A stochastic corporate claim value model with variable delay

To cite this article: Mahmoud A Eissa and Boping Tian 2018 J. Phys.: Conf. Ser. 1053 012018

View the article online for updates and enhancements.

You may also like

- Proof-of-principle experimental realization of a qubit-like qutrit-based quantum key distribution scheme
  Shuang Wang, Zhen-Qiang Yin, H F Chau et al.

- An efficient scheme for sampling fast dynamics at a low average data acquisition rate
  A Philippe, S Aime, V Roger et al.

- Round-robin differential quadrature phase-shift quantum key distribution
  Chun Zhou, Ying-Ying Zhang et al.
A stochastic corporate claim value model with variable delay

Mahmoud A Eissa\textsuperscript{1,2,3} and Boping Tian\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, Harbin Institute of Technology, Harbin, China
\textsuperscript{2}Department of Mathematics, Faculty of Science, Menoufia University, Menoufia, Egypt
\textsuperscript{3}Email: mahmoud.eisa@science.menofia.edu.eg

Abstract. Stochastic functional differential equation (SFDE) with constant delay has been used for modelling the price of a firm and European option. However, we can find some weak points such as bounded memory and availability of past event, when using delayed model with constant delay in some applications. Stochastic pantograph differential equation (SPDE) is very special SFDE with variable delay and unbounded memory. In this work, in order to overcome the weaknesses of delayed model with constant delay, we develop a nonlinear delayed model with variable delay for pricing corporate when claim value depends on firm value and time follows a nonlinear SPDE. We proved that, the proposed model is feasible. In addition, a random partial differential equation (RPDE) is derived which satisfied with the corporate claim of value depends on firm value and time. The solution of RPDE is evaluate the debt and equity of a firm. The proposed model with variable delay is more sufficiently flexible to fit real market data, particularly if practitioners are considering the future volatility depends on the past volatility with take into account the variable delay, which can reflect a historical image of volatility more randomly with unbounded memory.

1. Introduction

Recently, there is increasing interest in the equations with some kind of past-dependence; stochastic functional differential equation (SFDE). Since, they can give more realistic simulation for many dynamical systems than ordinary stochastic differential equation (OSDE).

These systems can modelled by the following form of SFDE
\begin{equation}
    dX(t) = F(X(t), X(t-L))dt + G(X(t), X(t-L))dW(t), \quad t \in [0, T]
\end{equation}

with initial data \( X(t) = \varphi(t), \ t \in [-L, 0] \), where the delay \( L > 0 \) is a positive constant with given drift function \( F \), diffusion function \( G \), and \( W(t) \) is a one-dimensional standard wiener process. Equation (1) also called stochastic delay differential equation (SDDE) with constant delay, which have expansively studied from theories and applications (See [1-8]).

Another kind of SFDE with proportional delay is usually referred to as stochastic pantograph differential equation (SPDE) with the form
\begin{equation}
    dX(t) = F(X(t), X(qt))dt + G(X(t), X(qt))dW(t), \quad t \in [0, T]
\end{equation}

where \( q \in (0,1) \) with initial value \( X(0) = X_0 \) is a real-valued random variable. SPDE (2) is special type of past-dependence equation with many special properties such as unbounded memory and variable delay time \( t - qt \), which namely pantograph delay and can write \( qt \) for simplicity, so that it is more difficult to deal with them.

