The finitistic dimension of a Nakayama algebra.

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Abstract: If $A$ is an artin algebra, Gélinas has introduced an interesting upper bound for the finitistic dimension $\text{fin-pro} \ A$ of $A$, namely the delooping level $\text{del} \ A$. We assert that $\text{fin-pro} \ A = \text{del} \ A$ for any Nakayama algebra $A$. This yields also a new proof that the finitistic dimension of $A$ and its opposite algebra are equal, as shown quite recently by Sen. For a cyclic Nakayama algebra with even finitistic dimension $d$ we show that $\Omega^d$ yields a bijection between the indecomposable injective modules $I$ with projective dimension $d$ such that the socle of $I$ has even or infinite projective dimension and the indecomposable projective modules $P$ with injective dimension $d$ such that the top of $P$ has even or infinite injective dimension.

Key words. Nakayama algebra. Finitistic dimension. Delooping level. Desuspending level. Torsionless modules. Resolution quiver. Coresolution quiver.

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1. Introduction.

Let $A$ be an artin algebra. The modules $M$ to be considered are usually left $A$-modules. Often (but not always) we will assume in addition that $M$ is finitely generated. We denote by $A^{\text{op}}$ the opposite algebra of $A$ (and stress that the right $A$-modules are just the (left) $A^{\text{op}}$-modules).

The projective dimension of a module $M$ will be denoted by $\text{pd} \ M$, its injective dimension by $\text{id} \ M$. We write $\text{fin-pro} \ A$ for the supremum of the projective dimension of finitely generated left modules of finite projective dimension, and similarly, $\text{fin-inj} \ A$ for the supremum of the injective dimension of finitely generated left modules of finite injective dimension. Note that $\text{fin-pro} \ A$ is usually called the (small) finitistic dimension of $A$ and we have $\text{fin-inj} \ A = \text{fin-pro} \ A^{\text{op}}$. The finitistic dimension lies at the center of the classical homological conjectures.

We will focus the attention to Nakayama algebra: An algebra $A$ is called Nakayama provided all indecomposable modules are serial (that means: they have a unique composition series). The module category of a Nakayama algebra is well understood and many questions which are difficult to deal with in general, can easily be answered for Nakayama algebras. But, actually, only very recently, it was shown by Sen [S] that $\text{fin-pro} \ A = \text{fin-inj} \ A$ for a Nakayama algebra $A$. This seems to be an important observation. We will provide here a new proof as well as some further information on fin-pro $A$.

Being interested in the finitistic dimension of an algebra $A$, Gélinas [Ge] has introduced a new invariant, the delooping level $\text{del} \ A$. If $M$ is a module, let $\Omega M$ be its (first) syzygy module (it is the kernel of a projective cover $PM \rightarrow M$, and sometimes also called the
“loop” module of $M$) and let $\Sigma M$ be its (first) suspension module (the cokernel of an injective envelope $M \to IM$). The delooping level $\text{del} S$ of a simple module $S$ is the smallest number $d \geq 0$ such that $\Omega^d S$ is a direct summand of $P \oplus \Omega^{d+1} M$, where $M$, $P$ are finitely generated modules with $P$ projective (and $\text{del} S = \infty$, if such a $d$ does not exist). Similarly, one may define the desuspending level $\text{des} S$ of $S$ as the smallest number $d \geq 0$ such that $\Sigma^d S$ is a direct summand of $I \oplus \Sigma^{d+1} M$ for some finitely generated modules $M$, $I$ with $I$ injective (and $\text{des} S = \infty$, if such a $d$ does not exist). By definition, $\text{del} A$ is the maximum of $\text{del} S$, where $S$ runs through the simple modules, and $\text{des} A$ is the corresponding maximum of the numbers $\text{des} S$. Of course, $\text{des} A = \text{del} A^{\text{op}}$.

If $A$ is a Nakayama algebra, we will show that $\text{fin-pro} A = \text{del} A$, thus, altogether

$$\text{fin-pro} A = \text{fin-inj} A = \text{del} A = \text{des} A.$$ 

**Outline of the paper.** Section 2 deals with arbitrary artin algebras. We will review the definition of the delooping level of a module, as introduced by Gélinas. We will show in 2.3: If $X$ is a submodule of a finitely generated module $Y$, then $\text{del} X \leq \text{pd} Y$. As a consequence, we get: if any simple module is a submodule of a finitely generated module of finite projective dimension, then $\text{del} A \leq \text{fin-pro} A$, see Theorem 2.4.

Section 3 restricts the attention to Nakayama algebra. Following Madsen [M], we show that if $A$ is Nakayama, then any simple module $S$ is a submodule of a finitely generated module of finite projective dimension. As a consequence, we can apply the general considerations of section 2, see 3.5.

In section 4 the cyclic Nakayama algebras with even finitistic dimension $d$ are considered in more detail. Such an algebras has always indecomposable injective modules $I$ with projective dimension $d$ such that the socle of $I$ has even or infinite projective dimension and we show that $\Omega^d$ provides a bijection between these modules $I$ and the indecomposable projective modules $P$ with injective dimension $d$ and top of $P$ having even or infinite injective dimension (Theorem 4.1). The proof uses some basic properties of monotone endofunctions of $Z$ which are presented in Appendix A.

In Appendix B, we provide an outline of Sen’s $\epsilon$-construction which allowed him to proof several interesting results using induction, for example the equality of the finitistic dimension of a Nakayama algebra $A$ and its opposite algebra [S2], but also, very recently, the equality $\text{fin-pro} A = \text{del} A$, see [S3]. In the center of this approach lies the fact that there are canonical bijections between several sets of modules. All these sets have cardinality $r$, where $r$ is the minimal number of relations needed to define $A$. Appendix C is devoted to these bijections.

In the final Appendix D, we consider two further invariants of a simple module $S$ which are related to the finitistic dimension, namely grade $S$ as well as the number $t$ such that $\Omega^t \Omega^t S$ is torsionless. These invariants have also been considered by Gélinas.

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2. The delooping level, the desuspending level.

2.1. Definitions. Let $M$ be a finitely generated module. The delooping level $\text{del} M$ of $M$ is the minimal number $d \geq 0$ such that $\Omega^d M$ belongs to $\text{add} A \oplus \Omega^{d+1} M'$ for some finitely generated module $M'$. (Note that if $\Omega^d M$ belongs to $\text{add} A \oplus \Omega^{d+1} M'$ for some module $M'$, then $\Omega^{d+1} M$ belongs to $\text{add} A \oplus \Omega^{d+2} M'$.) This definition is due to Gélinas [Ge]. As we have mentioned already, the delooping level of the algebra $A$ is defined as $\text{del} A = \max_S \text{del} S$, where $S$ are the simple modules.

There is the dual concept of the desuspending level $\text{des} M$ of a finitely generated module $M$, this is the minimal number $d \geq 0$ such that $\Sigma^d M$ belongs to $\text{add} D(A_A) \oplus \Sigma M'$ for some finitely generated module $M'$. Also, we put $\text{des} A = \max_S \text{des} S$, where $S$ are the simple modules. Clearly, $\text{des} A = \text{del} A^\text{op}$.

In addition, let $\text{Fin-pro} A$ be the supremum of $\text{pd} N$, where $N$ has finite projective dimension, but is not necessarily finitely generated (this is usually called the big finitely generated) module $N$, as well as $\text{Fin-inj} A$, the supremum of $\text{id} N$, where $N$ has finite injective dimension, but not necessarily finitely generated.

2.2. Proposition (Gélinas). Let $M$ be a finitely generated module. If there is a (not necessarily finitely generated) module $N$ with finite injective dimension $d \geq 1$ such that $\text{Ext}^d(M, N) \neq 0$, then $d \leq \text{del} M$.

Proof. Let $N$ be a module with $d = \text{id} N$ and $1 \leq d < \infty$. Then $\text{Ext}^{d+1}(X, N) = 0$ for all modules $X$. Let $M$ be a finitely generated module and assume that $\text{del} M < d$. By definition, $\Omega^{d-1} M$ belongs to $\text{add} A \oplus \Omega^d M'$, for some finitely generated module $M'$. It follows that $\text{Ext}^d(M, N) = \text{Ext}^1(\Omega^{d-1} M, N)$ belongs to $\text{add} \text{Ext}^1(A \oplus \Omega^d M', N)$, but

$$\text{Ext}^1(A \oplus \Omega^d M', N) = \text{Ext}^1(\Omega^d M', N) = \text{Ext}^{d+1}(M', N) = 0,$$

and therefore $\text{Ext}^d(M, N) = 0$, a contradiction. \hfill \Box

Corollary: $\text{Fin-inj} A \leq \text{del} A$.

Proof. We have to show: If $N$ is an arbitrary module with finite injective dimension $d$, then $d \leq \text{del} S$ for some simple module $S$. This is clear for $d = 0$. Thus, we can assume that $d \geq 1$. Since $\text{id} N = d$, there is a simple module $S$ with $\text{Ext}^d(S, N) \neq 0$. The Proposition yields $d \leq \text{del} S$. \hfill \Box

2.2'. The dual assertions.

Proposition. Let $M$ be a finitely generated module. If there is a (not necessarily finitely generated) module $N$ with finite projective dimension $d \geq 1$ which satisfies $\text{Ext}^d(N, M) \neq 0$, then $d \leq \text{des} M$.

Proof. Let $N$ be a module with projective dimension $d \geq 1$. Then $\text{Ext}^{d+1}(N, Y) = 0$ for all modules $Y$. Let $M$ be a finitely generated module with $\text{des} M < d$, thus $\Sigma^{d-1} M$ is a direct summand of a module of the form $I \oplus \Sigma^d M'$ where $I, M'$ are finitely generated modules with $I$ injective. Then $\text{Ext}^d(N, M) = \text{Ext}^1(N, \Sigma^{d-1} M)$ is a direct summand of $\text{Ext}^1(N, I \oplus \Sigma^d M')$. But $\text{Ext}^1(N, I \oplus \Sigma^d M') = \text{Ext}^{d+1}(N, M') = 0$, thus $\text{Ext}^d(N, M) = 0$. \hfill \Box
Corollary: Fin-pro $A \leq \text{des } A$.

Proof. We show: If $N$ is an arbitrary module with $\text{pd } M = d < \infty$, then $d \leq \text{del } S$ for some simple module $S$. This is clear for $d = 0$. Thus, we can assume that $d \geq 1$. Since $\text{pd } N = d$, there is a simple module $S$ with $\text{Ext}^d(S, N) \neq 0$. The Proposition yields $d \leq \text{del } S$. \qed

2.3. Proposition. If $X$ is a submodule of a finitely generated module $Y$, then

$$\text{del } X \leq \text{pd } Y.$$  

Proof. Let $\text{pd } Y = d$. We can assume that $d < \infty$. We apply the horseshoe lemma to the exact sequence $0 \to X \to Y \to Y/X \to 0$ and obtain an exact sequence

$$0 \to \Omega^d X \to P' \oplus \Omega^d Y \to \Omega^d (Y/X) \to 0,$$

where $P'$ is projective. Since $\Omega^d Y$ is projective, we see that the middle term $E = P' \oplus \Omega^d Y$ is projective and the projective cover $P(\Omega^d (Y/X))$ of $\Omega^d (Y/X)$ is a direct summand of $E$, say $E = P'' \oplus P(\Omega^d (Y/X))$ for some projective module $P''$. Therefore

$$\Omega^d X = P'' \oplus \Omega (\Omega^d (Y/X)) = P'' \oplus \Omega^{d+1} (Y/X).$$

This shows that $\text{del } X \leq d$. \qed

Remark. The inequality

$$\text{del } X \leq \inf \{ \text{pd } Y \mid Y \text{ finitely generated and } X \subseteq Y \}.$$

may be proper.

