Scaling Violation in $O(N)$ Vector Models

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Abstract

We investigate $O(N)$-symmetric vector field theories in the double scaling limit. Our model describes branched polymeric systems in $D$ dimensions, whose multicritical series interpolates between the Cayley tree and the ordinary random walk. We give explicit forms of residual divergences in the free energy, analogous to those observed in the strings in one dimension.
1. introduction

The method of $1/N$ expansion had provided us an insight into non-perturbative aspects of field theories, especially of gauge field theories [1]. Its utility, however, as a constructive definition of the string theory in less than one dimension is too great to be called as a by-product; matrix models in the double scaling limit [2] turned out to be equipped with the richest structure, and to share the physical content of the continuum theory [3].

In ref. [4], we have presented another example of non-perturbative treatment of random statistical objects — branched polymers via double-scaled $O(N)$-symmetric vector models. Our interest in these models was twofold: as a toy model for investigating non-perturbative features of bosonic strings in the branched-polymeric phase, and more practically, as a probe into the structure of two-dimensional gravity through the strong resemblance between the two theories. In particular, the fact that the Virasoro constraints on the $\tau$-function is characterized as a symmetry of the ‘string equation’ [5] is readily observed in its vector-model counterpart, owing to the simplicity and linearity of the latter [3].

The notion of double scaling limit renewed the approach to the $O(N)$-symmetric field theories which has been a fundamental set-up for the $1/N$ expansion. In refs. [7] a local field theory of the composite field was extracted via the double scaling limit, and the ‘canonical’ combination of scaling parameters was shown to be violated through the renormalization of ultraviolet divergences. See ref. [8] for the supersymmetric generalization. It is also pointed out [3] that the form of double-scaled field theory is universal, scarcely dependent upon the original $O(N)$ field theory, i.e. how we discretize and weight the branched polymers.

The purpose of this paper is to clarify the violation of the canonical scaling, in particular residual divergences in the free energy which is not explicitly worked out in refs. [7]. For completeness we review the double scaling limit of $O(N)$-vector field theories. Critical exponents of the theory, which fulfill Fisher’s relation, suggest that the statistical systems at hand are thought of multicritical generalizations of the Cayley tree. Then we state our main result on the scaling violations of the renormalized free energy at low-loop levels.

2. branched polymers and $O(N)$ vector models

We start by considering the one-dimensional reduction of Polyakov string:

$$F = \sum_{g=0}^{\infty} \kappa^{2g-2} \sum_{(BP_g)} e^{-t|BP|} \int [d^D X(\tau)] e^{-\frac{1}{2} \int d\tau \left( \frac{dX^\mu}{d\tau} \right)^2}. \quad (1)$$

Here the sum $(BP_g)$ is taken over the set of connected branched polymers with $g$-loops, $|BP|$ denotes the total intrinsic length of a branched polymer, $t$ the cosmological constant and $\tau$ parametrizes the polymer. In our context an arbitrary polymer should be weighted solely by its length and number of loops, irrespective of its way of branching.

The above definition is only formal unless accompanied with a definite choice of measure as well as of regularization. We employ random-lattice regularization of branched
polymers, where polymers are made of line segments (bonds) with the same lattice constant $\tilde{a}$ by joining arbitrary numbers of the ends at their vertices (molecules). Matter degrees of freedom in eq.(1) are assigned on the vertices of the polymer. Then the regularized partition function is defined as

$$F = \sum_{g=0}^{\infty} \sum_{(BP_g)^{2g-2}} \frac{1}{S(BP)} e^{-t_0 n_1(BP)} \prod_{i \in BP} d^D X_i e^{-\frac{1}{2} \sum_{(i,j)} |X_i - X_j|^2}$$

(2)

where $S(BP)$ denotes the symmetry factor of a branched polymer, $n_1(BP)$ the number of bonds in a polymer and $\sum_{(i,j)}$ the sum over adjacent vertices, and the subscript 0 refers to bare quantities. A possible generalization of this model is to introduce branch-units with more than three end points, with activities $\lambda_n$. This generalization is imported from the matrix-model realization of two-dimensional gravity where inclusion of equilateral polygons into the dynamical triangulation generates multi-critical behaviours.

