GEODESICS AND ISOMETRIC IMMERSIONS IN KIRIGAMI

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Abstract. Kirigami is the art of cutting paper to make it articulated and deployable, allowing for it to be shaped into complex two and three-dimensional geometries. The mechanical response of a kirigami sheet when it is pulled at its ends is enabled and limited by the presence of cuts that serve to guide the possible non-planar deformations. Inspired by the geometry of this art form, we ask two questions: (i) What is the shortest path between points at which forces are applied? (ii) What is the nature of the ultimate shape of the sheet when it is strongly stretched?

Mathematically, these questions are related to the nature and form of geodesics in the Euclidean plane with linear obstructions (cuts), and the nature and form of isometric immersions of the sheet with cuts when it can be folded on itself. We provide a constructive proof that the geodesic connecting any two points in the plane is piecewise polygonal. We then prove that the family of polygonal geodesics can be simultaneously rectified into a straight line by flat-folding the sheet so that its configuration is a (non-unique) piecewise affine planar isometric immersion.

1. Introduction

A thin rectangular sheet of paper pulled at its corners is almost impossible to stretch. Introducing a cut in its interior changes its topology, and thence changes its physical response. The corners can now be pulled apart as the sheet bends out of the plane, see Figure 1.1. The physical reason for this is that the geometric scale-separation associated with a sheet of thickness \( h \) and size \( L \) (where \( h \ll L \)), makes it energetically expensive to stretch and easy to bend, since the elastic potential energy of the sheet per unit area can be written as:

\[
U = Eh(\text{stretching strain})^2 + Eh^3(\text{curvature})^2,
\]

where the stretching strain and curvature characterize the modes of deformation of the sheet, and \( E \) is the elastic modulus of the material. Thus, as \( h/L \to 0 \), for given boundary conditions it is energetically cheaper to deform by bending (curving) rather than stretching, as can be observed readily with any thin sheet of any material. This observation and its generalizations are behind the Sino-Japanese art of kirigami (kiri = cut, gami = paper). Recently, this ability to make cuts in a sheet of paper that allow it to be articulated and deployed into complex two and three-dimensional patterns has become the inspiration for a new class of mechanical metamaterials [3, 1]. The geometrical and topological properties of the slender sheet-like structures, irrespective of their material constituents, can then be exploited to create functional structures on scales ranging from the nanometric [2] to centimetric and beyond [4, 5, 6].

Of the various mathematical and physical questions that arise from this ability to control the configurational degrees of freedom of the sheet using the geometry and topology of the cuts, perhaps the simplest is the following: if a sheet with random cuts was pulled at two points on the boundary, what is the nature of paths of stress transmission through the sheet? In the absence of cuts, the lines of force transmission are straight lines connecting the points, i.e. geodesics, but this needs to be revisited in the presence of obstructing cuts. One might ask about the nature of the paths of force transmission, i.e. the geodesics in this situation. The results of qualitative experiments with a sheet
of paper that has a single cut along the perpendicular bisector to the line joining the points of force application, are shown in Figure 1.1. For small forces, the sheet deforms into two conical regions that allow the edge of the cut to curve out of the plane, and when the forces are large enough, the ends of the cut become approximately collinear with the line joining the points of forcing. Observations of sheets with multiple cuts are suggestive of a generalization, namely that cuts cause the sheet to buckle out of the plane until a straight geodesic in $\mathbb{R}^3$ connects the points of force application. Furthermore, as the sheet thickness becomes vanishingly small, allowing the sheet to form sharp creases with a large curvature, the sheet can fold on itself and become flat again, as seen in Figure 1.1 (iii).

These observations suggest two conjectures:

(i) geodesics in a planar sheet with cuts are piecewise linear, i.e. they are polygonals;
(ii) on pulling at two points in a sheet with cuts, these polygonal geodesics straighten out by allowing the sheet to deform in the third dimension, which when flat-folded causes the geodesic to be rectified, leading to a configuration that is a piecewise affine isometric immersion.

Here, we prove the above two statements.

We point out that a combination of physical and numerical experiments can be used to characterize the geometric mechanics of kirigamized sheets as a function of the number, size, and orientation of cuts. This will be the topic of our forthcoming work [4], which in particular shows that by varying the geodesic lengths, one can shape the deployment trajectory of a sheet as a composition of developable units: flats, cylinders and cones, as well as control its compliance across orders of magnitude.
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## 2. The set-up and the main results of this paper

Let $\Omega \subset \mathbb{R}^2$ be a convex, bounded, planar domain and let $L$ be the union of finitely many closed segments contained in $\Omega$. We study the geodesic distance and the structure of geodesics in $\Omega \setminus L$.

Specifically, we work under the following setup:

In an open, bounded, convex set $\Omega \subset \mathbb{R}^2$, given is a graph $G$, consisting of $\bar{n} \geq 2$ vertices $V = \{a_i\}_{i=1}^{\bar{n}}$ and $n \geq 1$ edges $E = \{l_i\}_{i=1}^{n}$, represented by:

(S) \[ l_i = \{(1-t)a_j + ta_k; \; t \in [0,1]\} \text{ for some } a_j \neq a_k \in V. \]

We denote $L = \bigcup_{i=1}^{n} l_i$ and call $L$ the set of cuts. Without loss of generality, we further assume that $G$ is a planar graph, i.e. for all $i \neq j$ the intersection $l_i \cap l_j$ is either empty or consists of a single point that is a common vertex of $l_i$ and $l_j$.

The collection of cuts in $L$ is thus finite but completely arbitrary, i.e. the cuts may have any length and orientation, and are allowed to intersect each other.

We next define:

(G) \[ \text{Given two points } p \neq q \text{ that belong to the same connected component of } \bar{\Omega} \setminus L, \text{ we set:} \]

\[ \text{dist}(p,q) = \inf \{\text{length}(\tau); \; \tau : [0,1] \to \bar{\Omega} \setminus L \text{ piecewise } C^1 \text{ with } \tau(0) = p, \; \tau(1) = q\}. \]

Further, any piecewise $C^1$ curve $\sigma : [0,1] \to \bar{\Omega} \setminus L$ with $\sigma(0) = p, \; \sigma(1) = q$ is called a geodesic from $p$ to $q$ in $\bar{\Omega} \setminus L$, provided that:

1. $\text{length}(\sigma) = \text{dist}(p,q)$,
2. $\sigma$ is the uniform limit as $k \to \infty$, of a sequence of piecewise $C^1$ curves $\{\tau_k : [0,1] \to \bar{\Omega} \setminus L\}_{k=1}^\infty$, each satisfying $\tau_k(0) = p, \; \tau_k(1) = q$.

The above definition abuses the notion of a geodesic slightly, because it allows $\sigma$ to be not entirely contained in $\bar{\Omega} \setminus L$ (although we call it a geodesic in $\bar{\Omega} \setminus L$), that is, we allow cuts to be parts of $\sigma$. Figures 2.1 and 2.2 show a few examples of $G, p, q$ and the resulting geodesics. Here and below, by $p a_{i_1} a_{i_2} \ldots a_{i_k} q$ we denote the polygonal joining $p$ and $q$ through the consecutive points $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$.

Our first result is as follows:

**Theorem 2.1.** Assume (S) and let $p, q \in \bar{\Omega} \setminus L$ belong to one connected component of $\bar{\Omega} \setminus L$. Then there exists at least one geodesic from $p$ to $q$, as defined in (G). Each such geodesic $\sigma$ satisfies:

(i) $\sigma$ is a finite polygonal joining $p$ and $q$, with all its other vertices distinct and chosen from $V$,

(ii) for each $i = 1 \ldots n$, if $\sigma \cap l_i \neq \emptyset$ then either $l_i \subset \sigma$ or $\sigma \cap l_i \subset \{a_j, a_k\}$, where $l_i = a_j a_k$.

Our second result and the main contribution of this paper, is motivated by the general considerations in Section 1. We prove the existence of an isometric immersion of $\bar{\Omega} \setminus L$ into $\mathbb{R}^3$, which bijectively maps each geodesic between two chosen boundary points $p, q$ onto one segment in $\mathbb{R}^3$ of appropriate length (see Figure 2.3). More precisely, we have:

**Theorem 2.2.** Assume (S) and let $p, q \in \partial \bar{\Omega}$. Then, there exists a continuous and piecewise affine map $u : \bar{\Omega} \setminus L \to \mathbb{R}^3$ with the following properties:
Figure 2.1. Three configurations of $G, p, q$ with pairwise nonintersecting cuts: (i) $p = (0, -1), q = (1, 0), a_1 = (0, 1), a_2 = (0, -1)$ yield two geodesics: $\sigma_1 = p a_1 q, \sigma_2 = p a_2 q$; (ii) $p = (0, 0), q = (4, 0), a_1 = (2, (2^{3/2} - 1)^{1/2}), a_2 = (1, -1), a_3 = (3, -1), a_4 = (4, 0)$ yield three geodesics: $\sigma_1 = p a_1 q, \sigma_2 = p a_2 a_3 q, \sigma_3 = p a_2 a_4 q$; (iii) $p = (-1, 0), q = (1, 0), a_1 = (0, 1), a_4 = (0, -1)$ with $a_2, a_3 = (0, \pm \epsilon), a_5, a_6, a_7, a_8 = (\pm \epsilon, \pm (1 - \epsilon^{1/2}))$ for a sufficiently small $\epsilon > 0$ yield two geodesics: $\sigma_1 = p a_1 q, \sigma_2 = p a_4 q$.

Figure 2.2. Three configurations of $G, p, q$ with intersecting cuts: (i) $n = 4, \bar{n} = 5$ result in two geodesics: $\sigma_1 = p a_1 q, \sigma_2 = p a_5 q$, this configuration is minimal in the sense introduced in Section 4; (ii) $n = 7, \bar{n} = 6$ and two geodesics: $\sigma_1, \sigma_2$; (iii) $n = 4, \bar{n} = 5$ with $\text{length}(a_1 a_2 p) = \text{length}(p a_3)$, this is also a minimal configuration resulting in two geodesics $\sigma_1, \sigma_2$.

(i) $u$ is an isometry, i.e.: $(\nabla u)^T \nabla u = Id_2$ almost everywhere in $\Omega \setminus L$,
(ii) the image $u(\sigma)$ of every geodesic $\sigma$ from $p$ to $q$ in $\Omega \setminus L$, coincides with the segment $u(p)u(q)$.
In particular, $|u(p) - u(q)| = \text{length}(\sigma)$ for each geodesic $\sigma$ (as defined in (G)).

We prove Theorem 2.1 in section 3 and Theorem 2.2 in sections 4-8. Our proofs are constructive and describe: a specific algorithm to find the polygonal geodesics in Theorem 2.1, and a folding procedure that yields the isometric immersion $u$ in Theorem 2.2. Even when all cuts in $L$ are non-intersecting (i.e. the edges of the underlying graph $G$ are pairwise disjoint), the construction of $u$ is far from obvious. The general case requires a further refinement of the previous arguments, because of the completely arbitrary planar geometry of each connected component of $G$. 
The algorithm that yields the isometry $u$ in Theorem 2.2 consists of:

(i) identifying and sealing the portions of inessential cuts, which do not affect $\text{dist}(p, q)$;
(ii) ordering the geodesics and ordering the remaining cuts, that now form a new planar graph $G$ consisting of trees (i.e. $G$ is a forest);
(iii) constructing $u$ on each region between two consecutive trees and two consecutive geodesics;
(iv) constructing $u$ on regions within each tree;
(v) constructing $u$ on the exterior region that is not enclosed by any two geodesics.

The points (i) and (ii) above are introduced in sections 4 and 5, respectively. The main arguments towards (iii) in the simplified setting are presented in section 6. The general case is resolved in section 7, which carries the heaviest technical load of this paper. Section 8 completes the proofs and presents an example explaining the necessity of $p, q$ being located on the boundary of $\Omega$ in Theorem 2.2.

