CHARACTERIZATIONS OF HARDY-ORLICZ SPACES OF QUASICONFORMAL MAPPINGS

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ABSTRACT. An $H^p$-theory of quasiconformal mappings on $\mathbb{B}^n$ has already been established. By replacing $t^p$ with a general increasing growth function $\psi(t)$ we define the Hardy-Orlicz spaces of quasiconformal mappings and prove various characterizations of these spaces.

1. Introduction

Hardy-Orlicz spaces are a natural generalization of the Hardy spaces. Holomorphic functions on the unit disk in $\mathbb{C}$ belonging to Hardy-Orlicz spaces have been studied in [16], [10], [9], and [8]. For the higher dimensional case of holomorphic maps on the unit ball in $\mathbb{C}^n$ see for example [4], [14] and [6]. In this paper we are interested in the generalization of Hardy spaces of quasiconformal mappings on the unit ball in $\mathbb{R}^n$.

A quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{R}^n$ belongs to the Hardy space $H^p$ for a fixed $0 < p < \infty$ if the values $\int_{\mathbb{S}^{n-1}} |f(r\omega)|^p d\sigma$ are uniformly bounded for all $r \in [0, 1)$. Here $\sigma$ is the surface measure on $\mathbb{S}^{n-1}$. For results on these spaces see especially [2] and also [13] and [12]. We highlight in particular the following characterization theorem, proved as several results in [2].

Theorem A. Let $f$ be a quasiconformal mapping of $\mathbb{B}^n$, $f_i$ one of its component functions and fix $0 < p < \infty$. Then the following are equivalent:

(1) $f \in H^p$
(2) $f(\omega) \in L^p(\mathbb{S}^{n-1})$
(3) $f^*(\omega) \in L^p(\mathbb{S}^{n-1})$
(4) $f_i^*(\omega) \in L^p(\mathbb{S}^{n-1})$
(5) $\int_0^1 (1-r)^{n-2} M(r, f)^p dr < \infty$
(6) $\int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p-1} dx < \infty$.

The function $a_f(x)$ is an averaged version of the differential $Df(x)$, see Section 2 for its definition. The equivalence of (1) and (6) in Theorem A is the quasiconformal version of the following area characterization for $f$ conformal:

$$f \in H^p \text{ if and only if } \int_{\mathbb{B}^2} |f'(x)|^p (1-|x|)^{p-1} dx < \infty,$$

see [3]. The non-tangential maximal function $f^*$ and maximum modulus $M(r, f)$ are defined in Section 3.

Our main results generalize Theorem A to Hardy-Orlicz spaces of quasiconformal mappings. Let $\psi$ be a growth function; that is, a differentiable and strictly increasing function.
mapping \([0, \infty]\) to itself such that \(\psi(0) = 0\). Then a quasiconformal mapping \(f : \mathbb{R}^n \to \mathbb{R}^n\) belongs to the Hardy-Orlicz space \(H^\psi\) if there exists \(\delta > 0\) such that
\[
\sup_{0 < r < 1} \int_{S^n} \psi(\delta |f(r\omega)|)d\sigma(\omega) < \infty.
\]

Our first result shows that much of Theorem A extends to the Hardy-Orlicz setting without any additional restrictions on the growth of \(\psi\).

**Theorem 1.1.** Given a growth function \(\psi\) and a quasiconformal mapping \(f : \mathbb{R}^n \to \mathbb{R}^n\) the following are equivalent:

1. \(f(x) \in H^\psi\)
2. \(\psi(\delta_1 |f(\omega)|) \in L^1(S^{n-1})\) for some \(\delta_1 > 0\)
3. \(\psi(\delta_2 f^*(\omega)) \in L^1(S^{n-1})\) for some \(\delta_2 > 0\)
4. \(\int_0^1 (1-r)^{n-2} \psi(\delta_3 M(r,f))dr < \infty\) for some \(\delta_3 > 0\).

The characterizations involving \(a_f(x)\) and the component function \(f_i\) are extended to the Hardy-Orlicz spaces that have a doubling condition on the growth function \(\psi\) and its inverse.

**Theorem 1.2.** Let \(f\) be a quasiconformal mapping of \(\mathbb{R}^n\), \(f_i\) one of its component functions and \(\psi\) a growth function such that both \(\psi\) and \(\psi^{-1}\) are doubling. Then the following are equivalent:

1. \(f(x) \in H^\psi\)
2. \(\int_{\mathbb{R}^n} \psi(a_f(x)(1 - |x|)) \frac{dx}{1-|x|} < \infty\)
3. \(\psi(f^*_i(\omega)) \in L^1(S^{n-1})\)

The equivalences in Theorem 1.2 can fail if either \(\psi\) or \(\psi^{-1}\) is not doubling. For example, if \(f(z) = z\) then \(f\) belongs to every Hardy-Orlicz space \(H^\psi\). However, we can construct a growth function \(\psi\) such that the integral \(\int_{1/2}^1 \frac{\psi(1-r)}{1-r} dr\) diverges, thus failing the implication (1) \(\Rightarrow\) (2) for this \(f\). The inverse of the growth function from our example is not doubling. The implications (2) \(\Rightarrow\) (1) and (3) \(\Rightarrow\) (1) both fail for \(f(z) = \log(z+1)\) and an appropriate growth function \(\psi\) that is not doubling. See section 4 for details.

This paper is organized as follows. Section 2 includes notation, definitions and background lemmas necessary for the proofs of our main results. Section 3 focuses on the characterizations of Hardy-Orlicz spaces that hold for all growth functions, and Section 4 gives the results that hold for \(\psi\) satisfying additional growth conditions.

2. Preliminaries

We denote by \(B^n(x,r)\) the open ball in \(\mathbb{R}^n\) of radius \(r\) centered at \(x\) and write its boundary as \(S^{n-1}(x,r)\). We abbreviate \(B^n(0,1) = \mathbb{R}^n\), \(S^{n-1}(0,1) = S^{n-1}\) and let \(\omega_{n-1}\) denote the surface measure of \(S^{n-1}\). For each \(x \in \mathbb{R}^n\) we set \(B_x = B^n(x,(1-|x|)/2)\) and then define the cap \(S_x = \{ \frac{y}{|y|} : y \in B_x, y \neq 0\}\). Given \(\omega \in S^{n-1}\) let
\[
\Gamma(\omega) = \bigcup\{B_{t\omega} : 0 \leq t < 1\}
\]
be a Stolz cone at \(\omega\). Clearly \(x \in \Gamma(\omega)\) if and only if \(\omega \in S_x\).

