A Verifiable Quantum Secret Sharing Scheme Based on a Single Qubit

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Abstract

To detect frauds from some internal participants or external attackers, some verifiable threshold quantum secret sharing schemes have been proposed. In this paper, we present a new verifiable threshold structure based on a single qubit using bivariate polynomial. First, Alice chooses an asymmetric bivariate polynomial and sends a pair of values from this polynomial to each participant. Then Alice and participants implement in sequence unitary transformation on the \(d\)-dimensional quantum state based on unbiased bases, where those unitary transformations are contacted by this polynomial. Finally, security analysis shows that the proposed scheme can detect the fraud from external and internal attacks compared with the exiting schemes and is comparable to the recent schemes.

keywords  Verifiability  Mutually unbiased bases  Unitary transformation  Secret sharing scheme

1 Introduction

Quantum secret sharing is an important issue in quantum cryptography, which combines classical secret sharing and quantum theory and plays an important role in applied cryptography. It means that classic or quantum secret information can be divided into shares by a dealer among participants, so that only authorized participants can recover the secret, and any one or more unauthorized participants can not recover the secret. In a \((t,n)\)-threshold quantum secret sharing scheme, the dealer splits secret into \(n\) shares sending them to each of \(n\) participants where any set of \(t\) or more shareholders can recover the shared secret cooperatively, however less than \(t\) shareholders can not recover it. Hillery et al.\cite{1} firstly proposed quantum secret sharing (QSS) based on the quantum correlation of Greenberger-Horne-Zeilinger (GHZ) states in 1999. The main idea of this scheme is that an unknown quantum state is shared between two participants and only restored collaboratively. The fundamental theory of quantum determines that the QSS schemes are more secure than the classic ones. There are many QSS schemes \cite{2-9} that have been proposed. For example, a quantum secret sharing scheme was constructed using the product state rather than the entangled state in \cite{4}, such that the scheme is applicable when the number of participants is expanded. In addition, a \((n,n)\)-threshold quantum secret sharing scheme that shares multiple classic information was proposed based on a single photon in \cite{6}. Most of these schemes do not take into account two major security issues: during the secret distribution phase, an attacker may impersonate the dealer to send false information to the shareholders and during the recovery phase, some participants may provide false shares, so they cannot recover the correct secret. In 1985, Chor et al. \cite{10} proposed the concept of verifiable secret sharing (VSS) and gave a complete scheme that verifiable problem was solved effectively. The verifiable secret sharing scheme has attracted the attention of many scholars, because it can prevent dishonest participants from providing false information during the secret recovery stage, and it can also prevent external fraud from pretending to send false information to participants. With the advent of quantum algorithms \cite{11,12}, the theory of quantum secret sharing (VQSS) \cite{7-9} can be further developed. For example, an identity-based quantum signature encryption algorithm was constructed in \cite{8}, which makes the secret share and signature are safe under the choice of plaintext attack. What’s more, a verifiable \((t,n)\)-threshold QSS scheme with sequential communication was proposed in \cite{9}, in which the property of mutually unbiased bases is used \cite{13}. The quantum state is measured with the basis \(\{\mid \psi_i^{0}\}\) by the last participant Bob\(_t\) until the dealer Alice and all participants Bob\(_1, Bob_2, \ldots, Bob_t\) implement...
unitary transformation on the three single-bit quantum states sequentially. Then the measurement result is sent to each participant. In the secret recovery phase, participants exchange the information of unitary transformation so that each participant can recover secrets, where the first two qubits are used to share the two secrets, and the third single qubit is used as the verification information. This scheme combines the Shamir secret sharing scheme with sequential communication in the $d$-dimensional quantum system, and adds the third qubit as the authentication information, which easily identifies the spoof or attack. However, the scheme has two drawbacks: Firstly, when participants exchange classic information, they are vulnerable to external attackers, which will result in the scheme to be unsafe. Secondly, the third qubit used as verification information is wasted and the verification formula $p_1^0 = p_0^2 p_0^1 \mod d$ will be established with a certain probability, even if there are dishonest participants.

In this paper, the scheme in Ref. [9] is improved on these two issues and a new QSS scheme based on the property of mutually unbiased bases [13] is proposed. Compared with the original scheme, this paper has the following advantages:

(a). During the secret distribution phase, the distributor chooses an asymmetric binary polynomial to replace the original unary polynomial. The classical information of the participants’ unitary transformation is obtained by their respective secret shares, which will be supervised by the dealer so as to make sure that participants are honest. Consequently, the verifiability of the scheme is achieved with less authentication information.

