Complete positivity and dissipative factorized
dynamics

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Abstract

After reviewing the main properties of time-evolutions of open quantum systems, some considerations about the positivity of factorized Markovian dynamics for bipartite systems are made. In particular, it is shown that the positivity of the whole time-evolution in general does not ask for the complete positivity of the single system time-evolutions, if they are allowed to differ. However, they must be completely positive if one is a small perturbation of the other, which is the typical situation for open systems in a heat bath.

1 Introduction

The interaction of a quantum system with its surroundings is a source of irreversibility in its time-evolution: the usual unitary dynamics, given by the Schrödinger equation, has to be replaced by a more general dynamics irreversible and producing decoherence, in general not-Markovian \cite{1,2}. Under some rather broad assumptions the memory terms can be neglected, producing a time-evolution with semigroup structure. The form of such a time-evolution is uniquely fixed (Lindblad \cite{3}; Gorini, Kossakowski, Sudarshan \cite{4}) and it characterizes a wide variety of physical phenomena ranging from quantum optics \cite{5,6}, magnetic resonance \cite{7}, statistical mechanics, foundational aspects of quantum mechanics \cite{8} to elementary particle physics (see \cite{9} and references therein).

The consistency of the physical interpretation of the formalism strongly relies on the property of complete positivity (in the following CP) of the open system time-evolution \cite{2}. It turns out that this property, stronger than simple positivity, is indeed necessary when dealing with composite systems, that is systems built up from several possibly not-interacting subsystems. The necessity of CP shows up when there is an initial entanglement between them \cite{10,11}. In other words, the relevance of CP depends on statistical, not dynamical, properties of the considered composite system; for this reason

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we limit our attention to factorized dynamics, characterizing two not-interacting subsystems. In the Markovian case it has been shown that the positivity of the factorized map \( \gamma_t \otimes \gamma_t \) is a necessary and sufficient condition for the map \( \gamma_t \) being completely positive. In this contribution we explicitly show that this result does not generalize to two subsystems evolving under different Markovian time-evolutions \( \gamma_t^{(1)} \) and \( \gamma_t^{(2)} \): positivity of \( \gamma_t^{(1)} \otimes \gamma_t^{(2)} \) does not imply in general that both \( \gamma_t^{(1)} \) and \( \gamma_t^{(2)} \) are completely positive. However, if \( \gamma_t^{(2)} \) is a small perturbation of \( \gamma_t^{(1)} \), CP of both \( \gamma_t^{(1)} \) and \( \gamma_t^{(2)} \) is required.

In Section 2 a brief introduction to open quantum systems and their time-evolution is given, pointing out the assumptions leading to a Markovian approximation of the dynamics. We will be concerned only with this kind of time-evolutions. In Section 3 the properties of positivity and complete positivity of the dynamics are reviewed, stressing out the relevance of CP and its relation to quantum entanglement. Finally, in Section 4 we deal with factorized dynamics and discuss the main results of the present work.

2 Open quantum systems and their time-evolution

In the following S will denote the system we are interested in; it can be either a well defined physical system or an abstract \( n \leq +\infty \) level system. The outer system, called environment, will be denoted by E.

**Definition 1.** S is called closed if there is no interaction between S and E; otherwise open.

Physical states of S will be represented by statistical operators (or density matrices) \( \rho_S \), that is positive operators with unit trace acting on the Hilbert space of S, \( \mathcal{H}_S \):

\[
\rho_S = \rho_S^\dagger, \quad \rho_S \geq 0, \quad \text{Tr}(\rho_S) = 1. \tag{1}
\]

The statistical operators can be either pure (one-dimensional projectors) or mixed (convex linear superpositions of pure states); their spectral decomposition reads

\[
\rho_S = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \tag{2}
\]

where \( \{ |\psi_i\rangle \} \) is a basis for \( \mathcal{H}_S \) and the eigenvalues \( \lambda_i \) represent the weights of the statistical superposition; in fact, \( \lambda_i \geq 0 \) and \( \sum_i \lambda_i = 1 \), as a consequence of (1). Then (1) are fundamental in order to have a consistent statistical interpretation of the density matrices formalism which assigns to the \( \lambda_i \) the role of probabilities. The observables subject to measurement are represented by Hermitian and bounded linear operators \( A = A^\dagger \) and the mean value of \( A \) over the state \( \rho_S \) is given, in this formalism, by the trace operation:

\[
\langle A \rangle_{\rho_S} = \text{Tr}(A\rho_S). \tag{3}
\]