The systems with memory have growing interested in simulating phenomena in different fields of science, which has had a significant impact on the investigation of differential equation with delay.
incorporating memory or after-effect. These systems which take into account the influence of past event provide more realistic models to simulate phenomena that consider the time-lag or after effect. Especially, in studying of financial variables and predictions about their evolution. In general, a memory effect (influence past event) can be given in terms of delay function \( h(t) \) such that the value at current time \( t \) depends on the knowledge of their past (delay) time \( t - h(t) \). For simplicity, a routine way is to take a delay function \( h(t) = L, L > 0 \), which leads to a constant delay time \( t - L \) which is used in SDDE (1). In some applications there exist weaknesses for this choice (See [9]) such as; (a) the memory (past event) can’t be considered in \( t < L \). (b) the constant delay \( L \) is only suitable for short period memory (bounded memory). In order to overcome these weaknesses, especially for long time memory (unbounded memory), the memory effect should be a real time variable function. The simplest choice is \( h(t) = \delta t \) with \( 0 < \delta < 1 \), hence the delay time will be \( t - h(t) = (1 - \delta)t = qt \), where \( q = 1 - \delta \) and \( q \in (0,1) \). This treatment of variable delay leads to SPDE (2) which arises in different fields of science, the theoretical, numerical and applied studies were discussed in [10-14]. So, many results of SDDE (1) have been generalized for SPDE (2) to overcome the weaknesses of using constant delay in some applications.

Due to derivatives of credit market growth, the valuation of corporate claims has been considered an important issue in the financial mathematics field. Recently, this growth increases the need of developing and improving prediction models to be more sufficiently flexible to fit real market data. We can consider the study of corporate claims begun by Merton [15], who is the pioneer in deriving the corporate claims model which is close to Black-Scholes model for stock price [16]. However, the constant volatility, which was considered in the original Black-Scholes and Merton models, was questioned since the empirical evidence explained that volatility actually depends on time in the way that is not predictable (See [17, 18] and the references therein).

In order to reflect the real world perception of the market volatility, Hosbon et al. [19] introduced a new class of delayed model which considered stochastic volatility with constant delay that follows SDDE (1). [1, 20] discussed this model, which considered the influence of past dependence on the current and future state of the system and shown that the past event is an important feature of the option price. Kemajou et al. [3] discussed a corporate claim value as nonlinear SDDE with constant delay. In addition, a random partial differential equation (RPDE) which satisfied the corporate claim value was derived. Tambue et al. [6] provided a robust numerical methods to solve the delayed nonlinear model with constant delay for the corporate value, along with the corresponding RPDE modelling the debt and equity values of the corporate. Where, there exist weaknesses of the nonlinear delayed model with constant delay such as the bounded memory, availability of a past event (See [9]), furthermore, it is not fit the system when the practitioners need to consider the historical volatility more randomly. So, Arriojas et al. [1] have been introduced the nonlinear delayed model with variable delay for the stock price. In addition, the authors also developed a Black-Scholes formula for the option price.

In this work, the corporate claim value model is constructed follows a nonlinear SFDE with variable delay which has not yet been introduced. In order to reflect a real data market perceptions, we consider any claim value, which the value is a function of firm value and time, follows nonlinear SPDE (2). The variable delay time \( (qt) \) in equation (2) gives more advantages to the model such as unbounded memory (i.e. the past event can be considered for a long period time), also past event can be available for all intervals time, furthermore it can reflect a historical image of volatility more randomly. We prove the proposed model is feasible. In addition, we derive a RPDE such that its solution provide valuation form for equity and debt value of a firm in the last delay period interval.

The paper is organized as follows. In Section 2, notations and preliminaries are given. A nonlinear stochastic delay model with variable delay is constructed for corporate claim, and a RPDE is developed for claim, in Section 3. In Section 4, Evaluate of debt and equity of a firm is discussed. The conclusion and future work are given in Section 5.
2. Definitions and preliminary results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions. $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^+$ defines the set of all positive real numbers. Let $W(t), t \geq 0$ be a one-dimensional Brownian motion defined on the probability space adapted to the filtration $\mathcal{F}_t$ and independent of $\mathcal{F}_0$. Let $T > 0, \mathcal{L}^1$ denote the family of all $\mathbb{R}$-valued measurable $\{\mathcal{F}_t\}$-adapted processes, and $\mathcal{L}^2$ denote the family of all vector-valued measurable $\{\mathcal{F}_t\}$-adapted processes. Let the drift function $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given continuous functional, diffusion function $G: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $F$ and $G$ be both Borel measurable. Let $V_0$ be a $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variable.

