A thermodynamic approach to two-variable Ruelle and Selberg zeta functions via the Farey map

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Abstract

In this paper, we consider the transfer operator approach to the Ruelle and Selberg zeta functions associated with continued fraction transformations and the geodesic flow on the full modular surface. We extend the results by Ruelle and Mayer to two-variable zeta functions, $\zeta(q, z)$ and $Z(q, z)$. The $q$ variable plays the role of the inverse temperature and the introduction of the ‘geometric variable’ $z$ is essential in the tentative to provide a general approach, based on the Farey map, to the correspondence between the analytic properties of the zeta functions themselves, the spectral properties of a class of generalized transfer operators and the theory of a generalization of the three-term functional equations studied by Lewis and Zagier. The first step in this direction is a detailed study of the spectral properties of a family of signed transfer operators $P_{q, i}$ associated with the Farey map.

Keywords: transfer operators, Farey map, Gauss map, Selberg zeta function, Ruelle zeta function
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1. Introduction

The transfer operator approach to the Selberg zeta function for the full modular group $\text{PSL}(2, \mathbb{Z})$ introduced by Mayer [23] led to new interesting interactions between number theory and the thermodynamic approach to dynamical systems. The correspondence between
the zeros of the Selberg zeta function and the eigenfunctions of the transfer operators for the Gauss map has been subsequently studied in particular in connection with the theory of period functions for cusp and non-cusp forms on $PSL(2, \mathbb{Z})$ in [8, 9, 20]. In this paper we extend the approach in [23] to signed transfer operators $P^\pm_q$ for the Farey map, which is connected to the Gauss map by an induction procedure. This approach clarifies some aspects of the results in [23], and we are naturally led to the definitions of two-variable Ruelle and Selberg zeta functions for which extensions of Mayer results hold.

We now discuss in detail the content of the paper. The first issue is the study of the signed transfer operators $P^\pm_q$ for the Farey map defined in (2.1). The spectral properties of transfer operators for uniformly expanding maps of an interval are now well understood and depend crucially on the Banach space considered [3]. For sufficiently regular functions it turns out that the transfer operator is quasi-compact, hence the spectrum is made of isolated eigenvalues with finite multiplicity and the essential part, a disc of radius strictly smaller than the spectral radius. However, the essential spectral radius depends on the expanding constant $\rho$ of the map and on the degree of regularity of the functions. In particular, as $\rho \to 1$ from above, the essential spectral radius converges to the spectral radius. The Farey map $F$ is a prototype of a smooth intermittent map on the unit interval $[0, 1]$, being expanding everywhere except at the origin, a neutral fixed point. Hence for the Farey map $\rho = 1$, and classical approaches to the spectral properties of its transfer operator fail. As a matter of fact, using ad hoc techniques, the spectrum of the Farey transfer operator when acting on suitable spaces of holomorphic functions has been shown to have an absolutely continuous component given by the unit interval (see [5, 15, 30]).

Here we consider the family of signed generalized transfer operators $P^\pm_q$ associated with $F$ and defined in (2.4) and provide a detailed study of their spectral properties on a space of holomorphic functions on an open domain containing $(0, 1)$. In section 2, we prove our first main result which is a characterization of the eigenfunctions of $P^\pm_q$ with eigenvalues not embedded in the continuous spectrum (theorem 2.8). This is obtained in terms of an integral transform defined on weighted spaces $L^p((0, +\infty), m_q(t))$ with $dm_q(t) = t^{2\Re(q)-1}e^{-t}$. A similar approach has been used in [5, 13, 15, 28]. In [5] the Hilbert space $L^2((0, +\infty), m_q(t))$ has been studied in some detail and the resulting issues are used in this paper. In particular, a simple argument (proposition 2.1) shows that eigenfunctions of $P^\pm_q$ satisfy the three-term functional equation (2.6), which for $\lambda = 1$ is but the Lewis functional equation studied in [20], where a class of solutions of this equation is proved to be in one-to-one correspondence with the Maass cuspidal and non-cuspidal forms on the full modular group $PSL(2, \mathbb{Z})$. Some of our results, in particular corollary 2.10, are then generalizations to the case $\lambda \neq 1$ of results in [20] on the properties of solutions to the Lewis functional equation.

In the second part, we use an inducing procedure for $F$ which was first introduced in [29] for a general class of intermittent interval maps. The idea is to consider an induced map $G$ on the interval with respect to the first passage time at a subset of $[0, 1]$ away from the neutral fixed point, and to study the spectral properties of the transfer operators $Q$ of $G$. Then functional relations between $Q$ and $P$ allow one to translate the spectral properties of $Q$ into those of $P$ and vice versa. For applications of this method see also [15, 28, 30]. It is well known that the Farey map is related to number theory, and in particular to the continued fraction expansion of $x \in [0, 1]$. Moreover, if we define $G$ to be the induced map with respect to the first passage time at the interval $([\frac{1}{2}, 1]$, it turns out that $G$ is the Gauss map on the unit interval, see (3.1). In section 3, following this procedure, we define in (3.4) (see also (3.7)) a two-parameter family of transfer operators $Q_{q,z}$ associated with $G$ and look for functional relations between $Q_{q,z}$ and $P^\pm_q$. This is obtained in theorem 3.6 and allows us to obtain a one-to-one correspondence between all eigenfunctions to the eigenvalues $\pm 1$ of $Q_{q,z}$ and some eigenfunctions with eigenvalue $\frac{1}{z}$ of $P^\pm_q$ (corollary 3.7).
Let us recall that the properties of the transfer operators \( Q_{q,1} \) have been already studied in [9, 21–23], and also discussed in [20]. In particular, in [23] it is proved that the values of \( q \) for which there are eigenfunctions with eigenvalue \( \pm 1 \) of \( Q_{q,1} \) are in one-to-one correspondence with the zeros of the Selberg zeta function for the full modular group \( \text{PSL}(2, \mathbb{Z}) \). In turn, it is the content of Selberg trace formula that the zeros of the Selberg zeta function correspond to the Maass cusp forms and to the non-trivial zeros of the Riemann zeta function, which are related to the Maass non-cuspidal forms. One then obtains a relation between eigenfunctions to the eigenvalues \( \pm 1 \) of \( Q_{q,1} \) and a class of solutions of the Lewis three-term functional equation. This is discussed in [20] and proved without appealing to the Selberg zeta function.

It is one of the aims of this paper to obtain the latter relation from the eigenfunctions of the transfer operators \( P_{q}^{\pm} \), thus giving, in the spirit of [23], a thermodynamic formalism approach to the period functions studied in [20]. The first step is theorem 3.6 mentioned above. Note in particular the term \( \pm c\mu \) in (3.8). This term is indeed responsible for the restriction to a class of eigenfunctions of \( P_{q}^{\pm} \) in corollary 3.7, i.e. to a class of solutions of the Lewis functional equation. The second step is the main result of section 4, theorem 4.6. We first show that the transfer operators \( Q_{q,z} \) are of trace class on a suitable Banach space, hence the Fredholm determinants \( \text{det}(1 \mp Q_{q,z}) \) are well defined, and the traces can be computed as in [15, 21]. Hence we obtain (4.11), which is contained in [23] for \( z = 1 \), and leads to a definition of a two-variable Selberg zeta function \( Z(q, z) \). Altogether these steps show that the zeros of the function \( Z(q, z) \) are in one-to-one correspondence with the eigenfunctions of \( P_{q}^{\pm} \) with eigenvalue \( \frac{1}{z} \) when the constant \( c \) which appears in (3.8) (see also corollary 2.10) vanishes. In the case \( z \neq 1 \), as a corollary of our results we also obtain a new proof for the characterization of the nontrivial zeros of the function \( Z(q, 1) \) in terms of even and odd cusp forms given in [11]. This is the content of theorem 4.8 in section 4.1. In the case \( z \neq 1 \) it is proved in [28] that there exist eigenfunctions of \( P_{q}^{\pm} \) with real eigenvalue \( \lambda \in (1, 2) \) for real \( q \in (0, 1) \), hence there exist zeros for \( Z(q, z) \) with \( z \neq 1 \) (see remark 4.7).

In the same section we also discuss a two-variable Ruelle zeta function \( \zeta(q, z) \) for the Farey map, defined in (4.2). Our main result is equation (4.12) in theorem 4.6, which gives an expression of \( \zeta(q, z) \) in terms of the Fredholm determinants \( \text{det}(1 \mp Q_{q,z}) \), extending a previous result in [15]. For a discussion of multi-variable zeta functions in dynamical systems and number theory see [17].

Finally in section 5 we come back to the Farey map and its relations to number theory. Using a formal manipulation of (4.11) and (4.12), which is similar to the approach in [30], we obtain an expression for the zeta functions \( Z(q, z) \) and \( \zeta(q, z) \) as exponentials of power series whose coefficients are obtained as sums along branches of the Farey tree (theorem 5.3).

To conclude we remark that recently it has been introduced in [26, 27] a dynamical approach to the Selberg zeta function for any cofinite Hecke triangle group \( \Gamma \) by characterizing Maass cusp forms for \( \Gamma \) as solutions of a finite-term functional equation, in the same spirit as in [18, 20]. Moreover, the functional equation has been related to the eigenvalue equation of a transfer operator associated with a symbolic dynamics for the geodesic flow on the hyperbolic plane quotiented by \( \Gamma \). This approach can then be considered the analogous of our thermodynamic approach in the case \( z = 1 \) for the Hecke triangle groups. It is interesting to note that in the papers [26, 27] the authors find for the transfer operators of the dynamical systems they study exactly the same ‘structure’ as for the Farey and the Gauss map. Namely they first study a finite-term transfer operator for a ‘slow’ dynamical system which is not nuclear, and they ‘accelerate’ the system to obtain a transfer operator for which the Fredholm determinant is well defined. Other approaches to the Selberg zeta function for the cofinite Hecke triangle groups related to this paper can be found in [24, 25], where the authors considered transfer operators associated with maps related to suitable generalizations of the standard continued fractions.
2. Transfer operators for the Farey map

Let $F : [0, 1] \rightarrow [0, 1]$ be the Farey map defined by

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \quad (2.1)$$

To this map we can associate a family of signed generalized transfer operators $\mathcal{P}_q^\pm$, with $q \in \mathbb{C}$, $\Re(q) > 0$, whose action on a function $f : [0, 1] \rightarrow \mathbb{C}$ is given by a weighted sum over the values of $f$ on the set $F^{-1}(x)$, namely if

$$\mathcal{P}_0^q f(x) := \left( \frac{1}{x+1} \right)^{2q} f \left( \frac{x}{x+1} \right) \quad (2.2)$$

$$\mathcal{P}_1^q f(x) := \left( \frac{1}{x+1} \right)^{2q} f \left( \frac{1}{x+1} \right) \quad (2.3)$$

we define

$$\mathcal{P}_q^\pm f(x) := \mathcal{P}_0^q f(x) \pm \mathcal{P}_1^q f(x). \quad (2.4)$$

Since the operators $\mathcal{P}_q^\pm$ are defined by multiplication and composition with real analytic maps which extend to holomorphic maps on a neighbourhood of the interval $[0, 1]$, it is natural to consider the action of $\mathcal{P}_q^\pm$ on the space $\mathcal{H}(B)$ of holomorphic functions on the open domain

$$B := \left\{ x \in \mathbb{C} : \left| x - \frac{1}{2} \right| < \frac{1}{2} \right\}.$$

We point out that we still denote by $x$ the complex variable. We are interested in the spectral properties of these transfer operators, hence first of all we give a characterization of their eigenfunctions. To this aim we give some properties of the action of $\mathcal{P}_q^\pm$ on $\mathcal{H}(B)$. It is useful to also consider the involution $\mathcal{J}_q$ with

$$\mathcal{J}_q f(x) := \frac{1}{xq} f \left( \frac{1}{x} \right). \quad (2.5)$$

By definition of involution, any function $f \in \mathcal{H}(\{ \Re(x) > 0 \})$ can be written as a sum of eigenfunctions of $\mathcal{J}_q$, that is $f = \varphi_+ + \varphi_-$ with $\mathcal{J}_q \varphi_+ = \varphi_+$ and $\mathcal{J}_q \varphi_- = -\varphi_-$. 

**Proposition 2.1.**

(i) If $f \in \mathcal{H}(B)$ then $\mathcal{P}_q^\pm f \in \mathcal{H}(\{ \Re(x) > 0 \})$;

(ii) if $f \in \mathcal{H}(B)$ is an eigenfunction of $\mathcal{P}_q^\pm$ with eigenvalue $\lambda \in \mathbb{C} \setminus \{ 0 \}$, then $f \in \mathcal{H}(\{ \Re(x) > 0 \})$ and $\mathcal{J}_q f = \pm f$. In particular if $\mathcal{J}_q f = -f$ then $f(1) = 0$. Moreover

$$\lambda f(x) - f(x+1) = \left( \frac{1}{x+1} \right)^{2q} f \left( \frac{x}{x+1} \right), \quad \Re(x) > 0; \quad (2.6)$$

(iii) if $f \in \mathcal{H}(\{ \Re(x) > 0 \})$ satisfies (2.6) with $\lambda \in \mathbb{C} \setminus \{ 0 \}$, then $\varphi_\pm := \frac{1}{2} (f \pm \mathcal{J}_q f)$ satisfies $\mathcal{P}_q^\pm \varphi_\pm = \lambda \varphi_\pm$. 

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Proof.

(i) follows simply by the fact that if Re(x) > 0 then both \( \frac{x}{x+1} \) and \( \frac{1}{x+1} \) are in \( B \).

(ii) The first assertion follows by (i). For the second, by the definition of \( J_q \) given in (2.5) one easily checks that

\[ J_q P^\pm f = \pm P^\pm f. \]

Hence if \( f \) is an eigenfunction we can rewrite the previous expression substituting \( P^\pm f \) with \( \lambda f \) and the second assertion follows. Finally (2.6) is obtained using the fact that for eigenfunctions it holds

\[ \pm (P_{1,q} f)(x) = \pm (J_q f)(x + 1) = f(x + 1). \]

(iii) Let \( \varphi_+ \) and \( \varphi_- \) be defined as above. They satisfy \( f = \varphi_+ + \varphi_- \) and \( J_q \varphi_{\pm} = \pm \varphi_{\pm} \).

