Linear bound for abelian automorphisms of varieties of general type

(Version 1.0)

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The aim of this paper is to prove the following

Theorem 1. Let $G$ be a finite abelian group acting faithfully on a complex smooth projective variety $X$ of general type with numerically effective (nef) canonical divisor, of dimension $n$. Then

$$|G| \leq C(n)K_X^n,$$

where $C(n)$ depends only on $n$.

We refer to the Introduction of [Ca-Sch] for a nice account of the history for the study of bounds of automorphism groups of varieties of general type. The authors of that paper have also shown a polynomial bound for abelian automorphism groups.

To prove Theorem 1, the only major obstacle to a generalisation of our argument for surfaces $[X]$ is the lack of a theorem of minimal models in higher dimension: the basic idea is to find a pencil on $X$, whose general fibres are invariant under the action of $G$, then use induction on $n$. To do so one needs bounded globally generatedness of pluricanonical sheaves, and vanishing theorems. Unfortunately, these theorems currently exist only for varieties with extra conditions which are not preserved by fibres. Therefore we consider the problem for varieties in a more general category, as is done in [Ca-Sch]. Our main observation in Theorem 2 is that in the polynomial bound of Theorem 0.1 of [Ca-Sch], most copies of $d$ may be compensated by the ambient dimension $N$, leading thus to a linear bound.
All the coefficients in this paper are effective. As they are usually very big and probably much bigger than the reality, we have preferred not to write them down explicitly. Also no special effort has been made to optimise these coefficients, so as to give a more readable presentation.

§1. Preparation for induction

The main ingredient of the proof is the following theorem.

**Theorem 2.** Let $X \subset \mathbb{P}^N_C$ be a non-degenerated irreducible variety of dimension $n$ and degree $d$, with $\kappa(X) \geq 0$. Let $G$ be a finite abelian group acting linearly on $\mathbb{P}^{N-1}$, leaving $X$ invariant. Then

$$|G| \leq c(n, d/N) d,$$

where the coefficient $c(n, d/N)$ depends only on $n$ and $d/N$, and is increasing with both variables.

Theorem 1 follows directly from Theorem 2 and the following lemma, if we send $X$ into $\mathbb{P}_{p^m(X)}$ by the linear system $|mK_X|$ (take $d = m^nK^n_X$ and $N = p_m(X)$).

**Lemma 1.** For each positive integer $n$, there exist $M(n) \in \mathbb{N}$ and $a(n) \in \mathbb{R}^+$, such that:

Let $X$ be a smooth projective variety of general type with nef canonical divisor, of dimension $n$. Then there is an integer $m \leq M(n)$, such that $|mK_X|$ is free, $\Phi_{mK}$ is birational to its image, and

$$p_m(X) \geq a(n)m^nK^n_X.$$

**Proof.** We follow the argument of [Wi], Theorem 1.1. By [Ko], there is a $r = 2(n+2)(n+2)!$ such that $|mK_X|$ is free when $m \geq r$. As $\Phi_{rK_X}$ is then generically finite onto its image, we may take $n-1$ general divisors $D_1, \ldots, D_{n-1}$ in $|rK_X|$ such that the intersection $D_1 \cdots D_{n-1}$ is a smooth irreducible and reduced curve $C$.

By successively applying the vanishing theorem of Kawamata-Viehweg, it is easy to see that the natural map

$$H^0(X, ((n-1)r+2)K_X) \longrightarrow H^0(C, K_C + K_X|_C)$$

is surjective. As $\deg(K_C + K_X|_C) \geq 2g(C) + 1$, $K_C + K_X|_C$ is very ample on $C$, and

$$p_{(n-1)r+4}(X) \geq h^0(C, K_C + K_X|_C) \geq \frac{1}{2} \deg(K_C + K_X|_C) = \frac{1}{2}((n-1)r^n + 2r^{n-1})K^n_X.$$

This implies that $\Phi_{(n-1)r+2)K_X}$ is birational onto its image, so we may take

$$M(n) = (n-1)r + 2, \quad a(n) = \frac{1}{2} \left[ \frac{r}{(n-1)r + 2} \right]^{n-1}.$$ 

QED
§2. Proof of Theorem 2

First, we make the following remark which will be used in the argument.

