CLASSIFICATION OF ALL POISSON-LIE STRUCTURES ON AN INFINITE-DIMENSIONAL JET GROUP

BORIS A. KUPERSHMIDT

Department of Mathematics
The University of Tennessee Space Institute
Tullahoma, TN 37388
USA
e-mail: bkupersh@sparc2000.utsi.edu

OGNYAN S. STOYANOV

Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
USA
e-mail: stoyanov@math.rutgers.edu

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Abstract. A local classification of all Poisson-Lie structures on an infinite-dimensional group $G_\infty$ of formal power series is given. All Lie bialgebra structures on the Lie algebra $\mathcal{G}_\infty$ of $G_\infty$ are also classified.
Let $G_\infty$ be the group of formal power series in one variable $\{ x(u) = \sum_{i=1}^{\infty} x_i u^i \mid x_1 \neq 0 \}$, with a group multiplication $G_\infty \times G_\infty \to G_\infty$ being the substitution:

\begin{equation}
(xy)(u) := x(y(u)), \quad \text{or} \quad u \mapsto x(u),
\end{equation}

and with an identity $e$ the identity map $u \mapsto u$. The group $G_\infty$ is the group of formal diffeomorphisms of $\mathbb{R}^1$ which leave the origin fixed. It is a projective limit $G_\infty = \lim_{\leftarrow} G_n$, where $G_n$ are the finite-dimensional Lie groups of $n$-jets of the line at the origin. The multiplication in $G_n$ is again defined by the substitution (1): $(X_n Y_n)(u) := X_n(Y_n(u)) \mod u^{n+1}$, where $X_n(u)$ and $Y_n(u)$ are polynomials in $u$ of degree $n$. We define the space of smooth functions $C^\infty(G_\infty)$ to be the inductive limit $C^\infty(G_\infty) = \lim_{\rightarrow} C^\infty(G_n)$ of the spaces of smooth functions on the finite-dimensional groups $G_n$.

Following [1] we consider a multiplicative Poisson (Poisson-Lie) structure on $G_\infty$ to be the bilinear skew-symmetric map $\{ , \} : C^\infty(G_\infty) \times C^\infty(G_\infty) \to C^\infty(G_\infty)$ defined by

\begin{equation}
\{ f, g \} = \omega_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},
\end{equation}

for any $f, g \in C^\infty(G_\infty)$, such that the multiplication map $G_\infty \times G_\infty \to G_\infty$ is a Poisson map. Here $\omega_{ij} \in C^\infty(G_\infty)$, for any $i, j \in \mathbb{N}$, and a summation is assumed over repeated indices. The Poisson structure on $G_\infty \times G_\infty$ is taken to be the product Poisson structure. Note that the sum in (2) is finite since by definition $f$ and $g$ are functions of finite number of variables. Then the Jacobi identity for $\{ , \}$ implies that $\omega_{ij}$’s satisfy

\begin{equation}
\omega_{ij} \frac{\partial \omega_{kl}}{\partial x_i} + \omega_{ik} \frac{\partial \omega_{lj}}{\partial x_i} + \omega_{il} \frac{\partial \omega_{jk}}{\partial x_i} = 0,
\end{equation}

for any $j, k, l \in \mathbb{N}$. The multiplicativity of the Poisson brackets (2) ($\{ , \}$ being a 1-cocycle) means that $\omega_{ij}$’s must satisfy the following infinite system of functional
equations

\[(3b) \quad \omega_{ij}(xy) = \omega_{kl}(x) \frac{\partial z_i}{\partial x_k} \frac{\partial z_j}{\partial x_l} + \omega_{kl}(y) \frac{\partial z_i}{\partial y_k} \frac{\partial z_j}{\partial y_l}, \quad i, j \in \mathbb{N},\]

where \(z = xy\). Note again that the sums in the right hand side of (3b) are finite. This is immediately seen from the explicit formulae

\[z_k = \sum_{i=1}^{k} x_i \sum_{(\sum_{a=1}^{i} j_a) = k} y_{j_1} \cdots y_{j_i}, \quad k \geq 1,\]

for the coordinates of \(z\). From (3b) also follows that \(\omega_{ij}(e) = 0\).