Consider a firm whose price value $V(t)$ at time $t \in [0, T]$ is given by a nonlinear SPDE (2) as

$$dV(t) = F(V(t), V(qt)) dt + G(V(t), V(qt)) dW(t), \quad t \in [0, T]$$

with initial value $V(0) = V_0$ and $q \in (0, 1)$. By the definition of stochastic differential, equation (3) is equivalent to the following integral form

$$V(t) = V_0 + \int_0^t F(V(s), V(qs)) ds + \int_0^t G(V(s), V(qs)) dW(s), \quad t \in [0, T].$$

Definition 1. A $\mathbb{R}$-valid stochastic process $V = \{V(t)\}_{0 \leq t \leq T}$ is called solution of equation (3) if it has the following properties: $V$ is continuous and $\{\mathcal{F}_t\}$-adapted, $\{F(V(t), V(qt))\}_{0 \leq t \leq T} \in \mathcal{L}^1$ and $\{G(V(t), V(qt))\}_{0 \leq t \leq T} \in \mathcal{L}^2$, equation (4) holds for every $t \in [0, T]$ with probability 1.

A solution $V = \{V(t)\}_{0 \leq t \leq T}$ is said to be unique if and only if any other solution $\tilde{V} = \{\tilde{V}(t)\}_{0 \leq t \leq T}$ is distinguishable from $V$, that is $P(V(t) = \tilde{V}(t), \forall 0 \leq t \leq T) = 1$.

Remark 1. (Existence and uniqueness) We note from ([11], Theorems 2.2 and 2.6, respectively) that, if the functions $F$ and $G$ in SPDE (3) satisfy Lipschitz condition and linear growth condition, then there exist an unique solution $V(t)$ to equation (3). Furthermore, if the Lipschitz condition is replaced by local Lipschitz condition, then the existence and uniqueness results still hold.

3. Corporate claims model with variable delay

In order to fit a real data market and improve the prediction of a firm value, we construct an alternative stochastic delay model with variable delay for the price of a firm. We prove the proposed model is feasible. In addition, we derive a RPDE for claim. Through this paper, we consider the price value $V$ follows SPDE (3) and satisfying the following equation

$$dV(t) = (aV(t)V(qt) - C) dt + g(V(qt)) V(t) W(t),$$

with initial value $V(0) = V_0$, and $g: \mathbb{R} \to \mathbb{R}$. $\alpha$ is the riskless interest rate of return on the firm per unit time, $C$ is the total amount payout by the firm per unit time to either the shareholders or claims-holder (e.g., dividends or interest payments) if positive, and it is the net amount received by the firm from new financing if negative.

In the following result, we will ensure that the price valuation model (5) is feasible in the sense that they admit pathwise unique solution that $V(t) > 0$ almost surely for $t \in [0, T]$, with initial conditions $V(0) > 0$ and $g(V(qt)) = 0$ for all $V(qt) \in \mathbb{R}$.

Theorem 1. The firm price model (5) has a pathwise unique solution. Furthermore, if $C = 0$, the solution $V(t) > 0$ for a given initial value $V(0) > 0$ for all $t \geq 0$ and the solution is represented by the formula

$$V(t) = V(0) \exp \left\{ \alpha \int_0^t V(qs) \, ds - \frac{1}{2} \int_0^t g(V(qs))^2 \, ds + \int_0^t g(V(qs)) dW(s) \right\},$$

Proof.

