Semi-local nuclear forces from chiral EFT: State-of-the-art & challenges

Evgeny Epelbaum*, Hermann Krebs and Patrick Reinert

Abstract Recently, a new generation of nuclear forces has been developed in the framework of chiral EFT. An important feature of these potentials is a novel semi-local regularization approach that combines the advantages of a local regulator for long-range interactions with the convenience of an angle-independent nonlocal regulator for contact interactions. The authors discuss the key features of the semi-local two-nucleon potentials and demonstrate their outstanding performance in the two-nucleon sector by showing selected results up to fifth order in the EFT expansion. Also reviewed are applications to heavier systems, which are currently limited to third chiral order. This limitation reflects the conceptual difficulty in constructing a consistently regularized many-body forces and current operators and affects all currently available interactions. The authors outline possible ways to tackle this problem and discuss future directions in the field.

Introduction

In the past decade, a large number of nuclear potentials have been developed in the framework of chiral EFT. These interactions differ by the choices of degrees of freedom in the effective Lagrangian, the orders in the EFT expansion, the em-
ployed regulators, the values for the low-energy constants (LECs) and the strategies for their determination, the treatment of relativistic and isospin-breaking corrections and by many other aspects. It is, therefore, important to start with briefly summarizing the main principles and the general framework used to develop the semi-local momentum-space regularized (SMS) interactions of Refs. [1, 2].

- The expressions for nuclear forces and current operators are derived from the heavy-baryon effective Lagrangian for pions and nucleons via a perturbative expansion in powers of $Q \in \{ p/\Lambda_b, M_\pi/\Lambda_b \}$. Here, $M_\pi$ denotes the pion mass while $p \sim M_\pi$ stands for a typical three-momentum scale for low-energy few-nucleon processes under consideration. The breakdown scale of the chiral EFT expansion $\Lambda_b$ in the two-nucleon sector is estimated to be $\Lambda_b \sim 650$ MeV [3, 4, 5].

- Following Weinberg [6], the nucleon mass $m$ is treated as a heavier scale as compared to $\Lambda_b$, $m \sim \Lambda_b^2/\Lambda_\pi$. In Tab. 1 various types of contributions to the nuclear forces in this framework are listed, most of which have already been worked out using dimensional regularization to deal with divergent loop integrals. Notice that nuclear potentials and current operators are not uniquely defined, and their derivation is considerably more demanding than just calculating Feynman diagrams, see Refs. [7] for details.

- The resulting nuclear potentials are regularized with a finite cutoff $\Lambda \sim \Lambda_b$ [8, 9] that is chosen sufficiently soft to prevent the appearance of spurious deeply bound states. The functional form of the employed semi-local regulator will be specified in the next section. The authors of Ref. [1] do not allow for tuning the functional form of the regulator in specific partial waves to improve the description of experimental data. The SMS interactions discussed here are available for the cutoffs $\Lambda = 400, 450, 500$ and $550$ MeV. The residual dependence of observables on $\Lambda$ probes the impact of contact interactions beyond the accuracy level of the calculation and is used to validate the estimated truncation uncertainty. Throughout this chapter, the quoted truncation errors correspond to the Bayesian model $\bar{C}_{650}^{0.5, -10}$ from Ref. [10].

- For the pion-nucleon LECs, the values from the Roy-Steiner equation analysis of Ref. [11] are employed. LECs entering the NN contact interactions are determined from neutron-proton and proton-proton scattering data. Notice that 3 out of 15 LECs accompanying the NN contact interactions at fourth order (N$^3$LO) parametrize the off-shell dependence of the potential and therefore cannot be (reliably) determined in the NN sector. In Refs. [1, 2], the authors employed a convention to eliminate these off-shell terms via an appropriate unitary transformation. This implies that certain linear combinations of N$^4$LO short-range contributions to the three-nucleon force (3NF) and the NN charge density operator are enhanced and appear already at N$^3$LO [12]. This convention results in 21 isospin-invariant NN contact interactions in total at N$^3$LO and N$^4$LO, which can all be reliably determined from the neutron-proton and proton-proton experimental data as will be described below. Therefore, there seems to be no need to employ heavier nuclei to extract the corresponding LECs as done e.g. in Ref. [13]. Similarly, the LECs entering the 3NF at N$^3$LO are determined from 3N data.
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| LO ($Q^0$) | NLO ($Q^2$) | N^2LO ($Q^3$) | N^3LO ($Q^4$) | N^4LO ($Q^5$) |
|-------|-------|-------|-------|-------|
| 2NF 1π, NN [2] 2π, NN [7] 2π | 2π, 3π, NN [15] a | 2π, 3π, 2π, ring, NNN [8] | 2π, 1π-2π, ring, 1π-NN, 2π-NN | 2π, 1π-2π, ring, 1π-NN [unknown] b, 2π-NN b, NNN [13] |
| 3NF — — — | — | 2π, 1π-NN, 2π-NN | — | — c |
| 4NF — — — | 3π, 4π, 2π-NN | — | — c | 1π-NN-NN |

Table 1 Types of contributions to the nuclear forces at various orders in the EFT expansion using Weinberg’s power counting. 1π, 2π, 3π and 4π denote one-, two-, three- and four-pion exchange diagrams, respectively, while NN and NNN refer to the two- and three-nucleon contact interactions. For the relativistic corrections, the assignment $m \sim \Lambda^2 \beta / M_\pi$ is made. The subscripts in the square brackets indicate the numbers of LECs accompanying the two- and three-nucleon contact interactions. Isospin-violating interactions are not shown.

a 3 out of 15 operators do not contribute to the NN S-matrix in the Born approximation. The corresponding LECs can therefore not be (reliably) determined from NN data at this order.
b These topologies have not been worked out yet.
c These contributions have not been worked out yet.

• All charge-independence and charge-symmetry breaking contributions to the NN force up to N^4LO were included in the updated version of the original SMS potentials [1] described in Ref. [2].

Below, some of the key features of SMS NN potentials of Refs. [1,2] will be reviewed. In particular, the semi-local regularization approach and the partial wave analysis (PWA) of NN scattering data using the SMS chiral NN potentials will be discussed. The results for phase shifts and NN observables will be compared with alternative PWAs and with different chiral EFT potentials. The authors also discuss selected applications in the NN sector and for heavier systems and outline ongoing efforts towards developing consistent 3NFs and current operators beyond N^2LO.

