EQUIVARIANT $K$-THEORY CLASSES OF MATRIX ORBIT
CLOSURES

ANDREW BERGET AND ALEX FINK

Abstract. The group $G = \text{GL}_r(k) \times (k^\times)^n$ acts on $\mathbb{A}^{r \times n}$, the space of $r$-by-$n$ matrices: $\text{GL}_r(k)$ acts by row operations and $(k^\times)^n$ scales columns. A matrix orbit closure is the Zariski closure of a point orbit for this action. We prove that the class of such an orbit closure in $G$ equivariant $K$-theory of $\mathbb{A}^{r \times n}$ is determined by the matroid of a generic point. We present two formulas for this class. The key to the proof is to show that matrix orbit closures have rational singularities.

1. Introduction

Let $r$ and $n$ be integers, $r \leq n$, and $\mathbb{A}^{r \times n}$ the affine space of $r$-by-$n$ matrices with entries in an algebraically closed field $k$. We consider the left action of $G = \text{GL}_r(k)$ on $\mathbb{A}^{r \times n}$ by row operations, and the right action of $T^n = (k^\times)^n$ by scaling columns. Let $v \in \mathbb{A}^{r \times n}$ be a matrix, and consider $X_v^a = \text{GL}_r v T^n$, which is the orbit of $\text{GL}_r \times T^n$ through $v$. We call the Zariski closure $X_v = \overline{X_v^a}$ a matrix orbit closure, and it is our primary object of interest. Such varieties were studied in [BF17, BF18] and generalizations of them were studied in [LPST18, Li18].

Write $G = \text{GL}_r \times T^n$. We consider the Grothendieck group of $G$-equivariant coherent sheaves on $\mathbb{A}^{r \times n}$, denoted $K^G_0(\mathbb{A}^{r \times n})$. Since $\mathbb{A}^{r \times n}$ is an affine space, this group can be identified with the representation ring of $G$. As such, the class of a coherent sheaf on $\mathbb{A}^{r \times n}$ can be written as a Laurent polynomial with integer coefficients in variables $u_1, \ldots, u_r, t_1, \ldots, t_n$, which generate this representation ring (see Example 2.1). We view the class of $X_v$ in $K^G_0(\mathbb{A}^{r \times n})$ as a proxy for how complicated $X_v$ is. Essentially this class is also studied in the guise of the multigraded Hilbert series of $X_v$; either invariant is readily extracted from the other.

Our main goal is to prove the following result.

Theorem. Let $v \in \mathbb{A}^{r \times n}$ be any matrix.

(1) The class of $X_v$ in $K^G_0(\mathbb{A}^{r \times n})$ can be determined from the matroid of $v$ alone.
Assume that \( v \) is a rank \( r \) matrix, and denote its matroid by \( M \). Then, the sum

\[
\mathcal{K}(M) = \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - t_j/u_i) \cdot \frac{1}{\prod_{i=1}^{n-1} \left( 1 - t_{w_{i+1}}/t_{w_i} \right)},
\]

a priori a rational function, is a polynomial in \( u_1^{-1}, \ldots, u_r^{-1} \) and \( t_1, \ldots, t_n \), and it represents the class of \( X_v \) in \( K^G_0(\mathbb{A}^{r \times n}) \). Here \([r] = \{1, 2, \ldots, r\}\), \( S_n \) is the symmetric group on \([n]\), and for \( w = (w_1, \ldots, w_n) \in S_n \), \( B(w) \) is the lexicographically first basis of \( M \) in the list \( w \).

This resolves the main conjecture of the authors in [BF18, Conjecture 5.1], and generalizes recent results of Lee, Patel, Spink and Tseng [LPST18, Theorem 9.1] on equivariant Chow classes.

The key to proving this theorem is to show that \( X_v \) has rational singularities. This means, roughly, that the cohomological behavior of the structure sheaf of \( X_v \) does not change on desingularization. One can (partially) resolve \( X_v \) in at least two ways with distinct flavour, and the theorem exploits this: We prove that one resolution of \( X_v \) is rational, and deduce that a second one is. The latter resolution is the one used to construct the class of \( X_v \).

Once we have the rational singularities of \( X_v \), we give proofs of (1) and (2) taking two different paths. To obtain (1) we “lift” the class of a torus orbit closure in a Grassmannian \( Gr(r, n) \), which is known to be a matroid invariant by work of Speyer, to \( \mathbb{A}^{r \times n} \). To obtain (2), whose statement of course implies (1), we use a localization formula of Anderson, Gonzalez and Payne.

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2. Background on equivariant $K$-theory

Let $k$ be an algebraically closed field. A variety will be an integral scheme of finite type over $k$. Let $X$ be a variety with a $G$-action, where $G$ is a linear algebraic group. General references for the material discussed below are [CG97, Chapter 5] and [Mer05].

We let $K^G_0(X)$ denote the Grothendieck group of $G$-equivariant coherent sheaves over $X$. We let $K_0^G(X)$ denote the Grothendieck group of $G$-equivariant vector bundles over $X$. There is a natural group homomorphism, $K^G_0(X) \to K_0^G(X)$. Using the tensor product of vector bundles, $K^G_0(X)$ is a ring, and $K_0^G(X)$ is a module over $K^G_0(X)$. If $X$ is smooth then this map $K^0_0(X) \to K^G_0(X)$ is an isomorphism, as every equivariant coherent sheaf can be resolved by equivariant vector bundles.

If $f : Z \to X$ is a $G$-equivariant proper map then there is a pushforward $f_* : K^G_0(Z) \to K^G_0(X)$ defined by $f_*[\mathcal{F}] = \sum (-1)^i [R^if_*\mathcal{F}]$.

Let $R(G)$ denote the representation ring of $G$. Then $K^G_0(X)$ is a module over $R(G)$. If $X$ is an affine space we can identify $K^G_0(X) = K^G_0(G) = R(G)$.

It is important to emphasize that in our work, $G$ will always be a general linear group, a torus, or a product thereof. In this case $R(G)$ is easy to describe. Let $T$ be a maximal torus of $G$ and let $W$ be the Weyl group of $G$. The representation ring of $T$ is a Laurent polynomial ring $\mathbb{Z}[\text{Hom}(T, k^\times)]$. Then $R(G)$ is the ring of $W$-invariants of $R(T)$.

