$Z_N$-symmetry in gauge theories at finite temperatures.

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Abstract

The role of $Z_N$ symmetry in gauge theories at finite temperatures is discussed. This symmetry is studied in terms of $A_0$-effective potential. We consider two- and four-dimensional models where the question on physical interpretation of minima of $A_0$-effective potential can be considered exactly. It is shown that the correct result for the partition function can be obtained by summation of contributions near all minima of the $A_0$-effective potential.

PACS number(s): 11.10.Wx; 11.15.-q
1 Introduction

The Matsubara formalism is a convenient tool to study static properties of a theory at finite temperatures [1]. A quantum field theory partition function at finite temperatures can be represented in the following form

$$ Z = \int D\phi_i(t, x) e^{-\int_0^\beta dt \int d^3 x L_{\text{eucl.}}(\phi(t,x)_i)}, $$

where $L_{\text{eucl.}}$ is a Lagrangian in Euclidean space-time; the (anti)periodic boundary conditions are imposed on the (fermi) boson quantum fields: $\phi(t = 0, x) = \phi(t = \beta, x)$; $\beta$ is inverse temperature: $\beta = 1/T$.

It is important to note that due to the boundary conditions it is not possible to apply the gauge $A_0 = 0$. Indeed, let us consider a field configuration when $A_0 = \text{const}$ and $A_i$ are arbitrary gauge fields ($i = 1, 2, 3$). If we try to gauge off the constant gauge field $A_0$:

$$ A_0' = U(t)A_0U^\dagger(t) + \frac{i}{g}U(t)\partial_0U^\dagger(t) = 0; $$

$$ U(t) = e^{-igtA_0}; $$

then the boundary conditions for spatial components of gauge fields will be changed:

$$ A_i'(t = \beta, x) = U(t = \beta)A_i(t = \beta, x)U^\dagger(t = \beta) $$
$$ = U(t = \beta)A_i(t = 0, x)U^\dagger(t = \beta) $$
$$ = U(t = \beta)A_i'(t = 0, x)U^\dagger(t = \beta). $$

where $A_\mu = A_\mu^a t^a$, $t^a$ is a generator of a gauge group, $g$ is a coupling constant.

It is possible to keep the periodic boundary conditions only in the case when $[t^a, U(t = \beta)] = 0$ for any generator $t^a$ of a gauge group. It means that the theory is invariant under the gauge transformations $U^\dagger(t = 0, \vec{x})U(t = \beta, \vec{x})$ belonging to the center of the group. The center of the $SU(N)$ group is $\mathbb{Z}_N$-discrete subgroup and the boundary conditions are unchanged under discrete shifts of gauge fields corresponding to the gauge transformations from the center of the gauge group. In the case of $SU(2)$ gauge group the shift is $A_3^0 \rightarrow A_3^0 + \pi T/g$. It is important to note that the theory is not invariant under a general shift of the gauge field which does not correspond
to the gauge transformation from the center of a gauge group. It means that the effective potential for temporal component of the gauge fields is not trivial when \( T \neq 0 \). The effective potential \( V_{\text{eff}}(A_0) \) has a periodic structure and \( Z_N \)-symmetry can be formulated in terms of the order parameter:

\[
L(x) = \frac{1}{N_c} \mathcal{P} \text{Tr} \left( e^{ig \int_0^\beta A_0(t,x)dt} \right); \quad (4)
\]

here \( \mathcal{P} \) means P-ordering.

It was shown [2] that one- and two-loop effective potentials have global minima where

\[
< L(x) > = e^{i \frac{2\pi n}{N}}. \quad (5)
\]

General proof of this statement has been made in [3].

These minima are degenerated and correspond to the different but physically equivalent states of the Euclidean gauge theory with fields in adjoint representation. Fields in fundamental representation break \( Z_N \)-symmetry and \( Z_N \)-minima are not more degenerated.

It was a very attractive idea to interpretate such local minima of the effective potential as metastable phases of a hot gauge theory and to use the decay of such states in cosmology [4]. This interpretation of the local minima was strongly criticized [5] and it was pointed that this interpretation leads to a conclusion that it may exist physical metastable states with negative entropy. This remark is very important but it is not enough to exclude the existence of the states where the whole entropy is positive. The absence of such ”metastable states” has been demonstrated in 2-dimensional Schwinger model [6]. However this conclusion may be a specific property of 2-dimensional systems.