Moreover, one can easily checks that if \( f \) satisfies (2.6) with \( \lambda \in \mathbb{C} \setminus \{0\} \) than the same holds true for \( J_q f \). Hence also \( \varphi_{\pm} \) satisfy (2.6) with the same \( \lambda \), and their invariance properties under \( J_q \) imply the assertion.

\[ \blacksquare \]

Remark 2.2. Equation (2.6) has been studied in [18, 20] for \( \lambda = 1 \) in connection with Maass forms on the full modular group \( \text{PSL}(2, \mathbb{Z}) \). This relation with [18, 20] will be made clear in sections 3 and 4, see in particular corollary 3.7 and theorem 4.6. Moreover, part (iii) of the proposition implies that equation (2.6) has no solutions for \( |\lambda| \) bigger than both the spectral radii of \( P^\pm \).

We now introduce the functional analytic settings. In this section we are interested in defining a family of spaces \( \mathcal{H}_{q,\varphi} \) to which the eigenfunctions of \( P^\pm \) belong.

Throughout this paper we use the notation \( \xi := \text{Re}(q) \) and are assuming \( \xi > 0 \). Let \( L \) and \( L^{-1} \) denote the Laplace and inverse Laplace transform, respectively. We recall that

\[ \mathcal{L}[t^{\nu-1}](x) = \Gamma(v) x^{-v} \quad \mathcal{L}^{-1}[x^{-v}](t) = \frac{t^{\nu-1}}{\Gamma(v)} \quad t \in \mathbb{R}, \ v \in \mathbb{C} \setminus \mathbb{Z}, \]  

(2.7)

where \( \Gamma(v) \) is the usual Gamma function, \( t^\nu := t^a H(t) \), with \( H(t) \) the Heaviside function, and for \( \text{Re}(v) < 0 \) we are working with generalized functions. In the following when it causes no confusion we drop for brevity the independent variable in the Laplace and inverse Laplace transforms. Hence we use the notation

\[ \mathcal{L}[\phi(t)] := \mathcal{L}[\phi(t)](x) \quad \mathcal{L}^{-1}[f(x)] := \mathcal{L}^{-1}[f(x)](t). \]  

(2.8)

We now consider the integral transform \( B_q \) introduced in [15] and defined as

\[ \phi(t) \mapsto B_q[\phi](x) := \frac{1}{\xi^2q} \int_0^\infty e^{-\frac{x}{t}} e^x \phi(t) m_q(dt), \]  

(2.9)

where \( m_q \) is the absolutely continuous measure on \( \mathbb{R}^+ \) defined as \( m_q(dt) = t^{2\xi-1} e^{-t} dt \). As for the Laplace transform in the notation we drop the dependence on the variable.

Finally for each \( q \) with \( \xi := \text{Re}(q) > 0 \) we introduce the notation

\[ L^p(m_q) := \left\{ \phi : \mathbb{R}^+ \to \mathbb{C} : \int_0^\infty |\phi(t)|^p t^{2\xi-1} e^{-t} dt < \infty \right\} \]  

(2.10)

with the norm

\[ \|\phi\|_p := \left( \int_0^\infty |\phi(t)|^p t^{2\xi-1} e^{-t} dt \right)^{\frac{1}{p}}. \]  

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It is immediate to check that
\[ L^1(m_q) \ni \phi \mapsto B_q[\phi] \in \mathcal{H}(B) \]
and that \( B_q \) is continuous on \( L^1(m_q) \) with values on \( \mathcal{H}(B) \) with the standard topology induced by the family of supremum norms on compact subsets of \( B \). Indeed
\[ |B_q[\phi](x)| \leq |x|^{-2} ||\phi||_1 \quad \forall x \in B \]
hence \( B_q \) sends bounded sets of \( L^1(m_q) \) into bounded sets of \( \mathcal{H}(B) \). Moreover, since \( m_q(0, \infty) = \Gamma(2\xi) \), one has \( L^q(m_q) \subset L^1(m_q) \) for all \( p \in [1, \infty] \). We now study the spaces which contain the eigenfunctions of \( \mathcal{P}_q \) in terms of the \( B_q \) transform.

**Definition 2.3.** For \( \Re(q) > 0 \), \( q \neq \frac{1}{2} \), \( \mu \in \mathbb{C} \) and \( p \in [1, +\infty] \), let \( \mathcal{H}^p_{q,\mu} \) be the space of functions \( f(x) \) written as
\[ f(x) = \frac{c \mu^\frac{1}{2}}{x^{2q}} + B_q \left[ \frac{b}{t} + \phi(t) \right](x) \quad x \in B \]
for \( c, b \in \mathbb{C} \) and \( \phi \in L^p(m_q) \) as defined in (2.10).

**Remarks 2.4.** The definition of the spaces \( \mathcal{H}^p_{q,\mu} \) needs some comments. First of all to prove that definition 2.3 is well posed, we extend the definition of the \( B_q \) transform by means of (2.7) to the function \( \phi(t) = \frac{1}{t} \) as
\[ B_q \left[ \frac{1}{t} \right] = \frac{1}{x^{2q}} \int_0^\infty e^{-\frac{1}{t}} \frac{1}{t} t^{2q-1} dt = \frac{1}{x^{2q}} \mathcal{L}[t^{2q-2} \left( \frac{1}{x} \right)] = \Gamma(2q - 1) x^{-1} \]
which is well defined for \( \xi = \Re(q) > 0 \) and \( q \neq \frac{1}{2} \), whereas the first integral is defined only for \( \xi > \frac{1}{2} \). Hence, the condition \( q \neq \frac{1}{2} \) comes from the term \( \frac{1}{t} \). A typical example of functions to which we will apply the \( B_q \) transform is
\[ \frac{b}{t} + \phi(t) = \frac{e^{-t}}{1 - e^{-t}} \frac{a_0}{\Gamma(2q)} + \frac{e^{-t}}{1 - e^{-t}} \sum_{n \geq 1} \frac{a_n}{\Gamma(n + 2q)} t^n \quad t \in \mathbb{R}^+ \]
with \( \lim \sup_n (a_n)^\frac{1}{2} \leq 1 \), for which \( b = \frac{a_0}{\Gamma(2q)} \) and \( \phi \) is in \( L^2(m_q) \) with \( \phi(0) = \frac{a_0}{\Gamma(2q + 1)} - \frac{a_0}{\Gamma(1 + 2q)} \).

A further consideration is about the function to which we apply \( B_q \). This function is written as \( (\frac{1}{t} + \phi(t)) \) above, but could be written in many different ways, and actually some of them will be used below. However, the main feature of this term that we want to stress is that it can be divided into two parts with respect to the behaviour at the origin: the first part has a singularity at \( t = 0^+ \) of order \( t^{-1} \), hence in particular does not belong to \( L^p(m_q) \) for \( \xi \leq \frac{1}{2} \); the second part is instead in \( L^p(m_q) \).

**Proposition 2.5.** The transfer operators \( \mathcal{P}^\pm_q \) leave invariant the spaces \( \mathcal{H}^2_{q,\mu} \) for any \( \mu \in \mathbb{C} \).

In particular,
\[ \mathcal{P}_{0,q} \left( \frac{c \mu^\frac{1}{2}}{x^{2q}} + B_q[\psi] \right) = \frac{c \mu^\frac{1}{2}}{x^{2q}} + B_q[M(\psi)] \]
\[ \mathcal{P}_{1,q} \left( \frac{c \mu^\frac{1}{2}}{x^{2q}} + B_q[\psi] \right) = B_q \left[ c \frac{\mu^\frac{1}{2}}{x^{2q}} + N_q(\psi) \right] \]
where \( M \) and \( N_q \) are linear operators defined by
\[ M(\psi)(t) = e^{-t} \psi(t) \]
\[ N_q(\psi)(t) = \int_0^\infty \frac{J_{2q-1}(2\sqrt{st})}{(st)^{\frac{n-3}{2}}} \psi(s) m_q(ds) \]
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with $J_p$ the Bessel function of order $p$, and the function $\tilde{\phi}_{q, \mu}(t)$ defined by

$$\tilde{\phi}_{q, \mu}(t) := \mu \sum_{k=0}^{\infty} \frac{(\log \mu)^k}{\Gamma(k + 1) \Gamma(k + 2q)} t^k$$

(2.16)

letting $\tilde{\phi}_{q, 0}(t) \equiv 0$.

**Proof.** To prove (2.13), we study the action of the transfer operators on the different terms of a function $f \in \mathcal{H}_{q, \mu}^2$, for $q$ and $\mu$ fixed.

First of all, it is immediate that $P_{0,q} \left( \frac{\mu^i}{x^{2q}} \right) = \mu \frac{\mu^i}{x^{2q}}$ and $P_{1,q} \left( \frac{\mu^i}{x^{2q}} \right) = \mu^{i+1}$. It follows that the function $\tilde{\phi}_{q, \mu}$ we are looking for has to satisfy

$$\mu^{i+1} = B_q [\tilde{\phi}_{q, \mu}] = \frac{1}{x^{2q}} \mathcal{L}[t^{2q-1} \tilde{\phi}_{q, \mu}(t)] \left( \frac{1}{x} \right)$$

hence

$$\tilde{\phi}_{q, \mu}(t) = \frac{1}{x^{2q-1}} \mathcal{L}^{-1} \left[ \frac{\mu^{i+1}}{x^{2q}} \right] (t).$$

Expression (2.16) follows by a straightforward computation. It remains to prove that the function on the right-hand side of (2.16) admits a $B_q$ transform. If we differentiate term by term, it follows that for any $n \in \mathbb{N}$ there exists a positive polynomial $p_n(t)$ of degree $n$ such that

$$\frac{d}{dt} \tilde{\phi}_{q, \mu}(t) \leq p_n(t) + \frac{|\log \mu|}{n + 2q} |\tilde{\phi}_{q, \mu}(t)|.$$

From this it follows that the behaviour of $\tilde{\phi}_{q, \mu}$ at infinity is slower than $e^{\varepsilon t}$ for all $\varepsilon > 0$. Moreover, $\tilde{\phi}_{q, \mu}(0) = \frac{\mu}{\Gamma(2q)}$. Hence actually $\tilde{\phi}_{q, \mu} \in L^2(m_q)$.

For $f \in \mathcal{H}_{q, \mu}^2$ we write $f(x) = \frac{\mu^i}{x^{2q}} + B_q [\psi]$, where $\psi(t) = \frac{b}{\varepsilon} + \phi(t)$. The identities (2.13) have been proved in [10, proposition 2.3] for functions $\phi$ in $L^2(m_q)$. It remains to prove that the same holds for the term $\frac{\varepsilon}{\gamma}$. For $P_{0,q}$ we simply have

$$P_{0,q} \left( B_q \left[ \frac{1}{\gamma} \right] \right) = \frac{1}{x^{2q}} \int_0^{\infty} e^{-t} \frac{\varepsilon^{-1}}{t} t^{2q-1} \, dt = B_q \left[ \frac{\varepsilon^{-1}}{\gamma} \right].$$

For $P_{1,q}$ we have to prove

$$P_{1,q} \left( B_q \left[ \frac{1}{\gamma} \right] \right) = B_q \left[ N_q \left( \frac{1}{\gamma} \right) \right].$$

Using the power series expansion

$$J_p(x) = \frac{x^p}{2^p} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m! \Gamma(m + p + 1)}$$

for the Bessel function $J_{2q-1}$, we have

$$N_q \left( \frac{1}{\gamma} \right) = \int_0^{\infty} \frac{\varepsilon^{m+2q-2} e^{-\varepsilon}}{m! \Gamma(m+2q)} \, ds = \sum_{m=0}^{\infty} \frac{(-1)^m r^m}{m! \Gamma(m+2q)} L[s^{m+2q-2}]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+2q-1)}{m! \Gamma(m+2q)} \, r^m.$$
It follows that
\[ B_q \left[ N_q \left( \frac{1}{t} \right) \right] = \frac{1}{x^{2q}} \mathcal{L} \left[ t^{2q-1} N_q \left( \frac{1}{x} \right) \right] \]
\[ = \frac{1}{x^{2q}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+2q-1)}{m! \Gamma(m+2q)} \mathcal{L} \left[ t^{m+2q-1} \right] \left( \frac{1}{x} \right) \]
\[ = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+2q-1)}{m!} x^m = \Gamma(2q-1) (1+x)^{1-2q}. \]

Moreover,
\[ \mathcal{P}_{1,q} (B_q \left[ \frac{1}{t} \right]) = \int_0^\infty e^{-tx} e^{-t^{2q-2}} dt = \mathcal{L} \left[ e^{-t^{2q-2}} \right] = \Gamma(2q-1) (1+x)^{1-2q} \]
This completes the proof. \( \square \)

