ANGULAR MOMENTUM–MASS RELATION FOR DARK MATTER HALOS

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ABSTRACT

We study the empirical relation between an astronomical object’s angular momentum $J$ and mass $M$, $J = \beta M^\alpha$, the $J$–$M$ relation, using $N$-body simulations. In particular, we investigate the time evolution of the $J$–$M$ relation to study how the initial power spectrum and cosmological model affect this relation and to test two popular models of its origin—mechanical equilibrium and tidal torque theory (TTT). We find that in the $\Lambda$CDM model, $\alpha$ starts with a value of $\sim 1.5$ at high redshift $z$, increases monotonically, and finally reaches $5/3$ near $z = 0$, whereas $\beta$ evolves linearly with time in the beginning, reaches a maximum and decreases, and, finally, stabilizes. A three-regime scheme is proposed to understand this newly observed picture. We show that the TTT accounts for this time-evolution behavior in the linear regime, whereas $\alpha = 5/3$ comes from the virial equilibrium of halos. The $J$–$M$ relation in the linear regime contains the information of the power spectrum and cosmological model. The $J$–$M$ relations for halos in different environments and with different merging histories are also investigated to study the effects of a halo’s nonlinear evolution. An updated and more complete understanding of the $J$–$M$ relation is thus obtained.

Key words: cosmology: theory – dark matter – galaxies: evolution – galaxies: halos – methods: numerical

1. INTRODUCTION

The angular momentum–mass relation (or the $J$–$M$ relation) is a scaling relation between an astronomical object’s angular momentum $J$ and its mass $M$. It was first noticed by Brosche (1963) that for a wide range of astronomical objects, from planet–satellite systems to super clusters, their $J$ and $M$ follow an empirical relation $J \propto M^{5/3}$. Later follow-up works confirmed this power-law relation (see Carrasco et al. 1982 and references therein). Carrasco et al. (1982) then presented an updated version of the $J$–$M$ relation covering $\sim 30$ orders of magnitude in mass and $\sim 50$ orders of magnitude in angular momentum. Fall (1983) particularly studied the $J \propto M^{5/3}$ relation for spiral and elliptical galaxies. This relation was confirmed again by recent observational updates, e.g., Romanowsky & Fall (2012) and Fall & Romanowsky (2013). This $J \propto M^{5/3}$ relation was also observed for the cold dark matter halos in cosmological $N$-body simulations; see, e.g., Efstathiou & Jones (1979), Barnes & Efstathiou (1987), Sugerman et al. (2000), etc.

This simple and universal relation from observation and simulation is unusual and demands an explanation. Here, we briefly review two widely quoted explanations. The readers can refer to Li (1998) for another explanation from the global rotation of the universe.

Mechanical equilibrium. This explanation usually appears in astronomical papers (e.g., Ozernoy 1967; Carrasco et al. 1982). When a galaxy (halo) becomes virialized, its rotational energy $K$ and gravitational energy $U$ are linked by the virial theorem, $2K + U = 0$. Using $K \propto I\omega^2 \propto MR^2\omega^2$, $U \propto -GM^2/R$, and $M \propto R^3$, we can obtain $J \propto I\omega \propto MR^2\omega \propto M^{5/3}$. Here, $I$, $\omega$, and $R$ are the galaxy’s (halo’s) moment of inertia, average angular velocity, and radius, respectively, and $G$ is the gravitational constant. The key relation used in this explanation is the virial theorem, which implies that galaxies (halos) are in mechanical equilibrium.

Arguments from the tidal torque theory (TTT). In the TTT (Strömgren 1934; Hoyle 1951; Peebles 1969; Doroshkevich 1970; White 1984), a halo’s angular momentum is induced by the tidal torques from the surrounding inhomogeneities, and thus has a dependence on the halo’s moment of inertia and the tidal tensor. Since $J \propto MR^2$ and $M \propto R^3$, $J \propto I \propto M^{5/3}$ (Peebles 1969; White 1994). This scaling relation can also be addressed in detail by calculating the joint probability distribution of $J$ and $M$, $P(M, J)$; see, e.g., Catelan & Theuns (1996a). From the ensemble results of TTT, Catelan & Theuns (1996a) used the statistics of the initial density field to study $P(M, J)$, and found that $J$ is proportional to $M^{5/3}$ in the linear regime.

Although both explanations lead to the power index 5/3 in the observed $J$–$M$ relation, they differ in the origin of the index. The mechanical equilibrium argument states that the $J$–$M$ relation is established in the virialized stage, while TTT claims it is found in the linear stage. Further investigations are needed to find the exact origin of the $J$–$M$ relation. Furthermore, the orbital–merger scenario (Maller et al. 2002; Vittiksa et al. 2002; Peirani et al. 2004; D’Onghia & Navarro 2007) shows that the nonlinear evolution has significant effects on the halo angular momentum after the turnaround stage. Whether or not the nonlinear evolution affects the $J$–$M$ relation, however, is not addressed. Also, the $J$–$M$ relation is tightly related to the evolution of angular momentum, which in turn depends on the initial perturbations and cosmological model (TTT). How do the power spectrum and cosmological model affect the $J$–$M$ relation? This is an interesting question that deserves to be answered.

In this paper, we use $N$-body simulations to study the time evolution of the $J$–$M$ relation for protohalos. Here a protohalo is defined as a clump of matter that is destined to end up as a halo at redshift $z = 0$. Interestingly, $J \propto M^\alpha$ is found to be valid in the whole cosmological history, but with different $\alpha$ at different redshifts. This evolution behavior enables us to test the two possible explanations mentioned above. The $J$–$M$ relation in the linear regime is shown to depend on the initial power spectrum and cosmological model. The dependences of the $J$–$M$ relation on the environment and merging history are also studied to see the nonlinear evolution effects. We propose...
a three-regime scheme to explain the evolution of the J–M relation and give a more complete understanding for this relation.

The structure of the paper is as follows. In Section 2, we briefly review the TTT and derive the predictions for the J–M in the linear regime. We describe our N-body simulation details, halo finders, environment classification method, and merger tree constructions in Section 3. Section 4 presents our results. The summary and discussion are given in Section 5. The appendices summarize some numerical tests, including the simulation box size, resolution, halo finder, fitting method, and smoothing schemes in TTT.

2. J–M RELATION IN THE LINEAR REGIME

For the convenience of later discussion, we summarize some important steps of TTT from White (1984) in Section 2.1. We then derive the prediction for the J–M relation in the linear regime in Section 2.2.

2.1. Tidal Torque Theory

In the comoving Eulerian coordinate $x$, the total angular momentum of an object with respect to its center of mass $x_{\text{cm}}$ is

$$J(t) = a^2 \int_{V_0} \rho_{\text{com}}(x, t) (x - x_{\text{cm}}) \times \dot{x} \, dx,$$

where $a$ is the scale factor, $V_{\text{ce}}$ is the occupied region of the object in comoving Eulerian coordinate, and the comoving matter density can be expressed as

$$\rho_{\text{com}}(x, t) = \rho_0 [1 + \delta(x, t)].$$

Here, $S(q, t)$ is the displacement vector. The Jacobian transformation from $x$ to $q$ can be found by considering mass conservation. That is,

$$|Q(q)| = [1 + \delta(x, t)]^{-1}.$$  

With Equations (2)–(4), the angular momentum in the corresponding Lagrangian region $V_L$ can be expressed as

$$J(t) = a^2 \rho_0 \int_{V_L} (q - q_{\text{cm}} + S - S_{\text{cm}}) \times \dot{S} \, dq.$$

In this paper, we only consider first-order Lagrangian perturbations, i.e., the Zel’dovich approximation (Zel’dovich 1970):

$$S(q, t) = -D(t) \nabla \psi(q),$$

where $D(t)$ is the linear growth factor and $\psi(q)$ is the gravitational potential. Higher-order expressions can be found in Catelan & Theuns (1996b).

