SHARP RESULTS ON SAMPLING WITH DERIVATIVES IN BANDLIMITED FUNCTIONS

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Abstract. We discuss the problems of uniqueness, sampling and reconstruction with derivatives in the space of bandlimited functions. We prove that if

\[ X = \{ x_i : i \in \mathbb{Z} \} \ldots < x_{i-1} < x_i < x_{i+1} < \ldots \],

is a separated sequence of real numbers such that \( \delta := \sup_i (x_{i+1} - x_i) < \frac{\nu_k}{\sigma} \), then any bandlimited function of bandwidth \( \sigma \) can be reconstructed uniquely and stably from its nonuniform samples \( \{ f^{(j)}(x_i) : j = 0, 1, \ldots, k-1, i \in \mathbb{Z} \} \), where \( \nu_k \) denotes the Wirtinger-Cimmino constant. We also prove that if \( \delta \leq \frac{\nu_k}{\sigma} \), then \( X \) is a set of uniqueness for the space of bandlimited functions of bandwidth \( \sigma \) when the samples involving the first \( k-1 \) derivatives. As a by-product, we obtain the sharp the maximum gap condition for samples involving first derivative.

Key words and phrases : atomic systems, bandlimited functions, frames, Hermite interpolation, K-frames, Riesz basis, Cimmino’s inequality.

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1. Introduction

Let \( \mathcal{B}_\sigma \) denote the space of entire functions of exponential type \( \leq \sigma \) that are square integrable on the real axis. The classical Shannon’s sampling theorem states that every \( f \in \mathcal{B}_\sigma \) can be reconstructed from the sampling formula

\[
f(x) = \sum_{n \in \mathbb{Z}} f \left( \frac{n\pi}{\sigma} \right) \frac{\sin \sigma (x - n\pi/\sigma)}{\sigma (x - n\pi/\sigma)}. \]

In many applications, such as aircraft instrument communications, air traffic control simulation, or telemetry [7], one can consider the possibility of obtaining sampling expansion which involved sample values of a function and its derivatives. Fogel [7], Jagermann and Fogel [11], Linden and Abramson [13] extended the Shannon sampling theorem in this direction. They proved that if the values of \( f \) and its first \( R-1 \) derivatives are known on the sequence \( \{ r n \pi / \sigma : n \in \mathbb{Z} \} \), then every \( f \in \mathcal{B}_\sigma \) can be reconstructed from the sampling formula

\[
f(x) = \sum_{n \in \mathbb{Z}} \sum_{k=0}^{R-1} f_k(x_n) \frac{(x - x_n)^k}{k!} \left[ \frac{\sin \frac{\sigma}{R} (x - x_n)}{\frac{\sigma}{R} (x - x_n)} \right] \frac{\sigma}{R} (x - x_n)^R, \]

\]

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where \( x_n = rn\pi/\sigma \) and \( f_k(x_n) \) are linear combinations of \( f(x_n), f'(x_n), \ldots, f^{(k)}(x_n) \):

\[
 f_k(x_n) := \sum_{i=0}^{k} \binom{k}{i} \left( \frac{\sigma}{R} \right)^{k-i} \left[ \frac{d^{k-i}}{dx^{k-i}} \left( \frac{x}{\sin x} \right)^{R} \right]_{x=0}^{x_n} f^{(i)}(x_n).
\]

The problem of nonuniform sampling expansion involving derivatives is well studied in the literature. Rawn in [15] extended the Kadec 1/4-theorem for sampling involving derivatives. He proved that if \( \{ x_n : n \in \mathbb{Z} \} \) is a sampling sequence belonging to the class \( S = \{ (x_n) : |x_n - n| \leq d < 1/4R, \forall n \in \mathbb{Z} \} \), then any \( f \in B_\sigma \) can be recovered from its samples \( \{ f^{(j)}(x_n) : j = 0, 1, \ldots, R-1, n \in \mathbb{Z} \} \). The general nonuniform sampling problem involving derivatives was studied by Gröchenig and others in terms of Beurling density. They proved that if the Beurling-Landau density, \( D(X) \) of the set \( X \), is greater than \( \frac{\delta\xi}{\sigma} \), then any \( \sigma \)-bandlimited function \( f \) can be reconstructed uniquely and stably from its sample values \( \{ f^{(j)}(x_n) : j = 0, 1, \ldots, k-1, n \in \mathbb{Z} \} \). For further details, we refer the reader to [5, 10, 14].

The uniqueness problem is studied in the mathematical literature in the context of separation of zeros [6, 20]. In this paper, we examine the separation of real zeros of multiplicity \( k \) for the bandlimited functions. For a non-zero entire function \( f \), we define the longest zero-free interval of \( f \) of order \( k-1 \geq 0 \) as follows:

\[
 M_k(f) := \sup \{ b - a : f^{(l)}(x) \neq 0, x \in (a, b), l = 0, 1, \ldots, k-1 \}.
\]

If all the real zeros of \( f \) are bounded on the left or right, then \( M(f) = \infty \). Otherwise, the zeros of \( f \) can be arranged in a doubly infinite sequence \( \{ x_n : n \in \mathbb{Z} \} \) with \( x_n \leq x_{n+1} \). In this case,

\[
 M_k(f) := \sup \{ x_{n+1} - x_n : f^{(l)}(x_n) = 0, n \in \mathbb{Z}, l = 0, 1, \ldots, k-1 \}.
\]

We now define

\[
 d_k := \inf \{ M_k(f) : 0 \neq f \in B_\sigma \}.
\]

Walker in [20] proved that \( d_1 = \frac{\pi}{\eta} \). In this paper, we prove that \( d_2 = \frac{2\pi}{\eta} \) and hence we obtain a sharp result on uniqueness theorem involving first derivative for the bandlimited functions. We also provide lower and upper bounds for the constants \( d_k, k \geq 3 \).

