The Capacity of SWIPT Systems over Rayleigh-Fading Channels with HPA

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Abstract—In this paper, we study the fundamental limits of simultaneous information and power transfer over a Rayleigh fading channel in the presence of high-power amplifier (HPA) nonlinearity. In particular, a three-party communication system is considered, where a transmitter aims simultaneously conveying information to an information receiver and delivering energy to an energy harvester receiver. We study the information-energy capacity region and the associated input distribution under: i) average-power, peak-power (PP) constraints at the transmitter, b) HPA nonlinearity at the transmitter, and c) nonlinearity of the energy harvesting circuit at the energy receiver. By extending Smith’s mathematical framework [1], we show that the optimal input distribution under those constraints is discrete with a finite number of mass points. Moreover, we derive a closed-form expression of the capacity-achieving distribution for the low PP regime, where there is no trade-off between information and energy transfer. Finally, we show that HPA significantly reduces the information energy capacity region.

Index Terms—SWIPT, wireless power transfer, high-power amplifier, optimal input distribution, information-energy capacity region, HPA.

I. INTRODUCTION

Simultaneous information and power transfer (SWIPT) is a technology that exploits the duality of the radio frequency (RF) signals, which can carry both information and energy [2] through appropriate co-design and engineering. The idea of wireless power transfer (WPT) was first proposed by Tesla in the 20-th century [3], and now presents a promising solution for modern communication systems such as low-power short-range communication systems, sensor networks, machine-type networks, and body-area networks [4]. The notion of the information-energy capacity region for SWIPT systems, it was first formalized by Varshney [5] in the context of point-to-point scenarios. This work has been extended in [6] for a parallel links point-to-point channel. More recent works study the integration of SWIPT to more complex network topologies e.g., multiple access channel [7], interference channel [8], multiple-input multiple-output [9], multiple-antenna cellular networks [10], etc. A comprehensive overview of existing results in SWIPT for various fundamental multi-user channels is presented in [11].

The design of the WPT component is crucial in order to characterize SWIPT systems. Most of the literature assumes simple linear models for the RF energy harvester (EH) receiver [9], [12] to simplify analysis. However, one of the main particularities of a SWIPT network is that the WPT channel is highly nonlinear (in contrast to the linear information transfer channel). Recent studies take into account the nonlinearity of the rectification circuit, and study the impact of waveform design and/or input distribution on the achieved information energy capacity region. For instance, the work in [13] models the rectifier behavior and introduces a mathematical framework to design waveforms that exploit nonlinearity. This observation introduces a relevant question for SWIPT networks: “what is the fundamental limits of a SWIPT system with a non-linear EH receiver?”

The problem was first formalized in [14] by considering a truncated Taylor expansion series approximation for the diode’s characterization function over an additive white Gaussian noise (AWGN) channel. The authors have shown that the optimal input distribution is zero-mean complex Gaussian distribution, with an asymmetric power allocation for the real and the imaginary parts. However, a more general model was proposed in [15], by using the exact form of the diode’s characteristic function. The authors have extended Smith’s mathematical framework [1] and have shown that the optimal input distribution under the first the and second moments statistics as well as a peak power (PP) constraint at the transmitter, it is unique, discrete with a finite number of mass points.

On the other hand, experimental studies demonstrate that signals with high peak-to-average-power-ratio (PAPR) e.g., multi-sine, chaotic signals, white noise, etc [16], provide a higher direct-current (DC) output, in comparison to constant-envelop sinusoidal signals [17], [18]. However, signals with high PAPR are more sensitive to high-power amplifier (HPA) nonlinearities, which significantly degrade the quality of the communication [19], [20]. With the exception of a few studies (e.g., [21]), existing works do not consider the effects of HPA on SWIPT performance and assume that the HPA operates always in the linear regime. In addition, most of the aforementioned studies on SWIPT systems focus on simple AWGN channels and therefore the impact of the channel fading has not been investigated. To the best of the authors’ knowledge, this is the first work that takes into account the effect of HPA’s non-linearity on SWIPT systems over fading channels from an information theoretic standpoint.
Fig. 1. A SWIPT system over a fading channel with HPA at the transmitter and a non linear EH channel.

