A Unified Approach on the Local Power of Panel Unit Root Tests

Zhongwen Liang*
University at Albany, SUNY

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Abstract

In this paper, a unified approach is proposed to derive the exact local asymptotic power for panel unit root tests, which is one of the most important issues in non-stationary panel data literature. Two most widely used panel unit root tests known as Levin-Lin-Chu (LLC, Levin, Lin and Chu (2002)) and Im-Pesaran-Shin (IPS, Im, Pesaran and Shin (2003)) tests are systematically studied for various situations to illustrate our method. Our approach is characteristic function based, and can be used directly in deriving the moments of the asymptotic distributions of these test statistics under the null and the local-to-unity alternatives. For the LLC test, the approach provides an alternative way to obtain the results that can be derived by the existing method. For the IPS test, the new results are obtained, which fills the gap in the literature where few results exist, since the IPS test is non-admissible. Moreover, our approach has the advantage in deriving Edgeworth expansions of these tests, which are also given in the paper. The simulations are presented to illustrate our theoretical findings.

Keywords: local-to-unity; Edgeworth expansion; asymptotic moment; characteristic function.

JEL Classification: C12; C22; C23.

*Department of Economics, University at Albany, SUNY, Albany, NY 12222, USA.
E-mail: zliang3@albany.edu
1 Introduction

Since the circulation of working papers of Quah (1994), Breitung and Meyer (1994), and Levin and Lin (1992, unpublished manuscript), tremendous efforts have been made to construct and understand panel unit root tests. Levin et al. (2002) and Im et al. (2003) tests are among the ones that are the most widely used and influential, even though there are other tests proposed in the literature such as Breitung (2000), Ploberger and Phillips (2002), Moon and Phillips (2004), and Moon and Perron (2004). The LLC and IPS papers have received extremely high citations, which are among the most cited econometrics papers. There are excellent reviews in this area, such as Baltagi and Kao (2000), Phillips and Moon (2000), Choi (2006), Breitung and Pesaran (2008), and Westerlund and Breitung (2012).

To evaluate the performance of these test statistics, the local power is the major concern. Moon, Perron and Phillips (2007) gave comprehensive discussions on deriving the Gaussian power envelopes for different scenarios, especially with incidental intercepts or incidental trends. They showed that under the homogeneous alternative, some tests such as a t-test constructed in the paper and the optimal invariant test in Ploberger and Phillips (2002) could achieve the power envelope in different situations. Furthermore, they proposed the corresponding point optimal invariant panel unit root test for each scenario. However, the local asymptotic power of IPS test was not discussed but compared with other tests using simulations, since it is shown in Bowman (2002) to be non-admissible.

The majority of the literature on the local power of panel unit root tests rely on simulations, for example Maddala and Wu (1999) and Im et al. (2003), except some on the asymptotic limits such as Breitung (2000), Moon et al. (2006, 2007), Breitung and Pesaran (2008), Moon and Perron (2008), Harris et al. (2010), and Westerlund and Breitung (2012). In this paper, a new unified approach is proposed to explore the exact local asymptotic power of LLC and IPS tests, which utilizes the results of the Fredholm approach that targets directly on the characteristic functions of Dickey-Fuller tests, and that were extensively discussed in a series of papers by Nabeya and Tanaka (1988, 1990a, 1990b). Through this method, we are able to obtain the analytical forms of the local asymptotic power of LLC and IPS tests in different scenarios.
In this paper, following similar setups in Levin et al. (2002), Im et al. (2003) and Moon et al. (2007), we discuss three scenarios in nonstationary panel data models, i.e., (i) without fixed effects; (ii) with incidental intercepts; (iii) with both incidental intercepts and incidental trends. We consider both homogeneous and heterogeneous alternatives. The analytical local asymptotic power of both LLC test and IPS test are derived for all scenarios. Moreover, since we directly target on moments, one advantage of our method is to obtain the Edgeworth expansion for the LLC and IPS tests under both null and local-to-unity alternatives. The one term Edgeworth expansions for the LLC and IPS tests are also derived in the paper.

There is another strand of methods based on Fisher-type statistics such as Maddala and Wu (1999) and Choi (2001). As discussed in Bowman (2002), they are not admissible either. Until now, the local power of these tests are not very clear. Our method might also work for analyzing such tests. However, due to the extreme complexity of the problem, we leave it for further research. There is another literature on the second generation panel unit root tests for panel data models with cross section dependence, such as Bai and Ng (2004), Breitung and Das (2005, 2008) etc., see Hurlin and Mignon (2007), Breitung and Pesaran (2008), Westerlund and Breitung (2012) and references therein. Our method could also be used in obtaining the exact local asymptotic power in these settings, which we leave for future research.

The rest of paper is organized as follows. Section 2 introduces our unified approach in obtaining the asymptotic moments of test statistics involving unit root processes by summarizing the basic results of the Fredholm approach in Nabeya and Tanaka (1990a, 1990b) and extending these results to panel unit root data. Section 3 is devoted to derivations of the exact local asymptotic power of LLC and IPS tests under three different scenarios to illustrate our unified approach. In section 4 we obtain the one term Edgeworth expansions for both LLC and IPS tests by utilizing our approach. Section 5 gives the simulation results. Section 6 concludes the paper. The main steps of proofs are gathered in the Appendix. A comprehensive supplemental material of proofs is also available.
2 A unified approach

The traditional way to obtain the local asymptotic power is to derive the asymptotic limit of the test statistics under the local alternatives. However, this may lead to some expectations that are hard to compute if not possible at all, or it will keep the degenerate terms in the limiting expression which could be canceled out but cannot be seen directly. This is the case especially for the panel unit root tests. It will be clearer in the next two sections and the supplemental material. In contrast to the existing method, we propose a unified approach which is based on deriving the moments through the joint characteristic function.

To fix the idea, we describe it here. The typical panel unit root tests take the form as

\[ T_1 = \frac{N^{-1/2} \sum_{i=1}^{N} (A_i - E(A_i))}{\sqrt{N^{-1} \sum_{i=1}^{N} B_i}}, \quad \text{or} \quad T_2 = N^{-1/2} \sum_{i=1}^{N} \left( \frac{A_i}{\sqrt{B_i}} - E \left( \frac{A_i}{\sqrt{B_i}} \right) \right), \]

if we take the sequential limit and let \( T \) go to infinity. For instance, the LLC test takes the first expression, and the IPS test takes the second expression. Clearly, the limit of the first one under the local alternative is easier to evaluate, but not the second one.

We provide a unified approach here. Our idea is to compute the moments under the local alternatives. If we can derive the characteristic functions of \( T_1 \) and \( T_2 \), then the moments under both null and the local alternative can be obtained immediately. However, it’s not easy to derive it especially in the unit root case. Fortunately, the joint characteristic function of \((A_i, B_i)\) can be obtained using the Fredholm approach below. Then, our unified approach consists of two steps. The first step is to obtain the joint characteristic function or the joint moment generating function (m.g.f.) of \((N^{-1/2} \sum_{i=1}^{N} A_i, N^{-1} \sum_{i=1}^{N} B_i)\) for \( T_1 \) and that of \((A_i, B_i)\) for \( T_2 \), respectively. The second step is the calculation of the asymptotic moments of the test statistic based on the m.g.f. obtained from the first step. These will be better seen in the next section when the idea is illustrated using the LLC and IPS tests.

The first step of our approach can be accomplished using the Fredholm approach which is briefly summarized in the following. For more detailed discussions, the readers are referred to the excellent monograph by Tanaka (1996). The Fredholm approach with applications in deriving the characteristic functions of the Dickey-Fuller tests was systematically studied in a series of papers by Nabeya and Tanaka (1988, 1990a, 1990b) and Tanaka.
(1990). However, it seems these results are largely overlooked in later analysis of the local power in the panel unit roots context.

In unit root time series literature, we typically consider the following three setups:

Model 2.1: \( y_t = \rho y_{t-1} + \varepsilon_t, \quad (t = 1, 2, \ldots, T) \)

Model 2.2: \( y_t = \alpha + \rho y_{t-1} + \varepsilon_t, \quad (t = 1, 2, \ldots, T) \)

Model 2.3: \( y_t = \alpha + \beta t + \rho y_{t-1} + \varepsilon_t, \quad (t = 1, 2, \ldots, T) \)

where the initial value \( y_0 \) is assumed to be a constant or a random variable whose distribution is independent of \( T \), and \( \{\varepsilon_t\} \) is an i.i.d. sequence with \( (0, \sigma^2) \). It is well known that if the true value \( \rho = 1 \), when we consider the OLS estimator \( \hat{\rho}_i \) and the corresponding \( t \)-statistic \( t_{\hat{\rho}_i} \) for each model, we can obtain the following asymptotics:

Model 2.1: \( \frac{T(\hat{\rho}_1 - 1)}{V_1} \Rightarrow U_1, \quad t_{\hat{\rho}_1} \Rightarrow \frac{U_1}{\sqrt{V_1}}, \) \hspace{1cm} (1)

Model 2.2: \( \frac{T(\hat{\rho}_2 - 1)}{V_2} \Rightarrow U_2, \quad t_{\hat{\rho}_2} \Rightarrow \frac{U_2}{\sqrt{V_2}}, \) \hspace{1cm} (2)

Model 2.3: \( \frac{T(\hat{\rho}_3 - 1)}{V_3} \Rightarrow U_3, \quad t_{\hat{\rho}_3} \Rightarrow \frac{U_3}{\sqrt{V_3}}, \) \hspace{1cm} (3)

where \( \Rightarrow \) stands for weak convergence,

\[
U_1 = \int_0^1 W(r) dW(r) = \frac{1}{2}[W^2(1) - 1], \quad V_1 = \int_0^1 W^2(r) dr,
\]

\[
U_2 = \begin{vmatrix}
\int_0^1 W(r) dW(r) & \int_0^1 W(r) dr \\
\int_0^1 W(r) dr & 1
\end{vmatrix}, \quad V_2 = \begin{vmatrix}
\int_0^1 W^2(r) dr & \int_0^1 W(r) dr \\
\int_0^1 W(r) dr & 1
\end{vmatrix},
\]

\[
U_3 = 12 \begin{vmatrix}
\int_0^1 W(r) dW(r) & \int_0^1 W(r) dr & \int_0^1 rW(r) dr \\
\int_0^1 dW(r) & 1 & \frac{1}{2} \\
\int_0^1 r dW(r) & \frac{1}{2} & \frac{1}{12}
\end{vmatrix},
\]

\[
V_3 = 12 \begin{vmatrix}
\int_0^1 W^2(r) dW(r) & \int_0^1 W(r) dr & \int_0^1 rW(r) dr \\
\int_0^1 W(r) dr & 1 & \frac{1}{2} \\
\int_0^1 r dW(r) & \frac{1}{2} & \frac{1}{12}
\end{vmatrix},
\]

and \( \{W(t) : 0 \leq t \leq 1\} \) is a standard Brownian motion. These results were obtained by various authors, for example, the limiting expression for \( \hat{\rho}_1 \) in [1] was obtained by Chan
and Wei (1987) and Phillips (1987a), the limiting expression for \( t_{\hat{\rho}_1} \) in (1) was obtained by Phillips (1987a). The limiting expressions in (2) and (3) were obtained by Phillips and Perron (1988).

There was another strand of literature focusing directly on the limiting distributions of these statistics, for example, White (1958) derived the joint m.g.f. for \((U_1, V_1)\) as

\[
\phi_1(u, v) = e^{-u/2} \left( \cos(\sqrt{2}v) - u \frac{\sin(\sqrt{2}v)}{\sqrt{2}v} \right)^{-1/2}.
\]

Evans and Savin (1981) studied the moments of \( U_1/V_1 \) based on White’s result. Dickey and Fuller (1979, 1981) gave different expressions for the limit of \( t_{\hat{\rho}_1} \) and \( t_{\hat{\rho}_2} \) in (1) and (2).

Nabeya and Tanaka (1988, 1990a, 1990b) extended the idea on deriving the limiting distribution of \( T(\hat{\rho}_i - 1) \), for \( i = 1, 2, 3 \). First, they noticed that

\[
P(T(\hat{\rho}_i - 1) < x) \to P(xV_i - U_i > 0), \quad \text{as} \ T \to \infty.
\]

Denote \( Z_x = xV_i - U_i \). Then \( Z_x \) could be approximated by taking the limit of a quadratic form to reach the expression as

\[
Z_x = a^2 \int_0^1 \int_0^1 K_x(s, t) dW(s) dW(t) + b,
\]

where \( K_x(s, t) \) is the kernel associated with the eigenvalue integral equation

\[
f(t) = \lambda \int_0^1 K_x(s, t) f(s) ds
\]

which is of the Fredholm type. Second, given the expression of \( Z_x \), the characteristic function of \( Z_x/a^2 \) could be expressed following Anderson and Darling (1952) as

\[
\phi_x(\theta) = e^{ir\theta} [D_x(2i\theta)]^{-1/2},
\]

where \( r = b/a^2 \) and \( D_x(\cdot) \) is the Fredholm determinant of \( K_x(s, t) \). In the end, the limiting distribution of \( T(\hat{\rho}_i - 1) \) could be calculated by Imhof (1961)’s formula, i.e.,

\[
\lim_{T \to \infty} P(T(\hat{\rho} - 1) < x) = P(Z_x/a^2 > 0) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{\theta} \text{Im}(\phi_x(\theta)) d\theta.
\]

This is the so-called Fredholm approach. The key idea is to find the characteristic function of \( Z_x \) using the Fredholm determinant \( D_x(\cdot) \) associated with the kernel \( K_x(s, t) \). For models
1-3, the expressions of \( K_x(s,t) \), \( a \) and \( b \) can be derived. Nabeya and Tanaka (1988, 1990a, 1990b) obtained the expressions of the Fredholm determinants for various cases.

The above-mentioned results can be used in detecting unit roots. Results in (1), (2) and (3) give the asymptotic expressions of the OLS estimators and \( t \)-statistics under \( H_0 : \rho = 1 \). To consider the local power of these tests, the limiting distributions under the local alternatives are also very important. Chan and Wei (1987) and Phillips (1987b) unified the asymptotics through the local-to-unity alternatives \( H_1 : \rho = 1 - \frac{c}{T} \) or \( H_1 : \rho = e^c \), respectively. This unified framework was also adopted in Nabeya and Tanaka (1990a). We summarize their results here for the convenience of later reference. The following models were considered in Nabeya and Tanaka (1990a):

Model 2.1': \( y_t = \eta_t, \quad (t = 1, 2, \ldots, T) \)

Model 2.2': \( y_t = \beta_0 + \eta_t, \quad (t = 1, 2, \ldots, T) \)

Model 2.3': \( y_t = \beta_1 t + \eta_t, \quad (t = 1, 2, \ldots, T) \)

Model 2.4': \( y_t = \beta_0 + \beta_1 t + \eta_t, \quad (t = 1, 2, \ldots, T) \)

where \( \eta_t = \rho \eta_{t-1} + u_t, \rho = 1 - c/T, \) and \( \{u_t\} \) is a linear process such that

\[
\sum_{j=0}^{\infty} |\alpha_j| < \infty, \quad \sum_{j=0}^{\infty} \alpha_j \neq 0.
\]

Here, \( \{\varepsilon_t\} \) is a martingale difference process such that

\[
\frac{1}{T} \sum_{t=1}^{T} E(\varepsilon_t^2 | \mathcal{F}_{t-1}) \overset{p}{\to} \sigma^2,
\]

where \( \mathcal{F}_t = \sigma(\varepsilon_s, s \leq t) \). The initial value \( \eta_0 \) of \( \{\eta_t\} \) was assumed to be zero or a random variable whose distribution not depending on \( T \). The main results are given in the following two lemmas, which will be used in later discussion.

**Lemma 2.1 (Theorem 3 in Nabeya and Tanaka (1990a))** Let \( \hat{\rho}_j \) be the OLS estimator for Model 2.\( j' \) (\( j = 1, 2, 3, 4 \)) correspondingly, with \( \rho = 1 - c/T \). Then

\[
\lim_{T \to \infty} P(T(\hat{\rho}_j - 1)) = P \left( \left( \sum_{l=0}^{\infty} \alpha_l \right)^2 W_j(c, x) + \sum_{l=0}^{\infty} \alpha_l^2 > 0 \right) = P(W_j(c, x) + r > 0),
\]
where

\[ r = \sum_{l=0}^{\infty} \alpha_l^2 / \left( \sum_{l=0}^{\infty} \alpha_l \right)^2, \]

\[ W_j(c, x) = \int_0^1 \int_0^1 K_j(s, t; c, x) dW(s) dW(t), \]

and \( K_j(s, t; c, x) \) are defined as follows

\[ K_1(s, t; c, x) = A_x(s, t) - e^{-c(2-s-t)}, \]

\[ K_2(s, t; c, x) = A_x(s, t) - \frac{2x}{c^2} g(s) g(t) - e^{-c(2-s-t)} + \frac{1}{c} \left( e^{-c(1-s)} g(t) + e^{-c(1-t)} g(s) \right), \]

\[ K_3(s, t; c, x) = A_x(s, t) - \frac{6x}{c^4} (g(s) + ch(s))(g(t) + ch(t)) - \left( e^{-c(1-s)} - \frac{3}{c^2} (g(s) + ch(s)) \right) \times \left( e^{-c(1-t)} - \frac{3}{c^2} (g(s) + ch(t)) \right), \]

\[ K_4(s, t; c, x) = A_x(s, t) - x \left( \frac{8}{c^4} (c^2 - 3c + 3) g(s) g(t) - \frac{12}{c^4} (c - 2)(g(s) h(t) + g(t) h(s)) + \frac{24}{c^4} h(s) h(t) \right) \]

\[ - \left( e^{-c(1-s)} + \frac{2}{c} - \frac{6}{c^2} (g(s) + ch(s)) \right) \left( e^{-c(1-r)} + \frac{2}{c} g(t) - \frac{6}{c^2} (g(t) + ch(t)) \right) \]

\[ + \frac{4}{c^4} (3c(1-s) - (c + 3) g(s))(3c(1-t) - (c + 3) g(t)), \]

where \( A_x(s, t) = \frac{c}{e} (e^{-c|s-t|} - e^{-c(2-s-t)}), \; g(s) = 1 - e^{-c(1-s)}, \; \text{and} \; h(s) = s - e^{-c(1-s)}. \)

The characteristic functions of corresponding \( W_j(c, x) + r \) \((j = 1, 2, 3, 4)\) in Lemma 2.1 are given in the following lemma.

**Lemma 2.2 (Theorem 4 in Nabeya and Tanaka (1990a))** The characteristic functions of \( W_j(c, x) + r \) \((j = 1, 2, 3, 4)\) in Lemma 2.1 have the following expression

\[ \varphi_j(\theta; c, r, x) = e^{ir\theta} |D_j(2i\theta; c, x)|^{-1/2}, \]

where \( D_j(\lambda; c, x) \) is the Fredholm determinant associated with \( K_j(s, t; c, x) \), which is defined
as follows

\[
D_1(\lambda; c, x) = e^{-c} \left[ \cos(\mu) + (c + \lambda) \frac{\sin(\mu)}{\mu} \right],
\]

\[
D_2(\lambda; c, x) = e^{-c} \left[ \frac{\lambda^2 + 2\lambda x - c^2 \lambda - c^3 \sin(\mu) + c^2 \cos(\mu)}{\mu} + \frac{(2\lambda^2 - 4c\lambda x - 2c^2 \lambda) \cos(\mu) - 1}{\mu^4} \right],
\]

\[
D_3(\lambda; c, x) = e^{-c} \left[ -\frac{c^3 + (c^2 + 3c + 3)\lambda \sin(\mu)}{\mu^2} + \frac{3(c^2 + 3c + 3 + 2\lambda + 1)x\lambda}{\mu^4} \left( \frac{\sin(\mu)}{\mu} - \cos(\mu) \right) - \frac{c^2}{\mu^2} \cos(\mu) \right],
\]

\[
D_4(\lambda; c, x) = e^{-c} \left[ \frac{c^5 + (c^4 - 4(c^2 + 3c + 27)\lambda - 8x(c^2 - 3c - 3))\lambda \sin(\mu)}{\mu^4} \right. \\
\left. - \frac{24(c^4 - 8x\lambda^2 + 4(c + 1)(x^2 - 3)\lambda)\lambda}{\mu^6} \left( \frac{\sin(\mu)}{\mu} + \frac{\cos(\mu)}{\mu^2} - \frac{1}{\mu^4} \right) \right]
\]

\[
+ \left( \frac{c^4 + 8(c^3 + 2x) - 4(c^2 + 3c + 6)\lambda}{\mu^4} \right) \cos(\mu)
\]

\[
+ \frac{8(c^4 - 4(c^2 + 3c - 3)\lambda + 2c^2 x(c + 3))\lambda}{\mu^6} \right],
\]

where \( \mu = \sqrt{2\lambda x - c^2} \).

The Fredholm approach could not only be applied to the Dickey-Fuller tests, but could also be used for understanding other unit root tests, see Nabeya and Tanaka (1990b) for more discussions. The results in Lemma 2.2 serve as the basis for the discussion in the following sections, since the related joint characteristic functions of the panel unit root test statistics can be obtained with the adaption of the above-mentioned results.

3 Local powers

To illustrate our approach, the exact local asymptotic powers of LLC and IPS tests are obtained in this section. We consider the following general setting for the panel autoregressive model

\[
\begin{align*}
z_{it} &= d_{it} + y_{it}, \quad i = 1, \ldots, N; t = 0, 1, \ldots, T, \\
y_{it} &= \rho_i y_{i,t-1} + u_{it}, \\
d_{it} &= \beta_0 + \beta_1 t, \\
y_{i0} &= \xi_i, \\
\end{align*}
\]
where $d_{it}$ is the deterministic component with possible trending, $y_{it}$ is an autoregressive time series for each individual with possibility to be unit root processes, and $y_{i0}$ gives the random initial conditions. Specifically, the model includes three cases, i.e.,

Model 3.1: $\beta_{0i} = \beta_{1i} = 0$, for all $i = 1, \ldots, N$
Model 3.2: $\beta_{1i} = 0$, for all $i = 1, \ldots, N$
Model 3.3: no restrictions on $\beta_{0i}$ and $\beta_{1i}$.

Our goal would be testing the presence of a common unit root against local alternatives. The null and alternative hypotheses could be stated as follows.

$$H_0 : \rho_i = 1, \quad \text{for all } i,$$

$$H_1 : \rho_i < 1, \quad \text{for } M \text{ number of } i's,$$

where $M$ satisfies $\lim_{N \to \infty} M/N = p$, $0 < p \leq 1$. As a special case for the alternative hypothesis $H_1$, we can consider the homogeneous alternative

$$H_1' : \rho_1 = \rho_2 = \cdots = \rho_N = \rho < 1.$$

We discuss the LLC and IPS tests separately in the following subsections. We make the following assumptions before that.

**Assumption 1** The errors $u_{it}$ are i.i.d. with $(0, \sigma_{u,i}^2)$ over $t = 1, \ldots, T$ and are also independent across $i = 1, \ldots, N$. $\sup_i E(u_{it}^8) < M_1 < \infty$ and $\inf_i \sigma_{u,i}^2 \geq M_2 > 0$ for some constants $M_1$ and $M_2$.

**Assumption 2** The initial points $y_{i0}$ are i.i.d. with $E(y_{i0}^8) < M_3 < \infty$ for some constant $M_3$ and are independent of $\{u_{it}\}_{i=1}^T$ for all $i$.

**Assumption 3** For Model 3.1, let $\rho_i = 1 - c_i/(N^{1/2}T)$; for Model 3.2, let $\rho_i = 1 - c_i/(N^{1/2}T)$; for Model 3.3, let $\rho_i = 1 - c_i/(N^{1/4}T)$, where $c_i \geq 0$, $i = 1, \ldots, N$.

**Assumption 4** $1/T + 1/N + N/T \to 0$.

**Remark 3.1** Assumption 1 assumes i.i.d. errors, which is adopted for simplicity of the derivation. From Lemma 2.2 in Section 2 the i.i.d. errors could be relaxed to be linear processes with no essential impact on the results. Assumption 2 assumes the nonexplosive
initial conditions. We will relax this assumption and discuss the impact of initial conditions in the next section. Assumption 3 gives local-to-unity alternatives with different rates for different models and test statistics, which is well known in the literature. Assumption 4 is the same as Assumption 3 in Moon et al. (2007), which is required for the convergence of test statistics. The sequential convergence as $T \to \infty$ followed by $N \to \infty$ is adopted in this paper for convenience. The joint convergence could also be obtained with strengthened conditions. More comprehensive discussions could be found in Phillips and Moon (1999).

### 3.1 LLC test

Firstly, we discuss the LLC test. In Levin et al. (2002), the following models were considered.

Model 3.1': \[ \Delta y_{it} = \delta y_{i,t-1} + \varepsilon_{i,t}, \]

Model 3.2': \[ \Delta y_{it} = \alpha_0 + \delta y_{i,t-1} + \varepsilon_{i,t}, \]

Model 3.3': \[ \Delta y_{it} = \alpha_0 + \alpha_1 t + \delta y_{i,t-1} + \varepsilon_{i,t}, \] where $-2 < \delta \leq 0$, for $i = 1, \ldots, N$.

