1}{\sqrt{-g}} \partial (\sqrt{-g} \tilde{g}^{\mu\nu} \partial_\mu \omega), \quad (6) $$

For convenient we rewrite the action (1) in the form [33]:

$$ S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} f'R - U \right) + \int d^4x L_m(g_{\mu\nu}, \Psi_m), \quad (7) $$
where

\[ U = \frac{f' R - f}{2\kappa^2}. \]  

Using Eq. (5) and the relation \( \sqrt{-g} = \Omega^{-4} \sqrt{-\tilde{g}} \), the action \( S \) can be written as \[ S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} f' \Omega^{-2} (\tilde{\mathcal{R}} + 6 \tilde{\mathcal{E}} - 6 \tilde{g}^\mu\nu \partial_\mu \phi \partial_\nu \phi) - \Omega^{-4} U \right] + \int d^4x \mathcal{L}_m (\Omega^{-2} \tilde{g}^\mu\nu, \Psi_m). \]  

The integral \( \int d^4x \sqrt{-\tilde{g}} \Omega \) can be made to vanish on account of the Gauss’s theorem by using equation (6). Now if make a choice \[ f^2 = f', \]  
for \( f' > 0 \) and introduce a new scalar field \( \phi \) defined by

\[ \kappa \phi \equiv \sqrt{3/2} \ln f', \]  
then the action in the Einstein frame can be expressed as \[ S_E = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} \tilde{g}^\mu\nu \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + \int d^4x \mathcal{L}_m (F^{-1}(\phi) \tilde{g}^\mu\nu, \Psi_m), \]  
where

\[ V(\phi) = \frac{U}{f'^2} = \frac{f' R - f}{2\kappa^2 f'^2} \]  
is the potential of the field \( \phi \). Here the conformal factor is

\[ \Omega^2 = f' = \exp(\sqrt{2/3} \kappa \phi), \]  
which is field dependent. The Lagrangian density of the field \( \phi \) is given by

\[ \mathcal{L}_\phi = -\frac{1}{2} \tilde{g}^\mu\nu \partial_\mu \phi \partial_\nu \phi - V(\phi). \]  

From the matter action (12) it is clear that the scalar field \( \phi \) is directly coupled to matter in the Einstein frame, which can be seen more explicitly if we take the variation of the action (12) with respect to the field \( \phi \) [32]. However, we are not interested in this context, instead we ignore the matter field to derive the scalar field equations without coupling to the matter. Thus taking the variation of the action (12) with respect to the field \( \phi \), we obtain the field equations without the presence of matter as \[ \ddot{\phi} + 3 \dot{H} \phi + V(\phi) = 0, \]  
and

\[ \dot{H}^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]. \]  

Here the dots over \( \phi \) indicate the derivatives with respect to conformal cosmic time (i.e. cosmic time in Einstein frame) \( \tilde{t} \). \( \dot{H} = \frac{\dot{a}}{a} \) is the Hubble parameter in the Einstein frame, \( \dot{a}(t) \) being the scale factor in the same frame. If we consider the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with a cosmic time-dependent scale factor \( a(t) \) and a metric, viz.,

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) dx^2 \]  
in the Jordan frame, then the metric in the Einstein frame may be given as

\[ d\tilde{s}^2 = \Omega^2 ds^2 = f'(-dt^2 + a^2(t) dx^2) = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t}) d\tilde{x}^2, \]  
which leads to the relations of the cosmic time and scale factor in two frame as

\[ d\tilde{t} = \sqrt{f'} dt, \quad \tilde{a} = \sqrt{f} a, \quad \text{for} \quad \sqrt{f'} > 0. \]  

Finally we define the energy density and the pressure of the scalar field as \[ \tilde{\rho}_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \]  
and

\[ \tilde{P}_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \]  