For example, if $A$ is representation-finite, and $X$ is any finitely generated module, then $\text{del } X$ is, of course, finite. On the other hand, if $X$ is an injective module with infinite projective dimension, then $X \subseteq Y$ implies that also $\text{pd } Y$ is infinite (since $X$ is a direct summand of $Y$), thus the right hand side is $\infty$. A typical example of a representation finite algebra with an injective module with infinite projective dimension if the radical-square-zero algebra $A$ whose quiver has two vertices, say 1 and 2, and two arrows: a loop at 1 and an arrow $2 \to 1$. The simple module $S(2)$ is injective and has infinite projective dimension (since $\Omega S(2) = S(1)$ and $\Omega S(1) = S(1)$), Also, there are just five isomorphism classes of indecomposable modules.

2.3'. Dual assertion.

Proposition. If $X$ is a factor module of a finitely generated module $Y$. Then

$$\text{des } X \leq \text{id } Y.$$  

Proof. Let $d = \text{id } Y < \infty$. Let $U$ be a submodule of $Y$ with $Y/U = X$. Using the horseshoe lemma, the exact sequence $0 \to U \to Y \to X \to 0$ yields an an exact sequence
\[ 0 \to \Sigma^d U \to I' \oplus \Sigma^d Y \to \Sigma^d X \to 0, \] where \( I' \) is injective. Since \( \Sigma^d Y \) is injective, the middle term \( E = I' \oplus \Sigma^d Y \) is injective, thus the injective envelope \( I(\Sigma^d U) \) of \( \Sigma^d U \) is a direct summand of \( E \). Therefore \( \Sigma^d X = I'' \oplus \Sigma^{d+1} U \) for some injective module \( I'' \). This shows that \( \text{des } X \leq d \). \( \square \)

### 2.4. Theorem

Assume that every simple module \( S \) is a submodule of a finitely generated module \( M_S \) of finite projective dimension. Let \( d = \max_S \text{pd } M_S \). Then

\[ \text{Fin-inj } A \leq \text{del } A \leq d \leq \text{fin-pro } A. \]

**Proof.** The first inequality is Corollary 2.2. According to Proposition 2.3, we have \( \text{del } S \leq \text{pd } M_S \), thus \( \text{del } A = \max_S \text{del } S \leq \max \text{pd } M_S = d \). Finally, we have of course \( \text{pd } M_S \leq \text{fin-pro } A \), thus \( d = \max \text{pd } M_S \leq \text{fin-pro } A \). \( \square \)

Recall that an algebra is called left Kasch provided any simple module occurs as a left ideal. Thus, \( A \) is left Kasch if and only if any simple module is a submodule of a module of projective dimension \( d = 0 \). The case \( d = 0 \) of 2.4 is therefore just the well-known assertion that \( \text{Fin-inj } A = 0 \) for any Kasch algebra \( A \).

### 2.4’. Theorem

Assume that every simple module \( S \) is a factor module of a finitely generated module \( N_S \) of finite injective dimension. Let \( d' = \max_S \text{id } N_S \). Then

\[ \text{Fin-pro } A \leq \text{des } A \leq d' \leq \text{fin-inj } A. \]

**□**

### 2.5. Combination

Assume that every simple module \( S \) is a submodule of a finitely generated module \( M_S \) of finite projective dimension and also a factor module of a finitely generated module \( N_S \) of finite injective dimension. Let \( d = \max_S \text{pd } M_S \), and \( d' = \max_S \text{id } N_S \). Then

\[ \text{fin-pro } A = \text{fin-inj } A = \text{Fin-pro } A = \text{Fin-inj } A = \text{del } A = \text{des } A = d = d'. \]

**□**

### 3. Nakayama algebras

Let \( A \) be a cyclic Nakayama algebra (this means that \( A \) is connected and has no simple projective module, or, equivalently, that the quiver of \( A \) is a cycle). An indecomposable module \( M \) of finite projective dimension will be said to be even, or odd provided its projective dimension is even, or odd, respectively.

#### 3.1. The maps \( \psi \) and \( \gamma \)

We denote by \( \tau = D \text{Tr} \) the Auslander-Reiten translation, and write \( \tau^- = \text{Tr } D \). If \( S \) is a simple module, let \( \psi S = \tau^- \text{top } IS \). There is the dual concept: If \( S \) is a simple module, let \( \gamma S = \tau \text{soc } PS \). [The functions \( \psi \) and \( \gamma \) are important tools for dealing with cyclic Nakayama algebras; their study goes back to Gustafson [Gu]. They were used in many papers. The appendix A of this paper will focus the attention to further properties of \( \psi \) and \( \gamma \). In particular, one can define the coresolution quiver (or \( \psi \)-quiver) of \( A \). The \( \psi \)-paths are just the paths in the coresolution quiver. Similarly, using \( \gamma \), we obtain the resolution quiver (or \( \gamma \)-quiver) of \( A \).]
Lemma. (a) A simple module $U$ is in the image of $\psi$ if and only if $\text{pd} U \geq 2$. A simple module $T$ is in the image of $\gamma$ if and only if $\text{id} T \geq 2$.

(b) If $M$ is an indecomposable module, and $m \in \mathbb{N}$. Then either $\text{pd} M < 2m$ or else $\text{top} \Omega^{2m} M = \gamma^m \text{top} M$. Similarly, either $\text{id} M < 2m$ or else $\text{soc} \Sigma^{2m} M = \psi^m \text{soc} M$.

Proof. (a) We show the second assertion (the first follows by duality).

Let $T$ be simple with $\text{id} T \geq 2$. Then $\Sigma T$ is not injective, thus we have a proper inclusion $\Sigma T \subset I \Sigma T$. We denote by $S$ the top of $I \Sigma T$. Now all the indecomposable modules $M$ which properly include $\Sigma T$ have to be projective. In particular, $I \Sigma T$ is projective and therefore $I \Sigma T = \text{PS}$. By definition, $\gamma S = \tau \text{soc} \text{PS} = \tau \text{soc} I \Sigma T = \tau \tau^- T = T$. This shows that $T$ is in the image of $\gamma$.

Conversely, let $S$ be any simple module and $T = \gamma S$. We want to show that $\Sigma T$ is not injective. Now $\Sigma T$ and $\text{PS}$ have the same socle, thus both are submodules of the projective module $I \Sigma S$. Since $I \Sigma S$ is serial, its submodules are pairwise comparable. If $PS \subsetneq \Sigma T$, then $IT$ has a submodule $X$ of length $|PS| + 1$ which maps onto $PS$. But this is impossible since $PS$ is projective. It follows that there is the proper inclusion $\Sigma T \subset PS$. This shows that $\Sigma T$ is not injective, thus $\text{id} T \geq 2$.

(b) See [R2], section 3, Corollary. \(\square\)

3.2. $\psi$-paths (and $\gamma$-paths). A $\psi$-path of cardinality $m$ is a sequence $(S_1, S_2, \ldots, S_m)$ of simple modules with $S_{i+1} = \psi S_i$ for $1 \leq i < m$, it starts in $S_1$ and ends in $S_m$.

If $S$ is a simple module, let $a(S)$ be the supremum of the cardinality of the $\psi$-paths ending in $S$. We say that a simple module $S$ is $\psi$-cyclic provided $a(S) = \infty$, or, equivalently, provided there is some number $e \geq 1$ with $\psi^e S = S$. If $\psi(S) = T$, we say that $S$ is a $\psi$-predecessor of $T$. Thus, $a(T)$ is the maximum of $1 + a(S)$ with $S$ a predecessor of $T$. Finally, let $a(A)$ be the maximum of $a(S)$, where $S$ is simple and not $\psi$-cyclic. Note that $a(A)$ is also the maximum of $a'(S)$, where $S$ is simple and not $\gamma$-cyclic, see A.1 in the Appendix.

If $a(A) = 0$, then all simple modules are torsionless, thus all modules are torsionless, thus $A$ is self-injective. For the topics discussed in the paper, we usually may assume that $A$ is not self-injective.

There are the dual concepts: If $S$ is a simple module, let $\gamma S = \tau \text{soc} \text{PS}$. A $\gamma$-path of cardinality $m$ is a sequence $(S_1, S_2, \ldots, S_m)$ of simple modules with $S_{i+1} = \gamma S_i$ for $1 \leq i < m$, it starts in $S_1$ and ends in $S_m$. If $S$ is a simple module, let $a'(S)$ be the supremum of the cardinality of the $\gamma$-paths ending in $S$. We say that a simple module $S$ is $\gamma$-cyclic provided $a'(S) = \infty$, or, equivalently, provided there is some number $e \geq 1$ with $\gamma^e S = S$.

3.3. Cyclic Nakayama algebras have been investigated quite thoroughly by Madsen [M]. We will use many of his results, in particular:

Lemma (Madsen). Let $A$ be a cyclic Nakayama algebra and $M$ an indecomposable module.

(a) If $X$ is a subfactor of $M$, and $M$ is odd, then $X$ is odd and $\text{pd} X \leq \text{pd} M$.

(b) If $X$ is a subfactor of $M$, and $X$ is even, then $M$ is even and $\text{pd} M \leq \text{pd} X$. 
(c) A simple module $S'$ is a composition factor of $\Omega^2 S$ if and only if $S'$ is a $\psi$-predecessor of $S$ (and then the multiplicity of $S'$ in $\Omega^2 S$ is 1).

Proof. For (a) and (b), see [M], 2.2. For (c), see [M] 3.1. □

3.4. Maximum property of odd modules. An indecomposable module $M$ is odd if and only if all composition factors of $M$ are odd; and then $\text{pd} M$ is the maximum of $\text{pd} S$, where $S$ is a composition factor of $M$.

Proof. Let $M$ be indecomposable. First, assume that $M$ is odd. According to 3.3 (a), all composition factors $S$ of $M$ are odd and $\text{pd} S \leq \text{pd} M$. Of course, at least one of the composition factors must have $\text{pd} S = \text{pd} M$ (since the class of modules of projective dimension smaller $\text{pd} M$ is closed under extensions). Thus $\text{pd} M$ is odd. Conversely, assume that all composition factors of $M$ are odd. According to [M] Proposition 4.1, it follows that $\text{pd} M$ is the maximum of $\text{pd} S$, where $S$ is a composition factor of $M$. □

3.5. Proposition. Let $A$ be a cyclic Nakayama algebra, and $S$ a simple module.

(1) We have $a(S)$ finite if and only if $\text{pd} S$ is odd, and then $\text{pd} S = 2a(S) - 1$, in particular, $\text{pd} S < 2a(S)$. If $a(S)$ is infinite, then $\text{id} IS$ is even and $\text{id} IS \leq 2a(A)$.

