EIGENVALUES OF BLOCK STRUCTURED ASYMMETRIC RANDOM MATRICES

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ABSTRACT. We study the spectrum of an asymmetric random matrix with block structured variances. The rows and columns of the random square matrix are divided into $D$ partitions with arbitrary size (linear in $N$). The parameters of the model are the variances of elements in each block, summarized in $g \in \mathbb{R}^{D \times D}$. Using the Hermitization approach and by studying the matrix-valued Stieltjes transform we show that these matrices have a circularly symmetric spectrum, we give an explicit formula for their spectral radius and a set of implicit equations for the full density function. We discuss applications of this model to neural networks.

1. Introduction

Non-Hermitian random matrices have proven to be a useful theoretical tool to study, for example, physical and biological systems [15, 16]. The spectrum of different matrix models can be used to make statements and predictions for the behavior of those systems. There is often a constant tension between having to accurately represent the system of interest and wanting the matrix model to be tractable. In recent years this tension served as motivation to study the spectrum of generalized random matrix models that include structure [13, 19, 2].

Along these lines, recently we defined a square non-Hermitian random model by partitioning the matrix into a finite number of blocks and letting the variance of elements in each block be an independent parameter, see section 2 for a formal definition. We will refer to these here as the heterogeneous model, in contrast to the homogeneous model where all the elements are drawn from the same distribution (giving Girko’s circular law). The heterogeneous random matrix can be thought to represent the connectivity in a neural network with multiple cell-types, where the connectivity statistics are cell-type dependent [3].

Previously, by using a mean-field approach to study the heterogeneous network, we derived a formula for an effective parameter axis along which a critical point is located. As in the homogeneous network [15], at this critical point the network undergoes a phase transition and its dynamics transform from having a single stable fixed point to chaos. This parameter was identified as the spectral radius of the heterogeneous matrix. Thus we obtained a formula for the spectral radius but lacked information about its density.

Here, using the Hermitization approach and by studying the matrix-valued Stieltjes transform, we prove that the support of the spectral density is given by the formula in [3], and find a set of implicit equations for the density that can be solved.
numerically. For certain low dimensional parameterizations of the model it is possible to use these equations to approximate the parameter dependence of interesting quantities related to the spectrum, such as the mass on a given annulus.

It is possible to compute the spectrum of the matrix studied here when a finite rank perturbation is added. Knowledge of the spectrum is often necessary but not sufficient to understand the physical model’s behavior. In the example of neural networks with cell-type-dependent connectivity statistics that motivated this work, adding a finite rank perturbation allows treatment of matrices where all the elements in each column have the same sign. These are useful in modeling networks that obey Dale’s principle - stating that real neurons are either excitatory or inhibitory. However, the mean-field characterization of the dynamics in this model remains a subject for future research.

2. Main result

We now introduce our model and main result. Let $J_0^N$ be an $N \times N$ matrix with $i.i.d$ random entries with zero mean, variance $1/N$, and finite fourth moment. Let $g$ be a $D \times D$ matrix with real, positive entries. Let $\alpha$ be a $D$ dimensional vector such that $\alpha_i > 0$, $\sum_{i=1}^{D} \alpha_i = 1$ and let

$$c_i = \left\{ c \left| \frac{i}{N} \in \left( \sum_{d=1}^{c-1} \alpha_d, \sum_{d=1}^{c} \alpha_d \right) \right. \right\}$$

Let $X_N$ be an $N \times N$ random matrix whose $i,j$ entry is

$$X_{ij} = g_{c_i d_j} J_{ij}^0.$$

We denote by $X^{cd}$, for $1 \leq c, d \leq D$, the $[\alpha_c N] \times [\alpha_d N]$ matrix, which is the $cd^{th}$ block of $X_N$. Generally superscripts will refer to $O(D)$ quantities and subscripts to $O(N)$ quantities.

Our main result concerns the eigenvalues of $X_N$ in the limit $N \to \infty$ with $D$ fixed. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $X_N$ and let $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ be the empirical spectral measure of $X_N$.

Before stating our main result, we recall that $\partial_z = \frac{1}{2}(\partial_x - \sqrt{-1}\partial_y)$ and let $G$ be the $D \times D$ matrix with entries $G_{cd} = \alpha_c g^2_{cd}$ and $\hat{G}_{cd} = \alpha_c g^2_{dc}$. We denote by $\rho(G)$ the spectral radius of $G$, which is also the spectral radius of $\hat{G}$. The limiting density is characterized by the $D$ dimensional vectors, $a$, $\hat{a}$, and $b$, which will be functions of $z$, a point in the complex plane, and $\eta$ a regularization parameter.

**Theorem 2.1.** The empirical spectral measure of $X_N$ converges almost surely to a deterministic measure $\mu$. The density of $\mu$ is radially symmetric and its support has radius $\sqrt{\rho(G)}$. The density of $\mu$ at $|z| \leq \sqrt{\rho(G)}$ is

$$b_c(z, \eta) = \frac{z a_c(z, \eta)}{[Ga(z, \eta)]_c},$$

$$a_c(z, \eta) = \frac{[Ga(z, \eta)]_c + \eta}{|z|^2 - ([\hat{G}a(z, \eta)]_c + \eta) ([Ga(z, \eta)]_c + \eta)},$$

where
\[
\tilde{a}_c(z, \eta) = \frac{[\tilde{G}\tilde{a}(z, \eta)]_c + \eta}{|z|^2 - ([\tilde{G}\tilde{a}(z, \eta)]_c + \eta)([G\tilde{a}(z, \eta)]_c + \eta)}.
\]

The proof is divided into several steps. We characterize the limiting deterministic measure, \(\mu\) in section 3.1. We use Lemma 3.1 to prove the convergence of \(\mu_N \to \mu\) in sections 4 and 5. We show that the density of \(\mu\) is radially symmetric with given spectral radius in section 6.

