TORSION IN KHOVANOV HOMOLOGY
OF HOMOLOGICALLY THIN KNOTS

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ABSTRACT. We prove that every \( \mathbb{Z}_2 \)H-thin link has no \( 2^k \)-torsion for \( k > 1 \) in its Khovanov homology. Together with previous results by Eun Soo Lee \[L1\] \[L2\] and the author \[Sh\], this implies that integer Khovanov homology of non-split alternating links is completely determined by the Jones polynomial and signature. Our proof is based on establishing an algebraic relation between Bockstein and Turner differentials on Khovanov homology over \( \mathbb{Z}_2 \). We conjecture that a similar relation exists between the corresponding spectral sequences.

1. INTRODUCTION

Let \( L \) be an oriented link in the Euclidean space \( \mathbb{R}^3 \) represented by a planar diagram \( D \). In a seminal paper \[Kh1\], Mikhail Khovanov assigned to \( D \) a family of abelian groups \( \mathcal{H}^{i,j}(L) \), whose isomorphism classes depend on the isotopy class of \( L \) only. These groups are defined as homology groups of an appropriate (graded) chain complex \( \mathcal{C}(D) \) with integer coefficients. The main property of the Khovanov homology is that it categorifies the Jones polynomial. More specifically, let \( J_L(q) \) be a version of the Jones polynomial of \( L \) that satisfies the following skein relation and normalization:

\[
- q^{-2} J_{\bigtriangleup}(q) + q^2 J_{\bigtriangledown}(q) = (q - 1/q) J_{\bigcirc}(q); \quad J_{\bigcirc}(q) = q + 1/q. \tag{1.1}
\]

Then \( J_L(q) \) equals the (graded) Euler characteristic of the Khovanov chain complex:

\[
J_L(q) = \chi_q(\mathcal{C}(D)) = \sum_{i,j} (-1)^i q^j \mathcal{H}^{i,j}(L), \tag{1.2}
\]

where \( \mathcal{H}^{i,j}(L) = \text{rk}(\mathcal{H}^{i,j}(L)) \), the Betti numbers of \( \mathcal{H}(L) \). We explain the main ingredients of Khovanov’s construction in Section 2. The reader is referred to \[BN\], \[Kh1\] for a detailed treatment.

In this paper, we consider a more general setup by allowing \( \mathcal{C}(D) \) to consist of free \( R \)-modules, where \( R \) is a commutative ring with unity. This results in a Khovanov homology theory with coefficients in \( R \). We are mainly interested in the cases when \( R = \mathbb{Z}, \mathbb{Q}, \) or \( \mathbb{Z}_2 \).

1.A. Definitions (\[Sh\], cf. Khovanov \[Kh2\]). A link \( L \) is said to be homologically thin over a ring \( R \) or simply \( RH \)-thin if its Khovanov homology with coefficients in \( R \) is supported on two adjacent diagonals \( 2i - j = \text{const} \). A link \( L \) is said to

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be homologically slim or simply \( H \)-thin if it is \( H \)-thin and all its homology groups that are supported on the upper diagonal have no torsion. A link \( L \) that is not \( RH \)-thin is said to be \( RH \)-thick.

1.B. If a link is \(ZH\)-%thin, then it is \( QH\)-%thin as well by definition. Consequently, a \( QH\)-thick link is necessarily \( ZH\)-thick. If a link is \( H\)-%thin, then it is \( Z_mH\)-thin for every \( m > 1 \) by the Universal Coefficient Theorem.

1.C. Example. Most of the \( ZH\)-thin knots are \( H\)-thin. The first prime \( ZH\)-thin knot that is not \( H\)-thin is the mirror image of \( 16_{197566} \). It is also \( Z_2H\)-thick, see Figure T. In these tables, columns and rows are marked with \( i \)- and \( j \)-grading of the Khovanov homology, respectively. Only entries representing non-trivial groups are shown. An entry of the form \( a, b, c \) means that the corresponding group is \( Z^a \oplus Z_2^b \oplus Z_4^c \).

\( H\)-thin knots possess several important properties that were observed in [BN] and proved in [L1, L2, Sh]. We list them below.

1.D. Theorem (Lee [L1, L2]). Every oriented non-split alternating link \( L \) is \( H\)-thin and the Khovanov homology of \( L \) is supported on the diagonals \( 2i - j = \sigma(L) \pm 1 \), where \( \sigma(L) \) is the signature of \( L \).

