Radial extensions in fractional Sobolev spaces

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Abstract

Given \(f : \partial(-1,1)^n \rightarrow \mathbb{R}\), consider its radial extension \(T f(X) := f(X/\|X\|_{\infty}), \forall X \in [-1,1]^n \setminus \{0\}\). In “On some questions of topology for \(S^1\)-valued fractional Sobolev spaces” (RACSAM 2001), the first two authors (HB and PM) stated the following auxiliary result (Lemma D.1). If \(0 < s < 1, 1 < p < \infty\) and \(n \geq 2\) are such that \(1 < sp < n\), then \(f \rightarrow Tf\) is a bounded linear operator from \(W^{s,p}(\partial(-1,1)^n)\) into \(W^{s,p}((-1,1)^n)\). The proof of this result contained a flaw detected by the third author (IS). We present a correct proof. We also establish a variant of this result involving higher order derivatives and more general radial extension operators. More specifically, let \(B\) be the unit ball for the standard Euclidean norm \(\|\cdot\|\) in \(\mathbb{R}^n\), and set \(U_\alpha f(X) := |X|^{\alpha} f(X/|X|), \forall X \in \overline{B} \setminus \{0\}, \forall f : \partial B \rightarrow \mathbb{R}\). Let \(\alpha \in \mathbb{R}, s > 0, 1 \leq p < \infty\) and \(n \geq 2\) be such that \((s-\alpha)p < n\). Then \(f \rightarrow U_\alpha f\) is a bounded linear operator from \(W^{s,p}(\partial B)\) into \(W^{s,p}(B)\).

In [1], the first two authors stated the following

**Lemma 1.** ([1, Lemma D.1]) Let \(0 < s < 1, 1 < p < \infty\) and \(n \geq 2\) be such that \(1 < sp < n\). Let

\[
Q := (-1,1)^n. \tag{1}
\]

Set

\[
T f(X) := f(X/\|X\|_{\infty}), \forall X \in \overline{Q} \setminus \{0\}, \forall f : \partial Q \rightarrow \mathbb{R}; \tag{2}
\]

here, \(\|\cdot\|_{\infty}\) is the sup norm in \(\mathbb{R}^n\). Then \(f \rightarrow Tf\) is a bounded linear operator from \(W^{s,p}(\partial Q)\) into \(W^{s,p}(Q)\).

The argument presented in [1] does not imply the conclusion of Lemma 1. Indeed, it is established in [1] (see estimate (D.3) there) that

\[
|T f|_{W^{s,p}(Q)}^p \leq C \int_{\partial Q} \int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x-y\|_{\infty}^{n+sp}} d\sigma(x) d\sigma(y).
\]

However, this does not imply the desired conclusion in Lemma 1, for which we need the stronger estimate

\[
|T f|_{W^{s,p}(Q)}^p \leq C \int_{\partial Q} \int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x-y\|_{\infty}^{n-1+sp}} d\sigma(x) d\sigma(y).
\]

In what follows, we establish the following slight generalization of Lemma 1.

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Lemma 2. Let $0 < s \leq 1$, $1 \leq p < \infty$ and $n \geq 2$ be such that $sp < n$. Let $Q$, $T$ be as in (1)–(2). Then $f \mapsto Tf$ is a bounded linear operator from $W^{s,p}(\partial Q)$ into $W^{s,p}(Q)$.

Lemma 2 can be generalized beyond one derivative, but for this purpose it is necessary to work on unit spheres arising from norms smoother that $\| \cdot \|_{\infty}$. We consider for example maps $f : \partial B \to \mathbb{R}$, with $B :=$ the Euclidean unit ball in $\mathbb{R}^n$.

Lemma 3. Let $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ and $n \geq 2$ be such that $(s-a)p < n$. Then $f \mapsto U_af$ is a bounded linear operator from $W^{s,p}(\partial B)$ into $W^{s,p}(B)$.

It is possible to establish directly Lemma 2 by adapting some arguments presented in Step 3 in the proof of Lemma 4.1 in [2]. However, we will derive it from Lemma 3.

