Flat Holomorphic Connections and Picard-Fuchs Identities From $N = 2$ Supergravity

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ABSTRACT

We show that in special Kähler geometry of $N = 2$ space-time supergravity the gauge variant part of the connection is holomorphic and flat (in a Riemannian sense). A set of differential identities (Picard-Fuchs identities) are satisfied on a holomorphic bundle. The relationship with the differential equations obeyed by the periods of the holomorphic three form of Calabi-Yau manifolds is outlined.
1. Introduction

For the study of \( N = 1 \) superstring vacua in four space-time dimensions one is particularly interested in super conformal field theories (SCFT) with central charge \( c = 9 \) and worldsheet supersymmetry \((0, 2)\).\(^{[1]}\) For the subset of string vacua where this worldsheet supersymmetry is enlarged to \((2, 2)\) the low energy effective Lagrangian satisfies an additional constraint. The manifold spanned by certain massless scalar fields in the string spectrum (so called moduli) is promoted from being a Kähler manifold to a ‘special Kähler manifold’.\(^{[2]}\) Such a geometric structure first appeared in the context of \( N = 2 \) supergravity theories in four space-time dimensions as the scalar manifold spanned by the scalars of vectormultiplets.\(^{[3,4]}\) The moduli space of \((2, 2)\) SCFT with central charge \( c = 9 \) obeys a similar constraint as can be understood from the relation between heterotic and type II string theories\(^{[5,6]}\) or directly from SCFT Ward identities.\(^{[7]}\)

Calabi-Yau threefolds are an example of \((2, 2)\) SCFT and thus their moduli spaces are also special Kähler manifolds. This can also be derived by purely geometrical methods without ever referring to a SCFT.\(^{[8,9,10]}\)

Recently, for a specific Calabi-Yau manifold (a quintic in \( CP_4 \)) the exact Kähler and super-potential of the low energy theory was given without using the underlying SCFT.\(^{[11]}\) This was possible by explicitly constructing the mirror map\(^{[12]}\) of this Calabi-Yau moduli space. In the course of this investigation it was shown that the periods of the holomorphic threeform \( \Omega \) satisfy a linear differential equation. It was then realized that this equation is a particular case of Picard-Fuchs systems of differential equations, known to be obeyed by the periods of globally defined \( p \)-forms \((p = 3 \) for a threefold) from algebraic geometry.\(^{[13–15]}\) Furthermore, substantial information about \((2, 2)\) SCFT can be obtained from a purely topological sector of the theory.\(^{[16–19]}\) For \( c = 3 \) SCFT analogous differential equations were derived using methods developed in topological field theories (TFT).\(^{[20–22]}\)

Since the moduli space of Calabi-Yau-threefolds or more generally the moduli space of \((2, 2)\) SCFT of central charge \( c = 9 \) is constrained to display special geometry it should be possible to understand the Picard-Fuchs equations entirely within the framework of special geometry. This is the aim of this paper. We first briefly remind the reader of the properties of special Kähler geometry. Then we show that the Christoffel as well as the Kähler connection consist of two distinctively different pieces. A holomorphic
part is responsible for the transformation properties of the connection whereas the non-
holomorphic piece transforms like a tensor. The holomorphic part of the connection
turns out to be flat (in a Riemannian sense). We believe that this holomorphic flat
connection is the analogue of the flat connection encountered in TFT when it is restricted
to marginal deformations.\textsuperscript{[17,19,14]}

Equipped with this understanding we then show how special geometry implies a set of
covariant differential identities (which we call Picard-Fuchs identities). These identities
are entirely equivalent to the identity the Riemann tensor satisfies in special geometry
but are instead of purely holomorphic nature. Once the coefficient functions of these
identities are specified a non-trivial linear differential equation arises which is equivalent
to the Picard-Fuchs equation satisfied by the periods of the Calabi-Yau-manifold.

