Hermitian matrices with a bounded number of eigenvalues

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Abstract

Conjugation covariants of matrices are applied to study the real algebraic variety consisting of complex Hermitian matrices with a bounded number of distinct eigenvalues. A minimal generating system of the vanishing ideal of degenerate three by three Hermitian matrices is given, and the structure of the corresponding coordinate ring as a module over the special unitary group is determined. The method applies also for degenerate real symmetric three by three matrices. For arbitrary $n$ partial information on the minimal degree component of the vanishing ideal of the variety of $n \times n$ Hermitian matrices with a bounded number of eigenvalues is obtained, and some known results on sum of squares presentations of subdiscriminants of real symmetric matrices are extended to the case of complex Hermitian matrices.

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1 Introduction

Let $\mathbb{F}$ be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. For a matrix $A \in \mathbb{C}^{n \times n}$ denote $\bar{A}$ and $A^T$ the complex conjugate and transpose of $A$, respectively. Fix a positive integer $n \geq 2$, and let $\mathcal{M}$ be one of the following $\mathbb{F}$-subspaces of $\mathbb{C}^{n \times n}$:

(a) the Hermitian matrices $\text{Her}(n) = \{ A \in \mathbb{C}^{n \times n} \mid \bar{A} = A^T \}$
(b) the real symmetric matrices $\text{Sym}(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid A^T = A \}$
(c) all $n \times n$ complex matrices $\mathbb{C}^{n \times n}$
(d) the complex symmetric matrices $\text{Sym}(n, \mathbb{C}) = \{ A \in \mathbb{C}^{n \times n} \mid A^T = A \}$

For $k = 0, 1, \ldots, n - 1$ consider the following subset of $\mathcal{M}$:

$$\mathcal{M}_k := \{ A \in \mathcal{M} \mid \deg(m_A) \leq n - k \}$$

where $m_A$ stands for the minimal polynomial of the matrix $A$. Clearly $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_k \supset \mathcal{M}_{k+1}$, and for a fixed $k \in \{0, 1, \ldots, n - 1\}$ we have the inclusions

$$\left( \mathbb{C}^{n \times n} \right)_k \supset \text{Her}(n)_k$$
$$\cup \quad \cup$$
$$\text{Sym}(n, \mathbb{C})_k \supset \text{Sym}(n, \mathbb{R})_k$$

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We have also the equalities $\text{Her}(n)_k = (\mathbb{C}^{n \times n})_k \cap \text{Her}(n)$ and $\text{Sym}(n, \mathbb{R})_k = \text{Sym}(n, \mathbb{R})_k \cap \mathbb{R}^{n \times n}$. Obviously $M_k$ is the common zero locus in $M$ of the coordinate functions of the polynomial map

$$P_k : M \to \bigwedge^{n-k+1} M, \quad A \mapsto I_n \wedge A \wedge A^2 \wedge \cdots \wedge A^{n-k}$$

where $I_n$ is the $n \times n$ identity matrix and $\bigwedge^l M$ is the $l$th exterior power of $M$. In particular, $M_k$ is an affine algebraic subvariety of the affine space $M$, and it is natural to raise the following question:

**Question 1.1.** Do the coordinate functions of the polynomial map $P_k$ generate the vanishing ideal $I(M_k)$ in $F[M]$ of the affine algebraic subvariety $M_k \subset M$?

Above $F[M]$ is the coordinate ring of $M$, so $F = \mathbb{R}$ in cases (a), (b) whereas $F = \mathbb{C}$ in cases (c), (d), and $F[M]$ is a polynomial ring over $F$ in $\dim_F(M)$ variables. Recall that the vanishing ideal of $M_k$ is

$$I(M_k) := \{ f \in F[M] \mid f|_{M_k} \equiv 0 \} \triangleleft F[M]$$

We have $M_0 = M$, so $I(M_0)$ is the zero ideal, and $P_0$ is the zero map. From now on we focus on $M_{k+1}$ and $I(M_{k+1})$ where $k = 0, 1, \ldots, n - 2$.

Our original interest was in the real cases (a) and (b): then $F = \mathbb{R}$ and all $A \in M$ are diagonalizable with real eigenvalues, hence

$$M_{k+1} = \{ A \in M \mid A \text{ has at most } n - k - 1 \text{ distinct eigenvalues} \} \quad (1)$$

It follows from (1) that in the real cases $M_{k+1}$ (for $k = 0, 1, \ldots, n - 2$) is the zero locus of a single polynomial $s\text{Disc}_k \in \mathbb{R}[M]$, defined by

$$s\text{Disc}_k(A) := \sum_{1 \leq i_1 < \cdots < i_{n-k} \leq n} \prod_{1 \leq s < t \leq n-k} (\lambda_{i_s} - \lambda_{i_t})^2$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. Note that $s\text{Disc}_k(A)$ coincides with the $k$-subdiscriminant of the characteristic polynomial of $A$ (we refer to Chapter 4 of [1] for basic properties of subdiscriminants), and $s\text{Disc}_k$ is a homogeneous polynomial function on $M$ of degree $(n-k)(n-k-1)$. In the special case $k = 0$ we recover the discriminant $\text{Disc} = s\text{Disc}_0$. The ideal $I(M_{k+1})$ is generated by homogeneous elements (with respect to the standard grading on the polynomial ring $F[M] = \bigoplus_{d=0}^\infty F[M]_d$). In [6] it was deduced from the Kleitman-Lovász theorem (cf. Theorem 2.4 in [18]) that $\frac{1}{2} \deg(s\text{Disc}_k) = \binom{n-k}{2}$ is the minimal degree of a non-zero homogeneous component of $I(M_{k+1}) = \bigoplus_{d=0}^\infty I(M_k)_d$ (in fact [6] deals with the case $M = \text{Sym}(n, \mathbb{R})$ only, but the proof of Corollary 5.3 in loc. cit. works also for the case $M = \text{Her}(n)$, see Proposition 7.2 and Theorem 8.1 (i) in the present paper). Since the polynomial map $P_{k+1}$ is homogeneous of degree $\binom{n-k}{2}$, its coordinate functions are contained in the homogeneous component $I(M_{k+1})\binom{n-k}{2}$. So an affirmative answer to Question 1.1 would imply in particular that $I(M_{k+1})$ is generated by its minimal degree non-zero homogeneous component.

In Section 2 we observe that the Zariski closure of $\text{Her}(n)_k$ in the complex affine space $\mathbb{C}^{n \times n}$ is $(\mathbb{C}^{n \times n})_k$, and the Zariski closure of $\text{Sym}(n, \mathbb{R})_k$ in the complex affine space $\text{Sym}(n, \mathbb{C})$ is $\text{Sym}(n, \mathbb{C})_k$, see Proposition 2.3. This implies the following:
Corollary 1.2. Let $\mathcal{M}$ be $\text{Her}(n)$ respectively $\text{Sym}(n, \mathbb{R})$, and $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{M}$ its complexification $\mathbb{C}^{n \times n}$ respectively $\text{Sym}(n, \mathbb{C})$. We have the equality
\[ \mathcal{I}((\mathbb{C} \otimes_{\mathbb{R}} \mathcal{M})_k) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{I}(\mathcal{M}_k) \]
where we make the standard identification $\mathbb{C}[\mathbb{C} \otimes_{\mathbb{R}} \mathcal{M}] = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[\mathcal{M}]$.

Note that in the complex cases (c), (d) by the Hilbert Nullstellensatz the coordinate functions of $\mathcal{P}_k$ generate $\mathcal{I}(\mathcal{M}_k)$ up to radical, hence by Corollary 1.2 this holds also in the real cases, giving some evidence for an affirmative answer to Question 1.1:

Corollary 1.3. The coordinate functions of $\mathcal{P}_k$ generate $\mathcal{I}(\mathcal{M}_k)$ up to radical.

The answer to Question 1.1 is trivially yes for $k = 0$, $n$ arbitrary, and it is straightforward to check that the answer is yes for $k = n - 1$, $n$ arbitrary (since $\mathcal{M}_{n-1}$ consists of scalar matrices, so it is a linear subspace of $\mathcal{M}$, and thus its ideal is generated by linear polynomials). The smallest interesting case therefore is $n = 3$ and $k = 1$. The results of the present paper show in particular that the answer to Question 1.1 is yes also in this case, see Corollary 4.7 (i) and (iv).

By Corollary 1.2 it is sufficient to deal with the complex cases (c), (d). Then there is an action of a semisimple complex linear algebraic group $G$ on $\mathcal{M}$. Namely $G$ is the complex special linear group $\text{SL}(n, \mathbb{C})$ in case (c) and the complex special orthogonal group $\text{SO}(n, \mathbb{C})$ in case (d), acting by conjugation. In Section 3 we recall the notion of covariants and their relation to the algebra $\mathbb{C}[\mathcal{M}]^U$ of $U$-invariants on $\mathcal{M}$ (where $U$ is a maximal unipotent subgroup in $G$), and formulate Lemma 3.1 underlying our strategy to transfer information on relations between basic covariants to give the ideal of $G$-stable subsets in $\mathcal{M}$. Generators of the algebra of covariants on $\mathcal{M} = \mathbb{C}^{3 \times 3}$ were determined by Tange [27]. In Section 4 we recall this result (and provide a natural interpretation of the generators). It turns out that the algebra of $U$-invariants on $\mathcal{M}_1$ is isomorphic to a monomial subring of the three-variable polynomial ring, hence it is easy to determine its presentation, see Theorem 4.5. From this we deduce Corollary 4.7 describing a minimal generating system of $\mathcal{I}(\mathcal{M}_1)$ as well as the $G$-module structure of the minimal degree non-zero homogeneous component of $\mathcal{I}(\mathcal{M}_1)$. Moreover, in Section 5 we derive the formal character of the $G$-module $\mathbb{C}[\mathcal{M}_1]$, in particular, we compute the Hilbert series of the coordinate ring of $\mathcal{M}_1$ as a rational function, see Corollary 5.2. In Section 6 we show how the same method yields similar results (Corollary 6.4 and Corollary 6.5) for case (d): here the algebra of $U$-invariants on $\mathcal{M}$ can be obtained from classical results on covariants of binary quartic forms. Finally, in Sections 7 and 8 we generalize some of the constructions of Section 4 to arbitrary $n$. We derive some partial information on $\mathcal{I}((\mathbb{C}^{n \times n})_k)$ for arbitrary $n$ and $k$, and extend the results in [6] on sum of squares presentations of subdiscriminants of real symmetric matrices to the case of $n \times n$ Hermitian matrices.