It is evident from the above formalism that, in Einstein frame the scalar degree of freedom for a given \( f(R) \) model can be studied explicitly and conveniently. In the next section we will apply the results of this formalism to study our models of interest.
III. SCALAR FIELD DYNAMICS OF $f(R)$ GRAVITY MODEL

Using the formalism of the previous section, in this section we will study numerically different features of the scalar field and consequent cosmological implications of the power-law and the Starobinsky $f(R)$ gravity models as follows:

A. Power-law $f(R)$ gravity model

The general power-law $f(R)$ gravity model is given by

$$f(R) = \xi R^n,$$  \hspace{1cm} (23)

where $\xi$ and $n$ are model parameters. We use this $f(R)$ gravity model in this study because of its unique character for the existence of power-law solutions and hence absence of the singularity problem as mentioned in the Sec.I [63]. Here our main emphasis is to see the explicit behavior of scalar degree of freedom possessed by the $f(R)$ gravity model and its usefulness to study the cosmological evolution.

Using the equation (11), the scalar field $\phi$ corresponding to the model (23) can be expressed as

$$\phi = \sqrt{\frac{3}{2\kappa}} ln f' = \sqrt{\frac{3}{2\kappa}} ln(\xi n R^{n-1}).$$ \hspace{1cm} (24)

It is clear from this expression that, for the positive valued of $\xi$ and $n$ ($>1$), the scalar field $\phi$ is the increasing function of the Ricci curvature scalar $R$ and hence in this case, when $R \rightarrow \infty$, $\phi \rightarrow \infty$. This indicates that, under this condition there exist no singularity in the scalar field for any finite value of the curvature scalar $R$. Using this expression for $\phi$, the field potential (13) takes the form:

$$V(\phi) = \frac{n - 1}{2\kappa^2 n^2} \left( \frac{exp(\sqrt{2/3\kappa}\phi)}{\xi n} \right)^{\frac{2-n}{n-1}}.$$ \hspace{1cm} (25)

This equation for the field potential shows that the value of $n$ should be within two possible ranges: (i) $n < 1$ and (ii) $1 < n < 2$ for a viable field potential. For our numerical simulation in this work we consider the second range of the values of $n$ and the values of $\xi < 1$, in view of the above discussion for the scalar field $\phi$. We consider $\kappa = 1$ throughout this work.

![Fig. 1: The variation of field potential with respect to the field $\phi$. The first panel is for $n = 1.5$ with $\xi = 0.1$, 0.3 and 0.5, whereas the second panel is for $n = 1.5$ and 1.6 with $\xi = 0.1$. The value of $\kappa$ is taken as one.](image1.png)

![Fig. 1: The variation of field potential with respect to the field $\phi$. The first panel is for $n = 1.5$ with $\xi = 0.1$, 0.3 and 0.5, whereas the second panel is for $n = 1.5$ and 1.6 with $\xi = 0.1$. The value of $\kappa$ is taken as one.](image2.png)

Fig. 1 shows the variation of field potential with respect to the field $\phi$. The first panel is for $n = 1.5$ with $\xi = 0.1$, 0.3 and 0.5, whereas the second panel is the plot for $n = 1.5$ and 1.6 with $\xi = 0.1$. From this figure it is seen that the potential is well behaved function of the field and the potential is small for negative value of the field and it increases very fast as soon as the field $\phi$ become positive. This process is slowing down when the value of $\xi$ increases, which is more effective for lower value range of $\xi$, as clear from the left panel. On the other hand as seen from the second panel that there is no significant difference of the potentials for the different values of $n$ within the range of negative field, but the difference becomes more prominent as soon...
as the field $\phi$ acquired the positive values. The effect of $n$ on the value of the potential is similar to that for $\xi$, but the effect on higher positive value of $\phi$ is more noticeable in this case than the case for $\xi$.

