Wightman Functions in QCD

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ABSTRACT
The constraint imposed by Gauss’ law is used to show that the matrix elements of n-point Wightman Functions of gluon field and quark current operators at different space time points vanish when taken between physical states.
1. Introduction.

Christ and Lee[1,2] have warned against the common practice of starting from a formal path integral approach prior to choosing a gauge. In fact, for QCD they chose the (time)-axial (or temporal) gauge[3,4,5] which has a well defined Hamiltonian and then derived the Feynman rules for different gauges by appropriate coordinate transformations. In the temporal gauge, their construction of the path integral generating functional requires for consistency that it operate only between color singlet states.

For our purposes it will be convenient to work in the canonical formalism[5]. In section 2 we point out that just implementing the constraints, due to Gauss’ law, on physical states in the temporal gauge leads us to a pleasing result: all transition matrix elements of the color electric and magnetic fields as well as of color carrying currents between physical states vanish. This forecloses the possibility of any physical state with non-zero color appearing. In section 3 we show that two point Wightman functions of quark current and gluon fields in physical states vanish when the two points do not coincide, and in section 4 we extend the results to n-point functions.

2. The Physical States.

In the temporal gauge, $A_{0}^{a} = 0$, the color electric field

$$E_{i}^{a}(x, t) = - \partial A_{i}^{a}(x, t)/\partial t.$$  (1)

We will state the canonical commutation relations a little more carefully than is customary, so as to avoid inconsistencies. To this end let us consider the state vector $\Psi$ which is a member of an Hilbert space spanned by a complete set of normalizable functions of the vector potentials $A_{i}^{a}$, obeying appropriate boundary conditions. Let $\mathcal{F}$ be an operator such that $\mathcal{F}\Psi$ is also a member of this Hilbert space. Then the canonical equal time commutation relations are given by

$$[E_{i}^{a}(x, t), \mathcal{F}(t)] = \frac{i}{\delta A_{i}^{a}(x, t)} \delta \mathcal{F}(t).$$  (2)

The Gauss’ law operator $G^{a}(x, t)$

$$G^{a}(x, t) = \partial_{i}E_{i}^{a}(x, t) - g f^{abc}A_{b}^{i}(x, t)E_{c}^{i}(x, t) - J_{0}^{a}(x, t),$$  (3)

constraint on physical states $|\gamma\rangle$ reads[1,3,4,5,6,7,8,9,10]

$$G^{a}(x, t)|\gamma\rangle = 0.$$  (4)

Unlike the QED case, the Gauss operator $G^{a}$ generates rotations in color space and as such rotates the color electric $E_{i}^{a}$ and magnetic $B_{i}^{a}$ fields as well. For example,

$$[G^{a}(x, t), F_{\mu\nu}^{b}(x', t)] = ig f^{abc}F_{\mu\nu}^{c}(x, t)\delta^{3}(x - x').$$  (5)

where $F_{\mu\nu}^{a}$ is the color electromagnetic field tensor.

Taking the matrix element of Eq.(5) between arbitrary physical states $\gamma$ and $\gamma'$ and using Eq.(4) we find that

$$\langle\gamma'|F_{\mu\nu}^{a}(x, t)|\gamma\rangle = 0.$$  (6)

An identical answer ensues for the color current density $J_{\mu}^{a}$, viz.,

$$\langle\gamma'|J_{\mu}^{a}(x, t)|\gamma\rangle = 0.$$  (8)

Vanishing of all of the above can be satisfied only for physical states which are color singlets[11]. The equivalence of temporal and Coulomb gauges in QCD[1,2], would then imply that an identical result holds there as well. Note that it is only for gauge covariant quantities that one can meaningfully speak about their value. If they become null identically in one gauge they remain so in any other gauge. In contrast, we do not have a similar argument regarding non gauge covariant quantities such as the vector potential. We further
note that we have assumed there is no symmetry breaking, so that a similar argument cannot be applied to $SU(2) \times U(1)$.

3. Two-Point Wightman Functions.

Having deduced that the physical matrix elements of non-color singlet operators vanish, we next turn to constraining the matrix elements of color-singlet operators which are composites of other colored operators. It is not difficult to show that these are devoid of absorptive parts. By way of illustration consider the commutator of the product of YM fields $F_{\mu \nu}^b$ and $F_{\xi \sigma}^c$, with the Gauss law operator $G^n$. Using Eq.(5) and the fact that the $G^n(x)$ are independent of $x^0$, we find

$$f^{abc} [G^n(x), F_{\mu \nu}^b(y) F_{\xi \sigma}^c(z)] = 3ig F_{\mu \nu}^a(y) F_{\xi \sigma}^a(z) \{ \delta^3(x - y) - \delta^3(x - z) \}. \tag{9}$$

Taking Eq.(9) between arbitrary physical states $\gamma$ and $\gamma'$ and using Eq.(4) yields

$$\langle \gamma' | F_{\mu \nu}^a(y) F_{\xi \sigma}^a(z) | \gamma \rangle = 0 \tag{10}$$

when $y \neq z$. Employing the Lorentz invariance of the theory we then find that it is true for all $y^\mu \neq z^\mu$. A similar argument holds for the quark current operators either by themselves or in combination with $F_{\mu \nu}^a$.

4. N-point Functions.

We now construct operators $V^n(x)$ which are composites of current density and gluon fields at the same space time point and transform as vectors under the gauge group (for convenience). (In general they will also carry Lorentz indices as well, but which we now suppress). Let us then consider the following matrix elements in physical states of the commutators

$$< \alpha | [G^n(x), V^{b_1}(y_1)V^{b_2}(y_2) \cdots V^{b_n}(y_n)] | \beta > =$$

$$i g < \alpha | [f^{abc} V^c(y_1)V^{b_2}(y_2) \cdots V^{b_n}(y_n) \delta^3(x - y_1)$$

$$+ f^{abc} V^{b_1}(y_1)V^c(y_2) \cdots V^{b_n}(y_n) \delta^3(x - y_2)$$

$$+ \cdots + f^{abc} V^{b_1}(y_1)V^{b_2}(y_2) \cdots V^c(y_n) \delta^3(x - y_n)] | \beta > = 0 \tag{11}$$

using Eq.(4). Thus, if $y_i \neq y_j$ for all $i \neq j$ then each of the coefficients of the $\delta^3(x - y_i)$ must vanish. As before, the $y_i^0$ are arbitrary. Hence, once again, appealing to the Lorentz invariance of the theory, we find that these coefficients must vanish for space like, time like and light like separations of all the $y_i$.

Furthermore, there is only one $f^{abc}$ for a given $a$ and $b$. Thus, $f^{abc} V^c(y)$ is a single term and the $n$-point functions must vanish unless two or more of the $y_i$ are equal. In this latter case we may attempt to extract the singlet content at such points by contracting with $f^{abc}, d^{abc}$ and $\delta^{ab}$. If after all such contractions there remain terms with non zero color content at one or more points $y_i$ then they become matrix elements belonging to a smaller $n$ and hence they must also vanish. Matrix elements which have only singlet operators at different space time points need not vanish.

5. Conclusions.

The physical states of QCD are ones which satisfy Gauss’ law and hence have zero color. Furthermore, Wightman functions in physical states, which involve color non-singlet operators at different space time points, vanish identically.
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