ON THE TRIANGULATED CATEGORY OF FRAMED MOTIVES $\text{DFr}^{\text{eff}}(k)$

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ABSTRACT. The category of framed correspondences $\text{Fr}_r(k)$ was invented by Voevodsky [24] Section 2 in order to give another framework for $\text{SH}(k)$ more amenable to explicit calculations. Based on [24] and [7] Garkusha and the author introduced in [8] Section 2 a triangulated category of framed bispectra $\text{SH}^{\text{eff}}_\text{nis}(k)$. It is shown in [8] Section 2 that $\text{SH}^{\text{eff}}_\text{nis}(k)$ recover classical Morel–Voevodsky triangulated categories of bispectra $\text{SH}(k)$.

For any infinite perfect field $k$ a triangulated category of $\text{Fr}$-motives $\text{DFr}^{\text{eff}}(k)$ is constructed in the style of Voevodsky’s construction of the category $\text{DM}^{\text{eff}}(k)$. In our approach the Voevodsky category of Nisnevich sheaves with transfers is replaced with the category of $\text{Fr}$-modules. To each smooth $k$-variety $X$ the $\text{Fr}$-motive $M_{\text{Fr}}(X)$ is associated in the category $\text{DFr}^{\text{eff}}(k)$.

We identify the triangulated category $\text{DFr}^{\text{eff}}(k)$ with the full triangulated subcategory $\text{SH}^{\text{eff}}_\text{nis}(k)$ of the classical Morel–Voevodsky triangulated category $\text{SH}^{\text{eff}}_\text{fr}(k)$ of effective motivic bispectra [14]. Moreover, the triangulated category $\text{DFr}^{\text{eff}}(k)$ is naturally symmetric monoidal. Particularly, $M_{\text{Fr}}(X) \otimes_{\text{Fr}} M_{\text{Fr}}(Y) = M_{\text{Fr}}(X \times Y)$. The mentioned identification of the triangulated categories respects the symmetric monoidal structures on both sides.

We work with the derived category $\text{DFr}^{\text{eff}}(k)$ of bounded below $\text{Fr}$-modules rather than with the homotopy category $\text{SH}^{\text{eff}}_\text{nis}(k)$ of bispectra as in [8] Section 2.

1. INTRODUCTION

The Voevodsky triangulated category of motives $\text{DM}^{\text{eff}}(k)$ [23] provides a natural framework to study motivic cohomology. In this paper a new short approach to constructing the part $\text{SH}^{\text{eff}}_\text{fr}(k)$ of the classical triangulated category $\text{SH}(k)$ is presented providing the base field is infinite and perfect.

We work in the framework of strict V-spectral categories introduced in [4] Definition 2.4. The main new feature of our spectral category $\text{Fr}$ is that it is symmetric monoidal. It is also connective and Nisnevich excisive in the sense of [3]. Each $\pi_0(\text{Fr})$-presheaf $\mathcal{F}$ of Abelian groups is automatically a radditive framed presheaf of Abelian groups in the sense of [24]. By [6] Lemma 2.15 such an $\mathcal{F}$ is a $\mathbb{Z}F_\text{fr}(k)$-presheaf of Abelian groups in the sense of [6] 2.13. By [24] Lemma 4.5 and [6] Lemma 2.15 its associated Nisnevich sheaf $\mathcal{F}_\text{nis}$ is canonically a $\mathbb{Z}F_\text{fr}(k)$-presheaf of Abelian groups. If $\mathcal{F}$ is homotopy invariant and stable in the sense of [24] (see also [6] Def. 2.13, 2.14), then by [6] Thm. 1.1 the framed Nisnevich sheaf $\mathcal{F}_\text{nis}$ is strictly homotopy invariant and stable.

The main symmetric monoidal strict V-spectral category $\text{Fr}$ is constructed in Section 4. It is strict over infinite perfect fields. Denote by $\text{DFr}^{\text{eff}}(k)$ the full triangulated subcategory of $\text{SH}^{\text{fr}}_\text{nis}(\text{Fr})$ of bounded below $\text{Fr}$-modules. We also denote by $\text{DFr}^{\text{eff}}(k)$ the full triangulated subcategory of $\text{DFr}^{\text{eff}}(k)$ of those $\text{Fr}$-modules $M$ such that each $\mathbb{Z}F_\text{fr}(k)$-presheaf $\pi_i(M)|_{\mathbb{Z}F_\text{fr}(k)}$ is homotopy invariant and stable in the sense of [6] Def. 2.13, 2.14.

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We call $\mathbb{D} \mathbb{F}^{\text{eff}}_R(k)$ the triangulated category of $\mathbb{F}_R$-motives. The category $\mathbb{D} \mathbb{F}^{\text{eff}}_R(k)$ is naturally symmetric monoidal. For each $X \in \text{Sm}/k$ the $\mathbb{F}_R$-module

$$C_\ast(\mathbb{F}_R(X)) := |d \mapsto \text{Hom}(\Delta^d, \mathbb{F}_R(X))|$$

belongs to $\mathbb{D} \mathbb{F}^{\text{eff}}_R(k)$ and is called the $\mathbb{F}_R$-motive of $X$; $M_{\mathbb{F}_R}(X) \otimes_{\mathbb{F}_R} M_{\mathbb{F}_R}(Y) = M_{\mathbb{F}_R}(X \times Y)$.

The latter triangulated category is identified with the full triangulated subcategory $\text{SH}^{\text{eff}}_\ast(k)$ of the classical Morel–Voevodsky triangulated category $\text{SH}^{\text{eff}}_\ast(k)$ of effective motivic bispectra (this is the main result of the preprint). See Theorem 6.2.