Let $A = k[x_1, \ldots, x_m]$ and let $\mathbf{A} = \text{Spec}(A)$ be an affine space carrying a $G$-action. An equivariant coherent sheaf on $\mathbf{A}$ is described by an equivariant coherent $A$-module. We describe how to compute the class of such a sheaf in $K^G_0(\mathbf{A}) = R(G) \subset R(T)$. Let $T$ be the maximal torus of $G$ and $d$ its dimension, so that $R(T) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. The ring $k[x_1, \ldots, x_m]$ is graded by the character group of $T$, which is $\text{Hom}(T, k^\times) = \mathbb{Z}^d$. We assume that the grading is positive, in that the degrees of the variables $x_i$ lie in a common open half-space of $\mathbb{Q}^d \supset \mathbb{Z}^d$.

Let $M$ be a finitely generated, $G$-equivariant module over $k[x_1, \ldots, x_m]$. Then $M$ is a $\mathbb{Z}^d$-multigraded $k[x_1, \ldots, x_m]$-module. The multigraded Hilbert series of $M$ is

$$\text{Hilb}(M) = \sum_{a \in \mathbb{Z}^d} \dim_k(M_a) t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d} \in \mathbb{Z}[[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]].$$

Using an equivariant resolution, one can write the Hilbert series as a rational function $K(M)/\prod_{i=1}^n (1 - t^{\deg(x_i)})$, as explained in [MS05, Chapter 8], where $K(M) \in R(G) \subset R(T)$ is referred to as the $K$-polynomial of $M$. The $K$-polynomial $K(M)$ represents the class of the sheaf associated to $M$ in $K^G_0(\mathbf{A})$. 

When $M$ is the coordinate ring of a closed subvariety $Z \subseteq A$, we abuse notation and write $[Z]$ or $K(Z)$ for the class of the sheaf associated to $M$ in $K_0(G(A))$.

**Example 2.1.** Let $G = GL_r \times T^n$ and let $X = A^{r \times n}$. Then, the maximal torus $T$ of $G$ is $T^r \times T^n$, where $T^r$ is the diagonal maximal torus in $GL_r$. We have

$$R(T) = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}],$$

and,

$$R(G) = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]S_r,$$

where $S_r$ is the symmetric group on $[r]$, which permutes the $u$ variables.

Writing $A = k[x_{ij} : 1 \leq i \leq r, 1 \leq j \leq n]$ and $A^{r \times n}$ for $\text{Spec}(A)$, we have that $A^{r \times n}$ has a $G$-action as described in the introduction. We take the sign convention that the character of the $T$-action on the one-dimensional $k$-vector space spanned by $x_{ij}$ is $t_j/u_i$. Now, if $M$ is a $T$-equivariant $A$-module we can write

$$\text{Hilb}(M) = K(M) \prod_{j \in [n]} \prod_{i \in [r]} \frac{1}{1 - t_j/u_i}.$$

### 3. Background on Rational Singularities

A proper birational morphism $f : Z \to Y$ of varieties, where $Z$ is smooth, is called a resolution of singularities. It is called a rational resolution of singularities if

(i) $Y$ is normal, i.e., $O_Y \to f_*O_Z$ is an isomorphism, and

(ii) $R^mf_*O_Z = 0$ for $m > 0$.

We say that $Y$ has rational singularities if there exists a rational resolution of singularities $f : Z \to Y$. We refer the reader to Kollár and Mori [KM98, Section 5.1] for more on these singularities. The following well known results will be needed.

**Proposition 3.1.** If one resolution of singularities of $Y$ is rational, then every resolution is.

**Proposition 3.2.** Let $f : Z \to Y$ be a proper birational morphism, where both $Z$ and $Y$ have rational singularities. Then $f_*O_Z = O_Y$ and $R^mf_*O_Z = 0$ for $m > 0$.

**Proposition 3.3.** Let $Z$ be the total space of a vector bundle over $Y$. If $Y$ has rational singularities then $Z$ does too.
4. Rational singularities of matrix orbit closures

We now begin our study of matrix orbit closures. Matroids enter the story in this section. For those unfamiliar with matroids, a reference with viewpoint similar to our own is [Kat16]; a general text such as [Oxl11] may also be helpful.

The goal of this section is to prove the following theorem.

**Theorem 4.1.** Let \( v \in \mathbb{A}^{r \times n} \) be a rank \( r \) matrix. Then \( X_v \) has rational singularities.

To prove this, we immediately reduce to the case that \( v \) has no zero columns. Let \( p : \mathbb{A}^{r \times n} \rightarrow (\mathbb{P}^{r-1})^n \) denote the natural rational map. Then \( V = p(X_v) \) is the \( \text{GL}_r \) orbit closure of \( p(v) \) in \( (\mathbb{P}^{r-1})^n \). The class of \( V \) in the Chow ring of \( (\mathbb{P}^{r-1})^n \) can be described using a special case of work of Li [Li18, Theorem 1.1]. To describe Li’s result, recall that the Chow ring of \( (\mathbb{P}^{r-1})^n \) is isomorphic to \( \mathbb{Z}[t_1, \ldots, t_n]/(t_1^{r-1}, \ldots, t_n^{r-1}) \). Here, the class of \( t_i \) represents the class of a hyperplane in the \( i \)th factor.

For a matroid \( M \) with rank function \( \text{rk}_M \), define the set

\[
S(M) = \{ s \in \mathbb{N}^n : \sum_{i \in I} s_i < r \text{ rk}_M(I) \text{ for all } I \subseteq [n], \sum_{i=1}^{n} s_i = r^2 - 1 \}.
\]

The elements of \( S(M) \) are all of the lattice points of the Minkowski difference \( rP(M) - \text{conv}\{e_1, \ldots, e_n\} \), where \( P(M) \) is the basis polytope of \( M \) [LPST18, Theorem 5.3].

