Abstract. We show that for all $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that for all elliptic curves $E$ defined over a number field $F$ with $j(E) \in \mathbb{Q}$ we have
\[
\#E(F)[\text{tors}] \leq C(\epsilon)[F : \mathbb{Q}]^{5/2+\epsilon}.
\]
We pursue further bounds on the size of the torsion subgroup of an elliptic curve over a number field $E/F$ that are polynomial in $[F : \mathbb{Q}]$ under restrictions on $j(E)$. We give an unconditional result for $j(E)$ lying in a fixed quadratic field that is not imaginary of class number one as well as two further results, one conditional on GRH and one conditional on the strong boundedness of isogenies of prime degree for non-CM elliptic curves.

Notation

Let $\mathcal{P}$ be the set of prime numbers. For a commutative group $G$, we denote the subgroup of elements of order dividing $n$ by $G[n]$ and the torsion subgroup – i.e., the subgroup of elements of finite order – by $G[\text{tors}]$. For $\mathfrak{s} \subset \mathcal{P}$, we let $G[\mathfrak{s}^\infty]$ denote the subgroup of $G[\text{tors}]$ of elements with order divisible only by primes in $\mathfrak{s}$. If for a commutative group $G$ we have $G = G[n]$ for some $n \in \mathbb{Z}^+$ – as is the case if $G$ is finite – then the least such $n$ is the exponent $\exp G$ of $G$.

For a field $F$, let $\overline{F}$ be an algebraic closure. We denote by $\text{g}_F = \text{Aut}(\overline{F}/F)$ the absolute Galois group of $F$. For $n \in \mathbb{Z}^+$ and a field $F$ of characteristic 0, let $F(\mu_n)$ be the field obtained by adjoining to $F$ the $n$th roots of unity, and let $F^{cyc} = \bigcup_n F(\mu_n)$. Let $\chi_\ell : \text{g}_F \rightarrow \mathbb{Z}_\ell^\times$ be the $\ell$-adic cyclotomic character, and put
\[
F^{\ell\text{-cyc}} := \overline{F}^{\text{Ker} \chi_\ell} = \bigcup_{n \in \mathbb{Z}^+} F(\mu_{\ell^n}).
\]

1. Introduction

1.1. Known bounds on the torsion subgroup

For an elliptic curve $E$ defined over a number field $F$, the torsion subgroup $E(F)[\text{tors}]$ is finite. Moreover, by Merel’s strong uniform boundedness theorem [Me96], as we range over all degree $d$ number fields $F$ and all elliptic curves $E/F$, we have
\[
T(d) = \sup \#E(F)[\text{tors}] < \infty.
\]
Merel’s work gives an explicit upper bound on $T(d)$, which was improved by Oesterlé (unpublished, but see [De16]) and Parent [Pa99]. These lie more than an exponential

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1Here and throughout this paper we use the term “strong” to mean a bound that is uniform across elliptic curves defined over number fields of any fixed degree.
away from the best known lower bound, due to Breuer [Br10]:

\[ \inf_d \frac{T(d)}{d \log \log d} > 0. \]

Various authors have conjectured that Breuer’s lower bound is essentially sharp.

**Conjecture 1.1.** We have \( T(d) = O(d \log \log d) \).

The following weaker conjecture is more widely believed.

**Conjecture 1.2.** \( T(d) \) is polynomially bounded: there is \( B > 1 \) such that \( T(d) = O(d^B) \).

Even Conjecture 1.2 seems to lie out of current reach. Since the work of Merel, Osterlé and Parent, progress on understanding the asymptotic behavior of \( T(d) \) has come (only) by restricting the class of elliptic curves under consideration. For instance if we restrict to the case in which \( E \) has complex multiplication (CM) – and write \( T_{CM}(d) \) in place of \( T(d) \) when this restriction is made – breakthrough work of Silverberg [Si88, Si92] gave the asymptotically correct upper bound on the exponent, and recent work by the present authors [CP15, CP17] shows

\[ \limsup_d \frac{T_{CM}(d)}{d \log \log d} = e^{7\pi \sqrt{3}}. \]

If we instead restrict to the class of elliptic curves with integral moduli – i.e., with \( j \)-invariant lying in the ring \( \mathbb{Z} \) of algebraic integers – and write \( T_{IM}(d) \) in place of \( T(d) \), then Hindry-Silverman showed [HS99]

\[ T_{IM}(d) = O(d \log d). \]

1.2. **In pursuit of polynomial bounds**

In this paper we will pursue polynomial bounds on the size of the torsion subgroup in a different kind of restricted regime. We begin by stating the following result, which conveys the flavor with a minimum of technical hypotheses.

**Theorem 1.3.** Let \( \epsilon > 0 \). Then there is \( C = C(\epsilon) \) such that: for all degree \( d \) number fields \( F \) and all elliptic curves \( E/F \) arising via base extension from an elliptic curve \( (E_0)/\mathbb{Q} \), we have

\[ \exp E(F)[\text{tors}] \leq C d^{3/2+\epsilon}. \]

Now we make some comments:

- Of course we can assume that \( E \) does not have CM.
- For any elliptic curve \( E \) over a number field \( F \), we have

\[ \#E(F)[\text{tors}] \mid (\exp E(F)[\text{tors}])^2. \]

Thus, Theorem 1.3 implies a bound of \( O_\epsilon(d^{3+\epsilon}) \) on the size of the torsion subgroup itself. Later we will give an improvement of this bound.

- An easy quadratic twisting argument allows us to establish the bound (1.1) as we range over all elliptic curves \( E/F \) with \( j(E) \in \mathbb{Q} \). This serves to motivate the type of further result we would like to prove.

Let \( d_0 \in \mathbb{Z}^+ \). For a positive integer \( d \) that is divisible by \( d_0 \), let \( T_{d_0}(d) \) be the supremum of \( \#E(F)[\text{tors}] \) as \( E \) ranges over all elliptic curves defined over a degree \( d \) number field \( F \) such that \( [\mathbb{Q}(j(E)) : \mathbb{Q}] = d_0 \).
Conjecture 1.4. For each $d_0 \in \mathbb{Z}^+$, there are $B = B(d_0)$ and $C = C(d_0)$ such that

$$T_{d_0}(d) \leq Cd^B.$$  

Theorem 1.3 gives the case $d_0 = 1$ of Conjecture 1.4. At present we cannot prove Conjecture 1.4 unconditionally for any $d_0 \geq 2$, but we can make some progress in this direction, as shown by the following results.

