ON THE LI-YAU TYPE GRADIENT ESTIMATE OF LI AND XU

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ABSTRACT. In this paper, we obtain a Li-Yau type gradient estimate with time dependent parameter for positive solutions of the heat equation, so that the Li-Yau type gradient estimate of Li-Xu [4] are special cases of the estimate. We also obtain improvements of Davies’ Li-Yau type gradient estimate. The argument is different with those of Li-Xu [4] and Qian [9].

1. Introduction

In recent years, Li and Xu [4] obtained the following Li-Yau type gradient estimate with time dependent parameter.

Theorem 1.1. Let \((M^n, g)\) be a complete Riemannian manifold with Ricci curvature bounded from below by \(-k\), where \(k\) is a nonnegative constant. Let \(u \in C^\infty(M \times [0, T])\) be a positive solution of the heat equation

\[
\Delta u - u_t = 0.
\]

Then,

\[
\|\nabla f\|^2 - \left(1 + \frac{\sinh(kt) \cosh(kt) - kt}{\sinh^2(kt)}\right) f_t \leq \frac{nk}{2}[\coth(kt) + 1].
\]

and

\[
\|\nabla f\|^2 - \left(1 + \frac{2}{3}kt\right) f_t \leq \frac{n}{2t} + \frac{nk}{2} \left(1 + \frac{1}{3}kt\right)
\]

on \(M \times (0, T]\), where \(f = \log u\).

The estimate (1.2) and (1.3) of Li and Xu are of the same spirit of Hamilton’s Li-Yau type gradient estimate (see [3]):

\[
\|\nabla f\|^2 - e^{2kt} f_t \leq e^{4kt} \frac{n}{2t}.
\]

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We can compare the estimates (1.2), (1.3) and (1.4) with the Li-Yau-Davies gradient estimate [5, 2]:

\[(1.5) \quad \|\nabla f\|^2 - \alpha f_t \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 k}{4(\alpha - 1)}\]

for any \(\alpha > 1\) as follows. By comparing to the asymptotic behavior of the heat kernel as \(t \to 0\), we know that (1.5) is even not sharp in leading term. However, (1.2), (1.3) and (1.4) are all sharp in leading term as \(t \to 0\). When \(t \to \infty\), it is clear that (1.5) is better than (1.3) and (1.4). For (1.2), note that

\[(1.6) \quad 1 + \frac{\sinh(kt) \cosh(kt) - kt}{\sinh^2(kt)} \to 2(\triangle \alpha)\]

and

\[(1.7) \quad \frac{nk}{2} [\coth(kt) + 1] \to nk = \frac{n\alpha^2 k}{4(\alpha - 1)}\]

as \(t \to \infty\). So, the asymptotic behavior of (1.2) as \(t \to \infty\) is the same as (1.5) with \(\alpha = 2\). Note that (1.3) was first obtained in [1] by a different method.

Theorem 1.1 was later generalized by Qian [9] to the following general form.

**Theorem 1.2.** Let the notation be the same as in Theorem 1.1 with \(M\) closed. Then

\[(1.8) \quad \|\nabla f\|^2 - \alpha f_t \leq \varphi\]

with

\[(1.9) \quad \alpha = 1 + \frac{2k}{a(t)} \int_0^t a(s) ds\]

and

\[(1.10) \quad \varphi = \frac{nk}{2} + \frac{nk^2}{2a(t)} \int_0^t a(s) ds + \frac{n}{8a(t)} \int_0^t \frac{a'(s)}{a(s)} ds,\]

where \(a \in C^1([0, T])\) is a smooth function satisfying:

(A1) \(\forall t \in (0, T], \ a(t) > 0 \ and \ a'(t) > 0;\)

(A2) \(a(0) = 0 \ and \ \lim_{t \to 0} \frac{a(t)}{a'(t)} = 0;\)

(A3) \(\frac{a'^2}{a} \in L^1([0, T]).\)

For the complete noncompact case, some further technical conditions for the function \(a\) should be satisfied. See [9] for details. The estimates
and (1.3) are special cases of (1.8) with \(a(t) = \sinh^2(kt)\) and \(a(t) = t^2\) respectively. When, \(a(t) = t^{\frac{2}{\theta}} - 1\) with \(\theta \in (0, 1)\), one have

\[
\|\nabla f\|^2 - (1 + \theta kt)f_t \leq \frac{(2 - \theta)^2n}{16\theta(1 - \theta)t} + \frac{nk^2\theta t}{4} + \frac{nk}{2}
\]

for complete Riemannian manifolds with Ricci curvature bounded from below by \(-k\) where \(k\) is a positive constant.