The mentioned identification respects the symmetric monoidal structures on both sides.

It can be shown that the identification triangulated functor as in Theorem 6.2

$$M_{\text{SH}} : \mathbb{D} \mathbb{F}^{\text{eff}}_R(k) \to \text{SH}^{\text{eff}}_\ast(k)$$

takes the $\mathbb{F}_R$-motive $M_{\mathbb{F}_R}(X)$ of $X$ to the symmetric bispectrum $\Sigma G_m \Sigma S_1(X_i)$.

Sections 2 and 3 contain the materials of [4, Sections 2 and 3] adapted to the symmetric monoidal spectral category $\mathbb{F}_R$, which is defined in Section 4. In Section 4 the language of triangulated categories is used as opposed to the model categories language. This allows to state all constructions and results in a very explicit form. The main result here is Theorem 3.6. However it seems that this language does not allow to prove Theorem 6.2 (the main result of this preprint).

Also this language does not allow to state and prove the following true result: there is a triangulated equivalence of the triangulated categories

$$\text{SH}^{\text{mot}}(\mathbb{F}_R) \to \text{SH}^{\text{mot}}(k).$$

Triangulated subcategories $\text{SH}^{\text{mot}}(\mathbb{F}_R)$, $\mathbb{D} \mathbb{F}^{\ast}(k)$ and $\mathbb{D} \mathbb{F}^{\text{eff}}_R(k)$ are defined in Section 5. The main result of the preprint (Theorem 6.2) is stated in Section 6. Its proof is postponed to the next preprint.

Throughout the paper we denote by $\text{Sm}/k$ the category of smooth separated schemes of finite type over the base field $k$. The base field $k$ is supposed to be infinite and perfect. The paper [2] shows that there is no restriction on the characteristic of $k$.

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2. Preliminaries

We work in the framework of spectral categories and modules over them in the sense of Schwede–Shipley [19]. We start with preparations.

We follow [12, Definition 2.1.1, Remark 2.1.5]. A symmetric sequence of objects in a category $C$ is a functor $\Sigma \to C$, and the category of symmetric sequences of objects in $C$ is the functor category $C^{\Sigma}$. The category $\Sigma$ is a skeleton of the category of finite sets and isomorphisms. Hence every symmetric sequence has an extension, which is unique up to isomorphism, to a functor on the category of all finite sets and isomorphisms. We will use both viewpoints (often the second one).

Recall that symmetric spectra have two sorts of homotopy groups which we shall refer to as naive and true homotopy groups respectively following terminology of [18]. Precisely, the $k$th
naive homotopy group of a symmetric spectrum $X$ is defined as the colimit
\[ \hat{\pi}_k(X) = \operatorname{colim}_n \pi_{k+n}X_n. \]
Denote by $\gamma X$ a stably fibrant model of $X$ in $Sp^E$. The $k$-th true homotopy group of $X$ is given by
\[ \pi_kX = \hat{\pi}_k(\gamma X), \]
the naive homotopy groups of the symmetric spectrum $\gamma X$.

Naive and true homotopy groups of $X$ can considerably be different in general (see, e.g., [12, 13]). The true homotopy groups detect stable equivalences, and are thus more important than the naive homotopy groups. There is an important class of semistable symmetric spectra within which $\hat{\pi}_*$-isomorphisms coincide with $\pi_*$-isomorphisms. Recall that a symmetric spectrum is semistable if some (hence any) stably fibrant replacement is a $\pi_*$-isomorphism. Suspension spectra, Eilenberg–Mac Lane spectra, $\Omega$-spectra or $\Omega$-spectra from some point $X_n$ on are examples of semistable symmetric spectra (see [13]). Semistability is preserved under suspension, loop, wedges and shift.

A symmetric spectrum $X$ is $n$-connected if the true homotopy groups of $X$ are trivial for $k \geq n$. The spectrum $X$ is connective if it is $(-1)$-connected, i.e., its true homotopy groups vanish in negative dimensions. $X$ is bounded below if $\pi_iX = 0$ for $i < 0$.

**Definition 2.1.** (1) Following [19] a spectral category is a category $\mathcal{O}$ which is enriched over the category $Sp^E$ of symmetric spectra (with respect to smash product, i.e., the monoidal closed structure of [12 2.2.10]). In other words, for every pair of objects $o, o' \in \mathcal{O}$ there is a morphism symmetric spectrum $\mathcal{O}(o, o')$, for every object $o$ of $\mathcal{O}$ there is a map from the sphere spectrum $S$ to $\mathcal{O}(o, o)$ (the “identity element” of $o$), and for each triple of objects there is an associative and unital composition map of symmetric spectra $\mathcal{O}(o', o'') \wedge \mathcal{O}(o, o') \rightarrow \mathcal{O}(o, o'')$. An $\mathcal{O}$-module $M$ is a contravariant spectral functor to the category $Sp^E$ of symmetric spectra, i.e., a symmetric spectrum $M(o)$ for each object of $\mathcal{O}$ together with coherently associative and unital maps of symmetric spectra $M(o) \wedge \mathcal{O}(o', o) \rightarrow M(o')$ for pairs of objects $o, o' \in \mathcal{O}$. A morphism of $\mathcal{O}$-modules $M \rightarrow N$ consists of maps of symmetric spectra $M(o) \rightarrow N(o)$ strictly compatible with the action of $\mathcal{O}$. The category of $\mathcal{O}$-modules will be denoted by $\operatorname{Mod}\mathcal{O}$.

