The Effectiveness of Johnson-Lindenstrauss Transform for High Dimensional Optimization with Outliers

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Abstract

Johnson-Lindenstrauess (JL) Transform is one of the most popular methods for dimension reduction. In this paper, we study the effectiveness of JL transform for solving the high dimensional optimization problems with outliers. We focus on two fundamental optimization problems: k-center clustering with outliers and SVM with outliers. In general, the time complexity for dealing with outliers in high dimensional space could be very large. Based on some novel insights in geometry, we prove that the complexities of these two problems can be significantly reduced through JL transform. In the experiments, we compare JL transform with several other well known dimension reduction methods, and study their performances on synthetic and real datasets.

1. Introduction

Johnson-Lindenstrauss (JL) Transform (also called “random projection method”) is a popular tool for solving high-dimensional problems and has been extensively studied in different areas, such as optimization (Sarlos, 2006), compressive sensing (Baraniuk et al., 2006), and privacy preserving (Blocki et al., 2012). Comparing with the data-aware dimension reduction techniques (e.g., PCA and feature selection) (Cunningham & Ghahramani, 2015), JL transform is a data-oblivious technique that is more convenient to implement in practice, especially for distributed computing and streaming data. In general, given a set of \( n \) points in a \( d \)-dimensional Euclidean space, the well-known JL-Lemma shows that only \( O(\frac{\log n}{\epsilon^2}) \) dimensions, which is independent of \( d \), are sufficient to approximately preserve their pairwise distances.

Lemma 1 (Johnson & Lindenstrauss, 1984). Given \( \epsilon > 0 \),

\[ n \in \mathbb{Z}^+, \text{ and } k = O(\frac{\log n}{\epsilon^2}), \text{ for any set } P \text{ of } n \text{ points in } \mathbb{R}^d, \text{ there exists a mapping } f: \mathbb{R}^d \to \mathbb{R}^k \text{ such that for all } p, q \in P, \]

\[ ||p - q||^2 - ||f(p) - f(q)||^2 \leq \epsilon ||p - q||^2. \] (1)

Here, we use \( ||p - q|| \) to denote the Euclidean distance between two points \( p \) and \( q \). The mapping \( f \) in Lemma 1 is called the “JL transform”. Very recently, (Bartal et al., 2019) studied JL transform on several practical measures other than (1) in particular for the machine learning problems.

In the past decades, a large amount of articles focus on the construction methods of the mapping \( f \). Actually, a simple way is to build a \( k \times d \) matrix \( A \) where each entry is an independent Gaussian \( N(0, 1) \) random variable (Dasgupta & Gupta, 2003). (Achlioptas, 2003) proposed two different construction methods with entries belonging in \( \{0, \pm 1\} \); the implementation becomes easier especially for using SQL in a database environment. Inspired by Heisenbergs Uncertainty Principle, (Ailon & Chazelle, 2009) provided a faster implementation of JL transform by applying Walsh-Hadamard matrix. Further, a number of improvements on the sparsity of the matrix \( A \) have been proposed, such as (Kane & Nelson, 2014; Dasgupta et al., 2010).

1.1. Our Contributions

In this paper, we study the effectiveness of JL transform in particular for the optimization problems with outliers in high dimensions. Our motivation for considering outliers is twofold. (i) In this big data era, we apply machine learning methods to learn from data but the obtained dataset often contains a significant amount of outliers (Beyer & Sendhoff, 2007; Zimek et al., 2012; Moitra, 2018). Moreover, the data quality plays an important role that determines the final learning result to a great extent. (ii) As the rapid development of machine learning, the field of adversarial machine learning concerning about the potential vulnerabilities of the algorithms has attracted a great amount attentions (Huang et al., 2011; Kurakin et al., 2016; Biggio & Roli, 2018; Goodfellow et al., 2018). For example, a small set of outliers could be added by some adversarial attacker to make the decision boundary severely deviate and cause
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unexpected mis-classification (Biggio et al., 2012; Jagielski et al., 2018). Therefore, designing robust machine learning algorithms are urgently needed to meet these challenges. We consider two fundamental optimization problems in high dimensions: k-center clustering with outliers and support vector machine (SVM) with outliers.

However, existing methods for the problems of k-center clustering with outliers and SVM with outliers often have high complexities (more related works are shown in Section 2.2). In the past years, JL transform has been widely studied for reducing the time complexities for the clustering (Kerber & Raghvendra, 2014; Boutsidis et al., 2014; Cohen et al., 2015; Makarychev et al., 2019) and SVM (Arriaga & Vempala, 2006; Balcan et al., 2006; Kumar et al., 2008; Shi et al., 2012; Paul et al., 2014) problems. But its effectiveness for solving the case with outliers has never been studied, to the best of our knowledge.

To shed some light on our idea, consider k-center clustering with outliers as an example. Informally speaking, the problem is to find k balls to cover (1 − γ)n points of a given set of n points in \( \mathbb{R}^d \), where \( \gamma \in (0, 1) \), and minimize the maximum radius of the balls; the remaining \( \gamma n \) points are recognized as outliers. It is in fact a quite challenging combinatorial optimization problem, since there are an exponentially large number \( \binom{n}{\gamma n} \) of different possible cases. Moreover, if the instance is located in a high-dimensional space, the complexity will be even larger. To reduce the dimensionality, the main challenge is that the ball centers could appear anywhere in the space and hence the JL Lemma cannot directly preserve the resulting radius (note JL transform only preserves the \( \binom{n}{2} \) pairwise distances). Thus, the key is to prove that a good solution in the dimensionality-reduced space, in particular when outliers exist, can be efficiently mapped back to the original \( \mathbb{R}^d \) while the radius is still preserved.

