ON THE RANKS OF THE ADDITIVE AND
THE MULTIPLICATIVE GROUPS OF A BRACE

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Abstract. In [Bac16, Theorem 2.5] Bachiller proved that if $(G, \cdot, \circ)$
is a brace of order the power of a prime $p$ and the rank of $(G, \cdot)$
is smaller than $p - 1$, then the order of any element is the same
in the additive and multiplicative group. This means that in this
case the isomorphism type of $(G, \circ)$ determines the isomorphism
type of $(G, \cdot)$.

In this paper we complement Bachiller’s result in two directions.
In Theorem 2.2 we prove that if $(G, \cdot, \circ)$ is a brace of order the
power of a prime $p$, then $(G, \cdot)$ has small rank (i.e. $< p - 1$) if and
only if $(G, \circ)$ has small rank. We also provide examples of groups
of rank $p - 1$ in which elements of arbitrarily large order in the
additive group become of prime order in the multiplicative group.
When the rank is larger, orders may increase.

1. Introduction

Let $L/K$ be a finite field extension, and let $H$ be a finite cocom-
mutative $K$-Hopf algebra. $H$ defines a Hopf-Galois structure on $L/K$
if there exist a $K$-linear map $\mu: H \mapsto \text{End}_K(L)$ giving $L$ a left $H$
module algebra structure and inducing a $K$-vector space isomorphism
$K \otimes_K H \mapsto \text{End}_K(L)$. This notion was introduced by Chase
and Sweedler in [CS69]. Greither and Pareigis in [GP87] showed that finding
the Hopf-Galois structures can be reduced to a group-theoretic
problem.

In the particular case when $L/K$ is a Galois extension we can state
Greither and Pareigis results as follows.

Theorem. Let $L/K$ be a finite Galois extension of fields and let $\Gamma =
\text{Gal}(L/K)$. There is a bijective correspondence between the set of iso-
morphism classes of Hopf-Galois structures on $L/K$ and the set of reg-
ular subgroups $G$ of the group $S(\Gamma)$ of permutations on the set $\Gamma$, which

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are normalised by the image $\rho(\Gamma)$ of the right regular representation $\rho$ of $\Gamma$.

(In this paper we use the right regular representation $\rho$. In the literature it is more common to use the left regular representation $\lambda$.)

The groups $G$ and $\Gamma$ have the same cardinality but they need not be isomorphic. A Hopf-Galois structure $H$ is called of type $G$, if $G$ is the group associated to $H$ in the Greither-Pareigis correspondence.

Childs [Chi89] and Byott [Byo96] observed that the condition that $\rho(\Gamma)$ normalises $G$ can be reformulated by saying that $\Gamma$ is contained in the holomorph $\text{Hol}(G)$ of $G$, regarded as a subgroup of the group $S(G)$ of permutations on the set $G$, an advantage being that $\text{Hol}(G)$ is usually much smaller than $S(\Gamma)$. Therefore, the number of Hopf-Galois structures on $L/K$ of type $G$ can be computed in terms of the number of regular subgroups of the holomorph $\text{Hol}(G)$ of $G$, which are isomorphic to $\Gamma$ (see [Byo96, Corollary p. 3320]). In particular, $L/K$ admits a structure of type $G$ precisely when the holomorph $\text{Hol}(G)$ of $G$ contains a regular subgroup isomorphic to $\Gamma$.

The study of Hopf-Galois structures, or equivalently, of regular subgroups of holomorphs, is strictly related to the theory of skew (left) braces. In fact, if $G$ is a group with respect to the operation `·", classifying the regular subgroups of $\text{Hol}(G)$ is equivalent to determining the operations `◦" on $G$ such that $(G, \cdot, \circ)$ is a (right) skew brace [GV17a], that is, $(G, \circ)$ is also a group, and the two group structures on the set $G$ are linked by the identity

$$ (g \cdot h) \circ k = (g \circ k) \cdot k^{-1} \cdot (h \circ k), \quad (1.1) $$

for all $g, h, k \in G$. This connection was first observed by Bachiller in [Bac16 §2] and it is described in detail in the appendix to [SV18]. If $(G, \cdot, \circ)$ is a skew brace, $(G, \cdot)$ is its additive group, and $(G, \circ)$ its multiplicative group. A (left) brace can be defined as a skew brace with a commutative additive group, but the theory of braces predates that of skew braces [Rum07].

