Uncertainty-reality complementarity and entropic uncertainty relations

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Abstract
Reality of quantum observables, a feature of long-standing interest within foundations of quantum mechanics, has recently been quantified and deeply studied by means of entropic measures (Dieguez and Angelo 2018 Phys. Rev. A 97 022107). However, there is no state-independent 'reality trade-off' between non-commuting observables, as in certain systems all observables are real (Bilobran and Angelo 2015 Europhys. Lett. 112 40005). We show that the entropic uncertainty relation in the presence of quantum memory (Berta et al 2010 Nat. Phys. 6 659) perfectly supplements the discussed notion of reality, rendering trade-offs between reality and quantum uncertainty. State-independent complementarity inequalities involving entropic measures of both, uncertainty and reality, for two observables are presented.

Keywords: Shannon entropy, quantum reality, entropic uncertainty relations, quantum memory

1. Introduction

Shannon’s information theory [1], established 70 years ago and celebrated in the current special volume, has influenced physics across various domains. The range of applications for concepts and tools developed by Shannon is extraordinarily broad, likely because 'the information described by Shannon’s theory and measured by the Shannon entropy is not classical, but is neutral with respect to the physical theory that describes the systems used for its implementation’ [2]. As a consequence, several foundational and practical aspects of quantum mechanics, subsumed under the popular name of quantum information have been studied with the help of Shannon entropy and its relatives.
My favorite aspect of quantum mechanics, in which the Shannon entropy has brought a new quality, is the field of uncertainty relations (URs). Entropic uncertainty relations, pioneered in 1975 [3] for canonically conjugate pair of position and momentum, are a paradigm shift in the theory of URs. It is because values assumed by observables under discussion no longer play a role, instead, everything can be efficiently characterized by means of sole probability distributions. This conceptual novelty was soon adapted to the finite-dimensional (discrete) scenario [4] and substantially upgraded by the celebrated Maassen–Uffink (MU) bound [5]. Recently, several improvements (some on the conceptual side [6]) of the MU bound have been developed [7–15]—see also [16] and previous reviews on the topic [17, 18]. Let me mention in passing that ‘in between’ both cases (continuous and finite-dimensional) there are, so-called, coarse-grained observables for which the notion of uncertainty has also been well-captured with the Shannon [19, 20] and Rényi1 entropies [21–23] (for more results see the review [24]). At the end of the paper I will go back, for a moment, to both the continuous and coarse-grained scenario, while the main discussion will be conducted in the framework of finite-dimensional quantum systems.

I have devoted the whole paragraph to the description of the landscape formed by entropic URs, in order to emphasize how fruitful the concept of Shannon information entropy is. However, to unambiguously achieve this goal I shall point out that the Shannon and Rényi entropies are useful in various physical contexts, such as: entanglement detection [25–27], steering detection [28–30], quantum key distribution [31], quantum thermodynamics [32], resource theories [33] or Bell’s inequalities [34–37]—some without any direct connection to uncertainty relations (e.g. [32, 33, 36, 37]).

The last topic from the long list presented above, devoted to the physical underpinnings of the Bell-type experiments, brings the questions about reality of quantum observables. While reality in the above meaning is usually viewed as a competitor of quantum nonlocality (due to Bell’s theorem quantum mechanics violates Einstein’s local realism, so it must either be non-local or not real), thus being a qualitative notion, it is rarely discussed in a quantitative manner. Díeuez and Angelo [38] phrase that as follows: ‘...too little (if any) has been achieved with regard to formal connections between elements of reality and fundamental concepts such as information and quantum correlations.’ To partially fill this gap a measure of reality of quantum observables based on information entropies has been proposed in [39]. Deeper links between information and reality (also expressed in a kind of duality relation) were further investigated in [38]. In particular, it was shown that a completely positive trace preserving (CPTP) map, called monitoring, increases the reality of every observable [38].

The goal of the current contribution is to explore connections between the information-based measures of reality for quantum observables and entropic measures of uncertainty. The notion of reality, as being interrelated with the Bell’s inequalities and Einstein–Podolsky–Rosen paradox, must allow for a possibility of entanglement between the system under consideration and other parties. As a consequence, entropic URs suitable for the goal assumed are those with, so called, quantum memory [40]. In section 2 all necessary tools involving various entropic measures relevant for bipartite quantum systems are introduced, while in section 3 the main discussion of the uncertainty–reality complementarity takes place (appropriate inequalities are derived). In section 4 we go back to the monitoring quantum channel, as well as to continuous and coarse-grained observables.

