Comments on the NSVZ $\beta$ Functions in Two-dimensional $\mathcal{N} = (0, 2)$ Supersymmetric Models

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Abstract

The NSVZ $\beta$ functions in two-dimensional $\mathcal{N} = (0, 2)$ supersymmetric models are revisited. We construct and discuss a broad class of such models using the gauge formulation. All of them represent direct analogs of four-dimensional $\mathcal{N} = 1$ Yang-Mills theories and are free of anomalies. Following the same line of reasoning as in four dimensions we distinguish between the holomorphic and canonical coupling constants. This allows us to derive the exact two-dimensional $\beta$ functions in all models from the above class. We then compare our results with a few examples which have been studied previously.
1 Introduction and conclusion

The 2d/4d parallels are known and were used since the time of Polyakov who found asymptotic freedom (AF) in 2d non-linear sigma models [1], in analogy with AF in 4d Yang-Mills theories [2,3]. In the last three decades, 2d/4d correspondence acquired a much deeper meaning by virtue of supersymmetry. Much of non-perturbative dynamics in both 2d/4d supersymmetric gauge theories has been thoroughly understood and found to correspond to each other. By the “2d/4d correspondence” we mean here the cases in which either some of 2d/4d physics contents are exactly the same, e.g. the Alday-Gaiotto-Tachikawa (AGT) correspondence [4], or the dynamical behaviors in 2d and 4d coincide, for instance, the BPS spectra, certain correlation functions, dualities, etc., are identical [5–10]. Among these phenomena, an instructive example is provided by non-Abelian BPS vortex strings [11–13], both in 4d \(\mathcal{N} = 2\) and \(\mathcal{N} = 1\) gauge theories, whose low-energy dynamics are captured by 2d \(\mathcal{N} = (2,2)\) and heterotic \(\mathcal{N} = (0,2)\) sigma models respectively [14–18]. The above vortex strings present a “bridge” between 4d and 2d physics providing a quantitative explanation why the 2d dynamics are in correspondence with the dynamics in its 4d progenitor. This correspondence was established in a wide class of theories both from 2d and 4d directions, perturbatively and non-perturbatively [6, 19–24].

The goal of this paper is to derive NSVZ-like \(\beta\) functions [25–29] in general two-dimensional \(\mathcal{N} = (0,2)\) supersymmetric gauge theories adding new evidence for the 2d/4d correspondence. A number of 2d analogs of the NSVZ \(\beta\) functions were obtained in the past via both perturbative methods and instanton calculus in the \(\mathcal{N} = (0,2)\) CP\(^1\) model [24] and in a large class of heterotically deformed non-linear sigma models (NLSMs) which are deformations of their \(\mathcal{N} = (2,2)\) cousins [6]. Here we focus on another general class of \(\mathcal{N} = (0,2)\) gauged linear sigma models (GLSMs) and obtain the general form of the corresponding \(\beta\) functions. They have the same structure as the NSVZ \(\beta\) function in 4d. In those cases where comparison with the previous results is possible our newly derived GLSM \(\beta\) functions are identical to those of NLSMs. This is not surprising since the NLSMs studied previously can be embedded in GLSMs.

We want to emphasize not only the ubiquity of 2d/4d correspondence, but also the conspiracy of methodologies applicable to both 2d and 4d theories. Historically, 2d sigma models were considered as simplified toy models useful for understanding real world physics in 4d. Instead, in this paper, we follow the opposite direction, from 4d to 2d, establishing and using the 2d analog of the Konishi anomaly [30] and scaling anomalies in 2d \(\mathcal{N} = (0,2)\) gauge theories, à la Arkani-Hamed and Murayama in 4d \(\mathcal{N} = 1\) case [31]. This observation helps us relate holomorphic coupling constants
to canonic ones in 2d GLSMs thus trivializing derivation of their $\beta$ functions. The general master formula obtained in this paper is

$$
\beta(g^2) = -\frac{g^4}{4\pi} \sum_i q_i + \frac{1}{2} \sum_a \tilde{q}_a \gamma_a, 
$$

(1.1)
in the case of 2d $\mathcal{N} = (0,2)$ gauge theories with a single FI coupling

$$
\xi \equiv \frac{2}{g^2},
$$

where $q_i$’s are the $U(1)$ gauge charges of the bosonic matter fields, $\tilde{q}_a$ and $\gamma_a$’s are the $U(1)$ gauge charges and anomalous dimensions of the fermionic matter fields.

The paper is organized as follows: We will briefly review the building blocks of 2d $\mathcal{N} = (0,2)$ supersymmetric GLSMs in section 2 and a non-renormalization theorem for the FI coupling constants in section 3. We then explain the difference between holomorphic and canonical coupling constants both from the perspectives of the Konishi anomaly and the scaling anomalies of matter fields, and derive the master equation (1.1) in section 4. Finally, we apply the formula in several examples.

## 2 Two-dimensional $\mathcal{N} = (0,2)$ GLSMs

The $\mathcal{N} = (0,2)$ superspace is parametrized by 2d bosonic spacetime

$$
x^{\pm \pm} \equiv x^0 \pm x^1
$$

and their $\mathcal{N} = (0,2)$ fermionic partners $\theta^+$ and $\bar{\theta}^+$. The supercharges are defined in terms of these coordinates as follows:

$$
Q_+ \equiv \frac{\partial}{\partial \theta^+} + i\bar{\theta}^+ \partial_{++},
$$

$$
\bar{Q}_+ \equiv -\frac{\partial}{\partial \bar{\theta}^+} - i\theta^+ \partial_{++},
$$

(2.1)

where

$$
\partial_{++} \equiv 2\partial_{x^{++}}, \quad \partial_{--} \equiv 2\partial_{x^{--}}.
$$
Accordingly, the superderivatives are given by

\[ D_+ \equiv \frac{\partial}{\partial \theta^+} - i \bar{\theta}^+ \partial_{++}, \]
\[ \bar{D}_+ \equiv - \frac{\partial}{\partial \bar{\theta}^+} + i \theta^+ \partial_{++}, \]  

(2.2)

which satisfy the conditions

\[ D_+^2 = \bar{D}_+^2 = 0, \quad \{D_+, \bar{D}_+\} = 2i \partial_{++}. \]

With this notation, it is not difficult to build three types of supermultiplets to construct \( \mathcal{N} = (0, 2) \) GLSMs \[14, 32\].

