Radiation from a Charge Uniformly Accelerated for All Time

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Abstract

A recent paper of Singal [7] argues that a uniformly accelerated particle does not radiate, in contradiction to the consensus of the research literature over the past 30 years. This note points out some questionable aspects of Singal’s argument and shows how similar calculations can lead to the opposite conclusion.

1 Introduction

Over 40 years ago, it was a matter of controversy whether radiation would be observed from a charge which had been uniformly accelerated for all time. The situation around 1960 is described in [10], which concludes that “contrary to claims in some standard sources, (Pauli, Von Laue), a charge in uniform acceleration does radiate”. Subsequently a consensus for radiation seemed to have been reached (I know of no paper claiming the contrary within the past 30 years) until a recent paper of Singal presented a calculation which it interprets as proving that “there is no electromagnetic radiation from a uniformly accelerated particle” [7][p. 963].

This note analyzes this calculation and Singal’s interpretation. It observes that Singal’s method applied to a particle uniformly accelerated for a finite, but arbitrarily long, time leads to the opposite conclusion: such a particle does radiate in accordance with the Larmor law. The answer given by Singal’s method to the radiation question for uniform acceleration for all time is not a limit of
the corresponding answers for acceleration for arbitrarily large times. This inconsistency could be viewed as a paradox or as an indication of the unreliability of Singal’s method.

Other inconsistencies of the same nature have been long known. All such inconsistencies known to this author can be traced to mathematical ambiguities in formulating the radiation problem for a particle uniformly accelerated for all time, as opposed to the mathematically unambiguous problem of determining whether radiation would be observed from a particle uniformly accelerated for a very long, but finite, time.

We conclude by reviewing a related “paradox” (that calculation of radiated energy-momentum by integration over Bhabha and Dirac tubes give the same answer for asymptotically free particles but different answers for a particle uniformly accelerated for all time). The only known resolution to this paradox is to disallow (for good reasons) these calculational methods for particles which are not asymptotically free. We suggest that Singal’s paradox should be similarly resolved.

2 Summary of Singal’s method and results

Singal’s calculations are performed entirely within a fixed Lorentz frame which we’ll call the “laboratory frame”. Coordinates in the laboratory frame will be denoted \((t, x, y, z)\).

Consider a particle of charge \(e\) whose worldline is described in laboratory coordinates as a function of proper time \(\tau\) by

\[ \tau \mapsto g^{-1} (\sinh g\tau, \cosh g\tau, 0, 0) \]  

This describes a particle moving on the \(x\)-axis with uniform proper acceleration \(g\) in the positive \(x\)-direction, so that the particle comes to rest at \(x = g^{-1}\) at time \(t = 0\). Singal calculates the energy \(\mathcal{E}\) in its electromagnetic field at the laboratory time \(t = 0\) using the usual expression \((E^2 + B^2)/8\pi\) for the three-dimensional field energy density:

\[ \mathcal{E} = \int_W d^3v \left( \frac{E^2 + B^2}{8\pi} \right) \]

where \(E\) and \(B\) are respectively the electric and magnetic fields.

Of course, the integral is expected to diverge unless the region of integration \(W\) omits some ball containing the particle, since the corresponding integral for a stationary particle is \(e^2/2r_0\) when a ball of radius \(r_0\) centered at the particle is omitted. The ball which the paper chooses to omit is a ball of some given positive radius \(r_0\) with the particle at the center at laboratory time \(t = -r_0\).

In Singal’s terminology, the ball consists of all points at “retarded distance” no more than \(r_0\) from the particle. The “retarded distance” of \((t, x, y, z)\) from the
particle is defined as the laboratory-frame distance to the point where a past-directed light ray starting at \((t, x, y, z)\) will intersect the particle’s worldline. The region of integration \(W\) is taken to be the set of all points whose retarded distance \(R\) from the particle satisfies \(r_0 \leq R < \infty\).

This integration region \(W\) may also be described as the open half-space \(t = 0, x > 0\) with the above ball of radius \(r_0\) omitted. Only a half-space is covered because points with \(t = 0, x \leq 0\) cannot be connected to the wordline with a light ray. Despite this, the fields determined by the distributional Maxwell equations as given in [3] do not vanish on \(x = 0\) and Singal’s decision not to include them in the integration computing the field energy is one controversial aspect of his method. This will be discussed more fully below.

The result of Singal’s calculation [3] is that the total field energy \(E\) as defined above is given by

\[
E = \frac{e^2}{2r_0},
\]

which is the same as for a stationary particle. From this he concludes that no energy has been radiated up to time \(t = 0\).

