AUSSLANDER-REITEN TRIANGLES AND QUIVERS 
OVER TOPOLOGICAL SPACES

PETER JØRGENSEN

Abstract. In this paper, Auslander-Reiten triangles are introduced into algebraic topology, and it is proved that their existence characterizes Poincaré duality spaces.

Invariants in the form of quivers are also introduced, and Auslander-Reiten triangles and quivers over spheres are computed.

The quiver over the $d$-dimensional sphere turns out to consist of $d - 1$ components, each isomorphic to $\mathbb{Z}A_\infty$. So quivers are sufficiently sensitive invariants to tell spheres of different dimension apart.

0. Introduction

In this paper, two concepts from representation theory are introduced into algebraic topology: Auslander-Reiten triangles and invariants in the form of quivers (that is, directed graphs).

The highlights are that existence of Auslander-Reiten triangles characterizes Poincaré duality spaces (theorem 6.3), that Auslander-Reiten triangles and quivers over spheres can be computed (theorems 8.10 and 8.11), and that quivers are sufficiently sensitive invariants to tell spheres of different dimension apart (corollary 8.12).

After this very short survey, let me describe the paper at a more leisurely pace.

The idea to use methods from the representation theory of finite dimensional algebras in algebraic topology comes as follows:

If $k$ is a field and $X$ is a simply connected topological space with $\dim_k H^*(X; k) < \infty$, then the singular cochain differential graded algebra $C^*(X; k)$ is equivalent by a series of quasi-isomorphisms to a differential graded algebra $R$ which is finite dimensional over $k$, by the methods of [3, proof of thm. 3.6] and [11, exam. 6, p. 146].

Hence it seems obvious to try to study $R$ and thereby $C^*(X; k)$ with methods from the representation theory of finite dimensional algebras. A natural place to start is with the derived category of differential

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graded modules $D(R)$ which is the playing ground for homological algebra over $R$. Note that by [13 thm. III.4.2], the category $D(R)$ is equivalent to $D(C^*(X;k))$.

A number of concepts present themselves which are used to analyze the structure of derived categories in representation theory. I will concentrate on two important ones: Auslander-Reiten triangles and invariants in the form of quivers. Their definitions are recalled in 1.1 and 2.1 below, but let me make some remarks.

Auslander-Reiten triangles were introduced by Happel in [9], and are certain special triangles among the distinguished triangles in a triangulated category. They are the triangulated counterpart to Auslander-Reiten sequences which pervade representation theory, see for instance [2]. Not all triangulated categories have Auslander-Reiten triangles, but those that do enjoy many advantages. This is expounded in Happel’s book [10], but see also his papers [8] and [9].

The quiver of an additive category is an important structural invariant. The vertices are certain isomorphism classes in the category and the arrows are determined by certain morphisms. One can think of the quiver as an “X-ray image” of the category. Quivers of additive categories are used extensively in representation theory; an example is the so-called Auslander-Reiten quiver which is a tremendously useful tool, see [2].

Auslander-Reiten triangles and quivers are intimately connected: If a suitable triangulated category has Auslander-Reiten triangles, then they give enough information to compute the quiver of the category, see lemma 2.2 and they even give the quiver the extra structure of so-called stable translation quiver, see definition 2.3 and corollary 2.4.

Now, one can hope that the tools of Auslander-Reiten triangles and quivers will be as useful in studying the derived category $D(C^*(X;k))$ as they are in representation theory. This paper shows that at least something can be gained:

- Section 6 considers Auslander-Reiten triangles, and proves (essentially) that they exist in the category $D^c(C^*(X;k))$ if and only if the topological space $X$ has Poincaré duality over $k$ (theorem 6.3). Here $D^c(C^*(X;k))$ is the full subcategory of small objects of $D(C^*(X;k))$ (those where $\text{Hom}(M, -)$ commutes with set indexed coproducts).

- Section 7 considers the quiver of $D^c(C^*(X;k))$, and proves that it is a weak homotopy invariant of $X$ (proposition 7.1).

Section 8 applies the theory to spheres, and computes the Auslander-Reiten triangles and the quiver of $D^c(C^*(S^d;k))$ for $d \geq 2$ when the characteristic of $k$ is zero (theorems 8.10 and 8.11). The quiver consists of $d - 1$ components, each isomorphic to $\mathbb{Z}A_{\infty}$, and it is observed that
hence, the quiver of $D^c(C^*(S^d; k))$ is a sufficiently sensitive invariant to tell spheres of different dimension apart (corollary 8.12).

On the way to these results, the indecomposable objects of the category $D^c(C^*(S^d; k))$ are determined, and it is proved that each object is the coproduct of uniquely determined indecomposable objects (proposition 8.8). This gives a fairly accurate picture of $D^c(C^*(S^d; k))$ which may be of independent interest.

The initial sections 1 to 5 of the paper are organized as follows: Sections 1 and 2 briefly recall Auslander-Reiten triangles and quivers, and sections 3 to 5 develop the theory of Auslander-Reiten triangles over a general differential graded algebra $R$ which has the advantage of being typographically lighter than $C^*(X; k)$, and not mathematically harder.

Let me end the introduction by giving some notation.

Throughout the paper, $k$ denotes a field.

Differential Graded Algebras are abbreviated DGAs, and Differential Graded modules are abbreviated DG modules.

Standard notation is used for triangulated categories and for derived categories and functors of DG modules over DGAs. The suspension functor is denoted $\Sigma$ and the $i$'th cohomology functor is denoted $H^i$. The notation is cohomological (degrees indexed by superscripts, differentials of degree +1).

Module structures are occasionally emphasized by subscript notation. So for instance, $M_{R,S}$ indicates that $M$ has compatible right-structures over $R$ and $S$.

Let $S$ be a DGA over $k$.

$S^g$ denotes the graded algebra obtained by forgetting the differential of $S$, and if $M$ is a DG $S$-module then $M^g$ denotes the graded $S^g$-module obtained by forgetting the differential of $M$.

The opposite DGA of $S$ is denoted $S^\text{op}$, and is defined by having the product $s^\text{op} t = (-1)^{|s||t|} ts$. DG right-$S$-modules are identified with DG left-$S^\text{op}$-modules.

$D^c(S)$ denotes the full subcategory of the derived category $D(S)$ which consists of small $M$'s, that is, $M$'s so that $\text{Hom}(M, -)$ commutes with set indexed coproducts.

$D^f(S)$ denotes the full subcategory of $D(S)$ which consists of $M$'s with $\text{dim}_k H^i M < \infty$.

I write $D(-) = \text{Hom}_k(-, k)$.

This duality functor is defined on $k$-vector spaces. It can also be viewed as defined on modules, graded modules, or DG modules, and as such it interchanges left-modules and right-modules. The functor $D$ induces a
duality of categories

\[ \mathcal{D}^f(S) \xrightarrow{\mathcal{D}} \mathcal{D}^f(S^{\text{op}}). \]

Note that \( DS \) is a DG left/right-\( S \)-module, like \( S \) itself.

If \( \mathcal{D} \) is a triangulated category and \( M \) is an object of \( \mathcal{D} \), then an object of \( \mathcal{D} \) is said to be finitely built from \( M \) if it is in the smallest triangulated subcategory of \( \mathcal{D} \) which contains \( M \) and is closed under retracts.

1. Auslander-Reiten triangles

Sections 1 and 2 are introductory.

This section recalls the definition of Auslander-Reiten triangles and a few of their properties from [9].

Let \( \mathcal{D} \) be a \( k \)-linear triangulated category over the field \( k \), where each \( \text{Hom} \) space is finite dimensional over \( k \) and each indecomposable object has local endomorphism ring.

**Definition 1.1.** A distinguished triangle

\[ M \longrightarrow N \xrightarrow{\nu} P \xrightarrow{\pi} \]

in \( \mathcal{D} \) is called an Auslander-Reiten triangle in \( \mathcal{D} \) if

(i) \( M \) and \( P \) are indecomposable objects.

(ii) \( \pi \neq 0 \).

(iii) Each morphism \( N' \longrightarrow P \) which is not a retraction factors through \( \nu \).

Given an indecomposable object \( P \), there may or may not exist an Auslander-Reiten triangle as in the definition. But if there does, then it is determined up to isomorphism by [9, prop. 3.5(i)]. This allows the following definition.

