Bifurcation of straight-line librations

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Abstract

We study a class of 2-dimensional Hamiltonian systems

\[ H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) \]

in which the plane \( x = p_x = 0 \) is invariant under the Hamiltonian flow, so that straight-line librations along the \( y \) axis exist, and we also consider perturbations \( \delta H = \delta \cdot F(x, y, p_x, p_y) \) that preserve these librations. We describe a procedure for the analytical calculation of partial derivatives of the Poincaré map. These partial derivatives can be used to predict the bifurcation behavior of the libration, in particular to distinguish between transcritical and fork-like bifurcations, as was mathematically investigated in [1] and numerically studied in [2].

1 Introduction

We study 2-dimensional Hamiltonian systems

\[ H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) \]

with a potential satisfying

\[ \frac{\partial V}{\partial x}(0, y) = 0 \quad \text{for all} \quad y \in \mathbb{R}. \]

Then the \((y, p_y)\)-plane \( x = p_x = 0 \) is invariant under the Hamiltonian flow and thus the system will librate on the \( y \)-axis. Choose one of the libration families, parametrized by \( \varepsilon := E - E_0 \in I \) for some fixed energy \( E_0 \) and a suitable open interval \( I \). The family consists of closed orbits \( \gamma_\varepsilon(t) = (0, y_\varepsilon(t), 0, p_y(\varepsilon, t)) \), with \( p_y(\varepsilon, t) = \dot{y}(\varepsilon, t) \), which we let start, say, at their maximal value of \( y \), that is at the point \((0, y_{\max}(\varepsilon), 0, 0) \in \mathbb{R}^4\). Let \( T(\varepsilon) > 0 \) denote the period of \( \gamma_\varepsilon \).

For each \( \varepsilon \) we may use \( p_y = 0 \) as a Poincaré surface of section PSS at the starting point of the orbit, and the \((x, p_x)\)-plane as the PPSS, the projected Poincaré surface of section. As the canonical coordinates \( q \) and \( p \) in the PPSS, we may choose \( x \) and \( p_x \). Then the Poincaré map defines a symplectic family

\[ Q = Q(q, p, \varepsilon) \]
\[ P = P(q, p, \varepsilon) \]

on an open neighborhood of the \( \varepsilon \)-interval \( 0 \times 0 \times I \) in the \((q, p, \varepsilon)\)-space \( \mathbb{R}^3 \). Note that

\[ A := 0 \times 0 \times I = \{(0, 0, \varepsilon) \mid \varepsilon \in I\} \]

itself is a fixed point branch of this family, and we propose to study the bifurcations that may occur along this branch.
Among the questions we ask about these bifurcations, there is one concerning the behavior of a bifurcation under a small deformation of the Hamiltonian. Let $\delta$ denote a small deformation parameter and let us add a deformation term $\delta \cdot F(x, y, p_x, p_y)$ to the Hamiltonian:

$$H(x, y, p_x, p_y, \delta) = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) + \delta F(x, y, p_x, p_y),$$

and let $F$ satisfy the following ‘libration preserving condition’:

$$\frac{\partial F}{\partial p_x}(0, y, 0, p_y) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x}(0, y, 0, p_y) = 0. \quad (6)$$

Then for fixed $\delta$, the $(y, p_y)$-plane $x = p_x = 0$ will still be invariant under the Hamiltonian flow, so the system will still librate on the $y$-axis, and we ask how the bifurcation behavior found at $\delta = 0$ will change if we turn on the parameter $\delta$.

The symplectic family defined by the Poincaré map now depends on two parameters $\varepsilon$ and $\delta$:

$$Q = Q(q, p, \varepsilon, \delta) \quad \text{and} \quad P = P(q, p, \varepsilon, \delta). \quad (7)$$

Think of an arbitrary $\varepsilon_0$ being chosen. We will ask if the fixed point $(0, 0)$, of the undeformed system, is singular at $\varepsilon = \varepsilon_0$ and if so, what are the properties of the bifurcation and their behavior under deformations $\delta \neq 0$. Using [1], the answers to these questions could be read from those 38 partial derivatives up to third order of $P$ and $Q$ that involve the parameters $\varepsilon$ and $\delta$ at most in first order, at the single point $(0, 0, \varepsilon_0, 0)$ — if we only knew them. The purpose of the present note is to describe a procedure for the calculation of these partial derivatives of the Poincaré map from the potential $V(x, y)$ and the deformation term $F(x, y, p_x, p_y)$.

### 2 Numerical prerequisites

To start the procedure, for a given $\varepsilon_0$, we will need to know first of all the closed orbit $\gamma_{\varepsilon_0}$ itself, $\gamma_{\varepsilon_0}(t) = (0, y(t, \varepsilon_0), 0, p_y(t, \varepsilon_0))$, that is we have to solve the equation

$$\ddot{y} + \frac{\partial V}{\partial y}(0, y) = 0 \quad (8)$$

to the initial condition $y(0) = y_{\text{max}}(\varepsilon_0)$ and $\dot{y}(0) = 0$. The value $y_{\text{max}}(\varepsilon_0)$ satisfies

$$V(0, y_{\text{max}}(\varepsilon_0)) = E_0 + \varepsilon_0. \quad (9)$$

Within the chosen domain of libration it will be the larger of the two solutions of this equation and can be determined that way. The function $y(t) := y(t, \varepsilon_0)$ will be periodic of a period $T(\varepsilon_0) > 0$.

Furthermore, we will have to solve the linearized Hamiltonian equation along this closed orbit, that is we have to know the fundamental system $(\xi_1, \xi_2)$ of the linear equation

$$\ddot{\xi} + \frac{\partial^2 V}{\partial x^2}(0, y(t))\xi = 0 \quad (10)$$
to the initial condition
\[
\begin{pmatrix}
\xi_1(0) & \xi_2(0) \\
\dot{\xi}_1(0) & \dot{\xi}_2(0)
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
(11)
as well as the fundamental system \((\eta_1, \eta_2)\) of
\[
\ddot{\eta} + \frac{\partial^2 V}{\partial y^2}(0, y(t)) \eta = 0
\]
(12)
to the initial condition
\[
\begin{pmatrix}
\eta_1(0) & \eta_2(0) \\
\dot{\eta}_1(0) & \dot{\eta}_2(0)
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(13)
To say that these five functions \(y(t), \xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t)\) and their first derivatives must be ‘known’ means that they are known numerically on the whole period interval \([0, T(\varepsilon_0)]\). A computer program implementing the procedure for the calculation of the partial derivatives of the Poincaré map at \((0, 0, \varepsilon_0, 0)\) will have to treat them as known functions. But beyond that no further differential equations will have to be solved.

3 The Poincaré map

The Poincaré map is produced by the Hamiltonian flow. To facilitate the handling of higher partial derivatives, we will gradually shift from natural ‘speaking’ notations like \((x, y, p_x, p_y)\) to a simple enumeration of variables by upper indices. Partial derivatives by these variables will then be denoted by corresponding lower indices. We begin by writing
\[
\begin{align*}
a^1 &:= x \\
a^2 &:= y \\
a^3 &:= p_x \\
a^4 &:= p_y
\end{align*}
\]
(14)
for the independent variables in \(\mathbb{R}^4\) and
\[
x^i = x^i(t, a^1, a^2, a^3, a^4, \delta)
\]
(15)
with \(i = 1, \ldots, 4\), for the components of the Hamiltonian flow at a fixed \(\delta\), with initial conditions \(a^1, \ldots, a^4\):
\[
x^i(0, a^1, a^2, a^3, a^4, \delta) = a^i.
\]
(16)
For fixed \(\varepsilon\) and \(\delta\), the starting point in the PSS corresponding to a given point \((q, p)\) in the PPSS is described by
\[
\begin{align*}
a^1 &= q \\
a^2 &= y(q, p, \varepsilon, \delta) \\
a^3 &= p \\
a^4 &= 0,
\end{align*}
\]
(17)
where the $y$-component is defined implicitly by

$$
\frac{1}{2}p^2 + V(q, y(q, p, \varepsilon, \delta)) + \delta F(q, y(q, p, \varepsilon, \delta), p, 0) = \varepsilon + E_0
$$

and $y(0, 0, \varepsilon, 0) = y_{\text{max}}(\varepsilon)$. From this starting point, the flow line will travel for a time $T = T(q, p, \varepsilon, \delta) > 0$ until it hits the PSS $p_y = 0$ again, so implicitly this time is given by

$$
x^4(T(q, p, \varepsilon, \delta), q, y(q, p, \varepsilon, \delta), p, 0, \delta) = 0,
$$

in the notation (15) of the flow, and by the reference condition $T(0, 0, \varepsilon, 0) = T(\varepsilon)$, the period of $\gamma_\varepsilon$. The Poincaré map can now be described as

$$
Q(q, p, \varepsilon, \delta) = x^1(T(q, p, \varepsilon, \delta), q, y(q, p, \varepsilon, \delta), p, 0, \delta)
$$

$$
P(q, p, \varepsilon, \delta) = x^3(T(q, p, \varepsilon, \delta), q, y(q, p, \varepsilon, \delta), p, 0, \delta).
$$

Taking partial derivatives, we obtain the partial derivatives of the Poincaré map in terms of partial derivatives of the Hamiltonian flow and of partial derivatives of the flow time function $T(q, p, \varepsilon, \delta)$ and the starting point function $y(q, p, \varepsilon, \delta)$. That’s what we do next.

