

**TOWARD CLASSIFYING UNSTABLE THEORIES**

**SH500**

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**Abstract.** We prove a consistency results saying, that for a simple (first order) theory, it is easier to have a universal model in some cardinalities, than for the theory of linear order. We define additional properties of first order theories, the \( n \)-strong order property (\( SOP_n \) in short). The main result is that a first order theory with the 4-strong order property behaves like linear orders concerning existence of universal models.

**Key words and phrases.** Model theory, classification theory, stability theory, unstable theories, universal models, simple theories, Keisler’s order.

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§0 Introduction

Having finished [Sh:a], an important direction seems to me to try to classify unstable theories; i.e. to find meaningful dividing lines. In [Sh 10] two such were the strict order property and the independence property, their disjunction is equivalent to unstability (see [Sh:a], 4.7 = [Sh:c], 4.7, p.70). For theories without the independence property, we know $S(A)$ (and $S_{\Delta}(A)$) are relatively small (see [Sh 10], Keisler [Ke76], [Sh:a], III, §7, 3.7.4, II, §4.4.9, 4.10. Also for $\lambda > |T|$, $\{p \in S(A) : p$ does not split over some $B \subseteq A$ of cardinality $< \lambda$} is $\lambda$-dense (see [Sh:a], 7.5 = [Sh:c], 7.5, p.140).

Later this becomes interesting in the content of analyzing monadic logic (see Baldwin Shelah [BlSh 156] representation Baldwin [Bl]. By [Sh 197] if “no monadic expansion of $T$ has the independence property” is a significant dividing line.

Lately, some model theorist have become interested in finitary versions called VC dimensions, see Laskowski [Lw92], Macintyre [Mcxx] (good bound for the case ? Mcxx ? of expansion the real field).

More relevant to the present work is the tree property, which is weaker than the strict order property (in [Sh:c], III, p.171).

In [Sh:93] we try to investigate theories without the tree property, so called simple. This can be looked at as a weakening of stable, so: simple $\iff \kappa_{cdt}(T) < \infty \iff$ failure of the tree property $\iff$ suitable local ranks $< \infty$ are parallel of stable. We try to do the parallel to (parts of) Ch. II, III of [Sh:a], forking being generalized in some ways. But here instead showing the number of untrafilters of the Boolean Algebras of formulas $\varphi(x, \bar{a})$ over $A$ is small ($\le |A|^{|T|}$) we show that it can be decomposed to few subalgebras satisfying a strong chain condition. In this context we also succeed to get averages; but the Boolean algebras we get were derived from normal ones with a little twist. We did not start with generalizing the rest of [Sh:a] like supersimple (i.e. $\kappa_{cdt}(T) = \aleph_0$, equivalently suitable rank is $< \infty$). The test problem in [Sh:93] was trying to characterize the class of pairs

$$SP(T) = \left\{ (\lambda, \kappa) : \text{every model of } T \text{ of cardinality } \lambda \text{ has a} \kappa\text{-saturated elementary extensions of cardinality } \lambda \right\}.$$  

For simplicity we consider there only $\lambda = \lambda^{|T|} > 2^{|T|^{+\kappa}}$, $\kappa > |T|$ and $(\exists \mu)(\mu = \mu^{<\kappa} \le \lambda \le 2^\mu)$ (if this fails, see [Sh 576]). So by [Sh:93] for non-simple $T$, such $(\lambda, \kappa)$ is in $SP(T)$ iff $\lambda = \lambda^{<\kappa}$. If $\mu = \mu^{<\kappa} < \lambda = \lambda^{<\lambda}$, after suitable forcing preserving $\mu = \mu^{<\mu}$, not collapsing cardinals and making $2^\mu = \lambda$, we have under suitable generalization of MA, so $\kappa < \mu < \lambda < 2^\mu \Rightarrow (\lambda, \kappa) \in SP(T)$.

It seems much better to use just the cardinal arithmetic assumptions (not the generalizations of MA). This call to investigate problem of $P^{-}(n)$-amalgamation (see [Sh 87b], [Sh:c], XII, §5). For the case of $n = 3$ this means that

$\ast$ if $p_0(\bar{x}, \bar{y}), p_1(\bar{x}, \bar{z}), p_2(\bar{y}, \bar{z})$, complete types over $A$, each saying the two sequences of variables are “independent” in suitable ways (like nonforking) then we extend the union of the three (preserving “independence”).
Now $(*)_3$ can be proved. [Sh:93], Claim 7.8, p. 201, (3.5, p. 187). But the proof does not work for higher $n$, naturally counterexamples for the amalgamation should give counterexample to membership in $\text{SP}$. This was carried out by finding counterexamples in a wider framework: saturation inside $P$ in [Sh 126]; but we could still hope that for the “true” one there is a positive one.

For long, I was occupied elsewhere and not look into it, but eventually Hrushovski becomes interested (and through him, others) and we try to explain the relevant research below. Also, it could be asked if simple unstable theories “occurs in nature”, “are important to algebraic applications”. The works cited below forms a positive answer (note that, quite natural, those examples concentrate on the lower part of hierarchy, like strongly minimal or finite Morley rank).

On the one hand, Hrushovski, continuing [Sh 126], prove that there are simple theories with bad behaviour for $\mathcal{P}(n)$ so in the result above the cardinal arithmetic are not enough.

On the other hand, by Hrushovski Pillay [HrPi] in specific cases (finite ranks) relevant cases of $(*)_n$ are proved, for $n > 3$ under very specific conditions: for $n = 3$ more general; but the relationship with [Sh:93], 7.8 of $(*)$ was not clarified (in both cases the original rank does not work; the solution in [Sh:93] is to use $\text{dnwd}$ (= “do not weakly divide”), Hrushovski changes the rank replacing “contradictory” by having small rank: this seems a reasonable approach only for supersimple theories and was carried only for ones with finite rank, and it gives more information in other respects.

In Hrushovski [Hr1] let $\mathfrak{C}_0$ be the monster model for a strongly minimal theory with elimination of imaginaries, $A \subseteq \mathfrak{C}, A = \text{dcl} A$, such that every $p \in S^n(A)$ with multiplicity 1 is finitely satisfiable in $A$, now $\text{Th}(\mathfrak{C}, A)$ is simple (of rank 1) and we can understand PAC in general content. Hrushovski [Hr2] does parallel thing for finite rank.

We turn to the present work. First section deals with the existence of universal models. Note that existence of saturated models can be characterized nicely by stability (see [Sh:c]).

By Kojman Shelah [KjSh 409], the theory of linear order and more generally theories without the strict order property has universal models in “few” cardinals.

By [Sh 457] give a sufficient conditions for a consistency of “there is in $\mu^+$ a model of $T$ universal for models of $T$ of cardinality $\mu^+$”, which we use below.

The main aim is to show that all simple theories behave “better” in this respect than the theory of linear order. Specifically, it is consistent that $\aleph_0 < \lambda = \lambda^{<\lambda}, 2^{\lambda} > \lambda^+$, moreover, there is a club guessing $\langle C_\delta : \delta < \lambda^+, \text{cf}(\delta) = \lambda \rangle$, and every simple $T$ of cardinality $< \lambda$ has a model in $\lambda^{++}$ universal for $\lambda^+$. For this we represent results of [Sh:93] and do the things needed specifically for the use of [Sh 457]. See 1.4A(2).
§1 Simple theories have more universal models

We quote [Sh 457].

1.1 Lemma. Suppose

(A) $T$ is first order, complete for simplicity with elimination of quantifiers (or just inductive theory with the amalgamation and disjoint embedding property).

(B) $K_{ap}$ is a simple $\lambda$-approximation system such that every $M \in K_{ap}$ is a model of $T$ hence every $M_\Gamma$, where for $\Gamma \in K_{md}$ we let $M_\Gamma = \cup \{ M : M \in \Gamma \}$.

(C) Every model $M$ of $T$ of cardinality $\lambda^+$ can be embedded into $M_\Gamma$ for some $\Gamma \in K_{md}$.

Then

(a) in $V^P$, there is a model of $T$ of cardinality $\lambda^{++}$ universal for models of $T$ of cardinality $\lambda^+$.

(b) So in $V^P$, $\text{univ}(\lambda^+, T) \leq \lambda^{++} < 2^\lambda$.

Proof. Straightforward.

1.1 Fact. 1) Assume $M \prec N, \bar{a} \in {}^{\omega}N$, and $\triangle$ a finite set of formulas possibly with parameters from $M$. Then there is a formula $\psi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}, N)$ such that:

(*) for any $\bar{a}' \in M$ realizing $\psi(\bar{x}, \bar{b})$, we can find a $\triangle - 2$-indiscernible sequence $\langle \bar{a}_i : i \leq \omega \rangle$ such that: $\bar{a}_0 = \bar{a}', \bar{a}_{\omega+1} = \bar{a}$; hence we can find an indiscernible sequence $\langle \bar{a}_{\omega}^0 : i < \omega \rangle$ (in $C$) such that the $\triangle$-type of $\bar{a}_{\omega}^0 \bar{a}_1'$ is the same as that of $\bar{a}' \bar{a}$.

2) Assume $2^{\theta+|T|} \leq \kappa$ and $M \prec N$, moreover

$\bigotimes$ if $A \subseteq M, |A| \leq \kappa, \bar{a} \in {}^\theta N$ then one $\bar{a}' \in M$ realizes $\text{tp}(\bar{a}, A, N)$.

Then for any $\bar{a} \in {}^\theta N$ and $B \subseteq M, |B| \leq \theta$, there is $A \subseteq M < |A| \leq \kappa$, such that for every $\bar{a}' \in {}^\theta M$ realizing $\text{tp}(\bar{a}, A, N)$ there is a sequence $\langle \bar{a}_i : i \leq \kappa \rangle$ which is $2$-indiscernible over $B, a_0 = \bar{a}', a_\kappa = \bar{a}$; hence there is an indiscernible sequence $\langle \bar{a}_i' : i < \omega \rangle$ such that $\bar{a}_0' \bar{a}_1'$ realizes the same type as $\bar{a}' \bar{a}$ over $B$.

Proof. Obvious [notes on combination set theory].

1) Let $\langle p_i : i < k \rangle$ list the possible $\Delta$-types of sequences of length $\ell g(\bar{a}) + \ell(\bar{a})$, so $k < \omega$. For each $p_i, \psi_i(\bar{x}, \bar{b}_i) \in \text{tp}(\bar{a}, M, N)$ such that, if possible for no $\bar{a}' \in {}^{\ell g(\bar{a})} M$ realizing $\psi_i(\bar{x}, \bar{b}_i)$ do we have: $\bar{a}' \bar{a}$ realizes $p_i$; (if there..)

Now $\psi(\bar{x}, \bar{b}) =: \bigwedge_{i<k} \psi_i(\bar{x}, \bar{b}_i)$ is as required.

2) Similar.

3) $\langle p_i : i < k \rangle$ list the complete $2\ell g(\bar{a})$-types over $B$. Use $p_i \subseteq \text{tp}(\bar{a}, M, N), |p_i| \leq \theta + |T|$ instead $\psi_i(\bar{x}, \bar{b}_i), A = \bigcup_{i<k} \text{Dom } p_i$. 

1.2 Theorem. If $T$ is a complete simple (f.o.) theory, $|T| < \lambda$ then $T$ satisfies the assumption of 1.1 (hence its conclusions).

1.2A Remark. 1) We can get results for a theory $T$ of cardinality $\leq \lambda$ under stronger assumptions on $T$.
2) Though not always necessary, in this section we’ll assume that $T$ is simple.
3) Also this section is not written in a way focused on Theorem 1.2, but leisurely relook at simple theories.

Proof. Without loss of generality $T$ has elimination of quantifiers.

We first represent (in 1.3 - 1.10) the needed definitions and facts on simple theories from [Sh:93] (adding notation and some facts), then say a little more and prove the theorem. So for a while we work in a fixed $\kappa$-saturated model $C$ of $T$, $\kappa$ big enough. So $M, N$ denotes elementary submodels of $C$ of cardinality $< \kappa$, $A, B, C, D$ denote subsets of $C$ of cardinality $< \kappa$ and $a, b, c, d$ denote sequences of elements of $C$ of length $< \kappa$, usually finite. Let $a/B = \text{tp}(a, B) = \{\varphi(x, b) : b \in \omega \to B, \varphi \in L(T)\}$ is first order and $C \models \varphi[a, b]$.

1.3 Definition. 1) We say that “$p(\bar{x})$ does not weakly divide over $(r, B)$” (in short $p(\bar{x})$ does not weakly divide) over $(r, B)$; we write over $B$ when $r = p \upharpoonright B$, we write over $r$ if $B = \text{Dom}(r)$, where $r = r(\bar{x})$ is a type over $B$ (and $\bar{x}$ may be infinite) when: if $b \in B$ and $\psi = \psi(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ a formula (where $\ell g(\bar{x}^\ell) = \ell g(\bar{x}), \bar{x}^\ell$ with no repetition, $\langle \bar{x}^\ell : \ell = 1, n\rangle \psi(y)$ pair disjoint) and $\psi$ is finitely satisfiable (in $C$) then so is $[r \cup p]^{\psi}$ (see Definition 1.3(2) below).
2) If $\psi = \psi(\bar{x}^1, \ldots, \bar{x}^n)$ (possibly with parameters), $q = q(\bar{x})$ then $q^{\psi} = \{\psi\} \cup \bigcup_{\ell = 1}^{n} q(\bar{x}^\ell)$.
3) $p(\bar{x})$ divides over $A$ if for some formula $\psi(\bar{x}, \bar{a})$ we have $p \vdash q(\bar{x}, \bar{a})$ and for some indiscernible sequence $\langle \bar{a}_\ell : \ell < \omega \rangle$ over $A$, $\bar{a} = \bar{a}_0$, and $\{\varphi(\bar{x}, a_i) : i < \omega\}$ is $(< \omega)$-contradictory where a set $p$ of formulas is $n$-contradictory if for any distinct $\varphi_1, \ldots, \varphi_n \in p$, $\{\varphi_1, \ldots, \varphi_n\}$ is not realized (in $C$), and $(< \omega)$-contradictory means: $n$-contradictory for some $n$). We write “dnf” as shortening for “does not fork”.
4) The type $p$ fork over the set $A$ if for some $n < \omega$ and formulas $\varphi(\bar{x}, \bar{a}_\ell)$ for $\ell < n$ we have: $p \vdash \bigvee_{\ell < n} \varphi(\bar{x}, \bar{a}_\ell)$ and for each $\ell < n$ the formula $\varphi(\bar{x}, \bar{a}_\ell)$ divides over $A$.