Specifically, this paper studies the fundamental limits of SWIPT over a Rayleigh-fading channel by taking into account a memoryless HPA model at the transmitter. We consider a basic SWIPT system, where a transmitter simultaneously sends data to an information receiver and power to an EH receiver through a Rayleigh-fading channel; we consider average power (AP) and PP constraints at the transmitter as well as a non-linear power transfer channel. We characterize the information energy capacity region and we show that the associated capacity-achieving input distribution is unique, discrete, with a finite number of mass points. This study generalizes the result for the capacity-achieving input distribution of a discrete-memoryless Rayleigh-fading channel under AP, which has been studied in [22]. We show that HPA significantly reduces the information energy capacity region, while increasing the PP constraint enlarges the associated region. Finally, we study the optimal input distribution for the low PP regime, where no trade-off between information and energy transfer is observed.

II. SYSTEM MODEL

Consider a three part communication system, where a transmitter aims simultaneously convey information to an information receiver (IR) and energy to an EH receiver through a Rayleigh-fading channel. The IR converts the received signal to the baseband to decode the transmit information, while the EH receiver harvests energy from the received RF signal. In each channel use, the transmitter inputs a pulse-amplitude modulated signal \( x(t) = \sum_{k=-\infty}^{\infty} x[k] p(t-kT) \), with an average power \( P \), where \( p(t) \) is the rectangular pulse shaping filter (i.e., \( p(t) = 1 \) for \( 0 < t \leq T \), \( T \) is the symbol interval, and \( x[k] \) is the information symbol at time index \( k \), modeled as the realization of an independent and identically distributed (i.i.d) real random variable \( X \) with a cumulative distribution function \( F \). We assume a normalized symbol interval \( T = 1 \) and thus the measures of energy and power become identical and therefore are used equivalently.

The system model is depicted in Fig. 1. The transmitted amplitude-modulated signal \( x(t) \) is subjected to nonlinearities induced by the HPA; the output of the nonlinear HPA can be written as \( \hat{x}(t) = d(x[k]) \) (i.e., random variable \( \hat{X} = d(X) \)), where \( d(\cdot) \) denote the AM-to-AM conversion which is given by the considered solid state power amplifier (SSPA) HPA model [19] i.e.,

\[
d(r) = \frac{r}{1 + \left( \frac{r}{A} \right)^{2\beta}},
\]

where \( A \) is the output saturation voltage, and \( \beta \) represents the smoothness of the transition from the linear regime to the saturation.

A. Information transfer

From information transmission standpoint, we consider a memoryless discrete-time Rayleigh-fading channel proposed in [22]; the channel output at the receiver during the channel use \( t \) is given by

\[
y(t) = h_{1,t} \hat{x}(t) + Z_t,
\]

where \( \hat{x}(t) \) is the channel input induced by the HPA and \( h_{1,t} \) and \( Z_t \) are independent complex circular Gaussian random variables distributed as

\[
h_{1,t} \sim \mathcal{CN}(0, \sigma_1^2),
\]

\[
Z_t \sim \mathcal{CN}(0, \sigma_2^2).
\]

Since the phase of the fading parameter \( h_{1,t} \) is uniform, an equivalent channel with nonnegative input \( \hat{X} \) and nonnegative output \( Y \) was proposed in [22]. The conditional probability of this channel is given by

\[
p(y|\hat{x}) = \frac{1}{1 + \hat{x}^2} \exp \left( -\frac{y}{1 + \hat{x}^2} \right).
\]