**Gauge multiplets:**
The \( \mathcal{N} = (0, 2) \) gauge multiplet \( U_{--} = (A_{--}, \lambda_-, \bar{\lambda}_-, D) \) is real and adjoint-valued

\[ U_{--} = A_{--} - 2i \theta^+ \bar{\lambda}_- - 2i \bar{\theta}^+ \lambda_- + 2 \theta^+ \bar{\theta}^+ D \]  

(2.3)
in superfield formalism. Here

\[ A_{--} \equiv A_0 - A_1, \quad A_{++} \equiv A_0 + A_1 \]

are the 2d gauge fields, \( \lambda_- \) and \( \bar{\lambda}_- \) are the gaugino fields, and the real field \( D \) is auxiliary. The field \( A_{++} \) is an \( \mathcal{N} = (0, 2) \) singlet.

Next, we can promote superderivatives to be covariant, namely

\[ D_+ \equiv \frac{\partial}{\partial \theta^+} - i \bar{\theta}^+ \nabla_{++} \equiv \frac{\partial}{\partial \theta^+} - i \bar{\theta}^+ (\partial_{++} - i A_{++}), \]
\[ \bar{D}_+ \equiv - \frac{\partial}{\partial \bar{\theta}^+} + i \theta^+ \nabla_{++} \equiv - \frac{\partial}{\partial \bar{\theta}^+} + i \theta^+ (\partial_{++} - i A_{++}), \]
\[ \mathcal{D}_{--} \equiv \partial_{--} - iU_{--} = \nabla_{--} - 2 \theta^+ \bar{\lambda}_- - 2 \bar{\theta}^+ \lambda_- - 2i \theta^+ \bar{\theta}^+ D. \]  

(2.4)
The superfield strength of the gauge multiplet is given by

\[ \Upsilon_- = [\bar{D}_+, \mathcal{D}_{--}] = -2 \left( \lambda_- - i \theta^+ (D - iB) - i \theta^+ \bar{\theta}^+ D_{++} \lambda_- \right), \]  

(2.5)

where

\[ B = \partial_0 A_1 - \partial_1 A_0 - i [A_0, A_1] \]  

(2.6)
is the field strength of the $A_{\mu}$ field. The conjugated superfield $\bar{\Upsilon}_-$ is defined accordingly. The action of the gauge multiplet is as follows:

$$S_{\text{gauge}} = \frac{1}{8\epsilon^2} \text{Tr} \int d^2x \, d\theta^+ d\bar{\theta}^+ \bar{\Upsilon}_- \Upsilon_- = \frac{1}{\epsilon^2} \text{Tr} \int d^2x \left( \frac{1}{2} B^2 + i\bar{\lambda}_- \nabla_{++} \lambda_- + \frac{1}{2} D^2 \right).$$

(2.7)

Here $\epsilon^2$ is the gauge coupling. The corresponding NLSM can be obtained in the limit $\epsilon^2 \to \infty$.

**Chiral multiplets:**

The $\mathcal{N} = (0, 2)$ chiral multiplet $\Phi^i = (\phi^i, \psi^i_+)$ satisfies the usual chiral constraint

$$\bar{D}_+ \Phi^i = 0. \tag{2.8}$$

In the superfield formalism it is written as

$$\Phi^i = \phi^i + \sqrt{2} \theta^+ \psi^i_+ - i \theta^+ \bar{\theta}^+ \nabla_{++} \phi^i, \tag{2.9}$$

where

$$\nabla_\mu \phi^i = (\partial_\mu - iq_i A_\mu) \phi^i.$$ 

Moreover, $q_i$ is the charge of the field $\Phi^i$ with respect to the $U(1)$ gauge field. The action of the chiral multiplets can be written as

$$S_{\text{chiral}} = -\frac{i}{2} \int d^2x \, d\theta^+ d\bar{\theta}^+ \sum_i \bar{\Phi}_i \bar{D}_- \Phi^i$$

$$= \int d^2x \sum_i \left( -|\nabla_\mu \phi^i|^2 + i\bar{\psi}_+ i\nabla_\mu \psi^i_+ - \sqrt{2} q_i \bar{\phi} \lambda_- \psi^i_+ + \sqrt{2} q_i \bar{\psi}_+ \bar{\lambda}_- \phi^i + q_i \bar{\phi} \phi^i \right).$$

(2.10)

**Fermi multiplets:**

Another important matter superfield consists of a fermion $\chi^a_-$ and an auxiliary field $G^a$,

$$\left( \chi^a_-, G^a, \right) \in \Gamma^a_-.$$ 

(2.11)

It is not necessary chiral, but, instead, satisfies the constraint

$$\bar{D}_+ \Gamma^a_+ = \sqrt{2} E^a(\Phi), \tag{2.12}$$
where $E(\Phi)$ is an arbitrary holomorphic function with respect to chiral boson fields $\Phi$’s. In the superfield formalism, it can be expanded as

$$\Gamma^a_\pm = \chi^a_\pm - \sqrt{2}\theta^+ G^a - i\theta^+ \bar{\theta}^+ \nabla_{++} \chi^- - \sqrt{2}\bar{\theta}^+ E^a(\Phi).$$

(2.13)

The action for the fermi multiplet reduces to

$$S_{\text{fermi}} = \frac{1}{2} \int d^2x \, d\theta^+ d\bar{\theta}^+ \sum_a \overline{\Gamma}_-^a \Gamma_+^a$$

$$= \int d^2x \sum_{a,i} \left( i\bar{\chi}_{-a} \nabla_{++} \chi^a_{-} \right. \left. + |G^a|^2 - |E^a(\phi)|^2 - \bar{\chi}_{-a} \frac{\partial E^a}{\partial \phi^i} \psi^i_+ + \text{h.c.} \right).$$

(2.14)

Note that the gauge field strength $\Upsilon_-$ is a particular case of the fermi multiplets in the adjoint representation of the gauge group, satisfying

$$\bar{D}_+ \Upsilon_+ = 0.$$ 

(2.15)

**Superpotentials:**

Last but not least, we need to introduce superpotentials $J_a(\Phi)$ as holomorphic functions of chiral superfields, whose action reduces to a half of the superspace (accompanied by fermi multiplets $\Gamma^a_\pm$),

$$S_J = \frac{1}{\sqrt{2}} \sum_a \int d^2x \, d\theta^+ \Gamma_+^a J_a + \text{H.c.}$$

$$= \sum_a \int d^2x \, G^a J_a(\phi) + \sum_i \chi_{-a} \frac{\partial J^a}{\partial \phi^i} \psi^i_+ + \text{H.c.}. \quad (2.16)$$

Of the utmost interest is the Fayet-Iliopoulos (FI) term as a superpotential given by the gauge field strength, if it admits $U(1)$ factors,

$$S_{\tau} = \frac{1}{4} \text{Tr} \int d^2x \, d\theta^+ \tau \Upsilon_+ |_{\theta^+=0} + \text{h.c.} = \text{Tr} \int d^2x \left( -\xi D + \frac{\theta}{2\pi} B \right),$$

(2.17)

where for simplicity we only consider theories with a single FI term, and

$$\tau = \frac{\theta}{2\pi} + i\xi$$

(2.18)
is the complexified FI coupling constant.

