Relativistic Gamow Vectors

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Gamow vectors in non-relativistic quantum mechanics are generalized eigenvectors (kets) of self-adjoint Hamiltonians with complex eigenvalues \(E_R \mp i\Gamma/2\). Like the Dirac kets, they are mathematically well defined in the Rigged Hilbert Space \(\Phi \subset \mathcal{H} \subset \Phi^\times\). Gamow vectors are derived from the resonance poles of the S-matrix \(S_j(z)\) at \(z = z_R = E_R \mp i\Gamma/2\). They have a Breit-Wigner energy distribution, an exponential decay law, and are members of a basis vector expansion whose truncation gives the finite dimensional effective theories with a complex Hamiltonian matrix. They also have an asymmetric time evolution described by a semigroup generated by the Hamiltonian, which expresses a fundamental quantum mechanical arrow of time. These Gamow kets are generalized to relativistic Gamow vectors by extrapolating from the Galilei group to the Poincaré group. This leads to semigroup representations of the Poincaré group which are characterized by spin \(j\) and complex invariant mass square \(s = s_R = (M_R - \frac{1}{2}\Gamma)^2\). In these non-unitary representations \((j, s_R)\) the Lorentz subgroup is unitarily represented and the four-momenta are “minimally complex” in the sense that the four-velocity \(p_\mu = \frac{p_R}{\sqrt{\Gamma}}\) is real. The relativistic Gamow vectors have all the properties listed above for the non-relativistic Gamow vectors and are therefore ideally suited to describe relativistic resonances and quasistable particles with resonance mass \(M_R\) and lifetime \(\Gamma\).

I. INTRODUCTION

Following Wigner \[1\], an elementary relativistic quantum system, an elementary particle with mass \(m\) and spin \(j\) is in the mathematical theory described by the space of an unitary irreducible representation (UIR) of the Poincaré group \(\mathcal{P}\). From these UIR, the relativistic quantum fields are constructed \[2\]. More complicated relativistic systems are described by direct sums of UIR (for “towers” of elementary particles) or by direct products of UIR (for combination of two or more elementary particles) \[2\]. A direct product of UIR may be decomposed into a continuous direct sum (integral) of irreducible representations \[3,4\]. The UIR are characterized by three invariants \((m^2, j, \text{sign}(p_0))\), where \(j\) represents the spin and the real number \(m\) represents the mass of elementary particle (we restrict ourselves here to \(\text{sign}(p_0) = +1\)).

The UIR of the Poincaré group \(\mathcal{P}\) describe stable elementary particles (stationary systems). There is only a very small number of truly stable particles in nature and most relativistic (and also non-relativistic) quantum systems are decaying states (weakly or electromagnetically), or hadron resonances with an exponential decay law and finite lifetime \(\tau_R = \frac{1}{\Gamma}\) (in their rest frame) and a Breit-Wigner energy (at rest) distribution. The UIR of \(\mathcal{P}\) therefore describe only a few of the relativistic quantum systems in nature; for the vast majority of elementary particles listed in the Particle Physics Booklet \[5\], the UIR provide only a more or less approximate description. We want to present here a special class of (non-unitary) semi-group representations of \(\mathcal{P}\) which describe quasistable relativistic particles.

Phenomenologically, one always takes the point of view that resonances are autonomous quantum physical entities, and decaying particles are no less fundamental than stable particles. Stable particles are not qualitatively different from quasistable particles, but only quantitatively by a zero (or very small) value of \(\Gamma\). Therefore both stable and quasistable states should be described on the same footing. This has been accomplished in the non-relativistic case, where a decaying state is described by a generalized eigenvector of the (self adjoint, semi-bounded) Hamiltonian with a complex eigenvalue \(z_R = E_R - i\Gamma/2\) \[6\] called Gamow vectors. The stable state vectors with real eigenvalues \(E_S\) are the special case with \(\Gamma = 0\), i.e., \(z_R = E_R - i\Gamma/2 \rightarrow E_S\).

