Gravity on a parallelizable manifold
Exact solutions

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August 5, 2018

Abstract

The wave type field equation $\Box^a = \lambda \partial^a$, where $\partial^a$ is a coframe field on a space-time, was recently proposed to describe the gravity field. This equation has a unique static, spherical-symmetric, asymptotically-flat solution, which leads to the viable Yilmaz-Rosen metric. We show that the wave type field equation is satisfied by the pseudo-conformal frame if the conformal factor is determined by a scalar 3D-harmonic function. This function can be related to the Newtonian potential of classical gravity. So we obtain a direct relation between the non-relativistic gravity and the relativistic model: every classical exact solution leads to a solution of the field equation. With this result we obtain a wide class of exact, static metrics. We show that the theory of Yilmaz relates to the pseudo-conformal sector of our construction. We derive also a unique cosmological (time dependent) solution of the described type.

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The study of possible geometrical models of physical reality began soon after Einstein had proposed his theory of gravity field - general relativity (GR). All these classical field-theoretical generalizations of Einstein’s theory include some alternation of the primordial geometrical structure - the pseudo-\(\text{Riemannian manifold}\) [1],[2].

In order to describe the spinorial properties of the matter on the curved space-time one needs the existence of an orthonormal frame in every point of the manifold [1]. This result can be formulated mathematically in the form of the Geroch theorem [3]:

\textbf{Theorem 1.1:} \textit{A necessary and sufficient condition for a space-time} \(M\) \textit{(4D differential manifold with Lorentzian signature) to admit a spinor structure is that the orthonormal frame bundle} \(FM\) \textit{has a global cross-section.}

So one needs to consider two different geometrical structures on the differential manifold:

- \textit{frame structure} (a global cross-section of the frame bundle)
- \textit{metric structure} (a global cross-section of the second rank tensorial bundle).

These two structures are not complete independent - from one side one needs some metric to define the orthonormality of the frame, on the other side the metric can be obtained by the coordinate components of the orthonormal frame.

A question in the order is:

\textit{Which of these two structures should be taken to play a role of a primary dynamical variable for the gravity field?}

In the classical Einstein’s theory of GR the metric structure completely describes the gravity field. In most of the classical alternative theories (from Einstein-Cartan to MAG) [1], [3] the frame structure is used to describe the spinorial properties of the matter, but the pure gravitational sector is described by the metric structure only. Therefore, in the modern approach to the gravity-material system one needs to use two different geometrical structures described above.

In [3] we make an attempt to eliminate the metric from the status of an independent dynamical variable and use the frame structure as the only gravity variable. So we hope to construct some “frame gravity” instead of the traditional “metric gravity”. This model can be considered as one of the
teleparallelism class gravitational gauge field theories. We begin with the differential manifold $M$ endowed with a fixed cross-section of the coframe bundle $F M$ - coframe field $\{\vartheta^a, a = 0, ..., 3\}$. The manifold $M$ is also endowed with the Lorentzian signature of type $(+, -, -, -)$. It means that the Lorentzian scalar product is defined on the (co)tangent vector space in every point $x \in M$. However, the metric structure is not defined yet since the scalar products in distinct points are not connected. In other words we deal with the whole class of the Lorentzian signature metrics. The next step is to call the coframe $\{\vartheta^a\}$ - “a pseudo orthonormal coframe”. With this “magic name” we are able now to fix the metric structure of the manifold $M$. In this way we also obtain the unique Hodge dual map on the algebra of differential forms $\Lambda = \sum_{p=0}^{n} \Lambda^p$. We begin with the notation of Hodge dual map on an arbitrary vector space.

Definition 1.2: Let $V^*$ will be a $(n + 1)$-dimensional vector space with a basis $\{\vartheta^0, \ldots, \vartheta^n\}$. Hodge dual map is an $\mathbb{R}$-linear map which acts on monomial expressions of $\vartheta^a_j$ in the following way

$$ *(\vartheta^{a_0} \wedge \ldots \wedge \vartheta^{a_p}) = (-1)^s \vartheta^{a_{p+1}} \wedge \ldots \wedge \vartheta^{a_n}, $$

where all indices are different and are taken in such an order, that the sequence $\{a_0, \ldots, a_p, a_{p+1}, \ldots, a_n\}$ is an even permutation of the sequence $\{0, \ldots, n\}$. The integer $s$ is:

$$ s = \begin{cases} 0, & \text{if } 0 \text{ is among } a_0, \ldots, a_p \\ 1, & \text{if } 0 \text{ is among } a_{p+1}, \ldots, a_n. \end{cases} \tag{1} $$

