Thermal Hawking radiation of black hole with supertranslation field

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Abstract

Using the analytical solution for the Schwarzschild metric containing supertranslation field, we consider two main ingredients of calculation of the thermal Hawking black hole radiation: solution for eigenmodes of the d’Alambertian and solution of the geodesic equations for null geodesics. For calculation of Hawking radiation it is essential to determine the behavior of both the eigenmodes and geodesics in the vicinity of horizon. The equation for the eigenmodes is solved, first, perturbatively in the ratio \( O(C)/M \) of the supertranslation field to the mass of black hole, and, next, non-perturbatively in the near-horizon region. It is shown that in any order of perturbation theory solution for the eigenmodes in the metric containing supertranslation field differs from solution in the pure Schwarzschild metric by terms of order \( L^{1/2} = (1 - 2M/r)^{1/2} \). In the non-perturbative approach, solution for the eigenmodes differs from solution in the Schwarzschild metric by terms of order \( L^{1/2} \) which vanish on horizon. Using the simplified form of geodesic equations in vicinity of horizon, it is shown that in vicinity of horizon the null geodesics have the same behavior as in the Schwarzschild metric. As a result, the density matrices of thermal radiation in both cases are the same.

1 Introduction

Recently there was a renewed interest in asymptotic symmetries at the null infinity, the BMS symmetries [1, 2]. The group of symmetries of asymptotically flat gravity, the BMS group, extends the Poincare group and is the semi-direct product of the Lorentz group and the normal abelian subgroup of supertranslations which generalize translations. The BMS group can be enlarged to a group which also contains superrotations, singular supertranslations and local conformal transformations [3, 4, 5].

When acting on an asymptotically flat physical state the BMS group of diffeomorphisms transforms it to another physical state preserving the asymptotic flatness. The BMS group contains coordinate transformations which are pure gauge transformations and also diffeomorphisms that change supertranslation field in the metric and provide the mapping of a physical state to another physical state [6, 7].

In paper [6], was obtained an important physical result that provided a metric decreases fast enough at infinity, in some neighborhood of Minkowski vacuum the S matrix is invariant under an infinite-dimensional subgroup of \( BMS^+ \times BMS^- \), where \( BMS^\pm \) are the groups acting at the future and past null infinities. The correlated result is that in the low energy gravitational scattering, because of the supertranslation invariance of the S-matrix, the local energy is conserved at any angle. In papers [6, 8] connection of an (in principle) observable gravitational memory effect with the change of vacua under the action of the BMS transformations was elucidated and connection with the soft

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graviton theorem [9] was established. Possible relation of the extended BMS group to the black hole information problem was discussed in [10].

In papers [11, 7] a method to construct a metric containing a finite supertranslation field in the bulk was developed. The metric was obtained by exponentiation of infinitesimal supertranslation diffeomorphism. When applied to the Schwarzschild metric, solution generating technic of [7, 11] provides generalization of the Schwarzschild metric containing supertranslation field \( C(\theta, \varphi) \).

Using the analytical solution of paper [7] we consider calculation of ”hard” thermal Hawking radiation [12] of the Schwarzschild black hole with supertranslation field. In the present paper we consider the supertranslation field depending only on \( \theta \) : \( C = C(\theta) \). Calculations are performed perturbatively in the ratio \( O(C)/M \), where \( M \) is the mass of black hole.

Following the standard procedure of calculation of the Hawking radiation which assumes that the radiated particles are produced in the near-horizon region \([13, 14, 15]\), we study solutions for the eigenmodes of the d’Alambertian in the limit \( r \to 2M \). We find that in any order in perturbation theory in \( O(C)/M \), in the vicinity of horizon, solutions for the eigenmodes in the metric with supertranslation field differ from solutions in the pure Schwarzschild background by the terms of order \( L^{1/2}O(C^n/M^n) \) where \( L = (1 - 2M/r) \). Next, we show that the same result is valid for the non-perturbative solution, in which case the difference from the zero-order solution \( \psi_0 \) is \( L^{1/2}\varphi(\theta, r)\psi_0 \), where \( L^{1/2}\partial_r\varphi \) is finite in the limit \( L = 0 \).

Evolution of massless field is determined by its data on the past null infinity. To calculate the particle content at the future infinity in terms of excitations at the past infinity, the outgoing modes are traced back to the past null infinity and expanded in the basis of the incoming modes. To trace the outgoing modes to the past null infinity, we consider the null geodesics depending on \( r \) and \( \theta \) in the Schwarzschild background with the supertranslation field. The geodesic equations simplify in the vicinity of the horizon. We find that in the near-horizon region behavior of radial null geodesics is similar to those in the Schwarzschild background.

Collecting these results we find that the Bogolubov coefficients and the density matrix of thermal radiation in the metric with supertranslation field are the same as in the pure Schwarzschild background.

