Research Article

Some Properties of the Zero-Divisor Graphs of Idealization Ring $\Gamma(R(+)M)$

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1. Introduction

Graph theory is a very invariant tool to connect between application of mathematics and other fields. First, assume that $R$ is a commutative ring with unity and $M$ be an $R$-module. The ring of the idealization ring of $M$ in $R$ is $R(+)M = \{(r_1,n_1): r_1 \in R, n_1 \in M\}$ and any two elements $(r_1,n_1), (r_2,n_2) \in R(+)M$ which is defined by $(r_1,n_1)+(r_2,n_2) = (r_1+r_2,n_1+n_2)$ and $(r_1,n_1)(r_2,n_2) = (r_1r_2, r_1n_2+r_2n_1)$. For more details about the idealization ring, one can see in [1]. In [2, 3], Anderson, Livingston and Naseer proved that the graph $\Gamma(R)$ is connected with diameter at most 3. In [4] Axtell studied the zero-divisor of a commutative ring. The zero divisor graph of a commutative ring has been studied extensively by several authors [5–7].

Let $G$ be a simple connected undirected graph with the vertex set $V(G)$. Let $\Gamma(R)$ be a notation of the zero-divisor graph of a commutative ring $R$ whose vertices are the nonzero zero-divisors of $R$ i.e, $\mathbb{Z}^n(R)$, with $r_1$ and $s_1$ adjacent if $r_1 \neq s_1$ and $r_1s_1 = 0$ was introduced by I. Beck in [8], who linked some algebraic properties of $G$ with combinatorial properties of its zero-divisor graph.

Recently, Al-Labadi [9, 10], studied the properties of the zero-divisors graph of idealization ring such as when the zero-divisors graph of idealization ring is Planar graph, divisor graph and Eulerian graph and the independence number of the zero-divisor graph of the idealization ring. In this article we study the wiener index which is defined the sum of all distances between vertices of the graph and is denoted by $W(G)$.

We assume in this article the annihilator of $M$ is $\text{ann}(M) = \{r \in R | \forall m \in M: rm = 0\}$.

2. Wiener Index of the Graph $\Gamma(R(+)M)$

In this section, we compute the wiener index of the zero-divisor graph $\Gamma(R(+)M)$.

Al-Labadi M in [10], characterized the zero-divisor graph of idealization ring when $R$ is an integral domain.

Theorem 1. Let $R$ be an integral domain and $M$ be an $R$-module with $|M| = 2$. Then we have the following:

1. If $\text{ann}(M) = 0$, then the wiener index of the graph $\Gamma(R(+)M)$ is $W(\Gamma(R(+)M)) = 0$.
2. If $\text{ann}(M) \neq 0$, then the wiener index of the graph $\Gamma(R(+)M)$ is $W(\Gamma(R(+)M)) = 8|\text{ann}(Z_2)|$.

Proof. We have the following:

1. If $\text{ann}(M) = 0$, then by [10] we have $R \cong Z_2$, i.e, $\Gamma(Z_2(+)Z_2) = \{(0,1)\}$ is an empty graph.
Let $\Gamma (R(+M))$ be an integral domain and $M$ be an $R$ module with $|M| = 3$. Then we have the following:

1. If $\text{ann}(M) = 0$, then the wiener index of the graph $\Gamma (R(+M)) = W(\Gamma (R(+M))) = 2$. 

2. If $|\text{ann}(M)| \neq 0$, then $\Gamma (R(+M)) = \{(0,1), (r_1,0), (r_2,1): r \in \text{ann}(Z_3)\}$, where $Z_3$ is an idealization of the graph $\Gamma (R(+M))$. Hence, the wiener index of the graph $\Gamma (R(+M))$ is $W(\Gamma (R(+M))) = 2 \times |\text{ann}(Z_3)| + 2$. 

