Universal Rényi entanglement entropy of quasiparticle excitations

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Abstract – The Rényi entanglement entropies of quasiparticle excitations in the many-body gapped systems show a remarkable universal picture which is independent of the model, the quasiparticle momenta, and the connectedness of the subsystem. In this letter we calculate exactly the single-interval and double-interval Rényi entanglement entropies of quasiparticle excitations in the many-body gapped fermions, bosons, and XY chains and find additional contributions to the universal Rényi entanglement entropy in the excited states with quasiparticles of different momenta. The additional terms are different in different models, depend on the momentum differences of the quasiparticles, and are different for the single interval and the double interval. We derive the analytical Rényi entanglement entropy in the extremely gapped limit, matching perfectly the numerical results as long as either the intrinsic correlation length of the model or all the de Broglie wavelengths of the quasiparticles are small. When the momentum difference of any pair of distinct quasiparticles is small, the additional terms are non-negligible. We argue that the derived formulas can be applied for a wider range of models than those discussed here.

Using entanglement in the study of quantum many-body systems has been very fruitful in the last couple of decades. In particular, different measures of entanglement have been used extensively to study quantum phase transitions and different phases of the matter [1–4]. Not surprisingly, most of these studies were about the entanglement content of the ground state of the quantum many-body systems. There are also many studies regarding the entanglement in the excited states which one can divide to at least two categories. The first group studies the entanglement entropy in the highly excited states which is relevant in the study of local thermalisation after quantum quenches, see [5] and references therein. It is expected that the entanglement entropy in this regime typically follows the volume-law [6–9], see also [10–14] for the behavior of the average entanglement entropy of the excited states. The second group studies the low-lying excited states which is more relevant for the investigation of low temperature behavior [15–30]. In this regime one is interested in the excess entropy of the low-lying states compared to that of the ground state. When the system is integrable or in general when there is no particle production one can consider quasiparticle excitations whose entanglement shows a remarkable universal property which has a natural qubit interpretation [25, 26], see also [18, 20, 23]. In [25–27] it was shown that the entanglement entropy for the excited states composed of finite numbers of quasiparticles with finite De Broglie wavelengths or finite intrinsic correlation length is largely independent of the momenta and masses of the excitations, and of the geometry, dimension and connectedness of the entanglement region.

In this letter we calculate exactly the single-interval and double-interval Rényi entanglement entropies of quasiparticle excitations in the many-body gapped fermions, bosons, and XY chains beyond the regimes that were considered in [25–27]. For the excited states with more than one mode excited we prove that in the most generic circumstances apart from the universal terms there are extra terms. Especially, these additional terms have nontrivial dependence on the momenta of the excited quasiparticles. These terms are non-negligible when the momentum difference of at least a pair of distinct quasiparticles is small. Moreover, we argue that these extra terms are also universal though in a weaker sense. In the case of double intervals in the XY chain we find a novel universal formula. In all the cases we confirm the validity of our exact results by using exact numerical
calculations. The letter is organized as follows: we first list the essential definitions and the setup of our work. We then briefly review the universal entanglement discovered in [25–27]. Then we show our results regarding the single-interval and double-interval Rényi entanglement entropies of quasiparticle excitations in the many-body gapped fermions, bosons, and XY chains. Finally we comment on the universality and possible applications of our results.

Consider a quantum system in a pure state \( \rho \equiv |\psi\rangle \langle \psi| \), then divide it into a subsystem \( A \) and its complement \( B \). Tracing out the degrees of freedom of \( B \), one gets the reduced density matrix (RDM) of the subsystem \( A \), i.e., \( \rho_{A,\psi} = \text{tr}_B \rho \). Then the Rényi entanglement entropy is defined as

\[
S^{(n)}_{A,\psi} = -\frac{1}{n-1} \log \text{tr} \rho^n_{A,\psi}.
\]

(1)