(b). This scheme is supervised by the dealer to ensure that each participant is honest, thereby realizing the practicality and feasibility of the scheme.

(c). If a dishonest participant is found during the secret sharing phase, the dishonest participant will be found and removed. In the next secret sharing process, the dealer simply re-prepares a binary polynomial and a quantum state to recover the secret, which will prevent internal attacks.

(d). The classic information exchanged by the participants during the secret recovery phase can be protected by paired keys, so that the classic information will not be leaked, and external attacks can be prevented.

The organization of this paper is as follows: In Sect. 2, we give the preliminary knowledge related to bivariate polynomials and mutually unbiased bases as well as the construction of unitary transformation. In Sect. 3, we proposed a new verifiable $(t,n)$-threshold QSS scheme with sequential communication. The security of the scheme is analyzed in Sect. 4. Next, we give a comparison of the basic properties among our scheme and others in Sect. 5. Finally, the conclusion is given in Sect. 6.

2 Preliminaries

In this section, we briefly introduce the knowledge related to bivariate polynomials and mutually unbiased bases as well as the construction of unitary transformation.

2.1 Binary polynomials

A binary polynomial having degree $t-1$ for $x$ and $h-1$ for $y$ is defined as

$$ F(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2 + a_{12}xy^2 + a_{21}x^2y + a_{22}x^2y^2 + \cdots + a_{t-1,h-1}x^{t-1}y^{h-1} \mod d, \tag{1} $$

where $a_{ij} \in \mathbb{F}_2, i \in \{0, 1, \cdots, t-1\}, j \in \{0, 1, \cdots, h-1\}, x_i$ is the public information of Bob$_i$, $d$ is required to be an odd prime number in this paper. The verifiable secret sharing based on binary polynomial is called BVSS. One advantage of BVSSs is that it can provide shared keys for any two participants Bob$_i$ and Bob$_j$ in the information exchange process, so they can be protected. It can be divided into two categories:

(a). Verifiable secret sharing scheme based on symmetric bivariate polynomial (SBVSSs)[15-18]. For SBVSSs, the dealer Alice chooses a symmetric binary polynomial $F(x,y)$ and computes $F(x,y) \mod d$ and sending it to Bob$_i$ through the secure channel. Any two participants Bob$_i$ and Bob$_j$ compute $F(x_i,x_j)$ and $F(x_j,x_i)$ respectively, using the pair as a shared key between them.

(b). Verifiable secret sharing scheme based on asymmetric bivariate polynomial (ABVSSs) [14,15]. For ABVSSs, the dealer Alice chooses an asymmetric binary polynomial, computing $F(x_i,y) \mod d, F(x,y) \mod d$ and sending them to Bob$_i$ through the secure channel. According to $F(x_i,y) \mod d$ and $F(x,y) \mod d$, Bob$_i$ and Bob$_j$ calculate $F(x_i,x_j)$ respectively as a pairwise shared key between them, where $i, j \in \{1, 2, \cdots, n\}$. 

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2.2 Mutually unbiased bases

Two sets of standard orthogonal bases $A_1 = \{ |\varphi_1\rangle, |\varphi_2\rangle, \cdots, |\varphi_d\rangle \}$ and $A_2 = \{ |\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_d\rangle \}$ are defined over a $d$-dimensional complex space in Ref. [19, 20] if the following relationship is satisfied:

$$|\langle \varphi_i | \psi_j \rangle| = \frac{1}{\sqrt{d}}$$

(2)

If any two of the set of standard orthogonal bases $\{ A_1, A_2, \cdots, A_m \}$ in space are unbiased, then this set is called an unbiased bases set. Besides, it can be found $d + 1$ mutually unbiased bases if $d$ is an odd prime number. First, the computation base is expressed as $\{ |k\rangle | k \in D \}, D = \{ 0, 1, \cdots, d - 1 \}$, and the remaining groups can be expressed as:

$$|v^{(j)}_l\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} |k\rangle,$$

(3)

where $j$ and $l$ represent respectively the number of the mutually unbiased bases and the number of the vectors, $w = e^{\frac{2\pi i}{d}}, j \in D$. These mutually unbiased bases satisfy the following conditions:

$$|\langle v^{(j)}_l | v^{(j')}_{l'} \rangle| = \frac{1}{\sqrt{d}} j \neq j'.$$

(4)

