Finite volume cusp manifolds, Weyl law, Bérard remainder, scattering phase, resonances.

WEYL LAWS FOR MANIFOLDS WITH HYPERBOLIC CUSPS.

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Abstract. We give Weyl-type estimates on the natural spectral counting function for manifolds with exact hyperbolic cusps. We treat three different cases: without assumption on the compact part, assuming that periodic geodesics form a measure-zero set, and assuming the curvature is negative. In each case, we obtain the same type of remainder as in the corresponding case in the context of compact manifolds.

We also investigate the counting of resonances. In particular, we extend results of Selberg to the case of non-constant, negative curvature metrics, under a genericity assumption.

1. Introduction

In this paper, we aim to give precise spectral counting estimates for the Laplacian of some finite volume manifolds with cusps. Such a manifold $(M, g)$ can be decomposed as a compact part with boundary $M_0$, of dimension $d + 1$, and cusps $Z_1, \ldots, Z_\kappa$ with

\[(Z_i, g) \simeq \left( [a_i, +\infty) \times \left( \mathbb{R}^{d+1}/\Lambda_i \right)_\theta, \frac{dy^2 + d\theta^2}{y^2} \right), \]

where $\Lambda$ is a lattice in $\mathbb{R}^{d+1}$ of covolume 1. Such a manifold has finite volume. We denote the laplacian $\Delta$ with the analyst’s convention that $-\Delta \geq 0$. We let $y_0 = \max_i a_i$.

On a compact manifold, the Laplacian only has discrete spectrum, with eigenfunctions and eigenvalues. On the other hand, for a non-compact manifold, continuous spectrum appears. Thanks to the properties of the cusps (and in particular, the finite volume), cusp-manifolds come close to being compact, and they are in some sense a toy model to understand what exactly happens when creating a small non-compacity.

Before we can state the problem precisely, we have to introduce some terminology from Scattering Theory. What we learn from the theory is the following — see [LP76], then [CdV81, CdV83] and [Mül83, Mül86, Mül92]. The laplacian has both continuous and point spectrum. The discrete spectrum consists of a discrete set $\sigma_{pp}$ of real numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \ldots$. This set may be finite or infinite, and each eigenvalue has finite multiplicity. The continuous spectrum has multiplicity $\kappa$, the number of cusps. To each cusp is associated a meromorphic family of generalized eigenfunctions $E_i(s)$ called the Eisenstein functions. Additionally, the resolvent

\[ R(s) = (-\Delta - s(d - s))^{-1} \]

extends from $\Re s > d/2$ to $\mathbb{C}$ as a meromorphic family of operators $C_c^\infty \to C^\infty$. The poles are in $\{ \Re s \leq d/2 \} \cup [d/2, d]$. They form the so-called resonant set $\text{Res}(M, g)$. 

\[ \text{Res}(M, g) \]
If \( \frac{d^2}{4} + r^2 \) is a discrete eigenvalue \( \lambda_i \), then \( \frac{d}{2} + ir \) is in the resonant set. The other elements of the resonant set are the poles of the family of Eisenstein functions. They are also the set of poles of a meromorphic function \( \varphi \) called the Scattering Determinant, see §2.1 for a definition. On the axis \( \{ \Re s = \frac{d}{2} \} \), this function can be written \( \varphi(\frac{d}{2} + ir) = e^{2i\pi S(r)} \), where \( S(r) \) is a real, analytic function, called the Scattering Phase.

1.1. Long time behaviour of the wave equation and resonances. To each element \( s \) of the resonant set is associated a vector space of smooth (non necessarily \( L^2 \)) eigenfunctions. These eigenfunctions give fundamental solutions to several classical equations on the manifold — let us consider the wave equation.

\[
\left[ \frac{\partial^2}{\partial t^2} - \Delta - \frac{d^2}{4} \right] f = 0
\]

The number \( \eta(s) = \frac{d}{2} - \Re s \) is the parameter that describes how much the behaviour of these solutions differ from compact cases, in large time. The non-compactness of the manifold allows for energy leaking out of the manifold, and \( \eta \) is the rate of exponential decay of the solutions associated to that eigenspace.

As a consequence, to understand long time behaviour of solutions to the wave equation, one way is to determine the distribution of \( \eta(s) \) when \( s \in \text{Res}(M, g) \). In particular for values of \( \Im(s) \) — which is in general a frequency — around a given \( T \).

Instead of considering a cusp manifold, let us consider for a moment a manifold that would be a compactly supported small enough perturbation of the euclidean space. Then, with a similar formalism, one can consider its resonant set, and its resonances. Then one can prove that there is a zone of logarithmic size near the continuous spectrum with no resonances (see [Vali77]). That means in particular that solutions of the wave equation decrease exponentially in time, and the rate increases with the frequency. This can be related to the fact that there are no trajectories of the geodesic flow on that surface that are trapped.

On the other hand, for cusp manifolds, almost all trajectories are trapped in the manifold (they will pass through the compact part \( M_0 \) an infinite number of times). As a consequence, one expects a behaviour closer to the compact manifold, i.e, that there are resonances near the spectrum. In particular, although solutions to the wave equation may decrease exponentially in time, the rate of decay will be very small at high frequency.

1.2. Results. Let us state our main theorems. First, define the counting function for the point spectrum

\[(4) \quad N_{pp}(T) := \# \{ \lambda \in \sigma_{pp} \mid \lambda \leq \frac{d^2}{4} + T^2 \},\]

and the total counting function:

\[(5) \quad \tilde{N}(T) := N_{pp}(T) - S(T)\]

where \( S \) is the scattering phase, alluded to in the introduction, that we be defined in §2.1. Our first result is more of a remark:
**Theorem 1.** Let $M$ be a finite volume manifold with $\kappa$ cusps. Then the Fourier transform of $\tilde{N}'(T)$ is a tempered distribution, whose singular support is the lengths of periodic geodesics on $S^* M$.

This theorem is just a direct extension to manifolds with cusps of the result of Chazarain [Cha74] for compact manifolds. Concerning estimates of $\tilde{N}(T)$ under different assumptions on the manifold $M$, we get

**Theorem 2.** Let $M$ be a finite volume manifold with $\kappa$ cusps, of dimension $d + 1$. In decreasing order of generality,

- Without further assumption on $M$,
  \[
  \tilde{N}(T) = \frac{\text{vol} B^* M}{(2\pi)^{d+1}} T^{d+1} - \frac{\kappa}{\pi} T \log T + O(T^d).
  \]

- If one assumes that the set of periodic geodesics has measure zero in $S^* M$, then
  \[
  \tilde{N}(T) = \frac{\text{vol} B^* M}{(2\pi)^{d+1}} T^{d+1} - \frac{\kappa}{\pi} T \log T + \frac{\kappa(1 - \log 2)}{\pi} T + o(T^d).
  \]

- Finally, if $M$ is negatively curved, or in dimension 2, if the geodesic flow has no conjugate points,
  \[
  \tilde{N}(T) = \frac{\text{vol} B^* M}{(2\pi)^{d+1}} T^{d+1} - \frac{\kappa}{\pi} T \log T + \frac{\kappa(1 - \log 2)}{\pi} T + O\left(\frac{T^d}{\log T}\right).
  \]

Observe that the lower order terms are bigger than the remainder only when $d = 1$.

Selberg proved the last result for constant curvature surfaces (see [Sel80b] for a statement). Then Müller [Mü86] identified the leading term in the variable curvature case (and also in higher dimensions). For surfaces again, Parnovski [Par95] obtained the first and second estimate. The third estimate is the equivalent for cusp manifolds of the result of Bérard [Bér77] for compact manifold.

Selberg proved that for constant curvature surfaces, the resonances are contained in a vertical strip near $\{\Re s = 1/2\}$. I extended that result to a set of negatively curved metrics $\mathcal{G}(M)$ in [Bon15]. When there is only one cusp, $\mathcal{G}(M)$ is the whole set of negatively curved cusp metrics. When there are more than one cusp, it contains the constant curvature metrics, it is open in $C^2$ topology on metrics, and its complement has infinite codimension in $C^\infty$ topology. For every metric $g \in \mathcal{G}(M)$, there is $\delta > d/2$ such that for some $\epsilon > 0$, we have

\[
\#\{s \in \text{Res}(M, g) \mid \Re s \leq d - \delta, \ |s| \geq e^{-\epsilon \Re s} \} \text{ is finite.}
\]

As we will recall at the start of section 3, this result is related to the existence of several Dirichlet series $\{L_i\}$, and in particular the most important, $L_0$, that writes

\[
L_0(s) = \sum_{k \geq 0} a_k^0 e^{-s \ell_k}.
\]

The constants $a_k^0$ and $\ell_k$ have a dynamical interpretation in terms of scattered geodesics, see [Gui77, Bon15]. The first non vanishing term in the sum is $a_k^0 e^{-s \ell_k}$ (see the § after equation (45)). The set $\mathcal{G}(M)$ is the set of metrics such that $L_0$ is not identically zero. We prove here...
**Theorem 3.** Assume \( g \in \mathfrak{g}(M) \). Then we have the following estimate. First, for the resonance away from the vertical strip

\[
\# \{ s \in \text{Res}(M, g) \mid \Re s < d - \delta, \ |s| \leq T \} = \mathcal{O}(T).
\]

Actually, we have the more precise estimate: for every \( 0 < \epsilon < 1 \),

\[
\# \{ s \in \text{Res}(M, g) \mid \Re s < d - \delta, \ |s - d/2 - iT| \leq T^\epsilon \} = \mathcal{O}(T^\epsilon).
\]

Now, for the resonances in the strip,

\[
\sum_{\Re s \geq d - \delta, 0 \leq \Im s \leq T} d - 2\Re s = \frac{\kappa}{2\pi} T \log T - \frac{T}{\pi} \left( \frac{\kappa}{2} + \log |a_s^0| - \frac{d}{2} \right) + \mathcal{O}(\log T).
\]

and the main estimate,

\[
\# \{ s \in \text{Res}(M, g) \mid d - \delta < \Re s \leq d/2, 0 \leq \Im s \leq T \} = \frac{\text{vol}(B^*M)}{(2\pi)^{d+1}} T^{d+1} - \frac{\kappa}{\pi} T \log T + \left( \kappa(1 - \log 2) + \frac{\ell_s}{2} \right) \frac{T}{\pi} + \mathcal{O} \left( \frac{T^d}{\log T} \right).
\]

Again, for constant curvature surfaces, this is due to Selberg. Still for surfaces, but with variable curvature, Müller observed that the leading term in the counting function should be the same, and Parnovski gave a bound on the remainder of size \( \mathcal{O}(T^{3/2+\epsilon}) \). In [Bon14b], I proved that actually, the remainder is \( \mathcal{O}(T^{3/2}) \). This result can be generalized to any dimension, as the missing ingredient was the estimate (6) in higher dimension.

**Theorem 4.** Let \( M \) be a manifold with cusps of dimension \( > 2 \). Without further assumption,

\[
\# \{ s \in \text{Res}(M, g) \mid |s - d/2| \leq T \} = 2 \frac{\text{vol}(B^*M)}{(2\pi)^{d+1}} T^{d+1} + \mathcal{O}(T^d).
\]

We also have the local estimate

\[
\# \{ s \in \text{Res}(M, g) \mid |s - d/2 + iT| = \mathcal{O}(1) \} = \mathcal{O}(T^d).
\]

**Remark.** To the best of the author’s knowledge, there is no existing piece of mathematical technology that would enable one to give an improved upper bound on the size of the remainder in the Weyl laws stated above. That does not mean that these remainders are optimal. According to several standing conjectures in the case of negative curvature, the optimal size of the remainder could be several orders of magnitude smaller. However, it seems we are far from a new theorem in this direction, to this day.

1.3. **structure of the paper.** Section 2 will be devoted to the proof of theorem 1 and 2. The main emphasis is made on the third estimate in theorem 2. It is an elaboration on the techniques of Béard [Ber77], who gave an equivalent theorem for compact manifolds. His proof was a clever adaptation to variable curvature of ideas from the proof of Selberg (that uses the Trace Formula). Then we obtain the first part of theorem 2 as a corollary of the previous work. At last, recalling some facts from the article of Duistermaat and Guillemin [DG75], we sketch proofs for theorem 1 and the second estimate in theorem 2.
In section 3, we will turn to the proof of theorem 3. The proofs mostly use techniques from complex and harmonic analysis. For a good part, the reader acquainted with Selberg’s methods will recognize familiar tricks.

The proof of theorem 4 will be found in the last section.