The von Neumann entanglement entropy (the von Neumann entropy) \( S_{A,\psi} = -\text{tr} (\rho_{A,\psi} \log \rho_{A,\psi}) \) can be derived by taking the \( n \rightarrow 1 \) limit of the Rényi entanglement entropy. In this letter we study the entanglement entropy of a single interval and two disjoint intervals on a circular chain of length \( L \) as depicted in FIG. 1. The single-interval von Neumann and Rényi entanglement entropies in various many-body systems were calculated in the ground state [31–38] and excited states [15–28]. The double-interval von Neumann and Rényi entanglement entropies were studied in [39–50]. Following [16, 17], we use \( F^{(n)}_{A,\psi} \) to denote the difference between the Rényi entanglement entropy \( S^{(n)}_{A,\psi} \) in the excited state \( |\psi\rangle \) and the Rényi entanglement entropy \( S^{(n)}_{A,G} \) in the ground state \( |G\rangle \) as

\[
S^{(n)}_{A,\psi} = S^{(n)}_{A,G} = -\frac{1}{n-1} \log F^{(n)}_{A,\psi},
\]

(2)

where

\[
F^{(n)}_{A,\psi} = \frac{\text{tr} A \rho^n_{A,\psi}}{\text{tr} \rho^n_{A,G}}.
\]

(3)

We call \( F^{(n)}_{A,\psi} \) the Rényi entanglement entropy power.

Fig. 1: The subsystem of a single interval (left) and the subsystem of double interval (right) on a circular chain of \( L \) sites. For the single interval \( A \) the length is \( |A| = \ell \), and for the double interval \( A = A_1 \cup A_2 \) the lengths are \( |A_1| = \ell_1 \), \( |A_2| = \ell_2 \), \( |B_1| = d_1 \), \( |B_2| = d_2 \). For the single interval one can define a useful parameter \( x = \frac{x_1}{2} \), and for the double interval the parameters \( x_1 = \frac{x_1}{2}, x_2 = \frac{x_2}{2}, y_1 = \frac{y_1}{2}, y_2 = \frac{y_2}{2}, x = x_1 + x_2, y = y_1 + y_2 \).

The universal von Neumann and Rényi entanglement entropies in states of quasiparticle excitations were investigated in [25–28], see also [18,20,23]. One can denote the general quasiparticle excited state as \( |k_1^{s_1} k_2^{s_2} \cdots k_s^{s_s}\rangle \). There are \( s \) different quasiparticles with momenta \( k_i \), \( i = 1,2,\ldots,s \), and for each kind of quasiparticle with momentum \( k_i \) there are \( r_i \) excitations. There are totally \( R = \sum_{i=1}^{s} r_i \) excited quasiparticles. We use the convention that the momenta \( k_i \) are integers or half-integers depending on the boundary conditions of the circular chain, which is more convenient than convention in [25–28], where the physical momenta \( p_i = \frac{2\pi k_i}{L} \) are used. In the scaling limit \( L \rightarrow +\infty, \ell \rightarrow +\infty \) with fixed ratio \( x = \frac{\ell}{L} \), as well as the limit that all the momenta \( k_i \)'s are large with fixed finite physical momenta \( p_i = \frac{2\pi k_i}{L} \), as \( L \rightarrow +\infty \), the Rényi entanglement entropy power is universal [25–28]

\[
F^{(n)}_{A,p} = \prod_{i=1}^{s} \left\{ \sum_{a=0}^{r_i} \left[ C^a_{r_i} (-1)^{r_i-a} \right]^n \right\},
\]

(4)

where \( C^a_{r_i} \) is the binomial coefficient. Note that the universal Rényi entanglement entropy power is only valid for finite \( p_i \) in the limit \( L \rightarrow +\infty \). We write the universal Rényi entanglement entropy power as \( F^{(n)}_{A,p} \) instead of \( F^{(n)}_{A,k} \) to remind the reader of the different limit that was used in [25–27]. The above formula is valid as long as either the correlation length of the model \( \Delta \) or the maximal de Broglie wavelength of the excited quasiparticles is much smaller than the sizes of the subsystems [25–27]

\[
\min \left( \frac{1}{\Delta}, \max \left( \frac{L}{|k|} \right) \right) \ll \min(\ell, L - \ell),
\]

(5)

but one does not need to impose both. The universal Rényi entanglement entropy is independent of the model, the quasiparticle momenta and the connectedness of the subsystem. In this letter we will report that there are nontrivial additional contributions to the universal Rényi entanglement entropy of quasiparticle excitations in the fermionic, bosonic, and XY chains when the momenta \( k_i \) are general, i.e. that they can be small and/or close to each other.