One might try to resolve the paradox by noting that there is infinite energy in this field if the ball of radius \(r_0\) is not excised. However, this resolution is intuitively unsatisfactory because according to the classical picture of electromagnetic energy propagating at the speed of light (unity), any field energy within this ball must have been emitted between \(t = -r_0\) and \(t = 0\). If there is infinite energy in the fields at \(t = 0\) due to the infinite energy radiation from the infinite past to time 0, then at time 0 there ought to be infinite energy in the electromagnetic field outside the ball of radius \(r_0\). That is, \(E\) as calculated by Singal would still be expected to be infinite.

3 Singal’s method applied to a particle uniformly accelerated for a finite time

We shall suggest below that a particle uniformly accelerated for all time represents a situation too mathematically singular to be reliably treated with the sort of mathematical manipulations customary in this field. To clarify the singular mathematics associated with a particle uniformly accelerated for all time, we apply Singal’s method to a particle uniformly accelerated for only a finite time which can be arbitrarily long. We shall see that standard, nonsingular mathematics leads to the conclusion that the energy in the field at time \(t = 0\) goes to infinity as the beginning of acceleration is pushed back to the infinite past. In other words, according to Singal’s criterion for radiation (excess energy in the field at time 0 relative to the energy of a Coulomb field), a particle uniformly accelerated for a finite time does radiate.

\[\text{See also [3] and earlier references cited there and in [10]. We use [3] as a convenient modern reference for these fields.}\]
Now we begin the calculation of the energy in the field at \( t = 0 \) of a particle uniformly accelerated for a finite time using Singal’s method \([7]\), with which we assume the reader is familiar. Suppose the acceleration starts at laboratory time \( t = -R_0 < 0 \) in the distant past, \( R_0 \) being the retarded distance of the starting event, and continues to time \( t = R_0 \) in the future, the particle being otherwise at uniform velocity. The worldline during the interval of uniform acceleration will be taken as \([3]\), with uniform velocity otherwise.

Singal’s equation \([5]\) transforms the expression for the electric field \( E \) given in \([8]\) to a sum of two orthogonal terms:

\[
E = E_{\text{radial}} + E_{\text{trans}}
\]

\[
= \frac{n}{\gamma^2 R^2 (1 - \beta \cdot n)^2} + \frac{n \times \{n \times (\gamma \beta + \gamma^3 \beta R)\}}{\gamma^3 R^2 (1 - \beta \cdot n)^3}
\]

The notation is that of \([7]\) and \([8]\) except that the velocity of light is taken as unity: \( \beta \) is the particle’s spatial vector velocity at the “retarded” point connected by a forward-pointing light ray to the field point at which \( E \) is evaluated, \( \gamma := (1 - \beta^2)^{-1/2} \), \( n \) is the spatial unit vector pointing from the space coordinates of the retarded point to the space coordinates of the field point, and \( R \) is the “retarded distance” defined as the laboratory-frame distance of the retarded point from the field point. The “radial” first term of the right side will be abbreviated \( E_{\text{radial}} \), and the “transverse” second term \( E_{\text{trans}} \). The magnetic field \( B \) (which is zero in the context of \([7]\) but not here) is given by: \( B = n \times E \).

The field energy \( \mathcal{E} \) is a sum

\[
\mathcal{E} = \int \frac{E_{\text{radial}}^2}{8\pi} \, dv + \int \frac{E_{\text{trans}}^2}{8\pi} \, dv + \int \frac{B^2}{8\pi} \, dv
\]

The transverse term is easily integrated using Singal’s volume element \([7]\) [p. 964]

\[
\frac{dv}{8\pi \gamma_0^2} = 2\pi R^2 (1 - \beta \cos \theta) \sin \theta \, dR \, d\theta
\]

the result being:

\[
\int \frac{E_{\text{trans}}^2}{8\pi} \, dv = \frac{e^2 \beta_0^2}{8\pi \gamma_0^4} \int_0^\infty dR \int_0^{\pi} d\theta \frac{2\pi R^2 (1 - \beta_0 \cos \theta) \sin^3 \theta}{R^4 (1 - \beta_0 \cos \theta)^6}
\]

\[
= \frac{e^2 g^2 R_0^3}{3}
\]

where \( \beta_0 \) and \( \gamma_0 \) are respectively the velocity and corresponding \( \gamma \)-factor for the motion before the acceleration started, and we have used the relation \( \gamma_0^2 = 1 + g^2 R_0^2 \) to express the result in terms of \( R_0 \). The contribution of the magnetic field to the field energy is the same as \([3]\). The integral of the square of the
radial term is (due to a fortuitous cancellation of $\gamma$ factors in the integrand) identical, both in calculation and result, with Singal’s calculation for a particle uniformly accelerated for all time. The final result is:

$$\mathcal{E} = \int \frac{E^2 + B^2}{8\pi} dv$$

$$= \frac{e^2}{2r_0} + \frac{2e^2 g^2 R_0}{3}$$

(3)

(4)

The important point is that $\mathcal{E} \to \infty$ as $R_0 \to \infty$. If we take the result for a particle uniformly accelerated forever as a limit of this result as $R_0 \to \infty$, then the field energy in all space at time $t = 0$, excluding the above ball of radius $r_0$ surrounding the particle, is infinite as expected.

