Realizing useful quantum operations with high fidelity is a two-task quantum control problem wherein decoherence is to be suppressed and desired unitary evolution is to be executed. The dynamical decoupling (DD) approach to decoherence suppression has been fruitful but synthesizing DD fields with certain quantum control fields may be experimentally demanding. In the context of spin squeezing, here we explore an unforeseen possibility that continuous DD fields may serve dual purposes at once. In particular, it is shown that a rather simple configuration of DD fields can suppress collective decoherence and yield a $1/N$ scaling of the squeezing performance ($N$ is the number of spins), thus making spin squeezing more robust to noise and much closer to the so-called Heisenberg limit. The theoretical predictions should be within the reach of current spin squeezing experiments.

The feasibility of “dynamical decoupling” (DD) [1] in effectively isolating a quantum system from its environment has attracted great theoretical and experimental interests. Remarkable progress towards efficient protection of quantum states has been achieved [2-5]. In contrast, high-fidelity protection of quantum operations (such as quantum gates or quantum metrology schemes) is more challenging experimentally. As decoherence must be suppressed during quantum operations, it is natural to synthesize DD fields with other fields implementing a desired quantum operation. However, this bottom-up approach may require complicated coherent control fields. For example, explicit solutions to dynamically corrected quantum gates are sophisticated [6-8], and even a simple quantum metrology protocol, when combined with DD, already becomes a rather involving practice [8].

A top-down approach to the protection of useful quantum operations should be a worthy direction, along which we aim to better exploit the system’s own Hamiltonian under DD fields. In essence we are faced with a two-task problem: decoherence is to be suppressed and desired (almost) unitary evolution is to be executed. Is it possible to directly construct DD fields serving the dual tasks at once? Motivated by recent studies of decoherence effects on spin squeezing [10-17] and by recent exciting experiments of spin squeezing [18-19], here we use the spin squeezing context to give a positive answer to our question. That is, it is feasible to protect and enhance spin squeezing at the same time by searching for a special configuration of DD fields. The dual roles of DD fields arise from two facts: (i) DD fields modulate both system-bath interaction and the system Hamiltonian itself that describes the spin-spin interaction, and (ii) the system Hamiltonian itself under the modulation of DD fields may generate more useful quantum evolution. We emphasize that the enhancement in spin squeezing we achieve is not a secondary outcome of decoherence suppression. Rather, the enhanced spin squeezing is far superior to what can be normally achieved under “decoherence-free” conditions. Indeed, we predict a $1/N$ scaling of the obtained spin squeezing performance, where $N$ is the number of spins. Our theoretical results should be testable by modifying existing experiments. In addition, because spin squeezing is closely related to multi-partite entanglement [20-21], the results are also of interest to ongoing studies of entanglement protection [22-24].

Consider then a collection of $N$ identical spins (or qubits). In terms of the standard Pauli matrices, the dynamics can be described by collective angular momentum operators $J_k = \frac{1}{2} \sum_{m=1}^{N} \sigma_k^{(m)}$, with $k = x, y, z$. In spin squeezed states, quantum fluctuations of the collective angular momentum in one direction are significantly reduced at the price of increased uncertainty in another direction [27-28], thus offering higher precision in quantum metrology [24-30]. For instance, we can improve high-precision spectroscopy and atomic clocks which are currently limited by spin noise [31].

Using the angular momentum commutation relations, the two widely used measures of spin squeezing, $\xi_s^2$ and $\xi_R^2$, are found to be bounded by $1/N$ [32]. This fundamental limit to the amount of spin squeezing achievable reflects the Heisenberg precision limit in quantum measurement [28-32]. In practice, the achievable degree of squeezing is considerably worse than the $1/N$ limit for two main reasons. First, squeezing is in general degraded by decoherence or noise [34]. The environment tends to destroy squeezing, causing the sudden death of squeezing [13]. It may also change the optimal squeezing time window in an unpredictable way, leading to a non-optimal squeezing generation [19]. Second, Hamiltonians that can be implemented so far cannot reach the $1/N$ scaling in theory. For instance, in two recent experiments [18-19] based on two-mode Bose-Einstein condensate (BEC), the so-called one-axis twisting (OAT) Hamil-
tonian $H_{\text{OAT}} = \chi J_z^2$ is realized, which can at most generate $\xi^2 / N \approx 1/N^2$, not to mention decoherence effects. It is thus clear that protecting spin squeezing against decoherence and pushing spin squeezing towards the $1/N$ scaling would be of wide interest.