The integer $s$ in the definition above describes the signature of the vector space. Now by the linearity the definition can be extended to the algebra of the exterior forms on $V$. The vector space $V$ can be identified with the tangent space of the differential manifold $M$ and by the smooth structure on $M$ the Hodge dual map can be extended (as a smooth operation) to the whole algebraic bundle on $M$.

The unique defined metric tensor $g$, which makes the frame $\vartheta^a$ to be orthonormal, has the following components:

$$ g_{\mu\nu} = \eta^{ab} \vartheta^a_{\mu} \vartheta^b_{\nu}, \tag{2} $$

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1 We use the coframe bundle instead of the frame bundle in order to use the exterior algebra technique. It is obvious that all topological condition, such as the Geroch theorem hold true also in this situation.

2 We use the Latin indices to identify the different forms in the coframe and the Greek indices for coordinate components of a differential form. The ordinary summation convention is used.
where $\vartheta^a_\mu$ are the coordinate components of the differential form $\vartheta^a$ in a local coordinate system $\{x^\mu\}$: $\vartheta^a = \vartheta^a_\mu dx^\mu$.

In the framework of the metric structure the only natural covariant objects is the Riemann tensor and its traces - the Ricci tensor and the scalar curvature. However, for the frame structure we need some other natural covariant substances, which can be actually constructed using the natural operators on the algebra bundle on $M$. The most mathematical useful covariant second order differential operator is the Hodge - de Rham Laplacian

$$\Box \vartheta^a = (d \ast d \ast + \ast d \ast d) \vartheta^a$$

This operator commutes with the exterior derivative operator $d$ and with the Hodge dual map $\ast$. In the special case of a flat manifold it is the usual Laplace operator (for Euclidean signature) and the wave operator - d'Alembertian (for Lorentzian signature).

The field equation is declared $[5]$ in the following form

$$\Box \vartheta^a = \lambda \vartheta^a,$$

where $\lambda$ is some function of the frame $\vartheta^a$ and its exterior derivatives.

The equation $[5]$ represents a system of nonlinear PDE since the Laplacian operator $\Box$ itself depends on the particular choice of the coframe field $\{\vartheta^a\}$. Another useful forms of the field equation $[5]$ was proposed in $[3]$:

$$\left[ \Box + \frac{1}{4} \ast \left( \vartheta^\beta \wedge \Box \eta_\beta \right) \right] \eta_\alpha = 0,$$

or, alternatively:

$$\left[ - \frac{1}{4} \left( e_\beta \Box \vartheta^\beta \right) \right] \eta_\alpha = 0,$$

where $\eta_a = \eta_{ab} \vartheta^b$ is the down indexed basic 1-form and $e_a$ is the vector field dual to the frame field $\vartheta^a$.

In the special case of a spherical-symmetric static coframe the field equation $[5]$ has a unique asymptotic-flat solution. Namely, it is shown $[5]$ that the coframe:

$$\vartheta^0 = e^{-\frac{m}{r}} dt, \quad \vartheta^i = e^{\frac{m}{r}} dx^i \quad i = 1, 2, 3.$$  

provides a solution of the field equation $[5]$.

The correspondent metric is the Yilmaz-Rosen metric:

$$ds^2 = e^{-2m/r} dt^2 - e^{2m/r}(dx^2 + dy^2 + dz^2).$$

We use the notation of d’Alembertian $\Box$ in order to emphasize the Lorentzian signature of the manifold.
This metric is known to be in a good accordance with the observation data. The problem of derivation of the field equation (4) from some suitable variational principle is discussed in [5] and [6]. In the present work we generalize the coframe field (7) to the following form

\[ \vartheta^0 = e^{-f} dx^0, \quad \vartheta^i = e^f dx^i \quad i = 1, 2, 3, \] (9)

where \( f \) is an arbitrary function of coordinates. It is reasonable to call this coframe a pseudo-conformal coframe. The correspondent metric element will be

\[ ds^2 = e^{-2f} dt^2 - e^{2f} (dx^2 + dy^2 + dz^2). \] (10)

The metric of such a form is known in the classical theory of GR as the Majumdar-Papapetrou metrics. We are looking now for the conditions on the function \( f \), under which the coframe field (9) satisfies the field equation (4). It turns out that the function \( f \) must be spatial (elliptically) harmonic.

