BUNDLE-THEORETIC METHODS FOR HIGHER-ORDER
VARIATIONAL CALCULUS

Michał Jóźwikowski
Institute of Mathematics. Polish Academy of Sciences
Śniadeckich 8, PO box 21, 00-956 Warsaw, Poland

Mikołaj Rotkiewicz
Institute of Mathematics. Polish Academy of Sciences
Śniadeckich 8, PO box 21, 00-956 Warsaw, Poland
and
Faculty of Mathematics, Informatics and Mechanics. University of Warsaw
Banacha 2, 02-097 Warsaw, Poland

(Communicated by Manuel de León)

Abstract. We present a geometric interpretation of the integration-by-parts
formula on an arbitrary vector bundle. As an application we give a new geo-
metric formulation of higher-order variational calculus.

1. Introduction.

Our results. The main motivation of this paper is to clarify the geometry of
higher-order variational calculus. In order to study this topic we had to introduce
new geometric tools and prove some results which we briefly sketch below.

First, we observe that, given a vector bundle $\sigma : E \to M$, it is possible to char-
terize two canonical morphisms $\Upsilon_{k,\sigma} : \tilde{T}^{k,k}_\sigma \to \sigma$ and $\upsilon_{k,\sigma} : \tilde{T}^{k,k}_\sigma \to T^k\sigma$, where
the bundle of semi-holonomic vectors $\tilde{T}^{k,k}_\sigma$ consists of all elements of $T^kT^kE$ which
project to the elements of $T^{2k}M \subset T^kT^kM$ (i.e., to holonomic vectors). Morphisms
$\Upsilon_{k,\sigma}$ and $\upsilon_{k,\sigma}$ provide an elegant geometric description of the $k$th-order integration-
by-parts formula on the bundle $\sigma$. A more precise formulation is provided by The-
orem 3.3 below.

Then, we use this result to obtain a comprehensive and, to certain extent, sim-
pler geometric interpretation of the standard procedure of deriving both the Euler-
Lagrange equations (forces) and the boundary terms (generalized momenta) for
a $k$th-order variational problem. Our description is “natural” in the sense that
it mimics the standard way of deriving the Euler-Lagrange equations. We start,
namely, from a homotopy $\gamma(t,s)$ in $M$ and then transform the variation of the ac-
tion $t_{s=0}^1 \int_{t_0}^{t_1} L(t^k\gamma(t,s)) \, dt$ to extract the integral term and the boundary terms.
Here $t^k\gamma$ stands for the $k$th-jet of a curve $\gamma = \gamma(t)$ at a point $t \in \mathbb{R}$. In particular,

2010 Mathematics Subject Classification. 58A20, 58E30, 70H50.

Key words and phrases. Higher tangent bundles, variational calculus, higher-order Euler-
Lagrange equations, graded bundles, integration by parts, geometric mechanics.

This research was supported by Polish Ministry of Science and Higher Education under the
grant NN 201416839.
The geometric interpretation of the first step is the following. Vector $t^1_\gamma(t)$ is the tangent vector to the curve $\gamma$ at $t_0$. On the computational level this procedure consists basically of two steps:

- reversing the order of differentiation to get $t^1_\gamma(t,s)$ from $t^1_\gamma(t,0)$,
- performing the $k^{th}$-order integration by parts to extract $t^1_\gamma(t,0)$ from $t^1_\gamma(t,s)$.

The geometric interpretation of the first step is the following. Vector $t^1_\gamma(t,s)$ is the image of $t^1_\gamma(t,0)$ under the canonical flip $\kappa_k : T^kTM \rightarrow TT^kM$, hence reversing this operation requires applying the dual map $\varepsilon_k : T^*TT^kM \rightarrow T^*T^kM$ to the differential of the Lagrangian [2]. Concerning the second, we can apply the maps $\Upsilon_{k,\tau^*}$ and $v_{k-1,\tau^*}$ introduced by us (here $\tau^* : T^*M \rightarrow M$ stands for the cotangent fibration). In this way we obtain the geometric formula $\Upsilon_{k,\tau^*}(t^k\varepsilon_k(\text{d} L(t^k\gamma(t))))$ describing the force (integral term) along the trajectory $\gamma(t) = \gamma(t,0)$. Thus the Euler-Lagrange equations along $\gamma(t)$ read as

$$\Upsilon_{k,\tau^*}(t^k\varepsilon_k(\text{d} L(t^k\gamma(t)))) = 0.$$  

The geometric formula for the momentum (boundary term) is obtained in a similar way using the map $v_{k-1,\tau^*}$. The precise description is provided by Theorems 4.2 and 4.3.

**State of research, applications.** In the theory of jet bundles there exists a well established notion of semi-holonomic jets [19]. These objects share some similarities with our semi-holonomic vectors, however, in principle, are different, so the similarity of names shall not be confusing. Morphisms $\Upsilon_{k,\sigma}$ and $v_{k,\sigma}$ were so far, to our best knowledge, not present in the literature. On the contrary, the problem of geometric formulation of higher-order variational calculus has gained a lot of interest in the mathematical community and has many solutions (e.g., [3, 14, 24, 25]), briefly reviewed in Section 5.

Since the geometry of higher-order variational calculus is a well-established topic, one should ask about other applications of morphisms $\Upsilon_{k,\sigma}$ and $v_{k,\sigma}$. We believe that our results make it easier to generalize higher-order variational calculus and mechanics to the framework of algebroids. Apart from some very special cases (see [5]), we are not aware of any such generalization in the spirit of the classical papers quoted above. Our preprint [10] contains some results in this direction, which make use of the tools introduced here. We present them briefly in Section 5.

**Outline of the paper.** In the preliminary Section 2 we revise basic information on higher tangent bundles, vector bundles and their lifts, as well as canonical pairings between such objects. We also recall the notion of the canonical flip $\kappa_k : T^kTM \rightarrow TT^kM$ and its dual $\varepsilon_k : T^*TT^kM \rightarrow T^*T^kM$.

Section 3, with its central Theorem 3.3, contains our main results. We begin by introducing the notion of the bundle of semi-holonomic vectors $T^{m_1,\ldots,m_k}\sigma$ and the canonical projection $P_k : T^{(k)}\sigma := T^{1,\ldots,k-1}\sigma \rightarrow T^k\sigma$. In Theorem 3.3 we provide a geometric construction of the canonical maps $\Upsilon_{k,\sigma}$ and $v_{k,\sigma}$ which give a comprehensive geometric interpretation of the integration-by-parts procedure on a vector bundle $\sigma$. We also state Lemma 3.4 about the universality of the map $\Upsilon_{k,\sigma}$, whose proof is postponed to the Appendix.

In Section 4 we show how to apply our results to higher-order variational problems (Problem 4.1). In particular, in Theorem 4.2, we obtain the general formula for the variation of the action including the forces and the generalized momenta along the trajectory. As a corollary we prove Theorem 4.3, which characterizes the extremals of Problem 4.1 in terms of higher-order Euler-Lagrange equations and
general transversality conditions. We also give a geometric and local description of the Euler-Lagrange equations and generalized momenta.

Finally, in Section 5, we briefly discuss different approaches to the geometry of higher-order variational calculus and relate our work in this direction with the papers of Tulczyjew [20, 24]. Later we sketch some result from [10], which show an application of our results to higher-order variational problems on algebroids. We end with an example of Riemannian cubic polynomials.

2. Preliminaries.

Higher tangent bundles. Throughout the paper we shall work with higher tangent bundles and use the standard notation $T^k M$ for the $k$th tangent bundle of the manifold $M$. Points in the total space of this bundle will be called $k$-velocities.

An element represented by a curve $\gamma : [t_0, t_1] \to M$ at $t \in [t_0, t_1]$ will be denoted by $t^k \gamma (t)$ or $t^k \gamma (t)$.

The $k+1$st tangent bundle is canonically included in the tangent space of the $k$th tangent bundle (see, e.g., [21]):

$$\iota^{1,k} : T^{k+1} M \subset TT^k M, \quad t^{k+1}_{\ell = 0} \gamma (t) \longmapsto t^1_{\ell = 0} t^{k}_{\gamma (t + s)}.$$

The composition of this injection with the canonical projection $\tau_{\gamma^k} M : TT^k M \to T^k M$ defines the structure of the tower of higher tangent bundles:

$$T^k M \to T^{k-1} M \to T^{k-2} M \to \ldots \to TM \to M.$$

The canonical projections from higher to lower-order tangent bundles will be denoted by $\tau_k : T^k M \to T^s M$ (for $k \geq s$). Instead of $\tau_k : T^k M \to M$ we simply write $\tau^k$ and, instead of $\tau_0^1 : TM \to M$ just use the standard symbol $\tau$. The cotangent fibration is denoted by $\tau^* : T^* M \to M$.

Another important constructions are the iterated tangent bundles $T^{(k)} M := T \ldots TM$ and the iterated higher tangent bundles $T^{n_1} \ldots T^{n_r} M$. Elements of the latter will be called $(n_1, \ldots, n_r)$-velocities. For notational simplicity we shall also denote the iterated higher tangent functor $T^{n_1} T^{n_2} \ldots T^{n_r}$ by $T^{n_1 \cdots n_r}$.

Of our particular interest will be the bundles $T^k TM = T^{k,l} M$. These bundles admit natural projections to lower-order tangent bundles which will be denoted by $\tau_{(k,l)}^{(k',l')} : T^k T^l M \to T^{k'} T^l' M$ (for $k \geq k'$ and $l \geq l'$). (Iterated higher) tangent bundles are subject to a number of natural inclusions such as already mentioned $\iota^{1,k} : T^{k+1} M \subset TT^k M, \iota^{l,k} : T^{k+l} M \subset TT^k M, \iota^k : T^k M \subset T^{(k)} M$, etc., which will be used extensively.

Occasionally, when dealing with manifolds other than $M$, or when it can lead to confusions, we will add a suffix to the maps $\iota^* \ldots \tau^* \ldots$, etc., to emphasize which manifold we are working with, e.g., $\tau^k_l = \tau^{k,l} M$, etc.

Given a smooth function $f$ on a manifold $M$ one can construct functions $f^{(\alpha)}$ on $T^k M$, with $0 \leq \alpha \leq k$, the so called $(\alpha)$-lifts of $f$ (see [17]), defined by

$$f^{(\alpha)}(t^k \gamma (t)) := \left. \frac{d^\alpha}{dt^\alpha} \right|_{t=0} f(\gamma (t)). \quad (2.1)$$

The functions $f^{(k)} : T^k M \to \mathbb{R}$ and $f^{(1)} : TM \to \mathbb{R}$ are called the complete lift and the tangent lift of $f$, respectively. By iterating this construction we obtain functions $f^{(\alpha, \beta)} := (f^{(\beta)})^{(\alpha)}$ on $T^{k+1} M$ for $0 \leq \alpha \leq k$, $0 \leq \beta \leq l$, and, generally, functions $f^{(\epsilon_1, \ldots, \epsilon_r)}$ on $T^{n_1} \ldots T^{n_r} M$ for $0 \leq \epsilon_j \leq n_j$, $1 \leq j \leq r$. A coordinate system $(x^a)$ on $M$ gives rise to the, so-called, adapted coordinate systems $(x^a)^{(\alpha)}$ on $T^k M$. 
and \((x^{a,(e)})_\epsilon\) on \(T^{\alpha_1} \ldots T^{\alpha_r} M\) where the multi-index \(\epsilon = (\epsilon_1, \ldots, \epsilon_r)\) is as before, and \(x^{a,(\alpha)}, x^{a,(\epsilon)}\) are obtained from \(x^a\) by the above lifting procedure. Within this notation we easily find that the canonical inclusion \(\iota^k : T^k M \to T^{(k)} M\) is given by

\[
(\iota^k)^* (x^{a,(\epsilon)}) = x^{a,(\epsilon_1 + \ldots + \epsilon_k)},
\]

where \(\epsilon \in \{0, 1\}^k\).

Let us remark that in the definition of \(f^{(\alpha)}\) we follow the convention of [5, 20], whereas the original convention of [17] is slightly different, since it contains a normalizing factor \(\frac{1}{\alpha!}\) in front of the derivative.