These statistical systems of branched polymers are exactly realized as dual diagrams of Feynman graphs in the perturbative expansion of the following $O(N)$-symmetric vector model $[9]$

$$Z(\lambda) = \int [d^N \phi] \exp -\beta \left( \int d^D X V(\phi^2(X)) - \frac{\lambda}{4} \int d^D X d^D X' e^{-\frac{1}{2} |X - X'|^2} \phi^2(X) \phi^2(X') \right),$$

(3)

where $\phi = (\phi_a)$ is $N$-component real scalar field and

$$V(\phi^2) = \frac{1}{2} \phi^2 - \sum_{n \geq 2} \frac{\lambda_n}{2n} (\phi^2)^n.$$  

(4)

Here a dual diagram is meant in a sense of one-dimensional geometry, i.e. the 0-simplex ($(\phi^2)^n$ vertex) and 1-simplex (propagator) of Feynman graphs are interchanged with the 1-simplex (bond or branch unit) and 0-simplex (molecule) of branched polymers respectively. Namely, for the simplest potential $V(\phi^2) = \frac{1}{2} \phi^2$, we observe

$$\ln \frac{Z(\lambda)}{Z(0)} = \sum_{g=0}^{\infty} \sum_{(BP_g)^{2g-2}} \frac{1}{S(BP)} \prod_{i \in BP} d^D X_i \prod_{(i,j) \in BP} \left( \frac{\lambda}{\beta} e^{-\frac{1}{2} |X_i - X_j|^2} \right)$$

(5)

Since the divergent factor $\delta^D(0)$ (a redundant $\delta$-function in an index loop) is always associated with $\tilde{a}$ large, it can be absorbed into the definition of the bare parameters in
We then confirm \( F = \ln(Z(\lambda)/Z(0)) \) under identification of the parameters

\[
\kappa_0^2 = \frac{1}{N\delta^D(0)}, \quad t_0 = -\ln \left( \frac{N\delta^D(0)}{\beta} \lambda \right). \tag{6}
\]

3. Continuum Limit

Now we repeat the method developed in the zero-dimensional case: First we find a large-\( N \) saddle point of the effective potential. Then we scale the bare cosmological and string coupling constants towards the critical point simultaneously, in such a way that each loopwise infrared divergence of the free energy is compensated by the increasing power of \( N \) so that contributions from arbitrarily looped configurations survive the continuum limit. We expect a crucial difference to appear between the cases \( D = 0, 1 \) and \( D \geq 2 \), since in the latter case the canonical scaling determined by a naïve large-\( N \) expansion is altered by the renormalization procedure of ultraviolet divergences in the target space.

For brevity we take the simplest model \( V(\phi^2) = \frac{1}{2} \phi^2 \). After integrating over angular variables and exponentiating the jacobian involved, the partition function takes the form

\[
Z = \int [dx] \ e^{-S_{eff}[x]}, \quad S_{eff}[x] \equiv \beta \left\{ \int \frac{d^D X}{(2\pi)^{D/2}} \frac{1}{2} x(X) \right. \\
- \int \frac{d^D X}{(2\pi)^{D/2}} \frac{d^D X'}{(2\pi)^{D/2}} \frac{\lambda}{4} e^{-\frac{1}{4} |X-X'|^2} x(X)x(X') - \frac{N}{\beta} \delta^D(0) \int \frac{d^D X}{(2\pi)^{D/2}} \frac{1}{2} \ln \left( \frac{x(X)}{2} \right) \left\}.
\]

Here we have rescaled \( \phi^2 = x \) from eq.(3) by a factor \( (2\pi)^{-D/2} \) for later convenience.

The large-\( N \) saddle point equation of the effective potential reads

\[
\frac{N}{\beta} \delta^D(0) = x(X) - \lambda \int \frac{d^D X'}{(2\pi)^{D/2}} e^{-\frac{1}{4} |X-X'|^2} x(X)x(X'). \tag{8}
\]

Eq.(8) reduces to the saddle point equation of the one-vector model in \( m = 2 \) criticality, \( \Delta = (1-x/x_c)^2 \) \((\Delta \equiv 1-N\beta \delta^D(0))\) under the choice \( \lambda = 1/4 \) and \( x_c = 2 \) (position-independent saddle point \( x(X) = x_s \) is assumed). Note that the condition \( \delta S_{eff}/\delta x = \delta^2 S_{eff}/\delta x^2 = 0 \) which determines the critical values of \( \lambda \) and \( x \) leads to the masslessness of a boundstate associated with the composite field \( x = \phi^2 \).