Proof of Lemma 2 using Lemma 3

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi(X) := \begin{cases} |X| & \text{if } X \neq 0 \\ \|X\|_{\infty} & \text{if } X = 0 \end{cases}$, $\Lambda := \Phi|_{\overline{B}}$ and $\Psi := \Phi|_{\partial B}$.

Clearly, $\overline{\Lambda : \partial B \to \overline{Q}}$, $\Psi : \partial B \to \partial Q$ are bi-Lipschitz homeomorphisms and

$$Tf = [U_0(f \circ \Psi)] \circ \Lambda^{-1}. \quad (6)$$

Using (5) and the fact that $0 < s \leq 1$, we find that

$$f \mapsto f \circ \Psi$$ is a bounded linear operator from $W^{s,p}(\partial Q)$ into $W^{s,p}(\partial B) \quad (7)$$

and

$$g \mapsto g \circ \Lambda^{-1}$$ is a bounded linear operator from $W^{s,p}(B)$ into $W^{s,p}(Q). \quad (8)$$

We obtain Lemma 2 from (6)–(8) and Lemma 3 (with $a = 0$). The same argument shows that the conclusion of Lemma 2 holds for the unit sphere and ball of any norm in $\mathbb{R}^n$. \hfill \Box

Proof of Lemma 3

Consider $a$, $s$, $p$ and $n$ such that

$$a \in \mathbb{R}, s > 0, 1 \leq p < \infty, n \geq 2 \text{ and } (s-a)p < n. \quad (9)$$

Considering spherical coordinates on $B$, we obtain that

$$\|U_af\|_{L_p(B)}^p = \int_0^1 \int_{\partial B} r^{n-1} |U_af(rx)|^p \, d\sigma(x) \, dr$$

$$= \int_0^1 \int_{\partial B} r^{n-1+ap} |f(x)|^p \, d\sigma(x) \, dr = \frac{1}{n+ap} \|f\|_{L_p(\partial B)}^p. \quad (10)$$
Here, we have used the fact that, by (9), we have $n + ap > n - (s - a)p > 0$.
In view of (10), it suffices to establish the estimate

$$|U_a f|_{W^{s,p}(B)}^p = C \|f\|_{W^{s,p}(\partial B)}^p \quad \forall f \in W^{s,p}(\partial B),$$

for some appropriate $C = C_{a,s,p,n}$ and semi-norm $|\cdot|_{W^{s,p}(B)}$.

**Step 1. Proof of (11) when $0 < s < 1$.** We consider the standard Gagliardo semi-norm on $W^{s,p}(B)$. We have

$$|U_a f|_{W^{s,p}(B)}^p = \int_B \int_B \frac{|U_a f(X) - U_a f(Y)|^p}{|X - Y|^{n + sp}} \, dX dY
= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} \rho^{n-1} |U_a f(r x) - U_a f(r y)|^p \frac{1}{|r x - r y|^{n + sp}} \, d\sigma(x) d\sigma(y) \, dr \, d\rho
= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} \rho^{n-1} \frac{|r a f(x) - r a f(y)|^p}{|r x - r y|^{n + sp}} \, d\sigma(x) d\sigma(y) \, dr \, d\rho
= 2 \int_{\partial B} \int_{\partial B} \int_0^1 \frac{|r a f(x) - r a f(y)|^p}{|r x - r y|^{n + sp}} \, d\rho \, dr \, d\sigma(x) d\sigma(y).$$

With the change of variable $\rho = tr, t \in [0, 1]$, we find that

$$|U_a f|_{W^{s,p}(B)}^p = 2 \int_0^1 \rho^{n-(s-a)p-1} \, dr \int_{\partial B} \int_{\partial B} \int_0^1 t^{n-1} \frac{|f(x) - t a f(y)|^p}{|x - t y|^{n + sp}} \, dt \, d\sigma(x) d\sigma(y)
= \frac{2}{n-(s-a)p} \int_{\partial B} \int_{\partial B} \int_0^1 k(x,y,t) \, dtd\sigma(x) d\sigma(y),$$

with

$$k(x,y,t) := t^{n-1} \frac{|f(x) - t a f(y)|^p}{|x - t y|^{n + sp}}, \quad \forall x, y \in \partial B, \forall t \in [0,1].$$