From equation (5), we have

$$dV(t) = V(t) \left[ aV(qt) \, dt + g(V(qt)) \, dW(t) \right] - C \, dt,$$

$$V(0) = V_0, \quad t \in [0, T].$$

Define the semi-martingale

$$N(t) = \alpha \int_0^t V(qs) \, ds + \int_0^t g(V(qs)) \, dW(s), \quad t \in [0, T]$$

and its quadratic variation denote by

$$\langle N \rangle(t) = \alpha \int_0^t V(qs)^2 \, ds + \int_0^t g(V(qs))^2 \, ds, \quad t \in [0, T].$$
\[ [N,N](t) = \int_0^t g(V(qs))^2 \, ds, \quad t \in [0,T]. \]

Then (7) becomes
\[
dV(t) = V(t)dN(t) - Cdt, \quad V(0) = V_0, \quad t > 0,
\]
which has the unique solution
\[
V(t) = \left( \frac{1}{2} + \frac{1}{2} [N,N](t) \right) - Ct,
\]
\[
= V(0) \exp \left( \alpha \int_0^t V(qs) \, ds - \frac{1}{2} \int_0^t g(V(qs))^2 \, ds + \int_0^t g(V(qs))dW(s) \right) - Ct.
\]

If \( C = 0 \), the above equation is
\[
V(t) = V(0) \exp \left( \alpha \int_0^t V(qs) \, ds - \frac{1}{2} \int_0^t g(V(qs))^2 \, ds + \int_0^t g(V(qs))dW(s) \right),
\]
for \( t \in [0,T] \). This clearly implies that \( V(t) > 0 \) for all \( t \in [0,T] \) almost surely when \( V(0) > 0 \). Using the similar way, we can see that \( V(t) > 0 \) for all \( t \in [T,2T] \). By induction \( V(t) > 0 \) for all \( t \geq 0 \).

**Remark 2.** In Theorem 1 we proved that, \( V(0) \geq 0 \), (or \( V(0) > 0 \)) are only required to conclude that the solution of (5) satisfies \( V(t) \geq 0 \) for all \( t \geq 0 \) (or \( V(t) > 0 \) for all \( t \geq 0 \), respectively), which proved that feasibility of the price valuation model (5).

Following [15], we can derive a RPDE which must be satisfied by any security whose value can be written as a function of the value of the firm and time. Let corporate claim \( Y(t) \) at time \( t \) with \( Y(t) = U(V(t), t) \) follows a nonlinear SPDE
\[
dY(t) = (\alpha y(t) - C_y) \, dt + g_y(Y(t))Y(t) \, dW_y(t), \quad Y(t) = Y_0, \quad t \in [0,T],
\]
on a probability space \( (\Omega, \mathcal{F}, P) \). The constant \( \alpha_y \) is the riskless interest rate of return per unit time on this claim; \( C_y \) is the amount payout per unit time to this claim; \( g_y: \mathbb{R} \to \mathbb{R} \) is a continuous function representing the volatility function of the return on this claim per unit time; the initial value \( Y_0 \) is \( \mathcal{F}_0 \)-measurable \( \mathbb{R} \)-valued random variable. The process \( W_y \) is a one dimensional standard Brownian motion adapted to the filtration \( \{\mathcal{F}_t\}_{t \leq T} \).

**Assumption 1.** The value of the company is unaffected by how it is financed (The capital structure irrelevance principle). In addition, the market value of firm at time \( t \in [0,T] \), \( V(t) \) follows the nonlinear SPDE (3).

**Theorem 2.** Let Assumption 1 is satisfied and debt value accumulates interest compounded continuously at a rate \( r \), that is \( B(t) = B(0) e^{rt} \). For any claim whose value is a function of the firm value and time i.e. \( Y(t) = U(V(t), t) \), where \( U: \mathbb{R} \times [0,T] \to \mathbb{R} \) is continuous and that \( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial v} \) and \( \frac{\partial^2 u}{\partial v^2} \) exists and are continuous. Then the following RPDE should be satisfied;
\[
\frac{1}{2} g^2(V(qt))V^2(t) \frac{\partial^2 u}{\partial v^2} + (rV(t) - C) \frac{\partial u}{\partial v} + \frac{\partial u}{\partial t} + C_y - rU = 0, \quad (V(t), t) \in \mathbb{R}^+ \times (0,T)
\]

