New formulation of the finite depth free surface Green function

Zhi-Min Chen

Received: 15 October 2019 / Accepted: 1 June 2020 / Published online: 12 August 2020
© Springer Nature B.V. 2020

Abstract For a pulsating free surface source in a three-dimensional finite depth fluid domain, the Green function of the source presented by John [Communs. Pure Appl. Math. 3:45–101, 1950] is superposed as the Rankine source potential, an image source potential and a wave integral in the infinite domain \((0, \infty)\). When the source point together with a field point is on the free surface, John’s integral and its gradient are not convergent since the integration \(\int_{0}^{\infty}\) of the corresponding integrands does not tend to zero in a uniform manner as \(\kappa\) tends to \(\infty\). Thus evaluation of the Green function is not based on direct integration of the wave integral but is obtained by approximation expansions in earlier investigations. In the present study, five images of the source with respect to the free surface mirror and the water bed mirror in relation to the image method are employed to reformulate the wave integral. Therefore the free surface Green function of the source is decomposed into the Rankine potential, the five image source potentials and a new wave integral, of which the integrand is approximated by a smooth and rapidly decaying function. The gradient of the Green function is further formulated so that the same integration stability with the wave integral is demonstrated. The significance of the present research is that the improved wave integration of the Green function and its gradient becomes convergent. Therefore evaluation of the Green function is obtained through the integration of the integrand in a straightforward manner. The application of the scheme to a floating body or a submerged body motion in regular waves shows that the approximation is sufficiently accurate to compute linear wave loads in practice.

Keywords Added mass and damping coefficients · Evaluation of free surface Green function · Hess–Smith method · Potential flow · Radiation waves

1 Introduction

The understanding of wave-induced forces resulting from a wave-body motion is fundamental in hydrodynamics. In the linear potential flow theory, the velocity potential of the fluid motion problem is a harmonic function and can be represented as a solution of body boundary integral equation involving free surface Green function. The integral equation can be solved numerically by combining a boundary element method for the numerical integration of the free surface Green function or free surface sources distributed on wetted body surface (see, for example, Frank [1],

Z. -M. Chen (✉)
School of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China
e-mail: zmchen@szu.edu.cn
Lee and Sclavounos [2], Lee and Newman [3] for the infinite water depth). Thus it is fundamental for the evaluation of the free surface Green function (see, for example, the successful investigations given by Chakrabarti [4], Liang et al. [5], Newman [6], Noblesse [7], Ponizy et al. [8], Telste and Noblesse [9], Wu et al. [10] for the infinite depth case and John [11], Linton [12], Liu et al. [13], Newman [6], Pidcock [14] for the finite depth case).

For a radial symmetric body undergoing an oscillatory motion in a fluid of infinite water depth, its linear analytic solution can be approximated by a single free surface source rather than the boundary integral of free surface sources continuously distributed on the body surface. For a heaving or surging hemisphere, the velocity potential solution is decomposed into a free surface source located at the centre of the sphere and a wave-free potential, which is expanded in a series of Legendre polynomials and sinusoidal functions (see, for example, Havelock [15], Hulme [16] and Ursell [17]). The unknown source strength and expansion coefficients are determined by the boundary condition of the velocity potential on the hemisphere. This method also applies to the wave resistance problem (see Farell [18]) of a travelling spheroid and an oscillatory submerged sphere in a fluid (see Chatjigeorgiou [19], Liu et al. [20] of infinite water depth and Linton [22] of finite water depth). Satisfactory numerical solutions can also obtained from varieties of Rankine simple source methods on the wave-body motion for a pulsatating submerged sphere (see, for example, Cao et al. [23], Dawson [24], Feng et al. [25,26], Mantzaris [27], Yeung [28]) by using the dynamic and kinematic free surface boundary conditions rather than the Green function theory.

In the study of the pulsating free surface Green function, the author [29] showed the singular wave integral of the singular wave integral of the velocity potential on the hemisphere. This method also applies to the wave resistance problem (see Farell [18]) of a travelling spheroid and an oscillatory submerged sphere in a fluid (see Chatjigeorgiou [19], Liu et al. [20] of infinite water depth and Linton [22] of finite water depth). Satisfactory numerical solutions can also obtained from varieties of Rankine simple source methods on the wave-body motion for a pulsatating submerged sphere (see, for example, Cao et al. [23], Dawson [24], Feng et al. [25,26], Mantzaris [27], Yeung [28]) by using the dynamic and kinematic free surface boundary conditions rather than the Green function theory.

Consider a pulsating free surface source \( p = (\xi, \eta, \zeta) \), in a finite depth fluid domain \(-h < z < 0\), undergoing an oscillatory motion of a frequency \( \omega \). The velocity potential of the linear wave motion in the frequency domain with respect to a field point \( q = (x, y, z) \) is expressed as

\[
G = \frac{1}{4\pi} \text{Re}(Ge^{-i\omega t}), \quad i = \sqrt{-1},
\]

where \( G \) is a complex function expressed as

\[
G = \frac{1}{r} + \frac{1}{r_0} + G_1
\]

with \( G_1 \) a harmonic function in the fluid domain and

\[
r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \quad \text{and} \quad r_0 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z + \zeta + 2h)^2}.
\]

The function \( G \) is known as a finite depth free surface Green function or the fundamental solution of the Laplacian equation in the fluid domain associated with the free surface boundary condition and water bed boundary condition. As the Rankine source potential \( 1/r \) is the fundamental solution of the Laplacian equation in the whole three-dimensional domain, the function \( G \) is determined by the following the boundary value problem:

\[
\begin{align*}
\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial z^2} &= 0, \quad -h < z < 0, \\
\frac{\partial G}{\partial z} - vG \bigg|_{z=0} &= 0, \\
\frac{\partial G}{\partial z} \bigg|_{z=-h} &= 0, \\
\lim_{k \to \infty} \sqrt{R} \left( \frac{\partial G}{\partial R} - ivG \right) &= 0
\end{align*}
\]

for \( R = |(x, y) - (\xi, \eta)| \) the horizontal distance, \( g \) the gravitational acceleration and \( v \) the dimensional wave number \( \omega^2/g \). The Green function was obtained by John [11, Eq. (A9)] as

\[
G = \frac{1}{r} + \frac{1}{r_0} + \int_{L} \frac{2(v+k)e^{-kh}cosh k(\xi + h)cosh k(z + h)}{k sinh kh - v cosh kh} J_0(kR)dk
\]
for \( J_0 \) the Bessel function of the first kind. The wave integral pass \( L \) is illustrated in Fig. 1.