In order to complete this step, it thus suffices to establish the estimates

$$I_1 := \int_{\partial B} \int_{\partial B} \int_0^{1/2} k(x,y,t) \, dtd\sigma(x) d\sigma(y) \leq C \|f\|_{L^p(\partial B)}^p, \quad (12)$$
$$I_2 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|f(x) - f(y)|^p}{|x - t y|^{n + sp}} \, dt \, d\sigma(x) d\sigma(y) \leq C \|f\|_{W^{s,p}(\partial B)}^p, \quad (13)$$
$$I_3 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{(1-t^a)^p f(y)^p}{|x - t y|^{n + sp}} \, dt \, d\sigma(x) d\sigma(y) \leq C \|f\|_{L^p(\partial B)}^p, \quad (14)$$

here, $|\cdot|_{W^{s,p}(\partial B)}$ is the standard Gagliardo semi-norm on $\partial B$.

In the above and in what follows, $C$ denotes a generic finite positive constant independent of $f$, whose value may change with different occurrences.

Using the obvious inequalities

$$|x - t y| \geq 1 - t \geq 1/2, \quad \forall x, y \in \partial B, \quad \forall t \in [0,1/2],$$

$$|f(x) - t^a f(y)| \leq (1 + t^a)(|f(x)| + |f(y)|),$$

and the fact that, by (9), we have $n + ap > 0$, we find that

$$I_1 \leq C \int_0^{1/2} (t^{n-1} + t^{n-1+ap}) \, dt \|f\|_{L^p(\partial B)}^p \leq C \|f\|_{L^p(\partial B)}^p,$$
so that (12) holds.

In order to obtain (13), it suffices to establish the estimate

$$\frac{1}{\sqrt{|x-y|^{n+sp}}} \leq \frac{C}{|x-y|^{n-1+sp}}, \quad \forall x, y \in \partial B. \quad (15)$$

Set $A := (x, y) \in [-1, 1]$. If $A \leq 0$, then $|x-y| \geq 1$, $\forall t \in [1/2, 1]$, and then (15) is clear.

Assuming $A \geq 0$ we find, using the change of variable $t = A + (1-A^2)^{1/2} \tau$,

$$\int_{1/2}^{1} \frac{1}{|x-y|^{n+sp}} \, dt \leq \int_{R} \frac{1}{|x-y|^{n+sp}} \, dt = \int_{R} \frac{1}{(1-A^2)(n-1+sp)2} \, dt$$

and thus (15) holds again. This completes the proof of (13).

In order to prove (14), we note that

$$|1-t^n|^p \leq C(1-t)^p, \quad \forall t \in [1/2, 1],$$

and that the integral

$$J := \int_{1/2}^{1} \int_{\partial B} \frac{(1-t)^p}{|x-y|^{n+sp}} \, d\sigma(x) \, dt$$

does not depend on $y \in \partial B$.

By the above, we have

$$I_3 \leq C \int_{1/2}^{1} \int_{\partial B} \int_{\partial B} \frac{(1-t)^p |f(y)|^p}{|x-y|^{n+sp}} \, d\sigma(x) \, d\sigma(y) \, dt = C J \|f\|_{L^p(\partial B)},$$

and thus (14) amounts to proving that $J < \infty$. Since $J$ does not depend on $y$, we may assume that $y = (0, \ldots, 0, 1)$. Expressing $J$ in spherical coordinates and using the change of variable $t = 1 - \tau$, $\tau \in [0, 1/2]$, we find that

$$J = C \int_{1/2}^{1} \int_{0}^{\pi} \frac{\tau^p \sin^{n-1}\theta}{(\tau^2 + 4(1-\tau)\sin^2 \theta/2)^{(n+sp)/2}} \, d\theta \, d\tau.$$ 