2. Summary of special geometry

The metric \( g_{\alpha\bar{\beta}}, \alpha = 1, \ldots, n \) of a \( n \)-dimensional Kähler manifold is given by the
second derivative of a Kähler potential \( K \): \( g_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_\beta K (\equiv K_{\alpha\bar{\beta}}) \). \( K \) is a real function
of the complex coordinates \( z^\alpha \) and \( \bar{z}^{\bar{\alpha}} \). \( g_{\alpha\bar{\beta}} \) is invariant under the Kähler transformations
\( K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}) \). A field \( \Psi \) of Kähler weight \((k, \bar{k})\) transforms according
to \( \Psi \to \Psi e^{kf} e^{k\bar{f}} \). Its Kähler covariant derivative is defined by

\[
D_\alpha \Psi = (\nabla_\alpha - kK_\alpha)\Psi \\
\bar{D}_{\bar{\alpha}} \Psi = (\bar{\nabla}_{\bar{\alpha}} - \bar{k}\bar{K}_{\bar{\alpha}})\Psi
\]

where \( \nabla_\alpha \) denotes the covariant derivative with respect to reparametrisations of the
Kähler manifold. Thus \( K_\alpha, K_{\bar{\alpha}} \) act as connections for Kähler transformations. This is
a consequence of the fact that special manifolds are manifolds of restricted type. This
means that \( K_\alpha \) is an abelian connection of a holomorphic line bundle \( L \) whose first
Chern class is the Kähler class \((c_1(L) = [J])\).\textsuperscript{[2]}

Special geometry is defined by an additional constraint on the Kähler metric. This
constraint can be expressed as a covariant equation the Riemann tensor of the Kähler
manifold satisfies. It reads\textsuperscript{[4]}

\[
R^\delta_{\alpha\bar{\beta}\gamma} = g_{\alpha\bar{\beta}} \delta_\gamma^\delta + g_{\gamma\bar{\beta}} \delta_\alpha^\delta - C_{\alpha\gamma\epsilon} g^{\epsilon\delta} \overline{C}_{\bar{\beta}\bar{\delta}\epsilon} g^{\delta\bar{\epsilon}}
\]
where $C_{\alpha\beta\gamma}$ is of Kähler weight $(-1, 1)$, symmetric in its indices and satisfies

\[ \bar{D}_\epsilon C_{\alpha\beta\gamma} = 0, \]
\[ D_\epsilon C_{\alpha\beta\gamma} - D_\alpha C_{\epsilon\beta\gamma} = 0. \tag{3} \]

Using $D_\alpha e^K = 0$ eq. (3) can be solved by

\[ C_{\alpha\beta\gamma} = e^K W_{\alpha\beta\gamma}(z), \]
\[ W_{\alpha\beta\gamma} = D_\alpha D_\beta D_\gamma S(z, \bar{z}); \tag{4} \]
\[ \bar{\partial}_\epsilon W_{\alpha\beta\gamma} = 0. \]

Also, one can find $K$ and $S$ such that eq. (2) is satisfied in arbitrary coordinates.\[ ^{[23]} \]

This is achieved by introducing $n + 1$ holomorphic sections $X^A(z), A = 0, 1, \ldots, n$ which obey $\bar{\partial}_a X^A = 0$. In terms of $X^A$ the Kähler potential reads

\[ K = -\ln Y \]
\[ Y = X^A N_{AB} \bar{X}^B = X^A \bar{F}_A + \bar{X}^A F_A \tag{5} \]

where

\[ N_{AB}(X, \bar{X}) = F_{AB}(X) + \bar{F}_{AB}(\bar{X}) \]
\[ F_{AB}(X) = \partial_A \partial_B F(X) \tag{6} \]

and $F(X)$ is a homogeneous function of degree two. $C_{\alpha\beta\gamma}$ and $S$ are then given by

\[ W_{\alpha\beta\gamma} = \partial_\alpha X^A \partial_\beta X^B \partial_\gamma X^C F_{ABC} \]
\[ S = -\frac{1}{2} X^A N_{AB} X^B. \tag{7} \]

$X^A$ transforms under Kähler transformations with weight $(-1, 0)$, $\bar{X}^A$ with weight $(0, -1)$. $N_{AB}$ is invariant as a consequence of $F$ being homogeneous of degree two.