**Tangent lifts of vector bundles and canonical pairings.** Let us pass to another important tool for our analysis, namely the (iterated) higher tangent bundles of vector bundles. Let \(\sigma : E \to M\) be a vector bundle. It is clear that \(\sigma\) may be lifted to vector bundles \(T^k \sigma : T^k E \to T^k M\), \(T^{(k)} \sigma : T^{(k)} E \to T^{(k)} M\), etc. Let \(\sigma^* : E^* \to M\) be the bundle dual to \(\sigma\).

Throughout the paper we denote by \((x^a)\) the coordinates on the base \(M\), by \((y^i)\) the linear coordinates on fibers of \(\sigma\), and by \((\xi_j)\) the linear coordinates on fibers of the dual bundle \(\sigma^* : E^* \to M\). Natural weighted coordinates on (iterated higher) tangent lifts of \(\sigma\) and \(\sigma^*\) are constructed from \(x^a, y^i, \xi_j\) by the lifting procedure mentioned above. They are denoted by adding the proper degree to a coordinate symbol. Degrees will be denoted by bracketed small Greek letters: \((\epsilon), (\alpha), (\beta), \ldots\)

The natural pairing \(\langle \cdot, \cdot \rangle_{\sigma}\) between \(E\) and \(E^*\) induces a non-degenerate pairing between \(TE\) and \(TE^*\) over \(TM\):

\[
\langle \cdot, \cdot \rangle_{T_{\sigma}} := \langle \cdot, \cdot \rangle_{\sigma}^{(1)} : T(E^* \times M E) \simeq TE^* \times_{TM} TE \to \mathbb{R},
\]

i.e., \(\langle \cdot, \cdot \rangle_{T_{\sigma}}\) is the tangent lift of \(\langle \cdot, \cdot \rangle_{\sigma}\). In a similar way, \(\langle \cdot, \cdot \rangle_{\sigma}\) can be lifted to non-degenerate pairings on higher tangent prolongations of \(E\) and \(E^*\).

**Proposition 2.1.** Let \(\langle \cdot, \cdot \rangle_{\sigma} : E^* \times_M E \to \mathbb{R}\) be the natural pairing. Let us define inductively

\[
\langle \cdot, \cdot \rangle_{T^{(k)} \sigma} : T^{(k)} E^* \times_{T^{(k)} M} T^{(k)} E \to \mathbb{R}
\]

as the tangent lift of \(\langle \cdot, \cdot \rangle_{T^{(k-1)} \sigma}\) (for \(k \geq 1\)) and let

\[
\langle \cdot, \cdot \rangle_{T^k \sigma} : T^k E^* \times_{T^k M} T^k E \to \mathbb{R}
\]

be the restriction of \(\langle \cdot, \cdot \rangle_{T^{(k)} \sigma}\) to the product of subbundles \(T^k \sigma^* \subset T^{(k)} \sigma^*\) and \(T^k \sigma \subset T^{(k)} \sigma\). Then

(a) in the local coordinates \((x^{a,(\epsilon)}, y^{i,(\epsilon)})\) and \((x^{a,(\alpha)}, \xi_i^{(\alpha)})\), with \(\epsilon \in \{0, 1\}^k\) (resp. \((x^{a,(\alpha)}, y^{i,(\alpha)})\) and \((x^{a,(\alpha)}, \xi_i^{(\alpha)})\) with \(0 \leq \alpha \leq k\)) on \(T^{(k)} E\) and \(T^{(k)} E^*\) (resp. \(T^k E\) and \(T^k E^*\)),

\[
\left\langle (x^{a,(\epsilon)}, \xi_i^{(\alpha)}), (x^{a,(\epsilon)}, y^{i,(\epsilon)}) \right\rangle_{T^{(k)} \sigma} = \sum_{\epsilon \in \{0, 1\}^k} \sum_i \xi_i^{(\epsilon)} y^{i,(1, \ldots, 1) - (\epsilon)}, \tag{2.3}
\]

\[
\left\langle (x^{a,(\alpha)}, \xi_i^{(\alpha)}), (x^{a,(\epsilon)}, y^{i,(\alpha)}) \right\rangle_{T^k \sigma} = \sum_{\alpha \leq \alpha \leq k} \sum_{0 \leq \alpha \leq k} \left(\begin{array}{c} k \\ \alpha \end{array}\right) \xi_i^{(\alpha)} y^{i,(k-\alpha)}. \tag{2.4}
\]

(b) \(\langle \cdot, \cdot \rangle_{T^{(k)} \sigma}\) and \(\langle \cdot, \cdot \rangle_{T^k \sigma}\) are non-degenerate pairings.

(c) \(\langle \cdot, \cdot \rangle_{T^k \sigma} = \langle \cdot, \cdot \rangle^{(k)}_{\sigma}\), i.e., the pairing \(\langle \cdot, \cdot \rangle_{T^k \sigma}\) is the complete lift of \(\langle \cdot, \cdot \rangle_{\sigma}\) (up to the isomorphism \(T^k(E^* \times_M E) \simeq T^k E^* \times_{T^k M} T^k E)\).
Proof. We get (2.3) by iterated differentiation of the function \( \langle \cdot, \cdot \rangle_{\sigma} : ((x^a, \xi_i), (x^a, y^i)) \mapsto \sum_i \xi_i y^i \). Taking into account (2.2) we get (2.4). Since the tangent lift of a non-degenerated pairing is non-degenerated, so it is for \( \langle \cdot, \cdot \rangle_{T(k)^{i}} \). The non-degeneracy of the pairings \( \langle \cdot, \cdot \rangle_{T(k)^{i}} \) and \( \langle \cdot, \cdot \rangle_{T^k M} \) can also be easily seen from the above local formulas. Finally, (c) follows immediately from the definition of the complete lift and the local expression (2.4) of \( \langle \cdot, \cdot \rangle_{T^k M} \).

We remark that the above lifting procedure can be expressed in the framework of Weil functors and Frobenius algebras [13, 26].

**Canonical flip \( \kappa_k \) and its dual \( \varepsilon_k \):** It is well-known that the iterated tangent bundle \( T^k M \) admits an involutive double vector bundle isomorphism (called the canonical flip)

\[ \kappa : T \mapsto TTM, \]

which intertwines the projections \( \tau_{TM} : TTM \to TM \) and \( T \tau : TTM \to TM \) (see, e.g., [16]). Such a notion of canonical flip can be generalized to a family of isomorphisms

\[ \kappa_k : T^k \to TTT^k M, \quad t^k_{l=0} \sigma(t,s) \mapsto t^k_{l=0} \sigma(t,s), \]

which map the projection \( T^k \tau : T^k T^k M \to T^k M \) to \( \tau_{T^k M} : T^k T^k M \to T^k M \) over \( id_{T^k M} \) and \( T^k \tau : T^k T^k M \to T^k M \) to \( id_{T^k M} \). Morphisms \( \kappa_k \) can be also defined inductively as follows: \( \kappa_1 := \kappa \) and \( \kappa_k+1 := T\kappa_k \circ \kappa T^k M \big|_{T^{k+1} M} \),

i.e., as the unique morphism making the diagram

\[
\begin{array}{ccc}
T^k T M & \xrightarrow{T \kappa_k} & TTT^k M & \xrightarrow{T \kappa_{k+1}} & TTT^k M \\
\uparrow & & \uparrow & & \uparrow \\
T^{k+1} M & \xrightarrow{T \kappa_{k+1}} & TTT^k M \\
\end{array}
\]

commutative. The local description of \( \kappa_k \) is very simple. If \( x^{a,(\alpha,\epsilon)} \) are natural adapted coordinates on \( T^k T^k M \) and \( x^{a, (\epsilon, \alpha)} \) are natural adapted coordinates on \( T^k M \) (with \( 0 \leq \alpha < k \) and \( \epsilon = 0, 1 \)), then \( x^{a,(\alpha,\epsilon)} \) corresponds to \( x^{a,(\epsilon, \alpha)} \) via \( \kappa_k \).

Introduce now the dual \( \varepsilon_k : T^* T^k M \to T^k T^k M \) of the canonical flip \( \kappa_k \) defined via the equality,

\[ \langle \Psi, \kappa_k \circ V \rangle_{T^k M} = \langle \varepsilon_k \circ \Psi, V \rangle_{T^k M}, \]

where \( V \in T^k T^k M \) and \( \Psi \in T^* T^k M \) is a vector such that both pairings make sense (cf. [2]). Formula (2.6) shows that \( \kappa_k \) and \( \varepsilon_k \) are “adjoint” to each other with respect to the canonical pairings, as schematically shown by the commutative diagram

\[
\begin{array}{ccc}
T^k T M & \xrightarrow{\kappa_k} & T^k T M \\
\downarrow & & \downarrow \\
\langle \cdot, \cdot \rangle_{T^k T M} & \xrightarrow{\langle \cdot, \cdot \rangle_{T^k M}} & \langle \cdot, \cdot \rangle_{T^k T M} \\
\end{array}
\]

In the coordinates \( (x^{a,(\alpha)}, p_{a,(\alpha)}) = \partial_{x^{a,(\alpha)}} \) on \( T^* T^k M \) and \( (x^{a,(\alpha)}, p_{(\alpha)}) \) on \( T^k T^k M \) (adapted from standard coordinates \( (x^{a,(\alpha)}) \) on \( T^k M \), and \( (x^a, p_a) \) on \( T^k M \),
respectively), we find from (2.4) that

$$
\varepsilon_k \left( x^{a,\langle \alpha \rangle}, p_{a,\langle \alpha \rangle} \right) = \left( x^{a,\langle \alpha \rangle}, p^{\langle \alpha \rangle} = \left( k \right)^{-1} p_{a,\langle k-\alpha \rangle} \right).
$$

(2.7)

3. The main result. In this section the construction of the vector bundle morphisms $\Upsilon_{k,\sigma}$ and $\nu_{k,\sigma}$, associated with an integer $k$ and a vector bundle $\sigma : E \to M$, will be described. These morphisms are closely related with the geometric integration-by-parts procedure and will play a crucial role in the geometric construction of the Euler-Lagrange equations in the next Section 4.

Bundles of semi-holonomic vectors.

Definition 3.1. For non-negative numbers $n_1, \ldots, n_r \geq 0$ let denote $\bar{n} := n_1 + \ldots + n_r$. The set

$$
\tilde{T}^{n_1,\ldots,n_r} := (\tilde{\nu}_{\mathcal{M}})^* T^{n_1,\ldots,n_r} E = \{X \in T^{n_1,\ldots,n_r} E : T^{n_1,\ldots,n_r} \sigma(X) \in T^{\bar{n}} M \subset T^{n_1,\ldots,n_r} M\},
$$

consisting of all $(n_1, \ldots, n_r)$–velocities in $E$ projecting to $\bar{n}$–velocities (holonomic vectors) in $T^{n_1,\ldots,n_r} M$, is a vector subbundle of $T^{n_1,\ldots,n_r} \sigma : T^{n_1,\ldots,n_r} E \to T^{n_1,\ldots,n_r} M$. The restriction $\tilde{T}^{n_1,\ldots,n_r} \sigma$ of $T^{n_1,\ldots,n_r} \sigma$ to $\tilde{T}^{n_1,\ldots,n_r} E$ is the bundle of semi-holonomic vectors. Given a vector bundle $\sigma' : E' \to M'$ and a morphism $\phi : \sigma \to \sigma'$ we define $T^{n_1,\ldots,n_r} \phi : T^{n_1,\ldots,n_r} \sigma \to T^{n_1,\ldots,n_r} \sigma'$ as the restriction of $T^{n_1,\ldots,n_r} \phi$ to $T^{n_1,\ldots,n_r} E$. Thus $T^{n_1,\ldots,n_r}$ is a functor in the category of vector bundles.

In agreement with our previous notation we shall denote by $\bar{T}^{(k)} E = \bar{T}^{1,\ldots,1} E$ the subbundle of semi-holonomic velocities in $T^{(k)} E = T^{1,\ldots,1} E$.

In future considerations the bundles $\bar{T}^{k,k} E$ and $\bar{T}^{(k)} E$ will be of our special interest. Let us remark that although, in general, there is no canonical projection $T^{(k)} E \to T^k E$, there exists a natural projection $P_k : \bar{T}^{(k)} E \to T^k E$. It is defined as the left inverse to the canonical inclusion $\iota^k_k : T^k E \subset T^{(k)} E$ but considered as a map to $T^k E$, as is explained in the following proposition.

