THE USE OF THE MORSE THEORY TO ESTIMATE THE NUMBER OF NONTRIVIAL SOLUTIONS OF A NONLINEAR SCHRÖDINGER EQUATION WITH A MAGNETIC FIELD

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Abstract. Nonlinear Schrödinger equations with an external magnetic field and a power nonlinearity with subcritical exponent \( p \) are considered. It is established a lower bound to the number of nontrivial solutions to these equations in terms of the topology of the domains in which the problem is given if \( p \) is suitably close to the critical exponent \( 2^* = \frac{2N}{N - 2} \), \( N \geq 3 \). To prove this lower bound, based on a proof of a result of Benci and Cerami, it is provided an abstract result that establishes Morse relations that are used to count solutions.

1. Introduction. The relationship between existence and multiplicity of solutions to elliptic problems and the topology of the domains in which the problems are given has been investigated by many researchers since the works [9, 8] – see, for example, [7, 13, 15, 22]. In [9], Benci and Cerami studied the existence and multiplicity of solutions for the problem

\[
\begin{cases}
-\Delta u + \kappa u = |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1.1)

where \( \kappa \) is a fixed positive constant, \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( p \in (2, 2^*) \) and \( 2^* = \frac{2N}{N - 2} \) with \( N \geq 3 \). It was proved that (1.1) has at least \( \text{cat}(\Omega) \) positive solutions provided that \( \kappa \) is sufficiently large or \( p \) is sufficiently close to \( 2^* \), where \( \text{cat}(\Omega) \) denotes the Lusternik-Schnirelman category of \( \Omega \) in itself. Subsequently, Benci and Cerami in [8] showed that the number of positive solutions for a semilinear elliptic equations like

\[
\begin{cases}
-\varepsilon \Delta u + u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1.2)

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where $\varepsilon > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f$ is a continuous function with subcritical growth, depends on the Poincaré polynomial of the domain, that is, a lower estimate of the number of solutions can be performed entirely in terms of the Morse relations. More precisely, the authors proved that there exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*)$, problem 1.2 has at least $2\mathcal{P}_1(\Omega) - 1$ nontrivial solutions, where $\mathcal{P}_1(\Omega)$ denotes the Poincaré polynomial of $\Omega$.

The present paper, which is a revised version of [3], was mainly motivated by [8]. Whilst working on [4], by carefully examining the method used by Benci and Cerami in [8] to study some properties of the functional associated with 1.2 in order to apply the Morse relations, we have observed there is an abstract result behind this method providing these relations. This abstract result is in essence due to Benci and Cerami. What we do here is to write this result in a practical way to employ the method developed by Benci and Cerami.

In order to illustrate how this abstract result can be used, we study the existence and multiplicity for the following class of stationary Schrödinger equations with an external magnetic field

$$
\begin{aligned}
\left( \frac{1}{i} \nabla - A(x) \right)^2 u + \kappa u &= |u|^{p-2} u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
$$

where $\kappa$ is a positive parameter, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$, $i$ is the imaginary unit and $p \in (2, 2^*)$, $2^* = 2N/(N - 2)$. Here, we assume $A \in L^\infty(\Omega, \mathbb{R}^N)$. The Schrödinger operator shown in 1.3 is defined by

$$
\left( \frac{1}{i} \nabla - A(x) \right)^2 u = -\Delta u - \frac{2}{i} A(x) \nabla u + |A(x)|^2 u - \frac{1}{i} u \div A(x).
$$

These equations arise in some quantum model which considers the presence of an external magnetic field and have been extensively studied in recent years (see, e.g., [2, 5, 6, 10, 11, 12, 14, 15, 16, 17, 18, 20, 23, 24, 25, 26, 27, 28, 29, 30]).

In order to establish the abstract result, we need to fix some notations. Let $(E, \langle \cdot, \cdot \rangle)$ denote a real Hilbert space endowed with the induced norm $\| \cdot \|^2 = \langle \cdot, \cdot \rangle$. Let $I : E \to \mathbb{R}$ be a $C^2$ functional and let $\mathcal{M}$ be the Nehari manifold associated with $I$ given by

$$
\mathcal{M} := \{ u \in E \setminus \{ 0 \}; I'(u)u = 0 \}.
$$

Here, $I$ is assumed to be positive on $\mathcal{M}$ with

$$
b := \inf_{\mathcal{M}} I > 0. \tag{1.4}
$$

For $a \in \mathbb{R}$, consider the sets

$$
I^a := \{ u \in E; I(u) \leq a \} \text{ and } \mathcal{M}^a := \mathcal{M} \cap I^a.
$$

We can now state the above-mentioned abstract result. The proof proceeds along the same lines as in [8] (see also [7]). For the convenience of the reader, we repeat the main points of the proof in the Appendix.

**Theorem 1.1.** For $b$ given by 1.4, let $\delta \in (0, b)$. Suppose that

(i) $I(u) = \frac{1}{2}\| u \|^2 - \Psi(u)$, where $\Psi : E \to \mathbb{R}$ is such that $\Psi(0) = 0$ and $t \mapsto \Psi(tu)/t$ is strictly increasing in $(0, +\infty)$ and unbounded above, for every $u \in E \setminus \{ 0 \}$,
(ii) I satisfies the Palais-Smale condition and, for every $u \in E$, there exists a self-adjoint operator $L(u) : E \to E$ such that $H_1(u)(v, v) = \langle L(u)v, v \rangle_E$, for every $v \in E$, where $H_1$ is the Hessian form of $I$ at $u$,

(iii) The Nehari manifold $M$ is homeomorphic to the unit sphere in $E$,

(iv) There exist a regular value $b^* > 0$ of $I$, a nonempty set $\Theta \subset \mathbb{R}^N$ with smooth boundary and continuous applications $\Phi : \Theta^- \to M^{b^*}$, $\beta : \Theta^{b^*} \to \Theta^+$ such that $\beta \circ \Phi = \text{Id}_{\Theta^-}$, where $\text{Id}_{\Theta^-} : \Theta^- \to \Theta^-$ is the identity map of $\Theta^-$,

\[ \Theta^+ = \{ x \in \mathbb{R}^N; \text{dist}(x, \Theta) \leq r \} \quad \text{and} \quad \Theta^- = \{ x \in \Theta; \text{dist}(x, \partial \Theta) \geq r \}, \]

for some $r > 0$ such that $\Theta^+$ and $\Theta^-$ are homotopically equivalent to $\Theta$.

Suppose also that the set $K$ of critical points of $I$ is discrete. Then

\[ \sum_{u \in C_1} i_t(u) = tP_t(\Theta) + tQ(t) + (1 + t)Q_1(t) \quad (1.5) \]

and

\[ \sum_{u \in C_2} i_t(u) = t^2[P_t(\Theta) + Q(t) - 1] + (1 + t)Q_2(t), \quad (1.6) \]

where $i_t(u)$ is the generalized Morse index of $u$.

$C_1 := \{ u \in K; \delta < I(u) \leq b^* \}$, $C_2 := \{ u \in K; b^* < I(u) \}$,

$P_t(\Theta)$ is the Poincaré polynomial of $\Theta$ and $Q, Q_1, Q_2$ are polynomials with nonnegative integer coefficients.

Motivated by [9, 8] we obtain the following result:

**Theorem 1.2.** Suppose that the set $K$ of solutions of the problem 1.3 is discrete. Then there is a function $p : [0, +\infty) \to (2, 2^*)$ such that for every $p \in [p(\kappa), 2^*)$,

\[ \sum_{u \in K} i_t(u) = tP_t(\Omega) + t^2[P_t(\Omega) - 1] + Q(t), \]

where $Q$ is a polynomial with nonnegative integer coefficients, $P_t(\Omega)$ is the Poincaré polynomial of $\Omega$ and $i_t(u)$ is the generalized Morse index of $u$.