The harmonic function \( G_1 \) is an irregular wave integral due to the occurrence of the pole \( k_0 > 0 \), the positive root of the dispersion relation

\[
k \tanh k h = v. \tag{6}
\]

A well-known evaluation expansion of the Green function is given by John [11, pp. 93–95] as

\[
G = 2\pi \frac{v^2 - k_0^2}{h k_0^2 - h v^2 + v} \cosh k_0(z + h) \cosh k_0(\zeta + h)(Y_0(k_0 R) - i J_0(k_0 R)) + 4 \sum_{n=1}^{\infty} \frac{k_n^2 + v^2}{h k_n^2 + h v^2 - v} \cos k_n(z + h) \cos k_n(\zeta + h) K_0(k_n R), \tag{7}
\]

\[
\approx 2\pi \frac{v^2 - k_0^2}{h k_0^2 - h v^2 + v} \cosh k_0(z + h) \cosh k_0(\zeta + h)(Y_0(k_0 R) - i J_0(k_0 R)) + 4 \sum_{n=1}^{N_f} \frac{k_n^2 + v^2}{h k_n^2 + h v^2 - v} \cos k_n(z + h) \cos k_n(\zeta + h) K_0(k_n R) \tag{8}
\]

for \( N_f \) a suitable positive integer, \( Y_0 \) the Bessel function of the second kind, \( K_0 \) the modified Bessel function of the second kind and \( i k_n \) the roots of the dispersion relation (6) satisfying

\[
\pi \left( n - \frac{1}{2} \right) \leq k_n h \leq \pi n, \quad n \geq 1.
\]

For further understanding of this expansion, one may refer to Wehausen and Laitone [30, Eq. (13.19)] and Newman [6]. Equation (7) exhibits a simple form of the Green function evaluation, which however is not a harmonic function due to the inclusion of the Rankine source singular potential \( 1/r \). Thus it is numerically inefficient within finite boundary element discretization in wave-body motions. What is more, as noticed by Newman [6, p. 64], the series (7) is practically useless for small values of \( R/h \), since each summand contains a logarithmic singularity when \( R/h = 0 \). Successful developments on John’s evaluation series have been obtained by many authors (see, for example, Newman [6], Liu et al. [13], Linton [12], Pidcock [14]).

The purpose of the present study is to approximate the three-dimensional finite depth free surface Green function through direct integration together with its application to wave-body motions.

The integrand of the wave integral (5) has the asymptotic behaviour [31] (as \( k \to \infty \)):

\[
\frac{2(v + k)e^{-kh} \cosh k(\zeta + h) \cosh k(z + h)}{k \sinh kh - v \cosh kh} J_0(kR) \sim e^{k(z+\zeta)} \frac{2\sqrt{2} \cos (kR - \frac{\pi}{4})}{\sqrt{\pi kR}}. \tag{9}
\]

When both the field point \( q \) and the source point \( p \) tend to the free surface \( z = \zeta = 0 \), the wave integral (5) is not stable but oscillates with unbounded amplitude.

What is more, the wave integral stability for the gradient of the Green function is even worse. For example, the integrand of the horizontal derivative

\[
\partial_R G = \partial_R \left( \frac{1}{r} + \frac{1}{r_0} \right) - \int_{L} \frac{2(v + k)e^{-kh} \cosh k(\zeta + h) \cosh k(z + h)}{k \sinh kh - v \cosh kh} k J_1(kR) dk \tag{10}
\]
has the asymptotic behaviour (as \( k \to \infty \))

\[
\frac{2(v + k)e^{-kh} \cosh k(\zeta + h) \cosh k(z + h)}{k \sinh kh - v \cosh kh} kJ_1(kR) \sim \frac{2\sqrt{2k} \cos \left( kR - \frac{3\pi}{4} \right)}{\sqrt{\pi R}}
\]

(11)
on the free surface \( z = \zeta = 0 \).

Thus it is beneficial to provide a stable formulation of the Green function. To do so, we introduce a new formulation of the Green function expressed as

\[
G = \frac{1}{r} + \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + K
\]

(12)
with the wave integral as the limit of smooth function integrals

\[
K = \lim\limits_{\mu \to 0} \int_0^\infty \frac{[2v + (k + v)e^{-2kh}]\left[ e^{k(z + \zeta)} + e^{k(z - \zeta - 2h)} + e^{k(z + \zeta - 2h)} + e^{-k(z + \zeta + 4h)} \right] J_0(kR) dk}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)}
\]

(13)
and

\[
r_1 = \sqrt{R^2 + (z + \zeta)^2}, \quad r_2 = \sqrt{R^2 + (z - \zeta + 2h)^2},
\]

\[
r_3 = \sqrt{R^2 + (\zeta - z + 2h)^2}, \quad r_4 = \sqrt{R^2 + (z + \zeta + 4h)^2}.
\]

Similar, new formulation for the gradient Green function \( \nabla G \), which has the same wave integral stability with the new formulation of \( G \), is also obtained. For example, we have the horizontal derivative formulation

\[
\partial_R G = \partial_R \left( \frac{1}{r} + \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) - \lim\limits_{\mu \to 0} \int_0^\infty \frac{(2v^2 + (k + \nu)(k + \nu)e^{-2kh})\left[ e^{k(z + \zeta)} + e^{k(z - \zeta - 2h)} + e^{k(z + \zeta - 2h)} + e^{-k(z + \zeta + 4h)} \right] J_0(kR) dk}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)}
\]

(14)

\[
- \left( \frac{2vR}{r_1(r_1 + |z + \zeta|)} + \frac{2vR}{r_2(r_2 + |z - \zeta - 2h|)} + \frac{2vR}{r_3(r_3 + |\zeta - z - 2h|)} + \frac{2vR}{r_4(r_4 + |z + \zeta + 4h|)} \right).
\]

The parameter \( \mu \) is known as the Rayleigh artificial viscosity, which was used by Havelock [32,33] by employing the limit \( \mu \to 0 \) to show the uniqueness of the infinite water depth Green function of a free surface source advancing at a uniform speed. In the present study, however, we use the regular wave integral with \( \mu > 0 \) to cancel the singularity around the pole \( k = k_0 \) in the direct integration scheme rather than take the limit \( \mu \to 0 \) to describe the troublesome of the singularity in earlier investigations. For free surface wave damping under the effect of viscosity, one may refer to Lazauskas [34] and Spivak et al. [35].

The advantage of (13)–(14) is twofold. Firstly, the poor asymptotic behaviours (9) and (11) for the original integrands are improved, respectively, as

\[
\frac{[2v + (k + \nu)e^{-2kh}]\left[ e^{k(z + \zeta)} + e^{k(z - \zeta - 2h)} + e^{k(z + \zeta - 2h)} + e^{-k(z + \zeta + 4h)} \right] J_0(kR)}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)} \sim \frac{2\sqrt{2} \cos \left( kR - \frac{\pi}{4} \right)}{k \sqrt{\pi k R}}
\]

(15)
and

\[
\frac{(2v^2 + (k + \nu)(k + \nu)e^{-2kh})\left[ e^{k(z + \zeta)} + e^{k(z - \zeta - 2h)} + e^{k(z + \zeta - 2h)} + e^{-k(z + \zeta + 4h)} \right] J_1(kR)}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)} \sim \frac{2\sqrt{2} \cos \left( kR - \frac{3\pi}{4} \right)}{k \sqrt{\pi k R}}
\]

(16)
on the free surface \( z = 0 \) and \( \zeta = 0 \), so that the wave integrations in (13) and (14) are convergent for \( k \to \infty \). Secondary, the formalization implies that (13) and (14) with the limit \( \mu \to 0 \) can be replaced by those with a value \( 0 < \mu \ll 1 \) so that the wave integrals are on the straight line \( 0 \leq k < \infty \) rather than on the curve shown in Fig. 1.

Thus evaluation of the Green function can be obtained by directly integrating a smooth function involving a small value of \( \mu > 0 \). The numerical result is accurate in comparison with that given by John’s series (7). The direct integration evaluation is efficient in application to wave-body motions from very good agreement of the present method results with the semi-analytic results of Wang [20], Hulme [16] and Linton [12].