2.3 The construction of unitary transformation

Next, we introduce the two unitary transformations $X_d$ and $Y_d$ that we need to use in this paper. In Ref.[13], they can be expressed as:

$$X_d = \sum_{m=0}^{d-1} w^m |m\rangle \langle m|.$$  

(5)

Implementing on $|v^{(j)}_{l}\rangle$ in turn, we can obtain:

$$X_d^* Y_d^* |v^{(j)}_{l}\rangle = X_d^* \left( \sum_{m=0}^{d-1} w^{m^2} |m\rangle \langle m| \right) \left( \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} |k\rangle \right)$$

$$= \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} w^{m^2} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} w^{k(l+jk)} |k\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+x)+(j+y)k} |k\rangle$$

$$= |v^{(j+y)}_{l+x}\rangle.$$

(6)

For the convenience of expression, $X_d^* Y_d^*$ is denoted as $U_{xy}$, that is, $U_{xy} |v^{(j)}_{l}\rangle = |v^{(j+y)}_{l+x}\rangle$.

3 Verifiable $(t, n)$-threshold quantum secret sharing scheme

In this section, we construct a verifiable $(t, n)$-threshold quantum secret sharing scheme that includes an honest dealer Alice and shareholders Bob_1, Bob_2, \cdots, Bob_n.

3.1 Preparation phase

3.1.1 Alice randomly chooses an asymmetric bivariate polynomial of which the form is like formula mentioned in the previous section 2, where $a_{ij} \in \mathcal{F}_d, i, j \in \{0, 1, \cdots t - 1\}$.

3.1.2 Alice calculates $F(x, y)$ and $F(x, x_i)$ as the secret shares and sends it to Bob, through secure channel, where $x_i$ is the public information of Bob_i.

3.1.3 Alice prepares a $d$-dimensional quantum state $|\varphi\rangle = |\varphi_0\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$, and performs a unitary transformation $U_{p_0, q_0}$ on it, where $p_0 = s, q_0 = s - \sum_{i=1}^t F(x_i, 0), s = F(0, 0). p_0, q_0 \in \mathcal{F}_d$, $S$ is a secret.
3.2 Distribution phase

3.2.1 Alice sends the quantum state $|\phi\rangle_0 = U_{p_0,q_0} |\phi_0\rangle = |\phi^{p_0}_{q_0}\rangle$ that she has performed a unitary transformation on to Bob\(_1\) through the secure channel. Then Bob\(_1\) performs a unitary transformation $U_{p_1,q_1}$ on the obtained quantum state $|\phi\rangle_0$ to get $|\phi\rangle_1 = U_{p_1,q_1} |\phi^{p_0}_{q_0}\rangle = |\phi^{p_0+p_1}_{q_0+q_1}\rangle$, where $p_1 = F(x_1,x_1), q_1 = F(x_1,0), p_1,q_1 \in \mathcal{F}_d$. Next, the quantum state $|\phi\rangle_1$ performed by Bob\(_1\) is sent to Bob\(_2\).

3.2.2 The other participants Bob\(_i\) repeat the same operation of Bob\(_1\) in 3.2.1, that is, Bob\(_i\) performs a unitary transformation $U_{p_i,q_i}$ on the obtained quantum state $|\phi\rangle_{i-1}$ from Bob\(_{i-1}\), getting

$$|\phi\rangle_i = U_{p_i,q_i} \sum_{i_0=0}^{i-1} |q_i\rangle \sum_{i_0=0}^{i-1} |p_i\rangle = |\phi^{p_0}_{q_0}\rangle.$$  

Then he sends the quantum state $|\phi\rangle_i$ to Bob\(_{i+1}\) through the quantum safe channel until the last participant Bob\(_t\) performs the same operation and gets the final state

$$|\phi\rangle_t = U_{p_t,q_t} \sum_{i_0=0}^{t-1} |q_t\rangle \sum_{i_0=0}^{t-1} |p_t\rangle = |\phi^{p_0}_{q_0}\rangle,$$

where $p_i = F(x_i,x_i), q_i = F(x_i,0), p_i,q_i \in \mathcal{F}_d, i = 2,3,\ldots,t$.

3.2.3 Since $k_{ij} = F(x_i,x_j)$ is used as a shared key between participants Bob\(_i\) and Bob\(_j\), Bob\(_i\) can calculate $c'_i = E_{k_{ij}}(p_i), c''_j = E_{k_{ij}}(q_i)$ and send them to Bob\(_j\). After receiving the ciphertext $c'_i,c''_j$, he can infer $p_i = D_{k_{ij}}(c'_i), q_i = D_{k_{ij}}(c''_j)$, where $E_{k_{ij}}(p_i),E_{k_{ij}}(q_i)$ represents classical encryption of plaintext $p_i,q_i$. $D_{k_{ij}}(c'_i),D_{k_{ij}}(c''_j)$ represents classical decryption of ciphertext $c'_i,c''_j$, $i,j \in \{1,2,\ldots,t\}, i \neq j$.