Let \( \tilde{J}_L(q) = J_L(q)/(q + 1/q) \) be a renormalization of \( J_L(q) \) that equals 1 on the unknot instead of \( q + 1/q \).

1.E. Theorem (Lee [L2]). If \( L \) is an \( H\)-thin link, then its rational Khovanov homology \( H_Q(L) \) is completely determined, up to a grading shift, by the Jones polynomial \( J_L(q) \) of \( L \). In particular, the total rank of \( H(L) \) is given by \( \text{rk} H_Q(L) = |\tilde{J}_L(\sqrt{-1})| + 2c^{-1} \), where \( c \) is the number of components of \( L \) (cf. [G]).

1.F. Corollary. Every oriented non-split alternating link \( L \) has its rational Khovanov homology \( H_Q(L) \) completely determined by \( J_L(q) \) and \( \sigma(L) \).

1.G. Theorem ([Sh]). If \( L \) is an \( H\)-thin link, then its integer Khovanov homology \( \mathcal{H}(L) \) has no torsion elements of odd order and its Khovanov homology \( H_{Z_2}(L) \) with coefficients in \( Z_2 \) is completely determined, up to a grading shift, by the Jones polynomial \( J_L(q) \) of \( L \). In particular, the total dimension of \( H_{Z_2}(L) \) over \( Z_2 \) is given by \( \text{dim}_{Z_2}(H_{Z_2}(L)) = 2|\tilde{J}_L(\sqrt{-1})| \).

1.H. Corollary. Every oriented non-split alternating link \( L \) has its integer Khovanov homology \( \mathcal{H}(L) \) all but determined by \( J_L(q) \) and \( \sigma(L) \), except that one cannot distinguish between \( Z_{2^k} \) factors in the canonical decomposition of \( \mathcal{H}(L) \) for different values of \( k \).

Remark. [L2] and [Sh] contain much more information about the structure of \( H_Q(L) \) and \( H(L) \) of \( H\)-thin links than Theorems 1.E and 1.G, respectively. But we have no use of more general statements in this paper.

It was conjectured in [Sh] that \( \mathcal{H}(L) \) of \( H\)-thin links can only contain 2-torsion. Our main result is to prove this conjecture.

1.I. Theorem. Let \( L \) be a \( Z_2H\)-thin link. Then \( \mathcal{H}(L) \) contains no torsion elements of order \( 2^k \) for \( k > 1 \).

\(^{1}\)This denotes the non-alternating knot number 197566 with 16 crossings from the Knotscape knot table [HTB].
Knot $16_{197566}^n$ is $\mathbb{Q}$H-thin, but $\mathbb{Z}$H-thick and $\mathbb{Z}_2$H-thick.

Mirror image of the knot $16_{197566}^n$ is $\mathbb{Q}$H-thin and $\mathbb{Z}$H-thin, but $\mathbb{Z}_2$H-thick.

**Figure 1.** Integral Khovanov homology of the knot $16_{197566}^n$ and its mirror image.
1.J. Corollary. If $L$ is an $H$-slim link, then every non-trivial torsion element of $\mathcal{H}(L)$ has order 2. Consequently, every oriented non-split alternating link $L$ has its integer Khovanov homology completely determined by $J_L(q)$ and $\sigma(L)$.

We end this section with a long-standing conjecture from [Sh] that provides another motivation for studying 2-torsion in the Khovanov homology. Partial results in this directions were obtained in [AP, PPS, PS].

Conjecture 1. Khovanov homology of every non-split link except the trivial knot, the Hopf link, and their connected sums contains torsion elements of order 2.

This paper is organized as follows. In Section 2 we recall main definitions and facts about the Khovanov homology and auxiliary constructions that are going to be used in the paper. Section 3 contains brief overview of the Bockstein spectral sequence construction following [MC] as well as a proof of Theorem 1.I modulo technical Lemma 3.2.A whose proof is postponed until Section 4.

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2. Main ingredients and definitions

In this section we give a brief outline of the Khovanov homology theory following [Kh1]. We also recall required bits and pieces from [Tu] and [Sh].

2.1. Algebraic preliminaries. Let $R$ be a commutative ring with unity. In this paper, we are only interested in the cases $R = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_2$. If $M$ is a graded $R$-module, we denote its homogeneous component of degree $j$ by $M_j$. For an integer $k$, the shifted module $M\{k\}$ is defined as having homogeneous components $M\{k\}_j = M_{j-k}$. In the case when $M$ is free and finite dimensional, we define its graded dimension as the Laurent polynomial $\dim_q(M) = \sum_{j \in \mathbb{Z}} q^j \dim(M_j)$ in variable $q$.