Note this “mapping back” step is important, especially for clustering/classifying new coming data or extracting other useful information from the original space (Zhang et al., 2013). We also emphasize that most of the previous articles on using JL transform (even only for the case without outliers, e.g., (Balcan et al., 2006; Shi et al., 2012; Paul et al., 2014)) are in lack of the discussion on the mapping back step.

It is known that the “coreset” exists for k-center clustering (Badoiu et al., 2002). Namely, a small subset of the input data having size \( O(\frac{1}{\epsilon^2}) \) can approximately preserve the k balls (we refer the reader to the recent surveys (Munteanu & Schwiegelshohn, 2018; Feldman, 2020) for more details on coresets). Thus there are at most \( O\left(\binom{n}{0.5} \right) \) possible different coresets of a given instance (note that we cannot directly obtain the coreset due to the existence of outliers). Imagine to build a uniform dense enough grid inside the convex hull of each possible coreset and apply JL transform to the set of grid points. Through triangle inequality and the fact that JL transform is linear, we can solve the problem in the dimensionality-reduced space and achieve a good approximation in the original \( \mathbb{R}^d \). Actually, a similar idea has been studied by (Kerber & Raghvendra, 2014), and they proved that the resulting reduced dimensionality is at least

\[
\Omega\left(\frac{1}{\epsilon^3} \log \frac{1}{\epsilon} \log n\right).
\]

Based on some novel observations in geometry, we prove that the dimensionality actually can be further reduced to \( O\left(\frac{1}{\epsilon^2} \log n\right) \) that exactly matches the dimensionality in Lemma 1 and significantly improves the bound of (2). More importantly, it is a unified result that can speed up any existing k-center clustering with outliers algorithm. Suppose \( A \) is a given algorithm with running time \( G(n, d) \) for an instance of \( n \) points in \( \mathbb{R}^d \). For example, the algorithm (Charikar et al., 2001) has the running time \( G(n, d) = \Omega(n^2d) \), since it needs to read all the \( \binom{n}{2} \) pairwise distances. Then our result indicates that the running time can be reduced to be

\[
G(n, O(\frac{1}{\epsilon^2} \log n)) + O\left(\frac{k}{\epsilon} \cdot (n \cdot \frac{1}{\epsilon^2} \log n + d)\right) + \Gamma(n, d, \epsilon),
\]

where the term \( O\left(\frac{k}{\epsilon} \cdot (n \cdot \frac{1}{\epsilon^2} \log n + d)\right) \) is the time complexity for mapping the solution from \( \mathbb{R}^{O(\frac{1}{\epsilon^2} \log n)} \) to \( \mathbb{R}^d \), and \( \Gamma(n, d, \epsilon) \) is the time complexity of JL transform. In practice, the last two terms of (3) usually are negligible and much lower than the first part \( G(n, O(\frac{1}{\epsilon^2} \log n)) \).

Moreover, another immediate application of our dimension reduction result is to reduce the sample complexity for the uniform sampling based approach for k-center clustering with outliers (Huang et al., 2018; Ding et al., 2019). The details are shown in Remark 3. We also extend this dimension reduction result to the SVM with outliers problem and achieve the similar time complexity as (3). Moreover, this result is independent of any kernel method; namely, we can still reduce the dimensionality when using a kernel trick.

2. Preliminaries

Let \(|A|\) denote the number of points of a given point set \( A \) in \( \mathbb{R}^d \). We use \( B(c, r) \) to denote the ball centered at a point \( c \) with radius \( r > 0 \). Below, we introduce several definitions that will be used in the paper.

**Definition 1 (Minimum Enclosing Ball (MEB)).** Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), the MEB problem is to find a ball with minimum radius to cover all the points in \( P \). The resulting ball and its radius are denoted by \( MEB(P) \) and \( \text{rad}(P) \), respectively.
Definition 2 (k-Center Clustering with Outliers). Given a set $P$ of $n$ points in $\mathbb{R}^d$, $k \in \mathbb{Z}^+$, and $\gamma \in (0, 1)$, $k$-center clustering with outliers is to find a subset $P' \subseteq P$, where $|P'| = (1 - \gamma)n$, and $k$ centers $\{c_1, \cdots, c_k\} \subset \mathbb{R}^d$, such that $\max_{x \in P} \min_{1 \leq j \leq k} ||p - c_j||$ is minimized. If $k = 1$, the problem is also called MEB with outliers.

For the SVM problem, we consider one-class and two-class separately. Let $o$ be the origin of the $\mathbb{R}^d$ space. Given a point $x \in \mathbb{R}^d$, denote by $H_x$ the hyperplane passing through $x$ and being orthogonal to the vector $x - o$.

Definition 3 (One-class SVM with Outliers). Given a set $P$ of $n$ points in $\mathbb{R}^d$ and $\gamma \in (0, 1)$, the problem of one-class SVM with outliers is to find a point $x$ such that $|P \cap H_x^+| = (1 - \gamma)n$ and the (Euclidean) norm $||x||$ is maximized, where $H_x^+$ denotes the closed half space that is bounded by $H_x$ and excludes the origin $o$. The norm $||x||$ is also called the width of the margin separating $o$ and $H_x$.

Definition 4 (Two-class SVM with Outliers). Given two point sets $P_1$ and $P_2$ in $\mathbb{R}^d$ and two small parameters $\gamma_1, \gamma_2 \in (0, 1)$, the problem of two-class SVM with outliers is to find two subsets $P_1' \subset P_1$ and $P_2' \subset P_2$ with $|P_1'| = (1 - \gamma_1)|P_1|$ and $|P_2'| = (1 - \gamma_2)|P_2|$, and a margin separating $P_1'$ and $P_2'$, such that the width of the margin is maximized. The margin is the region bounded by two parallel hyperplanes $H^+$ and $H^-$.