Using the potential (25), the coupled field equations (16) and (17) are solved numerically for different values of $n$ and $\xi$ as mentioned above. The results of the numerical solution are shown in the Fig. 2. The first panel shows the numerical solutions for $n = 1.5$ with $\xi = 0.1, 0.2$ and $0.3$. The second panel is for $n = 1.5$ and $1.6$ with $\xi = 0.1$. In both cases the initial field value and the initial field velocity are taken as $1.0$ and $0.1$ respectively. As clear from the first panel that the value of the field $\phi$ falls towards to the negative values starting from its initial value as time increases. For the initial period of cosmic time $\phi$ falls very fast and this tendency slows down as time passage. For higher values of $\xi$ the overall tendency of the $\phi$ to fall towards the negative values is less in comparison to its lower values. This is similar to the nature of the field potential with respect to $\xi$. Form the second panel it is observed that the field falls faster initially for the smaller value of $n$ as compared to the higher value, but the situation reversed in the latter cosmic times. During the time before the situation is reversed, the difference of the field $\phi$ is not very high, however after this period the difference become increasingly significant as time elapses. We have also studied the effect of the initial field and field velocity on the time evolution of the field, whose results are shown in the last panel of the Fig. 2. This plot is for $n = 1.5$ and $\xi = 0.1$ with $\phi_0 = 1.0, 0.1$ and $\phi_0 = 0.01, 0.001$. We have seen that the initial values of $\phi$ and $\dot{\phi}$ does not have much noticeable effect on the time evolution of $\phi$. Initial value of $\phi$ has slight effect during initial period, which eliminates gradually in latter times. On the other hand the initial values of $\dot{\phi}$ does not have any effect on the time evolution of $\phi$.

From these numerical solutions we have calculated the scale factors $\dot{a}(\tilde{t})$ in Einstein frame for different values of $n$ and $\xi$ as mentioned above. The (cosmic) time evolution of the scale factors obtained from these calculation are shown in the the first two panels of the Fig. 3. The first panel of this figure shows the time evolution of the scale factor for $n = 1.5$ with different values of $\xi$. Whereas the second one is for $\xi = 0.1$ with two different values of $n$. The initial parameters are same as for the first two
panels of the Fig[2]. It is observed from these two panels of this figure that the \( f(R) = \xi R^n \) gravity model leads the very fast expansion of the Universe in the Einstein frame similar to the exponential expansion. The rate of expansion of the Universe slowed down gradually for higher values of \( \xi \) in comparison to its lower values. During initial period this process is slow, but becomes more significant as time passage. On the other hand the expansion is slower for lower value of \( n \) in comparison to the case for the higher value of \( n \) as clear from the second panel. In the initial period the difference of the values of the scale factor is very lass, but becomes more prominent in the latter times.

To see explicitly, whether the acceleration of the Universe always occurs simultaneously in the the Jordan frame, as in the case of the Einstein frame, we have calculated the cosmic time \( t \) and the scale factor \( a(t) \) in the Jordan frame using the equation \((20)\) for all three conditions of the first panel of the Fig[3] which is shown in the last panel of the same figure. This panel shows that, this power law model generates very fast expansion of the Universe is the Jordan frame also similar to that in the Einstein frame, but with a slightly different magnitude and in pattern in the parameter space. Where it is seen that, although the trend of variation of the scale factor, with respect to the parameter \( \xi \), remains same as in the case of the Einstein frame, the initial slowing down of the variation of the scale factor reduces considerably in the Jordan frame. Moreover, the magnitude of the acceleration of the Universe is slightly higher in the Jordan frame than that in the Einstein frame. Obviously, with the increasing value of \( n \), the expansion of the Universe in the Jordan frame will follow the pattern in accordance with the second plot shown for the Einstein frame.

