The Path Integral Formulation of Gravitational Thermodynamics

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I. Introduction

In the early 1970’s Bekenstein argued that black holes have entropy [1] and Hawking showed that black holes have temperature [2]. A few years later these conclusions were apparently confirmed by Gibbons and Hawking [3] who used path integral methods to evaluate the black hole partition function. However, the original analysis of Gibbons and Hawking is incorrect. The difficulty arises because, in the language of ordinary thermodynamics, black holes are unstable and have a negative heat capacity. On the other hand, any system that can be described by a partition function is necessarily stable and has a positive heat capacity.

The path integral approach of Gibbons and Hawking is also somewhat mysterious regarding the origin of black hole entropy. A number of authors have speculated that the large entropy of a black hole should be associated with a large number of internal states, hidden by the horizon, that are consistent with the few external parameters that characterize the black hole. Others have argued that black hole entropy can be associated with a large number of possible initial states that can collapse to form a given black hole. The path integral analysis does not support either of these views in an obvious way. The key element in the path integral calculation is the action of a static Euclidean black hole geometry. A static Euclidean black hole has no horizon, no interior region, and no information about the collapse process. The path integral calculation of the partition function seems to hint at a topological explanation for black hole entropy.

The first objective of this article is to show that the black hole partition function can be placed on a firm logical foundation by enclosing the black hole in a spatially finite “box” or boundary. The presence of the box has the effect of stabilizing the black hole and yields a system with a positive heat capacity. Without a finite box, or some other mechanism for stabilizing the black hole (such as a negative cosmological constant [4]), the partition function does not exist. It is, nevertheless, common practice to fix boundary conditions at infinity and treat the zero–loop approximation to the path integral as if it were a real partition function. Such an identification is logically unfounded and, at any rate, cannot be extended beyond the zero–loop approximation.

The second objective of this article is to explore the origin of black hole entropy. This is accomplished through the construction of a path integral

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expression for the density matrix for the gravitational field, and through an
analysis of the connection between the density matrix and the black hole
density of states. (The density matrix for a black hole has been studied
previously in Ref. [5].) Our results suggest that black hole entropy can be
associated with an absence of certain “inner boundary information” for the
system.

We begin in Sec. 2 with a review of the motivations for enclosing the
black hole in a box. In Sec. 3 we analyze the action for the gravitational
field in the presence of a spatially finite boundary. Section 4 contains a
discussion of the relationship between Lorentzian and Euclidean notation for
the gravitational action. Formal path integral expressions for the density
matrix, density of states, and partition function are constructed in Sec. 5. In
Sec. 6 we use these results to calculate the density of states for a black hole
and we discuss the origin of black hole entropy.

II. Finite Boundaries

Hawking’s analysis [2] shows that the temperature of a black hole, as
measured at spatial infinity, equals the surface gravity divided by $2\pi$. For a
Schwarzschild black hole of mass $M$, it follows that the inverse temperature
$\beta$ at infinity is $8\pi M$. On the other hand, the standard thermodynamical
definition of inverse temperature [6] is $\beta = \partial S(E)/\partial E$, where $S(E)$ is the
entropy function and $E$ is the thermodynamical internal energy. If the mass
at infinity $M$ and the internal energy $E$ are identified, then the relationship
$\partial S(E)/\partial E = 8\pi M$ can be integrated to yield $S(E) = 4\pi E^2$ (plus an additive
constant). This result is in complete agreement with the prediction made by
Bekenstein [1] that a black hole has entropy proportional to the area of its
event horizon.

The black hole entropy $S(E) = 4\pi E^2$ is a convex function of $E$. This
is characteristic of an unstable thermodynamical system [6]. In this case,
the instability arises because energy and temperature are inversely related
for black holes. Thus, if fluctuations cause a black hole to absorb an extra
amount of thermal radiation from its environment, its mass will increase and
its temperature will decrease. The tendency then is for the cooler black hole
to absorb even more radiation from its hotter environment, causing the black
hole to grow without bound.
These results can be reformulated within the context of statistical mechanics. First consider the canonical partition function $Z(\beta)$ for an arbitrary system. In general $Z(\beta)$ is a sum over quantum states weighted by the Boltzmann factor $e^{-\beta E}$. If $\nu(E)$ is the density of quantum states with energy $E$, then

$$Z(\beta) = \int dE \nu(E) e^{-\beta E}. \quad (2.1)$$

The partition function can also be expressed as

$$Z(\beta) = \int dE e^{-I(E)}, \quad (2.2)$$

where the “action” is defined by $I(E) \equiv \beta E - S(E)$ and the entropy function $S(E)$ is the logarithm of the density of states: $S(E) \equiv \ln \nu(E)$. (Dimensionful constant factors can be included in the logarithm as necessary.) The integral over $E$ can be evaluated in a steepest descents approximation by expanding the action $I(E)$ to quadratic order around the stationary points $E^*(\beta)$, which satisfy

$$0 = \left. \frac{\partial I}{\partial E} \right|_{E^*} = \beta - \left. \frac{\partial S}{\partial E} \right|_{E^*}. \quad (2.3)$$

The Gaussian integral associated with a stationary point $E^*$ will converge if the second derivative of the action at $E^*$ is positive:

$$\left. \frac{\partial^2 I}{\partial E^2} \right|_{E^*} = - \left. \frac{\partial^2 S}{\partial E^2} \right|_{E^*} > 0. \quad (2.4)$$

This condition shows that the entropy $S(E)$ should be a concave function at the extremum $E^*$ in order for the Gaussian integral to converge.

A further significance of the condition (2.4) can be seen as follows. In the steepest descents approximation, the expectation value of energy is $\langle E \rangle \equiv -\partial \ln Z/\partial \beta \approx E^*$ and the heat capacity is $C \equiv \partial \langle E \rangle/\partial \beta^{-1} \approx \partial E^*/\partial \beta^{-1}$. By differentiating Eq. (2.3) with respect to $\beta$, we find

$$1 = \left. \frac{\partial E^*}{\partial \beta} \frac{\partial^2 S}{\partial E^2} \right|_{E^*}. \quad (2.5)$$

Therefore the heat capacity is given by

$$C \approx -\beta^2 \left( \left. \frac{\partial^2 S}{\partial E^2} \right|_{E^*} \right)^{-1} = \beta^2 \left( \left. \frac{\partial^2 I}{\partial E^2} \right|_{E^*} \right)^{-1}. \quad (2.6)$$
Thus, we see that in the steepest descents approximation the convergence of the integral for the canonical partition function is equivalent to the thermodynamical stability of the system (the concavity of the entropy), which in turn is equivalent to the positivity of the heat capacity.

For the black hole in particular, the entropy \( S(E) = 4\pi E^2 \) is not a concave function of the internal energy and the integral for the partition function diverges. This can be seen in the path integral formalism as well. In that case, the partition function is expressed as a functional integral over Euclidean metrics with periodic time, where the time period equals the inverse temperature at infinity \[3\]. The action is extremized by a Euclidean black hole metric, but the integration about that stationary point diverges. Formally, such a divergent Gaussian integral yields an imaginary result. The Euclidean black hole extremum is properly interpreted as an instanton that dominates the semiclassical evaluation of the rate of black hole nucleation from flat space at finite temperature \[7\].