**Remark 2.2.** The assumption that the entries have a finite fourth moment is natural, under less assumptions one does not expect all of the eigenvalues of \(X_N\) to stay within the support of \(\mu\). Furthermore, a lengthy computation along the lines of [11, Section 6] can show that almost surely all there are no eigenvalues of \(X_N\) outside the support of \(\mu\). Finally, the appendix of [18] discusses a weak invariance principle for the eigenvalues of non-iid random matrices. These results can be used to reduce the assumptions made on the matrix.

**Remark 2.3.** Under certain restrictions, the matrix model we study here coincides with previously studied models. Rajan and Abbott [13] give an explicit formula for the spectrum in the case with \(D = 2\) and \(g\) that depends on only one index (i.e. \(g\) is equal to two copies of the same vector). Interestingly, one can show that when the entries of \(g\) depend only on their column or row index, for any \(D\), the spectral radius of \(X_N\) is equal to its Hilbert-Schmidt norm. This is in general not the case for our matrix model. When \(\text{rank}\{g\} = 1\), the matrices studied by Wei and Ahmadian et al. [19, 2] can be defined (under certain conditions) to coincide with our model.

![Figure 1](image-url)
3. Hermitization

An empirical spectral measure, $\mu$, of a Hermitian matrix is often studied by considering its Stieltjes transform, defined by $\int \frac{d\mu(z)}{z-x}$ for $z$ with positive imaginary part. Since the eigenvalues of non-normal matrices can be complex, it is difficult to directly study the Stieltjes transform of the empirical spectral measure. Spectral instablility of non-normal matrices introduces more difficulties in attempting to use standard hermitian techniques, see for example [5, Section 11.1]. To circumvent these issues we follow the method of Hermitization pioneered by Girko, [8], and since refined by many authors, see for example [4, 6, 18] and references within.

The Hermitization of $X_N$ is defined to be the $2N \times 2N$ matrix

$$H_N := \begin{pmatrix} 0 & X_N \\ X_N^* & 0 \end{pmatrix}.$$ 

We also define the Hermitization of $X_N - z$ as

$$H_N(z) := \begin{pmatrix} 0 & (X_N - zI_N)^* \\ (X_N - zI_N) & 0 \end{pmatrix}$$

from which we define the resolvent

$$R_N(q) := \left( \begin{pmatrix} -\eta I_N & X_N - zI_N \\ (X_N - zI_N)^* & -\eta I_N \end{pmatrix} \right)^{-1} = (H_N - q \otimes I_N)^{-1} \quad (3.1)$$

where $q = \begin{pmatrix} \eta & z \\ z & \eta \end{pmatrix}$.

Viewing $R_N(q)$ as an $2 \times 2$ block matrix with $N \times N$ sized matrix entries and taking the trace over each block leads to the $2 \times 2$ matrix-valued Stieltjes transform:

$$\Gamma_N(q) := \begin{pmatrix} a_N(q) & b_N(q) \\ c_N(q) & a_N(q) \end{pmatrix} := (I_2 \otimes \text{tr}_N)R_N(q)$$

where $\text{tr}_N := \frac{1}{N} \text{Tr}$ is the normalized trace of an $N \times N$ matrix. The $2 \times 2$ matrix $\Gamma_N(q)$ was introduced in the math literature in [6] and used in the physics literature previously.

We will often be interested in $\eta = \sqrt{-t}t$ for $t > 0$, in which case $c_N = \overline{b_N}$. When we wish to emphasize the dependence on $z$ or $\eta$ we will replace $(q)$ with $(z, \eta)$ in the argument of $\Gamma_N$, $a_N$, $b_N$ or $c_N$.

Let $\nu_{z,N}$ the empirical spectral measure of $H_N(z)$. From $\nu_{z,N}$ the singular values of $X_N - zI_N$ can be recovered by noting that if $\sigma$ is a singular value of $X_N - z$ then $\pm \sigma$ is an eigenvalue of $H_N(z)$. Since $R_N(z, \eta)$ is the resolvent of $H_N(z)$ at $\eta$, $a_N(z, \eta)$ is the Stieltjes transform of $\nu_{z,N}(x)$ at $\eta$.

By direct calculation

$$a_N(\eta, z) = \eta \text{tr}_N(\eta^2 - (X_N - z)(X_N - z)^*)^{-1}$$

In particular $a_N$ is purely imaginary and with positive imaginary part when $\eta = \sqrt{-t}t$ with $t > 0$.

By the Stieltjes inversion formula, $\nu_{z,N}$ can be recovered from $a_N(q)$. Furthermore, $\mu_N$ can be recovered from $\nu_{z,N}$ by taking the Laplacian of the logarithmic
potential, \( U_N(z) := -\int_C \log(s-z) d\mu_N(s) \), and using the following identities.

\[
2\pi \mu_N(z) = \Delta \int_C \log(s-z) d\mu_N(s) = \Delta \frac{1}{N} \sum_{i=1}^{N} \log(\lambda_i - z)
\]

\[
= \Delta \frac{1}{N} \log(\det(X_N - z)) = \Delta \int_0^\infty \log(x) d\nu_{z,N}(x)
\]

While \( a_N(q) \) characterizes \( \mu_N \), it can be difficult to compute its density from \( \mu_N \). But, the density can also be recovered from \( (X_N - z)^{-1} \), by noting that from Jacobi’s formula \( \partial_z U_N(z) = \frac{1}{2} \text{tr}_N((X_N - z)^{-1}) \), and then applying \( \partial_z \) to \( \text{tr}_N((X_N - z)^{-1}) \). As noted above the resolvent is not bounded uniformly, fortunately \( \Gamma_N \) gives a regularization of the resolvent.

