On a classical correspondence between K3 surfaces III

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Abstract

Let $X$ be a K3 surface, and $H$ its primitive polarization of the degree $H^2 = 8$. The moduli space of sheaves over $X$ with the isotropic Mukai vector $(2, H, 2)$ is again a K3 surface, $Y$. In [3] we gave necessary and sufficient conditions in terms of Picard lattice of $X$ when $Y$ is isomorphic to $X$. The proof of sufficient condition in [3], when $Y$ is isomorphic to $X$, used Global Torelli Theorem for K3 surfaces, and it was not effective.

Here we give an effective variant of these results: its sufficient part gives an explicit isomorphism between $Y$ and $X$.

We hope that our similar results in [4], [7], [8] for arbitrary primitive isotropic Mukai vector on a K3 surface also can be made effective.

1 Introduction

In [3] we had obtained the following result.

Theorem 1.1. Let $X$ be a K3 surface over $\mathbb{C}$ with Picard lattice $N(X)$, and $H \in N(X)$ is primitive, nef with $H^2 = 8$. Let $Y$ be the moduli space of sheaves on $X$ with the isotropic Mukai vector $v = (2, H, 2)$.

Then, $Y \cong X$ if there exists $h_1 \in N(X)$ such that the primitive sublattice $[H, h_1]_{pr}$ in $N(X)$ generated by $H$ and $h_1$ has an odd determinant (equivalently, $H \cdot [H, h_1]_{pr} = \mathbb{Z}$) and

$$h_1^2 = \pm 4 \quad \text{and} \quad h_1 \cdot H \equiv 0 \quad \text{mod} \ 2.$$ 

These conditions are necessary for $Y \cong X$ if the Picard number $\rho(X) = \text{rk } N(X) \leq 2$, and $X$ is a general K3 surface with its Picard lattice (i.e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is $\pm 1$).

The proof of Theorem 1.1 given in [3] used Global Torelli Theorem for K3 surfaces [9], and it was not effective; under conditions of Theorem 1.1, we had only proved existence of the isomorphism $Y \cong X$.

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The purpose of this paper is to prove the following effective variant of Theorem 1.1.

**Theorem 1.2.** Let $X$ be a K3 surface over $\mathbb{C}$ and $H \in N(X)$ is primitive, nef with $H^2 = 8$. Let $Y$ be the moduli space of sheaves on $X$ with the isotropic Mukai vector $v = (2, H, 2)$.

Then $Y \cong X$ with an explicit geometric isomorphism given by (3.8) and (3.15) in the proof below if there exists $h_1 \in N(X)$ such that the primitive sublattice $[H, h_1]_{pr}$ in $N(X)$ generated by $H$ and $h_1$ has an odd determinant (equivalently, $H \cdot [H, h_1]_{pr} = \mathbb{Z}$), and

$h_1^2 = \pm 4, \quad h_1 \cdot H \equiv 0 \mod 2, \quad (1.1)$

and

$h^0\mathcal{O}_X(h_1) = h^0\mathcal{O}_X(-h_1) = 0 \quad if \quad h_1^2 = -4. \quad (1.2)$

These conditions are necessary for $Y \cong X$ if the Picard number $\rho(X) \leq 2$, and $X$ is a general K3 surface with its Picard lattice (i.e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is $\pm 1$).

Here results by Tyurin [10] and by Ballico-Chiantini [1] are very useful. See Remark 3.1 about difference between conditions of Theorems 1.1 and 1.2.

In our papers [4], [7], [8], Theorem 1.1 was generalized to arbitrary primitive Mukai vector. We hope that similar considerations as in this paper will also permit to make these results effective.

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## 2 Reminding of the Main Result of [3]

We denote by $X$ an algebraic K3 surface over the field $\mathbb{C}$ of complex numbers. I.e. $X$ is a non-singular projective algebraic surface over $\mathbb{C}$ with the trivial canonical class $K_X = 0$ and the vanishing irregularity $q(X) = 0$.