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2. A Weyl asymptotics for the scattering phase

As announced, in this section, we will give Weyl-type asymptotics for the scattering phase, for negatively curved cusp manifolds (the third part of theorem 2 which was the original motivation for this paper). The main idea is to express the quantity we seek to evaluate as some integral involving the trace of the wave group. Then we use the Hadamard parametrix for the wave kernel to obtain an expansion.

In the case of compact manifolds, the trace of the wave group is a well defined distribution. However, for our non-compact manifolds, we have to change the definition of the trace and replace it by a 0-trace. This is explained in 2.1 where we prepare the stage for the rest of the proof. In the next section 2.2, we recall the construction of the Hadamard parametrix on the universal cover of the manifold. Then, in 2.3, we evaluate contributions from different elements of the fundamental group. Then comes the conclusion 2.4.

As a corollary of the proof, we obtain a Weyl law for general cusp manifolds (without assumption of curvature) by inspecting only the singularity at 0 of the wave-trace. At last, we sketch proofs for theorem 1 and part 2 of 2.

2.1. The wave trace and Weyl estimates.

2.1.1. Some spectral theory. Let \( \{u_\lambda\} \) be an orthogonal family of \( L^2 \) eigenfunctions of \( \Delta \) that generate the pure point spectrum part of \( L^2(M) \).

The Eisenstein functions \( E(s) = (E_1, \ldots, E_\kappa(s)) \) (alluded to in the introduction) form the only meromorphic family of eigenfunctions of \( M \) such that

\[
(-\Delta - s(d - s))E(s) = 0
\]

\[
E_i(s)_{0,Z_j}(y) = \delta_{ij} y^s + \phi_{ij}(s) y^{d-s}
\]

\[E - E_0^{y_0} \in L^2(M) \text{ for all } s \text{ such that } \Re s > d/2.\]

where \( E_j(s)_{0,Z_j} \) is the zeroth Fourier mode of \( E_i \) in \( Z_j \), and \( E_0^a \) is the function on \( M \) that equals the zeroth Fourier mode of \( E(s) \) for \( y > a \) in the cusps, and vanishes elsewhere. The matrix with coefficients \( \phi_{ij} \) is denoted \( \phi(s) \) (the Scattering Matrix) and its determinant is the scattering determinant. It is meromorphic, and from the definition of \( E \), we find

\[
\phi(s)\phi(d - s) = 1
\]
From this, we deduce
\begin{equation}
\varphi(s)\varphi(d-s) = 1.
\end{equation}
and that on \( \Re s = \frac{d}{2}, |\varphi(s)| = 1. \)

Then, one can show that for every \( f \in C^\infty_c(M) \) (see [MüI83, eq. 7.36]),
\begin{equation}
f = \sum_\lambda \langle f, u_\lambda \rangle u_\lambda + \frac{1}{4\pi} \sum_{i=1}^K \int_R \left\langle f, E_i \left( \frac{d}{2} + ir \right) \right\rangle E_i \left( \frac{d}{2} + ir \right) dr.
\end{equation}

2.1.2. The \( 0 \) trace. Now, we want to define a replacement for the trace of the wave group, and relate it to the spectral quantities. First, we observe that \( \text{Tr} \cdot \cos t\sqrt{-\Delta - d^2/4} \) is a tempered distribution in \( t \), valued in \( \mathcal{D}'(M) \).

Let \( \chi \in C^\infty_c(M) \). From (18),
\begin{equation}
\text{Tr} \chi \cos t\sqrt{-\Delta - d^2/4} = \sum_\lambda \cos t\sqrt{\lambda - d^2/4} \int_M \chi |u_\lambda|^2 \\
+ \frac{1}{4\pi} \sum_i \int_R \int_M \chi \left| E_i \left( \frac{d}{2} + ir \right) \right|^2 \cos tr dr,
\end{equation}
provided the RHS makes sense in \( \mathcal{S}'(\mathbb{R}) \), which we check now. The first term is the easiest one. Indeed, take \( \psi \in \mathcal{S}(\mathbb{R}) \). Then, integrating \( \psi \) against the first term, we find
\begin{equation}
\sum_\lambda \hat{\psi}(\sqrt{\lambda - d^2/4}) c_\lambda + \sum_\lambda \hat{\psi}(-\sqrt{\lambda - d^2/4}) c_\lambda
\end{equation}
where the \( c_\lambda \) are constants bounded by \( \|\chi\|_{\infty}/2 \). This defines a continuous functional on Schwarz functions because we already know there is at most \( CT^{d+1} \) eigenvalues smaller than \( T^2 \) (see [MüI86]).

Next, we turn to the second term. Assume \( \chi \) vanishes for \( y > y_1 \), with \( y_1 \geq y_0 \). We can bound \( \langle \chi E, E \rangle \) by \( \langle (1 - \Pi_{y_1}^0) E, E \rangle \), where \( \Pi_{y_1}^0 \) is the projection on the zeroth Fourier mode in \( y \geq y_1 \) — so that \( E_{y_1}^0 = \Pi_{y_1}^0 E \). Since the expression is a Fourier transform, it suffices to show that the following is in \( \mathcal{S}'(\mathbb{R}) \) :
\begin{equation}
r \mapsto \int_M \left| E \left( \frac{d}{2} + ir \right) - E_{y_1}^0 \left( \frac{d}{2} + ir \right) \right|^2.
\end{equation}
Setting \( y_0 = e^\tau \), from the Maass-Selberg relations (see equation (54) in the second half of the article), we find that
\begin{equation}
\int_M |E - E_0^\tau|^2 = 2\kappa_T - \frac{\varphi'}{\varphi} \left( \frac{d}{2} + ir \right) + \text{Tr} \frac{e^{2i\tau \phi^*} \phi(d/2 + ir) - e^{-2i\tau \phi} \phi(d/2 + ir)}{2ir}.
\end{equation}
In the RHS, the first term is a constant in \( r \), the second is \(-2\pi S'(r)\), and \( S \) is a tempered function. One can check that the last term is continuous at \( r = 0 \), hence a smooth, bounded function of \( r \) (for bounded \( \tau \)). This proves
\begin{lemma}
The trace
\begin{equation}
\text{Tr}(\cdot \cos t\sqrt{-\Delta - d^2/4}) \in \mathcal{D}'(M, \mathcal{S}'(\mathbb{R})).
\end{equation}
\end{lemma}
is well defined.

To compute the actual wave trace, we should integrate our distribution against 1 as a function on $\mathcal{M}$. However, this is not well defined. So we replace 1 by $1 - \Pi^{0}_{0}$, and we try to find a limit when $\tau \to \infty$. Actually, there is no limit but an expansion $\kappa \delta \tau + c + o(1)$, where $c$ is a constant (in the $\tau$ variable). We call $c$ the 0-Trace. For a more detailed exposition of the 0-Tr, one may consult [GZ97].

**Lemma 2.2** (Trace formula 1). We have, for $\psi \in \mathcal{S}(\mathbb{R})$ real and even,

$$\sum_{\lambda \in \sigma_{pp}} \hat{\psi}(\sqrt{\lambda - d^2/4}) - \frac{1}{2} \int S' \hat{\psi} + \frac{1}{4} \hat{\psi}(0) \text{Tr} \phi(d/2) = 0 \text{-Tr} \hat{\psi} \left( \sqrt{-\Delta - d^2/4} \right).$$

**Proof.** Let $\psi \in \mathcal{S}(\mathbb{R})$ be real and even. Using (19),

$$\text{Tr} 1_{\tau} \hat{\psi}(\sqrt{-\Delta - d^2/4}) = \int \text{Tr} 1_{\tau} \cos t \sqrt{-\Delta - d^2/4} \psi(t) dt$$

$$= \sum_{\lambda} \hat{\psi}(\sqrt{\lambda - d^2/4}(1 - c(\tau, \lambda)))$$

$$+ \kappa \tau \psi(0)$$

$$- \frac{1}{2} \int S' \hat{\psi}$$

$$+ \frac{1}{4\pi} \int \hat{\psi} e^{2i\tau r} \text{Tr} \phi^* + e^{-2i\tau r} \text{Tr} \phi dr.$$ 

The coefficients $c(\tau, \lambda)$ go to 0 as $\tau \to +\infty$. Denote the last integral by $I$. If we compute $\partial_{\tau} I$, we find

$$\partial_{\tau} I = \frac{1}{4\pi} \int \hat{\psi} \times \left( e^{2i\tau r} \text{Tr} \phi^* + e^{-2i\tau r} \text{Tr} \phi \right).$$

Since $\phi^*(d/2 + ir) = \phi(d/2 - ir)$,

$$\partial_{\tau} I = \frac{1}{2\pi} \psi \ast \overline{\text{Tr} \phi}(2\tau).$$

In particular, $I$ has limits for $\tau \to \pm\infty$. But, as $\psi$ is real and even, $\hat{\psi}$ also is. Hence, $I$ is real and $I(-\tau) = -I(\tau)$, so

$$2I(+\infty) = \int \partial_{\tau} I = \frac{1}{2} \hat{\psi}(0) \text{Tr} \phi(d/2).$$

We deduce that as $\tau \to \infty$,

$$\int \text{Tr} 1_{\tau} \cos t \sqrt{-\Delta - d^2/4} \psi(t) dt - \kappa \tau \psi(0) =$$

$$+ \sum_{\lambda} \hat{\psi}(\sqrt{\lambda - d^2/4}) - \frac{1}{2} \int S' \hat{\psi} + \frac{1}{4} \hat{\psi}(0) \text{Tr} \phi(d/2) + o(1).$$

This is what we wanted to prove. $\square$
If we replace \( \hat{\psi} \) by \( 1_{\{|r| \leq T\}} \) in the RHS of (20), we obtain the LHS in the equations of theorem 2. We will see later how to go from \( \hat{\psi} \) to \( 1_{\{|r| \leq T\}} \). However, for now, we want to compute asymptotics for the 0-trace of \( \cos t\sqrt{-\Delta - d^2/4} \), i.e. the LHS of (20). Let \( K(t, x, x') \) be the kernel of \( \cos t\sqrt{-\Delta - d^2/4} \).

Using dominated convergence, it is possible to replace \( 1_\tau \) by \( 1_{\{y \leq e^\tau\}} \) in the formula, and not change the result in (20). We deduce that

\[
(21) \sum_{\lambda} \hat{\psi}\left(\sqrt{\lambda - d^2/4}\right) - \frac{1}{2} \int S' \hat{\psi} + \frac{1}{4} \hat{\psi}(0) \operatorname{Tr} \phi \left(\frac{d}{2}\right) = \lim_{\tau \to \infty} \int_{y \leq e^\tau} \int_{\mathbb{R}} \psi(t) K(t, x, x) dx dt - \kappa \tau \psi(0).
\]

In the rest of the section, \( \psi \) is an even compactly supported Schwartz function. Since we want \( \hat{\psi} \) to approach \( 1_{\{|r| \leq T\}} \), eventually, we will take \( A > 0 \), and consider \( \psi \) of the following form

\[
(22) \psi(t) = \frac{1}{\pi} \frac{\sin(tT)}{t} \rho(At)
\]

where \( \rho \in C_c^\infty(\mathbb{R}) \) is even, compactly supported, and constant equal to 1 near 0. We will be interested in the scaling \( A \asymp 1 / \log T \) when the curvature is negative (or when there are no conjugate points when \( d = 1 \)), and \( A \asymp 1 \) in general.

### 2.2. The Hadamard parametrix

From now until section 2.4.2, we assume that the curvature of the manifold is non-positive (or again, the absence of conjugate points if \( d = 1 \)).

#### 2.2.1. Building the approximation on the universal cover

Originally, we wanted to use the usual Hadamard parametrix, just as in Bérard. However, doing this, a remainder appears, and it is not obvious that it is trace class. To avoid such a discussion, we found it was simpler to use a version of the Hadamard parametrix tweaked to our needs. Instead of being modelled on the Euclidean space, it is modelled on the hyperbolic space. For lack of a reference, we recall its construction.

First, we define an approximate kernel on the universal cover \( \widetilde{M} \) of \( M \). Let \( \widetilde{K} \) be the kernel of \( \cos t\sqrt{-\Delta - d^2/4} \) on \( \widetilde{M} \).