Once again by direct computation

\[
b_N(\eta, z) = -\text{tr}_N((X_N - z)(\eta^2 - (X_N - z)(X_N - z)^*)^{-1})
\]

so in the limit we recover the adjoint of the resolvent

\[
\lim_{\eta \to 0} b_N(\eta, z) = \text{tr}_N((X_N - z)^{-1}).
\]

The justification for this regularization to compute the limiting distribution is the content of the next theorem.

**Lemma 3.1** (Lemma 4.20, [6]). Let \( (X_N)_{N \geq 1} \) be a sequence of random matrices. Assume for all \( q \in \mathbb{H}_+ \), there exists

\[
\Gamma(q) = \begin{pmatrix} a(q) & b(q) \\ c(q) & a(q) \end{pmatrix} \in \mathbb{H}_+
\]

such that for a.a. \( z \in \mathbb{C}, \eta \in \mathbb{C}_+, \) with \( q = q(z, \eta) \),

(i) a.s. (respectively in probability) \( \Gamma_N(q) \) converges to \( \Gamma(q) \) as \( N \to \infty \)

(ii) a.s. (respectively in probability) \( \log \) is uniformly integrable for \( (\nu_{z,N})_{N \geq 1} \)

Then there exists a probability measure \( \mu \) such that

(j) a.s. (respectively in probability) \( \mu_N \to \mu \) as \( N \to \infty \)

(jj) a.s. (respectively in probability) \( \mu = \frac{1}{t} \lim_{q(z,\eta): t \to 0} \partial_z b(q) \).

For log to be uniformly integrable we require that for any \( \epsilon > 0 \)

\[
\lim_{t \to \infty} \sup_{N \geq 1} \mathbb{P} \left( \int_{|\log(x)| > t} |\log(x)| d\nu_{z,N}(x) > \epsilon \right) = 0. \tag{3.2}
\]

The remainder of this section and section 4 are devoted defining \( \Gamma(q) \) and showing that \( \Gamma_N(q) \to \Gamma(q) \) almost surely. In section 5, we show that log is uniformly integrable for \( (\nu_{z,N})_{N \geq 1} \).

### 3.1. Matrix-Valued Stieltjes transform.

Instead of directly working with the \( 2 \times 2 \) matrix, \( \Gamma_N(q) \), we use the block structure of \( X_N \) and consider a \( 2D \times 2D \) matrix-valued Stieltjes transform, formed by taking partial traces over each block of the resolvent. The resolvent \( R_N \) is divided into \( (2D)^2 \) blocks using the partitions coming from \( X_N \) in \( \begin{pmatrix} 0 & X_N \\ X_N & 0 \end{pmatrix} \). Namely, the \( cd^{th} \) block is of size \( \lfloor \alpha_c \mod D \rfloor \times \lfloor \alpha_d \mod D \rfloor \). The \( cd^{th} \) block will be labeled \( R^c_{ij} \), and its entries are \( R^c_{ij} \) for \( i, j \) running over the size of the block.
When possible we will omit the dependence of these matrices on $N$ and $q$. Let $M_N(q)$ be a $2D \times 2D$ matrix with $c,d$ entry

$$M_{N}^{cd}(q) := \frac{1}{\alpha_c} \text{tr}_N(R^{cd}(q)). \quad (3.3)$$

From this matrix-valued Stieltjes transform the actual Stieltjes transform can be recovered by taking a weighted trace, $a_N(q) = \sum_{c=1}^{D} \alpha_c M_{N}^{cc}(q)$.

To describe the limiting Stieltjes transform, we use the theory of operator-valued semicircular elements, also used in, for example [1, 9, 10]. In section 4, we show that $M_{N}(q)$ approximately satisfies a self consistent equation, from which we can estimate the difference between $M_{N}(q)$ and the actual solution to this self consistent equation. The solution to the fixed point equation is the operator-valued Stieltjes transform of an operator-valued semicircular element. This semicircular element is determined by a completely positive map $\Sigma$. The operator-valued Stieltjes transform of this element is the solution to the equation

$$M(\tilde{q}) = -(\tilde{q} + \Sigma(M(\tilde{q})))^{-1} \quad (3.4)$$

with positive imaginary part for $\tilde{q}$ with positive imaginary part. Recall the imaginary part of a matrix, $X$ is $\frac{1}{2\sqrt{-1}}(X - X^*)$. In [9], it is shown that there is a unique solution to (3.3) with this positivity condition. Furthermore, this solution can be found by iterating the map $W \rightarrow -(\tilde{q} + \Sigma(W))^{-1}$ and that the norm of the solution is bounded by $\|\text{Im}(\tilde{q})^{-1}\|$. When $\tilde{q}$ is taken to be $q \otimes I_d$, then $\text{Im}(\tilde{q}) = \text{Im}(\eta)I_{2d}$.

In our case, $M(\tilde{q})$ is a $2D \times 2D$ matrix and $\Sigma$ is a linear operator on $2D \times 2D$ matrices, defined by

$$\Sigma(A)_{cd} = \delta_{cd} \sum_{e=1}^{D} \alpha_e g_{ee}^2 A_{e+D,e+D}$$

for $1 \leq c,d \leq D$ and

$$\Sigma(A)_{cd} = \delta_{cd} \sum_{e=1}^{D} \alpha_e g_{ee}^2 A_{e,e}$$

for $D+1 \leq c,d \leq 2D$. In the derivation of this equation we will see that $\Sigma(A)$ is a diagonal matrix as a result of the independence between matrix entries of $X_N$.

