Estimations and Scaling Laws for Stellar Magnetic Fields

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Abstract
In rapidly rotating turbulence (i.e., a Rossby number much less than unity), the standard mixing length theory for turbulent convection breaks down. However, the Coriolis force enters the force balance such that the magnetic field eventually depends on rotation. By simplifying the self-sustained magnetohydrodynamics dynamo equations of electrically conducting fluid motion, with the aid of the theory of isotropic nonrotating or anisotropic rotating turbulence driven by thermal convection, we make estimations and derive scaling laws for stellar magnetic fields with slow and fast rotation. Our scaling laws are in good agreement with the observations.

Unified Astronomy Thesaurus concepts: Magnetic fields (994); Stellar physics (1621)

1. Motivation
Observations show that the fraction of stellar X-ray luminosity $L_X/L_{bol}$ increases with stellar rotation rate until it reaches saturation for sufficiently fast rotation (Wright et al. 2011; Reiners et al. 2014; Vidotto et al. 2014). Figure 1, extracted from Wright et al. (2011), shows this relation between $L_X/L_{bol}$ and rotation. The Rossby number $Ro = P_{rot}/τ$, the ratio of stellar rotation period to turbulent convection timescale, measures rotation. Wright et al. (2011) gives the scaling law $L_X/L_{bol} ∝ Ro^{-2}$ for the stage before saturation in which the field increases with rotation. The X-ray emission is caused by mass loss near the stellar surface, which arises from a surface magnetic field with open field lines. The surface field stems from the internal field, which is generated through dynamo in the convection zone, i.e., the magnetic field is amplified by shear and twist of field lines due to differential rotation and turbulent convection of electrically conducting fluid. Therefore, the fraction $L_X/L_{bol}$ represents the strength of the stellar magnetic field (Pevtsov et al. 2003; Vidotto et al. 2014).

To interpret the observations, the interface $α$–ω dynamo model (Montesinos et al. 2001) and the flux transport model have been proposed in the observational papers mentioned above. However, both models describe kinematic dynamo, i.e., fluid motion is prescribed but not coupled with magnetic field (Lorentz force or back reaction of field on flow is neglected), and cannot interpret a self-sustained turbulent dynamo in the stellar convection zone. Moreover, the mean-field dynamo has been widely used to interpret the field–rotation relation. For example, Blackman & Thomas (2015) used a kinematic $α$–ω mean-field dynamo model; Blackman & Owen (2016) used a dynamic mean-field dynamo model in which rotation, magnetic field, and mass loss were coupled; and Kitchatinov & Olemskoy (2015) used a Babcock–Leighton model, which is also an $α$–ω kinematic mean-field dynamo model. In addition to the mean-field dynamo, magnetic helicity has been introduced together with the mean-field model to interpret astrophysical dynamos (Vishniac & Cho 2001; Vishniac & Shapovalov 2014), or coronal activity (Blackman & Field 2000) and ejection (Blackman & Brandenburg 2003). However, the mean-field dynamo is a parameterized model, which solves only the magnetic induction equation in the absence of the fluid dynamics equation; that is, it does not solve the full magnetohydrodynamics (MHD) equations, especially in the lack of thermal convection. The flow is usually characterized by a differential rotation plus the parameter $α$, which arises from helical motion (Moffatt 1978). Although in Blackman & Brandenburg (2003) the back reaction of magnetic field on the parameter $α$ was considered, namely Lorentz force enters the expression of $α$, and in Blackman & Owen (2016) the effect of Lorentz force on differential rotation was considered, both did not solve the full MHD equations, especially thermal convection.

In this paper, we study the convection-driven dynamo by simplifying the self-sustained MHD equations, with the aid of the turbulence theory for thermal convection with or without rotation, to make estimations and derive scaling laws for magnetic energy at both slow and fast rotation rates, and then compare our predictions with the observations. We focus on the range in which magnetic field depends on rotation ($Ro > 0.1$ in Figure 1) and give a tentative interpretation about the saturation range ($Ro < 0.1$ in Figure 1).