**Theorem 4.2 ([Li18, Theorem 1.1]).** Let \( M \) denote the matroid of \( v \). The class of \( V \) in the Chow ring \( A^*((\mathbb{P}^{r-1})^n) \) is

\[
\sum_{s \in S(M)} \prod_{i=1}^{n} t_i^{r-1-s_i}.
\]

The variety \( (\mathbb{P}^{r-1})^n \) is a flag variety, and as such its Chow ring has a privileged generating set as a \( \mathbb{Z} \)-module consisting of the classes of Schubert varieties. In the case of \( (\mathbb{P}^{r-1})^n \) the Schubert varieties are products of linear subspaces in each \( \mathbb{P}^{r-1} \), whose Chow classes are exactly the monomials in \( t_1, \ldots, t_n \). It follows from **Theorem 4.2** that when the class of \( V \) is expressed in this Schubert basis, the coefficients involved are either 0 or 1. Thus, \( V \) is multiplicity free in the sense of Brion [Bri03], whose main theorem on such varieties is this.

**Theorem 4.3 ([Bri03, Theorem 1]).** Let \( \mathcal{F} \ell \) be a flag variety of a semisimple algebraic group. Let \( V \subset \mathcal{F} \ell \) be a subvariety whose Chow class is a linear
combination of classes of Schubert varieties, where all coefficients involved are 0 or 1. Then,

1. $V$ is arithmetically normal and Cohen-Macaulay in the projective embedding given by an ample line bundle on $F\ell$;
2. For any globally generated line bundle $\mathcal{L}$ on $F\ell$, the restriction map $H^0(F\ell, \mathcal{L}) \to H^0(V, \mathcal{L})$ is surjective. All higher cohomology groups $H^m(V, \mathcal{L})$, $m \geq 1$, vanish. If $\mathcal{L}$ is ample then $H^m(V, \mathcal{L}^{-1}) = 0$ for $m < \dim(V)$;
3. $V$ has rational singularities.

Proof. Only the third item is not part of the statement of [Bri03, Theorem 1]. For this item [Bri03, Remark 3] applies. Being a homogeneous space, $F\ell = G/P$ is a single $G$-orbit. So if $V$ contains a $G$-orbit, it is $F\ell$ itself; otherwise, $G \cdot V = F\ell$ has rational singularities since it is smooth.

Consider now the line bundle $\mathcal{L}_i$ on $(\mathbb{P}^{r-1})^n$ whose fiber over $(\ell_1, \ldots, \ell_n)$ is the line $\ell_i \subset A^r$. Note that $\mathcal{L}_i^{-1}$ is globally generated for all $i$ and the line bundle $\mathcal{L}_1^{-1-m_1} \otimes \cdots \otimes \mathcal{L}_n^{-1-m_n}$ is ample over $(\mathbb{P}^{r-1})^n$, provided that each $m_i$ is non-negative. We construct the vector bundle

$$E = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n,$$

which is a subbundle of the trivial bundle with fiber $(A^r)^n = A^{r \times n}$. Let $A = k[x_{ij} : 1 \leq i \leq r, 1 \leq j \leq n]$ be the coordinate ring of $A^{r \times n}$. We can identify $H^0((\mathbb{P}^{r-1})^n, \text{Sym}(E^*))$ with $A$.

**Proposition 4.4.** For $V = p(X_v)$, the natural map $A \to H^0(V, \text{Sym}(E^*))$ is surjective.

**Proof.** Decompose $\text{Sym}(E^*)$ as $\bigoplus_{(a_1, \ldots, a_n) \in \mathbb{N}^n} \mathcal{L}_1^{-a_1} \otimes \cdots \otimes \mathcal{L}_n^{-a_n}$. Since $\mathcal{L}_i^{-a_1} \otimes \cdots \otimes \mathcal{L}_n^{-a_n}$ is globally generated on $(\mathbb{P}^{r-1})^n$ we may apply Theorem 4.3(2), and the result follows. 

**Proposition 4.5.** Let $k[X_v]$ denote the coordinate ring of $X_v$. Then there is a $G$-equivariant isomorphism of $A$-modules, $H^0(V, \text{Sym}(E^*)) \approx k[X_v]$.

**Proof.** Recall that $A$ is multigraded by $\text{Hom}(T, k^*) = \mathbb{Z}^r \oplus \mathbb{Z}^n$, where the degree of $x_{ij}$ is $(-e_i, e_j)$. The prime ideal of $X_v$ is homogeneous for this grading, i.e. is generated by homogeneous elements, since $X_v$ is $G$- and hence $T$-invariant.

The identification $A = H^0((\mathbb{P}^{r-1})^n, \text{Sym}(E^*))$ is $G$-equivariant, and the restriction map $A \to H^0(V, \text{Sym}(E^*))$ is too. The kernel of the latter is generated by those homogeneous polynomials in $A$ whose restriction to the orbit $G \cdot v$
is zero; thus the kernel is the prime ideal of \(X_v\), which we have just seen is homogeneous. By Proposition 4.4 we obtain the desired result. □

Our proof of rational singularities now follows quite quickly from the following result of Kempf and Ramanathan.

**Theorem 4.6** (Kempf, Ramanathan [KR87, Theorem 1]). Let \(\mathcal{L}_1, \ldots, \mathcal{L}_n\) be globally generated line bundles on a complete variety \(X\) with rational singularities. If

1. \(H^i(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_n^{m_n}) = 0\) for all \(i > 0\) and all \(m_1, \ldots, m_n \in \mathbb{N}^n\) and
2. \(H^i(X, \mathcal{L}_1^{-1-m_1} \otimes \cdots \otimes \mathcal{L}_n^{-1-m_n}) = 0\) for all \(i < \dim(X)\) and \(m_1, \ldots, m_n \in \mathbb{N}^n\),

then the spectrum of the ring of sections \(\bigoplus_{(m_1, \ldots, m_n) \in \mathbb{N}^n} H^0(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_n^{m_n})\) has rational singularities.

**Proof of Theorem 4.1.** Theorem 4.6 applies to \(X_v\) by Theorem 4.3 and Proposition 4.5. □

5. **Borel-Weil-Bott theorem for torus orbits in Grassmannians**

In this section we use a variant of the Gel’fand-Macpherson correspondence to obtain a variant of the Borel-Weil-Bott theorem for \(T^n\) orbit closures in \(Gr(r, n)\).