When $\tau \in [0, 1/2]$ and $\theta \in [0, \pi]$, we have

$$\frac{\tau^p \sin^{n-1}\theta}{(\tau^2 + 4(1-\tau)\sin^2 \theta/2)^{(n+sp)/2}} \leq C \frac{\tau^p \sin^{n-1}\theta}{(\tau + \sin \theta/2)^{n+sp}} \leq C \frac{\tau^p \sin^{n-1}\theta/2 \cos \theta/2}{(\tau + \sin \theta/2)^{n+sp}} \leq C (\tau + \sin \theta/2)^{p-sp-1} \cos \theta/2.$$ 

Inserting the last inequality into the formula of $J$, we find that

$$J \leq C \int_{0}^{\pi} \int_{0}^{\tau} (\tau + \sin \theta/2)^{p-sp-1} \cos \theta/2 \, d\theta \, d\tau$$

$$= C \int_{0}^{\pi} \int_{0}^{\tau} (\tau + \xi)^{p-sp-1} \, d\xi \, d\tau < \infty.$$
the latter inequality following from $p - sp > 0$. This completes the proof of (14) and Step 1.

**Step 2. Proof of (11) when $s \geq 1$.** We will reduce the case $s \geq 1$ to the case $0 \leq s < 1$. Using the linearity of $f \mapsto U_{a}f^{a}$ and a partition of unity, we may assume with no loss of generality that $\text{supp} f$ is contained in a spherical cap of the form $\{x \in \partial B; |x - e| < \varepsilon\}$ for some $e \in \partial B$ and sufficiently small $\varepsilon$. We may further assume that $e = (0, 0, \ldots, 0, 1)$, and thus

$$f \in W^{s,p}(\partial B; \mathbb{R}), \quad \text{supp} f \subset \mathcal{E} := \{x \in \partial B; |x - (0, 0, \ldots, 0, 1)| < \varepsilon\}. \quad (16)$$

Let

$$\mathcal{S} := \{x \in \partial B; |x - (0, 0, \ldots, 0, 1)| \leq 2\varepsilon\} \text{ and } \mathcal{H} := \mathbb{R}^{n-1} \times \{1\}.$$

Consider the projection $\Theta$ with vertex 0 of

$$\mathbb{R}^{n} := \{X = (X', X_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; X_n > 0\}$$

onto $\mathcal{H}$, given by the formula $\Theta(X', X_n) = (X'/X_n, 1)$. The restriction $\Pi$ of $\Theta$ to $\mathcal{S}$ maps $\mathcal{S}$ onto $\mathcal{N} := \mathcal{B} \times \{1\}$, with

$$\mathcal{B} := \{X' \in \mathbb{R}^{n-1}; |X'| \leq r := 2\varepsilon \sqrt{1 - \varepsilon^{2}/(1 - 2\varepsilon^{2})}\},$$

and is a smooth diffeomorphism between these two sets. We choose $\varepsilon$ such that $r = 1/2$, and thus $\mathcal{B} \subset \{X' \in \mathbb{R}^{n-1}; \|X'\|_{\infty} \leq 1/2\}$.

Set

$$g(X') := \begin{cases} |(X', 1)|^{\alpha} f(\Pi^{-1}(X', 1)), & \text{if } X' \in \mathcal{B} \\ 0, & \text{otherwise} \end{cases}. \quad (17)$$

By the above, there exist $C, C' > 0$ such that for every $f \in W^{s,p}(\partial B)$ satisfying (16), the function $g$ defined in (17) satisfies

$$C\|g\|_{W^{s,p}(\mathbb{R}^{n-1})} \leq \|f\|_{W^{s,p}(\partial B)} \leq C'\|g\|_{W^{s,p}(\mathbb{R}^{n-1})}. \quad (18)$$

On the other hand, set $\mathcal{E} := \{(tY', t); Y' \in \mathcal{B}, t > 0\}$ and

$$V_{a}g(X', X_n) := \begin{cases} (X_n)^{\alpha} g(X'/X_n), & \text{if } (X', X_n) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases}. \quad (18)$$

Then we have $U_{a}F(X', X_n) = V_{a}g(X', X_n), \forall (X', X_n) \in \overline{B} \setminus \{0\}$.

