Dirac eigenvalues estimates
in terms of symmetric tensors

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Abstract: We review some recent results concerning lower eigenvalues estimates for the
Dirac operator [6, 7]. We show that Friedrich’s inequality can be improved via certain
well-chosen symmetric tensors and provide an application to Sasakian spin manifolds.

1 Introduction

Let \((M^n, g), n \geq 3\), be an \(n\)-dimensional closed Riemannian spin manifold. Let \((E_1, \ldots, E_n)\)
be a local orthonormal frame field on \((M^n, g)\). Then the spinor derivative \(\nabla\) and the Dirac
operator \(D\), acting on sections \(\psi \in \Gamma(\Sigma(M^n))\) of the spinor bundle \(\Sigma(M^n)\) over \((M^n, g)\),
are locally expressed as

\[
\nabla_X \psi = X(\psi) + \frac{1}{4} \sum_{i=1}^{n} E_i \cdot \nabla_X E_i \cdot \psi, \quad D\psi = \sum_{i=1}^{n} E_i \cdot \nabla_E_i \psi,
\]

respectively, where the dot ",\cdot\" indicates the Clifford multiplication [2, 4]. Friedrich proved
in [3] that the smallest eigenvalue \(\lambda_1\) of \(D\) satisfies

\[
\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_M S,
\]

(1.1)

where \(S\) is the scalar curvature of \((M^n, g)\). The limiting case of (1.1) occurs if and only if
\((M^n, g)\) admits a nontrivial spinor field \(\psi\) called Killing spinor, satisfying

\[
\nabla_X \psi = -\frac{\lambda_1}{n} X \cdot \psi,
\]

(1.2)

where \(X\) is an arbitrary vector field on \(M^n\). The simply-connected manifolds \((M^n, g)\)
admitting Killing spinors were classified by Bär [1], namely, the limiting manifold \((M^n, g)\)
must be either a standard \(n\)-sphere, an Einstein-Sasaki manifold, a 6-dimensional nearly
Kähler manifold or a 7-dimensional manifold with 3-form \(\phi\), \(\nabla \phi = *\phi\). Note that all of
these limiting manifolds are Einstein, since equation (1.2) allows a nontrivial solution \(\psi\)
only if \((M^n, g)\) is Einstein.

It has been found that inequality (1.1) is not optimal if \((M^n, g)\) allows certain geometric
structures, since the limiting case of (1.1) can not be attained [9, 10, 11]. For example,
Kirchberg proved for Kähler spin manifolds that the smallest eigenvalue \(\lambda_1\) of the Dirac
operator satisfies

\[
\lambda_1^2 \geq \frac{n + 2}{4n} \inf_M S \quad \text{for} \quad n \equiv 2 \mod 4
\]

(1.3)
and
\[ \lambda_1^2 \geq \frac{n}{4(n-2)} \inf_M S \quad \text{for} \quad n \equiv 0 \mod 4. \] (1.4)

Improvements of Friedrich’s inequality (1.1) do typically depend on additional geometric structures on the considered manifold \((M^n, g)\). The aim of this article is to review some new results in [6, 7], showing that Friedrich’s inequality can be improved via divergence-free symmetric tensors as well as Codazzi tensors (see Theorem 2.1 and 2.2). In the last section we discuss geometric implications of Theorem 2.1 over Sasakian spin manifolds.

2 Dirac eigenvalues estimates in terms of symmetric tensors

Throughout the article we fix some terminology.

**Definition 2.1** Let \( P \) be a first order self-adjoint elliptic operator on some closed Riemannian spin manifold. An eigenvalue \( \lambda \in \mathbb{R} \) of \( P \) is called the first eigenvalue if \( \lambda^2 \) is the smallest eigenvalue of \( P^2 \). An eigenspinor \( \varphi \) of \( P \) is called a first eigenspinor if its associated eigenvalue \( \lambda \) is the first eigenvalue of \( P \).

Evidently, the Killing spinors satisfying equation (1.2) are first eigenspinors of the Dirac operator. Let’s see one more example. The limiting case of inequality (1.3) occurs if and only if the coupled system

\[ D\psi = \lambda_1 \psi, \]
\[ \nabla_X \psi = -\frac{\lambda_1}{n+2} X \cdot \psi + \frac{\lambda_1}{n+2} J(X) \cdot \Omega \cdot \psi \] (2.1)

admits a nontrivial solution \( \psi \) called Kählerian Killing spinor, where \( \Omega \) is the Kähler form. Thus, the Kählerian Killing spinors are first eigenspinors of the Dirac operator.

