Instance-optimality in optimal value estimation: 
Adaptivity via variance-reduced $Q$-learning

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Abstract

Various algorithms in reinforcement learning exhibit dramatic variability in their convergence rates and ultimate accuracy as a function of the problem structure. Such instance-specific behavior is not captured by existing global minimax bounds, which are worst-case in nature. We analyze the problem of estimating optimal $Q$-value functions for a discounted Markov decision process with discrete states and actions and identify an instance-dependent functional that controls the difficulty of estimation in the $\ell_8$-norm. Using a local minimax framework, we show that this functional arises in lower bounds on the accuracy on any estimation procedure. In the other direction, we establish the sharpness of our lower bounds, up to factors logarithmic in the state and action spaces, by analyzing a variance-reduced version of $Q$-learning. Our theory provides a precise way of distinguishing “easy” problems from “hard” ones in the context of $Q$-learning, as illustrated by an ensemble with a continuum of difficulty.

1 Introduction

The need for data-driven decision-making has fueled tremendous interest in Markov decision processes and reinforcement learning (RL). Indeed, such techniques have found use cases across a wide range of application domains [TFR17, LFDA16, SHM16]. An intriguing fact is that in many applications, RL algorithms behave far better than the theoretical bounds provided by worst-case analyses would suggest. This gap provides impetus for a more refined instance-specific analysis, one which highlights the properties of a given instance that render it “easy” or “difficult.”

Instance-dependent analysis of RL algorithms has become of substantial interest in recent years [see, e.g., SJ19, ZB19, ZKB19, MMM14, PW20, KPR20]. By now, we have a fairly refined understanding of instance-dependence for policy evaluation problems, including work on temporal difference (TD) algorithms under the $\ell_2$-norm [BRS18, LS18, DSTM18], as well as bounds for the LSTD estimator in the $\ell_{\infty}$-norm [PW20]. A subset of the current authors [KPR20] provided a sharper instance-dependent $\ell_{\infty}$-bounds for a variance-reduced version of the TD(0) algorithm, and showed that this algorithm is optimal in a local non-asymptotic minimax sense.

For TD and LSTD methods, the underlying structure is linear in nature—in particular, it corresponds to solving a linear system—a property which greatly facilitates the analysis. In the current paper we undertake a similar instance-dependent analysis in the more challenging

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setting of Q-learning, for which the underlying updates are non-linear. Our main contributions are to identify a natural functional of the problem instance and show that it controls the fundamental difficulty of estimating optimal Q-value functions. We do so by establishing non-asymptotic lower bounds within a local minimax framework and matching those bounds, up to logarithmic factors, by analyzing a version of variance-reduced Q-learning [SWW+18, SWYV18, Wai19c].

This work is done in the context of Markov decision processes (MDPs) with a finite set of states $\mathcal{X}$ and a finite set of possible actions $\mathcal{U}$. We proceed to provide some background and notation to be able to introduce the functional that plays a central role in our analysis, and describe our contributions in more detail.

### 1.1 Some background

In a Markov decision process, the state $x$ evolves dynamically in time under the influence of the actions. More precisely, there is a collection of probability transition kernels, $\{P_u(\cdot \mid x) \mid (x, u) \in \mathcal{X} \times \mathcal{U}\}$, where $P_u(x' \mid x)$ denotes the transition to the state $x'$ when the action $u$ is taken at the current state $x$. In addition, an MDP is equipped with a reward function $r$ that maps every state-action pair, $(x, u)$, to a real number $r(x, u)$. The reward $r(x, u)$ is the reward received upon performing an action $u$ in the state $x$. Overall, a given MDP is characterized by the problem pair $(P, r)$, along with a discount factor $\gamma \in (0, 1)$.

A deterministic policy $\pi$ is a mapping $\mathcal{X} \to \mathcal{U}$: the quantity $\pi(x) \in \mathcal{U}$ indicates the action to be taken in the state $x$. The value of a policy is defined by the expected sum of discounted rewards in an infinite sample path. For a given policy $\pi$ and discount factor $\gamma \in (0, 1)$, the $Q$-function is given by

$$\theta^\pi(x, u) := \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k r(x_k, u_k) \mid x_0 = x, u_0 = u \right], \quad \text{where } u_k = \pi(x_k) \text{ for all } k \geq 1. \tag{1}$$

When both the state space $\mathcal{X}$ and action space $\mathcal{U}$ are finite, the $Q$-function $\theta$ can be conveniently represented as an element of $\mathbb{R}^{(|\mathcal{X}| \times |\mathcal{U}|)}$.

There are various observation models in reinforcement learning, and in this paper we study the generative setting in which we have the ability to draw next-state samples from the MDP when initialized with an arbitrary state-action pair $(x, u)$. More precisely, we are given a collection of $N$ i.i.d. samples of the form $\{(Z_k, R_k)\}_{k=1}^N$, where both $Z_k$ and $R_k$ are random matrices in $\mathbb{R}^{(|\mathcal{X}| \times |\mathcal{U}|)}$. For each state-action pair $(x, u)$, the entry $Z_k(x, u)$ is drawn according to the transition kernel $P_u(\cdot \mid x)$, whereas the entry $R_k(x, u)$ is a zero-mean random variable with mean $r(x, u)$ and $\sigma_r$-sub-Gaussian tails, corresponding to a noisy observation of the reward function. Here the rewards $\{R_k(x, u)\}_{(x, u)\in\mathcal{X}\times\mathcal{U}}$ are independent across the all state-action pairs, and the random rewards $\{R_k\}$ are independent of the randomness in $\{Z_k\}$.

Based on the observations, our goal is to estimate the optimal $Q$-value function $\theta^*$, along with an optimal policy $\pi^*$. From the classical theory of MDPs [Put14, SB18, Ber09], the optimal $Q$-function is a fixed point of the Bellman (optimality) operator $T$, a map from $\mathbb{R}^{(|\mathcal{X}| \times |\mathcal{U}|)}$ to itself given by

$$T(\theta)(x, u) := r(x, u) + \gamma \sum_{x' \in \mathcal{X}} P_u(x' \mid x) \max_{u' \in \mathcal{U}} \theta(x', u'), \tag{2}$$

and an optimal policy $\pi^*$ can be obtained from the optimal $Q$-function $\theta^*$ via the maximization $\pi^*(x) \in \arg \max_{u \in \mathcal{U}} \theta^*(x, u)$. In this paper, we measure the quality of a given estimate $\hat{\theta}$ in terms
of the $\ell_\infty$-norm error:

$$\|\hat{\theta} - \theta^*\|_\infty = \max_{(x,u)} |\hat{\theta}(x,u) - \theta^*(x,u)|. \quad (3)$$

### 1.2 Contributions of this paper

The main contribution of this paper is to show that for a given MDP, the difficulty of estimating the optimal $Q$-value function in $\ell_\infty$-norm is characterized by a particular functional of the problem instance $(P, r)$, defined here.

**An instance-dependent functional:** Given a sample $(Z, R)$ from our observation model, we can define the single-sample empirical Bellman operator

$$\hat{T}(\theta) := R(x,u) + \gamma \sum_{x' \in X} Z_u(x' \mid x) \max_{u' \in U} \theta(x', u'), \quad (4)$$

where we have introduced $Z_u(x' \mid x) := 1_{Z(x,u)=x'}$.

Note that for any fixed $Q$-function $\theta$, the difference $\hat{T}(\theta) - T(\theta)$ is a zero-mean random matrix, and a key object in this paper is the matrix $\nu_{\pi; P, r, \gamma}$ with entries

$$\nu(\pi; P, r, \gamma)(x,u) := \sqrt{\text{Var} \left( (I - \gamma P^\pi)^{-1} (\hat{T}(\theta^*) - T(\theta^*)) \right)}. \quad (5)$$

More explicitly, the quantity $P^\pi$ is a right-linear mapping of $\mathbb{R}^{|X| \times |U|}$ to itself, given by:

$$P^\pi \theta(x,u) := \sum_{x' \in X} P_u(x' \mid x) \cdot \theta(x', \pi(x')) \quad \text{for each} \ (x,u) \in X \times U, \quad (6)$$

and the square-root and variance operators in equation (5) are applied elementwise.

Let us provide some intuition as to why $\nu(\pi; P, r, \gamma)$ plays a fundamental role. The appearance of the zero mean term $\hat{T}(\theta^*) - T(\theta^*)$ is natural: it reflects the noise present in the empirical Bellman operator (4) as an estimate of the population Bellman operator (2). As for the pre-factor $(I - \gamma P^\pi)^{-1}$, by a von Neumann expansion we can write

$$(I - \gamma P^\pi)^{-1} = \sum_{k=0}^{\infty} (\gamma P^\pi)^k. \quad (5)$$

The sum of the powers of $\gamma P^\pi$ account for the compounded effect of an initial perturbation when following the Markov chain specified by the policy $\pi$.

**Upper and lower bounds:** With these definitions in place, the core of our work involves proving, via a combination of a lower and an upper bound (matching up to logarithmic factors), that the instance-specific difficulty of estimating the $Q$-function is captured by the quantity $\max_{\pi \in \Pi^*} \|\nu(\pi; P, r, \gamma)\|_\infty$. Here $\Pi^*$ denotes the set of all optimal policies for the MDP instance $(P, r)$. This functional exhibits a wide range of behaviors: in Example 1 to follow in Section 2.1.2, we exhibit a family of MDPs $(P_\lambda, r_\lambda)$, parameterized by a scalar $\lambda \geq 0$ such that

$$\max_{\pi \in \Pi^*} \|\nu(\pi; P_\lambda, r_\lambda, \gamma)\|_\infty \approx \left( \frac{1}{1 - \gamma} \right)^{\frac{1}{2} - \lambda}. \quad (5)$$
The setting $\lambda = 0$ recovers a “hard” instance, one for which the global minimax bound for estimation of $Q$-functions, known from past work [AMK13] on batched $Q$-learning, is sharp. On the other hand, as $\lambda$ grows, the problems in this family become progressively easier, so that the global minimax bound is no longer sharp.

In more detail, we prove a non-asymptotic lower bound, stated as Theorem 1 to follow, by adapting a particular definition of local minimax risk studied in past work on shape-constrained estimation [CL15]. The central challenge in this proof is that perturbations to the transition matrices of a given MDP change not only the transitions themselves, but also the structure of the optimal policies. In order to prove matching upper bounds, given the role of the empirical operators $T$ in our lower bound, which are used in the classical $Q$-learning algorithm [WD92, Tsi94, Sze97, JJS94], a natural thought would be to analyze this operator directly. However, it is known from past work [Wai19b] that the classical $Q$-learning algorithm is non-optimal, even when assessed when using the coarser metric of global minimax. Thus, in order to obtain a sharp upper bound, we turn to variance-reduced forms of $Q$-learning, as introduced in past work [SWW+18, SWWY18, Wai19c] and shown to be optimal in a globally minimax sense. Our main contribution is to show that under certain structural conditions and lower bounds on the sample size, there is a form of variance-reduced $Q$-learning that achieves our local minimax lower bound up to a logarithmic factor. These upper bounds, stated precisely in Theorem 2, confirm that our lower bound technique has extracted a useful form of instance dependence for estimating optimal $Q$-functions.

**Notation:** For a positive integer $n$, we use the shorthand $[n] := \{1, 2, \ldots, n\}$. For a finite set $S$, we use $|S|$ to denote its cardinality. We use $c_1, c_2, \ldots$ to denote universal constants that may change from line to line. For any pair of vectors or matrices $(v, w)$ with matching dimension(s), we write that $v \succeq w$ to imply $v - w$ has only positive entries, and $v \preceq w$ is defined similarly. We let $|u|$ denote the entrywise absolute value of a vector $u \in \mathbb{R}^n$ or a matrix $u \in \mathbb{R}^{m \times n}$; we use $|u|_+$ to denote the entry-wise positive part of $u$. For any vector or matrix $u$, we let $\|u\|_\infty$ denote the maximum absolute value taken over all entries of $u$, and $\|u\|_{\text{span}} = \max_j u_j - \min_j u_j$ denote the span seminorm. For a continuous operator $P : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$, we define its $\ell_\infty$-operator norm as $\|P\|_{\infty \to \infty} = \sup_{\|u\|_\infty = 1} \|Pu\|_\infty$. We often identify a $Q$-value function $\theta$ with its matrix representation and use $\|\theta\|_\infty$ to denote the infinity norm (i.e., largest entry in absolute terms). In the matrix representation of $\theta$, its rows and columns are indexed via an enumeration of the states and actions, respectively. We use the symbol $\succeq$ to denote a relation that holds up to logarithmic factors in the problem parameters.