2 Complex Zariski closure of the set of degenerate Hermitian matrices

The special linear group $\text{SL}(n, \mathbb{C})$ acts on $\mathbb{C}^{n \times n}$ by conjugation. Two matrices in $\mathbb{C}^{n \times n}$ are similar if they belong to the same $\text{SL}(n, \mathbb{C})$-orbit. A matrix in $\mathbb{C}^{n \times n}$ is diagonalizable if it is similar to a diagonal matrix. It is well known that the subset of diagonalizable matrices is Zariski dense in $\mathbb{C}^{n \times n}$. We need the following refinement:
Proposition 2.1. The diagonalizable elements constitute a Zariski dense subset in \((\mathbb{C}^{n \times n})_k\) for \(k = 0, 1, \ldots, n - 1\).

Proof. Note that any matrix in \((\mathbb{C}^{n \times n})_k\) has at most \(n - k\) distinct eigenvalues. If \(A \in (\mathbb{C}^{n \times n})_k\) has exactly \(n - k\) distinct eigenvalues, then \(m_A\) has no multiple roots, hence \(A\) is diagonalizable. Now suppose that \(A \in (\mathbb{C}^{n \times n})_k\) is not diagonalizable, hence in particular it has strictly less than \(n - k\) distinct eigenvalues. Moreover, for some eigenvalue \(\lambda\) of \(A\) the root factor \(x - \lambda\) has multiplicity \(r \geq 2\) in the minimal polynomial \(m_A\). We shall construct a polynomial map \(\mathbb{C} \to (\mathbb{C}^{n \times n})_k\), \(\varepsilon \mapsto A_\varepsilon\) such that \(A_0 = A\) and for all but finitely many \(\varepsilon\) the matrix \(A_\varepsilon\) has more eigenvalues than \(A\). Since \(m_A\) and the set of eigenvalues of \(A\) is an invariant of the \(SL(n, \mathbb{C})\)-orbit of \(A\), we may assume that \(A\) is in Jordan normal form. By assumption on the minimal polynomial of \(A\), it has a Jordan block \(J_\varepsilon(\lambda)\). In each such Jordan block of \(A\) replace the \((1,1)\)-entry by \(\lambda + \varepsilon\); the resulting matrix is \(A_\varepsilon\). When \(\lambda + \varepsilon\) is not an eigenvalues of \(A\), we have \(m_{A_\varepsilon} = \frac{x-\lambda-\varepsilon}{x-\lambda}m_A\), so \(A_\varepsilon \in (\mathbb{C}^{n \times n})_k\), and \(A_\varepsilon\) has one more eigenvalues than \(A\). Consequently \(A\) is contained in the Zariski closure of the subset of those elements in \((\mathbb{C}^{n \times n})_k\) that have more eigenvalues than \(A\). By a descending induction on the number of distinct eigenvalues of \(A\) one deduces the statement. \(\square\)

The complex orthogonal group

\[ O(n, \mathbb{C}) = \{ A \in \mathbb{C}^{n \times n} \mid AA^T = I_n \} \]

acts by conjugation on \(\mathbb{C}^{n \times n}\), and \(\text{Sym}(n, \mathbb{C})\) is an invariant subspace. Two matrices are orthogonally similar if they belong to the same \(O(n, \mathbb{C})\)-orbit. A matrix \(B\) is orthogonally diagonalizable if it is orthogonally similar to a diagonal matrix (this forces \(B \in \text{Sym}(n, \mathbb{C})\)). It is easy to see that the \(O(n, \mathbb{C})\)-orbit of a diagonal matrix coincides with its \(SO(n, \mathbb{C})\)-orbit, where

\[ SO(n, \mathbb{C}) = \{ A \in O(n, \mathbb{C}) \mid \det(A) = 1 \} \]

is the special orthogonal group.

Proposition 2.2. The orthogonally diagonalizable elements constitute a Zariski dense subset in \(\text{Sym}(n, \mathbb{C})_k\) for \(k = 0, 1, \ldots, n - 1\).

Proof. If \(A \in \text{Sym}(n, \mathbb{C})_k\) has \(n - k\) distinct eigenvalues, then it is diagonalizable, hence by Theorem 4.4.13 in [13] it is orthogonally diagonalizable. Thus by induction on the number of distinct eigenvalues, it suffices to prove that if \(A \in \text{Sym}(n, \mathbb{C})_k\) has less than \(n - k\) eigenvalues and is not diagonalizable, then it is contained in the Zariski closure of the subset of \(\text{Sym}(n, \mathbb{C})_k\) consisting of matrices having more eigenvalues than \(A\). Since the action of \(O(n, \mathbb{C})\) on \(\text{Sym}(n, \mathbb{C})\) preserves both the minimal polynomial and the number of eigenvalues of a matrix, in order to prove this claim it is sufficient to deal with \(A\) taken from a particular set of \(O(n, \mathbb{C})\)-orbit representatives in \(\text{Sym}(n, \mathbb{C})\). Any matrix in \(\mathbb{C}^{n \times n}\) is similar to a symmetric matrix (see Theorem 4.4.9 in [13]), and if two symmetric matrices are similar, then they are orthogonally similar (see Corollary 6.4.19 in [14]). We recall from [13] an explicit symmetric matrix in the similarity class of a Jordan block \(J_\varepsilon(\lambda)\). Denoting by \(E_{i,j}\) the matrix unit with \((i,j)\)-entry 1 and zeroes everywhere else, we have \(J_\varepsilon(\lambda) = \lambda I_r + N_r\) where \(N_r := \sum_{s,t} E_s,t\) is the nilpotent Jordan block. Define \(B_r := \frac{1}{\sqrt{2}}(I_r + i \sum_{s+t=r+1} E_{s,t})\) where \(i\) is the imaginary complex unit with \(i^2 = -1\). We have \(B_r B_r = I_r\) and \(B_r E_{s,t} B_r = \frac{1}{2}(E_{s,t} + E_{r+1-s,r+1-t} + i E_{r+1-s,t} - i E_{s,r+1-t})\). This shows
that $S_r(\lambda) := B_r J_r(\lambda) \overline{B}_r$ is symmetric. For $\varepsilon \in \mathbb{C}$ set

$$S_{r, \varepsilon}(\lambda) := \begin{cases} 
B_r (J_r(\lambda) - \varepsilon^2 E_{m+1,m}) \overline{B}_r, & \text{when } r = 2m \\
B_r (J_r(\lambda) + \varepsilon^2 (E_{m+1,m} + E_{m+2,m+1})) \overline{B}_r, & \text{when } 1 < r = 2m + 1 \\
S_1 (\lambda + \varepsilon) & \text{when } r = 1
\end{cases}$$

Then $S_{r, \varepsilon}(\lambda)$ is symmetric, and for $r > 1$ its characteristic polynomial is $k_{S_{r, \varepsilon}(\lambda)} = (x - \lambda - \varepsilon)(x - \lambda + \varepsilon)(x - \lambda)^{r-2} = \frac{(x-\lambda-\varepsilon)(x-\lambda+\varepsilon)}{(x-\lambda)^2} k_{S_r(\lambda)}$. Assume now that $A \in \text{Sym}(n, \mathbb{C})_k$ is not diagonalizable, and is block diagonal, with diagonal blocks of the form $S_l(\mu)$ with various $\mu \in \mathbb{C}$ and $l \in \mathbb{N}$. By assumption the minimal polynomial $m_A$ has a root factor $x - \lambda$ with multiplicity at least 2. Take for $A_\varepsilon$ the matrix obtained by replacing each block $S_r(\lambda)$ in $A$ by $S_{r, \varepsilon}(\lambda)$. Then $m_{A_\varepsilon}$ divides $\frac{(x-\lambda-\varepsilon)(x-\lambda+\varepsilon)}{(x-\lambda)^2} m_A$, so $A_\varepsilon \in \text{Sym}(n, \mathbb{C})_k$. Moreover, when none of $\lambda + \varepsilon$ and $\lambda - \varepsilon$ is an eigenvalue of $A$, then $A_\varepsilon$ has one or two more eigenvalues than $A$. This shows that $A$ is contained in the Zariski closure of the set of $\text{Sym}(n, \mathbb{C})_k$ consisting of matrices with more eigenvalues than $A$. So our claim is proved.

**Remark 2.3.** The proofs of Propositions 2.1 and 2.2 show that for any $A \in (\mathbb{C}^{n \times n})_k$ (respectively $A \in \text{Sym}(n, \mathbb{C})_k$) there are diagonalizable (respectively orthogonally diagonalizable) elements in $(\mathbb{C}^{n \times n})_k$ (respectively $\text{Sym}(n, \mathbb{C})_k$) arbitrarily close to $A$ with respect to the euclidean metric.

**Proposition 2.4.**

(i) The Zariski closure of $\text{Her}(n)_k$ in the complex affine space $\mathbb{C}^{n \times n}$ is $(\mathbb{C}^{n \times n})_k$.