In the standard Hilbert space formulation of quantum mechanics, such vectors do not exist and one had to employ the Rigged Hilbert Space (RHS) formulation of quantum mechanics. Dirac’s bras and kets are, mathematically, generalized eigenvectors with real eigenvalues, and Gamow vectors are generalizations of Dirac kets. They are described by kets \(\psi^G = |z_R^-\rangle \sqrt{2\pi\Gamma}\) with complex eigenvalue \(z_R = E_R - i\Gamma/2\), where \(E_R\) and \(\Gamma\) are respectively interpreted as resonance energy and width. Like Dirac kets, the Gamow kets are functionals of a Rigged Hilbert Space:

\[
\Phi_+ \subset H \subset \Phi^\times_+ : \quad \psi^G \in \Phi^\times_+ ,
\]
and the mathematical meaning of the eigenvalue equation $H^\times |z_R^\times\rangle = (E_R - i\Gamma/2)|z_R^\times\rangle$ is:

$$\langle H\psi|z_R^\times\rangle \equiv \langle \psi|H^\times|z_R^\times\rangle = z_R\langle \psi|z_R^\times\rangle \text{ for all } \psi \in \Phi_+.$$  \hspace{1cm} (2)

The conjugate operator $H^\times$ of the Hamiltonian $H$ is uniquely defined by the first equality in (2), as the extension of the Hilbert space adjoint operator $H^\dagger$ to the space of functionals $\Phi_+^\times$ on the space $\mathcal{H}$, the operators $H^\times$ and $H^\dagger$ are the same.

The non-relativistic Gamow vectors have the following properties:

1. They have an asymmetric (i.e., $t \geq 0$ only) time evolution and obey then an exponential law:

$$\psi_G^G(t) = e_{+}^{-iH^\times t}|E_R - i\Gamma/2^-\rangle = e^{-iE_R t}e^{-t^2/2}|E_R - i\Gamma/2^-\rangle, \text{ only for } t \geq 0.$$ \hspace{1cm} (3)

2. There is another Gamow vector $\tilde{\psi}_G^G = |E_R + i\Gamma/2^+\rangle \in \Phi_+^\times$, and another semigroup $e^{-iH^\times t}$ for $t \leq 0$ in another RHS $\Phi_- \subset \mathcal{H} \subset \Phi_+^\times$ (with the same $\mathcal{H}$) with the asymmetric evolution

$$\tilde{\psi}_G^G(t) = e_{-}^{-iH^\times t}|E_R + i\Gamma/2^+\rangle = e^{-iE_R t}e^{-t^2/2}|E_R + i\Gamma/2^+\rangle, \text{ only for } t \leq 0.$$ \hspace{1cm} (4)

3. The Gamow vectors have a Breit-Wigner energy distribution

$$\langle z \mathcal{E} |\psi_G^G\rangle = i\sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_R - i\Gamma/2}, \text{ for } -\infty < E < \infty,$$ \hspace{1cm} (5)

where $-\infty < E < \infty$ means that it extends to $-\infty$ on the second sheet of the S-matrix (whereas the standard Breit-Wigner extends to the threshold $E = 0$).

We want to present here a generalization of these non-relativistic Gamow vectors to the relativistic case.

In the non-relativistic case the inclusion of the degeneracy quantum numbers of energy, i.e., the extension of the Dirac-Lippmann-Schwinger kets

$$|E^\pm\rangle = |E\rangle + \frac{1}{E - H \mp i0}V|E\rangle = \Omega^\pm|E\rangle,$$

$$H|E^\pm\rangle = E|E^\pm\rangle; \text{ (}H - V)|E\rangle = E|E\rangle$$ \hspace{1cm} (6)

to the basis of the whole Galilei group is trivial.

For the two particle scattering states (direct product of two irreducible representations of the Galilei group $\mathfrak{g}$) one uses eigenvectors of angular momentum $(jj)$ for the relative motion and total momentum $p$ for the center of mass motion. Thus

$$|E^\text{tot}jj(l, s) \pm\rangle = |p\rangle \otimes |Ejjj_s \pm\rangle$$ \hspace{1cm} (7)

where $E^\text{tot} = \frac{p^2}{2m} + E$ (the Hamiltonian in (3) is $H = H^\text{tot} - \frac{p^2}{2m}$). The center-of-mass motion is usually separated by transforming to the center-of-mass frame, and there one uses in (3)

$$|p = 0\rangle \otimes |Ejjj_s \pm\rangle = |E, p = 0, jjj_s \pm\rangle = |E \pm\rangle.$$ \hspace{1cm} (8)

The generalized eigenvectors

$$|Ejjj_s \pm\rangle \in \Phi_+^\times \supset \mathcal{H} \supset \Phi_\pm$$ \hspace{1cm} (9)