**Definition 1.2.** Given an indecomposable object \( P \) of \( \mathcal{D} \). Suppose that there is an Auslander-Reiten triangle as in definition 1.1. Then \( M \) is called the Auslander-Reiten translate of \( P \) and denoted \( \tau P \). The operation \( \tau \) is called the Auslander-Reiten translation of \( \mathcal{D} \).

Note that \( \tau P \) is only defined up to isomorphism.

**Definition 1.3.** Suppose that for each indecomposable object \( P \) of \( \mathcal{D} \), there exists an Auslander-Reiten triangle as in definition 1.1. Then \( \mathcal{D} \) is said to have Auslander-Reiten triangles.
2. Quivers

This section recalls the definition of the quiver of an additive category and its connection with Auslander-Reiten triangles.

Let $D$ be an additive category.

A morphism $M \xrightarrow{\mu} N$ is called irreducible if it is neither a section or a retraction, but any factorization $\mu = \rho \sigma$ has either $\sigma$ a section or $\rho$ a retraction.

**Definition 2.1.** The quiver of $D$ has as vertices the isomorphism classes $[M]$ of indecomposable objects of $D$. It has one arrow from $[M]$ to $[N]$ when there is an irreducible morphism $M \to N$ and no arrows from $[M]$ to $[N]$ otherwise.

Now let $D$ be a $k$-linear triangulated category where each Hom space is finite dimensional over $k$ and each indecomposable object has local endomorphism ring.

In this case the quiver of $D$ and the Auslander-Reiten triangles in $D$ are intimately connected. The following result is immediate from [9, prop. 3.5].

**Lemma 2.2.** Let $M \to N \to P \to$ be an Auslander-Reiten triangle in $D$. Suppose that $N \cong \bigsqcup_j N_j$ is a splitting into indecomposable objects, and let $N'$ be some indecomposable object. Then the following statements are equivalent.

(i) There is an irreducible morphism $M \to N'$.

(ii) There is an irreducible morphism $N' \to P$.

(iii) There is a $j$ so that $N' \cong N_j$.

So if $D$ has Auslander-Reiten triangles, then knowledge of the Auslander-Reiten triangles gives full knowledge of the quiver of $D$.

Moreover, there is the notion of stable translation quiver.

**Definition 2.3.** A quiver is said to be a stable translation quiver if it is equipped with a map $\tau$ called the translation, which sends vertices to vertices in a way so that the number of arrows from $\tau[P]$ to $[N']$ equals the number of arrows from $[N']$ to $[P]$.

Lemma 2.2 implies that if $D$ has Auslander-Reiten triangles, then definition 2.3 is satisfied with $\tau[P] = [M]$. Note $[M] = [\tau P]$, where $\tau$ now stands for the Auslander-Reiten translation of $D$, see definition 1.2.

Hence the following result.

**Corollary 2.4.** If $D$ has Auslander-Reiten triangles, then the quiver of $D$ is a stable translation quiver with translation induced by the Auslander-Reiten translation of $D$ via $\tau[P] = [\tau P]$. 
3. Derived categories

Sections 3, 4, and 5 develop the theory of Auslander-Reiten triangles over a general DGA denoted $R$.

This section collects some lemmas on derived categories of DG modules.

Setup 3.1. In the rest of the paper, $R$ denotes a DGA over the field $k$ satisfying:

(i) $R$ is a cochain DGA, that is, $R^i = 0$ for $i < 0$.
(ii) $R^0 = k$.
(iii) $R^1 = 0$.
(iv) $\dim_k R < \infty$.

Note that $R/R \geq 1 \sim k$ is a DG left/right-$R$-module.

First a general result which holds by [11, thm. 5.3].

Lemma 3.2. Let $S$ be a DGA over $k$. Then the objects of $D^c(S)$ are exactly the ones which are finitely built from $S$.

The rest of this section deals with $R$, the DGA from setup 3.1. If $M$ is a DG left-$R$-module, then a semi-free resolution $F \rightarrow M$ is called minimal if the differential $\partial_F$ takes values in $R^{\geq 1}F$, whence $k \otimes_R F$ and $\text{Hom}_R(F, k)$ have vanishing differentials. (See [6, ch. 6] for general information on semi-free resolutions.) The following result is well known, see [5, appendix] or [7, appendix].

Lemma 3.3. Let $M$ be a DG left-$R$-module for which $u = \inf \{ i \mid H^i M \neq 0 \}$ is finite, and for which each $H^i M$ is finite dimensional over $k$.

(i) There is a minimal semi-free resolution $F \rightarrow M$ which has a semi-free filtration with quotients as indicated,

\[ \Sigma^{-u} R^{(\gamma_0)} \rightarrow \Sigma^{-u} R^{(\gamma_1)} \rightarrow \Sigma^{-u-1} R^{(\gamma_2)} \rightarrow \cdots \]

\[ 0 \subseteq F(0) \subseteq L(1) \subseteq F(1) \subseteq L(2) \subseteq F(2) \subseteq \cdots \subseteq F, \]

where superscripts $(\gamma_j)$ and $(\delta_j)$ indicate coproducts. Here each $\gamma_j$ and each $\delta_j$ is finite, and the induced map

\[ H^i F(m) \rightarrow H^i F \]

is an isomorphism for $i \leq u + m$. Moreover, I have $\gamma_0 \neq 0$.

(ii) In the construction from (i), I have

\[ F^2 \cong \prod_{j \leq -u} \Sigma^j (R^j)^{(\beta_j)}, \]
where each $\beta_j$ is finite.

**Remark 3.4.** It follows from lemma 3.2 that each $F(m)$ and each $L(m)$ in lemma 3.3(i) is in $\mathcal{D}(R)$, because each step in the semi-free filtration only adds finitely many $\Sigma^j R$’s.

The following truncation lemma uses that $R^0$ is $k$, and is an exercise in linear algebra.

**Lemma 3.5.**

(i) Let $M$ be a DG left-$R$-module for which $u = \inf\{i \mid H^i M \neq 0\}$ is finite. Then there exists an injective quasi-isomorphism of DG left-$R$-modules $U \to M$ with $U^j = 0$ for $j < u$.

(ii) Let $N$ be a DG left-$R$-module for which $v = \sup\{i \mid H^i N \neq 0\}$ is finite. Then there exists an surjective quasi-isomorphism of DG left-$R$-modules $N \to V$ with $V^j = 0$ for $j > v$.

The following two lemmas show that $\mathcal{D}^f(R)$ and $\mathcal{D}^c(R)$ are categories of the sort for which Auslander-Reiten triangles were defined in 1.1.

**Lemma 3.6.**

(i) Let $M$ and $N$ in $\mathcal{D}^f(R)$ be given. Then I have $\dim_k \text{Hom}_{\mathcal{D}(R)}(M, N) < \infty$.

(ii) If $M$ is an indecomposable object of $\mathcal{D}^f(R)$, then the endomorphism ring $\text{Hom}_{\mathcal{D}(R)}(M, M)$ is a local ring.

**Proof.** (i): If $N$ is isomorphic to zero in $\mathcal{D}^f(R)$, then part (i) of the lemma is trivial, so I can suppose that $N$ is not isomorphic to zero.

Let $F \to M$ and $G \to N$ be semi-free resolutions chosen according to lemma 3.3(i). Since I have $\dim_k R < \infty$, lemma 3.3(ii) implies $\dim_k F^j < \infty$ and $\dim_k G^j < \infty$ for each $j$.

As $N$ is in $\mathcal{D}^f(R)$ and is non-isomorphic to zero, the same holds for $G$, so $u = \inf\{i \mid H^i G \neq 0\}$ and $v = \sup\{i \mid H^i G \neq 0\}$ are finite. By using both parts of lemma 3.5 I can replace $G$ with a truncation $G'$ so that $G'$ is concentrated between degrees $u$ and $v$, and so that $G$ and $G'$ are connected by two quasi-isomorphisms. As $G'$ is a truncation of $G$, I have $\dim_k G'^j < \infty$ for each $j$, so altogether $\dim_k G' < \infty$ holds.

But $\dim_k F^j < \infty$ for each $j$ and $\dim_k G' < \infty$ imply

$$\dim_k \text{Hom}_R(F, G')^j < \infty$$

for each $j$, and so

$$\text{Hom}_{\mathcal{D}(R)}(M, N) \cong H^0(R\text{Hom}_R(M, N)) \cong H^0(\text{Hom}_R(F, G'))$$

also has $\dim_k \text{Hom}_{\mathcal{D}(R)}(M, N) < \infty$.