2 Pseudo conformal coframe

**Theorem 2.1:** The coframe

\[ \vartheta^0 = e^{-f} dx^0, \quad \vartheta^i = e^f dx^i \quad i = 1, 2, 3, \] (11)

where \( f = f(t, x, y, z) \) is an arbitrary scalar function on the manifold \( M \), provides the solution of the field equation

\[ \Box \vartheta^a = \lambda \vartheta^a, \] (12)

if and only if one of the following condition are satisfied:

- The function \( f = f(t, x, y, z) \) is static (is not depend on the time coordinate \( t \)) and spatial harmonic

\[ \triangle f = f_{xx} + f_{yy} + f_{zz} = 0. \] (13)

- The function \( f = f(t, x, y, z) \) is homogeneous (is not depend on the spatial coordinates \( x, y, z \)) and linear

\[ f = at. \] (14)
Proof:

The straightforward calculations of the Hodge-de Rham Laplacian for the coframe (11) give the following expressions (see Appendix 1):

\[ \vartheta^0 = e^{2f}(f_{xx} + f_{yy} + f_{zz} + f_x^2 + f_y^2 + f_z^2)\vartheta^0 + 3e^{2f}(f_{tt} + f_{tt})\vartheta^0 + 4(f_{xt} + f_x f_t)\vartheta^1 + 4(f_{yt} + f_y f_t)\vartheta^2 + 4(f_{zt} + f_z f_t)\vartheta^3. \]

\[ \vartheta^1 = \left[e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2)\right]\vartheta^1 + 4f_t f_x \vartheta^0. \]

\[ \vartheta^2 = \left[e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2)\right]\vartheta^2 + 4f_t f_y \vartheta^0. \]

\[ \vartheta^3 = \left[e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2)\right]\vartheta^3 + 4f_t f_z \vartheta^0. \]

In accordance with the field equation (4) the non-diagonal terms of the Laplacians have to vanish and we obtain two different possibilities

\[ f_t = 0 \]

or

\[ f_x = f_y = f_z = 0. \]

Let us first consider the static condition: \( f_t = 0. \)

The Hodge-de Rham Laplacians remain in the diagonal form

\[ \vartheta^0 = e^{2f}(f_{xx} + f_{yy} + f_{zz} + f_x^2 + f_y^2 + f_z^2)\vartheta^0, \quad (15) \]

\[ \vartheta^1 = e^{-2f}(-f_{xx} - f_{yy} - f_{zz} + f_x^2 + f_y^2 + f_z^2)\vartheta^1 \quad \text{etc.} \quad (16) \]

So the field equation (4) are satisfied if and only if

\[ \triangle f = f_{xx} + f_{yy} + f_{zz} = 0. \quad (17) \]

Let us consider now the homogeneous conditions \( f_x = f_y = f_z = 0. \)

The function \( f \) depends now only on the time coordinate \( t. \)

The Hodge-de Rham Laplacians are

\[ \vartheta^0 = 3e^{2f}(f_{tt} + f_t^2)\vartheta^0, \quad (18) \]

\[ \vartheta^1 = \left[e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2)\right]\vartheta^1 + 4f_t f_y \vartheta^0. \]

\[ \vartheta^2 = \left[e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2)\right]\vartheta^2 + 4f_t f_z \vartheta^0. \]

\[ \vartheta^3 = \left[e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2)\right]\vartheta^3 + 4f_t f_z \vartheta^0. \]
The field equation (4) is satisfied if and only if

\[ f_{tt} = 0 \]  

and the unique solution is \( f = at + b \).
The arbitrary constant \( b \) can be omitted by the suitable re-calibration of the time coordinate and we obtain the one-parametric solution \( f = at \).