## 2 Generalization of the Schwarzschild metric containing the supertranslation field

The generic final state of collapse of matter is a stationary space-time which metric is diffeomorphic to the Kerr metric, if deviation from the Kerr metric is small. In [7] was constructed a metric generalizing the Schwarzschild metric and containing supertranslation field. In the isotropic spherical coordinates in which the original Schwarzschild metric is written as

\[
\begin{align*}
    ds^2 &= -\frac{(1 - M/2\rho)^2}{(1 + M/2\rho)^2}dt^2 + (1 + M/2\rho)^4(d\rho^2 + \rho^2d\Omega^2),
\end{align*}
\]

the metric containing the supertranslation field \( C(\theta, \varphi) \) was obtained in a form

\[
    ds^2 = -\frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2}dt^2 + (1 + M/2\rho_s)^4 \left( d\rho^2 + ((\rho - E)^2 + U)\gamma_{AB} + (\rho - E)C_{AB}dz^Adz^B \right).
\]

Here

\[
    \rho_s(\rho, C) = \sqrt{(\rho - C - C_{00})^2 + D_ACD_A},
\]

\[
    \frac{1}{2}L = \frac{1}{2}L(\rho).
\]
$C_{00}$ is the lowest constant spherical harmonic of $C(\theta, \varphi)$. In the following we do not write $C_{00}$ explicitly understanding $C \rightarrow C - C_{00}$. Covariant derivatives $D_A$ are defined with respect to the metric on the sphere $ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. For technical simplicity we consider the case of $C$ depending only on $\theta$, $C = C(\theta)$.

The tensor $C_{AB}$ and the functions $U$ and $E$ are defined as

$$C_{AB} = -(2D_AD_B - \gamma_{AB}D^2)C,$$

$$U = \frac{1}{8}C_{AB}C^{AB},$$

$$E = \frac{1}{2}D^2C + C.$$

The function $C$ has dimension of mass.

The non-zero Christoffel symbols are $\Gamma^\varphi_\varphi \theta = \cos \theta / \sin \theta$, $\Gamma^\theta_\varphi \varphi = -\cos \theta \sin \theta$. For $C_{AB}$ we obtain

$$C_{\theta\theta} = -(C'' - C' \cot \theta)$$

$$C_{\varphi\varphi} = \sin^2 \theta (C'' - C' \cot \theta)$$

$$C_{\varphi\theta} = 0$$

Here prime denotes the derivative $\partial_\theta$. From (5)-(6) it follows that $C_{\varphi\varphi} = -\sin^2 \theta C_{\theta\theta}$. The functions $E(\theta)$ and $U$ are

$$E = \frac{1}{2} (C'' + C' \cot \theta) + C,$$

$$U = \frac{1}{4} (C'' - C' \cot \theta)^2.$$

We assume that $C(\theta)$ is such that the components of the metric are finite. In particular, this condition is fulfilled for $C(\theta) = \sum a_n P_n(\cos \theta)$, where $P_n(x)$ are Legendre polynomials. Note that despite the term $\cot \theta$ contained in the functions (5), (6), the components of the metric (17), (18) are finite.

Equation $\rho_s(\rho_H, C) = M/2$ defines location of the horizon $\rho_H$. For horizon to exist, condition $M^2/4 - D_ACD^A > 0$ should be fulfilled. In the case $C = C(\theta)$ condition of existence of horizon takes the form $M/2 > |C'(\theta)|$.

Determinant of the angular part of the metric (2), $(\rho - E)^2 - U$, vanishes on the surfaces $[7]$

$$\rho_{SH}\pm = E \pm \sqrt{U} = \frac{1}{2}((C'' + C' \cot \theta) \pm \frac{1}{2} (C'' - C' \cot \theta)).$$

The surfaces $\rho_{SH}\pm$ are located in the region $\rho < \rho_H$.

The class of models with a supertranslation fields $C(\theta)$ contains the Kerr solution in which case $C_{\theta\theta}(\theta) = a/\sin \theta$ [16, 17] and $C(\theta) = a_2 + a_1 \cos \theta + a \sin \theta$ with $a_1, a_2$ arbitrary.

Let us transform the metric (2) to the Schwarzschild variables introducing

$$r = \rho_s(\rho, C) \left(1 + \frac{M}{2\rho_s(\rho, C)}\right)^2.$$

Inverting this equation, we obtain

$$\rho_s(\rho, C) = \frac{1}{2}(r - M + \sqrt{r(r - 2M)}).$$
In variable $r$ horizon is located at the point $r = 2M$.

Let us introduce notations
\[ L = 1 - \frac{2M}{r}, \quad (10) \]
\[ K = r - M + rL^{1/2}, \quad \frac{dK}{dr} = \frac{K}{rL^{1/2}}. \quad (11) \]

Using the relation (9), we have
\[ \frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2} = L, \quad (1 + M/2\rho_s)^4 = \frac{4r^2}{K^2}. \quad (12) \]

From the equation (9) written as
\[ \sqrt{(\rho - C)^2 + D_A C D^A C} = \frac{K}{2}, \]
we obtain the expression for $\rho$ as a function of $r$ and $C$
\[ \rho = C + \frac{K}{2} \left( 1 - \frac{4(DC)^2}{K^2} \right)^{1/2}. \quad (13) \]

Let us denote
\[ b = \frac{2C'}{K}, \quad (14) \]
so that (13) becomes
\[ \rho = C + \frac{K}{2} \sqrt{1 - b^2}. \quad (15) \]

Because $K$ is the increasing function of $r$, $b$ has its maximum at $r = 2M$ equal to $2|C'|/M$. Relation (15) is meaningful for $|C'| < M/2$.