We start by considering an OAT system interacting with an environment, with the total Hamiltonian modeled by $H = H_0 + H_B + H_{\text{SB}}$, where $H_0 = H_{\text{OAT}} = \chi J_z^2$, $H_B$ is the Hamiltonian of the environment, and $H_{\text{SB}}$ represents the system-environment coupling. $H_{\text{SB}}$ is assumed to be

$$H_{\text{SB}} = B_x J_x + B_y J_y + B_z J_z,$$  \hspace{1cm} (1)

where the $B_k$ are arbitrary bath operators (or randomly fluctuating noise for a classical bath). Though coupling terms that are nonlinear in $J_k$ are not considered here, the $H_{\text{SB}}$ in Eq. (1) is already quite general insofar as it covers a broad class of problems with both dephasing and relaxation.

There is a standard route to seek a control Hamiltonian $H_c(t)$ that can effectively average out $H_{\text{SB}}$ and hence suppress decoherence. In particular we consider a continuous $H_c(t)$ of period $t_c$, whose time-ordered exponential defines a unitary operator $U_c(t) = \mathcal{T} \exp[-i \int_0^t H_c(t')dt']$ ($h = 1$ throughout), with $U_c(t+tc) = U_c(t)$. The Magnus expansion [25, 32] indicates that if

$$\int_0^{t_c} U_c(t)H_{\text{SB}}U_c(t)dt = 0,$$  \hspace{1cm} (2)

then to its first order $H_{\text{SB}}$ is suppressed. Extending previous studies for single-qubit and two-qubit systems [3, 37, 38], we choose $U_c(t) = e^{-2\pi i n_y J_y/t_c} e^{-2\pi i n_x J_x/t_c}$, (3)

where $n_x$ and $n_y$ are non-zero integers. For any $n_x \neq n_y$, $U_c(t)$ in Eq. (3) satisfies the first-order DD condition of Eq. (2). Qualitatively, such $U_c(t)$ causes the collective angular momentum operators to rapidly rotate in two independent directions and as a result, $H_{\text{SB}}$ is averaged out to zero. Using $i dU_c(t)/dt = H_c(t)U_c(t)$, we obtain the following DD control Hamiltonian,

$$H_c(t) = \omega n_y J_y + \omega n_x [J_x \cos(\omega y t) - J_z \sin(\omega y t)],$$  \hspace{1cm} (4)

where $\omega \equiv 2\pi/t_c$. The total system Hamiltonian $H_s(t) = H_{\text{OAT}} + H_c(t)$ then becomes

$$H_s(t) = \chi J_z^2 + \omega n_y J_y + \omega n_x [J_x \sin(\omega y t) - J_z \cos(\omega y t)].$$

To elaborate how $H_s(t)$ can be realized, we rotate the coordinate system along the $y$-axis by $\pi/2$, transforming $H_s(t)$ to $H'_s(t) = \chi J_z^2 + \omega n_y J_y - \omega n_x [J_z \sin(\omega y t) + J_z \cos(\omega y t)]$. We now comment on each term of $H'_s(t)$.

The first $J_z^2$ term describes spin-spin interaction, as is realized in experiments [15, 19]. The last term linear in $J_z$ can be realized by an oscillating energy bias using for example a time-dependent Zeeman shift. The $J_x$ and $J_y$ terms can be generated by use of electric-dipole interaction - considering a circularly polarized transition, a constant electric field along $y$ direction and an oscillating field along $x$ direction lead to the desired $J_x$ and $J_y$ terms [39].

With a continuous control Hamiltonian $H_c(t)$ implemented, decoherence can be well suppressed for sufficiently large $\omega$. Two observations are in order. First, as shown in Eq. (4), infinite DD solutions with different $(n_x, n_y)$ combinations are found. Second, the control Hamiltonian averages out $H_{\text{SB}}$ via fast modulations of $J_k$, so the system’s self-interaction term $J_z^2$ is necessarily modulated at the same time. One opportunity is then emerging: among all the DD solutions, can we enhance squeezing while suppressing decoherence?