The coframe (11) corresponds to the metric element

\[ ds^2 = e^{-2f} dt^2 - e^{2f} (dx^2 + dy^2 + dz^2) \]  

The curvature scalar for such a metric with a static harmonic function \( f \) is

\[ R = -8e^{-4f} (f_x^2 + f_y^2 + f_z^2) \]  

Note, that scalar curvature scalar is non positive for every particular choice of the harmonic function \( f \).
In the homogeneous case we obtain

\[ R = 16f_t^2 e^{8f} = 16a^2 e^{at} \]

and the scalar curvature is positive for all finite values of \( t \).

3 Static solutions

Static solutions of the pseudo-conformal type of the field equation (4) are determined by a particular choice of a harmonic scalar function \( f(x, y, z) \).

Note, that the scalar potential in the Newton theory of gravity (in vacuum) must be a harmonic function as well. So in framework of our model we obtain a direct relation between the classical (non-relativistic) gravity and the relativistic modification.

Every physical sensible classical solution of the Newton gravity has it’s respective counterpart as a solution of the relativistic field equation (4).

3.1 Spherical-symmetric solution

The Laplace equation (13) in the spherically-symmetry case has an unique asymptotically vanishing solution

\[ f = \frac{m}{r} \]  

\[ \Box \vartheta^1 = e^{2f} (f_{tt} + 3f_t^2) \vartheta^1 \]  

etc. (19)
where \( m \) is an arbitrary scalar constant. Consequently, we have the pseudo-conformal coframe (11) - a solution of the field equation (I):

\[
\vartheta^0 = e^{-\frac{m}{r}} dx^0 \quad \vartheta^i = e^{\frac{m}{r}} dx^i,
\]

which corresponds to the Yilmaz-Rosen metric:

\[
ds^2 = e^{-2\frac{m}{r}} dt^2 - e^{2\frac{m}{r}} (dx^2 + dy^2 + dz^2).
\]

This solution represents the gravity field of a single pointwise body with a mass \( m \).

The Tailor expansion of the line element (26) up to and including the order \( \frac{1}{r^2} \) takes the form

\[
ds^2 = \left(1 - \frac{2m}{r} + \frac{2m^2}{r^2} + \ldots\right) dt^2 - \left(1 + \frac{2m}{r} + \frac{2m^2}{r^2} + \ldots\right) (dx^2 + dy^2 + dz^2) . \tag{27}
\]

By comparison, the Schwarzchild line element, in the isotropic coordinates, is

\[
ds^2 = \left[\frac{1 - \frac{m}{r}}{1 + \frac{m}{2r}}\right]^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2) \tag{28}
\]

and it’s Tailor expansion up to the same order is

\[
ds^2 = \left(1 - \frac{2m}{r} + \frac{2m^2}{r^2} + \ldots\right) dt^2 - \left(1 + \frac{2m}{r} + \frac{3m^2}{2r^2} + \ldots\right) (dx^2 + dy^2 + dz^2). \tag{29}
\]

The difference between these two line elements is only in the second order term of the spatial part and in the third order term of the temporal part of the metric. It is impossible to distinguish, by the latest experiment techniques, between these two different line elements.

The curvature scalar for this metric is

\[
R = -2 \frac{m^2}{r^4} e^{-2\frac{m}{r}} . \tag{30}
\]

Note, that the curvature scalar is nonsingular for all permissible values of the radius \( r \) and vanishes only at the origin.

### 3.2 Solution with \( n \)-singular points

The Laplace equation (13) is linear so any linear combination of solutions provides a new solution. Therefore, we have a solution with the function \( f \) of the following form

\[
f = \sum_{i=0}^{n} \frac{m_i}{r_i} , \tag{31}
\]
where \( m_i \) are arbitrary scalar constants.
The coframe field (11) with such a choice of the harmonic function \( f \) corresponds to the following metric
\[
ds^2 = e^{-2 \sum_{i=0}^{n} \frac{m_i}{r_i}} dt^2 - e^{2 \sum_{i=0}^{n} \frac{m_i}{r_i}} (dx^2 + dy^2 + dz^2).
\] (32)

The solution can be interpreted as a metric of a static system of \( n \) pointwise bodies.
Thus, the field equation (4) has a solution with a static configuration of masses. Note, that the same type of solutions appear in classical field theories and in the Einstein gravity (Weyl solution).