Proposition 3.2. Consider a semi-holonomic vector $X \in \bar{T}^{(k)} E \subset T^{(k)} E$ lying over the $k$–velocity $v^k \in T^k M \subset T^{(k)} M$. Then the formula

$$
\langle P_k (X), \Psi \rangle_{T^k \sigma} = \langle X, \Psi \rangle_{T^{(k)} \sigma},
$$

where $\Psi \in T^k E^* \subset T^{(k)} E^*$ lies over $v^k$, defines a canonical projection $P_k : \bar{T}^{(k)} E \to T^k E$, i.e., $P_k \circ \iota^k_k = \text{id}_{T^k E}$. Moreover, $P_k$ is a vector bundle morphism and it can be expressed in local coordinates as

$$
P_k \left( x^{a,\langle \rho \rangle}, y^{\langle \rho \rangle} \right) = \left( x^{a,\langle \rho \rangle}, \bar{y}^{\langle \rho \rangle} \right),
$$

where $\bar{y}^{\langle \rho \rangle} = \left( k \right)^{-1} \sum_{|\rho| = \alpha} y^{\langle \rho \rangle}$ is the arithmetic average of all the coordinates of total degree $\alpha$.

Proof. It follows immediately from the properties of the non-degenerated pairing $\langle \cdot, \cdot \rangle_{T^k \sigma}$ (see Proposition 2.1).
The map $\Upsilon_{k,\sigma}$.

The result below describes the construction of a certain canonical and universal vector bundle morphism $\Upsilon_{k,\sigma}$ from $T^k\kappa$ to $\sigma$. The precise sense of the word "universal" will be given later in Lemma 3.4. Informally speaking, any other morphism $\tilde{T}^k\kappa \rightarrow \sigma$ can be derived in an easy way from $\tilde{T}^k\kappa$. In the next Section 4 we show that this morphism is directly connected with integration by parts in the procedure of deriving the Euler-Lagrange equations.

To fix some notation denote an element $\Phi \in \tilde{T}^k\kappa \subset T^k\kappa T^k E$ by $\Phi^{(k,k)}$ and its projections to lower-order velocities by

$$\Phi^{(m,n)} := \tau_{(m,n),E}^{(k,k)}(\Phi) \in T^m T^n E,$$

where $m, n \leq k$. Observe that since $\Phi$ lies over some $2k$–velocity, say $v^{2k} \in T^{2k} M$, then all the elements $\Phi^{(m,n)}$ project under $T^m T^n \sigma$ to a fixed $(m + n)$–velocity $v^{m+n} \in T^{m+n} M \subset T^m T^n M$ independently on the numbers in the sum $m + n$. In particular, different elements $\Phi^{(m,n)}$ with $m + n$ fixed, belonging a priori to different bundles $T^m T^n E$, can be added together in the vector bundle $T^{(m+n)} \sigma : T^{(m+n)} E \rightarrow T^{(m+n)} M$, which contains all of them. Denote by $\tilde{\Phi}$ the following element of $T^{(k)} E$:

$$\tilde{\Phi} := \Phi^{(0,k)} - \left( \frac{k}{1} \right) \Phi^{(1,k-1)} + \left( \frac{k}{2} \right) \Phi^{(2,k-2)} + \ldots + (-1)^k \Phi^{(k,0)}. \quad (3.1)$$

Similarly, define a morphism $\nu_{k,\sigma} : \tilde{T}^k\kappa \rightarrow T^k \sigma$ by the formula

$$\nu_{k,\sigma}(\Phi^{(k,k)}): = P_k \left[ \left( \frac{k+1}{1} \right) \Phi^{(0,k)} - \left( \frac{k+1}{2} \right) \Phi^{(1,k-1)} + \ldots + (-1)^{k+1} \left( \frac{k+1}{k+1} \right) \Phi^{(k,0)} \right], \quad (3.2)$$

where $P_k : \tilde{T}^{(k)} E \rightarrow T^k E$ was defined in Proposition 3.2.

Now we are ready to state the main result of this section.

**Theorem 3.3** (Bundle-theoretic integration by part). Let $\Phi = \Phi^{(k,k)} \in \tilde{T}^k\kappa E$ be a semi-holonomic vector projecting to $v^{2k} \in T^{2k} M$ and let $v^k := \tau^k_k(v^{2k}) \in T^k M$. Let $t^k \xi$ be any element in $T^k E^*$ which projects to $v^k$ under $T^k \sigma^*$. Then, if $\tilde{\Phi}$ is given by (3.1), the value of $\left\langle \tilde{\Phi}, t^k \xi \right\rangle_{T^k \sigma}$ does not depend on the choice of $t^k \xi$.

Hence it defines a canonical vector bundle morphism $\Upsilon_{k,\sigma} : \tilde{T}^k\kappa \rightarrow \sigma$ covering $\tau^k : T^k M \rightarrow M$ given by

$$\left\langle \Upsilon_{k,\sigma}(\Phi), \xi \right\rangle_{\sigma} := \left\langle \tilde{\Phi}, t^k \xi \right\rangle_{T^k \sigma} = \left\langle \sum_{j=0}^{k} (-1)^j \left( \frac{k}{j} \right) \Phi^{(j,k-j)}, t^k \xi \right\rangle_{T^k \sigma}. \quad (3.3)$$

Moreover,

(a) in coordinates, if $\Phi \sim (x^{a,\alpha}(t), y^{j,\beta,\gamma})$ where $0 \leq \alpha \leq 2k$ and $0 \leq \beta, \gamma \leq k$, then

$$\Upsilon_{k,\sigma}(\Phi) = \left( x^{a, \sum_{\alpha=0}^{k} (-1)^\alpha \left( \frac{k}{\alpha} \right) y^{i,\alpha,k-\alpha}} \right); \quad (3.4)$$

(b) $\Upsilon_{k,\sigma}$ satisfies the recurrence formulas

$$\Upsilon_{1,\sigma}(\Phi) = \nu_E \left[ \tau_E(\Phi) - T\tau_E(\Phi) \right], \quad (3.5)$$

$$\Upsilon_{k,\sigma} = \Upsilon_{1,\sigma} \circ \Upsilon_{k-1,TT\sigma} |_{\tilde{T}^{k-1} E}, \quad (3.6)$$
where \( \nu_E \) denotes the projection \( TE|_M \cong E \times_M TM \to E \);

(c) \( \nu_{k,\sigma} \) satisfies the recurrence formulas

\[
\nu_{0,\sigma} = \text{id}_{\sigma},
\]

\[
\nu_{k,\sigma} \left( \Phi^{(k,k)}, t^k \xi \right)_{T^k \sigma} = \left( \Upsilon_{k,\sigma} \left( \Phi^{(k,k)} \right), \xi \right)_{\sigma} + \left( \nu_{k-1,\sigma} \left( \Phi^{(k-1,k)} \right), t^{k-1} \xi \right)_{T^{k-1} \sigma},
\]

where in the last term we consider \( \Phi^{(k-1,k)} \) as a semi-holonomic vector in \( T^{k-1,k}TE \supset \tilde{T}^{k,k}E \);

(d) \( \Upsilon_{k,\sigma} \) and \( \nu_{k-1,\sigma} \) are related by the “bundle-theoretic integration by parts formula”

\[
\left( \Phi^{(k,0)}, t^k \xi \right)_{T^{k} \sigma} = \left( \Upsilon_{k,\sigma} \left( \Phi^{(k,k)} \right), \xi \right)_{\sigma} + \left( T \nu_{k-1,\sigma} \left( \Phi^{(k-1,k)} \right), t^{k} \xi \right)_{T^{k-1} \sigma},
\]

where in the last term we consider \( \Phi^{(k-1,k)} \) as a vector in \( T^{k-1,k}TE \supset \tilde{T}^{k,k}E \);

(e) morphism \( \Upsilon_{k,\cdot} \) commutes with the tangent functor \( T \) up to the canonical isomorphism \( \tilde{\kappa}_{k,k,E} \): \( T^{k,k}E \to \tilde{T}^{k,k}TE \), i.e., the diagram

\[
\begin{array}{ccc}
\tilde{T}^{k,k}TE & \xrightarrow{\Upsilon_{k,\cdot}} & TE \\
\tilde{T}^{k,k}E & \xrightarrow{\tilde{\kappa}_{k,k,E}} & T^{k}E \\
\end{array}
\]

is commutative;

(f) morphism \( \nu_{k,\cdot} \) commutes with the tangent functor \( T \) up to the canonical isomorphisms \( \tilde{\kappa}_{k,k,E} : \tilde{T}^{k,k}TE \to T^{k,k}TE \) and \( \kappa_{k,E} : T^{k}TE \to T^{k}E \), i.e., the diagram

\[
\begin{array}{ccc}
\tilde{T}^{k,k}TE & \xrightarrow{\nu_{k,\cdot}} & T^{k}TE \\
\tilde{T}^{k,k}E & \xrightarrow{\tilde{\kappa}_{k,k,E}} & T^{k}E \\
\end{array}
\]

is commutative.

Lemma 3.4 is a natural continuation of Theorem 3.3, but we decided to keep it separated since its proof is rather technical (see Appendix).

**Lemma 3.4 (Universality of \( \Upsilon_{k,\sigma} \)).** Let \( M \) be a connected manifold. Then any functorial vector bundle morphism \( (F_E, F_E) \)

\[
\begin{array}{ccc}
\tilde{T}^{k,k}E & \xrightarrow{\tilde{\kappa}_{k,k,E}} & T^{k}E \\
\tilde{M}^{2k} & \xrightarrow{\sigma} & M \\
\end{array}
\]

is a linear combination of the morphisms \( \Upsilon_{l,\sigma} \circ \tau^{(k)}_{l,E} \), where \( \tau^{(k)}_{l,E} : \tilde{T}^{k,k}E \to \tilde{T}^{l,l}E \) is the canonical projection induced by \( \tau^{(l,k)}_{l,E} : T^{k,k}E \to T^{l,l}E \) for \( 0 \leq l \leq k \).
Proof of Theorem 3.3. We shall prove first that $\Upsilon_{k,\sigma}$ is a well-defined mapping and, simultaneously, part (b). To this end, we proceed by induction with respect to $k$ for arbitrary $\sigma : E \to M$.

Consider $k = 1$. Take $\Phi = \Phi^{(1,1)} \in TTE$ and let $v^2 := TT\sigma(\Phi) \in T^2M$. $TT\sigma$ has the structure of a double vector bundle with the projections $TTE$ and $TTE$ onto $TE$. Vectors $\Phi^{(0,1)} = \tau_{TE}(\Phi)$ and $\Phi^{(1,0)} = \tau_{TE}(\Phi)$ project to the same vector $v^1 = \tau_{TM}(v^2) = T\tau(v^2)$ in $TM$ and to the same point $\tau_{E}(\Phi^{(0,1)}) = \tau_{E}(\Phi^{(1,0)}) = \Phi^{(0,0)}$ in $E$. It follows that their difference with respect to the vector bundle structure $T\sigma : TE \to TM$ belongs to $TE|_M \cong E \times_M TM$. Hence, from the properties of the pairing $\langle \cdot, \cdot \rangle_{T\sigma}$,

$$\left\langle \Phi^{(0,1)} - \Phi^{(1,0)}, t^1 \xi \right\rangle_{T\sigma} = \left\langle \nu_{E} \left( \Phi^{(0,1)} - \Phi^{(1,0)} \right), \xi \right\rangle_{\sigma}.$$

In other words $T\sigma_{1,\sigma}$ is well-defined and satisfies (3.5).