The preceding discussion shows that the canonical partition function \( Z(\beta) \) characterizes the thermal properties of thermodynamically stable systems. For unstable systems \( Z(\beta) \) can give information concerning the rate of decay from a quasi–stable configuration (such as “hot flat space” in the black hole example), but it cannot be used to define thermodynamical properties such as expectation values, fluctuations, response functions, etc. Thus, before the partition function can be used as a probe of black hole thermodynamics, it is first necessary to stabilize the black hole. It was recognized in Ref. \[8\] that a black hole is rendered thermodynamically stable by enclosing it in a spatially finite “box” or boundary whose walls are maintained at a finite temperature. In this case the energy and the temperature at the boundary are not inversely related because of the blueshift effect of temperature in a stationary gravitational field \[9\]. If the black hole absorbs an extra amount of thermal radiation from its environment, its energy will increase thereby increasing the gravitational blueshift. Although the temperature of the black hole as measured at infinity is decreased by such a fluctuation, the temperature as measured at the boundary can increase due to the enhanced blueshift effect. The hotter black hole will then give up its excess energy and return to its stable equilibrium configuration.

The stabilizing effect of a finite box can be confirmed by the following
simple analysis. Consider a Schwarzschild black hole of mass $M$ surrounded by a spherical boundary of radius $R$. The inverse temperature at infinity is $8\pi M$, so the inverse temperature at the boundary is blueshifted to $\beta = 8\pi M \sqrt{1 - 2M/R}$ [9]. On the other hand, the inverse temperature is defined by $\beta = \partial S(E)/\partial E$, where again $S(E)$ is the entropy as a function of internal energy $E$. Now, the entropy of the black hole, at least in the zero–loop (“classical”) approximation, depends only on the black hole size and is unaffected by the presence, absence, or proximity [8] of a finite box. Thus, we have $S(E) = 4\pi M^2$ as before. By equating the two expressions for inverse temperature we find

$$8\pi M \sqrt{1 - 2M/R} = \frac{\partial (4\pi M^2)}{\partial E} . \quad (2.7)$$

In this case the energy $E$ and the mass $M$ as measured at infinity do not coincide. Equation (2.7) can be integrated to yield [8,10]

$$E = R - R \sqrt{1 - 2M/R} , \quad (2.8)$$

where, for convenience, the integration constant has been chosen so that $E \to M$ in the limit $R \to \infty$ with $M$ fixed. The significance of this expression can be seen by expanding $E$ in powers of $GM/R$ (where Newton’s constant $G$ is set to unity), with the result $E = M + M^2/(2R) + \cdots$. This shows that the internal energy inside the box equals the energy at infinity $M$ minus the binding energy $-M^2/(2R)$ of a shell of mass $M$ and radius $R$. The binding energy $-M^2/(2R)$ is the energy associated with the gravitational field outside the box [10]. Also observe that the internal energy takes values in the range $0 \leq E \leq R$.

By solving Eq. (2.8) for $M$ as a function of $E$, we obtain the entropy function

$$S(E) = 4\pi E^2 (1 - E/(2R))^2 . \quad (2.9)$$

First, note that the derivative $\partial S/\partial E$ is a concave function of $E$ that vanishes at the extreme values $E = 0$ and $E = R$. It follows that $\partial S/\partial E$ has a maximum $\beta_{cr}$. For $\beta > \beta_{cr}$ the equation $\beta = \partial S/\partial E$ has no solutions for $E$. On the other hand, for $\beta < \beta_{cr}$, there are two solutions $E_1$ and $E_2$. At the larger of these two solutions, say, $E_2$, the second derivative $\partial^2 S/\partial E^2$ is negative and the stability criterion (2.4) is satisfied. At the smaller of these
two solutions, $E_1$, the second derivative $\partial^2 S/\partial E^2$ is positive and the stability criterion (2.4) is violated. These considerations indicate that for a small box at low temperature ($\beta > \beta_{cr}$), the equilibrium configuration consists of flat space. For a large box at high temperature ($\beta < \beta_{cr}$), the stable equilibrium configuration consists of a large black hole with energy $E_2$. The unstable black hole with energy $E_1$ is an instanton that governs the nucleation of black holes from flat space. In the limit $R \to \infty$, the stable black hole configuration is lost and only the instanton solution survives [8,4].

In the following sections, we will develop the formal functional integral expressions for the density of states, the density matrix, and the canonical partition function for the gravitational field with a spatially finite boundary $B$. For the partition function in particular, the inverse temperature will be fixed as a boundary condition on $B$. The use of a finite boundary has the following very important consequence: Because gravitational fields cause temperature to redshift and blueshift, one must allow for the temperature to be fixed to different values at different points on $B$. In other words, gravitating systems in thermal equilibrium are not characterized by a single temperature but instead by a temperature field on the boundary of the system [11,12]. Correspondingly, the partition function is actually a functional $Z[\beta]$ of the inverse temperature field on $B$. For some problems, such as the Schwarzschild black hole in a spherical box discussed above, it is possible to choose the temperature to be a constant on $B$. In those cases $B$ coincides with an isothermal surface for the system. However, experience with the Kerr black hole shows that this must be viewed as a particular choice of boundary conditions, not the most general choice. What happens in the Kerr case [11] is that the angular velocity of the black hole with respect to observers who are at rest in the stationary time slices enters as a “chemical” potential conjugate to angular momentum. It turns out that the constant temperature surfaces and the constant angular velocity surfaces do not coincide. Therefore it is necessary to allow for some thermodynamical data, either the temperature or the chemical potential or both, to vary across the boundary. This conclusion might seem disturbing at first, since traditionally one of the purposes of thermodynamics has been to provide a characterization of systems in terms of only a few parameters. Note, however, that the thermodynamic parameters such as $\beta$ are only given on a two–surface $B$. No assertion is made about the values of $\beta$ inside $B$. Thus the generalized
thermodynamical description required is still far simpler than that of the full configuration of the system. The thermodynamical formalism that results from a generalization to non–constant thermodynamical data also has a number of compelling features. In particular, the thermodynamical data are brought into direct correspondence with the canonical boundary data, and in the process an intimate connection between thermodynamics and dynamics is revealed [11–13].

III. The Action

Assume that the spacetime manifold \( \mathcal{M} \) is topologically the product of a spacelike hypersurface and a real line interval, \( \Sigma \times I \). The boundary of \( \Sigma \) is denoted \( \partial \Sigma = B \). The spacetime metric is \( g_{\mu \nu} \) with associated curvature tensor \( R^\sigma_{\mu \nu \rho} \) and derivative operator \( \nabla_\mu \). The boundary of \( \mathcal{M} \), \( \partial \mathcal{M} \), consists of initial and final spacelike hypersurfaces \( t' \) and \( t'' \), respectively, and a timelike hypersurface \( ^3B = B \times I \) joining these. The induced metric on the spacelike hypersurfaces \( t' \) and \( t'' \) is denoted by \( h_{ij} \), and the induced metric on \( ^3B \) is denoted by \( \gamma_{ij} \).