At last we have calculated the equation of state \( \tilde{w} = \rho_\phi / \tilde{P}_\phi \) in Einstein frame using equations \((21)\) and \((22)\) from the numerical solutions mentioned above, which is shown in the Fig[4]. Panels of this figure are for the same model parameters respectively as for the panels of the Fig[3]. It is observed from the figure that the equation of state \( \tilde{w} \) of the scalar field \( \phi \) for the model \((23)\) always less than \(-0.75\) for all cases of our study and for all times. For higher values of \( \xi \), \( \tilde{w} \) is smaller than that for smaller value of \( \xi \) during period for some initial times, but for latter times \( \tilde{w} \) becomes equal for all \( \xi \). But for higher value of \( n \), \( \tilde{w} \) is always smaller than that for the smaller value of \( n \) with a similar trend with respect to time and \( \xi \) as mentioned above. As the equation of state is always negative, therefore the model \((23)\) could produced the accelerated expansion of the Universe and it corresponds to dark energy, whose equation of state is \( w < -1/3 \) (in the Jordan frame also the equation of state will lay within this range as clear from the last panel of the Fig[3]). If we increase the values of \( n \) then the equation of state \( \tilde{w} \) in Einstein frame will approach to \(-1\). This implies that the model \((23)\) with higher values of \( n \) \( (n \rightarrow 2) \) corresponds to the cosmological constant \( \Lambda \).

**B. Starobinsky \( f(R) \) gravity model**

The Starobinsky \( f(R) \) gravity model with disappearing cosmological constant \((62)\) is,

\[
f(R) = R + \lambda R_0 \left[ \left(1 + \frac{R^2}{R_0^2}\right)^{-\frac{1}{n}} - 1 \right],
\]

where \( n, \lambda \) and \( R_0 \) are free parameters. The values of \( n \) and \( \lambda \) are greater than zero, whereas the value of \( R_0 \) is of the order of the presently observed cosmological constant \( \Lambda \) \((= 8\pi G\rho_{\text{vac}})\). This is carefully constructed model to give viable cosmology.
and to be satisfied with solar system and laboratory tests. Notwithstanding, it suffers from the singularity problem \cite{45}, which could be cured by adding a term \( \propto R^2 \) to it as observed in \cite{47,48} and consequently this observation was followed in \cite{44}. With this modification, the Starobinsky model \cite{27} takes the form:

\[
f(R) = R + \frac{\beta}{R_0} R^2 + \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right],
\]

(27)

where \( \beta \) is the positive constant parameter.

The scalar field embodied in the model \cite{26} which is obtained by using the equation \cite{11} can be written as

\[
\phi = \sqrt{\frac{3}{2} \lambda} |n f'| = \sqrt{\frac{3}{2} \lambda} \ln \left[ 1 - 2 \lambda n \left( \frac{R}{R_0} \right)^{1 + \frac{1}{n}} \right].
\]

(28)

From the observation of this equation it is quite clear that, for the given values of the parameters \( n \) and \( \lambda \), the scalar field \( \phi \) increases from the negative side with the increasing value of the Ricci curvature scalar \( R \). So when \( R \to \infty \), \( \phi \to \infty \), which is the point of singularity. However, if we express the scalar field associated with the Starobinsky modified model \cite{27}, it takes the form:

\[
\phi = \sqrt{\frac{3}{2} \lambda} |n f'| = \sqrt{\frac{3}{2} \lambda} \ln \left[ 1 + \frac{R}{R_0} \left\{ 2 \beta - 2 \lambda n \left( 1 + \frac{R^2}{R_0^2} \right)^{-(n+1)} \right\} \right].
\]

(29)

This equation indicates that for the large value of \( R \), the second term coming from \( \propto R^2 \) term of the model \cite{27} dominates over other terms in the expression and hence the situation as discussed above for the equation \cite{28} never arise here, but when \( R \to \infty \), \( \phi \to \infty \). Thus in this case for a finite curvature, we get finite value of the field \( \phi \). Nevertheless, in the context of our motivation, we will use the Strobinsky original model \cite{26} for the further discussion below. We will use the modified model \cite{27} only to see the effect on singularity in the scalar field potential.