**Remark 2.6.** Note that \( \mathcal{P}_{1,q} (H_{q,\mu}^2) \) is contained in the set of functions which are \( B_q \) transform of functions in \( L^2(m_q) \). This follows from the fact that \( \delta_{q,\mu} \in L^2(m_q) \), \( N_q(\phi) \in L^2(m_q) \) for any \( \phi \in L^2(m_q) \) (see [5]), and writing
\[ N_q \left( \frac{1}{t} \right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+2q-1)} t^m = \int_0^s s^{2q-2} e^{-s} ds = \Gamma(2q-1) - \int_s^\infty s^{2q-2} e^{-s} ds, \]
where the third term makes sense only for \( s > \frac{1}{2} \) and the last one for all \( s > 0 \) and \( t > 0 \). In particular, it follows from (2.17) that \( N_q (\frac{1}{t}) \) is bounded as \( t \to 0^+ \), and behaves like \( O(t^{1-2q}) \) as \( t \to +\infty \). Moreover, using the Dirac delta function \( \delta_{-\log \mu} \) we can write
\[ \frac{\mu^\frac{1}{2}}{x^{2q}} = B_q \left[ \frac{\delta_{-\log \mu} (t)}{t^{2q-1}} \right] \quad \text{and} \quad \frac{\mu^\frac{1}{2}}{x^{2q}} = B_q \left[ \frac{\delta_{-\log \mu} (t)}{t^{2q-1}} \right]. \]
and
\[ \tilde{\phi}_{q,\mu} (t) = \mu \frac{J_{2q-1} (2\sqrt{-t \log \mu})}{(\sqrt{-t \log \mu})^{\frac{1}{2}}} = N_q \left( \frac{\delta_{-\log \mu} (t)}{t^{2q-1}} \right), \]
where \( J_p \) denotes the Bessel function. Hence for the spaces
\[ K_{q,\mu}^2 := \left\{ \psi : \psi \in C \right\} \quad \text{one finds} \]
\[ H_{q,\mu}^2 = B_q[K_{q,\mu}^2] \]
and moreover
\[ \mathcal{P}_{0,q} \left( \frac{c \mu^\frac{1}{2}}{x^{2q}} + B_q [\psi] \right) = B_q \left[ M \left( \frac{\delta_{-\log \mu} (t)}{t^{2q-1}} + \psi \right) \right] \]
\[ \mathcal{P}_{1,q} \left( \frac{c \mu^\frac{1}{2}}{x^{2q}} + B_q [\psi] \right) = B_q \left[ N_q \left( \frac{\delta_{-\log \mu} (t)}{t^{2q-1}} + \psi \right) \right] \]
that is for any \( \chi \in K_{q,\mu}^2 \)
\[ \mathcal{P}_{0,q} (B_q [\chi]) = B_q [M(\chi)] \quad \text{and} \quad \mathcal{P}_{1,q} (B_q [\chi]) = B_q [N_q(\chi)] \]
The spaces \( K_{q,\mu}^2 \) can be written as
\[ K_{q,\mu}^2 = \text{Span}_C \left( \frac{\delta_{-\log \mu} (t)}{t^{2q-1}} \right) \oplus \text{Span}_C \left( \frac{1}{t} \right) \oplus L^2(m_q). \]
Putting together (2.19) and (2.20), we obtain that (2.6) implies that there exists a constant \( c \) to the Laplace transform gives the \( \mathcal{H}_{q,\mu} \) for \( \mathcal{P}_{0,q} \) but not for \( \mathcal{P}_{1,q} \). In fact \( N_q(\phi) \) is not defined for all functions \( \phi \in L^1(m_q) \).

**Remark 2.7.** Note that (2.13) also holds on \( \mathcal{H}_{q,\mu}^1 \) for \( \mathcal{P}_{0,q} \) but not for \( \mathcal{P}_{1,q} \). In fact \( N_q(\phi) \) is not defined for all functions \( \phi \in L^1(m_q) \).

**Theorem 2.8.** If \( f \in \mathcal{H}(B) \) and \( \mathcal{P}_{q}^\pm f = \lambda f \) with \( \lambda \in \mathbb{C} \setminus [0, 1) \) then \( f(x) \in \mathcal{H}_{q,\lambda}^2 \).

**Proof.** We use the properties of the inverse Laplace transform. In particular, we recall the following.

**Lemma 2.9.** A function \( u(x) \) is the Laplace transform of a generalized function if and only if there exists \( k \in \mathbb{R} \) such that \( u(x) \) is holomorphic in the half-plane \( \{ \text{Re}(x) > k \} \) and

\[
|u(x)| \leq M(1 + |x|)^m \quad \text{Re}(x) > k
\]

for some constants \( M, m \).

Let now \( f \in \mathcal{H}(B) \) satisfy \( \mathcal{P}_{q}^\pm f = \lambda f \) for some \( \lambda \in \mathbb{C} \setminus [0, 1) \). We study separately the cases \( \lambda = 1 \) and \( \lambda \in \mathbb{C} \setminus [0, 1] \). Let us first consider the case \( \lambda = 1 \). By proposition 2.1(ii), the function \( f \) satisfies equation (2.6), and we can write

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - 1)^n \quad \text{for } |x - 1| < 1,
\]

where the sequence \( \{a_n\} \) satisfies \( \limsup_n (a_n) \frac{1}{n} \leq 1 \), and the convergence is uniform on any compact set contained in \( |x - 1| < 1 \). By lemma 2.9, the right-hand side of equation (2.6) is the Laplace transform of a generalized function. Using (2.7) and (2.18) and known properties of the Laplace transform, we obtain

\[
\mathcal{L}^{-1}\left[ \frac{1}{x + 1} \right]^{2q} \left( \frac{x}{x + 1} \right) (t) = t_i^{2q-1} e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\Gamma(n + 2q)} t^n.
\]

The left-hand-side of equation (2.6) is then the Laplace transform of a generalized function. Moreover, from equation (2.6) with \( \lambda = 1 \) we obtain

\[
f(x + n) - f(x) \equiv \sum_{h=0}^{n-1} (f(x + h + 1) - f(x + h)) = - \sum_{h=0}^{n-1} \frac{f(x + h + 1)}{(x + h + 1)^{2q}}.
\]

Hence \( f \) satisfies the assumptions of lemma 2.9 for any \( k > 0 \). Hence

\[
\mathcal{L}^{-1}[f(x) - f(x + 1)] = (1 - e^{-t}) \mathcal{L}^{-1}[f].
\]

Putting together (2.19) and (2.20), we obtain that (2.6) implies that there exists a constant \( c \in \mathbb{C} \) such that

\[
f(x) = c + \mathcal{L} \left[ t_i^{2q-1} e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n + 2q)} \right] (x).
\]

Moreover, since by proposition 2.1(ii) it holds \( \mathcal{J}_q f = \pm f \), we can apply the operator \( \mathcal{J}_q \) to the right-hand side of (2.21) to have \( \pm f \). It is a straightforward computation that applying \( \mathcal{J}_q \) to the Laplace transform gives the \( \mathcal{B}_q \) transform defined in (2.9), hence

\[
f(x) = \pm \left[ \frac{c}{x^{2q}} + \mathcal{B}_q \left[ \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n + 2q)} \right] (x) \right]
\]

and the thesis follows for \( \lambda = 1 \). Let us now consider the case \( \lambda \in \mathbb{C} \setminus [0, 1] \). Let us first assume that \( f \) is bounded at \( x = 0 \). In this case, \( \mathcal{J}_q f = \pm f \) implies that \( |f(x)| = O(|x|^{-2q}) \) as \( \text{Re}(x) \to \infty \), hence \( f \) satisfies the
assumptions of lemma 2.9 with \( k = 0 \). In particular, using again the expansion (2.18), we can write equation (2.19) and the analogous of (2.20) which is 
\[
\mathcal{L}^{-1}[\lambda f(x) - f(x + 1)] = (\lambda - e^{-t}) \mathcal{L}^{-1}[f].
\]
Hence in this case we obtain 
\[
f(x) = \mathcal{L} \left[ \frac{1}{\lambda - e^{-t}} \sum_{n=0}^{\infty} (-1)^n a_n t^n \Gamma(n + 2q) \right] (x),
\]
(2.23)

Using again the same argument as before applying the involution \( \mathcal{J}_q \) to the right-hand side of (2.23), it follows that 
\[
f(x) = \mathcal{B}_q \left[ \frac{e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} (-1)^n a_n t^n \Gamma(n + 2q) \right] (x)
\]
(2.24)

hence \( f \in H^2_{q, \lambda} \) with \( c = b = 0 \), and 
\[
\phi(t) = \frac{e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} (-1)^n a_n t^n \Gamma(n + 2q),
\]
(2.25)

which can be written in the form (2.12) with \( \phi(0) = \frac{1}{\lambda^q} \sum_{n=0}^{\infty} a_n \Gamma(n + 2q) \). We finish the proof by studying the case \( \lambda \in \mathbb{C} \setminus [0, 1] \) when the function \( f(x) \) is not bounded at \( x = 0 \). Let us consider the function \( g(x) := \frac{f(x)}{e^{-x}} \). By equation (2.6) we can write 
\[
f(x + n) = \lambda^n f(x) - \sum_{h=0}^{n-1} \Lambda(n-h-1) f \left( \frac{x + h}{\lambda^{n-h+1}} \right) \frac{1}{(x+h+1)^{2q}},
\]
which implies that \( g \) satisfies 
\[
g(x + n) = g(x) - \sum_{h=0}^{n-1} \Lambda(n-h-1) f \left( \frac{x + h}{\lambda^{n-h+1}} \right) \frac{1}{(x+h+1)^{2q}}.
\]

Hence depending on whether \( |\lambda| \leq 1 \) or \( |\lambda| > 1 \), either \( f \) or \( g \) satisfy the assumptions of lemma 2.9 for any \( k > 0 \). Since equation (2.19) still applies, if \( |\lambda| \leq 1 \) we can repeat the same argument as above and obtain (2.23). If \( |\lambda| > 1 \), equation (2.6) implies that \( g \) satisfies 
\[
g(x) - g(x + 1) = \frac{1}{\lambda^{x+1}} \left( \frac{1}{x+1} \right)^{2q} f \left( \frac{x}{\lambda + 1} \right) = \sum_{n=0}^{\infty} (-1)^n a_n \frac{e^{-(x+1) \log \lambda}}{(x+1)^{n+2q}},
\]
(2.26)

where we have again used the expansion (2.18) for \( f \). Both the right-hand side of (2.26) and \( g \) satisfy the assumptions of lemma 2.9. Hence we can apply \( \mathcal{L}^{-1} \) to both sides of (2.26) and write the analogous of (2.20) for \( g \), to obtain that there exists \( c \in \mathbb{C} \) such that 
\[
g(x) = c + \mathcal{L} \left[ \frac{(t - \log \lambda)^{x+1} e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} (-1)^n a_n \frac{e^{-(x+1) \log \lambda}}{(n + 2q)} (t - \log \lambda)^n \right] (x),
\]
(2.27)

where we have used the notation \( (t - \log \lambda)_+ := (t - \log |\lambda|)_+ - i \arg \lambda \), and the relation 
\[
\mathcal{L}^{-1} \left[ \frac{e^{-(x+1) \log \lambda}}{(x+1)^{n+2q}} \right] (t) = \frac{e^{-(t - \log \lambda)_+ n+2q-1}}{1 - e^{-t}} \sum_{n=0}^{\infty} (-1)^n a_n \frac{e^{-(x+1) \log \lambda}}{(n + 2q)}.
\]

Now, since \( \mathcal{J}_q \) \( f = \pm f \), we obtain from (2.27) and the definition of (2.9) 
\[
f(x) = \pm \frac{\lambda^{x}}{x^{2q}} g \left( \frac{1}{x} \right) = \pm \left( \frac{c \lambda^{x}}{x^{2q}} + \mathcal{B}_q \left[ \frac{e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} (-1)^n a_n t^n \Gamma(n + 2q) \right] \right),
\]
(2.28)

and the thesis follows with \( b = 0 \) and \( \phi \) as in (2.25). \( \square \)
Corollary 2.10. If \( f \in \mathcal{H}(B) \) and \( P_q^\pm f = \lambda f \) with \( \lambda \in \mathbb{C} \setminus [0, 1) \) then it has the form

\[
f(x) = \frac{c \lambda^q}{x^{2q}} + \frac{\Gamma(2q-1)}{\Gamma(2q)} \frac{b}{x} + B_q[\phi],
\]

(2.29)

with \( c, b \in \mathbb{C} \) and \( \phi \in L^2(m_q) \) with \( \phi(0) \) finite and \( \phi(t) = \phi(0) + O(t) \) as \( t \to 0^+ \). Moreover, if \( \lambda \neq 1 \) then \( b = 0 \), if \( \lambda = 1 \) then \( b = f(1) \). If \( f \) is an eigenfunction of \( P_q^- \) then \( b = f(1) = 0 \).

Finally the function \( \phi \in L^2(m_q) \) is such that \( B_q[\phi] \) is bounded as \( x \to 0 \).