Write now $s = m + \sigma$, with $m \in \mathbb{N}$ and $0 \leq \sigma < 1$. When $s = m$, we consider, on $W^{s,p}(B)$, the semi-norm

$$|F|_{W^{s,p}(B)}^{p} = \sum_{\alpha \in \mathbb{N}^{n} \setminus \{0\} \atop |\alpha| \leq m} \|\partial^{\alpha} F\|_{L^{p}(B)}^{p}. \quad (19)$$

When $s$ is not an integer, we consider the semi-norm

$$|F|_{W^{s,p}(B)}^{p} = \sum_{\alpha \in \mathbb{N}^{n} \setminus \{0\} \atop |\alpha| \leq m} \|\partial^{\alpha} F\|_{L^{p}(B)}^{p} + \sum_{\alpha \in \mathbb{N}^{n} \setminus \{0\} \atop |\alpha| = m} |\partial^{\alpha} F|_{W^{s,p}(B)}^{p}. \quad (20)$$
(the semi-norm on $W^{\sigma,p}(B)$ is the standard Gagliardo one.)

By the above discussion, in order to obtain (11) it suffices to establish the estimate

$$|V_a g|_{W^{\sigma,p}(B)}^p \leq C \|g\|_{W^{\sigma,p}(\mathbb{R}^{n-1})}^p, \quad \forall g \in W^{\sigma,p}(\mathbb{R}^{n-1}) \text{ with } \text{supp } g \subset \mathcal{B}. \quad (21)$$

Let $\alpha \in \mathbb{N}^n \setminus \{0\}$ be such that $|\alpha| \leq m$. By a straightforward induction on $|\alpha|$, the distributional derivative $\partial^\alpha [V_a g]$ satisfies

$$\partial^\alpha [V_a g](X', X_n) = \sum_{|\beta'| \leq |\alpha|} V_{a-|\alpha|}[P_{a, |\beta'|} \partial^{\beta'} g](X', X_n) \text{ in } \mathcal{D}'(B \setminus \{0\}), \quad (22)$$

for some appropriate polynomials $P_{a, |\beta'|}(Y')$, $Y' \in \mathbb{R}^{n-1}$, depending only on $a \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$ and $\beta' \in \mathbb{N}^{n-1}$.

Thanks to the fact that $g(X'/X_n) = 0$ when $(X', X_n) \not\in \mathcal{C}$, we find that for any such $\alpha$ we have

$$\int_B |\partial^\alpha [V_a g]|^p \, dx \leq C \sum_{|\beta'| \leq |\alpha|} \int_{\mathcal{C} \cap Q} (X_n)^{(a-|\alpha|)p} |\partial^{\beta'} g(X'/X_n)|^p \, dX' dX_n$$

$$= C \frac{1}{n + (\alpha - |\alpha|)p} \sum_{|\beta'| \leq |\alpha|} \int_{\mathcal{B}} |\partial^{\beta'} g(Y')|^p \, dY'. \quad (23)$$

Here, we rely on

$$\int_0^1 (X_n)^{n-1+(\alpha - |\alpha|)p} \, dX_n = \frac{1}{n + (\alpha - |\alpha|)p} < \infty,$$

thanks to the assumption (9), which implies that $(|\alpha| - \alpha)p < n$.

Using (23), the fact that $V_a g \in W^{m,p}_{\text{loc}}(B \setminus \{0\})$ and the assumption that $n \geq 2$, we find that the equality (22) holds also in $\mathcal{D}'(B)$, that $V_a g \in W^{m,p}(B)$ and that

$$\|V_a g\|_{W^{m,p}(B)}^p \leq C \|g\|_{W^{m,p}(\mathbb{R}^{n-1})}^p, \quad \forall g \in W^{m,p}(\mathbb{R}^{n-1}) \text{ with } \text{supp } g \subset \mathcal{B}. \quad (24)$$

In particular, (21) holds when $s$ is an integer.