Proof. We have the following:

1. If $\text{ann}(M) = 0$, then by [10] we have $\Gamma (R(+M)) = \{(0,1), (0,2)\}$, where $Z_3$ is an idealization of the graph $\Gamma (R(+M))$. Hence, the wiener index of the graph $\Gamma (R(+M))$ is $W(\Gamma (R(+M))) = 2$.

2. If $|\text{ann}(M)| \neq 0$, then $\Gamma (R(+M)) = \{(0,1), (0,2), (r,m): r \in \text{ann}(Z_3)\}$. 

$|\text{ann}(Z_3)| = 2$, so $\Gamma (R(+M)) = \{(0,1), (0,2), (r,m): r \in \text{ann}(Z_3)\}$. 

Hence, the wiener index of the graph $\Gamma (R(+M))$ is $W(\Gamma (R(+M))) = 2 \times |\text{ann}(Z_3)| + 2$. 

Theorem 3. Let $R$ be an integral domain and $|M| \geq 4$ be an $R$– module. Then the wiener index of the graph $\Gamma (R(+M))$ is $W(\Gamma (R(+M))) = |M| - 1 + |M| - 2$. If $\text{ann}(M) = 0$, then $\Gamma (R(+M)) = \{(0,t): t \in M\}$ that is a complete graph. 

Proof. We have the following:

1. If $\text{ann}(M) = 0$, then by [10] we have $\Gamma (R(+M)) = \{(0,t): t \in M\}$ that is a complete graph. 

2. If $\text{ann}(M) \neq 0$, then $\Gamma (R(+M)) = \{(0,t): t \in M\}$ that is a complete graph. 

3. Wiener Index of the Graph $\Gamma (Z_N (+)Z_M)$

In this section, we compute the wiener index of the graph $\Gamma (Z_N (+)Z_M)$, where $Z_N (+)Z_M$ is called the idealization ring.

[9] Al-Labadi Manal presented the zero-divisors graph of idealization ring $Z_N (+)Z_M$.

First, we compute the wiener index of the graph $\Gamma (Z_N (+)Z_M)$, when $N = p$ and $M = p$ where $p$ is a prime number.

Theorem 4. If $M = p$ and $N = p^2$ where $p > 2$ is a prime number, then wiener index of the graph is $W(\Gamma (Z_N (+)Z_M)) = (p - 1)(p - 2)$. 

Proof. If $M = p$ and $N = p^2$ where $p > 2$ is a prime number, then we have $\Gamma (Z_p (+)Z_p) = \{(0,t): t \in Z_p\} \cup \{(kp,t): t \in Z_p, gcd(k,p) = 1\}$. So, the graph is a complete graph. Hence, the wiener index of the graph $\Gamma (Z_p (+)Z_p)$ is $W(\Gamma (Z_p (+)Z_p)) = \sum_{i=0}^{p-1} d(v_i, v_j) = (p - 1)(p - 2)$. 

Theorem 5. If $M = p$ and $N = p^2$ where $p > 2$ is a prime number, then the wiener index of the graph $\Gamma (Z_{p^2} (+)Z_p)$ is $W(\Gamma (Z_{p^2} (+)Z_p)) = (p - 1 + p\langle p + 1 \rangle)(p - 1 + p\langle p + 1 \rangle)$. 

Proof. If $M = p$ and $N = p^2$ where $p > 2$ is a prime number, then we have $\Gamma (Z_{p^2} (+)Z_p) = \{(0,m): m \in Z_p\} \cup \{(kp,t): t \in Z_p, gcd(k,p) = 1\}$. So, the graph is a complete graph. Hence, the wiener index of the graph $\Gamma (Z_{p^2} (+)Z_p)$ is $W(\Gamma (Z_{p^2} (+)Z_p)) = \sum_{i=0}^{p-1} d(v_i, v_j) = (p - 1 + p\langle p + 1 \rangle)(p - 1 + p\langle p + 1 \rangle)$.

Now, we characterize the sets of the zero-divisor graph for the following.

For $N = p^3$ and $M = p$ where $p$ is a prime number and $\alpha > 2$.