**SMS two-nucleon potentials up to N^4LO**

**Regularization and subtractions**

Semi-locally regularized nuclear potentials up to N^3LO were originally introduced in Ref. [3] and extended to fifth order (N^4LO) in Ref. [14]. The term "semi-local" refers to a local regularization method for the long-range interactions mediated by the exchange of a single or multiple pions in combination with a nonlocal (angle-independent) cutoff for contact terms. In the original papers [3,14], the local regulator was implemented in coordinate space. This somewhat *ad hoc* procedure was
Refs. [1, 2] use local regulator. For the (isospin invariant part of the) $\pi$-exchange potential $V_{1\pi}(q) = \alpha/(q^2 + M_\pi^2)$ with $q = p' - p$ being the momentum transfer, $p$ and $p'$ the initial and final momenta of the nucleons and $\alpha$ denoting the spin-momentum-isospin structure. Using local and nonlocal Gaussian-type regulators one obtains

$$V_{1\pi, \text{local}}(q) = \frac{\alpha}{q^2 + M_\pi^2} e^{-\frac{q^2 + M_\pi^2}{\Lambda^2}} = \frac{\alpha}{q^2 + M_\pi^2} - \frac{\alpha}{\Lambda^2} \frac{q^2 + M_\pi^2}{\Lambda^8} + \mathcal{O}(\Lambda^{-6}),$$

$$V_{1\pi, \text{nonlocal}}(q) = \frac{\alpha}{q^2 + M_\pi^2} e^{-\frac{q^2 + M_\pi^2}{\Lambda^2}} = \frac{\alpha}{q^2 + M_\pi^2} - \frac{\alpha}{\Lambda^2} \frac{p^2 + p'^2}{q^2 + M_\pi^2} + \mathcal{O}(\Lambda^{-4}),$$

where the last equalities hold for momenta below the cutoff $\Lambda$. The nonlocal regulator in the second line obviously affects the analytic structure of the potential by changing the residue of $V_{1\pi}(q)$ at $q^2 = -M_\pi^2$ and induces long-range finite-$\Lambda$ artefacts that need to be systematically taken care of at higher orders [15]. In contrast, the local regulator in the first line preserves the analytic structure of $V_{1\pi}(q)$, and all finite-$\Lambda$ artefacts have the form of contact interactions which are anyway present in the potential. This feature becomes particularly important when using soft cutoff values.

Local regulators can, in principle, be applied to contact interactions as well [16]. This then allows one to reduce the degree of nonlocality of the interactions, a particularly welcome feature for certain ab initio methods like e.g. the Quantum Monte Carlo technique. On the other hand, locally regularized contact terms cannot be formed into linear combinations that contribute to specific partial waves only, i.e. the one-to-one correspondence between the contact interactions and the partial waves as given by Eq. (A.2) of Ref. [1] is lost. This feature significantly complicates the determination of the corresponding LECs. A semi-local regulator allows one to preserve the analytic structure of the long-range interactions while at the same time keeping the simplicity of a non-local regulator $e^{-\frac{q^2 + M_\pi^2}{\Lambda^2}}$ for contact interactions.

The authors are now in the position to specify the form of the employed semi-local regulator. For the (isospin invariant part of the) $1\pi$-exchange the authors of Refs. [1, 2] use

$$V_{1\pi, \text{local}}(q) = -\frac{g_A^2}{4F_\pi^2} \tau_1 \cdot \tau_2 \left( \hat{\sigma}_1 \cdot \hat{q} \hat{\sigma}_2 \cdot \hat{q} + C \hat{\sigma}_1 \cdot \hat{\sigma}_2 \right) e^{-\frac{q^2 + M_\pi^2}{\Lambda^2}},$$

where $\hat{\sigma}_i$ ($\tau_i$) denote the Pauli spin (isospin) matrices of the nucleon $i$ while $g_A$ and $F_\pi$ are the axial-vector coupling constant of the nucleon and the pion decay constant, respectively. Here, the freedom to include in the definition of $V_{1\pi, \text{local}}(q)$ a (locally regularized) LO contact interaction is exploited to ensure that the Fourier transform of
the resulting spin-spin potential vanishes at \( r = 0 \). This fixes the subtraction constant \( C \) to

\[
C = -\frac{\Lambda}{3} \left( A^2 - 2M_\pi^2 \right) + 2\sqrt{\pi}M_\pi^3 e^{\frac{M_\pi^2}{2A}} \text{erfc} \left( \frac{M_\pi}{\sqrt{2A}} \right),
\]

where \( \text{erfc}(x) \) is the complementary error function.

For the \( 2\pi \)-exchange, the regulator can be easily implemented using the spectral representation. For example, the unregularized expression for the central \( 2\pi \)-exchange potential at NLO, \( V^{2\pi} = \mathbf{t}_1 \cdot \mathbf{t}_2 V_C(q) + \ldots \), can be written as a spectral integral

\[
W_C(q) = \frac{2}{\pi} \int_{2M_\pi}^{\infty} \frac{d\mu}{\mu^3} \eta_C(\mu) \frac{q^4}{\mu^2 + q^2},
\]

with the spectral function given by \[17\]

\[
\eta_C(\mu) = \sqrt{\frac{\mu^2 - 4M_\pi^2}{768\pi F_\pi^2 \mu}} \left[ 4M_\pi^4 (5g_\Lambda^4 - 4g_\Lambda^2 - 1) - \mu^2 (23g_\Lambda^4 - 10g_\Lambda^2 - 1) + \frac{48g_\Lambda^4 M_\pi^4}{4M_\pi^2 - \mu^2} \right].
\]

The corresponding locally regularized expression used in the SMS potentials of Refs. [1, 2] has the form

\[
W_{C, A}(q) = e^{-\frac{q^2}{2\Lambda^2}} \frac{2}{\pi} \int_{2M_\pi}^{\infty} \frac{d\mu}{\mu^3} \eta_C(\mu) \left[ \frac{q^4}{\mu^2 + q^2} + C_1(\mu) + C_2(\mu) \frac{q^2}{\mu^2} \right] e^{-\frac{\mu^2}{2\Lambda^2}}.
\]

Again, the short-range subtraction terms are chosen to minimize the admixtures of short-range interactions in the regularized \( 2\pi \)-exchange potential by enforcing \( W_{C, A} (r) \big|_{r=0} = 0 \) and \( \frac{d}{dr} W_{C, A} (r) \big|_{r=0} = 0 \), which leads to

\[
C_1(\mu) = \frac{2\Lambda \mu^2 (2\Lambda^4 - 4\Lambda^2 \mu^2 - \mu^4) + \sqrt{2\pi} \mu^5 e^{\frac{\mu^2}{2\Lambda^2}} (5\Lambda^2 + \mu^2) \text{erfc} \left( \frac{\mu}{\sqrt{2A}} \right)}{4\Lambda^5},
\]

\[
C_2(\mu) = -\frac{2\Lambda (6\Lambda^6 - 2\Lambda^4 \mu^4 - \mu^6) + \sqrt{2\pi} \mu^5 e^{\frac{\mu^2}{2\Lambda^2}} (3\Lambda^2 + \mu^2) \text{erfc} \left( \frac{\mu}{\sqrt{2A}} \right)}{12\Lambda^7}.
\]

To illustrate the effect of the regulator and to demonstrate the importance of maintaining the analytic structure of the interaction, Fig. 1 shows the ratio of the regularized to unregularized potential for different regulator choices. Notice that keeping either only the spectral regulator or the momentum-transfer cutoff affects the discontinuity across the left-hand cut and results in strong distortions of the potential that extend to large distances as shown in the figure. The difference between the green dashed-dotted line and red solid line representing the actual form of the SMS regulator visualizes the impact of the short-range subtraction terms \( \propto C_{1, 2}(\mu) \). Regularization of other long-range contributions is performed analogously, see Ref. [1] for details and the explicit expressions for the potential up to \( N^4\text{LO}^+ \). Here and in what follows, the “+” signifies the inclusion of 4 contact
interactions from N^3LO in the partial waves $^1F_3$, $^3F_3$, $^3F_2$ and $^3F_4$, which is necessary for performing a partial wave analysis of proton-proton data \[1\]. Notice that the corresponding LECs are of natural size and not enhanced.