**Lemma 2.** In theorem 2, we may replace "irreducible variety" by "subscheme whose components are of non-negative Kodaira dimension", under the extra condition that each component of $X$ contains a point with trivial stabiliser.

**Proof.** Let $X_1, \ldots, X_k$ be the irreducible components of $X$, $H_i$ the minimal subspace of $\mathbb{P}^{N-1}$ containing $X_i$, with $d_i = \deg(X_i)$, $N_i = \dim(H_i) + 1$. We have $\sum_i d_i = d$, $\sum_i N_i \geq N$ as $X$ is non-degenerate, hence there is an $i$ such that $d_i/N_i \leq d/N$. Let $G_i$ be the stabiliser of $X_i$. Clearly $[G : G_i] \leq d/d_i$, and $G_i$ acts faithfully on $H_i$ by hypothesis. Thus we are reduced to the study of the pair $(X_i \subset H_i, G_i)$.

Our basic tool is the induced linear actions of $G$ on the spaces $H^0(\mathcal{O}_X(m))$ for $m \geq 1$. By the hypothesis that $G$ is finite abelian, such an action is diagonalisable, hence $H^0(\mathcal{O}_X(m))$ has a basis consisting of semi-invariant vectors. Each semi-invariant corresponds to a character of $G$.

**Definition.** $H^0(\mathcal{O}_X(m))$ is called uniquely decomposable, if different semi-invariants correspond to different characters, or equivalently if there are no more than $h^0(\mathcal{O}_X(m))$ semi-invariants.

**Lemma 3.** We may assume that $H = H^0(\mathcal{O}_X(1))$ is uniquely decomposable.

**Proof.** We may obviously assume $N = \dim(H)$. Consider the decomposition of $H$ into eigenspaces $H = H_1 \oplus \cdots \oplus H_k$, with

$$N_1 \geq \ldots \geq N_r > 1, \quad N_{r+1} = \cdots = N_k = 1,$$

where $N_i = \dim(H_i)$. For each $i \leq r$, let $\pi_i : \mathbb{P}^{N-1} = \mathbb{P}(H^\vee) \longrightarrow \mathbb{P}(H_i^\vee)$ be the projection with centre $\mathbb{P}(H_i^\vee \perp)$, then let $T_i$ be a general hyperplane in $\mathbb{P}(H_i^\vee)$, and $Y_i$ the moving part of the divisor in $X$ cut out by $\pi_i^{-1}(T_i)$. Now fix $i$ such that $Y_i$ is of maximal degree among these divisors.

For an index $j \leq r$ with $j \neq i$, let $P_j$ be the minimal subspace of $\mathbb{P}(H_j^\vee)$ containing $\pi_j(Y_i)$. Then $P_j$ is at least a hyperplane of $\mathbb{P}(H_j^\vee)$, for otherwise $\deg(Y_j) > \deg(Y_i)$ as one sees by taking a general hyperplane containing $P_i$, taking into account that $Y_i$ being moving, no component of it can be contained in the fixed part of $\pi_j$.

This means that there is at most one section in $H_j$ vanishing on $Y_i$. Also, no section in $H_j$ vanishes on $Y_i$ if $j > r$, for such a section is proportionally rigid, therefore it would vanish on a Zariski dense open subset of $X$, contradicting the non-degeneratedness of $X$.