4 Discussion of Singal’s paradox

We propose that a reliable calculational method should have at least the following property: the answer to the question

“Does a particle uniformly accelerated for all time radiate?”

should be the same as the answer to

“Does a particle uniformly accelerated for a finite but arbitrarily long time radiate?”

The preceding section demonstrated that Singal’s method does not have this property.

Any method without this property is of questionable applicability to observational physics. Any physical meaning for uniform acceleration for all time must in practice be derived from approximations by acceleration for finite times. We will never observe a particle uniformly accelerated for all time, but we can hope to observe particles uniformly accelerated for very long times.

The above argument that Singal’s method is unreliable does not tell us why it is unreliable. What specific feature of these plausible calculations is suspect?

The most likely culprit seems Singal’s decision to limit the region $W$ of integration to the half-space $t = 0, x > 0$. Superficially this seems reasonable, since these are the only points on the hyperspace $t = 0$ which are causally connected to the particle—no light ray from the particle can reach the two-dimensional plane $t = 0, x = 0$ (and also no light ray from such a point can reach the particle).

However, it overlooks the curious fact that the electromagnetic field produced by a particle uniformly accelerated for all time (the retarded solution to the distributional Maxwell equations) does not vanish on this plane, and in fact is
highly singular there. This can be seen in Boulware’s expression \([3]\) for the field, which contains a \(\delta\)-function \(\delta(x + t)\).

We do not agree with Singal’s argument that it is legitimate to omit the plane \(t = 0, x = 0\) from the integration on the grounds that no radiation from the particle can reach this plane. We believe that it is inconsistent to omit the delta-function on \(x = 0\) from the fields for the purpose of computing the field energy. Although no light ray from the particle can reach this plane, the delta-function on the plane in Boulware’s expression (III.11) shows that “radiation” in the form of nonzero fields does in fact reach this plane—the delta-function field would not be there if the particle were not there!

The delta-function is part of the particle’s field, and it should not be surprising that if it is omitted in the energy calculation, the result is less energy than expected at \(t = 0\). Intuitively, at \(t = 0\), all the radiated energy is concentrated in the delta-function on the plane \(x = 0\).

We cannot prove this by completing the calculation because the standard expression for the field energy entails squaring this \(\delta\)-function, a mathematically ill-defined operation which leads to seemingly mathematically meaningless expressions like the \(\delta(0)\delta(x + t)\) factor in Boulware’s equation (IV.2) \([3]\) for the component \(T_{tt}\) of the energy-momentum tensor whose integral over space would normally give the energy in the plane. Unfortunately, \([3]\) does not furnish a mathematically meaningful interpretation for this expression.

5 Discussion of related paradoxes

It may be helpful to compare Singal’s paradox with a similar “paradox” whose resolution is well understood. In this “paradox”, two plausible methods for calculating a (not necessarily uniformly) accelerated particle’s energy radiation, one due to Dirac \([4]\) and the other to Bhabha \([1]\), give the same answer for an asymptotically free particle (such as one accelerated for only a finite time), but usually different answers for a particle which is not asymptotically free (in particular for a particle uniformly accelerated for all time).

It seems generally accepted that a charged particle which undergoes acceleration (not necessarily uniform) for only a finite time does radiate in accordance with the Larmor law. Specifically, if a particle of charge \(q\) with four-velocity \(u^i\) and acceleration \(a^i := du^i/d\tau\) is free (i.e., unaccelerated) for proper times \(\tau \leq \tau_1\) and \(\tau \geq \tau_2\), then it radiates total energy-momentum \(P^j\) given by:

\[
P^j = -2q^2/3 \int_{\tau_1}^{\tau_2} a^i a_i u^j d\tau.
\]  

Here vector components are with respect to an orthonormal basis for Minkowski space with metric \(g_{ij} = \text{diag}(1, -1, -1, -1)\), and the energy radiation is \(P^0\).

We emphasize that the hypothesis under which \([3]\) is unequivocally accepted is that the particle be free outside the proper time interval \(\tau_1 \leq \tau \leq \tau_2\). We
shall call this the “fundamental hypothesis”. The radiation expression (5) is accepted because all accepted calculational methods seem to lead to this conclusion under the fundamental hypothesis.