In this paper, we first obtain the following Li-Yau type gradient estimate for closed manifolds.

**Theorem 1.3.** Let the notation be the same as in Theorem 1.1 with \(M\) closed, and \(\lambda, \beta, \psi \in C^1((0, T])\) such that

- (B1) \(0 < \beta(t) < 1\) for any \(t \in (0, T]\);
- (B2) \(\lim_{t \to 0^+} \lambda(t) = 0\) and \(\lambda(t) > 0\) for any \(t \in (0, T]\);
- (B3) \(\frac{2\beta + \beta^\prime}{1 - \beta} - (\ln \lambda) > 0\) for any \(t \in (0, T]\);
- (B4) \(\limsup_{t \to 0^+} \psi(t) \geq 0\);
- (B5) \(\psi' + \frac{2\beta + \beta^\prime}{1 - \beta} \psi - \frac{n(2\beta + \beta^\prime)^2}{8\beta(1 - \beta)^2} = 0\) for any \(t \in (0, T]\).

Then,

\[
\beta\|\nabla f\|^2 - f_t \leq \psi
\]

on \(M \times (0, T]\).

We write the Li-Yau type estimate in the form (1.12) because it is more convenience for comparison. This form was also took in [11, 12, 13]. The Li-Yau-Davies estimate (1.5) written in this form is:

\[
\beta\|\nabla f\|^2 - f_t \leq \frac{n}{2\beta t} + \frac{nk}{4(1 - \beta)}.
\]

Comparing Theorem 1.3 to Theorem 1.2, we have \(\beta = \frac{1}{\alpha}\). Let \(\psi = \frac{1}{\alpha} \varphi\), by Lemma 2.2 in [9], we know that \(\psi\) satisfies (B5). So, if we choose \(\beta = \frac{1}{\alpha}\) satisfying (B1)–(B4), then Theorem 1.3 gives us the same conclusion of Theorem 1.2.

As a corollary of Theorem 1.3, we have the following general Li-Yau type estimate with similar spirit to that of Theorem 1.2.

**Corollary 1.1.** Let the notation be the same as in Theorem 1.1 with \(M\) closed and \(k > 0\), and \(b \in C^1((0, T])\) satisfy that

- (C1) \(\lim_{t \to 0^+} b(t) = 0\) and \(b'(t) > 0\) for any \(t \in (0, T]\);
- (C2) \(\frac{b^2}{b'} \in L^1([0, T])\);

Then

\[
\beta\|\nabla f\|^2 - f_t \leq \psi,
\]
where
\begin{equation}
\beta = 1 - \frac{2k}{b(t)e^{2kt}} \int_0^t b(s)e^{2ks} ds
\end{equation}
and
\begin{equation}
\psi = \frac{n}{8b} \int_0^t \frac{b^2}{b \beta} (s) ds.
\end{equation}

By direct computation, when
\begin{equation}
b(t) = (1 + \theta kt)t^{\frac{2}{n} - 1},
\end{equation}
one has
\begin{equation}
\beta(t) = \frac{1}{1 + \theta kt}.
\end{equation}
Moreover, $b$ satisfies (C1) and (C2) if and only if $\theta \in (0, 1)$. This gives us (1.11) for closed manifolds.

When $b(t) = \sinh^2(kt) + \cosh(kt) \sinh(kt) - kt$, by direct computation,
\begin{equation}
\beta = \frac{1}{1 + \frac{\sinh(kt) \cosh(kt) - kt}{\sinh^2(kt)}}.
\end{equation}
So, Corollary 1.1 also gives us (1.2) for closed manifolds.