(2) Similarly, $a'(S)$ finite if and only if $\text{id} S$ is odd, and then $\text{id} S = 2a'(S) - 1$, in particular, $\text{id} S < 2a(S)$. If $a'(S)$ is infinite, then $\text{id} PS$ is even and $\text{id} PS \leq 2a(A)$.

Proof. First we consider the case that $a(S)$ is finite. We show that $\text{pd} S = 2a(S) - 1$. It is obvious that $\text{pd} S = 1$ iff $\text{rad} PS$ is projective iff $S$ has no $\psi$-predecessor. Since $S$ is not projective, we now can assume that $\text{pd} S \geq 2$, thus $\Omega^2 S \neq 0$. According to 3.3 (c), the composition factors $S'$ of $\Omega^2 S$ are the $\psi$-predecessors $S'$ of $S$. Let $S'$ be a predecessor of $S$ such that $b = a(S')$ is maximal. Then, by definition of $a(S)$, we have $a(S) = 1 + b$. According to the Lemma above, we have $\text{pd} \Omega^2 S = 2b - 1$. Thus we get $\text{pd} S = 2 + \text{pd} \Omega^2 S = 2b + 1 = 2(a(S) - 1) + 1 = 2a(S) - 1$.

Conversely, let us show: If $\text{pd} S$ is odd, then $a(S)$ is finite. The proof is by induction. If $\text{pd} S = 1$, then $S$ cannot have a $\psi$-predecessor, thus $a(S) = 1$. If $\text{pd} S$ is odd and not 1, then $\Omega^2 S$ is odd. The composition factors $S'$ of $\Omega^2 S$ are the $\psi$-predecessors of $S$, see 3.3 (c), and $\text{pd} S'$ is an odd number with $\text{pd} S' \leq \text{pd} \Omega^2 S = \text{pd} S - 2$, according to 3.4. By induction, $a(S')$ is finite. Since $a(S')$ is finite for all $\psi$-predecessors $S'$ of $S$. This implies that also $a(S)$ is finite.
Now assume that \(a(S)\) is infinite. If \(IS\) is projective, then its projective dimension is zero, therefore even. Thus, we suppose that \(IS\) is not projective and show that \(\Omega IS\) has odd projective dimension.

On the left and on the right of \(S\) (at the lower boundary of he shaded areas) are the \(\psi\)-predecessors \(S'\) of \(S\) which are different from \(S\). For these modules \(S'\), we have \(a(S') < \infty\), thus, as we have seen already, they have odd projective dimension. The module \(\Omega IS\) has a filtration using such modules, thus 3.4 asserts that \(\text{pd} \Omega IS\) is odd, therefore \(\text{pd} IS\) is even. Since \(\text{pd} \Omega IS \leq 2a(A) - 1\), we have \(\text{pd} IS \leq 2a(A)\). This completes the proof of (1).

The assertions (2) follow by duality.

3.6. Summary.

(1) Let \(A\) be a Nakayama algebra. Any simple module is a submodule of an indecomposable module with finite projective dimension, and a factor module of an indecomposable module with finite injective dimension.

Proof. This is clear if the algebra is of finite global dimension. Thus, we can assume that \(A\) is a cyclic Nakayama algebra. Let \(S\) be a simple module. According to Proposition 3.4, \(S\) or \(IS\) has finite projective dimension, thus \(S\) is always a submodule of an indecomposable module with finite projective dimension. Since also \(A^\text{op}\) is a cyclic Nakayama algebra, there is the dual statement: Any simple module is a factor module of an indecomposable module with finite injective dimension.

(2) Let \(A\) be a Nakayama algebra. If \(S\) is simple, let \(M_S = IS\) provided \(S\) is \(\psi\)-cyclic, otherwise \(M_S = S\). Let \(N_S = PS\) provided \(S\) is \(\gamma\)-cyclic, otherwise \(N_S = S\). Let \(d = \max_S \text{pd} M_S\) and \(d' = \max_S \text{id} N_S\). Then

\[
\text{fin-pro} A = \text{fin-inj} A = \text{del} A = \text{des} A = d = d'.
\]

Proof. See 3.4 and 2.5.

3.7. Corollary. Always

\[2a(A) - 1 \leq \text{fin-pro} A \leq 2a(A).
\]

Proof. First, let us show that \(\text{fin-pro} A \leq 2a(A)\). Since \(d = \text{fin-pro} A\), we have \(\text{fin-pro} A = \text{pd} S\) for some simple module \(S\) which is not \(\psi\)-cyclic, or \(\text{fin-pro} A = \text{pd} IS\) for some \(\psi\)-cyclic module \(S\). Now, if \(S\) is not \(\psi\)-cyclic, then \(\text{pd} S = 2a(S) - 1 \leq a(A) - 1\). If
$S$ is $\psi$-cyclic and not projective, then $\text{pd } IS = \text{pd } \Omega IS + 1$, thus $\Omega IS$ is an odd module. According to the maximum principle, $\Omega IS$ is the maximum of $\text{pd } T$, where $T$ is a composition factor of $\Omega IS$, and $\text{pd } T \leq 2a(T) - 1 \leq 2a(A) - 1$, thus also $\text{pd } \Omega IS \leq 2a(A) - 1$. This shows that $\text{fin-pro } A \leq 2a(A)$.

In order to show that $2a(A) - 1 \leq \text{fin-pro } A$, we can assume that $a(A) \geq 1$. Since $a(A) \geq 1$, there is a simple module $S$, not $\psi$-cyclic, with $a(S) = a(A)$. Then $\text{pd } S = 2a(A) - 1$ and $\text{pd } S \leq \text{fin-pro } A$.

\[\square\]

3.8. Examples. As we have seen in 3.3 (a) and (b)): if $IS$ is odd, then $S$ is odd and $\text{pd } S \leq \text{pd } IS$; if $S$ is even, then $IS$ is even and $\text{pd } S \geq \text{pd } IS$. The proposition 3.4 shows in addition that we cannot have that both $S$ and $IS$ have infinite projective dimension. Note that all the remaining possibilities do occur, as the examples below show:

- $\text{pd } S_6 = \text{pd } IS_6$, both odd;
- $\text{pd } S_7 < \text{pd } IS_7$, both odd;
- $\text{pd } S_3$ odd, $\text{pd } IS_3 = \infty$;
- $\text{pd } S_1 > \text{pd } IS_1$, with $\text{pd } S_1$ odd, $\text{pd } IS_1$ even;
- $\text{pd } S_8 < \text{pd } IS_8$, with $\text{pd } S_x$ odd, $\text{pd } IS_x$ even;
- $\text{pd } S_4 = \infty$, $\text{pd } IS_4$ even;
- $\text{pd } S_2 = \text{pd } IS_2$, both even;
- $\text{pd } S_5 > \text{pd } IS_5$, both even.

Here are the examples:

The pictures exhibit Auslander-Reiten quivers (the Auslander-Reiten quiver of a cyclic Nakayama algebra lives on a cylinder — the dashed lines left and right have to be identified). A vertex of an Auslander-Reiten quiver is the isomorphism class of an indecomposable module $M$. Instead of drawing the vertex, we insert here the corresponding value $\text{pd } M$ (a natural number or $\infty$). The lower boundary of the Auslander-Reiten quiver of a Nakayama algebra consists of the simple modules: those which are of interest here, are labeled $S_1, \ldots, S_8$. The $\psi$-cyclic simple modules are encircled.

In particular, we see the following: Proposition 3.4 asserts that if $a(S)$ is infinite, then $\text{pd } IS$ is even, but the converse does not hold: $\text{pd } IS$ may be even, whereas $a(S)$ is finite, as the examples $S_1$ and $S_8$ show.
3.9. If $S$ is simple, then

$$\text{del } S \leq \min\{\text{pd } Y \mid Y \text{ indecomposable and } S \subseteq Y\} = \min\{\text{pd } S, \text{pd } IS\}$$

If $S$ is $\psi$-cyclic, then $\text{pd } S \geq \text{pd } IS$, thus $\text{del } S \leq \text{pd } IS$.

Proof. The inequality $\text{del } S \leq \min\{\text{pd } Y \mid Y \text{ indecomposable and } S \subseteq Y\}$ is due to 2.3. Let us consider the indecomposable modules $Y$ with $S \subseteq Y$; they are the non-zero submodules of $IS$. Thus, let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = IS$$

be a composition series of $IS$. Then there are indices $0 \leq u \leq v \leq m$ such that the modules $M_i$ with $1 \leq i \leq u$ are odd, those with $u + 1 \leq i \leq v$ have infinite projective dimension, and those with $v + 1 \leq i$ are even. According to 3.3 (a), $\text{pd } M_i \leq \text{pd } M_j$ for $1 \leq i \leq j \leq u$. According to 3.3 (b), $\text{pd } M_i \geq \text{pd } M_j$ for $v + 1 \leq i \leq j$. This shows that $\min_{1 \leq i \leq v} M_i = \min\{M_1, M_m\} = \min\{S, IS\}$.

If $S$ is $\psi$-cyclic, then $\text{pd } S \geq \text{pd } IS$. Namely, $\text{pd } S$ is either even or infinite. If $\text{pd } S$ is even, then we use 3.3 (b).

As we have mentioned, if $S$ is $\psi$-cyclic, then $\text{pd } S \geq \text{pd } IS$. But $\text{pd } S \geq \text{pd } IS$ (and thus $\text{pd } S > \text{pd } IS$ may also happen for $S$ odd. It happens, of course, in case $S$ is torsionless, since then $\text{pd } S > 0$ and $\text{pd } IS = 0$. Here is an example of $S$, with $\text{pd } S = 3$, $\text{pd } IS = 2$, thus $S$ is odd, not torsionless, and $\text{pd } S > \text{pd } IS$.

We do not know whether the inequality $\text{del } S \leq \min\{\text{pd } S, \text{pd } IS\}$ can be proper. But there is always a simple module $S$ with $\text{del } S = \text{fin-pro } A$ and if $\text{del } S = \text{fin-pro } A$, then $\text{del } S = \min\{\text{pd } S, \text{pd } IS\}$.

Proof. The first assertion follows directly from $\max_S \text{del } S = \text{fin-pro } A$. If $\text{del } S = \text{fin-pro } A$, then the inequalities

$$\text{del } S \leq \min\{\text{pd } S, \text{pd } IS\} \leq \text{fin-pro } A$$

yield the equality $\text{del } S = \min\{\text{pd } S, \text{pd } IS\}$. □

3.10. Historical Remarks. The essential properties of a Nakayama algebra which we use in order to apply the general theory are collected in Proposition 3.4. One should be aware that both assertions of 3.4 can be found (at least implicitly) in the literature. The odd modules have been studied quite carefully by Madsen. For the calculation of $\text{pd } IS$, where $S$ is a $\psi$-cyclic simple module, we should refer to Shen, in particular Lemma 3.5 (2) of [Sh3].

Actually, Shen [Sh3] provides a comprehensive study of the indecomposable injective modules with finite projective dimension in order to characterize the Nakayama algebras
which are Gorenstein (a Nakayama algebra is Gorenstein if and only if all injective modules have finite projective dimension if and only if all projective modules have finite injective dimension). It follows that a Nakayama algebra which is Gorenstein and has infinite global dimension always has even finitistic dimension. This has also been shown by Sen [Se], Corollary 4.13.