**Proof.** The form (2.29) follows from the proof of theorem 2.8 (see (2.22), (2.24) and (2.28)) and (2.11). That \( b = 0 \) if \( \lambda \neq 1 \) follows from (2.24) and (2.28) and that \( b = f(1) \) if \( \lambda = 1 \) follows from (2.22). That \( f(1) = 0 \) for eigenfunctions of \( P_q^- \) for any \( \lambda \) follows from proposition 2.1-(ii). We now have to prove boundedness of \( B_q[\phi] \) as \( x \to 0 \). We start with the case \( \lambda = 1 \). From equation (2.22), it follows that we have to prove boundedness of \( B_q[\psi] \) with

\[
\psi(t) := \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\Gamma(n+2q)} t^n - \frac{a_0}{\Gamma(2q)} \frac{1}{t}. \tag{2.30}
\]

Since for any \( \phi \in L^2(m_q) \) the definition of the \( B_q \) transform implies

\[
B_q[\phi]\left(\frac{1}{x}\right) = x^{2q} \mathcal{L}\left[ t^{2q-1} \phi \right](x),
\]

we need to show that \( x^{2q} \mathcal{L}\left[ t^{2q-1} \psi \right](x) \) is bounded as \( x \to \infty \). Since \( \limsup_n (a_n)^{\frac{1}{n}} \leq 1 \), the power series in (2.30) converges uniformly for \( t \in \mathbb{R} \). Hence

\[
\mathcal{L}\left[ t^{2q-1} \psi \right] = \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\Gamma(n+2q)} \mathcal{L}\left[ t^{n+2q-1} e^{-t} \right] + \frac{a_0}{\Gamma(2q)} \mathcal{L}\left[ t^{2q-1} e^{-t} - t^{2q-2} \right]. \tag{2.31}
\]

Moreover, since \( e^{-t} < 1 \) for \( t > 0 \), for all \( n \geq 1 \) and \( \xi > 0 \) we obtain

\[
\mathcal{L}\left[ t^{n+2q-1} e^{-t} \right] = \sum_{k=1}^{\infty} \mathcal{L}\left[ e^{-xt} t^{n+2q-1} \right] = \sum_{k=1}^{\infty} \frac{\Gamma(n+2q)}{(x+k)^{n+2q}} = \Gamma(n+2q) \zeta_H(n+2q, x+1), \tag{2.32}
\]

where \( \zeta_H(s, a) \) denotes the Hurwitz zeta function, and we have used uniform convergence of the series of functions in the first equality. The same argument works for \( n = 0 \) if \( \xi > \frac{1}{2} \), hence using analytic continuation of \( \zeta_H(2q, x+1) \) to \( \xi > \frac{1}{2}, q \neq \frac{1}{2} \), we write

\[
\mathcal{L}\left[ t^{2q-1} e^{-t} - t^{2q-2} \right] = \Gamma(2q) \zeta_H(2q, x+1) - \Gamma(2q-1) \frac{1}{x^{2q-1}}. \tag{2.33}
\]

Putting together (2.31), (2.32) and (2.33), we obtain

\[
x^{2q} \mathcal{L}\left[ t^{2q-1} \psi \right](x) = x^{2q} \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\Gamma(n+2q)} \zeta_H(n+2q, x+1) - \frac{a_0}{\Gamma(2q)} \Gamma(2q-1) \frac{x^{2q}}{x^{2q-1}}. \tag{2.34}
\]

To study the behaviour of (2.34) as \( x \to \infty \), we use the following formula (see [1, p 269]) valid for \( \text{Re}(n+2q) > 0, n+2q \neq 1 \)

\[
\zeta_H(n+2q, x+1) = \frac{1}{(x+1)^{n+2q}} + \frac{(x+1)^{1-n-2q}}{n+2q} - (n+2q) \int_0^\infty \frac{t - \lfloor t \rfloor}{(t+x)^{n+2q}+1} dt. \tag{2.35}
\]
Using (2.35), we write (2.34) as a sum of different terms. First, we isolate the $n = 0$ term and the last term in (2.34), which give

$$a_0 \frac{x^{2q}}{(x + 1)^{2q}} + a_0 \frac{x^{2q}}{2q - 1} (x + 1) - \frac{a_0}{2q - 1} x - a_0 2q x^{2q} \int_0^\infty \frac{t - \lfloor t \rfloor}{(t + x + 1)^{2q+1}} \, dt$$

$$= O(1) + \frac{a_0}{2q - 1} \left( \frac{x^{2q}}{(x + 1)^{2q}} - 1 \right) x + \frac{a_0}{2q - 1} x^{2q} + O(1) = O(1)$$

as $x \to \infty$, since

$$\left| \int_0^\infty \frac{t - \lfloor t \rfloor}{(t + x + 1)^{2q+1}} \, dt \right| \leq \frac{x^{2q} \int_0^\infty \frac{1}{(t + x + 1)^{2q+1}} \, dt}{2q - 1} = \frac{x^{2q}}{2q - 1} \frac{x^{2q}}{(x + 1)^{2q}}.$$

Now we come to the terms in (2.34) with $n \geq 1$. From (2.35) we obtain

$$\left| \sum_{n=1}^\infty (-1)^n a_n \xi_H(n + 2q, x + 1) \right| \leq \frac{x^{2q}}{(x + 1)^{2q}} \sum_{n=1}^\infty |a_n| (n + 2q - 1) (x + 1)^{n+1}$$

and the right-hand side is $O(1)$ as $x \to \infty$. Hence the thesis follows for $\lambda = 1$. For $\lambda \neq 1$, boundedness of $B_\lambda[\phi]$ follows from (2.24) and (2.28). Indeed in equation (2.24), the function $f(x)$ is bounded as $x \to 0$ by assumption, and $f(x) = B_\lambda[\phi]$. Moreover, from equation (2.28) it follows that the function $\phi$ is the same as in (2.24), hence $B_\lambda[\phi]$ is again bounded as $x \to 0$.

An example of eigenfunction of $P_\lambda^*$ with $\lambda = 1$ is the family of functions defined as

$$f_\lambda^+(x) = \frac{x \xi_H(2q)}{2} \left( 1 + \frac{1}{x^{2q}} \right) + \sum_{m,n \geq 1} \frac{1}{(mx+n)^{2q}}$$

for $\text{Re}(q) > 1$

where $\xi_H(s)$ denotes the Riemann zeta-function. It is shown in [20] that the family $\psi_\lambda^+(x) = \frac{x \xi_H(2q)}{\Gamma(2q)} f_\lambda^+$ can be analytically continued to $q \in \mathbb{C}$. In particular, $\psi_\lambda^+(x) = \frac{x}{2}$ is the invariant density of the Farey map and the only eigenfunction of $P_\lambda^*$ with $\lambda = 1$. We can write the functions $f_\lambda^+$ as in (2.29). First of all, note that

$$\sum_{m,n \geq 1} \frac{1}{(mx+n)^{2q}} = \frac{x^{2q}}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{1}{n^{2q} \left( \frac{m}{n} + \frac{1}{2} \right)^{2q}} = \frac{1}{\Gamma(2q) x^{2q}} \sum_{m,n \geq 1} \frac{1}{n^{2q}} \left[ t^{2q-1} e^{-\frac{t}{x}} \right] t^{\frac{1}{x}}$$

$$= \frac{1}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{1}{n^{2q}} B_q \left[ e^{-\frac{t}{x}} \right] = B_q \left[ \frac{1}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{e^{-\frac{t}{x}}}{n^{2q}} \right]$$

$$= B_q \left[ \frac{1}{\Gamma(2q)} \sum_{n \geq 1} \frac{e^{-\frac{t}{x}}}{n^{2q}} \right]$$

$$= B_q \left[ \frac{1}{\Gamma(2q)} \sum_{k \geq 0} \frac{B_k}{k!} t^k \left( \sum_{n \geq 1} \frac{1}{n^{k+2q-1}} \right) \right]$$

$$= B_q \left[ \frac{1}{\Gamma(2q)} \sum_{k \geq 0} \frac{B_k \xi_R(k + 2q - 1)}{k!} t^k \right].$$
where \([B_k]\) are the Bernoulli numbers, and the series converges for \(|t| < 2\pi\) since \(\zeta_R(k + 2q - 1) = O(1)\) as \(k \to \infty\). Hence we obtain

\[
f_q^+(x) = \frac{\zeta_R(2q)}{2} \frac{1}{x^2} + \frac{\Gamma(2q - 1)}{\Gamma(2q)} \frac{\zeta_R(k + 2q - 1)}{x} + B_q[\phi],
\]

where \(\phi\) is given for \(|t| < 2\pi\) by

\[
\phi(t) = \frac{\zeta_R(2q)}{2 \Gamma(2q)} + \frac{1}{\Gamma(2q)} \sum_{k \geq 1} B_k \zeta_R(k + 2q - 1) \frac{1}{k!}.
\]

Moreover, in [20] it is also proved that any eigenfunction of \(P_q^+\) with \(\lambda = 1\), when written as in (2.29), satisfies \(c = \alpha \frac{1}{2} \zeta_R(2q)\) and \(b = \alpha \zeta_R(2q - 1)\) for some \(\alpha \in \mathbb{C}\) (see [20, remark 1, p 246]). An example of eigenfunction of \(P_q^-\) with \(\lambda = 1\) is the family of functions

\[
f_q^-(x) = 1 - \frac{1}{x^2} - \frac{1}{x^2} + B_q \left[ \frac{1}{\Gamma(2q)} \right]
\]

in particular \(b = 0\) as stated in corollary 2.10. Moreover, any other eigenfunction of \(P_q^-\) with \(\lambda = 1\) can be written as in (2.29) with \(c = b = 0\). Indeed, \(b = 0\) since it is an eigenfunction of \(P_q^-\) as stated in corollary 2.10, and \(c\) can be eliminated by subtracting a multiple of \(f_q^+\).

### 3. Induced operators

The Farey map \(F\) has at least two induced versions. The first one is the well-known Gauss map \(G\) which is obtained by iterating \(F\) once plus the number of times necessary to reach the interval \([1/2, 1]\). The map \(G : [0, 1] \to [0, 1]\) is given by

\[
G(x) = \begin{cases} 
F^{\lfloor 1/x \rfloor}(x) = \left\{ \begin{array}{ll} 1/x & x > 0 \\
0 & x = 0. \end{array} \right. 
\end{cases}
\]

It is easily checked that the Gauss map acts on the continued fraction expansion of a number \(x \in [0, 1]\) as \(x = [a_1, a_2, a_3, \ldots]\) implies \(G(x) = [a_2, a_3, \ldots]\). We now introduce a family of operators related to the Gauss map. Recall that we assume \(\xi = \text{Re}(q) > 0\). Consider the spaces

\[
\mathcal{H}_q^{\mu} := \{ g \in \mathcal{H}_q^{\mu} : c = b = 0 \} \subset \mathcal{H}(B), \quad \xi > 0 \text{ and } p \in [1, +\infty].
\]

A function \(g\) in \(\mathcal{H}_q^{\mu}\) is a \(B_q\) transform of an \(L^p(m_q)\) function. Moreover, let

\[
\hat{\mathcal{H}}_q^{\mu} := \{ g \in \mathcal{H}_q^{\mu} : g = B_q[\phi] \text{ with } \phi(t) = \phi(0) + O(t^\varepsilon) \text{ for some } \varepsilon > 0 \text{ as } t \to 0^+ \}.
\]

Recall that from the definition of the \(L^p(m_q)\) spaces (see (2.10)), it follows that \(\mathcal{H}_q^{\mu} \subset \mathcal{H}_q^i\) and \(\hat{\mathcal{H}}_q^{\mu} \subset \hat{\mathcal{H}}_q^i\) for all \(p \geq 1.\)

We introduce the family of operators

\[
\mathcal{Q}_{q,z} : \mathcal{H}_q^i \to \mathcal{H}(B), \quad z \in \mathbb{C} \setminus [1, +\infty) \text{ and } \xi > 0
\]

\[
\mathcal{Q}_{q,1} : \hat{\mathcal{H}}_q^i \to \mathcal{H}(B), \quad \xi > \frac{1}{2}
\]

defined by

\[
g(x) = B_q[\phi](x) \mapsto (\mathcal{Q}_{q,z}g)(x) = z \mathcal{P}_{1,q} B_q \left[ (1 - z e^{-t})^{-1} \phi(t) \right](x). \tag{3.4}
\]

The operators are well defined since \((1 - z e^{-t})^{-1}\) \(\phi(t)\) is in \(L^1(m_q)\) for any \(\phi\) in \(L^1(m_q)\), being \((1 - z e^{-t})^{-1}\) in \(L^\infty\) for \(z \in \mathbb{C} \setminus [1, +\infty]\). If instead \(z = 1\) then \((1 - e^{-t})^{-1}\) \(\phi(t)\) is in \(L^1(m_q)\) if \(\phi(0)\) is finite since \(\frac{1}{z}\) is in \(L^1(m_q)\) for \(\xi > \frac{1}{2}\).
Remark 3.1. Note that \( Q_{q,z}(H_q^p) \) is contained in \( H_q^2 \) for all \( p \geq 2 \), since by remark 2.6 the set \( P_{1,q}(H_q^2) \) is in \( H_q^2 \). Hence, using proposition 2.5, the operators \( Q_{q,z} \) induce the operators \( Q_{q,z} : L^2(m_q) \to L^2(m_q) \) given by

\[
\phi \mapsto Q_{q,z}\phi = z N_q (1 - z M)^{-1} \phi. \tag{3.5}
\]

Note that for \( z = 1 \), the condition \( Q_{q,1}\phi = \phi \) is identical to the integral equation studied by Lewis [18].

Theorem 3.2. The operators \( Q_{q,z} \) admit the integral representation

\[
(Q_{q,z}B_q[\phi(t)])(x) = z \int_0^\infty \frac{e^{-t(x+1)}}{(1 - z e^{-t})} \phi(t) \, dt \in \mathcal{H}(B). \tag{3.6}
\]

In particular for \( \xi > 0 \) the function \( z \mapsto Q_{q,z}g \) is analytic in \( \mathbb{C} \setminus [1, +\infty) \) for any \( g \in \mathcal{H}_q^1 \).

Proof. The integral representation (3.6) is a straightforward consequence of definitions of \( B_q \) in (2.9) and \( P_{1,q} \) in (2.3). Moreover, since

\[
|e^{-t(x+1)}| \leq e^{-t} \quad \forall x \in B
\]

and \((1 - z e^{-t})^{-1} \phi(t)\) is in \( L^1(m_q) \) as defined in (2.10), the integral in (3.6) is finite. The analyticity of \( z \mapsto Q_{q,z}g \) on \( z \in \mathbb{C} \setminus [1, +\infty) \) for \( \xi > 0 \) follows by the uniform convergence of the integral representation. \( \square \)

Remark 3.3. Note that for \( \phi(t) = 1 \), which is in \( L^1(m_q) \) for \( \xi > 0 \), we have

\[
(Q_{q,z}B_q[1])(x) = z \Gamma(2q) \Phi(z, 2q, x + 1)
\]

where \( \Phi(z, s, a) \) is the Lerch zeta function, defined for \( |z| < 1 \) and \( \text{Re}(a) > 0 \) as

\[
\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}
\]

(see, e.g., [12, vol II, p 27]).

Theorem 3.4. If \( g \in \mathcal{H}_q^1 \) then we can write

\[
(Q_{q,z}g)(x) = \sum_{n \geq 1} \frac{z^n}{(x+n)^{2q}} g\left(\frac{1}{x+n}\right) \tag{3.7}
\]

for \( |z| < 1 \) if \( \xi > \frac{1}{2} \), and for \( |z| \leq 1 \) if \( \xi > 1 \).