We have to introduce a bit of notation. We follow Bérard (or, if you will, [BGM71]). If \( x, x' \in \widetilde{M}, \ r = d(x, x') \) is the riemannian distance between them. We also let

\[
\Theta(x, x') = \det T_{\exp^{-1} x'} \exp_x.
\]

For fixed \( x \), \( \partial / \partial r \) is the unitary vector field along geodesics from \( x \), and \( \Theta' = \partial_r \Theta \). From the proposition G.V.3 in [BGM71], we have

\[
\text{div} \left( \frac{\partial}{\partial r} \right) = \Delta r = -\left( \frac{\Theta'}{\Theta} + \frac{d}{r} \right).
\]

From this identity, one may compute \( \Theta \) in the case of the real hyperbolic space. It is only a function of \( r \), and we denote it by \( \Theta_0 \). One finds that

\[
\frac{\Theta'_0}{\Theta_0} = d \left( \coth r - \frac{1}{r} \right).
\]
so that

\[
\text{div} \left( \frac{\partial}{\partial r} \right) = - \left( \frac{\Theta'}{\Theta} - \frac{\Theta_0'}{\Theta_0} + \frac{d \cosh r}{\sinh r} \right).
\]

We define a family of tempered distributions on \( \mathbb{R} \) in the following way. If \( s \in \mathbb{R}, s_+ \) is its positive part. For \( \alpha \in \mathbb{C}, \)

\[
M_\alpha(s) = \begin{cases} 
\frac{s^\alpha}{\Gamma(|\alpha| + 1)} & \text{if } \Re \alpha > -1 \\
M_{\alpha-1}(s) = M'_\alpha(s).
\end{cases}
\]

In particular, we have \( sM_{\alpha-1}(s) = \alpha M_\alpha(s) \).

The wave equation propagates at speed 1. That is to say that \( \tilde{K}(t, x, x') \) is supported for \( d(x, x') \leq t \). Additionally, the singular part of the kernel is supported exactly on \( \{d(x, x') = t\} \) (see [Tay11]).

The point of the Hadamard parametrix is to expand the kernel of the wave operator in powers of \( (t^2 - r^2) + \). However, the first terms in the development have to be negative powers. To define them as distributions, we have to interpret \( (t^2 - r^2)^{-n} \) as \( M^{-n}(t^2 - r^2) \), if \( n \geq 0 \).

In the case of the real hyperbolic space, it is convenient to replace \( t^2 - r^2 \) by \( \cosh t - \cosh r \).

That is, we are looking for an expression of the kernel in the form

\[
\tilde{K}(t, x, x') = C_0 \sum_k \left( \frac{-1}{2} \right)^k f_k(x, x') \sinh |t| M_k(\cosh t - \cosh r),
\]

where we are summing over \( k_0 + N, k_0 \in \mathbb{R} \). Obviously such an expansion cannot be exact, but let us do a formal computation. Let \( \Box = \partial_t^2 - \Delta_{x'} - d^2/4 \), and \( \Box_0 = \partial_t^2 - \Delta_{x'} \). For \( t > 0, \)

\[
\Box \{ f_k \sinh tM_k \} = (-\Delta - d^2/4) f_k \sinh tM_k \\
+ f_k \sinh t\Box_0 M_k \\
+ f_k \sinh tM_k \\
+ 2 f_k \cosh t \partial_t M_k \\
- 2 \sinh t \nabla f_k \nabla M_k.
\]

We compute

\[
\nabla M_k = - \sinh r M_{k-1} \partial_r, \\
\partial_t M_k = \sinh t M_{k-1}, \\
\Box_0 M_k = \left[ k(\cosh t + \cosh r) + \sinh r \left( \frac{\Theta'}{\Theta} - \frac{\Theta_0'}{\Theta_0} \right) + d \cosh r \right] M_{k-1}.
\]

If we sum up the computations,

\[
0 = \Box \tilde{K} = C_0 \sum \left( \frac{-1}{2} \right)^k \sinh |t| M_k \times \left[ \left( -\Delta - d^2/4 + 1 + k(k + 2) \right) f_k \\
- \frac{1}{2} \left\{ f_{k+1} \left( 2g + d/2 \right) \cosh r + \sinh r \left( \frac{\Theta'}{\Theta} - \frac{\Theta_0'}{\Theta_0} \right) \right\} + 2 \sinh r \partial_x f_{k+1} \right].
\]
We deduce that
\[ f_k \left( \left( k + 1 + \frac{d}{2} \right) \cosh r + \frac{1}{2} \sinh r \left( \frac{\Theta'}{\Theta} - \frac{\Theta_0'}{\Theta_0} \right) \right) + \sinh r \partial_r f_k = (-\Delta + k^2 - d^2/4) f_{k-1}. \]

This is a nice family of transport equations. Now, we need to determine \( k_0 \). Since we require that the limit when \( t \to 0 \) is \( \delta(x, x') \), by a dimensional analysis, the sum has to start at \( k_0 = -d/2 - 1 \). The corresponding \( f_{k_0} \) is \( (\Theta_0/\Theta)^{1/2} \), up to a constant. We let \( u_k = f_{k_0+k} \) for \( k \in \mathbb{N} \).

The solution to the system is: for \( k > 0 \),
\[ u_0 = \sqrt{\frac{\Theta_0}{\Theta}}, \]
\[ u_k = \sqrt{\frac{\Theta_0}{\Theta}} \frac{1}{\sinh(r)^k} \int_0^r \sinh(s)^{k-1} \sqrt{\frac{\Theta}{\Theta_0}} (s) (-\Delta + (k-1-d/2)^2 - d^2/4) u_{k-1}(s) ds \]
where the parameter \( s \) describes the geodesic between \( x \) and \( x' \), travelled at speed 1. When the curvature around the geodesic from \( x \) to \( x' \) is constant equal to \(-1\), \( u_k \) vanishes at \((x, x')\) for \( k \geq 1 \), and \( u_0 = 1 \).

The following lemma is crucial.

**Lemma 2.3.** For all \( k \geq 0 \) and \( l \geq 0 \), \( \Delta_x^l u_k = O(1) e^{O(r)} \).

**Proof.** From the formula defining the \( u_k \), one can see that it suffices to prove the estimates for \( u_0 \) (proceeding by induction on \( k \)). Hence it suffices to prove the formula for \( 1/\sqrt{\Theta} \), which was done in the appendix in Bédard [Ber77]. \( \square \)

As the sum does not converge, we define the cutoff sums
\[ \tilde{K}_N(t, x, x') = C_0 \sum_{k=0}^N \left( -\frac{1}{2} \right)^k u_k(x, x') \sinh |t|M_{k-(d+2)/2}(\cosh t - \cosh r). \]

**Lemma 2.4.** There is a constant \( C_0 \) only depending on the dimension, so that for \( N \geq 0 \), \( x \in \tilde{M} \),
\[ \int \tilde{K}_N(t, x, x') \psi(x') dx' \to \psi(x) \text{ and } \int \partial_t \tilde{K}_N(t, x, x') \psi(x') dx' \to 0 \text{ as } t \to 0. \]
for all \( \psi \in C^\infty_c(B(x, \tilde{\rho})) \). Additionally, for \( N > (d+2)/2 \),
(26)
\[ \tilde{K}_N = C_0 \left( -\frac{1}{2} \right)^N \left[ (-\Delta + (N - d/2)^2 - d^2/4) u_N \right] \sinh |t|(\cosh t - \cosh r)^{N-(d+2)/2}. \]

The constant \( C_0 \) depending only on the dimension, we can compute its value in the case of constant \(-1\) curvature. But that value can be found (for example) in [BO94], page 360, in proposition 2.1. We get
\[ C_0 = \frac{1}{2} \frac{1}{\sqrt{2\pi}}. \]

The computations needed to check the limits at \( t \to 0 \) can be found in [Ber77]. \( \square \)
2.2.2. Summing over the fundamental group. Now, we denote by $\Gamma$ the fundamental group of $M$, and define for $N \geq 0$,

$$K_N(t, x, x') = \sum_{\gamma \in \Gamma} \tilde{K}_N(t, x, \gamma x').$$

For each $x, x'$, the number of non-vanishing terms is finite, so this is well defined. Since it is bi-invariant by $\Gamma$, it defines a kernel on $M$, that we still denote by $K_N$. The associated operator on $C^\infty_c(M)$ is denoted by $A_N$: for $f \in C^\infty_c(M)$,

$$A_N f(x) = \int_M K_N(x, x') f(x') dx'.$$

Lemma 2.5. For $N > 10d$, $R_N(t) := A_N(t) - \cos t \sqrt{-\Delta - d^2/4}$ is a continuous family of Trace class operators. We have the estimate

$$\text{Tr} R_N(t) = O(t^d) e^{O(|t|)}.$$

Proof. First, let $V_N(t) = \Box R_N(t)$. Then we have

$$R_N(t) = \int_0^t \sin \left[ \frac{(t-s)\sqrt{-\Delta - d^2/4}}{\sqrt{-\Delta - d^2/4}} \right] V_N(s) ds.$$

Since $\sin(s\sqrt{-\Delta - d^2/4})/\sqrt{-\Delta - d^2/4}$ is bounded with norm $O(1)$, it suffices to prove the same type of estimate for $V_N$. We need

Lemma 2.6. Let $L(x, x')$ be the kernel of some operator on $L^2(M)$. Assume that

$$C := \sum_{|\alpha| \leq 2d+3} \| y^{d/2} y^{d/2} \partial_{x,x'}^\alpha L\|_{L^1(M \times M)} < \infty.$$

Then the corresponding operator is trace class, and its trace norm is less than $C$. (Here, we have abused notations; y is a height function, corresponding with the usual function in the cusps, and being some positive constant in the compact part $M_0$). The derivatives are taken along unitary vector fields (for example, $y \partial_y$ and $y \partial_\theta$ in the cusps).

We will give the proof of this fact later. We try to estimate the quantity $C$ for $V_N$. This gives

$$\sum_{|\alpha| \leq 2d+3} \sum_{\gamma \in \Gamma} \int_{D \times D} |\partial_{x,x'}^\alpha \Box \tilde{K}_N(t, x, \gamma x')| dxdx'.$$

where $D$ is a fundamental domain for the action of $\Gamma$ on $\tilde{M}$. We can rewrite this as

$$\sum_{|\alpha| \leq 2d+3} \int_{D \times \tilde{M}} |\partial_{x,x'}^\alpha \Box \tilde{K}_N(t, x, x')| dxdx'.$$

Since the curvature is constant $-1$ in the cusp, $\Box \tilde{K}_N$ vanishes for $\gamma, \gamma'$ bigger than $y_0 e^{[t]}$. Hence, this is bounded by

$$O(1) e^{[t]d} \int_D \sum_{|\alpha| \leq 2d+3} \int_{\tilde{M}} |\partial_{x,x'}^\alpha \tilde{K}_N(t, x, x')| dx'.$$
Now, $D$ has finite volume, and we can use formula (26). The desired quantity is thus up to some constant less than
\[
e^{-O(|t|)} \sum_{\alpha \leq 2d+3} \int_{r \leq t} \left| \partial_{x,x'}^{\alpha} \left\{ (-\Delta + (k - d/2)^2 - d^2/4)u_N \right. \right. \\
\left. \left. \left( \cosh t - \cosh r \right)^{N-(d+2)/2} \right\} dx' \right|.
\]
Since the curvature of the manifold is bounded by below, we have uniform exponential estimates of the growth of the volume of balls in $\tilde{M}$. Using lemma 2.3, we deduce that
\[
\text{Tr} V_N(t) = O(1) e^{-O(|t|)}.
\]
For the continuity of the remainder, it suffices to observe that the same trace estimates hold for $\partial_t R_N$.

**Lemma 2.6.** We start by recalling the estimate 9.4 from Dimassi-Sjöstrand [DS99]. According to this, if $L^1$ is the kernel of an operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, the trace norm of the operator is bounded by
\[
\sum_{|\alpha| \leq 2n+1} \| \partial_{x,x'}^{\alpha} L_1 \|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}.
\]
We can use this to prove that if $L$ is supported only on $M_0$, the lemma holds. Hence, we only have to consider operators that map functions on the cusp to functions on $M_0$, or functions on the cusp to functions on the cusp. We will only do the former, as the two cases are very similar.