In the nondegenerate case, we have:

**Corollary 1.** Suppose that the solutions of problem (1.3) are nondegenerate. Then there is a function $p : [0, +\infty) \to (2, 2^*)$ such that for every $p \in [p(\kappa), 2^*)$, problem 1.3 has at least $2P_1(\Omega) - 1$ nontrivial solutions.

As noted in [8], the application of Morse theory can provide better information than the use of the category of Ljusternik-Schnirelman. Corollary 1 shows that 1.3 has at least $2P_1(\Omega) - 1$ nontrivial solutions. In the case where $\Omega$ is topologically trivial, it follows that $P_1(\Omega) = 1$, and this result does not give additional information about multiple solutions. On the other hand, when $\Omega$ is topologically rich, for example, if $\Omega$ is a domain with $\ell$ "holes", the number of nontrivial solutions is affected by $\ell$, even if the category of $\Omega$ is 2. Indeed, we have the following result (as in [7, Corolary IV.4.2]):

**Corollary 2.** Let $A$ and $C_i$, $i = 1, \ldots, \ell$ be contractible, open, smooth and bounded nonempty sets in $\mathbb{R}^N$. Suppose that $C_i$ are mutually disjoints, that is, $C_i \cap C_j = \emptyset$, for $i \neq j$, and $C_i \subset A$, for all $i = 1, \ldots, \ell$. Set

\[ \Omega = A \setminus \bigcup_{i} C_i. \]
Then there exists a function \( \bar{p} : [0, +\infty) \rightarrow (2, 2^*) \) such that for every \( p \in [\bar{p}(\kappa), 2^*) \), problem 1.3 has at least \( 2P_1(\Omega) - 1 = 2\ell + 1 \) solutions, if they are counted with their multiplicity.

One can also apply the abstract result to estimate the number of nontrivial solutions of the problem
\[
\begin{cases}
(-i\nabla - A)u + u = |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.7}
\]
where \( \lambda > 0 \) is a positive parameter, \( A(\cdot) := A(\cdot/\lambda) \), \( \Omega \lambda := \lambda\Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a bounded smooth domain and \( p \in (2, 2^*) \), in terms of the Poincaré polynomial of \( \Omega \lambda \), \( P_t(\Omega \lambda) \). In fact, in [2], the authors proved that if \( f \) is a superlinear function with subcritical growth, then for \( \lambda > 0 \) sufficiently large, the number of nontrivial solutions of the Dirichlet problem for equation 1.7 is at least the category of \( \Omega \). Combining the abstract result with arguments present in [2], it is possible to show that 1.7 has at least \( 2P_1(\Omega \lambda) - 1 \) nontrivial solutions provided that \( \lambda \) is sufficiently large. We observe that, unlike the case with no magnetic vector field \( A \), problem 1.7 cannot be written in the form 1.3, and hence these problems are different.

2. **Theorem 1.2.** This section is devoted to the proof of Theorem 1.2. Let \( E \) be a real Hilbert space defined as the closure of \( C_c^\infty(\Omega, \mathbb{C}) \) with respect to the norm induced by the inner product
\[
\langle u, v \rangle_\kappa := \text{Re} \left\{ \int_\Omega [\nabla_A u \nabla_A \overline{v} + \kappa u \overline{v}] \, dx \right\},
\]
where \( \text{Re} \) and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover, for every \( x \in \Omega \),
\[
\nabla_A u(x) := -i\nabla u(x) - A(x)u(x).
\tag{2.1}
\]
The norm induced by this inner product is
\[
\|u\|_\kappa^2 := \int_\Omega [||\nabla_A u||^2 + \kappa|u|^2] \, dx.
\]
As proved in Esteban and Lions [20], the so called diamagnetic inequality holds for every \( u \in E \), that is,
\[
||\nabla_A u|| \geq ||\nabla u||.
\]
Consequently, we have
\[
||u||_\kappa \geq ||u||, \quad \forall u \in E,
\]
where ||.|| denotes the gradient norm for \( H^1_0(\Omega, \mathbb{R}) \); this, in turn, implies \( E \hookrightarrow L^q(\Omega, \mathbb{C}) \) continuously for \( q \in [1, 2^*] \) and compactly for \( q \in [1, 2^*) \).

The functional associated with 1.3, \( I_{\kappa, p, \Omega} : E \rightarrow \mathbb{R} \), is given by
\[
I_{\kappa, p, \Omega}(u) = \frac{1}{2} \int_\Omega (||\nabla_A u||^2 + \kappa|u|^2) \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx, \quad \forall u \in E.
\]
By Sobolev embeddings and diamagnetic inequality, \( I_{\kappa, p, \Omega} \) is well defined. Further more, \( I_{\kappa, p, \Omega} \in C^2(E, \mathbb{R}) \), with
\[
I'_{\kappa, p, \Omega}(u) v = \text{Re} \left( \int_\Omega (\nabla_A u \nabla_A \overline{v} + \kappa u \overline{v} - |u|^{p-2}u \overline{v}) \, dx \right), \quad \forall u, v \in E.
\]
We recall that \( u \in E \) is a weak solution of 1.3 if
\[
\text{Re} \left( \int_{\Omega} (\nabla_A u \nabla_A v + \kappa u v) \, dx \right) = \text{Re} \left( \int_{\Omega} |u|^{p-2} u v \, dx \right), \quad \forall v \in E.
\]
Thus, every critical point of \( I_{\kappa,p,\Omega} \) is a weak solution of 1.3.

Next, we show the Palais-Smale condition for the functional \( I_{\kappa,p,\Omega} \).

**Proposition 1.** The functional \( I_{\kappa,p,\Omega} \) satisfies Palais-Smale condition, that is, every sequence \( (u_n) \subset E \) for which \( \sup_{n} |I_{\kappa,p,\Omega}(u_n)| < +\infty \) and \( I'_{\kappa,p,\Omega}(u_n) \to 0 \) as \( n \to \infty \), has a strongly convergent subsequence.

**Proof.** Let \( (u_n) \in E \) be a sequence satisfying
\[
\sup_{n \in \mathbb{N}} |I_{\kappa,p,\Omega}(u_n)| < \infty \quad \text{and} \quad I'_{\kappa,p,\Omega}(u_n) \to 0.
\]
Taking a subsequence if necessary, we can assume that \( I_{\kappa,p,\Omega}(u_n) \to d \) as \( n \to \infty \). Note that \( (u_n) \) is bounded in \( E \). Indeed, by hypothesis, we have, for \( n \) large enough,
\[
d + 1 + \|u_n\|_{\kappa} \geq I_{\kappa,p,\Omega}(u_n) - \frac{1}{p} I'_{\kappa,p,\Omega}(u_n) u_n = \left( \frac{1}{2} - \frac{1}{p} \right) \|u_n\|_{\kappa}^2.
\]
Hence, \( (u_n) \) is a bounded sequence. Thus, there is \( u \in E \) such that \( u_n \rightharpoonup u \) weakly in \( E \). Consequently, by Sobolev embedding, \( u_n \to u \) in \( L^p(\Omega, \mathbb{C}) \); passing to a subsequence if necessary, we can assume that \( u_n(x) \to u(x) \) almost everywhere in \( \Omega \) and \( |u_n(x)| \leq h(x) \) almost everywhere in \( \Omega \), for some \( h \in L^p(\Omega, \mathbb{C}) \). With these elements, it is standard to show that \( u \) is a critical point for \( I_{\kappa,p,\Omega} \). Moreover, since
\[
\|u_n - u\|_{\kappa}^2 = \left( I'_{\kappa,p,\Omega}(u_n) - I'_{\kappa,p,\Omega}(u) \right)(u_n - u) + \frac{1}{p} \text{Re} \left( \int_{\Omega} (|u_n|^{p-2} u_n - |u|^{p-2} u)((u_n - u)dx) \right),
\]
using once again the properties listed above and Hölder Inequality, we conclude that \( u_n \to u \) in \( E \). \( \square \)

It is straightforward to show that \( I_{\kappa,p,\Omega} \) satisfies the geometric hypotheses of the Mountain Pass Theorem. This fact and Proposition 1, for all \( p \in (2, 2^*) \) and \( \kappa > 0 \), guarantees problem 1.3 has a nontrivial solution \( u \in E \) such that \( I_{\kappa,p,\Omega}(u) = b_{\kappa,p,\Omega} \) and \( I'_{\kappa,p,\Omega}(u) = 0 \), where \( b_{\kappa,p,\Omega} \) denotes the mountain pass level of \( I_{\kappa,p,\Omega} \). Moreover, as in [31, Theorem 4.2],
\[
b_{\kappa,p,\Omega} := \inf_{u \in M_{\kappa,p,\Omega}} I_{\kappa,p,\Omega}(u),
\]
where \( M_{\kappa,p,\Omega} := \{ u \in E \setminus \{0\} ; I'_{\kappa,p,\Omega}(u)u = 0 \} \) is the Nehari manifold associated with \( I_{\kappa,p,\Omega} \).