(2) A spectral functor or a spectral homomorphism $F$ from a spectral category $\mathcal{O}$ to a spectral category $\mathcal{O}'$ is an assignment from $\operatorname{Ob}\mathcal{O}$ to $\operatorname{Ob}\mathcal{O}'$ together with morphisms $\mathcal{O}'(a, b) \rightarrow \mathcal{O}'(F(a), F(b))$ in $Sp^E$ which preserve composition and identities.

(3) The monoidal product $\mathcal{O} \otimes \mathcal{O}'$ of two spectral categories $\mathcal{O}$ and $\mathcal{O}'$ is the spectral category where $\operatorname{Ob}(\mathcal{O} \otimes \mathcal{O}') := \operatorname{Ob}\mathcal{O} \times \operatorname{Ob}\mathcal{O}'$ and $\mathcal{O} \otimes \mathcal{O}'((a, x), (b, y)) := \mathcal{O}'(a, b) \otimes \mathcal{O}'(x, y)$.

(3') A monoidal spectral category consists of a spectral category $\mathcal{O}$ equipped with a spectral functor $o : \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}$, a unit $u \in \operatorname{Ob}\mathcal{O}$, a $Sp^E$-natural associativity isomorphism and two $Sp^E$-natural unit isomorphisms. Symmetric monoidal spectral categories are defined similarly.

(4) A spectral category $\mathcal{O}$ is said to be connective if for any objects $a, b$ of $\mathcal{O}$ the spectrum $\mathcal{O}'(a, b)$ is connective.

(5) By a ringoid over $Sm/k$ we mean a preadditive category $\mathcal{R}$ whose objects are those of $Sm/k$ together with a functor
\[ \rho : Sm/k \rightarrow \mathcal{R}, \]
which is identity on objects. Every such ringoid gives rise to a spectral category $\mathcal{O}_{\mathcal{R}}$ whose objects are those of $Sm/k$ and the morphisms spectrum $\mathcal{O}_{\mathcal{R}}(X, Y)$, $X, Y \in Sm/k$, is the Eilenberg–Mac Lane spectrum $H\mathcal{R}(X, Y)$ associated with the abelian group $\mathcal{R}(X, Y)$. Given a map of schemes $\alpha$, its image $\rho(\alpha)$ will also be denoted by $\alpha$, dropping $\rho$ from notation.
(6) By a spectral category over $Sm/k$ we mean a spectral category $\mathcal{O}$ whose objects are those of $Sm/k$ together with a spectral functor

$$\sigma : \mathcal{O}_{\text{naive}} \to \mathcal{O},$$

which is identity on objects. Here $\mathcal{O}_{\text{naive}}$ stands for the spectral category whose morphism spectra are defined as

$$\mathcal{O}_{\text{naive}}(X,Y)_p = \text{Hom}_{Sm/k}(X,Y)_+ \wedge S^p$$

for all $p \geq 0$ and $X, Y \in Sm/k$.

It is straightforward to verify that the category of $\mathcal{O}_{\text{naive}}$-modules can be regarded as the category of presheaves $\text{Pre}^\Sigma(\text{Sm}/k)$ of symmetric spectra on $Sm/k$. This is used in the sequel without further comment.

Let $\mathcal{O}$ be a spectral category and let $\text{Mod} \mathcal{O}$ be the category of $\mathcal{O}$-modules. Recall that the projective stable model structure on $\text{Mod} \mathcal{O}$ is defined as follows (see [19]). The weak equivalences are the objectwise stable weak equivalences and fibrations are the objectwise stable projective fibrations. The stable projective cofibrations are defined by the left lifting property with respect to all stable projective acyclic fibrations.

Recall that the Nisnevich topology is generated by elementary distinguished squares, i.e. pullback squares

$$
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
$$

where $\varphi$ is etale, $\psi$ is an open embedding and $\varphi^{-1}(X \setminus U) \to (X \setminus U)$ is an isomorphism of schemes (with the reduced structure). Let $\mathcal{D}$ denote the set of elementary distinguished squares in $Sm/k$ and let $\mathcal{O}$ be a spectral category over $Sm/k$. By $\mathcal{O}\mathcal{D}$ denote the set of squares

$$
\begin{array}{ccc}
\mathcal{O}(-,U') & \longrightarrow & \mathcal{O}(-,X') \\
\downarrow & & \downarrow \\
\mathcal{O}(-,U) & \longrightarrow & \mathcal{O}(-,X)
\end{array}
$$

which are obtained from the squares in $\mathcal{D}$ by taking $X \in Sm/k$ to $\mathcal{O}(-,X)$. The arrow $\mathcal{O}(-,U') \to \mathcal{O}(-,X')$ can be factored as a cofibration $\mathcal{O}(-,U') \to \text{Cyl}$ followed by a simplicial homotopy equivalence $\text{Cyl} \to \mathcal{O}(-,X')$. There is a canonical morphism $A_{\mathcal{O}\mathcal{D}} := \mathcal{O}(-,U) \coprod_{\mathcal{O}(-,U')} \text{Cyl} \to \mathcal{O}(-,X)$.

**Definition 2.2** (see [3]). I. The **Nisnevich local model structure** on $\text{Mod} \mathcal{O}$ is the Bousfield localization of the stable projective model structure with respect to the family of projective cofibrations

$$\mathcal{A}_\mathcal{O} = \{\text{cyl}(A_{\mathcal{O}\mathcal{D}} \to \mathcal{O}(-,X))\}_{\mathcal{O}\mathcal{D}}.$$  

The homotopy category for the Nisnevich local model structure will be denoted by $\text{SH}^\text{nis}_\mathcal{O}$. In particular, if $\mathcal{O} = \mathcal{O}_{\text{naive}}$ then we have the Nisnevich local model structure on $\text{Pre}^\Sigma(\text{Sm}/k) = \text{Mod} \mathcal{O}_{\text{naive}}$ and we shall write $\text{SH}^\text{nis}_\mathcal{O}_{\text{naive}}$ to denote $\text{SH}^\text{nis}_\mathcal{O}_{\text{naive}}$.

