A Smoothing-Type Algorithm for Solving Monotone Weighted Complementarity Problems

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Abstract. The weighted complementarity problem is an extension of the standard finite dimensional complementarity problem. It is well known that the smoothing-type algorithm is a powerful tool of solving the standard complementarity problem. In this paper, we propose a smoothing-type algorithm for solving the weighted complementarity problem with a monotone function, which needs only to solve one linear system of equations and performs one line search at each iteration. We show that the proposed method is globally convergent under the assumption that the problem is solvable. The preliminary numerical results indicate that the proposed method is effective and robust for solving the monotone weighted complementarity problem.

Keywords: Weighted complementarity problem, smoothing-type algorithm, global convergence.

1. Introduction

Given an \( n \)-dimensional weighted vector \( w \geq 0 \) and a continuously differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the \textbf{weighted complementarity problem} is to find \((x, y) \in \mathbb{R}^{2n}\) such that

\[
(x, y) \geq 0, \quad y = f(x) \quad \text{and} \quad Xy = w, \tag{1}
\]

where \( X := \text{diag}(x) \in \mathbb{R}^{n \times n} \) is a diagonal matrix with the \( i \)-th diagonal entry being \( x_i \) for all \( i \in \{1, 2, \ldots, n\} \). When \( w = 0 \), problem (1) reduces to the standard complementarity problem, denoted by NCP, which has been studied extensively due to its wide applications [1–3]. When \( f \) is a linear function, the corresponding weighted complementarity problem is called a weighted linear complementarity problem; otherwise, it is called a weighted nonlinear complementarity problem. Moreover, when \( f \) is a monotone function, i.e., \((u - v)^\top [f(u) - f(v)] \geq 0\) holds for all \( u, v \in \mathbb{R}^n \), the corresponding problem (1) is called a monotone weighted complementarity problem.

The weighted complementarity problem was introduced by Potra [4]. Many equilibrium problems in economics, such as Fisher’s competitive equilibrium model, can be reformulated as a weighted complementarity problem, such as Potra [4] showed that the Fisher market equilibrium problem may be modeled as a weighted skew-symmetric linear complementarity problem, and particularly, it could be solved more efficiently than a complementarity problem.
model. Moreover, the linear programming and weighted centering problem, which was recently investigated by Anstreicher [5], can also be reformulated as a weighted complementarity problem. In [4], the author proposed a largest-step path following method and a predictor-corrector interior-point method for solving a class of monotone weighted complementarity problems. Recently, Potra [6] studied some theories and proposed an interior-point method for solving sufficient linear weighted complementarity problems; Zhang [7] presented a smoothing Newton method for solving weighted complementarity problems; and Chi et al. [8] gave some existence and uniqueness results of the weighted horizontal linear complementarity problem in the setting of Euclidean Jordan algebras.

The NCP can be reformulated as a system of parameterized smoothing equations in terms of some complementarity function [9, 10]. Instead of solving the original NCP, one solves the system of smoothing equations by some Newton-type method that iteratively find a solution to the system of equations while gradually reducing the smoothing parameter to zero. This is the so-called the smoothing-type algorithm. This kind of algorithm has been investigated extensively for solving various optimization problems, including linear complementarity problems [11–15], linear programs [16], nonlinear complementarity problems [17–22], variational inequalities [18,23], semidefinite complementarity problems [24,25], system of inequalities [11,26], symmetric cone complementarity problems [27,28], absolute value equations [29], and so on. Most known smoothing-type algorithms achieve their global convergence under an assumption that the solution set of the concerned problem is nonempty and bounded or some stronger conditions. In [20], the author proposed a smoothing-type algorithm for solving the monotone NCP, which is shown to be globally convergent under an assumption that the problem is solvable. Such an assumption is weaker than those required in the global convergence of smoothing-type algorithms (see, for example, [30]).

In this paper, based on the algorithmic framework of the one studied in [20] for the NCP, by using a symmetric perturbed smoothing function we propose a smoothing-type algorithm to solve the monotone weighted complementarity problem (1), which needs only to solve one linear system of equations and performs one line search at each iteration. Particularly, when \( f \) is a monotone function and the weighted complementarity problem (1) is solvable, we show that the proposed is globally convergent. The assumptions we used are weaker than most ones to ensure the global convergence of smoothing-type algorithms for the NCP. When the weighted complementarity problem (1) reduces to the standard complementarity problem, our proofs of main results given in this paper is simpler than those of the corresponding ones in [20].