Obviously, to determine the margin for one-class (two-class) SVM, we just need to find the normal vector $x$.

Remark 1. In the above Definition 2, 3, and 4, if setting $\gamma = 0$ ($\gamma_1 = \gamma_2 = 0$), the problems will become the ordinary $k$-center clustering and one-class (two-class) SVM.

Approximation ratio. Given an instance $P$ of MEB or $k$-center clustering with outliers (Definition 1 and 2), denote by $r_{\text{opt}}$ the radius induced by the optimum solution. Then, for any solution having radius $r \leq \lambda r_{\text{opt}}$ with $\lambda \geq 1$, we say that it is a $\lambda$-approximate solution. For the SVM with outliers problems (Definition 3 and 4), denote by $w_{\text{opt}}$ the width of the margin induced by the optimum solution. Then, for any solution having the margin width $w \geq \frac{1}{\lambda}w_{\text{opt}}$ with $\lambda \geq 1$, we say that it is a $\frac{1}{\lambda}$-approximate solution.

2.1. The MEB Algorithm and Gilbert Algorithm

We briefly introduce the MEB algorithm (Badoiu & Clarkson, 2003) (we call it BC’s algorithm) and Gilbert algorithm (Gilbert, 1966) in this section. They will be used as the sub-routines in our main algorithms. In fact, these two algorithms both fall under the umbrella of the Frank-Wolfe method (Frank & Wolfe, 1956), which has been systematically studied by (Clarkson, 2010).

BC’s Algorithm. Let $0 < \epsilon < 1$. Given a point set $S \subset \mathbb{R}^d$, the BC’s algorithm is an iterative procedure for computing $\text{MEB}(S)$. Initially, it selects an arbitrary point from $S$ and places it into an initially empty set $T$. In each of the following $\lceil 2/\epsilon \rceil$ iterations, the algorithm updates the center of $\text{MEB}(T)$ and adds to $T$ the farthest point from the current center of $\text{MEB}(T)$. Finally, the center of $\text{MEB}(T)$ yields a $(1 + \epsilon)$-approximation for $\text{MEB}(S)$. The selected set of $\lceil 2/\epsilon \rceil$ points (i.e., $T$) is called the coreset of MEB.

Gilbert algorithm. The algorithm is designed for solving the following polytope distance problem:

Assume $o$ is the origin and let $S$ be a given set of points in $\mathbb{R}^d$. The problem is to find a point $q$ inside the convex hull of $S$ (denoted as $\text{conv}(S)$) such that the norm $||q||$ is minimized. The obtained norm, i.e., $\min_{q \in \text{conv}(S)} ||q||$, is called the polytope distance between $o$ and $S$.

For any two points $p$ and $q \in \mathbb{R}^d$, denote by $p \parallel q$ the orthogonal projection of $p$ on the supporting line of segment $oq$. The norm $||p \parallel q||$ is called the ”projection distance” of $p$ to the origin $o$ on $oq$. Gilbert algorithm is a standard greedy algorithm that improves the current solution by selecting the point $p_i \in S$ having the smallest projection distance in each iteration. See Algorithm 1.

Definition 5 ($\epsilon$-Approximation of Polytope Distance). Let $q \in \text{conv}(S)$ and $\epsilon \in (0, 1)$. The point $q$ is an $\epsilon$-approximation of the polytope distance if $||q|| \leq \frac{1}{1 - \epsilon} ||p \parallel q||$ for any $p \in S$. 
Obviously, \( \min_{p \in S} \{ ||p - q|| \} \) is no larger than the real polytope distance. Thus, if \( q \) is an \( \epsilon \)-approximation, its norm \( ||q|| \) is no larger than \( \frac{1}{1-\epsilon} \) times the real polytope distance. Suppose \( D = \max_{p \in S} ||p - q|| \) and \( \rho \geq 0 \) is the polytope distance between \( o \) and \( S \), and let \( E = D^2 / \rho^2 \).

**Theorem 1** (Gärtner & Jaggi, 2009; Clarkson, 2010). For any \( \epsilon \in (0, 1) \), Algorithm 1 takes at most \( N = 2[2E/\epsilon] \) steps to achieve an \( \epsilon \)-approximation.

The selected set of points \( T = \{ p_i \mid 1 \leq i \leq 2[2E/\epsilon] \} \) is called the coreset of polytope distance.

**Remark 2.** A by-product of these two algorithms is that the solution, the approximate ball center of \( S \) or the point \( v_N \) inside \( \text{conv}(S) \), can be always represented by a convex combination of the obtained coreset \( T \). Moreover, the coefficients of this convex combination have been simultaneously generated after running the algorithm (since these coefficients are always recorded when updating the solution in each iteration). In our following algorithms, we will use these coefficients to bridge the original space \( \mathbb{R}^d \) and the low-dimensional space induced by the JL transform.

### 2.2. Other Related Works

**k-center clustering with outliers.** (Charikar et al., 2001) proposed a \( 3 \)-approximation algorithm for \( k \)-center clustering with outliers in arbitrary metrics. The time complexity of their algorithm is quadratic in data size. A following streaming \((4 + \epsilon)\)-approximation algorithm was proposed by (McCutchen & Khuller, 2008). (Chakrabarty et al., 2016) gave a \( 2 \)-approximation algorithm for metric \( k \)-center clustering with outliers based on the LP relaxation techniques. (Badoiu et al., 2002) proposed a coreset based approach but having an exponential time complexity if \( k \) is not a constant. Recently, (Ding et al., 2019) provided a greedy algorithm that yields a bi-criteria approximation (returning more than \( k \) clusters) based on the idea of the Gonzalez’s \( k \)-center clustering algorithm (Gonzalez, 1985).