### 3.3 Solid body solution

The Newton potential for a classical gravity field produced by a solid body (for example - massive ball of non-vanishing radius) can be described by an integral on the compact 3-dimensional domain:
\[
f(x, y, z) = \int_{V} \frac{\rho}{|r - r'|} dV',
\] (33)

where \( \rho = \rho(x, y, z) \) is a local mass density.
The function (33) is a scalar harmonic function. So we obtain a solution of the field equation (4) and consequently a metric element of the prescribed type (10). For a ball of radius \( R \) with a homogeneous distribution of mass we obtain
\[
f = \frac{M}{r},
\] (34)

where \( M \) is the total mass of the ball
\[
M = \int_{V'} \rho dV'
\]

Note two remarkable classical results that remain true also in our scheme:

- The external gravity field of a massive spherical body is equal to the field of the point with the mass equal to the mass of the ball and located in its center.
- The gravity field within a spherical cavity is zero.
3.4 Axial symmetric solution

The axial symmetric static solution of the Einstein equation in vacuum is given by the Weyl metric \[11\]. The metric element can be written as follows:

\[
ds^2 = e^{\sigma} dt^2 - e^{-\sigma} \left[ e^{\chi} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right],
\]  
(35)

where \(\sigma = \sigma(\rho, z)\) is a harmonic function, that is, satisfies the 2D-Laplace equation

\[
\Delta \sigma = \sigma_{\rho\rho} + \sigma_{zz} + \rho^{-1} \sigma_{\rho} = 0
\]  
(36)

and the function \(\chi = \chi(\rho, z)\) is given by the following two equations

\[
\chi_z = \rho \sigma_{\rho\rho} \sigma_z,
\]  
(37)

\[
\chi_{\rho} = \frac{1}{2} \rho (\sigma_{\rho}^2 - \sigma_z^2).
\]  
(38)

The consistency of the last two equation is guaranteed by (36).

Note, that the equations (37-38) can be solved for an every particular choice of a harmonic function \(\sigma\).

Let us return to the field equation (4). The axial symmetric static solution can be given by the pseudo conformal frame:

\[
\vartheta^0 = e^{-\frac{\sigma}{2}} dt \quad \vartheta^i = e^{\frac{\sigma}{2}} dx^i,
\]  
(39)

where \(\sigma\) is a harmonic function and in two dimensional case satisfies the equation (36).

The resulting metric is

\[
ds^2 = e^{-\sigma} dt^2 - e^{\sigma} \left( d\rho^2 + dz^2 + \rho^2 d\phi^2 \right).
\]  
(40)

Note that the metric (40) has the same form as the metric of Weyl with a vanishing function \(\chi\). So instead of the system (36,37,38) we have only one equation (36). Let us consider two particular solutions of the equation (36). It is easy to see that the function

\[
\sigma = \frac{m_1}{\sqrt{\rho^2 + (z-a)^2}} + \frac{m_2}{\sqrt{\rho^2 + (z+a)^2}}
\]  
(41)

satisfies this equation.

This function has two singular points on z-axis at \(z = -a\) and \(z = a\), thus it represents the gravity field of two massive pointwise bodies located at these points.
Another particular class of solutions of the equation (36) can be given by the following harmonic function \( \sigma \)

\[
\sigma = \delta \ln \left( \frac{z - a + R^{(-)}}{z + a + R^{(+)}} \right) = \delta \ln \left( \frac{R^{(-)} + R^{(+)}}{R^{(-)} + R^{(+)} + 2a} \right),
\]

(42)

where \( \delta \) is a dimensionless constant and \( a \) is a constant with dimension of length. The functions \( R^{(\pm)} \) are

\[
R^{(\pm)} = \left[ \rho^2 + (z \pm a)^2 \right]^{\frac{1}{2}}.
\]

(43)

In the framework of the Newtonian theory the function \( \sigma \) of a type (42) represents the gravity potential of an infinitesimally thin uniform rod with a density proportional to \( \delta \) and with a length equal to \( 2a \). The center of the rod is in the origin and its ends lying on \( z \)-axis at \( z = -a \) and \( z = a \). For such a choice of the function \( \sigma \) the metric element is

\[
ds^2 = \left( \frac{R^{(-)} + R^{(+)} + 2a}{R^{(-)} + R^{(+) + 2a}} \right)^\delta dt^2 - \left( \frac{R^{(-)} + R^{(+) - 2a}}{R^{(-)} + R^{(+)} + 2a} \right)^\delta \left( d\rho^2 + dz^2 + \rho^2 d\phi^2 \right).
\]