Differentiating (13), we obtain
\[ d\rho = \frac{K}{2} \left[ \left( b - \frac{bb'}{\sqrt{1 - b^2}} \right) d\theta + \frac{dr}{rL^{1/2}\sqrt{1 - b^2}} \right]. \quad (16) \]

Let us consider the components $g_{\theta\theta}$ and $g_{\phi\phi}$ in the metric (2). Noting that $U = C^2_{\theta\theta}/4$, we obtain
\[ g_{\theta\theta} = ((\rho - E)^2 + U)\gamma_{\theta\theta} + (\rho - E)C_{\theta\theta} = \left( \rho - E + \frac{C_{\theta\theta}}{2} \right)^2 = K^2 \frac{4}{4} (\sqrt{1 - b^2} - b')^2 \]
\[ g_{\phi\phi} = ((\rho - E)^2 + U)\gamma_{\phi\phi} + (\rho - E)C_{\phi\phi} = \left( \rho - E - \frac{C_{\theta\theta}}{2} \right)^2 \sin^2 \theta = \frac{K^2}{4} \sin^2 \theta (b \cot \theta - \sqrt{1 - b^2})^2. \]

The surfaces at which $g_{\theta\theta}$ and $g_{\phi\phi}$ vanish are located in the internal domain $r > 2M$.

Using the expressions (10), (12) and substituting the components of the metric (17), (18) in (2), we obtain the metric in Schwarzschild variables
\[ ds^2 = -Ldt^2 + \frac{dr^2}{L(1 - b^2)} + 2drd\theta \frac{b(\sqrt{1 - b^2} - b')r}{(1 - b^2)L^{1/2}} \]
\[ + d\theta^2 r^2 \frac{(\sqrt{1 - b^2} - b')^2}{1 - b^2} + d\varphi^2 r^2 \sin^2 \theta (b \cot \theta - \sqrt{1 - b^2})^2. \]

\[ \boxed{} \]
Determinant of the metric (19) is
\[ |g| = r^4 \sin^2 \theta \frac{(\sqrt{1 - b^2} - b')^2}{1 - b^2} (\sqrt{1 - b^2} - b \cot \theta)^2. \] (20)

Because of conditions $M/2 > (|C'|, \rho_{SH})$ we have $|g| > 0$ for all $\theta$ and $r > 2M$. The inverse metric is
\[
\begin{pmatrix}
-L^{-1} & 0 & 0 & 0 \\
0 & L & -L^{1/2}b[r(\sqrt{1 - b^2} - b')]^{-1} & 0 \\
0 & -L^{1/2}b[r(\sqrt{1 - b^2} - b')]^{-1} & [r(\sqrt{1 - b^2} - b')]^{-2} & 0 \\
0 & 0 & 0 & [r^2 \sin^2 \theta(\sqrt{1 - b^2} - b \cot \theta)]^{-1}
\end{pmatrix}
\] (21)

We introduce the following notations which we use below
\[
\sqrt{|g|} = \sqrt{g_0\hat{g}} = r^2 \sin \theta \sqrt{|g|}, \\
\sqrt{|\hat{g}|} = 1 + g_1 + g_2 + \cdots = \\
1 - (b' + b \cot \theta) - \left( \frac{b^2}{2} - b'b \cot \theta \right) + \cdots = 1 - \frac{D^2C}{K} + \left( \frac{(D^2C)^2 - C_{\theta\theta}^2}{K^2} - 2(DC)^2 \right) + \cdots 
\] (22)

\[ g_{rr} = \frac{1}{L} \hat{g}_{rr}, \quad g_{r\theta} = \frac{L^{1/2}}{r} \hat{g}_{r\theta}, \quad g^{r\theta} = -\frac{r}{L^{1/2}} \hat{g}^{r\theta}, \quad g_{\theta\theta} = r^2 \hat{g}_{\theta\theta}, \]
\] (23)

\[ \hat{g}^{r\theta} = g_1^{r\theta} + g_2^{r\theta} + \cdots = b + b' + \cdots = \frac{2C'}{K} + \frac{2(DC')^2}{K^2} + \cdots, \]
\] (24)

\[ f_1 = \sqrt{1 - b^2 - b'}, \]
\] (25)

\[ f_2 = \sin^2 \theta(\sqrt{1 - b^2} - b \cot \theta). \]
\] (26)

\[ f_2 = \sin^2 \theta(\sqrt{1 - b^2} - b \cot \theta). \]
\] (27)

\section{Equation for eigenmodes}

In this section we study solutions for the eigenmodes of the equation $\square(g)\psi = 0$ perturbatively and non-perturbatively in small parameter $O(C')/K \sim O(C)/M$ and prove that corrections to solutions in the pure Schwarzschild metric vanish in the vicinity of horizon $r = 2M$ as $L^{1/2} = (1 - 2M/r)^{1/2}$.