With the system decoupled from its environment, it can be shown that the system evolution operator is given by $U_s(t) \approx U_c(t)e^{-iHt/\hbar}$, (3), where the time-averaged Hamiltonian $\bar{H}$ is found to be

$$\bar{H} = \chi \frac{X}{t_c} \int_0^{t_c} U_c^\dagger(t)J_z^2U_c(t)dt.$$  \hspace{1cm} (5)

A straightforward though rather tedious calculation yields

$$U_c^\dagger(t)J_z^2U_c(t) = \frac{1}{2} \sin(2\omega x t) \sin^2(\omega y t)[J_x J_y + J_y J_z] + \frac{1}{2} \sin(2\omega y t) \cos(\omega x t)[J_x J_z + J_z J_x] + \frac{1}{2} \sin(2\omega y t) \sin^2(\omega y t)[J_x J_y + J_y J_x] + J_x^2 \cos^2(\omega x t) + J_y^2 \sin^2(\omega y t) + J_z^2 \sin^2(\omega x t) \sin^2(\omega y t).$$  \hspace{1cm} (6)

Using Eq. (6), one finds the time-averaged Hamiltonian $\bar{H}$ has two different forms. Specifically, if $n_x \neq 2n_y$, $\bar{H} = \frac{1}{4} J_z^2$ (up to a constant), which is just the original OAT Hamiltonian with the nonlinear coefficient scaled down by a factor of four. If $n_x = 2n_y$, which we call the “double-resonance” (DR) condition, we obtain (up to a constant)

$$\bar{H} = \bar{H}_{\text{DR}} = \chi \left( J_x^2 + J_x J_y + J_y J_x \right).$$  \hspace{1cm} (7)

Remarkably, $\bar{H}_{\text{DR}}$ is seen to be a mixture of a OAT Hamiltonian and a well-known two-axis twisting (TAT) Hamiltonian $H_{\text{TAT}} = \chi (J_x J_y + J_y J_x)$ [27, 28]. Since $H_{\text{TAT}}$ is known to produce the best scaling of squeezing, we are motivated to examine the squeezing performance of $\bar{H}_{\text{DR}}$, naturally obtained by one type of DD fields to fight against both relaxation and dephasing.
We consider an initial state $|J, -J\rangle$ describing all spins “pointing down”, that is, an eigenstate of $J_z$ with eigenvalue $-N/2$. To verify our expressions of $\bar{H}_{\text{DR}}$ and investigate its potential benefits we first switch off $H_s$ and $\xi$ to be essentially the same as using the OAT Hamiltonian (dot-dashed, red), $\bar{H}_{\text{DR}}$ (solid, dark blue), and $N_{\text{cyc}} = 20$ (dashed, blue) for $N = 10$ (i.e., $J = 5$). Note that the dynamics generated by $H_s(t)$ with $N_{\text{cyc}} = 20$ are almost indistinguishable from the dynamics generated by $\bar{H}_{\text{DR}}$. Here we use $n_x = 2$ and $n_y = 1$, and $t_{\text{min}}$ was found to be approximately 0.491.

We find the double-resonance condition $n_x = 2n_y$ is useful for spin squeezing, let us now turn to the full problem by switching on the system-environment coupling. For convenience we model $B_x$, $B_y$ and $B_z$ in Eq. (1) as three independent Gaussian colored noise processes, with the same inverse correlation time $\alpha$ and noise parameters $\sigma^2 = 20$.

![Figure 1](image1.png)

**FIG. 1.** (color online) Spin squeezing measure $\xi_S^2$ against time $t$ using the OAT Hamiltonian (dot-dashed, red), $\bar{H}_{\text{DR}}$ (solid, dark blue), and $H_s(t)$ with $N_{\text{cyc}} = 5$ (dotted, magenta) and $N_{\text{cyc}} = 20$ (dashed, blue) for $N = 10$ (i.e., $J = 5$). Note that the dynamics generated by $H_s(t)$ with $N_{\text{cyc}} = 20$ are almost indistinguishable from the dynamics generated by $\bar{H}_{\text{DR}}$. Here we use $n_x = 2$ and $n_y = 1$, and $t_{\text{min}}$ was found to be approximately 0.491.

![Figure 2](image2.png)

**FIG. 2.** (color online) Same as in Fig. 1 but this time we have $N = 100$ (i.e., $J = 50$) with $N_{\text{cyc}} = 10$ (dotted, magenta), $N_{\text{cyc}} = 30$ (dashed, blue), and $t_{\text{min}} \approx 0.0909$. It is also observed here that the dynamics generated by $H_s(t)$ with $N_{\text{cyc}} = 30$ are well captured by the dynamics generated by $\bar{H}_{\text{DR}}$ (solid, dark blue).