**Proposition 2.** The Nehari manifold \( M_{\kappa,p,\Omega} \) is diffeomorphic to the unit sphere of \( E \). Moreover, there is \( \delta = \delta(p) > 0 \), independent of \( \kappa > 0 \), such that for every \( u \in M_{\kappa,p,\Omega} \),
\[
\int_{\Omega} (|\nabla_A u|^2 + \kappa |u|^2) \, dx \geq \delta \quad \text{and} \quad I_{\kappa,p,\Omega}(u) \geq \delta.
\]

**Proof.** First of all, if \( u \in M_{\kappa,p,\Omega} \), then
\[
\int_{\Omega} (|\nabla_A u|^2 + \kappa |u|^2) \, dx = \int_{\Omega} |u|^p \, dx.
\]
and, consequently,
\[ I_{p,\kappa}(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega}(|\nabla A u|^2 + \kappa u^2) dx. \]

(2.3)

Diamagnetic inequality combined with Sobolev embedding implies
\[ \| u \|_\kappa^2 \leq \| u \|_\kappa^2 = \int_{\Omega} |u|^p dx \leq C_p p \left( \int_{\Omega} |\nabla |u|^2 dx \right)^{p/2} = C_p \| u \|_\kappa^p, \]

where \( C_p \) is the constant of the embedding \( H^1_0(\Omega, \mathbb{R}) \hookrightarrow L^p(\Omega, \mathbb{R}) \). Thus
\[ \| u \|_\kappa \geq \| u \| \geq C_p^{1/p^*} =: \delta_1, \]

from where it follows
\[ I_{k, p}(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \| u \|_\kappa^2 \geq \left( \frac{1}{2} - \frac{1}{p} \right) \delta_1^2 =: \delta, \forall u \in \mathcal{M}_{k, p}. \]

To conclude the proof, let \( U \) be the unit sphere in \( E \). For every \( u \in U \), let \( \xi(u) > 0 \) be the unique positive number such that
\[ \frac{d}{dt} I_{k, p}(tu)|_{t=\xi(u)} = 0. \]

This defines a \( C^1 \) function \( \xi : U \to (0, +\infty) \), by the Implicit Function Theorem. Thus, \( D : U \to \mathcal{M}_{k, p} \) given by
\[ D(u) = \xi(u)u \in \mathcal{M}_{k, p}. \]

is a \( C^1 \) diffeomorphism. \( \square \)

**Proposition 3.** \( I_{k, p}(\cdot)|_{\mathcal{M}_{k, p}} \) satisfies the Palais-Smale condition.

**Proof.** Let \( (u_n) \subset \mathcal{M}_{k, p} \) be a sequence satisfying
\[ \sup_{n \in \mathbb{N}} |I_{k, p}(u_n)| < \infty \quad \text{and} \quad \left( I_{k, p}(\cdot)|_{\mathcal{M}_{k, p}} \right)'(u_n) \to 0. \]

Taking a subsequence if necessary, we can assume that \( I_{k, p}(u_n) \to d \) as \( n \to \infty \). Arguing as in the proof of Proposition 1, \( (u_n) \subset E \) is bounded. Thus, there is \( u \in E \) such that \( u_n \rightharpoonup u \) weakly in \( E \). Consequently, \( u_n \to u \) in \( L^p(\Omega, \mathbb{C}) \). By [31, Proposition 5.12], for each \( n \in \mathbb{N} \), there is \( \mu_n \in \mathbb{R} \) such that
\[ I_{k, p}'(u_n) - \mu_n G_{k, p}'(u_n) = \left( I_{k, p}(\cdot)|_{\mathcal{M}_{k, p}} \right)'(u_n) = o_n(1), \]

(2.4)

where \( G_{k, p}(u) := I_{k, p}'(u)u \). Since \( u_n \in \mathcal{M}_{k, p} \), by Proposition 2, we obtain
\[ \lim_{n \to \infty} G_{k, p}'(u_n)u_n = \lim_{n \to \infty} (2 - p) \int_{\Omega} |u_n|^p dx \leq (2 - p)\delta < 0. \]

This and 2.4 implies that \( \mu_n \to 0 \) as \( n \to \infty \). The result follows from Proposition 1. \( \square \)

**Corollary 3.** If \( u \) is a critical point of \( I_{k, p} \) constrained to \( \mathcal{M}_{k, p} \), then \( u \) is a critical point of \( I_{k, p} \).

**Proof.** The proof proceeds along the same lines as the proof of Proposition 3. \( \square \)
2.1. **Behaviour of the minimax levels.** For any $p \in (2, 2^*)$ and $\kappa > 0$, let 

$$m_A(\kappa, p, \Omega) := \inf_{u \in E \setminus \{0\}} \frac{\int_{\Omega} (|\nabla A u|^2 + \kappa |u|^2) \, dx}{\left( \int_{\Omega} |u|^p \, dx \right)^{\frac{2}{p}}}.$$ 

We also consider 

$$m(\kappa, p, \Omega) := \inf_{u \in H^1_0(\Omega, \mathbb{R}) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + \kappa u^2) \, dx}{\left( \int_{\Omega} |u|^p \, dx \right)^{\frac{2}{p}}}.$$ 

Then $S = m(\kappa, 2^*, \Omega)$, where $S$ is the best constant of the embedding $H^1_0(\Omega, \mathbb{R}) \hookrightarrow L^{2^*}(\Omega, \mathbb{R})$, which is independent of $\Omega$ and $\kappa \geq 0$. Moreover, arguing as in [5, Theorem 1.1], one can show that

**Lemma 2.1.** For every $\kappa \geq 0$, $m_A(\kappa, 2^*, \Omega) = S$.

Thus, as a consequence of 2.2, 2.3 and Lemma 2.1, we obtain

$$b_{\kappa, p, \Omega} = \frac{1}{2} - \frac{1}{p} \quad m_A(\kappa, p, \Omega) \quad \text{and} \quad b_{\kappa, 2^*, \Omega} = \frac{1}{N} S^{\frac{N}{2}}. \quad (2.5)$$

The following lemma establishes a relation between $b_{\kappa, p, \Omega}$ and $\frac{1}{N} S^{\frac{N}{2}}$.