Do such structures exist on \(G_\infty\)? It is by no means obvious that such structures exist. For example:

(i) Let us consider the 3-dimensional factor group \(G_3 = G_\infty \mod u^n\), for \(n \geq 4\). Then there exists a Poisson-Lie structure on \(G_3\) described by

\[
\begin{align*}
\{x_1, x_2\} &= x_1 x_2 \\
\{x_1, x_3\} &= 4x_2^2 - 2x_1 x_3 \\
\{x_2, x_3\} &= 6 \frac{x_2^3}{x_1} - 5x_2 x_3,
\end{align*}
\]

where \(x_1, x_2, x_3\) are the coordinate functions on the group \(G_3\). However, this Poisson-Lie structure can not be extended to a Poisson-Lie structure on \(G_\infty\).

(ii) We conjecture that there are no non-trivial Poisson-Lie structures on the group of diffeomorphisms of \(S^1\).

Also a second question arises: If such structures exist, could they be classified? The answer to the first question is given by the following theorem.

**Theorem 1.** For every natural number \(d \in \mathbb{N}\), and every sequence \(M_d = (\mu_n)_{n=1}^\infty\), such that \(\mu_n = 0\) for \(1 \leq n \leq d\) and \(\mu_{d+1} \neq 0\), one has the following infinite-
parameter family of Poisson-Lie structures on $G_{\infty}$,

$$\omega_{ij}(x) = \sum_{p=1}^{i} \sum_{q=1}^{j} pxpqxq\lambda_{i-p+1,j-q+1} - \sum_{p=1}^{i} \sum_{q=1}^{j} \lambda_{pq} \sum_{k=1}^{p} r_{k}=i \sum_{l=1}^{q} s_{l}=j \lambda_{s,d+1} x_{r_{1}} \ldots x_{r_{p}} x_{s_{1}} \ldots x_{s_{q}},$$

where

$$\lambda_{mn} = \frac{1}{\mu_{d+1}} \left[ \mu_{m}\lambda_{d+1,n} - \mu_{n}\lambda_{d+1,m} \right] \quad \forall \ m, n \geq 1.$$

Here, $\lambda_{d+1,n}$ are given by rational functions $\lambda_{d+1,n} = \lambda_{d+1,n}(\mu_{d+1}, \ldots, \mu_{d+n})$ for $n \geq 1$, which are computed by the following recursive formula

$$\lambda_{d+1,n} = -\frac{1}{(d - n + 1)\mu_{d+1}} \left[ d\mu_{d+1}\mu_{n+d} - \sum_{s=1}^{n-1} (n+d-2s+1)\mu_{n+d-s+1}\lambda_{s,d+1} \right],$$

where $\lambda_{1,d+1} = \mu_{d+1}$, and there exists the following single relation between the $\mu_{n}$'s (with $n \geq d+1$)

$$\mu_{2d+1} = -\frac{1}{d\mu_{d+1}} \sum_{s=2}^{d} 2(d + 1 - s)\mu_{2d+2-s}\lambda_{s,d+1},$$

which are otherwise subject to no other restrictions. (We implicitly assume that $\lambda_{mn} = 0$ whenever $m < 1$ or $n < 1$.)

The classification is given by Corollary 2 below.

**Remark.** The relation (7) follows from (6) when $n = d + 1$, and this is the only value of $n$ for which the expression $(d - n + 1)\lambda_{d+1,n}\mu_{d+1}$ equals zero.

The proof of Theorem 1 is rather technical [5,6]. We confine ourselves to give here the main ideas and tools used. Let $\mathcal{V} = \{u, v, w, \ldots\}$ be a countable
set and let \( C^\infty(G_\infty)[[V]] \) be the ring of formal power series generated by \( V \) over \( C^\infty(G_\infty) \). Let \( X_i, i \in \mathbb{N}, \) be the coordinate functions on \( G_\infty \). That is, \( X_i(x) = x_i \) for \( x \in G_\infty \). Introduce the formal series \( X(u) := \sum_{i,j=1}^\infty X_i u^i \). Then \( x(u) = X(u)(x) = \sum_{i,j=1}^\infty x_i u^i \), and \( \omega_{ij} = \{X_i, X_j\} \). Define the formal series \( \Omega(u, v; X) := \sum_{i,j=1}^\infty \omega_{ij} u^i v^j \). Thus \( \Omega(u, v; x) \) is a generating series for the brackets \( \omega_{ij} \). After evaluation at \( x \in G_\infty \) one has

\[
\Omega(u, v; x) = \sum_{i,j=1}^\infty \omega_{ij}(x) u^i v^j.
\]