Moreover, by setting
\begin{equation}
b = a + 2k \int_0^t a(s) ds
\end{equation}
with $a$ in Theorem 1.2, it not hard to verify that $b$ satisfies (C1) and (C2). By direct computation, one has
\begin{equation}
\beta = \frac{1}{1 + \frac{2k}{a} \int_0^t a(s) ds},
\end{equation}
and Corollary 1.1 gives us Theorem 1.2.

For the complete noncompact case, similar with that of [9], we have to add more restricted assumptions.

**Theorem 1.4.** Let the notation be the same as in Theorem 1.1 with $M$ complete noncompact, and $\lambda, \beta, \psi \in C^1((0, T])$ such that
\begin{itemize}
  \item [(B1')] $\lim_{t \to 0^+} \beta(t) = 1$ and $0 < \beta(t) < 1$ for any $t \in (0, T]$;
  \item [(B2')] $\lim_{t \to 0^+} \lambda(t) = 0$, $\lambda'(t) > 0$ for any $t \in (0, T]$;
  \item [(B2'')] $\frac{\lambda}{1 - \beta}$ and $\beta'$ are bounded from above on $(0, T]$;
  \item [(B3')] there is some $\epsilon > 0$ such that $\frac{2k\beta + \beta'}{1 - \beta} - (1 + \epsilon)(\ln \lambda)' > 0$ for any $t \in (0, T]$;
  \item [(B4')] $\psi(t) \geq 0$ for any $t \in (0, T]$;
\end{itemize}
\[(B5) \quad \psi' + \frac{2k\beta + \beta'}{1 - \beta} \psi - \frac{n(2k\beta + \beta')^2}{8\beta(1 - \beta)^2} = 0 \text{ for any } t \in (0, T].\]

Then,
\[
(1.21) \quad \beta \|\nabla f\|^2 - f_t \leq \psi
\]
on $M \times (0, T]$.

Furthermore, for complete noncompact Riemannian manifolds, one has the following similar corollary with more restricted assumptions.

**Corollary 1.2.** Let the notation be the same as in Theorem 1.1 with $M$ complete noncompact and $k > 0$, and $b \in C^1((0, T])$ satisfy that

(C1) $\lim_{t \to 0^+} b(t) = 0$ and $b'(t) > 0$ for any $t \in (0, T]$;

(C2) $\frac{b'^2}{b} \in L^1([0, T])$;

(C3) there is a constant $\delta \in (0, 1)$ such that $\frac{b'}{b}$ is bounded from above on $(0, T]$;

(C4) $\frac{b' f_0 h_s ds}{b^2}$ is bounded from above on $(0, T]$.

Then
\[
(1.22) \quad \beta \|\nabla f\|^2 - f_t \leq \psi,
\]
where $\beta$ and $\psi$ are given in (1.15) and (1.16) respectively.

It is clear that
\[b(t) = (1 + \theta kt)^{\frac{2}{\theta} - 1}\]
and
\[b(t) = \sinh^2(kt) + \cosh(kt) \sinh(kt) - kt\]
also satisfy (C3) and (C4). So Corollary 1.2 also gives us (1.11) and Theorem 1.1 for complete noncompact Riemannian manifolds.

Finally, by using (1.11), we are able to obtain an improvement of the Li-Yau-Davies estimate (1.13) and the Li-Yau type gradient estimate in [11] for large time.

**Theorem 1.5.** Let the notation be the same as in Theorem 1.1 with $k > 0$. Then, for any $\beta \in (0, 1)$ and $t > \frac{1 - \beta}{k\beta}$,
\[
(1.23) \quad \beta \|\nabla f\|^2 - f_t \leq \frac{n(1 - \beta)}{16k(t - \frac{1 - \beta}{k\beta})t} + \frac{nk}{4(1 - \beta)}.
\]

This estimate is clearly better than (1.5) and the Li-Yau type estimate in [11] when time is large. In fact, by direct computation, we have the following straightforward corollary.