Theorem 1.5. Let $F_0$ be a quadratic number field that is not imaginary quadratic of class number one. (Thus, the discriminant of $F_0$ is not one of $-3, -4, -7, -8, -11, -19, -43, -67, -163$.) Then for all $\epsilon > 0$, there is $C = C(\epsilon, F_0)$ such that if $F$ is a degree $d$ number field and $E/F$ is an elliptic curve with $j(E) \in F_0$, we have

$$\exp E(F)[\text{tors}] \leq C d^{5/2 + \epsilon} \quad \text{and} \quad \#E(F)[\text{tors}] \leq C d^{5/2 + \epsilon}.$$  

Theorem 1.6. Assume the Generalized Riemann Hypothesis (GRH).\(^2\) Let $F_0$ be a number field that does not contain the Hilbert class field of any imaginary quadratic field. Then for all $\epsilon > 0$, there is $C = C(\epsilon, F_0)$ such that if $F$ is a degree $d$ number field and $E/F$ is an elliptic curve with $j(E) \in F_0$, we have

$$\exp E(F)[\text{tors}] \leq C d^{5/2 + \epsilon} \quad \text{and} \quad \#E(F)[\text{tors}] \leq C d^{5/2 + \epsilon}.$$  

For $d_0 \in \mathbb{Z}^+$, we introduce a hypothesis SI($d_0$) defined as follows.

SI($d_0$): There is prime $\ell_0 = \ell_0(d_0)$ such that for all primes $\ell > \ell_0$, the modular curve $X_0(\ell)$ has no noncuspidal non-CM points of degree $d_0$.

Theorem 1.7. If SI($d_0$) holds, then for all $\epsilon > 0$, there is $C = C(\epsilon, d_0)$ with

$$T_{d_0}(d) \leq Cd^{5/2 + \epsilon}.$$  

Remark 1.8. This paper is cognate to another work [CMP17], written in parallel, giving “typical bounds” on $\#E(F)[\text{tors}]$ for an elliptic curve $E/F$, under the same hypotheses as Theorems 1.5, 1.6 and 1.7.

1.3. Strategy of the proofs

In [Ar08], Arai showed that for each fixed prime $\ell$ and number field $F$, as we range over all non-CM elliptic curves $E_F$ there is a uniform upper bound on the index of the image of the $\ell$-adic Galois representation. In §2 we prove the strong form of this theorem by showing that the conclusion still holds as we range over all non-CM elliptic curves defined over all number fields of any fixed degree (Theorem 2.3).

A uniform upper bound on the index of the adelic Galois representation as we range over all non-CM elliptic curves defined over number fields of fixed degree $d_0$ would be – to say the least! – desirable. It implies SI($d_0$) but is so much stronger that the $d_0 = 1$ case was raised as an open problem in [Se72], has guided most subsequent work in the field, and remains open. Our approach to the $\ell$-adic version exploits the finiteness properties that $GL_2(\mathbb{Z}_l)$ enjoys by virtue of being an $\ell$-adic analytic group – finiteness properties that $GL_2(\hat{\mathbb{Z}})$ certainly does not possess.

From Theorem 2.3 we deduce Theorem 2.8: for each finite set of primes $s$, the quantity $\exp E(F)[s^{\infty}]$ is bounded by a polynomial in $|F : \mathbb{Q}(j(E))|$. Thus in order to bound $\exp E(F)[\text{tors}]$ it suffices to bound $\exp E(F)[\ell^{\infty}]$ for all sufficiently large primes.

The next ingredient is the following striking recent result.

\(^2\)Throughout, we use GRH to mean the Riemann Hypothesis for all Dedekind zeta functions.
Theorem 1.9 (Lozano-Robledo). Let $\ell > 2$ be a prime. Let $F_0$ be a number field, $\mathfrak{l}$ a prime ideal of $\mathbb{Z}_{F_0}$ lying over $\ell$, and $e(\ell/\mathfrak{l})$ the ramification index. Let $E_{/F_0}$ be a non-CM elliptic curve, and let $a \in \mathbb{Z}^+$ be such that $E$ admits no $F_0$-rational cyclic isogeny of degree $\ell^a$. Let $n \geq a$, and let $P \in E(F_0)$ have order $\ell^n$. Then there is an integer $1 \leq c \leq 12e(\ell/\mathfrak{l})$ and a prime $\mathcal{L}$ of $F_0(P)$ lying over $\mathfrak{l}$ such that the ramification index $e(\mathcal{L}/\mathfrak{l})$ is divisible by either $\frac{\varphi(\ell^n)}{\gcd(\varphi(\ell^n), ec^{n-1})}$ or by $\ell^n - a + 1$.

**Proof.** This is a simplified form of [LR15, Thm. 2.1].

To apply Theorem 1.9 to get uniform bounds on $\exp(E(F)[\text{tors}])$, we need finiteness results for rational $\ell$-isogenies. Thus Hypothesis $\text{SI}(d_0)$ intervenes naturally.

Corollary 1.10. Let $d_0 \in \mathbb{Z}^+$, and assume hypothesis $\text{SI}(d_0)$. There is a prime $\ell_0 = \ell_0(d_0)$ such that: for all number fields $F_0/\mathbb{Q}$ of degree $d_0$ and all primes $\ell > \ell_0$, if $\mathfrak{l}$ is a prime ideal of $\mathbb{Z}_{F_0}$ lying over $\ell$ and $E_{/F_0}$ is a non-CM elliptic curve, then for all $n \in \mathbb{Z}^+$, if $P \in E(F_0)$ is a point of order $\ell^n$, then there is $1 \leq c \leq 12d_0$ and a prime $\mathcal{L}$ of $F_0(P)$ lying over $\mathfrak{l}$ such that $e(\mathcal{L}/\mathfrak{l})$ is divisible by either $\frac{\varphi(\ell^n)}{\gcd(\varphi(\ell^n), ec^{n-1})}$ or by $\ell^n$.

**Proof.** This follows from Theorem 1.9 and the bound $e(\ell/\mathfrak{l}) \leq d_0$.

To proceed without assuming $\text{SI}(d_0)$ we need to restrict to weaker statements that are known or conditionally known. We make use of the following prior results.

Theorem 1.11 (Mazur [Ma78]). The hypothesis $\text{SI}(1)$ holds with $\ell_0(1) = 37$.

Theorems 1.7 and 1.11 imply Theorem 1.3.

Theorem 1.12 (Momose [Mo95]). Let $F_0$ be a quadratic field that is not imaginary quadratic of class number 1. There is a prime number $\ell_0 = \ell_0(F_0)$ such that for all primes $\ell > \ell_0$, no elliptic curve $E_{/F_0}$ admits an $F_0$-rational isogeny of degree $\ell$.

Theorem 1.13 (Larson-Vaintrob [LV14, Cor. 6.5]). Let $F_0$ be a number field that does not contain the Hilbert class field of any imaginary quadratic field. If the Generalized Riemann Hypothesis (GRH) holds, then the set of prime numbers $\ell$ such that some elliptic curve $E_{/F_0}$ admits an $F_0$-rational $\ell$-isogeny is finite.