The numerical aspects of nonuniform sampling expansion involving derivatives was studied by Gröchenig in [9]. If \( f^{(j)}(x), j = 0, \ldots, k-1 \) are sampled at uniform rate, it follows from Beurling density theorem that \( f \) can be reconstructed if the uniform gap is less than \( k\pi/\sigma \). From the above discussion, it is conjectured in [11, 9, 10] that if \( \sup_i (x_{i+1} - x_i) < \frac{k\pi}{\sigma} \), then any bandlimited function can be reconstructed uniquely and stably from its nonuniform samples \( \{ f^{(j)}(x_i) : j = 0, 1, \ldots, k-1, i \in \mathbb{Z} \} \). Gröchenig in [9] proved that if \( \sup_i (x_{i+1} - x_i) < \frac{\pi}{\sigma} \), then any bandlimited function can be reconstructed uniquely and stably from its nonuniform samples \( \{ f(x_i) : i \in \mathbb{Z} \} \). In [10], Razafinjatovo obtained a frame algorithm for reconstructing a function \( f \in B_\sigma \) from its nonuniform samples \( \{ f^{(j)}(x_i) : j = 0, 1, \ldots, k-1, i \in \mathbb{Z} \} \) with maximum gap condition, namely \( \sup_i (x_{i+1} - x_i) = \delta < \frac{\pi}{\sigma}((k-1)!/(2k-1)2k)^{1/k} \) using Taylor’s polynomial approximation. The authors in [11] improved the Razafinjatovo’s result using Hermite interpolation. Radha and the author in [14] obtained a better bound than the one given in [11] using Wirtinger-Sobolev inequality. While
this above conjecture for \( k \geq 2 \) is still unresolved for the past twenty five years, the case \( k = 2 \) is settled in this paper as follows.

**Theorem 1.1.** If \( \{x_i : i \in \mathbb{Z}\} \) is a separated set such that \( \sup_i (x_{i+1} - x_i) < \frac{2\pi}{\sigma} \), then there exists a Bessel sequence \( \{g_{1,i}, g_{2,i} : i \in \mathbb{Z}\} \) in \( B_\sigma \) such that

\[
f'(x) = \sum_{i \in \mathbb{Z}} f(x_i)g_{1,i}(x) + \sum_{i \in \mathbb{Z}} f'(x_i)g_{2,i}(x),
\]

for every \( f \in B_\sigma \). The function \( f \) can be computed using the relation

\[
f(x) = \int_{x_i}^{x} f'(t)dt + f(x_i).
\]

The paper is organized as follows. In section 2, we discuss some preliminary facts needed for the later sections. In section 3, we prove a uniqueness theorem for bandlimited function involving derivative samples. In section 4, we provide iterative reconstruction algorithms for the recovery of bandlimited functions from its samples involving derivatives.

2. Preliminaries

This section provides some useful terminology and results in order to prove our main results. Throughout this paper, we shall adopt the following notations: \( \mathcal{H} \) is a separable Hilbert space and \( \mathcal{L}(\mathcal{H}) \) is the space of bounded operators on \( \mathcal{H} \).

**Definition 2.1.** A sequence \( \{f_n : n \in \mathbb{Z}\} \) in \( \mathcal{H} \) is called a Riesz basis if it satisfies the following conditions:

(i) It is complete, i.e., \( \overline{\text{span}}\{f_n\} = \mathcal{H} \).

(ii) There exist two constants \( A, B > 0 \) such that

\[
A \sum_{n \in \mathbb{Z}} |d_n|^2 \leq \|\sum_{n \in \mathbb{Z}} d_nf_n\|_{\mathcal{H}}^2 \leq B \sum_{n \in \mathbb{Z}} |d_n|^2, \tag{2.1}
\]

for all \( (d_n) \in \ell^2(\mathbb{Z}) \).

**Definition 2.2.** A sequence \( \{f_n : n \in \mathbb{Z}\} \) in \( \mathcal{H} \) is called a frame if there exist two constants \( A, B > 0 \) such that for all \( f \in \mathcal{H} \) we have

\[
A\|f\|^2 \leq \sum_{n} |\langle f, f_n \rangle|^2 \leq B\|f\|^2. \tag{2.2}
\]

The sequence \( \{f_n : n \in \mathbb{Z}\} \) is called a Bessel sequence if at least the upper bound in (2.2) is satisfied. The numbers \( A, B \) are called the frame bounds.

Every Riesz basis is a frame. If \( \{f_n\} \) is a frame for \( \mathcal{H} \) with bounds \( A, B \), then

\[
Sf = \sum_{n} \langle f, f_n \rangle f_n
\]
is a positive self-adjoint invertible operator and called the frame operator associated with \( \{ f_n \} \). Let \( \rho = \frac{2}{A + B} \). Then \( f \) can be recovered by the frame algorithm: Set
\[
\begin{align*}
  f_0 &= 0, \\
  f_{n+1} &= f_n + \rho S(f - f_n), \quad n \geq 0.
\end{align*}
\]
Then we have \( \lim_{n \to \infty} f_n = f \). The error estimate after \( n \) iterations turns out to be
\[
\| f - f_n \| \leq \left( \frac{B - A}{B + A} \right)^n \| f \|.
\]
We refer the reader to [3, 21] for further details.

**Definition 2.3.** Let \( K \in \mathcal{L}(\mathcal{H}) \). A sequence of vectors \( \{ f_n : n \in \mathbb{Z} \} \) in \( \mathcal{H} \) is said to be a \( K \)-frame if there exist two constants \( A, B > 0 \) such that
\[
A\|K^* f\|_\mathcal{H}^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle_\mathcal{H}|^2 \leq B\| f \|_\mathcal{H}^2, \tag{2.3}
\]
for every \( f \in \mathcal{H} \). When \( K \) is the identity operator, it coincides with ordinary frames.