(ii): By part (i) and [10, 3.2], it is enough to see that idempotent morphisms in $\mathcal{D}^f(R)$ split. But by [3, prop. 3.2] they even do so in $\mathcal{D}(R)$ because $\mathcal{D}(R)$ is a triangulated category with set indexed coproducts. \qed
Lemma 3.7. There is the inclusion $\mathbb{D}^c(R) \subseteq \mathbb{D}^f(R)$.

Proof. This is clear by lemma 3.2 because $R$ is in $\mathbb{D}^f(R)$. □

Finally, some technicalities.

Lemma 3.8. Let $F$ and $N$ be DG left-$R$-modules with

$$F^\bullet \cong \prod_{j \leq -u} \Sigma^j (R^\bullet)^{(\beta_j)}$$

where each $\beta_j$ is finite, and with $N^j = 0$ for $j > v$. Then

$$\sup\{ i \mid H^i(\text{Hom}_R(F, N)) \neq 0 \} \leq -u + v.$$  

Proof. This follows since

$$\text{Hom}_R(F, N)^\bullet \cong \text{Hom}_R(F^\bullet, N^\bullet) \cong \prod_{j \leq -u} \Sigma^{-j}(N^\bullet)^{(\beta_j)}$$

is zero in degrees $> -u + v$, because the highest degree contribution to the product comes from $\Sigma^u(N^\bullet)^{(\beta_{-u})}$ which is certainly zero in degrees $> -u + v$. □

Lemma 3.9. Let $M$ and $N$ be in $\mathbb{D}^f(R)$. Then

$$\sup\{ i \mid H^i(\text{RHom}_R(M, N)) \neq 0 \}$$

$$= -\inf\{ i \mid H^iM \neq 0 \} + \sup\{ i \mid H^iN \neq 0 \}.$$  

Proof. If $M$ or $N$ is isomorphic to zero in $\mathbb{D}^f(R)$, then the equation just says $-\infty = -\infty$, so I can suppose that neither $M$ or $N$ is isomorphic to zero. Then $u = \inf\{ i \mid H^iM \neq 0 \}$ and $v = \sup\{ i \mid H^iN \neq 0 \}$ are finite.

By lemma 3.3(i), pick a semi-free resolution $F \twoheadrightarrow M$ with

$$F^\bullet \cong \prod_{j \leq -u} \Sigma^j (R^\bullet)^{(\beta_j)}.$$  

By lemma 3.5(ii), replace $N$ with a quasi-isomorphic truncation with $N^j = 0$ for $j > v$.

Since $\text{RHom}_R(M, N) \cong \text{Hom}_R(F, N)$ holds, what I must prove is

$$\sup\{ i \mid H^i(\text{Hom}_R(F, N)) \neq 0 \} = -u + v.$$  

Here $\leq$ follows from lemma 3.8, so it remains to show

$$H^{-u+v}(\text{Hom}_R(F, N)) \neq 0. \quad (1)$$

For this, note that the semi-free filtration of $F$ in lemma 3.3(i) gives that there is a semi-split exact sequence of DG left-$R$-modules,

$$0 \to \Sigma^{-u}R^{(\gamma_0)} \to F \to F' \to 0, \quad (2)$$
with $\gamma_0 \neq 0$. Here the left hand term is just $F(0)$, and $F'$ is the quotient $F/F(0)$. From the part of the semi-free filtration which continues up from $F(0)$ follows that $F'$ is graded free with
\[
(F')^\sharp \cong \prod_{j \leq -u} \Sigma^j (R^\natural)^{(\beta_j)}.
\] (3)

Since the sequence (2) is semi-split, applying $\text{Hom}_R(-, N)$ gives a short exact sequence of complexes,
\[
0 \to \text{Hom}_R(F', N) \to \text{Hom}_R(F, N) \to \Sigma^u N(\gamma_0) \to 0.
\]
The long exact cohomology sequence of this contains
\[
H^{−u+v}(\text{Hom}_R(F, N)) \to H^{−u+v}(\Sigma^u N(\gamma_0)) \to H^{−u+v+1}(\text{Hom}_R(F', N)).
\]
The middle term is $H^v(N(\gamma_0))$ which is non-zero. The last term is zero because lemma 3.8 and equation (3) imply
\[
\sup\{ i \mid H^i(\text{Hom}_R(F', N)) \neq 0 \} \leq −u + v.
\]
But then the first term is non-zero, proving equation (1).

**Lemma 3.10.** Let $N'$ in $D^f(R)$ and $v$ in $\mathbb{Z}$ be given. Then there exists a distinguished triangle in $D(R)$,
\[
F \to N' \to Q \to,
\]
so that $F$ is in $D^c(R)$ and so that $Q$ is in $D^f(R)$ with $\inf\{ i \mid H^i Q \neq 0 \} \geq v$.

**Proof.** Let me use lemma 3.3(i) to pick a semi-free resolution $G \to N'$. The semi-free filtration in lemma 3.3(i) gives $G(m) \to G$ with $H^i G(m) \to H^i G$ an isomorphism for $i \leq u + m$, and with $G(m)$ in $D^c(R)$ by remark 3.4. By picking $m$ large enough, I can arrange that $H^i G(m) \to H^i G$ is an isomorphism for $i \leq v$.

But then the composition $G(m) \to G \to N'$ also has $H^i G(m) \to H^i N'$ an isomorphism for $i \leq v$, and completing to a distinguished triangle
\[
G(m) \to N' \to Q \to,
\] (4)
the long exact cohomology sequence proves $\inf\{ i \mid H^i Q \neq 0 \} \geq v$.

So (4) can be used as the lemma’s $F \to N' \to Q \to$. □

### 4. Auslander-Reiten triangles over a DGA

Recall $R$, the DGA from setup 3.1. This section gives a criterion for the existence of Auslander-Reiten triangles in $D^f(R)$ (proposition 4.3), and a formula for Auslander-Reiten triangles when they exist (proposition 4.4).
Note that by lemmas 3.6 and 3.7 both $D^f(R)$ and $D^c(R)$ are categories of the sort for which Auslander-Reiten triangles were defined in 1.1, so the concept makes sense for them.

**Lemma 4.1.** Let $P$ be an indecomposable object of $D^c(R)$. Then there is an Auslander-Reiten triangle in $D^f(R)$,

$$\Sigma^{-1}(DR \otimes_R P) \to N \to P \to .$$

**Proof.** This is a consequence of the theory developed in [12]:

The natural equivalence

$$D(\text{Hom}_{D(R)}(P, -)) \simeq \text{Hom}_{D(R)}(-, DR \otimes_R P)$$

holds for $P$ equal to $R_R$, and therefore also holds for the given $P$ because $P$ is in $D^c(R)$ and therefore finitely built from $R_R$ by lemma 3.2.

Let

$$\Gamma = \text{Hom}_{D(R)}(P, P)$$

be the endomorphism ring of $P$, and rewrite the left hand side of (5) to

$$\text{Hom}_{\Gamma^\text{op}}(\text{Hom}_{D(R)}(P, -), D\Gamma).$$

This gives the natural equivalence

$$\text{Hom}_{\Gamma^\text{op}}(\text{Hom}_{D(R)}(P, -), D\Gamma) \simeq \text{Hom}_{D(R)}(-, DR \otimes_R P).$$

(6)

Now, since $P$ is an indecomposable object of $D^c(R)$ and hence of $D^f(R)$, the endomorphism ring $\Gamma$ is finite dimensional over $k$ and local by lemma 3.6. The unique simple left-$\Gamma$-module $rS = r(\Gamma/J(\Gamma))$ is also finite dimensional over $k$ and has only trivial submodules. Hence the dual module $(DS)_R$ has only trivial quotient modules, so must be the unique simple right-$\Gamma$-module, $(\Gamma/J(\Gamma))_R$. Moreover, the projective cover $r\Gamma \to rS$ dualizes to an injective envelope $(DS)_R \to (D\Gamma)_R$.

So $(D\Gamma)_R$ is the injective envelope of the unique simple right-$\Gamma$-module $(\Gamma/J(\Gamma))_R$. Therefore, by [12] def. 2.1, thm. 2.2, and lem. 2.3], the equivalence (6) implies that there is a distinguished triangle in $D(R)$,

$$\Sigma^{-1}(DR \otimes_R P) \to N \to P \to .$$

(7)

satisfying, among other things,

(i) $\Sigma^{-1}(DR \otimes_R P)$ is an indecomposable object of $D(R)$ (as is $P$ by assumption).

