nα Resonating-Group Calculation with a Quark-Model $G$-Matrix $NN$ Interaction

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We calculate nα phase-shifts and scattering observables in the resonating-group method, using the nuclear-matter $G$-matrix of an $SU_6$ quark-model $NN$ interaction. The $G$-matrix is generated in the recent energy-independent procedure of the quark-model $NN$ interaction with the continuous prescription for intermediate spectra, by assuming an appropriate Fermi momentum $k_F = 1.2$ fm$^{-1}$. The nα RGM interaction kernels are evaluated with explicit treatments of the nonlocality and momentum dependence of partial-wave $G$-matrix components. The momentum dependence of the $G$-matrix components is different for each of the nucleon-exchange and interaction types. Without introducing any artificial parameters except for $k_F$, the central and spin-orbit components of the nα Born kernel are found to have reasonable strengths under the assumption of a rigid translationally invariant shell-model wave function of the α-cluster. The characteristic behaviors of three different exchange terms, corresponding to knockout, heavy-particle pickup and nucleon-rearrangement processes, are essentially the same between the case of previous local effective $NN$ forces and the case of nonlocal $G$-matrix $NN$ interactions.

§1. Introduction

Microscopic cluster models have been successfully used to describe structure and reactions of light nuclear systems. For instance, low-energy nα scattering is well described by the nα resonating-group method (RGM) with various model spaces and effective nucleon-nucleon ($NN$) forces. These effective forces usually incorporate only the central and spin-orbit ($LS$) forces. The very strong one-pion exchange tensor force of the bare $NN$ interaction is renormalized to the $^3E$ central force effectively. The usage of these effective forces is justified largely by the success of the RGM calculations, in which the existence of ample experimental data for the $NN$ scattering and light nuclear systems are prerequisite. On the other hand, the experimental data in the hypernuclear systems are still not yet sufficient and basic baryon-baryon interactions are not well known because of the technical difficulties of strangeness experiments. From the theoretical side, some progress is made with the study of baryon-baryon interactions and with accurate calculational techniques to solve few-body systems. One of these attempts is our effort to construct a unified set of quark-model $B_8$-$B_8$ potentials\textsuperscript{1)} and to apply them to solve few-body systems.\textsuperscript{2)} We now need a procedure to link bare and effective interactions through some effective interaction theory such as the $G$-matrix formalism. In this paper we present such a calculational scheme for light nuclei, in which we introduce no intermediate localized effective potential, and directly use $G$-matrices in nuclear matter. First, we
reexamine the well-studied problem of \( n\alpha \) RGM calculations from the viewpoint of our method to establish the reliability to proceed to hyperon-nucleus systems. The latter applications are to be reported in a successive paper.

We will carry out \( n\alpha \) RGM calculation using a quark-model \( G \)-matrix \( NN \) interaction and a framework that has recently been developed for \( \alpha \)-cluster folding in the study of the baryon-octet (\( B_8 \)) \( \alpha \) interaction.\(^3\) In this framework, the partial-wave components of the \( G \)-matrix are explicitly used to generate the direct and knock-on\(^7\) terms of the \( n\alpha \) RGM kernel, without making any kind of local approximation for the \( G \)-matrix. The center-of-mass (c.m.) motion of two interacting nucleons in the \( n\alpha \) system is correctly treated for the Galilean non-invariant \( G \)-matrix interaction. The \( G \)-matrix and momentum-dependent single-particle (s.p.) potentials are predetermined by solving the Bethe-Goldstone equation in symmetric nuclear matter.\(^4\)

The only assumption is a constant Fermi momentum \( k_F \) for generating the \( G \)-matrix interaction that can be used as an effective interaction in light nuclear systems. We use the energy-independent version of the quark-model \( NN \) interaction to calculate the \( G \)-matrix, but the difference from the previous energy-dependent version in Ref. \(^4\) is very little. The original quark-model \( NN \) interactions, fss2 and FSS, used in this paper are a unified model for full octet-baryons,\(^1\) which have achieved accurate descriptions of the \( NN \) and \(YN \) scattering observables. In particular, the \( NN \) interaction of the most recent model fss2\(^5\) is sufficiently accurate in comparison with those of modern realistic meson-exchange models.

Since the \( n\alpha \) RGM kernel involves various nucleon-exchange terms, we need to extend the previous folding formula, starting from the transformation formula developed in Appendix A of Ref. \(^6\). In general, the interaction kernel involves five different interaction types for two-cluster configurations with a common width parameter of \((0s)\) clusters. Among them, the exchange terms, called the \( 1S \) and \( 1S' \) types in this paper, do not appear in the previous hyperon-\( \alpha \) interaction and need a special treatment in the present \( n\alpha \) study. These terms correspond to the so-called heavy-particle pickup process\(^7\) and play an important role in the posterior increase in the differential cross sections in the low-energy and intermediate-energy regions. On the other hand, the knock-on term specified by the \( 1D_- \) type gives an energy-dependent extra attraction to the direct potential specified by \( 0D_+ \), and contributes mainly to the forward direction in the usual RGM treatment with effective \( NN \) forces. In the present formalism, all these interaction kernels have their own momentum dependence for the Galilean non-invariant two-nucleon interaction. We can calculate the momentum dependence explicitly in the analytic form. A very important starting-energy dependence of the \( G \)-matrix is renormalized into this momentum dependence and the dependence to the relative momentum, and to the Fermi momentum \( k_F \) as well. Owing to the explicit evaluation of all the interaction kernels, the existence of the Pauli forbidden \((0s)\) state between the neutron and the \( \alpha \) cluster is strictly preserved. We will find that the present procedure gives rather reasonable strengths of the \( n\alpha \) interaction for both the central and LS com-

\(^{\text{3}}\) We will use in this paper the terminology “knock-on term” to specify the interaction terms responsible for the “knockout process” in the nuclear reaction mechanism discussed in Ref. \(^7\).
ponents, and reproduces reasonably well the empirical $S$-wave and $P$-wave phase shifts and the low-energy differential cross sections and polarizations below the neutron incident energy $E_n \sim 30$ MeV. For higher energies, we investigate the $n\alpha$ Born kernel and find that the characteristic behavior of three different groups of exchange terms, corresponding to knockout, heavy-particle pickup and nucleon-rearrangement processes, found in previous studies using local effective $NN$ forces,\textsuperscript{7} is essentially unchanged even in the present study with the nonlocal $G$-matrix interaction.

The $n\alpha$ RGM has been examined by many authors from many different viewpoints. In fact, early studies have naturally paid full attention to the adequacy of the model space and the $NN$ force used in the calculation. For example, Sugie \textit{et al.}\textsuperscript{8} studied the contribution of tensor force, trying to explain the large energy splitting of the $J^P = 3/2^-$ and $1/2^-$ states. Since the inherent $LS$ force of the $NN$ interaction has been found after this calculation, the authors did not include it and found that tensor force can account for only about 30% of the observed splitting. Later calculations by Kanada \textit{et al.}\textsuperscript{9} and Omojola\textsuperscript{10} included the $LS$ force and have found that it is important to deal with the $D$-state components of the $\alpha$ cluster and realistic $NN$ interactions. In these calculations, an approximate Hamada and Johnston potential is used in the Gaussian form. They reproduced the low-energy $n\alpha$ phase shifts reasonably well. The channel coupling effect of the $n\alpha$ and $d^3H$ configurations are examined by Heiss and Hackenbloch.\textsuperscript{11} Thompson and Tang\textsuperscript{7} analyzed the properties of the $n\alpha$ RGM exchange kernels for an effective central $NN$ force, and classified them into three different groups of terms, called the knockout, heavy-particle pickup and nucleon-rearrangement terms. Chwieroth \textit{et al.}\textsuperscript{12} and later calculations\textsuperscript{13,14} have clarified that the distortion effect of the $\alpha$ cluster also has an appreciable effect even in single-channel $n\alpha$ RGM calculation. Nevertheless, the effect is rather minor, in comparison with the other two-cluster systems, since the $\alpha$ particle is tightly bound. After all of these investigations, it is gradually recognized that single-channel $n\alpha$ RGM calculation, using a rigid $(0s)^4 \alpha$-cluster wave function and a simple effective $NN$ force of the central and $LS$ types, reproduces the low-energy $n\alpha$ scattering fairly well, except for some specific energy regions where other reaction channels open. This understanding cannot directly be reconciled with the recent \textit{ab initio} calculation by Nollett \textit{et al.},\textsuperscript{15} who claim that the correct $P$-wave spin-orbit splitting of the $n\alpha$ scattering phase shifts in the low-energy region can only be achieved with the effect of an appropriate three-nucleon interaction.

The organization of this paper is as follows. In the next section, we first recapitulate in Sec. 2.1 the standard RGM formulation in the momentum representation, together with the $G$-matrix calculation of the quark-model baryon-baryon interaction for symmetric nuclear matter. The full expressions of the $n\alpha$ exchange interaction kernels for the $G$-matrix $NN$ interaction are given in Sec. 2.2. The partial-wave decomposition is presented in Sec. 2.3. In Sec. 2.4, we discuss the selection of the starting energy in the $G$-matrix calculation. Numerical results of the $G$-matrix calculation and the $n\alpha$ RGM phase shifts are given in Sec. 3.1. The $n\alpha$ scattering cross sections and polarization for neutron incident energies less than 30 MeV are compared with experimental results in Sec. 3.2. The characteristic behavior of the $n\alpha$ Born amplitudes for higher energies is analyzed in Sec. 3.3. Section 4 is devoted to
the summary of this paper. The $n\alpha$ RGM Born kernels for a Gaussian-type effective $NN$ force are given in Appendix A. In Appendix B, we give the exchange interaction kernels of a Galilean non-invariant $NN$ interaction for general systems, composed of two $(0s)$-shell clusters.