Under the Zel’doovich approximation, the angular momentum is

$$J(t) = -a^2 \dot{D}(t) \rho_0 \int_{V_L} (q - q_{\text{cm}}) \times \nabla \psi(q) \, dq.$$

Further assuming the potential $\psi(q)$ to be smooth in the region $V_L$, we can approximate it using the Taylor expansion at the center-of-mass position up to second order:

$$\psi(q) \approx \psi(q_{\text{cm}}) + \frac{\partial \psi(q)}{\partial q_i} \bigg|_{q=q_{\text{cm}}} (q_i - q_{\text{cm},i}) + \frac{1}{2} \frac{\partial^2 \psi(q)}{\partial q_i \partial q_j} \bigg|_{q=q_{\text{cm}}} (q_i - q_{\text{cm},i})(q_j - q_{\text{cm},j}),$$

where the Einstein summation convention is used.

Substituting Equation (8) into (7), we obtain the major result of the TTT:

$$J(t) = -a^2 \dot{D}(t) \epsilon_{ijk} T_{ji} \dot{q}_k,$$

with the tidal tensor

$$T_{ji} = \frac{\partial^2 \psi(q)}{\partial q_i \partial q_j} \bigg|_{q=q_{\text{cm}}} ,$$

and the inertial tensor

$$I_k = \rho_0 \int_{V_L} (q_i - q_{\text{cm},i})(q_k - q_{\text{cm},k}) \, dq.$$  

Equation (9) tells us that in the linear regime, the angular momentum of a protohalo depends on its shape (represented by $I_k$) and the surrounding tidal torque (measured by $T_{ji}$) evolves according to $a^2 \dot{D}$, i.e., how the universe expands and how the perturbations grow. Specifically, the Levi-Civita symbol $\epsilon_{ijk}$ in Equation (9) implies that the angular momentum is produced due to the misalignment between the tidal tensor and inertial tensor. After the turnaround, the protohalo collapses to a virialized object (halo), TTT assumes that angular momenta are gained or lost during this nonlinear process. TTT has been tested using N-body simulations with relatively good agreement. See Sugerman et al. (2000) and Porciani et al. (2002a, 2002b) for recent testings.

The temporal part of $J(t)$, $a^2 \dot{D}$, depends on the cosmological model. Previous studies verified that for a de Sitter universe, or for the matter-dominated era in a ΛCDM model, the halo angular momentum grows linearly with time as $a(t)\dot{D}(t) = t$ (e.g., White 1984; Sugerman et al. 2000). We will test this temporal dependence for a quintessence dark energy model that has different expansion and structure growth rate as ΛCDM. When calculating the tidal tensor $T_{ji}$, the potential field (or density field) is smoothed with a smoothing scale equal to the protohalo scale (White 1984),

$$T_{ji} = -\frac{1}{(2\pi)^3} \int k_i k_j \hat{\psi}(k) \hat{W}(kr_e) e^{ikq_{\text{cm}} \cdot dk}. $$

Here, $\hat{\psi}(k)$ and $\hat{W}(kr_e)$ are the Fourier transforms of the potential function and window function, respectively. For the top-hat window function, the smoothing scale $r_e$ is usually set by $M = 4\pi \rho_0 R^3_3 / 3$. As pointed out by White (1984), this smoothing process is needed to keep the validity of the Zel’dovich approximation used in TTT. The Zel’dovich approximation requires $|\delta|^2 < 1$. However, inside a protohalo region, there may exist some smaller-scale perturbations with $|\delta|^2 > 1$ that need to be smoothed out. However, how to choose the value of $r_e$ is a nontrivial question. In Appendix D, we numerically test the choice of $r_e$ and show that the one usually adopted, $r_e = (3M/4\pi \rho_0)^{1/3}$, is the best choice.
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Table 1
Simulation Parameters of the ΛCDM Model

| Name       | Ω_m  | Ω_L | Ω_bh^2 | h   | N^3 | L_{box} (h^{-1} Mpc) | σ_8  | n_0 | Softening Length ℰ (h^{-1} kpc) | Realizations |
|------------|------|-----|--------|-----|-----|----------------------|-----|-----|----------------------------------|--------------|
| ΛCDM256   | 0.28 | 0.72| 0.024  | 0.7 | 256 | 100                  | 0.8 | 0.96| 3.0                             | 10           |
| ΛCDM512a  | 0.28 | 0.72| 0.024  | 0.7 | 512 | 100                  | 0.8 | 0.96| 5.0                             | 5            |
| ΛCDM512b  | 0.28 | 0.72| 0.024  | 0.7 | 512 | 200                  | 0.8 | 0.96| 20.0                            | 10           |

The formalism of White (1984) outlined above considers a random region V_i in the smooth density field that may not be a protogalaxy region. To study the angular momenta for density peaks, Catelan & Theuns (1996a) calculated the ensemble average of angular momentum with respect to the potential field ψ,

\[
\langle J^2 \rangle_\psi = \frac{1}{15\pi^2} a^4 D^4 \left( \mu_1^2 - 3\mu_2^2 \right) \int dk d^6 \psi(k) \tilde{W}(kR_s)^2. \tag{13}
\]

where \( \mu_1 = h_1 + I_2 + I_3 \), \( \mu_2 = h_1I_2 + I_1I_3 + I_1I_2 \), and \( I_1, I_2, I_3 \) are eigenvalues of the inertial tensor \( I_{ij} \). The term \( \mu_1^2 - 3\mu_2^2 \) depends on the statistical information of the density peaks. The potential power spectrum \( P_\psi(k) \) is defined as \( \langle \tilde{\psi}(k) \tilde{\psi}(k') \rangle_\psi = (2\pi)^3 \delta(k + k') P_\psi(k) \).

2.2. J–M Relation in the Linear Regime

The J–M relation is a statistical relation obtained from a large halo sample and has non-negligible scatterings. To calculate the linear theoretical predictions, we use the ensemble results of TTT (Equation (13)) and consider the simple scale-free models.

For a scale-free model with a density power spectrum \( P(k) = A k^n \) in the linear regime, the potential power spectrum \( P_\psi(k) \) is

\[
P_\psi(k) = A (4\pi G \rho_0)^2 k^{n-4}. \tag{14}
\]

Using a top-hat window function

\[
\tilde{W}(kR_s) = 3 \left[ \sin(kR_s) - kR_s \cos(kR_s) \right] / (kR_s)^3 \tag{15}
\]

and \( M = 4\pi \rho_0 R_s^3 / 3 \), we have

\[
\int dk d^6 \psi(k) \tilde{W}(kR_s) = 9A (4\pi G \rho_0)^2 \left( \frac{4\pi}{3} \rho_0 \right)^{1/2} M^{-1/2} I(n), \tag{16}
\]

where \( I(n) \equiv \int_0^\infty dx x^{n-4} \sin(x - x \cos x) x^2 \).