II. The **motivic model structure** on $\text{Mod} \mathcal{O}$ is the Bousfield localization of the Nisnevich local model structure with respect to the family of projective cofibrations

$$\mathcal{A}_\mathcal{O} = \{\text{cyl}(\mathcal{O}(-,X \times \mathbb{A}^1) \to \mathcal{O}(-,X))\}_{X \in Sm/k}.$$
The homotopy category for the motivic model structure will be denoted by $\text{SH}^{\text{mot}}_{S^1}$. In particular, if $\mathcal{O} = \mathcal{O}_{\text{naive}}$ then we have the motivic model structure on $\text{Pre}^\Sigma(Sm/k) = \text{Mod}_{\mathcal{O}_{\text{naive}}}^{\text{naive}}$ and we shall write $\text{SH}^{\text{mot}}_{S^1}(k)$ to denote $\text{SH}^{\text{mot}}_{S^1}(\mathcal{O}_{\text{naive}})$.

**Definition 2.3** (see [3]). I. We say that $\mathcal{O}$ is *Nisnevich excisive* if for every elementary distinguished square $Q$

\[
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow & & \downarrow \varphi \\
U & \longrightarrow & X
\end{array}
\]

the square $\mathcal{O}Q$ (2) is homotopy pushout in the Nisnevich local model structure on $\text{Pre}^\Sigma(Sm/k)$.

II. $\mathcal{O}$ is *motically excisive* if:

(A) for every elementary distinguished square $Q$ the square $\mathcal{O}Q$ (2) is homotopy pushout in the motivic model structure on $\text{Pre}^\Sigma(Sm/k)$ and

(B) for every $X \in Sm/k$ the natural map

$$\mathcal{O}(-,X \times \mathbb{A}^1) \to \mathcal{O}(-,X)$$

is a weak equivalence in the motivic model structure on $\text{Pre}^\Sigma(Sm/k)$.

Recall that a sheaf $\mathscr{F}$ of abelian groups in the Nisnevich topology on $Sm/k$ is *strictly $\mathbb{A}^1$-invariant* if for any $X \in Sm/k$, the canonical morphism

$$H^*_{\text{mot}}(X, \mathscr{F}) \to H^*_{\text{mot}}(X \times \mathbb{A}^1, \mathscr{F})$$

is an isomorphism.

**Definition 2.4.** Let $(\mathcal{O}, \circ, \text{pt})$ be a symmetric monoidal spectral category over $Sm/k$ together with the structure spectral functor $\sigma : \mathcal{O}_{\text{naive}} \to \mathcal{O}$ and an additive functor $\mathbb{Z}F_*(k) \xrightarrow{\epsilon} \pi_0 \mathcal{O}$. We say that $(\mathcal{O}, \circ, \text{pt}, \sigma, \epsilon)$ is a symmetric monoidal $V$-spectral category if

1. $\mathcal{O}$ is connective and Nisnevich excisive;
2. the structure map $\rho : Sm/k \to \pi_0 \mathcal{O}$ induced by $\sigma$ equals $\epsilon \circ \text{in}$, where $\text{in} : Sm/k \to \mathbb{Z}F_*(k)$ is the graphic functor.

**Remark 2.5.** Since $\mathcal{O}$ is connective and Nisnevich excisive, for each $\mathcal{O}$-module $M$ and each integer $i$ the presheaf $\pi_i(M)|_{Sm/k}$ is radditive (the restriction is taken via the $\rho$). That is $\pi_i(M)(\emptyset) = 0$ and $\pi_i(M)(X_1 \sqcup X_2) = \pi_i(M)(X_1) \times \pi_i(M)(X_2)$. Particularly, the functor $\pi_i(M)|_{\mathbb{Z}F_*(k)}$ is additive. So, $\pi_i(M)|_{\mathbb{Z}F_*(k)}$ is a presheaf of Abelian groups on $\mathbb{Z}F_*(k)$ in the sense of [6] Def. 2.13 (the restriction is taken via the $\epsilon$).

We note that if $(\mathcal{O}, \circ, \text{pt})$ is a symmetric monoidal spectral category over $Sm/k$, then for every $\mathcal{O}$-module $M$ and any smooth scheme $U$, the presheaf of symmetric spectra

$$\text{Hom}(U,M) := M(- \times U)$$

is an $\mathcal{O}$-module. Moreover, $M(- \times U)$ is functorial in $U$.

**Lemma 2.6.** Every symmetric monoidal $V$-spectral category $\mathcal{O}$ is motivically excisive.

**Proof.** Every symmetric monoidal $V$-spectral category is, by definition, Nisnevich excisive. Since there is an action of smooth schemes on $\mathcal{O}$, the fact that $\mathcal{O}$ is motivically excisive is proved similar to [3] 5.8. \qed
**Definition 2.7.** Let \( ((\mathcal{O}, \circ, pr), \sigma, \varepsilon) \) be a symmetric monoidal \( V \)-spectral category. Since it is both Nisnevich and motivically excisive, it follows from [4, 5.13] that the pair of natural adjoint functors

\[
\Psi_* : Pre^\Sigma(Sm/k) \rightleftarrows \Mod \mathcal{O} : \Psi^*
\]

induces a Quillen pair for the Nisnevich local projective (respectively motivic) model structures on \( Pre^\Sigma(Sm/k) \) and \( \Mod \mathcal{O} \). In particular, one has adjoint functors between triangulated categories

\[
(3) \quad \Psi_* : \SH^\nis(\mathcal{O}_{\text{naive}}) \rightleftarrows \SH^\nis(\mathcal{O}) : \Psi^* \quad \text{and} \quad \Psi_* : \SH^\mot(\mathcal{O}_{\text{naive}}) \rightleftarrows \SH^\mot(\mathcal{O}) : \Psi^*.
\]

3. **The triangulated category \( D\mathcal{O}^{eff}(k) \)**

In this section we work with a symmetric monoidal \( V \)-spectral category \( ((\mathcal{O}, \circ, pr), \sigma, \varepsilon) \) in the sense of Definition 2.4. We work in this section with the category \( \SH^\nis(\mathcal{O}) \) as in Definition 2.7.