**Partial wave analysis of NN scattering**

Once the chiral interaction has been derived and regularized, the numerical values of the LECs entering the potential need to be determined from experimental data. Whenever possible, such LECs are fixed from the simplest process they are contributing to. As already mentioned, the authors of Refs. \[1\] 2 use the subleading πN LECs which enter the two-pion exchange (TPE) starting at N^2LO from the recent Roy-Steiner equation analysis of πN scattering data in Ref. \[1\]. The isospin-invariant two-pion exchange potential is thus parameter-free in the two-nucleon (2N) system.

The corresponding isospin-breaking corrections to the pion-exchange potentials are also included, which are mostly parameter-free except for the now appearing charge-dependence of the leading πN coupling constant in the one-pion exchange and a particular contribution to the TPE at N^3LO. For these terms, one has to distinguish between three different coupling constants $f^{0}_{π^0nn}$, $f^{0}_{π^0np}$ and $f^{π^0}_{π^0nn}$ for the interactions between protons and neutrons with neutral or charged pions, respectively. Extractions from πN data are only available for $f^{π^0}_{π^0np}$ and therefore the authors employ their own determination \[2\] of all three coupling constants from neutron-proton and proton-proton data using the N^4LO interaction discussed here.

It remains to determine the contact interaction LECs in the short-range part of the potential. As has become customary for higher-order chiral interactions in recent years, the authors of Refs. \[1\] 2 also fit them directly to neutron-proton and
proton-proton scattering data. Many elements of this determination are shared with the aforementioned extraction of the charged-dependent $\pi N$ coupling constants of Ref. [2], but in contrast the latter uses a Bayesian approach to integrate the cutoff $\Lambda$ and the contact interaction LECs out of the probability density to obtain a unique value for each of the coupling constant. Once the $\pi N$ coupling constants have been fixed, the authors choose a set of fixed values of $\Lambda$ and adjust the contact LECs for each of them to arrive at a fully specified set of parameters. In addition to the scattering data, the exact reproduction of the deuteron binding energy $B_d = 2.224575(9)\text{ MeV}$ [18] and of the coherent neutron-proton scattering length $b_{np} = -3.7405(9)\text{ fm}$ [19] is imposed as constraints on the fit.

The required machinery for a high-precision fit of the 2N scattering data has been worked out by the Nijmegen group, culminating in their seminal 1993 PWA of Ref. [20]. One important element for the accurate description of the scattering observables is the inclusion of the appropriate long-range electromagnetic interactions, especially at low energies and/or small forward angles (and for proton-proton scattering also large backward angles). The treatment of electromagnetic interactions by the Nijmegen group in Ref. [20] has become de facto standard when calculating scattering observables, and it is employed in the presented analysis as well. The second element required for a statistically satisfactory description of 2N scattering data is the removal of data sets that are not compatible with the bulk of the database. The so-called 3σ-criterion established by the Nijmegen group for rejecting such outlier data has been employed in nearly all subsequent PWAs.

In order to perform a reliable statistical testing of the data sets, the employed nuclear interaction has to be able to achieve a near-perfect description of the mutually consistent data. The results for the $N^4\text{LO}^+$ interaction of the first version of the SMS interaction in Ref. [1] showed that a $\chi^2$/datum $\sim 1$ description of the data could indeed be achieved and encouraged the authors to subsequently perform their own data selection to arrive at a database of mutually consistent scattering data. Such a data selection has been performed in the energy range of $E_{\text{lab}} = 0$–300 MeV in Ref. [2] where it is compared against the database of the recent 2013 Granada PWA of Ref. [21]. A complete listing can be found in Ref. [22]. The same database is also used in the fits of the contact interactions discussed here, but the energy range is slightly lowered below the pion-production threshold to $E_{\text{lab}} = 280\text{ MeV}$.

Tab. 2 shows the $\chi^2$/datum values for the description of the neutron-proton and proton-proton database of the fitted SMS $N^4\text{LO}^+$ interaction for all considered values of the cutoff $\Lambda$. The database up to $E_{\text{lab}} = 280\text{ MeV}$ consists of 2845 individual neutron-proton and 2081 individual proton-proton data points including estimated data set normalizations. For details regarding the definition of the $\chi^2$ measure and the estimation of normalizations, see Ref. [1]. The excellent description of the scattering data (especially for the cutoffs $\Lambda = 450$ and 500 MeV) and the independently performed data selection qualify these results to be regarded as a PWA of 2N scattering.

The focus of a PWA is the accurate determination of the phase shifts and mixing angles which parametrize the on-shell scattering amplitude. Indeed, the modern determinations of the phase shift vary only by a small amount relative to their abso-
lute sizes and are therefore well-known. This is especially true for the proton-proton phase shifts where the precise scattering data constrain the phase shifts very well, and all recent PWAs agree well with each other. The authors therefore focus below on the results for neutron-proton phase shifts, where the lower precision of the neutron-proton data compared to the proton-proton data and the different assumptions about isospin-breaking (IB) effects in the nuclear interactions lead to a greater variation in the phase shifts.

In order to better visually inspect the differences between selected recent determinations of the neutron-proton phase shifts, Fig. 2 shows their difference to the SMS N4LO+ results for the most accurate cutoff $\Lambda = 450$ MeV. In particular, the authors compare to the results of the Nijmegen 1993 [20], Gross & Stadler 2008 [24], Granada 2013 [21] and Granada 2017 [23] PWAs. Also shown is a Bayesian estimation of the uncertainty due to the truncation of the chiral expansion along with the combined statistical uncertainties of all parameters. The latter are dominated by the uncertainties from the nucleon-nucleon system while the errors of the $\pi N$ LECs of Ref. [11] are small. For details regarding the employed uncertainty quantification see Ref. [22].

The uncertainties of the phase shifts at low energies up to $E_{\text{lab}} \sim 100–150$ MeV are dominated by the statistical errors, whereas the truncation uncertainty becomes dominant at higher energies. Even larger, however, is the variation between the considered PWAs in many cases. In particular, consider the S- and P-waves. There are some differences in the assumptions about isospin-breaking between the considered analyses: The Nijmegen and Granada 2013 analyses allow for a charge-dependent short-range interaction only in the $^1S_0$ channel, whereas isovector P- and higher partial waves only take into account the pion mass difference in the $1\pi$-exchange. In contrast, the analysis described above and the Granada 2017 one allow for short-range charge dependence in both S- and isovector P-waves. The determination from neutron-proton data is also reflected in the statistical uncertainties of the $^1S_0$ and the isovector P-wave phase shifts, which are 2–3 times larger than the corresponding proton-proton phase shift uncertainties. Lastly, the Gross-Stadler PWA is fitted to neutron-proton data only.

| $E_{\text{lab}}$ bin | $\Lambda = 400$ MeV | $\Lambda = 450$ MeV | $\Lambda = 500$ MeV | $\Lambda = 550$ MeV |
|----------------------|---------------------|---------------------|---------------------|---------------------|
| neutron-proton scattering data |
| 0 – 100              | 1.069               | 1.061               | 1.060               | 1.062               |
| 0 – 200              | 1.085               | 1.074               | 1.069               | 1.075               |
| 0 – 280              | 1.113               | 1.060               | 1.048               | 1.055               |
| proton-proton scattering data |
| 0 – 100              | 0.876               | 0.860               | 0.866               | 0.875               |
| 0 – 200              | 0.933               | 0.909               | 0.918               | 0.942               |
| 0 – 280              | 0.956               | 0.932               | 0.950               | 0.989               |

