EQUIVALENCE OF CONTACT GLUING MAPS IN SUTURED FLOER HOMOLOGY

RYAN LEIGON AND FEDERICO SALMOIRAGHI

Abstract. We show that the contact gluing map of Honda, Kazez, and Matić has a natural algebraic description. In particular, we establish a conjecture of Zarev, that his gluing map on sutured Floer homology is equivalent to the contact gluing map.

1. Introduction

The contact invariant in Heegaard Floer homology has profoundly advanced our understanding of contact geometry in dimension three. Its first iteration was introduced by Ozsváth and Szabó [OS05] for closed contact 3-manifolds, while Honda, Kazez, and Matić [HKM09] extended it to contact 3-manifolds $(M,\xi)$ with convex boundary. In the latter setting the contact invariant is denoted $EH(\xi)$ and takes values in the sutured Floer homology $\text{SFH}(-M, -\Gamma)$ of the manifold $M$ with reversed orientation and dividing set $\Gamma$ induced by $\xi$. These and other versions of the contact invariant have proven to be extremely useful in different contexts. Ghiggini [Ghi05, Ghi06b, Ghi06a] has used them to study the fillability of closed 3-manifolds. Invariants commonly referred to as LOSS invariants for Legendrian and transverse knots were introduced by Lisca, Ozsváth, Stipsicz, and Szabó [LOSS09]; connections with the invariant defined by Honda, Kazez, and Matić were first shown by Stipsicz and Vértesi [SV09], and extended by Etnyre, Vela-Vick, and Zarev [EVZ17].

The importance of contact invariants makes understanding their behaviour under various topological operations desirable. In [HKM08] Honda, Kazez, and Matić constructed a map $\Phi_\xi$, commonly referred to as the HKM map, associated to a proper inclusion of sutured manifolds $(M, \Gamma) \subset (M', \Gamma')$ and a compatible contact structure $\xi$ on the complement $M' \setminus \text{Int}(M)$. When $M' \setminus \text{Int}(M)$ has no connected components disjoint from $\partial M'$, the gluing map takes the form

$$\Phi_\xi : \text{SFH}(-M, -\Gamma) \rightarrow \text{SFH}(-M', -\Gamma').$$

This map preserves the contact invariant $EH$ in the following way. If $\zeta$ is a contact structure on $(M', \Gamma')$ compatible with $\xi$, then

$$\Phi_\xi(EH(\zeta)) = EH(\zeta \cup \xi).$$

The HKM map is also functorial in the sense that it satisfies identity and composition laws; see Section 6 of [HKM08].

Because of these properties, HKM maps play an important role in both sutured Floer theory and contact geometry. Of note, Juhász [Juh16] employed them in his construction of cobordism maps on sutured Floer homology and Golla [Gol15] used HKM maps to relate

\footnote{Homology groups are over $\mathbb{F}_2$-coefficients through the entirety of this paper.}
the LOSS invariant for Legendrian knots in $S^3$ with the invariant defined by Honda, Kazez, and Matić. HKM maps can often track contact invariants under cut-and-paste operations. In this regard, Massot [Mas12] used them to systematically produce isotopy classes of universally tight, torsion-free contact structures with vanishing Ozsváth–Szabó contact invariant. Recently, Juhász and Zemke [JZ20] gave an alternate description of the HKM map in terms of contact handles and used it to prove several results about cobordisms maps in sutured Floer homology.

Outside the realm of contact geometry, Zarev [Zar11] defined a gluing operation $\cup F$ for sutured manifolds $(M_1, \Gamma_1), (M_2, \Gamma_2)$ with suitable sutured subsurfaces $\mathcal{F} \subset \partial M_1, \mathcal{F} \subset \partial M_2$. He also defined an associated gluing map which takes the form of a pairing $\Psi_{\mathcal{F}}: \text{SFH}(M_1, \Gamma_1) \otimes \text{SFH}(M_2, \Gamma_2) \rightarrow \text{SFH}((M_1, \Gamma_1) \cup F (M_2, \Gamma_2))$.

If $M_1$ is equipped with a contact structure $\xi$ compatible with $\Gamma_1$, then we can define a map which is formally similar to the HKM map $\Psi_{\xi}: \text{SFH}(-M_2, -\Gamma_2) \rightarrow \text{SFH}((-M_1, -\Gamma_1) \cup -\mathcal{F} (-M_2, -\Gamma_2))$ by $\Psi_{\xi}(y) = \Psi_{-\mathcal{F}}(\text{EH}(\xi) \otimes y)$. We will refer to $\Psi_{\xi}$ as the bordered contact gluing map, since Zarev used bordered sutured theory to define the map $\Psi_{\mathcal{F}}$.

Due to its algebraic nature, it is not clear a priori whether this map deserves the moniker of contact gluing map. Indeed, it is not apparent from the definition that $\Psi_{\xi}$ sends contact invariants to contact invariants. Despite this, Zarev claimed that the HKM and bordered contact gluing maps are equivalent. The main result of this paper affirms this conjecture; we give a rough formulation here and the precise statement in Theorem 6.3.

**Theorem 1.** Suppose that $\Phi_{\xi}$ is the HKM map for a proper inclusion $(M, \Gamma) \subset (M', \Gamma')$ of sutured manifolds with no isolated components and compatible contact structure $\xi$. There is a sutured contact manifold $(M'', \Gamma'', \xi'')$ and graded isomorphisms $f, g$ such that $(M', \Gamma') \cong (M'', \Gamma'') \cup_{\mathcal{F}} (M, \Gamma)$, the contact structures $\xi$ and $\xi''$ have equivalent contact handle decompositions, and the following diagram commutes.

\[
\begin{array}{ccc}
\text{SFH}(-M, -\Gamma) & \xrightarrow{f} & \text{SFH}(-M, -\Gamma) \\
\downarrow \Phi_{\xi} & & \downarrow \Psi_{\xi''} \\
\text{SFH}(-M', -\Gamma') & \xrightarrow{g} & \text{SFH}((-M'', -\Gamma'') \cup_{\mathcal{F}} (-M, -\Gamma))
\end{array}
\]

The strategy for attacking this theorem is as follows. We first describe the HKM maps for contact handle attachments in terms of certain diagrammatic maps for contact handles. We then identify these diagrammatic maps with bordered contact gluing maps for contact handle attachments. Next, we prove that an arbitrary bordered contact gluing map is a composition of such handle attachment maps; along the way, we must show that $\Psi_{\xi}$ maps contact invariants to contact invariants and satisfies an appropriate identity law. Combining all these results with the functorial properties of the HKM map yields Theorem 1.

**Remark 1.1.** Note that the hypothesis of Theorem 1 requires that $M' \setminus \text{int}(M)$ have no components disjoint from $\partial M'$ (no “isolated components”); this ensures that $\xi$ can be decomposed using only contact 1-and 2-handles.

**Remark 1.2.** It is natural to ask whether the graded isomorphisms in Theorem 1 can be chosen to be identity maps. As of the writing of this paper, the map $\Psi_{\mathcal{F}}$ is only known to
be well-defined up to maps induced by graded homotopy equivalence, since it is defined using bordered sutured theory. A stronger version of Theorem 1 would require naturality of bordered sutured Floer homology.

Theorem 1 has several interesting consequences. First, we obtain a bordered version of the HKM map.

**Corollary 1.3.** Let $\mathcal{M} = (M, \gamma, \mathcal{F}, \mathcal{Z})$ and $\mathcal{M}' = (M', \gamma', \mathcal{F}, \mathcal{Z})$ be bordered sutured manifolds with $M \subset M'$ and $M' \setminus \text{int}(M)$ a sutured manifold equipped with a compatible contact structure $\xi$. Then there exists a map of type-D structures induced by $\xi$

$$\Phi_\xi : \text{BSD}(-\mathcal{M}) \to \text{BSD}(-\mathcal{M}')$$

which is natural with respect to gluing bordered sutured manifolds along $\mathcal{F}$.

Similar statements can be formulated for type-A structures and the various bimodule structures involved in the bordered sutured theory. Of note, Etnyre, Vela-Vick, and Zarev [EVZ17] used this corollary as a fundamental tool to prove their alternate characterization of the minus version of knot Floer homology. As Mathews [Mat19] points out, this corollary also has interesting implications for the relationship between sutured Floer homology and the contact category defined by Honda and Tian [HT16].

Theorem 1 also provides an alternate proof of Juhász and Zemke’s description of contact gluing maps for contact handle attachments found in [JZ20]. Honda, Kazez, and Matić’s original description of their gluing map required Heegaard diagrams to satisfy a number of technical conditions, collectively referred to as contact-compatibility. Even for contact handle attachments, the use of contact-compatible Heegaard diagrams often renders the associated HKM gluing map uncomputable on the level of chain complexes. As a result, most prior applications relied only on formal properties of the HKM maps. However, there are a priori diagram-dependent maps $\sigma_i$ associated to contact handle attachments which are simple to compute at the level of chain complexes; see Section 2. It is possible to describe a given HKM map in terms of diagrammatic maps; see Lemma 2.8. It turns out that these diagrammatic maps are well-defined up to graded isomorphism. This means that an arbitrary diagrammatic map induces the HKM up to graded isomorphism.

**Corollary 1.4.** The diagrammatic maps $\sigma_i$ are well-defined up to graded isomorphism. In particular, any diagrammatic map induces the corresponding HKM map up to graded isomorphism.

Finally, we use Corollary 1.4 to provide an algorithm for computing the HKM map combinatorially following Plamenevskaya’s work in [Pla07].

**Corollary 1.5.** Given a proper inclusion of sutured manifolds $(M, \Gamma) \subset (M', \Gamma')$ with no isolated components and compatible contact structure $\xi$ on $M' \setminus \text{int}(M)$, there are nice Heegaard diagrams $\mathcal{H}, \mathcal{H}'$ for $(-M, -\Gamma), (-M', -\Gamma')$ respectively and an inclusion of complexes $\text{SFC}(\mathcal{H}) \to \text{SFC}(\mathcal{H}')$ which can be used to compute the HKM map $\Phi_\xi : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M', -\Gamma')$ up to graded isomorphism.

**Organization.** In Section 2, we introduce the diagrammatic maps and relate them to the corresponding HKM maps; see Lemma 2.8. In Section 3, we review the bordered sutured theory that goes into Zarev’s gluing map $\Psi_F$ on sutured Floer homology. We define the associated contact gluing map $\Psi_\xi$ in Section 4 and prove that it satisfies the composition
Section 5 reviews more background on making concrete computations with the bordered gluing map and using the formula in Lemma 5.1 in particular. We relate the diagrammatic maps with the bordered contact gluing map in Section 6 by Lemma 6.1 and Lemma 6.2 and use them to prove two versions of Theorem 1. We discuss corollaries and applications in Section 7.

Acknowledgment. The authors would sincerely like to express their gratitude to their advisor Shea Vela-Vick for introducing them to the subject, for suggesting the present problem, and for his patient support throughout the development of the project. Without his guidance this work would not have been possible. They are also indebted to Mike Wong for his invaluable encouragement and suggestions for improving various versions of the manuscript. His wisdom has contributed to the quality of the exposition and the spirits of the authors. Finally, the authors would like to thank John Etnyre, Ko Honda, and Robert Lipshitz, for the illuminating conversations.

2. HKM Handle Attachments

The goal of this section is to describe the HKM maps associated to contact $i$-handle attachments in terms of the diagram-dependent maps $\sigma_i$. The ideas described here are familiar to experts, though we record them for the sake of clarity and self-containment.

Remark 2.1. Note that the discussion in this section parallels many ideas in [JZ20]. In particular, our padded contact handles are analogous their Morse-type contact handles and Lemma 2.8 is analogous to their Theorem 5.8.

We assume familiarity with some basic ideas in contact geometry, particularly in convex surface theory. Throughout this paper, the contact structures we consider are defined up to isotopy. As such, we make implicit use of Giroux’s uniqueness and flexibility theorems; see [Gir91]. We also assume familiarity with Heegaard diagrams arising from partial open books, though we review some key features in Subsection 2.2 to clarify notation. We refer to such diagrams as partial open book diagrams; we will represent a partial open book diagram as in [HKM08] by the quotient of a surface $S$ with embedded $\beta$-arcs and $\alpha$-arcs. The full Heegaard surface $\Sigma$ is formed by identifying pairs of intervals in $\partial S$, which we
shade black, so that each arc becomes a closed curve after identification. All our pictures represent diagrams \((\Sigma, \alpha, \beta)\) for \((M, -\Gamma)\) with the positive orientation of \(\Sigma\) as depicted in Figure 7. We reverse the roles of \(\alpha\)-curves and \(\beta\)-curves, and we think of these as diagrams \((\Sigma, \beta, \alpha)\) for \((-M, -\Gamma)\), as is standard when working with partially open books.

\[\begin{array}{c}
\begin{array}{c}
\text{Figure 2. A diagrammatic contact 1-handle attachment.}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Figure 3. A diagrammatic contact 2-handle attachment.}
\end{array}
\end{array}\]

2.1. Diagrammatic maps for handles and bypasses. We need to define diagrammatic maps associated to contact handle attachments in order to relate the HKM and bordered contact gluing maps. We refer to the work of Giroux [Gir91] and Ozbagci [Ozb11] for the notion of contact handles in dimension three. Contact 1- and 2-handles are shown in Figure 1 along with their dividing sets. Denote the result of attaching a contact \(i\)-handle \(h^i\) to \((M, \Gamma, \xi)\) by \((M^i, \Gamma^i, \xi^i)\). Note too that we can attach a contact handle to a balanced sutured manifold \((M, \Gamma)\). Identify a neighborhood of \(\partial M\) with \(\partial M \times [-1, 0]\) and let \(\xi_G\) be the \([-1, 0]\)-invariant contact structure compatible with \(\Gamma\). Then we can attach \(h^i\) to \(\partial M = \partial M \times \{0\}\), and the induced dividing set will make the result a balanced sutured manifold, denoted \((M^i, \Gamma^i)\).

Let \(H = (\Sigma, \beta, \alpha)\) be a diagram for \((-M, -\Gamma)\). Recall that we recover \((-M, -\Gamma)\) from \(H\) by attaching 2-handles along \((\beta \times \{0\}) \cup (\alpha \times \{1\})\) to \((\Sigma \times [0, 1], \partial \Sigma \times \{\frac{1}{2}\})\). With this identification, a contact 1-handle attachment along \((\{p\} \times \{\frac{1}{2}\}) \cup (\{q\} \times \{\frac{1}{2}\}) \subset \Gamma\), corresponds to attaching a 2-dimensional 1-handle to \(\Sigma\) along \(\{p, q\}\) to obtain a new surface \(\Sigma_1\). Then \(H_1 = (\Sigma_1, \beta, \alpha)\) is a diagram for \((-M_1, -\Gamma_1)\); see Figure 2. A contact 2-handle attached along a curve \(\delta\) with \(\delta \cap \Gamma = \{p\} \times \{\frac{1}{2}\} \cup (\{q\} \times \{\frac{1}{2}\}) \subset \Gamma\), effects the following change at the level of diagrams. Attach a 2-dimensional 1-handle to \(\Sigma\) along \(\{p, q\}\) to obtain a new surface \(\Sigma_2\); call the core of the 1-handle \(\lambda\). Define arcs \(b = \delta \cap R_+(\Gamma)\) and \(a = \delta \cap R_-(\Gamma)\), and let \(\beta_0 = b \cup \lambda\), \(\alpha_0 = a \cup \lambda\). Perturb these curves so that \(\beta_0\) and \(\alpha_0\) intersect a single time.

\(^2\)While \(\Sigma\) is usually defined by gluing a copy of a subsurface \(Q \subset S\) to \(S\) along \(\partial Q \cap \partial S\), the condition that the \(\alpha\)-arcs in \(S\) form a basis, guarantees that the diagram \((S \cup Q, \beta, \alpha)\) is diffeomorphic to \((S/\sim, \beta, \alpha)\), where \(\sim\) is the relation which identifies pairs of intervals in \(\partial S\).
positively in the 1-handle. Then \( H_2 = (\Sigma_2, \beta \cup \beta_0, \alpha \cup \alpha_0) \) is a diagram for \((-M_2, -\Gamma_2)\); see Figure 3.

We refer to these operations on diagrams as \textit{diagrammatic contact handle attachments}. Furthermore, if \( \mathcal{H} \) is a partial open book diagram compatible with \( \xi \), then the result of a diagrammatic handle attachment \( \mathcal{H}_i \) is a partial open book diagram compatible with \( \xi_i = \xi \cup h^i \); see the proof of Proposition 2.6 or [HKM09] for a more detailed discussion. Additionally, if \( x \) is the canonical collection of intersection points representing \( \text{EH}(\xi) \), then \( x \) represents \( \text{EH}(\xi_1) \) and \( (x, x_0) \) represents \( \text{EH}(\xi_2) \). In the case of the 2-handle, call \( x_0 \) the \textit{preferred intersection point} of \( \mathcal{H}_2 \); this parallels the terminology in Subsection 2.2.

In general, a diagrammatic handle attachment induces an obvious inclusion of the associated sutured Floer complexes.