In recent years, these different approaches concurred to construct a rich theory. A number of papers are devoted to enumerating Hopf-Galois structures on Galois extensions of degree of a particular form ([Byo96] Koh98 Byo04 Chi05 Zen18 AB20d AB20a AB20b CDC20). Part of the literature is devoted to understanding the group-theoretic relation between $\Gamma$ and $G$ for a Galois extension with Galois group isomorphic to $\Gamma$ could admit a Hopf-Galois structure of type $G$ or not ([FCC12], Byo13, Byo15, Bac16, Tsa19, Nas19, TQ20).

In the language of skew braces, this correspond to understanding how properties of $(G, \cdot)$ influence those of $(G, \circ)$, and vice versa, when $(G, \cdot, \circ)$ is a skew brace.

In [Bac16, Theorem 2.5] Bachiller proved that if $(G, \cdot, \circ)$ is a brace of order the power of a prime $p$ and the rank of $(G, \cdot)$ is lower than
p − 1, then the order of any element is the same in the additive and multiplicative group. This means that in this case the isomorphism type of \((G, \cdot)\) determines the isomorphism type of \((G, \cdot)\). Bachiller’s Theorem 2.5 generalises \([FCC12, \text{Theorem } 1]\) using a similar argument.

In this paper we adopt the same point of view as in \([FCC12]\) and \([Bac16]\) and study some relations between the additive and the multiplicative group of a brace, focusing on the order of the elements. Our first result is Theorem 2.2 in which we prove that the rank of \((G, \cdot)\) is lower than \(p − 1\) if and only if the rank of \((G, \circ)\) is lower than \(p − 1\) (see Definition 2.1). This result builds upon \([Bac16, \text{Theorem } 2.5]\).

When the rank of \((G, \cdot)\) is \(p − 1\), we show in Proposition 2.11 that the orders of elements may only decrease when going from \((G, \cdot)\) to \((G, \circ)\); in Proposition 2.10 we provide examples in which elements of arbitrarily large order in the additive group become of prime order in the multiplicative group. When the rank is larger than \(p − 1\), orders of elements may also increase.

The paper is enriched with some corollaries in which we specify the consequences of our result in the Hopf-Galois context.

Section 2 contains the statements of our results, and a description of the method of the gamma function (see \([CCDC20]\)) we use. In Section 3 we prove Theorem 2.2 and Proposition 2.11. In Section 5 we prove Proposition 2.10.

2. Statements

Given a group \(G = (G, \cdot)\), write \(S(G)\) for the group of permutations on the set \(G\). Let \(\rho\) be the right regular representation of \(G\),

\[
\rho : G \to S(G) \\
g \mapsto (x \mapsto x \cdot g)
\]

The (permutational) holomorph \(\text{Hol}(G)\) is the normaliser

\[
\text{Hol}(G) = N_{S(G)}(\rho(G))
\]

of the image \(\rho(G)\) of \(\rho\) in \(S(G)\). \(\text{Hol}(G)\) is isomorphic to the (abstract) holomorph \(\text{Aut}(G) \rtimes G\) of \(G\).

In the holomorph of an abelian \(p\)-group \(G\) of order at least \(p^3\) one can find (non-abelian) regular subgroups which are not isomorphic to \(G\). In the other direction, by \([Bac16, \text{Theorem } 2.5]\) (quoted below as Theorem A), when the group \(G\) is small, then the isomorphism class of a regular subgroup of \(\text{Hol}(G)\) does determine the isomorphism class of \(G\).

Let \(N\) be a regular subgroup of \(S(G)\). The map \(\nu : G \to N\), that takes \(g \in G\) to the unique element \(\nu(g)\) of \(N\) such that \(1^{\nu(g)} = g\) is a bijection. (We write the action of permutations as exponents.) Using \(\nu\) for transport of structure from \(N\) to \(G\) yields a group operation “\(\circ\)”
on $G$ such that $\nu(g \circ h) = \nu(g)\nu(h)$, so that \\
$\nu : (G, \circ) \to N$ \\
is an isomorphism. Moreover $g^{\nu(h)} = g \circ h$, for $g, h \in G$.