(ii) The Zariski closure of $\text{Sym}(n, \mathbb{R})_k$ in the complex affine space $\text{Sym}(n, \mathbb{C})$ is $\text{Sym}(n, \mathbb{C})_k$.

**Proof.** (i) The special unitary group

$$SU(n) := \{ A \in \mathbb{C}^{n \times n} \mid AA^T = I_n, \quad \det(A) = 1 \}$$

is Zariski dense in the complex linear algebraic group $SL(n, \mathbb{C})$. Note that the subset $\text{Her}(n)_k$ in $\mathbb{C}^{n \times n}$ is $SU(n)$-stable, hence its Zariski closure is $SL(n, \mathbb{C})$-stable. Therefore by Proposition 2.1 it is sufficient to show that the Zariski closure of $\text{Her}(n)_k$ contains the set $X$ of all complex diagonal matrices with at most $n - k$ distinct diagonal entries. Let $L$ be an irreducible component of $X$. Then $L$ is an $n - k$-dimensional linear subspace, spanned by its intersection with the space $D$ of real diagonal matrices. Now $L \cap D \subset \text{Her}(n)_k$, and the Zariski closure of the real linear subspace $L \cap D$ is obviously its $\mathbb{C}$-linear span $L$. Thus $X$ is contained in the Zariski closure of $\text{Her}(n)_k$.

The proof of (ii) is similar: the real special orthogonal group $SO(n)$ is Zariski dense in $SO(n, \mathbb{C})$, hence the Zariski closure of $\text{Sym}(n, \mathbb{R})_k$ is $SO(n, \mathbb{C})$-stable. Now use Proposition 2.2 and conclude in the same way as above. 

**3 Covariants and $G$-stable ideals**

Let $G$ be a connected reductive linear algebraic group over the base field $\mathbb{C}$ (like $SL_n(\mathbb{C})$ or $SO(n, \mathbb{C})$). Fix a maximal unipotent subgroup $U$ in $G$, and a maximal torus $T$ in $G$ normalizing $U$. We need to recall some basic facts from highest weight theory (cf. e.g. [8, 9, 21]): by a $G$-module we mean a rational $G$-module. Any $G$-module $V$ is spanned by $T$-eigenvectors. A (non-zero) $T$-eigenvector $v$ is called a weight vector, and the character
\( \lambda : T \to \mathbb{C}^\times \) given by \( t \cdot v = \lambda(t)v \) is called its weight. A \( U \)-invariant weight vector is called a highest weight vector. A highest weight vector generates an irreducible \( G \) submodule. Moreover, an irreducible \( G \)-module contains a unique (up to scalar multiples) highest weight vector.

Our proof of Corollary \[4.7\] and \[6.4\] is based on the following general observation. Let \( M \) be an affine \( G \)-variety with coordinate ring \( \mathbb{C}[M] \). It is a \( G \)-module via \( (g \cdot f)(x) = f(g^{-1}x) \) for \( g \in G, f \in \mathbb{C}[M], x \in M \). The algebra \( \mathbb{C}[M]^U \) of \( U \)-invariant polynomial functions on \( M \) is finitely generated by \([12]\) (see Theorem 9.4 in \[11\] or \[7\]). Let \( u_1, \ldots, u_r \) be generators of the algebra \( \mathbb{C}[M]^U \).

**Lemma 3.1.** For any Zariski closed \( G \)-stable subset \( X \) in \( M \), the vanishing ideal \( \mathcal{I}(X) \) is generated as a \( G \)-stable ideal in \( \mathbb{C}[M] \) by \( f_j(u_1, \ldots, u_r), j = 1, \ldots, m \), where \( f_1, \ldots, f_m \) generate as an ideal in the \( r \)-variable polynomial ring the kernel of the \( \mathbb{C} \)-algebra homomorphism \( \varphi : \mathbb{C}[x_1, \ldots, x_r] \to \mathbb{C}[X]^U \) given by \( x_i \mapsto u_i|_X \) (the restriction of \( u_i \) to \( X \)), \( i = 1, \ldots, r \).

**Proof.** Denote by \( \eta \) the \( \mathbb{C} \)-algebra homomorphism \( \mathbb{C}[M]^U \to \mathbb{C}[X]^U \) given by restriction of functions on \( M \) to \( X \). Obviously we have \( \varphi = \eta \circ \Psi \), where \( \Psi : \mathbb{C}[x_1, \ldots, x_r] \to \mathbb{C}[M]^U \) is the \( \mathbb{C} \)-algebra surjection given by \( x_i \mapsto u_i \) \( i = 1, \ldots, r \). Hence \( \ker(\eta) = \Psi(\ker(\varphi)) \). On the other hand, clearly \( \ker(\eta) = \mathcal{I}(X)^U \). Recall that any \( G \)-submodule of \( \mathbb{C}[M] \) is the sum of finite dimensional irreducible \( G \)-submodules, each summand containing a non-zero \( U \)-invariant element (a highest weight vector). Therefore any \( G \)-submodule of \( \mathbb{C}[M] \) is generated by its \( U \)-invariant elements. In particular, \( \mathcal{I}(X) \) is generated by \( \mathcal{I}(X)^U \) as a \( G \)-module.

**Remark 3.2.** The map \( \eta : \mathbb{C}[M]^U \to \mathbb{C}[X]^U \) is surjective. Indeed, the maximal torus \( T \) acts rationally on \( \mathbb{C}[M]^U \) and on \( \mathbb{C}[X]^U \), and these spaces are spanned by weight vectors (i.e. \( T \)-eigenvectors). Thus it is sufficient to show that any weight vector \( h \in \mathbb{C}[X]^U \) is contained in the image of \( \eta \). Since \( h \) is \( U \)-invariant, it is a highest weight vector in \( \mathbb{C}[X] \), hence generates an irreducible \( G \)-submodule \( V \) in \( \mathbb{C}[X] \). Thus \( \mathcal{I}(X) \) has an irreducible \( G \)-module direct complement \( V' \) in the inverse image of \( V \) under the natural surjection \( \mathbb{C}[M] \to \mathbb{C}[X] \). Take a highest weight vector \( h' \) in \( V' \), so \( h' \in \mathbb{C}[M]^U \), and \( \eta(h') \) is a nonzero scalar multiple of \( h \).

By a covariant \( f \) on \( M \) we mean a non-zero \( G \)-equivariant polynomial map \( f : M \to V \), where \( V \) is a finite dimensional (rational) \( G \)-module. The non-zero covariant \( f \) is irreducible if \( V \) is an irreducible \( G \)-module. In this case the comorphism of \( f \) restricts to an embedding \( f^* \) of the \( G \)-module \( V^* \) into the coordinate ring \( \mathbb{C}[M] \), and we shall denote by \( f^U \in \mathbb{C}[M]^U \) the unique (up to scalar multiples) highest weight vector in \( f^*(V^*) \). Conversely, a non-zero \( T \)-eigenvector in \( \mathbb{C}[M]^U \) generates an irreducible \( G \)-submodule \( W \) in \( \mathbb{C}[M] \), and the map \( M \to W^* \) sending \( m \in M \) to the linear functional \( W \to \mathbb{C}, w \mapsto w(m) \) is an irreducible covariant. So an irreducible covariant determines (up to scalar multiples) a non-zero \( T \)-eigenvector in \( \mathbb{C}[M]^U \), and vice versa. The algebra \( \mathbb{C}[M]^U \) is sometimes called therefore the algebra of covariants on \( M \). We shall write \( \text{Cov}_G(M, V) \) for the set of covariants \( f : M \to V \); it is naturally a module over the algebra \( \mathbb{C}[M]^G \) of polynomial invariants on \( M \).
4 Covariants of $3 \times 3$ matrices

In Sections 3 and 4, set $\mathcal{M} := \mathbb{C}^{3 \times 3}$ and $G := SL(3, \mathbb{C})$ acting by conjugation on $\mathcal{M}$. We take the maximal unipotent subgroup $U$ of $G$ consisting of the unipotent upper triangular matrices, normalized by the maximal torus $T$ consisting of the diagonal matrices in $G$. Generators of the algebra $\mathbb{C}[\mathcal{M}]^U$ were determined by Tange [27], Section 3. Here we give a natural interpretation of all the generators, by presenting some natural covariants $f$ on $\mathcal{M}$ such that the corresponding $f^U$ (with the notation introduced in Section 3) provide the generators found in [27]. We shall identify the group $\text{Char}(T)$ of rational characters of the maximal torus $T$ in $G$ with $\mathbb{Z}^2$, such that $\lambda \in \mathbb{Z}^2$ corresponds to the character of $T$ given by $\text{diag}(z_1, z_2, z_1^{-1}z_2^{-1}) \mapsto z_1^{\lambda_1}z_2^{\lambda_2}$. The possible highest weights correspond to $\{\lambda \in \mathbb{Z}^2 \mid \lambda_1 \geq \lambda_2 \geq 0\}$, and denote by $V^\lambda$ the irreducible $G$-module with highest weight $\lambda$. The $G$-module $V^{(2,1)}$ can be realized as

$$V^{(2,1)} \cong \mathcal{N} := \{ f \in \mathcal{M} \mid \text{tr}(f) = 0 \}$$

We start with the covariant $\mathcal{M} \rightarrow \mathcal{N}$ given by

$$c_1 : A \mapsto A - \frac{1}{3} \text{tr}(A)I_3$$

(2)

where $\text{tr}$ is the usual trace function. Define a second covariant $\mathcal{M} \rightarrow \mathcal{N}$ by

$$c_2 := c_1 \circ \partial_2 \circ c_1 \quad \text{where} \quad \partial_2 : \mathcal{N} \rightarrow \mathcal{M}, \quad A \mapsto A^2$$

