On the stress–energy tensor of a null shell in Einstein–Cartan gravity

S Khakshournia¹ and R Mansouri²,³

¹ Nuclear Science and Technology Research Institute (NSTRI), Tehran, Iran
² Department of Physics, Sharif University of Technology, Tehran, Iran
³ Institute for Studies in Physics and Mathematics (IPM), Tehran, Iran

E-mail: skhakshour@aeoi.org.ir and mansouri@ipm.ir

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Abstract
The stress–energy tensor of a matter shell whose history coincides with a null hypersurface in the Einstein–Cartan gravity is revisited. It is demonstrated that with a proper choice for the torsion discontinuity taken to be orthogonal to the null hypersurface and consistent with the antisymmetric property of torsion tensor, the modified expression for the asymmetric stress–energy tensor is automatically tangent to the hypersurface. The main differences with respect to a previous work are addressed.

Keywords: Null shell, surface stress–energy tensor, Einstein–Cartan gravity

1. Introduction

The Einstein–Cartan (EC) theory is a natural generalization of general relativity that accounts for the presence of spacetime torsion which could be emanating from the spinning properties of the matter distribution. Hence, dynamics of the spacetime is determined by torsion being triggered by the intrinsic angular momentum of the matter, and the curvature being stemmed from the mere presence of matter [1–3].

The first generalization of the junction conditions for a null hypersurface within the EC gravity was made by Bressange [4] (see also [5]). He extended the unified description of shells of any type, including the null case, provided by Barrabes–Israel [6] in the presence of a non-symmetric connection. In his approach the contorsion discontinuity, naturally appearing in the expression for the shell’s stress–energy tensor, is taken to be orthogonal to the hypersurface, resulting in a modification to the surface stress–energy tensor. According to this modification, the expression for the stress–energy tensor of the shell is formally the same one as in general relativity but the tensor representing the jump of the transverse derivatives of the metric is replaced with a non-symmetric one splitting into a Riemann part and a Cartan part. Recently, using the Bressange’s approach [7], the junction conditions of two generic spacetimes through
a non-lightlike hypersurface in the context of $f(R)$ gravity with torsion has been derived via the definition of a suitable effective extrinsic curvature tensor which splits into a Riemann part and a Cartan one.

There is another approach to generalize the junction conditions for a singular hypersurface within the EC gravity applied to the Braneworld scenarios [8]. Within this approach, it is the torsion discontinuity which is assumed to be orthogonal to the hypersurface. It turns out that the expression for the shell’s stress–energy tensor includes a term associated with the torsion sector of the connection. This term vanishes if we require that the surface stress–energy tensor be tangent to the hypersurface. As a result the Darmois–Israel junction conditions remain unchanged in the presence of torsion.

Motivated by the approach presented in [8], we assume an orthogonal torsion discontinuity across a null shell to find the surface stress–energy tensor in the presence of torsion.

Conventions. Natural geometrized units, $G = c = 1$, are used throughout the paper. The null hypersurface is denoted by $\Sigma$. We use square brackets $[F]$ to denote the jump of any quantity $F$ across $\Sigma$. Latin indices range over the intrinsic coordinates of $\Sigma$ denoted by $\xi^a$, and Greek indices over the coordinates of the 4-manifolds. As we are going to work with distributional valued tensors, there may be terms in a tensor quantity $F$ proportional to some $\delta$-function distribution. These terms are indicated by $\tilde{\tilde{F}}$.

2. The Einstein–Cartan gravity

In a Riemann–Cartan spacetime manifold, the torsion tensor is defined by the antisymmetric component of the affine connection as

$$ T^\sigma_{\mu \nu} = \Gamma^\sigma_{\nu \mu} - \Gamma^\sigma_{\mu \nu}. \tag{1} $$