Fermionic chain: We consider the chain of spinless fermions

\[
H = \sum_{j=1}^{L} \left[ \begin{array}{cc} \lambda (a_j^\dagger a_j - \frac{1}{2}) & -\frac{\gamma}{2} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) \\ -\frac{\gamma}{2} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) & \lambda (a_{j+1}^\dagger a_{j+1} - \frac{1}{2}) \end{array} \right],
\]

(6)

where the number of the total sites \( L \) is an even integer. In this letter we only consider the excitations in the Neveu-Schwarz sector, i.e. that we impose the anti-periodic boundary conditions on the fermions. The Hamiltonian

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can be diagonalized in terms of the lowering and raising operators $c_k^+, c_k^-$ [53–55]

$$H = \sum_k \varepsilon_k (c_k^+ c_k - \frac{1}{2}),$$

(7)

with the spectrum $\varepsilon_k = \sqrt{\left(\lambda - \cos \frac{2\pi k}{\ell}\right)^2 + \gamma^2 \sin^2 \frac{2\pi k}{\ell}}$ and momenta $k = \frac{1}{\ell}, \frac{2}{\ell}, \cdots, \frac{1}{2}$, $\frac{3}{2}, \cdots, \frac{\ell - 1}{2}$. The ground state $|G\rangle$ is annihilated by all the lowering operators $c_k |G\rangle = 0$, and the excited states are generated by applying the raising operators with different momenta on the ground state

$$|k_1 k_2 \cdots k_n\rangle = c_{k_1}^+ c_{k_2}^+ \cdots c_{k_n}^+ |G\rangle.$$  

(8)

We first consider the single-interval case as shown in the left panel of FIG. 1. In the extremely gapped limit $\lambda \to +\infty$, the ground state has zero Rényi entanglement entropy $S_\ell^{(n)}|G\rangle = 0$. In the limit $L \to +\infty$, $\ell \to +\infty$, $k_1 \to +\infty$ with fixed $x = \frac{\ell}{\ell}$, $p_i = \frac{2\pi k_i}{\ell}$, the Rényi entanglement power in the excited state $|k_1 k_2 \cdots k_n\rangle$ takes a universal form [25–28]

$$\mathcal{F}^{(n)}_{A, p_{1, 2, \ell}} = \left[x^n + (1 - x)^n\right].$$

(9)

In the extremely gapped limit $\lambda \to +\infty$, the excited state $|k_1 k_2 \cdots k_n\rangle$ can be written in terms of subsystem excitations. In the basis of the subsystem modes, the RDM is of finite dimension and we get analytically the Rényi entanglement entropy. We call it the subsystem mode method, which is efficient for both analytical and numerical calculations, and we show details of the method in [54, 55]. The von Neumann and Rényi entanglement entropies of the ground state and excited states can also be calculated numerically from exact diagonalization, i.e. the correlation matrix method [33].

For the state with a single quasiparticle $|k\rangle$, we get the same analytical and numerical results as the universal Rényi entanglement entropy. However, for the state with two different quasiparticles $|k_1 k_2\rangle$ excited, we get the new result with additional contribution

$$\mathcal{F}^{(2)}_{A, k_1 k_2} = \mathcal{F}^{(2), \text{univ}}_{A, p_{1, 2}} + 8x(1 - x)\alpha_{k_1 - k_2}^2 + 4\alpha_{k_1 - k_2}^4,$$

(10)

where

$$\alpha_k = \frac{\sin(\pi k\ell/L)}{\ell \sin(\pi k/L)}.$$  

(11)

Similar formula can be also derived for $\mathcal{F}^{(n)}_{A, k_1 k_2}$ with a general integer $n$ as we show the details in [54]. We have not taken into account any approximation except considering the extremely gapped limit, and so these analytical results are exact for any $L$, $\ell$, $k_1$, $k_2$. This is different from the limit in which the universal Rényi entanglement entropy power $\gamma$ was derived in [25–27], where all $L$, $\ell$, $k_1$, $k_2$, $|k_1 - k_2|$ are large. For a finite fixed $k = k_1 - k_2$, the factor $\alpha_k$ is scale invariant in the limit $L \to +\infty$, $\ell \to +\infty$ with fixed $x = \frac{\ell}{\ell}$

$$\alpha_k = \frac{\sin(\pi kx)}{\pi k}.$$  

(12)

