Harmonic analysis/Functional analysis

A trace formula for functions of contractions and analytic operator Lipschitz functions

Une formule de trace pour les fonctions de contraction et les fonctions analytiques opérateurs-lipschitziennes

Mark Malamud\textsuperscript{a,b}, Hagen Neidhardt\textsuperscript{c}, Vladimir Peller\textsuperscript{d,b}

\textsuperscript{a} Institute of Applied Mathematics and Mechanics, NAS of Ukraine, Slavyansk, Ukraine
\textsuperscript{b} RUDN University, 6 Miklukho-Maklay St., Moscow, 117198, Russia
\textsuperscript{c} Institut für Angewandte Analysis und Stochastik, Mohrenstr. 39, 10117 Berlin, Germany
\textsuperscript{d} Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

\textbf{A R T I C L E  I N F O}

Article history:
Received 6 April 2017
Accepted after revision 7 June 2017
Available online 3 July 2017
Presented by Gilles Pisier

\textbf{A B S T R A C T}

In this note, we study the problem of evaluating the trace of $f(T) - f(R)$, where $T$ and $R$ are contractions on a Hilbert space with trace class difference, i.e. $T - R \in S_1$, and $f$ is a function analytic in the unit disk $\mathbb{D}$. It is well known that if $f$ is an operator Lipschitz function analytic in $\mathbb{D}$, then $f(T) - f(R) \in S_1$. The main result of the note says that there exists a function $\xi$ (a spectral shift function) on the unit circle $\Gamma$ of class $L^1(\Gamma)$ such that the following trace formula holds: $\text{trace}(f(T) - f(R)) = \int_{\Gamma} f'(\zeta) \xi(\zeta) d\zeta$, whenever $T$ and $R$ are contractions with $T - R \in S_1$, and $f$ is an operator Lipschitz function analytic in $\mathbb{D}$.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\textbf{R É S U M É}

Nous considérons dans cette note le problème qui consiste à trouver le trace de $f(T) - f(R)$, où $T$ et $R$ sont des contractions dans un espace hilbertien et $f$ est une fonction analytique dans le disque unité $\mathbb{D}$. Il est bien connu que, si $f$ est une fonction analytique dans $\mathbb{D}$ qui est opérateurs-lipschitzienne, la différence $T - R$ est de classe trace, c’est-à-dire que si $T - R \in S_1$, alors $f(T) - f(R) \in S_1$. Le résultat principal de cette note établit qu’il existe une fonction $\xi$ (une fonction de décalage spectral) sur le cercle unité $\Gamma$ dans l’espace $L^1(\Gamma)$ pour laquelle la formule de trace suivante est vraie : $\text{trace}(f(T) - f(R)) = \int_{\Gamma} f'(\zeta) \xi(\zeta) d\zeta$ pour n’importe quelle fonction $f$ opérateurs-lipschitzienne et analytique dans $\mathbb{D}$.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Version française abrégée

La fonction de décalage spectral pour des couples d'opérateurs auto-adjoints a été introduite par I.M. Lifshits dans [11]. M.G. Krein considère dans [7] le cas le plus général. Soient $A$ et $B$ des opérateurs auto-adjoints (pas nécessairement bornés) dont la différence $A - B$ est de classe trace, c'est-à-dire que $A - B \in S_1$. Il est démontré dans [7] qu'il existe une fonction $\xi = \xi_{A,B}$ réelle dans $L^1(\mathbb{R})$ (qui dépend de $A$ et $B$) pour laquelle la formule de trace suivante est vraie :

$$\text{trace} \left( f(A) - f(B) \right) = \int_{\mathbb{R}} f'(t) \xi_{A,B}(t) \, dt \quad (1)$$

pour chaque fonction $f$ différentiable sur $\mathbb{R}$ telle que la dérivée $f'$ de $f$ est la transformée de Fourier d'une mesure complexe borélienne sur $\mathbb{R}$. La fonction $\xi$ s'appelle la fonction de décalage spectral pour le couple $(A, B)$. M.G. Krein a posé dans [7] le problème qui consiste à décrire la classe de fonctions $f$ pour lesquelles la formule de trace ci-dessus est vraie pour tous les couples d'opérateurs auto-adjoints $(A, B)$ tels que $A - B \in S_1$.