(ii) $\pi \neq 0$.

(iii) Each morphism $N' \to P$ in $D(R)$ which is not a retraction factors through $\nu$. 

(In [12], the triangle (7) is called an Auslander-Reiten triangle, but his definition of this concept differs from mine.)

Moreover, (7) is in $\text{D}^f(R)$: As $P$ is finitely built from $R$, it follows that $DR \otimes_R P$ is finitely built from $R(\text{D}R)$. But then $DR \otimes_R P$ is in $\text{D}^f(R)$ because $R(\text{D}R)$ is in $\text{D}^f(R)$. And $P$ is also in $\text{D}^f(R)$ by lemma 3.7. So both end terms in (7) are in $\text{D}^f(R)$, and the long exact cohomology sequence then proves the same for the middle term.

Together, these properties of the distinguished triangle (7) imply that it is an Auslander-Reiten triangle in $\text{D}^f(R)$; cf. definition 1.1. □

Lemma 4.2. If

$$M \to N \xrightarrow{\nu} P \xrightarrow{\pi}$$

is an Auslander-Reiten triangle in $\text{D}^c(R)$, then it is also one in $\text{D}^f(R)$.

Proof. When viewed in $\text{D}^c(R)$, the Auslander-Reiten triangle (8) is characterized by satisfying conditions (i) to (iii) of definition 1.1. Clearly, when viewed in $\text{D}^f(R)$, the triangle again satisfies conditions (i) and (ii). It remains to check condition (iii).

So suppose that

$$N' \xrightarrow{\nu'} P$$

is a non-retraction in $\text{D}^f(R)$. I must show that $\nu'$ factors through $\nu$, which is equivalent to

$$\pi \nu' = 0.$$  \hspace{1cm} (9)

To prove this, let me first write $v = \sup \{ i \mid H^iM \neq 0 \}$. This is finite because $M$ is indecomposable in $\text{D}^c(R)$, hence not isomorphic to zero. By lemma 3.10 there is a distinguished triangle in $\text{D}(R)$,

$$F \xrightarrow{\varphi} N' \to Q \rightarrow,$$

with $F$ in $\text{D}^c(R)$ and $Q$ in $\text{D}^f(R)$ with

$$\inf \{ i \mid H^iQ \neq 0 \} \geq v.$$  \hspace{1cm} (11)

Here I claim

$$\pi \nu' \varphi = 0,$$  \hspace{1cm} (12)

which is a first approximation to equation (9). To see this, note that as $F$ is in $\text{D}^c(R)$ and as (8) is an Auslander-Reiten triangle in $\text{D}^c(R)$, it is enough to see that $\nu' \varphi$ is not a retraction. But it is not for if there were a section $P \xrightarrow{\sigma} F$ with $(\nu' \varphi) \sigma = 1_P$, then $\nu' (\varphi \sigma) = 1_P$ would mean that $\nu'$ had the section $\varphi \sigma$, but $\nu'$ is not a retraction.
Next, note \(\sup\{ i \mid H^i(\Sigma M) \neq 0 \} = \sup\{ i \mid H^i M \neq 0 \} - 1 = v - 1\). Using this and equation (11) proves \(\leq\) in
\[
\sup\{ i \mid H^i(R\text{Hom}_R(Q, \Sigma M)) \neq 0 \} \equiv -\inf\{ i \mid H^iQ \neq 0 \} + \sup\{ i \mid H^i(\Sigma M) \neq 0 \} \\
\leq -v + v - 1 \\
= -1,
\]
where (a) is by lemma 3.9. Hence the = in
\[
\text{Hom}_{D(R)}(Q, \Sigma M) \cong H^0(R\text{Hom}_R(Q, \Sigma M)) = 0. \quad (13)
\]
However, the distinguished triangle (10) gives a long exact sequence containing
\[
\text{Hom}_{D(R)}(Q, \Sigma M) \longrightarrow \text{Hom}_{D(R)}(N', \Sigma M) \longrightarrow \text{Hom}_{D(R)}(F, \Sigma M),
\]
where \(\pi'\) is an element in the middle term. The right hand map sends \(\pi'\) to \(\pi'\phi\) which is zero by equation (12). So \(\pi'\phi\) is in the image of the left hand map, and this image is zero by equation (13). This proves equation (9).

**Proposition 4.3.** The category \(D^c(R)\) has Auslander-Reiten triangles if and only if \(R(DR)\) is in \(D^c(R)\).

**Proof.** On one hand, suppose that \(R(DR)\) is in \(D^c(R)\). Let \(P\) be an indecomposable object of \(D^c(R)\). Then lemma 4.1 gives an Auslander-Reiten triangle in \(D^f(R)\). In the present situation, I claim that the triangle is in fact in \(D^c(R)\), from which follows readily that it is an Auslander-Reiten triangle in \(D^c(R)\); cf. definition 1.1.

To see this, note that as \(P\) is in \(D^c(R)\), it is finitely built from \(DR\) by lemma 3.2 whence \(DR \otimes_R P\) is finitely built from \(R(DR)\). But since \(R(DR)\) is in \(D^c(R)\), it is also finitely built from \(R\). All in all, \(DR \otimes_R P\) is finitely built from \(R\), so is in \(D^c(R)\). But as both \(DR \otimes_R P\) and \(P\) are in \(D^c(R)\), so is the middle term in the distinguished triangle from lemma 4.1 so the triangle is in \(D^c(R)\).

On the other hand, suppose that \(D^c(R)\) has Auslander-Reiten triangles. Let \(R \cong \coprod_j R_j\) be a splitting into indecomposable objects of \(D^c(R)\); such a splitting clearly exists since \(\dim_k HR < \infty\). Now, for each \(j\) there is an Auslander-Reiten triangle in \(D^c(R)\),
\[
M_j \longrightarrow N_j \longrightarrow R_j \longrightarrow,
\]
and by lemma 4.2 this is even an Auslander-Reiten triangle in \(D^f(R)\).

Also for each \(j\) there is an Auslander-Reiten triangle in \(D^f(R)\),
\[
DR \otimes_R R_j \longrightarrow N'_j \longrightarrow R_j \longrightarrow,
\]
by lemma 4.1.
However, the two Auslander-Reiten triangles have the same right
hand end term, \( R_j \), so by [9, prop. 3.5(i)] they are isomorphic. In
particular, the left hand end terms are isomorphic, so
\[
M_j \cong DR_L \otimes_R R_j.
\]
Hence
\[
\prod_j M_j \cong \prod_j DR_L \otimes_R R_j \cong DR_L \otimes_R \prod_j R_j \cong DR_L \otimes_R R \cong_R (DR),
\]
and here the left hand side is in \( D^c(R) \) so \( R(DR) \) is also in \( D^c(R) \). □

The following result complements lemma 4.1.

Proposition 4.4. Suppose that \( D^c(R) \) has Auslander-Reiten triangles.
(i) Let \( P \) be an indecomposable object of \( D^c(R) \). Then there is an
Auslander-Reiten triangle in \( D^c(R) \),
\[
\Sigma^{-1}(DR_L \otimes_R P) \rightarrow N \rightarrow P \rightarrow .
\]
(ii) The Auslander-Reiten translation of \( D^c(R) \) is given by
\[
\tau(-) = \Sigma^{-1}(DR_L \otimes_R -).
\]
Proof. (i): The distinguished triangle here is the one from lemma 4.1,
so is an Auslander-Reiten triangle in \( D^f(R) \). The first part of the proof
of proposition 4.3 shows that it is also an Auslander-Reiten triangle in
\( D^c(R) \) provided \( R(DR) \) is in \( D^c(R) \). And this holds by proposition 4.3
because \( D^c(R) \) has Auslander-Reiten triangles.
(ii): This is immediate from part (i); cf. definition 1.2 □

5. Poincaré duality DGAs

Recall \( R \), the DGA from setup 3.1. This section considers the sit-
uation where \( R(DR) \) is in \( D^c(R) \) and \( (DR)_R \) is in \( D^c(R^{op}) \), cf. proposition 4.3. Theorem 5.1 shows that this is equivalent to \( HR \) having
Poincaré duality.

Note that by the proof of theorem 5.1 it is also equivalent to \( R \) being
a so-called Gorenstein DGA; cf. [5].