\footnote{For (essentially) self adjoint $H$, $H^\dagger$ is equal to (the closure of) $H$; but we shall use the definition (2) also for unitary operators $\mathcal{U}$ where $\mathcal{U}^\times$ is the extension of $\mathcal{U}^\dagger$, but not of $\mathcal{U}$.}
with
\[ H^x |E_{jj3}^\pm\rangle = E |E_{jj3}^\pm\rangle, \quad 0 \leq E < \infty, \] (10)
where \( E \) runs along the cut on the positive real axis of the 1-st sheet of the \( j \)-th partial S-matrix, are the scattering states. The proper eigenvectors of \(|\rangle\rangle\) with \( E = -|E_n| \) at the poles on the negative real axis on the 1-st sheet are the bound states \(|E_{n,jj3}\rangle\rangle \in \Phi\). By the Galilei transformation one can transform these vectors to arbitrary momentum \( p \); \( E \) and \( p \) are not intermingled by Galilei transformations.

To obtain the non-relativistic Gamow kets one analytically continues the Dirac-Lippmann-Schwinger ket \(|\rangle\rangle\) into the second sheet of the \( j \)-th partial S-matrix to the position of the resonance pole \(|z_R = E_R - i\Gamma/2, jj3\rangle\rangle\) and obtains the following representation \([\rangle\rangle\rangle\):
\[ |z_R = E_R - i\Gamma/2, jj3\rangle\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE |E_{jj3}^-\rangle\rangle \frac{1}{E - z_R}. \] (11)
A Galilei transformation can boost this Gamow ket to any real momentum \( p \)
\[ |p, z_R, jj3\rangle\rangle = U(p)|0\rangle \otimes |z_R jj3\rangle\rangle. \]
However, complex momenta cannot be obtained in this way since the Galilei transformations commute with the intrinsic energy operator \( H \).

In the relativistic case the Lorentz transformation – in particular Lorentz boosts – intermingle energy \( E^{\text{tot}} = p^0 \) and momenta \( p^m, m = 1, 2, 3 \). Thus complex energy or complex mass also leads to complex momenta. This has led in the past to consider complicated complex momentum representations of the Poincaré group \( \mathcal{P} \). To restrict this unwieldy set of Poincaré group representations we will consider a special class of “minimally complex” irreducible representations of \( \mathcal{P} \) to describe relativistic resonances and decaying elementary particles. Our construction will also lead to complex momenta \( \hat{p}_\mu \equiv \frac{p_\mu}{m} \) remain real. This construction was motivated by a remark of D. Zwanziger \([8]\) and is based on the fact that the 4-velocities \( \hat{\mathbf{v}}_\mu \) furnish as valid a basis for the representation space of \( \mathcal{P} \) as the usual Wigner basis of momentum eigenvectors \(|p_{j3}(m, j)\rangle\rangle\). This means every state of an UIR \((m, j), (\phi \in \mathcal{H}(m, j) \subset \Phi^x, \) where \( \Phi \) denotes the space of well-behaved vectors and \( \Phi^x \) the space of kets for the Hilbert space \( \mathcal{H}(m, j) \) of an UIR), can be written according to Dirac’s basis vector decomposition as
\[ \phi = \sum_{j3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{p}, j3\rangle\langle j3, \hat{p}|\phi \] (12)
where we have chosen the invariant measure
\[ d\mu(\hat{p}) = \frac{d^3\hat{p}}{2\hat{p}^0} = \frac{1}{m^2} \frac{d^3p}{2E(p)}, \quad \hat{p}^0 = \sqrt{1 + \hat{p}^2}. \] (13)
As a consequence of (13), the \( \delta \)-function normalization of these velocity-basis vectors is
\[ \langle \xi, \hat{p} | \hat{p}', \xi' \rangle = 2\hat{p}^0 \delta^3(\hat{p} - \hat{p}') \delta_{\xi\xi'} = 2\hat{p}^0 m^2 \delta^3(p - p') \delta_{\xi\xi'}. \] (14)
Here, |\( \hat{p}, j3\rangle\rangle \in \Phi^x \) are the eigenkets of the 4-velocity operator \( \hat{P}_\mu = P_\mu M^{-1} \) and \( \phi_{j3}(\hat{p}) = \langle j3, \hat{p}|\phi \) represents the 4-velocity distribution of the vector \( \phi \). The 4-velocity eigenvectors are often more useful for physical reasoning, because 4-velocities seem to fulfill rather good approximation “velocity super-selection rules” which the momenta do not \([9]\).