**Definition 2.** By time evolution we mean a continuous one-parameter family of linear operators, \( \{ \gamma_t \} \), mapping states into states: \( \rho_S(t) = \gamma_t[\rho_S(0)] \), where \( \rho_S(0) \), \( \rho_S(t) \) are the statistical operators representing the system at the initial and at a later time \( t \), respectively.
For example, the time-evolution of a closed system $S$ reads

$$\rho_S(t) = \gamma_t[\rho_S(0)] = U_t\rho_S(0)U_t^\dagger, \quad U_t = e^{-iH_S t}$$

where $H_S$ is the Hamiltonian of $S$. In differential form we get the Liouville-von Neumann equation

$$\dot{\rho}_S(t) = L_H[\rho_S(t)] = -i[H_S, \rho_S(t)]$$

generated by the Liouville superoperator $L_H[\cdot] = -i[H_S, \cdot]$.

If we assume the total system $T = S + E$ to be closed, the time-evolution for the open system $S$ is obtained tracing over the environment degrees of freedom:

$$\rho_S(t) = \gamma_t[\rho_S(t)] = \text{Tr}_E[U_t\rho_T(0)U_t^\dagger], \quad U_t = e^{-iH_T t},$$

where $H_T = H_S + H_E + H_I$ is the Hamiltonian of the total system $T$, $H_E$ the Hamiltonian of $E$ and $H_I$ the interaction term. The differential form of (6), called generalized master equation (GME), is

$$\dot{\rho}_S(t) = -i[H_{S\text{eff}}, \rho_S(t)] + \int_0^t K(t-u)\rho_S(u)du + \mathcal{I}(t).$$

Some comments concerning (7): $H_{S\text{eff}} = H_S + \text{Tr}_E(H_I\rho_E)$ is the redefined Hamiltonian of $S$, containing a Lamb-shift term ($\rho_E$ is a reference state of $E$). The convolution term in (7) depends on the kernel $K$ and contains the whole history of $\rho_S$ from initial time to $t$ (memory term). The inhomogeneity $\mathcal{I}$ is related to initial correlations between $S$ and $E$ (explicit expressions of $K$ and $\mathcal{I}$ can be found in [13]). If we switch off the interaction, $H_I = 0$, eq. (7) reduces to the Liouville-von Neumann equation since $K = 0$, $\mathcal{I} = 0$ and also the Lamb-shift term drops out.

It is difficult to deal with the dynamics generated by (7) because of the convolution term. However, in many physical situations it is possible to approximate it with a Markovian dynamics, that is a memoryless dynamics characterized by a linear generator $L$ that generalizes the Liouville superoperator:

$$\dot{\rho}_S(t) = L[\rho_S(t)] = L_{H\text{eff}}[\rho_S(t)] + L_D[\rho_S(t)].$$

Besides the redefined Hamiltonian part $L_{H\text{eff}}[\cdot] = -i[H_{S\text{eff}}, \cdot]$, the additional contribution $L_D$ has appeared. Its form is uniquely fixed by a theorem that in the finite-dimensional Hilbert space case is as follows [4]:

**Theorem 1.** For a finite dimensional Hilbert space $(\dim\mathcal{H}_S = n)$, $L_D$ in (8) is given by

$$L_D[\rho_S] = \sum_{i,j=1}^{n^2-1} c_{ij} \left[ F_i^\dagger \rho_S F_j - \frac{1}{2} \left< F_j^\dagger F_i, \rho_S \right> \right]$$

where $\{F_i, i = 1, n^2-1\}$ is an orthonormal basis for the space of traceless $n \times n$ matrices: $\text{Tr}(F_i^\dagger F_j) = \delta_{ij}$, $\text{Tr}(F_i) = 0$ and $C = [c_{ij}]$ is a self-adjoint matrix.

The assumptions underlying the Markovian approximation (8) are: a) weak interaction between $S$ and $E$; b) absence of initial correlation: $\rho_T(0) = \rho_S(0) \otimes \rho_E$. The generated time-evolution,
Indeed, consider the time-evolution of the composite system \( S + S \) positivity-preserving. Denoting by \( M \) interest evolving under \( \gamma \) condition for \( \gamma \) be positive for any initial state. But, according to Theorem 2, this is a necessary and sufficient point in the next section. The one-parameter (time) family of maps \( \{ \gamma_t, t \geq 0 \} \) is a semigroup of transformations since it satisfies the forward in time composition law: \( \gamma_t \circ \gamma_s = \gamma_{t+s} \forall t, s \geq 0 \).