$L_0 = \{(0,1)(0,2), \ldots, (0,p - 1)\}$ and $|L_0| = p - 1.

$L_1 = \{(kp,t): t \in Z_p, gcd(k,p) = 1\}$. 

For $1 \leq s \leq \alpha - 1$ and $|L_s| = \phi(p^{s+1})$, where $\phi$ is an Euler function.

Theorem 6. If $M = p$ and $N = p^s$ where $p$ is a prime number and $\alpha > 2$, then the wiener index of the graph is $W(\Gamma (Z_{p^s} (+)Z_p)) = \sum_{k=0}^{p-1} d(v_k, v_j) = \sum_{k=0}^{p-1} d(v_k, v_j)$, where $0 \leq k \leq p - 1$. 

Figure 1: $|\text{ann}(Z_2)| = 1$. 

Figure 2: $|\text{ann}(Z_2)| = 1$. 

Figure 3: $|\text{ann}(Z_3)| = 1$. 

Figure 4: $|\text{ann}(Z_4)| = 1$.
Proof. Let \( M = p \) and \( N = p^2 \), \( \alpha > 2 \). Then the zero divisor graph of idealization ring \( \mathbb{Z}/p \mathbb{Z} \) is \( \mathbb{Z}^* \) \((\mathbb{Z}/p \mathbb{Z})^*\) = \( \{(0, m) : m \in \mathbb{Z}/p \mathbb{Z} \} \cup L_1 \cup L_2 \).

For any vertex \( v_i \in L_0 \), let \( v_j \in L_j \), \( 0 \leq j \leq \alpha - 1 \). Then we can define:

\[
\beta_0 = \sum_{v_i \in L_0, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).
\]

Then \( \beta_0 = (p-1) \left( \sum_{j=0}^{\alpha-1} (L_j) + L_0 \right) - 1 \). So,

\[
\beta_0 = (p-1) \left( \sum_{j=0}^{\alpha-1} p\phi(p^{\alpha-j}) + p - 2 \right) \cdots (1)
\]

For any vertex \( v_i \in L_1 \), let \( v_j \in L_j \), \( 0 \leq j \leq \alpha - 1 \). Then we can define:

\[
\beta_1 = \sum_{v_i \in L_1, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).
\]

Then

\[
\beta_1 = (|L_1|) \left( |L_0| + |L_{\alpha-1}| + 2 \sum_{j=1}^{\alpha-2} (L_j) \right) \cdots (2)
\]

And so on, for any vertex \( v_i \in L_{(a/2)} \), let \( v_j \in L_j \), \( 0 \leq j \leq \alpha - 1 \). Then we can define:

\[
\beta_{(a/2)} = \sum_{v_i \in L_{(a/2)}, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).
\]

Then

\[
\beta_{(a/2)} = \left( \sum_{i=0}^{a/2} |L_{a-i}| + 2 \sum_{j=1}^{a/2} |L_j| \right) \left| L_{(a/2)} \right| \cdots (3)
\]

For any vertex \( v_i \in L_{(a/2)} \), let \( v_j \in L_j \), Then we can define:

\[
\beta_{(a/2)} = \sum_{v_i \in L_{(a/2)}, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).
\]

Then

\[
\beta_{(a/2)} = \left( \sum_{i=0}^{a/2} |L_{a-i}| + 2 \sum_{j=1}^{a/2} |L_j| \right) \left| L_{(a/2)} \right| \cdots (4)
\]

Hence from (1)–(3) and (4) the wiener index of the graph is \( W(\Gamma(Z\mathbb{Z}/p \mathbb{Z})) = \sum_{k=0}^{2} \beta_k \).

Now, we characterize the sets of the zero-divisor graph for the following.

