SMOOTH EXTENDABILITY OF PROPER HOLOMORPHIC MAPPINGS

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In [9] Ch. Fefferman proved that any biholomorphic mapping \( f: \Omega_1 \to \Omega_2 \) between strictly pseudoconvex \( C^\infty \)-smooth domains in \( C^n \) extends smoothly to the boundary. Subsequently, the proof of this result has been simplified considerably by S. Webster [14, 15], E. Ligocka [12], St. Bell [1, 2]. And it was St. Bell who realized the importance of the following regularity condition of the Bergman projection for the proof of such extendability results:

**Definition.** A domain \( \Omega \subset \subset C^n \) is said to satisfy condition \( R \) for its Bergman projection operator \( P \) if for any positive integer \( s \) there is an integer \( N \) such that \( P \) is a bounded linear operator from \( W_0^{s+N}(\Omega) \) to \( H^s(\Omega) \).

(Here \( W_0^s(\Omega) \) denotes as usual the closure of \( C_0^\infty(\Omega) \) in the Sobolev \( s \)-Norm \( \| \cdot \|_s \) with respect to the volume Lebesgue-measure on \( \Omega \) and \( H^s(\Omega) \) is the space of holomorphic functions on \( \Omega \) with finite \( \| \cdot \|_s \)-norm.)

Since condition \( R \) is a consequence of subelliptic estimates for the \( \bar{\partial} \)-Neumann problem, it is known to be satisfied for instance in the following cases:

1. \( \Omega \) strictly pseudoconvex, \( C^\infty \)-smooth (J. J. Kohn [10]);
2. \( \Omega \) pseudoconvex, \( C^{\omega} \)-smooth (J. J. Kohn [11], K. Diederich, J. E. Fornaess [6]).

The new methods allowed to generalize Fefferman's result. It is now known that a biholomorphic mapping \( f: \Omega_1 \to \Omega_2 \) extends smoothly up to the boundary if \( \Omega_1 \) and \( \Omega_2 \) are \( C^\infty \)-smooth and, in addition, both satisfy condition \( R \) [1] or both are pseudoconvex and at least one satisfies condition \( R \) [2].

The result which we wish to announce deals with the case of proper holomorphic mappings and is contained in the

**Theorem.** Let \( \Omega_1, \Omega_2 \subset \subset C^n \) be \( C^\infty \)-smooth pseudoconvex domains and suppose that \( \Omega_1 \) satisfies condition \( R \). Then any proper holomorphic mapping \( f: \Omega_1 \to \Omega_2 \) extends smoothly up to the boundary.

For unbranched mappings \( f \) this result is contained in K. Diederich and J. E. Fornaess [7]. This also includes the case of \( \Omega_1, \Omega_2 \) being strictly pseudoconvex and \( f \) proper holomorphic since any such \( f \) is necessarily unbranched, S. Pincuk [13]. Under different, more restrictive assumptions on \( \Omega_1 \) and \( \Omega_2 \) the result

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was obtained by S. Bell in [4 and 5]. More detailed information about the
history will be given in the paper containing also all the proofs.

Our proof is based on the following transformation formula for the Bergman
projection proved by S. Bell [3]:

**Proposition 1.** Let \( f: \Omega_1 \to \Omega_2 \) be a proper holomorphic mapping
between bounded domains in \( \mathbb{C}^n \) and let \( P_t \) denote the Bergman projection on
\( \Omega_1 \). Then one has

\[
P_t(u \cdot (h \circ f)) = u \cdot (P_t(h \circ f))
\]

for all \( h \in L^2(\Omega_2) \). Here \( u = \det f' \).

As a consequence of this, S. Bell [3] obtains

**Proposition 2.** Under the assumptions of the theorem one has \( u \cdot h \circ f \in A^\infty(\Omega_1) \) for all \( h \in A^\infty(\Omega_2) \), in particular \( u \in A^\infty(\Omega_1) \).