Assuming \( I_1 = B, M^{5/3} \), we obtain \( \mu_1^2 - 3\mu_2^2 = B^2 M^{10/3} \), where \( B \) and \( M \) are constants that depend on the protogalaxi shapes. Equation (13) becomes

\[
\langle J | \psi \rangle = \frac{48A}{5} B G \rho_0 \left( \frac{4\pi}{3} \rho_0 \right)^{1/2} I(n)^{1/2} a^2 D \times M^{2/5 - 2}. \tag{17}
\]

Equation (17) implies that in the linear regime, for a model with scale-free \( P(k) \), the J–M relation has a constant power exponent

\[
\alpha = \frac{7}{6} - \frac{n}{6}. \tag{18}
\]

and a time-dependent coefficient

\[
\beta(t) \propto a^2 D. \tag{19}
\]

Specifically, in the ΛCDM model, matter dominates in this regime, \( D \sim a \sim t^{3/2} \), and thus \( \beta(t) \propto t \).

If we ignored the scale (or mass) dependence of \( \int d^6 \psi(k) \tilde{W}(kR_s)^2 \), then \( \langle J | \psi \rangle \propto (\mu_1^2 - 3\mu_2^2)^{1/2} M^{5/3} \). This is how the previous arguments in TTT explain the observed J–M relation. However, the smoothing scale \( R_s \) in the smoothing potential is related to a protohalo’s mass as \( M = 4\pi \rho_0 R_s^3 / 3 \), and this introduces an additional mass dependence into TTT’s predicted angular momentum. Therefore, when considering the J–M relation in the linear regime, we cannot ignore this dependence. It leads to a deviation of \( \alpha \) from 5/3 in the linear regime.

Equations (18) and (19) are our predictions for the J–M relation in the linear regime. We will test them in Section 4.

3. NUMERICAL METHODS

3.1. N-body Simulation

We used the public TreePM code GADGET2 (Springel 2005) to perform all simulations. The initial conditions were generated using grid uniform particle distribution and Zel’dovich approximation. The simulations were divided into three groups: ΛCDM, scale-free, and quintessence dark energy models.

The simulation parameters for ΛCDM model are summarized in Table 1. It is known that a finite simulation box size could lower halos’ spins and affect the mass function (see, e.g., Bagla & Ray 2005; Power & Knebe 2006). This might affect the J–M relation and should be checked. We found that \( L_{box} \geq 100 h^{-1} \) Mpc gave converged results (see Appendix A). In this paper, we only show the results of ΛCDM simulations that have a larger box size \( L_{box} = 200 h^{-1} \) Mpc and thus better statistics of high-mass halos. Other simulations give similar results.

For scale-free simulations, we set up the initial conditions as in Knollmann et al. (2008). But instead of starting at the same scale factor \( a_0 \), our simulations began at different \( a \) and stopped at the same \( a = 1 \) in order to offer a direct comparison to our results from the ΛCDM models. Especially, when compared to the ΛCDM or quintessence dark energy models with the time variable \( t \), we use the corresponding \( t \) in the Einstein–de Sitter cosmology for scale-free models. To normalize the power spectrum, we chose the characteristic nonlinear mass \( M_{NL} \approx 36,000 \) particles at \( a = 1 \) for all simulations. The starting scale factor \( a_0 \) is set by requiring the integral power inside the box \( \sigma^2_{box}(a_0) = (2\pi)^{-3} \int dk P(k, a_0) = 0.15^2 \) so that the simulation started with all scales in the linear regime. The normalization \( A \) of the scale-free power spectrum \( P(k) = A k^n \) and \( a_0 \) for different simulations are listed in Table 2.
Table 2

Simulation Parameters of Scale-free Models

| Name   | $n$ | $\Omega_m$ | $\Delta$ | $a_i$ | $N^0$ | Realizations |
|--------|-----|------------|----------|-------|-------|--------------|
| SF-0.50| -0.50 | 1.0       | 3358.69  | 6.76E-4 | 256$^3$ | 4            |
| SF-1.00| -1.00 | 1.0       | 1603.62  | 1.69E-3 | 256$^3$ | 4            |
| SF-1.50| -1.50 | 1.0       | 674.74   | 4.36E-3 | 256$^3$ | 4            |
| SF-2.00| -2.00 | 1.0       | 239.27   | 1.16E-2 | 256$^3$ | 4            |

For the homogeneous dynamical dark energy simulation, we use the AS quintessence model (Albrecht & Skordis 2000), which has a significant portion of dark energy in early times and thus a notably different growth factor $D(t)$ from the ΛCDM model (see Section 4.1). We adopt the parameterization formula for quintessence dark energy's equation of state in Corasaniti & Copeland (2003), i.e.,

$$w_Q(a) = w_Q^0 + \left(w_Q^m - w_Q^0\right) \times \frac{1 + e^{-m\Delta_m}}{1 + e^{-m\Delta}} \times \frac{1 - e^{-w_Q^0\Delta_m}}{1 - e^{-w_Q^0\Delta}},$$

(20)

with parameters $w_Q^0 = -0.96$, $w_Q^m = -0.01$, $a_i = 0.53$, and $\Delta_m = 0.13$. Other cosmological parameters in this simulation are the same as ΛCDM512b. We started an AS model simulation with the same initial conditions as a ΛCDM512b run. Therefore, the output differences give a direct and clean comparison between the growth of angular momenta in two cosmologies.

3.2. Halo Identification

We adopted the AMIGA Halo Finder (AHF; Knollmann & Knebe 2009) to extract halos in our simulation outputs. The virial overdensity parameter $\Delta_{vir}(z)$ is set according to

$$\Delta_{vir}(z) = 18\pi^2 + 82x - 39x^2,$$

(21)

where $x = \Omega_m(z) - 1$ (Bryan & Norman 1998). We have tested that the $J-M$ relation results are not sensitive to $\Delta_{vir}$ for a wide range of its values. We excluded subhalos in our analysis since subhalos usually are tidally disrupted and their angular momenta vary violently. We have used the friends-of-friends (Davis et al. 1985) halo finder to cross check the AHF results, and their $J-M$ relation results were consistent with each other.

In order to determine the minimum particle number $N_{\text{min}}$ to define a halo for angular momentum studies, we performed a resolution test and found that $N_{\text{min}} = 200-400$ is needed to obtain converged results (see Appendix B). In this paper, we conservatively choose $N_{\text{min}} = 400$.

To study the time evolution of the $J-M$ relation, we identified halos at $z = 0$ and traced the particles within these halos back to the earlier time. Protohalos are defined as the configurations of these particles in earlier time (see Figure 1).

A halo’s angular momentum and mass are calculated as

$$J(t) = \sum_i m_i \left[r_i(t) - r_c(t)\right] \times \left[v_i(t) - v_c(t)\right]$$

(22)

and

$$M = \sum_i m_i,$$

(23)

respectively. Here the summation is over all particles within a protohalo. $r_c(t)$ and $v_c(t)$ are the center-of-mass position and velocity. Notice that in this trace-back picture, the halo mass $M$ is a constant.

We used two independent methods to fit the $J-M$ relation: all points fitting (APF) and mass bins fitting (MBF). The details of these methods are described in Appendix C. They showed consistent results. In the text, if not mentioned, we only show results using the MBF method.