Demanding that the metric tensor is covariantly constant, i.e. $\nabla_\sigma g_{\mu \nu} = 0$, the following decomposition of the asymmetric connection can be done:

$$ \Gamma^\sigma_{\mu \nu} = \Gamma^\sigma_{\mu \nu}^* + K^\sigma_{\mu \nu}, \tag{2} $$

where $\Gamma^\sigma_{\mu \nu}^*$ denotes the Christoffel symbols and $K^\sigma_{\mu \nu}$ are the components of the contorsion or defect tensor of the connection given in terms of the components of the torsion by

$$ K^\sigma_{\mu \nu} = \frac{1}{2}(T^\mu_{\sigma \nu} + T^\nu_{\sigma \mu} - T^\sigma_{\mu \nu}), \tag{3} $$

with $K^\sigma_{\mu \nu} = -K^\sigma_{\nu \mu}$. The EC field equations are

$$ G_{\mu \nu} = 8\pi T_{\mu \nu}, \tag{4} $$

$$ T^\mu_{\nu \sigma} + \delta^\mu_{\sigma} T^\rho_{\sigma \rho} - \delta^\mu_{\rho} T^\rho_{\nu \rho} = 8\pi S^\mu_{\nu \sigma}, \tag{5} $$

where $S^\mu_{\nu \sigma}$ is the spin tensor representing the density of intrinsic angular momentum in the matter distribution related to the torsion tensor in a purely algebraic way. It must be noted that although equation (4) is apparently identical to its general relativistic counterpart, here both $G_{\mu \nu}$ and $T_{\mu \nu}$ are generally asymmetric due to the presence of torsion.

3. The null shell formalism

Let $x^\mu$ be an admissible coordinate system in a coordinate neighborhood that includes the null hypersurface $\Sigma$ extending into both Riemann–Cartan spacetimes $\mathcal{M}^\pm$. The parametric
A class of solutions for the Einstein-Maxwell system in the context of a scalar field is presented. The solutions are found for a class of potentials that allow for the existence of solitary waves. The equations of motion are derived from the variational principle, and the self-dual condition is imposed on the electromagnetic field. The solutions are obtained using a numerical method based on a modified wavelet transform. The results are presented in the form of contour plots and color maps, which illustrate the behavior of the solutions in both the spatial and temporal domains. The solutions exhibit interesting features, such as the formation of shock waves and the existence of stable solitary waves. The method of the modified wavelet transform is shown to be effective in obtaining accurate and stable solutions. The results are compared with those obtained using other methods, and the agreement is good. The solutions are relevant to various applications in physics, such as in the study of plasma waves and cosmology.
\[
\alpha^{-1}\tilde{R}_{\mu\nu} = \frac{1}{2}(\gamma_{\alpha\rho}n_\rho n_\sigma - \gamma_{\nu\alpha}n_\mu n_\sigma + \gamma_{\mu\rho}n_\alpha n_\sigma - \gamma_{\mu\sigma}n_\alpha n_\rho)
- \frac{1}{2}(\zeta_{\alpha\rho}n_\rho n_\sigma - \zeta_{\nu\alpha}n_\mu n_\sigma - \zeta_{\mu\rho}n_\alpha n_\sigma + \zeta_{\mu\sigma}n_\alpha n_\rho).
\]

The singular part \(\tilde{R}_{\mu\nu}\) of the Ricci tensor can be determined by contracting two indices in (14), yielding
\[
\alpha^{-1}\tilde{R}_{\mu\nu} = \frac{1}{2}(\gamma_{\alpha\rho}n_\rho n_\sigma - \gamma_{\nu\alpha}n_\mu n_\sigma + \gamma_{\mu\rho}n_\alpha n_\sigma - \gamma_{\mu\sigma}n_\alpha n_\rho) - \frac{1}{2}(\zeta_{\alpha\rho}n_\rho n_\sigma - \zeta_{\nu\alpha}n_\mu n_\sigma - \zeta_{\mu\rho}n_\alpha n_\sigma + \zeta_{\mu\sigma}n_\alpha n_\rho),
\]
and also the Ricci scalar
\[
\alpha^{-1}\tilde{R} = \alpha^{-1}g^{\mu\nu}\tilde{R}_{\mu\nu} = -\gamma_{\sigma\nu}n^\sigma n^\nu.
\]

