Lower bounds of Lipschitz constants on foliations

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Abstract
In this paper we consider Llarull’s theorem in the foliation case and get a lower bound of the Lipschitz constant of the map $M \to S^n$ in the foliation case under the spin condition.

1 Introduction

In [2], M. Gromov conjectured the following.

Conjecture 1.1 (Gromov) Let $g$ be a Riemannian metric on $S^n$ such that $g \geq g_0$ where $g_0$ is the standard metric of constant curvature. Then the scalar curvature $k_g$ must become small somewhere, more precisely, $\inf k_g \leq c(n)k_{g_0}$, where $c(n) \leq 1$ is a constant that depends on the dimension $n$, with best constant when $c(n) = 1$.

A map $f : M \to N$ between Riemannian manifolds is said to be $\varepsilon$-contracting if $\|f_*v\| \leq \varepsilon\|v\|$ for tangent vectors $v$ to $M$ (cf. [4]).

The normalized scalar curvature of a manifold $M$ of dimension $n$ is defined as

$$\tilde{k} = \frac{k}{n(n-1)},$$

where $k$ is the usual scalar curvature.

In [6], M. Llarull proved the following theorem which confirmed Gromov’s conjecture.

Theorem 1.2 ([6]) Let $M$ be a compact Riemannian spin manifold of dimension $n$. Suppose there exists a 1-contracting map $f : (M, g) \to (S^n, g_0)$ of non-zero degree. Then either there exists $x \in M$ with $\tilde{k}_g(x) < 1$, or $M \equiv S^n$ and $f$ is an isometry.

Recall that for a map $f : M \to S^n$ the Lipschitz constant (cf. [3,4]) is defined by

$$\text{Lip}(f) = \sup_{x_1 \neq x_2} \frac{\text{dist}_{S^n}(f(x_1), f(x_2))}{\text{dist}_M(x_1, x_2)}.$$  \hfill (1.1)

In [3, Sect. 3], Gromov pointed out that Theorem 1.2 is related to the problems of the Lipschitz constants of the maps $M \to S^n$.
Let $F$ be an integrable subbundle of the tangent vector bundle $TM$ of a closed smooth manifold $M$. Let $g^F$ be a metric on $F$, and $k^F_g \in C^\infty(M)$ be the associated leafwise scalar curvature (cf. [8, (0.1)]). Let $\tilde{k}^F_g$ be the normalized leafwise scalar curvature, i.e.

$$\tilde{k}^F_g = \frac{k^F_g}{rkF(rkF - 1)}.$$

In this paper, we prove the following theorem which partly generalizes Theorem 1.2 in the foliation case.

**Theorem 1.3** Let $M$ be a closed Riemannian manifold of dimension $n$ and $F$ be a foliation on $M$. Suppose $TM$ or $F$ is spin and there exists a smooth map $f : (M, g) \to (S^n, g_0)$ of non-zero degree such that for any $v \in F$, $\|f_*(v)\| \leq \|v\|$. Then there exists $x \in M$ with $\tilde{k}^F_g(x) \leq 1$.

From Theorem 1.3, one sees that the leafwise scalar curvature is also related to the lower bounds of the Lipschitz constants. If $\tilde{k}^F_g > 1$, then there exists $v \in F$, such that $\|f_*(v)\| > \|v\|$ on a small neighborhood. So from Theorem 1.3, one has the following theorem.

**Theorem 1.4** Let $M$ be a closed Riemannian manifold of dimension $n$ and $F$ be a foliation on $M$. Suppose $TM$ or $F$ is spin and $\tilde{k}^F_g > 1$, then for smooth maps $f : (M, g) \to (S^n, g_0)$ of non-zero degree, we have $\text{Lip}(f) > 1$.

Our proof of Theorem 1.3 combines the methods in [6,8] and [9]. It is based on deforming (twisted) sub-Dirac operators on the Connes fibration. It will be carried out in Sect. 2.

**2 Proof of theorem 1.3**

In this section, we give a proof of the main theorem. We give the details for the case $TM$ is spin, $F$ is spin is similar.

If there does not exist any point such that $\tilde{k}^F_g \leq 1$, then there exists $\delta > 0$ such that $\tilde{k}^F_g(x) - 1 \geq \delta$, for any $x \in M$.