**GLSM action:**
Overall we assemble all the above ingredients and arrive at the action of $\mathcal{N} = (0, 2)$ supersymmetric GLSM,

$$ S = S_{\text{gauge}} + S_{\text{chiral}} + S_{\text{fermi}} + S_{\tau} + \text{H.c.}. $$

Here and below, without loss of generality, we will consider theories in which the superpotentials are limited to FI terms. Importantly, for such theories to be consistent at the quantum level (i.e. free of internal anomalies), we need to impose constraints on the representations of the chiral and fermi multiplets to get rid of the gauge anomalies, see also in [33],

$$ U(1) \text{ gauge}: \quad \sum_i q_i^2 = \sum_a \tilde{q}_a^2, $$

$$ \text{non-Abelian gauge}: \quad \sum_i t_2(i) = t_2(A) + \sum_a t_2(a), $$

where $q_i$ and $\tilde{q}_a$ are $U(1)$ gauge charges of chiral and fermi multiplets, $t_2$ is the dual Coxeter number, and “$i$”, “$a$” and “$A$” denote the Reps. of chiral, fermi and gauge multiplets.

### 3 A non-renormalization theorem for the holomorphic coupling $\tau$

In 2$d$ gauge theories, the gauge coupling $e$ has dimension of mass, and is thus super-renormalizable. For energy scale $\mu \ll e$, the gauge multiplets will be non-dynamical and we arrive at NLSMs. Therefore the only sensible parameter in the theory is its FI coupling constant $\tau$, which is marginal and runs at the quantum level. In much the same way as with the gauge couplings in 4$d$ $\mathcal{N} = 1$ gauge theories, the 2$d$ FI parameter $\tau$, as the coupling of the $\mathcal{N} = (0, 2)$ superpotential, is subject to a non-renormalization theorem and receives at most one-loop correction (see e.g. [28]) . We will follow [28][31] in reviewing the relevant argument.
From eq. (2.19), we see that the action $S$ depends on $\tau$ holomorphically. It is convenient to use the notation

$$2\pi i \tau = -2\pi \xi + i \theta \equiv -\frac{4\pi}{g^2} + i \theta.$$  \hspace{1cm} (3.1)

Let us ask ourselves: when we change the cutoff from $M_0$ to $\mu$, how the coupling $2\pi i \tau(\mu)$ (in the Wilsonian sense) changes to keep the low-energy physics intact. To answer this question, let us examine an ansatz

$$2\pi i \tau(\mu) = 2\pi i \tau(M_0) + f\left(2\pi i \tau(M_0), \log \frac{M_0}{\mu}\right).$$  \hspace{1cm} (3.2)

It is worth noting that a $2\pi$ shift of the $\theta$ angle leads no change of physics, therefore at most,

$$f\left(2\pi i \tau(M_0), \log \frac{M_0}{\mu}\right) \longrightarrow f\left(2\pi i \tau(M_0), \log \frac{M_0}{\mu}\right) + 2\pi i F\left(\log \frac{M_0}{\mu}\right), \text{ for } \theta \rightarrow \theta + 2\pi,$$

where function $F\left(\log \frac{M_0}{\mu}\right)$ can only take integer values. Furthermore because $F(0) = 0$, by continuity we conclude that function $f$ is periodic respect to the $\theta$ angle. Therefore the $\beta$ function for $2\pi i \tau$,

$$\beta(2\pi i \tau) = \frac{\partial}{\partial \mu} (2\pi i \tau(\mu)) = \frac{\partial f}{\partial \mu},$$  \hspace{1cm} (3.4)

is periodic with respect to $\theta$ and admits a Fourier expansion,

$$\beta(2\pi i \tau) = \sum_{n \geq 0} b_n e^{2\pi i n \tau}.$$  \hspace{1cm} (3.5)

It is clear that in perturbation theory we can only have non-negative integer values of $n$ appearing in the expansion (3.5). Also, in the perturbative regime we at most have $b_0$ nonzero, i.e.

$$\beta(2\pi i \tau) = b_0,$$  \hspace{1cm} (3.6)

It perturbation theory it is obvious that all $b_n$'s with $n = 1, 2, 3, \ldots$ vanish. Hence the non-renormalization theorem of the absence of higher loops is proven for the holomorphic coupling.

Non-perturbatively, one needs to apply the anomalous $R$-symmetry of $\mathcal{N} = (0, 2)$, which guarantees that the $\theta$ angle receives no quantum corrections at all.
Consequently $\beta(2\pi i\tau)$ is independent of $\text{Im}(2\pi i\tau)$, and, simultaneously is holomorphic in $2\pi i\tau$. It implies that $\beta(2\pi i\tau)$ can only be a constant, i.e. eq. (3.6) holds both perturbatively and non-perturbatively.

Before proceeding to the discussion of the canonical coupling $\tau_c$ in next sections, let us first calculate $b_0$ that would be used latter. It can be easily obtained by inspecting the $D$ term of the action (2.19),

$$S_D = \int d^2x \left( \frac{1}{2e^2} D^2 - \xi D + \sum_i q_i \bar{\phi}_i D\phi^i \right). \quad (3.7)$$

From (3.7) we see that the real part of $\tau$ receives a tadpole one-loop correction. The tadpole graph emerges through contracting $\phi$ and $\bar{\phi}$. As a result,

$$\xi(\mu) = \xi(M_0) - \frac{\sum_i q_i}{2\pi} \log \left( \frac{M_0}{\mu} \right), \quad (3.8)$$

which implies, in turn, that

$$\beta(\xi) = \frac{\sum_i q_i}{2\pi}, \quad \text{or, say, } \beta(g^2) = -\frac{\sum_i q_i}{4\pi}, \quad (3.9)$$

and

$$b_0 = -\sum_i q_i.$$

## 4 From the holomorphic to canonic coupling

As known from [28], all higher order loops in the gauge coupling renormalization appear in passing from the holomorphic to canonic coupling from the $Z$ factors of the matter fields (which are converted into the anomalous dimensions in the $\beta$ functions). To see how this happens we must convert the kinetic terms of the matter fields into (2.17) by virtue of anomalies. In other words, we must take into account a subtle difference between the Wilsonian Lagrangian and 1PI irreducible functional (see [25–28]).