Assume now that the assertion holds for every $l \leq k - 1$ and every vector bundle $\sigma$. Observe that, for every $l$,

$$\widetilde{T}^{l,E} \ni \Phi \mapsto \sum_{i=0}^{l} (-1)^i \binom{l}{i} \Phi^{(i,l-i)} \in T^{l,E}$$

is a functorial vector bundle morphism over $\tau_{E}^{2,l} : T^{2,l}M \to T^{l,l}M \subset T^{l,l}M$ being the combination, with constant coefficients, of functorial vector bundle morphisms $\Phi \mapsto \Phi^{(i,l-i)}$. In other words, given a vector bundle $\sigma' : E' \to M'$ and a vector bundle morphism $\alpha : \sigma \to \sigma'$, it holds

$$T^{l,E} \alpha \left( \sum_{i=0}^{l} (-1)^i \binom{l}{i} \Phi^{(i,l-i)} \right) = \sum_{i=0}^{l} (-1)^i \binom{l}{i} \left( \left( T^{l,E} \alpha \right) \Phi \right)^{(i,l-i)}.$$

This fact, combined with the functoriality of $\langle \cdot, \cdot \rangle_{T\sigma}$ and the inductive assumption, guarantees that

$$\alpha(T_{L,\sigma}(\Phi)) = T_{L,\sigma'}((T^{l,E} \alpha)\Phi)$$

(3.13)

for every $l \leq k - 1$. In particular, we may take as $\alpha$ in (3.13), the following vector bundle morphisms $T_{TE} : TTE \to TE$ over $T_{TM}, \Psi \mapsto \Psi^{(1,0)}$ and $T_{TE} : TTE \to TE$ over $T_{TM}, \Psi \mapsto \Psi^{(0,1)}$ to get

$$\left( T_{k-1,TE}(\Phi) \right)^{(1,0)} = T_{k-1,TE} \left( \Phi^{(k,k-1)} \right),$$

(3.14)

$$\left( T_{k-1,TE}(\Phi) \right)^{(0,1)} = T_{k-1,TE} \left( \Phi^{(k-1,1)} \right),$$

(3.15)

where we treat $\Phi^{(k,k)}$ as an element of $T^{k-1,k-1}E$, while $\Phi^{(k,k-1)}$ and $\Phi^{(k-1,k)}$ as elements of $T^{k-1,k-1}E$. Now for the pairing (3.3) of our interest we can write:

$$\left\langle \Phi^{(0,k)} - \binom{k}{1} \Phi^{(1,k-1)} + \binom{k}{2} \Phi^{(2,k-2)} + \ldots + (-1)^k \Phi^{(k,0)}, t^k \xi \right\rangle_{T^{(k)}\sigma} =$$

$$\left\langle \Phi^{(0,k)} - \binom{k-1}{1} \Phi^{(1,k-1)} + \binom{k-1}{2} \Phi^{(2,k-2)} + \ldots + (-1)^{k-1} \Phi^{(k-1,1)}, t^k \xi \right\rangle_{T^{(k)}\sigma}$$

$$- \left\langle \Phi^{(1,k-1)} - \binom{k-1}{1} \Phi^{(2,k-2)} + \binom{k-1}{2} \Phi^{(3,k-3)} + \ldots + (-1)^k \Phi^{(k,0)}, t^k \xi \right\rangle_{T^{(k)}\sigma}.$$

Thanks to our inductive assumption the right-hand side of the latter equals

$$\left\langle T_{k-1,TE} \left( \Phi^{(k,k-1)} \right), t^1 \xi \right\rangle_{T\sigma} = - \left\langle T_{k-1,TE} \left( \Phi^{(k-1,k)} \right), t^1 \xi \right\rangle_{T\sigma}.$$
Now we can use equalities (3.14) and (3.15) to bring the above expression to the form
\[ \left\langle \left( T_{k-1} \mathcal{T}T_{\sigma} \Phi \right)^{(0,1)} - \left( T_{k-1} \mathcal{T}T_{\sigma} \Phi \right)^{(1,0)} , t^1 \xi_1 \right\rangle_{\sigma} \stackrel{(3.5)}{=} \left\langle T_{1,\sigma} \left( T_{k-1} \mathcal{T}T_{\sigma} \Phi \right) , \xi_1 \right\rangle_{\sigma}. \]
This sums up to formula (3.6) and assures that \( \Upsilon_{k,\sigma} \) is indeed correctly defined.

The proof of (a) is simple. Locally, \( \Phi \) and \( t^k \xi \) (in canonical induced graded coordinates on \( T^k E \) and \( T^k E^* \)) are given by
\[ \Phi \sim \left( x^i, y_j, (\alpha, \beta) \right) \quad \text{and} \quad t^k \xi \sim \left( x^i, (\alpha), \xi_j^{(\beta)} \right), \]
where \( \alpha = 0, 1, \ldots, 2k \) and \( \beta, \beta' = 0, 1, \ldots, k \).

We have shown that \( \left\langle \Phi, t^k \xi \right\rangle_{T^{(k)} \sigma} \), which has a coordinate form of a polynomial in \( \xi_i^{(\beta)} \) and \( y_j^{(\beta', \beta'')} \), actually does not depend on \( \xi_i^{(\beta)} \) for \( \beta \geq 1 \). Hence from (3.3) we get that \( \Upsilon_{k,\sigma} (\Phi) = (x^i, \mathbf{y}^i) \), where \( \mathbf{y}^i \) is a \( \xi_i^{(0)} \)-coefficient in the coordinate expression for \( \left\langle \Phi, t^k \xi \right\rangle_{T^{(k)} \sigma} \). By (2.3) we know that \( \mathbf{y}^i \) is the coordinate of total degree \( k \) in \( \Phi \). We conclude that the formula (3.4) for \( \Upsilon_{k,\sigma} \) is true.

To prove (c) we use the standard decomposition of binomial coefficients \( \binom{k+1}{i+1} = \binom{k}{i} + \binom{k}{i+1} \) to obtain
\[ \begin{align*}
\binom{k+1}{1} \Phi^{(0,k)} - \binom{k+1}{2} \Phi^{(1,k-1)} + \cdots + (-1)^k \binom{k+1}{k+1} \Phi^{(k,0)} &= \\
\left[ \binom{k}{0} \Phi^{(0,k)} - \binom{k}{1} \Phi^{(1,k-1)} + \cdots + (-1)^k \binom{k}{k} \Phi^{(k,0)} \right] + \\
\left[ \binom{k}{1} \Phi^{(0,1+k-1)} - \binom{k}{2} \Phi^{(1,1+k-2)} + \cdots + (-1)^{k-1} \binom{k}{k} \Phi^{(k-1,1+0)} \right].
\end{align*} \]

In the first expression on the right-hand side of this equality we easily recognize formula (3.3), whereas, in the second, formula (3.2) \( k-1 \).

The proof of (d) is very similar. We decompose
\[ \Phi^{(0,k)} = \left[ \binom{k}{0} \Phi^{(0,k)} - \binom{k}{1} \Phi^{(1,k-1)} + \cdots + (-1)^k \binom{k}{k} \Phi^{(k,0)} \right] + \\
\left[ \binom{k}{1} \Phi^{(1,0+k-1)} - \binom{k}{2} \Phi^{(1+1,k-2)} + \cdots + (-1)^{k-1} \binom{k}{k} \Phi^{(k+1,1-0)} \right]. \]
Now on the right-hand side of this equality we easily recognize formulas (3.3) and (3.2) \( k-1 \).

Finally, to prove (e) and (f), observe that the tangent functor \( T \) commutes (up to canonical isomorphisms) with the projections \( \tau_{i,k-1} \) and \( P_k \). Thus it commutes (up to canonical isomorphisms) with \( \Upsilon_{k,\cdot} \) and \( u_k \cdot \) which are linear combinations of the compositions of the mentioned projections.

**Remark 3.5.** Note that even though in formula (3.4) only the coordinates of the highest degree \( k \) matter, the coordinates of lower degrees may be non-trivial. For example, if \( k = 2 \), we have \( \Phi = \Phi^{(0,2)} - 2 \Phi^{(1,1)} + \Phi^{(2,0)} \in T^2 E \) and \( \Phi^{(0,2)} \in T^2 E \), interpreted as an element of \( T^2 E \) by means of \( \iota^{1,1} : T^2 E \to T^2 E \), equals to...
\[ \left( x^0, x^1, x^2, y^0, y^1, y^2 \right). \] Similarly for \( \Phi^{(2,0)} \). Therefore, in coordinates, \( \Phi \) reads
\[
\Phi \sim \left( x^0, x^1, x^2, y^0, y^1, y^2 \right),
\]
where \( y^0 = 0, y^1 = -y^0, y^2 = y^1, y^2 = y^1 \) and \( y^2 = y^1 - 2y^0 \).

**Remark 3.6.** Observe that, for any \( k \) and \( l \), the bundle \( \mathcal{T}^{l,l}E \) is the pullback of \( \mathcal{T}^{l,l}E \) with the canonical inclusion \( \mathcal{T}^{l,l}E \). Any semi-holonomic vector \( X \in \mathcal{T}^{k+l}E \) lying over \( \mathcal{T}^{2k+2l}M \) is mapped via this inclusion to an element lying over an element in \( \mathcal{T}^{2k+2l}M \). Thus we have the canonical inclusion \( \mathcal{T}^{k+l}E \subset \mathcal{T}^{l,l}E \). Therefore the inductive formula (3.6) can be described as follows:

\[
\begin{array}{ccc}
\tilde{\mathcal{T}}^{k,k} E \ar@{^{(}->}[r] & \tilde{\mathcal{T}}^{k-1,k-1} E \ar@{^{(}->}[r] & \mathcal{T}^{1,1} E \\
\mathcal{Y}_{k,\sigma} \ar@{^{(}->}[u] & \mathcal{Y}_{k-1,1,\sigma} \ar@{^{(}->}[u] & \\
E \ar@{^{(}->}[u] & \mathcal{T}^{1,1} E \ar@{^{(}->}[u] & \\
\end{array}
\]

Expressing \( \mathcal{Y}_{k,\sigma} \) as a composition of morphisms \( \mathcal{Y}_{k,\sigma} \), according to (3.6), we can obtain a more general formula \( \mathcal{Y}_{k+l,\sigma} \), i.e.,

\[
\begin{array}{ccc}
\tilde{\mathcal{T}}^{k+l,k+l} E \ar@{^{(}->}[r] & \tilde{\mathcal{T}}^{k,k} E \ar@{^{(}->}[r] & \mathcal{T}^{l,l}E \\
\mathcal{Y}_{k+l,\sigma} \ar@{^{(}->}[u] & \mathcal{Y}_{k,\sigma} \ar@{^{(}->}[u] & \\
E \ar@{^{(}->}[u] & E \ar@{^{(}->}[u] & \\
\end{array}
\] (3.16)

Similarly, formula (3.8) generalizes to

\[
\begin{aligned}
&\left\langle u_{k+l,\sigma} \left( \Phi^{(k+l,k+l)} \right), t^{k+l} \xi \right\rangle_{T^{k+l}E} = \\
&\left\langle u_{k,\sigma} \left( \mathcal{Y}_{k,\sigma} \Phi^{(k+l,k+l)} \right), t^k \xi \right\rangle_{T^kE} + \left\langle u_{l-1,k+l+1,\sigma} \left( \Phi^{(l-1+l+k)} \right), t^{k+l} \xi \right\rangle_{T^{k+l}E}.
\end{aligned}
\] (3.17)

The proof is left to the reader.

In light of (3.9), we can interpret formulas (3.16) and (3.17) as follows: integration by parts \( k + l \) times can be obtained as the composition of \( k \) times and \( l \) times integration by parts.

4. **Applications to variational calculus.** In this section we use Theorem 3.3 to give a geometric construction of the force and momentum in higher-order variational calculus (Theorem 4.2). As a consequence, in Theorem 4.3, we obtain necessary and sufficient conditions (Euler-Lagrange equations and transversality conditions) for a curve to be an extremal of the higher-order variational Problem 4.1.
Formulation of the problem. Consider a smooth Lagrangian function \( L : T^k M \to \mathbb{R} \) and the associated action
\[
\gamma \mapsto S_L(t^k \gamma) = \int_{t_0}^{t_1} L(t^k \gamma(t)) \, dt,
\]
where \( \gamma : [t_0, t_1] \to M \) is a path and \( t^k \gamma(t) \in T_{\gamma(t)}^k M \) its \( k \)th prolongation. The set of admissible paths \( t^k \gamma \) will be denoted by \( ADM([t_0, t_1], T^k M) \).