Let us now consider a nondegenerate symmetric (0, 2)-tensor field \( \beta \) on \((M^n, g)\) and define the \( \beta \)-twist \( D_\beta \) of the Dirac operator \( D \) by

\[ D_\beta \psi = \sum_{i=1}^n \beta^{-1}(E_i) \cdot \nabla_{E_i} \psi = \sum_{i=1}^n E_i \cdot \nabla_{\beta^{-1}(E_i)} \psi, \]

where \( \beta \) was identified with the induced (1,1)-tensor \( \beta \) via \( \beta(X, Y) = g(X, \beta(Y)) \). Recall that a symmetric \((0, 2)\)-tensor field \( \beta \) is called

(i) a divergencefree tensor if \( \text{div}(\beta) = \sum_{i=1}^n (\nabla_{E_i} \beta)(E_i) = 0 \).
(ii) a Codazzi tensor if \( (\nabla_X \beta)(Y, Z) = (\nabla_Y \beta)(X, Z) \) holds for all vector fields \( X, Y, Z \).

Let \( (\cdot, \cdot) := \text{Re} \langle \cdot, \cdot \rangle \) denote the real part of the standard Hermitian product \( \langle \cdot, \cdot \rangle \) on the spinor bundle \( \Sigma(M) \) over \( M^n \). Let \( \alpha \) be a 1-form on \( M^n \) induced by a nondegenerate symmetric tensor \( \beta \) and spinor fields \( \phi, \psi \in \Gamma(\Sigma) \) via

\[ \alpha(X) = \langle \phi, \beta^{-1}(X) \cdot \psi \rangle. \]
Then
\[ \text{div}(\alpha) = -(D_\beta \phi, \psi) + (\phi, D_\beta \psi) + (\phi, \text{div}(\beta^{-1}) \cdot \psi). \]

Consequently, if \( \beta^{-1} \) is divergencefree, then \( D_\beta \) is a self-adjoint elliptic operator of first order and hence its spectrum is discrete and real.

We have proved in [6, 7] the following theorems.

**Theorem 2.1** Let \((M^n, g)\) be an \(n\)-dimensional closed Riemannian spin manifold. Let \(\beta\) be such a nondegenerate symmetric tensor on \(M^n\) that both \(\text{div}(\beta^{-1}) = 0\) and \(\text{tr}(\beta^{-1}) = 0\) vanish identically. Let \(\lambda_1 \in \mathbb{R}\) and \(\overline{\lambda}_1 \in \mathbb{R}\) be the first eigenvalue of \(D\) and \(D_\beta\), respectively. Then we have
\[ \lambda_1^2 \geq \inf_M \left\{ \frac{nS}{4(n-1)} + \frac{\overline{\lambda}_1^2}{(n-1)|\beta^{-1}|^2} + \frac{n\triangle(|\beta^{-1}|^2)}{2(n-1)|\beta^{-1}|^2} \right\}. \]

The limiting case occurs if and only if there exists a spinor field \(\psi_1\) on \((M^n, g)\) with the following properties:

(i) The differential equation
\[ \nabla_X \psi_1 = -\frac{\lambda}{n} X \cdot \psi_1 - \frac{\overline{\lambda}}{|\beta^{-1}|^2} \beta^{-1}(X) \cdot \psi_1 \]
holds for some constants \(\lambda, \overline{\lambda} \in \mathbb{R}\) and for all vector fields \(X\).

(ii) \(\psi_1\) is a first eigenspinor of both \(D\) and \(D_\beta\).

