CLASSICAL AND QUANTUM DYNAMICS IN TRANSVERSE GEOMETRY OF RIEMANNIAN FOLIATIONS

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ABSTRACT. First, we survey some results on classical and quantum dynamical systems associated with transverse Dirac operators on Riemannian foliations. Then we illustrate these results by two examples of Riemannian foliations: a foliation given by the fibers of a fibration and a linear foliation on the two-dimensional torus.

INTRODUCTION

In classical mechanics, a particle, moving on a compact manifold $M$, is described by a point of the phase space, which is the cotangent bundle $T^*M$ of $M$. The evolution of the particle in the phase space is governed by the Hamilton equations of motion. In particular, a Riemannian metric $g$ considered as a function on $T^*M$ is the Hamiltonian of a free particle on $M$, and the corresponding motion is given by the geodesic flow on $T^*M$.

In quantum mechanics, a particle on a compact manifold $M$ is described by a function in the Hilbert space $L^2(M)$ called the wave function. The evolution of the quantum particle is determined by the Schrödinger equation. The Hamiltonian of a free quantum particle, moving on a Riemannian manifold $(M, g)$, is the Laplace-Beltrami operator $\Delta_g$ associated with the Riemannian metric $g$.

It would be more convenient for us to consider Heisenberg’s picture of quantum mechanics, which deals with observable quantities and their dynamics. For a free particle on a compact manifold $M$, classical observables are real-valued functions on the phase space $T^*M$, and quantum observables are pseudodifferential operators on $M$ considered as (unbounded) self-adjoint operators in $L^2(M)$. The quantum evolution of pseudodifferential operators is described by the Heisenberg equations of motion. The classical evolution of their principal symbols is induced by the action of the geodesic flow. The quantum and the classical evolutions are related by the Egorov theorem.

The purpose of this paper is to discuss classical and quantum dynamical systems for a particular class of Riemannian singular spaces, namely, for the leaf space $M/F$ of a Riemannian foliation $\mathcal{F}$ on a compact manifold $M$. In this case, it is very natural to use ideas and notions of noncommutative geometry by A. Connes [3]. The conormal bundle $N^*\mathcal{F}$ of $\mathcal{F}$ carries a natural foliation, $\mathcal{F}_N$, so that the leaf space $N^*\mathcal{F}/\mathcal{F}_N$ of this foliation can be naturally considered as the cotangent bundle of $M/\mathcal{F}$. Using constructions of noncommutative geometry, one can associate to the singular space $N^*\mathcal{F}/\mathcal{F}_N$ some noncommutative algebra, denoted in the paper by $C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$, which plays the role of an algebra of smooth functions on...
this space, that is, of the algebra of classical observables. The algebra of quantum observables is given by the algebra of transverse pseudodifferential operators \( \Psi^{0,-\infty} (M, F) \) on \( M \). We consider quantum Hamiltonians defined by transverse Dirac operators. Then the associated classical dynamics on \( C^\infty_{pr \text{ op}} (G_{F_N}, |T G_N|^{1/2}) \) is induced by the action of the transverse geodesic flow on \( N^*F \), and the quantum evolution of transverse pseudodifferential operators is related with the classical evolution by a version of Egorov’s theorem for transversally elliptic operators stated in [12, 13].

These results are closely related to the reduction theory for quantum Hamiltonian systems with symmetry (see, for instance, [7, 8, 9, 22, 24] and references therein).

We believe that the results discussed in the paper will play some important role in further investigations of spectral theory of transversally elliptic operators on foliated manifolds as well as in the study of asymptotic spectral problems for elliptic operators on foliated manifolds, such as adiabatic limits.

The paper is organized as follows. In Section 1 we review the results on classical and quantum dynamics in transverse geometry of Riemannian foliations mentioned above. In Section 2 we discuss two examples of Riemannian foliations: a foliation given by the fibers of a fibration and a linear foliation on the two-dimensional torus.

1. Definitions and main results

Throughout in the paper, \((M, F)\) is a compact foliated manifold, \( \dim M = n, \dim F = p, p + q = n \). We will consider foliated charts \( \kappa : U \subset M \to \mathbb{R}^p \times \mathbb{R}^q \) on \( M \) with coordinates \((x, y) \in \mathbb{R}^p \times \mathbb{R}^q \) (\( I \) is the open interval \((0, 1)\)) such that the restriction of \( F \) to \( U \) is given by the sets \( y = \text{const} \). We will also use the following notation: \( TF \) is the tangent bundle of \( F \); \( Q = TM/TF \) is the normal bundle of \( F \); \( N^*F \) is the conormal bundle of \( F \).

1.1. Classical phase space. To define the cotangent bundle of the leaf space \( M/F \) of the foliation \((M, F)\), one can proceed as follows. The conormal bundle \( N^*F \) carries a natural foliation, \( F_N \), called the linearized or the lifted foliation. One can define it by constructing its foliated atlas. Given a foliated chart \( \kappa : U \subset M \to \mathbb{R}^p \times \mathbb{R}^q \) on \( M \) with coordinates \((x, y) \in \mathbb{R}^p \times \mathbb{R}^q \), there is the corresponding chart in \( T^*M \) with coordinates written as \((x, y, \xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^q \). In these coordinates, the restriction of the conormal bundle \( N^*F \) to \( U \) is given by the equation \( \xi = 0 \). So we have a chart \( \kappa_\kappa : U_1 \subset N^*F \to \mathbb{R}^p \times \mathbb{R}^q \) on \( N^*F \) with the coordinates \((x, y, \eta) \in \mathbb{R}^p \times \mathbb{R}^q \). This chart is a foliated chart on \( N^*F \) for the linearized foliation \( F_N \), and the restriction of \( F_N \) to \( U_1 \) is given by the level sets \( y = \text{const}, \eta = \text{const} \).

If the foliation \( F \) is given by the fibers of a fibration \( f : M \to B \) over a compact manifold \( B \), then the leaf space \( N^*F/F_N \) of the linearized foliation \( F_N \) coincides with the cotangent bundle \( T^*B \) of \( B \). Therefore, for an arbitrary foliation \( F \), the leaf space \( N^*F/F_N \) can be naturally considered as the cotangent bundle of the leaf space \( M/F \).

Remark 1. This construction can be viewed a particular example of the symplectic reduction in the sense of [16, Chapter III, Section 14] (see also [17, 18]).

Let \((X, \omega)\) be a symplectic manifold, and \( Y \) a submanifold of \( X \) such that the 2-form \( \omega_Y \) induced by \( \omega \) on \( Y \) is of constant rank. Let \( F_Y \) be the characteristic foliation of \( Y \) relative to \( \omega_Y \) (that is, \( TF_Y \) is the skew-orthogonal complement...
symplectic form $\omega$ to be the reduced symplectic manifold associated with $Y$. The symplectic manifold $(Z, \omega_Z)$ is said to be the reduced symplectic manifold associated with $Y$. In a particular case when the submanifold $Y$ is the preimage of a point under the momentum map associated with the Hamiltonian action of a Lie group, the symplectic reduction associated with $Y$ is the Marsden-Weinstein symplectic reduction. In the case under consideration, one can consider the symplectic reduction associated with the coisotropic submanifold $Y = N^*F$ in the symplectic manifold $X = T^*M$. The corresponding characteristic foliation $F_Y$ is the linearized foliation $F_N$, and the leaf space $N^*F/F_N$ plays a role of the reduced phase space $Z$.

In the general case, the leaf space $N^*F/F_N$ is not a smooth manifold. We will use the ideas of the noncommutative geometry in the sense of A. Connes and introduce a noncommutative algebra, $C^*_prop(G_{F_N}, |T^*F_N|^{1/2})$, as a noncommutative analogue of an algebra of smooth functions on $N^*F/F_N$.

First, we recall several notions (for more details, see e.g. [14] and references therein). Let $\gamma : [0,1] \to M$ be a continuous leafwise path in $M$ with the initial point $x = \gamma(0)$ and the final point $y = \gamma(1)$ and $T_0$ and $T_1$ arbitrary smooth submanifolds (possibly, with boundary), transversal to the foliation, such that $x \in T_0$ and $y \in T_1$. Sliding along the leaves of the foliation $F$ determines a diffeomorphism $H_{T_0,T_1}(\gamma)$ of a neighborhood of $x$ in $T_0$ to a neighborhood of $y$ in $T_1$, called the holonomy map along $\gamma$. The differential of $H_{T_0,T_1}(\gamma)$ at $x$ gives rise to a well-defined linear map $Q_x \to Q_y$, which is independent of the choice of transversals $T_0$ and $T_1$. This map is called the linear holonomy map and denoted by $dh_\gamma$. The adjoint of $dh_\gamma$ yields a linear map $dh_\gamma^* : N^*F_y \to N^*F_x$.

The holonomy groupoid $G$ of $F$ consists of $\sim_h$-equivalence classes of continuous leafwise paths in $M$, where we set $\gamma_1 \sim_h \gamma_2$, if $\gamma_1$ and $\gamma_2$ have the same initial and final points and the same holonomy maps. The set of units $G^{(0)}$ is the manifold $M$. The multiplication in $G$ is given by the product of paths. The corresponding source map $s : G \to M$ and range map $r : G \to M$ are given by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. Finally, the diagonal map $\Delta : M \to G$ takes any $x \in M$ to the element in $G$ given by the constant path $\gamma(t) = x, t \in [0,1]$. To simplify the notation, we will identify $x \in M$ with $\Delta(x) \in G$. For any $x \in M$ the map $s$ maps $G^x$ onto the leaf $L_x$ through $x$. The group $G^x_\nu$ coincides with the holonomy group of $L_x$. The map $s : G^x \to L_x$ is the covering map associated with the group $G^x_\nu$, called the holonomy covering.