The operator $\square(g)$ is
\[
\square = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu = \\
g^{tt} \partial_t^2 + \frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} g^{rr} \partial_r + \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} g^{r\theta} \partial_\theta + \frac{1}{\sqrt{-g}} \partial_\varphi \sqrt{-g} g^{r\varphi} \partial_\varphi + \\
+ \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} g^{\theta\theta} \partial_\theta + \frac{1}{\sqrt{-g}} \partial_\varphi \sqrt{-g} g^{\varphi\varphi} \partial_\varphi,
\] (28)

or explicitly
\[
\square \psi = -L^{-1} \partial_t^2 \psi + \frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} L \partial_r \psi - \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} \frac{L^{1/2}b}{r f_1} \partial_\theta \psi \\
- \frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} \frac{L^{1/2}b}{r f_1} \partial_r \psi + \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} (r f_1)^{-2} \partial_\theta \psi + \frac{1}{\sqrt{-g}} \partial_\varphi \sqrt{-g} (r f_2)^{-2} \partial_\varphi \psi = 0.
\] (29)
First, we solve the equation $\Box \psi = 0$ perturbatively in $O(C)/K$ taking $\psi = \psi_0 + \psi_1 + \psi_2 + \cdots$ and $\Box = \Box_0 + \Box_1 + \Box_2 + \cdots$, where subscripts denote the orders in $O(C)/K$. We have

$$\Box \psi = \Box_0 \psi_0 + (\Box_1 \psi_0 + \Box_0 \psi_1) + (\Box_2 \psi_0 + \Box_0 \psi_2 + \Box_1 \psi_1) + \cdots. \quad (30)$$

Expanding the terms in (29) in the series in $O(C)/K$, and using notations (22)-(27), we have

$$\frac{1}{\sqrt{-g}}\partial_r \sqrt{-g} L \partial_r \psi = [L \partial_r + L(2/r) + (\partial_r L) + L(\partial_r g_1) + L((\partial_r g_2 - g_1(\partial_r g_1)) + \cdots] \partial_r \psi \quad (31)$$

$$\frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} \frac{L^{1/2} b}{r f_1} \partial_\psi = \left(\frac{L^{1/2}}{r}\right) \left[ g^r_1 \partial_\theta + (\cot \theta g^r_1 + (\partial_\theta g^r_1) + \cdots \right] \partial_\psi \quad (32)$$

$$\frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} \frac{L^{1/2} b}{r f_1} \partial_\theta \psi = \left(\frac{L^{1/2}}{r}\right) \left[ g^r_1 \partial_r + (2/r) g^r_1 + (\partial_r g^r_1) + \cdots \right] \partial_\theta \psi \quad (33)$$

Numerical calculations [18] of the Hawking radiation in different modes and theoretical considerations [19] have shown that the main part of the energy of radiation is contained in the $s$-wave. Because of that, first, in the main order, we consider the $s$-wave eigenmode and, next, at the end of the Section, comment on the modes with $l > 0$.

In the zero order in $O(C)/K$, written in tortoise variable $r_\ast = r + 2M \ln(r/2M - 1)$, the operator $\Box_0 \psi$ is

$$\Box_0 \psi = (rL)^{-1} \left[ -\partial^2_t + \partial^2_{r_\ast} + L \left(\frac{2M}{r^3} - \frac{\hat{K}^2(\theta, \varphi)}{r^2}\right) \right] r \psi, \quad (34)$$

where $\hat{K}^2$ is the angular momentum operator. In calculation of the Hawking radiation it is important to determine the behavior of the eigenmodes in the vicinity of the horizon [13, 15, 19]. We look for solution of the equation for the eigenmodes in the region $r - 2M \ll M$ and set $r = 2M$ in the functions with regular behavior at $r = 2M$.

When acting on $\psi_0$ the derivatives $\partial_\psi$ produce the terms with powers of $L^{-1}$ which grow up as $r \to 2M$. In the leading order in $L^{-1}$ we have

$$\partial_r \frac{e^{i\omega r}}{r} \sim \frac{e^{i\omega r} \cdot i\omega}{rL}, \quad \partial^2_r \frac{e^{i\omega r}}{r} \sim -\left(\frac{\omega^2}{L^2} + \frac{2iM\omega}{L^2 r^2}\right) \frac{e^{i\omega r}}{r}. \quad (35)$$

Acting on $O(C^n)/K^n$ derivatives $\partial_r$ produce the factor $L^{-1/2}$

$$\partial_r \frac{O(C^n)}{K^n} = -n \frac{O(C^n)}{K^n r L^{1/2}}. \quad (36)$$

In the region $r \sim 2M$, neglecting in the operator $\Box_0$ the small term $L2M/r^3$, solution of the Eq. (34) for the $s$-mode is obtained as

$$\psi_0 = \frac{e^{i\omega(t \pm r_\ast)}}{\sqrt{4\pi r}}. \quad (37)$$

Let us consider the equation (30) in the first order $\Box_1 \psi_0 + \Box_0 \psi_1 = 0$. Using (31), (32) and (22), (24), we obtain (51) the terms without the derivatives $\partial_\theta \psi_0$ and $\partial_\varphi \psi_0$ in the operator $\Box_1 \psi_0$ as

$$\left[ L(\partial_r g_1) + \left( -\frac{L^{1/2}}{r} \right) (\cot \theta g^r_1 + (\partial_\theta g^r_1)) \right] \partial_r \psi_0 =$$

$$\left[ L\partial_r(-b' - b \cot \theta) - \left(\frac{L^{1/2}}{r}\right) (b' + b \cot \theta) \right] \partial_r \psi_0 = 0 \quad (38)$$
where we used the formula \( \partial L \partial b = L \partial_K (2C/K) \partial_r K = -b L^{1/2} / r \). The remaining terms in the operator \( \Box_1 \psi_0 \) contain derivatives \( \partial_\theta \) and \( \partial_\varphi \) acting on \( \psi_0 \) and vanish. The equation \( \Box_1 \psi_0 + \Box_0 \psi_1 = 0 \) reduces to \( \Box_0 \psi_1 = 0 \) and yields \( \psi_1 = \psi_0 \).