**Definition 2.4.** Let \( K \in \mathcal{L}(\mathcal{H}) \). A sequence of vectors \( \{ f_n : n \in \mathbb{Z} \} \) in \( \mathcal{H} \) is said to be an atomic system for \( K \) if the following statements hold:

(i) \( \{ f_n : n \in \mathbb{Z} \} \) is a Bessel sequence in \( \mathcal{H} \).

(ii) There exists \( B > 0 \) such that for every \( f \in \mathcal{H} \) there exists \( d_f = (d_n) \in \ell^2(\mathbb{Z}) \) such that \( \|d_f\|_2 \leq B\| f \|_\mathcal{H} \) and \( Kf = \sum_{n \in \mathbb{Z}} d_n f_n \).

**Theorem 2.1.** ([8]) A sequence of vectors \( \{ f_n : n \in \mathbb{Z} \} \) in \( \mathcal{H} \) is an atomic system for \( K \) if and only if it is a \( K \)-frame for \( \mathcal{H} \). In this case, there exists a Bessel sequence \( \{ g_n : n \in \mathbb{Z} \} \) such that
\[
Kf = \sum_{n \in \mathbb{Z}} \langle f, g_n \rangle f_n \quad \text{and} \quad K^* f = \sum_{n \in \mathbb{Z}} \langle f, f_n \rangle g_n,
\]
for every \( f \in \mathcal{H} \).

For an integrable function \( f \), the Fourier transform \( \hat{f} \) of \( f \) is defined by
\[
\hat{f}(w) := \int_{\mathbb{R}} f(x) e^{-2\pi iwx} \, dx, \quad w \in \mathbb{R}.
\]
If \( f \in L^1 \cap L^2(\mathbb{R}) \), one has the Plancherel formula \( \| f \|_2 = \| \hat{f} \|_2 \). As \( L^1 \cap L^2(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \), the definition of Fourier transform is extended to functions in \( L^2(\mathbb{R}) \).

For \( \sigma > 0 \), let \( B_\sigma \) denote the space of all \( \sigma \)-bandlimited functions, i.e.,
\[
B_\sigma = \left\{ f \in L^2(\mathbb{R}) : \text{supp} \hat{f} \subseteq \left[ -\frac{\sigma}{2\pi}, \frac{\sigma}{2\pi} \right] \right\}.
\]
The celebrated theorem of Paley-Wiener says that \( B_\sigma \) coincides with the space of entire functions of exponential type \( \leq \sigma \). Moreover, \( B_\sigma \) is a reproducing kernel
Hilbert space with reproducing kernel $\mathcal{K}(x, y) = \frac{\sin \sigma(x-y)}{\sigma(x-y)}$. The following inequality is well-known in the literature.

**Theorem 2.2** (Bernstein’s inequality). If $f \in B_\sigma$, then $f' \in B_\sigma$ and
\[
\|f^{(k)}\|_2 \leq \sigma^k \|f\|_2. \tag{2.4}
\]

**Definition 2.5.** Let $X = \{x_i : i \in \mathbb{Z}\}$ be a sequence of real or complex numbers. Then

(i) $X$ is said to be a set of uniqueness of order $k - 1$ for $B_\sigma$ if
\[
f^{(l)}(x_i) = 0, \quad n \in \mathbb{Z}, \quad l = 0, 1, \ldots, k - 1,
\]
implies that $f \equiv 0$.

(ii) $X$ is said to be a stable set of sampling of order $k - 1$ for $B_\sigma$ if there exist constants $A, B > 0$ such that
\[
A\|f\|_2^2 \leq \sum_{i \in \mathbb{Z}} \sum_{l=0}^{k-1} |f^{(l)}(x_i)|^2 \leq B\|f\|_2^2, \tag{2.5}
\]
for all $f \in B_\sigma$.

(iii) $X$ is said to be a set of interpolation of order $k - 1$ for $B_\sigma$ if the interpolation problem
\[
f^{(l)}(x_i) = c_{i\ell}, \quad n \in \mathbb{Z}, \quad l = 0, 1, \ldots, k - 1,
\]
has a solution $f \in B_\sigma$ for every square summable sequence $\{c_{i\ell} : i \in \mathbb{Z}\}, l = 0, 1, \ldots, k - 1$.

**Definition 2.6.** A set $X = \{x_i : i \in \mathbb{Z}\}$ is said to be a stable set of sampling of order $(m, k - 1)$ for $B_\sigma$ with respect to the weight $\{w_{i\ell} \in \mathbb{R}^* : \ell = 0, \ldots, k - 1, i \in \mathbb{Z}\}$ if there exist constants $A, B > 0$ such that
\[
A\|f^{(m)}\|_2^2 \leq \sum_{i \in \mathbb{Z}} \sum_{\ell=0}^{k-1} w_{i\ell}|f^{(\ell)}(x_i)|^2 \leq B\|f\|_2^2, \tag{2.6}
\]
for all $f \in B_\sigma$.

Let $X = \{x_n\}$ be sequence of distinct real numbers. To each $X$ and $m \in \mathbb{N}$, we associate an exponential system
\[
\mathcal{E}(X; m) := \{e^{2\pi i x_n t}, e^{2\pi i x_n t}, \ldots, e^{2\pi i x_n t} : x_n \in X\}.
\]
It is well known that $\mathcal{E}(m\pi/\sigma, \mathbb{Z}, m)$ is a Riesz basis for $L^2[-\frac{\pi}{\sigma}, \frac{\pi}{\sigma}]$. Arguing as in
[17], we can prove that $X$ is a set of uniqueness and interpolation of order $m - 1$ for $B_\sigma$ if and only if $\mathcal{E}(X, m)$ is a Riesz basis for $L^2[-\frac{\pi}{\sigma}, \frac{\pi}{\sigma}]$. N. Levinson in
[12] proved that the completeness of the system $\mathcal{E}(X; m)$ is unaffected if we replace finitely many points $x_n$ by the same number of points $y_n \notin X$. R. M. Young in
[21] showed that if $\{f_n\}$ is a Riesz basis for $\mathcal{H}$ and $\{g_n\}$ is a complete sequence in $\mathcal{H}$ such that
\[
\sum \|f_n - g_n\| < \infty,
\]
then $\{g_n\}$ is a Riesz basis for $\mathcal{H}$. Consequently, we have the following
Theorem 2.3 (Replacement Theorem). The Riesz basis property of the system \( \mathcal{E}(X; m) \) is unaffected if we replace finitely many points \( x_n \) by the same number of points \( y_n \notin X \).