**Robust SVM and SVM with outliers.** SVM is a fundamental tool for classification (Chang & Lin, 2011). For the ordinary SVM problem, a number of methods have been proposed, such as (Cortes & Vapnik, 1995; Platt, 1999; Crisp & Burges, 1999; Scholkopf et al., 2000; Tsang et al., 2005). In recent years, several methods considered the case that a significant fraction of outliers are mixed in the dataset; existing approaches include (Xu et al., 2006; Yang et al., 2010; Suzumura et al., 2014; Ding & Xu, 2015; Xu et al., 2017). Moreover, one-class SVM has also been used as an efficient approach for outlier recognition (Erfani et al., 2016; Schölkopf et al., 1999).

Beyond the aforementioned JL transform based dimension reduction methods for SVM in Section 1.1, a number of other methods (e.g., feature selection) were also proposed for speeding up the SVM procedure in the past (Rahimi & Recht, 2008; Paul et al., 2015). However, to the best of our knowledge, it is still unclear about their quality guarantees when the dataset contains outliers.

### 3. Minimum Enclosing Ball and \( k \)-Center Clustering with Outliers

To explain our idea more clearly, we first consider the MEB with outliers problem (i.e., the case \( k = 1 \) of Definition 2), and then extend the idea to solve the general \( k \)-center clustering with outliers problem.

#### 3.1. MEB with Outliers

Let \((P, \gamma)\) be an instance of the problem of MEB with outliers in \( \mathbb{R}^d \). Our high-level idea is as follows. First, we apply the JL transform \( f \) to \( P \) and solve the new instance \((f(P), \gamma)\) by any existing algorithm for MEB with outliers. Suppose the obtained ball center is \( \bar{c} \) and it can be represented by a convex combination of a subset of \( f(P) \). Then, we transform the center \( \bar{c} \) back to the original space \( \mathbb{R}^d \) using the coefficients of the convex combination (see Remark 2). See the details in Algorithm 2.

**Theorem 2.** Suppose the algorithm \( A \) used in Step 1 of Algorithm 2 yields a \( \lambda \)-approximate solution of MEB with outliers with \( \lambda \geq 1 \). The returned point \( f^{-1}(\bar{c}) \) of Algorithm 2 yields a \( \lambda \sqrt{(1+\epsilon)^2 / (1-\epsilon)} \)-approximate solution for the original instance \((P, \gamma)\) in \( \mathbb{R}^d \) with constant probability.

When \( \epsilon \) is small enough, the approximation factor \( \lambda \sqrt{(1+\epsilon)^2 / (1-\epsilon)} \approx \lambda \). The exact time complexity of BC’s algo-
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...
Algorithm 3 \(k\)-center clustering with outliers

**Input:** A set \(P\) of \(n\) points in \(\mathbb{R}^d\), \(k \in \mathbb{Z}^+\), and \(\gamma, \epsilon \in (0, 1)\).

1. Apply the JL transform \(f\) to reduce the dimensionality \(d\) to be \(O(\frac{1}{\epsilon^2} \log n)\), and run any existing algorithm \(A\) for \(k\)-center clustering with outliers on the new instance \((f(P), \gamma)\).

2. Let \(f(C_1), f(C_2), \ldots, f(C_k)\) be the \(k\) clusters of \(f(P)\) obtained by the algorithm \(A\). \(\sum_{j=1}^{k} |f(C_j)| = (1-\gamma)n\).

3. Compute a \((1+\epsilon)\)-approximate MEB for each \(f(C_j)\) through BC’s algorithm, and denote by \(\bar{c}_j\) the obtained ball center.

   - Each center \(\bar{c}_j\) is represented as a convex combination of the points of \(f(C_j)\), say \(\bar{c}_j = \sum_{q \in f(C_j)} \alpha_q q\) where \(\sum_{q \in f(C_j)} \alpha_q = 1\) and \(\alpha_q \geq 0\) for all \(q \in f(C_j)\).

**Output:** The \(k\) points \(f^{-1}(\bar{c}_j) = \sum_{q \in f(C_j)} \alpha_q f^{-1}(q)\) for \(1 \leq j \leq k\), as the \(k\) cluster centers.

The proof of Theorem 2 can be extended for Theorem 3. We also use \(r_{opt}\) and \(r'_{opt}\) to denote the optimal radii of the instances \((P, \gamma)\) and \((f(P), \gamma)\), respectively. Then, we have the same claim of (7), \(r'_{opt} \leq \sqrt{1+\epsilon} r_{opt}\). Consequently, the inequality (11) is replaced by: for any \(1 \leq j \leq k\) and \(p \in C_j\),

\[
||p - f^{-1}(\bar{c}_j)||^2 \leq \frac{(1+\epsilon)^3}{1-\epsilon} \lambda_{opt}^2 .
\]

Thus, the set \(\bigcup_{j=1}^{k} C_j\) is covered by the union of the \(k\) balls \(\bigcup_{j=1}^{k} B(f^{-1}(\bar{c}_j), \sqrt{\frac{(1+\epsilon)^3}{1-\epsilon}\lambda_{opt}^2})\), i.e., \(\{f^{-1}(\bar{c}_1), f^{-1}(\bar{c}_2), \ldots, f^{-1}(\bar{c}_k)\}\) yields a \(\sqrt{\frac{(1+\epsilon)^3}{1-\epsilon}}\)-approximate solution for the instance \((P, \gamma)\), and \(\{C_1, C_2, \ldots, C_k\}\) are the \(k\) clusters.