3 Possibility and complete positivity

For the sake of the statistical interpretation, the time-evolution \( \gamma_t \) we have described must be positivity-preserving. Denoting by \( M_n(\mathbb{C}) \) the algebra of \( n \times n \) complex matrices, for the map \( \gamma_t : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) we have the following definitions:

**Definition 3.** \( \gamma_t \) is said to be positive (or positivity-preserving) if and only if it preserves the positivity of the states it acts on:

\[
\gamma_t[\rho S] \geq 0 \quad \forall \rho S \geq 0.
\] (10)

A stronger property is as follows:

**Definition 4.** \( \gamma_t \) is said to be completely positive (CP) if and only if its lifting \( \gamma_t \otimes I_d : M_n(\mathbb{C}) \otimes M_d(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_d(\mathbb{C}) \) is positive \( \forall d \in \mathbb{N} \), where \( I_d \) is the \( d \times d \) identity.

CP is stronger than positivity since this latter is obtained with the particular choice \( d = 1 \). The following theorem of Choi gives a practical characterization of completely positive maps [14], saying that checking \( d = n \) is enough:

**Theorem 2.** The map \( \gamma_t : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) is completely positive if and only if the map \( \gamma_t \otimes I_n : M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \) is positive.

The standard form of completely positive maps was obtained by Stinespring:

**Theorem 3.** \( \gamma_t \) is CP if and only if there is a set of bounded operators \( \{ V^{(t)}_i \} \) such that

\[
\gamma_t[\rho S] = \sum_i V^{(t)}_i \rho S V^{(t)}_i \quad \forall \rho S \in M_n(\mathbb{C}).
\] (11)

We claim that positivity is not enough for a correct interpretation of the quantum formalism. Indeed, consider the time-evolution of the composite system \( S + S_n \), where \( S \) is the system of interest evolving under \( \gamma_t \) and \( S_n \) is an arbitrary \( n \)-level system (where \( n = \text{dim} \mathcal{H}_S \)). Suppose in particular that \( S_n \) is characterized by the null Hamiltonian \( H_n = 0 \) and that \( S \) and \( S_n \) do not interact, \( H_I = 0 \), then the time-evolution of \( S + S_n \) is given by \( \gamma_t \otimes I_n \). In order to preserve the positivity of the eigenvalues of a statistical operator representing \( S + S_n \), this dynamics must be positive for any initial state. But, according to Theorem 2, this is a necessary and sufficient condition for \( \gamma_t \) being completely positive.

In different terms, if \( \gamma_t \) is only positive and not completely positive, then there surely exists an entangled state \( \rho \) of \( S + S_n \) such that at some \( t, \gamma_t \otimes I_n[\rho] \) has negative eigenvalues and cannot be interpreted as a state.

**Definition 5.** Given a composite system \( S_1 + S_2 \), the state \( \rho \) representing it is called separable if and only if \( \rho = \sum_i p_i \rho^1_i \otimes \rho^2_i \), where \( \rho^1_i, \rho^2_i \) are statistical operators characterizing systems \( S_1 \), respectively \( S_2 \), and \( p_i \geq 0, \sum_i p_i = 1 \).
The only states in $S + S_n$ that could exhibit an unphysical evolution (i.e. negative eigenvalues) are entangled states, since $\gamma_t \otimes I_n$ certainly preserves the positivity of separable states when $\gamma_t$ is positivity-preserving. Then, whereas positivity guarantees the physical consistency of evolving states of single systems, CP prevents inconsistencies in entangled composite systems.

It is possible to give a general characterization of completely positive maps having the semigroup structure \[4\]; instead, positivity has so far not got a complete characterization.

**Theorem 4.** The time-evolution generated by $\mathcal{L}$ with $L_D$ as in \[3\] consists of completely positive maps if and only if the matrix $C$ defined in \[9\] is positive definite.

Note that the time-evolution of a closed system, eq. \[4\], is completely positive since it has the Stinespring form.