Let us define the $(2(n + 1)$ dimensional symplectic) holomorphic vector $V(z) = (X^A(z), iF_A(z))$. It is straightforward to show that as a consequence of eqs. (5)-(7)
\( V(z) \) satisfies the following set of identities (and its complex conjugate)\(^{[23]}\)

\[
\begin{align*}
    D_\alpha V &= U_\alpha \\
    D_\alpha U_\beta &= C_{\alpha\beta\gamma} g^{\gamma\bar{\gamma}} U_{\bar{\gamma}} \\
    D_\alpha \bar{U}_{\bar{\beta}} &= g_{\alpha\bar{\beta}} V \\
    D_\alpha \nabla &= 0
\end{align*}
\] (8)

where the first equation defines \( U_\alpha \).\(^*\) Eq. (2) now follows from eqs. (8). Thus (8) is an alternative way to characterize special geometry.

Finally, there is a particularly simple set of coordinates given by \( z^\alpha = X^a/X^0, a = 1, \ldots, n. \)\(^{[3,23]}\) In these so called ‘special coordinates’ \( K \) and \( C_{\alpha\beta\gamma} \) reduce to\(^\dagger\)

\[
\begin{align*}
    K &= -\ln \left[ 2(F + \bar{F}) - (F_\alpha - \bar{F}_\alpha)(z^\alpha - \bar{z}^{\bar{\alpha}}) \right] \\
    C_{\alpha\beta\gamma} &= Y^{-1} F_{\alpha\beta\gamma}.
\end{align*}
\] (9)

We will see in the next section that these coordinates are the flat coordinates for a holomorphic connection.

As we remarked in the introduction special Kähler geometry arises in \( N = 2 \) supergravity theories in four space-time dimensions as the manifold spanned by the scalars in the vectormultiplets.\(^{[3,4]}\) The moduli space of \((2,2)\) SCFT (and thus the moduli space of Calabi-Yau-threefolds) also obeys eq. (2) where \( C_{\alpha\beta\gamma} \) are the Yukawa couplings (the structure constants of the chiral ring\(^{[24]}\)) of the associated matter multiplets.\(^{[7]}\)

3. HOLOMORPHIC CONNECTION

Before we turn to the study of eqs. (8) let us first make a few observations about the connections in special geometry. We already remarked that in addition to the usual Christoffel connection \( \Gamma^\gamma_{\alpha\beta} \) there also is a Kähler connection given by \( K_\alpha \). However, due to the additional constraint on the Kähler potential (eqs. (2),(8)) we find that both

\* The exact same set of identities were derived in a Calabi-Yau context in ref. 9,10.
\dagger With a further (Kähler) gauge choice one can set \( X^0 = 1 \). However, special coordinates do not require this gauge.
connections enjoy further properties in special geometry. These are most easily displayed by introducing Kähler invariant functions $t^a$ as follows

$$ t^a(z) = \frac{X^a(z)}{X^0(z)}, \quad a = 1, \ldots, n. \quad (10) $$

In terms of $t^a$ and $X^0$ the Kähler connection $K_\alpha$ decomposes into a sum of a purely holomorphic term and a non-holomorphic piece

$$ K_\alpha(z, \bar{z}) = \hat{K}_\alpha(z) + \mathcal{K}_\alpha(z, \bar{z}) \quad (11) $$

where

$$ \mathcal{K}_\alpha(z, \bar{z}) = e^a_\alpha(z) K_a(z, \bar{z}) \equiv e^a_\alpha(z) \frac{\partial}{\partial t^a} K(t(z), \bar{t}(\bar{z})) $$

$$ \hat{K}_\alpha(z) = - \partial_\alpha \ln X^0(z) $$

$$ e^a_\alpha(z) = \partial_\alpha t^a(z) \quad (12) $$

The significance of the holomorphic piece $\hat{K}_\alpha$ is that it transforms as a connection whereas $\mathcal{K}_\alpha$ is invariant under Kähler transformation:

$$ K_\alpha \rightarrow K_\alpha + f_\alpha, \quad \hat{K}_\alpha \rightarrow \hat{K}_\alpha + f_\alpha, \quad \mathcal{K}_\alpha \rightarrow \mathcal{K}_\alpha. \quad (13) $$