Let us introduce the groupoid $G_{F_N}$ as the set of all $(\gamma, \nu) \in G \times N^*F$ such that $r(\gamma) = \pi(\nu)$. The source map $s_N : G_{F_N} \to N^*F$ and the range map $r_N : G_{F_N} \to N^*F$ are defined as $s_N(\gamma, \nu) = dh_\gamma^*(\nu)$ and $r_N(\gamma, \nu) = \nu$. We have a map $\pi_G : G_{F_N} \to G$ given by $\pi_G(\gamma, \nu) = \gamma$. If $F$ is a Riemannian foliation, $G_{F_N}$ coincides with the holonomy groupoid of the linearized foliation $F_N$, but, for a general foliation, these two groupoids may be different. Observe also that the leaf of $F_N$ through $\nu \in N^*F$ can be described as the set of all points $dh_\gamma^*(\nu) \in N^*F$, where $\gamma \in G, r(\gamma) = \pi(\nu)$ (here $\pi : T^*M \to M$ is the bundle map).

The groupoid $G_{F_N}$ carries a natural codimension $q$ foliation $G_N$. The leaf of $G_N$ through a point $(\gamma, \nu) \in G_{F_N}$ is the set of all $(\gamma', \nu') \in G_{F_N}$ such that $\nu$ and $\nu'$ lie in the same leaf in $F_N$. For any vector bundle $V$ on $M$, denote by $|V|^{1/2}$ the associated half-density vector bundle. Let $|T^*G_N|^{1/2}$ be the line bundle of leafwise
half-densities on $G_N^F$, with respect to the foliation $G_N^F$. It is easy to see that

$$|T G_N^F|^{1/2} = r_N^* (|T F_N^F|^{1/2}) \otimes s_N^* (|T F_N^F|^{1/2}),$$

where $s_N^* (|T F_N^F|^{1/2})$ and $r_N^* (|T F_N^F|^{1/2})$ denote the lifts of the line bundle $|T F_N^F|^{1/2}$ of leafwise half-densities on $N^* F$ via the source and the range mappings $s_N$ and $r_N$ respectively.

Let $x_0 : U \to I^p \times I^0, x_1 : U' \to I^p \times I^q$ be two foliated charts, let $T_0 = \varphi_{x_0}^{-1} (\{0\} \times I^0), T_1 = \varphi_{x_1}^{-1} (\{0\} \times I^0)$ be the corresponding local transversals, and let $\pi_0 : T_0 \to I^q, \pi_1 : T_1 \to I^q$ be the corresponding diffeomorphisms. The foliated charts $x_0$ and $x_1$ are called compatible, if, for any $m \in U$ and $m' \in U'$ with $\pi_0 (m) = \pi_1 (m')$, there is a leafwise path $\gamma$ from $m$ to $m'$ such that the corresponding holonomy diffeomorphism $H_{T_0 T_1} (\gamma)$ considered as a local diffeomorphism of $I^q$ is the identity in a neighborhood of $\pi_0 (m) = \pi_1 (m')$.

For any pair of compatible foliated charts $x_0$ and $x_1$ denote by $W (x_0, x_1)$ the subset in $G$, consisting of all $\gamma \in G$ such that $s (\gamma) = \varphi_0^{-1} (x, y) \in U$ and $r (\gamma) = \varphi_1^{-1} (x', y) \in U'$ for some $(x, x', y) \in I^p \times I^p \times I^q$ and the corresponding holonomy diffeomorphism $H_{T_0 T_1} (\gamma)$ considered as a local diffeomorphism in $I^q$ is the identity in a neighborhood of $y \in I^q$. There is a coordinate map $\Gamma : W (x_0, x_1) \to I^p \times I^p \times I^q$, which takes an element $\gamma \in W (x_0, x_1)$ as above to the corresponding triple $(x, x', y) \in I^p \times I^p \times I^q$. As shown in [2], the coordinate neighborhoods $W (x_0, x_1)$ form an atlas of a $(2p + q)$-dimensional manifold (in general, non-Hausdorff and non-paracompact) on $G$. Moreover, the groupoid $G$ is a smooth groupoid.

Now let $x : U \subset M \to I^p \times I^q, x_1 : U' \subset M \to I^p \times I^q$, be two compatible foliated charts on $M$. Then the corresponding foliated charts $x_n : U_1 \subset N^* F \to I^p \times I^q 	imes \mathbb{R}^q, (x_1)_n : U'_1 \subset N^* F \to I^p \times I^q \times \mathbb{R}^q$, are compatible with respect to the foliation $F_N$. So they define a foliated chart $V$ on the foliated manifold $(G_{F_N}, G_N^F)$ with the coordinates $(x, x', y, \eta) \in I^p \times I^q \times I^q \times \mathbb{R}^q$, and the restriction of $G_N$ to $V$ is given by the level sets $y = \text{const}, \eta = \text{const}$.

A section $\sigma \in C^\infty (G_{F_N}, |T G_N^F|^{1/2})$ is said to be properly supported, if the restriction of the map $r_N : G_{F_N} \to N^* F$ to supp $\sigma$ is a proper map. One can introduce the structure of involutive algebra on the space $C^\infty_{\text{prop}} (G_{F_N}, |T G_N^F|^{1/2})$, smoothly, properly supported sections of $|T G_N^F|^{1/2}$ by the formulas

\begin{equation}
\sigma_1 \ast \sigma_2 (\gamma, \nu) = \int_{(\gamma_1, \nu_1)(\gamma_2, \nu_2) = (\gamma, \nu)} \sigma_1 (\gamma_1, \nu_1) \sigma_2 (\gamma_2, \nu_2), \quad \gamma \in G_{F_N},
\end{equation}

$$\sigma^* (\gamma, \nu) = \sigma ((\gamma, \nu)^{-1}), \quad \gamma \in G_N^F,$$

where $\sigma, \sigma_1, \sigma_2 \in C^\infty_{\text{prop}} (G_{F_N}, |T G_N^F|^{1/2})$. The formula for $\sigma_1 \ast \sigma_2$ should be interpreted in the following way. The composition $(\gamma, \nu) = (\gamma_1, \nu_1)(\gamma_2, \nu_2)$ is defined if $\nu_2 = dh_{\gamma_1} (\nu_1)$ and equals $(\gamma, \nu) = (\gamma_1 \gamma_2, \nu_2)$. Then we have

$$\sigma_1 (\gamma_1, \nu_1) \sigma_2 (\gamma_2, \nu_2) \in |T_{\nu_1} F_N |^{1/2} \otimes |T_{dh_{\gamma_1} (\nu_1)} F_N |^{1/2} \otimes |T_{\nu_2} F_N |^{1/2} \otimes |T_{dh_{\gamma_2} (\nu_2)} F_N |^{1/2} \cong |T_{\nu_1} F_N |^{1/2} \otimes |T_{\nu_2} F_N |^{1/2} \otimes |T_{dh_{\gamma_1} \gamma_2 (\nu)} F_N |^{1/2},$$

and, integrating the $|T_{\nu} F_N |^{1/2}$-component of $\sigma_1 (\gamma_1, \nu_1) \sigma_2 (\gamma_2, \nu_2)$ over the set of all $\nu_2 \in N^* F$ of the form $\nu_2 = dh_{\gamma_1} (\nu_1)$, which coincides with the leaf of $F_N$ through $\nu$, we get a well-defined section of the bundle $r_N^* (|T F_N |^{1/2}) \otimes s_N^* (|T F_N |^{1/2}) = |T G_N^F|^{1/2}$. 
As mentioned above, the algebra $C^\infty_{prop}(G\mathcal{F}_N, |T\mathcal{G}_N|^{1/2})$ plays a role of the non-commutative analogue of an algebra of functions on the leaf space $N^*\mathcal{F}/\mathcal{F}_N$.

1.2. Transversal pseudodifferential operators. Our quantum phase space is a certain completion of the algebra $\Psi^*_{\infty}(M, \mathcal{F}, E)$ of transversal pseudodifferential operators introduced in [11]. So we will briefly recall the definition of this algebra.

Let $E$ be a complex vector bundle over $M$ of rank $r$. We will consider pseudodifferential operators, acting on half-densities. Let $C^\infty(M, E)$ denote the space of smooth sections of the vector bundle $E \otimes |TM|^{1/2}$, $L^2(M, E)$ the Hilbert space of square integrable sections of $E \otimes |TM|^{1/2}$, $\mathcal{D}'(M, E)$ the space of distributional sections of $E \otimes |TM|^{1/2}$, $\mathcal{D}'(M, E) = C^\infty(M, E)'$. Finally, let $\Psi^m(M, E)$ denote the standard classes of pseudodifferential operators, acting on $C^\infty(M, E)$.

For any $k_A \in S^m(I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathcal{C}^r))$, define an operator $A : C^\infty_c(I^p, \mathcal{C}^r) \to C^\infty(I^p, \mathcal{C}^r)$ by the formula

$$Au(x, y) = (2\pi)^{-q} \int e^{i(x-y')\eta}k_A(x, x', y, \eta)u(x', y') \, dx' \, dy' \, d\eta,$$

where $u \in C^\infty_c(I^p, \mathcal{C}^r)$, $x \in I^p$, $y \in I^q$. The function $k_A$ is called the complete symbol of $A$. As usual, we will consider only classical (or polyhomogeneous) symbols, that is, those symbols, which can be represented as an asymptotic sum of homogeneous (in $\eta$) components.

Let $\kappa : U \subset M \to I^p \times I^q$, $\kappa_1 : U' \subset M \to I^p \times I^q$ be a pair of compatible foliated charts on $M$ equipped with trivializations of the bundle $E$ over them. Any operator $A$ of the form (2) with the Schwartz kernel, compactly supported in $I^n \times I^n$, determines an operator $A : C^\infty_c(U, E|_{U'}) \to C^\infty_c(U', E|_{U'})$, which extends to an operator in $C^\infty(M, E)$ in a trivial way. The resulting operator is called an elementary operator of class $\Psi^m_{\infty}(M, \mathcal{F}, E)$.