Theorem 5.1. With \( d = \sup\{ i \mid H^i R \neq 0 \} \), the following conditions
are equivalent.
(i) \( R(DR) \) is in \( D^c(R) \) and \( (DR)_R \) is in \( D^c(R^{op}) \).
(ii) There are isomorphisms of graded \( HR \)-modules
\[
HR(DHR) \cong \Sigma^d HR \quad \text{and} \quad (DHR)_{HR} \cong (\Sigma^d HR)_{HR}.
\]
Proof. To facilitate the proof, here are three more conditions each of
which is equivalent to the ones in the theorem.
(iii) \( \dim_k \text{Ext}_{R(R)}(k, R) < \infty \) and \( \dim_k \text{Ext}_{R^{op}}(k, R) < \infty \).
(iv) There are isomorphisms of graded $k$-vector spaces $\text{Ext}_R(k, R) \cong \Sigma^{-d} k$ and $\text{Ext}_{R^\text{op}}(k, R) \cong \Sigma^{-d} k$.

(v) There are isomorphisms $R(DR) \cong R(\Sigma^d R)$ in $\mathcal{D}(R)$ and $(DR)_R \cong (\Sigma^d R)_R$ in $\mathcal{D}(R^\text{op})$.

(i) $\Rightarrow$ (iii): Duality gives

$$\text{Ext}_{R^\text{op}}(k, R) \cong \text{Ext}_R(DR, k) \cong \text{Ext}_R(DR, k) = (*) \quad (14)$$

When (i) holds, lemma 3.2 implies that $R(DR)$ is finitely built from $R^R$, and then $\text{Ext}_{R^\text{op}}(k, R)$ is finite dimensional over $k$ since $\text{Ext}_R(R, k) \cong k$ is finite dimensional over $k$. Equation (14) then shows that $\text{Ext}_{R^\text{op}}(k, R)$ is finite dimensional over $k$. This gives half of (iii), and the other half follows by symmetry.

(iii) $\Rightarrow$ (i): Let $F \rightarrow R(DR)$ be a minimal semi-free resolution picked according to lemma 3.3(i). Continuing the computation from equation (14) gives

$$(*) = H(R\text{Hom}_R(DR, k)) \cong H(\text{Hom}_R(F, k)) \cong \text{Hom}_{R^R}(F^2, k^2) \quad (15)$$

where the last $\cong$ is by minimality of $F$. When (iii) holds, $\text{Ext}_{R^\text{op}}(k, R)$ is finite dimensional over $k$, and equations (14) and (15) then show that $\text{Hom}_{R^R}(F^2, k^2)$ is finite dimensional over $k$. This means that there are only finitely many summands $\Sigma^j R^2$ in $F^2$, so the semi-free filtration of $F$ in lemma 3.3(i) must terminate after finitely many steps. So $F$ and therefore $R(DR)$ is finitely built from $R^R$, whence $R(DR)$ is in $\mathcal{D}^c(R)$. This gives half of (i), and the other half follows by symmetry.

(iii) $\Rightarrow$ (iv): Assume (iii). The proof that (iii) implies (i) considered a minimal semi-free resolution $F \rightarrow R(DR)$ obtained from lemma 3.3(i), and proved that the semi-free filtration of $F$ in 3.3(i) terminates after finitely many steps. But then $F$ must be bounded because $\dim_k R < \infty$ implies that $R$ itself is bounded. Now, the dual of $F \rightarrow R(DR)$ is

$$R^R \cong D(R(DR)) \rightarrow DF,$$

and this is a $K$-injective resolution of $R^R$ where $DF$ is bounded because $F$ is.

Also, lemma 3.3 gives that $Rk$ has a semi-free resolution $G \rightarrow Rk$ with

$$G^2 \cong \prod_{j \leq 0} \Sigma^j (R^2)^{(\beta_j)} \quad (16)$$

and each $\beta_j$ finite.

The existence of these resolutions implies that the canonical morphism

$$Rk \rightarrow R\text{Hom}_{R^\text{op}}(R\text{Hom}_R(k, R), R) \quad (17)$$
is an isomorphism by \([1, \text{sec. } 1, \text{thm. } 1]\). Hence
\[
0 = \sup \{ i \mid H^i(Rk) \neq 0 \}
= \sup \{ i \mid H^i(\text{RHom}_{R^{op}}(\text{RHom}_R(k, R), R)) \neq 0 \}
\]
\[
\overset{(a)}{=} -\inf \{ i \mid H^i(\text{RHom}_R(k, R)) \neq 0 \} + \sup \{ i \mid H^iR \neq 0 \}
= -\inf \{ i \mid H^i(\text{RHom}_R(k, R)) \neq 0 \} + d,
\]
where (a) follows from lemma \([3.9]\). The lemma can be used because (iii) implies that \(\text{RHom}_R(k, R)\) is in \(D^f(R^{op})\), while \(R_R\) is certainly in \(D^f(R^{op})\). This shows
\[
\inf \{ i \mid H^i(\text{RHom}_R(k, R)) \neq 0 \} = d.
\]
On the other hand,
\[
\sup \{ i \mid H^i(\text{RHom}_R(k, R)) \neq 0 \}
\overset{(b)}{=} -\inf \{ i \mid H^iR \neq 0 \} + \sup \{ i \mid H^iR \neq 0 \}
= d,
\]
where (b) is again by lemma \([3.9]\).

The last two equations show that \(H(\text{RHom}_R(k, R))\) is concentrated in degree \(d\). Lemma \([5.5]\) now implies that \(\text{RHom}_R(k, R)\) itself is isomorphic in \(D(R^{op})\) to a DG right-\(R\)-module concentrated in degree \(d\). This DG right-\(R\)-module must have the form \(\Sigma^{-d}k\), so I get
\[
\text{RHom}_R(k, R) \cong \Sigma^{-d}k\,
\]
Inserting this into equation (17) proves \(\alpha = 1\), so all in all
\[
\text{RHom}_R(k, R) \cong \Sigma^{-d}k\,
\]
holds. Taking cohomology gives half of (iv). The other half follows by symmetry.

(iv) \(\Rightarrow\) (iii): This is clear.

So now, the equivalence of (i), (iii), and (iv) is established. I close the proof by establishing the equivalence of (ii), (iv), and (v).

(ii) \(\Rightarrow\) (iv): This is immediate from the Eilenberg-Moore spectral sequence
\[
E_2^{pq} = \text{Ext}^p_{HR}(k, HR)^q \Rightarrow \text{Ext}^{p+q}_R(k, R)
\]
as found in \([5, 1.3(2)]\), and the corresponding spectral sequence over \(R^{op}\).

(iv) \(\Rightarrow\) (v): Equation (14) gives that (iv) implies
\[
\text{Ext}_R(DR, k) \cong \Sigma^{-d}k.
\]
Using a minimal semi-free resolution of \(R(DR)\), it is easy to see that this implies half of (v), and the other half follows by symmetry.

(v) \(\Rightarrow\) (ii): This follows by taking cohomology. \(\square\)
Theorem 5.1 and proposition 4.3 combine to give:

**Corollary 5.2.** With $d = \text{sup} \{ i \mid H^i R \neq 0 \}$, the following conditions are equivalent.

(i) Both $D^c(R)$ and $D^c(R^{\text{op}})$ have Auslander-Reiten triangles.

(ii) There are isomorphisms of graded $HR$-modules $\text{HR}(DHR) \cong \text{HR}(\Sigma^d HR)$ and $(DHR)_{HR} \cong (\Sigma^d HR)_{HR}$.

6. **Auslander-Reiten triangles over a topological space**

Sections 6, 7, and 8 form the topological part of this paper. They develop the theory of Auslander-Reiten triangles and quivers over topological spaces, and apply the theory to spheres.

This section proves that existence of Auslander-Reiten triangles characterizes Poincaré duality spaces (theorem 6.3), and gives a formula for Auslander-Reiten triangles when they exist (proposition 6.4). Theorem 6.3 is the first main result of this paper.

**Setup 6.1.** In sections 6, 7, and 8, singular cohomology and singular cochain DGAs are only considered with coefficients in the field $k$. So when $X$ is a topological space, $\text{H}^*(X; k)$ and $C^*(X; k)$ are abbreviated to $\text{H}^*(X)$ and $C^*(X)$. Moreover, $D(C^*(X; k))$ is abbreviated to $D(X)$, and this is combined freely with other adornments. So for instance, $D^c(X^{\text{op}})$ stands for $D^c(C^*(X; k)^{\text{op}})$.