3.3. Environment Classification

We used the Hessian matrix method (Hahn et al. 2007) to classify the cosmic web. The Hessian matrix

$$H_\theta(r) = \frac{\partial^2 \rho_\theta(r)}{\partial r_i \partial r_j}$$

(24)

was calculated from the density field smoothed with a Gaussian kernel (smoothing scale $R_s = 2.1 h^{-1} \text{Mpc}$). The eigenvalues of $H_\theta(r)$ are then calculated for each halo in its center-of-mass position. A halo is classified as cluster/filament/sheet/void type if it has 0/1/2/3 positive eigenvalues (i.e., the classification threshold $\lambda_h = 0$).

Here, we use the density field to classify the cosmic web (see also Zhang et al. 2009). One can use other fields such as the potential field, velocity divergence field, and velocity shear field (e.g., Hahn et al. 2007; Hoffman et al. 2012; Cautun et al. 2013).

3.4. Merger Trees

In our simulations, there were 30 snapshots ranging from $z = 5$ to $z = 0$ with time intervals of 0.1–0.5 Gyr. To construct merging histories, we identify halos in each of 30 snapshots with a minimum particle number of 20. Then, progenitor halos in snapshot $n$ that merge to form a halo in the subsequent snapshot $n + 1$ (target halo) are identified by locating particles of the target halo in halos of snapshot $n$. We call the progenitor halo that contributes most particles to the target halo as “mother” and the ones contributing less as “satellites.”

Notice that there is no satellite for some halos. It implies that these halos increase their masses by small accretions. Also, for some halos—especially high-redshift and low-mass ones—we may not be able to find their mothers because their progenitors are too small to show up in our halo catalog. We only use those halos whose progenitors can be traced back to $z > 2$. 

To study the dependence of the $J-M$ relation on the halo merging history, we divided all halos at $z = 0$ (with $N_{\text{min}} = 400$) into two groups, major merger (MM) and minor merger (mM), according to two parameters: the satellite-to-mother mass ratio $r_m$ (defined as the mass ratio between the largest satellite halo and the mother halo) and merger redshift $z_m$. If a merger event with $r_m \geq r_{\text{th}}$ occurs for $z_m < z_{\text{th}}$ ($r_{\text{th}}$ and $z_{\text{th}}$ are the given threshold parameters), then we mark it as an MM. Otherwise, it is labeled as an mM.
4. RESULTS

4.1. J–M Relation for Protohalos

We write the J–M relation as

\[ \frac{J}{J_0} = \beta \left( \frac{M}{M_0} \right)^\alpha, \]

where \( J_0 = 10^{10} h^{-2} M_\odot \text{ kpc km s}^{-1} \) and \( M_0 = 10^{10} h^{-1} M_\odot \).

We find that in the ΛCDM model, at all redshifts, the J–M relation for protohalos (or halos at \( z = 0 \)) can be well fitted as a power law (Figure 2), with \( \alpha(t) \) increasing from \( \sim 1.5 \) to 5/3, and \( \beta(t) \) evolving linearly with time in the beginning and reaching a constant finally (Figure 3). This time-evolution behavior can be understood using a three-regime scheme.

(1) Linear regime. In this stage, all protohalos in our catalog still evolve linearly. We adopt one of the methods in Sugerman et al. (2000) to estimate the halo turnaround time \( t_T \) as the earliest time that half of particles have negative radial physical velocity. The probability distribution of \( t_T \) is plotted in Figure 4.

In our ΛCDM halo sample, almost all halos reach turnaround during \( t = 1–5 \) Gyr. Only 0.3% of halos have turnaround time less than \( t = 1 \) Gyr. As a result, we conservatively estimate the time period of the linear regime as \( t < 0.5 \) Gyr (or \( z > 10 \)) for our ΛCDM halo sample. According to the discussion in Section 2.2, in this regime, \( \alpha \) remains constant and \( \beta \propto t \). This is confirmed by our simulation results (Figure 3).

To help us understand the evolution behaviors of \( \alpha(t) \) and \( \beta(t) \), especially to test our predictions of Equations (18) and (19) in the linear regime, we look at scale-free and AS quintessence dark energy (AS-QCDM; see Section 3.1) simulations. Their J–M relations are shown in Figure 5.

Different models have different \( \alpha \) in the linear regime, \( \alpha_{\text{lin}} \). For scale-free simulations, the \( \alpha_{\text{lin}}-n \) relation from simulations is shown in Figure 6. It can be fitted as \( \alpha_{\text{lin}} = -0.17n + 1.11 \), which has a deviation of \( \sim 0.05 \) in the y-intercept from the theoretical prediction \( \alpha_{\text{lin}} = -n/6 + 7/6 \) (Equation (18)). This comes from the underlying moment of inertia–mass relation (J–M relation). We have used \( I \propto M^{5/3} \) when deriving Equation (18), assuming protohalos have similar triaxial ratios. However, the simulated protohalos follow a slightly different relation, \( I \propto M^{1.56\pm0.01} \), since their triaxial ratios are usually not perfectly similar. After taking into account such an effect, the numerical results agree with our prediction.

We can use the effective index \( n_{\text{eff}}(k) = d \ln P(k)/d \ln k \) to understand the value of \( \alpha_{\text{lin}} \) in the ΛCDM and AS-QCDM models. For the protohalo scale (\( \sim 1 h^{-1} \text{Mpc} \)) in such models, \( n_{\text{eff}} \sim -2.0 \), and Equation (18) gives \( \alpha_{\text{lin}} \sim 1.5 \).

The linear regimes span different periods in different models (Figure 5). This can be understood by looking at the halo turnaround time \( t_T \) in different models, as shown in Figure 4. A scale-free model with less negative \( n \) has more power in small-scale perturbations and thus make halos turn around earlier.
Although we start with the same power spectrum in the ΛCDM and AS-QCDM models, halos in the AS-QCDM model tend to have larger \( t_T \), since the AS-QCDM model contains a larger fraction of dark energy and thus a faster expansion rate at high redshifts, consequently delaying the halo turnaround time.

In the linear regime, scale-free models follow a similar \( \beta \sim t \) as the ΛCDM model since both of them are matter dominated at high redshifts. However, the AS-QCDM model has a significantly different growth factor \( D(t) \) from ΛCDM in the linear regime. As we can see from the lower panel of Figure 5, the evolution behaviors are similar to the ΛCDM case (dotted). In the linear regime, scale-free models’ \( \beta \) evolve approximately to \( \beta \sim t \) (dashed line), while \( \beta \) of the AS-QCDM model varies as \( \beta \sim t^{0.9} \) (dashed–dotted line).
in the linear regime, $\beta_{\text{AS-QCDM}} \sim t^{0.9}$. This is consistent with the numerically calculated $J \sim a^2D \sim t^{0.9132}$, and thus supports Equation (19) (see Table 3). We can also look at the time evolution of each halo’s angular momentum, which has the same dependence on $a^2D$ according to TTT. Assuming $J(t) \sim \bar{r}^5$, we fit $\gamma$ for each protohalo in its linear regime and obtain a Gaussian probability distribution $p(\gamma)$ for all halos, shown in Figure 7. The mean of $\gamma$ in each cosmological model agrees with the TTT prediction as expected.

Since the time-evolution behaviors of the $J-M$ relations in the AS-QCDM and scale-free models are qualitatively similar to that of $\Lambda$CDM (Figure 5), in the following discussion, we will mainly present the $\Lambda$CDM results. Similar arguments and explanations can be applied to the AS-QCDM and scale-free models.