Below we will discuss two alternative (but related) derivations, through the Konishi anomaly [30] and through the scale anomaly [31].

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1 As in the $4d$ case, the tadpole correction appears if and only if $\sum_i q_i \neq 0$. 


4.1 The Konishi anomaly in $\mathcal{N} = (0, 2)$ GLSM

It is not difficult to establish the 2d analog of the Konishi anomaly. To this end, as an example, we will consider the operator $\sum_a \bar{\Gamma}_- a \Gamma^a$ appearing in (2.14) (assuming that $E^a = 0$). Classically, the equation of motion for this operator is

$$D_+ \left( \sum_a \bar{\Gamma}_- a \Gamma^a \right) = 0 .$$

(4.1)

This follows, e.g. from inspection of the $\bar{\theta}^+$ component. However, at the quantum level this particular component contains a well-known anomaly in the derivative of the $\chi^-$ current, see more details in appendix B and also [34], analogous to the triangle anomaly in the axial current in 4d.

$$\partial_{++} \left( \sum_a \bar{\chi}^- a \chi^a \right) = \sum_a \bar{\tilde{q}}_a B \bigg|_{U(1)} \frac{2\pi}{\pi B} \left| \bar{U}(1) \right|,$$

(4.2)

where $B$ is defined in (2.6). Note that the relative coefficient between $D$ and $B$ in (2.5) is rigidly fixed by $\mathcal{N} = (0, 2)$ supersymmetries. Needless to say, that the full derivative in the $\bar{U}(1)$ part does not appear in the action classically (it can be dropped). However, at the quantum level we can establish the following relations (after evolving the action from $M_0$ down to $\mu$),

$$\Delta \mathcal{L}_\Gamma(\mu) = -\frac{1}{2} Z_{\text{fermi}} \int d\bar{\theta}^+ d\theta^+ \left( \bar{\Gamma}_- a \Gamma^a \right) = -\frac{1}{2} \int d\bar{\theta}^+ D_+ \left( \bar{\Gamma}_- a \Gamma^a \right)$$

$$= i \frac{Z_{\text{fermi}}}{2} \partial_{++} \left( \sum_a \bar{\chi}^- a \chi^a \right) = i Z_{\text{fermi}} \sum_a \frac{\bar{\tilde{q}}_a}{4\pi} B \bigg|_{U(1)}$$

$$= i Z_{\text{fermi}} \sum_a \frac{\bar{\tilde{q}}_a}{8\pi} \left( \int d\bar{\theta}^+ \chi^- + \int d\theta^+ \bar{\chi}^- \right) \bigg|_{U(1)},$$

(4.3)

where in the last step, we uplifted the equation to the level of superspace, cf. (2.17). The $\Upsilon_-$ part gives the evolution of the wave function renormalization of fermion $\Gamma^a_-$ to the FI-coupling constant $\tau$, see also eq. (4.7). Adding the one-loop tadpole graph and differentiating over $\mu/\partial \mu$ we arrive at the $\bar{\tilde{q}}_a \gamma_a$ term in (1.1).

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The triangle anomalous graph in four dimensions is replaced in two dimensions by a diangle graph. That’s why the right-hand side in (4.2) is linear in $\bar{\tilde{q}}_a$. 

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4.2 Scaling anomalies

Now we would like to discuss the $2d \mathcal{N} = (0, 2)$ $\beta$ function along the the lines of [31]. It is true that the holomorphic $\tau$ only receives one-loop correction, however, because of the normalization point running down from $M_\Omega$ to $\mu$, the kinetic terms of the matter fields will receive a wave function renormalization,

$$
\sum_i \bar{\Phi}_i \mathcal{D}_- \Phi^i \rightarrow \sum_i Z_i(\mu) \bar{\Phi}_i \mathcal{D}_- \Phi^i, \quad \sum_a \bar{\Gamma}_- \Gamma^a \rightarrow \sum_a Z_a(\mu) \bar{\Gamma}_- \Gamma^a, \tag{4.4}
$$

see section 4.1 for $\bar{\Gamma}_- \Gamma^a$.

To keep all matter fields canonically normalized, we need to change field variables, i.e. redefine

$$
\Phi^i \equiv \frac{1}{\sqrt{Z_i(\mu)}} \Phi'^i, \quad \Gamma^a_- \equiv \frac{1}{\sqrt{Z_a(\mu)}} \Gamma'^a_. \tag{4.5}
$$

However, such rescaling will result in anomalous Jacobians from the functional measure. Formally we have

$$
[d \Phi^i] = \left[ d \left( \frac{1}{\sqrt{Z_i(\mu)}} \Phi'^i \right) \right] = \text{sDet} \left( \frac{1}{\sqrt{Z_i(\mu)}} \right) [d \Phi'^i] = [d \Phi'^i] e^{-\frac{1}{2} \log Z_i(\mu) s\text{Tr}_{\Phi^i} 1},
$$

$$
[d \Gamma^a_-] = \left[ d \left( \frac{1}{\sqrt{Z_a(\mu)}} \Gamma'^a_\cdot \right) \right] = \text{sDet} \left( \frac{1}{\sqrt{Z_a(\mu)}} \right) [d \Gamma'^a_-] = [d \Gamma'^a_-] e^{-\frac{1}{2} \log Z_a(\mu) s\text{Tr}_{\Gamma^a_-} 1}, \tag{4.6}
$$

where “sDet” and “sTr” denote the super-determinant and super-trace, respectively. The super-trace is superficially vanishing due to supersymmetries. Nevertheless, in a non-trivial gauge field background, we can show that they give rise to terms proportional to the $U(1)$ field strength $\Upsilon_\cdot$. More specifically,

$$
\text{sTr}_{\Phi^i} 1 = -i \frac{q_i}{8\pi} \int d^2x \, d\theta^+ \, \Upsilon_- |_{\theta^+ = 0}, \quad \text{sTr}_{\Gamma^a_-} 1 = i \frac{\tilde{q}_a}{8\pi} \int d^2x \, d\theta^+ \, \Upsilon_- |_{\theta^+ = 0}. \tag{4.7}
$$

The derivation of this formula is presented in appendix B. Therefore, the holomorphic $\tau$ will receive non-holomorphic corrections from wave function renormalizations,

$$
\tau \rightarrow \tau_c = \tau + \sum_i i \frac{q_i}{4\pi} \log Z_i(\mu) - \sum_a i \frac{\tilde{q}_a}{4\pi} \log Z_a(\mu). \tag{4.8}
$$

The anomalous dimensions of $\Phi^i$ and $\Gamma^a_\cdot$ are given by

$$
\gamma_i = -\mu \frac{\partial}{\partial \mu} \log Z_i(\mu), \quad \text{and} \quad \gamma_a = -\mu \frac{\partial}{\partial \mu} \log Z_a(\mu). \tag{4.9}
$$
and they are non-holomorphic. This statement is in one-to-one correspondence with the NSVZ $\beta$ function in four dimensions.