As given in [30], the original formulas (5) or (10) on the curved integration line can be written as a Cauchy-principle-value (CPV) integral plus \( \pi i \) times the residue of the integrand at the pole \( k = k_0 \). A CPV integral part close to the pole \( k_0 \) may be obtained by the method of Monacella [36] using the property

\[
\text{(CPV)} \quad \int_{k_0-a}^{k_0+a} \frac{dk}{k-k_0} = 0, \tag{17}
\]

although the function \( 1/(k \tanh(kh) - v) \) is even with respect to \( k \). However, the integration close to the infinity is not convergent on the free surface due to (9) and (11). Therefore, (5) or (10) cannot be integrated directly in the rigorous analysis manner. Thus it is necessary to use other approximations (see, for example, [6,11]).

2 New formulation of the free surface Green function \( G \)

It is convenient to use the inverse Hankel transformation or the inverse Fourier transformation in the polar coordinate system of the horizontal plane (see, for example, [37, p. 384])

\[
\frac{1}{\sqrt{R^2 + Z^2}} = \int_0^\infty \frac{1}{k} e^{kz} J_0(kR)kdk = \mathcal{H}^{-1} \left( \frac{1}{k} e^{kz}, \ z < 0 \right). \tag{18}
\]

It follows from (18) that, for \( z + \zeta < 0 \),

\[
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \int_0^\infty \left( e^{k(z+\zeta)} + e^{k(z-\zeta-2h)} + e^{k(-z+\zeta-2h)} + e^{-k(z+\zeta+4h)} \right) J_0(kR)dk
\]

\[
= \int_0^\infty 4e^{-2kh} \cosh k(\zeta + h) \cosh k(z + h) J_0(kR)dk.
\]

This yields that

\[
\int_L \frac{2(v + k)e^{-kh} \cosh k(\zeta + h) \cosh k(z + h)}{k \sinh kh - v \cosh kh} J_0(kR)dk - \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right)
\]

\[
= \int_L \left( \frac{(v + k)}{(1 + e^{-2kh})(k \tanh kh - v)} - 1 \right) 4e^{-2kh} \cosh k(\zeta + h) \cosh k(z + h) J_0(kR)dk
\]

\[
= \int_L \left[ 2v + (k + v)e^{-2kh} \right] \left( e^{k(z+\zeta)} + e^{k(z-\zeta-2h)} + e^{k(-z+\zeta-2h)} + e^{-k(z+\zeta+4h)} \right]
\]

\[
(1 + e^{-2kh})(k \tanh kh - v)
\]

\[
J_0(kR)dk.
\]

Note that exponential functions and the Bessel function are continuous along the positive real line and thus independent of the integral pass change in Fig. 1. Hence it remains to check the analytical behaviour of the following singular integral on the lower half circle around \( k = k_0 \):

\[
\int_{|k-k_0|=\epsilon, \ \text{Im}(k) \leq 0} \frac{dk}{k \tanh kh - v} = \frac{1}{\tanh k_0h} \int_{|k-k_0|=\epsilon, \ \text{Im}(k) \leq 0} \frac{dk}{k - k_0} = \frac{i\pi}{\tanh k_0h} \tag{19}
\]

for a given constant \( \epsilon > 0 \) sufficiently small. In contrast, we have the integral
The function $f(k) = \left(1/(k \tanh k h - v - i\mu)\right)$ for $h = 1, v = 1, k_0 = 1.19965$ and $\mu = 0.0005$; c and d profile of the smooth function $f$ around $k = k_0$.

This together with (19) implies that, for any small $\epsilon > 0$,

$$
\int_{k_0-\epsilon}^{k_0+\epsilon} \frac{dk}{k \tanh k h - v - i\mu} \approx \frac{1}{\tanh k_0 h} \int_{k_0-\epsilon}^{k_0+\epsilon} \frac{dk}{k - k_0 - \frac{i\mu}{\tanh k_0 h}} = \frac{1}{\tanh k_0 h} \ln \frac{\epsilon \tanh k_0 h - i\mu}{-\epsilon \tanh k_0 h - i\mu}
$$

$$
= \frac{i\pi}{\tanh k_0 h} + \frac{1}{\tanh k_0 h} \ln \left(1 + \frac{-2i\mu}{\epsilon \tanh k_0 h + i\mu}\right).
$$

(20)

This together with (19) implies that, for any small $\epsilon > 0$,

$$
\int_{|k-k_0|<\epsilon, \text{Im}(k)\leq0} \frac{dk}{k \tanh k h - v} \equiv \lim_{\mu \to 0^+} \int_{k_0-\epsilon}^{k_0+\epsilon} \frac{dk}{k \tanh k h - v - i\mu}
$$

and thus we have the desired formulation (13).

The behaviour of the function $1/(k \tanh k h - v - i\mu)$ is displayed in Fig. 2. Although this function is becoming sharper and unbounded as $\mu \to 0$, its real part close to $k_0$ is symmetric with respect to the point $k = k_0$, while the imaginary part tends to the dirac delta function at the vicinity of the root $k = k_0$. Thus the integral of the function around $k_0$ or the area bounded by the function and the real line as given by (20) is always meaningful. Therefore, the integral of the real part in the vicinity of $k_0$ is zero and the integral of the imaginary part around $k_0$ remains constant for $\mu$ sufficiently small. However, as shown in Fig. 2, the sharpness of the function increases when the parameter $\mu > 0$ decreases. Thus denser meshgrid points around $k_0$ are necessary for smaller $\mu > 0$. 

Springer
Moreover, the dispersion relation (6) becomes the deep water dispersion relation \( k - \nu = 0 \) when \( h\nu > 7 \), since \( \tanh(k_0h) = 1 \) numerically for \( k_0h > 7 \). Thus compared with the infinite water depth case, the main difference of the finite water depth case is defined by the integral on the integral domain \([0, 7/\nu]\).

Next, we consider the vertical derivative of \( K \). By (18), it shows that

\[
\partial_z K - 2\nu \left( \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right)
= \int_L \left( \frac{[2\nu + (k + \nu)e^{-2kh}]k}{(1 + e^{-2kh})(k \tanh kh - \nu)} - 2\nu \right) \left( e^{k(z+\xi)} - e^{k(z-\xi-2h)} + e^{k(z+\xi-2h)} - e^{-k(z+\xi+4h)} \right) J_0(kR) \, dk
= \int_L \frac{2\nu^2 + (k + 2\nu)(k + \nu)e^{-2kh}}{(1 + e^{-2kh})(k \tanh kh - \nu)} \left( e^{k(z+\xi)} - e^{k(z-\xi-2h)} + e^{k(z+\xi-2h)} - e^{-k(z+\xi+4h)} \right) J_0(kR) \, dk.
\]

Hence, by (13), we have

\[
\partial_z G = -\partial_z \left( \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right) - 2\nu \left( \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right)
= \lim_{\mu \to 0^+} \int_0^\infty \frac{2\nu^2 + (k + 2\nu)(k + \nu)e^{-2kh}}{(1 + e^{-2kh})(k \tanh kh - \nu - i\mu)} \times \left( e^{k(z+\xi)} - e^{k(z-\xi-2h)} + e^{k(z+\xi-2h)} - e^{-k(z+\xi+4h)} \right) J_0(kR) \, dk.
\]

Finally, for the Bessel function \( J_1(s) = -(dJ_0(s)/ds) \), we have

\[
\partial_R K + \int_0^\infty 2\nu \left( e^{k(z+\xi)} + e^{k(z-\xi-2h)} + e^{k(z-\xi-2h)} + e^{-k(z+\xi+4h)} \right) J_1(kR) \, dk
= -\int_L \left( \frac{[2\nu + (k + \nu)e^{-2kh}]k}{(1 + e^{-2kh})(k \tanh kh - \nu)} - 2\nu \right) \left( e^{k(z+\xi)} + e^{k(z-\xi-2h)} + e^{k(z-\xi-2h)} + e^{-k(z+\xi+4h)} \right) J_1(kR) \, dk
= -\int_L \frac{2\nu^2 + (k + 2\nu)(k + \nu)e^{-2kh}}{(1 + e^{-2kh})(k \tanh kh - \nu)} \left( e^{k(z+\xi)} + e^{k(z-\xi-2h)} + e^{k(z+\xi-2h)} + e^{-k(z+\xi+4h)} \right) J_1(kR) \, dk.
\]

