A New Probabilistic Algorithm for Approximate Model Counting

Cunjing Ge\textsuperscript{1,3}, Feifei Ma\textsuperscript{1,2,3(✉)}, Tian Liu\textsuperscript{4}, Jian Zhang\textsuperscript{1,3}, and Xutong Ma\textsuperscript{3,5}

\textsuperscript{1} State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing, China
\{gecj,maff,zj\}@ios.ac.cn
\textsuperscript{2} Laboratory of Parallel Software and Computational Science, Institute of Software, Chinese Academy of Sciences, Beijing, China
\textsuperscript{3} University of Chinese Academy of Sciences, Beijing, China
\textsuperscript{4} School of Electronics Engineering and Computer Science, Peking University, Beijing, China
\textsuperscript{5} Technology Center of Software Engineering, Institute of Software, Chinese Academy of Sciences, Beijing, China

Abstract. Constrained counting is important in domains ranging from artificial intelligence to software analysis. There are already a few approaches for counting models over various types of constraints. Recently, hashing-based approaches achieve success but still rely on solution enumeration. In this paper, a new probabilistic approximate model counter is proposed, which is also a hashing-based universal framework, but with only satisfiability queries. A dynamic stopping criteria, for the new algorithm, is presented, which has not been studied yet in previous works of hashing-based approaches. Although the new algorithm lacks theoretical guarantee, it works well in practice. Empirical evaluation over benchmarks on propositional logic formulas and SMT\textsc{(BV)} formulas shows that the approach is promising.

1 Introduction

Constrained counting, the problem of counting the number of solutions for a set of constraints, is important in theoretical computer science and artificial intelligence. Its interesting applications in several fields include program analysis \cite{17,18,20,21,28,30}, probabilistic inference \cite{12,31}, planning \cite{14} and privacy/confidentiality verification \cite{19}. Constrained counting for propositional formulas is also called model counting, to which probabilistic inference is easily reducible. However, model counting is a canonical \#P-complete problem, even

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for polynomial-time solvable problems like 2-SAT [37], thus it presents fascinating challenges for both theoreticians and practitioners.

There are already a few approaches for counting solutions over propositional logic formulas and SMT(BV) formulas. Recently, hashing-based approximate counting achieves both strong theoretical guarantees and good scalability [29]. The use of universal hash functions in counting problems began in [34,36], but the resulting algorithm scaled poorly in practice. A scalable approximate counter \textit{ApproxMC} in [10] scales to large problem instances, while preserving rigorous approximation guarantees. \textit{ApproxMC} has been extended to finite-domain discrete integration, with applications to probabilistic inference [4,7,15]. It was improved by designing efficient universal hash functions [8,25] and reducing the use of NP-oracle calls from linear to logarithmic [11].

The basic idea in \textit{ApproxMC} is to estimate the model count by randomly and iteratively cutting the whole space down to a small “enough” cell, using hash functions, and sampling it. The total model count is estimated by a multiplication of the number of solutions in this cell and the ratio of the whole space to the small cell. To determine the size of the small cell, which is essentially a small-scale model counting problem with the model counts bounded by some thresholds, a model enumeration in the cell is adopted. In previous works, the enumeration query was handled by transforming it into a series of satisfiability queries, which is much more time-consuming than a single satisfiability query. An algorithm called \textit{MBound} [23] only invokes satisfiability query once for each cut. Its model count is determined with high precision by the number of cuts down to the boundary of being unsatisfiable. However, this property is not strong enough to give rigorous guarantees, and \textit{MBound} only returns an approximation of upper or lower bound of the model count.

In this paper, a new hashing-based approximate counting algorithm, with only satisfiability query, is proposed. Dynamic stopping criterion for the algorithm to terminate, once meeting the criterion of accuracy, is presented, which has not been proposed yet in previous works of hashing-based approaches. Theoretical insights over the efficiency of a prevalent heuristic strategy called leapfrogging are also provided. The new algorithm works well in practice but does not provide theoretical guarantees, since it builds on an assumption of a correlation between the model count and the probability of the hashed formula being unsatisfiable, which has not been proved yet.

The proposed approach is a general framework easy to handle various types of constraints. Prototype tools for propositional logic formulas and SMT(BV) formulas are implemented. An extensive evaluation on a suite of benchmarks demonstrates that (i) the approach significantly outperforms the state-of-the-art approximate model counters, including a counter designed for SMT(BV) formulas, and (ii) the dynamic stopping criterion is promising.

The rest of this paper is organized as follows. Preliminary material is in Sect. 2, related works in Sect. 3, the algorithm in Sect. 4, analysis in Sect. 5, experimental results in Sect. 6, and finally, concluding remarks in Sect. 7.
2 Preliminaries

Let $F(x)$ denote a propositional logic formula on $n$ variables $x = (x_1, \ldots, x_n)$. Let $S$ and $S_F$ denote the whole space (the space of assignments) and the solution space of $F$, respectively. Let $\#F$ denote the cardinality of $S_F$, i.e. the number of solutions of $F$.

$(\epsilon, \delta)$-bound To count $\#F$, an $(\epsilon, \delta)$ approximation algorithm, $\epsilon > 0$ and $\delta > 0$, is an algorithm which on every input formula $F$, outputs a number $\hat{Y}$ such that $\Pr[(1 + \epsilon)^{-1}\#F \leq \hat{Y} \leq (1 + \epsilon)\#F] \geq 1 - \delta$. Such an algorithm is called a $(\epsilon, \delta)$-counter and the bound is called a $(\epsilon, \delta)$-bound [26].

Hash Function. Let $\mathcal{H}_F$ be a family of XOR-based bit-level hash functions on the variables of a formula $F$. Each hash function $H \in \mathcal{H}_F$ is of the form $H(x) = a_0 \bigoplus_{i=1}^{n} a_i x_i$, where $a_0, \ldots, a_n$ are Boolean constants. In the hashing procedure $\text{Hashing}(F)$, a function $H \in \mathcal{H}_F$ is generated by independently and randomly choosing $a_i$s from a uniform distribution. Thus for an assignment $\alpha$, it holds that $\Pr_{H \in \mathcal{H}_F}(H(\alpha) = \text{true}) = \frac{1}{2}$. Given a formula $F$, let $F_i$ denote a hashed formula $F \land H_1 \land \cdots \land H_i$, where $H_1, \ldots, H_i$ are independently generated by the hashing procedure.