When a constant is written as \(C = C(a,b,...)\) it means that the value of \(C\) depends only on the values of \(a,b,...\). The values of constants may change from line to line in a sequence of inequalities without explicit mention or special notation. We use the symbol \(A \approx B\) to denote that there exists a constant \(C\) such that
\[
\frac{A}{C} \leq B \leq CA.
\]
A homeomorphism of a domain $\Omega$ in $\mathbb{R}^n$ into $\mathbb{R}^n$ is $K$-quasiconformal if $f$ belongs to the Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ and $|Df(x)|^n \leq K|J_f(x)|$ for almost every $x \in \Omega$. In this paper all quasiconformal mappings will have as domain $\mathbb{R}^n$.

The quasiconformal analogue of Beurling’s theorem, [5 Theorem 4.4], says that given a quasiconformal mapping $f : \mathbb{H}^n \rightarrow \mathbb{R}^n$ the radial limit
\[
f(\omega) := \lim_{r \rightarrow 1} f(r\omega)
\]
exists for almost every $\omega \in S^{n-1}$.

An important tool for us will be the modulus of curve families. Given a family of curves $\Gamma$ in $\mathbb{R}^n$ the modulus $\text{Mod}(\Gamma) \in [0, \infty]$ is
\[
\text{Mod}(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^n \, dm,
\]
where the infimum is taken over all admissible Borel functions $\rho : \mathbb{R}^n \rightarrow [0, \infty]$. A non-negative Borel function $\rho$ on $\mathbb{R}^n$ is considered admissible if $\int_\gamma \rho \, ds \geq 1$ for each locally rectifiable $\gamma \in \Gamma$. We collect some basic results about modulus of curve families here; their proofs can be found in [17].

First, the modulus of a curve family is quasi-invariant under quasiconformal mappings. More precisely, if $f : \Omega \rightarrow \mathbb{R}^n$ is $K$-quasiconformal then $\text{Mod}(\Gamma)/K \leq \text{Mod}(f\Gamma) \leq K\text{Mod}(\Gamma)$ for every family of curves $\Gamma \subset \Omega$. Here $f\Gamma = \{ f \circ \gamma : \gamma \in \Gamma \}$.

The exact value of the modulus can be calculated for certain families of curves. Let $\Gamma$ be the collection of radial segments joining $S(0, r), 0 < r < 1$, to a Borel set $E \subset S^{n-1}$. Then
\[
\text{Mod}(\Gamma) = \sigma(E)(\log(1/r))^{1-n}.
\]
Moreover, if $\Gamma$ is a family of curves with each $\gamma \in \Gamma$ joining $S^{n-1}(x, r)$ to $S^{n-1}(x, R)$, $0 < r < R$, then we have the upper bound
\[
\text{Mod}(\Gamma) \leq \frac{\omega_{n-1}}{(\log(R/r))^{n-1}}.
\]

We will make repeated use of the following lemma (cf. [17 Theorem 18.1]), which is a direct result of modulus estimates.

**Lemma 2.1.** Let $f : \mathbb{H}^n \rightarrow \Omega$ be $K$-quasiconformal. There is a constant $C$ depending only on $n$ and $K$ so that for each $x \in \mathbb{H}^n$
\[
diam(f(B_x))/C \leq d(f(x), \partial \Omega) \leq Cdiam(f(B_x)).
\]
Moreover, $f(B_x)$ contains a ball of radius $d(f(x), \partial \Omega)/C$, centered at $f(x)$.

The following lemmas are proved similarly using modulus estimates.

**Lemma 2.2.** Let $f$ be a $K$-quasiconformal mapping of $\mathbb{H}^n$ onto $\Omega \subset \mathbb{R}^n$. For each $x \in \mathbb{H}^n$ and $M > 1$
\[
\sigma(\{ \omega \in S_x : |f(\omega) - f(x)| > Md(f(x), \partial \Omega) \}) \leq C\sigma(S_x)(\log M)^{1-n},
\]
where the constant $C$ depends only on $n, K$.

**Proof.** Abbreviate $d = d(f(x), \partial \Omega)$ and $E = \{ \omega \in S_x : |f(\omega) - f(x)| > Md \}$ and first assume that $|x| < 1/4$. Let $\Gamma_E$ be the set of radial segments with one endpoint in $E$ and the other in $B_x \cap S(0, 1/4)$. Then $\text{Mod}(\Gamma_E) = \sigma(E)(\log 4)^{1-n}$ and by Lemma 2.1 there exists a constant $C = C(n, K)$ such that each curve in $f(\Gamma_E)$ joins $S(f(x), Cd)$ with $S(f(x), Md)$. If $2 \leq C$ and $C^2 < M$ then
\[
\sigma(E)(\log 4)^{1-n} = \text{Mod}(\Gamma_E) \leq K\text{Mod}(f(\Gamma_E)) \leq C\omega_{n-1}(\log M)^{1-n}
\]
and if $1 < M \leq C^2$ then we have anyway
\[ \sigma(E) \leq \omega_{n-1}(\log C^2)^{n-1}(\log M)^{1-n}. \]

For the other case when $1/4 \leq |x|$ set $\Gamma_E$ to be the collection of radial segments with one endpoint in $E$ and the other endpoint in $B_x \cap S(0, |x|)$. Then $\text{Mod}(\Gamma_E) = \sigma(E)(\log \frac{1}{|x|})^{1-n}$ and as above each curve in $f(\Gamma_E)$ joins $S(f(x), Cd)$ with $S(f(x), Md)$. Calculating like before, the case $1 < M \leq C^2$ is trivial and assuming $C^2 < M$ we get that
\[ \sigma(E)(\log 1/|x|)^{1-n} \leq C \omega_{n-1}(\log M)^{1-n}. \]

Noting that $(\log 1/|x|)^{n-1} \approx \sigma(S_x)$ we are done. \hfill \Box

**Lemma 2.3.** Let $f : \mathbb{B}^n \to \mathbb{R}^n$ be $K$-quasiconformal with $f(x) \neq 0$ for all $x \in \mathbb{B}^n$, $\phi$ a growth function and $\delta > 0$. There is an absolute constant $C$ such that for each $x \in \mathbb{B}^n$ and $M > 1$,
\[ \sigma(\{\omega \in S_x : \phi(\delta |f(\omega)|) < \phi(\delta |f(x)|/M)\}) \leq C \sigma(S_x)(\log M)^{-1}. \]

**Proof.** Let $x \in \mathbb{B}^n$ and $\omega \in S_x$. It suffices to prove the inequality in the instance that $\delta = 1$ and $\phi$ is the identity map on $[0, \infty]$, since growth functions are strictly increasing.