**2.1 The dimension of $M$ is even**

Over $(S^{2n}, g_0)$, we have the spinor bundle (cf. [5])

$$E_0 = P_{Spin_{2n}}(S^{2n}) \times \lambda \ Cl_{2n},$$

with the induced metric and connection from $(S^{2n}, g_0)$. Fix $x \in S^{2n}$ and choose local pointwise orthonormal tangent vector fields around $x, \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}\}$ such that $(\nabla \varepsilon_k)_x = 0$. Let $\omega_0$

$$\omega_0 = i^n \varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_{2n}.$$

Then $\omega_0$ gives the splitting

$$E_0 = E_0^+ \oplus E_0^-,$$

into the $+1$ and $-1$ eigenspaces of $\omega_0$.

Fix $p \in M$, such that $f(p)$ is a regular value. Then as in [6], we can choose the frames near $p$ and $f(p)$ as follows. Let $\{f_1, \ldots, f_{2n}\}$ be a $g$-orthonormal tangent frame near $p \in M$
such that \((\nabla f_k)_{\rho} = 0\) for each \(k\) and \(\{f_1, \ldots, f_{rkF}\}\) be a basis of \(F\). Let \(\{e_1, \ldots, e_{2n}\}\) be a \(g_0\)-orthonormal tangent frame near \(f(p) \in S^{2n}\) such that \((\nabla e_k)_{(p)} = 0\) for each \(k\). Moreover, the bases \(\{f_1, \ldots, f_{2n}\}\) and \(\{e_1, \ldots, e_{2n}\}\) can be chosen so that \(e_j = \lambda_j f_* f_j\), \(1 \leq j \leq rkF\) for appropriate \(\{\lambda_j\}_{j=1}^{rkF}\).

Following [1, Sect. 5] (cf. [8, Sect. 2.1]), let \(\pi : M \rightarrow M\) be the Connes fibration over \(M\) such that for any \(x \in M\), \(M_x = \pi^{-1}(x)\) is the space of Euclidean metrics on the linear space \(T_x M / F_x\). Let \(T^V M\) denote the vertical tangent bundle of the fibration \(\pi : M \rightarrow M\). Then it carries a natural metric \(g(T^V M)\).

By using the Bott connection on \(TM / F\), which is leafwise flat, one lifts \(F\) to an integrable subbundle \(\mathcal{F}\) of \(TM\). Then \(g^F\) lifts to a Euclidean metric \(g^F = \pi^* g^F\) on \(\mathcal{F}\).

Let \(\mathcal{F}^\perp_1 \subset TM\) be a subbundle, which is transversal to \(\mathcal{F} \oplus T^V M\), such that we have a splitting \(TM = (\mathcal{F} \oplus T^V M) \oplus \mathcal{F}^\perp_1\). Then \(\mathcal{F}^\perp_1\) can be identified with \(TM/(\mathcal{F} \oplus T^V M)\) and carries a canonically induced metric \(g^{\mathcal{F}^\perp_1}\). We denote \(\mathcal{F}^\perp_2 = T^V M\).

Set \(E = f^* E_0\). Let \(E = \pi^* E\) be the lift of \(E\) which carries the lifted Hermitian metric \(g^E = \pi^* g^E\) and the lifted Hermitian connection \(\nabla^E = \pi^* \nabla^E\). Let \(R^E = (\nabla^E)^2\) be the curvature of \(\nabla^E\).

For any \(\beta, \varepsilon > 0\), following [8, (2.15)], let \(g^{T^V M}_{\beta, \varepsilon}\) be the metric on \(TM\) defined by the orthogonal splitting,

\[
TM = \mathcal{F} \oplus \mathcal{F}^\perp_1 \oplus \mathcal{F}^\perp_2, \quad g^{T^V M}_{\beta, \varepsilon} = \beta^2 g^\mathcal{F} \oplus \frac{g^{\mathcal{F}^\perp_1}}{\varepsilon^2} \oplus g^{\mathcal{F}^\perp_2}.
\]