On the contrary, it is vanishing in the limit $|k| \to +\infty$. For a large momentum difference $|k_1 - k_2|$, the Rényi entanglement power $\mathcal{F}^{(2)}_{A, k_1 k_2}$ (10) approaches the universal Rényi entanglement entropy power $\mathcal{F}^{(2), \text{univ}}_{A, p_{1, 2}}$. We compare the universal results of the Rényi entanglement entropy power, the results with nontrivial additional terms, and the numerical results in FIG. 2(a). There are non-negligible additional terms when the momentum difference $|k_1 - k_2|$ is small. The results with nontrivial additional terms match perfectly the numerical results. We also calculated analytically the Rényi entanglement entropy power $\mathcal{F}^{(n)}_{A, k_1 k_2 k_3}$ for the states with three different quasiparticles $|k_1 k_2 k_3\rangle$. The results are reported in [54]. In FIG. 2(b) we compare the universal results, the results with additional terms, and the numerical results. The results with additional terms match perfectly the numerical ones.

We then consider the double-interval case, as shown in the right panel of FIG. 1. In the extremely gapped limit, we calculated the analytical Rényi entanglement entropy power in the single-particle, double-particle, and triple-particle states. The double-interval Rényi entanglement entropy power in the single-particle state is the same as the universal result, while there are nontrivial additional contributions to the Rényi entanglement entropy power in the double-particle and triple-particle states. We compare the universal results, the ones with additional terms, and the numerical results in FIG. 2(c) and 2(d). There are perfect matches between the new results and the numerical ones. Especially, note that the universal double-interval Rényi entanglement entropy power is independent of $y_1 = \frac{2\ell}{\ell}$, however we see nontrivial dependence on $y_1$ for the new and the numerical results.
The universal Rényi entanglement entropy power (9) leads to the universal von Neumann entanglement entropy [25–28]

\[ S_{A,p_1,p_2}^{\text{univ}}(n) = s[-x \log x - (1 - x) \log(1-x)]. \]  

We could analytically evaluate the excited state von Neumann entanglement entropy. For example, in the double-particle state we get the single-interval von Neumann entanglement entropy

\[ S_{A,k_1k_2} = \frac{(x - \alpha_{k_1} - \alpha_{k_2}) \log(x + \alpha_{k_1} - \alpha_{k_2}) - (x - \alpha_{k_1} - \alpha_{k_2}) \log(x - \alpha_{k_1} - \alpha_{k_2}) - (1 - x - \alpha_{k_1} - \alpha_{k_2}) \log(1 - x - \alpha_{k_1} - \alpha_{k_2}) - (1 - x + \alpha_{k_1} - \alpha_{k_2}) \log(1 - x + \alpha_{k_1} - \alpha_{k_2})}{\alpha_k}, \]  

with the definition of \( \alpha_k \) (11). We compare the universal single-interval von Neumann entropy entropy, the new results and the numerical results in the double-particle and triple-particle states, as shown in Fig. 3. One could see the necessity of the additional contributions to the von Neumann entanglement entropy of the excited states with small momentum differences.

![Fig. 3: The universal single-interval von Neumann entropy, the new analytical results with additional corrections (solid lines) and the numerical results (empty circles) in the double-particle (left) and triple-particle (right) states in the extremely gapped fermionic chain. We use different colors for different momenta. We have used \( \lambda = +\infty \), \( L = 64 \), \( k_1 = \frac{1}{3} \).

Bosonic chain: Here we consider the periodic bosonic chain, i.e., the harmonic chain

\[ H = \frac{1}{2} \sum_{j=1}^{L} [p_j^2 + m^2 q_j^2 + (q_j - q_{j+1})^2], \]

with the total number of sites \( L \) being an even integer and the identification \( q_{L+1} = q_1 \). The Hamiltonian in terms of the bosonic lowering and raising operators is

\[ H = \sum_k \varepsilon_k \left( b_k^\dagger b_k + \frac{1}{2} \right), \]

with the spectrum \( \varepsilon_k = \sqrt{m^2 + 4 \sin^2 \frac{2k}{L}} \) and momenta \( k = 1, \frac{L}{2}, \ldots, -1, 0, 1, \ldots, \frac{L}{2} \). The ground state \( |G \rangle \) is annihilated by all the lowering operators, i.e., \( b_k |G \rangle = 0 \). The excited states are constructed by applying various raising operators on the ground state

\[ |k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s} \rangle = \frac{(b_1^{r_1})^{k_1}(b_2^{r_2})^{k_2} \cdots (b_s^{r_s})^{k_s}}{\sqrt{r_1! r_2! \cdots r_s!}} |G \rangle \]

