Inverse problems in atmospheric science and their application

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Abstract. This paper reviews various kinds of inverse problems in atmospheric science and oceanography, and introduces powerful methods to treat these problems – variational data assimilation (VAR) and improved discrepancy principle, and discusses some essential difficulties in VAR. Due to the ill-posedness of these problems, the regularization method is also applied, i.e., additional terms are added to the cost functional as a stabilized functional with physical meaning. Inversions of four specific problems, such as the inversion of one-dimensional sea temperature model, the inversion of parameters in an ENSO cycle model, the inversion of wind field with single-Doppler data, and the inversion of satellite remote sensing, indicate that, adoption of the regularization method in VAR will overcome the ill-posedness, constrain calculational oscillations in iteration, and speed up convergence of solutions.

1. Introduction

It is well known that numerical prediction of atmospheric and oceanic motions is reduced to solving a set of nonlinear partial differential equations with initial and boundary conditions, which is often called direct problems\cite{1}. The direct problems are essentially, given models (equations and initial and boundary conditions ), to seek solutions and make prediction. In recent years, a variety of methods have been proposed to boost accuracy of numerical weather prediction, such as VAR, etc.

Data assimilation is, according to Talagrand and Thacker et al\cite{3-6}, using all the available information (e.g., observational data from satellites, radars, and GPS, etc.) to determine as accurately as possible the state of the atmospheric or oceanic flow. The 4-D variational data assimilation (4-D VAR), is to study assimilation through variational analytical method.

VAR was suggested in the 1980s, which is a combination of variational analysis and optimal control theory of partial differential equations (PDE). Variational data assimilation takes models as constraints, and calculates the gradient of cost functional with respect to initial conditions, boundary conditions, and parameters by the adjoint method, so that the solutions to models fit observations as accurately as possibly.

Specifically, the aim of VAR is to obtain initial and boundary conditions and parameters of models statically or dynamically from observational data, which includes a variety of cases. Such as, given a model, retrieving parameters and initial conditions of the model using observations over a period; retrieving lateral boundary conditions for a limited-area numerical prediction model using observations, and so on. Generally speaking, VAR problem is characterized by
ill-posedness, and belongs to the category of inverse problems[2]. In the present paper, the data assimilation problem in meteorology and physical oceanography is re-examined using the variational optimal control approaches in combination with the regularization techniques. To overcome the difficulty of ill-posedness, an additional term is added to the cost functional as a stabilized functional with physical meaning.

2. Performance of 4-D VAR and existing key difficulties

As an example, we consider inversion of an initial value condition (IVC) for the following problem

\[
\begin{aligned}
\frac{\partial X}{\partial t} &= F(t, X) \\
X|_{t=0} &= u, \quad \text{BVC}
\end{aligned}
\]

We hope to seek an optimal IVC so as to make the functional \( J[u] \) minimal.

\[
J[u] = \frac{1}{2} \int_0^T \| X - X^{obs} \|_{L^2(\Omega)}^2 dt = \min!
\]

The main part of the algorithm is to calculate the functional gradient \( \nabla_u J \), by which \( u^* \) can be worked out by the steepest descent method. This can be done in the following three steps:

**Step1**: derivation of the following tangent linear model (TLM)

Let \( u \) be disturbed to \( u + \alpha \hat{u} \), and the corresponding solution of equation(1) are \( X(t) \) and \( \hat{X}(t) \). Set \( \hat{X} = \lim_{\alpha \to 0} \frac{\tilde{X} - X}{\alpha} \), and assume that \( F \) is differentiable. Thus the TLM for \( \hat{X} \) is

\[
\begin{aligned}
\frac{\partial \hat{X}}{\partial t} &= F'(t, \hat{X}) \hat{X} \\
\hat{X}|_{t=0} &= \hat{u}, \quad \text{homogeneous BVC}
\end{aligned}
\]

The linear equation (3) establishes the relationship between \( \hat{X} \) and \( \hat{u} \). The solution of (3) is \( \hat{X} = R(t, 0) \hat{u} \), where \( R(t, 0) \) is the resolvent operator of TLM.