A typical calculation might surround the particle’s worldline by some sort of three-dimensional tube and obtain $P^j$ as the integral of the Hodge dual of the energy-momentum tensor $T^{ij}$ over this tube. For example, this is how Dirac originally obtained the Lorentz-Dirac equation (4).

Expressed in three-dimensional language, such a calculation surrounds the particle with a two-dimensional surface (usually a sphere of a given radius $r$ relative to some appropriate reference frame), integrates the Poynting vector over the surface to obtain the rate of energy radiation, and then integrates the radiation rate over all time to obtain the total radiation.

There are a number of reasonable ways to choose the tube or surface, but the fundamental hypothesis guarantees that the result $P^j$ of the calculation is the same for all accepted methods known to this author. This is because the covariant divergence $\partial_i T^{ij}$ vanishes. Two different tubes can be smoothly joined at the ends assuming that the acceleration vanishes at the ends, and then the vanishing divergence together with Gauss’s Theorem implies that the calculated $P^j$ is independent of the tube.

Many of the seeming contradictions in the literature of the problem of radiation from a uniformly accelerated charge can be traced to overlooking the “fundamental hypothesis”. For example, Bhabha (1) calculated the radiated energy-momentum from an arbitrarily accelerated particle using a tube built from “retarded spheres” $S_\tau$ (obtained by imagining the particle emitting a flash of light at proper time $\tau$ and letting it expand to a sphere of a given radius $r$ relative to the particle’s rest frame at $\tau$), and the result was the above (5). Dirac (4) used a different tube generated by evolving through time $\tau$ two-dimensional spheres $\bar{S}_\tau$ of radius $r$ with the particle at the center, where the spheres are taken relative to the particle’s rest frame at proper time $\tau$. His calculation yielded (in the limit $r \to 0$) the different result

$$P^j = -\frac{2q^2}{3} \int_{\tau_1}^{\tau_2} \left( a^i a_i u^j + da^j / d\tau \right) d\tau . \quad (6)$$

The derivations of both of these results are of a high standard of rigor, and seem universally accepted. They obviously agree under the fundamental hypothesis, but ignoring this hypothesis results in a “paradox”.

For uniform acceleration (meaning that the acceleration is in a fixed spatial direction with $a^i a_i$ constant and $\tau_1 := -\infty$, $\tau_2 := \infty$), the Bhabha and Dirac methods do not agree. In this case, the integrand of (6) is well known to

\[^2\text{Space limitations preclude a proper treatment of this important point here, but a general discussion can be found in Chapter 4 of [4], along with full definitions of the Bhabha and Dirac “tubes” to be briefly described below and the associated radiation calculations. Discussion of this point as it applies specifically to uniform acceleration can be found in the Internet archive [6].}\]
vanish identically but the energy component of (6) is strictly positive when $a \neq 0$. Thus for uniform acceleration for all time, the Dirac calculation gives zero energy radiation, while the Bhabha calculation gives infinite energy radiation.

Obviously, both cannot be correct. The only known resolution of this “paradox” is that the usual justifications for these methods (in particular, mass renormalization) require that $a(\tau_1) = 0 = a(\tau_2)$, and since the methods are of comparable plausibility and give inconsistent results in the absence of this assumption, both should be disallowed.

6 Conclusions

So far as we know, no one questions that a charged particle which is uniformly accelerated for a finite time does radiate energy. In particular, Singal’s method yields the usual Larmor radiation expression for this situation.

However, some calculational methods, including Singal’s, predict zero energy radiation when applied to a particle uniformly accelerated for all time. Other equally plausible methods (such as integration over a Bhabha tube) predict the contradictory result of infinite radiation in accordance with the Larmor law.

We suggest that uniform acceleration for all time should be recognized as too mathematically singular to be treated reliably with the sort of mathematical manipulations customary in this field. One indication of this is the delta-function in the field at a point not causally connected to the particle and the seemingly mathematically meaningless term in the energy-momentum tensor corresponding to its square.

The situation might be compared to ambiguities in calculation of limits which perplexed 19’th century mathematicians and spurred the development of rigorous foundations for calculus. For example, if the series $\sum_{n=1}^{\infty}(-1)^n$ is summed as $[-1+1]+[-1+1]+\cdots$, the result is 0, whereas summing it as $-1+1+(-1)+1+(-1)+\cdots$ gives $-1$. If one attempts to apply algebraic rules valid for finite sums to infinite sums without any proof of their validity, one can expect to obtain such seemingly paradoxical results. There is much evidence that a charge uniformly accelerated forever is similarly a mathematically singular situation in which algebraic manipulations conventional in mathematical physics can be expected to lead to contradictory results.

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