**Corollary 1.3.** Let the notation be the same as in Theorem 1.1 with $k > 0$. Then,
for any $\gamma > \frac{1-\beta}{16k}$ and $t > \frac{\gamma(1-\beta)}{k\beta(\gamma - \frac{1-\beta}{16})}$,

\begin{equation}
(1.24) \quad \beta\|\nabla f\|^2 - f_t \leq \frac{\gamma n}{t^2} + \frac{nk}{4(1-\beta)};
\end{equation}

(2) for any $\gamma > 0$ and $t > \frac{1-\beta}{16k\gamma} + \frac{1-\beta}{k\beta}$,

\begin{equation}
(1.25) \quad \beta\|\nabla f\|^2 - f_t \leq \frac{\gamma n}{t} + \frac{nk}{4(1-\beta)};
\end{equation}

(3) for any $\gamma > 0$ and $\theta \in (1, 2)$, there is a positive constant $T_0(k, \beta, \theta, \gamma)$ such that for any $t > T_0$,

\begin{equation}
(1.26) \quad \beta\|\nabla f\|^2 - f_t \leq \frac{\gamma n}{t^\theta} + \frac{nk}{4(1-\beta)}.
\end{equation}

Although Theorem 1.3 (or Theorem 1.4) and Corollary 1.1 (or Corollary 1.2) are similar to Li-Xu’s estimate (Theorem 1.1) and Qian’s generalization (Theorem 1.2), the proofs of Theorem 1.3 (or Theorem 1.4) and Corollary 1.1 (or Corollary 1.2) are different to those of Li-Xu [4] and Qian [9], where Li-Xu and Qian applied the maximum principle to $\Delta - \frac{\partial}{\partial t} (\beta\|\nabla f\|^2 - f_t - \psi)$ which is more in the spirit of Perelman [6] (see also [7, 8]), while we simply apply the maximum principle to $\lambda(\beta\|\nabla f\|^2 - f_t - \psi)$ which is similar to that of Li-Yau [5].

2. Li-Yau type gradient estimate

We first prove Theorem 1.3.

**Proof of Theorem 1.3** Let $G = \beta\|\nabla f\|^2 - f_t - \psi$, and $L = \Delta - \partial_t$. Note that

\begin{equation}
(2.1) \quad Lf = -\|\nabla f\|^2,
\end{equation}

\begin{equation}
(2.2) \quad Lf_t = -2\langle \nabla f_t, \nabla f \rangle
\end{equation}

and

\begin{equation}
(2.3) \quad L\|\nabla f\|^2 = 2\|\nabla f\|^2 + 2Ric(\nabla f, \nabla f) - 2\langle \nabla\|\nabla f\|^2, \nabla f \rangle.
\end{equation}

Then, by noting that

\begin{equation}
(2.4) \quad \|\nabla f\|^2 = -\frac{1}{1-\beta}(\Delta f + G + \psi),
\end{equation}

where $F = \|\nabla f\|^2 - \alpha f_t - \varphi$ which is more in the spirit of Perelman [6] (see also [7, 8]), while we simply apply the maximum principle to $\lambda(\beta\|\nabla f\|^2 - f_t - \psi)$ which is similar to that of Li-Yau [5].
we have

\[
LG = \beta L \|\nabla f\|^2 - \beta' \|\nabla f\|^2 - Lf_t + \psi'
\]

\[
\geq \beta \left( \frac{2}{n} (\Delta f)^2 - 2k \|\nabla f\|^2 - 2\langle \nabla \|\nabla f\|^2, \nabla f \rangle \right) - \beta' \|\nabla f\|^2 + 2\langle \nabla f_t, \nabla f \rangle + \psi'
\]

\[
= \frac{2\beta}{n} (\Delta f)^2 - (2k\beta + \beta') \|\nabla f\|^2 + \psi' - 2\langle \nabla G, \nabla f \rangle
\]

\[
= \frac{2\beta}{n} (\Delta f)^2 + \frac{2k\beta + \beta'}{1 - \beta} (\Delta f + G + \psi) + \psi' - 2\langle \nabla G, \nabla f \rangle
\]

\[
\geq \frac{2k\beta + \beta'}{1 - \beta} G + \psi' + \frac{2k\beta + \beta'}{1 - \beta} \psi - \frac{n(2k\beta + \beta')^2}{8\beta(1 - \beta)^2} - 2\langle \nabla G, \nabla f \rangle.
\]

where we have used that \( ax^2 + bx \geq -\frac{b^2}{4a} \) when \( a > 0 \).