**Step2**: Determining of the directional derivative of \( J \) along the direction \( \hat{u} \)

The G-derivative is

\[
J'[u; \hat{u}] = (\nabla_u J, \hat{u}) = \int_\Omega \nabla_u J \cdot \hat{u} d\Omega
\]

On the other hand, the G-derivative can be expressed as from (2)

\[
J'[u; \hat{u}] = \lim_{\alpha \to 0} \frac{1}{2\alpha} \int_0^T \int_\Omega \left[ (\hat{X} - X^{obs})^2 - (X - X^{obs})^2 \right] d\Omega dt = \int_0^T (X - X^{obs}, \hat{X}) dt
\]

Hence

\[
\int_0^T (X - X^{obs}, \hat{X}) dt = (\nabla_u J, \hat{u})
\]

**Step3**: Introducing the adjoint system

Multiplying (3) by the adjoint variable \( p \), and integrating on \([0, T] \times \Omega \), yields to

\[
\int_0^T \int_\Omega \frac{\partial \hat{X}}{\partial t} p d\Omega dt = \int_0^T (F'(t, X) \hat{X}, p) dt
\]

After some manipulation, we have:
\[
(\dot{X}(T), p(T)) - (\dot{X}(0), p(0)) + \int_0^T \left( \dot{X}, -\frac{\partial p}{\partial t} \right) dt = \int_0^T \left( \dot{X}, F^s(t, X) p \right) dt \quad (5)
\]

Setting
\[
\begin{cases}
-\frac{\partial p}{\partial t} = F^s(t, X) p + (X - X^{obs}) \\
p(T) = 0, \quad \text{conjugating BVC}
\end{cases}
\quad (6)
\]

then,
\[
(\nabla_u J, \ddot{u}) = \int_0^T \left( X - X^{obs} \right) \dot{X} dt = (p(0), \ddot{u})
\]

from which, we have
\[
\nabla_u J = p(0) \quad (7)
\]

There exist some essential difficulties in 4-DVAR.

[I] Ill-posedness. During iteration, the cost functional often oscillates and decrease slowly, and thus lead to low accuracy. This phenomenon results from ill-posedness of the problem. One way to overcome this difficulty is to apply the regularization method, i.e., to introduce stabilization functionals and regularization parameters into the cost functional. However, due to the complexity of problems, it is a difficult problem how to choose proper stabilization functionals with physical backgrounds and regularization parameters

[II] Incomplete observation problem. In some cases, especially in oceans, observations are insufficient, and available data are often obtained through ship, satellite, sounding balloons, and sparse observation stations, which will lead to instability of calculation.

[III] Determination of lateral boundary conditions. In past years, 4-DVAR has been successful for global models. But the determination of lateral boundary conditions is still a difficult problem and unsolved well for assimilation of limited area models. It is known that boundaries include closed and open ones. The former is physical boundary, and the latter is the artificial one. Lu and browning[7] suggested, for example, that physical magnitudes from a larger scale model be taken as boundary conditions in limited area models, which inevitably results in errors. Gustanfsson[8] pointed out that, if lateral boundary conditions are inaccurate, VAR will give rise to bad results. It is a very difficult problem how to modify boundary conditions.

[IV] The adjoint method holds only for differentiable systems, while parameterizations of physical processes are adopted in usual prediction models, which contains uncontinuity of physical processes (called as “on-off” [9-12]). For systems containing non-differentiable physical processes, a new method must be developed.

[V] Assimilation of data from radar, satellite and GPS. Let’s take an example to illustrate this. The grid lengths of usual prediction models are large, while data of radar are distributed densely. Then, there exists a problem of matching between data of radar and regular observational data, i.e., a problem of combination of data of radar and regular observational data. For regions with dense data, if all data are used, calculation is gigantic; if one part of data is used, some useful information is omitted. How to combine various data properly is a very hard task. In recent years, Kalman filtering has been applied universally in numerical weather prediction, just because it makes full use of available information. Nevertheless, there are some difficulties for Kalman filtering to be applied to 4-DVAR, such as, how to give prediction and observational operators, choice of initial values, and so on.
3. Assimilation of complete and incomplete data and the regularization

For illustrational purpose, we will use a one-dimensional heat-diffusion model formulated for describing the vertical distribution of sea temperature over time [13]. The governing equation is:

\[
\left\{ \begin{array}{l}
\frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial T}{\partial z} \right) - \frac{v I_0}{\rho_0 c_p} \exp(-\gamma z) \\
T|_{t=0} = U(z), \quad K \frac{\partial T}{\partial z} + \frac{I_0}{\rho_0 c_p} |_{z=0} = \frac{Q(t)}{\rho_0 c_p}, \quad K \frac{\partial T}{\partial z} |_{z=H} = 0 \end{array} \right. \tag{8}
\]

Where \( T = T(t,z) \) is sea temperature, \( K = K(t,z) \) the vertical eddy diffusion coefficient, \( \rho_0 \) the sea water density, \( c_p \) sea water specific capacity, \( v \) the light diffusion coefficient, \( H \) the depth of ocean upper layer, \( I_0 \) is the transmission component of solar radiation at sea surface, and \( Q(t) \) the net heat flux at sea surface.