We denote by $N(X)$ the Picard lattice (i.e. the lattice of 2-dimensional algebraic cycles) of $X$. By $\rho(X) = \text{rk} \ N(X)$ we denote the Picard number of $X$. By

$$T(X) = N(X)^{\perp}_H (X, \mathbb{Z}) \quad (2.1)$$

we denote the transcendental lattice of $X$.

For a Mukai vector $v = (r, c_1, s)$ where $r, s \in \mathbb{Z}$ and $c_1 \in N(X)$, we denote by $Y = M_X(r, c_1, s)$ the moduli space of stable (with respect to some ample $H' \in N(X)$) rank $r$ sheaves on $X$ with first Chern classes $c_1$, and Euler characteristic $r + s$.

By results of Mukai [5], [6], under suitable conditions on the Chern classes, the moduli space $Y$ is always deformations equivalent to a Hilbert scheme of 0-dimensional cycles on $X$ (of same dimension).
In [3] we had considered the case of the isotropic Mukai vector $v = (2, H, 2)$ with $H^2 = 8$ and $H$ nef and primitive, and we had looked for conditions on the Picard lattice $N(X)$ which imply that $Y \cong X$. One of our main results in [3] was the following Theorem.

**Theorem 2.1.** Let $X$ be a K3 surface over $\mathbb{C}$ and $H \in N(X)$ is primitive, nef with $H^2 = 8$. Let $Y$ be the moduli space of sheaves on $X$ with the isotropic Mukai vector $v = (2, H, 2)$.

Then, $Y \cong X$ if there exists $h_1 \in N(X)$ such that the primitive sublattice $[H, h_1]_{pr}$ in $N(X)$ generated by $H$ and $h_1$ has an odd determinant (equivalently, $H \cdot [H, h_1]_{pr} = \mathbb{Z}$) and

$$h_1^2 = \pm 4 \quad \text{and} \quad h_1 \cdot H \equiv 0 \mod 2.$$

These conditions are necessary for $Y \cong X$ if the Picard number $\rho(X) \leq 2$, and $X$ is a general K3 surface with its Picard lattice (i.e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is $\pm 1$).

The proof of Theorem 2.1 in [3] used Global Torelli Theorem for K3 surfaces [9], and it was not effective.

The purpose of this paper is to give an effective variant of Theorem 2.1 which does not use Global Torelli Theorem for K3 surfaces. This variant gives an explicit isomorphism $Y \cong X$. In [3] we had only proved existence of such isomorphism.

### 3 An effective (without use of Global Torelli Theorem for K3) variant of Theorem 2.1

Here we prove the following effective variant of Theorem 2.1.

**Theorem 3.1.** Let $X$ be a K3 surface over $\mathbb{C}$ and $H \in N(X)$ is primitive, nef with $H^2 = 8$. Let $Y$ be the moduli space of sheaves on $X$ with the isotropic Mukai vector $v = (2, H, 2)$.

Then $Y \cong X$ with an explicit geometric isomorphism given by (3.8) and (3.15) in the proof below if there exists $h_1 \in N(X)$ such that the primitive sublattice $[H, h_1]_{pr}$ in $N(X)$ generated by $H$ and $h_1$ has an odd determinant (equivalently, $H \cdot [H, h_1]_{pr} = \mathbb{Z}$), and

$$h_1^2 = \pm 4, \quad h_1 \cdot H \equiv 0 \mod 2,$$

and

$$h^0\mathcal{O}_X(h_1) = h^0\mathcal{O}_X(-h_1) = 0 \quad \text{if} \quad h_1^2 = -4. \quad (3.2)$$