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Let \( P \) be the minimal subspace of \( \mathbb{P}^{N-1} \) containing \( Y_i \), with \( N' = \dim(P) + 1 \). As \( Y_i \) is invariant under the action of \( G \), \( P \) is the intersection of \( N - N' \) invariant hyperplanes. So from the above, we get \( n' \geq N - r \geq N/2 \). We also have
\[
n' = \dim(Y_i) = n - 1, \quad d' = \deg(Y_i) \leq d,
\]
and \( Y_i \subset P \) verifies the conditions of Lemma 2 from the easy addition formula of Kodaira dimensions. By induction,
\[
|G| \leq c(n', d'/N') \leq c(n - 1, 2d/N) d.
\]

QED

We note by \( D(X) \) the vector space \( \text{div}(X) \otimes \mathbb{Q} \), where \( \text{div}(X) \) is the additive group of Weil divisors on \( X \) (without taking linear equivalence). We are interested in the finite subset \( \Sigma \subset D(X) \) formed by the images of the \( N \) invariant hyperplane sections, \( H^0(\mathcal{O}_X(1)) \) being uniquely decomposable. For any positive integer \( m \) and \( p_1, \ldots, p_m \in \Sigma \), the sum \( p_1 + \cdots + p_m \) corresponds to an invariant divisor in \( |\mathcal{O}_X(m)| \), hence to a semi-invariant of \( H^0(\mathcal{O}_X(m)) \). Our aim is to find an \( m \) such that semi-invariants of this kind outnumber \( h^0(\mathcal{O}_X(m)) \), so that \( H^0(\mathcal{O}_X(m)) \) is not uniquely decomposable.

**Definition.** Let \( \Sigma \) be a finite set in a vector space. The **dimension** of \( \Sigma \) is the dimension of the convex hull of \( \Sigma \). Also, we define
\[
m\Sigma = \left\{ \sum_{i=1}^{m} p_i | p_i \in \Sigma \right\}
\]
to be the set of all sums of \( m \) points in \( \Sigma \). The cardinal of \( \Sigma \) will be denoted by \( |\Sigma| \).

**Lemma 4.** If \( \Sigma \) is a finite set of dimension \( \delta \), then
\[
|m\Sigma| \geq \binom{m + \delta - 1}{\delta} |\Sigma| - \binom{m + \delta - 1}{\delta - 1} (m - 1)
\]
for any \( m \geq 2 \).

**Proof.** Choose a point \( p_0 \in \Sigma \) which is a summet of the convex hull of \( \Sigma \), and let \( \Sigma' = \Sigma \setminus \{p_0\} \). When \( |\Sigma| \geq \delta + 1 \), we may assume that \( \Sigma' \) is also of dimension \( \delta \). We can find \( \delta \) points \( p_1, \ldots, p_\delta \in \Sigma' \) such that the convex hull of \( \{p_0, p_1, \ldots, p_\delta\} \) is a simplex polyhedron whose intersection with the convex hull of \( \Sigma' \) is just the face generated by \( p_1, \ldots, p_\delta \). In particular, all points of the form \( \sum_{i=0}^{\delta} n_i p_i \) with \( n_0 > 0, n_i \geq 0 \), \( \sum_{i=0}^{\delta} n_i = m \) are in \( m\Sigma \) but not \( m\Sigma' \). As the number of such points equals the number of \( \delta + 1 \)-partitions of \( m - 1 \), we get
\[
|m\Sigma| \geq |m\Sigma'| + \binom{m + \delta - 1}{\delta}.
\]
The lemma follows by induction. QED
Remark. The inequality of the lemma remains true when $\Sigma$ is a set of dimension $> \delta$: one has only to take a generic projection of $\Sigma$ to a space of dimension $\delta$.

Lemma 5. Let $\Lambda$ be a linear system of affine dimension $N$ on a projective variety $X$. Let $\{D_1, \ldots, D_k\}$ be a set of generators of $\Lambda$, and $p_i$ the point in $D_i$ corresponding to $D_i$ for each $i$. Let $\delta$ be the dimension of the set $\{p_1, \ldots, p_k\}$. Then map $\Phi_{\Lambda}: X \longrightarrow \mathbb{P}^{N-1}$ factors through a rational map $\psi: X \longrightarrow \mathbb{P}^\delta$.