Le problème de Krein a été résolu récemment dans [17] ; la classe de fonctions ci-dessus coïncide avec la classe de fonctions opérateurs-lipschitziennes sur $\mathbb{R}$. Rappelons qu'une fonction $f$ continue sur $\mathbb{R}$ est dite opérateurs-lipschitzienne si on a

$$\|f(A) - f(B)\| \leq \text{const} \|A - B\| \quad (2)$$

pour tous les opérateurs auto-adjoints $A$ et $B$.

Dans [8], M.G. Krein a introduit la fonction de décalage spectral pour les couples d'opérateurs unitaires dont la différence est de classe trace. Il a démontré que, pour chaque couple $(U, V)$ d'opérateurs unitaires pour lesquels $U - V \in S_1$, il existe une fonction $\xi_{U,V}$ dans l'espace $L^1(\mathbb{T})$ (qui s'appelle une fonction de décalage spectral pour le couple $(U, V)$) telle que

$$\text{trace} \left( f(U) - f(V) \right) = \int_{\mathbb{T}} f'(\xi) \xi_{U,V}(\xi) \, d\xi \quad (3)$$

pour chaque fonction $f$ différentiable dont la dérivée a une série de Fourier absolument convergente.

Le problème qui consiste à décrire la classe maximale de fonctions $f$ pour lesquelles la formule (3) s'applique pour tous les couples $(U, V)$ d'opérateurs unitaires avec $U - V \in S_1$ a été résolu récemment dans [3]. Notamment, il a été démontré dans [3] que la classe en question coïncide avec la classe de fonctions opérateurs-lipschitziennes sur le cercle $\mathbb{T}$.

Dans cette note, nous considérons le cas des fonctions des contractions sur l'espace hilbertien. Rappelons qu'on dit qu'un opérateur $T$ sur l'espace hilbertien est une contraction si $\|T\| \leq 1$.

Le résultat principal de cette note est le théorème suivant :

**Théorème.** Pour chaque couple $(T,R)$ de contractions sur l'espace hilbertien dont la différence $T - R$ est de classe trace il existe une fonction $\xi = \xi_{T,R}$ de l'espace $L^1(\mathbb{T})$ — une fonction de décalage spectral pour $T$ et $R$ — pour laquelle la formule de trace suivante

$$\text{trace} \left( f(T) - f(R) \right) = \int_{\mathbb{T}} f'(\xi) \xi(\xi) \, d\xi \quad (4)$$

s'applique pour toutes les fonctions $f$ opérateurs-lipschitziennes et analytiques dans $\mathbb{D}$.

Remarquons que la classe des fonctions opérateurs-lipschitziennes et analytiques dans $\mathbb{D}$ est la classe maximale de fonctions pour lesquelles la formule (4) est vraie pour toutes les contractions $T$ et $R$ dont la différence est de classe trace.

1. Introduction

The notion of spectral shift function was introduced by physicist I.M. Lifshits in [11]. It was M.G. Krein who generalized in [7] this notion to a most general situation. Namely, if $A$ and $B$ are (not necessarily bounded) self-adjoint operators on a Hilbert space with trace class difference (i.e. $A - B \in S_1$), then it was shown in [7] that there exists a unique real function $\xi = \xi_{A,B}$ in $L^1(\mathbb{R})$, the spectral shift function for the pair $(A, B)$, such that trace formula (1) holds for all functions $f$ that are differentiable on $\mathbb{R}$ and whose derivative $f'$ is the Fourier transform of a complex Borel measure.

Krein observed in [7] that the right-hand side of (1) makes sense for arbitrary Lipschitz functions $f$, and he posed the problem of describing the maximal class of functions $f$, for which trace formula (1) holds for an arbitrary pair $(A, B)$ of self-adjoint operators with $A - B \in S_1$.