The class $\Psi^m_{\infty}(M, \mathcal{F}, E)$ consists of all operators $A$ in $C^\infty(M, E)$, which can be represented in the form

$$A = \sum_i A_i + K,$$

where $A_i$ are elementary operators of class $\Psi^m_{\infty}(M, \mathcal{F}, E)$, corresponding to a pair $\kappa_i, \kappa_i'$ of compatible foliated charts, $K \in \Psi^{-\infty}(M, E)$.

The principal symbol of the operator $A$ given by (2) is the leafwise half-density

$$\sigma(A)(x, x', y, \eta) = k_{A,m}(x, x', y, \eta)|dx|^{1/2}|dx'|^{1/2},$$

where $k_{A,m}$ is the degree $m$ homogeneous component of the complete symbol $k_A$.

The global definition of the principal symbol is given as follows. Let $\pi^*E$ denote the lift of the vector bundle $E$ to $\tilde{T}^*M = T^*M \setminus 0$ via the bundle map $\pi : \tilde{T}^*M \to M$. Put $\tilde{N}^*\mathcal{F} = N^*\mathcal{F} \setminus 0$. Denote by $\tilde{\mathcal{F}}_N$ the restriction of $\mathcal{F}_N$ and by $G_{\tilde{\mathcal{F}}_N}$ the groupoid associated with $\tilde{\mathcal{F}}_N$. Let $\mathcal{L}(\pi^*E)$ be the vector bundle on $G_{\tilde{\mathcal{F}}_N}$, whose fiber at a point $(\gamma, \nu) \in G_{\tilde{\mathcal{F}}_N}$ is the space $\mathcal{L}((\pi^*E)_{s_N(\gamma, \nu)}, (\pi^*E)_{r_N(\gamma, \nu)})$ of linear maps from $(\pi^*E)_{s_N(\gamma, \nu)}$ to $(\pi^*E)_{r_N(\gamma, \nu)}$. Consider the space $C^\infty_{\text{prop}}(G_{\tilde{\mathcal{F}}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ of smooth, properly supported sections of $\mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}$. One can introduce the structure of involutive algebra on $C^\infty_{\text{prop}}(G_{\tilde{\mathcal{F}}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ by formulas similar to (1). Let $S^m(G_{\tilde{\mathcal{F}}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ be the space of all $s \in C^\infty_{\text{prop}}(G_{\tilde{\mathcal{F}}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ homogeneous of degree $m$ with respect to the action of $\mathbb{R}$ given by the multiplication in the fibers of the vector bundle $\pi_G : G_{\mathcal{F}_N} \to G$. 

The principal symbol $\sigma(A)$ of an operator $A \in \Psi^{-\infty}(M, \mathcal{F}, E)$ given in local coordinates by the formula (3) is globally defined as an element of the space $S^m(G_{\mathcal{F}N}, \mathcal{L}(\pi^* E) \otimes T\mathcal{G}_N^{1/2})$. Thus, we have the half-density principal symbol mapping

$$\sigma : \Psi^{-\infty}(M, \mathcal{F}, E) \to S^m(G_{\mathcal{F}N}, \mathcal{L}(\pi^* E) \otimes T\mathcal{G}_N^{1/2}),$$

which satisfies

$$\sigma(AB) = \sigma(A)\sigma(B), \quad \sigma(A^*) = \sigma(A)^*$$

for any $A \in \Psi^{m_1,-\infty}(M, \mathcal{F}, E)$ and $B \in \Psi^{m_2,-\infty}(M, \mathcal{F}, E)$.

1.3. Riemannian foliations. We will consider a particular class of Hamiltonians associated with a naive analogue of a Riemannian metric on the leaf space $M/\mathcal{F}$. It only exists for a particular class of foliations, called Riemannian foliations.

Let $M$ be a compact manifold equipped with a foliation $\mathcal{F}$, $g_M$ a Riemannian metric on $M$. Let $H = T\mathcal{F}^\perp$ be the orthogonal complement of the tangent bundle of $\mathcal{F}$. So we have the direct sum decomposition

$$(4) \quad TM = T\mathcal{F} \oplus H.$$ 

There are natural isomorphisms $H \cong Q$ and $H^* \cong Q^* \cong N^*\mathcal{F}$. One can also decompose the metric $g_M$ into the sum of its leafwise and transverse components:

$$(5) \quad g_M = g_\mathcal{F} + g_H.$$

In a foliated chart with coordinates $(x, y) \in I^p \times I^q$ the transverse part $g_H$ can be written as $g_H = \sum_{\alpha\beta} g_{\alpha\beta}(x, y)\theta^\alpha \theta^\beta$, where $\theta^\alpha \in H^*$ is the (unique) lift of $dy^\alpha$ under the projection $I^p \times I^q \to I^q$. The metric $g_M$ is called bundle-like, if in any foliated chart the components $g_{\alpha\beta}$ are independent of $x$, $g_{\alpha\beta}(x, y) = g_{\alpha\beta}(y)$. Equivalently, one can say that $g_M$ is bundle-like, if, for any leafwise continuous path $\gamma$ from $x$ to $y$, the corresponding linear holonomy map $dh_{\gamma} : Q_x \to Q_y$ is an isometry (see, for instance, [20, 21] for more details). The foliation $\mathcal{F}$ is called Riemannian, if there exists a bundle-like metric on $M$. In the sequel, we will always assume that $\mathcal{F}$ is a Riemannian foliation and $g_M$ is a bundle-like metric.

Let $P_\mathcal{F}$ (resp. $P_H$) denotes the orthogonal projection operator of $TM$ onto $T\mathcal{F}$ (resp. $H$). Denote by $\nabla^L$ the Levi-Civita connection on $TM$ defined by $g_M$. The following formulas define a connection $\nabla$ in $H$:

$$(6) \quad \nabla_X N = P_H[X, N], \quad X \in C^\infty(M, T\mathcal{F}), \quad N \in C^\infty(M, H)$$

$$\nabla_X N = P_H\nabla^L_X N, \quad X \in C^\infty(M, H), \quad N \in C^\infty(M, H).$$

Remark that the first identity in (6) yields a canonical flat connection in $H$, defined along the leaves of $\mathcal{F}$, which exists for an arbitrary foliation and is called the Bott connection. It turns out that $\nabla$ depends only on the transverse part of the metric $g_M$ and preserves the inner product of $H$. It will be called the transverse Levi-Civita connection.

Let $\omega_\mathcal{F}$ denote the leafwise Riemannian volume form of $\mathcal{F}$. Let $f \in H_x$ and let $\tilde{f} \in C^\infty(M, H)$ denote any infinitesimal transformation of $\mathcal{F}$, which coincides with $f$ at $x$. By the definition of infinitesimal transformation, the local flow generated by $\tilde{f}$ preserves the foliation and gives rise to a well-defined action on $\Lambda^p T^* \mathcal{F}$. The mean curvature vector field $\tau \in C^\infty(M, H)$ of $\mathcal{F}$ is defined by the identity

$$L_f \omega_\mathcal{F} = g_M(\tau, \tilde{f}) \omega_\mathcal{F}.$$
If \(e_1, e_2, \ldots, e_p\) is a local orthonormal frame in \(TF\), then
\[
\tau = \sum_{i=1}^{p} P_H (\nabla_{e_i} e_i).
\]

1.4. **Transverse Dirac operators.** For any \(x \in M\), denote by \(Cl(Q_x)\) the complex Clifford algebra of \(Q_x\). Recall that, relative to an orthonormal basis \(\{f_1, f_2, \ldots, f_q\}\) of \(Q_x\), \(Cl(Q_x)\) is the complex algebra generated by 1 and \(f_1, f_2, \ldots, f_q\), satisfying the relations
\[
f_\alpha f_\beta + f_\beta f_\alpha = -2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \ldots, q.
\]

Consider the \(\mathbb{Z}_2\)-graded vector bundle \(Cl(Q)\) over \(M\) whose fiber at \(x \in M\) is \(Cl(Q_x)\). This bundle is associated to the principal \(SO(q)\)-bundle \(O(Q)\) of oriented orthonormal frames in \(Q\), \(Cl(Q) = O(Q) \times_{O(q)} Cl(\mathbb{R}^q)\). Therefore, the transverse Levi-Civita connection \(\nabla\) induces a natural leafwise flat connection \(\nabla^{Cl(Q)}\) on \(Cl(Q)\) which is compatible with the multiplication and preserves the \(\mathbb{Z}_2\)-grading on \(Cl(Q)\).

If \(\{f_1, f_2, \ldots, f_q\}\) is a local orthonormal frame in \(T^H M\), and \(\omega_{\alpha\beta}\) is the coefficients of the connection \(\nabla\): \(\nabla_{f_\alpha} f_\beta = \sum_{\gamma} \omega_{\alpha\beta}^\gamma f_\gamma\), then
\[
(7) \quad \nabla^{Cl(Q)} f_\alpha = f_\alpha + \frac{1}{4} \sum_{\gamma=1}^{q} \omega_{\alpha\beta}^\gamma c(f_\beta)c(f_\gamma),
\]
where \(c(a)\) denotes the action of an element \(a \in C^\infty(M, Cl(Q))\) on \(C^\infty(M, Cl(Q))\) by pointwise left multiplication.