3.3 Measurement phase

3.3.1 After receiving $p_i,q_i$, by the binary Lagrange interpolation formula, Bob\(_j\) can calculate

$$s' = \sum_{i=1}^{t} \left( F(x_i,0) \prod_{k=1}^{t} \frac{x_k}{x_k - x_i} \right) = \sum_{i=1}^{t} \left( q_i \prod_{k=1, k \neq i}^{t} \frac{x_k}{x_k - x_i} \right)$$

where $i,j \in \{1,2,\ldots,t\}$. At first, the last participant Bob\(_t\) performs a unitary transformation $U_{p_t,q_t}$ on the obtained quantum state $|\phi\rangle_{t-1}$ to obtain $|\phi\rangle_t$, and then he select bases $\sum_{i=1}^{t} \left\{ v'_i(\phi) \right\}$ to measure the quantum state $|\phi\rangle_t$ with the measurement result denoted as $R'$. Next he could calculate $c = E_{k_{it}}(R')$ and sent it to other participants Bob\(_i\).

3.3.2 After receiving the ciphertext $c$ from the last participant Bob\(_t\), Bob\(_i\) calculates $R' = D_{k_{it}}(c)$, where $E_{k_{it}}(R')$ is the classic encryption of $R'$ sent by Bob\(_t\), and $D_{k_{it}}(c)$ is the classic decryption of ciphertext $c$, where $i = 1,2,\ldots,t-1$.

3.4 Testing phase

3.4.1 For the security of the scheme, the quantum states are randomly selected and detected by Alice during the process of transmitting the quantum states. Alice requires Bob, to send the calculated $s'$ to her and checks whether it is satisfied. If satisfied, the participants are honest and the scheme continues because the following formula is true:

$$\sum_{j=0}^{i} q_j = \left( s - \sum_{j=1}^{i} q_j \right) + \sum_{j=1}^{i} q_j \mod d = s.$$

If one or some of the participants calculate $s' \neq s$ is not satisfied, it indicates that the participant has fraudulent behavior, which can be divided into two cases. If $s' = s$ calculated by the last participant is
not satisfied, the scheme terminates. If it is found that one or some of the previous participants Bob_1, Bob_2, ..., Bob_{i-1} dissatisfaction, then it move on to the next step.

3.4.2 As long as Alice checks each two participant’s p_i, p_j that send by Bob_i and received by Bob_j, she can find out which participant has fraudulent behavior, and remove it in the next round of secret sharing scheme, where i, j ∈ {1, 2, ..., t}, i ≠ j.

3.4.3 Alice checks each participant’s received R' sent by the last participant Bob_t and examine if the following is established:

\[ R' = R = p_0 + p_1 + \cdots + p_t = S + F(x_1, x_1) + \cdots + F(x_t, x_t). \] (11)

If it is established, the scheme continues; if at least one participant receives R' such that does not hold, indicating that the last participant Bob_t is dishonest and the scheme is terminated, so Bob_t is removed in the next round of secret sharing.

3.5 Recovery phase

In order to restore the original secret, Bob_i can calculate

\[ p_0 = R - \sum_{i=1}^{t} p_i \] (12)

and obtain the secret p_0 = S. If Alice wants to share multiple secrets, then repeat the above process, where i = 1, 2, ..., t.

4 Correctness and security

In this section, we mainly account for the correctness analysis and the security of our scheme against four primary attack: dishonest participant attacks, the intercept-and-resend attack, entangle-and-measure attack and collusion attack.

4.1 Correctness analysis

After all participants Bob_1, Bob_2, ..., Bob_t complete their operations, the final quantum state is

\[ |\phi\rangle_t = \left( \prod_{k=0}^{t} U_{p_k, q_k} \right) |\phi\rangle = \left| \sum_{k=0}^{t} q_k \right| \sum_{k=0}^{t} p_k \] (13)

Based on the binary Lagrange interpolation formula, they can calculate s after exchanging classic information p_i, q_i. The last participant Bob_t selects the basis to measure the final state and the measured result R = \sum_{i=0}^{t} p_i is sent to each participant after being encrypted by the shared key, so each participant can recover the secret p_0 = S, where s = \sum_{i=0}^{t} q_i mod d.