It was Farforovskaya who proved in [5] that there exist self-adjoint operators $A$ and $B$ with $A - B \in S_1$ and a Lipschitz function $f$ on $\mathbb{R}$ such that $f(A) - f(B) \notin S_1$. Thus, trace formula (1) cannot be generalized to the class of all Lipschitz functions $f$. In [13] and [14], it was shown that trace formula (1) holds for all functions $f$ in the (homogeneous) Besov class $B^1_{\infty,1}(\mathbb{R})$.
Krein’s problem was completely solved recently in [17]. It was shown in [17] that the maximal class of functions $f$, for which (1) holds whenever $A$ and $B$ are (not necessarily bounded) self-adjoint operators with trace class difference coincides with the class of operator Lipschitz functions $f$ on $\mathbb{R}$. Recall that $f$ is called an operator Lipschitz function if inequality (2) holds for arbitrary self-adjoint operators $A$ and $B$. We refer the reader to [2] for detailed information on operator Lipschitz functions.

Later M.G. Krein introduced in [8] the notion of spectral shift function for pairs of unitary operators with trace class difference. He proved that for a pair $(U, V)$ of unitary operators with $U - V \in \mathcal{S}_1$, there exists a function $\xi = \xi_{U, V}$ in $L^1(\mathbb{T})$ (a spectral shift function for the pair $(U, V)$) such that trace formula (3) holds for an arbitrary differentiable function $f$ on the unit circle $\mathbb{T}$ whose derivative has absolutely convergent Fourier series. Note that $\xi$ is unique modulo an additive constant; it can be normalized by the condition $\int_\mathbb{T} \xi(\xi) d\xi = 0$.

An analog of the result of [17] was obtained in [3]. It was proved in [3] that the maximal class of functions $f$, for which trace formula (3) holds for arbitrary unitary operators $U$ and $V$ with trace class difference coincides with the class of operator Lipschitz functions on the unit circle; this class can be defined by analogy with operator Lipschitz functions on $\mathbb{R}$. Note that the method used in [17] does not work in the case of unitary operators. We denote the class of operator Lipschitz functions on $\mathbb{T}$ by $\text{OL}_\mathbb{T}$.

In this note we consider the case of functions of contractions. Recall that an operator $T$ on a Hilbert space is called a contraction if $\|T\| \leq 1$. For a contraction $T$, the Sz.-Nagy–Foiaş functional calculus associates with each function $f$ in the disk-algebra $\mathcal{C}_A$, the operator $f(T)$. The functional calculus $f \mapsto f(T)$ is linear and multiplicatrive and $\|f(T)\| \leq \max\{|f(\zeta)|: \zeta \in \mathbb{C}, |\zeta| \leq 1\}$ (von Neumann’s inequality). As usual, $\mathcal{C}_A$ stands for the space of functions analytic in the unit disk $\mathbb{D}$ and continuous in the closed unit disk. The purpose of this note is to obtain analogs of the above-mentioned results of [7,8,17] and [3] for functions of contraction.

We are going to prove the existence of a spectral shift function for pairs $(T_0, T_1)$ of contractions with trace class difference. This is an integrable function $\xi$ on the unit circle $\mathbb{T}$ such that

$$\text{trace} \left( f(T_1) - f(T_0) \right) = \int f'(|\xi|)(\xi) d\xi \quad (5)$$

for all analytic polynomials $f$. Such a function $\xi$ is called a spectral shift function for the pair $(T_0, T_1)$. It is unique up to an additive in the Hardy class $H^1$. In other words, if $\xi$ is a spectral shift function for $(T_0, T_1)$, then all spectral shift functions for the pair $(T_0, T_1)$ are given by $[\xi + h : h \in H^1]$.

The second principal result of this note is that the maximal class of functions $f$ in $\mathcal{C}_A$, for which formula (5) holds for all such pairs $(T_0, T_1)$ coincides with the class of operator Lipschitz functions analytic in $\mathbb{D}$. We say that a function $f$ analytic in $\mathbb{D}$ is called operator Lipschitz if

$$\|f(T) - f(R)\| \leq \text{const} \|T - R\|$$

for contractions $T$ and $R$. We denote the class of operator Lipschitz functions analytic in $\mathbb{D}$ by $\text{OL}_A$. It is well known that if $f \in \text{OL}_A$, then $f \in \mathcal{C}_A$ and $\text{OL}_A = \mathcal{OL}_\mathbb{D} \cap \mathcal{C}_A$ (see [6] and [2]).