Consider the gravitational action

\[
S^1 = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g} (R - 2\Lambda) + \frac{1}{\kappa} \int_{t'} d^3x \sqrt{h} K - \frac{1}{\kappa} \int_{^3B} d^3x \sqrt{-\gamma} \Theta. \tag{3.1}
\]

Here, \( \kappa \) is \( 8\pi \) times Newton’s constant and \( \Lambda \) is the cosmological constant. For simplicity, we have omitted matter contributions to the action. The symbol \( \int_{t'} d^3x \) denotes an integral over the boundary element \( t' \) minus an integral over the boundary element \( t'' \). The function \( K \) is the trace of the extrinsic curvature \( K_{ij} \) for the boundary elements \( t' \) and \( t'' \), defined with respect to the future pointing unit normal \( u^\mu \). Likewise, \( \Theta \) is the trace of the extrinsic curvature \( \Theta_{ij} \) of the boundary element \( ^3B \), defined with respect to the outward pointing unit normal \( n^\mu \).

Under variations of the metric the action (3.1) varies according to

\[
\delta S^1 = \text{(terms that vanish when the equations of motion hold)}
\]


\[\text{We use latin letters } i, j, k, \ldots \text{ as indices both for tensors on } ^3B \text{ and for tensors on a generic hypersurface } \Sigma. \text{ The two uses of such indices can be distinguished by the context in which they occur.}\]
\[ + \int_{t'}^{t''} d^3x P_{ij} \delta h_{ij} + \int_{3B} d^3x \pi_{ij} \delta \gamma_{ij} - \frac{1}{\kappa} \int_{B'} d^2x \sqrt{\sigma} \delta \alpha. \]  

The coefficient of \( \delta h_{ij} \) in the boundary terms at \( t' \) and \( t'' \) is the gravitational momentum

\[ P_{ij} = \frac{1}{2\kappa} \sqrt{h} (Kh^{ij} - K^{ij}). \]  

Likewise, the coefficient of \( \delta \gamma_{ij} \) in the boundary term at \( 3B \) is

\[ \pi_{ij} = -\frac{1}{2\kappa} \sqrt{-\gamma} (\Theta \gamma^{ij} - \Theta^{ij}). \]  

Equation (3.2) also includes integrals over the “corners” \( B'' = t'' \cap 3B \) and \( B' = t' \cap 3B \) whose integrands are proportional to the variation of the “angle” \( \alpha = \sinh^{-1}(u \cdot n) \) between the unit normals \( u^\mu \) of the hypersurfaces \( t'' \) and \( t' \) and the unit normal \( n^\mu \) of \( 3B \). The determinant of the two–metric on \( B' \) or \( B'' \) is denoted by \( \sigma \).

The action \( S^1 \) yields the classical equations of motion when the induced metric on \( 3B \), \( t' \), and \( t'' \) and the angle \( \alpha \) at \( B' \) and \( B'' \) are held fixed in the variational principle. In general, the functional \( S = S^1 - S^0 \), where \( S^0 \) is a functional of the fixed boundary data, also yields the classical equations of motion. For simplicity we define \( S^0 \) to be a functional of \( \gamma_{ij} \) only. The variation \( \delta S \) then differs from \( \delta S^1 \) of Eq. (3.2) only in that \( \pi_{ij} \) is replaced by \( \pi_{ij} - (\delta S^0 / \delta \gamma_{ij}) \).

Now foliate the boundary element \( 3B \) into two–dimensional surfaces \( B \) with induced two–metrics \( \sigma_{ab} \). The three–metric \( \gamma_{ij} \) can be written according to the familiar Arnowitt–Deser–Misner decomposition as

\[ \gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt), \]  

where \( N \) is the lapse function and \( V^a \) is the shift vector. The corresponding variation of \( \gamma_{ij} \) is [10]

\[ \delta \gamma_{ij} = (-2u_i u_j / N) \delta N + (-2\sigma_{a[i}u_{j]} / N) \delta V^a + (\sigma^a_{\delta_i} \sigma^b_{\delta_j}) \delta \sigma_{ab}, \]  

where \( u_i \) is the unit normal of the slices \( B \) and \( \sigma^i_a = \delta^i_a \) projects covariant tensors from \( 3B \) to the slices \( B \). With this result, the contribution to the
variation of $S$ from the boundary element $3B$ becomes
\[
\delta S|_{3B} = \int_{3B} d^3x (\pi^{ij} - (\delta S^0 / \delta \gamma_{ij})) \delta \gamma_{ij}
\]
\[
= \int_{3B} d^3x \sqrt{\sigma} \left( -\varepsilon \delta N + j_a \delta V^a + (N/2) s^{ab} \delta \sigma_{ab} \right),
\]
where the coefficients of the varied fields are defined by
\[
\varepsilon = \frac{2}{N \sqrt{\sigma}} u_i \pi^{ij} u_j + \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta N},
\]
\[
j_a = -\frac{2}{N \sqrt{\sigma}} \sigma_{ai} \pi^{ij} u_j - \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta V^a},
\]
\[
s^{ab} = \frac{2}{N \sqrt{\sigma}} \sigma^b_i \pi^{ij} \sigma^a_j - \frac{2}{N \sqrt{\sigma}} \frac{\delta S^0}{\delta \sigma_{ab}}.
\]
The leading terms in Eqs. (3.8)–(3.10) can be rewritten in terms of an extrinsic curvature $k_{ab}$ that is defined by parallel transporting the unit normal $n^\mu$ of $3B$ across a two–dimensional slice $B$. Thus, $k_{ab}$ is the extrinsic curvature of $B$ considered as the boundary $B = \partial \Sigma$ of a spacelike hypersurface $\Sigma$ whose unit normal $u^\mu$ is chosen orthogonal to $n^\mu$. (The case in which $u^\mu n_\mu \neq 0$ has been considered in Refs. [16,17], and will be treated in Ref. [15].) Also let $P^{ij}$ denote the gravitational momentum for the hypersurfaces $\Sigma$ that are “orthogonal” to $3B$, and let $a_\mu = u^\nu \nabla_\nu u_\mu$ denote the acceleration of the unit normal $u_\mu$ for this family of hypersurfaces. The resulting expressions are [10]
\[
\varepsilon = \frac{1}{\kappa} k - \varepsilon_0,
\]
\[
j_i = -\frac{2}{\sqrt{h}} \sigma_{ij} P^{jk} n_k - (j_0)_i,
\]
\[
s^{ab} = \frac{1}{\kappa} (k^{ab} + (n_\mu a^\mu - k) \sigma^{ab}) - (s_0)^{ab}.
\]
In these equations, we have expressed the terms proportional to the functional derivatives of $S^0$ as $\varepsilon_0$, $(j_0)_i$, and $(s_0)^{ab}$. We will assume that $S^0$ is a linear functional of the lapse $N$ and shift $V^a$, so that $\varepsilon_0$ and $(j_0)_i$ are functionals of the two–metric $\sigma_{ab}$ only [10]. Also note that the indices in Eq. (3.12) refer to the hypersurface $\Sigma$. Thus, $j_i = j_a \sigma^a_i$ where $\sigma^a_i = \delta^a_i$ projects tensors from $\Sigma$ to $B$, and $\sigma_{ij} = \sigma^a_i \sigma^a_j$. 