4.2 Security analysis

The security of the scheme is analyzed in this section.

4.2.1 Participant attack

One or some of the participants use the random numbers replace the real ones for the unitary operation during the unitary transformation phase. It is checked whether s' calculated by each participant satisfies s' = s in the testing phase is true, so that dishonest participants are found. This indicates that the participant attack is invalid. Therefore the secret cannot be recovered.
### Table 1: comparison among QSS schemes

| Access structure | Lu[9] | Qin[20] | New |
|------------------|-------|---------|------|
| Dimension of space | $d$-threshold | $d$-threshold | $d$-threshold |
| Quantum state | Three qubits | Two qubits | Single-qubit |
| Quantum operations | The unitary operation | The unitary operation | The unitary operation |
| Method | Lagrange interpolation, MUB | Lagrange interpolation | Lagrange interpolation, MUB |
| Verification of secret | Verifiable equation | Hash function | Supervision of the dealer |

**4.2.2 Intercept-and-resend attack**

We suppose that the eavesdropper Eve intercepts the quantum state $|\phi_k^i\rangle$ sent by Bob$_i$ to Bob$_{i+1}$, but he does not know any information about the measurement basis. He can only choose the correct measurement basis with the probability of $1/d$. In addition, the measurement result is

$$S = \sum_{k=1}^{i-1} p_i. \quad (14)$$

If Eve does not know the basis chosen by the participant before, he can only infer the secret of the dealer with the probability of $1/d$. In short, Eve cannot obtain the secret with a probability of exceeding $1/d$ in intercept-and-resend attack.

**4.2.3 Entangle-and-measure Attack**

The eavesdropper Eve entangles the auxiliary quantum state onto the transmitted quantum state or replaces the quantum state with a new entangled state, but the entanglement exchange causes the quantum state to be in a mixed state. There is no way to distinguish, so he cannot obtain any information about the measurement basis and participant’s $p_i, q_i$. This will also make formula and are false, so it will be found by Alice.

**4.2.4 Collusion attack**

Participant collusion is a more destructive attack than an external attack. It is assumed that in the worst case only the dealer Alice and one participant are honest, and the remaining $t-1$ participants will conduct a collusion attack that they exchange their $p_i$ only to get $\sum_{i=1}^{t-1} p_i$ but could not get the original secret $p_0$. Therefore, it is invalid with collusion attack.

**5 Comparison**

Here, we give a comparison of the basic properties in our scheme and other $d$-dimensional QSS schemes. The dealer in Lu’s scheme [9] can share three secrets by delivering three identical states sequentially among participants. To recover the secret, they perform proper unitary operations on a vector of a set of MUBs and the qubits are measured in an appointed basis by the last participant. After that, they exchange the random numbers that are embedded in the qubits and vulnerable to eavesdropper to recover the secrets. Besides, a verifiable $(t, n)$-threshold quantum secret sharing scheme is proposed by using-dimensional Bell state and the Lagrange interpolation. The scheme is verified by Hash function with less verification information. We give a new verifiable QSS scheme based on single qubit in this paper by using sequential communication of a single quantum $d$-dimensional system and the Lagrange interpolation. A detailed comparison of these schemes is presented in Table 1.

So it can be seen that

(a) Our scheme achieves the verifiability with less verification information under the supervision of the dealer Alice.

(b) The scheme adds shared keys to protect the share information that needs to be exchanged, thereby making this scheme be more secure for external attacks.
6 Conclusion

In this paper, a new verifiable \((t, n)\)-threshold quantum secret sharing scheme based on binary polynomial and mutually unbiased bases is proposed in combination with the Ref.[9]. Compared with the original scheme, our scheme makes a dealer supervise whether each participant is honest, thereby achieving verifiability. If the dishonest participants were found during the testing phase, he can be eliminated to prevent internal attacks. This paper also uses a binary polynomial to add a pair of shared keys to ensure confidentiality. Participants can be protected by paired keys when exchanging classic information in the recovery phase, which is more secure than the original one with preventing external attacks. The security analysis illuminates that our scheme can resist the participants attack, the intercept-and-resend attack, the entangle-and-measure attack and the collusion attack. Therefore, the security of the scheme is significantly improved. Compared with other existing verifiable schemes, the verifiable mechanism based on the supervision of the dealer is implemented in this paper, which will not waste extra quantum states with less authentication information.

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