These conditions are necessary for $Y \cong X$ if the Picard number $\rho(X) \leq 2$, and $X$ is a general K3 surface with its Picard lattice (i.e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is $\pm 1$).
Proof. The ‘necessary’ part of the proof is the same as in \[3\]. Let us assume that \(\rho(X) \leq 2\), \(X\) is general (i.e., the automorphism group of the transcendental periods \((T(X), H^{2,0}(X))\) is \(\pm 1\)), and \(Y \cong X\). Since \(Y \cong X\), periods of \(Y\) and \(X\) must be isomorphic. We have shown in \[3\] that periods of \(Y\) and \(X\) are isomorphic if and only if \(H \cdot N(X) = \mathbb{Z}\) (Mukai condition), and there exists \(h_1 \in N(X)\) such that

\[
h_1^2 = \pm 4, \quad H \cdot h_1 \equiv 0 \mod 2.
\]

Thus, we obtain exactly the conditions of Theorem 2.1 or conditions of Theorem 3.1 except (3.2). The determinant of the Gram matrix of \(H\) and \(h_1\) is equal to

\[
H^2 h_1^2 - (H \cdot h_1)^2 = \pm 32 - (H \cdot h_1)^2 \neq 0.
\]  

(3.3)

Thus, \(\rho(X) = 2\), and \(N(X) = [H, h_1]_{pr}\) is a 2-dimensional lattice \((\rho(X) = 1\) never happens if \(Y \cong X\)). In \((3)\), Proposition 3.2.1 and Theorems 3.2.2, 3.2.3) we have shown that if \(N(X)\) has only elements \(h_1\) satisfying (3.1) with \(h_1^2 = -4\), then \(N(X)\) has no elements \(\delta\) with \(\delta^2 = -2\). Since any irreducible curve \(C\) on a K3 surface has \(C^2 \geq -2\) (it is well-known and obvious) and \(N(X)\) is an even lattice, it then follows that any effective element of \(N(X)\) has a non-negative square. Then (3.2) is automatically valid.

Now let us consider the ‘sufficient’ part of the proof of Theorem 3.1 which used Global Torelli Theorem for K3 surfaces \(9\) and was not effective in \(3\).

We have simple

Lemma 3.1. Let \(X\) be a K3 surface and \(H \in N(X)\) a primitive element with \(H^2 = 8\).

Then existence of \(h_1 \in N(X)\) satisfying conditions of Theorem 2.1 i.e.

\[
(\ h_1^2 = \pm 4, \ H \cdot h_1 \equiv 0 \mod 2, \ H \cdot [H, h_1]_{pr} = \mathbb{Z}\ )
\]

is equivalent to

\[
\exists D \in N(X) \text{ such that Mukai vector } v_1 = (2, H + 2D, \pm 1) \text{ is isotropic } (3.5)
\]

i.e. \((H + 2D)^2 = \pm 4\).

The relation between \((3.4)\) and \((3.5)\) is just

\[
h_1 = H + 2D.
\]

Proof. Assume (3.4) is valid. The determinant of Gram matrix of \(H\) and \(h_1\) is equal to \(\pm 32 - (H \cdot h_1)^2 \neq 0\). It follows that \([H, h_1]_{pr}\) is a 2-dimensional sublattice in \(N(X)\).

Since \(H\) is primitive in \(N(X)\), then \(H \notin 2[H, h_1]_{pr}\). Since \((h_1/2)^2 = \pm 1\) and \(N(X)\) is even lattice, then \(h_1 \notin 2[H, h_1]_{pr}\). If \(H - h_1 \notin 2[H, h_1]_{pr}\,\text{ it then follows that } H + 2[H, h_1]_{pr}, h_1 + 2[H, h_1]_{pr}\,\text{ give a basis of } [H, h_1]_{pr}\,\text{ mod } 2 = [H, h_1]_{pr}/2[H, h_1]_{pr}\). Then \(H \cdot [H, h_1]_{pr} \equiv \{H^2, \ H \cdot h_1\} \equiv 0 \mod 2\) which
contradicts $H \cdot [H, h_1] = Z$. Thus, $h_1 = H + 2D$ where $D \in [H, h_1] \subset N(X)$. It follows the condition (3.8).