3. Proof of Theorem 2.1

Given $p, q \in \bar{\Omega} \setminus L$ and a piecewise $C^1$ curve $\tau : [0, 1] \to \Omega \setminus L$ with $\tau(0) = p$, $\tau(1) = q$, we first demonstrate a general procedure to produce a finite polygonal $\sigma$ which joins $p$ and $q$, whose other vertices are (not necessarily distinct) points in $V$, which satisfies condition (ii) in (G), and such that:

$$\text{length}(\sigma) \leq \text{length}(\tau).$$

Applying this procedure to curves $\tau$ with $\text{length}(\tau) \leq \text{dist}(p, q) + 1$ yields a family of polygonals with the listed properties, each of them having number of edges bounded by:

$$\frac{\text{dist}(p, q) + 1}{\min_{a_j \neq a_k} |a_j - a_k|}.$$

Hence, all geodesics from $p$ to $q$ in $\Omega \setminus L$ are precisely the length-minimizing polygons among such (finitely many) polygonals. We further show that any length-minimizing polygonal satisfying condition (ii) of (G) cannot pass through the same vertex in $V$ multiple times. Theorem 2.1 is then a direct consequence of these statements.

Without loss of generality, the path $\tau$ has no self-intersections. We construct $\sigma$ by successive replacements of portions of $\tau$ by segments, as follows:
1. It is easy to show that for all \( t > 0 \) sufficiently small there holds: \( p\tau(t) \subset \Omega \setminus L \). Set \( \tau_1 = \tau \) and define:

\[
t_1 = \sup \{ t \in (0, 1); p\tau_1(s) \subset \Omega \setminus L \text{ for all } s \in (0, t) \}, \quad q_1 = \tau_1(t_1).
\]

There further holds: \( t_1 \in (0, 1) \) and \( p\tau_1 \) is a geodesic from \( p \) to \( q_1 \). If \( q_1 = q \) then we set \( \sigma = p\tau_1 \) and stop the process. Otherwise, by construction, the segment \( p\tau_1 \) must contain some of the vertices in \( V \). Call \( p_1 \) the closest one of these points to \( q_1 \) and note that \( p_1 \neq q_1 \). Consider the concatenation of the segment \( p\tau_1q_1 \) and the curve \( \tau_1|_{[t_1, 1]} \). After re-parametrisation, it yields a piecewise \( C^1 \) curve \( \tau_2 : [0, 1] \to \Omega \), with the property that \( \tau_2((0, 1]) \subset \Omega \setminus L \) and also:

\[
|p - p_1| + \text{length}(\tau_2) \leq \text{length}(\tau).
\]

2. We inductively define a finite sequence of endpoints \( \{p_i\}_{i=2}^k \subset V \) and a sequence of piecewise \( C^1 \) curves \( \{\tau_i : [0, 1] \to \Omega\}_{i=2}^{k+1} \), by applying the procedure in Step 1 to curve \( \tau_i \) and points \( \tau_i(t_i) = p_i \) and \( q_i \), until \( q_{k+1} = q \) so that \( p\tau_iq_i \) is a geodesic from \( p_k \) to \( q \). Along the way, we get: \( \tau_i(0) = p_i \neq p_{i-1}, \quad \tau_i(1) = q_i \), \( \tau_i((0, 1]) \subset \Omega \setminus L \), and:

\[
\text{the sequence } \left\{ |p - p_1| + \sum_{j=2}^{i} |p_j - p_{j-1}| + \text{length}(\tau_{i+1}) \right\}_{i=1}^{k} \text{ is non-increasing.}
\]

Also, the subset of \( \Omega \) enclosed by the concatenation of \( p\tau_1 \ldots p\tau_k \) with the portion of the curve \( \tau \) between \( p \) and \( q_i \), contains no cuts in its interior. Consequently, each polygonal \( p\tau_1 \ldots p\tau_k \) is a uniform limit of \( C^1 \) curves contained in \( \Omega \setminus L \).

3. We finally define: \( \sigma = p\tau_1 \ldots p\tau_k \).

The above process indeed terminates in a finite number \( k \) of steps, because the length of each polygonal \( p\tau_1 \ldots p\tau_k \) is bounded by \( \text{length}(\tau) \), and at each step this length increases by at least: \( \min_{a_j \neq a_k} |a_j - a_k| > 0 \). See Figure 3.1 for an example of \( L, p, q, \tau \) and the resulting polygonal \( \sigma \).

The following observation concludes the proof of Theorem 2.1

**Lemma 3.1.** Let \( \sigma \) be a geodesic from \( p \) to \( q \) in \( \Omega \setminus L \), as in (G). Then for every \( a_i \in V \), there holds \( a_i = \sigma(t) \) for at most one \( t \in (0, 1) \).

**Proof.** We argue by contradiction and assume that a geodesic polygonal \( \sigma \) passes through some vertex \( a_i \in V \) at least twice. Without loss of generality, we take \( a_i \) to be the first vertex in \( \sigma \) (counting from \( p \)) with this property. Consider the portion of \( \sigma \) containing the first and second occurrences of \( a_i \), namely: \( \overline{ai_0ai_1ai_2 \ldots ai_{j} ai_{i+1}} \), and consider the approximating curve \( \tau \) as in definition (G). From the approximate length-minimizing property of \( \tau \), it follows that both angles \( \angle(ai_0ai_1ai_2) \) and \( \angle(ai_{i} ai_{i+1}) \) must be at least \( \pi \). Consequently, they are both equal to \( \pi \). Another application of the same minimality condition yields that at least one of the segments \( \overline{ai_0ai_i} \) and \( \overline{ai_i ai_{i+1}} \) must be a cut in \( L \). This contradicts with \( p, q \notin L \) and \( a_i \) being the first multiple vertex of \( \sigma \).

4. **Proof of Theorem 2.2** Step 1: sealing the inessential cuts

Assume (S) and let \( p, q \) be two distinct points belonging to one connected component of \( \bar{\Omega} \setminus L \). We describe a procedure which “seals” portions of cuts in \( L \) without decreasing the geodesic distance
between \( p \) and \( q \) in \( \Omega \setminus L \). First, consider \( i = 1 \ldots n \) and \( j,k = 1 \ldots \bar{n} \) so that \( l_i = a_ja_k \). Given \( t \in [0,\text{length}(l_i)] \), define the altered endpoint of the cut \( l_i \):

\[
a_j(t) = (1-t)a_j + ta_k.
\]

Let \( L(t) \) be the new set of cuts in which \( l_i \) has been replaced by \( l_i(t) = a_j(t)a_k \), while all other cuts are left unchanged (this construction alters the \( V \) of the underlying graph \( G \) as well). We have the following observation:

**Lemma 4.1.** With the above notation, the geodesic distance between \( p \) and \( q \) in \( \Omega \setminus L(t) \):

\[
t \mapsto \text{dist}_t(p,q) = \inf \{ \text{length}(\tau); \ \tau: [0,1] \to \Omega \setminus L(t) \text{ piecewise } C^1, \ \text{with } \tau(0) = p, \tau(1) = q \}
\]

is left-continuous as a function of \( t \in [0,1] \), and right-continuous in \( t \in (0,1] \). It is also right-continuous at \( t = 0 \) when \( a_j \) is not the end-point of any other cut in \( L \) besides \( l_i \).

**Proof.** Step 1. To prove the asserted left-continuity, take a sequence \( \{t_m \in (0,1)\}_{m=1}^{\infty} \) that is strictly increasing to some \( t_0 > 0 \). It is clear that \( \text{dist}_{t_0}(p,q) \leq \liminf_{m \to \infty} \text{dist}_{t_m}(p,q) \), because \( \Omega \setminus L(t_0) \subset \Omega \setminus L(t_m) \) so that \( \text{dist}_{t_0}(p,q) \leq \text{dist}_{t_m}(p,q) \) for all \( m \).

For the reverse bound, fix \( \epsilon > 0 \) and let \( \tau: [0,1] \to \Omega \setminus L(t_0) \) be piecewise \( C^1 \) with \( \tau(0) = p, \tau(1) = q \), and such that \( \text{length}(\tau) \leq \text{dist}_{t_0}(p,q) + \epsilon \). We observe that if \( \tau \) intersects \( L(t_m) \), it must do so within \( l_i(t_m) \setminus l_i(t_0) \). Since for sufficiently large \( m \) there holds: \( \text{length}(l_i(t_m)) - \text{length}(l_i(t_0)) < \epsilon \), it follows that there exists \( \tau_{\epsilon}: [0,1] \to \Omega \setminus L(t_m) \) which is a local modification of \( \tau \), increasing its length by at most \( 2\epsilon \). Here, we are taking advantage of the fact that \( a_j(t_m) \) is not the endpoint of
any other cut besides $l_i(t_m)$ in $L(t_m)$. Consequently, we get:

$$\text{dist}_{\tau_m} (p,q) \leq \text{length} (\tau) \leq \text{dist}_{\tau_0} (p,q) + 3 \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, this implies: $\limsup_{m \to \infty} \text{dist}_{\tau_m} (p,q) \leq \text{dist}_{\tau_0} (p,q)$.

**Step 2.** To show right-continuity of the function $\text{dist}_{\tau} (p,q)$ at $t_0 \in (0,1)$, let $\{t_m \in (0,1)\}_{m=1}^\infty$ that is strictly decreasing to $t_0$. As in Step 1, we get: $\limsup_{m \to \infty} \text{dist}_{\tau_m} (p,q) \leq \text{dist}_{\tau_0} (p,q)$. In virtue of Theorem 2.1 for each $m$ there holds:

$$\text{dist}_{\tau_m} (p,q) = \text{length} (\overline{pa_{i_1,m}(t_m)a_{i_2,m}(t_m)\ldots a_{i_k(m),m}(t_m)q}),$$

where for $s \neq i$ we set $a_s(t) = a_s$. Since the number of finite sequences of distinct indices chosen among $\{1 \ldots n\}$ equals $\sum_{k=1}^n k!$ and it is finite, it follows that at least one of such sequences $(i_1, i_2 \ldots i_k)$ represents the order of the vertices in a geodesic polygonal as above, for infinitely many $t_m$-s. Passing to a subsequence if necessary, we may thus write:

$$\text{dist}_{\tau_m} (p,q) = \text{length} (\overline{pa_{i_1,m}(t_m)a_{i_2,m}(t_m)\ldots a_{i_k(m),m}(t_m)q}) \quad \text{for all} \quad m.$$

We emphasize that at most one of the vertices changes as $m \to \infty$ and all others remain fixed. Further, as $m \to \infty$, the geodesics $\overline{pa_{i_1,m}(t_m)a_{i_2}(t_m)\ldots a_{i_k}(t_m)q}$ converge to the polygonal $\sigma = \overline{pa_1 \ldots a_k q}$ that satisfies condition (ii) of (G). Consequently:

$$\lim_{m \to \infty} \text{dist}_{\tau_m} (p,q) = \text{length} (\sigma) \geq \text{dist}_{\tau_0} (p,q).$$

This concludes the proof of the lemma. The same argument is valid at $t_0 = 0$ under the indicated condition on $a_j$. 

**Figure 4.1.** Different minimal configurations resulting from the original graph $G$ in (i), obtained by the sealing procedure upon changing the order of edges in $E$ and vertices in $V$: (ii) and (iii) yield two geodesics, while (iv) and (v) yield three geodesics.

We now define an inductive procedure in which lengths of all cuts are decreased as much as possible. Any resulting configuration $G, V, L$ (see Figure 4.1 for examples) will be called *minimal*. 

1. Fix \( i = 1 \), write \( l_1 = \overline{a_ja_k} \) and define:
\[
   t_1 = \sup \{ t \in [0, \text{length}(l_1)]; \ \text{dist}_t(p, q) = \text{dist}_0(p, q) \},
\]
where \( \text{dist}_t(p, q) \) is as in Lemma 4.1. Replace the endpoint \( a_j \) by \( a_j(t_1) \), and replace the cut \( l_1 \) by the segment \( l_1(t_1) = \overline{a_j(t_1)a_k} \). If \( a_j(t_1) = a_k \) then we remove \( l_1 \) altogether. Consider the problem of finding an isometric immersion \( u_1 \) with properties (i), (ii) in Theorem 2.2, for the same points \( p, q \) but with \( L \) replaced by \( L_1 = L(t_1) \). Then \( u = u_1|_{\Omega \setminus L} \) is a continuous, piecewise affine map fulfilling Theorem 2.2.

2. Write now \( l_1 = \overline{a_ka_j} \) and let \( t_2 \) be defined as above, where we decrease the length of the already modified cut \( l_1 \) starting from the so far unaltered vertex \( a_k \), up to \( a_k(t_2) \). Replace \( l_1 \) by \( l_2(t_2) = \overline{a_k(t_2)a_j} \) or remove it altogether in case \( a_k(t_2) = a_j \). Call the new set of cuts \( L_2 \).