*Table 2* $\chi^2$/datum values of the N4LO+ potential for the description of neutron-proton and proton-proton data and for all considered values of the cutoff $\Lambda$. The energy bin for $E_{\text{lab}} = 0 – 280$ MeV corresponds to the fitting energy range.
Fig. 2 Differences of the neutron-proton phase shifts of selected partial-wave analyses to the chiral SMS N^{4\text{LO}} phase shifts of this work for $\Lambda = 450$ MeV. Black circles, purple down triangles, blue up triangles and green squares denote the results of the Nijmegen 1993 [20], the Granada 2013 [21], the Granada 2017 [23] and the Gross-Stadler [24] PWAs, respectively, and the corresponding error bars denote their statistical uncertainties (if provided by the original publication). The peach- and light blue-colored bands show the truncation uncertainty and the combined statistical uncertainties of the N^{4\text{LO}} result due to the NN and πN LECs, respectively.
One significant change upon introduction of the additional short-range IB in P-waves can be seen in the $^3P_1$ channel, where the analysis by the authors and the Granada 2017 one find a phase shift that is up to $2^\circ$ smaller in magnitude than for the Nijmegen and Granada 2013 PWAs. The impact of the IB effects is also supported by the fact that the fits the authors performed without the additional P-wave charge dependence are in good agreement with the latter. However, the behavior of the corresponding Gross-Stadler phase shift, which is completely determined by neutron-proton data, is puzzling. The situation is less clear in other P-waves, where there is e.g. notable variation in the maximum of the $^3P_0$ phase shift around $E_{lab} \sim 50$ MeV. Based on the statistical error, the Granada 2017 result at that energy constitutes a $7\sigma$ deviation from the result of the present analysis. The authors also found statistically significant differences in the low-energy behavior of the $^3S_1$ phase shift. The obtained phase shift is in very good agreement with the Nijmegen analysis, whereas the other PWAs obtain slightly smaller phase shifts in the range of $E_{lab} = 0–100$ MeV. For the Granada 2013 analysis, this amounts to a $11\sigma$ deviation from the result obtained by the authors at $E_{lab} = 25$ MeV, which presumably also manifests itself in a different result for the deuteron asymptotic S-state normalization $A_S = 0.8829$ fm$^{-1/2}$ compared to other analyses.

Regarding higher partial-waves in Fig. 2 it is worth pointing out that D- and F-waves are parametrized with one isospin-invariant short-range LEC each. The results for $^3G_3$ and $\varepsilon_3$, however, are predictions based on the long-range potential alone. While the deviation of $\sim 1^\circ$ of the mixing angle $\varepsilon_3$ may appear large, it should be noted that the mixing angle itself is comparatively large and reaches $\sim 6–7^\circ$ at $E_{lab} = 300$ MeV.

Finally, consider the description of the scattering data in comparison to other available high-precision potentials. Tab. 3 gives the $\chi^2$/datum values of the most recent chiral N$^3$LO$^+$ potentials, i.e. the SMS potential of this work and the non-locally regularized Entem-Machleidt-Nosyk (EMN) potential of Ref. [26], for all available cutoff values. When comparing the numbers, one should keep in mind

### Table 3

| $E_{lab}$ bin | EMN        | SMS         |
|---------------|------------|-------------|
|               | 450 MeV    | 500 MeV     | 550 MeV     | 400 MeV    | 450 MeV     | 500 MeV     | 550 MeV     |
| 0–100         | 1.302      | 1.113       | 1.235       | 1.008      | 1.021       | 1.070       | 1.140       |
| N$^3$LO       | 0–200      | 1.549       | 1.284       | 1.426      | 1.182       | 1.353       | 1.595       | 1.904       |
| 0–300$^a$     | 2.354      | 1.503       | 1.691       | 1.601      | 2.524       | 3.903       | 5.831       |
| N$^3$LO$^+$   | 0–100      | 1.156       | 1.084       | 1.140      | 1.001       | 0.990       | 0.991       | 0.996       |
| 0–200         | 1.219      | 1.136       | 1.238       | 1.023      | 1.007       | 1.008       | 1.021       |
| 0–300         | 2.019      | 1.203       | 1.315       | 1.063      | 1.013       | 1.015       | 1.042       |

$^a$ The SMS N$^3$LO potentials are fitted to the scattering data up to $E_{lab} = 200$ MeV.
that the SMS interaction was explicitly fitted to this particular database while the
database employed for the EMN potentials slightly differs from the one employed
here, in particular with respect to the neutron-proton data. The shown results
nevertheless give a reliable qualitative idea about the accuracy of the interactions. This is
supported by the results of Tab. 4 which give the corresponding $\chi^2$/datum values for
selected high-precision semi-phenomenological potentials. Although the Nijmegen
potentials (NijmI, NijmII and Reid93) and the CD-Bonn potential have been fitted
to older 1993 and 1999 databases, respectively, they hold up quite well when com-
pared to the authors’ own data selection. It should be noted that when restricted to
data before 1993/1999 the database used in Ref. 26 should be identical to these
of the semi-phenomenological potentials. These values also show that in the energy
range $E_{\text{lab}} = 0–300$ MeV, the SMS $N^4\text{LO}^+$ interaction achieves or even exceeds the
precision of the most sophisticated phenomenological potentials.

For the sake of completeness, Tab. 3 also provides a comparison of the SMS
and EMN chiral potentials at the $N^3\text{LO}$ level. When comparing the description of
the data, one should keep in mind that the $N^3\text{LO}$ SMS chiral potentials have been
fitted to the NN scattering data up to $E_{\text{lab}} = 200$ MeV only. Notice further that the
EMN potentials employ partial-wave dependent functional form of the regulator for
contact interactions (and have 15 order-$Q^4$ contact interactions as compared to 12
terms in the SMS interactions).

**Selected applications in the NN sector**

As an example for the description of scattering data, the SMS $N^4\text{LO}^+$ results are
shown for the cutoff $\Lambda = 450$ MeV along with the experimental data for selected
proton-proton observables around $E_{\text{lab}} = 143$ MeV in Fig. 3. The results of the Ni-
jmegen PWA, the EMN potential with the central cutoff $\Lambda = 500$ MeV and the
CD-Bonn potential are also provided for the purpose of comparison. The differential
cross section data of Ref. 29 shown in the left panel has been used in Ref. 11
to illustrate the importance of F-waves in the description of high-precision proton-
proton observables. The differential cross section data of Fig. 3 are not included in
the authors’ own database selection, and the shown $N^4\text{LO}^+$ results are thus predic-
tions instead of being fitted. Nevertheless, the data are well described within the

| $E_{\text{lab}}$ bin | CD Bonn | Nijm I | Nijm II | Reid93 |
|------------------|---------|--------|---------|--------|
| 0 – 100          | 1.015   | 1.000  | 1.008   | 1.004  |
| 0 – 200          | 1.030   | 1.037  | 1.050   | 1.057  |
| 0 – 300          | 1.042   | 1.061  | 1.070   | 1.078  |

Table 4 $\chi^2$/datum for the description of the combined neutron-proton and proton-proton scattering
data up to $E_{\text{lab}} = 300$ MeV of the semi-phenomenological CD-Bonn 27 and Nijmegen NijmI,
NijmII and Reid93 potentials 28.
truncation uncertainties at $N^4\LO$. There exists some variation in the predictions of the differential cross section among the shown results by the other groups, especially when compared with the good agreement for the two spin observables $P$ and $D$. It should be noted that no definite conclusions regarding the accuracy of the different results can be made based on the data in Fig. 3 alone. The authors have chosen to show the data with their estimated norms from Ref. [1], which are in good agreement with the Nijmegen PW A. The norms of these data sets estimated in the CD-Bonn result.