Finally, if $(C,d) = (\cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \to \cdots)$ is a (co)chain complex of graded free $R$-modules such that all differentials $d^i$ are graded of degree 0 with respect to the internal grading, we define its graded Euler characteristic as $\chi_q(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_q(C^i)$.

Remark. One can think of a chain complex of graded $R$-modules as a bigraded $R$-module where the homogeneous components are indexed by pairs of numbers $(i,j) \in \mathbb{Z}^2$. Under this point of view, the differentials are graded of bidegree $(0,1)$.

Let $A = R[X]/X^2$, the ring of truncated polynomials. As an $R$-module, $A$ is freely generated by 1 and $X$. We put grading on $A$ by specifying that $\deg(1) = 1$ and $\deg(X) = -1$. In other words, $A \simeq R\{1\} \oplus R\{-1\}$, and $\dim_q(A) = q + q^{-1}$. At the same time, $A$ is a (graded) commutative algebra with the unit 1 and multiplication $m : A \otimes A \to A$ given by

$$m(1 \otimes 1) = 1, \quad m(1 \otimes X) = m(X \otimes 1) = X, \quad m(X \otimes X) = 0. \quad (2.1)$$

Algebra $A$ can also be equipped with comultiplication $\Delta : A \to A \otimes A$ defined as

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X. \quad (2.2)$$

It follows directly from the definition that $m$ and $\Delta$ are graded maps with

$$\deg(m) = \deg(\Delta) = -1. \quad (2.3)$$
positive crossing

negative crossing

positive marker

negative marker

Figure 2. Positive and negative crossings

Figure 3. Positive and negative markers and the corresponding resolutions of a diagram.

Remark. Together with a counit map \( \varepsilon : A \to R \) given by \( \varepsilon(1) = 0 \) and \( \varepsilon(X) = 1 \), \( A \) has a structure of a commutative Frobenius algebra over \( R \), see [Kh3].

2.2. Khovanov chain complex. Let \( L \) be an oriented link and \( D \) its planar diagram. We assign a number \( \pm 1 \), called sign, to every crossing of \( D \) according to the rule depicted in Figure 2. The sum of these signs over all the crossings of \( D \) is called the writhe number of \( D \) and is denoted by \( w(D) \).

Every crossing of \( D \) can be resolved in two different ways according to a choice of a marker, which can be either positive or negative, at this crossing (see Figure 3). A collection of markers chosen at every crossing of a diagram \( D \) is called a (Kauffman) state of \( D \). For a diagram with \( n \) crossings, there are, obviously, \( 2^n \) different states. Denote by \( \sigma(s) \) the difference between the numbers of positive and negative markers in a given state \( s \). Define

\[
i(s) = \frac{w(D) - \sigma(s)}{2}, \quad j(s) = \frac{3w(D) - \sigma(s)}{2}.
\]

Since both \( w(D) \) and \( \sigma(s) \) are congruent to \( n \) modulo 2, \( i(s) \) and \( j(s) \) are always integer. For a given state \( s \), the result of the resolution of \( D \) at each crossing according to \( s \) is a family \( D_s \) of disjointly embedded circles. Denote the number of these circles by \( |D_s| \).

For each state \( s \) of \( D \), let \( \mathcal{A}(s) = A^{\otimes |D_s|}\{j(s)\} \). One should understand this construction as assigning a copy of algebra \( A \) to each circle from \( D_s \), taking the tensor product of all of these copies, and shifting the grading of the result by \( j(s) \). By construction, \( \mathcal{A}(s) \) is a graded free \( R \)-module of graded dimension \( \dim_q(\mathcal{A}(s)) = q^{i(s)}(q + q^{-1})^{|D_s|} \). Let \( \mathcal{C}^i(D) = \bigoplus_{j(s) = i} \mathcal{A}(s) \) for each \( i \in \mathbb{Z} \). It is easy to check (see [BN, Kh1]) that \( \chi_q(\mathcal{C}(D)) = J_L(q) \), that is, the graded Euler characteristic of \( \mathcal{C}(D) \) equals the Jones polynomial of the link \( L \).