B. Power transfer

At the EH receiver, the contribution of the noise is assumed to be negligible, hence, the representation of received signal at the EH receiver (in the baseband) is given by

\[
s(t) = h_2 \hat{x}(t),
\]

where \( h_2 \in \mathbb{R} \) is the channel fading for the link between the transmitter and the EH receiver; it is assumed to be static in this work. In practical SWIPT environments, the EH receivers are located close to the transmitter (e.g., a line of sight link). Hence, we can assume that the channel fading \( h_2 \) (associated to the power transfer) is a constant [15]. Let \( \mathcal{E} : \mathbb{R} \to \mathbb{R}_+ \) be the function that determines the average energy harvested [15], which is given by

\[
\mathcal{E}(\hat{X}) = \mathbb{E} \left[ I_0 \left( \sqrt{2} Bh_2 |\hat{X}| \right) \right],
\]

where \( I_0(\cdot) \) is the modified Bessel function of the first kind and order zero and \( B \) is a constant that depends on the characteristics of the rectification circuit. The EH constraint is reduced as

\[
\mathcal{E}(\hat{X}) \geq E_{\text{req}}.
\]

C. Problem formulation

The main objective is to maximize the average mutual information between \( X \) and \( Y \) subject to AP and PP constraints on the transmit symbols \( X \), and a minimum harvested power constraints at the EH receiver. The optimization problem can be written as

\[
\sup_{F \in \mathcal{F}} I(F) = \int \int p(y|x) \log \frac{p(y|x)}{p(y; F)} dydF(x),
\]

subject to \( \mathbb{E}[X^2] \leq P \),

\[
\mathcal{E}(\hat{X}) \geq E_{\text{req}}.
\]
Theorem 1. The capacity $C$ is achieved by a unique input distribution $F^*$, i.e.,

$$C = \sup_{F \in \Omega} I(F) = I(F^*).$$

Proof: The proof is presented in [23].

Note that in the case where the EH receiver is omitted, the problem is reduced to the information capacity of a Rayleigh-fading channel [22]; in this case, it has been shown that the optimal input distribution is discrete with a finite number of mass points, even without considering the AP constraint.

Corollary 1. The strong duality holds for the optimization problem in (14), i.e., there exist constants $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that

$$C = \sup_{F \in \Omega} I(F) - \lambda_1 g_1(F) - \lambda_2 g_2(F),$$

Proof: The proof follows similar arguments as [15] and [22] and is presented in [23].

The following theorem establishes a necessary and sufficient condition on the optimal input distribution.

**Corollary 2.** Let $E_0$ be the point of increase of a distribution $F^*$, then $F^*$ is the optimal input distribution, if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, such that

$$\lambda_1 (x^2 - P) - \lambda_2 \left( I_0(\sqrt{2} Bh_2 x) - E_{\text{req}} \right) + C - \int p(y|x) \log \frac{p(y|x)}{p(y; F^*)} dy \geq 0,$$

for all $x$, with equality if $x \in E_0$.

Proof: The proof follows the same arguments as [15] and [22] and is presented in [23].

By using the invertible change of variables, i.e., $s = \frac{1}{1+s^2}$, we have

$$p(y|s) = s \exp(-ys), \quad s \in (0, 1].$$

The following proposition holds by using the new random variable $S$.

**Proposition 1.** $S^*$ is the optimal input distribution on $(0, 1]$, if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, such that

$$\lambda_1 \left( \frac{1}{s} - 1 - P \right) - \lambda_2 \left( I_0(\sqrt{2} Bh_2 \sqrt{\frac{1}{s} - 1}) - E_{\text{req}} \right) + C - \log s + 1 + \int_0^\infty s e^{-sy} \log p(y; F^*) dy \geq 0,$$

for all $s \in (0, 1]$, with equality if $S^*(x) \neq 0$.

In the following, we will show that the equality in Proposition 1 cannot be satisfied in a set that has an accumulation point, hence the support of $S^*$ must be discrete. The discreteness property of the optimal input distribution is given by the following theorem.