4 Taking derivatives of the Poincaré map

We are now unifying the notation of the independent variables of the flow, currently written as $(t, a^1, a^2, a^3, a^4, \delta)$, to $(a^0, \ldots, a^5)$. Derivatives are denoted by lower indices, so for instance $x^1_{02}$ would mean

$$
x^1_{02} = \frac{\partial^2 x^1}{\partial t \partial (a^2)} = \frac{\partial \dot{x}^1}{\partial (a^2)}
$$

and so on. Next, we write the two components $Q$ and $P$ of the Poincaré map as compositions

$$
Q = x^1 \circ Z
$$

$$
P = x^3 \circ Z,
$$

or $Q(q, p, \varepsilon, \delta) = x^1(Z^0(q, p, \varepsilon, \delta), \ldots, Z^5(q, p, \varepsilon, \delta))$ and analogously for $P$, where the six components $Z^0, \ldots, Z^5$ are given, according to (20), by

$$
Z^0(q, p, \varepsilon, \delta) := T(q, p, \varepsilon, \delta)
$$

$$
Z^1(q, p, \varepsilon, \delta) := q
$$

$$
Z^2(q, p, \varepsilon, \delta) := y(q, p, \varepsilon, \delta)
$$

$$
Z^3(q, p, \varepsilon, \delta) := p
$$

$$
Z^4(q, p, \varepsilon, \delta) := 0
$$

$$
Z^5(q, p, \varepsilon, \delta) := \delta.
$$

Finally, let us also enumerate the independent coordinates $(q, p, \varepsilon, \delta)$ in the product of the PPSS with the parameter plane by writing

$$
u^1 := q
$$

$$
u^2 := p
$$

$$
u^3 := \varepsilon
$$

$$
u^4 := \delta
$$

(24)
again denoting partial derivatives by lower indices, like in $Z_3^2 = \frac{\partial Z_3^2}{\partial u_3} = \frac{\partial y}{\partial u}$. I shall try to be consistent in using greek letters $\lambda, \mu, \ldots$ for the $a$-related indices that run from 0 to 5 and use roman letters $\ell, m, \ldots = 1, 2, 3, 4$ for indices refering to the $u$-variables.

The partial derivatives we are after, like $P_{qq\varepsilon} = (x^3 \circ Z)_{113}$, can now neatly be written in terms of partial derivatives of flow, flow time and starting point functions as

$$(x^i \circ Z)_{\ell} = x^i Z_{\ell}^\lambda$$

$$(x^i \circ Z)_{\ell m} = x^i Z_{m}^\lambda Z_{n}^\mu + x^i Z_{\ell}^\lambda Z_{m}^\mu + x^i Z_{m}^\lambda Z_{n}^\mu + x^i Z_{\ell}^\lambda Z_{m}^\mu) + x^i Z_{\ell m n}.$$

That’s just the chain rule, so it holds everywhere. But for our application, we need to know the left hand sides only at the point $(u^1, u^2, u^3, u^4) = (0, 0, \varepsilon_0, 0)$, and so on the right hand side we want to know the $Z_{\lambda}^\star$ at $(0, 0, \varepsilon_0, 0)$ and the $x_{\lambda}^\star$ at $(T(\varepsilon_0), 0, y_{\text{max}}(\varepsilon_0), 0, 0, 0)$.

### 5 Derivatives of the starting point function

First we will take care of the derivatives $Z_{\lambda}^\star$. For $\lambda = 1, 3, 4, 5$ this is easy, because for those $\lambda$ the $Z_{\lambda}^\star$ are given by

$$Z_{1}^1 = Z_{2}^3 = Z_{4}^5 = 1$$

$$Z_{1}^3 = Z_{2}^3 = Z_{4}^5 = Z_{5}^5 = 0$$

for all other lower indices $\ast$, in particular for all derivatives of order $\geq 2$. It remains to determine the derivatives $Z_{\lambda}^0$ and $Z_{\lambda}^2$ of $Z^0 = T(u^1, \ldots, u^4)$ and $Z^2 = y(u^1, \ldots, u^4)$.

In the present section we will calculate $Z_{\lambda}^2$ up to second order.

The defining equation (18) for the starting point function $y(u^1, \ldots, u^4)$ in the $u$-notation becomes

$$\frac{1}{2} u^2 \cdot u^2 + V(u^1, y(u^1, \ldots, u^4)) + u^4 F(u^1, y(u^1, \ldots, u^4), u^2, 0) = u^3 + E_0. \quad (27)$$

As a first step, we will determine the eight derivatives

$$y_1, y_2, y_3 \quad \text{and} \quad y_{11}, y_{12}, y_{22}, y_{13}, y_{23}$$

at $(0, 0, \varepsilon_0, 0)$. Since here the variable $u^4 = \delta$ is not involved, we may put $u^4 = 0$.

Taking first derivatives by $u^1, u^2, u^3$ gives

$$V_1 + V_2 y_1 = 0$$

$$u^2 + V_2 y_2 = 0$$

$$V_2 y_3 = 1$$

$(29)$
at all \((u^1, u^2, u^3, 0)\). Differentiating further, we get
\[
V_{11} + V_{12}y_1 + (V_{21} + V_{22}y_1)y_1 + V_{2}y_{11} = 0
\]
\[
V_{12}y_1 + V_{22}y_2y_1 + V_{2}y_{12} = 0
\]
\[
1 + V_{22}y_2 + V_{2}y_{22} = 0
\]
\[
(V_{21} + V_{22}y_1)y_3 + V_{2}y_{31} = 0
\]
\[
V_{2}y_3 + V_{2}y_{32} = 0
\]
again at all \((u^1, u^2, u^3, 0)\). Specializing now to \((0, 0, \varepsilon_0, 0)\) and using our assumption \(\frac{\partial V}{\partial x}(0, y) \equiv 0\) on the potential, which implies
\[
V_1(0, y) = V_{12}(0, y) \equiv 0
\]
for all \(y\), we obtain
\[
y_1 = y_2 = y_12 = y_{13} = y_{23} = 0
\]
\[
\text{and } y_3 = \frac{1}{V_2}
\]
\[
y_{11} = -\frac{V_{11}}{V_2}
\]
\[
y_{22} = -\frac{1}{V_2}
\]
at \((0, 0, \varepsilon_0, 0)\) as some first ‘end results’ on the derivatives of the starting point function in terms of the derivatives \(V_2(0, y_{\text{max}}(\varepsilon_0))\) and \(V_{11}(0, y_{\text{max}}(\varepsilon_0))\) of the potential.

Let us now consider the derivatives \(y_4, y_{14}, \text{and } y_{24}\) involving the deformation parameter \(u^4 = \delta\). Differentiating (27) first by \(u^4\) and then in addition by \(u^1\) resp. \(u^2\) we obtain
\[
V_{2}y_4 + F = 0
\]
\[
(V_{21} + V_{22}y_1)y_4 + V_{22}y_4 + F_1 + F_{2}y_1 = 0
\]
\[
V_{2}y_4 + V_{2}y_{42} + F_{2}y_2 = 0
\]
at all \((u^1, u^2, u^3, u^4)\). Specializing again to \((0, 0, \varepsilon_0, 0)\) and using (31), (32) and the assumption (6) about \(F(x, y, p_x, p_y)\), we get
\[
y_{14} = y_{24} = 0
\]
\[
\text{and } y_4 = -\frac{F(0, y_{\text{max}}(\varepsilon_0), 0, 0)}{V_2(0, y_{\text{max}})}.
\]