3. Having constructed \( L_{2i} \) for some \( 1 \leq i < n \), consider the next cut \( l_{i+1} = \overline{a_ja_k} \) and define:
\[
   t_{2i+1} = \sup \{ t \in [0, \text{length}(l_{i+1})]; \ \text{dist}_t(p, q) = \text{dist}_0(p, q) \},
\]
where \( \text{dist}_t(p, q) \) is taken with respect to the previously obtained set of cuts \( L_{2i} \). Replace the endpoint \( a_j \) by \( a_j(t_{2i+1}) \) and replace the cut \( l_{i+1} \subset L_{2i} \) by \( l_{i+1}(t_{2i+1}) \). This defines the new collection of cuts \( L_{2i+1} \subset L_{2i} \).

4. In the same manner, by possibly modifying the endpoint \( a_k \) of the already considered cut \( l_{i+1} \), we construct the new set of cuts \( L_{2i+2} \subset L_{2i+1} \).

5. We finally set:
\[
   \bar{L} = L_{2n}.
\]
As in Step 1 of the algorithm, this ultimate collection \( \bar{L} \subset L \) of cuts in \( \Omega \) has the property that the validity of Theorem 2.2 for the configuration \( p, q, \bar{L} \) implies its validity for the original configuration \( p, q, L \).

Informally speaking, the above procedure starts by moving the first endpoint vertex of \( l_1 \) toward its second vertex, whereas we start “sealing” the portion of the cut \( l_1 \) left behind. The length of the geodesics connecting \( p \) and \( q \) may drop initially, in which case we leave the configuration unchanged. Otherwise, the geodesic distance is continuously nonincreasing, in view of Lemma 4.1 (it may initially remain constant). We stop the sealing process when the aforementioned distance becomes strictly less than the original one, and label the new position point as the new vertex endpoint of \( l_1 \). In the next step, we move the remaining endpoint along \( l_1 \) toward the (new) first endpoint and repeat the process, thus possibly sealing the cut \( l_1 \) further. The procedure is carried out for each \( l_i \) in the given order \( i = 1, 2, \ldots, n \). We now claim that the distance between \( p \) and \( q \) cannot be further decreased, upon repeating the same process for the newly created configuration.

**Lemma 4.2.** With respect to the cuts in \( \bar{L} = \bigcup_{i=1}^n \bar{l}_i \), for any \( i = 1 \ldots n \), any of the endpoint vertices of \( \bar{l}_i \), and any \( t > 0 \) there holds:
\[
   \text{dist}_t(p, q) < \text{dist}_0(p, q).
\]

**Proof.** Denote \( d = \text{dist}_t(p, q) \) as above. At the \((2i - 1)\)-th step of construction of \( \bar{L} \), we have:
\[
   d \leq \text{dist}_{t_{2i-1}+t}(p, q),
\]
because \( d \) corresponds to the geodesic distance between \( p \) and \( q \) in the complement of the cut set \( \bar{L} \) with \( \bar{l}_i \) further decreased, while \( \text{dist}_{t_{2i-1}+t}(p, q) \) corresponds to the geodesic distance in the subset of
the aforementioned complement, obtained by enlarging all cuts \( \{ l_j \}_{j>1} \) to their original lengths in \( L \).

On the other hand, directly by construction of \( \tilde{L} \) we get:

\[
\text{dist}_{t_{2i-1} +1}(p,q) < \text{dist}_{t_{2i-1}}(p,q) = \text{dist}_0(p,q).
\]

This ends the proof of the lemma. \( \blacksquare \)

**Corollary 4.3.** The set of cuts \( \tilde{L} \) constructed above coincides with the set of edges \( \tilde{E} \) of the modified graph \( \tilde{G} \), with the new set of vertices \( \tilde{V} \), which have the following properties:

1. \( \tilde{G} \) has no loops, and consequently it is a forest, consisting of finitely many trees,
2. each vertex in \( \tilde{V} \) that is an endpoint of only one edge in \( \tilde{E} \) (i.e. a leaf of the forest \( \tilde{G} \)), is a vertex of some geodesic \( \sigma \) from \( p \) to \( q \) in \( \Omega \setminus \tilde{L} \).

**Proof.**

**Step 1.** To prove (i), we show that \( \mathbb{R}^2 \setminus \tilde{L} \) must be connected. Indeed, in the opposite case, the boundary of the connected component \( R_1 \) of \( \mathbb{R}^2 \setminus \tilde{L} \) containing \( p \) and \( q \), must contain a cut \( \overline{a_j a_k} \) that is also a part of the boundary of some other connected component \( R_2 \) of \( \mathbb{R}^2 \setminus \tilde{L} \). By the minimality property of \( \tilde{L} \) in Lemma 4.2, it follows that the sealing procedure with respect to the indicated cut \( \overline{a_j a_k} \) and its endpoint \( a_j \) results in the decrease of \( \text{dist}_t(p,q) \) for any \( t > 0 \). Consequently, there exists a piecewise \( C^1 \) curve \( \tau : [0,1] \to \Omega \setminus \tilde{L}(t) \) with \( \tau(0) = p, \tau(1) = q \) and \( \text{length}(\tau) < \text{dist}_0(p,q) \), where this last distance is taken in \( \Omega \setminus \tilde{L} \). The curve \( \tau \) must both enter and exit \( R_2 \) through the segment \( \overline{a_j a_k} \). This means that \( \tau \) may be further shortened by replacing its portion contained in the aforementioned interior region by an appropriate straight segment. The resulting curve is \( \tau : [0,1] \to \Omega \setminus \tilde{L} \), with:

\[
\text{length}(\tau) < \text{length}(\tau) < \text{dist}_0(p,q),
\]

which is a contradiction.

**Step 2.** Let \( a_i \in \tilde{V} \) be as requested in (ii). Consider the modified endpoints \( a_i(1/m) \) and the cut collections \( \tilde{L}(1/m) \) as described in the sealing algorithm. As in the proof of Lemma 4.1 there must exist a finite sequence \( (i_1, \ldots, i_k) \) such that:

\[
\sigma_m = \overline{a_{i_1}(1/m)a_{i_2}(1/m) \ldots a_{i_k}(1/m)}
\]

is a geodesic from \( p \) to \( q \) in \( \Omega \setminus \tilde{L}(1/m) \) for infinitely many \( m \)-s. By the maximality assertion in Lemma 4.2, there must be: \( i \in \{ i_1, \ldots, i_k \} \). But then Lemma 4.1 yields:

\[
\text{dist}_0(p,q) = \lim_{m \to \infty} \text{dist}_{1/m}(p,q) = \lim_{m \to \infty} \text{length}(\sigma_m) = \text{length}(\overline{pa_{i_1} \ldots a_{i_k}q}).
\]

Consequently, \( \overline{pa_{i_1} \ldots a_{i_k}q} \) is a geodesic from \( p \) to \( q \) in \( \Omega \setminus \tilde{L} \), whose existence is claimed in (ii). \( \blacksquare \)

**Remark 4.4.** When the minimal configuration \( \tilde{L} \) consists of disjoint segments \( \{ l_i \}_{i=1}^n \), then each geodesic \( \sigma \) from \( p \) to \( q \) in \( \Omega \setminus \tilde{L} \) does not contain any cuts. Indeed, assume by contradiction that \( l \subset \sigma \), for some cut \( l \subset \tilde{L} \). Denote \( \tilde{L} = \tilde{L} \setminus l \), then by the maximality condition in Lemma 4.2, there exists a piecewise \( C^1 \) curve \( \tau \) from \( p \) to \( q \) in \( \Omega \setminus \tilde{L} \) satisfying \( \text{length}(\tau) < \text{length}(\sigma) \). Hence, \( \tau \) must intersect \( l \) at only one point which we call \( x \). Consider two piecewise \( C^1 \) curves: the curve \( \tau_1 \) obtained by concatenating the portion of \( \tau \) from \( p \) to \( x \), with the portion of \( \sigma \) from \( x \) to \( q \), and the curve \( \tau_2 \) obtained by concatenating the portion of \( \sigma \) from \( p \) to \( x \), with the portion of \( \tau \) from \( x \) to \( q \). One of these curves, say \( \tau_1 \), must satisfy:

\[
\text{length}(\tau_1) < \text{length}(\sigma).
\]

But then one can approximate \( \tau_1 \) by another piecewise \( C^1 \) curve \( \tilde{\tau}_1 : [0,1] \to \Omega \setminus \tilde{L} \) (see Figure 4.2 (i)), to the effect that \( \text{length}(\tilde{\tau}_1) < \text{length}(\sigma) \), which contradicts \( \sigma \) being a geodesic. Note that in
the general case of $\bar{L}$ supported on the minimal graph $G$ with vertex degrees possibly exceeding 1, the above property is no more true (see Figure 4.2 (ii)).

**Figure 4.2.** Concatenating and shortening of the geodesic in Remark 4.4. In (i), the turning vertices of the base polygonals $\sigma$ and $\tau$ are indicated by, respectively, dashes and mid-markers. The concatenated shortened polygonal $\tau_1$ is in blue; it can be approximated by a polygonal $\bar{\tau}_1$ with values in $\Omega \setminus \bar{L}$, by means of a segment (in light blue) that avoids $\bar{l}$. In (ii) the displayed configuration of cuts is minimal, yet cuts are not separated. There are three geodesic polygonals $\{\sigma_i\}_{i=1}^3$ from $p$ to $q$.

5. **Proof of Theorem 2.2.** Step 2: ordering the geodesics

Assume (S) and let $p, q$ be two distinct points belonging to one connected component of $\bar{\Omega} \setminus L$. In the previous section we showed that, without loss of generality, the set of cuts $L = \bigcup_{n=1}^N l_i$ satisfies assertions in Corollary 4.3. From now on, we work assuming these additional properties and denote by $(G, V, L)$ a minimal configuration (instead of the notation $(\bar{F}, \bar{V}, \bar{L})$ used in section 4).

The (finite, nonempty) set of all geodesics from $p$ to $q$ in $\Omega \setminus L$ has a partial order relation $\preceq$ in:

(O) Given two geodesics from $p$ to $q$ in $\Omega \setminus L$:

$$\sigma_1 = pa_{i_1}a_{i_2}\ldots a_{i_k}q, \quad \sigma_2 = pa_{j_1}a_{j_2}\ldots a_{j_s}q,$$

we write $\sigma_1 \preceq \sigma_2$ provided that the concatenated polygonal:

$$\sigma = \sigma_1 * (\sigma_2)^{-1} = pa_{i_1}a_{i_2}\ldots a_{i_k}qa_{j_s}a_{j_s-1}\ldots a_{j_1}p$$

is the boundary of (finitely many) open bounded connected regions in $\mathbb{R}^2$, and moreover $\sigma$ is oriented counterclockwise with respect to all of these regions.

**Lemma 5.1.** In the above setting, we have:

(i) there exist the unique geodesic $\sigma_{\text{min}}$ and the unique geodesic $\sigma_{\text{max}}$ such that $\sigma_{\text{min}} \preceq \sigma \preceq \sigma_{\text{max}}$ for all geodesics $\sigma$ from $p$ to $q$ in $\Omega \setminus L$; we call $\sigma_{\text{min}}$ the least and $\sigma_{\text{max}}$ the greatest geodesic,

(ii) there exists a chain of geodesics $\sigma_1 \preceq \sigma_2 \preceq \ldots \preceq \sigma_N$, such that $\sigma_1 = \sigma_{\text{min}}$, $\sigma_N = \sigma_{\text{max}}$ and that the consecutive geodesics cover each other, i.e. for all $i = 1 \ldots N - 1$ there holds: $\sigma_i \neq \sigma_{i+1}$ and if $\sigma_i \preceq \sigma \preceq \sigma_{i+1}$ for some other geodesic $\sigma$, then $\sigma = \sigma_i$ or $\sigma = \sigma_{i+1}$.

**Proof.** Step 1. For the least geodesic statement in (i), it suffices to show that if $\sigma_1, \sigma_2$ are two minimal elements for the partial order $\preceq$, then necessarily $\sigma_1 = \sigma_2$. To this end, we will construct a geodesic $\sigma$ with $\sigma \preceq \sigma_1$ and $\sigma \preceq \sigma_2$. The statement for the greatest geodesic follows by a symmetric argument.