**Lemma 2.2.** For any given $\kappa \geq 0$, the following limit holds:

$$\lim_{p \to 2^*} b_{\kappa, p, \Omega} = \frac{1}{N} S^{\frac{N}{2}}.$$ 

**Proof.** Fix $\kappa \geq 0$. For $2 \leq r < s \leq 2^*$ and $u \in E$, $|u|_{r, \Omega} \leq |\Omega|^{\frac{r-s}{s-r}} |u|_{s, \Omega}$, where $| \cdot |_{r, \Omega}$ denotes the norm of $L^r(\Omega, \mathbb{R})$ or $\mathbb{C}$, and $|\Omega|$, the Lebesgue measure of $\Omega$. Thus,

$$\int_{\Omega} \frac{(|\nabla A u|^2 + \kappa |u|^2) \, dx}{|u|^2_{r, \Omega}} \geq |\Omega|^{-2(s-r)} \int_{\Omega} \frac{(|\nabla A u|^2 + \kappa |u|^2) \, dx}{|u|^2_{s, \Omega}}. \quad (2.6)$$

Taking $r = p$, $s = 2^*$ and the infimum over all $u \in E \setminus \{0\}$, we obtain, by Lemma 2.1,

$$m_A(\kappa, p, \Omega) \geq \frac{|\Omega|^{-2(s-r)}}{|\Omega|^{-2(2^*-p)}} S. \quad (2.7)$$

On the other hand, taking $r = 2$, $s = p$ and using similar arguments, we conclude

$$m_A(\kappa, p, \Omega) \leq |\Omega|^{\frac{p-2}{2^*-p}} m_A(\kappa, 2, \Omega). \quad (2.8)$$

By 2.7 and 2.8, \{m_A(\kappa, p, \Omega); p \in (2, 2^*)\} is a bounded set. Then, let

$$M := \limsup_{p \to 2^*} m_A(\kappa, p, \Omega) \quad \text{and} \quad m := \liminf_{p \to 2^*} m_A(\kappa, p, \Omega).$$

We claim that $M = S = m$. Indeed, by 2.7 and the definitions of $M$ and $m$,

$$M \geq m \geq \liminf_{p \to 2^*} |\Omega|^{\frac{2(s-r)}{2^*-p}} S = S.$$
Suppose by contradiction that $M > S$. Let $\epsilon \in (0, M - S)$. By the definition of $m_A(\kappa, p, \Omega)$ and Lemma 2.1, there is $\overline{u} \in E$ such that

$$\frac{\int_{\Omega} \left( |\nabla \overline{u}|^2 + \kappa |\overline{u}|^2 \right) dx}{\left( \int_{\Omega} |\overline{u}|^p dx \right)^{\frac{2}{p}}} < S + \frac{\epsilon}{2}. $$

On the other hand, as the function $p \mapsto |\overline{u}|^p_{p, \Omega}$ is continuous, for $p$ sufficiently close to $2^*$, we have

$$\frac{\int_{\Omega} \left( |\nabla \overline{u}|^2 + \kappa |\overline{u}|^2 \right) dx}{\left( \int_{\Omega} |\overline{u}|^p dx \right)^{\frac{2}{p}}} < \frac{\epsilon}{2}. $$

Thus, for $p$ sufficiently close to $2^*$,

$$m_A(\kappa, p, \Omega) \leq \frac{\int_{\Omega} \left( |\nabla \overline{u}|^2 + \kappa |\overline{u}|^2 \right) dx}{\left( \int_{\Omega} |\overline{u}|^p dx \right)^{\frac{2}{p}}} < \frac{\epsilon}{2} + \frac{S + \epsilon}{2},$$

that is, $M = \liminf_{p \to 2^*} m_A(\kappa, p, \Omega) < S + \epsilon < M$, which is a contradiction. Hence $S = M$ and, then, $M = m = S$. The desired equality follows by 2.5.

Without loss of generality, we can assume that $0 \in \Omega$. Let $r > 0$ be such that $B_r(0) \subset \Omega$ and the sets

$$\Omega^+_r := \{ x \in \mathbb{R}^N; \text{dist}(x, \Omega) \leq r \} \quad \text{and} \quad \Omega^-_r := \{ x \in \Omega; \text{dist}(x, \partial \Omega) \geq r \}$$

are homotopically equivalent to $\Omega$.

In the following, for any $\kappa \geq 0$ and $p \in (2, 2^*)$, we consider the functional $J_{\kappa, p} : H^1_0(B_r(0), \mathbb{R}) \to \mathbb{R}$ given by

$$J_{\kappa, p}(u) = \frac{1}{2} \int_{B_r(0)} (|\nabla u|^2 + \kappa u^2) dx - \frac{1}{p} \int_{B_r(0)} |u|^p dx, \quad \forall u \in H^1_0(B_r(0), \mathbb{R}).$$

By [31], $J_{\kappa, p} \in C^2(H^1_0(B_r(0), \mathbb{R}), \mathbb{R})$ and its mountain pass level, denoted here by $c_{\kappa, p}$, satisfies

$$c_{\kappa, p} = \inf_{u \in N_{\kappa, p}} J_{\kappa, p}(u),$$

where

$$N_{\kappa, p} := \{ u \in H^1_0(B_r(0), \mathbb{R}) \setminus \{0\}; J'_{\kappa, p}(u)u = 0 \}$$

is the corresponding Nehari manifold. Moreover, $J_{\kappa, p}$, as well as $J_{\kappa, p}|_{N_{\kappa, p}}$, satisfies the Palais-Smale condition and, using Ekeland’s Variational Principle, there is a positive function $u_{\kappa, p} \in N_{\kappa, p}$ such that $J_{\kappa, p}(u_{\kappa, p}) = c_{\kappa, p}$ and $J'_{\kappa, p}(u_{\kappa, p}) = 0$. By Schwarz symmetrization, we conclude that $u_{\kappa, p}$ is also a radially symmetric function.

As for $c_{\kappa, p}$, putting $m(\kappa, p) := m(\kappa, p, B_r(0))$, one can show that, as in 2.5 and Lemma 2.2,

$$c_{\kappa, p} = \left( \frac{1}{2} - \frac{1}{p} \right) m(\kappa, p)^{\frac{2}{p^*}} \quad \text{and} \quad \lim_{p \to 2^*} c_{\kappa, p} = \frac{1}{N} S_N^{\frac{N}{p^*}}.$$
Thus, by Lemma 2.1,
\[
\lim_{p \to 2^*} c_{\kappa, p} = \lim_{p \to 2^*} b_{\kappa, p, \Omega} = \frac{1}{N} S_{\kappa}^\infty.
\] (2.9)

For \( x, y \in \Omega \), let \( \tau_\rho(x) := \sum_{j=1}^N A_j(y)x_j \), where \( A_j(y) \), \( x_j \) are the \( j \)-th coordinate of \( A(y) \) and \( x \), respectively. Since, by Proposition 2, \( M_{\kappa, \rho, \Omega} \) is diffeomorphic to the unit sphere of \( E \), there is a unique positive number \( t_{\kappa, \rho, y} \) such that \( t_{\kappa, \rho, y} e^{i\tau_\rho} u_{\kappa, \rho}(\cdot - y) \in M_{\kappa, \rho, \Omega} \) – in fact,
\[
t_{\kappa, \rho, y} = \xi (\| e^{i\tau_\rho} u_{\kappa, \rho}(\cdot - y) \| / \| e^{i\tau_\rho} u_{\kappa, \rho}(\cdot - y) \|_\kappa),
\]
where \( \xi \) is the function defined in the proof of Proposition 2. Now, define the function \( \Phi_{\kappa, \rho} : \Omega^- \to M_{\kappa, \rho, \Omega} \) as
\[
[\Phi_{\kappa, \rho}(y)](x) = \begin{cases} t_{\kappa, \rho, y} e^{i\tau_\rho}(x) u_{\kappa, \rho}(\cdot - y), & x \in B_r(y), \\ 0, & x \in \Omega \setminus B_r(y). \end{cases}
\]

**Lemma 2.3.** For a fixed \( \kappa \geq 0 \),
\[
\lim_{p \to 2^*} \max_{y \in \Omega^-} \left| I_{\kappa, \rho, \Omega}(\Phi_{\kappa, \rho}(y)) - \frac{1}{N} S_{\kappa}^\infty \right| = 0.
\]

**Proof.** Fix \( \kappa \geq 0 \). We begin by noticing that for any sequences \( (y_n)_n \subset \Omega^- \) and \( (p_n)_n \subset [2, 2^*) \) such that \( p_n \to 2^* \), we have
\[
I_{\kappa, p_n, \Omega}(\Phi_{\kappa, p_n}(y_n)) \to \frac{1}{N} S_{\kappa}^\infty \quad \text{as} \quad n \to \infty.
\] (2.10)