\textbf{Definition 2.2.} Let \( \mathcal{H} \) be a Heegaard diagram for \((-M, -\Gamma)\). Let \( \mathcal{H}_i \) be a diagram for \((-M_i, -\Gamma_i)\) obtained from \( \mathcal{H} \) by a diagrammatic contact \( i \)-handle attachment. For \( i = 1 \), the associated \textit{diagrammatic map} is the map \( \sigma_1 : \text{SFC}(\mathcal{H}) \to \text{SFC}(\mathcal{H}_1) \) given by \( \sigma_1(y) = (y) \). For \( i = 2 \), the associated \textit{diagrammatic map} is the map \( \sigma_2 : \text{SFC}(\mathcal{H}) \to \text{SFC}(\mathcal{H}_2) \) given by \( \sigma_2(y) = (y, x_0) \).

Note that these maps depend on a choice of diagram, and it is not clear a priori whether or not they induce well-defined maps on \( \text{SFH} \). We will show in the course of this paper that the diagrammatic maps for \( i \)-handle attachments are well-defined up to graded homotopy equivalence and induce maps on homology which are equivalent to the HKM contact gluing maps for the corresponding handle attachments; see Corollary 1.4. Note that Juhasz and Zemke have shown in [JZ20] that this is the case, though our proof is independent from theirs.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{positive_trivial_bypass.png}
\caption{A diagrammatic positive trivial bypass.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{negative_trivial_bypass.png}
\caption{A diagrammatic negative trivial bypass.}
\end{figure}

In the course of identifying these diagrammatic maps with the HKM map, we will also need maps associated to trivial bypasses. Attaching a bypass can be viewed as attaching a
contact 1-handle followed by a contact 2-handle; as such, trivial bypasses have associated diagrammatic maps. A diagrammatic positive trivial bypass attachment is depicted in Figure 4 and a diagrammatic negative trivial bypass attachment is depicted in Figure 5.

Definition 2.3. Let $\mathcal{H}$ be a Heegaard diagram for $(-M, -\Gamma)$, and let $\mathcal{H}_{12}$ be a diagram for $(-M, -\Gamma)$ obtained from $\mathcal{H}$ by a diagrammatic trivial bypass attachment. The associated diagrammatic map is the map $\tau : \text{SFC}(\mathcal{H}) \to \text{SFC}(\mathcal{H}_{12})$ given by $\tau(y) = (y, x_0)$.

We will need the following standard computation; a stronger version can be found in [HKM08].

Lemma 2.4. The map $\tau : \text{SFC}(\mathcal{H}) \to \text{SFC}(\mathcal{H}_{12})$ associated to a diagrammatic trivial bypass is a graded isomorphism of complexes.

Proof. For a positive bypass, $x_0$ is the only intersection point in the new $\alpha$-curve; see Figure 4. For a negative bypass, $x_0$ is the only intersection point in the new $\beta$-curve; see Figure 5. In either case, the map $\tau$ thus gives a one-to-one correspondence between generators of $\mathcal{H}$ and generators of $\mathcal{H}_{12}$. Furthermore, holomorphic disks in $\mathcal{H}_{12}$ correspond exactly to holomorphic disks in $\mathcal{H}$. □

2.2. HKM maps for contact handle attachments. Since HKM maps are only defined for proper inclusions $(M, \Gamma) \subset (M', \Gamma')$ of sutured manifolds, diagrammatic handle attachments do not obviously fit into the HKM framework. We will introduce proper inclusions associated to contact handle attachments which allow us to relate diagrammatic and HKM contact handle attachment maps.

Definition 2.5. Given a sutured manifold $(M, \Gamma)$, let the padding be the sutured contact manifold $(P = \partial M \times [0, 1], -\Gamma \sqcup \Gamma, \xi_\Gamma)$, where $-\Gamma \sqcup \Gamma = -\Gamma \times \{0\} \sqcup \Gamma \times \{1\}$ and $\xi_\Gamma$ is the $[0, 1]$-invariant contact structure compatible with $-\Gamma \sqcup \Gamma$. A padded contact $i$-handle $(P_i, -\Gamma \sqcup \Gamma_i, \xi_i)$ is the result of attaching a contact $i$-handle to $\partial M \times \{1\}$ in $(P, -\Gamma \sqcup \Gamma, \xi_\Gamma)$.

We will often omit the contact structure $\xi_i$ from the notation and write $(P_i, -\Gamma \sqcup \Gamma_i)$ for a padded $i$-handle. The result of attaching a padded contact handle to $(M, \Gamma)$ is diffeomorphic to $(M_i, \Gamma_i)$. Moreover, the inclusion $M \subset M \cup P_i$ has an associated HKM map $\Phi_i : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M_i, -\Gamma_i)$.

Before identifying diagrammatic and HKM handle attachment maps, we will need to review the pertinent features of the definition of the HKM map. To give context and
set notation, we first recall how partial open books are constructed following [HKM09]; see also [EO11]. Here and throughout this paper, $N(L)$ denotes a small neighborhood of a submanifold $L$. If $L$ is a Legendrian graph, $N(L)$ is a standard neighborhood unless otherwise indicated.

Given a sutured contact manifold $(M,\Gamma,\xi)$, a partial open book decomposition is determined by a graph $K \subset M$ satisfying the conditions of Theorem 1.1 of [HKM09]. This guarantees that $M \setminus N(K)$ is product disk decomposable so that we can write $M \setminus N(K) = H_1 = (S \times [-1,1])/\sim$ and $N(K) = H_2 = (Q \times [1,2])/\sim$, where $Q \subset S$ and the relations $\sim$ collapse each interval of the form $\{s\} \times [-1,1], s \in \partial S$ and $\{q\} \times [1,2], q \in \partial Q \cap \partial S$ to a single point; the respective dividing sets are the images of $\partial S \times [-1,1]$ and $\partial Q \times [1,2]$ under $\sim$. This decomposition determines a monodromy map $h: Q \times \{2\} \to S \times \{-1\}$ with $h|_{Q \cap S} = \text{Id}_{Q \cap S}$ so that $(M, \Gamma)$ is diffeomorphic to $H_1 \cup_{Q \times \{1\}} H_2/\sim h$.

The Heegaard surface for an associated partial open book diagram is the image of $-S \times \{-1\} \cup Q \times \{1\}$ in the quotient. Curves are determined by a choice of arcs $\{a_1,\ldots,a_m\}$ in $Q$ which form a basis for $(S \setminus Q,\partial S)$ in the sense that $(S \times \{1\}) \cup a_k$ deformation retracts to $\partial S$. For each $k$, let $b_k$ be a perturbation of $a_k$ such that $\partial b_k$ is $a_k$ moved in the $\partial S$ direction and $b_k \cap a_k$ is a single intersection point. The sets of full curves $\alpha, \beta$ are given by the images of the $a_k \times \{1\} \cup a_k \times \{-1\}$ and $b_k \times \{1\} \cup h(b_k) \times \{-1\}$ respectively.

Observe that the image of $-\Sigma = ((\partial Q \setminus \partial S) \times \{1\}) \cup (\partial S \setminus \partial Q) \times \{-1\}$ corresponds to $-\Gamma$ in the quotient, so that the diagram $(\Sigma, \alpha, \beta)$ is a balanced Heegaard diagram for $(M, -\Gamma)$. Then $(\Sigma, \beta, \alpha)$ is a diagram for $(-M, -\Gamma)$; we call any such diagram constructed this way a partial open book diagram.

![Figure 7](image_url)

**Figure 7.** On the left a partial open book $M = (S \times [-1,1] \cup Q \times [1,2])/\sim$. On the right the Heegaard surface embedded in $(M, \Gamma)$; the black arrows represent the positive orientation of $\Sigma$.

When defining the HKM map, one uses diagrams $\mathcal{H} = (\Sigma, \beta, \alpha)$ for $(-M, -\Gamma)$ and $\mathcal{H}' = (\Sigma' \supset \Sigma, \beta' \cup \alpha', \alpha \cup \alpha')$ for $(-M', -\Gamma')$ which satisfy a number of technical conditions collectively referred to as contact-compatibility; in this context, $\mathcal{H}$ is called a contact-compatible diagram and $\mathcal{H}'$ is called a contact-compatible extension of $\mathcal{H}$. The idea is that the these diagrams will mimic partial open book diagrams to a large degree, with
some necessary adaptations. We will need to understand some of these conditions, which we now describe, following section 4 of [HKM08]. Note that there are some notational differences; the roles of objects decorated by single primes and undecorated objects are reversed, and we use $Q$ instead of $P$.

Parametrize a neighborhood of $\partial M \subset M'$ by $\partial M \times [-1,1]$, where $\partial M = \partial M \times \{0\}$ and $V = \partial M \times [-1,0]$ is contained in $M$. Note that the sutures $\Gamma$ determine a $[-1,0]$-invariant contact structure $\zeta$ on $V$. The Heegaard splitting for a contact-compatible diagram of $(-M, -\Gamma)$ is determined by a choice of an embedded graph $K$ whose restriction to $V$ satisfies the conditions in Theorem 1.1 of [HKM09] with respect to $\zeta$. (The conditions in Theorem 1.1 of [HKM09] guarantee that a Legendrian graph determines a compatible partial open book decomposition.) In particular, $K|_V$ is Legendrian and the handlebodies $V \setminus N(K)$ and $N(K)|_V$ are each product disk decomposable with respect to the sutures induced by $\zeta$.

The essential choices made in constructing a contact-compatible extension of $\mathcal{H}$ are as follows. Let $W = \partial M \times [0,1]$, and note that $\xi$ can be perturbed to restrict to the $[0,1]$-invariant contact structure on $W$. The extension is determined by a Legendrian graph $K_W \cup K'' \subset M' \setminus \text{Int}(M)$, with $K_W \subset W$ and $K'' \subset M = M' \setminus \text{Int}(M \cup W)$. (If one compares with [HKM08], a standard neighborhood of our $K_W$ is the complement of a standard neighborhood of their $K''$.) Both $K_W$ and $K''$ are required to satisfy the conditions of Theorem 1.1 of [HKM09] on $W$ and $M''$ with respect to $\xi$. This splits $M'$ into handlebodies $H_1 = M \setminus N(K \cup K_W \cup K'')$ and $H_2 = N(K \cup K_W \cup K'');$ however, $H_1$ and $H_2$ need not be product disk decomposable, since $K$ is not necessarily Legendrian for some contact structure on $M \setminus V$. If we restrict our attention to $V \cup W \cup M''$, we have an honest contact manifold with contact structure $\zeta \cup \xi$. The restrictions $J_1$ and $J_2$ of the handlebodies $H_1$ and $H_2$ to $V \cup W \cup M''$ are product disk decomposable and can be written as $J_1 = S' \times [-1,1]/\sim$ and $J_2 = Q' \times [1,2]/\sim$, with $Q' \subset S'$ and $\sim$ the relation which collapses each interval $\{s'\} \times [-1,1], s' \in \partial S'$ and $\{q'\} \times [-1,1], q' \in \partial Q'$ to a point, just as for an ordinary partial open book decomposition.

Define the full Heegaard surface $\Sigma'$ to be the complement of $R_+ (\Gamma')$ in $\partial H_1$. More explicitly, $\Sigma'$ is given by $\partial (M \setminus (N(K \cup V)) \cup (S' \times \{1\}) \cup Q' \times \{1\})$. The curves in $\beta'$ and $\alpha'$ are defined to be the curves in $\beta$ and $\alpha$ together with additional curves $\beta'', \alpha''$ that are generated in the following way. Chose a collection of arcs $\{a_1', ..., a_m'\}$ on $Q' \times \{1\}$ which form a basis for $(S' \setminus Q, R_+(\Gamma'))$ in the sense that $(S' \setminus Q) \cup \bigcup_{k=1}^m a_k''$ deformation retracts to $R_+ (\Gamma')$; here, $S'$ is identified with $S' \times \{1\}$, while $Q$ is the restriction of $Q' \times \{1\}$ to $V \subset M$. For each $k$, let $b_k''$ be a perturbation of $a_k''$ such that $\partial b_k''$ is $\partial a_k''$ moved in the $\partial S'$-direction and $b_k'' \cap a_k''$ is a single intersection point, as is standard procedure when constructing a partial open book diagram. Call the collection of these intersection points the preferred intersections of the extension, and denote it by $x''$. The arcs $\{b_k''\}, \{a_k''\}$ are completed to full sets of curves $\beta'', \alpha''$ in $\mathcal{H}'$ using the monodromy map $h : Q' \times \{2\} \to S' \times \{1\}$ obtained from the identification of $V \cup W \cup M''$ with $J_1 \cup h J_2$. More explicitly, $\beta''$ and $\alpha''$ are the respective images of $b_1'' \times \{1\} \cup h (b_2'' \times \{2\})$ and $a_1'' \times \{1\} \cup a_2'' \times \{-1\}$ under the relation $\sim$.

2.3. Identifying diagrammatic and HKM maps. In this section we identify the diagrammatic and HKM contact handle attachment maps. We will show that a diagrammatic handle attachment can be chosen to respect contact-compatibility and the preferred intersections.
Figure 8. Left: \((M', \Gamma')\) and \(\mathcal{H}'\); Right: \((M'_1, \Gamma'_1)\) and \(\mathcal{H}'_1\)

Figure 9. Left: \((M'_2, \Gamma'_2)\); Center: \(M'_2 \setminus N(L)\); Right: \(\mathcal{H}'_2\)

**Proposition 2.6.** Let \(\mathcal{H} = (\Sigma, \beta, \alpha)\) be a contact-compatible diagram for \((-M, -\Gamma)\) and \(\mathcal{H}' = (\Sigma', \beta \cup \beta'', \alpha \cup \alpha'')\) be a contact compatible extension of \(\mathcal{H}\) for \((-M', -\Gamma')\). Let \((M'_i, \Gamma'_i)\) be the result of attaching a contact i-handle to \((M', \Gamma')\). Then there is a contact-compatible extension \(\mathcal{H}'_i\) of \(\mathcal{H}\) for \((-M'_i, -\Gamma'_i)\) which is obtained from \(\mathcal{H}'\) by a diagrammatic handle attachment.

**Proof.** The graphs involved in the definition of a contact-capable extension are all required to satisfy a number of conditions; however, \(K''\) need only

1. satisfy the conditions of Theorem 1.1 of [HKM09] with respect to \((M'', \xi|_{M''})\)
2. have each component intersect \(\Gamma'\) at least twice
3. have \(\partial K''\) disjoint from \(\partial K_W\).

Attaching a contact handle to \(\partial M'\) only alters \(K''\) and the choice of basis arcs, so it suffices to verify that there are modifications of \(K''\) and \(\{a''_k\}\) which satisfy the properties describe above. The reader may verify that no other conditions in contact-compatibility are affected, and may be safely swept under the rug.

Given a 1-handle attachment \((M'_i, \Gamma'_i)\), the feet of the 1-handle can be isotoped to be disjoint from \(\partial K''\), since \(K''\) meets \(\partial M'\) in a finite number of points by Theorem 1.1 of [HKM09]. The graph \(K'' \subset M' \subset M'_i\) still satisfies the conditions of Theorem 1.1 of [HKM09] with respect to \(\xi_1 = \xi \cup h^1\), since one can cut \(M''_i\) along the cocore of \(h^1\) to recover \(M''\), i.e. a system of compressing disks for \(M''_i \setminus N(K'')\) can be obtained by adding
the cocore of $h^1$ to a system of compressing disks for $M'' \setminus N(K'')$. The new Heegaard surface is obtained from $\Sigma'$ by attaching a strip at the attaching sphere of $h^1$, since that is the corresponding change on $S''$; see Figure 8. The basis $\{a_k''\}_{k=1}^m$ for $H'$ is still a basis for the new extension. Since the monodromy is unaffected, the curves also remain the same. This new extension is thus exactly a diagrammatic $1$-handle attachment $H_1'$.

Given a $2$-handle attachment $(M'_2, \Gamma'_2)$ with attaching curve $\delta$, the attaching region can be isotoped to be disjoint from $K''$, since $K''$ meets $\partial M'$ in a finite number of points by Theorem 1.1 of [HKM09]. Let $L$ be a Legendrian approximation of the cocore of $h^2$ with $\partial L \subset \Gamma'_2$. The graph $K'' \cup L$ satisfies the conditions of Theorem 1.1 of [HKM09] with respect to the contact structure $\xi_2$ obtained by attaching the contact $2$-handle $h^2$ to $\xi$, since $M'' \setminus N(K'' \cup L)$ is contactomorphic to $M'' \setminus N(K'')$. Note that $K'' \cup L$ also satisfies the other two conditions listed above. Let $S'_2$ play the role of $S'$ for the new extension. While $S'_2$ is diffeomorphic to $S'$, there is an additional strip in $\Sigma'_2$ due to the new component $\partial N(L) \setminus \partial h^2$ of $Q'_2$ coming from $L$; see Figure 9. This strip does not deformation retract to a subset of $R_+(\Gamma'_2)$, so the arcs $\{a_k''\}_{k=1}^m$ no longer form a basis. Let $\mu$ be a meridian of $\partial N(L) \setminus \partial h^2$, and define $a'_0$ to be the arc $R_+(\partial N(L) \setminus \partial h^2) \cap \mu$. The collection $\{a_k'\}_{k=0}^m$ is now a basis for the new extension.