So, it is clear that we have to find a clear physical interpretation of such minima in the case of a simple 4-dimensional model. Below we give a physical explanation of the so-called ”metastable states” in the case of \( U(1) \) gauge theory at finite temperatures with two type of charged particles. The main conclusion which can be made from this model is that the correct expression for partition function is a sum of partition functions near minima of the effective potential. It means that there are no negative entropy and negative specific heat for these systems: they are just negative corrections to the entropy and specific heat. It is also demonstrated that it is not correct to consider \( < L > \) as an order parameter which confirms argumentation of [8].
At the end of the paper we consider $2-d$ Schwinger model at finite temperatures. It is demonstrated that in this model the effective potential for the constant field has a periodical structure in contrast to the effective potential for variable gauge fields. That is a property of 2-dimensional systems.

2 Partition Function and Order Parameter

Let us start our consideration from the formal derivation of eq.(1) in the case of $U(1)$ gauge theory. For simplicity we consider a compact 3-dimensional space. The partition function of a gauge theory in the gauge $A_0 = 0$ has the following form:

$$Z = \sum_n <n|e^{-\hat{H}t}|n>$$

$$\sim \int \mathcal{D}A_\mu \mathcal{D}E_i \mathcal{D}a e^{-\int_0^\beta dt \int d^3x (H + iE_i \partial_\mu A_\mu + i\alpha (\partial_i E_i - e\rho) - ieA_\mu j_\mu)}$$  (6)

where $H = \frac{1}{2} (E^2 + B^2)$ is a Hamiltonian; $E_i = \partial_0 A_i$ are electric fields; $B_i = \frac{1}{2} \epsilon_{ijk} \partial_j A_k$ are the magnetic fields; $\rho$ is a charge density, $j_i$ is an electric current, $e$ is an electric charge.

The partition function (6) has the Gauss law constraint:

$$\int \mathcal{D}a e^{i \int_0^\beta dt \int d^3x (\partial_i E_i - \rho)} \sim \delta[\partial_i E_i - e\rho]$$.  (7)

After integration by parts in eq.(6) of the term with $\partial_i E_i$ it is not difficult to show that in the case of compact space the partition function has a form of Euclidean gauge theory with periodic boundary conditions imposed on the gauge fields:

$$Z = \int \mathcal{D}A_\mu e^{-\int_0^\beta dt \int d^3x (\frac{1}{4} F_{\mu \nu}^2 - ieA_\mu j_\mu)}$$  (8)

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a strength tensor, and $A_0 = \alpha$. This formula can be obtained in the case of nonabelian gauge theory with a replacement $\partial_\mu \rightarrow D_\mu = \partial_\mu - igA_\mu$.

One can represent $\alpha$-field in the following form:

$$\alpha(t, x) = \alpha_0(t) + \alpha_x(t, x)$$  (9)
where fields $\alpha_0$ and $\alpha_x$ belong to different orthogonal functional subspaces. It means that any field configuration $\alpha_x(t, x)$ is orthogonal to an arbitrary function $\alpha_0$ and:

$$\int d^3x \alpha_x(t, x) = 0$$  \hspace{1cm} (10)

The integration in the $\alpha_0$-subspace leads us to a conclusion (in the case of compact $x$-space case) that

$$\int \mathcal{D}\alpha_0 e^{i \int_0^\beta dt \int d^3x_0 (\partial_\mu E_\mu - e\rho)}$$

$$= \int \mathcal{D}\alpha_0 e^{-ie\int_0^\beta dt \int d^3x_0 (t, x, t) \rho(x, t)} \sim \delta \left[ e \int d^3x \rho(x, t) \right]$$  \hspace{1cm} (11)

which means that only neutral states give nonzero contribution.

Let us consider the order parameter (4) in the case of $U(1)$ theory with light electrons ($T \gg m$) where $m$ is a mass of particles with charge $e$.