Now, if we use the expression \cite{28} for \( \phi \) in a region where \( R/R_0 \gg 1 \), the field potential \cite{13} for the Starobinsky model \cite{26} becomes,

\[
V(\phi) = -\lambda \left[ \frac{(2n + 1) \left( 1 - \exp(\sqrt{2/3} \phi) \right)^{2n}}{2n \exp(\sqrt{8/3} \phi) 2^{n+1}} - 1 \right].
\]

(30)

Similarly, using the expression \cite{29} for \( \phi \) in the same region, the field potential \cite{13} for the modified model \cite{27} to a good approximation can be written as

\[
V(\phi) = \frac{\beta \left( \frac{\exp(\sqrt{2/3} \phi) - 1}{2\beta} \right)^2 - \lambda \left[ \frac{\exp(\sqrt{2/3} \phi) - 1}{2\beta} \right]^{-2n} - 1}{2^{n+1} \exp(\sqrt{8/3} \phi) 2^{n+1}}.
\]

(31)

The expression for the field potential \cite{30} shows that for the validity of the Starobinsky model \cite{26} the values of \( n \) and \( \lambda \) must be greater than zero, but may take any values freely beyond it in contrast with the power-law model \cite{23}. However, for our numerical work on this model, we have taken \( n = 1 \) and 2 with the values of \( \lambda = 1.2, 1.5, 2.0 \) and 4.0. We have followed the same numerical procedures for this model also that have been used for the case of power-law model.

The behaviours of the scalar field potential \cite{30} with respect to the field \( \phi \) associated with the Starobinsky model \cite{26} for different values of \( n \) and \( \lambda \) as mentioned above are shown in the top three panels of the Fig.5. We observed that for smaller values of \( \lambda \), when the field \( \phi \) is rolling from negative to zero, the potential increases very fast from negative to its maximum positive value as the field reached its zero value and then falls off rapidly to a almost flat region near its zero value when the field increases again from its zero value. This behaviour of the potential for such \( \lambda \) values form a pattern like the \( \lambda \)-pattern. Beyond this range of smaller values of \( \lambda \), the \( \lambda \)-pattern of the potential vanishes gradually and the potential always remain positive for all values of the field \( \phi \). The range of this smaller values of \( \lambda \) depends on the value of \( n \), higher the value of \( n \), smaller is this range. Thus for higher values of \( n \) and \( \lambda \), the potential falls off from a very high value to its almost flat region near zero. Moreover, for all values of \( n \) and \( \lambda \) the potential becomes flat at a particular high positive value of \( \phi \) depending on the value of \( \lambda \), higher the value of \( \lambda \), slightly higher this value of \( \phi \). It should be noted that, the unusual behaviour of the field potential around the zero value of field \( \phi \) for the lower and higher values of \( \lambda \) depending on the value of \( n \) is the exhibition of the kind of singularity behaviour of the field \( \phi \), as mentioned above.
The general pattern of time evolution of the field is that the field magnitude increases for a period before the field of their smaller values. It should be noted that for the given parameter \(\lambda\), the field potential initially increases very fast to its highest peak value from an initial minimum and then falls off rapidly from its initial positive value to its minimum. However, in the case of \(\beta = 0.8\), initial value of the field potential increases very fast from higher negative value to its positive peak value around the value of the \(\phi = 1\), which shows some unusual nature. However, in maximum situations we may avoid the singularity behaviour of the scalar field potential by using the modified Starobinsky model (27). Moreover, it should be noted that, scalar field \(\phi\) associated with this model is always positive as we have pointed above already. As already mentioned above, still we are using the Starobinsky original model (26) to continue our discussions further, in view of our motivation of this work.