### 4 Factorized dynamics

In this section we consider factorized Markovian dynamics for bipartite systems $S = S_1 + S_2$. Both subsystems are assumed to be described by $n$-dimensional Hilbert spaces $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$; their time-evolutions are assumed to be semigroups of linear maps denoted by $\{\gamma_t^{(1)}, t \geq 0\}$ and the dynamics of the whole system to be given by $\{\gamma_t^{(1)} \otimes \gamma_t^{(2)}, t \geq 0\}$. We have seen that $\gamma_t^{(1)} \otimes I_n$ positive implies $\gamma_t^{(1)}$ completely positive and the same holds for $I_n \otimes \gamma_t^{(2)}$. We now study what conditions the request that $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ be positive puts on $\gamma_t^{(1)}, \gamma_t^{(2)}$. This question is justified since there are physical situations, subject to experimental investigation, characterized by dissipative Markovian evolutions in the factorized form \[9\].

Quantum open systems can be thought of as being in contact with a heat bath at some temperature $T$; if they are bipartite, not interacting among themselves and only weakly with the environment, their joint time-evolution may be modeled by $\gamma_t \otimes \gamma_t$. However, due to local fluctuations, it may be that their dynamics be as if they were in contact with baths at slightly different temperatures so that a realistic description of the evolution is given by $\gamma_t \otimes \gamma_t^\ell$ with $\gamma_t$ a perturbation of $\gamma_t$.

It is thus possible to give more physical flavour to the rather abstract notion of CP, offering stronger evidences to its necessity in the description of open systems dynamics.

The generator of $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ is given by $L = L_1 \otimes I_n + I_n \otimes L_2$, where $L_{1,2}$ generate the elementary evolutions $\gamma_t^{(1),(2)}$. Without loss of generality we neglect any Hamiltonian contribution in these generators (such contribution is in fact completely irrelevant for the following discussion), then they are expressed as in \[9\] with self-adjoint $n \times n$ matrices $C_1 = [c_{ij}^{(1)}]$ and $C_2 = [c_{ij}^{(2)}]$. The generator $L$ is thus

\[ L[\rho_S] = \sum_{i,j=1}^{n^2-1} \left( c_{ij}^{(1)} \left[ F_i \otimes I_n \rho_S F_j^\dagger \otimes I_n - \frac{1}{2} \{ F_j^\dagger F_i \otimes I_n, \rho_S \} \right] + c_{ij}^{(2)} \left[ I_n \otimes F_i \rho_S I_n \otimes F_j^\dagger - \frac{1}{2} \{ I_n \otimes F_j^\dagger F_i, \rho_S \} \right] \right) \]  

(12)
and the corresponding $2n \times 2n$ matrix $C$ can be represented in block diagonal form

$$ C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}. \quad (13) $$

If both the subsystems do evolve under the same time-evolution $\gamma_t$, a strong equivalence can be obtained \cite{[12]}

**Theorem 5.** If $\{\gamma_t, t \geq 0\}$ is a semigroup of linear maps over the states of $M_n(\mathbb{C})$, then the semigroup $\{\gamma_t \otimes \gamma_t, t \geq 0\}$ of linear maps over the states of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is positivity-preserving if and only if $\{\gamma_t, t \geq 0\}$ is made of completely positive maps.

This result offers a new approach to CP, without reference to arbitrary $n-$level systems usually introduced in order to give a physical interpretation of CP (see the discussion in the previous section). It turns out that if $\gamma_t$ is positive but not completely positive, there exists an entangled state in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ mapped by $\gamma_t \otimes \gamma_t$ into a non-positive operator.

Does this result hold when the two time-evolutions $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ are different? The answer is no, as we will shortly prove by a counterexample, i.e. a positive factorized map $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ with $\gamma_t^{(2)}$ violating CP. We set $n = 2$ (that is, $\dim \mathcal{H}_{S_1} = \dim \mathcal{H}_{S_2} = 2$) and define

$$ C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (14) $$

Using the coherence-vector representation for $\rho_S \in M_2(\mathbb{C})$,

$$ \rho_S = \sum_{\mu=0}^{3} \rho^\mu \sigma_\mu, \quad \rho^\mu \in \mathbb{R}, \quad \mu = 0, \ldots, 3 \quad (15) $$

where $\{\sigma_i, i = 1, 2, 3\}$ are the Pauli matrices and $\sigma_0 = I_2$, we get the action of the elementary time-evolutions on $\rho_S \in M_2(\mathbb{C})$:

$$ \gamma_t^{(1)}[\rho_S] = \rho^0 \sigma_0 + e^{-4t}(\rho^1 \sigma_1 + \rho^2 \sigma_2 + \rho^3 \sigma_3), $$

$$ \gamma_t^{(2)}[\rho_S] = \rho^0 \sigma_0 + \rho^1 \sigma_1 + e^{-4t} \rho^2 \sigma_2 + \rho^3 \sigma_3. \quad (16) $$