By iterating the map $W \rightarrow -(q \otimes I_d + \Sigma(W))^{-1}$, with initial point $q \otimes I_d$, we see the solution will be of the form

$$M(q) = \begin{pmatrix}
a_1 & \cdots & b_1 \\
\vdots & \ddots & \vdots \\
a_D & \cdots & b_D \\
c_1 & \cdots & c_D \\
\cdots & \ddots & \cdots \\
c_D & \cdots & \hat{a}_D
\end{pmatrix} \quad (3.5)$$

with the empty entries equal to zero. Furthermore when $\eta = it$, the diagonal entries will be purely imaginary, with positive imaginary part.

Inverting the matrix on the right side of (3.4) with our particular $\Sigma$ gives a system of equations for the entries of $M$ (written below). The first $D$ diagonal entries are
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acted on by the previously introduced matrix $G$, with entries $G_{cd} = \alpha g^2_{cd}$, and the rest of the diagonal entries are acted on by $\tilde{G}$ be the $D \times D$ matrix with entries $\tilde{G}_{cd} = \alpha g^2_{dc}$. 

$$a_c = \frac{[Ga]_c + \eta}{|z|^2 - (\tilde{G}a)_c + \eta)([Ga]_c + \eta)}$$

(3.6)

$$\hat{a}_c = \frac{[\tilde{G}a]_c + \eta}{|z|^2 - (\tilde{G}a)_c + \eta)([Ga]_c + \eta)}$$

(3.7)

and

$$b_c = \frac{-z}{|z|^2 - (\tilde{G}a)_c + \eta)([Ga]_c + \eta)}$$

(3.8)

for $1 \leq c \leq D$.

Using Lemma 3.1 the limiting measure $\mu = \frac{-1}{\pi} \lim_{\eta \to 0} \partial_z \sum_{c=1}^{D} \alpha_c b_c (z, \sqrt{-1}t)$. If $\|Ga\| \to 0$ as $\eta \to 0$ then $b \to 1/z$ whose derivative with respect to $z$ vanishes, otherwise $b_c \to -z a_c$.

**Remark 3.2.** In Fig. 1 we compare the full spectral density and the radial part computed numerically from the implicit equations (3.6-3.7) to direct diagonalization of instantiations with $D = 2$ and $D = 3$. The density is computed from the above equations by choosing a grid of $z$ values and, for each point, iterating the maps for $a, \hat{a}$ until convergence. The solutions to these equations are substituted into (3.8), and finally the density is obtained by computing the gradient weighted by the appropriate model parameters.

**4. Derivation of fixed point equation**

In this section we show $M_N(q)$ approximately solves equation (3.4) with the given $\Sigma$, from which the difference $\|M_N(q) - M(q)\|$ will be estimated.

We first introduce some notation. Let $R^{(k)}(q)$ be the resolvent of $H_N(z)$ with the $k^{th}$ row and column of each block set to zero, let $R^{cd(k)}(q)$ be the $c,d$ block of this matrix, and let $M^{cd(k)}_N(q) := \frac{1}{\alpha_c} \text{tr}_N(R^{cd(k)})(q)$. Let $R_{N;11}(q)$ be the $2D \times 2D$ matrix with entries that are the $(1,1)$ entry of each block.

$$R_{N;11} := \begin{pmatrix} R^{11}_{11} & R^{12}_{11} & \cdots & R^{2D,2D}_{11} \\ R^{21}_{11} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ R^{2D,1}_{11} & \cdots & \cdots & R^{2D,2D}_{11} \end{pmatrix}$$

We recall Schur’s complement for computing entries of the inverse of a matrix. Given a block matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then the entries of the upper left block of the inverse can be computed via the formula:

$$(X^{-1})_{11} = (A - CD^{-1}B)^{-1}_{11}$$

or more generally, for a set $I$ such that $\{1, \ldots, N\} = I \cup I^c$

$$(X^{-1})_{1,I} = (X_{I,I} - X_{I,I^c}(X_{I^c,I^c})^{-1}X_{I^c,I})^{-1}$$
We apply Schur’s complement to \( H_N - q \otimes I_N \) using the set \( I \) formed by taking the index of the first entry of each block, leading to an expression for \( R_{N;11} \).

The main estimate of this section is the contents of the next proposition. By Vitali’s convergence theorem, it suffices to prove convergence of \( \Gamma_N \) for \( \eta = it \), with \( 0 < t < T \), for some large \( T \). It is straightforward to show that \( \| X_N \| < T \) almost surely for \( T \) sufficiently large (see for instance [17, Section 2.3]), so we only need to prove convergence for \( z \) in a compact set.

**Proposition 4.1.** Let \( T > 0 \) by sufficiently large and \( |z| < T \), \( \eta = it \) such that \( 0 < |t| < T \). Then almost surely \( \| M_N(q) - M(q) \| = O(|\text{Im}(\eta)|^{-5}N^{-1/4}) \).

This estimate is not optimal, but will suffice for our purposes. It would be interesting to see if optimal bounds could be obtained as in [7].

Following a similar argument to [12], Proposition 4.1 will follow from Lemma 4.3 and Lemma 4.4. We begin by deriving an equation for \( \mathbb{E}[M_N(q)] \). Then bounding the resulting error term gives a bound on \( \| \mathbb{E}[M_N(q)] - M_N(q) \| \). We finish the section by bounding \( \| \mathbb{E}[M_N(q)] - M_N(q) \| \).