With the use of integration by parts and (18), we have

\[
\int_0^\infty 2\nu \left( e^{k(z+\xi)} + e^{k(z-\xi-2h)} + e^{k(z-\xi-2h)} + e^{-k(z+\xi+4h)} \right) J_1(kR) \, dk
= -\frac{2\nu}{R} \int_0^\infty \left( e^{k(z+\xi)} + e^{k(z-\xi-2h)} + e^{k(z-\xi-2h)} + e^{-k(z+\xi+4h)} \right) dJ_0(kR)
= \frac{8\nu}{R} + \frac{2\nu}{R} \int_0^\infty \left( (z + \xi)e^{k(z+\xi)} + (z - \xi - 2h)e^{k(z-\xi-2h)} + (\xi - z - 2h)e^{k(z+\xi-2h)} - e^{k(z+\xi+4h)} \right) J_0(kR) \, dk
= \frac{8\nu}{R} + \frac{2\nu}{R} \left( \frac{z + \xi + z - \xi - 2h}{r_1} + \frac{z - \xi - 2h}{r_2} + \frac{\xi - z - 2h}{r_3} + \frac{-z - \xi - 4h}{r_4} \right)
= \frac{2\nu R}{r_1(r_1 + |z + \xi|)} + \frac{2\nu R}{r_2(r_2 + |z - \xi - 2h|)} + \frac{2\nu R}{r_3(r_3 + |\xi - z - 2h|)} + \frac{2\nu R}{r_4(r_4 + |z + \xi + 4h|)}.
\]
This together with (19) and (22) implies that

\[
\partial_R K + \frac{2v R}{r_1(r_1 + |z + \xi|)} + \frac{2v R}{r_2(r_2 + |z - \xi - 2h|)} + \frac{2v R}{r_3(r_3 + |\xi - z - 2h|)} + \frac{2v R}{r_4(r_4 + |z + \xi + 4h|)}
= - \lim_{\mu \to 0^+} \int_0^{\infty} \frac{2v^2 + (k + 2v)(k + v)e^{-2kh}}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)} \left( e^{k(z + \xi)} + e^{k(z - \xi - 2h)} + e^{k(\xi - z - 2h)} + e^{-k(z + \xi + 4h)} \right) J_1(kR) \, dk.
\]

This gives the formulation for the horizontal derivative of the Green function

\[
\partial_R G - \partial_R \left( \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right)
= - \lim_{\mu \to 0^+} \int_0^{\infty} \frac{2v^2 + (k + 2v)(k + v)e^{-2kh}}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)} \left( e^{k(z + \xi)} + e^{k(z - \xi - 2h)} + e^{k(\xi - z - 2h)} + e^{-k(z + \xi + 4h)} \right) J_1(kR) \, dk
- \left( \frac{2v R}{r_1(r_1 + |z + \xi|)} + \frac{2v R}{r_2(r_2 + |z - \xi - 2h|)} + \frac{2v R}{r_3(r_3 + |\xi - z - 2h|)} + \frac{2v R}{r_4(r_4 + |z + \xi + 4h|)} \right). \tag{23}
\]

When \( h \to \infty \), the formulation reduces to the infinite depth Green function

\[
G = \frac{1}{r} + \frac{1}{r_1} + \lim_{\mu \to 0^+} \int_0^{\infty} \frac{2ve^{k(z + \xi)}J_0(kR) \, dk}{k - v - i\mu}, \tag{24}
\]

\[
\partial_\xi G = \partial_\xi \left( \frac{1}{r} + \frac{1}{r_1} \right) + \frac{2v}{r_1} + \lim_{\mu \to 0^+} \int_0^{\infty} \frac{2v^2e^{k(z + \xi)}J_0(kR) \, dk}{k - v - i\mu}, \tag{25}
\]

\[
\partial_R G = \partial_R \left( \frac{1}{r} + \frac{1}{r_1} \right) - \frac{2v R}{r_1(r_1 + |z + \xi|)} - \lim_{\mu \to 0^+} \int_0^{\infty} \frac{2v^2e^{k(z + \xi)}J_1(kR) \, dk}{k - v - i\mu} \tag{26}
\]

3 Direct integration of the Green function

Let \( K^\mu \) denote the wave integral of (13) involving \( \mu > 0 \). By (13), (21) and (23), we have

\[
K = \lim_{\mu \to 0^+} K^\mu, \quad \partial_\xi K = \lim_{\mu \to 0^+} \partial_\xi K^\mu, \quad \partial_R K = \lim_{\mu \to 0^+} \partial_R K^\mu.
\]

Therefore we may numerically take

\[
K \approx K^\mu, \quad \partial_\xi K \approx \partial_\xi K^\mu, \quad \partial_R K \approx \partial_R K^\mu
\]

for \( \mu > 0 \) sufficiently small.

Note that the wave integral \( K^\mu \) and those of \( \partial_\xi K^\mu \) and \( \partial_R K^\mu \) are convergent even for the limit case \( z = \xi = 0 \) or the integration \( \int_{k_{\text{max}}}^{\infty} \) of the corresponding integrands is uniformly small for a large \( k_{\text{max}} \). The infinite integration domain \( 0 < k < \infty \) is thus truncated by finite integration domain \( 0 < k < k_{\text{max}} \) for a suitable number \( k_{\text{max}} \). For simplicity, we may use the direct integration in the approximation manner

\[
K \approx \sum_{j=0}^{N_k} \int_k^{k_{j+1}} \frac{2v + (k + v)e^{-2k_jh}}{(1 + e^{-2kh})(k \tanh kh - v - i\mu)} \left[ e^{k_j(z + \xi)} + e^{k_j(z - \xi - 2h)} + e^{k_j(\xi - z - 2h)} + e^{-k_j(z + \xi + 4h)} \right] J_0(kR) \, dk
+ \sum_{j=1}^{N_k} \frac{1}{\tanh k_{j+1}h} \ln \frac{k_{j+1} \tanh k_{j+1}h - v - i\mu}{k_j \tanh k_jh - v - i\mu} \tag{27}
\]

\[\square\] Springer
New formulation of the finite depth free surface Green function

Here \( (k_j)_{j=1}^{N_k} \) represents a set of meshgrid points of the truncation domain \( 0 < k < k_{\text{max}} \) and is sufficiently dense so that the numerator of integrand together with the exponential function \( e^{-2kh} \) and the hyperbolic function \( \tanh(kh) \) are approximately constant on \([k_j, k_{j+1}]\).