**Theorem 2.** The optimal input distribution that achieves the capacity in (9) is discrete with a finite number of mass points.

Proof: The proof is presented in Appendix A.

## III. MAIN RESULTS

### A. Discreteness of the optimal input distribution

In this section, we study the properties of the capacity achieving distribution, i.e., the solution of the optimization problem in (9). By extending the mathematical framework proposed in [1], Theorem 1 establishes the existence and the uniqueness of the optimal input distribution. By using the Lagrangian Theorem, the dual equivalent problem is given by Corollary 2. Furthermore, we give necessary and sufficient conditions for the optimal input distribution in Corollary 2. Finally, we show that the capacity-achieving input distribution is discrete in Theorem 2.

**Theorem 1.** The capacity $C$ is achieved by a unique input distribution $F^*$, i.e.,

$$C = \sup_{F \in \Omega} I(F) = I(F^*).$$

Proof: The proof is presented in [23].

Note that in the case where the EH receiver is omitted, the problem is reduced to the information capacity of a Rayleigh-fading channel [22]; in this case, it has been shown that the optimal input distribution is discrete with a finite number of mass points, even without considering the AP constraint.

**Corollary 1.** The strong duality holds for the optimization problem in (14), i.e., there exist constants $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that

$$C = \sup_{F \in \Omega} I(F) - \lambda_1 g_1(F) - \lambda_2 g_2(F),$$

Proof: The proof follows similar arguments as [15] and [22] and is presented in [23].

The following theorem establishes a necessary and sufficient condition on the optimal input distribution.

**Corollary 2.** Let $E_0$ be the point of increase of a distribution $F^*$, then $F^*$ is the optimal input distribution, if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, such that

$$\lambda_1 (x^2 - P) - \lambda_2 \left( I_0(\sqrt{2} Bh_2 x) - E_{\text{req}} \right) + C - \int p(y|x) \log \frac{p(y|x)}{p(y; F^*)} dy \geq 0,$$

for all $x$, with equality if $x \in E_0$.

Proof: The proof follows the same arguments as [15] and [22] and is presented in [23].

By using the invertible change of variables, i.e., $s = \frac{1}{1+s^2}$, we have

$$p(y|s) = s \exp(-ys), \quad s \in (0, 1].$$

The following proposition holds by using the new random variable $S$.

**Proposition 1.** $S^*$ is the optimal input distribution on $(0, 1]$, if there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, such that

$$\lambda_1 \left( \frac{1}{s} - 1 - P \right) - \lambda_2 \left( I_0(\sqrt{2} Bh_2 \sqrt{\frac{1}{s} - 1}) - E_{\text{req}} \right) + C - \log s + 1 + \int_0^\infty s e^{-sy} \log p(y; F^*) dy \geq 0,$$

for all $s \in (0, 1]$, with equality if $S^*(x) \neq 0$.

In the following, we will show that the equality in Proposition 1 cannot be satisfied in a set that has an accumulation point, hence the support of $S^*$ must be discrete. The discreteness property of the optimal input distribution is given by the following theorem.

**Theorem 2.** The optimal input distribution that achieves the capacity in (9) is discrete with a finite number of mass points.

Proof: The proof is presented in Appendix A.
conditions which are satisfied by the optimal input distribution for some $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ are

$$i(x; F_N^*) \leq C + \lambda_1(x^2 - P) - \lambda_2(\mathcal{E}(x) - E_{req}),$$
$$i(x^*_1; F_N^*) = C + \lambda_1(x^*_{1}^2 - P) - \lambda_2(\mathcal{E}(x^*_1) - E_{req}).$$

(21)

Denote by $g$ the following function i.e.,

$$g(w, F_N^*) = i(w; F_N^*) - \lambda_1(w^2 - P) + \lambda_2(\mathcal{E}(w) - E_{req}).$$

(22)

It has been shown in [24] that a point of increase (except $A$) is a local maximum for the function $g(x, F_N^*)$, and hence

$$\frac{\partial g(w, F_N^*)}{\partial w} \bigg|_{w=x_1} = 0.$$

(23)

Unfortunately, a closed form solution for (23) is cumbersome even for the simpler case (AWGN channel). However, in the next remark, we are able to characterize a particular mass point of the optimal input distribution.