Corollary 1.14. Let $F_0$ be a number field that does not contain the Hilbert class field of any imaginary quadratic field. If $[F_0 : \mathbb{Q}] \geq 3$, assume GRH. Then there is a prime $\ell_0 = \ell_0(F_0)$ such that: for all primes $\ell > \ell_0$, if $\mathfrak{l}$ is a prime ideal of $\mathbb{Z}_{F_0}$ lying over $\ell$, and $E_{/F_0}$ is an elliptic curve, then for all $n \in \mathbb{Z}^+$, if $P \in E(F_0)$ is a point of order $\ell^n$, then there is an integer $1 \leq c \leq 12e(\ell/\mathfrak{l})$ and a prime $\mathcal{L}$ of $F_0(P)$ lying over $\mathfrak{l}$ such that $e(\mathcal{L}/\mathfrak{l})$ is divisible by either $\frac{\varphi(\ell^n)}{\gcd(\varphi(\ell^n), ec^{n-1})}$ or by $\ell^n$.

**Proof.** Combine Theorems 1.9, 1.12 and 1.13.

In §3.2 we use Corollaries 1.10 and 1.14 to get the polynomial bounds on $\exp(E(F)[\text{tors}])$ of Theorems 1.5, 1.6 and 1.7. Via (1.2) this immediately gives a polynomial bound on $\#E(F)[\text{tors}]$. However in §3.3 we improve this bound using an analysis of cyclotomic characters, completing the proofs of Theorems 1.3, 1.5, 1.6 and 1.7.
The main goal of this section is to prove the following: Fix a prime number \( \ell \) and a positive integer \( d \).

2. Bounded index results for the \( \ell \)-adic Galois representation

2.1. Statement of the strong Arai theorem

Let \( F \) be a number field, and let \( E/F \) be a non-CM elliptic curve. Let \( \hat{\rho}: \mathfrak{g}_F \rightarrow \text{Aut} \, TE \cong \text{GL}_2(\hat{\mathbb{Z}}) \) denote the adelic Galois representation of \( E/F \), and for a prime \( \ell \), let \( \rho_{\ell}\infty: \mathfrak{g}_F \rightarrow \text{Aut} \, T_{\ell}E \cong \text{GL}_2(\mathbb{Z}_\ell) \) denote the \( \ell \)-adic Galois representation of \( E/F \). Then \( \det \rho_{\ell}\infty = \chi_\ell \) and \( F^{\ell\text{-cyc}} = F^{\text{Ker} \chi_\ell} \), so

\[
\rho_{\ell}\infty(\mathfrak{g}_F^{\ell\text{-cyc}}) = \rho_{\ell}\infty(\mathfrak{g}_F) \cap \text{SL}_2(\mathbb{Z}_\ell). \tag{2.1}
\]

By a result of Serre [Se72], the image \( \hat{\rho}(\mathfrak{g}_F) \) is open in \( \text{GL}_2(\hat{\mathbb{Z}}) \) – equivalently, has finite index. Thus for each prime \( \ell \) the \( \ell \)-adic image \( \rho_{\ell}\infty(\mathfrak{g}_F) \) has finite index in \( \text{GL}_2(\mathbb{Z}_\ell) \) and \( \rho_{\ell}\infty \) is surjective for all but finitely many primes \( \ell \). As mentioned above, a uniform adelic open image theorem is the ultima Thule of this field, but Arai has proved a uniform \( \ell \)-adic open image theorem.

**Theorem 2.1** (Arai [Ar08]). Let \( F \) be a number field, and let \( \ell \) be a prime. There is \( I \in \mathbb{Z}^+ \) such that for every non-CM elliptic curve \( E/F \), the image \( \rho_{\ell}\infty(\mathfrak{g}_F) \) of the \( \ell \)-adic Galois representation has index at most \( I \) in \( \text{GL}_2(\mathbb{Z}_\ell) \).

**Remark 2.2.** Arai states Theorem 2.1 slightly differently. For \( n \in \mathbb{Z}^+ \), put

\[
\mathcal{U}(n) := \text{Ker} \,(\text{GL}_2(\mathbb{Z}_\ell) \rightarrow \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})).
\]

Then each \( \mathcal{U}(n) \) is an open subgroup of \( \text{GL}_2(\mathbb{Z}_\ell) \), and each open subgroup \( \Gamma \) of \( \text{GL}_2(\mathbb{Z}_\ell) \) contains \( \mathcal{U}(n) \) for all sufficiently large \( n \). The least such \( n \) is called the level of \( \Gamma \). Then Arai proves: for a number field \( F \) and a prime \( \ell \), there is \( n = n(F, \ell) \) such that for every non-CM elliptic curve \( E/F \), the level of \( \rho_{\ell}\infty(\mathfrak{g}_F) \) is at most \( n \).

The statement in terms of the level immediately implies the statement in terms of the index. The reverse implication holds because (cf. Lemma 2.5a) the intersection of all open subgroups of \( \text{GL}_2(\mathbb{Z}_\ell) \) of index at most \( I \) is an open subgroup of \( \text{GL}_2(\mathbb{Z}_\ell) \).

The main goal of this section is to prove the following strong form of Arai’s theorem.

**Theorem 2.3.** Fix a prime number \( \ell \) and a positive integer \( d \).

a) As we range over all non-CM elliptic curves \( E/F \) defined over number fields of degree \( d \), there is an absolute bound on the index of the image of the \( \ell \)-adic Galois representation \( \rho_{\ell}\infty(\mathfrak{g}_F) \) in \( \text{GL}_2(\mathbb{Z}_\ell) \).

b) Moreover, for all but finitely many \( j \)-invariants we have

\[
[\text{GL}_2(\mathbb{Z}_\ell) : \rho_{\ell}\infty(\mathfrak{g}_F)] \leq \frac{3200d^2}{7}.
\]

**Remark 2.4.**

a) Theorem 2.3a) is a quick consequence of results of Cadoret and Tamagawa [CT12, CT13]. Namely, we apply [CT13, Thm. 1.1] with \( k = \mathbb{Q} \) to the family of elliptic curves \( E \rightarrow X := \mathbb{P}^1 \setminus \{0, 1728, \infty\} \) given by

\[
E_j : y^2 + xy - x^3 + \frac{36}{j-1728}x + \frac{1}{j-1728} = 0.
\]
of [CT12, §5.1.3] – this family is geometrically Lie perfect by [CT12, Thm. 5.1]. The conclusion is that, for each \( d \in \mathbb{Z}^+ \), for each fixed prime \( \ell \) and positive integer \( d \), there is \( B(\ell, d) \in \mathbb{Z}^+ \) such that for all but finitely many closed points \( j \in X \) of degree at most \( d \), the index of the image of the \( \ell \)-adic Galois representation on \((E_j)_{/\mathbb{Q}(j)} \) in \( \text{GL}_2(\mathbb{Z}_\ell) \) is at most \( B(\ell, d) \). By Serre’s open image theorem the result extends to all non-CM \( j \)-invariants of degree at most \( d \) with some absolute bound, say \( \tilde{B}(\ell, d) \).