2. Estimation and Scaling Laws
How planetary magnetic fields depend on physical properties (e.g., density, radius, rotation, etc.) has been extensively studied; a good summary can be found in the review paper by Christensen (2010). In Christensen et al. (2009) a scaling law for the magnetic fields of rotating planets and stars is proposed with the aid of the standard mixing length theory for isotropic nonrotating turbulence. Although this scaling law is for a rotating planet or star, it is independent of rotation itself, because the energy equation that was used to derive this scaling does not involve rotation, i.e., the Coriolis force does not enter the energy equation (we will see this later). However, as we have already shown in the last section, observations confirm that stellar magnetic fields depend on rotation. We will illustrate the reason for this in this section, i.e., the standard mixing length theory breaks for rapidly rotating turbulence. Here fast or slow rotation is evaluated with Rossby number, i.e., whether it is greater or less than unity. For planetary fields, the mixing length cannot exceed the depth of the convection
is the gas constant. On the right-hand side of (1) the terms are pressure force, Coriolis force, Lorentz force, buoyancy force, and viscous force. Performing \( v \cdot (1) + B \cdot (2), \) we obtain the energy equation

\[
\frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + \frac{B^2}{2\mu} \right) = -\nabla \cdot A
\]

\[+ \delta \rho \mathbf{g} \cdot \mathbf{v} - D_v - \frac{J^2}{\sigma}. \tag{3}\]

The left-hand side is the rate of total energy. On the right-hand side, vector \( \mathbf{A} \) is the total flux, consisting of kinetic energy flux \((\rho v^2/2)\mathbf{v})\), pressure energy flux \(p\mathbf{v})\), and Poynting flux \(\mathbf{E} \times \mathbf{B}/\mu\). The second term is the power of buoyancy force. The third term, \(D_v\), is viscous dissipation. The last term is ohmic dissipation, where \(\sigma = 1/(\mu\eta)\) is electric conductivity. We take the volume average for (3) in the convection zone, i.e., \((1/V)\int (3) dV\). When dynamo saturates, the total energy is statistically steady so that the left-hand side vanishes. By divergence theorem \(\nabla \cdot \mathbf{A}\) vanishes because the net flux across the surface is very small, that is, the energy loss from the surface due to stellar wind and Poynting flux is negligible compared to the buoyancy energy in the interior. Although mass loss due to stellar wind or magnetic helicity may be noticeable (Blackman & Brandenburg 2003), we think that the magnetic energy that mass loss carries away is a tiny fraction of the magnetic energy generated by convection dynamo in the interior. Viscous dissipation is very small compared to ohmic dissipation. Thus, the two terms are left to balance each other:

\[
\langle \delta \rho \mathbf{g} \cdot \mathbf{v} \rangle \approx \left\langle \frac{J^2}{\sigma} \right\rangle, \tag{4}\]

where bracket denotes volume average. Equation (4) states that the power of the buoyancy force is almost equal to the ohmic dissipation rate, which is exactly the essence of convection dynamo. It should be noted that Coriolis force due to rotation does not enter the energy equation, since it is perpendicular to fluid velocity.

Next we introduce two length scales, the mixing length \(l\) for turbulent momentum transfer, i.e., the size of the largest turbulent eddies, and the length scale \(l_B\) of the magnetic field. The mixing length is usually assumed to be twice the pressure scale height:

\[
l \approx \frac{2p}{dp/dr} \approx \frac{2p}{\rho g} \approx \frac{2RT}{\mu_m g}, \tag{5}\]

where hydrostatic balance \(dp/dr \approx \rho g\) and equation of state for ideal gas \(p = (\mathcal{R}/\mu_m)T\) are employed (\(\mathcal{R}\) is the gas constant and \(\mu_m\) is the mean molecular weight). To obtain the estimation for \(l_B\) we return back to (2). The three terms are estimated as follows. The magnetic field temporally varies on the convective timescale \(l/v\), i.e., \(\partial \mathbf{B}/\partial t \sim v\mathbf{B}/l\). The magnetic induction term takes effect on the large length scale \(l\), i.e., \(\nabla \times (\mathbf{v} \times \mathbf{B}) \sim v\mathbf{B}/l\), which is comparable to the time-derivative term. The magnetic diffusion term takes effect on the small length scale \(l_B\), i.e., \(\eta \nabla^2 \mathbf{B} \sim \eta \mathbf{B}/l_B^2\). This two-scale analysis with respect to \(l\) and \(l_B\) is widely used in dissipative systems (e.g., our system with magnetic diffusion or ohmic dissipation). The three terms are of a comparable order of magnitude, and this balance immediately yields the estimation