**2 Main results**

We proceed to provide precise statements of the main results of this paper, along with a discussion of some of their consequences. In Section 2.1, we define a notion of a local non-asymptotic minimax risk, and then state Theorem 1, which provides such a lower bound for estimating optimal $Q$-value functions. In Section 2.2, we turn to the complementary problem of deriving achievable results. Theorem 2 shows that under certain structural conditions on the policies, there is a form of variance-reduced $Q$-learning that achieves the local minimax risk up to logarithmic factors.
2.1 Instance-dependent lower bounds

In this section, we state a non-asymptotic lower bound for estimating optimal $Q$-function in the $\ell_{\infty}$-norm. This lower bound, to be stated in Theorem 1, is instance-dependent, meaning that it depends on the particular instance of the MDP $(P, r)$ at hand. This dependence should be contrasted with classical global minimax bounds, which are oblivious to such local properties.

The starting point of our lower bound development is the two-point framework introduced by Cai and Low [CL15] for local minimax bounds for nonparametric shape-constrained inference; here we adapt it to our current setting. Focusing on the $\ell_{\infty}$-norm error metric, the local non-asymptotic minimax risk for $\theta$ at an instance $P$ is defined as

$$M_N(P) := \sup_{P'} \inf_{\hat{\theta}} \max_{Q \in (P, P')} \sqrt{N} \cdot \mathbb{E}_Q \left[ \| \hat{\theta}_N - \theta(Q) \|_\infty \right].$$

Here the infimum is taken over all estimators $\hat{\theta}_N$ that are measurable functions of the $N$ i.i.d. observations drawn according to our observation model (see Section 1.1).

The intuition underlying the definition (7) is that given an instance $P$, the adversary that defines the instance-dependent non-asymptotic risk $M_N(P)$ behaves as follows: it extracts the hardest alternative $P'$ relative to $P$, and then measures the worst-case risk over $P$ and this alternative $P'$.

2.1.1 Lower bounds for $Q$-function estimation

We now turn to the statement of some lower bounds for estimating the optimal $Q$-function. Recall the definition (6) of the operator $P^\pi$, along with the functional $\nu(\pi; P, r, \gamma)$ from equation (5). We let $\nu^2(\pi; P, r, \gamma)$ denote the matrix obtained by taking squares entrywise. Our first step is to provide a decomposition of this matrix into two separate components, corresponding to the noisiness in the reward function observation and transition matrix observations, respectively.

In order to deal with the latter source of noise, with a slight abuse of notation, we use the observed matrix $Z$ to define a stochastic analog of $P^\pi$—namely, the (random) right-linear operator $(Z^\pi \theta)(x, u) = \sum_{x' \in X} Z_u(x' | x) \cdot \theta(x', \pi(x'))$, where $Z_u(x' | x) := 1_{Z(x,u)=x'}$. (8)

By assumption, the randomness in our observations of the reward and transitions are independent, so that for any optimal\footnote{Optimality of $\pi$ is required so that $T(\theta^*) = r + \gamma P^\pi \theta^*$, with a similar relation for the empirical Bellman operator.} policy $\pi$, we have the decomposition

$$\nu^2(\pi; P, r)(x, u) = \gamma^2 \rho^2(\pi; P, r)(x, u) + \sigma^2(\pi; P, r)(x, u).$$

Here we define

$$\rho^2(\pi; P, r) := \text{Var} \left( (I - \gamma P^\pi)^{-1} (Z^\pi - P^\pi) \theta^* \right),$$

and

$$\sigma^2(\pi; P, r) := \text{Var} \left( (I - \gamma P^\pi)^{-1} (R - r) \right),$$

where we compute the variances in an elementwise sense.
Theorem 1. There exists a universal constant \( c > 0 \) such that for any instance \( \mathcal{P} = (P, r) \), the local non-asymptotic minimax risk is lower bounded as

\[
\mathcal{M}_N(\mathcal{P}) \geq c \max_{\pi \in \Pi} \| \nu(\pi; P, r, \gamma) \|_{\infty}.
\] (10a)

This bound is valid for all sample sizes \( N \) that satisfy the lower bound

\[
N \geq N_0 := \max \left\{ \frac{2\gamma^2}{(1 - \gamma)^2}, \frac{2\|\theta^*\|_{\text{span}}^2}{(1 - \gamma)^2_\star^{\|P^*\|_{\infty}}} \right\},
\]

where \( \pi^* \in \arg \max_{\pi \in \Pi_\star} \| \nu(\pi; P, r, \gamma) \|_{\infty}. \)

We prove this theorem in Section 3. The main take-away is that the functional \( \max_{\pi \in \Pi_\star} \| \nu(\pi; P, r, \gamma) \|_{\infty} \) controls the local minimax risk. In order to gain intuition for this claim, it is worth exploring the range of possible behaviors exhibited by this functional.

2.1.2 Exploring the range of possible behaviors

One point of comparison is between the instance-dependent lower bound from Theorem 2 with the existing minimax lower bounds for \( Q \)-learning. Azar et al. [AMK13] provided a global minimax lower bound on the \( \ell_\infty \)-norm error for estimating the optimal \( Q \)-function. For a \( \gamma \)-discounted MDP, they showed that the \( \ell_\infty \)-error of any procedure is lower bounded by the quantity

\[
\frac{1}{1 - \gamma} \cdot \frac{1}{\sqrt{N}},
\]

up to logarithmic factors in dimension.

This lower bound is optimal in a globally minimax sense, and it is worthwhile understanding the properties of instances that exhibit this worst-case behavior—that is, instances for which \( \max_{\pi \in \Pi_\star} \| \nu(\pi; P, r, \gamma) \|_{\infty} \approx \frac{1}{(1 - \gamma)^{\gamma}} \). It is also worthwhile understanding the properties of problems that are much “easier” than this worst-case theory would suggest. The following construction, which takes inspiration from [PW20, KPR+20], allows us to explore this continuum.

Example 1 (A continuum of local minimax risks). Consider an MDP with two states \( \{x_1, x_2\} \), two actions \( \{u_1, u_2\} \), and with transition functions and reward functions given by

\[
P_{u_1} = \begin{bmatrix} p & 1 - p \\ 0 & 1 \end{bmatrix}, \quad P_{u_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad r = \begin{bmatrix} 1 & 0 \\ \tau & 0 \end{bmatrix}.
\]

(11)

We assume that there is no randomness in the rewards. Here, the pair \( (p, \tau) \) along with the discount factor \( \gamma \) are parameters of the construction, and we consider a sub-family of these parameters indexed by a scalar \( \lambda \geq 0 \). For any such \( \lambda \) and discount factor \( \gamma \in (\frac{1}{4}, 1) \), consider the settings

\[
p = \frac{4\gamma - 1}{3\gamma}, \quad \text{and} \quad \tau = 1 - (1 - \gamma)^{\lambda}.
\]

With these choices, the optimal \( Q \)-function \( \theta^* \) takes the form

\[
\theta^* = \begin{bmatrix} \frac{1}{4} \cdot \frac{3 + \tau}{1 - \gamma} & \frac{\gamma \cdot 3 + \tau}{1 - \gamma} \\ \frac{1}{1 - \gamma} & \frac{1}{1 - \gamma} \end{bmatrix}
\]
with an unique optimal policy \( \pi^*(x_1) = \pi^*(x_2) = u_1 \). We can then compute that
\[
\max_{\pi^* \in \Pi} \| \nu(\pi^*; P_\lambda, r_\lambda) \|_\infty = c \cdot \left( \frac{1}{1 - \gamma} \right)^{1.5 - \lambda}. \tag{12}
\]
See Appendix A for the details of this calculation.

Substituting into equation (10a) yields that the local minimax risk is lower bounded as
\[
\mathcal{R}_N(P) \geq c \frac{1}{(1 - \gamma^{1.5 - \lambda})}. \tag{13}
\]
Consequently, for \( \lambda > 0 \), our lower bounds suggest it should be possible to estimate the optimal \( Q \)-function more accurately by a factor \((1 - \gamma)^\lambda\); note that this difference is particularly significant for values of the discount factor \( \gamma \) that are close to one.

### 2.2 Instance-dependent upper bounds

Thus far, we have stated some instance-dependent lower bounds on the sample complexity of estimating \( Q \)-value functions. As we saw in the preceding Example 1, these lower bounds exhibit a wide range of behavior depending on the structure of the transition functions, discount parameter and reward functions. However, these differences in the lower bounds are only interesting if we can show that they are optimal, meaning that there is a (hopefully practical) algorithm that matches the behavior predicted by the lower bounds.

In this section we close this gap, in particular via a careful analysis of variance-reduced \( Q \)-learning (or VR-QL for short). Variance-reduced forms of \( Q \)-learning have been proposed and shown to be globally minimax in previous work [SWW+18, SWWY18, Wai19c]; the version analyzed here is motivated from [Wai19c]. In Theorem 2, we show that the VR-QL algorithm is instance-optimal up to logarithmic factors under two different sets of assumption.

#### 2.2.1 From standard to variance-reduced \( Q \)-learning

The classical \( Q \)-learning algorithm is a stochastic approximation algorithm for estimating the unique fixed point \( \theta^* \) of the Bellman operator \( T \). Recall the definition (4) of the empirical Bellman operator \( \hat{T}_k \). At each iteration \( k = 1, 2, \ldots \), standard \( Q \)-learning performs an update of the form
\[
\theta_{k+1} = (1 - \alpha_k) \theta_k + \alpha_k \hat{T}_k(\theta_k), \tag{13}
\]
where \( \alpha_k \in (0, 1) \) is a stepsize parameter. Appropriately decaying choices of the stepsize ensure that the estimate \( \theta_k \) converges to \( \theta^* \). Unfortunately, the convergence rate is known to be non-optimal, failing to achieve the global minimax rate [Wai19b], let alone the finer-grained instance-dependent requirements in this paper. This non-optimality has to do with the rate at which variance accumulates as the procedure is run.

Variance reduction is a general principle that can be applied to stochastic approximation schemes so as to accelerate their convergence. Here we describe the variance-reduced version of \( Q \)-learning that we analyze here. Similar to standard variance-reduced schemes for stochastic optimization [see, e.g., JZ13], the algorithm consists of a sequence of epochs. Within each epoch, we run a re-centered version of the \( Q \) update. The re-centering is done in such a way, using a Monte Carlo approximation of the population Bellman operator \( T \), so that the re-centered updates have lower variance. We leave the details of the epochs and Monte Carlo to Section 2.2.4; here let us describe the basic form of the updates within a given epoch.

Suppose that we run the algorithm using a total of \( M \) epochs. At epoch \( m \), the algorithm uses a re-centering point \( \bar{\theta}_m \) in order to re-center the update, where \( \bar{\theta}_m \) acts as the current
best estimate of $\theta^*$. Ideally, we should re-center the operator $\hat{T}_k$ using the quantity $T(\bar{\theta}_m)$, but we lack the access to it; instead, we use the Monte Carlo approximation

$$T_{N_m}(\bar{\theta}_m) := \frac{1}{N_m} \sum_{i \in D_m} \hat{T}_i(\bar{\theta}_m).$$

(14)

Given the pair $(\bar{\theta}_m, T_{N_m}(\theta))$ and a stepsize parameter $\alpha \in (0, 1)$, we define the variance-reduced Q-learning update as follows:

$$\theta \mapsto \mathcal{V}_k(\theta; \alpha, \bar{\theta}_m, T_{N_m}) := (1 - \alpha)\theta + \alpha \left\{ \hat{T}_k(\theta) - \hat{T}_k(\bar{\theta}_m) + T_{N_m}(\bar{\theta}_m) \right\},$$

(15)

where the operator $\hat{T}_k$ is independent of the set of operators $\{\hat{T}_i\}_{i \in D_m}$ used to compute the Monte Carlo approximation $T_{N_m}$. As a result, the stochastic operator $\hat{T}_k$ is independent of the re-centering quantity $T_{N_m}(\bar{\theta}_m)$. See Section 2.2.4 for the details on how the epoch lengths and re-centering sample sizes $D_m$ are chosen.

### 2.2.2 Non-asymptotic guarantees for variance-reduced Q-learning

In this section, we state some non-asymptotic guarantees for the VR-QL algorithm. We provide guarantees under two conditions, both of which involve the structure of the set of optimal policies. We begin by introducing some definitions that underlie these two conditions.