As in [CCDC20, Section 2], to which we refer for further details, we have that the regular subgroup $N$ normalises $\rho(G)$ if and only if there is a function $\gamma : G \to \text{Aut}(G)$ such that \\
$\nu(g) = \gamma(g)\rho(g), \text{ for } g \in G. \quad (2.1)$

We thus have \\
$h \circ g = h^{\nu(g)} = h^{\gamma(g)\rho(g)} = h^{\gamma(g)} \cdot g.$

The functions $\gamma : G \to \text{Aut}(G)$ appearing in (2.1) are characterised \\
by the functional equation \\
$\gamma(h^{\gamma(g)} \cdot g) = \gamma(h)\gamma(g)$ for $h, g \in G.$

As in [CCDC20], we call this equation the gamma functional equation, \\
and refer to the functions $\gamma$ that satisfy it as gamma functions.

A (right) skew brace [GV17b] is a triple $(G, \cdot, \circ)$, where $G$ is a set, \\
“$\cdot$” and “$\circ$” are two group operations on $G$, and the following brace axiom holds for $a, b, c \in G$ \\
$$((a \cdot b) \circ c) \cdot c^{-1} = (a \circ c) \cdot c^{-1} \cdot (b \circ c) \cdot c^{-1}. \quad (2.2)$$

If $(G, \cdot, \circ)$ is a skew brace, then (2.2) and the fact that $(G, \circ)$ is a group yield that for all $c \in G$ the maps $G \to G$ given by \\
$\gamma(c) : a \mapsto (a \circ c) \cdot c^{-1}$

are automorphisms of $G$, and that $\gamma : G \to \text{Aut}(G)$ is a gamma function. Conversely, if $\gamma : G \to \text{Aut}(G)$ is a gamma function, then $(G, \cdot, \circ)$ is a skew brace, where $a \circ b = a^{\gamma(b)} \cdot b$, for $a, b \in G$. Moreover, the set $N = \{\nu(g) : g \in G\}$ of the functions \\
$$\nu(g) : G \to G$$  

$$x \mapsto x \circ g$$

is a regular subgroup of $\text{Hol}(G)$ isomorphic to $(G, \circ)$. (See for instance the discussion after Theorem 2.2 in [CCDC20].)

In the following, given a group $G = (G, \cdot)$, we will make use without 

further mention of the equivalence described in this section among the following concepts:

1. a regular subgroup $N$ of $\text{Hol}(G)$,
2. a gamma function on $G$,
3. a skew brace with additive group $(G, \cdot)$.

Note that braces predate skew braces [Rum07, GV17b], but a brace may be defined as a skew brace whose additive group is abelian.

If $(G, \cdot, \circ)$ is a (skew) brace, we refer to $(G, \cdot)$ as its additive group, 
and to $(G, \circ)$ as its multiplicative group.
In the paper [Bac16] Bachiller, generalizing the main result of [FCC12], proved that if \((G, \cdot, \circ)\) is a brace of order the power of a prime \(p\), and the rank of the abelian group \((G, \cdot)\) is less than \(p - 1\) then the isomorphism type of \((G, \circ)\) determines the isomorphism type of \((G, \cdot)\). This result depends on the fact that in these braces, the order of any element is the same in the additive and multiplicative group.

**Theorem A** ([Bac16, Theorem 2.5]). Let \(p\) be a prime, and let \((G, \cdot, \circ)\) be a brace of order a power of the prime \(p\), with \((G, \cdot)\) of rank \(< p - 1\).

Then each element has the same order in \((G, \cdot)\) and \((G, \circ)\).

Moreover, if \((G, \cdot)\) is abelian, then \((G, \cdot)\) and \((G, \circ)\) are isomorphic.

In her recent paper [Cre20], Teresa Crespo considers the same question in the Hopf Galois context. In the language of braces her result states that if \((G, \cdot, \circ)\) is a brace of order \(p^n\) where \(p \geq 3\) is a prime and \(n < p\), then the isomorphism type of \((G, \circ)\) determines the isomorphism type of \((G, \cdot)\). Her result [Cre20, Theorem 6] improves Theorem A in the case when \(p\) is odd and \((G, \cdot)\) is elementary abelian.

In this paper we complement Theorem A in two directions. In Theorem 2.2 we give a sort of converse of Theorem A, and in Proposition 2.11 we consider the case when the rank of \((G, \cdot)\) is \(p - 1\), improving [Cre20, Theorem 6].