b) Rouse outlined a proof of Theorem 2.3a) on MathOverflow [R-MO]. His methods would yield a version of Theorem 2.3b).

c) Theorem 2.3a) is sufficient for our applications. Nevertheless we want to include a proof of Theorem 2.3b). First, it seems interesting that in a natural case\(^3\) of [CT13, Thm. 1.1] we can get a bound that is – after omitting a finite set of \( j \)-invariants that depends on \( \ell \) and \( d \) – explicit and independent of \( \ell \). Second, the proof of [CT13, Thm. 1.1] takes about 25 pages, whereas the outline of [R-MO] is 13 lines. Our argument is about 2.5 pages; readers may appreciate having a proof of this intermediate length. Finally, in [CT13, Thm. 1.2], Cadoret-Tamagawa state a result of Frey [Fr94] but omit Frey’s assumption that \( X(k) \neq \emptyset \). This is easily remedied by using a variant on Frey’s result from [Ci09], and our argument shows how to do this.

2.2. Group theoretic preliminaries

**Lemma 2.5.** Let \( \ell \) be a prime, and let \( G \) be an infinite \( \ell \)-adic analytic group.

a) \( G \) is topologically finitely generated. A subgroup of \( G \) is open iff it has finite index. For all \( I \in \mathbb{Z}^+ \), there are only finitely many index \( I \) subgroups of \( G \).

b) Every open subgroup of \( G \) has at least one and finitely many maximal proper open subgroups.

c) Let \( \mathcal{F} \) be a set of open subgroups of \( G \) such that \( \mathcal{F} \) contains all but finitely many open subgroups of \( G \). Then every element of \( \mathcal{F} \) is contained in a maximal element, and \( \mathcal{F} \) has finitely many maximal elements.

d) For \( I \in \mathbb{Z}^+ \), the family \( \mathcal{F}_I \) of open subgroups \( H \subset G \) with \( [G : H] > I \) satisfies the hypotheses of part c) and thus has at least one and finitely many maximal elements.

**Proof.** a) Lazard has shown that an \( \ell \)-adic analytic group is topologically finitely generated and that a subgroup of \( G \) is open iff it has finite index [La65]. Moreover, every topologically finitely generated profinite group has only finitely many open subgroups of any given finite index [FJ, Lemma 16.10.2].\(^4\)

b) Every nontrivial profinite group has a proper open subgroup, and any such group is contained in a maximal proper subgroup. And every open subgroup \( U \) of \( G \) is again

\(^3\)In [CT12], the authors identify Arai’s work as a motivation for their own.

\(^4\)Each finite index subgroup of a topologically finitely generated profinite group is open [NS07].
an \(\ell\)-adic analytic group, so by [S-LMW, pp. 148–149] the Frattini subgroup \(\Phi(U)\) is open and thus \(U\) has only finitely many maximal proper open subgroups.

c) We may assume \(G \notin \mathcal{F}\). Since \(G\) is infinite and profinite, \(\mathcal{F} \neq \emptyset\). As for any group, the set of finite index subgroups of \(G\), partially ordered under inclusion, satisfies the ascending chain condition, hence so does \(\mathcal{F}\) and every element of \(\mathcal{F}\) is contained in a maximal element. To show that there are only finitely many maximal elements of \(G\) it suffices to find a finite subset \(S \subset F\) so that there is \(H \in S\) such that \(K \subseteq H\), for then the maximal elements of \(\mathcal{F}\) are the maximal elements of \(S\).

Let \(S\) be the set of elements \(H \in \mathcal{F}\) such that \(H\) is a maximal proper open subgroup of an open subgroup \(P\) of \(G\) such that \(P \notin \mathcal{F}\). By assumption the set of such subgroups \(P\) is finite, so \(S\) is finite by part b). Because \(G \notin \mathcal{G}\), for \(K \in \mathcal{F}\), the set of subgroups \(Q\) with \(K \subseteq Q \subseteq G\) and \(Q \notin \mathcal{F}\) is finite and nonempty; choose a minimal element \(Q\). Indeed, let \(H\) be a degree \(d\) number field, and let \(E/F\) be an elliptic curve. Let \(\ell\) be a prime number. Let \(G = \rho_{\ell}^\infty(\mathfrak{g}_F)\) be the image of the \(\ell\)-adic Galois representation on \(E\) and let \(H = G \cap SL_2(\mathbb{Z}_\ell)\). Then we have

\[
\text{GL}_2(\mathbb{Z}_\ell) : G \leq \text{d}[\text{SL}_2(\mathbb{Z}_\ell) : H].
\]

\[\text{(2.2)}\]

Proof. Step 1: Let \(G\) be a group, let \(N\) be a normal subgroup of \(G\), and let \(q: G \to G/N\) be the quotient map. Then we have

\[
[G : H] \leq [N : H \cap N][G/N : q(H)].
\]

\[\text{(2.3)}\]

Indeed, let \(X \subseteq G\) have cardinality larger than \([N : H \cap N][G/N : q(H)]\). By the Pigeonhole Principle, there is \(Y \subseteq X\) of cardinality larger than \([N : H \cap N] [H : H]\) such that for all \(y_1, y_2 \in Y\), we have \(q(y_2y_1^{-1}) \in q(H)\), so \(y_2y_1^{-1} \in q^{-1}(q(H)) = HN\). So there are \(y_1 \neq y_2\) such that \(y_1H = y_2H\).