The relativistic Gamow vectors will therefore be defined, not as momentum eigenvectors, but as 4-velocity eigenvectors in the direct product space of UIR spaces for the decay products of the resonances \( R \). We want to obtain the relativistic Gamow vectors from the pole term of the relativistic S-matrix in complete analogy to the way the non-relativistic Gamow vectors were obtained \([6]\). In the absence of a vector space description of a resonance, we shall also in the relativistic theory define the unstable particle by the pole of the analytically continued partial S-matrix with angular momentum \( j = j_R \) at the value \( s = s_R \equiv (M_R - i\Gamma/2)^2 \) of the invariant mass square variable (Mandelstam variable) \( s = (p_1 + p_2 + \cdots)^2 = E_R^2 - p_R^2 \), where \( p_1, p_2, \ldots \) are the momenta of the decay products of \( R \) \([10]\). This means that the mass \( M_R \) and lifetime \( \hbar/T_R \) or the complex invariant mass \( w_R = (M_R - i\Gamma/2) = \sqrt{s_R} \), in addition to spin \( j_R \), are the intrinsic properties that define a quasistable relativistic particle \([6]\).

\( ^2\)Conventionally and equivalently one often writes
In order to make the analytic continuation of the partial S-matrix with angular momentum \( j \), we need the angular momentum basis vectors

\[
|\hat{p}j_j(\omega_j)\rangle = \int \frac{d^4\hat{p}_1}{2E_1} \frac{d^4\hat{p}_2}{2E_2} |\hat{p}_1\hat{p}_2|m_1m_2\rangle \langle \hat{p}_1\hat{p}_2|m_1m_2|\hat{p}j_j(\omega_j)\rangle
\]

for any \((m_1 + m_2)^2 \leq \omega^2 < \infty\) \( j = 0, 1, \ldots \)

in the direct product space of the decay products of the resonance \( R \)

\[
\mathcal{H} \equiv \mathcal{H}(m_1, 0) \otimes \mathcal{H}(m_2, 0) = \int_0^\infty ds \sum_{j=0}^\infty \oplus \mathcal{H}(s, j),
\]

where \( s = \omega^2 \), the Mandelstam variable defined above.

For simplicity, we have assumed here that there are two decay products, \( R \rightarrow \pi_1 + \pi_2 \) with spin zero, described by the irreducible representation spaces \( \mathcal{H}^{\pi_i}(m_i, s_i = 0) \) of the Poincaré group \( \mathcal{P} \).

The kets \( |\hat{p}j_j(\omega_j)\rangle \) are eigenvectors of the 4-velocity operators

\[
\hat{P}_\mu = (P^{\mu}_1 + P^{\mu}_2)M^{-1}, \quad M^2 = (P^{\mu}_1 + P^{\mu}_2)(P^{\mu}_1 + P^{\mu}_2)
\]

with eigenvalues

\[
\hat{p}^\mu = \left( \hat{E} = \frac{E}{p} = \sqrt{1 + \hat{p}^2}, \quad \text{and} \quad w^2 = s. \right)
\]

In here \( \hat{P}_\mu \) are the 4-velocity operators in the one particle spaces \( \mathcal{H}^{\pi_i}(m_i, s_i) \) with eigenvalues \( \hat{p}^\mu = \frac{p^\mu}{m_i} \).

To obtain the Clebsch-Gordan coefficients \( \langle \hat{p}_1\hat{p}_2|m_1m_2|\hat{p}j_j(\omega_j)\rangle \) in \( \mathcal{H} \), one follows the same procedure as given in the classic papers \( [11,3,4] \) for the Clebsch-Gordan coefficients in the Wigner (momentum) basis. This has been done in \( [12] \). The result is

\[
\langle \hat{p}_1\hat{p}_2|m_1m_2|\hat{p}j_j(\omega_j)\rangle = 2\hat{E}(\hat{p})\delta^3(p - r)\delta(w - \epsilon)Y_{jjj}(\epsilon)\mu_j(w^2, m_1^2, m_2^2)
\]

with \( \epsilon^2 = r^2 = (p_1 + p_2)^2, \quad r = p_1 + p_2, \)

where \( \mu_j(w^2, m_1^2, m_2^2) \) is a function that fixes the \( \delta \)-function “normalization” of \( |\hat{p}j_j(\omega_j)\rangle \). The unit vector \( e \) in \( (19) \) is chosen to be in the c.m. frame the direction of \( \hat{p}_1^{\text{cm}} = -\frac{m_2}{m_1}\hat{p}_2 \).