It turns out that as in the case of functions of self-adjoint operators and functions of unitary operators, the maximal class of functions, for which trace formula (5) holds for all pairs of contractions $(T_0, T_1)$ with trace class difference coincides with the class $\text{OL}_A$.

To obtain the results described above, we combine two approaches. The first one is based on double operator integrals with respect to semi-spectral measures. It leads to a trace formula trace $\left( f(T) - f(R) \right) = \int_{\mathbb{T}} f'(|\zeta|) d\nu(\zeta)$ for a Borel measure $\nu$ on $\mathbb{T}$.

The second approach is based on an improvement of a trace formula obtained in [12] for functions of dissipative operators.

2. Double operator integrals and a trace formula for arbitrary functions in OL_A

Double operator integrals

$$\iint \Phi(x, y) dE_1(x)Q dE_2(y)$$

were introduced by Birman and Solomyak in [4]. Here $\Phi$ is a bounded measurable function, $E_1$ and $E_2$ are spectral measures on a Hilbert space and $Q$ is a bounded linear operator. Such double operator integrals are defined for arbitrary bounded measurable functions $\Phi$ if $Q$ is a Hilbert–Schmidt operator. If $Q$ is an arbitrary bounded operator, then for the double operator integral to make sense, $\Phi$ has to be a Schur multiplier with respect to $E_1$ and $E_2$, (see [13] and [2]).

In this note we deal with double operator integrals with respect to semi-spectral measures

$$\iint \Phi(x, y) d\mathcal{E}_1(x)Q d\mathcal{E}_2(y).$$
Such double operator integrals were introduced in [15] (see also [16]). We refer the reader to a recent paper [2] for detailed information about double operator integrals.

If \( T \) is a contraction on a Hilbert space \( \mathcal{H} \), it has a minimal unitary dilation \( U \), i.e. \( U \) is a unitary operator on a Hilbert space \( \mathcal{K} \), \( \mathcal{K} \supset \mathcal{H} \), \( T^n = P_\mathcal{H} U^n |_{\mathcal{H}} \) for \( n \geq 0 \), and \( \mathcal{K} \) is the closed linear span of \( U^n \mathcal{H}, n \in \mathbb{Z} \) (see [20]). Here \( P_\mathcal{H} \) is the orthogonal projection onto \( \mathcal{H} \). The semi-spectral measure \( \mathcal{E}_T \) of \( T \) is defined by

\[
\mathcal{E}_T(\Delta) \overset{\text{def}}{=} P_\mathcal{H}E_U(\Delta)|_{\mathcal{H}},
\]

where \( E_U \) is the spectral measure of \( U \) and \( \Delta \) is a Borel subset of \( \mathbb{T} \). It is well known that \( T^n = \int_T \xi^n d\mathcal{E}_T(\xi), n \geq 0 \).

If \( f \in \mathcal{OL}_A \), then the divided difference \( \mathcal{D} f \),

\[
(\mathcal{D} f)(\zeta, \tau) \overset{\text{def}}{=} (f(\zeta) - f(\tau))((\zeta - \tau)^{-1}, \zeta, \tau \in \mathbb{T},
\]

is a Schur multiplier with respect to arbitrary Borel (semi-)spectral measures on \( \mathbb{T} \) and

\[
f(T_1) - f(T_0) = \int_{\mathbb{T} \times \mathbb{T}} (\mathcal{D} f)(\zeta, \tau) d\mathcal{E}_T(\zeta)(T_1 - T_0) d\mathcal{E}_T(\tau)
\]

for an arbitrary pair of contractions \((T_0, T_1)\) with trace class difference, see [2].