The following four results are the main ingredients to prove our main results.

Theorem 2.4 (Schmidt's inequality). (\[2\]). If \( p_n(x) \) is a polynomial of degree \( \leq n \), then

\[
\int_a^b |p'_n(x)|^2 \, dx \leq \mu_n (b-a)^{-2} \int_a^b |p_n(x)|^2 \, dx,
\]

(2.7)

where \( \mu_n = \frac{n(n+1)(n+2)(n+3)}{2} \).

Theorem 2.5 (Cimmino's inequality). (\[2\]). If \( f \) is \( r \) times continuously differentiable in \([a, b]\) with \( f^{(l)}(a) = f^{(l)}(b) = 0, 0 \leq l \leq r-1 \), then

\[
\int_a^b |f^{(k)}(x)|^2 \, dx \leq \left( \frac{b-a}{\lambda_{r,k}} \right)^{2r-2k} \int_a^b |f^{(r)}(x)|^2 \, dx, \quad 0 \leq k \leq r-1,
\]

(2.8)

where \( \lambda_{r,k}^{2r-2k} \) is the first eigenvalue of the boundary value problem

\[
u^{(2r)}(x) - \lambda (-1)^{r+k} u^{(2k)}(x) = 0, \quad \lambda > 0, \quad x \in [0, 1],
\]

\[u^{(k)}(0) = u^{(k)}(1) = 0, \quad 0 \leq k \leq r-1, \quad u \in C^{2r}[0, 1].\]

In (2.8) equality holds if and only if \( f(x) \) is the first eigenfunction of the boundary value problem \( u^{(2r)}(x) - \lambda (-1)^{r+k} u^{(2k)}(x) = 0, \quad u^{(k)}(a) = u^{(k)}(b) = 0, \quad 0 \leq k \leq r-1.\)

Theorem 2.6. ([9], [16]). Let \( A \) be a bounded operator on a Hilbert space \( \mathcal{H} \) that satisfies

\[
\| f - Af \|_\mathcal{H} \leq C \| f \|_\mathcal{H},
\]

for every \( f \in \mathcal{H} \) and for some \( C, 0 < C < 1 \). Then \( A \) is invertible on \( \mathcal{H} \) and \( f \) can be recovered from \( Af \) by the following iteration algorithm. Setting

\[
f_0 = Af \quad \text{and} \quad f_{n+1} = f_n + A(f - f_n), \quad n \geq 0,
\]

we have \( \lim \limits_{n \to \infty} f_n = f \). The error estimate after \( n \) iterations is

\[
\| f - f_n \|_\mathcal{H} \leq C^{n+1} \| f \|_\mathcal{H}.
\]

Theorem 2.7 (Hermite Interpolation Formula). ([18]). Let \( f \) be \( r \) times continuously differentiable in \([a, b]\) and \( \xi, \eta \in [a, b] \). Then the Hermite interpolation polynomial \( H_{2r+1}^{(j)}(x) \) of degree \( 2r + 1 \) such that \( H_{2r+1}^{(j)}(y) = f^{(j)}(y) \), for \( y = \xi, \eta, 0 \leq j \leq r \), is given by

\[
H_{2r+1}(\xi, \eta; f; x) = \sum_{k=0}^{r} A_{0k}(x)f^{(k)}(\xi) + \sum_{k=0}^{r} A_{1k}(x)f^{(k)}(\eta),
\]

(2.9)
where

\[
A_{0k}(x) = (x - \eta)^{r+1}(x - \xi)^k \frac{1}{k!} \sum_{s=0}^{r-k} \frac{1}{s!} g_0^{(s)}(\xi)(x - \xi)^s,
\]

\[
A_{1k}(x) = (x - \xi)^{r+1}(x - \eta)^k \frac{1}{k!} \sum_{s=0}^{r-k} \frac{1}{s!} g_1^{(s)}(\eta)(x - \eta)^s,
\]

\[
g_0(x) = (x - \eta)^{-(r+1)},
\]

\[
g_1(x) = (x - \xi)^{-(r+1)}.
\]

Using Cimmino’s inequality, the following error estimate

\[
\int_a^b |f^{(r-1)}(x) - H_{2r-1}^{(r-1)}(x)|^2 \, dx \leq \left( \frac{b-a}{\nu_r} \right)^2 \int_a^b |f^{(r)}(x)|^2 \, dx,
\]

(2.10)

is proved in [2], where \(\nu_r = \lambda_{r,r-1}\). We explicitly mention certain values for the constants \(\nu_r\) as given in [2].

\[\nu_1 = \pi, \quad \nu_2 = 2\pi, \quad \nu_3 = 8.9868.\]

3. A Uniqueness Theorem

Let us define the sample sets \(\Lambda_N = \{\lambda_n(l) : l \in \mathbb{Z}\}, \, N \geq 1\) as follows:

\[
\lambda_n(l) := \begin{cases} 
\frac{k\pi l}{\sigma} & \text{if } |l| > N, \\
0 & \text{if } l = 0, \\
\text{sgn}(l)(2l-1)\frac{k\pi(N+1)}{\sigma(2N+1)} & \text{if } 1 \leq |l| \leq N.
\end{cases}
\]

It is clear that \(\Lambda_N\) is equal to \(\frac{k\pi}{\sigma}\mathbb{Z}\) except at finite number of points. Then it follows from Replacement Theorem that \(\Lambda_N\) is an interpolating set of order \(k-1\) for \(\mathcal{B}_\sigma\).