**Proof.**
We can prove our theorem in the same idea and similar process in [3, 15]. We note that the \( \alpha_y, g_y, dW_y \) have parallels with the corresponding \( \alpha, g, dW \) in SPDE (5). Knowing that \( V(t) \) in SPDE (3) is an Itô process which can rewrite in a simple form as
\[
dV(t) = F \, dt + GdW(t),
\]
where \( U: \mathbb{R} \times [0,T] \to \mathbb{R} \) is continuous and that \( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial v} \) and \( \frac{\partial^2 u}{\partial v^2} \) exists and are continuous. So, the Itô formula allow us to get the following
\[
dY(t) = \int \frac{\partial^2 u}{\partial v^2} G^2 \, dt + \frac{\partial u}{\partial v} \, dV + \frac{\partial u}{\partial t} \, dt.
\]
\[
= \left[ \frac{1}{2} \frac{\partial^2 u}{\partial v^2} G^2 + \frac{\partial u}{\partial v} F + \frac{\partial u}{\partial t} \right] \, dt + \frac{\partial u}{\partial v} \, GdW(t).
\]
\[
= \left[ \frac{1}{2} g^2(V(qt))V^2(t) \frac{\partial^2 u}{\partial v^2} + (aV(t) + C) \frac{\partial u}{\partial v} + \frac{\partial u}{\partial t} \right] \, dt + g(V(qt))V(t) \frac{\partial u}{\partial v} \, dW(t).
\]
Since the SPDE (5) has the unique solution according to Theorem 1, the coefficients of the corresponding terms in equation (8) and (10) are almost sure equal. Then, we can get the following properties
\[
\alpha_y = \frac{1}{2}g^2(V(qt)\sqrt{v^2(t)\frac{\partial^2 u}{\partial v^2}} + (av'(t)v(qt) - C)\frac{\partial u}{\partial v} + C_y, \tag{11}
\]
\[
g_y(Y(qt)) = \frac{g(V(qt))\sqrt{v(t)}}{u(v(t))}, \tag{12}
\]
\[
dW_y(t) = dW(t). \tag{13}
\]
Following the self-financing and replication strategy [1], let \(z_1, z_2, \) and \(z_3\) be the instantaneous number corresponding to the amount invested in the firm, security, and riskless debt, respectively. Let the instantaneous return of the portfolio is \(dx\) and assume \(z_1 + z_2 + z_3 = 0\) (i.e. the total investment in the portfolio is zero), we have
\[
dx = z_1 \frac{dv(t)}{v(t)} + z_2 \frac{dv(t) + C_y dt}{v(t)} + z_3 r dt,
\]
\[
= z_1 \left[ (av'(t)v(qt) - C) dt + g'(V(qt))v'(t) dW(t) \right] + z_2 C_y dt
\]
\[
+ z_2 \left[ (av'(t)v(qt) - C) dt + g'(V(qt))v'(t) dW(t) \right] + z_3 C_y dt + z_3 r dt,
\]
\[
= z_1 \alpha V(qt) dt + z_1 g'(V(qt)) dW(t) + z_2 \alpha_y dt + z_2 g_y(Y(qt)) dW_y(t) - (z_1 + z_2) r dt.
\]
From equation (13), Yield
\[
dx = \left[ z_1 \left( \alpha V(qt) - r \right) + z_2 \left( \alpha_y - r \right) \right] dt + \left[ z_1 g'(V(qt)) + z_2 g_y(Y(qt)) \right] dW(t). \tag{14}
\]
It is well known that, the return of the portfolio is non-stochastic and there is no arbitrage condition which leading to the following system
\[
z_1 \left( \alpha V(qt) - r \right) + z_2 \left( \alpha_y - r \right) = 0,
\]
\[
z_1 g'(V(qt)) + z_2 g_y(Y(qt)) = 0.
\]
A non-trivial solution \((z_i \neq 0)\) to this system exists if and only if
\[
\frac{\alpha V(qt) - r}{g'(V(qt))} = \frac{\alpha_y - r}{g_y(Y(qt))}. \tag{15}
\]
But from equation (11) and (12) substituting for \(\alpha_y\) and \(g_y(Y(qt))\), we obtain the right hand side in equation (15) as follows
\[
\frac{\alpha V(qt) - r}{g'(V(qt))} = \frac{1}{2}g^2(V(qt))\sqrt{v^2(t)\frac{\partial^2 u}{\partial v^2}} + (av'(t)v(qt) - C)\frac{\partial u}{\partial v} + C_y - r U(V(t), t).
\]
By rearranging terms and simplifying, we get
\[
a V(t)V(qt)\frac{\partial v}{\partial v} - r V(t)\frac{\partial v}{\partial v} = \frac{1}{2}g^2(V(qt))V^2(t)\frac{\partial^2 u}{\partial v^2} + (a V(t)v(qt) - C)\frac{\partial u}{\partial v} + \frac{\partial u}{\partial t} + C_y - r U(V(t), t). \tag{16}
\]
Therefore, we can rewrite equation (16) as the following RPDE for \(U\)
\[
\frac{1}{2}g^2(V(qt))V^2(t)\frac{\partial^2 u}{\partial v^2} + (r V(t) - C)\frac{\partial u}{\partial v} + \frac{\partial u}{\partial t} + C_y - r U = 0.
\]