Indeed, for simplicity, we will write
\[
t_{\kappa, p_n, y_n} = t_n, \quad I_{\kappa, p_n, \Omega} = I_n, \quad \Phi_{\kappa, p_n}(y_n) = \Phi_n(y_n) \quad \text{and} \quad u_{\kappa, p_n} = u_n.
\]

By the definitions of \( \nabla A \), in 2.1, and \( \Phi_n \), since \( u_n \) has compact support contained in \( B_r(0) \), using a change of variables, we obtain
\[
I_n(\Phi_n(y_n)) = \frac{1}{2} \int_\Omega (|\nabla A(\Phi_n(y_n))|^2 + \kappa |\Phi_n(y_n)|^2) dx - \frac{1}{p_n} \int_\Omega |\Phi_n(y_n)|^p dx
\]
\[
= \frac{t_n^2}{2} \int_{B_r(0)} |A(y_n) - A(y_n + x)|^2 |u_n|^2 dx + \frac{t_n^2}{2} \int_{B_r(0)} (|\nabla u_n|^2 + \kappa |u_n|^2) dx - \frac{t_n^2}{p_n} \int_{B_r(0)} u_n^p dx
\]
\[
= \frac{t_n^2}{2} \int_{B_r(0)} |A(y_n) - A(y_n + x)|^2 |u_n|^2 dx + J_{\kappa, p_n}(t_n u_n)
\]
\[
\leq \frac{t_n^2}{2} \int_{B_r(0)} |A(y_n) - A(y_n + x)|^2 |u_n|^2 dx + J_{\kappa, p_n}(u_n)
\]
\[
= \frac{t_n^2}{2} \int_{B_r(0)} |A(y_n) - A(y_n + x)|^2 |u_n|^2 dx + c_{\kappa, p_n}.
\]

On the other hand, by diamagnetic inequality,
\[
I_n(\Phi_n(y_n)) = I_n(t_n e^{i\tau_\rho} u_n(\cdot - y_n)) \geq I_n(e^{i\tau_\rho} u_n(\cdot - y_n))
\]
\[
\geq \frac{1}{2} \int_\Omega (|\nabla (e^{i\tau_\rho} u_n(x - y_n))|^2 + \kappa |e^{i\tau_\rho} u_n(x - y_n)|^2) dx
\]
\[
- \frac{1}{p} \int_\Omega |e^{i\tau_\rho} u_n(x - y_n)|^p dx = J_{\kappa, p_n}(u_n) = c_{\kappa, p_n}.
\]
By hypothesis, we have
\[
\frac{t_n^2}{2} \int_{B_r(0)} |A(y_n) - A(x + y_n)|^2 |u_n|^2 \, dx = o_n(1).
\]
(2.11)

We begin by showing that \( u_n \to 0 \) weakly in \( H_0^1(B_r(0), \mathbb{R}) \) and \( (t_n)_n \) is a bounded sequence. In fact, since \( u_n \in \mathcal{N}_{r,p_n} \) is such that \( J_{c,p_n}(u_n) = c_{r,p_n} \),
\[
\int_{B_r(0)} (|\nabla u_n|^2 + \kappa |u_n|^2) \, dx = \left( \frac{1}{2} - \frac{1}{p_n} \right)^{-1} c_{r,p_n}.
\]
(2.12)

From 2.9 and 2.12, the sequence \((u_n)_n \subset H_0^1(B_r(0), \mathbb{R})\) is bounded. Thus, there is \( v \in H_0^1(B_r(0), \mathbb{R}) \) such that
\[
\begin{aligned}
&u_n \to v \text{ weakly in } H_0^1(B_r(0), \mathbb{R}) \text{ as } n \to \infty, \\
&u_n \to v \text{ in } L^q(B_r(0), \mathbb{R}), \text{ for each } q \in [1, 2^*), \text{ as } n \to \infty, \\
&u_n(x) \to v(x) \text{ almost everywhere in } B_r(0) \text{ as } n \to \infty.
\end{aligned}
\]
(2.13)

By hypothesis, \( u_n \) is a solution of
\[
\begin{cases}
-\Delta u + \kappa u = u_n^{p_n-1} & \text{ in } B_r(0), \\
u = 0 & \text{ on } \partial B_r(0).
\end{cases}
\]
Consequently, for any \( \psi \in C_c^\infty(B_r(0), \mathbb{R}) \),
\[
\int_{B_r(0)} (\nabla u_n \nabla \psi + \kappa u_n \psi) \, dx = \int_{B_r(0)} u_n^{p_n-1} \psi \, dx.
\]
By 2.13, as \( n \to \infty, \)
\[
\int_{B_r(0)} (\nabla u_n \nabla \psi + \kappa u_n \psi) \, dx \to \int_{B_r(0)} (\nabla v \nabla \psi + \kappa v \psi) \, dx.
\]
(2.14)

Since \((u_n^{p_n-1})_n\) is a bounded sequence in \( L^{2^*_t}(\Omega, \mathbb{R}) \) and \( u_n^{p_n-1}(x) \to v^{2^*-1}(x) \) almost everywhere in \( \Omega \), it follows that
\[
u_n^{p_n-1} \to v^{2^*-1} \text{ weakly in } L^{2^*_t}(\Omega, \mathbb{R}).
\]
Consequently, as \( n \to \infty, \)
\[
\int_{B_r(0)} u_n^{p_n-1} \psi \, dx \to \int_{B_r(0)} v^{2^*-1} \psi \, dx, \quad \forall \psi \in H_0^1(B_r(0), \mathbb{R}).
\]
Therefore, the last limit and 2.14 imply that \( v \in H_0^1(B_r(0), \mathbb{R}) \) is a solution of
\[
\begin{cases}
-\Delta u + \kappa u = u^{2^*-1} & \text{ in } B_r(0), \\
u = 0 & \text{ on } \partial B_r(0).
\end{cases}
\]

By Pohozaev’s identity, \( v \equiv 0 \) in \( B_r(0), \) and so we can rewrite the limits 2.13. By the definition of \( t_n, \) we have
\[
\int_{B_r(0)} |A(y_n) - A(x + y_n)|^2 |u_n|^2 \, dx + \int_{B_r(0)} |\nabla u_n|^2 + \kappa |u_n|^2 \, dx
\]
\[
= \int_{\Omega} |A(x) - A(y_n)|^2 |u_n(x - y_n)|^2 \, dx + \int_{\Omega} |\nabla u_n(x - y_n)|^2 \, dx + \kappa |u_n(x - y_n)|^2 \, dx
\]
\[
= \int_{\Omega} |(\nabla A(\rme^{ir_n(x)y_n}) u_n(x - y_n))|^2 + \kappa |\rme^{ir_n(x)y_n} u_n(x - y_n)|^2 \, dx
\]
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\[ t_n p_n^{-2} \int_\Omega |e^{i \tau_n(x)} u_n(x - y_n)|^{p_n} \, dx = t_n p_n^{-2} \int_{B_r(0)} |u_n|^{p_n} \, dx; \]

since \( u_n \in N_{\kappa, p, n} \), we get

\[ \int_{B_r(0)} |A(y_n) - A(x + y_n)|^2 |u_n|^2 \, dx = (t_n p_n^{-2} - 1) \int_{B_r(0)} (|\nabla u_n|^2 + \kappa |u_n|^2) \, dx. \quad (2.15) \]

Moreover, by 2.12 and 2.9, there is \( \delta^* > 0 \) such that, for \( n \) large enough,

\[ \int_{B_r(0)} (|\nabla u_n|^2 + \kappa |u_n|^2) \, dx \geq \delta^*. \quad (2.16) \]

By 2.15, 2.16 and 2.13, since \( v \equiv 0 \), we have \( t_n \to 1 \); using 2.13 again and the boundedness of \( (t_n)_n \), we obtain 2.11, which implies 2.10. Finally, since \( (y_n)_n \subset \Omega^- \) is an arbitrary sequence, the desired limit holds.