Given an MDP instance $(P, r)$, we define the optimality gap

$$\Delta := \min_{\pi \in \Pi_{\text{opt}}} \|\theta^* - \{r + \gamma P^{\pi} \theta^*\}\|_{\infty},$$

(16)

where $\theta^*$, $\Pi^*$, and $\Pi$, respectively, denote the optimal Q-function, the set of optimal policies, and the set of all policies for MDP $(P, r)$. Observe that the scalar $\Delta$ captures the difficulty in detecting the set of optimal policies. In other words, when $\Delta$ is small, it is hard to distinguish an optimal policy from a suboptimal policy.

Our second set of conditions involves the family of right-linear operators $\{P^{\pi} : \pi \in \Pi\}$ defined in equation (6). For any Q-value function $\theta$, we say that a policy $\pi$ is greedy with respect to $\theta$ if $\pi(x) \in \arg\max_{u \in \mathcal{U}} \theta(x, u)$ for all $x \in \mathcal{X}$. Note that any policy $\pi^*$ that is greedy with respect to the optimal Q-value function $\theta^*$ is an optimal policy. We say that the operators satisfy a Lipschitz condition if there is a constant $L$ such that for any Q-value function $\theta$ and associated greedy-optimal policy $\pi$, we have

$$\| (P^{\pi} - P^{\pi^*}) (\theta - \theta^*) \|_{\infty} \leq L \|\theta - \theta^*\|_{\infty}^2. $$

(17)

Intuitively, this condition means that the operator difference $P^{\pi} - P^{\pi^*}$ is small whenever the underlying Q-value functions that induce the policies are close. Conditions of this type were introduced by Puterman and Brumelle [PB79] in their classical analysis of the convergence rates of policy iteration algorithms for exact dynamic programming.

With these definitions in place, we can specify the two settings under which we provide upper bounds on the VR-QL algorithm:

**Setting UNQ:** There is a unique optimal policy $\pi^*$, and the sample size $N$ is lower bounded as

$$\frac{N}{(\log N)^2} \geq c_2 \log(D/\delta) \cdot \frac{1 + \|r\|_{\infty} + \sigma r \sqrt{T - \gamma})^2}{(1 - \gamma)^3} \cdot \max\{1, \frac{1}{\Delta^2 (1 - \gamma)^3}\} \quad \text{for some } \beta > 0.$$

(18)
Setting LIP: The Lipschitz condition (17) condition holds, and the sample size is lower bounded as

$$\frac{N}{(\log N)^2} \geq c_2 \log(D/\delta) \frac{(1 + \|r\|_\infty + \sigma_\xi \sqrt{1 - \gamma})^2}{(1 - \gamma)^{3 + \beta}} \cdot \min\{1, \frac{L^2}{(1 - \gamma)\Delta^2}\}$$

for some $\beta > 0$. (19)

In all cases, we assume that we are given an initial point $\hat{\theta}_1$ such that

$$\|\hat{\theta}_1 - \theta^*\|_\infty \leq \frac{\|r\|_\infty}{\sqrt{1 - \gamma}}.$$

Such an initial condition has already been used in the literature [Wai19c], and it can be ensured by first running Algorithm VR-QL for a total of $\frac{1}{(1 - \gamma)^3}$ samples (up to logarithmic factor corrections).

**Theorem 2.** Under either settings (UNQ) or (LIP), there are choices of epoch parameters such that given any discount parameter $\gamma \in [\frac{1}{2}, 1)$ and an initial point $\hat{\theta}_1$ satisfying the initialization condition (20), Algorithm VR-QL run for $M : = \log_4 \left( \frac{N(1 - \gamma)^2}{8 \log(16D/\delta) \log N} \right)$ epochs yields an estimate $\hat{\theta}_{M+1}$ such that

$$\|\hat{\theta}_{M+1} - \theta^*\|_\infty \leq c_0 \cdot \sqrt{\frac{\log_4(8DM/\delta)}{N}} \cdot \max_{\pi^* \in \Pi^*} \|\nu(\pi^*; P, r, \gamma)\|_\infty + c_1 \cdot \frac{\log_4(8DM/\delta)}{N} \cdot \frac{\|\theta^*\|_{\text{span}}}{1 - \gamma},$$

with probability exceeding $1 - \delta$. (21)

See Section 4 for the proof of this claim.

**Comparing the upper and lower bounds:** Assuming the sample size lower bound from Theorem 1 are valid, we see that the second term in the bound (21) is of smaller order. In this case, the upper bound from Theorem 2 and the lower bound from Theorem 1 matches, and we conclude that the VR-QL algorithm is instance optimal.

Although the guarantee (21) involves the same $1/\sqrt{N}$ rate and complexity term $\max_{\pi^* \in \Pi^*} \|\nu(\pi^*; P, r, \gamma)\|_\infty$ as the lower bound in Theorem 1, it should be noted that the sample size lower bounds required for Theorem 2 are more stringent than that in Theorem 1. Moreover, our lower bound does not require the side conditions—either the unique optima or Lipschitz conditions—that are imposed in Theorem 2. Closing these remaining differences between the two results is a worthwhile goal for future work.

### 2.2.3 Confirming the theoretical predictions

Some numerical experiments are helpful in order to illustrate instance-adaptive behavior guaranteed by Theorem 2. Recall the family of MDPs (11) from Example 1. Suppose that we set $\lambda = 0.5$ and for each choice of $\gamma \in (1/2, 1)$, we collect $N = \left\lceil \frac{16/32}{9} \frac{1}{(1 - \gamma)^3} \right\rceil$ samples, and then run the VR-QL algorithm over a range of discount parameters $\gamma$, using the settings from Theorem 2 and Section 2.2.4, thereby obtaining an estimate $\hat{\theta}_{M+1}$.

Figure 1(a) plots the evolution of $\log \ell_\infty$-norm error of the estimate over time as the algorithm proceeds; the form of these curves show the epoch-based nature of the convergence. See
Section 2.2.4 for more details on the parameters of the epochs, including the base parameter illustrated here. Plotted as blue circles in panel (b) of Figure 1 are the logarithm of the $\ell_\infty$-norm error of the final output; that is, $\log \| \hat{\theta}_{M+1} - \theta^* \|_\infty$, versus the logarithm of the discount complexity, $\log(1/(1 - \gamma))$. Each point in this plot represents an average over 1000 trials.

In terms of theory, with the settings given above, existing worst-case bounds [AMK13, Wai19c] predict that the log $\ell_\infty$-norm error remains constant as the log discount complexity grows; accordingly, we have plotted a dotted red line with slope zero to illustrate the worst-case guarantee. On the other hand, for the MDP instance (11) with $\lambda = 0.5$, a simple calculation yields that for the instance (11) the suboptimality gap $\Delta$ satisfies $\Delta = 1 - \frac{(1-\gamma)^{\lambda}}{4} \geq \frac{3}{4}$. In our experiment, we set the sample size to be $N = \left\lceil \frac{32}{(1-\gamma)^{\lambda} \cdot \frac{4^3}{3^2}} \right\rceil \geq \frac{32}{(1-\gamma)^{\lambda} \cdot 2^2}$; as a result, the bounds from Theorems 1 and 2 are valid.

![Figure 1](image-url)

**Figure 1.** (a) $\lambda = 0.5$, $N = \left\lceil \frac{32}{(1-\gamma)^{\lambda} \cdot \frac{4^3}{3^2}} \right\rceil \gamma = 0.9$. Illustration of the qualitative behavior of Algorithm VR-QL applied on the MDP (1) along with instance dependent and the worst case bounds. The figure plots the log $\ell_\infty$-error $\| \hat{\theta}_{M+1} - \theta^* \|_\infty$ against the log discount complexity factor $\log(\frac{1}{1-\gamma})$ with $\lambda = 0.5$. We have also plotted the least-squares fit through these points, and the instance-dependent lower bound from Theorem 1, the instance-dependent upper bound from Theorem 2, and the worst-case bound [Wai19c]. (b) Behavior of the VR-QL algorithm with different choices of the base $b$. The plot demonstrates that different choices of the base $b$ yield similar behavior.

With the setting $\lambda = 0.5$, our calculations from Example 1 yield

$$\max_{\pi^* \in \Pi^*} \| \nu(\pi^*; P, r, \gamma) \|_\infty \approx \left( \frac{1}{1-\gamma} \right)^{-0.5}.$$ 

Thus, with the choice of sample size $N$ given above, our theory predicts that the log $\ell_\infty$-norm error should exhibit the scaling

$$\log \| \hat{\theta}_{M+1} - \theta^* \|_\infty \approx \log \left( \frac{1}{\sqrt{N}} \max_{\pi^* \in \Pi^*} \| \nu(\pi^*; P, r, \gamma) \|_\infty \right) \approx c - 0.5 \log \left( \frac{1}{1-\gamma} \right),$$

where $c$ is a constant. In Figure 1(b), we plot the lower bound from Theorem 1 as a solid red line, and the upper bound from Theorem 2 as a dashed green line. (While these lines both
have slope $-0.50$, the intercept term $c$ is different due to the additional logarithmic factors in dimension present in the upper bound.

In order to test how the empirical behavior conforms to the theoretical prediction, we did an ordinary least-squares fit of the log $\ell_\infty$-norm error versus the log discount complexity; this fit yields a line with slope $\beta = -0.45$, and is plotted in solid blue. This test shows good agreement between the theoretical prediction and the practical behavior.

### 2.2.4 Details of the epochs and procedure

In this section, we provide the complete details of the algorithm used in our version of variance-reduced $Q$-learning.

**A single epoch:** A single epoch of the overall variance-reduced QL algorithm involves repeated applications of the basic variance-reduced update $V_k$ from equation (15). The epochs are indexed with integers $m = 1, 2, \ldots, M$, where $M$ corresponds to the total number of epochs to be run. Each epoch $m$ requires the following four inputs:

- an element $\theta$, which is chosen to be the output of the previous epoch $m - 1$;
- a positive integer $K$ denoting the number of steps within the given epoch;
- a positive integer $N_m$ denoting the batch size used to calculate the Monte Carlo update (14);
- a set of fresh operators $\{\hat{T}_i\}_{i \in \mathcal{C}_m}$, with $|\mathcal{C}_m| = N_m + K$. The set $\mathcal{C}_m$ is partitioned into two subsets having sizes $N_m$ and $K$, respectively. The first subset, of size $N_m$, which we call $\mathcal{D}_m$, is used to construct the Monte Carlo approximation (14). The second subset, of size $K$ is used to run the $K$ steps within the epoch.

We summarize a single epoch in pseudocode form in Algorithm SingleEpoch.

| Algorithm SingleEpoch | RunEpoch ($\theta; K, N_m, \{\hat{T}_i\}_{i \in \mathcal{C}_m}$) |
|-----------------------|--------------------------------------------------|
| 1: Given (a) Epoch length $K$, (b) Re-centering vector $\theta$, (c) Re-centering batch size $N_m$, (d) Operators $\{\hat{T}_i\}_{i \in \mathcal{C}_m}$ | |
| 2: Compute the re-centering quantity $\mathbf{T}_{N_m}(\theta) := \frac{1}{N_m} \sum_{i \in \mathcal{D}_m} \hat{T}_i(\theta)$ | |
| 3: Initialize $\theta_1 = \theta$ | |
| 4: for $k = 1, 2, \ldots, K$ do | |
| 5: Compute the variance-reduced update: $\theta_{k+1} = V_k(\theta_k; \alpha_k, \theta, \mathbf{T}_{N_m})$ with stepsize $\alpha_k = \frac{1}{1 + (1 - \gamma)k}$. | |
| 6: end for | |
| 7: return $\theta_{K+1}$ | |

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**Overall algorithm:** The overall algorithm, denoted by VR-QL for short, has five inputs: (a) an initialization $\theta_1$, (b) an integer $M$, denoting the number of epochs to be run, (c) an integer $K$, denoting the length of each epoch, (d) a sequence of batch sizes $\{N_m\}_{m=1}^M$, denoting the number of operators used for re-centering in the $M$ epochs, and (e) sample batches $\{\hat{T}_i\}_{i \in \mathcal{C}_m}^M$ to be used in the $M$ epochs. Given these five inputs, the overall procedure can be summarized as in Algorithm VR-QL.