§2. Formulation

2.1. Lippmann-Schwinger RGM and the $G$-matrix calculation of symmetric nuclear matter

For the correct treatment of the c.m. motion, it is most convenient to formulate the RGM in the momentum representation, which we call the Lippmann-Schwinger RGM (LS-RGM). In this approach, we write the RGM equation in the Schrödinger-type equation

$$\left[\varepsilon - T_r - V^{\text{RGM}}(\varepsilon)\right] \chi = 0,$$

where $T_r$ is the kinetic-energy operator for the relative motion of two clusters, $\varepsilon = E - E_{\text{int}}$ with $E_{\text{int}}$ being the internal energy, and $V^{\text{RGM}}(\varepsilon) = V_{\text{D}} + G + \varepsilon K$ the sum of the direct potential $V_{\text{D}}$, the exchange kernel $G$, and the energy-dependent term $\varepsilon K$, inherent in the RGM formalism. We use the notation $K$ for the exchange normalization kernel, which is defined through

$$N = \langle \phi^{\text{int}} | {\mathcal A}' | \phi^{\text{int}} \rangle = 1 + X_N M_N = 1 - K. $$

(2.2)

In the $n\alpha$ RGM, the spin-isospin factor for the exchange normalization kernel is $X_N = -1$, and $M_N$ is the corresponding spatial part. The total wave function is expressed as $\Psi = {\mathcal A}' \{ \phi^{\text{int}} \chi \}$ using the relative wave function $\chi$ in Eq. (2.1). Here, $\phi^{\text{int}}$ is the internal wave function and $\mathcal{A}'$ the antisymmetrization operator between two clusters. For the $n\alpha$ system, these are given by $\phi^{\text{int}} = \phi_\alpha \xi_{0\frac{1}{2}}$ and $\mathcal{A}' = 1 - \sum_{i=1}^{4} P_{(i5)} \to 1 - 4P_{(45)}$ using the internal wave function of the $\alpha$ cluster $\phi_\alpha = \phi^{\text{orb}}_\alpha \xi_{00}$ with $\phi^{\text{orb}}_\alpha$ being the spatial part of the translationally invariant $(0s)^4$ harmonic-oscillator (h.o.) wave function without the c.m. motion. The spin-isospin wave functions are denoted by $\xi_{00}$ and $\xi_{\frac{1}{2},\frac{1}{2}}$ for the $\alpha$ cluster and the fifth nucleon, respectively. In this particular case, we have one Pauli-forbidden state $|u_{(0s)}\rangle$ satisfying

$$K |u_{(0s)}\rangle = |u_{(0s)}\rangle \quad \text{and} \quad \mathcal{A}' \{ \phi^{\text{int}} u_{(0s)} \} = 0.$$

(2.3)

The exchange kernel $G = G^K + G^V$ is composed of the exchange kinetic-energy kernel $G^K$ and the exchange interaction kernel $G^V$ for the total Hamiltonian consisting of

$$H = \sum_{i=1}^{5} t_i - T_G + \sum_{i<j}^{5} v_{ij} $$

(2.4)

in the $n\alpha$ RGM. Equation (2.1) is then solved in the form of the Lippmann-Schwinger equation for the half-off shell $T$-matrix with discretized momentum mesh points, which is fully spelled out in Ref. 6).
The main task in the LS-RGM is therefore to calculate the plane-wave matrix elements of various exchange kernels, which we call the RGM Born kernels. For the interaction kernels, we separate the two-nucleon force into the spatial and spin-isospin parts, using \( v_{ij} = u_{ij} w_{ij} \). The direct potential \( V_D \) and the exchange interaction kernel \( G^V \) are obtained from the general expression

\[
M(q_f, q_i) = \langle \delta(X_G) e^{iq_f \cdot r} \phi^\text{int} | \sum_{i<j} v_{ij} A' | 1 \cdot e^{iq_i \cdot r} \phi^\text{int} \rangle
= \sum_{xT} X_{xT} M_{xT}(q_f, q_i) .
\]

(2.5)

Here, the spatial integral \( M_{xT}(q_f, q_i) \) and the spin-isospin factor \( X_{xT} \) are defined by

\[
M_{xT}(q_f, q_i) = \langle \delta(X_G) z_x e^{iq_f \cdot r} \phi^\text{orb} | u_{ij} | 1 \cdot e^{iq_i \cdot r} \phi^\text{orb} \rangle ,
X_{xT} = C_x \langle z_x | \sum_{i<j} w_{ij} | \xi \rangle \quad \text{with} \quad (i, j) \in T .
\]

(2.6)

In the \( n\alpha \) RGM, \( \phi^\text{orb} = \phi^\text{orb}_\alpha \) and \( \xi = \xi_0 \xi_{1\alpha} \). These kernel components are specified by the number of exchanged nucleons, \( x = 0, 1 \), with \( C_0 = 0, C_1 = -4 \), and various interaction types \( T \). More specifically, \( x = 0 \) with \( z_0 = 1 \) corresponds to the direct terms including \( T = E \) and \( D_+ \) interaction types and \( x = 1 \) with \( z_1 = P_{(45)} \) the one-nucleon exchange terms. The interaction types, \( T = E, S, S', D_+ \) and \( D_- \), correspond to some specific \( (i,j) \) pairs of the nucleons.\(^{16}\) When an effective \( NN \) force is used, these Born kernels are most easily calculated using a general transformation formula developed in Appendix A of Ref. 6). For Gaussian-type effective \( NN \) forces with the central form

\[
v^{(C)} = u(r) w = v_0 e^{-\kappa r^2} (W + BP_\sigma - HP_\tau - MP_\sigma P_\tau) ,
\]

(2.7)

and the LS form

\[
v^{(LS)} = u^{LS}(r) w^{LS} = v_0^{LS} e^{-\kappa r^2} (W - HP_\tau) (L \cdot S) ,
\]

(2.8)

the final results of the \( n\alpha \) Born kernels are given in Appendix A for completeness.

The quark-model baryon-baryon interaction is formulated in a similar way to the \( n\alpha \) RGM.\(^1\) In this case, the internal wave function is a product of two three-quark clusters \((3q)-(3q)\). We solve the G-matrix equation

\[
G_{NN}(p, q; K, \omega, k_F) = V_{NN}^\text{RGM}(p, q) + \frac{1}{(2\pi)^3} \int d k V_{NN}^\text{RGM}(p, k) \frac{Q(k, K, k_F)}{e(k, K; \omega)} G_{NN}(k, q; K, \omega, k_F) ,
\]

(2.9)

using the energy-independent Born kernel \( V_{NN}^\text{RGM}(p, q) \) for the \((3q)-(3q)\) system.\(^4\) Here, \( V_{NN}^\text{RGM}(p, q) \) is defined by\(^17\)

\[
V_{NN}^\text{RGM}(p, q) = V_D(p, q) + G(p, q) + W(p, q) ,
\]

(2.10)
with
\[
W = \frac{1}{\sqrt{N}} (T_r + V_D + G) \frac{1}{\sqrt{N}} - (T_r + V_D + G) .
\] (2.11)

Details of this energy-independent treatment of the quark-model baryon-baryon interaction in the \( G \)-matrix formalism will be published elsewhere. In Eq. (2.9), we use the angle-averaged Pauli operator \( Q(k, K, k_F) \), and the energy denominator \( e(k, K; \omega) \) is given by
\[
e(k, K; \omega) = \omega - E_N(k_1) - E_N(k_2),
\] (2.12)
with the starting energy \( \omega \) expressed as
\[
\omega = E_N(q_1) + E_N(q_2).
\] (2.13)

The s.p. momenta \( q_1 \) and \( q_2 \) (\( k_1 \) and \( k_2 \)) are related to the relative momentum \( q (k) \) and the c.m. momentum \( K \) through the conventional relationship \( q = (q_1 - q_2)/2 \) and \( K = q_1 + q_2 \ (k = (k_1 - k_2)/2 \) and \( K = k_1 + k_2) \). The nucleon s.p. potential \( U_N(q_1) \) included in the s.p. energy
\[
E_N(q_1) = \frac{\hbar^2}{2M_N}q_1^2 + U_N(q_1)
\] (2.14)
is determined self-consistently in the standard procedure. In Eq. (2.14), \( M_N \) is the nucleon mass. The procedure to include the s.p. potential even in the intermediate spectra in Eq. (2.12) is called a continuous prescription.

2.2. Interaction kernels for \( G \)-matrix \( NN \) interaction

In this subsection, we derive the interaction Born kernel for the \( G \)-matrix \( NN \) interaction. The starting point is the invariant \( G \)-matrix\(^3\) expressed as\(^4\)
\[
G^L_{NN}(p, p’; K, \omega, k_F)
= \frac{1}{2} \langle [NN]_{I_1} | G(p, p'; K, \omega, k_F) - G(p, -p'; K, \omega, k_F) P_\sigma P_\tau | [NN]_{I_1} \rangle
= g^L_0 + g^L_{ss}(\sigma_1 \cdot \sigma_2) + h^L_0 i\vec{n} \cdot (\sigma_1 + \sigma_2) + \cdots .
\] (2.15)

Here \( \vec{n} = |p' \times p|/(p'p \sin \theta) \), and the invariant functions \( g^L_0 \) (central), \( g^L_{ss} \) (spin-spin), \( h^L_0 \) (LS), etc. are functions of \( p = |p|, \ p' = |p'| \), and \( \cos \theta = (\vec{p} \cdot \vec{p'}) \), as well as the \( G \)-matrix parameters \( K, \omega \) and \( k_F \). These are expressed by the partial-wave components of the \( NN \) \( G \)-matrix as in Appendix D of Ref. 6). In the following, we will focus on the momentum dependence of the \( G \)-matrix, and keep only the parameter \( K \) in \( G^L_{NN}(p, p'; K, \omega, k_F) \), since the explicit dependence depends on the interaction type. As in Ref. 3), it is convenient to write the isospin dependence of the invariant \( G \)-matrix as
\[
G_{NN}(p, p'; K) = G^L_{NN}(p, p'; K) \frac{1 + P_r}{2} + G^L_{NN}(p, p'; K) \frac{1 - P_r}{2} .
\] (2.16)