Assume \( v, c_p \) and \( \rho_0 \) are known constants, \( U(z) \), \( Q(t) \), \( K(t,z) \) and \( I_0 \) are not known exactly, e.g. they have unknown errors and need to be determined by data assimilation. Now a set of observations of sea temperature \( T^{\text{obs}} \) is given on the whole domain or is taken only at sea surface, i.e. \( T^{\text{obs}}(t,0) \). We consider two cases of assimilation.

3.1. Assimilation of complete data

Let \( T^{\text{obs}} \) be given on the whole domain. The improved cost functional is defined as follows:

\[
J[U,k,Q,I_0] = \frac{1}{2} \int_0^\tau \int_0^H (T - T^{\text{obs}})^2 \, dz \, dt + \frac{\gamma^2}{2} \int_0^\tau \int_0^H K(t,z) \left( \frac{\partial T}{\partial z} \right)^2 \, dz \, dt \tag{9}
\]

Where \( \frac{\gamma^2}{2} \int_0^\tau \int_0^H K(t,z) \left( \frac{\partial T}{\partial z} \right)^2 \, dz \, dt \) is a stable functional, and \( \gamma \) is a regularization parameter.

Then, the adjoint equations and adjoint boundary conditions are:

\[
\left\{ \begin{array}{l}
-\frac{\partial p}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial p}{\partial z} \right) - \gamma^2 \frac{\partial}{\partial z} \left( K \frac{\partial T}{\partial z} \right) + (T - T^{\text{obs}}) \\
p|_{t=\tau} = 0, \quad \left[ \frac{\partial p}{\partial z} - \gamma^2 \frac{\partial T}{\partial z} \right] |_{z=0} = 0, \quad \left[ \frac{\partial p}{\partial z} \right] |_{z=H} = 0 \end{array} \right. \tag{10}
\]

\[
\nabla_U J = p(0,0) \\
\nabla_K J = -\frac{1}{2} \gamma^2 \left( \frac{\partial T}{\partial z} \right)^2 \\
\nabla_Q J = -\frac{\rho_0 c_p}{K} p(t,0) \\
\nabla_{I_0} J = \int_0^\tau \frac{p(t,0)}{\rho_0 c_p} \, dt - \frac{v}{\rho_0 c_p} \int_0^\tau \int_0^H \exp(-vz) \cdot p(t,z) \, dt \, dz 
\]

Then by using the Newton methods, the optimal values of \( U(z) \), \( Q(t) \), \( K(t,z) \) and \( I_0 \) can be obtained.

In order to test the theoretical results above, one experiment was performed by the numerical method. The ideal version of the model (8) is given below

\[
\left\{ \begin{array}{l}
\frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial T}{\partial z} \right) + f(t,z), \quad (t,z) \in (0,1) \times \left( 0, \frac{\pi}{2} \right) \\
T|_{t=0} = U(z), \quad K \frac{\partial T}{\partial z} + \frac{I_0}{\rho_0 c_p} |_{z=0} = \frac{Q(t)}{\rho_0 c_p}, \quad K \frac{\partial T}{\partial z} |_{z=\frac{\pi}{2}} = 0 \end{array} \right. \tag{12}
\]
Figure 1. The iteration process of estimation of eddy diffusion coefficient cost functionals. (a) The $J$ (note that the two cost functionals shown in this figure are both calculated according to eq(9) in order to compare the efficiency of the two methods); (b) The variation norm of eddy diffusion coefficient error $E_K$ during the iteration process.

where $f(t, z) = \sin[\cos(t) - \sin(t)]$, $Q(t) = \text{const}$, $U(z) = \sin(z)$ and eddy diffusion coefficient is $k(t, z) = 1$, and the ideal model (12) has the analytical solution $T(t, z) = \sin(z) \cos(t)$. We take the true solution $T(t, z) = \sin(z) \cos(t)$ as the observation data, and add different perturbations to the first guess of initial conditions and eddy diffusion coefficient, and then the assimilation process is performed. The space is discretized into $N-1$ equal intervals, here $N = 11$, and the number of time steps is $M, M = 10$. The time interval is 0.01. Set

$$E_U = \left\| U^i - U^T \right\| = \left[ \frac{1}{N} \sum_{j=1}^{N} \left( U^i_j - U^T_j \right)^2 \right]^{\frac{1}{2}}, E_K = \left\| K^i - K^T \right\| = \left[ \frac{1}{MN} \sum_{l=1}^{M} \sum_{j=1}^{N} \left( K^i_{jl} - K^T_{jl} \right)^2 \right]^{\frac{1}{2}}$$

where superscript “$i$” stands for the $i$th iteration and superscript “$T$” stands for the true solution.