The multiplicativity (3b) of the Poisson brackets on \( G_\infty \) is equivalent to \( \Omega(u, v; x) \) satisfying the following functional equation

\[
\Omega(u, v; xy) = \Omega(y(u), y(v); x) + \Omega(u, v; y) x'(y(u)) x'(y(v)).
\]

Here \( x' \) denotes the derivative of \( x \) with respect to its argument. The general solution of (8) is given by

\[
\Omega(u, v; x) = \varphi(u, v) x'(u) x'(v) - \varphi(x(u), x(v)),
\]

where \( \varphi(u, v) \) is a formal series in \( u, v \) subject to the conditions:

(i) \( \varphi(u, v) \) is divisible by \( u \) and \( v \),

(ii) \( \varphi(u, v) = -\varphi(v, u) \).

The map \( \{,\} : C^\infty(G_\infty) \times C^\infty(G_\infty) \to C^\infty(G_\infty) \) induces a map \( \{,\} : C^\infty(G_\infty)[[V]] \times C^\infty(G_\infty)[[V]] \to C^\infty(G_\infty)[[V]] \). In particular one has

\[
\{X(u), X(v)\} = \sum_{i,j=1}^\infty \{X_i, X_j\} u^i v^j = \Omega(u, v; X).
\]

Then the Jacobi identities (3a) are equivalent to the single equation

\[
\{X(w), \{X(u), X(v)\}\} + \{X(u), \{X(v), X(w)\}\} + \{X(v), \{X(w), X(u)\}\} = 0,
\]

which, after a short calculation using the explicit formula (9), implies that \( \varphi(u, v) \) must satisfy the following functional partial differential equation

\[
\varphi(u, v) \left[ \partial_u \varphi(w, u) + \partial_v \varphi(w, v) \right] + c.p. = 0.
\]
Thus the content of Theorem 1 is a description of all solutions of (10) satisfying (i) and (ii). The relation between (9) and (4) is $\varphi(u, v) = \sum_{i,j=1}^{\infty} \lambda_{ij} u^i v^j$. The solution (4) completely describes the space of solutions of (10). An equivalent description can be given as follows.

**Theorem 1a.** For each $d \in \mathbb{N}$, and any (formal series) $f_d(u), g_d(u)$ such that $f_d'(u)g_d(u) - f_d(u)g_d'(u) = -d\mu_{d+1}f_d(u)$, where $\mu_{d+1} \neq 0$ is an arbitrary parameter, and $f_d$ has a zero of order $d+1$ at $u = 0$, there is a solution of (10) given by

$$\varphi_d(u, v) = \frac{1}{\mu_{d+1}} \left[ f_d(u)g_d(v) - f_d(v)g_d(u) \right].$$

The set of all solutions of (10) is described in this way.

**Corollary 1.** A subclass of the above family is the following countable family of Poisson-Lie structures. For each $d \in \mathbb{N}$, choosing $M_d = (0, 1, 0, 0, \ldots)$, one has

$$\omega_{ij}(x) = (i - d)jx_j x_{i-d} - i(j - d)x_i x_{j-d} +$$

$$+ x_i \sum_{\sum_{k=1}^{d+1} s_k = j} x_{s_1} \cdots x_{s_{d+1}} - x_j \sum_{\sum_{k=1}^{d+1} s_k = i} x_{s_1} \cdots x_{s_{d+1}},$$

for every $i, j \geq 1$. (We adopt the convention that $x_i = 0$ whenever $i < 1$).

This family corresponds to the set of solutions $\varphi(u, v) = uv(u^d - v^d)$, $d \in \mathbb{N}$, of (10).

Let $i_\infty : G_\infty \to G_\infty$ be the inversion map defined by $i_\infty(x) = x^{-1}$ for every $x \in G_\infty$.