Similarly, we have the following evaluation:

\[
\partial_\xi K = -2\nu \left( \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right) 
\approx \sum_{j=1}^{N_k} \int_{k_j}^{k_{j+1}} 2\nu^2 + (k + 2\nu)(k + \nu)e^{-2kh} \left( \frac{e^{k(z+\xi)} - e^{k(z-\xi-2h)} + e^{k(z-\xi-2h)} - e^{k(z+\xi+4h)}}{1 + e^{-2kh}} \right) J_0(kR) dk 
\approx \sum_{j=1}^{N_k} \left[ 2\nu^2 + (k_j + 2\nu)(k_j + \nu)e^{-2k_j h} \left( \frac{e^{k_j(z+\xi)} + e^{k_j(z-\xi-2h)} + e^{k_j(z-\xi-2h)} - e^{-k_j(z+\xi+4h)}}{1 + e^{-2k_j h}} \right) \right] 
\times \frac{1}{\tanh k_{j+1} h} \ln \frac{k_{j+1} h - v - i\mu}{k_j h - v - i\mu} J_0(k_j R) \tag{28}
\]

and

\[
\partial_R K + \left( \frac{2\nu R}{r_1(r_1 + |z + \xi|)} + \frac{2\nu R}{r_2(r_2 + |z - \xi - 2h|)} + \frac{2\nu R}{r_3(r_3 + |z - \xi - 2h|)} + \frac{2\nu R}{r_4(r_4 + |z + \xi + 4h|)} \right) 
\approx -\sum_{j=1}^{N_k} \left[ 2\nu^2 + (k_j + 2\nu)(k_j + \nu)e^{-2k_j h} \left( \frac{e^{k_j(z+\xi)} + e^{k_j(z-\xi-2h)} + e^{k_j(z-\xi-2h)} - e^{-k_j(z+\xi+4h)}}{1 + e^{-2k_j h}} \right) \right] 
\times \frac{1}{\tanh k_{j+1} h} \ln \frac{k_{j+1} h - v - i\mu}{k_j h - v - i\mu} J_1(k_j R) \tag{29}
\]

To improve the accuracy of the evaluation formulas (27)–(29), we may use \( \tanh(k_0 h) \) instead of \( \tanh(k_{j+1} h) \) whenever \( k_0 \in [k_j, k_{j+1}] \).

With the use of (29), we have the evaluation for the horizontal partial derivatives

\[
\partial_\xi K = \partial_R K \frac{\xi - x}{R} \quad \text{and} \quad \partial_\eta K = \partial_R K \frac{\eta - y}{R}. \tag{30}
\]

Here we do not use integral approximation methods such as trapezoidal rule and Simpson rules, which are not designed for dealing with the integration of a sharp function close to the dirac delta function. In order to cancel the huge positive integration area with the huge negative integration area displayed in Fig. 2a, c and to compute the constant integration area bounded by the dirac delta like function in Fig. 2b, d in a tiny integral interval \([k_0 - \epsilon, k_0 + \epsilon]\), it is convenient to integrate the sharp function \( 1/(k \tanh(k_0 h) - v - i\mu) \) around the pole \( k = k_0 \), under the condition that the meshgrid is sufficiently dense.

In the computation for Fig. 3, we use the uniform meshgrid

\[
k_j = \left( \frac{(j-1)k_{\text{max}}}{N_k} \right)
\]

for simplicity, where the integer \( N_k \) increases with \( \mu \). To derive the suitable results in Fig. 3, we have to take \( N_k = 20000 \) for \( \mu = 0.0005 \) and \( N_k = 5000 \) for \( \mu = 0.001 \). However, it is also possible to obtain suitable result for \( \mu = 0.01 \) and \( N_k = 500 \). For the result of John’s expansion, we take \( N_j = 10 \) as the expansion result remains almost unchanged for \( N_j > 10 \). The roots \( k_0 \) and \( i_k \) of the dispersion equation are determined by the Newton iteration method. The Bessel functions \( J_0, Y_0 \) and \( K_0 \) are evaluated by the polynomial expansions in [31]. The non-dimensional horizontal variable \( R/h \) on the interval \([0, 2.5]\) takes 25 ordinates. In computing \( G \) and \( \partial_R G \) by using John’s expansion formula (8) via a Fortran 90 code in an PC of i5-4460 CPU@3.20GHz, the elapsed time is
less than 1 second. On the other hand, the elapsed time for the computation of $G$ and $\partial_R G$ through the formulas (12), (27) and (29) with $\mu = 0.001$ and $N_k = 5000$ is less than 1 second as well.

Actually, $K^\mu$ remains little changed for small $\mu \leq 0.01$. For displaying purpose, Fig. 3 shows that $G$ and $\partial_R G$ with $\mu = 0.001$ and 0.0005 in the present evaluation are almost the same with those given by John’s series (7). Thus we mainly use the value $\mu = 0.001$ in our computations.

With the use of the approximation (27)–(29), the evaluation of the Green function becomes simple but robust. To help understanding the present Green function evaluation, selected numerical results are displayed in Fig. 4, which shows the accuracy of the present evaluation in comparison with John’s expansion (7) for the non-dimensional wave number $h \nu$ at the moderate value $h \nu = 4$ and the small value $h \nu = 0.1$.

4 The Green function method in the wave-body motion problem

Consider a three-dimensional body undergoing periodic oscillatory motion with a constant frequency $\omega$ in the fluid $-h < z < 0$, the velocity potential of the linearized oscillatory fluid motion problem can be represented as

$$\Phi = \text{Re}(\phi e^{-i\omega t}),$$  \hspace{1cm} (32)

where the complex potential $\phi$ is a harmonic function satisfying the free surface boundary condition (2) and the water bed boundary condition (3). Thus $\phi$ is determined by the boundary integral equation

$$\phi(q) + \frac{1}{4\pi} \int_S \phi(p) n_p \times \nabla_p G(q, p) dS_p = \frac{1}{4\pi} \int_S G(q, p) n_p \times \nabla \phi(p) dS_p$$  \hspace{1cm} (33)

for the field point $q$ in the fluid domain together with the linear body boundary condition

$$n_p \times \nabla \phi(p) = -i\omega n_\alpha$$  \hspace{1cm} (34)

Here $S$ is the average wetted body surface and $n_p = n(p) = (n_1, n_2, n_3)(p)$ represents the normal vector field at $p \in S$ and pointing into the fluid domain. The body undergoes heave motion for $\alpha = 3$, sway motion for $\alpha = 2$ and surge motion for $\alpha = 1$.

When the field point in the fluid domain $D$ tends to the wetted body boundary $S$, Eq. (33) reduces to the boundary integral equation

$$\phi(q) + \frac{1}{4\pi} \lim_{q' \to q} \int_S \phi(p) n_p \times \nabla_p G(q', p) dS_p = \frac{1}{4\pi} \int_S G(q, p) n_p \times \nabla \phi(p) dS_p, \quad q \in S,$$  \hspace{1cm} (35)
Fig. 4 Comparison of the numerical Green function results given, respectively, by the present method (solid lines, \( \mu = 0.001 \)) and John’s method [11] (circle lines, \( N_J = 10 \)) defined by (8). Moreover, for the present method, \( k_{\text{max}} = 8 \) and \( N_k = 5000 \) whenever \( h\nu = 4 \), but \( k_{\text{max}} = 6 \) and \( N_k = 3000 \) whenever \( h\nu = 0.1 \)

which is approximated by the finite boundary element discretization equation system

\[
\phi(q_{i,j}) + \frac{1}{4\pi} \sum_{I=1}^{N} \sum_{J=1}^{M} \phi(p_{I,J}) \lim_{q' \in D, q' \to q_{i,j}} \int_{\text{panel}_{I,J}} n_p \times \nabla_p G(q', p) \, dS_p
\]
by using the surface mesh discretization

\[ S \approx \sum_{i=1}^{N} \sum_{j=1}^{M} \text{panel}_{i,j} \]

defined by meshgrid points \( p_{I,J} \) with \( I = 1, \ldots, N + 1 \) and \( J = 1, \ldots, M + 1 \). Here \( q_{I,J} \) presents the centre point of \( \text{panel}_{I,J} \).