**Algorithm VR-QL**

1: Given (a) Initialization $\theta_1$, (b) Number of epochs, $M$, (c) Epoch length $K$, (d) Re-centering sample sizes $\{N_m\}_{m=1}^M$, (e) Sample batches $\{\hat{T}_i\}_{i \in \mathcal{C}_m}$ for $m = 1, \ldots, M$
2: Initialize at $\theta_1$
3: for $m = 1, 2, \ldots, M$ do
4: $\theta_{m+1} = \text{RunEpoch}(\theta_m; K, N_m, \{\hat{T}_i\}_{i \in \mathcal{C}_m})$
5: end for
6: return $\theta_{M+1}$ as final estimate

**Settings for Theorem 2:** Given a tolerance probability $\delta \in (0, 1)$ and the number of available i.i.d. samples $N$, we run Algorithm VR-QL with a total of $M : = \log_4 \left( \frac{N(1-\gamma)^2}{8 \log((16D/\delta) \log N)} \right)$ epochs, along with the following parameter choices:

- **Re-centering sizes:**
  \[ N_m = c_1 \frac{4^m}{(1-\gamma)^2} \cdot \log_4(16MD/\delta) \] (22a)

- **Sample batches:**
  Partition the $N$ samples to obtain $\{\hat{T}_i\}_{i \in \mathcal{C}_m}$ for $m = 1, \ldots, M$ (22b)

- **Epoch length:**
  \[ K = \frac{N}{2M}. \] (22c)

3 **Proof of Theorem 1**

Given an MDP instance $\mathcal{P} = (\mathcal{P}, r)$, we start by introducing the following two classes of alternative MDPs:

- $\mathcal{S}_1 = \{\mathcal{P}' = (\mathcal{P}', r') \mid r' = r\}$, and $\mathcal{S}_2 = \{\mathcal{P}' = (\mathcal{P}', r') \mid \mathcal{P}' = \mathcal{P}\}$. (23)

We consider the restricted version of the local minimax risk at the instance $\mathcal{P}'$ to the classes $\mathcal{S}_i$:

\[ \mathbb{M}_N(\mathcal{P}; \mathcal{S}_i) : = \sup_{\mathcal{P}' \in \mathcal{S}_i} \max_{\theta_N, \mathcal{Q} \subset (\mathcal{P}, \mathcal{P}')} \mathbb{E}_{\mathcal{P}}[\sqrt{N} \| \theta_N - \theta(\mathcal{Q}) \|_\infty], \quad i = 1, 2. \] (24)

The main part of the proof involves showing that there exists a universal constant $c > 0$ such that

\[ \mathbb{M}_N(\mathcal{P}; \mathcal{S}_1) \geq c \cdot \max_{\pi \in \Pi} \| \gamma \rho(\pi; \mathcal{P}, r) \|_\infty, \quad \text{and} \]

\[ \mathbb{M}_N(\mathcal{P}; \mathcal{S}_2) \geq c \cdot \max_{\pi \in \Pi} \| \sigma(\pi; \mathcal{P}, r) \|_\infty, \] (25a) (25b)
where \( \Pi^* \) denotes the optimal policy set for \((P, r)\). We can then conclude

\[
\mathfrak{M}_N(P) \geq \max\{\mathfrak{M}_N(P; S_1), \mathfrak{M}_N(P; S_2)\}
\]

\[
\geq \frac{1}{2} (\mathfrak{M}_N(P; S_1) + \mathfrak{M}_N(P; S_2))
\]

\[
\geq \frac{c}{2} \max_{\pi \in \Pi^*} \|\gamma \rho_0(\pi; P, r)\|_\infty + \frac{c}{2} \max_{\pi \in \Pi^*} \|\sigma(\pi; P, r)\|_\infty
\]

\[
\geq \frac{c}{2} \max_{\pi \in \Pi^*} \|\nu(\pi; P, r)\|_\infty.
\]

The last inequality above follows from the decomposition (9a). It remains to prove the claims (25a) and (25b). More precisely, the core of our proof involves proving the following two lemmas:

**Lemma 1.** For all \( S \in \{S_1, S_2\} \), we have that \( \mathfrak{M}_N(P; S) \geq \frac{1}{2} \mathfrak{M}_N(P; S) \) where we define

\[
\mathfrak{M}_N(P; S) := \sup_{P \in S} \left\{ \sqrt{N} \cdot \|\theta(P) - \theta(P')\|_\infty \mid d_{\text{Hel}}(P, P') \leq \frac{1}{2\sqrt{N}} \right\}.
\]

This lemma follows as a fairly straightforward consequence of the standard reduction from estimation to testing; see Appendix B.1 for the details.

Our next lemma requires more effort to prove, and leverages the specific structure of the problem at hand:

**Lemma 2.** Given any MDP instance \( P = (P, r) \):

(a) There exists an instance \( P_1 = (P', r) \in S_1 \) such that \( d_{\text{Hel}}(P, P_1) \leq \frac{1}{2\sqrt{N}} \) and

\[
\sqrt{N} \cdot \|\theta(P) - \theta(P_1)\|_\infty \geq c \cdot \max_{\pi \in \Pi^*} \|\gamma \rho_0(\pi; P, r)\|_\infty.
\]

(b) There exists an instance \( P_2 = (P, r') \in S_2 \) such that \( d_{\text{Hel}}(P, P_2) \leq \frac{1}{2\sqrt{N}} \) and

\[
\sqrt{N} \cdot \|\theta(P) - \theta(P_2)\|_\infty \geq c \cdot \max_{\pi \in \Pi^*} \|\sigma(\pi; P, r)\|_\infty.
\]

Note that the bounds (25a)–(25b) stated in Theorem 1 follow by combining the claims of Lemmas 1 and 2. The remainder of our proof focuses on establishing Lemma 2.

### 3.1 Proof of Lemma 2

In this section, we prove the two parts of Lemma 2.

#### 3.1.1 Proof of Lemma 2(a)

Throughout the proof, we use \( z \) to denote a generic element of the state-action set \( \mathcal{X} \times \mathcal{U} \). Let \( \theta \) be the true Q-function for the MDP \( P = (P, r) \). We adopt the shorthands

\[
\pi_1 \in \arg\max_{\pi \in \Pi^*} \|\rho(\pi; P, r)\|_\infty, \quad \bar{z} \in \arg\max_{z \in \mathcal{X} \times \mathcal{U}} \rho(\pi_1; P, r), \quad \bar{\rho}(z) := \rho(\pi_1; P, r)(z),
\]

\[
U := (I - \gamma P^{\pi_1})^{-1} \quad \text{and} \quad \varphi^2(z) := \text{Var}(Z^{\pi_1}(z)).
\]
Additionally, note that $U$ is a linear transformation from $\mathbb{R}^{|X| \times |U|}$ to itself, so we can express the action of $U$ on $\theta$ as

$$(U\theta)(z) = \sum_{z' \in X \times U} U_{z,z'} \theta(z').$$

Note moreover that

$$\varphi^2(z) = \sum_{x'} P_{x',z}(\theta(x', \pi_1(x')) - (P^{\pi_1} \theta)(z))^2 \quad \text{and} \quad \hat{\rho}^2(z) = \sum_{z'} (U_{z,z'})^2 \varphi^2(z'). \quad (27)$$

With these definitions, we now define $\bar{P}_{y,z}$ as follows (we will prove that this choice is a valid probability transition kernel shortly):

$$\bar{P}_{y,z} = P_{y,z} + \frac{1}{\hat{\rho}(z)\sqrt{2N}} P_{y,z} U_{y,z} \cdot (\theta(y, \pi_1(y)) - (P^{\pi_1} \theta)(z)). \quad (28)$$

Here, we have used the shorthand $P_{y,z} = P_{u}(y \mid x)$, where $z = (x, u) \in X \times U$. Let $\theta := \theta(P, r)$, and $\bar{\theta} := \theta(\bar{P}, r)$ be the optimal $Q$ functions for MDP instances $(P, r)$ and $(\bar{P}, r)$ respectively. In the rest of the proof, we use the following properties of $\bar{P}$.

**Lemma 3.** For any MDP $P = (P, r)$ and the optimal policy $\pi_1$ defined in equation (26), the corresponding $\bar{P}$ has the following properties:

(a) The $\bar{P}$ is a probability transition kernel.

(b) The MDP instances $P_i = (P, r)$ and $P_1 = (\bar{P}, r)$ satisfy $d_{Hel}(P, P_1) \leq \frac{1}{2\sqrt{N}}$ and $\|P^{\pi_1} - \bar{P}^{\pi_1}\|_{\infty} \leq \frac{1}{\sqrt{2N}}$.

(c) Each entry of $(I - \gamma \bar{P}^{\pi_1})^{-1}[\bar{P}^{\pi_1} - P^{\pi_1}]\theta$ is non-negative.

See Appendix B.2 for a proof of this lemma.

Equipped with these tools, we are now ready to lower bound the norm $\|\theta - \bar{\theta}\|_{\infty}$. The optimal $Q$-functions $\theta$ and $\bar{\theta}$ satisfy the following Bellman equations:

$$\theta = r + \gamma P^{\pi_1} \theta \quad \text{and} \quad \bar{\theta} = r + \gamma \bar{P}^{\pi_1} \bar{\theta}, \quad (29)$$

where $\pi_1 \in \Pi^*$ is the optimal policy that achieves $\max_{\pi \in \Pi^*} \|\rho(\pi; P, r)\|_{\infty}$, and $\bar{\pi}$ is an optimal policy for $(\bar{P}, r)$. By the optimality of policy $\bar{\pi}$ and the $Q$-function $\bar{\theta}$, we have the entrywise inequality $\bar{P}^{\pi_1} \bar{\theta} \geq P^{\pi_1} \bar{\theta}$, which implies $(I - \gamma \bar{P}^{\pi_1}) \bar{\theta} \geq (I - \gamma \bar{P}^{\pi_1}) \theta = r$. Thus, using the identity $A^{-1}_1 - A^{-1}_0 = A^{-1}_1 (A_0 - A_1) A^{-1}_0$ for invertible operators $A_0$ and $A_1$, we have

$$\bar{\theta} - \theta \geq \left[ (I - \gamma \bar{P}^{\pi_1})^{-1} - (I - \gamma P^{\pi_1})^{-1} \right] r$$

$$= (I - \gamma \bar{P}^{\pi_1})^{-1} \left[ (I - \gamma P^{\pi_1}) - (I - \gamma \bar{P}^{\pi_1}) \right] (I - \gamma P^{\pi_1})^{-1} r$$

$$= \gamma (I - \gamma P^{\pi_1})^{-1} \left[ P^{\pi_1} - \bar{P}^{\pi_1} \right] (I - \gamma P^{\pi_1})^{-1} r$$

$$+ \gamma (I - \gamma P^{\pi_1})^{-1} \left[ \bar{P}^{\pi_1} - P^{\pi_1} \right] \theta$$

$$= \gamma (I - \gamma P^{\pi_1})^{-1} \left[ \bar{P}^{\pi_1} - P^{\pi_1} \right] \theta$$

$$+ \gamma (I - \gamma P^{\pi_1})^{-1} (I - \gamma \bar{P}^{\pi_1})^{-1} (\bar{P}^{\pi_1} - P^{\pi_1}) \theta,$$
where the final equation follows from the Bellman optimality condition (29). Lemma 3(c) guarantees that the entries of \((I - \gamma \Pi^g)^{-1} [\Pi^g - \Pi^g] \theta\) are non-negative, and therefore we conclude
\[
\|\tilde{\theta} - \theta\|_\infty \geq \gamma \| (I - \gamma \Pi^g)^{-1} [\Pi^g - \Pi^g] \theta \|_\infty - \gamma \| (I - \gamma \Pi^g)^{-1} - (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty. \tag{30}
\]

Consider the second term \(T_2 := \| (I - \gamma \Pi^g)^{-1} - (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty\). We have
\[
T_2 = \|(I - \gamma \Pi^g)^{-1} (I - \gamma \Pi^g) - I\|_\infty \cdot \|(I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty \\
\leq \| (I - \gamma \Pi^g)^{-1} (I - \gamma \Pi^g) - I \|_\infty \cdot \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty \\
= \gamma \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \|_\infty \cdot \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty \\
\leq \frac{\gamma}{1 - \gamma} \| \Pi^g - \Pi^g \|_\infty \cdot \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty \\
\leq \frac{\gamma}{2} \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty,
\]
where the last inequality uses Lemma 3(b) and the first part of the minimum sample size assumption (10b). Combining this result with the bound (30) we conclude
\[
\|\tilde{\theta} - \theta\|_\infty \geq \frac{\gamma}{2} \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta \|_\infty.
\]

With this result in hand, substituting the value of the transition kernel \(\Pi\) from equation (28) and recalling the definition of state-action pair \(z\) from equation (26) we have
\[
\sqrt{N} \cdot \|\tilde{\theta} - \theta\|_\infty \geq \frac{\gamma \sqrt{N}}{2} \cdot \| (I - \gamma \Pi^g)^{-1} (\Pi^g - \Pi^g) \theta(z) \geq (i) \gamma \frac{\sqrt{N}}{4 \tilde{\rho}(z)} \sum_z \| U_{z,z} \|_{\notin} (\Pi^g - \Pi^g) \theta(z) \\
\geq (ii) \frac{\gamma \tilde{\rho}(z)}{4} = \frac{1}{4} \max_{\pi \in \Pi^g} \| \gamma \rho(\pi; \Pi, r) \|_\infty,
\]
where step (i) follows by substituting the value of the transition kernel \(\Pi\) (cf. Proof of Lemma 3 part (c)), and step (ii) follows by using the expression (27). This completes the proof of part (a) of Lemma 2.