The group \(G = GL_r \times T^n\) acts on \(Gr(r, n) = GL_n/P\), where \(T^n\) is the maximal torus of \(GL_n\) and \(GL_r\) acts trivially. Thus, a \(T^n\) orbit closure in \(Gr(r, n)\) is the same thing as a \(G\) orbit closure.

Let \(S\) be the rank \(r\) tautological bundle over the Grassmannian \(Gr(r, n)\). Its fiber over a subspace is precisely that subspace. Recall that the Borel-Weil-Bott theorem for the Grassmannian describes the cohomology groups of various Schur functors applied to \(S^*\). It says, in a weakened form, that the higher cohomology groups of such bundles vanish and gives a formula for their global sections. Our variant of this result is below.

**Theorem 5.1.** Let \(Y\) be a \(T\) orbit closure in \(Gr(r, n)\). Let \(S\) be the tautological bundle over \(Gr(r, n)\), and \(S^\lambda\) be a Schur functor where \(\lambda\) is a partition with at most \(r\) parts. Then for all \(m \geq 1\),

\[
H^m(Y, S^\lambda(S^*)) = 0.
\]

The result will follow by applying Weyman’s geometric method [Wey03, Chapter 5] (cf. [KR87, p. 355, Condition I’]), knowing in advance that \(X_v\) has rational singularities.
In this section we assume that $v$ has rank $r$. This means its row span represents a point in the Grassmannian $Gr(r, n)$.

Let $Y$ be the torus orbit closure through the row space of $v$; $Y$ is a normal toric variety. Let $Z$ be the total space of the vector bundle $S^\oplus_r$ restricted to $Y$, which is a subbundle of the trivial bundle with fiber $(A^n)^r = A^{r \times n}$. We put a $GL_r$ action on $S^\oplus_r$ using the left action on fibers; since this commutes with the natural $T^n$ action we see that $S^\oplus_r$ is a $G$-equivariant vector bundle over $Y$. We have a commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{i} & A^{r \times n} \times Y \\
\downarrow{s'} & & \downarrow{s} \\
X_v & \xrightarrow{i'} & A^{r \times n}
\end{array}
$$

Here the vertical arrows are projections to the first factor, $\pi$ is projection to the second factor and other maps are inclusions. All these maps are $G$-equivariant.

**Proposition 5.2.** The higher direct images $R^m s_* O_Z$, $m \geq 1$, vanish and $s_* O_Z$ is isomorphic to $k[X_v]$ as a $G$-equivariant $A$-module.

**Proof.** The map $s' : Z \to X_v$ is a partial desingularization, in that it is proper and a birational isomorphism: The inverse is the map $u \mapsto (u, \text{rowSpan}(u))$, defined over the set of full rank matrices in $X_v$. Since $Y$ is a toric variety it has rational singularities and thus $Z$ has rational singularities too, by Proposition 3.3. It follows from Proposition 3.2 that $R^m s'_* O_Z = 0$ for $m \geq 1$ and that $s'_* O_Z = O_{X_v}$.

The result follows by viewing these as statements about $G$-equivariant $A$-modules (i.e., applying $i'_*$).

**Proof of Theorem 5.1.** By [Wey03, Theorem 5.1.2(b)] we may identify $R^m s_* O_Z$ with $H^m(Y, \text{Sym}((S^{\oplus_r})^*))$, which is zero for $m \geq 1$. Now use the Cauchy formula [Wey03, Theorem 2.3.2] to write

$$
\text{Sym}((S^{\oplus_r})^*) = \bigoplus_{\lambda} S^\lambda(k^r) \otimes S^\lambda(S^*)
$$

the sum over partitions $\lambda$ with at most $r$ parts. The result follows from the additivity of global sections over direct sums and the linear independence of the Schur functors.

6. **Matroid invariance of the equivariant $K$ class of $X_v$**

To prove the first part of the theorem in the introduction, the matroid invariance of the class of $X_v$ for *any* matrix $v$, we consider the case when $v$ has rank
By [BF18, Proposition 6.6], it is enough to prove the theorem in this special case. We thus assume without loss of generality that \( v \) has rank \( r \) for the remainder of our article.

In this section we give a formula for the equivariant \( K \)-class of \( X_v \) in terms of the class of \( Y \), the \( T \) orbit closure in \( Gr(r, n) \) through the row space of \( v \).

That this formula solves the problem of the matroid invariance of the class of \( X_v \) follows from a result of Speyer.

**Theorem 6.1** (Speyer [Spe09, proof of Prop. 12.5]). The class of a torus orbit closure \( Y \) in \( K^T_0(Gr(r, n)) \) depends only of the matroid of a point in the big orbit of \( Y \).

As mentioned above, the class of \( Y \) can be expressed as a linear combination of classes of Schubert varieties. These have lifts to \( A^{r \times n} \), known as *matrix Schubert varieties* [KM05], which we will use in an analogous expression for the class of \( X_v \).

Let \( \Omega_\lambda \) be a Schubert variety in \( Gr(r, n) \), and let \( X^r_\lambda \) denote the locus of \( r \)-by-\( n \) matrices in \( A^{r \times n} \) whose row space lies in \( \Omega_\lambda \). The closure of \( X^r_\lambda \) in \( A^{r \times n} \), denoted \( X_\lambda \), is a matrix Schubert variety.

**Theorem 6.2.** Let \( v \) be a rank \( r \) matrix.

1. The class of the structure sheaf of \( X_v \) in \( K^G_0(A^{r \times n}) \) can be determined from the matroid of \( v \) alone.
2. The class of \( X_v \) can be written as a \( \mathbb{Z}[t^\pm 1, \ldots, t^\pm n] \)-linear combination of the classes of matrix Schubert varieties, \( X_\lambda \), where \( \lambda_1 \leq n - r \).