4. Modules of maximal finite projective or injective dimension.

Let \( A \) be a cyclic Nakayama algebra and \( a = a(A) \). If the finitistic dimension \( \text{fin-pro} A \) of \( A \) is even, then section 3 shows that \( \text{fin-pro} A = 2a \) and this happens if and only if one of the following equivalent conditions is satisfied:

(a) There is an indecomposable injective module \( I \) with \( \psi \)-cyclic socle such that \( \text{pd} I = 2a \).
(b) There is an indecomposable projective module \( P \) with \( \gamma \)-cyclic top such that \( \text{id} P = 2a \).

In this section we want to look more carefully at the relationship between (a) and (b).

4.1. Theorem. Let \( A \) be a cyclic Nakayama algebra, let \( a = a(A) \). Then \( \Omega^{2a} \) provides a bijection between the indecomposable injective modules \( I \) with \( \psi \)-cyclic socle such that \( \text{pd} I = 2a \), and the indecomposable projective modules \( P \) with \( \gamma \)-cyclic top such that \( \text{id} P = 2a \). The inverse of this bijection is \( \Sigma^{2a} \).

We should recall that a simple module is \( \psi \)-cyclic if and only if its projective dimension is not odd, and \( \gamma \)-cyclic if and only if its injective dimension is not odd. Thus, we deal with a bijection between sets of simple modules defined by specifying the projective or injective dimension of the simple modules, their projective covers and their injective envelopes.

The proof of Proposition will be given in 4.4.

Let us illustrate the Proposition by two examples.

Example 1, with \( a = 2 \).

\[
\begin{array}{c}
\begin{array}{cccc}
0 & 0 & T & IT \\
3 & 5 & 4 & 3
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{cccc}
0 & 0 & S & PS \\
4 & 1 & 3 & 2
\end{array}
\end{array}
\]

mod \( A \) with pd \( M \)

mod \( A \) with id \( 5 \)

(On the left, the \( \psi \)-cyclic simple modules have been encircled; on the right the \( \gamma \)-cyclic simple modules.)

The simple module \( T = 5 \) is the only simple module with \( \text{pd} IT = 4 \) and \( 5 \) is \( \psi \)-cyclic. There are two simple modules \( S \) with \( \text{id} PS = 4 \), namely \( S = 4 \) and \( S = 5 \), but only \( S = 5 \) is \( \gamma \)-cyclic. thus the proposition deals with the bijection \( \gamma^2 : \{5\} \rightarrow \{5\} \) and the non-vanishing of \( \text{Ext}^4(I5, P5) \).

Here is the minimal projective resolution \( P_\bullet(I5) \) of \( I5 \), which is also a minimal injective coresolution \( I_\bullet(P5) \) of \( P5 \):

\[
P_\bullet(I5) = I_\bullet(P5) : \quad 0 \rightarrow P5 \rightarrow P3 \rightarrow P2 \rightarrow P1 \rightarrow P3 \rightarrow I5 \rightarrow 0.
\]
Example 2, with $a = 3$.

\[
\begin{array}{c}
\begin{array}{c}
\text{mod } A \text{ with } \text{pd } M \\
\begin{array}{cccccccc}
8 & 6 & 5 & 1 & 2 & 3 & 4 & 5 \\
0 & 6 & 0 & 2 & 3 & 0 & 2 & 0
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{mod } A \text{ with } \text{id } M \\
\begin{array}{cccccccc}
8 & 6 & 5 & 1 & 2 & 3 & 4 & 5 \\
0 & 6 & 0 & 2 & 3 & 0 & 2 & 0
\end{array}
\end{array}
\end{array}
\end{array}
\]

The simple module $T = 8$ is $\psi$-cyclic and $\text{pd } IT = 6$, the same module $S = 8$ is also $\gamma$-cyclic with $\text{id } PS = 6$. The Proposition highlights the bijection $\gamma^3 : \{8\} \to \{8\}$ and the non-vanishing of $\text{Ext}^6(I S, P S)$; if $P_\bullet(I S)$ is a minimal projective resolution of $I S$, then $P_0(I S) = P S$, and if $I_\bullet(P S)$ is a minimal injective coresolution of $P S$, then $I_0(I S) = I S$. Here are $P_\bullet(I S)$ and $I_\bullet(P S)$ (for better comparison, we write $P T$ instead of $I$ if $I$ is a projective-injective module with top $T$):

\[
P_\bullet(I S) : \quad 0 \to P S \to P 3 \to P 5 \to P 7 \to P 1 \to P 7 \to I S \to 0,
\]

\[
I_\bullet(P S) : \quad 0 \to P S \to P 4 \to P 3 \to I 3 \to P 1 \to P 7 \to I S \to 0.
\]

4.2. Lemma. If $M$ is indecomposable and $\text{pd } M = 2t$, then $\Omega^{2t} M = P(\gamma^t \text{top } M)$.

Proof. If $X$ is indecomposable and not projective, then also $\Omega X$ is indecomposable. Thus, if $M$ indecomposable and $\text{pd } M = 2s$, then $\Omega^{2s} M$ is indecomposable and projective, thus equal to $P(S)$ where $S = \text{top } \Omega^{2s} M$. Now we use 3.2 (b). \qed

4.3. Lemma. Let $A$ be a cyclic Nakayama algebra with $a = a(A)$. Let $T$ be a $\psi$-cyclic simple module with $\text{pd } IT = 2a$. Let $S = \gamma^a \text{top } IT$. Then $S$ is $\gamma$-cyclic simple, $PS = \Omega^{2a} IT$ and $\text{id } PS = 2a$.

Proof. Let $S = \gamma^a \text{top } IT$. Then $S$ is, of course, $\gamma$-cyclic, since it is in the image of $\gamma^0$. If $\text{pd } IT = 2a$, then Lemma 4.2 (b) asserts that $\Omega^{2a} IT = PS$. In particular, $\text{Ext}^{2a}(IT, PS) \neq 0$, thus $\text{id } PS \geq 2a$. On the other hand, the dual assertion of 3.4 asserts that $\text{id } PS \leq 2a$, since $S$ is $\gamma$-cyclic. Thus, we see that $\text{id } PS = 2a$. \qed

4.4. Proof of Proposition 4.1. Let $A$ be a cyclic Nakayama algebra, let $a = a(A)$.

Let $T$ be a $\psi$-cyclic simple module $T$ with $\text{pd } IT = 2a$. According to Lemma 4.3, $S = \gamma^a \text{top } IT$ is $\gamma$-cyclic simple, $PS = \Omega^{2a} IT$ and $\text{id } PS = 2a$. Thus, $S = \text{top } PS = \text{top } \Omega^{2a} IT$. This shows that $\text{top } \Omega^{2a} IT = S = \gamma^a \text{top } IT$. Since $\psi T = \tau^{-1} \text{top } IT$, we see that $\text{top } IT = \tau \psi T$, thus $S = \gamma^{a} \tau \psi T$. Altogether we see: we have $S = \text{top } \Omega^{2a} IT = \gamma^{a} \tau \psi T$ and $S$ is $\gamma$-cyclic with $\text{id } PS = 2a$.

By duality, starting with a $\gamma$-cyclic module $S$ with $\text{id } PS = 2a$, there is the module $T = \text{soc } \Sigma^{2a} PS = \psi^{a} \tau^{-1} \psi S$ and $T$ is $\psi$-cyclic and $\text{pd } T = 2a$.

It remains to be seen that these constructions are mutually inverse. In order to see this, we only have to show that given non-isomorphic $\psi$-cyclic simple modules $T, T'$ with $\text{pd } IT = 2a = \text{pd } IT'$, the modules $\gamma^a \tau \psi T$ and $\gamma^a \tau \psi T'$ are non-isomorphic.

With $T$ also $\psi T$ is $\psi$-cyclic. According to the Corollary in Appendix A.2, the restriction of $\gamma^a \tau$ to the set of $\psi$-cyclic modules is injective. Since $\psi$ is a permutation of the $\psi$-cyclic
modules, the restriction of $\gamma^a \tau \psi$ to the set of $\psi$-cyclic modules is injective. This completes the proof.

Appendix A. The functions $\psi$ and $\gamma$.

A.1. Proposition. Let $A$ be a cyclic Nakayama algebra. For all $t \geq 0$, we have

$$\psi^t \gamma^t \psi^t = \psi^t \quad \text{and} \quad \gamma^t \psi^t \gamma^t = \gamma^t.$$ 

The proof will be given in A.4, using some basic properties of monotone endofunctions of $\mathbb{Z}$, see A.3.

Corollary 1. Let $A$ be a cyclic Nakayama algebra. For all $t \geq 0$, the map

$$\gamma^t : \text{Im} \psi^t \to \text{Im} \gamma^t$$

is a bijection with inverse $\psi^t$. In particular, we have $|\text{Im} \psi^t| = |\text{Im} \gamma^t|$. □

Recall that $a(A)$ is the smallest natural number $a$ with $\text{Im} \psi^a = \text{Im} \psi^{a+1}$. Using duality, the smallest natural number $b$ with $\text{Im} \gamma^b = \text{Im} \gamma^{b+1}$ is $a(A^{\text{op}})$.

We denote by $c(A)$ the number of $\psi$-cyclic modules. Thus, using duality, $c(A^{\text{op}})$ is the number of $\gamma$-cyclic $A$-modules.

Corollary 2. Let $A$ be a cyclic Nakayama algebra. Then

$$a(A) = a(A^{\text{op}}) \quad \text{and} \quad c(A) = c(A^{\text{op}}).$$

Proof. We have $\text{Im} \psi^0 \supseteq \text{Im} \psi^1 \supseteq \text{Im} \psi^2 \supseteq \cdots$. If $S$ is a simple module, then, by definition, $a(S) = t$ provided $S$ belongs to $\text{Im} \psi^{t-1} \setminus \text{Im} \psi^t$, and $a(S) = \infty$ provided $S$ belongs to $\bigcap_t \text{Im} \psi^t$.

Similarly, $\text{Im} \gamma^0 \supseteq \text{Im} \gamma^1 \supseteq \text{Im} \gamma^2 \supseteq \cdots$. If $S$ is a simple module, let $a'(S) = t$ provided $S$ belongs to $\text{Im} \gamma^{t-1} \setminus \text{Im} \gamma^t$, and let $a'(S) = \infty$ provided $S$ belongs to $\bigcap_t \text{Im} \gamma^t$. The simple module $S$ is $\gamma$-cyclic, provided $S$ belongs to $\text{Im} \gamma^t$ for all $t$, thus provided $a'(S) = \infty$. Using duality, we see that $a(A^{\text{op}})$ is the maximum of the numbers $a'(S)$, where $S$ is simple and not $\gamma$-cyclic.

Since $\gamma^t$ provides a bijection between $\text{Im} \psi^t$ and $\text{Im} \gamma^t$, it provides a bijection from the set of isomorphism classes $[S]$ of simple modules with $a(S) \geq t$ onto the set of isomorphism classes $[S]$ of simple modules with $a'(S) \geq t$. It follows that $a(A) = a(A^{\text{op}})$.