\(^3\) The invariant \( NN \) \( G \)-matrix in Eq. (2.15) is defined without the factor 2, shown in Eq. (2.4) of Ref. 3), since the exchange terms are explicitly calculated in the present \textit{no} RGM formalism.
Table I. Spin-isospin factors $X_{xT}^{O\Omega}$ for the invariant species, $\Omega = 0$ (g0), ss (gss), and LS (h0), in Eq. (2.17). The factors for $xT = 1E$, 1S, and 1S$'$ are obtained from $X_{xT}^{0E} = X_{xT}^{0S} = X_{xT}^{0\Omega} = (1/2)X_{xT}^{1E}$. For the LS term, $X_{xT}^{LS} = X_{xT}^{LS} \cdot 2S$ is assumed with the neutron spin operator $S$.

| I   | $X_{xT}^{0E}$ | $X_{xT}^{0S}$ | $X_{xT}^{0\Omega}$ |
|-----|----------------|----------------|-------------------|
| 0   | 1              | 1              | $-\frac{1}{2}$    |
| 1 ss| -3             | 0              | $-\frac{3}{2}$    |
| LS  | -              | 1              | -1               |
| 0 ss| 3              | 0              | $\frac{3}{2}$    |
| LS  | -              | 1              | 1                |

We find it convenient to separate the isospin multiplicity factor $(2I + 1)$ and define the spin-isospin factors in Eq. (2.6) by

$$X_{xT}^{O\Omega} = (2I + 1) X_{xT}^{O\Omega}.$$  \hfill (2.17)

The interaction species $\Omega = 0$, ss and LS, correspond to the invariant functions, $g_0$, $g_{ss}$ and $h_0$, respectively. Table I lists $X_{xT}^{O\Omega}$ for each of these species.

The spatial integrals are obtained by assuming a general Galilean non-invariant interaction

$$\langle p_1, p_2 | u | p_1', p_2' \rangle = \delta(K - K') \frac{1}{(2\pi)^3} u(k', q'; |K|),$$  \hfill (2.18)

where $u$ is $g_0^i$, $g_{ss}^i$, or $h_0^i \hat{m}$. Here the relative momentum $p$ and the total momentum $K$ (and also $p'$ etc. with primes) are related to $p_1$ and $p_2$ by $p = (p_1 - p_2)/2$ and $K = p_1 + p_2$, respectively, and a further transformation from $p$ and $p'$ to $k' = p - p'$ and $q' = (p + p')/2$ is applied. The necessary spatial integrals for the no$\alpha$ system is obtained from more general expressions given in Appendix B for systems of two (0s)-shell clusters.

For the Galilean non-invariant $G$-matrix interaction, the subtraction of the internal-energy part of the $\alpha$ cluster involves a subtle problem. In the total c.m. system of the no$\alpha$ system, the 0E-type spatial integral, involving the $\alpha$-cluster internal-energy contribution, becomes momentum-dependent. Similarly, the 1E-exchange-type spatial integral also involves a momentum dependence besides the exchange normalization kernel $M_N(q_f, q_i)$. These are explicitly given by

$$M_{0E}(q_f, q_i) = (2\pi)^3 \delta(k) E_{0E}^V(q), \quad M_{1E}(q_f, q_i) = M_N(q_f, q_i) E_{1E}^V(q),$$  \hfill (2.19)

where $k = q_f - q_i$, $q = (q_f + q_i)/2$, and $E_{xE}^V(q)$ with $x = 0, 1$ are calculated from

$$E_{0E}^V(q) = \left( \frac{1}{2\pi \nu} \right)^{\frac{3}{2}} \int K e^{-\frac{1}{\nu}(K - \frac{1}{2}q)^2} \mathcal{E}(K),$$

$$E_{1E}^V(q) = \left( \frac{3}{4\pi \nu} \right)^{\frac{3}{2}} \int K e^{-\frac{1}{\nu}(K - \frac{1}{2}q)^2} \mathcal{E}(K),$$  \hfill (2.20)
with
\[
\mathcal{E}(K) = \frac{1}{(2\pi)^3} \left( \frac{1}{\pi \nu} \right)^{\frac{1}{2}} \int dp \, dp' \, \exp \left\{ -\frac{1}{2\nu} \left( p^2 + p'^2 \right) \right\} \, g(p, p'; K). \tag{2.21}
\]

Here \( g(p, p'; K) = u(k', q'; K) \) and \( \nu \) is the h.o. width parameter for the \((0s)^4\) \(\alpha\) cluster. Owing to the different \(q\) dependence involved in the \(0E\) and \(1E\) types, the standard subtraction of the \(\alpha\)-cluster internal energy in \(V_{\text{RGM}}^{\nu}(\varepsilon)\) is not complete and is given by
\[
V_{\text{RGM}}^{\nu}(\varepsilon) = V_D + G^K + \tilde{G}^V + \varepsilon K ,
\]
\[
V_D = X_{0D^+} M_{0D^+} ,
\]
\[
\tilde{G}^V = \frac{1}{2} X_{0E} \left( \tilde{M}_{1E} - M_{1S} - M_{1S'} \right) + X_{1D^-} M_{1D^-} ,
\]
\[
\tilde{M}_{1E} = 2M_{0E} - M_{1E} - 2M_{0E}^{(0)} (1 - K) . \tag{2.22}
\]

Here we have omitted \(\Omega\) and the isospin sum for simplicity. In deriving Eq. (2.22), we have assumed that only the \(q = 0\) part of \(E_{0E}^{\nu}(q)\) contributes to the \(\alpha\) internal energy and defined the relative energy \(\varepsilon\) using this zero-momentum \(\alpha\) energy. In Eq. (2.22), \(M_{0E}^{(0)}\) implies
\[
M_{0E}^{(0)}(q_f, q_i) = (2\pi)^3 \delta(k) E_{0E}^V(0) . \tag{2.23}
\]

Since the delta function part in \(\tilde{M}_{1E}\) of Eq. (2.22) is inconvenient for the exchange interaction kernel, we assume
\[
\tilde{M}_{1E}(q_f, q_i) = M_N(q_f, q_i) \left[ 2E_{0E}^V(q) - E_{1E}^V(q) \right] , \tag{2.24}
\]
without violating the redundancy property of the interaction kernel. In fact, the \(q\) dependence in \(E_{0E}^V(q)\) and \(E_{0E}^V(q)\) is very weak for the wide range of \(q\) and the difference between \(E_{0E}^V(q)\) and \(E_{1E}^V(q)\) is also very small.

Some simplification takes place in the RGM calculation using partial-wave components of the invariant interaction. We can easily show that for each isospin component the direct term \((0D^+\text{ type})\) and the knock-on term \((1D^-\text{ type})\) give the same contribution, when the central \((\nu = 0)\) and spin-spin \((\nu = ss)\) components are added up. For the central \(1E\text{-}, 1S\text{-}\) and \(1S'\text{-}\)type spatial integrals, only the \(S\)-wave component of the two-nucleon interaction contributes to the exchange kernel. Thus, by incorporating the spin-isospin factors given in Table II we find for the central \(\alpha\) RGM kernel (we remove the tilde of \(\tilde{M}_{1E}\) using the approximation in Eq. (2.24))
\[
V_{\text{RGM}}^{\nu}(\varepsilon) = (V_D + G^V) + G^K + \varepsilon K ,
\]
\[
V_D + G^V = 2 \sum_{I=0}^{1} (2I + 1) \left( M_{0D^+}^{(0I)} + M_{1E}^{0I} - M_{1S}^{0I} - M_{1S'}^{0I} \right) , \tag{2.25}
\]
where \(\Omega I = 0I\) spatial functions are calculated from the \(g_0^I\) (central) component in Eq. (2.15). Similarly, the \(LS\) RGM kernel is obtained only from the \(0D^+\text{-type}}
contribution using the $h_0^1$ component in Eq. (2.13). The internal energy of the $\alpha$ cluster is given by
\[ E_\alpha(q) = 3 \frac{3h^2}{2M} + 4 \left[ 3E_{0E}^{V01}(q) + E_{0E}^{V00}(q) \right] + 2e^2 \sqrt{\frac{\nu}{\pi}}, \] (2.26)
where $E_{0E}^{V0I}(q)$ is the central component of $E_{0E}^{V}(q)$ with an isospin $I = 0, 1$.

To calculate the RGM kernel, it is sometimes convenient to use either of the forms $g(p, p'; K)$ and $u(k', q'; K)$, which are equal to each other with $k' = p - p'$ and $q' = (1/2)(p + p')$. The kernel expressions become sometimes simple, if we use $k = q_f - q_i$ and $q = (1/2)(q_f + q_i)$ instead of $q_f$ and $q_i$. For example, the exchange normalization kernel $M_N(q_f, q_i) = K(q_f, q_i)$ and the exchange kinetic-energy kernel $G^N(q_f, q_i)$ are given in Eqs. (A.1) and (A.2). The $E$-type integrals are given in Eqs. (2.19) - (2.21). For the direct term, we obtain
\[ M_{0D+}(q_f, q_i) = e^{-\frac{3h^2}{2M}} \left( \frac{8}{3\pi^2} \right)^{\frac{3}{2}} \int dq' \exp \left\{ -\frac{8}{3\nu} \left( q' - \frac{5}{3}q \right)^2 \right\} \]
\[ \times u(k, q'; 2|q - q'|), \] (2.27)
and the $S$- and $S'$-type integrals are given by
\[ M \left\{ \begin{array}{c} 1S \\ 1S' \end{array} \right\}(q_f, q_i) = M_N(q_f, q_i) \left( \frac{3}{2\pi^3} \right)^{\frac{3}{2}} \]
\[ \times \int dp \int dp' \exp \left\{ \begin{array}{c} -\frac{1}{2\nu} (5p^2 + p'^2) + \frac{4}{3} p (q_f + \frac{1}{4}q_i) - \frac{4}{3\nu} (q_f + \frac{1}{4}q_i)^2 \\ -\frac{1}{2\nu} (p^2 + 5p'^2) + \frac{4}{3} p' (\frac{1}{4}q_f + q_i) - \frac{4}{3\nu} (\frac{1}{4}q_f + q_i)^2 \end{array} \right\} \]
\[ \times \left\{ \begin{array}{c} g(p, p'; 2|q_f - p|) \\ g(p, p'; 2|q_i - p'|) \end{array} \right\}, \] (2.28)