The singular part \(\tilde{G}_{\mu\nu}\) of the Einstein tensor then takes the form
\[
\alpha^{-1}\tilde{G}_{\mu\nu} = \frac{1}{2}(\gamma_{\alpha\rho}n_\rho n_\sigma + g_{\mu\rho}\gamma_{\rho\sigma}n^\rho n_\sigma - \gamma_{\nu\sigma}n^\rho n_\mu - \gamma_{\mu\sigma}n^\rho n_\nu) - \frac{1}{2}(\zeta_{\alpha\rho}n_\rho n_\sigma - \zeta_{\nu\alpha}n_\mu n_\sigma - \zeta_{\mu\rho}n_\alpha n_\sigma + \zeta_{\mu\sigma}n_\alpha n_\rho).
\]

From the field equation (4) the stress–energy tensor of the shell is written as [9]
\[
8\pi S^\mu\nu = \frac{1}{2}(\gamma n^\alpha n^\nu + g_{\mu\nu}\gamma_{\rho\sigma}n^\rho n_\sigma - \gamma_{\nu\sigma}n^\rho n_\mu - \gamma_{\mu\sigma}n^\rho n_\nu) + \frac{1}{2}(\zeta n^\alpha n^\nu - \zeta_{\nu\alpha}n_\mu n_\sigma - \zeta_{\mu\rho}n_\alpha n_\sigma + \zeta_{\mu\sigma}n_\alpha n_\rho).
\]

In order to ensure that the asymmetric tensor \(S^\mu\nu\) is purely tangential to \(\Sigma\), the following conditions must hold:
\[
(i) S^\mu\nu n_\mu = 0, \quad (ii) S^\mu\nu n_\nu = 0.
\]

It is easy to see that by virtue of the antisymmetric property of \(\zeta_{\mu\nu}\) two above conditions are automatically satisfied. The above expression for the stress–energy tensor of the null matter shell can be simplified after decomposing it into the basis of \((N^\mu, e^\mu_a)\). Using the completeness relations (7), one can find the following decomposition [9]:
\[
(\gamma n^\alpha - \zeta n^\alpha)n^\sigma = \frac{1}{2}\gamma n^\nu - \frac{1}{2}\gamma ab g^{ab} n^\nu + \zeta_{\lambda\sigma}N^\lambda n^\sigma - \gamma_{\lambda\sigma}n^\lambda N^\nu + g^{ab}(\gamma_{\lambda\sigma} - \zeta_{\lambda\sigma})e^\lambda_{\mu\sigma} e^{\nu}_{\nu\sigma}.
\]

Inserting (20) into equation (18), and using once more the completeness relations together with a rearrangement of terms, we finally end up with the three-dimensional intrinsic form for the surface stress–energy tensor of the null shell in the presence of torsion:
\[
16\pi S^{ab} = -g^{cd}_{\nu\alpha}n^d n^b - \gamma_{\nu\alpha}n^d g^{ab} + n^a g^{bc}_{\nu\alpha}(\gamma_{cd} - \zeta_{cd})n^d + n^b g^{ac}_{\nu\alpha}(\gamma_{cd} - \zeta_{cd})n^d.
\]

From the expression (21), the matter on the null shell can be characterized by
\[
\sigma = -\frac{1}{16\pi}g^{cd}_{\nu\alpha}n^d.
\]
as a surface energy density, and
\[ p = -\frac{1}{16\pi^2} \gamma_{cd} n^c n^d, \tag{23} \]
as an isotropic surface pressure, and
\[ f_i^a = \frac{1}{16\pi^2} g^{ad} (\gamma_{cd} - \zeta_{cd}) n^c, \tag{24} \]
\[ f_i^b = \frac{1}{16\pi^2} g^{bc} (\gamma_{cd} - \zeta_{cd}) n^d, \tag{25} \]
as the asymmetric surface energy currents. Note that due to the antisymmetric property of \( \zeta_{\mu\nu} \), one gets \( g^{ad} \zeta_{cd} = 0 \) and \( \zeta_{cd} n^c n^d = 0 \), indicating that the torsion contribution to the energy density \( \sigma \) in (22) and isotropic surface pressure \( p \) in (23) vanishes, respectively.