1 The Rényi entropy is the most important generalization of the Shannon entropy, which appears after relaxation of one from the axioms originally assumed by Shannon [1].
2. Preliminaries

Before we move to the main part we need to fix the notation and define the objects of interest. We consider a mixed state $\rho_{AB}$ acting on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We make no assumptions about dimensionality of both Hilbert spaces, denoted by $d_{A/B} = \dim \mathcal{H}_{A/B}$, though in the main part we will concentrate on finite-dimensional systems. The reduced density matrix of the second subsystem (after the partial trace) is $\rho_B = \text{Tr}_A (\rho_{AB})$.

Let $X$ be a non-degenerate observable acting on $\mathcal{H}_A$ and let $|x_i\rangle$ for $i = 1, \ldots, d_A$ be the eigenstates of $X$, so that $\sum_{i=1}^{d_A} |x_i\rangle \langle x_i| = \mathbf{1}_A$. The post-measurement state is described by the density matrix

$$\rho_{XB} = \Phi_X (\rho_{AB}) = \sum_{i=1}^{d_A} (|x_i\rangle \langle x_i| \otimes \mathbf{1}_B) \rho_{AB} (|x_i\rangle \langle x_i| \otimes \mathbf{1}_B).$$

The map $\Phi_X (\cdot)$ is completely positive and trace preserving. In an analogous way we define $\rho_{XB}$ to be the post-measurement state relevant for the second observable $Y$, again acting on $\mathcal{H}_A$.

The assumed notation might suggest that the state $\rho_{XB}$ acts on $\mathcal{H}_X \otimes \mathcal{H}_B$, and that $\Phi_X : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_X \otimes \mathcal{H}_B$, which in fact is formally consistent (the same applies to $Y$). However, for simplicity and without loss of generality we set $\mathcal{H}_X = \mathcal{H}_A = \mathcal{H}_Y$. Still, the first label of the bipartite density matrix encodes the information whether we deal with the original state ($A$) or post-measurement states ($X$ or $Y$).

We further use several entropic quantities based on the von-Neumann entropy $S = -\text{Tr} (\rho \ln \rho)$ equal to the Shannon entropy $H$ calculated for the spectrum of the density matrix in question. In particular, we denote:

$$H (AB) = S (\rho_{AB}),$$

and consequently $H (XB) = S (\rho_{XB})$. For the reduced state we employ the notation $H (B) = S (\rho_B)$.

Differences of the above quantities form the conditional Shannon (von Neumann) entropies:

$$H (A|B) = H (AB) - H (B) \equiv S (\rho_{AB}) - S (\rho_B),$$

$$H (X|B) = H (XB) - H (B) \equiv S (\rho_{XB}) - S (\rho_B),$$

and similarly for $Y$. Note that since the state $\rho_{AB}$ can, in principle, be entangled, the entropy $H (A|B)$ can assume negative values. However, the post-measurement state (only discussed for $X$; the same applies to $Y$) can also be written as

$$\rho_{XB} = \sum_{i=1}^{d_A} p_i |x_i\rangle \langle x_i| \otimes \sigma_B^{(x)} ,$$

with $p_i$ and $\sigma_B^{(x)}$ being the probabilities associated with $|x_i\rangle \langle x_i| \otimes \mathbf{1}_B$ and the resulting states of the second subsystem respectively. As $\rho_{XB}$ in the form (4) is manifestly separable we know that [41] $H (X|B) \geq 0$. As a consequence, $H (X|B)$ is considered as an entropic notion of uncertainty in the bipartite setting [40], often referred to as equipped with quantum memory. Note that if $d_A = 1$ this quantity simply reduces to the ordinary Shannon entropy of the probabilities $p_i$, present in all standard entropic URs.

The memory-assisted entropic UR for the pair of observables $X$ and $Y$ is of the form [40]

$$H (X|B) + H (Y|B) \geq q + H (A|B),$$

where
with \( q \) being the MU lower bound for \( X \) and \( Y \) [5], i.e. \( q = -2 \ln c \) with \( c = \max_{i,j} |\langle x_i | y_j \rangle| \), or its improvements [10]. The major qualitative difference with respect to the case without quantum memory is due to the presence of \( H(A|B) \) on the right hand side. For a sufficiently entangled state the lower bound in (5) is trivial—equal to 0. More importantly, this trivial bound can be saturated, what means that one can have \( H(X|B) = 0 = H(Y|B) \).