**Remark 6.2.** Recall $R$, the DGA from setup 3.1. If $S$ is a DGA which is equivalent by a series of quasi-isomorphisms to $R$, then by [13, thm. III.4.2] the derived categories $D(S)$ and $D(R)$ are equivalent. Hence the results of sections 3, 4, and 5 on derived categories apply to $S$.

In particular, if $X$ is a simply connected topological space with $\text{dim}_k \text{H}^*(X) < \infty$, then $C^*(X)$ is equivalent by a series of quasi-isomorphisms to a DGA satisfying the conditions of setup 3.1 by the methods of [5, proof of thm. 3.6] and [6, exam. 6, p. 146]. This DGA can be used as $R$ in setup 3.1, so the results of sections 3, 4, and 5 on derived categories apply to $C^*(X)$.

By this remark and lemmas 3.3 and 3.7, if $X$ is a simply connected topological space with $\text{dim}_k \text{H}^*(X) < \infty$, then $D^c(X)$ and $D^c(X^{\text{op}})$ are categories of the sort for which Auslander-Reiten triangles were defined in 1.1, so the concept makes sense for them.

**Theorem 6.3.** Let $X$ be a simply connected topological space with $\text{dim}_k \text{H}^*(X) < \infty$. Then the following conditions are equivalent.

(i) $X$ has Poincaré duality over $k$.

(ii) Both $D^c(X)$ and $D^c(X^{\text{op}})$ have Auslander-Reiten triangles.
Proof. Remark 6.2 gives that corollary 5.2 applies to \( C^* \) \((X)\), the singular cochain DGA of \( X \) with coefficients in \( k \). For this particular DGA, condition (ii) of corollary 5.2 simply says that \( X \) has Poincaré duality over \( k \). So the present theorem follows. \( \square \)

**Proposition 6.4.** Let \( X \) be a simply connected topological space with \( \dim_k H^*(X) < \infty \) which has Poincaré duality over \( k \), and write \( d = \sup \{ i \mid H^i(X) \neq 0 \} \).

(i) Let \( P \) be an indecomposable object of \( D^c(X) \). Then there is an Auslander-Reiten triangle in \( D^c(X) \),

\[
\Sigma^{d-1} P \longrightarrow N \longrightarrow P \longrightarrow .
\]

(ii) The Auslander-Reiten translation of \( D^c(X) \) is given by

\[
\tau(-) = \Sigma^{d-1}(-).
\]

Proof. (i): Theorem 6.3 gives that \( D^c(X) \) has Auslander-Reiten triangles. Remark 6.2 gives that proposition 4.4(i) applies to \( C^*(X) \). Hence there is an Auslander-Reiten triangle in \( D^c(X) \),

\[
\Sigma^{-1}(DC^*(X) \otimes_{c^*(X)} P) \longrightarrow N \longrightarrow P \longrightarrow .
\]

But it is easy to see from Poincaré duality for \( X \) over \( k \) that \( DC^*(X) \) is isomorphic to \( \Sigma^d C^*(X) \) in the derived category of DG left/right-\( C^*(X) \)-modules. So in fact, the Auslander-Reiten triangle is the one given in the proposition.

(ii): This is immediate from part (i); cf. definition 1.2. \( \square \)

7. The Quiver over a Topological Space

Recall the conventions from setup 6.1. When \( X \) is a topological space, I can consider the quiver of \( D^c(X) \). Moreover, when \( X \) is simply connected with \( \dim_k H^*(X) < \infty \) and with Poincaré duality over \( k \), then \( D^c(X) \) has Auslander-Reiten triangles by theorem 6.3 so the quiver of \( D^c(X) \) is a stable translation quiver by corollary 2.4.

**Proposition 7.1.** The quiver of \( D^c(X) \) is a weak homotopy invariant of \( X \).

Moreover, if \( X \) is restricted to simply connected topological spaces with \( \dim_k H^*(X) < \infty \) which have Poincaré duality over \( k \), then the quiver of \( D^c(X) \), viewed as a stable translation quiver, is a weak homotopy invariant of \( X \).

Proof. If \( X \) and \( X' \) have the same weak homotopy type, then \( C^*(X) \) and \( C^*(X') \) are equivalent by a series of quasi-isomorphisms as follows from [6, thm. 4.15 and its proof]. Hence \( D(X) \) and \( D(X') \) are equivalent categories by [13, thm. III.4.2], and so the same holds for \( D^c(X) \) and \( D^c(X') \). This implies both parts of the proposition. \( \square \)
8. Spheres

Recall the conventions from setup 6.1. The \(d\)-dimensional sphere \(S^d\) has Poincaré duality over any field, so for \(d \geq 2\) the category \(\mathcal{D}^c(S^d)\) has Auslander-Reiten triangles by theorem 6.3.

This section determines the Auslander-Reiten triangles in \(\mathcal{D}^c(S^d)\) for \(d \geq 2\) when \(k\) has characteristic zero (theorem 8.10). As a consequence follows the determination of the quiver of \(\mathcal{D}^c(S^d)\) (theorem 8.11), and it is observed that the quiver is a sufficiently sensitive invariant to tell spheres of different dimension apart (corollary 8.12). These are the paper’s second main results.

To determine the Auslander-Reiten triangles, I must first determine the possible end terms, that is, the indecomposable objects of \(\mathcal{D}^c(S^d)\). This requires some preparations which take up most of this section. The method is to set up in lemma 8.4 an equivalence of categories between \(\mathcal{D}^c(S^d)\) and another category whose indecomposable objects turn out to be tractable by lemma 8.6. Transporting these objects through the equivalence then gives the indecomposable objects in \(\mathcal{D}^c(S^d)\) in proposition 8.8.

Setup 8.1. In this section, \(d \geq 2\) is always assumed.

Let \(A\) be the graded algebra \(k[T]\) with \(\deg T = -d + 1\), and view \(A\) as a DGA over \(k\) with vanishing differential.

Now \(A/A^{\leq -1} \cong k\) can be viewed as a DG right-\(A\)-module, \(k_A\). Let \(F \to k_A\) be a \(K\)-projective resolution.

Let \(\mathcal{E} = \text{Hom}_{A^{op}}(F, F)\) be the endomorphism DGA of \(F\).

The point of this setup is that there is a nice connection between the derived categories of \(\mathcal{E}\) and \(A\). Here \(\mathcal{E}\) is interesting because it turns out to be equivalent by a series of quasi-isomorphisms to \(C^*(S^d)\) when the characteristic of \(k\) is zero. The algebra \(A\) is not so interesting in itself, but is needed because it is more computationally tractable than \(\mathcal{E}\) and \(C^*(S^d)\).

The connection between the derived categories of \(\mathcal{E}\) and \(A\) can be obtained with the methods of [4] which work because \(k_A\) is a small object of \(\mathcal{D}(A^{op})\), as one easily checks (see also setup 8.2). It takes the following form: \(F\) acquires the structure \(F_{A,\mathcal{E}}\) in a canonical way, and there are quasi-inverse equivalences of categories,

\[
\begin{array}{c}
\mathcal{T} \\
\xrightarrow{F \otimes -} \\
\xleftarrow{\text{LHom}_{A^{op}}(F,-)} \\
\mathcal{D}(\mathcal{E}),
\end{array}
\]

where \(\mathcal{T}\) is a certain full triangulated subcategory of \(\mathcal{D}(A^{op})\) which contains \(k_A\).

Since \(k_A\) is in \(\mathcal{T}\), so is every object finitely built from \(k_A\). It is easy to check that such objects are exactly the ones in \(\mathcal{D}(A^{op})\). Moreover,
under the above equivalences, the object $k_A$ in $\mathcal{T}$ corresponds to the object

$$\mathrm{RHom}_{A^\text{op}}(F, k_A) \cong \mathrm{RHom}_{A^\text{op}}(F, F) \cong \mathcal{E}$$

in $\mathcal{D}(\mathcal{E})$, so objects finitely built from $k_A$ correspond to objects finitely built from $\mathcal{E}$. By lemma 3.2 these are exactly the objects of $\mathcal{D}^c(\mathcal{E})$.

So the above equivalences restrict to quasi-inverse equivalences

$$\mathcal{D}^c(A^\text{op}) \xrightarrow{\mathrm{RHom}_{A^\text{op}}(F, -)} \mathcal{D}^c(\mathcal{E}). \quad (18)$$

To go on, it is convenient to make a specific choice of $F$.