![Figure 3](image3.png)

**FIG. 3.** (color online) For $J = 5$, spin squeezing measure $\xi_S^2$ against time $t$ using the bare OAT Hamiltonian without noise (dot-dashed, red), $\bar{H}_{\text{DR}}$ (solid, dark blue), the OAT Hamiltonian with noise but without DD fields (dashed, magenta), and the OAT Hamiltonian in the presence noise and the DD fields with $n_x = 2$, $n_y = 1$, $t_{\text{min}} \approx 0.491$, and $N_{\text{cyc}} = 20$ (dotted blue line, which is almost on top of the solid line). An average over 2000 sample paths of the noise was taken. The noise parameters are $\alpha = 2$ and $\sigma^2 = 20$. 

![Diagram](image4.png)
variance $\sigma^2$. We numerically compute the dynamics of squeezing for $H_s(t)$ in the presence of noise and then compare it with that generated by $H_{OAT}$, with noise or without noise. As shown in Figs. 3 and 4, $H_s(t)$ with noise yields much better squeezing than $H_{OAT}$ in the absence of noise. This may be understood as an outcome of decoherence suppression. On the other hand, $H_s(t)$ with noise also generates better squeezing than $H_{OAT}$ in the absence of noise. Hence our DD fields have played one more role in addition to decoherence suppression. Note also that the time to obtain maximum squeezing is in excellent agreement with that obtained from $H_{DR}$, hence avoiding decoherence effects on the optimal squeezing time and also confirming again the usefulness of $H_{DR}$ in predicting the optimal squeezing time.

Having shown how DD fields may suppress decoherence and enhance the spin squeezing generation as a unitary process, we finally investigate how the squeezing performance of $H_{DR}$ scales with $N$. Within the validity regime of $H_{DR}$ as an effective Hamiltonian (for describing the dynamics associated with the OAT Hamiltonian in the presence of noise and continuous DD fields), the scaling of the squeezing performance of $H_{DR}$ with $N$ represents to what degree our DD fields can protect and enhance spin squeezing. Calculations for even larger values of $N$ then become necessary. We first compare the performance of $H_{DR}$ with what is known to give the best scaling behavior, namely, the TAT Hamiltonian $H_{TAT} = \chi (J_x J_y + J_y J_x)$. Significantly, although $H_{DR}$ produces slightly less squeezing than $H_{TAT}$, two close and parallel lines describing their respective performance are seen in Fig. 5 indicating that both cases give the $\xi_S^2 \sim 1/N$ scaling. By contrast, Fig. 4 also presents the $1/N^{2/3}$ scaling of the OAT Hamiltonian [also see the inset of Fig. 6 for a comparison of two different scalings]. The DD fields under the double-resonance condition hence allows squeezing to occur in the presence of noise and in the mean time brings about a squeezing enhancement factor of $N^{1/3}$, which is in principle unlimited as $N$ increases.

It is also interesting to note how this work differs from a recent proposal for realizing TAT Hamiltonian by applying a designed pulse sequence to a OAT Hamiltonian [40]. While our starting point is continuous DD fields for decoherence suppression, the short control pulses considered in Ref. 40 do not average out $H_{SB}$ in Eq. (1) to zero. Further, the effective Hamiltonian $H_{DR}$ found here under a double-resonance condition is a mixture of OAT and TAT Hamiltonians. To our knowledge, $H_{DR}$ is a newly found, physically motivated Hamiltonian that can generate the $\xi_S^2 \sim 1/N$ scaling.

To conclude, by considering a class of continuous fields to suppress both dephasing and relaxation in the dynamics of spin squeezing, we are able to identify a special type of DD solutions that can effectively yield a previously unknown spin squeezing Hamiltonian, generating the $1/N$ scaling of squeezing performance in the presence of an environment. With their dual roles in decoherence suppression and in generating more useful quantum evolution identified, the found DD fields are appealing from an experimental point of view. Our results should be able to help design new experimental studies of spin squeezing based on one-axis twisting Hamiltonians (such as those using two-mode BEC). Indeed, by exploiting system’s own spin-spin interaction Hamiltonian under the modulation of continuous DD fields, we expect to see other
interesting DD designs that can carry out desired quantum operations while protecting quantum coherence.

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[1] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998).
[2] G. S. Uhrig, Phys. Rev. Lett. 98, 100504 (2007).
[3] W. Yang et al., Frontiers of Physics 6, 2 (2011).
[4] M. J. Biercuk et al., Nature (London) 458, 996 (2009).
[5] J. Du et al., Phys. Rev. Lett. 102, 080501 (2009); K. Khodjasteh et al., Phys. Rev. Lett. 104, 090501 (2010).
[6] L. Viola et al., Phys. Rev. Lett. 83, 4888 (1999).
[7] K. Khodjasteh and L. Viola, Phys. Rev. Lett. 102, 080501 (2009).