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From its definition through Eq. (3.7), \( \sqrt{\sigma \varepsilon} \) equals minus the time rate of change of the action, where changes in time are controlled by the lapse function \( N \) on \( \mathcal{B} \). Thus, \( \varepsilon \) is identified as an energy surface density for the system [10]. Likewise, we identify \( j_i \) as the momentum surface density and \( s^{ab} \) as the spatial stress [10]. The fact that the relevant stress–energy–momentum quantities live on the boundary of space is a consequence of the fact that the gravitational field on the boundary \( \mathcal{B} \) "knows" about the entire matter–energy content of space interior to \( \mathcal{B} \). Also note that the total energy \( E \) can be defined by integrating \( \sqrt{\sigma \varepsilon} \) over the two–surface \( \mathcal{B} \). The result is precisely the energy discussed in Sec. 2. In particular, if \( \mathcal{B} \) is a spherical boundary in a static time slice of a Schwarzschild black hole, the total energy \( E \) is given by Eq. (2.8). (This assumes \( S^0 \) is chosen such that \( \varepsilon_0 = -1/(4\pi R) \).)

## IV. Lorentzian and Euclidean Notation

The action \( S \) discussed in the previous section can be viewed as a functional of complex metrics. In particular \( S \) is the action for both Lorentzian and Euclidean metrics. We adopt the point of view that, strictly speaking, there is no distinction between the "Lorentzian action" and the "Euclidean action", or between the "Lorentzian equations of motion" and the "Euclidean equations of motion". Of course, a particular solution of the classical equations of motion can be Lorentzian or Euclidean. But for the action functional itself, the only distinction between Lorentzian and Euclidean is simply one of notation. In Sec. 3 we have used what might be called Lorentzian notation: the action \( S \) is defined with the convention that \( \exp(iS) \) is the phase in the path integral; the volume elements for \( \mathcal{M} \) and \( \mathcal{B} \) are written as \( \sqrt{-g} \) and \( \sqrt{-\gamma} \), respectively; the lapse function associated with the foliation of \( \mathcal{M} \) into hypersurfaces \( \Sigma \) is defined by \( N \equiv \sqrt{-1/g^{tt}} \). (The lapse function that appears in Eq. (3.5) is the restriction of this spacetime lapse to the boundary element \( \mathcal{B} \). It is defined by \( N \equiv \sqrt{-1/\gamma^{tt}} \).) Therefore \( S, \sqrt{-g}, \sqrt{-\gamma}, \) and \( N \) are real for Lorentzian metrics and imaginary for Euclidean metrics. We can re–express the action in Euclidean notation by making the following changes. Define a new action functional by \( I[g] \equiv -iS[g] \) so that the phase in the path integral is given by \( \exp(-I) \). Also rewrite the volume elements for \( \mathcal{M} \) and \( \mathcal{B} \) as \( \sqrt{g} \equiv i\sqrt{-g} \) and \( \sqrt{\gamma} \equiv i\sqrt{-\gamma} \), respectively, and define a
new lapse function by \( \bar{N} \equiv \sqrt{1/g^{tt}} \equiv i\sqrt{-1/g^{tt}} \equiv iN \).

It is also convenient to redefine the timelike unit normal of the slices \( \Sigma \).

In Lorentzian notation, the unit normal is defined by \( u_\mu \equiv -\bar{N}\delta_\mu^t \) and satisfies \( u \cdot u = -1 \). A new unit normal is defined by \( \bar{u}_\mu \equiv \bar{N}\delta_\mu^t \equiv iN\delta_\mu^t \equiv -iu_\mu \), and satisfies \( \bar{u} \cdot \bar{u} = +1 \). In some contexts it is also useful to define a new extrinsic curvature \( \bar{K}_{\mu\nu} \) in terms of the normal \( \bar{u}_\mu \). \( \bar{K}_{\mu\nu} \) is related to the old extrinsic curvature \( K_{\mu\nu} \) by \( \bar{K}_{\mu\nu} \equiv -\left( \delta_\sigma^\mu - \bar{u}_\sigma \bar{u}_\mu \right) \nabla_\sigma \bar{u}_\nu \equiv i(\delta_\sigma^\mu + u_\sigma u_\mu) \nabla_\sigma u_\nu \equiv -iK_{\mu\nu} \).

In turn, \( \bar{K}_{\mu\nu} \) can be used to define a new gravitational momentum \( \bar{P}^{ij} \) that is related to the momentum of Eq. (3.3) by \( \bar{P}^{ij} \equiv -iP^{ij} \). We will, however, continue to use the old notations \( K^{ij} \) and \( P^{ij} \).

In addition to the notational changes described above, we will also define a new shift vector by \( \bar{V}_i \equiv ig_{ti} \equiv iV_i \). This notation is a departure from the standard Euclidean notation in the sense that \( \bar{V}_i \) is imaginary for Euclidean metrics. One of the motivations for this change is the following. Apart from surface terms, the gravitational Hamiltonian is a linear combination of constraints built from the gravitational canonical data with the lapse function and shift vector as coefficients. In conjunction with the new notation \( \bar{N} \), \( \bar{V}_i \) for the lapse and shift, we choose to continue to denote the gravitational canonical data by \( h_{ij} \), \( P^{ij} \), as mentioned above. Then the constraints are unaffected by the change in notation, and the Hamiltonian can be written as \( H[N, V] \equiv -iH[\bar{N}, \bar{V}] \). The overall factor of \(-i\) that appears in this relationship is precisely what is required for the connection between the evolution operator \( e^{-iHt} \) in particle mechanics and the density operator \( (e^{-iH\beta}) \) in ordinary statistical mechanics). When the gravitational field is coupled to other gauge fields, such as Yang–Mills or electrodynamics, it is natural to redefine the Lagrange multipliers associated with the gauge constraints as well [11].

With our new notation, Eq. (3.1) becomes

\[
I^1 = -\frac{1}{2\kappa} \int_M \ d^4x \sqrt{g} (\mathcal{R} - 2\Lambda) - \frac{i}{\kappa} \int_{t' \to t} d^3x \sqrt{\bar{h}} K + \frac{1}{\kappa} \int_{3B} d^3x \sqrt{\gamma} \Theta
\]

\[ \tag{4.1} \]

\[ ^3\text{A bit of care is required in defining the square roots. For example, the appropriate definition of } \sqrt{-g} \text{ is obtained by taking the branch cut in the upper half complex plane, say, along the positive imaginary axis. Then the imaginary part of } \sqrt{-g} \text{ is negative. Correspondingly, the appropriate definition of } \sqrt{g} \text{ is obtained by taking the branch cut along the negative imaginary axis. Then the imaginary part of } \sqrt{g} \text{ is positive.} \]
and Eq. (3.2) becomes

$$\delta I^1 = \text{(terms that vanish when the equations of motion hold)}$$

$$-i \int_{t'}^{t''} d^3x P^{ij} \delta h_{ij} - i \int_{3B} d^3x \pi^{ij} \delta \gamma_{ij} + \frac{1}{\kappa} \int_{B'}^{B''} d^2x \sqrt{\sigma} \delta \bar{\alpha}. \quad (4.2)$$