Taking into account the $\delta$-function (11) we can conclude that the average value for the order parameter with a charge $e/2$ is equal to zero:

$$<L(A_0(\vec{x})> = <\mathcal{P}\exp \left( \frac{ie}{2} \int_0^\beta A_0(t, \vec{x}) dt \right) >= 0,$$  \hspace{1cm} (12)

because of the fact that the presence of the Wilson line in the path integral is equivalent to consideration of a system with one heavy particle with a charge $1/2$ and it is not possible to create a neutral system with one particle with a charge $1/2$ plus any large but finite number of particles with charge 1.

In the imaginary time formalism one can reproduce this result only in the case if we make $A_0$-integration in the whole functional subspace $\alpha_0$. In high temperature limit we can apply saddle point approximation near the points where $<L> = \pm 1$:

$$<L> = Z(L)/Z = \frac{Z(L)_{A_0=0} + Z(L)_{A_0=2\pi T/e}}{Z_{A_0=0} + Z_{A_0=2\pi T/e}} = 0.$$  \hspace{1cm} (13)

Here

$$Z(L) = \int \mathcal{D}A_\mu L(A_0) \exp \left( -\int_0^\beta dt \int d^3x \frac{1}{4} F_{\mu\nu}^2 \right)$$  \hspace{1cm} (14)

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Due to the $Z_2$ symmetry (in terms of the order parameter $L$ with semi-integer charge) this result is exact. The relation \[ Q = \int d^3 x \rho(x) = 0, \] does not mean that we have a confinement but it is just the consequence of the neutrality condition.

So, it is clear that it is not correct to consider the value $< L >$ as an order parameter. It confirms argumentation of \[8\] that instead of $< L >$ the asymptotic of the correlator

\[ < L(\vec{x})L^\dagger(0) >_{\vec{x} \to \infty} \]  

can be considered as a correct order parameter of a theory.

Note that to obtain the correct result $< L > = 0$ for the Wilson line with an arbitrary fractional charge one have to sum up over all minima of the effective potential. Only in this case it is possible to obtain the correct result for the partition function.

It is easy to construct a model with so-called "metastable state". It is $U(1)$ model with two types of charged particles: the first particles have a mass $m \ll T$ and integer charge $e$ and the second ones have a mass $M \gg T$ and semi-integer charge $e/2$.

The order parameter $< L >$ with semi-integer charge is not equal to zero because of the presence of the massive particles with charge $e/2$. The neutrality condition in this case means that only states with odd numbers of particles with charge $e/2$ give nonzero contribution into the partition function. It is clear that the order parameter $< L >$ will be suppressed by a factor $\sim e^{-m/T}$.

It is possible to reproduce this result in the imaginary time formalism:

\[ < L > = \frac{Z(L)_{A_0=0} + Z(L)_{A_0=2\pi T/e}}{Z_{A_0=0} + Z_{A_0=2\pi T/e}} \approx \frac{Z_{A_0=0} - Z_{A_0=2\pi T/e}}{Z_{A_0=0} + Z_{A_0=2\pi T/e}} \approx V \int \frac{d^3 k}{(2\pi)^3} e^{-\sqrt{k^2 + M^2}/T}; \]  

where $V$ is 3-dimensional volume of a system. This result is exact in the limit when $\int \frac{d^3 k}{(2\pi)^3} e^{-\sqrt{k^2 + M^2}/T} \ll V^{-1}$. Nonzero result for in (16) appears due to the difference for a free energy at $A_0 = 0$ and $A_0 = \pi/(e\beta)$. It is important
to note that the correct result (16) for the partition function can be obtained only after integration over the all values of $A_0$.

According to the neutrality condition only the states with even numbers of heavy particles give nonzero contribution to the partition function. It is possible to check that the total contribution of heavy particles into the partition function in present model has the following structure:

$$Z = \frac{1}{2} [Z(0) + Z(1) + Z(2) + \ldots] + [Z(0) - Z(1) + Z(2) + \ldots]$$

$$= Z(0) + Z(2) + \ldots$$  \hspace{1cm} (17)

where $Z(n)$ is a partition function of the system with $n$-heavy particles, and the first term of eq. (17) in square brackets corresponds to $A_0 = 0$ minimum of the effective potential and the second one to the local minimum of the potential where $<L> = -1$.