We need to leave the notion of uncertainty for a moment, and turn to the concept of reality. In [38, 39] it was proposed that reality of an arbitrary observable \( X \) results in the equality \( \varrho_{AB} = \varrho_{X\bar{B}} \), i.e. the post-measurement state is the same as was the state before the measurement. Consequently, one could quantify the degree of irreality of the discussed observable in terms of distinguishability measures on the space of quantum states. Again we can observe how influential the contributions of Shannon are, since probably the most popular choice for the distinguishability measure (both in classical and quantum scenario) is the (quantum) relative entropy \( D(\rho||\sigma) \). Due to a special relationship between original and post-measurement states their relative entropy assumes a particularly handy form. One thus defines the ‘measure’ of irreality of an observable \( X \), given the state \( \varrho_{AB} \) as [38, 39]

\[
\mathfrak{I} (X|\varrho_{AB}) = D(\varrho_{X\bar{B}}||\varrho_{AB}) = S(\varrho_{X\bar{B}}) - S(\varrho_{AB}) \equiv H(X|B) - H(A|B).
\]

As is the relative entropy, \( \mathfrak{I} (X|\varrho_{AB}) \) is non-negative and vanishes if and only if the observable in question is real according to the definition provided above. We shall point out that the definition of \( \mathfrak{I} \) does not lead to a distinguished choice for the measure of reality (denoted, e.g. by \( \mathfrak{R} \)). In particular, there is no obvious argument selecting the form \( \mathfrak{R} = \text{Const} - \mathfrak{I} \), with the constant chosen, for instance, as \( \log d_A \). However, the ‘change of reality’ happening in any process shall obey the conservation law \( \Delta \mathfrak{R}(X) + \Delta \mathfrak{I}(X) = 0 \) [38]. Note that both changes do only depend on the observable \( X \), as processes causing the change do in principle affect the quantum state. In section 4 we go back to \( \Delta \mathfrak{R}(X) \) while discussing the monitoring map, offering the reader an insight into the physical meaning of the quantities under consideration.

### 3. Uncertainty–reality complementarity

For a single observable \( X \) we have two quantities, supposed to grasp its distinct quantum-mechanical aspects. While \( H(X|B) \) quantifies the uncertainty, \( \mathfrak{I}(X|\varrho_{AB}) \) shall describe the amount of irreality of \( X \). Apart from them, we also have \( H(A|B) \) which takes care of intrinsic properties of \( \varrho_{AB} \) such as entanglement, being completely independent from the observable in question.

We immediately observe that these three measures are not independent. Instead, they are in a basic linear relationship

\[
\mathfrak{I} (X|\varrho_{AB}) = H(X|B) - H(A|B).
\]

The above relation does not diminish the meaning of \( \mathfrak{I} (X|\varrho_{AB}) \), as it merely gives new interpretation to \( H(A|B) \). If the observable \( X \) is less uncertain than irreal, then the state \( \varrho_{AB} \) is necessarily entangled as \( H(A|B) \) is negative.

If we consider the second observable \( Y \) and write down its counterpart of equation (7), we find that

\[
H(X|B) - \mathfrak{I}(X|\varrho_{AB}) = H(Y|B) - \mathfrak{I}(Y|\varrho_{AB}),
\]

where both sides of (8) are equal to \( H(A|B) \). As a consequence, the four quantities: \( H(X|B), H(Y|B), \mathfrak{I}(X|\varrho_{AB}) \) and \( \mathfrak{I}(Y|\varrho_{AB}) \) are not mutually independent. Still, every pair of measures, \( H(X|B) \) and \( \mathfrak{I}(Y|\varrho_{AB}) \) for example, or even an arbitrary triple, are not generically interrelated.
We are in position to write-down inequalities which involve both, the uncertainty and the irreality. As sometimes happens in the topic of wave-particle duality\(^2\), the results presented below are more a consequence of algebraic manipulations, rather than deep mathematical theorems used to prove original entropic URs, such as \(L_p - L_q\) \cite{3} or \(l_p - l_q\) \cite{5} norm inequalities. Still, similarly to the wave-particle duality problem, the obtained results bring interesting interpretations.