4. Evaluation debt and equity of a firm
We consider a corporate claim value of market whose value a function of a firm value and time as a corporate of debt and equity of a firm. In this section, we provide the valuation form for equity and debt value of a firm. In order to evaluate the debt and equity of a firm, the equation (9) must be satisfied by some specific initial and boundary conditions. So, the value of a firm can be presented as follows
\[
V(t) = u(V(t), t) + U(V(t), t), \tag{17}
\]
were \( u(V(t), t) \), and \( U(V(t), t) \) are the value of the equity, and debt, respectively at any time \( t \in [qT, T] \), and \( r \) is the interest risk rate.

Without a lack of generalization, we will consider the simple case of corporate debt and equity and therefore give the following assumption

**Assumption 2.** Let a firm is financed by a single class of debt and equity. In addition, for the bond issue the contract has the following restrictions:

- At maturity time \( T \), amount \( B(T) \) must be paid by a firm to debt holders.
- If a firm can’t make the payment at \( T \), the debtholders take over the company and the equity holders lose their investment.
- A firm is not allowed to coupon payment nor pay cash dividend during the option life (i.e. the maturity of the debt and \( C = C_y = 0 \)).

Under Assumption 2, and following [3], it is easy to get the following valuation form with initial and boundary conditions for debt and equity linked by equation (9)

\[
\begin{align*}
\frac{1}{2} g^2(V(qt))V^2(t) \frac{\partial^2 u}{\partial v^2} + rV(t) \frac{\partial u}{\partial v} + \frac{\partial u}{\partial t} + C_y - ru &= 0, \quad 0 < t < T, \\
u(V, T) &= \max(V - B(T), 0), \quad V > 0, \\
u(0, t) &= 0, \quad u(V, t) \sim V - B(T)e^{-r(T-t)}, \text{ as } V \to \infty.
\end{align*}
\]

And

\[
\begin{align*}
\frac{1}{2} g^2(V(qt))V^2(t) \frac{\partial^2 u}{\partial v^2} + rV(t) \frac{\partial u}{\partial v} + \frac{\partial u}{\partial t} - ru &= 0, \quad 0 < t < T, \\
U(V, T) &= \min(V, B(T)), \quad V > 0, \\
U(0, t) &= 0, \quad U(V, t) \sim B(T)e^{-r(T-t)}, \text{ as } V \to \infty.
\end{align*}
\]

### 5. Conclusions and future work

We consider especial type of stochastic functional differential equations (SFDE) with variable delay and unbounded memory; stochastic pantograph differential equation (SPDE), to modelling the corporate claim value. The feasibility of the proposed model is proved. We derive a random partial differential equation (RPDE) whose solution provide valuation of equity and debt of a firm. In this work, the delayed model with variable delay is considered to overcome the weaknesses of the delayed model with constant delay. Furthermore, the variable delay model can be more useful when the practitioners need to consider the historical image of volatility more randomly to predict the future state of the system.