In the experiment, the initial condition was kept as true. A perturbation is added to the eddy diffusion coefficient $U_0(z) = \sin(z), \quad K = 1 + 0.2z, \quad \gamma = 0$ and $\gamma = 0.001$

The results are shown in Figure 1.

3.2. Local data assimilation

The forward model is the same as (8), the differences are that the observations are taken only at sea surface, i.e., $T^{\text{obs}}(t, 0)$ are given[14]. The cost function are defined

$$J[U, K] = \frac{1}{2} \int_0^\tau (T(t, 0) - T^{\text{obs}})^2 dt + \frac{\gamma^2}{2} \int_0^\tau \int_0^H K(t, z) \left( \frac{\partial T}{\partial z} \right)^2 dz dt \tag{13}$$

The remaining steps are the same as those in assimilation of complete data, and omitted here. The other work related to VAR and the regularization method can refer to the papers[15-20].

4. ENSO cycle and parameters inversion

The El Nino-Southern Oscillation (ENSO) is the most prominent interannual oscillation of tropical climate system, and therefore draws great attention [21-25]. In the following, we study it from the inverse problem viewpoint.
A nonlinear dynamic system describing ENSO is \[24\]:

\[
\begin{align*}
\dot{T} &= a_1 T - a_2 h + a_3 T(T - \mu h) + f_1 \\
\dot{h} &= b(2h - T) + f_2 \\
T|_{t=t_0} &= T_0 \\
|h|_{t=t_0} &= h_0
\end{align*}
\]

(14)

Where the variable \(T\) represents (non-dimensional) sea surface temperature anomaly (SSTA), variable \(h\) represents (non-dimensional) thermocline depth anomaly. The parameter \(b\) is a monotonous function of the air-sea coupling coefficient; \(f_i (i = 1, 2)\) are external forcing; \(a_i\) and \(\mu\) are constants, where \(a_1 = -0.97, a_2 = -2.84, a_3 = 0.14\).

Using the observation data set TAO (Tropical Atmosphere and Oceans), we can obtain the time series of \(T^{\text{obs}}\) and \(h^{\text{obs}}\). Figure 2.(a) is the time series of \(T^{\text{obs}}\) (solid line) and \(h^{\text{obs}}\) (dotted line). Figure 2.(b) is the phase orbit of \(T^{\text{obs}}\) and \(h^{\text{obs}}\) (Running clockwise as the time goes on).

From Figure 2.(b), we see that the distribution of \(T^{\text{obs}}\) and \(h^{\text{obs}}\) is quasi-periodic during the years (1997-1998). Now, we seek the optimal parameter \(b\) and external forcing \(f_1 (t)\) and \(f_2 (t)\), such that the solutions \((T, h)\) satisfies:

\[
J[\mu, b, T_0, h_0, f_1, f_2] = \frac{1}{2} \int_{t_0}^{t_e} \left[ \left( T - T^{\text{obs}} \right)^2 + W \left( h - h^{\text{obs}} \right)^2 \right] dt + \frac{\gamma}{2} \left[ \left( T - T^{\text{obs}} \right)^2 + W \left( h - h^{\text{obs}} \right)^2 \right] |_{t_e}
\]

\[
= \min!
\]

(15)

where \(\frac{1}{2} \left[ \left( T - T^{\text{obs}} \right)^2 + W \cdot \left( h - h^{\text{obs}} \right)^2 \right] |_{t_e}\) is the terminal control term, \(\gamma\) the control parameter and \(W\) weight function. In order to assure that the values of \(T - T^{\text{obs}}\) and \(h - h^{\text{obs}}\) have the same effect in the functional, the value of \(W\) is 1.2, which is equal to the norm of \(T\) and \(h\).