Step 2: The \(\ell\)-adic cyclotomic character \(\chi_\ell: \mathfrak{g}_Q \to \hat{\mathbb{Z}}_\ell^\times\) is surjective, so for any degree \(d\) number field \(F\) and elliptic curve \(E/F\), we have \([\mathbb{Z}_\ell^\times : \det \rho_{\ell}^\infty(\mathfrak{g}_F)] \mid d\). Applying (2.3) with \(G = \text{GL}_2(\mathbb{Z}_\ell), N = \text{SL}_2(\mathbb{Z}_\ell), H = G\) and using (2.1), we get (2.2). \[\Box\]

2.3. Proof of Theorem 2.3

Step 1: Put \(I := [1600d^2/7]\). Let \(F\) be a number field, let \(E/F\) be a non-CM elliptic curve, and let \(G = \rho_{\ell}^\infty(\mathfrak{g}_F)\) be the image of the \(\ell\)-adic Galois representation on \(E/F\). Some quadratic twist \(E'\) of \(E\) has \(-1 \in G\). Put \(G' := \rho_{\ell}^\infty(\mathfrak{g}_F)\). Then \([\text{GL}_2(\mathbb{Z}_\ell) : G] \leq 2[\text{GL}_2(\mathbb{Z}_\ell) : G']\), so it suffices to work with \(E'\). We may also assume that \([\text{GL}_2(\mathbb{Z}_\ell) : G'] > I\). Applying Lemma 2.5 with \(G = \text{GL}_2(\mathbb{Z}_\ell)\) we get that \(G'\) is contained in one of finitely many open subgroups \(\Gamma \subseteq \text{GL}_2(\mathbb{Z}_\ell)\) with \([\text{GL}_2(\mathbb{Z}_\ell) : \Gamma] > I\). So it suffices to bound \([\text{GL}_2(\mathbb{Z}_\ell) : G']\) while assuming that \(G' \subset \Gamma\) for a fixed \(\Gamma\).

Step 2: Let \(\Gamma \subset \text{GL}_2(\mathbb{Z}_\ell)\) be an open subgroup with

\[
[\text{GL}_2(\mathbb{Z}_\ell) : \Gamma] > I.
\]

\[\text{(2.4)}\]
Put
\[ \Sigma := \Gamma \cap \text{SL}_2(\mathbb{Z}_\ell). \]
Then \( \overline{\Sigma} := \Gamma \cap \text{SL}_2(\mathbb{Z}) \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Let \( P\overline{\Sigma} \) be the image of \( \overline{\Sigma} \) in \( \text{PSL}_2(\mathbb{Z}) \), and put \( D_{\overline{\Sigma}} := \lbrack \text{PSL}_2(\mathbb{Z}) : P\overline{\Sigma} \rbrack \). Since \( -1 \in G' \subset \Gamma \), we have \( \lbrack \text{PSL}_2(\mathbb{Z}) : P\overline{\Sigma} \rbrack = \lbrack \text{SL}_2(\mathbb{Z}) : \Sigma \rbrack \). Using (2.2) we get
\[ D_{\overline{\Sigma}} \geq \frac{\lbrack \text{GL}_2(\mathbb{Z}_\ell) : \Gamma \rbrack}{d}. \]

Step 3: Let \( Q(\Gamma) \) be the finite subextension of \( \mathbb{Q}^{\text{cyc}}/\mathbb{Q} \) corresponding to the open subgroup \( \det \Gamma \subset \mathbb{Z}_\ell^\times \). Then there is a modular curve \( X_{\Gamma} \) that is defined and geometrically integral over \( Q(\Gamma) \), and such that the base extension of \( X_{\Gamma} \) to \( \overline{\mathbb{Q}} \) is the compact Riemann surface \( \overline{\Gamma}/\mathbb{H} \). If for an elliptic curve \( E/F \) we have \( G' = \rho_{\ell^\infty}(g_F) \subset (\Gamma, -1) \), then \( Q(\Gamma) \subset F \), and there is an induced point on \( X_{\Gamma}(F) \) and thus a closed point on \( X_{\Gamma} \) of degree dividing \( d \). Let \( d_{\Gamma} \) be the gonality of the curve \( X_{\Gamma} \) – the least positive degree of a map \( X_{\Gamma} \to \mathbb{P}^1 \) defined over \( Q(\Gamma) \) – and let \( d_{\overline{\Gamma}} \) be the gonality of \( (X_{\Gamma})/\overline{\mathbb{Q}} \), so
\[ d_{\Gamma} \leq d_{\overline{\Gamma}}. \]

We claim that \( X_{\Gamma} \) has only finitely many closed points of degree dividing \( d \). Indeed, if not then by [Cl09, Thm. 5] we have
\[ d_{\Gamma} \leq 2d. \]
On the other hand, by a theorem of Abramovich [Ab96, Thm. 0.1], we have
\[ d_{\overline{\Gamma}} \geq \frac{7}{800} D_{\overline{\Sigma}}. \]
Putting these results together, we get the upper bound
\[ \lbrack \text{GL}_2(\mathbb{Z}_\ell) : \Gamma \rbrack \leq \frac{1600}{7} d^2, \]
and thus
\[ \lbrack \text{GL}_2(\mathbb{Z}_\ell) : \Gamma \rbrack \leq I, \]
contradicting (2.4). Thus the set of \( j \)-invariants of non-CM elliptic curves over degree \( d \) number fields such that the index of the image of the \( \ell \)-adic Galois representation exceeds \( \frac{3200}{7} d^2 \) is finite. (Here we have multiplied by 2 to get back from \( G' \) to \( G \).) Let the exceptional \( j \)-invariants be \( j_1, \ldots, j_M \in \mathbb{Q} \). For \( 1 \leq i \leq M \), choose an elliptic curve \( E/\mathbb{Q}(j_i) \) with \( j \)-invariant \( j_i \), and let \( I_i \) be the index of the image of the \( \ell \)-adic Galois representation (by Serre’s open image theorem, each \( I_i \) is finite). Now let \( E/F \) be a non-CM elliptic curve defined over a degree \( d \) number field \( F \) such that the index of the image of the \( \ell \)-adic Galois representation exceeds \( \frac{3200}{7} d^2 \). Then \( j(E) = j(E_i) \) for some \( i \). There is a number field \( K \supset F \) such that \( E/K \cong (E_i)/K \) and \( [K : \mathbb{Q}] \leq 2d^2 \), and thus the index of the image of the \( \ell \)-adic Galois representation of \( E/F \) is at most \( 2d^2 I_i \). So for any non-CM elliptic curve over any degree \( d \) number field for which the image of the \( \ell \)-adic Galois representation is contained in \( \Gamma \), the index of the image of the \( \ell \)-adic Galois representation is at most
\[ \max \left( \frac{3200}{7} d^2, \max_{1 \leq i \leq M} 2d^2 I_i \right). \]
2.4. A consequence

**Theorem 2.8.** Let $d_0 \in \mathbb{Z}^+$, and let $s \subset \mathcal{P}$ be finite. There is $C = C(d_0, s) \in \mathbb{Z}^+$ such that if $E_{/F}$ is a non-CM elliptic curve with $[\mathbb{Q}(j(E)) : \mathbb{Q}] = d_0$, then

$$\exp E(F)[s^\infty] \leq C[F : \mathbb{Q}(j(E))]^{\frac{1}{2}}.$$ 

**Proof.** We consider non-CM elliptic curves $E$ defined over number fields $F$ such that $F_0 := \mathbb{Q}(j(E))$ is a number field of degree $d_0$. In case $F = F_0$, as we range over all $E_{/F_0}$, by Theorem 2.3 the index of the image of the $\ell$-adic Galois representation is uniformly bounded. Thus there is $v_\ell \in \mathbb{N}$ such that for all such $E_{/F_0}$, we have

$$\text{ord}_\ell[\text{GL}_2(\mathbb{Z}_\ell) : \rho_{E, \ell, \infty}(g_{F_0})] \leq v_\ell.$$ 