A very similar structure also occurs for $\Gamma_{\alpha \beta}^{\gamma}$. Let us first compute the metric in terms of $t^a$ and $X^0$:

$$ g_{\alpha \bar{\beta}} = e^a_\alpha e_{\bar{\beta}}^b g_{a \bar{b}} \equiv e^a_\alpha e_{\bar{\beta}}^b \frac{\partial}{\partial t^a} \frac{\partial}{\partial \bar{t}^b} K(z, \bar{z}) \quad (14) $$

(We see that $e^a_\alpha(z)$ can be viewed as a holomorphic vielbein.) As a consequence the Christoffel connection ($\Gamma_{\alpha \beta}^{\gamma} \equiv g_{\alpha \bar{\gamma}} g^{-1} (\bar{\gamma} \gamma)$) also splits in the following way

$$ \Gamma_{\alpha \beta}^{\gamma}(z, \bar{z}) = \hat{\Gamma}_{\alpha \beta}^{\gamma}(z) + T_{\alpha \beta}^{\gamma}(z, \bar{z}) \quad (15) $$

where

$$ T_{\alpha \beta}^{\gamma}(z, \bar{z}) = e^a_\alpha e_{\bar{\beta}}^b g_{a \bar{b}} g^{-1} c^{1 \gamma} \quad (16) $$

Again the holomorphic $\hat{\Gamma}_{\alpha \beta}^{\gamma}(z)$ transforms as a connection under reparametrisations whereas $T_{\alpha \beta}^{\gamma}$ enjoys tensorial transformation properties. Thus in both cases the transformation properties of the connections are entirely carried by the holomorphic objects.
\( \hat{K} \) and \( \hat{\Gamma} \). Thus it is possible to introduce a purely holomorphic covariant derivative \( \hat{D} \) where \( \Gamma \) and \( K \) are replaced by \( \hat{\Gamma} \) and \( \hat{K} \). As we will see in the next section the holomorphic (Picard-Fuchs) identities precisely use \( \hat{D} \).

The (holomorphic) metric for which \( \hat{\Gamma} \) is the connection reads

\[
\hat{g}_{\alpha\beta} = e^a_{\alpha} e^b_{\beta} \eta_{ab} \tag{17}
\]

where \( \eta_{ab} \) is a constant (invertible) symmetric matrix. Note that \( \hat{g}_{\alpha\beta} \) has two holomorphic indices in contrast to the Kähler metric \( g_{\alpha\bar{\beta}} \). From eq. (16) and (17) it comes as no surprise that \( \hat{\Gamma} \) also satisfies

\[
\hat{R}^\gamma{}_{\delta\alpha\beta} = \partial_\delta \hat{\Gamma}^\gamma{}_{\alpha\beta} - \partial_\alpha \hat{\Gamma}^\gamma{}_{\delta\beta} + \hat{\Gamma}^\mu{}_{\alpha\beta} \hat{\Gamma}^\gamma{}_{\mu\delta} - \hat{\Gamma}^\mu{}_{\delta\beta} \hat{\Gamma}^\gamma{}_{\mu\alpha} = 0 \tag{18}
\]

which means that it is a flat Riemannian connection. The flat coordinates are exactly the special coordinates \( (t^a = z^a) \) introduced in the last section. In these coordinates we find

\[
e^a_{\alpha} = \delta^a_{\alpha}, \quad \hat{\Gamma}^\gamma{}_{\alpha\beta} = 0, \quad \hat{g}_{\alpha\beta} = \eta_{\alpha\beta}. \tag{19}
\]

(The gauge choice \( X^0 = 1 \) then implies \( \hat{K}_\alpha = 0 \).)

Before we turn to the derivation of the differential equation let us collect a few more formulas which we use in the next section. We observe that eq. (2) can also be expressed as

\[
\partial_\beta \left( \Gamma^\delta{}_{\alpha\gamma} - K_\alpha \delta^\delta_{\gamma} - K_\gamma \delta^\delta_{\alpha} + T^\delta{}_{\alpha\gamma} \right) = 0 \tag{20}
\]

where

\[
T^\delta{}_{\alpha\gamma} = C_{\alpha\gamma\epsilon} g^{\epsilon\zeta} (D_\delta D_{\epsilon} D_{\zeta} ) g^{\delta\bar{\delta}} = Y^{-1} D_{\alpha} X^A D_\gamma X^C D_{\beta} X^B F_{ABC} g^{\bar{\delta}\delta}. \tag{21}
\]