**Theorem 2.2** Let \((M^n, g)\) be an \(n\)-dimensional closed Riemannian spin manifold and consider a nondegenerate Codazzi tensor \(\beta\) such that \(\text{tr}(\beta^{-1}) = 0\) vanishes identically. Denote by \(\overline{g}\) the metric induced by \(\beta\) via \(\overline{g}(\beta(X), \beta(Y)) = g(\beta(X), \beta(Y))\) and by \(\overline{D}\) the Dirac operator of \(\overline{g}\). Let \(\lambda_1 \in \mathbb{R}\) and \(\overline{\lambda}_1 \in \mathbb{R}\) be the first eigenvalue of the Dirac operators \(D\) and \(D_\beta\), respectively. Then we have
\[ \lambda_1^2 \geq \inf_M \left\{ \frac{nS}{4(n-1)} + \frac{\overline{\lambda}_1^2}{(n-1)|\beta^{-1}|^2} + \frac{n\triangle(|\det(\beta^{-1})||\beta^{-1}|^2)}{2(n-1)|\det(\beta^{-1})||\beta^{-1}|^2} \right\}. \]

The limiting case occurs if and only if there exists a spinor field \(\psi_1\) on \((M^n, g)\) with the following properties:

(i) The differential equation
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holds for some constants \(\lambda, \overline{\lambda} \in \mathbb{R}\) and for all vector fields \(X\).

(ii) \(\psi_1\) is a first eigenspinor of both \(D\) and \(D_\beta\).
3 Dirac eigenvalues estimates over Sasakian manifolds

In this section we will apply Theorem 2.1 to Sasakian manifolds. Consider a manifold $M^{2m+1}$ of odd dimension $n = 2m + 1$. An almost contact metric structure $(\phi, \xi, \eta, g)$ of $M^{2m+1}$ consists of a (1,1)-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$, and a metric $g$ with the following properties:

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The fundamental 2-form $\Phi$ of the contact structure is a 2-form defined by

$$\Phi(X, Y) = g(X, \phi Y).$$

An almost contact metric structure $(\phi, \xi, \eta, g)$ of $M^{2m+1}$ becomes a Sasakian structure if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X$$

holds for all vector fields $X, Y$. A Sasakian manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called eta-Einstein if the Ricci curvature tensor $\text{Ric}$ satisfies

$$\text{Ric} = \kappa g + \tau \eta \otimes \eta$$

for some constants $\kappa, \tau \in \mathbb{R}$ with $\kappa + \tau = 2m$. Any eta-Einstein Sasakian manifold is necessarily of constant scalar curvature $S$ and we can rewrite eta-Einstein condition (3.1) as

$$\text{Ric} = \left(\frac{S}{n-1} - 1\right) g + \left(n - \frac{S}{n-1}\right) \eta \otimes \eta, \quad n = 2m + 1.$$

From now on we assume that any Sasakian manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ we consider has a fixed spin structure. An important property of a Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is that the spinor bundle $\Sigma(M)$ splits under the action of the fundamental 2-form $\Phi$ as follows.

**Lemma 3.1** Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be an almost contact metric manifold with spin structure and fundamental 2-form $\Phi$. Then the spinor bundle $\Sigma$ splits into the orthogonal direct sum $\Sigma = \Sigma_0 \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_m$ with

(i) $\Phi|_{\Sigma_r} = \sqrt{-1}(2r - m)I$, $\dim(\Sigma_r) = \binom{m}{r}$ ($0 \leq r \leq m$),

(ii) $\xi|_{\Sigma_0 \oplus \Sigma_2 \oplus \Sigma_4 \oplus \cdots} = (\sqrt{-1})^{2m+1}I$, $\xi|_{\Sigma_1 \oplus \Sigma_3 \oplus \Sigma_5 \oplus \cdots} = -(\sqrt{-1})^{2m+1}I$,

where $I$ stands for the identity map. Moreover, the bundles $\Sigma_0$ and $\Sigma_m$ can be defined by

$$\Sigma_0 = \{ \psi \in \Sigma : \phi(X) \cdot \psi + \sqrt{-1}X \cdot \psi + (-1)^m \eta(X)\psi = 0 \text{ for all vectors } X \},$$

$$\Sigma_m = \{ \psi \in \Sigma : \phi(X) \cdot \psi - \sqrt{-1}X \cdot \psi - \eta(X)\psi = 0 \text{ for all vectors } X \}.$$

In particular, we have the formulas

$$\xi \cdot \psi = (-1)^m \sqrt{-1} \psi_0, \quad \Phi \cdot \psi = -m \sqrt{-1} \psi_0, \quad \psi_0 \in \Sigma_0,$$

$$\xi \cdot \psi = \sqrt{-1} \psi_m, \quad \Phi \cdot \psi = m \sqrt{-1} \psi_m, \quad \psi_m \in \Sigma_m.$$
Over Sasakian spin manifolds, a special class of spinors deserves attention.