The bottom two panels of the Fig. 5 show as an example the behaviours of the field potential (31) with respect field \(\phi\) associated with the modified Starobinsky model (27) for different values of \(\lambda\) and \(\beta\) with \(n = 2\). It is clear from these panels that, there is no \(\lambda\)-pattern or very large value of the field potential for any value of the parameter \(\lambda\). From the bottom left panel, which is for \(\beta = 0.5\), we see that for some lower value of the parameter \(\lambda\), the field potential increases slowly from its minimum to maximum value at a particular value of the field \(\phi\) and then remains constant for all values of the field. Above such values of the parameter \(\lambda\), the field potential initially increases very fast to its highest peak value from a initial minimum and then falls to same minimum value at same value of the field \(\phi\) as in the case of the lower value of the parameter \(\lambda\) to behave in a similar fashion for all parameter \(\lambda\). The value of initial minimum and the peak of the field potential increase with increasing value of the parameter \(\lambda\). From the bottom right panel (\(\lambda = 4.0\)) it is clear that, the initial behaviour of the field potential depends on the the value of the parameter \(\beta\). For \(\beta = 0.8\), initial value of the field potential increases very fast from higher negative value to its positive peak value around the value of the \(\phi = 1\), which shows some unusual nature. However, in maximum situations we may avoid the singularity behaviour of the scalar field potential by using the modified Starobinsky model (27). Moreover, it should be seen that, field \(\phi\) associated with this model is always positive as we have pointed above already. As already mentioned above, still we are using the Starobinsky original model (26) to continue our discussions further, in view of our motivation of this work.

The Fig 5 shows the results of numerical solution of field equations (16) and (17) using the field potential (31) for different values of \(n\) and \(\lambda\). The initial values of field \(\phi\) and field velocity \(\dot{\phi}\) are same as their values used in the case of power-law model. The general pattern of time evolution of the field is that the field \(\phi\) falls off rapidly from its initial positive value to its minimum negative value and then oscillates for sometimes with a decreasing amplitude before becomes steady with a negative value. For higher values of \(\lambda\) and \(n\) the field is legging behind the field for the lower values of these parameters by shifting its minimum towards more negative side. This indicates that the magnitude of the field is smaller for higher values of \(\lambda\) and \(n\) in comparison to the field of their smaller values. It should be noted that for the given \(\lambda\) or the given \(n\) this behaviour of the field \(\phi\) is observed for different values of \(n\) or \(\lambda\). However, in the case of \(\lambda = 4.0\) for \(n = 2\) a slight different behaviour of the field is observed as seen from the top right panel of this figure wherein the field falls from higher initial values in contrast with the behaviour for other cases. The initial field velocity has noticeable effect on the time evolution of the field as it is observed from the bottom right panel of this figure that for the lower value of the initial field velocity the field magnitude increases for a period before the
field \( \phi \) becomes steady with the same value attained for other initial conditions.

The study of the scale factor evolution in the Einstein frame using the scalar degree of freedom of this model, as we have studied in the case of the power-law \( f(R) \) model, is shown in the first two panels of the Fig. 7. It is clear from the first panel of this figure that, during a considerable initial period the expansion of the Universe is slower and after that period the Universe experiences very fast accelerated expansion similar to the exponential expansion under this model. This initial period becomes shorter as the value of \( \lambda \) increases. Moreover, for a given value of \( \lambda \) this slower expansion period of the Universe is shorter for the higher value of \( n \) as shown in the second panel of this figure. That is higher the value of \( n \) and \( \lambda \), the earlier is the period of accelerated expansion of the Universe. Thus this model could explain the late time accelerated expansion of the Universe with the suitable values of \( n \) and \( \lambda \).

For the same purpose as in the case of power law \( f(R) \) gravity model, here also we have calculated the cosmic time \( t \) and the scale factor \( a(t) \) in the Jordan frame for all conditions of the first panel of the Fig. 7. The results of the calculation are shown in the last panel of the first figure. It shows that, the Starobinsky \( f(R) \) gravity model (26) produces the accelerated expansion of the Universe in the Jordan frame also with exactly similar patterns for the values of the parameter \( \lambda \) as in the case of the Einstein frame, but with a many times higher in magnitude, which increases with increasing cosmic time.