### 4. SVM with Outliers

To solve the SVM with outliers problems, we introduce an important geometric result below. Let \(\Delta y_1 y_2\) be a right triangle in a two-dimensional plane, where \(o\) is the origin, \(y_1 = (a, 0)\), and \(y_2 = (a, b)\) with \(a, b > 0\) (see Figure 2). Suppose the points \(y_1\) and \(y_2\) are moved to the new locations \(y'_1 = (a_0, 0)\) and \(y'_2 = (a', b')\), respectively; they form a new triangle \(\Delta y'_1 y'_2\).

**Lemma 3.** Let \(\delta\) be a small positive value. If the three sides of the new triangle \(\Delta y'_1 y'_2\) satisfy

\[
\begin{align*}
||y'_1 - o||^2 &\leq ||y_1 - o||^2 \pm \delta, \\
||y'_2 - o||^2 &\leq ||y_2 - o||^2 \pm \delta, \\
||y'_1 - y'_2||^2 &\leq ||y_1 - y_2||^2 \pm \delta,
\end{align*}
\]

then we have \(a' \geq \frac{2a^2 - 3\delta}{2\sqrt{a^2 + \delta}}\).

**Proof.** From (13), we directly have

\[
\begin{align*}
(a' - a_0)^2 + (b')^2 &\leq a^2 + b^2 \pm \delta; \quad (14) \\
(a' - a_0)^2 + (b')^2 &\leq a^2 + b^2 \pm \delta; \quad (15) \\
(a')^2 + (b')^2 &\leq (a^2 + b^2) \pm \delta. \quad (16)
\end{align*}
\]

Through (15) and (16), we have

\[
(a^2 + b^2) - \delta \leq (a')^2 + (b')^2 \\
= (a' - a_0)^2 + (b')^2 + 2a_0a' - (a_0)^2 \leq b^2 + \delta + 2a_0a' - (a_0)^2. \\
\rightarrow a^2 + a_0^2 \leq 2a_0a' + 2\delta. \quad (17)
\]

Further, we have

\[
2a^2 - 3\delta \leq \frac{a^2 - 2\delta + a_0^2}{\text{by (14)}} \leq 2a_0a' \leq 2\sqrt{a^2 + \delta a'}. \quad (18)
\]

Finally, we have the following inequality from (18), \(a' \geq \frac{2a^2 - 3\delta}{2\sqrt{a^2 + \delta}}\). So we complete the proof. \(\square\)

### 4.1. One-class SVM with Outliers

To present our result, we need to relate polytope distance to one-class SVM through the following lemmas. Let \(S \subset \mathbb{R}^d\) and \(x \neq o\) be a point in \(\mathbb{R}^d\).

**Lemma 4** ([Gärtner & Jaggi, 2009; Clarkson, 2010]). If \(x\) is the optimal solution of the polytope distance from \(o\) to \(S\), then \(x\) is also the optimal normal vector for the one-class SVM on \(S\), i.e., the hyperplane \(H_x\) yields the maximum margin separating the origin and \(S\).

**Lemma 5** ([Ding & Xu, 2015]). Suppose \(x\) is an \(\epsilon\)-approximation of the polytope distance (see Definition 5). Let \(y = (1-\epsilon)x\). Then the hyperplane \(H_y\) separates the
We set the convex hull of \( \{ (1 - \gamma)n \text{ points of } f(P) \} \) formed from the origin and \( S \), and its width, i.e., the norm \( ||y|| \), is at least \((1 - \epsilon)\) times the maximum width.

Lemma 5 suggests that we can find an approximate solution for one-class SVM through Gilbert Algorithm. We further show how to apply it to the case with outliers. Given an instance \(( P, \gamma) \) of one-class SVM with outliers, let \( D = \max_{p,q \in P} ||p - q|| \) be the diameter of \( P \). Suppose that \( x_{opt} \) is the normal vector yielding the optimal separating margin for the instance \(( P, \gamma) \) and \( P_{opt} \) is the set of induced inliers. For convenience, let \( \rho = ||x_{opt}|| \) and \( E = D^2/\rho^2 \).

**Theorem 4.** We set \( \epsilon_0 \in (0, 1) \) and \( \epsilon = \frac{1}{2} \frac{\epsilon_0 \rho}{E + 1} \) in Algorithm 4. Suppose the algorithm \( A \) used in Step 1 yields a 1/\( \lambda \)-approximate solution of SVM with outliers with \( \lambda \geq 1 \). The returned vector \( f^{-1}(\bar{x}) \) of Algorithm 4 yields a \( \frac{1}{\lambda}(1 - \epsilon_0)^2 \)-approximate solution for the instance \(( P, \gamma) \) with constant probability.

**Proof.** For any point \( p \in P_{opt} \), denote by \( \tau(p) \) the projection of \( p \) on the vector \( x_{opt} \). Then, we focus on the right triangle \( \Delta o \tau(p) p \) and its image, \( \Delta o f(\tau(p)) f(p) \), in the lower dimensional space via the JL Transform \( f \) (w.l.o.g., we assume the two spaces share the same origin \( o \)). See Figure 3. From Lemma 4, we know that \( x_{opt} \) should be inside \( conv(P_{opt}) \). Consequently, the triangle \( \Delta o \tau(p) p \) is inside the convex hull of \( \{ o \} \cup P_{opt} \). Moreover, the set \( \{ o \} \cup P_{opt} \) is the normal vector yielding the optimal separating margin for the instance \(( P, \gamma) \) and \( P_{opt} \) is the set of induced inliers. For convenience, let \( \rho = ||x_{opt}|| \) and \( E = D^2/\rho^2 \).