First of all, if we apply equation (7) to the uncertainty relation (5) and utilize (8) we get

\[
\Im \langle X|\varrho_{AB}\rangle + H(Y|B) + \Im \langle Y|\varrho_{AB}\rangle \geq q. 
\]  

(9)

The above inequality does only involve the bound \(q\) which is state-independent. This is a substantial qualitative difference with respect to the bound in (5), which due to its dependence on \(\varrho_{AB}\) could be trivial. In other words, in the presence of quantum memory, the uncertainty relation (9) seems to encapsulate the original meaning of the entropic URs. To substantiate that claim and flash more light on the uncertainty–reality complementarity expressed by (9), we find two additional inequalities closely related to the former one.

The first inequality is obtained by rewriting (5) in terms of both measures of irreality. Due to the relation (7) and its counterpart for \(Y\), not written-down explicitly, we get

\[
\Im \langle X|\varrho_{AB}\rangle + \Im \langle Y|\varrho_{AB}\rangle \geq q - H(A|B), 
\]  

(10)

which is the uncertainty relation for (ir)reality measures of two observables. One immediately observes that the lower bounds in (5) and (10) differ ‘only’ by the sign of the conditional entropy contribution. This new UR, even though derived directly from (5), is in a kind of duality relationship with its ancestor. Entangled states characterized by negative \(H(A|B)\) can decrease the sum of uncertainties (sometimes trivializing the lower bound), however, they also result in a joint lack of reality.

To make the observed duality even more visible, we finally take the sum of (5) and (10)

\[
H(X|B) + \Im \langle X|\varrho_{AB}\rangle + H(Y|B) + \Im \langle Y|\varrho_{AB}\rangle \geq 2q. 
\]  

(11)

We again face the uncertainty relation with state-independent lower bound. Importantly, the inequality (11) can be saturated in completely opposite (dual in some sense) cases. To this end we set \(d_A = d_B = d\), and let \(X\) and \(Y\) be complementary (mutually unbiased) observables for which \(c = 1/\sqrt{d}\), so that \(q = \ln d\).

For a pure, maximally entangled state we easily find \(H(AB) = 0\) as \(\varrho_{AB}\) is pure and \(H(B) = q\) because \(\varrho_B = d^{-1}1_B\) as a consequence of maximal entanglement in \(\varrho_{AB}\). Thus \(H(A|B) = -q\). We can also calculate the post-measurement state (here for the observable \(X\))

\[
\varrho_{XB} = \frac{1}{d} \sum_{i=1}^{d} \langle x_i | \otimes | \overline{x}_i \rangle \langle \overline{x}_i | \otimes | x_i \rangle, 
\]  

(12)

with \(\overline{x}_i\) being the complex conjugate (coefficiencewise) of the ket \(x_i\) with respect to the Schmidt basis relevant for \(\varrho_{AB}\). Thus \(H(XB) = q\), and also \(H(YB) = q\) because the state \(\varrho_{YB}\) would clearly have the same eigenvalues (\(d\)-times degenerate eigenvalue equal to \(1/d\)) as \(\varrho_{XB}\). Taking all the above partial results together we get

\[
H(X|B) = H(Y|B) = 0, \quad \Im \langle X|\varrho_{AB}\rangle = \Im \langle Y|\varrho_{AB}\rangle = q. 
\]  

(13)

We can see that the combined UR (11) is saturated.

As the second example we consider the ‘dual’ case in which \(\varrho_{AB} = d^{-2}1_{AB}\) is the maximally mixed state. Clearly \(\varrho_{XB} = \varrho_{AB}\) for any \(X\), what was already pointed out in \cite{39}, so that

\(^2\)Note that also here, the information entropies were successfully applied \cite{42}.\]
both measures $\Im$ are zero. On the contrary, both uncertainties $H(X|B)$ and $H(Y|B)$ are equal to $q$, even though the reduced state $\varrho_B$ is exactly the same as in the case of the pure, maximally entangled state. Simply, the von Neumann entropy for all possible post-measurement states is equal to $2q$, which is the entropy of $\varrho_{AB}$. Again, the lower bound in (11) becomes saturated, even though uncertainty and irreality swapped their contributions.

4. Discussion

In the last section I sketched a way in which one discovers that the measure of irreality is complementary to the entropic measure of uncertainty. Equations (9)–(11) are to a large extent equivalent to each other, though each UR conveys a different physical interpretation. To shortly repeat: in (9) we find the state-independent UR similar in spirit to the seminal results by Maassen and Uffink; the bound in (10) is dual to the one present in the quantum memory-assisted UR in equation (5); the UR (11) is the four-term uncertainty relation which for any pair of observables becomes saturated in both opposite regimes of pure, maximally entangled states and maximally mixed, fully separable states.