The adjoint system and the gradients of the functional are as follows:
\[
\begin{align*}
\frac{dp_1}{dt} &= (-a_1 - 2a_3 T + a_3 \mu h)p_1 + bWp_2 - \left(T - T^{obs}\right) \\
\frac{dp_2}{dt} &= -2bWp_2 + (a_2 + a_3 \mu T)p_1 - W\left(h - h^{obs}\right) \\
p_1 \mid t=t_e &= -\gamma \left(T - T^{obs}\right) \\
p_2 \mid t=t_e &= -\gamma W\left(h - h^{obs}\right)
\end{align*}
\]

(16)

\[
\begin{align*}
\nabla \mu J &= a_3 \mu h T p_1 \\
\nabla b J &= -W \left(2h - T\right) p_2 \\
\nabla f_1 J &= -p_1 \\
\nabla f_2 J &= -W p_2
\end{align*}
\]

(17)

In order to show the effectiveness of the terminal control items, two comparable experiments have been made. In the first experiment, \(\gamma = 0\) while in the second one \(\gamma = 0.4\). The results are illustrated in Figure 3.

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**Figure 3.** Comparison of the phase orbit of \(T\) and \(h\) distilled from TAO (Between the years of 1997 and 1998) and predicted by the model

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5. **Two-dimensional wind retrievals from single-Doppler data**

Modern weather Doppler radars are important tools to investigate microscale and mesoscale weather systems, but their direct observations are limited to the reflectivity and the radial-velocity component. So, it is necessary to obtain the two-dimensional or three-dimensional wind fields by retrieval methods.

In this section, we combine the variational adjoint method with regularization methods to perform retrievals of two-dimensional wind and related parameters from single-Doppler radar data in polar coordinate system. The method uses the mass continuity equation as weak constraints, in addition to the strong constraint of reflectivity conservation equation. The wind retrievals are classified into two cases: wind fields without and with vortex. The former is called the clear air mode, and the latter is called the storm mode. After the experiments with artificial data, natural two-dimensional wind fields are retrieved with low-altitude radar data from Nanjing Doppler radar, and results are encouraging and promising.

5.1. **Theoretical analyses**

The two-dimensional reflectivity conservation equation in polar coordinate system follows:
\[
\begin{aligned}
&\frac{\partial \eta}{\partial t} + \frac{v_r}{r} \frac{\partial \eta}{\partial r} + \frac{v_\alpha}{r} \frac{\partial \eta}{\partial \alpha} = k_1 \frac{\partial^2 \eta}{\partial r^2} + 1 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \eta}{\partial \alpha^2} \eta, (t, r, \alpha) \in (0, T) \times (r_1, r_2) \times (\alpha_1, \alpha_2) \\
&\eta|_{t=0} = \phi(r, \alpha), \eta|_{r=r_1} = \varphi_1(t, \alpha), \eta|_{r=r_2} = \varphi_2(t, \alpha), \eta|_{\alpha=\alpha_1} = \psi_1(t, r), \eta|_{\alpha=\alpha_2} = \psi_2(t, r)
\end{aligned}
\]  

(18)

Here \(t, r, \alpha\) denotes time, radial distance and azimuth respectively. The radial directions in the polar coordinates will be coincident with radar beam, if we assume the radar situation point is the polar point. \(\eta(t, r, \alpha)\) denotes radar reflectivity, and \((v_r, v_\alpha, k)\) are radial and tangent velocity and coefficient of eddy viscosity respectively. \((0, T) \times (r_1, r_2) \times (\alpha_1, \alpha_2)\) is the retrieval time window and area. \(\phi(r, \alpha), \varphi_1(t, \alpha)\) and \(\psi_1(t, r)(i = 1, 2)\) are initial-boundary conditions given by the reflectivity observations \(\eta^{\text{obs}}\).

The two-dimensional wind fields and the coefficient of eddy viscosity \((v_r, v_\alpha, k)\) will be retrieved or estimated from the Doppler radar data \(\eta^{\text{obs}}, v_r^{\text{obs}}\) containing errors, with the model (18) as physical constraints. Because three unknown variables are to be retrieved from just one equation, the solution of (18) must be ill-posed. That is to say, solutions do not always exist, or they aren’t determined uniquely when they exist, or they become unstable when they exist uniquely. To overcome the ill-posedness of solutions, the cost function can be defined as follows:

\[
J[v_r(r, \alpha), v_\alpha(r, \alpha), k] = J_1 + J_2 + J_3
\]  

(19)

Here

\[
\begin{aligned}
J_1 &= \frac{1}{2} \int_0^T \int_{r_1}^{r_2} \int_{\alpha_1}^{\alpha_2} (\eta - \eta^{\text{obs}})^2 d\Omega + \frac{\gamma_1}{2} \int_0^T \int_{r_1}^{r_2} \int_{\alpha_1}^{\alpha_2} \left(v_r - v_r^{\text{obs}}\right)^2 d\Omega \\
J_2 &= \frac{\gamma_2}{2} \int_0^T \int_{r_1}^{r_2} \int_{\alpha_1}^{\alpha_2} \left(\frac{\partial (rv_r)}{\partial r} + \frac{\partial v_\alpha}{\partial \alpha}\right)^2 d\Omega \\
J_3 &= \frac{\gamma_3}{2} \int_0^T \int_{r_1}^{r_2} \int_{\alpha_1}^{\alpha_2} \left(\|\nabla v_r\|^2 + |\nabla v_\alpha|^2\right) d\Omega
\end{aligned}
\]  