The difficulty in deriving the theorem from this statement is to show that
the functions \( u \cdot h \circ f \in A^\infty(\Omega_1) \) can be divided by \( u \) in \( A^\infty(\Omega_1) \) even close to
the cluster points of the branching locus \( X \) of \( f \) at \( b\Omega_1 \). This, certainly, would
be hopeless if the Jacobian determinant \( u \) could vanish to infinite order at such
points. But we can show

**Lemma 1.** In the situation of the theorem the Jacobian determinant \( u \) of
\( f \) does not vanish to infinite order at any point of \( \overline{\Omega_1} \).

For the proof we assume that \( u \) does vanish to infinite order at \( q \in b\Omega_1 \)
and take an arbitrary nontangential cone \( K \) in \( \Omega_1 \) with vertex at \( q \). The function
\[
u_1(z) := \prod_{\xi \in f^{-1}(q)} u(\xi) \quad \text{for} \quad \xi \in \Omega_1 \times
\]
extends to a bounded holomorphic function on \( \Omega_1 \) which vanishes in \( \Omega_1 \) exactly
on \( X_1 := f^{-1}(f(X)) \) and at \( q \) goes to zero with infinite order. There is a bounded
holomorphic function \( u_2 \) on \( \Omega_2 \) with \( u_1 = u_2 \circ f \) and \( \{u_2 = 0\} = f(X_1) = X_2 \).
Notice, that on \( \Omega_2 \setminus X_2 \) are locally defined inverse mappings \( F_1, \ldots, F_m \)
of \( f \), the Jacobian determinants of which are denoted by \( U_k \). One has

\[
u_2 \cdot \prod_{k=1}^{m} U_k \equiv 1.
\]

Furthermore, it is well known that there are an \( \eta > 0 \) and positive constants \( c, C \)
such that

\[
c \operatorname{dist}^{1/\eta}(z; b\Omega_1) \leq \operatorname{dist}(f(z), b\Omega_2) \leq C \operatorname{dist}^{\eta}(z, b\Omega_1)
\]
on \( \Omega_1 \). By considering for each fixed \( k = 1, \ldots, n \), the elementary symmetric
functions of the \( k \)th coordinate functions of \( F_j \), \( j = 1, \ldots, m \), which are globally
defined and bounded on $\Omega_2$, applying the Schwarz lemma to them and studying
the zeros of the polynomial in one variable with these coefficients we can show
using (2) that for large positive integers $N$ the preimage of a polynomial neigh­
borhood of $X_2$ of the form

$$V_M^N := \{w \in \Omega_2 : \text{dist}(w, X_2) < \text{dist}^N(w, b\Omega_2)\}$$

is contained in a polynomial neighborhood $V_M^1$ of $X_1$ in $\Omega_1$ of the same form.
Here $M < N$, but $M$ goes to infinity if $N$ does. Next by using the Blaschke con­
dition for the zero set $X_1$ of $u_1$ one proves that for large $M$

$$\lim_{r \to 0} \frac{A(V_M^1 \cap K \cap B_r)}{A(K \cap B_r)} = 0$$

(3)

where $A(\ )$ means euclidean volume and $B_r$ is the ball of radius $r$ around $q$.
Therefore, there is a sequence $(z^k) \subset K$, $z^k \to q$, such that $\Omega_2 \setminus X_2$ contains for
each $k$, a relatively “large” ball around $f(z^k)$. This allows us to show that all
$U_j(f(z^k))$ have to be small compared to a fixed negative power of

$$\text{dist}(f(z^k), b\Omega_2).$$

Because of (1) and (2) we obtain that $u_1(z^k)$ stays above a certain fixed power
of $\text{dist}(z^k, b\Omega_1)$. This contradicts our assumption. $\square$

The next step is the division by $u$. We show

**LEMMA 2.** Let $h$ be a bounded holomorphic function on $\Omega_1$ such that
$u \cdot h^N \in A^\infty(\Omega_1)$ for all positive integers $N$, then $h \in A^\infty(\Omega_1)$.