Proof. Using definition (3.4), write formally

\[
(Q_{q,z}g)(x) = z P_{1,q}B_q \left[ \sum_{n=0}^{\infty} z^n e^{-nt} \phi(t) \right]
\]

\[
= \sum_{n=0}^{\infty} z^{n+1} \int_0^\infty e^{-t(x+1)} t^{2q-1} e^{-nt} \phi(t) \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{z^{n+1}}{(x+n+1)^{2q}} B_q \left[ \phi \right] \left(\frac{1}{x+n+1}\right)
\]

\[
= \sum_{n \geq 1} \frac{z^n}{(x+n)^{2q}} g\left(\frac{1}{x+n}\right).
\]
The conditions for convergence follow from conditions for uniform convergence of the series in the second equality.

**Remark 3.5.** From (3.7) it follows that the operators \( Q_{q,1} \) coincide with the generalized transfer operators of the Gauss map \( G \) for functions \( g \in \tilde{H}^1_q \).

We now study the relations between \( Q_{q,z} \) and \( P_q^\pm \).

**Theorem 3.6.** Let \( f \in \mathcal{H}^1_{q,\mu} \) with \( z = \frac{1}{\mu} \in \mathbb{C} \setminus (1, \infty) \). Then for \( \xi = \text{Re}(q) > \frac{1}{2} \)

\[
(1 + Q_{q,z}) \left(1 - z P_{0,q}\right) f = (1 - z P^\pm_\mu) f \pm \epsilon B^r_q.
\]

If \( \xi \leq \frac{1}{2}, q \neq \frac{1}{2}, \) then equality (3.8) holds for \( f \in \mathcal{H}^1_{q,1} \), and if \( \mu \neq 1 \) for functions \( f \in \mathcal{H}^1_{q,\mu} \) with \( b = 0 \).

**Proof.** First of all, we show that the left-hand side of (3.8) is well defined. This follows using Proposition 2.5 and Remark 2.7, and writing

\[
(1 - z P_{0,q}) f = \left(1 - z P_{0,q}\right) \left( e^{\mu x_{2q}} + B_q \left[ \frac{b}{t} + \phi(t) \right] \right) = \frac{e^{\mu x_{2q}}}{x_{2q}} + B_q \left[ \frac{b}{t} + \phi(t) \right] - z \left( \phi(t) (1 - e^{-t}) + \frac{b (1 - e^{-t})}{t} \right).
\]

The last term is in \( \mathcal{H}^1_q \) since the function in square brackets is in \( L^1(m_q) \) for any \( \phi \in L^1(m_q) \) and \( \xi > 0 \). The first term is in \( L^1(m_q) \) for \( \xi > \frac{1}{2} \), and vanishes for \( \xi \leq \frac{1}{2} \) since if \( \mu \neq 1 \) we have \( b = 0 \). Hence we can now apply \((1 - Q_{q,z}) \). Assume first \( \mu \neq 1 \) and \( \xi \leq \frac{1}{2} \), then it follows that

\[
(1 + Q_{q,z}) \left(1 - z P_{0,q}\right) f = (1 + Q_{q,z}) B_q \left[ \phi(t) (1 - e^{-t}) \right] = B_q \left[ \phi(t) (1 - e^{-t}) \right] - z \left( \phi(t) (1 - e^{-t}) \right)^{-1} \times \left( \phi(t) (1 - e^{-t}) \right) = (1 - z P_{0,q} - z P_{1,q}) B_q[\phi] = (1 - z P^\pm_q) f,
\]

where we have used again Proposition 2.5 and Remark 2.7. The case \( \mu \neq 1 \) and \( \xi > \frac{1}{2} \) follows in the same way, by writing \( f = B_q[\tilde{\phi}] \) with \( \tilde{\phi} = \frac{2}{3} + \phi \in L^1(m_q) \). In the case \( \mu = 1 \) instead we obtain

\[
(1 + Q_{q,1}) \left(1 - P_{0,q}\right) f = (1 + Q_{q,1}) B_q \left[ \phi(t) (1 - e^{-t}) + \frac{b (1 - e^{-t})}{t} \right] = B_q \left[ \phi(t) (1 - e^{-t}) + \frac{b (1 - e^{-t})}{t} \right] - P_{1,q} B_q \left[ \phi(t) (1 - e^{-t}) + \frac{b (1 - e^{-t})}{t} \right] = (1 - P_{0,q} - P_{1,q}) B_q \left[ \frac{b}{t} + \phi \right] = (1 - P^\pm_q) f
\]

where we have used again Proposition 2.5 and Remark 2.7.  \( \square \)
Corollary 3.7. Let \( z \in \mathbb{C} \setminus (1, \infty) \). The operator \( Q_{q,z} \) has an eigenfunction \( g \in \mathcal{H}^1_q \) with eigenvalue \( \lambda_Q = \pm 1 \) if and only if \( P^\pm_q \) has an eigenfunction \( f \in \mathcal{H}^1_{q,z} \) with eigenvalue \( \lambda_P = \frac{1}{z} \) and term \( c = 0 \). Moreover, the eigenfunctions \( g \) and \( f \) satisfy

\[
g = f - z P_{0,q} f. \tag{3.9}\]

Proof. If \( f \) is in \( \mathcal{H}^1_{q,z} \) satisfies \( P^+_q f = \lambda f \), with \( \lambda \neq 1 \) then we can apply theorem 3.6, since \( b = 0 \) by corollary 2.10. Then by (3.8) \((1 - \frac{1}{z} P_{0,q}) f \) is an eigenfunction of \( Q_{q,z} \) with eigenvalue \( \lambda_Q = 1 \) if \( c = 0 \). The same follows in the case \( \lambda = 1 \).

In contrast, if \( g = B_q[\phi] \in \mathcal{H}^1_q \) satisfies \( Q_{q,z} g = g \) for \( z \neq 1 \), then the function

\[
f := B_q[(1 - z e^{-t})^{-1} \phi] \in \mathcal{H}^1_q \]

satisfies by (3.4)

\[
z P_{1,q} f = g = (1 - z P_{0,q}) f,
\]

hence it is an eigenfunction of \( P^+_q \) with eigenvalue \( \lambda_P = \frac{1}{z} \) and \( c = 0 \). If \( z = 1 \) we can repeat the same argument using the fact that \( Q_{q,1} \) is defined on \( \mathcal{H}^1_q \), hence \( g = B_q[\phi] \) with \( \phi(t) = \phi(0) + O(t^r) \) for some \( r > 0 \) as \( t \to 0^+ \). Hence we can write \( \frac{\phi(t)}{1 - e^{-t}} = \frac{b}{t} + \tilde{\phi}(t) \), with \( b = \phi(0) \) and \( \tilde{\phi} \in L^1(m_q) \) for all \( \xi > 0 \), and \( f \in \mathcal{H}^1_{q,1} \) with \( c = 0 \). \( \square \)

Corollary 3.8. Let \( z \in \mathbb{C} \setminus (1, \infty) \). The eigenfunctions \( g = B_q[\phi] \) of \( Q_{q,z} \) with eigenvalue \( \lambda_Q = \pm 1 \) are in \( \mathcal{H}^2_q \) and are bounded at \( x = 0 \) with \( g(0) = \Gamma(2q) \phi(0) \) if \( z = 1 \), whereas \( g(0) = (1 - z) \Gamma(2q) \phi(0) \) if \( z \neq 1 \).

Proof. Let \( z \neq 1 \). By corollary 3.7, from any \( g = B_q[\phi] \in \mathcal{H}^1_q \) an eigenfunction of \( Q_{q,z} \) with eigenvalue \( \lambda_Q = \pm 1 \), we obtain an eigenfunction \( f \in \mathcal{H}^1_q \) of \( P^+_q \) with eigenvalue \( \lambda_P = \frac{1}{z} \) and such that \( c = 0 \) when written as in (2.29), and (3.9) holds. Moreover, by corollary 2.10, we know that \( f \in \mathcal{H}^2_{q,z} \) if \( b = 0 \), hence actually \( f \in \mathcal{H}^2_q \). Since \( f = B_q[(1 - z e^{-t})^{-1} \phi] \) and \((1 - z e^{-t})^{-1} \in L^\infty \), then \( \phi \in L^2(m_q) \) and \( g \in \mathcal{H}^2_q \). If \( z = 1 \), we can repeat the same argument with few changes. In this case \( f = B_q[(1 - e^{-t})^{-1} \phi] \) and

\[
\frac{\phi(t)}{1 - e^{-t}} = \frac{\phi(0)}{t} + \tilde{\phi}(t) \quad \text{with} \quad \tilde{\phi}(t) = \frac{\phi(t) - \phi(0)}{1 - e^{-t}} + \frac{\phi(0)}{1 - e^{-t}} - \frac{\phi(0)}{t}.
\]

From corollary 2.10 it follows that \( \tilde{\phi}(t) \in L^2(m_q) \), hence \( \phi \in L^2(m_q) \) and \( g \in \mathcal{H}^2_q \). Moreover, from (2.6) and (3.9) it follows that \( g(x) = z f(x + 1) \), hence \( g(0) = z f(1) \). Finally theorem 2.8 shows that the eigenfunction \( f \) is written as in (2.22), (2.24) and (2.28). Let \( z = 1 \), using (3.9) we write \( g = B_q[\phi] \) with

\[
\phi(t) = e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^{n} a_n t^n}{\Gamma(n + 2q)}
\]

with \( \lim sup_n \sqrt[n]{|a_n|} \leq 1 \). Since for \( n \geq 0 \)

\[
B_q \left[ e^{-t} t^n \right] = \frac{1}{x^{2q}} \int_0^\infty e^{-t(x + 1)} t^{n+2q-1} \, dt = \frac{1}{x^{2q}} \zeta(t^{n+2q-1}) \left( \frac{1}{x + 1} \right)
\]

\[
= \Gamma(n + 2q) \frac{1}{x^{2q}} \left( \frac{x}{x + 1} \right)^{n+2q}
\]

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it follows that
\[ g(x) = \frac{1}{(x + 1)^2q} \sum_{n=0}^{\infty} (-1)^n a_n \left( \frac{x}{x + 1} \right)^n \] (3.10)
which implies in particular \( g(0) = \Gamma(2q)\phi(0) \). The same argument can be used for \( z \neq 1 \). □

We finish this section by considering a second induced map from (2.1). It is the Fibonacci map \( H \) which is defined by iterating \( F \) once plus the number of times necessary to reach the interval \([0, 1/2]\). The map \( H \) is defined by
\[ H(x) = \begin{cases} \frac{S_{2n+1}x - S_n}{S_{2n+2} - S_{2n+1}x} & \text{if } x \in \left[ \frac{S_{2n+1}}{S_{2n+2}}, \frac{S_{2n+2}}{S_{2n+3}} \right] \\ \frac{S_{2n+1} - S_n}{S_{2n+2} - S_{2n+1}} & \text{if } x \in \left( \frac{S_{2n+1}}{S_{2n+2}}, \frac{S_{2n+2}}{S_{2n+3}} \right) \end{cases} \] (3.11)
where \( \{S_n\}_{n \geq 0} \) are the Fibonacci numbers with \( S_0 = 0, S_1 = 1 \). \( H \) is defined by iterating \( F \) once plus the number of times necessary to reach the interval \([0, 1/2]\). The map \( H \) is defined by
\[ H(x) = \begin{cases} \frac{S_{2n+1}x - S_n}{S_{2n+2} - S_{2n+1}x} & \text{if } x \in \left[ \frac{S_{2n+1}}{S_{2n+2}}, \frac{S_{2n+2}}{S_{2n+3}} \right] \\ \frac{S_{2n+1} - S_n}{S_{2n+2} - S_{2n+1}} & \text{if } x \in \left( \frac{S_{2n+1}}{S_{2n+2}}, \frac{S_{2n+2}}{S_{2n+3}} \right) \end{cases} \] (3.11)
where \( \{S_n\}_{n \geq 0} \) are the Fibonacci numbers with \( S_0 = 0, S_1 = 1 \). The corresponding operators are obtained by exchanging the roles of \( P_{0,q} \) and \( P_{1,q} \) in (3.4). We recall from [5] that the operators \( N_q \) on \( L^2(m_q) \) are of trace class with spectrum
\[ \sigma(N_q) = \{0\} \cup \{-1\}^k \alpha^{2(qk)} \] (3.12)
where \( \alpha = \frac{\sqrt{5}-1}{2} \) and each eigenvalue is simple. Hence we introduce on \( \mathcal{H}_q^4 \) the second family of operators
\[ \mathcal{R}_{q,z} : \mathcal{H}_q^4 \rightarrow \mathcal{H}(B), \quad z \in \mathbb{C} \setminus \{-1\}^k \alpha^{2(qk)} \] (3.13)
defined by
\[ g(x) = B_q[\phi(t)] \longmapsto (\mathcal{R}_{q,z}g)(x) = z P_{0,q} B_q \left[ (1 - z N_q)^{-1} \phi(t) \right] \] (3.13)
By (3.12) the operators are well defined and we obtain the following.

**Theorem 3.9.** For \( \xi > 0 \), the operator-valued function \( z \mapsto \mathcal{R}_{q,z} \) is meromorphic in \( \mathbb{C} \) with simple poles at \( \{-1\}^k \alpha^{-2(qk)} \) \( k \geq 0 \). Moreover, for all \( g \in \mathcal{H}_q^4 \) one has
\[ (\mathcal{R}_{q,z}g)(x) = \sum_{n \geq 1} \frac{z^n}{(S_{n+1}x + S_n)^2q} g \left( \frac{S_{n+1}x + S_n}{S_{n+1}x + S_n} \right) \] (3.14)
which, by the growth property of the Fibonacci numbers, is absolutely convergent for \( |z| < \alpha^{-2\xi} \).

**Proof.** The first part follows from the definition of \( \mathcal{R}_{q,z} \) in (3.13) and (3.12). Equation (3.14) follows instead by writing
\[ (\mathcal{R}_{q,z}g)(x) = (z P_{0,q} (1 - z P_{1,q})^{-1} g)(x) = \sum_{n \geq 1} z^n (P_{0,q} P_{1,q}^{n-1} g)(x) \]
and using
\[ (P_{1,q}^{n-1} g)(x) = \frac{1}{(S_{n-1}x + S_n)^2q} g \left( \frac{S_{n-1}x + S_n}{S_{n-1}x + S_n} \right) \]
which can be proved by induction. □

**Remark 3.10.** From (3.14) it follows that the operators \( \mathcal{R}_{q,1} \) coincide with the generalized transfer operators of the Fibonacci map \( H \) for functions \( g \in \mathcal{H}(B) \).