Figure 3. Because the JL transform \( f \) is linear mapping, the point \( f(\tau(p)) \) is located on the line determined by the vector \( f(x_{opt}) \). The point \( f(p) \) has projection distance \( a' \) to the origin \( o \).

is covered by a ball with radius at most \( D + \rho \). Therefore, through the inequality (5) of Lemma 2, we have

\[
||f(\tau(p)) - o||^2 \in ||\tau(p) - o||^2 \pm \epsilon(D + \rho)^2;
||f(p) - o||^2 \in ||p - o||^2 \pm \epsilon(D + \rho)^2;
||f(\tau(p)) - f(p)||^2 \in ||\tau(p) - p||^2 \pm \epsilon(D + \rho)^2.
\]

We apply Lemma 3 to \( \Delta o \tau(p) p \) by letting \( y_1 = \tau(p) \), \( y_2 = p \), \( y_1' = f(\tau(p)) \), \( y_2' = f(p) \), and \( \delta = \epsilon(D + \rho)^2 \). For convenience, we use the same notations as Lemma 3 and have

\[
a' \geq \frac{2\alpha^2 - 3\delta}{2\sqrt{\alpha^2 + \delta}} = \sqrt{\rho^2 + \delta} - \frac{5\delta}{2\sqrt{\rho^2 + \delta}}.
\]

Note that \( \alpha = ||\tau(p)|| \geq \rho \) and \( \epsilon = \frac{1}{2} \frac{\epsilon_0 \rho}{E + 1} \) with \( E = D^2/\rho^2 \); then we have

\[
\rho \geq \sqrt{\rho^2 + \delta} - \frac{5\delta}{2\sqrt{\rho^2 + \delta}}
\]

\[
\geq \rho - \frac{5\delta}{2\rho}
\]

\[
\geq \rho - \frac{\epsilon_0}{2} \frac{(D + \rho)^2}{E + 1} \frac{1}{\rho}
\]

\[
= (1 - \epsilon_0)\rho.
\]

(20) implies that for any point \( p \in P_{opt} \), \( f(p) \)'s projection on the vector \( f(x_{opt}) \) has distance \( \geq (1 - \epsilon_0)\rho \) to the origin. That is, the vector \( f(x_{opt}) \) yields a solution for the instance \(( f(P), \gamma) \) with separating margin width at least \((1 - \epsilon_0)\rho \). Further, because \( v \) yields a \( \frac{1}{\lambda} \)-approximate solution for \(( f(P), \gamma) \) (step 2 of Algorithm 4), we have

\[
w_v \geq \frac{1}{\lambda} (1 - \epsilon_0)\rho
\]

where \( w_v \) denotes the margin width induced by \( v \). Also, since \( \bar{x} \) is an \( \epsilon_0 \)-approximation of the polytope distance problem for the instance \( f(S) \), through Lemma 5 we know that the vector \( \bar{x} \) yields a margin separating the origin and \( f(S) \) with the width

\[
||| (1 - \epsilon_0)\bar{x} ||| \geq (1 - \epsilon_0)||v|| \geq \frac{1}{\lambda} (1 - \epsilon_0)^2 \rho.
\]
Algorithm 5 Two-class SVM with outliers

Input: Two point sets $P_1$ and $P_2$ in $\mathbb{R}^d$ with $n = |P_1 \cup P_2|, \epsilon, \epsilon_0 > 0$, and $\gamma_1, \gamma_2 \in (0, 1)$.

1. Apply the JL transform $f$ to reduce the dimensionality $d$ to be $O(\frac{d}{\epsilon^2} \log n)$, and run any existing algorithm $A$ for SVM with outliers on the new instance $(f(P_1), f(P_2), \gamma_1, \gamma_2)$.

2. Suppose the normal vector returned by $A$ is $v$. Then, we project all the points of $f(P_1)$ and $f(P_2)$ to the vector $v$, and find the margin along $v$ such that exactly $(1 - \gamma_1)|P_1|$ points of $f(P_1)$ and $(1 - \gamma_2)|P_2|$ points of $f(P_2)$ are separated. These points form the sets of inliers $f(S_1)$ and $f(S_2)$, respectively.

3. Compute an $\epsilon_0$-approximate solution for the polytope distance between $f(S_1)$ and $f(S_2)$ through Gilbert algorithm, and denote by $\bar{x}$ the obtained point in $MD(f(S_1), f(S_2))$.

   - In Gilbert algorithm, $\bar{x}$ is represented as a convex combination of the points of $f(S)$ where $S = S_1 \cup S_2$, say $\bar{x} = \sum_{q \in f(S)} \alpha_q q$ where $\sum_{q \in f(S)} \alpha_q = 1$ and $\alpha_q \geq 0$ for all $q \in f(S)$.

Output: $f^{-1}(\bar{x}) = \sum_{q \in f(S)} \alpha_q f^{-1}(q)$ as the normal vector.

Then, we consider the inverse mapping $f^{-1}$ from $\mathbb{R}^{O(\frac{d}{\epsilon^2} \log n)}$ to $\mathbb{R}^d$. For any $q \in f(S)$, consider the right triangle $\Delta \alpha(q) q$ and its image, $\Delta f^{-1}(\tau(q)) f^{-1}(q)$ in $\mathbb{R}^d$, where $\tau(q)$ is the projection of $q$ on the vector $\bar{x}$. Since $\tau(q) \in conv\{o \cup f(S)\}$, it can be represented as a convex combination of $o \cup f(S)$; therefore, we can define its image $f^{-1}(\tau(q))$ as the convex combination of $o \cup S$ with the same coefficients. So we can apply the same manner for proving the above (20) with replacing $f$ and $P_{opt}$ by $f^{-1}$ and $f(S)$ respectively. For any point $q \in f(S)$, $f^{-1}(q)$'s projection on the vector $f^{-1}(\bar{x})$ has distance at least

$$
(1 - \epsilon_0)(1 - \epsilon_0)\bar{x} ||(1 - \epsilon_0)\bar{x}|| \geq \frac{1}{\lambda}(1 - \epsilon_0)^3 \rho
$$

(23)

to the origin by (22). In other words, the vector $f^{-1}(\bar{x})$ yields a $\frac{1}{\lambda}(1 - \epsilon_0)^3$-approximation for the instance $(P_{opt}, \gamma_1, \gamma_2)$ and $S$ is the set of inliers.