Note that one of the term of eq. (17) looks like a partition function of particles with a wrong statistic. The same wrong statistic in distribution functions appears in the case of nonabelian theory with matter in fundamental representation. So it becomes clear that the physical sense of this local minimum: it is not a metastable state with such wrong statistic but only the correction to the partition function.

However there is a possibility to interpritate these states as a metastable phases of a theory in the case when the compact coordinate is chosen in a spatial direction. In this case such domain walls can be considered as physical objects because in this case we do not consider a sum over all physical states which is essential in the case of a hot gauge theory.

3 Schwinger Model

In Schwinger Model we are able to determine the thermodynamical properties exactly.

The partition function has the following form:

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int_0^\beta dt \int dx \left( \frac{1}{4} F_{\mu\nu}^2 + i\bar{\psi}\hat{D}\psi \right)};$$  \hspace{1cm} (18)

where $\hat{D} = \gamma_\mu (\partial_\mu - ieA_\mu)$. 
The simplest way to find the effective action for gauge fields $A_\mu$ in this model is to apply the local axial transformation:

$$\psi(t, x) \rightarrow e^{i\gamma_3\phi(t, x)}\psi(t, x)$$
$$\bar{\psi}(t, x) \rightarrow \bar{\psi}(t, x)e^{i\gamma_3\phi(t, x)}$$

(19)

The determinant of fermions after the transformation (19) have the following form:

$$\det i\hat{D}_{A_\mu} \rightarrow \det i\hat{D}_{A'_\mu}$$

(20)

where

$$A'_\mu = A_\mu + \frac{i}{e} \varepsilon_{\mu\nu}\partial_\nu\phi(t, x).$$

(21)

The strength tensor of the $A'_\mu$ field is

$$F'_{\mu\nu} = F_{\mu\nu} - \frac{i}{e} \partial^2\phi(t, x)$$

(22)

So, it is clear that if we choose that

$$\phi(t, x) = \frac{i}{-\partial^2} F$$

(23)

where $F = \frac{1}{2} \varepsilon_{\mu\nu} F_{\mu\nu}$ then the determinant will be equal to the determinant of free fermions in the presence of a constant gauge fields with zero strength tensor.

It means that the determinant is

$$\det(i\hat{D})_{A_\mu} = \det(i\hat{D})_{A_\mu=const}J$$

(24)

where $J$ is the Jacobian of the axial transformation (19) (anomaly):

$$J = e^{i\frac{\pi}{2} \int_0^\beta dt \int dx F\phi}$$

(25)

It is important to remark that the constant mode of the $\phi$ is excluded.

It was noted in the Introduction that we can not gauge away the constant gauge field. Thus, after integration over fermion fields the partition function (18) have the following form:

$$Z = Z_F(A_0 = const) \int \mathcal{D}A_\mu e^{-\int_0^\beta dt \int dx \left(\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{4}F_{\mu\nu}^2 - \frac{1}{2}F\right)}$$

(26)
where $Z_F$ is a partition function for fermion field in the presence of constant
gauge field with $F = 0$.

It is possible to rewrite the eq.(26) in more suitable form by adding the
auxiliary scalar field $\phi$:

$$Z = Z_F(A_0 = \text{const})Z_B^{-1}(m = 0, \phi \neq \text{const}) \int \mathcal{D}A_\mu \mathcal{D}\phi e^{-\int_0^\beta dt \int dx \left( \frac{1}{4} F_{\mu \nu}^2 + i \frac{e}{\pi} F \phi + \frac{1}{2} (\partial_\mu \phi^2) \right)}; \quad (27)$$

where $Z_B$ is a partition function for a free massless boson field and the
constant mode of field $\phi$ is excluded. Otherwise, the integration over this
constant mode leads us to a conclusion that the topological charge of the
system is equal to zero:

$$\int_0^\beta dt \int dx F = 0 \quad (28)$$

Now let us go back to the eq.(26). Let us try to find the effective potential
for the field $A_0(x)$. The effective potential for the constant part of this field
has a periodic structure and comes from $Z_F(A_0)$ only:

$$V_{\text{eff}}(A_0 = \text{const}) = \frac{e^2}{\pi} (A_0^2)_{\text{mod } 2\pi T/e} \quad (29)$$

At the same time the exact result for the effective potential of the nonzero
modes of the field $A_0(x)$ has no a periodic structure and can be obtained
from effective Lagrangian:

$$V_{\text{eff}}(A_0 \neq \text{const}) = \frac{e^2}{\pi} A_0^2 \quad (30)$$

It is a specific property of $2 - d$ theory. In the case of $4 - d$ theory this
factorization property is absent.