As described in the previous Sec.III(A) for the power-law \( f(R) \) gravity model, we have calculated in the same way the equation of state of the scalar field related with the Starobinsky \( f(R) \) gravity model (26). The results of this numerical calculation is shown in the Fig. 8. The patterns of time evolution of the equation state of the scalar field for different values of \( \lambda \) and \( n \) are almost similar to the patterns of time evolution of the field \( \phi \) for theses parameters. That is the equation of state is also oscillate with decreasing amplitude during a period after falling from its initial positive value and then after that period it remains steady with time. During a initial period depending on the value of \( \lambda \) and \( n \) the equation of state falls from \( 1 \) to \(-1\), which indicates that during this period the scalar field corresponding to the model (26) make a transition from free scalar field \((w = 1)\) through matter \((w \approx 0, \) radiation \((w = 1/3)\) and dark energy \((w < -1/3)\) to cosmological constant \(\Lambda \) \((w = -1)\). During the oscillation
FIG. 7: First two panels: Evolution of the scale factor in the Einstein frame obtained from the numerical solutions of equations (16) and (17) using the field potential (30) for different values of \( n \) and \( \lambda \). The first panel is for \( n = 1 \) with different values of \( \lambda \). Whereas, the second panel is for \( \lambda = 2.0 \) with two values of \( n \). In both these panel \( \phi_0 = 1.0 \) and \( \dot{\phi}_0 = 0.1 \) are used. Last panel: Evolution of the scale factor in the Jordan frame obtained from the first panel by using the equation (20).

period the scalar field behaves as matter to the cosmological constant and finally behaves as the cosmological constant \( \Lambda \) after completing this period depending on the values of the parameters \( \lambda \) and \( n \). There is a special behaviour of the equation of state for the \( n = 2 \) and \( \lambda = 4.0 \) during the initial period. In contrast to the behaviour of the equation of states for the other pairs of parameters, the equation of state of this pair of parameters initially make transit from the dark energy state to other states and then make transition to the cosmological constant state. However for latter periods the behaviour of equation of state for this pair of parameters is similar to the other cases.

FIG. 8: Equation of state of the scalar field in the Einstein frame obtained from the numerical solutions of equations (16) and (17) using the field potential (30) for different values of \( n \) and \( \lambda \). The first two panel is for \( n = 1 \) and 2 respectively with different values of \( \lambda \). Whereas the last panel is for \( \lambda = 4.0 \) with two vales of \( n \). All initial conditions are same as in the previous figure.

IV. CONCLUSION

We have investigated the scalar degrees of freedom and it’s related cosmological implications of two \( f(R) \) gravity models, viz., (i) the power-law \( f(R) \) gravity model \( \xi R^n \) and (ii) the Starobinsky disappearing cosmological constant \( f(R) \) gravity model for different values of the parameters of these models. This study is basically on the theoretical interests rather than the observational consequences. The power-law \( f(R) \) gravity model is considered due to the existence it’s power-law solutions and consequently absence of singularity problem [63]. On the other hand the Starobinsky \( f(R) \) gravity model with the disappearing cosmological constant [62] is a standard \( f(R) \) gravity model which provide viable cosmology by satisfying the solar system and the laboratory tests. As the scalar degree of freedom of \( f(R) \) gravity model can be treated explicitly in the Einstein frame, so we used this frame for this study by conformal transformations from the Jordan frame.

It is found that the field potential of the power-law \( f(R) \) gravity model is the well behaved function of the the field \( \phi \) representing the scalar degree of freedom of the model and increases very fast as \( \phi \) increases, which depends on the model parameters \( \xi \) and \( n \) such that for the lower value of them such process is faster than higher value. From the numerical simulation
of the field equations, we found that $\phi$ falls towards the negative values as time increases, which is more faster during the initial period of time than late times. The rate of falling the value of the field slower for the higher values of the parameter $\xi$. However $n$ does not have much impact as $\xi$ has on the behaviour of field $\phi$ with respect to time. Initial conditions on the field and field velocity have negligible effect on the nature of the time evolution of $\phi$.