Note that with (26), (32) and (34) we have determined all partial derivatives \(Z_{\ell}^{\lambda}\) and \(Z_{\ell m}^{\lambda}\) at \((0, 0, \varepsilon_0, 0)\) for \(\lambda \geq 1\) and at most one of the indices \(\ell\) and \(m\) being 3 or 4. For convenience, let us collect these results. First for the ‘easy’ \(\lambda\)’s. Here the first derivatives are, everywhere:

| \(Z_{\ell}^{\lambda}\) | \(\ell=1\) | \(\ell=2\) | \(\ell=3\) | \(\ell=4\) |
|-----------------| --------| --------| --------| --------|
| \(\lambda=1\)   | 1       | 0       | 0       | 0       |
| \(\lambda=3\)   | 0       | 1       | 0       | 0       |
| \(\lambda=4\)   | 0       | 0       | 0       | 0       |
| \(\lambda=5\)   | 0       | 0       | 0       | 1       |
The higher derivatives therefore are zero, in particular \( Z_{\ell m}^\lambda = 0 \) for \( \lambda = 1, 3, 4, 5 \). For \( \lambda = 2 \) we have found at \((u^1, u^2, u^3, u^4) = (0, 0, \varepsilon_0, 0)\):

\[
\begin{align*}
Z_1^2 &= 0 \\
Z_2^2 &= 0 \\
Z_3^2 &= \frac{1}{V_2} \\
Z_4^2 &= -\frac{F}{V_2} \\
Z_{11}^2 &= -\frac{V_{11}}{V_2} \\
Z_{12}^2 &= 0 \\
Z_{22}^2 &= -\frac{1}{V_2} \\
Z_{13}^2 &= 0 \\
Z_{23}^2 &= 0 \\
Z_{14}^2 &= 0 \\
Z_{24}^2 &= 0,
\end{align*}
\]

(36)

where the derivatives of \( V \) have to be taken at \((x, y) = (0, y_{\text{max}}(\varepsilon_0))\), the value of \( F \) at \((x, y, p_x, p_y) = (0, y_{\text{max}}(\varepsilon_0), 0, 0)\). In the next section, we turn to the remaining case \( \lambda = 0 \).

6 Derivatives of the flow time function

Although we are interested in the partial derivatives of \( x^i \circ Z \) for \( i = 1, 3 \) only, we will also become involved with the \( p_y \)-component \( x^4(a^0, \ldots, a^5) \) of the flow, because the defining condition (19) of the flow time function \( Z_0^0 \) is

\[ x^4 \circ Z \equiv 0. \tag{37} \]

This is also the reason, by the way, why the linear equation (12) will come up in the calculations. — From (37), using the first equation of (25) for \( i = 4 \), we obtain

\[
x_0^4 Z_0^{\lambda} = -x_0^4 Z_1^{\lambda} - x_0^4 Z_2^{\lambda} - x_0^4 Z_3^{\lambda} - x_0^4 Z_4^{\lambda} - x_0^4 Z_5^{\lambda}.
\]

(38)

The \( Z_\ell^\lambda \) on the right hand side are known at \((0, 0, \varepsilon_0, 0)\). Correspondingly, the \( x_0^4 \), for \( \lambda = 0, \ldots, 5 \) are meant to be taken at \((a^0, \ldots, a^5) = (T(\varepsilon_0), 0, y_{\text{max}}(\varepsilon_0), 0, 0, 0)\). We still have to determine them, but we certainly know \( x_0^4 \) there, since by the Hamiltonian equations

\[
x_0^4(t, 0, y_{\text{max}}(\varepsilon_0), 0, 0, 0) = \dot{p}_y(t) = -\frac{\partial V}{\partial y}(0, y(t, \varepsilon_0))
\]

(39)

along the closed orbit \( \gamma_{\varepsilon_0} \), and therefore

\[
x_0^4(T(\varepsilon_0), 0, y_{\text{max}}(\varepsilon_0), 0, 0, 0) = -V_2(0, y_{\text{max}}(\varepsilon_0)). \tag{40}
\]
So from (35), (36) and (38) we get

\[
\begin{align*}
Z_0^1 & = \frac{1}{V_2} x_1^4 \\
Z_0^2 & = \frac{1}{V_2} x_3^4 \\
Z_0^3 & = \frac{1}{V_2 V_2} x_2^4 \\
Z_0^4 & = -\frac{F}{V_2 V_2} x_2^4 + \frac{1}{V_2} x_5^4.
\end{align*}
\]

Similarly, from the second equation of (25) for \( i = 4 \) we now have

\[
\begin{align*}
Z_0^0 & = \frac{1}{V_2} (x_1^4 Z_1^0 + x_2^4 Z_2^0 + x_3^4 Z_3^0 + x_4^4 Z_4^0 + x_5^4 Z_5^0 + x_\lambda^4 Z_\lambda^0 Z_\mu^0 ) \\
& = \frac{1}{V_2} (x_2^4 Z_2^0 + x_\lambda^4 Z_\lambda^0 Z_\mu^0 )
\end{align*}
\]

at \((0, 0, \varepsilon_0, 0)\), with the \( x_\lambda^0 \) to be taken at \( (T(\varepsilon_0), 0, y_{\max}(\varepsilon_0), 0, 0, 0) \), as before. And here we leave it for now: the \( Z_\lambda^0 \) on the right hand side, at \((0, 0, \varepsilon_0, 0)\), are all known from (35), (36) and (41). A computer will understand (42) as given, and for us, there is no point in writing out the formula in great length before we know more about the \( x_\lambda^0 \), in particular before we know which of them will vanish anyway.

The same reasoning applies to the last \( Z_\lambda^0 \) that are still missing, namely the third derivatives \( Z_\ell^0 \) and \( Z_\ell^0 \) of the starting point and flow time functions, which might be needed in the computation of the third derivatives of the Poincaré map. In fact they will not be needed, because they enter the third equation of (25) for \( i = 1, 3 \) with coefficients \( x_2^2 \) and \( x_1^0 \), which will soon be seen to vanish for \( i = 1, 3 \). This is just one of the details of the problem to calculate all the \( x_\lambda^0 (T(\varepsilon_0), 0, y_{\max}(\varepsilon_0), 0, 0, 0) \) that we need. To this problem our bifurcation analysis is now reduced and it will be solved in the remaining sections.

7 Equations for the flow derivatives

The four components of the Hamiltonian flow, see (15) and (16), are written currently as

\[
x^i = x^i(a^0, \ldots, a^5) \quad \text{for} \quad i = 1, 2, 3, 4
\]

with \( a^0 \) denoting the time \( t \) and \( a^5 = \delta \), while \( (a^1, \ldots, a^4) \) is the initial point. As a bookkeeping device we now introduce a ‘fifth flow component’ by

\[
x^5(a^0, \ldots, a^5) := a^5.
\]

But we also use \( x^1, \ldots, x^5 \) as the names of the independent variables of the Hamil-
tonian vector field $\vec{v} = (v^1, \ldots, v^4)$, which is then given by

$$
\begin{align*}
  v^1(x^1, \ldots, x^5) &= x^3 + x^5 F_3(x^1, \ldots, x^4) \\
  v^2(x^1, \ldots, x^5) &= x^4 + x^5 F_4(x^1, \ldots, x^4) \\
  v^3(x^1, \ldots, x^5) &= -V_1(x^1, x^2) - x^5 F_1(x^1, \ldots, x^4) \\
  v^4(x^1, \ldots, x^5) &= -V_2(x^1, x^2) - x^5 F_2(x^1, \ldots, x^4).
\end{align*}
$$

(45)

The Hamilton equations become $\dot{x}^r(\vec{a}) = v^r(\vec{x}(\vec{a}))$ for $r=1,2,3,4$, and as in (25) we obtain

$$
\begin{align*}
  \dot{x}^r_\lambda &= v^r_i x^i_\lambda \\
  \dot{x}^r_{\lambda\mu} &= v^r_{ij} x^i_\lambda x^j_\mu + v^r_i x^i_\lambda \\
  \dot{x}^r_{\lambda\mu\nu} &= v^r_{ijk} x^i_\lambda x^j_\mu x^k_\nu + v^r_{ij} (x^i_\lambda x^j_\mu + x^i_\mu x^j_\lambda + x^i_\nu x^j_\lambda) + v^r_i x^i_\lambda
\end{align*}
$$

(46)

The summation indices $i, j, k$ run from 1 to 5, while any $\lambda, \mu, \nu \in \{0,1,2,3,4,5\}$ are admitted. The equations hold everywhere, that is the flow components $x^i$ and their derivatives may be taken at any $\vec{a} := (a^0, \ldots, a^5)$, the vector field components $v^r$ and their derivatives then at the corresponding $\vec{x}(\vec{a}) = (x^1(\vec{a}), \ldots, x^5(\vec{a}))$. Also note that the time derivative of the flow is the partial derivative by $a^0$, so on the left hand sides we might have written $x^r_{0\lambda}$, $x^r_{0\lambda\mu}$, $x^r_{0\lambda\mu\nu}$ instead of $\dot{x}^r_\lambda$, $\dot{x}^r_{\lambda\mu}$, $\dot{x}^r_{\lambda\mu\nu}$.