Consider the sequence \((c_\lambda)_{\lambda \in \Lambda_N}\) defined as

\[
c_\lambda := \begin{cases} 
1 & \text{if } \lambda = 0, \\
0 & \text{if } \lambda \in \Lambda_N - \{0\}.
\end{cases}
\]

Then there exists a non-zero function \(g_N \in \mathcal{B}_\sigma\) such that \(g_N^{(l)}(\lambda) = c_\lambda, 0 \leq l \leq k-1\).

Notice that \(M_k(g_N) = \frac{2k(N+1)\pi}{(2N+1)\sigma}\) converges to \(\frac{k\pi}{\sigma}\) as \(N \to \infty\). Therefore, \(d_k \leq \frac{k\pi}{\sigma}\).

**Theorem 3.1.** If a non-zero function \(f \in \mathcal{B}_\sigma\) has infinitely many zeros on the real axis of multiplicity \(k\), then there exists at least one pair of consecutive zeros whose distance apart is greater than \(\frac{\nu_k}{\sigma}\). Consequently, \(\frac{\nu_k}{\sigma} \leq d_k \leq \frac{k\pi}{\sigma}\).
Proof. Let a non-zero function \( f \in \mathcal{B}_\sigma \) have infinitely many zeros \( x_j \)'s of multiplicity \( k \) on the real line such that \( x_j < x_{j+1}, \ j \in \mathbb{Z} \) and \( \bigcup_{j \in \mathbb{Z}} [x_j, x_{j+1}] = \mathbb{R} \). If possible, there exists \( M \leq \frac{\nu_k}{\sigma} \) such that \( x_{j+1} - x_j \leq M \), for every \( j \). Since \( f^{(l)}(x_j) = f^{(l)}(x_{j+1}) = 0 \), for every \( j \in \mathbb{Z}, \ 0 \leq l \leq k - 1 \), by Cimmino’s inequality

\[
\int_{x_j}^{x_{j+1}} |f^{(k-1)}(x)|^2 \, dx < \left( \frac{x_{j+1} - x_j}{\nu_k} \right)^2 \int_{x_j}^{x_{j+1}} |f^{(k)}(x)|^2 \, dx. \tag{3.1}
\]

Notice that the inequality is strict; otherwise if the equality holds, then \( f(x) \) coincides with the first eigenfunction of the following boundary value problem:

\[
u^2 u^{(2k)}(x) + \lambda u^{(2k-2)}(x) = 0, \ x \in [x_i, x_{i+1}], \\
u^{(l)}(x_i) = u^{(l)}(x_{i+1}) = 0, \ 0 \leq l \leq k - 1, \ u \in C^{2k}[x_i, x_{i+1}].
\]

Since \( f \) and the eigenfunction are entire, \( f \) coincides with the eigenfunction on the whole real axis. Moreover, the eigenfunction is a finite linear combination of \( \{x^l e^{\lambda x} \cos \mu_l x, x^l e^{\lambda x} \sin \mu_l x, l = 0, 1, 2, \ldots \} \). (For example, see the case \( k = 2 \) in [19]). Clearly, it is not square integrable on the real axis. Hence \( f \not\in L^2(\mathbb{R}) \), which is impossible. Summing over all \( j \) in (3.1), we get

\[
\int_{\mathbb{R}} |f^{(k-1)}(x)|^2 \, dx < \sum_j \left( \frac{x_{j+1} - x_j}{\nu_k} \right)^2 \int_{x_j}^{x_{j+1}} |f^{(k)}(x)|^2 \, dx \\
\leq \left( \frac{M}{\nu_k} \right)^2 \int_{\mathbb{R}} |f^{(k)}(x)|^2 \, dx.
\]

Taking square root on both sides, we get

\[
\|f^{(k-1)}\|_2 < \frac{M}{\nu_k} \|f^{(k)}\|_2. \tag{3.2}
\]

On the other hand, by Bernstein’s inequality,

\[
\|f^{(k)}\|_2 \leq \sigma \|f^{(k-1)}\|_2. \tag{3.3}
\]

Since \( f \) is nonconstant entire function which is square integrable on the real axis, \( f^{(k-1)} \) is non-zero. Combining (3.2) and (3.3), we get \( M > \frac{\nu_k}{\sigma} \) which is a contradiction.

\[\square\]

Corollary 3.1. If \( X = \{x_i : i \in \mathbb{Z}\} \) is sequence of real numbers such that \( \sup_i (x_{i+1} - x_i) \leq \frac{\nu_k}{\sigma} \), then \( X \) is a set of uniqueness of order \( k - 1 \) for \( \mathcal{B}_\sigma \).

Since \( \nu_2 = \lambda_{2,1} = 2\pi \), we obtain the following sharp results.

Corollary 3.2. The constant \( d_2 \) is equal to \( \frac{2\pi}{\sigma} \).
Corollary 3.3. If \( X = \{ x_i : i \in \mathbb{Z} \} \) is sequence of real numbers such that \( \sup (x_{i+1} - x_i) \leq \frac{2\pi}{\vartheta} \), then \( X \) is a set of uniqueness of order 1, i.e., if \( f(x_i) = f'(x_i) = 0 \), for all \( i \in \mathbb{Z} \), then \( f \equiv 0 \).