Let \( M \) be an \( \mathcal{O} \)-module. By Remark 2.5 its \( \pi_0 \mathcal{O} \)-presheaves \( \pi_i(M) \) restricted via the \( \varepsilon \) to the additive category \( \mathbb{Z}F_\mathcal{O}(k) \) are \( \mathbb{Z}F_\mathcal{O}(k) \)-presheaves of Abelian groups in the sense of [6, Def. 2.13]. Thus, by [24, Lemma 4.6] and [6, Cor. 2.17] the associated Nisnevich sheaf \( \pi_i^\nis(M) \) is canonically a \( \mathbb{Z}F_\mathcal{O}(k) \)-presheaves of Abelian groups (possibly it is not a \( \pi_0 \mathcal{O} \)-presheaf).

We shall often work with simplicial \( \mathcal{O} \)-modules \( M[\bullet] \). The **realization** of \( M[\bullet] \) is the \( \mathcal{O} \)-module \( |M| \) defined as the coend

\[
|M| = \Delta[\bullet]_+ \wedge M[\bullet] : \Delta \times \Delta^{op} \to \Mod \mathcal{O}.
\]

Here \( \Delta[n] \) is the standard simplicial \( n \)-simplex.

Recall that the simplicial ring \( k[\Delta] \) is defined as

\[
k[\Delta] = k[x_0, \ldots, x_n]/(x_0 + \cdots + x_n - 1).
\]

By \( \Delta \) we denote the cosimplicial affine scheme \( \Spec(k[\Delta]) \). Given an \( \mathcal{O} \)-module \( M \), we set

\[
C_*(M) := [\Hom(\Delta, M)].
\]

Note that \( C_*(M) \) is an \( \mathcal{O} \)-module and is functorial in \( M \). **Our \( C_*(M) \) is different of \( C_*(M) \) used in [4] Sect. 3.**

**Definition 3.1.** (Definition 3.3 in [4]). The \( \mathcal{O} \)-motive \( M_\mathcal{O}(X) \) of a smooth algebraic variety \( X \in Sm/k \) is the \( \mathcal{O} \)-module \( C_*(\mathcal{O}(-, X)) \). We say that an \( \mathcal{O} \)-module \( M \) is bounded below if for \( i < 0 \) the Nisnevich sheaf \( \pi_i^\nis(M) \) is zero. \( M \) is \( n \)-connected if \( \pi_i^\nis(M) \) are trivial for \( i \leq n \). \( M \) is connective is it is \((−1)\)-connected, i.e., \( \pi_i^\nis(M) \) vanish in negative dimensions.

**Definition 3.2.** (4). Denote by \( \Mod_\mathcal{O}(\mathcal{O}) \) the full subcategory of of bounded below \( \mathcal{O} \)-modules. Denote by \( D\mathcal{O}_-(k) \) the full triangulated subcategory of \( \SH^\nis(\mathcal{O}) \) of bounded below \( \mathcal{O} \)-modules. We also denote by \( D\mathcal{O}^{eff}_-(k) \) the full triangulated subcategory of \( D\mathcal{O}_-(k) \) of those \( \mathcal{O} \)-modules \( M \) such that each \( \pi_0 \mathcal{O} \)-presheaf \( \pi_i(M) \) regarded via the functor \( \varepsilon \) as a \( \mathbb{Z}F_\mathcal{O}(k) \)-presheaf of Abelian groups is homotopy invariant and stable in the sense of [6, Def. 2.13, 2.14].

The category \( D\mathcal{O}^{eff}_-(k) \) is an analog of Voevodsky’s triangulated category \( DM^{eff}_-(k) \).

**Lemma 3.3.** (Corollary 3.4 in [4]). If an \( \mathcal{O} \)-module \( M \) is bounded below (respectively \( n \)-connected) then so is \( C_*(M) \). In particular, the \( \mathcal{O} \)-motive \( M_\mathcal{O}(X) \) of any smooth algebraic variety \( X \in Sm/k \) is connective.

**Remark 3.4.** By Lemma 5.3 the assignment \( M \mapsto C_*(M) \) is a functor \( C_* : \Mod_\mathcal{O}(\mathcal{O}) \to \Mod_\mathcal{O}(\mathcal{O}) \).
Lemma 3.5 (Compare with Lemma 3.5 in [4]). The functor $C_\circ : \text{Mod}_- \mathcal{O} \to \text{Mod}_- \mathcal{O}$ respects local equivalences and induces a triangulated endofunctor

$$C_\circ : D\mathcal{O}_-(k) \to D\mathcal{O}_-(k)$$

Theorem 3.6 (Compare with Theorem 3.5 in [4]). Let $(\mathcal{O}, \circ, pt)$ be a symmetric monoidal $V$-spectral category. Consider the full triangulated subcategory $\mathcal{T}$ of $SH^{mis}(\mathcal{O})$ generated by the compact objects cone $(\mathcal{O}(-, X \times A^1) \to \mathcal{O}(-, X))$, $X \in Sm/k$. Then the triangulated endofunctor