**Definition 2.1.** Let \(\mathcal{G}\) be a finite \(p\)-group, for a prime \(p\). Its rank \(r_p\) is the maximum \(r\) such that \(\mathcal{G}\) has a subgroup of exponent \(p\) and order \(p^r\). We say that \(\mathcal{G}\) has small rank if \(r_p < p - 1\)

Let \(\mathcal{G}\) be a finite group, and let \(p\) be a prime dividing its order. We say that has small p-rank if a Sylow \(p\)-subgroup of \(\mathcal{G}\) has small rank.

We say that \(\mathcal{G}\) has small rank if it has a small p-rank for each prime \(p\) dividing its order.

The following theorem will be proved in Section 3.

**Theorem 2.2.** Let \(p\) be a prime, and let \((G, \cdot, \circ)\) be a brace of order a power of the prime \(p\).

Then \((G, \cdot)\) has small rank if and only if \((G, \circ)\) has small rank.

When these conditions hold, \((G, \cdot)\) and \((G, \circ)\) have the same rank, and the same number of elements of each order.

**Remark 2.3.** Yakov Berkovich has shown in [Ber00, Proposition 1.6(a)] (see also the paragraph preceding Proposition 7.8 of [Ber02], and [Ber05, Lemma 3(a)]) that a finite \(p\)-group of small rank is regular in the sense of Philip Hall [Hal34].

The following corollaries of Theorem 2.2 will be proved in Section 3.

The first one generalises [Cre20, Theorem 6] for the case of groups of rank \(< p - 1\) and is indeed a consequence of [Bac16, Theorem 2.5].

**Corollary 2.4.** Let \(p\) be a prime, and let \(G_1, G_2\) be two abelian groups of order the same power of a prime \(p\), with \(G_1\) of small rank.
Let $N_i$ be a regular subgroup of the holomorph $\text{Hol}(G_i)$ of $G_i$, for $i = 1, 2$.

If $N_1 \cong N_2$, then $G_1 \cong G_2$. In particular, $G_2$ has also small rank.

**Corollary 2.5.** Let $L/K$ be a Galois extension of degree a power of a prime $p$, let $\text{Gal}(L/K) = \Gamma$, and let $r$ denotes the rank of $\Gamma$. Then

1. If $r < p - 1$ (that is, $\Gamma$ has small rank) and $L/K$ admits abelian Hopf-Galois structures, then all of them are of the same type and the group $G$ associated to these structures is determined by $\Gamma$ and has the same rank as $\Gamma$. In particular, if $\Gamma$ is abelian, then every abelian Hopf Galois structure on $L/K$ is of type $\Gamma$.

2. If $r \geq p - 1$ (that is, $\Gamma$ does not have small rank), then every abelian group giving a Hopf-Galois structure on $L/K$ has rank $\geq p - 1$.

**Remark 2.6.** Corollary 2.5 delimits the types of the possible abelian Hopf-Galois structure on a Galois extension of prime power order, dividing the abelian $p$-groups in two rigid classes, accordingly with their rank.

However, it gives no indication on the question of whether a Galois extension with a non-abelian Galois group $\Gamma$ does admit at least an abelian Hopf-Galois structure. This is the same as asking whether the (non-abelian) groups $\Gamma$ can be the multiplicative group of a brace. A necessary condition is provided by Byott, who showed in [Byo15, Theorem 1] that for $L/K$ could admit a nilpotent Hopf-Galois structure its Galois Group $\Gamma$ must be solvable. In particular this implies that the multiplicative group of a brace is solvable (see also [ESS99, Theorem 2.15]).

In the paper [Bac16] Bachiller considered the converse, asking whether any finite solvable group is the multiplicative group of a brace, giving to this question a negative answer. In fact, he provided an example of a $p$-group $\Gamma_0$ of order $p^{10}$ with all elements of order $p$ which, for some large value of the prime $p$, is not the multiplicative group of a brace. Therefore, the Galois extensions with Galois group isomorphic to $\Gamma_0$ do not admit abelian Hopf-Galois structures. Since all $p$-groups are realisable as Galois group over $\mathbb{Q}$, then this actually happens also in the number field context.