Since the arc $b'_0$ is completed to a full $\beta$-curve using the core of the new strip attached along $\delta \cap \Gamma'$, this new extension is an exactly a diagrammatic $2$-handle attachment $H'_2$. □

**Proposition 2.7.** Let $H$, $H'$, and $H'_1$ be as above, and $\phi_{\xi} : \text{SFC}(H) \to \text{SFC}(H')$ be the map $y \to (y, x'')$ which induces the HKM map as described above. Then the map $\sigma_i \circ \phi_{\xi} : \text{SFC}(H) \to \text{SFC}(H'_1)$ induces the HKM map $\Phi_{\xi_i} : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M'_1, -\Gamma'_1)$.

**Proof.** For $i = 1$, the preferred intersections $x''$ of $H'$ are also the preferred intersections of $H'_1$. The HKM map $\Phi_{\xi_1}$ is thus induced by the chain map $y \to (y, x''', x_0)$, which factors as $\sigma_1 \circ \phi_{\xi}$.

For $i = 2$, the preferred intersections of $H'_2$ are $x'' \cup x_0$, where $x_0 = b'_0 \cap a_0''$, the preferred intersection of the diagrammatic $2$-handle. The HKM map $\Phi_{\xi_2}$ is thus induced by the chain map $y \to (y, x''', x_0)$, which factors as $\sigma_2 \circ \phi_{\xi}$. □

We are now ready to relate diagrammatic and HKM maps.

**Lemma 2.8.** Let $(M, \Gamma)$ be a balanced sutured manifold. There is a contact-compatible diagram $H$ for $(-M, -\Gamma)$, a contact-compatible extension $H'$ for $(-(M \cup P_i), -\Gamma)$, a diagrammatic handle attachment $H'_i$ for $(-(M \cup P_i), -\Gamma)$, and a graded homotopy equivalence $\phi_{\xi_i} : \text{SFC}(H) \to \text{SFC}(H')$ such that the induced map on homology

$$
(\sigma_i \circ \phi_{\xi_i})_* : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-(M \cup P_i), -\Gamma) \approx \text{SFH}(-M_i, -\Gamma_i)
$$

is the HKM map $\Phi_{\xi_i}$.

**Proof.** From the proof of Theorem 6.1 in [HKM08], there is a contact compatible extension $H'$ of a contact-compatible diagram $H$ such that the map $\phi_{\xi_i} : \text{SFC}(H) \to \text{SFC}(H')$ which induces $\Phi_{\xi_i}$ is a graded homotopy equivalence induced by a composition of diagrammatic trivial bypass maps. The lemma then follows immediately from **Proposition 2.7.** □

### 3. Background on Zarev’s Gluing Map

We now turn our attention to Zarev’s bordered gluing map $\Psi_F$. In this section we review the requisite bordered sutured Floer theory for understanding the structure of this
gluing map, though the interested reader should look in [Zar11] for a full treatment of the topological and algebraic objects involved. Readers familiar with bordered sutured theory and Zarev’s gluing map should skip to Section 4, where we define the associated contact gluing map \( \Psi_\xi \) and prove that it satisfies the composition law necessary for the proof of Theorem 1.

We deviate from Zarev’s notational conventions in the following ways. Throughout, we reserve script letters for partially sutured manifolds, bordered sutured manifolds, and sutured surfaces. We avoid using them for honest sutured manifolds, except when the margins of the page force us to compress notation. We distinguish between suture sets on sutured manifolds and dividing sets on sutured surfaces by using a lower case letter for the former and the latter is denoted \( W \cup F \). As mentioned in Subsection 3.1, we denote the mirror of a partially sutured manifold \( W \) by \( W \), instead of \( W \). If \( H \) is a diagram for a sutured manifold \( (M, \Gamma) \), then Zarev treats \( \text{SFC}(M, \Gamma) \) as the \( A_\infty \)-homotopy type of the complex \( \text{SFC}(H) \); morphisms between sutured Floer complexes are then defined up to \( A_\infty \)-homotopy equivalence. We will instead work with representatives of these invariants, treating sutured Floer complexes as honest graded \( \mathbb{F}_2 \)-vector spaces, and graded chain maps as morphisms. Similarly, we treat the bordered invariants \( \text{BSA}, \text{BSD}, \text{BSAA} \) as honest Type-A, Type-D, and Type-AA structures, instead of their \( A_\infty \)-homotopy classes.

3.1. Some basic objects. We first review some topological objects which play an important role in bordered sutured theory. A **sutured surface** is an oriented surface \( F \) with boundary and a number of signed points \( \Lambda \subset \partial F \) which alternate in sign as one traverses each component of \( \partial F \); each component of \( \partial F \) is required to intersect \( \Lambda \). A sutured surface \( F = (F, \Lambda) \) has associated sutured surfaces \( -F = (-F, -\Lambda) \) and \( F = (-F, \Lambda) \). A **dividing set** on a sutured surface is a collection of embedded, oriented arcs and closed curves \( \gamma \subset F \) such that \( \partial \gamma = -\Lambda \) and such that each component of \( F \setminus \gamma \) can be oriented so that the induced orientation on the boundary agrees with the orientation of \( \gamma \). Components where this orientation agrees with the orientation on \( F \) constitute the \( R_+(\gamma) \) region, while the other components constitute the \( R_-(\gamma) \) region. Unless otherwise indicated, we assume that all sutured surfaces are **non-degenerate**, i.e. they have no closed components.

A **partially sutured manifold** \( W = (W, \gamma, F) \) is a manifold \( M \) whose boundary is the union of two sutured surfaces with common boundary, one of which (usually referred to as the “sutured part of the boundary”) has a dividing set \( \gamma \) and the other being \( F \). If \( W_1 = (W_1, \gamma_1, F_1 \cup F_2) \) and \( W_2 = (W_2, \gamma_2, -F_2 \cup F_3) \) are partially sutured manifolds, then we can **glue** or **concatenate** them along \( F_2 \) to obtain

\[
W_1 \cup_{F_2} W_2 = (W_1 \cup_F W_2, \gamma_1 \cup_{\Lambda_2} \gamma_2, F_1 \cup F_3).
\]

If \( W = (W, \gamma, F) \) is a partially sutured manifold, we denote \((-W, -\gamma, -F)\) by \(-W\) and \((-W, \gamma, F)\) by \(\overline{W}\). Note that this differs from Zarev’s notation, where there is no notation for the former and the latter is denoted \(-W\). In the special case of an honest sutured manifold, i.e. \( F = (\emptyset, \emptyset) \), we have \(- (M, \Gamma) = (-M, -\Gamma) \), and \((M, \Gamma) = (-M, \Gamma) \).

If \((M, \Gamma)\) is a sutured manifold, and \( W = (W, \gamma, F) \) is a partially sutured manifold with \( W \subset M, \text{Int}(F) \subset M, \partial W \setminus \text{Int}(F) \subset \partial M, \) and \( \gamma \subset \Gamma \), we denote the partially sutured manifold \((\overline{M} \setminus \overline{W}, \overline{\Gamma} \setminus \text{Int}(\gamma), -F)\) by \((M, \Gamma) \setminus W\).

A **bordered sutured manifold** \((M, \gamma, F, Z)\) is a partially sutured manifold whose non-sutured part \( F \) of the boundary is parametrized by an arc diagram \( Z \). We discuss arc diagrams in Subsection 5.1.
Let \( \mathcal{F} = (F, \Lambda) \) be a sutured surface and \( \gamma \) a dividing set on \( \mathcal{F} \). The cap associated to \( \gamma \) is the partially sutured manifold \((F \times [0, 1], (\Lambda \times [0, 1]) \cup (\gamma \times \{1\}), -\mathcal{F})\). The caps associated to contact 1-and 2-handle attachments are shown in Figure 10.

The positive twisting slice \( TW_{\mathcal{F},+} \) for \( \mathcal{F} \) is the partially sutured manifold \((F \times [0, 1], \gamma, -\mathcal{F} \cup \partial F)\), where \( \gamma \) is obtained by giving \( \Lambda \times [0, 1] \) a minimal fractional Dehn twist in the \( \partial F \)-direction. The positive twisting slices for contact 1-and 2-handles are shown in Figure 11. The negative twisting slice \( TW_{\mathcal{F},-} \) for \( \mathcal{F} \) is defined the same way, except that one twists in the \(-\partial F\)-direction. Note that \(-TW_{\mathcal{F},+} \cong TW_{\mathcal{F},-}\).

Given a partially sutured manifold \( \mathcal{W} \) with non-sutured boundary \( \mathcal{F} \), the double \( D(\mathcal{W}) \) of \( \mathcal{W} \) is the sutured manifold \( \mathcal{W} \cup \mathcal{F} \cup \mathcal{W} \cup \mathcal{F} \). Suppose \( \mathcal{W} \) is the cap for a dividing set \( \gamma \) on \( \mathcal{F} \), and consider the the \([0,1]\)-invariant contact structure \( \xi_\gamma \) on \( F \times [0,1] \) associated to \( \gamma \). Then \( \xi_\gamma \) induces a dividing set \( \Gamma_\gamma \) on \( F \times [0,1] \) such that \( (F \times [0,1], \Gamma_\gamma) \) is diffeomorphic to \( D(\mathcal{W}) \) as sutured manifolds. Note that \( \Gamma_\gamma \) is a dividing set in the sense of convex surface theory, not in the sense of sutured surfaces discussed above.

### 3.2. Gluing along sutured surfaces

We now review the operation of gluing sutured manifolds in the sense of [Zar11] and its connection with contact geometry.

Let \( \mathcal{F} = (F, \Lambda) \) be a sutured surface with dividing set \( \gamma \). Let \((M_1, \Gamma_1)\) and \((M_2, \Gamma_2)\) be sutured manifolds with embeddings \((F, \gamma) \subset (\partial M_1, \Gamma_1)\) and \((-F, \gamma) \subset (\partial M_2, \Gamma_2)\). The embedding \( F \to \partial M_1 \) extends to an embedding \( \mathcal{W} \to M_1 \), where \( \mathcal{W} \) is the cap for the dividing set \( \gamma \) on \( \mathcal{F} \). Similarly, \(-F \to \partial M_2 \) extends to an embedding \( \mathcal{W} \to M_2 \). Zarev defines the result of gluing \((M_1, \Gamma_1)\) to \((M_2, \Gamma_2)\) along \((\mathcal{F}, \gamma)\) to be the manifold \((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2) := ((M_1, \Gamma_1) \setminus \mathcal{W}) \cup_{\mathcal{F}} TW_{\mathcal{F},+} \cup_{\mathcal{F}} ((M_2, \Gamma_2) \setminus \mathcal{W})\), where \( TW_{\mathcal{F},+} \) is the positive twisting slice associated to \( \mathcal{F} \). We refer to the gluing operation above as a bordered gluing to distinguish it from a proper inclusion of sutured manifolds; a bordered gluing operation is shown in Figure 12. Note that the manifold \((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)\) is independent of the dividing set \( \gamma \); however, the bordered gluing map defined below will depend on it.

Bordered gluing operations describe how to glue the sutured manifolds underlying contact structures with convex boundary; however, there is a necessary orientation reversal. Let \((M, \Gamma, \xi)\) be a connected sutured contact manifold and let \( \mathcal{F} = (F, \Lambda) \) be a sutured surface with \((F, \partial F, \Lambda) \subset (M, \partial M, \Gamma)\); \((\text{Int}(F)) \subset \text{Int}(M)\); \( F \) convex with respect to \( \xi \); and \( \partial F \) Legendrian. If \( M \setminus F \) has two connected components, then its closure may be decomposed as sutured contact manifolds \((M_1, \Gamma_1, \xi_1) \cup (M_2, \Gamma_2, \xi_2)\), where \( \xi_i = \xi|_{M_i} \) and...
Figure 11. The positive twisting slices for contact 1- and 2-handles.

Figure 12. A bordered gluing operation along an annulus. The caps $W$ and $\overline{W}$ are a neighborhood of the shaded region.

$\Gamma_i$ is the dividing set induced by $\xi_i$ after smoothing corners. Suppose the normal vector for $F$ in $M$ points out of $M_1$ and into $M_2$. If $W = (W, \Gamma_1|_{F}, -F)$ is the cap for $F$, then we can write

$$(M, \Gamma) \cong ((M_1, \Gamma_1) \setminus W) \cup_F \mathcal{T}W_{F, -} \cup_{-F} ((M_2, \Gamma_2) \setminus \overline{W}).$$

We will use the notation $\xi = \xi_1 \cup_{F} \xi_2$ to describe the relationship among these contact structures.

Note that the twist of dividing sets which arises from smoothing corners is negative, instead of the positive twist used in the bordered gluing operation. This is consistent with the orientation reversal associated to the contact invariant, since we have

$$(-M_1, \Gamma_1) \cup_F (-M_2, \Gamma_2) = ((-M_1, \Gamma_1) \setminus -W) \cup_{-F} \mathcal{T}W_{F, -} \cup_{-F} ((-M_2, \Gamma_2) \setminus \overline{W}),$$

which is just

$$-[((M_1, \Gamma_1) \setminus W) \cup_F \mathcal{T}W_{F, -} \cup_{-F} ((M_2, \Gamma_2) \setminus \overline{W})] = (-M, -\Gamma).$$

Thus, it makes sense for the contact invariants $\text{EH}(\xi_1) \in \text{SFH}(-M_1, -\Gamma_1)$ and $\text{EH}(\xi_2) \in \text{SFH}(-M_2, -\Gamma_2)$ to be paired by the bordered gluing map to $\text{EH}(\xi_1 \cup_{F} \xi_2)$; see Lemma 4.12.

**Remark 3.1.** Note that we are using the symbol $\cup_F$ for the distinct operations of gluing partially sutured manifolds along components of their non-sutured boundaries, gluing balanced sutured manifolds along sutured subsurfaces of their boundaries with dividing sets, and gluing contact structures along convex surfaces. We rely on context to distinguish them.
Zarev shows that if \((M_1, \Gamma_1)\) and \((M_2, \Gamma_2)\) are balanced, so is \((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)\). He also defines a gluing map

\[
\text{SFH}(M_1, \Gamma_1) \otimes \text{SFH}(M_2, \Gamma_2) \xrightarrow{\Psi_{\mathcal{F}, \gamma}} \text{SFH}((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)),
\]
associated to the bordered gluing operation. To avoid confusion with the HKM contact gluing map, we will refer to this map as the \textit{bordered gluing map}. Also, when \(\mathcal{F}\) is embedded in the boundary of a sutured manifold, we will assume unless otherwise stated that the dividing set is the restriction of the sutures to \(\mathcal{F}\); in this case we omit the dividing set from the notation and write \(\Psi_{\mathcal{F}}\) for the bordered gluing map.

**Remark 3.2.** The bordered gluing map is well-defined up to maps induced by graded homotopy equivalence. It is not known to be natural with respect to holomorphic triangle maps in the way that the HKM map is.

The bordered gluing operation is a special case of the \textit{join operation} found in [Zar11]. Similarly, the bordered gluing map is a special case of the \textit{join map}. We will use the fact the the bordered gluing map possesses the following properties of the join.

**Theorem 3.3 (Zarev).** The bordered gluing map \(\Psi_{\mathcal{F}}\) satisfies the following properties:

1. **(Symmetry)** Let \((M_1, \Gamma_1), (M_2, \Gamma_2)\) be sutured manifolds with embeddings \(\mathcal{F} \subset \partial M_1, \mathcal{F} \subset \partial M_2\). There are graded isomorphisms \(f, g\) such that the following diagram commutes

\[
\begin{array}{ccc}
\text{SFH}(M_1, \Gamma_1) \otimes \text{SFH}(M_2, \Gamma_2) & \xrightarrow{f} & \text{SFH}(M_2, \Gamma_2) \otimes \text{SFH}(M_1, \Gamma_1) \\
\downarrow \Psi_{\mathcal{F}} & & \downarrow \Psi_{\mathcal{F}} \\
\text{SFH}((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)) & \xrightarrow{g} & \text{SFH}((M_2, \Gamma_2) \cup_{\mathcal{F}} (M_1, \Gamma_1)))
\end{array}
\]

2. **(Associativity)** Let \((M_1, \Gamma_1), (M_2, \Gamma_2), (M_3, \Gamma_3)\) be sutured manifolds and \(\mathcal{F}_1, \mathcal{F}_2\) be sutured surfaces with embeddings \(\mathcal{F}_1 \hookrightarrow \partial M_1, (\mathcal{F}_1 \sqcup \mathcal{F}_2) \hookrightarrow \partial M_2, \text{ and } \mathcal{F}_2 \hookrightarrow \partial M_3\). Write \(\mathcal{M}_k = (M_k, \Gamma_k)\). The following diagram commutes up to graded isomorphism:

\[
\begin{array}{ccc}
\text{SFH}(\mathcal{M}_1) \otimes \text{SFH}(\mathcal{M}_2) \otimes \text{SFH}(\mathcal{M}_3) & \xrightarrow{\Psi_{\mathcal{F}_2}} & \text{SFH}(\mathcal{M}_1) \otimes \text{SFH}(\mathcal{M}_2 \cup_{\mathcal{F}_2} \mathcal{M}_3) \\
\downarrow \Psi_{\mathcal{F}_1} & & \downarrow \Psi_{\mathcal{F}_1} \\
\text{SFH}(\mathcal{M}_1 \cup_{\mathcal{F}_1} \mathcal{M}_2) \otimes \text{SFH}(\mathcal{M}_3) & \xrightarrow{\Psi_{\mathcal{F}_2}} & \text{SFH}(\mathcal{M}_1 \cup_{\mathcal{F}_1} \mathcal{M}_2 \cup_{\mathcal{F}_2} \mathcal{M}_3)
\end{array}
\]

3. **(Identity)** Let \((M, \Gamma)\) be a sutured manifold with embedded cap \(W\) for \(\mathcal{F}\), and let \(\mathcal{D}(W)\) be the double of \(W\). There is a distinguished class \([\Delta] \in \text{SFH}(\mathcal{D}(W))\), called the diagonal class such that the map

\[
\Psi_{\mathcal{F}}(\cdots, [\Delta]) : \text{SFH}(M, \Gamma) \rightarrow \text{SFH}((M, \Gamma) \cup_{\mathcal{F}} \mathcal{D}(W))
\]

is a graded isomorphism.