Now we replace the sub-Dirac operator constructed in [8, (2.16)] by the obvious twisted (by \(E\)) analogue ([9, (1.3)])

\[
D^E_{\mathcal{F} \oplus \mathcal{F}^\perp_1, \beta, \varepsilon} : \Gamma \left( S_{\beta, \varepsilon} \left( \mathcal{F} \oplus \mathcal{F}^\perp_1 \right) \otimes \Lambda^* \left( \mathcal{F}^\perp_2 \right) \otimes E \right) \rightarrow \Gamma \left( S_{\beta, \varepsilon} \left( \mathcal{F} \oplus \mathcal{F}^\perp_1 \right) \otimes \Lambda^* \left( \mathcal{F}^\perp_2 \right) \otimes E \right).
\]

(2.3)

Take a metric on \(TM / F\). This is equivalent to taking an embedded section \(s : M \hookrightarrow M\) of the Connes fibration \(\pi : M \rightarrow M\). Then we have a canonical inclusion \(s(M) \subset M\).

For any \(p \in \mathcal{M} \setminus \{M\}\), we connect \(p\) and \(s(\pi(p))\) in \(s(M)\) by the unique geodesic in \(\mathcal{M}_{\pi(p)}\). Let \(\sigma(p) \in \mathcal{F}^\perp_2 |_p\) denote the unit vector tangent to this geodesic. Let \(\rho(p) = d\mathcal{M}_{\pi(p)}(p, s(\pi(p)))\) denote the length of this geodesic.

Let \(f : [0, 1] \rightarrow [0, 1]\) be a smooth function such that \(f(t) = 0\) for \(0 \leq t \leq \frac{1}{4}\), while \(f(t) = 1\) for \(\frac{1}{2} \leq t \leq 1\). Let \(h : [0, 1] \rightarrow [0, 1]\) be a smooth function such that \(h(t) = 1\) for \(0 \leq t \leq \frac{3}{4}\), while \(h(t) = 0\) for \(\frac{7}{8} \leq t \leq 1\).

For any \(R > 0\), denote

\[
\mathcal{M}_R = \{ p \in \mathcal{M} : \rho(p) \leq R \}. \tag{2.4}
\]

Then \(\mathcal{M}_R\) is a smooth manifold with boundary.

On the other hand, the following formula holds on \(\mathcal{M}_R\) (cf. [8, (2.28)], [9, (1.4)])

\[
\left( D^E_{\mathcal{F} \oplus \mathcal{F}^\perp_1, \beta, \varepsilon} \right)^2 = -\Delta^E_{\beta, \varepsilon} + \frac{k^E}{4\beta^2} + \frac{1 + \frac{\varepsilon^2}{\beta^2}}{2\beta^2} \sum_{i, j = 1}^{rkF} c_{\beta, \varepsilon}(\beta f_i) c_{\beta, \varepsilon}(\beta^{-1} f_j) R^E(f_i, f_j) + O_R \left( \frac{1 + \frac{\varepsilon^2}{\beta^2}}{\beta^2} \right), \tag{2.5}
\]

where \(-\Delta^E_{\beta, \varepsilon} \geq 0\) is the corresponding Bochner Laplacian, \(k^E = \pi^* k^E\) and \(f_1, \ldots, f_{rkF}\) is an orthonormal basis of \((\mathcal{F}, g^\mathcal{F})\).
Since [6, Lemma 4.3] and [6, Lemma 4.5] hold for fixed \((i, j)\), proceeding as the computations in [6, Lemmas 4.3, 4.5], for any \(\phi \in \Gamma(S_{\beta, \varepsilon}(F \oplus F_1^+) \otimes \Lambda^* (F_2^+) \otimes \mathcal{E})\) supported in \(\mathcal{M}_R\), one has

\[
\left\langle \frac{1}{2\beta^2} \sum_{i,j=1}^{\text{rk} F} R^\mathcal{E} (f_i, f_j) c_{\beta, \varepsilon} (\beta^{-1} f_i) c_{\beta, \varepsilon} (\beta^{-1} f_j) \phi, \phi \right\rangle \geq -\frac{1}{4\beta^2} \text{rk} F (\text{rk} F - 1) \| \phi \|^2.
\] (2.6)

Then by (2.5) and (2.6), for any \(\phi \in \Gamma(S_{\beta, \varepsilon}(F \oplus F_1^+) \otimes \Lambda^* (F_2^+) \otimes \mathcal{E})\) supported in \(\mathcal{M}_R\), one gets

\[
\left\langle \left( D^\mathcal{E}_{F \oplus F_1^+, \beta, \varepsilon} \right)^2 \phi, \phi \right\rangle \geq -\frac{1}{4\beta^2} \text{rk} F (\text{rk} F - 1) \left( \tilde{k}_g^\mathcal{E} - 1 \right) \| \phi \|^2 + O_R \left( \frac{1}{\beta} + \varepsilon^2 \frac{1}{\beta^2} \right) \| \phi \|^2.
\] (2.7)

From (2.7), proceeding as the proof of [8, Lemma 2.4], one can get the following analogue inequality of [8, (2.22)].