For \( N = p_1 p_2 \) and \( M = p_1 \) where \( p_1 \) and \( p_2 \) are a prime numbers.

\[
L_0 = \{(0, 1), (0, 2), \ldots, (0, p_1 - 1) \} \text{ and } |L_0| = p_1 - 1.
\]

\[
L_1 = \{(k, p_1) : t \in \mathbb{Z}/p_1 \mathbb{Z}, \gcd(k, p_2) = 1 \} \text{ and } |L_{p_1}| = p_1 \phi(p_2), \text{ where } \phi \text{ is an Euler function i.e., } |L_{p_1}| = p_1 - 1.
\]

\[
L_2 = \{(k, p_2) : t \in \mathbb{Z}/p_2 \mathbb{Z}, \gcd(k, p_1) = 1 \} \text{ and } |L_{p_2}| = p_2 \phi(p_1), \text { where } \phi \text{ is an Euler function i.e., } |L_{p_2}| = p_2 - 1.
\]

Theorem 7. If \( M = p_1 \) and \( N = p_1 p_2 \) where \( p_1 \) and \( p_2 \) are a prime numbers, then wiener index of the graph is

\[
W(\Gamma\left(Z\mathbb{Z}/p_1 \mathbb{Z}\right)) = \sum_{k=0}^{2} \beta_k.
\]

\[
\beta_k = \sum_{v_i \in L_k, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j) \text{ where } k = 0, 1 \text{ and } 2.
\]

Proof. Let \( M = p_1 \) and \( N = p_1 p_2 \) where \( p_1 \) and \( p_2 \) are prime numbers. Then the zero divisor graph of idealization ring \( \mathbb{Z}/p_1 \mathbb{Z} \) \((\mathbb{Z}/p_1 \mathbb{Z})^*\) = \( \{(0, m) : m \in \mathbb{Z}/p_1 \mathbb{Z} \} \cup L_1 \cup L_2 \).

For any vertex \( v_i \in L_0 \), let \( v_j \in L_j \) for all \( j \in \{0, 1, 2\} \).

Then we can define

\[
\beta_0 = \sum_{v_i \in L_0, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).\]

Then

\[
\beta_0 = (p_1 - 1) \left( \sum_{j=0}^{\alpha-1} \phi(p_2^{\alpha-j}) + p_1 - 1 \right) \cdots (5)
\]

For any vertex \( v_i \in L_1 \), let \( v_j \in L_j \) for all \( j \in \{0, 1, 2\} \).

Then we can define

\[
\beta_1 = \sum_{v_i \in L_1, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).\]

Then

\[
\beta_1 = (|L_1|) \left( |L_0| + |L_{\alpha-1}| + 2 \sum_{j=1}^{\alpha-2} (L_j) \right) \cdots (2)
\]

And so on, for any vertex \( v_i \in L_{(a/2)} \), let \( v_j \in L_j \), \( 0 \leq j \leq \alpha - 1 \). Then we can define:

\[
\beta_{(a/2)} = \sum_{v_i \in L_{(a/2)}, v_j \neq v_i} \sum_{j=0}^{\alpha-1} d(v_i, v_j).
\]

Then

\[
\beta_{(a/2)} = \left( \sum_{i=0}^{a/2} |L_{a-i}| + 2 \sum_{j=1}^{a/2} |L_j| \right) \left| L_{(a/2)} \right| \cdots (3)
\]

Hence from (5) and (6) the wiener index of the graph is \( W(\Gamma\left(Z\mathbb{Z}/p_1 \mathbb{Z}\right)) \) = \( \sum_{k=0}^{2} \beta_k \).

4. When is the Graph \( \Gamma(\mathbb{Z}/(\mathbb{Z}/p_1 \mathbb{Z})) \) Hamiltonian?

In this section, we determine when the graph \( \Gamma(\mathbb{Z}/(\mathbb{Z}/p_1 \mathbb{Z})) \) is a Hamiltonian graph.

Definition 1. A Hamiltonian cycle is a cycle that visits each vertex only once time.

Definition 2. A Hamiltonian graph is a graph that contains a Hamiltonian cycle.

We begin with the following lemma when \( \mathbb{R} \) is an integral domain.