2.3. Partial-wave expansion

The partial-wave decomposition of the $n\alpha$ RGM kernels derived in the preceding section can be carried out for two different types of the partial-wave decomposition of the invariant $G$-matrix $NN$ interaction:
\[ g(p, p'; K) = \sum_{\ell=0}^{\infty} (2\ell + 1) g_\ell(p, p'; K) P_\ell(\hat{p} \cdot \hat{p}'), \]
\[ u(k', q'; K) = \sum_{\lambda=0}^{\infty} (2\lambda + 1) u_\lambda(k', q'; K) P_\lambda(\hat{k}' \cdot \hat{q}'), \] (2.29)
for the central part and
\[ h(p, p'; K) = \sum_{\ell=1}^{\infty} (2\ell + 1) h_\ell(p, p'; K) P_\ell^1(\hat{p} \cdot \hat{p}'), \]
\[ u^{LS}(k', q'; K) = \sum_{\lambda=1}^{\infty} (2\lambda + 1) u^{LS}_\lambda(k', q'; K) P_\lambda^1(\hat{k}' \cdot \hat{q}'), \] (2.30)
for the LS part. Here, the $u$-type partial-wave components are calculated from
\[
g_0^I(p, p'; K) \left\{ g_0^I(p, p'; K) \right\} = \frac{1}{4} \sum_{J,S} \left( \frac{2J + 1}{2\ell + 1} \right) \left\{ \frac{1}{3} [2S(S + 1) - 3] \right\} G_{J,S}^I(p, p'; K) \\
(\ell = 0, 1, 2, \ldots),
\]
and
\[
h_0^I(p, p', K) = -\frac{1}{4} \left[ \frac{1}{\ell(\ell + 1)} G_{1,1}^I(p, p'; K) + \frac{2\ell - 1}{\ell(2\ell + 1)} G_{1,0}^I(p, p'; K) \right. \\
- \left. \frac{2\ell + 3}{(\ell + 1)(2\ell + 1)} G_{1,1}^I(p, p'; K) \right] (\ell = 1, 2, 3, \ldots),
\]
using the transformation formula (see Eq. (2.22) of Ref. 3)
\[
u_\lambda^\Omega(k, q; K) = \sum_{i,j} \sum_{\ell = 0}^{\infty} (2\ell + 1) F_{i,j}^{\Omega,\lambda}(k, q) g_\ell^\Omega(p, p; K) \quad (\Omega = C, LS).
\]

Here, $g_\ell^\Omega(p, p; K)$ is assigned to $g_0^I(p, p'; K)$ or $g_0^I(p, p'; K)$ in Eq. (2.31) for the central part and to $h_0^I(p, p'; K)$ for the LS part. In Eq. (2.31), the prime on the summation symbol indicates that the sum is taken only for quantum numbers such as $J$ and $S$ that satisfy the generalized Pauli principle $(-1)^{\ell+S+1} = -1$, and the angular-momentum coupling $(\ell S)J$. Similarly, the LS component involves only odd $\ell$ for $I = 1$ and even $\ell$ for $I = 0$. As discussed in Ref. 3), $u_\lambda(2q', k/2; K)$ needed in the calculation of $M_{1D_+}(k, q) = M_{1D_+}(q_f, q_i)$ is obtained by simply changing $g_\ell(p, p'; K)$ to $(-1)^{\ell} g_\ell(p, p'; K)$ in Eq. (2.32). This is a consequence of the symmetry property satisfied by the coefficients, $F_{i,j}^{\Omega,\lambda}(k, q)$, given in Eq. (2.23) of Ref. 3). Because of this symmetry, we only need to calculate the partial wave components for $M_{0D_+}(q_f, q_i)$ for the spin-independent central component $g_0^I(p, p'; K)$, as shown in Eq. (2.25).

For the $E$-type RGM kernel, only the $S$-wave components of the $G$-matrix interaction contribute to the RGM kernel:
\[
E(K) = \frac{2}{\pi} \left( \frac{1}{\pi\nu} \right)^{\frac{3}{2}} \int p^2 \, dp \, p'^2 \, dp' \, \exp \left\{ -\frac{1}{2\nu} \left( p^2 + p'^2 \right) \right\} g_0(p, p'; K),
\]
and
\[
E_{0E}(q) = 4\pi \left( \frac{3}{4\pi\nu} \right)^{\frac{3}{2}} \int_0^\infty K^2 \, dK \, e^{-\frac{1}{2\nu} (K - \frac{1}{2}q)^2} i_0 \left( \frac{1}{2\nu} qK \right) E(K),
\]
\[
E_{1E}(q) = 4\pi \left( \frac{3}{4\pi\nu} \right)^{\frac{3}{2}} \int_0^\infty K^2 \, dK \, e^{-\frac{3}{4\nu} (K - \frac{3}{2}q)^2} i_0 \left( \frac{2}{\nu} qK \right) E(K),
\]
where $i_\lambda(x) = e^{-x} i_\lambda(x) = e^{-x} i_\lambda j_\lambda(-ix)$ is employed. For the $0D_+$-type RGM kernel, the partial wave decomposition is carried out, including the momentum dependence of $K = 2|q - q'|$:
\[
M_{0D_+}(k, q) = \sum_{\lambda = 0}^\infty (2\lambda + 1) M_{0D_+ \lambda}(k, q) P_\lambda(\tilde{k} \cdot \tilde{q}).
\]
The result is
\[
\mathcal{M}_{0D+\lambda}(k, q) = 4\pi e^{-\frac{3}{32\nu}k^2} \left(\frac{8}{3\pi\nu}\right)^{\frac{3}{2}} \int_{-1}^{1} dq' \frac{1}{2} \int_{0}^{\infty} q'^2 dq' \int_{0}^{\infty} \frac{d x}{x} e^{-\frac{3}{32\nu}(q' - \frac{3}{8}q)^2} \times e^{-\frac{3}{8\nu}q'(1-x)} u_{\lambda}(k, q', 2\sqrt{q^2 + q'^2 - 2qq'x}) P_{\lambda}(x) .
\]

This term (and also \(M_{1E}^{0I}\)) term of Eq. (2.24) is transformed back to the partial wave component \(M_{0D+\lambda}(p_f, p_i)\) by the inverse transformation of Eq. (2.32). The partial-wave components of the \(LS\) term \(M_{0D+\lambda}^{0I}(p_f, p_i)\) are similarly obtained. For the \(S\) type (and also for \(S'\) type), it is easiest to calculate \(M_{1S}(q_f, q_i)\) directly from
\[
M_{1S}(q_f, q_i) = \sum_{\ell=0}^{\infty} (2\ell + 1) M_{1S}(q_f, q_i) P_{\ell}(q_f \cdot \tilde{q}_i) .
\]

We obtain
\[
M_{1S}(q_f, q_i) = (-1)^\ell \frac{2}{\pi} \left(\frac{2}{\nu}\right)^{\frac{3}{2}} e^{-\frac{3}{32\nu}(3q_f^2 + \tilde{q}_i^2)} \int_{0}^{\infty} p'^2 dp' e^{-\frac{1}{2\nu}p'^2}
\times \int_{0}^{1} K^2 dK e^{-\frac{3}{32\nu}K^2 + \frac{1}{\nu}(q_f + q_i)K} e^{-\frac{1}{2\nu}q_fK} \times \frac{1}{2} \int_{-1}^{1} \frac{d x}{x} e^{-\frac{1}{2\nu}q_f K(1-x)} g_0 \left(\frac{q_f^2 + K^2/4 - q_f K x, p'}{K} \right) P_{\ell}(x) .
\]

Here again, we only find the contribution from the \(S\)-wave components of the \(G\)-matrix interaction. If we modify \(\sqrt{q_f^2 + K^2/4 - q_f K x} \rightarrow q_f\) in the third line of Eq. (2.38), it becomes \(\tilde{\tau}_\ell(q_f K/2\nu)\) \(g_0(q_f, p'; K)\). The \(S'\)-type component is obtained from the symmetry discussion as
\[
M_{1S'}(q_f, q_i) = M_{1S}(q_i, q_f) .
\]

With all of these contributions, the partial-wave component of the RGM Born kernel for the \(n\alpha\) system is given by
\[
V^{\text{RGM}}_{\ell}(q_f, q_i; \varepsilon) = V^C_{\ell}(q_f, q_i; \varepsilon) + V^{LS}_{\ell}(q_f, q_i) (\ell \cdot S)_{J\ell} ,
\]
\[
V^C_{\ell}(q_f, q_i; \varepsilon) = G^K_{\ell}(q_f, q_i) + \varepsilon K_{\ell}(q_f, q_i)
+ 2 \sum_{I=0}^{1} (2I + 1) \left[M_{0D+\ell}^{0I}(q_f, q_i) + M_{1E}^{0I}(q_f, q_i) - M_{1S}^{0I}(q_f, q_i) - M_{1S'}^{0I}(q_f, q_i)\right] ,
\]
\[
V^{LS}_{\ell}(q_f, q_i) = 4 \sum_{I=0}^{1} (2I + 1) M_{0D+\ell}^{LS I}(q_f, q_i) ,
\]
where \((\ell \cdot S)_{J\ell} = \ell/2\) for \(J = \ell + 1/2\) and \(-\ell/2\) for \(J = \ell - 1/2\), and \(K_{\ell}(q_f, q_i)\) and \(G^K_{\ell}(q_f, q_i)\) are given in Eq. (A.3).
2.4. Selection of starting energies of G-matrix