For $t \geq a(A)$, the map $\gamma^t$ provides a bijection from the set of $\psi$-cyclic simple modules onto the set of $\gamma$-cyclic modules. This shows that $c(A) = c(A^{\text{op}})$. □

A.2. Proposition. Let $A$ be a cyclic Nakayama algebra. For all $t \geq 0$, we have

$$\psi^t \tau - \gamma^t \tau \psi^t = \psi^t \quad \text{and} \quad \gamma^t \tau \psi^t \tau - \gamma^t = \gamma^t.$$ 

The proof will be similar to the proof of Proposition A.1, see A.5. As in A.1, there is the following Corollary.
**Corollary.** Let \( A \) be a cyclic Nakayama algebra. For all \( t \geq 0 \), the map
\[
\gamma^t \tau: \text{Im} \psi^t \to \text{Im} \gamma^t
\]
is a bijection with inverse \( \psi^t \tau^- \).

Proof. The image of \( \gamma_t \tau \) is contained in \( \text{Im} \gamma_t \), thus \( \gamma^t \tau \) maps \( \text{Im} \psi^t \) into \( \text{Im} \gamma^t \). Similarly, \( \psi^t \tau^- \) maps \( \text{Im} \gamma^t \) into \( \text{Im} \psi^t \). The equalities mentioned in A.2 yield the assertion.

A.3. A (set-theoretical) function \( f: \mathbb{Z} \to \mathbb{Z} \) is said to be **monotone** provided \( i > j \) in \( \mathbb{Z} \) implies \( f(i) \geq f(j) \).

**Lemma.** Let \( f, g: \mathbb{Z} \to \mathbb{Z} \) be monotone functions with \( fg(i) \geq i \geq gf(i) \) for all \( i \in \mathbb{Z} \). Then, for all \( t \geq 0 \), we have \( f^t g^t(i) \geq i \geq g^t f^t(i) \), and \( f^t g^t f^t = f^t \) and \( g^t f^t g^t = g^t \).

Proof. First, we show by induction on \( t \) that \( f^t g^t(i) \geq i \) for all \( i \in \mathbb{Z} \). The case \( t = 1 \) is one of the assumptions. Thus, assume that we know already for some \( t \) that \( f^t g^t(i) \geq i \) for all \( i \). Replacing \( i \) by \( g(i) \), we obtain \( f^t g^{t+1}(i) \geq g(i) \). Since \( f \) is monotone, we get \( f^{t+1} g^{t+1}(i) \geq fg(i) \). Altogether we see that \( f^{t+1} g^{t+1}(i) \geq fg(i) \geq i \). Similarly, we see that \( i \geq g^t f^t(i) \) for all \( i \), and all \( t \geq 0 \).

Applying \( f^t \) to \( i \geq g^t f^t(i) \), the monotony gives \( f^t(i) \geq f^t g^t f^t(i) \). On the other hand, we take \( f^t g^t(i) \geq i \) and replace \( i \) by \( f^t(i) \), this yields \( f^t g^t f^t(i) \geq f^t(i) \). Altogether we have \( f^t g^t f^t(i) \geq f^t(i) \geq f^t g^t f^t(i) \), thus \( f^t g^t f^t = f^t \). Similarly, we see that \( g^t f^t g^t = g^t \).

A.4. Now, let \( A \) be a cyclic Nakayama algebra. In order to use A.3, we need to work with the universal cover \( \tilde{A} \) of the algebra \( A \) and a covering functor \( \pi: \text{mod} \tilde{A} \to \text{mod} A \) (see [Ga] and related papers by Bongartz-Gabriel and Gordon-Green). The quiver of \( \tilde{A} \) is \( \mathbb{Z} \) (we consider the integers \( \mathbb{Z} \) as a quiver with vertex set \( \mathbb{Z} \) and with arrows \( i \to i+1 \) for all \( i \in \mathbb{Z} \)). We have to fix a simple module \( S = S(0) \), define \( S(i) = \tau^i S \) and use as covering map \( \pi: i \mapsto S(i) \). (Warning: this convention means that in our drawing of the Auslander-Reiten quiver of \( A \) (or better of \( \tilde{A} \)) the integers which are used in order to index the simple modules **increase** when going from right to left.)

Let \( \tilde{\psi}(i) = i - |I(i)| \) and \( \tilde{\gamma}(i) = i + |P(i)| \). Then \( \tilde{\psi} \) is a covering of \( \psi \), and \( \tilde{\gamma} \) is a covering of \( \gamma \) (this means: \( \pi \tilde{\psi} = \psi \pi \) and \( \pi \tilde{\gamma} = \gamma \pi \)). Similarly, let \( \tilde{\tau}(i) = i + 1 \); thus \( \tilde{\tau} \) is a covering of \( \tau \).

(1) The functions \( \tilde{\psi} \) and \( \tilde{\gamma} \) are monotone.

Proof. Let \( i > j \) in \( \mathbb{Z} \). If \( i - j \geq |I(i)| \), then \( \psi(i) = i - |I(i)| \geq j > j - |I(j)| = \psi(j) \). Thus, we can assume that \( i - j < |I(i)| \). Since \( 0 < i - j < |I(i)| \), there is a submodule \( U \) of \( I(i) \) of length \( i - j \) and \( I(i)/U \) has socle \( j \).
Since $I(i)/U$ is a module with socle $j$, it is a submodule of $I(j)$, thus $|I(j)| \geq |I(i)/U| = |I(i)| - |U| = |I(i)| - i + j$. This shows that $\psi(i) = i - |I(i)| \geq j - |I(j)| = \psi(j)$. This shows that $\psi$ is monotone.

In the same way, or using duality, one sees that $\widetilde{\gamma}$ is monotone. \hfill $\square$

(2) If $i \in \mathbb{Z}$, then $\widetilde{\gamma} \geq i \geq \widetilde{\gamma} \psi i$.

Proof. We show that $|I(i)| \geq |P(\widetilde{\psi}i)|$. Assume, for the contrary, that $|I(i)| < |P(\widetilde{\psi}i)|$. Then $P(\widetilde{\psi}i)$ has a submodule $V$ of length $|I(i)| + 1$. Note that the socle of $V$ has to be $i$.

This implies that $I(i)$ is a submodule of $V$, and, of course, a proper submodule. This is impossible, since $I(i)$ is injective.

Since $|I(i)| \geq |P(\widetilde{\psi}i)|$, we see that $i \geq i - |I(i)| + |P(\widetilde{\psi}i)| = \widetilde{\psi}(i) - |P(\widetilde{\psi}i)| = \widetilde{\gamma} \widetilde{\psi}(i)$. In the same way, or using duality, one sees that $\widetilde{\gamma} \psi i \geq i$. \hfill $\square$

Proof of Proposition A.1. The assertions (1) and (2) show that we can apply Lemma A.3 to the functions $f = \widetilde{\psi}$ and $g = \widetilde{\gamma}$. We get $\psi^t \widetilde{\gamma^t} \psi^t = \psi^t$ as well as $\widetilde{\gamma^t} \psi^t \widetilde{\gamma^t} = \widetilde{\gamma^t}$, for all $t \geq 0$. But $\widetilde{\gamma^t} \psi^t \psi^t = \widetilde{\gamma^t}$ implies that $\psi^t \gamma^t \psi^t = \psi^t$. Similarly, $\widetilde{\gamma^t} \psi^t \widetilde{\gamma^t} = \widetilde{\gamma^t}$ implies that $\gamma^t \psi^t \gamma^t = \gamma^t$. \hfill $\square$

A.5. As a second application of A.3, we show Proposition A.2.

(1) The function $\widetilde{\gamma} \psi \widetilde{\gamma}$ is monotone.

Proof. This follows directly from A.2, Lemma 1 and the fact that $\widetilde{\gamma}$ and $\widetilde{\gamma}^-$ are, of course, monotone. \hfill $\square$

(2) If $i \in \mathbb{Z}$, then $\widetilde{\gamma} \psi \widetilde{\gamma} \psi(i) \geq i \geq \widetilde{\psi} \widetilde{\gamma} \psi \widetilde{\gamma}(i)$.

Proof. First, let us show that $\widetilde{\gamma} \psi \widetilde{\gamma} \psi(i) \geq i$. Let $j = \widetilde{\gamma} \psi(i)$. Always, $I(i)$ is a factor module of $P(j)$ and $x = \widetilde{\gamma} \psi(j)$ is the socle of $P(j)$. Thus $x \geq i$. 

\begin{center}
\begin{tikzpicture}
\node (I) at (0,0) [circle,fill,inner sep=2pt] {}; \node (Ii) at (-1,-1) [circle,fill,inner sep=2pt] {}; \node (Ii) at (1,-1) [circle,fill,inner sep=2pt] {};
\node (Pi) at (0,-2) [circle,fill,inner sep=2pt] {};
\node (V) at (0,-3) [circle,fill,inner sep=2pt] {};
\node (Z) at (2,-4) [circle,fill,inner sep=2pt] {};
\node (Z) at (-2,-4) [circle,fill,inner sep=2pt] {};
\node (Pi) at (0,-2) [circle,fill,inner sep=2pt] {};
\node (Pi) at (0,-2) [circle,fill,inner sep=2pt] {};
\draw[-stealth] (I) -- (V);
\draw[-stealth] (V) -- (Z);
\draw[-stealth] (I) -- (Z);
\draw[-stealth] (Ii) -- (Ii);
\draw[-stealth] (Ii) -- (Pi);
\draw[-stealth] (Pi) -- (V);
\end{tikzpicture}
\end{center}
Second, we show that $i \geq \widetilde{\psi}(i)$. We start with $i$ and $\tau(i) = i + 1$ and let $j = \widetilde{\psi}(i + 1)$. Note that $j = \text{soc} P(i+1)$. It follows that $P(i+1) \subseteq I(j)$. If $P(i+1) = I(j)$, then $i = \widetilde{\psi}(j)$. Otherwise, $P(i+1)$ is a proper submodule of $I(j)$ and then $i < \widetilde{\psi}(j)$.

\begin{center}
\begin{tikzpicture}
    \node (i) at (0,0) {$i$};
    \node (i1) at (1,0) {$i+1$};
    \node (j) at (-1,-1) {$j$};
    \node (z) at (2,-1) {$\mathbb{Z}$};
    \node (i2) at (3,0) {$\widetilde{\psi}(j)$};

    \draw (i) -- (i1);
    \draw (i) -- (j);
    \draw (i) -- (i2);
    \draw (i1) -- (j);
    \draw (i1) -- (i2);
    \draw (j) -- (i2);

    \node (p) at (1,-2) {$P(i+1)$};
    \node (i3) at (1,-3) {$I(j)$};
    \draw (i2) -- (p);
    \draw (p) -- (i3);

    \draw (j) -- (p);
    \draw (i1) -- (p);
    \draw (i) -- (p);
    \draw (i2) -- (i1);
    \draw (i2) -- (i);
    \draw (i2) -- (j);

    \end{tikzpicture}
\end{center}

□

Proof of Proposition A.2. The functions $f = \tau - \gamma\tau$ and $g = \psi$ are monotone, according to A.5 (1) and A.4 (1), respectively. According to A.5 (2), we see that the functions $f$ and $g$ satisfy the conditions of Lemma A.3. Thus, for all $t \geq 0$, we have $\psi^t\tau - \gamma^t\psi^t = \psi^t(\tau - \gamma\tau)^t\psi^t = \psi^t$ and $\tau - \gamma^t\tau\psi^t\tau - \gamma^t\tau = (\tau - \gamma\tau)^t\psi^t(\tau - \gamma\tau)^t = (\tau - \gamma\tau)^t = \tau - \gamma^t\tau$. Multiplying the latter equality from the left by $\tau$, from the right by $\tau^t$, we get $\gamma^t\tau\psi^t\tau - \gamma^t = \gamma^t$, as required.