Up to taking some charts, we can assume $L$ is an operator from $L^2(Z)$ to $L^2(U)$ where $U$ is an open set in $\mathbb{R}^{d+1}$. We take $\chi \in C^\infty_c([1,3])$, so that
\[
\sum_{i \geq 1} \chi(2^{-n}x) = 1, \text{ for } x > 2.
\]
Then we consider the operators $L\chi_n$ with kernel $L_n := L(x, x')\chi(2^{-n}y')$. We want to estimate the trace norm of each of these pieces. Each of them is an operator from $L^2(\{2^n \leq y \leq 32^n\})$ to $L^2(U)$. We define $T_n f(y, \theta) = 2^{-n/2} f(2^{-n}y, \theta)$. This defines a unitary operator from $L^2(\{2^n \leq y \leq 32^n\})$ to $L^2(\{1 \leq y \leq 3\})$. The kernel of $L_n T_n^{-1}$ is
\[
L_n'(x, x') = 2^{n/2} L_n(x; 2^n y', \theta').
\]
Then, we have
\[
\|L_n\|_{\text{Tr}} = \|L_n T_n^{-1}\|_{\text{Tr}} \leq 2^{n/2} \sum_{|\alpha| \leq 2d+3} \| \partial_{x,x'}^{\alpha} L_n' \|_{L^1(U \times \{1 \leq y \leq 3\})}.
\]
Summing over $n$ yield exactly the desired estimate.

With lemma 2.5 in mind, we rewrite the RHS in (21) as
\[
\left\{ \lim_{\tau \to \infty} \int_{y \leq e^\tau} \int \psi(t) K_N(t, x) dx dt - \kappa \tau \psi(0) \right\} + \int \psi(t) \text{Tr} R_N(t) dt.
\]
Let $D_\tau \subset D$ be the part of $D \subset \tilde{M}$ that projects to $\{y \leq e^\tau\}$ in $M$. The expression in brackets in the equation above is

\[
\lim_{\tau \to \infty} \left\{ \int_{D_\tau} \sum_{\gamma \in \Gamma} \int \psi(t)K_N(t,x,\gamma x)dt\,dx \right\} - \kappa \tau \psi(0).
\]

2.3. Estimation of contributions to the trace. The main term in the last expression corresponds to $\gamma = 1$, the diagonal term. To estimate the other terms, we recall some facts on the action of $\Gamma$. First,

\[
R_0 := \inf \left\{ d(x,\gamma x) \mid x \in \tilde{M}, \gamma \neq 1, \pi(x,\gamma x) \not\subset Z_i, i = 1 \ldots \kappa \right\} > 0.
\]

By $\pi(x,\gamma x)$ we refer to the projection of the geodesic between $x$ and $\gamma x$ in $M$. By $\{\pi(x,\gamma x) \not\subset Z_i, i = 1 \ldots \kappa\}$ we mean that this geodesic of $M$ does not remain in any one cusp.

Next, if $d(x,\gamma x) < R_0$ with $\gamma \neq 1$, then $x$ has to be in an open set $\tilde{U}_0$ of $\tilde{M}$ that is isometric to $U_{y_0} := \{x = (y,\theta) \mid y > y_0\}$ in the half-space model for the hyperbolic space, with the half-space hyperbolic metric $ds^2 = y^{-2}dx^2$. Then the restriction of the action of $\gamma$ to $U_{y_0}$ is a translation in the horizontal direction.

The set of $\gamma$ that have such behaviour when restricted to $\tilde{U}_0$ is the set of $\gamma$ that preserves $\tilde{U}_0$, and it is a subgroup isomorphic to $\Lambda_i$. The $i$ index refers to the index of the cusp onto which $x$ is projected under $\tilde{M} \to M$. (Recall that $\Lambda_i$ is the fundamental group of the cusp $Z_i$). From now on, we will see $\Lambda_i$ as a lattice in $\mathbb{R}^d$.

We let $\theta$ be the horizontal coordinate ($\theta \in \mathbb{R}^d$). For future reference, we know that

\[
d((y,0),(y,\theta)) = 2 \text{arcsinh} \frac{|\theta|}{2y}.
\]

Last, using the convergence of Poincaré sums, it is classical to observe that for given $x \in \tilde{M}$, the number of $\gamma$ such that $d(x,\gamma x) \leq t$ and such that $\pi(x,\gamma x)$ does not remain in a cusp, is bounded by $Ce^{O(|t|)}$, where the constant $C > 0$ does not depend on $x$.

Also recall that

$$\psi(t) = \frac{\sin Tt}{\pi t} \rho(At),$$

where $A \asymp (1/\log T)$; $\rho$ is smoothly compactly supported in $]-1,1[$, is even, and flat equal to 1 around 0. With these assumptions, the last term in (28) can be bounded:

$$\int \psi(t) \operatorname{Tr} R_N(t)dt = e^{O(1/A)}.$$

2.3.1. The diagonal term. We prove

Lemma 2.7. Assume that $\psi$ takes the form above. Then,

$$\lim_{\tau \to \infty} \int_{D_\tau} \int \psi(t)K_N(t,x,x)dx\,dt = P(T) + O(1)e^{O(1/A)},$$

where $P(T) = c_0 T^{d+1} + \cdots + c_k T^{d+1-2k} + \ldots$. 
We find a new expression

Proof. We consider the big integral in (32), and first make the change of variables $C$

Lemma 2.8.

The functions $x \mapsto u_k(x,x)$ are bounded, hence integrable. Let us separate the time integral into two parts, using $1 = \rho(t) + 1 - \rho(t)$. When $k > d/2$, the part of the integral supported away from 0 is $O(1)$ as $T \to +\infty$. When $k > d/2$, it can be bounded above by

$$\int_0^{1/A} e^{t(k-d/2)} dt = O(1)e^{O(1/A)}.$$ 

We are left with the following integrals

$$\int \frac{\sin tT}{\pi t} \rho(t) \sinh |t|M_{k-(d+2)/2} \cosh t - 1) dt.$$ 

When $t \to 0$, $\cosh t - 1 \sim t^2/2$, so this is the Fourier transform of $M_{k-d/2-1}(t^2) \times W(t) \times \epsilon(t)$ where $W \in C_c^\infty(\mathbb{R})$ is even and $\epsilon(t)$ is the sign of $t$. Hence it has an expansion in powers of $T$ as $T \to \infty$, the main power being $T^{d+1-2k}$, and all powers having the same parity.

2.3.2. The contribution of the cusps. From the discussion on points $x$ such that $d(x, \gamma x) < R_0$, we know that each cusp $Z_i$ contributes a term to the trace

(32)

$$\lim_{\tau \to \infty} -\psi(0)\tau + \int_{y_0}^{e^\tau} \frac{dy}{y^{d+1}} C_0 \sum_{\gamma \in \Lambda_i, \gamma \neq 0} \int \psi(t) \sinh |t|M_{-d/2-1} \left[ \cosh t - 2 \left( \frac{|\gamma|}{2y} \right)^2 - 1 \right] dt.$$ 

Lemma 2.8. If one replaces $y_0$ by 0 in equation (32), the result is

$$-\frac{T}{\pi} \log T + \frac{C_1(\Lambda_i)}{\pi} T + O(1).$$ 

Here, the constant $C_1(\Lambda_i)$ only depends on the lattice $\Lambda_i$:

(33)

$$C_1(\Lambda_i) = 1 + C(d) + \gamma(\Lambda_i) - \gamma.$$ 

The constants are defined in the proof, see equations (34), (36) and (37). When $d = 1$, $C_1(\mathbb{Z}) = 1 - \log 2$. When $d > 1$, this contribution will be smaller than the $O(T^d)$ remainder.

Proof. We consider the big integral in (32), and first make the change of variables $u = |\gamma|/y$. We find a new expression

$$C_0 \sum_{\gamma \in \Lambda_i, \gamma \neq 0} \frac{1}{|\gamma|^2} \int_{|\gamma|=r}^{\infty} u^{d-1} du \int \psi(t) \sinh |t|M_{-d/2-1} \left[ \cosh t - 1 - \frac{u^2}{2} \right] dt.$$
Let $\tilde{\psi}(v) = \psi(t(v))$, where $\cosh t - 1 = v$. We can rearrange the above expression in the following way (beware of the integration by parts):

$$-2C_0 \int M_{-d/2}(v) \int u^{d-1} \tilde{\psi}' \left( v + \frac{u^2}{2} \right) \sum_{\gamma \neq 0, \ |\gamma| \leq ue\tau} \frac{1}{|\gamma|^d} du dv.$$ 

Using simple arguments of comparison between series and integrals, one finds that for fixed $u \neq 0$ and $\tau \to +\infty$,

$$\sum_{\gamma \neq 0, \ |\gamma| \leq ue\tau} \frac{1}{|\gamma|^d} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \left\{ \log(ue\tau) + \gamma(\Lambda_i) + o(1) \right\},$$

where $\gamma(\Lambda_i)$ is a constant a priori only depending on $\Lambda_i$ (when $d = 1$, it is just the Euler-Mascheroni constant). When $\tau \to \infty$, we hence find an expression of the form $a\tau + b$. The expression of $a$:

$$-2C_0 \frac{2\pi^{d/2}}{\Gamma(d/2)} \int M_{-d/2}(v) \int u^{d-1} \tilde{\psi}' \left( v + \frac{u^2}{2} \right) \mu du dv.$$ 

Now, let $w = u^2/2$. The last expression becomes

$$-\int M_{-d/2}(v) \int M_{d/2-1}(w) \tilde{\psi}' (v + w) dw dv.$$ 

Now, we make a distinction. If $d$ is even, integrating by parts, since $M_{-1} = \delta$, this reduces to $\tilde{\psi}(0) = \psi(0)$. When $d$ is odd, the result is the same. Indeed, integrating by parts, we find

$$\int_{v>0, w>0} \frac{1}{\sqrt{vw}} \tilde{\psi}' (v + w) dv dw = -\frac{1}{\pi} \int_0^{+\infty} \tilde{\psi}' (z) \int_0^{z} \frac{dv}{\sqrt{v(z-v)}} = \tilde{\psi}(0).$$

As could be expected, it is the contribution from the cusps that creates the divergence as $\tau \to +\infty$. Now, remember we are looking for the result of the renormalization, that is, the constant $b$. Its expression is

$$-2C_0 \frac{2\pi^{d/2}}{\Gamma(d/2)} \int M_{-d/2}(v) \int u^{d-1} \tilde{\psi}' \left( v + \frac{u^2}{2} \right) [\log u + \gamma(\Lambda_i)] du dv.$$ 

Changing again the parameter with $w = u^2/2$, this is found equal to

$$-\int M_{-d/2}(v) \int M_{d/2-1}(w) \tilde{\psi}' (v + w) \left[ \frac{1}{2} \log w + \frac{1}{2} \log 2 + \gamma(\Lambda_i) \right] dw dv.$$ 

The constant term contributes in the final expression by

$$\frac{T}{\pi} \left[ \frac{1}{2} \log 2 + \gamma(\Lambda_i) \right].$$

On the other hand, in the contribution from the log term, change variables again with $v + w = V$, and $w = Vx$ with $x \in [0, 1]$, and $V \in \mathbb{R}^+$. It becomes