$$C_\circ : D\mathcal{O}_-(k) \to D\mathcal{O}_-(k)$$

as in Lemma 3.5 lands in $D\mathcal{O}^\text{eff}_-(k)$. The kernel of $C_\circ$ is $\mathcal{T}_- := \mathcal{T} \cap D\mathcal{O}_-(k)$. Moreover, $C_\circ$ is left adjoint to the inclusion functor

$$i : D\mathcal{O}^\text{eff}_-(k) \to D\mathcal{O}_-(k)$$

and $D\mathcal{O}^\text{eff}_-(k)$ is triangle equivalent to the quotient category $D\mathcal{O}_-(k)/\mathcal{T}_-$. 

4. THE MAIN SYMMETRIC MONOIDAL STRICT $V$-SPECULAR CATEGORY

We construct in this section our main symmetric monoidal strict $V$-spectral category $(\mathbb{F}r, \circ, pt)$. 

First construct a spectral category $\mathbb{F}r$. Its objects are those of $Sm/k$. To each pair $Y, X \in Sm/k$ we assign a symmetric spectrum $\mathbb{F}r(Y, X)$. The latter is described as follows. Its terms are the functors $A \mapsto \mathbb{F}r(Y, X)_A = Fr_A(Y, X \otimes S^A)$ (here $A$ runs over the category of finite sets and their isomorphisms). The structure maps are defined by the obvious compositions

$$\varepsilon_{A,B} : Fr_A(Y, X \otimes S^A) \otimes S^B \to Fr_A(Y, X \otimes S^A \otimes S^B) \to Fr_{A \otimes B}(Y, X \otimes S^A \otimes S^B).$$

For each triple $Z, Y, X \in Sm/k$ there is an obvious symmetric spectra morphism

$$\circ_{Z,Y,X} : \mathbb{F}r(Y, X) \otimes \mathbb{F}r(Z, Y) \to \mathbb{F}r(Z, X)$$

( the composition law).

It is uniquely determined by simplicial set morphisms $\mathbb{F}r(Y, X)_A \otimes \mathbb{F}r(Z, Y)_B \to \mathbb{F}r(Z, X)_{A \otimes B}$ which on $n$-simplices are given by the set maps

$$Fr_A(Y, X \otimes (S^A)_n) \otimes Fr_B(Z, Y \otimes (S^B)_n) \to Fr_{A \otimes B}(Z, X \otimes (S^A)_n \otimes (S^B)_n).$$

In details, the set map is given by

$$(\alpha, \beta) \mapsto (\alpha \otimes id_{(S^B)_n}, \beta) \mapsto (\alpha \otimes id_{(S^B)_n}) \circ \beta.$$ 

For each $X \in Sm/k$ the identity morphism $id_X$ gives rise to the symmetric spectra morphism $u_X : S \to \mathbb{F}r(X, X)$. We formed a spectral category $\mathbb{F}r$ and a spectral functor $\sigma : O_{\text{naive}} \to \mathbb{F}r$, which is identity on objects. The pair $(\mathbb{F}r, \sigma)$ is a spectral category over $Sm/k$ in the sense of Definition 2.1(6).

Equip now the spectral category $\mathbb{F}r$ with a spectral functor $\circ : \mathbb{F}r \otimes \mathbb{F}r \to \mathbb{F}r$ (taking $(X_1, X_2)$ to $X_1 \times X_2$), a unit $u \in \mathbb{F}r$, a $S^p$-natural associativity isomorphism $\alpha$ and two $Sp^2$-natural unit isomorphisms $u_l, u_r$ and a twist isomorphism $tw : \mathbb{F}r \otimes \mathbb{F}r \to \mathbb{F}r \otimes \mathbb{F}r$ and a spectral functor isomorphism $\Phi : \circ \to \circ \circ tw$ such that the data

$$(\mathbb{F}r, \circ, tw, \Phi, u, a, u_l, u_r)$$

form a symmetric monoidal spectral category.
First construct the spectral functor \( \diamond \). On objects it takes an object \((X_1, X_2) \in \text{Sm}/k \times \text{Sm}/k\) to \(X_1 \times X_2 \in \text{Sm}/k\). To construct \( \diamond \) on morphisms it suffices to construct certain symmetric spectra morphisms

\[
\diamond_{(V,Y), (U,X)} : \text{Fr}(V, U) \land \text{Fr}(Y, X) \to \text{Fr}(V \times Y, U \times X)
\]

and check that they satisfy the expected properties. To construct the morphism \( \diamond_{(V,Y), (U,X)} \) it is sufficient to construct simplicial set morphisms

\[
\otimes_{(V,Y), (U,X), A, B} : \text{Fr}(V, U)_A \land \text{Fr}(Y, X)_B \to \text{Fr}(V \times Y, U \times X)_{A \cup B}
\]

subjecting the known properties. The latter are given on \( n \)-simplices by the exterior product maps

\[
\otimes_{(V,Y), (U,X), A, B, n} : \text{Fr}_A(V, U \otimes (S^n)^A) \land \text{Fr}_B(Y, X \otimes (S^n)^B) \to \text{Fr}_{A \cup B}(V \times Y, (U \times X) \otimes (S^{n|A\cup B})^B).
\]

We constructed the spectral functor \( \diamond \).

Second we take the point \( pt := \text{Spec}(k) \) as the unit of the spectral category \( \text{Fr} \) and we skip constructions of desired \( a, u_i, u_r \) (they are obvious).