A transverse Clifford module is a complex vector bundle \(\mathcal{E}\) on \(M\) endowed with a fiberwise action of the bundle \(Cl(Q)\). We will denote the action of \(a \in Cl(Q_x)\) on \(s \in \mathcal{E}_x\) as \(c(a)s \in \mathcal{E}_x\). A transverse Clifford module \(\mathcal{E}\) is called self-adjoint if it endowed with a Hermitian metric such that the operator \(c(f) : \mathcal{E}_x \to \mathcal{E}_x\) is skew-adjoint for any \(x \in M\) and \(f \in Q_x\). Any transverse Clifford module \(\mathcal{E}\) carries a natural \(\mathbb{Z}_2\)-grading \(\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-\).

A Hermitian connection \(\nabla^\mathcal{E}\) on a transverse Clifford module \(\mathcal{E}\) is called a Clifford connection if it is compatible with the Clifford action, that is, for any \(f \in C^\infty(M, H)\) and \(a \in C^\infty(M, Cl(Q))\),
\[
[\nabla^\mathcal{E}_f, c(a)] = c(\nabla^\mathcal{E}_f c(a)).
\]

A self-adjoint transverse Clifford module \(\mathcal{E}\) equipped with a leafwise flat Clifford connection \(\nabla^\mathcal{E}\) is called a transverse Clifford bundle. For any transverse Clifford bundle \((\mathcal{E}, \nabla^\mathcal{E})\), we define the operator \(D^\mathcal{E}'\) acting on the sections of \(\mathcal{E}\) as the composition
\[
\begin{align*}
C^\infty(M, \mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} C^\infty(M, Q^* \otimes \mathcal{E}) = C^\infty(M, Q \otimes \mathcal{E}) \xrightarrow{c} C^\infty(M, \mathcal{E}).
\end{align*}
\]

Here we identify the bundle \(Q\) and \(Q^*\) by means of the metric \(g_M\). The transverse Dirac operator \(D_\mathcal{E}\) is defined as
\[
D_\mathcal{E} = D^\mathcal{E}' - \frac{1}{2} c(\tau).
\]

If \(f_1, \ldots, f_q\) is a local orthonormal frame for \(H\), then
\[
D_\mathcal{E} = \sum_{\alpha=1}^{q} c(f_\alpha) \left( \nabla^\mathcal{E}_f - \frac{1}{2} g_M(\tau, f_\alpha) \right).
\]

This operator is odd with respect to the natural \(\mathbb{Z}_2\)-grading on \(\mathcal{E}\).
Denote by $(\cdot, \cdot)_x$ the inner product in the fiber $E_x$ over $x \in M$. Then the inner product in $L^2(M, E)$ is given by the formula

$$(s_1, s_2) = \int_M (s_1(x), s_2(x))_x \omega_M, \quad s_1, s_2 \in L^2(M, E),$$

where $\omega_M = \sqrt{\det g} dx$ denotes the Riemannian volume form on $M$. As shown in [13], the transverse Dirac operator $D_E$ is formally self-adjoint in $L^2(M, E)$.

The transverse Dirac operators were introduced in [4, 5] (see [13, 15] for further references).

We will use the Riemannian volume form $\omega_M$ to identify the half-densities bundle with the trivial one. So the action of $D_E$ on half-densities is defined by

$$D_E(u|\omega_M|^{1/2}) = (D_E u)|\omega_M|^{1/2}, \quad u \in C^\infty(M, E).$$

**Example 2.** Assume that $F$ is transversely oriented and the normal bundle $Q$ is spin. Thus the $SO(q)$ bundle $O(Q)$ of oriented orthonormal frames in $Q$ can be lifted to a $Spin(q)$ bundle $O'(Q)$ so that the projection $O'(Q) \to O(Q)$ induces the covering projection $Spin(q) \to SO(q)$ on each fiber.

Let $F(Q), F_+(Q), F_-(Q)$ be the bundles of spinors

$$F(Q) = O'(Q) \times_{Spin(q)} S, \quad F_{\pm}(Q) = O'(Q) \times_{Spin(q)} S_{\pm}.$$ 

Since $\dim Q = q$ is even, $\text{End} F(Q)$ is as a bundle of algebras over $M$ isomorphic to the Clifford bundle $Cl(Q)$. The transverse Levi-Civita connection $\nabla$ lifts to a leafwise flat Clifford connection $\nabla^{F(Q)}$ on $F(Q)$. So $F(Q)$ is a transverse Clifford bundle.

More generally, one can take a Hermitian vector bundle $W$ equipped with a leafwise flat Hermitian connection $\nabla^W$. Then $F(Q) \otimes W$ is a transverse Clifford bundle: the action of $a \in C^\infty(M, Cl(Q))$ on $C^\infty(M, F(Q) \otimes W)$ is given by $c(a) \otimes 1$ ($c(a)$ denotes the action of $a$ on $C^\infty(M, F(Q))$, and the product connection $\nabla^{F(Q) \otimes W} = \nabla^{F(Q)} \otimes 1 + 1 \otimes \nabla^W$ on $F(Q) \otimes W$ is a Clifford connection.

**Example 3.** Another example of a transverse Clifford bundle associated with a transverse almost complex structure on $(M, \mathcal{F})$, a transverse Clifford module $\Lambda^{0,*}$, is described in [15].

**Example 4.** The decomposition (4) induces a bigrading on $AT^*M$:

$$AT^*M = \bigoplus_{i,j=1}^n \Lambda^{i,j}T^*M, \quad \Lambda^{i,j}T^*M = \Lambda^i T^*F^* \otimes \Lambda^j H^*.$$ 

In this bigrading, the de Rham differential $d$ can be written as

$$d = d_F + d_H + \theta,$$

where $d_F$ and $d_H$ are first order differential operators (called the tangential de Rham differential and the transversal de Rham differential accordingly), and $\theta$ is a zero order differential operator.

By definition, the transverse signature operator is a first order differential operator in $C^\infty(M, \Lambda H^*)$ given by

$$D_H = d_H + d_H^*.$$
Consider the vector bundle $\mathcal{E} = \Delta H^* \otimes \mathbb{C}$ equipped with a natural structure of a transverse Clifford bundle. As shown in [13], for the associated transverse Dirac operator $D_\mathcal{E}$, we have

$$D_\mathcal{E} = d_H + d_H^* + \frac{1}{2}(\varepsilon_{\tau^*} + i_{\tau^*}),$$

where $\varepsilon_{\tau^*}$ denotes the exterior product by the covector $\tau^* \in Q^*_\mathcal{E}$ dual to $\tau$, $i_{\tau^*}$ the interior product by $\tau$. So we see that the transverse signature operator $D_H$ coincides with $D_\mathcal{E}$ if and only if $\tau = 0$, that is, all the leaves are minimal submanifolds.

1.5. Quantum dynamics. Let $D_\mathcal{E}$ be a transverse Dirac operator associated with a transverse Clifford bundle $(\mathcal{E}, \nabla^\mathcal{E})$. The operator $D_\mathcal{E}$ is essentially self-adjoint with initial domain $C^\infty(M, \mathcal{E})$. Define an unbounded linear operator $(D_\mathcal{E})$ in the space $L^2(M, \mathcal{E})$ as

$$(D_\mathcal{E}) = (D_\mathcal{E}^2 + 1)^{1/2}.$$ 

By the spectral theorem, the operator $(D_\mathcal{E})$ is well-defined as a positive, self-adjoint operator in $L^2(M, \mathcal{E})$. It can be shown that the Sobolev space $H^1(M, \mathcal{E})$ is contained in the domain of $(D_\mathcal{E})$ in $L^2(M, \mathcal{E})$.

By the spectral theorem, the operator $(D_\mathcal{E})$ defines a strongly continuous group $e^{it(D_\mathcal{E})}$ of bounded operators in $L^2(M, \mathcal{E})$. Consider a one-parameter group $\Phi_t$ of $*$-automorphisms of the algebra $\mathcal{L}(L^2(M, \mathcal{E}))$ defined by

$$\Phi_t(T) = e^{it(D_\mathcal{E})}T e^{-it(D_\mathcal{E})}, \quad T \in \mathcal{L}(L^2(M, \mathcal{E})), \quad t \in \mathbb{R}.$$ 

By the spectral theorem, the operator $(D_\mathcal{E})^s = (D_\mathcal{E}^2 + 1)^{s/2}$ is a well-defined positive self-adjoint operator in $\mathcal{H} = L^2(M, \mathcal{E})$ for any $s \in \mathbb{R}$, which is unbounded if $s > 0$. For any $s \geq 0$, denote by $\mathcal{H}^s$ the domain of $(D_\mathcal{E})^s$, and, for $s < 0$, $\mathcal{H}^s = (\mathcal{H}^{-s})^*$. Put also $\mathcal{H}^\infty = \bigcap_{s \geq 0} \mathcal{H}^s$, $\mathcal{H}^{-\infty} = (\mathcal{H}^\infty)^*$. It is clear that $\mathcal{H}^s(M, \mathcal{E}) \subset \mathcal{H}^r$ for any $s \geq 0$ and $\mathcal{H}^r \subset \mathcal{H}^{r+s}(M, \mathcal{E})$ for any $s < 0$. In particular, $C^\infty(M, \mathcal{E}) \subset \mathcal{H}^s$ for any $s$.

We say that a bounded operator $A$ acting on $\mathcal{H}^\infty$ belongs to $\mathcal{L}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ (resp. $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$), if, for any $s$ and $r$, it extends to a bounded (resp. compact) operator from $\mathcal{H}^s$ to $\mathcal{H}^r$, or, equivalently, the operator $(D_\mathcal{E})^r A(D_\mathcal{E})^{-s}$ extends to a bounded (resp. compact) operator in $L^2(M, \mathcal{E})$. It is easy to see that $\mathcal{L}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ is an involutive subalgebra in $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ is its ideal. We also introduce the class $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$, which consists of all operators from $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ such that, for any $s$ and $r$, the operator $(D_\mathcal{E})^r A(D_\mathcal{E})^{-s}$ is a trace class operator on $L^2(M, \mathcal{E})$. It should be noted that any operator $K$ with smooth kernel belongs to $L^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$.