To apply (25), we will only need to know the $x^r_\star(T(\varepsilon_0), 0, \max(\varepsilon_0), 0, 0, 0)$. But in order to determine these numbers, we will also have to consider the functions $x^r_\star(t, 0, \max(\varepsilon_0), 0, 0, 0)$ on the interval $[0, T(\varepsilon_0)]$, for which we now introduce the notation

$$
x^i_\star(t) := x^i_\star(t, 0, \max(\varepsilon_0), 0, 0, 0).
$$

(47)

Correspondingly, we write

$$
v^r_\star(t) := v^r_\star(\vec{x}(t, 0, \max(\varepsilon_0), 0, 0, 0)).
$$

(48)

Then from (47) we get

$$
\begin{align*}
  \dot{x}^r_\lambda(t) &= v^r_i(t) x^i_\lambda(t) \\
  \dot{x}^r_{\lambda\mu}(t) &= v^r_{ij}(t) x^i_\lambda(t) x^j_\mu(t) + v^r_i(t) x^i_\lambda(t) \\
  \dot{x}^r_{\lambda\mu\nu}(t) &= v^r_{ijk}(t) x^i_\lambda(t) x^j_\mu(t) x^k_\nu(t) + v^r_{ij}(t) x^i_\lambda(t) x^j_\mu(t) + v^r_{ij}(t) x^j_\lambda(t) x^i_\mu(t) + v^r_i(t) x^i_\lambda
\end{align*}
$$

(49)

from which we will now proceed to determine the functions (47). We only consider $r = 1, \ldots, 4$ as there is no need to write equations for $x^5_\star(t)$, since of course

$$
x^5_\star(t) \equiv 1 \quad \text{and} \quad x^5_\star(t) \equiv 0.
$$

(50)

for all other partial derivatives of $x^5(\vec{a}) = a^5$.
8 Calculation of the first order flow derivatives

Note that all the $v^r_\lambda(t)$ on the right hand sides of (49) are known functions in the sense agreed upon in section 2 on the numerical prerequisites, since

$$\tilde{x}(t, 0, y_{\text{max}}(\varepsilon_0), 0, 0, 0) = (0, y(t), 0, \dot{y}(t), 0),$$

which is to be used in (48). In particular, let us tabulate the first derivatives $v^r_\lambda(t)$:

| $v^r_\lambda(t)$ | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ |
|------------------|-------|-------|-------|-------|-------|
| $r=1$            | 0     | 0     | 1     | 0     | 0     |
| $r=2$            | 0     | 0     | 0     | 0     | $F_4(t)$ |
| $r=3$            | $-V_{11}(t)$ | 0 | 0 | 0 | 0 |
| $r=4$            | 0     | $-V_{22}(t)$ | 0 | 0 | $-F_2(t)$ |

Here we write $F_4(t) := F_4(0, y(t), 0, \dot{y}(t))$ and $V_{11}(t) := V_{11}(0, y(t))$ and so on, in line with the notation introduced in (47) and (48). Note that $F_1(t) = F_3(t) = 0$ by our assumption (6) on $F(x, y, p_x, p_y)$. As a first consequence of (49), we see that the $x^r_\lambda(t)$ for $r, \lambda \in \{1, 2, 3, 4\}$ satisfy the homogeneous linear differential equations

$$\dot{x}_\lambda^1 - x_\lambda^3 = 0$$
$$\dot{x}_\lambda^3 + V_{11}(t)x_\lambda^1 = 0$$

and

$$\dot{x}_\lambda^2 - x_\lambda^4 = 0$$
$$\dot{x}_\lambda^4 + V_{22}(t)x_\lambda^2 = 0$$

These are just the first order systems corresponding to (10) and (12), and the initial conditions, as we see from (16), are

$$x^r_\lambda(0) = 1 \quad \text{if} \quad r = \lambda, \quad \text{and}$$
$$x^r_\lambda(0) = 0 \quad \text{if} \quad r \neq \lambda.$$  

But this shows that the $x^r_\lambda(t)$ for $r, \lambda = 1, \ldots, 4$ are known functions from the numerical prerequisites, more precisely

$$\begin{pmatrix} x_1^1(t) & x_1^3(t) \\ x_1^3(t) & x_1^3(t) \end{pmatrix} = \begin{pmatrix} \xi_1(t) & \xi_2(t) \\ \dot{\xi}_1(t) & \dot{\xi}_2(t) \end{pmatrix}$$

and

$$\begin{pmatrix} x_2^2(t) & x_2^2(t) \\ x_2^4(t) & x_4^4(t) \end{pmatrix} = \begin{pmatrix} \eta_1(t) & \eta_2(t) \\ \dot{\eta}_1(t) & \dot{\eta}_2(t) \end{pmatrix},$$

while

$$x^r_\lambda(t) = 0 \quad \text{for} \quad r + \lambda \quad \text{odd}, \quad \lambda \in \{1, 2, 3, 4\}.$$  

What about $\lambda = 0$ and $\lambda = 5$? We know $x^r_0(t) = \dot{x}^r(t) = v^r(t)$, so from (45) we have

$$x_0^1(t) = 0$$
$$x_0^2(t) = \dot{y}(t)$$
$$x_0^3(t) = 0$$
$$x_0^4(t) = -V_2(t).$$
Looking at $\lambda = 5$, we see from (47) and (52) that (53) is satisfied also in this case, but (54) has to be replaced by the \textit{inhomogeneous} system

$$\begin{align*}
\dot{x}_5^2 - x_5^4 &= F_4(t) \\
\dot{x}_5^4 + V_{22}(t)x_5^2 &= -F_2(t).
\end{align*}$$

The initial conditions are $x_5^2(0) = 0$, as we see again from (16). In particular we have

$$x_5^3(t) = x_5^3(t) = 0,$$

and since we have the fundamental matrix (57) of the homogeneous system (54), we obtain the the remaining two functions $x_5^2(t)$ and $x_5^4(t)$ by \textit{variation of constants} as

$$\begin{pmatrix} x_5^2(t) \\ x_5^4(t) \end{pmatrix} = \begin{pmatrix} x_5^2(t) & x_5^4(t) \\ x_5^4(t) & x_5^4(t) \end{pmatrix} \int_0^t d\tau \begin{pmatrix} x_5^4(\tau) & -x_5^4(\tau) \\ -x_5^4(\tau) & x_5^4(\tau) \end{pmatrix} \begin{pmatrix} F_4(\tau) \\ -F_2(\tau) \end{pmatrix}.$$ 

Note that with (50) and (56)-(62) all flow derivatives $x_\lambda^i(t)$ of first order are now determined. Among them the $x_5^2(t)$ and $x_5^4(t)$ have been seen to vanish for $i = 1, 3$, as announced at the end of section 6, and so there is in fact no need to determine third derivatives $Z_{\ell mn}^0$ and $Z_{\ell mn}^0$ of the starting point and flow time functions.

9 Calculation of higher order flow derivatives

Just as the first equation of (49) led to (53) and (54), so the other equations of (49) show that the higher order flow derivatives $x_\mu^i(t)$ satisfy differential equations

$$\begin{align*}
\dot{x}_\mu^1 - x_\mu^3 &= g_\mu^1(t) \\
\dot{x}_\mu^3 + V_{11}(t)x_\mu^1 &= g_\mu^3(t)
\end{align*}$$

and

$$\begin{align*}
\dot{x}_\mu^2 - x_\mu^4 &= g_\mu^2(t) \\
\dot{x}_\mu^4 + V_{22}(t)x_\mu^2 &= g_\mu^4(t),
\end{align*}$$

with

$$g_{\lambda\mu}(t) = v_{ij}(t)x_\lambda^i(t)x_\mu^j(t) \quad \text{and}$$

$$g_{\lambda\mu
nu}(t) = v_{ijk}(t)x_\lambda^i(t)x_\mu^j(t)x_\nu^k(t)$$

$$+ v_{ij}(t)(x_\mu^i(t)x_\nu^j(t) + x_\nu^i(t)x_\mu^j(t)) + x_\mu^i(t)x_\nu^j(t)x_\lambda^k(t).$$