(20)

\(\eta\) is the reflectivity field predicted from model (18). \((v_r, v_\alpha, k)\) are retrieved variables. \(\eta^{\text{obs}}, v_r^{\text{obs}}\) are observations from radar. \(d\Omega = rdrd\alpha dt\) and \(\nabla = \partial_r \partial r + j \partial_\alpha / \partial \alpha\) is the gradient operator. \(\gamma_1, \gamma_2, \gamma_3\) and \(\gamma_4\) are weighting coefficients, and these values are chosen so that all cost function terms have the same order of magnitude after being multiplied by the coefficients. \(J_1\) is observations data constraint term measuring the misfit between radar observations and retrieved or predicted variables. \(J_2\) is the weak constraint of two-dimensional mass continuity equation, which measures the extent that mass conservation are satisfied by the estimated variables. \(J_3\) is called as the regularization term or spatial smooth term, due to the fact that natural wind fields always change slowly in space. The main effect of it is to eliminate false and high frequency component in the retrieved winds and to overcome the ill-posedness in the retrieving processes. The smaller the cost function is, the more the predicted variable \(\eta\) and retrieved ones \((v_r, v_\alpha, k)\) are consistent with observations and background fields; and the physical law are more satisfied by the retrieved ones; and also it shows that the more correct the retrieved winds and source and sink terms or coefficient of eddy viscosity are. The ultimate goal for retrieval is to optimally estimate three unknowns \((v_r, v_\alpha, k)\) with the precondition that the model is met maximally through the minimization of the cost function. In a word, the variational adjoint method is used to retrieve winds and related parameters, which is similar to four-dimensional VAR in meteorology. Namely, the optimal control variables are determined by the use of all available information including observations, background fields and known physical laws which are expressed by models. There is no essential difference between data assimilation and retrieval except that the former is an initial value problem and the latter is a model parameter identification one.
The gradient of $J$ with respect to the unknowns $(v_r, v_\alpha, k)$ can be calculated by variational adjoint method. For convenience, the inner product is defined: $\langle f, g \rangle = \int_0^T \int_{r_1}^{r_2} \int_{\alpha_1}^{\alpha_2} (f \cdot g) d\Omega$. The gradients are given as follows:

$$
\begin{align*}
\nabla_{v_r} J &= \int_0^T \left\{ \gamma_1 (v_r - v_r^{\text{obs}}) - p \frac{\partial \eta}{\partial r} - \gamma_2 \frac{\partial (r v_r)}{\partial r} + \partial v_\alpha - \gamma_3 \frac{1}{r} \frac{\partial (r v_r)}{\partial \alpha} + \frac{\partial v_r}{\partial \alpha} \right\} dt \\
\nabla_{v_\alpha} J &= \int_0^T \left\{ -p \frac{\partial \eta}{r \partial \alpha} - \gamma_2 \frac{\partial (r v_r)}{r \partial \alpha} + \frac{\partial v_\alpha}{r \partial \alpha} - \gamma_3 \frac{1}{r} \frac{\partial (r v_r)}{r \partial \alpha} + \frac{\partial v_\alpha}{r \partial \alpha} \right\} dt \\
\nabla_k J &= \langle p, \left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \alpha^2} \right) \eta \rangle
\end{align*}
$$

(21)

5.2. Retrieval with real data

In this section, the two-dimensional wind fields are retrieved from real Doppler radar data. The data collected by the Nanjing Doppler radar on 8 July 2003 are used for testing. Nanjing Doppler radar is located at the top of the LongWang mountain. The situated point is at the height of 138.8 m and its longitude and latitude is (118°24'33''E, 32°11'38''N). The reflectivity data can reach a range of 460 km and its resolution is 1 km. The radial-wind data can reach a range of 230 km and its resolution is 250 m. The time elapsed for the entire volume scan is about 6 minutes which contains 9 elevation scans from low to high in the precipitation mode. The reflectivity, radial-wind and velocity band width data are collected in each elevation. So, the Doppler radar data are of high resolution in both time and space comparing with conventional data. The raw datasets are selected for the following period: 15 : 28 – 15 : 58, 8 July 2003. The period covered six sequential volume scans. The raw data from the lowest elevation angle (0.3°) are used for retrieval. The polar coordinates $(r, \alpha)$ are centered at the radar point. The radial and azimuthal resolution for reflectivity is chose as grid resolution $(\Delta r, \Delta \alpha) = (1 km, 0.9865^\circ)$. The retrieval area is set as follows:

$125 km \leq r \leq 125 km + 24 \Delta r, 92.06^\circ \leq \alpha \leq 92.06^\circ + 14 \Delta \alpha$ where it contains fewer holes than other places.