In the case of the Starobinsky $f(R)$ gravity model with disappearing cosmological constant the field potential shows a peculiar behaviour with respect to the field $\phi$. When the field increases from its negative value to zero, the field potential increases from its lowest negative value to the maximum positive value and then falls off very fast to the flat region near to its zero value as the field increases from zero to higher positive values for a lower value range of the parameter $\lambda$. This range of the lower value of the parameter $\lambda$ decreases with increasing the value of $n$. This behaviour of the field potential form a pattern similar to the $\lambda$-pattern. The $\lambda$-pattern gradually vanishes with increasing values of the parameters $\lambda$ and $n$. For higher values of the field $\phi$ the values of field potential for different $\lambda$ and $n$ parameters are almost identical with a slight dependence on the value of the parameter $\lambda$. All unusual behaviours of the field potential can be avoided by adding a term $\propto R^2$ to the Starobinsky model. The numerical solution for the time evolution of the field $\phi$ for this model shows that the field falls off rapidly from its maximum positive value to its minimum negative value in the earlier period and then starts oscillating with a decreasing amplitude with time for a period before become steady with time taking a particular negative value. This pattern of time evolution of the field $\phi$ depends on the model parameters in such a way that with increasing values of $\lambda$ and $n$ the field $\phi$ leg behind its values for lower model parameters with decreasing amplitudes to attain steady value much earlier.

The study of the time evolution of the scale factor in the Einstein frame shows that the Universe is accelerating very fast for both of our $f(R)$ gravity models, which depends on values of the parameters of the models. For the case of the power-law $f(R)$ model the scale factor depends on the parameters $\xi$ and $n$ in such a way that accelerating process slows down for higher values of $\xi$ and for lower values of $n$. Whereas in the case of the Starobinsky $f(R)$ model the accelerating process becomes more faster for the higher value of both parameters $\lambda$ and $n$. Moreover, in this case there is a initial period depending on the value of $\lambda$ in which the expansion of the Universe is very slow. This initial period is shorter for the higher value of the parameter $\lambda$ and does not depends on the parameter $n$. After this initial period the Universe is expanding with a very fast acceleration process. This indicates that this model with the suitable model parameters $\lambda$ and $n$ is very effective to give the late time accelerated expansion of the Universe. We found that, the Universe expands almost in similar fashions in the Jordan frame also as in the case of the Einstein frame for both these models, but with slightly higher in magnitude for the power law model and with much higher in magnitude for the Starobinsky model.

The equation of state of the scalar field for the power-law model is always negative and less than $-1/3$, which corresponds to the behaviour of the dark energy that produces the accelerated expansion of the Universe. The equation of state of the field for this model tends to $-1$ as the value of $n$ increases towards higher values, nearly equal to $2$. Thus for such values of $n$, the scalar field for the model behave as the cosmological constant $\Lambda$. The same study for the scalar field of the Starobinsky model shows that the equation of state oscillates with a decreasing amplitude with time for a period before falling from its maximum value around $1$ to its minimum value at $-1$. After this period the equation of state takes the value equal to $-1$. This implies that the scalar field related with the Starobinsky model also behaves like the cosmological constant $\Lambda$. The behavior of the time evolution and the dependence of the equation of state on the model parameters $\lambda$ and $n$ are almost similar to the field $\phi$ itself.

At last it is interesting to mention that, our study can be applied to more complicated, unified (inflation plus cosmic acceleration) $f(R)$ models of the form given in [70], to see more explicitly the stages of the inflation and the cosmic acceleration of our Universe by using the scalar degree of freedom in Einstein frame. From the observational point of view, the results can be easily transformed to Jordan frame (which is clear from our discussion in the previous sections), as by convention the Jordan frame is considered as physical frame, although is still pre-mature to say, which frame is the realistic one, as far as the present cosmological observations are concerned. We are planing to work on these models and issues in near future.

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