Fig. 4 shows the neutron-proton total cross sections in the energy range $E_{\text{lab}} = 0$-300 MeV. Similar to Fig. 3, the SMS $N^4\LO$ result for $\Lambda = 450$ MeV is compared with other predictions and selected experimental data in the left panel. The authors show here relative values as the total cross section changes considerably in this energy range and the shown differences are small compared to that scale. The right panel exemplifies the uncertainty quantification. Here, the relative sizes of the truncation error, the statistical error due to the parameters determined from NN data and the statistical error due to the $\pi N$ LECs of Ref. [11] are shown. As expected based on the phase shift differences in Fig. 2, the NN statistical error can become large at smaller energies below $E_{\text{lab}} = 100$ MeV while the truncation error dominates at higher energies. The $\pi N$ statistical error is about one order of magnitude smaller than the NN statistical one.
Fig. 4 Neutron-proton total cross section in the range $E_{\text{lab}} = 0–300$ MeV. The left panel shows the total cross sections divided by the SMS N$^4$LO$^+$ results for $\Lambda = 450$ MeV. The light blue points are the experimental data of Ref. [33] corrected for an estimated norm of 1.002. Peach-colored bands represent the truncation uncertainty for the 68% DoB. For the remaining notation of this panel see Fig 3. The right panel compares the different uncertainties of the SMS N$^4$LO$^+$ results for $\Lambda = 450$ MeV: Black solid, red dashed and blue dashed-dotted lines denote the truncation uncertainty and the statistical uncertainties due to the parameters fitted from the NN system and the πN LECs of Ref. [11], respectively.

Table 5 Deuteron binding energy $E_b$, asymptotic S-state normalization $A_S$, asymptotic D/S-state ratio $\eta$, matter radius $r_m$, leading contribution to the quadrupole moment $Q_0$ and D-state probability $P_D$ for the SMS potential at N$^4$LO$^+$ for all values of the cutoff. The first error is the statistical uncertainty with respect to both NN and πN parameters and the second error is the truncation uncertainty (only provided for the observables $A_S$ and $\eta$).

| $\Lambda$ (MeV) | $B_b$ (MeV) | $A_S$ (fm$^{-1/2}$) | $\eta$ | $r_m$ (fm) | $Q_0$ (fm$^2$) | $P_D$ (%) |
|-----------------|-------------|----------------------|--------|------------|----------------|----------|
| $400$           | 2.2246      | 0.8842 (3)(5)        | 0.0260 (1)(2)(0) | 1.9647 (7) | 0.271 (2)     | 4.25     |
| $450$           | 2.2246      | 0.8846 (3)(5)        | 0.0261 (2)(0) | 1.9662 (6) | 0.275 (2)     | 4.79     |
| $500$           | 2.2246      | 0.8848 (3)(5)        | 0.0263 (2)(0) | 1.9674 (6) | 0.279 (2)     | 5.29     |
| $550$           | 2.2246      | 0.8851 (3)(6)        | 0.0265 (2)(0) | 1.9686 (6) | 0.282 (2)     | 5.73     |
| $550$ Empirical | 2.2246      | 0.8854 (3)(6)        | 0.0266 (2)(0) | 1.9686 (6) | 0.282 (2)     | 5.73     |

*The deuteron binding energy has been taken as input in the fit.

The excellent reproduction of experimental data of the SMS N$^4$LO$^+$ interaction also extends to the deuteron bound state, whose properties are shown in Tab. 5 for all considered cutoff values. The authors have limited the estimation of the truncation error to (non-fitted) observables and thus provide them only for $A_S$ and $\eta$. $r_m$ and $Q_0$ are related to the corresponding moments of the probability density distribution and thus determined solely by the deuteron wave function. These quantities are not measurable but constitute the leading contributions to the observable structure radius and the quadrupole moment, which will be discussed below.
It is interesting to test the novel SMS 2NF and more generally the employed chiral EFT framework by calculating the deuteron electromagnetic form factors (FF). The magnetic FF of the deuteron requires the calculation of the expectation value of the current density operator, whose consistently regularized 2N contributions are only available at N^4LO. On the other hand, the charge and quadrupole FFs \( G_C(Q) \) and \( G_Q(Q) \), with \( Q^2 = -q^2 > 0 \) and \( q \) denoting the four-momentum of the virtual photon, depend on the isoscalar charge density operator that has been worked out in chiral EFT to a high accuracy. In Fig. 5 the results for the charge and quadrupole FF of the deuteron are shown at N^4LO. Here, the empirical results were used for the nucleon form factors to parametrize the single-nucleon contributions to the charge density operator without relying on the chiral expansion. The relativistic corrections and the contributions of the 2N charge density operators were also taken into account. The latter depend on two LECs that have been fixed from the experimental data for \( G_C(Q) \) and \( G_Q(Q) \), see Fig. 5. With all LECs being determined as described above, a prediction for the deuteron structure radius \( r_{str} \) and the quadrupole moment \( Q_d = G_Q(0) \) was made. The structure radius denotes the contribution to the deuteron charge radius, which is related to the FF \( G_C \) via \( r^2 = -\frac{dG_C(Q^2)}{dQ^2} \big|_{Q^2=0} \), that arises from the nuclear binding mechanism. Up to the so-called Darwin term, \( r_{str} \) can be interpreted as the charge radius of the deuteron made out of structure-less nucleons. See Ref. [38] for more details and Ref. [39] for the interpretation of the FF in terms of the charge density distribution.

The final predictions for the deuteron structure radius and quadrupole moment in Ref. [38] read

\[
r_{str} = 1.9729^{+0.0015}_{-0.0012} \text{ fm}, \quad Q_d = 0.2854^{+0.0038}_{-0.0017} \text{ fm}^2, \tag{6}
\]

where the quoted errors include the statistical uncertainties of various LECs, the uncertainty in the parametrizations of the nucleon FFs and the estimated N^4LO truncation uncertainty. These values are to be compared with the determinations from

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{The deuteron charge (left panel) and quadrupole (right panel) form factors calculated at N^4LO for the cutoff \( \Lambda = 500 \) MeV. Bands between dashed red lines correspond to a 1\( \sigma \) error in the determination of the 2N short-range contributions to the charge density operator. For references to the experimental data and their parametrization shown by the black solid circles see Ref. [38].}
\end{figure}
laser spectroscopy experiments \[40, 41\]:

\[ r_{\text{str}}^{\text{exp}} = 1.97507(78) \text{ fm}, \quad Q_d^{\text{exp}} = 0.285699(23) \text{ fm}^2. \] (7)

Here, the quoted value for \( r_{\text{str}}^{\text{exp}} \) was obtained using the mean square neutron radius of \( r_n^2 = -0.114(3) \text{ fm}^2 \). Alternatively, the prediction for \( r_{\text{str}}^{\text{exp}} \) was used in combination with the very precise experimental data on the deuteron-proton charge-radius difference from Ref. \[40\] to update the value of the mean square neutron radius \( r_n^2 = -0.105 \pm 0.006 \text{ fm}^2 \).