Satisfiability Query. Let $\text{Solving}(F)$ denote the satisfiability query of a formula $F$. With a target formula $F$ as input, the satisfiability of $F$ is returned by $\text{Solving}(F)$.

Enumeration Query. Let $\text{Counting}(F, p)$ denote the bounded solution enumeration query. With a constraint formula $F$ and a threshold $p$ ($p \geq 2$) as inputs, a number $s$ is returned such that $s = \min(p - 1, \#F)$. Specifically, 0 is returned for unsatisfiable $F$, or $p = 1$ which is meaningless.

SMT(BV) Formula. SMT(BV) formulas are quantifier-free and fixed-size that combine propositional logic formulas with constraints of bit-vector theory. For example, $\neg(x + y = 0) \lor (x = y << 1)$, where $x$ and $y$ are bit-vector variables, $<<$ is the shift-left operator. It can be regarded as a propositional logic formula $\neg A \lor B$ that combines bit-vector constraints $A \equiv (x + y = 0)$ and $B \equiv (x = y << 1)$. To apply hash functions to an SMT(BV) formula, a bit-vector is bit-blasted to a set of Boolean variables.

3 Related Works

[3] showed that almost uniform sampling from propositional constraints, a closely related problem to constrained counting, is solvable in probabilistic polynomial time with an NP oracle. Building on this, [10] proposed the first scalable approximate model counting algorithm $\text{ApproxMC}$ for propositional formulas. $\text{ApproxMC}$ is based on a family of 2-universal bit-level hash functions that compute XOR of randomly chosen propositional variables. In the current work, this family of hash functions is adopted, which was shown to be 3-independent in [24], and is
Algorithm 1

1: function ApproxMC($F$, $T$, $pivot$)  
2:     for 1 to $T$ do  
3:         $c$ ← ApproxMCCore($F$, $pivot$)  
4:             if ($c \neq 0$) then AddToList($C$, $c$)  
5:     end for  
6:     return FindMedian($C$)  
7: end function  
8: function ApproxMCCore($F$, $pivot$)  
9:     $F_0 ← F$  
10:     for $i ← 0$ to $∞$ do  
11:         $s$ ← Counting($F_i$, $pivot + 1$)  
12:             if ($0 \leq s \leq pivot$) then return $2^i s$  
13:         $H_{i+1} ← Hashing(F)$  
14:         $F_{i+1} ← F_i ∧ H_{i+1}$  
15:     end for  
16: end function

revealed to potentially possess better properties than expected by the experimental results and the theoretical analysis in the current work.

The sketch framework of ApproxMC [10,13] is listed as Algorithm 1. Its inputs are a formula $F$ and two accuracy parameters $T$ and $pivot$, where $T$ determines the number of times ApproxMCCore is invoked, and $pivot$ determines the threshold of the enumeration query. The function ApproxMCCore starts from the formula $F_0$, iteratively calls Counting and Hashing over each $F_i$, to cut the space (cell) of all models of $F_0$ using random hash functions, until the count of $F_i$ is no larger than $pivot$, then breaks out of the loop and adds the approximation $2^i s$ into list $C$. The main procedure ApproxMC repeatedly invokes ApproxMCCore and collects the returned values, at last returning the median number of list $C$. The general algorithm in [8] is similar to Algorithm 1, but cuts the cell with dynamically determined proportion instead of the constant $\frac{1}{2}$, due to the word-level hash functions. [11] improves ApproxMCCore via binary search to reduce the number of enumeration queries from linear to logarithmic. This binary search improvement is orthogonal to our approach.

A recent work [1] considered a special family of shorter XOR-constraints to improve the efficiency of SAT solving while preserving rigorous guarantee. This improvement of hash functions is also orthogonal to our approach as we use hash functions and SAT solving as black boxes. However, it is unknown whether there exist similar theoretical results like [1].

4 Algorithm

In this section, a new hashing-based algorithm for approximate model counting, with only satisfiability queries, will be proposed, building on an assumption of a probabilistic approximate correlation between the model count and the probability of the hashed formula being unsatisfiable.
Let $F_d = F \land H_1 \land \cdots \land H_d$ be a hashed formula resulted by iteratively hashing $d$ times independently over a formula $F$. $F_d$ is unsatisfiable if and only if no solution of $F$ satisfies $F_d$, thus $\Pr_{F_d}(F_d \text{ is unsat}) = \Pr_{F_d}(F_d(\alpha) = \text{false}, \alpha \in S_F)$. Assume we have

$$\Pr_{F_d}(F_d \text{ is unsat}) \approx (1 - 2^{-d})^{\#F}. \tag{1}$$

Then based on Eq. (1), an approximation of $\#F$ is achieved by taking logarithm on the value of $\Pr_{F_d}(F_d \text{ is unsat})$, which is estimated in turn by sampling $F_d$. This is the general idea of our approach. The pseudo-code is presented in Algorithm 2. The inputs are the target formula $F$ and a constant $T$ which determines the number of times GetDepth invoked. GetDepth calls Solving and Hashing repeatedly until an unsatisfiable formula $F_{\text{depth}}$ is encountered, and returns the depth. Every time GetDepth returns a depth, the value of $C[i]$ is increased, for all $i < \text{depth}$. At line 9, the algorithm picks a number $d$ such that $C[d]$ is close to $T/2$, since the error estimation fails when $C[d]/T$ is close to 0 or 1. The final result is returned by the formula $\log_{1 - 2^{-d}} \frac{\text{counter}}{T}$ at line 11.