Set $E = \{\omega \in S_x : |f(\omega)| < |f(x)|/M\}$, and choose the curve families $\Gamma_E$ like in the proof of Lemma 2.2, taking separately the cases $|x| \leq 1/4$ and $1/4 < |x| < 1$. Each curve belonging to $f(\Gamma_E)$ will have one endpoint in $B(f(x), C|f(x)|)$ and the other in $\mathbb{R}^n \setminus B(f(x), |f(x)|/M)$, for some absolute constant $C$. The desired upper bound follows using the same modulus of curve family techniques as in the proof of Lemma 2.2. \hfill \Box

The average derivative of a $K$-quasiconformal mapping, introduced by Astala and Gehring [1], is defined as
\[ a_f(x) = \exp\left(\int_{B_x} \log J_f(y) \frac{dm}{n|B_x|}\right). \]

Here $|B_x|$ is the $n$-measure of $B_x$. The mean value property implies that $a_f = |Df|$ if $f$ is conformal. The following lemma is proved in [1], and it is an example of how $a_f(x)$ can take the place of $|f'(x)|$ in quasiconformal analogues of statements originally proved for conformal mappings on the unit disk.

**Lemma 2.4.** Let $f : \mathbb{B}^n \to \Omega$ be $K$-quasiconformal. There is a constant $C$ depending only on $n, K$ so that for each $x \in \mathbb{B}^n$
\[ d(f(x), \partial \Omega)/C \leq a_f(x)(1 - |x|) \leq Cd(f(x), \partial \Omega) \]

and
\[ \frac{1}{C} \left( \frac{1}{|B_x|} \int_{B_x} |Df(y)|^n dm \right)^{1/n} \leq a_f(x) \leq C \left( \frac{1}{|B_x|} \int_{B_x} |Df(y)|^n dm \right)^{1/n}. \]

The following is a consequence of Lemmas 2.1 and 2.3. See [2] Lemma 2.5 for details.

**Lemma 2.5.** Let $f : \mathbb{B}^n \to \Omega$ be $K$-quasiconformal. Suppose that $u > 0$ satisfies
\[ u(x) \approx u(y) \]
for each $x \in \mathbb{B}^n$ and all $y \in B_x$. Let $0 < q \leq n$ and $p \geq q$. Then
\[ \int_{\mathbb{B}^n} a_f^n(x)u(x)dx \approx \int_{\mathbb{B}^n} a_f^{p-q}|Df(x)|^q u(x)dx. \]
3. Characterizations of $H^\psi$

With each quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{R}^n$ we associate its maximum modulus function

$$M(r, f) = \{\sup |f(x)| : |x| = r\}, \quad r \in (0, 1).$$

**Theorem 3.1.** Let $f$ be a $K$-quasiconformal mapping of $\mathbb{B}^n$ and $\psi$ a growth function. Then the following are equivalent:

$$\psi(\delta_1 |f(\omega)|) \in L^1(S^{n-1}) \text{ for some } \delta_1 > 0.\quad (3.1)$$

$$\int_0^1 (1 - r)^{n-2} \psi(\delta_2 M(r, f))dr < \infty \text{ for some } \delta_2 > 0.\quad (3.2)$$

**Proof.** First suppose $(3.2)$ holds for some $\delta > 0$. We can assume that $f(0) = 0$. We will show in this case that there exists a constant $C = C(n, K)$ such that

$$\int_{S^{n-1}} \psi(\frac{\delta}{2} |f(\omega)|)d\sigma(\omega) \leq C \int_0^1 (1 - r)^{n-2} \psi(\delta M(r, f))dr.\quad (3.3)$$

To prove $(3.3)$ we rewrite the integral on the left as

$$\int_{S^{n-1}} \psi(\frac{\delta}{2} |f(\omega)|)d\sigma(\omega) = \int_0^\infty \psi'(\lambda)\sigma(\{\omega \in S^{n-1} : \frac{\delta}{2} |f(\omega)| > \lambda\})d\lambda.$$

Let $E = \{\omega \in S^{n-1} : \frac{\delta}{2} |f(\omega)| > \lambda\}$ for a fixed $\lambda > 0$. We obtain an upper bound on $\sigma(E)$ using modulus of curve families as follows.

There exists a unique $r = r(\lambda)$ such that

$$\delta M(r, f) = \lambda.$$

Let $\Gamma_E$ be the family of radial segments connecting $B(0, r)$ to $E$. Then

$$M(\Gamma_E) = \frac{\sigma(E)}{(\log(1/r))^{n-1}} \geq \frac{\sigma(E)}{2^{n-1}(1 - r)^{n-1}},$$

as long as $1/2 < r < 1$.