**Theorem 2.** The map $i_\infty : G_\infty \to G_\infty$ is an anti-Poisson map.

In other words one has $\{f, g\}(i_\infty(x)) = -\{f, g\}(x)$, for every $f, g \in C^\infty(G_\infty)$ and $x \in G_\infty$. The proof uses only the explicit form (9) of the brackets on $G_\infty$. Let
\( \overline{X}(u) \) be the inverse of \( X(u) \). Then one has \( \overline{X}(X(u)) = u \), and \( X(\overline{X}(u)) = u \), as well as \( \overline{X}'(X(u))X'(u) = 1 \), and \( X'(\overline{X}(u))\overline{X}'(u) = 1 \). On the other hand

\[
0 = \{u, X(v)\}
= \{\overline{X}(X(u)), X(v)\}
= \{\overline{X}(w), X(v)\}|_{w=X(u)} + \overline{X}'(w)|_{w=X(u)}\{X(u), X(v)\}.
\]

Therefore,

\[
(12) \quad \{X(v), \overline{X}(w)\}|_{w=X(u)} = \overline{X}'(w)|_{w=X(u)}\{X(u), X(v)\}.
\]

Also, one has the chain of identities

\[
0 = \{v, \overline{X}(w)\}|_{w=X(u)}
= \{\overline{X}(X(v)), \overline{X}(w)\}|_{w=X(u)}
= \{\overline{X}(s), \overline{X}(w)\}|_{s=X(v), w=X(u)} + \overline{X}'(s)|_{s=X(v)}\{X(v), \overline{X}(w)\}|_{w=X(u)}.
\]

Using (9) and (12), one rewrites the last identity as

\[
0 = \varphi(X(v), X(u))\overline{X}'(X(v))\overline{X}'(X(u)) - \varphi(v, u)
+ \overline{X}'(X(v))\overline{X}'(X(u)) [\varphi(u, v)X'(u)X'(v) - \varphi(X(u), X(v))]
= \{\overline{X}(s), \overline{X}(w)\}|_{s=X(v), w=X(u)} + \varphi(u, v) - \overline{X}'(w)\overline{X}'(s)\varphi(w, s).
\]

Thus,

\[
\{\overline{X}(w), \overline{X}(s)\} = -[\overline{X}'(w)\overline{X}'(s)\varphi(w, s) - \varphi(\overline{X}(w), \overline{X}(s))],
\]

and this concludes the proof.

**Remark.** In the beginning of the theory of Poisson-Lie groups, the property of the inversion map \( i : G \to G \) to be anti-Poisson was considered as an axiom [1,4]. However, for finite-dimensional groups this property can be deduced from the other axioms [5,6]. For infinite-dimensional groups such deduction is not likely.
To show that formula (4) provides all Poisson-Lie structures on \(G_\infty\), we now turn to the Lie algebra \(G_\infty\) of \(G_\infty\). Let \(\{e_n\}_{n\geq 0}\) be a basis of \(G_\infty\), and let \(\alpha\) be a 1-cochain \(\alpha: G_\infty \to G_\infty \hat{\wedge} G_\infty\), which we write in the above basis as \(\alpha(e_n) = \sum_{i,j=0}^{\infty} \alpha_{ij}^n e_i \wedge e_j\), where \(\alpha\) takes values in the completed tensor product \(G_\infty \hat{\otimes} G_\infty = \bigoplus_{n=1}^{\infty} \left( \otimes_{i+j=n} G_i \otimes G_j \right)\), where each \(G_i\) is a one-dimensional subspace of \(G_\infty\) spanned by \(e_i\). The Lie algebra structure on \(G_\infty\) is given by

\[ [e_n, e_m] = (n - m)e_{n+m} \quad \forall \ n, m \geq 0. \]