**Definition 3.1** A nontrivial spinor field \( \psi \) on Sasakian spin manifold \((M^{2m+1}, \phi, \xi, \eta, g)\) is called an *eta-Killing spinor* with Killing pair \((a, b)\) if it satisfies

\[
\nabla_X \psi = a X \cdot \psi + b \eta(X) \xi \cdot \psi
\]

(3.2)

for some real numbers \(a, b \in \mathbb{R}, \ a \neq 0\), and for all vector fields \(X\).

Note that if \(b = 0\), then equation (3.2) reduces to equation (1.2). Moreover, any eta-Killing spinor with Killing pair \((a, b)\) is an eigenspinor of the Dirac operator with eigenvalue \(\lambda = -(2m + 1)a - b\).

Now we summarize some basic relations between the Killing pair \((a, b)\) of an eta-Killing spinor and the geometry of the Sasakian manifold. For proofs for Propositions 3.1-3.4 we refer to [5, 7]. In the following we will often write \(n\) to mean the dimension \(2m + 1\) of the manifold \(M^{2m+1}\).

**Proposition 3.1** Let \((M^{2m+1}, \phi, \xi, \eta, g), m \geq 2\), be a Sasakian spin manifold and suppose that it admits an eta-Killing spinor \(\psi\) with Killing pair \((a, b)\), where both \(a \neq 0\) and \(b \neq 0\) are nonzero. Then \((M^{2m+1}, \phi, \xi, \eta, g)\) is eta-Einstein with scalar curvature \(S = 4n(n-1)a^2 + 8(n-1)ab\). Moreover, all the possible values for \(a, b\) can be expressed in terms of the scalar curvature as

\[
(a, b) = \left( \frac{1}{2}, \frac{-n}{4} + \frac{S}{4(n-1)} \right), \quad \left( -\frac{1}{2}, \frac{n}{4} - \frac{S}{4(n-1)} \right),
\]

and the following statements are true:

(i) If \( (a, b) = \left( \frac{1}{2}, \frac{-n}{4} + \frac{S}{4(n-1)} \right) \), then \( m \equiv 0 \mod 2 \) and \( \psi \in \Gamma(\Sigma_0) \) is a section in \( \Sigma_0 \).

(ii) If \( (a, b) = \left( -\frac{1}{2}, \frac{n}{4} - \frac{S}{4(n-1)} \right) \) and \( m \equiv 0 \mod 2 \), then \( \psi \in \Gamma(\Sigma_m) \) is a section in \( \Sigma_m \).

(iii) If \( (a, b) = \left( -\frac{1}{2}, \frac{n}{4} - \frac{S}{4(n-1)} \right) \) and \( m \equiv 1 \mod 2 \), then \( \psi \in \Gamma(\Sigma_0) \cup \Gamma(\Sigma_m) \) is a section in \( \Sigma_0 \) or in \( \Sigma_m \).

**Proposition 3.2** Let \((M^{2m+1}, \phi, \xi, \eta, g), m \geq 2\), be a simply-connected Sasakian spin manifold. Suppose that \((M^{2m+1}, \phi, \xi, \eta, g)\) is eta-Einstein. Then, in case

(i) \( m \equiv 0 \mod 2 \), there exists an eta-Killing spinor \( \psi_0 \in \Gamma(\Sigma_0) \) with Killing pair \( \left( \frac{1}{2}, \frac{-n}{4} + \frac{S}{4(n-1)} \right) \) as well as an eta-Killing spinor \( \psi_m \in \Gamma(\Sigma_m) \) with Killing pair \( \left( -\frac{1}{2}, \frac{n}{4} - \frac{S}{4(n-1)} \right) \).

(ii) \( m \equiv 1 \mod 2 \), there exist two eta-Killing spinors \( \psi_0, \psi_m \) with Killing pair \( \left( -\frac{1}{2}, \frac{n}{4} - \frac{S}{4(n-1)} \right) \) such that \( \psi_\alpha \) is a section in the bundle \( \Sigma_\alpha \) (\( \alpha = 0, m \)).