The rest of this paper is organized as follows. In Section II, we give a reformulation of the weighted complementarity problem (1) and propose a smoothing-type algorithm for solving the weighted complementarity problem (1). In Section III, we show the global convergence of the algorithm. The preliminary numerical results are reported in Section IV, and conclusions are given in the last section.

Throughout this paper, the subscript \( ^\top \) denotes transpose, \( \mathbb{R}^n \) denotes the space of \( n \)-dimensional real column vectors, and \( \mathbb{R}^n_+ \) (respectively, \( \mathbb{R}^{2n}_+ \)) denotes the nonnegative (respectively, positive) orthant in \( \mathbb{R}^n \). We denote \( I = \{1,2,\ldots,n\} \) and \( K = \{0,1,2,\ldots,\} \). For any vector \( u \), we denote by \( u_i \) the \( i \)th component of \( u \). For any vectors \( u, v \in \mathbb{R}^n \), we write \( (u^\top, v^\top)^\top \) as \( (u, v) \) for simplicity. For any \( (\mu, x, y), (\mu_k, x^k, y^k) \in \mathbb{R}_+ \times \mathbb{R}^{2n} \), we always use the following notation: \( z = (\mu, x, y) \) and \( z^k = (\mu_k, x^k, y^k) \).

### 2. A smoothing-type algorithm

Given \( c \in \mathbb{R} \). We define a smoothing function \( \phi_c \) as

\[
\phi_c(\mu, a, b) = (1 + \mu)(a + b) - \sqrt{(1 - \mu)^2(a - b)^2 + 4c + 4\mu}, \quad \forall (\mu, a, b) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},
\]

which is an extension of the one introduced in [19]. We have the following results.
Lemma 1 For any \((\mu, a, b, c, d) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R},\) the following results hold.

(a) \(\phi_c(0, a, b) = 0 \iff a \geq 0, b \geq 0\) and \(ab = c.\)

(b) \(\phi_c(\mu, a, b, d) = d \iff a + \mu b - d/2 > 0, \mu a + b - d/2 > 0\) and \((a + \mu b - d/2)(\mu a + b - d/2) = \mu + c.\)

Proof. The result in (a) holds directly from the definition of \(\phi_c.\) Moreover, it is easy to see that \(\phi_c(\mu, a, b, d) = d\) if and only if \(\phi_{c+\mu}(0, a + \mu b - d/2, \mu a + b - d/2) = 0.\) Thus, by using the result in (a) we can obtain the result in (b) holds.

Given \(w \in \mathbb{R}^n_+.\) For any \((\mu, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n,\) we define

\[
H_w(z) = H_w(\mu, x, y) := \begin{pmatrix} \mu \\ y - f(x) \\ \Phi_w(z) \end{pmatrix},
\]

where \(\Phi_w(z) = \begin{pmatrix} \phi_{w_1}(\mu, x_1, y_1) \\ \phi_{w_2}(\mu, x_2, y_2) \\ \vdots \\ \phi_{w_n}(\mu, x_n, y_n) \end{pmatrix}.\) (2)

Suppose that \(f\) is a continuously differentiable function, then, for any \(z \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n,\) the function \(H_w\) is continuously differentiable at \(z.\) We use \(H_w(z)\) to denote the Jacobian matrix of \(H_w\) at \(z \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n,\) then, similar to the analysis given in [19], it is not difficult to verify that \(H_w(z)\) is nonsingular at \(z \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n.\)

From (2) and Lemma 1(a), we have that \(H_w(z) = 0\) if and only if \(\mu = 0\) and \((x, y)\) solves the problem (1). Thus, we may apply some Newton-type method to solve the system of smoothing equations \(H_w(z) = 0\) at each iteration, and make \(H_w(z)\) tend to zero so that a solution of (1) can be found.