Then the map \(\alpha\) equips \(G_\infty\) with a Lie bialgebra structure [1] iff

\[
\begin{align*}
(i) \quad & \tau \circ \alpha = -\alpha \\
(ii) \quad & \alpha([a, b]) = a.\alpha(b) - b.\alpha(a), \quad a, b \in G_\infty, \\
(iii) \quad & [1 \otimes 1 \otimes 1 + (\tau \otimes 1)(1 \otimes \tau) + (1 \otimes \tau)(\tau \otimes 1)](1 \otimes \alpha) \circ \alpha = 0,
\end{align*}
\]

where \(\tau\) is the transposition map \(\tau: G_\infty \hat{\otimes} G_\infty \to G_\infty \hat{\otimes} G_\infty\) defined by \(\tau(a \otimes b) = b \otimes a\), for any \(a, b \in G_\infty\), and the dot stands for the action of \(G_\infty\) on \(G_\infty \hat{\wedge} G_\infty\) induced by the adjoint action of \(G_\infty\) on itself. In the case when \(\alpha\) is a 1-coboundary one has \(\alpha(a) = a.r\), where \(r \in G_\infty \hat{\wedge} G_\infty\) is a 0-cochain referred to as the classical \(r\)-matrix [1,3]. In the latter case, (iii) above is equivalent to \(a. < r, r > = 0\), for any \(a \in G_\infty\). Here \(< r, r > := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]\), and \([r^{12}, r^{13}] := \sum_{i,j,k,l=0}^{\infty} r^{ij} r^{kl} [e_i, e_k] \wedge e_j \wedge e_l\), etc., where \(r = \sum_{i,j=0}^{\infty} r^{ij} e_i \wedge e_j\).

**Theorem 3.** The first cohomology group \(H^1(G_\infty, G_\infty \hat{\wedge} G_\infty) = 0\). That is, all 1-cocycles \(\alpha: G_\infty \to G_\infty \hat{\wedge} G_\infty\) are coboundaries.

This can be proven by analyzing the infinite system of linear equations

\[ \alpha([e_n, e_m]) = e_n.\alpha(e_m) - e_m.\alpha(e_n), \quad n, m \geq 0, \]

using its symmetries, and inductive arguments. Then an analysis of the system of equations \(e_n. < r, r > = 0, n \geq 0\), shows that it is equivalent to \(< r, r > = 0\) (CYBE,
the classical Yang-Baxter equation). This turned out to be a specific property of the algebra $G_\infty$ [5,6]. Thus, all Lie bialgebra structures on $G_\infty$ are given by solutions of the classical Yang-Baxter equation. Moreover the following theorem holds.

**Theorem 4.** There is a one-to-one correspondence between the coboundary Lie bialgebra structures on $G_\infty$ and the Poisson-Lie structures (4) on $G_\infty$, the correspondence being given by $r^{ij} = \lambda_{i+1,j+1}$, for every $i, j \geq 0$. Thus, for each $d \in \mathbb{N}$ we have the following infinite-parameter family of Lie bialgebra structures on $G_\infty$ given by $\alpha(e_n) = \sum_{i,j=0}^{\infty} \alpha_{n}^{ij} e_i \wedge e_j$, where

$$\alpha_{n}^{ij} = (2n - i)r^{i-j} + (2n - j)r^{i-j-n} = (2n - i)\lambda_{i-n+1,j+1} + (2n - j)\lambda_{i+1,j-n+1}, \quad \forall n, i, j \geq 0,$$

and $\lambda_{nm}$ are subject to the same conditions as described in Theorem 1.

**Corollary 2.** Thus, Theorem 1 describes all Poisson-Lie structures on the group $G_\infty$.

The proof of Theorem 4 consists of showing that each Lie bialgebra structure on $G_\infty$ can be integrated to a unique Poisson-Lie structure on the group $G_\infty$. To show this one has to show that the following infinite system of linear partial differential equations

$$\sum_{i=j}^{n} (i+1-j)x_{i+1-j} \frac{\partial \omega_{mn}}{\partial x_{i}} = \omega_{m+1-j,n}(x)(m+1-j) + \omega_{m,n+1-j}(x)(n+1-j) + \sum_{k=1}^{m} \sum_{l=1}^{n} \alpha_{n}^{kl}(m+1-k)(n+1-l)x_{m+1-k}x_{n+1-l},$$