**Proposition 3.3** Let \((M^3, \phi, \xi, \eta, g)\) be a 3-dimensional Sasakian spin manifold and suppose that it admits an eta-Killing spinor \( \psi \) with Killing pair \((a, b)\), where \( a \neq 0 \) and \( b \neq 0 \).
Then $(M^3, \phi, \xi, \eta, g)$ is eta-Einstein with constant scalar curvature $S = 24a^2 + 16ab$. Moreover, all the possible values for $a, b$ can be expressed in terms of the scalar curvature as

$$(a, b) = \left( -\frac{1}{2}, \frac{3}{4} - \frac{S}{8} \right), \left( -\frac{2 + \sqrt{4 + 2S}}{4}, \frac{4 - \sqrt{4 + 2S}}{4} \right),$$

$$\left( -\frac{2 - \sqrt{4 + 2S}}{4}, \frac{4 + \sqrt{4 + 2S}}{4} \right).$$

**Proposition 3.4** Let $(M^3, \phi, \xi, \eta, g)$ be a simply-connected Sasakian spin manifold of dimension 3 and suppose that the scalar curvature $S$ of $g$ is constant. Then,

(i) there exist two eta-Killing spinors $\psi_0, \psi_1$ with Killing pair $(-\frac{1}{2}, \frac{3}{4} - \frac{S}{8})$ such that $\psi_\alpha$ is a section in the bundle $\Sigma_\alpha$ ($\alpha = 0, 1$).

(ii) If $S \geq -2$, there exists an eta-Killing spinor $\psi \in \Gamma(\Sigma = \Sigma_0 \oplus \Sigma_1)$ with Killing pair $\left( -\frac{2 + \sqrt{4 + 2S}}{4}, \frac{4 - \sqrt{4 + 2S}}{4} \right)$.

We are now ready to apply Theorems 2.1 to Sasakian spin manifolds. The resulting inequality (3.3) clearly improves inequality (1.1).

**Proposition 3.5** Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 1$, be a closed Sasakian spin manifold. Let $\beta^{-1}$ be a nondegenerate symmetric tensor field on $M^{2m+1}$ defined by $\beta^{-1} = \frac{2}{n} I - 2\xi \otimes \eta$. (Note that $\text{div}(\beta^{-1}) = 0$ and $\text{tr}(\beta^{-1}) = 0$.) Let $\lambda_1 \in \mathbb{R}$ and $\lambda_1' \in \mathbb{R}$ be the first eigenvalue of $D$ and $D_\beta$, respectively. Then we have

$$\lambda_1^2 \geq \frac{n S_{\text{min}}}{4(n-1)} + \frac{n^2 \lambda_1'^2}{4(n-1)^2},$$

(3.3)

where $S_{\text{min}}$ denotes the minimum of the scalar curvature. The limiting case of (3.3) occurs, in case

(i) $n \geq 5$, if and only if there exists an eta-Killing spinor $\psi_1$ with Killing pair

$$\left( \frac{1}{2}, \frac{-n}{4} + \frac{S}{4(n-1)} \right), \left( \frac{1}{2}, \frac{n}{4} - \frac{S}{4(n-1)} \right),$$

(3.4)

such that $\psi_1$ is a first eigenspinor of both $D$ and $D_\beta$.

(ii) $n = 3$, if and only if there exists an eta-Killing spinor $\varphi_1$ with Killing pair

$$\left( \frac{-2 + \sqrt{4 + 2S}}{4}, \frac{4 - \sqrt{4 + 2S}}{4} \right)$$

(3.5)

such that $\varphi_1$ is a first eigenspinor of both $D$ and $D_\beta$.

Let $(M^{2m+1}, \phi, \xi, \eta, g)$, $m \geq 1$, be a closed Sasakian spin manifold with positive scalar curvature $S > 0$. From inequality (3.3) we see that the first eigenvalue $\lambda_1 \neq 0$ is necessarily nonzero. The statement for the limiting case of (3.3) then gives rise to a natural question:
Is every eta-Killing spinor with Killing pair (3.4) or (3.5) a first eigenspinor of the Dirac operator?

We have recently found that answer to the question in 3-dimensional case is positive [8], but the question in higher dimensional case is still open.

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