In the second order in \( O(C)/K \) the equation (30) is \( \Box_2 \psi_0 + \Box_1 \psi_1 + \Box_0 \psi_2 = 0 \) which reduces to \( \Box_2 \psi_0 + \Box_0 \psi_2 = 0 \). The terms without derivatives over angular variables acting on \( \psi_0 \) are

\[
\Box_2 \psi_0 = \left[ L((\partial_r g_2 - g_1 (\partial_r g_1)) + \left( -\frac{L^{1/2}}{r} \right) (\cot \theta g_2^\theta + (\partial_\theta g_1)^\theta + (\partial_\varphi g_2^\varphi)) \right] \partial_r \psi_0 \tag{39}
\]

Both terms in the square brackets are of order \( L^{1/2} \). In the leading order in \( L^{-1} \) we have

\[
\Box_2 \psi_0 = \frac{\pm i\omega F(2) (\theta, r)}{L^{1/2} K^2} \psi_0, \tag{40}
\]

(51) where \( F(2) = O(C^2) \), and the explicit form of \( F(2) \) is irrelevant for us. The equation \( \Box_0 \psi_2 = -\Box_2 \psi_0 \) has the following structure

\[
\frac{1}{rL} \left[ -\partial_t^2 + \partial_{r_*}^2 + L \left( \frac{2M}{r^3} - \frac{\dot{K}^2(\theta, \varphi)}{r^2} \right) \right] r \psi_2 = \mp i\omega \frac{F(2)}{L^{1/2} K^2} e^{i\omega(t \pm r_*)} \tag{41}
\]

Looking for a solution in the form \( r \psi_2 = f e^{i\omega(t \pm r_*)} \), in the leading order in \( L^{-1} \) we obtain

\[
\partial_t^2 f \pm 2i\omega \partial_r f = \pm i\omega L^{1/2} \frac{F(2)}{K^2} \tag{42}
\]

In the region \( r \approx 2M \) approximately \( r_* = 2M \ln L \). Solving the equation (42), we have

\[
\psi_2 = \pm i\omega \frac{F(2)}{K^2} \frac{L^{1/2} (4M)^2}{1 \pm 8i\omega M} \psi_0. \tag{43}
\]

In the higher orders we proceed by induction. The equation for \( \psi_n \) is

\[
\Box_0 \psi_n + \sum_{k+r=n, k,r \neq 0} \Box_k \psi_r + \Box_n \psi_0 = 0. \tag{44}
\]

Let \( \psi_r \) have the following structure

\[
\psi_r = L^{1/2} F(O(C^k/K^k), \theta, r) \psi_0, \quad k \lesssim r, \tag{45}
\]

The operators \( \Box_k, \ k > 2 \) can be presented in a form

\[
L(\partial_r F_1) \partial_r + \partial_\theta L^{1/2} F_2 \partial_\theta + \partial_r L^{1/2} F_3 \partial_\theta, \tag{46}
\]

where \( F_i = F_i(O(C^k/K^k), \theta, r) \sim O(L^0) \). When acting on the functions (45) each term in (44) yields an expression of the form (45) without the prefactor \( L^{1/2} \) i.e. the result is of order \( O(L^0) \). The operator \( \Box_n \) is of the same structure as (46), but when acting on the function \( \psi_0 \), it produces an expression of order \( L^{-1/2} \). Eq. (44) reduces to \( \Box_0 \psi_n + \Box_n \psi_0 = 0 \) which is of the functional form similar to (41) and yields solution

\[
\psi_n \sim L^{1/2} O(C^n/K^n) \psi_0.
\]
The origin of this result can be traced back to the form of d’Alambertian (29) in which the \((t, r)\) part is the same as in the Schwarzschild metric and the additional \((r, \theta)\) terms contain the factor \(L^{1/2}\).