(2) **Nonlinear regime.** After the linear regime, some protohalos (especially the small mass ones) start to evolve nonlinearly. For our $\Lambda$CDM halo catalog, this regime ranges from $t = 0.5$ Gyr to present.

In this regime, $\alpha$ increases monotonically while $\beta$ reaches a maximum and decreases a little. Note that even when almost all halos have reached turnaround (e.g., in Figure 4, 99.95% of $\Lambda$CDM halos reached turnaround after $t = 6$ Gyr), the $J-M$ relation still evolves. This is different from TTT’s prediction. We conclude that nonlinear effects play an important role in the time evolution of the $J-M$ relation.

The evolution of $\beta$ is similar to that of a halo’s angular momentum (Sugerman et al. 2000; Porioni et al. 2002a). The decrease of a halo’s angular momentum, or $\beta$ in the $J-M$ relation, is due to its nonlinear interactions with the surrounding matter that lead to the redistribution of angular momenta.

(3) **Virial regime.** Once the halos become virialized and if they experience no merger events, their angular momenta stop evolving, and thus the $J-M$ relation becomes stable, with $\alpha$ and $\beta$ both becoming constants. In particular, $\alpha$ approaches $5/3$, which can be explained using the mechanical equilibrium argument.

To quantify the virialization of halos at $z = 0$, we use the offset parameter defined as

$$s = \left| \frac{r_{mb} - r_{cm}}{R_{vir}} \right|,$$

where $r_{mb}$, $r_{cm}$, and $R_{vir}$ are the position of the most bound particle within a halo, the center of mass of a halo, and the halo’s virial radius, respectively. Relaxed halos have small $s$; halos having $s < 0.1$ are usually regarded as relaxed (e.g., D’Onghia & Navarro 2007). In our $z = 0$ halo sample ($\Lambda$CDMS512b simulation), $\log s$ distributes normally with a mean of $-1.12$ and standard deviation of $0.28$.

### Table 3

| Model          | $\gamma$ Fitted from | $x$ | $y$ | $2x + y - 1$ | Protohalos |
|----------------|-----------------------|-----|-----|--------------|------------|
| $\Lambda$CDM  |                        | 0.6667 | 0.6667 | 1.0001 | 1.00 ± 0.02 |
| AS-QCDM       |                        | 0.6684 | 0.5764 | 0.9132 | 0.91 ± 0.02 |

### Figure 7

Probability distribution function (PDF) of the fitted $\gamma$ for $\Lambda$CDM (thick solid) and AS-QCDM (thick dashed) models, fitted as a Gaussian PDF $p(\gamma) = (1/\sqrt{2\pi} \Sigma)^{-1/2} \exp[-(\gamma - \mu)^2/2\Sigma^2]$ shown by the thin solid and dashed lines, respectively. The best-fit $(\mu, \Sigma)$ are $(1.00, 0.02)$ and $(0.91, 0.02)$ for the $\Lambda$CDM and AS-QCDM models, respectively. Numerically calculated TTT predictions, $\gamma = 2x + y - 1$, are marked with arrows.

As a complementary way to quantify the relaxation of halos, we also calculate the virial parameter

$$\eta = \frac{2K}{\langle U \rangle},$$

where $K = \sum_{i} m_{i} \dot{v}_{i}^2/2$ and $U = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} - \frac{Gm_{i}m_{j}}{r_{ij}}$ are the halo’s kinetic and potential energy. According to the virial theorem, $\eta$ becomes 1 when an isolated object relaxes. For our halo catalog at $z = 0$, the mean (median) value of $\eta$ is 1.11 (1.08), with a standard deviation of 0.16. The distribution of $s$ and $\eta$ for our halo sample indicates that most halos are close to being virialized at $z = 0$.

To see more explicitly the correlation between $\alpha = 5/3$ and virialization, we divide the halos at $z = 0$ into two subsets: $s < 0.1$ and $s > 0.1$ and fit the $J-M$ relation for them separately. The best fits are $\alpha = 1.65 \pm 0.01$, $\log \beta = 1.82 \pm 0.03$ for $s < 0.1$ halos and $\alpha = 1.75 \pm 0.02$, $\log \beta = 1.74 \pm 0.04$ for $s > 0.1$ halos. The threshold value of 0.1 here is not special. Changing this threshold value for $s$ does not change the conclusion that $\alpha$ becomes 5/3 for virialized halos, but is significantly different from 5/3 for non-virialized ones.

In addition, we plot in Figure 8 the time evolution of the virial parameter. Especially, to illustrate the correlation between the evolution of $\alpha$ and $\eta$ more clearly, we use future halos identified at $a = 4$ (or $z = -0.75$, $t = 35.65$ Gyr) since the majority of them will be fully virialized. For $a = 4$ halos, we trace the particles back and perform the same fitting for the $J-M$ relation, as for halos identified at $a = 1$. As shown in Figure 8(a), when most halos become virialized, that is, the mean $\bar{\eta} \sim 1$ and standard deviation $\sigma$ becomes small enough, $\alpha$ reaches a stable value $\sim 5/3$. Notice that the mean $\bar{\eta}$ reaches 1 at $a \sim 0.8$, but $\alpha$ is still varying at this moment. This is due to the fact that there are still some halos that are far from virialization, as shown by the relatively large standard deviation $\sigma_\eta$. For example, at $a = 1$, $\sigma_\eta/\bar{\eta} = 10.0\%$, while at
a = 4, σf/̄J = 3.5%. As time evolves, ̄J gets closer to 1 and the dispersion becomes smaller (Figure 8(b)).

With this three-regime scheme, we can understand the observed time evolution of the J–M relation from N-body simulations. Especially, we show clearly that the observed exponent α = 5/3 correlates with virialization. On the other hand, TTT is able to explain the J–M relation in the linear regime if we consider the effects from smoothing the potential term. In the linear regime, α depends on the power index of the power spectrum, whereas the time evolution of β contains the information of the underlying cosmological model. The three-regime scheme can also be used to understand the J–M relations for halos in different environments and with different merging histories, as we will discuss in Sections 4.2 and 4.3.

4.2. Dependence on Environments

The time evolution of the J–M relations for halos in clusters, filaments, and sheets are shown in Figure 9. In our ΛCDM simulations, there are too few void halos to perform a reliable fit for the J–M relation, and thus we do not discuss them here. The numbers (fractions) of cluster, filament, sheet, and void halos at z = 0 are 14,000 (45.15%), 16,000 (51.60%), 1000 (3.22%), and 100 (0.33%), respectively, in a ΛCDM12b simulation with at least 200 halo particles (M ≥ 9.2 × 10¹¹h⁻²M☉). Note that we only perform environmental classifications on the halos at z = 0 and not as a function of redshift.

From Figure 9, we can see that filament and sheet halos have a larger α in the linear regime. This is due to the deviation of the power index from 5/3 in the underlying I–M relation (Table 4). Cluster halos experience more nonlinear effects and their protohalos usually have more complicated and non-similar shapes. Their I–M relation deviates more from a power index of 5/3, which leads to a larger deviation of α from 7/6 – n_eff/6 for their J–M relation.