In previous calculations of $B_{\Lambda\Lambda}$ interactions with $B_{\Lambda} = \Lambda$, $\Sigma$ and $\Xi$, we used the starting-energy dependence of the G-matrix according to the rule of the nuclear matter calculation. Namely, we used the angle-averaging procedure over the relative momentum $\bm{q}$ (which is the integral variable) in the relationship, $\bm{K} = \bm{q}_1 + \bm{q}_2$ and $\bm{q} = (1/2)(\bm{q}_1 - \bm{q}_2)$, under the constraint $|\bm{q}_2| < k_F$. This procedure yields $K = 2[q_1^2 + q^2 - q_1 q(1 + [-1|x|1])]^{1/2}$ and $q_2 = \sqrt{K^2/2 + 2q^2 - q_1^2}$. Here, $x$ is defined by $x = (q_1^2 + 4q^2 - k_F^2)/(4q_1q)$ and $[-1|x|1] = \max\{-1, \min\{x, 1\}\}$. The starting-energy $\omega$ is then calculated from

$$\omega = \frac{\hbar^2}{M_N}q^2 + U_N(q_1) + U_N(q_2), \quad (2.41)$$

after the subtraction of the conserved c.m. energy $(\hbar^2/2M_N)K^2$. The assignments $\bm{q}$ to $\bm{q}_1$ and $\bm{q}'$ to $\bm{q}$ in Eq. (2.27) (and similar prescription in the $1S$- and $1S'$ terms in Eq. (2.28)) clearly destroy the redundancy property of the $n\alpha$ RGM kernel. Although the $K$ dependence is explicitly treated in the partial wave decomposition, we need at least one more vector to uniquely specify the momenta $q_1$ and $q_2$ even under the assumption of the angle-averaging procedure. Here we use a set of $\bm{K}$ and $\bm{q}$ to specify the starting energy $\omega$. This method, however, involves some ambiguity, since the definitions of $\bm{q}_1$ and $\bm{q}$ differ for each type. The explicit expressions in Eqs. (2.27)-(2.28) indicate that the roles of $\bm{q}$ and $\bm{q}'$ in the $0D_+$ and $1D_-$ types are taken over by $\bm{q}_f$ and $\bm{p}$ in the $1S$ type and by $\bm{q}_i$ and $\bm{p}'$ in the $1S'$ type. Such a choice of three different sets of the G-matrix interaction parameters apparently causes a problem. Here, we choose the $0D_+$-type definitions for $\bm{q}_1$ and $\bm{q}$ as a standard set. Namely, we assume that Eq. (2.18) is explicitly given by

$$\langle \bm{p}_1, \bm{p}_2|u|\bm{p}'_1, \bm{p}'_2 \rangle = \delta(\bm{K} - \bm{K}') \frac{1}{(2\pi)^3} g(\bm{p}, \bm{p}'; \omega(|\bm{p} + \bm{p}'|/2), K, K, k_F). \quad (2.42)$$

In practical calculation, we first assume $q = |\bm{q}|$ and $K = |\bm{K}|$ and calculate the G-matrix using the angle-averaging procedure of $\bm{K}$ under the constraint $|\bm{K}/2 - \bm{q}| < k_F$. Then the rule of the partial-wave decompositions in $g$ and $u$ is given by

$$g(\bm{p}, \bm{p}'; \omega(q, K), K, k_F) = \sum_{\ell} (2\ell + 1) g_{\ell}(p, p'; \omega(q, K), K, k_F) P_{\ell}(\tilde{\bm{p}}, \tilde{\bm{p}}')$$

$$= u(k', q'; \omega(q, K), K, k_F)$$

$$= \sum_{\lambda}(2\lambda + 1) u_{\lambda}(k', q'; \omega(q, K), K, k_F) P_{\lambda}(\tilde{\bm{k}}', \tilde{\bm{q}}'). \quad (2.43)$$

We pick up the $q = q'$ portion in the last line of Eq. (2.43) and define

$$\tilde{u}_{\lambda}(k', q'; K, k_F) = u_{\lambda}(k', q'; \omega(q', K), K, k_F). \quad (2.44)$$

The full G-matrix interaction is constructed using

$$\tilde{u}(k', q'; K, k_F) = \sum_{\lambda}(2\lambda + 1) \tilde{u}_{\lambda}(k', q'; K, k_F) P_{\lambda}(\tilde{\bm{k}}', \tilde{\bm{q}}')$$

$$= \tilde{g}(\bm{p}, \bm{p}'; K, k_F) = \sum_{\ell} (2\ell + 1) \tilde{g}_{\ell}(p, p'; K, k_F) P_{\ell}(\tilde{\bm{p}}, \tilde{\bm{p}}'), \quad (2.45)$$

$$\tilde{g}(\bm{p}, \bm{p}'; K, k_F) = \sum_{\ell} (2\ell + 1) \tilde{g}_{\ell}(p, p'; K, k_F) P_{\ell}(\tilde{\bm{p}}, \tilde{\bm{p}}'), \quad (2.45)$$
which is equivalent to
\[ g(p, p'; \omega(q', K), K, k_F) = u(k', q'; \omega(q, K), K, k_F) . \] (2.46)

Since the \( q' \) dependence in the starting-energy part is absorbed in the relative momenta in the bra and ket sides of the \( G \)-matrix, we can apply the previous whole formalism to the \( G \)-matrix interactions \( \tilde{u} \) and \( \tilde{g} \). Note that we only need the \( S \)-wave component for \( \tilde{g} \), since only this component is required for the 0\( E\), 1\( E\), 1\( S\) and 1\( S'\) types.

The angle average over \( K \) in the \( G \)-matrix calculation is carried out similarly to that in the previous case. We start from
\[ \omega(q, K) = \frac{\hbar^2}{M_N} q^2 + U_N (K/2 + q) + U_N (K/2 - q) , \] (2.47)

and make the angle-averaging over \( K \) under the constraint \(|K/2 - q| < k_F\). Then we find
\[ \omega(q, K) = \frac{\hbar^2}{M_N} q^2 + U_N \left( \sqrt{K^2/4 + q^2 + Kq\langle x \rangle} \right) \]
\[ + U_N \left( \sqrt{K^2/4 + q^2 - Kq\langle x \rangle} \right), \] (2.48)

where
\[ \langle x \rangle = \frac{1}{2}(1 + [-1|x|1]) = \begin{cases} 1 & \text{for } x > 1 \\ \frac{1}{2}(1 + x) & \text{for } -1 < x < 1 \\ 0 & \text{for } x < -1 \end{cases} \] (2.49)

with
\[ x = \frac{1}{Kq} \left( \frac{1}{4} K^2 + q^2 - k_F^2 \right). \] (2.50)

In two special cases, we find
\[ \omega(q, K) = \begin{cases} \frac{\hbar^2}{M_N} q^2 + U_N(\sqrt{K^2/4 + q^2}) + U_N(\sqrt{K^2/4 + q^2}) & \text{for } K/2 + q < k_F \\ \frac{\hbar^2}{M_N} q^2 + U_N(K/2 + q) + U_N(|K/2 - q|) & \text{for } |K/2 - q| > k_F \end{cases} . \] (2.51)

§3. Results and discussion

3.1. \( NN \) \( G \)-matrix and \( n\alpha \) phase shifts

Figure 1 shows the nucleon s.p. potential of the model fss2, obtained by \( G \)-matrix calculation for symmetric nuclear matter. The Fermi momentum \( k_F = 1.20 \) fm\(^{-1}\) is assumed in the continuous prescription for intermediate spectra. The solid curve
Fig. 1. Comparison of the nucleon s.p. potentials predicted by the renormalized RGM kernel and the $\varepsilon K$ prescription. The model is fss2 and $k_F = 1.20$ fm$^{-1}$ is assumed in the continuous prescription for intermediate spectra.

(ren. RGM) is obtained from the renormalized RGM kernel and the dashed curve ($\varepsilon K$) from the energy-dependent RGM kernel with the explicit $\varepsilon K$ term. We find that the off-shell transformation by $1/\sqrt{N}$ gives a rather minor modification for the $NN$ $G$-matrix, giving a slightly repulsive effect to the nucleon s.p. potential. We use the prescription of the renormalized RGM to calculate the $n\alpha$ Born kernel in this paper. The internal energy of the $\alpha$ cluster, calculated from Eq. (2.26) with $q = 0$ and $\nu = 0.257$ fm$^{-2}$, is $E_{\alpha}(0) = -26.5$ MeV.

Figure 2 shows the $n\alpha$ phase shifts of some low partial waves, predicted by fss2, for the neutron incident energies, $E_n = 0 - 30$ MeV. The $G$-matrix calculation in symmetric nuclear matter is carried out using the s.p. potential in Fig. 1 and the prescription for the starting energies discussed in Sec. 2.4. The Fermi momentum $k_F = 1.20$ fm$^{-1}$ corresponds to 70% of the normal saturation density ($0.7 \rho_0$). The h.o. width parameter $\nu = 0.257$ fm$^{-2}$ used for the $(0s)^4 \alpha$ cluster reproduces the root-mean-square (rms) radius of the $\alpha$ particle. Phase-shift solutions are obtained by solving the Lippmann-Schwinger equation for the $n\alpha$ Born kernel in Eq. (2.40). In Fig. 2(a), we find that the $n\alpha$ phase shifts in $L_J = S_{1/2}$, $P_{3/2}$ and $P_{1/2}$ states are well reproduced, although the $S_{1/2}$ phase shift might be slightly too attractive at higher energies. For higher partial waves with $D_{5/2,3/2}$ and $F_{7/2,5/2}$ in Figs. 2(b) and (c), the central attraction is not sufficiently large, although the spin-orbit splitting seems to be reasonably good. The $D_{3/2}$ phase shift is particularly too low in Fig. 2(b) (and also in Fig. 3(b)). Part of the reason for the missing attraction in the $D_{3/2}$ channel is because the specific distortion effect of the $d + {}^3\text{H}$ channel is not included in the calculation, whose threshold opens at $E_n \sim -22$ MeV.

For comparison, we show in Figs. 3(a) - (c) the $n\alpha$ phase shifts obtained by
Fig. 2. $\alpha\alpha$ RGM phase shifts predicted by the quark-model $G$-matrix interaction by $f_{ss2}$ for $k_F = 1.20$ fm$^{-1}$. (a) $S_{1/2}$, $P_{3/2}$ and $P_{1/2}$ states. (b) $D_{5/2}$ and $D_{3/2}$ states. (c) $F_{7/2}$ and $F_{5/2}$ states. The $(0s)^4$ shell-model wave function with the h.o. size parameter $\nu = 0.257$ fm$^{-2}$ is used for the $\alpha$-cluster wave function. The experimental data are taken from Refs. 21) and 22).