### 3.1.2 Proof of Lemma 2(b)

Borrowing the notation from part (a) of the proof, let \(z\) denote a generic element of the state-action set \(\mathcal{X} \times \mathcal{U}\). Let \(\pi_2 \in \arg\max_{\pi \in \Pi^g} \| \sigma(\pi; \Pi, r) \|_\infty\). We use the shorthands
\[
\sigma^2(z) := \max_{\pi \in \Pi^g} \| \sigma(\pi; \Pi, r) \|_\infty^2 = \| \sigma(\pi_2; \Pi, r) \|_\infty^2, \quad \text{and} \quad U := (I - \gamma \Pi^g)^{-1} \tag{31}
\]
We define our perturbed reward function to be
\[
\tilde{r}(z) = r(z) + \frac{1}{\sigma(z) \sqrt{2N}} U_{z,z} \sigma^2 \quad \text{for} \quad z \in \mathcal{X} \times \mathcal{U}. \tag{32}
\]
For $\mathcal{P}_2 := (\mathcal{P}, \bar{r})$, a short computation shows that the Hellinger distance between the components of the instance pair $(\mathcal{P}, \mathcal{P}_2)$ takes the form

$$d_{\text{Hel}} (\mathcal{P}, \mathcal{P}_2)^2 \leq D_{KL}(\mathcal{N}(r, \sigma_r^2 \mathbf{I}) \mid \mathcal{N}(\bar{r}, \sigma_r^2 \mathbf{I})) = \frac{1}{2\sigma_r^2} \| r - \bar{r} \|_2^2.$$ 

Substituting the value of the reward $\bar{r}$ from equation (32) yields

$$d_{\text{Hel}} (\mathcal{P}, \mathcal{P}_2)^2 \leq \frac{1}{2\sigma_r^2} \| \bar{r} - r \|_2^2 = \frac{1}{\sigma^2(\bar{z})} \cdot 4N \sum_z (U_{\bar{z}, z})^2 \sigma_r^2 = \frac{1}{4N},$$

where the last equality uses the definition of the term $\sigma^2(\bar{z})$, i.e.,

$$\sigma^2(\bar{z}) = \sum_{z'} (U_{\bar{z}, z'})^2 \sigma_r^2.$$ \hspace{1cm} (33)

It remains to prove a lower bound on the $\ell_\infty$-norm between the optimal $Q$-functions for instances $\mathcal{P}$ and $\mathcal{P}_2$.

Let $\theta := \theta(\mathcal{P}, r)$, and $\bar{\theta} := \theta(\mathcal{P}, \bar{r})$ be the optimal $Q$ functions for MDP instances $\mathcal{P} := (\mathcal{P}, r)$ and $\mathcal{P}_2 := (\mathcal{P}, \bar{r})$, respectively. Note that $\theta$ and $\bar{\theta}$ satisfy the Bellman equations

$$\theta = r + \gamma \mathbf{P}^{\pi_\theta} \theta, \quad \text{and} \quad \bar{\theta} = \bar{r} + \gamma \mathbf{P}^{\bar{\pi}} \bar{\theta},$$ \hspace{1cm} (34)

where $\bar{\pi}$ is an optimal policy for the MDP instance $(\mathcal{P}, \bar{r})$. By the optimality of policy $\bar{\pi}$, we have the entrywise inequality $\mathbf{P}^{\bar{\pi}} \theta \geq \mathbf{P}^{\pi_\theta} \theta$; as a result, we have

$$(\mathbf{I} - \gamma \mathbf{P}^{\pi_\theta}) \bar{\theta} \geq \bar{r} \implies \bar{\theta} \geq (\mathbf{I} - \gamma \mathbf{P}^{\pi_\theta})^{-1} \bar{r},$$

where the last step uses the fact that $(\mathbf{I} - \gamma \mathbf{P}^{\pi_\theta})^{-1}$ is entry-wise non-negative. Combining the last inequality with the Bellman equation (34) we have that

$$\bar{\theta} - \theta \geq (\mathbf{I} - \gamma \mathbf{P}^{\pi_\theta})^{-1}(\bar{r} - r)$$ \hspace{1cm} (35)

and that

$$\|(\mathbf{I} - \gamma \mathbf{P}^{\pi_\theta})^{-1}(\bar{r} - r)\|_\infty \geq (\mathbf{I} - \gamma \mathbf{P}^{\pi_\theta})^{-1}(\bar{r} - r)(\bar{z}) = \frac{1}{\sigma(\bar{z})\sqrt{2N}} \sum_z (U_{\bar{z}, z})^2 \sigma_r^2 \sigma(\bar{z})^2$$

$$\geq \frac{\sigma(\bar{z})}{\sqrt{2N}}.$$ 

where the last equality uses the relation (33). Putting together the pieces, we have shown that

$$\|\bar{\theta} - \theta\|_\infty \geq \frac{\sigma(\bar{z})}{\sqrt{2N}} = \frac{1}{\sqrt{2N}} \cdot \max_{\pi \in \Pi^*} \sigma(\pi; \mathcal{P}, r)\|_\infty,$$

as desired.
4 Proof of Theorem 2

In the section, we provide a proof of the upper bounds stated in Theorem 2. Throughout the proof, we adopt the following shorthands

\[ \kappa = \frac{\|\theta^*\|_{\text{span}}}{(1-\gamma)} \cdot \log(8DM/\delta), \quad \tau^* = \max_{\pi^* \in \Pi^*} \|\nu(\pi^*; P, r)\|_{\infty} \cdot \sqrt{\log(8DM/\delta)}, \]

and

\[ \tau_{\text{max}} = \frac{1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma}}{(1-\gamma)^{1.5}} \cdot \sqrt{\log(8DM/\delta)}. \] (36)

4.1 Proof of Theorem 2(a)

Our proof is based on the following two lemmas characterizing the behavior of VR-QL across epochs.

**Lemma 4.** Under the assumptions of Theorem 2, for each epoch \( m = 1, \ldots, M \), we have

\[ \|\hat{\theta}_{m+1} - \theta^*\|_{\infty} \leq \|\hat{\theta}_m - \theta^*\|_{\infty} + c \left( \frac{\tau_{\text{max}}}{\sqrt{N_m}} + \frac{\kappa}{N_m} \right), \] (37)

with probability at least \( 1 - \frac{\delta}{M} \).

Lemma 4 follows by an argument similar to that used in the proof of Theorem 1 of the paper [Wai19c], so we omit the details here. See also the proof of Lemma 5 for some relevant arguments.

**Lemma 5.** Under the assumptions of Theorem 2, for epochs \( m \) such that the re-centering sample size \( N_m \) satisfies the bound

\[ N_m \geq \log_4(8DM/\delta) \left( \frac{(1+\|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma})^2}{\Delta^2(1-\gamma)^3} \right), \]

we have

\[ \|\hat{\theta}_{m+1} - \theta^*\|_{\infty} \leq \|\hat{\theta}_m - \theta^*\|_{\infty} + c \cdot \left( \frac{\tau^*}{\sqrt{N_m}} + \frac{\kappa}{N_m} \right), \] (38)

with probability at least \( 1 - \frac{\delta}{M} \).

See Section 4.1.2 for the proof of Lemma 5.

4.1.1 Completing the proof

Using the two lemmas above, we can now complete the proof of Theorem 2(a). Recalling the epoch sample size formula (22a), we see that the bound (38) holds for all epochs

\[ m \geq m^* := \log_2 \frac{1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma}}{\Delta \sqrt{1 - \gamma}}. \]
Observe that the minimum sample size requirement from Theorem 2 ensures that $M \geq m^*$. Now, applying the recursions (38) and (37) we obtain

\[
\| \tilde{\theta}_{M+1} - \theta^* \|_\infty \leq \| \tilde{\theta}_M - \theta^* \|_\infty + \frac{\tau^*}{\sqrt{N_M}} + \kappa \left( \sum_{k=0}^{M-m^*} \frac{\tau^*}{16^k \sqrt{N_{M-k}}} + \frac{\kappa}{16^k \sqrt{N_{M-k}}} \right)
\]

\[
= \left( \sum_{k=0}^{M-m^*} \frac{\tau^*}{16^k \sqrt{N_{M-k}}} + \frac{\kappa}{16^k \sqrt{N_{M-k}}} \right) + c \cdot \sum_{k=0}^{M} \frac{\kappa}{16^k N_{M-k}}.
\]

Inequality (i) follows via repeated application of the recursion (38), inequality (ii) follows via repeated application of the recursion (37), and inequality (iii) utilizes the relation $N_{M-k} \cdot 4^k = N_M$ (cf. definition (22a)). Via the union bound, the above inequalities hold simultaneously with probability at least $1 - \delta$. Next, note that by our choice of $N_m$, we have the inequality $2N_M \leq N \leq \frac{3}{2}N_M$. Putting together the pieces, we conclude that

\[
\| \tilde{\theta}_{M+1} - \theta^* \|_\infty \leq c \cdot \tilde{\theta}_1 - \theta^* \|_\infty + \frac{\log^2((8D/\delta) \cdot \log N)}{N^2(1 - \gamma)^4} + \frac{(1 + \| r \|_\infty + \sigma_r \sqrt{1 - \gamma})^4 \cdot \log^2((8D/\delta) \cdot \log N)}{(1 - \gamma)^{1.5} \sqrt{N}} \cdot \frac{\log(8DM/\delta)}{N} \cdot \frac{\log(8DM/\delta)}{N} \cdot \frac{\| \theta^* \|_{\text{span}}}{1 - \gamma}.
\]

Substituting the lower bound condition

\[
\frac{N}{\log^2(N)} \geq c \log(D/\delta) \left( \frac{1 + \| r \|_\infty + \sigma_r \sqrt{1 - \gamma})^2}{(1 - \gamma)^2} \right) \cdot \max \left\{ 1, \frac{1}{\Delta^2 \cdot (1 - \gamma)^3} \right\}
\]

yields the claimed bound. All that remains is to verify the choice of batch sizes $\{N_m\}_{m=1}^M$ is a valid choice, i.e., we need to verify that the algorithm VR-QL with parameter choices (22) uses at most $N$ samples. Recall that the total number of samples used in the $M$ epochs is given by $KM + \sum_{m=1}^{M} N_m$. Substituting the values of $N_m$ and $M$ from equations (22) we obtain

\[
KM + \sum_{m=1}^{M} N_m \leq c \cdot \log(8DM/\delta) \cdot \sum_{m=1}^{M} \frac{4^m}{(1 - \gamma)^2} + \frac{N}{2} \leq c' \cdot \log(8DM/\delta) \cdot \frac{4^M}{(1 - \gamma)^2} \leq \frac{N}{2} + \frac{N}{2} \leq N.
\]

This completes the proof of Theorem 2(a).
Comment on the lower-order terms: Here, we argue that the first two terms in the right-hand side of the bound (39) are of lower order. A careful look at the proof reveals that for any $p \geq 1$ by increasing our choice of $N_m$ by a constant factor depending on $p$, we can bound the first term by

$$c_1 \cdot \frac{\|\tilde{\theta}_1 - \theta^*\|_\infty}{N^p} \cdot \frac{\log^p((8D/\delta) \cdot log N)}{(1 - \gamma)^{2p}},$$

and the second term by

$$c_2 \cdot \frac{(1 + r\|\|_\infty + \sigma r \sqrt{1 - \gamma})^{3q+1}}{(1 - \gamma)^{1.5} \sqrt{N}} \cdot \frac{\log^{2q}((8D/\delta) \cdot log N)}{N^{3q/2}(1 - \gamma)^{\frac{9q}{2} \Delta^{3q}}},$$

where $q = \frac{2}{3} - \frac{1}{7}$, and $(c_1, c_2)$ are universal constants only depending on $(p, q)$. The number of samples satisfies $N \geq \frac{(1 + r\|\|_\infty + \sigma r \sqrt{1 - \gamma})^2}{\Delta^2(1 - \gamma)^{3q+3}}$ by assumption, and consequently, the two terms can be made arbitrarily small by increasing $(p, q)$ appropriately. The equation (39) displays the bound for the pair $(p, q) = (2, 1)$.