A.6. The case $t = 1$. Note that $\text{Im} \psi$ are just the simple modules with projective dimension different from 1. Similarly, $\text{Im} \gamma$ are just the simple modules with injective dimension different from 1. The bijection between $\text{Im} \psi = \{T \mid \text{pd} T \neq 1\}$ and $\text{Im} \gamma = \{S \mid \text{id} S \neq 1\}$ given by $\psi S = T$ and $\gamma T = S$ can also be seen also as bijections of $\text{Im} \psi$ and $\text{Im} \gamma$ with the set of “valleys” of the “roof” of $A$ (see [R2]); such a “valley” is an indecomposable module which is both the radical of a projective module as well of the form $IS/S$, where $S$ is simple:

\begin{center}
\begin{tikzpicture}
    \node (s) at (0,0) {$S$};
    \node (t) at (2,0) {$T$};
    \node (i) at (1,2) {$IS$};
    \node (p) at (1,1) {$PT$};

    \draw (s) -- (i);
    \draw (t) -- (i);
    \draw (s) -- (p);
    \draw (t) -- (p);

    \node (g) at (0,-1) {$\gamma T$};
    \node (psi) at (2,-1) {$\psi S$};
    \draw (s) -- (g);
    \draw (t) -- (psi);

    \end{tikzpicture}
\end{center}

Of course, if $A$ is a cyclic Nakayama algebra, then the number of “valleys” is the same as the number of “peaks” in the roof (these are the indecomposable modules which are both projective and injective), and is also the minimal number of admissible relations which are needed to define $A$.

A.7. Historical remark. Some of the effects of the interrelation between $\psi$ and $\gamma$ were already discussed by Shen [Sh3].

The assertion that there are as many $\psi$-cyclic simple modules as there are $\gamma$-cyclic simple modules was shown by Shen in [Sh2].

A.8. The $\psi$-quiver and the $\gamma$-quiver of a Nakayama algebra.
Using $\gamma$, one can define the resolution quiver (or $\gamma$-quiver) of $A$, see \cite{R2}: its vertices are the isomorphism classes $[S]$ of the simple modules $S$, and there is an arrow $[S] \to [T]$ provided $T = \gamma(S)$. Of course, the $\gamma$-paths are just the paths in the resolution quiver.

Dually, using $\psi$, one can define the coresolution quiver (or $\psi$-quiver) of $A$: its vertices are the isomorphism classes $[S]$ of the simple modules $S$, and there is an arrow $[S] \to [T]$ provided $T = \psi(S)$. The $\psi$-paths are just the paths in the coresolution quiver.

Note that the coresolution quiver of $A$ is just the resolution quiver of $A^{\text{op}}$.

**Warning.** The $\psi$-quiver and the $\gamma$-quiver of a Nakayama algebra $A$ have many properties in common. Several such properties are mentioned above. In addition, Shen \cite{Sh1} has shown, that also the number of components are equal. However, the cardinalities of the components of the $\gamma$-quiver may be different from the cardinalities of the components of the $\psi$-quiver as the following example shows:

\[
\begin{array}{c}
\text{mod } A \\
\begin{array}{ccccccc}
1 & 6 & 5 & 4 & 3 & 2 & 1 \\
\hline
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & \end{array}
\end{array}
\quad
\begin{array}{cccc}
1 & \leftrightarrow & 5 & \\
3 & \leftrightarrow & 6 & \leftrightarrow 2 \\
\psi\text{-quiver of } A \\
\gamma\text{-quiver of } A
\end{array}
\]

**Appendix B. Sen's $\epsilon$-construction.**

It seems to be worthwhile to add a short outline of a reduction method which was introduced and very skillfully used by Sen \cite{S1, S2, S3}, since it relates to the functions $\gamma$ and $\psi$ discussed in sections 3 and 4, as well as in Appendix A. Let $A$ be a cyclic Nakayama algebra. Sen considers the subcategory $\mathcal{F}$ of modules $M$ with what he calls “syzygy filtrations”: this is an abbreviation of saying that $M$ has a filtration whose factors are second syzygies of simple modules (they are denoted below by $\Delta T$).

**B.1.** Three sets of simple modules will play a role: $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ (see also Appendix C). Here, $\mathcal{S}$ is the set of simple torsionless modules. Note that an indecomposable module $M$ is torsionless if and only if $\text{soc } M$ belongs to $\mathcal{S}$. Then, $\mathcal{T}$ is the set of simple modules $T$ with $\text{id } T \geq 2$; these are the simple modules in $\text{Im } \gamma$. Finally, $\mathcal{U}$ is the set of simple modules $U$ with $\text{id } U \geq 2$; these are the simple modules in $\text{Im } \psi$. A simple module $S$ belongs to $\mathcal{S}$ if and only if $\tau S$ belongs to $\mathcal{T}$.

Let $\mathcal{F}$ be the full subcategory of all modules with top in $\text{add } \mathcal{T}$ and socle in $\text{add } \mathcal{S}$. Thus, an indecomposable module $M$ belongs to $\mathcal{F}$ if and only of top $M$ belongs to $\mathcal{S}$ and $M$ is torsionless.

**B.1. The modules $\Delta T$.** For $T \in \mathcal{T}$, let $\Delta T$ be the smallest non-zero factor module of $PT$ which is torsionless (since $PT$ is torsionless, $\Delta T$ exists, and is, of course, uniquely determined by $T$).

(a) The module $\Delta T$ has a unique composition factor which belongs to $\mathcal{T}$, namely $T = \text{top } \Delta T$. Similarly, $\Delta T$ has a unique composition factor which belongs to $\mathcal{S}$, namely $\tau^{-1} T = \text{soc } \Delta T$. 

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(b) If $C$ is any simple module, there is a unique $T \in \mathcal{T}$ such that $C$ is a composition factor of $\Delta T$ and the multiplicity of $C$ in $\Delta T$ is 1.

(c) If $T \in \mathcal{T}$, then

$$\Delta T = \Omega^2 \psi T,$$

thus $\Delta T = \Omega^2 U$, where $U = \psi T \in \mathcal{U}$. Conversely, if $U$ is simple, then either $\text{pd} U \leq 1$ and $\Omega^2 U = 0$, or else $U \in \mathcal{U}$, and then $T = \gamma U$ belongs to $\mathcal{T}$ and $\psi T = U$. It follows that the modules $\Delta T$ are just the non-zero modules which are second syzygy modules of simple modules.

Note that the relevance of the modules $\Omega^2 U$ mentioned in (c) was stressed already by Madsen [M].

Proof. (a) First, we show that $\text{top} \Delta T$ is the only composition factor of $\Delta T$ which belongs to $\mathcal{T}$. Namely, assume, for the contrary, that there are submodules $X' \subset X \subset \Delta T$ with $X/X' \in \mathcal{T}$. Then $\text{soc}(\Delta T/X) \simeq \tau^{-1} \text{top} X = \tau^{-1}(X/X')$ is in $\mathcal{S}$. This implies that $\Delta T/X$ is torsionless. But this contradicts the minimality of $\Delta T$.

Next, we show that $\text{soc} \Delta T$ is the only composition factor of $\Delta T$ which belongs to $\mathcal{S}$. If $0 \subset X' \subset X \subset \Delta T$ are submodules such that $X/X'$ is in $\mathcal{S}$, then $\text{top} X' \simeq \tau \text{soc}(\Delta T/X') = \tau(X/X')$ belongs to $\mathcal{T}$. But this contradicts the first assertion.

(b) Let $S$ be simple. Let $c \geq 1$ be minimal with $\tau^{-c} S$ torsionless. Thus $T = \tau^{-c+1} S = \tau(\tau^{-c} S)$ belongs to $\mathcal{T}$, whereas the modules $\tau^i T$ with $1 \leq i < c$ do not belong to $\mathcal{T}$. It follows that $\Delta T$ has length at least $c$ and that $S$ is a composition factor of $\Delta T$. Since the top $T$ of $\Delta T$ has Jordan-Hölder multiplicity 1, any composition factor of $\Delta T$ has multiplicity 1.

(c) Let $T \in \mathcal{T}$ and $U = \psi T$. Then $\text{pd} U \geq 2$ and $\gamma U = T$. Since $\text{pd} U \geq 2$, it follows from 3.2 (b) that $\top \Omega^2 U = \gamma U = T$. Of course, $\Omega^2 U$ is torsionless, thus we see that $\Delta T$ is a factor module of $\Omega^2 U$. According to 3.2 (c), only the socle of $\Omega^2 U$ can be torsionless, thus $\Delta T = \Omega^2 U$.

Conversely, let $U$ be simple. If $\text{pd} U \leq 1$, then $\Omega^2 U = 0$. Thus, let $\text{pd} U \geq 2$ and $T = \gamma U$. Thus, $T \in \mathcal{T}$ and $T = \gamma U$. $\square$

B.2. The category $\mathcal{F}$. The modules $\Delta T$ with $T \in \mathcal{T}$ are pairwise orthogonal bricks and $\mathcal{F}$ is the full subcategory of all modules which have a filtration with factors of the form $\Delta T$ with $T \in \mathcal{T}$.

The category $\mathcal{F}$ is an extension-closed exact abelian category with simple objects the modules $\Delta T$ with $T \in \mathcal{T}$. The category $\mathcal{F}$ is a serial length category with

$$\tau_\mathcal{F} \Delta T = \Delta(\text{soc} \Delta T),$$

provided $\Delta T$ is not projective. (Here, $\tau_\mathcal{F}$ is the Auslander-Reiten translation inside the category $\mathcal{F}$.)

Proof. Let $T \in \mathcal{T}$. If $f: \Delta T \to \Delta T$ is a non-zero non-invertible endomorphism, then the image of $f$ is a proper submodule of $\Delta T$ with top $T$, a contradiction. Also, if $T', T$ are non-isomorphic modules in $\mathcal{T}$, and $f: \Delta T' \to \Delta T$ is a non-zero homomorphism, then the image of $f$ is a submodule of $\Delta T$ with top $T'$, again a contradiction. Thus, we see that the modules $\Delta T$ with $T \in \mathcal{T}$ are pairwise orthogonal bricks.
Next, we show that an indecomposable module $M$ with top in $T$ and socle in $S$ has a filtration with factors of the form $\Delta T$ with $T \in T$. The proof is by induction on the length of $M$. Let $M$ be a module with top in $T$ and socle in $S$. Say let top $M = T'$. Then $M$ is a factor module of $PT'$. Since soc $M$ is torsionless, also $M$ is torsionless. By definition of $\Delta T'$ we see that $\Delta T'$ is a factor module of $PT'$, say $M = PT'/X$ for some submodule $X$ of $PT'$. Now top $X = \tau \text{soc}(PT'/X) = \tau \text{soc} M$, thus top $X$ is in $T$. By induction, $X$ has a filtration with factors of the form $\Delta T$.

It follows that $F$ is the extension closure of the class of modules $\Delta T$ with $T \in T$, and that $F$ is an extension-closed exact abelian subcategory, whose simple objects are just the objects $\Delta T$ with $T \in T$, see for example [R1].