The initial guesses for wind fields are all zeroes and initial value for the coefficient of eddy diffusion is 100.0. Each weight is specified as follows: $\gamma_1 = 1.80, \gamma_2 = 0.05, \gamma_3 = 0.50$. The retrieved value for the coefficient of eddy viscosity is 150.1 and retrieved wind field is shown in Figure 4.

![Figure 4. Retrieved wind field from real observation data](image-url)
As shown in the Figure 4, the retrieved wind field is even and uniform in the whole and similar to real wind fields. The retrieved wind direction is consistent with the qualitative analysis of radar PPI figure. The accuracy for retrieved pattern and magnitude of wind field need to be verified by the dual-Doppler radar data.

6. Inversion of satellite remote sensing

If the atmosphere scatter effect is ignored, then the infrared radiance of the earth’s atmosphere system that goes to satellite infrared sensors is[30-31]:

\[ R = \varepsilon B_s \tau_s - \int_0^{p_s} B \tau(0, p) + (1 - \varepsilon) \int_0^{p_s} B \tau(p, p_s) + R', \]  

(22)

where \( R \) is the spectral radiance of a channel, \( B \) is Plank function, \( \tau \) is the total atmosphere transmittance above the pressure level, \( \varepsilon \) is surface emissivity, subscript \( s \) indicates surface value of physical quantities, \( R' \) is reflected radiation of sun. If the radiance of all the spectral channels \( R \) is known, then eq.(22) can be written as a nonlinear operator equation:

\[ F(X) = Y \]  

(23)

where vector \( X = X(T, q, T_s, ... \) contains atmosphere temperatures and water vapor mixed ratios of levels and surface temperature, vector \( Y \) contains \( N \) satellite brightness temperature (i.e., channel numbers are \( N \)).

Assume \( X_0 \) is the first guess of atmosphere profile and denote \( Y_0 = F(X_0) \). Then, two parts are processed as follows

6.1. Linearize the nonlinear operator equation at \( X_0 \) to determine \( \gamma_1 \).

Consider the following two linearized equations:

\[ F(X) \approx F(X_0) + F'(X_0) \cdot (X - X_0) = Y_0 + F' \cdot (X - X_0) = Y^m, \]  

(24)

\[ F(X_f) = Y_f, \]  

(25)

where \( F' = F'(X_0) \), \( Y_f \) denote accurate data, \( Y^m \) is observation date.

The iterative schemes of \( X(\gamma_n) \) and \( \gamma_n \) are shown in following. From Newton iteration method we know

\[ \gamma_n = \gamma_{n-1} - \frac{G(\gamma_{n-1})}{G'(\gamma_{n-1})} \]  

(26)

where

\[
\begin{align*}
G(\gamma_{n-1}) &= [F(X(\gamma_{n-1}) - Y^m)]^T E^{-1} [F(X(\gamma_{n-1}) - Y^m)] \\
G'(\gamma_{n-1}) &= 2 \left\langle \frac{dX(\gamma_{n-1})}{d\gamma}, F^T E^{-1} (F(X(\gamma_{n-1})) - Y^m) \right\rangle \\
dX(\gamma_{n-1}) &= - G(\gamma_{n-1}) + (\gamma_{n-1} I + F^T E^{-1} F')^{-2} F^T E^{-1} (Y^m - Y_0) \\
X(\gamma_n) &= X_0 + ((\gamma_n I + F^T E^{-1} F')^{-1} \cdot F^T E^{-1} (Y^m - Y_0))
\end{align*}
\]  

(27)
and in the iteration process, the following scheme is adopted,

\[
(\gamma_0, X_0) \xrightarrow{eq.(27,c). (27,d)} X(\gamma_0), \frac{dX(\gamma_0)}{d\gamma} \xrightarrow{eq.(26)} \gamma_1 \xrightarrow{eq.(27,c). (27,d)} (X(\gamma_1), \frac{dX(\gamma_1)}{d\gamma}) \tag{28}
\]

\[
\xrightarrow{eq.(26)} \gamma_2, \ldots, \tag{29}
\]

so \((\gamma_n, X(\gamma_n))\) is obtained, and \(X(\gamma_n) \rightarrow X^\sigma, \gamma_n \rightarrow \gamma_f, (n \rightarrow +\infty)\)