4. **The Bordered Contact Gluing Map**

The goal of this section is to define the bordered contact gluing map and prove that it satisfies functorial properties akin to the HKM map.
4.1. The bordered contact gluing map. We will need to introduce a technical condition on the dividing set \( \gamma \) in order to ensure that the bordered contact gluing map possesses the desired functorial properties found in Subsection 4.2.

Definition 4.1. A dividing set \( \gamma \) on a sutured surface \( F \) is **disk-decomposable** if the underlying sutured manifold of the \( I \)-invariant contact structure for \( \gamma \) on \((F \times I)\) has a product disk decomposition.

Definition 4.2. Let \((M'', \Gamma'')\) and \((M, \Gamma)\) be sutured manifolds which can be glued along a sutured surface \( F \) with disk-decomposable dividing set \( \gamma \). Let \( \xi \) be a compatible contact structure on \( M'' \). The associated **bordered contact gluing map**

\[
\Psi_\xi : \text{SFH}(-M, -\Gamma) \to \text{SFH}((-M'', -\Gamma'') \cup \_F (-M, -\Gamma))
\]

is defined to be \( \Psi_\xi([y]) = \Psi_{\_F}(\text{EH}(\xi), [y]). \)

If \((M'', \Gamma'')\) is a contact handle \( h^i \) with attaching region \( F_i \), then we denote this map by \( \Psi_i : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M_i, -\Gamma_i). \)

Remark 4.3. Note that the dividing sets for contact handle attachments are disk-decomposable; see Subsection 5.2.

Remark 4.4. Note that some restriction on the dividing set is necessary, since the identity property Lemma 4.6 is false when the \( I \)-invariant contact structure is overtwisted. We conjecture that the bordered contact gluing map map satisfies Lemma 4.6, Lemma 4.12, and Lemma 4.13 whenever the \( I \)-invariant contact structure associated to the dividing set is tight. It would suffice to prove Lemma 4.7 for such dividing sets, since the rest of the arguments carry through.

Lemma 4.5. **The map** \( \Psi_\xi \) **is well-defined up to graded isomorphism.**

Proof. This is immediate from the fact that \( \Psi_{\_F} \) is defined up to graded isomorphism.

4.2. Properties of the bordered contact gluing map. We will now prove that \( \Psi_\xi \) satisfies identity, EH-preservation, and composition laws analogous to the ones satisfied by the HKM-map. The composition law Lemma 4.13 is necessary for the proof of Theorem 1, and our proof the composition law requires the other two properties.

Lemma 4.6. Let \((M, \Gamma)\) be a sutured manifold and \( F \subset \partial M \) be a sutured surface with disk-decomposable dividing set \( \gamma \). Let \( \mathcal{W} \) be the cap for \( -\gamma \) on \( -F \) and let \( \gamma \) be the \( I \)-invariant contact structure for \( -\gamma \) on the double \( D(\mathcal{W}) \). Then

\[
\Psi_{\xi, \gamma} : \text{SFH}(-M, -\Gamma) \to \text{SFH}(D(\mathcal{W}) \cup \_F (-M_i, -\Gamma_i))
\]

is a graded isomorphism.

Proof. This follows immediately from the identity and symmetry properties of the bordered gluing map along with Lemma 4.7.

Lemma 4.7. Let \( \mathcal{W} \) be the cap for a disk-decomposable dividing set \( \gamma \) on a sutured surface \( F \). Let \( \xi \) be the \( I \)-invariant contact structure on \( F \times I \). Then the diagonal class \([\Delta] \in \text{SFH}(D(\mathcal{W}))\) is the contact class \(\text{EH}(\xi)\).

Proof. The diagonal class is always non-trivial by [Zar11]. Because the dividing set is disk-decomposable, \( \text{SFH}(D(\mathcal{W})) \) has rank 1 and the contact invariant is non-trivial. Thus, the contact invariant and the diagonal class are the same.
We now wish to prove that the bordered contact gluing map preserves the contact invariant. To do this, we rely on the identification of the bordered contact handle maps with HKM handle maps provided by Lemma 6.1 and Lemma 6.2. This shows that each bordered contact handle map \( \Psi_i \) preserves the contact invariant. We then need to leverage associativity of the bordered gluing map; see Theorem 3.3. However, we can only do this with contact handles which have disjoint attaching regions, and this cannot always be achieved when the attaching curve of a 2-handle runs over a 1-handle. To get around this problem, we now introduce a version of the padded handles of Subsection 2.2 adapted to our needs. We will prove that the bordered contact maps associated to these modified handle attachments are equivalent to the \( \Psi_i \).

Let \((M, \Gamma)\) be a balanced sutured manifold and \( \xi \) be the \([-1, 0]\)-invariant contact structure in a closed neighborhood \( N(\partial M) \cong \partial M \times [-1, 0] \) of the boundary \( \partial M \times \{0\} \). Let \( \Delta \) be a subsurface of \( \partial M \) such that \( \gamma = \Gamma|_{\Delta} \) is a dividing set for the (possibly degenerate) sutured surface \((\Sigma, -\partial \gamma)\). Let \( \xi_\gamma \) be the \([0, 1]\)-invariant contact structure on \( \Sigma \times [0, 1] \) compatible with \( \gamma \).

If \( h^i \) is a contact handle whose attaching region \( A \) is contained in \( \Delta \), let \( L \) be a Legendrian graph of the form \( \{q_k\} \times [0, 1] \) where each \( q_k \) is a point in \( \gamma \) disjoint from \( A \) and each component of \( \gamma \) contains at least one \( q_k \). The punctured padding associated to \((\Sigma, A)\) and \( L \) is the contact manifold \(( (\Sigma \times [0, 1]) \setminus N(L), \xi^L_\gamma \)\), where \( \xi^L_\gamma \) is the restriction of \( \xi_\gamma \) to \((\Sigma \times [0, 1]) \setminus N(L)\). We denote the underlying sutured manifold by \((P^L, \Gamma^L)\), where \( \Gamma^L \) is the dividing set induced by \( \xi^L_\gamma \). The punctured padded contact \( i \)-handle associated to \((\Sigma, h^i)\), and \( L \) is \( ((\Sigma \times [0, 1]) \setminus N(L), \xi^L_\gamma) \cup_{A \times \{1\}} h^i \); we denote the associated sutured contact manifold by \((P_i^L, \Gamma_i^L, \xi^L_i)\). The sutured surface \((\Sigma \setminus N(\{q_j\}), -\partial(\gamma \setminus N(\{q_j\})))\) is denoted \( \mathcal{F}^L \). Note that \( \xi^L_i \cup_A h^i \) is contactomorphic to \( \xi^L_i \) relative to \( \partial \Sigma \times [-1, -\frac{1}{2}] \).

**Remark 4.8.** Note that a contact handle decomposition of 1-and 2-handles can be described by a punctured padded contact handle decomposition. If a contact structure \( \xi \) can be factored into padded handles \((P_{11}, \xi_{11}), \ldots, (P_{1n}, \xi_{1n})\) attached to an \( I \)-invariant contact structure \((P, \xi)\), we can choose Legendrians \( L_j \subset P_i \) so that each \( L_j \) can be extended through \( P_{i, j+1} \cup \ldots \cup P_{in} \) without intersecting the attaching region of any contact handle; see Figure 13. One can do this in such a way that \( \xi \) restricted to \( P \cup P_{11} \cup \ldots \cup P_{1n} \) is contactomorphic to \( \xi \). One can also perform a similar construction if each padded handle \((P_{ij}, \xi_{ij})\) is replaced by \( (\Sigma_j \times [j, j+1], \xi_{ij}) \cup h^i_j \), where each \( \Sigma_j \) is a (possibly degenerate) sutured surface with dividing set \( \gamma_j \). We employ this latter case in Lemma 4.12 and Lemma 4.13; details are left to the reader.

In order to make use of these modified contact handles, we need to to know that the dividing set \( \Gamma^L|_{\mathcal{F}^L} = \gamma \setminus N(\{q_j\}) \) is disk-decomposable.

**Lemma 4.9.** Let \( \Sigma \) be a (possibly degenerate) sutured surface with dividing set \( \gamma \), and let \( \{p_k\} \) be a finite collection of points which has non-empty intersection with each closed component of \( \gamma \). Let \( F \) be the surface \( \Sigma \setminus N(\{p_k\}) \). Then \((F, -\partial \gamma|_F)\) is a non-degenerate sutured surface and the dividing set \( \gamma|_F \) is disk-decomposable.

**Proof.** \((F, -\partial \gamma|_F)\) is non-degenerate by construction. Let \( \xi \) be the \([0, 1]\)-invariant contact structure on \( F \times [0, 1] \) compatible with \( \gamma|_F \) and \( \Gamma_\xi \) be the sutures induced by \( \xi \). Note that \( \xi \) is tight, since \( \gamma|_F \) has no closed components.
We can choose a collection of disjoint embedded arcs \( \{ a_i \} \subset F \setminus \gamma |_F \) with \( \partial a_i \subset \partial F \) such that \( F \cup a_i \) is a number of disks. The collection \( \{ a_i \times [0, 1] \} \subset F \times [0, 1] \) determines a product disk decomposition of \( \xi \). (Each \( \partial (a_i \times [0, 1]) \) intersects \( \Gamma_{\xi} \) twice because of smoothing at the corners.) Thus, \( \text{SFH}(F \times [0, 1], \Gamma_{\xi}) \) has rank one.

Note that punctured padded handles depend on a choice of subsurface \( \Sigma \) and Legendrian graph \( L \). We will show that this ambiguity does not affect the associated gluing maps. To prove this and the remaining properties of \( \Psi_{\xi} \), we assume the following consequence of Lemma 6.1 and Lemma 6.2; we may do this, since the proof of these lemmas does not depend on any of the results in this section.

\[ \text{Figure 13. Decomposing a contact structure into padded handle attachments. A choice of Legendrians which determine a punctured padded handle decomposition are indicated by dashed lines.} \]

**Lemma 4.10.** Let \((M_i, \Gamma_i, \xi_i)\) be a sutured contact manifold obtained from \((M, \Gamma, \xi)\) by a contact \( i \)-handle attachment, and let \( \Psi_i : \text{SFH}(-M, -\Gamma) \to (-M_i, -\Gamma_i) \) be the associated bordered contact gluing map. Then \( \Psi_i(\text{EH}(\xi)) = \text{EH}(\xi_i) \) up to graded isomorphism.

**Proof.** By Lemma 2.8, Lemma 6.1, and Lemma 6.2, the bordered contact gluing map \( \Psi_i \) equals the corresponding HKM-map \( \Phi_i \) up to graded isomorphism. Since \( \Phi_i \) preserves the contact class, so does \( \Psi_i \) up to graded isomorphism. \( \Box \)

Now we show that the bordered contact gluing maps for punctured padded handle attachments and ordinary contact handle attachments agree.

**Lemma 4.11.** Let \((M, \Gamma)\) be a sutured manifold, \( h^i \) be a contact \( i \)-handle, and \((P^L_i, \Gamma^L_i, \xi^L_i)\) be a punctured padded contact \( i \)-handle. Under the identification \((M, \Gamma) \cong (P^L_i, \Gamma^L_i) \cup_{F^L} (M, \Gamma)\) the corresponding bordered contact gluing maps

\[ \Psi_i, \Psi^L_{\xi_i} : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M_i, -\Gamma_i) \]

are equal up to graded isomorphism.
Proof. Split \((P^L_i, \Gamma^L_i, \xi^L_i)\) into a contact handle \(h^i\) and the \(I\)-invariant contact structure \(\xi^L_i\) on the punctured padding \((P^L, \Gamma^L)\). By associativity of the bordered gluing map, the following diagram commutes up to graded isomorphism:

\[
\begin{array}{c}
\text{SFH}(h^i) \otimes \text{SFH}(P^L, -\Gamma^L) \otimes \text{SFH}(-M, -\Gamma) \xrightarrow{\Psi_{-F_i}} \text{SFH}(P^L_i, -\Gamma^L_i) \otimes \text{SFH}(-M, -\Gamma) \\
\downarrow \Psi_{-F^L} \\
\text{SFH}(h^i) \otimes \text{SFH}(P^L \cup -M, -\Gamma) \xrightarrow{\Psi_{-F_i}} \text{SFH}(P^L_i \cup -M, \Gamma_i)
\end{array}
\]

By Lemma 4.10, the bordered gluing map \(\Psi_{-F_i}\) on the top sends \((\text{EH}(h^i)), \text{EH}(\xi^L_i))\) to \(\text{EH}(\xi^L_i))\), up to graded isomorphism. Evaluating on the contact elements yields the following commutative diagram:

\[
\begin{array}{c}
\text{SFH}(-M, -\Gamma) \xrightarrow{\text{Id}} \text{SFH}(-M, -\Gamma) \\
\downarrow \Psi_{\xi^L_i} \\
\text{SFH}(P^L \cup -M, -\Gamma) \xrightarrow{\Psi_{\xi^L_i}} \text{SFH}(P^L_i \cup -M, \Gamma_i)
\end{array}
\]

The map \(\Psi_{\xi^L_i}\) is a graded isomorphism by Lemma 4.6, since the dividing set on \(F^L\) is disk-decomposable by Lemma 4.9. Since the diagram commutes up to graded isomorphism, the lemma follows. \(\square\)

We now show that the bordered contact gluing map preserves the contact class.

**Lemma 4.12.** Let \((M, \Gamma)\) and \((M'', \Gamma'')\) be sutured manifolds with compatible contact structures \(\zeta\) and \(\xi\) respectively, and let \((M', \Gamma') = (M'', \Gamma'') \cup F (M, \Gamma)\) for some sutured surface \(F\) with disk-decomposable dividing set. Then

\[
\Psi_{\xi}(\text{EH}(\zeta)) = \text{EH}(\xi \cup F \zeta)
\]

up to graded isomorphism.

Proof. Decompose \((M'', \Gamma'')\) into a number of punctured padded handles \((P^L_1, \Gamma^L_1), \ldots, (P^L_n, \Gamma^L_n)\) and let us consider two particular compositions of bordered gluing maps

\[
\text{SFH}(P^L_{i_n}, -\Gamma^L_{i_n}) \otimes \ldots \otimes \text{SFH}(P^L_{i_1}, -\Gamma^L_{i_1}) \otimes \text{SFH}(-M, -\Gamma) \rightarrow \text{SFH}(-M', -\Gamma')
\]

corresponding to gluing in different orders. One composition corresponds to attaching the punctured padded handles sequentially to \((M, \Gamma)\), while another composition corresponds to attaching the punctured padded handles sequentially to \((P^L_{i_1}, \Gamma^L_{i_1})\) to get \((M'', \Gamma'')\) before finally gluing to \((M, \Gamma)\). By Theorem 3.3, these compositions give rise to the following diagram which commutes up to graded isomorphism:

\[
\begin{array}{c}
\text{SFH}(P^L_{i_n}, -\Gamma^L_{i_n}) \otimes \ldots \otimes \text{SFH}(P^L_{i_1}, -\Gamma^L_{i_1}) \otimes \text{SFH}(-M, -\Gamma) \\
\downarrow \\
\text{SFH}(-M'', -\Gamma'') \otimes \text{SFH}(-M, -\Gamma) \xrightarrow{\Psi_{-F}} \text{SFH}(-M', -\Gamma')
\end{array}
\]

By Lemma 4.11 and Lemma 4.10, each punctured padded handle attachment map preserves the contact invariant up to graded isomorphism. The vertical arrow thus maps
$\text{EH}(\xi_{n}) \otimes \ldots \otimes \text{EH}(\xi_{1}) \otimes \text{EH}(\zeta)$ to $\text{EH}(\xi) \otimes \text{EH}(\zeta)$, while the diagonal arrow maps $\text{EH}(\xi_{n}) \otimes \ldots \otimes \text{EH}(\xi_{1}) \otimes \text{EH}(\zeta)$ to $\text{EH}(\xi \cup_{\mathcal{F}} \zeta)$. Since $\Psi_{\xi}$ is the evaluation of $\Psi_{-\mathcal{F}}$ on $\text{EH}(\xi)$, it must send $\text{EH}(\xi)$ to $\text{EH}(\xi \cup_{\mathcal{F}} \zeta)$ up to graded isomorphism. }

We can finally prove the composition law which we need in our proof of Theorem 1.