**Algorithm 2. Satisfiability Testing based Approximate Counter (STAC)**

1: function STAC($F$, $T$)
2:   initialize $C[i]$s with zeros
3:   for $t \leftarrow 1$ to $T$ do
4:     depth $\leftarrow$ GetDepth($F$)
5:     for $i \leftarrow 0$ to depth $-1$ do
6:       $C[i] \leftarrow C[i] + 1$
7:     end for
8:   end for
9:   pick a number $d$ such that $C[d]$ is closest to $T/2$
10:  counter $\leftarrow T - C[d]$
11:  return $\log_{1 - 2^{-d}} \frac{\text{counter}}{T}$ /* return 0 when $d = 0$ */
12: end function
13: function GetDepth($F$)
14:   $F_0 \leftarrow F$
15:   for $i \leftarrow 0$ to $\infty$ do
16:     $b \leftarrow$ Solving($F_i$)
17:     if ($b$ is false) then return $i$
18:     $H_{i+1} \leftarrow$ Hashing($F_i$)
19:     $F_{i+1} \leftarrow F_i \land H_{i+1}$
20:   end for
21: end function

Note that our approach is based on Eq. (1) which is only an assumption. In Sect. 5, we provide theoretical analysis, including the bound of the approximation and the correctness of algorithm, based on the hypothesis. Then in Sect. 6, experimental results on an extensive set of benchmarks show that the approximation given by our approach fits the bound well. It indicates that Eq. (1) is probably true as it is a reasonable explanation to the positive results.
**Dynamic Stopping Criterion.** The essence of Algorithm 2 is a randomized sampler over a binomial distribution. The number of samples is determined by the value of $T$, which is pre-computed for a given $(\epsilon, \delta)$-bound, and we loosen the value of $T$ to meet the guarantee in theoretical analysis. However, it usually does not loop $T$ times in practice. A variation with dynamic stopping criterion is presented in Algorithm 3.

**Algorithm 3.** STAC with Dynamic Stopping Criterion

1: `function` STAC\_DSC($F$, $T$, $\epsilon$, $\delta$)
2: \hspace{1em} initialize $C[i]$s with zeros
3: \hspace{1em} for $t \leftarrow 1$ to $T$ do
4: \hspace{2em} `depth` $\leftarrow$ GetDepth($F$)
5: \hspace{2em} for $i \leftarrow 0$ to `depth` − 1 do
6: \hspace{3em} $C[i] \leftarrow C[i] + 1$
7: \hspace{2em} end for
8: \hspace{1em} for each $d$ that $C[d] > 0$ do
9: \hspace{2em} $q \leftarrow \frac{t-C[d]}{t}$
10: \hspace{2em} $M \leftarrow \log_{1-2^{-d}} q$
11: \hspace{2em} $U \leftarrow \log_{1-2^{-d}} (q - z_{1-\delta} \sqrt{\frac{q(1-q)}{t}})$
12: \hspace{2em} $L \leftarrow \log_{1-2^{-d}} (q + z_{1-\delta} \sqrt{\frac{q(1-q)}{t}})$
13: \hspace{2em} if $U < (1+\epsilon)M$ and $L > (1+\epsilon)^{-1}M$ then
14: \hspace{2em} \hspace{2em} return $M$
15: \hspace{2em} end if
16: \hspace{2em} end for
17: \hspace{1em} end for
18: `end function`

Lines 2 to 7 is the same as Algorithm 2, still setting $T$ as a stopping rule and terminating whenever $t = T$. Line 8 to 16 is the key part of this variation, calculating the binomial proportion confidence interval $[L, U]$ of an intermediate result $M$ for each cycle. A commonly used formula $q \pm z_{1-\delta} \sqrt{\frac{q(1-q)}{t}}$ [5,38] is adopted, which is justified by the central limit theorem to compute the $1 - \delta$ confidence interval. However, it becomes invalid for small sample size or proportion close to 0 or 1. In practice, we also considered some improvements, e.g., Wilson score interval [40]. The exact count $#F$ lies in the interval $[L, U]$ with probability $1 - \delta$. Combining the inequalities presented in line 13, the interval $[(1+\epsilon)^{-1}M, (1+\epsilon)M]$ is the $(\epsilon, \delta)$-bound (if the assumption of Formula 1 holds). So the algorithm terminates when the condition in line 13 comes true. The time complexity of Algorithm 3 is still the same as the original algorithm, though it usually terminates earlier.

**Satisfiability and Enumeration Query.** The bounded counting can be done by negating solution and calling SAT oracle repeatedly, which is employed by
ApproxMC. In practice, enumerating solutions in this way is not very efficient. In evaluation section, experimental results show that the average number of SAT calls of ApproxMC is usually 20 to 30 times to STAC. It may also cause problems while extending to other kinds of formulas. For example, for linear integer arithmetic formula, inserting solution negation clauses will exponentially increase the number of calls of LIP solver.

**Leap-frogging Strategy.** Recall that GetDepth is invoked $T$ times with the same arguments, and the loop of line 15 to 20 in the pseudo-code of GetDepth in Algorithm 2 is time consuming for large $i$. A heuristic called leap-frogging to overcome this bottleneck was proposed in [9,10]. Their experiments indicate that this strategy is extremely efficient in practice. The average depth $\bar{d}$ of each invocation of GetDepth is recorded. In all subsequent invocations, the loop starts by initializing $i$ to $\bar{d} - k \cdot \text{offset}$, where $k \geq 1$. Note that if $F_i$ is unsatisfiable, the algorithm repeatedly decreases $i$ by increasing $k$ and check the satisfiability of the new $F_i$, until a proper initialization $i$ is found for satisfiable $F_i$. In practice, the constant offset is usually set to 5. Theorem 3 in Sect. 5 shows that the depth computed by GetDepth lies in an interval $[d, d + 7]$ with probability over 90%. So it suffices to invoke Solving in constant time since the second iteration.

## 5 Analysis

In this section, we assume Eq. (1) holds. Based on this assumption, theoretical results on the error estimation of our approach are presented. For lack of space, we omit proofs in this section.