With this notation, we will show that we may take

$$C = \left(2\prod_{\ell \in s} \ell^{2+v_\ell}\right)^{\frac{1}{2}}.$$ 

Now let $E_{/F}$ be as above, and suppose $E(F)$ has a point of order $N = \prod_{\ell \in s} \ell^{a_\ell}$. Let $s' = \{\ell \in s \mid a_\ell > 0\}$. We also suppose, temporarily, that $E$ arises by base extension from an elliptic curve defined over $F_0$. Let $\ell \in s'$. Since $E(F)$ has a point of order $\ell^{a_\ell}$, this forces the image of the $\ell$-adic Galois representation to lie in a subgroup conjugate to

$$\Gamma_1(\ell^{a_\ell}) := \begin{bmatrix} 1 + \ell^{a_\ell} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ \ell^{a_\ell} \mathbb{Z}_\ell & \mathbb{Z}_\ell \end{bmatrix}.$$ 

Since $[\text{GL}_2(\mathbb{Z}_\ell) : \Gamma_1(\ell^{a_\ell})] = \ell^{2a_\ell - 2}(\ell^2 - 1)$, we get

$$\ell^{2a_\ell - 2}(\ell^2 - 1) \mid [\text{GL}_2(\mathbb{Z}_\ell) : \rho_{E, \ell, \infty}(g_{F})] \mid [F : F_0][\text{GL}_2(\mathbb{Z}_\ell) : \rho_{E, \ell, \infty}(g_{F_0})]$$

and thus

$$\ell^{2a_\ell - 2 - v_\ell} \mid [F : F_0].$$

(Here and below, we write $a \mid b$ for rational numbers $a, b$ whenever $b = aq$ for some $q \in \mathbb{Z}$.) Compiling these divisibilities across all $\ell \in s'$, we get

$$N^2 \prod_{\ell \in s'} \ell^{2 + v_\ell} \mid [F : F_0]$$

and thus

$$N \leq \left(\prod_{\ell \in s'} \ell^{2 + v_\ell}\right)^{\frac{1}{2}} [F : F_0]^{\frac{1}{2}} \leq \left(\prod_{\ell \in s} \ell^{2 + v_\ell}\right)^{\frac{1}{2}} [F : F_0]^{\frac{1}{2}}.$$ 

Now suppose that $E_{/F}$ does not necessarily arise by base extension from an elliptic curve over $F_0$. Nevertheless there is an elliptic curve $(E_0)_{/F_0}$ and a quadratic extension $F'/F$ such that $E_{/F'} \cong (E_0)_{/F'}$. Since $\exp E(F)[s^\infty] \mid \exp E(F')[s^\infty]$, applying the previously addressed special case with $F'$ in place of $F$ gives

$$N \leq \left(\prod_{\ell \in s} \ell^{2 + v_\ell}\right)^{\frac{1}{2}} [F' : F_0]^{\frac{1}{2}} = \left(2\prod_{\ell \in s} \ell^{2 + v_\ell}\right)^{\frac{1}{2}} [F : F_0]^{\frac{1}{2}}.$$ 

$\square$
Remark 2.9. Theorem 2.8 is sharp up to the value of the constant. Indeed, for a non-CM elliptic curve \( E \) defined over a number field \( F_0 \) and any prime \( \ell \), since there are \((\ell^2 - 1)\ell^{2n - 2}\) points of order \( \ell^n \) on \( E(\overline{F}) \), there is a field extension \( F_n/F_0 \) of degree at most \((\ell^2 - 1)\ell^{2n - 2} < \ell^{2n}\) such that \( E(F_n) \) has a point of order \( \ell^n \).

3. The proofs of Theorems 1.5, 1.6 and 1.7

3.1. An easy lemma

Lemma 3.1. Let \( F \) be a complete discretely valued field, with residue characteristic \( p \geq 0 \). Let \( \ell \neq p \) be a prime number, and let \( E/F \) be an elliptic curve over \( F \) with semistable reduction. Then \( e(F(E[\ell])/F) \mid \ell \).

Proof. Case 1: Suppose \( E/F \) has good reduction. Then by the Néron-Ogg-Shafarevich criterion, since \( \ell \neq p \) the extension \( F(E[\ell])/F \) is unramified: \( e(F(E[\ell])/F) = 1 \).

Case 2: Suppose \( E/F \) has multiplicative reduction. Then there is an unramified quadratic extension \( F'/F \) such that \( E_{F'} \) admits an analytic uniformization, or in other words is a Tate curve in the sense of [Si94, §5.3]: \( E_{F'} \cong \mathbb{G}_m/(q^{-\ell}) \). It follows that \( F'(E[\ell]) = F'/(\mu_\ell, q^{\frac{1}{\ell}}) \). Put \( F'' = F'/(\mu_\ell) \). Since \( F'/F \) is unramified and \( \ell \neq p \), the extension \( F''(q^{\frac{1}{\ell}})/F'' \) is Galois and

\[ e(F(E[\ell])/F) = e(F''(q^{\frac{1}{\ell}})/F'') | [F''(q^{\frac{1}{\ell}}): F'']. \]

By Kummer theory, we have

\[ [F''(q^{\frac{1}{\ell}}): F''] = \begin{cases} 1 & \text{if } q \text{ is an } \ell \text{th power in } F'', \\ \ell & \text{otherwise}. \end{cases} \]

\[ \square \]

3.2. Bounding the exponent

For the sake of a uniform presentation, we begin by fixing some notation. In the case of Theorems 1.5 and 1.6, we let \( F_0 \) be as in the theorem statement, put \( d_0 = [F_0 : \mathbb{Q}] \), and define \( \ell_0 \) as in Corollary 1.14. In the case of Theorem 1.7, we let \( F_0 \) be any number field of the given degree \( d_0 \), and we define \( \ell_0 \) as in Corollary 1.10.

Let \( E \) be a non-CM elliptic curve over over a degree \( d \) number field \( F \) having \( j(E) \in F_0 \). (For CM curves, any of [Si92], [HS99], [CP15, CP17] yield stronger results.) It suffices to show that there is a constant \( C \) with

\[ \exp E(F)[\text{tors}] \leq Cd_0^{3/2 + \epsilon}, \]

where \( C \) is a function of \( \ell_0, d_0 \), and \( \epsilon \). Note that since \( \ell_0 \) is determined by \( F_0 \) in the case of Theorems 1.5 and 1.6, the constant \( C \) depends on \( F_0 \) and \( \epsilon \) in those theorems. Since \( \ell_0 \) is determined by \( d_0 \) in the case of Theorem 1.7, the constant \( C \) depends only on \( d_0 \) and \( \epsilon \) in that situation.