**Proof of Theorem 6.2(1).** Let \( Y \subset Gr(r, n) \) be the torus orbit closure through the row span of \( v \). Recall that \( G = GL_r \times T \). Then \( G \) acts on the Grassmannian \( Gr(r, n) \), where the \( GL_r \)-factor acts trivially. So there is a natural isomorphism of \( R(G) = R(GL_r) \otimes R(T) \)-modules,

\[
R(GL_r) \otimes K^G_0(Gr(r, n)) \approx K^G_0(Gr(r, n)).
\]

Under this isomorphism \( 1 \otimes [O_Y] \) maps to \([O_Y] \).

Now, we use [Mer05, Corollary 12] to see that the projection map \( \pi : A^{r \times n} \times Gr(r, n) \to Gr(r, n) \) induces a pullback isomorphism

\[
\pi^* : K^G_0(Gr(r, n)) \to K^G_0(A^{r \times n} \times Gr(r, n)).
\]

The pullback of \( O_Y \) is \( O_{A^{r \times n} \times Y} \). We now multiply by the class of the vector bundle \( S^{\oplus r} \), which is say, the class of its locally free sheaf of sections. The result is the class of \( O_Z \), which is the class of the sheaf of sections of the restriction of \( S^{\oplus r} \) to \( Y \). Now we apply \( s_* \) to \([O_Z] \), which by Proposition 5.2, gives \([O_{X_v}] \). In summary,

\[
[O_{X_v}] = s_*([S^{\oplus r}] \cdot \pi^*[O_Y]),
\]
and since the right side is determined by the matroid of $v$, by Theorem 6.1, so is the left. \hfill $\Box$

**Proof of Theorem 6.2(2).** In $K^G_0(Gr(r, n))$ write $[\mathcal{O}_Y] = \sum_{\lambda} c_{\lambda}[\Omega_{\lambda}]$, where the sum is over partitions $\lambda$ with $\lambda_1 \leq n - r$ and $c_{\lambda} \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ (since $GL_r$ acts trivially on $Y$). A matrix Schubert variety $X_{\lambda}$ has rational singularities [KM05, Theorem 2.4.3] and is (partially) resolved by the total space of the vector bundle $S^\oplus r$ restricted to $\Omega_{\lambda}$, just as we have done for $X_v$ in part (1). Thus,

$$[\mathcal{O}_{X_v}] = s_*([S^\oplus r] \cdot \pi^*[\mathcal{O}_Y])$$

$$= s_*\left([S^\oplus r] \cdot \pi^* \sum_{\lambda} c_{\lambda}[\Omega_{\lambda}]\right)$$

$$= \sum_{\lambda} c_{\lambda}s_*([S^\oplus r] \cdot \pi^*[\Omega_{\lambda}]) = \sum_{\lambda} c_{\lambda}[\mathcal{O}_{X_{\lambda}}].$$

7. **Consequences of matroid invariance**

In this section we state the consequences of the matroid invariance of $[X_v]$, which we studied further in [BF18]. Let $v_1, \ldots, v_n$ denote the columns of the matrix $v$. Write $G(v)$ for the span in $(k^r)^{\otimes n}$ of the tensors

$$(gv_1) \otimes (gv_2) \otimes \cdots \otimes (gv_n), \quad g \in GL_r.$$

Clearly $G(v)$ is a representation of $GL_r$.

**Corollary 7.1.** The class of $G(v)$ in $R(GL_r)$ is determined by the matroid of $v$.

Let $S(v)$ denote the span in $(k^r)^{\otimes n}$ of the tensors

$$v_{w_1} \otimes v_{w_2} \otimes \cdots \otimes v_{w_n}, \quad w \in S_n.$$

Clearly $S(v)$ is a representation of $S_n$.

**Corollary 7.2.** The class of $S(v)$ in $R(S_n)$ is determined by the matroid of $v$.

Describing the irreducible decomposition of these representations was the motivation for studying $X_v$ and its class in $K$-theory. For the proofs of the corollaries, we refer the reader to [BF18].
8. Explicit formula for the $K$-class of $X_v$

In this section we apply a result of Anderson, Gonzales and Payne to compute a formula for the class of $X_v$ in $K^0_G(\mathbb{A}^{r \times n})$. This class is represented as the numerator of the Hilbert series $\text{Hilb}(k[X_v])$, as explained in Section 2.

Let $E$ be a finite dimensional $T$-equivariant vector bundle over $Y$. We use the same symbol $E$ for its sheaf of sections. The equivariant Euler characteristic of $E$ is $\chi_T(E) = \sum_i (-1)^i [H^i(Y, E)]$, where $[-]$ means to compute the character as a representation of $T$.

Since we have shown the $G$-equivariant identification $H^0(Y, \text{Sym}((S^* \oplus r)) = k[X_v]$, and the higher cohomology groups of $\text{Sym}((S^* \oplus r))$ on $Y$ vanish, we can compute the Hilbert series of $k[X_v]$ by computing the $T$-equivariant Euler characteristic of $\text{Sym}((S^* \oplus r))$.

To state the formula for $\chi_T(E)$ we let $T_B Y$ be the Zariski tangent space of $Y$ at a fixed point $B$, and let $C_B \subset T_B Y$ be the tangent cone. The localization formula [And17, Corollary 6.4] states that there is an equality in $Q(u_1, \ldots, u_r, t_1, \ldots, t_n)$, the field of fractions of $R(T)$,

$$\chi_T(E) = \sum_{B \in Y^T} [E_B] \cdot \text{Hilb}(C_B),$$

where the sum is over $T$-fixed points $B$ of $Y$. Above, $\text{Hilb}(C_B) \in Q(u_1, \ldots, u_r, t_1, \ldots, t_n)$ is the Hilbert series of $C_B \subset T_B$ and $E_B$ is the fiber of $E$ over $B$. The formula applies in our case since $Y$ has finitely many $T$-fixed points and the trivial character does not appear in $[T_B Y]$. The Hilbert series $\text{Hilb}(C_B)$ are referred to as “equivariant multiplicities” in loc. cit.