Clearly, we can see that Model 3.1', Model 3.2' and Model 3.3' are equivalent to Model 3.1, Model 3.2 and Model 3.3, respectively, under the null hypothesis (5) and the homogeneous alternative (7), i.e.,

\[ \tilde{H}_0 : \delta = 0, \quad \text{v.s.} \quad \tilde{H}_1 : \delta < 0. \]

LLC proposed a three-step pooled OLS estimation of $\delta$ and showed that the pooled $t$-statistic converges to $N(0, 1)$ under $H_0$ for Model 1 with some regularity conditions, and converges to $N(0, 1)$ under $H_0$ for Model 2 and Model 3 with corrections for the means and additional conditions.

Here, we focus on the local-to-unity alternatives such as those given in Assumption 3. Our goal is to achieve analytical expressions for the local asymptotic power. For simplicity of exposition, we focus on the prototype LLC test statistics. The construction of the LLC tests are discussed in the following. For Model 3.1', we can estimate $\delta$ by

\[
\hat{\delta}_1 = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} \Delta y_{i,t} / \hat{\sigma}_{\varepsilon_{i,t}}^2)}{\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1}^2 / \hat{\sigma}_{\varepsilon_{i,t}}^2)},
\]
Thus, as $T \to \infty$, the asymptotic normality could be obtained by applying the standard Central Limit Theorem (CLT). To this end, we derive the joint m.g.f. in the first step. By substituting

$$
\tilde{\beta}_1 = \frac{\hat{\beta}_1}{(\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1}/\sigma_{\varepsilon 1,i})^{-1/2})}
$$

(9)

Under $H_1$ that is specified in Assumption 3, we have

$$
\hat{t}_{\delta,1} = -\frac{1}{N^{1/2}T} \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} c_i (y_{i,t-1}/\sigma_{\varepsilon 1,i})^2}{\sqrt{\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1}/\sigma_{\varepsilon 1,i})^2}} + \frac{N^{-1/2}T^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} \varepsilon_{i,t}/\sigma_{\varepsilon 1,i})}{\sqrt{N^{-1}T^{-2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1}/\sigma_{\varepsilon 1,i})^2}}.
$$

(10)

From Phillips (1987b), we have that as $T \to \infty$

$$
T^{-1/2}(y_{i,[Tr]} - y_{i,0}) \Rightarrow \begin{cases} 
\sigma_{\varepsilon,i} W_i(r) & \text{for } c_i = 0, \\
\sigma_{\varepsilon,i} \int_0^r e^{-c_i N^{-1/2}(r-s)} dW_i(s) & \text{else.}
\end{cases}
$$

(11)

Thus, as $T \to \infty$

$$
\hat{t}_{\delta,1} \Rightarrow -\frac{1}{N^{-1} \sum_{i=1}^{N} c_i} \int_0^1 K_{i,c_i}(r)^2 dr + \frac{N^{-1/2} \sum_{i=1}^{N} \int_0^1 K_{i,c_i}(r) dW_i(r)}{\sqrt{N^{-1} \sum_{i=1}^{N} \int_0^1 K_{i,c_i}(r)^2 dr}} \equiv \frac{N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{1i}}{\sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{1i}}},
$$

(12)

where $K_{i,c_i}(r) = \int_0^r e^{-c_i N^{-1/2}(r-s)} dW_i(s)$.

Now, we illustrate our approach to achieve the exact local asymptotic power. The idea is to calculate the asymptotic moments of the statistic under the local-to-unity alternatives. The asymptotic normality could be obtained by applying the standard Central Limit Theorem (CLT). To this end, we derive the joint m.g.f. in the first step. By substituting

$$
\theta = iu/2, \quad x = -v/u \quad \text{and } \quad c = c_i N^{-1/2}
$$

into $\varphi_1(\theta; c, 1, x)$ in Lemma 2.2, we obtain the joint m.g.f. for $(\tilde{U}_{1i}, \tilde{V}_{1i})$

$$
\psi_{1,i}(u, v) = e^{-\frac{u}{2}} \left[ e^{-c_i N^{-1/2}} \left[ \cos \sqrt{\frac{2v - c_i^2 N^{-1}}{2v - c_i^2 N^{-1}}} \right] \right]^{-1/2}.
$$

(13)

Hence, the joint m.g.f. for $(N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{1i}, N^{-1} \sum_{i=1}^{N} \tilde{V}_{1i})$ is

$$
\phi_1(u, v) = e^{-\frac{N}{2} \sum_{i=1}^{N} c_i N^{-1/2}} \left[ \prod_{i=1}^{N} \cos \sqrt{\frac{2v - c_i^2 N^{-1}}{2v - c_i^2 N^{-1}}} \right]^{-1/2}.
$$
Hence, combining (12) and (14) and plugging in (15), we have
\[ T \frac{\hat{\delta}}{\hat{\sigma}_1} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\bar{U}_{1i}}{\bar{V}_{1i}} \] (15)\]

The asymptotic moments \( \hat{\delta}_1 \) can be directly calculated using this formula. From Taylor expansion (see the detailed derivations in the Appendix), we have
\[
\frac{\partial}{\partial u} \phi_1(u, -v) \bigg|_{u=0} = -\frac{1}{2} e^{-v/2} N^{-1} \sum_{i=1}^{N} c_i + O(N^{-1/2}).
\]

Hence, combining (12) and (14) and plugging in (15), we have
\[
E(\hat{\delta}_1) = E \left( \frac{N^{-1/2} \sum_{i=1}^{N} \frac{\bar{U}_{1i}}{\bar{V}_{1i}}}{\sqrt{N^{-1} \sum_{i=1}^{N} \frac{\bar{U}_{1i}}{\bar{V}_{1i}}}} \right) = \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \phi_1(u, -v) \bigg|_{u=0} dv
\]
\[
= - \left( N^{-1} \sum_{i=1}^{N} c_i \right) \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-v/2} dv + O_p(N^{-1/2}) = -\frac{\bar{c}}{\sqrt{2}} + O_p(N^{-1/2}),
\]
where \( \bar{c} = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} c_i \). This moment gives the asymptotic bias that leads to the local power. The result is summarized in Theorem 3.1 below.

Next, we consider Model 3.2', \( \hat{\delta} \) can be estimated by the fixed-effects estimator
\[
\hat{\delta}_2 = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})(\Delta y_{i,t} - \bar{\Delta} y_{i,t})/\hat{\sigma}_{2,i}^2]}{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{\sigma}_{2,i}^2]},
\]
where \( \hat{\sigma}_{2,i} = \sqrt{\frac{1}{T-1} \sum_{t=2}^{T} (\Delta y_{i,t} - \hat{\alpha}_0 - \hat{\delta}_2 y_{i,t-1})^2} \) is a consistent estimator for \( \sigma_{\epsilon,i} \) with that \( \hat{\alpha}_0 \) and \( \hat{\delta}_2 \) are OLS estimators from the individual time series, \( \bar{y}_{i,t-1} = (T-1)^{-1} \sum_{s=2}^{T} y_{i,s-1} \), and \( \bar{\Delta} y_{i,t} = (T-1)^{-1} \sum_{s=2}^{T} \Delta y_{i,s} \). However, \( \hat{\delta}_2 \) is biased, since under the null hypothesis, by (2) as \( T \to \infty \)
\[
T \hat{\delta}_2 = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})(\epsilon_{i,t} - \bar{\epsilon}_{i,t})/\hat{\sigma}_{2,i}^2]}{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{\sigma}_{2,i}^2]}
\]
\[
= \frac{N^{-1} \sum_{i=1}^{N} \int_{0}^{1} W_i^\mu(r) dW_i(r)}{N^{-1} \sum_{i=1}^{N} \int_{0}^{1} W_i^\mu(r)^2 dr} \overset{def}{=} \frac{N^{-1} \sum_{i=1}^{N} U_{2i}}{N^{-1} \sum_{i=1}^{N} V_{2i}}
\]
where \( W_i^\mu(r) = W_i(r) - \int_{0}^{1} W_i(s) ds, \bar{\epsilon}_{i,t} = (T-1)^{-1} \sum_{s=2}^{T} \epsilon_{i,s} \) and \( E(U_2) = -1/2, Var(U_2) = 1/12, E(V_2) = 1/6, Var(V_2) = 1/45 \) from Table 1 in Levin et al. (2002) or the derivations.
in the supplementary material. As discussed in Moon and Perron (2008), there are several ways for bias correction. The first way is to correct the overall bias for the whole test statistic as what was proposed in Levin and Lin (1992). With this type of bias correction, the \( t \)-statistic is given by

\[
\hat{t}_{\delta,21} = \frac{\sqrt{5}}{2} \left( \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{\hat{\delta}_2}{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{\varepsilon_{i,t}}^2]} \right)^{-1/2} + \sqrt{\frac{15N}{8}}. \quad (17)
\]

Under the corresponding specification of \( H_1 \) in Assumption 3, we have

\[
\hat{t}_{\delta,21} = -\frac{\sqrt{5}}{2N^{1/2}T} \sum_{i=1}^{N} \sum_{t=2}^{T} c_i \left( \frac{(y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{\varepsilon_{i,t}}^2}{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{\varepsilon_{i,t}}^2]} \right) + \frac{\sqrt{5} N^{-1/2} T^{-1}}{2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left[ \left( (y_{i,t-1} - \bar{y}_{i,t-1})(\varepsilon_{i,t} - \bar{\varepsilon}_{i,t}) / \hat{\sigma}_{\varepsilon_{i,t}}^2 \right) + \frac{1}{2} \right] \right)
\]

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{\varepsilon_{i,t}}^2 \right) \right)
\]

\[
-\frac{\sqrt{5}}{2} \left( \frac{\sqrt{N(T-1)}}{2T \sqrt{N^{-1} T^{-2}}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{\varepsilon_{i,t}}^2 \right) - \sqrt{\frac{3N}{2}}
\]

Furthermore, as \( T \to \infty \)

\[
\hat{t}_{\delta,21} \Rightarrow -\frac{\sqrt{5}}{2} \sum_{i=1}^{N} c_i \int_{0}^{1} K_{i,c_i}(r) \sigma_{i,c_i}^2 dr + \frac{\sqrt{5}}{2} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}(r) \sigma_{i,c_i}^2 dr
\]

\[
+ \frac{1}{2} \sqrt{5} N^{-1/2} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}(r) dr - \sqrt{\frac{3N}{2}} \right) \right)
\]

\[
+ \frac{1}{2} \sqrt{5} N^{-1/2} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}(r) dr - \sqrt{\frac{3N}{2}} \right) \right)
\]

\[
= \frac{\sqrt{5}}{2} N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{2i} + \sqrt{\frac{15N}{8}}
\]

(18)

where \( K_{i,c_i}(r) = K_{i,c_i}(r) - \int_{0}^{1} K_{i,c_i}(s) ds \).

Our unified approach can be applied here. Substituting \( \theta = \frac{iu}{2} \), \( x = -v/u \) and \( c = c_i N^{-1/2} \) into \( \psi_2(\theta; c, 1, x) \) in Lemma 2.2 we have the joint m.g.f. for \( (\tilde{U}_{2i}, \tilde{V}_{2i}) \) as

\[
\psi_{2,i}(u, v) = e^{-\frac{1}{2}} \left[ e^{-c_i N^{-1/2}} \left[ \frac{u^2 + 2v + c_i^2 N^{-1} - u - c_i^2 N^{-3/2} \sin \sqrt{2v - c_i^2 N^{-1}}}{2v - c_i^2 N^{-1}} \right] \right]^{-1/2}.
\]

(19)
Combining (18), (14) and (20), it implies

\[ \phi_2(u, v) = e^{-\frac{\sqrt{5} u}{\sqrt{N}}} \left[ e^{-\frac{1}{2} \sum_{i=1}^{N} c_i N^{-\frac{1}{2}} \left( \frac{2u^2 + c_i^2 N^{-1} u}{2} - c_i^3 N^{-3/2} \sin \sqrt{\frac{2u}{N} - c_i^2 N^{-1}} \right) - c_i^2 N^{-1} \cos \sqrt{\frac{2u}{N} - c_i^2 N^{-1}} + \left( \frac{2u^2}{N} - 4c_i N^{-1/2} \frac{u}{N} + 2c_i^2 N^{-1} \right) \frac{u}{\sqrt{N}} \cos \sqrt{\frac{2u}{N} - c_i^2 N^{-1}} - 1 \right]^{-1/2}. \]

From Taylor expansion (see the Appendix), we have

\[ \frac{\partial}{\partial u} \phi_2(u, v) \bigg|_{u=0} = -\frac{\sqrt{5}}{2} e^{-v/6} - \frac{1}{24} ve^{-v/6} N^{-1} \sum_{i=1}^{N} c_i + O_p(N^{-1/2}). \] (20)

Combining (18), (14) and (20), it implies

\[ E \left( \frac{\sqrt{5} \sum_{i=1}^{N} \bar{U}_{2i}}{2 \sqrt{N^{-1} \sum_{i=1}^{N} \bar{V}_{2i}}} \right) = \frac{\sqrt{5}}{2} \frac{1}{\sqrt{\Gamma(\frac{1}{2})}} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \phi_2(u, v) \bigg|_{u=0} dv \]

\[ = -\frac{\sqrt{5}}{2} \frac{\sqrt{N}}{2 \sqrt{\pi}} \int_{0}^{\infty} e^{-v/6} dv - \left( \frac{1}{2} \sum_{i=1}^{N} c_i \right) \frac{\sqrt{5}}{2} \frac{1}{24 \sqrt{\pi}} \int_{0}^{\infty} \frac{ve^{-v/6}}{\sqrt{v}} dv + O_p(N^{-1/2}) \]

\[ = -\frac{\sqrt{15N}}{8} - \frac{1}{8} \sqrt{\frac{15}{2} \bar{c} + O_p(N^{-1/2})}. \]

Therefore, we have

\[ E \left( \hat{t}_{\delta,21} \right) = E \left( \frac{\sqrt{5} \sum_{i=1}^{N} \bar{U}_{2i}}{2 \sqrt{N^{-1} \sum_{i=1}^{N} \bar{V}_{2i}}} \right) + \sqrt{\frac{15N}{8}} = -\frac{1}{8} \sqrt{\frac{15}{2} \bar{c} + O_p(N^{-1/2})}, \]

which gives the local power.

Another way to correct the bias is to correct the overall bias for \( \hat{\delta}_2 \). With this type of bias correction, the \( t \)-statistic is given by

\[ \hat{t}_{\delta,22} = \sqrt{\frac{10}{17} \left( \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{z_2,i}^2 \right) - 3 / 2}. \] (21)
Under $H_1$ given in Assumption 3, we have

$$\hat{t}_{\delta,22} = -\sqrt{\frac{10}{17}} \frac{1}{N^{1/2}T} \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{c_i[(y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{\sigma}_{\varepsilon,i}^2]}{\sqrt{\sum_{i=1}^{N} \sum_{t=2}^{T}[(y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{\sigma}_{\varepsilon,i}^2]}}$$

$$+ \sqrt{\frac{10}{17}} \frac{N^{-1/2}T^{-1}}{N^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})(\varepsilon_{i,t} - \bar{\varepsilon}_{i,t})/\hat{\sigma}_{\varepsilon,i}^2] + \frac{1}{2}}$$

$$- \sqrt{\frac{10}{17}} \sqrt{\frac{N(T-1)}{2T}} - 3\sqrt{N} \left( \frac{N^{-1/2} \sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{\sigma}_{\varepsilon,i}^2]}{\sqrt{N^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{\sigma}_{\varepsilon,i}^2]}} \right).$$

Moreover, as $T \to \infty$, we have

$$\hat{t}_{\delta,22} \Rightarrow -\sqrt{\frac{10}{17}} \frac{1}{N^{1/2}T} \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{c_i[f_0 K_{i,c}(r)^2 dr]}{\sqrt{N^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} f_0 K_{i,c}(r)^2 dr}}$$

$$+ \sqrt{\frac{10}{17}} \frac{3\sqrt{N}}{\sqrt{N^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} f_0 K_{i,c}(r)^2 dr}} \left[ N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{2i} + 3\sqrt{N} \sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{2i}} \right].$$

The local asymptotic power of $\hat{t}_{\delta,22}$ can be obtained in the same way using our approach with an additional relationship on the moments. It can shown that

$$E \left( V^{n-\alpha} \right) = \frac{(-1)^n}{\Gamma(\alpha)} \int_{0}^{\infty} v^{\alpha-1} \frac{\partial^n}{\partial v^n} [\phi(0,-v)] dv. \quad (23)$$

From Taylor expansion, we have

$$\frac{\partial}{\partial v} [\phi_2(0,-v)] = -\frac{1}{6} e^{-v/6} + \frac{e^{-v/6}}{12N^{3/2}} \sum_{i=1}^{N} c_i + O(N^{-1}). \quad (24)$$

Thus, combining (22), (14), (24) and (23), we have

$$E \left( \hat{t}_{\delta,22} \right) = E \left( \frac{10}{17} \sqrt{N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{2i}} + 3\sqrt{N} \sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{2i}} \right)$$

$$= \frac{10}{17} \frac{1}{\Gamma(1/2)} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} [\phi_2(u,-v)] \left. \right|_{u=0} dv - \frac{10}{17} \frac{3\sqrt{N}}{\Gamma(1/2)} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial v} [\phi_2(0,-v)] dv$$

$$= -\frac{1}{2} \sqrt{15} \tilde{c} + O_p(N^{-1/2}). \quad (25)$$
The result in (25) implies the result in Theorem 3.1 with the standard CLT.

Finally, for Model 3.3', we can estimate δ by the following estimator

\[ \hat{\delta}_3 = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1}) - \hat{\beta}_{1i}(t - \bar{t})][\Delta y_{i,t} - \bar{\Delta y}_{i,t}]/\hat{\sigma}_{3,i}^2}{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1}) - \hat{\beta}_{1i}(t - \bar{t})]^2/\hat{\sigma}_{3,i}^2}, \tag{26} \]

where \( \hat{\beta}_{1i} = \frac{\sum_{s=1}^{T} (s-s)(\Delta y_{is} - \bar{\Delta y}_{i})}{\sum_{s=1}^{T} (s-s)^2} \), \( \hat{\sigma}_{3,i} = \sqrt{(T-1)^{-1} \sum_{t=2}^{T} (\Delta y_{it} - \hat{\sigma}_{3i} - \hat{\alpha}_{0i} - \hat{\alpha}_{1i} t - \hat{\delta}_{3i} y_{it})^2} \) is a consistent estimator for \( \sigma_{3,i} \), and \( \hat{\alpha}_{0i}, \hat{\alpha}_{1i} \) and \( \hat{\delta}_{3i} \) are OLS estimators for each cross section. It is well known that \( \hat{\delta}_3 \) is also biased, since under the null hypothesis as \( T \to \infty \),

\[
T \hat{\delta}_3 = \sum_{i=1}^{N} \left( \sum_{t=1}^{T} \frac{(t - \bar{t})^2 (\sum_{i=1}^{N} (y_{i,t-1} - \bar{y}_{i,t-1})(\epsilon_{it} - \bar{\epsilon}_{it})) - (\sum_{i=1}^{N} (t - \bar{t})(y_{i,t-1} - \bar{y}_{i,t-1})(\epsilon_{it} - \bar{\epsilon}_{it}))}{\sum_{i=1}^{N} (\sum_{t=1}^{T} (t - \bar{t})^2 (\sum_{i=1}^{N} (y_{i,t-1} - \bar{y}_{i,t-1})^2 - (\sum_{i=1}^{N} (t - \bar{t})(y_{i,t-1} - \bar{y}_{i,t-1})^2)/\hat{\sigma}_{3,i}^2} \right) \]
\[
\Rightarrow \frac{N^{-1} \sum_{i=1}^{N} \left[ \int_0^1 W_t^i(r)dr - 12 \int_0^1 (r - \frac{1}{2}) W_t^i(dr) \right]}{N^{-1} \sum_{i=1}^{N} \left[ \int_0^1 W_t^i(r)^2 dr - 12 \int_0^1 (r - \frac{1}{2}) W_t^i(dr) \right]^2} \overset{d}{=} \frac{N^{-1} \sum_{i=1}^{N} U_{3i}}{N^{-1} \sum_{i=1}^{N} V_{3i}},
\]

where \( E(U_3) = -1/2, Var(U_3) = 1/60, E(V_3) = 1/15, Var(V_3) = 11/6300 \) from Table 1 in Levin et al. (2002).

Similar as what we discussed for Model 3.2', there are different ways for bias correction. The first way is to correct the overall bias for the whole test statistic. With this type of bias correction, the \( t \)-statistic is given by

\[
\hat{\delta}_{31} = \sqrt{\frac{448}{277} \left( \sum_{i=1}^{N} \sum_{t=1}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1}) - \sum_{s=1}^{T} (s-s)(y_{is} - \bar{y}_s)(t-\bar{t})^2/\hat{\sigma}_{3,i}^2) \right)^{-1/2}} \]

Following the corresponding specification of \( H_1 \) in Assumption 3, and taking \( T \to \infty \), we have as given in the Appendix

\[
\hat{\delta}_{31} \overset{d}{=} \sqrt{\frac{448}{277} \left( \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{U_{3i}}{\bar{V}_{3i}} \right)^{-1/2} + \sqrt{\frac{1680N}{277}}}. \tag{27} \]

To derive the local asymptotic power of \( \hat{\delta}_{31} \), we apply the same approach. The joint m.g.f. \( \psi_{3,i}(u, v) \) of \( (\bar{U}_{3i}, \bar{V}_{3i}) \) can be obtained from Lemma 3.2 as given in the Appendix. Hence, the joint m.g.f. \( \phi_3(u, v) \) for \( (N^{-1/2} \sum_{i=1}^{N} \bar{U}_{3i}, N^{-1/2} \sum_{i=1}^{N} \bar{V}_{3i}) \) can be obtained by the relationship

\[
\phi_3(u, v) = \prod_{i=1}^{N} \psi_{3,i}(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}).
\]
From Taylor expansion, we have

\[
\frac{\partial}{\partial u} \phi_3(u, -v) \bigg|_{u=0} = -\frac{\sqrt{N}}{2} e^{-v/15} - \frac{1}{840} v e^{-v/15} N^{-1} \sum_{i=1}^{N} c_i^2 + O_p(N^{-1/4}).
\]

(29)

Hence, by combining (28), (14) and (29), we get

\[
E \left( \hat{t}_{\delta,31} \right) = E \left( \sqrt{\frac{448}{277}} \frac{N^{-1/2} \sum_{i=1}^{N} \hat{U}_{3i}}{\sqrt{N}} \right) + \sqrt{\frac{448}{277}} \sqrt{\frac{15N}{4}} = -\frac{1}{14} \sqrt{\frac{105}{277}} \bar{c}^2 + O_p(N^{-1/4}),
\]

where \( \bar{c}^2 = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} c_i^2 \). This implies the result in Theorem 3.1 with the standard CLT.

Another way to correct the bias is to correct the overall bias for \( \hat{\delta} \). With this type of bias correction, the \( t \)-statistic is given by

\[
\hat{t}_{\delta,32} = \sqrt{\frac{112}{193}} \frac{N^{-1/2} \sum_{i=1}^{N} \hat{U}_{3i}}{\sqrt{N}} \frac{\hat{\delta} + \frac{15}{277} \sum_{i=1}^{N} \hat{V}_{3i}}{N^{-1/2} \sum_{i=1}^{N} \hat{V}_{3i}}.
\]

(30)

Based on the specification of \( H_1 \) given in Assumption 3, we have as \( T \to \infty \),

\[
\hat{t}_{\delta,32} \overset{d}{\to} \sqrt{\frac{112}{193}} \left[ \frac{N^{-1/2} \sum_{i=1}^{N} \hat{U}_{3i}}{\sqrt{N}} \frac{15 \sqrt{N}}{2} \sqrt{N^{-1} \sum_{i=1}^{N} \hat{V}_{3i}} \right].
\]

(31)

Further, we have

\[
\frac{\partial}{\partial v} \phi_3(0, -v) = -\frac{1}{15} e^{-v/15} + \frac{e^{-v/15}}{420 N^{3/2}} \sum_{i=1}^{N} c_i^2 + O(N^{-3/4}).
\]

(32)

Therefore, combining (31), (14), (23) and (32), we have

\[
E \left( \hat{t}_{\delta,32} \right) = E \left( \sqrt{\frac{112}{193}} \left[ \frac{N^{-1/2} \sum_{i=1}^{N} \hat{U}_{3i}}{\sqrt{N}} \frac{15 \sqrt{N}}{2} \sqrt{N^{-1} \sum_{i=1}^{N} \hat{V}_{3i}} \right] \right)
\]

\[
= \sqrt{\frac{112}{193}} \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \phi_3(u, -v) \bigg|_{u=0} dv - \sqrt{\frac{112}{193}} \frac{15 \sqrt{N}}{2} \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial v} \phi_3(0, -v) \bigg|_{u=0} dv
\]

\[
= -\frac{\sqrt{15}}{56} \sqrt{\frac{112}{193}} + O_p(N^{-1/4}),
\]

which further implies the result in Theorem 3.1.