We use “dnf” as shortening for “does not fork”.
5) The type $p$ is finitely satisfiable (finitely satisfiable) in $A$ (or in $I$) if every finite subset $p'$ of $p$ is realized by some sequence from $A$ (or a member of $I$).
6) If $D$ is an ultrafilter on $\text{Dom}(D) = I$ (where all members of $I$ have the same length, say $m$) then $Ae(B, D) =: \{\varphi(x, \bar{a}) : \bar{b} \in \text{Dom}(D) : \varphi(b, \bar{a}) \in D\}$.  

1.4 Definition. We say “$a/A$ (or $\text{tp}(\bar{a}, A)$) weakly divides over $B$” if $B \subseteq A$ and $\text{tp}(a, A)$ weakly divides over $(\text{tp}(\bar{a}, B), B)$, (similarly for does not weakly divide).

1.4A Remark. 1) An equivalent formulation is “$a/A$ is an extension of $a/B$ with the same degree for most $(\triangle, \lambda_0, b)$”; see 1.5(8) below.
2) On “divides”, “fork”, “weakly divide” see [Sh:93], Def.1.1,1.2, 2.7(2) respectively. On the first two also [Sh:a], but there the focus is on stable theories. On “finitely satisfiable” see [Sh:a], Ch.VII, § 4. We present here most of their properties, ignoring mainly the connections with suitable degrees and indiscernibility and the derived Boolean algebras of formulas (satisfying chain conditions). For stable $T$ the notions of Def. 1.3 collapse becoming equivalent (finitely satisfiable - only when the set is a model, see [Sh:a], Ch.III).

Basic properties are (mostly check directly, but 0A,6,8,9 are quoted).

1.5 Claim. 0) [Implications] If $p$ divides over $A$ then $p$ forks over $A$.
0A) If $p$ forks over $A$ then $p$ weakly divides over $(p \upharpoonright A, \emptyset)$, (by [Sh:93],2.11(1), p.184, in its proof we have rely not only of [Sh:93],2.10(2) + 2.9(3) but also on [Sh:93],2.4(3)).

0B) If a type $p$ is finitely satisfiable in $A$ then $p$ does not fork over $A$.
1) [Monotonicity] If $B \subseteq A_1 \subseteq A_2 (\subseteq C)$, $\bar{a}/A_2$ does not widely divide over $B$.
1A) If $p$ does not divide over $A$, $A \subseteq A_1$ and $p_1 \subseteq p$ or at least $p \vdash p_1$ then $p_1$ does not divide over $A_1$.
1B) If $p$ does not fork over $A$, $A \subseteq A_1$ and $p_1 \subseteq p$ or at least $p \vdash p_1$ then $p_1$ does not fork over $A_1$.
1C) If $p$ does not widely divide over $(r, A)$, $A \subseteq A_1$, $r_1 \vdash r$ and $p_1 \subseteq p$ or at least $p \vdash p_1$ then $p_1$ does not widely divide over $(r_1, A_1)$.
2) [Local character] $\bar{a}/A$ does not widely divide over $B$ iff for every finite subsequence $\bar{a}'$ of $\bar{a}$ and finite subset $A'$ of $A$, $\bar{a}'/(A' \cup B)$ does not widely divide over $B$.
2A) The type $p$ does not weakly divide over $(r, B)$ iff every finite $p' \subseteq p$ $dnwd$ over $(r, B)$.
2B) The type $p$ does not divide over $A$ iff for every finite $p' \subseteq p$ does not divide over $A$, iff some finite conjunction $\varphi$ of members of $p$ satisfies the requirement in Definition 1.3(1).
2C) The type $p$ does not fork over $A$ iff every finite $p' \subseteq p$ does not fork over $A$.
3) [More monotonicity] Assume $Rang(\bar{a}') = Rang(\bar{a}'')$, then $\bar{a}' / A$ $dnwd$ over $B$ iff $\bar{a}'' / A$ $dnwd$ over $B$.
3A) If $B \subseteq A$, $Rang \bar{a}'' \subseteq acl(B \cup \bar{a}')$ and $\bar{a}' \setminus A$ $dnwd$ over $B$ then $\bar{a}'' / A$ $dnwd$ over $B$.
3B) Similarly to 3), 3A) for “does not divide” and for “does not fork” and for “$dnwd$ over $(r, B)$”.
4) [Transitivity] If $A_0 \subseteq A_1 \subseteq A_2$ and $\bar{a}/A_{\ell+1}$ $dnwd$ over $A_\ell$ for $\ell = 0, 1$ then $\bar{a}/A_2$ $dnwd$ over $A_0$.
5) [Extendability] If $B \subseteq A \subseteq A^+$, $p$ an $m$-type over $A$ and $p$ does not fork over $B$ then $p$ has an extension $q \in S^m(A^+)$ which does not fork over $B$ (clear or see [Sh:93],2.11(3)).
5A) If $p$ is finitely satisfiable in $A$ and $(Dom p) \cup A \subseteq B$ then we can extend $p$ to a complete type over $B$ finitely satisfiable in $A$.
6) [Trivial nice behaviour] $\bar{a}/A$ does not fork over $A$ (by [Sh:93],2.11(2)).
6A) For a set $A$ and an $m$-type $p$ we have: $p$ does not widely divide over $(p, A)$ (check).
6B) Every $m$-type over $M$ is finitely satisfiable in $M$.
7) [Continuity] If $p_1$ does not widely divide over $(r_i, B_i)$ for $i < \delta$ and $i < j < \delta \Rightarrow p_i \subseteq p_j$ and $r_i \subseteq r_j$ and $B_i \subseteq B_j$, then $\bigcup_{i<\delta} p_i$ does not widely divide
over \((\bigcup r_i \cup B_i)\).

7A) If \(\langle A_i : i < \delta \rangle\) is increasing, \(\langle B_i : i < \delta \rangle\) is increasing and \(C/B_i\) is finite satisfiable in \(A_i\) for each \(i < \delta\) then \(C/\bigcup_{i<\delta} B_i\) is finite satisfiable in \(\bigcup_{i<\delta} A_i\).

[Why? if \(p \subseteq C/\bigcup_{i<\delta} B_i\) is finite then for some \(j\) it is over \(B_j\) hence \(\subseteq C/B_j\) is satisfiable in \(A_j\) hence is satisfiable in \(\bigcup_{i<\delta} A_i\).]

8) [Degree] Let \(x_m = \langle x_i : \ell < m \rangle, E_m\) be an ultrafilter on \(\Omega_m =: (\langle \Delta, k \rangle : \Delta = \Delta(x_m) \subseteq L(T), k < \omega\) such that for every \((\Delta_0, k_0) \in \Omega_m\) the following set belongs to \(E_m\):

\[
\{ (\Delta, k) \in \Omega_m : \Delta \subseteq \Delta_0 \text{ and } k_0 < k \}.
\]

If \(p(\bar{x})\) is a type over \(A, \ell g(\bar{x}) = m\), then for some complete type \(q(\bar{x})\) over \(A\) extending \(p\) for the \(E_m\)-majority of \((\Delta(x), k)\) we have \(D(q(\bar{x}), \Delta, \aleph_0, k) = D(p(\bar{x}), \Delta, \aleph_0, k)\) (by [Sh:93],2.2(5),p.182; of course, we can use infinite \(\bar{x}\)).

In such a case we say: \(q(\bar{x})\) is an \(E_m\)-nonforking extension of \(p(\bar{x})\) or \(q(\bar{x})\) \(E_m\)-does not fork over \(p(\bar{x})\). If \(p(\bar{x}) \in S^m(A)\) (so \(p = q \upharpoonright A\) we may replace “over \(p(\bar{x})\)” by “over \(A\”).

9) [Additivity] If for every \(\alpha < \alpha^*\) the type \(tp(\bar{b}^\alpha, \bar{a} \cup A \cup \bigcup_{\beta < \alpha} \bar{b}^\beta)\) does not divide over \(A \cup \bigcup_{\beta < \alpha} \bar{b}^\beta\) then \(tp(\bigcup_{\beta < \alpha^*} \bar{b}^\beta, \bar{a} \cup A)\) does not divide over \(A\) (by [Sh:93],1.5,p.181).

10) [Finitely satisfiable is average] Let \(\ell g(\bar{x}) = m\) and \(p = p(\bar{x})\) a type. Then \(p\) is finitely satisfiable in \(I\) iff for some ultrafilter \(D\) over \(I\) we have \(p \subseteq Av(D, \text{Dom } p)\).

11) If \(D\) is an ultrafilter on \(I\), then \(Av(D, A)\) belongs to \(S^m(A)\) and is a finitely satisfiable in \(I\).

1.6 Claim. 1) [small basis] If \(p \in S^c(A)\) and \(B_0 \subseteq A\) then for some \(B\) we have:

\[
\begin{align*}
(\alpha) & \quad B_0 \subseteq B \subseteq A \\
(\beta) & \quad |B| \leq |\bar{c}| + |T| + |B_0| \\
(\gamma) & \quad p \text{ does not wind downward over } (p \upharpoonright B, B).
\end{align*}
\]

2) If \(\bar{a}/(A \cup \bar{b})\) dnd over \(A\) and \(A \subseteq A^+ \subseteq ac(\bar{a}) = \bar{a}'\) realizes \(\bar{a}/A\) then there is \(\bar{b}'\) (of the same length as \(\bar{b}\)) such that:

\[
\begin{align*}
(\alpha) & \quad \bar{a}' \subseteq A^+ \text{ and } \bar{a}'/A = \bar{a}/A \quad \text{then } \bar{b}'/A = \bar{b}' \bar{a}/A.
\end{align*}
\]

3) [weak symmetry] If \(\bar{a}/(A \cup \bar{b})\) dnd over \(A\) and then \(\bar{b}/(A \cup \bar{a})\) dnd over \(A\).

4) Assume \(A \subseteq B \cap C\) (all \(\subseteq \bar{c}\)) and \(C/B\) is finitely satisfiable in \(A\) hence \(A = B \cap C\). Then \(B/C\) dnd over \((B/A, A)\).

Proof. 1) By [Sh:93],3.3,p.186.

2) By [Sh:93],2.13,p.185 we can get clause (a).

3) By [Sh:93],2.14,p.185 it dnd.

4) Straightforward (e.g. use 1.5(10)).
1.7 Theorem. If $M < N < C, \|M\| = \mu, \|N\| = \mu^+, |T| < \kappa, \mu = \mu^{<\kappa}$, then there are $M^+ < N^+$ such that $N < N^+, M < M^+, \|M^+\| = \mu, \|N^+\| = \mu^+$ and:

\begin{enumerate}[(*)]
\item if $B \subseteq A \subseteq N, B \subseteq M, |A| < \kappa, C \subseteq C, |C| < \kappa$ and $A/(B \cup C)$ dwd over $B$ then there is $C' \subseteq M^+$ realizing $C/(B \cup A)$.  
\end{enumerate}

Proof. Clearly we can prove $(*)_1, (*)_2$ separately. Now $(*)_2$ is immediate from 1.6(2). As for $(*)_1$, this is proved in [Sh:93], §4 (read [Sh:93], 4.13, 4.14, 4.15 p. 193 there, so we use [Sh:92], Theorem 3.1 which says that a Boolean algebra of cardinality $\lambda^+$ satisfying the $\kappa$-c.c., $\lambda^+ < \kappa = \lambda$ is $\lambda^+$-centered, i.e. the union of $\leq \lambda$ ultrafilters, so if $\kappa > 2^{|T|}$ we are done which is enough for our main theorem (1.2 when $\lambda > |T|$). Actually repeating the proof of [Sh:93], Theorem 3.1 in the circumstances of [Sh:93], §4 show that $\kappa > |T|$ is enough).  \(\square_{1.7}\)

1.8 Definition. 1) $K^0_{\lambda}$ be

$$\left\{ \bar{M} : \bar{M} = \langle M_i : i < \lambda^+ \rangle \text{ is } \prec \text{-increasing continuous, each } M_i \text{ a model of } T \right\}.$$  

2) $\leq^0$ is the following partial order on $K^0_{\lambda}$: $\bar{M}^1 \leq^0 \bar{M}^2$ if for $i < j < \lambda^+$ we have $M_i^1 \prec M_i^2$.

1.9 Observation. 1) If $(\bar{M}^\alpha : \alpha < \delta)$ is an $\leq^0$-increasing chain (in $K^0_{\lambda}$) and $\delta < \lambda^+$ then it has a lub $\bar{M} : M_i = \bigcup_{\alpha < \delta} M^\alpha_i$.  
2) If $M$ is a model of $T$ of cardinality $\lambda^+$, then for some $\bar{M} \in K^0_{\lambda}, M = \bigcup_{i < \lambda^+} M_i$.  
3) If $\bar{M}, \bar{N} \in K^0_{\lambda}$ and $\bigcup_{\alpha < \lambda^+} M_\alpha \prec \bigcup_{\alpha < \lambda^+} N_\alpha$ then for some club $E$ of $\lambda^+$ for every $\alpha \in E : M_\alpha \prec N_\alpha$ and $N_\alpha/ \bigcup_{\beta < \lambda^+} M_\beta$ dwd over $M_\beta$.

Proof. 1) Immediate.  
2) Use 1.5(2) + 1.6(1).  \(\square_{1.9}\)

Using 1.7 and 1.5(5A) $\lambda^+ \times \lambda$ we get

1.10 Observation. Assume $\lambda = \lambda^{<\kappa}$. For every $\bar{M} \in K^0_{\lambda}$ there is an $\leq^0$-increasing continuous sequence $\langle N_\zeta : \zeta \leq \lambda \rangle$, in $K^0_{\lambda}$, (so $\bar{N}_0 = \langle N_\zeta : \alpha < \lambda^+ \rangle, \bar{N}_0 = \bar{M}$.
such that (letting \( N_\kappa = \bigcup_{\alpha < \lambda^+} N_\kappa^\alpha \)) and (fixing \( \chi \), letting \( \overline{a}_{\zeta,\alpha} \) an enumeration of \( |N_\kappa^\alpha| \) of length \( \lambda \)) we can add: every type definable in \((H(\chi), \in, \mathcal{E}_\chi)\) from \((\bar{N}_\kappa : \kappa \in \zeta), (N_\kappa^{\alpha+1} : \alpha < \beta), (\overline{a}_{\zeta,\alpha} : \alpha < \lambda^+) \) and \((\alpha_{\zeta,\alpha,j} : j < i) \) and finitely many ordinals \( < \lambda \) is realized in \( N_{\beta+1}^\xi \), hence:

\((*)_1\) if \( \alpha < \lambda^+, \zeta \leq \lambda, \text{ cf}(\zeta) \in \{\lambda, 1\}, \text{ cf}(\alpha) \in \{\lambda, 1\}, \)
\( B \subseteq A \subseteq N_{\zeta-1}, B \subseteq N_\kappa^{-1}, |A| < \kappa, p \in S^m(A) \) and \( p \) dnwd over \((p \upharpoonright B, \emptyset)\) then \( p \) is realized in \( N_\kappa^\xi \).
(Note: \( \lambda - 1 = \lambda \)).