It suffices to consider \( R_{N;11} \) because

\[
\mathbb{E}[M_N] = \mathbb{E}[R_{N;11}]
\]

by exchangeability.

Then, by Schur’s complement

\[
R_{N;11} = - \left( -H_{11} + q \otimes I_d + H_{11}^{(1)} R_N^{(1)} H_{11}^{(1)} \right)^{-1}
\]

(4.1)

Where \( H_{11} \) is the \( 2D \times 2D \) with scalar entries

\[
H_{11} = \begin{pmatrix}
0 & X_{11}^{1,1} & \ldots & X_{11}^{1,D} \\
\vdots & \ddots & \ddots & \vdots \\
X_{11}^{D,1} & \ldots & X_{11}^{D,D} & 0 \\
X_{11}^{1,D} & \ldots & X_{11}^{D,D}
\end{pmatrix}
\]

and \( H_{11}^{(1)}, H_{11}^{(1)} \) are \( 2D \times 2D \) with vector entries

\[
H_{11}^{(1)} = \begin{pmatrix}
0 & X_{11}^{1,1(1)} & \ldots & X_{11}^{1,D(1)} \\
\vdots & \ddots & \ddots & \vdots \\
X_{11}^{D,1(1)} & \ldots & X_{11}^{D,D(1)} & 0 \\
X_{11}^{1,D(1)} & \ldots & X_{11}^{D,D(1)}
\end{pmatrix}
\]
Recalling the definition of $\Sigma$, we see that $E$ and $X$ then averaging with respect to the first row and column of $X$ leads to the equation:

\[ \text{Eigenvectors of (4.1) leads to the equation:} \]

and $X_{1,c,d}^{e,d(1)}$ is the first row (column) of the $cd^{th}$ block of $X_N$ with the first entry set to zero.

Each entry of \( H_1^{(1)} R_N^{(1)} H_1^{(1)} \) is a sum of quadratic forms.

Since the set \( \{X_{1,c,d}^{e,d(1)}\}_{1 \leq c,d \leq D} \) is independent from \( \{X_{1,c,d}^{e,d(1)}\}_{1 \leq c,d \leq D} \),

\[ \text{E}[H_1^{(1)} R_N^{(1)} H_1^{(1)}] = 0, \text{ when exactly one of } c,d \text{ is less than or equal to } D. \]

If $c,d$ are both less than or equal to $D$ or both greater than $D$ then \( (H_1^{(1)} R_N^{(1)} H_1^{(1)}) \) can have a non-zero expectation. We consider case when $1 \leq c,d \leq D$, the other case is the similar.

\[
(H_1^{(1)} R_N^{(1)} H_1^{(1)})^{cd} = \sum_{1 \leq e,f \leq D} X_{1,e}^{c,e} R_{e+f+D(1)}^{c,f} X_{1,f}^{d,f} \\
= \sum_{1 \leq e,f \leq D} \sum_{1 \leq i,j \leq K} X_{11}^{e,e} R_{ij}^{e+f+D(1)} X_{1j}^{d,f}.
\]

Then averaging with respect to the first row and column of $X_N$ gives:

\[
\text{E}[\sum_{1 \leq e,f \leq D} \sum_{1 \leq i,j \leq K} X_{11}^{e,e} R_{ij}^{e+f+D(1)} X_{1j}^{d,f}] = \delta_{e,d} D \sum_{e=1}^{D} \text{E}[\text{Tr}(R_{e+f+D(1)})] g_{x}^{2} N.
\]

Recalling the definition of $\Sigma$, we see that $\text{E}[H_1^{(1)} R_N^{(1)} H_1^{(1)}] = \Sigma(\text{E}[M_N])$. Taking the expectation of (4.1) leads to the equation:

\[
\text{E}[M_N] = \text{E}[R_N^{11}] = -(q \otimes I_d + \Sigma(\text{E}[M_N]) + Z_N)^{-1}
\]

where

\[
Z_N = (H_1^{(1)} R_N^{(1)} H_1^{(1)}) - \Sigma(\text{E}[M_N]) - H_{11}.
\]

**Lemma 4.2.** The expectation of the norm of the error term $Z_N = (H_1^{(1)} R_N^{(1)} H_1^{(1)}) - \Sigma(\text{E}[M_N]) - H_{11}$ is bounded by $\text{E}[\|Z_N\|] = O(N^{-1/2} |\text{Im}(\eta)|^{-1})$.

**Proof.** We rewrite

\[
Z_N = (H_1^{(1)} R_N^{(1)} H_1^{(1)}) - \Sigma(M_N^{(1)}) + \Sigma(M_N^{(1)}) - \Sigma(M_N) + \Sigma(M_N) - \Sigma(\text{E}[M_N]) - H_{11}
\]

By the triangle inequality, it suffices to bound each term individually. Furthermore, since $Z_N$ is a matrix, to show its norm goes to zero, it suffices to show each entry goes to zero individually. By assumption $\text{E}[\|H_{11}\|] = O(N^{-1/2})$. 

\[
H_1^{(1)} = \begin{pmatrix}
0 & X_{1}^{1,1(1)} & \cdots & X_{1}^{1,D(1)} \\
X_{1}^{1,1(1)} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & X_{1}^{D,1(1)} \\
X_{1}^{1,D(1)} & \cdots & \ddots & 0
\end{pmatrix}
\]
Applying $\Sigma$ only changes the value of the constants.