Lemma 1. Let \( \mathbb{R} \) be an integral domain and \( M \) be an \( \mathbb{R} \)-module with \(|\mathbb{M}| = 2 \). Then the graph \( \Gamma(\mathbb{R}(\mathbb{Z}/p_1 \mathbb{Z})) \) is not a Hamiltonian graph.

Proof. We have the following:

(1) If \( \text{ann}(\mathbb{M}) = 0 \), then by [10], we have \( \mathbb{R} \equiv \mathbb{Z}_2 \) i.e., \( \Gamma(\mathbb{Z}_2(\mathbb{Z}/p_1 \mathbb{Z})) = \{(0, 1), (r_0, 1), (r_1, 1), r_i \in \text{ann}(\mathbb{Z}_2)\} \), So, cannot find a cycle between any vertices. Therefore, it is not a Hamiltonian graph.

Lemma 2. Let \( \mathbb{R} \) be an integral domain and \( M \) be an \( \mathbb{R} \)-module with \(|\mathbb{M}| = 3 \). Then the graph \( \Gamma(\mathbb{R}(\mathbb{Z}/p_1 \mathbb{Z})) \) is a Hamiltonian graph.
Proof. We have the following:

1. If \( \text{ann}(M) = 0 \), then by [10] we have \( R \equiv \mathbb{Z} \), i.e., \( \varGamma(\mathbb{R}(+)\mathbb{Z}_3) = \{(0, 1), (0, 2)\} \) that is not a Hamiltonian graph.

2. If \( \text{ann}(M) \neq 0 \), then \( \varGamma(\mathbb{R}(+)\mathbb{Z}_3) = \{(0, 1), (0, 2), (r, m) : r \in \text{ann}(\mathbb{Z}_3), m \in \mathbb{Z}_3\} \). So, we can not find a cycle between all vertices in the graph \( \varGamma(\mathbb{R}(+)M) \) that the graph \( \varGamma(\mathbb{R}(+)M) \) is not a Hamiltonian graph.

Now, we generalize the result by the following theorem.

**Theorem 8.** Let \( R \) be an integral domain and \( |M| \geq 4 \) be an \( R- \) module. Then the graph \( \varGamma(\mathbb{R}(+)M) \) is Hamiltonian, if \( \text{ann}(M) = 0 \), not Hamiltonian, if \( \text{ann}(M) \neq 0 \).

Proof. We have the following:

1. If \( \text{ann}(M) = 0 \), then by [10] we have \( \varGamma(\mathbb{R}(+)M) = \{(0, t) : t \in M\} \) that is a complete graph. So, the graph \( \varGamma(\mathbb{R}(+)M) \) is Hamiltonian graph.

2. If \( \text{ann}(M) \neq 0 \), then by [10] we have \( \varGamma(\mathbb{R}(+)M) = \{(0, t), (r, m) : r \in \text{ann}(\mathbb{M}) \text{ and } t \in M\} \). So, we can not find the cycle between all vertices in the graph \( \varGamma(\mathbb{R}(+)M) \) that is not a Hamiltonian graph.

5. **When is the Graph \( \varGamma(Z_N(+)Z_M) \) Hamiltonian?**

In this section, we determine when the graph \( \varGamma(Z_N(+)Z_M) \) is a Hamiltonian graph.

**Theorem 9.** If \( M = p \) and \( N = p \) where \( p \geq 2 \) is a prime number, then the graph \( \varGamma(Z_N(+)Z_M) \) is a Hamiltonian graph.

Proof. If \( M = p \) and \( N = p \) where \( p \geq 2 \) is a prime number, then we have \( \varGamma(Z_p(+)Z_p) = \{(0, t) : t \in Z_p^{*}\} = \{(0, 1), (0, 2), \ldots, (0, p - 1)\} \) which is a complete graph. So, the graph \( \varGamma(Z_p(+)Z_p) \) is a Hamiltonian graph.