The above-mentioned results are summarized in the following theorem.
**Theorem 3.1** Under Assumptions 1 to 4, when $T \to \infty$ followed by $N \to \infty$, we have the following asymptotic results.

(a) For Model 3.1', $\hat{\delta}_1 \Rightarrow N(0, 1) - \frac{\bar{c}}{\sqrt{2}}$;

(b) For Model 3.2', $\hat{\delta}_{21} \Rightarrow N(0, 1) - \frac{1}{8} \sqrt{\frac{15}{2}} \bar{c}$, and $\hat{\delta}_{22} \Rightarrow N(0, 1) - \frac{1}{2} \sqrt{\frac{15}{17}} \bar{c}$;

(c) For Model 3.3', $\hat{\delta}_{31} \Rightarrow N(0, 1) - \frac{1}{14} \sqrt{\frac{105}{277}} \bar{c}$, and $\hat{\delta}_{32} \Rightarrow N(0, 1) - \frac{\sqrt{15}}{56} \sqrt{\frac{112}{193}} \bar{c}$.

**Remark 3.2** Clearly, all of these results are obtained in a unified way. The same results were also obtained by different authors, for example, Moon et al. (2007), Moon and Perron (2004), Moon and Perron (2008), and Westerlund and Breitung (2012), using the computation of the expectations, which is different from our approach. The comparison with our approach is given in the supplementary material. Also, from Moon et al. (2007), we know that none of these tests would achieve the power envelope under the heterogeneous alternatives, and $\hat{\delta}_1$ can achieve the optimal power for Model 3.1' under the homogeneous alternative $H'_1$, but not $\hat{\delta}_{21}$, $\hat{\delta}_{22}$, $\hat{\delta}_{31}$ and $\hat{\delta}_{32}$ for Model 3.2' and Model 3.3' even under the homogeneous alternative. Also, clearly $\hat{\delta}_{22}$ and $\hat{\delta}_{32}$ have larger local powers than $\hat{\delta}_{21}$ and $\hat{\delta}_{31}$, respectively. Our approach provides an alternative way. The advantage of our approach can be better appreciated when we consider the IPS test.

### 3.2 IPS test

The IPS test is also one of the most widely used panel unit root tests. However, so far the literature on the local power of the IPS test is rare. One of our major contributions in this paper is to derive the analytical local asymptotic power of the IPS test for different scenarios. The advantage of our approach can be better seen in this section. The model in (4) was considered in Im et al. (2003). The idea is to form the standardized test statistics from the OLS estimation of each individual time series.

For Model 3.1, the $t$ test statistic of $\rho_i$ is constructed by running the OLS estimation for each cross section. Therefore, we have

$$\hat{\rho}_i = \frac{\sum_{t=2}^{T} z_{i,t-1} z_{i,t}}{\sum_{t=2}^{T} z_{i,t-1}^2}, \quad \hat{\delta}_i = \frac{\hat{\rho}_i - 1}{\hat{\sigma}_{ul,i}(\sum_{t=2}^{T} z_{i,t-1}^2)^{-1/2}},$$

(33)
where $\hat{\sigma}_{u,i} = \sqrt{(T - 1)^{-1} \sum_{t=2}^{T} (z_{i,t} - \hat{\rho}_i z_{i,t-1})^2}$ is a consistent estimator for $\sigma_{u,i}$. Under $H_0$, the asymptotics of $\hat{t}_i$ is given in (11).

The IPS test statistic is constructed as the standardized statistic of t-statistic, i.e.

$$Z = \frac{\sqrt{N}[N^{-1} \sum_{i=1}^{N} \hat{t}_i - E(t_0)]}{\sqrt{Var(t_0)}},$$

where $E(t_0)$ and $Var(t_0)$ are the mean and the variance from the limiting distribution of the corresponding Dickey-Fuller statistic, respectively. We can find the approximated values of $E(t_0) = -0.42309565$ and $\sqrt{Var(t_0)} = 0.98111424$ from Table 4 in Nabeya (1999).

Under $H_1$, we have

$$\hat{t}_i = -\frac{c_i}{N^{1/2} \hat{\sigma}_{u,i}} \sqrt{T^{-2} \sum_{t=2}^{T} y_{i,t-1}^2} + \frac{T^{-1} \sum_{t=2}^{T} y_{i,t-1} u_{i,t}}{\hat{\sigma}_{u,i} \sqrt{T^{-2} \sum_{t=2}^{T} y_{i,t-1}^2}}.$$  

From (11), we have that as $T \to \infty$

$$\hat{t}_i \Rightarrow -\frac{c_i}{N^{1/2}} \sqrt{\int_0^1 K_{i,c_i}(r)^2 dr} + \frac{\int_0^1 K_{i,c_i}(r) dW_i(r)}{\sqrt{\int_0^1 K_{i,c_i}(r)^2 dr}} = \frac{\tilde{U}_{1i}}{\sqrt{V_{1i}}}. \quad (34)$$

Using Taylor expansion, we could get the formal expression as

$$Z = \frac{\sqrt{N}[N^{-1} \sum_{i=1}^{N} \hat{t}_i - E(t_0)]}{\sqrt{Var(t_0)}} \Rightarrow N(0,1) - \tilde{c} \left[ E \left( \sqrt{\int_0^1 W(r)^2 dr} \right) \right. 
+ E \left( \frac{\int_0^1 W(s) dW(s) r dr}{\sqrt{\int_0^1 W(r)^2 dr}} \right) - E \left( \frac{\int_0^1 W(r)^3 dW(r) r dr}{\sqrt{\int_0^1 W(r)^2 dr}} \right) \left. \right] / \sqrt{Var(t_0)}.$$  

However, this is not very informative, since the expectations in this expression could not be calculated easily, which has to rely on simulations.

Our approach can be readily applied here. In the first step, the joint m.g.f is directly given in (13). Further, we get

$$\frac{\partial}{\partial u} \psi_{1,i}(u,v) \bigg|_{u=0} = -\frac{1}{2} \left[ \cosh \sqrt{2v} - c_i N^{-\frac{1}{2}} \left( \cosh \sqrt{2v} - \frac{\sinh \sqrt{2v}}{\sqrt{2v}} \right) + O(N^{-1}) \right]^{-1/2} 
+ \frac{1}{2} \left[ \cosh \sqrt{2v} - c_i N^{-\frac{1}{2}} \left( \cosh \sqrt{2v} - \frac{\sinh \sqrt{2v}}{\sqrt{2v}} \right) + O(N^{-1}) \right]^{-3/2} 
\times \left( \frac{\sinh \sqrt{2v}}{\sqrt{2v}} - c_i N^{-1/2} \frac{\sinh \sqrt{2v}}{\sqrt{2v}} + O(N^{-1}) \right).$$
Therefore, by the change of variable as \( x = \sqrt{2}v \) and Taylor expansion, with the formula in (14) we have

\[
E(Z) = (Var(t_0))^{-1/2} N^{1/2} \sum_{i=1}^{N} (E(\hat{t}_i) - E(t_0))
\]

\[
= (Var(t_0))^{-1/2} N^{1/2} \sum_{i=1}^{N} \left( \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \psi_{1,i}(u, -v) \bigg|_{u=0} dv - E(t_0) \right)
\]

\[
= -(Var(t_0))^{-1/2} \frac{\bar{c}}{2\sqrt{2\pi}} \int_{0}^{\infty} (\cosh(x))^{-1/2} \left( 1 - \frac{2\sinh(x)}{x \cosh(x)} + \frac{3(\sinh(x))^2}{x^2(\cosh(x))^2} \right) dx + O_p(N^{-1/2}),
\]

where

\[
E(t_0) = -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (\cosh(x))^{-1/2} \left( 1 - \frac{\sinh(x)}{x \cosh(x)} \right) dx
\]

is also given in Nabeya (1999).

For Model 3.2, the \( t \) test statistic of \( \rho_i \) is constructed by running the OLS estimation for each cross section. We have

\[
\hat{\rho}_i^\mu = \frac{\sum_{t=2}^{T}(z_{i,t-1} - \bar{z}_{i,t-1})(z_{i,t} - \bar{z}_{i,t})}{\sum_{t=2}^{T}(z_{i,t-1} - \bar{z}_{i,t-1})^2},
\]

\[
\hat{\mu}_i = \frac{\hat{\rho}_i^\mu - 1}{\hat{\sigma}_{u,i}^2} (\sum_{t=2}^{T}(z_{i,t-1} - \bar{z}_{i,t-1})^2)^{-1/2},
\]

where \( \bar{z}_{i,t-1} = (T - 1)^{-1} \sum_{s=2}^{T} z_{i,s-1}, \bar{z}_{i,t} = (T - 1)^{-1} \sum_{s=2}^{T} z_{i,s}, \)

\[
\hat{\sigma}_{u,i}^2 = \sqrt{(T - 1)^{-1} \sum_{t=2}^{T} (z_{i,t} - \hat{\alpha}_i - \hat{\rho}_i^\mu z_{i,t-1})^2}
\]

is a consistent estimator for \( \sigma_{u,i} \), and \( \hat{\alpha}_i \) is the OLS estimator from each cross section. Under \( H_0 \), the asymptotics of \( \hat{\mu}_i \) is given in (2).

The IPS test statistic is constructed as the standardized statistic of \( t \)-statistic, i.e.

\[
Z^\mu = \frac{\sqrt{N} \sum_{i=1}^{N} \hat{\mu}_i - E(t_0^\mu)}{\sqrt{Var(t_0^\mu)}},
\]

where \( E(t_0^\mu) \) and \( Var(t_0^\mu) \) are the mean and the variance from the limiting distribution of the corresponding Dickey-Fuller statistic for the model with an intercept, respectively. The
approximated values of $E(t_0^\mu) = -1.53296244$ and $\sqrt{Var(t_0^\mu)} = 0.84025086$ are given in Table 4 in Nabeya (1999).

Under $H_1$, we have

$$\hat{i}_t = -\frac{c_i}{N^{1/2}\hat{\sigma}_{u,i}} \sqrt{T^{-2} \sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 + \frac{T^{-1} \sum_{t=2}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})(u_{i,t} - \bar{u}_{i,t})}{\hat{\sigma}_{u,i} \sqrt{T^{-2} \sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2}},$$

where $\bar{u}_{i,t} = (T - 1)^{-1} \sum_{s=2}^{T} u_{i,s}$. Moreover, as $T \to \infty$

$$\hat{i}_t \Rightarrow -\frac{c_i}{N^{1/2}} \sqrt{\int_0^1 K_{i,c_i}^\mu(r)^2 dr + \int_0^1 K_{i,c_i}^\mu(r) dW_i(r) \defeq \tilde{U}_{2i}}. \quad (37)$$

From the expression in (19) and the derivations in the Appendix, we have

$$\frac{\partial}{\partial u} \psi_{2,i}(u, -v) \bigg|_{u=0} = -\frac{1}{2} \left[ \frac{\sinh \sqrt{2v}}{\sqrt{2v}} - c_i N^{-\frac{1}{2}} \left( \frac{\sinh \sqrt{2v}}{\sqrt{2v}} - v^{-1}(\cosh \sqrt{2v} - 1) \right) + O(N^{-1}) \right]^{-1/2} + O(N^{-1}).$$

Applying our approach, we have

$$E(Z^\mu) = (Var(t_0^\mu))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} (E(\hat{i}_t^\mu) - E(t_0^\mu))$$

$$= (Var(t_0^\mu))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} \left( \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \psi_{2,i}(u, -v) \bigg|_{u=0} dv - E(t_0^\mu) \right)$$

$$= -(Var(t_0^\mu))^{-1/2} \frac{\bar{c}}{2\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} \left( 1 - \frac{2(\cosh(x) - 1)}{x \sinh(x)} \right) dx + O(N^{-1/2}), \quad (38)$$

where

$$E(t_0^\mu) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} dx,$$

is given in (7) in Nabeya (1999).

For Model 3.3, The $t$ test statistic of $\rho_i$ is constructed by running the OLS estimation for each cross section. We have

$$\rho_i^\mu = \frac{(\sum_t (t - \bar{t})^2)(\sum_t (z_i,t-1 - \bar{z}_{i,t-1})(z_{it} - \bar{z}_{it})) - (\sum_t (t - \bar{t})(z_i,t-1 - \bar{z}_{i,t-1}))(\sum_t (t - \bar{t})(z_{it} - \bar{z}_{it}))}{(\sum_t (t - \bar{t})^2)(\sum_t (z_i,t-1 - \bar{z}_{i,t-1})^2) - (\sum_t (t - \bar{t})(z_i,t-1 - \bar{z}_{i,t-1}))^2}, \quad (39)$$

22
where all the summations are taken over 2 to $T$, and
\[ \tilde{t} = \frac{1}{T-1} \sum_{s=2}^{T} s = \frac{T + 2}{2}, \quad \tilde{z}_{i,t-1} = \frac{1}{T-1} \sum_{s=2}^{T} z_{i,s-1}, \quad \tilde{z}_{it} = \frac{1}{T-1} \sum_{s=2}^{T} z_{i,s}. \]

Then the t-statistic is given by
\[ \hat{t}_i^t = \frac{\hat{\rho}_t - 1}{\hat{\delta}_{u,i} \sqrt{\frac{\sum_i (t-i)^2}{(\sum_i(z_{i,t-1}-\bar{z}_{i,t-1})^2) - (\sum_i(t-i)(z_{i,t-1}-\bar{z}_{i,t-1}))^2}}, \] (40)

where $\hat{\delta}_{u,i} = \sqrt{(T-1)^{-1} \sum_{t=2}^{T}(z_{it} - \hat{\alpha}_i - \hat{\gamma}_it - \hat{\rho}_t z_{i,t-1})^2}$ is a consistent estimator for $\sigma_{u,i}$, and $\hat{\alpha}_i$ and $\hat{\gamma}_i$ are OLS estimators from each cross section. Under $H_0$, the asymptotics of $\hat{t}_i^t$ is given in [3].

The IPS test statistic is constructed as the standardized statistic of t-statistic, i.e.
\[ Z^\tau = \frac{\sqrt{N}[N^{-1} \sum_{i=1}^{N} \hat{t}_i^t - E(t_0^t)]}{\sqrt{\text{Var}(t_0^t)}}, \]
where $E(t_0^t)$ and $\text{Var}(t_0^t)$ are the mean and the variance from the limiting distribution of the Dickey-Fuller statistic for the model with both an intercept and an time trend, respectively. The approximated values of $E(t_0^t) = -2.18135582$ and $\sqrt{\text{Var}(t_0^t)} = 0.74990847$ are given in Table 4 in Nabeya (1999).

Under $H_1$, we have as $T \to \infty$
\[ \hat{t}_i^\tau \Rightarrow -c_i \frac{N^{1/4}}{\sqrt{N^{-1} \sum_{i=1}^{N} \hat{t}_i^t - E(t_0^t)}} \left[ \int_0^1 K_{i,c_i}(r)^2 dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K_{i,c_i}(r) dr \right)^2 \right] \]
\[ + \int_0^1 K_{i,c_i}(r) dW_i(r) - 12 \int_0^1 (r - \frac{1}{2}) K_{i,c_i}(r) dr \int_0^1 (r - \frac{1}{2}) dW_i(r) \]
\[ \left\{ \frac{1}{\sqrt{\int_0^1 K_{i,c_i}(r)^2 dr}} \left[ \int_0^1 K_{i,c_i}(r)^2 dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K_{i,c_i}(r) dr \right)^2 \right] \right\} \overset{\text{def}}{=} \frac{\tilde{U}_{3i}}{\sqrt{\tilde{V}_{3i}}}. \] (41)

From the expression of $\psi_{3,i}(u, v)$ which is given in the Appendix, we obtain
\[ \frac{\partial}{\partial u} \psi_{3,i}(u, v) \bigg|_{u=0} = -\frac{1}{2} \left[ \frac{4v(5c_i^2 N^{-1/2} - 6)}{(-2v - c_i^2 N^{-1/2})^2} + \frac{1}{(-2v - c_i^2 N^{-1/2})^2} + \frac{4(6c_i^2 N^{-1/2}v)}{(-2v - c_i^2 N^{-1/2})^3} \right]^{-1/2} + O(N^{-1/2}). \]
Applying our approach, we have

\[
E(Z^*) = (\text{Var}(t_0^*))^{-1/2}N^{1/2}N^{-1} \sum_{i=1}^{N} \left( E(t_0^i) - E(t_0^0) \right)
\]

\[
= (\text{Var}(t_0^*))^{-1/2}N^{1/2}N^{-1} \sum_{i=1}^{N} \left( \frac{1}{\Gamma(\frac{3}{2})} \int_{0}^{\infty} \frac{1}{v^{3/2}} \frac{\partial}{\partial u} \psi_{3,i}(u,-v) \bigg|_{u=0} dv - E(t_0^i) \right)
\]

\[
= (\text{Var}(t_0^*))^{-1/2} \left( \frac{c^2}{\sqrt{2\pi}} \int_{0}^{\infty} [3f_{22}(x)]^{-3/2} \left( -\frac{\sinh(x)}{x^3} + \frac{9\cosh(x)}{x^4} - \frac{33\sinh(x)}{x^5} + \frac{48(\cosh(x) - 1)}{x^6} \right) dx \right)
\]

\[
+ O(N^{-1/4})
\]

where

\[
f_{22}(x) = 4 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4} [\cosh(x) - 1] \right), \quad \text{and} \quad E(t_0^0) = -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} [3f_{22}(x)]^{-1/2} dx
\]

are given in equation (7) and page 147 in Nabeya (1999).

We summarize these results in the following theorem.

**Theorem 3.2** Under the assumptions 1 to 4, when \( T \to \infty \) followed by \( N \to \infty \), we have the following asymptotic results.

(a) For Model 3.1,

\[
Z \Rightarrow N(0,1) - (\text{Var}(t_0))^{-1/2} \frac{c}{2\sqrt{2\pi}} \int_{0}^{\infty} (\cosh(x))^{-1/2} \left( 1 - \frac{2\sinh(x)}{x\cosh(x)} + \frac{3(\sinh(x))^2}{x^2(\cosh(x))^2} \right) dx;
\]

(b) For Model 3.2, \( Z^u \Rightarrow N(0,1) - (\text{Var}(t_0^u))^{-1/2} \frac{c}{2\sqrt{2\pi}} \int_{0}^{\infty} (\sinh(x)/x)^{-1/2} \left( 1 - \frac{2(\cosh(x) - 1)}{x\sinh(x)} \right) dx;
\]

(c) For Model 3.3, \( Z^\tau \Rightarrow N(0,1) - (\text{Var}(t_0^\tau))^{-1/2} \left( \frac{c^2}{\sqrt{2\pi}} \int_{0}^{\infty} [3f_{22}(x)]^{-3/2} \left( \frac{\sinh(x)}{x^3} - \frac{9\cosh(x)}{x^4} + \frac{33\sinh(x)}{x^5} - \frac{48(\cosh(x) - 1)}{x^6} \right) dx \right).

The more detailed proofs are delegated in the Appendix.

**Remark 3.3** The results in Theorem 3.2 give the analytical forms of the asymptotic distributions of IPS tests under the local-to-unity alternatives, which imply the exact local asymptotic power of these tests. These results are new in the literature, which fills the gap for the IPS tests. Moreover, we can see that \( Z \) and \( Z^u \) have the local power in the neighborhood of unity with the order of \( N^{-1/2}T^{-1} \), but \( Z^\tau \) only has the local power in the undefined area.
neighborhood of unity with the order of $N^{-1/4}T^{-1}$. These are consistent with the general order results obtained in Moon et al. (2007).

The integrals in Theorem 3.2 can be evaluated numerically to further simplify the results. We adopt the numerical calculations stated in Nabeya (1999) to achieve the accuracy up to eight decimal places. From the numerical integrations using MATLAB, we have

$$
\frac{1}{2\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} \left( 1 - \frac{2}{x \sinh(x)} \right) dx \approx 0.58198749,
$$

$$
\frac{1}{2\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} \left( 1 - \frac{2(cosh(x) - 1)}{x \sinh(x)} \right) dx \approx 0.23431142,
$$

$$
\frac{1}{\sqrt{2\pi}} \int_0^\infty \left[ 3f_{22}(x) \right]^{-3/2} \left( \frac{\sinh(x)}{x^3} - \frac{9 \cosh(x)}{x^4} + \frac{33 \sinh(x)}{x^5} - \frac{48(cosh(x) - 1)}{x^6} \right) dx \approx 0.02854706,
$$

Therefore, we obtain the following corollary.

**Corollary 3.3** Under Assumptions 1-4, when $T \to \infty$ followed by $N \to \infty$, we have that

(a) For model 3.1, $Z \Rightarrow N(0, 1) - 0.58198749\hat{c}/\sqrt{\text{Var}(t_0)}$;

(b) For Model 3.2, $Z^\nu \Rightarrow N(0, 1) - 0.23431142\hat{c}/\sqrt{\text{Var}(t^\nu_0)}$;

(c) For Model 3.3, $Z^\tau \Rightarrow N(0, 1) - 0.02854706\hat{c}^2/\sqrt{\text{Var}(t^\tau_0)}$.

**Remark 3.4** Compared with the results in Moon et al. (2007), we can see that the IPS tests would not achieve the power envelope for any case. Moreover, the IPS tests have lower power than the LLC tests in all scenarios. In Section 5, we calculate the theoretical local asymptotic powers for different cases based on this corollary.

## 4 Edgeworth expansion

One advantage of our approach is to derive the Edgeworth expansion of panel unit root statistics, since our approach can be used to calculate the moments directly. Hall (1992) gave comprehensive discussions of conditions and results on the Edgeworth expansion. For
LLC tests, Theorem 2.2 in Hall (1992) can be applied. The detailed derivations are collected in the Appendix. For \( \hat{t}_{\delta,1} \), under \( H_0 \) we have the one term Edgeworth expansion as

\[
F_{1n}(x) = \Phi(x) + \frac{\sqrt{2}}{3} N^{-1/2} \phi(x) + O(N^{-1}).
\]  

(46)

Under \( H_1 \), we have

\[
F_{1n,c}(x) = P \left( \hat{t}_{\delta,1} + \frac{\widetilde{c}}{\sqrt{2}} - \frac{\sqrt{2}}{6} \widetilde{c}^2 N^{-1/2} + O(N^{-1}) \leq x \right)
\]

\[
= F_{1n}(x) - \frac{\sqrt{2} \widetilde{c}}{12} N^{-1} \phi(x) + O(N^{-1}).
\]  

(47)

Similarly, for \( \hat{t}_{\delta,21} \), under \( H_0 \) we have the one term Edgeworth expansion as

\[
F_{21n}(x) = \Phi(x) + \frac{3\sqrt{30}}{40} N^{-1/2} \phi(x) - \frac{3\sqrt{30}}{560} N^{-1/2} (x^2 - 1) \phi(x) + O(N^{-1}).
\]  

(48)

Under \( H_1 \), we have

\[
F_{21n,c}(x) = P \left( \hat{t}_{\delta,21} + \frac{1}{8} \sqrt{\frac{15}{2}} \widetilde{c} - \frac{67\sqrt{30}}{1920} \widetilde{c}^2 N^{-1/2} + O(N^{-1}) \leq x \right)
\]

\[
= F_{21n}(x) - \frac{3\sqrt{30} \widetilde{c}}{160} N^{-1} \phi(x) - \frac{11\sqrt{30} \widetilde{c}}{17920} N^{-1} (x^2 - 1) \phi(x) + O(N^{-1}).
\]  

(49)

Moreover, for \( \hat{t}_{\delta,22} \), under \( H_0 \) we have the one term Edgeworth expansion as

\[
F_{22n}(x) = \Phi(x) + \frac{\sqrt{1020}}{85} N^{-1/2} \phi(x) - \frac{27\sqrt{1020}}{20230} N^{-1/2} (x^2 - 1) \phi(x) + O(N^{-1}).
\]  

(50)

Under \( H_1 \), we have

\[
F_{22n,c}(x) = P \left( \hat{t}_{\delta,22} + \frac{1}{2} \sqrt{\frac{15}{17}} \widetilde{c} - \frac{31\sqrt{1020}}{4080} \widetilde{c}^2 N^{-1/2} + O(N^{-1}) \leq x \right)
\]

\[
= F_{22n}(x) - \frac{3\sqrt{1020} \widetilde{c}}{680} N^{-1} \phi(x) - \frac{13\sqrt{1020} \widetilde{c}}{20230} N^{-1} (x^2 - 1) \phi(x) + O(N^{-1}).
\]  

(51)

Using the same method, for \( \hat{t}_{\delta,31} \), under \( H_0 \) we have the one term Edgeworth expansion as

\[
F_{31n}(x) = \Phi(x) + \frac{33}{56} \sqrt{\frac{105}{277}} N^{-1/2} \phi(x) - \frac{491}{15512} \sqrt{\frac{105}{277}} N^{-1/2} (x^2 - 1) \phi(x) + O(N^{-1}).
\]  

(52)
Under $H_1$, we have

$$F_{31n,c}(x) = P \left( \hat{t}_{\delta,21} + \frac{1}{14} \sqrt{\frac{105}{277}} \epsilon^2 + \frac{1}{12} \sqrt{\frac{105}{277}}^3 N^{-1/4} + O(N^{-1/2}) \leq x \right)$$

$$= F_{31n}(x) - \frac{59}{3136} \sqrt{\frac{105}{277}} N^{-1} \phi(x) + \frac{118445}{955392} \sqrt{\frac{105}{277}} N^{-1}(x^2 - 1)\phi(x) + O(N^{-1}),$$

(53)

where $\epsilon^3 = \lim_{N \to \infty} N^{-1} \sum_{i=1}^N \epsilon_i^3$.

