Gauge algebra of irreducible theories in the Sp(2)-symmetric BRST formalism

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Abstract

An explicit solution to classical master equations of the Sp(2)-symmetric Hamiltonian BRST quantization scheme is presented in the case of irreducible gauge theories. A realization of the observable algebra is constructed.

1 Introduction

Let $M$ be a phase space with the phase coordinates $p_i, q^i$ $i = 1, \ldots, n$, and the Poisson bracket $\{ \ldots \}$. Let $T_\alpha = T_\alpha(p, q), \alpha = 1, \ldots, m, m < n$, be first class constraints of a Hamiltonian system,

$$\{T_\alpha, T_\beta\} \approx 0.$$ 

The weak equality $\approx$ means equality on the constraint surface

$$\Sigma: \quad T_\alpha = 0.$$ 

Let $P$ denote the Poisson algebra of first class functions,

$$P = \{f(p, q) \mid \{f, T_\alpha\} \approx 0\},$$

and let

$$J = \{u(p, q) \mid u \approx 0\}.$$
Elements of the Poisson algebra $P/J$ are called classical observables. The Hamiltonian $H_0(p, q)$ is assumed to be a first class function. These definitions correspond to the Dirac quantization without gauge fixing \[1\].

There are different realizations of $P/J$. In the gauge fixing method \[2\] auxiliary constraints are introduced and the original Poisson algebra is replaced by the Dirac one. The algebra $P/J$ is isomorphic to a quotient Dirac bracket algebra. Some other realizations of the observable algebra without extending of the original phase space were given in Refs. \[3\], \[4\].

In the hamiltonian BRST theory with the BRST charge $\Omega$ the realization of $P/J$ looks like $U/V$ where $U$ is the space of solutions to the equation

$$\{\Omega, \Phi\} = 0$$

with certain boundary conditions \[5\].

In the present paper we study the observable algebra of irreducible gauge theories in the framework of the Sp(2)-symmetric BRST formalism. The extended BRST symmetry was discovered in Refs. \[6\], \[7\]. In the Hamiltonian formalism it is generated by the charges $\Omega^a$, $a = 1, 2$, satisfying the master equations

$$\{\Omega^a, \Omega^b\} = 0. \quad (1)$$

Observables are determined by solutions to the equation

$$\{\Omega^a, \Phi\} = 0. \quad (2)$$

A solution to the generating equations \[1\], \[2\] for rang-1 theories was found in Ref. \[8\]. In the general case an algorithm for the construction of a solution to these equations was given in Refs. \[9\], \[10\] (see also \[11\]). However, the problem of finding the charges $\Omega^a$ and observables has not been solved. The goal of this paper is to present an explicit solution to Eqs. \[1\], \[2\] and construct a realization of $P/J$.

The paper is organized as follows. In section 2, we review the master equations of the Sp(2)-symmetric Hamiltonian BRST theory and introduce notations. An explicit formula for $\Omega^a$ is given in section 3. The realization of the observable algebra is described in section 4.

In what follows the Grassmann parity and new ghost number of a function $X$ are denoted by $\epsilon(X)$ and $\text{ng}h(X)$, respectively. The constraints are supposed to be of definite Grassmann parity $\epsilon(T_a) = \epsilon_a$. For a function $X^{a_1a_2...a_n}$

$$X^{\{a_1a_2...a_n\}} = X^{a_1a_2...a_n} + \text{cycl. perm.} (a_1, a_2, \ldots, a_n).$$
2 Master equations

An extended phase space of the theory under consideration is parametrized by the canonical variables
\[ G = (p_i, q^i; P_{\alpha a}, C^{\alpha a}; \lambda_\alpha, \pi^\alpha), \]

\[ \epsilon(p_i) = \epsilon(q^i) = \epsilon_i, \quad \epsilon(P_{\alpha a}) = \epsilon(C^{\alpha a}) = \epsilon_\alpha + 1, \quad \epsilon(\lambda_\alpha) = \epsilon(\pi^\alpha) = \epsilon_\alpha, \]

\[ \text{ngh}(p_i) = \text{ngh}(q^i) = 0, \quad \text{ngh}(P_{\alpha a}) = -1, \quad \text{ngh}(C^{\alpha a}) = 1, \]

\[ \text{ngh}(\pi^\alpha) = 2, \quad \text{ngh}(\lambda_\alpha) = -2. \tag{3} \]

The Poisson bracket is given by
\[ \{X, Y\} = \frac{\partial X}{\partial q^i} \frac{\partial Y}{\partial p_i} + \frac{\partial X}{\partial C_\alpha^{a}} \frac{\partial Y}{\partial P_{\alpha a}} + \frac{\partial X}{\partial \pi^\alpha} \frac{\partial Y}{\partial \lambda_\alpha} - (-1)^{\epsilon(X)\epsilon(Y)}(X \leftrightarrow Y). \]

Derivatives with respect to the generalized momenta \( p, P, \lambda \) are understood as left-hand, and those with respect to the generalized coordinates \( q, C, \pi \) as right-hand ones.