\((*)_2\) if \( \alpha \leq \beta < \gamma < \lambda^+, \gamma \) is non-limit, \( B \subseteq A \subseteq A^+ \subseteq M_\gamma, |A^+| < \kappa, a \in M_\gamma, \text{ tp}(a, A \cup M_\alpha) \) dnwd over \( B \) then either for some \( a' \in N_\gamma \) we have:
\( \text{ tp}(a, A^+ \cup M_\alpha) = \text{ tp}(a', A \cup M_\alpha) \) and \( \text{ tp}(a', A \cup N_\beta) \) dnwd over \( B \) or there is no such \( a' \in \mathcal{E} \).

\((*)_3\) for \( \gamma \) non-limit, \( \zeta \leq \lambda, \text{ cf}(\zeta) \in \{1, \lambda\} \) we have:
\( N_\kappa^\xi \) is \( \kappa \)-saturated (so when \( \kappa = \lambda \) it is saturated).

1.11 Definition. Let \( A, B, C \) be given \((\subseteq \mathcal{E})\).

0) \( A \leq_B^0 C \) means that for every \( \bar{b} \subseteq B, \bar{b}/(A \cup C) \) dnwd over \((\frac{1}{A}, A)\).

1) \( A \leq_B^1 C \) means that for every \( \bar{c} \subseteq C, \bar{c}/(A \cup B) \) dnwd over \((\frac{1}{A}, \emptyset)\).

2) \( A \leq_B^2 C \) means there is an increasing continuous sequence \( \langle A - \alpha : \alpha < \beta \rangle \) such that:
\( A = A_0, A \cup C = A_\beta \) and
\[ \begin{align*}
\alpha \text{ an even ordinal} & \Rightarrow A_\alpha \leq_B^0 A_{\alpha+1} \\
\alpha \text{ an odd ordinal} & \Rightarrow A_\alpha \leq_B^{-1} A_{\alpha+1}.
\end{align*} \]

3) \( A \leq_B^3 C \) means that for some \( C', C \subseteq C' \) and \( A \leq_B^3 C' \).

4) \( A \leq_B^4 C \) means that for some increasing continuous sequence \( \langle A_\alpha : \alpha < \beta \rangle \) we have:
\( A = A_0, A \cup C = A_\beta \) and \( A_\alpha \leq_B^2 A_{\alpha+1} \).

1.12 Claim. 0) \( A \leq_B^e A \) for \( e = -1, 0, 1, 2, 3 \).

1) \( A \leq_B^e C \) iff \( A \leq_{A \cup B}^e A \cup C \) for \( e = -1, 0, 1, 2, 3 \).

2) If \( A \not\subseteq B \subseteq B \cup A \) and \( A \leq_B^e C \) then \( A \leq_{B_1}^e C \) for \( e = -1, 0, 1, 2, 3 \).

3) For \( e = 1, 3 \) we have: \( \leq_B^e \) is a partial order.

4) If \( e = 1, 3 \) and \( \langle A_\alpha : \alpha < \beta \rangle \) is increasing continuous and \( A_\alpha \leq_B^e A_{\alpha+1} \) for \( \alpha < \beta \) then \( A_0 \leq_B^e A_\beta \).

5) For \( e^1, e^2 \in \{(-1, 1), (0, 1), (1, 2), (2, 3)\} \), we have: \( A \leq_B^{e^1} C \) implies \( A \leq_B^{e^2} C \).

6) If for every \( \bar{b} \subseteq B, \bar{b}/(A \cup C) \) is finitely satisfiable in \( A \) then \( A \leq_B^e C \).

[Why? By 1.6(4) and Definition 1.11(1)].
7) If $A \lesssim_B C$ and $C' \subseteq C$ then $A \lesssim_B C'$.
[Why? Read Definition 1.11(2)].
8) $A \lesssim_B C$ iff $A \lesssim_{C'} B$.
[Why? Read the definitions].

1.13 Claim. Let $M \prec N$ and $M \subseteq A$. Then the following are equivalent:

(a) $M \lesssim^3_N A$
(b) there are $M_0 \prec M_1 \prec M_2$ such that:
(i) $M = M_0$
(ii) the type $tp_*(N, M_1)$ is finitely satisfiable in $M_0$ and the type $tp_*(M_2, M_1 \cup N)$ is finitely satisfiable in $M_1$
(iii) for some elementary map $f, f(A) \subseteq M_2$ and $f \upharpoonright N = \text{id}$
(c) like (b) with $\|M_2\| \leq |T| + |A|$ (d) $M \lesssim^3_N A$.

13A Remark. 1) Clause (ii) of (b) implies $M_0 \lesssim^0 N \lesssim^{-1}_N M_2$.
2) An equivalent formulation of (b) is
   
   (b)' for some $M_0, M_1, M_2, f$ we have $M = M_0 \leq f(N)$, $M_1 \leq f(N)$, $M_2$, $f \upharpoonright M_0 = \text{id}_{M_0}, f(A) \subseteq M_2$.

3) Another formulation is
   
   (b)'' like (b)' but $f = \text{id}_{A \cup N}$.

Proof. (c) ⇒ (b) Trivial.
(b) ⇒ (c).
By the Lowenkeim Skolem argument.

(b) ⇒ (d)

By 1.12(6) clearly $M_0 \lesssim^0_N M_1$ and similarly $M_1 \lesssim^0_N M_2$, hence by 1.2(8) we have $M_1 \lesssim^{-1}_N M_2$. Hence by 1.12(5), $M_e \lesssim^e_N M_{e+1}$ (for $e = 0, 1$), so by 1.12(3) $M_0 \lesssim^1_N M_2$ hence by Definition 1.11(3) (and clause (iii) of 1.13(b)), $M = M_0 \lesssim^3_N A$ as required.

(d) ⇒ (a) Trivial (by 1.12(5)).

So the only (and main) part left is:

(a) ⇒ (b) We know $M \lesssim^3_N A$, by 1.12(1) without loss of generality $M \subseteq A$, hence there is an increasing continuous sequence $\langle A_\varepsilon : \varepsilon \leq \zeta \rangle$ such that: $A_0 = M, A_\zeta = A$ and $A_\varepsilon \lesssim^3_N A_{\varepsilon+1}$. By the Definition of $\lesssim^3_N$ there is an increasing continuous sequence $\langle B_\varepsilon, i : i \leq i_\varepsilon \rangle$ such that $B_{\varepsilon,0} = A_\varepsilon, A_{\varepsilon+1} \subseteq B_{\varepsilon,i_\varepsilon}$ and $B_{\varepsilon,i} \leq N$ (where for $i < i_\varepsilon$ we have $\ell(\varepsilon) \in \{-1, 0\}$ and $\varepsilon = \ell(\varepsilon) \mod 2$). Let $\theta = 2^{|T| + |N| + \sum_{\varepsilon < \zeta} (|i_\varepsilon| + |B_{\varepsilon,i_\varepsilon}|)}^+$ and choose regular $\mu = \mu^\theta$. 

We choose by induction on $\alpha < \mu^+, M, N, \alpha$ such that: medskip
(i) $M_\alpha < C$ is increasing continuous
(ii) $\|M_\alpha\| = \mu, M \subseteq M_0$.
(iii) $f_\alpha$ is an elementary mapping, Dom($f_\alpha$) = $N$, Rang($f_\alpha$) = $N_\alpha$ and $f_\alpha \upharpoonright M = \text{id}_M$
(iv) $\text{tp}_\alpha(N_\alpha, M_\alpha)$ is finitely satisfiable in $M$
(v) $N_\alpha \subseteq M_{\alpha+1}$
(vi) $M_{\alpha+1}$ is $\theta^+$-saturated.

There is no problem to carry the definition. (First choose $M_\alpha$: if $\alpha = 0$ to satisfy (i) + (ii), if $\alpha$ is a limit ordinal, as $\bigcup \beta < \alpha M_\beta$, and if $\alpha = \beta + 1$ to satisfy (i) + (ii) + (v).

Second choose $f_\alpha, N_\alpha$ satisfying (iii) + (iv) which exists by 1.5(10) + (11)).

By using 1.7, $\lambda^+$ times we can find $M^+ = (M^+_\alpha : \alpha < \lambda^+)$ such that:

(A) $M^+$ is an increasing continuous sequence of elementary submodels of $C$
(B) $\|M^+_\alpha\| \leq \mu, M_\alpha < M^+_\alpha$
(C), if $\alpha < \beta < \mu^+, B_1 \subseteq M_\alpha$ and $\text{cf}(\alpha) = \mu, B_1 \subseteq B_2 \subseteq M_\beta, |B_2| < \theta, C \subseteq C$ and $C/B_2$ dnwd over $(C/B_2, \emptyset)$ (equivalently, for every finite $\bar{c} \subseteq C, \bar{c}/B_2$ dnwd over $\bar{c}/B_1, \emptyset$) then $C/B_2$ is realized in $M_\alpha$
(C)2 similarly, but we replace the dnwd assumption by “$B_2/(C \cup C)$ dnwd over $(B_2/B_1, B_1)$”.

[Note: we use (\*)1 from 1.7 for (C)1 and (\*)2 from 1.7 for (C)2].

Now let $M = \bigcup_{\alpha < \mu^+} M_\alpha, M^+ = \bigcup_{\alpha < \mu^+} M^+_\alpha$; and let $E = \{\delta < \mu^+ : \delta$ a limit ordinal and $(M^+_\delta, M_\delta) \prec (M^+, M)\}$. Clearly $E$ is a club of $\mu^+$ and

(\*) $\delta \in E \Rightarrow \text{tp}(M^+_\delta, M)$ is finitely satisfiable in $M_\delta$.

Choose $\delta \in E$ of cofinality $\mu$. Now we choose $g_\varepsilon$ by induction on $\varepsilon \leq \zeta$ such that:

(a) $g_\varepsilon$ an elementary mapping
(\beta) Dom($g_\varepsilon$) = $N \cup A_\varepsilon$
(\gamma) $g_\varepsilon$ is increasing continuous in $\varepsilon$
(\delta) $g_\varepsilon \upharpoonright N = f_\delta$
(\varepsilon) Rang($g_\varepsilon \upharpoonright A_\varepsilon$) $\subseteq M^+_\delta$.

If we succeed, then we get the desired conclusion (i.e. prove clause (b)).

[Why? First note that in clause (b) we can omit $f \upharpoonright N = \text{id}$ by $f \upharpoonright M = \text{the identity}$ if in clause (ii) we use $f(N)$; we call this (b)', Now (b)' holds with $M, M_\delta, M^+_\delta, g_\zeta$ here standing to $M_0, M_1, M_2, f$ there). So it is enough to carry the induction on $\varepsilon$. For $\varepsilon = 0$ let $g_\varepsilon = f_\delta$, and for $\varepsilon$ a limit ordinal let $g_\varepsilon = \bigcup_{\zeta < \varepsilon} g_\zeta$; lastly for $\varepsilon$ a successor ordinal say $\varepsilon = \xi + 1$, we choose $g_{\varepsilon,i}$ by induction on $i \leq \iota_\varepsilon$ such that:

(a)' $g_{\varepsilon,i}$ an elementary mapping
(\beta)' Dom($g_{\varepsilon,i}$) = $N \cup B_{\varepsilon,i}$
(\gamma)' $g_{\varepsilon,i}$ is increasing continuous in $i$
(\delta)' $g_{\varepsilon,0} = g_\varepsilon$
(\varepsilon)' Rang($g_{\varepsilon,i}$) $\subseteq M^+_\delta$.
If we succeed then \( g_{ε,i} \upharpoonright A_{ε+1} \) is as required. So it is enough to carry the induction on \( i \). For \( i = 0 \) let \( g_{ε,i} = g_{ε} \), for \( i \) limit let \( g_{ε,i} = \bigcup_{j<i} g_{ε,j} \) and for \( i \) a successor ordinal say \( j + 1 \), use clause \((C)_{1}\) in the choice of \( M^+_{α} \) if \( j \) even, remembering Definition 1.11(1) and use clause \((C)_{2}\) in the choice of \( M^+_{α} \) if \( j \) is odd remembering Definition 1.11(0).

\[ \square \text{1.13} \]

**1.13A Claim.** If \( M < N, M \leq N \) for \( \ell = 1, 2 \) then there are \( M^+, f_1, f_2 \) such that: \( M \prec M^+, M \leq N \) and for \( \ell = 1, 2 f_\ell \) is an elementary mapping, \( \text{Dom}(f_\ell) = N \cup A_\ell, f_\ell \upharpoonright N = \text{id}_N, f_\ell(A_\ell) \subseteq M^+ \).

**Proof.** Same proof as 1.13 (just shorter).

**1.14 Definition.** 1) Let \( K_{0\ell}^pr = \{(M,N) : M \prec N \prec \mathcal{C}\} \) and \((M_1,N_1) \leq^* (M_2,N_2) \) if \((M_ε,N_ε) \in K_{0\ell}^pr\) for \( e = 1, 2 \) and \( M_1 \prec M_2, N_1 \prec N_2 \) and \( M_1 \leq N_2 \) if \( M_2 \) (equivalently, \( M_1 \leq N_2 \) by (1.13)).

2) We define \((M_1,N_1) \leq_{fs} (M_2,N_2)\) similarly replacing \( M_1 \leq N_2 \) by \( N_1/M_2 \) is finitely satisfiable in \( M_1 \).

**1.15 Claim.** 1) \( \leq^* \) is a partial order on \( K_0^pr \).

2) If \((M_α,N_α) : α < β \) is increasing continuous and \((M_α,N_α) \leq^* (M_{α+1},N_{α+1}) \) for \( α < β \) then \((M_0,N_0) \leq^* (M_β,N_β) \).