\[
l_B \approx l \left( \frac{v}{\eta} \right)^{1/2} = R_m^{-1/2}, \tag{6}\]

where the magnetic Reynolds number is defined on the large length scale \(l\), i.e., \(R_m = vl/\eta\). Ampere’s law, \(\nabla \times \mathbf{B} = \mu \mathbf{J}\), yields the estimation \(J \approx B/(\mu l)\). Inserting \(J \approx B/(\mu l_B)\) and (6) into (4), we obtain

\[
\left\langle \frac{B^2}{\mu} \right\rangle \approx \left\langle \delta \rho \mathbf{g} \cdot \mathbf{l} \right\rangle, \tag{7}\]

which states the equipartition between magnetic energy and buoyancy energy. Equation (7) is what we will use to estimate magnetic energy in the following.

We study two cases: one is slow rotation, with \(Ro > 1\), and the other fast rotation, with \(Ro < 1\). With slow rotation we adopt the standard mixing length theory

\[
\delta \rho \mathbf{g} \cdot \mathbf{l} \approx \nu v^2, \tag{8}\]
which states the equipartition between buoyancy energy and kinetic energy. Combining Equations (7) and (8), we find that the three energies (magnetic, buoyancy, kinetic) are of the same order of magnitude. If we change the view from energy to force, (8) indicates the balance between buoyancy force and inertial force, i.e., \( \delta \rho g \approx \rho v^2/l \). With slow rotation, this force balance can be applied. However, with fast rotation, inertial force is negligible compared to Coriolis force. Then the force balance is built between buoyancy force and Coriolis force, i.e., \( \delta \rho g \approx \rho v_\perp \Omega \), where \( v_\perp \) is velocity perpendicular to rotation axis (the parallel component has no contribution to Coriolis force). In the following we will see that this force balance in rapidly rotating turbulence is numerically validated. Moreover, the length scale in estimation (7) will be the eddy length scale parallel to rotational axis \( l_{by} \); we will also address this point in the following.

With slow rotation, we combine (7) and (8) to find \( (B^2/\mu) \approx (\rho \delta \varpi^2) \). Now we need to estimate convective velocity \( \nu \). Instead of \( \delta \rho \), which cannot be observed, we use heat flux, \( F \), which is related to luminosity (an observable quantity) to measure convection:

\[
F = \rho c_p \delta \varpi \nu = c_p T \delta \rho \nu, \tag{9}
\]

where the thermodynamics relation \( \delta \varpi/T = -\delta \rho/\rho \) for ideal gas at constant pressure is employed (what we are concerned with is magnitude, so that we omit the minus sign). Then (5), (8) and (9) combine to yield the estimation

\[
\nu \approx \left(\frac{F_c}{\rho}\right)^{1/3}, \tag{10}
\]

where \( c_p = 2.5 \mathcal{R}/\mu m \) in the convection zone is employed. The estimation for convective velocity (10) has already been validated by numerical simulations (Chan & Sofia 1996; Cai 2014). Inserting (10) into \( (B^2/\mu) \approx (\rho \delta \varpi^2) \), we obtain the estimation for magnetic energy with slow rotation:

\[
\left\langle \frac{B^2}{\mu} \right\rangle \approx \left(\frac{\rho \delta \varpi^2}{\mu} \right)^{3} \mathcal{R}^{-2}. \tag{11}
\]

which is similar to Christensen et al. (2009), i.e., magnetic energy is independent of rotation rate. More strictly speaking, Christensen et al. (2009) chose the length scale in (7) to be the minimum of the mixing length (5) and the depth of convection zone. For most planets with small size, this length scale will be the depth of the convection zone; but for most stars it will be the mixing length. Therefore, for a rapidly rotating planet, this scaling law independent of rotation is fine because the depth of the convection zone does not depend on rotation. However, for a rapidly rotating star, this length scale will be \( l_{by} \), which strongly depends on rotation (we will see how strong this dependence is in the following). That is the reason why Christensen et al. (2009) cannot well interpret the stellar field–rotation relation (Wright et al. 2011).