Let $Y, Z$ be $X_{1e}^d$ or $X_{1e}^d$ for some $c, d$

$$\mathbb{E}[\|Y^* R_N^{a,b(1)} Z - \delta_{Y,Z} \text{tr}_N(R_N^{b(1)})\|] \leq C \left( \frac{1}{N} \text{tr}_N(R_N^{a,b(1)} R_N^{a,b(1)}) \right)^{1/2} \leq C |\text{Im}(\eta)|^{-1}.$$

To bound $\Sigma(M_N^{(1)} - M_N)$, we use that the normalized partial trace of a matrix is bounded by its norm times its rank, giving the bound:

$$|M_N^{d}(q) - M_N^{d(k)}(q)| \leq \frac{4D |\text{Im}(\eta)|^{-1}}{N}.$$  
Equation (4.2)

Applying $\Sigma$ only changes the value of the constants.

Finally, the concentration of measure estimate, Proposition 4.4 below, we have the bound $\mathbb{E}[\|\Sigma(M_N - \mathbb{E}[M_N])\|] = O(N^{-1/2} |\text{Im}(\eta)|^{-1}).$

\[\square\]

Having bounded the error term, we now bound the difference between $\mathbb{E}[M_N(q)]$ and $M(q)$. Before we begin, recall that there is a unique solution to equation (3.4) with positive imaginary part.

**Lemma 4.3.** Let $M_N(q)$ be as in (3.3) and $M(q)$ be as in (3.3), then

$$\|\mathbb{E}[M_N(q)] - M(q)\| = O(|\text{Im}(\eta)|^{-5} N^{-1/2})$$

**Proof.** We assume that $|\text{Im}(\eta)| > 4^{1/4} N^{-1/8}$ otherwise the trivial bound of $|\text{Im}(\eta)|^{-1}$ on the norm of resolvent gives the desired estimate.

Rewriting

$$\mathbb{E}[M_N] = \mathbb{E}[-(q \otimes I_d + \Sigma(\mathbb{E}[M_N]) + Z_N)^{-1}]$$

$$= -(q \otimes I_d + \Sigma(\mathbb{E}[M_N]))^{-1} + (q \otimes I_d + \Sigma(\mathbb{E}[M_N]))^{-1} \mathbb{E}[Z_N (q \otimes I_d + \Sigma(\mathbb{E}[M_N]) + Z_N)^{-1}]$$

Let

$$Z'_N = (q \otimes I_d + \Sigma(\mathbb{E}[M_N]))^{-1} Z_N (q \otimes I_d + \Sigma(\mathbb{E}[M_N]) + Z_N)^{-1}.$$

For $N$ large, $\mathbb{E}[\|Z_N (q + \Sigma(\mathbb{E}[M_N]) + Z_N)^{-1}\|] \leq 1/2$ so $\mathbb{E}[1 + Z_N R_{11}]$ is invertible, implying

$$\|(q \otimes I_d + \Sigma(\mathbb{E}[M_N]))^{-1}\| \leq 2|\text{Im}(\eta)^{-1}|$$

and $\|Z'_N\| \leq C|\text{Im}(\eta)^{-2}||Z_N||$.

Let $q_N := q \otimes I_d + \Sigma(Z'_N)$, which has imaginary part bigger that $|\text{Im}(\eta)|/2 > 0$ for large $N$. Then

$$\mathbb{E}[M_N(q)] - Z'_N = -(q_N + \Sigma(\mathbb{E}[M_N(q)] - Z'_N))^{-1}$$

then by uniqueness of the solution to (3.4)

$$\mathbb{E}[M_N(q)] - Z_N = M(q_N)$$

Then

$$\|\mathbb{E}[M_N(q)] - M(q)\| = \|M(q_N) - M(q) + Z_N\| = O(|\text{Im}(\eta)|^{-5} N^{-1/2}).$$

\[\square\]

We conclude with a concentration estimate.
Lemma 4.4. Let $M_N(q)$ be as in (3.3) then
\[ \|M_N(q) - \mathbb{E}[M_N(q)]\| \leq N^{-1/4} |\text{Im}(\eta)|^{-1} \]

Proof. We will use McDiarmid’s inequality, which states:
Suppose $x_1, x_2, \ldots, x_n$ are independent and assume that $f$ satisfies
\[ \sup_{x_1, x_2, \ldots, x_n, x'_i} |f(x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)| \leq c_i \quad \text{for} \quad 1 \leq i \leq n \]
then any $\varepsilon > 0$:
\[ \Pr(|f(x_1, x_2, \ldots, x_n)| - f(x_1, x_2, \ldots, x_n) \geq \varepsilon) \leq 2 \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^{n} c_i^2}\right) \]
We apply the inequality choosing $x_i$ to be the $i$th column of $X_N$. Let $M'_N(q)$ be the same as $M_N(q)$ except with the $i$th column of $X_N$ resampled, then using that the difference between $X_N$ and its resampled version is rank one, gives the bound:
\[ \|M_N(q) - M'_N(q)\| \leq \eta N^{-1/4} |\text{Im}(\eta)|^{-1} \]
and McDiarmid’s inequality implies
\[ \Pr(||M_N(q) - \mathbb{E}[M_N(q)]\| > \varepsilon) \leq 2 e^{-2N|\text{Im}(\eta)|^{-2} \varepsilon^2} \]
or
\[ \Pr(||M_N(q) - \mathbb{E}[M_N(q)]\| > N^{-1/4} |\text{Im}(\eta)|^{-2}) \leq e^{-N^{1/2}} \]
the Borel-Cantelli lemma completes the proof. 