4.2. Two-class SVM with outliers

Consider the connection between polytope distance and two-class SVM. Let $Q_1$ and $Q_2$ be two point sets in $\mathbb{R}^d$. The Minkowski Difference $MD(Q_1, Q_2)$ of $conv(Q_1)$ and $conv(Q_2)$ is the set of all difference vectors, i.e., $MD(Q_1, Q_2) = \{ u - v \mid u \in conv(Q_1), v \in conv(Q_2) \}$. Note that $MD(Q_1, Q_2)$ is also a convex polytope.

Lemma 6. ([Gärtner & Jaggi, 2009]) Finding the shortest distance between two polytopes $conv(Q_1)$ and $conv(Q_2)$ is equivalent to finding the polytope distance from the origin to $MD(Q_1, Q_2)$.

Lemma 6 tells us that to find the maximum margin separating two point sets $Q_1$ and $Q_2$, we only need to find the maximum margin separating the origin and $MD(Q_1, Q_2)$. However, directly computing $MD(Q_1, Q_2)$ takes quadratic time $O(|Q_1||Q_2|d)$. Actually, we do not need to explicitly compute $MD(Q_1, Q_2)$ for obtaining its polytope distance to the origin. In each iteration of the Gilbert algorithm, we just select the point $p_i$ that has the closest projection to $o$.

In $MD(Q_1, Q_2)$, this point $p_i$ should be the difference vector $q_i - q_i'$, where $q_i \in Q_1$ and $q_i' \in Q_2$, and

$$p_i \mid_v = q_i \mid_v - q_i' \mid_v.$$

Therefore, $q_i$ should be the point having the closest projection to $o$ from $Q_1$ and $q_i'$ should be the point having the farthest projection to $o$ from $Q_2$. As a consequence, we just need to select the points $q_i$ and $q_i'$ in each iteration, and can still compute the polytope distance between $o$ and $MD(Q_1, Q_2)$ in linear time. Given a point $x \in \mathbb{R}^d$, we use $M_x$ to denote the margin that is orthogonal to $x$ and separates $Q_1$ and $Q_2$.

Lemma 7. Let $x \in MD(Q_1, Q_2)$ be an $\epsilon$-approximation of the polytope distance yielded by Gilbert Algorithm for some small constant $\epsilon \in (0, 1)$. Then $M_x$ separates $Q_1$ and $Q_2$, and its width is at least $(1 - \epsilon)$ times that of the maximum width.

Finally, we have the algorithm for solving the two-class SVM with outliers problem.

Theorem 5. Let $\epsilon_0 \in (0, 1)$ and $\epsilon = \frac{1}{2} \epsilon_0 \frac{\rho}{\sqrt{\lambda}}$. Suppose the algorithm $A$ used in Step 1 of Algorithm 5 yields a $1/\lambda$-approximate solution of SVM with outliers with $\lambda \geq 1$. The returned vector $f^{-1}(\bar{x})$ of Algorithm 5 yields a $\frac{1}{\lambda}(1 - \epsilon_0)^3$-approximate solution for the instance $(P_1, P_2, \gamma_1, \gamma_2)$ with constant probability.

5. Experiments

All the experimental results were obtained on a Windows workstation with 2.8GHz Intel Core i5-840 and 8GB main memory; the algorithms were implemented in Matlab R2018b. For each instance, we repeat the experiment 50 times and report the average result and the corresponding standard deviation.

5.1. k-Center Clustering with Outliers

We use the popular 3-approximation algorithm ([Charikar et al., 2001]) as the algorithm $A$ in Algorithm 3. We compare
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Figure 4. The experimental results of \(k\)-center clustering with outliers and one-class SVM with outliers on synthetic datasets.

JL transform (named JLT) (Achlioptas, 2003) with several other widely studied dimension reduction methods, PCA, approximate PCA, and the SVD based feature selection (named PCA, APPROX-PCA, FS accordingly) (Boutsidis et al., 2014).

We generate the synthetic dataset in \(\mathbb{R}^{4000}\) with \(n = 10^4\), \(z = 10\%n\), and \(k = 20\). We randomly pick \(k\) points as the cluster centers inside a hypercube of side length 500; around each center, we generate a cluster following a Gaussian distribution with standard deviation 20; we keep the total number of points to be \(n - z\) and uniformly generate \(z\) outliers at random outside the MEBs of these \(k\) clusters. We keep the dimension reduction rate to be \(\{6\%, 8\%, 10\%, 12\%, 14\%\}\) and repeat the experiment 15 times for each instance; we show the average normalized running time and radius (over the result without dimension reduction) in Figure 4(a)-(b). The obtained normalized radii of these 4 methods are very close (the difference range is within \(1\%\)), and JLT achieves the lowest running time among these 4 methods (which is around \(20\%\) of the running time without dimension reduction). The standard deviations of normalized radius are 0.54 (JLT), 0.26 (PCA), 1.67 (APPROX-PCA), and 1.44 (FS). So APPROX-PCA and FS appear very unstable, and PCA is the stabllest one though it takes the largest running time.