Let \( |\text{panel}_{I,J}| \) denote the area of \( \text{panel}_{I,J} \). The influence coefficients can be calculated as

\[ \int_{\text{panel}_{I,J}} G(q_{i,j}, p) dS_p \approx \frac{1}{4\pi} \int_{\text{panel}_{I,J}} \left( \frac{1}{|q_{i,j} - p|} + \sum_{l=0}^{4} \frac{1}{r_l(q_{i,j}, p)} \right) \ dS_p + \frac{1}{4\pi} K(q_{i,j}, p_{I,J}) |\text{panel}_{I,J}| \]

and

\[ \lim_{q' \in D, q' \rightarrow q_{I,J}} \int_{\text{panel}_{I,J}} n_p \times \nabla_p G(q, p) dS_p - \lim_{q' \in D, q' \rightarrow q_{I,J}} \int_{\text{panel}_{I,J}} n_p \times \nabla_p \frac{1}{|q - p|} dS_p \]

\[ \approx \int_{\text{panel}_{I,J}} \sum_{l=0}^{4} n_p \times \nabla_p \frac{1}{r_l(q_{i,j}, p)} dS_p + n_p \times \nabla_p K(q_{i,j}, p_{I,J}) |\text{panel}_{I,J}|. \]

Here \( \nabla_p K \) is evaluated by (28)–(30),

\[ \lim_{q' \in D, q' \rightarrow q_{I,J}} \int_{\text{panel}_{I,J}} n_p \times \nabla_p \frac{1}{|q - p|} dS_p = \int_{\text{panel}_{I,J}} n_p \times \nabla_p \frac{1}{|q_{i,j} - p|} dS_p, \quad \text{when} \ (i, j) \neq (I, J), \]

and

\[ \lim_{q' \in D, q' \rightarrow q_{I,J}} \int_{\text{panel}_{I,J}} n_p \times \nabla_p \frac{1}{|q - p|} dS_p = \lim_{\zeta \rightarrow 0^+} \int_{S_{I,J}} \frac{1}{\sqrt{\xi^2 + \eta^2 + (z - \tilde{\zeta})^2}} \left. d\tilde{\xi} d\tilde{\eta} \right|_{\tilde{\zeta} = 0} = 2\pi, \]

where we have used the coordinate transformation by transform \( \text{panel}_{I,J} \) in the global coordinate system on to the panel \( S_{I,J} \) centred at \((0, 0, 0)\) in a local coordinate system \((\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})\).

The panel integrals of \((1/(|q_{i,j} - p|))\), \(1/r_l\) and their normal derivatives are given by the Hess–Smith quadrilateral integral method [38–40]. Therefore, with the use of the Green function approximation (27)–(29) in (37) and (38), the evaluation of the influence coefficients is obtained. Thus the algebraic equation system (36) can be numerically solved by the Gaussian elimination scheme.

### 5 Added mass and damping coefficients

To validate the Green function evaluation in practice and understand wave-induced loading to an oscillating body in waves, we calculate numerically added mass and damping coefficients for the Green function method in wave-
New formulation of the finite depth free surface Green function

Fig. 5 Cross sections \((y = 0)\) of a submerged sphere of radius \(a\) centred at \((0, 0, -1.5a)\) and a floating sphere of radius \(a\)

body motions. Consider firstly a sphere of radius \(a\) submerged in the fluid \(-h < z < 0\). This sphere is centred at \((0, 0, -h_0)\) with \(h_0/a = 1.5\) (see Fig. 5). For the numerical velocity potential solution \(\phi = \phi_{\alpha}\) \((\alpha = 1, 2, 3)\) of the boundary value problem (34) and (35), the linear hydrodynamic pressure is expressed as

\[
p_{\alpha} = -\rho \frac{\partial \phi_{\alpha}}{\partial t} = \omega \rho \text{Re} \left( i \phi_{\alpha} e^{-i\omega t} \right)
\]

for \(\rho\) the fluid density. This defines the hydrodynamic wave force exerted on the average wetted body surface \(S:\)

\[
F_{\alpha,\alpha} = \int_{S} p_{\alpha} n_{\alpha} dS
\]

and the non-dimensional added mass and damping coefficients \(A_{\alpha,\alpha}\) and \(B_{\alpha,\alpha}\):

\[
A_{\alpha,\alpha} + iB_{\alpha,\alpha} = \frac{1}{\omega V} \int_{S} i\phi_{\alpha} n_{\alpha} dS \approx \frac{1}{\omega V} \sum_{i=1}^{N} \sum_{j=1}^{M} i\phi_{\alpha}(q_{i,j}) n_{\alpha}(q_{i,j}) |\text{panel}_{i,j}|.
\]

(41)

Here \(V\) is the volume of the moving body with the wetted body surface \(S\). Especially, \(V = 4\pi a^3/3\) for the submerged sphere and \(V = 2\pi a^3/3\) for the floating hemisphere.

Selected results of added mass and damping coefficients of heave and surge motions at different \(h/a\) values are displayed in Fig. 6. For the deep water case \(h/a = 10\), the present method results coincide with the semi-analytical results of Linton [22], where sway rather than surge motion is calculated. However, for the radial symmetric body, the sway motion is the same with the surge motion. Actually, for the submerged sphere oscillating at the water depth \(h_0/a = 1.5\), the influence of the water bed \(h/a = 10\) is negligible due to the comparison of the results with the semi-analytical results of Wang [20] in infinite water depth situation. Figure 6 also shows good agreement for the present method and Linton [22] results at \(h/a = 10/3\).

Next, for the comparison with the semi-analytic results of Hulme [16] at infinite water depth, we consider a floating hemisphere (see Fig. 5) undergoing, respectively, heave and surge motions in deep water depth. The comparison is illustrated in Fig. 7. The present method results exhibit irregular frequencies at \(\nu a = 2.6\) for heaving hemisphere motion and of \(\nu a = 4\) for the surging hemisphere motion. As is well known (see, Frank [1] John [11]) that the combination of panel method and free surface Green function gives rise to irregular frequencies in a high frequency range when a floating body undergoes oscillatory motions. Various methods exist (see, for example, Lee and Sclavounos [2] Ursell [41], Lee et al. [42] and Zhu and Lee [43]) to remove non-physical irregular frequencies. We may also use simple correction method by interpolating regular frequency data as given by the author [45] so that the smooth interpolation data for the deep water depths present excellent agreement with the results of Hulme [16]. It should be noted that the surging hemisphere oscillates in the horizontal direction and thus is not very sensitive with the water depth. Here we take \(h/a = 8\) in Fig. 7. Actually, the numerical results remain the same for \(h/a = 4\). This case is comparable with the Smith effect. However, the heaving hemisphere oscillates in the vertical direction; we have to take much deep water depth such as \(h/a = 25\) (see Fig. 7) to reach the infinite water depth results of Hulme [16].
Fig. 6 Added mass and damping coefficients of heaving and surging sphere of radius $a$ submerged at the depth of $h_0 = 1.5a$ and derived by the present method and the semi-analytical methods of Wang [20] for the infinite water depth $-h \to -\infty$ and Linton [22] for finite water bed $z = -h$ values, which are only available in [22] for $\nu a < 0.8$ when $h/a = 10$. 
Fig. 7 Added mass and damping coefficients of heaving and surging floating hemisphere of radius $a$ derived by the present method and the semi-analytical methods of Hulme [16] for the infinite water depth $-h \rightarrow -\infty$

6 Discussion and conclusion

To solve a wave-body motion problem through a Green function method, it is necessary to evaluate the Green function $G$ and its gradient $\nabla G$. It is known that it is troublesome in the calculation of $G$ and it is even much worse for the approximation of $\nabla G$.