5. Logarithmic Integrability

By Markov’s inequality, to prove (3.2) it suffices to show
\[ \limsup_{N \to \infty} \mathbb{P}\left(\int_{0}^{\infty} x^p d\nu_{N,z}(x)\right) < \infty \quad \text{and} \quad \limsup_{N \to \infty} \mathbb{P}\left(\int_{0}^{\infty} x^{-p} d\nu_{N,z}(x)\right) < \infty \]
for some $p > 0$. Let $s_{N,z} \leq s_{N-1,z} \leq \ldots \leq s_{1,z}$ be the ordered singular values of $X_N - z$. To prove the first inequality, let $p = 2$. Then
\[ \int_{0}^{\infty} x^p d\nu_{N,z}(x) = \frac{1}{N} \sum_{i=1}^{N} s_{i,z}^2 \leq |z|^2 + \frac{1}{N} \sum_{i=1}^{N} s_{i,0}^2 = |z|^2 + \frac{1}{N} \text{Tr}(XX^*) \]
Which is almost surely finite by the strong law of large numbers.

The second inequality will follow from Corollary 5.2 and Lemma 5.3. To prove Corollary 5.2 we use the following theorem:

Theorem 5.1 ([18], Theorem 2.1). Let $a, c_1$ be positive constants, and let $x$ be a complex-valued random variable with non-zero finite variance. Then there are positive constants $b$ and $c_2$ such that the following holds: if $N_n$ is the random matrix of order $n$ whose entries are iid copies of $x$, and $M$ is a deterministic matrix of order $n$ with spectral norm at most $n^{c_1}$, then,
\[ \mathbb{P}(\sigma_n(M + N_n) \leq n^b) \leq c_2 n^{-a} \]
Corollary 5.2. Let $a, c'_{1}$ be positive constants. Let $D$ be an $n \times n$ random matrix with iid entries and finite variance and $M$ be an $n \times n$ deterministic matrix of spectral norm at most $n^{c'_{3}}$. Let $A$ be an $(N - n) \times (N - n)$ matrix, $B$ be an $(N - n) \times n$ matrix, and $C$ be an $n \times (N - n)$ matrix. Assume there exist $b''$ and an event occurring with probability $1 - N^{-a}$ on which $A^{-1}$ has norm bounded by $N^{b''}$ and $\|B\|, \|C\|$ have norm $O(1)$. Furthermore assume the $D$ is independent of $A, B$ and $C$.

Let $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and let $M = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$, then there are positive constants $b'$ and $c'_{2}$ (depending on $a, c'_{1}$ and $b''$) such that

$$\mathbb{P}(\sigma_{N}(M + Y) \leq N^{b'}) \leq c'_{2}N^{-a}.$$ 

Proof. Recall that the least singular value of a matrix $X$ can be characterized as $\min_{\|u\|=1} \|Xu\|$ or $\|X^{-1}\|$. Let $u$ be a vector such that $\|Yu\| = \sigma_{N}(Y)$ and partition $u$ as $\begin{pmatrix} u_{1} \\ u_{2} \end{pmatrix}$, where $u_{1}$ is $N - n$ dimensional and $u_{2}$ is $n$ dimensional. Let $v := Yu$ and partition $v$ similarly as $\begin{pmatrix} v_{1} \\ v_{2} \end{pmatrix}$. Expressing $v = Yu$ terms of the blocks leads to $Au_{1} + Bu_{2} = v_{1}$ and $Cu_{1} + (D + M)u_{2} = v_{2}$.

If $\|u_{2}\| = o(N^{-b})$ then $Au_{1} = v_{1} + o(N^{-b})$ and by our assumption on the least singular value of $A$ we conclude, $\|v_{1}\| \geq cN^{-b''}$ with probability $1 - N^{-a}$.

Otherwise, using the assumption that there is an event of probability $1 - N^{-a}$ on which $A$ is invertible, we solve for $u_{1}$. Substituting into the second equation gives $(CA^{-1}B + M + D)u_{2} = v_{2} - A^{-1}v_{1}$.

Since $CA^{-1}B$ is independent of $D$ and has norm bounded by $N^{b''}$ with probability $1 - N^{-a}$, Theorem 5.1 implies $\|v_{2} - A^{-1}v_{1}\| \geq n^{-b}\|u_{2}\|$. Implying that $\|v\| \geq \frac{1}{2}n^{-2b-b''}$.

It is well known that with probability $1 - O(N^{-a})$ the operator norm of a matrix with iid entries is $O(1)$, see for instance [17, Section 2.3]. Concatenating such matrices can only increase the operator norm by a constant factor. Thus the corollary can inductively be applied to the matrices formed by the $k$ upper left blocks for $1 \leq k \leq D$ of $X_{N}$ giving the almost sure polynomial bound on the least singular value.

The following lemma gives control on moderate singular values.

Lemma 5.3. There exist $a \gamma > 0$ and $C > 0$ such that $\nu_{z, N}([0, I]) \leq CI$ for every $I \geq N^{-\gamma}$.

Proof. We first observe that $1_{\{|x| \leq t\}}(x) \leq 2t \text{Im}(1/(x - it))$. Furthermore $a(q)$ is bounded in the upper half plane, therefore choosing $q$ such that $\eta = it$ gives

$$\nu_{z, N}([0, t]) \leq 2t \text{Im}(a_{N}(q)) \leq |2t((t^{-3}N^{-1/4}) - a(q))|$$

this term is bounded by $Ct$ for all $t > N^{-1/20}$.
Then by Corollary 5.2, there exist a $b > 0$, and by Lemma 5.3, there exist a $\gamma > 0$, such that almost surely,
\[
\int_0^\infty x^{-p} \, d\nu_{N,z} = \frac{1}{N} \sum_{i=1}^{N} s_{i,z}^{-p} \leq \frac{1}{N} \sum_{i=1}^{N-N^{-\gamma}} s_{i,z} + \frac{1}{N} \sum_{i=N-N^{-\gamma}}^{N} s_{i,z} \leq C \frac{1}{N} \sum_{i=1}^{N} \left( \frac{i}{N} \right)^{-p} + N^{-\gamma} N^b p
\]

The first term uses the moderate singular value bounded from Lemma 5.3. This Riemann sum is finite for $0 < p < 1$. The second term comes from the least singular value bound and is $o(1)$ for $p$ sufficiently small.