What are the initial conditions? Again from (16) we see that

$$x_\lambda^i(0) = x_\lambda^i(0) = 0 \quad \text{if} \quad \lambda, \mu, \nu \neq 0$$

and therefore by variation of constants we get

$$\begin{pmatrix} x_\mu^1(t) \\ x_\mu^3(t) \end{pmatrix} = \begin{pmatrix} x_\mu^1(0) & x_\mu^3(0) \\ x_\mu^3(0) & x_\mu^3(0) \end{pmatrix} \int_0^t d\tau \begin{pmatrix} x_\mu^3(\tau) & -x_\mu^3(\tau) \\ -x_\mu^3(\tau) & x_\mu^3(\tau) \end{pmatrix} \begin{pmatrix} g_\mu^1(\tau) \\ g_\mu^3(\tau) \end{pmatrix}.$$
and
\[
\begin{pmatrix}
  x^2_\ast(t) \\
  x^4_\ast(t)
\end{pmatrix} = \begin{pmatrix}
  x^2_\ast(t) & x^2_\ast(t) \\
  x^4_\ast(t) & x^4_\ast(t)
\end{pmatrix} \int_0^t d\tau \begin{pmatrix}
  -x^2_\ast(\tau) & -x^2_\ast(\tau) \\
  -x^2_\ast(\tau) & x^2_\ast(\tau)
\end{pmatrix} \begin{pmatrix}
  g^2_\ast(\tau) \\
  g^4_\ast(\tau)
\end{pmatrix}
\]
(68)

for all indices $\ast = \lambda \mu$ and $\ast = \lambda \mu \nu$ with $\lambda, \mu, \nu \neq 0$. But do we know the functions $g^i_\ast(t)$ in all these cases? Let us look at (65). The $v^r_\ast(t)$ are all known, see (48) and (51). The $x^r_\lambda(t)$ have been determined in section 8, so we know all $g^r_{\lambda \mu}(t)$ and hence the $x^r_{\lambda \mu}(t)$ for $\lambda, \mu \neq 0$ from (67) and (68). These in turn give us, now for $\lambda, \mu, \nu \neq 0$, the $g^r_{\lambda \mu \nu}(t)$ by (65) and the $x^r_{\lambda \mu \nu}(t)$ from (67) and (68).

It remains to determine the $x^r_{\lambda \mu}(t)$ and $x^r_{\lambda \mu \nu}(t)$ in those cases where one or several of the indices are zero. The values of these functions at $t = T(\varepsilon_0)$ might also be needed in (25) for the calculation of the partial derivatives of the Poincaré map. The index 0 denotes the time derivative. Knowing the $x^r_0(t)$ from (59) we derive $x^r_{00}(t)$ and $x^r_{000}(t)$ as
\[
\begin{align*}
  x^1_{00}(t) &= 0 \\
  x^2_{00}(t) &= \ddot{y}(t) = -V_2(t) & \text{by (8)} \\
  x^3_{00}(t) &= 0 \\
  x^4_{00}(t) &= -V_{22}(t)\ddot{y}(t), \\
  x^1_{000}(t) &= 0 \\
  x^2_{000}(t) &= -V_{22}(t)\ddot{y}(t) \\
  x^3_{000}(t) &= 0 \\
  x^4_{000}(t) &= -V_{222}(t)\ddot{y}(t)^2 + V_2(t)V_{22}(t).
\end{align*}
\]
(69)

For $\lambda, \mu \neq 0$ the $x^r_{0\lambda}(t)$ are obtained by (53), (54) and (58), (60), (61) from the known $x^r_\lambda(t)$ and similarly the $x^r_{0\lambda \mu}(t)$ by (63), (64) and the first equation of (65) from the $x^r_\lambda(t)$ and $x^r_{\lambda \mu}(t)$. Finally, differentiating (53), (54), (58), (60) and (61) we see that for $\lambda \in \{1, 2, 3, 4\}$
\[
\begin{align*}
  x^1_{00\lambda}(t) &= -V_{11}(t)x^1_\lambda(t) \\
  x^2_{00\lambda}(t) &= -V_{22}(t)x^2_\lambda(t) \\
  x^3_{00\lambda}(t) &= -V_{112}(t)\ddot{y}(t)x^1_\lambda(t) + V_{11}(t)x^3_\lambda(t) \\
  x^4_{00\lambda}(t) &= -V_{222}(t)\ddot{y}(t)x^2_\lambda(t) + V_{22}(t)x^4_\lambda(t),
\end{align*}
\]
(70)
in particular $x^r_{00\lambda}(t) = 0$ for $r + \lambda$ odd and $r, \lambda \in \{1, 2, 3, 4\}$, and
\[
\begin{align*}
  x^1_{005}(t) &= 0 \\
  x^2_{005}(t) &= -V_{22}(t)x^2_5(t) + F_{42}(t)\ddot{y}(t) + F_{44}(t)V_2(t) \\
  x^3_{005}(t) &= 0 \\
  x^4_{005}(t) &= -V_{222}(t)\ddot{y}(t)x^2_5(t) + V_{22}(t)x^4_5(t) - F_{22}(t)\ddot{y}(t) - F_{24}(t)V_2(t).
\end{align*}
\]
(71)

In principle we now have all we need to calculate the partial derivatives of the Poincaré map.
10 Summary of the procedure

Once the numerical prerequisites of section 2 are established, we get the $x_i^r(t)$ as described in section 8 almost without further calculation, the only exceptions are $x_2^2$ and $x_4^4$, for which the integral (62) has to be evaluated. As explained in section 9, we also have the $x_i^r(\lambda \mu)(t)$ in those cases where at most one of the indices $\lambda, \mu, \nu$ is different from zero. Next determine the $x_i^r(\lambda \mu)(t)$ for $\lambda, \mu \neq 0$ by calculating $g_i^r(\lambda \mu)(t)$ from the first equation of (65) and applying (67) and (68). Then we get $x_{0\lambda \mu}(t)$ for $\lambda, \mu \neq 0$ by (63),(64) and by the first equation of (65) without new integration. Also the flow time derivatives $Z_{\ell m}^0$ of (42) are now known. Finally, we now have the $g_i^r(\lambda \mu \nu)(t)$ for $\lambda, \mu, \nu \neq 0$ from the second equation of (65) and we can calculate the corresponding $x_i^r(\lambda \mu \nu)(t)$ as the integrals (67) and (68). Taking values at $t = T(\varepsilon_0)$ of all these functions and applying (25), we obtain the 38 partial derivatives of the Poincaré map at $(0, 0, \varepsilon_0, 0)$ we wanted.

For the computer, these instructions may be good enough, but a person might want to see step by step what is going on. For this we have some choice in which order to proceed. We will first describe all those steps that are not connected with the deformation question.

11 The undeformed system

Step 1. Choose the potential $V(x, y)$ to be studied, with $\frac{\partial V}{\partial x}(0, y) \equiv 0$, choose one of its libration families on the $y$-axis and a reference point $E_0$ for the energy parameter $\varepsilon = E - E_0$. Choose a fixed $\varepsilon_0$ at which the bifurcation behavior of the libration shall be predicted.

Step 2. Set up a first part of the numerical prerequisites, namely $y(t), \xi_1(t), \xi_2(t)$ and their first derivatives, as described in section 2, including the period $T(\varepsilon_0)$. Define the four functions $x_i^r(t)$ on $[0, T(\varepsilon_0)]$ with $r, \lambda \in \{1, 3\}$ by (56).

Step 3. Collect the Jacobian matrix of the Poincaré map at $(0, 0, \varepsilon_0)$, or monodromy matrix of our librating orbit, as

$$
\begin{pmatrix}
Q_q & Q_p \\
P_q & P_p
\end{pmatrix}
= \begin{pmatrix}
x_1^1(T(\varepsilon_0)) & x_3^1(T(\varepsilon_0)) \\
x_3^3(T(\varepsilon_0)) & x_3^3(T(\varepsilon_0))
\end{pmatrix},
$$

(72)

according to (25), (26) and (55). If the trace $Q_q + P_p$ is different from $+2$, the fixed point is regular and the orbit will not bifurcate. In this case the procedure may stop here, since then we might not be interested in the higher derivatives. But $Q_q + P_p = 2$ is not a technical necessity for going on.

Step 4. Now we will calculate the $\varepsilon$-derivative $\text{Tr}'_{A}(\varepsilon_0) = Q_{q\varepsilon} + P_{p\varepsilon}$ of the trace. From (25) we find

$$
Q_{q\varepsilon} = \frac{1}{V_2 V_2} x_1^3 x_4^4 + \frac{1}{V_2} x_{12}^1
$$

and

$$
P_{p\varepsilon} = \frac{-V_1}{V_2 V_2} x_3^1 x_4^4 + \frac{1}{V_2} x_{32}^3
$$

(73)
at \((0, 0, \varepsilon_0)\). Here we need the remaining functions \(\eta_1(t)\) and \(\eta_2(t)\) of the prerequisites: they define the \(x^i_\lambda(t)\) on \([0, T(\varepsilon_0)]\) with \(r, \lambda \in \{2, 4\}\) by (57). Apart from the factor \(x^1_2(T(\varepsilon_0))\) in the first summand, they are needed as functions on \([0, T(\varepsilon_0)]\) to calculate \(x^1_{12}\) and \(x^3_{32}\) in the second summand by integration (67), because the inhomogeneities \(g^1_{\lambda\mu}(t)\) and \(g^3_{\lambda\mu}(t)\) for \(\lambda \in \{1, 3\}\) and \(\mu \in \{2, 4\}\) turn out by (65) to be

\[
g^1_{\lambda\mu}(t) = 0, \quad g^3_{\lambda\mu}(t) = -V_{112}(t)x^1_\lambda(t)x^2_\mu(t). \tag{74}
\]

If step 3 has shown \(\varepsilon_0\) to be singular \((\text{Tr}_A(\varepsilon_0) = 2)\), then after completion of step 4 we know if it is a cross-bifurcation, that is if \(\text{Tr}'_A(\varepsilon_0) \neq 0\).