In order to make \( \mathcal{C}(D) \) into a graded complex, we need to define a (graded) differential \( d^i : C^i(D) \to C^{i+1}(D) \) of degree 0. Let \( s_+ \) and \( s_- \) be two states of \( D \) that differ at a single crossing, where \( s_+ \) has a positive marker while \( s_- \) has a negative one. We call two such states adjacent. In this case, \( \sigma(s_-) = \sigma(s_+) - 2 \) and, consequently, \( i(s_-) = i(s_+) + 1 \) and \( j(s_-) = j(s_+) + 1 \). Consider now the resolutions of \( D \) corresponding to \( s_+ \) and \( s_- \). One can readily see that \( D_{s_+} \) is obtained from \( D_{s_-} \) by either merging two circles into one or splitting one circle into two (see Figure 4). We define \( d_{s_+:s_-} : \mathcal{A}(s_+) \to \mathcal{A}(s_-) \) as either \( m \) or \( \Delta \).
2.3. Turner spectral sequence. In the case of $R = \mathbb{Z}_2$, Paul Turner [Tu] defined another differential $d_T : C_{\mathbb{Z}_2}(D) \rightarrow C_{\mathbb{Z}_2}(D)$ on the Khovanov chain complex over $\mathbb{Z}_2$. Its definition follows the one above for $d$ almost verbatim except that the multiplication and comultiplication maps are different:

\[
m_T(1 \otimes 1) = m_T(1 \otimes X) = m_T(X \otimes 1) = 0, \quad m_T(X \otimes X) = X;
\]

\[
\Delta_T(1) = 1 \otimes 1, \quad \Delta_T(X) = 0.
\]

Main properties of $d_T$ are summarized in the theorem below.

2.3.A. Theorem (Turner [Tu]). Let $L$ be a link with $c$ components represented by a diagram $D$. Then

1. $d_T$ is a differential on $C_{\mathbb{Z}_2}(D)$ of bidegree $(1,2)$;
2. $d_T$ commutes with $d$ and induces a well-defined differential $d_T^r$ of bidegree $(1,2)$ on the Khovanov homology groups $\mathcal{H}_{\mathbb{Z}_2}(L)$;
3. $(C_{\mathbb{Z}_2}(D), d, d_T)$ has a structure of a double complex and, hence, results in a spectral sequence $\{(E^r_T, d_r)\}_{r \geq 0}$ with $E^0_T = (C_{\mathbb{Z}_2}(D), d)$ and $E^1_T = (\mathcal{H}_{\mathbb{Z}_2}(L), d_T)$, where the bidegree of $d_r$ equals $(1,2r)$;
4. $\{(E^r_T, d_r)\}$ converges to $E^\infty_T$ that has the total dimension over $\mathbb{Z}_2$ equal $2^c$;
5. if $L$ is $\mathbb{Z}_2$-thin, then $\{(E^r_T, d_r)\}$ collapses at the second page, that is, $d_r = 0$ for $r \geq 2$ and $E^2_T = E^\infty_T$.
2.3.B. Definition. The spectral sequence from Theorem 2.3.A is called the Turner spectral sequence for a link $L$.

2.4. Generators of $C(D)$ and another differential on $\mathcal{H}_{Z_2}(L)$. Let $L$ be a link represented by a diagram $D$ and let $s$ be its Kauffman state. Then $A(s)$ is freely generated as an $R$-module by $2^{|D_s|}$ generators of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_{|D_s|}$, where each $a_i$ is either 1 or $X$.

Define an $R$-linear homomorphism $\nu_s : A(s) \to A(s)$ of degree 2 by specifying that on a generator of $A(s)$ it equals the sum of all the possibilities to replace an $X$ in this generator with 1. For example, if $|D_s| = 3$, then $\nu_s(X \otimes X \otimes 1) = 1 \otimes X \otimes 1 + X \otimes 1 \otimes 1, \nu_s(1 \otimes X \otimes 1) = 1 \otimes 1 \otimes 1$ and $\nu_s(1 \otimes 1 \otimes 1) = 0$. We extend this to a map $\nu : C(D) \to C(D)$ of bidegree $(0, 2)$ as $\nu = \sum \nu_s$.

Restrict now our attention to the case of $R = Z_2$. Main properties of $\nu$ over $Z_2$ are listed below. They are proved in [Sh].

2.4.A. Theorem ([Sh]). Let $L$ be a link and $D$ its diagram. Then

1. $\nu$ is a differential on $C_{Z_2}(D)$ of bidegree $(0, 2)$;
2. $\nu$ commutes with $d$ and, hence, induces a differential $\nu^*$ of bidegree $(0, 2)$ on $\mathcal{H}_{Z_2}(L)$;
3. both $\nu$ and $\nu^*$ are acyclic;
4. if $L$ is $Z_2$-thin, then $\nu^*$ establishes an isomorphism between the two non-trivial diagonals of $\mathcal{H}_{Z_2}(L)$.