This is proved by a rather technical inductive procedure. We will, there­
fore indicate here only how the continuity of $h$ follows. We fix a point $q \in b\Omega_1$
and a generic transverse complex line $H$ through $q$ such that $u \mid H \cap \Omega_1$ vanishes
at $q$ only to finite order $k$. We can assume that $q = 0$ and

$$H = \{z_2 = \cdots = z_n = 0\}.$$

For $p \in b\Omega_1$ near 0 we denote by $H(p)$ the complex line through $p$ parallel
to $H$. It is easy to see that $h \mid \Omega_1 \cap H_p$ is $C^\infty$ up to $b\Omega_1 \cap H_p$ near 0. This
gives us an extension of $h$ to $b\Omega_1$ near 0. We may assume that $h(0) = 0$. We
denote

$$g(s) := \partial s g / \partial z_1^s$$

and use the following observation.

**LEMMA 3.** For each $k$ there exists a sequence of polynomials $P_j, j = 1, 2,$
$3, \ldots,$ in $u$ and $h$ and their derivatives, such that for all $N > k$

$$(uh^N)(k) = u^{(k)}h^N + \left(\sum_{j=1}^k N!P_j\right)h^{N-k}.$$ (4)

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The proof goes by induction over \( k \) and is straightforward. Let us now assume that there is a sequence \( S = (p_s) \subset \Omega_1, p_s \to 0, \) such that \( h(p_s) \to \gamma \neq 0. \) For \( N > k \) we have

\[
(uh^N)^{(k)} \to 0 \quad \text{on } S,
\]

since this function is in \( A^\infty \) and its restriction to \( H(0) \cap \Omega_1 \) goes to zero at 0. Therefore, we get from (4)

\[
u^{(k)}h^k + \left( \sum_{j=1}^{k} N^j \lambda_j \right) \to 0 \quad \text{on } S.
\]

Next we prove inductively for \( 0 \leq \kappa \leq k: \)

\((A_{\kappa})\) There are coefficients \( a^{(k)}_{j,s}, j = 1, \ldots, k, s = 1, 2, 3, \ldots, \) such that for all \( N > k \)

\[
u^{(k)}h^k(p_s) + \sum_{j=1}^{k} N^j a^{(k)}_{j,s} \to 0 \quad \text{as } s \to \infty.
\]

For \( \kappa = k \) the choice \( a^{(k)}_{j,s} := \lambda_j(p_s) \) satisfies \((A_k)\) because of (6). If \((A_{\kappa})\) for some \( 1 \leq \kappa \leq k \) has been reached, one may put

\[
a^{(k-1)}_{j,s} := a^{(k)}_{j,s} \left( 1 - 2^{-k} (1 - 2^{-\kappa})^{-1} \right).
\]

That this satisfies \((A_{\kappa-1})\) follows from the fact that

\[\left( 1 - 2^{-\kappa} \right)^{-1} \left[ \nu^{(k)}h^k(p_s) + \sum_{j=1}^{k} N^j a^{(k)}_{j,s} - 2^{-\kappa} \left( \nu^{(k)}h^k(p_s) + \sum_{j=1}^{k} (2N)^j a^{(k)}_{j,s} \right) \right] \to 0 \]

as \( s \to \infty. \) The statement \((A_0)\) now gives \( \nu^{(k)}h^k \to 0 \) on \( S, \) a contradiction to \( h \to \gamma \neq 0 \) on \( S. \) \( \square \)

An easy consequence of our theorem is the following statement. An independent and simpler proof of it has been given in [8].

**Corollary.** Let \( f: \Omega_1 \to \Omega_2 \) be proper holomorphic, \( \Omega_1, \Omega_2 \) \( C^\infty \)-smooth and pseudoconvex. Suppose that \( \Omega_1 \) satisfies condition \( R \) (e.g. \( \Omega_1 \) strictly pseudoconvex). Then the branching locus of \( f \) does not cluster at any strictly pseudoconvex boundary point. In particular, \( f \) is unbranched if the \((2n - 3)\)-dimensional Hausdorff measure of the set of weakly pseudoconvex boundary points of \( \Omega_1 \) is zero.

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