We write: $\sigma_1 = pa_{i_1}a_{i_2}\ldots a_{i_k}q$ and $\sigma_2 = pa_{j_1}a_{j_2}\ldots a_{j_s}q$. Observe first that $\sigma_1$ and $\sigma_2$ cannot have a common point $x \notin \{p, q\}$ that is not a vertex in $V$, unless they have a common edge $a_{i_m}a_{i_{m+1}} = \ldots$
Lemma 5.2. In the above setting, let $\sigma_1 \preceq \sigma_2 \ldots \preceq \sigma_N$ be as in Lemma 5.1 (ii). For each $r = 1 \ldots N - 1$, let $R_r$ be the open, bounded region enclosed by the concatenation $\sigma_r \ast (\sigma_{r+1})^{-1}$. We set $R_0 = \Omega \setminus \bigcup_{r=1}^{N-1} R_r$ to be the exterior region relative to the concatenation $\sigma_1 \ast (\sigma_N)^{-1}$. Then, for each tree $T$ that is a connected component of $G$, there holds:

\[ \text{length}(\tau_1) = \text{length}(\tau_2) = \text{length}(\sigma_1) = \text{length}(\sigma_2). \]

Similarly to the construction in the proof of Corollary 4.3, see also Figure 4.2, we could then approximate $\tau_1$ (and also $\tau_2$) by a piecewise $C^1$ curve $\tau : [0,1] \rightarrow \Omega \setminus L$ with length($\tau$) strictly less than the four coinciding lengths above. This would contradict $\sigma$-s being geodesics.

Let $a_{i_m} = a_{j_l}$ be the first common vertex of $\sigma_1$ and $\sigma_2$, beyond $p$. If $i_m = i_1$ and $j_l = j_1$, then we include $pa_{i_1}$ as the starting portion of $\sigma$; otherwise we choose $pa_{i_1}a_{i_2} \ldots a_{i_m}$ in case the concatenation $pa_{i_1}a_{j_1-1} \ldots a_{j_l}$ has the counterclockwise orientation with respect to the bounded open region it encloses, or $pa_{j_1}a_{j_2} \ldots a_{j_l}$ in the reverse case. Let $a_{i_m} = a_{j_l}$ be the second common vertex of $\sigma_1$ and $\sigma_2$, beyond $a_{i_m}$; we choose the least of $a_{i_m} \ldots a_{i_m}$ and $a_{j_l} \ldots a_{j_l}$, as above, to be concatenated with the previous portion of $\sigma$. By such inductive procedure, we obtain a required geodesic $\tau$ that satisfies $\tau \preceq \sigma_1$ and $\tau \preceq \sigma_2$. From minimality, it follows that $\sigma = \sigma_1 = \sigma_2$, and so $\sigma = \sigma_{\min}$ is the least element for $\preceq$.

Step 2. To prove (ii), we set $\sigma_1 = \sigma_{\min}$ and $\sigma_N = \sigma_{\max}$ for some $N \geq 2$. If $\sigma_N$ covers $\sigma_1$, then $\sigma_1 \preceq \sigma_N$ is the required chain. Otherwise, there exists a geodesic $\sigma \not\in \{\sigma_1, \sigma_N\}$ such that $\sigma_1 \preceq \sigma \preceq \sigma_N$. If $\sigma$ covers $\sigma_1$ then we write $\sigma_2 = \sigma$, if it is covered by $\sigma_N$ then we set $\sigma_{N-1} = \sigma$. If none of the above holds, there must exist a geodesic $\tau \not\in \{\sigma_1, \sigma, \sigma_N\}$ such that:

$\sigma_1 \preceq \tau \preceq \sigma$ or $\sigma \preceq \tau \preceq \sigma_N$.

We continue in this fashion until the process is stopped, which will occur in finitely many steps due to the finite number of geodesics from $p$ to $q$ in $\Omega \setminus L$.

**Figure 5.1.** Examples of sequence of geodesics produced in Lemma 5.1 diagram (i) refers to the configuration $L, p,q$ in Figure 2.1 (ii) where the resulting sequence consists of the following geodesics: $\sigma_1 = \sigma_{\min}$ in blue, $\sigma_2$ in red and $\sigma_3 = \sigma_{\max}$ in black; in a more complex diagram (ii) the sequence consists of: $\sigma_1 = \sigma_{\min}$ in blue, $\sigma_2$ in red, $\sigma_3$ in green, $\sigma_4$ in brown and $\sigma_5 = \sigma_{\max}$ in black.
(i) $T$ has nonempty intersection with the interior of exactly one region $R_r$, 
(ii) if $T \subset R_r$ for $r = 1 \ldots N - 1$, then $T$ has vertices on both $\sigma_r$ and $\sigma_{r+1}$.

Moreover, if $p,q \in \partial \Omega$, then there are no trees in $R_0$.

**Proof.** Step 1. If $T$ violated the condition in (i) then there would exist a path $\alpha \subset T$ and two distinct points $A,B \in \alpha$ (which are not necessarily the vertices in $V$) such that $A \in R_{r-1}$, $B \in R_r$ for some $r = 1 \ldots N$ (where we set $R_N = R_0$), and such that the portion of $\alpha$ between $A$ and $B$ crosses the geodesic $\sigma_r$. This would contradict condition (ii) in definition (G).

Step 2. To prove (ii), assume without loss of generality that all vertices of a maximal tree $T \subset \tilde{R}_r$ belong to $\sigma_r$. Call $A$ the leaf of $T$ that is closest to $p$ along $\sigma_r$, and $B$ the leaf that is closest to $q$. By Corollary 4.3 (ii) and since each tree has at least two leaves, there must be $A \neq B$. Consider the (unique) path $\alpha \subset T$ connecting $A$ and $B$. If $\alpha \subset \sigma_r$, then there would be $T = \alpha$. Also, in this case $\alpha$ together with edges of $\sigma_r$ immediately preceding $A$ and immediately succeeding $B$ would form a straight segment, contradicting the minimality of $G$. Thus, $\alpha$ passes through $R_r$.

Call $A'$ the first vertex on $\alpha$ whose immediate successor belongs to $R_r$, and $B'$ the last vertex whose immediate predecessor belongs to $R_r$; there may be $A' = A$ or $B' = B$. Call $\alpha' = A'a_{i_1} \ldots a_{i_l}B' \subset \alpha$ the unique path in $T$ connecting $A'$ with $B'$, and denote by $D \subset R_r$ the region enclosed by the concatenation of $\alpha'$ and the portion of $\sigma_r$ between $A'$ and $B'$.

![Figure 5.2](image-url)

**Figure 5.2.** Notation in the proof of Lemma 5.2, Step 2: in (i) the curve $\tau_k$ enters the (shaded) region $D$, hence the indicated vertex $a_{i_s} \in \alpha'$ is of type I from left; in (ii), existence of a shortening path $\bar{\tau}_k$ which exits $D$ via the removed edge portion preceding $a_{i_s}$ in $\alpha'$, implies that $a_{i_s}$ is of type II from left.

We now label each vertex $a_{i_s} \in \alpha' \setminus \sigma_r$ as **type I/II from left**, provided that there exists a sequence of piecewise $C^1$ paths $\{\tau_k : [0,1] \to \Omega \setminus L(1/k)\}_{k=1}^{\infty}$ with $\tau_k(0) = p$, $\tau_k(1) = q$ and $\text{length}(\tau_k) < \text{dist}(p,q)$, where $L(1/k)$ denotes the modified cut set $L$ in which the edge $a_{i_{s-1}}a_{i_s}$ in the graph $G$ is replaced by the shortened segment $a_{i_{s-1}}a_{i_s}(1/k)$ with $a_{i_s}(1/k) = a_{i_s} - \frac{1}{k}(a_{i_s} - a_{i_{s-1}})$. Further, we request that:

- **Type I from left:** each $\tau_k$ enters $D$, only once, through the removed segment portion $a_{i_s}(1/k)a_{i_s}$.
- **Type II from left:** each $\tau_k$ exits the region $D$, only once, through $a_{i_s}(1/k)a_{i_s}$.

Similarly, we label $a_{i_s} \in \alpha' \setminus \sigma_r$ as **type I/II from right**, when there exists a sequence of piecewise

- **Type I from right:** each $\bar{\tau}_k$ enters $D$, only once, through the removed segment portion $a_{i_s}a_{i_s}(1/k)$.
- **Type II from right:** each $\bar{\tau}_k$ exits $D$, only once, through $a_{i_s}a_{i_s}(1/k)$.
In the definitions above (see diagrams in Figure 5.2), we set $a_{i_k} = A'$ and $a_{i_{k+1}} = B'$. By the minimality of $G$, each $a_i$ must be of type I or type II from left (it may be both), and it must be of type I or type II from right (it may be both).

**Step 3.** We claim that $a_i$ has to be of type I from left. We argue by contradiction and hence assume that $a_i$ is of type II from left. Note that the length of the portion of the shortening curve $\tau_k$ between $p$ and the exit point from $D$ is strictly larger than the distance from $p$ to $a_i$ in $\Omega \setminus L$, because all the internal (with respect to $R_r$) angles along $\alpha$ from $A$ to $A'$ are not greater than $\pi$, whereas the angle at $A'$ is strictly smaller than $\pi$. Concatenating with the remaining portion of $\tau_k$ and taking the limit $k \to \infty$, it follows that there is a geodesic from $p$ to $q$ in $\Omega \setminus L$ passing through $a_i$. This contradicts the fact that $a_i \notin \sigma_k$, and proves the claim.

By a similar argument, we can show that if $a_i$ is of type I from left, then it is also of type I from right. We argue by contradiction and hence assume that $a_i$ is of type II from right. Consider the curves $\tau_k$ and $\tau_k$ corresponding to the two assumed properties of $a_i$; they must intersect at some point $C$ occurring after $\tau_k$ enters $D$ and before $\tau_k$ exits from $D$. Define the curves: $\eta_k$ as the concatenation of the portion of $\tau_k$ from $p$ to $C$ with the portion of $\tau_k$ from $C$ to $q$, and $\bar{\eta}_k$ as the concatenation of the portion of $\tau_k$ from $p$ to $C$ with the portion of $\tau_k$ from $C$ to $q$. Since $\bar{\eta}_k \subset \Omega \setminus L$, it follows that $\text{length}(\bar{\eta}_k) \geq \text{dist}(p,q)$. Consequently:

$$\text{length}(\eta_k) = \text{length}(\tau_k) + \text{length}(\bar{\tau}_k) - \text{length}(\bar{\eta}_k) < \text{dist}(p,q).$$

The only possibility for this when taking the limit $k \to \infty$, we obtain the existence of a geodesic from $p$ to $q$ in $\Omega \setminus L$ passing through $a_i$. This contradicts the fact that $a_i$ is not on any geodesic.

![Figure 5.3. Concatenating curves $\tau_k$ and $\tau_k$ at the intersection $C$ in the proof of Lemma 5.2 Step 3, when the region $D$ has multiple connected components.](image)

Finally, we argue that if $a_i$ is of type I from right, then the next vertex $a_j$ on $\alpha'$ that belongs to $R_r$ must be of type I from left as well. If not, then $a_j$ is of type II from left and we can define the point $C$ and the concatenated curves $\eta_k$ and $\bar{\eta}_k$ as in the previous reasoning. Again, $\text{length}(\bar{\eta}_k) \geq \text{dist}(p,q)$, so $\text{length}(\eta_k) < \text{dist}(p,q)$. However, we may replace the portion of $\eta_k$ between the entry point of $\tau_k$ to $D$, and the exit point of $\bar{\tau}_k$ from $D$, by a shorter curve (see Figure 5.3) which is completely contained in $\Omega \setminus L$. Indeed, when $i' = i_{k+1}$, then the said curve may follow the segment $a_{i'}a_{i'} \subset \alpha'$. When $i' \neq i_{k+1}$, then the portion of the polygonal $\alpha'$ between $a_i$ and $a_i'$ has all internal angles (with respect to $R_r$) not greater than $\pi$, so one can simply take the geodesic from $a_i$ to $a_i'$ in $\Omega \setminus L$. 
As a consequence and passing to the limit with \( k \to \infty \), we obtain a geodesic from \( p \) to \( q \) in \( \Omega \setminus L \) passing through \( a_{i_1} \) and \( a_{i_{d'}} \). This contradicts \( a_{i_1}, a_{i_{d'}} \) not being on any geodesic.

**Step 4.** Applying the observations from Step 3, it follows that the vertex \( a_{i_1} \) must be of type I from right. However this is impossible by a symmetric argument to \( a_{i_1} \) not being of type II from left. This ends the proof of (ii). In case \( p, q \in \partial \Omega \), the region \( R_0 \) consists of two connected components, and hence any tree \( T \in R_0 \) would have vertices either only on \( \sigma_1 \) or only on \( \sigma_N \). By the same arguments as above, this is impossible, which implies the final statement of the lemma.