4. RECONSTRUCTION ALGORITHMS

Let \( \mathcal{D} \) denote the differentiation operator on \( \mathcal{B}_\sigma \). By Bernstein inequality, it is a bounded operator on \( \mathcal{B}_\sigma \). If \( f \in \mathcal{B}_\sigma \), then \( |f(x)| \to 0 \) as \( |x| \to \infty \) which implies that \( \int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)g'(x)dx \). Therefore, \( \mathcal{D} \) is a skew-Hermitian operator on \( \mathcal{B}_\sigma \). Let us define \( \mathcal{B}_{\sigma}^{k-1} : = \mathcal{D}^{k-1}(\mathcal{B}_\sigma) \). Clearly \( \mathcal{B}_{\sigma}^{k-1} \) is a subspace (not necessarily closed) of \( L^2(\mathbb{R}) \).

Let \( P \) be the orthogonal projection of \( L^2(\mathbb{R}) \) onto \( \mathcal{B}_{\sigma}^{k-1} \). Now assume that \( f \) and its first \( k-1 \) derivatives \( f', \ldots, f^{(k-1)} \) are sampled at a sequence \( (x_i)_{i \in \mathbb{Z}} \). Define the approximation operator for \( f \in \mathcal{B}_\sigma \)

\[
A[f^{(k-1)}] = P \left( \sum_{i \in \mathbb{Z}} \left( H_{2k-1}^{(k-1)}(x_i, x_{i+1}, f; \cdot) \right) \chi_{[x_i, x_{i+1}]} \right),
\]

where \( H_{2k-1}(x_i, x_{i+1}, f; \cdot) \) denotes the Hermite interpolation of \( f \) in the interval \( [x_i, x_{i+1}] \). Since \( f^{(k-1)} = P f^{(k-1)} = P \left( \sum_{i \in \mathbb{Z}} f^{(k-1)} \chi_{[x_i, x_{i+1}]} \right) \) for all \( f \in \mathcal{B}_\sigma \) and the characteristic functions \( \chi_{[x_i, x_{i+1}]} \) have mutually disjoint support, it can be easily shown that

\[
\| f^{(k-1)} - A[f^{(k-1)}] \|_2^2 \leq \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} |f^{(k-1)}(x) - H_{2k-1}^{(k-1)}(x_i, x_{i+1}, f; x)|^2 dx.
\]

If \( \sup (x_{i+1} - x_i) = \delta \), then it follows from (2.10) that

\[
\| f^{(k-1)} - A[f^{(k-1)}] \|_2^2 \leq \sum_{i \in \mathbb{Z}} \left( \frac{\delta}{v_k} \right)^2 \int_{x_i}^{x_{i+1}} |f^{(k)}(x)|^2 dx
\]

\[
= \left( \frac{\delta}{v_k} \right)^2 \| f^{(k)} \|_2^2
\]

\[
\leq \left( \frac{\delta}{v_k} \right)^2 \sigma^2 \| f^{(k-1)} \|_2^2,
\]

(4.1) using Bernstein’s inequality. As \( \| A f^{(k-1)} \|_2 \leq \| f^{(k-1)} - A f^{(k-1)} \|_2 + \| f^{(k-1)} \|_2 \), it follows from the inequality (4.1) that the operator \( A \) is bounded on \( \mathcal{B}_{\sigma}^{k-1} \). Hence the operator \( A \) can be uniquely extended to a bounded operator \( \tilde{A} \) on \( \mathcal{B}_{\sigma}^{k-1} \) such that

\[
\| g - \tilde{A} g \|_2 \leq \left( \frac{\delta \sigma}{v_k} \right) \| g \|_2,
\]

(4.2)
written as

\[ f \in B_{\sigma}^k. \]

If \( \delta < \frac{\nu_k}{\sigma} \), then \( \frac{\delta \sigma}{\nu_k} < 1 \) which implies that the operator \( \tilde{A} \) is invertible on \( B_{\sigma}^k \). Thus we can obtain the following result as a corollary of Theorem 2.6.

**Theorem 4.1.** Suppose that \( f \) and its first \( k - 1 \) derivatives \( f', \ldots, f^{(k-1)} \) are sampled at a sequence \( (x_i)_{i \in \mathbb{Z}} \). If \( \delta < \frac{\nu_k}{\sigma} \), then any \( f \in B_{\sigma} \) can be reconstructed from the sample values \( \{f^{(j)}(x_i)F : j = 0, 1, \ldots, k-1, i \in \mathbb{Z}\} \) using the following iteration algorithm. Set

\[
\begin{align*}
  f_0 &= A f^{(k-1)} = P \left( \sum_{i \in \mathbb{Z}} H_{2k-1}^{(k-1)}(x_i, x_{i+1}, f; \cdot) \chi_{[x_i, x_{i+1}]} \right), \\
  f_{n+1} &= f_n + \tilde{A}(f^{(k-1)} - f_n), \quad n \geq 0,
\end{align*}
\]

where \( H_{2k-1}(x_i, x_{i+1}, f; \cdot) \) denotes the Hermite interpolation of \( f \) in the interval \([x_i, x_{i+1}]\). Then we have \( \lim_{n \to \infty} f_n = f^{(k-1)} \). The error estimate after \( n \) iterations becomes

\[
\|f^{(k-1)} - f_n\|_2 \leq \left( \frac{\delta \sigma}{\nu_k} \right)^{(n+1)} \|f^{(k-1)}\|_2.
\]

As a consequence of the above theorem, we construct a frame (or atomic system) for differentiation operator on \( B_{\sigma} \). Let \( c_{i,l} = \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+1})^{2l+1}}{(2l+1)!^2} \, dx \). This can also be written as

\[
c_{i,l} = \frac{(x_{i+1} - x_i)^{2l+1}}{(2l+1)!^2} = \int_{x_i}^{x_{i+1}} \frac{(x - x_i)^{2l}}{l!^2} \, dx.
\]

Let \( X \) be separated by a constant \( \gamma > 0 \), i.e., assume that \( \inf_{i \neq j} |x_i - x_j| \geq \gamma \). It is proved in [4] that

\[
\frac{1}{2kC(k)} \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} |H_{2k-1}(x_i, x_{i+1}, f; x)|^2 \, dx
\]

\[
\leq \sum_{i \in \mathbb{Z}} \sum_{l=0}^{k-1} |f^{(l)}(x_i)|^2 (c_{i,l} + c_{i-1,l}) \leq B \|f\|_2^2,
\]

(4.3)

where \( B = 2 \left( \sum_{l=0}^{k-1} \frac{(\delta \sigma)^{2l}}{l!^2} \right)^{e^2+\sigma^2} \) and \( C(k) = \left[ \sum_{s=0}^{k-1} \binom{k+s-1}{s} \right]^2 \).