Differentiating $\log \mu$ on both sides of eq.(4.8) and using eq.(3.9), we have

$$\beta(\tau_c) = i \left( \sum_{i} q_i \frac{1}{2\pi} - \sum_{i} \frac{q_i}{4\pi} \gamma_i + \sum_a \frac{q_a}{4\pi} \gamma_a \right).$$

(4.10)

In terms of coupling constant

$$\text{Im}(\tau_c) = \xi_c \equiv \frac{2}{g_c^2}$$

(4.11)

we have

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi} \left( \sum_{i} q_i - \frac{1}{2} \sum_{i} q_i \gamma_i + \frac{1}{2} \sum_a \tilde{q}_a \gamma_a \right).$$

(4.12)

Furthermore, from eq.(3.7), the $\beta$ function of $g_c^2$, or say, $\xi$, is nothing other than the wave function renormalization of chiral multiplets, i.e.

$$\gamma_i = \frac{\beta(g_c^2)}{g_c^2}. \qquad (4.13)$$

Using it, we arrive at the master formula,

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi} \left( \sum_{i} q_i + \frac{1}{2} \sum_a \tilde{q}_a \gamma_a \right) \left( 1 - \sum_{a, \tilde{a}} \frac{q_{a, \tilde{a}}}{8\pi} g_c^2 \right).$$

(4.14)

Remark: The gauge multiplets have no contribution to the $\beta$ function, because $\tau_c$ is associated with the $U(1)$ factor gauge group, with respect to which the gauge multiplet is $U(1)$ neutral.

## 5 Examples

In this section, we will apply eq.(1.1) in various examples.
5.1 $\mathcal{N} = (2, 2) \mathbb{C}\mathbb{P}^{N-1}$ model

For $\mathcal{N} = (2, 2)$ supersymmetries, the $\mathcal{N} = (0, 2)$ chiral and fermi multiplets are combined to an $\mathcal{N} = (2, 2)$ chiral multiplet. We have

$$q_i = \bar{q}_a, \quad \text{and} \quad Z_i = Z_a, \quad \text{for} \quad i = a = 1, 2, \ldots$$

(5.1)

Therefore the holomorphic $\tau$ and canonical $\tau_c$ coincide, and the $\beta$-function terminates at one-loop, in terms of $g_c^2$

$$\beta(g_c^2) = -\sum q_i g_c^4$$

(5.2)

Especially, for a $U(1)$ gauge theory with all $q_i = 1$, we have the standard $\mathcal{N} = (2, 2) \mathbb{C}\mathbb{P}^{N-1}$ sigma model, and its $\beta$-function is

$$\beta(g_c^2) = -\frac{N}{4\pi} g_c^4.$$  

(5.3)

5.2 $\mathcal{N} = (0, 2) \mathbb{C}\mathbb{P}^{N-1}$ model

We can deform the previous $\mathcal{N} = (2, 2) \mathbb{C}\mathbb{P}^{N-1}$ model by deleting part of $\mathcal{N} = (2, 2) U(1)$ field strength, considered in [16]. In the language $\mathcal{N} = (0, 2)$ supersymmetries, the $\mathcal{N} = (2, 2) U(1)$ field strength $\Sigma_{(2,2)}$ can be decomposed as,

$$\Sigma_{(2,2)} = \Sigma_{(0,2)} \oplus \Upsilon_-, \quad (5.4)$$

where the $\Sigma_{(0,2)}$ is a $\mathcal{N} = (0, 2)$ chiral superfield and $\Upsilon_-$ is the $\mathcal{N} = (0, 2)$ fermi multiplet as the field strength of $U(1)$ gauge multiplet. $\mathcal{N} = (2, 2)$ chiral multiplet $\Phi^{i}_{(2,2)}$ also admits a decomposition as

$$\Phi^{i}_{(2,2)} = \Phi^{i} \oplus \Gamma^i_-, \quad (5.5)$$

and the $\mathcal{N} = (0, 2)$ fermi multiplet $\Gamma^i_-$ satisfy the constraint

$$\mathcal{D}_+ \Gamma^i_- \propto \Sigma_{(0,2)} \Phi^i.$$  

(5.6)

Now, if we delete $\Sigma_{(0,2)}$, the deformed theory will have only $\mathcal{N} = (0, 2)$ supersymmetry, and the fermi multiplets satisfy

$$\mathcal{D}_+ \Gamma^i_- = 0.$$  

(5.7)

\[\text{Exactly the same occurs in 4d Yang-Mills [28][29].}\]
Its β function turns out to be
\[ \beta(g_c^2) = -\frac{Ng_c^4}{4\pi} \left( 1 + \frac{1}{2} \gamma \right) \left( 1 - \frac{k}{8\pi g_c^2} \right), \]
where γ denotes the anomalous dimension of Fermi multiplet \( \Gamma_- \). We want to further comment that, in [24], the authors also considered a type of deformed \( \mathcal{N} = (0, 2) \) CP\(^1\) model at the level of NLSM, which is different from ours. However, we do see that the β functions of the two models are similar. To compare the difference between our model and that in [24], we discuss its non-linear formalism in appendix A.