Here, we have defined $\bar{\alpha} \equiv \cos^{-1}(\bar{u} \cdot n)$ so that $\delta \bar{\alpha} \equiv i \delta \alpha$. Thus, $\bar{\alpha}$ is the angle between the unit normals $\bar{u}$ and $n$ of the boundary elements $t''$ (or $t'$) and $3B$. The full action $I \equiv -iS$ differs from $I^1$ by a term $I^0 \equiv -iS^0$ that is a functional of the metric $\gamma_{ij}$ on $3B$. The contribution to $\delta I$ from the boundary element $3B$ is obtained from Eq. (3.7). Putting this together with Eq. (4.2) yields

$$\delta I = \text{(terms that vanish when the equations of motion hold)}$$

$$-i \int_{t'}^{t''} d^3x P^{ij} \delta h_{ij} + \frac{1}{\kappa} \int_{B'}^{B''} d^2x \sqrt{\sigma} \delta \bar{\alpha}$$

$$+ \int_{3B} d^3x \sqrt{\sigma} \left( \varepsilon \delta \bar{N} - j_a \delta \bar{V}^a - (\bar{N}/2)s^{ab} \delta \sigma_{ab} \right). \quad (4.3)$$

The stress–energy–momentum quantities $\varepsilon$, $j_i$, and $s^{ab}$ from Eqs. (3.11)–(3.13) are given by

$$\varepsilon = \frac{1}{\kappa} k - \frac{1}{\sqrt{\sigma}} \frac{\delta I^0}{\delta \bar{N}}, \quad (4.4)$$

$$j_i = -\frac{2}{\sqrt{\bar{N}}} \sigma_{ij} P^{jk} n_k + \frac{\sigma_a}{\sqrt{\sigma}} \frac{\delta I^0}{\delta \bar{V}^a}, \quad (4.5)$$

$$s^{ab} = \frac{1}{\kappa} (k^{ab} - k \sigma^{ab}) + \left( \frac{n^i \partial_i \bar{N}}{\kappa \bar{N}} \right) \sigma^{ab} + \frac{2}{\bar{N} \sqrt{\sigma}} \frac{\delta I^0}{\delta \sigma_{ab}}. \quad (4.6)$$

In Eq. (4.6) we have written the $n_\mu$ component of the acceleration of the unit normal $u_\mu$ as $n_\mu a^\mu \equiv (n^i \partial_i N)/N \equiv (n^i \partial_i \bar{N})/\bar{N}$. Also note that the action $I$ in canonical form is [10,12]

$$I = \int_{\mathcal{M}} d^4x \left( -iP^{ij} \dot{h}_{ij} + \bar{N} \mathcal{H} + \bar{V}^a \mathcal{H}_a \right) + \int_{3B} d^3x \sqrt{\sigma} \left( \bar{N} \varepsilon - \bar{V}^a j_a \right). \quad (4.7)$$

Here, $\mathcal{H}$ and $\mathcal{H}_a$ are the Hamiltonian and momentum constraints.

Since $I^0$ (like $S^0$) is functionally linear in the lapse $\bar{N}$ and shift $\bar{V}^a$, then $\varepsilon$ and $j_i$ are functions of the canonical variables only. Moreover, $s^{ab}$ depends on
the lapse and shift only through ratios such as \((\partial_t \bar{N})/\bar{N}\) and \(\bar{V}^a/\bar{N}\). These observations are important for the following reason. Consider the metric \(g_L\) for a stationary Lorentzian geometry, written in stationary coordinates. Given such a metric, we can compute the stress–energy–momentum quantities \(\varepsilon, j_i,\) and \(s^{ab}\) for the surface \(\mathcal{B}\) that is defined by taking a two–surface \(B\) within one of the stationary time slices and extending this surface along the stationary time direction. Now consider the generally complex metric \(g_C\) obtained by substituting \(t \rightarrow -it\) in the stationary Lorentzian metric \(g_L\). Under this transformation the lapse function \(\bar{N}\) and shift vector \(\bar{V}^i\) are changed from real to imaginary, and the canonical variables \(h_{ij}, P^{ij}\) are invariant. That is, the substitution \(t \rightarrow -it\) generates a complex metric \(g_C\) with \(\bar{N}, \bar{V}^i, h_{ij},\) and \(P^{ij}\) real. (The metric will be Euclidean if \(V^i = 0\).) If we now compute \(\varepsilon, j_i,\) and \(s^{ab}\) for the surface \(\mathcal{B}\) for the complex metric, the results will be the same as the results for the original Lorentzian metric. This observation is the key to understanding the relationship between the results of a functional integral over complex (or Euclidean) metrics and the real Lorentzian spacetime that it describes [12]. For example, suppose we compute the partition function as a functional integral and find that it is extremized by a certain stationary complex (or Euclidean) black hole metric \(g_C\). The partition function characterizes the system in terms of its thermal properties, such as its temperature, its expectation value for energy, etc. Can we conclude that the physical system described by this partition function is a real Lorentzian black hole simply because the path integral is extremized by a complex (or Euclidean) black hole? The answer is yes, precisely because the stress–energy–momentum for the stationary complex black hole \(g_C\) coincides with the stress–energy–momentum of the related stationary Lorentzian black hole \(g_L\). Thus, for example, when we compute the expectation value of energy from the partition function, it will coincide in the zero–loop approximation with the energy surface density \(\varepsilon\) of the complex black hole \(g_C\), which in turn characterizes the energy surface density of a real Lorentzian black hole \(g_L\). In this way we can conclude that the partition function indeed provides a description of the thermal properties of a physical black hole.

V. Functional Integrals

\(^4\)The boundary data for the partition function is chosen to be stationary on \(\mathcal{B}\), so the extremum of the action can be expected to be stationary as well.
A path integral constructed from an action $I$ is a functional of the quantities that are held fixed in the variational principle $\delta I = 0$. What are held fixed in the variational principle are the quantities that appear varied in the boundary terms of $\delta I$. The fixed boundary data for the action of Eq. (4.3) are the metric $h_{ij}$ on $t'$ and $t''$, the angle $\bar{\alpha}$ at the corners $B'$ and $B''$, and the lapse function $\bar{N}$, shift vector $\bar{V}^a$, and two–metric $\sigma_{ab}$ on $3B$. In the path integral, the gauge invariant part of the data on $3B$ corresponds to the inverse temperature $\beta$, chemical potential $\omega^a$, and two–geometry of the boundary $B$. These are grand canonical boundary conditions.

The inverse temperature is defined in terms of the boundary data on $3B$ by [12]

$$\beta = \int dt \, \bar{N}\bigg|_B .$$  

(5.1)

In geometrical terms, this is the proper distance between $t'$ and $t''$ as measured along the curves in $3B$ that are orthogonal to the slices $B$. Thus, $\beta$ is the $t$–coordinate invariant part of the lapse function $\bar{N}$. Note that $\beta$ is a function on the space boundary $B$, as anticipated in the discussion at the end of Sec. 2.