Each curve in $f(\Gamma_E)$ connects $B(0, \lambda/\delta)$ to $\mathbb{R}^n \setminus B(0, 2\lambda/\delta)$, and so

$$M(f\Gamma_E) \leq \frac{\omega_{n-1}}{(\log 2)^{n-1}}.$$

Since $M(\Gamma_E) \leq KM(f\Gamma_E)$ we have

$$\sigma(E) \leq C(n, K)(1 - r)^{n-1}$$
whenever \( \delta M(r, f) = \lambda \) and \( 1/2 < r < 1 \). This estimate and an application of Fubini’s theorem now give

\[
\int_0^\infty \psi'(\lambda)\sigma(\{\omega \in \mathbb{S}^{n-1} : \frac{\delta}{2}|f(\omega)| > \lambda\})d\lambda \leq \\
\leq \sigma(\mathbb{S}^{n-1})\psi(\delta M(1/2, f)) + C(n, K) \int_0^\infty \psi'(\lambda)(1 - r(\lambda))^{n-1}d\lambda = \\
= \sigma(\mathbb{S}^{n-1})\psi(\delta M(1/2, f)) + C(n, K) \int_0^\infty \psi'(\lambda) \int_{r(\lambda)}^1 (1 - t)^{-n-2}dt d\lambda = \\
= \sigma(\mathbb{S}^{n-1})\psi(\delta M(1/2, f)) + C(n, K) \int_0^1 (1 - t)^{-n-2} \int_0^{\delta M(t, f)} \psi'(\lambda)d\lambda dt \leq \\
\leq C(n, K) \int_0^1 (1 - t)^{-n-2}\psi(\delta M(t, f))dt,
\]

which gives (3.3).

For the converse direction assume there exists \( \delta > 0 \) such that

\[
\int_{\mathbb{S}^{n-1}} \psi(\delta|f(\omega)|)d\sigma(\omega) < \infty,
\]

and choose points \( x_k \in \mathbb{B}^n \) with \( |x_k| = r_k = 1 - 2^{-k} \) and \( |f(x_k)| = M(r_k, f), k = 1, 2, \ldots \).

Given any \( \epsilon > 0 \) we have

\[
\int_0^1 (1 - r)^{-n-2}\psi(\epsilon M(r, f))dr \leq 2^n \sum_{k=1}^\infty (2^{-k})^{n-1}\psi(\epsilon M(r_k, f)) = 2^n \int_{\mathbb{B}^n} \psi(\epsilon|f(x)|)d\mu,
\]

where \( d\mu(x) = \sum_{k=1}^\infty (1 - |x|)^{n-1}\delta_{x_k}. \) The measure \( \mu \) is clearly a Carleson measure, and assuming \( f(x) \neq 0 \) for all \( x \in \mathbb{B}^n \) we can apply Lemma 3.3 below to obtain constants \( C_1 \) and \( C_2 \) not depending on \( f \) or \( \delta \) such that

\[
(3.4) \quad \int_{\mathbb{B}^n} \psi(\epsilon|f(x)|/C_1)d\mu \leq C_2 \int_{\mathbb{S}^{n-1}} \psi(\epsilon|f(\omega)|)d\sigma.
\]

By choosing \( \epsilon = \delta/C_1 \) the proof is finished in this case. The case when \( 0 \notin f(\mathbb{B}^n) \) is handled by applying the result to \( g(x) = f(x) - y_0 \) for some fixed \( y_0 \in \mathbb{R}^n \setminus f(\mathbb{B}^n). \)

We now give some results involving maximal functions, which we need for proving Lemma 3.3. Given a quasiconformal mapping \( f \) on \( \mathbb{B}^n \) its nontangential maximal function is defined as

\[
f^*(\omega) = \sup_{x \in \Gamma(\omega)} |f(x)|, \quad \omega \in \mathbb{S}^{n-1}.
\]

Clearly \( \psi(\delta f^*(\omega)) \in L^1(\mathbb{S}^{n-1}) \) implies \( \psi(\delta|f(\omega)|) \in L^1(\mathbb{S}^{n-1}) \). The Hardy-Littlewood maximal function and one of the modulus estimates from Section 2 help us prove the reverse implication as stated in Theorem 3.2.

**Theorem 3.2.** Let \( \psi \) be a growth function and \( f : \mathbb{B}^n \to \mathbb{R}^n \) a \( K \)-quasiconformal mapping such that \( 0 \notin f(\mathbb{B}^n) \). There exist constants \( C_1 = C_1(n, K) \) and \( C_2 = C_2(n) \) such that

\[
\int_{\mathbb{S}^{n-1}} \psi(\delta/C_1 f^*(\omega))d\sigma \leq C_2 \int_{\mathbb{S}^{n-1}} \psi(\delta|f(\omega)|)d\sigma
\]

whenever \( \delta > 0. \)
Proof. We may assume there is \( \delta > 0 \) such that \( \int_{\mathbb{S}^{n-1}} \psi(\delta|f(\omega)|)d\sigma < \infty \). Let \( \phi = \psi^{1/2} \). By Lemma 2.3 there is a constant \( C_1 = C_1(n, K) \) such that

\[
\sigma(\{\omega \in S_x : \phi(\delta|f(\omega)|) \geq \phi(\delta|f(x)|/C_1)\}) \geq \sigma(S_x)/2
\]

for every \( x \in \mathbb{B}^n \). Thus

\[
\int_{S_x} \phi(\delta|f(\omega)|)d\sigma \geq \phi(\delta|f(x)|/C_1)\sigma(\{\omega \in S_x : \phi(\delta|f(\omega)|) \geq \phi(\delta|f(x)|/C_1)\})
\]

\[
\geq \phi(\delta|f(x)|/C_1)\frac{\sigma(S_x)}{2}
\]

for each \( x \in \mathbb{B}^n \). Let \( M \) denote the non-centered Hardy-Littlewood maximal function on \( \mathbb{S}^{n-1} \); that is,

\[
Mg(\omega) = \sup_{r>0} \frac{1}{|\mathbb{S}^{n-1} \cap B(\omega, r)|} \int_{\mathbb{S}^{n-1} \cap B(\omega, r)} g d\sigma,
\]

where \( g \in L^1(\mathbb{S}^{n-1}) \). It follows from the previous inequality that

\[
\phi(\delta f^*(\omega)/C_1) \leq 2M(\phi(\delta|f|)(\omega).
\]