The normalization of the basis vectors \( (13) \) is chosen to be

\[
\langle \hat{p}'j'_j(w'j')|\hat{p}j_j(\omega_j)\rangle = 2\hat{E}(\hat{p})\delta(p' - \hat{p})\delta_{j'j}\delta(s - s')
\]

where \( \hat{E}(\hat{p}) = \hat{p} = \frac{1}{w}\sqrt{w^2 + \hat{p}^2} \equiv \frac{1}{w}E(p, w). \)

This determines the weight function \( \mu_j(w^2, m_1^2, m_2^2) \) to

\[
|\mu_j(w^2, m_1m_2)|^2 = \frac{2m_1^2m_2^2w^2}{\sqrt{\lambda(1, (\frac{m_1}{w})^2, (\frac{m_2}{w})^2)}}
\]

\[

s_R \equiv M_R^2 - iM_R\Gamma_R = M_R^2 \left(1 - \frac{1}{4} \left(\frac{\Gamma_R}{M_R}\right)^2\right) - iM_R\Gamma_R
\]

and calls \( M_\rho = M_R\sqrt{1 - \frac{1}{4} \left(\frac{\Gamma_R}{M_R}\right)^2} \) the resonance mass and \( \Gamma_\rho = \Gamma_R \left(1 - \frac{1}{4} \left(\frac{\Gamma_R}{M_R}\right)^2\right)^{-1/2} \) its width. For the \( \rho \) meson \( \left(\frac{\Gamma_R}{M_R}\right)^2 \approx 0.03 \) and for most other resonances it is an order of magnitude or more smaller than this.

\[^3\text{Though our discussions apply with obvious modifications to the general case of} \]

\[1 + 2 + 3 + \cdots \rightarrow R_i \rightarrow 1' + 2' + 3' + \cdots,\]

these generalizations lead to enormously more complicated equations.
where \( \lambda \) is defined by \[ 11 \]

\[
\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac).
\]

The basis vectors \[ 15 \] are the eigenvectors of the free Hamiltonian \( H_0 = P_0^+ + P_0^2 \)

\[
H_0^\pm |\hat{p}j_3(wj)\rangle = E|\hat{p}j_3(wj)\rangle, \quad E = w\sqrt{1 + \hat{p}^2}.
\]

The Dirac-Lippmann-Schwinger scattering states are obtained, in analogy to \[ 6 \] (cf. also \[ 3 \] sect. 3.1) by:

\[
|\hat{p}j_3(wj)\rangle = \Omega^\pm |\hat{p}j_3(wj)\rangle
\]

where \( \Omega^\pm \) are the Møller operators. For the basis vectors at rest, \[ 24 \] is given by the solution of the Lippmann-Schwinger equation

\[
|0j_3(wj)\rangle = \left(1 + \frac{1}{w - H \pm i\epsilon}V\right)|0j_3(wj)\rangle.
\]

They are eigenvectors of the exact Hamiltonian \( H = H_0 + V \)

\[
H|0j_3(wj)\rangle = \sqrt{s}|0j_3(wj)\rangle, \quad (m_1 + m_2)^2 \leq s < \infty.
\]

The vectors \( |\hat{p}j_3(wj)\rangle \) are obtained from the basis vectors at rest \( |0j_3(wj)\rangle \) by the boost (rotation-free Lorentz transformation) \( U(L(\hat{p})) \) whose parameters are the 4-velocities \( \hat{p}\mu \). The generators of the Lorentz transformations are the interaction-incorporating observables

\[
P_0 = H, \quad P^\mu, \quad J_{\mu\nu},
\]

i.e., the exact generators of the Poincaré group (\[ 11 \] sec. 3.3). These vectors \( |\hat{p}j_3(wj)\rangle \) in \[ 24 \], or \( |0j_3(wj)\rangle \) in \[ 23 \] when boosted by \( U(L(\hat{p})) \), which for the fixed value \( jw \) span an irreducible representation space of the Poincaré group with the “exact generators”, will be used for the definition of the relativistic Gamow vectors.