Recall that in Algorithm 2, $\#F$ is approximated by a value $\log_{1-2^{-d}} \frac{\text{counter}}{T}$. Let $q_d$ denote the value of $(1 - 2^{-d})\#F$. We obtain that $\Pr(F_d \text{ is unsat}) = q_d$ for a randomly generated formula $F_d$. This is justified by Eq. (1). Since the ratio $\frac{\text{counter}}{T}$ in Algorithm 2 is a proportion of successes in a Bernoulli trial process, which is used to estimate the value of $q_d$. Then $\text{counter}$ is a random variable following a binomial distribution $\mathbb{B}(T, q_d)$.

**Theorem 1.** Let $z_{1-\delta}$ be the $1-\delta$ quantile of $N(0,1)$ and

$$T = \max \left( \left[ \left( \frac{z_{1-\delta}}{2q_d(1-q_d)} \right)^2 \right], \left[ \left( \frac{z_{1-\delta}}{2(q_d(1+\epsilon)-1-q_d)} \right)^2 \right] \right).$$

(2)

Then $\Pr\left[ \log_{1-2^-d} \frac{\text{counter}}{T} \leq (1 + \epsilon)\#F \right] \leq 1 - \delta$.

**Proof.** By above discussions, the ratio $\frac{\text{counter}}{T}$ is the proportion of successes in a Bernoulli trial process which follows the distribution $\mathbb{B}(T, q_d)$. Then we use the approximate formula of a binomial proportion confidence interval $q_d \pm z_{1-\delta} \sqrt{q_d(1-q_d)}$, i.e., $\Pr[q_d - z_{1-\delta} \sqrt{q_d(1-q_d)} \leq \frac{\text{counter}}{T} \leq q_d + z_{1-\delta} \sqrt{q_d(1-q_d)}]$.
1 − δ. The log function is monotone, so we only have to consider the following two inequalities:

\[
\log_{1 - 2^{-a}} (q_d - z_1 - \delta \sqrt{q_d(1 - q_d)} T) \leq (1 + \epsilon) \# F, \tag{3}
\]

\[
(1 + \epsilon)^{-1} \# F \leq \log_{1 - 2^{-a}} (q_d + z_1 - \delta \sqrt{q_d(1 - q_d)} T). \tag{4}
\]

We first consider Eq. (3). By substituting \( \log_{1 - 2^{-a}} q_d \) for \( \# F \), we have

\[
\log_{1 - 2^{-a}} (q_d - z_1 - \delta \sqrt{q_d(1 - q_d)} T) \leq (1 + \epsilon) \log_{1 - 2^{-a}} q_d \]

which implies

\[
q_d - z_1 - \delta \sqrt{q_d(1 - q_d)} T \geq q_d^{(1 + \epsilon)} \]

\[
q_d(1 - q_d) \geq z_1 - \delta \sqrt{q_d(1 - q_d)} T \]

\[
T \geq \left( \frac{z_1 - \delta}{q_d(1 - q_d)} \right)^2 q_d(1 - q_d). \]

Since \( 0 \leq q_d \leq 1 \), we have \( \sqrt{q_d(1 - q_d)} \leq \frac{1}{2} \). Therefore, \( T = \left[ \left( \frac{z_1 - \delta}{2q_d(1 - q_d)} \right)^2 \right] \geq \left( \frac{z_1 - \delta}{q_d(1 - q_d)} \right)^2 q_d(1 - q_d). \)

We next consider Eq. (4). Similarly, we have

\[
\log_{1 - 2^{-a}} (q_d + z_1 - \delta \sqrt{q_d(1 - q_d)} T) \geq (1 + \epsilon)^{-1} \log_{1 - 2^{-a}} q_d \]

which implies

\[
q_d + z_1 - \delta \sqrt{q_d(1 - q_d)} T \leq q_d^{1/(1 + \epsilon)} \]

\[
T \geq \left( \frac{z_1 - \delta}{q_d^{1/(1 + \epsilon)} - q_d} \right)^2 q_d(1 - q_d). \]

So Eq. (2) implies Eqs. (3) and (4).

Theorem 1 shows that the result of Algorithm 2 lies in the interval \([(1 + \epsilon)^{-1} \# F, (1 + \epsilon) \# F]\) with probability at least \( 1 - \delta \) when \( T \) is set to a proper value. So we focus on the possible smallest value of \( T \) in subsequent analysis.

The next two lemmas are easy to show by derivations.

**Lemma 1.** \( \frac{z_1 - \delta}{2x(1 - x^2)} \) is monotone increasing and monotone decreasing in \([(1 + \epsilon)^{-1}, 1]\) and \([0, (1 + \epsilon)^{-1}]\) respectively.

**Lemma 2.** \( \frac{z_1 - \delta}{2(x^{1/(1 + \epsilon)} - x)} \) is monotone increasing and monotone decreasing in \([(1 + \epsilon)^{-1 + \epsilon}, 1]\) and \([0, (1 + \epsilon)^{-1 + \epsilon}]\) respectively.

**Theorem 2.** If \( \# F > 5 \), then there exists a proper integer value of \( d \) such that \( q_d \in [0.4, 0.65] \).
Proof. Let \( x \) denote the value of \( q_d = (1 - \frac{1}{2^x})^\#F \), then we have \( (1 - \frac{1}{2^x})^\#F = \left( \frac{1}{2} + \frac{x^{\#F}}{2} \right)^\#F \). Consider the derivation

\[
\frac{d}{d\#F} \left( \frac{1}{2} + \frac{x^{\#F}}{2} \right)^\#F = \left( \frac{1}{2} + \frac{x^{\#F}}{2} \right)^\#F \ln \left( \frac{1}{2} + \frac{x^{\#F}}{2} \right) \frac{x^{\#F}}{2} \ln x \frac{d}{d\#F} (\#F^{-1}).
\]

Note that \( \left( \frac{1}{2} + \frac{x^{\#F}}{2} \right)^\#F \) and \( \frac{x^{\#F}}{2} \) are the positive terms and \( \ln \left( \frac{1}{2} + \frac{x^{\#F}}{2} \right) \), \( \ln x \) and \( \frac{d}{d\#F} (\#F^{-1}) \) are the negative terms. Therefore, the derivation is negative, i.e., \( (\frac{1}{2} + \frac{x^{\#F}}{2})^\#F \) is monotone decreasing with respect to \( \#F \). In addition, \( (\frac{1}{2} + \frac{1}{x^5})^5 \) is the upper bound when \( \#F \geq 5 \).