Third we construct the twist spectral categories isomorphism \( tw : \text{Fr} \land \text{Fr} \to \text{Fr} \land \text{Fr} \). On objects it takes \((X_1, X_2)\) to \((X_2, X_1)\). On morphisms it is determined by certain symmetric spectra isomorphisms

\[
tw_{(V,Y), (U,X)} : \text{Fr}(V, U) \land \text{Fr}(Y, X) \to \text{Fr}(Y, X) \land \text{Fr}(V, U).
\]

In turn the \( tw_{(V,Y), (U,X)} \) is determined by the family of simplicial set isomorphisms (switching factors)

\[
tw^C_{A, B} : \text{Fr}(V, U)_A \land \text{Fr}(Y, X)_B \to \text{Fr}(Y, X)_B \land \text{Fr}(V, U)_A.
\]

Here for each finite set \( C \) the ordered pairs \((A, B)\) run over all subsets \( A \subseteq C, B \subseteq C \) such that \( A \cup B = C \) and \( A \cap B = \emptyset \).

Finally we construct the desired spectral functor isomorphism \( \Phi : \diamond \to \diamond \circ tw \). It is the assignment \( (V, Y) \mapsto \Phi(V, Y) = [\tau_{V,Y} : V \times Y \to Y \times X] \). Here the switching factors isomorphism \( \tau_{V,Y} \) is regarded as a point in \( Fr_0(V \times Y, Y \times V) \). So, it is regarded as a symmetric spectra morphism \( \Phi(V, Y) \to \text{Fr}(V \times Y, Y \times V) \). It’s easy to check that \( \Phi \) is a spectral functor isomorphism indeed.

We left to the reader to check that the data \((\text{Fr}, \diamond, tw, \Phi, u, a, u_l, u_r)\) form a symmetric monoidal spectral category.

5. Properties of the main spectral category

Let \((\text{Fr}, \diamond, pt), (\sigma : \sigma_{\text{naif}} \to \text{Fr})\) be the symmetric monoidal spectral category over \( \text{Sm}/k \) as in Section 4.

**Lemma 5.1.** There is an additive functor \( \mathbb{Z}F_*(k) \to \pi_0(\text{Fr}) \) such that the data \( ((\sigma, \diamond, pt), (\sigma, \varepsilon)) \) is a symmetric monoidal \( V \)-spectral category in the sense of Definition 2.3.

Applying now Lemma 2.6 we get the following

**Corollary 5.2.** The symmetric monoidal spectral category \((\text{Fr}, \diamond, pt, tw, \Phi, u, a, u_l, u_r)\) as in Section 4 is Nisnevich and Motivically excisive in the sense of \( [3] \) (see Definition 2.3).

The following definition is just Definition 3.1 adapted to the category \( \text{Mod Fr} \).
**Definition 5.3.** The $\mathbb{F}_r$-motive $M_{\mathbb{F}_r}(X)$ of a smooth algebraic variety $X \in Sm/k$ is the $\mathbb{F}_r$-module $C_*(\mathbb{F}_r(-,X))$. We say that an $\mathbb{F}_r$-module $M$ is bounded below if for $i \leq 0$ the Nisnevich sheaf $\pi_i^{\text{eff}}(M)$ is zero. $M$ is $n$-connected if $\pi_i^{\text{eff}}(M)$ are trivial for $i \leq n$. $M$ is connective is it is $(-1)$-connected, i.e., $\pi_i^{\text{eff}}(M)$ vanish in negative dimensions.

**Definition 5.4.** Denote by $\text{Mod}_{\mathbb{F}_r}$ the full subcategory of of bounded below $\mathbb{F}_r$-modules. Denote by $\text{Dfr}_{\mathbb{F}_r}(-)$ the full triangulated subcategory of $SH^{\text{nis}}(\mathbb{F}_r)$ of bounded below $\mathbb{F}_r$-modules. We also denote by $\text{Dfr}_{\mathbb{F}_r}^e(k)$ the full triangulated subcategory of $\text{Dfr}_{\mathbb{F}_r}(-)$ of those $\mathbb{F}_r$-modules $M$ such that each $\mathbb{Z}_F(k)$-presheaf $\pi_i(M)|_{\mathbb{Z}_F(k)}$ is homotopy invariant and stable in the sense of [6 Def. 2.13, 2.14].

In certain sense $\text{Dfr}_{\mathbb{F}_r}^e(k)$ is an analog of Voevodsky’s triangulated category $DM^{\text{eff}}(k)$ [23].

**Definition 5.5.** The triangulated category $\text{Dfr}_{\mathbb{F}_r}^e(k)$ is called the triangulated category of effective $\mathbb{F}_r$-motives.

One can prove the following

**Theorem 5.6.** There is a natural triangulated equivalence between the triangulated categories $\text{Dfr}_{\mathbb{F}_r}^e(k)$ and the Voevodsky category $SH^{^e\text{ff}}(k)$.

A sketch of a proof of this result will be presented in the next section.

6. **TRIANGULATED EQUIVALENCES $SH^{^e\text{ff}}(k) \equiv \text{Dfr}_{\mathbb{F}_r}^e(k)$**

We construct in this section triangulated equivalences (quasi-inverse to each the other)

$$M_{\text{eff}}^\mathbb{F}_r : SH^{^e\text{ff}}(k) \equiv \text{Dfr}_{\mathbb{F}_r}^e(k) : M_{\text{eff}}^\mathbb{F}_r.$$  

To construct these functors we need preliminaries. Let $G^\Lambda_m \in \Delta^{op}(Fr_0(k))$ be as in [7, Notation 8.1]. Let $G^{\Lambda^m}_m$ be the $n$th monoidal power $G^{\Lambda^m}_m$ be as in [7 Notation 8.1]. The category $\text{Pre}_{G^{\Lambda^m}_m}(Sm/k)$ of presheaves of symmetric bispectra can be regarded as the category of symmetric $G^{\Lambda^m}_m$-spectra in the category $\text{Mod}_{\mathcal{G}_{\text{naive}}}$ of presheaves of symmetric spectra (see Definition 2.11).