with general momenta \( k_i \). In the infinitely massive limit \( m \to +\infty \), the ground state has vanishing Rényi entanglement entropy \( S_{A,G}^{(n)} = 0 \). The universal excited Rényi entanglement entropy with large \( k_i \) is (4). Similar to the fermionic chain, in the extremely gapped limit \( m \to +\infty \), we write the excited state \( |k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s} \rangle \) with general \( k_i \), in terms of subsystem excitations, and calculate the excited state Rényi entanglement entropy power analytically from the subsystem mode method. The excited state Rényi entanglement entropy can be also calculated from the wave function method [25, 26].

For both the single-interval and double-interval cases in the excited state of \( r \) quasiparticles with equal momenta \( |k_i^2 \rangle \), we get the same analytical and numerical results as the universal Rényi entanglement entropy power. For a single interval in the state with two different quasiparticles \( |k_1 k_3 \rangle \), we get the Rényi entanglement entropy power with additional terms

\[ F_{A,k_1 k_2}^{(2)} = F_{A,p_1,p_2}^{(2),\text{univ}} + 4(1 - 2x)^2 \alpha_{k_1}^2 - 4x \alpha_{k_1} + 4 \alpha_{k_1} + \frac{\lambda}{2} \alpha_{k_2}^2, \]  

where \( \alpha_k \) is defined in (11). Similar formulas can be also derived for \( F_{A,k_1 k_2}^{(n)} \) with larger \( n \) and also for a state with three different quasiparticles \( |k_1 k_2 k_3 \rangle \), we calculated \( F_{A,k_1 k_2 k_3}^{(n)} \) with nontrivial additional contributions to the universal results in all the cases. The double-interval Rényi entanglement entropy power in the double-particle and triple-particle states \( F_{A,k_1 k_2}^{(n)} \) and \( F_{A,k_1 k_2 k_3}^{(n)} \) were also calculated analytically which in all the cases we found additional contributions to the universal term. Although not shown here the results of the Rényi entanglement entropy power with nontrivial additional terms match perfectly with the results from the parafraction method [54].

**XY chain:** The XY chain has the Hamiltonian

\[ H = -\sum_{j=1}^{L} \left( \frac{1 + \gamma}{4} \sigma_j^x \sigma_{j+1}^x + \frac{1 - \gamma}{4} \sigma_j^y \sigma_{j+1}^y + \frac{\lambda}{2} \sigma_j^z \right), \]

with the Pauli matrices \( \sigma_j^{x,y,z} \). It is related to the fermionic chain (6) by the Jordan-Wigner transformation [51–53]

\[ a_j = \left( \prod_{i=1}^{j-1} \sigma_i^z \right) \sigma_j^+, \]

with \( \sigma_j^z \equiv \frac{1}{2}(\sigma_j^x + i \sigma_j^y) \) and properly taking care of the boundary conditions. The ground and excited states in the XY chain can be constructed in the same way as those in the fermionic chain.

For a single interval, the Rényi entanglement entropy power is the same as that of the fermionic chain. However, it is different for the double-interval case [42, 43]. In the extremely gapped limit, we use the subsystem mode method and obtain exactly the double-interval Rényi entanglement entropy power \( F_{A,1}^{(n)} \), \( F_{A,2}^{(n)} \), and \( F_{A,3}^{(n)} \), respectively.
\( \delta F_{A_1 A_2, k_1 k_2}^{(n)} \) – Especially, we get for example the new universal results
\[
F_{A_1 A_2, p_1 p_2}^{(2), \text{univ}} = (x^2 + y^2)^2 - 16 x_1 x_2 y_1 y_2, \quad (21)
\]
\[
F_{A_1 A_2, p_1 p_2}^{(3), \text{univ}} = (x^3 + y^3)^2 - 24 x_1 x_2 y_1 y_2 x_3 y_3, \quad (22)
\]
which are different from (4). The double-interval Rényi entanglement entropy was calculated numerically in [39, 40, 42–44], and we adopt the effective method in [44]. The analytical and numerical results of the Rényi entanglement entropy power are compared in FIG. 4, and we see perfect matches. In FIG. 4 we just choose \( x_1 = x_2 = \frac{1}{2} \) as an example, and there are similar results for other values of \( x_1, x_2 \). In the limit of large momentum difference \( |k_1 - k_2| \), the numerical results approach to the new universal results (21) and (22) instead of the old one (4). The universal result (4) has an elegant quantum information interpretation in terms of entangled qubits [25–28], which breaks down for the double-interval Rényi entanglement entropy in the double-particle state in the XY chain (21) and (22) even in the limit that all the momenta and the momentum difference are large. Although quite remarkable, this is not surprising as the local excitations in the XY chain are the same as the ones in neither the fermionic nor the bosonic chain. In fact, as stated in [25, 26], the validity of the universal Rényi entanglement entropy therein requires that the quasiparticles are localized quantum excitations, while it is not the case in the XY chain.