Since \( M \) is a bounded operator on \( L^2(\mathbb{S}^{n-1}) \), we have that

\[
\int_{\mathbb{S}^{n-1}} \psi(\delta f^*(\omega)/C_1)d\sigma = \int_{\mathbb{S}^{n-1}} \phi^2(\delta f^*(\omega)/C_1)d\sigma \leq 4 \int_{\mathbb{S}^{n-1}} M^2(\phi(\delta|f|)(\omega)d\sigma
\]

\[
\leq C_2 \int_{\mathbb{S}^{n-1}} \phi^2(\delta|f(\omega)|))d\sigma = C_2 \int_{\mathbb{S}^{n-1}} \psi(\delta|f(\omega)|))d\sigma,
\]

which completes the proof. \( \square \)

We introduce Carleson measures in order to finally prove the lemma we used in the proof of Theorem 3.1. A measure \( \mu \) on \( \mathbb{B}^n \) is called a \textit{Carleson measure} if there exists a constant \( C(\mu) > 0 \) such that

\[
\mu(B^n \cap B(\omega, r)) \leq C(\mu)r^{-n-1}
\]

for all \( \omega \in \mathbb{S}^{n-1} \) and all \( r > 0 \). The infimum of all such constants \( C(\mu) \) is called the Carleson norm of \( \mu \) and is denoted as \( \alpha_\mu \).

**Lemma 3.3.** Let \( \psi \) be a growth function, \( f : \mathbb{B}^n \to \mathbb{R}^n \) a \( K \)-quasiconformal mapping such that \( f(x) \neq 0 \) for all \( x \in \mathbb{B}^n \), and \( \mu \) a Carleson measure on \( \mathbb{B}^n \). Then there exist constants \( C_1 = C_1(n, K) \) and \( C_2 = C_2(\alpha_\mu, n) \) such that

\[
\int_{\mathbb{B}^n} \psi(\delta|f(x)|/C_1)d\mu \leq C_2 \int_{\mathbb{S}^{n-1}} \psi(\delta|f(\omega)|))d\sigma
\]

whenever \( \delta > 0 \).

**Proof.** First let \( \epsilon > 0 \) be arbitrary and set \( E(\lambda) = \{x \in \mathbb{B}^n : \epsilon|f(x)| > \lambda\} \) and \( U(\lambda) = \{\omega \in \mathbb{S}^{n-1} : \epsilon f^*(\omega) > \lambda\} \). Then \( U(\lambda) \) is an open set and we can use the generalized form of the Whitney decomposition [7, Theorem III.1.3] to write

\[
U(\lambda) = \bigcup_{k=1}^{\infty} S_{x_k}
\]

where the points \( x_k \in \mathbb{B}^n \) are chosen so that each \( \omega \in U(\lambda) \) is contained in at most \( N = N(n) \) caps \( S_{x_k} \) and \( (1 - |x_k|)/C \leq d(S_{x_k}, \partial U(\lambda)) \leq C(1 - |x_k|) \). Here \( C \) is an absolute constant and the distance is measured in the spherical distance on \( \mathbb{S}^{n-1} \).
If \( |f(x)| > \lambda \), then \( \epsilon f^*(\omega) > \lambda \) for all \( \omega \in S_x \), so \( E(\lambda) \) is contained in the union of \( B(x_k/|x_k|, C(1 - |x_k|)) \), \( k = 1, 2, \ldots \) where \( C \) is an absolute constant. Hence by the properties of the measure \( \mu \) and the decomposition of \( U(\lambda) \) we get
\[
\mu(E(\lambda)) \leq \sum_{k=1}^{\infty} \mu(B(x_k/|x_k|, C(1 - |x_k|)) \cap \mathbb{B}^n)
\]
\[
\leq C \sum_{k=1}^{\infty} \sigma(S_{x_k}) \leq C \sigma(U(\lambda)).
\]
Here \( C \) depends on \( n \) and the Carleson norm of \( \mu \).

Therefore,
\[
\int_{\mathbb{B}^n} \psi(|f(x)|) d\mu = \int_{0}^{\infty} \psi'(\lambda) \mu(E(\lambda)) d\lambda
\]
\[
\leq C \int_{0}^{\infty} \psi'(\lambda) \sigma(U(\lambda)) d\lambda
\]
\[
= C \int_{\mathbb{B}^{n-1}} \psi(\epsilon f^*(\omega)) d\sigma.
\]
Now applying Theorem 3.2 and choosing \( \epsilon \) appropriately completes the proof. \( \square \)

We finish this section with the short proof of Theorem 1.1.

**Proof of Theorem 1.1.** Theorem 3.1 gives the equivalence of (2) and (4). By definition (3) implies (1), by Fatou’s Lemma (1) gives (2), and in the case that \( f(x) \neq 0 \) on \( B^n \), Theorem 3.2 tells us that (2) implies (3). The other case is obtained by applying the result to an appropriate translation of \( f \). \( \square \)

4. **Results with additional growth conditions on \( \psi \)**

A growth function \( \psi \) is called **doubling** if there exists a constant \( C \) such that \( \psi(2t) \leq C \psi(t) \) for all \( t \in [0, \infty] \). We refer to the infimum of all such \( C \) as the doubling constant of \( \psi \). We will make use of the following property of doubling growth functions: if \( s \geq 1 \) then by choosing the smallest integer \( k \) such that \( s^n = 1 \) and using the monotonicity of \( \psi \) we have that \( \psi(st) \leq C^k \psi(t) \) for all \( t \in [0, \infty] \).

We next prove the lemmas needed for Theorem 1.2. We start by giving a family of Carleson measures on \( \mathbb{B}^n \).

**Lemma 4.1.** Let \( f \) be a \( K \)-quasiconformal mapping on \( \mathbb{B}^n \) such that \( f(x) \neq 0 \) for all \( x \in \mathbb{B}^n \). If \( \psi \) is a growth function such that both \( \psi \) and \( \psi^{-1} \) are doubling then the measure \( \mu \) defined by \( d\mu = \frac{\psi(a_f(x)(1-|x|))}{\psi(|f(x)|)} \frac{dx}{1-|x|} \) is a Carleson measure on \( \mathbb{B}^n \).