Similarly we can (and will) consider a category of symmetric $G^{\Lambda^m}_m$-spectra in the category $\text{Mod}_{\mathbb{F}_r}$ It follows from [3, 5.13] that there is a pair of natural adjoint functors

$$\Phi_s : \text{Pre}_{G^{\Lambda^m}_m}(Sm/k) = Sp_{G^{\Lambda^m}_m}(\text{Mod}_{\mathcal{G}_{\text{naive}}}) \rightleftarrows Sp_{G^{\Lambda^m}_m}(\text{Mod}_{\mathbb{F}_r}) : \Phi^s$$

There is another pair of adjoint functors

$$\Sigma_{Fr(G^{\Lambda^m}_m)}^m : \text{Mod}_{\mathbb{F}_r} \rightleftarrows Sp_{G^{\Lambda^m}_m}(\text{Mod}_{\mathbb{F}_r}) : \Omega_{Fr(G^{\Lambda^m}_m)}^m.$$  

Here $Fr(G^{\Lambda^m}_m)$ stands for the simplicial scheme $G^{\Lambda^m}_m$.

For each $\mathbb{F}_r$-module $M$ consider the $\mathbb{F}_r$-module $C_s(M) := |\text{Hom}(\Delta, M)|$ as in Section 3. By Lemma 3.5 and Theorem 3.6 the endo-functor $C_s : \text{Mod}_{\mathbb{F}_r} \rightarrow \text{Mod}_{\mathbb{F}_r}$ induces a triangulated functor $C_s : \text{Dfr}_{\mathbb{F}_r}(-) \rightarrow \text{Dfr}_{\mathbb{F}_r}^e(-)$. By Theorem 3.6 the pair of triangulated functors

$$(4) \quad C_s : \text{Dfr}_{\mathbb{F}_r}(-) \rightleftarrows \text{Dfr}_{\mathbb{F}_r}^e(-) : i$$

is a pair of adjoint triangulated functors (here $i$ is the inclusion functor).

Let $Fr(n) = M_{\mathbb{F}_r}(G^{\Lambda^m}_m)$ be the $\mathbb{F}_r$-motive of $G^{\Lambda^m}_m$. For each cofibrant object $E$ in the projective
model structure on $\text{Mod} \mathcal{F} \mathcal{r}$ put $E(n) = E \otimes \mathcal{F} \mathcal{r}(n)$. It is a cofibrant object in the projective model structure on $\text{Mod} \mathcal{F} \mathcal{r}$. Clearly, $\Sigma_{\mathcal{F} \mathcal{r}(1)}^\infty (E) := (E, E(1), E(2), ...)$ is naturally an object of $\text{Sp}_{G_m}^\Sigma (\text{Mod} \mathcal{F} \mathcal{r})$.

**Definition 6.1.** Let $E \mapsto E^c$ be the cofibrant replacement in the projective model structure on $\text{Pre}_{G_m}^\Sigma (\text{Mod} \mathcal{F} \mathcal{r})$. Put
\[
M_{\mathcal{F} \mathcal{r}}^c(E) = (C_\ast \circ \Omega_{\mathcal{F} \mathcal{r}(1)}^\infty \circ \Phi_\ast)(E^c) = \Omega_{G_m}^\infty C_\ast \mathcal{F} \mathcal{r}(E^c) \in \text{Mod} \mathcal{F} \mathcal{r}.
\]
Let $\delta^c$ be the cofibrant replacement in the projective model structure on $\text{Mod} \mathcal{F} \mathcal{r}$. Put
\[
M_{SH}(\delta^c) = \Phi_\ast (\Sigma_{\mathcal{F} \mathcal{r}(1)}^\infty (\delta^c)) \in \text{Pre}_{G_m}^\Sigma (\text{Mod} \mathcal{F} \mathcal{r}) .
\]
Thus,
\[
M_{SH}(\delta^c) = \text{the object } \Sigma_{\mathcal{F} \mathcal{r}(1)}^\infty (\delta^c) \text{ of } \text{Sp}_{G_m}^\Sigma (\text{Mod} \mathcal{F} \mathcal{r}) \text{ regarded as an object in } \text{Pre}_{G_m}^\Sigma (\text{Mod} \mathcal{F} \mathcal{r}).
\]

A proof of the following result is postponed to the next preprint. It can be given in the spirit of the proofs as in [8, Section 2].

**Theorem 6.2.**
• The functor $M_{SH}$ induces a triangulated equivalence
\[
M_{SH}^{eff} : \text{D} \text{Pre}_{G_m}^c / (k) \to \text{SH}_{G_m}^{eff} (k)
\]
between these triangulated categories;

• A triangulated functor $M_{eff}^\mathcal{F} \mathcal{r} : \text{SH}_{G_m}^{eff} (k) \to \text{D} \text{Pre}_{G_m}^c / (k)$

quasi-inverse to $M_{SH}^{eff}$ is induced by the functor
\[
M_{fr}^\mathcal{F} \mathcal{r} : \text{Pre}_{G_m}^\Sigma (\text{Mod} \mathcal{F} \mathcal{r}) \to \text{Mod} \mathcal{F} \mathcal{r}.
\]

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