We assume that \( T_\alpha \) are independent and satisfy the regularity conditions. This means that there are some functions \( F_{\alpha'}(p, q) \), such that \( (F_{\alpha'}, T_\alpha) \) can be locally taken as new coordinates in the original phase space. This assumption allows changing variables: \( (p, q) \rightarrow \xi = (\xi_\alpha, \xi_{\alpha'}) \),

\[ \xi_\alpha = T_\alpha(p, q), \quad \xi_{\alpha'} = F_{\alpha'}(p, q). \]

In what follows we use only the phase variables \((\xi, P, C, \lambda, \pi)\). The constraint surface \( \Sigma \) looks like
\[ \xi_\alpha = 0. \]

The Poisson bracket is denoted by \( \{\ldots, \ldots\}' \).

With respect to the new variables Eq. (1) takes the form
\[ \{\Omega^\alpha, \Omega^{\beta'}\}' = 0, \tag{4} \]
where \( \Omega^a(\xi, P, C, \lambda, \pi) = \Omega^a(p, q, P, C, \lambda, \pi) \). The charges \( \Omega^a \) also satisfy the conditions
\[
\varepsilon(\Omega^a) = 1, \quad \text{ngh}(\Omega^a) = 1,
\]
\[
\frac{\partial \Omega^a}{\partial C_{ab}} \bigg|_{C=\pi=P=\lambda=0} = \xi_\alpha \delta^a_b, \quad \frac{\partial \Omega^a}{\partial \pi^\alpha} \bigg|_{C=\pi=\lambda=0} = \varepsilon^{ab} \mathcal{P}_{ab}, \quad (5)
\]
where
\[
\varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
One can write
\[
\Omega^a = \Omega^a_1 + \Pi^a, \quad (6)
\]
where
\[
\Omega^a_1 = \xi_\alpha C^\alpha + \varepsilon^{ab} \mathcal{P}_{ab} \pi^\alpha, \quad \Pi^a = \sum_{n \geq 2} \Omega^a_n, \quad \Omega^a_n \sim C^{m-n} \pi^m. \quad (7)
\]

Let \( N \) be the counting operator
\[
N = \xi_\alpha \frac{\partial}{\partial \xi_\alpha} + \mathcal{P}_{aa} \frac{\partial}{\partial \mathcal{P}_{aa}} + \lambda_\alpha \frac{\partial}{\partial \lambda_\alpha},
\]
and let
\[
W^a = \xi_\alpha \frac{\partial}{\partial \mathcal{P}_{aa}} + \varepsilon^{ab} \mathcal{P}_{ab} \frac{\partial}{\partial \lambda_\alpha} + (-1)^{\varepsilon_\alpha} \varepsilon^{ab} \pi^\alpha \frac{\partial}{\partial C_{ab}},
\]
\[
\Gamma_a = \mathcal{P}_{aa} \frac{\partial}{\partial \xi_\alpha} - \varepsilon_{ab} \lambda_\alpha \frac{\partial}{\partial \mathcal{P}_{ab}}, \quad M = \Gamma_a W^a,
\]
where \( \varepsilon^{ab} \varepsilon_{bc} = \delta^a_c \). Then
\[
W^a W^b = 0, \quad \Gamma_{\{a} \Gamma_{b\}} = 0, \quad W^a \Gamma_b + \Gamma_b W^a = \delta^a_b N,
\]
\[
NW^a = W^a N, \quad NT_a = \Gamma_a N,
\]
\[
M^2 W^a = NW^a, \quad \Gamma_a M^2 = NT_a M,
\]
\[
M^n = (2^{n-1} - 1) N^{n-2} M^2 - (2^{n-1} - 2) N^{n-1} M, \quad n \geq 3. \quad (8)
\]
Substituting (3) in (1) we get
\[
W^\{a \Pi^b\} + F^{ab} + A^\{a \Pi^b\} + \{\Pi^a, \Pi^b\}' = 0, \quad (9)
\]
where
\[
F^{ab} = C^{\alpha a} \{\xi_\alpha, \xi_\beta \}' C^{\beta b}, \quad A^a = C^{\alpha a} \{\xi_\alpha, \cdot \}'.
\]
3 Solving the master equations