Remark. $\nu^*$ helps to establish the fact that $\mathcal{H}_{Z_2}(L)$ is isomorphic to a direct sum of two copies of the reduced Khovanov homology of $L$ over $Z_2$, see [Sh].

3. Relations between differentials on Khovanov homology over $Z_2$

3.1. Bockstein spectral sequence. We start this sections by recalling main definitions and properties of the Bockstein spectral sequence following Chapter 10 of [MC]. Our setup differs slightly from the one in [MC] since we consider abstract chain complexes, while [MC] deals with singular homology of topological spaces. Nonetheless, all the relevant statements still hold true as long as we work with finitely generated complexes of free Abelian groups. We state all the results for an arbitrary prime $p$, but will only need them for $p = 2$.

Let $C = \left( \cdots \rightarrow C^{i-1} \rightarrow C^i \rightarrow C^{i+1} \rightarrow \cdots \right)$ be a (co)chain complex of free Abelian groups and let $p$ be a prime number. A short exact sequence $0 \to Z_p \xrightarrow{\times p} Z_{p^2} \xrightarrow{\text{mod } p} Z_p \to 0$ of coefficient rings induces a short exact sequence $0 \to C \otimes Z_p \xrightarrow{\times p} C \otimes Z_{p^2} \xrightarrow{\text{mod } p} C \otimes Z_p \to 0$ of complexes that, in turn, results in a long exact sequence of the corresponding (co)homology groups. Let $\beta : H(C; Z_p) \to H(C; Z_p^r)$ be a connecting homomorphism in this long exact sequence. It has homological degree 1. We call $\beta$ the Bockstein homomorphism on $H(C; Z_p^r)$.

3.1.A. Theorem (cf. Theorem 10.3 and Proposition 10.4 of [MC]). Let $C$ be a finitely generated (co)chain complex of free Abelian groups. Then there exists a singly-graded spectral sequence $\{ (B_r, b_r) \}_{r \geq 1}$ with $(B_1, b_1) = (H(C; Z_p), \beta)$ that converges to $(H(C) / \text{torsion}) \otimes Z_p$. Moreover, $B_r \simeq \text{Im} (H(C; Z_{p^r}) \xrightarrow{\times p^{r-1}} H(C; Z_{p^r}))$, a graded subgroup of $H(C; Z_{p^r})$, and $b_r$ can be identified with the connecting homomorphism induced on $H(C; Z_{p^r})$ from a short exact sequence of coefficient rings $0 \to Z_{p^r} \to Z_{p^2r} \to Z_{p^r} \to 0$. Also, $\text{deg}(b_r) = 1$ for every $r \geq 1$. 
3.1.B. Definition. \{(B_r, b_r)\} is called the Bockstein spectral sequence of \(H(C; \mathbb{Z}_p)\) and \(b_r\) is called the \(r\)-th order Bockstein homomorphism.

3.1.C. Corollary. The Bockstein spectral sequence \(\{(B_r, b_r)\}\) collapses at the \(r\)-th page if and only if \(H(C)\) has no torsion elements of order \(2^k\) for \(k \geq r\).

Proof. \(\{(B_r, b_r)\}\) collapses at the \(r\)-th page if and only if \(\dim_{\mathbb{Z}_p} B_r = \dim_{\mathbb{Z}_p} B_\infty\). By Theorem 3.1.A \(\dim_{\mathbb{Z}_p} B_r\) equals the number of \(\mathbb{Z}_p^r\) factors in \(H(C; \mathbb{Z}_p)\) and \(\dim_{\mathbb{Z}_p} B_\infty\) equals the number of \(\mathbb{Z}\) factors in \(H(C)\). The rest follows from the Universal Coefficient Theorem.

3.1.D. The Bockstein differential \(\nu\) has another description that is more useful for our purposes. Namely, if \(b : H(C; \mathbb{Z}_p) \to H(C)\) is the connecting homomorphisms induced from a short exact sequence of coefficients \(0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0\), then \(\beta = b \mod p\).

3.2. Proof of the main result. Let \(p = 2\) from now on.

3.2.A. Lemma. Let \(L\) be a link. Then \(d_L^* = \beta \circ \nu^* + \nu^* \circ \beta\) on \(\mathcal{H}_{\mathbb{Z}_2}(L)\).

We postpone the proof of this lemma until the next section.