**Theorem 4.2.** If \( X = \{x_i\} \) is a separated set such that \( \delta < \frac{\nu_k}{\sigma} \), then for every \( f \in B_{\sigma} \), we have

\[
A \|D^{k-1}f\|_2^2 \leq \sum_{i \in \mathbb{Z}} \sum_{l=0}^{k-1} |f^{(l)}(x_i)|^2 (c_{i,l} + c_{i-1,l}) \leq B \|f\|_2^2,
\]

(4.4)
where \( A = \left(1 - \frac{\delta \sigma}{\nu_k}\right)^2 \frac{1}{2kC(k)} \frac{\gamma^{2(k-1)}}{\mu^{2k-1}} \), \( B = 2 \left(\sum_{l=0}^{k-1} \frac{(\delta \sigma)^2 l^2}{l!^2}\right) e^{\delta \sigma + \sigma^2} \), and
\[
C(k) = \left[ \sum_{s=0}^{k-1} \frac{(k + s - 1)!^2}{s!}\right]^2,
\]
i.e., \( X \) is a stable set of sampling of order \((k-1, k-1)\) for \( \mathcal{B}_\sigma \) with respect to the weight \( \{c_{il} + c_{i-1,l} : l = 0, \ldots, k - 1, i \in \mathbb{Z}\} \).

**Proof.** Recall \( \tilde{A}f^{(k-1)} = Af^{(k-1)} = P \left(\sum_{i \in \mathbb{Z}} H^{(k-1)}_{2k-1}(x_i, x_{i+1}, f; \cdot) \chi_{[x_i,x_{i+1}]}\right) \). Then
\[
\|f^{(k-1)}\|_2^2 = \|\tilde{A}^{-1} \tilde{A}f^{(k-1)}\|_2^2 \\
\leq \|\tilde{A}^{-1}\|^2 \|Af^{(k-1)}\|_2^2 \\
\leq (1 - \|I - \tilde{A}\|^{-2}) \|Af^{(k-1)}\|_2^2 \\
\leq \left(1 - \frac{\delta \sigma}{\nu_k}\right)^{-2} \|Af^{(k-1)}\|_2^2. \tag{4.5}
\]

We now estimate the value of \( \|Af^{(k-1)}\|_2 \).
\[
\|Af^{(k-1)}\|_2^2 \leq \left\|\sum_{i \in \mathbb{Z}} H^{(k-1)}_{2k-1}(x_i, x_{i+1}, f; \cdot) \chi_{[x_i,x_{i+1}]}\right\|_2^2 \\
\leq \int_{\mathbb{R}} \left|\sum_{i \in \mathbb{Z}} H^{(k-1)}_{2k-1}(x_i, x_{i+1}, f; x) \chi_{[x_i,x_{i+1}]}(x)\right|^2 dx \\
\leq \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} \left|H^{(k-1)}_{2k-1}(x_i, x_{i+1}, f; x)\right|^2 dx \\
\leq \frac{\mu^{2k-1}}{\gamma^{2(k-1)}} \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} \left|H^{(k-1)}_{2k-1}(x_i, x_{i+1}, f; x)\right|^2 dx, \tag{4.6}
\]
by Schmidt’s inequality. Hence the desired inequality \((4.4)\) follows from \((4.3)\) and \((4.6)\). \( \square \)

**Corollary 4.1.** If \( \{x_i\} \) is a separated set such that \( \delta < \frac{\nu_k}{\sigma} \), then there exists a Bessel sequence \( \{g_{i,l} : i \in \mathbb{Z}, l = 0, \ldots, k - 1\} \) in \( \mathcal{B}_\sigma \) such that
\[
f^{(k-1)}(x) = \sum_{i \in \mathbb{Z}} \sum_{l=0}^{k-1} f^{(l)}(x_i) g_{i,l}(x),
\]
for every \( f \in \mathcal{B}_\sigma \). The function \( f \) can be computed using the relation
\[
f^{(l-1)}(x) = \int_{x_i}^{x} f^{(l)}(t) dt + f^{(l-1)}(x_i), \ l = k - 1, k - 2, \ldots, 1.
\]
Proof. Since $D$ is a skew-Hermitian operator on $B_\sigma$ and $f^{(r)}(x) = (-1)^r \langle f, K_x^{(r)} \rangle$, the result follows from Theorem 2.1.

We know that $\nu_2 = 2\pi$. Consequently, we obtain our Theorem 4.1 as a particular case of the above corollary. We do not know how to find the Bessel sequence given in Corollary 4.1. Since the differentiation operator is not bounded below, we cannot apply the frame reconstruction algorithm also. However, we are going to apply the frame algorithm for a large subclass of $B_\sigma$.