**Remark 1.** *The input distribution has a necessary mass point at zero; by contradiction, we assume that $0 < x_1$, then it can be shown

$$\frac{\partial g(w, F_N^*)}{\partial w} \bigg|_{w=x_1} < 0.$$*

Hence, $x_1$ is not a point of increase for the function $g(x, F_N^*)$ according to (23).

In the following, we study the behavior of the optimal input distribution at the transition point, where the binary distribution gives away to a ternary [24]. Specifically, we give a closed-form expression of the optimal input distribution on the transition point, where the binary distribution is not longer optimal. This is a critical point in SWIPT systems, since the binary distribution maximizes both information and energy transfer simultaneously and therefore there is not a trade-off between them [21]. For simplicity, we assume that the AP constraint is active, for a low PP constraint, the optimal input distribution is binary and is characterized by $x^*$ and $q$, i.e.,

$$x^* = (0, x_1),$$

(25a)

$$q = \frac{1 - \frac{P}{x_1^*}}{x_1^*},$$

(25b)

Note that if

$$\frac{\partial I(F_N^*)}{\partial x_1} > 0,$$

then we have $x_1 = A$. Let us assume that at the amplitude value $A$, the binary distribution is not longer optimal. Thus at the amplitude value $A + \Delta A$, a new mass point appears denoted by $x_2^* \in [0, A]$, which satisfies

$$i(x_2^*; F_N^*) = C + \lambda_1(x_2^* - P) - \lambda_2(\mathcal{E}(x_2^*) - E_{req}).$$

(27)

Let $q$ the transition probability associated to the mass point $x_2^*$, then the optimal input distribution $F_N^*$ is characterized by

$$q^* = \left(1 - \frac{P}{x_1 + \Delta x_1}\right) - \frac{1 - \frac{x_2^*}{x_1 + \Delta x_1}}{\left(\frac{x_2^*}{x_1 + \Delta x_1}\right)^2},$$

(28)

$$x^* = (0, x_2^*, x_1 + \Delta x_1).$$

(29)

**Remark 2.** *Note at that for low PP constraints, there is not a trade-off between information and energy transfer, since the optimal input distribution is binary and hence it maximizes both information and energy transfer simultaneously. By increasing the PP constraint, we show that there exist an amplitude value $A$, in which the binary distribution is not longer optimal and hence, a trade-off between information and energy transfer is observed. The transition point $A$ at the low PP regime, satisfies the following condition

$$\mathcal{I}(F_A^*(A + \Delta A)) > \mathcal{I}(F_A^*(A + \Delta A)).$$

Hence by choosing $\Delta$ to be small enough ($\Delta A \to 0$), we obtain a sufficient condition for the transition point $A$ [25].*

**IV. Numerical Results**

In the previous section, we characterized the optimal input distribution for a Rayleigh-fading channel with AP, PP, and EH constraints in the presence of HPA. Now, we numerically evaluate the information-energy capacity region by using a numerical solver such CVX [26]. Fig. 2 shows the information-energy capacity region for different PP constraints. The corresponding region is obtained by solving the optimization problem in (9). A trade-off is observed between the information rate transmitted to the information decoder and the energy delivered to the EH receiver; this trade-off becomes evident since for higher EH constraints, the transmitter selects
Fig. 3. Effect of the HPA on the Information-energy capacity region; \( A = 5 \), \( \beta = 1 \), \( B = 0.5 \), \( P = 30 \text{ dB} \).

a symbol with a higher amplitude and thus it degrades the information transfer performance. Another interesting remark is that for low amplitude constraints, there is no trade-off between the two objectives. The optimal input distribution for this regime is binary, hence it maximizes both the information and the energy transfer simultaneously (Remark 2).