By an admissible variation of an admissible path \( t^k \gamma(t) \) we will understand a curve \( \delta t^k \gamma(t) \in T_{t^k \gamma(t)} T^k M \). Observe that every admissible variation \( \delta t^k \gamma(t) \) can be obtained from a homotopy \( \chi(t, \cdot) : [t_0, t_1] \times (-\epsilon, \epsilon) \to M \) such that \( \chi(t, 0) = \gamma(t) \) and \( \delta t^k \gamma(t) = \dot{t}^1_{s=0} \chi(t, s) \). We see that \( \delta t^k \gamma \) is generated by \( \delta \gamma(t) = \dot{t}^1_{s=0} \chi(t, s) \in T\gamma(t) M \), i.e., a vector field along \( \gamma(t) \), in the sense that
\[
\delta t^k \gamma = \kappa_k(\dot{t}^1 \delta \gamma).
\]

So, \( \delta \gamma \) can be called a generator of the variation \( \delta t^k \gamma \). Given an admissible variation \( \delta t^k \gamma \) we may define the differential of the action \( S_L \) in the direction of this variation:
\[
\left\langle d S_L(t^k \gamma), \delta t^k \gamma \right\rangle := \int_{t_0}^{t_1} \left\langle \frac{d}{dt} L(t^k \gamma(t)), \delta t^k \gamma(t) \right\rangle_{T^k M} \, dt.
\]

Define now a natural projection
\[
\mathcal{P} : ADM([t_0, t_1], M) \ni t^k \gamma \mapsto (t^{k-1} \gamma(t_0), t^{k-1} \gamma(t_1)) \in T^{k-1} M \times T^{k-1} M,
\]
which sends an admissible path \( t^k \gamma \) to the pair consisting of its initial and final \((k-1)\)-velocity. Its tangent map \( T\mathcal{P} \) sends a variation \( \delta t^k \gamma \in T_{t^k \gamma} T^k M \) to the pair \( (\delta t^{k-1} \gamma(t_0), \delta t^{k-1} \gamma(t_1)) \in TT^{k-1} M \times TT^{k-1} M \).

Now we are ready to formulate the following variational problem.

**Problem 4.1** (Variational problem). For a given Lagrangian function \( L : T^k M \to \mathbb{R} \) and a submanifold \( S \subset T^{k-1} M \times T^{k-1} M \), which represents the admissible boundary values of \((k-1)\)-velocities, find all curves \( \gamma : [t_0, t_1] \to M \) such that their \( k \)th prolongation \( t^k \gamma \) satisfies
\[
\left\langle d S_L(t^k \gamma), \delta t^k \gamma \right\rangle = 0 \quad \text{for every admissible variation } \delta t^k \gamma \text{ such that } T\mathcal{P}(\delta t^k \gamma) \in TS.
\]

Let us comment the above formulation. In our approach to variational problems we study the behavior of the differential of the action functional in the directions of admissible variations (differential approach), rather to compare the values of the action on nearby trajectories (integral approach). Hence, solutions of Problem 4.1 are only the critical, not extremal, points of the action functional (4.1). The philosophy of understanding a variational problem as the study of the differential of the action restricted to the sets of admissible trajectories and admissible variations allows one to treat the unconstrained and constrained cases in a unified way (see, e.g., [7, 9, 11]).

**Higher-order variational calculus.**

Let \( S_L \) be the action functional (4.1) and \( \delta t^k \gamma \) the variation (4.2). Define the force \( F_{L, \gamma}(t) \in T^* M \) and the momentum \( M_{L, \gamma}(t) \in T^{k-1} T^* M \) along \( \gamma(t) \) by
\[
F_{L, \gamma}(t) = \mathcal{T}_{k, T^*} (t^k \Lambda_L(t^k \gamma(t))),
\]
\[
M_{L, \gamma}(t) = \upsilon_{k-1, T^*} (t^{k-1} \lambda_L(t^k \gamma(t)));
\]
where \( \Lambda_L := \varepsilon_k \circ d L : T^k M \to T^k T^* M \) and \( \lambda_L := t_{k-1, T^*}^k \circ \Lambda_L : T^k M \to T^{k-1} T^* M \).
Theorem 4.2. The differential of the action $S_L$ in the direction of the variation $\delta t^k \gamma$ equals
\[
\left< dS_L(t^k \gamma), \delta t^k \gamma \right> = \int_{t_0}^{t_1} \left< F_{L, \gamma}(t), \delta \gamma(t) \right>_\tau \, dt + \left< M_{L, \gamma}(t), t^{k-1} \delta \gamma(t) \right>_{T^{k-1} \tau} \bigg|_{t_0}^{t_1},
\]
(4.5)

Theorem 4.3 below is an immediate consequence of formula (4.5).

Theorem 4.3. A curve $\gamma$ is a solution of Problem 4.1 if and only if it satisfies the following Euler-Lagrange (EL) equation
\[
F_{L, \gamma}(t) = 0
\]
(4.6)
and the transversality conditions
\[
(-\varepsilon_{k-1}^{-1}(M_{L, \gamma}(t_0)), \varepsilon_{k-1}^{-1}(M_{L, \gamma}(t_1))) \in T^* T^{k-1} M \times T^* T^{k-1} M \text{ annihilates } TS.
\]
(4.7)

Proof of Theorems 4.2 and 4.3. Let us calculate the variation of the action $S_L$ in the direction $\delta t^k \gamma$:
\[
\left< dL(t^k \gamma(t)), \delta t^k \gamma(t) \right>_{\tau_k M} \overset{(4.2)}{=} \left< dL(t^k \gamma(t)), \kappa_k (t^k \delta \gamma(t)) \right>_{\tau_k M} \overset{(2.6)}{=} \varepsilon_k \circ dL(t^k \gamma(t)), t^k \delta \gamma(t) \right>_{T^* M} = \left< \Lambda_L(t^k \gamma(t)), t^k \delta \gamma(t) \right>_{T^* M}.
\]

Now we can use formula (3.9) with $\Phi^{k,k} = t^k \Lambda_L(t^k \gamma(t)), \Phi^{(0,k)} = \Lambda_L(t^k \gamma(t))$ and $\Phi^{(k,k-1)} = t^k \lambda_L(t^k \gamma(t))$, since the element $\Phi^{(k,k)} = t^k \Lambda_L(t^k \gamma(t)) \in T^k T^k T^* M$ is a semi-holonomic vector as it projects to $t^k t^{k \gamma} = t^{2k_\gamma} \in T^{2k} M$. We get
\[
\left< \Lambda_L(t^k \gamma(t)), t^k \delta \gamma(t) \right>_{T^* M} \overset{(3.9)}{=} < \Upsilon_{k,\tau^*} \left( t^k \Lambda_L(t^k \gamma(t)) \right), \delta \gamma(t) \right>_{\tau^*} + \left< T_{\nu_{k-1,\tau^*}} \left( t^k \lambda_L(t^k \gamma(t)) \right), t^k \delta \gamma(t) \right>_{T^{(k,\tau^*)}}.
\]

In the first summand we recognize the force $F_{L, \gamma}(t)$ defined by (4.3). To the second we can apply the equality
\[
T_{\nu_{k-1,\tau^*}} \left( t^k \lambda_L(t^k \gamma(t)) \right) = T_{\nu_{k-1,\tau^*}} \left( t^k t^{k-1} \lambda_L(t^k \gamma(t)) \right) = t^1 \left[ \nu_{k-1,\tau^*} \left( t^{k-1} \lambda_L(t^k \gamma(t)) \right) \right].
\]

We conclude that
\[
\left< \Lambda_L(t^k \gamma(t)), t^k \delta \gamma(t) \right>_{T^* M} = \left< F_{L, \gamma}(t), \delta \gamma(t) \right>_{\tau^*} + t^1 \left[ \nu_{k-1,\tau^*} \left( t^{k-1} \lambda_L(t^k \gamma(t)) \right) \right], t^1 t^{k-1} \delta \gamma(t) \right>_{T^{*k-1} \tau^*} = \left< F_{L, \gamma}(t), \delta \gamma(t) \right>_{\tau^*} + \frac{d}{dt} \left[ \nu_{k-1,\tau^*} \left( t^{k-1} \lambda_L(t^k \gamma(t)) \right) \right], t^{k-1} \delta \gamma(t) \right>_{T^{*k-1} \tau^*} = \left< F_{L, \gamma}(t), \delta \gamma(t) \right>_{\tau^*} + \frac{d}{dt} \left[ M_{L, \gamma}(t), t^{k-1} \delta \gamma(t) \right]_{T^{*k-1} \tau^*},
\]
where the momentum $M_{L, \gamma}(t)$ is defined by (4.4). Thus the variation at $t$ reads
\[
\left< dL(t^k \gamma(t)), \delta t^k \gamma(t) \right>_{\tau_k M} = \left< F_{L, \gamma}(t), \delta \gamma(t) \right>_{\tau^*} + \frac{d}{dt} \left[ M_{L, \gamma}(t), t^{k-1} \delta \gamma(t) \right]_{T^{*k-1} \tau^*}.
\]
(4.8)

Integrating the above expression over $[t_0, t_1]$ we get (4.5), concluding the proof of Theorem 4.2. Theorem 4.3 follows easily, as $\delta t^{k-1} \gamma(t) = \kappa_{k-1}(t^{k-1} \delta \gamma(t))$ and $\varepsilon_{k-1}$ is dual to $\kappa_{k-1}$, in light of equation (2.6). $\square$
Therefore, we calculate the coordinates of $\Lambda_L$ function, whereas dashed arrows are maps defined only along the images of $\gamma$. Above we used dotted arrows to denote the objects associated with the Lagrangian function.

Remark 4.4. The process of constructing the EL equations (4.6) starting from the Lagrangian function $L$ can be followed on the diagram

\[
\text{ker } \Upsilon_{k,\tau^*} \quad (4.9)
\]

Similarly, the geometric construction of the momenta (4.4) corresponds to the diagram

\[
\text{ker } \Upsilon_{k,\tau^*} \quad (4.10)
\]

Above we used dotted arrows to denote the objects associated with the Lagrangian function, whereas dashed arrows are maps defined only along the images of $\gamma(t)$.

Note also that the map $\Upsilon_{k,\tau^*}$ allows us to define higher-order EL equations with external forces. Namely, given an external force, i.e., a map $F : [t_0, t_1] \to T^*M$, we can consider equation

\[
\Upsilon_{k,\tau^*}(t^k_\Lambda_L(t^k\gamma(t))) = F(t).
\]

When $F(t) = 0$, the equation above reduces to the EL equation (4.6).

Local form of the forces and momenta. We shall now derive the local form of the force (4.3) and momentum (4.4).

Let us first calculate the force. Consider a trajectory $\gamma(t) \sim (x^a(t))$. The differential $dL(t^k\gamma(t)) \in T^*T^kM$ is given by $p_{a,(\alpha)} = \partial L / \partial x^a(\alpha)(t^k\gamma(t))$, hence using (2.7) we calculate the coordinates of $\Lambda_L(t^k\gamma(t)) \in T^kT^*M$, namely,

\[
p_{a,(\alpha)}(\Lambda_L(t^k\gamma(t))) = \left( \begin{array}{c} k \\ \alpha \end{array} \right)^{-1} \frac{\partial L}{\partial x^a(\alpha)(\bar{k}-\alpha)}(t^k\gamma(t)), \quad x^a(\alpha)(\Lambda_L(t^k\gamma(t))) = \frac{d^a x^a(t)}{dt^\alpha}.
\]

Therefore, $p_{a,(\alpha)}(t^k_\Lambda L(t^k\gamma(t))) = \left( \begin{array}{c} k \\ \alpha \end{array} \right)^{-1} \frac{\partial L}{\partial x^a(\alpha)(\bar{k}-\alpha)}$, and using (3.4) we find that our formula (4.3) of the force takes the following well-known form

\[
F_{L,\gamma}(t)_a = \sum_{\alpha=0}^k (-1)^\alpha \frac{d^\alpha}{dt^\alpha} \left( \frac{\partial L}{\partial x^a(\alpha)}(t^k\gamma(t)) \right).
\]

Concerning momentum, a direct (i.e., by means of formulas (3.2) and (4.10)) derivation of its local form is a quite complicated computational task, so that we will obtain it by using formula (4.8), instead.