**Proof.** The doubling properties of \( \psi \) imply that there exist \( p, q \geq 1 \) such that
\[
\frac{\psi(a)}{\psi(b)} \leq 2^p \left( \frac{a^p}{b^p} + \frac{a^{1/q}}{b^{1/q}} \right)
\]
for all \( a, b \in (0, \infty) \). Indeed, when \( b \leq a \), there exists \( k \in \mathbb{N} \) such that
\[
\psi(a) \leq C^{k+1} \psi(b),
\]
and by choosing \( p \) large enough we obtain
\[
\frac{\psi(a)}{\psi(b)} \leq 2^p \frac{a^p}{b^p}.
\]
The other case is obtained similarly using the doubling property of $\psi^{-1}$. Thus

$$\int_{\mathbb{B}^n} \frac{\psi(a_f(x)(1 - |x|))}{\psi(|f(x)|)} \frac{dx}{1 - |x|} \leq 2^p \left( \int_{\mathbb{B}^n} a_f(x)^{1/q} |f(x)|^{-1/q}(1 - |x|)^{1/q-1} dx + \int_{\mathbb{B}^n} a_f(x)^p |f(x)|^{-p}(1 - |x|)^{p-1} dx \right).$$

To show that these integrals are bounded by a constant that depends only on $n, K, p,$ and $q$, let $\epsilon > 0$. By Holder’s inequality

$$\int_{\mathbb{B}^n} a_f(x)^{1/q} |f(x)|^{-1/q}(1 - |x|)^{1/q-1} dx \leq 2^p \left( \int_{\mathbb{B}^n} a_f(x)^n |f(x)|^{-n}(1 - |x|)^{n\epsilon q} dx \right)^{1/qn} \left( \int_{\mathbb{B}^n} (1 - |x|)^{(1/q-1-\epsilon)n/(n-1/q)} dx \right)^{(n-1/q)/n}.$$

We can choose $\epsilon > 0$, depending only on $q, n$, so that the latter integral converges. Since $f(x) \neq 0$ on $\mathbb{B}^n$, Lemma 2.1 implies that $|f(y)|^{-1} \approx |f(x)|^{-1}$ for all $y \in B_x$. Then by applying Lemma 2.5 the distortion inequality $|Df(x)|^n \leq KJf(x)$ and a change of variables we obtain

$$\int_{\mathbb{B}^n} a_f(x)^n |f(x)|^{-n}(1 - |x|)^{n\epsilon q} dx \leq C \int_{f(\mathbb{B}^n)} \frac{1}{|y|^n} (1 - |f^{-1}(y)|)^{n\epsilon q} dy,$$

where the constant $C$ depends on $n, K$ only. A result of Minioiwz [11, Theorem 1] shows there are constants $C, b$ depending on $n, K$ only so that

$$\frac{1}{C} (1 - |x|)^b \leq \frac{|f(x)|}{|f(0)|} \leq C (1 - |x|)^{-b}$$

for all $x \in \mathbb{B}^n$. By integrating over $f(\mathbb{B}^n) \cap B(0, |f(0)|)$ and $f(\mathbb{B}^n) \setminus B(0, |f(0)|)$ separately, switching to polar coordinates and inserting the Minioiwz result we have

$$\int_{f(\mathbb{B}^n)} \frac{1}{|y|^n} (1 - |f^{-1}(y)|)^{n\epsilon q} dy \leq C \int_0^{\delta} \frac{r^{n-1}}{r^n} \frac{r^\delta}{|f(0)|^\delta} dr + \int_{\delta}^{\infty} \frac{r^{n-1}}{r^n} \frac{1}{r^\delta} dr,$$

with $C$ and $0 < \delta \leq 1$ depending only on $n, K, q$. These integrals are clearly finite and give us the desired universal upper bound for the integral involving exponent $1/q$.

The estimate for the integral $\int_{\mathbb{B}^n} a_f(x)^p |f(x)|^{-p}(1 - |x|)^{p-1} dx$ is similar, noting that if $p > n$ then

$$\int_{\mathbb{B}^n} \frac{a_f(x)^p(1 - |x|)^{p-1}}{|f(x)|^p} dx = \int_{\mathbb{B}^n} \frac{a_f(x)^n(1 - |x|)^{p-1} a_f(x)^{p-n}}{|f(x)|^n} \frac{dx}{|f(x)|^p} \leq C \int \frac{a_f(x)^n(1 - |x|)^{n-1}}{|f(x)|^n} dx$$

by Lemmas 2.1 and 2.4 and the assumption on $f$. It follows that there is a constant $M = M(n, K, q, p)$ such that

$$\int_{\mathbb{B}^n} \frac{\psi(a_f(x)(1 - |x|))}{\psi(|f(x)|)} \frac{dx}{1 - |x|} \leq M,$$

where $f$ is any map satisfying the assumptions of the Lemma.

To finish showing that $\mu$ is a Carleson measure let $g : \mathbb{B}^n \rightarrow \mathbb{R}^n$ be $K$-quasiconformal with $g(x) \neq 0$ on $\mathbb{B}^n, \omega \in S^{n-1},$ and $r > 0$. By what we have already shown we can
assume \( r < 1/4 \). Let \( T_{r\omega} \) denote a Möbius automorphism of \( \mathbb{B}^n \) that maps \((1 - r)\omega\) to 0 and \( \mathbb{B}^n \cap B(\omega, r) \) onto the lower hemisphere of \( \mathbb{B}^n \). By setting \( f(x) = g(T_{r\omega}(x)) \) we have

\[
\int_{\mathbb{B}^n \cap B(\omega, r)} \frac{\psi(a_y(y)(1 - |y|))}{\psi(|y|)} \frac{dy}{1 - |y|} \leq C r^{n-1} \int_{\mathbb{B}^n \cap B(\omega, r)} \frac{\psi(C a_y(T_{r\omega}(y))(1 - |T_{r\omega}(y)|))}{\psi(|f(T_{r\omega}(y))|(1 - |T_{r\omega}(y)|))} J_{T_{r\omega}} dy = C r^{n-1} \int_{\mathbb{B}^n} \frac{\psi(C a_z(z)(1 - |z|))}{\psi(|f(z)|)} \frac{dz}{1 - |z|} \leq C M r^{n-1},
\]

which is what we needed to show. \( \square \)

**Lemma 4.2.** Let \( \psi \) be a doubling growth function and \( f \) a quasiconformal mapping on \( \mathbb{B}^n \). If

\[
\int_{\mathbb{B}^n} \psi(a_f(x)(1 - |x|)) \frac{dx}{1 - |x|} < \infty
\]

then \( \psi(\sup_{x \in \Gamma(\omega)} (a_f(x)(1 - |x|))) \in L^1(\mathbb{S}^{n-1}) \).