We now compute $\chi_T(\text{Sym}^m((S^* \oplus r)))$ using this formula. The $T$-fixed points of $Y$ correspond to the bases $B$ of the matroid $M$ of $v$. Over a fixed point $B$, we can compute the character of $\text{Sym}((S^* \oplus r))$ to be

$$\frac{1}{\prod_{j \in B} \prod_{i \in [r]} \frac{1}{1 - t_i/u_j}}$$

The equivariant multiplicities were computed in [BF17, Lemma 5.2] as

$$\text{Hilb}(C_B) = \sum_{(w_1, \ldots, w_n)} \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_i+1}/t_{w_i}},$$

where the sum is over those permutations $w \in S_n$ whose lexicographically first basis is $B$. Putting all this together and condensing the summation over fixed
points and permutations gives
\[
\text{Hilb}(k[X_v]) = \chi_T(Sym((S^*)^{\oplus r}) = \sum_{w \in S_n} \prod_{j \in B(w)} \prod_{i \in [r]} \frac{1}{1 - t_j/u_i} \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}},
\]
where \(B(w)\) denotes the lexicographically first basis of the matroid \(M\) occurring in the list \(w = (w_1, \ldots, w_n)\). Finally, to obtain the \(K\)-theory class \(K(k[X_v])\) we multiply by \(\prod_{j \in [n]} \prod_{i \in [r]} (1 - t_j/u_i)\).

**Theorem 8.1.** The class of \(X_v\) in \(K^G_0(A^{r \times n})\) is given by the formula
\[
K(k[X_v]) = \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - t_j/u_i) \cdot \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}}.
\]

Note that the left side is \emph{a priori} a Laurent polynomial in \(u_1^{-1}, \ldots, u_r^{-1}\) and \(t_1, \ldots, t_n\). We have thus shown the claims of the second part of the theorem from the introduction.

**Example 8.2.** Let \(v\) represent the matroid \(M\) of rank \(r = 2\) on \(n = 4\) elements with bases
\[
\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\}.
\]
We first evaluate the equivariant multiplicities. One way to do this is afforded by the proof of [BF17, Lemma 5.2]: they are Hilbert series of affine toric varieties, and therefore lattice point enumerators of cones. The cones in question are the tangent cones to \(P(M)\). For this \(M\), the toric variety \(Y\) is smooth except at the fixed point span\(\{e_1, e_2\}\), so the tangent cones other than \(C_{\{1,2\}}\) are unimodular simplicial cones. We get
\[
\text{Hilb}(C_{\{1,3\}}) = \frac{1}{(1 - t_2/t_3)(1 - t_4/t_3)(1 - t_2/t_1)},
\]
and
\[
\text{Hilb}(C_{\{1,2\}}) = \frac{1 - t_3t_4/t_1t_2}{(1 - t_3/t_2)(1 - t_4/t_2)(1 - t_3/t_1)(1 - t_4/t_1)},
\]
and the other three equivariant multiplicities are images of \(\text{Hilb}(C_{\{1,3\}})\) under transposing 1 and 2 and/or transposing 3 and 4.

The reader may check that the same rational functions are obtained from (1). To facilitate this we describe the permutations \(w\) that achieve \(B = B(w)\) for each basis \(B\). If \(\{w_1, w_2\}\) is a basis of \(M\), then \(B(w) = \{w_1, w_2\}\); this gives four permutations for each basis. The \(4! - 5 \cdot 4 = 4\) permutations unaccounted for are 3412, 3421, 4312, and 4321, whose lexicographically first bases are \(\{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\) respectively.
Using these equivariant multiplicities, the sum in Theorem 8.1 works out to
\begin{equation}
K(k[X_v]) = 1 - t_3t_4/u_1u_2.
\end{equation}

Elements 3 and 4 are parallel, so $M$ is a parallel extension of the uniform matroid $U_{2,3}$, and therefore the above class can also be computed from Theorem 9.3 or Proposition 9.5 of [BF18]. The procedure described in [BF18, Theorem 9.3] is to apply a Demazure divided difference operator $\delta_3$ to the class of the matroid $U_{2,3} \oplus U_{0,1}$, which is $(1 - t_4/u_1)(1 - t_4/u_2)$ since the associated matrix orbit closure is a linear subspace of $\mathbb{A}^{2 \times 4}$. Application of the divided difference gives
\begin{align*}
\delta_3((1 - t_4/u_1)(1 - t_4/u_2)) &= (1 - t_4/u_1)(1 - t_4/u_2)/1 - t_4/t_3 \\
&= 1 - t_3t_4/u_1u_2.
\end{align*}

We check the agreement of this class with Theorem 4.2, first deriving the Chow class featuring in that theorem from (2). The class of $X_v$ in $K_0^{T^n}(\mathbb{A}^{r \times n})$, forgetting the $\text{GL}_r$-action, can be computed by evaluating the $u_i^{-1}$ at 1. The resulting class is $1 - t_3t_4$. The class of $V = p(X_v)$ in $K_0((\mathbb{P}^{r-1})^n) = \mathbb{Z}[t_1, \ldots, t_n]/((1 - t_1)^r, \ldots, (1 - t_n)^r)$ is represented by this same polynomial. Finally, [MS05, Section 8.5] asserts that the class of $V$ in the Chow ring $A^*((\mathbb{P}^{r-1})^n)$ is the sum of terms of lowest degree after substituting $1 - t_i$ for $t_i$ ($i \in [n]$) in the last $K$-class. The substitution yields
\begin{equation}
1 - (1 - t_3)(1 - t_4) = t_3 + t_4 - t_3t_4
\end{equation}
so the Chow class of $V$ is $t_3 + t_4$.

For the other side of the comparison we must compute the set $S(M)$. As singletons have rank 1 in $M$ it follows that $S(M) \subseteq \{0,1\}^4$, and of the remaining inequalities defining $S(M)$ the only one not automatically satisfied once $\sum_{i=1}^4 s_i = 2^2 - 1 = 3$ is $s_3 + s_4 < 2$. Therefore
\begin{equation}
S(M) = \{(1,1,0,1), (1,1,1,0)\},
\end{equation}
from which Theorem 4.2 also produces the Chow class $t_3 + t_4$.