The chemical potential is defined in terms of the boundary data on $3B$ by [12]

$$\omega^a = \frac{\int dt \, \bar{V}^a\bigg|_B}{\int dt \, \bar{N}\bigg|_B} = \frac{\int dt \, V^a\bigg|_B}{\int dt \, N\bigg|_B} .$$

(5.2)

For a complex metric $g_C$, $\beta\omega^a$ has a geometrical interpretation as the proper distance along the $a$–coordinate line by which the spatial coordinates are shifted between $t'$ and $t''$. For a Lorentzian metric $g_L$, $\omega^a$ has the following interpretation. Recall that the shift vector $V^a$ on $3B$ is the velocity of the spatial coordinate system with respect to the observers who are at rest in the constant time slices $B$. It is convenient to tie the spatial coordinates to the local motion of the physical system [11,12], so that $V^a$ gives the local velocity of the system in terms of coordinate time. Thus, we see that the chemical potential $\omega^a$ is the proper velocity of the physical system as measured with respect to observers who are at rest at the system boundary $B$. In the case of axisymmetric boundary data, the only nonzero component of $\omega^a$ is the proper angular velocity of the system.
The functional integral constructed from the action $I$ is

$$\rho[h'', h'; \bar{\alpha}'', \bar{\alpha}'; \beta, \omega, \sigma] = \int Dg e^{-I[g]},$$  \hspace{1cm} (5.3)$$

where $h''$ and $h'$ denote the metrics on $t''$ and $t'$, and $\bar{\alpha}''$ and $\bar{\alpha}'$ denote the angles at the corners $B''$ and $B'$. We will consider this expression as a functional integral over the class of complex metrics for which $\bar{N}, \bar{V}^i$, and $h_{ij}$ are real.

The path integral (5.3) is the grand canonical density matrix for the gravitational field in a box $B$. The grand canonical partition function, denoted $Z[\beta, \omega, \sigma]$, is obtained by tracing over the initial and final configurations. In the path integral language, this amounts to performing a periodic identification so that the manifold topology becomes $\mathcal{M} = \Sigma \times S^1$. In addition, $\bar{\alpha}''$ and $\bar{\alpha}'$ should be chosen so that the total angle $\bar{\alpha}'' + \bar{\alpha}'$ equals $\pi$. This insures that the boundary $\partial\mathcal{M}$ is smooth when the initial and final hypersurfaces are joined together. Thus, the grand canonical partition function can be written as

$$Z[\beta, \omega] = \int Dh \rho[h, h; \bar{\alpha}'', \bar{\alpha}'; \beta, \omega, \sigma] \bigg|_{\bar{\alpha}'' + \bar{\alpha}' = \pi}. \hspace{1cm} (5.4)$$

The right hand side of this expression apparently depends on the angle difference $\bar{\alpha}'' - \bar{\alpha}'$. However, we expect that with the periodic identification, $\bar{\alpha}'' - \bar{\alpha}'$ is pure gauge and in a more detailed analysis would be absent from the path integral.

One can consider various density matrices and partition functions corresponding to different combinations of thermodynamical variables, where one variable is selected from each of the conjugate pairs. For example, in ordinary statistical mechanics the thermodynamically conjugate variables might consist of the inverse temperature $\beta$ and energy $E$, and the chemical potential $\mu$ and particle number $N$. Then the grand canonical partition function is $Z(\beta, \mu)$, the canonical partition function is $Z(\beta, N)$, and the microcanonical partition function (the density of states) is $\nu(E, N)$. These partition functions are related to one another by Laplace and inverse Laplace transforms, where each transform has the effect of switching the functional dependence from some thermodynamical variable (such as $\beta$) to its conjugate (such as $E$).
When the gravitational field is included in the description of the system, all of the thermodynamical data can be expressed as boundary data \cite{18,12}. In the path integral formalism, the effect of the Laplace and inverse Laplace transforms is simply to add or subtract certain boundary terms from the action. Thus, for example, the action $I_m$ appropriate for microcanonical boundary conditions just differs from the action $I$ by boundary terms:

$$I_m = I - \int_{3B} d^3x \sqrt{\sigma} \left( \bar{N} \varepsilon - \bar{V}^a j_a \right).$$  \hspace{1cm} (5.5)

From Eq. (4.3) we see that the contribution to $\delta I_m$ from the boundary element $3B$ is

$$\delta I_m|_{3B} = \int_{3B} d^3x \left( -\bar{N} \delta(\sqrt{\sigma} \varepsilon) + \bar{V}^a \delta(\sqrt{\sigma} j_a) - (\bar{N} \sqrt{\sigma}/2) s^{ab} \delta \sigma_{ab} \right).$$  \hspace{1cm} (5.6)

The path integral constructed from $I_m$ is a functional of the quantities that appear varied in the boundary terms of $\delta I_m$. Thus, the microcanonical density matrix is

$$\rho_m[h'', h'; \bar{\alpha}'', \bar{\alpha}'; \varepsilon, j, \sigma] = \int Dg e^{-I_m[g]},$$  \hspace{1cm} (5.7)

The trace of this density matrix is the density of states for the gravitational field in a box,

$$\nu[\varepsilon, j, \sigma] = \int Dh \rho_m[h, h; \bar{\alpha}'', \bar{\alpha}'; \varepsilon, j, \sigma]|_{\bar{\alpha}'' + \bar{\alpha}' = \pi}. \hspace{1cm} (5.8)$$

When combined, Eqs. (5.7) and (5.8) yield a path integral expression for the density of states. If the contours of integration for $\bar{N}$ and $\bar{V}^i$ are rotated from the real to the imaginary axis, then $\nu[\varepsilon, j, \sigma]$ is expressed as a functional integral over the class of metrics with $N$, $V^i$, and $h_{ij}$ real. In other words, $\nu[\varepsilon, j, \sigma]$ can be expressed as a path integral over real Lorentzian metrics \cite{12}.

\section{VI. Black Hole Entropy}

In the previous section the partition function $Z[\beta, \omega, \sigma]$ and the density of states $\nu[\varepsilon, j, \sigma]$ were constructed as functional integrals over the gravitational field on manifolds whose topologies are necessarily $\Sigma \times S^1$. This would seem to be an unavoidable consequence of deriving $Z[\beta, \omega, \sigma]$ and $\nu[\varepsilon, j, \sigma]$
from traces of density matrices because the density matrices $\rho$ and $\rho_m$ are defined in terms of functional integrals on manifolds $\mathcal{M}$ with product topology $\Sigma \times I$. However, experience has shown that for a black hole, the functional integrals for the partition function and density of states are extremized by a complex (or Euclidean) metric on the manifold $R^2 \times S^2$ [3,12]. Thus, one would expect the black hole contribution to the density of states to come from a path integral that is defined on a manifold with topology $R^2 \times S^2$. In this section we will show how the black hole density of states can be related to the microcanonical density matrix.