**Step 5.** Is this cross-bifurcation transcritical? To answer this question, we need the monodromy matrix (72) from step 3 and the second partial derivatives of \(Q\) and \(P\) by the variable \(q\) and \(p\) at \((0, 0, \varepsilon_0)\), that is the \((x^i \circ Z)_{\ell m}\) for \(i \in \{1, 3\}\) and \(\ell, m \in \{1, 2\}\), to see if \(\tilde{P}^q_{qe} \neq 0\), where the ‘tilde’ denotes adapted coordinates. By (26) and from our knowledge of the \(x^i_\lambda\), the second equation of (25) reads

\[
\begin{align*}
Q_{qq} &= x^1_{11} \\
Q_{qp} &= x^1_{13} \\
Q_{pp} &= x^1_{33} \\
P_{qq} &= x^3_{11} \\
P_{qp} &= x^3_{13} \\
P_{pp} &= x^3_{33}
\end{align*} \tag{75}
\]

at \(t = T(\varepsilon_0)\). The \(x^i_{\lambda\mu}(t)\) for \(i, \lambda, \mu \in \{1, 3\}\) have to be calculated from (67) by integration with inhomogeneities

\[
\begin{align*}
g^1_{\lambda\mu}(t) &= 0 \\
g^3_{\lambda\mu}(t) &= -V_{111}(t)x^1_\lambda(t)x^1_\mu(t). \tag{76}
\end{align*}
\]

**Step 6.** If the cross-bifurcation is not transcritical, then we are interested in

\[
\varepsilon''_B(0) = \frac{3Q_{qq}\tilde{P}^{-1}_{qp} - 3Q_{pp}\tilde{P}^{-1}_{qq}}{3Q_{pp}\tilde{P}^{-1}_{qe}}, \tag{77}
\]

since the bifurcation is fork-like if and only if \(\varepsilon''_B(0) \neq 0\), and its sign and absolute value describe geometric properties of the fork. We need information beyond the first five steps only for \(\tilde{P}^q_{qe}\) and \(\tilde{P}^{-1}_{qq}\). In the present step 6 we will take care of \(\tilde{P}^{-1}_{qe}\). For this, we only have to complete step 4 by the calculation of \(Q_{pe}\) and \(P_{qe}\), which turn out to be

\[
\begin{align*}
Q_{pe} &= \frac{1}{V_2V_3}x^3_3 x^4_2 + \frac{1}{V_2}x^3_{32} \\
P_{qe} &= -\frac{V_{11}}{V_2V_3}x^1_1 x^4_2 + \frac{1}{V_2}x^3_{12}. \tag{78}
\end{align*}
\]
similar to (73), with \( x^1_{32} \) and \( x^3_{12} \) determined by integration (67) with inhomogeneities given by (74).

**Step 7.** To calculate \( \tilde{P}_{qqq} \), we will need the third order partial derivatives of \( Q \) and \( P \) by \( q \) and \( p \). As a preparation we will now extend steps 4 and 6 to the determination of all \( x^r_{\lambda\mu}(t) \) for \( r, \lambda, \mu \in \{1, 2, 3, 4\} \) by (65) and (67), (68). In all these cases we have

\[
g^1_{\lambda\mu}(t) = g^2_{\lambda\mu}(t) = 0 \quad (79)
\]

and

\[
g^3_{\lambda\mu}(t) = \begin{cases} -V_{111}(t)x^1_{\lambda}(t)x^1_\mu(t) & \text{if } \lambda, \mu \in \{1, 3\} \\ -V_{112}(t)x^1_{\lambda}(t)x^2_\mu(t) & \text{if } \lambda \in \{1, 3\} \text{ and } \mu \in \{2, 4\} \\ 0 & \text{if } \lambda, \mu \in \{2, 4\} \end{cases} \quad (80)
\]

\[
g^4_{\lambda\mu}(t) = \begin{cases} -V_{112}(t)x^1_{\lambda}(t)x^1_\mu(t) & \text{if } \lambda, \mu \in \{1, 3\} \\ 0 & \text{if } \lambda \in \{1, 3\} \text{ and } \mu \in \{2, 4\} \\ -V_{222}(t)x^2_{\lambda}(t)x^2_\mu(t) & \text{if } \lambda, \mu \in \{2, 4\} \end{cases}
\]

Note that therefore

\[
x^1_{\lambda\mu}(t) = x^3_{\lambda\mu}(t) = 0 \quad \text{if } \lambda, \mu \in \{2, 4\} \quad \text{and}
\]

\[
x^2_{\lambda\mu}(t) = x^4_{\lambda\mu}(t) = 0 \quad \text{if } \lambda \in \{1, 3\} \quad \text{and} \quad \mu \in \{2, 4\} \quad (81)
\]

To calculate the other functions \( x^r_{\lambda\mu}(t) \) for \( r, \lambda, \mu \in \{1, 2, 3, 4\} \) by (67) and (68) would be step 7.

**Step 8.** It is now a suitable moment to improve upon (42) and write out the second order flow time derivatives as

\[
Z^0_{11} = \frac{1}{V_2} x^4_{11} - \frac{V_{11}}{V_2 V_2} x^4_{2}
\]

\[
Z^0_{12} = \frac{1}{V_2} x^4_{13}
\]

\[
Z^0_{22} = \frac{1}{V_2} x^4_{33} - \frac{1}{V_2 V_2} x^4_{2}
\]

\[
Z^0_{13} = 0
\]

\[
Z^0_{23} = 0 ,
\]

again at \((0, 0, \varepsilon_0, 0)\), with the \( x^4_{\ast} \) to be taken at \( t = T(\varepsilon_0) \) in the notation introduced with (47). We will need these numbers when we apply the third equation of (25) to calculate \( Q_{qqq}, Q_{qpp}, \ldots, P_{ppp} \). But before we can do this, we have to determine the \( x^r_{\lambda\mu\nu}(T(\varepsilon_0)) \) for \( r, \lambda, \mu, \nu \in \{1, 2, 3, 4\} \).
Step 9. This step is parallel to and based on step 7. We read the inhomogeneities $g_{\lambda\mu\nu}(t)$ from the second equation of (65) and then refer to (67), (68) to determine the $x_{\lambda\mu\nu}^{r}(t)$ by integration. Again

$$g_{\lambda\mu}^{1}(t) = g_{\lambda\mu}^{2}(t) = 0$$ (83)

for all $\lambda, \mu, \nu \in \{1, 2, 3, 4\}$, and the $g_{\lambda\mu\nu}^{3}(t)$ are given in the four cases a)-d) as follows.

a) If $\lambda, \mu, \nu \in \{1, 3\}$, then

$$g_{\lambda\mu\nu}^{3}(t) = -V_{1111}(t)x_{\lambda}^{1}(t)x_{\mu}^{1}(t)x_{\nu}^{1}(t)$$

$$-V_{111}(t)(x_{\lambda}^{1}(t)x_{\mu}^{1}(t) + x_{\mu}^{1}(t)x_{\nu\lambda}^{1}(t) + x_{\nu}(t)x_{\lambda\mu}^{1}(t))$$

$$-V_{112}(t)(x_{\lambda}^{1}(t)x_{\mu\nu}^{2}(t) + x_{\mu}^{1}(t)x_{\nu\lambda}^{2}(t) + x_{\nu}(t)x_{\lambda\mu}^{2}(t)),$$ (84)

b) if $\lambda, \mu \in \{1, 3\}$ and $\nu \in \{2, 4\}$, then

$$g_{\lambda\mu\nu}^{3}(t) = -V_{1112}(t)x_{\lambda}^{1}(t)x_{\mu}^{1}(t)x_{\nu}^{2}(t)$$

$$-V_{111}(t)(x_{\lambda}^{1}(t)x_{\mu\nu}^{1}(t) + x_{\mu}^{1}(t)x_{\nu\lambda}^{1}(t))$$

$$-V_{112}(t)x_{\nu}^{2}(t)x_{\lambda\mu}^{1}(t),$$ (85)

c) if $\lambda \in \{1, 3\}$ and $\mu, \nu \in \{2, 4\}$, then

$$g_{\lambda\mu\nu}^{3}(t) = -V_{1122}(t)x_{\lambda}^{1}(t)x_{\mu\nu}^{2}(t)x_{\nu}^{2}(t)$$

$$-V_{112}(t)(x_{\lambda}^{1}(t)x_{\mu\nu}^{1}(t) + x_{\mu}^{2}(t)x_{\nu\lambda}^{1}(t) + x_{\nu}(t)x_{\lambda\mu}^{1}(t)),$$ (86)

d) if $\lambda, \mu, \nu \in \{2, 4\}$, then

$$g_{\lambda\mu\nu}^{3}(t) = 0.$$ (87)