Moreover, for $\hat{t}_{\delta,32}$, under $H_0$ we have the one term Edgeworth expansion as

$$F_{32n}(x) = \Phi(x) + \frac{11}{28} \sqrt{\frac{105}{193}} N^{-1/2} \phi(x) - \frac{397}{5405} \sqrt{\frac{105}{193}} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}).$$

(54)

Under $H_1$, we have

$$F_{32n,c}(x) = P \left( \hat{t}_{\delta,22} + \frac{\sqrt{15}}{56} \sqrt{\frac{112}{193}} \epsilon^2 + \frac{1}{24} \sqrt{\frac{105}{193}}^3 N^{-1/4} + O(N^{-1/2}) \leq x \right)$$

$$= F_{32n}(x) - \frac{151}{9408} \sqrt{\frac{105}{193}} N^{-1} \phi(x) + \frac{84829}{665728} \sqrt{\frac{105}{193}} N^{-1}(x^2 - 1)\phi(x) + O(N^{-1}).$$

(55)

Next, we consider IPS tests. Theorem 2.1 in Hall (1992) can be applied directly. For $Z$, under $H_0$, we have the standard one term Edgeworth expansion as

$$G_{1n}(x) = P(Z \leq x) = \Phi(x) - \frac{\lambda_1}{6} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}),$$

(56)

where $\lambda_1 = E(t_0 - E(t_0))^3/(Var(t_0))^{3/2} = [E(t_0)^3 - 3E(t_0)^2E(t_0) + 2(E(t_0))^3]/(Var(t_0))^{3/2}$ is given in the Appendix. Under $H_1$, we have the one term Edgeworth expansion as

$$G_{1n,c}(x) = P \left( Z + N^{1/2} \sum_{i=1}^N (E(\hat{t}_i) - E(t_0)) \leq x \right)$$

$$= \Phi(x) - \frac{\lambda_{1,c}}{6} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}),$$

(57)

where $\lambda_{1,c} = [E(\hat{t}_i)^3 - 3E(\hat{t}_i)^2E(\hat{t}_i) + 2(E(\hat{t}_i))^3]/(Var(t_0))^{3/2}$ is also given in the Appendix.

Similarly, for $Z^\mu$, under $H_0$, we have the standard one term Edgeworth expansion as

$$G_{2n}(x) = P(Z^\mu \leq x) = \Phi(x) - \frac{\lambda_2}{6} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}).$$

(58)
where \( \lambda_2 = E(t_0^\mu - E(t_0^\mu))^3/(\text{Var}(t_0^\mu))^{3/2} \). Under \( H_1 \), we have the one term Edgeworth expansion as

\[
G_{2n,c}(x) = P \left( Z^\mu + N^{1/2} \sum_{i=1}^N (E(t_i^\mu) - E(t_0^\mu)) \leq x \right) = \Phi(x) - \frac{\lambda_{2,c}}{6} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}), \tag{59}
\]

where

\[
\lambda_{2,c} = [E(t_i^\mu)^3 - 3E(t_i^\mu^2)E(t_i^\mu) + 2(E(t_i^\mu))^3]/(\text{Var}(t_0^\mu))^{3/2}.
\]

For \( Z^\tau \), under \( H_0 \), we have the standard one term Edgeworth expansion as

\[
G_{3n}(x) = P (Z^\tau \leq x) = \Phi(x) - \frac{\lambda_3}{6} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}). \tag{60}
\]

where \( \lambda_3 = E(t_0^\tau - E(t_0^\tau))^3/(\text{Var}(t_0^\tau))^{3/2} \). Under \( H_1 \), we have the one term Edgeworth expansion as

\[
G_{3n,c}(x) = P \left( Z^\tau + N^{1/2} \sum_{i=1}^N (E(t_i^\tau) - E(t_0^\tau)) \leq x \right) = \Phi(x) - \frac{\lambda_{3,c}}{6} N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}), \tag{61}
\]

where

\[
\lambda_{3,c} = [E(t_i^\tau)^3 - 3E(t_i^\tau^2)E(t_i^\tau) + 2(E(t_i^\tau))^3]/(\text{Var}(t_0^\tau))^{3/2}.
\]

Further, we can evaluate the numerical values of integrals in (56) to (61) following the way we discussed early. The results are stated in the following corollary.

**Corollary 4.1** Under the Assumptions 1, 2 and 4, when \( T \to \infty \) followed by \( N \to \infty \), we have the one term Edgeworth expansion as follows.

(a) For Model 3.1, for \( Z \),

under \( H_0 \), \( G_{1n}(x) = P (Z \leq x) \approx \Phi(x) - 0.0416N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}) \),

under \( H_1 \), \( P \left( Z + N^{1/2} \sum_{i=1}^N (E(t_i) - E(t_0)) \leq x \right) = \Phi(x) - 0.0416N^{-1/2}(x^2 - 1)\phi(x) + 0.0672\bar{c}N^{-1}(x^2 - 1)\phi(x) + O(N^{-1}); \)

28
(b) For Model 3.2, for $Z^\mu$,

\[
\begin{align*}
&\text{under } H_0, \quad G_{2n}(x) = P(Z^\mu \leq x) \approx \Phi(x) - 0.0364N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}), \\
&\text{under } H_1, \quad P\left(Z^\mu + N^{1/2} \sum_{i=1}^{N} (E(\hat{t}^\mu_i) - E(t^\mu_0)) \leq x\right) = \Phi(x) - 0.0364N^{-1/2}(x^2 - 1)\phi(x) \\
&\quad + 0.0354cN^{-1}(x^2 - 1)\phi(x) + O(N^{-1});
\end{align*}
\]

(c) For Model 3.3, for $Z^\tau$,

\[
\begin{align*}
&\text{under } H_0, \quad G_{3n}(x) = P(Z^\tau \leq x) \approx \Phi(x) - 0.0095N^{-1/2}(x^2 - 1)\phi(x) + O(N^{-1}), \\
&\text{under } H_1, \quad P\left(Z^\tau + N^{1/2} \sum_{i=1}^{N} (E(\hat{t}^\tau_i) - E(t^\tau_0)) \leq x\right) = \Phi(x) - 0.0095N^{-1/2}(x^2 - 1)\phi(x) \\
&\quad + 0.0058c^2N^{-1}(x^2 - 1)\phi(x) + O(N^{-1}).
\end{align*}
\]

5 Monte Carlo simulations

In this section, we provide some simulations to illustrate our theoretical results. The simulations on LLC tests were presented in Moon et al. (2007). Therefore, we only focus on the IPS tests. The following data generating processes similar to that in Moon et al. (2007) are adopted.

\[
\begin{align*}
&z_{it} = b_{0i} + b_{1i}t + y_{it}, \\
y_{it} = \left(1 - \frac{c_i}{n^\alpha T}\right)y_{i,t-1} + \sigma_i e_{it}, \\
y_{i,0} = 0, \quad b_{0i}, b_{1i}, e_{it} \sim i.i.d. N(0, 1), \quad \sigma_i^2 \sim U[0.5, 1.5].
\end{align*}
\]

Several different cases are considered where $c_i$ follows different distributions, i.e., (1) $c_i \sim iid U[0, 1]$; (2) $c_i \sim iid U[0, 8]$; (3) $c_i \sim iid \chi^2(1)$; (4) $c_i \sim iid \chi^2(6)$. Moreover, $N$ and $T$ are selected from \{25, 100, 1000\} and \{50, 100, 250\}, respectively. The results at 5% significance level are reported with 2,000 replications. More simulations can be conducted with similar results.

Firstly, the theoretical values of the local asymptotic powers are evaluated in Table 1 based on Corollary 3.3. Clearly, we can see IPS tests have lower power than LLC tests.
For $\hat{t}_{\delta,1}, \hat{t}_{\delta,21}, \hat{t}_{\delta,22}, Z$ and $Z^\mu$, the local asymptotic powers are increasing as $\bar{c}$ increases. For $\hat{t}_{\delta,31}, \hat{t}_{\delta,32}$ and $Z^\tau$, the local asymptotic powers are increasing as $\bar{c}^2$ increases. This can also be seen in Figure 1 where the theoretical local asymptotic powers of LLC and IPS tests are drawn given the values of $\bar{c}$ and $c_i \equiv c$ for all $i$.

Table 1: Theoretical local asymptotic powers of LLC and IPS tests at 5% level

| $c_i \sim iid$ | LLC | IPS |
|-----------------|-----|-----|
|                 | $\hat{t}_{\delta,1}$ | $\hat{t}_{\delta,21}$ | $\hat{t}_{\delta,22}$ | $\hat{t}_{\delta,31}$ | $\hat{t}_{\delta,32}$ | $Z$ | $Z^\mu$ | $Z^\tau$ |
| $U[0, 1]$       | 0.0983 | 0.0703 | 0.0792 | 0.0515 | 0.0518 | 0.0888 | 0.0661 | 0.0513 |
| $U[0, 8]$       | 0.8817 | 0.3914 | 0.5924 | 0.2398 | 0.3012 | 0.7666 | 0.2982 | 0.2025 |
| $\chi^2(1)$     | 0.1741 | 0.0963 | 0.1199 | 0.0651 | 0.0685 | 0.1464 | 0.0859 | 0.0629 |
| $\chi^2(6)$     | 0.9953 | 0.6587 | 0.8796 | 0.6794 | 0.8116 | 0.9722 | 0.5112 | 0.5723 |

Secondly, the simulated local asymptotic powers are reported in Table 2 for different models. The values are consistent with the theoretical values reported in Table 1. The difference between the theoretical values and simulated values is due to the well known finite sample bias of the unit root tests, see for instance Phillips (2012) and Hansen (2014).
Finally, in Table 2, the local power is increasing as \( N \) increases. Also, for \( Z \) and \( Z^\mu \), the local power is increasing as \( \bar{c} \) increases. For \( Z^\tau \), the local power is increasing as \( c^2 \) increases.

| Table 2: Local power of IPS tests \( Z \) and \( Z^\mu \) in the neighborhood of unity with order \( N^{-\alpha T^{-1}} \) |
|---|---|---|---|---|---|---|---|---|
| \( N \) | \( c_i \sim \) | \( T=50 \) | \( T=100 \) | \( T=250 \) | \( T=50 \) | \( T=100 \) | \( T=250 \) |
| 25 | \( U[0,1] \) | 0.0800 | 0.0785 | 0.0820 | 0.0545 | 0.0685 | 0.0745 | 0.0510 | 0.0605 | 0.0540 |
| | \( U[0,8] \) | 0.5985 | 0.5540 | 0.5160 | 0.1560 | 0.1535 | 0.1845 | 0.1190 | 0.0850 | 0.1105 |
| | \( \chi^2(1) \) | 0.1520 | 0.0875 | 0.1430 | 0.0645 | 0.0665 | 0.0720 | 0.0510 | 0.0625 | 0.0615 |
| | \( \chi^2(6) \) | 0.8345 | 0.8075 | 0.8115 | 0.2685 | 0.2535 | 0.2465 | 0.2750 | 0.1440 | 0.2155 |
| 100 | \( U[0,1] \) | 0.0745 | 0.0865 | 0.0795 | 0.0615 | 0.0615 | 0.0565 | 0.0565 | 0.0485 | 0.0500 |
| | \( U[0,8] \) | 0.6150 | 0.6090 | 0.6555 | 0.1810 | 0.1995 | 0.2120 | 0.1635 | 0.1230 | 0.1345 |
| | \( \chi^2(1) \) | 0.1070 | 0.1260 | 0.1380 | 0.0835 | 0.0655 | 0.0705 | 0.0615 | 0.0550 | 0.0605 |
| | \( \chi^2(6) \) | 0.9465 | 0.9035 | 0.9135 | 0.2780 | 0.3205 | 0.3500 | 0.2430 | 0.3135 | 0.2645 |
| 1000 | \( U[0,1] \) | 0.0640 | 0.0715 | 0.0825 | 0.0490 | 0.0575 | 0.0730 | 0.0520 | 0.0440 | 0.0530 |
| | \( U[0,8] \) | 0.6635 | 0.7365 | 0.7035 | 0.2330 | 0.2520 | 0.2390 | 0.1535 | 0.1645 | 0.1575 |
| | \( \chi^2(1) \) | 0.1035 | 0.1285 | 0.1550 | 0.0695 | 0.0780 | 0.0820 | 0.0610 | 0.0530 | 0.0585 |
| | \( \chi^2(6) \) | 0.9175 | 0.9610 | 0.9725 | 0.3675 | 0.4065 | 0.4040 | 0.3410 | 0.3710 | 0.3550 |

6 Conclusion

In this paper, we propose a unified approach to study the local asymptotic power of panel unit root tests. We use two most widely used panel unit root tests to illustrate our method, i.e. LLC and IPS tests. We demonstrate how to apply our approach to achieve the exact local asymptotic power of LLC and IPS tests for a variety of scenarios. Moreover, the Edgeworth expansion of these test statistics can also be achieved with our approach. Our approach can also be extended to other panel unit root tests.
Appendix

A.1 Proofs of Theorem 3.1 and Theorem 3.2

We give the detailed proofs of Theorem 3.1 and Theorem 3.2 in the following.

Proof of Theorem 3.1(a): Our goal is to calculate $E \left( \tilde{t}_{\delta,1} \right) = E \left( \frac{N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{1i}}{\sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{1i}}} \right)$ under $H_1$. Substituting $\theta = iu/2$, $x = -v/u$ and $c = c_iN^{-1/2}$ into $\varphi_1(\theta; c, 1, x)$ in Lemma 2.2, we have the joint m.g.f. for $(\tilde{U}_{1i}, \tilde{V}_{1i})$ as

$$\psi_{1,i}(u, v) = e^{-\frac{u}{2}} \left[ e^{-c_iN^{-\frac{1}{2}}} \left[ \cos \left( 2v - c_i^2N^{-1} \right) + (c_iN^{-1/2} - u) \frac{\sin \left( 2v - c_i^2N^{-1} \right)}{\sqrt{2v - c_i^2N^{-1}}} \right] \right]^{-1/2}. \quad (A.1)$$

Hence, the joint m.g.f. for $\left( N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{1i}, N^{-1} \sum_{i=1}^{N} \tilde{V}_{1i} \right)$ is

$$\phi_1(u, v) = \prod_{i=1}^{N} \psi_{1,i} \left( \frac{u}{\sqrt{N}}, \frac{v}{N} \right) = e^{-\frac{N}{2} \sum_{i=1}^{N} \frac{c_iN^{-\frac{1}{2}}}{N}} \prod_{i=1}^{N} \left[ \cos \left( \frac{2v}{N} - c_i^2N^{-1} \right) + (c_iN^{-1/2} - u) \frac{\sin \left( \frac{2v}{N} - c_i^2N^{-1} \right)}{\sqrt{\frac{2v}{N} - c_i^2N^{-1}}} \right]^{-1/2}. \quad (A.1)$$

Then, we have

$$\frac{\partial}{\partial u} \phi_1(u, -v) \bigg|_{u=0} = -\frac{\sqrt{N}}{2} \left[ e^{-\sum_{i=1}^{N} c_iN^{-\frac{1}{2}}} \prod_{i=1}^{N} \left[ \cos \left( \frac{2v}{N} - c_i^2N^{-1} \right) + c_iN^{-1/2} \frac{\sin \left( \frac{2v}{N} - c_i^2N^{-1} \right)}{\sqrt{\frac{2v}{N} - c_i^2N^{-1}}} \right] \right]^{-1/2}$$

$$-\frac{1}{2} \left[ e^{-\sum_{i=1}^{N} c_iN^{-\frac{1}{2}}} \prod_{i=1}^{N} \left[ \cos \left( \frac{2v}{N} - c_i^2N^{-1} \right) + c_iN^{-1/2} \frac{\sin \left( \frac{2v}{N} - c_i^2N^{-1} \right)}{\sqrt{\frac{2v}{N} - c_i^2N^{-1}}} \right] \right]^{-1/2}$$

$$\times \left( \sum_{i=1}^{N} \cos \left( \frac{2v}{N} - c_i^2N^{-1} \right) + c_iN^{-1/2} \frac{\sin \left( \frac{2v}{N} - c_i^2N^{-1} \right)}{\sqrt{\frac{2v}{N} - c_i^2N^{-1}}} \right). \quad (A.2)$$
From Taylor expansion, we have

\[
\frac{\partial}{\partial u} \phi_1(u, -v) \bigg|_{u=0} = -\frac{N}{2} e^{-\sum_{i=1}^N c_i N + \frac{1}{2} \sum_{i=1}^N c_i^2} \sum_{i=1}^N \log \left( 1 + c_i N^{-1/2} + \frac{1}{2} (2v + c_i^2) + \frac{c_i N^{-3/2}}{6} (2v + c_i^2) + O(N^{-2}) \right) -1/2
\]

Further, combining (A.2) and (A.3), we have

\[
\frac{\partial}{\partial u} \phi_1(u, -v) \bigg|_{u=0} = -\frac{N}{2} e^{-\sum_{i=1}^N c_i N + \frac{1}{2} \sum_{i=1}^N c_i^2} \sum_{i=1}^N \log \left( 1 + c_i N^{-1/2} + \frac{1}{2} (2v + c_i^2) + \frac{c_i N^{-3/2}}{6} (2v + c_i^2) + O(N^{-2}) \right) -1/2
\]

\[
\times \left( -\frac{N}{2} e^{-\sum_{i=1}^N c_i N + \frac{1}{2} \sum_{i=1}^N c_i^2} \sum_{i=1}^N c_i - N^{-1/2} \left( \frac{v}{3} + \frac{1}{6} N^{-1/2} \sum_{i=1}^N c_i^2 \right) + \frac{v}{\sqrt{N}} + \frac{1}{2N^{3/2}} \sum_{i=1}^N c_i^2 - \frac{1}{N^{3/2}} \sum_{i=1}^N c_i^2 + O(N^{-1}) \right)
\]

\[
\frac{\partial}{\partial u} \phi_1(u, -v) \bigg|_{u=0} = -\frac{N}{2} e^{-\sum_{i=1}^N c_i N + \frac{1}{2} \sum_{i=1}^N c_i^2} \sum_{i=1}^N \log \left( 1 + c_i N^{-1/2} + \frac{1}{2} (2v + c_i^2) + \frac{c_i N^{-3/2}}{6} (2v + c_i^2) + O(N^{-2}) \right) -1/2
\]

\[
\times \left( -\frac{N}{2} e^{-\sum_{i=1}^N c_i N + \frac{1}{2} \sum_{i=1}^N c_i^2} \sum_{i=1}^N c_i - N^{-1/2} \left( \frac{v}{3} + \frac{1}{6} N^{-1/2} \sum_{i=1}^N c_i^2 \right) + \frac{v}{\sqrt{N}} + \frac{1}{2N^{3/2}} \sum_{i=1}^N c_i^2 - \frac{1}{N^{3/2}} \sum_{i=1}^N c_i^2 + O(N^{-1}) \right)
\]
From Sawa (1972), we have
\[
E \left( \frac{U^p}{V^q} \right) = \frac{1}{\Gamma(q)} \int_0^\infty v^{q-1} \frac{\partial^p}{\partial u^p} \phi(u, -v) \bigg|_{u=0} \, dv. \tag{A.5}
\]
Hence, from (A.5) and plugging in (A.4), we get
\[
E \left( 
\hat{t}_{\delta,1} \right) = E \left( \frac{N^{-1/2} \sum_{i=1}^N \tilde{U}_{1i} \tilde{V}_{1i}}{N^{-1/2} \sum_{i=1}^N \tilde{V}_{1i}} \right)
= \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \phi(u, -v) \bigg|_{u=0} \, dv
\]
Hence, the joint m.g.f. for $(\tilde{U}_{2i}, \tilde{V}_{2i})$ as
\[
\psi_{2,i}(u, v) = e^{\frac{1}{2} \left[ e^{-c_i N^{-1/2}} \left[ \frac{u^{2} + 2v + c_i^2 N^{-1} u - c_i^2 N^{-3/2}}{2v - c_i^2 N^{-1}} \right] \right]}
\]
Hence, the joint m.g.f. for $(N^{-1/2} \sum_{i=1}^N \tilde{U}_{2i}, N^{-1} \sum_{i=1}^N \tilde{V}_{2i})$ is
\[
\phi_2(u, v) = \prod_{i=1}^N \psi_{2,i} \left( \frac{u}{\sqrt{N}}, \frac{v}{N} \right)
\]
Then, we have
\[
\frac{\partial}{\partial u} \phi_2(u, -v) \bigg|_{u=0}
= -\frac{\sqrt{N}}{2} \left[ e^{-\sum_{i=1}^N c_i N^{-1/2}} \prod_{i=1}^N \left[ \frac{-\frac{2N}{N} - c_i^2 N^{-3/2}}{-\frac{2N}{N} - c_i^2 N^{-1}} \frac{1}{\sqrt{-\frac{2N}{N} - c_i^2 N^{-1}}} - \frac{1}{\sqrt{-\frac{2N}{N} - c_i^2 N^{-1}}}
\]
\]
Proof of Theorem 3.1(b): Substituting $\theta = iu/2$, $x = -v/u$ and $c = c_i N^{-1/2}$ into $\varphi_2(\theta; c, 1, x)$ in Lemma 2.2.2, we have the joint m.g.f. for $(\tilde{U}_{2i}, \tilde{V}_{2i})$ as
\[
\psi_{2,i}(u, v) = e^{\frac{1}{2} \left[ e^{-c_i N^{-1/2}} \left[ \frac{u^{2} + 2v + c_i^2 N^{-1} u - c_i^2 N^{-3/2}}{2v - c_i^2 N^{-1}} \right] \right]}
\]
Hence, the joint m.g.f. for $(N^{-1/2} \sum_{i=1}^N \tilde{U}_{2i}, N^{-1} \sum_{i=1}^N \tilde{V}_{2i})$ is
\[
\phi_2(u, v) = \prod_{i=1}^N \psi_{2,i} \left( \frac{u}{\sqrt{N}}, \frac{v}{N} \right)
\]
Then, we have
\[
\frac{\partial}{\partial u} \phi_2(u, -v) \bigg|_{u=0}
= -\frac{\sqrt{N}}{2} \left[ e^{-\sum_{i=1}^N c_i N^{-1/2}} \prod_{i=1}^N \left[ \frac{-\frac{2N}{N} - c_i^2 N^{-3/2}}{-\frac{2N}{N} - c_i^2 N^{-1}} \frac{1}{\sqrt{-\frac{2N}{N} - c_i^2 N^{-1}}} - \frac{1}{\sqrt{-\frac{2N}{N} - c_i^2 N^{-1}}}
\]
\]
Further, combining (A.7) and (A.8), we have

\[
\frac{\partial}{\partial u}\phi_2(u, -v)\bigg|_{u=0} = -\sqrt{N} + \frac{1}{12N^{3/2}} \sum_{i=1}^{N} c_i^2 + \frac{1}{12N^2} \sum_{i=1}^{N} c_i^3 + O(N^{-5/2})
\]

From Taylor expansion, we have

\[
\frac{-2v}{N} - c_i N^{-3/2} \sin\left(\frac{-2v}{N} - c_i N^{-3/2}\right) + \frac{c_i^2 N^{-3/2}}{2} \cos\left(\frac{-2v}{N} - c_i N^{-3/2}\right) = 1 + c_i N^{-1/2} + \frac{1}{6N}(2v + 3c_i^2) + \frac{c_i N^{-3/2}}{6}(v + c_i^2) + O(N^{-2}).
\]

Thus,

\[
\log \left(1 + c_i N^{-1/2} + \frac{1}{6N}(2v + 3c_i^2) + \frac{c_i N^{-3/2}}{6}(v + c_i^2) + O(N^{-2})\right) = c_i N^{-1/2} + \frac{v}{3N} + \frac{1}{2} c_i^2 N^{-1} + \frac{1}{6} c_i v N^{-3/2} + \frac{1}{6} c_i^3 N^{-3/2} - \frac{1}{2} \left[c_i^2 N^{-1} + \frac{2c_i v}{3} N^{-3/2} + c_i^3 N^{-3/2}\right] + \frac{1}{3} c_i^3 N^{-3/2} + O(N^{-2})
\]