Let \( G' \) be the set of variables \( \xi, P, C, \lambda, \pi, \) and let \( \mathcal{V} \) be the space of the formal power series in the variables \( G' \) which vanish on \( \Sigma \) at \( P = \lambda = 0 \). The space \( \mathcal{V} \) splits as

\[
\mathcal{V} = \bigoplus_{m \geq 1} \mathcal{V}_m.
\]

with \( NX = mX \) for \( X \in \mathcal{V}_m \). It is clear that the operator \( N : \mathcal{V} \to \mathcal{V} \) is invertible. Let \( S^n, n \geq 1 \), denote the space of the functions \( X^{a_1 \ldots a_n} \in \mathcal{V} \) which are symmetric under permutation of any indices. Equations (3) imply that \( \Omega^a \in S^1 \), and \( \{\Omega^a, \Omega^b\}' \in S^2 \).

Define the operators \( W : S^n \to S^{n+1} \), and \( \Gamma : S^{n+1} \to S^n \), as

\[
(WX)^a = W^a X, \quad (\Gamma X) = \Gamma^a X^a, \quad n = 0,
\]

\[
(WX)^{a_1 \ldots a_{n+1}} = W^{a_1} X^{a_2 \ldots a_{n+1}}, \quad (\Gamma X)^{a_1 \ldots a_n} = \Gamma^a X^{a_1 \ldots a_n a}, \quad n \geq 1,
\]

where \( S^0 = \mathcal{V} \). For \( X \in S^0 \) we set

\[
\Gamma X = 0.
\]

One can directly verify that

\[
W^2 = 0, \quad \Gamma^2 = 0, \quad \Gamma M = (M - N)\Gamma,
\]

\[
WM = (M + N)W, \quad (\Gamma W + W\Gamma)X = (nN + M)X, \quad (10)
\]

where \( X \in S^n, n \geq 0 \). By using (8), we get

\[
(nN + M)^{-1} = \frac{1}{n} N^{-1} - \frac{1}{n(n + 2)(n + 1)} ((n + 3)M N^{-2} - M^2 N^{-3}), \quad n \geq 1.
\]

Let \( Q : S^n \to S^n \) be defined by

\[
QX = \frac{1}{6} (11N^{-1} - 6MN^{-2} + M^2 N^{-3})X, \quad X \in S^0,
\]

\[
QX = (nN + M)^{-1} X, \quad X \in S^n, \quad n \geq 1.
\]
Then $W^+ = Q\Gamma$ is a generalized inverse of $W$

$$WW^+ W = W.$$  \hfill (11)

From (10) it follows that

$$(W^+)^2 = 0,$$

and for any $X \in S^n$, $n \geq 1$,

$$X = W^+WX + WVW^+X.$$  \hfill (12)

Here

$$V = \frac{1}{n(n+2)(n+1)} \left( n(n^2+4n+6)I - (n-4)MN^{-1} - 2M^2N^{-2} \right),$$

and $I$ is the identity map.

For any $X \in S^1$, $Y \in S^n$, $n \geq 1$, we define the bracket $[, ,] : S^1 \times S^n \rightarrow S^{n+1}$ as

$$[X, Y]^{a_1...a_n} = \{X^{\{a_1}, Y^{a_2...a_n}\}}.$$  \hfill (1)

Then (9) can be written as

$$W\Pi + F + A\Pi + [\Pi, \Pi] = 0,$$  \hfill (13)

where

$$\Pi = (\Pi^1, \Pi^2), \quad F = (F^{ab}), \quad (A\Pi)^{ab} = A^{(a}\Pi^{b)}.$$

Applying the operator $WW^+$ to (13), we get

$$W\Pi + WW^+(F + A\Pi + [\Pi, \Pi]) = 0.$$  \hfill (14)

From (14) it follows that

$$\Pi = Y - W^+(F + A\Pi + [\Pi, \Pi]),$$  \hfill (15)

where

$$Y \in S^1, \quad WY = 0, \quad Y = \sum_{n \geq 2} Y^{(n)}, \quad Y^{(n)} \sim C^{n-m}x^m.$$
If $\Pi$ is a solution to (15) then

$$W^+\Pi = W^+\Upsilon.$$  \hfill (16)

Let $\langle \ldots \rangle : S^1 \times S^1 \rightarrow S^1$ be defined by

$$\langle X_1, X_2 \rangle = -\frac{1}{4}(I + W^+A)^{-1}W^+([X_1, X_2] + [X_2, X_1]),$$ \hfill (17)

where

$$(I + W^+A)^{-1} = \sum_{m \geq 0}(-1)^m(W^+A)^m.$$\hfill (16)