**Beyond the NN system**

**SMS three-nucleon force at N^2LO**

The leading 3NF at N^2LO is generated by the tree-level topologies visualized in Fig. 6, see also Tab. I. It is straightforward to regularize the corresponding expressions using the semi-local cutoff in momentum space and employing the same convention for the subtraction terms as in the SMS 2NF of Ref. II:

\[
V_{\Lambda}^{\text{NN}} = \frac{g_A^2}{8F^2_{\pi}} \left\{ \frac{\hat{q}_1 \cdot \hat{q}_1 \cdot \hat{q}_3 \cdot \hat{q}_3}{(q_1^2 + M_\Lambda^2)(q_3^2 + M_\Lambda^2)} \left[ T_{13}(2c_3 \hat{q}_1 \cdot \hat{q}_3 - 4c_4 M_\Lambda^2) + c_4 T_{132} \hat{q}_1 \times \hat{q}_3 \cdot \hat{\sigma}_2 \right] \right. \\
+ C \frac{\hat{q}_1 \cdot \hat{q}_1}{q_1^2 + M_\Lambda^2} \left[ 2c_3 T_{13} \hat{\sigma}_3 \cdot \hat{q}_3 + c_4 T_{132} \hat{q}_1 \times \hat{\sigma}_3 \cdot \hat{\sigma}_2 \right] \\
+ C \frac{\hat{q}_3 \cdot \hat{q}_3}{q_3^2 + M_\Lambda^2} \left[ 2c_3 T_{13} \hat{\sigma}_1 \cdot \hat{q}_3 + c_4 T_{132} \hat{q}_1 \times \hat{\sigma}_3 \cdot \hat{\sigma}_2 \right] \\
+ C^2 \left[ c_3 T_{13} \hat{\sigma}_1 \cdot \hat{\sigma}_3 + c_4 T_{132} \hat{q}_1 \times \hat{\sigma}_3 \cdot \hat{\sigma}_2 \right] \left[ e^{-\frac{q_1^2 + M_\Lambda^2}{\Lambda^2}} e^{-\frac{q_3^2 + M_\Lambda^2}{\Lambda^2}} \right] \\
\left. - \frac{g_\Lambda D}{8F^2_{\pi}} T_{13} \left[ \frac{\hat{q}_3 \cdot \hat{q}_3}{q_3^2 + M_\Lambda^2} \hat{\sigma}_1 \cdot \hat{q}_3 + C \hat{\sigma}_1 \cdot \hat{\sigma}_3 \right] e^{-\frac{q_1^2 + q_3^2}{\Lambda^2}} e^{-\frac{q_3^2 + M_\Lambda^2}{\Lambda^2}} \right. \\
+ \frac{1}{2} E T_{12} e^{-\frac{q_1^2 + q_3^2}{\Lambda^2}} e^{-\frac{q_3^2 + M_\Lambda^2}{4\Lambda^2}} + 5 \text{ permutations}, \tag{8} \]
Nucleon-deuteron scattering

Three-nucleon scattering and bound state observables are calculated by solving the Faddeev equations in momentum space in the partial wave basis, see Ref. [42] for details. The partial wave decomposition of a general 3NF is carried out numerically as detailed in Ref. [43].

The leading 3NF depends on the LECs $D$ and $E$, whose values are extracted from low-energy 3N observables. Specifically, following Refs. [44, 42], the authors first require that the $^3$H binding energy is correctly reproduced. This constraint fixes the value of the LEC $E$ for a given value of $D$. To reliably determine this remaining LEC, it is essential to employ observables that are not strongly correlated with the $^3$H binding energy [45]. The authors of Ref. [44] have studied the constraints imposed on the value of the LEC $D$ by the experimental data for the Nd doublet scattering length, the total Nd scattering cross section and the differential cross section minimum in elastic Nd scattering at several energies. It was found that the high-precision experimental data of Ref. [46] for the differential cross section at the proton energy of $E_p = 70$ MeV, see Fig. 7 provide a particularly strong constraint on $D$. This finding is in line with the known sensitivity of the cross section minimum to 3NF effects, see [47] and references therein. The LECs $D$ and $E$ determined from the $^3$H binding energy and the Nd cross section minimum at 70 MeV were shown to result in a consistent description of all observables mentioned above, see Fig. 2 of [44] and Fig. 2 of [43].

Having determined the LECs $D$ and $E$ as described above, it is interesting to consider selected predictions for Nd scattering observables up to $N^3$LO. In Fig. 7 it is demonstrated that the NLO and $N^2$LO results for the differential cross section and selected polarization observables in elastic Nd scattering at 70 MeV are in agreement with the experimental data. For the analyzing powers $A_{yy}$ and $A_{xz}$, the discrepancies between the calculations and the experimental data are comparable with the truncation errors at $N^2$LO. These discrepancies are not resolved by higher-order corrections to the 2NF [10] and thus indicate the important role played by subleading 3NF contributions. The residual cutoff dependence of the calculated 3NF is, in general, compatible with the estimated truncation errors. In particular, the results for $\Lambda = 500$ MeV shown in Fig. 7 are very similar to those using $\Lambda = 450$ MeV and shown in Fig. 3 of Ref. [42]. The dependence of the obtained predictions on the different choices of semi-local regulators (i.e., coordinate-space versus momentum-space) and the subtraction conventions for the 3NF is also fully consistent with the
truncation uncertainty, see [42, 44] for details. The estimated truncation errors for Nd scattering observables were further validated by analyzing the contributions of selected higher-order short-range 3NF terms in Ref. [10].

Finally, in Fig. 8 the predictions for the total cross section at $E_N = 70$ and 135 MeV are shown for all four cutoff values. These calculations are based on the SMS 2NF at LO, NLO, N^2LO, N^3LO and N^4LO from Ref. [11]. Starting from N^2LO, the leading 3NF contributions specified in Eq. (8) are also taken into account. For each combination of the 2NF and 3NF and for each cutoff value, the LECs $D$ and $E$ are fixed to reproduce the $^3$H binding energy and the Nd cross section minimum at 70 MeV as described above. Since the 3NF is included only at N^2LO, one can regard the predictions based on the 2NF at N^3LO and N^4LO+ as alternative N^2LO calculations when estimating the truncation error. The total cross section is underestimated by $\sim 3.5\%$ ($\sim 7\%$) at $E_N = 70$ MeV ($E_N = 135$ MeV) when using the high-precision 2NF at N^4LO+ alone. Similar discrepancies that tend to increase with the energy were also observed in calculations based on high-precision phenomenological 2N potentials [48]. Adding the leading 3NF is essential to bring the chiral EFT predictions at N^2LO in agreement with the data. Moreover, the 3NF contributions to the total cross section appear to be comparable to the NLO truncation errors in agreement with the Weinberg power counting [6]. Also the differences between the N^2LO and N^3LO predictions that originate from the N^3LO contributions to the 2NF are comparable with the estimated N^2LO truncation errors.
Fig. 8 Predictions for the Nd total cross section at 70 MeV (left panel) and 135 MeV (right panel) based on the SMS chiral interactions at different orders (shown by solid symbols with error bars). 3NF is included at N^2LO only. Error bars show the EFT truncation uncertainty (68% DoB intervals). For the incomplete calculations at N^3LO and N^4LO, the quoted errors are the N^2LO truncation uncertainties. Gray open symbols without error bars show the results based on the 2NF only. Horizontal bands are experimental data from Ref. [48].