We close the above discussion by pointing out that in case \( p, q \notin \partial \Omega \), there may be a tree (or even multiple trees) in \( R_0 \), with vertices both on \( \sigma_1 \) and \( \sigma_N \) (see Figure 8.1 in section 8). The next main result of this section allows for the lexicographic ordering of the connected components of \( \Omega \setminus L \). Namely, we have (see example in Figures 5.4 and 5.5):

**Lemma 5.3.** In the above setting, let \( \sigma_1 \leq \sigma_2 \ldots \leq \sigma_N \) be as in Lemma 5.1 (ii). Fix \( r = 1 \ldots N - 1 \) and consider the region \( R_r \) enclosed between two consecutive geodesics \( \sigma_r = \overline{p_{i_1} a_{i_2} \ldots a_{i_{q_r}}} q_r \) and \( \sigma_{r+1} = \overline{p_{j_1} a_{j_2} \ldots a_{j_{q_{r+1}}}} q_{r+1} \), as in Lemma 5.2. Consider further the set of maximal trees \( \{ T_m \}_{m=1}^s \) which are the connected components of \( G \) contained in \( R_r \). Then we have:

(i) the ordering \( T_1, \ldots, T_s \) can be made so that each leaf of \( T_i \) on \( \sigma_r \) (respectively \( \sigma_{r+1} \)) precedes each leaf of \( T_j \) on \( \sigma_r \) (resp. \( \sigma_{r+1} \)), when \( i < j \).

The region \( R_r \setminus L \) is the union of \( s + 1 \) (open) polygons \( \{ P_m \}_{m=0}^s \) and of additional families of polygons \( \{ Q_m \}_{m=1}^s \), described as follows:

(ii) we denote \( \alpha^{left}_{m} = p_1 \overline{p_{i_1} p_{i_2}} \) and \( \alpha^{right}_m = \overline{q_{i+1} q_{i+2}} \), where \( p_1, q_{i+1} \) are two common vertices of \( \sigma_r \) and \( \sigma_{r+1} \), such that \( \overline{p_{i_1} \ldots p_{i_j}} = \overline{p_{j_1} \ldots p_{j_p}} \) and \( \overline{q_{i} \ldots q_{j}} = \overline{q_{j} \ldots q_{j_q}} \) (we take the last, along \( \sigma_r \), vertex with the said property to be \( p_1 \) and the first vertex to be \( q_{i+1} \)). For each \( m = 1 \ldots s \) we denote \( \alpha^{right}_{m-1} \) (respectively, \( \alpha^{left}_m \)) the unique path in \( T_m \) joining its first (resp. its last) vertex on \( \sigma_r \) with its first (resp. the last) vertex on \( \sigma_{r+1} \), both counting from \( p_1, q_{i+1} \). Note that there may be \( \alpha^{left}_m = \alpha^{right}_{m-1} \). Then, the boundary of each \( P_m \) consists of paths \( \alpha^{left}_m, \alpha^{right}_m \) and of the intermediate portions of \( \sigma_r \) and \( \sigma_{r+1} \) which are concave with respect to \( P_m \). Namely, all interior angles of \( P_m \) which are not on \( \alpha^{left}_m \cup \alpha^{right}_m \) are not less than \( \pi \). Finally, there are no cuts in \( P_m \).

(iii) each family \( Q_m \) consists of finitely many polygons \( Q_m \), that are the connected components of \( R_r \setminus L \) enclosed between \( \alpha^{left}_m, \alpha^{right}_{m-1} \) and the portions of \( \sigma_r \) and \( \sigma_{r+1} \). The boundary of each \( Q_m \) consists of a single path within \( T_m \) plus a single portion of the geodesic \( \sigma_r \) or \( \sigma_{r+1} \). It has all interior angles not at vertices belonging of \( T_m \) concave with respect to \( R_m \).

**Proof.** For (i), consider first \( \sigma_r \) and recall that each vertex \( a_{i_1}, \ldots, a_{i_{d}} \) is an endpoint of some cut, which belongs to some maximal tree \( T \subset G \). If \( T \) extends inside the region \( R_r \), then it must have vertices on both \( \sigma_r \) and \( \sigma_{r+1} \), by Lemma 5.2. The same reasoning can be applied to cuts emanating from \( \sigma_{r+1} \). We can now order the trees \( \{ T_m \}_{m=1}^s \), based on how many vertices (along \( \sigma_r \) and \( \sigma_{r+1} \)) separate their leaves from \( p \). This ordering is well defined, as trees are non-intersecting. Assertions (ii) and (iii) follow directly by construction and since \( \sigma_r, \sigma_{r+1} \) are geodesics.

Concluding, we see that the assumption \([S]\) may be replaced by the following modified setup:
Figure 5.4. Partitions \( \{P_m\}_{m=0}^s \) defined in Lemma 5.3: (i) depicts partition of the region \( R_1 \) corresponding to Figure 5.1 (i), while (ii) depicts the region \( R_3 \) in Figure 5.1 (ii). In both figures the trees \( T_m \) coincide with paths \( \alpha_{m-1}^{\text{right}} = \alpha_m^{\text{left}} \) that are single cuts, and consequently all intermediate polygonal collections \( Q_m \) are empty.

Figure 5.5. Polygons \( \{P_m\}_{m=0}^s \) and polygon families \( \{Q_m\}_{m=1}^s \) defined in Lemma 5.3: (i) corresponds to the unique region \( R_1 \) in Figure 2.2 (iii) with paths: \( \alpha_0^{\text{left}} = pp \), \( \alpha_1^{\text{right}} = a_2a_5a_3 \), \( \alpha_1^{\text{left}} = a_4a_5a_3 \), \( \alpha_1^{\text{right}} = qq \); (ii) is a general diagram depicting the partition of the region \( R_r \).

The set of cuts \( L \) satisfies assertions of Corollary 4.3. The chain of geodesics \( \{\sigma_r\}_{r=1}^N \) from \( p \) to \( q \) in \( \Omega \setminus L \), satisfies condition in Lemma 5.1 (ii) with respect to the partial order in (O). In agreement with Lemma 5.3 the set \( \Omega \setminus L \) is partitioned into \( N \) regions \( \{R_r\}_{r=0}^N \):

(i) for each \( r = 1 \ldots N - 1 \), the “interior” bounded region \( R_r \) which is enclosed by \( \sigma_r * (\sigma_{r+1})^{-1} \), and partitioned into polygonal sub-regions \( \{P_m\}_{m=0}^s \cup \{Q_m\}_{m=1}^s \) corresponding to the consecutive trees \( \{T_m\}_{m=0}^s \) (we suppress the dependence on \( r \) in this notation), as specified in Lemma 5.3

(ii) the “exterior” region \( R_0 = \Omega \setminus \bigcup_{r=1}^{N-1} R_r \).

We also define the segment \( I = 0, \text{length}(\sigma_1)e_1 \subset \mathbb{R}^3 \).
6. Proof of Theorem 2.2, a simplified case. Step 3: isometric immersion on interior regions between consecutive cuts

Assume (S) and let \( p, q \) be two distinct points in \( \Omega \setminus L \). In view of the results in previous sections, the goal is to construct an isometry \( u \) as in Theorem 2.2 separately on each \( R_r \) identified in (S1). We first concentrate on the interior case \( r = 1 \ldots N - 1 \), while in section 8 we address the case \( r = 0 \).

In this section we treat a simplified scenario in which all trees \( T_1, \ldots, T_s \) consist of single cuts; note that this occurs, in particular, if all cuts in the original graph \( G \) are non-intersecting:

**Lemma 6.1.** Assume (S) and (S1). Fix \( r = 1 \ldots N - 1 \) and further assume that:

\[
T_m = \bar{l}_m = \overline{a_m a_{m+1}} \quad \text{for all } m = 1 \ldots s.
\]

Then there exists a continuous, piecewise affine isometric immersion \( u : R_r \setminus \bigcup_{m=1}^s \bar{l}_m \to \mathbb{R}^3 \), with:

\[
u(p) = 0, \quad u(q) = \text{length}(\sigma_1)e_1, \quad u(\sigma_r) = u(\sigma_{r+1}) = 1.
\]

**Proof.** We will inductively find the matching isometric immersions \( u \) of the consecutive polygons \( \{ P_m \}_{m=0}^s \). Note that polygons in \( \bigcup_{m=1}^s Q_m \) are absent in the presently discussed case.

**Step 1.** On \( P_0 \), we first fold its “top” part so that the image of the portion of \( \sigma_{r+1} \) from \( p_1 \) to the endpoint \( B_1 \) of the cut \( \bar{l}_1 = \overline{A_1 B_1} \) coincides with the sub-interval:

\[
\text{length}(\overline{p a_1 \ldots p_1})e_1, \text{length}(\overline{p a_i \ldots B_1})e_1 \subset I.
\]

This can be achieved because all internal angles of \( \sigma_{r+1} \) at vertices between (but not including) \( p_1 \) and \( B_1 \) are at least \( \pi \). A symmetric fold construction can be performed on the “bottom” part of \( P_0 \), along the boundary portion contained in \( \sigma_r \).

As a result, the vector \( u(B_1) - u(A_1) \) equals \( (\text{length}(\overline{p a_i \ldots B_1}) - \text{length}(\overline{p a_i \ldots A_1}))e_1 \) and we consecutively have to find an isometric immersion of the polygon \( P_1 \) with the property that writing \( \bar{l}_2 = \overline{A_2 B_2} \) with \( A_2, B_2 \in \sigma_r, B_2 \in \sigma_{r+1} \), the length of the vector \( u(B_2) - u(A_2) \) is prescribed, and that the images of portions of: geodesic \( \sigma_r \) between \( A_1 \) and \( A_2 \), and of geodesic \( \sigma_{r+1} \) between \( B_1 \) and \( B_2 \), are contained in \( \mathbb{R}e_1 \).

**Step 2.** Assume that \( u \) has been constructed on \( P_1 \cup \ldots P_{m-1} \), for some \( m \leq s - 1 \). Consider the polygon \( P_m \) and the two related closed convex sets \( S_A \) and \( S_B \). The set \( S_B \) is defined by specifying its boundary to consist of: the portion of \( \sigma_{r+1} \) between the endpoints \( B_m \) and \( B_{m+1} \) of the cuts \( \bar{l}_m, \bar{l}_{m+1} \), respectively, and of the segment \( \overline{B_mB_{m+1}} \). The boundary of the set \( S_A \) is: the portion of \( \sigma_r \) between the remaining endpoints \( A_m \) and \( A_{m+1} \) of the cuts \( \bar{l}_m, \bar{l}_{m+1} \), and of the segment \( \overline{A_mA_{m+1}} \).

We note that the interior of the defined sets may be empty; for example if \( \overline{A_mA_{m+1}} \subset \sigma_r \) then \( S_A = \overline{A_mA_{m+1}} \subset \sigma_r \).

Since \( S_A \) and \( S_B \) are closed, convex and disjoint, there exist precisely two lines \( \xi_A, \xi_B \) which are supporting to both sets. Each of these lines intersects \( S_A \) and \( S_B \) either at a single vertex (which may be \( A_m \) or \( A_{m+1} \) for \( S_A \) and \( B_m \) or \( B_{m+1} \) for \( S_B \)) or along the whole segment which is the union of some consecutive edges in \( \sigma_r, \sigma_{r+1} \). Denote \( \bar{B}_m' \in (\xi_A \cup \xi_B) \cap \sigma_{r+1} \) the vertex which is closest to \( B_m \) along \( \sigma_{r+1} \), and let \( \bar{B}_{m+1}' \in (\xi_A \cup \xi_B) \cap \sigma_{r+1} \) be the vertex that is closest to \( B_{m+1} \) (along \( \sigma_{r+1} \)). Similarly, define \( \bar{A}_m', \bar{A}_{m+1}' \in (\xi_A \cup \xi_B) \cap \sigma_r \) as vertices on \( \sigma_r \) which are closest (possibly equal) to, respectively, \( A_m \) and \( A_{m+1} \) along \( \sigma_r \).