In addition, filament and sheet halos’ J–M relations become stable earlier than cluster halos. For example, the filament halos’ α stabilizes to a value near 5/3 at a ≈ 0.7, while the α – a curve for cluster halos reaches a plateau at a ≈ 0.9. A similar behavior can be observed for log β. This is due to the fact that filament and sheet halos tend to locate in relatively low density regions (see, e.g., Hahn et al. 2007), experience less nonlinear effects, and enter the equilibrium regime earlier. However, halos of each classified type still can span a wide range of densities, and this is likely the cause for the large scatters of sheet halos, which have a relatively small number, in Figure 9. Note that we only study the case of λb = 0 here. A different λb can lead to different fractions of classified types, as shown by Forero-Romero et al. (2009).

The environmental dependences of J–M relations in the ASQCDM and scale-free models are qualitatively similar to the ΛCDM case.

Thus, by dividing the halos into different environments, we can see clearly how the nonlinear effects affect the J–M relation. We have also shown that the differences of J–M relations in different environments can be explained using the three-regime scheme.

4.3. Dependence on Merging Histories

We study the J–M relation for mM and MM halos with different threshold parameters rth = 1/6, 1/5, 1/3 and zth = 0.5, 1.0, 2.0. The results are shown in Figure 10. To be simple and clear, we only plot the cases of rth = 1/6 and zth = 0.5, 2.0. Other cases lead to similar conclusions.

α has a larger initial value for the mM halo subset and becomes stable earlier compared to the MM halo subset. For both types of halos, α tends to be larger for higher zth. These can be understood as follows. (1) MM halos usually have more complicated protoshapes and thus larger deviations from 5/3 for the power index of the I–M relation. This leads to a larger deviation for α from 7/6 – n_eff/6 for their J–M relation.

In addition, filament and sheet halos’ J–M relations become stable earlier than cluster halos. For example, the filament halos’ α stabilizes to a value near 5/3 at a ≈ 0.7, while the α – a curve for cluster halos reaches a plateau at a ≈ 0.9. A similar behavior can be observed for log β. This is due to the fact that filament and sheet halos tend to locate in relatively low density regions (see, e.g., Hahn et al. 2007), experience less nonlinear effects, and enter the equilibrium regime earlier. However, halos of each classified type still can span a wide range of densities, and this is likely the cause for the large scatters of sheet halos, which have a relatively small number, in Figure 9. Note that we only study the case of λb = 0 here. A different λb can lead to different fractions of classified types, as shown by Forero-Romero et al. (2009).

The environmental dependences of J–M relations in the ASQCDM and scale-free models are qualitatively similar to the ΛCDM case.

Thus, by dividing the halos into different environments, we can see clearly how the nonlinear effects affect the J–M relation. We have also shown that the differences of J–M relations in different environments can be explained using the three-regime scheme.
MM increase to 1.62 ± 0.01 and 1.64 ± 0.02, which are closer to \(5/3\).

5. SUMMARY AND DISCUSSIONS

We have used N-body simulations to study the time evolution of the \(J-M\) relation. From our results, a picture of the origin and evolution of the \(J-M\) relation in the \(\Lambda\)CDM model emerges.

At high redshifts, when all halos in our sample still evolve linearly, \(\alpha\) is a constant of \(\sim 1.5\) and \(\beta\) increases linearly with time. We show that this can be explained using the TTT if we carefully consider the mass dependence introduced by the smoothing of the potential field (Equations (18) and (19)), needed for keeping the validity of the Zel'dovich approximation and Taylor approximation up to the second order. In the nonlinear regime, \(\alpha\) increases monotonically and \(\beta\) gradually reaches a maximum and decreases. Finally, in the virial regime when the majority of halos become virialized, \(\alpha\) becomes a constant \(5/3\) and \(\beta\) stabilizes.

| Type    | Cluster | Filament | Sheet     |
|---------|---------|----------|-----------|
| \(w\)   | 1.54 ± 0.01 | 1.58 ± 0.01 | 1.58 ± 0.02 |

Figure 9. Time evolution of \(\alpha\) and \(\beta\) for halos in different environments in the \(\Lambda\)CDM model. Solid, dashed, and dotted curves show the results for cluster, filament, and sheet halos, respectively. The shaded regions are standard deviations among realizations. There are few void halos in our simulation, and thus they are not included.

Figure 10. Similar to Figure 9, but here we show the time evolution of \(J-M\) relations for minor merger (mM) and major merger (MM) halos. To be clear, we only plot the results with \(r_{\text{th}} = 1/6\) and \(z_{\text{th}} = 0.5, 2.0\). Cases for other \(r_{\text{th}}\) and \(z_{\text{th}}\) are similar.

Figure 11. \(J-M\) relations for different simulations at \(z = 0\). For clarity, we only plot results for \(f_{\text{chop}} = 1.0\) (square) and 0.1 (circle), the two extreme cases. Here, we have divided the halos into several bins according to their masses. The points and error bars in the plot show us the means and standard deviations of log \(J\) and log \(M\) in different bins.

This time-evolution picture enables us to discriminate among possible explanations. We show that the empirically observed \(\alpha = 5/3\) is consistent with the mechanical equilibrium of halos.
On the other hand, TTT successfully explains the $J-M$ relation in the linear regime. Halos in different environments and with different merging histories show different time-evolution behaviors of the $J-M$ relation. The nonlinear effects drive the $J-M$ relation in the linear regime to the one we observed. Antonuccio-Delogu et al. (2010) also looked at the evolution of the $J-M$ relation and found that $\alpha$ is compatible to $5/3$ at high redshift but becomes slightly smaller than $5/3$ recently (see their Figure 2). However, one should note that their $J-M$ relations are fitted from halo samples identified at different redshifts, which are different from ours, the trace-back picture.

The three-stage scheme implies that for different cosmologies, in the linear regime, $\alpha$ has different values according to the initial power spectrum and protohalo's shapes, and $\beta$ evolves with different rates depending on the scale factor and growth rate. Thus, in the linear stage, the $J-M$ relation is quite sensitive to the underlying cosmological model. In the nonlinear regime, the evolution of the $J-M$ relation depends on the details of nonlinear collapse, mergers, and other nonlinear effects in a cosmology. In a cosmological model with more halos in the denser environment and experiencing MMs, the $J-M$ relation will take more time to reach the stable state. When all halos become virialized and go through no merger events, the corresponding $J-M$ relation stabilizes. $\alpha$ will lose the memory of the initial power spectrum and background cosmology and has a universal value of $5/3$. Whether $\beta$ in the virial stage depends on the initial power spectrum and background cosmology is an interesting question. There is no exact analytical theory to calculate the final spin of a halo. It is shown that the spin parameters $\lambda$ (Peebles 1971; Bullock et al. 2001) for virialized halos have no substantial dependence on the initial conditions and background cosmology (see, e.g., Barnes & Efstathiou 1987; Bullock et al. 2001; Macciò et al. 2008; Carlesi et al. 2012, etc.). Here, in the virial regime, the dependence of $\beta$ on the background cosmology is also weak (see Figure 5). Recently, Lee et al. (2013) showed that

| $f_{\text{chop}}$ | $\alpha$ | $\log \beta$ |
|------------------|---------|--------------|
| 1.0              | 1.654 ± 0.004 | 1.863 ± 0.010 |
| 0.75             | 1.649 ± 0.004 | 1.874 ± 0.010 |
| 0.5              | 1.652 ± 0.004 | 1.867 ± 0.010 |
| 0.25             | 1.628 ± 0.004 | 1.918 ± 0.010 |
| 0.1              | 1.582 ± 0.004 | 2.002 ± 0.010 |

Figure 12. Relation between $r_\lambda = \lambda_n/\lambda_{512}$ and halo particle number $N$ (left panel), and the relation between $\theta$ (the angle between $J_n$ and $J_{512}$) and halo particle number $N$ (right panel) for different low-resolution simulations at $z = 0$. The square dots are the mean of $r_\lambda$ or $\theta$ in different bins, and the associated error bars are $1\sigma$ standard deviations.