In [38] the CPTP map $\mathcal{M}_Y^\epsilon$, referred to as monitoring and defined through its action

$$\mathcal{M}_Y^\epsilon : \varrho_{AB} \mapsto (1 - \epsilon) \varrho_{AB} + \epsilon \varrho_{YB},$$

(14)

has been investigated. A compelling physical intuition behind this map is offered in [38], where interaction with an appropriate ancilla is deeply discussed. For our purpose it is enough to notice that $[\mathcal{M}_Y]^n = [\mathcal{M}_Y]^{-1}(1 - \epsilon)^n$, so that several instances of the monitoring map’s action do simply increase $\epsilon$, pushing the initial state towards $\varrho_{YB}$.

As already mentioned, the map (14) has extensively been studied in [38], in particular, it was shown that

$$\Im (X|\varrho_{AB}) \geq \Im (X|\mathcal{M}_Y^\epsilon [\varrho_{AB}]),$$

(15)

for every pair of observables $X$ and $Y$, so that the monitoring (with respect to any observable) increases the reality of every other observable, i.e. $\Delta \Re (X) \geq 0$ for (14).

As $\text{Tr}_A (\mathcal{M}_Y^\epsilon [\varrho_{AB}]) = \varrho_B$, and $\Phi_Y (\mathcal{M}_Y^\epsilon [\varrho_{AB}]) = \varrho_{YB}$, we can see that the uncertainty of $Y$ for the state $\varrho_{AB}$, $H(Y|B)$, remains the same for the monitored state $\mathcal{M}_Y^\epsilon [\varrho_{AB}]$. As a consequence, equation (9) written down for $\mathcal{M}_Y^\epsilon [\varrho_{AB}]$ reads

$$\Im (X|\mathcal{M}_Y^\epsilon [\varrho_{AB}]) + H(Y|B) \geq q.$$ 

(16)

In other words, irreality of $X$ with respect to the quantum state monitored by $Y$ is lower-bounded by $q - H(Y|B)$. For example, if $X$ and $Y$ are taken to be complementary, i.e. such that $q = \ln d_A$, the bound is very restrictive as $H(Y|B) \leq \ln d_B$ by construction. In particular, for the monitoring map (14) equation (16) tells us that

$$\Delta \Re (X) = - \Delta \Im (X) = \Im (X|\varrho_{AB}) - \Im (X|\mathcal{M}_Y^\epsilon [\varrho_{AB}]) \leq \Im (X|\varrho_{AB}) + H(Y|B) - q.$$ 

(17)

If the state $\varrho_{AB}$ is pure and maximally entangled, so that $H(Y|B) = 0$, the bound (16) prevents the reality from a severe growth driven by monitoring based on a ‘sufficiently complementary’ observable. Is the observable $Y$ fully complementary to $X$ (see discussion around equation (13)) the level of reality remains constant.

At the end, we shall go back to the continuous [3] and the coarse-grained [20–24] scenarios. Two recent, mathematically-oriented contributions [43, 44] extend the URs relevant for the above cases, in order to include quantum-memory effects. In [43], a continuous counterpart of equation (5)
\( h(x|B) + h(p|B) \geq \ln 2\pi + H(A|B), \quad (18) \)

has been derived. By \( h(x|B) \) and \( h(p|B) \) we denote continuous conditional Shannon entropies for position and momentum probability densities respectively. It is interesting that in the case without quantum memory, state-independent part of the lower bound is sharper (equal to \( \ln e\pi \) [3]), however, the quantum-memory UR in the form of equation (18) can also be saturated (i.e. \( \ln 2\pi \) in equation (18) is optimal) [45].

Despite mathematical subtleties related to infinite dimension of the Hilbert space, one could consider a direct generalization of the major results of this paper. However, as pointed out in [44] it is better to define all conditional entropies first in the coarse-grained setting and then take the fine-graining limit in which the coarse-graining widths tend to 0. It might happen that a difference of entropic measures which diverge in this limit does exist and has a physical meaning. In this formally safer scenario, however, one finds memory-assisted entropic URs which differ from (18), mainly because the involved quantities are also different. Thus, while it would be interesting to merge the concept of reality (now for continuous and coarse-grained observables) with entropic measures of uncertainty, this task requires much more care and attention. For now, it will be left as a future open problem.

Staying with the questions for further research, one shall point out that uncertainty in the presence of quantum memory is also well-captured by the information exclusion principle [10, 46, 47], expressed in terms of mutual information. All results derived here could gain additional interpretations, while rewritten that way.

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