**Setup 8.2.** Consider the morphism

$$\Sigma^{d-1}k[T] \rightarrow k[T], \quad \Sigma^{d-1}1_{k[T]} \rightarrow T$$

of DG right-modules over $A = k[T]$. Its mapping cone is easily seen to be a minimal $K$-projective resolution of $k_A$, and from now on I will use this mapping cone as $F$.

Observe

$$F^\natural \cong \Sigma \Sigma^{d-1}k[T]^\natural \amalg k[T]^\natural \cong \Sigma^d A^\natural \amalg A^\natural. \quad (19)$$

Now I can prove:

**Lemma 8.3.** Suppose that $k$ has characteristic zero. Then $\mathcal{E}$ is equivalent by a series of quasi-isomorphisms to $C^*(S^d)$.

*Proof.* The sphere $S^d$ is a so-called formal space, so since $k$ has characteristic zero, $C^*(S^d)$ is equivalent by a series of quasi-isomorphisms to $\mathbb{H}^*(S^d)$ viewed as a DGA with vanishing differential (see [1, exam. 1, p. 142]). Hence it is enough to see that $\mathcal{E}$ is equivalent by a series of quasi-isomorphisms to $\mathbb{H}^*(S^d)$ viewed as a DGA with vanishing differential.

$\mathbb{H}^*(S^d)$ is a very simple DGA: It has a copy of $k$ in degree zero, spanned by $1_{\mathbb{H}^*(S^d)}$, and another copy of $k$ in degree $d$, spanned by some element, say $S$.

The cohomology of $\mathcal{E}$ is

$$H\mathcal{E} = H(\mathrm{Hom}_{A^\text{op}}(F, F)) \cong H(\mathrm{Hom}_{A^\text{op}}(F, k_A)) = (*),$$

and as $F$ is minimal, this is

$$(* ) \cong \mathrm{Hom}_{A^\text{op}}(F, k_A)^\natural \cong \mathrm{Hom}_{(A^\text{op})^\natural}(F^\natural, k^\natural) \cong \mathrm{Hom}_{(A^\text{op})^\natural}(\Sigma^d A^\natural \amalg A^\natural, k^\natural) \cong \Sigma^{-d} k^\natural \oplus k^\natural,$$

where (a) is by equation (19). So $H\mathcal{E}$ also has copies of $k$ in degrees 0 and $d$. 
Let $e$ be a cycle in $E^d$ whose cohomology class spans the copy of $k$ in degree $d$ of $H\mathcal{E}$. It is now easy to check that

$$H^\bullet(S^d) \longrightarrow \mathcal{E}; \quad 1_{H^\bullet(S^d)} \mapsto 1_{\mathcal{E}}, \quad S \mapsto e$$

is a quasi-isomorphism of DGAs, proving the lemma. □

From lemma 8.3 and [13, thm. III.4.2] follows that $D_c(E)$ and $D_c(S^d)$ are equivalent. Combining this with equation (18) gives the next result.

**Lemma 8.4.** Suppose that $k$ has characteristic zero. Then there are quasi-inverse equivalences of categories,

$$D^f(A^{op}) \xrightarrow{\cong} D^c(S^d).$$

Let me now determine the indecomposable objects of $D^f(A^{op})$.

**Definition 8.5.** For each $m \geq 0$ the element $T^{m+1}$ generates a DG ideal $(T^{m+1})$ in $k[T]$, so I can define a DG right-module over $A = k[T]$ by

$$Y_m = k[T]/(T^{m+1}).$$

**Lemma 8.6.** Up to isomorphism, the indecomposable objects of the category $D^f(A^{op})$ are exactly the (positive and negative) suspensions

$$\Sigma^j Y_m$$

with $j$ in $\mathbb{Z}$ and $m \geq 0$.

**Proof.** When $K$ is a graded right-$A^2$-module, let $\delta K$ denote $K$ viewed as a DG right-$A$-module with vanishing differential. I claim that

$$K \mapsto \delta K$$

induces a bijective correspondence between the isomorphism classes of $k$-finite dimensional graded indecomposable right-$A^2$-modules and the isomorphism classes of indecomposable objects of $D^f(A^{op})$.

For this, first note that if $M$ is a DG right-$A$-module, then the cohomology $HM$ is a graded right-$HA$-module. But $A$ has vanishing differential, so $HA$ is just $A^2$, so $HM$ is a graded right-$A^2$-module. Now in fact, I have that $M$ and $\delta HM$ are quasi-isomorphic. This is easy to prove directly; it is also a well known manifestation of $A^2$ being graded hereditary. (This means that any graded submodule of a graded projective module is again graded projective. The algebra $A^2$ is graded hereditary because it is a polynomial algebra on one generator.) So I have $M \cong \delta HM$ in $D(A)$.

Also, if $K$ is a graded right-$A^2$-module, then I have $K \cong H\delta K$.

Observe that this does not set up an equivalence of categories, as the isomorphism $M \cong \delta HM$ is not natural. However, it does show that
$K \mapsto \delta K$ induces a bijective correspondence between the isomorphism classes of graded right-$A^\sharp$-modules and the isomorphism classes of $D(A^{\mathrm{op}})$.

Now, if $M$ is an indecomposable object of $D^f(A^{\mathrm{op}})$, then by the above I have $M \cong \delta HM$ in $D^f(A^{\mathrm{op}})$. If $HM \cong K_1 \amalg K_2$ were a non-trivial decomposition, then

$$M \cong \delta HM \cong \delta(K_1 \amalg K_2) \cong \delta K_1 \amalg \delta K_2$$

would clearly be a non-trivial decomposition in $D^f(A^{\mathrm{op}})$, a contradiction. So $HM$ is a $k$-finite dimensional graded indecomposable right-$A^\sharp$-module.

On the other hand, if $K$ is a $k$-finite dimensional graded indecomposable right-$A^\sharp$-module, then a similar argument shows that $\delta K$ is an indecomposable object of $D^f(A^{\mathrm{op}})$.

So $K \mapsto \delta K$ also induces a bijective correspondence between isomorphism classes of indecomposables, as claimed.

However, the finitely generated graded indecomposable right-$A^\sharp$-modules are exactly the (positive and negative) suspensions of graded cyclic right-$A^\sharp$-modules. This is a manifestation of $A^\sharp$ being a principal ideal domain, see [14, p. 9] for the ungraded case. The $k$-finite dimensional among these modules are

$$\Sigma^j(k[T]/(T^{m+1}))$$

with $j$ in $\mathbb{Z}$ and $m \geq 0$.

By the above correspondence, up to isomorphism, the indecomposable objects of $D^f(A^{\mathrm{op}})$ are then

$$\delta \Sigma^j(k[T]/(T^{m+1}))$$

with $j$ in $\mathbb{Z}$ and $m \geq 0$. And these are exactly the objects $\Sigma^jY_m$. □

Transporting the $\Sigma^jY_m$'s through the equivalence of lemma 8.4 at last gives the indecomposable objects of $D^c(S^d)$.

**Definition 8.7.** Suppose that $k$ has characteristic zero. For each $m \geq 0$ I let $Z_m$ be the object of $D^c(S^d)$ obtained by transporting $Y_m$ through the equivalence of lemma 8.4.

**Proposition 8.8.** Suppose that $k$ has characteristic zero.

(i) Up to isomorphism, the indecomposable objects of $D^c(S^d)$ are exactly the (positive and negative) suspensions

$$\Sigma^j Z_m$$

with $j$ in $\mathbb{Z}$ and $m \geq 0$.