It remains to calculate $\tau_F \Delta T$. Let $\pi: PT \to \Delta T$ be a projective cover. If $\Delta T$ is not projective, then Ker($f$) is an indecomposable module, which belongs to $F$ and has $\Delta T'$ with $T' = \tau \text{soc} \Delta T$ as a factor object (its top in $F$). Thus $\tau_F \Delta T = \Delta(T')$. \hfill $\square$

Note: The quiver of $\epsilon(A)$ does not have to be a cycle! For example, for

![Quiver Diagram]

$F$ consists just of the projective module $P2$. Thus, $F$ is a semisimple category and its quiver is just a singleton (thus of type $A_1$).

The subcategory $F$ of mod $A$ is closed under projective covers. It follows that the abelian category $F$ has enough projective objects. The indecomposable projective objects in $F$ are the modules $PT$ with $T \in T$.

Proof that $F$ is closed under projective covers: if $M$ is indecomposable and in $F$, say with top $T$, then $PM = PT$ also belongs to $F$. \hfill $\square$

The module $G = \bigoplus_{T \in T} PT$ is a progenerator of $F$. It follows that $\epsilon(A) = (\text{End } G)^{op}$ is a Nakayama algebra and there is a categorical equivalence

$$ F \simeq \text{mod } \epsilon(A). $$

\hfill $\square$

B.3. The $\gamma$-quiver of $\epsilon(A)$. We assume that none of the modules $\Delta T$ is projective, thus $\epsilon(A)$ is again a cyclic Nakayama algebra. Using the categorical equivalence $F \simeq \text{mod } \epsilon(A)$, the $\gamma$-quiver of $\epsilon(A)$ may be considered as a quiver with vertex set the set of modules $\Delta T$ with $T \in T$ and with arrows $\Delta T \to \gamma_F(\Delta T)$. There is the following observation:

Lemma. If $T \in T$, then

$$ \gamma_F(\Delta T) = \Delta(\gamma T). $$
Proof. Let \( \text{soc}_F P_F(\Delta T) = \Delta(T') \) for some \( T' \). Then \( \text{soc} \Delta(T') = \text{soc} P T \). Since \( \gamma_F = \tau_F \text{soc}_F P_F(-) \) and \( \gamma = \tau \text{soc} P(-) \), we see that

\[
\gamma_F(\Delta T) = \tau_F \text{soc}_F P_F(\Delta T) = \tau_F \Delta(T') = \Delta(\tau \text{soc} \Delta(T')) = \Delta(\tau \text{soc} P T) = \Delta(\gamma T).
\]

This shows: Let \( \Gamma(A) \) be the \( \gamma \)-quiver of \( A \) and \( \Gamma(A)|\text{Im} \gamma \) its restriction to \( \text{Im} \gamma \). Then the map \( T \mapsto \Delta(T) \) identifies \( \Gamma(A)|\text{Im} \gamma \) with the \( \gamma_F \)-quiver of \( \epsilon(A) \).

**B.4. The subcategory \( F \) and the class of reflexive modules.**

**Proposition.** The modules in \( F \) are reflexive.

**Proof.** Assume that \( M \) belongs to \( F \). We want to show that \( M \) is reflexive. We can assume that \( M \) is indecomposable and not projective. Now \( M \) is torsionless, thus \( IM \) is projective. Let \( 0 \neq P \subseteq IM \) be minimal projective (this means: projective, and \( \text{rad} P \) is not projective). Since \( M \) is a non-projective submodule of \( IM \), we see that \( M \) is a proper submodule of \( P \). It follows that the inclusion map \( u: M \to P \) is a minimal left add \( A \)-approximation, therefore \( \bigcup M = \text{Cok} u \). Now \( \text{top} M \) belongs to \( T \), therefore \( \text{soc} \text{Cok} u = \tau^- \text{top} M \) is torsionless, see (x). It is well-known that a module \( M \) is reflexive if and only if both \( M \) and \( \bigcup M \) are torsionless (see for example [RZ], Theorem 1.5).

Let \( R \) be the full subcategory of all reflexive modules. Let \( R_0 \) be the full subcategory of all reduced reflexive modules (a module \( M \) is said to be reduced provided \( M \) has no non-zero projective direct summand).

**Corollary.** We have

\[
R_0 \subseteq \{ \Omega^2 X \mid X \in \text{mod} A \} \subseteq F \subseteq \{ P \oplus \Omega^2 X \mid X \in \text{mod} A, P \in \text{add} A A \} = R.
\]

**Proof.** First, let \( M \) be indecomposable, reflexive and non-projective. If \( X = \bigcup M \), then \( M = \Omega^2 X \). This shows that \( R_0 \subseteq \{ \Omega^2 X \mid X \in \text{mod} A \} \).

Next, let us show that any module of the form \( \Omega^2 M \) belongs to \( F \). On the one hand, \( \Omega^2 M \) is torsionless. On the other hand, either \( \Omega^2 M = 0 \), or else \( \text{top} \Omega^2 M = \gamma \text{top} M \), thus \( \text{top} \Omega^2 M \in T \). This shows that \( M \in F \).

Second, let \( M \in F \). We want to show that \( M = P \oplus \Omega^2 X \) for some modules \( P, X \) with \( P \) projective. We can assume that \( M \) is indecomposable and not projective. As we saw, \( M \) is reflexive. Thus, \( M = \Omega^2 X \) for some module \( X \).

**Examples.** All three inclusions mentioned in the Corollary may be proper.
Example 1: Here is $\mathcal{R}_0$ the zero category, whereas there is an indecomposable module of the form $\Omega^2 X$, namely the projective $P2$.

In example 2, the indecomposable modules of the form $\Omega^2 X$ are just 1 and $\text{rad} P1$ (marked by bullets). The subcategory $\mathcal{F}$ contains also $P1$ and $P2$, but not $P3$. Note that $\mathcal{F}$ is closed under extensions, whereas the subcategories $\{\Omega^2 X \mid X \in \text{mod } A\}$ and $\{P \oplus \Omega^2 X \mid X \in \text{mod } A, P \in \text{add } A\}$ are not closed under extensions (the extension closure of $\{\Omega^2 X \mid X \in \text{mod } A\}$ is $\mathcal{F}$, and the extension closure of $\{P \oplus \Omega^2 X \mid X \in \text{mod } A, P \in \text{add } A\}$ contains $3 \oplus P2$).

Warning. Some mathematical papers (for example Auslander-Reiten [AR1], [AR2]) denote by $\Omega^t(\text{mod } A)$ the full subcategory of all modules of the form $P \oplus \Omega^t X$, with $P$ projective (and not, what one would expect, the full subcategory of all modules of the form $\Omega^t X$, where $X$ is a module, or its additive closure) and consider the question whether this subcategory $\Omega^t(\text{mod } A)$ is closed under extension or not. The examples above show that subcategories related to this subcategory may be closed under extensions whereas $\Omega^t(\text{mod } A)$ itself is not.

Appendix C. Some canonical bijections.

As an addition to appendix B, let us focus the attention to several further sets of modules which are in natural bijection to the subsets $S, T, U$ mentioned above. The category $\mathcal{F}$ is defined by using as building blocks the modules $\Omega^2 U$ with $U \in U$, thus one may draw the attention also to the modules $\Omega U$ with $U \in U$, we call them “valley” modules.

We say that an indecomposable module $V$ is a valley provided that $V = \text{rad} P = I/\text{soc } I$, where $P$ is projective and $I$ is injective. We say that an indecomposable module $Z$ is a peak provided that $Z$ is both projective and injective. (Looking at the roof of the Auslander-Reiten quiver of $A$, we see that valleys and peaks alternate, of course.)

An indecomposable projective module $P$ is said to be minimal provided that $\text{rad } P$ is not projective. An indecomposable injective module $I$ is said to be minimal provided that $I/\text{soc } I$ is not injective.

Some relevant bijections. We consider the modules marked above by bullets: these are modules which determine each other uniquely. Thus, we look at the set $\mathcal{T}$ of simple
modules \( T \) with \( \text{id} T \geq 2 \); the set \( U \) of simple modules \( U \) with \( \text{pd} U \geq 2 \); the set \( S \) of simple torsionless modules; the set of modules \( \Delta T \) with \( T \in \mathcal{T} \); the set of valley modules, the set of peak modules, the set of minimal injective modules and finally the set of minimal projective modules (the sets are arranged roughly in the same way as the modules are located in the Auslander-Reiten quiver): All these sets have the same cardinality, and any arrow in the following pictures exhibits a canonical bijection. Actually, the set \( S \) of simple torsionless modules appears twice in the picture, since two different bijections between \( S \) and \( \mathcal{T} \) are relevant in our setting: if \( T \in \mathcal{T} \), then, on one hand, the socle of \( \Delta T \) is torsionless and simple, on the other hand, also \( \tau^{-} T \) is torsionless and simple. These bijections are combined when we look at the Auslander-Reiten translation \( \tau_{F} \) of \( \mathcal{F} \), see (4).

\[
\begin{align*}
\mathcal{T} &= \{ T \text{ simple, } \text{id} T \geq 2 \} \\
\mathcal{S} &= \{ S \text{ simple, torsionless} \} \\
\{ U \text{ simple, } \text{pd} U \geq 2 \} &= \mathcal{U} \\
\{ \text{peak module} \} &= \{ Z \} \\
\{ \text{minimal injective} \} &= \{ I \} \\
\{ \text{minimal projective} \} &= \{ P \} \\
\{ \text{valley module} \} &= \{ V \} \\
\{ \text{rad} \} &= \{ \text{rad} \} \\
\{ \text{soc} \} &= \{ \text{soc} \} \\
\{ \text{top} \} &= \{ \text{top} \} \\
\{ \text{psi} \} &= \{ \text{psi} \} \\
\{ \text{gamma} \} &= \{ \text{gamma} \} \\
\{ \text{Omega} \} &= \{ \text{Omega} \} \\
\end{align*}
\]

Here, \( \pi(I) = I / \text{soc} I \) for \( I \) injective.

The display shows the interrelation between a large number of sets of cardinality \( r \), where \( |\mathcal{T}| = r \). However one should be aware that even this display is not yet complete. For example, similar to the torsionless simple modules, also the divisible simple modules play a role (a module is said to be divisible provided it is a factor module of an injective module). And, parallel to \( \Delta T = \Omega V \), where \( V \) is a valley, also the modules \( \Sigma W \) should be taken into account. But we have refrained from doing so in order to keep the display manageable.

In general if \( W \) is a set of simple modules and has cardinality \( r \) (such as \( \mathcal{S} \), \( \mathcal{T} \), \( \mathcal{U} \) or the set \( \mathcal{D} \) of divisible modules), also the sets \( \{ PW \mid W \in \mathcal{W} \} \), \( \{ \text{rad} PW \mid W \in \mathcal{W} \} \), \( \{ IW \mid W \in \mathcal{W} \} \) and \( \{ IW / \text{soc} \mid W \in \mathcal{W} \} \) are sets of cardinality \( r \).