Table 5
Best Fits of the $J-M$ Relation for Different $f_{\text{chop}}$ Simulations
modified gravity could spin up galactic halos with $M \lesssim 10^{11} h^{-1} M_{\odot}$. It will be interesting to see whether or not modified gravity has great effects on the virial $J$–$M$ relation.

Although our simulations are only for dark matter particles, some behaviors of the $J$–$M$ relation we found can be generalized to baryonic matter. For example, in the linear regime, we expect that $\alpha$ for baryonic matter also depends on its initial power spectrum and the protogalaxy’s shapes. $\beta$ also increases proportionally to $a D$. In the virial regime, once galaxies become virialized, their $J$–$M$ relation remains unchanged. $\alpha$ equals $5/3$ because it is a result of virialization. Indeed, the spin parameters for baryonic and dark components have been shown to correlate in cosmological hydrodynamic simulations (van den Bosch et al. 2002; Chen et al. 2003; Sharma & Steinmetz 2005; Gottlöber & Yepes 2007; Kimm et al. 2011). However, in the nonlinear regime, due to the diverse baryonic physics, such as radiative cooling, star formation, supernovae, and active galactic nucleus feedback, etc., the evolution of galaxies’ spins, disks, spin alignments, and other properties is quite complicated (see, e.g., Bailin & Steinmetz 2005; Bailin et al. 2005; Libeskind et al. 2007; Schäfer 2009; Roškar et al. 2010; Schewtschenko & Macciò 2011), and protogalaxies’ $J$–$M$ relation in this stage might have different behaviors from the dark matter and needs further investigation. The baryonic processes in the nonlinear regime are also key elements to explain the observed offset between the spirals and ellipticals’ $J$–$M$ relations, as discussed in Romanowsky & Fall (2012).

Another question related to the baryon physics is how the angular momentum transfer between dark and baryonic matter affects our results. It has been shown that from the hydrodynamical simulations, the baryonic physics mainly spins up the inner part of a halo, and has minor effects on the whole halo’s spin (e.g., Bett et al. 2010; Bryan et al. 2013). Thus, we also expect that our results about the dark matter halos’ $J$–$M$ relation should not change significantly when one adds baryon physics into the simulations.
Although it was discovered in 1960s, the $J$–$M$ relation is still an ongoing research topic in observations (e.g., Romanowsky & Fall 2012; Fall & Romanowsky 2013), and a complete theoretical explanation is needed. Here, we give an updated picture of this relation for the dark matter part.

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### APPENDIX A

#### NUMERICAL BOX SIZE STUDIES

Previous studies (e.g., Bagla & Ray 2005; Power & Knebe 2006) showed that small numerical box size would reduce the number of massive halos and lower the halos’ spin parameters. Therefore, we expect that the numerical box size should affect the $J$–$M$ relation. To study its effects and find out a suitable box size, we performed several $N$-body simulations using the method of Power & Knebe (2006). In this method, we chopped the long wavelength perturbations in different degrees to mimic different box sizes $L_{\text{chop}}$. The smallest wave vector $k_{\text{min}}$ depends on the chopping factor $f_{\text{chop}}$ as

$$k_{\text{min}} = \frac{2\pi}{L_{\text{chop}}} = \frac{2\pi}{f_{\text{chop}} L_{\text{box}}},$$

where $L_{\text{box}}$ is the box size.

#### Table 6

| $f$   | Mean | Median | Dispersion$^*$ | Mean | Median | Dispersion |
|-------|------|--------|----------------|------|--------|------------|
| 0.5   | 0.32 | 0.24   | 0.18           | 61.0 | 52.2   | 37.5       |
| 0.8   | 0.72 | 0.63   | 0.32           | 39.3 | 29.4   | 28.6       |
| 1.0   | 0.93 | 0.82   | 0.42           | 38.7 | 28.7   | 28.6       |
| 1.1   | 1.04 | 0.91   | 0.47           | 38.8 | 28.7   | 28.6       |
| 1.2   | 1.16 | 1.01   | 0.54           | 39.3 | 28.9   | 29.2       |
| 1.5   | 1.38 | 1.14   | 0.67           | 44.7 | 35.0   | 32.2       |
| 2.0   | 1.45 | 1.11   | 0.79           | 57.7 | 48.8   | 39.3       |
| 5.0   | 1.59 | 1.14   | 0.96           | 78.2 | 74.1   | 47.6       |

Note.

$^*$ The dispersions in this table are the 68.3% confidence intervals. It is calculated by $\sigma_x = \frac{x_{84.2\%} - x_{15.9\%}}{2}$, where $x_{84.2\%}$ and $x_{15.9\%}$ are the values corresponding to 84.2% and 15.9% in the cumulative distribution of $x$.

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Figure 16. TTT predictions with different smoothing scales $R = fR_0$ at the linear regime ($z = 100$).

Table 6

TTT Predictions of the Spin Magnitude and Direction with Different Smoothing Scales $R = fR_0$ and $R = R_{\text{chop}}^m$ (at $z = 100$)

| $f$   | Mean | Median | Dispersion$^*$ | Mean | Median | Dispersion |
|-------|------|--------|----------------|------|--------|------------|
| 0.5   | 0.40 | 0.33   | 0.22           | 52.9 | 43.4   | 38.8       |
| 1.2   | 1.37 | 1.13   | 0.67           | 44.7 | 34.9   | 32.5       |

$^*$ The dispersions in this table are the 68.3% confidence intervals. It is calculated by $\sigma_x = \frac{x_{84.2\%} - x_{15.9\%}}{2}$, where $x_{84.2\%}$ and $x_{15.9\%}$ are the values corresponding to 84.2% and 15.9% in the cumulative distribution of $x$.

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estimation of $N_{\text{min}}$, we performed several simulations with different resolutions. All parameters are the same for these simulations, except for the particle number $N_p$. In our simulations, $N_p = 512^3, 256^3, 128^3,$ and $64^3$. Other parameters are $\Omega_m = 0.28$, $\Omega_{\Lambda} = 0.72$, $\Omega_b h^2 = 0.024$, $h = 0.7$, $L_{\text{box}} = 100 h^{-1}$ Mpc, $\sigma_8 = 0.8$, $n_s = 0.96$, and comoving softening length $\epsilon = 10.0 h^{-1}$ kpc. The same scale perturbations in all of these simulations have the same phases, and thus it allows us to compare the halos by one-on-one mapping.

We first identified halos using AHF in all simulations with $\Delta_{\text{vir}} = 98$ and $N_{\text{min}} = 10$, and then we mapped the halos in $N_p = 64^3, 128^3, 256^3$ simulations to halos in the $N_p = 512^3$ simulation by requiring that each corresponding halo pair has similar locations and masses. Due to having less particles to sample the density field in low-resolution simulations and the noise from the halo finder, the halos in low-resolution simulations usually do not have perfectly identical positions and masses as those in high-resolution simulations. Thus, it is a nontrivial task to map the halos.