(3)

Recall that the defining representation of $G$ on $\mathbb{C}^3$ is irreducible and is isomorphic to $V^{(1,0)}$, its dual is $(\mathbb{C}^3)^* \cong V^{(1,1)}$. The symmetric powers of $\mathbb{C}^3$ and $(\mathbb{C}^*)^3$ are also irreducible, we have $S^3(\mathbb{C}^3) \cong V^{(3,0)}$ and $S^3(\mathbb{C}^3)^* \cong V^{(3,3)}$. Think of $S^3(\mathbb{C}^3)^*$ as the space of homogeneous cubic polynomial functions on $\mathbb{C}^3$, and define a covariant

$$c_3 : \mathcal{M} \rightarrow S^3(\mathbb{C}^3)^*, \quad A \mapsto (x \mapsto \text{det}(x|Ax|A^2x))$$

(4)

where for $x \in \mathbb{C}^3$ and $A \in \mathcal{M}$ we write $(x|Ax|A^2x)$ for the $3 \times 3$ matrix whose columns are $x, Ax, A^2x$ and det is the determinant. For $g \in G$ we have

$$(c_3(gAg^{-1}))(x) = \text{det}(x|gAg^{-1}x|gA^2g^{-1}x) = \text{det}(g|g^{-1}x|Ag^{-1}x|A^2g^{-1}x)$$

$$= c_3(A)(g^{-1}x) = (g \cdot c_3(A))(x)$$

showing that $c_3$ is indeed a covariant. Moreover, it is non-zero (e.g. take for $A$ a diagonal matrix with distinct eigenvalues), hence is an irreducible covariant. Identify $(\mathbb{C}^3)^*$ with the space of row vectors $\{x^T \mid x \in \mathbb{C}^3\}$ in the standard way. Think of $S^3(\mathbb{C}^3)$ as the space of homogeneous cubic polynomial functions on $(\mathbb{C}^3)^*$, and similarly to the construction of $c_3$, define the irreducible covariant

$$c_4 : \mathcal{M} \rightarrow S^3(\mathbb{C}^3), \quad A \mapsto (x^T \mapsto \text{det} \left( \begin{pmatrix} x^T \\ x^T A \\ x^T A^2 \end{pmatrix} \right))$$

(5)

It is well known that the algebra $\mathbb{C}[\mathcal{M}]^G$ of polynomial invariants is generated by the following three algebraically independent elements:

$$d_1 : A \mapsto \text{tr}(A), \quad d_2 : A \mapsto \frac{1}{6} \text{tr}(c_1(A)^2), \quad d_3 : A \mapsto \frac{1}{2} \text{det}(c_1(A))$$

(the scalars $\frac{1}{6}$ and $\frac{1}{2}$ above are chosen in order to make certain later formulae simpler).
Proposition 4.1. The algebra \( \mathbb{C}[\mathcal{M}]^U \) is generated by the seven elements \( d_i \) \( (i = 1, 2, 3) \) and \( c_j^U \) \( (j = 1, 2, 3, 4) \).

Proof. This is a restatement of Proposition 2 from [27] giving seven explicit \( T \)-eigenvectors generating \( \mathbb{C}[\mathcal{M}]^U \). To see this one just has to write down explicit expressions for the \( c_j^U \) in terms of the coordinate functions on \( \mathcal{M} \).

We shall view \( \mathbb{C}[\mathcal{N}] \) as a subalgebra of \( \mathbb{C}[\mathcal{M}] \) via the embedding \( f \mapsto f \circ c_1 \). Clearly \( \mathbb{C}[\mathcal{M}] \) is a polynomial ring over \( \mathbb{C}[\mathcal{N}] \) generated by \( d_1 \). Moreover,

\[
\text{for each } f \in \{ d_2, d_3, c_1, c_2, c_3, c_4 \} \text{ we have that } f = f \circ c_1
\]

hence \( f^U \in \mathbb{C}[\mathcal{N}] \). Thus Proposition 4.1 can be restated as follows:

Proposition 4.2. We have \( \mathbb{C}[\mathcal{M}]^U = \mathbb{C}[\mathcal{N}]^U [d_1] \) and \( \mathbb{C}[\mathcal{N}]^U \) is generated by \( d_2, d_3, c_1^U, c_2^U, c_3^U, c_4^U \).

Proposition 4.3. The coordinate functions of \( c_3 \) and \( c_4 \) are contained in the \( \mathbb{C} \)-subspace of \( \mathbb{C}[\mathcal{M}] \) spanned by all the coordinate functions of \( \mathcal{P}_1 : \mathcal{M} \to \Lambda^3 \mathcal{M}, A \mapsto I_3 \wedge A \wedge A^2 \).

Proof. This follows from the Cauchy-Binet formula and the following two matrix equalities, where \( e_1, e_2, e_3 \) are the standard basis vectors in \( \mathbb{C}^3 \), \( a_1, a_2, a_3 \) are the columns of a \( 3 \times 3 \) matrix \( A \), \( b_1, b_2, b_3 \) are the columns of a \( 3 \times 3 \) matrix \( B \), and \( x \in \mathbb{C}^3 \):

\[
\begin{pmatrix}
  x_1 I_3 & x_2 I_3 & x_3 I_3
\end{pmatrix}_{3 \times 9}
\begin{pmatrix}
  e_1 & a_1 & b_1 \\
  e_2 & a_2 & b_2 \\
  e_3 & a_3 & b_3
\end{pmatrix}_{3 \times 3}
= \begin{pmatrix}
  x & A x & B x
\end{pmatrix}_{3 \times 3}
\]

Now we turn to the affine subvariety \( \mathcal{M}_1 \subset \mathcal{M} \). Restriction of functions on \( \mathcal{M} \) to \( \mathcal{M}_1 \) gives the natural surjection

\[ \mathbb{C}[\mathcal{M}] \rightarrow \mathbb{C}[\mathcal{M}_1] \]

onto the coordinate ring \( \mathbb{C}[\mathcal{M}_1] = \mathbb{C}[\mathcal{M}] / \mathcal{I}(\mathcal{M}_1) \) of the affine algebraic variety \( \mathcal{M}_1 \). We shall write \( \overrightarrow{d}_j, \overrightarrow{c}_j \), respectively \( \overrightarrow{c}_j^U \), for the restriction to \( \mathcal{M}_1 \) of \( d_i, c_j \), respectively \( c_j^U \). The covariant \( \overrightarrow{c}_1 \) maps \( \mathcal{M}_1 \) onto

\[ \mathcal{N}_1 := \mathcal{M}_1 \cap \mathcal{N} \]

hence induces an embedding of \( \mathbb{C}[\mathcal{N}_1] \) as a subalgebra of \( \mathbb{C}[\mathcal{M}_1] \). Furthermore, \( \mathcal{M}_1 = \mathcal{N}_1 \oplus \mathcal{I}_3 \), hence

\[ \mathbb{C}[\mathcal{M}_1] = \mathbb{C}[\mathcal{N}_1]\overrightarrow{[\overrightarrow{d}_1]} \]

is a polynomial ring generated by \( \overrightarrow{d}_1 \) over the subalgebra \( \mathbb{C}[\mathcal{N}_1] \), and \( \overrightarrow{d}_2, \overrightarrow{d}_3, \overrightarrow{c}_1 \in \mathbb{C}[\mathcal{N}_1] \).

Proposition 4.4. We have the following equalities for covariants on \( \mathcal{M}_1 \):

\[ \overrightarrow{c}_3 = 0, \quad \overrightarrow{c}_4 = 0, \quad \overrightarrow{d}_2^3 = \overrightarrow{d}_3^2, \quad \overrightarrow{d}_3 \overrightarrow{c}_1 = \overrightarrow{d}_2 \overrightarrow{c}_2 \]

(the last equality is understood in the \( \mathbb{C}[\mathcal{M}_1]^G \)-module \( \text{Cov}_G(\mathcal{M}_1, \mathcal{N}) \)).
Proof. The equalities \( \overline{c}_3 = 0 = \overline{c}_1 \) follow from Proposition 4.3 and the fact that \( \mathcal{P}_1 \) maps \( \mathcal{M}_1 \) to zero. Since diagonalizable elements in \( \mathcal{M}_1 \) constitute a Zariski dense subset in \( \mathcal{M}_1 \) by Proposition 2.1, it is sufficient to check vanishing of the polynomial maps \( d_2^3 - d_3^3 \) and \( d_3 c_1 - d_2 c_2 \) on diagonalizable elements in \( \mathcal{M}_1 \). Therefore by Proposition 4.4 and the covariance property it is sufficient to check vanishing of the above covariants on the diagonal matrices \( D(z) := \text{diag}(z, z, -2z) \) where \( z \in \mathbb{C} \). Now we have
\[
d_2(D(z)) = z^2, \quad d_3(D(z)) = -z^3, \quad c_1(D(z)) = D(z), \quad c_2(D(z)) = -z D(z)
\]
so the desired relations obviously hold. \( \square \)

Recall that the elements \( c_i^U \) are determined only up to non-zero scalar multiples; according to Proposition 4.4 it is possible to normalize \( c_1^U \) and \( c_2^U \) so that the equality
\[
\overline{d_3 c_1^U} = \overline{d_2 c_2^U}
\]
holds, and from now on we assume that \( c_1^U \) and \( c_2^U \) were chosen so that \( 8 \) holds. The standard \( \mathbb{N}_0 \)-grading on the polynomial algebra \( \mathbb{C}[\mathcal{M}] \) and the grading by the group \( \text{Char}(T) = \mathbb{Z}^2 \) of rational characters defined by the action of the maximal torus \( T \subset G \) can be combined to a bigrading by \( \mathbb{N}_0 \times \text{Char}(T) \): we say that \( f \in \mathbb{C}[\mathcal{M}] \) is \textit{bihomogeneous of bidegree} \( \text{bideg}(f) = (n, \lambda) \) if \( f(zA) = z^n f(A) \) for all \( A \in \mathcal{M} \) and \( z \in \mathbb{C} \), and \( t \cdot f = t_1^{\lambda_1} t_2^{\lambda_2} f \) for any \( \text{diag}(t_1, t_2, t_1^{-1} t_2^{-1}) \in T \). Clearly the algebras \( \mathbb{C}[\mathcal{M}]^U, \mathbb{C}[\mathcal{M}_1]^U, \mathbb{C}[\mathcal{M}_1]^U, \mathbb{C}[\mathcal{N}_1]^U \) all inherit the bigrading from \( \mathbb{C}[\mathcal{M}] \).