By substituting the saddle point solution \( x(X) = x_s = 2(1-\Delta^{1/2}) \), we obtain the leading singular term of the free energy per unit volume \( (2\pi)^{D/2} \)

\[
F^{(0)} \sim -\frac{1}{3} N\delta^D(0) \Delta^{3/2} + \text{regular terms} \tag{9}
\]

We can also generate multicritical behaviours characterized by \( \gamma^{(0)} = 1 - 1/m \) by fine-tuning the potential.

The effective action in eq.(7) is expanded in the \( 1/N \) series around this saddle point. In terms of \( \tilde{x}(X) = x(X) - x_s \) it reads

\[
S_{eff}[\tilde{x}] = \beta \int \frac{d^D X}{(2\pi)^{D/2}} \left( -\frac{1}{3} \Delta^{3/2} + \frac{1}{32} (\partial_\mu \tilde{x})^2 + \frac{1}{8} \frac{\Delta^{1/2} x^2}{48} \right) + \text{(higher order terms in } \tilde{x}, \text{ its derivatives and } \Delta). \tag{10}
\]
This seemingly local expression follows from the Taylor expansion of $\tilde{x}(X')$ in the nonlocal term in eq.(4),

$$\int \frac{d^D X'}{(2\pi)^{D/2}} e^{-\frac{1}{2}|X-X'|^2} \tilde{x}(X') = \tilde{x}(X) + \frac{1}{2} \partial^2 \tilde{x}(X) + \cdots.$$ 

Now we introduce the lattice spacing $a$ in the target space, with which a dimensionful (physical) length $\tilde{X}$ is defined as $\tilde{X} = a \cdot X$. Then the form of the expression (10) suggests that the effective action can be made nontrivial and finite in renormalized quantities (in the sense of continuous polymers) under the following scaling limit:

$$\begin{cases}
\beta = \tilde{a}^{(D-6)/4} \cdot \kappa^{-2} \\
\Delta = \tilde{a} \cdot t \\
\tilde{x} = \tilde{a}^{1/2} \cdot (-4\kappa)\varphi
\end{cases}$$

accompanied with the identification between the intrinsic and extrinsic lattice constants by $\tilde{a} = a^4$ approaching zero. The first equation in (11) requires the dimension of the space to be smaller than six in order for $\beta$ to diverge to infinity so that large-$N$ expansion is meaningful. Then higher order terms in eq.(10) are suppressed by positive powers of $a$, to render the effective action to a local field theory

$$S_{\text{eff}} [\varphi] = \int \frac{d^D \tilde{X}}{(2\pi)^{D/2}} \left( -\frac{1}{3} \tilde{a}^{3/2} + \frac{1}{2} (\tilde{\partial}_\mu \varphi)^2 + 2t^{1/2} \varphi^2 + \frac{4\kappa}{3} \varphi^3 \right).$$

The correlation length of branched polymers in the target space is characterized by the mass of the $\varphi$-field, $m(t) = 2t^{1/4}$. This immediately shows that the (external) Hausdorff dimension $D_H$, defined as a diverging rate of the correlation length $\xi(t) \sim t^{-1/D_H}$ in the critical region, is equal to 4. We observe that these value of critical exponents ($\gamma = \frac{1}{2}$ and $\nu = 1/D_H = \frac{1}{4}$) are coincident with those of the Cayley tree, as naturally expected from the form of the $\varphi^3$ field theory derived in the scaling limit. We point out, moreover, that under the translation $\varphi \rightarrow \varphi - 2t^{1/2}/\kappa$ the effective action takes the typical form

$$S_{\text{eff}} [\varphi] = \int \frac{d^D \tilde{X}}{(2\pi)^{D/2}} \left( \frac{1}{2} (\tilde{\partial}_\mu \varphi)^2 - \frac{t}{\kappa} \varphi + \frac{4\kappa}{3} \varphi^3 \right)$$

of a massless scalar field theory perturbed by the most infrared-divergent operator $\varphi^1$, coupled to the cosmological constant; it reduces to the zero-dimensional ‘effective action’ in the Laplace-transformed solution to the chain equation in ref.[4] if the $\varphi$-field is translationally invariant.