**Lemma 4.13.** Let $(M_{1}, \Gamma_{1})$, $(M_{2}, \Gamma_{2})$, and $(M_{3}, \Gamma_{3})$ be sutured manifolds with compatible contact structures $\xi$ on $(M_{1}, \Gamma_{1})$ and $\xi'$ on $(M_{2}, \Gamma_{2})$. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are sutured surfaces with disk decomposable dividing sets such that the bordered gluing $(M_{1}, \Gamma_{1}) \cup_{\mathcal{F}} (M_{2}, \Gamma_{2}) \cup_{\mathcal{F}'} (M_{3}, \Gamma_{3})$ is defined. Then

$$\Psi_{\xi \cup_{\mathcal{F}} \xi'} = \Psi_{\xi} \circ \Psi_{\xi'}$$

up to graded isomorphism.

**Proof.** Write $M_{k} = (M_{k}, \Gamma_{k})$. The following diagram commutes up to graded isomorphism by associativity of Zarev’s bordered gluing map.

$$\begin{array}{ccc}
\text{SFH}(-M_{1} \cup -M_{2} \cup -M_{3}) & \xrightarrow{\Psi_{-\mathcal{F}'} \Psi_{-\mathcal{F}}} & \text{SFH}(-M_{1} \cup -M_{2} \cup -M_{3}) \\
\downarrow \Psi_{-\mathcal{F}} & & \downarrow \Psi_{-\mathcal{F}} \\
\text{SFH}(-M_{1} \cup_{-\mathcal{F}} -M_{2} \cup -M_{3}) & \xrightarrow{\Psi_{-\mathcal{F}'} \Psi_{-\mathcal{F}}} & \text{SFH}(-M_{1} \cup_{-\mathcal{F}} -M_{2} \cup_{-\mathcal{F}'} -M_{3})
\end{array}$$

By Lemma 4.12, the map $\Psi_{-\mathcal{F}}$ on top sends $\text{EH}(\xi) \otimes \text{EH}(\xi') \otimes [y]$ to $\text{EH}(\xi \cup_{\mathcal{F}} \xi') \otimes [y]$ for any cycle $y$. By evaluating on contact elements, we obtain the following diagram which also commutes up to graded isomorphism.

$$\begin{array}{ccc}
\text{SFH}(-M_{3}) & \xrightarrow{\Psi_{\xi} \Psi_{\xi'} \text{Id}} & \text{SFH}(-M_{2} \cup_{-\mathcal{F}'} -M_{3}) \\
\downarrow \text{Id} & & \downarrow \Psi_{\xi} \\
\text{SFH}(-M_{3}) & \xrightarrow{\Psi_{\xi \cup_{\mathcal{F}'} \xi} \text{Id}} & \text{SFH}(-M_{1} \cup_{-\mathcal{F}} -M_{2} \cup_{-\mathcal{F}'} -M_{3})
\end{array}$$

□

5. **Background on Computing the Bordered Gluing Map**

This section contains the background necessary for computing the bordered contact gluing map in the case of a contact handle attachment. We will need this in Subsection 6.1 when identifying the bordered map with the diagrammatic maps of Subsection 2.1.

5.1. **Arc diagrams, bordered algebras, and bordered sutured diagrams.** Computing the bordered gluing map explicitly requires one to identify a particular generator in a certain Heegaard diagram for the twisting slice $\mathcal{TW}_{-\mathcal{F},+}$; see Lemma 5.1. This section is devoted to the background necessary to construct this diagram and identify the generator. Those familiar with Auroux-Zarev diagrams or willing to take this identification on faith may skip to Section 6 and simply refer to Figure 16 and Lemma 5.1 when proving Lemma 6.2. We first review arc diagrams and describe the bordered algebras associated to contact handles.

An **arc diagram** $\mathcal{Z}$ is a union of oriented intervals $Z \subseteq \mathbb{Z}_{i}$, an even number of marked points $\{z_{j}\} \subset \text{Int}(Z)$, and a 2-to-1 function $\mu : \{z_{j}\} \to \mathbb{Z}_{>0}$. Points $z_{j}$ and $z_{k}$ are said to be **matched** if $\mu(z_{j}) = \mu(z_{k})$. We require that the result of surgery on $Z$ along all pairs of
Figure 14. Arc diagrams $Z_1$ (Far left) and $Z_2$ (Middle right) for contact 1-and 2-handles. The sutured surfaces $F_1$ (Middle left) and $F_2$ (Far right) they parametrize are also depicted. Note that these diagrams have $\alpha$-type, though we have drawn the arcs blue since we will be working with Heegaard diagrams of the form $(\Sigma, \beta, \alpha)$.

matched points $\mu^{-1}(l)$ have no closed components. An oriented subinterval of $Z$ from one marked point to another (not necessarily matched) marked point is called a Reeb chord. Arc diagrams also come with a type: $\alpha$-type or $\beta$-type. Given $Z$, there is an arc diagram $-Z$ obtained by changing the orientation of each $Z_i$, and another arc diagram $Z$ obtained by changing the type.

The graph $G(Z)$ of an arc diagram is the ribbon graph obtained by attaching arcs $\{a_l\}$ to $Z$ so that $\partial a_l = \mu^{-1}(l)$. If $Z$ is $\alpha$-type, the arcs are attached on the left with respect to the orientation on the intervals $Z_i$; if $Z$ is $\beta$-type, the arcs are attached on the right. The graph $G(Z)$ is obtained by reversing the cyclic ordering around each vertex in $\cup \partial a_l$. Alternatively, $G(Z)$ is $G(-Z)$ as a ribbon graph.

An arc diagram $Z$ parametrizes a sutured surface $F = (F, \Lambda)$ if there is an embedding $G(Z) \subset F$ such that $\partial (\cup Z_i) = \Lambda$ as oriented manifolds, and $F$ deformation retracts onto $G(Z)$. If $Z$ parametrizes $F$, then $-Z$ parametrizes $-F$ and $\overline{Z}$ parametrizes $\overline{F}$.

The bordered algebra $A = A(Z)$ associated to an arc diagram $Z$ is generated as an $F_2$-vector space by certain formal sums of strand diagrams; multiplication corresponds to concatenation of strand diagrams. $A$ is a differential graded algebra over the ground ring $I \subset A$ consisting of idempotents of $A$.

There is a preferred basis of idempotents which is in one-to-one correspondence with collections of arcs of $Z$; we denote the basis idempotent corresponding to $\{a_{i_1}, \ldots, a_{i_p}\}$ by $\iota_{i_1, \ldots, i_p}$. The basis of idempotents can be extended to a basis of $A$ by adding algebra elements associated to collections of Reeb chords; we will abuse notation and write $\rho$ for the algebra element associated to a single Reeb chord $\rho$. (We will not need to consider algebra elements associated to collections of multiple Reeb chords.)

The algebra and the idempotents split as $A = \bigoplus_i A(i)$ and $I = \bigoplus_i I(i)$, each indexed by the number of strands in the associated diagrams. (Note that we are following Zarev’s convention for indexing summands of the algebra, which deviates from the convention in [LOT15].) Since we only need to understand the bordered algebras for the arc diagrams we have chosen for contact 1-and 2-handles, we omit the general definition here, and explicitly describe the salient features of the algebras we need. The interested reader can find formal
Figure 15. Strand diagrams corresponding to basis elements of $A_2^1$.

The two leftmost diagrams correspond to basis idempotents, while the others correspond to Reeb chords. Multiplication is given by concatenating, i.e. $\rho_1 \cdot \rho_2 = \rho_{12}$.

definitions in [LOT15] and [Zar11]. Note that the bordered algebras $A(-Z)$ and $A(Z)$ are isomorphic to the opposite algebra $A(Z)^{op}$, which is $A(Z)$ with multiplication reversed.

Let $Z_1$ be the arc diagram on the left of Figure 14 which parametrizes the sutured surface $F_1 = (F_1, \Lambda_1)$ for a contact 1-handle. The associated bordered algebra $A_1 = A(Z_1)$ has rank 1; the only non-trivial summand is $A_1(0) = I_1(0)$, with basis element the idempotent $\iota_\emptyset$ corresponding to the strand diagram with no strands.

Let $Z_2$ be the arc diagram on the right of Figure 14 which parametrizes the sutured surface $F_2 = (F_2, \Lambda_2)$ for a contact 2-handle. The associated bordered algebra $A_2 = A(Z_2)$ has rank 9 with three non-trivial summands. Just as for $Z_1$, the summand $A_2(0) = I_2(0)$ has rank 1, with basis element the idempotent $\iota_\emptyset$ corresponding to the strand diagram with no strands. The summand $A_2(1)$ has rank 5, with basis elements corresponding to the strand diagrams depicted in Figure 15. The ground ring $I_2(1)$ is generated by the basis idempotents $\iota_1, \iota_2$. Each of these corresponds to a formal sum of two strand diagrams, each such sum is represented by a superposition of the two diagrams with dashed lines. The nontrivial actions by idempotents are

\[
\iota_2 \cdot \rho_1 \cdot \iota_1 = \rho_1 \quad \iota_1 \cdot \rho_2 \cdot \iota_2 = \rho_2 \quad \iota_2 \cdot \rho_{12} \cdot \iota_2 = \rho_{12},
\]

while the only other non-trivial multiplication is $\rho_1 \cdot \rho_2 = \rho_{12}$. Multiplication is given by concatenating strand diagrams. The summand $A_2(2)$ has rank 3; it does not appear in our computations.

Furthermore, each algebra is equipped with a grading and a differential; we will not need to consider either.

Suppose $Z^\alpha$ is an $\alpha$-type arc diagram and $Z^\beta$ is a $\beta$-type arc diagram. A bordered sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, Z^\alpha, Z^\beta)$ consists of

- an oriented surface $\Sigma$ without closed components
- a collection $\alpha = \alpha^c \cup \alpha^a$ of pairwise disjoint, properly embedded curves $\alpha^c$ and arcs $\alpha^a$ in $\Sigma$
- another collection $\beta = \beta^c \cup \beta^a$ of pairwise disjoint, properly embedded curves $\beta^c$ and arcs $\beta^a$ in $\Sigma$
- embeddings $(\overline{G(Z^\alpha)}, Z^\alpha) \subset (\Sigma, \partial \Sigma)$ and $(\overline{G(Z^\beta)}, Z^\beta) \subset (\Sigma, \partial \Sigma)$ such that $Z^\alpha$ and $Z^\beta$ are disjoint, the orientation on each component of $Z^\alpha \sqcup Z^\beta$ agrees with the orientation on $\partial \Sigma$, the arcs of $G(Z^\alpha)$ agree with the arcs of $\alpha^a$, the arcs of $G(Z^\beta)$ agree with the arcs of $\beta^a$, and no component of $\Sigma \setminus \alpha$ or $\Sigma \setminus \beta$ is disjoint from $\partial \Sigma \setminus (Z^\alpha \sqcup Z^\beta)$.
Let $A = A(\mathcal{Z}^\alpha)$ and $B = A(\mathcal{Z}^\beta)$ be the bordered algebras with respective subrings of idempotents $I^\alpha \subset A$, $I^\beta \subset B$. We can choose to consider the bordered module $\hat{\text{BSAA}}(\mathcal{H})$ as an $(I^\alpha)^{op}$, $I^\beta$-bimodule generated by tuples of intersection points which occupy each curve in $\alpha^c \cup \beta^c$ exactly once and each arc in $\alpha^a \cup \beta^a$ at most once. If a generator $x$ occupies arcs which correspond to $a_{i_1}, \ldots, a_{i_p} \subset G(\mathcal{Z}^\alpha)$ and $b_{j_1}, \ldots, b_{j_q} \subset G(\mathcal{Z}^\beta)$, then we have the algebra action $t_{i_1,\ldots,i_p} \cdot x \cdot t_{j_1,\ldots,j_q} = x$. Since these are the unique basis idempotents which act nontrivially on $x$, we will also write this equation as $t_L(x) \cdot x \cdot t_R(x) = x$.

The differential and $A_\infty$-algebra actions are determined by counting certain pseudoholomorphic curves. The $A_\infty$-module actions are maps

$$m_{i|j|j}: (A^{op})^i \otimes \hat{\text{BSAA}}(\mathcal{H}) \otimes B^j \to \hat{\text{BSAA}}(\mathcal{H})$$

which satisfy certain relations, where each tensor product is over $(I^\alpha)^{op}$ or $I^\beta$ as appropriate.

One similarly defines Type-A and Type-D structures $\text{BSA}$, BSD for diagrams of the form $(\Sigma, \alpha, \beta, Z^\alpha, \emptyset)$ or $(\Sigma, \alpha, \beta, \emptyset, Z^\beta)$. Just as for $\text{BSAA}$, these invariants are generated as $\mathbb{F}_2$-vector spaces over tuples of intersection points which occupy each full curve exactly once and each arc in at most once.

A bordered sutured diagram is nice, if every component of $\Sigma \setminus (\alpha \cup \beta)$ is one of the following:

1. a region with non-trivial intersection with $(\partial \Sigma) \setminus (Z^\alpha \cup Z^\beta)$; referred to as a basepoint region or suture region
2. a bigon with no side in $Z^\alpha \cup Z^\beta$
3. a quadrilateral with at most one side in $Z^\alpha \cup Z^\beta$.

In this case the differential of $\hat{\text{BSAA}}(\mathcal{H})$ is given by counting each region which is an embedded bigon or rectangle in the interior of the Heegaard surface, and which is disjoint from the basepoint regions. The $A_\infty$-module actions are given by counting each region which is an embedded rectangle with one side a Reeb chord $\rho$ in $Z^\alpha \cup Z^\beta \subset \partial \Sigma$ and the other three sides in $\text{Int}(\Sigma)$. More precisely, let $x = (x_0, z_1, \ldots, z_n)$ and $y = (y_0, z_1, \ldots, z_n)$ be generators in a nice bordered diagram $\mathcal{H}$ with basis idempotent actions $\iota_{L}(x) \cdot x \cdot \iota_{R}(x) = x$ and $\iota_{L}(y) \cdot y \cdot \iota_{R}(y) = y$, and let $\rho$ be a Reeb chord in $Z^\alpha$. Suppose there is a rectangular region $D$ which has corners $\{x_0, y_0\} \cup \partial \rho$, goes out of $x_0$, goes into $y_0$, and has no $z_k$ in its interior. Then $D$ contributes a $y$ term to the action $m_{0|1|1}(x, \iota_{L}(x) \cdot \rho \cdot \iota_{R}(y))$. If $\rho$ is a chord in $Z^\beta$ instead, then $D$ contributes a $y$ term to the action $m_{1|0|0}(\iota_{L}(y) \cdot \rho \cdot \iota_{R}(x), x)$. The only other non-trivial actions in either case come from compatible idempotents, i.e. $m_{0|1|1}(x, \iota_{R}(x)) = m_{1|0|0}(\iota_{L}(x), x) = x$.

As an example, we explicitly compute some actions of the $A_\infty$-bimodule $\hat{\text{BSAA}}(\mathcal{H}_{AZ})$ associated to the nice diagram $\mathcal{H}_{AZ} = (\Sigma, \beta, \alpha, Z_2 \cup Z_2)$ on the right hand side of Figure 16. We will give $\text{BSAA}(\mathcal{H}_{AZ})$ a right action by $A(-Z_2) \approx A(Z_2)^{op}$ and a left action by $A(Z_2) \approx A(Z_2)^{op}$. Note that the usual roles of $\alpha$ and $\beta$ curves are reversed and $Z_2$ has $\alpha$-type.

The region $D_1$ contributes the action $m_{1|1|0}(\rho_1, z_1) = \rho_1 \cdot z_1 = z_4$. The region $D_1 \cup D_2$ contributes the action $m_{1|1|0}(\rho_1, z_1) = \rho_1 \cdot z_1 = z_3$. The region $D_5$ contributes the action $m_{0|1|1}(z_2, \rho_1) = z_2 \cdot \rho_1 = z_3$. The region $D_3 \cup D_4$ contributes $m_{0|1|1}(z_4, \rho_2) = z_4 \cdot \rho_2 = z_5$. The region $D_1 \cup D_3 \cup D_4$ contributes no action, since it has two sides which hit the boundary.
In fact, $\overline{\text{BSAA}}(\mathcal{H}_{AZ})$ is isomorphic to the dual algebra $A(-Z_2)^\vee$ as an $A_\infty$-bimodule; see page 122 of [Zar11] for the general relationship between the actions of a bimodule $M$ and its dual $M^\vee$. This can be seen by identifying $z_1$ with $\rho_1^2$, $z_2$ with $\rho_2^1$, $z_3$ with $\iota_2^\vee$, $z_4$ with $\rho_2^2$, and $z_5$ with $\iota_1^\vee$. Under this identification, the actions described above are dual to the multiplications $\rho_2 \cdot \rho_1 = \rho_1^2$, $\iota_2 \cdot \rho_1^2 = \rho_1^2$, $\rho_1^2 \cdot \iota_2 = \rho_1$, and $\rho_2^1 \cdot \iota_1 = \rho_2$ in $A(-Z_2) \approx A(Z_2)^\vee$. The reader may verify that the remaining actions of $\overline{\text{BSAA}}(\mathcal{H}_{AZ})$ and $A(-Z_2)^\vee$ correspond exactly.

We will see in the next section that understanding this identification is necessary for computing the bordered gluing map.

### 5.2. Computing the bordered gluing map with Heegaard diagrams

In this section we discuss how to compute the bordered gluing map in the special case of contact handle attachments.