Proof of Theorem 3.2. Recall that our task is to prove that if \(L\) is \(\mathbb{Z}_2\)-H-thin, then \(\mathcal{H}(L)\) contains no torsion elements of order \(2^k\) for \(k > 1\). By Corollary 3.1.C it is enough to show that the Bockstein spectral sequence collapses at the second page, that is, \(\dim_{\mathbb{Z}_2} B_2 = \dim_{\mathbb{Z}_2} B_\infty = \text{rk} \mathcal{H}_Q(L)\). Now, \(\text{rk} \mathcal{H}_Q(L) = |J_L(\sqrt{-1})| + 2c_1\), where \(c\) is the number of components of \(L\), by Theorem 1.B and \(\dim_{\mathbb{Z}_2} B_1 = \dim_{\mathbb{Z}_2}(\mathcal{H}_{\mathbb{Z}_2}(L)) = 2|J_L(\sqrt{-1})|\) by Theorem 1.C. Since \(b_1 = \beta\), it only remains to show that \(\text{rk} \mathcal{H}_{\mathbb{Z}_2}(L) = \frac{1}{2}(\dim_{\mathbb{Z}_2} B_1 - \dim_{\mathbb{Z}_2} B_2) = \frac{1}{2}|J_L(\sqrt{-1})| - 2c_1\).

Since \(L\) is \(\mathbb{Z}_2\)-H-thin, the Turner spectral sequence collapses at the second page, that is, \(\dim_{\mathbb{Z}_2} E_2^T = \dim_{\mathbb{Z}_2} E_\infty^T = 2^c\) by Theorem 2.3.A. Since \(E_2^T = \mathcal{H}_{\mathbb{Z}_2}(L)\) and \(d_1 = d_T^*\), we have that \(\text{rk} \mathcal{H}_{\mathbb{Z}_2}(d_T^*|_{\mathcal{H}_{\mathbb{Z}_2}}) = \frac{1}{2}(\dim_{\mathbb{Z}_2} E_2^T - \dim_{\mathbb{Z}_2} E_2^T) = |J_L(\sqrt{-1})| - 2c_1\).

Let \(\mathcal{H}_t\) and \(\mathcal{H}_u\) be (graded) subgroups of \(\mathcal{H}_{\mathbb{Z}_2}(L)\) supported on the lower and upper diagonals, respectively. Then \(\mathcal{H}_{\mathbb{Z}_2}(L) \simeq \mathcal{H}_t \oplus \mathcal{H}_u\) since \(L\) is \(\mathbb{Z}_2\)-H-thin. Finally, we observe that \(\nu^*|_{\mathcal{H}_t}\) is an isomorphism between \(\mathcal{H}_t\) and \(\mathcal{H}_u\) by Theorem 2.3.A. \(\nu^*|_{\mathcal{H}_u} = 0\), \(\beta|_{\mathcal{H}_t} = 0\), and \(\text{rk} \mathcal{H}_{\mathbb{Z}_2}(\beta|_{\mathcal{H}_u})\) since \(\beta\) is trivial everywhere else. We conclude that \(\text{rk} \mathcal{H}_{\mathbb{Z}_2}(d_T^*|_{\mathcal{H}_t}) = \text{rk} \mathcal{H}_{\mathbb{Z}_2}(\beta \circ \nu^*|_{\mathcal{H}_t}) = \text{rk} \mathcal{H}_{\mathbb{Z}_2}(\beta|_{\mathcal{H}_u}) = \text{rk} \mathcal{H}_{\mathbb{Z}_2}(\beta|_{\mathcal{H}_u})\). Hence, \(\text{rk} \mathcal{H}_{\mathbb{Z}_2}(d_T^*) = \text{rk} \mathcal{H}_{\mathbb{Z}_2}(d_T^*|_{\mathcal{H}_t}) + \text{rk} \mathcal{H}_{\mathbb{Z}_2}(d_T^*|_{\mathcal{H}_u}) = 2\text{rk} \mathcal{H}_{\mathbb{Z}_2}(\beta)\).

Proof of Theorem 3.2 suggests that there is a deeper relation between Turner and Bockstein spectral sequences on \(\mathcal{H}_{\mathbb{Z}_2}(L)\). We end this section with a couple of conjectures.

Conjecture 2. There exists an algebraic relation between Turner and Bockstein differentials on higher pages in the corresponding spectral sequences. This relation should involve higher order generalizations of \(\nu\).