Since in 2 dimensions $\rho_S$ is positive if and only if $\det(\rho_S) \geq 0$, we see that $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ are positivity-preserving. Indeed $\det(\rho_S) \geq 0$ if and only if $\sum_{i=1}^{3} [\rho^i(t)]^2 \leq 1/4$, but $\sum_{i=1}^{3} [\rho^i(0)]^2 \leq 1/4$ (since we start with a positive operator) and from \cite{[14]} any component $\rho^i$ is fixed or exponentially decreasing in time under both $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$. Then $\det(\gamma_t^{(1), (2)}[\rho_S(t)]) \geq 0 \forall t \geq 0$ and finally $\gamma_t^{(1)}, \gamma_t^{(2)}$ are positive.

From Theorem 4 we deduce that $\gamma_t^{(1)}$ is completely positive, $\gamma_t^{(2)}$ is not ($C_1 \geq 0, C_2 \not\geq 0$).

Consider now $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$. Since $C_2$ as in \cite{[14]} is not positive, the evolution is not completely positive\footnote{Note that an arbitrary $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ is completely positive if and only if $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ are completely positive.}. Is it positive? In order to prove that it is indeed so, we can restrict our attention to
its action on pure states in $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. In fact, if a linear time-evolution is positive on pure states, it must be positive also on mixed states. Then $\rho_S = |\psi\rangle\langle\psi|$, with $|\psi\rangle \in \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}$. It is convenient to use the Schmidt decomposition

$$|\psi\rangle = \sum_{i=1,2} \sqrt{\mu_i} |\phi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle, \quad |\phi_i^{(1)}\rangle \in \mathcal{H}_{S_1}, |\phi_i^{(2)}\rangle \in \mathcal{H}_{S_2}, \quad \langle \phi_i^{(a)} | \phi_j^{(a)} \rangle = \delta_{ij}, \ a = 1, 2$$

with $\mu_{1,2} \geq 0$, $\mu_1 + \mu_2 = 1$. Then, since (10) holds in general for any matrix in $M_2(\mathbb{C})$ (and not only for density matrices), we can compute $\rho_S(t) = \gamma_t^{(1)} \otimes \gamma_t^{(2)} [\rho_S]$; writing $\mu = \mu_1$ and defining $\alpha, \varphi$ such that $\langle \phi_i^{(2)} | \sigma_2 | \phi_i^{(2)} \rangle = \cos \alpha, \langle \phi_2^{(2)} | \sigma_2 | \phi_1^{(2)} \rangle = e^{i\varphi} \sin \alpha$, we get

$$\rho_S(t) = e^{-4t} \rho_S + \frac{1}{2} (1 - e^{-4t}) \left[ \sigma_0 \otimes \left( \begin{array}{cc} \mu & 0 \\ 0 & 1 - \mu \end{array} \right) - Z_t \otimes \sigma_2 \right] \quad (18)$$

where

$$Z_t = \left( \begin{array}{cc} \frac{1}{2} \cos \alpha (2\mu - 1 + e^{-4t}) & e^{i\varphi} \sin \alpha e^{-4t} \sqrt{\mu(1 - \mu)} \\ e^{-i\varphi} \sin \alpha e^{-4t} \sqrt{\mu(1 - \mu)} & \frac{1}{2} \cos \alpha (2\mu - 1 - e^{-4t}) \end{array} \right). \quad (19)$$

After diagonalization of $Z_t$, we can compute the doubly degenerate eigenvalues of the second term in the r.h.s. of (18):

$$z_{\pm}(t) = \frac{1}{4} (1 - e^{-4t}) \left( 1 \pm \sqrt{1 - (1 - e^{-8t})[1 - \sin^2 \alpha(1 - 2\mu)^2]} \right); \quad (20)$$

they are both positive for any time $t$ as a consequence of $\mu \in [0, 1], \sin \alpha \in [-1, 1]$. $\rho_S(t)$ is the sum of two positive definite matrices, then it is positive for any time $t$ and for any initial state $\rho_S$ and $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ is positivity-preserving $\forall t \geq 0$.