**Minimal sets of relations.** Let \( r \) be the cardinality of \( \mathcal{T} \). When looking at the large number of sets of modules which are in canonical bijection with \( \mathcal{T} \), thus of cardinality \( r \), one should be aware that it is another set of cardinality \( r \) which really is the decisive one, namely any minimal set of relations which defines \( A \). This is Sen’s approach and it is definitely the relevant one. It leads immediately to an intrinsic choice of indices. Unfortunately, it seems that the use of these indices turns out to be not easy to digest, at least at a first reading. This is the reason that in contrast to Sen, we have started in a
different way, namely with the set $\mathcal{T}$. As we saw, there is a canonical bijection between the set $\mathcal{T}$ and the set $\mathcal{V}$ of valleys $V$, and it is the set $\mathcal{V}$ which has to be considered as a natural setting for dealing with relations. We recall that relations can be, of course, interpreted as non-zero elements of $\text{Ext}^2$, say of $\text{Ext}^2(\cdot,\cdot)$. Any valley $V$ yields an exact sequence

$$0 \to T \to IT \xrightarrow{f} PU \to U \to 0$$

with $V = \text{Im} f$. Its equivalence class in $\text{Ext}^2(U,T)$ is non-zero. In this way, the set of valleys correspond to a minimal set of relations which defines $A$.

Appendix D. Further invariants related to the finitistic dimension.

In order to get hold of fin-pro $A$, Gélinas discusses in the paper [G1] not only del $S$, but also the grade of simple modules $S$ (thus the depth of $A$), and he looks for numbers $t \geq 0$ such that $\Omega^t \Omega^t S$ is torsionless. Examples of Nakayama algebras can be used in order to get a better understanding of these settings.

D.1. The grade of the simple modules and the depth of the algebra.

Let $M$ be a module. The grade $\text{grade} M$ is defined as follows. We have $\text{grade} M = 0$ iff $\text{Hom}(M,A) \neq 0$, and $\text{grade} M = d \geq 1$ iff $\text{Hom}(M,A) = 0$, $\text{Ext}^i(M,A) = 0$ for $1 \leq i < d$ and $\text{Ext}^d(M,A) \neq 0$. The depth of $A$ is the maximum of grade $S$, where $S$ are the simple modules.

Proposition (Jans–Gélinas). Let $S$ be a simple module with finite grade $S = d \geq 1$. There are modules $N$ with injective dimension $d$ which satisfy $\text{Ext}^d(S,N) \neq 0$, for example $N = \tau \Omega^{d-1} S$.

Proof. Let $P_\bullet$ be a minimal projective resolution of $S$ and consider its truncation

$$P_d \xrightarrow{f_d} P_{d-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \to 0.$$  

Here, $P_0 = P(S)$ is a projective cover of $S$. For all $0 \leq i < d$, the cokernel of $f_{i+1}$ is $\Omega^i S$. In particular, $X = \Omega^{d-1} S$ is the cokernel of $f_d$. We form the $A$-dual

$$P_d^* \xleftarrow{f_d^*} P_{d-1}^* \to \cdots \to P_1^* \xleftarrow{f_1^*} P_0^* \to 0.$$  

This is an exact sequence, since $\text{Ext}^i(S,A) = 0$ for $0 \leq i < d$. By definition, $\text{Tr} X$ is the cokernel of $f_d^*$. Thus, we have obtained a minimal projective resolution of the right module $\text{Tr} X$

$$0 \to \text{Tr} X \to P_d^* \to P_{d-1}^* \to \cdots \to P_1^* \to P_0^* \to 0.$$  

We apply $D$ to this sequence and get a minimal injective coresolution of $N = \tau \Omega^{d-1} S = D \text{Tr} \Omega^{d-1} S = D \text{Tr} X$ with $DP_0^* = DP(S)^* = I(S)$. This shows that $\text{id} N = d$ and $\text{Ext}^d(S,N) \neq 0$.

Corollary 1. For any simple module $S$, we have grade $S \leq \text{del} S$. 

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Proof. This is trivially true if grade $S = 0$. Thus we can assume that grade $S = d \geq 1$. The proposition asserts that there is a module $N$ with injective dimension $d$ which satisfies $\text{Ext}^d(S,N) \neq 0$. According to Proposition 2.2, we get $d \leq \text{del} S$. □

**Corollary 2.**

$$\text{depth} A \leq \text{del} A.$$ □

**D.2. The case of Nakayama algebras.**

**Lemma.** Let $A$ be Nakayama and let $S$ be simple. Then $\text{del} S = 0$ iff grade $S = 0$ iff $S$ is torsionless. Next, $\text{del} S = 1$ iff grade $S = 1$ iff $S$ is not torsionless, and pd $S = 1$, and with projective dimension equal to 1). Finally, $\text{del} S = 2$ implies that grade $S = 2$.

Proof. If $\Omega S$ is non-zero, then $\text{Ext}^1(S,\Omega S) \neq 0$. Thus, if $\Omega S$ is non-zero and projective, then grade $S \leq 1$. Conversely, let grade $S = 1$, and assume that $\Omega S$ is not projective. Then $\Omega S \to P(S) \to S$ is not an $\Omega$-sequence, but this implies that rad $P(S) = \Omega S \to P(S)$ is not an add $A$-approximation, thus $P(S)$ is not a minimal projective. But this implies that rad $P(S)$ is projective, a contradiction.

Next, let $\text{del} S = 2$. Then grade $S \geq 2$. But in general, grade $S \leq \text{del} S$. Thus grade $S = 2$. □

**Examples.** There are cyclic Nakayama algebras of arbitrarily large finite global dimension with depth equal to 2.

Let $m \geq 2$ and $n = 2m$. Let $A_n$ be the cyclic Nakayama algebra with Kupisch series $(2,3,4,5,\cdots,m+1,m+1,m+2)$. We denote the simple modules by $S_i$ with $1 \leq i \leq n$, where $P(S_1) = 2$ and $S_{i+1} = \tau^-S_i$. Thus, $|P(S_{2i-1})| = i+1$ and $|P(S_{2i})| = i+2$ for all $i \geq 1$. In particular, we see that pd $S_{2i} = 1$ for all $i \geq 1$. Let us look at the simple modules with odd index. Since $\Omega S_1 = S_n = S_{2m}$ has projective dimension 1, we have pd $S_1 = 2$. Since $\Omega^2 S_3 = S_n = S_{2m}$ has projective dimension 1, we have pd $S_3 = 3$. For $i \geq 2$, we have $\Omega^2 S_{2i+1} = S_{i-1}$. This shows that $A_n$ has finite global dimension. Also,

$$\text{pd} S_{2t-3} = 2t - 2, \quad \text{for} \quad t \geq 2.$$ (The proof is by induction on $t \geq 2$. As we know, pd $S_1 = 2$. We have $\Omega S_{2t+1-3} = P(S_{2t+1-4})/\text{soc}$ and soc $P(S_{2t+1-4}) = S_{2t-3}$. By induction, we assume that pd $S_{2t-3} = 2t - 2$, therefore pd $S_{2t+1-3} = 2 + \text{pd} S_{2t-3} = 2t = 2(t + 1) - 2$, as we want to show.) Thus the global dimension of $A_n$ gets arbitrarily large.

It remains to be seen that depth $A_n = 2$. Since the simple modules with even index have projective dimension 1, we have grade $S_{2i} \leq 1$ for all $i$. We have $\Omega S_{2i+1} = \text{rad} P(S_{2i+1}) = P(S_{2i})/\text{soc}$ and there is a non-split exact sequence

$$0 \to P(S_{2i-1}) \to P(S_{2i}) \oplus P(S_{2i-1})/\text{soc} \to P(S_{2i})/\text{soc} \to 0$$

(an Auslander-Reiten sequence), thus

$$\text{Ext}^2(S_{2i+1}, P(S_{2i-1})) = \text{Ext}^1(P(S_{2i})/\text{soc}, P(S_{2i-1})) \neq 0.$$
This shows that grade $S_{2i+1} \leq 2$ (and actually, grade $S_{n-1} = 2$).

Here is the case $n = 8$. On the left, there is the Auslander-Reiten quiver of $A_8$ with the function $pd$, and below the indices of the simple modules. On the right we highlight the module $S_7$ and the Auslander-Reiten sequence which shows that $\text{Ext}^2(S_7, P(S_5)) \neq 0$.

\[ \text{D.3. The modules } \mathcal{U}^t \Omega^t S. \]

If $M$ is a module, let $\mathcal{U}M = \text{Tr} \Omega \text{Tr} M$ (note that $\mathcal{U}$ is a stable functor; it was considered by Auslander and Reiten in [AR], and is left adjoint to $\Omega$). One can show that $\mathcal{U}M$ is just the cokernel of a minimal left add $A$-approximation of $M$, see [RZ], 4.4.

If $S$ is a simple module, Gélinas looks for a non-negative number $t$ such that $\mathcal{U}^t \Omega^t S$ is torsionless. Namely, Theorem 1.13 of Gélinas [Ge1] asserts: If $\mathcal{U}^t \Omega^t M$ is torsionless, then $\operatorname{del} M \leq t$. There is the following immediate consequence (the “torsionfreeness criterion” 1.14): If there is $t \geq 0$ such that $\mathcal{U}^t \Omega^t S$ is torsionless, then $\operatorname{del} A \leq \sup n_S$.

Here is an example of a simple module $S$ such that no module $\mathcal{U}^t \Omega^t S$ with $t \geq 0$ is torsionless.

We have $\mathcal{U} \Omega S = S$ and $\mathcal{U}^t \Omega^t S = X$ for $t \geq 2$, and both modules $S$ and $X$ are not torsionless. On the right, we have marked by bullets the modules $\Omega^t S$ with $0 \leq t \leq 4$ and stress that $\Omega^3 S = \Omega^2 S$. Actually, we write $T = \Omega^2 S$. Also, we have inserted $X = \mathcal{U}^2 T$ as well as the projective cover $PT$ of $T$.

Proof. One checks immediately that $\mathcal{U} \Omega S = S$. Thus, let us show that $\mathcal{U}^t \Omega^t S = X$ for $t \geq 2$. It is easy to see that we have: (a) $\Omega^3 T = T$. (b) $\mathcal{U} \Omega T = T$. and (c) $\mathcal{U}^2 \Omega^2 T = T$. We define (d) $X = \mathcal{U}^2 T$. Then we have (e) $\mathcal{U} X = T$ (a minimal add $A$-approximation $X \to PT$ is not a monomorphism, but this does not matter!). It follows from (d) and (e) that we have (f) $\mathcal{U}^3 T = T$.

Now, let $t \geq 2$. If $t = 3s$ with $s \geq 1$, then

$$\mathcal{U}^t \Omega^t S = \mathcal{U}^2 \mathcal{U}^{3(s-1)}(\mathcal{U} \Omega) \Omega^{3(s-1)} T = X$$
using (a), (b), (f) and (d). Next, for \( t = 3s + 1 \) with \( s \geq 1 \), we get
\[
\mathcal{U}^t \Omega^t S = \mathcal{U}^2 \mathcal{U}^{3(s-1)}(\mathcal{U}^2 \Omega^2) \Omega^{3(s-1)} T = X
\]
using (a), (c), (f) and (d). Finally, for \( t = 3s + 2 \) with \( s \geq 0 \), we get
\[
\mathcal{U}^t \Omega^t S = \mathcal{U}^2 \mathcal{U}^{3s} \Omega^{3s} T = X
\]
using (a), (f) and (d). □

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