Further, combining (A.7) and (A.8), we have

\[
\frac{\partial}{\partial u}\phi_2(u, -v)\bigg|_{u=0} = -\sqrt{N} - \frac{1}{2} \left[\sum_{i=1}^{N} c_i N^{-1/2} \sum_{i=1}^{N} \log \left(1 + c_i N^{-1/2} + \frac{1}{6N}(2v + 3c_i^2) + \frac{c_i N^{-3/2}}{6}(v + c_i^2) + O(N^{-2})\right)\right]^{-1/2}
\]

\[
-\frac{1}{2} \left[\sum_{i=1}^{N} c_i N^{-1/2} \sum_{i=1}^{N} \log \left(1 + c_i N^{-1/2} + \frac{1}{6N}(2v + 3c_i^2) + \frac{c_i N^{-3/2}}{6}(v + c_i^2) + O(N^{-2})\right)\right]^{-1/2}
\]

\[
\times \left(-\frac{1}{12N^{3/2}} \sum_{i=1}^{N} c_i^2 + \frac{1}{12N^2} \sum_{i=1}^{N} c_i^3 + O(N^{-5/2})\right)
\]

\[
= -\sqrt{N} - \frac{1}{2} \left[\sum_{i=1}^{N} c_i N^{-1/2} \sum_{i=1}^{N} \log \left(1 + c_i N^{-1/2} + \frac{1}{6N}(2v + 3c_i^2) + \frac{c_i N^{-3/2}}{6}(v + c_i^2) + O(N^{-2})\right)\right]^{-1/2}
\]

\[
-\frac{1}{2} \left[\sum_{i=1}^{N} c_i N^{-1/2} \sum_{i=1}^{N} \log \left(1 + c_i N^{-1/2} + \frac{1}{6N}(2v + 3c_i^2) + \frac{c_i N^{-3/2}}{6}(v + c_i^2) + O(N^{-2})\right)\right]^{-1/2}
\]

\[
\times \left(-\frac{1}{12N^{3/2}} \sum_{i=1}^{N} c_i^2 + \frac{1}{12N^2} \sum_{i=1}^{N} c_i^3 + O(N^{-5/2})\right)
\]
It can be shown

\[ \frac{\partial}{\partial v} = \frac{\partial}{\partial v} \]

Therefore,

\[ E \left( \tilde{t}_{\delta,21} \right) = E \left( \frac{\sqrt{5}}{2} \frac{N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{2i}}{\sqrt{N-1} \sum_{i=1}^{N} \tilde{V}_{2i}} \right) + \sqrt{\frac{15N}{8}} = -\frac{1}{8} \sqrt{\frac{15}{2}} \tilde{c} + O_p(N^{-1/2}). \]

Next, we consider \( \tilde{t}_{\delta,22} \). We have

\[ \frac{\partial}{\partial v} \left[ \phi_2(0, -v) \right] \]

It can be shown

\[ E \left( V^{n-\alpha} \right) = \frac{(-1)^n}{\Gamma(\alpha)} \int_0^\infty v^{\alpha-1} \frac{\partial^n}{\partial v^n} [\phi(0, -v)] \, dv. \]
Hence, from (A.5) and (A.11), and plugging in (A.9) and (A.10), we receive

\[
E\left(\sqrt{\frac{10}{17}}\left[\frac{\sqrt{N^{-1/2}}\sum_{i=1}^{N}\tilde{U}_{2i}}{\sqrt{N^{-1}\sum_{i=1}^{N}\tilde{V}_{2i}}} + 3\sqrt{N}\left[\frac{N^{-1}\sum_{i=1}^{N}\tilde{V}_{2i}}{\sqrt{N^{-1}\sum_{i=1}^{N}\tilde{V}_{2i}}}\right]\right]\right)
\]

\[
= \sqrt{\frac{10}{17}}\Gamma(\frac{3}{2}) \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \phi_2(u, -v) \left|_{u=0}\right. dv - \sqrt{\frac{10}{17}} \Gamma(\frac{1}{2}) \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial v} \left[\phi_2(0, -v)\right] dv
\]

\[
= -\sqrt{\frac{10}{17}} \int_{0}^{\infty} \frac{e^{-v/6}}{\sqrt{v}} dv - \left(\sum_{i=1}^{N} c_i\right) \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{v e^{-v/6}}{v} dv
\]

\[
= -\sqrt{\frac{10}{17} \sqrt{2}} \int_{0}^{\infty} e^{-v/6} dv - \left(\sum_{i=1}^{N} c_i\right) \int_{0}^{\infty} \frac{v e^{-v/6}}{\sqrt{v}} dv + O_p(N^{-1/2})
\]

\[
= -\sqrt{30N} \frac{13}{34} - \frac{1}{4} \sqrt{\frac{15}{17}} \hat{c} + \sqrt{30N} \frac{13}{34} - \frac{1}{4} \sqrt{\frac{15}{17}} \hat{c} + O_p(N^{-1/2}) = -\frac{1}{2} \sqrt{\frac{15}{17}} \hat{c} + O_p(N^{-1/2}).
\]

Therefore,

\[
E\left(\hat{t}_{\delta, 22}\right) = E\left(\sqrt{\frac{10}{17}}\left[\frac{N^{-1/2}\sum_{i=1}^{N}\tilde{U}_{2i}}{\sqrt{N^{-1}\sum_{i=1}^{N}\tilde{V}_{2i}}} + 3\sqrt{N}\left[\frac{N^{-1}\sum_{i=1}^{N}\tilde{V}_{2i}}{\sqrt{N^{-1}\sum_{i=1}^{N}\tilde{V}_{2i}}}\right]\right]\right) = -\frac{1}{2} \sqrt{\frac{15}{17}} \hat{c} + O_p(N^{-1/2}).
\]

**Proof of Theorem 3.1(c):** Following the discussion in Section 3.1, we have

\[
\hat{t}_{\delta, 31} = \sqrt{\frac{448}{977}} \left(\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} - \bar{y}_{i,t-1}) - \frac{\hat{\delta}_3}{\sigma_{\tilde{z}, i}}\right) + \sqrt{\frac{1680N}{277}}.
\]

Following the corresponding specification of \(H_1\) in Assumption 3, we have

\[
\hat{t}_{\delta, 31} = -\sqrt{\frac{448}{277}} \frac{1}{N^{1/4}T} \sum_{i=1}^{N} \left[\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 - \frac{\sum_{t=1}^{T} y_{i,t-1}}{\sum_{t=1}^{T}}\right] / \sigma_{\tilde{z}, i}^{-1/2}
\]

\[
+ \sqrt{\frac{448}{277}} \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \left[\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 - \frac{\sum_{t=1}^{T} y_{i,t-1}}{\sum_{t=1}^{T}}\right] / \sigma_{\tilde{z}, i}^{-1/2}\right)^{-1/2}
\]

\[
\times \left[ \sum_{i=1}^{N} \left(\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1}) (\varepsilon_{ti} - \bar{y}_{i,t-1}) \right) - \left(\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})\right) \right] / \sigma_{\tilde{z}, i}^{-1/2}
\]

\[
- \frac{\sqrt{N}}{2} \left(\sum_{i=1}^{N} \left[\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2 - \frac{\sum_{t=1}^{T} y_{i,t-1}}{\sum_{t=1}^{T}}\right] / \sigma_{\tilde{z}, i}^{-1/2}\right) - \frac{15N}{4}.
\]

37
Moreover, as \( T \to \infty \), we have

\[
\hat{d}_{3,1} \Rightarrow -\sqrt{\frac{448}{277}} \sqrt{N^{-3/4}} \sum_{i=1}^{N} c_i \left[ \int_0^1 K_{i,c_1}^{\mu}(r) dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K_{i,c_1}^{\mu}(r) dr \right)^2 \right]
\]

\[
+ \sqrt{\frac{448}{277}} \sqrt{N^{-1/2}} \sum_{i=1}^{N} \left( \int_0^1 K_{i,c_1}^{\mu}(r) dW_i(r) - 12 \int_0^1 (r - \frac{1}{2}) K_{i,c_1}^{\mu}(r) dr \int_0^1 (r - \frac{1}{2}) dW_i(r) + \frac{1}{2} \right)
\]

\[
- \sqrt{\frac{448}{277}} \sqrt{N^{-1/2}} \sum_{i=1}^{N} \left[ \int_0^1 K_{i,c_1}^{\mu}(r) dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K_{i,c_1}^{\mu}(r) dr \right)^2 \right]
\]

\[
\left( 2 \sqrt{N^{-1}} \sum_{i=1}^{N} \left[ \int_0^1 K_{i,c_1}^{\mu}(r) dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K_{i,c_1}^{\mu}(r) dr \right)^2 \right] \right) - \sqrt{\frac{15N}{4}}
\]

\[
def \sqrt{\frac{448}{277}} \sqrt{N^{-1}} \sum_{i=1}^{N} \frac{\mu_i}{\sqrt{N}} + \sqrt{\frac{448}{277}} \sqrt{\frac{15N}{4}}.
\]

(A.13)

Substituting \( \theta = iu/2, x = -v/u \) and \( c = c_1N^{-1/4} \) into \( \varphi_4(\theta; c, 1, x) \) in Lemma 2.2, we have the joint m.g.f. for \((\tilde{U}_{3i}, \tilde{V}_{3i})\) as

\[
\psi_{3,i}(u, v) = e^{- \frac{N}{\sqrt{N}} \sum_{i=1}^{N} c_1N^{-1/4}} \left[ (c_1N^{-1/4})^5 - (c_1N^{-1/4})^4 u - 4((c_1N^{-1/4})^2 + 3c_1N^{-1/4} + 27)u^2 - 8v((c_1N^{-1/4})^2 - 3c_1N^{-1/4} - 3) \right]
\]

\[
\times \left[ \left( 2u - c_1N^{-1/2} \right)^3 \right] + 24((c_1N^{-1/4})^4 u + 8u^2 - 4(c_1N^{-1/4} + 1)(u^2 - 3u^2)) \left( \sin \frac{2u - c_1N^{-1/2}}{\sqrt{2u - c_1N^{-1/2}}} \right)
\]

\[
+ \cos \frac{2u - c_1N^{-1/2}}{\sqrt{2u - c_1N^{-1/2}}} \left( \frac{c_1N^{-1}}{2u - c_1N^{-1/2}} \right) - \frac{1}{2} \left( \frac{c_1N^{-1}}{2u - c_1N^{-1/2}} \right)^3 \right)
\]

\[
\times \left[ \frac{c_1N^{-1}}{2u - c_1N^{-1/2}} - 4c_1N^{-1}u + 4c_1N^{-1/2} + 3c_1N^{-1/4} - 3u^2 - 2c_1N^{-1/2}v(c_1N^{-1/4} + 3) \right]^{-1/2}
\]

\[
\left( \frac{2u - c_1N^{-1/2}}{2u - c_1N^{-1/2}} \right)^3
\]

(A.14)

Hence, the joint m.g.f. for \((N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{3i}, N^{-1} \sum_{i=1}^{N} \tilde{V}_{3i})\) is

\[
\phi_{3,i}(u, v) = \prod_{i=1}^{N} \psi_{3,i}\left( \frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}} \right)
\]

\[
= e^{- \frac{N}{\sqrt{N}} \sum_{i=1}^{N} c_1N^{-1/4}} \left[ (c_1N^{-1/4})^5 - (c_1N^{-1/4})^4 u - 4((c_1N^{-1/4})^2 + 3c_1N^{-1/4} + 27)u^2 - 8v((c_1N^{-1/4})^2 - 3c_1N^{-1/4} - 3) \right]
\]

\[
\times \left[ \left( 2u - c_1N^{-1/2} \right)^3 \right] + 24((c_1N^{-1/4})^4 u + 8u^2 - 4(c_1N^{-1/4} + 1)(u^2 - 3u^2)) \left( \sin \frac{2u - c_1N^{-1/2}}{\sqrt{2u - c_1N^{-1/2}}} \right)
\]

\[
+ \cos \frac{2u - c_1N^{-1/2}}{\sqrt{2u - c_1N^{-1/2}}} \left( \frac{c_1N^{-1}}{2u - c_1N^{-1/2}} \right) - \frac{1}{2} \left( \frac{c_1N^{-1}}{2u - c_1N^{-1/2}} \right)^3 \right)
\]

\[
\times \left[ \frac{c_1N^{-1}}{2u - c_1N^{-1/2}} - 4c_1N^{-1}u + 4c_1N^{-1/2} + 3c_1N^{-1/4} - 3u^2 - 2c_1N^{-1/2}v(c_1N^{-1/4} + 3) \right]^{-1/2}
\]

\[
\left( \frac{2u - c_1N^{-1/2}}{2u - c_1N^{-1/2}} \right)^3
\]
Then, we have

\[
\frac{\partial}{\partial u} \phi_3(u, -v) \bigg|_{u=0} = \frac{-\sqrt{N}}{2} e^{-\sum_{i=1}^{N} e^{-\frac{1}{2}v_i N^{-1/2}}} \prod_{i=1}^{N} \left[ \left( c_i N^{-1/4} \right)^5 + \frac{8v}{N} \left( c_i N^{-1/4} \right)^2 - 3c_i N^{-1/4} - 3 \right] \frac{\sin \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} \\
+ 24(-4c_i N^{-1/4} + 1) \frac{v^2}{N} \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \left( \sin \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} + \cos \frac{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} - \frac{1}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} \right) \\
+ \left( \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^2} - \frac{8(c_i^3 N^{-3/4} \frac{2v}{N})}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right) \cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} - \frac{4(2c_i^2 N^{-1/2} + \frac{v}{N} (c_i N^{-1/4} + 3))}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right]^{-1/2} \\
+ \left( \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^2} - \frac{8(c_i^3 N^{-3/4} \frac{2v}{N})}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right) \cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} - \frac{4(2c_i^2 N^{-1/2} + \frac{v}{N} (c_i N^{-1/4} + 3))}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right]^{-1/2} \\
\times \sum_{i=1}^{N} \left[ \left( c_i N^{-1/4} \right)^5 + \frac{8v}{N} \left( c_i N^{-1/4} \right)^2 - 3c_i N^{-1/4} - 3 \right] \frac{\sin \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} \\
+ 24(-4c_i N^{-1/4} + 1) \frac{v^2}{N} \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \left( \sin \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} + \cos \frac{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} - \frac{1}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} \right) \\
+ \left( \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^2} - \frac{8(c_i^3 N^{-3/4} \frac{2v}{N})}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right) \cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} - \frac{4(2c_i^2 N^{-1/2} + \frac{v}{N} (c_i N^{-1/4} + 3))}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right]^{-1} \\
\times \left( -c_i N^{-1/4} \right)^4 \frac{1}{\sqrt{N}} \sin \frac{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} + 24(c_i N^{-1/4})^4 \frac{1}{\sqrt{N}} \left( \sin \frac{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} \right) \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^2} \\
+ \frac{\cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} - \frac{4c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \left( \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right) \cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} \right) \\
\times \left( -c_i N^{-1/4} \right)^4 \frac{1}{\sqrt{N}} \sin \frac{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} + 24(c_i N^{-1/4})^4 \frac{1}{\sqrt{N}} \left( \sin \frac{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} \right) \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^2} \\
+ \frac{\cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} - \frac{4c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \left( \frac{c_i^4 N^{-1}}{(-\frac{2v}{N} - c_i^2 N^{-1/2})^3} \right) \cos \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} \right) \right). 
\] (A.15)

From Taylor expansion, we have

\[
(c_i N^{-1/4})^5 + \frac{8v}{N} ((c_i N^{-1/4})^2 - 3c_i N^{-1/4} - 3) \sin \sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}} \\
\left( -\frac{2v}{N} - c_i^2 N^{-1/2} \right)^2 \frac{\sqrt{2v}{N} - c_i^2 N^{-1/2}}{\sqrt{\frac{2v}{N} - c_i^2 N^{-1/2}}} 
\]
Further, we have

\[
\log \left( 1 + c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + \frac{2v}{15N} + \frac{1}{24} c_i^4 N^{-1} + \frac{2}{15} c_i v N^{-5/4} + \frac{1}{120} c_i^5 N^{-5/4} + \frac{13 c_i^2 v}{210} N^{-3/2} + \frac{c_i^6}{720} N^{-3/2} + O(N^{-7/4}) \right)
\]

\[
= c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + \frac{2v}{15N} + \frac{1}{24} c_i^4 N^{-1} + \frac{2}{15} c_i v N^{-5/4} + \frac{1}{120} c_i^5 N^{-5/4} + \frac{13 c_i^2 v}{210} N^{-3/2} + \frac{c_i^6}{720} N^{-3/2} + O(N^{-7/4})
\]

and combining (A.15) and (A.16), we have

\[
\left. \frac{\partial}{\partial u} \phi(u, v) \right|_{u=0} = -\frac{\sqrt{N}}{2} \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/4} \frac{2v}{15N} \sum_{i=1}^{N} \log \left( 1 + c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + \frac{2v}{15N} + \frac{1}{24} c_i^4 N^{-1} + \frac{2}{15} c_i v N^{-5/4} + \frac{1}{120} c_i^5 N^{-5/4} + \frac{13 c_i^2 v}{210} N^{-3/2} + \frac{c_i^6}{720} N^{-3/2} + O(N^{-7/4}) \right)} \right]^{-1/2}
\]

\[
- \frac{1}{2} \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/4} \frac{2v}{15N} \sum_{i=1}^{N} \log \left( 1 + c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + \frac{2v}{15N} + \frac{1}{24} c_i^4 N^{-1} + \frac{2}{15} c_i v N^{-5/4} + \frac{1}{120} c_i^5 N^{-5/4} + \frac{13 c_i^2 v}{210} N^{-3/2} + \frac{c_i^6}{720} N^{-3/2} + O(N^{-7/4}) \right)} \right]^{-1/2}
\]

\[
- \frac{1}{2} \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/4} \frac{2v}{15N} \sum_{i=1}^{N} \log \left( 1 + c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + \frac{2v}{15N} + \frac{1}{24} c_i^4 N^{-1} + \frac{2}{15} c_i v N^{-5/4} + \frac{1}{120} c_i^5 N^{-5/4} + \frac{13 c_i^2 v}{210} N^{-3/2} + \frac{c_i^6}{720} N^{-3/2} + O(N^{-7/4}) \right)} \right]^{-1/2}
\]

\[
- \frac{1}{2} \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/4} \frac{2v}{15N} \sum_{i=1}^{N} \log \left( 1 + c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + \frac{2v}{15N} + \frac{1}{24} c_i^4 N^{-1} + \frac{2}{15} c_i v N^{-5/4} + \frac{1}{120} c_i^5 N^{-5/4} + \frac{13 c_i^2 v}{210} N^{-3/2} + \frac{c_i^6}{720} N^{-3/2} + O(N^{-7/4}) \right)} \right]^{-1/2}
\]
Based on the specification of $H_1$ given in Assumption 3, we have

\[
\hat{t}_{\delta,32} = -\sqrt{\frac{112}{193}N^{1/4}T} \sqrt{\sum_{i=1}^{N} \left[ \frac{\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{T} (t-\bar{t})^2} - \frac{\sum_{i=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{T} (t-\bar{t})^2} \right]} / \sigma^2_{\hat{c},i} \]

(A.18)

Based on the specification of $H_1$ given in Assumption 3, we have

\[
\hat{t}_{\delta,32} = -\sqrt{\frac{112}{193}N^{1/4}T} \sqrt{\sum_{i=1}^{N} \left[ \frac{\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{T} (t-\bar{t})^2} - \frac{\sum_{i=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{T} (t-\bar{t})^2} \right]} / \sigma^2_{\hat{c},i} \]

(A.18)

Therefore,

\[
E \left( \frac{\sqrt{\sum_{i=1}^{N} \tilde{U}_{3i}}}{\sqrt{N-1} \sum_{i=1}^{N} \tilde{V}_{3i}} \right) = \sqrt{\frac{448}{277}} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \left. \frac{\partial}{\partial u} \phi_3(u,-v) \right|_{u=0} dv
\]

\[
= -\sqrt{\frac{448}{277}} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-v/15}}{\sqrt{v}} dv - \left( N^{-1} \sum_{i=1}^{N} c_i^2 \right) \sqrt{\frac{448}{277}} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-v/15}}{\sqrt{v}} dv + O_p(N^{-1/4})
\]

\[
= -\sqrt{\frac{448}{277}} \sqrt{\frac{15N}{4}} - \frac{1}{14} \sqrt{\frac{105}{277}} c^2 + O_p(N^{-1/4}).
\]

Next, we consider $\hat{t}_{\delta,32}$. Recall that

\[
\hat{t}_{\delta,32} = \sqrt{\frac{112}{193}N^{1/4}T} \left[ \frac{\delta_3 + \frac{15}{27}}{\sqrt{\sum_{i=1}^{N} \left( (y_{i,t-1} - \bar{y}_{i,t-1}) - \frac{\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{N} (t-\bar{t})^2} \right)^2} } / \sigma^2_{\hat{c},i} \right]^{-1/2}.
\]

(A.18)
Hence, we have that as $T \to \infty$

$$i_{\delta,32} \Rightarrow -\sqrt{\frac{12}{193}N^{-3/4}} \sum_{i=1}^{N} c_i \left[ \int_0^1 K^\mu_{i,c_i}(r)^2 dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K^\mu_{i,c_i}(r) dr \right)^2 \right]$$

$$+ \sqrt{\frac{12}{193}N^{-1/2}} \sum_{i=1}^{N} \left[ \int_0^1 K^\mu_{i,c_i}(r)^2 dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K^\mu_{i,c_i}(r) dr \right)^2 \right] \int_0^1 (r - \frac{1}{2}) dW_i(r) + \frac{1}{2}$$

$$- \sqrt{\frac{12}{193}N^{-1/2}} \sum_{i=1}^{N} \left[ \int_0^1 K^\mu_{i,c_i}(r)^2 dr - 12 \left( \int_0^1 (r - \frac{1}{2}) K^\mu_{i,c_i}(r) dr \right)^2 \right]$$

$$\equiv \sqrt{\frac{12}{193}N^{-1/2}} \sum_{i=1}^{N} \tilde{U}_{3i} \left[ \tilde{V}_{3i} + \frac{15\sqrt{N}}{2} \sqrt{N-1} \sum_{i=1}^{N} \tilde{V}_{3i} \right].$$

(A.19)

We have

$$\frac{\partial}{\partial v} [\phi_3(0, -v)]$$

$$= \frac{\partial}{\partial v} \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/4} \frac{4}{4}} \exp \left( \sum_{i=1}^{N} \log \left( 1 + c_i N^{-1/4} + \frac{1}{2} c_i^2 N^{-1/2} + \frac{1}{6} c_i^3 N^{-3/4} + N^{-1} \left( \frac{2v}{15} + \frac{c_i^4}{24} \right) + N^{-5/4} \left( \frac{2c_i v}{15} + \frac{c_i^5}{120} \right) + N^{-3/2} \left( \frac{13c_i^2 v}{210} + \frac{c_i^6}{720} \right) + O(N^{-7/4}) \right) \right] \right]^{-1/2}.$$
Proof of Theorem 3.2(a):

Let

Therefore,

\[
\frac{\partial}{\partial v} \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/2} + \sum_{i=1}^{N} c_i N^{-1/4} + \sum_{i=1}^{N} c_i^2 N^{-3/2} + O(N^{-3/4})} \right] ^{-1/2}
\]

\[
\frac{\partial}{\partial v} \left( \left[ e^{2v} N^{-3/2} \sum_{i=1}^{N} c_i^2 N^{-3/2} + O(N^{-3/4}) \right] ^{-1/2} - \frac{1}{2} \left[ e^{2v} N^{-3/2} \sum_{i=1}^{N} c_i^2 N^{-3/2} + O(N^{-3/4}) \right] ^{-1/2} \right)
\]

\[
= -\frac{1}{15} e^{-v/15} + \frac{e^{-v/15} - \frac{1}{15} v e^{-v/15}}{420N^{3/2}} \sum_{i=1}^{N} c_i^2 + O(N^{-3/4}).
\]

Hence, from (A.5) and (A.11), and plugging in (A.17) and (A.20), we have

\[
E \left( \sqrt{\frac{112}{193}} \left[ \frac{N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{3i}}{\sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{3i}}} + \frac{15 \sqrt{N}}{2} \sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{3i}} \right] \right)
\]

\[
= \sqrt{\frac{112}{193}} \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \phi_3(u, -v) \left. \right|_{u=0} dv - \sqrt{\frac{112}{193}} \frac{15 \sqrt{N}}{2} \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \left[ \phi_3(0, -v) \right] dv
\]

\[
= -\sqrt{\frac{112}{193}} \frac{\sqrt{N}}{2} \int_{0}^{\infty} \frac{1}{\sqrt{v}} e^{-v/15} dv - \left( N^{-1} \sum_{i=1}^{N} c_i^2 \right) \sqrt{\frac{112}{193}} \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{v e^{-v/15}}{\sqrt{v}} dv
\]

\[
+ \sqrt{\frac{112}{193}} \frac{\sqrt{N}}{2} \int_{0}^{\infty} \frac{1}{\sqrt{v}} e^{-v/15} dv - \left( N^{-1} \sum_{i=1}^{N} c_i^2 \right) \sqrt{\frac{112}{193}} \frac{1}{\Gamma(\frac{1}{2})} \left( \int_{0}^{\infty} \frac{e^{-v/15}}{\sqrt{v}} dv - \frac{1}{15} \int_{0}^{\infty} \frac{v e^{-v/15}}{\sqrt{v}} dv \right)
\]

\[+O_p(N^{-1/4})
\]

\[
= -\sqrt{\frac{112}{193}} \frac{15N}{4} - \frac{\sqrt{15}}{112} \sqrt{\frac{112}{193}} c^2 + \sqrt{\frac{112}{193}} \sqrt{\frac{15N}{4}} - \frac{\sqrt{15}}{112} \sqrt{\frac{112}{193}} c^2 + O_p(N^{-1/4}).
\]

Therefore,

\[
E (\hat{t}_{\delta,32}) = E \left( \sqrt{\frac{112}{193}} \left[ \frac{N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{3i}}{\sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{3i}}} + \frac{15 \sqrt{N}}{2} \sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{3i}} \right] \right)
\]

\[
= -\frac{\sqrt{15}}{56} \sqrt{\frac{112}{193}} c^2 + O_p(N^{-1/4}).
\]

Proof of Theorem 3.2(a): Let

\[
F_i = \int_{0}^{1} W_i(r)^2 dr,
\]

\[
G_i = -2c_i N^{-1/2} \int_{0}^{1} W_i(r) \int_{0}^{r} W_i(s) ds dr + O_p(N^{-1}).
\]
We have that

\[
\frac{(F_i + G_i)^{-1/2}}{\sqrt{F_i}} - \frac{G_i}{2\sqrt{F_i^3}} + O_p(N^{-1})
\]

\[
= \frac{1}{\sqrt{\int_0^1 W_i(r)^2dr}} + \frac{2c_i N^{-1/2} \int_0^1 W_i(r) \int_0^r W_i(r) dsdr}{2 \sqrt{\left(\int_0^1 W_i(r)^2dr\right)^3}} + O_p(N^{-1}).
\]

Hence,

\[
\widehat{t}_i \Rightarrow \frac{\tilde{U}_{1i}}{\sqrt{V_{1i}}} = -\frac{c_i}{N^{1/2}} \sqrt{\int_0^1 W_i(r)^2 dr} + \frac{\int_0^1 W_i(r) dW_i(r) - c_i N^{-1/2} \int_0^1 W_i(r) dsdr}{\sqrt{\int_0^1 W_i(r)^2 dr}} + O_p(N^{-1}).
\]

From the standard CLT and LLN, we have

\[
Z = \sqrt{\frac{N(N-1) \sum_{i=1}^N \hat{t}_i - E(t_0)}{\text{Var}(t_0)}} \Rightarrow N(0, 1) - \tilde{c} \left[ E \left( \sqrt{\int_0^1 W(r)^2 dr} \right) + E \left( \frac{\int_0^1 W(r) dsdr}{\sqrt{\int_0^1 W(r)^2 dr}} \right) - E \left( \frac{\int_0^1 W(r) dW(r) \int_0^1 W(s) dsdr}{\sqrt{\left(\int_0^1 W(r)^2dr\right)^3}} \right) \right] / \sqrt{\text{Var}(t_0)}.
\]

However, this is not very informative, since the expectations in this expression could not be calculated easily, which has to rely on simulations.