Proof. We may obviously assume $k = N$. Let $A$ be the additive subgroup of $D(X)$ of rank $\delta$ generated by all the vectors $v_{ij} = p_i - p_j$, and $\{u_1, \ldots, u_\delta\}$ a basis of $A$. Let $f_i$ be the rational function on $X$ corresponding to $u_i$. We have a rational map $\psi: X \longrightarrow \mathbb{P}^\delta$ defined by $(1 : f_1 : \cdots : f_\delta)$.

Now for any $i$ and $j$, we have $v_{ij} = \sum_k n_{ijk} u_k$ with $n_{ijk} \in \mathbb{Z}$, so if $g_{ij}$ is the rational function corresponding to $v_{ij}$, we have $g_{ij} = \prod_{k=1}^\delta f_k^{n_{ijk}}$. $\Phi_{\Lambda}$ is defined, say, by $(1 : g_{21} : \cdots : g_{\delta 1})$. Therefore if we define a rational map

$$\alpha: \mathbb{P}^\delta \longrightarrow \mathbb{P}^{N-1}$$

by

$$\alpha(1 : x_1 : \cdots : x_\delta) = (1 : \prod_{k=1}^\delta x_k^{n_{11k}} : \prod_{k=1}^\delta x_k^{n_{21k}} : \cdots : \prod_{k=1}^\delta x_k^{n_{\delta 1k}}),$$

we have $\Phi_{\Lambda} = \alpha \circ \psi$ birationally. QED

Lemma 6. Let $h_m(d, n) = \left(\binom{m+n-1}{n} d + \binom{m+n-2}{n-2}\right)$. For each positive integer $m$, we have

$$h^0(\mathcal{O}_X(m)) \leq h_m(d, n).$$

Proof. This follows by double induction on $m$ and $n$, and the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X(m-1)) \longrightarrow H^0(\mathcal{O}_X(m)) \longrightarrow H^0(\mathcal{O}_Y(m)),$$

where $Y$ is a general hypereplane section of $X$. By additivity of Kodaira dimensions, the induction on $n$ starts from curves of genus $\geq 1$. QED

Lemma 7. Let $m$ be an integer greater than or equal to $(n+1)\frac{d}{N-n-1} - n$. Then $H^0(\mathcal{O}_X(m))$ is not uniquely decomposable under the induced action of $G$. 

— 5 —
Proof. Let $\Sigma \in D(X)$ be the finite set corresponding to invariant hyperplane sections. As different points in $m\Sigma$ correspond to non-proportional semi-invariants in $H^0(\mathcal{O}_X(m))$, we have only to show $|m\Sigma| > h_m(d, n)$. But we have $|m\Sigma| \geq \left(\frac{m+n}{n+1}\right) N - \left(\frac{m+n}{n}\right)(m-1)$ by Lemma 4, as the dimension of $\Sigma$ is at least $n+1$ due to Lemma 5. The rest is straightforward computation. QED

Now to use induction on $n$ to prove Theorem 2, choose an $m < (n+1)(n+2)\frac{d}{N}$ as in Lemma 7, $F$ a general memeber of the moving part of a pencil in $|mD|$ corresponding to a plane in $H^0(\mathcal{O}_X(m))$ consisting of semi-invariants. $F$ is invariant under the action of $G$. As $F$ is moving while $X$ has only finitely many invariant hyperplane sections due to Lemma 3, $F$ is non-degenerated in $\mathbb{P}^{N-1}$. Therefore the couple $(F \subset \mathbb{P}^{N-1}, G)$ satisfies the conditions of Lemma 2, with dim$(F) = n - 1$ and deg$(F) \leq [(n+1)(n+2)\frac{d}{N}] d$, so

$$|G| \leq \left[c \left(n-1, (n+1)(n+2)\left(\frac{d}{N}\right)^2\right)(n+1)(n+2)\frac{d}{N}\right] d$$

by induction.

Finally, taking into account the proof of Lemma 3, Theorem 2 is shown by setting

$$c(n, \frac{d}{N}) = \max \left\{c(n-1, 2\frac{d}{N}), c \left(n-1, (n+1)(n+2)\left(\frac{d}{N}\right)^2\right)(n+1)(n+2)\frac{d}{N}\right\}.$$ 

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