Inserting eqs. (15) and (11) into eq. (20) we learn that \( T^\delta{}_{\alpha\gamma} \) can be further decomposed into

\[
T^\delta{}_{\alpha\gamma} = K_\alpha \delta^\delta_{\gamma} + K_\gamma \delta^\delta_{\alpha} - T^\delta{}_{\alpha\gamma} \tag{22}
\]

Of course, this can also be verified directly from the definitions of \( T^\delta{}_{\alpha\gamma} \) and \( K_\alpha \).
Another flat (non-holomorphic) connection exists on the flat symplectic bundle of ref. 9. What we find here is that a flat holomorphic connection exists on a $n$-dimensional space rather than a $2n + 2$ dimensional space as in ref. 9. In the deformation theory of the Hodge structure of Calabi-Yau-threefolds the components of the symplectic vector $V$ are precisely the periods of the holomorphic three form $\Omega$. We believe that $\hat{\Gamma}$ is the analogue of the flat connection of $N = 2$ TFT when it is restricted to the marginal deformations.\textsuperscript{[17,19,14]} (In general the connection of the TFT acts in the space of all topological deformations not only the marginal ones.)

4. \textbf{Holomorphic covariant Picard-Fuchs identities}

Let us come back to the set of identities given in eq. (8). As we remarked above they are equivalent to the identity (2) the Riemann tensor of special geometry satisfies. The Riemann tensor identity turns into a non-trivial differential equation for the Kähler potential $K$ if the Yukawa couplings $C_{\alpha\beta\gamma}$ are specified. In the same spirit the identities (8) turn into a non-trivial differential equation once the coefficient functions are specified. As we will see in a moment the difference is that the latter leads to holomorphic differential equations which are exactly the Picard-Fuchs equations of the Calabi-Yau-manifold\textsuperscript{[25,11,13–15]} and the analogous equations in TFT.\textsuperscript{[20–22]}

First we consider the case where $C_{\alpha\beta\gamma} = 0$ holds.\footnote{This is not an unrealistic case but does occur on certain subspaces of the moduli space of (2, 2) SCFT.} Then eqs. (8) reduce to

$$D_\alpha D_\beta V(z) = (\partial_\alpha \partial_\beta + \Lambda^\gamma_{\alpha\beta} \partial_\gamma + \Sigma_{\alpha\beta})V(z) = 0$$

(23)

where

$$\Lambda^\gamma_{\alpha\beta} = K_{\alpha\delta} \delta^\gamma_\delta + K_{\beta\delta} \delta^\gamma_\delta - \Gamma^\gamma_{\alpha\beta},$$

$$\Sigma_{\alpha\beta} = K_{\alpha\beta} + K_{\alpha}K_{\beta} - \Gamma^\gamma_{\alpha\beta}K_\gamma.$$  

(24)

As we just discussed eq.(23) is an identity if we use the explicit form of $K$ given by eqs. (5)-(7). However, the coefficients $\Lambda^\gamma_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ satisfy two noteworthy features. First, they transform under coordinate and Kähler transformations as to render eq. (23) covariant. This is obvious from eq. (23) but can also be checked explicitly in eq. (24).
Secondly, they satisfy $\bar{\partial}_{\bar{\gamma}} \Lambda_{\alpha\beta} = \bar{\partial}_{\bar{\gamma}} \Sigma_{\alpha\beta} = 0$ which holds as a consequence of eq. (2). Again, this has to be the case since $V(z)$ is holomorphic by definition. The considerations of the last section explain how it is possible to have manifestly holomorphic $\Lambda, \Sigma$ which at the same time transform appropriately: they can only depend on the holomorphic connections $\hat{K}$ and $\hat{\Gamma}$. Indeed, inserting eqs. (11) and (15) into (24) one verifies

$$
\Lambda_{\alpha\beta}^\gamma = \hat{\Lambda}_{\alpha\beta}^\gamma + \hat{\Sigma}_{\alpha\beta}^\gamma = \hat{\Sigma}_{\alpha\beta}^\gamma + \hat{\Sigma}_{\alpha\beta}^\gamma.
$$