Let \( F = \lambda G \). Then,

\[
LF = \lambda LG - (\ln \lambda)' F \geq \left( \frac{2k\beta + \beta'}{1 - \beta} - (\ln \lambda)' \right) F - 2\langle \nabla F, \nabla f \rangle.
\]

By (B1),(B2) and (B4), we know that

\[
\lim \inf_{t \to 0^+} F \leq 0.
\]

Then, by maximum principle, we complete the proof of the theorem.

Next, we come to prove Corollary 1.1.

**Proof of Corollary 1.1.** By the expression (1.15), it is clear that \( \beta(t) < 1 \) for \( t \in (0, T] \). On the other hand, by integration by parts,

\[
\beta(t) = \frac{1}{b(t)e^{2kt}} \int_0^t b(s)e^{2ks} ds > 0
\]

for \( t \in (0, T] \). So, (B1) is satisfied. Let \( \lambda = \sqrt{b} \), it is clear that (B2) is satisfied. Moreover, by direct computation, we have

\[
\frac{2k\beta + \beta'}{1 - \beta} = (\ln b)'
\]

which implies that

\[
\frac{2k\beta + \beta'}{1 - \beta} - (\ln \lambda)' = \frac{1}{2} (\ln b)' > 0
\]
for $t \in (0, T]$ by (C1). So (B3) is satisfied. (B4) is clearly true by expression (1.16) of $\psi$. Finally, (B5) can be verified by direct computation. So, by Theorem 1.3, we complete the proof of the corollary. □

The proof of Theorem 1.4 uses the classical cut-off argument of Li-Yau [5]. First recall the following existence of cut-off functions.

**Lemma 2.1.** Let $(M^n, g)$ be a complete noncompact Riemannian manifold with Ricci curvature bounded from below. Then, there is a constant $C_1 > 1$ such that for any $p \in M$ and $R > 1$, there is a smooth function $\rho_R$ on $M$ satisfying:

1. $0 \leq \rho_R \leq 1$;
2. $\rho_R|_{B_p(R)} \equiv 1$ and $\text{supp}\rho_R \subset B_p(C_1 R)$;
3. $\|\nabla \rho_R\|^2 \leq C_1 R^{-2} \rho_R$ and $\Delta \rho_R \geq -C_1 R^{-1}$ on $M$.

**Proof.** Let $r$ be a smooth function on $M$, such that

\[
\begin{cases}
C_2^{-1} (1 + d(p, x)) \leq r(x) \leq C_2 (1 + d(p, x)) \\
\|\nabla r\| \leq C_2 \\
|\Delta r| \leq C_2
\end{cases}
\]  

all over $M$, where $C_2 > 1$ is some constant. The existence of such a function can be found in [10]. Let $\eta$ be a smooth function on $[0, +\infty)$ with (i) $\eta(t) = 1$ for $t \in [0, 1]$, (ii) $\eta(t) = 0$ for $t \geq 2$ and (ii) $\eta' \leq 0$. Let $\rho_R(x) = \eta^2 \left( \frac{r(x)}{2C_2 R} \right)$. It is not hard to check that $\rho_R$ satisfies the requirements of the lemma with

$$C_1 = \max\{4C_2^2, \max(|\eta'| + |\eta|^2 + |\eta''|)\}.$$  

□

We are now ready to prove Theorem 1.4.

**Proof.** We will proceed by contradiction. Let $F$ and $G$ be the same as in the proof of Theorem 1.3. Then, by (2.5), we have

\[
LG \geq \frac{2\beta}{n} (\Delta f)^2 - (2k\beta + \beta') \|\nabla f\|^2 + \psi' - 2\langle \nabla G, \nabla f \rangle
\]

\[=
\frac{2\beta}{n} (\|\nabla f\|^2 - f_1^2) - (2k\beta + \beta') \|\nabla f\|^2 + \psi' - 2\langle \nabla G, \nabla f \rangle
\]

\[=
\frac{2\beta}{n} (G + \psi + (1 - \beta)\|\nabla f\|^2)^2 - (2k\beta + \beta') \|\nabla f\|^2 + \psi' - 2\langle \nabla G, \nabla f \rangle
\]