Let $(\mathcal{W}, -Z) = (W, \gamma, -\mathcal{F}, -Z)$ be the bordered cap for a sutured surface $\mathcal{F} = (F, \Lambda)$, where $\mathcal{F}$ is parametrized by $\mathcal{Z}$. Now, suppose that sutured manifolds $(M_1, \Gamma_1)$ and $(M_2, \Gamma_2)$ can be glued along $\mathcal{F}$; this is equivalent to saying that there are embeddings $(F, \Lambda) \subset (\partial M_1, \Gamma_1)$ and $(-F, \Lambda) \subset (\partial M_2, \Gamma_2)$ which extend to embeddings $W \subset (M_1, \Gamma_1)$ and $\overline{W} \subset (M_2, \Gamma_2)$. Then $(M_1, \Gamma_1) \setminus W$ and $(M_2, \Gamma_2) \setminus \overline{W}$ are partially sutured manifolds with sutured surfaces $\mathcal{F}$ and $\overline{\mathcal{F}}$, so that $((M_1, \Gamma_1) \setminus W, \mathcal{Z})$ and $((M_2, \Gamma_2) \setminus \overline{W}, \overline{\mathcal{Z}})$ are bordered sutured manifolds. Note that the positive twisting slice $(TW_{\mathcal{F}+}, -\mathcal{Z} \cup -\overline{\mathcal{Z}})$ is also a bordered sutured manifold.

Choose bordered sutured Heegaard diagrams $\mathcal{H}_W$ for $(\mathcal{W}, -\mathcal{Z})$, $\mathcal{H}_U$ for $((M_1, \Gamma_1) \setminus W, \mathcal{Z})$, and $\mathcal{H}_V$ for $((M_2, \Gamma_2) \setminus \overline{W}, \overline{\mathcal{Z}})$. These determine Type-D structures $U = \overline{\text{BSD}}(\mathcal{H}_U)$ and $V = \overline{\text{BSD}}(\mathcal{H}_V)$, along with a Type-A structure $W = \overline{\text{BSA}}(\mathcal{H}_W)$. The mirror image $\mathcal{H}_{-W}$ of $\mathcal{H}_W$ is a Heegaard diagram for $\overline{W}$ and the corresponding Type-A structure is $\overline{\text{BSA}}(\mathcal{H}_{-W}) = W^\vee$. Let $A = A(\mathcal{Z})$. There is a nice diagram $\mathcal{H}_{AZ}$ for $(TW_{\mathcal{F}+}, -\mathcal{Z} \cup -\overline{\mathcal{Z}})$, discovered independently by Auroux and Zarev, such that $\overline{\text{BSAA}}(\mathcal{H}_{AZ}) \cong A^\vee$ as $A_\infty$-bimodules over $A$. The Auroux-Zarev diagrams associated to contact handle attachments are depicted in Figure 16; see Proposition 12.4.2 in [Zar11] for the general construction as well as [Aur10] and [LOT11] for computations.

Now, we have

$$SFC(M_1, \Gamma_1) \otimes SFC(M_2, \Gamma_2) \simeq U \boxtimes W \otimes W^\vee \boxtimes V,$$

and

$$SFC((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)) \simeq U \boxtimes A^\vee \boxtimes V.$$

With these identifications, [Zar11] gives an explicit formula for computing the bordered gluing map when the bordered module $W = \overline{\text{BSA}}(W)$ has a single generator and trivial structure maps; in this case $W$ is called elementary. Note that if $\mathcal{H} = (\Sigma, \alpha, \beta, Z^\alpha, Z^\beta)$ is a bordered sutured diagram with a single generator and all regions adjacent to the intervals $Z^\alpha \sqcup Z^\beta \subset \partial \Sigma$ are basepoint regions, then the bordered modules for $\mathcal{H}$ are elementary.

**Lemma 5.1.** (Zarev) Let $U, V, W, A$ be as above. Suppose that $W$ is an elementary module with unique generator $w$. The bordered gluing map $\Psi_{\mathcal{F}}$ is induced by the map $\psi_{\mathcal{F}} : U \boxtimes W \otimes W^\vee \boxtimes V \to U \boxtimes A^\vee \boxtimes V$ defined by

$$\psi_{\mathcal{F}}(u \boxtimes w \otimes w^\vee \boxtimes v) = u \boxtimes \iota_L(w)^\vee \boxtimes v,$$
Figure 16. Left: $\mathcal{H}_{AZ}$ for $Z_1$; Right: $\mathcal{H}_{AZ}$ for $Z_2$. Each point is additionally labeled by the basis element it corresponds to under the identification $\widehat{\text{BSAA}}(\mathcal{H}_{AZ}) \cong A(-Z_2)^{\vee}$. The orientations of the Reeb chords agrees with the boundary orientation, i.e. the chords on the right are oriented upward, while the chords on the left are oriented downward.

Figure 17. Decomposition of a contact 1-handle (Left) and a contact 2-handle (Right) into a cap and a bordered contact handle. The bordered part is shaded and parametrizing arc diagrams are depicted.

where $u \in U, v \in V$ are arbitrary and $\iota_L(w)$ is the basis idempotent in $A$ which acts non-trivially on $w$.

Figure 17 shows how to decompose each contact handle $-h^i$ into a bordered cap $(-W_i, Z_i)$ and a bordered contact handle $(-h^i \setminus -W_i, -Z_i)$, along with the parametrizations of $F_i$ and $-F_i$ in each case. Figure 18 shows the diagrams $\mathcal{H}_U \cup \mathcal{H}_W$ corresponding to this decomposition. Suppose that $\mathcal{H}$ is a Heegaard diagram for the manifold $(-M, -\Gamma)$ to which the contact handle $-h^i$ is attached. After some Heegaard moves, we may arrange that a neighborhood of the attaching region in $-M$ be represented by a copy of $\mathcal{H}_{-W}$ embedded in $\mathcal{H}$, so that $\mathcal{H} = \mathcal{H}_{-W} \cup \mathcal{H}_V$. To attach $-h^i$, one forms the concatenation $\mathcal{H}_U \cup \mathcal{H}_{AZ} \cup \mathcal{H}_V$. Note that, for each contact handle, the bordered module $\widehat{\text{BSA}}(\mathcal{H}_W)$ is elementary, so that we can use Lemma 5.1 for computations.
Figure 18. $\mathcal{H}_U \cup \mathcal{H}_W$ for a contact 1-handle (Left) and a contact 2-handle (Right).

Figure 19. Gluing a contact 1-handle.

6. Proof of the Main Theorem

The goal of this section is the proof of Theorem 1. We will start by relating the diagrammatic and bordered contact gluing maps for contact handles.

6.1. Diagrammatic and bordered contact gluing maps.

Lemma 6.1. Given an admissible diagram $\mathcal{H}$ for $(-M, -\Gamma)$, there is a diagrammatic handle attachment $\mathcal{H}'$ for $(-M_1, -\Gamma_1)$ such that the diagrammatic map $\sigma_1 : \text{SFC}(\mathcal{H}) \to \text{SFC}(\mathcal{H}')$ induces the bordered contact gluing map $\Psi_1 : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M_1, -\Gamma_1)$ on homology.

Proof. Use the diagram $\mathcal{H}_U \cup \mathcal{H}_W$ shown in Figure 18 for $-h^1$. Note that any admissible Heegaard diagram $\mathcal{H}$ for $(-M, -\Gamma)$ splits as $\mathcal{H} = \mathcal{H}_W \cup \mathcal{H}_V$, where $\mathcal{H}_W$ is a neighborhood of the attaching region of $h^1$ in $\mathcal{H}$.

The intersection point $e$ represents the sole generator $\text{EH}(h^1)$ of $\text{SFH}(-h^1)$; it is also the generator of $\text{BSD}(\mathcal{H}_U)$. The diagram $\mathcal{H}_W$ for the cap has no curves, so the generator $w$ of $\text{BSD}(\mathcal{H}_W)$ is the empty element. (In bordered sutured theory, a diagram with no generator is allowed; in this case the bordered invariants have rank 1, generated by the empty set.)

We have $\iota_{\emptyset} \cdot w = w$, where $\iota_{\emptyset}$ is the unique generator of $A(-Z_1)$; it corresponds to the strand diagram with no strands. Fix an arbitrary cycle $y \in \text{SFC}(\mathcal{H})$ and note that it corresponds to $(w^\vee, y) \in \text{SFC}(\mathcal{H}_W \cup \mathcal{H}_V)$. Since $\text{BSD}(\mathcal{H}_W)$ is an elementary module, we can use Lemma 5.1 to see that the bordered contact gluing map is induced by

$$\psi_{-\mathcal{F}}(e \otimes y) = \psi_{-\mathcal{F}}((e, w) \otimes (w^\vee, y)) = (e, \iota_{\emptyset}, y).$$
Since the diagram $\mathcal{H}_{AZ}$ for the twisting slice has no curves, $\iota_{\sigma_2}$ is the empty set, and we have $(e, \iota_{\sigma_2}, y) = (e, y)$. Now, destabilizing yields the diagram for $(-M, -\Gamma)$ with a 1-handle attached to the Heegaard surface, exactly as in the case of the diagrammatic map. The map $(e, y) \to y$ is a graded homotopy equivalence, so that $\Psi_1[y] = [y] = [\sigma_1(y)]$.

Note that $\mathcal{H}' = \mathcal{H}_U \cup \mathcal{H}_{AZ} \cup \mathcal{H}_V$ is admissible, since its periodic domains correspond to the periodic domains of $\mathcal{H}$.

**Lemma 6.2.** Given an admissible diagram $\mathcal{H}_1$ for $(-M, -\Gamma)$, there are admissible diagrams $\mathcal{H}_3$ for $(-M, -\Gamma)$, $\mathcal{H}_4$ for $(-M_2, -\Gamma_2)$, and $\mathcal{H}_6$ for $(-M_2, -\Gamma_2)$; and graded isomorphisms $f, g$ making the following diagram commute.

$$
\begin{array}{ccc}
\text{SFH}(\mathcal{H}_1) & \xrightarrow{f} & \text{SFH}(\mathcal{H}_3) \\
\downarrow{\sigma_2} & & \downarrow{\Psi_2} \\
\text{SFH}(\mathcal{H}_6) & \xleftarrow{g} & \text{SFH}(\mathcal{H}_4)
\end{array}
$$

**Proof.** We will show how to compute $\Psi_2$ from a given diagram for $\sigma_2$ by a sequence of diagrams $\{\mathcal{H}_i\}$. For the most part, the diagram $\mathcal{H}_{i+1}$ will differ from $\mathcal{H}_i$ by a number of Heegaard moves, so we have graded isomorphisms $\Psi_{\mathcal{H}_i, \mathcal{H}_{i+1}} : \text{SFH}(\mathcal{H}_i) \to \text{SFH}(\mathcal{H}_{i+1})$.
induced by holomorphic triangle counts; $f$ and $g$ will be compositions of such maps. To ease discussion, we defer the computations of these triangle maps to Subsection 6.3.

Let $H_1$ be an admissible Heegaard diagram for $(-M, -\Gamma)$ and use the diagram $H_U \cup H_W$ shown in Figure 18 for $-h^2$. The tuple $(c, w)$ represents the unique generator $EH(h^2)$ of $SFH(-h^2)$, while $w$ is the sole generator for $\hat{\text{BSA}}(H_W)$. Referencing Figure 16, we have $\iota_2 \cdot w = w$. The dashed lines in Figure 20 are the curves $\alpha_0, \beta_0$ for the diagrammatic 2-handle attachment. Fix a cycle $y \in SFC(H_1)$.

In order to compute the bordered contact gluing map, we must find a way to realize $(-M, -\Gamma)$ as $-W_2 \cup_F ((-M, -\Gamma) \setminus W_2)$ at the level of Heegaard diagrams, where $W_2$ is the cap for the 2-handle. For this purpose, we perform two stabilizations. The first is depicted on the right side of Figure 20 so that $\beta_{g+1}$ agrees with $\beta_0$. The second is depicted in Figure 21. This yields a diagram $H_2$ where $\alpha_{g+2}$ agrees with $\alpha_0$ outside a neighborhood of $\beta_0$. (One can think of these stabilizations as attaching trivial bypasses along arcs which approximate the attaching curve for the 2-handle; see Figure 21.) The map $\Psi_{H_1, H_2}$ sending $[y]$ to $[(w^\vee, y_0, y)]$ is induced by a well-defined isomorphism of graded complexes. Handleslide $\beta_{g+1}$ over $\beta_{g+2}$ to obtain the diagram $H_3$. The triangle count $\Psi_{H_2, H_3}$ sends $[(w^\vee, y_0, y)]$ to $[(w^\vee, y_0, y)]$.
Figure 24. $\mathcal{H}_5$; the regions shaded grey represent terms in $\partial(y, y_0, z_1)$.

Figure 25. $\mathcal{H}_6$

The right hand side of Figure 22 shows how to split $\mathcal{H}_3$ as $\mathcal{H}_3 = \mathcal{H}_W \cup \mathcal{H}_V$. Performing a bordered gluing with the 2-handle yields the diagram $\mathcal{H}_4 = \mathcal{H}_U \cup \mathcal{H}_{AZ} \cup \mathcal{H}_V$. Note that $z_3$ corresponds to the basis idempotent $\iota_2$ which acts non-trivially on $w$; compare with Figure 16. The associated bordered gluing map is induced by

$$\psi^\prime_F((c, w) \otimes (w^\vee, y_0, y)) = (c, z_3, y_0, y).$$

After gluing, we destabilize to obtain $\mathcal{H}_5$ as depicted in Figure 24. The part of $\mathcal{H}_5$ which represents $-h^2 \cup TW_{-, F}$ before and after the destabilization are shown in Figure 23. This destabilization is equivalent to performing two handleslides followed by a trivial destabilization. The triangle count $\Psi_{\mathcal{H}_4, \mathcal{H}_5}$ sends $[(c, z_3, y_0, y)]$ to $[(z_3, y_0, y)]$.

Since $y$ is a cycle, we have $\partial(z_1, y_0, y) = (z_3, y_0, y) + (z_2, x_0, y)$; this can be seen from Figure 24, where the shaded regions correspond to the terms on the right. Thus, $[(z_3, y_0, y)] = [(z_2, x_0, y)]$.

Now, perform a small Hamiltonian isotopy to remove $z_1$ and $z_3$, handleslide $\beta_{g+1}$ over $\beta_{g+2}$, and perform a trivial destabilization to obtain $\mathcal{H}_6$; this is the diagram for a diagrammatic 2-handle attachment on $\mathcal{H}_1$. The associated triangle count $\Psi_{\mathcal{H}_5, \mathcal{H}_6}$ sends $[(z_2, x_0, y)]$ to $[(x_0, y)]$.

By composing these maps, we see that the bordered contact gluing map from $\mathcal{H}_1$ to $\mathcal{H}_6$ is

$$[y] \xrightarrow{\Psi_{\mathcal{H}_1, \mathcal{H}_2}} [(w^\vee, y_0, y)] \xrightarrow{\Psi_{\mathcal{H}_2, \mathcal{H}_4}} [(w^\vee, y_0, y)] \xrightarrow{\psi_2} [(c, z_3, y_0, y)] \xrightarrow{\Psi_{\mathcal{H}_4, \mathcal{H}_5}} [(z_3, y_0, y)] = [(z_2, x_0, y)] \xrightarrow{\Psi_{\mathcal{H}_5, \mathcal{H}_6}} [(x_0, y)] = [(y, x_0)],$$
which is the map on homology induced by $\sigma_2$.

Note that all the diagrams we have used are admissible by the following argument. In general, let $\alpha \subset \alpha$ be a curve in a diagram which abuts a suture region on each side. Without loss of generality, suppose that $\alpha$ appears in the boundary of a periodic domain $D$, and the multiplicity of $D$ on each region to the left of $\alpha$ is non-negative. Since there is a basepoint region $D$ on the left where $D$ has multiplicity zero; let $a = \partial D \cap \alpha$. Let $D'$ be the region with $\partial D' \cap \alpha = -a$. Then $D$ must have negative multiplicity on $D'$.