**Theorem 2.1.** Let \( f \in \mathcal{OL}_A \) and let \( T_0 \) and \( T_1 \) be contractions on a Hilbert space and \( T_t = T + t(R - T), 0 \leq t \leq 1 \). Then

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(f(T_{t+\varepsilon}) - f(T_t)) = \int_{\mathbb{T} \times \mathbb{T}} (\mathcal{D} f)(\zeta, \tau) d\mathcal{E}_T(\zeta)(T_1 - T_0) d\mathcal{E}_T(\tau)
\]

in the strong operator topology, where \( \mathcal{E}_t \) is the semi-spectral measure of \( T_t \).

It can be shown that if \( T_1 - T_0 \in \mathcal{S}_1 \), then

\[
f(T_1) - f(T_0) = \int_0^1 Q_t \, dt,
\]

where \( Q_t \) is the right-hand side of (6), and \( Q_t \in \mathcal{S}_1 \) for every \( t \in [0, 1] \). The integral can be understood in the sense of Bochner in the space \( \mathcal{S}_1 \). It can be shown that trace \( Q_t = \int_T f'(\zeta) \, d\nu_t(\zeta) \), where \( \nu_t \) is defined by \( \nu_t(\Delta) \overset{\text{def}}{=} \text{trace}((T - R)\mathcal{E}_t(\Delta)) \). We can define now the Borel measure \( \nu \) on \( \mathbb{T} \) by

\[
\nu \overset{\text{def}}{=} \int_0^1 \nu_t \, dt,
\]

which can be understood as the integral of the vector-function \( t \mapsto \nu_t \), which is continuous in the weak-star topology in the space of complex Borel measures on \( \mathbb{T} \).

**Theorem 2.2.** Let \( T_0 \) and \( T_1 \) be contractions on Hilbert space such that \( T_1 - T_0 \in \mathcal{S}_1 \). Then

\[
\text{trace} \left( f(T_1) - f(T_0) \right) = \int_T f'(\zeta) \, d\nu(\zeta)
\]

for every \( f \in \mathcal{OL}_A \), where \( \nu \) is the Borel measure defined by (7).

**3. A spectral shift function for a pair of contractions with trace class difference**

In this section we obtain the existence of a spectral shift function for pairs of contractions with trace class difference.

**Theorem 3.1.** Let \( T_0 \) and \( T_1 \) be contractions on Hilbert space with trace class difference. Then there exists a complex function \( \xi \) in \( L^1(\mathbb{T}) \) such that for an arbitrary analytic polynomial \( f \),

\[
\text{trace} \left( f(T_1) - f(T_0) \right) = \int_T f'(\zeta) \xi(\zeta) \, d\zeta.
\]

Moreover, if \( T_0 \) is a unitary operator, we can find such a function \( \xi \) that also satisfies the requirement \( \text{Im} \xi \leq 0 \). On the other hand, if \( T_1 \) is a unitary operator, we can add the requirement \( \text{Im} \xi \geq 0 \).
Remark. It is not true in general that for a pair of contractions with trace class difference, there exists a real spectral shift function. However, this is true under certain assumptions. In particular, if $\xi$ is a spectral shift function and $\xi \log(1 + |\xi|) \in L^1(\mathbb{T})$, then we can find a real spectral shift function for the same pair of contractions. The same conclusion holds if $\xi$ is a spectral shift function that belongs to the weighted space $L^p(T, w)$, where $1 < p < \infty$ and $w$ satisfies the Muckenhoupt condition $(A_p)$.

To prove Theorem 3.1, we can improve Theorem 3.14 of [12] and deduce Theorem 3.1 from that improvement with the help of the Cayley transform. On the other hand, Theorem 3.1 allows us to obtain a further improvement of Theorem 3.14 of [12] and obtain the following result:

Theorem 3.2. Let $L_0$ and $L_1$ be maximal dissipative operators such that

\[(L_1 + iI)^{-1} - (L_0 + iI)^{-1} \in S_1.\]  

Then there exists a complex measurable function $\omega$ (a spectral shift function for $(L_0, L_1)$) such that

\[
\int |\omega(t)|(1 + t^2)^{-1} dt < \infty,
\]

for which the following trace formula holds:

\[
\text{trace} \left( (L_1 - \lambda I)^{-1} - (L_0 - \lambda I)^{-1} \right) = -\int \omega(t)(t - \lambda)^{-2} dt, \quad \text{Im} \lambda < 0.
\]

Moreover, if $L_0$ is self-adjoint, there exists a function $\omega$ satisfying (11) and (12) such that $\text{Im} \omega \geq 0$ on $\mathbb{R}$, while if $L_1$ is self-adjoint, there exists a function $\omega$ satisfying (11) and (12) such that $\text{Im} \omega \leq 0$ on $\mathbb{R}$.