**Remark 2.7.** In [Cre20, Remark 9] Crespo points out that a Galois extension with Galois group $C_9 \times C_3 \times C_3$ has Hopf Galois structure of types $C_2^3$ and $C_3^4$, in addition to the classical one. This shows that under the hypothesis of Corollary 2.5 (2), a Galois extension can have abelian Hopf-Galois structures of different types.

**Corollary 2.8.** Let $(G, \cdot, \circ)$ be a brace of finite order.

Let $p$ be a prime divisor of the order $G$. 

Then the $p$-rank of $(G, \cdot)$ is small (that is, $< p - 1$) if and only if the $p$-rank of $(G, \circ)$ is small. In this case the Sylow $p$-subgroups of $(G, \circ)$ and $(G, \cdot)$ have the same number of elements of each order.

In particular, $(G, \cdot)$ has small rank if and only if $(G, \circ)$ has small rank. If this is the case, and $(G, \circ)$ is abelian then $(G, \cdot) \cong (G, \circ)$.

**Corollary 2.9.** Let $L/K$ be a finite Galois extension and let $\text{Gal}(L/K) = \Gamma$.

If $\Gamma$ has small rank, and $L/K$ admits abelian Hopf-Galois structures, then all of them are of the same type $G$, and the group $G$ is determined by $\Gamma$.

In particular, if $\Gamma$ is abelian, then every abelian Hopf Galois structure on $L/K$ is of type $\Gamma$.

The following proposition shows that the second part of Theorem 2.2 fails when $(G, \cdot)$ (or equivalently $(G, \circ)$) does not have small rank.

**Proposition 2.10.** For every $k > 1$, there is a brace $(G, \cdot, \circ)$ of $p$-power order, with $(G, \cdot)$ of rank $p - 1$, with the following properties.

1. $(G, \circ)$ is non-abelian.
2. There is a maximal subgroup $H$ of $(G, \cdot)$, such that every element of $H$ has the same order in $(G, \cdot)$ and $(G, \circ)$.
3. Every element $g \in G \setminus H$ has order $p^k$ in $(G, \cdot)$, and order $p$ in $(G, \circ)$.

In the examples of Proposition 2.10 the order of an element does not increase when going from $(G, \cdot)$ to $(G, \circ)$. This is actually a general fact, as shown in the following

**Proposition 2.11.** Let $(G, \cdot, \circ)$ be a brace of $p$-power order, with $(G, \cdot)$ of rank $p - 1$.

Then the order of an element does not increase when going from $(G, \cdot)$ to $(G, \circ)$.

In the case when $(G, \cdot)$ is elementary abelian the previous proposition gives the following corollary, which was already covered in [Cre20].

**Corollary 2.12.** Let $(G, \cdot, \circ)$ be a brace of $p$-power order, with $(G, \cdot)$ isomorphic to $C_p^m$ with $m \leq p - 1$. Then each element has the same order in $(G, \cdot)$ and $(G, \circ)$. In particular if $(G, \circ)$ is abelian, then it is isomorphic to $(G, \cdot)$.

When the rank of $G$ reaches $p$, the order of an element may increase when going from $(G, \cdot)$ to $(G, \circ)$, as shown for instance by [FCC12, Example 8].

The braces $(G, \cdot, \circ)$ of Proposition 2.10 are bi-skew braces ([Chi19][Car20]), so that $(G, \circ, \cdot)$ is a skew brace (with non-abelian additive group $(G, \circ)$) in which the order of an element may increase when going from $(G, \circ)$ to $(G, \cdot)$.
3. PROOFS OF THEOREM 2.2 AND PROPOSITION 2.11

For a finite $p$-group $(H, \ast)$, and $i \geq 0$, we will denote by $\Omega_i(H, \ast)$ the set of elements of $(H, \ast)$ of order dividing $p^i$.

We begin with proving Proposition 2.11 which is the particular case $m = p - 1$ of the following lemma.

**Lemma 3.1.** Let $p$ be a prime.
Let $G = (G, \cdot)$ be a finite abelian $p$-group of $p$-rank $m$. Let $(G, \cdot, \circ)$ be a brace.
If $m \leq p - 1$, then for each $i \geq 0$ one has

$$\Omega_{i+1}(G, \cdot) \subseteq \Omega_{i+1}(G, \circ). \quad (3.1)$$

In other words, the order of an element does not increase when going from $(G, \cdot)$ to $(G, \circ)$.