We begin by considering the manifold $\mathcal{M} = \Sigma \times I$, where $\Sigma$ is topologically a thick spherical shell ($S^2 \times I$). The boundary $\partial \Sigma = B$ consists of two disconnected surfaces, an inner sphere $B_i$ and an outer sphere $B_o$. The boundary element $3B$ consists of disconnected surfaces as well, $3B_i = B_i \times I$ and $3B_o = B_o \times I$. The results of the previous sections can be applied in constructing various density matrices for the gravitational field on $\Sigma$. We wish to consider the particular density matrix $\rho_*$ that is defined through the path integral by the action

$$I_* = I - \int_{3B_i} d^3x \sqrt{\sigma} \left( \bar{N}_\varepsilon - \bar{V}^a j_a \right) + \int_{3B_o} d^3x \sqrt{\sigma} \bar{N} \bar{s}^a_a / 2 .$$

(6.1)

$I_*$ differs from the action $I$ of Sec. 4 by boundary terms that are not the same for the two disconnected parts of $3B$. The contributions to the variation $\delta I_*$ from $3B_i$ and $3B_o$ are

$$\delta I_*|_{3B_i} = \int_{3B_i} d^3x \left( (\sqrt{\sigma} \varepsilon) \delta \bar{N} - (\sqrt{\sigma} j_a) \delta \bar{V}^a + \sigma_{ab} \delta (\bar{N} \sqrt{\sigma} s^a_b / 2) \right) ,$$

(6.2)

$$\delta I_*|_{3B_o} = \int_{3B_o} d^3x \left( -\bar{N} \delta (\sqrt{\sigma} \varepsilon) + \bar{V}^a \delta (\sqrt{\sigma} j_a) - (\bar{N} \sqrt{\sigma} s^a_b / 2) \delta \sigma_{ab} \right) .$$

(6.3)

Comparison with Eq. (5.6) shows that at the outer boundary element $3B_o$ we have chosen microcanonical boundary conditions. At the inner boundary element $3B_i$ we have chosen completely open boundary conditions. By “completely open” we mean that none of the traditional “conserved” quantities like energy, angular momentum, or area is fixed. Thus, all of the stress–energy–momentum quantities $\varepsilon$, $j_i$, and $s^{ab}$ are allowed to fluctuate on the inner boundary element while their conjugates, the inverse temperature, chemical potential, and spatial stress, are held fixed. The microcanonical boundary
conditions and the completely open boundary conditions are precisely opposite in this respect.

Observe that the spatial stress tensor \( s^{ab} \) can be split into its trace \( \theta = s^{ab} \sigma_{ab} \equiv s^a_a \), which is the sum of the normal components of stress ("pressures"), and the shear stresses \( \eta^{ab} = s^{ab} - \theta \sigma^{ab}/2 \). Then the contribution from \( 3B_i \) to \( \delta I_* \) becomes

\[
\delta I_* \big|_{3B_i} = \int_{3B_i} d^3x \left( (\sqrt{\sigma} \varepsilon) \delta \tilde{N} - (\sqrt{\sigma} j_a) \delta \tilde{V}^a - \sigma_{ab} \delta (\tilde{N} \sqrt{\sigma} \eta^{ab}) - \sqrt{\sigma} \delta (\tilde{N} \theta) \right).
\]

(6.4)

Thus, the fixed data on \( 3B_i \) consist of \( \tilde{N}, \tilde{V}^a, \tilde{N} \sqrt{\sigma} \eta^{ab}, \) and \( \tilde{N} \theta \), and the density matrix constructed as a path integral from \( I_* \) is

\[
\rho_*[h'', \alpha'', \alpha'; \varepsilon, j, \sigma; \tilde{N}, \tilde{V}, \tilde{N} \sqrt{\sigma} \eta, \tilde{N} \theta] = \int Dg e^{-I_*[g]}.
\]

(6.5)

Here, it is understood that the data \( \varepsilon, j_a, \) and \( \sigma_{ab} \) are fixed on the outer boundary element \( 3B_o \) and the data \( \tilde{N}, \tilde{V}^a, \sqrt{\sigma} \eta^{ab}, \) and \( \theta \) are fixed on the inner boundary element \( 3B_i \). Also, \( \alpha'' \) represents the angles at both disconnected parts of \( B'' \) and likewise for \( \alpha' \).

The black hole density of states \( \nu_*[\varepsilon, j, \sigma] \) is obtained from the trace of the density matrix \( \rho_* \) along with the following special choice of data on the inner boundary element \( 3B_i \):

\[
\begin{align*}
\tilde{N} &= 0, \\
\tilde{V}^a &= 0, \\
\tilde{N} \sqrt{\sigma} \eta^{ab} &= 0, \\
\tilde{N} \theta &= -\frac{4\pi}{\kappa (t'' - t')}.
\end{align*}
\]

(6.6) \( \quad \) (6.7) \( \quad \) (6.8) \( \quad \) (6.9)

In Eq. (6.9), \( (t'' - t') \) is just the range of coordinate time \( t \). In addition to the above conditions, the sum of angles \( \tilde{\alpha}'' + \tilde{\alpha}' \) must equal \( \pi \) so that the boundary is smooth. The resulting expression for the black hole density of states is

\[
\nu_*[\varepsilon, j, \sigma] = \int dh \rho_*[h, h; \tilde{\alpha}'', \tilde{\alpha}'; \varepsilon, j, \sigma; 0, 0, 0, -4\pi/\kappa(t'' - t')] \bigg|_{\tilde{\alpha}'' + \tilde{\alpha}' = \pi}.
\]

(6.10)
First we will discuss the geometrical meaning of the data (6.6)–(6.9) and show that Eq. (6.10) indeed gives the correct expression for black hole entropy. We will then discuss the physical implications of this result.

With the periodic identification, condition (6.6) fixes to zero the length of the circles on the inner boundary \( \partial^3 B_i = B_i \times S^1 \) that are orthogonal to the slices \( B_i \). That is, \( \vec{N} = 0 \) on the inner boundary causes the hole in the planes defined by the unit normals \( u_\mu \) and \( n_\mu \) to become closed. This condition therefore seals the opening in the manifold, changing the manifold topology to \( R^2 \times S^2 \). The sealed inner boundary of the manifold is called the bolt [19].

The condition (6.7) is not mandatory, but is a convenient choice. In terms of a physical black hole metric, this condition fixes the spatial coordinate system to be co–rotating with the black hole [11,12]. The chemical potential is then related to the shift vector on the outer boundary element \( B_o \) according to the relationship (5.2). Without the condition (6.7), the chemical potential and the shift vector on \( B_o \) are not related in the simple way (5.2). Condition (6.8) is chosen to be consistent with condition (6.6). Finally, condition (6.9) ensures that the geometry is smooth at the bolt. This can be seen by writing Eq. (6.9) in detail. From Eq. (4.6) we have

\[
- \frac{\vec{N}}{\kappa} k + \frac{2}{\kappa} (n^i \partial_i \vec{N}) + \frac{2}{\sqrt{\sigma}} \sigma_{ab} \delta I^0_{ab} = - \frac{4\pi}{\kappa (t' - t)} ,
\]

which, in light of conditions (6.6) and (6.7), becomes \((t'' - t')(n^i \partial_i \vec{N}) = -2\pi\). Now, \((t'' - t')(n^i \partial_i \vec{N})\) is just minus the rate of change of proper circumference with respect to proper radius for the circles on the inner boundary \( \partial^3 B_i \) that are orthogonal to the slices \( B_i \). (The minus sign appears because \( n^i \) is the unit normal to \( B_i \) that points outward from \( \Sigma \).) Thus, condition (6.9) fixes the ratio of circumference to radius to \( 2\pi \). This ensures that the spacetime four–geometry is smooth as the inner boundary is sealed.