To this end, consider a generator of the variation $\delta \gamma(t) \sim (x^a(t), \delta x^a(t))$ and its $k^{th}$ tangent lift $t^k\gamma(t) \sim (x^a(\alpha)(t), \delta x^a(\alpha)(t))_{\alpha=0,\ldots,k}$, where $x^a(0)(t) = x^a(t)$, $\delta x^a(0)(t) = \delta x^a(t)$ and $x^a(\alpha+1)(t) = \frac{d}{dt} x^a(\alpha)(t)$, $\delta x^a(\alpha+1)(t) = \frac{d}{dt} \delta x^a(\alpha)(t)$, for $\alpha = 0, \ldots, k - 1$. Let the momentum $M_{L,\gamma}(t)$ along the trajectory $\gamma(t)$ be locally
Thus given by \( \left( x^{\alpha,(\alpha)}(t), \left( \frac{k-1}{\alpha} \right)^{-1} p_{a,(k-1-\alpha)}(t) \right) \). Thus, by (2.7) \( \varepsilon_{k-1}^{-1} (M_{L,\gamma}(t)) \sim (x^{\alpha,(\alpha)}(t), p_{a,(\alpha)}(t)) \in T^*T^k M \). Locally

\[
\left\langle M_{L,\gamma}(t), t^{k-1} \delta \gamma(t) \right\rangle_{T^{k-1}T^*} = \left\langle \varepsilon_{k-1}^{-1} (M_{L,\gamma}(t)), \delta t^{k-1} \gamma(t) \right\rangle_{T^{k-1}T^*} M
\]

\[
= \sum_{a=0}^{k-1} \sum_{\alpha=0}^{k} p_{a,(\alpha)}(t) \delta x^{a,(\alpha)}(t).
\]

Thus

\[
\frac{d}{dt} \left\langle M_{L,\gamma}(t), t^{k-1} \delta \gamma(t) \right\rangle_{T^{k-1}T^*} = \sum_{a=0}^{k} \sum_{\alpha=0}^{k} \left( p_{a,(\alpha-1)}(t) + \dot{p}_{a,(\alpha)}(t) \right) \delta x^{a,(\alpha)}(t),
\]

where in the last formula we take \( p_{a,(k)}(t) = p_{a,(1)}(t) = 0 \). The left-hand side of (4.8) equals

\[
\sum_{a=0}^{k} \sum_{\alpha=0}^{k} \frac{\partial L}{\partial x^{a,(\alpha)}} (t^{k} \gamma(t)) \delta x^{a,(\alpha)}(t),
\]

so from (4.11) we get

\[
\sum_{a=0}^{k} \sum_{\alpha=0}^{k} \delta x^{a,(\alpha)}(t) \left( p_{a,(\alpha-1)}(t) + \dot{p}_{a,(\alpha)}(t) \right) = \sum_{a=0}^{k} \sum_{\alpha=0}^{k} \frac{\partial L}{\partial x^{a,(\alpha)}} (t^{k} \gamma(t)) \delta x^{a,(\alpha)}(t) - \sum_{a=0}^{k} F_{L,\gamma}(t) \delta x^{a,(0)}(t).
\]

Since at a fixed time \( t \) the variation \( \delta x^{a,(\alpha)}(t) \) can be arbitrary, above equation splits into the following set of linear equations

\[
p_{a,(k-1)}(t) = \frac{\partial L}{\partial x^{a,(k-1)}} (t^{k} \gamma(t)),
\]

\[
\dot{p}_{a,(k-1)}(t) + p_{a,(k-2)}(t) = \frac{\partial L}{\partial x^{a,(k-2)}} (t^{k} \gamma(t)),
\]

\[
\vdots
\]

\[
\dot{p}_{a,(1)}(t) + p_{a,(0)}(t) = \frac{\partial L}{\partial x^{a,(1)}} (t^{k} \gamma(t)),
\]

\[
\dot{p}_{a,(0)}(t) = \frac{\partial L}{\partial x^{a,(0)}} (t^{k} \gamma(t)) - F_{L,\gamma}(t),
\]

whose unique solution is

\[
p_{a,(\alpha)}(t) = \sum_{\beta=0}^{k-1-\alpha} (-1)^{\beta} \frac{d^\beta}{dt^\beta} \left( \frac{\partial L}{\partial x^{a,(\alpha+\beta+1)}} (t^{k} \gamma(t)) \right), \quad \text{for } \alpha = 0, \ldots, k-1;
\]

(4.12)

i.e., the well-known formula for momenta.

5. Examples and perspectives.

Tulczyjew’s approach to higher-order geometric mechanics. The problem of geometric formulation of higher-order variational calculus on a manifold \( M \) has a few solutions. The first approach is due to Tulczyjew [20, 21], who gave a geometric construction of a map \( \mathcal{E} : \text{Sec}(T^*T^\infty M) \rightarrow \text{Sec}(T^*T^\infty M) \) such that \( \mathcal{E}(d L) = 0 \) are the higher-order Euler-Lagrange equations for any Lagrangian function \( L : T^k M \rightarrow \mathbb{R} \). Tulczyjew expressed his construction using the language of derivations and infinite jets. The latter allowed him to cover all orders \( k \) by a
unique operator. Later Tulczyjew’s theory was interestingly extended by Crampin, Sarlet and Cantrijn [4].

Another approach was inspired by Tulczyjew’s papers [22, 23] on the first order mechanics. The idea was to generate the EL equations from a Lagrangian submanifold. Two similar solutions were given by Crampin [3] and de Leon and Lacomba [14]. They constructed the equations from a Lagrangian submanifold in $T^k M$ or $TT^k M$ generated by the Lagrangian $L$.

All these solutions have, however, some drawbacks. First of all, they describe only a part of the Lagrangian formalism, namely the EL equations, whereas the full structure of variational calculus should contain also momenta (boundary terms). Secondly, the correctness of these constructions is checked in coordinates. Note, however, that it is not the coordinate expression that defines the EL equations, but the opposite: we deduce the right local expression from the proper variational principle. Therefore a fully satisfactory geometric construction should somehow explain the steps performed while deriving the known form of the equations (as it is in our approach described in the previous Section 4), not give a black-box answer.

In later years Tulczyjew [24] extended his work to give a full description of higher-order Lagrangian formalism (i.e., including momenta) in the language of derivations. Another approach was communicated to us by Grabowska [6], who derived the higher-order Lagrangian formalism (i.e., including momenta) in the language of secondary calculus in [25]. The values $F_n$ of $d_{F_n}$ on $\Omega^k$ are, to some extent, similar to these of [3, 14], where the canonical inclusion $T^k M \subset TT^k M$ was also used. Another quite general approach to the topic was presented by A.M. Vinogradov and his collaborators in the framework of secondary calculus (see Section 3 of [25] and the references therein).

Below we relate Tulczyjew’s approach to our results from the previous Section 4.

**Comparison with Tulczyjew’s formulas.** In [21, 24] Tulczyjew introduced maps

$$\mathcal{E} := \sum_{n=0}^{k} \frac{(-1)^n}{n!} (\tau_{k+n}^{2k})^* (d_T)^n \iota_{F_n} : \Omega^*(T^k M) \to \Omega^*(T^{2k} M)$$

and

$$\mathcal{P} := \sum_{n=1}^{k} \frac{(-1)^n}{n!} (\tau_{k+n-1}^{2k-1})^* (d_T)^{n-1} \iota_{F_n} : \Omega^*(T^k M) \to \Omega^*(T^{2k-1} M)$$

defined by means of two basic derivations, namely:

(i) the total derivative $d_T : \Omega^*(T^s M) \to \Omega^*(T^{s+1} M)$ being a $d^*$-derivation characterized by $d_T f^{(s)} = f^{(s+1)}$ for $0 \leq s \leq k$, where $f^{(s)}$ denotes the $s$-th lift of a smooth function $f$ on $M$ as defined in (2.1), and

(ii) the $i^*$-derivations $\iota_{F_n} : \Omega^*(T^k M) \to \Omega^*(T^k M)$, associated with the canonical (nilpotent) endomorphism $F_n : TT^k M \to TT^k M$ of the tangent bundle,

$$F_n (\sum_{s=0}^{k} \gamma(s, t) t^s) := \sum_{s=0}^{k} \gamma(s, t) t^s$$.

Recall that a derivation $a$ of the algebra of differential forms $\Omega^*(M)$ is called a $d^*$-derivation (resp. $i^*$-derivation) if $a$ commutes with the de Rham differential (resp. if $a$ vanishes on $\Omega^0(M) = \mathcal{C}^\infty(M)$) [21], see also [13], chapter 8). As the algebra $\Omega^*(M)$ is generated by $\Omega^0(M)$ and $\Omega^1(M)$, hence a $d^*$-derivation (resp. $i^*$-derivation) is fully determined by its values on $\Omega^0(M)$ (resp. $\Omega^1(M)$). The values of $d_T$ on $\Omega^0(T^k M)$ are given in the above characterization of $d_T$, while for $\iota_{F_n}$ one
defines \( \langle u, F_n v \rangle := \langle \mu, F_n v \rangle \) for a 1-form \( \mu \) and a tangent vector \( v \) on \( T^k M \). Note that \( F_n = F_1^n \) and \( F_1 \) is the canonical higher almost tangent structure on \( T^k M \) [15].

It turns out that the form \( \mathcal{E}(d L) \in \Omega^1(T^{2k} M) \) is vertical with respect to the projection \( \tau^{2k} : T^{2k} M \to M \) and that the form \( \mathcal{P}(d L) \in \Omega^1(T^{2k-1} M) \) is vertical with respect to \( \tau^{2k-1}_{k-1} : T^{2k-1} M \to T^{k-1} M \). Therefore taking the appropriate vertical parts of these forms we can define operators

\[
\mathcal{E}L : T^{2k} M \to T^* M \quad \text{and} \quad \mathcal{P}L : T^{2k-1} M \to T^* T^{k-1} M.
\]

Formulas \( \mathcal{E}L(t^{2k} \gamma) = \Upsilon_{k, \tau^*} \left( \varepsilon k\Lambda_L (t^{k} \gamma) \right) \) and

\[
\mathcal{P}L(t^{2k-1} \gamma) = \varepsilon \varepsilon_{k-1} \left( (u_{k-1, \varepsilon^*} (t^{k-1} \varepsilon \Lambda_L (t^{k} \gamma))) \right)
\]

relate Tulczyjew’s constructions to ours.

**Applications to mechanics on algebroids.** In [10] we showed that with every almost-Lie algebroid structure on the bundle \( \sigma : E \to M \), one can canonically associate an infinite tower of graded bundles

\[
\ldots \to E^k \to E^{k-1} \to \ldots \to E^1 = E \quad (5.1)
\]

equipped with a family of graded-bundle relations

\( \varsigma_k : T^k E \to T^k \).

The relation \( \varsigma_k \) is of special kind – it is, namely, a dual of a vector bundle morphism \( \varepsilon_k : T^* E^k \to T^k E^* \). A natural example of such a structure is provided by a higher tangent bundle \( E^k = T^k M \) together with the canonical flip \( \varsigma_k : T^k TM \to T^k M \).

Another example is \( E^k := T^k G \), the higher tangent space at the identity \( e \in G \) of a Lie group \( G \). Both examples should be considered as the extreme cases of what should we call a *higher algebroid*. Except for \( k = 1 \) there is no Lie bracket on sections of \( E^k \). It is the relation \( \varsigma_k \) which is responsible for the algebraic structure on \( E^k \). More general examples can be obtained by reducing higher tangent bundles of Lie groupoids.

Given a smooth function \( L : E^k \to \mathbb{R} \) one can naturally define a variational problem on \( E^k \). Such problems cover, on one hand, the standard variational problems like Problem 4.1 (in which case \( E^k = T^k M \)), and, on the other hand, the reductions of invariant higher-order variational problems on Lie groupoids. We showed in [10] that for such problems an analog of Theorem 4.2 holds as well. Thus, we can characterize the variation of an action by means of the force \( \Upsilon_{k, \sigma^*} \left( (t^{k} \varepsilon \Lambda_L (d L(t^{k} \gamma(t)))) \right) \) and momentum \( v_{k-1, \sigma^*} \left( (t^{k-1} \tau_{k-1}^* \varepsilon \Lambda_L (d L(t^{k} \gamma(t)))) \right) \), where now \( \varepsilon_k : T^* E^k \to T^k E^* \) is the dual of the relation \( \varsigma_k \), \( \tau_{k-1}^* : E^k \to E^{k-1} \) is the tower projection (5.1), \( \sigma^* : E^* \to M \) is the dual of \( \sigma \), while \( \Upsilon_{k, \sigma^*} \) and \( v_{k, \sigma^*} \) are the maps introduced in Section 3.