2.2. Estimates involving the barycenter function. Consider the function \( \beta : \mathcal{M}_{\kappa, p, \Omega} \to \mathbb{R}^N \), the barycenter function, defined as

\[ \beta(u) = \frac{\int_\Omega x |u|^2 \, dx}{\int_\Omega |u|^2 \, dx}. \]

Our next objective is to state and prove the first result on the barycenter function. We will use the symbol \( \mathcal{M}(\mathbb{R}^N) \) to denote the space of finite Radon measures with sign on \( \mathbb{R}^N \), endowed with the norm \( \|\nu\|_{\mathcal{M}(\mathbb{R}^N)} := |\nu|(\mathbb{R}^N) \), where \( |\nu| \) denotes the total variation of \( \nu \in \mathcal{M}(\mathbb{R}^N) \). This space is the dual of \( C_0(\mathbb{R}^N, \mathbb{R}) \), the space of continuous functions that vanishes at infinity, that is, the space of continuous functions for which the set \( \{x \in \mathbb{R}^N; |f(x)| \geq \epsilon \} \) is compact, for every \( \epsilon > 0 \). Thus, if \( (\nu_n)_n \) is a bounded sequence of Radon measures, then there is \( \nu \in \mathcal{M}(\mathbb{R}^N) \) satisfying \( \nu_n \rightharpoonup^* \nu \), where \( \rightharpoonup^* \) represents weakly-* convergence. For details, see [21, Section 7.3].

**Proposition 4.** For fixed \( \kappa \geq 0 \), there are \( \epsilon(\kappa) > 0 \) and \( p^*(\kappa) \in (2, 2^*) \) such that, for \( p \in [p^*(\kappa), 2^*) \), if \( u \in \mathcal{M}_{\kappa, p, \Omega} \) and \( I_{\kappa, p, \Omega}(u) \leq \frac{1}{N} S^{\frac{2}{p}} + \epsilon(\kappa) \), then \( \beta(u) \in \Omega^+_\kappa \).

**Proof.** Fix \( \kappa \geq 0 \). By 2.9, for \( p \) close enough to \( 2^* \), the set

\[ \left\{ u \in \mathcal{M}_{\kappa, p, \Omega}; I_{\kappa, p, \Omega}(u) \leq \frac{1}{N} S^{\frac{2}{p}} + \epsilon \right\} \]

is nonempty. Suppose, by contradiction, that the result is false. Thus, there are sequences \( (p_n)_n, (\epsilon_n)_n \), with \( p_n \in (2, 2^*), p_n \to 2^*, \epsilon_n > 0, \epsilon_n \to 0 \), and \( u_n \in \mathcal{M}_{\kappa, p_n, \Omega} \) such that

\[ I_{\kappa, p_n, \Omega}(u_n) \leq \frac{1}{N} S^{\frac{2}{p}} + \epsilon_n \text{ and } \beta(u_n) \notin \Omega^+_\kappa, \forall n \in \mathbb{N}. \quad (2.17) \]

On the other hand, 2.9 gives

\[ \liminf_{n \to \infty} I_{\kappa, p_n, \Omega}(u_n) \geq \lim_{n \to \infty} b_{\kappa, p_n, \Omega} = \frac{1}{N} S^{\frac{2}{p}}. \]

Hence, the last two inequalities lead to

\[ \lim_{n \to \infty} I_{\kappa, p_n, \Omega}(u_n) = \frac{1}{N} S^{\frac{2}{p}}; \]

this, in turn, with 2.3, gives us

\[ \lim_{n \to \infty} \int_\Omega (|\nabla A u_n|^2 + \kappa |u_n|^2) \, dx = S^{\frac{2}{p}}. \]
The limit above yields
\[
\lim_{n \to \infty} \frac{\int_{\Omega} (|\nabla A u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{p_n} \, dx \right)^{\frac{2}{p_n}}} = \lim_{n \to \infty} \left( \int_{\Omega} (|\nabla A u_n|^2 + \kappa|u_n|^2) \, dx \right)^{-\frac{2}{p_n}} = S.
\]

Using the diamagnetic inequality and the last limit, we get
\[
\limsup_{n \to \infty} \frac{\int_{\Omega} (|\nabla u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{p_n} \, dx \right)^{\frac{2}{p_n}}} \leq \lim_{n \to \infty} \frac{\int_{\Omega} (|\nabla A u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{p_n} \, dx \right)^{\frac{2}{p_n}}} = S \quad (2.18)
\]

The limit 2.18 implies that, for \( \delta_1 > 0 \) to be chosen later, there is \( n_1 \in \mathbb{N} \) such that for \( n \geq n_1 \),
\[
\frac{\int_{\Omega} (|\nabla u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{p_n} \, dx \right)^{\frac{2}{p_n}}} \leq S + \delta_1. \quad (2.19)
\]

Now, for \( \delta_2 > 0 \) to be also chosen later, there is \( n_2 \in \mathbb{N} \) such that for \( n \geq n_2 \),
\[
\frac{\int_{\Omega} (|\nabla u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{p_n} \, dx \right)^{\frac{2}{p_n}}} \leq \frac{\int_{\Omega} (|\nabla u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{p_n} \, dx \right)^{\frac{2}{p_n}}} + \delta_2. \quad (2.20)
\]

Indeed,
\[
\lim_{n} \int_{\Omega} |u_n|^{p_n} \, dx = \lim_{n} \int_{\Omega} (|\nabla A u_n|^2 + \kappa|u_n|^2) \, dx = S^{\frac{2}{p_n}},
\]

thus \( |u_n|^{p_n} \to S^{\frac{2}{p_n}} \) as \( n \to \infty \). By Hölder inequality, \( |u_n|^{2^*_p} \leq |\Omega|^{\alpha_n} |u_n|^{2^*_p} \), with \( \alpha_n \to 0 \) as \( n \to \infty \); then, \( |\Omega|^{\alpha_n} \to 1 \) as \( n \to \infty \). Finally, given \( \zeta > 0 \), for all \( n \) sufficiently large,
\[
\frac{1}{|u_n|^{2^*_p}} \leq |\Omega|^{\alpha_n} \frac{1}{|u_n|^{2^*_p}} \leq (1 + \zeta) \frac{1}{|u_n|^{2^*_p}} \leq \frac{1}{|u_n|^{2^*_p}} + \zeta \frac{1}{|u_n|^{2^*_p}},
\]

and 2.20 follows.

From 2.19 and 2.20, for \( n \geq \max n_j \), we have
\[
S \leq \frac{\int_{\Omega} (|\nabla u_n|^2 + \kappa|u_n|^2) \, dx}{\left( \int_{\Omega} |u_n|^{2^*_p} \, dx \right)^{\frac{2}{2^*_p}}} \leq S + \delta_1 + \delta_2. \quad (2.21)
\]

We claim that there is \( \eta > 0 \) such that
\[
v \in H_0^1(\Omega, \mathbb{R}), \quad \frac{\int_{\Omega} (|\nabla v|^2 + \kappa v^2) \, dx}{\left( \int_{\Omega} |v|^{2^*_p} \, dx \right)^{\frac{2}{2^*_p}}} \leq S + \eta \Rightarrow \beta(v) \in \Omega_+^\circ. \quad (2.22)
\]
Indeed, suppose, by contradiction, that 2.22 does not hold. Thus, there are \((v_n)_n \subset H^1_0(\Omega, \mathbb{R})\) and \(\eta_n \to 0\) such that
\[
\int_\Omega \left( |\nabla v_n|^2 + \kappa |v_n|^2 \right) dx \leq S + \eta_n, \text{ with } \beta(v_n) \notin \Omega_r^+.
\]