$$-\frac{1}{2} \int \tilde{\psi}'(V) \int M_{-d/2}(1-x) M_{d/2-1}(x) \log(Vx) dx dV.$$
This integral is well defined as we are taking the product of distributions that are not singular at the same points, and the result is a compactly supported distribution (in $x$). From here, the next step is to compute the integrals
\[
\int M_{-d/2} (1 - x) M_{d/2-1}(x) dx \quad \text{and} \quad \int M_{-d/2} (1 - x) M_{d/2-1}(x) \log(x) dx.
\]
After $\lfloor d/2 \rfloor$ integration by parts, we find that the first one was already computed, and is equal to 1. Hence, we have a final term
\[
\frac{1}{2} \int_0^{+\infty} \tilde{\psi}'(V) \log V dV.
\]
The other $x$ integral becomes, when $d$ is even,
\[
\frac{1}{d/2 - 1} + \cdots + \frac{1}{2} + 1 = 2 \sum_{k=1}^{d/2-1} \frac{1}{d - 2k}
\]
On the other hand, when $d$ is odd, it is
\[
2 \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{d - 2k} + \frac{1}{\pi} \int \frac{\log x dx}{\sqrt{x(1-x)}} = 2 \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{d - 2k} - 2 \log 2.
\]
Call this constant $2\mathcal{C}(d)$. The final contribution will be
\[
\mathcal{C}(d) \tilde{\psi}(0) = \frac{\mathcal{C}(d) T}{\pi}.
\]
Before we conclude, we have to estimate the contribution from (35), which contains the $T \log T$ term. We come back to the $t$ variable, with $\cosh t - 1 = V$.
\[
-\frac{1}{2\pi} \int_0^{+\infty} \frac{d}{dt} \left\{ \sin T t \frac{\rho(At)}{t} \coth \frac{t}{T} \right\} \log \left[ 2 \sinh^2 \frac{t}{2} \right] dt
\]
This gives a $T \log 2/(2\pi)$ term, and
\[
-\frac{1}{\pi} \int_0^{+\infty} \frac{d}{dt} \left\{ \sin T t \frac{\rho(At)}{t} \coth \frac{t}{T} \right\} \log \left[ \sinh \frac{t}{2} \right] dt
\]
We can insert in the differentiated expression $1 = \rho(t) + (1 - \rho(t))$. The second term is a $o(1/T)$, as we recognize the Fourier transform of a smooth, $L^2$ function, whose derivative is in $L^1$. We are left with
\[
-\frac{1}{\pi} \int_0^{+\infty} \frac{d}{dt} \left\{ \sin T t \frac{\rho(t)}{t} \coth \frac{t}{T} \right\} \log \left[ \sinh \frac{t}{2} \right] dt
\]
Now, we change variables $u = tT$, integrate by part and find
\[
\frac{1}{\pi} \lim_{\epsilon \to 0^+} \left\{ \int_{\epsilon}^{+\infty} \frac{1}{2} \sin \frac{u}{2} \rho(u/T) \coth \frac{u}{2T} du + T \frac{\sin \epsilon}{\epsilon} \log \left[ \sinh \frac{\epsilon}{2T} \right] \right\}
\]
We recover the main term $-T \log(2T)/\pi$, and
\[
\frac{1}{\pi} \lim_{\epsilon \to 0^+} \left\{ \int_{\epsilon}^{+\infty} \frac{1}{2} \sin \frac{u}{2} \rho(u/T) \coth \frac{u}{2T} du + T \log(\epsilon) \right\}
But, as $T \to \infty$, 
\[
\frac{1}{2} \rho(u/T) \coth \frac{u}{2T} = \frac{T}{u} + (1 - \rho(u/T)) \frac{T}{u} + \mathcal{O}(1) \rho(u/T) \frac{u}{T}.
\]
Both the $(1 - \rho)$ and the $u/T$ term will only contribute by $\mathcal{O}(1)$. So the last integral we have to compute is 
\[
\int_{\epsilon}^{+\infty} \sin u \frac{1}{u^2} du + \log(\epsilon)
\]
To compute this last constant, one may use Cauchy’s theorem, shifting the contour of integration to $i\mathbb{R}^+$; this gives $1 - \gamma$. Then, the value of the constant $b$ is found to be 
\[
-\frac{T}{\pi} \log T + \frac{T}{\pi} \left[ \frac{\log \mathcal{F}}{2} + \gamma(\Lambda) + \mathcal{C}(d) + \frac{\log \mathcal{F}}{2} - \log 2 + 1 - \gamma \right] + \mathcal{O}(1).
\]
In particular, for $d = 1$, we do get to $1 - \log 2$ (which is the value computed by Selberg).

2.3.3. The other terms. There are two contributions left to compute. The first one is
\[
(38) \quad C_0 \int_0^{y_0} \frac{dy}{y^\alpha + 1} \sum_{\gamma \in \Lambda^+} \int \psi(t) \sinh |t| M_{-d/2-1} \left[ \cosh t - 2 \left( \frac{|\gamma|}{2y} \right)^2 - 1 \right] dt.
\]
The other one is the sum over $k = 1, \ldots, N$ of (up to some constants)
\[
(39) \quad \int_D \sum_{\gamma \neq 1, d(x, \gamma x) > R_0} u_k(x, \gamma x) \int \psi(t) \sinh |t| M_{k-d/2-1} (\cosh t - \cosh d(x, \gamma x)) dt dx.
\]
These are remainder terms, as we will see. The first step is the following: assume $R > R_0$, then we consider
\[
I := \int \psi(t) \sinh |t| M_{k-d/2-1} (\cosh t - \cosh R) dt.
\]
We insert the cutoff $1 = \rho(t - R) + 1 - \rho(t - R)$. The part $1 - \rho(t - R)$ only contributes $\mathcal{O}(T^{-\infty}) e^{\mathcal{O}(R)}$ (it suffices to integrate by parts, and recall that the only $R$’s that contribute are $\mathcal{O}(1/A)$). We are left with the $\rho(t - R)$ part. In the interval where this integral is supported, we can write
\[
\cosh t - \cosh R = (t - R) \sinh RW(t - R, R),
\]
with $W(t - R, R)$ uniformly bounded (and $W(0, R) = 1$). Hence
\[
M_{k-d/2-1} (\cosh t - \cosh R) = \sinh(R)^{k-d/2-1} \widetilde{W}(t - R, R) M_{k-d/2-1}(t - R).
\]
As a consequence, $I$ is the imaginary part of
\[
\frac{\sinh(R)^{k-d/2}}{R} \int e^{iT(R+u)} \rho(A(R + u)) \rho(u) \frac{\sinh(R + u)}{\sinh(R)(1 + u/R)} \widetilde{W}(u, R) M_{k-d/2-1}(u) du.
\]
We deduce that
\[
I = \mathcal{O}(T^{d/2-k}) e^{\mathcal{O}(R)} 1(R \leq 1/A).
\]
(Actually, $I$ has an expansion in powers of $T^{-1}$).
From this bound, we deduce that the quantities in (38) and (39) contribute to the trace by $O(T^{d/2}) e^{O(1/A)}$. Indeed, first for (38), that contribution is bounded by $$O(T^{d/2}) \int_0^{y_0} \frac{dy}{y^{d+1}} \sum_{\gamma \in \mathbb{Z}^d} e^{O(|\gamma|/y)} 1(A|\gamma| \leq y) = O(T^{d/2}) e^{O(1/A)} \sum_{\gamma \in \mathbb{Z}^d} \int_0^{y_0} \frac{dy}{y^{d+1}}.$$ It suffices to see that $$\sum_{\gamma \in \mathbb{Z}^d} \int_0^{y_0} \frac{dy}{y^{d+1}} = O(A^{-d}) \sum_{0 < |\gamma| \leq y_0/A} |\gamma|^{-d} = O(A^{-d} \log A).$$

Now, for (39), we use the fact that the number on non vanishing terms in the sum is $O(1)e^{O(1/A)}$, and lemma 2.3 directly.

2.4. The conclusion. We can now complete the proof.

2.4.1. When the curvature is negative. When the curvature is negative, we gather all the different pieces above:

$$\sum_{\lambda} \hat{\psi} \left( \sqrt{\lambda - \frac{d^2}{4}} \right) - \frac{1}{2} \int S' \hat{\psi} + \frac{1}{4} \hat{\psi}(0) \text{Tr} \left( \frac{d}{2} \right) = \lim_{\tau \to \infty} \int_{y \leq e^\tau} \int \psi(t)K(t,x,x)dxdt - \kappa \tau \psi(0),$$

$$= \lim_{\tau \to \infty} \int_{y \leq e^\tau} \int \psi(t)K_N(t,x,x)dxdt - \kappa \tau \psi(0) + O(1)e^{O(1/A)},$$

$$= (c_0 T^{d+1} + \ldots + c_k T^{d+1-2k} + \ldots) - \frac{\kappa T}{\pi} \log T + \frac{\kappa(1 - \log 2)T}{\pi} + O(T^{d/2})e^{O(1/A)}.$$

When taking $A$ to be a sufficiently big multiple of $1/\log T$, we can get $e^{O(1/A)} = O(T^\epsilon)$ for any fixed $\epsilon > 0$. Recall $\tilde{N}(T)$ is the counting function defined in (5):

$$\tilde{N}(T) = \sum_{\lambda} \frac{1}{\lambda} \left[ \sqrt{\lambda - \frac{d^2}{4}} \leq T \right] - \frac{1}{2} \int_{-T}^T S'.$$

From the definition of $\psi$, we deduce that the quantity in the LHS of (40) is, up to $O(1)$,

$$\int \tilde{N}(T + u) \frac{1}{A} \tilde{\rho} \left( \frac{u}{A} \right) du.$$ 

From equation (57) (in the second part of the article), we deduce that $\tilde{N} = P + f$ where $P$ is a $O(T^{d+1})$ polynomial and $f$ is an increasing function. We can choose $\tilde{\rho}$ to be non-negative, so we have for $x > A/2$,

$$\int [f(T + x + u) - f(T + u)] \frac{1}{A} \tilde{\rho} \left( \frac{u}{A} \right) du \geq \int_{-1/2}^0 [f(T + x + Au) - f(T + Au)] \tilde{\rho}(u) du \geq c(f(T + x - A/2) - f(T)).$$

by repeating the argument for $f(T) - f(T - x)$ ($x < -A/2$) we deduce that for all $x \in \mathbb{R}$, $|f(T + x) - f(T)| = O(T^{d}/\log T + xT^{d} + x^{d+1}).$
Now, the same is true for $\tilde{N}(T)$, and we find

$$\tilde{N}(T) - \int \tilde{N}(T+Au)\hat{\rho}(u)du = O(1) \int \hat{\rho}(u)[AT^d u + A^{d+1} u^{d+1} + AT^d]du = O(T^d/\log T).$$

This ends the proof of the third estimate in theorem 2. Indeed, to identify the constant $c_0$ in the RHS of (40), it suffices to check the equivalent given by Müller in [Mül86]. That is,

$$c_0 = \frac{\text{vol } M}{(4\pi)^{(d+1)/2}\Gamma(d/2 + 3/2)} = \frac{\text{vol } B^* M}{(2\pi)^{d+1}}.$$  

2.4.2. Without assumptions on the curvature. Now, we do not assume anymore that the curvature is negative, or that there are no conjugate points. First, we give the usual Hörmander bound on the remainder in all generality, and then, we turn to some Duistermaat-Guillemin type results.

A Hörmander type remainder The compact part of the manifold always has a positive injectivity radius $r > 0$. Hence, we can still build a Hadamard parametrix for times $|t| < r$. One can check that all the arguments above apply, albeit replacing $A = c/\log T$ by a fixed $A > 0$ sufficiently large. From that, one deduces that with no assumptions on the curvature,

$$\int \tilde{N}(T+u)\hat{\rho}(u)du = cT^{d+1} - \frac{\kappa T}{\pi} \log T + O(T^d).$$

Using the same argument as above, one finds $\tilde{N}(T) = cT^{d+1} + \frac{T}{\pi} \log T + O(T^d)$, which proves part 1 of theorem 2.

A Duistermaat-Guillemin approach. Let us follow the argument in the classical article [DG75], itself inspired by the earlier work of Chazarain [Cha74] and Hörmander [Hör68] (among others). The point of the article is to observe that for a compact manifold $X$, $\exp\{it\sqrt{-\Delta+c}\}$ is a Fourier Integral Operator (when $c$ is some positive constant). The canonical relation associated to the operator is

$$C = \{(t,\tau),(x,\xi),(y,\eta) \in T^*\mathbb{R} \setminus \{0\} \times (T^*X \setminus \{0\})^2 \mid (y,\eta) = \varphi_t(x,\xi), \tau + |\xi| = 0\}.$$  

From there, they deduce that the singularities of $\text{Tr } e^{it\sqrt{-\Delta+c}} \in S'(\mathbb{R})$ are the lengths of periodic geodesics of the geodesic flow.

Following [DG75], let us denote $Q = \sqrt{-\Delta - d^2/4}$. The arguments to prove that $U(t) := e^{itQ}$ is a FIO, are based on the fact that $Q$ is pseudo-differential, and local arguments. According to [Bon14a], with respect to the appropriate Pseudo-differential calculus, $Q$ is also pseudo-differential in the case of a cusp manifold. This enables one to use the local considerations in [DG75] to show that, given $y_*, 0 < T > 0$, the behaviour of $U(t)$ on $\{|t| \leq T, y(x) \leq y_*, d(x,x') \leq |t|\}$ (which is a compact set of $\mathbb{R} \times M \times M$) is that of a FIO, with the same relation. See for example [Zel92], for similar considerations.

To prove the results we seek, we are only interested in $\cos tQ$, which has finite speed of propagation. This ensures that, again, the contributions to the 0-trace of the cusps will be the same as computed in section 23.2. That is, the cusps merely modify the singularity at zero of the Fourier transform of $\tilde{N}'$, but do not create new singularities.
This proves in particular that the singularities of the Fourier transform of $\tilde{N}'$ are indeed the length of closed geodesics (theorem 1).