Note that in the Lemma the $\Omega_i(G, \cdot)$ are characteristic subgroups of the abelian group $(G, \cdot)$. On the other hand, the $\Omega_i(G, \circ)$ need not be subgroups of $(G, \circ)$; in the examples of Section 5 no $\Omega_i(G, \circ)$ is a subgroup of $(G, \circ)$, for $0 < i < \log_p(\exp(G))$. In these examples we have $m = p - 1$; however, when $m < p - 1$ Remark 2.3 implies that the $\Omega_i(G, \circ)$ are also subgroups of $(G, \circ)$.

**Proof of Lemma 3.1.** For $g \in G$, write

$$\delta(g) = -1 + \gamma(g) \in \text{End}(G, \cdot).$$

It is immediate to see that for $g \in G$ we have, for the $p$-th power $g^{op}$ of $g$ in $(G, \circ)$,

$$g^{op} = g^{\gamma(g)^{p-1} + \cdots + \gamma(g) + 1}.$$

from which we obtain the formula

$$g^{op} = g^{p+\binom{p}{2}\delta(g)+\cdots+\binom{p}{p-1}\delta(g)^{p-2}} g^{\delta(g)^{p-1}}. \quad (3.2)$$

(We write simply $g^{p}$, $g^{\binom{p}{2}}$, etc. for the powers of $g$ in $(G, \cdot)$.)

We first assume $m \leq p - 1$, and prove (3.1).

Let us start with the case $i = 0$. Let $g \in \Omega_1(G, \cdot)$. Then in (3.2) we have $g^{op} = g^{\delta(g)^{p-1}}$. Now $\gamma(G)\Omega_1(G, \cdot)$ is a finite $p$-group, therefore nilpotent, with $\Omega_1(G, \cdot)$ a normal subgroup of it. Therefore for all $i \geq 1$ we have

- either $[\Omega_1(G, \cdot), \gamma(G), \ldots, \gamma(G)] = 1$,
- or $[\Omega_1(G, \cdot), \gamma(G), \ldots, \gamma(G)] > [\Omega_1(G, \cdot), \gamma(G), \ldots, \gamma(G)]_{i-1}$.

Since $\Omega_1(G, \cdot)$ has order at most $p^{p-1}$, we obtain

$$[\Omega_1(G, \cdot), \gamma(G), \ldots, \gamma(G)]_{p-1} = 1.$$
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Now note that for $h, g \in G$, one has that $h^\delta(g)$ equals the commutator $[h, \gamma(g)]$ in the (abstract) holomorph of $G$. We thus obtain $g^\delta(g)^{p-1} = 1$, and we are done.

Proceeding by induction, we take $i \geq 1$, and assume

$$\Omega_i(G, \cdot) \subseteq \Omega_i(G, \circ),$$

and we prove

$$\Omega_{i+1}(G, \cdot) \subseteq \Omega_{i+1}(G, \circ).$$

Let $g \in \Omega_{i+1}(G, \cdot)$. We have $g^p \in \Omega_i(G, \cdot)$, so that by (3.2) the following are equivalent

1. $g^p \in \Omega_i(G, \cdot)$, and
2. $g^{\delta(g)^{p-1}} \in \Omega_i(G, \cdot)$.

Now $M = \Omega_{i+1}(G, \cdot)/\Omega_i(G, \cdot)$ is an elementary abelian section of $G$, invariant under automorphisms, of order at most $p^{i-1}$, so $g^{\delta(g)^{p-1}} \in \Omega_i(G, \cdot)$. We have obtained that $g^p \in \Omega_i(G, \cdot) \subseteq \Omega_i(G, \circ)$, so that $g \in \Omega_{i+1}(G, \circ)$. □

We recall the following observation from [CCDC20]

**Lemma 3.2** ([CCDC20, Proposition 2.6]). Let $G = (G, \cdot)$ be a finite group, let $H \subseteq G$ and let $\gamma$ be a GF on $G$.

Any two of the following conditions imply the third one:

1. $H \leq G$;
2. $(H, \circ) \leq (G, \circ)$;
3. $H$ is $\gamma(H)$-invariant.

If these conditions hold, then $(H, \circ)$ is isomorphic to a regular subgroup of $\text{Hol}(H)$.