Let us consider solutions with higher harmonics, \(l > 0\). Approximate solution of the equation \(\Box \psi_{l0} = 0\) in the main order in \(L^{-1}\) with the \(l\)-th harmonic is

\[
\psi_{l0} = \frac{e^{i\omega(t+\tau)}}{\sqrt{4\pi r}} P_l(\cos \theta).
\]

Now the terms containing \(\partial_\theta \psi_{l,k}\) are nonbibitem-zero. In the next order, in the operator \(\Box_1 \psi_{l-1,0}\), there appears the new term

\[
\frac{1}{\sqrt{g}} \partial_r \sqrt{g} \partial_\theta \psi_{l-1,0} = \frac{1}{r^2} \partial_r r^2 \frac{L^{1/2}}{r} \partial_\theta \psi_{l-1,0}.
\]

Because \(\partial_\theta\) does not change the order in \(L\), this term is of order \(L^{1/2}\) and can be neglected. In the next orders we find similar situation. New terms with the derivative \(\partial_\theta\) do not increase powers of \(L^{-1}\). As a result, we obtain solution of the form

\[
\psi_l \sim L^{1/2} \frac{F_n(\omega)}{K^n} \psi_0.
\]

Above we assumed that \(C(\theta) = \sum P_n(\cos^2 \theta)\). Because the action of \(\partial_\theta\) on \(P_n(\cos^2 \theta)\) produces factor \(n\), in the higher orders in \(O(C)/K\) in the terms \(\Box_k \psi_n\), in principle, accumulate powers of \(n\). Trying to sum all orders, we encounter the problem of convergence. To avoid this problem, we consider non-perturbative solution of the equation (30).

We present \(\Box\) as the sum \(\Box_0 + \Box\), where \(\Box_0\) is the zero-order part of d’Alambertian and \(\Box\) written in notations (22)-(27) is

\[
\Box = \frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} L \partial_r - \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} \frac{L^{1/2}}{r f_1} \partial_r - \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} \frac{L^{1/2}}{r f_1} \partial_\theta - r^{-2}(f_1^2 - 1)(\partial_\theta^2 + \cot \theta \partial_\theta) + \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g}(r f_1)^{-2} \partial_\theta + r^{-2}(f_2^{-2} - 1) \partial_\varphi^2.
\]

We look for a solution \(\psi\) in the form \(\psi_0 + \hat{\psi}\), where \(\psi_0\) is the zero-order \(s\)-mode. Equation \(\Box \psi = 0\) takes the form

\[
\Box_0 \hat{\psi} + \Box \psi_0 + \Box \hat{\psi} = 0 \tag{48}
\]

We take an ansatz for \(\hat{\psi}\) in a form \(L^{1/2} \varphi \psi_0\), where \(\varphi\) is sufficiently smooth function and \(L^{1/2} \partial_r \varphi\) is finite in the limit \(L = 0\). We have

\[
\partial_r \hat{\psi} = \partial_r (L^{1/2} \varphi \psi_0) = \text{bibitem} \left( \frac{M}{r^2} \pm i \omega \right) L^{-1/2} \varphi \psi_0 + L^{1/2} (\partial_r \varphi) \psi_0. \tag{49}
\]

Let us consider \(\Box \psi_0\). The terms with derivatives \(\partial_\theta\) and \(\partial_\varphi\) acting on \(\psi_0\) yield zero. We have

\[
\Box \psi_0 = \frac{1}{\sqrt{-g}} L(\partial_r \sqrt{-g}) \partial_r \psi_0 + \frac{1}{\sqrt{-g}} \partial_\theta \sqrt{-g} \frac{L^{1/2}}{r f_1} \partial_r \psi_0 = \pm i \omega L^{-1/2}(F_1 + F_2) \psi_0, \tag{50}
\]

where

\[
\frac{1}{\sqrt{-g}} L(\partial_r \sqrt{-g}) \partial_r \psi_0 = \pm i \omega L^{-1/2} F_1 \psi_0
\]
and
\[
\frac{1}{\sqrt{-g}} \partial_{\theta} \sqrt{-g} \frac{L^{1/2} b}{r f_1} \partial_{r} \psi_0 = \pm i \omega L^{-1/2} F_2 \psi_0
\]

Next, let us consider the term \( \hat{\Box} \hat{\psi} \). The \((rr)\) term is
\[
L \frac{\partial_r \sqrt{-g}}{\sqrt{-g}} \partial_r \hat{\psi} = F_1 \left( \frac{M}{r^2} \pm i \omega \right) L^{-1/2} \dot{\varphi} \psi_0 + F_1 L^{1/2} \partial_r \varphi \hat{\psi}_0.
\]

The \((\theta r)\) term is
\[
\frac{1}{\sqrt{-g}} \partial_{\theta} \sqrt{-g} \frac{L^{1/2} b}{r f_1} \partial_{\theta} \hat{\psi} = L^{1/2} [(\partial_b F_2) \partial_{\theta} \hat{\psi} + F_2 \partial_r \partial_{\theta} \hat{\psi}] = L^{-1/2} \left( \frac{M}{r^2} \pm i \omega \right) [(\partial_b F_2) \varphi + F_2 (\partial_{\theta} \varphi)] \hat{\psi}_0,
\]
and the \((r \theta)\) term is
\[
\frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} \frac{L^{1/2} b}{r f_1} \partial_{r} \hat{\psi} = L^{-1/2} F_3 \partial_{r} \hat{\psi} + L^{1/2} F_2 \partial_r \partial_{\theta} \hat{\psi}.
\]

Collecting the leading in \( L^{-1} \) terms, we obtain Eq.(48) as
\[
\hat{\Box} \hat{\psi} + L^{-1/2} (F_1 + F_2) i \omega \psi_0 + L^{-1/2} (\Psi_1 \dot{\varphi} \psi_0 + \Psi_2 \partial_{\theta} \varphi \psi_0) = 0.
\]

Eq.(51) has the same functional form as Eq.(41), and we conclude that the ansatz \( \hat{\psi} = L^{1/2} \varphi \psi_0 \) yields a solution of (51).