4 Conclusions

The main question considered in present paper is the physical sense of so-
called "metastable states" in hot gauge theories. It was shown that in $4 - d$
QED such local minima of the effective potential can not be treated as a
physical states. The only sense of such minima is a way to remove charged
states from the consideration. It was shown that in the case of the simplest
\(U(1)\) theory this conclusion can be proved exactly. It was demonstrated that we have to integrate over all values of static \(A_0\) fields to obtain a physical result. It has been shown that the mean value of Polyakov operator cannot be considered as an order parameter. It confirms an argumentation of Ref.\[8\].

We have to emphasize that a simple naive picture of the confinement-deconfinement transition in terms of transitions of \(A_0\) gauge field from one minima to another is not correct. The counter-example is the \(U(1)\ 4-d\) model where it was shown that at low temperatures when these transitions are not suppressed and the ”order parameter” is equal to zero. This zero value for \(<L>\) means just the absence of charged states at low temperatures and can not be treated as a confinement.

In the case of the Schwinger model all results can be obtained exactly. It was demonstrated that the periodic structure of the effective potential is a property of a constant nondynamical mode of the gauge field \(A_0\) only. The effective potential for the nonconstant gauge fields has no periodic structure.

These two examples clearly show that it is not correct to consider local minima of the effective potential as metastable phases of theory at finite temperatures. Nevertheless such a minima of effective potentials may appear in gauge theories with one compact space coordinate. In this case these minima of the effective potential can be considered as real metastable states.

5 Acknowledgements

The author thanks J.-P. Blaizot and A.V. Smilga for stimulating discussions. This research was sponsored in part by the INTAS Grant 93-0283, CRDF Grant RP-2-132, and Swiss Grant 7SUPJ048716.

References

[1] J.I. Kapusta, Finite Temperature Field Theory, (Cambridge University Press, Cambridge, England 1989).

[2] N. Weiss, Phys.Rev \textbf{D24} (1981) 475; \textbf{D25} (1982) 423; R. Anishetty, J.Phys. \textbf{B10} (1985) 439; K.J. Dahlem, Z.Phys. \textbf{C29} (1985) 553. V.M.
Belyaev and V.L. Eletsky JETP Lett. 50 (1989) 55; V.M. Belyaev
Phys.Lett. 254B (1991) 153.

[3] A. Gocksch and R. Pisarski, Nucl.Phys. B402 (1993) 657.

[4] K. Kajantie, in Brookhaven 1991, Proceedings, Hot summer daze (1991)
68.

[5] V.M. Belyaev, I.I. Kogan, G.W. Semenoff, and Nathan Weiss, Phys.Lett.
B277 (1992) 331; A.V. Smilga Acta Phys.Polon. B25 (1994) 73.

[6] T. Bhattacharya, A. Gocksch, C.P. Korthals-Altes, R.D. Pisarski,
Nucl.Phys. B383 (1992) 497; C.P. Korthals Altes, N. Jay Watson,
Phys.Rev.Lett. 75 (1995) 2799.

[7] S. Bronoff, C.P. Korthals Altes, CERN-TH-96-164, In Minneapolis
1996, Continuous advances in QCD 185-195. e-Print Archive: hep-lat/9607016.

[8] A.V. Smilga, preprint TPI-MINN-96/23, NUC-MINN-96/12-T (1996),
to be published in Phys.Rep., E-print Archive: hep-ph/9612347.

[9] D. Gross, I.R. Klebanov, A. Matytsin, A.V. Smilga, Nucl.Phys. B461
(1996) 109.