Now that it is straightforward to extract an $m$-th multicritical field theory

$$S_{\text{eff}} [\varphi] = \int \frac{d^D \tilde{X}}{(2\pi)^{D/2}} \left( \frac{1}{2} (\tilde{\partial}_\mu \varphi)^2 - \frac{t}{\kappa} \varphi + \frac{2m^{m-1}}{m+1} \varphi^{m+1} \right)$$

out of a vector model with a generalized potential by fine-tuning $(m-1)$ parameters as in ref.[4]. Here the scaling limit

$$\begin{cases}
\beta = \tilde{a}^{((m-1)D-2(m+1))/(2m)} \cdot \kappa^{-2} \\
\Delta = \tilde{a} \cdot t \\
\tilde{x} = \tilde{a}^{1-1/m} \cdot (-4\kappa)\varphi
\end{cases}$$
is accompanied with the identification between the lattice constants 
\( \tilde{a} = a^{2m/(m-1)} \rightarrow 0 \).
Again the double scaling limit is possible only for \( D < 2(m + 1)/(m - 1) \), i.e., exactly
when the resulting local field theory is superrenormalizable.

The external Hausdorff dimension is also read from eq. (14) and is equal to

\[ D_H = \frac{2m}{m - 1}, \]  

(16)

which coincides with the results of different approaches in refs. [10, 11]. We note that
this result seems to suggest that the theory approaches the ordinary random walk (where
\( D_H = 2 \)) in the limit \( m \rightarrow \infty \). On the other hand, since the Feynman propagator of scalar
field theories behaves as \( |\tilde{X}|^{-(D-2)} \) at the vanishing mass for large \( |\tilde{X}| \), the anomalous
dimension \( \eta \) is equal to zero, irrespective of order of the criticality. It is instructive to
point out that by combining \( \gamma(0) = 1 - 1/m, \nu = (m - 1)/2m \) and \( \eta = 0 \) we can confirm
that Fisher’s relation

\[ \gamma = \nu(2 - \eta) \]  

(17)

between critical exponents is fulfilled for any branched-polymeric systems of our kinds, as
long as these exponents are well-defined.

4. renormalization and residual divergences

We have identified superrenormalizability with a criterion for the applicability of the
double scaling limit; the \( m \)-th critical continuum theory exists iff dimensionality of the
target space satisfies \( D < 2(m + 1)/(m - 1) \).

So far we have constrained ourselves to the *canonical* scaling limit of \( O(N) \) vector field
theories. This does not suffice to render the contribution of an arbitrarily looped configu-
ration of polymers finite, because of ultraviolet divergences in the target space with dimen-
sionality \( D \geq 2 \). In fact, the cosmological constant \( t \) in the effective action (14) must not
be a constant, but be dependent upon the cutoff \( \Lambda \sim a^{-1} \) in such a way that the counter-
term part of \( S_{\text{eff}}[\varphi] \) cancels divergences originating from its physical part. This means
that the canonical double scaling limit (13) where the combination \( \beta \Delta(2(m+1)-(m-1)D)/2m \)
is kept fixed to \( \kappa^{-2}t(2(m+1)-(m-1)D)/2m \) is violated through the renormalization procedure
[7]. In the following we clarify the situation by paying attention to the *residual* divergences
in the free energy which is not explicitly stated in the above references.

As illuminating examples of the scaling violation, we consider \( \varphi^3 \) field theories (12) at
\( D = 1, 2, 3 \) and 4. For convenience we employ the following form of the effective action

\[ S_{\text{eff}}[\varphi] = \int d^D \tilde{X} \left( -\frac{1}{3}\frac{t^{3/2}}{\tilde{\kappa}^2} + \frac{1}{2}(\tilde{\partial}_\mu \varphi)^2 + 2t^{1/2}\varphi^2 + \frac{4\tilde{\kappa}}{3}\varphi^3 \right) \]  

(18)

by rescaling \( \varphi \rightarrow (2\pi)^{D/4} \varphi \) \( (\tilde{\kappa} \equiv (2\pi)^{D/4}\kappa) \).
For dimensions $D < 6$, possible primitively divergent graphs are zero-, one- and (the momentum-independent part of) two-point functions. Thus we should separate the bare action \((18)\) into the physical and the counter-term part appropriately as

\[
S_{\text{eff}} = S_{\text{phys}} + S_{\text{c.t.}} = \int d^D \tilde{X} \left( \frac{1}{2} (\tilde{\partial}_\mu \varphi_r)^2 + 2 t_r^{1/2} \varphi_r^2 + \frac{4 \tilde{\kappa}}{3} \varphi_r^3 \right) + \int d^D \tilde{X} \left( -c_0 - c_1 \varphi_r - \frac{1}{2} c_2 \varphi_r^2 \right). \tag{19}
\]

Here \(\varphi_r\) is the renormalized field and \(t_r\) is the renormalized mass parameter\(^\dagger\).