The only remaining detail is to prove Lemma 5.

4.1.2 Proof of Lemma 5

Recall that the update within an epoch takes the form (cf. SingleEpoch)

$$\theta_{k+1} = (1 - \alpha_k)\theta_k + \alpha_k \{ \hat{T}_k(\theta) - \hat{T}_k(\hat{\theta}_m) + T_{N_m}(\hat{\theta}_m) \},$$

where $\hat{\theta}_m$ represents the input into epoch $m$. We define the shifted operators and noisy shifted operators for epoch $m$ by

$$J(\theta) = T(\theta) - T(\hat{\theta}_m) + T_{N_m}(\hat{\theta}_m) \quad \text{and} \quad \hat{J}_k(\theta) = \hat{T}_k(\theta) - \hat{T}_k(\hat{\theta}_m) + T_{N_m}(\hat{\theta}_m). \quad (40)$$

Since both of the operators $T$ and $\hat{T}_k$ are $\gamma$-contractive in the $\ell_\infty$-norm, the operators $J$ and $\hat{J}_k$ are also $\gamma$-contractive operators in the same norm. Let $\hat{\theta}_m$ denote the unique fixed point of the operator $J$. The roadmap of the proof is to show that at the end of epoch $m$, the estimate $\theta_{K+1}$ is close to the fixed point $\hat{\theta}_m$ for sufficiently large value of the epoch length $K$ and that the fixed point $\hat{\theta}_m$ is closer to $\theta^*$ than the epoch initialization $\hat{\theta}_m$ for sufficiently large $N_m$.

The proof of Lemma 5 relies on two auxiliary lemmas that formalize this intuition. Lemma 6 characterizes the progress of Algorithm VR-QL within an epoch, and Lemma 7 addresses the progress of Algorithm VR-QL over the epochs.

Lemma 6. Given an epoch length $K$ lower bounded as $K \geq c_2 \frac{\log(ND/\delta)}{(1 - \gamma)^3}$, we have

$$\|\theta_{K+1} - \hat{\theta}_m\|_\infty \leq \frac{1}{33} \|\hat{\theta}_m - \theta^*\|_\infty + \frac{1}{33} \|\hat{\theta}_m - \theta^*\|_\infty,$$

with probability exceeding $1 - \frac{\delta}{2M}$.

Lemma 6 is borrowed from the paper [KPR+20]; see the proof of Lemma 2 in that paper for details.

Our next lemma bounds the difference between the epoch fixed point $\hat{\theta}_m$ and the optimal value function $\theta^*$.
Lemma 7. Assume that $N_m$ satisfies the bound $N_m \geq c \log_4(8DM/\delta) \frac{(1+|r|_{\infty} + \sigma \sqrt{1-\gamma})^2}{\Delta^2(1-\gamma)^s}$. Then we have

$$\|\hat{\theta}_m - \theta^*\|_\infty \leq \|\hat{\theta}_m - \theta^*\|_\infty + c_4 \left( \frac{\tau^*}{\sqrt{N_m}} + \frac{\kappa}{N_m} \right),$$

with probability exceeding $1 - \frac{\delta}{2M}$.

See Appendix C.1 for a proof of this lemma.

With these two auxiliary results in hand, completing the proof of Lemma 5 is relatively straightforward. By the triangle inequality, we have

$$\|\hat{\theta}_{m+1} - \theta^*\|_\infty = \|\theta_{K+1} - \hat{\theta}_m\|_\infty + \|\hat{\theta}_m - \theta^*\|_\infty \leq \left( \frac{1}{32} \|\hat{\theta}_m - \theta^*\|_\infty + \frac{1}{32} \|\hat{\theta}_m - \theta^*\|_\infty \right) + \|\hat{\theta}_m - \theta^*\|_\infty$$

$$\leq \frac{1}{32} \|\hat{\theta}_m - \theta^*\|_\infty + \frac{33}{32} \|\hat{\theta}_m - \theta^*\|_\infty$$

$$\leq \frac{1}{16} \|\hat{\theta}_m - \theta^*\|_\infty + \frac{c}{\sqrt{N_m}} \left( \frac{\tau^*}{\sqrt{N_m}} + \frac{\kappa}{N_m} \right).$$

Here inequality (i) follows from Lemma 6, whereas inequality (ii) follows from Lemma 7. Finally, the two bounds hold jointly with probability at least $1 - \frac{\delta}{2M}$ via a union bound.

4.2 Proof of Theorem 2(b)

This argument follows the same structure as the proof of part (a) of Theorem 2; we retain the same shorthands from equation (36). Our proof uses Lemma 4 along with the following modification of Lemma 5.

Lemma 8. Under the conditions of Theorem 2(b), for epochs $m$ such that the re-centering sample size $N_m$ satisfies the bound $N_m \geq \log_4(8DM/\delta) \frac{(1+|r|_{\infty} + \sigma \sqrt{1-\gamma})^2}{\Delta^2(1-\gamma)^s}$, we have

$$\|\hat{\theta}_{m+1} - \theta^*\|_\infty \leq \frac{\|\hat{\theta}_m - \theta^*\|_\infty}{16} + \frac{c}{\sqrt{N_m}} \left( \frac{\tau^*}{\sqrt{N_m}} + \frac{\kappa}{N_m} \right),$$

with probability at least $1 - \frac{\delta}{2M}$.

See Appendix C.3 for a proof of this lemma.

Observe that Lemma 8 holds for all epochs $m \geq m^* := \log_2 \frac{\frac{L^2(1+|r|_{\infty} + \sigma \sqrt{1-\gamma})^2}{\Delta^2(1-\gamma)^s}}{\Delta^2(1-\gamma)^s}$. Invoking Lemma 4 for all $m < m^*$ and Lemma 8 for all epochs $m \geq m^*$, and doing a calculation similar to the proof of part (a), yields

$$\|\hat{\theta}_{M+1} - \theta^*\|_\infty \leq \frac{\|\hat{\theta}_1 - \theta^*\|_\infty}{16^M} + \frac{c}{\sqrt{N_M}} \left( \frac{\tau_{\text{max}}}{\sqrt{N_M}} + \frac{\tau^*}{\sqrt{N_M}} + \frac{\kappa}{N_M} \right),$$

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with probability exceeding \(1 - \delta\). Finally, our choice of the epoch size \(N_m\) (cf. definition (22a)) ensures \(2N_M \leq N \leq \frac{8}{3}N_M\), and substituting the values of the triple \((N_m, m^*, M)\) we conclude that

\[
\|\hat{\theta}_{M+1} - \theta^*\|_\infty \leq c \cdot \|\hat{\theta}_1 - \theta^*\|_\infty \cdot \frac{\log^2((8D/\delta) \cdot \log N)}{N^2(1 - \gamma)^4} + c \cdot \frac{L^3(1 + \|r\|_\infty + \sigma_r \sqrt{1 - \gamma})^4 \cdot \log^2((8D/\delta) \cdot \log N)}{(1 - \gamma)^{1.5} \sqrt{N}} \\
+ c \cdot \left(\sqrt{\frac{\log_4(8DM/\delta)}{N}} \max_{\pi^* \in \Pi^*} \|\nu(\pi^*; P, r, \gamma)\|_\infty + \frac{\log_4(8DM/\delta)}{N} \cdot \|\theta^*\|_{\text{span}} \right),
\]

**Comment on the lower-order terms:** For any \(p \geq 1\) by increasing our choice of \(N_m\) by a constant factor depending on \(p\), we can bound the first term via

\[
c_1 \cdot \frac{\|\hat{\theta}_1 - \theta^*\|_\infty \cdot \log^p((8D/\delta) \cdot \log N)}{(1 - \gamma)^{2p}},
\]

and the second term by

\[
c_2 \cdot \frac{L^{3q}(1 + \|r\|_\infty + \sigma_r \sqrt{1 - \gamma})^{3q+1} \cdot \log^{3q/2}(8D/\delta) \cdot \log N)}{(1 - \gamma)^{1.5} \sqrt{N} \cdot N^{3q/2}(1 - \gamma)^{3q/2}},
\]

with \(q = \frac{2}{p} - \frac{1}{3}\). The number of samples satisfies \(N \geq \frac{L^{2(1+\|r\|_\infty + \sigma_r \sqrt{1 - \gamma})^2}}{(1 - \gamma)^{\frac{5}{2}}(1 - \gamma)^{3q/2}}\) by assumption, so the two can be made arbitrarily small by increasing \((p, q)\) appropriately. This completes the proof of Theorem 2(b).

## 5 Discussion

In this paper, we presented an analysis of Q-learning through the instance-dependent framework in the synchronous setting. For \(\gamma\)-discounted MDPs with finite state space \(\mathcal{X}\) and action space \(\mathcal{U}\), we have proved a local non-asymptotic lower bound for estimating the Q-function dependent on a functional \(\nu(\cdot; P, r)\) of the MDP instance \((P, r)\) that measures the variance of solving for the Q-function. In addition, we have provided an analysis of a form of variance-reduced Q-learning and obtained instance-dependent guarantees on the \(\ell_\infty\)-error for sample sizes \(\frac{N}{\log N} \geq c \cdot \frac{\log(D/\delta)}{(1 - \gamma) \Delta^2}\) and \(\frac{N}{\log N} \geq c \cdot \frac{\log(D/\delta)}{(1 - \gamma)^3}\) on Lipschitz MDPs that match the corresponding lower bound, establishing instance-optimality. We conjecture that optimality of Algorithm VR-QL still remains true for general MDPs and sample sizes \(N \geq \frac{\log(D/\delta)}{(1 - \gamma)^3}\), and is left for further endeavours.

## A Calculations for Example 1

Here we derive the bound (12). Letting \(V^*\) denote the value function of the optimal policy \(\pi^*\), we have

\[
(Z^{\pi^*} - P^{\pi^*})\theta = \left[ (Z_{u_1} - P_{u_1})V^* \right] 0 \right].
\]

(43)
Letting $W = (I - \gamma P_{u_1})^{-1}(Z_{u_1} - P_{u_1})\theta_{\pi^*}$ and solving for $(I - \gamma P^{\pi^*})Y = \gamma(Z^{\pi^*} - P^{\pi^*})\theta$ gives

$$Y = \gamma \cdot \left[ \begin{array}{c} W \\ \gamma W \end{array} \right].$$

(44)

Thus, we have

$$\|\nu(\pi^*; P_\lambda, r_\lambda)\|_\infty : = \max_{(x,u)} \|\nu(\pi^*; P_\lambda, r_\lambda)(x,u)\| \leq \frac{1}{(1 - \gamma)^{1.5-\lambda}}.$$

The second equality above follows from the definition (5) of the matrix $\nu(\pi^*; P_\lambda, r_\lambda)$, and the last step via some simple calculations.

**B Auxiliary lemmas for Theorem 1**

In this section, we prove the auxiliary lemmas that are the used in the proof of Theorem 1.