In the following statement \( \mathbb{C}[z^2, z^3, D, zD] \) stands for the subalgebra of the two-variable polynomial ring \( \mathbb{C}[z, D] \) generated by the monomials \( z^2, z^3, D, zD \) (the notation \( z, D \) for the indeterminates is inspired by the proof of Proposition 4.4), and \( \mathbb{C}[x_0, x_1, x_2, x_3, x_4] \) is a five-variable polynomial algebra.

**Theorem 4.5.** (i) The algebra \( \mathbb{C}[\mathcal{M}_1]^U \) is a polynomial ring generated by \( \overline{d}_1 \) over \( \mathbb{C}[\mathcal{N}_1]^U \).

(ii) There is a \( \mathbb{C} \)-algebra isomorphism \( \eta : \mathbb{C}[z^2, z^3, D, zD] \rightarrow \mathbb{C}[\mathcal{N}_1]^U \) with
\[
\eta : z^2 \mapsto \overline{d}_2, \quad z^3 \mapsto \overline{d}_3, \quad D \mapsto \overline{c}_1^U, \quad zD \mapsto \overline{c}_2^U.
\]

(iii) The kernel of the natural surjection \( \varphi : \mathbb{C}[x_0, x_1, x_2, x_3, x_4] \rightarrow \mathbb{C}[\mathcal{M}_1]^U, x_0 \mapsto \overline{d}_1, x_1 \mapsto \overline{d}_2, x_2 \mapsto \overline{d}_3, x_3 \mapsto \overline{c}_1^U, x_4 \mapsto \overline{c}_2^U \) is generated as an ideal by
\[
x_1^3 - x_2^2, \quad x_1 x_4 - x_2 x_3, \quad x_4^2 - x_1 x_3^2, \quad x_2 x_4 - x_1^2 x_3.
\]

Proof. Statement (i) follows from \( 7 \). As explained in Remark 3.2, the natural surjection \( \mathbb{C}[\mathcal{N}] \rightarrow \mathbb{C}[\mathcal{N}_1] \) restricts to a surjection \( \mathbb{C}[\mathcal{N}]^U \rightarrow \mathbb{C}[\mathcal{N}_1]^U \). Hence by Propositions 4.2 and 4.4 \( \mathbb{C}[\mathcal{N}_1]^U \) is generated by \( \overline{d}_2, \overline{d}_3, \overline{c}_1^U, \overline{c}_2^U \). The variety \( \mathcal{N}_1 \) is irreducible, as by Proposition 2.1 it is the Zariski closure of \( G \cdot \{D(z) \mid z \in \mathbb{C} \} \) (with the notation of the proof of Proposition 4.4). Thus the coordinate ring \( \mathbb{C}[\mathcal{N}_1] \) is a domain, and by \( 8 \) we have the equality \( \overline{c}_2^U = \overline{c}_1^U \overline{d}_3 / \overline{d}_2 \) in the function field \( \mathbb{C}(\mathcal{N}_1) \). The proof of Proposition 4.4 shows that the map \( z^2 \mapsto \overline{d}_2, z^3 \mapsto \overline{d}_3 \) extends to a \( \mathbb{C} \)-algebra isomorphism \( \mathbb{C}[z^2, z^3] \rightarrow \mathbb{C}[\overline{d}_2, \overline{d}_3] \subset \mathbb{C}[\mathcal{N}_1] \). This extends to a \( \mathbb{C} \)-algebra surjection \( \tilde{\eta} : \mathbb{C}[z^2, z^3, D] \rightarrow \mathbb{C}[\overline{d}_2, \overline{d}_3, \overline{c}_1^U] \) with \( D \mapsto \overline{c}_1^U \). We claim that \( \tilde{\eta} \) is an isomorphism. Indeed, define a bigrading on the polynomial algebra \( \mathbb{C}[z, D] \) by setting \( \text{bideg}(z) := (1, (0, 0)) \) and \( \text{bideg}(D) := (1, (2, 1)) \). Then \( \tilde{\eta} \) is a homomorphism of bigraded algebras, so \( \ker(\tilde{\eta}) \) is spanned by bihomogeneous elements. Now observe that the
bihomogeneous components of \( \mathbb{C}[z^2, z^3, D] \) are one-dimensional, each is spanned by a monomial \((z^2)^i(z^3)^j D^k\), and these monomials are not mapped to zero, since \( d_2, d_3, c_1' \) are nonzero, and \( \mathbb{C}[d_2, d_3, c_1'] \) is a domain. The isomorphism \( \eta \) extends to an isomorphism between the fields of fractions of \( \mathbb{C}[z^2, z^3, D] \) and \( \mathbb{C}[d_2, d_3, c_1'] \), and this latter field isomorphism restricts to the desired \( \mathbb{C} \)-algebra isomorphism \( \eta : \mathbb{C}[z^2, z^3, D, zD] \to \mathbb{C}[d_2, d_3, c_1', c_1'd_3/d_2] \). Thus (ii) is proved.

To prove (iii), by (ii) it is sufficient to show that the given four polynomials generate the kernel of the natural surjection \( \phi : \mathbb{C}[x_1, x_2, x_3, x_4] \to \mathbb{C}[z^2, z^3, D, zD] \) given by \( x_1 \mapsto z^2, x_2 \mapsto z^3, x_3 \mapsto D, x_4 \mapsto zD \). The given four polynomials are indeed in the kernel of \( \phi \), and it is easy to see that modulo the ideal generated by them, any monomial in \( \mathbb{C}[x_1, x_2, x_3, x_4] \) can be rewritten as a linear combination of the monomials
\[
\{ x_3^i x_4, \ x_1^i x_3^j, \ x_1^i x_2 x_3^j \mid i, j = 0, 1, \ldots \}.
\]
Now \( \phi \) maps bijectively the above set of monomials onto \( \{ z^k D^l \mid (k, l) \neq (1, 0) \} \), which is a basis of \( \mathbb{C}[z^2, z^3, D, zD] \). This implies the claim. \( \square \)

**Corollary 4.6.** As a \( G \)-stable ideal, \( \mathcal{I}(M_1) \) is generated by \( c_3' \) and \( c_4' \).

**Proof.** We apply Lemma 3.1 by Propositions 4.2, 4.4 and by Theorem 4.5 we conclude that \( \mathcal{I}(M_1) \) is generated as a \( G \)-stable ideal by \( c_3', c_4', d_2 - d_3, d_2 c_3^2 - d_3 c_4^2, (c_3')^2 - d_2 (c_4')^2, d_3 c_2^2 - d_2 c_3^2 \). It is easy to verify by computer (we used the computer algebra system [3]) that the latter four elements of \( \mathbb{C}[M] \) are contained in the ideal generated by the coordinate functions of \( c_3 \) (the linear span of these coordinate functions is the \( G \)-module generated by \( c_3' \)), hence the result follows. \( \square \)

**Corollary 4.7.** (i) The ideal \( \mathcal{I}(M_1) \) is generated by its degree 3 homogeneous component \( \mathcal{I}(M_1)_3 \).

(ii) The 20 coordinate functions of \( c_3 \) and \( c_4 \) constitute a \( \mathbb{C} \)-basis in \( \mathcal{I}(M_1)_3 \).

(iii) As a \( G \)-module \( \mathcal{I}(M_1)_3 \) is isomorphic to \( S^3(\mathbb{C}^3) \oplus S^3(\mathbb{C}^3)^* \).

(iv) The coordinate functions of \( \mathcal{P}_1 : M \to \bigwedge^3 M \) span \( \mathcal{I}(M_1)_3 \).

**Proof.** Since \( c_3' \) and \( c_4' \) are homogeneous of degree three, it follows trivially from Corollary 4.6 that \( \mathcal{I}(M_1) \) is generated by its degree three homogeneous component, so (i) is proved. The covariants \( c_3 \) and \( c_4 \) are non-zero, irreducible, and map \( M \) into non-isomorphic \( G \)-modules. It follows that their coordinate functions are linearly independent, so both (ii) and (iii) hold by Corollary 4.6 and by construction of \( c_3 \), \( c_4 \). Finally, (iv) follows from (ii) and Proposition 4.3. \( \square \)

**Remark 4.8.** In [16] for any complex simple Lie group \( G \) the authors construct a \( G \)-submodule in the minimal degree non-zero homogeneous component of the vanishing ideal of the subset of singular elements in the Lie algebra of \( G \), and determine its \( G \)-module structure. For the special case \( G = SL(n, \mathbb{C}) \) this subspace coincides with the space spanned by the coordinate functions of \( \mathcal{P}_1 \).