3) If \( M \prec N \) and \( M \leq N \) for some \((M_1,N_1)\) we have: \( A \subseteq M_2 \) and \((M,N) \leq^* (M_1,N_1) \in K_0^pr \).

**Proof.** 1) If \((M_0,N_0) \leq^* (M_1,N_1) \leq^* (M_2,N_2)\) then

- \((i) \) \( M_0 \subseteq M_1 \subseteq M_2 \) and \( N_0 \subseteq N_1 \subseteq N_2 \)
- \((ii) \) \( M_0 \leq N_0, M_1 \) and \( M_2 \)
- \((iii) \) \( M_1 \leq N_1, M_2 \)
- \((iv) \) \( M_2 \subseteq N_2 \)

By 1.12(2) + (1) and clause \((iii)\) above

- \((v) \) \( M_1 \leq N_2, M_2 \)

by 1.12(5) we have (by \((ii)\) and \((v)\) respectively)

- \((ii)' \) \( M_0 \leq N_0, M_1 \)
- \((v)' \) \( M_1 \leq N_2, M_2 \)

hence by 1.13

- \((vii) \) \( M_0 \leq N_2, M_2 \)

hence \((M_0,N_0) \leq^* (M_2,N_2)\) holds by \((i)\), \((iv)\) and \((vii)\).

2) Similarly using 1.12(4) + (1.13).

3) Use 1.13 (see 1.13a(3)) so there are \( M_0 \prec M_1 \prec M_2 \) such that \( M = M_0, N/M_1 \) is in \( M_0, M_2/(M_1 \cup N) \) is in \( M_1 \) and \( A \subseteq M_2 \). So by 1.1 x \( M_0 \leq M_1, M_1 \leq N_1 \) \( M_2 \) hence (see 1.12(4)) \( M_0 \leq^* M_1 \) \( M_2 \) hence for any \( N^*, M_2 \cup N \subseteq N^* < \mathcal{C} \) we have \((M,n) \leq^* (M_2,N^*) \in K_0^pr \).
1.16 Definition. \( K^{pr}_2 = \{(M, N) : \text{the pair } (M, N) \in K^{pr}_0 \text{ and if } (M, N) \leq^* (M', N') \in K^{pr}_0 \text{ then } M'/N \text{ is fs in } M\} \).

1.17 Claim. If \((M, N) \in K^{pr}_0\) then for some \((M', N')\) we have:

(a) \((M, N) \leq (M', N') \in K^{pr}_0\)

(b) \(|N'| \leq |N| + |T|\)

(c) \((M', N') \in K^{pr}_1\) i.e.

if \((M', N') \leq^* (M'', N'')\) then \(M''/N'\) is fs in \(M'.\)

Proof. Let \(\mu = |N| + |T|\), assume the conclusion fails. We now choose by induction on \(\alpha < \mu^+\), \((M_\alpha, N_\alpha)\) such that:

(i) \((M_0, N_0) = (M, N)\)

(ii) \((M_\alpha, N_\alpha) \in K^{pr}_0, \|N_\alpha\| \leq \mu\)

(iii) \(\beta < \alpha \Rightarrow (M_\beta, N_\beta) \leq^* (M_\alpha, N_\alpha)\)

(iv) for limit \(\delta\) we have \((M_\delta, N_\delta) = \left( \bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right)\)

(v) \(M_{\alpha+1}/N_\alpha\) is not fs in \(M_\alpha\).

For \(\alpha = 0\) see (i) for \(\alpha\) limit see (iv) and 1.15(2) if \(\alpha = \beta + 1\) find \((M_\beta, N_\beta)\) satisfying \((M_\beta, N_\beta) \leq^* (M_\alpha, N_\alpha)\) and satisfying (v). By Lowenheim Skolem argument without loss of generality \(\|N_\alpha\| \leq \mu\) and by 1.15(1) also clause (iii) holds. For a club of \(\delta < \mu^+\) we get contradiction to clause (v). \(\square_{1.17}\)

1.18 Fact. 1) If \((M, N) \in K^{pr}_1\) and \((M', N') \in K^{pr}_0\) and \((M, N) \leq^* (M', N')\) then \((M, M') \leq^* (N, N')\).

2) If \((M, N) \in K^{pr}_1\) and \(M \leq^2 N\) then \(A/N\) is fs in \(M\).

Proof. 1) By 1.17 we know \(M'/N\) is fs in \(M\) hence by 1.13, \((b) \Rightarrow (d)\) we know \(M \leq^2_M N\) which give the desired conclusion.

2) By 1.13A.

1.19 Claim. 1) If \((M_\alpha, N_\alpha) \in K^{pr}_1\) for \(\alpha < \delta\) and \(\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle\) is \(<^*\)-increasing then for \(\alpha < \delta\)

\[ (M_\alpha, N_\alpha) \leq^* \left( \bigcup_{i \leq \alpha} M_i, \bigcup_{i \leq \alpha} N_i \right) \in K^{pr}_1. \]

2) If \((M_\alpha, N_\alpha) \in K^{pr}_1\) and \(\langle (M_\alpha, N_\alpha) : \alpha \leq \delta \rangle\) is \(<^*\)-increasing then

\[ \left( \bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right) \in K^{pr}_1 \text{ and } \left( \bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right) \leq^* (M_\delta, N_\delta). \]

Proof. We prove both together by induction on \(\delta\).

0) By the induction hypothesis without loss of generality \(\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle\) is
increasing continuous.

1) Clearly \((M_\alpha, N_\alpha) \leq^* \left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \in K^0_\delta\) (see 1.17(2)). Suppose \(\bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \leq^* (M, N)\). So by 1.17(1), for \(\alpha < \delta, (M_\alpha, N_\alpha) \leq^* (M, N)\), but \((M_\alpha, N_\alpha) \in K^0_\delta\) hence \(M/N_\alpha\) is fs in \(M_\alpha\). But this implies \(M/\bigcup_{\alpha<\delta} N_\alpha\) is fs in \(\bigcup_{\alpha<\delta} M_\alpha\) by 1.5(7A).

2) As we are proving by induction on \(\delta\); without loss of generality \(((M_\alpha, N_\alpha) : \alpha < \delta)\) is \(\leq^*\)-increasing continuous, so by part (1), \((M_\alpha, N_\alpha) \leq^* \left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \in K^0_\delta\) for \(\alpha < \delta\). Now for \(\alpha < \delta, (M_\alpha, N_\alpha) \leq^* (M_\delta, N_\delta)\) and \((M_\alpha, N_\alpha) \in K^0_\delta\) clearly \(M_\delta/N_\alpha\) is fs in \(M_\alpha\) hence by 1.5(7A), \(M_\delta/\bigcup_{\alpha<\delta} N_i\) is fs in \(\bigcup_{\alpha<\delta} M_i\), hence by 1.13

\[
\bigcup_{i<\delta} M_i \leq^3 \bigcup_{\alpha<\delta} N_i \bigcup_{i<\delta} N_i \text{ hence } \left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \leq^* (M_{\delta+1}, N_{\delta+1}). \quad \square_{1.19}
\]

Now we want to apply 1.1. Toward this (for \(\lambda\) as there) we define:

**1.20 Definition.** 1) \(K^0_{ap} = K^0_{ap} = K^0_{ap}[T] = K^0_{ap}[T, \lambda]\) is the set of models \(M\) of \(T\) with universe \(\subseteq \lambda^+\) and cardinality \(< \lambda\) such that: \(M \cap \lambda \neq \emptyset\) and \(0 < \alpha < \lambda^+\) implies \(M \upharpoonright (\lambda \times \alpha) \prec M\). For such \(M\) let \(\text{Dom}(M) = \{\alpha < \lambda^+ : [\lambda \times \alpha, \lambda \times \alpha + \lambda) \cap M \neq \emptyset\}\). We now define \(K^0_{ap}\) by: \(M \leq K^0_{ap}\) \(N\) if (both are in \(K^0_{ap}\) and \(M \prec N\) and): for every \(\alpha \in (0, \lambda^+)\), \(M \upharpoonright (\lambda \times \alpha) \leq^2 M/I_{(\lambda \times \alpha + \lambda)} \upharpoonright (\lambda \times \alpha)\).

1.21 Observation. So \(M \leq K^0_{ap}\) \(N\) if both are in \(K^0_{ap}\), \(M < N\) and for \(\alpha \in (0, \lambda^+)\) we have \((M \upharpoonright (\lambda \times \alpha), M \upharpoonright (\lambda \times \alpha + \lambda)) \leq^* (N \upharpoonright (\lambda \times \alpha), N \upharpoonright (\lambda \times \alpha + \lambda))\).

**1.22 Claim.** 1) \(\leq K^0_{ap}\) is a partial order on \(K^0_{ap}\).
[Why? By 1.15(1)].

2) If \(\{M_i : i < \delta\}\) is \(\leq K^0_{ap}\)-increasing, \(\sum_{i<\delta} \|M_i\| < \lambda\) then \(M_i \leq K^0_{ap} \bigcup_{j<\delta} M_j \in K^0_{ap}\).
[Why? By 1.15(2)].

**1.23 Claim.** Let \(\bar{N}^\zeta, N_\zeta\) (for \(\zeta \leq \lambda\)) as be as in 1.10. Let \(E \subseteq \lambda^+\) be a thin enough club of \(\lambda^+, \{\varepsilon(\alpha) : \alpha < \lambda^+\}\) enumerate \(\{0\} \cup E, H\) a 1-to-1 map from \(N_\lambda\) onto \(\lambda^+\) mapping \(N^\zeta_{\varepsilon(\alpha)}\) onto \(\lambda \times \alpha\). Let \(N^*_\alpha = H(N^\zeta_{\varepsilon(\alpha)}), N^* = \bigcup_{\alpha<\lambda} N^*_\alpha\).

1) If \(M \in K^0_{ap}\) then there is a lawful \(f\) (see [Sh 457],4.4) which is an elementary embedding of \(M\) into \(N^*\) such that for \(\alpha \in \text{Dom}(M), f(M \upharpoonright (\lambda \times \alpha)) < f(M \upharpoonright (\lambda \times \alpha + \lambda))\).

Proof. Straightforward.
[Saharon: put old proof of 1.27 from AP here?]
But we want more, not only universality but also homogeneity.

1.24 Definition. $K_{ap}^1 = K_{ap}^1[T, \lambda]$ is the set of $M \in K_{ap}^0$ such that for every $\alpha \in (0, \lambda^+]$, if $\neg(M \subseteq \lambda \times \alpha)$ then $(M \upharpoonright (\lambda \times \alpha), M) \in K_{ap}^{pp}$.

Let $\leq_{K_{ap}^1}$ be $\leq_{K_{ap}^0} \upharpoonright K_{ap}^1$.

1.25 Claim. 1) $\leq_{K_{ap}^1}$ is a partial order on $K_{ap}^1$.
[Why? By 1.22 and Definition 1.2].

2) If $\langle M_i : i < \delta \rangle$ is $\leq_{K_{ap}^1}$-increasing, $\sum_{1<\delta} \|M_i\| < \lambda$ then $M_i \leq_{K_{ap}^1} \bigcup_{j<\delta} M_j \in K_{ap}^1$.
[Why? By 1.23(1) and 1.19(1)].

1.26 Claim. Let $\check{N}_\zeta, N_\zeta$ (for $\zeta \leq \lambda$) be as in 1.10. Let $E \subseteq \lambda^+$ be a thin enough club of $\lambda^+, \{\varepsilon(\alpha) : \alpha < \lambda^+\}$ enumerate $\{0\} \cup E$, $H$ a 1-to-1 map from $N_\lambda$ onto $\lambda^+$ mapping $N_{\varepsilon(\alpha)}^\lambda$ onto $\lambda \times \alpha$. Let $N_\alpha^* = H(N_{\varepsilon(\alpha)}^\lambda), N^* = \bigcup_{\alpha<\lambda} N_\alpha^*$.

1) If $M \in K_{ap}^1$ then there is a lawful $f$ (see [Sh 457], 4.1) which is an elementary embedding of $M$ into $N^*$ such that for $\alpha \in \text{Dom}(M), f(M \upharpoonright (\lambda \times \alpha)) < f(M \upharpoonright (\lambda \times \alpha + \lambda))) N^* \upharpoonright (\lambda \times \alpha)$.

2) If $M_0 \leq_{K_{ap}^1} M_1$ and $(M_0, f_0)$ is as in part (1) then we can find $f_1, f_0 \subseteq f_1$ such that $(M_1, f_1)$ is as in part (1). Moreover, if $f_0 \cup (f_1 \upharpoonright (M_1 \upharpoonright (M_1 \upharpoonright (\lambda \times \alpha))))$ has been determined we can continue.

1.27 Amalgamation Claim. Assume $M_0 \leq_{K_{ap}^0} M_\ell$ for $\ell = 1, 2$ and (for simplicity) $|M_1| \cap |M_2| = |M_0|$. Then there is $M \in K_{ap}^0$ such that $M_1 \leq_{K_{ap}^0} M$ and $M_2 \leq_{K_{ap}^0} M$.

Proof. Follows from 1.26(1) + (2) (Q: domain?)

1.28 Claim. $(K_{ap}, \leq^*)$ is a $\lambda$-system (see [Sh 457], §4).

Proof. Check.

1.29 Claim. $(K_{ap}, \leq^*)$ is simple (see [Sh 457], §4).

Proof. Included in the proof of amalgamation (see last clause of 1.26(2)).

1.30 Claim. If $M$ is a model of $T$ of cardinality $\lambda^+$ then for some $\Gamma \in K_{ap}^{md}, M$ can be elementarily embedded into $M_\Gamma$.

Proof. Use 1.10 with $M = \bigcup_{\alpha<\lambda} M_\alpha$, so we get $N^*, N_{\alpha}^*(\alpha < \lambda)$ as in 1.18. Check.