Step 0: We first reduce to a special case. Suppose that there is a \( C' > 0 \) with the property that for all elliptic curves \( E/F \) obtained by base extension from an elliptic curve \( (E_0)/F_0 \), we have

\[ \exp E(F)[\text{tors}] \leq C'[F : \mathbb{Q}]^{3/2 + \epsilon}. \]

Now let \( E/F \) be an elliptic curve with \( j(E) \in F_0 \). Then there is an elliptic curve \( (E_0)/F_0 \) with \( j(E) = j(E_0) \) and a quadratic extension \( F'/F \) such that \( E_{F'} \cong (E_0)/F' \).
Let \( d = [F : \mathbb{Q}] \) and \( d' = [F' : \mathbb{Q}] \). Then
\[
\exp E(F)[\text{tors}] \mid \exp E(F')[\text{tors}] \leq C' \cdot d^{3/2+\epsilon} \leq (C' \cdot 2^{3/2+\epsilon}) \cdot d^{3/2+\epsilon}.
\]
So we may choose \( C = C' \cdot 2^{3/2+\epsilon} \). If \( C' \) depends only on \( \ell_0, d_0 \), and \( \epsilon \), then so does \( C \).

Step 1: Now suppose that \( E \) is obtained by base change from an elliptic curve defined over \( F_0 \), which for notational simplicity we continue to denote by \( E \). Write
\[
E(F)[\text{tors}] = \prod_{\ell \in \mathcal{P}} E(F)[\ell^{\infty}] \]
and, for \( \ell \in \mathcal{P} \),
\[
E(F)[\ell^{\infty}] \cong \mathbb{Z}/\ell^{a_\ell} \mathbb{Z} \times \mathbb{Z}/\ell^{b_\ell} \mathbb{Z}
\]
for integers \( a_\ell, b_\ell \) satisfying \( 0 \leq a_\ell \leq b_\ell \). We will partition \( \{ \ell \in \mathcal{P} \mid b_\ell \geq 1 \} \) into two classes \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \), get bounds on \( \exp E(F)[\mathcal{S}_1^{\infty}] \) and \( \exp E(F)[\mathcal{S}_2^{\infty}] \), and multiply them to get a bound on \( \exp E(F)[\text{tors}] \).

Step 2: Put \( \mathcal{S}_1 = \{ \ell \in \mathcal{P} \mid \ell \leq \ell_0 \} \). By Theorem 2.8, we have
\[
\exp E(F)[\mathcal{S}_1^{\infty}] \leq C_1(d_0, \ell_0)[F : F_0]^{1/2} \leq C_1(d_0, \ell_0)d^{1/2}.
\]

Step 3: Let
\[
\mathcal{S}_2 = \{ \ell \in \mathcal{P} \mid \ell > \ell_0 \text{ and } b_\ell \geq 1 \}.
\]
List the elements of \( \mathcal{S}_2 \) in decreasing order:
\[
\ell_1 > \ell_2 > \cdots > \ell_k.
\]
Suppose that \( \ell \in \mathcal{S}_2 \) and \( P \in E(F) \) is a point of order \( \ell^{b_\ell} \). By Corollaries 1.10 and 1.14, for each prime \( \ell \) of \( F_0 \) lying above \( \ell \), there is a positive integer \( c \leq 12d_0 \) and a prime \( \mathcal{L} \) of \( F_0(P) \) lying above \( \ell \) with \( e(\mathcal{L}/\ell) \) divisible by either \( \varphi(\ell^{b_\ell})/\gcd(\varphi(\ell^{b_\ell}), c) \) or \( \ell^{b_\ell} \). Thus, \( e(\mathcal{L}/\ell) \) is divisible by either \( \varphi(\ell^{b_\ell})/\gcd(\varphi(\ell^{b_\ell}), (12d_0)!)) \) or by \( \ell^{b_\ell} \).

For simplicity, from now on we write the exponent on \( \ell_i \) as \( b_i \) rather than \( b_{\ell_i} \). Let \( P_1, \ldots, P_k \) be \( F \)-rational torsion points of orders \( b_1, \ldots, b_k \). Choose \( D_1, \ldots, D_k \) with
\[
D_i \in \{ \varphi(\ell_i^{b_i})/\gcd(\varphi(\ell_i^{b_i}), (12d_0)!)), \ell_i^{b_i} \}.
\]
and such that the field \( F_0(P_i) \) has a prime \( \mathcal{L}_i \) above \( \ell_i \) with \( e(\mathcal{L}_i/\ell_i) \) divisible by \( D_i \).

We introduce the sequence of fields \( K_{-1} = F_0, K_0 = F_0(E[12]), K_1 = K_0(P_1), K_2 = K_1(P_2), \ldots, K_k = K_{k-1}(P_k) \). By a result of Raynaud, \( E/K_0 \) has everywhere semistable reduction; see e.g. [SZ95, Thm. 3.5]. Since \( K_k \subset F(E[12]) \), we have
\[
(3.1) \quad \prod_{i=0}^k [K_i : K_{i-1}] = [K_k : F_0] \leq [F(E[12]) : \mathbb{Q}] = [F(E[12]) : F] \cdot d \leq 4608d.
\]
(We used here that \( \# \text{GL}_2(\mathbb{Z}/12\mathbb{Z}) = 4608 \).) Moreover,
\[
(3.2) \quad [K_1 : K_0][K_0 : K_{-1}] = [K_1 : F_0] \geq [F_0(P_1) : F_0] = \frac{1}{d_0}[F_0(P_1) : \mathbb{Q}] \geq \frac{D_i}{d_0}.
\]
Now let us look at \([K_i : K_{i-1}]\) where \(i \geq 2\). For notational simplicity, let \(\ell = \ell_i\). Since \(K_i \supset F_0(P_i)\), we know there is a prime \(\mathcal{L}\) of \(K_i\) above \(\ell\) for which \(e(\mathcal{L}/\ell)\) is divisible by \(D_i\). Define prime ideals \(\mathcal{L}_j\) of \(\mathbb{Z}_{K_j}\), for \(j = -1, 0, 1, \ldots, i\), by

\[
\mathcal{L}_j = K_j \cap \mathcal{L}.
\]

Thus, \(\mathcal{L}_i = \mathcal{L}\), and

\[
D_i \mid e(\mathcal{L}/\ell) = e(\mathcal{L}_{i-1}/\ell)e(\mathcal{L}_0/\mathcal{L}_{i-1}) \prod_{j=1}^{i} e(\mathcal{L}_j/\mathcal{L}_{j-1}).
\]