### 4 Isotropic geodesics

In this section we consider a class of the isotropic geodesics in the metric (19) depending on \( r \) and \( \theta \) and show that in the vicinity of horizon they differ from geodesics in the Schwarzschild background by the terms of order \( L^{1/2} = (1 - 2M/r)^{1/2} \). Following the standard treatment [20] we start from the Lagrangian corresponding to the metric (19) which we present as
\[
2 \mathcal{L} = -L \dot{r}^2 + \frac{\dot{r}^2}{r L} \dot{g}_{rr} + 2 \frac{\dot{r}^2 \dot{\theta}}{r L} \dot{g}_{r \theta} + r^2 \left[ \dot{\varphi}^2 \varphi \theta \dot{\varphi} + \varphi^2 \sin^2 \theta \dot{\varphi} \dot{\varphi} \right].
\]

Derivatives are taken with respect to an affine parameter on geodesic. The Lagrange equations following from the Lagrangian yield the geodesic equations for \( r(\tau), t(\tau) \) and \( \theta(\tau), \varphi(\tau) \). The equations for \( r(\tau) \) and \( \theta(\tau) \) are
\[
\frac{\dot{r}}{L} \dot{g}_{rr} - \frac{\dot{r}^2}{r^2 L^2} \dot{g}_{rr} + \frac{E^2}{r^2 L^2} \dot{g}_{rr} + \frac{\dot{r}^2}{2L} \partial_r \dot{g}_{rr} + \frac{\dot{\theta}^2}{L} \partial_\theta \dot{g}_{rr} + \\
+ \frac{\dot{r} r}{L^{1/2}} \dot{g}_{r \theta} + \dot{\theta}^2 \left( \frac{r}{L^{1/2}} \partial_\theta \dot{g}_{r \theta} - r^2 \partial_\theta \dot{g}_{r \theta} - \frac{r^2}{2} \partial_r \dot{g}_{r \theta} \right) - \varphi^2 \left( \varphi \partial_\varphi \dot{g}_{r \varphi} + \frac{r^2}{2} \partial_r \dot{g}_{r \varphi} \right) = 0,
\]
\[
\ddot{\theta}^2 \varphi \dot{g}_{\varphi \varphi} + 2 \dot{\theta}^2 \varphi \dot{g}_{\varphi \varphi} + r^2 \dot{\theta}^2 \partial_\theta \varphi \dot{g}_{\varphi \varphi} + \frac{r^2}{2} \dot{\theta} \dot{g}_{\varphi \varphi} + \frac{\dot{r} r}{L^{1/2}} \dot{g}_{r \theta} - \frac{\dot{r}^2}{r L^{3/2}} \dot{g}_{r \theta} + \\
+ \frac{\dot{r}^2 r}{L^{1/2}} \partial_r \dot{g}_{r \theta} - \frac{\dot{r}^2}{2L} \partial_\theta \dot{g}_{r \theta} - \varphi^2 r^2 (\sin \theta \cos \theta \dot{g}_{\varphi \varphi} + \sin^2 \theta \partial_\theta \dot{g}_{\varphi \varphi}) = 0.
\]
Here $\partial_r \hat{g}_{ij} = K \partial_K \hat{g}_{ij}/L^{1/2}r$. Because $t$ and $\varphi$ are cyclic variables, we solved the corresponding equations and set in the Lagrange equations $\dot{t} = E/L$ and $r^2 \dot{\varphi} \sin^2 \theta = B = \text{const}$. We consider the geodesics with $B = 0$.

For the isotropic geodesics equation $\mathcal{L} = 0$ is the first integral of the system of geodesic equations

$$
\frac{E^2}{2L} + \frac{\dot{r}^2}{2L} \hat{g}_{rr} + \frac{r \dot{r} \dot{\theta}}{L^{1/2}} \hat{g}_{r\theta} + \frac{r^2}{2} \dot{\theta}^2 \hat{g}_{\theta \theta} = 0.
$$

(55)

In its general form the system of equations is intractable. To proceed, we consider the near-horizon region $r \to 2M$. In this limit $L \to 0$. Examining the equations (53), (54) and (55) we see that in the near-horizon limit we can look for a solution in the form

$$
\dot{r} = C + C_1 L^{1/2} + \cdots,
$$

(56)

$$
\dot{\theta} = \frac{A}{L^{1/2}} + A_1 + \cdots, \quad \ddot{\theta} = -\frac{A \dot{r}}{r^2 L^{3/2}} + \cdots.
$$

With the ansatz (56) the leading in $L^{-1}$ terms in (53) and (54) are

$$
\frac{E^2}{r^2 L^2} - \frac{\dot{r}^2}{r^2 L^2} \hat{g}_{rr} + \frac{\dot{r} \dot{\theta}}{L^{1/2}} \hat{g}_{r\theta} = 0,
$$

(57)

$$
\ddot{r} t^2 \hat{g}_{\theta \theta} - \frac{\dot{r}^2}{r L^{1/2}} \hat{g}_{r\theta} = 0.
$$

(58)