Observe that \( \xi_A \) is precisely the line through \( \bar{A}_m' \) and \( \bar{B}_m' \) and that it intersects the closures of the cuts \( \bar{l}_m \) and \( \bar{l}_{m+1} \). By consecutive folding as in Step 1, one constructs an isometric immersion \( v \).
of $P_m$ with the property that its both boundary polygonal sides: from $B_m$ to $B_{m+1}$ and from $A_m$ to $A_{m+1}$ are mapped onto $\xi_A$. By a further rotation, we may ensure that $\xi_A = R e_1$. Then: \[ v(B_m) - v(A_m) = \left(-\text{length}(B_m \ldots B_{m+1}) + \text{length}(A_m \ldots A_{m+1})\right) e_1 = \alpha_v e_1. \]

Similarly, by folding on the line $\xi_B$ through $B_m$ and $A_{m+1}$, one obtains an isometric immersion $w$ of $P_m$ with both polygonal sides (namely, the sides distinct from $\bar{l}_m$ and $\bar{l}_{m+1}$) mapped on $\xi_B$. By a further rotation, we ensure that $\xi_B = R e_1$, so that there holds: \[ w(B_m) - w(A_m) = \left(-\text{length}(B_m \ldots B_{m+1}) - \text{length}(A_m \ldots A_{m+1})\right) e_1 = \alpha_w e_1. \]

**Step 3.** We now estimate the length of the vector $u(B_m) - u(A_m)$ that we need to achieve, and that is determined through previous steps in the construction. There clearly holds: \[
(6.1) \quad u(B_m) - u(A_m) = \left(\text{length}(pa_{j_1} \ldots B_m) - \text{length}(pa_{j_1} \ldots A_m)\right) e_1.
\]

Since $P_m$ contains no cuts in its interior, the polygonal: $pa_{i_1} \ldots A_m A_m' B_m' \ldots B_{m+1} \ldots a_{j_1} q$ (this polygonal follows the portion of $\sigma_r$ up to $A_m'$, then switches to $\sigma_{r+1}$ along the segment $A_m' B_{m+1}' \subset P_m$, and continues to $q$ along $\sigma_{r+1}$) cannot be shorter than $\text{length}(\sigma_{r+1})$. Equivalently, we obtain: \[
\text{length}(pa_{i_1} \ldots A_m) + \text{length}(A_m \ldots A_m') + \text{length}(A_m' B_m') \\
\geq \text{length}(pa_{j_1} \ldots B_{m+1}') = \text{length}(pa_{j_1} \ldots B_m) + \text{length}(B_m \ldots B_{m+1}').
\]

By (6.1) the above yields: \[
\langle u(B_m) - u(A_m), e_1 \rangle \leq \alpha_v.
\]
By a parallel argument, in which we concatenate \( \sigma_{r+1} \) up to \( B'_m \) with the segment \( B'_m \alpha'_{m+1} \) and then with the portion of \( \sigma_r \) from \( A'_{m+1} \) to \( q \), there follows the bound:

\[
\text{length}(p_{a_j} \ldots B'_m) + \text{length}(B_m \ldots B'_m) + \text{length}(B'_m \alpha'_{m+1}) \\
\geq \text{length}(p_{a_i} \ldots A_m) + \text{length}(A_m \ldots A'_{m+1}),
\]

so (6.1) results in:

\[
\langle u(B_m) - u(A_m), e_1 \rangle \geq \alpha_w.
\]

Step 4. Let now \( \xi \) be any line passing through the intersection point \( \xi_A \cap \xi_B \) and disjoint from the interiors of \( S_A \) and \( S_B \) (see Figure 6.1 (ii)). There exist exactly one line \( \xi(A) \) which is supporting to the convex set \( S_A \) and parallel to \( \xi \), and exactly one line \( \xi(B) \) supporting to \( S_B \) and parallel to \( \xi \). As before, we may fold the top portion of \( P_m \) so that the image of \( B_m \ldots B'_m \) is a segment within \( \xi(B) \), and fold the bottom portion of \( P_m \) so that the image of \( A_m \ldots A'_{m+1} \) is a segment in \( \xi(A) \). We now perform two more folds, which map both \( \xi(A) \), \( \xi(B) \) onto \( \xi \), plus a rigid rotation that maps \( \xi \) onto \( \mathbb{R} e_1 \). Call the resulting isometric immersion \( u_\xi \) and observe that:

the function \( \xi \mapsto u_\xi(B_m) - u_\xi(A_m) \) is continuous.

Since \( u_{\xi_A} = v \) and \( u_{\xi_B} = w \), the intermediate value theorem implies that \( \langle u_\xi(B_m) - u_\xi(A_m), e_1 \rangle \) achieves an arbitrary value within the interval:

\[
[\alpha_v, \alpha_w] = \left[ \langle u_{\xi_A}(B_m) - u_{\xi_A}(A_m), e_1 \rangle, \langle u_{\xi_B}(B_m) - u_{\xi_B}(A_m), e_1 \rangle \right].
\]

In conclusion, there exists a line \( \xi \) such that the corresponding \( u_\xi \) on \( P_m \) gives:

\[
u_\xi(B_m) - u_\xi(A_m) = u(B_m) - u(A_m).
\]

We set \( u|_{P_m} \equiv u_\xi \).

Step 5. The final step is to construct \( u \) on \( P_0 \). This can be done by the same folding technique described in Step 1 for \( P_0 \). We then note that \( \langle u(B_s) - u(A_s), e_1 \rangle \) automatically equals:

\[
\text{length}(A_s \ldots a_{j_1}q_1) - \text{length}(B_s \ldots a_{i_1}q_1),
\]

because \( \text{length}(\sigma_r) = \text{length}(\sigma_{r+1}) \). The proof is done.

7. Proof of Theorem 2.2, the general case. Step 3: Isometric immersion on interior regions between and within consecutive trees

In this section we exhibit a procedure of constructing an isometric immersion on \( R_r \), in the general setting \( [S1] \). Namely, we prove the following version of Lemma 6.1

**Lemma 7.1.** Assume \( [S] \), \( [S1] \) and fix \( r = 1 \ldots N - 1 \). Then, there exists a continuous, piecewise affine isometric immersion \( u \) of \( R_r \setminus \bigcup_{m=1}^s T_m \) into \( \mathbb{R}^3 \), which satisfies:

\[
u(p) = 0, \quad u(q) = \text{length}(\sigma_1)e_1, \quad u(\sigma_r) = u(\sigma_{r+1}) = I.
\]

*Proof.* We will inductively find the matching isometric immersions (always denoted by \( u \)) of the consecutive polygons in \( P_0 \), \( \{Q_m \cup P_m\}_{m=1}^s \). Recall that we have defined two families of cut paths within each tree \( T_m \): the path \( \alpha_m \) joining vertices \( A'_m \in \sigma_r \) with \( B'_m \in \sigma_{r+1} \), and the path \( \alpha_m \) joining vertices \( A'_m \in \sigma_r \) with \( B'_m \in \sigma_{r+1} \).
Step 1. Thus, the polygon $P_0$ is bounded by the concatenation of: the portion of $\sigma_{r+1}$ from $p_1$ to $B_0^{right}$, with $\alpha_0^{right}$, with the portion of $\sigma_r$ from $A_0^{right}$ to $p_1$. We first fold the indicated portion of $\sigma_{r+1}$ so that it coincides with the sub-interval:

$$\text{length}(p_{a_1} \ldots p_i)e_1, \text{length}(p_{a_1} \ldots B_0^{right})e_1 \subset I.$$ 

This can be achieved as all internal angles of $\sigma_{r+1}$ at vertices between $p_1$ and $B_0^{right}$ are at least $\pi$. A symmetric folding can be done along the portion of $\sigma_r$ within the boundary of $P_0$, see Figure 7.1 (i). This construction is similar to Step 1 in the proof of Lemma 6.1 (i).

As a result, the vector $u(B_0^{right}) - u(A_0^{right})$ equals $(\text{length}(p_{a_1} \ldots B_0^{right}) - \text{length}(p_{a_1} \ldots A_0^{right}))e_1$ and we consecutively have to find an isometric immersion of each region in the family $Q_1$, with the property that the vector $u(B_{0}^{right}) - u(A_{0}^{right})$ is prescribed, and that the images of the portion of $\sigma_r$ between $A_{0}^{right}$ and $A_{1}^{left}$, and of $\sigma_{r+1}$ between $B_{0}^{right}$ and $B_{1}^{left}$, are contained in $Re_1$.

**Figure 7.1.** The folding patterns in the proof of Lemma 7.1: (i) corresponds to Step 1 and the polygon $P_0$, the arrows indicate the directions of folding; (ii) corresponds to Step 2 and the collection of polygons $Q_m$ within the tree $T_m$.

Step 2. Assume that $u$ has been constructed on $P_0 \cup Q_1 \cup \ldots P_{m-1}$ for some $m \leq s$. Consider the tree $T_m$ and the corresponding family of polygons $Q_m$ enclosed between the paths of cuts $\sigma_{m-1}^{right}$, $\alpha_m^{left}$ (both contained in $T_m$) and the portions of $\sigma_r$ (respectively $\sigma_{r+1}$) between the vertices $A_{m-1}^{right}$ and $A_m^{left}$ (resp. $B_{m-1}^{right}$ and $B_m^{left}$), see Figure 7.1 (ii). By Lemma 5.3 (iii), each polygon $Q_m \subset Q_m$ has an isometric immersion $u$ in which the image of its boundary portion included in $\sigma_r \setminus T_m$ (or in $\sigma_{r+1} \setminus T_m$) is a segment on $Re_1$. This construction consists of a collection of simple foldings as in Step 1 and Figure 7.1 (i), that can be implemented because all the internal (with respect to $R_r$) angles of $Q_m^f$ at the vertices $\sigma_r \setminus T_m$ and $\sigma_{r+1} \setminus T_m$, are at least $\pi$.

Step 3. Assume that $u$ has been constructed on $P_0 \cup Q_1 \cup \ldots P_{m-1} \cup Q_m$ for some $m \leq s - 1$. We now aim at describing $u$ on the polygon $P_m$; note that the vector $u(B_m^{left}) - u(A_m^{left})$ is a prescribed, by the previous steps of the proof, scalar multiple of $e_1$. The construction below is based on the ideas
of Steps 2-4 in the proof of Lemma [6.1] however the present setting of trees $T$ replacing the single cuts $\tilde{l}$ requires taking care of the additional details below.

Call $\xi_A$ (respectively $\xi_B$) the shortest path in $P_m$ that joins $A_m^{\text{left}}$ with $B_m^{\text{right}}$ (resp. $A_m^{\text{right}}$ with $B_m^{\text{left}}$). We now estimate the length of the vector $u(B_m^{\text{left}}) - u(A_m^{\text{left}})$, namely:

$$\langle u(B_m^{\text{left}}) - u(A_m^{\text{left}}), e_1 \rangle = \text{length}(p_{a_1} \ldots B_m^{\text{left}}) - \text{length}(p_{a_1} \ldots A_m^{\text{left}}).$$

Since there are no cuts in the interior of $P_m$, it follows that the concatenation of the portion of $\sigma_r$ from $p$ to $A_m^{\text{left}}$ with $\xi_A$ and then with $\sigma_{r+1}$ from $B_m^{\text{right}}$ to $q$, cannot be shorter than $\sigma_{r+1}$. Equivalently:

$$\text{length}(p_{a_1} \ldots A_m^{\text{left}}) + \text{length}(\xi_A) \geq \text{length}(p_{a_1} \ldots B_m^{\text{right}}).$$

Also, concatenating $\sigma_{r+1}$ up to $B_m^{\text{left}}$ with $\xi_B$ and then with the portion of $\sigma_r$ from $A_m^{\text{right}}$ to $q$, yields:

$$\text{length}(p_{a_1} \ldots B_m^{\text{left}}) + \text{length}(\xi_B) \geq \text{length}(p_{a_1} \ldots A_m^{\text{right}}).$$

The three last displayed bounds imply that:

$$\langle u(B_m^{\text{left}}) - u(A_m^{\text{left}}), e_1 \rangle \in [\alpha_w, \alpha_v],$$

(7.1)

where $\alpha_v = \text{length}(\xi_A) - \text{length}(B_m^{\text{left}} \ldots (\sigma_{r+1}) \ldots B_m^{\text{right}})$,

$$\alpha_w = \text{length}(\xi_B) - \text{length}(A_m^{\text{left}} \ldots (\sigma_r) \ldots A_m^{\text{right}}).$$

**Step 4.** Since $\xi_A$ has no self-intersections, it divides $P_m$ into two connected components, and the endpoints $B_m^{\text{left}}, A_m^{\text{right}}$ of $\xi_B$ belong to the closures of the distinct components. Hence $\xi_A \cap \xi_B \neq \emptyset$.