Thus we have exhibited an explicit example of a positive map $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ whose constituent maps $\gamma_t^{(1)}, \gamma_t^{(2)}$ are not both completely positive.

Nevertheless, we will shortly show that, if $\gamma_t^{(2)}$ slightly differs from $\gamma_t^{(1)}$, then $\gamma_t^{(1)}, \gamma_t^{(2)}$ both completely positive is a necessary condition for the positivity of $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$. A preliminary result is the following.

**Lemma 1.** Let the semigroup $\{ \gamma_t^{(1)} \otimes \gamma_t^{(2)} : t \geq 0 \}$ over the states of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ have $\gamma_t^{(1)}$, $\gamma_t^{(2)}$ generated by $L_1, L_2$ respectively; it consists of positivity-preserving maps only if $C_1 + C_2 \geq 0$, where $C_1, C_2$ are the matrices of coefficients characterizing the generators $L_1, L_2$ as in (9).

**Proof:** From positivity preservation it follows that

$$\mathcal{G}_{\phi,\psi}(t) = \langle \phi | (\gamma_t^{(1)} \otimes \gamma_t^{(2)}) | \psi \rangle \langle \psi | \phi \rangle \geq 0 \quad \forall |\phi\rangle, |\psi\rangle \in \mathbb{C}^n \times \mathbb{C}^n. \quad (21)$$

Choosing $\langle \phi | \psi \rangle = 0$ it must be $\mathcal{L}_{\phi,\psi} = d\mathcal{G}_{\phi,\psi}(t)/dt|_{t=0} \geq 0$ and thus

$$\mathcal{L}_{\phi,\psi} = \langle \phi | (L_1 \otimes I_n + I_n \otimes L_2) | \psi \rangle \langle \psi | \phi \rangle \geq 0. \quad (22)$$
Let \{ |j \rangle, j = 1, \ldots, n \} an orthonormal basis of \( \mathbb{C}^n \), then

\[
\mathcal{L}_{\phi, \psi} = \sum_{i,j=1}^{n^2-1} \left\{ c^{(1)}_{ij} \left[ \text{Tr}(\Phi \Phi^\dagger F_i) \text{Tr}(\Phi^\dagger F_j^\dagger) \right] + c^{(2)}_{ij} \left[ \text{Tr}(\Phi^\dagger \Psi^\dagger F_i) \text{Tr}(\Psi F_j^\dagger) \right] \right\} \geq 0
\]  

where \( C_1 = [c^{(1)}_{ij}], C_2 = [c^{(2)}_{ij}] \), \( \{ F_i, i = 1, \ldots, n^2 - 1 \} \) are the traceless matrices appearing in Theorem 1 and \( \Psi = [\psi_{ij}], \Phi = [\phi_{ij}] \) are the matrices of the coefficients of the expansion of \( |\psi\rangle \) and \( |\phi\rangle \) on the basis \( \{|j\rangle \otimes |k\rangle; j, k = 1, \ldots, n \} \). Because of the orthogonality between \( |\psi\rangle \) and \( |\phi\rangle \), \( \text{Tr}(\Phi^\dagger \Psi^\dagger) = 0 \).

Now, for any \( |\xi\rangle = (\xi_1, \ldots, \xi_{n^2-1})^T \in \mathbb{C}^{n^2-1} \), consider the traceless \( n \times n \) matrix \( W = \sum_{i=1}^{n^2-1} \xi_i F_i \) and impose \( W = \Phi \Psi^\dagger \). Since \( W \) and its transpose \( W^T \) are similar to each other \( [15] \), define \( \Phi \) such that \( \Phi^{-1} W \Phi = W^T \). \( \Psi \) is then fixed by \( \Psi^\dagger \Phi = W^T \) and equation (23) becomes

\[
\mathcal{L}_{\phi, \psi} = \sum_{i,j=1}^{n^2-1} (c^{(1)}_{ij} + c^{(2)}_{ij}) \xi_i^* \xi_j \geq 0
\]

that is \( \langle \xi | (C_1 + C_2) |\xi\rangle \geq 0 \) \( \forall |\xi\rangle \in \mathbb{C}^{n^2-1} \). Therefore \((C_1 + C_2)\) must be positive definite.

\[\blacksquare\]

We will now consider the situation of a bipartite system evolving according to a semigroup of the form \( \{ \gamma_t \otimes \gamma^\varepsilon_t, t \geq 0 \} \) where \( \gamma^\varepsilon_t \) is a perturbation of \( \gamma_t \), namely, if \( \gamma_t \) is generated by \( L \), \( \gamma^\varepsilon_t \) is generated by \( L_\varepsilon = L + \varepsilon \Lambda \) where \( \varepsilon \in \mathbb{R} \) and \( \Lambda \) acts on \( M_n(\mathbb{C}) \).