**Example: Riemannian cubic polynomials.** Let us consider one of the simplest, but interesting, second-order variational problem: given an integer \( n \geq 2 \) and points \( a = x_1 < x_2 < \ldots < x_n = b \) on the real line \( \mathbb{R} \), and values \( y_1, y_2, \ldots, y_n, v_a, v_b \in \mathbb{R} \), find an \( f \in C^2([a, b]) \) such that \( f(x_i) = y_i \) for \( 1 \leq i \leq n \) and \( f^{(i)}(a) = v_a, f^{(i)}(b) = v_b \) which minimizes the integral \( \int_a^b f''(x)^2 dx \). This problem has a unique solution, called a *complete cubic spline* [1], which is a piece-wise cubic polynomial \( P \) determined uniquely by the following properties: it is a polynomial of degree \( \leq 3 \) on each interval \( [x_i, x_{i+1}] \), \( 1 \leq i \leq n - 1 \), \( P'(a) = v_a, P'(b) = v_b \), and it has continuous second derivatives at each “slope” \( x_2, \ldots, x_{n-1} \). Note that the EL equations (4.11) read as \( f^{(i)} = 0 \), i.e., \( f \) is locally a polynomial of degree \( \leq 3 \).
Following [18] we define a smooth curve \( \gamma : \mathbb{R} \to M \), denote by \( D_t \) the covariant derivative \( \nabla_{\dot{\gamma}}(t) \) along \( \gamma \). Following [18] we define

\[
L(t^2)_{\dot{\gamma}(t)} := g(\gamma(0), D_t|_{t=0} \dot{\gamma}(t), D_t|_{t=0} \dot{\gamma}(t)).
\]

(5.2)

Locally, for \( \gamma(t) \sim (x^a(t)) \),

\[
D_t|_{t=0} \dot{\gamma}(t) = (\ddot{x}^c + \Gamma^c_{ab}(x) \dot{x}^a \dot{x}^b) \partial_{x^c},
\]

where \( \Gamma^c_{ab}(x) \) are the Christoffel symbols of the metric \( g \) and \( \dot{x}^a \) are the second-order derivatives of \( x^a(t) \) at \( t = 0 \). Therefore (5.2) is indeed a function on \( T^2M \).

We shall now compute the second-order EL equations associated with the Lagrangian \( L \) given by (5.2). To this end, recall two fundamental properties of the Levi-Civita connection:

\[
\nabla_X Y - \nabla_Y X = [X, Y],
\]

(5.3)

\[
Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),
\]

(5.4)

which hold for any vector fields \( X, Y, Z \) on the manifold \( M \).

Note that although the Lie bracket \( [X, Y] \) is defined for vector fields, to calculate its value at a point \( p \in M \) it is enough to know vectors \( X(Y)(p) := TY \circ X(p) \in T_YM \) and \( Y(X)(p) \in T_XM \) (see, e.g., [13]). Accordin, we introduce the following notion: for a vector \( X \in T_pM \) and a vector \( A \in T_XM \) lying over \( Y := T\gamma_M(A) \) by an \( A \)-extension of \( X \) around \( p \) we will understand any (local) vector field \( \tilde{X} \) on \( M \) such that \( \tilde{X}(p) = X \) and \( Y(X)(p) = A \). This means that \( A \) is tangent to the graph of \( \tilde{X} \) at \( X \).

Consider now a curve \( \gamma(t) \in M \) and any generator \( \delta \gamma(t) \in T_{\gamma(t)}M \) of an admissible variation \( \delta t^2 \gamma(t) = \kappa_2(t^2 \delta \gamma(t)) \). Let \( \tilde{\gamma} \) be any \( \delta(t^1 \gamma)(t) \)-extension, in the aforementioned sense, of \( t^1 \gamma \) and let \( \delta \gamma \) be any \( t^1 \delta \gamma(t) \)-extension of \( \delta \gamma(t) \) along \( \gamma(t) \) (in particular \( \delta \gamma = \delta \gamma \) along \( \gamma(t) \)). It follows immediately from [13] (or [10], Proposition 2.2) that

\[
[\delta \gamma, \tilde{\gamma}] = 0 \quad \text{along} \quad \gamma(t).
\]

(5.5)

We shall now compute the differential of the action \( S_L \) in the direction of \( \delta t^2 \gamma(t) \). Note that \( g(\nabla_{\tilde{\gamma}} \tilde{\gamma}, \nabla_\gamma \tilde{\gamma}) \) is a function on \( M \) coinciding with \( L(t^2 \gamma) = g(\nabla_{\dot{\gamma}} \dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \) along \( \gamma(t) \), hence

\[
\langle dS_L(t^2 \gamma), \delta t^2 \gamma \rangle = \int_{t_0}^{t_1} \delta \gamma g(\nabla_{\dot{\gamma}} \dot{\gamma}, \nabla_\gamma \tilde{\gamma}) \, dt \quad (5.4)
\]

\[
= 2 \int_{t_0}^{t_1} g(\nabla_{\delta \gamma} \nabla_\gamma \tilde{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \, dt + 2 \int_{t_0}^{t_1} g(R(\delta \gamma, \dot{\gamma}) \dot{\gamma}, \nabla_\gamma \tilde{\gamma}) + g(\nabla_\gamma \nabla_{\delta \gamma} \tilde{\gamma} + \nabla_{[\delta \gamma, \dot{\gamma}]} \tilde{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \, dt,
\]

where \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \) is the Riemann tensor of \( g \). By (5.5) and by the standard symmetry property \( g(R(X, Y)Z, T) = g(R(T, Z)Y, X) \), the later equals

\[
2 \int_{t_0}^{t_1} g(R(\nabla_{\dot{\gamma}} \dot{\gamma}) \dot{\gamma}, \delta \gamma) + g(\nabla_\gamma \nabla_{\delta \gamma} \tilde{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) \, dt.
\]
Let us transform the last integrand as follows
\[
g \left( \nabla_\gamma \nabla_\delta \tilde{\gamma}, \nabla_\gamma \tilde{\gamma} \right) = g \left( \nabla_\gamma \nabla_\delta \tilde{\gamma}, \nabla_\gamma \tilde{\gamma} \right) = \frac{1}{2} \left( \gamma g \left( \nabla_\gamma \delta \gamma, \nabla_\gamma \tilde{\gamma} \right) - g \left( \nabla_\gamma \delta \gamma, \nabla_\gamma \nabla_\gamma \tilde{\gamma} \right) \right)
\]
\[
= \frac{1}{2} \left( \gamma g \left( \nabla_\gamma \delta \gamma, \nabla_\gamma \tilde{\gamma} \right) - g \left( \delta \gamma, \nabla_\gamma \nabla_\gamma \tilde{\gamma} \right) + g \left( \delta \gamma, \nabla_\gamma \tilde{\gamma} \right) \right).
\]
This gives us
\[
\langle d S_L(t^2 \gamma), \delta t^2 \gamma \rangle = \frac{1}{2} \int_{t_0}^{t_1} g \left( R(\nabla_\gamma \tilde{\gamma}, \tilde{\gamma} \gamma) + \nabla_\gamma \tilde{\gamma} \nabla_\gamma \tilde{\gamma}, \delta \gamma \right) dt + 2 \left[ g \left( \nabla_\gamma \delta \gamma, \nabla_\gamma \tilde{\gamma} \right) - g \left( \delta \gamma, \nabla_\gamma \nabla_\gamma \tilde{\gamma} \right) \right] \bigg|_{t_0}^{t_1}.
\]
We see that the boundary term depends only on \( t^1 \delta \gamma \), hence comparing the last expression with (4.5) we get that \( \mathcal{T}_{2,t} \left( t^2 A_L(t^2 \gamma) \right) = g \left( R(\nabla_\gamma \tilde{\gamma}, \tilde{\gamma} \gamma) + \nabla_\gamma \tilde{\gamma} \nabla_\gamma \tilde{\gamma}, \gamma \right) \). Consequently, the EL equations for \( L \) read
\[
D^2 \gamma(t) + R(D \gamma(t), \gamma(t)) \gamma(t) = 0, \tag{5.6}
\]
in agreement with [18]. Solutions of (5.6) are called Riemannian cubic polynomials.

**Appendix A. Proof of Lemma 3.4.**

Let us explain that the functoriality of a vector bundle morphism \( \overline{T}^{k,k} E \to E \) means that actually we have a family \( F = \{ F_E \} \) of vector bundle morphisms as in (3.12), parameterized by vector bundles \( \sigma : E \to M \), such that for any morphism \( f : E_1 \to E_2 \) between vector bundles \( \sigma_i : E_i \to M_i, i = 1,2 \), we have
\[
F_{E_2} \circ \overline{T}^{k,k} f = f \circ F_{E_1}. \tag{A.1}
\]

We shall derive the local coordinate form of \( F_E \). Observe first that the base map of \( F_E \), denoted by \( F_E : T^{k,k} M \to M \), has to be functorial as well. In other words \( F_E \) is a natural transformation between Weil functors \( T^{k,k} \) and \( \text{Id} \). It follows from [12] that such a transformation corresponds to a unique homomorphism between the corresponding Weil algebras, namely \( h : \mathbb{D}^{2k} \to \mathbb{R}[\nu]/\langle \nu^{2k+1} \rangle \to \mathbb{R} \) given by \( 1 \mapsto 1 \) and \( \nu \mapsto 0 \). We conclude that \( F_E = \tau^{2k} \) is the canonical bundle projection.

Every vector bundle can be locally described as \( E = M \times V \), where \( M \) is the base and \( V \) is the model fiber. In this case \( T^{k,k} E = T^{2k} M \times T^{k} T^{k} V \) and hence \( F_E \) must be of the form \( F_E(v^{2k}, X) = (\tau^{2k}(v^{2k}), L_{v^{2k}}(X)) \), where \( L_{v^{2k}} : T^{k} T^{k} V \to V \) is a linear map, \( v^{2k} \in T^{2k} M \) and \( X \in T^{k} T^{k} V \). Consider now a morphisms of the form \( f = f_0 \times \text{id}_V : M \times V \to M \times V \), where \( f_0 : M \to M \) is a smooth function. It follows from (A.1) that \( L_{v^{2k}} \) does not depend on \( v^{2k} \) and hence, locally, \( F_E \) is of the form
\[
F_E(v^{2k}, X) = (\tau^{2k}(v^{2k}), L(X)),
\]
where \( L : T^{k} T^{k} V \to V \) is a linear map.

Now we shall find the coordinate form of \( L \). Since every vector space \( V \) is a vector bundle over a one-point base, it follows that \( F \) induces a functorial morphism \( F_V : T^{k,k} V \to T^{k} T^{k} V \). But \( T^{k} T^{k} V \simeq D^{k,k} \otimes V \), canonically, where \( D^{k,k} \) denotes the Weil algebra \( D^{k,k} = \mathbb{R}[\nu, \nu']/\langle \nu^{k+1}, \nu'^{k+1} \rangle \). Any functorial linear map \( D^{k,k} \otimes V \to V \) is of the form \( f \otimes \text{id}_V \) for some fixed linear map \( f : D^{k,k} \to \mathbb{R} \). If we
denote \( c_{\alpha\beta} := f(\nu^\alpha \nu^\beta) \), then we conclude that general local form of \( F_E : \tilde{T}^{k,k}E \to E \) is
\[
F_E \left( x^{\alpha(0)}, y^{(\alpha,\beta)} \right) = \left( x^\alpha = x^{\alpha(0)}, y^i = \sum_{0 \leq \alpha, \beta \leq k} c_{\alpha\beta} y^{i(\alpha,\beta)} \right),
\]
where \((x^{\alpha(r)}, y^{(\alpha,\beta)})\) are the adapted coordinates on \( \tilde{T}^{k,k}E \) (as defined in Preliminaries) induced from the standard coordinates \((x^\alpha, y^i)\) on \( E \) and we have underlined the coordinates in the co-domain.