(44)

### 4 Homogeneous solution

The second choice of the function \( f = at \) in the theorem gives a homogeneous solution depending on the time coordinate. The corresponding metric element is

\[
ds^2 = e^{-2at} dt^2 - e^{2at} (dx^2 + dy^2 + dz^2).
\]

(45)

The curvature scalar is (see Appendix)

\[
R = 18a^2 e^{2at}.
\]

(46)

Observe, that for the negative values of the arbitrary parameter \( a \) the curvature scalar is equal to \( 18a^2 \) for \( t = 0 \) and vanishes for \( t \to \infty \). So the solution describes a world that is expanding exponential from a finite radius of order \( \frac{1}{a} \) to infinity with the time coordinate \( t \).

For \( a > 0 \) we have an evolution of an inverse type and this solution can not be consistent with the observations.

Using a new time coordinate

\[
\tau = \pm \frac{1}{a} e^{-at}
\]

(47)

the metric element can be rewritten as

\[
ds^2 = d\tau^2 - \frac{1}{a^2 \tau^2} (dx^2 + dy^2 + dz^2).
\]

(48)

The time \( \tau \) is the proper time at each point in the space.
5  The theory of Yilmaz

Yilmaz [7], [8] was presented a theory of gravitation in which the basic dynamical variable is a scalar field $\phi$. The metric tensor $g_{\mu\nu}$ is not an independent dynamical variables, but a function of $\phi$. The field equations of the theory are the following ones:

the Einstein field equation

$$R_{\nu}^{\mu} - \frac{1}{2} \delta_{\nu}^{\mu} R = 8\pi T_{\nu}^{\mu}$$

(49)

and the Laplace equation for the scalar field

$$g^{\mu\nu} \phi_{,\mu\nu} = 0$$

(50)

The tensor $T_{\nu}^{\mu}$ is the usual energy-momentum tensor for the scalar field

$$T_{\nu}^{\mu} = \frac{1}{8\pi} (2g^{\mu\lambda} \phi_{,\mu\lambda} - \delta_{\nu}^{\mu} g_{\lambda\tau} \phi_{,\lambda\tau})$$

(51)

The main result of the Yilmaz approach is as follows:

The field equation (50) can be solved by the following special form of the metric tensor

$$g_{00} = e^{-2\phi}; \quad g_{ii} = -e^{2\phi}, \quad i = 1, 2, 3$$

(52)

where $\phi$ is scalar function of spatial coordinates $x, y, z$.

In this case the field equation (50) satisfies automatically and the equation (50) reduces to the Newtonian equation

$$\triangle \phi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0$$

(53)

The unique asymptotically vanishing harmonic function with one singular point produce the metric of Yilmaz, which is in a good accordance with the observations.

The analysis above shows that the magic result of Yilmaz reproduces in the pseudo-conformal sector of solutions of the field equation (3).

Acknowledgments. The author is very grateful to Professor F.W. Hehl for his excellent hospitality and stimulation discussions in the University of Cologne. I would like also to thank Professor S. Kaniel for constant support and many useful discussions.
A Calculations of Laplacians

Let us calculate the Hodge-de Rham Laplacian for the coframe (11).

The exterior derivative of the differential form $\vartheta^0$ is

$$d\vartheta^0 = e^{-f}dt \wedge (f_x dx + f_y dy + f_z dz)$$

$$= e^{-f}\vartheta^0 \wedge (f_x \vartheta^1 + f_y \vartheta^2 + f_z \vartheta^3).$$