Recall that a closed densely defined operator $L$ is called dissipative if $\text{Im}(Lx, x) \geq 0$ for every $x$ in its domain. It is called a maximal dissipative operator if it does not have a proper dissipative extension.

Remark. In the case when $L_0 - L_1 \in S_1$, Theorem 3.2 can be specified. Namely, it was shown in [12] (Theorem 4.11) that a spectral shift function $\omega$ can be chosen in $L^1(\mathbb{R})$.

Note also that Theorem 3.1 improves earlier results in [1] and [19], while Theorem 3.2 improves Theorem 3.14 of [12] (the latter imposes the additional assumption $\rho(L_0) \cap C_+ \neq \emptyset$) and also improves and complements earlier results in [18] and [9] (see [12] for details).

4. The main result

Now we are able to state the main result of this note.

Theorem 4.1. Let $T_0$ and $T_1$ be contractions satisfying $T_1 - T_0 \in S_1$ and let $\xi$ be a spectral shift function for $(T_0, T_1)$. Then for every $f \in \text{OL}_A$ the following trace formula holds

\[
\text{trace} \left( f(T_1) - f(T_0) \right) = \int_{\mathbb{T}} f'(\zeta)\xi(\zeta) d\zeta.
\]

Indeed, by Theorem 3.1, formula (13) holds for analytic polynomials $f$. Combining this fact with formula (8), we see that the measure $\nu$ is absolutely continuous with respect to normalized Lebesgue measure and differs from the measure $\xi \, dz$ by an absolutely continuous measure with Radon–Nikodym density in $H^1$.

Remark. It is easy to see that the condition that $f$ has to be operator Lipschitz is not only sufficient for formula (13) to hold for arbitrary pairs of contractions $(T_0, T_1)$ with trace class difference, but also necessary. Indeed, it is well known (see [2]) that if $f$ is not operator Lipschitz, then there exist unitary operators $U$ and $V$ such that $U - V \in S_1$, but $f(U) - f(V) \notin S_1$.

By applying Cayley transform, we can deduce now from Theorem 4.1 the following analog of it for dissipative operators.

Theorem 4.2. Let $L_0$ and $L_1$ be maximal dissipative operators satisfying (10). Suppose that $f$ is a function analytic in the upper half-plane and such that the function

\[
\zeta \mapsto f(i(1 - \zeta)(1 + \zeta)^{-1}), \quad \zeta \in \mathbb{D},
\]

is such that $f(U) - f(V) \notin S_1$. Then

\[
\text{trace} \left( f(L_1) - f(L_0) \right) = \int_{\mathbb{T}} f'(\zeta)\xi(\zeta) d\zeta.
\]
belongs to OLₐ. Then \( f(L₁) - f(L₀) \in S₁ \) and

\[
\text{trace } (f(L₁) - f(L₀)) = \int f'(t) \omega(t) \, dt,
\]

where \( \omega \) is a spectral shift function for the pair \((L₀, L₁)\).

**Remark.** In the case when \( L₁ - L₀ \in S₁ \) and \( \omega \in L²(\mathbb{R}) \), it can be shown that formula (14) holds for all operator Lipschitz functions in the upper half-plane (see [2] for a discussion of the class of such functions).

Finally, we mention the paper [10], in which a trace formula for pairs of bounded operators with trace class difference is obtained for functions holomorphic in a neighbourhood of the spectra in terms of integration over a contour containing the spectra.