Similarly the inhomogeneities $g_{\lambda\mu\nu}^{4}(t)$:

a) If $\lambda, \mu, \nu \in \{1, 3\}$, then

$$g_{\lambda\mu\nu}^{4}(t) = -V_{1112}(t)x_{\lambda}^{1}(t)x_{\mu}^{1}(t)x_{\nu}^{1}(t)$$

$$-V_{112}(t)(x_{\lambda}^{1}(t)x_{\mu\nu}^{1}(t) + x_{\mu}^{1}(t)x_{\nu\lambda}^{1}(t) + x_{\nu}(t)x_{\lambda\mu}^{1}(t)),$$ (88)

b) if $\lambda, \mu \in \{1, 3\}$ and $\nu \in \{2, 4\}$, then

$$g_{\lambda\mu\nu}^{4}(t) = -V_{1122}(t)x_{\lambda}^{1}(t)x_{\mu\nu}^{2}(t)x_{\nu}^{2}(t)$$

$$-V_{112}(t)(x_{\lambda}^{1}(t)x_{\mu\nu}^{1}(t) + x_{\mu}^{2}(t)x_{\nu\lambda}^{1}(t))$$

$$-V_{222}(t)x_{\nu}^{2}(t)x_{\lambda\mu}^{2}(t),$$ (89)

c) if $\lambda \in \{1, 3\}$ and $\mu, \nu \in \{2, 4\}$, then

$$g_{\lambda\mu\nu}^{4}(t) = 0,$$ (90)
d) if \( \lambda, \mu, \nu \in \{2, 4\} \), then

\[
g_{\lambda\mu\nu}^4(t) = -V_{2222}(t)x_2^2(t)x_2^2(t)x_2^2(t)
- V_{2222}(t)(x_2^2(t)x_{\mu\nu}(t) + x_2^2(t)x_{\nu\lambda}(t) + x_2^2(t)x_{\lambda\mu}(t)).
\]  

(91)

Note that in particular

\[
x_{1\lambda\mu}(t) = x_{1\lambda\mu}(t) = 0 \quad \text{if} \quad \lambda, \mu, \nu \in \{2, 4\}
\]

\[
x_{2\lambda\mu}(t) = x_{2\lambda\mu}(t) = 0 \quad \text{if} \quad \lambda \in \{1, 3\} \quad \text{and} \quad \mu, \nu \in \{2, 4\}.
\]  

(92)

The other functions \( x_{r\lambda\mu}(t) \) for \( r, \lambda, \mu, \nu \in \{1, 2, 3, 4\} \) have to be calculated by integrations (67) and (68) in step 9.

**Step 10.** We can now write down the eight missing numbers \( Q_{qqq}, Q_{qpp}, \ldots, P_{ppp} \), thereby completing the first part of our program, the part that is not concerned with deformation. The \( Z_{\ell m}^2 \) and \( Z_{\ell m}^0 \) with \( \ell, m \in \{1, 2\} \) in the formulas are taken from (36) and (82). The third equation of (25) gives:

\[
\begin{align*}
Q_{qqq} &= x_{111}^1 + 3x_{121}^1 Z_{11}^2 + 3x_{11}^3 Z_{11}^0 \\
Q_{qpp} &= x_{113}^1 + x_{121}^1 Z_{12}^1 + x_{32}^3 Z_{11}^0 + 2x_{1}^1 Z_{12}^0 \\
Q_{bpp} &= x_{133}^1 + x_{122}^1 Z_{22}^1 + x_{1}^3 Z_{22}^0 + 2x_{3}^1 Z_{12}^0 \\
Q_{ppp} &= x_{333}^1 + 3x_{322}^1 Z_{22}^1 + 3x_{1}^3 Z_{22}^0.
\end{align*}
\]  

(93)

at \( t = T(\varepsilon_0) \).

12 Deformation

**Step 11.** Choose a deformation term \( F(x, y, p_x, p_y) \) satisfying the libration preserving condition (6). No new ‘numerical prerequisites’ are required, we can start right away calculating \( x_5^1(t) \) and \( x_5^3(t) \) from (62), which is the same as (68) with \( g_5^2(t) = F_4(t) \) and \( g_5^2(t) = -F_2(t) \). For the other \( x_5^1(t) \) see (50) and (61).

**Step 12.** Now we can determine the sixteen functions \( x_{r\lambda}(t) \) for \( r, \lambda \in \{1, 2, 3, 4\} \). Again we derive the corresponding \( g_{r\lambda}(t) \) from (65). It turns out that \( g_{r\lambda}(t) = 0 \) if \( r + \lambda \) is odd and hence also

\[
x_{r\lambda}(t) = 0 \quad \text{if} \quad r + \lambda \quad \text{is odd}.
\]  

(94)

For \( \lambda \in \{1, 3\} \) we get

\[
\begin{align*}
g_{1\lambda}(t) &= F_{11}(t)x_{1}(t) + F_{13}(t)x_{3}(t) \\
g_{3\lambda}(t) &= -F_{11}(t)x_{1}(t) - F_{13}(t)x_{3}(t) - V_{112}(t)x_{1}(t)x_{3}(t),
\end{align*}
\]  

(95)
and if \( \lambda \in \{2, 4\} \), then
\[
\begin{align*}
    g_{2\lambda}^1(t) &= F_{24}(t)x_\lambda^2(t) + F_{44}(t)x_\lambda^4(t) \\
    g_{4\lambda}^1(t) &= -F_{22}(t)x_\lambda^2(t) - F_{24}(t)x_\lambda^4(t) - V_{222}(t)x_\lambda^2(t)x_\lambda^2(t).
\end{align*}
\] (96)

**Step 13.** Now we collect the needed flow time and starting point derivatives at \((0, 0, \varepsilon_0, 0)\) that involve the deformation parameter \( u^4 = \delta \), namely the \( Z_4^\lambda \) and \( Z_{\ell4}^\lambda \) for \( \lambda = 0, 2 \) and \( \ell = 1, 2 \). From (36), (41) and (42) we get
\[
Z_4^2 = -\frac{F}{V_2},
\]
\[
Z_4^0 = \frac{1}{V_2}x_5^4 - \frac{F}{V_2V_2}x_2^4
\] (97)
and
\[
\begin{align*}
    Z_{14}^2 &= Z_{24}^2 = 0 \\
    Z_{14}^0 &= \frac{1}{V_2}x_{15}^4 \\
    Z_{24}^0 &= \frac{1}{V_2}x_{35}^4
\end{align*}
\] (98)
at \((0, 0, \varepsilon_0, 0)\).

**Step 14.** We can now calculate those second derivatives of the Poincaré map in which the deformation parameter is involved:
\[
\begin{align*}
    Q_{q\delta} &= x_{15}^1 + x_{12}^1Z_4^2 + x_1^3Z_4^0 \\
    Q_{p\delta} &= x_{35}^1 + x_{32}^1Z_4^2 + x_3^3Z_4^0 \\
    P_{q\delta} &= x_{15}^3 + x_{12}^3Z_4^2 - V_{11}x_{1}^1Z_4^0 \\
    P_{p\delta} &= x_{35}^3 + x_{32}^3Z_4^2 - V_{11}x_{3}^1Z_4^0
\end{align*}
\] (99)
at \( t = T(\varepsilon_0) \). But before we reach the third derivatives of \( Q \) and \( P \) involving \( \delta \), we have to take one more step.

**Step 15.** We have to determine the \( x_{\lambda\mu5}^\lambda(T(\varepsilon_0)) \) for \( \lambda, \mu \in \{1, 2, 3, 4\} \), again by integration (67) and (68), with inhomogeneities \( g_{\lambda\mu5}^\lambda(t) \) as follows.