We define two parameters related to position and mass differences as

$$
\tau_{\text{pos}} = \sqrt{(x_n - x_{512})^2 + (y_n - y_{512})^2 + (z_n - z_{512})^2},
$$

$$
\tau_{\text{mass}} = \frac{|M_n - M_{512}|}{M_{512}},
$$

where $x_n, y_n, z_n, R_n$, and $M_n$ are the $x-, y-, z-$ positions, virial radius, and mass of a halo in the $N_p = n^3$ simulation. We first map the halos with $\tau_{\text{pos}} \lesssim 0.05$ and $\tau_{\text{mass}} \lesssim 0.05$ and take them out from our halo catalogs. We then gradually increase the threshold values of $\tau_{\text{pos}}$ and $\tau_{\text{mass}}$, map the halos in the remaining halo catalogs that satisfy the new conditions, and remove them from the halo catalog. With the maximum threshold values of $\tau_{\text{pos, th}} = 2.0$ and $\tau_{\text{mass, th}} = 0.6$, $\sim 90\%$ of halos in the low-resolution halo catalog can be mapped into the high-resolution ones. To test the effects of mis-mapping, we have varied the maximum threshold values and found that our conclusion in the following does not change.

The spin parameters $\lambda$ are calculated and used to obtain the ratio parameter

$$
r_\lambda = \frac{\lambda_n}{\lambda_{512}},
$$

where $n = 64, 128, 256$ for $N_p = 64^3, 128^3, 256^3$ simulations, respectively.

The dependence of $r_\lambda$ on halo particle number $N$ (or mass) is shown in Figure 12 (left panel). Low-resolution halos (small $N$) have a trend to overestimate the magnitudes of angular momenta. In particular, halos with $\sim 20$ particles have an average overestimate of $r_\lambda \approx 2.0$. This is a numerical artifact and we can use it to find a suitable $N_{\text{min}}$. To give an average estimate of the spin parameter within 20% accuracy level, $N_{\text{min}} \approx 200$ is required. In our ΛCDM512b simulations, we conservatively use halos with $N_{\text{min}} = 400$ to study the $J-M$ relation (within 10% accuracy level).

Although only the magnitudes of angular momenta are considered in the $J-M$ relation, as a reference and for interest, we also present the direction dependence on the halo resolution in Figure 12 (right panel), where $\theta$ is the angle between $J_x$ and

**APPENDIX B**

**HALO RESOLUTION STUDIES**

What resolution is needed for studying halos’ angular momenta? Or, what is the suitable minimum particle number $N_{\text{min}}$ for a halo to give converged angular momentum? To our knowledge, there is a wide range for $N_{\text{min}}$ (from $\sim 50$ to $\sim 1000$) used in the literature. Here, to obtain a better
$J_{12}$. For halos with $N \geq 200$, on average, the directions of angular momenta in high and low resolutions agree with each other to within 45°. For halos with $N \geq 400$, on average $\theta \leq 30°$ is obtained.

The resolution effects on the $J$–$M$ relation are shown in Figure 13. Low-resolution halos tend to bend the $J$–$M$ relation upward due to the fact that low-resolution halos overestimate the magnitudes of angular momenta. Halos with $N \geq 200$ tend to give a converged $J$–$M$ relation as compared to high-resolution simulations.

### APPENDIX C
TESTS OF FITTING METHODS

When fitting the $J$–$M$ relation, we use two independent methods. (1) APF. For every simulated realization, we performed a linear least square fitting for all data points in the log $J$–log $M$ plane, as most of scaling relation studies did. The final results were obtained from the mean and standard deviation among all realizations. (2) MBF. In this method, we first divided data points into several equal-sized bins according to halos’ logarithmic masses. Then, for each bin, we calculated the mean values and standard deviations of both log $J$ and log $M$. For those bins with a small number of data points, we use the bootstrap sampling method to get a better estimation. Finally, we used the total least square fitting method (Krystek & Anton 2007) to fit the $J$–$M$ relation from the bin means by setting weights as the reciprocal of squared bin standard deviations. Like the APF method, the final results were obtained from averaging over all realizations.

Examples of APF and MBF can be found in Figures 2 and 11. Comparison between these fitting methods is shown in Figure 14. The maximum differences are smaller than 0.5% for both $\alpha$ and log $\beta$. We conclude that they give consistent fitting results.

### APPENDIX D
SMOOTHING IN TTT

In this appendix, we test the smoothing method used in TTT. A smoothing process (Equation (12)) is necessary in TTT for two reasons.

First, as pointed out by White (1984), the validity of the Zel’dovich approximation requires $|\delta^2| < 1$. However, inside a protohalo region, there may exist some smaller-scale perturbations with $|\delta^2| > 1$. To keep $|\delta^2| < 1$ for all scales within a protohalo during the whole period before the turnaround, we need to smooth perturbations with a scale equal to the protohalo scale.

Second, truncating the Taylor expansion at second order requires that the smoothing scale $R$ of a protohalo should be comparable to its size. As shown in Figure 15, using a smaller smoothing scale $R_1$, we can see more smaller hills and valleys that lead to failure of the Taylor approximation at certain points such as $d_0$. Similarly, one can expect that too large a smoothing scale is not acceptable either. Only with a smoothing scale comparable to the protohalo’s size ($R = R_0$) can one approximate $\psi(q)$ better within the whole protohalo.

In practice, a top-hat smoothing function with scale $R_0 = (3M/4\pi \rho_0)^{1/3}$ is often used. Here, we test this smoothing scheme by comparing the results with different smoothing scales: $R = fR_0$, $R = R_0^m$ and a globally constant smoothing scale $R_{\text{global}}$. Note that $R_0$ is the scale of a protohalo in the Lagrangian space, which depends only on the mass of the final virialized halo, and thus it is not a function of redshift. In the following, we use the particle distribution at $z = 100$ (the initial condition) to test the prediction of TTT because the Zel’dovich approximation is expected to hold at high redshifts.

The probability distribution of $J_{NB}/J_{TTT}$ and for $R = R_0$ are shown in Figure 16, where $J_{NB}$ is a protohalo’s angular momentum measured from $N$-body simulation (Equation (22)), $J_{TTT}$ is the angular momentum predicted by TTT (Equation (9)), and $\theta$ is the angle between $J_{NB}$ and $J_{TTT}$. With a smaller smoothing scale ($f < 1.0$), TTT overestimates the magnitude of angular momentum and gives a poorer prediction of the spin direction. Using a larger scale ($f > 1.0$), TTT underestimates the angular momentum magnitude and fails to predict its direction. More results can be found in Table 6, together with the results for $R = R_0^m$.

If we use a globally constant smoothing scale $R_{\text{global}}$ (for example, $R_{\text{global}}$ is the median length scale in our protohalo sample), then TTT overestimates (underestimates) the angular momenta of high (low) mass protohalos, as shown in Figure 17. Therefore, a globally constant smoothing scale is not suitable for TTT either.

Among all these smoothing schemes, $R = R_0$ works best. We conclude that the smoothing of the potential is a key ingredient for TTT. Without potential smoothing, TTT fails to give acceptable predictions (see Figure 16 and Table 6 for the case of $f = 0$). The smoothing scale $R = R_0$ introduces an additional mass dependence into TTT’s predicted angular momentum and leads to a deviation of $\alpha$ from 5/3 in the linear regime.

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