It also shows that if the measure of the set of periodic geodesics with length $\ell$ is zero, then the contribution to $\tilde{N}(T)$ of the singularity at $\ell$ will be $o(T^d)$ (just as in theorem 3.5 in [DG75]). Hence, if we smooth out $\tilde{N}(T)$ by convolution with $\hat{\rho}_A$ with arbitrary large fixed $A$, we obtain a $o(T^d)$ remainder. This is enough to obtain part 2 of theorem 2.

3. Counting resonances in negative curvature

Now, we turn to the second part of the main theorem. In this whole section, we assume that the curvature of $g$ is negative in $M$. According to the relation (17), we deduce that counting the poles of $\varphi(s)$ in $\{\Re s < d/2\}$ is the same as counting the zeroes of $\varphi$ in $\{\Re s > d/2\}$. So, from now on, we will work in $\{\Re s > d/2\}$, and will be interested in the zeroes of $\varphi$.

From [Bon15], we recall that there constants $\delta_g > d/2$ and $T^\#_0$, $T^\#_1$, ..., that converge absolutely for $\Re s > \delta_g$ such that when $\Re s > \delta_g$

$$\varphi(s) = s^{-\kappa d/2} \left( L_0(s) + \frac{1}{s} L_1(s) + \cdots + O \left( \frac{e^{-s T^\#_0}}{s^N} \right) \right).$$

We can denote the coefficients of the $L_i$’s in the following way:

$$L_i(s) = \sum_k a_k^i e^{-s\ell_k}, \text{ with } \ell_0 < \ell_1 < \ldots, \text{ for } \Re s > \delta_g.$$

It is possible that $a_0^0$ vanishes, so we let $n^*$ be the smallest integer $n$ such that $a_n^0 \neq 0$. Then we let $\ell_* = \ell_{n^*}$ and $a_0^{n^*} = a_{n^*}^0$. I gave a proof that the set $\mathcal{G}(M)$ of negatively curved metrics $g$ such that $L_0$ is not identically zero (i.e, $n^* < \infty$), is open in $C^2$ topology, contains the constant curvature metrics, and its complement has infinite codimension in $C^\infty$ topology.

Assuming $g \in \mathcal{G}(M)$, there is a $b > \delta$ such that $L_0$ and $\varphi$ do not vanish on the line $\{\Re s = b\}$ (this comes from (17), as explained in [Bon15]).

Let us introduce $\mathcal{D}_b$ the set of Dirichlet series whose absolute abscissa of convergence is strictly smaller than $b$, and are bounded for $\Re s > b$. That, is series of the form

$$\sum_{k \geq 0} \frac{a_k}{\lambda_k^s} \text{ where } \lambda_k \in \mathbb{R}, \lambda_0 < \lambda_1 < \ldots,$$

that converge absolutely for $\Re s \geq b$. Also consider $\mathcal{D}_b^0$ those in $\mathcal{D}_b$ that tend to zero as $\Re s \to \infty$ (i.e $\lambda_0 > 1$).

From the assumption on $\varphi$ and $L_0$, if $b$ was taken large enough, there exist $\tilde{L}_0, \tilde{L}_1$ in $\mathcal{D}_b^0$, such that for $\Re s = b$

$$(P1) \quad \Re \frac{\varphi'}{\varphi} = -\ell_* + \Re \tilde{L}_0 + \mathcal{O} \left( \frac{1}{s} \right),$$

and

$$(P2) \quad \log |\varphi(s)| = -\frac{\kappa}{2} \log |s| - b\ell_* + \log |a_0^0| + \Re \tilde{L}_1 + \mathcal{O} \left( \frac{1}{s} \right).$$
These are the properties that we will actually use in the proof.

3.1. Preliminaries. Before we turn to the actual proof, we recall some useful facts. First, we review lemmas of harmonic analysis useful when counting the zeroes of holomorphic functions. Then, we recall the Maass-Selberg formula, and give a full factorization for the scattering determinant.

3.1.1. Some lemmas from harmonic analysis. Let us start with some abstract lemmas on zeros of holomorphic functions. Take a function holomorphic in a neighbourhood of a half plane \( \{ \Re z \geq a \} \). All sums are over the zeros of \( F \), denoted by \( z = \beta + i\gamma \) — following Selberg’s notations. When a zero is sitting on the boundary of the counting box, it is counted with half multiplicity.

Lemma 3.1 (Carleman’). Assume \( b > a \) and \( T > 0 \), and assume that \( F \) does not vanish on \( \{ \Re z = b \} \). Then

\[
2\pi \sum_{\beta > b, |z - b| < T} \log \frac{T}{|z - b|} = \int_{-\pi/2}^{\pi/2} \log |F(b + Te^{i\theta})| d\theta - \pi \log |F(b)|
\]

\[
+ \int_{-T}^{T} \log \frac{T}{|t|} \Re \frac{F'(b + it)}{F(b + it)} dt.
\]

(47)

Now, additionally assume that \( a = d/2 \), that \( |F| = 1 \) on the axis \( \{ \Re s = d/2 \} \), and that \( F \) is real on the real axis.

Lemma 3.2 (Counting in big rectangles). For \( T > 0 \),

\[
2\pi \sum_{d/2 \leq \beta \leq b, 0 \leq \gamma \leq T} (T - \gamma)(\beta - d/2) = \int_{0}^{T} \Re \frac{F'(b + it)}{F(b + it)}(b - d/2)(T - t) dt
\]

\[
+ \int_{d/2}^{b} \log \frac{|F(x + iT)|}{|F(x)|} (x - d/2) dx
\]

\[
- \int_{0}^{T} \log |F(b + it)|(T - t) dt.
\]

(48)

Lemma 3.3 (Counting in small rectangles). Take \( c > 0 \). For \( T > 0 \),

\[
2\pi \sum_{-\pi/c \leq \gamma - T \leq \pi/c, d/2 \leq \beta \leq b} \cos(c(\gamma - T)) \sinh(c(\beta - d/2)) =
\]

\[
\int_{-\pi/c}^{\pi/c} \sinh(c(b - d/2)) \cos(cT) \frac{F'}{F}(b + iT + it) dt
\]

\[
- c \int_{-\pi/c}^{\pi/c} \cosh(c(b - d/2)) \cos(cT) \log |F(b + iT + it)| dt
\]

\[
+ c \int_{d/2}^{b} \log( |F(x + iT + i\pi/c)|, |F(x + iT - i\pi/c)| \sinh(c(x - d/2)) dx.
\]

(49)
Proof. These three counting lemmas are obtained by considering the fact that $\log |F|$ is a harmonic function where $F$ does not vanish. Hence, if $u$ is another harmonic function on some open set $\Omega$, such that $F$ does not vanish on $\partial \Omega$, by Stoke's theorem,

$$2\pi \sum_{z \in \Omega, F(z) = 0} u(z) = \int_{\partial \Omega} u \partial_{\nu} \log |F| - \log |F| \partial_{\nu} u.$$ 

If $F$ vanishes on the boundary of $\Omega$, by removing small half disks around those zeros, one find that they are counted with multiplicity 1/2, in a similar formula.

For 3.1, we consider $u(z) = -\log |z - b|/T$, and integrate on the boundary of the half-disk. For (48), we take $u(z) = (T - \Re z)(\Re z - 1/2)$, and finally $u(z) = \cos(c(\Im s - T)) \sinh(c(\Re z - 1/2))$ for (49).

The estimates on counting in boxes are similar to equations (1.1) in [Sel89b], and lemma 14, p. 319 in [Sel89a]. The Carleman's lemma is reminiscent of the usual Carleman theorem [Tit58 §3.7]. Last of this section is Lemma 3.4.

**Lemma 3.4.** Let $L \in \mathcal{D}_b^0$ be real on $\mathbb{R}$. Then, as $T \to \infty$,

$$\int_0^T \Re L(b + it) dt = \mathcal{O}(1).$$

**Proof.** Since $L$ converges absolutely in the region we are considering, we can write

$$L(b + it) = \frac{c_0}{\lambda_0^T} + \frac{c_1}{\lambda_1^T} + \cdots + \frac{c_k}{\lambda_k^T} + \cdots$$

where the $c_k$’s are real, the $\lambda_k$’s are real, ordered, and strictly greater than 1, and the sum converges normally. So we can estimate

$$\int_0^T \Re L = \sum_k c_k \frac{\sin(T \log \lambda_k)}{\log \lambda_k}.$$ 

Since all $\lambda_k$’s are bigger than $\lambda_0 > 1$, and since $\sum |c_k| < \infty$, we conclude. □

3.1.2. The Maass-Selberg relations.

**Lemma 3.5** (Maass Selberg). Take $y > 0$ big enough. The scattering determinant satisfies

$$|\varphi(\sigma + it)| \leq y^{\kappa(2\sigma - d)} \left( \sqrt{1 + \frac{(\sigma - d/2)^2}{t^2}} + \frac{\sigma - d/2}{|t|} \right)^\kappa$$

when $\sigma > d/2$.

In particular, $|\varphi(s)|$ is bounded in the vertical strip $\{d/2 < \Re s < b\}$, away from eventual poles in $[d/2, d]$.

**Proof.** To prove this, we use the Eisenstein series. For each cusps $Z_i$, recall $E_i(s)$ is a meromorphic family of smooth functions on $M$ that satisfy

$$-\Delta E_i(s) = s(d - s)E_i(s).$$

(52)
Additionally, the zeroth Fourier coefficient of $E_i(s)$ in cusp $Z_j$ equals

$$f_{ij}(y, s) = \delta_{ij}y^s + \phi_{ij}(y)y^{d-s}. \quad (53)$$

We denote by $W(s, y)$ the matrix whose coefficients are the $f_{ij}$. Let $\Pi^*_y$ be the projection on functions whose zero Fourier mode vanishes for $\{y > y_0\}$ in all cusps. Then we define $G_{ij}^y(s) = \Pi^*_y E_i(s)$. We set to prove the Maass-Selberg formula. Actually, the proof of constant curvature works out identically. We differentiate (52) with respect to $s$, and we use Stoke’s formula again to obtain

$$(d - 2s) \int G_i^y(s)G_j^y(s) + 2i\Im(s(d - s)) \int \partial_s G_i^y(s)W^y(s) = \sum_k [\partial_s f_{ik}\partial_y f_{jk} - f_{jk}\partial_y \partial_s f_{ik}] .$$

The sum in the RHS is the $(ij)$ coefficient of the matrix

$$\partial_s W\partial_y W^* - \partial_s \partial_y W.W^* = \partial_s (W.\partial_y W^* - \partial_y W.W^*).$$

In the LHS, it is $\partial_s [2i\Im(s(d - s)) \int G_i^yG_j^y]$. We deduce that there is an anti-meromorphic function $A(s)$ such that

$$2i\Im(s(d - s)) \int G_i^yG_j^y = A(s) + W.\partial_y W^* - \partial_y W.W^*.$$ 

Let $V(s)$ be the matrix with coefficients $\int G_i^yG_j^y$. Elementary computations give

$$2i(d - 2\Re s)\Im V(s) = A(s) + (2\Re s - d)(y^{-2\Im s}\phi - y^{-2\Im s}\phi^*) + 2i\Im(y^{d-2\Re s}\phi^* - y^{2\Re s - d}).$$

We deduce that $A(s)$ vanishes on the unitary axis $2\Re s = d$, and thus has to vanish identically. (Maass-Selberg Relation)

$$V(s) = \frac{y^{2\Im s}\phi^* - y^{-2\Im s}\phi}{2i\Im s} + \frac{y^{2\Re s - d} - y^{d-2\Re s}\phi^*}{2\Re s - d} \quad \text{for } \Re s > d/2.$$

The matrix on the LHS is non-negative, so that as a hermitian matrix,

$$\phi\phi^* \leq y^{2(2\Re s-d)} + \frac{2\Re s - d}{\Im s} \frac{y^{2s}\phi^* - y^{2\Re s - d}}{2i\Im s} \quad \text{for } \Re s > d/2.$$

We deduce that

$$\phi\phi^* \leq y^{2(2\Re s-d)} \left( \sqrt{1 + \left( \frac{\Re s - d/2}{\Im s} \right)^2} + \frac{\Re s - d/2}{|\Im s|} \right)^2.$$

This formula is true as long as we are still in the cusp, with constant curvature. That is, we cannot take $y$ arbitrarily small. \hfill \Box

Observe that taking the limit $\Re s \to d/2$, we find

$$V \left( \frac{d}{2} + it \right) = 2\log y \left( \frac{d}{2} + it \phi \right) - \frac{1}{2} \left( \phi^* + \phi \phi^* \right).$$

Since $\phi$ is unitary on the unitary axis, $\phi^* \phi$ is self-adjoint, and we recover the classical

$$\int G_i^yG_j^y \left( \frac{d}{2} + it \right) = 2\kappa \log y - \frac{\phi^*}{\phi} \left( \frac{d}{2} + it \right) + \text{Tr} \frac{y^{2it}\phi^* - y^{-2it}\phi}{2it}.$$ 

$$\int G_i^yG_j^y \left( \frac{d}{2} + it \right) = 2\kappa \log y - \frac{\phi^*}{\phi} \left( \frac{d}{2} + it \right) + \text{Tr} \frac{y^{2it}\phi^* - y^{-2it}\phi}{2it}.$$
3.1.3. **Factorization.** To readers accustomed to scattering theory in one dimension, the following lemma will not be a surprise

**Lemma 3.6.** There is a polynomial $Q$ of order at most $2\lfloor d/2 \rfloor + 1$, such that

$$\varphi(s) = \varphi\left(\frac{d}{2}\right) e^{iQ(s)} \prod_{\rho \text{ resonance}} \frac{s - d + \rho}{s - \rho}$$

Additionally, $Q$ is real for $\Re s = d/2$, and $Q(s) + Q(d - s)$ is constant. We also have

$$\sum_{\rho \text{ resonance}} \frac{d - 2\Re \rho}{|\rho - d/2|^2} < \infty.$$  

This lemma is due to Müller [Müll92] in the case of surfaces (and Selberg for hyperbolic surfaces). Actually, the proof of Müller easily extends to the higher dimensionnal case. We will not discuss it further.