Fig. 3. $\alpha\alpha$ RGM phase shifts predicted by the Minnesota 3-range force with $u = 0.94687$ and $\nu = 0.257$ fm$^{-2}$. The $LS$ force is KKNN $\times$ 1.5. (See the text.)

the standard RGM calculation, using an effective $NN$ force. In this calculation, the Minnesota three-range force (MN3R)$^{18}$ with the Majorana exchange mixture $u = 0.94687$ is used for the central force. The $\alpha\alpha$ phase shifts are well reproduced in the $\alpha\alpha$ RGM calculation using this effective $NN$ force.$^{19}$ For the $LS$ force, the two-range Gaussian $LS$ force by Kanada et al.$^{20}$ is used with the spin-isospin
coefficients \( W = 0.5 \) and \( H = -0.5 \) (no \( ^3E \) \( LS \)). If we use the force parameters given in Table I of Ref. 20), the \( LS \) splitting of the \( P_{3/2} \) and \( P_{1/2} \) states becomes too small. This is because these authors introduced the \( D \)-wave component of the \( \alpha \) cluster and an extra contribution to the \( LS \) splitting originates from the two-nucleon tensor force. If we multiply the strength of this \( LS \) force by a factor 1.5, it gives the correct magnitude, as seen in Fig. 3(a). We call this set of effective \( NN \) forces \( MN3R+1.5KKNN \). Although the results in Figs. 2 are not as good as those of \( MN3R+1.5KKNN \) in Fig. 3, it is clear that our quark-model \( NN \) interaction gives a reasonable description of the \( n\alpha \) scattering through the \( G \)-matrix approach of the bare interaction.

3.2. Scattering cross sections and polarization

Since the agreement of the calculated phase shifts with the empirical values determined from the \( R \)-matrix analyses\(^{21,22} \) is not complete, we examine the scattering cross sections and polarization of the \( n\alpha \) scattering directly with the experiment. Figure 4 shows the \( n\alpha \) total cross sections up to \( E_n = 30 \) MeV. The prominent peak structure at \( E_n = 1 - 2 \) MeV is due to the sharp \( P_{3/2} \) resonance. Although the calculated results at around \( E_n = 4 - 6 \) MeV are slightly small, the agreement from 12 to 18 MeV is satisfactory. We also examine the differential cross sections and polarization in Figs. 5 and 6 at some available energies. We obtain a fair agreement between the calculational and experimental results except in the threshold energy region, \( E_n \sim 22 \) MeV, for the decay to the \( d + ^3H \) channel. For higher energies, we need to introduce imaginary potentials.

![Fig. 4. Calculated \( n\alpha \) total cross sections by fss2, compared with the experiment. The experimental data are taken from Ref. 23).](image-url)
Resonating-Group Calculation with a Quark-Model $G$-Matrix $NN$ int.

\[ ^4\text{He}(n,n)^4\text{He} \]
\[ E_n=0.840 \text{ MeV} \]

\[ ^4\text{He}(n,n)^4\text{He} \]
\[ E_n=16.4 \text{ MeV} \]

\[ ^4\text{He}(n,n)^4\text{He} \]
\[ E_n=23.7 \text{ MeV} \]

Fig. 5. Differential cross sections of the $n\alpha$ scattering predicted by fss2. The experimental data are taken from Refs. 24) and 25).

\[ P(\xi \eta) \]
\[ E_n=1.01 \text{ MeV} \]

\[ P(\xi \eta) \]
\[ E_n=17.7 \text{ MeV} \]

\[ P(\xi \eta) \]
\[ E_n=30.3 \text{ MeV} \]

Fig. 6. Polarization of the $n\alpha$ scattering predicted by fss2. The experimental data are taken from Refs. 26) and 27).
3.3. Analysis of Born amplitudes

Since we calculated the $n\alpha$ Born kernel using only the real part of the $G$-matrix, the success of the $n\alpha$ single-channel RGM calculation is qualitative for energies greater than 30 MeV. Here, we analyze the roles of the various exchange terms of the RGM central kernels from the $G$-matrix $NN$ interaction. Such an analysis was carried out by Thompson and Tang\(^7\) for a simple one-term Gaussian $NN$ interaction. They found that three different groups of the nuclear exchange terms have characteristic behavior of various reaction processes. First, the direct (0\(D_+\)-type) and knock-on (1\(D_-\)-type) terms, specified by $f_D$ and $f_1$ terms in the Born amplitudes below, respectively, have large amplitudes at the forward angle, $\theta = 0$, characteristic of the Wigner-type potential. Secondly, the heavy-particle pickup (1\(S\) and 1\(S'\)) term $f_2$ and the nucleon-rearrangement amplitude $f_3$ are important at backward angles. They are mainly responsible for the existence of large scattering cross sections at

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Fig. 7. $n\alpha$ Born amplitudes of the $G$-matrix $NN$ interaction by fss2 with $k_F = 1.20$ fm\(^{-1}\) and $\nu = 0.257$ fm\(^{-2}\) at the c.m. energies $E_{c.m.} = 50$ and 100 MeV.

Fig. 8. $n\alpha$ Born amplitudes of the MN3R force with $\nu = 0.94687$ and $\nu = 0.257$ fm\(^{-2}\) at the c.m. energies $E_{c.m.} = 50$ and 100 MeV.
these angles. In the G-matrix formalism, these Born amplitudes are given by

\begin{align}
    f_D + f_1 &= -\frac{\mu}{2\pi\hbar^2} 2 \sum_{\ell=0}^{\infty} \sum_{I=0}^{1} (2\ell + 1)(2I + 1) M_{0D+}^{0I}(q, q) P_\ell(\cos \theta) , \\
    f_2 &= \frac{\mu}{2\pi\hbar^2} 4 \sum_{\ell=0}^{\infty} \sum_{I=0}^{1} (2\ell + 1)(2I + 1) M_{1S}^{0I}(q, q) P_\ell(\cos \theta) , \\
    f_3 &= -\frac{\mu}{2\pi\hbar^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ G_{\ell}^{K}(q, q) + 2 \sum_{I=0}^{1} (2I + 1) M_{1E}^{0I}(q, q) + \varepsilon K_\ell(q, q) \right] \\
    &\times P_\ell(\cos \theta) .
\end{align}

Here, \( \mu = (4/5)M_N \) is the reduced mass, \( q \) is the relative momentum determined from \( \varepsilon = (\hbar^2/2\mu)q^2 \), and \( \theta \) is the c.m. angle for the \( n\alpha \) scattering. In Eq. (3.1), the partial-wave sum is actually taken up to \( \ell = 8 \). We plot \( f_D + f_1 \), \( f_2 \), and \( f_3 \) in Fig. [7] for the c.m. energies \( E_{\text{c.m.}} = 50 \) and 100 MeV. For the effective local \( NN \) forces, these Born amplitudes are analytically calculated as in Eqs. (30) - (34) of Ref. 7). We show the results of the MN3R force in Fig. 9 for comparison. In the MN3R force, the \( f_D \) term is a Fourier transform of the momentum-independent local potential and \( f_1 \) is that of the nonlocal RGM kernel with a smaller amplitude. Since the \( f_D \) and \( f_1 \) terms are not separated for the G-matrix interaction, the sum \( f_D + f_1 \) is compared with the predictions by fss2. The amplitudes \( f_2 \) and \( f_3 \) arise as a consequence of the antisymmetrization procedure and are closely related to the nucleon-exchange effect in the backward angular region. Figures 7 and 8 show that the angular and energy dependences of these two different \( NN \) interactions are very similar to each other, in spite of the strong nonlocality of the G-matrix interaction.

§ 4. Summary

The \( n\alpha \) system is one of the most successful examples, in which microscopic RGM calculations using effective \( NN \) forces give a good description of the experimental data. It is, therefore, very interesting to examine if this result is still valid when more realistic \( NN \) interactions based on bare interactions are employed in G-matrix formalism. Here, we studied the \( n\alpha \) system by our previous technique, in which baryon-octet \( (B_8) \) \( \alpha \) Born kernels are calculated with explicit treatments of the nonlocality and the center-of-mass (c.m.) motion between \( B_8 \) and \( \alpha \). Since the \( n\alpha \) system involves the antisymmetrization of nucleons due to the Pauli principle, we needed to extend the previous techniques for the direct and knock-on terms of the hyperon \( \alpha \) interaction to other nucleon-exchange and interaction types such as the \( S \) and \( S' \) types. We found that the explicit treatment of the Galilean non-invariant interaction gives some definite recoil effect to the c.m. momentum of two interacting nucleons, involved in the G-matrix, for each particular interaction type of the RGM kernels. If one uses the invariant G-matrix as the input \( NN \) interaction for the \( n\alpha \) RGM kernel, both of the direct potential and the knock-on interaction kernel become nonlocal, and give the same contributions for each isospin component of the
NN interaction with \( I = 0 \) or \( 1 \). This is a common feature of two-cluster systems composed of a single nucleon and a nucleus.