(ii) Each object of $D^c(S^d)$ is the coproduct of uniquely determined indecomposable objects.
For each $m \geq 0$ the object $Z_m$ in $D^c(S^d)$ has
\[ H^i Z_m = \begin{cases} 
  k & \text{for } i = -m(d-1) \text{ and } i = d, \\
  0 & \text{otherwise.}
\end{cases} \]

Proof. (i): This is clear from lemma 8.6 and definition 8.7.
(ii): Remark 6.2 gives that lemmas 3.6 and 3.7 apply to $C^*(S^d)$. Hence $D^c(S^d)$ is a Krull-Schmidt category by [9, 3.1], so (ii) holds.
(iii): It is easy to see that there is a distinguished triangle in $D(A^{op})$,
\[ \Sigma^{(m+1)(d-1)} A \rightarrow A \rightarrow Y_m \rightarrow . \]

It is also easy to prove
\[ H^i (RHom_{A^{op}}(F, A)) \cong \begin{cases} 
  k & \text{for } i = d, \\
  0 & \text{otherwise.}
\end{cases} \]  

Applying $RHom_{A^{op}}(F, -)$ to the distinguished triangle (20) gives a distinguished triangle in $D(E)$,
\[ \Sigma^{(m+1)(d-1)} RHom_{A^{op}}(F, A) \rightarrow RHom_{A^{op}}(F, A) \rightarrow RHom_{A^{op}}(F, Y_m) \rightarrow, \]

and the long exact cohomology sequence and equation (21) then prove
\[ H^i (RHom_{A^{op}}(F, Y_m)) = \begin{cases} 
  k & \text{for } i = -m(d-1) \text{ and } i = d, \\
  0 & \text{otherwise.}
\end{cases} \]  

Now, to transport $Y_m$ through the equivalence of lemma 8.4 means first to transport it through the equivalence (18), secondly to transport the resulting object through the equivalence induced by lemma 8.3. The first of these steps gives $RHom_{A^{op}}(F, Y_m)$ whose cohomology is in equation (22). And the second step leaves the cohomology unchanged, viewed as a graded $k$-vector space. This proves the proposition’s formula for $H^i Z_m$. \[ \square \]

Remark 8.9. It is easy to see that $C^*(S^d)$ itself is an indecomposable object of $D^c(S^d)$. By proposition 8.8 parts (i) and (iii), the only possibility is
\[ Z_0 \cong C^*(S^d) \]
in $D^c(S^d)$.

Now to the second main results of this paper, which sum up the theory in the case of spheres. Recall from setup 8.1 the condition $d \geq 2$.

Theorem 8.10. Suppose that $k$ has characteristic zero.
(i) In the category $D^c(S^d)$, there is an Auslander-Reiten triangle
\[ \Sigma^{d-1} Z_0 \rightarrow Z_1 \rightarrow Z_0 \rightarrow \]
and an Auslander-Reiten triangle

\[ \Sigma^{d-1} Z_n \longrightarrow \Sigma^{d-1} Z_{n-1} \oplus Z_{n+1} \longrightarrow Z_n \longrightarrow \]

for each \( n \) with \( n \geq 1 \), where the \( Z \)'s are the indecomposable objects from definition 8.7. Each Auslander-Reiten triangle is a (positive or negative) suspension of one of these.

(ii) The Auslander-Reiten translation of \( D^c(S^d) \) is given by

\[ \tau(-) = \Sigma^{d-1}(-). \]

Proof. (i): By [9, prop. 3.5(i)], Auslander-Reiten triangles are determined up to isomorphism by their right hand end terms. The right hand end terms are indecomposable objects by definition, so in the present case have the form \( \Sigma^j Z_m \) with \( j \in \mathbb{Z} \) and \( m \geq 0 \) by proposition 8.8(i). So to prove part (i) of the theorem, it is clearly enough to see that the Auslander-Reiten triangles with right hand end terms \( Z_m \) for \( m \geq 0 \) are as claimed.

By proposition 6.4(i), the left hand end terms of the Auslander-Reiten triangles are as claimed in the theorem, so let me consider the middle terms. First the Auslander-Reiten triangle ending in \( Z_0 \),

\[ \Sigma^{d-1} Z_0 \longrightarrow N \longrightarrow Z_0 \xrightarrow{\pi} . \quad (23) \]

By definition the morphism \( \pi \) is non-zero. But by remark 8.9 this morphism is \( C^*(S^d) \xrightarrow{\pi} \Sigma^d C^*(S^d) \). This makes it easy to compute the long exact cohomology sequence of (23) and get

\[ H^i N \approx \begin{cases} \kappa & \text{for } i = -(d-1) \text{ and } i = d, \\ 0 & \text{otherwise.} \end{cases} \]

But \( N \) is the coproduct of uniquely determined indecomposable objects of \( D^c(S^d) \) by proposition 8.8(ii), and by 8.8 parts (i) and (iii), the only possibility is \( N \cong \mathbb{Z}_1 \).

Next the Auslander-Reiten triangle ending in \( Z_n \),

\[ \Sigma^{d-1} Z_n \longrightarrow N \longrightarrow Z_n \xrightarrow{\pi} , \quad (24) \]

with \( n \geq 1 \). There can be no retract \( \Sigma^j C^*(S^d) \longrightarrow Z_n \), for else \( Z_n \) would be a direct summand in the indecomposable object \( \Sigma^j C^*(S^d) \cong \Sigma^j Z_0 \). Hence each morphism \( \Sigma^j C^*(S^d) \xrightarrow{\gamma} Z_n \) has \( \pi \gamma = 0 \). But this shows \( H\pi = 0 \), so the long exact cohomology sequence of (24) splits into short exact sequences. So using proposition 8.8(iii), the cohomology of \( N \) can be read off as

\[ H^i N \approx \begin{cases} \kappa & \text{for } i \in \{-n+1(d-1), -n(d-1), 1, d\}, \\ 0 & \text{otherwise.} \end{cases} \]

Proposition 8.8(ii) says that \( N \) is the coproduct of uniquely determined indecomposable objects of \( D^c(S^d) \). Comparing the cohomology of \( N \) with the cohomology of the indecomposable objects, obtained from 8.8.
parts (i) and (iii), leaves only two possibilities: \( N \) is either \( \Sigma^{d-1}Z_{n-1} \oplus Z_{n+1} \) or \( \Sigma^{d-1}Z_n \oplus Z_n \).

However, let me suppose by induction that the Auslander-Reiten triangle ending in \( Z_{n-1} \) is as claimed in the theorem, hence has a summand \( Z_n \) in its middle term. By lemma 2.2 (iii) \( \Rightarrow \) (ii), this implies that there is an irreducible morphism \( Z_n \rightarrow Z_{n-1} \). Hence there is an irreducible morphism \( \Sigma^{d-1}Z_n \rightarrow \Sigma^{d-1}Z_{n-1} \), and by lemma 2.2 (i) \( \Rightarrow \) (iii), this implies that \( \Sigma^{d-1}Z_{n-1} \) is a direct summand of \( N \). So \( N \) must be \( \Sigma^{d-1}Z_{n-1} \oplus Z_{n+1} \), proving the theorem.

(ii): This is immediate from part (i), or from proposition 6.4(ii). □

If a category has Auslander-Reiten triangles, then knowledge of the Auslander-Reiten triangles gives full knowledge of the quiver of the category by lemma 2.2. Also in this case, the quiver is a stable translation quiver with translation induced by the Auslander-Reiten translation of the category, by corollary 2.4. Applying this to the data from theorem 8.10 gives the following.

**Theorem 8.11.** Suppose that \( k \) has characteristic zero. Then the quiver of the category \( \mathcal{D}^c(S^d) \) consists of \( d-1 \) components, each isomorphic to \( \mathbb{Z}\mathbb{A}_\infty \). The component containing \( Z_0 \cong C^*(S^d) \) is

\[
\begin{array}{c}
\cdots \Sigma^{-2(d-1)}Z_4 \\
\vdots \\
\Sigma^{-2(d-1)}Z_3 \\
\cdots \Sigma^{-(d-1)}Z_2 \\
\cdots \Sigma^{-(d-1)}Z_1 \\
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\end{array}
\]

where the unbroken arrows are the arrows of the quiver and the dotted arrows indicate the action of the translation induced by the Auslander-Reiten translation of \( \mathcal{D}^c(S^d) \).

Finally, the following corollary is clear from theorem 8.11.

**Corollary 8.12.** Suppose that \( k \) has characteristic zero. Then the quiver of \( \mathcal{D}^c(S^d) \) is a sufficiently sensitive invariant to tell different \( S^d \)'s apart.

**Acknowledgement.** Theorem 6.3 is inspired by Happel’s result [8, thm. 3.4], which considers a finite dimensional algebra \( \Lambda \) and says roughly that \( \mathcal{D}^c(\Lambda) \) has Auslander-Reiten triangles if and only if \( \Lambda \) is Gorenstein. This is related to theorem 6.3 because the differential
graded analogue of the Gorenstein property is Poincaré duality (see section 5). I thank Henning Krause for directing my attention to [8].

The diagrams were typeset with Paul Taylor’s diagrams.tex.

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