**Theorem 10.** If \( M = p \) and \( N = p^2 \) where \( p \geq 2 \) is a prime number, then the graph \( \varGamma(Z_p(+)Z_{p^2}) \) is a Hamiltonian graph.

Proof. If \( M = p \) and \( N = p^2 \) where \( p \geq 2 \) is a prime number, then we have \( \varGamma(Z_p(+)Z_{p^2}) = \{(0, m) : m \in Z_p^*\} \cup L_0 \), so the graph is a complete graph. Hence, the graph \( \varGamma(Z_p(+)Z_{p^2}) \) is a Hamiltonian graph.

Now, we investigate when \( N = p^a \) and \( M = p \) where \( p \) is a prime number and \( a > 2 \).

**Theorem 11.** If \( M = p \) and \( N = p^a \), where \( a > 2 \) and \( p \) is a prime number, then the graph \( \varGamma(Z_p(+)Z_{p^a}) \) is not a Hamiltonian graph.

Proof. If \( M = p \) and \( N = p^a \) where \( a > 2 \) and \( p \) is a prime number, then we have \( \varGamma(Z_p(+)Z_{p^a}) \) = \{(0, m) : m \in \mathbb{Z}_p^*\} \cup L_0 \), so the number of connected components of graph \( \varGamma(Z_{p^a}(+)Z_{p^a}) - M \) is not less than of the order \( |M| \), by theorem in [7] where \( M = \{L_0 \cup L_{1a} \cup L_{2a} \cup \cdots \cup L_{a-1}\} \). Hence, the graph \( \varGamma(Z_{p^a}(+)Z_{p^a}) \) is not a Hamiltonian graph.

**Theorem 12.** If \( M = p_1 \) and \( N = p_1p_2 \) where \( p_1 \) and \( p_2 \) are a prime numbers, then the graph \( \varGamma(Z_{p_1}(+)Z_{p_2}) \) is not a Hamiltonian graph.

Proof. Let \( M = p_1 \) and \( N = p_1p_2 \) where \( p_1 \) and \( p_2 \) are prime numbers. Then the zero divisor graph of idealization ring \( \varGamma(Z_{p_1}(+)Z_{p_2}) \) is \( Z^* \{Z_{p_1}(+)Z_{p_2}\} = \{(0, m) : m \in \mathbb{Z}_{p_1}^*\} \cup L_0 \). For any vertex \( v_i \in L_0 \) is adjacent only to any other vertex in \( L_1 \) and any vertex in \( L_1 \cup L_2 \) is adjacent only to vertex in \( L_0 \). Hence, we can not find a cycle between all vertices in the graph \( \varGamma(Z_{p_1}(+)Z_{p_2}) \) that is not a Hamiltonian graph. See Figure 3.

6. **Clique Number of the Graph \( \varGamma(\mathbb{R}(+)M) \)**

In this section, we compute the clique number of the graph \( \varGamma(\mathbb{R}(+)M) \).

**Definition 3.** A graph \( G \) is called a maximal clique if there is no clique with more vertices.

**Definition 4.** The clique number of the graph \( G \) is the number of vertices in a maximum clique in the graph \( G \) which is denoted by \( \omega(G) \).

**Theorem 13.** Let the ring \( R \) be an integral domain and \( M \) be an \( R- \) module with \( |M| = 2 \). Then we have the following:

1. If \( \text{ann}(M) = 0 \), then the clique number of the graph \( \varGamma(\mathbb{R}(+)M) \) is \( \omega(\varGamma(\mathbb{R}(+)M)) = 1 \).
Theorem 16. Let $R$ be an integral domain and $M$ be an $R$-module. Then

$$W(\Gamma(R(+)M)) =
\begin{cases}
0, & \text{if } \text{ann}(Z_2) = 0, |M| = 2,
8|\text{ann}(Z_2)|, & \text{if } \text{ann}(M) \neq 0, |M| = 2,
2, & \text{if } \text{ann}(Z_2) = 0, |M| = 3,
24|\text{ann}(Z_2)| + 2 & \text{if } \text{ann}(M) \neq 0, |M| = 3.
\end{cases}$$