1.31 Proof of 1.2. Use the above claims.
§2 on the Strong Order Properties and Finitary Versions

2.0 Discussion. By [Sh 457], for some non-simple (first order complete) the answer to the following is yes:

\[ \bigoplus_T \text{ if } \lambda = \lambda^\kappa > |T|, 2^\lambda = \lambda^+, \text{ is there a } (\lambda\text{-complete),} \]
\[ \lambda^+\text{-c.c. forcing notion } Q, \models Q \text{ "univ}(\lambda^+, T) \leq \lambda^{++} < 2^{\lambda^+}? \]
\[ \bigoplus' T \text{ and by } [?] \text{ if } \lambda = \lambda^\kappa > |T|, 2^\lambda = \lambda^+ \text{ is there a } \lambda\text{-complete } \lambda^+\text{-c.c. forcing notion } Q, \models Q \text{ "univ}(\lambda^+, T) = 1, \lambda^+ < 2^{\lambda^+}? \]

We know that for theories \( T \) with the strict order property the answer is no (by [K]Sh 409), or see [Sh 457], §3). We would like to characterize the answer by a natural property of \( T \) (hence show that the answer to all reasonable variants is the same, e.g. does not depend on \( \lambda, \bigoplus_T \equiv \bigoplus_T' \), etc.) So the results we mention above give a lower bound (simple theories + \( T_{\text{ref}} + T_{\text{ref}} \)) and an upper bound (failure of the strict order property) to the family of \( T \)'s with a positive answer. However, we can lower the upper bound. We suggest below a strictly weaker property. From another point of view, a major theme of citeSh:a, [Sh:c] was to find natural dividing lines for the family of first order theories (so the main ones there were stable, superstable and also \( \text{NTOP, depthess NOTOP} ) \). Now [Sh:93] suggests another one: simplicity.

Note that the negation of simple, the tree property has been touched upon in [Sh:a] and also \( \text{NTOP, deepness NOTOP} \). Now [Sh:93] suggests another one: simplicity. For the family of first order theories (so the main ones there were stable, superstable to the family of \( T \)'s with a positive answer. However, we can lower the upper bound. We suggest below a strictly weaker property. From another point of view, a major theme of citeSh:a, [Sh:c] was to find natural dividing lines

We know that for theories \( T \) with the strict order property the answer is no (by [K]Sh 409), or see [Sh 457], §3). We would like to characterize the answer by a natural property of \( T \) (hence show that the answer to all reasonable variants is the same, e.g. does not depend on \( \lambda, \bigoplus_T \equiv \bigoplus_T' \), etc.) So the results we mention above give a lower bound (simple theories + \( T_{\text{ref}} + T_{\text{ref}} \)) and an upper bound (failure of the strict order property) to the family of \( T \)'s with a positive answer. However, we can lower the upper bound. We suggest below a strictly weaker property. From another point of view, a major theme of citeSh:a, [Sh:c] was to find natural dividing lines

2.1 Definition. \( T \) has the strict order property if some formula \( \varphi(\bar{x}, \bar{y}) \) (with \( \ell g\bar{x} = \ell g\bar{y} \)) define in some model \( M \) of \( T \), a partial order with infinite chains.

2.2 Definition. 1) A first order complete \( T \) has the strong order property if some sequence \( \bar{\varphi} = (\varphi_n(\bar{x}^n; \bar{y}^n) : n < \omega) \) of formulas exemplifies it which means that for every \( \lambda \):

\[ (\ast)_{\bar{\varphi}}(a) \quad \ell g\bar{x}^n = \ell g\bar{y}^n \text{ are finite, } \bar{x}^n \text{ an initial segment of } \bar{x}^{n+1} \]
\[ \bar{y}^n \text{ an initial segment of } \bar{y}^{n+1} \]
\[ (b) \quad T, \varphi_{n+1}(\bar{x}^{n+1}, \bar{y}^{n+1}) \models \varphi_n(\bar{x}^n, \bar{y}^n) \]
\[ (c) \text{ for } m \leq n, -(\bar{x}^n, 0 \ldots \bar{x}^{n+1-1})[\Lambda\{\varphi_n(\bar{x}^n, \bar{x}^n, k) : k = \ell + 1 \text{ mod } m\}] \text{ belongs to } T \]
\[ (d) \text{ there is a model } M \text{ of } T \text{ and } \bar{a}_n \in M \text{ (of length } \bar{y}^n \text{, for } n < \omega, \alpha < \lambda) \text{ such that } \bar{a}_\alpha = \bar{a}_{\alpha+1} \models \ell g\bar{y}^n \text{ and } \models \varphi_n[\bar{a}_\beta, \bar{a}_\alpha] \text{ for } n < \omega \text{ and } \alpha < \beta < \lambda. \]
2) The finitary strong order property is defined similarly but $\bar{x}^n = \bar{x}, \bar{y}^n = \bar{y}^n$.
3) We use the shorthand SOP, FSOP and for the negation NSOP, NFSOP (similarly later for NSOPn).

2.3 Claim. 1) The strict order property implies finitary strong order property which implies the strong order property.
2) There is a first order complete $T$, which has the strong order property (even the finitary one) but not the strict order property.
3) Also some first order complete $T$ has the strong order property but not the finitary strong order property, i.e. no $\langle \varphi_n(\bar{x}, \bar{y}) : n < \omega \rangle$ exemplifies it (i.e. with $\ell \bar{x}_n$ constant).

Proof. 1) Immediate.
2) For $\ell \leq n < \omega$ let $<_{n,\ell}$ be a two-place relation. Let $<_n = <_{n,0}$. Let $T_0$ say:

(a) $x <_{n, m-1} y \Rightarrow x <_{n, m} y$
(b) $x <_{n, n} y$
(c) $\neg(x <_{n, n-1} x)$
(d) if $\ell + k + 1 = m \leq n$ then $x <_{n, \ell} y \& y <_{n, k} z \Rightarrow x <_{n, m} z$.

We shall now prove that $T_0$ has the amalgamation property; it also has the point embedding property (as the latter is easier we leave its checking to the reader).

Now suppose $M_i \models T_0, M_0 \subseteq M_i$ for $i = 0, 1, 2$ and $M_1 \cap M_2 = M_0$. We define a model $M$: its universe is $M_1 \cup M_2$ and

$$<_{n, m} = \{(a, b) \in M \times M : \text{if } m < n \text{ then for some } i \in \{1, 2\} \text{ we have :}
(a, b) \in <_{n, m}^{M_i} \text{ or } a \in M_i \setminus M_0, b \in M_{3-i} \setminus M_0
$$
and for some $c \in M_0$ and $\ell, k$ we have :
$$m = \ell + k + 1, (a, c) \in <_{n, \ell}^{M_i}, (c, b) \in <_{n, k}^{M_{3-i}} \}.$$}

Now clearly $M$ extends $M_1$ and $M_2$: trivially $<_{n, m}^M | M_i = <_{n, m}^{M_i}$. Is $M$ a model of $T_0$? Let us check.

Clause (a) holds: For $x, y \in M_i$ as $M_i \subseteq M$; for $i = 1, 2$ and $x \in M_i \setminus M_0, y \in M_{3-i} \setminus M_0$, without loss of generality $m < n$; let $c \in M$ witness $(a, b) \in <_{n, m-1}^M$ i.e. for some $\ell, k$ we have $\ell + k + 1 = m - 1, (a, c) \in <_{n, \ell}^{M_i}$ and $(c, b) \in <_{n, k}^{M_{3-i}}$.

Now by clause (a) applied to $M_i, (a, c) \in <_{n, \ell}^{M_i}$ now apply the definition to get $(a, b) \in <_{n, (\ell+1)+k+1}^M$. Now by clause (a) applied to $<_{n, \ell+1}^M$ we say: “if $m < n$ then ...” so if $n = m$ there is no requirement.

Clause (c): As $M_i \subseteq M$ and $M_i \models T_0$. 

\[ \text{TOWARD CLASSIFYING UNSTABLE THEORIES SH500} \]
Clause (d): Check by cases, i.e. for some $i \in \{1, 2\}$ one of the following cases hold.

1. $\{x, y, z\} \subseteq M_i$:
   - use “$M_i$ is a model of $T_0$ and $M_i$ a submodel of $M$”.
2. $\{x, y\} \subseteq M_i, \{y, z\} \subseteq M_{3-i}$:
   - use the definition of $<_{n, m}$.
3. $y \in M_i \setminus M_0, \{x, z\} \subseteq M_{3-i} \setminus M_0$.

As $x <_{n, \ell} y$ there are $\ell_1, \ell_2$ and $x_1 \in M_0$ such that $x <_{n, \ell_1} x_1, x_1 <_{n, \ell_2} y$ and $\ell_1 + \ell_2 + 1 = \ell$.

As $y <_{n, k} z$ there are $k_1, k_2$ and $z_1 \in M_0$ such that $y <_{n, k_1} z_1, z_1 <_{n, k_2} z, k_1 + k_2 + 1 = k$. In $M_i$ we have $x_1 <_{n, \ell_2} y <_{n, k_1} z_1$ hence $x_1 <_{n, \ell_2 + k_1 + 1} z_1$ and as $\{x_1, z_1\} \subseteq M_0 \subseteq M_i$ clearly $x_1 <_{n, \ell_2 + k_1 + 1} z_1$. Now in $M_{3-i}$ we have $x <_{n, \ell_1}$, $x_1 <_{n, \ell_1} x_1 <_{n, \ell_2 + k_1 + 1} z_1$ hence $x <_{n, \ell_1 + \ell_2 + k_1 + 2} z_1$ so $x <_{n, \ell_1 + \ell_2 + k_1 + 2} z_1 <_{n, k_2} z$ hence $x <_{n, \ell_1 + \ell_2 + k_1 + 2} z$ but $\ell_1 + \ell_2 + k_1 + 2 = \ell + k + 1 = m$ so $x <_{n, m} z$ as required.

4. $y \in M_i \setminus M_0, x \in M_{3-i} \setminus M_0, z \in M_0$.

   Similar to case (3) but with no $x_1$.

5. $y \in M_i \setminus M_0, x \in M_0, z \in M_{3-i} \setminus M_i$.

   Similar to case (3) but with no $z_1$.

Let $T$ be the model completion of $T^0$, easy to check that it exists and has elimination of quantifiers. Let $\varphi_n(x, y) = \bigwedge_{\ell \leq n} x <_{\ell} y$ (remember $x <_{\ell} y$ means $x <_{\ell, \delta} y$) now $\langle \varphi_n : n < \omega \rangle$ exemplifies that $T$ has the (finitary) strong order property. On the other hand we shall show that for every $n(*) < \omega$ the theory $T_{n(*)} := T \upharpoonright \{<_{n, \ell} : \ell \leq n \leq n(*)\}$ does not have the strict order property (as $T = \bigcup_{n < \omega} T_n$, this clearly implies that $T$ does not have the strict order property).

First note that also $n(*)$ has elimination of quantifiers and then check directly.

3. Let $T^0$ say:

   a. $P_n$ (for $n < \omega$) are pairwise disjoint ($P_n$ unary predicates)
   b. $F_n$ a partial one place function from $P_{n+1}$ into $P_n$
   c. $<_{n, \ell}$ are two-place relations on $P_n$ for $\ell \leq n < \omega$; and let $<_{n, m} = <_{n, 0}$

   $\alpha$ $x <_{n, m-1} y \rightarrow x <_{n, m} y$
   $\beta$ $P_n(x) \& P_n(y) \rightarrow x <_{n, m} y$
   $\gamma$ $\neg(x <_{n, m-1} x)$
   $\delta$ if $\ell + k + 1 = m \leq n$ then: $x <_{n, \ell} y \& y <_{n, k} z \rightarrow x <_{n, m} z$

   d. $x <_{n+1, \ell} y \rightarrow F_n(x) <_{n, \ell} F_n(y)$.

Again $T$ will be the model completion of $T^0$ and it has elimination of quantifiers and we shall use $\bar{x}_n = (x_i : i < n), \bar{y}_n = (y_i : i < n)$ and $\varphi_n(\bar{x}_n, \bar{y}_n) = \bigwedge_{i < n} F_i(x_{i+1}) = x_i \& \bigwedge_{i < n} F_i(y_{i+1}) = y_i \& \bigwedge_{i < n} x_i < i y_i$. □2.3
2.4 Claim. 1) The following are equivalent (for $\lambda \geq |T|$):

- (A) $T$ has the strong order property
- (B) there is a $\lambda^+$-saturated model $M$ of $T$, a $L_{\infty, \lambda^+}$-formula $\varphi = \varphi(x, y), \varepsilon = \ell g x = \ell g y \leq \lambda$, possible with $\leq \lambda$ parameters, such that in $M, \varphi$ defines a partial linear order with a chain of length $\geq \beth_2(\lambda)^+$. 2) The following are equivalent ($\lambda \geq |T|$):

- (A$'$) $T$ has the finitary strong order property
- (B$'$) like (B)$\lambda$ but $\varepsilon < \omega$.

Proof. 1) (A$'$) $\Rightarrow$ (B$'$) Straight: for a given $\varphi = \langle \varphi_n(x, y) : n < \omega \rangle$, let $\bar{x}, \bar{y}$ be the limit of $\bar{x}_n, \bar{y}_n$ respectively and write $\psi^x(\bar{x}, \bar{y}) = \bigvee (\exists \bar{z}_0, \ldots, \bar{z}_m)[\bar{z} = \bar{z}_0 \& \bar{y} = \bar{z}_m \& \bigwedge_{\ell < m} \varphi_\omega(\bar{z}_\ell, \bar{z}_{\ell+1})]$ where $\varphi_\omega(\bar{x}, \bar{y}) = : \bigwedge_n \varphi_n(\bar{x}, \bar{y})$.

(B$'$) $\Rightarrow$ (A) Let $\bar{a}_\alpha \in \forall M$ for $\alpha < \beth_2(\lambda)^+$ form a chain. Without loss of generality the order $\varphi$ defines is strict (i.e. $\vdash \varphi(\bar{x}, \bar{x})$) and no parameters (just add them to the $\bar{a}_\alpha$'s). By Erdos Rado theorem without loss of generality for some type $q = q(\bar{x}, \bar{y})$ for all $\alpha < \beta < \omega$ the sequence $\bar{a}_\alpha \vec{a}_\beta$ realizes $q$. For every $\nu, \bigcup \{q(\bar{x}_\ell, \bar{x}_k) : k = \ell + 1 \mod n$ and $k, \ell < n\}$ cannot be realized in $M$ (as if $\bar{b}_0 \cdots \bar{b}_{n-1}$ realizes if we get a contradiction to “$\varphi(\bar{x}, \bar{y})$ defines a strict partial order”). By saturation there is $\varphi^0_n(\bar{x}, \bar{y}) \in q(\bar{x}, \bar{y})$ such that $\{\varphi^0_n(\bar{x}_\ell, \bar{x}_k) : k = \ell + 1 \mod n$ and $k, \ell < n\}$ is not realized in $M$. The rest should be clear.