For the free surface Green function with respect to a free surface oscillating source in a fluid of infinite depth, the function $G$ is expressed as the sum of the Rankine simple Green function $1/r$, its image $1/r_1$ with regarding to the average free water surface and a singular wave integral. Similarly, the finite depth Green function is the sum of $1/r$ and its image $1/r_0$ with respect to the fluid bottom $z = -h$ and a singular wave integral $G_1$ (see John [11]):

$$G = \frac{1}{r} + \frac{1}{r_0} + G_1. \quad (42)$$

Under this formulation, the integrands of $G_1$ and $\nabla G_1$ on the free surface have the asymptotic behaviours (as $k \rightarrow \infty$)

$$O\left(\frac{\cos(kR - \frac{\pi}{4})}{\sqrt{kR}}\right) \text{ and } O\left(\sqrt{\frac{k}{R}} \cos\left(kR - \frac{\pi}{4}\right)\right), \quad (43)$$

respectively, as given in (9) and (11).

The principal difference between the infinite depth and finite depth Green functions is that the former has the single free surface mirror $z = 0$, while the latter has the additional water bed mirror $z = -h$ in relation to the image method. Due to the reflection in between the two mirrors, for a free surface source point $p = (\xi, \eta, \zeta)$ in the fluid domain $-h < z < 0$, we need the following five images of $p$:

$$p_0 = (\xi, \eta, -\zeta - 2h),$$

the image of $p$ with respect to the mirror $z = -h$,
\( p_1 = (\xi, \eta, -\zeta), \) the image of \( p \) with respect to the mirror \( z = 0 \),
\( p_2 = (\xi, \eta, \zeta - 2h), \) the image of \( p_1 \) with respect to the mirror \( z = -h \),
\( p_3 = (\xi, \eta, \zeta + 2h), \) the image of \( p_0 \) with respect to the mirror \( z = 0 \),
\( p_4 = (\xi, \eta, -\zeta - 4h), \) the image of \( p_3 \) with respect to the mirror \( z = -h \).

Therefore, in the present formulation, the Green function with respect to a field point \( q = (x, y, z) \) is expressed as
\[
G = \frac{1}{|q - p|} + \sum_{n=0}^{4} \frac{1}{|q - p_n|} + K. \tag{44}
\]
Here \( K \) is approximated by a wave integral, of which the integral is smooth and decay rapidly in the infinite frequency integral domain. This also gives rise to the new formulation of the vertical and horizontal derivatives, for \( r = |q - p| \) and \( r_n = |q - p_n| \),
\[
\partial_\zeta G = \partial_\zeta \left( \frac{1}{r} + \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) + 2\nu \left( \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right) + K_1,
\]
\[
\partial_R G = \partial_R \left( \frac{1}{r} + \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) + K_2
\]
\[
+ \frac{2\nu R}{r_1(r_1 + |z + \zeta|)} + \frac{2\nu R}{r_2(r_2 + |z - \zeta - 2h|)} + \frac{2\nu R}{r_3(r_3 + |\zeta - z - 2h|)} + \frac{2\nu R}{r_4(r_4 + |z + \zeta + 4h|)}, \tag{45}
\]
with \( K_1 \) and \( K_2 \) the wave integrals defined, respectively, by (21) and (23). As illustrated in (15) and (16), the integrands of the wave integrals \( K, K_1 \) and \( K_2 \) have the same asymptotic behaviours \( O(1/k^{3/2}) \) when both the field point \( q \) and the source point \( p \) are on the free surface. Thus the divergence for the wave integrals from John’s formula becomes the convergence for the modified wave integrals. This together with the use of the artificial parameter \( \mu \), direct integral of \( K, K_1 \) and \( K_2 \) can be performed on the straight line \( 0 < k < \infty \).

The Green function formula (5) involves explicitly the Rankine source potential \( 1/r \), the Rankine image source potential \( 1/r_0 \) and the wave integral \( G_1 \). The integral is divergent at the infinity because the other four Rankine image source potentials \( 1/r_1, ..., 1/r_4 \) are contained implicitly in \( G_1 \). We therefore separate them from \( G_1 \) to form \( K \), which becomes convergent at the infinity.

With the presence of the solid mirror \( z = -h \) and the free surface mirror \( z = 0 \), the source \( p \) in the fluid domain has infinitely many images due to continued reflections in between the mirrors. However, only the five images are useful in the new wave integral formulation. If a Rankine source is in between two parallel solid mirrors without the free surface effect, the Green function of this problem can be presented in a form involving infinitely many source images (see [44]). This is different to the present problem due to the absence of free surface effect and thus wave integral in [44].

Validation of the present method is provided through comparison between the present method results and the results from John [11] series together with the added mass and damping coefficients results of Linton [22] and Wang [20] on submerged oscillatory sphere and Hulme [16] on oscillatory floating hemisphere in waves.

This research is motivated by the direct integration approach of the wave integral [45] on double wave integral and [29] on single wave integral approximations for the three-dimensional infinite depth Green function with application to an oscillatory body motion in waves. This study is originated from [46] on two-dimensional vortex Green function method for a travelling body in fluid.

Nowadays, numerical computation of a linear hydrodynamic problem is no longer a time consuming job due to the popularization of high computing capacity computers. Nevertheless, the Green function evaluation due to the presence of an irregular wave integral is still known to be troublesome and sophisticated mathematical approximation theories are supposed to be employed to attack the evaluation [6,7,10,47,48]. The present investigation however shows that the singularity is removable and an accurate evaluation of the Green function is obtainable by an elementary integration in a straightforward manner.
Acknowledgements  This work was partially supported by NSFC of China (No. 11571240).

Appendix A: Derivation of John’s Green function

For the completion of the analysis, we follow John [11] to show the derivation of the Green function

\[ G = \frac{1}{r} + \frac{1}{r_0} + G_1. \]

The use of the Hankel transformation (18) produces

\[ \mathcal{H}\left(\frac{1}{r} + \frac{1}{r_0}\right) = \frac{1}{k} \left(e^{-k|z-\zeta|} + e^{-k|z+\zeta+2h|}\right). \]

Then applying the Hankel transformation to the Laplace equation (1) and employing the bottom boundary condition (3), we have

\[ \mathcal{H}(G_1) = \frac{1}{k} A_0(k) \cosh k(z + h) \quad \text{(A.1)} \]

for a function \( A_0 \). Therefore applying again the Hankel transformation to the free surface condition (2), we have, for \( z \) close to 0,

\[ 0 \approx (\partial_z - v)\mathcal{H}(G_1) + \partial_z \mathcal{H}\left(\frac{1}{r} + \frac{1}{r_0}\right) - v\mathcal{H}\left(\frac{1}{r} + \frac{1}{r_0}\right) \]

\[ = \frac{1}{k} A_0(k) [k \sinh k(z + h) - v \cosh k(z + h)] - \frac{k + v}{k} \left(e^{-k(z-\zeta)} + e^{-k(z+\zeta+2h)}\right). \quad \text{(A.2)} \]

This implies that

\[ A_0(k) = \frac{2(v + k)e^{-kh} \cosh k(\zeta + h)}{k \sinh kh - v \cosh kh}. \]

Therefore the desired Green function is obtained by rewriting (A.1) as

\[ G_1 = H^{-1}\left(\frac{1}{k} A_0(k) \cosh k(z + h)\right), \]

or the desired Green function

\[ G = \frac{1}{r} + \frac{1}{r_0} + \int_{L} \frac{2(v + k)e^{-kh} \cosh k(\zeta + h) \cosh k(z + h)}{k \sinh kh - v \cosh kh} J_0(kR)dk. \quad \text{(A.3)} \]

The integral pass \( L \) passing beneath the pole is determined by the asymptotic behaviour (4).