**Theorem 5.** [13] Let $D_\mathcal{E}$ be a transverse Dirac operator associated with a transverse Clifford bundle $(\mathcal{E}, \nabla^\mathcal{E})$. For any $K \in \Psi^{-\infty}(M, \mathcal{F}, \mathcal{E})$, there exists an operator $K(t) \in \Psi^{-\infty}(M, \mathcal{F}, \mathcal{E})$ such that $\Phi_t(K) \rightarrow K(t)$, $t \in \mathbb{R}$, is a smooth family of operators of class $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$.

By this theorem, we have a well-defined one-parameter automorphism group

$$\Phi_t : \Psi^0(\mathcal{H}^{-\infty}, \mathcal{F}, \mathcal{E}) \rightarrow \Psi^0(\mathcal{H}^{-\infty}, \mathcal{F}, \mathcal{E}), \quad t \in \mathbb{R},$$

of the completed algebra of transverse pseudodifferential operators defined by

$$\hat{\Psi}^0(\mathcal{H}^{-\infty}, \mathcal{F}, \mathcal{E}) = \Psi^0(\mathcal{H}^{-\infty}, \mathcal{F}, \mathcal{E}) + \mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty).$$

It describes the quantum dynamics associated with the quantum Hamiltonian $(D_\mathcal{E})$. 


1.6. Classical dynamics. In this Section, we give a definition of the transverse geodesic flow associated with a bundle-like metric \(g\) on a compact foliated manifold \((M, \mathcal{F})\). This notion is a particular case of the bicharacteristic flow associated with a first order transversally elliptic operator (see [12] [13]).

Denote by \(f_t\) the geodesic flow of the Riemannian metric \(g\) on \(T^*M\). Since \(g\) is bundle-like, the flow \(f_t\) preserves the conormal bundle \(N^*\mathcal{F}\), and its restriction to \(N^*\mathcal{F}\) (denoted also by \(f_t\)) preserves the foliation \(\mathcal{F}_N\), that is, takes any leaf of \(\mathcal{F}_N\) to a leaf. Moreover, one can show existence of a flow \(F_t\) on \(G_{\mathcal{F}_N}\) such that \(s_N \circ F_t = f_t \circ s_N, r_N \circ F_t = f_t \circ r_N\), which preserves the foliation \(\mathcal{G}_N\).

Let \(X\) be the generator of the geodesic flow \(f_t\). Then it is tangent to \(N^*\mathcal{F}\) and determines a vector field on \(N^*\mathcal{F}\), denoted also by \(X\). Since \(f_t\) preserves the foliation \(\mathcal{F}_N\), \(X\) is an infinitesimal transformation of \(\mathcal{F}_N\), and there exists a vector field \(\mathcal{H}\) on \(G_{\mathcal{F}_N}\) such that \(ds_N(\mathcal{H}) = X\) and \(dr_N(\mathcal{H}) = X\). Then the vector field \(\mathcal{H}\) is the generator of the flow \(F_t\).

In a foliation chart, \(X\) is given by

\[
X = \sum_{j=1}^p \left( \partial_{\xi_j} \sqrt{g} \frac{\partial}{\partial x_j} - \partial_{x_j} \sqrt{g} \frac{\partial}{\partial \xi_j} \right) + \sum_{k=1}^q \left( \partial_{y_k} \sqrt{g} \frac{\partial}{\partial y_k} - \partial_{y_k} \sqrt{g} \frac{\partial}{\partial \eta_k} \right),
\]

and \(\mathcal{H}\) is given by

\[
\mathcal{H}(x, x', y, \eta) = \sum_{j=1}^p \partial_{\xi_j} \sqrt{g}(x, y, 0, \eta) \frac{\partial}{\partial x_j} + \sum_{j=1}^p \partial_{x_j} \sqrt{g}(x', y, 0, \eta) \frac{\partial}{\partial x_j'},
\]

\[
+ \sum_{k=1}^q \left( \partial_{y_k} \sqrt{g}(y, \eta) \frac{\partial}{\partial y_k} - \partial_{y_k} \sqrt{g}(y, \eta) \frac{\partial}{\partial \eta_k} \right),
\]

\[
(x, x', y, \eta) \in I^p \times I^p \times I^q \times \mathbb{R}^q.
\]

The flow \(F_t^*\) on \(C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})\) induced by \(F_t\) is called the transverse geodesic flow.

Now consider a transverse Clifford bundle \((\mathcal{E}, \nabla^\mathcal{E})\). The pull-back of the connection form of \(\nabla^\mathcal{E}\) by the projection \(\pi: N^*\mathcal{F} \rightarrow M\) yields a connection \(\nabla^{N^*\mathcal{E}}\) on \(N^*\mathcal{E}\). The parallel transport along the orbits of the transverse geodesic flow \(f_t\) on \(N^*\mathcal{F}\) associated with the connection \(\nabla^{N^*\mathcal{E}}\) defines a one-parameter automorphism group \(\alpha_t\) of the vector bundle \(\pi^*\mathcal{E}\), covering the flow \(f_t\), and the induced flow \(\alpha_t^*\) on \(C^\infty(N^*\mathcal{F}, \pi^*\mathcal{E})\). In its turn, this flow induces the flow \(\text{Ad}(\alpha_t)^*\) on the space \(C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*\mathcal{E}) \otimes |T\mathcal{G}_N|^{1/2})\), which satisfies

\[
\frac{d}{dt} \text{Ad}(\alpha_t)^* \sigma = \nabla^{\mathcal{H}} \sigma, \quad \sigma \in C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*\mathcal{E}) \otimes |T\mathcal{G}_N|^{1/2}).
\]

This flow will be called the transverse parallel transport. One can show that

\[
\text{Ad}(\alpha_t)^* \circ s_N^* = s_N^* \circ \alpha_t^*, \quad \text{Ad}(\alpha_t)^* \circ r_N^* = r_N^* \circ \alpha_t^*.
\]

The following theorem proved in [13] (for scalar operators, see [12]) relates the quantum dynamics defined by a transverse Dirac operator with the corresponding classical dynamics.

**Theorem 6.** Let \(D_E\) be a transverse Dirac operator associated with a transverse Clifford bundle \((\mathcal{E}, \nabla^\mathcal{E})\). For any \(K \in \Psi^{m,-\infty}(M, \mathcal{F}, \mathcal{E})\) with the principal symbol...
\[ \sigma \in S^m(G_{\mathcal{F}}, \mathcal{L}(\pi^*\mathcal{E}) \otimes |T\mathcal{G}_N|^{1/2}), \text{ the principal symbol } \sigma_t \in S^m(G_{\mathcal{F}}, \mathcal{L}(\pi^*\mathcal{E}) \otimes |T\mathcal{G}_N|^{1/2}) \text{ of the operator } K(t) \text{ is given by } \]
\[ \sigma_t = \text{Ad}(\alpha_t)^*\sigma. \]

Remark 7. The construction of the transversal geodesic flow provides an example of what can be called noncommutative symplectic (or, maybe, better, Poisson) reduction in the setting discussed above.

As in Remark 1, let \((X, \omega)\) be a symplectic manifold, and \(Y\) a submanifold of \(X\) such that the 2-form \(\omega_Y\) induced by \(\omega\) on \(Y\) is of constant rank. Let \(\mathcal{F}_Y\) be the characteristic foliation of \(Y\) relative to \(\omega_Y\). Suppose that the foliation \(\mathcal{F}_Y\) is simple and \((B, \omega_B)\) is the associated reduced symplectic manifold. If \(Y\) is invariant under the Hamiltonian flow of a Hamiltonian \(H \in C^\infty(X)\) (this is equivalent to the fact that \((dH)|_Y\) is constant along the leaves of the characteristic foliation \(\mathcal{F}_Y\)), there exists a unique function \(\hat{H} \in C^\infty(B)\), called the reduced Hamiltonian, such that \(H|_Y = \hat{H} \circ p\). Furthermore, the map \(p\) projects the restriction of the Hamiltonian flow of \(H\) to \(Y\) to the reduced Hamiltonian flow on \(B\) defined by the reduced Hamiltonian \(\hat{H}\) (see, for instance, [15 Chapter III, Theorem 14.6]).

In the case under consideration, one can consider as above the symplectic reduction associated with the cosotropic submanifold \(Y = N^*\mathcal{F}\). The Hamiltonian of the geodesic flow on \(T^*M\) determined by a bundle-like metric \(g_M\) on \(M\) satisfies the above invariance condition, and, if the foliation \(\mathcal{F}\) is simple, one can apply the symplectic reduction procedure mentioned above, that gives us the geodesic flow on the cotangent bundle of the base as the corresponding reduced Hamiltonian flow (see also Section 2.1). For a general foliation, following the ideas of noncommutative geometry, one can consider the corresponding reduced Hamiltonian flow as the flow \(F_t^*\) on the algebra \(C^\infty_{prop}(G_{\mathcal{F}}, |T\mathcal{G}_N|^{1/2})\). Following the ideas of [11][23], one can interpret the algebra \(C^\infty_{prop}(G_{\mathcal{F}}, |T\mathcal{G}_N|^{1/2})\) as a noncommutative Poisson manifold and the flow \(F_t^*\) as a noncommutative Hamiltonian flow.