We now address the polynomiality of the rational function
\begin{equation}
K(M) = \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - t_j/u_i) \cdot \prod_{i=1}^{n-1} \frac{1}{1 - t_{w_{i+1}}/t_{w_i}}.
\end{equation}
for arbitrary matroids $M$. 
Theorem 8.3. For any matroid $M$ of rank $r$ on $[n]$, $\mathcal{K}(M)$ is a polynomial in $u_1^{-1}, \ldots, u_r^{-1}, t_1, \ldots, t_n$.

Proof. For a fixed permutation $w \in S_n$ let $B \mapsto [B = B(w)]$ be the indicator function of $B(w)$, and write $D(w)$ for $\prod_{i=1}^{n-1} \frac{1}{1-t_{w_i+1}/t_{w_i}}$. Then,

$$\mathcal{K}(M) = \sum_{w \in S_n} \prod_{j \notin B(w)} \prod_{i \in [r]} (1 - t_j/u_i) \cdot D(w)$$

$$= \sum_{B \in \binom{[n]}{r}} \sum_{w \in S_n} \prod_{j \notin B} \prod_{i \in [r]} (1 - t_j/u_i) \cdot D(w) \cdot [B = B(w)].$$

If $B = \{w_1, \ldots, w_r\}$ with $i_1 < \cdots < i_r$, then $B = B(w)$ if and only if $\text{rk}_M(\{w_1, \ldots, w_i\}) = \max\{j : i_j \leq i\}$ for all $1 \leq i \leq n$. Therefore, viewed as a function of $M$, the function $[B = B(w)]$ is valuative on matroid polytope subdivisions by [DF10, Proposition 5.3]. In the expansion of $\mathcal{K}(M)$ above the only factors depending on $M$ are the expressions $[B = B(w)]$, so $\mathcal{K}(M)$ is a $\mathbb{Q}(u_1, \ldots, u_r, t_1, \ldots, t_n)$-linear combination of valuative functions, and is therefore valuative itself.

The dual to the abelian group of valuative functions is spanned by Schubert matroids for all orderings of the ground set $[n]$ [DF10, Theorem 5.4] (see also the discussion after that work’s Theorem 6.3). That is, if $M$ is an arbitrary matroid, then there exist Schubert matroids $M_1, \ldots, M_k$ and integers $a_1, \ldots, a_k$ such that $\mathcal{K}(M) = \sum_{i=1}^k a_i \mathcal{K}(M_i)$. All Schubert matroids are representable over any infinite field, so each $\mathcal{K}(M_i)$ is a polynomial because it is the $K$-theory class of a sheaf. Thus $\mathcal{K}(M)$ is a polynomial too. $\square$

9. Positivity properties

In this section we address positivity of the class $\mathcal{K}(k[X_v])$ in the sense of Anderson, Griffeth and Miller [AGM11]. We assume that our coefficient field $k$ is the field of complex numbers in this section.

There are two types of positivity that $\mathcal{K}(k[X_v])$ could exhibit, being a simultaneous lift of the class of the $T^n$-equivariant subvariety $Y \subset Gr(r, n)$ as well as the class of the $T^r$-equivariant subvariety $p(X_v) \subset (\mathbb{P}^{r-1})^n$.

In order to state the first positivity result we need to be explicit about which (matrix) Schubert varieties we use to express our classes. For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ we define $X_\lambda$ to be the set of matrices $m \in \mathbb{A}^{r \times n}$ whose column selected submatrix $m_{1,2,\ldots,\lambda_r-j+1+j-1}$ has rank at most $j$, for $1 \leq j \leq r$. 
Proposition 9.1. Let \( v \in \mathbb{A}^{r \times n} \) be a rank \( r \) matrix. Write \( \mathcal{K}(k[X_v]) \) in terms of matrix Schubert varieties,

\[
\mathcal{K}(k[X_v]) = \sum_{\lambda: \lambda_1 \leq n-r} c_\lambda [\mathcal{O}_{X_\lambda}].
\]

Then, the Laurent polynomials \((-1)^{\text{codim}(X_\lambda) - \text{codim}(X_\lambda)} c_\lambda \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\) are polynomials in \( t_2/t_1 - 1, t_3/t_4 - 1, \ldots, t_n/t_{n-1} - 1 \) with positive integer coefficients.

Proof. By the proof of Theorem 6.2(2), it suffices to prove the class of the toric variety \( Y \subseteq Gr(r, n) \) has the analogous property in \( K^T_n(Gr(r, n)) \). This follows from [AGM11, Corollary 5.1]. \( \square \)

Example 9.2. Let \( v \in \mathbb{A}^{2 \times 4} \) be a generic matrix, whose matroid is therefore the uniform matroid \( U_{2,4} \). We have

\[
\mathcal{K}(k[X_v]) = 1 - u_1^{-2}u_2^{-2}t_1t_2t_3t_4,
\]

which can be seen from either Theorem 8.1, [BF18, Proposition 5.2] or the fact that \( X_v \) is a hypersurface in \( \mathbb{A}^{2 \times 4} \). We expand the above polynomial in terms of the classes of the matrix Schubert varieties and obtain

\[
1 - u_1^{-2}u_2^{-2}t_1t_2t_3t_4 = t_1^{-1}t_4[\mathcal{O}_{X_{(2,1)}}] - t_1^{-1}t_4[\mathcal{O}_{X_{(3)}}] - t_1^{-1}t_4[\mathcal{O}_{X_{(1,1)}}] + (t_1^{-1}t_4 + t_1^{-1}t_2^{-1}t_3t_4)[\mathcal{O}_{X_{(1)}}] - (t_1^{-1}t_2^{-1}t_3t_4 - 1)[\mathcal{O}_{X_{(3)}}].
\]

Writing \( \beta_i = t_{i+1}/t_i - 1, 1 \leq i \leq 3 \), we see that

\[
t_1^{-1}t_4 = (\beta_1 + 1)(\beta_2 + 1)(\beta_3 + 1)
\]

and

\[
t_1^{-1}t_2^{-1}t_3t_4 = (\beta_1 + 1)(\beta_2 + 1)^2(\beta_3 + 1)
\]

have positive coefficients in the \( \beta_i \), and the right hand side of the latter has 1 as a term so that \( t_1^{-1}t_2^{-1}t_3t_4 - 1 \) is positive in the \( \beta_i \) as well. These computations verify the result of Proposition 9.1.