In principle, we can deal with the momentum dependence and the starting-energy dependence of the \( G \)-matrix according to the explicit expressions derived in this study. The treatment of the Fermi momentum \( k_F \) and the starting energy \( \omega \), however, needs careful treatment. Local density approximation, usually assumed in heavier systems, cannot be justified for \((0s)\)-shell nuclei like the \( \alpha \) particle. The single-particle (s.p.) energy used in the definition of the starting energy is not well defined in the \( n\alpha \) cluster model. We assumed a constant \( k_F \) to generate \( G \)-matrix \( NN \) interaction for \( n\alpha \) scattering. The starting energy \( \omega \) is determined from the local momentum \( q \) and the c.m. momentum \( K \) of the two interacting nucleons using angle-averaging procedure over \( K \) under the constraint \(|K/2 - q| < k_F\). The \( G \)-matrix calculation is carried out in the continuous prescription for intermediate spectra using the energy-independent version of the quark-model \( NN \) interaction \text{fss2}. We found that, in the present framework, \( k_F = 1.20 \text{ fm}^{-1} \) is a most favorable choice for generating an appropriate strength of the \( n\alpha \) interaction. A larger \( k_F \) say \( k_F = 1.35 \text{ fm}^{-1} \), gives less attractive \( n\alpha \) interaction, and a smaller \( k_F \) gives more attractive \( n\alpha \) interaction. We carried out the angular-momentum projection of the \( G \)-matrix \( NN \) interaction explicitly including the \( K \) dependence. Owing to this procedure, the existence of the Pauli-forbidden \((0s)\) state for the \( S \)-wave relative motion is strictly preserved, and the admixture of the redundant component of the RGM formalism is completely eliminated.

With these treatments of \( G \)-matrix parameters, we found that the central and spin-orbit components of the \( n\alpha \) Born kernel have reasonable strengths under the assumption of a rigid translationally invariant \((0s)^4\) shell-model wave function of the \( \alpha \) cluster. The \( n\alpha \) phase shifts in the energy region, \( E_n \leq 30 \text{ MeV} \), are reasonably reproduced for the \( S_{1/2}, P_{3/2} \) and \( P_{1/2} \) states, while the central attraction is somewhat too weak for higher partial waves. The direct comparison of the differential cross sections and polarization with experiment shows that these higher partial waves do not markedly impair the fit to the experimental data, except for the energy region where the \( d + ^3\text{H} \) channel opens. In the higher energy region of up to 100 MeV in the c.m. system, we compared the Born amplitudes from the \( G \)-matrix \( NN \) interaction with those from an effective \( NN \) force, the Minnesota three-range force. We found that characteristic behaviors of three different groups of exchange terms, the direct and knock-on terms, heavy-particle pickup terms, and nucleon-rearrangement terms, are essentially the same between these two approaches.

Note that the appropriate strength of the \( n\alpha \) central attraction in the present calculation depends largely on how we deal with the strong starting-energy dependence in the \( G \)-matrix calculation. The present procedure is one of the possible procedures for preserving the redundancy property of RGM formalism and still dealing with the recoil effect of the \( \alpha \) cluster explicitly in the \( G \)-matrix \( NN \) interaction. Much simpler treatments are indeed possible, by assuming some appropriate relative momentum \( q \) and the c.m. momentum \( K \) in the starting energy \( \omega(q, K) \). We can find the most appropriate \( k_F \), which strongly correlates with the approximate reproduction of \( \alpha \)-cluster internal energy. In all of such calculations, however, the strength of the \( n\alpha \)
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The exchange normalization kernel of the \( n\alpha \) system in the momentum representation is given by

\[
K(q_f, q_i) = f(\theta) = \left( \frac{8\pi}{3\nu} \right)^{\frac{3}{2}} \exp \left\{ -\left( \frac{3}{32\nu} k^2 + \frac{25}{24\nu} q^2 \right) \right\} = \left( \frac{8\pi}{3\nu} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{3\nu} \left[ \frac{17}{16} (q_f^2 + q_i^2) + q_f \cdot q_i \right] \right\},
\]

where \( \nu \) is the h.o. size parameter of the \( \alpha \) cluster. In many cases, the transformation from \( q_f \) and \( q_i \) to the momentum transfer \( k \) and the local momentum \( \mathbf{q} \) via \( k = q_f - q_i \) and \( \mathbf{q} = \frac{1}{2} (q_f + q_i) \) is convenient for simplifying the expressions for the kernels. We also use the notation \( f(\theta) \) for Eq. (A.1), although it is a function of \( q_f = |q_f|, q_i = |q_i| \) and \( \cos \theta = (\mathbf{q}_f \cdot \mathbf{q}_i) \). The Born kernel for the exchange kinetic-
yields the spin-flavor factors in Eq. (2.6) are calculated to be

$$G^K(q_f, q_i) = \frac{3\hbar^2\nu}{2M_N} f(\theta) \left[ 1 - \frac{2}{3\nu} \left( \frac{5}{3} q^2 + \frac{1}{4} k^2 \right) \right], \quad (A.2)$$

with $M_N$ being the nucleon mass. The partial-wave decomposition of these kernels yields

$$K_\ell(q_f, q_i) = (-1)^\ell \left( \frac{8\pi}{3\nu} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{3\nu} \left[ \frac{17}{16} (q_f^2 + q_i^2) - q_f q_i \right] \right\} \times \frac{1}{3\nu} q_f q_i \left( \frac{1}{3\nu} q_f q_i \right),$$

$$G_e^K(q_f, q_i) = (-1)^\ell \left( \frac{8\pi}{3\nu} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{3\nu} \left[ \frac{17}{16} (q_f^2 + q_i^2) - q_f q_i \right] \right\} \times \frac{3\hbar^2\nu}{2M_N} \left\{ \left[ 1 + \frac{2}{3} \ell - \frac{4}{9\nu} (q_f^2 + q_i^2) \right] \frac{1}{3\nu} q_f q_i \left( \frac{1}{3\nu} q_f q_i \right) + \frac{2}{9\nu} q_f q_i \right\}, \quad (A.3)$$

where $i_\lambda(x) = i_\lambda j_\lambda(-ix) = e^{x} i_\lambda(x)$. For the Gaussian central $NN$ force in Eq. (2.7), the spin-flavor factors in Eq. (2.6) are calculated to be

$$X_{OE} = X_d + X_e, \quad X_{OD+} = \frac{1}{2} X_d, \quad X_{1E} = X_{1S} = X_{1S'} = -\frac{1}{2} (X_d + X_e), \quad X_{1D-} = \frac{1}{2} X_e, \quad X_d = 8W + 4B - 4H - 2M, \quad X_e = 8M + 4H - 4B - 2W. \quad (A.4)$$

By using these factors, $V_D$ and $G^N$ are obtained as

$$V_D(q_f, q_i) = v_0 \frac{1}{2} X_d \left( \frac{\pi}{\kappa} \right)^{\frac{3}{2}} \exp \left\{ -\frac{k^2}{4} \left( \frac{3}{8\nu} + \frac{1}{\kappa} \right) \right\},$$

$$G^N(q_f, q_i) = v_0 \frac{1}{2} (X_d + X_e) [f_E(\theta) - f_S(\theta) - f_{S'}(\theta)] + v_0 \frac{1}{2} X_e f_{D-}(\theta), \quad (A.5)$$

with

$$f_T(\theta) = f(\theta) \times \left\{ \begin{array}{ll}
\left( \frac{1}{1 + \frac{5}{2}} \right)^{\frac{3}{2}} \\
\left( \frac{1}{1 + \frac{2\nu}{\kappa}} \right)^{\frac{3}{2}} \exp \left\{ \frac{2\nu}{1 + \frac{2\nu}{\kappa}} \left( \frac{5}{3} + \frac{1}{2} k^2 \right) \right\} \\
\left( \frac{1}{1 + \frac{5}{2}} \right)^{\frac{3}{2}} \exp \left\{ \frac{2\nu}{1 + \frac{2\nu}{\kappa}} \frac{50}{1 + 2\nu} q^2 \right\} \\
\left( \frac{1}{1 + \frac{5}{2}} \right)^{\frac{3}{2}} \exp \left\{ \frac{2\nu}{1 + \frac{2\nu}{\kappa}} \frac{50}{1 + 2\nu} q^2 \right\}
\end{array} \right. \quad \text{for } T = \left\{ \begin{array}{l}
E \\
S \\
D_+ \\
D_-
\end{array} \right. \quad (A.6)$$
The \( S' \)-type function \( f_{S'}(\theta) \) is obtained from \( f_S(\theta) \) by simply replacing \( k \) by \(-k\). For the Gaussian \( LS \) force in Eq. (2.15), the spin-isospin factors are given by

\[
X_{0D_+}^{LS} = (4W - 2H)S, \quad X_{1D_-}^{LS} = (4H - 2W)S, \quad (A.7)
\]

where \( S \) is the total spin of the \( n\alpha \) system. The \( LS \) Born kernels are given by

\[
V_{D}^{LS}(q_f, q_i) = v_0^{LS}(4W - 2H) \left( \frac{\pi}{k} \right)^{\frac{3}{2}} \exp \left\{ -\frac{k^2}{4} \left( \frac{3}{8\nu} + \frac{1}{\kappa} \right) \right\} \frac{5}{16\kappa} \mathbf{n} \cdot \mathbf{S},
\]

\[
G^{LS}(q_f, q_i) = v_0^{LS}(2W - 4H) f_{D_-}(\theta) \left( \frac{1}{1 + \frac{8\kappa}{8\nu}} \right) \frac{5}{8\nu} \mathbf{n} \cdot \mathbf{S}, \quad (A.8)
\]

where \( \mathbf{n} = [\mathbf{q} \times \mathbf{k}] = [\mathbf{q}_i \times \mathbf{q}_f] \). For the effective \( NN \) force, the direct potentials become local in the coordinate representation:

\[
V_{D}^{C}(r) = v_0 \frac{1}{2} X_d \left( \frac{1}{1 + \frac{3\kappa}{8\nu}} \right) \frac{3}{2} \exp \left\{ -\frac{\kappa}{1 + \frac{3\kappa}{8\nu}} r^2 \right\},
\]

\[
V_{D}^{LS}(r) = v_0^{LS} (4W - 2H) \left( \frac{1}{1 + \frac{3\kappa}{8\nu}} \right) \frac{5}{8} \frac{5}{8} \exp \left\{ -\frac{\kappa}{1 + \frac{3\kappa}{8\nu}} r^2 \right\}, \quad (A.9)
\]

where the \( LS \) potential is \( V_{D}^{LS}(r) \ell \cdot \mathbf{S} \). The internal energy of the \( \alpha \) cluster is given by

\[
E_{\alpha}^{\text{int}} = 3 \cdot \frac{3h^2\nu}{2M_N} + v_0 (X_d + X_e) \left( \frac{\nu}{\nu + \kappa} \right)^{\frac{3}{2}} + 2e^2 \sqrt{\frac{\nu}{\pi}}. \quad (A.10)
\]