**B.1 Proof of Lemma 1**

This proof uses standard arguments, in particular following the usual avenue of reducing estimation to testing [Bir83, Wai19a]. For completeness, we provide the details here. We use $\theta$ and $\theta'$ to denote the optimal $Q$-functions for problem $P$ and $P'$ respectively. We first lower bound the minimax risk over $P, P'$ by the averaged risk as follows:

$$\inf_{\hat{\theta}} \max_{Q \in \{P, P'\}} \mathbb{E}_P \left[ \|\hat{\theta} - \theta(Q)\|_\infty \right] \geq \frac{1}{2} \left( \mathbb{E}_P \left[ \|\hat{\theta}_N - \theta\|_\infty \right] + \mathbb{E}_{(P')^N} \left[ \|\hat{\theta}_N - \theta'\|_\infty \right] \right).$$

Here $P^N$ is a product measure that yields $N$ i.i.d. samples from $P$. Then, for any $\delta \geq 0$, we have by Markov’s inequality

$$\mathbb{E}_P \left[ \|\hat{\theta}_N - \theta\|_\infty \right] + \mathbb{E}_{(P')^N} \left[ \|\hat{\theta}_N - \theta'\|_\infty \right] \geq \delta \left[ P^N \left( \|\hat{\theta}_N - \theta\|_\infty \geq \delta \right) + (P')^N \left( \|\hat{\theta}_N - \theta'\|_\infty \geq \delta \right) \right].$$

Define $\delta_{01} : = \frac{1}{2}\|\theta - \theta'\|_\infty$, we have

$$\|\hat{\theta}_N - \theta\|_\infty < \delta_{01} \implies \|\hat{\theta}_N - \theta'\|_\infty > \delta_{01},$$

yielding

$$\mathbb{E}_P \left[ \|\hat{\theta}_N - \theta\|_\infty \right] + \mathbb{E}_{(P')^N} \left[ \|\hat{\theta}_N - \theta'\|_\infty \right] \geq \delta_{01} \left[ 1 - P^N \left( \|\hat{\theta}_N - \theta\|_\infty < \delta_{01} \right) + (P')^N \left( \|\hat{\theta}_N - \theta'\|_\infty \geq \delta_{01} \right) \right]$$

$$\geq \delta_{01} \left[ 1 - P^N \left( \|\hat{\theta}_N - \theta'\|_\infty \geq \delta_{01} \right) + (P')^N \left( \|\hat{\theta}_N - \theta\|_\infty \geq \delta_{01} \right) \right]$$

$$\geq \delta_{01} \left[ 1 - \|P^N - (P')^N\|_{TV} \right]$$

$$\geq \delta_{01} \left[ 1 - \sqrt{2}d_{\text{Hel}}(P^N, (P')^N)^2 \right].$$

Via the tensorization property of the Hellinger distance for independent random variables we have

$$d_{\text{Hel}}(P^N, (P')^N)^2 = 1 - \left( 1 - d_{\text{Hel}}(P, P')^2 \right)^N \leq N d_{\text{Hel}}(P, P')^2.$$
Putting together the pieces, we have that
\[ \inf_{\theta_N} \max_{Q \in \mathcal{P}, \mathcal{P}'} \mathbb{E}_Q[\|\theta - \theta(Q)\|_\infty] \geq \frac{1}{4} \|\theta(P) - \theta(P')\|_\infty \cdot \left(1 - \sqrt{2N} \cdot d_{\text{Hel}}(P, P')^2\right). \]

The desired result then follows from taking a supremum over all positive alternative \( \mathcal{P}' \in \mathcal{S} \) and a simple calculation.

B.2 Proof of Lemma 3

We devote a subsection to each of the three parts of this lemma.

B.2.1 Proof of Lemma 3(a)

In order to establish that \( \bar{P} \) is a transition kernel, we observe that
\[ \sum_{x'} \bar{P}_{x',z} = 1 + \frac{1}{\bar{\rho}(z) \sqrt{2N}} U_{z,z} \cdot \left( \sum_{x'} P_{x',z} (\theta(x', \pi_1(x')) - (\mathcal{P}_{\pi_1} (z)) \right) = 1, \]

where the last equality above follows by noting that \( (\mathcal{P}_{\pi_1} (z)) = \sum_{x'} P_{x',z} (\theta(x', \pi_1(x'))) \). To check non-negativity of entries of \( \bar{P} \) note we have \( |U_{z,z}| \leq \frac{1}{1 - \gamma} \), and \( 2\|\theta\|_{\text{span}} \leq |\theta(x', \pi_1(x')) - (\mathcal{P}_{\pi_1} (z))| \).

Combining the last two observation along with the sample size requirement \( (10b) \) implies
\[ \bar{P}_{x',z} \geq 1 - \frac{1}{\bar{\rho}(z) \sqrt{2N}} \cdot \frac{\|\theta\|_{\text{span}}}{1 - \gamma} \geq 0, \]

establishing that \( \bar{P} \) defines a valid set of transition kernels.

B.2.2 Proof of Lemma 3(b)

The proof of part (b) follows by first providing a general upper bound on the Hellinger distance \( d_{\text{Hel}}(P, P_1) \), and then substituting the values of instances \( P \) and \( P_1 \). Concretely, we prove
\[ d_{\text{Hel}}^2(P, P_1) \overset{(a)}{=} \frac{1}{2} \sum_{z,x'} (P_{x',z} - \bar{P}_{x',z})^2 \leq \overset{(b)}{=} \frac{1}{4N}. \]

With this result in hand, the claimed bound on \( \|\bar{P}_{\pi_1} - \mathcal{P}_{\pi_1}\|_{\infty \rightarrow \infty} \) is immediate. Indeed,
\[ \|\mathcal{P}_{\pi_1} - \mathcal{P}_{\pi_1}\|_{\infty \rightarrow \infty} \leq \sum_{z,x'} (P_{x',z} - \bar{P}_{x',z})^2 \leq \sum_{z,x'} (P_{x',z} - \bar{P}_{x',z})^2 \leq \frac{1}{2N}. \]

It remains to prove the bounds \( (45a) \) and \( (45b) \).

Proof of \( (45a) \): We use \( (Z, R) \) (respectively \( (Z', R') \)) to denote a sample drawn from the distribution \( P \) (respectively \( P' \)), and \( P_Z, P_R \) (respectively \( P'_Z, P'_R \)) to denote the marginal distribution of \( Z, R \) (respectively \( Z', R' \)). By independence of \( Z \) and \( R \) (and likewise for \( Z', R' \)) we have
\[ P = P_Z \otimes P_R, \quad \text{and} \quad P' = P'_Z \otimes P'_R. \]
Let \( P' = (P', r') \in S_1 \) (so \( r' = r \)). Via the independence between \( Z \) and \( R \), we have
\[
d_H^2(P, P') = d_H^2(P_Z, P_Z').
\] (47)
For state-action pairs \((x, u)\), \( Z(x, u) \) are independent (and likewise for \( Z' \)) so
\[
d_H^2(P_Z, P_Z') = 1 - \prod_{x, u} \left(1 - d_H(P_{Z(x,u)}, P_{Z'(x,u)})\right)^2 \leq \sum_{x, u} d_H^2(P_{Z(x,u)}, P_{Z'(x,u)}).
\]
Note that \( Z(x, u) \) and \( Z'(x, u) \) have multinomial distribution with parameters \( P_u(\cdot | x) \) and \( P_u'(\cdot | x) \) respectively. Therefore,
\[
d_H^2(P_{Z(x,u)}, P_{Z'(x,u)}) \leq \frac{1}{2} D_{\text{KL}} \left( P_{Z'(x,u)} \parallel P_{Z(x,u)} \right) = \frac{1}{2} \sum_{x', z} \frac{(P_{x', z} - P_{x', z}')^2}{P_{x', z}}.
\]
**Proof of (45b):** We have
\[
\sum_{z, x'} \frac{(P_{x', z} - P_{x', z})^2}{P_{x', z}} = \frac{1}{2N \bar{\rho}^2(z)} \sum_z \sum_{x'} P_{x', z} (U_{z, z})^2 (\theta(x', \pi_1(x')) - (P^{\pi_1} \theta)(z))^2
\]
\[
= \frac{1}{2N \bar{\rho}^2(z)} \sum_z (U_{z, z})^2 \left( \sum_{x'} P_{x', z} (\theta(x', \pi_1(x')) - (P^{\pi_1} \theta)(z))^2 \right)
\]
\[
= \frac{1}{2N \bar{\rho}^2(z)} \sum_z (U_{z, z})^2 \varphi^2(z)
\]
Equality (i) follows from the definition
\[
\varphi^2(z) = \text{Var} (Z^{\pi_1} \theta(z)) = \sum_{x'} P_{x', z} (\theta(x', \pi_1(x')) - (P^{\pi_1} \theta)(z))^2,
\] (48)
whereas equality (ii) follows from the definition (26), which ensures that
\[
\bar{\rho}^2(z) = \text{Var} (\Gamma - \gamma P^{\pi_1})^{-1} Z^{\pi_1} \theta(z)) = \sum_{z'} (U_{z, z'})^2 \varphi^2(z').
\]

**B.2.3 Proof of Lemma 3(c)**

The entries of the matrix \( U := (\Gamma - \gamma P^{\pi_1})^{-1} \) are positive, so that it suffices to show that the vector \( (P^{\pi_1} - P^{\pi_1}) \theta \) is entry-wise positive. We have
\[
(P^{\pi_1} - P^{\pi_1}) \theta(z) = \sum_{x'} (P_{x', z} - P_{x', z}) \theta(x', \pi_1(x'))
\]
\[
= \sum_{x'} (P_{x', z} - P_{x', z}) (\theta(x', \pi_1(x')) - (P^{\pi_1} \theta)(z))
\]
\[
= \frac{1}{\bar{\rho}(z) \sqrt{2N}} U_{z, z} \sum_{x'} P_{x', z} (\theta(x', \pi_1(x')) - (P^{\pi_1} \theta)(z))^2
\]
\[
= \frac{1}{\bar{\rho}(z) \sqrt{2N}} U_{z, z} \varphi^2(z) \geq 0,
\]
where the second equality follows from the fact that \( \sum_{x'} P_{x', z} = \sum_{x'} P_{x', z} = 1 \), the third equality follows by substituting the value of \( \Gamma \) from equation (28), and the equality in the last line follows from the definition (48). This completes the proof of part (c).
C Auxiliary lemmas for Theorem 2

In this section, we prove the auxiliary lemmas that are used in the proof of Theorem 2.

C.1 Proof of Lemma 5

This section is devoted to the proof of Lemma 5 which underlies the proof of Theorem 2. In order to simplify notation, we drop the epoch number \( m \) from \( p_{\theta}^m \) and \( \theta_m \) throughout the remainder of the proof. Let \( p_{\pi} \) and \( \pi_{\lessdot} \) denote the greedy policies with respect to the \( Q \)-functions \( p_{\theta} \) and \( \theta_{\lessdot} \), respectively. Concretely,

\[
\pi^*(x) = \arg \max_{u \in \mathcal{U}} \theta^*(x, u) \quad \hat{\pi}(x) = \arg \max_{u \in \mathcal{U}} \hat{\theta}(x, u). \tag{49}
\]

Ties in the arg max are broken by choosing the action \( u \) with smallest index.

By the optimality of the policies \( p_{\pi} \) and \( \pi_{\lessdot} \) for the \( Q \)-functions \( p_{\theta} \) and \( \theta_{\lessdot} \), respectively, we have

\[
\theta_{\lessdot} = r + \gamma p_{\pi}^\pi \theta^* \quad \text{and} \quad \hat{\theta} = \hat{r} + \gamma p_{\pi} \hat{\theta}, \quad \text{where} \quad \hat{r} := r + T_{N_m}(\hat{\theta}) - T(\hat{\theta}). \tag{50}
\]

In order to complete the proof, we require the following auxiliary result.

Lemma 9. We have

\[
\| (I - \gamma P^{\pi^*})^{-1} (\hat{r} - r) \|_\infty \leq \frac{\| \hat{\theta} - \theta^* \|_\infty}{33} + 4 \cdot \sqrt{\frac{\log_4(8DM/\delta)}{N_m}} \cdot \max_{\pi \in \Pi^\pi} \| \nu(\pi; P, r) \|_\infty \]

\[
+ 4 \cdot \frac{\log_4(8DM/\delta)}{N_m} \cdot \frac{\| \theta^* \|_{\text{span}}}{(1 - \gamma)},
\]

with probability exceeding \( 1 - \frac{\delta}{8M} \).

See Appendix C.2 for the proof.

It remains to prove that under the assumptions of Lemma 5, the following bound holds with probability \( 1 - \frac{\delta}{M} \):

\[
\| \hat{\theta} - \theta \|_\infty \leq \| (I - \gamma P^{\pi^*})^{-1} (\hat{r} - r) \|_\infty. \tag{51}
\]

We establish this claim by first showing that the policy \( \hat{\pi} \) is an optimal policy, which is achieved in the following lemma.