Thus, if one is looking for a periodic domain with no negative coefficients, one can erase any curve (not just an $\alpha$-curve) which abuts the basepoint region on both sides. In each of the diagrams $H_2$ through $H_6$, we can erase a number of such curves to obtain $H_1$ with a number of strips attached to the boundary; the periodic domains of each diagram obtained this way are exactly the periodic domains of $H_1$. Since $H_1$ is admissible by hypothesis, so are $H_2$ through $H_6$.

\[ \square \]

6.2. Proof of Theorem 1. We are now ready to prove the main theorem. The precise statement is as follows.

**Theorem 6.3.** Suppose that $\Phi_\xi$ is the HKM map for a proper inclusion $(M, \Gamma) \subset (M', \Gamma')$ of sutured manifolds with no isolated components and compatible contact structure $\xi$. There is a sutured contact manifold $(M'', \Gamma'', \xi'')$, a sutured surface $F = (F, \Lambda)$, and graded isomorphisms $f, g$ such that

1. $(M', \Gamma') \cong (M'', \Gamma'') \cup_F (M, \Gamma)$
2. $(M' \setminus \text{Int}(M), \xi)$ is contactomorphic to $(M'' \cup_F (\partial M \times [0, 1]), \xi'' \cup_F \xi_\Gamma)$, where $\xi_\Gamma$ is the $[0, 1]$-invariant contact structure compatible with $\Gamma$
3. the dividing set $\Gamma|_F$ is disk-decomposable as defined in Subsection 4.1
4. the following diagram commutes

\[
\begin{array}{ccc}
\text{SFH}(-M, -\Gamma) & \xrightarrow{f} & \text{SFH}(-M, -\Gamma) \\
\downarrow\Phi_\xi & & \downarrow\Phi''
\end{array}
\]

\[
\begin{array}{ccc}
\text{SFH}(-M', -\Gamma') & \xrightarrow{g} & \text{SFH}((-M'', -\Gamma'') \cup_{-F} (-M, -\Gamma))
\end{array}
\]

**Proof of Theorem 1.** Decompose $\xi$ into a sequence of contact handles $h^1, \ldots, h^n$ attached to $\partial M \times \{1\}$ in $(\partial M \times [0, 1], -\Gamma \cup \Gamma, \xi_\Gamma)$. We can factor the HKM map as $\Phi_\xi = \Phi_{i_n} \circ \cdots \circ \Phi_{i_1}$, where each $\Phi_{i_j}$ is the HKM-map for attaching a padded handle $(P_{i_j}, \Gamma_{i_j}, \xi_{i_j})$.

We can also decompose $\xi$ as a sequence of punctured padded handles $(P_{i_1}^{L_1}, \Gamma_{i_1}^{L_1}, \xi_{i_1}^{L_1}), \ldots, (P_{i_n}^{L_n}, \Gamma_{i_n}^{L_n}, \xi_{i_n}^{L_n})$ attached to $(\partial M \times [0, 1], -\Gamma \cap \Gamma, \xi_\Gamma)$ along a surface $F \subset \partial M \times \{1\}$. Let $(M'', \Gamma'', \xi'')$ be the sphere obtained by attaching $(P_{i_2}^{L_2}, \Gamma_{i_2}^{L_2}, \xi_{i_2}^{L_2}), \ldots, (P_{i_n}^{L_n}, \Gamma_{i_n}^{L_n}, \xi_{i_n}^{L_n})$ to $(P_{i_1}^{L_1}, \Gamma_{i_1}^{L_1}, \xi_{i_1}^{L_1})$.

By construction, $(M'', \Gamma'') \cup_F (M, \Gamma)$ can be decomposed as a number of punctured padded handle attachments $(M, \Gamma)$ with bordered contact gluing maps $\Psi_{i_1}, \ldots, \Psi_{i_n}$, where each $\Psi_{i_j}$ corresponds to $(P_{i_j}^{L_j}, \Gamma_{i_j}^{L_j}, \xi_{i_j}^{L_j})$. By Lemma 4.11, these are also the maps for the $h_{i_j}$ up to graded isomorphism. The bordered contact gluing map $\Psi''$ factors as $\Psi'' = \Psi_{i_n} \circ \cdots \circ \Psi_{i_1}$ up to graded isomorphism by Lemma 4.13.

By Lemma 2.8, Lemma 6.1, and Lemma 6.2 the gluing maps $\Phi_{i_j}$ and $\Psi_{i_j}$ equal $(\sigma_{i_j})$, up to graded isomorphism for each $j$. By the composition law for the HKM map and Lemma 4.13, $\Phi_\xi = \Psi''$ up to graded isomorphism.
While we have couched Theorem 1 in terms of constructing a bordered contact map from an HKM map, we can also go in the other direction.

**Theorem 6.4.** Let \((M'', \Gamma'') \cup_F (M, \Gamma)\) be a bordered gluing, where the dividing set \(\Gamma|_F\) on \(F\) is disk-decomposable, and let \(\xi''\) be a compatible contact structure on \(M''\). There is a sutured manifold \((M', \Gamma')\), a proper inclusion \((M, \Gamma) \subset (M', \Gamma')\), a compatible contact structure \(\xi\) on \(M' \setminus \text{Int}(M)\), and graded isomorphisms \(f', g'\) such that

1. \((M', \Gamma') \cong (M'', \Gamma'') \cup_F (M, \Gamma)\)
2. \((M' \setminus \text{Int}(M), \xi)\) is contactomorphic to \((M'' \cup_F (\partial M \times [0, 1]), \xi'' \cup_F \xi_{\Gamma})\), where \(\xi_{\Gamma}\) is the \([0, 1]\)-invariant contact structure compatible with \(\Gamma\)
3. the proper inclusion has no isolated components
4. the following diagram commutes

\[
\begin{array}{ccc}
\text{SFH}(-M, -\Gamma) & \xrightarrow{f'} & \text{SFH}(-M, -\Gamma) \\
\downarrow{\Psi_{\xi''}} & & \downarrow{\Phi_{\xi}} \\
\text{SFH}\left((-M'', -\Gamma'' \cup_F (-M, -\Gamma))\right) & \xrightarrow{g'} & \text{SFH}(-M', -\Gamma')
\end{array}
\]

**Proof.** Let \((M', \Gamma') = (M'', \Gamma'') \cup_F \times \{1\} (M \cup_M \times \{0\}) (\partial M \times [0, 1], \Gamma)\) and let \(\xi = \xi'' \cup_F \times \{1\} \xi_{\Gamma}\). It is clear that the first two conditions are satisfied by construction.

Furthermore, any isolated component of the proper inclusion \((M, \Gamma) \subset (M', \Gamma')\) would correspond to a component of \(M''\) whose boundary is entirely contained in \(F\), or a closed component of \(M''\). Since bordered gluings are only allowed along proper sutured subsurfaces, the former case is not allowed, while the latter is excluded by hypothesis.

Since \(\xi\) and \(\xi''\) are constructed from \(I\)-invariant contact structures by attaching corresponding contact handles, we can argue as in the proof of Theorem 1 that \(\Phi_{\xi}\) and \(\Psi_{\xi''}\) are equal up to graded isomorphism. \(\square\)

### 6.3. Triangle counts

We now verify that the maps in the proof of Lemma 6.2 are graded isomorphisms. All these maps are induced by the standard holomorphic triangle counts associated to handle slides. For \(\alpha\)-handle slides, triangle counts induce chain maps of the form \(\text{SFC}(\Sigma, \beta, \alpha) \otimes \text{SFC}(\Sigma, \alpha, \alpha') \to \text{SFC}(\Sigma, \beta, \alpha')\), while triangle counts induce maps of the form \(\text{SFC}(\Sigma, \beta', \beta) \otimes \text{SFC}(\Sigma, \beta, \alpha) \to \text{SFC}(\Sigma, \beta', \alpha)\) for \(\beta\)-handle slides. We color \(\alpha\)-curves red and \(\alpha'\)-curves orange; we color \(\beta\)-curves blue and \(\beta'\)-curves purple.

We will decorate generators in each \((\Sigma, \beta, \alpha')\) or \((\Sigma, \beta', \alpha)\) with primes and leave generators in each \((\Sigma, \beta, \alpha)\) undecorated. We relabel after each computation, i.e. the generator \((c', z'_2, y'_0, y')\) in \(H_{4,4.5}\) is the generator \((c, z_3, y_0, y)\) in \(H_{4,5.5}\). We denote the highest degree element in each \((\Sigma, \alpha, \alpha')\) or \((\Sigma, \beta', \beta)\) by \(\Theta\) and the lowest degree element by \(\eta\).

We will make use of the following standard fact in our computations. Let \(D\) and \(D'\) be regions in a Heegaard diagram with a common edge in an oriented \(\alpha\)-curve \(\alpha_0\), and let \(D\) be the domain of a Whitney polygon with holomorphic representative. If \(\alpha_0\) abuts a basepoint region on the left and a basepoint region on the right, then the multiplicities of \(\partial D\) on \(D\) and \(D'\) differ by at most 1. In particular, if the common edge is contained in \(\partial D\) and \(D\) is a basepoint region, then \(D\) has multiplicity one on \(D'\).

**Proposition 6.5.** \(\Psi_{\mathcal{H}_2, \mathcal{H}_4}([w, y_0, y]) = [(w', y'_0, y')]\).
Proof. Consider Figure 26. Note that any triangle exiting $y_0$ must have multiplicity 1 on $D$ and multiplicity 0 on $D'$. There is a unique such triangle, and it sends $y_0$ to $y'_0$. We can erase the curves which contain $y_0$ or $y'_0$, and the new diagram is the diagram for a small Hamiltonian isotopy. The corresponding triangle map is chain homotopic to the identity, and thus sends $[(w, y)]$ to $[(w', y')]$. We then have $\Psi_{\mathcal{H}_2, \mathcal{H}_3}([(w, y_0, y)]) = [(w', y'_0, y')]$. \hfill $\square$

Proposition 6.6. $\Psi_{\mathcal{H}_3, \mathcal{H}_4}(c, z_3, y_0, y) = [(z'_3, y_0, y')]$.

Proof. Let $\mathcal{H}_{4,5}$ be the diagram obtained by handlesliding $\alpha'_{g+2}$ over $\alpha_{g+3}$ and consider the triple diagram in Figure 27 for this move. Any triangle for $\Psi_{\mathcal{H}_4, \mathcal{H}_{4,5}}$ exiting $z_3$ must have must have multiplicity 0 on $D_1$ and $D_3$ and multiplicity 1 on $D_2$; this follows from the observation that the curve $\alpha'_{g+2}$ abuts a basepoint region on both the left and the right. (This cannot be seen in Figure 27, but it is true in the complete diagram.) There is a unique such triangle; it enters $z'_3$. Erasing curves yields a diagram for a small Hamiltonian isotopy, so that $\Psi_{\mathcal{H}_4, \mathcal{H}_{4,5}}([(c, z_3, y_0, y)]) = [(c', z'_3, y'_0, y')]$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure26.png}
\caption{The diagram for $\Psi_{\mathcal{H}_2, \mathcal{H}_3}$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure27.png}
\caption{The diagram for $\Psi_{\mathcal{H}_4, \mathcal{H}_{4,5}}$.}
\end{figure}
EQUIVALENCE OF CONTACT GLUING MAPS IN SUTURED FLOER HOMOLOGY

Figure 28. The diagrams for $\Psi_{H_{4.5}}$.

Figure 29. $\Psi_{H_5,H_{5.5}}$ and $\Psi_{H_{5.5},H_6}$.

To compute $\Psi_{H_{4.5},H_5}$, consider the diagram on the left of Figure 28. Any triangle exiting $z_3$ must have multiplicity 0 on $D_1$ and $D_3$ and multiplicity 1 on $D_2$. There is a unique such triangle; it enters $z'_3$. Erasing curves yields the diagram on the right of Figure 28. Since no triangle in our count can enter $\eta$, all triangles must have multiplicity 0 on $D'$. There is a unique such triangle exiting $c'$; its domain is the region $D$, and it enters $c'_3$. Erasing curves yields a diagram for a small Hamiltonian isotopy, and the corresponding triangle map sends $[(c, z_3, y_0, y)]$ to $[(c'_3, y'_0, y')]$. We now obtain $H_5$ by performing a trivial destabilization. The associated map sends $[(c, z_3, y_0, y)]$ to $[(z'_3, y'_0, y')]$ so that $\Psi_{H_{4.5},H_5}[(c, z_3, y_0, y)] = [(z'_3, y'_0, y')]$. □

Proposition 6.7. $\Psi_{H_5,H_{6}}[(z_2, x_0, y)] = [(x'_0, y')]$.

Proof. Let $H_{5.5}$ be the diagram obtained by performing the obvious finger move on $H_5$; diagrams for $\Psi_{H_5,H_{5.5}}$ and $\Psi_{H_{5.5},H_6}$ are shown in Figure 29. On the left, any triangle exiting $x_0$ must have multiplicity one on $D$ and $D'$. There is a unique such triangle; it enters $x'_0$. On the right, any triangle exiting $x_0$ must have multiplicity one on $D_2$ and multiplicity zero on $D_1$ and $D_3$. There is a unique such triangle; it enters $x'_0$.

In each diagram in Figure 29, erasing curves yields a diagram for a small Hamiltonian isotopy. Each corresponding triangle map sends $[(z_2, x_0, y)]$ to $[(z'_2, x'_0, y')]$. We then obtain $H_6$ by performing a trivial destabilization; the corresponding map sends $[(z_2, x_0, y)]$ to $[(x'_0, y')]$. □
Note that the diagrams used in Proposition 6.5, Proposition 6.6, and Proposition 6.7 are all admissible, since the admissibility argument at the end of Lemma 6.2 holds for triply-periodic domains in triple diagrams. Each of the diagrams can be reduced to a diagram for a small Hamiltonian isotopy by erasing curves which abut suture regions on both sides. Since the diagram for a small Hamiltonian isotopy in an admissible diagram is itself admissible, so is each diagram in Proposition 6.5, Proposition 6.6, and Proposition 6.7.

7. Applications

Our first corollary is the extension of contact gluing maps to the bordered sutured category, Corollary 1.3 from Section 1. This corollary is an essential ingredient in [EVZ17].

**Corollary 7.1.** Let \( \mathcal{M} = (M, \gamma, \mathcal{F}, \mathcal{Z}) \) and \( \mathcal{M}' = (M', \gamma', \mathcal{F}, \mathcal{Z}) \) be bordered sutured manifolds with \( M \subset M' \) and \( M' \setminus \text{int}(M) \) a sutured manifold equipped with a compatible contact structure \( \xi \) and no isolated components. Let \( \mathcal{H}_\mathcal{M} \) and \( \mathcal{H}_{\mathcal{M}'} \) be admissible diagrams for \( -\mathcal{M} \) and \( -\mathcal{M}' \), respectively. Then there exists a map of type-D structures induced by \( \xi \)

\[
\phi_\xi : \text{BSD}(\mathcal{H}_\mathcal{M}) \to \text{BSD}(\mathcal{H}_{\mathcal{M}'}),
\]

satisfying the following property. If \( \mathcal{N} = (N, \gamma_N, -\mathcal{F}, -\mathcal{Z}) \) is a bordered sutured manifold with diagram \( \mathcal{H}_N \) for \( -N \), then the map

\[
\phi_\xi \otimes \text{Id}_{\text{BSA}(\mathcal{H}_\mathcal{N})} : \text{BSA}(\mathcal{H}_\mathcal{M}) \otimes \text{BSD}(\mathcal{H}_N) \to \text{BSA}(\mathcal{H}_{\mathcal{M}'}) \otimes \text{BSD}(\mathcal{H}_N)
\]

induces the contact gluing map \( \Phi_\xi : \text{SFH}(-\mathcal{M} \cup_{-\mathcal{F}} -\mathcal{N}) \to \text{SFH}(-\mathcal{M}' \cup_{-\mathcal{F}} -\mathcal{N}) \) up to graded homotopy equivalence.

Similar statements hold for the type-A and bimodule structures found in the bordered sutured theory.

**Proof.** Bordered gluing operations and the bordered gluing map extend to the bordered sutured category. Using the notation of Subsection 5.2, a bordered gluing map

\[
\Psi_{\mathcal{F}} : \text{SFH}(M_1, \Gamma_1) \otimes \text{SFH}(M_2, \Gamma_2) \to \text{SFH}((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2))
\]

is induced by a map of the form

\[
\text{Id}_U \otimes \nabla_W \otimes \text{Id}_V : U \otimes W \otimes W^\vee \otimes V \to U \otimes A^\vee \otimes V.
\]

Zarev [Zar11] proves that \( \Psi_{\mathcal{F}} \) is well-defined by showing that the algebraic join \( \nabla_W : W \otimes W^\vee \to A^\vee \) is well-defined up to \( A_\infty \)-homotopy, and using the fact that \( (\otimes V, \otimes \text{Id}_V) \) is a \( dg \)-functor from the category of \( A_\infty \)-modules to the category of chain complexes; see [LOT15] for explanation.

Note that the bordered gluing operation \( \cup_{\mathcal{F}} \) as discussed in Subsection 3.2 extends to partially sutured and bordered sutured manifolds. The role of one or both \( (M_i, \Gamma_i) \) can be replaced by \( \mathcal{M}_i = (M_i, \gamma_i, \mathcal{F}_i, \mathcal{Z}) \) with \((\pm \mathcal{F}_i, \gamma_i) \subset (\partial M_i) \setminus \mathcal{F}_i \). As in Theorem 1, there is a sutured manifold \((M'', \Gamma'') \) such that \((M'', \Gamma'') \cup_{\mathcal{F}_i} \mathcal{M} \cong \mathcal{M}' \), where the dividing set on \( \mathcal{F}_i \) is disk-decomposable. Furthermore, we can choose \( \mathcal{F}_i \) such that the dividing set has no closed components, so that we can choose a parametrization of \( \mathcal{F}_i \) by an arc diagram \( \mathcal{Z}' \) so that the dividing set is elementary with respect to \( \mathcal{Z}' \). Let \( A' \) be the associated bordered algebra. Let \( \mathcal{H}_{\mathcal{W}'} \) be a diagram for the cap \( \mathcal{W}' \) for the dividing set on \( \mathcal{F}_i \); let \( \mathcal{H}_{\mathcal{U}'} \) be a diagram for \((M'', \Gamma'') \setminus \mathcal{W}'' \) and let \( \mathcal{H}_{\mathcal{U}' V} \) be a diagram for \((-\mathcal{M} \setminus \overline{\mathcal{W}}) \). Let \( \mathcal{U}', \mathcal{V}' \), and \( \mathcal{W}' \) be the bordered modules \( \text{BSD}(\mathcal{H}_{\mathcal{W}'}) \), \( \text{BSD}(\mathcal{H}_{\mathcal{U}' V}) \), and \( \text{BSA}(\mathcal{H}_{\mathcal{W}'}) \) respectively. Note
that $W'$ has a single generator $w$, since the dividing set on $\mathcal{F}'$ is elementary by Proposition 15.1.2 in [Zar11]. We can define a map

$$\psi_{-\mathcal{F}'} = \text{Id}_{U'} \boxtimes \nabla_{W'} \boxtimes \text{Id}_{V'} : U' \boxtimes W' \boxtimes (W')^\vee \boxtimes V' \to U' \boxtimes (A')^\vee \boxtimes V'.$$

which is well-defined up to graded homotopy equivalence of Type-D structures, since by Lemma 2.3.13 of [LOT15], $\boxtimes \text{Id}_{V'}$ is an $A_\infty$-functor.