2. Examples

2.1. Riemannian submersions. Suppose that a foliation \(\mathcal{F}\) on a compact manifold \(M\) is given by the fibers of a fibration \(p : M \to B\) over a compact manifold \(B\). Then, for any \(x \in M\), \(N^*_x\mathcal{F}\) coincides with the image of the cotangent map \(dp(x)^*: T^*_p(x)B \to T^*_xM\). The inverse maps \((dp(x)^*)^{-1} : N^*_x\mathcal{F} \to T^*_p(x)B\) determine a fibration \(N^*\mathcal{F} \to T^*B\) whose fibers are the leaves of the linearized foliation \(\mathcal{F}_N\). Thus, \(N^*\mathcal{F}\) is diffeomorphic to the fiber product
\[ M \times_B T^*B = \{(x, \xi) \in M \times T^*B : p(x) = \pi(\xi)\} \]
with a diffeomorphism \(M \times_B T^*B \xrightarrow{\pi} N^*\mathcal{F}\), given by
\[ (x, \xi) \in M \times_B T^*B \mapsto dp(x)^*(\xi) \in N^*_x\mathcal{F}. \]

The holonomy groupoid \(G\) of \(\mathcal{F}\) is the fiber product
\[ M \times_B M = \{(x, y) \in M \times M : p(x) = p(y)\}, \]
where \(s(x, y) = y, r(x, y) = x\). Similarly, the holonomy groupoid \(G_{\mathcal{F}_N}\) is the fiber product \(N^*\mathcal{F} \times T^*B N^*\mathcal{F}\), which consists of all \((\nu_x, \nu_y) \in N^*_x\mathcal{F} \times N^*_y\mathcal{F}\) such that \((x, y) \in M \times_B M\) and \((dp(x)^*)^{-1}(\nu_x) = (dp(x)^*)^{-1}(\nu_y)\), with \(s_N(\nu_x, \nu_y) = \)
\[\nu_y, r_N(\nu_x, \nu_y) = \nu_x.\] On the other hand, \(N^* F \times_{T^* B} N^* F\) is also diffeomorphic to the fiber product
\[G \times_B T^* B = \{(x, y, \xi) \in M \times M \times T^* B : p(x) = p(y) = \pi_B(\xi)\}.\]
A diffeomorphism \(G \times_B T^* B \overset{\cong}{\longrightarrow} N^* F \times_{T^* B} N^* F\) can be defined as
\[(x, y, \xi) \in G \times_B T^* B \mapsto (dp(x)^*(\xi), dp(y)^*(\xi)) \in N^* F \times_{T^* B} N^* F.\]
The foliation \(G_N\) is given by the fibers of the fibration
\[(x, y, \xi) \in G_N \cong G \times_B T^* B \mapsto \xi \in T^* B.\]

As in Remark 1, consider the cotangent bundle \(T^* M\) as a symplectic manifold equipped with the canonical symplectic structure, and \(N^* F\) as its closed coisotropic submanifold. Then the linearized foliation \(F_N\) coincides with the characteristic foliation of this coisotropic submanifold. It is well-known (see, for instance, [6]) that the fiber product \(N^* F \times_{T^* B} N^* F\) is a canonical relation in \(T^* M\), which is often called the flowout of the coisotropic submanifold \(N^* F\). For a complex vector bundle \(E\) on \(M\), the algebra of Fourier integral operators in \(C^\infty(M, E)\) associated with this canonical relation coincides with the algebra \(\Psi^{\star, \infty}(M, F, E)\). It is a particular case of a general construction described in [6].

In this case, the corresponding classes of symbols \(S^m(G_{F_N}, L(\pi^* E) \otimes |TG_N|^{1/2})\) can be described as follows. For any \(\xi \in T^* B\), let \(\Psi^{\star, \infty}((N^* F)_\xi, (\pi^* E)_\xi)\) be the involutive algebra of all smoothing operators, acting on \(C^\infty((N^* F)_\xi, (\pi^* E)_\xi)\), where \((N^* F)_\xi\) is the fiber of the fibration \(N^* F \to T^* B\) at \(\xi\) and \((\pi^* E)_\xi\) is the restriction of \(\pi^* E\) to \((N^* F)_\xi\). Consider a field \(\Psi^{\star, \infty}(N^* F, \pi^* E)\) of involutive algebras on \(T^* B\) whose fiber at \(\xi \in T^* B\) is \(\Psi^{\star, \infty}((N^* F)_\xi, (\pi^* E)_\xi)\). For any section \(K\) of the field \(\Psi^{\star, \infty}(N^* F, \pi^* E)\), the Schwartz kernels of the operators \(K_\xi\) in \(C^\infty((N^* F)_\xi, (\pi^* E)_\xi)\) determine a well-defined section \(\sigma_K\) of \(L(\pi^* E) \otimes |TG_N|^{1/2}\) over \(G_{F_N} \cong N^* F \times_{T^* B} N^* F\). We say that the section \(K\) is smooth, if the corresponding section \(\sigma_K\) is smooth. This defines an algebra isomorphism of \(\Psi^{\star, \infty}(N^* F, \pi^* E)\) with \(C^\infty(G_{F_N}, L(\pi^* E) \otimes |TG_N|^{1/2})\).

A Riemannian metric \(g_M\) on \(M\) is bundle-like if and only if there exists a Riemannian metric \(g_B\) on \(B\) such that, for any \(x \in M\), the tangent map \(dp\) induces an isometry from \((H_x, g_M)\) to \((T_{p(x)}B, g_B)\), or, equivalently, \(p : (M, g_M) \to (B, g_B)\) is a Riemannian submersion. Then the transverse geodesic flow \(f_t\) of \(g_M\) projects under the map \(N^* F \to T^* B\) to the geodesic flow \(f^B_t\) of \(g_B\) that implies commutativity of the following diagram

\[
\begin{array}{ccc}
N^* F & \xrightarrow{f_t} & N^* F \\
\downarrow & & \downarrow \\
T^* B & \xrightarrow{f^B_t} & T^* B
\end{array}
\]

Commutativity of this diagram allows us to lift the flow \(f_t\) to the flow \(F_t\) on the holonomy groupoid \(G_{F_N} \cong N^* F \times_{T^* B} N^* F\) in the following way:

\[F_t(\nu_x, \nu_y) = (f_t(\nu_x), f_t(\nu_y)), \quad (\nu_x, \nu_y) \in N^* F \times_{T^* B} N^* F.\]

If \(W\) is a Clifford bundle on \(B\) equipped with a Clifford connection \(\nabla^W\) and \(D_W\) is the associated Dirac operator acting on \(C^\infty(B, W)\), then its lift \(p^* W\) to \(M\) has a natural structure of a transverse Clifford bundle, the pull-back of the connection \(\nabla^W\) is a leafwise flat, transverse Clifford connection on \(p^* W\), and the associated
transverse Dirac operator can be considered as a natural lift of $D_W$ to an operator in $C^\infty(M,p^*W)$.

Remark 8. We refer the reader to [13] for a discussion of a particular case when the fibration $p : M \to B$ is a principal $K$-bundle with a compact group $K$.

Remark 9. Since any compact reduced orbifold is diffeomorphic to the quotient of a compact manifold by an action of a compact Lie group with finite isotropy groups, Theorems 5 and 6 imply the Egorov theorem for Dirac-type operators on compact reduced Riemannian orbifolds (see [13] for more details).

2.2. Linear foliation on the two-torus. Consider the linear foliation $F_\theta$ on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, whose leaves are the images of the lines $L(u_0,v_0) = \{(u_0 + \tau, v_0 + \theta \tau) : \tau \in \mathbb{R}\}$, $(u_0, v_0) \in \mathbb{R}^2$, under the projection $\mathbb{R}^2 \to \mathbb{T}^2$.

The tangent space $\tau \mathbb{T}^2$ is described as

$$N^*F_\theta = \{(u, v, p_u, p_v) \in T^*\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2 : p_u + \theta p_v = 0\},$$

that is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$ with coordinates $(u, v, p_u)$. The leaves of the lifted foliation $F_N$ are the images of the lines $L(u_0,v_0,p_0^\theta) = \{(u_0 + \tau, v_0 + \theta \tau, p_0^\theta) : \tau \in \mathbb{R}\}$, $(u_0, v_0, p_0^\theta) \in \mathbb{R}^2 \times \mathbb{R}$, under the natural projection $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$.

If $\theta \in \mathbb{Q}$, then all the leaves of $F_\theta$ are closed and diffeomorphic to the circle $S^1$, and the foliation $F_\theta$ is determined by the fibers of a foliation $\mathbb{T}^2 \to S^1$. Thus this is a particular case of the situation discussed in Section 2.1. The reduced phase space can be described as $T^*S^1 \cong S^1 \times \mathbb{R}$ with the coordinates $y = v - \theta u \in S^1$ and $\eta = p_v \in \mathbb{R}$, and the reduced symplectic structure is given by the two-form $\omega = dy \wedge d\eta$.