\[=
\frac{2\beta}{n} (G + \psi + X)^2 - \frac{2k\beta + \beta'}{1 - \beta} (\psi + X) + \frac{n(2k\beta + \beta')^2}{8\beta(1 - \beta)^2} - 2\langle \nabla G, \nabla f \rangle
\]

\[=
\frac{2\beta}{n} G^2 + \frac{4\beta}{n} G (\psi + X) + \frac{2\beta}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{4\beta(1 - \beta)} \right)^2 - 2\langle \nabla G, \nabla f \rangle
\]
where $X = (1 - \beta)||\nabla f||^2$ and we have substituted (B5) into the inequality. Moreover,

\[(2.13)\]
\[
LF = \lambda LG - (\ln \lambda)'F \\
\geq \frac{2\beta}{n\lambda} F^2 + \frac{4\beta}{n} F(\psi + X) + \frac{2\lambda\beta}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{4\beta(1 - \beta)} \right)^2 - (\ln \lambda)'F \\
- 2\langle \nabla F, \nabla f \rangle.
\]

Suppose that $F(p, t_0) > 0$ for some $p \in M$ and $t_0 \in (0, T]$. For each $R > 1$, let $\rho_R$ be the cut-off function in Lemma 2.1. Let $Q_R = \rho_R F$. Then, by (2.13),

\[(2.14)\]
\[
LQ_R = \rho_R LF + F \Delta \rho_R + 2\langle \nabla \rho_R, \nabla F \rangle \\
\geq \frac{2\beta}{n\lambda \rho_R} Q^2_R + \frac{4\beta}{n} Q_R(\psi + X) + \frac{2\lambda\rho_R\beta}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{4\beta(1 - \beta)} \right)^2 - (\ln \lambda)'Q_R \\
- 2\langle \rho_R \nabla F, \nabla f \rangle + F \Delta \rho_R + 2\langle \nabla \rho_R, \nabla F \rangle.
\]

By (B1'), (B2') and (B4'), there is a $\tilde{t}_R \in (0, t_0)$ small enough such that

\[(2.15)\]
\[
\max_{x \in M} Q_R(x, \tilde{t}_R) < F(p, t_0) = Q_R(p, t_0).
\]

Let $(x_R, t_R)$ be the maximum point of $Q_R$ in $M \times [\tilde{t}_R, T]$. By (2.15), we have $t_R > \tilde{t}_R$ and $Q_R(x_R, t_R) > 0$. Then,

\[(2.16)\]
\[
\nabla F(x_R, t_R) = -F(x_R, t_R)\rho_R^{-1}\nabla \rho_R(x_R)
\]

and

\[(2.17)\]
\[
0 \geq LQ_R(x_R, t_R).
\]
So, by (2.14), and multiplying \( \lambda(t_R)\rho_R(x_{R_i}) \) to (2.17), at the point \((x_{R_i}, t_{R_i})\), we have

\[
(2.18) \quad 0 \geq \frac{2\beta}{n}Q_R^2 + \frac{4\beta}{n}Q_R\lambda\rho_R(\psi + X) + \frac{2\beta\lambda^2\rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{{4\beta(1 - \beta)}} \right)^2 - \lambda'\rho_R Q_R
- 2\lambda(\rho_R^2 \nabla F, \nabla f) + \lambda Q_R \Delta \rho_R + 2\lambda(\nabla \rho_R, \rho_R \nabla F).
\]

\[
= \frac{2\beta}{n}Q_R^2 + \frac{4\beta}{n}Q_R\lambda\rho_R(\psi + X) + \frac{2\beta\lambda^2\rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{{4\beta(1 - \beta)}} \right)^2 - \lambda'\rho_R Q_R
+ 2\lambda Q_R(\nabla \rho_R, \nabla f) + \lambda Q_R \Delta \rho_R - 2\lambda\|\nabla \rho_R\|^2 F,
\]