What is important is that we are not necessarily allowed to fix the renormalization coefficients \(c_i(a)\) so that all divergences in bubble diagrams are canceled; it is because the constant and the mass term in the bare action \((18)\), to be equated to eq.(19) through a shift in \(\varphi\) by a divergent constant, are related as they have been extracted from the original vector model. Since finiteness of correlation functions always takes preference over that of vacuum bubbles in the renormalization procedure, the free energy generically contains residual divergences.

**One Dimension**

In one dimension the continuum theory is convergent so that the canonical double scaling where \(\beta \Delta^{5/4}\) is kept fixed is maintained. The free energy par unit volume reads

\[
F = \sum_{g=0}^{\infty} \tilde{\kappa}^{2g-2} t_r^{g-\frac{3}{2}} f_g
\]

\[
= -\frac{1}{3} \tilde{\kappa}^{-2} t_r^{3/2} + t_r^{1/4} - \frac{11}{72} \tilde{\kappa}^{2} t_r^{-1} - \cdots \tag{20}
\]

and the susceptibility exponents for the \(g\)-looped polymers are equal to \(\gamma^{(g)} = \frac{1}{2} + \frac{5}{4} g\).

**Two Dimensions**

In two dimensions the one-loop tadpole is the only primitively divergent correlation function, and among vacuum bubbles the one-loop bubble is divergent. The divergent tadpole is canceled by the counterterm

\[
c_1(a) = -\frac{\tilde{\kappa}}{2\pi} \ln(16 t_r a^4) \tag{21}
\]

and \(c_2 = 0\). Equating eq.(19) with eq.(18), we find that the renormalized quantities are related to the bare ones by

\[
\varphi = \varphi_r + \frac{t_r^{1/2} - t_r^{1/2}}{2\tilde{\kappa}}, \quad t = t_r - \frac{\tilde{\kappa}^2}{2\pi} \ln(16 t_r a^4). \tag{22}
\]

After renormalization the free energy is completely finite for each loop and is given by

\[
F = \sum_{g=0}^{\infty} \tilde{\kappa}^{2g-2} t_r^{g-\frac{3}{2}} f_g
\]

\[
= -\frac{1}{3} \tilde{\kappa}^{-2} t_r^{3/2} + \frac{1}{2\pi} t_r^{1/2} - \frac{\alpha}{6\pi^2} \tilde{\kappa}^{2} t_r^{-1/2} - \cdots \tag{23}
\]

\(^\dagger\) Here the term ‘renormalized’ is meant in a sense of target space, but not of branched polymers.
where \( \alpha \equiv \int_0^1 du(-\ln u)/(u^2-u+1) = 1.1719 \cdots \). The susceptibility exponents for the \( g \)-looped polymers, defined with respect to \( t_r \), are equal to \( \gamma^{(g)} = \frac{1}{2} + g \).

If we recover the bare variables (in a sense of branched polymers) \( \beta \) and \( \Delta \) by eq.(11), the second equation in eq.(22) reads
\[
\beta \Delta = \frac{1}{4 \pi} \ln \beta + \frac{t_r}{2 \hat{\kappa}^2} - \frac{1}{4 \pi} \ln \frac{t_r}{2 \hat{\kappa}^2}. \tag{24}
\]
Thus we observe that the canonical scaling \( \beta \Delta = \text{fixed} \) is violated, instead we have to tune \( \beta \Delta / \ln \beta \) to \( 1/4 \pi \) in order to render the continuum theory finite. We remark that only the critical value of the combination \( \beta \Delta / \ln \beta \) (or equivalently \( \beta \Delta / |\ln \Delta| \)) is universal, while the way how its deviation from \( 1/4 \pi \) is related to the renormalized parameter \( t_r \) depends on the regularization and renormalization schemes employed.

The logarithmic violation of the canonical combination of scaling parameters is also observed in the non-perturbative theory of strings in one dimension [12, 13]. It is instructive to point out the similarities between the two cases, that scaling violations are induced by the tadpole of massless modes — \( \varphi \)-field in our case and the tachyon field in the string theory which becomes massless in one dimension — in effectively two-dimensional theories; both represent the fluctuations of \( (O(N)- \) and \( U(N)\)-)invariant composite fields.