Algorithm 1 (A smoothing-type algorithm)

Given \(w \in \mathbb{R}^n_+.\) Choose \(\delta, \sigma \in (0, 1), \mu_0 > 0, (x^0, y^0) \in \mathbb{R}^{2n}.\) Set \(z^0 := (x^0, y^0).\) Choose \(\beta > 1\) such that \(\|H_w(z^0)\| \leq \beta \mu_0.\) Set \(\epsilon^0 := (1, 0, \ldots , 0) \in \mathbb{R}^{1+2n}, k := 0.\)

Step 1 If \(\|H_w(z^k)\| = 0,\) stop.

Step 2 Find \(\Delta z^k := (\Delta \mu_k, \Delta x^k, \Delta y^k) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n\) from

\[
H_w(z^k) + H'_w(z^k)\Delta z^k = (1/\beta)\|H_w(z^k)\|\epsilon^0.
\]

Step 3 Let \(\lambda_k\) be the maximum in \(\{1, \delta, \delta^2, \ldots \} \) such that

\[
\|H_w(z^k + \lambda_k \Delta z^k)\| \leq [1 - \sigma(1 - 1/\beta)\lambda_k]\|H_w(z^k)\|.
\]

Step 4 Set \(z^{k+1} := z^k + \lambda_k \Delta z^k, k := k + 1.\) Go to Step 1.

Algorithm 1 needs only to solve one linear system of equations and performs one line search at each iteration, which is simpler than many smoothing-type algorithms for solving problem (1) with \(w = 0.\) In fact, similar algorithmic framework was proposed in [20] for solving problem (1) with \(w = 0.\) Thus, similar to the analysis given in [20], we may obtain several basic properties about Algorithm 1, which are given as follows.

Lemma 2 Suppose that \(f\) is a continuously differentiable monotone function, and the sequence \(\{z^k = (\mu_k, x^k, s^k)\}\) is generated by Algorithm 1. Then,

(a) the sequence \(\{\|H_w(z^k)\|\}\) is nonnegative and monotonically decreasing;

(b) \(\{z^k\} \subset \mathcal{N}(\beta) := \{z \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n : \|H_w(z)\| \leq \beta \mu\};\)

(c) the sequence \(\{\mu_k\}\) is nonnegative and monotonically decreasing;

(d) Algorithm 1 is well-defined;

(e) \(\lim_{k \to \infty} H_w(z^k) = 0\) and \(\lim_{k \to \infty} \mu^k = 0.\)

Proof. The proofs of results (a)-(c) are similar to those in [20, Lemma 3.1]; the proof of result (d) is similar to the one in [20, Remark 2.1(iii)(iv)]; and the proof of result (e) is similar to those in [20, Lemmas 3.2 and 3.3]. We omit the proofs here.
3. Global Convergence

In this section, we show the global convergence of Algorithm 1.

**Theorem 1** Suppose $f$ is a continuously differentiable monotone function and the solution set of the weighted complementarity problem (1) is nonempty. Let the sequence $\{z^k = (\mu_k, x^k, y^k)\}$ be generated by Algorithm 1. Then, the sequence $\{z^k\}$ is bounded, and every accumulation point of $\{(x^k, y^k)\}$ is a solution of problem (1).

**Proof.** If the sequence $\{z^k\}$ is bounded, then by Lemma 2(e) and the continuity of $H_w$, it follows that every accumulation point of $\{(x^k, y^k)\}$ is a solution of problem (1). Thus, we only need to show the boundedness of the sequence $\{z^k\}$.

First, we show that the sequence $\{\mu_k\}$ is bounded. This can be easily obtained from Lemma 2(c).

Second, we show that the sequence $\{x^k\}$ is bounded. Suppose $\{x^k\}$ is unbounded, we will derive a contradiction. Since the solution set of the weighted complementarity problem (1) is nonempty, there exists a vector $(x^*, y^*) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that

$$x_i^* \geq 0, \ y_i^* = f_i(x^*) \geq 0, \ x_i^*y_i^* = w_i, \ \forall i \in I.$$  \hspace{1cm} (3)

For any $k \in \mathcal{K}$, we define

$$u^k := [y^k - f(x^k)]/\mu_k; \ v^k := [\Phi_w(z^k) + \mu_k x^k]/\mu_k.$$  \hspace{1cm} (4)

Then, by Lemma 2(b) it follows that $\|H_w(z^k)\| \leq \beta \mu_k$ for all $k \in \mathcal{K}$.