(25)

Thus eq. (23) is covariant and holomorphic and the coefficients $\Lambda$ and $\Sigma$ depend on the holomorphic connections defined in the last section. Thus, if these holomorphic coefficients are given eq. (23) turns into a non-trivial differential equation for $V$. Of course in this simple case the solution is already known. In special coordinates (and in the gauge $X^0 = 1$) eq. (23) simplifies further and reads $\partial_\alpha \partial_\beta X^A = \partial_\alpha \partial_\beta F_A = 0$ which is consistent with our input $C_{\alpha\beta\gamma} = F_{\alpha\beta\gamma} = 0$. The solutions are (at least locally) the homogeneous spaces $SU(1,n)/SU(n) \times U(1)$ with $F = c_{AB} X^A X^B$ where $c_{AB}$ is some constant symmetric matrix.

Let us turn to the case of non vanishing $C_{\alpha\beta\gamma}$. Now $D_{\alpha} D_{\beta} V$ is non-zero and $\Lambda, \Sigma$ are no longer holomorphic. Instead they satisfy

$$
\Lambda_{\alpha\beta}^\gamma = \hat{\Lambda}_{\alpha\beta}^\gamma + T_{\alpha\beta}^\gamma,
$$

$$
\Sigma_{\alpha\beta} = \hat{\Sigma}_{\alpha\beta} + \hat{T}_{\alpha\beta} K_{\gamma} + \Sigma_{\alpha\beta}^{(3)}
$$

(26)

where

$$
\bar{\partial}_{\bar{\gamma}} \Sigma_{\alpha\beta}^{(3)} = -T_{\alpha\beta} g_{\gamma\bar{\gamma}}.
$$

(27)

Now a holomorphic identity appears if one uses the full set (8). This is particularly simple if one considers a one dimensional manifold. In this case eqs. (8) are equivalent.

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† Globally, there can be discrete identifications which leads to the quotient of the homogeneous space with the duality group. The latter is related to the monodromy group of eq. (23).\textsuperscript{[13,14]}
to
\[ DDW^{-1}DDV = \sum_{n=0}^{4} a_n(z) \partial^n V = 0. \] (28)

The coefficients \( a_n \) are given by‡

\[ a_4 = W^{-1} \]
\[ a_3 = 2\partial W^{-1} \]
\[ a_2 = W^{-1}(\partial \Lambda - \Lambda^2 + 2\Sigma) + \partial W^{-1} \Lambda + \partial^2 W^{-1} \]
\[ a_1 = W^{-1}(\partial^2 \Lambda + 2\partial \Sigma - 2\Lambda \partial \Lambda) + \partial W^{-1}(2\Sigma + 2\partial \Lambda - \Lambda^2) + \partial^2 W^{-1} \Lambda \]
\[ a_0 = W^{-1}(\Sigma^2 - \Sigma \partial \Lambda - \Lambda \partial \Sigma + \partial^2 \Sigma) + \partial W^{-1}(2\partial \Sigma - \Lambda \Sigma) + \partial^2 W^{-1} \Sigma \]

where

\[ \Lambda = 2\partial K - \Gamma \]
\[ \Sigma = \partial^2 K + (\partial K)^2 - \Gamma \partial K. \] (30)

As before the \( a_n \)'s have the appropriate transformation properties and satisfy \( \bar{\partial}a_n = 0 \) as a consequence of eq. (2). Inserting eq. (15) and (11) into (29) we find that in all coefficients \( a_n \) \( K \) and \( \Gamma \) are replaced by there holomorphic pieces \( \hat{K} \) and \( \hat{\Gamma} \). Thus in eq. (29) instead of \( \Lambda \) and \( \Sigma \) appear

\[ \hat{\Lambda} = 2\partial \hat{K} - \hat{\Gamma} \]
\[ \hat{\Sigma} = \partial^2 \hat{K} + (\partial \hat{K})^2 - \hat{\Gamma} \partial \hat{K}. \] (31)

Now the meaning of eq. (28) becomes transparent. As it stands it is a covariant identity in special geometry. The coefficient functions \( a_n \) are holomorphic functions of the Yukawa couplings \( W \) and the holomorphic connections \( \hat{K}, \hat{\Gamma} \) (or equivalently the einbein \( e \) and \( X^0 \)). If these three quantities are given as 'data' eq. (28) turns into a linear differential equation for \( V \). (Given \( a_n \) eqs. (29) also imply a non-linear differential equation for \( e \) and \( X^0 \).)