Conjecture 3. If the Turner spectral sequence on \(\mathcal{H}_{\mathbb{Z}_2}(L)\) collapses at the \(r\)-th page, then the Bockstein one collapses at the \(r\)-th page as well. In particular, if \(\mathcal{H}_{\mathbb{Z}_2}(L)\) is supported on \(r\) adjacent diagonals, then \(\mathcal{H}(L)\) does not have torsion elements of order \(2^k\) for \(k \geq r\).
4. Proof of Lemma 3.2.1

Recall that we have to prove that $d_\nu^T = \beta \circ \nu^* + \nu^* \circ \beta$ on $\mathcal{H}_{Z_2}(L)$. For a (co)chain $c \in C_{Z_2}(D)$ and a generator $x$ of $\mathcal{C}(D)$ (see Section 2.1), denote by $c|_x$ the coefficient of $x$ in $c$. We say that $x$ is from $c$ and write $x \in c$ if $c|_x = 1$ and we say that $x$ is not from $c$ and write $x \notin c$ otherwise. Finally, we denote the Kauffman state that corresponds to $x$ by $s(x)$.

Fix a bigrading $(i, j)$ on $\mathcal{C}(D)$ so that $d_\nu^T : \mathcal{C}^{i,j}_{Z_2}(D) \to \mathcal{C}^{i+1,j+2}_{Z_2}(D)$. For a (co)homology class $[c]_2 \in H^{i,j}_{Z_2}(L)$ represented by a (co)cycles $c \in C^{i,j}_{Z_2}$, the connecting homomorphism $b$ from $[c]_2$ is defined as $b([c]_2) = \frac{1}{2} d(c) \in H^{i+1,j+1}(L)$, where $c$ is lifted to $\mathcal{C}(D)$ so that $d(c)$ makes sense. It follows that

$$\beta([c]_2) = \left[ \frac{1}{2} d(c) \right]_2 \in H^{i+1,j+1}_{Z_2}(L), \quad (4.1)$$

where $[\cdot]_2$ means taking the homology class over $Z_2$. Let $\delta : \mathcal{C}^{i,j}(D) \to \mathcal{C}^{i+1,j+2}(D)$ be the homomorphism defined as $\delta = d \circ \nu + \nu \circ d$. Formula (4.1) implies that

$$(\beta \circ \nu^* + \nu^* \circ \beta)([c]_2) = \left[ \frac{1}{2} \delta(c) \right]_2 = \left[ \frac{1}{2} \sum_{x \in c} \delta(x) \right]_2. \quad (4.2)$$

Given a generator $x$ of $\mathcal{C}^{i,j}(D)$, and a generator $z$ of $\mathcal{C}^{i+1,j+2}(D)$, denote by $L^x_s$ the set of all generators $y$ of $\mathcal{C}^{i+1,j+2}(D)$ such that $y \in d(x)$ and $z \notin d(y)$. Similarly, denote by $U^z_s$ the set of all generators $y'$ of $\mathcal{C}^{i+1,j+2}(D)$ such that $y' \in \nu(x)$ and $z \notin d(y')$. We denote elements of $L^x_s$ and $U^z_s$ by $x \to y \to z$ and $x \to y' \to z$, respectively. Since $\nu$ does not change the underlying Kauffman states of the generators, $L^x_s$ and $U^z_s$ are empty when states $s(x)$ and $s(z)$ are not adjacent. Hence, we are going to assume that $s(x)$ and $s(z)$ are adjacent states from now on. By definition, $\delta(x) = \varepsilon^x_s(|L^x_s| + |U^z_s|)$, where $|\cdot|$ denotes the cardinality of a set and $\varepsilon^x_s = \varepsilon(s(x), s(z))$, see Definition 2.2.1. Let $L^x_s \subset L^x_s$ consist of all $(x \to y \to z) \in L^x_s$ such that the circle of $s(y)$ where $X$ is replaced with 1 in $z$ does not pass through the crossing at which $s(x)$ and $s(y)$ differ. We define $U^z_s \subset U^z_s$ similarly. Let $\hat{L}^x_s = L^x_s \setminus \hats^x_s$ and $\hat{U}^z_s = U^z_s \setminus \hat{U}^z_s$. There is a natural bijection between sets $\hat{L}^x_s$ and $\hat{U}^z_s$ since changes made under $d$ and $\nu$ take place on circles that do not interfere with each other. Hence, $|\hat{L}^x_s| = |\hat{U}^z_s|$. We list elements of $\hat{L}^x_s$ and $\hat{U}^z_s$ for all possible $x$ and $z$ in Figure 3 where we omit common parts of $x$, $y$, $\nu(x)$ and $\nu(y)$. We classify pairs $(x, z)$ as being of four different types, $A$, $B$, $C_m$ and $C_\Delta$, depending on the outcome (see Figure 3). Let $t(x, z)$ denote the type of $(x, z)$. Then $|\hat{L}^x_s| = |\hat{U}^z_s| = 0$ if $t(x, z) = A$ and $|\hat{L}^x_s| = |\hat{U}^z_s| = 1$ if $t(x, z) = B$. If $t(x, z) = C_m$, then $|\hat{L}^x_s| = 0$ and $|\hat{U}^z_s| = 2$, and if $t(x, z) = C_\Delta$, then $|\hat{L}^x_s| = 2$ and $|\hat{U}^z_s| = 0$. It follows that