**Appendix B**

---The momentum dependence of the interaction kernels for systems of two (0s)-shell clusters---

In this Appendix, we will show the explicit momentum dependence that appears in the interaction kernels of the Galilean non-invariant two-nucleon interaction for systems of two (0s)-shell clusters. For the two-nucleon interaction in Eq. (2.18), the spatial part of the interaction kernel defined in Eq. (2.19) is given by

\[
M_{xT}(q_f, q_i) = M_x^N(q_f, q_i)
\]

\[
\times \frac{1}{(2\pi)^3} \int d\mathbf{k}' \exp \left\{ -\left( 1 + \frac{\alpha}{2\mu} \right) \frac{1}{4\nu} k'^2 - \frac{1}{2\sqrt{\gamma}} \mathbf{v} \cdot \mathbf{k}' \right\}
\]

\[
\times \left( \frac{1}{\pi\nu} \frac{1}{1 - \alpha/2\mu} \right)^{3/2} \int dq' \exp \left\{ -\frac{1}{1 - \alpha/2\mu} \frac{1}{4\nu} \left( q' + \frac{\epsilon}{2\mu} k' + \frac{\nu}{2\sqrt{\gamma}} A \right)^2 \right\}
\]

\[
\times \left( \frac{A}{A - 2} \frac{1}{4\pi\nu} \frac{1}{1 - \beta/2\mu} \right)^{3/2} \int d\mathbf{P} \exp \left\{ -\frac{A}{A - 2} \frac{1}{\beta/2\mu} \frac{1}{4\nu} (P - P_0)^2 \right\}
\]

\[
\times u(k', q'; |P|). \quad (B.1)
\]
Here $M^N_q(q_f, q_i)$ is the normalization kernel given by

$$\begin{align*}
M^N_q(q_f, q_i) &= \left( \frac{2\pi}{\gamma} \frac{1}{1 - \tau^2} \right) \frac{3}{2} \exp \left\{ -\frac{1}{2\gamma} \left( \frac{1 - \tau q^2 + 1 + \tau k^2}{1 - \tau 4 k^2} \right) \right\} , \\
&= (B.2)
\end{align*}$$

with $\tau = 1 - x/\mu$, $k = q_f - q_i$ and $q = (q_f + q_i)/2$. We assume the mass numbers $A_1$ and $A_2$ with $A_1$, $A_2 \leq 4$ for the two clusters, $A = A_1 + A_2$ is the total mass number and $\mu = A_1 A_2 / A$ is the reduced mass number. Almost all of the coefficients and vectors appearing in Eq. (B.1) are defined in Eq. (A.14) of Ref. 6). The $P$ integral is used to suitably treat the c.m. motion of the interacting two nucleons. The new parameters $P_0$ and $\beta$ are parametrized as

$$\begin{align*}
P_0 &= \frac{1}{\mu (1 - \tau^2)(1 - \alpha/2\mu)} \tilde{P}_0 , \\
\beta &= \frac{1}{(1 - \tau^2)(1 - \alpha/2\mu)} A_2 \tilde{\beta} ,
\end{align*}$$

with $\tilde{P}_0$ and $\tilde{\beta}$ given in Table II.

It sometimes happens that the formula in Eq. (B.1) cannot be used because of the divergence of the coefficients. For example, all the direct terms with $x = 0$ and many of the $A_1 = 1$ or $A_2 = 1$ cases should be treated separately. For the direct

| $T$     | $\tilde{P}_0$ | $\tilde{\beta}$ |
|---------|---------------|------------------|
| $E_{11}$ | $\frac{4}{A_1} q$ | $8 \frac{4A_2}{A_1 A_2}$ |
| $E_{22}$ | $\frac{4}{A_1} q$ | $8 \frac{4A_2}{A_1 A_2}$ |
| $E_{12}$ | $2x \left( \frac{1}{A_1} - \frac{1}{A_2} \right) q + \left( 2 - \frac{x}{\gamma} \right) k$ | $4 \left( 1 - \frac{2x}{\gamma} \right)$ |
| $E_{21}$ | $2x \left( \frac{1}{A_1} - \frac{1}{A_2} \right) q - \left( 2 - \frac{x}{\gamma} \right) k$ | $4 \left( 1 - \frac{2x}{\gamma} \right)$ |
| $S_1$   | $\left( 1 - \frac{2x}{\gamma} \right) \left( q' + \frac{1}{2} k' \right) + \left( \frac{1 - 3x}{A_1} + \frac{1}{A_2} \right) q + \left( 2 - \frac{1}{A_1} - \frac{x}{\gamma} \right) \frac{1}{2} k$ | 1 |
| $S_2$   | $\left( 1 - \frac{2x}{\gamma} \right) \left( q' + \frac{1}{2} k' \right) + \left( \frac{1 - 3x}{A_1} + \frac{1}{A_2} \right) q + \left( 2 - \frac{1}{A_1} - \frac{x}{\gamma} \right) \frac{1}{2} k$ | 2 |
| $S'_1$  | $\left( 1 - \frac{2x}{\gamma} \right) \left( q' - \frac{1}{2} k' \right) + \left( \frac{1 - 3x}{A_1} + \frac{1}{A_2} \right) q - \left( 2 - \frac{1}{A_1} - \frac{x}{\gamma} \right) \frac{1}{2} k$ | 1 |
| $S'_2$  | $\left( 1 - \frac{2x}{\gamma} \right) \left( q' - \frac{1}{2} k' \right) + \left( \frac{1 - 3x}{A_1} + \frac{1}{A_2} \right) q - \left( 2 - \frac{1}{A_1} - \frac{x}{\gamma} \right) \frac{1}{2} k$ | 2 |
| $D_1$   | $2x \left( \frac{1}{A_1} - \frac{1}{A_2} \right) (q' + q)$ | 3 |
| $D_-$   | $(x - 1) \left( \frac{1}{A_1} - \frac{1}{A_2} \right) (k' + 2q)$ | 4 |

1) $1 - \frac{2A_1}{A_1^2} + 4 \frac{2x}{\gamma} \left( \frac{A_1}{A_1} - 1 \right)$
2) $1 - \frac{2A_1}{A_1^2} + 4 \frac{2x}{\gamma} \left( \frac{A_1}{A_2} - 1 \right)$
3) $2x \left( \frac{1}{A_1} - \frac{1}{A_2} \right) \frac{A_2 - A_1}{A}$
4) $2(x - 1) \left( \frac{1}{A_1} - \frac{1}{A_2} \right) \frac{A_2 - A_1}{A}$
When A terms, we obtain
\[ M_{0E_{\alpha}}(q_f, q_i) = \delta(k) \left( \frac{1}{\pi \nu} \right)^{\frac{3}{2}} \int \frac{d \mathbf{k}' d \mathbf{q}'}{A_{\alpha}} \exp \left\{ -\frac{1}{\nu} \left( q'^2 + \frac{1}{4} k'^2 \right) \right\} \]
\times \left( \frac{1}{1 - 2/A_{\alpha}} \frac{1}{4 \pi \nu} \right)^{\frac{3}{2}} \int d \mathbf{P} \exp \left\{ -\frac{1}{1 - 2/A_{\alpha}} \frac{1}{4 \nu} \left( P - \frac{2}{A_{\alpha}} q \right)^2 \right\}
\times u(k', q'; |\mathbf{P}|) , \quad (B.4)

and
\[ M_{0D_{\alpha}}(q_f, q_i) = \exp \left\{ -\left( 1 - \frac{1}{2\mu} \right) \frac{1}{4\nu} k^2 \right\} \left( \frac{1 - \frac{1}{2\mu}}{1 - \frac{1}{2\mu} \pi \nu} \right)^{\frac{3}{2}} \int d \mathbf{P} d \mathbf{q}' \exp \left\{ -\frac{1}{1 - 2\mu} \frac{1}{\nu} \left( q' - \frac{1}{2\mu} q \right)^2 \right\}
- \frac{1 - \frac{1}{2\mu}}{(1 - 1/A_1)(1 - 1/A_2)} \frac{1}{4 \nu} \left[ P - \frac{1}{1 - 2\mu} \left( \frac{1}{A_1} - \frac{1}{A_2} \right) (q' - q)^2 \right]
\times u(k, q'; |\mathbf{P}|) . \quad (B.5)

When \( A_1 = 1 \) or \( A_2 = 1 \), Eq. (B.5) cannot be used either. In this case, we obtain
\[ M_{0D_{\alpha}}(q_f, q_i) = \exp \left\{ -\left( 1 - \frac{1}{2\mu} \right) \frac{1}{4\nu} k^2 \right\} \left( \frac{1 - \frac{1}{2\mu}}{1 - \frac{1}{2\mu} \pi \nu} \right)^{\frac{3}{2}} \int d \mathbf{q}' \exp \left\{ -\frac{1}{1 - 2\mu} \frac{1}{\nu} \left( q' - \frac{1}{2\mu} q \right)^2 \right\} u(k, q'; |2(q - q')|)
\text{ for } A_1 = 1 \text{ or } A_2 = 1 . \quad (B.6)

The knock-on term kernel \( M_{1D_{\alpha}}(q_f, q_i) \) can be obtained from Eqs. (B.5) and (B.6) by simply changing \( u(k', q'; |\mathbf{P}|) \) to \( u(2q', k'/2; |\mathbf{P}|) \). Finally, \( 1 - \beta/2\mu \) becomes zero for the S- and S'-type interaction kernels in the no\( \alpha \) case. In such a case, we can use the limit formula
\[ \lim_{\kappa \to 0} \left( \frac{1}{\kappa \pi} \right)^{\frac{3}{2}} e^{-\alpha^2/\kappa} = \delta(x) . \quad (B.7) \]

The last two lines of Eq. (B-1) become simply \( u(k', q'; |\mathbf{P}_0|) \) using this procedure.

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