Then we have

$$\Pi = \Pi_0 + \frac{1}{2}\langle \Pi, \Pi \rangle,$$ \hfill (18)

where

$$\Pi_0 = (I + W^+A)^{-1}(\Upsilon - W^+F).$$

Let us show that a solution to (18) satisfies (13). We shall use the approach of ref. [12]. The Jacobi identities

$$\{\Omega^a, \{\Omega^b, \Omega^c\}\} + \text{cycl. perm. } (a, b, c) = 0,$$

imply

$$[\Omega, G] = 0,$$ \hfill (19)

where $\Omega = (\Omega^a)$ and $G$ is left-hand side of (13),

$$G = W\Pi + F + A\Pi + [\Pi, \Pi].$$

Equation (19) can be written as

$$WG + AG + [\Pi, G] = 0.$$ \hfill (20)

Consider (20), where $\Pi$ is a solution to (15), with the boundary condition

$$W^+G = 0.$$ \hfill (21)
Applying $W^+$ to (20), and using (21), we get

$$G + W^+(AG + [\Pi, G]) = 0.$$  

From this by iterations it follows that $G = 0$.

For checking (21) we have

$$W^+G = W^+W\Pi + W^+(F + A\Pi + [\Pi, \Pi]) = W^+W\Pi + \Upsilon - \Pi,$$

and therefore by (12) and (16), $W^+G = 0$.

To obtain an explicit expression for $\Pi$ we introduce the functions

$$\langle \ldots \rangle : (S^1)^m \to S^1, \quad m = 1, 2, \ldots,$$

which recursively defined by $\langle X \rangle = X$,

$$\langle X_1, \ldots, X_m \rangle = \frac{1}{2} \sum_{r=1}^{m-1} \sum_{1 \leq i_1 < \ldots < i_r \leq m} \langle \langle X_{i_1}, \ldots, X_{i_r} \rangle, \langle X_1, \ldots, \hat{X}_{i_1}, \ldots, \hat{X}_{i_r}, \ldots, X_m \rangle \rangle$$

if $m = 3, 4, \ldots$, where $\hat{X}$ means that $X$ is omitted. Recall that $\langle X_1, X_2 \rangle$ is defined in (17). Using induction on $m$ one easily verifies that $\langle X_1, \ldots, X_m \rangle$ is an $m$–linear symmetric function.

For $m \geq 2, 1 \leq i, j \leq m$, let

$$P_{ij}^m : (S^1)^m \to (S^1)^{m-1}$$

be defined by

$$P_{ij}^m(X_1, \ldots, X_m) = \langle \langle X_i, X_j \rangle, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_m \rangle.$$ 

If $X \in S^1$ is given by

$$X = P_{i_2}^2 P_{i_3}^3 \cdots P_{i_{m-2}}^{m-2} P_{i_{m-1}}^{m-1} P_{i_1 j_1}^m (X_1, \ldots, X_m)$$

for some $(i_1, j_1), \ldots, (i_{m-2}, j_{m-2})$, we say that $X$ is a descendant of $(X_1, \ldots, X_m)$. A descendant of $X \in S^1$ is defined as $X$. One can show that $\langle X_1, \ldots, X_m \rangle$ equals the sum of all the descendants of $(X_1, \ldots, X_m)$ [13]. For example,

$$\langle X_1, X_2, X_3 \rangle = \langle \langle X_1, X_2 \rangle, X_3 \rangle + \langle \langle X_1, X_3 \rangle, X_2 \rangle + \langle \langle X_2, X_3 \rangle, X_1 \rangle.$$ 

This remembers Wick’s theorem in QFT.
We can now write the general solution to eq. (18). It is given by

\[ \Pi = \langle e^{\Pi_0} \rangle, \]

where

\[ \langle e^{\Pi_0} \rangle = \sum_{m \geq 0} \frac{1}{m!} \langle \Pi_0^m \rangle, \quad \langle \Pi_0^0 \rangle = 0, \quad \langle \Pi_0^n \rangle = \langle \Pi_0, \ldots, \Pi_0 \rangle_{n \text{ times}}, \quad n \geq 1. \]

Finally, we get

\[ \Omega = \Omega_1 + \langle e^{\Pi_0} \rangle, \]

where \( \Omega_1 = (\Omega_1^0) \).