The relativistic Gamow kets are obtained by analytically continuing the Dirac-Lippmann-Schwinger kets \[ 23 \] or \[ 24 \] into the second (or higher) sheet of the \( j \)-th partial wave S-matrix \( S_{jR}(s) \) to the position of the resonance pole at \( s_R = (M_R - i\Gamma/2)^2 \). This can be done for any value of \( \hat{p} \) (and \( j_3 \)), e.g., for \( \hat{p} = 0 \). In complete analogy to the non-relativistic case one obtains the relativistic kets

\[
|\hat{p}j_3(s_{RjR})^-\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} ds |\hat{p}j_3(s_{jR})^-\rangle \frac{1}{s - s_R}.
\]

The Lorentz transformations \( \Lambda \) are represented by unitary operators \( U(\Lambda) \):

\[
U(\Lambda)|\hat{p}j_3(s_{RjR})^-\rangle = \sum_{j_3} D_{j_3, j_3}^{j\mu}(R(\Lambda, \hat{p}))|\Lambda \hat{p}j_3(s_{RjR})^-\rangle,
\]

where \( R(\Lambda, \hat{p}) = L^{-1}(\Lambda\hat{p})\Lambda L(\hat{p}) \) is the Wigner rotation. In particular for the rotation free Lorentz boost

\[
U(L(\hat{p}))|\hat{p} = 0, j_3(s_{RjR})^-\rangle = |\hat{p}j_3(s_{RjR})^-\rangle
\]

because the boost is a function of the real \( \hat{p}\mu \) and not of the complex \( p^\mu \):

\[
L^\mu_\nu = \left(\begin{array}{ccc}
\frac{\hat{p}^\mu}{m} & -\frac{p^\mu}{m} \\
\delta^k_n & \frac{n \cdot \delta^k_n}{1 + \frac{n \cdot \hat{p}}{m}}
\end{array}\right), \quad L(\hat{p})\left(\begin{array}{c}1 \\ 0 \\ 0 \\ 0\end{array}\right) = \hat{p}.
\]

The relativistic Gamow kets \[ 28 \] are generalized eigenvectors of the invariant mass squared operator \( M^2 = P_\mu P^\mu \) with eigenvalue \( s_R \)

\[
|\psi^-|^2|\hat{p}j_3(s_{RjR})^-\rangle = s_R|\psi^-|^2|\hat{p}j_3(s_{RjR})^-\rangle \quad \text{for every } \psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times.
\]
also represents relativistic Gamow states with the complex mass $s_R = (M_R - i\Gamma/2)^2$ and have the representation (28).

This then implies, much like in the RHS theory of non-relativistic Gamow vectors, that the time translation of the decaying state is represented by a semigroup, e.g., at rest,

$$e^{-iH\tau|\hat{p} = 0, j_R(s_R, R^-)} = e^{-imR\tau}e^{-\Gamma\tau/2}|\hat{p} = 0, j_R(s_R, R^-)\rangle \text{ for } \tau \geq 0 \text{ only}$$

(33)

where $\tau$ is time in the rest system.

Thus relativistic Gamow states are representations of $\mathcal{P}$ with spin $j_R$ and complex mass $s_R = (M_R - i\Gamma/2)^2 = m^2 - im_\rho \Gamma_\rho$, for which the Lorentz subgroup is unitarily represented. They are obtained from the resonance pole of the relativistic partial S-matrix $S_{j_R}(s)$, have an exponential time evolution at rest with lifetime $h/\Gamma$ and have – due to their association to the S-matrix pole and to the Hardy class spaces – a Breit-Wigner energy distribution

$$a_j = \frac{Bm\Gamma}{s - (m - i\Gamma/2)^2}, \quad -\infty < s < +\infty.$$  

(34)

These are all features which one may welcome or accept for states that are to describe relativistic resonances. In addition, they have a semigroup time evolution expressing a fundamental quantum mechanical arrow of time.

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$$\langle p_1, (s_1)_3, p_2, (s_2)_4|E^{tot}, p_{j_3}(s, l)\rangle = N_3^{(p - p_1 - p_2)6(E^{tot} - \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2})} \times \sum_{s_3, s_4} \langle s_1(s_1)s_2(s_2)\rangle|s_3(s_3)s_4(s_4)|Y_{j_3}(e)$$

where $e$ is the unit vector in the direction of $p$ and $N$ is a suitable normalization factor.
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