Substituting the ansatz (56), we obtain

$$
L^{-2} [E^2 - C^2 \hat{g}_{rr} - r A C \hat{g}_{r\theta}] = 0,
$$

(59)

$$
L^{-3/2} [C \hat{g}_{r\theta} + r A \hat{g}_{\theta \theta}] = 0.
$$

(60)

and (55) is

$$
L^{-1} [C^2 \hat{g}_{rr} - E^2 + 2 r C A \hat{g}_{r\theta} + r^2 A^2 \hat{g}_{\theta \theta}] = 0.
$$

(61)

Here we have introduced $\hat{g}_{ij} = \hat{g}_{ij}|_{r=2M}$. Note that $\hat{g}_{ij}$ are the components of the metric without the factors $L$ (see (24). Using the relation

$$
\hat{g}_{rr} \hat{g}_{\theta \theta} - \hat{g}_{r\theta}^2 = \hat{g}_{\theta \theta},
$$

(62)

from the Eqs.(57) and (58) we obtain

$$
C^2 = E^2,
$$

(63)

$$
A \approx -\frac{C \hat{g}_{r\theta}}{2M \hat{g}_{\theta \theta}}.
$$

(64)

Because of the relation (62) Eq. (61) turns to identity.

In the limit $b = 0$ the Lagrangian reduces to that of the Schwarzschild metric. In the spherically-symmetric metrics trajectories of the geodesics are located in a plane going through the symmetry center. Position of the plane depends on the initial conditions. $\theta$ and $\varphi$ are coordinates in a coordinate system with the origin located at the symmetry center. Position of the plane depends on the initial conditions. Solution of the radial geodesic equations with the initial conditions $\theta(\tau_0) = \pi/2$, $\dot{\theta}(\tau_0) = 0$, $\varphi(\tau_0) = \dot{\varphi}(\tau_0) = 0$ is

$$
\ddot{\tau} = E^2 = \dot{\tau}^2
$$

(65)

$$
\theta(\tau) = \pi/2,
$$

$$
\varphi(\tau) = 0.
$$
Comparing solution (63), (64) with that in the Schwarzschild metric and adjusting the integration constants, we obtain

\[ r = 2M + E\tau, \]
\[ t = t_0 + E\tau, \]
\[ \theta \simeq \frac{\pi}{2} - \frac{g_{r\theta}}{2g_{\theta\theta}} \sqrt{\frac{\tau E}{2M}} = \frac{\pi}{2} - \frac{g_{r\theta}}{2g_{\theta\theta}} L^{1/2}. \]

In the vicinity of horizon solution (66) differ from (65) by the terms of order \( L^{1/2} \).

5 Conclusions

A black hole emerging as a result of the collapse can be considered as practically stationary, the surface of the collapsing body approaches the horizon as \( r - 2M \sim conste^{-1/2M} \) with a very small characteristic time \( \sim 2GM/c^3 \) in dimensionful units [21], and propagation of the wave packets can be treated as propagation in the background of the stationary black hole. Evolution of the massless field is determined by the data at the past null infinity \( I^- \) in the basis \( \{u_i^{(-)}\} \). Alternatively the field \( \varphi \) can be expanded at the hypersurface \( \Sigma^+ = I^+ \oplus H^+ \) where \( I^+ \) is the future null infinity and \( H^+ \) is the event horizon

\[ \varphi = \sum_i (b_i u_i^{(+)} + b_i^+ u_i^{(+)*} + c_i q_i + c_i^+ q_i^*) \],

where \( \{u_i^{(+)}\} \) is the orthonormal set of modes which contain at the \( I^+ \) only positive frequencies and \( \{q_i\} \) is the orthonormal set of solutions of the wave equation which contains no outgoing components [13]. The modes \( \{u_i^{(+)}\} \) traced back to \( I^- \) can be expanded in terms of the modes \( \{u_i^{(-)}\} \)

\[ u_i^{(+)} = \int d\omega \left( \alpha_{\omega\omega'} u_i^{(-)} + \beta_{\omega\omega'} u_i^{(-)*} \right). \]

The Bogolubov coefficients \( \beta_{\omega\omega'} \) calculated at the surface \( I_- \) are

\[ \beta_{\omega\omega'} = i \int_{I_-} 4\pi r^2 d\omega \left( \frac{\partial}{\partial u(v)} \right) \left( \frac{\partial}{\partial u^*(v)} \right). \]

Actual calculation of \( \beta_{\omega\omega'} \) is performed in the limit \( r \to 2M \). Because in the near-horizon region the geodesics have the same form as in the Schwarzschild background the modes \( u^{(+)} \) traced back to \( I^- \) are piled in the near-horizon region the same as in the Schwarzschild case. At the horizon, the additional parts in the modes depending on the supertranslation field are proportional to \( L^{1/2} = (1 - 2M/r)^{1/2} \) and vanish. This is an accuracy of conventional calculations of the Hawking effect (cf. [15]). Thus, the Bogolubov coefficients and the density matrix of thermal radiation are the same as in the Schwarzschild background.

Acknowledgments

I am grateful to L. Slad and M. Smolyakov for useful discussions.

This work was partially supported by the Ministry of Science and Education of Russian Federation under project 01201255504.
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