Now assume (3.8) is valid. We put $h_1 = H + 2D$. Then $h_1^2 = (H+2D)^2 = \pm 4$, $h_1 \cdot H = H^2 + 2(H \cdot D) \equiv 0 \pmod{2}$. We have

$$(H + 2D)^2 = 8 + 4(H \cdot D) + 4D^2 = \pm 4$$

where $D^2 \equiv 0 \pmod{2}$ since $N(X)$ is even. It follows $H \cdot D \equiv 1 \pmod{2}$. Since $H \cdot H = 8$, it follows $H \cdot [H, h_1] = Z$. We obtain the condition (3.9).

This finishes the proof of Lemma 3.1. \hfill $\Box$

To give an effective proof of Theorem 3.1, we now should consider two cases.

**The case $h_1^2 = 4$ of Theorem 3.1.** By Lemma 3.1, this is equivalent to the Mukai vector $v_1 = (2, h_1 = H + 2D, 1)$ is isotropic for some $D \in N(X)$. (3.7)

Then $Y = M_X(v) \cong M_X(v_1)$ under tensorization by $\mathcal{O}_X(D)$. By general results (see e.g. [10], Chapter II, Section 4) $M_X(v_1) \cong M_X(w_1) \cong X$ where $w_1 = (1, h_1, 2)$. This gives an explicit isomorphism

$$Y = M_X(2, H, 2) \cong M_X(2, H + 2D, 1) \cong M_X(1, H + 2D, 2) \cong X.$$ (3.8)

**The case $h_1^2 = -4$ of Theorem 3.1.** By Lemma 3.1, this is equivalent to the Mukai vector $v_1 = (2, h_1 = H + 2D, -1)$ is isotropic for some $D \in N(X)$, (3.9) and

$$h^0 \mathcal{O}_X(H + 2D) = h^0 \mathcal{O}_X(-H - 2D) = 0.$$ (3.10)

Let us take $D \in N(X)$ satisfying these conditions. Changing $h_1$ by $-h_1$ is equivalent to changing $D$ by $-H - D$. Replacing $h_1$ by $-h_1$ if necessary, we can assume that $H \cdot h_1 = H \cdot (H + 2D) = 8 + 2H \cdot D \geq 0$. Equivalently, $H \cdot D \geq -4$. From $(H + 2D)^2 = 8 + 4H \cdot D + 4D^2 = -4$ and $D^2 \equiv 0 \pmod{2}$, it follows that $H \cdot D$ is always odd (and $h_1 \cdot D$ as well). Thus, we can even assume more:

$$H \cdot D > -4.$$ (3.11)

From $h^0 \mathcal{O}_X(h_1) = 0$ and $h^2 \mathcal{O}_X(h_1) = h^0 \mathcal{O}_X(-h_1) = 0$ and Riemann-Roch Theorem for K3, we obtain that $\chi \mathcal{O}_X(h_1) = 0$ and $h^1 \mathcal{O}_X(h_1) = 0$.

Let $p \in X$ be a point and $\mathcal{I}_p$ its sheaf of ideals. Since $\mathcal{I}_p \subset \mathcal{O}_X$ and $h^0 \mathcal{O}_X(h_1) = 0$, then $h^0 \mathcal{I}_p(1) = 0$. By the exact sequence of $\mathcal{I}_p \subset \mathcal{O}_X$, we also obtain $h^1 \mathcal{I}_p(h_1) = h^1 \mathcal{O}_p(H + 2D) = h^0 \mathcal{O}_p(h_1) = 1$. Then $(H^1 \mathcal{I}_p(H + 2D))^* \cong \text{Ext}^1(\mathcal{I}_p(H + D), \mathcal{O}_X(-D))$ is one-dimensional, and we can construct a rank 2 bundle $\mathcal{E}$ given by the non-trivial extension