Let \(w_n := v_n/|v_n|_{2^*}.\) Thus, \((w_n)_n \subset H^1_0(\Omega, \mathbb{R})\) is a bounded sequence. Hence, there are \(u \in H^1_0(\Omega, \mathbb{R})\) and finite positive measures \(\mu, \nu \in \mathcal{M}(\mathbb{R}^N)\) verifying, for some subsequence,
\[
\begin{align*}
|w_n| & \to u \text{ weakly in } D^{1,2}(\mathbb{R}^N, \mathbb{R}) \text{ as } n \to \infty, \\
|\nabla w_n - \nabla u|^2 & \rightharpoonup \mu \text{ weakly-* in } \mathcal{M}(\mathbb{R}^N) \text{ as } n \to \infty, \\
|w_n - u|^2 & \rightharpoonup \nu \text{ weakly-* in } \mathcal{M}(\mathbb{R}^N) \text{ as } n \to \infty, \\
w_n(x) & \to u(x) \text{ almost everywhere in } \Omega \text{ as } n \to \infty,
\end{align*}
\]
where, when necessary, we consider \(u\) and \(w_n\) functions defined on \(\mathbb{R}^N\), extended by zero outside of \(\Omega\). By Concentration-Compactness Lemma (e.g. [31, Lemma 1.40]), since \(w_n\) has support contained in \(\Omega\),
\[
S = |\nabla u|_{2^*}^2 + \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}, \quad 1 = |u|_{2^*}^2 + \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}, \quad \|\nu\|_{\mathcal{M}(\mathbb{R}^N)}^2 \leq S^{-1} \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}.
\]

Employing the arguments in [31, Theorem 1.41], \(u \in D^{1,2}(\mathbb{R}^N, \mathbb{R})\) is identically zero, thus \(\nu\) and \(\mu\) are concentrated at a single point \(y \in \overline{\Omega}\) and satisfy \(\|\nu\|_{\mathcal{M}(\mathbb{R}^N)} = S^{-1} \|\mu\|_{\mathcal{M}(\mathbb{R}^N)}\). Let \(\Gamma: \mathbb{R}^N \to \mathbb{R}^N\) and \(\Upsilon: \mathbb{R}^N \to \mathbb{R}\) be continuous functions with compact support such that \(\Gamma(x) = x\) and \(\Upsilon(x) = 1\) for every \(x\) in a neighborhood of \(\overline{\Omega}\). Using these functions, we derive
\[
\beta(v_n) = \beta(w_n) = \frac{\int_\Omega x |w_n|^{2^*} \, dx}{\int_\Omega |w_n|^{2^*} \, dx} = \frac{\int_{\mathbb{R}^N} \Gamma(x) |w_n|^{'2^*} \, dx}{\int_{\mathbb{R}^N} \Upsilon(x) |w_n|^{2^*} \, dx}.
\]
Hence, since \(|w_n|^{2^*} \rightharpoonup \nu\) weakly-*,
\[
\beta(v_n) \to \frac{\nu(\{y\}) \Gamma(y)}{\nu(\{y\}) \Upsilon(y)} = y \in \overline{\Omega},
\]
contradicting the fact that \(\beta(v_n) \notin \Omega_r^+\). Hence, 2.22 holds. For \(\eta\) given by 2.22, take in 2.21 \(\delta_1 = \delta_2 = \frac{\delta}{2}\). Observing that \(\beta(|u_n|) = \beta(u_n)\), we have
\[
\beta(u_n) \in \Omega_r^+,
\]
which contradicts 2.17 and completes the proof.

For any \(\kappa \geq 0\) fixed, consider \(\epsilon(\kappa) > 0\) given by Proposition 4. Define
\[
\epsilon^* = \epsilon^*(\kappa) := \frac{1}{N} S_\Omega^\kappa + \epsilon(\kappa) \quad \text{(2.23)}
\]
and the set
\[
\mathcal{M}_{\kappa,p,\Omega}^\epsilon := \{ u \in \mathcal{M}_{\kappa,p,\Omega}; I_{\kappa,p,\Omega}(u) \leq \epsilon^* \}.
\]

**Corollary 4.** For fixed \(\kappa \geq 0\), there is \(\overline{\kappa}(\kappa) \in (2, 2^*)\) such that, for each \(p \in [\overline{\kappa}(\kappa), 2^*)\),
\[
\Phi_{\kappa,p}(\Omega_r^-) \subset \mathcal{M}_{\kappa,p,\Omega}^- \text{ and } \beta(\mathcal{M}_{\kappa,p,\Omega}^\epsilon) \subset \Omega_r^+.
\]
Proof. Follows immediately from Lemma 2.3 and Proposition 4. \hfill \Box

2.3. Proofs of Theorem 1.2 and Corollary 1. We are now ready to conclude the proofs of Theorem 1.2 and Corollary 1. The key ingredient for the first demonstration is the verification of Theorem 1.1.

Proof of Theorem 1.2. Fix $\kappa \geq 0$, and take $p \in [\overline{p}, 2^*)$, for $\overline{p} = \overline{p}(\kappa)$ given by Corollary 4. Let $\mathcal{K}$ be the set of critical points of $I_{\kappa,p,\Omega}$. Suppose that $\mathcal{K}$ is discrete. We begin by observing that condition (i) is a consequence of the definition of $I_{\kappa,p,\Omega}$, for $\Psi$ given by $\Psi(u) = \frac{1}{p} \int_{\Omega} |u|^p \, dx$. Using that the Hessian form of $I_{\kappa,p,\Omega}$ at $u$ is given by

$$H_{I_{\kappa,p,\Omega}}(u)(v,w) = \langle v, w \rangle_\kappa - (p - 1) \int_{\Omega} |u|^{p-2} \Re(w\overline{v}) \, dx, \quad \forall v, w \in E,$$

we have that $H_{I_{\kappa,p,\Omega}}(u)$ is a bounded symmetric bilinear form, for every $u \in E$. The Riesz representation produces a self-adjoint operator $L(u) : E \to E$ such that $H_{I_{\kappa,p,\Omega}}(u)(v,v) = \langle L(u)v, v \rangle_\kappa$. This and Proposition 1 imply that condition (ii) holds. By Proposition 2, the Nehari manifold $\mathcal{M}_{\kappa,p,\Omega}$ is homeomorphic to the unit sphere of $E$, which implies (iii) and condition 1.4. Consider $\epsilon^*$ given by 2.23.

Now, we claim that $\epsilon^*$ is a regular level of $I_{\kappa,p,\Omega}$. Indeed, we recall that $\epsilon^* = \frac{1}{\kappa} S^\frac{\kappa}{2} + \epsilon(\kappa)$, where $\epsilon(\kappa)$ is given by Proposition 4. If every $\eta \in (0, \epsilon(\kappa)]$ is such that $\frac{1}{\kappa} S^\frac{\kappa}{2} + \eta$ is a critical level of $I_{\kappa,p,\Omega}$, then we have infinite solutions and the result is trivially satisfied. Thus, without loss of generality, we can assume that there is $\eta \in (0, \epsilon(\kappa))$ such that $\frac{1}{\kappa} S^\frac{\kappa}{2} + \eta$ is a regular level of $I_{\kappa,p,\Omega}$, and we can rework the previous steps with this number instead of $\epsilon^*$, if necessary.

By Corollary 4, for $p \in [\overline{p}, 2^*)$, the maps $\Phi_{\kappa,p} : \Omega_+^r \to \mathcal{M}_{\kappa,p,\Omega}^r$ and $\beta : \mathcal{M}_{\kappa,p,\Omega}^r \to \Omega_+^r$ are continuous and satisfy $\beta \circ \Phi_{\kappa,p} = \text{Id}_{\Omega_+^r}$, where, by construction, $\Omega_+^r, \Omega_-^r$ are homotopically equivalent to $\Omega$; thus, (iv) holds. Consequently, by Theorem 1.1, we have

$$\sum_{u \in \mathcal{C}_1} i_t(u) = tP_t(\Omega) + tQ(t) + (1 + t)Q_1(t)$$

and

$$\sum_{u \in \mathcal{C}_2} i_t(u) = t^2[P_t(\Omega) + Q(t) - 1] + (1 + t)Q_2(t),$$

where, for $\sigma \in (0, \delta)$, $\delta > 0$ given by Proposition 2,

$$\mathcal{C}_1 := \{ u \in \mathcal{K} ; \sigma < I_{\kappa,p,\Omega}(u) \leq \epsilon^* \}, \quad \mathcal{C}_2 := \{ u \in \mathcal{K} ; \epsilon^* < I_{\kappa,p,\Omega}(u) \}.$$ 

Then

$$\sum_{u \in \mathcal{K}} i_t(u) = tP_t(\Omega) + t^2[P_t(\Omega) - 1] + Q_3(t),$$

where $Q_3$ is a polynomial with nonnegative integer coefficients. The proof of Theorem 1.2 is complete. \hfill \Box

Proof of Corollary 1. Suppose that every critical point of $I_{\kappa,p,\Omega}$ is nondegenerate. By general Morse theory (see, e.g., [7, Theorem 1.5.8]),

\[ i_t(u) = t^m(u), \quad \text{for all } u \in \mathcal{K}, \]

where $m(u)$ is the numerical Morse index of $u$, and the result follows from Theorem 1.2 by taking $t = 1$. \hfill \Box
Appendix A. In this appendix, we give the proof of Theorem 1.1. We begin by showing a relation between the sets Θ and \( \mathcal{M}^b \).