Now, we apply our approach. From (A.1), we have

\[
\frac{\partial}{\partial u} \psi_{1,1}(u, -v) \bigg|_{u=0} = -\frac{1}{2} \left[ e^{-c_i N^{-\frac{1}{2}}} \left[ \cos \sqrt{2v - c_i^2 N^{-1}} + c_i N^{-1/2} \sin \frac{\sqrt{2v - c_i^2 N^{-1}}}{\sqrt{2v - c_i^2 N^{-1}}} \right] \right]^{-1/2}
\]

\[
-\frac{1}{2} \left[ e^{-c_i N^{-\frac{1}{2}}} \left[ \cos \sqrt{2v - c_i^2 N^{-1}} + c_i N^{-1/2} \sin \frac{\sqrt{2v - c_i^2 N^{-1}}}{\sqrt{2v - c_i^2 N^{-1}}} \right] \right]^{-3/2} \left( -\frac{\sin \sqrt{2v - c_i^2 N^{-1}}}{\sqrt{2v - c_i^2 N^{-1}}} \right) e^{-c_i N^{-\frac{1}{2}}}
\]

\[
= -\frac{1}{2} \left[ \left( 1 - c_i N^{-\frac{1}{2}} + O(N^{-1}) \right) \left( \cosh \frac{\sqrt{2v}}{\sqrt{2v}} + c_i N^{-1/2} \sinh \frac{\sqrt{2v}}{\sqrt{2v}} + O(N^{-1}) \right) \right]^{-1/2}
\]

\[
+ \frac{1}{2} \left[ \left( 1 - c_i N^{-\frac{1}{2}} + O(N^{-1}) \right) \left( \cosh \frac{\sqrt{2v}}{\sqrt{2v}} + c_i N^{-1/2} \sinh \frac{\sqrt{2v}}{\sqrt{2v}} + O(N^{-1}) \right) \right]^{-3/2}
\]

\[
\times \left( \frac{\sinh \frac{\sqrt{2v}}{\sqrt{2v}} - c_i N^{-1/2} \sinh \frac{\sqrt{2v}}{\sqrt{2v}} + O(N^{-1})}{\sqrt{2v}} + O(N^{-1}) \right)
\]

44
\[-\frac{1}{2} \left[ \cosh \sqrt{2v} - c_i N^{-\frac{1}{2}} \left( \cosh \sqrt{2v} - \frac{\sinh \sqrt{2v}}{\sqrt{2v}} \right) + O(N^{-1}) \right]^{-1/2}
\]
\[+ \frac{1}{2} \left[ \cosh \sqrt{2v} - c_i N^{-\frac{1}{2}} \left( \cosh \sqrt{2v} - \frac{\sinh \sqrt{2v}}{\sqrt{2v}} \right) + O(N^{-1}) \right]^{-3/2} \left( \frac{\sinh \sqrt{2v}}{\sqrt{2v}} - c_i N^{-\frac{1}{2}} \frac{\sinh \sqrt{2v}}{\sqrt{2v}} + O(N^{-1}) \right).\]

Recall that \( \hat{t}_i = \frac{E_i}{\sqrt{V_i}} \). Therefore, applying (A.5) and by the change of variable as \( x = \sqrt{2v} \) and Taylor expansion, we have

\[E(Z)\]
\[= (\text{Var}(t_0))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} (E(\hat{t}_i) - E(t_0))\]
\[= (\text{Var}(t_0))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} \left( \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \psi_{1,i}(u, -v) \right) \left|_{u=0} \right. \text{dv} - E(t_0) \right)
\[= (\text{Var}(t_0))^{-1/2} N^{1/2} \left( -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (\cosh(x))^{-1/2} \left( 1 - \frac{\sinh(x)}{x \cosh(x)} \right) \right.
\[\left. + \frac{1}{2} \left( N^{-1} \sum_{i=1}^{N} c_i \right) N^{-1/2} (\cosh(x))^{-1/2} \left( 1 - \frac{2 \sinh(x)}{x \cosh(x)} + \frac{3(\sinh(x))^2}{x^2(\cosh(x))^2} \right) \right) \text{dx} + O(N^{-1}) - E(t_0) \right)
\[= -(\text{Var}(t_0))^{-1/2} \frac{c}{2\sqrt{2\pi}} \int_{0}^{\infty} (\cosh(x))^{-1/2} \left( 1 - \frac{2 \sinh(x)}{x \cosh(x)} + \frac{3(\sinh(x))^2}{x^2(\cosh(x))^2} \right) \text{dx} + O_p(N^{-1/2}),\]

where
\[E(t_0) = -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (\cosh(x))^{-1/2} \left( 1 - \frac{\sinh(x)}{x \cosh(x)} \right) \text{dx}.\]

**Proof of Theorem 3.2(b):** Let

\[F_i = \int_{0}^{r} W_{i}^\mu(r)^2 \text{dr},\]
\[G_i = -2c_i N^{-1/2} \int_{0}^{r} W_{i}^\mu(r) \left( \int_{0}^{r} W_{i}(s) \text{ds} - \int_{0}^{r} \int_{0}^{r} W_{i}(s) \text{ds} \text{dt} \right) \text{dr} + O_p(N^{-1}).\]

We have that

\[(F_i + G_i)^{-1/2}\]
\[= \frac{1}{\sqrt{F_i}} - \frac{G_i}{2 \sqrt{F_i^3}} + O_p(N^{-1})\]
\[= \frac{1}{\sqrt{\int_{0}^{r} W_{i}^\mu(r)^2 \text{dr}}} + \frac{2c_i N^{-1/2} \int_{0}^{r} W_{i}^\mu(r) \left( \int_{0}^{r} W_{i}(s) \text{ds} - \int_{0}^{r} \int_{0}^{r} W_{i}(s) \text{ds} \text{dt} \right) \text{dr}}{2 \sqrt{(\int_{0}^{r} W_{i}^\mu(r)^2 \text{dr})^3}} + O_p(N^{-1}).\]
Hence,

\[
\hat{t}_i^\mu \Rightarrow \frac{U_{2i}}{\sqrt{V_{2i}}} = \frac{c_i}{N^{1/2}} \sqrt{\int_0^1 W_\mu(r)^2 dr + \frac{f_0^1 W_\mu(r) W_i(r) - c_i N^{-1/2} \int_0^1 \left( \int_0^r W_i(s) ds - \int_0^1 W_i(s) ds + \int_0^1 W_i(s) ds \right) dr}{\sqrt{\int_0^1 W_\mu(r)^2 dr}} + O_p(N^{-1}).}
\]

From the standard CLT and LLN, we have

\[
Z^\mu = \frac{\sqrt{N} \left( \sum_{i=1}^N \hat{t}_i^\mu - E(t_0^\mu) \right)}{\sqrt{\text{Var}(t_0^\mu)}} \Rightarrow N(0, 1) - \tilde{c} \left[ E \left( \sqrt{\int_0^1 W_\mu(r)^2 dr} \right) + E \left( \frac{\int_0^1 \left( \int_0^r W(s) ds - \int_0^1 W(s) ds + \int_0^1 W(s) ds \right) dr}{\sqrt{\int_0^1 W_\mu(r)^2 dr}} \right) \right] / \sqrt{\text{Var}(t_0^\mu)},
\]

which is almost impossible to evaluate.

Now, we apply our approach. From (A.6), we have

\[
\frac{\partial}{\partial u} \psi_{2i}(u, -v) \bigg|_{u=0} = -\frac{1}{2} \left[ e^{-c_i N^{-1/2}} \left[ -2v - c_i^2 N^{-3/2} \sin \sqrt{-2v - c_i^2 N^{-1}} - c_i^2 N^{-1} \cos \sqrt{-2v - c_i^2 N^{-1}} \right] - \frac{2c_i^2 N^{-1}}{(-2v - c_i^2 N^{-1})^2} \left[ -2v - c_i^2 N^{-3/2} \sin \sqrt{-2v - c_i^2 N^{-1}} - c_i^2 N^{-1} \cos \sqrt{-2v - c_i^2 N^{-1}} \right] \right]^{-1/2} \left[ \frac{2c_i^2 N^{-1}}{(-2v - c_i^2 N^{-1})^2} \left[ -2v - c_i^2 N^{-3/2} \sin \sqrt{-2v - c_i^2 N^{-1}} - c_i^2 N^{-1} \cos \sqrt{-2v - c_i^2 N^{-1}} \right] \right]^{-3/2} \left( \frac{c_i^2 N^{-1}}{(-2v - c_i^2 N^{-1})^2} \sin \sqrt{-2v - c_i^2 N^{-1}} \right)
\]

\[
+ 2c_i N^{-1} \cos \sqrt{-2v - c_i^2 N^{-1}} \left[ (1 - c_i N^{-1/2} + O(N^{-1})) \left( \frac{\sin \sqrt{2v}}{\sqrt{2v}} + c_i N^{-1/2} [\cosh \sqrt{2v} - 1] + O(N^{-1}) \right) \right]^{-1/2}.
\]

46
\[-\frac{1}{2} \left( 1 - c_i N^{-\frac{1}{2}} + O(N^{-1}) \right) \left( \frac{\sinh \sqrt{2v}}{\sqrt{2v}} + c_i N^{-1/2} v^{-1} \cosh \sqrt{2v} - O(N^{-1}) \right)^{-3/2} O(N^{-1}) \]

\[ = -\frac{1}{2} \left[ \frac{\sinh \sqrt{2v}}{\sqrt{2v}} - c_i N^{-\frac{1}{2}} \left( \frac{\sinh \sqrt{2v}}{\sqrt{2v}} - v^{-1} \cosh \sqrt{2v} - 1 \right) + O(N^{-1}) \right]^{-1/2} + O(N^{-1}). \]

Recall that \( \hat{t}_i^\mu = \frac{\tilde{t}_{i \mu}}{\sqrt{v_i}} \), applying \([A.5]\) and by the change of variable as \( x = \sqrt{2v} \) and Taylor expansion, we have

\[ E(Z^\nu) \]

\[ = (Var(t_0^\mu))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} (E(\hat{t}_i^\mu) - E(t_0^\mu)) \]

\[ = (Var(t_0^\mu))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} \left( \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \psi_{2,i}(u, -v) \bigg|_{u=0} \right) dv - E(t_0^\mu) \]

\[ = (Var(t_0^\mu))^{-1/2} N^{1/2} \left( -\frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} + \frac{1}{2} \left( \frac{N^{-1} \sum_{i=1}^{N} c_i}{N^{-1/2}} \right) \right) N^{-1/2} \]

\[ \times \left( \frac{\sinh(x)}{x} \right)^{-1/2} \left( 1 - \frac{2(cosh(x) - 1)}{x \sinh(x)} \right) dx + O(N^{-1}) - E(t_0^\mu) \]

\[ = -\frac{Var(t_0^\mu)}{2\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} \left( 1 - \frac{2(cosh(x) - 1)}{x \sinh(x)} \right) dx + O(N^{-1/2}), \]

where

\[ E(t_0^\mu) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \frac{\sinh(x)}{x} \right)^{-1/2} dx. \]

**Proof of Theorem 3.2(c):** Plugging \([10]\) into \([39]\) and applying (12), we have that as \( T \to \infty \)

\[ \hat{t}_i^\tau = \frac{\hat{\delta}_1^\tau - 1}{\hat{\sigma}_{\tau,i}} \sqrt{ \frac{\sum_t(t-i)^2}{(\sum_t(t-i)^2)(\sum_t(z_{i,t-1} - \bar{z}_{i,t-1})^2 - (\sum_t(t-i)(z_{i,t-1} - \bar{z}_{i,t-1}))^2} } \]

\[ = \frac{1}{\hat{\sigma}_{\tau,i}} \left( \delta_1 - 1 \right) \sqrt{ \frac{\left( \sum_t(z_{i,t-1} - \bar{z}_{i,t-1})^2 \right)^2}{\sum_t(t-i)^2} - \left( \sum_t(t-i)(z_{i,t-1} - \bar{z}_{i,t-1}))^2 \right)^2 } \]

\[ + \frac{1}{\sqrt{\sum_t(t-i)^2}} \sqrt{\left( \sum_t(t-i)^2 \right)^2 \left( \sum_t(z_{i,t-1} - \bar{z}_{i,t-1})^2 - (\sum_t(t-i)(z_{i,t-1} - \bar{z}_{i,t-1}))^2 \right)^2 } \]

47
\[
\left(\sum_t (t \bar{t} (z_{i,t-1} - \bar{z}_{i,t-1})) (\sum_t (t \bar{t} ((1 - \delta_t) \beta_{1,i} (t \bar{t}) + (\varepsilon_{it} - \bar{\varepsilon}_{it})))
\right)
\sqrt{\sum_t (t \bar{t})^2 \left(\sum_t (t \bar{t})^2 \left(\sum_t (z_{i,t-1} - \bar{z}_{i,t-1})^2 - (\sum_t (t \bar{t}) (z_{i,t-1} - \bar{z}_{i,t-1})^2
\right)\right)}
\]

\[
= \frac{1}{\sigma_{\varepsilon,i}} \left( -\frac{c_i}{T N^{1/4}} \sqrt{\sum_t (t \bar{t})^2 \left(\sum_t (t \bar{t})^2 \left(\sum_t (z_{i,t-1} - \bar{z}_{i,t-1})^2 - (\sum_t (t \bar{t}) (z_{i,t-1} - \bar{z}_{i,t-1})^2
\right)\right)}\right)
\]

\[
\left(\sum_t (t \bar{t} (y_{i,t-1} - \bar{y}_{i,t-1})) (\sum_t (t \bar{t} ((1 - \delta_t) \beta_{1,i} (t \bar{t}) + (\varepsilon_{it} - \bar{\varepsilon}_{it})))
\right)
\sqrt{\sum_t (t \bar{t})^2 \left(\sum_t (t \bar{t})^2 \left(\sum_t (y_{i,t-1} - \bar{y}_{i,t-1})^2 - (\sum_t (t \bar{t}) (y_{i,t-1} - \bar{y}_{i,t-1})^2
\right)\right)}
\]

\[
\Rightarrow \frac{-c_i}{N^{1/4}} \int_0^1 K_{\mu,i_{\varepsilon}}(r)^2 dr - 12 \left(\int_0^1 (r - \frac{1}{2}) K_{\mu,i_{\varepsilon}}(r)^2 dr\right)^2
\]

\[
\int_0^1 K_{\mu,i_{\varepsilon}}(r)^2 dr - 12 \left(\int_0^1 (r - \frac{1}{2}) K_{\mu,i_{\varepsilon}}(r)^2 dr\right)^2 = \tilde{U}_{3i}
\]

where \(K_{\mu,i_{\varepsilon}}(r) = K_{i_{\varepsilon},i}(r) - \int_0^1 K_{i_{\varepsilon},i}(s) ds\), and \(K_{i_{\varepsilon},i}(r) = \int_0^r e^{-c_i N^{-1/4} (r-s)} dW_i(s)\).

Now, we apply our approach. From (A.14), we have

\[
\frac{\partial}{\partial u} \psi_{3,i}(u, v) \bigg|_{u=0}
= -\frac{1}{2} \left[ e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right) \right]
\]

\[
+ \frac{24 (-4(c_i N^{-1/4} + 1) v^2)}{(-2 v - c_i^2 N^{-1/2})^2} \left( \sin \left( -2 v - c_i^2 N^{-1/2} \right) + \cos \left( -2 v - c_i^2 N^{-1/2} \right) \right)
\]

\[
+ \frac{1}{(-2 v - c_i^2 N^{-1/2})^2} \left( \frac{8(c_i N^{-3/4} v^2)}{(-2 v - c_i^2 N^{-1/2})^2} \cos \left( -2 v - c_i^2 N^{-1/2} - 4 \left( -2 v - c_i^2 N^{-1/2} \right) \right) \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
+ \frac{24 (-4(c_i N^{-1/4} + 1) v^2)}{(-2 v - c_i^2 N^{-1/2})^2} \left( \sin \left( -2 v - c_i^2 N^{-1/2} \right) + \cos \left( -2 v - c_i^2 N^{-1/2} \right) \right)
\]

\[
+ \frac{1}{(-2 v - c_i^2 N^{-1/2})^2} \left( \frac{8(c_i N^{-3/4} v^2)}{(-2 v - c_i^2 N^{-1/2})^2} \cos \left( -2 v - c_i^2 N^{-1/2} - 4 \left( -2 v - c_i^2 N^{-1/2} \right) \right) \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]

\[
\times e^{-c_i N^{-1} - \frac{1}{2}} \left( (c_i N^{-1/4})^5 + 8 (c_i N^{-1/4})^2 - 3 c_i N^{-1/4} - 3 \right) \sin \left( -2 v - c_i^2 N^{-1/2} \right)
\]
\[ \cos \sqrt{-2v - c_i^2 N^{-1/2}} + \frac{8c_i^2 N^{-1} - 3c_i N^{-1/4} - 3}{(-2v - c_i^2 N^{-1/2})^2} \sin \sqrt{-2v - c_i^2 N^{-1/2}} + \frac{1}{(-2v - c_i^2 N^{-1/2})^3} \cos \sqrt{-2v - c_i^2 N^{-1/2}} \]

Further, from Taylor expansion, we have

\[
\frac{\partial}{\partial u} \psi_{3,i}(u, -v) \bigg|_{u=0} = -\frac{1}{2} \left[ (1 - c_i N^{-1/4} + 1 \frac{c_i^2 N^{-1/2} N^{-1/2}}{2}) \frac{8c_i^2 N^{-1/2} - 3c_i N^{-1/4} - 3}{(-2v - c_i^2 N^{-1/2})^2} \sin \sqrt{-2v - c_i^2 N^{-1/2}} + \frac{1}{(-2v - c_i^2 N^{-1/2})^3} \cos \sqrt{-2v - c_i^2 N^{-1/2}} \right] + O(N^{-3/4}) + O(N^{-1})
\]

Recall that \( \hat{\tau}_i^T = \frac{\bar{v}_{0i}}{\sqrt{V_{0i}}} \), applying (A.35) and by the change of variable as \( x = \sqrt{2v} \) and a Taylor expansion, we have

\[
E(Z^*) = (\text{Var}(\hat{\tau}_{0i}^T))^{-1/2} N^{1/2} N^{-1} \sum_{i=1}^{N} (E(\hat{\tau}_i^T) - E(\hat{\tau}_{0i}^T))
\]

49
\[
\begin{align*}
\text{Var}(t^*_0) & = (N^{-1/2}) \left( - \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[ 3f_{22}(x) + \left( \frac{N-1}{2}\sum_{i=1}^N c^2_i \right) N^{-1/2} \left( -\frac{2\sinh(x)}{x^3} + \frac{18 \cosh(x)}{x^4} - \frac{66 \sinh(x)}{x^5} \right) dx + O(N^{-3/4}) \right]^{1/2}
+ \frac{96 \cosh(x) - 96}{x^6} \right) + O(N^{-1/4})
\end{align*}
\]

\[
\begin{align*}
\text{Var}(t^*_0) & = (N^{-1/2}) \left( - \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[ 3f_{22}(x) \right]^{1/2} dx + \left( \frac{N-1}{2}\sum_{i=1}^N c^2_i \right) N^{-1/2} \left( -\frac{2\sinh(x)}{x^3} + \frac{18 \cosh(x)}{x^4} - \frac{66 \sinh(x)}{x^5} \right) dx + O(N^{-3/4}) \right)
+ \frac{18 \cosh(x) - 66 \sinh(x)}{x^4} + \frac{96 \cosh(x) - 96}{x^6} \right) + O(N^{-1/4})
\end{align*}
\]

\[
\begin{align*}
\text{Var}(t^*_0) & = (N^{-1/2}) \left( - \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[ 3f_{22}(x) \right]^{1/2} dx + \left( \frac{N-1}{2}\sum_{i=1}^N c^2_i \right) N^{-1/2} \left( -\frac{2\sinh(x)}{x^3} + \frac{18 \cosh(x)}{x^4} - \frac{66 \sinh(x)}{x^5} \right) dx + O(N^{-3/4}) \right)
+ \frac{18 \cosh(x) - 66 \sinh(x)}{x^4} + \frac{96 \cosh(x) - 96}{x^6} \right) + O(N^{-1/4})
\end{align*}
\]

where \( f_{22}(x) = 4 \left( \frac{1}{x^2} \sinh(x) - \frac{2}{x^3} [\cosh(x) - 1] \right) \), and \( E(t^*_0) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty [3f_{22}(x)]^{1/2} dx \)
from equation (7) and page 147 in Nabeya (1999).

### A.2 Edgeworth expansion

We give more detailed derivations below for the Edgeworth expansions of LLC and IPS test. For LLC tests, Theorem 2.2 in Hall (1992) can be applied. For \( \hat{t}_{\delta,1} \), we have the one-term Edgeworth expansion under \( H_0 \) or \( H_1 \) as

\[
F_{1n}(x) = P \left( \hat{t}_{\delta,1} - \frac{N^{1/2} E(U_{1i})}{\sqrt{E(V_{1i})}} \leq x \right) = \Phi(x) - \frac{1}{2} N^{-1/2} \phi(x) [l_{11} E(U_{1i}^2) + 2l_{12} E(U_{1i} V_{1i}) + l_{22} E(V_{1i}^2)]
- \frac{1}{6} N^{-1/2} H_2(x) \phi(x) [3l_{11} E(l_1 \tilde{U}_{1i}^2 + l_2 \tilde{U}_{1i} \tilde{V}_{1i})]^2 + 3l_{12} E(l_1 \tilde{U}_{1i} \tilde{V}_{1i} + l_2 \tilde{V}_{1i}^2)]^2
+ 6l_{12} E(l_1 \tilde{U}_{1i}^2 + l_2 \tilde{U}_{1i} \tilde{V}_{1i}) E(l_1 \tilde{U}_{1i} \tilde{V}_{1i} + l_2 \tilde{V}_{1i}^2) + E(l_1 \tilde{U}_{1i} + l_2 \tilde{V}_{1i})^3] + O(N^{-1}),
\]

(A.21)

where \( \tilde{U}_{1i} = U_{1i} - E(U_{1i}), \tilde{V}_{1i} = V_{1i} - E(V_{1i}), l_1 = (E(V_{1i}))^{-1/2}, l_2 = \frac{1}{2} E(U_{1i})/(E(V_{1i}))^{-3/2}, l_{11} = 0, l_{12} = \frac{1}{2} (E(V_{1i}))^{-3/2}, l_{22} = \frac{1}{2} E(U_{1i})/(E(V_{1i}))^{-5/2}, H_2(x) = x^2 - 1, \) and \( \phi(x) \) is the density function of the standard normal distribution.