‡ The dependence of \( a_3 \) on \( W \) was also found in ref.11.
From eq. (29) we learn that the \( a_n \) are related in the following form

\[
a_3 = 2\partial a_4, \quad a_1 = \partial a_2 - \frac{1}{2}\partial^2 a_3.
\] (32)

Indeed, for the specific example of the quintic in \( CP_4 \) the \( a_n \)'s (in the Landau-Ginzburg coordinates) are given by\,[11,13,14]

\[
a_4 = 1 - \psi^5 \quad a_3 = -10\psi^4 \quad a_2 = -25\psi^3
\]

\[
a_1 = -15\psi^2 \quad a_0 = -\psi
\] (33)

and they obey (32).

In special coordinates (with the additional gauge choice \( X^0 = 1 \)) eq. (28) takes a particularly simple form, its coefficients read

\[
a_4 = W^{-1}, \quad a_3 = 2\partial W^{-1},
\]

\[
a_2 = \partial W^{-2}, \quad a_1 = a_0 = 0.
\] (34)

The reason for the vanishing of \( a_1 \) and \( a_0 \) is that in these coordinates \( \partial X^0 = 0 \) and \( \partial X^1 = 1 \) holds. Thus for consistency \( a_0 = a_1 = 0 \) is required. (Also, we can use (34) and \( W = \partial^3 F, V = (1, X^1, i(2F - X^1\partial F), i\partial F) \) to verify that (28) is identically satisfied.)

Finally, it is possible to remove the Kähler connection \( \hat{K} \) from eq. (28) altogether. If we rescale \( V \) by \( 1/X^0 \) and define a Kähler invariant \( \mathcal{V} = V/X^0 \) the differential equation reads

\[
\sum_{n=0}^{4} a'_n(z) \partial^n \mathcal{V} = 0.
\] (35)

In this basis \( a'_n \) is independent of \( \hat{K} \) and all \( X^0 \) dependence appears through the ratio \( W/(X^0)^2 \). In this basis \( a_0 = 0 \) holds which again is required for consistency. Inverting this procedure produces a relation between \( a_0 \) and \( a_1, a_2, a_3 \).
5. Conclusion

In this letter we found that the connections in special geometry can be split into a holomorphic piece which transforms as a connection and a non-holomorphic piece with tensorial transformation properties. This fact allows for the existence of holomorphic covariant identities in special geometry. When certain ‘data’ (the Yukawa coupling and the holomorphic connection) are given these identities turn into linear differential equations which are the exact analogue of the Picard-Fuchs equations for the periods of the holomorphic threeform $\Omega$ in Calabi-Yau manifolds.

We treated in some detail the case of a one dimensional Kähler manifold but clearly our considerations can be extended to arbitrary dimension. From eq. (8) it is obvious that again holomorphic identities occur whose coefficient function depend on the holomorphic connections.

Our method is particularly powerful when applied to deformation theory of Calabi-Yau threefolds and $\mathcal{N} = 2$ TFT. Indeed, it allows to obtain Picard-Fuchs equations in arbitrary coordinate systems and to give a precise relation between the Kähler geometry of the moduli space and the flat geometry of the deformation of the chiral ring restricted to marginal operators. Various issues, discussed recently in the literature can be further investigated and better understood such as relation of Picard-Fuchs systems to $W$-algebras and target space duality symmetry.

Acknowledgements:

It is a pleasure to thank R. D’Auria, A.Ceresole, M. Cvetič, W. Lerche, M. Peskin, R. Schimmrigk, D. Smit and N. Warner for useful discussions and in particular P. Candelas and X. de la Ossa for communicating some unpublished work. J.L. thanks the Aspen Center of Physics for providing a stimulating atmosphere where this investigation started.

Some of the computations were performed using Maple.
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