Lemma 10. Given two \( Q \)-functions \( \theta^* \) and \( \hat{\theta} \) and the associated optimal policies \( \pi^* \) and \( \hat{\pi} \), we have

\[
P^{\hat{\pi}} \theta^*(x, u) \geq P^{\pi^*} \theta^*(x, u) - 2 \| \hat{\theta} - \theta^* \|_\infty \quad \text{for all} \quad (x, u) \in \mathcal{X} \times \mathcal{U}.
\]

Moreover, if the batch size satisfies the lower bound \( N_m \geq c_3 \frac{(1 + |r|_\infty + \sigma_r \sqrt{1 - \gamma})^2}{(1 - \gamma)^3} \cdot \frac{\log(4DM^2/\delta)}{\Delta^2} \),

then \( \hat{\pi} \) is an optimal policy with probability at least \( 1 - \frac{\delta}{M} \). Hence, under the unique optimal policy (UNQ) condition or Lipschitz (LIP) condition, we have \( P^{\hat{\pi}} = P^{\pi^*} \).
We prove this lemma in Appendix C.4. In order to prove the bound (51), it suffices to prove the following elementwise inequalities:

\[ \theta^* - \hat{\theta} \leq (I - \gamma \Pi^*)^{-1}(r - \hat{\tau}) \quad \text{(52a)} \]
\[ \hat{\theta} - \theta^* \leq (I - \gamma \hat{\Pi}^*)^{-1}(r - \hat{\tau}) \quad \text{(52b)} \]

Indeed, we have

\[ |\theta^* - \hat{\theta}| \leq \max\{|\theta^* - \hat{\theta}|_+, |\hat{\theta} - \theta^*|_+\} \leq |(I - \gamma \Pi^*)^{-1}(r - \hat{\tau})|, \]

where the maximum operator \( \max\{\cdot, \cdot\} \) is applied entry-wise. Combining the last two bounds with Lemma 10, and using the lower bound assumption on the epoch sample size \( N_m \) we obtain

\[ |\theta^* - \hat{\theta}| \leq \max\{|\theta^* - \hat{\theta}|_+, |\hat{\theta} - \theta^*|_+\} \leq |(I - \gamma \Pi^*)^{-1}(r - \hat{\tau}), (I - \gamma \hat{\Pi}^*)^{-1}(r - \hat{\tau})| \leq |(I - \gamma \Pi^*)^{-1}(r - \hat{\tau})|, \]

where the last inequality uses \( \Pi^* = \hat{\Pi} \) (cf. Lemma 10). It remains to prove the bounds (52a) and (52b).

**Proof of bounds (52a) and (52b):** By optimality of policies \( \hat{\pi} \) and \( \pi^* \) for Q-functions \( \hat{\theta} \) and \( \theta^* \), respectively, we have

\[ \theta^* = r + \gamma \Pi^* \theta^* \geq r + \gamma \Pi^* \hat{\theta} \quad \text{and} \quad \hat{\theta} = \hat{\tau} + \gamma \hat{\Pi} \hat{\theta} \geq \hat{\tau} + \gamma \hat{\Pi} \theta^*. \quad \text{(53)} \]

This yields:

\[ \theta^* - \hat{\theta} = r - \hat{\tau} + \gamma (\Pi^* \theta^* - \hat{\Pi} \hat{\theta}) \leq r - \hat{\tau} + \gamma \Pi^* (\theta^* - \hat{\theta}). \quad \text{(54)} \]

Rearranging the last inequality, and using the non-negativity of the entries of \( (I - \gamma \Pi^*)^{-1} \) we conclude

\[ (\theta^* - \hat{\theta}) \leq (I - \gamma \Pi^*)^{-1}(r - \hat{\tau}). \]

This completes the proof of the bound (52a). The proof of bound (52b) is similar.

**C.2 Proof of Lemma 9**

Recall the definition \( \hat{\tau} := \hat{R} + \gamma (\hat{Z} - \Pi \hat{\theta}) \), where \( \Pi \) a policy greedy with respect to \( \hat{\theta} \); that is, \( \Pi(x) = \arg \max_{\hat{a} \in A} \hat{\theta}(x, u) \), where we break ties in the arg max by choosing the action \( u \) with smallest index. We have

\[ \|(I - \gamma \Pi^*)^{-1}(\hat{\tau} - r)\|_\infty \leq \|(I - \gamma \Pi^*)^{-1} \left\{ (\hat{R} - r) + \gamma (\hat{Z} - \Pi \hat{\theta}) \theta^* \right\} \|_\infty \]
\[ + \gamma \|(I - \gamma \Pi^*)^{-1} \left\{ (\hat{Z} - \Pi \hat{\theta}) - (\Pi \theta^* - \Pi \theta^*) \right\} \|_\infty. \]

Observe that the random variable \( \hat{R} \) and \( \hat{Z} \) are averages of \( N_m \) i.i.d. random variables \( \{R_i\} \) and \( \{\hat{Z}_i\} \), respectively. Additionally, entrywise, the random reward is a Gaussian random variable with variance \( \sigma^2 \), and by the generative model assumption, the randomness in the random rewards \( \{R_i\} \) is independent of the randomness in \( \{\hat{Z}_i\} \). Consequently, applying
Hoeffding’s bound on the term involving \( \{R_i\} \), a Bernstein bound on the term involving \( \{\hat{Z}_i\} \) and a union bound yields the following result which holds with probability at least \( 1 - \frac{\delta}{4M} \):

\[
\|(I - \gamma P^{\pi^*})^{-1} \left\{ \left( \hat{\bar{R}} - r \right) + \gamma (\hat{\bar{Z}}^{\pi^*} - P^{\pi^*}) \theta^* \right\} \|_\infty \\
\leq \frac{4}{\sqrt{N_m}} \cdot \|\nu(\pi^*; P, r)\|_\infty \cdot \sqrt{\log_4(8DM/\delta)} + \frac{4\|\theta^*\|_{\text{span}}}{(1 - \gamma)N_m} \cdot \log_4(8DM/\delta) \\
\leq \frac{4}{\sqrt{N_m}} \cdot \max_{\pi \in \Pi} \|\nu(\pi; P, r)\|_\infty \cdot \sqrt{\log_4(8DM/\delta)} + \frac{4\|\theta^*\|_{\text{span}}}{(1 - \gamma)N_m} \cdot \log_4(8DM/\delta).
\]

Finally, for each state-action pair \((x, u)\) the random variable \( (\hat{Z}^{\pi^*} - \hat{\bar{Z}}^{\pi^*})_c(x, u) \) has expectation \( (P^{\pi^*} - P^{\pi^*})(x, u) \) with entries uniformly bounded by \( 2\|\theta - \theta^*\|_\infty \). Consequently, by a standard application of Hoeffding’s inequality combined with the lower bound \( N_m \geq c_3 \frac{4^m}{1 - \gamma^2} \log_4(8DM/\delta) \), we have

\[
\frac{\gamma}{1 - \gamma} \cdot \| (\hat{Z}^{\pi^*} - \hat{\bar{Z}}^{\pi^*})_c - (P^{\pi^*} - P^{\pi^*}) \|_\infty \leq \frac{\|\theta - \theta^*\|_\infty}{33},
\]

with probability exceeding \( 1 - \frac{\delta}{4M} \). The statement then follows from combining these two high-probability statements with a union bound.

### C.3 Proof of Lemma 8

By Lemma 6 and Lemma 9, it suffices to show

\[
\|\hat{\theta}_m - \theta^*_m\|_\infty \leq \frac{1}{2} \| (I - \gamma P^{\pi^*})^{-1} (\hat{\bar{r}} - r) \|_\infty
\]

Recalling the bounds (52a)–(52b), we have

\[
\|\hat{\theta}_m - \theta^*\|_\infty \leq \| (I - \gamma P^{\pi^*})^{-1} (\hat{\bar{r}} - r) \|_\infty + \gamma \| (I - \gamma P^{\pi^*})^{-1} (\hat{P}^{\pi} - P^{\pi^*}) (\hat{\theta}_m - \theta^*) \|_\infty \\
\leq \| (I - \gamma P^{\pi^*})^{-1} (\hat{\bar{r}} - r) \|_\infty + \frac{L\gamma}{1 - \gamma} \|\hat{\theta}_m - \theta^*\|_\infty
\]

where the last inequality uses the Lipschitz condition (17). If we can show \( \frac{L\gamma}{1 - \gamma} \|\hat{\theta}_m - \theta^*\|_\infty \leq \frac{1}{2} \), we are done. In order to do so, we require the following auxiliary result:

**Lemma 11.** Given a batch size \( N_m \) lower bounded as \( N_m \geq c_3 \frac{\log_4(8DM/\delta)}{(1 - \gamma)^2} \), we have

\[
\|\hat{\theta}_m - \theta^*\|_\infty \leq c_1 \cdot \frac{1 + \|r\|_\infty + \sigma_r \sqrt{1 - \gamma}}{\sqrt{N_m(1 - \gamma)^{1.5}}} \cdot \log_4(8DM^2/\delta)
\]

with probability at least \( 1 - \frac{\delta}{4M} \).

With the above lemma at hand and using \( N_m \geq c \log_4(8DM/\delta) \frac{L^2(1 + \|r\|_\infty + \sigma_r \sqrt{1 - \gamma})^2}{(1 - \gamma)^3} \), we conclude

\[
\|\hat{\theta}_m - \theta^*\|_\infty \leq \frac{1 - \gamma}{2L},
\]

as desired. It remains to prove Lemma 11.
Proof of Lemma 11: This proof exploits the result of Lemma 4, that with probability at least $1 - \frac{\delta}{3\mathcal{M}}$, we have

$$
\|\hat{\theta}_m - \theta^*\|_{\infty} \leq \frac{\|\hat{\theta}_m - \theta^*\|_{\infty}}{33} + 1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma} \sqrt{\frac{\log(8\Delta^2/\delta)}{N_m}} + \frac{\log(8\Delta^2/\delta)}{N_m} \cdot \|\theta^*\|_{\text{span}} \cdot \frac{1 - \gamma}{1 - \gamma}.
$$

(55)

Following an argument similar to the proof of Theorem 2, we have

$$
\|\hat{\theta}_{m+1} - \theta^*\|_{\infty} \leq \frac{\|\hat{\theta}_m - \theta^*\|_{\infty}}{16} + c \left\{ \frac{1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma} \sqrt{\frac{\log(8\Delta^2/\delta)}{N_m}} + \frac{\log(8\Delta^2/\delta)}{N_m} \cdot \|\theta^*\|_{\text{span}}}{(1 - \gamma)^{1.5}} \right\}
$$

(i)

$$
\|\hat{\theta}_{1} - \theta^*\|_{\infty} \leq \frac{1}{16^m} + 2c \left\{ \frac{1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma} \sqrt{\frac{\log(8\Delta^2/\delta)}{N_m}} + \frac{\log(8\Delta^2/\delta)}{N_m} \cdot \|\theta^*\|_{\text{span}}}{(1 - \gamma)^{1.5}} \right\},
$$

(ii)

(56)

with probability at least $1 - \frac{\delta}{3\mathcal{M}}$. Inequality (ii) follows by recursing the first inequality; the last inequality uses the initialization condition $\|\hat{\theta}_1 - \theta^*\|_{\infty} \leq \frac{|r|_{\infty}}{\sqrt{1 - \gamma}}$ and $N_m \geq \frac{4^m}{(1 - \gamma)^2}$. Combining the bounds (55) and (56) and using the bounds $\|\theta^*\|_{\infty} \leq \frac{|r|_{\infty}}{\sqrt{1 - \gamma}}$ and $\|\theta^*\|_{\text{span}} \leq 2\|\theta^*\|_{\infty}$, we find that

$$
\|\hat{\theta}_m - \theta^*\|_{\infty} \leq 8c \cdot \frac{1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma}}{\sqrt{N_m(1 - \gamma)^{1.5}}} \cdot \log(8\Delta^2/\delta),
$$

with probability at least $1 - \frac{\delta}{3\mathcal{M}}$. This completes the proof.

C.4 Proof of Lemma 10

The proof of this lemma exploits the optimality of the policies $\pi^*$ and $\hat{\pi}$ with respect to the $Q$-functions $\theta^*$ and $\hat{\theta}$, respectively. Accordingly, we have for all state action pair $(x, u) \in \mathcal{X} \times \mathcal{U}$

$$
P_{\pi^*}(x, u) = P_{\pi^*}(x, u) + P_{\pi^*}(x, u) - P_{\pi^*}(x, u)
$$

$$
= P_{\pi^*}(x, u) + P_{\pi^*}(x, u) - P_{\pi^*}(x, u) - \|\theta^* - \hat{\theta}\|_{\infty}
$$

(57)

The first inequality follows from the optimality of the policy $\pi^*$ with respect to the $Q$-function $\pi$. This completes the proof of the first part of the lemma.

Turning to the second part, invoking Lemma 11 with a batch size $N_m \geq \frac{1 + \|r\|_{\infty} + \sigma_r \sqrt{1 - \gamma}^2}{(1 - \gamma)^2} \cdot \frac{\log(\Delta^2/\delta)}{\Delta^2}$ guarantees that

$$
2\|\theta^* - \hat{\theta}\|_{\infty} \leq \Delta.
$$
This inequality, combined with the bound (57) and the definition of the optimality gap ∆,
implies that ̂π is an optimal policy, and hence P̂π = Pπ* under the unique policy or Lipschitz
assumptions.

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