References

1. Frank W (1967) Oscillation of cylinders in or below the free surface of deep fluids, Report 2375. Naval Ship Research Development Center, Bethesda
2. Lee CH, Sclavounos PD (1989) Removing the irregular frequencies from integral equations in wave–body interactions. J Fluid Mech 207:393–418
3. Lee CH, Newman JN (2004) Computation of wave effects using the panel method. In: Chakrabarti SK (ed) Numerical models in fluid–structure interaction. WIT Press, Southampton
4. Chakrabarti SK (2001) Application and verification of deep water Green function for water waves. J Ship Res 45:187–196
5. Liang H, Wu H, Noblesse F (2018) Validation of a global approximation to the Green function of diffraction radiation in deep water. Appl Ocean Res 74:80–86
6. Newman JN (1985) Algorithms for the free-surface Green functions. J Eng Math 19:57–67
7. Noblesse F (1982) The Green function in the theory of radiation and diffraction of regular water waves by a body. J Eng Math 16:137–169
8. Ponizy B, Noblesse F, Ba M, Guibbaud M (1994) Numerical evaluation of free-surface Green function. J Ship Res 38:193–202
9. Telste JG, Noblesse F (1986) Numerical evaluation of the Green function of water-wave radiation and diffraction. J Ship Res 30:69–84
10. Wu H, Zhang C, Zhu Y, Li W, Wan D, Noblesse F (2017) A global approximation to the Green function for diffraction radiation of water waves. Eur J Mech B Fluids 65:54–64
11. John F (1950) On the motion of floating bodies: II. Simple harmonic motions. Commun Pure Appl Math 3:45–101
12. Linton CM (1999) Rapidly convergent representations for Green functions for Laplace’s equation. Proc R Soc A 455:1767–1797
13. Liu Y, Ishashita H, Hu C (2015) A calculation method for finite depth free-surface green function. Int J Nav Archit Ocean Eng 7:375–389
14. Pidcock MK (1985) The calculation of Green functions in three dimensional hydrodynamic gravity wave problems. Int J Numer Meth Fluids 5:891–909
15. Havelock T (1955) Waves due to a floating hemi-sphere making periodic heaving oscillations. Proc R Soc Lond A 231:1–7
16. Hulme A (1982) The wave forces on a floating hemisphere undergoing forced periodic oscillations. J Fluid Mech 121:443–463
17. Ursell F (1949) On the heaving motion of a circular cylinder on the surface of a fluid. Quart J Mech Appl Math 2:218–231
18. Furel C (1973) On the wave resistance of a submerged spheroid. J Ship Res 17:1–11
19. Chatjigeorgiou IK (2013) The analytic solution for hydrodynamic diffraction by submerged prolate spheroids in infinite water depth. J Eng Math 81:47–65
20. Wang S (1986) Motions of a spherical submarine in waves. Ocean Eng. 13:249–271
21. Wu GX, Eatock Taylor R (1987) The exciting force on a submerged spheroid in regular waves. J Fluid Mech 182:411–426
22. Linton CM (1991) Radiation and diffraction of water waves by a submerged sphere in finite depth. Ocean Eng 18:61–74
23. Cao Y, Schultz W, Beck R (1991) Three-dimensional desingularized boundary integral methods for potential problems. Int J Numer Meth Fluids 12:785–803
24. Dawson CW (1977) A practical computer method for solving ship wave problems. In Proceedings of 2nd International Conference on Numerical Ship Hydrodynamics, University of California, Berkeley, pp 30–38
25. Feng A, Chen ZM, Price WG (2014) A Rankine source computation for three dimensional wave–body interactions adopting a nonlinear body boundary condition. Appl Ocean Res 47:313–321
26. Feng A, Chen ZM, Price WG (2015) A continuous desingularized source distribution method describing wave–body interactions of a large amplitude oscillatory body. J Offshore Mech Arctic Eng 137:021302
27. Mantzaris DA (1998) A Rankine panel method as a tool for the hydrodynamic design of complex marine vehicles, Ph.D. Thesis, MIT
28. Yeung RW (1981) Added mass and damping of a vertical cylinder in finite depth waters. Appl Ocean Res 3:119–133
29. Chen ZM (2019) Straightforward integration for free surface Green function and body wave motions. Eur J Mech B Fluids 74:10–18
30. Wehausen JV, Laitone EV (1960) Surface waves. In: Flugge S, Truesdell C (eds) Fluid dynamics iii in handbuch der physik, vol 9. Springer, Berlin, pp 446–778
31. Abramowitz M, Stegun IA (1965) Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover, New York
32. Havelock TH (1928) Wave resistance. Proc R Soc Lond A 118:24–33
33. Havelock TH (1932) The theory of wave resistance. Proc R Soc Lond A 138:339–348
34. Lazauskas LV (2009) Resistance, wave-making and wave-decay of thin ships, with emphasis on the effects of viscosity. Ph.D. Thesis, The University of Adelaide
35. Spivak B, Vanden-Broeck J-M, Miloh T (2002) Free-surface wave damping due to viscosity and surfactants. Eur J Mech B Fluids 21:207–224
36. Monacella VJ (1967) On ignoring the singularity in the numerical evaluation of Cauchy Principal Value integrals, Hydromechanics Laboratory Research and Development Report 2356
37. Watson GN (1944) A treatise on the theory of Bessel functions. Cambridge University Press, Cambridge
38. Hess JL, Smith AMO (1962) Calculation of potential flow about arbitrary three-dimensional bodies, Report No. E.S. 40622, Douglas Aircraft Co., Inc. Aircraft Division, Long Beach, CA
39. Hess JL, Smith AMO (1966) Calculation of potential flow about arbitrary bodies. Prog Aerospace Sci 8:1–138
40. Newman JN (1986) Distributions of sources and normal dipoles over a quadrilateral panel. J Eng Math 20:113–126
41. Ursell F (1981) Irregular frequencies and the motion of floating bodies. J Fluid Mech 105:143–156
42. Lee CH, Newman JN, Zhu X (1996) An extended boundary integral equation method for the removal of irregular frequency effects. Int J Numer Meth Fluids 23:637–660
43. Zhu X, Lee CH (1994) Removing the irregular frequencies in wave-body interactions. In Proceedings of the 9th International Workshop on Water Waves and Floating Bodies, Japan, pp 245–249
44. Breit SR (1991) The potential of a Rankine source between parallel planes and in a rectangular cylinder. J Eng Math 25:151–163
45. Chen ZM (2015) Regular wave integral approach to numerical simulation of radiation and diffraction of surface waves. Wave Motion 52:171–182
46. Chen ZM (2012) A vortex based panel method for potential flow simulation around a hydrofoil. J Fluids Struct 28:378–391
47. Hess JL, Wilcox DC (1969) Progress in the solution of the problem of a three-dimensional body oscillating in the presence of a free surface—Final technical report, McDonnell Douglas Company Rep. DAC 67647
48. Peter MA, Meylan MH (2004) The eigenfunction expansion of the infinite depth free surface Green function in three dimensions. Wave Motion 40:1–11

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.