### THREE DIMENSIONS

In three dimensions the one-loop tadpole is again the only primitively divergent correlation function, and the two-loop bubble in addition to the one-loop is also divergent. The divergent tadpole is canceled by the counterterm
\[
c_1(a) = \frac{2 \hat{\kappa}}{\pi^2} (a^{-1} - \pi t_r^{1/4}) \tag{25}
\]
and \( c_2 = 0 \). We find that, by renormalizing \( t \) as
\[
t = t_r + \frac{2 \hat{\kappa}}{\pi^2} (a^{-1} - \pi t_r^{1/4}), \tag{26}
\]
the free energy takes the almost finite form
\[
F = \sum_{g=0}^\infty \tilde{\kappa}^{2g-2} t_r^{\frac{3}{2} - \frac{4g}{3}} f_g
= -\frac{1}{3} \tilde{\kappa}^{-2} t_r^{3/2} + \frac{1}{3\pi} \tilde{\kappa}^{3/4} + \frac{1}{6\pi^2} \hat{\kappa}^2 \ln(16t_r a^4) - \cdots \tag{27}
\]
with one logarithmically divergent term \( F_2 \), originating from the above mentioned graph. The susceptibility exponents for the \( g \)-looped polymers are equal to \( \gamma^{(g)} = \frac{1}{2} + \frac{1}{4} g \).

If we recover the bare variables, eq.(24) reads
\[
\beta \Delta = \frac{1}{\pi^2} + \beta^{-1/3} \left( \frac{t_r}{(2 \hat{\kappa})^{3/2}} - \frac{1}{2 \pi} \frac{t_r^{1/4}}{(2 \hat{\kappa})^{3/4}} \right). \tag{28}
\]
Thus we observe that the canonical scaling \( \beta \Delta^{3/4} = \text{fixed} \) is violated with a power in \( \beta \). We have not yet encountered a counterpart of the powerlike violation of scaling in the non-perturbative theory of strings.
Four Dimensions

In four dimensions the one-loop tadpole and the self energy are primitively divergent correlation functions, and one-, two- and three-loop bubbles contain divergences. The divergence in correlation functions are canceled by the counterterms

\[ c_1(a) = \frac{\kappa}{4\pi^2} \left( a^{-2} + 2t_r^{1/2} \ln \left( 16t_r a^4 \right) \right), \quad c_2(a) = \frac{\kappa^2}{\pi^2} \ln \left( 16t_r a^4 \right). \]  

We find that, by renormalizing \( t \) as

\[ t = t_r + \frac{\kappa^2}{4\pi} a^{-2} + \frac{\kappa^4}{16\pi^2} \ln^2(16t_r a^4), \]  

the free energy takes the almost finite form

\[ F = \sum_{g=0}^{\infty} \kappa^{2g-2} t_r^{\frac{3}{2} - \frac{1}{2} g} f_g \]

\[ = -\frac{1}{3} \kappa^{-2} t_r^{3/2} + \frac{t_r}{8\pi^2} \left( \frac{1}{3} \ln(16t_r a^4) - 1 \right) - \cdots \]  

with three divergent terms (logarithmically divergent \( F_1 \), quadratically divergent \( F_2 \) and logarithmically divergent \( F_3 \)), originating from the above mentioned graphs but diverging more mildly than the unrenormalized ones. The susceptibility exponents for the \( g \)-looped polymers are equal to \( \gamma^{(g)} = \frac{1}{2} + \frac{1}{2} g \).

If we recover the bare variables, eq. (30) reads

\[ \beta \Delta = \frac{1}{8\pi^2} + \beta^{-1} \left( \frac{t_r}{64\kappa^4} + \frac{1}{1024\pi^4} \ln^2 \left( \frac{t_r}{64\kappa^2} \beta^{-2} \right) \right). \]  

Thus we observe that the canonical scaling \( \beta \Delta^{1/2} = \) fixed is violated with a power in \( \beta \).

Up to now we have confirmed the following expression of the susceptibility exponent (defined with respect to \( t_r \))

\[ \gamma^{(g)}_{m,D} = 1 - \frac{1}{m} + \left( 1 + \frac{1}{m} - \frac{m - 1}{2m} D \right) g, \]  

advocated in ref. [11] holds for all theories in concern. According to the above formula, the exponent \( 2 - \gamma \) happens to be a nonnegative integers for several low-loop levels (\( \gamma^{(2)}_{2,D=3} = \gamma^{(3)}_{2,D=4} = 2 \) and \( \gamma^{(1)}_{2,D=4} = 1 \)). In these cases we have observed logarithmic residual divergences in the free energies as it should be, otherwise they could not depend nontrivially upon the renormalized cosmological constant \( t_r \). We note that this situation is shared by the strings in one dimension, where sphere and torus free energies (with exponents \( \gamma = 0 \) and 2, respectively) are logarithmically divergent in the double scaling limit.

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