Finally, Fig. 3 highlights the effect of the HPA non-linearity on the information-energy capacity region. It can be seen that there is a gap between the two regions; this is mainly due to the negative effect of the HPA.

V. Conclusion

In this paper, we studied the fundamental limits of a SWIPT system over a Rayleigh-fading channel with non-linear EH, and by taking into account the non-linearity imposed by the HPA. We proved that the input distribution that maximizes the information-energy capacity region is unique, discrete, with a finite number of mass points. Also, we proposed a mathematical framework to study the capacity achieving distribution for low PP constraints, where there is not a trade-off between information and energy transfer. Finally, we have shown that the information energy capacity region increases by relaxing the PP constraint, while the HPA significantly degrades the performance of both objectives.

Appendix A
Proof of Theorem 2

Assuming that \( S^* \) is not discrete, then the support of \( S^* \) has a limit point by the Bolzano-Weierstrass theorem [27]. Denote by \( h : z \rightarrow h(z) \) the following function

\[
h(z) = \lambda_1 \left( \frac{1}{s} - 1 - P \right) - \lambda_2 \left( I_0(\sqrt{2} Bh_2) - E_{\text{req}} \right) + C - \log z + 1 + \int_0^\infty ze^{-z} \log p(y) dy, \quad z \in \mathcal{D},
\]

with \( \mathcal{D} \) defined by \( \Re(z) > 0 \). By extending the necessary and sufficient conditions of Proposition 1 to the complex domain, we have

\[
h(z) = 0, \quad z \in \text{Supp}(S^*).
\]

Recall that the support of \( S^* \) has an accumulation point and the function \( h(z) \) is analytic over the domain \( \mathcal{D} \), hence by applying the identity theorem [1]

\[
h(z) = 0, \quad z \in \mathcal{D}.
\]

By using the expression in (33), then for all \( s \in [0, 1] \), we have

\[
\int_0^\infty se^{-s} \log p(y) dy = -\frac{1}{s} \left[ \lambda_1 \left( \frac{1}{s} - 1 - P \right) - \lambda_2 \left( I_0(\sqrt{2} Bh_2) - E_{\text{req}} \right) + C - \log s + 1 \right].
\]

The left hand side in (34) is the unilateral Laplace transform of the function \( \log p(y) \), while the right-hand side (without the Bessel function) can be recognized as the Laplace transform of

\[
-\lambda_1 y + [\lambda_1 (1 + a) - C - 1 - C_E] - \log y,
\]

where \( C_E \) is Euler’s constant. The modified Bessel function is given by

\[
I_0 \left( \sqrt{2} Bh_2 \right) \left( \frac{1}{s} - 1 \right) = \sum_{n=0}^\infty a_n \left( \frac{1}{s} - 1 \right)^n
\]

with \( a_n = \frac{(Bh_2/\sqrt{2})^2n}{n!} \). By using the fact that

\[
\mathcal{L}^{-1} \left( \frac{1}{s^k} \right) = \frac{y^{n-1}}{n!},
\]

and by taking into account the uniqueness of the Laplace transform for continuous functions with a bounded variation, the following holds

\[
p(y) = K \exp(-\lambda_1 y) \times \exp \left( \lambda_2 \sum_{n=0}^\infty a_n \sum_{k=0}^n \frac{n!}{(k+1)!} \frac{y^k}{(k+1)} (\frac{1}{s} - 1)^{n-k} \right).
\]

For every \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), we have

\[
\int_0^\infty p(y) dy > \infty,
\]

hence, \( p(y) \) cannot be a probability distribution, and \( \text{Supp}(S^*) \) cannot have an accumulation point; which means that the optimal input distribution is discrete.