A consequence we used in part 1 is the following:

$$2\pi S'(T) = iQ'(d/2 + iT) + \sum_{\rho \text{ resonance}} \frac{2\Re \rho - d}{(d - \Re \rho/2)^2 + (T - \Im \rho)^2}$$

Observe how the second summand in the RHS is a negative function.

3.2. **Two estimates.** Before proving the actual Weyl law in the strip, we give two remarkable bounds on other counting quantities. First, we deal with resonances that are not in the strip, and then we give an asymptotics for a weighted counting function. The latter is crucial for the proof of the Weyl estimate.

3.2.1. **Counting resonances far from the spectrum.** In this section, we only use \((P1)\). We deduce from lemma \([3.1]\) applied to $\varphi$ that

$$2\pi \sum_{\Re s > b, |s - b| < T} \frac{T}{|s - b|} \log \frac{\varphi(b + it)}{\varphi(b + i(T + t))} dt + \int_{-\pi/2}^{\pi/2} \log |\varphi(b + e^{i\theta})| d\theta - \pi \log |\varphi(b)|.$$ 

From the Maass-Selberg estimate (lemma \([3.3]\)), we deduce that the second term is $O(T)$ (the log of $|\varphi|$ is bounded by a $O(T)$ term, and a log sin $\theta$). The third is a constant, and the first one is $O(T)$ by \((P1)\) (because \((P1)\) implies that $\Re \varphi'/\varphi$ is bounded on $\{\Re s = b\}$). Hence

$$\# \{\rho \text{ resonance} \mid \Re \rho < d - b, |\rho| \leq T \} = O(T)$$

Still using the Carleman’ estimate, we can give an additionnal statement. Let $\alpha \leq 1$. Then, for $\epsilon < 1$, we have

$$2\pi \sum_{\Re s > b, |s - (b + iT)| < (\epsilon T)^{\alpha}} \frac{(\epsilon T)^{\alpha}}{|s - (b + iT)|} = \int_{-(\epsilon T)^{\alpha}}^{(\epsilon T)^{\alpha}} \log |\varphi(b + iT + \theta)| d\theta - \pi \log |\varphi(b + iT)|.$$
According to property \((P1)\), the first term is \(O((\epsilon T)^\alpha)\). Using property \((P2)\), we see that the last term is \(O(\log T)\). According to lemma 3.5, we find that the second term is less than
\[
O(1) + C \int_{-\pi/2}^{\pi/2} d\theta \left( (\epsilon T)^\alpha \cos \theta + \log \left( 1 + \frac{(\epsilon T)^\alpha}{T(1-\epsilon)} \right) \right).
\]

We deduce that
\[
# \{ \rho \mid \Re \rho > b, \ |\rho - b - iT| \leq (\epsilon T)^\alpha \} = O((\epsilon T)^\alpha).
\]

### 3.2.2. Counting with weights in vertical strips.

Now, we will use both \((P1)\) and \((P2)\). Using lemma 3.5 again, we see that there is a constant \(C > 0\) such that
\[
\int_{b-d/2}^{b} \log |\varphi(x+iT)| (x-d/2) dx \leq C.
\]
The counting in small rectangles enables us to also have a lower bound. Indeed, we proceed as in Selberg [Sel89b, p.21] The LHS in (49) is always positive, so we write
\[
c \int_{d/2}^{b} \log(|\varphi(x+iT+i\pi/c)||\varphi(x+iT-i\pi/c)|) \sinh(c(x-d/2)) dx >
\]
\[
- \int_{-\pi/c}^{\pi/c} \sinh(c(b-d/2)) \cos(ct) \frac{\Re \varphi'}{\varphi}(b+iT+it) dt
\]
\[
+ c \int_{-\pi/c}^{\pi/c} \cosh(c(b-d/2)) \cos(ct) \log |\varphi(b+iT+it)| dt
\]
Property \((P1)\) implies that the first term in the RHS is \(O(1)\) (\(\Re \varphi'/\varphi\) is bounded on \(\Re s = b\)). Then, \((P2)\) implies that the second term is \(O(\log T)\) (since \(\log |\varphi| = O(\log T)\) on \(\Re s = b\)).

Using the upper bound \((60)\) on \(\log |\varphi(z)|\) given by Maass-Selberg, we find for some constant \(C > 0\) depending on \(c\),
\[
\int_{d/2}^{b} \log(|\varphi(x+iT+i\pi/c)|) \sinh(c(x-d/2)) dx > -C \log T.
\]

Now, we use again the upper bound of \(\varphi\): for some \(c' > 0\), \(|\varphi(x+iT+i\pi/c)| \leq 1/c'\) for all \(d/2 < x < b\) and \(T \in \mathbb{R}\). So that for some constant \(C > 0\),
\[
\int_{d/2}^{b} \log(|\varphi(x+iT+i\pi/c)|c')(x-d/2) dx \geq C \int_{d/2}^{b} \log(|\varphi(x+iT+i\pi/c)|c') \sinh(c(x-d/2)) dx.
\]

We conclude that
\[
\int_{d/2}^{b} \log |\varphi(x+iT)|(x-d/2) dx = O(\log T).
\]

Applying \((49)\) (counting in small rectangles) again, still with the notation that the zeroes of \(\varphi\) are \(s = \beta + i\gamma\), we see that
\[
\sum_{\beta \leq b \atop T \leq \gamma \leq T+1} \beta - d/2 = O(\log T).
\]
Now, we apply equation (48) (counting in big rectangles) for $\varphi$ at $T$ and at $T + 1$, and we subtract the two equalities. We obtain

$$2\pi \sum_{d/2 \leq \beta \leq b \atop 0 \leq \gamma \leq T} (\beta - d/2) = \int_0^T \Re \varphi'(b + it)(b - d/2)dt$$

$$+ \int_{d/2}^b \log \frac{|\varphi(x + iT + i)|}{|\varphi(x + iT)|} (x - d/2)dx$$

$$- \int_0^T \log |\varphi(b + it)| dt.$$  \hfill (63)

where the remainder $R$ is

$$R = - \sum_{d/2 \leq \beta \leq b \atop T \leq \gamma \leq T + 1} (T + 1 - \gamma)(\beta - d/2)$$

$$+ \int_0^1 \Re \varphi'(b + i(T + 1 - t))t(b - d/2)dt$$

$$- \int_0^1 \log |\varphi(b + i(T + 1 - t))| dt.$$

In $R$, the first term is $O(\log T)$ by (62). The second one is $O(1)$ by (P1), and the last one is $C \log T + O(1)$ by (P2).

In the RHS of (63), equation (61) proves that the second term is $O(\log T)$. For the first and third terms, we can use lemma 3.4, (P1) and (P2) to see that

$$\pi \sum_{d/2 \leq \beta \leq b \atop 0 \leq \gamma \leq T} \beta - d/2 = \frac{\kappa}{2} T \log T - \left( \frac{\kappa}{2} + \log a_0 + \frac{d}{2}\ell_\star \right) T + O(\log T).$$

for some constant $C$. This is the estimate we sought.

3.3. A Weyl law in the vertical strip. To obtain a Weyl law for the resonances, we follow again Selberg’s ideas. Let

$$N(T) := \# \left\{ s \in \text{Res}(M, g) \ \middle|\ d - b < \Re s \leq \frac{d}{2}, \ 0 \leq \Im s \leq T \right\}.$$  \hfill (64)

The first step is the following

**Lemma 3.7.** The number of resonances in a rectangle with vertices $d/2 + iT \pm 1/\log T$ and $b + iT \pm 1/\log T$ is bounded by $O(T^d/\log T)$.

**Proof.** First, we observe that

$$S(T + 2/\log T) - S(T - 2/\log T) = O(T^d/\log T)$$

$$+ \frac{1}{2\pi} \sum_{\rho \text{ resonance}} \int_{T - 2/\log T}^{T + 2/\log T} \frac{2\Re \rho - d}{(\Im \rho - t)^2 + (\Re \rho - d/2)^2} dt$$

...
In the RHS, all the terms in the sum are positive, so
\[
\frac{1}{2\pi} \sum_{|\rho - d/2 + iT| \leq 2/\log T} \int_{T/2/\log T}^{T+2/\log T} \frac{d - 2\Re \rho}{(3\Re - t)^2 + (\Re \rho - d/2)^2} dt \\
\leq O(T^d/\log T) + S(T - 2/\log T) - S(T - 2/\log T).
\]
However, when \(|\rho - d/2 + iT| \leq 2/\log T\),
\[
\int_{T/2/\log T}^{T+2/\log T} \frac{2\Re \rho - d}{(3\Re - t)^2 + (\Re \rho - d/2)^2} dt \geq \pi.
\]
Whence we deduce that the number of resonances in the half ball of radius \(2/\log T\), centered at \(d/2 + iT\) is bounded by
\[
\tilde{N}(T + 2/\log T) - \tilde{N}(T - 2/\log T) + O(T^d/\log T) = O(T^d/\log T),
\]
according to the Weyl law proved in the first part of the article. To finish the proof of our lemma, it suffices to prove that
\[
\# \left\{ \rho \left| d/2 + \frac{\sqrt{3}}{2 \log T} < \Re \rho < b, \ |\Im \rho - T| \leq 1/\log T \right. \right\} = O(T^d/\log T).
\]
However, according to (62) this quantity is bounded above by
\[
\frac{2 \log T}{\sqrt{3}} \sum_{|\Im \rho - T| \leq 1/\log T} \Re \rho - d/2 = O(\log^2 T).
\]
and this is a better bound than we need. \(\square\)

Now, consider the rectangle \(R_T\) with vertices \(d/2, b, d/2 + iT\) and \(b + iT\). If we integrate \(\varphi'/\varphi\) along its boundary, we obtain the number of zeroes \(\beta + i\gamma\) of \(\varphi\) in \(R_T\). This gives
\[
N(T) := N_{pp}(T) + \sum_{\beta \leq b, 0 \leq \gamma \leq T} 1 = N_{pp}(T) - \int_0^T S'(t) dt \\
- \frac{1}{2\pi} \int_0^T \Re \left[ \frac{\varphi'}{\varphi} (b + it) \right] dt \\
+ \frac{1}{2\pi} \int_{d/2}^b \Im \left[ \frac{\varphi'}{\varphi} (\sigma + iT) - \frac{\varphi'}{\varphi} (\sigma) \right] d\sigma.
\]
(65)

From the result of the first part of the article, the first term in the RHS is
\[
\frac{\text{vol}(B^* \mathcal{M})}{(2\pi)^{d+1}} T^{d+1} - \frac{\kappa T}{\pi} \log T + \frac{\kappa(1 - \log 2)}{\pi} T + O(T^d/\log T).
\]
(66)
Using (P1), one can see that the second term is \(\ell_s T/(2\pi) + O(\log T)\).