By using (4) and (2), we have that for any $k \in \mathcal{K}$,

$$(x^k - x^*)^\top [f(x^k) - f(x^*)] \geq 0; \ (x^k - v^k/2)^\top [f(x^k) - f(v^k/2)] \geq 0.$$  \hspace{1cm} (5)

Furthermore, we define two sequences $\{\bar{x}^k\}$ and $\{\bar{y}^k\}$ by

$$\bar{x}^k := x^k + \mu_k y^k - \mu_k v^k/2 \text{ and } \bar{y}^k := \mu_k x^k + y^k - \mu_k v^k/2, \ \forall k \in \mathcal{K}.$$  \hspace{1cm} (6)

Then, it follows from Lemma 1(b) and (4) that

$$x_i^k \geq 0, \ y_i^k \geq 0, \ \bar{x}_i^k \bar{y}_i^k = \mu_k + w_i, \ \forall i \in I, \forall k \in \mathcal{K},$$  \hspace{1cm} (7)

and hence, $(\bar{x}^k)^\top \bar{y}^k = n\mu_k + \sum_{i=1}^{n} w_i$. Thus, by using (3), (6) and (7), we have

$$n\mu_k = (\bar{x}^k)^\top \bar{y}^k - \sum_{i=1}^{n} w_i = (\bar{x}^k)^\top \bar{y}^k - (x^*)^\top y^* \geq (x^k - x^*)^\top (\bar{y}^k - y^*) \geq (x^k - x^*)^\top (y^k - y^*) + (x^k - x^*)^\top (\mu_k x^k - \mu_k v^k/2) + (\mu_k y^k - \mu_k v^k/2)^\top (y^k - y^*) + (\mu_k y^k - \mu_k v^k/2)^\top (\mu_k x^k - \mu_k v^k/2).$$  \hspace{1cm} (8)

By (4) and (5), we have that for any $k \in \mathcal{K}$,

$$(x^k - x^*)^\top (y^k - y^*) = (x^k - x^*)^\top [f(x^k) - f(x^*)] + (x^k - x^*)^\top (\mu_k y^k) \geq \mu_k (x^k - x^*)^\top u^k;$$  \hspace{1cm} (9)

and by (4) and (5), we have that for any $k \in \mathcal{K}$,

$$(\mu_k y^k - \mu_k v^k/2)^\top (\mu_k x^k - \mu_k v^k/2) \geq \mu_k^2 (x^k - v^k/2)^\top [f(v^k/2) - v^k/2 + \mu_k u^k].$$  \hspace{1cm} (10)
Moreover, it follows that for any $k \in \mathcal{K}$,
\[
\left(\mu_k y^k - \mu_k v^k / 2\right)^\top (y^k - y^*) = \mu_k \alpha(x^k),
\]
where $\alpha(x^k) := \|y^k - y^*\|^2 + (y^* - \mu_k v^k / 2)^\top (y^k - y^*)$. Thus, by combining (9) with (10) and (11), it follows from (8) that for any $k \in \mathcal{K}$,
\[
n \geq \alpha(x^k) + (x^k - x^*)^\top u^k + (x^k - x^*)^\top \left( x^k - v^k / 2 \right)
\]
\[
+ \mu_k \left( x^k - v^k / 2 \right)^\top \left[ f(v^k / 2) - v^k / 2 + \mu_k u^k \right].
\]

Since the sequence $\{x^k\}$ is unbounded, it follows that the right-hand side of (12) tends to $+\infty$ as $k \to +\infty$; while the left-hand side of (12) is a constant. A contradiction. Therefore, the sequence $\{x^k\}$ is bounded.

Third, we show that the sequence $\{y^k\}$ is bounded. By the boundedness of the sequence $\{x^k\}$ and the continuity of the function $f$, it follows that $\{f(x^k)\}$ is bounded. Furthermore, by (4) we have that $y^k = \mu_k u^k + f(x^k)$ for all $k \in \mathcal{K}$. This as well as the boundedness of sequences $\{\mu_k\}, \{u^k\}$ and $\{f(x^k)\}$ imply that $\{y^k\}$ is bounded.

Therefore, we obtain that the sequence $\{z^k\}$ is bounded.

The basic idea of the proof given in Theorem 1 is similar to those in [20, Lemma 3.4 and Theorem 3.1], however, the proof of Theorem 1 is more simpler than those of Lemma 3.4 and Theorem 3.1 in [20].