To get more information on the coefficients \( c_{\alpha\beta} \) it will be enough to consider the case \( E = M \times \mathbb{R} \) with \( V = \mathbb{R} \). Consider the vector bundle morphism by \( \tilde{\phi} : M \times \mathbb{R} \to M \times \mathbb{R} \) given by \( \tilde{\phi}(x, y) = (x, \phi(x) \cdot y) \), where \( \phi \in C^\infty(M) \). We are looking for possible \( F_E \) such that the following diagram
\[
\xymatrix{ \tilde{T}^{k,k}E \ar[r]^{F_E} \ar[d]_{\tilde{\phi}} & E \ar[d]^{\phi} \\
\tilde{T}^{k,k}E \ar[r]^{F_E} & E }
\]
commutes. In our case, \( \tilde{T}^{k,k}E = T^{2k} \mathbb{R} \times T^{k}k \mathbb{R} \) and the commutativity of (A.2) reads as
\[
\sum_{0 \leq \alpha, \beta \leq k} c_{\alpha\beta} y^{(\alpha,\beta)} = \phi(x) \cdot \sum_{0 \leq \alpha, \beta \leq k} c_{\alpha\beta} y^{(\alpha,\beta)},
\]
where \((x^{(r)}, y^{(\alpha,\beta)}) \mapsto (x^{(r)}, y^{(\alpha,\beta)})\) is the coordinate expression of the morphism \( \tilde{\phi} \). To obtain the coordinate expression of \( T^k \mathbb{R} \times T^k \mathbb{R} : \tilde{T}^{k,k}E \to T^{k,k}E \), for \( E = M \times \mathbb{R} \) and \( M = \mathbb{R} \), which sends the class \([\gamma]\) of a map \( \gamma : \mathbb{R} \times \mathbb{R} \to E \) to the class \([\tilde{\phi} \circ \gamma]\) in \( T^{k,k}E \), we write
\[
x(\tilde{\phi}(\gamma(s,t))) = x(\gamma(s,t)) = \sum_{0 \leq \alpha, \beta \leq k} x^{(\alpha,\beta)}([\gamma]) \frac{s^\alpha t^\beta}{\alpha! \beta!} + o(s^k, t^k),
\]
\[
y(\tilde{\phi}(\gamma(s,t))) = y(x(s,t)) y(\gamma(s,t)) = \left( \phi(x) + \sum_{r=1}^\infty \frac{\phi^{(r)}(x)}{r!} h^r + o(h^k) \right) \left( \sum_{0 \leq \alpha, \beta \leq k} y^{(\alpha,\beta)}([\gamma]) \frac{s^\alpha t^\beta}{\alpha! \beta!} + o(s^k, t^k) \right),
\]
where \( x = x(\gamma(0,0)) = x^{(0,0)}([\gamma]) \) and \( h = x(\gamma(s,t)) - x(\gamma(0,0)) \) as in (A.4). Clearly, \( y^{(\alpha,\beta)} \) is the coefficient of \( s^\alpha t^\beta /\alpha! \beta! \) in \( y(\tilde{\phi}(\gamma(s,t))) \). For example, in case \( k = 2 \) we find that
\[
y^{(2,1)} = y^{(2,1)}(\phi(x)) + 2y^{(1,1)}x^{(1,0)} \phi'(x) + y^{(2,0)}x^{(0,1)} \phi'(x) + \ldots
\]
For any \( l, m \geq 0 \), \( T^l T^m E \) is a three-fold graded bundle with bases \( E, T^l E \) and \( T^m E \) [8]. Its algebra of multi-homogeneous functions, which is a subalgebra of all smooth functions on \( T^l T^m E \), is \( \mathbb{Z}^3 \)-graded. For example, \( y^{(\alpha,\beta)} \) is of degree 1 with respect to the vector bundle structure over \( T^l T^m M \) and of degrees \( \alpha \) and \( \beta \) with respect to the bundle structures over \( T^m E \) and \( T^l E \), respectively. Of course, \( T^k \mathbb{R} \tilde{\phi} \) preserves this grading. Restricting \( T^k T^k \tilde{\phi} \) to \( \tilde{T}^{k,k}E \) means just replacing \( x^{(\alpha,\beta)} \) with \( x^{(\alpha+\beta)} \).

Thus we get
\[
y^{(\alpha,\beta)} = y^{(\alpha,\beta)}(\phi(x)) + \alpha y^{(\alpha-1,\beta)}x^{(1)} \phi'(x) + \beta y^{(\alpha,\beta-1)}x^{(1)} \phi'(x) + \ldots
\]
where, in case $\alpha = 0$ or $\beta = 0$, it is enough to put 0 instead of $y^{(-1,\beta)}$ or $y^{(\alpha,-1)}$. By comparing the $\phi'(x)$ coefficients in the above expression and separating the terms with respect to the gradation we find that the necessary condition for (A.3) is

$$\sum_{0 \leq \alpha, \beta \leq k, \alpha + \beta = s} c_{\alpha \beta} (\alpha y^{(\alpha-1,\beta)} + \beta y^{(\alpha,\beta-1)}) = 0,$$

for any $0 \leq s \leq 2k$, with the convention that $y^{(\alpha,\beta)} = 0$ whenever $\alpha$ or $\beta$ is negative or greater than $k$. This gives a recurrence relation between the coefficients $c_{\alpha \beta}$ when the sum $\alpha + \beta = s$ fixed. Namely, $(\alpha + 1)c_{(\alpha+1)(\beta-1)} + \beta c_{\alpha \beta} = 0$ if $0 \leq \alpha \leq k - 1$ and $1 \leq \beta \leq k$. Moreover, $c_{\alpha k} = 0$ for $1 \leq \alpha \leq k$. It follows that for $0 \leq s \leq k$ the vector $(c_0, c_1(s-1), \ldots, c_s)$ is proportional to $((s)_0, -(1), (2), \ldots, \pm (s))$ and $c_{\alpha \beta} = 0$ whenever $\alpha + \beta > k$. Hence,

$$\sum_{\alpha, \beta} c_{\alpha \beta} y^{(\alpha,\beta)} = \sum_{s=0}^{k} a_s \sum_{\alpha + \beta = s} (-1) \binom{s}{\alpha} y^{(\alpha,\beta)},$$

for some coefficients $a_0, \ldots, a_k \in \mathbb{R}$. Therefore, $F_E$ must have the desired form being a linear combination of morphisms $\Upsilon_{k,\sigma}$ given locally by (3.4). \qed

A slight modification of the proof above shows that $\Upsilon_{k,\sigma}$ has no canonical extension to $T^k T^k E$.

**Corollary A.1.** Any functorial vector bundle morphism

$$\xymatrix{T^k T^k E \ar[r]^{F_E} & E \ar[d] \ar[r]^{F_E} & E}$$

is proportional to $\tau_{(k,k)} : T^k T^k E \to E$ covering $\tau^{(k,k)} : T^k T^k M \to M$. In particular, $\Upsilon_{k,\sigma}$ has, in general, no canonical extension to a functorial vector bundle morphism on $T^k T^k E$.

**Proof.** Just go over the same reasoning of the proof of Lemma 3.4. By considering a trivial vector bundle $E = M \times V$ and endomorphisms of the form $f = f_0 \times \text{id}_V : E \to E$, where $f_0 \in C^\infty(M)$, we find that a general local form of $F_E : T^k T^k E \to E$ has to be

$$F_E(x^{a,(\alpha,\beta)}, y^{i,(\alpha,\beta)}) = (\tilde{x}^a = x^{a,(0,0)}, \tilde{y}^i = \sum_{0 \leq \alpha, \beta \leq k} c_{\alpha \beta} y^{i,(\alpha,\beta)}).$$

In the same way we find that the coefficients $c_{\alpha \beta}$ have to satisfy (A.3), where $(x^{(\alpha,\beta)}, y^{(\alpha,\beta)}) \mapsto (x^{(\alpha,\beta)}, y^{(\alpha,\beta)})$ is the coordinate expression of the morphism $T^k T^k \tilde{\phi}$, where $\tilde{\phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, (x, y) \mapsto (x, \phi(x) \cdot y)$ for some function $\phi \in C^\infty(\mathbb{R})$. In the transformation expression for $y^{(\alpha,\beta)}$ we separate terms with respect to the grading and collect the terms which contain $\phi'(x)$. Then we find easily that (A.3) implies that $c_{\alpha,\beta} = 0$ unless $\alpha = \beta = 0$. Therefore $F_E = c_{0,0} \cdot \tau^{(k,k)}_{E}$, as claimed. \qed

**Acknowledgments.** This research was supported by Polish Ministry of Science and Higher Education under the grant NN 201416839.

The authors are grateful for professors Paweł Urbański and Janusz Grabowski for reference suggestions. We wish to thank especially the second of them for reading...
the manuscript and giving helpful remarks. We are also grateful for the reviewers for many helpful suggestions.

REFERENCES

[1] C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.
[2] F. Cantrijn, M. Crampin, W. Sarlet and D. Saunders, The canonical isomorphism between $T^sT^tM$ and $T^sT^tM$, *C. R. Acad. Sci. Paris*, 309 (1989), 1509–1514.
[3] M. Crampin, Lagrangian submanifolds and the Euler-Lagrange equations in higher-order mechanics, *Lett. Math. Phys.*, 19 (1990), 53–58.
[4] M. Crampin, W. Sarlet and F. Cantrijn, Higher-order differential equations and higher-order Lagrangian mechanics, *Math. Proc. Cambridge Phillos. Soc.*, 99 (1986), 565–587.
[5] F. Gay-Balmaz, D. D. Holm, D. M. Meier, T. S. Ratiu and F. Vialard, Invariant higher-order variational problems, *Comm. Math. Phys.*, 309 (2012), 413–458.
[6] K. Grabowska, private communication, 2012.
[7] K. Grabowska and J. Grabowski, Variational calculus with constraints on general algebroids, *J. Phys. A: Math. Theor.*, 41 (2008), 175204.
[8] J. Grabowski and M. Rotkiewicz, Graded bundles and homogeneity structures, *J. Geom. Phys.*, 62 (2011), 21–36.
[9] X. Gracia, J. Martin-Solano and M. Munoz-Lecenda, Some geometric aspects of variational calculus in constrained systems, *Rep. Math. Phys.*, 51 (2003), 127–148.
[10] M. Jóźwikowski and M. Rotkiewicz, Prototypes of higher algebroids with application to variational calculus, preprint arXiv:1306.3379.
[11] M. Jóźwikowski and W. Respondek, A comparison of vakonomic and nonholonomic variational problems with applications to systems on Lie groups, preprint arXiv:1310.8528.
[12] I. Kolar, Weil bundles as generalized jet spaces, in *Handbook of Global Analysis*, Elsevier Sci. B. V., Amsterdam, 1214 (2008), 625–664.
[13] I. Kolar, P. W. Michor and J. Slovak, *Natural Operations in Differential Geometry*, Springer, Berlin, 1993.
[14] M. de Leon and E. Lacomba, Lagrangian submanifolds and higher-order mechanical systems, *J. Phys. A*, 22 (1989), 3809–3820.
[15] M. de Leon and P. R. Rodrigues, Higher order almost tangent geometry and non-autonomous Lagrangian dynamics, in *Proceedings of the Winter School ‘Geometry and Physics’*, Circolo Matematico di Palermo, Palermo (1987), 157–171.
[16] K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, CUP, Cambridge, 2005.
[17] A. Morimoto, Liftings of tensor fields and connections to tangent bundles of higher order, *Nagoya Math. J.*, 40 (1970), 99–120.
[18] L. Noakes, G. Heinzinger and B. Paden, Cubic splines on curved surfaces, *IMA J. Math. Control Inform.*, 6 (1989), 465–473.
[19] D. J. Saunders, *The Geometry of Jet Bundles*, Lecture Notes Math., 142, CUP, 1989.
[20] W. Tulczyjew, Sur la différentielle de Lagrange, *C. R. Acad. Sci. Paris Serie A*, 280 (1975), 1205–1208.
[21] W. Tulczyjew, The Lagrange differential, *Bull. Acad. Polon. Sci.*, 24 (1976), 1089–1096.
[22] W. Tulczyjew, Les sous-variétés lagrangiennes et la dynamique hamiltonienne, *C. R. Acad. Sci. Paris Serie A*, 283 (1976), 15–18.
[23] W. Tulczyjew, Les sous-variétés lagrangiennes et la dynamique lagrangienne, *C. R. Acad. Sci. Paris*, 283 (1976), 675–678.
[24] W. Tulczyjew, Evolution of Ehresmann’s jet theory, in *Geometry and topology of manifolds: The mathematical legacy of Charles Ehresmann*, Banach Centre Publications, 76, Warsaw, 2007, 159–176.
[25] L. Vitagliano, The Lagrangian-Hamiltonian formalism for higher-order field theories, *J. Geom. Phys.*, 60 (2010), 857–873.
[26] A. Weil, Théorie des points proches sur les varietes différentiables, in *Colloque de géometrie différentielle*, CNRS, Strasbourg (1953), 111–117.

Received August 2013; revised February 2014.

E-mail address: mjozwikowski@gmail.com
E-mail address: mrotkiew@mimuw.edu.pl