### 5.3 Heterotically deformed \( \mathcal{N} = (0, 2) \) CP\(^{N-1}\) model

We can also consider a further deformation from the \( \mathcal{N} = (0, 2) \) CP\(^{N-1}\) model discussed above, by adding an additional gauge singlet \( \mathcal{N} = (0, 2) \) fermi multiplet,
\[ \Omega_- = \eta_+ - \sqrt{2} \theta^+ H - i\theta^+ \bar{\theta}^+ \bar{\eta}_- , \]
(5.9)
to the \( \mathcal{N} = (0, 2) \) model, with the corresponding deformed term in the action,
\[ S_\Omega = \int d^2x d\theta^+ d\bar{\theta}^+ \left( \frac{1}{2} \bar{\Omega} \Omega + \frac{\kappa}{2} \bar{\Phi}_i \Gamma_i \Omega + \text{h.c.} \right) , \]
(5.10)
where \( \kappa \) is an additional coupling. It is crucial to note that, since we start from the \( \mathcal{N} = (0, 2) \) model, all fermi multiplets satisfy
\[ \bar{D}_+ \Omega_- = \bar{D}_+ \bar{\Omega}_- = 0 . \]
(5.11)
This constraint turns out to be important, because it guarantees that the interaction term can be recast in half superspace as,
\[ \frac{\kappa}{2} \int d^2x d\theta^+ d\bar{\theta}^+ \bar{\Phi}_i \Gamma_i \bar{\Omega}_- = \frac{\kappa}{2} \int d^2x d\theta^+ \bar{D}_+ \bar{\Phi}_i \Gamma_i \bar{\Omega}_- . \]
(5.12)
It was argued in [23] that this type of interaction is subject to a “D-term” non-renormalization theorem in 2d, see also [6]. Therefore, the holomorphic coupling constant \( \kappa \) is not renormalized. Here we pause and remark that, if one tries to perform the heterotic deformation from \( \mathcal{N} = (2, 2) \) CP\(^{N-1}\) GLSM, there would be no non-renormalization theorem to protect the coupling \( \kappa \), because in the \( \mathcal{N} = (2, 2) \) case, \( \bar{D}_+ \Gamma_i \propto \Sigma_{(0,2)} \Phi^i \), see eq.(5.6). This differs from the situation in [6].
where the heterotic deformation is indeed performed on $\mathcal{N} = (2, 2)$ $\mathbb{CP}^{N-1}$ NLSM, because the superderivative acting on the fermi multiplet in NLSM automatically vanishes.

Since the coupling $\kappa$ receives no renormalization, we thereby will focus on the $\beta$ function of $\xi$, or say $g_c^{-2}$, in the presence of the coupling constant $\kappa$. Let us first write down the action in components,

$$S_\Omega = \int d^2x \left( i \bar{\eta}_- \partial_{++} \eta_- + \bar{H} H \right)$$

$$+ \kappa \int d^2x \left( i \nabla_{++} \bar{\phi}_i \chi_- \eta_- + G^i \bar{\psi}_{++} \eta_- - H \bar{\psi}_{++} \chi_- \right) + \text{h.c.} \, .$$

(5.13)

The key observation, see also [6], is that the evolution of the interaction term $i\kappa \nabla_{++} \bar{\phi}_i \chi_- \eta_-$ and its Hermitian conjugate will give a finite shift to the kinetic term of $\phi^i$, i.e.

$$\left\langle \kappa \int d^2x \left( i \nabla_{++} \bar{\phi}_i \chi_- \eta_- \right) , \bar{\kappa} \int d^2y \left( i \nabla_{++} \phi^i \chi_- \eta_- \right) \right\rangle = - \frac{\kappa^2}{4\pi Z_\chi Z_\eta} \int d^2x |\nabla \phi^i|^2 \, ,$$

(5.14)

where we take fermions as quantum fluctuations and bosons as a background. We write the wave function renormalizations of $\chi_-$ and $\eta_-$ explicitly. It was argued in [6] that this $|\kappa|^2$ iteration is limited to one-loop in the computation of the quantum correction in the instanton background. Here we have a similar situation – our 2d GLSM admits an (anti-)vortex background, say,

$$\nabla_{\bar{z}} \bar{\phi}_i = 0 \, , \text{ or } \nabla_{\bar{z}} \phi^i = 0 \, ,$$

(5.15)

where $\nabla_{\bar{z}}$ is the Euclidean continuation of $\nabla_{++}$. In this background, the iteration of $|\kappa|^2$ will not enter higher loops. Nevertheless, the wave function renormalization of the fields $\psi^i_-$ and $\eta_-$ will still enter higher loops evaluation. Therefore, we define a new coupling,

$$h^2 \equiv \frac{|\kappa|^2}{Z_\chi Z_\eta} \, ,$$

(5.16)

whose $\beta$ function is given by

$$\beta(h^2) = \mu \frac{\partial}{\partial \mu} h^2 = h^2 (\gamma_\chi + \gamma_\eta) \, ,$$

(5.17)

where

$$\gamma_\chi = -\mu \frac{\partial}{\partial \mu} \log Z_\chi(\mu) , \text{ and } \gamma_\eta = -\mu \frac{\partial}{\partial \mu} \log Z_\eta(\mu) \, ,$$

(5.18)
are the anomalous dimension of the fields $\chi_i^-$ and $\eta^-$. 

Now we assemble this additional contribution to the one-loop correction of the holomorphic coupling $\xi$. The imaginary part of eq. (4.8) is thus modified as

$$\frac{2}{g_c^2} = \frac{2}{g^2} - \frac{h^2}{4\pi} + \frac{N}{4\pi} \log Z_\phi(\mu) - \frac{N}{4\pi} \log Z_\chi(\mu).$$ (5.19)

Differentiating with respect to the running scale $\mu$, and using eqs. (4.13) and (5.17), we arrive at the $\beta$ function for $g_c^2$ in the heterotically deformed $\mathcal{N} = (0, 2)$ $\mathbb{CP}^{N-1}$ GLSM,

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi} N \left(1 + \frac{\gamma_{\chi}}{2}\right) - \frac{h^2(\gamma_{\chi} + \gamma_{\eta})}{1 - \frac{N}{8\pi} g_c^2}. \quad (5.20)$$

Finally, we can compare eq. (5.23) to the master formula in [6]. In [6], the kinetic term of the fermion $\chi_i^-$ (in their notation, it was $\psi^i_R$) is non-linearly coupled to the bosonic field $\phi^i$. It makes the definition of the wave function renormalizations of the two theories different up to a scale factor $g_c^2$, i.e.

$$Z_\chi \text{here} = g_c^2 Z_\chi \text{there}. \quad (5.21)$$

Therefore it leads us to define

$$h'^2 = h^2 g_c^2, \quad \text{and} \quad \gamma'_{\chi} = \gamma_{\chi} + \frac{\beta(g_c^2)}{g_c^2}. \quad (5.22)$$

Under these new definition, we exactly reproduce the master formula in [6],

$$\beta(g_c^2) = -\frac{g_c^4}{4\pi} N g_c^2 \left(1 + \frac{\gamma'_{\chi}}{2}\right) - h'^2(\gamma'_{\chi} + \gamma_{\eta}) \frac{1}{1 - \frac{h'^2}{4\pi}}. \quad (5.23)$$