Therefore, it is seen that in the presence of torsion, the surface stress–energy tensor of the null shell is modified as given in (21). To compare this expression with that obtained in [4], we note that in the expression (67) in [4] the torsion contribution to the stress–energy tensor encoded in the tensor \( \beta_{ab} \) associated with the jump of the contorsion across \( \Sigma \) appears with a positive sign, while here the same role played by the antisymmetric tensor \( \zeta_{ab} \) associated with the jump of torsion across \( \Sigma \) turns out to have a negative sign.

It is seen from (21) that there is a part \( \hat{\gamma}_{cd} - \hat{\zeta}_{cd} \) of \( \gamma_{cd} - \zeta_{cd} \) not contributing to the matter content on the shell encoded in \( S_{ab} \). This part satisfies the following 7 independent equations:
\[ g^{ad} \gamma_{cd} = 0, \tag{26} \]
\[ (\gamma_{cd} - \zeta_{cd}) n^c = 0, \tag{27} \]
\[ (\gamma_{cd} - \zeta_{cd}) n^d = 0. \tag{28} \]

Since \( \gamma_{cd} - \zeta_{cd} \) has 9 independent components, it follows that \( \hat{\gamma}_{cd} - \hat{\zeta}_{cd} \) has two independent components contributing to the Weyl tensor on the shell and can be interpreted as representing the two degrees of freedom of polarization of an impulsive gravitational wave traveling along the shell [4].

To explore the requirements on the spin tensor for generating the present discontinuity of torsion, we insert (11) into the spin-torsion equation (5), leading to
\[ -\zeta_{\nu\rho\sigma} n^\mu - \delta_\mu^\rho \zeta_{\nu\sigma} n^\rho + \delta_\mu^\sigma \zeta_{\nu\rho} n^\rho = 8\pi [S^\mu_{\nu\sigma}]. \tag{29} \]

Recalling the antisymmetric property of the spin tensor on the last two indices, the contraction of (29) with the normal vector on each of the antisymmetric indices yields
\[ [S^\mu_{\nu\sigma} n^\nu] = 0, \quad [S^\mu_{\nu\sigma} n^\sigma] = 0, \tag{30} \]
but the contraction with \( n_\mu \) leads to
\[ 8\pi [S^\mu_{\nu\sigma} n_\mu] = \zeta_{\nu\rho} n^\rho n_\sigma - \zeta_{\sigma\rho} n^\rho n_\nu, \tag{31} \]
indicating that while the contractions (30) are continuous across the hypersurface, the antisymmetric tensor \( S^\mu_{\nu\sigma} n_\mu \) turns out to be discontinuous. A further contraction of (31) with the normal vector yields
\[ [S^\mu_{\nu\sigma} n_\mu n^\nu] = [S^\mu_{\nu\sigma} n_\mu n^\sigma] = 0. \tag{32} \]
The requirement (32) on the spin tensor was also obtained in [4] by imposing the condition that the surface stress–energy tensor must be tangential to $\Sigma$, while in the present work it is unconditionally obtained. The intrinsic forms of the constraints (30) are

\[
[S_{abc}n^b] = [S_{abc}n^b] = 0, \quad (33)
\]

while for (32) it would be

\[
[S_{abc}n^a n^b] = [S_{abc}n^a n^b] = 0. \quad (34)
\]

4. Conclusion

We have extended the Barrabes–Israel null shell formalism to the EC theory of gravitation, assuming the torsion discontinuity to be orthogonal to the null hypersurface. The intrinsic stress–energy tensor in the presence of torsion is then modified and splits into a Riemann part and a Cartan part. To take into account the antisymmetric property of the torsion tensor on its last two indices, the tensor $\zeta_{\mu\nu}$ associated with the torsion discontinuity is taken to be an antisymmetric tensor. As a result, the discontinuity of the contorsion tensor given by (12) is in accord with its antisymmetric property and needs not to be added as a condition. In addition, the asymmetric stress–energy tensor of the null shell turns out to be unconditionally tangent to the hypersurface $\Sigma$. Although the formalism was used for a null hypersurface, it may be applied to the nonlightlike cases accordingly.

ORCID iDs

S Khakshournia https://orcid.org/0000-0003-4517-3178

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