Let \( x = 0.4 \), then \( (1 - \frac{1}{2^x})^\#F \leq (\frac{1}{2} + \frac{0.4^{1/5}}{2})^5 \approx 0.65 \). Since \( (1 - \frac{1}{2^x})^\#F = 0 \) and \( \lim_{d \to +\infty} (1 - \frac{1}{2^x})^\#F = 1 \) and \( (1 - \frac{1}{2^x})^\#F \) is continuous with respect to \( d \), we consider the circumstances close to the interval \([0.4, 0.65]\). Assume there exists an integer \( \sigma \) such that \( (1 - \frac{1}{2^x})^\#F < 0.4 \) and \( (1 - \frac{1}{2^x})^\#F > 0.65 \). According to the intermediate value theorem, we can find a value \( \epsilon > 0 \) such that \( (1 - \frac{1}{2^x})^\#F = 0.4 \). Obviously, we have \( (1 - \frac{1}{2^x})^\#F \leq 0.65 \) which is contrary to the monotone decreasing property.

From Theorem 2 and Lemmas 1 and 2, it suffices to consider the results of Eq. (2) when \( q_d = 0.4 \) and \( q_d = 0.65 \). For example, \( T = 22 \) for \( \epsilon = 0.8 \) and \( \delta = 0.2 \), \( T = 998 \) for \( \epsilon = 0.1 \) and \( \delta = 0.1 \), etc. We therefore pre-computed a table of the value of \( T \). The proof of next theorem is omitted.

**Theorem 3.** There exists an integer \( d \) such that \( q_d < 0.05 \) and \( q_{d+7} > 0.95 \).

Let \( \text{depth} \) denote the result of \( \text{GetDepth} \) in Algorithm 2. Then \( F_d \) is unsatisfiable only if \( \text{depth} \geq \text{depth} \). Theorem 3 shows that there exists an integer \( d \) such that \( \text{Pr} (\text{depth} < d) < 0.05 \) and \( \text{Pr} (\text{depth} < d + 7) > 0.95 \), i.e., \( \text{Pr} (d \leq \text{depth} \leq d + 7) > 0.9 \). So in most cases, the value of \( \text{depth} \) lies in an interval \([d, d + 7]\]. Also, it is easy to see that \( \log_2 \#F \) lies in this interval as well. The following theorem is obvious now.

**Theorem 4.** Algorithm 2 runs in time linear in \( \log_2 \#F \) relative to an NP-oracle.

6 Evaluation

To evaluate the performance and effectiveness of our approach, two prototype implementations \( \text{STAC\_CNF} \) and \( \text{STAC\_BV} \) with dynamic stopping criterion for propositional logic formulas and SMT(BV) formulas are built respectively. We considered a wide range of benchmarks from different domains: grid networks, plan recognition, DQMR networks, Langford sequences, circuit synthesis, random 3-CNF, logistics problems and program synthesis [8, 10, 27, 33]¹. For lack of

¹ Our tools \( \text{STAC\_CNF} \) and \( \text{STAC\_BV} \) and the suite of benchmarks are available at [https://github.com/bearben/STAC](https://github.com/bearben/STAC).
space, we only list a part of results here. All our experiments were conducted on a single core of an Intel Xeon 2.40 GHz (16 cores) machine with 32 GB memory and CentOS6.5 operating system.

6.1 Quality of Approximation

Recall that our approach is based on Eq. (1) which has not been proved. So we would like to see whether the approximation fits the bound. We experimented 100 times on each instance.

Table 1. Statistical results of 100-times experiments on STAC_CNF ($\epsilon = 0.8$, $\delta = 0.2$)

| Instance     | $n$   | $\#F$     | $[1.8^{-1}\#F, 1.8\#F]$ | Freq. | $\bar{t}$ (s) | $\bar{T}$ | $\bar{Q}$ |
|--------------|-------|------------|--------------------------|-------|---------------|-----------|-----------|
| special-1    | 20    | $1.0 \times 10^6$ | $[5.8 \times 10^5, 1.9 \times 10^6]$ | 82    | 0.3           | 12.2      | 86.7      |
| special-2    | 20    | 1          | $[0.6, 1.8]$             | 86    | 0.6           | 12.6      | 37.6      |
| special-3    | 25    | $3.4 \times 10^7$ | $[1.9 \times 10^7, 6.0 \times 10^7]$ | 82    | 11.2          | 11.8      | 90.1      |
| 5step        | 177   | $8.1 \times 10^4$ | $[4.5 \times 10^4, 1.5 \times 10^5]$ | 90    | 0.1           | 11.9      | 80.5      |
| blockmap_05_01 | 1411  | $6.4 \times 10^2$ | $[3.6 \times 10^2, 1.2 \times 10^3]$ | 84    | 1.1           | 12.0      | 73.8      |
| blockmap_05_02 | 1738  | $9.4 \times 10^6$ | $[5.2 \times 10^6, 1.7 \times 10^7]$ | 89    | 12.7          | 11.8      | 87.7      |
| blockmap_10_01 | 11328 | $2.9 \times 10^6$ | $[1.6 \times 10^6, 5.2 \times 10^6]$ | 83    | 80.3          | 12.0      | 85.0      |
| fs-01        | 32    | $7.7 \times 10^2$ | $[4.3 \times 10^2, 1.4 \times 10^3]$ | 80    | 0.02          | 12.6      | 76.2      |
| or-50-10-10-UC-20 | 100  | $3.7 \times 10^6$ | $[2.0 \times 10^6, 6.6 \times 10^6]$ | 77    | 7.7           | 12.0      | 86.1      |
| or-60-10-10-UC-40 | 120  | $3.4 \times 10^6$ | $[1.9 \times 10^6, 6.1 \times 10^6]$ | 91    | 3.5           | 12.1      | 86.0      |

In Table 1, column 1 gives the instance name, column 2 the number of Boolean variables $n$, column 3 the exact counts $\#F$, and column 4 the interval $[1.8^{-1}\#F, 1.8\#F]$. The frequencies of approximations that lie in the interval $[1.8^{-1}\#F, 1.8\#F]$ in 100 times of experiments are presented in column 5. The average time consumptions, average number of iterations, and average number of SAT query invocations are presented in columns 6, 7 and 8 respectively, which also indicate the advantages of our approach.