Similar to Theorem 1, note that there is a compatible contact structure $\xi''$ on $M''$ such that $\xi$ is contactomorphic to $\xi''$ attached to an $I$-invariant contact structure. Since $U' \boxtimes W'$ is homotopy equivalent to $\text{SFC}(-M'', -\Gamma'')$, we can define $\phi_{\xi} : (W')^\vee \boxtimes V \to U' \boxtimes (A')^\vee \boxtimes V'$ by

$$\phi_{\xi}(w^\vee \boxtimes y) = \psi_{-\mathcal{F}'}(x \boxtimes w, w^\vee \boxtimes y),$$

where $x \boxtimes w$ represents $\text{EH}(\xi'')$.

Denote $\widehat{\text{BSD}}(\mathcal{H}_N)$ by $X$. Since $(W')^\vee \boxtimes V' \boxtimes X \simeq \text{SFC}(\mathcal{H}_M \cup_{\mathcal{F}} \mathcal{H}_N)$ and $U' \boxtimes (A')^\vee \boxtimes V' \boxtimes X \simeq \text{SFC}(\mathcal{H}_M' \cup_{\mathcal{F}} \mathcal{H}_N)$, the bordered contact gluing map $\Psi_{\mathcal{F}_i}$ is induced by evaluating

$$\text{Id}_{U'} \boxtimes \nabla_{W'} \boxtimes \text{Id}_{V'} \boxtimes \text{Id}_{X}$$

on $x \boxtimes w$, while $\phi_{\xi} \boxtimes \text{Id}_X$ is the evaluation of

$$\text{Id}_{U'} \boxtimes \nabla_{W'} \boxtimes \text{Id}_{V'} \boxtimes \text{Id}_X$$

on $x \boxtimes w$. It is clear that these maps are equal up to homotopy equivalence since $\text{Id}_{V'} \boxtimes \text{Id}_X$ is homotopic to $\text{Id}_{V'} \boxtimes \text{Id}_X$. (This is from Lemma 2.3.13 in [LOT15].)

Note that by Theorem 1, $\Psi_{\mathcal{F}_i}$ agrees with the contact gluing map $\Phi_{\xi}$ up to graded isomorphism. Also, since $\psi_{-\mathcal{F}'}$ is defined up to graded homotopy equivalence, so is $\phi_{\xi}$, so we can define $\phi_{\xi} : \widehat{\text{BSD}}(\mathcal{H}_M) \to \widehat{\text{BSD}}(\mathcal{H}_M')$ by using homotopy equivalences $W' \boxtimes V' \simeq \widehat{\text{BSD}}(\mathcal{H}_M)$ and $U' \boxtimes (A')^\vee \boxtimes V' \simeq \widehat{\text{BSD}}(\mathcal{H}_M')$.

This completes the proof for type-$D$ structures. Similar arguments hold for type-$A$ modules and bimodules.

We also obtain an independent proof of Juhász and Zemke’s result that diagrammatic maps agree with the corresponding contact gluing maps; see [JZ20]. This is Corollary 1.4 in Section 1.

**Corollary 7.2.** Given a diagram $\mathcal{H}$ for $(-M, -\Gamma)$, there is a diagram for $(-M_i, -\Gamma_i)$ such that the HKM map $\Psi_{\mathcal{F}_i} : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M_i, -\Gamma_i)$ is induced by the diagrammatic map $\sigma_{\mathcal{F}_i}$ up to graded isomorphism. Furthermore, given any other diagrammatic attachment for $(-M_i, -\Gamma_i)$, the associated diagrammatic map $\sigma'_i$ also induces $\Phi_{\mathcal{F}_i}$ up to graded isomorphism.

**Proof.** The first statement is immediate from Lemma 2.8. By Lemma 6.1 and Lemma 6.2, both $(\sigma_i)_*$ and $(\sigma'_i)_*$ are equal to bordered contact gluing maps $\Psi_{\mathcal{F}_i}$ and $\Psi_{\mathcal{F}_i}'$, respectively, up to graded isomorphism. Since the map $\Phi_{-\mathcal{F}_i}$ is well-defined, $\Psi_i$ and $\Psi_i'$ are equal up to graded isomorphism. Then $(\sigma_i)_*$ and $(\sigma'_i)_*$ are also equal up to graded isomorphism. □

As a final application, we use Corollary 7.2 to show that the HKM map can be computed using nice diagrams by extending Plamenevskaya’s application of the Sarkar–Wang algorithm in [Pla07]. This is Corollary 1.5 in Section 1.
Corollary 7.3. Suppose that \((-M, -\Gamma) \subset (-M', -\Gamma')\) is a proper inclusion of sutured manifolds with no isolated components and contact gluing map \(\Phi_\xi\). Given an admissible diagram \(\mathcal{H}_0\) for \((-M, -\Gamma)\), there is a diagram \(\mathcal{H}_2\) for \((-M', -\Gamma')\), a nice diagram \(\mathcal{H}_2^{\text{nice}}\) for \((-M, -\Gamma)\), a nice diagram \(\mathcal{H}_2^{\text{nice}}\) for \((-M', -\Gamma')\), a map \(\phi_\xi^{\text{nice}} : \text{SFC}(\mathcal{H}_2^{\text{nice}}) \to \text{SFC}(\mathcal{H}_2^{\text{nice}})\) of the form \(y \mapsto (y, x_0)\), and graded isomorphisms \(f, g\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{SFH}(\mathcal{H}_0) & \xrightarrow{f} & \text{SFH}(\mathcal{H}_2^{\text{nice}}) \\
\downarrow{\Phi_\xi} & & \downarrow{\phi_\xi^{\text{nice}}} \\
\text{SFH}(\mathcal{H}_2) & \xrightarrow{g} & \text{SFH}(\mathcal{H}_2^{\text{nice}})
\end{array}
\]

Some discussion is in order before we begin the proof. By Corollary 1.4, the HKM map can be factored into diagrammatic maps \(\Phi_\xi = (\sigma_{i_n} \circ \ldots \circ \sigma_{i_1})_*\) up to graded isomorphism. Let \(\mathcal{H}\) be a diagram for \((-M, -\Gamma)\) and \(\mathcal{H}'\) a diagram for \((-M', -\Gamma')\) obtained by diagrammatic handle attachments. The composition \(\sigma_{i_n} \circ \ldots \circ \sigma_{i_1}\) takes the form \(\phi_\xi(y) = (y, x_0)\), where \(x_0\) is the collection of preferred intersections of the 2-handles. We wish to show that this situation can be realized by nice diagrams and that the preferred intersections in the nice diagrams correspond to \(x_0\) up to graded homotopy equivalence.

Note that we cannot just perform diagrammatic handle attachments on a given nice diagram for \((M, \Gamma)\), since a diagrammatic contact 2-handle attachment on a nice diagram need not result in a nice diagram. We will instead find a nice diagram for \((M', \Gamma')\) and work backwards to obtain a suitable diagram for \((M, \Gamma)\).

As in [Pla07], the difficulty is showing that the Sarkar–Wang algorithm may be applied in a way that preserves \(x_0\). More precisely, we wish to show there is a diagram which is commutative up to homotopy,

\[
\begin{array}{ccc}
\text{SFC}(\mathcal{H}) & \xrightarrow{\psi} & \text{SFC}(\mathcal{H}_2^{\text{nice}}) \\
\downarrow{\phi_\xi} & & \downarrow{\phi_\xi^{\text{nice}}} \\
\text{SFC}(\mathcal{H}') & \xrightarrow{\psi'} & \text{SFC}(\mathcal{H}_2^{\text{nice}})
\end{array}
\]

where \(x_0^{\text{nice}}\) is the preferred intersection in \((\mathcal{H}_2^{\text{nice}})'\) and \(\psi, \psi'\) are triangle maps induced by Heegaard moves in the Sarkar–Wang algorithm. In [Pla07], they key idea is to show that one can apply Sarkar–Wang without performing a finger move around a full \(\beta\)-curve. This implies that no handleslide maps arise, and the preferred intersections are preserved by the maps which arise in the course of the algorithm.

In our setting, we cannot always rule out finger moves around full \(\beta\)-curves in general, but we can rule out finger moves around \(\beta\)-curves coming from contact 2-handle attachments. Recall that attaching a contact 2-handle adds a strip to the Heegaard surface. If a finger move which enters this strip does not cross the corresponding \(\alpha\)-curve, then it can be isotoped outside the strip. If it crosses the \(\alpha\)-curve, it must terminate there, since the \(\alpha\)-curve borders a basepoint region on one side. In this case, we can also isotope the finger move outside the strip; see Figure 30. This means that we can apply Sarkar–Wang without handlesliding over any of the new \(\beta\)-curves.

This is sufficient for our purposes, since if we perform a handleslide over a \(\beta\)-curve coming from \(\mathcal{H}\) or a finger move, then the diagram for the triangle count in the strip looks like Figure 31. It is easy to see that these maps all preserve the preferred intersections.
Proof. Let $\mathcal{H}_0 = (\Sigma, \beta, \alpha)$ be a diagram for $(-M, -\Gamma)$ and apply the Sarkar–Wang algorithm to obtain a nice diagram $\mathcal{H}_0^{nice}$. Decompose $(-M', -\Gamma')$ as $(-M, -\Gamma) \cup_j -h_1^1 \cup_k -h_2^2$. We attach diagrammatic 1-handles to form a diagram $\mathcal{H}_1^{nice}$ for $(-M, -\Gamma) \cup_j -h_1^1$ which is nice, since we have only modified the sutured region.

Now, attach diagrammatic 2-handles to $\mathcal{H}_1^{nice}$ to obtain a diagram $\mathcal{H}_2 = (\Sigma', \beta \cup \beta_0, \alpha \cup \alpha_0)$ for $(-M', -\Gamma')$. Apply Sarkar–Wang again to get a nice diagram $\mathcal{H}_2^{nice} = (\Sigma', \beta' \cup \beta'_0, \alpha \cup \alpha_0)$. As discussed above, we can do this so that all finger moves are performed in the complement of the strips for the contact handles, $\Sigma' \setminus \text{Int}(\Sigma)$. Thus, the triangle map $\psi_{\mathcal{H}_2,\mathcal{H}_2^{nice}}$ sends generators of the form $(y, x_0)$ to generators of the form $(z, x_0^{nice})$.

Now, let $\mathcal{H}_n^{nice} = (\Sigma, \beta', \alpha)$ be the Heegaard diagram for $(-M, -\Gamma)$ obtained by removing all the diagrammatic handles from $\mathcal{H}_2^{nice}$. Note that it is nice, since it differs from $\mathcal{H}_0^{nice}$ by some finger moves which appear in the Sarkar–Wang algorithm. Consider the triple diagram $(\Sigma', \beta \cup \beta_0, \beta' \cup \beta'_0, \alpha \cup \alpha_0)$ for $\Psi_{\mathcal{H}_2,\mathcal{H}_2^{nice}}$; let $x_0$ and $x_0^{nice}$ be the preferred intersections in $\beta_0 \cap \alpha_0$ and $\beta'_0 \cap \alpha_0$ respectively. Since we only care about the image of the contact gluing map, we can restrict the triangle count for $\Psi_{\mathcal{H}_2,\mathcal{H}_2^{nice}}$ to the subspace of $\text{SFC}(\mathcal{H}_2)$ generated by tuples of the form $(y, x_0)$. This restricted count is the same as the full triangle count for $\Psi_{\mathcal{H}_0,\mathcal{H}_0^{nice}}$ arising from $(\Sigma, \beta, \beta', \alpha)$, so that for a cycle $y$ in $\text{SFC}(\mathcal{H}_0)$, we have $\psi_{\mathcal{H}_2,\mathcal{H}_2^{nice}}(y, x_0) = (\psi_{\mathcal{H}_0,\mathcal{H}_0^{nice}}(y), x_0^{nice})$. We can define a map
\( \phi^\text{nice} : \text{SFC}(\mathcal{H}^\text{nice}) \to \text{SFC}(\mathcal{H}_2^\text{nice}) \) by

\[
\phi^\text{nice}_\xi(\psi_{\mathcal{H}_0, \mathcal{H}^\text{nice}}(y)) = (\psi_{\mathcal{H}_0, \mathcal{H}^\text{nice}}(y), x^\text{nice}_0),
\]

so that for any cycle \( z \in \text{SFC}(\mathcal{H}^\text{nice}) \) we have

\[
(\phi^\text{nice}_\xi)_*[z] = [(z, x^\text{nice}_0)].
\]

By construction, \( (\phi^\text{nice}_\xi)_* \) has the desired form and is equal to the HKM map \( \Phi_\xi \) up to graded isomorphism. \( \square \)

References

[Aur10] Denis Auroux, *Fukaya categories and bordered Heegaard-Floer homology*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 917–941. MR 2827825

[EO11] Tolga Etgü and Burak Ozbagci, *On the relative Giroux correspondence*, Low-dimensional and symplectic topology, Proc. Sympos. Pure Math., vol. 82, Amer. Math. Soc., Providence, RI, 2011, pp. 65–78. MR 2768654

[EVZ17] John B. Etnyre, David Shea Vela-Vick, and Rumen Zarev, *Sutured Floer homology and invariants of Legendrian and transverse knots*, Geom. Topol. 21 (2017), no. 3, 1469–1582. MR 3650078

[Ghi05] Paolo Ghiggini, *Strongly fillable contact 3-manifolds without Stein fillings*, Geom. Topol. 9 (2005), 1677–1687. MR 2175155

[Ghi06a] _______, *Infinitely many universally tight contact manifolds with trivial Ozsváth-Szabó contact invariants*, Geom. Topol. 10 (2006), 335–357. MR 2224460

[Ghi06b] _______, *Ozsváth-Szabó invariants and fillability of contact structures*, Math. Z. 253 (2006), no. 1, 159–175. MR 2206641

[Gir91] Emmanuel Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991), no. 4, 637–677. MR 1129802

[Gol15] Marco Golla, *Comparing invariants of Legendrian knots*, Quantum Topol. 6 (2015), no. 3, 365–402. MR 3392959

[HKM08] Ko Honda, William H. Kazez, and Gordana Matić, *Contact structures, sutured Floer homology and TQFT*, Preprint, 2008.

[HKM09] _______, *The contact invariant in sutured Floer homology*, Invent. Math. 176 (2009), no. 3, 637–676. MR 2501299

[HT16] Ko Honda and Yin Tian, *Contact category of disks*, Preprint, 2016.

[Juh16] András Juhász, *Cobordisms of sutured manifolds and the functoriality of link Floer homology*, Adv. Math. 299 (2016), 940–1038. MR 3519484

[LOT15] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston, *Bimodules in bordered Heegaard Floer homology*, Geom. Topol. 19 (2015), no. 2, 525–724. MR 3336273

[LOT11] Patrick Massot, *Infinitely many universally tight torsion free contact structures with vanishing Ozsváth-Szabó contact invariants*, Math. Ann. 353 (2012), no. 4, 1351–1376. MR 2944032

[Mat19] Daniel V. Mathews, *Strand algebras and contact categories*, Geom. Topol. 23 (2019), no. 2, 637–683. MR 3939043

[OS05] Peter Ozsváth and Zoltán Szabó, *Heegaard Floer homology and contact structures*, Duke Math. J. 129 (2005), no. 1, 39–61. MR 2153455

[Ozb11] Burak Ozbagci, *Contact handle decompositions*, Topology Appl. 158 (2011), no. 5, 718–727. MR 2774055
[Pla07] Olga Plamenevskaya, A combinatorial description of the Heegaard Floer contact invariant, Algebr. Geom. Topol. 7 (2007), 1201–1209. MR 2350279

[SV09] András I. Stipsicz and Vera Vértesi, On invariants for Legendrian knots, Pacific J. Math. 239 (2009), no. 1, 157–177. MR 2449016

[Zar11] Rumen Zarev, Bordered Sutured Floer Homology, ProQuest LLC, Ann Arbor, MI, 2011, Thesis (Ph.D.)–Columbia University. MR 2941830

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
E-mail address: cleigo2@lsu.edu
URL: http://www.math.lsu.edu/~cleigo2/

Department of Mathematics, The Israel Institute of Technology, Haifa, Israel
E-mail address: salmoiraghi@campus.technion.ac.il