**Proof.** Fix \( \omega \in \mathbb{S}^{n-1} \) and let \( x \in \Gamma(\omega) \). Then there exists a constant \( C \) depending on \( n, K \) and the doubling constant of \( \psi \) such that

\[
\psi(a_f(x)(1 - |x|)) \leq \frac{C}{(1 - |x|)^n} \int_{B_x} \psi(a_f(y)(1 - |y|)) dy \leq C \int_{\Gamma(\omega)} \psi(a_f(y)(1 - |y|)) \frac{dy}{(1 - |y|)^n}.
\]

Now if

\[
v(\omega) = \psi^{-1} \left( C \int_{\Gamma(\omega)} \psi(a_f(x)(1 - |x|)) \frac{dx}{(1 - |x|)^n} \right)
\]

then \( \psi(v(\omega)) \in L^1(\mathbb{S}^{n-1}) \). Indeed, given any function \( u \) integrable on \( \mathbb{B}^n \), Fubini’s Theorem gives us

\[
\int_{\mathbb{S}^{n-1}} \int_{\Gamma(\omega)} u(y)(1 - |y|)^{1-n} d\sigma dy = \int_{\mathbb{B}^n} u(y)(1 - |y|)^{1-n} \int_{\mathbb{S}^{n-1}} \chi_{\Gamma(\omega)}(y) d\sigma dy \\
\approx \int_{\mathbb{B}^n} u(y) dy.
\]

The claim follows with \( u(y) = \frac{\psi(a_f(y)(1 - |y|))}{1 - |y|} \).

The estimates above showed that

\[
\sup_{x \in \Gamma(\omega)} (a_f(x)(1 - |x|)) \leq v(\omega),
\]

and so the proof is finished. \( \square \)

**Lemma 4.3.** Let \( f \) be a quasiconformal mapping of \( \mathbb{B}^n \) and \( \psi \) a growth function that is doubling. If there is a function \( v(\omega) \) such that \( \psi(v(\omega)) \in L^1(\mathbb{S}^{n-1}) \) and

\[
\sup_{x \in \Gamma(\omega)} d(f(x), \partial f(\mathbb{B}^n)) \leq C v(\omega)
\]

for almost every \( \omega \in \mathbb{S}^{n-1} \) and some constant \( C \), then \( \psi(|f(\omega)|) \in L^1(\mathbb{S}^{n-1}) \).
Proof. We can assume $f(0) = 0$. Let $U(\lambda) = \{\omega \in S^{n-1} : f^*(\omega) > \lambda\}$. Like in the proof of Lemma 3.3 we can write $U(\lambda)$ as the union of caps $S_{x_j}$

$$U(\lambda) = \cup S_{x_j}$$

so that the caps have uniformly bounded overlap and

$$(1 - |x_j|)/C \leq d(S_{x_j}, \partial U(\lambda)) \leq C(1 - |x_j|).$$

Suppose $\omega \in S_{x_j}$ with $v(\omega) \leq \gamma$. By Lemma 2.1, our assumption on $d(f(x), \partial f(B^\mathbb{R}^n))$, and our decomposition of $U(\lambda)$, there is a constant $C = C(n, K)$ such that $d(f(x_j), \partial f(B^\mathbb{R}^n))$, $\text{diam} f(B_{x_j}) \leq C\gamma$, and there is $\omega' \in S^{n-1} \setminus U(\lambda)$ with $d(\omega, \omega') \leq C(1 - |x_j|)$. It follows that

$$|f(x_j)| \leq \lambda + C\gamma.$$

Now let $M > 1$, $\gamma = \frac{\lambda}{(M+1)C}$ and suppose $\omega \in S_{x_j}$ with $v(\omega) \leq \gamma$ and $|f(\omega)| > 2\lambda$. Then

$$|f(\omega) - f(x_j)| \geq |f(\omega)| - |f(x_j)| > \lambda - C\gamma = MC\gamma \geq Md(f(x_j), \partial \Omega),$$

and so by Lemma 2.2

$$\sigma(\{\omega \in S_{x_j} : |f(\omega)| > 2\lambda \text{ and } v(\omega) \leq \gamma\}) \leq \sigma(\{\omega \in S_{x_j} : |f(\omega)| - |f(x_j)| > Md(f(x_j), \partial \Omega)\}) \leq C\sigma(S_{x_j})(\log M)^{1-n}.$$ 

Note if we are in the case that $U(\lambda) = S^{n-1}$ then $U(\lambda) = S_0$, and with our assumption that $f(0) = 0$ the inequality holds anyway.

If $|f(\omega)| > 2\lambda$ then by continuity $\omega \in U(\lambda)$, and so

$$\sigma(\{\omega \in S^{n-1} : |f(\omega)| > 2\lambda\}) \leq \sigma(\{\omega \in U(\lambda) : |f(\omega)| > 2\lambda \text{ and } v(\omega) \leq \gamma\}) + \sigma(\{\omega \in S^{n-1} : v(\omega) > \gamma\}) \leq C\sum_j \sigma(S_{x_j})(\log M)^{1-n} + \sigma(\{\omega \in S^{n-1} : v(\omega) > \gamma\}) \leq C\sigma(U(\lambda))(\log M)^{1-n} + \sigma(\{\omega \in S^{n-1} : v(\omega) > \gamma\}).$$

Thus

$$\int_{S^{n-1}} \frac{1}{2} |f(\omega)| d\sigma = \int_0^\infty \psi'(\lambda) \sigma(\{\omega \in S^{n-1} : |f(\omega)| > 2\lambda\}) d\lambda \leq \int_0^\infty \psi'(\lambda)(C\sigma(U(\lambda))(\log M)^{1-n} + \sigma(\{\omega \in S^{n-1} : v(\omega) > \frac{\lambda}{(M+1)C}\})) d\lambda = C(\log M)^{1-n} \int_{S^{n-1}} \psi(f^*(\omega)) d\sigma + \int_{S^{n-1}} \psi((M+1)Cv(\omega)) d\sigma \leq C(n, K)(\log M)^{1-n} \int_{S^{n-1}} \psi(f^*(\omega)) d\sigma + C(M, n, K, C_\psi) \int_{S^{n-1}} \psi(v(\omega)) d\sigma,$$

where $C_\psi$ denotes the doubling constant of $\psi$.