Under \( H_0 \), we can calculate the moments of \( U_{1i} \) and \( V_{1i} \) from (A.1) with \( c_i = 0 \), which give \( E(U_{1i}) = 0, E(V_{1i}) = 1/2, E(U_{1i}^2) = 1/2, \) and \( E(U_{1i} V_{1i}) = 1/3 \). Moreover, these imply
Then we have the skewness of 

Further, from (A.21), we have

The computation could be carried out using the symbolic calculations in MATLAB.

Next, we consider IPS tests. Theorem 2.1 in Hall (1992) can be applied directly. For Z, under H₀, applying (A.5) to (A.1) with \( c_i = 0 \), we have

Then we have the skewness of \( t_0 \) as

The standard one term Edgeworth expansion could be obtained
as (56). Under $H_1$, applying (A.5) to (A.1), we have

$$E(\hat{t}_i) = E(t_0) - \frac{\bar{c}}{2\sqrt{2\pi}} N^{-1/2} \int_0^\infty (\cosh(x))^{-1/2} \left(1 - \frac{2\sinh(x)}{x \cosh(x)} + \frac{3(\sinh(x))^2}{x^2(\cosh(x))^2} \right) dx + O(N^{-1}),$$

$$E(\hat{t}_i)^2 = E(t_0)^2 + \frac{\bar{c}}{8} N^{-1/2} \int_0^\infty x(\cosh(x))^{-1/2} \left(1 - \frac{3\sinh(x)}{x \cosh(x)} + 9 \frac{(\sinh(x))^2}{x \cosh(x)} \right) dx - 15 \left(\frac{\sinh(x)}{x \cosh(x)}\right)^3 dx + O(N^{-1}),$$

$$E(\hat{t}_i)^3 = E(t_0)^3 - \frac{\sqrt{2}\bar{c}}{16\sqrt{\pi}} N^{-1/2} \int_0^\infty x^2(\cosh(x))^{-1/2} \left(1 - \frac{4\sinh(x)}{x \cosh(x)} + 18 \frac{(\sinh(x))^2}{x \cosh(x)} \right) dx - 60 \left(\frac{\sinh(x)}{x \cosh(x)}\right)^3 + 105 \left(\frac{\sinh(x)}{x \cosh(x)}\right)^4 dx + O(N^{-1}).$$

The one term Edgeworth expansion could be obtained as (57) with

$$\lambda_{1, c} = [E(\hat{t}_i)^3 - 3E(\hat{t}_i)^2E(\hat{t}_i) + 2(E(\hat{t}_i))^3]/(Var(t_0))^{3/2}.$$

Similarly, for $Z^\mu$, under $H_0$, applying (A.5) to (A.6) with $c_i = 0$, we have

$$E(t_0^\mu) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \left(\frac{\sinh(x)}{x}\right)^{-1/2} dx,$n

$$E(t_0^\mu)^2 = \frac{1}{4} \int_0^\infty x \left(\frac{\sinh(x)}{x}\right)^{-1/2} \left(1 + \frac{4}{x^2} - \frac{8(\cosh(x) - 1)}{x^3 \sinh(x)} \right) dx,$n

$$E(t_0^\mu)^3 = -\frac{\sqrt{2}}{8\sqrt{\pi}} \int_0^\infty x^2 \left(\frac{\sinh(x)}{x}\right)^{-1/2} \left(1 + \frac{12}{x^2} - \frac{24(\cosh(x) - 1)}{x^3 \sinh(x)} \right) dx.$$

Then we have the skewness of $t_0^\mu$ as $\lambda_2 = E(t_0^\mu - E(t_0^\mu))^3/(Var(t_0^\mu))^{3/2}$. The standard one term Edgeworth expansion could be obtained as (58). Under $H_1$, applying (A.3) to (A.6), we have

$$E(\hat{t}_i^\mu) = E(t_0^\mu) - \frac{\bar{c}}{2\sqrt{2\pi}} N^{-1/2} \int_0^\infty \left(\frac{\sinh(x)}{x}\right)^{-1/2} \left(1 - \frac{2(\cosh(x) - 1)}{x \sinh(x)} \right) dx + O(N^{-1}),$$

$$E(\hat{t}_i^\mu)^2 = E(t_0^\mu)^2 + \frac{\bar{c}}{8} N^{-1/2} \int_0^\infty x \left(\frac{\sinh(x)}{x}\right)^{-1/2} \left(1 - \frac{2(\cosh(x) - 1)}{x \sinh(x)} + \frac{4}{x^2} - \frac{32(\cosh(x) - 1)}{x^3 \sinh(x)} + \frac{48(\cosh(x) - 1)^2}{x^4(\sinh(x))^2} \right) dx + O(N^{-1}),$$

52
\[ E(\hat{t}_i^\mu)^3 = E(t_0^\mu)^3 - \frac{\sqrt{2}c}{16\sqrt{\pi}} N^{-1/2} \int_0^\infty x^2 \left( \frac{\sinh(x)}{x} \right)^{-1/2} \left( 1 - \frac{2(\cosh(x) - 1)}{x \sinh(x)} + \frac{12}{x^2} \right) dx + O(N^{-1}). \]

The one term Edgeworth expansion could be obtained as \[ \text{(59)} \] with

\[ \lambda_{2,c} = \left[ E(t_0^\mu)^3 - 3E(t_0^\mu)^2E(t_0^\mu) + 2(E(t_0^\mu))^3 \right] / (\text{Var}(t_0^\mu))^{3/2}. \]

For \( Z^\tau \), under \( H_0 \), applying \( (A.15) \) to \( (A.14) \) with \( c_i = 0 \), we have

\[ E(t_0^\tau) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \left[ 12 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4} \cosh(x) - 1 \right) \right]^{-1/2} dx, \]

\[ E(t_0^\tau)^2 = \frac{1}{4} \int_0^\infty x \left[ 12 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4} \cosh(x) - 1 \right) \right]^{-1/2} \left( 1 + 4 \left( \frac{24}{x^7} + \frac{1}{x^5} \right) \sinh(x) \right. \]

\[ - \left( \frac{24}{x^8} + \frac{8}{x^6} \right) \cosh(x) + \frac{24}{x^8} - \frac{4}{x^6} \right) dx, \]

\[ E(t_0^\tau)^3 = -\frac{\sqrt{2}}{8\sqrt{\pi}} \int_0^\infty x^2 \left[ 12 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4} \cosh(x) - 1 \right) \right]^{-1/2} \left( 1 + 12 \left( \frac{24}{x^7} + \frac{1}{x^5} \right) \sinh(x) \right. \]

\[ - \left( \frac{24}{x^8} + \frac{8}{x^6} \right) \cosh(x) + \frac{24}{x^8} - \frac{4}{x^6} \right) dx. \]

Then we have the skewness of \( t_0^\tau \) as \( \lambda_3 = E(t_0^\tau - E(t_0^\tau))^3 / (\text{Var}(t_0^\tau))^{3/2} \). The standard one term Edgeworth expansion could be obtained as \[ \text{(60)}. \] Under \( H_1 \), applying \( (A.15) \) to \( (A.14) \), we have

\[ E(\hat{t}_i^\tau) = E(t_0^\tau) - \frac{c}{\sqrt{2\pi}} N^{-1/2} \int_0^\infty \left[ 12 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4} \cosh(x) - 1 \right) \right]^{-3/2} \left( \frac{\sinh(x)}{x^3} \right. \]

\[ - \frac{9 \cosh(x)}{x^4} + \frac{33 \sinh(x)}{x^5} - \frac{48(\cosh(x) - 1)}{x^6} \right) dx + O(N^{-1}), \]

\[ E(\hat{t}_i^\tau)^2 = E(t_0^\tau)^2 + \frac{c^2}{4} N^{-1/2} \int_0^\infty x \left[ 12 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4} \cosh(x) - 1 \right) \right]^{-3/2} \left( \frac{\sinh(x)}{x^3} \right. \]

\[ - \frac{9 \cosh(x)}{x^4} + \frac{\sinh(x)}{x^5} + \frac{232 \cosh(x)}{x^6} + \frac{176}{x^7} - \frac{1080 \sinh(x)}{x^7} + \frac{2496 \cosh(x)}{x^8} \]

\[ - \frac{192}{x^8} - \frac{4608 \sinh(x)}{x^9} + \frac{4608(\cosh(x) - 1)}{x^{10}} \right) dx + \frac{c^2 N^{-1/2}}{\sqrt{\pi}} \int_0^\infty x \left[ 12 \left( \frac{1}{x^3} \sinh(x) \right. \right. \]

\[ - \frac{9 \cosh(x)}{x^4} + \frac{\sinh(x)}{x^5} + \frac{232 \cosh(x)}{x^6} + \frac{176}{x^7} - \frac{1080 \sinh(x)}{x^7} + \frac{2496 \cosh(x)}{x^8} \]

\[ - \frac{192}{x^8} - \frac{4608 \sinh(x)}{x^9} + \frac{4608(\cosh(x) - 1)}{x^{10}} \right) dx + O(N^{-1}). \]
\[-\frac{2}{x^4}[\cosh(x) - 1]\]^{-5/2} \left( \frac{9 \sinh(x)}{x^3} - \frac{39 \cosh(x)}{x^4} + \frac{99 \sinh(x)}{x^5} + \frac{12}{x^4} - \frac{144(\cosh(x) - 1)}{x^6} \right) \\
\times \left( \frac{12 \sinh(x)}{x^5} - \frac{96 \cosh(x)}{x^6} + \frac{288 \sinh(x)}{x^7} - \frac{48}{x^6} - \frac{288(\cosh(x) - 1)}{x^8} \right) \, dx + O(N^{-1}),

\[E(\hat{t}_i^3) = E(t_0^3) - \frac{\sqrt{2c^2}}{8\sqrt{\pi}} N^{-1/2} \int_0^\infty x^2 \left[ 12 \left( \frac{1}{x^3} \sinh(x) - \frac{2}{x^4}[\cosh(x) - 1] \right) \right]^{-3/2} \left( \frac{\sinh(x)}{x^3} \right. \right.

\[\left. - \frac{9 \cosh(x)}{x^4} - \frac{63 \sinh(x)}{x^5} + \frac{792 \cosh(x)}{x^6} + \frac{432}{x^6} - \frac{3240 \sinh(x)}{x^7} + \frac{7488 \cosh(x)}{x^8} \right. \right.

\[\left. - \frac{576}{x^8} - \frac{13824 \sinh(x)}{x^9} + \frac{13824(\cosh(x) - 1)}{x^{10}} \right) \, dx - \frac{3\sqrt{2c^2}}{2\sqrt{\pi}} N^{-1/2} \int_0^\infty x^2 \left[ 12 \left( \frac{1}{x^3} \sinh(x) \right. \right.

\[\left. \left. - \frac{2}{x^4}[\cosh(x) - 1] \right) \right]^{-5/2} \left( \frac{9 \sinh(x)}{x^3} - \frac{39 \cosh(x)}{x^4} + \frac{99 \sinh(x)}{x^5} + \frac{12}{x^4} - \frac{144(\cosh(x) - 1)}{x^6} \right) \\
\times \left( \frac{12 \sinh(x)}{x^5} - \frac{96 \cosh(x)}{x^6} + \frac{288 \sinh(x)}{x^7} - \frac{48}{x^6} - \frac{288(\cosh(x) - 1)}{x^8} \right) \, dx + O(N^{-1}).

The one term Edgeworth expansion could be obtained as \([61]\) with

\[\lambda_{3,c} = [E(\hat{t}_i^3) - 3E(\hat{t}_i^2)E(\hat{t}_i) + 2(E(\hat{t}_i^3))^2]/(Var(\hat{t}_i^3))^{3/2}.\]

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Supplemental Material to “A Unified Approach on the Local Power of Panel Unit Root Tests”

Zhongwen Liang
Department of Economics
University at Albany, SUNY

1 Comparison with the existing method

To compare the difference between our approach and the existing method, the brief discussion is given in the following. For \( \hat{t}_{\delta,1} \), by the standard Law of Large Numbers (LLN) and CLT, we have

\[
\hat{t}_{\delta,1} \Rightarrow -\frac{N^{-1} \sum_{i=1}^{N} c_i E \left( \int_{0}^{1} W_i(r)^2 dr \right)}{\sqrt{E \left( \int_{0}^{1} W_i(r)^2 dr \right)}} + \frac{N^{-1/2} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}(r)dW_i(r)}{\sqrt{N^{-1} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}(r)^2 dr}} \Rightarrow -\frac{\tilde{c}}{\sqrt{2}} + N(0,1), \tag{S.1}
\]

where \( \tilde{c} = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} c_i \), and by noticing that \( E \left( \int_{0}^{1} W_i(r)^2 dr \right) = \int_{0}^{1} r dr = 1/2 \). This coincides with the result in Theorem 3.1, and was obtained in Moon et al. (2007) and Westerlund and Breitung (2012).

Next, we consider \( \hat{t}_{\delta,21} \) and \( \hat{t}_{\delta,22} \). We have as \( T \to \infty \)

\[
\hat{t}_{\delta,21} \Rightarrow -\frac{\sqrt{3} N^{-1} \sum_{i=1}^{N} c_i \int_{0}^{1} K_{i,c_i}^\mu(r)^2 dr}{2 \sqrt{N^{-1} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}^\mu(r)^2 dr}} + \frac{\sqrt{3} N^{-1/2} \sum_{i=1}^{N} \left( \int_{0}^{1} K_{i,c_i}^\mu(r)dW_i(r) + \frac{1}{2} \right)}{\sqrt{N^{-1} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}^\mu(r)^2 dr}}
\]

\[
-\frac{\sqrt{3} N^{-1} \sum_{i=1}^{N} c_i \int_{0}^{1} K_{i,c_i}^\mu(r)^2 dr}{2 \sqrt{N^{-1} \sum_{i=1}^{N} \int_{0}^{1} K_{i,c_i}^\mu(r)^2 dr}} - \frac{3N}{2},
\]

where \( K_{i,c_i}^\mu(r) = K_{i,c_i}(r) - \int_{0}^{1} K_{i,c_i}(s)ds \), and

\[
K_{i,c_i}(r) = \int_{0}^{r} e^{-c_i N^{-1/2}(r-s)} dW_i(s) = W_i(r) - c_i N^{-1/2} \int_{0}^{r} e^{-c_i N^{-1/2}(r-s)} W_i(s)ds
\]

\[
= W_i(r) - c_i N^{-1/2} \int_{0}^{r} W_i(s)ds + c_i^2 N^{-1} \int_{0}^{r} (r-s) W_i(s)ds + O_p(N^{-3/2}),
\]

\[
K_{i,c_i}^\mu(r) = W_{i}^\mu(r) - c_i N^{-1/2} \left( \int_{0}^{r} W_i(s)ds - \int_{0}^{1} \int_{0}^{t} W_i(s)dsdt \right)
\]

\[
+ c_i^2 N^{-1} \left( \int_{0}^{r} (r-s) W_i(s)ds - \int_{0}^{1} \int_{0}^{t} (r-s) W_i(s)dsdt \right) + O_p(N^{-3/2}),
\]
where $W_i^\mu(r) = W_i(r) - \int_0^{1} W_i(s)ds$.

Hence,
\[
\begin{align*}
\int_0^{1} K_{i,c_i}^\mu(r)dW_i(r) &= \int_0^{1} W_i^\mu(r)dW_i(r) - c_i N^{-1/2} \int_0^{1} \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dW_i(r) \\
&\quad + c_i N^{-1} \int_0^{1} \left( \int_0^{r} (r-s)W_i(s)ds - \int_0^{1} \int_0^{t} (t-s)W_i(s)dtds \right) dW_i(r) + O_p(N^{-3/2}),
\end{align*}
\]
and
\[
\int_0^{1} K_{i,c_i}^\mu(r)^2 dr = \int_0^{1} W_i^\mu(r)^2 dr - 2c_i N^{-1/2} \int_0^{1} W_i^\mu(r) \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dr \\
&\quad + c_i N^{-1} \int_0^{1} \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right)^2 dr \\
&\quad + 2c_i N^{-1} \int_0^{1} W_i^\mu(r) \left( \int_0^{r} (r-s)W_i(s)ds - \int_0^{1} \int_0^{t} (t-s)W_i(s)dtds \right) dr + O_p(N^{-3/2}).
\]

Thus, we have
\[
\begin{align*}
&-\sqrt{N} \frac{N^{-1} \sum_{i=1}^{N} c_i \int_0^{1} K_{i,c_i}^\mu(r)^2 dr}{\sqrt{2} \sqrt{N^{-1} \sum_{i=1}^{N} \int_0^{1} K_{i,c_i}^\mu(r)^2 dr}} \sqrt{N} \frac{N^{-1/2} \sum_{i=1}^{N} \left( \int_0^{1} K_{i,c_i}^\mu(r)dW_i(r) + \frac{1}{2} \right)}{\sqrt{N^{-1} \sum_{i=1}^{N} \int_0^{1} K_{i,c_i}^\mu(r)^2 dr}} \\
&= \frac{-\sqrt{N}}{2} \left( \sqrt{N} \frac{N^{-1} \sum_{i=1}^{N} c_i \int_0^{1} W_i^\mu(r)^2 dr}{\sqrt{2} \sqrt{N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr}} + O_p(N^{-1/2}) \right) \\
&\quad + \sqrt{\frac{N}{2}} \left( N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr - 2N^{-3/2} \sum_{i=1}^{N} c_i \int_0^{1} W_i^\mu(r) \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dr + O_p(N^{-1}) \right)^{-1/2} \\
&\quad \times \left( N^{-1/2} \sum_{i=1}^{N} \left( \int_0^{1} W_i^\mu(r)dW_i(r) + \frac{1}{2} \right) - N^{-1} \sum_{i=1}^{N} c_i \int_0^{1} \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dW_i(r) + O_p(N^{-1/2}) \right) \\
&\quad - \sqrt{\frac{N}{2}} \left( \sqrt{N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr} - \frac{\sqrt{\frac{N}{2}}}{\sqrt{\frac{N}{2}}} - \frac{\sqrt{\frac{N}{2}}}{\sqrt{\frac{N}{2}}} \right) \left( N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr - \frac{1}{6} - 2N^{-3/2} \sum_{i=1}^{N} c_i \int_0^{1} W_i^\mu(r) \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dr + O_p(N^{-1}) \right)^{1/2} \\
&\quad + O_p(N^{-1}) + \sqrt{\frac{N}{2}} \sum_{j=1}^{\infty} \frac{\sqrt{\pi} \left( \frac{1}{2} - \frac{j}{2} \right)}{\Gamma(-\frac{1}{2} - j)(j + 2)!} \left( N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr - \frac{1}{6} \right) \\
&\quad - 2N^{-3/2} \sum_{i=1}^{N} c_i \int_0^{1} W_i^\mu(r) \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dr + O_p(N^{-1}) \left( \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r) dW_i(r) + \frac{1}{2} \right)^{1/2} \\
&= \frac{-\sqrt{N}}{2} \sqrt{N^{-1} \sum_{i=1}^{N} c_i \int_0^{1} W_i^\mu(r)^2 dr} + \sqrt{\frac{N}{2}} \left( N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr \right)^{-1/2} \left( N^{-1/2} \sum_{i=1}^{N} \left( \int_0^{1} W_i^\mu(r)dW_i(r) + \frac{1}{2} \right) \right) \\
&\quad - N^{-1} \sum_{i=1}^{N} c_i \int_0^{1} \left( \int_0^{r} W_i(s)ds - \int_0^{1} \int_0^{t} W_i(s)dtds \right) dW_i(r) + \frac{\sqrt{N}}{4} \left( N^{-1} \sum_{i=1}^{N} \int_0^{1} W_i^\mu(r)^2 dr \right)^{-3/2}.
\end{align*}
\]
\[
x \left( \frac{N^{-1} \sum_{i=1}^{N} \left( \int_{0}^{1} W_{i}^{2}(r)dr \right) + \frac{1}{2} \right) \left[ \frac{2N^{-1} \sum_{i=1}^{N} c_i \int_{0}^{1} W_{i}^{2}(r) \left( \int_{0}^{t} W_{i}(s)ds - \int_{0}^{t} W_{i}(s)dsdt \right) \right] \\
+ \sqrt{\frac{3}{8}} N^{-1/2} \sum_{i=1}^{N} \left( \int_{0}^{1} W_{i}^{2}(r)^{2}dr - \frac{1}{6} \right) - \sqrt{\frac{3}{8}} N^{-1/2} \sum_{i=1}^{N} c_i \int_{0}^{1} W_{i}^{2}(r) \left( \int_{0}^{t} W_{i}(s)ds - \int_{0}^{t} W_{i}(s)dsdt \right) \\
+ O_p(N^{-1/2})
\]
\[
\Rightarrow N(0,1) - \sqrt{\frac{3}{24}} - \sqrt{\frac{\nu}{2}} \left( E(\int_{0}^{1} W_{\mu}(r)^{2}dr) \right)^{-1/2} \left[ E(\int_{0}^{1} W_{\mu}(r) \int_{0}^{t} W(r)dsdt)dr \right] \\
- \sqrt{\frac{3}{2}} \nu E(\int_{0}^{1} W_{\mu}(r) \left( \int_{0}^{t} W_{\pi}(s)ds - \int_{0}^{t} W_{\pi}(s)dsdt \right)dr) = N(0,1) - \frac{1}{8} \sqrt{\frac{3}{2} \nu},
\]

(S.2)

where

\[
E \left( \int_{0}^{1} W_{\mu}(r)^{2}dr \right) = E \left( \int_{0}^{1} W(r)^{2}dr \right) - E \left( \int_{0}^{1} W(r)dr \right)^2
\]
\[
= \frac{1}{2} - E \left( \int_{0}^{1} \int_{0}^{1} W(s)W(r)dsdr \right)
\]
\[
= \frac{1}{2} - \int_{0}^{1} \int_{0}^{1} (s \wedge r)dsdr = \frac{1}{2} - \int_{0}^{1} \int_{0}^{r} sdsdr - \int_{0}^{1} \int_{r}^{1} rdsdr = \frac{1}{6},
\]

and

\[
E \left( \int_{0}^{1} \left( \int_{0}^{r} W(s)ds - \int_{0}^{t} W(s)dsdt \right) \right) \]
\[
= E \left( \int_{0}^{1} \int_{0}^{r} W(s)dsdr - \int_{0}^{t} \int_{0}^{r} W(s)dsdt W(1) \right)
\]
\[
= E \left( \int_{0}^{r} W(s)dsW(r) \right) - \int_{0}^{t} \int_{0}^{t} W(s)dsdt W(1)
\]
\[
= E \left( \int_{0}^{r} W(s)dsW(r) \right) - \int_{0}^{t} W(r)^2dr - \int_{0}^{t} \int_{0}^{t} W(s)dsdt W(1)
\]
\[
= \int_{0}^{1} E[W(s)W(1)]ds - \int_{0}^{1} E[W(r)^2]dr - \int_{0}^{1} \int_{0}^{t} E[W(s)W(1)]dsdt
\]
\[
= \int_{0}^{1} sds - \int_{0}^{r} rdr - \int_{0}^{1} sdsdt = -\frac{1}{6},
\]

\[
E \left( \int_{0}^{1} W_{\mu}(r) \left( \int_{0}^{r} W(s)ds - \int_{0}^{t} W(s)dsdt \right) \right)
\]
\[
= E \left[ \int_{0}^{1} W(r) \int_{0}^{r} W(s)dsdr - \int_{0}^{t} W(r)dr \int_{0}^{t} \int_{0}^{r} W(s)dsdt - \int_{0}^{1} W(r)dr \int_{0}^{t} \int_{0}^{r} W(s)dsdr \right]
\]
\[
+ \int_{0}^{1} W(r)dr \int_{0}^{t} \int_{0}^{r} W(s)dsdt
\]
\[
= E \left[ \int_{0}^{1} W(r) \int_{0}^{r} W(s)dsdr - \int_{0}^{1} W(r)dr \int_{0}^{t} \int_{0}^{t} W(s)dsdt \right]
\]
\[
\begin{align*}
\hat{t}_{\delta,22} & = N(0,1) - \sqrt{\frac{5}{51} - \frac{10}{17}} \left( E\left( \int_0^1 W^\mu(r)^2 dr \right)^{-1/2} \left( \bar{c} E\left( \int_0^1 W(s) ds - \int_0^t W(s) ds dt \right) dW(r) \right) \right) \\
& - 6 \sqrt{\frac{10}{17}} e \left( E\left( \int_0^1 W^\mu(r)^2 dr \right)^{-1/2} E\left( \int_0^t W^\mu(r) \left( \int_0^r W(s) ds - \int_0^t W(s) ds dt \right) dr \right) \right) \\
& = N(0,1) - \frac{1}{2} \sqrt{\frac{15}{17}} \bar{c}, \quad (S.3)
\end{align*}
\]

The results \((S.2)\) and \((S.3)\) were obtained in Moon and Perron (2008), where we have slightly different derivations here.