Consider a Riemannian metric $g$ on $\mathbb{T}^2$:

$$g = a \, du^2 + 2b \, du \, dv + c \, dv^2, \quad a, b, c \in C^\infty(\mathbb{T}^2), \quad ac - b^2 > 0.$$ Let $\omega_{\mathbb{T}^2}$ be the Riemannian volume form on $\mathbb{T}^2$:

$$\omega_{\mathbb{T}^2} = \sqrt{\det g_{\mathbb{T}^2}} \, du \, dv. \quad \det g_{\mathbb{T}^2} = ac - b^2.$$ The tangent space $T_{(u,v)}F_\theta \subset T_{(u,v)}\mathbb{T}^2 = \mathbb{R}^2$ of $F_\theta$ is spanned by $(1, \theta) \in \mathbb{R}^2$, and its orthogonal complement $H_{(u,v)}$ is spanned by $-(b(u,v) + c(u,v)\theta), a(u,v) + b(u,v)\theta) \in \mathbb{R}^2$. The decomposition (5) holds with

$$g_F = \frac{(a + b\theta)du + (b + c\theta)dv)^2}{a + 2b\theta + c\theta^2}, \quad g_H = \frac{ac - b^2}{a + 2b\theta + c\theta^2}(dv - \theta du)^2.$$ Therefore, the metric $g$ is bundle-like iff

$$\left(\frac{\partial}{\partial u} + \theta \frac{\partial}{\partial v}\right) a_H = 0,$$

where $a_H$ is the positive smooth function on $\mathbb{T}^2$ defined by

$$a_H^2 = \frac{ac - b^2}{a + 2b\theta + c\theta^2}.$$ In particular, if $\theta$ is irrational, then $g$ is bundle-like iff $a_H \equiv \text{const.}$
There is an orthonormal base \{e_F, e_H\} in \(T\mathbb{T}^2\) given by

\[
e_F = \frac{1}{\sqrt{a + 2b\theta + c\theta^2}} (\frac{\partial}{\partial u} + \theta \frac{\partial}{\partial v}) \in F,
\]

\[
e_H = \frac{1}{\sqrt{a + 2b\theta + c\theta^2}} (-\frac{b + c\theta}{\sqrt{ac - b^2}} \frac{\partial}{\partial u} + (a + b\theta) \frac{\partial}{\partial v}) \in H.
\]

The dual orthonormal base \{E_F, E_H\} in \(T^*\mathbb{T}^2\) is given by

\[
E_F = \frac{1}{\sqrt{a + 2b\theta + c\theta^2}} ((a + b\theta)du + (b + c\theta)dv),
\]

\[
E_H = \frac{\sqrt{ac - b^2}}{\sqrt{a + 2b\theta + c\theta^2}} (-\theta du + dv).
\]

The decomposition \([\mathbf{8}]\) holds with

\[
d_F f = e_F(f) \cdot E_F, \quad d_H f = e_H(f) \cdot E_H, \quad \theta(f) = 0, \quad f \in C^\infty(\mathbb{T}^2).
\]

The adjoint \(d_H^*\) of the operator \(d_H : C^\infty(\mathbb{T}^2) \rightarrow C^\infty(\mathbb{T}^2, H^*)\) is given by

\[
d_H^*(g \cdot E_H) = e_H^*(g) = -e_H(g) + F \cdot g,
\]

where \(F \in C^\infty(\mathbb{T}^2)\) is given by

\[
F = \frac{1}{\sqrt{ac - b^2}} \left[ (\frac{\partial}{\partial u} \left( \frac{b + c\theta}{\sqrt{a + 2b\theta + c\theta^2}} \right) - \frac{\partial}{\partial v} \left( \frac{a + b\theta}{\sqrt{a + 2b\theta + c\theta^2}} \right)) \right].
\]

The function \(F\) is closely related with the mean curvature vector \(\tau\):

\[
\tau = F e_H.
\]

For any \((u, v) \in \mathbb{T}^2\), the complex Clifford algebra \(Cl(Q_{(u,v)})\) is the complex algebra generated by \(1\) and \(e_H\) with the relation \(e_H^2 = -1\). Thus, the vector bundle \(Cl(Q)\) over \(\mathbb{T}^2\) is trivial, and the sections \(1 \in C^\infty(\mathbb{T}^2)\) and \(e_H \in C^\infty(\mathbb{T}^2, Q)\) determine a natural trivialization of this bundle. We also have \(\nabla_{\tau} e_H = 0\), and

\[
\nabla_{\tau}^{Cl(Q)} = e_H.
\]

Consider the transverse Clifford bundle \(\mathcal{E}\) defined in Example\([\mathbf{4}]\). Thus, we have \(\mathcal{E} = \Lambda H^* \otimes \mathbb{C} \cong \mathbb{T}^2 \times \mathbb{C}^2\). It is equipped with the trivial connection \(\nabla^{\mathcal{E}}\), and the action of \(Cl(Q_{(u,v)})\) on \(\mathcal{E}_{(u,v)} \cong \mathbb{C}^2\) is given by

\[
c(e_H(u, v)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The transverse Dirac operator \(D_{\mathcal{E}}\) associated with \(\mathcal{E}\) is a matrix-valued first order differential operator acting on \(C^\infty(\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}^2)\):

\[
D_{\mathcal{E}} = \begin{pmatrix} 0 & -e_H + \frac{1}{2}F \\ e_H - \frac{1}{2}F & 0 \end{pmatrix}.
\]

The geodesic flow \(f_t\) on \(T^*\mathbb{T}^2\) is the Hamiltonian flow of the function

\[
h(u, v, p_u, p_v) = (G_{(u,v)}(p_u, p_v))^{1/2},
\]

where \(G\) is the induced metric on \(T^*\mathbb{T}^2\) given by

\[
G_{(u,v)}(p_u, p_v) = \frac{1}{ac - b^2(c p_u^2 - 2b p_u p_v + a p_v^2)}, \quad (p_u, p_v) \in T^*_{(u,v)} \mathbb{T}^2.
\]
The infinitesimal generator \( X \) of \( f_t \) is given by

\[
X = (G(u,v)(p_u,p_v))^{-1/2}p_u - \frac{bp_v}{ac-b^2} \frac{\partial}{\partial u} + (G(u,v)(p_u,p_v))^{-1/2} - \frac{bp_u + ap_v}{ac-b^2} \frac{\partial}{\partial v}
\]

\[
- \frac{1}{2} (G(u,v)(p_u,p_v))^{-1/2} \frac{\partial G(u,v)(p_u,p_v)}{\partial p_u} \frac{\partial}{\partial p_u}
\]

\[
- \frac{1}{2} (G(u,v)(p_u,p_v))^{-1/2} \frac{\partial G(u,v)(p_u,p_v)}{\partial p_v} \frac{\partial}{\partial p_v}
\]

The restriction of \( G \) to \( N^* F_\theta \) is given by

\[
G(u,v)(-\theta p_v, p_v) = \frac{1}{a_H^2} p_v^2, \quad p_v \in \mathbb{R}.
\]

It follows that

\[
X(p_u + \theta p_v) = \frac{1}{2} (G(u,v)(p_u,p_v))^{-1/2} \left( \frac{\partial}{\partial u} + \theta \frac{\partial}{\partial v} \right) G(u,v)(p_u,p_v),
\]

that immediately implies that, if the metric is bundle-like, then the flow \( f_t \) preserves \( N^* F_\theta \). We also see that in this case the restriction of \( X \) to \( N^* F_\theta \) is given by

\[
X = -a_H \frac{b + c \theta}{ac - b^2} \text{sign } p_v \frac{\partial}{\partial u} + a_H \frac{a + \theta b}{ac - b^2} \text{sign } p_v \frac{\partial}{\partial v} + a_H^{-2} \frac{\partial a_H}{\partial p_v} |p_v| \frac{\partial}{\partial p_v}.
\]

We have

\[
X(v - \theta u) = -\frac{1}{a_H} \text{sign } p_v,
\]

that is constant along the leaves of \( F_N \), because \( a_H \) depends only on \( v - \theta u \). It follows that the flow \( f_t \) takes each leaf of \( F_N \) to a leaf.

So if \( \theta \in \mathbb{Q} \), the reduced dynamic on the reduced phase space \( T^* S^1 \) with coordinates \( y = v - \theta u \) and \( \eta = p_v \) is determined by the reduced Hamiltonian

\[
h(y, \eta) = \frac{1}{a_H |y|} |\eta|:
\]

\[
\frac{dy}{dt} = -\frac{1}{a_H(y)} \text{sign } \eta, \quad \frac{d\eta}{dt} = a_H^{-2} \frac{\partial a_H}{\partial y} |\eta|.
\]

If \( \theta \notin \mathbb{Q} \), the foliation \( F_\theta \) is given by the orbits of a free action of \( \mathbb{R} \) on \( T^2 \), and its holonomy groupoid is described as follows: \( G = T^2 \times \mathbb{R}, G^{(0)} = T^2, s(u,v,\tau) = (u-t, v - \theta \tau), r(u,v,\tau) = (u, v), (u,v) \in T^2, \tau \in \mathbb{R}, and the product of (u_1, v_1, \tau_1) and (u_2, v_2, \tau_2) is defined if u_2 = u_1 - \tau_1, v_2 = v_1 - \theta \tau_1 and equals

\[
(u_1, v_1, \tau_1)(u_2, v_2, \tau_2) = (u_1, v_1, \tau_1 + \tau_2).
\]

Remark 10. If \( \theta \in \mathbb{Q} \), then the linear foliation on \( T^2 \) is given by the orbits of a free group action of \( S^1 \) on \( T^2 \), and its holonomy groupoid coincides with the crossed product groupoid \( G = T^2 \times S^1 \).

The holonomy groupoid \( G_{F_N} \) is described as follows: \( G_{F_N} = T^2 \times \mathbb{R}, G_{F_N}^{(0)} = T^2 \times \mathbb{R}, s_N(u,v,p_v,\tau) = (u-t, v - \theta \tau, p_v), r_N(u,v,p_v,\tau) = (u, v, p_v), (u,v) \in T^2 \times \mathbb{R}, \tau \in \mathbb{R}, and the product of (u_1, v_1, p_{v,1}, \tau_1) and (u_2, v_2, p_{v,2}, \tau_2) is defined if u_2 = u_1 - \tau_1, v_2 = v_1 - \theta \tau_1, p_{v,1} = p_{v,2}(= p_v), and equals

\[
(u_1, v_1, p_v, \tau_1)(u_2, v_2, p_v, \tau_2) = (u_1, v_1, p_v, \tau_1 + \tau_2).
\]

There is a natural trivialization of the line bundle \( T F_N \) denoted by \( d\tau \). Let \( \omega_F \in C^\infty(T^2, TF_N) \) be the leafwise Riemannian volume form:

\[
\omega_F(u,v,p_v) = \sqrt{\det g_F(u,v)} \, d\tau, \quad \det g_F = a + 2b\theta + c\theta^2.
\]
An arbitrary section \( \sigma \in C^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2}) \) can be written as

\[
(12) \quad \sigma(u, v, p_v, \tau) = k(u, v, p_v, \tau)|\omega_F(u, v, p_v)|^{1/2}[\omega_F(u - \tau, v - \theta \tau, p_v)]^{1/2},
\]

\( (u, v) \in T^2, \quad p_v \in \mathbb{R}, \quad \tau \in \mathbb{R}, \)

It is properly supported if and only if \( k \) has compact support as a function of \( \tau \) for any \( (u, v, p_v) \).