The Hodge dual of this expression is

$$*d\vartheta^0 = e^{-f} (f_x \vartheta^2 \wedge \vartheta^3 - f_y \vartheta^1 \wedge \vartheta^3 + f_z \vartheta^1 \wedge \vartheta^2)$$

$$= e^f (f_x dy \wedge dz - f_y dx \wedge dz + f_z dx \wedge dy).$$

The exterior derivative of this expression is

$$d*d\vartheta^0 = e^{-2f} (f_{xx} + f_{yy} + f_{zz} + f_x^2 + f_y^2 + f_z^2)\vartheta^{123}$$

$$+ (f_{xt} + f_x f_t)dt \wedge dx \wedge dy - (f_{yt} + f_y f_t)dt \wedge dx \wedge dz$$

$$+ (f_{zt} + f_z f_t)dt \wedge dx \wedge dy$$

$$= e^{-2f} (f_{xx} + f_{yy} + f_{zz} + f_x^2 + f_y^2 + f_z^2)\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$$

$$+ (f_{xt} + f_x f_t)\vartheta^1 \wedge \vartheta^2 + (f_{yt} + f_y f_t)\vartheta^0 \wedge \vartheta^3 + (f_{zt} + f_z f_t)\vartheta^0 \wedge \vartheta^1.$$
The exterior differential is

\[ d \ast d \ast \vartheta^0 = 3e^f \left[(f_t^2 + f_{tt}) dt + (f_t f_x + f_{tx}) dx + (f_t f_y + f_{ty}) dy \right. \]

\[ + (f_t f_z + f_{tz}) dz \]

\[ = 3e^f (f_t^2 + f_{tt}) \vartheta^0 + 3(f_t f_x + f_{tx}) \vartheta^1 + 3(f_t f_y + f_{ty}) \vartheta^2 \]

\[ + 3(f_t f_z + f_{tz}) \vartheta^3. \]

The Hodge-de Rham Laplacian of the differential form \( \vartheta^0 \) is

\[ \square \vartheta^0 = e^{-2f}(f_{xx} + f_{yy} + f_{zz} + f_x^2 + f_y^2 + f_z^2) \vartheta^0 + 3e^f (f_t^2 + f_{tt}) \vartheta^0 \]

\[ + 4(f_{xt} + f_x f_t) \vartheta^1 + 4(f_{yt} + f_y f_t) \vartheta^2 + 4(f_{zt} + f_z f_t) \vartheta^3. \]

Let us calculate the Hodge-de Rham Laplacian of the differential form \( \vartheta^1 \).

\[ d \vartheta^1 = d(e^f dx) = e^f (f_t dt \wedge dx - f_y dx \wedge dy - f_z dx \wedge dz) \]

\[ = e^f f_t \vartheta^{01} - e^{-f} f_y \vartheta^{12} - e^{-f} f_z \vartheta^{13}. \]

The Hodge dual of this expression is

\[ \ast d \vartheta^1 = e^f f_t \vartheta^{23} + e^{-f} f_y \vartheta^{03} - e^{-f} f_z \vartheta^{02} \]

\[ = e^f f_t dy \wedge dz + e^{-f} f_y dt \wedge dz - e^{-f} f_z dt \wedge dy. \]

The exterior derivative of this expression is

\[ d \ast d \vartheta^1 = e^f (f_{tt} + 3f_t^2) dt \wedge dy \wedge dz + e^f (f_{tx} + 3f_t f_x) dx \wedge dy \wedge dz \]

\[ + e^{-f} (f_{xy} - f_x f_y) dx \wedge dt \wedge dz + e^{-f} (f_{yy} - f_y^2) dy \wedge dt \wedge dz \]

\[ - e^{-f} (f_{xz} - f_x f_z) dx \wedge dt \wedge dy - e^{-f} (f_{zz} - f_z^2) dz \wedge dt \wedge dy \]

\[ = e^f (f_{tt} + 3f_t^2) \vartheta^{023} + (f_{tx} + 3f_t f_x) \vartheta^{123} - e^{-f} (f_{xy} - f_x f_y) \vartheta^{013} \]

\[ - e^{-f} (f_{yz} - f_y f_z) \vartheta^{023} - e^{-f} (f_{xy} - f_x f_y) \vartheta^{012} - e^{-f} (f_{zz} - f_z^2) \vartheta^{023}. \]

Consequently

\[ \ast d \ast \vartheta^1 = \left[ e^f (f_{tt} + 3f_t^2) - e^{-f} (f_{yy} - f_y^2) - e^{-f} (f_{zz} - f_z^2) \right] \vartheta^1 \]

\[ + (f_{tx} + 3f_t f_x) \vartheta^0 + e^{-f} (f_{xy} - f_x f_y) \vartheta^2 + e^{-f} (f_{xz} - f_x f_z) \vartheta^3. \]