\[
\geq \frac{2\beta}{n}Q_R^2 + \frac{4\beta}{n}Q_R\lambda\rho_R(\psi + X) + \frac{2\beta\lambda^2\rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{{4\beta(1 - \beta)}} \right)^2 - \lambda'\rho_R Q_R
- 2C_1R^{-1} \left( \frac{\lambda}{1 - \beta} \right)^\frac{1}{2} Q_R(\lambda\rho_R X)^\frac{1}{2} - 3C_1R^{-1}\lambda Q_R
\]

\[
\geq \frac{2\beta}{n}Q_R^2 + \frac{4\beta}{n}Q_R\lambda\rho_R(\psi + X) + \frac{2\beta\lambda^2\rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{{4\beta(1 - \beta)}} \right)^2 - \lambda'\rho_R Q_R
- 2C_3R^{-1}Q_R(\lambda\rho_R X)^\frac{1}{2} - C_3R^{-1}Q_R
\]

\[
\geq \frac{2\beta}{n}Q_R^2 + \left( \frac{4\beta}{n} - C_3R^{-1} \right) Q_R\lambda\rho_R(\psi + X) + \frac{2\beta\lambda^2\rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{{4\beta(1 - \beta)}} \right)^2
- \lambda'\rho_R Q_R - 2C_3R^{-1}Q_R.
\]

where we have used Lemma 2.1 (B2' \( \frac{1}{2} \)), (B4') and the fact

\[ 2Q_R(\lambda\rho_R X)^\frac{1}{2} \leq Q_R + Q_R(\lambda\rho_R X). \]

Here \( C_3 = C_1 \max \left\{ \sup_{[0, T]} \sqrt{\frac{\lambda}{1 - \beta}}, 3\max_{[0, T]} \lambda \right\} \).

Next, we divide the proof into three cases to draw a contradiction.

1. There is a sequence \( R_i \to +\infty \) as \( i \to \infty \), such that \( \rho_{R_i}(x_{R_i}) \to 0 \) as \( i \to \infty \). Then, by (2.18), we have

\[
(2.19) \quad 0 < F(p, t_0) \leq Q_{R_i}(x_{R_i}, t_{R_i}) \leq \frac{n}{2\beta(t_{R_i})} \left( \lambda'(t_{R_i})\rho_{R_i}(x_{R_i}) + 2C_3R_i^{-1} \right)
\]

when \( i \) is sufficiently large. By (B1'), (B2' \( \frac{1}{2} \)) and (B3'), we know that \( \min_{[0, T]} \beta(t) > 0 \) and \( \lambda' \) is bounded from above on \((0, T]\). So, taking \( i \to \infty \) in (2.19) gives us a contradiction.

2. There is a sequence \( R_i \to +\infty \) as \( i \to \infty \), such that \( \lambda'(t_{R_i}) \to 0 \) as \( i \to \infty \). Then, similarly as in case (1), by (2.19), we can draw a contradiction.
(3) If there is a positive constant \( \epsilon_0 > 0 \) such that \( \rho_R(x_R) \geq \epsilon_0 \) and \( \lambda'(t_R) \geq \epsilon_0 \) when \( R \) is sufficiently large. Then, by (2.18), at the point \( (x_R, t_R) \),

\[
(2.20) \quad 0 \geq \frac{2\beta}{n} Q_R^2 + \left( \frac{4\beta}{n} - C_3 R^{-1} \right) Q_R \lambda \rho_R (\psi + X) + \frac{2\beta \lambda' \rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{4\beta(1-\beta)} \right)^2 \\
- (1 + C_4 R^{-1}) \lambda' \rho_R Q_R \\
\geq \frac{2\beta}{n} Q_R^2 + \left( \frac{4\beta}{n} - C_3 R^{-1} \right) Q_R \lambda \rho_R \left( \psi + X - \frac{n(2k\beta + \beta')}{4\beta(1-\beta)} \right) \\
+ \frac{2\beta \lambda' \rho_R^2}{n} \left( \psi + X - \frac{n(2k\beta + \beta')}{4\beta(1-\beta)} \right)^2 + (1 - C_5 R^{-1}) \frac{2k\beta + \beta'}{1-\beta} \lambda \rho_R Q_R \\
- (1 + C_4 R^{-1}) \lambda' \rho_R Q_R \\
\geq (1 - C_5 R^{-1}) \lambda \left( \frac{2k\beta + \beta'}{1-\beta} - \frac{1 + C_4 R^{-1}}{1 - C_5 R^{-1}} (\ln \lambda)' \right) \rho_R Q_R,
\]