1a) If \( \lambda, \mu \in \{1, 3\} \), then
\[
\begin{align*}
    g_{1\mu5}^\lambda(t) &= F_{113}(t)x_\lambda^1(t)x_\mu^1(t) \\
    &+ F_{133}(t)\left[x_\lambda^1(t)x_\mu^3(t) + x_\lambda^3(t)x_\mu^1(t)\right] \\
    &+ F_{333}(t)x_\mu^3(t)x_\mu^3(t) \\
    &+ F_{13}(t)x_\mu^1(t) + F_{33}(t)x_\mu^3(t),
\end{align*}
\] (100)
1b) if \( \lambda \in \{1, 3\} \) and \( \mu \in \{2, 4\} \), then
\[
g_{\lambda \mu 5}^1(t) = F_{233}(t)x_3^3(t)x_\mu^3(t) + F_{334}(t)x_3^3(t)x_\mu^4(t)
+ F_{123}(t)x_1^3(t)x_\mu^2(t) + F_{134}(t)x_1^3(t)x_\mu^4(t)
+ F_{13}(t)x_{1\mu}(t) + F_{33}(t)x_{3\mu}(t),
\]
(101)

1c) if \( \lambda, \mu \in \{2, 4\} \), then
\[
g_{\lambda \mu 5}^1(t) = 0.
\]
(102)

2a) If \( \lambda, \mu \in \{1, 3\} \), then
\[
g_{\lambda \mu 5}^2(t) = F_{114}(t)x_1^1(t)x_\mu^1(t)
+ F_{134}(t)\left[x_1^1(t)x_\mu^3(t) + x_1^3(t)x_\mu^1(t)\right]
+ F_{334}(t)x_3^3(t)x_\mu^3(t)
+ F_{24}(t)x_{3\mu}(t) + F_{44}(t)x_{4\mu}(t),
\]
(103)

2b) if \( \lambda \in \{1, 3\} \) and \( \mu \in \{2, 4\} \), then
\[
g_{\lambda \mu 5}^2(t) = 0,
\]
(104)

2c) if \( \lambda, \mu \in \{2, 4\} \), then
\[
g_{\lambda \mu 5}^2(t) = F_{224}(t)x_2^3(t)x_\mu^2(t)
+ F_{244}(t)\left[x_2^3(t)x_\mu^4(t) + x_2^4(t)x_\mu^2(t)\right]
+ F_{444}(t)x_{4\mu}(t) + F_{44}(t)x_{4\mu}(t),
\]
(105)

3a) If \( \lambda, \mu \in \{1, 3\} \), then
\[
g_{\lambda \mu 5}^3(t) = -V_{1112}(t)x_1^1(t)x_\mu^2(t)x_5^2(t)
- V_{111}(t)x_1^1(t)x_\mu^1(t)x_5^3(t)
- V_{112}(t)x_1^3(t)x_\mu^1(t)x_5^2(t) + x_5^3(t)x_\mu^1(t)
- F_{111}(t)x_{1\mu}(t)x_1^1(t)
- F_{113}(t)x_1^3(t)x_\mu^1(t)
- F_{133}(t)x_3^3(t)x_\mu^3(t)
- F_{11}(t)x_{1\mu}(t) - F_{13}(t)x_{3\mu}(t),
\]
(106)
3b) if \( \lambda \in \{1, 3\} \) and \( \mu \in \{2, 4\} \), then
\[
g^3_{\lambda \mu 5}(t) = -V_{1122}(t)x^1_\lambda(t)x^2_\mu(t)x^3_5(t) \\
- V_{112}(t) \left[ x^1_\lambda(t)x^2_\mu(t) + x^2_\mu(t)x^1_5(t) + x^2_5(t)x^1_\lambda(t) \right] \\
- F_{112}(t)x^1_\lambda(t)x^2_\mu(t) - F_{114}(t)x^4_\lambda(t)x^1_\mu(t) \\
+ F_{123}(t)x^3_\lambda(t)x^2_\mu(t) + F_{134}(t)x^3_\lambda(t)x^4_\mu(t) \\
- F_{11}(t)x^4_\lambda(t) - F_{13}(t)x^3_\mu(t),
\]
(107)

3c) if \( \lambda, \mu \in \{2, 4\} \), then
\[
g^3_{\lambda \mu 5}(t) = 0.
\]
(108)

4a) If \( \lambda, \mu \in \{1, 3\} \) then
\[
g^4_{\lambda \mu 5}(t) = -V_{1122}(t)x^1_\lambda(t)x^1_\mu(t)x^2_5(t) \\
- V_{112}(t) \left[ x^1_\lambda(t)x^1_\mu(t) + x^1_\mu(t)x^1_5(t) \right] \\
- V_{222}(t)x^2_\lambda(t)x^2_\mu(t) \\
- F_{112}(t)x^1_\lambda(t)x^1_\mu(t) - F_{233}(t)x^3_\lambda(t)x^3_\mu(t) \\
- F_{123}(t) \left[ x^1_\lambda(t)x^3_\mu(t) + x^3_\lambda(t)x^1_\mu(t) \right] \\
- F_{22}(t)x^2_\lambda(t) - F_{24}(t)x^4_\mu(t),
\]
(109)

4b) if \( \lambda \in \{1, 3\} \) and \( \mu \in \{2, 4\} \), then
\[
g^4_{\lambda \mu 5}(t) = -V_{222}(t)x^2_\mu(t)x^2_5(t),
\]
(110)

4c) if \( \lambda, \mu \in \{2, 4\} \), then
\[
g^4_{\lambda \mu 5}(t) = -V_{2222}(t)x^2_\lambda(t)x^2_\mu(t)x^2_5(t) \\
- V_{222}(t) \left[ x^2_\lambda(t)x^2_\mu(t) + x^2_\mu(t)x^2_5(t) + x^2_5(t)x^2_\lambda(t) \right] \\
- F_{222}(t)x^2_\lambda(t)x^2_\mu(t) \\
- F_{224}(t) \left[ x^2_\lambda(t)x^4_\mu(t) + x^4_\lambda(t)x^2_\mu(t) \right] \\
- F_{244}(t)x^4_\lambda(t)x^4_\mu(t) \\
- F_{22}(t)x^2_\lambda(t) - F_{24}(t)x^4_\mu(t),
\]
(111)

Note that in particular
\[
x^1_{\lambda \mu 5}(t) = x^3_{\lambda \mu 5}(t) = 0 \quad \text{if} \quad \lambda, \mu \in \{2, 4\}.
\]
(112)
Step 16. This then will be the last step in the procedure to determine the 38 numbers we set out to calculate. Only six of them remain, and these are given by

\begin{align*}
Q_{qq\delta} &= x_{115}^1 + x_{112}^1 Z_4^2 + x_{111}^1 Z_4^0 + 2x_{11}^3 Z_{14}^0 \\
Q_{qp\delta} &= x_{135}^1 + x_{123}^1 Z_4^2 + x_{13}^3 Z_4^0 + x_{1}^3 Z_{14}^0 + x_{1}^3 Z_{24}^0 \\
Q_{pp\delta} &= x_{335}^1 + x_{233}^1 Z_4^2 + x_{33}^3 Z_4^0 + 2x_{3}^3 Z_{24}^0 \\
P_{qq\delta} &= x_{115}^3 + x_{112}^3 Z_4^2 - V_{11} x_{11}^1 Z_4^0 - 2V_{11} x_{1}^1 Z_{14}^0 \\
P_{qp\delta} &= x_{135}^3 + x_{123}^3 Z_4^2 - V_{11} x_{13}^1 Z_4^0 - V_{11} (x_{3}^1 Z_{14}^0 + x_{1}^1 Z_{24}^0) \\
P_{pp\delta} &= x_{335}^3 + x_{233}^3 Z_4^2 - V_{11} x_{13}^1 Z_4^0 - 2V_{11} x_{3}^1 Z_{24}^0
\end{align*}

(113)

at \( t = T(\varepsilon_0) \), as before, or explicitly: the left hand sides are meant to be taken at \((q,p,\varepsilon,\delta) = (0,0,\varepsilon_0,0)\), thus on the right hand sides the \( x^r_*(t) \) at \( t = T(\varepsilon_0) \), the \( V_*(x,y) \) at \((x,y) = (0,y_{\text{max}}(\varepsilon_0))\), and the \( Z^r_\ast \) are also to be taken at \((q,p,\varepsilon,\delta) = (0,0,\varepsilon_0,0)\), as in (41), (42), (82), (97), (98) where they have been calculated from \( x^r_*(T(\varepsilon_0)) \), \( V_*(0,y_{\text{max}}(\varepsilon_0)) \) and various \( F_*(0,y_{\text{max}}(\varepsilon_0),0,0) \).

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