For the dual form we have

\[ \ast \vartheta^1 = \vartheta^{023} = e^f dt \wedge dy \wedge dz. \]

The exterior derivative of this expression is

\[ d \vartheta^1 = -e^f f_x dt \wedge dx \wedge dy \wedge dz = -e^{-f} f_x \vartheta^{0123}. \]
The Hodge dual is

\[ *d \ast \vartheta^1 = -e^{-f}f_x. \]

The exterior derivative is

\[ d \ast d \ast \vartheta^1 = -e^{-f}(f_{xx} - f_x^2)dx - e^{-f}(f_{xy} - f_x f_y)dy - e^{-f}(f_{xyz} - f_x f_y f_z)dz \\
- e^{-f}(f_{xt} - f_x f_t)dt \\
= -(f_{xt} - f_x f_t)\vartheta^0 - e^{-2f}(f_{xx} - f_x^2)\vartheta^1 - e^{-2f}(f_{xy} - f_x f_y)\vartheta^2 \\
- e^{-2f}(f_{xz} - f_x f_z)\vartheta^3. \]

The Hodge-de Rham Laplacian of the differential form \( \vartheta^1 \) is

\[ \Box \vartheta^1 = \left[ e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2) \right] \vartheta^1 \\
+ 4 f_t f_x \vartheta^0. \]

For the Hodge-de Rham Laplacian of the differential forms \( \vartheta^2 \) and \( \vartheta^3 \) the expressions are similar:

\[ \Box \vartheta^2 = \left[ e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2) \right] \vartheta^2 \\
+ 4 f_t f_y \vartheta^0. \]

\[ \Box \vartheta^3 = \left[ e^{2f}(f_{tt} + 3f_t^2) - e^{-2f}(f_{xx} + f_{yy} + f_{zz} - f_x^2 - f_y^2 - f_z^2) \right] \vartheta^3 \\
+ 4 f_t f_z \vartheta^0. \]

## B Curvature

For calculation of the curvature tensor we use the formulas \[^4\] for non-vanishing components of Ricci tensor in the case of a “diagonal” metrics. Let the components of the metric tensor will be as following \[^5\]

\[ g_{ii} = e_i e^{2F_i}, \quad g_{ij} = 0 \quad \text{for} \quad i \neq j \quad (54) \]

where \( e_i = \pm 1 \).

The components of Ricci tensor are:

\[ R_{ik} = \sum_{l \neq i,k} \left( F_{i,k} F_{k,i} + F_{i,k} F_{l,i} - F_{l,j} F_{k,k} - F_{l,i,k} \right) \quad \text{for} \quad i \neq k \quad (55) \]

\[^5\]In all formulas above the summation over repeated indices is not used.
\[ R_{ii} = \sum_{l \neq i} \left[ F_{i,l} F_{i,i} - F_{l,i}^2 - F_{l,i,i} \right] + e_i e_l e^{2(F_i - F_l)} \left( F_{i,l} F_{i,l} - F_{i,l,l}^2 - F_{i,l} \sum_{m \neq i,l} F_{m,l} \right) \] (56)

Thus we have for the pseudo conformal metric (10)

\[ R_{00} = -6f_t^2 - 3f_{tt} - e^{-4f} \Delta f \] (57)
\[ R_{11} = e^{4f} (4f_t^2 - f_{tt}) - 2f_x^2 - \Delta f \] (58)
\[ R_{22} = e^{4f} (4f_t^2 - f_{tt}) - 2f_y^2 - \Delta f \] (59)
\[ R_{33} = e^{4f} (4f_t^2 - f_{tt}) - 2f_z^2 - \Delta f \] (60)

The curvature scalar is

\[ R = 18f_t^2 e^{2f} - 2e^{-2f}(\Delta f + f_x^2 + f_y^2 + f_z^2) \] (61)

Static solution satisfying the equation \( \Delta f = 0 \) gives

\[ R = -2e^{-2f}(f_x^2 + f_y^2 + f_z^2). \] (62)

Note, that the curvature scalar is non-positive.

For the homogeneous solution we have

\[ R = 18f_t^2 e^{2f} \] (63)

Note, that the curvature scalar is nonnegative.

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