where \( C_4 = \frac{2C_3}{\epsilon_0} \) and \( C_5 = \frac{4nC_3}{\min_{[0,T]} \beta} \). Moreover, by (B2') and (B3), when \( R \) is sufficiently large,

\[
(2.21) \quad \frac{2k\beta + \beta'}{1-\beta} - \frac{1 + C_4 R^{-1}}{1 - C_5 R^{-1}} (\ln \lambda)' > 0.
\]

So,

\[
(2.22) \quad 0 < F(p, t_0) \leq Q_R(x_R, t_R) \leq 0
\]

when \( R \) is sufficiently large. This is a contradiction.

This completes the proof of the theorem. \( \square \)

We next come to prove Corollary 1.2.

**Proof of Corollary 1.2.** Note that

\[
(2.23) \quad \beta(t) = 1 - \frac{2k \int_0^t b(s) e^{2ks} ds}{b(t) e^{2kt}} \geq 1 - 2kt,
\]

by (C1), and it has been shown in the proof of Corollary 1.1 that \( 0 < \beta(t) < 1 \) for \( t \in (0, T] \). So (B1') is satisfied.

Let \( \lambda = b^{1-\delta} \). Then, (B2') is satisfied by (C1). By (2.9),

\[
(2.24) \quad \frac{2k\beta + \beta'}{1-\beta} - (1 + \delta)(\ln \lambda)' = \delta^2 (\ln b(t))' > 0
\]
for $t \in (0, T]$. So, (B3') is also satisfied. Moreover,
\[
\begin{align*}
\lambda & = \frac{b^{2-\delta} e^{2kt}}{2k \int_0^t b(s)e^{2ks}ds} \\
& \leq \frac{1}{2k} e^{2kT} \frac{b^{2-\delta}}{\int_0^t b(s)ds} \\
& = \frac{2 - \delta}{2k} e^{2kT} \int_0^t b^{1-\delta}b'(s)ds
\end{align*}
\]
(2.25)
is bounded from above by (C3), and by (2.9),
\[
\begin{align*}
\beta' \leq (1 - \beta)(\ln b)' = \frac{2kb' \int_0^t b(s)e^{2ks}ds}{b^2 e^{2kt}} \leq \frac{2kb' \int_0^t b(s)ds}{b^2}
\end{align*}
\]
(2.26)
is bounded from above by (C4). So (B21) is satisfied.

Finally, by the expression (1.16) and direct computation, (B4') and (B5) is clearly satisfied. So, by Theorem 1.4, we complete the proof of the Corollary. \(\square\)

Finally, we come to prove Theorem 1.5.

Proof of Theorem 1.5. For each $\beta_0 \in (0, 1)$ and $t_0 \geq \frac{1 - \beta_0}{k\beta_0}$, let $\theta_0 = \frac{1 - \beta_0}{k\beta_0} \in (0, 1)$. Then $\beta_0 = \frac{1}{1 + \theta_0 kt}$. In (1.11), let $\beta(t) = \frac{1}{1 + \theta_0 kt}$ and $t = t_0$, we have
\[
\begin{align*}
\beta_0 \|
abla f \|^2 - f_t & \leq \frac{(2 - \theta_0)^2 n\beta_0}{16\theta_0(1 - \theta_0)t_0} + \frac{n\beta_0^2 \theta_0 t_0}{4} + \frac{nk\beta_0}{2} \\
& = \frac{n(2k\beta_0 t_0 + \beta_0 - 1)^2 \beta_0}{16(1 - \beta_0)(k\beta_0 t_0 + \beta_0 - 1)t_0} + \frac{nk(\beta_0 + 1)}{4} \\
& = \frac{n(1 - \beta_0)}{16k(t_0 - \frac{1 - \beta_0}{k\beta_0})t_0} + \frac{nk}{4(1 - \beta_0)}.
\end{align*}
\]
(2.27)
This completes the proof of the theorem. \(\square\)

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