We turn now to the proof of Theorem 2.2. As we noted above, the “if” part of this theorem is [Bac16, Theorem 2.5].

Conversely, we have to prove that if $(G, \circ)$ has small rank, then $(G, \cdot)$ has also small rank.

Consider the finite $p$-group $K = \gamma(G)G$. Since $\Omega_1(G)$ is a characteristic subgroup of $G$, and $G$ is a normal subgroup of $K$, it follows that $\Omega_1(G)$ is a normal subgroup of $K$. Since $K$ is nilpotent, and $\Omega_1(G) \leq K$, there is a central series of $K$ which goes through $\Omega_1(G)$. Refining this series to a principal series, we obtain that for each divisor of the order of $\Omega_1(G)$ there is a subgroup of $\Omega_1(G)$ which is normal in $K$, and thus in particular invariant under $\gamma(G)$. Thus if the rank of $(G, \cdot)$ is greater than or equal to $p - 1$, so that the order of $\Omega_1(G)$ is greater than or equal to $p^{p-1}$, then there will be a subgroup $T$ of $\Omega_1(G)$ of order $p^{p-1}$ and exponent $p$, invariant under $\gamma(G)$.

Now, Lemma 3.2 ensures that $(T, \circ)$ is a subgroup of $(G, \circ)$, and by Proposition 2.11 of Section 2 $(T, \circ)$ has exponent $p$. This gives a contradiction.
4. PROOFS OF THE COROLLARIES

Proof of Corollary 2.4. A regular subgroup \( N_i \) of \( \text{Hol}(G_i) \) corresponds to a brace \((G_i, \cdot, \circ)\) with additive group \( G_i \) and multiplicative group \((G_i, \circ) \cong N_i \). By Theorem 2.2 if the rank of \( G_1 \) is \( < p-1 \) then the group \( N_1 \) determines the structure of \( G_1 \). Therefore, \( N_1 \cong N_2 \) implies \( G_1 \cong G_2 \), and clearly the rank of \( G_2 \) is the same as the rank of \( G_1 \). □

Proof of Corollary 2.5. Let \( G \) be an abelian group defining a Hopf Galois structure on \( L/K \). Then we define on \( G \) an operation \( \cdot \) such that \( \Gamma \cong (G, \cdot, \circ) \) and \((G, \cdot, \circ)\) is a brace. The corollary follows from Theorem 2.2 applied to \((G, \cdot, \circ)\). □

For the next proof, note that Lemma 3.2 can be reformulated as follows in the language of skew braces:

Corollary 4.1. Let \((G, \cdot, \circ)\) be a skew brace and let \((H, \cdot)\) be a subgroup \((G, \cdot)\).
Then \((H, \cdot, \circ)\) is a sub-skew brace of \((G, \cdot, \circ)\) if and only if \((H, \cdot)\) is invariant under \( \gamma(H) \).

Proof of Corollary 2.8. For \( p \) a prime divisor of \(|G|\), let \( G_p \) denote the Sylow \( p \)-subgroup of \((G, \cdot)\). Since \( G_p \) is characteristic, by Lemma 3.2 and Corollary 4.1 \((G_p, \circ)\) is a subgroup, actually a Sylow \( p \)-subgroup, of \((G, \circ)\) and \((G_p, \cdot, \circ)\) is a sub-brace of \((G, \cdot, \circ)\). Theorem 2.2 applied to \((G_p, \cdot, \circ)\) gives the first part, and applied to all prime divisors of \(|G|\) gives the second one. □

The proofs of Corollaries 2.9 and 2.12 are immediate.

5. PROOF OF PROPOSITION 2.10

In this section we construct the examples of Proposition 2.10. These are based on the unique pro-\( p \) group of maximal class, whose construction we now recall.

Let \( p \) be a prime, and \( \mathbb{Z}_p \) be the ring of \( p \)-adic integers. Let \( \omega \) be a primitive \( p \)-th root of unity. \( \omega \) has minimal polynomial
\[ x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}_p[x] \]
over \( \mathbb{Z}_p \), so that the ring \( \mathbb{Z}_p[\omega] \), when regarded as a \( \mathbb{Z}_p \)-module, is free of rank \( p-1 \).