Thanks to lemma 3.7 to prove the Weyl estimate of resonances, it now suffices to find for each \(T\) a \(T'\) such that \(T - T' = O(1/\log T)\), and such that the Weyl estimate holds for \(T'\). That is, to obtain the result we seek, we only have to prove
Lemma 3.8. For \( T \) big enough, there is a \( T' \) such that \( T - T' = O(1/\log T) \), and

\[
\int_{d/2}^{b} \Re \varphi' \sigma \, d\sigma = Q_2(T') + O \left( T^{d/2} \log T \right) = O(T^d / \log T).
\]

where \( Q_2 \) is a polynomial of order at most \( 2|d/2| - 1 \).

This estimate shows that, to some extent, improving the remainder in the continuous Weyl law of part 1 automatically improves the Weyl law for the resonances. This was pointed out by Selberg.

Proof. Again, we use the decomposition 3.6 for \( \varphi'/\varphi \) as a sum over the resonances and a polynomial term. We will deal first with the polynom. Then we will bound the contribution from resonances out of the strip, and then we will come back to resonances in the strip.

The polynomial contribution is

\[
\int_{d/2}^{b} \Re Q'(\sigma + iT') \, d\sigma = \Re [Q(b + iT') - Q(d/2 + iT')].
\]

However, since \( Q \) is polynomial of order at most \( 2|d/2| + 1 \), and real for \( \Re s = d/2 \), the RHS here is a polynomial of order at most \( 2|d/2| - 1 \). Let us call it \( Q_2(T) \).

Next, each resonance contributes to the imaginary part of \( \varphi'/\varphi \) by

\[
f_\rho(s) := \Im \left[ \frac{1}{s - d + \rho} - \frac{1}{s - \rho} \right] = \frac{2\Re \rho - d}{|s - \rho|^2} \frac{3(s - \rho)}{|s - d + \rho|^2}.
\]

For the resonances that are not in the strip, for \( s \in iT + [d/2, b] \), \( |s - \rho| \approx |s - d + \rho| \), so we have the bound

\[
\int_{d/2}^{b} f_\rho(\sigma + iT) \, d\sigma = O(1) \frac{|2\Re \rho - d| (T - \Im \rho)}{|d/2 + iT - \rho|^4}.
\]

Assume that \( |d/2 + iT - \rho| > T^{-1/6}|d/2 - \rho| \). Then the LHS is bounded by

\[
O(1) \sqrt{T} \frac{2\Re \rho - d}{|\rho - d/2|^3} = \sqrt{T} O \left( \frac{2\Re \rho - d}{|\rho - d/2|^2} \right).
\]

Denote \( s_T := d/2 + iT \). Since \( \sum_\rho (2\Re \rho - d)/|\rho - d/2|^2 < \infty \) the contribution from the resonances that are not in the strip is

\[
O(1) \sqrt{T} + \sum_{|s_T - \rho| \leq T^{-1/6}|\rho - d/2|, \Re \rho > b} \frac{(2\Re \rho - d)(T - \Im \rho)}{|s_T - \rho|^4}.
\]

But, for \( T \) big enough, \( |s_T - \rho| \leq T^{-1/6}|\rho - d/2| \) implies \( |s_T - \rho| \leq C T^{-1/6}|s_T - d/2| = C T^{5/6} \), for some positive constant \( C \). We count separately the resonances with \( |s_T - \rho| \leq \sqrt{T} \) and the others.

The ones that are in the half annulus \( \{ |\sqrt{T} \leq |s_T - \rho| \leq C T^{5/6} \} \) each contribute by

\[
O(1) \frac{(2\Re \rho - d)(T - \Im \rho)}{|s_T - \rho|^4} = O(T^{-1})
\]

and according to (59), there are at most \( O(T^{5/6}) \), so their total contribution is \( O(T^{1/6}) \).
For the resonances out of the strip that satisfy $|s_T - \rho| \leq \sqrt{T}$, from [50], we know there are at most $O(\sqrt{T})$, and for them $|s_T - \rho| > \log T$ (thanks to the logarithmic size of the resonance-free zone). In particular, their total contribution is $O(\sqrt{T}/\log^2 T)$. Hence, the total contribution of resonances out of the strip is $O(\sqrt{T})$.

Now, we turn to resonances in the strip. Here, we follow Selberg’s argument closely. Let $H > 1$, and assume $H < |s_T - \rho| < 2H$. Then $|\Im(s_T - \rho)| \approx |s_T - \rho|$ and

$$\sum_{H < |s_T - \rho| < 2H} f_\rho = \frac{O(1)}{H^3} \sum_{H < |s_T - \rho| < 2H} (2\Re \rho - d)$$

$$= \frac{O(1)}{H^3} [H \log T + H \log H] = O \left( \frac{\log T + \log H}{H^2} \right)$$

Summing this for $H = 2^n$, $n \geq 0$ proves that

$$\sum_{|s_T - \rho| > 1, \Re \rho < b} f_\rho = O(\log T).$$

So far, we have proved that for all $T$,

$$\int_{d/2}^{b} \sum_{\Im \rho < 1, \Re \rho < b} (\sigma + iT) d\sigma = \sum_{|s_T - \rho| < 1, \Re \rho < b} \int_{d/2}^{b} f_\rho(\sigma + iT) d\sigma + O(T^d/\log T).$$

Now comes the delicate part. Let $\ell(T)$ be a monotonous integer valued function such that $\ell(T) \to +\infty$, and $h(T)$ also monotonous, such that $h(T) \to 0$. Also assume that $h^* = \ell h = O(1/\log T)$. Then, from lemma 3.7

$$\# \{\rho, \Re \rho \leq b, T \leq \Im \rho \leq T + h^*\} = O(T^d/\log T)$$

By the pigeonhole principle, there is some integer $0 \leq v \leq \ell - 1$ so that there are at most $O(T^d/\log T)/\ell$ resonances in the strip with $T + vh \leq \Im \rho \leq T + (v + 1)h$. We let $T' = T + (v + 1/2)h$.

Consider $\rho$ in the strip such that $|\Im(\rho - T')| > h$. Then

$$\int_{d/2}^{b} f_\rho(\sigma + iT') d\sigma = \int_{0}^{b-d/2} \frac{(2\Re \rho - d)(\Im(\rho - T')) \sigma d\sigma}{|\rho - \sigma| - d/2 - iT'|^2|\sigma + d/2 + iT' - d + \rho|^2}$$

$$= \int_{0}^{(b-d/2)/(|\Im(\rho - T')|)} \frac{(2\Re \rho - d)(\Im(\rho - T')) \sigma d\sigma}{(1 + (\sigma + d/2 - \Re(\rho - T'))^2)(1 + (\sigma - d/2 - \Re(\rho - T'))^2)}$$

$$\leq \int_{0}^{(b-d/2)/(|\Im(\rho - T')|)} \frac{(2\Re \rho - d)(\Im(\rho - T')) \sigma d\sigma}{1 + \sigma^2}$$

$$\leq O(1)(2\Re \rho - d)(\Im(\rho - T'))^{-1}(1 + \log |\Im(\rho - T')|).$$

Observe how this is $(2\Re \rho - d)O(h^{-1}|\log h|)$. If we sum this, we get

$$\sum_{h < |\Im(\rho - T')| < 1, \Re \rho < b} \int_{d/2}^{b} f_\rho(\sigma + iT') d\sigma = \frac{|\log h|}{h} \sum_{|\Im(\rho - T')| < 1, \Re \rho < b} 2\Re \rho - d = O \left( \frac{|\log T| \log h}{h} \right).$$
The remaining terms are those with $|\Im \rho - T| < h$. For them, the best bound we can give for their individual contribution is $O(1)$. Indeed, their contribution is the variation of the argument of $(s - d + \rho)/(s - \rho)$ as $s$ goes from $d/2 + iT'$ to $b + iT'$ along a horizontal line. This variation is at most $\pi$. Hence

$$\sum_{|\Im \rho - T| < h} f_\rho(\sigma + iT')d\sigma = \frac{1}{\ell} O(T^d / \log T).$$

Combining all the above,

$$\Im \int_{d/2}^{b} \frac{\varphi'(\sigma + iT')}{\varphi}(d\sigma) = Q_2(T) + O\left(\sqrt{T} + \log T + \frac{\log h \log \log T}{h} + \frac{T^d}{\ell \log T}\right),$$

where $Q_2$ is a polynomial of order at most $2 \lfloor d/2 \rfloor - 1$. Now, we have to choose $\ell$ and $h$ so that $\ell \to +\infty$, $h \to 0$ and $\ell h = O(1/\log T)$. Consider $\ell = T^{d/2} \log^{-2} T$ and $h = T^{-d/2} \log T$. Then we have

$$\frac{\log T \log h}{h} + \frac{T^d}{\ell \log T} = O(T^{d/2} \log T).$$

\[\square\]

4. Counting with no assumption

Now, we will explain shortly how to obtain the theorem 4. The argument is very similar to the one in [Bon14b], itself based on the computations p282 in [Müller92].

The first observation is that since

$$\sum_{\rho \text{ resonance}} \frac{d - 2\Re \rho}{|\rho - d/2|^2} < \infty$$

we directly have for any $\epsilon > 0$.

(69) \# \{\rho \text{ resonance} \mid \Re \rho \leq d/2 - \epsilon, \, |\rho - d/2| \leq T\} = o(T^2).

Since we work in dimension $> 2$, we only have to count the resonances in a strip $\{d/2 - \epsilon < \Re \rho \leq d/2\}$.

According to the formula (57), together with (6), for $T > 0$,

(70) \# \{r_i \mid d^2/4 + r_i^2 \text{ eigenvalue, } T \leq r_i \leq T + 1\} + \frac{1}{2\pi} \sum_{\rho \text{ resonance}} \int_T^{T+1} \frac{d - 2\Re \rho}{|\rho - d/2 + it|^2} dt = O(T^d).

But each resonance contributes by a positive term in the LHS, so

$$\sum_{\Re \rho > d/2 - \epsilon, \, 1/3 \leq \Im \rho - T \leq 2/3} \int_T^{T+1} \frac{d - 2\Re \rho}{|\rho - d/2 + it|^2} dt = O(T^d).$$

Each of the terms in the LHS is larger than $4 \arctan(1/3\epsilon)$. Taking $\epsilon$ small enough, we find the local part of the theorem:

(71) \# \{\rho \text{ resonance} \mid \Re \rho > d/2 - \epsilon, \, |\Im \rho - T| = O(1)\} = O(T^d).
Now, the global version of (70) is
\[
\left\{ r_i \mid d^2/4 + r_i^2 \text{ eigenvalue, } |r_i| \leq T \right\} + \frac{1}{2\pi} \sum_{\rho \text{ resonance}} \int_{-T}^{T} \frac{d - 2\Re \rho}{|\rho - d/2 + it|^2} dt = \text{vol}(B^*M)_{T^{d+1}} - \frac{\kappa T}{\pi} \log T + O(T^d)
\]
(72)

But, as in the equation 4.9 in [M¨ ul92], we find
\[
\int_{-T}^{T} \frac{d - 2\Re \rho}{|\rho - d/2 + it|^2} dt = 2 \arctan \left( \frac{d - 2\Re \rho}{|\rho - d/2|^2} T \right) \left( 1 - \frac{T^2}{|\rho - d/2|^2} \right)^{-1}
\]
+ \begin{cases} 0 & \text{if } |\rho - d/2| > T \\ 2\pi & \text{else} \end{cases}
\]

We can rewrite (72) as
\[
\# \{ s \text{ pole of } \mathcal{R}(s) \mid |s - d/2| \leq T \} = \frac{\text{vol}(B^*M)}{(2\pi)^{d+1}} T^{d+1} - \frac{\kappa T}{\pi} \log T + O(T^d) + R(T),
\]
where
\[
R(T) = \frac{1}{\pi} \sum_{s \text{ resonance}} \arctan \left( \frac{d - 2\Re \rho}{|\rho - d/2|^2} T \right) \left( 1 - \frac{T^2}{|\rho - d/2|^2} \right)^{-1}.
\]

In this sum, the resonances with $|\rho - d/2| - T > 1$ contribute by $O(T^2)$. Indeed, then,
\[
\left| 1 - \frac{T^2}{|\rho - d/2|^2} \right|^{-1} \leq 2T.
\]
Hence their total contribution is (recall $|\arctan(x)| \leq |x|$)
\[
O(1) \sum_{|\rho - d/2| - T > 1} \frac{d - 2\Re \rho}{|\rho - d/2|^2} T^2 = O(T^2).
\]

From (61) and (71), we know that there are at most $O(T^d)$ resonances with $|\rho - d/2| - T \leq 1$, and so we conclude that $R(T) = O(T^d)$.
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