where $1 \leq j \leq n$, and $m, n \in \mathbb{N}$, has a unique solution. Here $\alpha_{n}^{ij}$ are the coalgebra structure constants of $G_\infty$. The above system can be obtained by differentiating (3b) with respect to $y$ and setting $y = e$, in which case $\alpha_{n}^{ij} = \frac{\partial \omega_{ij}}{\partial y_{n}} |_{y=e}$. The
existence of a solution is furnished by Theorem 1 since any solution of (3b) is a solution of the above system. To show that it is unique one shows inductively that the corresponding homogeneous system
\[
\sum_{i=j}^{n} (i+1-j)x_{i+1-j} \frac{\partial \omega_{mn}}{\partial x_i} = \omega_{m+1-j,n}(x)(m+1-j) + \omega_{m,n+1-j}(x)(n+1-j)
\]
has only the trivial solution.

A subfamily of Lie bialgebra structures that corresponds to the family of Poisson-Lie structures (11) is given by
\[
(13) \quad \alpha_d(e_n) = 2ne_d \wedge e_n - 2(n-d)e_0 \wedge e_{d+n},
\]
for each \(d \in \mathbb{N}\). The entries of the \(r\)-matrix in this case are \(r^{ij} = \delta_{i+1}^1 \delta_{j+1}^{d+1} - \delta_{i+1}^{d+1} \delta_{j+1}^1 = \lambda_{i+1,j+1} \). The family (13) of Lie bialgebra structures on \(G_\infty\) had been found and studied in [2,7]. Also, we describe below a 1-parameter family, \(\alpha_{d,\lambda}\), of Lie bialgebra structures, for each \(d \geq 1\), of which the family (13) is a subfamily obtained after the specialization \(\lambda = 0\). Namely,
\[
\alpha_{d,\lambda}(e_n) = 2 \sum_{i=d+n}^{\infty} (2n - i) \lambda^{i-(n+d)}(d-1)^i-(n+d)e_0 \wedge e_i - 2n \sum_{i=d}^{\infty} \lambda^{i-d}(d-1)^i-d e_i \wedge e_n
\]
\[
+ 2 \sum_{i=d+n}^{\infty} \sum_{j=1}^{d-1} (2n - i) \lambda^{i+j-(n+d)}(d-1)^{i+j-(n+d+1)} e_i \wedge e_j
\]
\[
+ 2 \sum_{i=d}^{\infty} \sum_{j=n+1}^{d+n-1} (2n - j) \lambda^{i+j-(n+d)}(d-1)^{i+j-(n+d+1)} e_i \wedge e_j.
\]

Again, the right-hand-side of the above formula is an element of the completed tensor product \(G_\infty \hat{\otimes} G_\infty\). This family corresponds to the following solution of (10):
\[
\varphi_{d,\lambda}(u, v) = \frac{1}{1 - (d-1)\lambda u}[1 - (d-1)\lambda v] \left\{ uv(v^d - u^d) + \lambda du^2v^2(u^{d-1} - v^{d-1}) \right\}.
\]

We conclude by noting that as a consequence of Theorem 4 the equation (10) is a functional realization of the classical Yang-Baxter equation for \(G_\infty\).

The complete proofs of the above results will be published elsewhere [5,6].
References

1. V.G. Drinfel’d, *Hamiltonian Structures on Lie Groups, Lie Bialgebras and the Geometric Meaning of the Classical Yang-Baxter Equations*, Soviet Math. Dokl. **27** (1983), 68.

2. W. Michaelis, *A Class of Infinite-Dimensional Lie Bialgebras Containing the Virasoro Algebra*, Advances in Mathematics **107** (1994), 365-392.

3. M.A. Semenov-Tian-Shansky, *What is a Classical r-matrix?,* Funct. Anal. and its Applications **17** (1983), 259-272.

4. M.A. Semenov-Tian-Shansky, *Dressing Transformations and Poisson Group Actions*, Publ. RIMS **21** (1985), 1237-1260.

5. O.S. Stoyanov, *Poisson-Lie Structures on Infinite-Dimensional Jet Groups and Quantum Groups Related to Them*. (submitted)

6. O.S. Stoyanov, Ph.D. Thesis, Virginia Polytechnic Institute and State University, 1993.

7. E.J. Taft, *Witt and Virasoro Algebras as Lie Bialgebras*, Journal of Pure and Applied Algebra, **87** (1993), 301-312.