Assume next that $s$ is not an integer. In view of (18), (22) and (24), estimate (21) will be a consequence of

$$|V_h (P h)|_{W^{\sigma,p}(B)}^p \leq C \|h\|_{W^{\sigma,p}(\mathbb{R}^{n-1})}^p, \quad \forall h \in W^{\sigma,p}(\mathbb{R}^{n-1})$$

$$\text{with } \text{supp } h \subset \mathcal{B}, \quad (25)$$

under the assumptions

$$0 < \sigma < 1, \quad 1 \leq p < \infty, \quad n \geq 2, \quad (\sigma - b)p < n \quad (26)$$

and

$$P \in C^\infty(\mathbb{R}^{n-1}). \quad (27)$$

(Estimate (25) is applied with $b := a - m$, $P := P_{a,|\beta'|}$ and $h := \partial^{\beta'} g.$)
In turn, estimate (25) follows from Step 1. Indeed, consider \( k : \partial B \to \mathbb{R} \) such that \( \text{supp} \, k \subseteq \mathbb{B} \) and \( U_b k = V_b [P h] \). (The explicit formula of \( k \) can be obtained by “inverting” the formula (17).) By Step 1 and (18), we have

\[
|V_b[P h]|_{W^{s,p}(B)}^p \leq |U_b k|_{W^{s,p}(B)}^p \leq C \| k \|_{W^{s,p}(\partial B)}^p \leq C \| P h \|_{W^{s,p}(\mathbb{R}^{n-1})}^p \leq C \| h \|_{W^{s,p}(\mathbb{R}^{n-1})}^p.
\]

This completes Step 2 and the proof of Lemma 3. \( \square \)

Finally, we note that the assumptions of Lemma 3 are optimal in order to obtain that \( U_a f \in W^{s,p}(B) \).

**Lemma 4.** Let \( a \in \mathbb{R} \), \( s > 0 \), \( 1 \leq p < \infty \) and \( n \geq 2 \). Assume that for some measurable function \( f : \partial B \to \mathbb{R} \) we have \( U_a f \in W^{s,p}(B) \). Then:

1. \( f \in W^{s,p}(\partial B) \).
2. If, in addition, \( U_a f \) is not a polynomial, we deduce that \( (s-a)p < n \).

**Proof.** 1. Let \( G : (1/2, 1) \times \partial B \to \mathbb{R} \), \( G(r, x) := r^{-a} U_a f (r x) \). If \( U_a f \in W^{s,p}(B) \), then \( G \in W^{s,p}((1/2, 1) \times \partial B) \). In particular, we have \( G(r, \cdot) \in W^{s,p}(\partial B) \) for a.e. \( r \). Noting that \( G(r, x) = f(x) \), we find that \( f \in W^{s,p}(\partial B) \).

2. Let

\[ \Omega_j := \{ x \in \mathbb{R}^n ; 2^{-j-1} < |x| < 2^{-j} \}, \quad j \in \mathbb{N}. \]

We consider on each \( \Omega_j \) a semi-norm as in (19)–(20). Assuming that \( U_a f \) is not a polynomial, we have \( |U_a f|_{W^{s,p}(\Omega_0)} > 0 \). By scaling and the homogeneity of \( U_a f \), we have

\[ |U_a f|_{W^{s,p}(\Omega_j)}^p = 2^{-j[(s-a)p-n]} |U_a f|_{W^{s,p}(\Omega_0)}^p. \]

Assuming that \( U_a f \in W^{s,p}(B) \), we find that

\[ \infty > |U_a f|_{W^{s,p}(B)}^p \geq \sum_{j \geq 0} |U_a f|_{W^{s,p}(\Omega_j)}^p = \sum_{j \geq 0} 2^{-j[(s-a)p-n]} |U_a f|_{W^{s,p}(\Omega_0)}^p > 0, \]

so that \( (s-a)p < n \). \( \square \)

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