4. Numerical Experiments

In this section, we implement Algorithm 1 in Matlab for solving the following problem:

\textbf{Problem.} Consider the weighted complementarity problem:

\[
(x, y) \geq 0, \quad y = Mx + q \quad \text{and} \quad Xy = w,
\]

which is randomly generated in the following way: Given positive integers $m$ and $n$, take $M := A^\top A$ with $A := 0.1 \ast \text{rand}(m, n)$, $q := \text{rand}(n, 1)$, and $x^* := \text{rand}(n, 1)$, and set $y^* := M \ast x^* + q$ and $w := X^* y^*$.

Obviously, this problem is a monotone weighted linear complementarity problem. Throughout our experiments, the parameters used in Algorithm 1 are chosen as $\delta := 0.6$, $\sigma := 0.00001$, and $\mu_0 := 0.001$. Let $m$ and $n$ be chosen according to ones listed in Table 1. Take the starting point $x^0 = -0.5 \ast \text{rand}(n, 1)$. Set $y^0 := M \ast x^0 + q$ and $z^0 := (\mu_0, x^0, y^0)$. Take $\beta := \|H_w(z^0)\| / \mu_0$. We used $\|H_w(z^k)\| \leq 10^{-6}$ as the stopping rule.

For every case about the choices of $m$ and $n$, we test the problem ten times, and the numerical results are listed in Table 1, where NI denotes the average number of iterations; NF denotes the average number of function evaluations for the function $H(z^k)$; Valv, Vals and Vall denote, respectively, the average value, the smallest value and the largest value of $\|H(z^k)\|$ when Algorithm 1 terminates; and Cpuv, Cpus and Cpul denote, respectively, the average CPU time, the smallest CPU time and the largest CPU time in seconds.

From Table 1, it can be seen that Algorithm 1 is effective for solving the weighted linear complementarity problems in the sense that every problem can be successfully solved in a small number of iterations and short CPU time; and the algorithm is robust in the sense that different problems with the same size can be successfully solved with the same number of iterations, and small differences between the smallest value and the largest value of $\|H(z^k)\|$ and and between the smallest CPU time and the largest CPU time. We have also tested some other problems, the performances are similar.
Table 1. The test results

| m   | n   | NI  | NF  | Valv | Vals | Vall | Cpuv | Cpus | Cpul |
|-----|-----|-----|-----|------|------|------|------|------|------|
| 50  | 50  | 5   | 6   | 1.87e-10 | 1.22e-10 | 3.30e-10 | 0.109 | 0.063 | 0.188 |
| 50  | 100 | 5   | 6   | 3.67e-07  | 2.89e-07  | 4.27e-07  | 0.136 | 0.094 | 0.188 |
| 100 | 50  | 5   | 6   | 2.03e-07  | 1.73e-07  | 2.48e-07  | 0.114 | 0.063 | 0.172 |
| 100 | 100 | 6   | 7   | 5.98e-09  | 4.67e-09  | 7.27e-09  | 0.152 | 0.125 | 0.250 |
| 300 | 300 | 8   | 9   | 8.92e-09  | 8.13e-09  | 9.90e-09  | 0.480 | 0.391 | 0.547 |
| 500 | 500 | 9   | 10  | 6.13e-09  | 5.74e-09  | 6.61e-09  | 1.147 | 1.060 | 1.300 |
| 500 | 1000| 10  | 11  | 5.82e-10  | 5.41e-10  | 6.11e-10  | 5.575 | 5.420 | 6.140 |
| 1000| 1000| 10  | 11  | 3.32e-07  | 3.24e-07  | 3.42e-07  | 5.505 | 5.270 | 5.940 |
| 1000| 500 | 10  | 11  | 2.20e-10  | 2.04e-10  | 2.32e-10  | 1.270 | 1.110 | 1.720 |

5. Conclusions
In this paper, we proposed a smoothing-type algorithm for solving weighted complementarity problems, and showed that the proposed algorithm is globally convergent under a weak assumption. The preliminary numerical results indicate that the algorithm is effective and robust for solving the monotone weighted complementarity problem. As future research issues, it is worth investigating the theoretical properties of weighted complementarity problems and some numerical methods for solving large-scale weighted complementarity problems.

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