Theorem 17. Let $R$ be an integral domain and $M$ be an $R$-module, $|M| \geq 4$. Then

$$W(\Gamma(R(+)M)) =
\begin{cases}
(|M| - 1)(|M| - 2), & \text{if } \text{ann}(M) = 0, (|M| - 1)(|M| - 2 + |M|\text{ann}(M)) + |M|\text{ann}(M)(|M| - 1) + 2(|M|\text{ann}(M)| - 1)), & \text{if } \text{ann}(M) \neq 0.
\end{cases}$$

Theorem 18. Let $Z_N(+)Z_M$ be the ring of the idealization ring. Then

$$W(\Gamma(Z_N(+)Z_M)) =
\begin{cases}
(p - 1)(p - 2), & \text{if } N = M = p,
(p - 1 + p(p - 1))(p(p - 1) + p - 2), & \text{if } M = p, N = p^2.
\end{cases}$$

Now, when the zero-divisor graph is a Hamiltonian graph.

Theorem 19. Let $R$ be an integral domain. Then the zero-divisor graph $(Z_N(+)Z_M)$ is a Hamiltonian graph if $\text{ann}(M) = 0$ and $|M| \geq 4$.

Theorem 20. Let $Z_N(+)Z_M$ be the ring of the idealization ring. Then the zero-divisor graph $\Gamma(Z_N(+)Z_M)$ is a Hamiltonian graph if $N = p$ or $N = p^2$.

The clique number for the zero-divisor graph $\Gamma(R(+)M)$ when $R$ is an integral domain.

Theorem 21. Let $R$ be an integral domain and $M$ be an $R$-module. Then the clique number of the graph $\Gamma(R(+)M)$ is $\omega(\Gamma(R(+)M)) = |M| - 1$.

In the future work can be ask the following questions:

1. What is the clique number for the zero-divisor graph of idealization ring $\Gamma(R(+)M)$ when $R$ is not integral domain?
2. What is the geodetic number of the zero-divisors graph of idealization ring and the complement?

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. Harary, *Graph Theory*, Addison-Wesley, Boston, MA, USA, 1972.
[2] D. D. Anderson and M. Naseer, “Beck’s c of a commutative ring,” *Journal of Algebra*, vol. 159, no. 2, pp. 500–514, 1993.
[3] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," Journal of Algebra, vol. 217, no. 2, pp. 434–447, 1999.

[4] M. Axtell and J. Stickles, "Zero-divisor graphs of idealizations," Journal of Pure and Applied Algebra, vol. 204, no. 2, pp. 235–243, 2006.

[5] G. Chatrand, R. Muntz, V. Saenpholphat, and P. Zhang, "Which graph are divisor graphs?" Congr. Number, vol. 151, pp. 189–200, 2001.

[6] P. Erdos, R. Freud, and N. Hegavari, "Arithmatical properties of permutations of integers," Acta Mathematica Hungarica, vol. 41, pp. 169–176, 1983.

[7] J. Huckaba, "Commutative rings with zero divisors," in Monographs Pure Applied Mathematics Marcel Dekker, Basel, New York, USA, 1988.

[8] I. Beck, "Coloring of commutative rings," Journal of Algebra, vol. 116, no. 1, pp. 208–226, 1988.

[9] M. Al-Labadi, E. M. Almuhur, A. Shatarah et al., "Eulerian graph of some special idealization rings," Advances in Mathematics: Scientific Journal, vol. 10, no. 3, pp. 1833–1837, 2021.

[10] M. Al-Labadi, E. M. Almuhur, and A. Alboustanji, "International Conference on Information Technology (ICIT)," in Proceedings of the Independence Number and Covering Vertex Number of \( \mathbb{R}(+M) \), pp. 70–74, Jordan, Amman, May 2021.