Under the dynamic stopping criterion, the counts returned by our approach should lie in an interval $[1.8^{-1}\#F, 1.8\#F]$ with probability 80% for $\epsilon = 0.8$ and $\delta = 0.2$. The statistical results in Table 1 show that the frequencies are around 80 for 100-times experiments which fit the 80% probability. The average number of iterations $\bar{T}$ listed in Table 1 is smaller than the loop termination criterion $T = 22$ which is obtained via Formula 2, indicating that the dynamic stopping technique significantly improves the efficiency. In addition, the values of $\bar{T}$ appear to be stable for different instances, hinting that there exists a constant upper bound on $T$ which is irrelevant to instances.

Intuitively, our approach may start to fail on “loose” formulas, i.e., with an “infinitesimal” fraction of non-models. Instance special-1 and special-3 are
such “loose” formulas where special-1 has $2^{20}$ models with only 20 variables and special-3 has $2^{25} - 1$ models with 25 variables. Instance special-2 is another extreme case which only has one model. The results in Table 1 demonstrate that STAC\_CNF also works fine on these extreme cases.

**Table 2.** Statistical results of 100-times experiments on STAC\_CNF ($\epsilon = 0.2, \delta = 0.1$)

| Instance         | $n$ | $\#F$          | $[1.2^{-1}\#F, 1.2\#F]$ | Freq. | $\bar{t}$ (s) | $T$   | $Q$   |
|------------------|-----|----------------|--------------------------|-------|---------------|-------|-------|
| special-1        | 20  | $1.0 \times 10^6$ | $[8.7 \times 10^5, 1.3 \times 10^6]$ | 86 | 4.0 | 179 | 1023 |
| special-2        | 20  | 1              | $[0.8, 1.2]$             | 91 | 0.1 | 179 | 540  |
| special-3        | 25  | $3.4 \times 10^7$ | $[2.8 \times 10^7, 4.0 \times 10^7]$ | 91 | 138 | 178 | 1029 |
| 5step            | 177 | $8.1 \times 10^4$ | $[6.8 \times 10^4, 9.8 \times 10^5]$ | 96 | 1.9 | 190 | 1078 |
| blockmap\_05\_01 | 1411| $6.4 \times 10^2$ | $[5.3 \times 10^2, 7.7 \times 10^2]$ | 94 | 17.1 | 190 | 1069 |
| blockmap\_05\_02 | 1738| $9.4 \times 10^6$ | $[7.9 \times 10^6, 1.1 \times 10^7]$ | 87 | 281 | 193 | 1088 |
| blockmap\_10\_01 | 11328| $2.9 \times 10^6$ | $[2.4 \times 10^6, 3.5 \times 10^6]$ | 93 | 1371 | 180 | 1034 |
| fs-01            | 32  | $7.7 \times 10^2$ | $[6.4 \times 10^2, 9.2 \times 10^2]$ | 91 | 0.1 | 172 | 975  |
| or-50-10-10-UC-20 | 100 | $3.7 \times 10^6$ | $[3.1 \times 10^6, 4.4 \times 10^6]$ | 90 | 140 | 166 | 925  |
| or-60-10-10-UC-40 | 120 | $3.4 \times 10^6$ | $[2.8 \times 10^6, 4.1 \times 10^6]$ | 92 | 66 | 167 | 949  |

**Table 3.** Statistical results of 100-times experiments on STAC\_BV ($\epsilon = 0.8, \delta = 0.2$)

| Instance         | TB | $\#F$          | $[1.8^{-1}\#F, 1.8\#F]$ | Freq. | $\bar{t}$ (s) | $T$   | $Q$   |
|------------------|----|----------------|--------------------------|-------|---------------|-------|-------|
| FINDpath1        | 32 | $4.1 \times 10^6$ | $[2.3 \times 10^6, 7.3 \times 10^6]$ | 83 | 27.5 | 12.4 | 88.0 |
| queue            | 16 | $8.4 \times 10^4$ | $[4.7 \times 10^4, 1.5 \times 10^5]$ | 75 | 1.7 | 12.0 | 70.6 |
| getopPath2       | 24 | $8.1 \times 10^3$ | $[4.5 \times 10^3, 1.5 \times 10^4]$ | 88 | 2.7 | 12.2 | 79.5 |
| coloring\_4     | 32 | $1.8 \times 10^9$ | $[1.0 \times 10^9, 3.3 \times 10^9]$ | 76 | 51.9 | 12.0 | 96.1 |
| FISCHER2-7-fair  | 240| $3.0 \times 10^4$ | $[1.7 \times 10^4, 5.4 \times 10^4]$ | 79 | 149 | 11.8 | 79.8 |
| case2            | 24 | $4.2 \times 10^6$ | $[2.3 \times 10^6, 7.6 \times 10^6]$ | 79 | 16.5 | 12.4 | 89.3 |
| case4            | 16 | $3.3 \times 10^4$ | $[1.8 \times 10^4, 5.9 \times 10^4]$ | 87 | 2.2 | 12.5 | 85.2 |
| case7            | 18 | $1.3 \times 10^5$ | $[7.3 \times 10^4, 2.4 \times 10^5]$ | 83 | 2.9 | 12.4 | 84.1 |
| case8            | 24 | $8.4 \times 10^6$ | $[4.7 \times 10^6, 1.5 \times 10^7]$ | 82 | 14.4 | 12.1 | 91.1 |
| case11           | 15 | $1.6 \times 10^4$ | $[9.1 \times 10^3, 2.9 \times 10^4]$ | 76 | 2.1 | 12.0 | 81.2 |

We considered another pair of parameters $\epsilon = 0.2, \delta = 0.1$. Then the interval should be $[1.2^{-1}\#F, 1.2\#F]$ and the probability should be 90%. Table 2 shows the results on such parameter setting. The frequencies that the approximation lies in interval $[1.2^{-1}\#F, 1.2\#F]$ are all around or over 90 which fits the 90% probability.