For any \( \sigma_j \in C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2}) \), of the form \( (12) \) with some \( k_j, j = 1, 2 \), their product \( \sigma = \sigma_1 \ast \sigma_2 \) is written in the form \( (12) \) with

\[
(k_1 \ast k_2)(u, v, p_v, \tau)
= \int_{-\infty}^{\infty} k_1(u, v, p_v, \tau_1)k_2(u - \tau_1, v - \theta \tau_1, p_v, \tau - \tau_1)\sqrt{|\det g_F(u - \tau_1, v - \theta \tau_1)|}d\tau_1,
\]

\( (u, v) \in T^2, \quad p_v \in \mathbb{R}, \quad \tau \in \mathbb{R}. \)

The infinitesimal generator \( \mathcal{H} \) of the induced flow \( F_t \) on \( G_{\mathcal{F}_N} \) is given by

\[
\mathcal{H} = -a_H \frac{b(u, v) + c(u, v)\theta}{a(u, v)c(u, v) - b(u, v)^2}\text{sign } p_v \frac{\partial}{\partial u}
+ a_H \frac{a(u, v)c(u, v) - b(u, v)^2}{a(u, v)c(u, v) - b(u, v)^2}\text{sign } p_v \frac{\partial}{\partial v}
+ a_H \frac{b(u - \tau, v - \theta \tau) + c(u - \tau, v - \theta \tau)\theta}{a(u - \tau, v - \theta \tau)c(u - \tau, v - \theta \tau) - b(u - \tau, v - \theta \tau)^2}
- \frac{b(u, v) + c(u, v)\theta}{a(u, v)c(u, v) - b(u, v)^2}\text{sign } p_v \frac{\partial}{\partial \tau}.
\]

We have

\[
\mathcal{L}_X |\omega_F|^{1/2}(u, v, p_v) = \frac{1}{2} F(u, v)|\omega_F|^{1/2}(u, v, p_v),
\]

where the mean curvature \( F \) is given by \( (11) \). Thus, for any \( \sigma \in C^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2}) \) of the form \( (12) \), we have

\[
(13) \quad \mathcal{L}_X \sigma(u, v, p_v, \tau) = \left( \mathcal{H} + \frac{1}{2} F(u, v) + \frac{1}{2} F(u - \tau, v - \theta \tau) \right) k(u, v, p_v, \tau) \times
\]

\[
|\omega_F(u, v, p_v)|^{1/2}[\omega_F(u - \tau, v - \theta \tau, p_v)]^{1/2}, \quad (u, v) \in T^2, \quad p_v \in \mathbb{R}, \quad \tau \in \mathbb{R},
\]

The bundle \( \pi^*\mathcal{E} \) is a trivial two-dimensional complex bundle over \( N^*\mathcal{F}_\theta \cong T^2 \times \mathbb{R} \). Therefore, an element \( \sigma \in C^\infty(\mathcal{G}_{\mathcal{F}_N}, \mathcal{L}(\pi^*\mathcal{E}) \otimes |T\mathcal{G}_N|^{1/2}) \) can be written as

\[
\sigma(u, v, p_v, \tau) = k(u, v, p_v, \tau)|\omega_F(u, v, p_v)|^{1/2}[\omega_F(u - \tau, v - \theta \tau, p_v)]^{1/2},
\]

\( (u, v, p_v, \tau) \in \mathcal{G}_{\mathcal{F}_N} \cong T^2 \times \mathbb{R} \times \mathbb{R}, \)

where \( k \) is a smooth function with values in the space \( M_2(\mathbb{C}) \) of complex \( 2 \times 2 \) matrices. Since the Clifford connection \( \nabla^\mathcal{E} \) is trivial, the action of \( \mathcal{H} \) on such a \( \sigma \) is defined by the formula \( (13) \) with

\[
(\mathcal{H}k)_{\alpha\beta} = \mathcal{H}(k_{\alpha\beta}), \quad \alpha, \beta = 1, 2.
\]

The class \( \Psi^{d, -\infty}(T^2, \mathcal{F}_\theta, T^2 \times \mathbb{C}^2) \), \( d \in \mathbb{R} \), can be described as follows. Suppose that a polyhomogeneous symbol \( k \in S^d(\mathbb{R}^4 \times \mathbb{R}, M_2(\mathbb{C})) \) satisfies the conditions:
(1) for any \((m, n) \in \mathbb{Z}^2\),

\[ k(u + m, v + n, u' + m, v' + n, \eta) = k(u, v, u', v', \eta), \quad (u, v, u', v', \eta) \in \mathbb{R}^4 \times \mathbb{R}; \]

(2) there exists \(R > 0\) such that

\[ k(u, v, u', v', \eta) = 0, \quad (u, v, u', v', \eta) \in \mathbb{R}^4 \times \mathbb{R}, \quad (u - u')^2 + (v - v')^2 > R^2. \]

Define an operator \(K : C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times \mathbb{C}^2) \to C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times \mathbb{C}^2)\) by the formula

\[ Kf(u, v) = \int e^{i(v - v' - \theta(u - u'))\eta} k(u, v, u', v', \eta) f(u', v') \sqrt{\det g_F(u', v')} \, du' \, dv' \, d\eta. \]

By \((14)\) and \((15)\), the operator \(K\) takes any \(\mathbb{Z}^2\)-invariant function from \(C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times \mathbb{C}^2)\) to a \(\mathbb{Z}^2\)-invariant function from \(C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times \mathbb{C}^2)\), and determines an operator, acting on \(C^\infty(\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{C}^2)\).

It is clear that any operator from \(\Psi^{d, -\infty}(\mathbb{T}^2, F^\sigma, \mathbb{T}^2 \times \mathbb{C}^2)\) can be written in this form. On the other hand, one can show that any operator \(K\) defined by \((14)\) belongs to \(\Psi^{d, -\infty}(\mathbb{T}^2, F^\sigma, \mathbb{T}^2 \times \mathbb{C}^2)\), and its principal symbol is given by

\[ \sigma(K)(u, v, p_\sigma, \tau) = \sum_{(m, n) \in \mathbb{Z}^2} k_d(u + m, v + n, u - \tau, v - \theta \tau, p_\sigma) \times \]

\[ \times |\omega_F(u, v, p_\sigma)|^{1/2} |\omega_F(u - \tau, v - \theta \tau, p_\sigma)|^{1/2}, \quad (u, v, p_\sigma, \tau) \in G_{F^\sigma} \cong \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}, \]

where \(k_d\) is the (degree \(d\)) homogeneous component of \(k\).

The orbits of the vector field \(e_H\) determine a foliation, \(F^\sigma\), transverse to \(F^\sigma\). It is easy to see that, for any natural \(m\), the space \(\mathcal{H}^m\) coincides with the space of all \(f \in L^2(\mathbb{T}^2)\) such that \(c_0^j f \in L^2(\mathbb{T}^2)\) for \(j = 1, \ldots, m\). Thus, for any \(s \in \mathbb{R}\), the space \(\mathcal{H}^s\) coincides with the anisotropic Sobolev space \(H^{0, s}(\mathbb{T}^2, F^\sigma)\) \((13)\).

Now Theorems \((4)\) and \((5)\) read as follows.

**Theorem 11.** Let \(D_E\) be the transverse Dirac operator defined by \((11)\).

(1) For any \(K \in \Psi^{m, -\infty}(\mathbb{T}^2, F^\sigma, \mathbb{T}^2 \times \mathbb{C}^2)\), there exists an operator \(K(t) \in \Psi^{m, -\infty}(\mathbb{T}^2, F^\sigma, \mathbb{T}^2 \times \mathbb{C}^2)\) such that \(\Phi_t(K) - K(t), t \in \mathbb{R}\), is a smooth family of operators of class \(\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)\).

(2) If \(\sigma \in S^m(G_{F^\sigma}, \mathcal{L}(\pi^* E) \otimes |T G_N|^{1/2})\) is the principal symbol of \(K\):

\[ \sigma(u, v, p_\sigma, \tau) = k(u, v, p_\sigma, \tau)|\omega_F(u, v, p_\sigma)|^{1/2}|\omega_F(u - \tau, v - \theta \tau, p_\sigma)|^{1/2}, \]

then the principal symbol \(\sigma_t \in S^m(G_{F^\sigma}, \mathcal{L}(\pi^* E) \otimes |T G_N|^{1/2})\) of \(K(t)\):

\[ \sigma_t(u, v, p_\sigma, \tau) = k(t, u, v, p_\sigma, \tau)|\omega_F(u, v, p_\sigma)|^{1/2}|\omega_F(u - \tau, v - \theta \tau, p_\sigma)|^{1/2}, \]

is the solution of the equation

\[ \frac{\partial k}{\partial t}(t, u, v, p_\sigma, \tau) = \left(\mathcal{H} + \frac{1}{2} F(u, v) + \frac{1}{2} F(u - \tau, v - \theta \tau)\right) k(t, u, v, p_\sigma, \tau), \]

satisfying the condition \(k(0, u, v, p_\sigma, \tau) = k(u, v, p_\sigma, \tau)\).
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