With fast rotation, turbulence is anisotropic and has a large-scale columnar structure along the rotation axis. The formation of such a two-dimensional turbulence structure is caused by propagation of inertial waves at its group velocity (inertial waves are induced by Coriolis force). Suppose that an eddy with an initial size \( l \), namely, the mixing length in the absence of rotation, is elongated at the group velocity of inertial waves \( c_g \approx \Omega \) until the eddy turnover timescale \( \tau \approx 1/v_\perp \) (i.e., the lifetime of an eddy with size \( l \), where \( v_\perp \) is the turbulent velocity perpendicular to the rotation axis. This eddy will grow until its length parallel to rotation axis reaches \( l_\parallel \approx c_g \tau \approx \Omega t/l_\perp \) when the eddy is destroyed in turbulence at its lifetime \( \tau \). The most recent numerical simulations of rotating turbulent convection find that turbulent velocity is suppressed by fast rotation (Cai 2021), i.e., the velocity parallel to rotation axis scales as \( v_\parallel \approx v \Omega t/l_\perp \) and the velocity perpendicular to rotation axis scales as \( v_\perp \approx v \Omega t/l_\parallel \) (\( v_\perp \) is even more suppressed than \( v_\parallel \)).

The observations show that \( L_X = L_{\text{bol}} \) and the scaling law is \( L_X \propto \Omega^{-1} \) (or \( L_X \propto \Omega^{-1/2} \) and \( \Omega \propto \Omega^{-1} \)) and \( l_{by} \propto \Omega^{-1/2} \), which indicates that this force balance is reliable. We now estimate magnetic energy \( (B^2/\mu) \approx (\rho \delta \varpi^2) \). We can validate the scaling law, i.e., \( l_{by} \) instead of \( l \) in (6) and (7), because magnetic induction tends to take effect on the larger length scale \( l_{by} \) (\( \propto B_0 \Omega^{-1/3} \)). Inserting the scaling \( \delta \rho \propto \Omega^{-1/3} \) (or \( v_\perp \propto \Omega^{-1/2} \) and \( \Omega \propto \Omega^{-1} \)) and \( l_{by} \propto \Omega^{-1/2} \), we immediately obtain the estimation for magnetic energy with fast rotation:

\[
\left\langle \frac{B^2}{\mu} \right\rangle \approx \left(\frac{\rho \delta \varpi^2}{\mu} \right)^{3} \mathcal{R}^{-2}. \tag{12}
\]

With fast rotation, magnetic energy depends on rotation rate and faster rotation (smaller \( \Omega \)) corresponds to stronger magnetic field.

3. Comparison with Observations

As stated in Section 1, the fraction of X-ray luminosity \( L_X/L_{\text{bol}} \) represents the strength of the magnetic field. The observations clearly show that faster rotation indeed corresponds to stronger \( L_X/L_{\text{bol}} \) and the scaling law is \( L_X \propto \Omega^{-2} \) (Wright et al. 2011). On the other hand, observations with different techniques give different relations of the X-ray luminosity and surface field, i.e., \( L_X/L_{\text{bol}} \propto B_{\text{surf}}^{1.61} \) or \( L_X/L_{\text{bol}} \propto B_{\text{surf}}^{2.25} \) (Vidotto et al. 2014). Surface field \( B_{\text{surf}} \) is proportional to the volume-averaged internal field \( B \), such that observations yield a scaling law of the internal magnetic field \( \langle B \rangle \propto \Omega^{-1/2} \) or \( \langle B \rangle \propto \Omega^{-0.89} \). Our prediction (12) yields \( \langle B \rangle \propto \Omega^{-1/2} \), which is in good agreement with the observations.

The observations show that \( L_X/L_{\text{bol}} \), or equivalently the magnetic field, saturates at sufficiently low \( \Omega < 0.01 \) (it should be noted that \( \Omega = \mathcal{P}_{\text{rot}}/\tau \) used for the observations differs from our definition \( \Omega = \mathcal{P}_{\text{rot}}/\tau \) by a factor 2\( \pi \). The saturation mechanism might be tentatively interpreted in this way: \( l_{by} \) grows to its maximum, namely the depth of the stellar convection zone, at which buoyancy energy saturates, and consequently magnetic energy saturates. According to the scaling law derived in the last section, the aspect ratio of the columns is proportional to \( \Omega^{-1/2} \), so that when \( \Omega \) reaches \( \sim 0.01 \) the column is very thin with its aspect ratio \( \sim 10^2 \), i.e., column height (stellar convection zone depth) \( \sim 10^2 \) km and column width less than 1 km. These thin columns seem to be
unstable, though fast rotation can stabilize them (Chandrasekhar 1961).

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