## 5 Hilbert series

Following [2] we introduce the graded multiplicity series of \( \mathbb{C}[M_1] \) as follows:
\[
M(\mathbb{C}[M_1]; q_1, q_2, t) := \sum_{d=0}^{\infty} \sum_{\lambda \in \text{Char}(T)} m(d, \lambda) q_1^{\lambda_1} q_2^{\lambda_2} t^d \in \mathbb{Z}[q_1, q_2][[t]] \tag{9}
\]
where \( m(d, \lambda) \) denotes the multiplicity of the irreducible \( G \)-module \( V^\lambda \) as a summand in the degree \( d \) homogeneous component of \( \mathbb{C}[\mathcal{M}_1] \).

**Corollary 5.1.** We have the equality

\[
M(\mathbb{C}[\mathcal{M}_1]; q_1, q_2, t) = \frac{1 - t + t^2 + q_1^2 q_2 t^2 - q_1^2 q_2 t^3}{(1 - t)^2(1 - q_1^2 q_2 t)}
\]

**Proof.** By the discussion at the beginning of Sections 3 and 4, \( M(\mathbb{C}[\mathcal{M}_1]; q_1, q_2, t) \) is nothing but the bigraded Hilbert series of \( \mathbb{C}[\mathcal{M}_1] \) (with respect to the bigrading by \( \mathbb{N}_0 \times \text{Char}(T) \) introduced before Theorem 4.5). By Theorem 4.5 (i) and (ii), \( (1 - t)^{-3} \eta(z^j D^k) \) where \( i, j, k \in \mathbb{N}_0, (j, k) \neq (1, 0) \) is a \( \mathbb{C} \)-vector space basis in \( \mathbb{C}[\mathcal{M}_1] \), and the basis element corresponding to \((i, j, k)\) is bihomogeneous of bidegree \((i + j + k, (2k, k))\). Consequently, \( M(\mathbb{C}[\mathcal{M}_1]; q_1, q_2, t) = \frac{1}{(1 - t)^2(1 - q_1^2 q_2 t)} - \frac{t}{1 - t} \).

The Hilbert series of a multigraded vector space in general is the generating function of the dimensions of its multihomogeneous components. In particular, the Hilbert series of the bigraded algebra \( \mathbb{C}[\mathcal{M}_1] \) is

\[
H(\mathbb{C}[\mathcal{M}_1]; q_1, q_2, t) := \sum_{d=0}^{\infty} \sum_{\lambda \in \text{Char}(T)} a(d, \lambda) q_1^{\lambda_1} q_2^{\lambda_2} t^d \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][[t]]
\]

(10)

where \( a(d, \lambda) \) is the multiplicity of the 1-dimensional \( T \)-module with character \( \lambda \) in the degree \( d \) homogeneous component of \( \mathbb{C}[\mathcal{M}_1] \). The series (9) and (10) are related by

\[
H(\mathbb{C}[\mathcal{M}_1]; q_1, q_2, t) = \sum_{d=0}^{\infty} \sum_{\lambda \in \text{Char}(T)} m(d, \lambda) \text{tr}(\text{diag}(q_1, q_2, q_1^{-1} q_2^{-1}))_{V^\lambda} t^d
\]

where \( \text{diag}(q_1, q_2, q_1^{-1} q_2^{-1})_{V^\lambda} \) is the linear transformation of \( V^\lambda \) corresponding to \( \text{diag}(q_1, q_2, q_1^{-1} q_2^{-1}) \in T \subset G \) under the representation on \( V^\lambda \). Setting \( q_3 := q_1^{-1} q_2^{-1} \) and denoting by \( S_3 \) the symmetric group of degree 3, we have

\[
\text{tr}(\text{diag}(q_1, q_2, q_1^{-1} q_2^{-1}))_{V^\lambda} = \sum_{\pi \in S_3} \text{sign}(\pi) q_{\pi(1)}^{\lambda_1+2} q_{\pi(2)}^{\lambda_2+1} (q_1 - q_2)(q_1 - q_3)(q_2 - q_3)
\]

Consequently, still using the notation \( q_3 := q_1^{-1} q_2^{-1} \) we have

\[
H(\mathbb{C}[\mathcal{M}_1]; q_1, q_2, t) = \sum_{\pi \in S_3} \text{sign}(\pi) q_{\pi(1)}^2 q_{\pi(2)} M(\mathbb{C}[\mathcal{M}_1]; q_{\pi(1)}, q_{\pi(2)}, t) \prod_{1 \leq i < j \leq 3}(q_i - q_j)
\]

from which (after substituting \( q_1 = q_2 = 1 \)) one can easily compute the ordinary Hilbert series

\[
H(\mathbb{C}[\mathcal{M}_1]; t) := \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(\mathbb{C}[\mathcal{M}_1]_d) t^d
\]

where \( \mathbb{C}[\mathcal{M}_1]_d \) stands for the degree \( d \) homogeneous component of the graded algebra \( \mathbb{C}[\mathcal{M}_1] \):

**Corollary 5.2.** We have the equality

\[
H(\mathbb{R}[\text{Her}(3)]; t) = H(\mathbb{C}[\mathcal{M}_1]; t) = \frac{1 + 3t + 6t^2 - 10t^3 + 10t^4 - 5t^5 + t^6}{(1 - t)^6}
\]
6 Symmetric $3 \times 3$ matrices

In this section set $\mathcal{M} := \text{Sym}(3, \mathbb{C})$ and $G := SO(3, \mathbb{C})$ acting by conjugation on $\mathcal{M}$. Again denote $\mathcal{N}$ the subset of trace zero matrices in $\mathcal{M}$, and $\mathcal{N}_1 := \mathcal{M}_1 \cap \mathcal{N}$. We may restrict the covariants on $\mathbb{C}^{3 \times 3}$ introduced in Section 4 to its subspace $\mathcal{M}$ of symmetric complex $3 \times 3$ matrices; we keep the same notation $d_i$, $c_j$ for the resulting $G$-equivariant polynomial maps on $\mathcal{M}$. An essential difference compared to the case of $\mathbb{C}^{3 \times 3}$ is that now $c_4 = c_3$. Moreover, $S^3(\mathbb{C}^3)^*$ is not an irreducible $\mathcal{M}_1$-module. The maximal torus $T$ in $\mathcal{M}_1$ has rank 1, i.e. $T \cong \mathbb{C}^\times$, so $\text{Char}(T) = \mathbb{Z}$, where the character $T \to \mathbb{C}^\times$, $t \mapsto t^n$ is identified with $n \in \mathbb{Z}$. The possible highest weights are the non-negative integers, we shall denote by $V^{(n)}$ the irreducible $G$-module with highest weight $n$; it has dimension $2n + 1$, and for $t \in T = \mathbb{C}^\times$ we have $\text{tr}(t|_{V^{(n)}}) = t^n + t^{n-1} + \cdots + t^{-n}$. With this notation we have

$$S^3(\mathbb{C}^3)^* = V^{(3)} + V^{(1)}$$

where $V^{(3)}$ is the kernel of the Laplace operator $\Delta := \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ restricted to $S^3(\mathbb{C}^3)^*$.

Proposition 6.1. The covariant $c_3$ is non-zero and maps $\mathcal{M}$ into $\ker(\Delta|_{S^3(\mathbb{C}^3)})$.

Proof. Since $\Delta$ is a $G$-equivariant operator, and by Proposition 2.2 there is a Zariski dense subset in $\mathcal{M}_1$ consisting of $G$-orbits of diagonal matrices, it suffices to show that $\Delta(c_3(A)) = 0$ for any diagonal $A \in \mathcal{M}$. Now we have $c_3(\text{diag}(a_1, a_2, a_3)) = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)x_1x_2x_3$. This shows that $c_3$ is non-zero, and since $\Delta(x_1x_2x_3) = 0$, the second claim also follows.

From now on we shall view $c_3$ as an irreducible covariant $c_3 : \mathcal{M} \to V^{(3)}$. Moreover, $c_1, c_2 : \mathcal{M} \to \mathcal{N} \cong V^{(2)}$ are independent irreducible covariants. The covariant $c_1$ induces an embedding of $\mathbb{C}[\mathcal{N}]$ as a subalgebra of $\mathbb{C}[\mathcal{M}]$.

Denote by $U$ a maximal unipotent subgroup of $\mathcal{M}_1$ normalized by $T$. The algebra of covariants on $\mathcal{M}$ is known classically from the theory of covariants of binary forms. The result in our notation can be stated as follows:

Proposition 6.2. (i) The algebra $\mathbb{C}[\mathcal{M}]^U$ is a polynomial ring generated by $d_1$ over $\mathbb{C}[\mathcal{N}]^U$.

(ii) The algebra $\mathbb{C}[\mathcal{N}]^U$ is generated by $d_2, d_3, c_1, c_2, c_3$.

Proof. Recall the well-known isomorphism $SO(3, \mathbb{C}) \cong SL(2, \mathbb{C})/\{\pm I_2\}$, so $G$-modules can be thought of as representations of the special linear group $SL(2, \mathbb{C})$ with $-I_2$ in the kernel. This way the conjugation action of $G$ on $\mathcal{N}$ can be identified with the natural $SL(2, \mathbb{C})$-representation on the space of binary quartic forms. Generators (and relations) for the algebra of covariants of binary quartics were determined in nineteenth century invariant theory (see e.g. [11] or [21]). There are two algebraically independent invariants, one of degree 2 and 3. The covariant $c_2$ corresponds to the Hessian covariant $\text{Hess}$ mapping the binary quartic $Q = \sum_{i=0}^{4} a_i x^i y^{4-i}$ to the binary quartic $\text{Hess}(Q) := \text{det}(\frac{\partial^2 Q}{\partial x^i \partial x^j}, \frac{\partial^2 Q}{\partial x^i \partial y^j})$. The covariant $c_3$ corresponds to the map sending the binary quartic $Q$ to the Jacobian of $Q$ and its Hessian, which is the binary sextic $\text{Jac}(Q, \text{Hess}(Q)) := \text{det}(\frac{\partial^2 Q}{\partial y^i \partial y^j}, \frac{\partial^2 \text{Hess}(Q)}{\partial y^i \partial y^j})$.