6. Analysis of the limiting measure

Let $\lambda$ be the Perron-Frobenius eigenvalue of $G$, recall this is also the spectral radius of $G$.

**Theorem 6.1.** The support of $\mu$ is a disk with radius $\sqrt{\lambda}$.

Before starting the proof, we begin by recalling (3.6) in the homogeneous case where all of the entries of $X_N$ have the same variance. Let $a_{\text{iid}}$ be the solution to (3.6) in the case that there is just one block of size $N$ with variance $\sigma^2$, and $h_{\text{iid}} = \sqrt{-1} a_{\text{iid}}$ then
\[
h_{\text{iid}} = \frac{\sigma^2 h_{\text{iid}} + t}{|z|^2 + (\sigma^2 h_{\text{iid}} + t)^2}
\]
which has solution as $t \to 0$
\[
\lim_{t \to 0} h_{\text{iid}} = \begin{cases} \frac{1}{\sigma^2 \sqrt{\sigma^2 - |z|^2}} & \text{if } |z|^2 \leq \sigma^2 \\ 0 & \text{otherwise} \end{cases}
\]

In our proof, we manipulate the equations defining $a$ to resemble the iid case, from which the spectral radius can be extracted.

**Proof.** The equations, (3.6), only depend on $|z|$ and hence correspond to a radially symmetric measure.

Fix $|z|^2 \leq \lambda$ we will now show there exist a $C_\varepsilon > 0$ such that $\lim_{t \to 0} |a(it,z)| > C_\varepsilon$. In what follows many of the variables depend on $\eta$ and $z$ but this dependence will often be suppressed.

Let $h_c := a_c / \sqrt{-1}$. Since $a_c$ is purely imaginary with non-negative imaginary part, $h_c \geq 0$. Let $h$ be the vector with $c^{th}$ entry $h_c$ and define $\hat{h}$ similarly. Multiplying (3.6) by $\sqrt{-1}$ leads to the equations:
\[
h_c = \frac{[G h_c] + t}{([G h_c] + t)([G h_c] + t) + |z|^2}, \quad \hat{h}_c = \frac{[\hat{G} h_c] + t}{([\hat{G} h_c] + t)([\hat{G} h_c] + t) + |z|^2} \quad (6.1)
\]
It will be more convenient to work with these equations, as all the quantities are real. Since $|a(it,z)| = \sum_{c=1}^{D} \alpha_c h_c(it,z) = \sum_{c=1}^{D} \alpha_c \hat{h}_c(it,z)$ it suffices to show at least one of the $h_c$ or $\hat{h}_c$ is bounded from below for $t$ small.

Let $v_c$ be the Perron-Frobenius eigenvector of $G$, normalized such that $v_c^* = h_c$ and $\tilde{v}_c$ be the Perron-Frobenius eigenvector of $\hat{G}$, normalized such that $\tilde{v}_c^* = \hat{h}_c$.

Then we rewrite $[G h_c]$ as:
where \( \tilde{\eta} \) as desired.

For each \( t \), let \( c \) be an index such that \( v^c_d \leq v^d_d \) for all \( d \), implying \( v^d_d - v^c_d \geq 0 \) for all \( d \).

Using this \( c \), let \( t_1 = t + \sum_{d \neq c} G_{cd}(v^d_d - v^c_d) \) and \( t_2 = t + \sum_{d \neq c} \tilde{G}_{cd}(v^d_d - v^c_d) \).

Substituting into (6.1) gives

\[
Gh = \lambda h + \sum_{d \neq c} G_{cd}(h_d - v^c_d) = \lambda h + \sum_{d \neq c} G_{cd}(v^d_d - v^c_d)
\]

for all \( c \), \( d \).

Let \( \lambda \) be the function from \( R^{2D} \rightarrow R^{2D} \) defined by

\[
|t| = \frac{1}{|z|^2} (|Gh|_c + t)
\]

for \( 1 \leq c \leq D \) and

\[
|t| = \frac{1}{|z|^2} (|\tilde{G}h|_c + t)
\]

for \( d + 1 \leq c \leq 2D \).

If \( (h, \tilde{h}) \) is a solution to (6.1) then \( T((h, \tilde{h})) = (h, \tilde{h}) \) and therefore \( T^k((h, \tilde{h})) = (h, \tilde{h}) \). Since \( |z|^2 > \rho(G) \), for any \( \epsilon > 0 \) there exist a \( k \) such that \( \|G^k\|/|z|^k \leq \epsilon \).

Then using that the denominator is always greater than 1, we bound \( T^k(h, \tilde{h}) \) in terms of \( T^{k-1}(h, \tilde{h}) \) by

\[
T^k(h, \tilde{h}) \leq \frac{1}{|z|^2} (GT^{k-1}(h, \tilde{h}) + t)
\]

Iterating this estimate leads to

\[
T^k(h, \tilde{h}) \leq \left( \frac{1}{|z|^2} \right)^k (G^k h) + \frac{1}{|z|^2} (I + G/T^2 + \cdots G^k/T^{2k}) \tilde{t} \leq \epsilon + Ct
\]

where \( \tilde{t} \) is vector with all entries equal to \( t \). So we conclude

\[
0 \leq \|(h, \tilde{h})\| = \|T^k((h, \tilde{h}))\| \leq Ct
\]

as desired. \( \square \)
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