The ring \( \mathbb{Z}_p[\omega] \) is a discrete valuation ring, with maximal ideal \( I = (\omega - 1) \). Consider the automorphism \( \alpha \) of the group \( E = (\mathbb{Z}_p[\omega], +) \) given by multiplication by \( \omega \). Clearly \( \alpha \) has order \( p \) in \( \text{Aut}(E) \).

The infinite pro-\( p \)-group of maximal class is
\[ M = \langle \alpha \rangle \ltimes E. \]

For \( p = 2 \) this is the infinite pro-2-dihedral group, in which all elements outside \( E \) have order 2. In general, we have the following
Fact 5.1. All elements \( m \in M \setminus E \) have order \( p \).

This fact is a statement about the abstract holomorph \( \text{Aut}(E) \times E \), and then clearly an analogous fact holds true in the permutational holomorph.

Proof. If \( g \in E \) and \( 0 < i < p \), we have

\[
(\alpha^i g)^p = \alpha^{\frac{p^i}{p-1}} g^{\alpha(p-1) + \alpha(p-2) + \ldots + \alpha + 1}
= g^{\alpha(p-1) + \alpha(p-2) + \ldots + \alpha + 1}
= (\omega^i(p-1) + \omega^i(p-2) + \ldots + \omega^i + 1)g
= 0
\]
since for \( 0 < i < p \), \( \omega^i \) is a conjugate of \( \omega \), that is it has the same minimal polynomial.

Consider the group morphism \( \gamma : E \rightarrow \text{Aut}(E) \) which has kernel \( I \), and then takes the value \( \gamma(1) = \alpha \) on 1.

Since for \( g \in E \) one has

\[
[g, \alpha] = g^{-1+\alpha} = (-1 + \omega)g \in I,
\]
we have \( [E, \gamma(E)] = \ker(\gamma) \), and thus \( \gamma \) is a gamma function, according to [CCDC20, Lemma 2.13].

Such a gamma function \( \gamma \) thus defines

1. a group operation

\[
g \circ h = g^{\gamma(h)} \cdot h
\]
on \( E \) such that \( (E, \cdot, \circ) \) is a brace, and

2. equivalently a regular subgroup \( N \) of \( \text{Hol}(E) \) given by

\[
N = \{ \gamma(g)\rho(g) : g \in E \}.
\]

Writing \( \nu(g) = \gamma(g)\rho(g) \), we have \( h^{\nu(g)} = h \circ g \), and the map \( \nu : (E, \cdot, \circ) \rightarrow N \) is an isomorphism of groups.

Clearly “\( \cdot \)” and “\( \circ \)” coincide on \( I = \ker(\gamma) \). We now prove that every element \( g \in E \setminus I \) has order \( p \) in \( (E, \circ) \). Since \( \nu \) is an isomorphism of groups, this is equivalent to showing that all elements \( \gamma(g)\rho(g) \) of \( N \) with \( \gamma(g) \neq 1 \) have order \( p \), and this is Fact 5.1 above.

The structure of \( (E, \circ) \) is easily seen. Write \( u = 1 \) in \( E \) for clarity. \((I, \circ) \cong (I, +)\) is an abelian normal subgroup of \( (E, \circ) \), and then for \( h \in I \) one has, keeping in mind that \( \gamma(h) = 1 \) and \( \gamma(u) = \alpha \),

\[
u^{-1} \circ h \circ u = -u^{\gamma(u)-1} \gamma(h) \gamma(u) + h \gamma(u) + u = h \gamma(u) = h^\alpha = \omega h.
\]
Here \( u^{-1} \) is the inverse of \( u \) in \( (E, \circ) \), and we are using [CCDC20 Lemma 2.10].

To finish the proof of Proposition 2.10, consider, for a given \( k > 1 \), the quotient group

\[
(G, +) = E/I^k(p-1).
\]
\(\alpha\) induces an automorphism of \(G\), which we still call \(\alpha\). Write \(H = I/I^{k(p-1)}\). As above, the function \(\gamma : G \rightarrow \text{Aut}(G)\) that has kernel \(H\), and such that \(\gamma(u + I^{k(p-1)}) = \alpha\), is a gamma function, which defines an operation “\(\circ\)” on \(G\). As above, the elements of \(G\) have the same order in \((H, +)\) and \((H, \circ)\), while the elements of \(G \setminus H\) have order \(p^k\) in \((G, +)\), and order \(p\) in \((G, \circ)\).

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