We would like to combine the integral involving $f^*(\omega)$ with the left hand side of the inequality, but since both of the integrals could be infinite we finish the proof with a convergence argument and an application of Theorem 3.2. If we set $f_t(x) = f(tx)$, for each $0 < t < 1$, then $\sup_{x \in \Gamma(\omega)} d(f_t(x), \partial f(B^\mathbb{R}^n)) \leq \sup_{x \in \Gamma(\omega)} d(f(x), \partial f(B^\mathbb{R}^n))$. Applying
Theorem 1.1 we have

\[ \int_{S^{n-1}} \psi\left(\frac{1}{2} |f(\omega)|\right) d\sigma \leq C \int_{S^{n-1}} \psi(v(\omega)) d\sigma + C < \infty. \]

These lemmas together with some of the results in the previous sections give our theorem.

**Proof of Theorem 1.2.** We first show the equivalence of (1) and (3). Theorem 1.1 implies that (1) \( \Rightarrow \) (3) for all growth functions. For the reverse implication we only need to suppose \( \psi \) is doubling and that (3) holds. We claim there is a constant \( C = C(n, K) \) such that

\[ d(f(x), \partial \Omega) \leq Cf^*_i(\omega) \]

for all \( \omega \in S^{n-1} \) and each \( x \in \Gamma(\omega) \). By Lemma 2.1 it is enough to check that this is the case for each \( x = t\omega, 0 < t < 1 \). By Lemma 2.1 \( f(B_x) \) contains a ball of radius \( d(f(x), \partial \Omega)/C \) centered at \( f(x) \), with \( C \) depending only on \( n, K \). So there is \( y \in B_x \) such that

\[ d(f(x), \partial \Omega) = C|f_i(y) - f_i(x)| \leq 2Cf^*_i(\omega), \]

which gives the claim. Then Lemma 3.3 and Theorem 1.1 imply that \( f \in H^v \).

We now show that (1) and (2) are equivalent. Assume (1) and also that \( 0 \notin f(\mathbb{B}^n) \). The measure \( \mu \) given by \( d\mu = \frac{\psi(a_f(x)(1-|x|))}{\psi(|f(x)|)} \frac{dx}{1-|x|} \) is a Carleson measure on \( \mathbb{B}^n \) by Lemma 4.1. Then by Lemma 3.3 there are absolute constants \( C_1 \) and \( C_2 \) such that

\[ \int_{\mathbb{B}^n} \psi(a_f(x)(1-|x|)) \frac{dx}{1-|x|} = \int_{\mathbb{B}^n} \psi(|f(x)|) d\mu \leq C_1 \int_{S^{n-1}} \psi(C_2 |f(\omega)|) d\sigma. \]

The integral on the right is finite, by Theorem 1.1, which implies (2). If we assume (2) then

\[ \int_{S^{n-1}} \psi\left(\sup_{x \in \Gamma(\omega)} (a_f(x)(1-|x|))\right) d\sigma < \infty \]

by Lemma 4.2. Then \( \psi(|f(\omega)|) \in L^1(S^{n-1}) \) by Lemmas 2.1 and 3.3 and finally by Theorem 1.1 we have \( f \in H^v \).

We give more details here on the examples mentioned in the introduction section regarding how Theorem 1.2 may fail if either the growth function or its inverse is not doubling. In our first example let \( f \) be the identity mapping on \( \mathbb{B}^n \) and choose the growth function \( \psi \) to be

\[ \psi(t) = \begin{cases} \frac{1}{\log t^2}, & t < 1/2 \\ \frac{2t}{1}, & t \geq 1/2. \end{cases} \]

This \( \psi \) is doubling but \( \psi^{-1} \) is not. Also, we have that \( f \in H^v \) clearly, but

\[ \int_{\mathbb{B}^n} \psi(a_f(x)(1-|x|)) \frac{dx}{1-|x|} = C + C \int_{1/2}^1 \frac{1}{r \log \frac{1}{1-r}} dr, \]

which is not finite. Thus the implication (1) \( \Rightarrow \) (2) fails for this example.
Now let \( f(z) = \log(z + 1), z \in \mathbb{B}^2 \), and \( \psi(t) = e^{t^2} - 1, t \in [0, \infty) \). Then \( \psi \) is a growth function that is not doubling. The second coordinate function of \( f \) is a bounded function, and so \( \psi(f_2^*(\omega)) \in L^1(\mathbb{S}^1) \). However, since
\[
M(r, f) \geq \log \frac{1}{1 - r}
\]
for all \( r \in (0, 1) \) and
\[
\int_0^1 \psi \left( \delta \log \frac{1}{1 - r} \right) \, dr
\]
diverges given any \( \delta, f \notin H^\psi \) by Theorem 1.1. This example shows how the implication (3) \( \Rightarrow \) (1) can fail when the growth function is not doubling.

Finally, let \( f(z) = \log(z + 1), z \in \mathbb{B}^2 \), and choose
\[
\psi(t) = \begin{cases} 
2et, & t \leq 1 \\
e^{t^2} + e, & t > 1,
\end{cases}
\]
as the growth function. This \( \psi \) is not doubling, and, like above, \( f \notin H^\psi \) by Theorem 1.1. However,
\[
\int_{\mathbb{B}^n} \psi(a_f(x)(1 - |x|)) \frac{dx}{1 - |x|} = \int_{\mathbb{B}^n} \psi \left( \frac{1 - |x|}{|x + 1|} \right) \frac{dx}{|x + 1|} = \int_{\mathbb{B}^n} \frac{dx}{|x + 1|} \int_{B(1,1)} \frac{dx}{|x|} < \infty,
\]
which shows that (2) does not imply (1) for this example.

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