**6.2. Improved Discrepancy Principle (IDP) and the program of the iteration process.**

Set \((\gamma_0, X_0)\) as the initial value, i.e., linearize eq.(23) at \(X_0\) to obtain \((\gamma_1, X_1)\) as the above scheme, where \(X_1 = X(\gamma_0)\). Then set \((\gamma_1, X_1)\) as the initial value, i.e., linearize eq.(23) at \(X_1\) to obtain \((\gamma_2, X_2)\) as the above, where \(X_2 = X(\gamma_1)\) and so on. With the nonlinear effect of \(F\) considered, we have

\[
X_{n+1} = X_n + (\gamma_n I + F'F^{-1}(F'(X_n))^{-1} \cdot F'F^{-1}) \cdot E^{-1} |Y^m - F(X_n)| \tag{30}
\]

\[
\gamma_{n+1} = \gamma_n - \frac{G_n}{G'_n}, \tag{31}
\]

where

\[
\begin{align*}
G_n &= [F(X_{n+1}) - Y^m]^T E^{-1} [F(X_{n+1}) - Y^m] - \sigma^2 \\
G'_n &= 2 \left\langle \frac{dX_{n+1}}{d\gamma}, F'F^{-1}(F'(X_n))^{-1} [F(X_{n+1}) - Y^m] \right\rangle \\
&= 2 [F(X_{n+1}) - Y^m]^T E^{-1} F'(X_n) \left[ \gamma_n I + F'F^{-1}F'F^{-1} \right]^{-2} \\
&\cdot F'F^{-1} \cdot E^{-1} [F(X_{n+1}) - Y^m]
\end{align*} \tag{32}
\]

In actual process, when \(\|X_{n+1} - X_n\| \leq \xi\), the iterative process is stopped, where \(\xi\) is a small positive value; and in the calculation process, iterate for two times are enough.

Now, we apply IDP to the inverse test of a group of GOES-8 practical observation data. We take one month data from June 18 to July 17, 1997, and only choose the data related to radio sound data in 56 specimens. Radiation transfer model (RTM) is high-speed model of CIMMS (Cooperative Institute for Meteorological Satellite Studies, University of Wisconsin). We take the result of numerical predictive model of NCEP (National Center Enviromental Predictive, USA) as the initial guess value. In the inverse process, we divided atmospheric pressure into 42 layers from 0.1hpa to 1050hpa. The observation error consists of instrumental noise and model error of 0.2k. We took two tests. First, we chose empirical parameter \(\gamma\), which obtain optimum parameter \(\gamma=0.1\) with statistical model to do the first test of minimum information method. Second, we chose \(\gamma=0.1\) as \(\gamma_0\), and by IDP proceeded to inverse it. In general, the iterative times is about three or four, then we compared the inverse results with radio sound data results, and obtained root mean square errors (RMSE).

Figure 5.(a) is RMSE for each layer temperature by using two methods. Because it is difficult to get additional information of numerical predictive temperature for GOES, so it is difficult for us to improve initial guess value. The results by IDP, however, show that the accuracy of temperature profile is better than that in use.

Figure 5.(b) is RMSE of relative humidity profile by the two methods. It is clear that the accuracy of relative humidity is better than that by empirical inversion.
Figure 5. (a) Temperature RMSE of first guess, IDP and empirical retrievals from GOES-8 observations. (b) Relative humidity of first guess, IDP and empirical retrievals from GOES-8 observations.

7. Conclusions

There are a variety of inverse problems in atmospheric science and oceanography. This paper introduces the powerful methods to treat these problems – VAR (adjoint method), and improved discrepancy principle, and discusses some essential difficulties in VAR. Due to the ill-posedness of these problems, the regularization method is applied, i.e., additional terms are added to the cost functional as a stabilized functional with physical meaning. Four specific inversion problems, such as the inversion of one-dimensional sea temperature model, the inversion of parameters in an ENSO cycle model, the inversion of wind field with single-Doppler data, and the inversion of satellite remote sensing, indicate that, adoption of the regularization method in VAR , will overcome the ill-posedness, constrain calculational oscillations in iteration, speed up convergence of solutions, and enhance accuracy of solutions.

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