The research of the first author is partially supported by the Ministry of Education and Science of the Russian Federation (agreement number 02.03.21.0008); the research of the third author is partially supported by NSF grant DMS 1300924 and by the Ministry of Education and Science of the Russian Federation (agreement number 02.03.21.0008).

**References**

[1] V.M. Adamjan, H. Neidhardt, On the summability of the spectral shift function for pair of contractions and dissipative operators, J. Oper. Theory 24 (1990) 187–205.
[2] A.B. Aleksandrov, V.V. Peller, Operator Lipschitz functions, Russ. Math. Surv. 71 (4) (2016) 605–702.
[3] A.B. Aleksandrov, V.V. Peller, Krein’s trace formula for unitary operators and operator Lipschitz functions, Funct. Anal. Appl. 50 (3) (2016) 167–175.
[4] M.S. Birman, M.Z. Solomyak, Double Stieltjes Operator Integrals, Problems of Math. Phys., vol. 1, Leningrad Univ., 1966, pp. 33–67 (in Russian). English transl.: Topics Math. Phys., vol. 1, Consultants Bureau Plenum Publishing Corporation, New York, 1967, pp. 25–54.
[5] Yu.B. Farforovskaya, An example of a Lipschitzian function of selfadjoint operators that yields a nonnuclear increase under a nuclear perturbation, Zap. Nauč. Semin. POMI 30 (1972) 146–153 (in Russian).
[6] E. Kissin, V. Shulman, On fully operator Lipschitz functions, J. Funct. Anal. 253 (2) (2007) 711–728.
[7] M.G. Krein, On a trace formula in perturbation theory, Mat. Sb. 33 (1953) 597–626 (in Russian).
[8] M.G. Krein, On perturbation determinants and a trace formula for unitary and self-adjoint operators, Dokl. Akad. Nauk SSSR 144 (2) (1962) 268–271 (in Russian).
[9] M.G. Krein, Perturbation determinants and a trace formula for some classes of pairs of operators, J. Oper. Theory 17 (1987) 129–187.
[10] H. Langer, Eine Erweiterung der Spurformel der Störungstheorie, Math. Nachr. 30 (1965) 123–135.
[11] L.M. Lifshits, On a problem in perturbation theory connected with quantum statistics, Usp. Mat. Nauk 7 (1952) 171–180 (in Russian).
[12] M. Malamud, H. Neidhardt, Trace formulas for additive and non-additive perturbations, Adv. Math. 274 (2015) 736–832.
[13] V.V. Peller, Hankel operators in the theory of perturbations of unitary and self-adjoint operators, Funkc. Anal. Prilozh. 19 (2) (1985) 57–51 (in Russian). English transl.: Funkt. Anal. Appl. 19 (1985) 111–123.
[14] V.V. Peller, Hankel operators in the perturbation theory of unbounded self-adjoint operators, in: Analysis and partial differential equations, in: Lect. Notes Pure Appl. Math., vol. 122, Dekker, New York, 1990, pp. 529–544.
[15] V.V. Peller, For which \( f \) does \( A - B \in S_2 \) imply that \( f(A) - f(B) \in S_2 \)?, Oper. Theory 24 (1987) 289–294, Birkhäuser.
[16] V.V. Peller, Differentiability of functions of contractions, in: Linear and Complex Analysis, in: AMS Translations, Ser. 2, vol. 226, AMS, Providence, 2009, pp. 109–131.
[17] V.V. Peller, The Lifshits–Krein trace formula and operator Lipschitz functions, Proc. Amer. Math. Soc. 144 (2016) 5207–5215.
[18] A.V. Rybkin, The spectral shift function for a dissipative and a selfadjoint operator, and trace formulas for resonances, Mat. Sb. (N. S.) 125 (167) (1984) 420–430.
[19] A.V. Rybkin, A trace formula for a contractive and a unitary operator, Funkc. Anal. Prilozh. 21 (4) (1987) 85–87.
[20] B. Sz.-Nagy, C. Foiaş, Analyse harmonique des opérateurs de l’espace de Hilbert, Akadémiai Kiadó/Masson et Cie, Budapest/Paris, 1967.