For \(\hat{t}_{\delta,31}\) and \(\hat{t}_{\delta,32}\), using the similar derivations, we have

\[
\begin{align*}
\hat{t}_{\delta,31} & = N(0,1) - \sqrt{\frac{448}{277} N^{1/4} \bar{c} E[A - 12B_2^2] + \frac{E[B_1 - 12B_6B_8]}{\sqrt{E[A - 12B_2^2]}}} + \sqrt{\frac{448 \bar{c}^2 E[B_{10} - 12B_9B_8]}{277 \sqrt{E[A - 12B_2^2]}}} \\
& + \sqrt{\frac{448}{277} \times \frac{15}{4} \bar{c}^2 \left[ E[B_3 - 12B_6^2] + 2E[B_7 - 12B_5B_9] \right]} = N(0,1) - \frac{1}{14} \sqrt{\frac{105 \bar{c}^2}{277}}, \quad (S.4)
\end{align*}
\]

where \(\bar{c}^2 = \lim_{N \to \infty} N^{-1} \sum_{i=1}^N c_i^2\).

\[
\begin{align*}
A & = \int_0^1 W^\mu(r)^2 dr, \\
B_1 & = \int_0^1 \left\{ \int_0^r W(s) ds - \int_0^t W(s) ds dt \right\} dW(r), \\
B_2 & = \int_0^1 W^\mu(r) dW(r), \\
B_3 & = \int_0^1 \left\{ \int_0^r W(s) ds - \int_0^t W(s) ds dt \right\}^2 dr, \\
B_4 & = \int_0^1 W^\mu(r) \left\{ \int_0^r W(s) ds - \int_0^t W(s) ds dt \right\} dr, \\
B_5 & = \int_0^1 (r - \frac{1}{2}) W^\mu(r) dr,
\end{align*}
\]
\[ B_6 = \int_0^1 (r - \frac{1}{2}) \left\{ \int_0^r W(s) ds - \int_0^t \int_0^r W(s) ds dt \right\} dr, \]
\[ B_7 = \int_0^1 W^\mu(r) \left\{ \int_0^r (r - s) W(s) ds - \int_0^t \int_0^s (t - s) W(s) ds dt \right\} dr, \]
\[ B_8 = \int_0^1 (r - \frac{1}{2}) dW(r) = \frac{1}{2} W(1) - \int_0^1 W(r) dr, \]
\[ B_9 = \int_0^1 (r - \frac{1}{2}) \left\{ \int_0^r (r - s) W(s) ds - \int_0^t \int_0^s (t - s) W(s) ds dt \right\} dr, \]
\[ B_{10} = \int_0^1 \left\{ \int_0^r (r - s) W(s) ds - \int_0^t \int_0^s (t - s) W(s) ds dt \right\} dW(r), \]

and \( W^\mu(r) = W(r) - \int_0^1 W(s) ds \). From the complicated calculations, we have \( E[A - 12B_6^2] = \frac{1}{17} \), \( E[B_1 - 12B_6B_8] = -\frac{1}{15} \), \( E[B_4 - 12B_5B_6] = 0 \), \( E[B_{10} - 12B_9B_8] = 0 \), \( E[B_3 - 12B_6^2] = \frac{1}{225} \), and \( E[B_7 - 12B_5B_9] = -\frac{1}{1425} \). Furthermore, we have

\[ \hat{\delta}_{3,2} \]
\[ \Rightarrow N(0, 1) - \sqrt{\frac{112}{193}} \frac{N^{1/4} c E[A - 12B_6^2] + E[B_1 - 12B_6B_8]}{\sqrt{E[A - 12B_6^2]}} + \sqrt{\frac{112}{193}} \frac{c^2 E[B_{10} - 12B_9B_8]}{E[A - 12B_6^2]} \]
\[ + \sqrt{\frac{112}{193}} \frac{15 c^2 [E[B_3 - 12B_6^2] + 2E[B_7 - 12B_5B_9]]}{2 \sqrt{E[A - 12B_6^2]}} = N(0, 1) - \frac{15}{56} \sqrt{\frac{112}{193}} c^2. \] (S.5)

The results in (S.4) and (S.5) were also obtained in Moon and Perron (2004) and Moon et al. (2007).

### 2 Bias correction in LLC test

For LLC tests, in addition to the two bias correction methods mentioned in the paper, there is another way to correct the bias for Model 3.2' and Model 3.3'. First, we consider Model 3.2'. The additional way to correct the bias is to only correct the numerator’s bias. However, the test constructed in this way does not have power in the neighborhood of unity with order \( N^{-1/2} T^{-1} \) but with order \( N^{-1/4} T^{-1} \) as shown in Moon and Perron (2008).

We need to modify Assumption 3 to

**Assumption 3'** Let \( \delta_i = 1 - c_i / (N^{1/4} T) \) where \( c_i \geq 0, i = 1, \ldots, N \).

With this type of bias correction, the t-statistic is given by

\[ \tilde{t}_{\delta,23} = \sqrt{2} \frac{\hat{s}_2 + \sqrt{N}(T - 1)/(2(\sum_{i=1}^N \sum_{t=2}^T (y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{s}_{\delta,ii}))}{(\sum_{i=1}^N \sum_{t=2}^T (y_{i,t-1} - \bar{y}_{i,t-1})^2/\hat{s}_{\delta,ii})^{-1/2}}. \]
Under Assumption 3',

\[ \tilde{t}_{\delta,23} = \sqrt{2} \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} c_i [(y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{e,i}^2]}{N^{1/4} T} \sqrt{\sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{e,i}^2]} + \sqrt{2} \frac{N^{-1/2} T^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( (y_{i,t-1} - \bar{y}_{i,t-1}) (\varepsilon_{i,t} - \bar{\varepsilon}_{i,t}) / \hat{\sigma}_{e,i}^2 + \frac{1}{2} \right)}{\sqrt{N^{-1} T^{-2} \sum_{i=1}^{N} \sum_{t=2}^{T} [(y_{i,t-1} - \bar{y}_{i,t-1})^2 / \hat{\sigma}_{e,i}^2]}}. \]

Hence, as \( T \to \infty \)

\[ \tilde{t}_{\delta,23} \rightarrow \sqrt{2} \frac{N^{-3/4} \sum_{i=1}^{N} c_i \int_0^1 \tilde{K}_{i,c_i}(r)^2 dr}{\sqrt{N^{-1} \sum_{i=1}^{N} \int_0^1 \tilde{K}_{i,c_i}(r)^2 dr}} + \sqrt{2} \frac{N^{-1/2} \sum_{i=1}^{N} \left( \int_0^1 \tilde{K}_{i,c_i}(r) dW_i(r) + \frac{1}{2} \right)}{\sqrt{N^{-1} \sum_{i=1}^{N} \int_0^1 \tilde{K}_{i,c_i}(r)^2 dr}} \]

\[ \epsilon \equiv \sqrt{2} \frac{N^{-1/2} \sum_{i=1}^{N} (\tilde{U}_{2i} + 1/2)}{\sqrt{N^{-1} \sum_{i=1}^{N} \tilde{V}_{2i}}}. \quad (S.6) \]

where \( \tilde{K}_{i,c_i}(r) = \tilde{K}_{i,c_i}(r) - \int_0^r \tilde{K}_{i,c_i}(s) ds \), and \( \tilde{K}_{i,c_i}(r) = \int_0^r e^{-c_i N^{-1/4}(r-s)} dW_i(s) \).

Our approach can also be applied here. Substituting \( \theta = iu/2 \), \( x = -v/u \) and \( c = c_i N^{-1/4} \) into \( \varphi_2(\theta; c, 1, x) \) in Lemma 2.2, we have the joint m.g.f. for \( (\tilde{U}_{2i}, \tilde{V}_{2i}) \) as

\[ \tilde{\psi}_2(u,v) = e^{-c_i N^{-1/4} \left[ \frac{u^2 + 2v + c_i^2 N^{-1/2} \sin \left( \frac{\sqrt{2v - c_i^2 N^{-1/2}}}{\sqrt{2v - c_i^2 N^{-1/2}}} \right) - c_i^2 N^{-1/2} \cos \left( \frac{\sqrt{2v - c_i^2 N^{-1/2}}}{\sqrt{2v - c_i^2 N^{-1/2}}} \right)}{2v - c_i^2 N^{-1/2}} \right]} \]

\[ + (2u^2 - 4c_i N^{-1/4}u + 2c_i^2 N^{-1/2}u) \cos \left( \frac{\sqrt{2v - c_i^2 N^{-1/2}}}{(2v - c_i^2 N^{-1/2})^2} \right) \right\}^{-1/2}. \quad (S.7) \]

Hence, the joint m.g.f. for \( (N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{2i}, N^{-1/2} \sum_{i=1}^{N} \tilde{V}_{2i}) \) is

\[ \tilde{\phi}_2(u,v) = \left( \psi_2 \left( \frac{u}{\sqrt{N}}, \frac{v}{N} \right) \right)^N \]

\[ = e^{-\frac{N}{2} \sum_{i=1}^{N} c_i N^{-1/4} \prod_{i=1}^{N} \left( \frac{u^2 + 2v + c_i^2 N^{-1/2} \frac{u}{\sqrt{N}} - c_i^2 N^{-3/4}}{2v - c_i^2 N^{-1/2}} \right) \sin \left( \frac{\sqrt{2v - c_i^2 N^{-1/2}}}{\sqrt{2v - c_i^2 N^{-1/2}}} \right)} \]

\[ - c_i^2 N^{-1/2} \cos \left( \frac{\sqrt{2v - c_i^2 N^{-1/2}}}{(2v - c_i^2 N^{-1/2})^2} \right) \right\}^{-1/2}. \]

Further, we have

\[ \frac{\partial}{\partial u} \tilde{\phi}_2(u, -v) \bigg|_{u=0} = -\sqrt{N} e^{-v/6} - \frac{1}{24} v e^{-v/2} N^{-3/4} \sum_{i=1}^{N} c_i \]

\[ + \frac{v e^{-v/6}}{40N} \sum_{i=1}^{N} c_i^2 + \frac{e^{-v/6}}{24N} \sum_{i=1}^{N} c_i^2 + O(N^{-1/4}), \quad (S.8) \]
and

\[
\tilde{\phi}_2(0, -v) = \left[ e^{-\sum_{i=1}^{N} c_i N^{-1/4} \frac{1}{T} \sum_{i=1}^{N} \log \left( 1 + c_i N^{-1/4} + \frac{1}{N} c_i v N^{-5/4} + \frac{8}{T} N^{-3/2} + O(N^{-7/4}) \right)} \right]^{-1/2}
\]

\[
= e^{-v/6} + \frac{v e^{-v/6}}{12 N^{5/4}} \sum_{i=1}^{N} c_i - \frac{v e^{-v/6}}{20 N^{3/2}} \sum_{i=1}^{N} c_i^2 + O(N^{-3/4}).
\]

From (S.6), applying (14) and (23) and plugging in (S.8) and (S.9), we have

\[
E(\hat{t}_{\delta, 23}) = E \left( \sqrt{2} \frac{N^{-1/2} \sum_{i=1}^{N} \left( \hat{U}_{2i} + \frac{1}{2} \right)}{\sqrt{N^{-1} \sum_{i=1}^{N} V_{2i}}} \right)
\]

\[
= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \left| \frac{1}{\sqrt{v}} \frac{\partial}{\partial u} \tilde{\phi}_2(u, -v) \right|_{u=0} dv + \frac{\sqrt{N}}{\sqrt{2} \Gamma(\frac{1}{2})} \int_{0}^{\infty} \left| \frac{1}{\sqrt{v}} \tilde{\phi}_2(0, -v) \right| dv
\]

\[
= \left( N^{-1} \sum_{i=1}^{N} c_i^2 \right) \frac{\sqrt{2}}{24 \sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-v/6}}{\sqrt{v}} dv + O_p(N^{-1/4}) = \frac{\sqrt{3}}{12} c^2 + O_p(N^{-1/4}).
\]

Next, we consider Model 3.3'. Similarly, we can construct the test statistic by only correcting the numerator's bias. However, the test constructed in this way does not have power in the neighborhood of unity with order \(N^{-1/4}T^{-1}\) but in the neighborhood of unity with order \(N^{-1/8}T^{-1}\).

We need to modify Assumption 3 to

**Assumption 3''** Let \(\delta_i = 1 - c_i/(N^{1/8}T)\) where \(c_i \geq 0, i = 1, \ldots, N\).

With this type of bias correction, the \(t\)-statistic is given by

\[
\tilde{t}_{\delta, 33} = 2 \frac{\tilde{\delta}_3 + \sqrt{NT}/(2 \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{i,t-1}) - \frac{\sum_{i=1}^{N} (s-\bar{s}) (y_{i,t-1} - \bar{y}_{i,t-1})}{\sum_{i=1}^{N} (s-\bar{s})^2} (t - \bar{t})^2 / \hat{\sigma}_i^2)}{\sqrt{\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{i,t-1} - \bar{y}_{i,t-1}) - \frac{\sum_{i=1}^{N} (s-\bar{s}) (y_{i,t-1} - \bar{y}_{i,t-1})}{\sum_{i=1}^{N} (s-\bar{s})^2} (t - \bar{t})^2 / \hat{\sigma}_i^2}} - 1/2
\]

Under Assumption 3'', we have

\[
\tilde{t}_{\delta, 33} = -2 \frac{N^{-1/8} T \sum_{i=1}^{N} c_i \left[ \frac{\sum_{i=1}^{N} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{N} (t - \bar{t})^2} \right] / \hat{\sigma}_i^2}{\sqrt{\sum_{i=1}^{N} \left[ \frac{\sum_{i=1}^{N} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{N} (t - \bar{t})^2} \right] / \hat{\sigma}_i^2}} + 2 \left( N^{-1} \sum_{i=1}^{N} T^{-2} \left[ \frac{\sum_{i=1}^{N} (y_{i,t-1} - \bar{y}_{i,t-1})^2}{\sum_{i=1}^{N} (t - \bar{t})^2} \right] / \hat{\sigma}_i^2 \right)^{-1/2}
\]

64
\[
\times \left( N^{-1/2} \sum_{i=1}^{N} \left[ T^{-1} \left( \sum_{t}(y_{i,t-1} - \bar{y}_{i,t-1})(\varepsilon_{it} - \bar{\varepsilon}_{it}) \right) - \left( \sum_{t}(t - \bar{t})^2 \right)^{-1} \right] \times \left( \sum_{t}(t - \bar{t})(y_{i,t-1} - \bar{y}_{i,t-1}) \left( \sum_{t}(t - \bar{t})(\varepsilon_{it} - \bar{\varepsilon}_{it}) \right) / \sigma^2_{x,t} \right) \frac{1}{2} \right) .
\]

Furthermore, as \( T \to \infty \)
\[
\tilde{t}_{33} \Rightarrow -2 \frac{N^{-5/8} \sum_{i=1}^{N} c_i \left[ f_0^1 \tilde{K}^{\mu}_{i,c_i}(r)^2 dr - 12 \left( f_0^1 (r - \frac{1}{2}) \tilde{K}^{\mu}_{i,c_i}(r) dr \right)^2 \right]}{\sqrt{N^{-1} \sum_{i=1}^{N} \left[ f_0^1 \tilde{K}^{\mu}_{i,c_i}(r)^2 dr - 12 \left( f_0^1 (r - \frac{1}{2}) \tilde{K}^{\mu}_{i,c_i}(r) dr \right)^2 \right]} + 2 \frac{N^{-1/2} \sum_{i=1}^{N} \left( f_0^1 \tilde{K}^{\mu}_{i,c_i}(r)^2 dr - 12 \left( f_0^1 (r - \frac{1}{2}) \tilde{K}^{\mu}_{i,c_i}(r) dr \right)^2 \right]}{\sqrt{N^{-1} \sum_{i=1}^{N} \left[ f_0^1 \tilde{K}^{\mu}_{i,c_i}(r)^2 dr - 12 \left( f_0^1 (r - \frac{1}{2}) \tilde{K}^{\mu}_{i,c_i}(r) dr \right)^2 \right]}} \quad \text{(S.10)}
\]

where \( \tilde{K}^{\mu}_{i,c_i}(r) = \tilde{K}_{i,c_i}(r) - \int_{0}^{L} \tilde{K}_{i,c_i}(s) ds \), and \( \tilde{K}_{i,c_i}(r) = \int_{0}^{r} e^{-c_i N^{-1/8} (r - s)} dW_i(s) \).

Our approach can also be applied here. Substituting \( \theta = iu/2 \), \( x = -v/u \) and \( c = c_i N^{-1/8} \) into \( \varphi_4(\theta; c, 1, x) \) in Lemma 2.2, we have the joint m.g.f. for \((\tilde{U}_{3i}, \tilde{V}_{3i})\) as
\[
\psi(u, v) = e^{-\frac{u}{2}} \left( -c_i N^{-1/8} \right)^4 \left[ \frac{(c_i N^{-1/8})^4 - (c_i N^{-1/8})^4 u - 4(c_i N^{-1/8})^2 + 3c_i N^{-1/8} + 27u} (2v - c_i N^{-1/4})^2 \right] \times \frac{\sin \left( \sqrt{2v - c_i N^{-1/4}} \right) + 24 \left( (c_i N^{-1/8})^4 u + 8uv - 4(c_i N^{-1/8} + 1)(u^2 - 3u^2) \right) \left( \frac{\sin \left( \sqrt{2v - c_i N^{-1/4}} \right)} (2v - c_i N^{-1/4})^3 \right)} \times \frac{1}{2v - c_i N^{-1/4}} \frac{(c_i N^{-1/2})^4 - 8(c_i N^{-1/4} u - c_i N^{-1/8} + 3u)(u^2 + 3u^2) \left( \frac{\sin \left( \sqrt{2v - c_i N^{-1/4}} \right)} (2v - c_i N^{-1/4})^3 \right)^2}{(2v - c_i N^{-1/4})^3} . \quad \text{(S.11)}
\]

Hence, the joint m.g.f. for \( (N^{-1/2} \sum_{i=1}^{N} \tilde{U}_{3i}, N^{-1} \sum_{i=1}^{N} \tilde{V}_{3i}) \) is
\[
\tilde{\psi}_3(u, v) = \left( \psi_3 \left( \frac{\tilde{U}_{3i}}{\sqrt{N}}, \frac{\tilde{V}_{3i}}{\sqrt{N}} \right) \right)^N = e^{-\frac{u}{2}} \left( -c_i N^{-1/8} \right)^4 \left[ \frac{(c_i N^{-1/8})^4 - (c_i N^{-1/8})^4 u - 4(c_i N^{-1/8})^2 + 3c_i N^{-1/8} + 27u} (2v - c_i N^{-1/4})^2 \right] \times \frac{\sin \left( \sqrt{2v - c_i N^{-1/4}} \right) + 24 \left( (c_i N^{-1/8})^4 u + 8uv - 4(c_i N^{-1/8} + 1)(u^2 - 3u^2) \right) \left( \frac{\sin \left( \sqrt{2v - c_i N^{-1/4}} \right)} (2v - c_i N^{-1/4})^3 \right)} \times \frac{1}{2v - c_i N^{-1/4}} \frac{(c_i N^{-1/2})^4 - 8(c_i N^{-1/4} u - c_i N^{-1/8} + 3u)(u^2 + 3u^2) \left( \frac{\sin \left( \sqrt{2v - c_i N^{-1/4}} \right)} (2v - c_i N^{-1/4})^3 \right)^2}{(2v - c_i N^{-1/4})^3} . \quad \text{(S.11)}
\]
Furthermore, we have

\[
\frac{\partial \tilde{\phi}_3(u, -v)}{\partial u} \bigg|_{u=0} = -\sqrt{\frac{N}{2}}e^{-v/15} - \frac{1}{840}ve^{-v/15}N^{-3/4}\sum_{i=1}^{N}c_i^2 \\
+ \frac{17}{60480}ve^{-v/15}N^{-1}\sum_{i=1}^{N}c_i^4 + \frac{1}{1440}e^{-v/15}N^{-1}\sum_{i=1}^{N}c_i^4 + O_p(N^{-1/8}) \tag{S.12}
\]

and

\[
\tilde{\phi}_3(0, -v) = \left[ e^{-\sum_{i=1}^{N}c_iN^{-\frac{1}{6}}} \exp \left( \sum_{i=1}^{N} \log \left( 1 + c_iN^{-1/8} + \frac{1}{2}c_i^2N^{-1/4} + \frac{1}{6}c_i^3N^{-3/8} + N^{-1/2}c_i^4 \frac{1}{24} + N^{-5/8}c_i^5 \frac{1}{120} \right) \\
+ N^{-3/4}c_i^6 \frac{1}{720} + N^{-7/8}c_i^7 \frac{7}{7!} + \frac{2v}{15N} + N^{-9/8}c_i^8 \frac{8!}{15} + N^{-9/8}\frac{2c_i}{v} \frac{9!}{15} + N^{-5/4}\frac{13c_i^2}{210} + N^{-5/4}c_i^4 \frac{10!}{10!} \\
+ N^{-11/8}\frac{11c_i^3}{630} + N^{-11/8}\frac{c_i^{11}}{11!} + N^{-3/2}\frac{13c_i^4}{3024} + N^{-3/2}\frac{c_i^{12}}{12!} + O(N^{-13/8}) \right) \right]^{-1/2} \\
= e^{-v/15} + \frac{ve^{-v/15}}{420N^{5/4}}\sum_{i=1}^{N}c_i^2 - \frac{17ve^{-v/15}}{30240N^{3/2}}\sum_{i=1}^{N}c_i^4 + O(N^{-5/8}) . \tag{S.13}
\]

From \(S.10\), applying (14) and (23) and plugging in (S.12) and (S.13), we have

\[
E \left( \tilde{t}_{\delta, 33} \right) = E \left( \frac{N^{-1/2}\sum_{i=1}^{N} \left( \tilde{U}_{3i} + \frac{1}{2} \right)}{\sqrt{N^{-1}\sum_{i=1}^{N} \tilde{V}_{3i}}} \right) \\
= 2 \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \frac{\partial \tilde{\phi}_3(u, -v)}{\partial u} \bigg|_{u=0} \ dv + \frac{\sqrt{N}}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\infty} \frac{1}{\sqrt{v}} \tilde{\phi}_3(0, -v) dv \\
= -\sqrt{N} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v/15} \sqrt{v} dv - \left( N^{-5/6} \sum_{i=1}^{N} \frac{c_i^2}{2} \frac{1}{420\sqrt{\pi}} \int_{0}^{\infty} ve^{-v/15} \sqrt{v} dv \right. \\
+ \left. \left( N^{-1} \sum_{i=1}^{N} \frac{c_i^4}{8} \right) \frac{17}{30240\sqrt{\pi}} \int_{0}^{\infty} ve^{-v/15} \sqrt{v} dv \right) \\
+ \sqrt{N} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v/15} \sqrt{v} dv + \left( N^{-5/6} \sum_{i=1}^{N} \frac{c_i^2}{2} \frac{1}{420\sqrt{\pi}} \int_{0}^{\infty} ve^{-v/15} \sqrt{v} dv \right. \\
+ \left. \left( N^{-1} \sum_{i=1}^{N} \frac{c_i^4}{8} \right) \frac{17}{30240\sqrt{\pi}} \int_{0}^{\infty} ve^{-v/15} \sqrt{v} dv \right) \\
- \left( N^{-1} \sum_{i=1}^{N} \frac{c_i^4}{8} \right) \frac{17}{30240\sqrt{\pi}} \int_{0}^{\infty} ve^{-v/15} \sqrt{v} dv + O_p(N^{-1/8}) \\
= \frac{\sqrt{15}}{720}e^{-v} + O_p(N^{-1/8}) ,
\]
where \(e^4 = \lim_{N \to \infty} N^{1/4} \sum_{i=1}^{N} c_i^4\). 

66
The results are summarized in the following proposition.

**Proposition S.1** Under Assumptions 1, 2 and 4, when $T \to \infty$ followed by $N \to \infty$, we have the following asymptotic results.

(a) For Model 3.2', with the additional Assumption 3', $\hat{t}_{\delta,23} \Rightarrow N(0,1) + \frac{\sqrt{7}}{12}c^2$;

(b) For Model 3.3', with the additional Assumption 3'', $\hat{t}_{\delta,33} \Rightarrow N(0,1) + \frac{\sqrt{15}}{720}c^3$.

**Remark S.1** From Assumptions 3' and 3'', we can see that $\hat{t}_{\delta,23}$ and $\hat{t}_{\delta,33}$ have local power in the neighborhood of unity with a slower rate, compared with those of $\hat{t}_{\delta,21}$, $\hat{t}_{\delta,22}$, $\hat{t}_{\delta,31}$, and $\hat{t}_{\delta,32}$, respectively. In addition, $\hat{t}_{\delta,23}$ and $\hat{t}_{\delta,33}$ have the local power on the right tail rather than the left tail.