2) Left to the reader. 2.4.5

2.5 Definition. 1) $T$ has the $n$-stronger order property ($\text{SOP}_{n}$) if there is a formula $\varphi(\bar{x}, \bar{y})$ having this property for $T$ which means: $\ell g \bar{x} = \ell g \bar{y}$ (allowing parameters changes nothing) and there is a model $M$ of $T$ and $\bar{a}_k \in \ell g^2 M$ for $k < \omega$ such that:

- (a) $M \models \varphi[\bar{a}_k, \bar{a}_m]$ for $k < m < \omega$
- (b) $M \models \neg \exists \bar{z}_0 \cdots \bar{z}_{n-1}(\bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n$ and $k = \ell + 1 \mod n\})$.

2) “$T, \varphi(\bar{x}, \bar{y})$ have the SOP$_{\leq n}$” is defined similarly except that in (b) we replace $n$ by each $m \leq n$.

2.6 Claim. SOP $\Rightarrow$ SOP$_{n+1}$, SOP$_{n+1}$ $\Rightarrow$ SOP$_n$, SOP$_{\leq n+1}$ $\Rightarrow$ SOP$_{\leq n}$ and SOP$_n$ $\Leftrightarrow$ SOP$_{\leq n}$ for any given $T$, (we did not say “for any $\varphi$”).

Proof. The first clause is immediate. The second clause is straight too: let $\varphi(\bar{x}, \bar{y}), M, \langle \bar{a}_m : m < \omega \rangle$ exemplify SOP$_{n+1}$ and without loss of generality the sequence $\langle \bar{a}_m : m < \omega \rangle$ is an indiscernible sequence. Does $M \models (\exists \bar{x}_0, \ldots, \bar{x}_{n-1})[\bar{x}_0 = \bar{a}_1 \& \bar{x}_{n-1} = \bar{a}_0 \& \bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n$ and $k = \ell + 1 \mod n\}]$? If the answer is yes we can replace $\bar{a}_1$ by $\bar{a}_2$ (by indiscernibility), let $\bar{c}_0, \ldots, \bar{c}_{n-1}$ be as required above on $\bar{x}_0, \ldots, \bar{x}_{n-1}$ and $\bar{b}_0 =: \bar{a}_1, \bar{b}_1 =: \bar{a}_2(= \bar{c}_0), \bar{b}_2 =: \bar{c}_1, \ldots, \bar{b}_{n-1} =: \bar{c}_{n-2}$, $\bar{b}_n =: \bar{c}_{n-1} = \bar{a}_0$; now they satisfy the requirement mentioned in (b) of 2.5(1) on
\[\bar{x}_0, \ldots, x_n \text{ (for SOP}_{n+1})\text{, contradicting clause (b) of 2.5(1). So assume "no" and now } \varphi^*(\bar{x}, \bar{y}) \text{ have SOP}_n \text{ for } T \text{ where: } \varphi^*(\bar{x}, \bar{y}) := \varphi(\bar{x}, \bar{y}) \& \neg(\exists ! \bar{x}_0, \ldots, x_{n-1})[\bar{x}_0 = \bar{x} \& \bar{x}_1 = \bar{y} \& \big\{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell \mod n\}\}.\]

As for SOP\(_n \Leftrightarrow SOP\leq_n\), the implications \(\rightarrow\) is really included in the proof above, (i.e. by it, if \(\langle \bar{a}_\ell : \ell < \omega, \varphi_n \rangle\) exemplifies SOP\(_n\), for some \(\varphi_{n-1}\) we have \(\langle \bar{a}_\ell : \ell < \omega, \varphi_{n-1} \rangle\) exemplifies SOP\(_{n-1}\) (with \(n, n-1\) here corresponding to \(n+1, n\) there), and we can define \(\varphi_{n-2}, \cdots\) similarly; now \(\langle \bar{a}_\ell : \ell < \omega, \bigwedge i \varphi_i \rangle\) exemplifies SOP\(_n\). The implication \(\Leftarrow\) is trivial. Now the third clause SOP\(_{\leq n+1} \Rightarrow SOP\leq_n\) is trivial (read the definition).

\[\Box \text{2.6}\]

**2.7 Claim.** Let \(T\) be complete. If \(T\) has SOP\(_3\) then \(T\) has the tree property (i.e. is not simple).

**Proof.** Let \(\kappa = \text{cf}(\kappa) > |T|\) and \(\lambda > \kappa\) be a strong limit singular cardinal of cofinality \(\kappa\). Let \(J = \text{"}_\lambda I = \{\eta \in \text{"}_\lambda : \eta(i) = 0 \text{ for every } i < \kappa \text{ large enough}\}\). Let \(\varphi(\bar{x}, \bar{y})\) exemplify the SOP\(_3\). By the definition we can find a model \(M\) of \(T\) and \(\bar{a}_\eta \in M\) (for \(\eta \in J\)) such that:

\((*)\) \(\eta <_{\ell \chi} \nu \text{ in } I \Rightarrow M \models \varphi[\bar{a}_\eta, \bar{a}_\nu]\).

Without loss of generality \(||M|| \geq \lambda, M\) is \(\kappa^+\)-saturated. So for every \(\eta \in \text{"}_\lambda \{\eta\}\) \(\setminus I\) we can find \(\bar{a}_\eta \in M\) such that it realizes \(p_\eta = \{\varphi(\bar{a}_{\eta(i)}^{a_{\eta(i)}}, \bar{x}) \& \varphi(\bar{x}, \bar{a}_{\eta(i)}^{(\eta(i)+1)}\eta_{\eta(i)+1}) : i < \kappa\}\). But if \(\eta_1 <_{\ell \chi} \eta_2 \in \text{"}_\lambda \{\eta\}\) then we can find \(\nu, \rho \in I\) such that:

\(\eta_1 <_{\ell \chi} \nu <_{\ell \chi} \rho <_{\ell \chi} \eta_2\) and \(\varphi(\bar{x}, \bar{a}_\nu) \in p_{\eta_1}, \varphi(\bar{a}_\rho, \bar{x}) \in p_{\eta_2}\) and by (*) we have \(M \models \varphi[\bar{a}_\nu, \bar{a}_\rho]\), so \(p_{\eta_1} \cup p_{\eta_2}\) is contradictory (by clause (b) of 2.5(1) for “\(\varphi\) have SOP\(_3\)”). So \(\langle p_\eta : \eta \in \text{"}_\lambda \{\eta\}\rangle\) are pairwise contradictory, \(|p_\eta| = \kappa\), and \(\lambda^\kappa > \lambda = \lambda^{< \kappa} > 2^{|T|}\) and \(\bigcup \{\text{Dom } p_\eta : \eta \in \text{"}_\lambda \{\eta\}\}\) has cardinality \(\leq \lambda\) and \(\kappa \geq |T|\).

By [Sh:a], III,7.7 = [Sh:c], III,7.7, p.141 this implies that \(T\) has the tree property.

\[\Box \text{2.7}\]

**2.8 Claim.** 1) The theory \(T_n := T \upharpoonright \{<_{n, \ell} : \ell \leq n\}\) from 2.3(2) has SOP\(_n\) but not SOP\(_{n+1}\).

2) \(T^\text{mc}_{\text{tr}}\), the model completion of the theory of triangle free graphs has SOP\(_3\) but not SOP\(_4\).

3) For \(n \geq 3\) the model completion \(T^\text{mc}_n = T^\text{mc}_{\text{def}(n)}\) of the theory \(T_n = T^\text{def}_n\) of graphs (= directed graphs, no loops or multiple edge for simplicity) with no directed circle of length \(\leq n\) has SOP\(_n\) but not SOP\(_{n+1}\).

4) For odd \(n \geq 3\), the model completion \(T^\text{mc}_n = T^\text{mc}_{\text{def}(n)}\) of the theory \(T_n = T^\text{def}_n\) of graphs with no odd circle of length \(\leq n\), has SOP\(_n\) but not SOP\(_{n+1}\).

5) For \(n \geq 3\), the model completion \(T^\text{mc}_{\text{cl}(n)}\) of the theory \(T_n = T^\text{cl}(n)\) of graphs with no circles of length \(\leq n\), has SOP\(_n\) but not SOP\(_4\).

6) The theory \(T^\text{cl}_{\text{tr}}\) (see [Sh.457]) does not have SOP\(_3\) (but is not simple).
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2.8A Remark. 1) Note that \( \text{univ}(\lambda, T_{\text{mc}}(\ell)) = \text{univ}(\lambda, T_{\text{cf}}(\ell)) \).
2) For those theories, \( D(T_{\text{mc}}) \) is an uncountable; they have no universal model in \( \lambda < 2^{2^{\aleph_0}} \).

Proof. 1) Proved really in 2.3.
2) This is included in part (5).
3), 4), 5) We discuss the existence of model completion later; note that the meaning of \( T_n \) depends on the part we are proving.

Let \( xRy \) mean \( (x, y) \) is an edge; when we say \( (x, y) \) is an edge, for graphs we mean \( \{x, y\} \) is an edge. Let \( \bar{y} = (y_\ell : \ell < n) \), \( \varphi(\bar{x}, \bar{y}) = \bigwedge_{\ell < n-1} x_\ell R y_{\ell+1} \& x_{n-1} R y_0 \). First we note there \( T_n \models \neg(\exists x_0, \ldots, x_{n-1}) \bigwedge \{ \varphi(\bar{x}, \bar{x}_k) : \ell, k < n, k = \ell + 1 \text{ mod } n \} \), otherwise there are \( M \models T_n \) and \( \bar{a}_\ell = \langle a_{\ell,0}, \ldots, a_{\ell,n-1} \rangle \in {}^n M \) as forbidden but then \( a_{0,0}, a_{1,1}, \ldots, a_{n-1,n-1} \) is a circle, so in all cases this is impossible.

For parts 3), 4) let \( M \) be the following model of \( T_n \): elements
\[ a_{i}^{\ell}(i < \omega, \ell < n) \subseteq R = \{a_{i}^{\ell}, a_{j}^{\ell+1} : i < j < \omega, \ell < n - 1\} \cup \{(a_{i}^{n-1}, a_{0}^{n}) : i < j < \omega\} \] (but for graphs we put all such pairs and the inverted pair as \( R \) should be symmetric and irreflexive relation). For part (5) for \( R \) is not symmetric the absence of any circle should be clear, \( M \models a_{i}^{n}Ra_{k}^{n} \rightarrow i < k < n + 1 \) mod \( n \), so \( T_{\text{def}(n)}, T_{\text{cf}(n)} \) (and \( T_{\text{mc}(n)} \)) has even or long and \( T_{\text{def}(n)} \) (and \( T_{\text{cf}(n)} \)) has SOF.

Let \( n = 3 \). Now \( T_{\text{def}(3)} = T_{\text{cf}(3)} \) so we can ignore part (5). Also \( T_{\text{def}(n)}, T_{\text{cf}(n)} \) has the amalgamation property and joing embedding property. Thus, it is enough to show that \( T_{\text{mc}(n)} \) fails the SOF. As \( T_{\text{mc}} \) lacks elimination of quantifiers the reader can check directly that \( T_n \) does not have SOF.

Let \( n > 3 \). Though \( T_{\text{mc}}(n) \) does not have elimination of quantifiers, every formula is equivalent to a Boolean combination of formulas of the form: \( x = y, xRy \) for \( m < n, \psi_m(x, y) =: (\exists x_0, \ldots, x_m)[x = x_0 \& \cdots \& x_m \bigwedge_{\ell < m} x_\ell R x_{\ell+1}] \) (i.e. the distance from \( x \) to \( y \) is \( \leq m \), directed from \( x \) to \( y \) in the case of di-graphs). For part (5) of 2.8, we should add for \( \ell < m < n/2, \ell > 0 \) a partial function \( F_m, \ell \) defined by: \( F_m,\ell(x, y) = z \) iff there are \( t_0, \ldots, t_m \) with no repetition such that \( x = t_0, y = t_m, z = t_\ell \) and \( \bigwedge_{\ell < m} t_\ell t_{\ell+1} \) and lastly \( \psi_m,\ell(x, y) =: (\exists z)[F_m,\ell(x, z) = z] \). Let \( T_n \) be the set of obvious (universal) axioms for those relations. Then easily \( T_n \) has amalgamation and has model completion, \( T_n \) which has elimination of quantifiers (but the closure of a finite set under those functions may be infinite). Moreover, assume \( M \models T_n, (a_m : m < \omega) \) is an indiscernible sequence in \( M, a_m = \langle a_{1}^{m} : \ell < k\rangle \), with \( k < \omega \). Then there is \( w \subseteq k \) such that \( a_{1}^{n} = a_{1}^{n+1} \leftrightarrow \ell \in \omega \) and without loss of generality \( \ell_1 < \ell_2 \Rightarrow a_{1}^{\ell_1} \neq a_{1}^{\ell_2} \). Let for \( u \subseteq \omega, M_u \) be the submodel of \( \mu \) generated by \( \bigcup_{m \in u} a_m \) for parts (3), (4), \( M_u = \bigcup\{M_v : v \subseteq u \text{ and } |v| \leq 1\} \) so
things are simple. By the indiscreetibility (increasing the $\bar{a}_m$’s e.g. taking $\omega$ blocks) without loss of generality

$(\ast)$  $M_u \cap M_v = M_{uv}$ and the universe of $M_{(m)}$ is the range of $\bar{a}_m$.

Let $m = n$ for parts (3), (4) of 2.8, $m = 3$ for part (5). For part (5) note: the distance between $a_{i,o}^0, a_{i,1}^1$ is $> n$.

[Why? If not there is a path $C^{n,j}$ of length $\leq \frac{n}{3}$ for $a_{i,o}^0$ to $a_{i,1}^1$, now $C^{0,3} \cup C^{1,2} \cup C^{1,4} \cup C^{0,4}$ is a circle of length $\leq n$, may cross itself but still there is a too small circle.]