Proposition 9.3. Let \( v \in \mathbb{A}^{r \times n} \) have no columns equal to zero. We may write

\[
\mathcal{K}(k[X_v]) = \sum_{\alpha} d_\alpha \prod_{j=1}^{n} \prod_{i=1}^{\alpha_j} (1 - t_j/u_i),
\]

where the sum ranges over compositions \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( 0 \leq \alpha_i \leq r - 1 \), and the coefficients \( d_\alpha \) are Laurent polynomials in \( u_1, \ldots, u_r \). Then, \((-1)^{\text{codim}(X_v) - \sum \alpha_i} d_\alpha\) is a polynomial in \( u_1/u_2 - 1, u_2/u_3 - 1, \ldots, u_{r-1}/u_r - 1 \) with positive integer coefficients.
Proof. As discussed in Section 4, \( (\mathbb{P}^{r-1})^n \) can be regarded as a flag variety. Hence, \( K^{Tr}_0 ((\mathbb{P}^{r-1})^n) \) has a \( \mathbb{Z}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}] \)-module basis given by the classes of Schubert varieties, which now are products of linear spaces indexed by compositions \( \alpha \) of length \( n \) whose parts \( \alpha_i \) satisfy \( 0 \leq \alpha_i \leq r - 1 \). The Schubert variety \( \Sigma_\alpha \) of \( \alpha \) consists of those points \( (p_1, \ldots, p_n) \) where \( (p_i)_j = 0 \) for \( 0 \leq j \leq \alpha_i \), and has codimension \( \sum_i \alpha_i \). The corresponding affine analog \( X_\alpha \) of \( \Sigma_\alpha \) clearly has rational singularities since it is a linear subspace of \( \mathbb{A}^{r \times n} \).

The \( K \)-class of \( X_\alpha \subset \mathbb{A}^{r \times n} \) is readily computed to be \( \prod_{j=1}^n \prod_{i=1}^{\alpha_j} (1 - t_j/u_i) \).

When we work in \( K^{Tr}_0 ((\mathbb{P}^{r-1})^n) \), we have

\[
[\mathcal{O}_{p(X_\alpha)}] = \sum_\alpha d_\alpha [\mathcal{O}_{\Sigma_\alpha}]
\]

to which we can apply \([\text{AGM11, Corollary 5.1}]\) to conclude that \( (-1)^{\text{codim}(X_\alpha) - \text{codim}(\Sigma_\alpha)}d_\alpha \) is a polynomial in \( u_1/u_2 - 1, u_2/u_3 - 1, \ldots, u_{r-1}/u_r - 1 \) with positive integer coefficients. Our goal is to lift this result to \( \mathcal{K}(k[X_v]) \), following the proof of Theorem 6.2.

Let \( q_i \) be the projection of \( \mathbb{A}^{r \times n} \times (\mathbb{P}^{r-1})^n \) to its \( i \)-th factor, and let \( \mathcal{E} \subset \mathcal{A}^{r \times n} \) be the vector bundle of Section 4. As before if \( X \subset \mathbb{A}^{r \times n} \) has rational singularities (and does not lay in the space of matrices with a zero column) then \( q_{1*} ([\mathcal{E}] \cdot q_2^*[\mathcal{O}_{p(X)}]) = [\mathcal{O}_X] \). Then, as all the varieties below have rational singularities,

\[
[\mathcal{O}_{X_v}] = q_{1*} ([\mathcal{E}] \cdot q_2^*[\mathcal{O}_{p(X_v)}])
\]

\[
= q_{1*} \left( [\mathcal{E}] \cdot q_2^* \sum_\alpha d_\alpha [\mathcal{O}_{\Sigma_\alpha}] \right)
\]

\[
= \sum_\alpha d_\alpha q_{1*} ([\mathcal{E}] \cdot q_2^*[\mathcal{O}_{\Sigma_\alpha}])
\]

\[
= \sum_\alpha d_\alpha [\mathcal{O}_{X_\alpha}].
\]

The result follows. \( \square \)

Example 9.4. We continue the above example with a generic 2-by-4 matrix \( v \) and expand

\[ \mathcal{K}(k[X_v]) = 1 - u_1^{-2}u_2^{-2}t_1t_2t_3t_4, \]

as described in Proposition 9.3. We have

\[ 1 - u_1^{-2}u_2^{-2}t_1t_2t_3t_4 = 1 - (u_1/ u_2)^2 \sum_\alpha (-1)^{|\alpha|-1} \prod_{j: \alpha_j > 0} (1 - t_j/u_1) \]

where the sum is over compositions \( \alpha \), with \( \alpha_i \leq 1 \) for all \( i \). The coefficient of \( 1 = [\mathcal{O}_{X_{(0,0,0,0)}}] \) on the right hand side is \( -(u_1/ u_2)^2 - 1 \), and the remaining
coefficients are greater by unity, \( \pm (u_1/u_2)^2 \). Since
\[
(u_1/u_2)^2 - 1 = (u_1/u_2 - 1)^2 + 2(u_1/u_2 - 1),
\]
we have verified the proposition.

By Theorem 8.3 we know that for any matroid \( M \) of rank \( r \) on \( n \) elements, \( K(M) \) can be expanded in terms of either the classes of the matrix Schubert varieties \( X_\lambda, \lambda \) a partition fitting in a \( r \)-by-(\( n-r \)) box, or the classes of the matrix Schubert varieties \( X_\alpha \), where \((\alpha_1, \ldots, \alpha_n)\) satisfies \( 0 \leq \alpha_i \leq r - 1 \). For a matroid \( M \) that is not realizable over a field it is unknown if \( K(M) \) will exhibit the positivity properties of Propositions 9.1 and 9.3.

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**Western Washington University, Bellingham, WA, USA**

*E-mail address:* andrew.berget@wwu.edu

**Queen Mary University of London, London, UK**

*E-mail address:* a.fink@qmul.ac.uk