Since \(e(\mathcal{L}_{i-1}/\ell) \leq d_0\) and \(e(\mathcal{L}_0/\mathcal{L}_{i-1}) \mid [F_0(E[12]) : F_0] \mid 4608\), it follows that

\[
(3.3) \quad D_i \mid d_0! \cdot 4608 \cdot \prod_{j=1}^{i} e(\mathcal{L}_j/\mathcal{L}_{j-1}).
\]

Suppose that \(1 \leq j < i\). We have

\[
K_j = K_{j-1}(P_j) \subset K_{j-1}(E[\ell_j^b]).
\]

Hence, \(e(\mathcal{L}_j/\mathcal{L}_{j-1})\) divides the ramification index of \(\mathcal{L}_{j-1}\) in \(K_{j-1}(E[\ell_j^b])\). Note that \(\mathcal{L}_{j-1}\) lies above the rational prime \(\ell = \ell_i\), which is distinct from \(\ell_j\) (since \(j < i\)). By Lemma 3.1, the ramification index of \(\mathcal{L}_{j-1}\) in \(K_{j-1}(E[\ell_j])\) divides \(\ell_j\). Moreover,

\[
[K_j^{-1}(E[\ell_j^b]) : K_j^{-1}(E[\ell_j])]
\]

is a power of \(\ell_j\). (The image of the representation on the \(\ell_j^b\)-torsion lands in the kernel of the natural map \(\text{GL}_2(\mathbb{Z}/\ell_j^b\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/\ell_j\mathbb{Z})\), an \(\ell_j\)-group.) Thus, the ramification index of \(\mathcal{L}_{j-1}\) in \(K_{j-1}(E[\ell_j])\) is a power of \(\ell_j\). So \(e(\mathcal{L}_j/\mathcal{L}_{j-1})\) is also a power of \(\ell_j\).

The definition of \(D_i\) and the ordering of the primes \(\ell_1, \ldots, \ell_k\) implies that \(D_i\) is coprime to \(\ell_1, \ldots, \ell_{i-1}\). Since \(e(\mathcal{L}_j/\mathcal{L}_{j-1})\) is a power of \(\ell_j\) for \(j < i\), (3.3) implies that

\[
D_i \mid d_0! \cdot 4608 \cdot e(\mathcal{L}/\mathcal{L}_{i-1}), \quad \text{whence} \quad e(\mathcal{L}/\mathcal{L}_{i-1}) \geq \frac{D_i}{4608 \cdot d_0!}.
\]

Therefore,

\[
[K_i : K_{i-1}] \geq e(\mathcal{L}_i/\mathcal{L}_{i-1}) \geq \frac{D_i}{4608 \cdot d_0!} \quad \text{(for} \ 2 \leq i \leq k\).
\]

From the definition of \(D_i\), it is easy to see that every

\[
D_i \geq \frac{\ell_i^b}{2 \cdot (12d_0)!}.
\]

Combining these estimates with (3.1) and (3.2), we find that

\[
(3.4) \quad 4608d \geq \prod_{i=0}^{k} [K_i : K_{i-1}] = [K_0 : K_{-1}][K_1 : K_0] \prod_{2 \leq i \leq k} [K_i : K_{i-1}]
\]

\[
\geq \frac{D_1}{d_0} \cdot \prod_{2 \leq i \leq k} \frac{D_i}{4608 \cdot d_0!} \geq \prod_{i=1}^{k} \frac{\ell_i^b}{9216 \cdot (12d_0)!^2}.
\]

Put

\[
Z_0 = 9216 \cdot (12d_0)!^2.
\]
Let $Z > Z_0$ be a constant (depending on $d_0$ and $\epsilon$) to be specified momentarily. If $\ell_i^{b_i} > Z$, then the $i$th factor in the last displayed product on $i$ is at least $Z/Z_0$. Hence, there can be at most $\frac{\log(4608d)}{\log(Z/Z_0)}$ such values of $i$. There are at most $\pi(Z)$ values of $i$ with $\ell_i^{b_i} \leq Z$, where $\pi(\cdot)$ is the usual prime-counting function. So

$$k \leq \pi(Z) + \frac{\log(4608d)}{\log(Z/Z_0)}.$$

We may now deduce from (3.4) that

$$\prod_{i=1}^{k} \ell_i^{b_i} \leq Z_0^k \cdot 4608d \leq Z_0^{\pi(Z)} \cdot 4608^{1+\log(Z)/\log(Z/Z_0)} \cdot d^{1+\log(Z)/\log(Z/Z_0)}.$$

We fix $Z$ large enough, in terms of $d_0$ and $\epsilon$, to make $\frac{\log(Z_0)}{\log(Z/Z_0)} < \epsilon$; then

$$\exp E(F)[S_2^\infty] = \prod_{i=1}^{k} \ell_i^{b_i} \leq C_2(d_0, \epsilon) d^{1+\epsilon}.$$

Step 4: Putting the contribution from $S_1$ and $S_2$ together,

$$\exp E(F)[\text{tors}] \leq C_1(d_0, \ell_0) d^{1/2} \cdot C_2(d_0, \epsilon) d^{1+\epsilon} \leq C(d_0, \ell_0, \epsilon) d^{3/2+\epsilon},$$

as desired.

### 3.3. From the exponent to the order

Let $F$ be a set, each element of which is an elliptic curve defined over a number field $E/F$, and such that for some $B > 0$ and all $E/F \in F$ we have

$$\exp E(F)[\text{tors}] = O([F : \mathbb{Q}]^B).$$

(The implied constant is allowed to depend on $F$, but not on the choice of $E/F \in F$.)

Let $b = \exp E(F)[\text{tors}]$. Then there is a positive integer $a$ dividing $b$ such that

$$E(F)[\text{tors}] \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}.$$

Since $E$ has full $a$-torsion over $F$, the field $F$ contains $\mathbb{Q}(\mu_a)$, so that $[F : \mathbb{Q}] \geq \varphi(a)$. It follows (cf. [HW08, Thms. 327, 328]) that for all $\epsilon > 0$, we have

$$a = O_{\epsilon}([F : \mathbb{Q}]^{1+\epsilon}).$$

So

$$\# E(F)[\text{tors}] = ab = O_{\epsilon}([F : \mathbb{Q}]^{B+1+\epsilon}),$$

the implied constant depending on $\epsilon$ (and $F$) but not on the choice $E/F \in F$.

Applying this with $B = \frac{3}{2}$ gives a bound $O_{\epsilon}([F : \mathbb{Q}]^{3/2+\epsilon})$ on the size of the torsion subgroup, completing the proofs of Theorems 1.5, 1.6 and 1.7.

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