Analogously to \( Q_{q,z} \), we now study the relations between \( \mathcal{R}_{q,z} \) and \( P_q^\pm \).
Theorem 3.11. Let \( f \in \mathcal{H}_q^{\pm} \), with \( c = b = 0 \). Then for \( z = \frac{1}{n} \in \mathbb{C} \setminus \{(−1)^k \alpha^{−2(q+k)}\} \)

\[
(1 \mp R_{q,z}) \left( 1 - z \mathcal{P}_{1,q} \right) f = (1 - z \mathcal{P}_{q}^\pm) f.
\] (3.15)

Proof. First of all, that the left-hand side of (3.15) is well defined follows easily from definitions. Note that in this case we need \( c = b = 0 \) to be sure that \( (1 - z \mathcal{P}_{1,q}) f \) is in \( \mathcal{H}_q^{\pm} \). Hence let \( f = B_q[\phi] \) with \( \phi \in L^2(m_q) \). Applying \( (1 - R_{q,z}) \) it follows

\[
(1 \mp R_{q,z}) \left( 1 - z \mathcal{P}_{1,q} \right) f = (1 \mp R_{q,z}) B_q \left[ (1 - z N_q) \phi(t) \right] = (1 \mp R_{q,z}) B_q \left[ (1 - z N_q - z M) \phi(t) \right] = (1 - z \mathcal{P}_{q}^\pm) B_q[\phi].
\]

We remark that using the power series expansion (3.14), one can prove that relation (3.15) holds for all \( f \in \mathcal{H}(B) \) for \( |z| < \alpha^{-2\xi} \).

Corollary 3.12. Let \( z \in \mathbb{C} \setminus \{ (−1)^k \alpha^{−2(q+k)} \} \). The operator \( R_{q,z} \) has an eigenfunction \( g \in \mathcal{H}_q^{\pm} \) with eigenvalue \( \lambda_R = \pm 1 \) if and only if \( \mathcal{P}_{q}^\pm \) has an eigenfunction \( f \) with eigenvalue \( \lambda_P = \frac{1}{z} \) and \( f \in \mathcal{H}_{q,ib} \) with \( c = b = 0 \).

Proof. The proof is as in corollary 3.7.

Putting together corollaries 3.7 and 3.12, we obtain the following.

Corollary 3.13. Let \( z \in \mathbb{C} \setminus \{ (1, \infty) \cup \{ (−1)^k \alpha^{−2(q+k)} \} \} \). Then \( f \in \mathcal{H}_{q,ib}^{\pm} \) with \( c = b = 0 \) is an eigenfunction of \( \mathcal{P}_{q}^\pm \) with eigenvalue \( \lambda_P = \frac{1}{z} = \mu \) if and only if

\[
f(x) = h_0(x) + h_1(x),
\]

with \( h_0 \) and \( h_1 \) eigenfunctions of \( Q_{q,z} \) and \( R_{q,z} \), respectively, in \( \mathcal{H}_q^{\pm} \), with eigenvalues \( \lambda_Q = \lambda_R = \pm 1 \).

Proof. Simply use that \( h_0 := (1 - z \mathcal{P}_{0,q}) f \) and \( h_1 := (1 - z \mathcal{P}_{1,q}) f \).

Remark 3.14. Recall that relation (3.15) can be extended to functions in \( \mathcal{H}(B) \), and that corollary 3.7 holds for functions \( f \in \mathcal{H}_{q,ib}^{\pm} \) with \( b \) not necessarily vanishing. An example of an eigenfunction as in corollary 3.13 is given for \( q = 1 \) by

\[
\frac{1}{x} = \frac{1}{1+x} + \frac{1}{x(x+1)}
\]

4. Two-variable zeta functions of Ruelle and Selberg

The first two-variable zeta function we are interested in can be written in terms of the Gauss map as

\[
Z(q, z) := \exp \left( - \sum_{n \geq 1} \sum_{x \in \mathbb{G}^{2n}(x)} \frac{\left| (G^{2n}(x))^{-q} \right|}{1 - \left| (G^{2n}(x))^{-1} \right|} \right).
\] (4.1)

For \( z = 1 \) and \( \xi = \text{Re}(q) > 1/2 \) the function \( Z(q, 1) \) coincides with the Selberg zeta function for the full modular group. This follows from the well-known one-to-one correspondence between the length spectrum (with multiplicities) of the modular surface \( PSL(2, \mathbb{Z}) \setminus \mathbb{H} \) and
the set of values $\log |(G^2)'(x)|$ (see, e.g., [32]). Another zeta function which naturally comes into play [31] is the Ruelle zeta function of the Farey map $F$, defined for $|z|$ small enough by

$$
\zeta(q, z) := \exp \left( \sum_{n \geq 1} \frac{1}{n} \sum_{x \in F^n(x)} |(F^n)'(x)|^{-q} \right).
$$

We shall study these functions by means of the operator-valued functions dealt with in the previous section. Our approach is similar in spirit to that used in [23] for $Z(q, G)$. But first we describe the correspondence between the periodic points of the map $F$ and those of its induced versions $G$ (3.1) and $H$ (3.11). Denoting $\text{Per}_F, \text{Per}_G$ and $\text{Per}_H$ the corresponding subsets of $[0, 1]$ we have

$$
\text{Per}_F \{0\} \cup \{\alpha\} = \text{Per}_G \{\alpha\} = \text{Per}_H \{0\}.
$$

From the definitions we immediately see that whenever $x$ belongs to either of these sets its continued fraction expansion has to be periodic, which we write $x = [a_1, \ldots, a_n]$. Denoting $p_F(x), p_G(x)$ and $p_H(x)$ the corresponding periods we have

$$
p_F(x) = \sum_{i=1}^{n} a_i, \quad p_G(x) = n, \quad p_H(x) = p_F(x) - \#\{i \in [1, n] : a_i = 1\}.
$$

In other words, if for a given map $T : [0, 1] \to [0, 1]$ we define the partition function

$$
Z_n(q, T) := \sum_{x = T^n(x)} |(T^n)'(x)|^{-q}
$$

then

$$
Z_n(q, F) = 1 + \sum_{m=1}^{n} n Z_m(q, G) = \alpha^{2q^n} + \sum_{m=1}^{n} Z_m(q, H).
$$

Let, moreover,

$$
\lambda(q) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(q, F).
$$

It follows from thermodynamic formalism that the above limit exists for all $q \in \mathbb{R}$ and is a differentiable and monotonically decreasing function for $q \in (-\infty, 1)$, with $\lim_{q \to 1^-} \lambda(q) = 1$ and $\lambda(q) = 1$ for all $q \geq 1$ [29]. In particular, for $q \in (-\infty, 1)$ the function $\zeta(q, z)$ converges absolutely for $|z| < 1/\lambda(q)$ and has a simple pole at $z = 1/\lambda(q)$.

Our aim is now to express both $Z(q, z)$ and $\zeta(q, z)$ in terms of Fredholm determinants of the operators $Q_{q,z}$ introduced in section 3. Following Mayer [21], we restrict the operators $Q_{q,z}$ to the Banach space $A_\infty(D)$ of functions which are holomorphic on $D$ and continuous on $\overline{D}$ where

$$
D := \{x \in \mathbb{C} : |x - 1| < \frac{3}{4} - \varepsilon\}
$$

for small $\varepsilon > 0$, equipped with the sup-norm. The space $A_\infty(D)$ is the set of holomorphic functions on which it is natural to study the spectral properties of $Q_{q,z}$ written as in theorem 3.4. Note that in [21] the set $D$ was used as above with $\varepsilon = 0$. However, here we do not know whether eigenfunctions $g$ of $Q_{q,z}$ are bounded at $x = -\frac{1}{2}$ (see (3.10)), hence we set $\varepsilon > 0$.

We now prove that

**Proposition 4.1.** If $g$ is in $A_\infty(D)$ then $g$ is in $\mathcal{H}^2_q$ and $g = B_q[\phi_q]$ with $\phi_q(t) = \phi_q(0) + O(t)$ as $t \to 0^+$. 
In particular since \( A_\infty(D) \subset \tilde{\mathcal{H}}^2 \) we can study the action of \( Q_{q,z} \) on \( A_\infty(D) \) for all \( z \in \mathbb{C} \setminus (1, +\infty) \) and \( \xi > 0 \).

**Proof of proposition 4.1.** We recall the definition of the generalized Laguerre polynomials

\[
L_n^{2q-1}(t) = \sum_{m=0}^{n} \frac{\Gamma(n+2q)}{\Gamma(m+2q) (n-m)!} \frac{(-t)^m}{m!}, \quad \xi = \text{Re}(q) > 0 \quad (4.4)
\]

which satisfy

\[
B_q[L_n^{2q-1}](x) = \frac{\Gamma(n+2q)}{n!} (-1)^n (x-1)^n. \quad (4.5)
\]

Let now \( g(x) \) be in \( A_\infty(D) \). It can be expressed as a power series

\[
g(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \quad a_n \in \mathbb{C}
\]

with

\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} \leq \left( \frac{3}{2} - \varepsilon \right)^{-1}. \quad (4.6)
\]

Hence from (4.5) we can write

\[
g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} B_q[L_n^{2q-1}](x).
\]

Letting

\[
\phi_q(t) := \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_n^{2q-1}(t)
\]

we need to prove that \( \phi_q \in L^2(m_q) \) and that \( g = B_q[\phi_q] \).

That \( \phi_q \in L^2(m_q) \) follows from the following estimates on \( \|L_n^{2q-1}(t)\|_{L^2(m_q)} \). Using computations from [33], we have

\[
\|L_n^{2q-1}(t)\|^2_{L^2(m_q)} = \int_{0}^{\infty} L_n^{2q-1}(t) L_n^{2q-1}(t) t^{2\xi-1} e^{-t} dt
\]

\[
= B_q[\sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_n^{2q-1}(t) t^{2\xi-1} e^{-t} dt]
\]

\[
= \sum_{k=0}^{n} \frac{(-n)^\xi}{\Gamma(2\xi)} \Gamma(-2k\xi - n + 1) \Gamma(2\eta - n + 1 + k)
\]

\[
\leq \frac{1}{(n-k)^2 + 4\eta^2}.
\]

It follows that for \( \xi > 0 \) and \( \eta \neq 0 \)

\[
\epsilon \leq \frac{n^2}{\min\{1, 4\eta^2\}} \sum_{k=0}^{n} \frac{\epsilon^2 - 1}{k} \leq \frac{(n+2\xi)^{2\xi+2}}{\min\{1, 4\eta^2\}}.
\]
Hence
\[ \sum_{n=0}^{\infty} \left\| \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_2^{q-1}(t) \right\|_{L^2(m_q)} \leq \sum_{n=0}^{\infty} \frac{|a_n n!|}{|\Gamma(n+2q)|} \frac{\sqrt{\Gamma(2\xi)} (n + 2\xi)^{\xi+1}}{|\Gamma(2\eta)| n! \sqrt{\min[1, 4q^2]}} < \infty \] (4.8)
by (4.6) and since
\[ \frac{|\Gamma(n+2\eta)|}{|\Gamma(n+2q)|} \leq n \quad \forall n \geq 1 \]
by standard estimates. If instead $\xi > 0$ and $\eta = 0$ then (4.7) reads
\[ \|L_2^{q-1}(t)\|_{L^2(m_q)}^2 = \frac{\Gamma(n+2q)}{n!} \]
and (4.8) follows again. Condition (4.8) implies that $\phi_q \in L^2(m_q)$. Moreover, it satisfies
\[ \phi_q(0) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_2^{q-1}(0) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \frac{\Gamma(n+2q)}{\Gamma(2q)n!} \in \mathbb{C} \]
\[ \lim_{t \to 0^-} \frac{\phi(t) - \phi(0)}{t} = -\sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \frac{\Gamma(n+2q)}{\Gamma(2q+1)(n-1)!} \in \mathbb{C}. \]
To finish the proof we have to show that $g(x) = B_q[\phi_q](x)$ for $x \in B$. This follows from
\[
\left| B_q[\phi_q] - \sum_{n=0}^{N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} B_q[L_2^{q-1}] \right| \leq \int_{0}^{\infty} \left| e^{-\frac{t}{2}} t^{q-1} \sum_{n>N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_2^{q-1}(t) \right| dt \leq \int_{0}^{\infty} \left| e^{-\frac{t}{2}} t^{q-1} \sum_{n>N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_2^{q-1}(t) \right| dt = \int_{0}^{\infty} \left| \sum_{n>N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_2^{q-1}(t) \right| m_q(dt) \]
and the last term vanishes as $N \to \infty$ since $\phi_q \in L^2(m_q) \subset L^1(m_q)$. \hfill \Box

We now recall that by (3.10) in the proof of corollary 3.8, the eigenfunctions $g \in \mathcal{H}_q^2$ of $Q_{q,z}$ are in $A_{\infty}(D)$. Hence we obtain the following.

**Theorem 4.2.** The operator-valued function $q \mapsto Q_{q,z} : A_{\infty}(D) \rightarrow \mathcal{H}(B)$ is analytic in $\text{Re}(q) > 0$ for $z \in \mathbb{C} \setminus \{1, \infty\}$, and in $\text{Re}(q) > \frac{1}{2}$ for $z = 1$. Moreover, for $z = 1$, the function $q \mapsto Q_{q,1}$ can be extended to a meromorphic function in $\text{Re}(q) > 0$ with a simple pole at $q = \frac{1}{2}$ with residue the operator $g \mapsto (R^1_q g)(x) = g(0)$.