$$0 \to \mathcal{O}_X(-D) \to \mathcal{E} \to \mathcal{I}_p(H + D) \to 0,$$ (3.12)
equivalently a rank 2 bundle $\mathcal{E}(D)$

$$0 \to \mathcal{O}_X \to \mathcal{E}(D) \to \mathcal{I}_p(H + 2D) \to 0,$$  
(3.13)

and $\mathcal{E}$ is a rank 2 bundle with $c_1 = H$ and $c_2 = 4$. The bundle $\mathcal{E}$ is semistable since $\mathcal{E}(D)$ is so. If $L = \mathcal{O}_X(L) \subset \mathcal{E}(D)$ is such that $L \cdot H > (H \cdot h_1)/2 > 0$ then $L$ is not contained in the image of the map $\mathcal{O}_X \to \mathcal{E}(D)$. Hence the image of the inclusion of $L \subset \mathcal{E}(D)$ under the projection $\mathcal{E} \to \mathcal{I}_p(H + 2D)$ gives a non zero map $L \to \mathcal{I}_p \otimes \mathcal{O}_X(H + 2D)$ and $h_1 = L + L'$ with $L'$ effective. Then $h_1 - L = L'$ is effective and we have $h^0\mathcal{O}_X(h_1 - L) \leq h^0\mathcal{O}_X(h_1) = 0$ which is absurd. Indeed the last vanishing follows by the exact sequence

$$0 \to \mathcal{O}_X(h_1 - L) \to \mathcal{O}_X(h_1) \to \mathcal{O}_L(h_1) \to 0$$  
(3.14)

Thus $\mathcal{E} \in M_X(2, H, 2)$. Since $h^0\mathcal{I}_p(H + 2D) = 0$, by (3.13) we obtain that $h^0\mathcal{E}(D) = 1$. Thus, (3.12) is defined by a unique (up to proportionality) non-zero section of $\mathcal{E}(D)$, and $p$ is the zero locus of this section. Thus the constructed using (3.13) and (3.12) map

$$X \to M_X(2, H, 2) = Y$$  
(3.15)

has the degree one, and it defines an explicit isomorphism $Y \cong X$.

This finishes the proof of Theorem 3.1. □

**Remark 3.1.** Difference between Theorems 2.1 and 3.1 is in condition (3.2) which means that both elements $h_1$ with $h_1^2 = -4$ should be also not effective.

If there exists $h_1' \in N(X)$ with $(h_1')^2 = -4$, then $N(X)$ has plenty of elements $h_1$ with $h_1^2 = -4$ such that both $\pm h_1$ are not effective.

Really, the nef cone $NEF(X)$ of $X$ is a fundamental chamber for the group $W(-2)(X)$ generated by reflections in all elements $\delta \in N(X)$ with $\delta^2 = -2$. It follows that there exists $w \in W(-2)(X)$ such that $h_1 = w(h_1')$ divides $NEF(X)$ in two open parts: there exist two nef elements $H_1, H_2 \in NEF(X)$ such that $H_1 \cdot h_1 < 0$ and $H_2 \cdot h_1 > 0$ (such elements $\pm h_1$ are called not pseudo-effective). It follows that both elements $\pm h_1$ are not effective.

Thus, to satisfy conditions of Theorem 3.1 for the case $h_1^2 = -4$, one should look first for not pseudo-effective $\pm h_1$ satisfying $h_1^2 = -4$, $h_1 \cdot H \equiv 0 \mod 2$ and $h_1 \cdot [H, h_1]_{pr} = \mathbb{Z}$. All not pseudo-effective elements $\pm f \in N(X)$ with negative square $f^2 < 0$ satisfy the geometric condition (3.2), and there are plenty of them.

In our papers [4], [7], [8], Theorem 2.1 was generalized to arbitrary primitive Mukai vector $v = (r, H, s)$ where $r, s \in \mathbb{N}$ and $H^2 = 2rs$. We hope that similar considerations as here also permit to make these results effective. We hope to consider that in further publications.
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