$$\sum_{x \in c} \varepsilon^x_s |\hat{L}^x_s| = \sum_{x \in c} \varepsilon^x_s + \sum_{x \in c, t(x, z) = B} 2\varepsilon^x_s; \quad (4.3)$$

$$\sum_{x \in c} \varepsilon^x_s |\hat{U}^z_s| = \sum_{x \in c} \varepsilon^x_s + \sum_{x \in c, t(x, z) = C_m} 2\varepsilon^x_s.$$
Therefore, for every \([c]_2 \in \mathcal{H}_{\Delta}^{i,j}(L)\) and every generator \(z\) of \(\mathcal{C}^{i+1,j+2}(D)\) we have:

\[
\frac{1}{2} \delta(c)|_z = \frac{1}{2} \sum_{x \in C} \delta(x)|_z \\
= \frac{1}{2} \sum_{x \in C} \varepsilon^x_x (\|L^x_x\| + |\widetilde{U}^x_x|) \\
= \frac{1}{2} \sum_{x \in C} \varepsilon^x_x (\|L^x_x\| + |\widetilde{U}^x_x|) + \frac{1}{2} \sum_{x \in C} \varepsilon^x_x (|\widetilde{L}^x_x| + |\widetilde{U}^x_x|) \\
= \sum_{x \in C} \varepsilon^x_x |\widetilde{L}^x_x|_{t(x) = B} + \sum_{x \in C} \varepsilon^x_x + \sum_{x \in C} \varepsilon^x_x \cdot \sum_{x \in C} \varepsilon^x_x \cdot \sum_{x \in C} \varepsilon^x_x .
\]

(4.4)

where the last equality follows from (4.3) and the fact that \(\|L^x_x\| = \|\widetilde{U}^x_x\|\).

Since \(c\) is a (co)cycle modulo 2, for every generator \(y\) of \(\mathcal{C}^{i+1,j}(D)\) we have that \(d(c)|_y\) is an even number. It follows that \(\sum_{x \in C} \varepsilon^x_x \cdot |L^x_x|\) is even as well. From (4.3) we have:

\[
\sum_{x \in C} \varepsilon^x_x \cdot |L^x_x| = \sum_{x \in C} \varepsilon^x_x \cdot |\widetilde{L}^x_x| + \sum_{x \in C} \varepsilon^x_x \cdot |\widetilde{L}^x_x| = \sum_{x \in C} \varepsilon^x_x \cdot |\widetilde{L}^x_x| + \sum_{x \in C} \varepsilon^x_x + 2 \sum_{x \in C} \varepsilon^x_x .
\]

(4.5)
Therefore, $\sum_{x \in c} \varepsilon^x_\Delta \varepsilon^x_{\Delta_m} + \sum_{x \in c} \varepsilon^x_{\Delta_m} t(x, z) = 2N$ for some $N \in \mathbb{Z}$.

It follows from (4.4) that

$$\frac{1}{2} \delta(c) \equiv 2N + \sum_{x \in c} \varepsilon^x_\Delta \varepsilon^x_{\Delta_m} t(x, z) \equiv \sum_{x \in c} 1 + \sum_{x \in c} 1 \mod 2 \quad \text{(4.6)}$$

by the definition (2.5) of $d_T$.

Since this is true for every generator $z$ of $\mathcal{C}^{i+1,j+2}(D)$, we have that

$$d_T^*(|c|_2) = \left[ \frac{1}{2} \delta(c) \right]_2 = (\beta \circ \nu^* + \nu^* \circ \beta)(|c|_2) \quad \text{(4.7)}$$

for every $|c|_2 \in \mathcal{H}^{i,j}_2(L)$ and, hence, $d_T^* = \beta \circ \nu^* + \nu^* \circ \beta$, as desired. \qed

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