Acknowledgments

The research of J.C. is supported in part by the Chinese Academy of Sciences (CAS) Hundred-Talent Program and by Project 11747601 supported by National Natural Science Foundation of China. The work of M.S. is supported in part by DOE grant de-sc0011842.
In this appendix, we transform the action of the deformed $\mathcal{N} = (0,2)$ $\mathbb{CP}^{N-1}$ model of section 5.2 into the corresponding NLSM version. The NLSM can be obtained by integrating out the gauge multiplet of its GLSM cousin at the energy scale $\mu \ll e$. Then, one can study the model in the geometric formalism. First, by integrating the $D$ term, eq.(3.7), one finds the potential

$$V_D = \left( \sum_i \bar{\phi}_i \phi_i - \xi \right)^2. \tag{A.1}$$

On the level of NLSM, it constrains all bosonic fields on $S^{2N-1}$, i.e. $\phi^i$ must satisfy the equation

$$\sum_i \bar{\phi}_i \phi^i - \xi = 0. \tag{A.2}$$

On the other hand, integrating the gaugino fields $\lambda_-$ and $\bar{\lambda}_-$ in eq.(2.10), we see that the fermion fields $\psi_+^i$ are subject to constraints

$$\sum_i \bar{\phi}_i \psi_+^i = 0, \tag{A.3}$$

implying that $\psi_+^i$'s live on the tangent bundle of the manifold. In fact, we can rewrite eqs.(A.2) and (A.3) together in terms of superfields,

$$\sum_i \Phi_i \Phi^i - \xi = 0. \tag{A.4}$$

To obtain the $\mathbb{CP}^{N-1}$ model, we need to also take account of the $U(1)$ gauge imposed on $\Phi^i$'s. We can use this gauge to fix one of the chiral multiplet, say the $N$-th field $\Phi^N$, to have its bosonic field real,

$$\Phi^N = \varphi + \sqrt{2} \theta^+ \kappa_+ + \cdots, \tag{A.5}$$

where $\varphi$ now is a real boson, and $\kappa_+$ is its superpartner that is still a complex Weyl fermion. Further we define the gauge invariant coordinates,

$$Z^i = z^i + \sqrt{2} \theta^+ \zeta_+^i \equiv \Phi^i_{\Phi^N}, \quad \text{for} \quad i = 1, 2, \ldots, N - 1, \tag{A.6}$$

from which we find

$$z^i = \frac{\phi^i}{\varphi}, \quad \text{and} \quad \zeta_+^i = \frac{1}{\varphi} \left( \psi_+^i - \frac{\phi^i}{\varphi} \kappa_+ \right). \tag{A.7}$$
Now, we can solve for $\Phi^i$ in terms of $Z^i$. From eq. (A.4), we express $\Phi^N$ as

$$|\Phi^N|^2 = \frac{\xi}{1 + \overline{Z}_i Z^i},$$

(A.8)

or, in components,

$$\varphi = \frac{\sqrt{\xi}}{\sqrt{1 + \overline{z}_i z^i}} \equiv \frac{\sqrt{\xi}}{\rho}, \quad \kappa_+ = -\frac{\sqrt{\xi}}{\rho^2} \overline{z}_i \zeta_+^i.$$  

(A.9)

We then solve

$$\phi^i = \frac{\sqrt{\xi}}{\rho^2} \overline{z}_i \zeta_+^i \quad \text{and} \quad \psi_+^i = \frac{\sqrt{\xi}}{\rho^2} \left( \delta_j^i - \frac{1}{\rho^2} \overline{z}_i \zeta^j_+ \right) \zeta_+^j, \quad \text{for } i = 1, 2, \ldots N - 1.$$  

(A.10)

Next, we integrate out the gauge fields $A_\mu$ in eqs. (2.10) and (2.14), and find

$$A_{++} = \frac{i\xi}{2\rho^2} \left( \partial_{++} \overline{z}_i z^i - \overline{z}_i \partial_{++} z^i \right) + i g_{ij} \overline{\zeta}_+^j \zeta_+^i,$$

$$A_{--} = \frac{i\xi}{2\rho^2} \left( \partial_{--} \overline{z}_i z^i - \overline{z}_i \partial_{--} z^i \right) + i \overline{\chi}_- \chi^a_+,$$

(A.11)

where, to distinguish the fermi multiplet $\Gamma^a$ from the bosonic one $\Phi^i$, we use the Latin letter “$a$” to label them, with

$$i = 1, 2, \ldots, N - 1 \quad \text{and} \quad a = 1, 2, \ldots, N.$$  

Moreover,

$$g_{ij} = \frac{\xi}{\rho^2} \left( \delta_{ij} - \frac{1}{\rho^2} \overline{z}_i \zeta^j_+ \right),$$

(A.12)

is the standard Fubini-Study metric on $\mathbb{C}P^{N-1}$. The bosonic part of the gauge field is in fact the $U(1)$ piece of the holonomy group $U(N - 1)$ of $\mathbb{C}P^{N-1}$[34], and couple to the left moving fermion $\chi^a$. It implies that the left mover lives on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{C}P^{N-1}$.

Using eqs. (A.9), (A.10) and (A.11), we can recast the eqs. (2.10), (2.14) and (2.17) to obtain the NLSM action

$$S_{\text{NLSM}} = \int d^2 x (g_{ij} \partial_\mu \overline{z}_i \partial^\mu z^j + i g_{ij} \overline{\zeta}_+^j \nabla_{--} U(N-1) \zeta_+^i + i \overline{\chi}_- \chi^a_+ \nabla_{++} U(1) \chi^a_-)

+ 2 \left( g_{ij} \overline{\zeta}_+^j \zeta_+^i \right) \left( \overline{\chi}_- \chi^a_+ \right),$$

(A.13)
where
\[ \nabla^U (N-1) \zeta_i^+ \equiv d\zeta_i^+ + \Gamma_{jk}^i dz^j \psi^k _+ , \text{ with } \Gamma_{jk}^i = g^f \partial_k g_{ji} , \]
\[ \nabla^U (1) \chi_a^- \equiv d\chi_a^- + i \omega \chi_a^- , \text{ with } \omega = \frac{i \xi}{2 \rho^2} (d \bar{z}_i z^i - \bar{z}_i dz^i) . \] (A.14)

One can clearly see that unlike \( \mathcal{N} = (2, 2) \) \( \mathbb{C}P^{N-1} \) case, the deformed model has all its left movers living on \( \mathcal{O}(-1)^{\oplus N} \). We remark here that at the level of NLSM, the study of isometry/holonomy anomalies is easy. The \( N-1 \) right movers \( \zeta_i^+ \) living on tangent bundle of \( \mathbb{C}P^{N-1} \) contribute to the anomaly proportional to the first Chern class of \( T \mathbb{C}P^{N-1} \),
\[ A_{\zeta^+} = c_1 (T \mathbb{C}P^{N-1}) = \frac{N}{4\pi} d\omega . \] (A.15)

On the other hand, the \( N \) left movers \( \chi_a^- \) on \( \mathcal{O}(-1)^{\oplus N} \) contribute
\[ A_{\chi^-} = - \frac{N}{4\pi} d\omega . \] (A.16)

Therefore, the deformed model is anomaly-free as its GLSM cousin, for more details see [34].