In the future work, we will provide a robust numerical methods to approximate the nonlinear delayed model with variable delay for corporate claim value along with RPDE. In addition, using the financial real corporate data, we will forecast and compare the numerical solutions from constant delay model and variable delay model.

### Acknowledgments

This research was partly financed by National Nature Science Fund in China (NSFC) Grant No. 91646106. Additionally, this work is supported by the Higher Education Commission of Egypt.

### References

[1] Arriojas M, Hu Y and Mohammed S 2007 A Delayed Black and Scholes Formula Journal of Stochastic Analysis and Applications 25 (2) 471-492

[2] Buckwars E 2000 Introduction to the numerical analysis of stochastic delay differential equations Journal of computational and applied mathematics 125(1) 297-307

[3] Kemajou E and Mohammed S E 2012 A stochastic delay model for pricing debt and loan guarantees: theoretical results arXiv preprint arXiv 1210 0570

[4] Mao X 1996 Razumikhin-type theorems on exponential stability of stochastic functional differential equations Stochastic Processes and Their Application 65(2) 233-250
[5] Mohammed S E 1984 Stochastic Functional Differential Equations Res. Notes Math., Pitman, London 1984
[6] Tambue A, Brown E K and Mohammed S 2015 A stochastic delay model for pricing debt and equity: Numerical techniques and applications Communications in Nonlinear Science and Numerical Simulation 20(1) 281-297
[7] Zong X, Wu F and Huang C 2015 Theta schemes for SDDEs with non-globally Lipschitz continuous coefficients Journal of Computational and Applied Mathematics 278 258-277
[8] Eissa M A and Tian B 2017 Lobatto-Milstein Numerical Method in Application of Uncertainty Investment of Solar Power Projects Energies 10 (1) 43
[9] Liu C S 2016 Basic theory of a class of linear functional differential equations with multiplication delay arXiv preprint arXiv 1605 06734
[10] Baker C T and Buckwar E 2000 Continuous \( \theta \)-methods for the stochastic pantograph equation Electronic Transactions on Numerical Analysis 11 131-151
[11] Fan Z, Liu M and Cao W 2007 Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations Journal of Mathematical Analysis and Applications 325(2) 1142-1159
[12] Xiao Y, Eissa M A and Tian B 2018 Convergence and stability of split-step theta methods with variable step-size for stochastic pantograph differential equations International Journal of Computer Mathematics 95(5) 939-960
[13] M A, Xiao Y and Tian B 2016 Convergence and Stability of Two Classes of Theta Methods with Variable Step Size for a Stochastic Pantograph Differential Equations Journal of Advanced Mathematics and Application 5(2) 95-106
[14] Zhang H, Xiao Y and Guo F 2014 Convergence and stability of a numerical method for nonlinear stochastic pantograph equations Journal of the Franklin Institute 351(6) 3089-3103
[15] Merton R C 1974 On the Pricing of Corporate Debt: The Risk Structure of Interest Rates Journal of Finance 29 (2) 449-470
[16] Black F and Scholes M 1973 The pricing of options and corporate liabilities Journal of political economy 81(3) 637-654
[17] Dumas B, Fleming J and Whaley R E 1998 Implied volatility functions: Empirical tests The Journal of Finance 53(6) 2059-2106
[18] Scott L O 1987 Option pricing when the variance changes randomly: Theory, estimation, and an application Journal of Financial and Quantitative analysis 22(4) 419-438
[19] Hobson D G and Rogers L C 1998 Complete models with stochastic volatility Mathematical Finance 8(1) 27-48
[20] Mao X and Sabanis S 2013 Delay geometric Brownian motion in financial option valuation Stochastics An International Journal of Probability and Stochastic Processes 85(2) 295-320