Similarly to Section 4, write $\overline{d_i}, \overline{c_j}$ for the restriction to $\mathcal{M}_1$ of $d_i, c_j$. Since $\mathcal{M}, \mathcal{M}_1, \mathcal{N}, \mathcal{N}_1$ are all subsets of the set denoted by the same symbol in Section 4 and $\overline{d_i}$ and $\overline{c_j}$ are
restrictions of the corresponding functions from Section 1 as a corollary of Proposition 4.4 we obtain that exactly the same relations hold with the new scenario. Moreover, the statement of Theorem 4.5 remains valid, with verbatim the same proof.

**Corollary 6.3.** As an $SO(3, \mathbb{C})$-stable ideal, $\mathcal{I}(\mathcal{M}_1)$ is generated by $c_3^U$.

**Proof.** We apply Lemma 3.1 by Proposition 6.2 and the new versions of Proposition 4.4 and Theorem 4.5 discussed in the above paragraph we conclude that $\mathcal{I}(\mathcal{M}_1)$ is generated as a $G$-stable ideal by $c_3^U$, $d_3^U - d_3^V$, $d_2^U - d_2^V$, $(c_1^U)^2 - d_2(c_1^V)^2$, $d_3c_2^U - d_2c_1^U$. We know already from Corollary 4.6 that the elements of $\mathbb{C}[\mathbb{C}^{3\times 3}]$ denoted by the same symbols as the latter four elements are contained in the ideal generated by the coordinate functions of $c_3$ (defined on $\mathbb{C}^{3\times 3}$). Applying the natural surjection $\mathbb{C}[\mathbb{C}^{3\times 3}] \to \mathbb{C}[\mathcal{M}]$ given by restriction of functions to $\mathcal{M} \subset \mathbb{C}^{3\times 3}$ we conclude that the ideal generated by the coordinate functions of $c_3$ (interpreted as a covariant on $\mathcal{M}$) contain the latter four elements of $\mathbb{C}[\mathcal{M}]$. So our statement follows, since the coordinate functions of $c_3$ span the $SO(3, \mathbb{C})$-module generated by $c_3^U$. □

**Corollary 6.4.** (i) The ideal $\mathcal{I}(\mathcal{M}_1)$ is generated by its degree 3 component $\mathcal{I}(\mathcal{M}_1)_3$.

(ii) The 7 coordinate functions of $c_3 : \mathcal{M} \to V^{(3)}$ constitute a $\mathbb{C}$-basis in $\mathcal{I}(\mathcal{M}_1)_3$.

(iii) As an $SO(3, \mathbb{C})$-module, $\mathcal{I}(\mathcal{M}_1)_3 \cong V^{(3)}$, the space of 3-variable spherical harmonics of degree 3.

(iv) The coordinate functions of $\mathcal{P}_1 : \mathcal{M} \to \bigwedge^3 \mathcal{M}$ span $\mathcal{I}(\mathcal{M}_1)_3$.

**Proof.** Since $c_3^U$ is homogeneous of degree 3, it follows trivially from Corollary 6.3 that $\mathcal{I}(\mathcal{M}_1)$ is generated by its degree three homogeneous component, so (i) follows. The covariant $c_3$ is non-zero and irreducible, hence its coordinate functions are linearly independent, so both (ii) and (iii) hold by Corollary 6.3 and by construction of $c_3$. Finally, (iv) follows from (ii) and Proposition 4.3. □

The multiplicity series of $\mathbb{C}[\mathcal{M}_1]$ is

$$M(\mathbb{C}[\mathcal{M}_1]; q, t) = \sum_{d=0}^{\infty} \sum_{n=0}^{\infty} m(d, n) q^n t^d$$

where $m(d, n)$ denotes the multiplicity of the irreducible $SO(3, \mathbb{C})$-module $V^{(n)}$ as a summand in $\mathbb{C}[\mathcal{M}_1]$. The present variant of Theorem 4.5 yields

$$M(\mathbb{C}[\mathcal{M}_1]; q, t) = \frac{1}{1 - t} \left( \frac{1}{(1 - t)(1 - q^2t)} - t \right)$$

(11)

The trace of $q \in T \in \mathbb{C}^*$ acting on $V^{(n)}$ is $q^{1/2} q^{n/2} - q^{-1/2} q^{-n}$, hence the Hilbert series of $\mathbb{C}[\mathcal{M}_1]$ bigraded by $\mathbb{N}_0 \times \text{Char}(T)$ is

$$H(\mathbb{C}[\mathcal{M}_1]; q, t) = \frac{q^{1/2} M(\mathbb{C}[\mathcal{M}_1]; q, t) - q^{-1/2} M(\mathbb{C}[\mathcal{M}_1]; q^{-1}, t)}{q^{1/2} - q^{-1/2}}$$

**Corollary 6.5.** The Hilbert series $H(\mathbb{C}[\mathcal{M}_1]; t) := \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(\mathbb{C}[\mathcal{M}_1]_d) t^d$ equals

$$H(\mathbb{R}[\text{Sym}(3, \mathbb{R})]; t) = H(\mathbb{C}[\mathcal{M}_1]; t) = \frac{1 + 2t + 3t^2 - 3t^2 + t^4}{(1 - t)^4}$$

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Finally we point out a connection between Corollary 6.4 and coincident root loci. Denote by $\text{Pol}_d(\mathbb{C}^2)$ the $SL(2, \mathbb{C})$-module of binary forms of degree $d$. Up to non-zero scalar multiples there is a unique $SL(2, \mathbb{C})$-module isomorphism $\varphi : N \to \text{Pol}_d(\mathbb{C}^2)$ (where we view the $SO(3, \mathbb{C})$-module $N$ an $SL(2, \mathbb{C})$-module via the surjection $SL(2, \mathbb{C}) \to SO(3, \mathbb{C})$.

As we pointed out in the proof of Proposition 6.2, the covariant $c_2$ corresponds to the Hessian covariant $\text{Hess} : \text{Pol}_4(\mathbb{C}) \to \text{Pol}_4(\mathbb{C})$, and $c_3$ corresponds to the covariant $\text{Pol}_4(\mathbb{C}) \to \text{Pol}_6(\mathbb{C})$ $Q \mapsto \text{Jac}(Q, \text{Hess}(Q))$. It is well known that the zero locus of the coefficient space of the latter covariant is the subset of binary quartics that are the square of a binary quadric (see [3]), whence by Corollary 6.4 we conclude:

**Proposition 6.6.** The $SL(2, \mathbb{C})$-equivariant vector space isomorphism $\varphi : N \to \text{Pol}_4(\mathbb{C})$ maps the set $N_1$ of trace zero symmetric matrices with a minimal polynomial of degree at most 2 onto the set of binary quartics that are the square of a binary quadric.

In fact it is known that the coefficients of $\text{Jac}(Q, \text{Hess}(Q))$ generate the vanishing ideal of the set of binary quartics that are the square of a quadric, see [3], where this is stated (without the concrete computational details), after an explanation of a general method for the study of ideals of coincident root loci in the space of binary forms of degree $d$. So it would be possible to derive our Corollary 6.4 from this result with the aid of Proposition 4.1 in [3] and Proposition 4.3 of the present paper. For further results on coincident root loci see the papers [3], [22], [28] (and the references therein).

### 7 Real forms and sums of squares

Recall that the compact real form $SU(n)$ is Zariski dense in the complex affine algebraic group $SL(n, \mathbb{C})$, hence an irreducible $SL(n, \mathbb{C})$-module remains irreducible over $SU(n)$. For a compact real Lie group $G$ and a finite dimensional complex $G$-module $V$ denote $V_\mathbb{R}$ the realification of $V$, and for a finite dimensional real $G$-module $W$, its complexification is $\mathbb{C} \otimes_\mathbb{R} W$. The realification $S^n(\mathbb{C}^n)_\mathbb{R}^*$ of the $n$th symmetric power of the dual of the natural $SU(n)$-module $\mathbb{C}^n$ is irreducible as a real representation of $SU(n)$, whereas its complexification splits as

$$\mathbb{C} \otimes_\mathbb{R} S^n(\mathbb{C}^n)_\mathbb{R}^* \cong S^n(\mathbb{C}^n)^* \oplus S^n(\mathbb{C}^n)$$

as a complex $SU(n)$-module. Set

$$c : \text{Her}(n) \to S^n(\mathbb{C}^n)^*_\mathbb{R}, \quad A \mapsto (\underline{x} \mapsto \det(\underline{x}|A\underline{x}) \ldots |A^{n-1}\underline{x}))$$  \hspace{1cm} (12)

where for $\underline{x} \in \mathbb{C}^n$ and $A \in \text{Her}(n)$ we write $(\underline{x}|A\underline{x}) \ldots |A^{n-1}\underline{x})$ for the $n \times n$ matrix whose columns are $\underline{x}$, $A\underline{x}$, $A^2\underline{x}$, $\ldots$, $A^{n-1}\underline{x}$, and $S^n(\mathbb{C}^n)^*$ is identified with the space of homogeneous forms of degree $n$ on $\mathbb{C}^n$. For a diagonal matrix $A = \text{diag}(a_1, \ldots, a_n)$ we have

$$c(A)(\underline{x}) = x_1 \ldots x_n \prod_{1 \leq i < j \leq n} (a_j - a_i)$$  \hspace{1cm} (13)

hence $c$ is non-zero. We obtain the following statement:

**Proposition 7.1.** The $2\binom{2n-1}{n-1}$ real coordinate functions of $c$ span an $SU(n)$-submodule in $\mathcal{I}(\text{Her}(n)_1)(\underline{x})$ isomorphic to $S^n(\mathbb{C}^n)^*_\mathbb{R}$.