4 Realization of observables

With respect to the variables \( (\xi, \mathcal{P}, C, \lambda, \pi) \) Eq. (2) takes the form

\[ \{ \Omega^a, \Phi' \}' = 0, \]

where \( \Phi'(\xi, \mathcal{P}, C, \lambda, \pi) = \Phi(p, q, \mathcal{P}, C, \lambda, \pi) \). The boundary conditions read

\[ \text{ngh}(\Phi') = 0, \quad \Phi'|_{\mathcal{C} = \pi = 0} = \Phi_0, \quad \bar{\Gamma}(\Phi' - \Phi_0) = 0, \]

where \( \Phi_0(\xi) \in P, \Gamma = \epsilon^{ab} \Gamma_a \Gamma_b \).

The function \( \Phi' \) can be written as

\[ \Phi' = \Phi_0 + K, \quad K = \sum_{n \geq 1} \Phi^{(n)}, \quad \Phi^{(n)} \sim C^{n-m} \pi^m. \]

Substituting (25) in (23), we get

\[ W^a K + \{ \Omega^a, \Phi_0 \}' + A^a K + \{ \Pi^a, K \}' = 0, \]

For \( X \in S^1, Y \in S^0 \) denote

\[ [X, Y]^a = \{ X^a, Y \}', \quad (\text{ad} X) Y = [X, Y]. \]

Then (26) can be written in the form

\[ WK + [\Omega, \Phi_0] + AK + [\Pi, K] = 0. \]
By using (11) we get
\[ K + W^+([\Omega, \Phi_0] + AK + [\Pi, K]) = Y, \quad (28) \]
where
\[ Y \in S^0, \quad WY = 0, \quad \text{ngh}(Y) = 0. \]

Let us denote \( \bar{W} = \epsilon_a \epsilon^b W^a W^b \). Then
\[ \bar{W} \bar{\Gamma} - \bar{\Gamma} \bar{W} = 4N^2 - 2MN, \]
from which it follows that for any \( X \in \mathcal{V} \)
\[ X = \frac{1}{2} MN^{-1} X + \frac{1}{4} (\bar{W} \bar{\Gamma} - \bar{\Gamma} \bar{W}) N^{-2} X. \quad (29) \]

The boundary conditions (24) imply \( \bar{\Gamma} K = 0 \), and therefore \( \bar{\Gamma} Y = 0 \), since \( \bar{\Gamma} W^+ = 0 \). By using (29) we get \( Y = 0 \).

Solving (28) for \( K \) yields
\[ K = -(I + W^+ (A + \text{ad} \Pi))^{-1} W^+ [\Omega, \Phi_0]. \quad (30) \]

We must now show that (30) satisfies (27).

The Jacobi identities for the functions \( \Omega^a, \Phi^b \) imply
\[ \{ \Omega^a, \{ \Omega^b, \Phi^c \}' \} + \{ \Omega^b, \{ \Omega^a, \Phi^c \}' \} = 0. \quad (31) \]

Let \( R = (R^a) \) denote left-hand side of (27),
\[ R = WK + [\Omega, \Phi_0] + AK + [\Pi, K]. \quad (32) \]

Then (31) takes the form
\[ WR + AR + [\Pi, R] = 0. \quad (33) \]

It is easily verified that if \( K \) satisfies (26) then \( W^+ K = W^+ \Upsilon \), and
\[ W^+ R = 0. \quad (34) \]

We note that \( K \in S^0 \) and \( R \in S^1 \). Consider (33) and (31), where \( K \) satisfies (26). By using (12), we get
\[ R = -W^+ (AR + [\Pi, R]). \]
From this it follows that $R = 0$.

We conclude that the solution to Eqs. (23), (24) is given by

$$\Phi' = L \Phi_0,$$

(35)

where

$$L = I - (I + W^+ (A + \text{ad} \Pi))^{(-1)} W^+ \text{ad} \Omega.$$  

The operator $L$ is invertible. The inverse $L^{-1}$ is given by

$$L^{-1} \Phi' = \Phi'\big|_{C=\pi=0}.$$  

Equation (35) establishes a one-to-one correspondence between first class functions and solutions to Eqs. (23), (24).

Let us denote by $L(A)$ the image of $A \subset P$ under the mapping $L$. For $\Phi_1', \Phi_2' \in L(P)$

$$\{\Phi_1', \Phi_2'\}'\big|_{C=\pi=0} = \{\Phi_1|_{C=\pi=0}, \Phi_2|_{C=\pi=0}\}',$$

$$(\Phi_1' \Phi_2')|_{C=\pi=0} = \Phi_1'|_{C=\pi=0} \Phi_2'|_{C=\pi=0}.$$  

This means that $L(P)$ and $P$ are isomorphic as Poisson algebras, and therefore $L(P)/L(J)$ gives a realization of classical observables.

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