![Figure 7.2](image-url) Two types of regions $P_m$ (shaded) and the supporting polygons $\xi_A$, $\xi_B$ in the proof of Lemma [6.1] Step 4: in (i) the intersection $\xi_A \cap \xi_B$ consists of a single point $E$; in (ii) $\xi_A$ and $\xi_B$ intersect along the polygonal $a_1 a_2 a_3$.

Define $C_m^{\text{left}}$ to be the vertex at which $\xi_A$ detaches itself from $\sigma_r$ ($C_m^{\text{left}}$ may be equal to $A_m^{\text{left}}$) and $C_m^{\text{right}}$ to be the vertex where $\xi_B$ detaches from $\sigma_r$. Similarly, we have the detachment vertices $D_m^{\text{left}} \in \sigma_{r+1} \cap \xi_B$ and $D_m^{\text{right}} \in \sigma_{r+1} \cap \xi_A$. We observe in passing, that $\xi_A$ and $\xi_B$ used in the proof of Lemma [6.1] are precisely the portions of $\xi_A, \xi_B$ that we use presently, between $C_m^{\text{left}}, D_m^{\text{right}}$ and
 głównej strategii. We now argue that \( C^{\text{left}} \) precedence or equals to \( C^{\text{right}} \) along \( \sigma_r \) (with the usual order from \( p \) to \( q \)). Assume by contradiction that \( C^{\text{right}} \) strictly precedes \( C^{\text{left}} \) and call \( \gamma \) the line spanned by the edge interval of \( \sigma_r \), that precedes \( C^{\text{left}} \). By the minimizing length of \( \xi_A \), its portion right after the detachment from \( \sigma_r \), stays in the half-plane \( S \) that is on the same side of \( \gamma \) as \( \sigma_r \cap \partial P_m \).

Assume first that \( \alpha_m^{\text{left}} \) has no intersection with \( \xi_A \). In this case, \( \xi_A \) is the straight line from \( C^{\text{left}} \) up to \( D^{\text{right}} \), and \( \alpha_m^{\text{right}} \subset S \). Thus, \( \alpha_m^{\text{right}} \) has no intersection with \( \xi_B \) and both \( \xi_B \) and \( \alpha_m^{\text{left}} \) are contained in \( S \), with \( \xi_B \) being a straight line from \( C^{\text{right}} \) to \( D^{\text{left}} \). The fact that both \( D^{\text{left}} \neq D^{\text{right}} \) are in \( \sigma_r \cap \partial P_m \) entails that the convexity of \( \sigma_{r+1} \cap \partial P_m \) and its disjointness from \( \sigma_r \).

It hence follows that the polygonal \( \xi_A \) must have a common vertex with \( \alpha_m^{\text{left}} \). Let \( a \) be the first such vertex (in order from \( C^{\text{left}} \) to \( D^{\text{right}} \)), it necessarily belongs to \( S \). Then \( D^{\text{left}} \) is contained in the region bounded by the concatenation of the portion of \( \alpha_n^{\text{left}} \) from \( a \) to \( A_m^{\text{left}} \), with \( \sigma_r \) from \( A_m^{\text{left}} \) to \( C^{\text{left}} \), with the straight segment of \( \xi_A \) from \( C^{\text{left}} \) to \( a \). Further, \( B_m^{\text{right}} \) cannot belong to the said region, unless \( B_m^{\text{right}} = a \). This again contradicts the convexity of \( \sigma_{r+1} \) between \( D^{\text{left}} \) and \( B_m^{\text{right}} \).

So indeed \( C^{\text{left}} \) (respectively \( D^{\text{left}} \)) precedes or equals \( C^{\text{right}} \) (resp. \( D^{\text{right}} \)) along \( \sigma_r \) (resp. \( \sigma_{r+1} \)).

Step 5. We now make further observation about the supporting polygonals \( \xi_A, \xi_B \). Firstly, \( \xi_A \) (respectively \( \xi_B \)) have common vertices only with \( \alpha_m^{\text{left}} \) (resp. \( \alpha_m^{\text{right}} \)) from its endpoint on \( \sigma_r \), up to its first intersection point \( E \) with \( \xi_B \) (resp. \( \xi_A \)). Indeed, the first time that \( \xi_A \) encounters \( \alpha_m^{\text{left}} \) it must also encounter \( \xi_B \), and the first time \( \xi_B \) encounters \( \alpha_m^{\text{left}} \) it must encounter \( \xi_A \) as well.

Secondly, the angles formed by \( \xi_A \) or \( \xi_B \) at these common vertices (up to \( E \)), interior with respect to the polygon with vertices \( C^{\text{left}}, E, C^{\text{right}} \) and with boundary along the appropriate portions of \( \sigma_r \), \( \xi_A \) and \( \xi_B \), are at least \( \pi \). This fact is an easy consequence of \( \xi_A, \xi_B \) being shortest paths.

The two symmetric statements to those made above, are likewise valid for the portions of \( \xi_A, \xi_B \) from their endpoints on \( \sigma_{r+1} \) up to their respective first intersection point \( E' \) (there may be \( E' = E \)). Our last statement is that between \( E \) and \( E' \), the polygonal \( \xi_A, \xi_B \) coincide. For otherwise, \( \xi_A \) and \( \xi_B \) would be non-intersecting between some common vertices \( \bar{E} \neq E' \), hence bounding a polygon whose at least one angle other than \( \bar{E}, E', E' \) would be less than \( \pi \) (with respect to the polygon’s interior). This would result in \( \alpha_m^{\text{left}} \) or \( \alpha_m^{\text{right}} \) having a vertex at the indicated angle, and hence necessarily intersecting the opposite boundary portion (of the polygon), which is a contradiction.

The two scenarios of \( \xi_A \cap \xi_B \) consisting of a single intersection point, and of a polygonal with vertices in \( V \) (we recall that \( V \) is the set of vertices of the graph \( G \)) are presented in Figure 7.2.

Step 6. In this and the next Step we assume that:

\[
E = E' \notin V.
\]

We claim that there exists a folding pattern on \( P_m \) such that:

(i) the polygonal \( C^{\text{right}} a_{h_1} \ldots a_{h_t} D^{\text{left}} \subset \xi_B \) has its image contained in some straight line \( \xi_B \),

(ii) the same polygonal has its image length unchanged from the original length,

(iii) the images of the portions of geodesics \( \sigma_r \cap \partial P_m, \sigma_{r+1} \cap \partial P_m \) are their rigid motions.

For this construction, we write \( E \in \bar{a}_{h_t} a_{h_{t+1}} \) and first consider the polygonal \( E a_{h_{t+1}} a_{h_{t+2}} D^{\text{left}} \) (see Figure 7.3 (i) with \( t = 2 \)). We start at the vertex \( a_{h_{t+1}} \) and perform the simple fold of the half-line that extends \( a_{h_{t+1}} a_{h_{t+2}} \) beyond \( a_{h_{t+1}} \), intersected with \( P_m \), onto the line spanned by the edge \( 
\bar{a}_{h_t} a_{h_{t+1}} \). In doing so, we take advantage of the fact that both indicated lines intersect \( \alpha_m^{\text{right}} \) before they may intersect \( \sigma_{r+1} \). Next, we fold by bisecting the angle at the vertex \( a_{h_{t+2}} \): the half-line extending \( \bar{a}_{h_{t+3}} a_{h_{t+2}} \) beyond \( a_{h_{t+2}} \) becomes the part of half-line that extends the (previously
modified) segment $\overline{a_{h_{z+1}}a_{h_{z+2}}}$. We continue in this fashion, until we align $\overline{D^{left}a_{h_{z}}}$ with (previously modified) $\overline{a_{h_{z-1}}a_{h_{z}}}$. Similarly, we consider the polygonal $C^{right}a_{h_{1}} \cdots a_{h_{t}}E$, and starting from $a_{h_{t}}$ we align $\overline{a_{h_{t-1}}a_{h_{t+1}}}$ with $\overline{a_{h_{t}}a_{h_{t+1}}}$: the final fold in this construction is that of half-line extending $C^{right}$ beyond $a_{h_{1}}$, intersected with $P_{m}$, onto the line spanned by $\overline{a_{h_{1}}a_{h_{2}}}$. As a result, the portion of $\xi_{B}$ between $C^{right}$ and $D^{left}$ has been straightened onto the line $\overline{\xi_{B}}$ spanned by the segment $\overline{a_{h_{t}}a_{h_{t+1}}}$.

We now fold both portions of geodesics $\sigma_{r} \cap \partial P_{m}$ and $\sigma_{r+1} \cap \partial P_{m}$ onto $\overline{\xi_{B}}$, using their concavity, in the same manner as was done in the proof of Lemma 6.1 in the simplified context of section 6. By a further rotation we may exchange $\overline{\xi_{B}}$ into $\mathbb{R}e_{1}$ and denote the resulting isometric immersion of $P_{m}$ by $w$. Recalling the notation in (7.1), it directly follows that:

$$w(B_{m}^{left}) - w(A_{m}^{left}) = \alpha_{w}e_{1}.$$  

Similarly, by folding $(\sigma_{r} \cup \sigma_{r+1}) \cap \partial P_{m}$ onto the line $\overline{\xi_{A}}$ obtained as the straightening of the portion of $\xi_{A}$ from $C^{left}$ to $D^{right}$, we get an isometric immersion $v$ of $P_{m}$ with the property that:

$$v(P_{m}^{left}) - v(A_{m}^{left}) = \alpha_{v}e_{1}.$$  

**Figure 7.3.** Elements of the proof of Lemma 7.1: diagram (i) depicts construction of the isometric immersion $w$ on the region $R_{r}$ as in Figure 7.2 (i). The arrows indicate the consecutive folds and the resulting straightenings of the intermediate polygonal $C^{right}a_{h_{1}}a_{h_{2}}a_{h_{3}}a_{h_{4}}D^{left} \subset \xi_{B}$ to the line $\overline{\xi_{B}}$, and the projections of the boundary portions of $\sigma_{r}$, $\sigma_{r+1}$ onto $\overline{\xi_{B}}$ in Step 6; diagram (ii) depicts the direction of rotating from $\overline{\xi_{B}}$ to $\overline{\xi_{A}}$, through the intermediate direction lines $\xi$ in Step 7. The intersection point of the given half-line $\xi_{1}$ with the polygonal $\xi_{B}$ is called $a$ (provided it exists).

**Step 7.** Consider the family of lines $\{\xi\}$ obtained by rotating $\overline{\xi_{B}}$ around $E$ onto $\overline{\xi_{A}}$. The direction of rotation (see Figure 7.3 (ii)) is so that the half-line from $E$ through $a_{h_{t+1}}$ gets rotated onto the half-line from $E$ to the vertex on $\overline{\xi_{A}}$ that is closest to $E$ between $E$ and $C^{left}$, without passing through $D^{right}$ and $C^{right}$ along the way. For each such line $\xi$ we will describe the folding and the resulting isometry $u_{\xi}$ on $P_{m}$, with the property that the function:

$$\xi \mapsto u_{\xi}(B_{m}^{left}) - u_{\xi}(A_{m}^{left})$$
is continuous and that \( u_{\xi_B} = w, \ u_{\xi_A} = v \). By a further rotation, we may map \( \xi \) onto \( \mathbb{R} e_1 \) and hence conclude that the scalar function \( \xi \mapsto \langle u_\xi(B_{m}^{\text{left}}) - u_\xi(A_{m}^{\text{left}}), e_1 \rangle \) attains all values in the interval \([\alpha_w, \alpha_v]\). In virtue of (7.1), this will end the proof of Lemma 7.1 under the assumption (7.2).

Fix \( \xi \) as above and denote by \( \xi^1 \) the half-line emanating from \( E \) which is the rotated image of the half-line obtained by extending \( \overline{Ea_{h_{i+1}}} \subset \xi_B \) beyond \( a_{h_{i+1}} \). We also denote \( \xi^2 = \xi \setminus \xi^1 \). Now, if \( \xi^1 \) intersects the portion of the polygonal \( \xi_B \) between \( E \) and \( D_{m}^{\text{left}} \) (we will refer to this intersection point by calling it \( a \) ), we utilize the same folding construction as in Step 6, but we replace the portion of \( \xi_B \) between \( E \) and \( a \) by the segment \( \overline{Ea} \subset \xi \). Observe that \( \overline{Ea} \) must intersect the boundary \( \alpha_m^{\text{left}} \) of \( P_m \) before it reaches \( a \). Thus there exists a simple fold which results in rotating (around \( a \)) of the edge of \( \xi_B \) containing \( a \), onto \( \xi \), and in such a way that the position of \( a \) remains unchanged, and that \( \sigma_{r+1} \cap \partial P_m \) is also only transformed via a rigid motion. We then continue the straightening procedure of \( \xi_B \) beyond \( a \) as before.