**Theorem 6.** In the setting sketched above, there always exists an interval \( I_0 = [0, \varepsilon_0] \), such that \( \forall \varepsilon \in I_0 \), the set \( \{ \gamma_t \otimes \gamma^\varepsilon_t, t \geq 0 \} \) is made of positivity-preserving maps if and only if the semigroups \( \{ \gamma_t, t \geq 0 \} \) and \( \{ \gamma^\varepsilon_t, t \geq 0 \} \) are completely positive.

**Proof:** The if part is trivial, we will prove only the necessity.

To start with, we show that the non-perturbed maps \( \{ \gamma_t, t \geq 0 \} \) must be completely positive.

Choosing \( \varepsilon = 0 \) we obtain the set \( \{ \gamma_t \otimes \gamma_t, t \geq 0 \} \); it is made of positive maps if and only if \( \{ \gamma_t, t \geq 0 \} \) is made of completely positive maps (Theorem 5).

Let \( C \) and \( C_\varepsilon \) be the Hermitian matrices associated to \( L, L_\varepsilon \) respectively (Theorem 1). By the previous result and following Theorem 4, \( C \) must be positive definite. Since \( L_\varepsilon = L + \varepsilon \Lambda \), it follows \( C_\varepsilon = C + \varepsilon \Gamma \), where \( \Gamma \) is a \( (n^2-1) \times (n^2-1) \) matrix, satisfying \( \Gamma = \Gamma^\dagger \).

If the semigroup \( \gamma_t \otimes \gamma^\varepsilon_t, t \geq 0 \) is positivity-preserving \( \forall \varepsilon \in [0, \varepsilon'] \), by Lemma 1 \( C + C_\varepsilon \geq 0 \), whence

\[
\langle \xi | (C + C_\varepsilon) |\xi\rangle = 2 \langle \xi | C |\xi\rangle + \varepsilon \langle \xi | \Gamma |\xi\rangle \geq 0 \quad \forall |\xi\rangle \in \mathbb{C}^{n^2-1}.
\]  

Then

\[
2 \langle \xi | C_\varepsilon |\xi\rangle = 2 \langle \xi | C |\xi\rangle + \varepsilon \langle \xi | \Gamma |\xi\rangle \geq 0;
\]  

we conclude that \( \langle \xi | C_\varepsilon |\xi\rangle \geq 0 \) \( \forall \varepsilon \in I_0 = [0, \varepsilon_0], \varepsilon_0 \geq \varepsilon'/2 \), \( \forall |\xi\rangle \in \mathbb{C}^{n^2-1} \) and then \( C_\varepsilon \geq 0 \). Therefore \( \forall \varepsilon \in I_0 \) we have not only the positivity of the semigroup \( \{ \gamma_t \otimes \gamma^\varepsilon_t, t \geq 0 \} \) but also the complete positivity of both the maps \( \{ \gamma^\varepsilon_t, t \geq 0 \} \) and \( \{ \gamma_t, t \geq 0 \} \).

\[\blacksquare\]
5 Conclusions

There are many physical systems whose irreversible time-evolutions are suitably described by factorized dynamics $\gamma_t \otimes \gamma_t$. Usually, they are bipartite systems living in the same environment and evolving under irreversible dynamics because of their (weak) interaction with it. In this work we have addressed the problem of a non-uniform environment, causing the two subsystems to evolve under possibly different time-evolutions, $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$.

We have shown that, while the minimal request of positivity of the factorized dynamics $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ does not in general imply the CP of the constituent maps, however, they must be so if $\gamma_t^{(2)}$ is a perturbation of $\gamma_t^{(1)}$ as in the case under experimental circumstances when treating systems in a heat bath.

We can thus conclude that, when small temperature fluctuations occur, in order to avoid physical inconsistencies in the time-evolution $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ of bipartite systems in heat baths, $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ have to be completely positive maps.

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