**Lemma A.1.** Under the assumptions of Theorem 1.1, we have

\[
\mathcal{P}_t(\mathcal{M}^b) = \mathcal{P}_t(\Theta) + Q(t),
\]

where \( Q \) is a polynomial with nonnegative coefficients.

**Proof.** We observe that \( \Phi \) induces a homomorphism \( (\Phi)_k : H_k(\Theta^-) \to H_k(\mathcal{M}^b^-) \) between the \( k \)-th homology groups. Since \( \Phi \) is a injective function, so is \( (\Phi)_k \). Hence, \( \dim H_k(\Theta^-) \geq \dim H_k(\mathcal{M}^b^-) \), and the result follows from the definition of the Poincaré polynomials and the fact that \( \Theta^- \) and \( \Theta \) are homotopically equivalent. \( \Box \)

**Lemma A.2.** Let \( \delta \in (0, b) \) and let \( a \in (\delta, \infty] \) be a noncritical level of \( I \). Then

\[
\mathcal{P}_t(I^a, I^\delta) = t \mathcal{P}_t(I^a).
\]

**Proof.** The proof proceeds along the same lines as the proof of [8, Lemma 5.2]. \( \Box \)

**Lemma A.3.** Let \( \delta \) be as in Lemma A.2. Then

\[
\mathcal{P}_t(I^b, I^\delta) = t \mathcal{P}_t(I^\delta + t) + Q(t)
\]

(A.1)

and

\[
\mathcal{P}_t(E, I^\delta) = t \mathcal{P}_t(E) = t,
\]

(A.2)

where \( Q \) is a polynomial with nonnegative coefficients.

**Proof.** By assumption, \( b^* \) is a regular value. Applying Lemma A.2, for \( a = b^* \), and Lemma A.1, we get A.1. Using the fact that \( \mathcal{M} \) is homeomorphic to the unit sphere in \( E \), which is known to be contractible, we have that \( \mathcal{M} \) is contractible. Hence, \( \dim H^k(\mathcal{M}) = 1 \) if \( k = 0 \), and \( \dim H^k(\mathcal{M}) = 0 \) if \( k \neq 0 \). The identity A.2 follows from Lemma A.2 with \( a = +\infty \) and the fact that \( \mathcal{M} \) is contractible. \( \Box \)

**Lemma A.4.** Let \( \delta \) be as in Lemma A.2. Then

\[
\mathcal{P}_t(E, I^{b^*}) = t^2[\mathcal{P}_t(\Theta) + Q(t) - 1],
\]

(A.3)

where \( Q \) is a polynomial with nonnegative coefficients.

**Proof.** We follow Benci and Cerami [8] in considering the exact sequence

\[
\cdots \to H_k(E, I^\delta) \xrightarrow{j_k} H_k(E, I^{b^*}) \xrightarrow{\partial_k} H_{k-1}(I^{b^*}, I^\delta) \xrightarrow{i_k} H_{k-1}(E, I^\delta) \to \cdots
\]

From A.2, we obtain \( \dim H_k(\mathcal{M}) = 0 \), for every \( k \neq 1 \). Combining this with the fact that the sequence is exact, we obtain that \( \partial_k \) is an isomorphism for every \( k \geq 3 \).

Hence,

\[
\dim H_k(E, I^{b^*}) = \dim H_{k-1}(I^{b^*}, I^\delta), \forall k \geq 3.
\]

(A.4)

For \( k = 2 \), we have

\[
\cdots \to H_2(E, I^\delta) \xrightarrow{j_2} H_2(E, I^{b^*}) \xrightarrow{\partial_2} H_1(I^{b^*}, I^\delta) \xrightarrow{i_2} H_1(E, I^\delta) \to \cdots
\]

Since the homomorphism induced by the canonic projection \( j_2 \) is surjective and \( \dim H_2(E, I^\delta) = 0 \), by A.2, we have

\[
H_2(E, I^{b^*}) = j_2(H_2(E, I^\delta)) = \{0\}.
\]

(A.5)

For \( k = 1 \),

\[
\cdots \to H_1(I^{b^*}, I^\delta) \xrightarrow{i_1} H_1(E, I^\delta) \xrightarrow{j_1} H_1(E, I^{b^*}) \xrightarrow{\partial_1} H_0(I^{b^*}, I^\delta) \to \cdots
\]
Using that $E$ is a connected set, we have
\[ H_0(E, I^{b^*}) = 0. \] (A.6)
We now claim that $i_1$ is an isomorphism. Indeed, as $\Theta \neq \emptyset$ and $\dim H_0(\Theta)$ is the number of connected components of the set $\Theta$, we have $H_0(\Theta) \neq \{0\}$. By A.1, $H_1(I^{b^*}, I^\delta) \neq \{0\}$. From A.2, we obtain $\dim H_1(E, I^\delta) = 1$. Since $i_1$ is injective, it follows that $\dim H_1(I^{b^*}, I^\delta) = 1$, and so $i_1$ is an isomorphism. Hence, as $j_1$ is surjective, we get
\[ \dim H_1(E, I^{b^*}) = 0. \] (A.7)
Combining Lemma A.3 with A.4-A.7, we have
\[ \mathcal{P}_1(E, I^{b^*}) = \sum_{k \geq 3} t^k \dim H_k(E, I^{b^*}) \]
\[ = \sum_{k \geq 3} t^k \dim H_{k-1}(I^{b^*}, I^\delta) = t \sum_{k \geq 3} t^{k-1} \dim H_{k-1}(I^{b^*}, I^\delta) \]
\[ = t \left[ \mathcal{P}_1(I^{b^*}, I^\delta) - t \dim H_1(I^{b^*}, I^\delta) - \dim H_0(I^{b^*}, I^\delta) \right] \]
\[ = t^2 \left[ \mathcal{P}_1(\Theta) + Q(t) - 1 \right], \]
which completes the proof of Lemma A.4.
\[ \square \]

A.1. Proof of Theorem 1.1. Now we are able to conclude proof of Theorem 1.1.

Proof of Theorem 1.1. By (ii), $I$ satisfies the Palais-Smale condition and, for a nondegenerate critical point $u$ of $I$, the linear operator $L(u)$ associated to $H_I(u)$ is a Fredholm operator with index 0. By [7, Example I.5.1], we can use [7, Theorem I.5.9] and Lemmas A.3 and A.4 to get
\[ \sum_{u \in \mathcal{C}_1} i_1(u) = \mathcal{P}_1(I^{b^*}, I^\delta) + (1 + t)Q_1(t) = t \left[ \mathcal{P}_1(\Theta) + Q(t) \right] + (1 + t)Q_1(t) \]
and
\[ \sum_{u \in \mathcal{C}_2} i_1(u) = \mathcal{P}_1(E, I^{b^*}) + (1 + t)Q_2(t) = t^2 \left[ \mathcal{P}_1(\Theta) + Q(t) - 1 \right] + (1 + t)Q_2(t). \] \[ \square \]

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