We also conducted 100-times experiments on SMT(BV) problems and the results show that STAC\_BV is also promising. Table 3 similarly shows the results
of 100-times experiments on STAC_BV. Its column 2 gives the sum of widths of all bit-vector variables (Boolean variable is counted as a bit-vector of width 1) instead. The statistical results demonstrate that the dynamic stopping criterion is also promising on SMT(BV) problems.

6.2 Performance Comparison with $(\epsilon, \delta)$-counters

We compared our tools with ApproxMC2 [11] and SMTApproxMC [8] which are hashing-based $(\epsilon, \delta)$-counters. Both STAC_CNF and ApproxMC2 use CryptoMiniSAT [35], an efficient SAT solver designed for XOR clauses. STAC_BV and SMT-ApproxMC use the state-of-the-art SMT(BV) solver Boolector [6].

![Fig. 1. Performance comparison between STAC_CNF and ApproxMC2](image1)

![Fig. 2. Performance comparison between STAC_BV and SMTApproxMC](image2)

We first conducted experiments with $\epsilon = 0.8, \delta = 0.2$ and 8 hours timeout which are also used in evaluation in previous works [8,11]. Figure 1 presents a comparison on performance between STAC_CNF and ApproxMC2. Each point represents an instance, whose $x$-coordinate and $y$-coordinate are the running times of STAC_CNF and ApproxMC2 on this instance, respectively. The figure is in logarithmic coordinates and demonstrates that STAC_CNF outperforms ApproxMC2 by about one order of magnitude. Figure 2 presents a similar comparison on performance between STAC_BV and SMTApproxMC, showing that STAC_BV outperforms SMTApproxMC by one or two orders of magnitude. Furthermore, the advantage enlarges as the scale grows.

Table 4 presents more experimental results with $(\epsilon, \delta)$ parameters other than $(\epsilon = 0.8, \delta = 0.2)$. Nine pairs of parameters were experimented. “Time Ratio” represents the ratio of the running times of ApproxMC2 to STAC_CNF. “#Calls Ratio” represents the ratio of the number of SAT calls of ApproxMC2 to STAC_CNF. The results show that ApproxMC2 gains advantage as $\epsilon$ decreases and STAC_CNF gains advantage as $\delta$ decreases. On the whole, ApproxMC2 gains advantage when
Table 4. Performance comparison between STAC-CN\(F\) and ApproxMC2 with different pairs of \((\epsilon, \delta)\) parameters

| \((\epsilon, \delta)\) | Instance | blockmap | fs-01 | 5step | ran5 | ran6 | ran7 |
|-------------------------|----------|----------|-------|-------|------|------|------|
| \((0.8, 0.3)\)          | Time Ratio | 1.11     | 3.99  | 1.22  | 3.00 | 3.83 | 6.53 | 8.24 | 5.57 |
|                         | #Calls Ratio | 22.60 | 39.02 | 17.91 | 19.12 | 23.11 | 22.53 | 21.28 | 23.68 |
| \((0.8, 0.2)\)          | Time Ratio | 1.84     | 6.16  | 2.44  | 2.80 | 6.05 | 9.61 | 15.41 | 7.37 |
|                         | #Calls Ratio | 26.70 | 34.68 | 25.16 | 33.46 | 27.24 | 33.35 | 38.22 | 30.94 |
| \((0.8, 0.1)\)          | Time Ratio | 2.27     | 7.36  | 3.72  | 5.25 | 12.62 | 9.60 | 9.54 | 8.19 |
|                         | #Calls Ratio | 44.88 | 48.26 | 40.01 | 49.40 | 43.03 | 46.12 | 44.84 | 52.63 |
| \((0.4, 0.3)\)          | Time Ratio | 0.75     | 1.37  | 0.42  | 3.00 | 5.04 | 1.97 | 2.31 | 2.74 |
|                         | #Calls Ratio | 17.75 | 36.20 | 14.69 | 16.40 | 27.63 | 21.07 | 27.34 | 24.13 |
| \((0.4, 0.2)\)          | Time Ratio | 0.77     | 1.44  | 0.86  | 4.50 | 7.70 | 2.82 | 1.77 | 3.02 |
|                         | #Calls Ratio | 20.91 | 26.35 | 29.16 | 26.72 | 40.66 | 26.49 | 27.82 | 28.94 |
| \((0.4, 0.1)\)          | Time Ratio | 1.08     | 2.57  | 1.29  | 4.90 | 7.09 | 3.84 | 3.43 | 3.11 |
|                         | #Calls Ratio | 37.16 | 46.28 | 39.40 | 31.99 | 39.36 | 41.02 | 35.88 | 34.11 |
| \((0.2, 0.3)\)          | Time Ratio | 0.42     | 0.47  | 0.23  | 5.08 | 3.79 | 1.26 | 1.14 | 1.81 |
|                         | #Calls Ratio | 13.75 | 20.82 | 24.35 | 13.37 | 19.74 | 25.20 | 19.19 | 20.06 |
| \((0.2, 0.2)\)          | Time Ratio | 0.57     | 0.92  | 0.26  | 8.42 | 3.37 | 2.07 | 1.50 | 2.45 |
|                         | #Calls Ratio | 21.80 | 29.62 | 25.60 | 21.83 | 21.59 | 25.88 | 22.72 | 22.98 |
| \((0.2, 0.1)\)          | Time Ratio | 0.87     | 0.92  | 0.44  | 16.69 | 3.17 | 3.61 | 2.27 | 2.60 |
|                         | #Calls Ratio | 27.86 | 29.91 | 33.36 | 34.17 | 31.58 | 40.81 | 29.01 | 29.90 |

\(\epsilon\) and \(\delta\) both decrease. Note that the numbers of SAT calls represent the complexity of both algorithms. In Table 4, #Calls Ratio is more stable than Time Ratio among different pairs of parameters and also different instances. It indicates that the difficulty of NP-oracle is also an important factor of running time performance.