**Proof.** We use the integral representation of theorem 3.2 and proposition 4.1 to write for each $g \in A_{\infty}(D)$
\[ (Q_{q,z}g)(x) = z \int_{0}^{\infty} \frac{e^{-\frac{t}{2}} t^{q-1}}{1 - ze^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} L_2^{q-1}(t) dt, \] (4.9)
where
\[ g(x) = \sum_{n=0}^{\infty} a_n (x - 1)^n \]
and \([a_n]\) satisfies (4.6). Consider first the case \(z \in \mathbb{C} \setminus [1, \infty)\), for which \(|1 - z e^{-t}|\) is bounded by a constant for all \(t \in [0, \infty)\). Moreover, since
\[
|e^{-r(t+1)}| \leq e^{-t} \quad \forall x \in B
\]
we can argue as in the end of proposition 4.1 to write
\[
(Q_{q,z}g)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - z e^{-t}} L_{n+1}^{2q-1}(t) \, dt.
\]
Since
\[
\sum_{n=0}^{\infty} \sup_{x \in B} \left| \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - z e^{-t}} L_{n+1}^{2q-1}(t) \, dt \right|
\]
\[
\leq \sum_{n=0}^{\infty} \frac{|a_n| n!}{\Gamma(n+2q)} \left( \int_0^\infty \frac{e^{-r t} t^{2q-1}}{|1 - z e^{-t}|^q} \, dt \right)^{\frac{1}{2}} \|L_n^{2q-1}\|_{L^2(m_\eta)} < \infty
\]
as in (4.8), the operators \(Q_{q,z}\) are bounded and to conclude we need to show that for any bounded domain \(C \subset \{\text{Re}(q) > 0\}\) it holds
\[
\sum_{n=0}^{\infty} \sup_{q \in C} \sup_{x \in B} \left| \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - z e^{-t}} L_{n+1}^{2q-1}(t) \, dt \right| < \infty.
\]
From (4.7) and arguing as in (4.8), we have
\[
\sum_{n=0}^{\infty} \sup_{q \in C} \sup_{x \in B} \left| \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - z e^{-t}} L_{n+1}^{2q-1}(t) \, dt \right|
\]
\[
\leq \sum_{n=0}^{\infty} \sup_{q \in C} \frac{|a_n| n!}{\Gamma(n+2q)} \left( \int_0^\infty \frac{e^{-r t} t^{2q-1}}{|1 - z e^{-t}|^q} \, dt \right)^{\frac{1}{2}} \|L_n^{2q-1}\|_{L^2(m_\eta)}
\]
\[
\leq \frac{1}{\min_{\eta \in \mathbb{R}^+} |1 - z e^{-\eta}|} \sum_{n=0}^{\infty} \sup_{q \in C} \frac{|a_n| \Gamma(2\xi)}{|\Gamma(2\eta)| \sqrt{\min\{1, 4\eta\}}} (n + 2\xi)^{\xi+2} < \infty
\]
in the case \(C \cap \{\text{Im}(q) = 0\} = \emptyset\), and using the value of \(|L_n^{2q-1}\|_{L^2(m_\eta)}\) in the real case if \(C \cap \{\text{Im}(q) = 0\} \neq \emptyset\), where we recall that \(|\Gamma(2\eta)| \eta\) is bounded at \(\eta = 0\), having the Gamma function a simple pole at 0. This shows that the result holds for \(z \in \mathbb{C} \setminus [1, \infty)\).

Let us consider now the case \(z = 1\). We rewrite (4.9) as
\[
(Q_{q,z}g)(x) = \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - e^{-t}} \phi_q(t) \, dt
\]
\[
= \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - e^{-t}} \phi_q(0) \, dt + \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{1 - e^{-t}} \frac{\phi_q(t) - \phi_q(0)}{t} \, dt
\]
\[
= \Phi(1, 2q, x + 1) g(0) + \int_0^\infty \frac{e^{-r(t+1)} t^{2q-1}}{(1 - e^{-t})/t} \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \frac{L_n^{2q-1}(t) - L_n^{2q-1}(0)}{t} \, dt.
\]
where \(\Phi(z, 2q, x + 1)\) denotes the Lerch zeta function (see remark 3.3). For the second term we show that we can repeat the same argument as above. Indeed \((1 - e^{-t})/t\) is a bounded function on \([0, \infty)\) and for any \(\delta > 0\)
\[
\left| \frac{L_n^{2q-1}(t) - L_n^{2q-1}(0)}{t} \right| \leq \min\left\{ \frac{1}{\delta} \sqrt{\frac{L_n^{2q-1}(t)^2}{t} + \frac{|L_n^{2q-1}(0)|^2}{t}}, \frac{1}{\delta} \frac{|\Gamma(n+2q)|}{n!} (1 + \delta)^n \right\}
\]
if \(t \geq \delta\)
\[
\frac{1}{\delta} \frac{|\Gamma(n+2q)|}{n!} (1 + \delta)^n \quad \text{if } t \leq \delta.
\]
from which it follows that
\[
\left\| \frac{L_n^{2q-1}(t) - L_n^{2q-1}(0)}{t} \right\|_{L^2(m_q)} \leq \frac{1}{\delta} \left( \| L_n^{2q-1} \|_{L^2(m_q)} + \sqrt{\Gamma(2q)} |L_n^{2q-1}(0)| (1 + \delta)^n \right).
\]

Finally, using (4.8) and \( L_n^{2q-1}(0) = \Gamma(n+2q) \), by choosing \( \delta \) small enough such that
\[
\limsup_{n \to \infty} \frac{|a_n|}{n!} (1 + \delta) < 1,
\]
which is possible by (4.6), we obtain
\[
\sum_{n=0}^{\infty} \frac{|a_n| n!}{\Gamma(n+2q)} \left\| \frac{L_n^{2q-1}(t) - L_n^{2q-1}(0)}{t} \right\|_{L^2(m_q)} < \infty
\]
and from this we can prove as above that
\[
q \to \int_0^\infty e^{-t(x+1)} t^{2q-1} \frac{(1-e^{-t})}{(1-e^{-t})^t} \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \frac{L_n^{2q-1}(t) - L_n^{2q-1}(0)}{t} \, dt
\]
is analytic on \( \text{Re}(q) > 0 \). The proof follows from well-known properties of the Lerch zeta function \( \Phi_1(1, 2q, x+1) \).
\[\square\]

Moreover, we recall the following.

**Theorem 4.3** ([5]). For \( \text{Re}(q) > 0 \), the operators \( P_{1,q} \) and \( N_q \) on the spaces \( \mathcal{H}_q^2 \) and \( L^2(m_q) \), respectively, are of trace class.

From this, theorem 4.2 and remark 3.1, where we defined the induced operators \( Q_{q,z} \) (3.5), we immediately obtain the following.

**Corollary 4.4.** The operators \( Q_{q,z} \) on \( A_\infty(D) \) are of trace class. Moreover, for \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \text{Re}(q) > 0 \), and for \( \text{Re}(q) > \frac{1}{2} \) if \( z = 1 \), it holds
\[
\text{trace}(Q_{q,z}) = \text{trace}(Q_{q,z}) = z \int_0^\infty \frac{L_n^{2q-1}(2t)}{t^{2q-1}} (1 - z e^{-t})^{-1} m_q(dt) \tag{4.10}
\]
using (2.15).

Applying Fredholm theory [14] to the operators \( Q_{q,z} \), and putting together theorems 3.2 and 4.2, we conclude the following.

**Corollary 4.5.** For \( z \in \mathbb{C} \setminus [1, \infty) \) the Fredholm determinants \( q \mapsto \det(1 \pm Q_{q,z}) \) are analytic functions in \( \text{Re}(q) > 0 \). For \( \text{Re}(q) > 0 \) the Fredholm determinants \( z \mapsto \det(1 \pm Q_{q,z}) \) are analytic functions in \( z \in \mathbb{C} \setminus [1, \infty) \). For \( z = 1 \) the determinants \( q \mapsto \det(1 \pm Q_{q,1}) \) are analytic functions in \( \text{Re}(q) > \frac{1}{2} \) with a meromorphic extension to \( \text{Re}(q) > 0 \) with a simple pole at \( q = \frac{1}{2} \).

Using corollary 4.5 we can express \( Z(q, z) \) and \( \zeta(q, z) \) in terms of the Fredholm determinants \( \det(1 \pm Q_{q,z}) \). By (4.10), this is an easy generalization of results from [15, section 4] and [21, 23]. More precisely we have...
Theorem 4.6. For $z \in \mathbb{C} \setminus [1, \infty)$ and $\text{Re}(q) > 0$ one has
\[
Z(q, z) = \det \left[ (1 - Q_{q,z})(1 + Q_{q,z}) \right]
\]
and
\[
\zeta(q, z) = (1 - z)^{-\frac{1}{2}} \frac{\det(1 + Q_{q,z})}{\det(1 - Q_{q,z})}
\]
which are analytic, respectively, meromorphic functions. Moreover,
\[
(q, z) \mapsto \zeta(q, z) Z(q, z) = (1 - z)^{-\frac{1}{2}} \frac{\det(1 + Q_{q,z})}{\det(1 - Q_{q,z})}
\]
is analytic in $\{z \in \mathbb{C} \setminus [1, \infty]\} \times \{q \in \mathbb{C} : \text{Re}(q) > 0\}$.

For $z = 1$ the function $Z(q, 1)$ satisfies (4.11) and is analytic for $\text{Re}(q) > \frac{1}{2}$. Moreover, it can be continued to $\text{Re}(q) > 0$ as a meromorphic function with a simple pole at $q = \frac{1}{2}$.

The case $z = 1$ for the Ruelle zeta function $\zeta(q, z)$ is more delicate as is evident from the term $(1 - z)^{-\frac{1}{2}}$ in (4.12). However, at the end of section 5 we show that it is possible to define $\zeta(q, 1)$ for $\text{Re}(q) > 1$.

Remark 4.7. As explained above, the case $q$ real has been studied in the framework of applications of thermodynamic formalism to the Farey map. In particular, the spectrum of $P^+_q$ on $\mathcal{B}_q[L^2(m_q)]$ has been studied in [28]. Using numerical techniques it has been proved there that for $q \in (0, 1)$ there is only one eigenvalue $\lambda(q) = (1, 2)$, with the rest of the spectrum given by the interval $[0, 1]$ which is purely continuous (see also [5]). Whereas there are no eigenvalues for $q \geq 1$ and the spectrum is purely continuous. By corollary 3.7 and theorem 4.6 this implies that for $q \in (0, 1)$ the function $z \mapsto \zeta(q, z)$ has only one simple pole at $z = \frac{1}{\lambda(q)}$ and no poles for $q \geq 1$, and, respectively, the function $z \mapsto Z(q, z)$ has only one single zero at $z = \frac{1}{\lambda(q)}$ for $q \in (0, 1)$ and no zeros for $q \geq 1$.

4.1. Remarks on the case $z = 1$

We now give some remarks on the relations with the works of Mayer and Lewis–Zagier, who have studied the case $z = 1$. We first mention that in [23] the meromorphic extension for $Z(q, 1)$ that we obtained in theorem 4.6 is given to all the complex $q$-plane. Moreover, (4.11) and (4.12) give an explicit connection between zeros of $Z(q, z)$, respectively, zeros or poles of $\zeta(q, z)$, and the existence of eigenfunctions $g \in A_\infty(D)$ for $Q_{q,z}$ with eigenvalue $\lambda_Q = \pm 1$. By corollary 3.7 this turns out to be a connection with eigenfunctions of $P^+_q$ with eigenvalues $\lambda_P = \frac{1}{2}$. These connections have been proved in [23] for the operators $Q_{q,i}$, and in [20] for the operators $P^+_q$ with $\lambda_P = 1$ (see also proposition 2.1). For recent developments of Lewis–Zagier theory see [6, 7].

Finally, the characterization of the eigenfunctions for $P^+_q$ given in corollary 2.10 together with results from [20] leads to a new proof of the following result from [11].

Theorem 4.8. Even, respectively odd, spectral zeros of $Z(q, 1)$ correspond to eigenfunctions of $P^+_q$, respectively $P^-_q$. Moreover, the zeros $q$ in $\text{Re}(q) > 0$ with the Riemann zeta function $\zeta_R(2q) = 0$ correspond to eigenfunctions of $P^+_q$, hence of $Q_{q,i}$ with eigenvalue $\lambda_Q = 1$.

Proof. By corollary 3.7 and theorem 4.6, the zeros of $Z(q, 1)$ correspond to eigenfunctions of $P^+_q$ with eigenvalue $\lambda_P = 1$ and term $c = 0$ in (2.29). From corollary 2.10 it also follows that for eigenfunctions of $P^-_q$ it holds $b = 0$. Hence corollary 2.10 and (2.6) imply that
\[
P^-_q f = f \quad \text{with } c = 0 \quad \Rightarrow \quad f(x) = \begin{cases} O(1) & x \to 0 \\ O(x^{-2\text{Re}(q)}) & x \to \infty. \end{cases}
\]
For eigenfunctions of $P_q^+$ instead two cases are possible, $b = 0$ and $b \neq 0$. We have

\[
P_q^+ f = f \quad \text{with} \quad c = b = 0 \Rightarrow f(x) = \begin{cases} \mathcal{O}(1) & x \to 0 \\ \mathcal{O}(x^{-2\Re(q)}) & x \to \infty \end{cases}
\]

\[
P_q^+ f = f \quad \text{with} \quad c = 0, b \neq 0 \Rightarrow f(x) = \begin{cases} \sim \frac{b}{2q - 1} & x \to 0 \\ \sim \frac{b}{2q - 1} x^{1 - 2\Re(q)} & x \to \infty \end{cases}
\]

Applying theorem 2 and the subsequent corollary from [20] it follows that all eigenfunctions of $P_q^-$ with $c = 0$ are period functions associated with cusp forms or Maass wave forms. Moreover from proposition 2.1-(ii) it follows that they are odd period functions, in the sense of [20], hence are associated with odd cusp forms. The same holds for all eigenfunction of $P_q^+$ with $c = b = 0$ which are associated with even cusp forms. This concludes the proof for the spectral zeros of $Z(q, 1)$. The previous argument also implies that the other zeros of the Selberg zeta function with $\Re(q) > 0$ necessarily correspond to eigenfunctions of $P_q^+$ with $c = 0$ and $b \neq 0$. For example, it is well known that the zero at $q = 1$ corresponds to the eigenfunction $f(x) = \frac{1}{x}$ of $P_q^+$ and the zeros satisfying $\zeta_R(2q) = 0$ correspond to the eigenfunctions $f_q^+(x)$ defined in (2.36) (see [20, p 256]).