The same proof as for Theorem 4.1 in [6] yields the following:
Proposition 7.2. Up to non-zero scalar multiples sDisc$_k$ $\in \mathbb{R}[\text{Her}(n)]$ is the only SU($n$)-invariant element in the degree $(n-k)(n-k-1)$ homogeneous component of $\mathcal{I}(\text{Her}(n), k+1)$, and there are no non-zero SU($n$)-invariants in $\mathcal{I}(\text{Her}(n), k+1)$ of degree less than $(n-k)(n-k-1)$.

By Lemma 2.1 in [5] this yields:

Corollary 7.3. The discriminant Disc $\in \mathbb{R}[\text{Her}(n)]$ can be written as the sum of $2^{(2n-1)}$ squares.

Remark 7.4. (i) The study of sum of squares representations of the discriminant of real symmetric matrices goes back to Kummer and Borchardt (see some references in [5], whose approach was inspired by [17]). A relation to the entropic discriminant is established in [26]. A sum of squares presentation of the discriminant of Hermitian matrices was shown by Newell [20], Ilyushechkin [15], Parlett [23]. Corollary 7.3 significantly reduces the number of summands in these presentations.

(ii) Sum of squares presentations of discriminants for the isotropy representation of Riemannian symmetric spaces were studied by Gorodski [10] (and also in [25]). In particular, it is proved in [10] that the discriminant associated to the symmetric space $Sp(n, \mathbb{R})/U(n)$ is the sum of $2^{(2n-1)}$ squares (the corresponding representation of $U(n)$ is the action $X \mapsto gXg^T$ on Sym$(n, \mathbb{C})$). This number coincides with the number appearing in Corollary 7.3 above, but the associated symmetric space (and representation) is different, it is $SL(n, \mathbb{C})/SU(n)$ in our case.

(iii) Similarly to [12] consider the $SO(n)$-equivariant polynomial map

$$c_R : \text{Sym}(n, \mathbb{R}) \rightarrow \mathbb{R}^n(\mathbb{R}^n)^*,$$ $A \mapsto (\varphi \mapsto \det(\varphi \cdot A\varphi | A\varphi | \ldots | A^{n-1}\varphi))$

Since the $SO(n)$-orbit of any $A \in \text{Sym}(n, \mathbb{R})$ contains a diagonal matrix and the Laplace operator $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is $SO(n)$-equivariant, formula [13] shows that the image of $c_R$ is contained the space $\mathcal{H}^n(\mathbb{R}^n) := \mathbb{R}^n(\mathbb{R}^n)^* \cap \ker(\Delta)$ of $n$-variable spherical harmonics of degree $n$. Note that $SO(n)$-modules are self-dual. This shows that the $\mathbb{R}$-subspace of the coordinate functions of $c_R$ span an $SO(n)$-submodule in $\mathcal{I}(\text{Sym}(n, \mathbb{R}))$ isomorphic to $\mathcal{H}^n(\mathbb{R}^n)$. Thus we obtained a more direct proof of the first statement of Theorem 6.2 from [5] than the proof given in loc. cit..

8 Subdiscriminants of Hermitian matrices

In this section we extend the results of [6] on real symmetric matrices to the case of Hermitian matrices. In particular, we generalize Proposition 7.1 and Corollary 7.3 of the present paper for the $k$-subdiscriminant of Hermitian matrices with arbitrary $k$. Let $U$ denote the subgroup of upper unitriangular matrices in $SL(n, \mathbb{C})$ acting by conjugation on $\mathbb{C}^{n \times n}$, and $T$ the subgroup of diagonal matrices in $SL(n, \mathbb{C})$. We identify $\mathbb{Z}^{n-1}$ with the group of rational characters of $T$: $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}$ corresponds to diag$(z_1, \ldots, z_{n-1}, (z_1 \ldots z_{n-1})^{-1}) \mapsto z_1^{\lambda_1} \ldots z_{n-1}^{\lambda_{n-1}}$. The irreducible $SL(n, \mathbb{C})$-modules are labeled by $\lambda \in \mathbb{Z}^{n-1}$ with $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0$. We denote by $V^\lambda$ the irreducible $SL(n, \mathbb{C})$-module with highest weight $\lambda$.

First we present a family of highest weight vectors in $\mathbb{C}[\mathbb{C}^{n \times n}]$ (introduced by Tange [27]) in the spirit of Section 7. For an $n \times n$ matrix $B$, $1 \leq i_1 < \cdots < i_s \leq n$, and $1 \leq j_1 < \cdots < j_t \leq n$ denote by $B_{i_1, \ldots, i_s}^{j_1, \ldots, j_t}$ the $s \times t$ submatrix of $B$ obtained by omitting the
rows of index other than \(i_1, \ldots, i_s\) and the columns of index other than \(j_1, \ldots, j_t\). Define \(f_k \in \mathbb{C}[\mathbb{C}^{n \times n}] (k = 1, \ldots, n - 1)\) by

\[
f_k(A) := \text{det}([Ae_1 | A^2 e_1 | \ldots | A^{n-k} e_1]_{k+1, k+2, \ldots, n})
\]

where \(e_1 := [1, 0, \ldots, 0]^T\), \(A \in \mathbb{C}^{n \times n}\). For \(g \in U\) we have \(g^{-1} e_1 = e_1\), so

\[
(g^{-1} \cdot f_k)(A) = f_k(gAg^{-1}) = \text{det}([ge_1 | gA^2 e_1 | \ldots | gA^{n-k} e_1]_{k+1, \ldots, n})
\]

\[
= \text{det}(g_{k+1, \ldots, n}[Ae_1 | A^2 e_1 | \ldots | A^{n-k} e_1]_{k+1, \ldots, n}) = f_k(A)
\]

by multiplicativity of the determinant and since \(g_{k+1, \ldots, n}\) is upper unitriangular, hence has determinant 1. Thus \(f_k\) is \(U\)-invariant. Moreover, we have

\[
\text{diag}(z_1, \ldots, z_{n-1}, (z_1 \ldots z_{n-1})^{-1}) \cdot f_k = z_1^{n-k+1} z_2 z_3 \ldots z_k f_k
\]

So \(f_k\) is a highest weight vector in \(\mathbb{C}[\mathbb{C}^{n \times n}]\) of weight \((n - k + 1, 1^{k-1})\) (we write \(1^r\) for the sequence \(1, \ldots, 1\) with \(r\) terms), therefore it generates an irreducible \(SL(n, \mathbb{C})\)-module isomorphic to \(V^{(n-k+1, 1^{k-1})}\). For an irreducible complex \(SL(n, \mathbb{C})\)-module \(V^\lambda\) write \(V^\lambda_{\mathbb{R}}\) for \(V^\lambda\) viewed as a real representation of \(SU(n)\).

We obtain the following extension of Proposition 7.1 and Corollary 7.3 which correspond to the special case \(k = 0\):

**Theorem 8.1.** Let \(n \geq 3\) and \(0 \leq k \leq n - 3\) be integers.

(i) For \(d < \binom{n-k}{2}\) the degree \(d\) homogeneous component of \(\mathcal{I}(\text{Her}(n)_{k+1}) \cap \mathbb{R}[\text{Her}(n)]\) is zero.

(ii) The degree \(\binom{n-k}{2}\) homogeneous component of \(\mathcal{I}(\text{Her}(n)_{k+1})\) contains an irreducible real \(SU(n)\)-submodule isomorphic to \(V^{(n-k, 1^k)}_{\mathbb{R}}\).

(iii) The \(k\)-subdiscriminant \(\text{sDisc}_k \in \mathbb{R}[\text{Her}(n)]\) can be written as the sum of \(2 \dim_{\mathbb{C}} V^{(n-k, 1^k)}\) squares.

**Proof.** (i) follows from Lemma 2.1 in [5] and Proposition 7.2 and the latter two statements together with (ii) imply (iii) as well.

In order to prove (ii) note that for \(A \in (\mathbb{C}^{n \times n})_{k+1}\) the matrices \(I_n, A, \ldots, A^{n-k-1}\) are linearly dependent, hence

\[
\text{det}([e_1 | Ae_1 | A^2 e_1 | \ldots | A^{n-k-1} e_1]_{1, k+2, \ldots, n}) = 0.
\]

By elementary properties of the determinant the left hand side coincides with \(f_{k+1}(A)\). This shows that the degree \(\binom{n-k}{2}\) highest weight vector \(f_{k+1}\) constructed above belongs to \(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{I}(\text{Her}(n)_{k+1}) = \mathcal{I}((\mathbb{C}^{n \times n})_{k+1})\). Thus the complexification of \(\mathcal{I}(\text{Her}(n)_{k+1})\) contains the irreducible complex \(SL(n, \mathbb{C})\)-module \(V^{(n-k, 1^k)}\). View \(V^{(n-k, 1^k)}\) as an irreducible complex \(SU(n)\)-module. It is not self-conjugate, hence its realification \(V^{(n-k, 1^k)}_{\mathbb{R}}\) is an irreducible real \(SU(n)\)-module. Consequently, \(\mathcal{I}(\text{Her}(n)_{k+1})\) contains an \(SU(n)\)-submodule \(V^{(n-k, 1^k)}_{\mathbb{R}}\) (it is spanned by the real and imaginary parts of a \(\mathbb{C}\)-basis of the \(SU(n)\)-module generated by \(f_k\) in \(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[\text{Her}(n)]\)).
The Weyl dimension formula provides an explicit expression for \( \dim_{\mathbb{C}}(V^{(n-k,1^k)}) \), see for example page 303 in [9].

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