![Fig. 4: The double-interval Rényi entanglement entropy power in the double-particle state in the extremely gapped XY chain, including the old universal results (red dotted lines), the new universal results (blue dashed lines), the new universal results plus additional terms (solid lines), and the numerical results (empty circles). We also use different colors for different momenta. We have set \( \lambda = +\infty, L = 64, k_1 = \frac{1}{2}, x_1 = x_2 = \frac{1}{2} \).](image)

**Discussion:** In this letter we have used the extremely gapped fermionic and bosonic Hamiltonians and calculated exactly the Rényi entanglement entropy in the quasiparticle excited states. In addition to the universal terms derived for large \( k_1 \) in [25–28] we found extra terms that cannot be ignored in the limit of small difference between the excited momenta. Technically, the additional term comes from the (anti)commuting relations of the local modes, and we could call them “exchange term”. The exchange term is larger when the quasiparticles have closer but distinct momenta. These extra terms are also valid beyond the regime they are derived. For example, we realized that for the free fermionic Hamiltonian (6) and the XY chain (19) the result (10) remains intact also for the Hamiltonian with general parameters \( \gamma, \lambda \) and momenta \( k_1, k_2 \) as long as the condition (5) is satisfied. It is expected that if one considers a generic Hamiltonian \( H = H_0 + \lambda \sum_{j=1}^L (a_j^\dagger a_j - \frac{1}{2}) \) in the large \( \lambda \) limit one recovers the result (10). We also note that the magnon states considered in the supplemental material of [25] leads to the same result as ours after considering the limit that \( k_i \)'s are general. The same argument can be also applied for bosonic systems which supports the idea of the universality of our results. Our universality argument can be also generalized to the double interval situation in the case of the fermionic and bosonic chains. However, the double interval case in the XY chain ends up to have a completely novel behavior. Instead of the universal results discovered in [25] we found a new universal term with new additional terms. The difference from the fermionic chain is due to the fact that the local degrees of freedom of the XY chain are the Pauli matrices not the spinless fermions. We expect the equations (21) and (22) to be also universal as far as one considers double interval in quantum spin-\( \frac{1}{2} \) chains.

Although we have derived the analytical Rényi entanglement entropy in the extremely gapped limit, we also checked numerically that the Rényi entanglement entropy powers with nontrivial additional terms in the bosonic and the fermionic chains also apply to the slightly gapped and even critical chains as long as all the excited quasiparticles have large momenta. As \( L \to +\infty \), in the fermionic and XY chains we fit numerically for several examples and get
\[
[1 - F_{A, K}^{(n)}(\text{finite } \lambda)]/F_{A, K}^{(n)}(\lambda = +\infty) \sim \frac{1}{L}, \quad (23)
\]
and in the bosonic chain we get
\[
[1 - F_{A, K}^{(n)}(\text{finite } m)]/F_{A, K}^{(n)}(m = +\infty) \sim \frac{1}{L^2}. \quad (24)
\]
We show more details in [54].

One highlight of the paper is the momentum dependence of the Rényi and entanglement entropies in the quasiparticle excited states, improving the results in [18, 20, 23, 25–27]. The momentum independence of the von Neumann entanglement entropy has inspired the studies of local quantum quenches in [56, 57], which can possibly be revisited with the new results in this letter. The quasiparticle excited states can be engineered experimentally [58, 59], and it is interesting to measure the quasiparticle excite state Rényi entanglement entropy and compare the data with our momentum-dependent results. The universal Rényi and entanglement entropies we have obtained can also be used as a benchmark to other methods, for example the tensor network techniques [60–63]. The techniques to calculate the Rényi and
entanglement entropies are also useful to calculate the subsystem Schatten and trace distances [55].

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