For given $\varepsilon > 0$, we introduce a closed subspace in $L^2(\mathbb{R})$

$$B_{\sigma,\varepsilon} := \{ f \in L^2(\mathbb{R}) : \text{supp} (\hat{f}) \subseteq \left[ -\frac{\sigma}{2\pi}, -\varepsilon \right] \cup \left[ \varepsilon, \frac{\sigma}{2\pi} \right] \}. $$

Clearly $B_{\sigma,\varepsilon} \subseteq B_\sigma$. Since $\| f' \| = \| \hat{f}' \| = 2\pi |iw\hat{f}| \geq 2\pi \varepsilon \| f \|$, the differentiation operator is bounded below on $B_{\sigma,\varepsilon}$. Consequently, it follows from Theorem 4.2 that if $\delta < \frac{\nu_k}{\sigma}$, there exist $A_\varepsilon, B > 0$ such that

$$A_\varepsilon \| f \|_2^2 \leq \sum_{i \in \mathbb{Z}} \sum_{l=0}^{k-1} \left( -1 \right)^l f^{(l)}(x_i) \left( c_{i,l} + c_{i-1,l} \right) \leq B \| f \|_2^2, $$

(4.7)

for every $f \in B_{\sigma,\varepsilon}$. Recall that $B_\sigma$ is a reproducing kernel Hilbert space with reproducing kernel $K(x, y) = \frac{\sin (x-y)}{\sigma (x-y)}$. i.e., every $f \in B_\sigma$ can be written as

$$f(x) = \int_{\mathbb{R}} f(t) \frac{\sin (t-x)}{\sigma (t-x)} \, dt = \langle f, K_x \rangle, $$

(4.8)

where $K_x(t) = K(t,x)$. Moreover, $f^{(r)}(x) = (-1)^r \langle f, K_x^{(r)} \rangle$. Hence, if $\delta < \frac{\nu_k}{\sigma}$, it follows from Theorem 4.2 that the family $\{ \sqrt{c_{i,l} + c_{i-1,l}} K_x^{(l)} : l = 0, 1, \ldots, k-1, i \in \mathbb{Z} \}$ is a frame for $B_{\sigma,\varepsilon}$ with frame bounds $A_\varepsilon$ and $B$. This leads us to the following reconstruction algorithm for $f \in B_{\sigma,\varepsilon}$ from its samples values.

**Frame Algorithm:**

Set $S_k f := \sum_{i \in \mathbb{Z}} \sum_{l=0}^{k-1} (-1)^l f^{(l)}(x_i) (c_{i,l} + c_{i-1,l}) K_x^{(l)}$. and $\rho = \frac{2}{A_\varepsilon + B}$. Define

$$f_0 = 0, $$

$$f_{n+1} = f_n + \rho S_k (f - f_n), \quad n \geq 0. $$

Then we have $\lim_{n \to \infty} f_n = f$. The error estimate after $n$ iterations turns out to be

$$\| f - f_n \|_2 \leq \left( \frac{B - A_\varepsilon}{B + A_\varepsilon} \right)^n \| f \|_2. $$

**References**

1. Adcock, B., Gataric, M., and Hansen, A. C. Density theorems for nonuniform sampling of bandlimited functions using derivatives or bunched measurements, J. Fourier Anal. Appl., 23, 1311-1347, 2017.
2. Agarwal, R and Wong, P. Error inequalities in polynomial interpolation and their applications, vol. 262, Kluwer, Dordrecht, 1993.
3. Aldroubi, A and Gröchenig, K. Non-uniform sampling and reconstruction in shift-invariant spaces, SIAM Rev., 43(4), 585-620, 2001.
4. Antony Selvan, A and Radha, R. Separation of zeros, a Hermite interpolation based and a frame based reconstruction algorithms for bandlimited functions, Sampl. Theory Signal Image Process, 15, 21-35, 2016.
5. Beurling, A. In Carleson, L., Ed., A. Beurling. Collected Works, vol. 2, Birkhäuser, Boston, 341-365, 1989.
6. Cluine, J., Rahman, Q. I., and Walker, W. J. On entire functions of exponential type bounded on the real axis, J. Lond., Math., Soc. (2), 61(1), 163-176, 2000.
7. Fogel, L. J. A note on the sampling theorem, IRE Trans. Inform. Theory, IT-1, 47-48,1955.
8. Gavruta, L. Frames for operators, Appl. Comput. Harmon. Anal., 32, 139-144, 2012.
9. Gröchenig, K. Reconstruction algorithms in irregular sampling, Math. Comp., 59(199), 181-194, 1992.
10. Gröchenig, K., Romero, J. L., and Stöckler, J. Sharp results on sampling with derivatives in shift-invariant spaces and multi-window Gabor frames, Constr. Approx., 51(1), 1-25, 2020.
11. Jagerman, D. L and Fogel, L. J. Some general aspects of the sampling theorem, IRE Trans. Inform. Theory, IT-2, 139-146, 1956.
12. Levinson, N. Gap and density theorems, Amer. Math. Soc., Colloq. Publications, vol.26, 1940.
13. Linden, D. A and Abramson, N. M. A generalization of the sampling theorem, Inform. Contr, 3, 26-31, 1965.
14. Landau, H. J. Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math., 117, 37-52, 1967.
15. Rawn, M. D. A stable nonuniform sampling expansion involving derivatives, IEEE Trans. Inform. Theory, 36, 1223-1227, 1989.
16. Razafinjatovo, H. N. Iterative reconstructions in irregular sampling with derivatives, J. Fourier Anal. Appl., 1(3), 281-295, 1995.
17. Seip, K. On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$, J. Funct. Anal., 139, 131-160, 1995.
18. Spitzbart. A. A Generalization of Hermite’s interpolation formula, Amer. Math. Monthly, 67(1), 42-46, 1960.
19. Tcheng, Tchou-Yun. Sur les inégalités différentielles, Paris, 1934, 33pp.
20. Walker, W. J. The separation of zeros for entire functions of exponential type, J. Math. Anal. Appl., 122, 257-259, 1987.
21. Young, R. M. An introduction to non-harmonic Fourier series, Academic press, New York-London, 1980.

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