Heavier systems

The SMS chiral interactions have also been applied by the LENPIC collaboration to predict the ground and excited state energies and radii of light and medium-mass nuclei. In Fig. 9, the results for the ground state energies of selected nuclei up to A = 12 are shown using the NLO 2NF (left symbols), the N^2LO 2NF (middle symbols) and the N^2LO 2NF in combination with the N^2LO 3NF (right symbols) for the cutoff $\Lambda = 450$ MeV as a representative example. Notice that the calculated energies are pure predictions since the LECs in the nuclear Hamiltonian are determined from the 2N and 3N systems only. For nuclei with $A \leq 10$, the leading 3NF significantly improves the agreement with the data by increasing the binding energy. For heavier nuclei, the N^2LO Hamiltonian is systematically too attractive (but the ground state energies of both $^{12}$B and $^{12}$C are still in agreement with the data within 1.5$\sigma$). This tendency continues and increases with the mass number [42]. The origin of the systematic overbinding in heavier nuclei is currently under investigation by the LENPIC collaboration. More results for $p$-shell nuclei based on semi-local chiral EFT interactions can be found in Refs. [49, 50, 44, 42].

Towards consistent regularization beyond the 2N system

The missing three- and four-nucleon forces and exchange current operators at N^3LO and beyond are becoming more and more of a bottleneck for ab initio low-energy nuclear theory. Although the N^3LO and even some of the N^4LO contributions to
the many-body forces and currents have been worked out using dimensional regularization (DR), see Tab. 1, the existing expressions cannot be directly employed in few-nucleon calculations due to the inconsistencies caused by combining the dimensional and cutoff regularizations [7]. Below, an explicit example will be given to demonstrate such an inconsistency for the 3NF regularized in a naive way using both (semi-) local and nonlocal cutoffs.

**Statement of the problem**

Both the 2NF and 3NF need to be regularized in order to obtain a well defined solution of the Faddeev equations. High-momentum components in the integrals appearing in the iterations of the Faddeev equation generate contributions involving positive powers and logarithms of the cutoff which diverge in the $\Lambda \rightarrow \infty$ limit and are supposed to get absorbed by the available short-range interactions. The momentum dependence of such contact interactions beyond the 2N sector is, however, severely constrained by the spontaneously broken chiral symmetry of QCD. In particular, in the limit of exact chiral symmetry (i.e., for $M_\pi \rightarrow 0$), only derivative pion couplings are allowed in the effective Lagrangian according to the Goldstone theorem. In the 2N sector, the tree-level short range interactions do not involve any pion couplings, and their momentum dependence is therefore not restricted by the chiral
symmetry. This is in contrast to the $D$-like 3NF interactions, which are constrained by the chiral symmetry as visualized in Fig. [b]. These constraints lead to inconsistencies that plague calculations involving the 3NF derived using DR and regularized additionally by multiplying with a local or nonlocal cutoff. As will be exemplified below, the resulting mismatch between the two regularization schemes can not be compensated by shifting the values of the available LECs.

To be specific, the authors focus here on the example discussed in Ref. [7] and consider the relativistic correction to $2\pi$-exchange 3NF proportional to $g_A^4$ [51]:

$$V_{3N}^{2\pi,1/m} = i \frac{g_A^4}{32mF^2_\pi} \frac{\vec{q}_1 \cdot \vec{q}_3}{(q_1^2 + M^2_\pi)(q_3^2 + M^2_\pi)} \tau_1 \cdot (\tau_2 \times \tau_3)(2\vec{k}_1 \cdot \vec{q}_3 + 4\vec{k}_3 \cdot \vec{q}_3) + i[\vec{q}_1 \times \vec{q}_3] \cdot \vec{q}_2 + 5 \text{ permutations},$$

(9)

with $\vec{k}_i = (\vec{p}_i + \vec{p}_j)/2$. Consider now the first iteration of these N$^3$LO contributions with the LO $1\pi$-exchange 2N potential

$$V_{2N}^{1\pi} = - \left( \frac{g_A}{2F_\pi} \right)^2 \tau_1 \cdot \tau_2 \frac{\vec{q}_2 \cdot \vec{q}_1}{q^2 + M^2_\pi}.$$ 

(10)

Regularization of these 2NF and 3NF is achieved by multiplying them with a local Gaussian cutoff

$$V_{3N,\Lambda}^{2\pi,1/m} = V_{3N}^{2\pi,1/m} e^{-\frac{q^2 + M^2_{\text{cutoff}}}{\Lambda^2}}, \quad V_{2N,\Lambda}^{1\pi} = V_{2N}^{1\pi} e^{-\frac{q^2 + M^2_{\text{cutoff}}}{\Lambda^2}}.$$ 

(11)

Performing a large-$\Lambda$ expansion leads to

$$V_{3N,\Lambda}^{2\pi,1/m} G_0 V_{2N,\Lambda}^{1\pi} + V_{2N,\Lambda}^{1\pi} G_0 V_{3N,\Lambda}^{2\pi,1/m} = \Lambda \frac{g_A^4}{128\sqrt{2} \pi^{3/2} F^6_\pi} \left( \tau_2 \cdot \tau_3 - \tau_1 \cdot \tau_3 \right) \frac{\vec{q}_2 \cdot \vec{q}_3 \cdot \vec{q}_1}{q_3^2 + M^2_\pi} - \Lambda \frac{g_A^4}{96\sqrt{2} \pi^{3/2} F^6_\pi} \tau_1 \cdot \tau_3 \frac{\vec{q}_3 \cdot \vec{q}_1}{q_3^2 + M^2_\pi} + \ldots,$$

(12)

where $G_0$ is the free resolvent operator and the ellipses refer to all permutations of the nucleon labels and to terms that are finite in the $\Lambda \to \infty$-limit. The last term on the right-hand side (rhs) of Eq. (12) has the form of the $D$-term of the N$^3$LO 3NF, and it therefore can be absorbed into a redefinition of the LEC $D$. In contrast, the first term on the rhs of Eq. (12) can not be absorbed into a redefinition of the LECs entering the N$^3$LO 3NF since the corresponding structure is not allowed by the chiral symmetry. One therefore expects this problematic term to cancel against some other contribution in the 3N amplitude. The only other term with the desired combination of the LECs is the $1\pi$-$2\pi$-exchange 3NF at N$^3$LO $\propto g_A^4$, whose expression was derived in Ref. [52] using DR. Had one used the same cutoff regularization also in the calculation of this 3NF, one would obtain a linearly divergent term.
\[ V_{3N,\Lambda}^{2\pi-1\pi} = -\Lambda \frac{g_A^4}{128 \sqrt{2} \pi^{3/2} F_\pi^6} (\tau_2 \cdot \tau_3 - \tau_1 \cdot \tau_3) \frac{\vec{q}_2 \cdot \vec{\sigma}_2 \vec{q}_3 \cdot \vec{\sigma}_3}{q_3^2 + M_\pi^2} \]
\[ - \Lambda \frac{g_A^4}{32 \sqrt{2} \pi^{3/2} F_\pi^6} \tau_1 \cdot \tau_3 \frac{\vec{q}_3 \cdot \vec{\sigma}_3 \vec{q}_3 \cdot \vec{\sigma}_1}{q_3^2 + M_\pi^2} + \ldots , \] (13)