We also test the algorithms on two real-world datasets. The CIFAR-10 dataset (Krizhevsky, 2009) consists of \(n = 60,000\) color images of \(k = 10\) classes (e.g., airplane, bird), with each class having 6000 images and each image being represented by a \(3072\)-dimensional vector. The MNIST dataset (Y. LeCun & P. Haffner, 1998) contains \(n = 60,000\) handwritten digit images from 0 to 9 (i.e., \(k = 10\)), where each image is represented by a \(784\)-dimensional vector. For each dataset, we randomly add \(10\%n\) outliers outside the MEBs of the clusters. We show the resulting normalized radius and running time in Figure 5 and the average standard deviation of normalized radius in Table 1. Similar to the results on the synthetic datasets, JLT and PCA achieve the performances much more stable than the other two methods, and JLT is the fastest one.

We also compare the three different JL transform methods from (Dasgupta & Gupta, 2003; Achlioptas, 2003; Ailon & Chazelle, 2009), which are named JLT-GAUSSIAN, JLT-BINARY, and JLT-FAST. The results are shown in Figure 6 and Table 2. The results suggest that JLT-BINARY is the stabllest and fastest one, though it achieves worse normalized radius comparing with the other two methods.

5.2. SVM with Outliers

One-class SVM with outliers. We use the one-class SVM algorithm from (Chang & Lin, 2011) as the algorithm \(A\) in Algorithm 4. We randomly generate a set of \(n = 2 \times 10^4\) points in \(\mathbb{R}^{1000}\) following a Gaussian distribution with standard deviation \(\sqrt{10^3}\). Also, we run the algorithm 15 times with cross-validations; specifically, the data is randomly partitioned into two equal-sized parts respectively for training and testing. To generate the training dataset with outliers, we first compute the hyperplane \(H\) by the algorithm \(A\), and randomly add \(10\% \ast (n/2)\) outliers on the other side of \(H\); then we run Algorithm 4. The results are shown in Figure 4 (c)-(d). The standard deviations of accuracy are as follows, \(5.6 \ast 10^{-3}\) (JLT), \(6.2 \ast 10^{-3}\) (PCA), \(8.3 \ast 10^{-3}\) (APPROX-PCA), and \(7.9 \ast 10^{-3}\) (FS), which suggest that JLT’s performance is the stabllest among the 4 methods.

We also take the real-world datasets MNIST (“0” and “1” classes) and GISETTE (Guyon et al., 2005) (two classes of \(n = 13500\) points in \(\mathbb{R}^{5000}\)) to conduct our experiments. Each class of MNIST and GISETTE forms an individual instance. We run the algorithm with cross-validations; specifically, the dataset is randomly partitioned into two equal-sized parts respectively for training and testing. To generate outliers, we first compute the hyperplane \(H\) by running the algorithm \(A\) on the training data, and randomly add \(10\%n\) outliers on the other side of \(H\); then we run Algorithm 4.

Two-class SVM with outliers. We use the LIBLINEAR SVM algorithm from (Chang & Lin, 2011) as the algorithm \(A\). Similar to the one-class case, we also run the algorithm with cross-validations. To generate outliers, we simply exchange the labels of randomly picked \(10\%n\) pairs of data items for each instance.

We show the resulting accuracy and normalized running time in Figure 7 & 8 and the average standard deviation of accuracy in Table 3.
Finally, similar to $k$-center clustering with outliers, we compare the three different JL transform methods JLT-GAUSSIAN, JLT-BINARY, and JLT-FAST. The results are shown in Figure 9 & 10 and Table 4.

In general, similar to the experimental results for clustering, JL and PCA are more stable, and JLT-BINARY outperforms the other two JL transform methods in terms of running time and stability.

### 6. Future Work

We study the effectiveness of JL transform for solving two fundamental high-dimensional problems. Following this work, it is worth considering other robust optimization problems. Also, though we use the (approximate) PCA and feature selection techniques as the baselines in our experiments, it is still unclear about their theoretical quality guarantees in terms of the objective functions when outliers exist.

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Table 1. The average standard deviations of four dimension reduction methods.

| Datasets  | JLT   | FS   | PCA  | APPROX-PCA |
|-----------|-------|------|------|-------------|
| CIFAR-10  | 0.2829| 0.8344| 0.0386| 0.5082      |
| MNIST     | 0.1463| 0.3286| 0.0137| 0.2275      |

Table 2. The standard deviations of three JL transform methods.

| Datasets  | JLT-GAUSSIAN | JLT-BINARY | JLT-FAST |
|-----------|--------------|------------|----------|
| CIFAR-10  | 0.8046       | 0.2829     | 0.8424   |
| MNIST     | 0.2888       | 0.1463     | 0.2920   |

Table 3. The average standard deviations of four dimension reduction methods.

| Datasets  | One-calss SVM | Two-calss SVM |
|-----------|---------------|---------------|
|           | JLT | FS | PCA | APPROX-PCA | JLT | FS | PCA | APPROX-PCA |
| MNIST     | 0.0091 | 0.0181 | 0.0017 | 0.0082 | 0.0394 | 0.1943 | 0.0452 | 0.1726 |
| GISETTE   | 0.0095 | 0.0275 | 0.0012 | 0.0117 | 0.0352 | 0.0664 | 0.0330 | 0.0413 |

Table 4. The standard deviations of three JL transform methods.

| Datasets  | One-calss SVM | Two-calss SVM |
|-----------|---------------|---------------|
|           | JLT-GAUSSIAN | JLT-BINARY | JLT-FAST | JLT-GAUSSIAN | JLT-BINARY | JLT-FAST |
| MNIST     | 0.0152 | 0.0091 | 0.0179 | 0.1041 | 0.0394 | 0.1144 |
| GISETTE   | 0.0127 | 0.0095 | 0.0145 | 0.0525 | 0.0392 | 0.0647 |
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Figure 7. Different dimension reduction methods for one-class SVM with outliers.

Figure 8. Different dimension reduction methods for two-class SVM with outliers.

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