We can now define models $N_{(\ell)}$ (for $\ell < n + 1$), $N_{(\ell, \ell + 1)}(\ell < n)$ and $N_{(n, 0)}$ and isomorphisms $h_\ell, g_\ell(\ell < n + 1)$ such that:

(a) for $\ell < n + 1, h_\ell$ an isomorphism from $M_{(\ell)}$ onto $N_{(\ell)}$
(b) for $\ell < n, g_\ell$ an isomorphism $M_{(\ell, \ell + 1)}$ onto $N_{(\ell, \ell + 1)}$ extending $h_\ell, h_{\ell + 1}$.
(c) for $\ell = n, g_\ell$ an isomorphism from $M_{(n, n + 1)}$ onto $N_{(n, 0)}$ extending $h_n$ and $h_0 \circ f$ where $f$ is the isomorphism from $M_{(n + 1)}$ onto $M_{(0)}$ taking $\bar{a}_{n + 1}$ onto $\bar{a}_0$.
(d) $N_{\emptyset} = : g_\ell(M_0)$ does not depend on $\ell$.
(e) $N_u \cap N_\nu = N_{u \cap \nu}$ if $u, \nu$ are among $\emptyset, \{\ell\}, \{m, m + 1\}, \{n, 0\}(\ell < n + 1, m < n)$.

Now,

$\otimes$ There is a model of $T_n^2$ extending all $N_{(\ell, \ell + 1)}, N_{(n, 0)}(\ell < n)$.

This is enough for showing that $T_n^1$ lacks the SOP$_{m + 1}$.

Lastly the reader can check that $T_n^{mc}$ has SOP$_3$ [choose $k \in (\frac{n}{4}, n), \bar{a}_\ell = (a_{\ell, 0}, \ell_1 < \ell_2 \Rightarrow \varphi_k(a_{\ell, 0}, a_{\ell_2, 0})].$ \hspace{1cm} \square_{2.8}

2.9 Theorem. Let $T$ be first order complete, $\lambda \geq |T|$ and $T$ has the SOP$_3$. Then:

1) $T$ is maximal in the Keisler order $\preceq_\lambda$, i.e. for a regular filter $D$ on $\lambda$ and some (= every) model $M \models T$ we have $M^\lambda / D$ is $\lambda^+$-saturated if $D$ is a good ultrafilter.

2) Moreover, in 2.10 $T$ is $\preceq^d$-maximal, (see Definition 2.10 below).

We delay the proof.

Remark. The order $\preceq$ was introduced and investigated by Keisler [Ke76]; further investigated in [Sh 42], [Sh:a], CH.VI, new version [Sh:c], CH.VI. The following is a generalization.

2.10 Definition. 1) For models $M_0, M_1$ we say $M_0 \preceq_\lambda M_1$ if the following holds: for some model $\mathfrak{B}_0$ in which $M_0, M_1$ are interpreted (so $M_0 = M_0^{\mathfrak{B}_0}$), for every elementary extension $\mathfrak{B}$ of $\mathfrak{B}_0$, which is $(\emptyset, \|\tau(M_0)\| + \|\tau(M_1)\|)^+-$saturated we have: $[M_1^{\mathfrak{B}}$ is $\lambda^+$-saturated $\Rightarrow M_0^{\mathfrak{B}}$ is $\lambda^+$-saturated].

2) $M_0 \preceq^* M_1$ if for every $\lambda \geq \emptyset, \|\tau(M_0)\| + \|\tau(M_1)\|$ we have $M_0 \preceq_\lambda M_1$.

3) Using the superscript $\ell$ instead of $*$ means in the saturation we use only $\varphi$-types for some $\varphi = \varphi(x, y)$ (so any $\varphi$ is O.K., but for each type $\varphi$ is constant) and omit the saturation demand on $\mathfrak{B}$.

4) For complete theories $T_1, T_2$ we say $T_1 \preceq_\lambda^* T_2$ if for every model $M_1$ of $T_1$ for some model $M_2$ of $T_2, M_1 \preceq_\lambda M_2$. Similarly for $T_1 \preceq_\lambda^* T_2, T_1 \preceq_\lambda^* T_2$. 
Let \( \varphi \) then \( M_I \) find a finite be such that \( T(B) \) as there \([references]\).

1) - 4) Obvious.

Proof. 1) - 4) Obvious.
5) The proof of \([Sh:a],[VI,2.6] = [Sh:c],[VI,2.6, p.337\) gives this, too.
6), 7), 8) As there [references].

2.11 Observation. 1) In 2.11(1) we can just use \( \mathfrak{B}_0 \) of the form
\((H(\chi), \in, <^*_\chi, M_0, M_1) \) with \( \chi \) strong limit.
2) \( <^*_\chi \) is a partial order, also \( <^*_\alpha, <^* \) are partial orders; \( M <^*_\chi M \), and if \( M_0 \) is interpretable in \( M_1 \) then \( M_0 <^*_\chi M_1 \).
2A) For models of countable vocabulary, similar statements hold for \( <^* \) (without the countability if \( |\tau(M_1)| > |\tau(M_0)| + |\tau(M_2)| + \aleph_0 \), we can get a silly situation).\(^1\)
3) If \( \lambda \geq \aleph_0 + |\tau(M_0)| + |\tau(M_1)| \) then: \( M_0 <^*_\chi M_1 \) iff for every finite \( \tau \subseteq \tau(M_0), M_0 \upharpoonright \tau \equiv \tau^* M_1 \).
4) \( M_1 \upharpoonright \lambda^* \lambda_2 \Rightarrow M_1 <^*_\lambda M_2 \).
5) Parallel results hold for theories.
6) Any (complete first order) theory of any infinite linear order is \( \subseteq \)-maximal hence \( <^*_\chi \)-maximal for every \( \lambda \geq |T| + \aleph_0 \).
7) All countable stable theories without the f.c.p. (e.g. \( T = \text{Th}(\omega_1 =) \) are \( \subseteq \)-equivalent.
8) All countable stable theories with the f.c.p. are equivalent
\((e.g. T_{eq} = \text{Th}(\bigcup_n \{n\} \times n), E) \) where \( E \) is equally of first coordinates).

9) If \( T_1 \) is countable unstable, then
\( T = \frac{\frac{\frac{\frac{\lambda}{\lambda}}{\lambda}}{\lambda}}{\lambda} \) \( T_{eq} \) \( \lambda \geq 2^{\aleph_0 \Rightarrow T \equiv \lambda^* \lambda} \) \( \Rightarrow T <^*_\lambda T_{eq} \).

Proof. 1) - 4) Obvious.
5) The proof of \([Sh:a],[VI,2.6] = [Sh:c],[VI,2.6,p.337\) gives this, too.
6), 7), 8) As there [references]. \( \square \) 2.11

2.12 Proof of Theorem 2.9(1),(2). Without loss of generality \( \tau(T) \) is finite. Remember: if \( T' \) has infinite linear orders as models then it is \( \subseteq \)-maximal. Let \( J \) be a dense linear order, such that:

(a) \( J \) has a closed interval which is \( I \)
(b) for any regular \( \mu_1, \mu_2 \leq |J|, J \) has an interval isomorphic to \( [(1) \times \mu_1) \cup (2) \times \mu_2) \) ordered by \( (i_1, \alpha_1) <_J (i_2, \alpha_2) \iff (i_1 = 1 & i_2 = 2) \bigvee (i_1 = 1, i_2 & \alpha_1 < \alpha_2) \bigvee (i_1 = 2 = t_2 & \alpha_1 > \alpha_2) \).

Let \( \varphi(x, y) \) exemplify the SOP3. Let \( M \) be a model of \( T \) and \( F : I \rightarrow \ell g \mathfrak{B} \) be such that \( I \models \eta < \nu \Rightarrow M \models \varphi[F(\eta), F(\nu)] \) and for every \( c \in M^\mathfrak{B} \) we can find a finite \( I' \subseteq I \) such that: if \( t_1, t_2 \) \( I \) \( I' \) then \( M^\mathfrak{B} \models \varphi[F(t_1), c] \equiv \varphi[F(t_2), c] \) and \( M^\mathfrak{B} \models \varphi[c, F(t_1)] \equiv \varphi[c, F(t_1)] \). Let \( \mathfrak{B}_0 = (H(\chi), \in, <^*_\chi, J, I, F, M) \) and \( \mathfrak{B} \) be a model, \( j \) an elementary embedding of \( \mathfrak{B}_0 \) into \( \mathfrak{B} \) such that \( M^* \models M^\mathfrak{B} \upharpoonright L(T) \) is locally \( \lambda^+ \)-saturated but \( I^\mathfrak{B} = j(I) \) is not \( \lambda^+ \)-saturated (for 2.9: \( \mathfrak{B}^* = \mathfrak{B}^{\lambda \upharpoonright D} \)).

As \( j(I) \) is not \( \lambda^+ \)-saturated, we can find \( \lambda_0, \lambda_1 \leq \lambda \) and \( \alpha_i^j \in j(I) \) (for \( i < \lambda, \ell < 2 \)) such that:

\( \alpha \) \( \lambda^\mathfrak{B} \models \alpha_i^0 < \alpha_j^0 \) for \( i < j < \lambda_0 \)

\(^1\)so to overcome this, we may in Definition 2.10(2) replace “every \( \lambda > \cdots \)” by “every large enough \( \lambda \)”
(β) \( I^\mathfrak{B} \models a_i^j > a_j^i \) for \( i < j < \lambda_1 \)
(γ) \( I^\mathfrak{B} \models a_0^j < a_j^0 \) for \( i < \lambda_0, j < \lambda_1 \)
(δ) \( I^\mathfrak{B} \models \neg (\exists x) [\bigwedge_{i < \lambda_0, j < \lambda_1} a_i^j < x < a_j^i] \).

Clearly \( \{ \varphi(a_0^i, \bar{x}), \varphi(\bar{x}, a_1^i) : i < \lambda_0, j < \lambda_1 \} \) is finitely satisfiable in \( \bar{j}(M) \). Now as \( \bar{j}(M) \) is locally \( \lambda^+ \)-saturated there is \( \bar{a} \in \bar{t}^\mathfrak{E}(M*) \) such that \( M \models \varphi(a_0^i, \bar{a}) \land \varphi(a_1^i, \bar{a}) \) for \( i < \lambda_0, j < \lambda_1 \). In \( \mathfrak{B} \) we can define:

\[
I^\mathfrak{B} \big[ \bar{a} \big] = \left\{ \eta \in I^\mathfrak{B} : \text{there is } \nu \in I^\mathfrak{B} \text{ such that } I^\mathfrak{B} \models \varphi(\bar{a}, \bar{a}) \right\}
\]

\[
I^\mathfrak{B} \big[ \bar{a} \big] = \left\{ \eta \in I^\mathfrak{B} : \text{there is } \nu \in I^\mathfrak{B} \text{ such that } I^\mathfrak{B} \models \varphi(\bar{a}, \bar{a}) \right\}
\]

\[
\text{and } j(M) = \varphi(\bar{a}, \bar{a}).
\]

Clearly

(a) \( I^\mathfrak{B} \big[ \bar{a} \big] \) is an initial segment of \( j(I) \) which belongs to \( \mathfrak{B} \).
(b) \( I^\mathfrak{B} \big[ \bar{t} \big] \) is an end segment of \( j(I) \) which belongs to \( \mathfrak{B} \).
(c) For every \( i < \lambda_0 \)
\[
(\star)_0 a_0^i \in I^\mathfrak{B} \big[ j, t \big]
\]
(d) for every \( j < \lambda_1 \)
\[
(\star)_1 a_0^1 \notin I^\mathfrak{B} \big[ t \big]
\]
(e) By the choice of \( \varphi \)
\[
(\star)_3 I^\mathfrak{B} \big[ \bar{t} \big] \cap I^\mathfrak{B} \big[ \bar{t} \big] = \emptyset.
\]

If for some \( c \in \mathfrak{B}, \mathfrak{B} \models \varphi(\bar{c}, \bar{c}) \) and \( (\forall x \in I_{-}[t]) (c \leq t, x) \) we are done. So \( \mathfrak{B} \) thinks \( I^\mathfrak{B} \big[ \bar{t} \big] \big[ I^\mathfrak{B} \big[ \bar{t} \big] \big] \) is a Dedekind cut, so let \( \mathfrak{B} \models \varphi(\bar{t}, \bar{t}) \). If \( \mu_1, \mu_2 \) are infinite, we use clause (b) of the choice of \( \bar{J} \) (and the choice of \( \mathfrak{B} \)). We are left with the case where \( \mu_1 = 1 < \mu_2 \) (the other case is the same). Use what \( \mathfrak{B} \) “thinks” is a \( (t_1, t_2) \) Dedekind cut of \( J \) to show \( \mu_2 \geq \mu_t + 1 \) a contradiction. \( \square_{2.9} \)

2.13 Theorem. 1) The theorems on non-existence of a universal model in \( \lambda \) for linear order from [KjSh 409], [Sh 457],§3 hold for any theory with SOP4.
2) We can use embedding (not necessarily elementary) if \( \varphi(\bar{x}, \bar{y}) \) is quantifier free or even existential.

Proof. We concentrate on the case \( \lambda \) is regular and part (1). We will concentrate on the new part relative to [KjSh 409]. Let \( \varphi(\bar{x}, \bar{y}) \) exemplify SOP\(_{<4} \) (exists by
2.6(1)) in a complete first order theory $T$. Without loss of generality $\ell g \bar{x} = \ell g \bar{y} = 1$ and $T \vdash \neg \varphi(\bar{x}, \bar{x})$.

Let $M$ be a model of $T$ with universe $\lambda$, $I$ a linear order, $a_s \in M, M \models \varphi[a_s, a_t]$ for $s <_I t$ (from $I$).

We do not have a real Dedekind cut (as $\varphi(x, y)$ is not transitive), but we use replacements. Now for every $b \in M$, let $I^-[b] = \{ t : M \models \varphi[a_t, b] \}$ and $I^+[b] = \{ t : M \models \varphi[b, a_t] \}$. As $\varphi$ exemplifies also SOP$_{\leq 3}$ clearly the following is satisfied:

\[(*) \quad s \in I^-[b] \text{ & } t \in I^+[t] \Rightarrow s < t \]

(if $t < s$ from a counterexample, $b, t, s$ gives a contradiction).

Note: $I^-[a_s] = \{ t : t <_I s \}$, $I^+[a_s] = \{ t : s <_I t \}$.

Let $P = \{ a_s : s \in I \}$, $<_s = \{ (a_s, a_t) : s <_I t \}$ and

$J^-|t, a| = \{ s \in I : s <_I t, a_s \in \alpha \}$, $J^+|t, a| = \{ s \in I : s <_I t, a_s \in \alpha \}$

(remember: $|M|$, the universe of $M$, is $\lambda$, $\alpha = \{ \beta : \beta < \alpha \}$).