where the ellipses refer to terms which are finite in the \( \Lambda \to \infty \) limit that coincides with the DR result of Ref. [52]. As expected, the problematic term in Eq. (12) cancels exactly by the first term on the rhs of Eq. (13), while the second term can, again, be absorbed into a redefinition of the LEC \( D \). Clearly, the cancellation is only operative if one uses the same cutoff regularization in the derivation of the N\(^3\)LO 3NF and in the Faddeev equation. A naive approach by multiplying the N\(^3\)LO 3NF expressions, derived using DR, with some cutoff regulators obviously fails to ensure the cancellation of the chiral-symmetry-violating UV divergences and results in an uncontrolled approximation for the amplitude, which cannot be renormalized.

It is important to emphasize that the above inconsistency is by no means restricted to the usage of local regulators for long-range interactions. Indeed, repeating the same exercise using a nonlocal regulator of a Gaussian type,
\[ V_{3N,\Lambda}^{2\pi,1/m} = V_{3N}^{2\pi,1/m} e^{-\frac{r_{12}^2 + r_{13}^2}{\alpha^2}} e^{-3 \frac{q_1^2 + q_2^2}{4 \alpha^2}} \]
\[ V_{2N,\Lambda}^{1\pi} = V_{2N}^{1\pi} e^{-\frac{r_{12}^2 + r_{13}^2}{\alpha^2}} , \] (14)

one obtains for the first iteration of the Faddeev equation with the 1\(\pi\)-exchange 2NF being antisymmetrized in the 12-subsystem:
\[ V_{3N,\Lambda}^{2\pi,1/m} G_0 V_{2N,\Lambda}^{1\pi} + V_{2N,\Lambda}^{1\pi} G_0 V_{3N,\Lambda}^{2\pi,1/m} \]
\[ \quad = \Lambda \frac{g_A^4}{1536 (2\pi)^{3/2} F_\pi^6} \left( \frac{7\vec{k}_1 - 3\vec{k}_2}{\tau_1 \cdot \tau_2} \right) \frac{(7\vec{k}_1 - 3\vec{k}_2) \cdot \vec{\sigma}_3 \vec{q}_3 \cdot \vec{\sigma}_1}{q_3^2 + M_\pi^2} \]
\[ - \Lambda \frac{g_A^4}{384 (2\pi)^{3/2} F_\pi^6} \frac{\vec{q}_3 \cdot \vec{\sigma}_3 \vec{q}_3 \cdot \vec{\sigma}_1}{q_3^2 + M_\pi^2} + \ldots . \] (15)

Again, the first term on the rhs of Eq. (15) violates the chiral symmetry and can not be absorbed into a redefinition of the LEC \( D \). The non-locality of the regulator thus does not cure the problem. It actually introduces additional complications by affecting the analytic structure of the long-range potentials and making the derivation of consistent cutoff-regularized 3NF technically more demanding.

The issue with inconsistent regularization affects not only three- and more-nucleon forces, but it is also relevant for exchange currents at and beyond N\(^3\)LO. In particular, analogous considerations for the axial vector current at N\(^3\)LO demonstrate the appearance of chiral-symmetry-violating UV divergences when mixing the DR and cutoff regularization [53].
Possible solutions

The above inconsistencies can be rectified by using the same regulator in the derivation of the nuclear forces and currents and iterations of the dynamical equation. Such a regulator has to respect all the relevant symmetries. One option is to implement the regulator at the level of the effective Lagrangian. One can, in particular, require for the regularized pion propagator to take the form \( \exp\left(-\frac{q^2 + M_\pi^2}{\Lambda^2}\right)/(q^2 + M_\pi^2) \). This can be achieved by adding specific higher-derivative terms to the effective Lagrangian, which disappear in the limit \( \Lambda \to \infty \). Such higher-derivative regularization was introduced by Slavnov a long time ago to regularize the nonlinear sigma model [54]. This idea can be used to construct a \( \Lambda \)-dependent effective Lagrangian for pions that is manifestly invariant under global chiral transformations and yields long-range nuclear interactions regularized in a local way. On top of it, one can employ a nonlocal higher-derivative regularization for contact interactions. The implementation of these ideas in the 2N and 3N sectors is in progress.

Another possibility to implement a symmetry preserving regulator is given by the so-called gradient flow regularization approach proposed originally by Lüscher [55], see also [56, 57]. The idea behind this method is similar to that of the stochastic quantization by introducing a fifth dimension. The original pion field that depends on space-time coordinates is to be replaced by a field that depends, in addition, on a fictitious time \( t \). This field satisfies a gradient flow equation and reduces to the original pion field in the limit \( t \to 0 \). Chiral perturbation theory with this kind of regulator was discussed in Ref. [58], see also a related work in Ref. [59].

Last but not least, one can also employ a lattice regularization in chiral EFT. The regularized version of the effective pion Lagrangian on the lattice can be found in [60]. Nuclear forces and currents regularized in this way are guaranteed to respect the underlying symmetries and may be particularly useful for ongoing efforts to extend nuclear lattice EFT simulations to higher orders [61, 62].

Summary and outlook

To summarize, this chapter focused on the new generation of nuclear interactions derived from chiral EFT using semi-local regulators [3, 14, 1, 2]. At \( N^4\)LO\(^+\), the highest order available, the resulting SMS potentials were used to perform, for the first time, a full-fledged partial wave analysis of proton-proton and neutron-proton scattering data in the framework of chiral EFT [2, 22]. The resulting near perfect description of NN data up to the pion production threshold leaves little room for improvement and suggests no need to extend the EFT expansion beyond \( N^4\)LO\(^+\) given the available NN data. The recent high accuracy calculation of the deuteron charge and quadrupole form factors [37, 38] were also briefly reviewed.

The novel SMS interactions have been used to analyze 3N scattering observables and selected properties of light and medium-mass nuclei [10, 49, 50, 44, 42]. The
accuracy of these studies is limited by that of the 3NF, which is only available at N^2LO. At this chiral order, the predicted Nd scattering observables and ground state energies of nuclei with A ≤ 12 agree with the data within truncation error.

N^3LO and some of the N^4LO contributions to the 3NF and 4NF have already been derived using dimensional regularization [63, 64, 52, 51, 65, 66, 67], see Tab. [1]. Unfortunately, these expressions cannot be employed to calculate observables. This is because mixing the dimensional and cutoff regularizations when calculating scattering amplitudes violates the chiral symmetry and results in uncontrolled approximations beyond the 2N system. This issue is not restricted to a particular type of cutoff regulator and applies to local, semi-local and nonlocal cutoffs. It also plagues calculations involving exchange currents at N^3LO and beyond. A solution of this challenge requires a complete re-derivation of many-body forces and exchange currents using a cutoff regulator that respects the underlying symmetries such as e.g. the higher-derivative regularization [54]. Work along this line is in progress and will open an avenue for performing high-accuracy chiral EFT calculations beyond the 2N system.

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