5. Connections with Farey fractions

We now use (4.11) and (4.12) to give expressions for the generalized Selberg and Ruelle zeta functions as exponentials of Dirichlet series. By (4.11), (4.12) and theorem 3.6 we can write for $z \in \mathbb{C} \setminus [1, \infty)$ and $\Re(q) > 0$

\[
Z(q, z) = \det \left[ 1 - z P_{1,q} \left( 1 - z P_{0,q} \right)^{-1} \right] \det \left[ 1 + z P_{1,q} \left( 1 - z P_{0,q} \right)^{-1} \right] \quad (5.1)
\]

\[
\zeta(q, z) = (1 - z)^{-1} \frac{\det \left[ 1 + z P_{1,q+1} \left( 1 - z P_{0,q+1} \right)^{-1} \right]}{\det \left[ 1 - z P_{1,q} \left( 1 - z P_{0,q} \right)^{-1} \right]} \quad (5.2)
\]

The existence of the right-hand sides can be justified as in [30] for $\Re(q) > \frac{1}{2}$. Expression (5.1) can be also given for $Z(q, 1)$ and it follows from the uniform convergence of the series representation of $Q_{q,z}$ on $A_\infty(D)$ given in theorem 3.4. We now perform a formal calculation which gives a connection between the zeta functions $\zeta(q, z)$ and $Z(q, z)$ and the Farey fractions. Let us consider the matrices

\[
\phi_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}
\]

as elements of the group $GL(2, \mathbb{Z})$ acting on $\mathbb{C}$ as Möbius transformations, see [2]. Then $(P_{1,q} f)(x) = (\phi'_1(x))^q f(\phi_1(x))$. Thus, each term in the expansion of $(P_q^+)^n$ can be represented in terms of a product of the matrices $\phi_i$. At the same time, the Farey fractions can be represented by means of a subgroup of $SL(2, \mathbb{Z})$ with generators

\[
L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
Lemma 5.1. Expanding

\[(P_q^n + (P_{-q})^n - 2(P_{0,q})^n)\]

one obtains \(2(2^n - 1)\) terms which are twice all the possible combinations of \(n\) factors involving \(L\) and \(R\) and starting with \(L\), without the term \(L^n\).

Proof. The only products which do not cancel out are those where \(P_{1,q}\) appears an even number of times (counted twice). On the other hand we point out that if \(K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) then \(K^2 = Id\), \(L = \phi_0\) and \(LK = KR = \phi_1\). We thus have

\[\phi_1 \phi_0 \ldots \phi_0 \phi_1 = L K \ldots L K R \ldots R R.\]

The thesis now easily follows. □

Proposition 5.2. Let \(A\) be the matrix corresponding to the term

\[(P_{0,q})^{n_1}(P_{1,q})^{n_2} \ldots (P_{i,q})^{n_l},\]

with \(\sum_{j=1}^{l} n_j = n > 1\) and \(i = (1 + (-1)^{l})/2\). Then, setting \(T := \text{trace}(A)\), we have \(T > 2\) and

\[\text{trace } [(P_{0,q})^{n_1}(P_{1,q})^{n_2} \ldots (P_{i,q})^{n_l}] = \frac{1}{\sqrt{T^2 - 4}} \left( \frac{2}{T + \sqrt{T^2 - 4}} \right)^{2q-1} \cdot\]

Proof. The proof of the first assertion amounts to a straightforward verification. Let \(V\) be the composition operator acting as \((V f)(x) = \varphi(x)f(\psi(x))\). If \(\psi(x)\) is holomorphic in a disc and has there a unique fixed point \(\bar{x}\) with \(|\psi'(\bar{x})| < 1\) then \(V\) is of the trace-class with \(\text{trace}(V) = \frac{\varphi(\bar{x})}{1 - \varphi'(\bar{x})}\). The thesis follows by applying this relation to the operators under consideration. □

We are now going to make use of the correspondence between products of matrices \(L\) and \(R\) and the Farey fractions. Let us consider the Farey tree \(F\). We recall that every rational number \(\frac{a}{b} \in (0, 1)\) appears exactly once in \(F\). One may therefore identify \(\frac{a}{b}\) with the path on \(F\) which reaches it starting from the root node \(\frac{1}{2}\) (first row), which in turn can be encoded as a matrix product in the following way: first recall that every rational number \(\frac{a}{b} \in (0, 1)\) has a unique finite continued fraction expansion \(\frac{a}{b} = [a_1, \ldots, a_k]\) with \(a_k > 1\), and one may define the rank of \(\frac{a}{b}\) as

\[\frac{a}{b} = [a_1, \ldots, a_k] \implies \text{rank } \left( \frac{a}{b} \right) = \sum_{i=1}^{k} a_i - 1.\]

It turns out that \(\frac{a}{b}\) has rank \(n\) if and only if it belongs to \(F_{n+1} \setminus F_n\). Furthermore, we may uniquely decompose \(\frac{a}{b}\) as

\[\frac{a}{b} = \frac{a' + a''}{b' + b''} \quad \text{with} \quad b'a - a'b = ba'' - ab'' = b'a'' - a'b'' = 1.\]

The neighbours \(\frac{a'}{b'}\) and \(\frac{a''}{b''}\) are the parents of \(\frac{a}{b}\) in \(F\) and we may accordingly identify

\[\frac{a}{b} \simeq \left( \frac{a''}{b''} \frac{a'}{b'} \right) \in \mathcal{Z}\]
where
\[ Z := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : 0 < a \leq c, 0 \leq b < d \right\}. \]

Clearly \( \frac{1}{2} \simeq L \) and, more generally,
\[ \frac{a}{b} \simeq L \prod_{i} M_i, \tag{5.3} \]

where the number of terms in the product \( L \prod_{i} M_i \) is equal to \( \text{rank}(\frac{a}{b}) \) and \( M_i = L \) or \( R \) according to whether the \( i \)th turn, along the descending path in \( F \) which starts from the root node \( \frac{1}{2} \) and reaches \( \frac{a}{b} \), goes to the left or to the right. Using (5.3) one may then define a map \( T : F \rightarrow \mathbb{N} \)
\[ T \left( \frac{a}{b} \right) := \text{trace} \left( L \prod_{i} M_i \right). \tag{5.4} \]

Note, moreover, that the set \( \tilde{F}_n := F_{n+1} \setminus F_n = \left\{ \frac{a}{b} \in F : \text{rank} \left( \frac{a}{b} \right) = n \right\} \)
has \( 2^{n-1} \) elements which are in a one-to-one correspondence with the (equal pairs of) elements in the expansion dealt with in lemma 5.1 plus \( \left\{ \frac{1}{n+1} \right\} \). We now obtain an expression for the \( Z(q, z) \) and \( \zeta(q, z) \) as exponentials of power series whose coefficients are computed along lines of \( F \) (see [19] for \( Z(q, 1) \)).

**Theorem 5.3.** For \( \text{Re}(q) > 1 \) and \( |z| \leq 1 \) the two-variable zeta functions \( Z(q, z) \) and \( \zeta(q, z) \) can be written as
\[ Z(q, z) = \exp \left( - \sum_{n \geq 2} \frac{z^n}{n} \Lambda_n(q) \right), \quad \zeta(q, z) = \exp \left( \sum_{n \geq 1} \frac{z^n}{n} \Xi_n(q) \right) \]
with
\[ \Lambda_n(q) := \sum_{\frac{a}{b} \in \tilde{F}_n \setminus \left\{ \frac{1}{n+1} \right\}} \left( \frac{2}{T^2(\frac{a}{b}) - 4} \right)^{2q-1} \left( \frac{2}{(T(\frac{a}{b}) + \sqrt{T^2(\frac{a}{b}) - 4})} \right)^{2q}, \]
\[ \Xi_n(q) = \sum_{\frac{a}{b} \in \tilde{F}_n} \left[ \left( \frac{2}{T(\frac{a}{b}) + \sqrt{T^2(\frac{a}{b}) - 4}} \right)^{2q} + \left( \frac{2}{T(\frac{a}{b}) + \sqrt{T^2(\frac{a}{b}) + 4}} \right)^{2q} \right] \]
where the map \( T \) is defined in (5.3) and (5.4).

**Proof.** We first obtain the expansion for \( Z(q, z) \). A formal manipulation of (5.1) gives
\[ Z(q, z) = \det[(1 - zP^+_q)(1 - zP^-_q)(1 - zP_{0,q})^{-2}] \tag{5.5} \]
which is well defined by lemma 5.1.

We now only have to prove that \( \sum_{n \geq 2} \frac{z^n}{n} \Lambda_n(q) \) converge for \( \text{Re}(q) > 1 \) and \( |z| \leq 1 \). Since then the assertion follows from (5.5), proposition 5.2 and the definition of the map \( T \) in (5.3) and (5.4). Set
\[ \gamma(k, n) := \# \left\{ \frac{a}{b} \in \tilde{F}_n \setminus \left\{ \frac{1}{n+1} \right\} : T \left( \frac{a}{b} \right) = k \right\} \]
and let $\mathcal{M}$ be the free multiplicative monoid generated by the matrices $L$ and $R$. The function

$$\Psi(k) = \#\{X \in \mathcal{M} : \text{trace}(X) \leq k\}$$

has been recently studied in the literature and the asymptotic behaviour

$$\Psi(k) = \frac{k^2 \log k}{\zeta_R(2)} + O(k^2)$$

has been found (see [14, 16]). Let us decompose $\Psi(k)$ as $\Psi(k) = \Psi_L(k) + \Psi_R(k)$ where $\Psi_L(k)$ (respectively, $\Psi_R(k)$) is obtained restricting to the elements of $\mathcal{M}$ which start with $L$ (respectively $R$). Note that

$$\Psi_L(k) = \sum_{j \leq k} \sum_{n=2}^{j-1} \gamma(j, n).$$

Moreover, using $R^k K = KL^k$ one easily realizes that if

$$\frac{a}{b} \simeq L \prod_i M_i = \left( \begin{array}{cc} a'' & a' \\ b'' & b' \end{array} \right)$$

then, setting $\overline{M}_i = L$ if $M_i = R$ and vice versa, we have

$$\frac{b}{a} \simeq R \prod_i \overline{M}_i = \left( \begin{array}{cc} b' & b'' \\ a' & a'' \end{array} \right).$$

Therefore, $T(\frac{a}{b}) = T(\frac{b}{a})$ and

$$\sum_{j \leq k} \sum_{n=2}^{j-1} \gamma(j, n) = \frac{k^2 \log k}{2\zeta_R(2)} + O(k^2).$$

This implies that if

$$\alpha(k) := \sum_{n=2}^{k-1} \frac{\gamma(k, n)}{n}$$

then

$$\sum_{j \leq k} \alpha(j) = O(k^2 \log k)$$

hence

$$\sum_{n \geq 2} \frac{1}{n} \Lambda_n(q) = \sum_{k=3}^{\infty} 2 \frac{\alpha(k)}{\sqrt{k^2 - 4}} \left( \frac{2}{k + \sqrt{k^2 - 4}} \right)^{2q-1}$$

converges absolutely for $\text{Re}(q) > 1$. The case $|z| < 1$ easily follows.

To obtain the expansion for $\zeta(q, z)$, we again start from the following formal manipulation of (5.2):

$$\zeta(q, z) = \frac{\det[(1 - z P_{q+1}^{-1})(1 - z P_{q}^{-1})^{-1}(1 - z P_{0,q+1})^{-1}(1 - z P_{0,q})]}{(1 - z)}. \quad (5.6)$$

Now, given $\epsilon > 0$ let us consider the perturbation $M_{q, \epsilon}$ of the operator $M$ in (2.14) acting as

$$(M_{q, \epsilon} \psi)(t) = \frac{e^{-\left(\frac{t}{1+\epsilon}\right)^{q}}}{(1+\epsilon)^{q}} \psi \left( \frac{t}{1+\epsilon} \right).$$
Reasoning as in the proof of [13, proposition 4.5] one easily sees that for all $\epsilon > 0$ the operator $M_{q,\epsilon}$ is of trace class on $L^2(m_q)$ and its spectrum is given by $\sigma(M_{q,\epsilon}) = \{(1 + \epsilon)^{-q-n}\}_{k \geq 0}$, each eigenvalue being simple. Therefore,
\[
\text{trace } (M_{q,\epsilon}^n - M_{q+1,\epsilon}^n) = (1 + \epsilon)^{-q-n}.
\]
Let, moreover, $P_{0,q}^\epsilon$ be the corresponding operator on $\mathcal{H}_q^2$. A short calculation then gives
\[
\lim_{\epsilon \to 0^+} \det [(1 - z P_{0,q+1}^-)^{-1}(1 - z P_{0,q}^-)] = \lim_{\epsilon \to 0^+} \left(1 - \frac{z}{1 + \epsilon}\right)^{(1+\epsilon)^{-q}} = 1 - z
\]
and therefore, letting $P_{\pm,q}^\epsilon := P_{0,q}^\epsilon \pm P_{1,q}^\epsilon$,
\[
\zeta(q, z) = \lim_{\epsilon \to 0^+} \det [(1 - z P_{q+1,q}^-)(1 + z P_{q,q}^+)] = \zeta(q, 1) Z(q, 1) \text{ turns out to be analytic for } \Re(q) > 1.
\]

We remark that from theorem 5.3, in particular from (5.7), we obtain a definition for $\zeta(q, 1)$ for $\Re(q) > 1$, which is not immediate from (4.12). Hence we can extend theorem 4.6 to the case $q = 1$, in particular the function $q \mapsto \zeta(q, 1) Z(q, 1)$ turns out to be analytic for $\Re(q) > 1$.

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