6.3 Performance Comparison with Bounding and Guarantee-Less Counters

Since our approach is not a \((\epsilon, \delta)\)-counter in theory, we also compared STAC-CN\(F\) with bounding counters (SampleCount [22], MBound [23]) and guarantee-less counters (ApproxCount [39], SampleTreeSearch [16]). Table 5 shows the experimental results.

For SampleCount, we used \(\alpha = 2\) and \(t = 3.5\) so that \(\alpha t = 7\), giving a correctness confidence of \(1 - 2^{-7} = 99\%\). The number of samples per variable setting, \(z\), was chosen to be 20. Our results show that the lower-bound approximated by SampleCount is smaller than exact count \(#F\) by one or more orders of magnitude. We tried larger \(z\), such as \(z = 100\) and \(z = 1000\), but still failed to obtain...
| Instance | #\( n \) | SampleCount | ApproxCount | SampleTreeSearch | L-bound | Models | Time | L-bound | Time | Models | Time |
|----------|---------|-------------|-------------|-----------------|--------|--------|------|--------|------|--------|------|
| blockmap-50-01 | 1411 | 640 | ≈ 0.1 s | 1 s | > 11 s | > 64 s | 5 s | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| blockmap-50-02 | 1738 | 93 × 10^6 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| blockmap-100-01 | 11326 | 29.9 × 10^9 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| blockmap-150-01 | 33035 | 41 m | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| f3-01 | 32 | 768 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| PLAN RECOGNITION | 177 | 8.1 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| tile-1 | 352 | 7.3 × 10^8 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| tile-3 | 577 | 2.2 × 10^11 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| lang12 | 576 | 2.2 × 10^5 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| LANGFORD PROBS. | 1024 | 4.7 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| DQMR NETWORKS | 200 | 3.4 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| or-100-20-60-UC-60 | 100 | 3.1 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| or-50-20-10-UC-40 | 300 | 3.1 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| or-50-20-10-UC-30 | 100 | 6.2 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| or-60-10-10-UC-30 | 120 | 1.9 × 10^5 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| or-70-10-10-UC-40 | 140 | 1.7 × 10^5 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| or-70-10-10-UC-30 | 140 | 1.7 × 10^5 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| RANDOM 3-CNF | 30 | 1.2 × 10^6 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| ran12 | 40 | 3.5 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| ran27 | 50 | 1.3 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
| ran44 | 60 | 1.1 × 10^4 | ≈ 0.1 s | 9 s | > 11 s | > 64 s | 5 m | > 21.1 × 10^2 | 3 m | > 7.1 × 10^6 | 3 s |
a lower-bound larger than \( \#F/10 \). Moreover, there are some wrong approximations on DQMR networks problems, e.g., or-100-20-6-UC-60 only has \( 2.8 \times 10^7 \) models but SampleCount returns a lower-bound \( \geq 1.1 \times 10^{29} \). SampleCount is more efficient on Langford problems and random 3-CNF problems, but weaker on problems with a large number of variables, such as blockmap problems.

For MBound, we used \( \alpha = 1 \) and \( t = 7 \) so that \( \alpha t = 7 \), also giving a correctness confidence of \( 1 - 2^{-7} = 99\% \). MBound also employs a family of XOR hashing functions which is similar to the function used by our approach. The size of XOR constraints \( k \) should be no more than half of the number of variables \( n \), i.e., \( k \leq n/2 \). We found that XOR constraints start to fail as \( k < n/2 \). So in our experiments, \( k \) was chosen to be close to \( n/2 \). Since MBound can only check the bound and may return failure as the bound is too close to the exact count, we implemented a binary search to find the best lower-bound verified by MBound. The results in Table 5 are the best lower-bounds and the running times of the whole binary search procedure. Though the lower-bounds are better than SampleCount, they are still around \( \#F/10 \). Similar to our approach, the running times of MBound are also quite relevant to the size of \( \#F \).

For ApproxCount, we manually increased the value of “cutoff” as ApproxCount requires. Note that ApproxCount calls exact model counter Cachet [32] and Relsat [2] after formula simplifications, so it sometimes returns the exact counts, such as blockmap_05_01, blockmap_05_02, 5step and tire-1. On Langford problems and DQMR networks problems, wrong approximations were provided. On other instances, the results show that STAC_CNFS usually outperforms ApproxCount.

For SampleTreeSearch, we used its default setting about the number of samples, which is a constant. The results show that it is very efficient and provides good approximations. Our approach only outperforms SampleTreeSearch on blockmap problems which consist of a large number of variables. However, there is a lack of analysis on the accuracy of the approximation of SampleTreeSearch, i.e., no explicit relation between the number of samples and the accuracy.

7 Conclusion

In this paper, we propose a new hashing-based approximate algorithm with dynamic stopping criterion. Our approach has two key strengths: it requires only one satisfiability query for each cut, and it terminates once meeting the criterion of accuracy. We implemented prototype tools for propositional logic formulas and SMT(BV) formulas. Extensive experiments demonstrate that our approach is efficient and promising. Despite that we are unable to prove the correctness of Eq. (1), the experimental results fit it quite well. This phenomenon might be caused by some hidden properties of the hash functions. To fully understand these functions and their correlation with the model count of the hashed formula might be an interesting problem to the community. In addition, extending the idea in this paper to count solutions of other formulas is also a direction of future research.
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