On the other hand, if the first intersection point \( a \) of \( \xi^1 \) with \( \xi_B \) occurs between \( D_{m}^{\text{left}} \) and \( B_{m}^{\text{left}} \), we again take advantage of the fact that the open segment \( \overline{Ea} \) must intersect \( \alpha_m^{\text{left}} \); this allows for a single simple fold which rotates the segment edge of \( \xi_B \), to which \( a \) belongs (this edge must be contained in \( \xi^1 \) around \( a \) and onto \( \xi \)). In both so far described cases, the geodesic portion \( \sigma_{r+1} \cap \partial P_m \) may be subsequently folded onto \( \xi \), due to its concavity and the fact that its one point (\( D_{m}^{\text{left}} \) in the former case, \( a \) in the latter) already belongs to \( \xi^1 \).

In the third case when \( \xi^1 \) has no intersection with \( \xi_B \) beyond \( E \) (hence \( \xi^1 \cap \sigma_{r+1} \cap \partial P_m = \emptyset \)), we utilize the construction from proof of Lemma 6.1. This entails identifying the line \( \gamma \) that is parallel to \( \xi \) and supporting to \( \sigma_{r+1} \cap \partial P_m \). We then first fold \( \sigma_{r+1} \cap \partial P_m \) onto \( s \), and then fold \( \gamma \) onto \( \xi \). This, again, can be done without altering \( \sigma_r \cap \partial P_m \) beyond possibly applying a rigid motion to it, because both lines \( \xi^2 \) and \( \gamma \) intersect \( \alpha_m^{\text{right}} \) before they possibly intersect \( \sigma_r \cap \partial P_m \).

Rotating \( \xi^1 \) further, we have it eventually pass through \( A_{m}^{\text{left}} \), then \( C_{m}^{\text{left}} \), then intersect the polygonal \( \xi_A \setminus \sigma_r \), and finally coincide with the appropriate half-line in \( \xi_A \). In each of these listed scenarios, we perform the corresponding (in the reverse order of appearance) folding construction relative to the polygonal \( \xi_A \) rather than \( \xi_B \). In the same fashion, we define the folding patterns relative to the half-line \( \xi^2 \). This concludes the definition of each \( u_\xi \) in case (7.2).

Step 8. Note that \( E \neq E' \) implies \( E, E' \in V \) (see Figure 7.2 (ii)). In this Step we assume that:

\[
E = E' \in V \quad \text{or} \quad E \neq E'.
\]

When \( E \neq E' \), we first perform several simple folds which straighten the polygonal \( \xi_A \cap \xi_B \) into a segment with endpoints that we continue to denote \( E \) and \( E' \), and such that the line spanned by the new \( \overline{EE'} \) enters each of the two angles between the pairs of distinct edges in \( \xi_A \) and \( \xi_B \), emanating from \( E \) and \( E' \) (see Figure 7.4 (i)). As a result, the new polygonals \( \xi_A \) and \( \xi_B \) have the same convexity properties as in case (7.2). This allows for applying the folding construction in Step 7, relative to the center point \( (E + E')/2 \) and the projection lines \( \xi_B = \xi_A \) spanned by \( \overline{EE'} \).

When \( E = E' \), we first perform one simple fold at the vertex \( E \), which either: (i) makes the two edges of \( \xi_B \) with common vertex \( E \) collinear, and keeps the angle (internal to \( P_m \)) between the two edges of \( \xi_A \) adjacent to \( E \) not smaller than \( \pi \); or (ii) makes the two edges of \( \xi_A \) with common vertex \( E \) collinear, and keeps the angle between the two edges of \( \xi_B \) adjacent to \( E \) not smaller than \( \pi \). In what follows we will assume, without loss of generality, the former scenario as in Figure 7.4 (ii).
Figure 7.4. Construction in Step 8 of the proof of Lemma 7.1: (i) depicts the result of the initial folding, straightening $\xi_A \cap \xi_B$ into segment $EE'$, applied to $P_m$ in Figure 7.2 (ii); diagram (ii) indicates the initial folding in case $E = E'$.

We let $\bar{\xi}_B$ to be the line spanned by the segment of $\xi_B$ passing through $E$ and $\bar{\xi}_A$ to be spanned by the segment with vertex $E$ and the successive vertex along $\xi_A$ towards $C$ left. We also call $\xi_2^A$ the half-line from $E$ through the successive vertex along $\xi_A$ towards $D$ right. We now apply the construction from Steps 6 and 7, where we rotate the line $\bar{\xi}_B$ onto $\bar{\xi}_A$ around $E$ and perform a family of foldings onto each intermediate line $\xi$. In case $\xi_2^A \not\subset \bar{\xi}_A$, the same construction is applied to each half-line emanating from $E$ and intermediate to $\bar{\xi}_A$ and $\xi_2^A$, completed by an extra simple fold at $E$ that aligns the said lines. This ends the definition of each $u_{\xi}$ in case (7.3).

Step 9. The final step is to construct $u$ on $P_s$. This can be done by the same folding technique as in Step 1. We also get: $\langle u(B_{s}^{left}) - u(A_{s}^{left}), e_1 \rangle = length(A_{s}^{left} \ldots a_i q_1) - length(B_{s}^{left} \ldots a_i q_1)$, because $length(\sigma_r) = length(\sigma_{r+1})$. This ends the proof of Lemma 7.1.

8. PROOF OF THEOREM 2.2. Step 4: ISOMETRIC IMMERSION ON THE EXTERIOR REGION. A COUNTEREXAMPLE WHEN $p, q \notin \partial \Omega$

In this section, we first construct an isometric immersion $u$ on the remaining region $R_0$.

Lemma 8.1. Assume $[5]$ and $[51]$. If $p, q \in \partial \Omega$ then there exists a continuous, piecewise affine isometric immersion $u$ of $R_0$ into $\mathbb{R}^3$, satisfying:

$$u(p) = 0, \quad u(q) = length(\sigma_1)e_1, \quad u(\sigma_1) = u(\sigma_N) = I.$$

Proof. By Lemma [52] we have: $R_0 \cap L = \emptyset$ and both the least and the greatest geodesics $\sigma_1, \sigma_N$ are convex, i.e. the region $\Omega \setminus R_0$ has all the (internal) angles at the vertices distinct from $p, q$, not greater than $\pi$. Indeed, consider an intermediate vertex $A \notin \{p, q\}$ of $\sigma_{min}$. If the internal angle at $A$ was strictly larger than $\pi$, then the cut $l = AB \in E$ emanating from $A$ would have to point inside
$R_0$, as otherwise $\sigma_{\min}$ could be shortened, contradicting the fact that it is a geodesic. The argument for $\sigma_{\max}$ is similar. One can now apply the usual sequence of simple folds to obtain $u$ on $R_0$. 

The proof of Theorem 2.2 is now complete. Note that the constructed isometric immersion $u$ consists exclusively of planar folds and returns the image that is a subset of $\mathbb{R}^2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Examples of minimal configurations with the external region $R_0$ containing non-sealable cuts: in (i) the graph $G$ consists of three trees $\{T_i\}_{i=1}^3$. There are two geodesics $\sigma_1, \sigma_2$ from $p$ to $q$ in $\Omega \setminus L$, a single internal region $R_1$ (which contains $T_2, T_3$) and the external region $R_0$ which contains $T_1$. Note that $T_1$ has leaves on both $\sigma_1$ and $\sigma_2$; in (ii) $R_0$ contains two “nested” trees $T_1, T_2$, while there are five more trees in $R_1$; in (iii) $R_0$ contains two trees $T_1, T_2$ (not “nested”); in (iv) we assume that $|pa_1| < |a_2a_3| - |a_2x|$, for minimality. There is a single tree $T_1$ in $R_0$ and a single tree $T_2$ in $R_1$. The two geodesics are: $\sigma_1 = pa_2a_3a_4q$ and $\sigma_2 = pa_1a_5a_6q$.}
\end{figure}

The fact that $R_0 \cap L = \emptyset$ is directly related to the assumption that $p, q \in \partial \Omega$. Indeed, examples in Figure 8.1 show there may be (even multiple) external trees, necessarily with vertices on both $\sigma_1$ and $\sigma_N$, when the said assumption is removed. This type of configuration may also be used to show that an isometric immersion $u$ of $\Omega \setminus L$ with the property that the Euclidean distance between $u(p)$ and $u(q)$ equals the geodesic distance from $p$ to $q$ in $\Omega \setminus L$, may in general not exist.
Figure 8.2. A configuration of $G, \Omega$ and $p, q \in \Omega$ for which the conclusion of Theorem 2.1 fails. The set of vertices of $G$ is: $V = \{a_1 = (0,0), a_2 = (1,0), a_3 = (\frac{1}{\sqrt{2}} - c, \frac{1}{\sqrt{2}}), a_4 = (\frac{1}{\sqrt{2}} - c, \alpha)\}$ and $p = (-c,0), q = (\frac{1}{2}, \frac{1}{2})$. The set of cuts is: $L = T_1 \cup T_2$, where $T_1$ is the single exterior tree and $T_2$ is the single interior tree. For every $\frac{1}{\sqrt{2}} - \frac{1}{2} < c < \frac{1}{\sqrt{2}}$ there exists $0 < \alpha < \frac{1}{2}$ so that there are two geodesics $\sigma_1 = pa_1a_2q$, $\sigma_2 = pa_3a_4q$ satisfying: $\text{length}(\sigma_1) = \text{length}(\sigma_2) = 1 + \frac{1}{\sqrt{2}} + c$. When additionally $c < \sqrt{2} - 1$, then the above configuration is minimal.

Consider the example in Figure 8.2. It is easy to check that $\text{dist}_{\Omega \setminus L}(p,q) = c + 1 + \frac{1}{\sqrt{2}} = \text{length}(\sigma_1) = \text{length}(\sigma_2)$, when:

$$\alpha = \alpha(c) = \frac{1 - \frac{1}{\sqrt{2}} + (1 - \sqrt{2})c}{2c + 1}.$$ 

Also, the constraints $\frac{1}{\sqrt{2}} - \frac{1}{2} < c < \frac{1}{\sqrt{2}}$ and $0 < \alpha < \frac{1}{2}$ (implying that the interior region $R_1$ has exactly the shape indicated in Figure 8.2) hold, in particular, when taking:

$$\frac{1}{\sqrt{2}} - \frac{1}{2} < c < \sqrt{2} - 1. \tag{8.1}$$

The minimality of the configuration $L = T_1 \cup T_2$ is guaranteed by requesting that: $\text{length}(pa_1pa_2q) < \text{dist}(p,q)$, which upon a simple calculation reduces to: $c < \sqrt{2} - 1$, guaranteed in (8.1).

We now claim that there is no isometry $u$ of $\Omega \setminus (T_1 \cup T_2)$, which straightens the polygonal $\bar{a_1a_2q}$. This is because otherwise there would be:

$$\text{length}(\bar{a_1a_2q}) = |u(q) - u(a_1)| \leq \text{dist}_{\Omega \setminus R_1}(a_1, q).$$

However, the inequality above is violated when the tree $T_1$ approximates closely the polygonal path $\bar{a_1pa_2q}$, in view of the bound $\text{length}(\bar{a_1pa_2q}) < \text{length}(\bar{a_1a_2q})$ which again follows from (8.1).

9. Discussion

Our two geometrical theorems are inspired by simple observations of the mechanical response of a sheet of paper that has cuts in it, valid only in the limit when the sheet is mapped to itself via a piecewise (non-unique) affine map that is isometric to the plane. To remove this non-uniqueness, we must account for the energetic penalty of deforming a sheet of small but finite thickness, by bending
it out of the plane. When this physical fact is accounted for, a kirigamized sheet will deform into a complex shape constituted of conical and cylindrical structures glued together.

Understanding the mechanics and mathematics of these objects, while also solving the inverse problem of how to design the number, size, orientation and location of the cuts, remain open problems. And while we have limited ourselves to the study of Euclidean case, our study naturally raises questions about the nature and form of geodesics in non-Euclidean surfaces with co-dimension one obstructions, and higher-dimensional generalizations that might be relevant for traffic, fluid flow and stress transmission in continuous and discrete geometries.

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