The correctness of the prescription (6.5), (6.10) can be confirmed by considering the evaluation of the functional integral for \( \nu_\epsilon [\varepsilon, j, \sigma] \), where the data \( \varepsilon, j_a, \sigma_{ab} \) on the outer boundary \( \partial^3 B_o \) correspond to a stationary black hole. That is, let \( \varepsilon, j_a, \sigma_{ab} \) be the stress–energy–momentum for a topologically spherical two–surface \( B_o \) within a time slice of a stationary Lorentzian black hole solution \( g_L \) of the Einstein equations. In the path integral for \( \nu_\epsilon [\varepsilon, j, \sigma] \), fix this data on each slice \( B_o \) of the outer boundary \( \partial^3 B_o \). The path integral can be evaluated semiclassically by searching for metrics that
extremize the action $I_*$ and satisfy the conditions at both $^3B_o$ and $^3B_i$. One such metric will be the complex metric $g_c$ that is obtained by substituting $t \to -it$ in the Lorentzian black hole solution $g_L$. This complex metric will satisfy the boundary conditions at $^3B_o$ because, as discussed at the end of Sec. 4, the substitution $t \to -it$ does not affect the stress–energy–momentum quantities $\varepsilon, j_a$, and $\sigma_{ab}$. The complex metric also will satisfy the conditions at $^3B_i$, where $B_i$ coincides with the intersection of the stationary time slices and the black hole horizon for the Lorentzian metric. This follows from the observation that the lapse function corresponding to the natural stationary time–slicing vanishes on the horizon of a stationary black hole. Likewise, the corresponding shift vector vanishes on the horizon for co–rotating spatial coordinates. Also, with an appropriate choice of period $t'' - t'$ for coordinate time, the complex metric will describe a smooth geometry at the bolt. (A single choice of period suffices for all points on $B_i$, because $n^i \partial_i \bar{N}$ is proportional to the surface gravity of the stationary black hole and is therefore constant across $B_i$.)

In the zero–loop approximation to the path integral, the density of states becomes

$$\nu_*[\varepsilon, j, \sigma] \approx e^{-I_*[g_c]}.$$  

From Eqs. (4.7) and (6.1), we see that the action $I_*$ written in canonical form is

$$I_* = \int_{\mathcal{M}} d^4x \left( -iP^{ij} \dot{h}_{ij} + \bar{N} \mathcal{H} + \bar{V}^{i} \mathcal{H}_i \right) + \int_{^3B_i} d^3x \sqrt{\sigma} \left( \bar{N} \varepsilon - \bar{V}^a j_a + \bar{N} \theta / 2 \right).$$  

In evaluating this action at the solution $g_c$, the term proportional to $\dot{h}_{ij}$ vanishes because of stationarity and the constraints $\mathcal{H}$ and $\mathcal{H}_i$ vanish because the equations of motion are satisfied. The remaining terms in $I_*$ can be found from the boundary conditions (6.6)–(6.9), which yield

$$I_*[g_c] = -\frac{2\pi}{\kappa} \int_{^3B_i} d^2x \sqrt{\sigma}.$$  

The integral that remains is just the area of the bolt, or the area $A_H$ of the
black hole event horizon. Thus, in the approximation (6.12), the entropy is

\[ S[\varepsilon, j, \sigma] = \ln \nu_\ast[\varepsilon, j, \sigma] \approx \frac{2\pi}{\kappa} A_H. \tag{6.15} \]

With \( \kappa = 8\pi \), this is the standard result \( S = A_H/4 \) for the black hole entropy.

It might appear that the conditions (6.6)–(6.9) have been artificially contrived in an effort to obtain the result (6.15). On the contrary, these conditions are “no boundary” conditions in the sense that they are precisely the conditions needed to seal the opening \( 3B_i \) in the manifold \( \mathcal{M} = \Sigma \times S^1 \), and convert the manifold topology to \( R^2 \times S^2 \). Thus, with these conditions there is no inner boundary in the spacetime four–geometry. Equations (6.6)–(6.9) are more properly called regularity conditions, rather than boundary conditions. Accordingly, the path integral defined through Eqs. (6.5) and (6.10) is not to be viewed as a functional of any inner boundary data. (We took a slightly different viewpoint in Refs. [11,12] which will be discussed in detail in [15].) This is why the argument of the density of states (6.10) includes only the conditions on the outer boundary \( 3B_o \).

For a general choice of data on \( 3B_i \), the object defined by the trace of the density matrix (6.5) would have to be treated as a hybrid partition function with completely open boundary conditions on \( 3B_i \) and microcanonical boundary conditions on \( 3B_o \). That is, for a general choice of data \( \bar{N}, \bar{V}^a, \sqrt{\sigma} \eta^{ab}, \theta \) on \( 3B_i \), the trace of the density matrix \( \rho_\ast \) yields

\[
Z_\ast[\varepsilon, j, \sigma; \bar{N}, \bar{V}^a, \sqrt{\sigma} \eta^{ab}, \theta] = \int dh \rho_\ast[h, h; \bar{\alpha}''', \bar{\alpha}'; \varepsilon, j, \sigma; \bar{N}, \bar{V}^a, \sqrt{\sigma} \eta^{ab}, \theta] \bigg|_{\bar{\alpha}''' + \bar{\alpha}' = \pi}. \tag{6.16}
\]

In order to compute the density of states in this case, we would first perform a series of inverse Laplace transforms to change the data on \( 3B_i \) from completely open conditions to microcanonical conditions. The inverse Laplace transforms would have the effect of canceling the boundary terms in the action (6.13) and changing the action to the microcanonical action \( I_m \) of Eq. (5.5). This action is zero for any stationary solution, including stationary black holes. So in the zero–loop approximation, the entropy so computed

\footnote{The corresponding result for an arbitrary diffeomorphism invariant theory of gravitational and matter fields has been derived in Ref. [20].}
would be zero. This holds true even if the action is extremized by a black hole solution.

The difference between the density of states as derived from the partition function $Z_*$ and the black hole density of states $\nu_*$ is precisely in the presence or absence of an inner boundary $^3B_i$ and its associated data. Thus, one is led to the view that the black hole entropy derived from $\nu_*$ is associated with a lack or absence of boundary information. The entropy derived from $\nu_*$ is greater than the entropy derived from $Z_*$ because in the case of $\nu_*$ there is less spacetime boundary data information: less information, more entropy.

We should remark that our present interpretation does not support the view [5] that the entropy derived from $\nu_*$ arises from an integration over interior boundary data. In fact, by integrating over interior data in the density of states derived from $Z_*$, one merely recovers the partition function $Z_*$. The black hole density of states $\nu_*$ should be viewed as the result of a path integral over certain gravitational fields (spacetimes) with no inner boundaries whatsoever, not as the result of an integration over inner boundary conditions.

The above conclusions will be developed in greater detail in forthcoming publications [15].

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