Hence $C =: \{ \delta < \lambda : (M, P, <^*, <) \upharpoonright \delta < M^+ := (M, P, <^*, <) \}$ is a club of $\lambda$.

Clearly:

\[(**) \quad \text{let } \delta \in C, b \in M \cap \delta; \]

(i) if $(I^-[b], <_I)$ has cofinality $< \lambda$, then $I^-[b] \cap M_6$ is $<_I$-cofinal in it

(ii) if $(I^+[b], >_I)$ has cofinality $< \lambda$, then $I^+[b] \cap M_6$ is $(>_I)$-cofinal in it

(iii) if there is $t$ such that $I^-[b] \leq_I t \leq_I I^+[b]$ then there is such $t \in M \cap \delta$.

Now suppose that $\delta_1 < \delta_2$ are in $C, t(\ast) \in I, a_{t(\ast)} \in P \setminus \delta_2$.

**Case 1:** For some $s(\ast) \in M \cap \delta_2$ we have $(\forall s \in J^-|t(\ast), \delta_1|)(s <_I s(\ast) <_I t(\ast))$. Let $b =: a_{s(\ast)}$. Hence for every $c \in M \cap \delta_1$: if $\varphi(c, a_{t(\ast)})$ then for every $t', t''$ satisfying $t'' <_I t(\ast) <_I t', t' \in I, a_{t'} <_s a_{t''} \in \delta_1$ we have $M^+ \models (\exists x)[x \in P \& \varphi(c, x) \& a_{t'} <^* x <^* a_{t''}]$. Clearly (or see the middle of the proof of case 2 below) necessarily for arbitrary $<_I$-large $t \in J^-|t(\ast), \delta_1|$ we have $\varphi[c, a_t]$ but for any such $t, \varphi[a_t, a_{s(\ast)}]$ i.e. $\varphi[a_t, b]$ hence

\[ (\ast)_1 (\forall c \in M \cap \delta_1)[\varphi(c, a_{s(\ast)})] \rightarrow (\exists y \in M \cap \delta_1)[\varphi(c, y) \& \varphi(y, b)]. \]

Of course,

\[ (\ast)_2 b \in M \cap \delta_2 \]

\[ (\ast)_3 \varphi[b, a_{t(\ast)}]. \]

Note: those three properties speak on $M, \delta_1, \delta_2, a_{t(\ast)}, b$ but not on $I, <^*, P, <$.

**Case 2:** For no $s(\ast) \in I, a_{s(\ast)} \in \delta_2$ do we have $(\forall s \in J^-|t(\ast), \delta_1|)(s <_I s(\ast) <_I t(\ast))$

we assume:

\[ (A) \quad \{ a_s : s \in I, a_s \in \delta_2, s <_I t(\ast) \} \text{ is not definable in } M^+ \upharpoonright \delta_2. \]
We shall now show that for no \( b \in M \cap \delta_2 \) do we have \((*)_1 + (*)_2 + (*)_3\), so assume \( b \) is like that and we shall get a contradiction.

Without loss of generality \((A)\) holds. By \((*)_3\) we have \( \varphi[b, a_{t(*)}] \) hence for arbitrarily \( < I \)-large \( t \in J^-[t(*), \delta_2] \) we have \( \varphi[b, a_t] \); choose such \( t_0 \).

[Why? Otherwise \( I^+[b] \cap J^-[t(*), \delta_2] \) is bounded say by some \( t^* \), so \( \theta(x, b, a_{t^*}) =: x \in P \) & \( (\exists y) [y \in P \land y \leq^* x \land \varphi(b, y) \land t^* <^* x] \) define in \( M^+ \) a set which is an end segment of \( (P, <^*) \), include \( t(*) \) (check) but so \( s \in \delta_2, s <^* t(*) \). So in \( M^+ \upharpoonright \delta_2 \) it defines the set \( \{ a_s : s \in J^+[t(*), \delta_2] \} \) hence \( \neg \theta(x, b, a_{t^*}) \) define in \( M^+ \upharpoonright \delta_2 \) the set \( \{ a_s : s \in J^-[t(*), \delta_2] \} \), hence by the assumption of the case, \( M^+ \upharpoonright \delta_2 \) satisfies:

\[
(\forall z)[z \in P \land \neg \theta(z, b, a_{t^*}) \rightarrow (\exists y < \delta_1)(y \in P \land z \leq^* y \land \neg \theta(z, b, a_{t^*})]
\]

contradicting \((A)\) above.

So by the assumption of the case (i.e. that \( t_0 < \delta_1 \) cannot serve as \( s(*) \) and \( t_0 < I t(*) \) for some \( t_1 \in J^-[t(*), \delta_1] \) we have \( t_0 < I t_1 \) and clearly \( t_1 < I t(*) \) hence \( \varphi[a_{t_1}, a_{t(*)}] \). So by \((*)_1\) applied with \( a_{t_1} \), standing for \( c \) for some \( y \in M \cap \delta_1 \) we have \( \varphi[a_{t_1}, y] \land \varphi[y, b] \). Now \( b, a_{t_0}, a_{t_1}, y \) contradicts “\( \varphi(x, y) \) exemplifies SOP\(_4\)”.

Hence we get together:

\( \Theta \) if \( \delta_1 < \delta_2 \in C, a_{t(*)} \in P \setminus \delta_2 \) (so \( t(*) \in I \)) then the following conditions are equivalent:

- (\( \alpha \)) for some \( s(*) \in \delta_2 \), (so necessarily \( s(*) \neq t(*) \)) we have \( (\forall s \in I)[a_s \in \delta_1 \Rightarrow s < I s(*) \equiv s < I t(*)] \) we have:
  - (A) \( \{ a_s : s \in I, a_s \in \delta_2 s < I t(*) \} \) is definable in \( M^+ \upharpoonright \delta_2 \)
    (with parameters)
  - (B) \( \{ a_s : s \in I, a_s \in \delta_2, t(*) < I s \} \) is definable in \( M^+ \upharpoonright \delta_2 \)
    (with parameters)

- (\( \beta \)) for some \( b \in \delta_2 \) the conditions \((*)_1, (*)_2, (*)_3\) above holds for \( \varphi(x, y) \) or for \( \varphi^-(x, y) \) where \( \varphi^-(x, y) = \varphi(y, x) \).

**Proof.** If clause \((\alpha)\) holds and \( s(*) < I t(*) \) holds use Case 1 above. If clause \((\alpha)\) holds and \( s(*) < I t(*) \) fails, then \( t(*) < I s(*) \) inverts the order of \( I \), use \( \varphi^- \) and now apply Case 1 above. So assume \( \neg(\alpha) \). We first want to apply Case 2 to prove there is no \( b \) satisfying \((*)_1, (*)_2, (*)_3\). For this we need clause (A) there. We claim it holds.

[Why? Assume \( \bar{d} \in (M^2 \upharpoonright \delta_2), \psi \) a first order formula (in the vocabulary of \( M^+ \)), such that for every \( e \in M^+ \upharpoonright \delta_2 \) we have: \( M^+ \upharpoonright \delta_2 \models \psi[e, \bar{d}] \) if \( e = \{ a_s : s \in I, a_s \in \delta_2, s < I t(*) \} \). So \( M^+ = (\exists z)[P(z) \land (\forall y)(y < \delta_1 \land P(y) \Rightarrow y <^* z \equiv \psi[y, \bar{d}])] \) as \( z \mapsto a_{t(*)} \) satisfies it, but \( M^+ \upharpoonright \delta_2 \times M^+ \) hence there is \( z^* \in \delta_2 \) satisfying this. So \( z^* \in P \) hence for some \( s(*) \), \( z^* = a_s(*) \), so \( s(*) \) contradicts the assumptions \( \neg(\alpha) \). So we have proved the failure of the first possibility from clause (\( \beta \)). The second is proved similarly inverting the order of \( I \), using \( \varphi^- \) (noting that this transformation preserves the statement (A) from Case 2).
2.14 Definition. Let $M$ be a model with universe $\lambda$ and $\varphi(x, y)$ a formula exemplifying SOP$^4$ (possibly with parameters) let $\varphi^+(x, y) = \varphi(x, y), \varphi^−(x, y) = \varphi(y, x)$. Assume $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a club system, $S \subseteq \lambda$ stationary, guessing club$^2$ (i.e. for every club $E$ of $\lambda$ for stationarily many $\delta < \lambda, \delta \in S, C_\delta \subseteq E$)

(a) for $x \in |M|$ and $\delta \in S$ let

\[
\text{inv}_\varphi(x, C_\delta, M) = \left\{ \alpha \in \text{nacc} C_\delta : \delta_2 = \alpha, \delta_1 = \sup(C_\delta \cap \alpha) \right\}
\]

(\text{well defined),}

for some $b$ conditions $(*)_1, (*)_2, (*)_3$ of Case 1

holds for $\varphi^+$ or for $\varphi^−$

(b) $\text{Inv}_\varphi(C_\delta, M) = \{\text{inv}(x, C_\delta, M) : x \in M\}$

$\text{INV}_\varphi(M, C) = \{\text{Inv}_\varphi(C_\delta, M) : \delta \in S\}$

$\text{INV}_\varphi(M < C) = \text{INV}_\varphi(M, C)/\text{id}^\circ(C)$ where:

2.15 Definition. $\text{id}^\circ(C) = \{S' \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ the set of } \delta \in S' \cap S \text{ for which } C_\delta \subseteq E \text{ is not stationary}\}.$

2.16 Observation. If $M' \cong M''$ are models of $T$ and both have universe $\lambda$ in $M$ then $\text{INV}_\varphi(M', \bar{C}) = \text{INV}_\varphi(M'', \bar{C})$ so $\text{INV}_\varphi(M, \bar{C})$ can be defined for any model of cardinality $\lambda$.

Proof. Let $f$ be from $M'$ onto $M''$, so $f$ is a permutation of $\lambda$. So $E_0 = \{\delta < \lambda : \delta \text{ a limit ordinal}, f \text{ maps } \delta \text{ onto } \delta\}$. Assume $C_\delta \subseteq E$, then for $x \in M' \setminus \delta, \delta \in S$ we have $\text{inv}(x, C_\delta, M') = \text{inv}_\varphi(f(x), C_\delta, M'').$

[Why? Read $(*)_1, (*)_2, (*)_3$. Hence $\text{Inv}_\varphi(C_\delta, M') \in \text{Inv}_\varphi(C_\delta, M'')$. By the definition of $\text{id}^\circ(C)$ we are done. □]

2.17 Observation. If $\bar{C} = \langle C_\delta : \delta \in S \rangle, S \subseteq \lambda$ stationary, $C_\delta \subseteq \delta = \sup(C_\delta), C_\delta$ closed, $I$ a linear order with the set of elements being $\lambda$ we let:

(a) for $x \in \lambda, \delta \in S, \text{inv}(x, C_\delta, I) = \{\alpha \in \text{nacc}(C_\delta) : \text{there are } y, z \in \alpha \text{ such that } y <_I x <_I z \text{ such that } (\forall s)(s \in \sup(C_\delta \cap \alpha) \Rightarrow s <_I y \text{ or } z <_I s)\}$

(b) $\text{Inv}(C_\delta, I) = \{\text{inv}(x, C_\delta, I) : x \in M\}$

(c) $\text{INV}(I, \bar{C}) = \{\text{Inv}(C_\delta, I) : \delta \in S\}$

(d) $\text{INV}(I, \bar{C}) = \text{INV}(I, \bar{C})/\text{id}^\circ(\bar{C})$

2.18 Observation. $\text{INV}(I, \bar{C}) = \text{INV}(I', \bar{C})$ if $I \cong I'$, so actually it is well defined for any linear order with cardinality $\lambda$.

2.19 Observation. If $M$ is a model with universe $\lambda$ and $\varphi, \langle a_s : s \in I \rangle$ as above and, $\emptyset \notin \text{id}^\circ(\bar{C})$ then $\text{INV}(I, \bar{C}) \subseteq \text{INV}_\varphi(M, \bar{C})$ i.e. for some club $E$ of $\lambda, \delta \in S$ & $C_\delta \subseteq E \Rightarrow \text{Inv}(C_\delta, I) \subseteq \text{Inv}_\varphi(C_\delta, M)$.

$^2$otherwise dull
Proof. By \(\oplus\) above.

Conclusion of the proof of 2.13. As in [KjSh 409]. \(\square_{2.13}\)

2.20 Claim. For a complete \(T\), the following are equivalent:

(a) \(T\) does not have SOP\(_3\)

(b) if in \(C\), \(\langle \bar{a}_i : i < \alpha \rangle\) is an indiscernible sequence, \(\alpha\) infinite and \(\{\varphi(\bar{x}, \bar{y}, \psi(\bar{x}, \bar{y}))\}\) contradictory and for each \(j\) for some \(b_j\) we have \(i < j \Rightarrow \models \varphi[\bar{a}, \bar{b}, \bar{a}_i]\) and \(i > j \Rightarrow \models \psi[\bar{b}, \bar{a}_i]\) then for \(i < j\) we have \((\exists \bar{x})(\varphi(\bar{x}, \bar{a}_j) \& \psi(\bar{x}, \bar{a}_i))\)

(c) in clause (b) we replace the conclusion: for every finite disjoint \(u, v \subseteq \omega\) we have \((\exists \bar{x}) \left( \bigwedge_{i \in u} \varphi(\bar{x}, \bar{a}_i) \& \bigwedge_{j \in v} \psi(\bar{x}, \bar{a}_j) \right)\).

Proof. \((c) \Rightarrow (b)\): Trivial.

\((c) \Rightarrow (b)\): Choose counterexample with \(|u \cup v|\) minimal, assume \(\alpha > \omega + |u \cup v|\).

\((a) \Rightarrow (b)\): Straight by the Definition of SOP\(_3\), etc.

\((b) \Rightarrow (a)\): Without loss of generality \(\langle \bar{a}_i : i < \alpha \rangle\) is an indiscernible sequence. Now we cannot find \(\bar{c}_0, \bar{c}_1, \bar{c}_2\) such that \(\bar{c}_0 \models \bar{c}_1, \bar{c}_1 \models \bar{c}_2, \bar{c}_2 \models \bar{c}_0\) realizes the same type as \((\bar{a}_0 \bar{b}_0) \langle (\bar{a}_1 \bar{b}_1)\rangle\), so SOP\(_3\) is exemplified. \(\square_{2.20}\)
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