B Scaling anomalies: technicalities

In this Appendix we explain the technique to compute the anomalous Jacobian in section 4.2, say \( s \text{Tr}_\Phi \) and \( s \text{Tr}_\Gamma \) in eq.(4.6). A careless treatment of the chiral multiplet \( \Phi^i = (\phi^i, \psi^i_+ ) \) seemingly tells us that
\[ s \text{Tr}_\Phi 1 = \text{Tr}_\phi 1 - \text{Tr}_\psi_+ 1 = 0 . \] (B.1)

One has to regularize the above super-trace by introducing regulators. To find a proper regulator, it is sufficient to look at the equation of motion of the superfield \( \Phi^i \) which enters the action \( S_{\text{chiral}} \), see eq.(2.10),
\[ \mathcal{D}_+ \mathcal{D}_- \Phi^i = \cdots . \] (B.2)

We need to further act by \( \mathcal{D}_+ \) to project the operator equation into the half chiral superspace, i.e.
\[ \mathcal{D}_+ \mathcal{D}_+ \mathcal{D}_- \Phi^i = \mathcal{D}_+ (\cdots) . \] (B.3)
After some algebra, we find
\[
\bar{D} + D + D_{\pm} \Phi^i \propto \left( \nabla_{\mu}^2 + q_i D \right) \phi^i + \sqrt{2} \theta^+ \left( \nabla_{\mu}^2 + iq_i B \right) \psi^i_+ + \cdots .
\] (B.4)

Therefore, the super-trace eq. (B.1) is regularized as
\[
sTr_{\Phi} \mathbf{1} = \lim_{M^2 \to \infty} \left( \text{Tr}_{\phi^i} e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 + q_i D \right)} - \text{Tr}_{\psi^i_+} e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 + iq_i B \right)} \right).
\] (B.5)

For trivial fields $D$ and $B$, the above trace is surely zero. But now let us turn on a non-zero but constant $D$ and $B$ backgrounds. We have
\[
\text{Tr}_{\phi^i} e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 + q_i D \right)} = \int d^2 x \left< x \mid e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 + \frac{1}{M^2} (q_i D + \mathcal{O}(A_\mu)) + \mathcal{O} \left( \frac{1}{M^4} \right) \right)} \mid x \right>,
\]
\[
= \frac{1}{4\pi} \int d^2 x \left( M^2 + (q_i D + \mathcal{O}(A_\mu)) + \mathcal{O} \left( \frac{1}{M^2} \right) \right),
\]
\[
\text{Tr}_{\psi^i_+} e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 + iq_i B \right)} = \int d^2 x \left< x \mid e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 + \frac{1}{M^2} (iq_i B + \mathcal{O}(A_\mu)) + \mathcal{O} \left( \frac{1}{M^2} \right) \right)} \mid x \right>,
\]
\[
= \frac{1}{4\pi} \int d^2 x \left( M^2 + (iq_i B + \mathcal{O}(A_\mu)) + \mathcal{O} \left( \frac{1}{M^2} \right) \right).
\] (B.6)

Therefore, putting $M^2 \to \infty$, we arrive at
\[
sTr_{\Phi} \mathbf{1} = \frac{q_i}{4\pi} \int d^2 x \left(D - iB\right),
\] (B.7)

or, in superspace,
\[
sTr_{\Phi} \mathbf{1} = -\frac{i q_i}{8\pi} \int d^2 x \, d\theta^+ \mid \bar{\theta}^+ = 0 .
\] (B.8)

Similarly, for fermi multiplet $\Gamma^a$, we also impose $\bar{D} + D_{\pm} D_{\pm}$ upon $\Gamma_-$ and find,
\[
\bar{D} + D_{\pm} D_{\pm} \Gamma^a = \bar{D} + D_{\pm} D_{\pm} \Gamma^a + \bar{D} + (\bar{\Upsilon}_- \Gamma_-)
\]
\[
\propto \left( \nabla_{\mu}^2 - i\bar{q}_a B \right) \chi_- - \sqrt{2} \theta^+ \left( \nabla_{\mu}^2 - \bar{q}_a D \right) G^a + \bar{D} + (\bar{\Upsilon}_- \Gamma_-) + \{B.9\}
\]

Thus we regularize the super-trace of the fermi multiplet as
\[
sTr_{\Gamma^a} \mathbf{1} = \lim_{M^2 \to \infty} \left( -\text{Tr}_{\chi_-} e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 - i\bar{q}_a B \right)} + \text{Tr}_{G^a} e^{\frac{1}{M^2} \left( \nabla_{\mu}^2 - \bar{q}_a D \right)} \right)
\]
\[
= -\frac{\bar{q}_a}{4\pi} \int d^2 x \left(D - iB\right) + i \frac{\bar{q}_a}{8\pi} \int d^2 x \, d\theta^+ \mid \bar{\theta}^+ = 0 ,
\] (B.10)
From eq. (B.8) and (B.10), we establish the relation between canonical coupling $\tau_c$ and holomorphic $\tau$ in eq. (4.8), i.e.

$$
\tau_c = \tau + \sum_i i \frac{q_i}{4\pi} \log Z_i(\mu) - \sum_a i \tilde{q}_a \frac{4\pi}{4\pi} \log Z_a(\mu)
$$

We further remark that, as a consistency check, given a complexified $U(1)$ rotation of the chiral or fermi matter, e.g.

$$
\Phi^i \rightarrow e^{\alpha} \Phi^i,
$$

the anomalous Jacobian takes the form

$$
J(\alpha) = e^{\alpha(sTr \Phi^i)} = e^{\alpha \frac{\mu}{2\pi} \int d^2 x (D - iB)}.
$$

For real $\alpha$, such as the wave function renormalization or a scale transformation, the anomalous Jacobian only gives a correction to the $D$ term, because $\text{Im} J(\alpha)$ cancels with the contribution from $\bar{\Phi}^i$. It simply signals that fermions do not contribute to the one-loop $\beta$-function. On the other hand, for imaginary $\alpha$, it is equivalent to a chiral rotation. We see that $J(\alpha)$ and its conjugation only contribute to the flux $B$ term, which gives us the correct chiral anomaly from the chiral fermions $\psi_+^i$ (section 4.1).
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