**Lemma 2.1** There exist \(C_0, R_0 > 0\), such that for any (fixed) \(R \geq R_0\), when \(\beta, \varepsilon > 0\) (which may depend on \(R\)) are small enough, for any \(\phi \in \Gamma(S_{\beta, \varepsilon}(F \oplus F_1^+) \otimes \Lambda^* (F_2^+) \otimes \mathcal{E})\) supported in \(\mathcal{M}_R\), one has

\[
\left\| \left( D^\mathcal{E}_{F \oplus F_1^+, \beta, \varepsilon} + \frac{1}{\beta} \tilde{c} (\sigma) \right) \phi \right\| \geq \frac{C_0 \sqrt{\delta}}{\beta} \| \phi \|.
\] (2.8)

Next we recall the construction of the operator \(P^\mathcal{E}_{R, \beta, \varepsilon}\) from [8].

Let \(\partial \mathcal{M}_R\) bound another oriented manifold \(\mathcal{N}_R\) so that \(\bar{\mathcal{N}}_R = \mathcal{M}_R \cup \mathcal{N}_R\) is a closed manifold. Let \(H\) be a Hermitian vector bundle over \(\mathcal{M}_R\) such that \((S_{\beta, \varepsilon}(F \oplus F_1^+) \otimes \Lambda^* (F_2^+) \otimes \mathcal{E})_+ \oplus H\) is a trivial vector bundle near \(\partial \mathcal{M}_R\), under the identification \(\tilde{c}(\sigma) + \text{Id}_H\).

By obviously extending the above trivial vector bundles to \(\mathcal{N}_R\), we get a \(\mathbb{Z}_2\)-graded Hermitian vector bundle \(\xi = \xi_+ \oplus \xi_-\) over \(\bar{\mathcal{N}}_R\) and an odd self-adjoint endomorphism \(V = v + v^* \in \Gamma(\text{End}(\xi))\) (with \(v : \Gamma(\xi_+) \to \Gamma(\xi_-), v^*\) being the adjoint of \(v\)) such that

\[
\xi_\pm = (S_{\beta, \varepsilon}(F \oplus F_1^+) \otimes \Lambda^* (F_2^+) \otimes \mathcal{E})_\pm \oplus H
\] (2.9)

over \(\mathcal{M}_R\), \(V\) is invertible on \(\mathcal{N}_R\) and

\[
V = \frac{1}{\beta} \tilde{c}(\sigma) + \text{Id}_H
\] (2.10)
on \(\mathcal{M}_R\), which is invertible on \(\mathcal{M}_R \setminus \mathcal{M}_R^C\).

Recall that \(h(\tilde{c})\) vanishes near \(\partial \mathcal{M}_R\). We extend it to a function on \(\bar{\mathcal{N}}_R\) which equals to zero on \(\mathcal{N}_R\), and we denote the resulting function on \(\bar{\mathcal{N}}_R\) by \(\tilde{h}_R\). Let \(\pi_{\bar{\mathcal{N}}_R} : \bar{\mathcal{N}}_R \to \bar{\mathcal{N}}_R\) be the projection of the tangent bundle of \(\bar{\mathcal{N}}_R\). Let \(\gamma_{\bar{\mathcal{N}}_R} \in \text{Hom}(\pi_{\bar{\mathcal{N}}_R}^*, \xi_+, \pi_{\bar{\mathcal{N}}_R}^*, \xi_-)\) be the symbol defined by

\[
\gamma_{\bar{\mathcal{N}}_R}(p, w) = \pi_{\bar{\mathcal{N}}_R}^* \left( \sqrt{-1} \bar{h}_R^2 c_{\beta, \varepsilon}(w) + v(p) \right) \text{ for } p \in \bar{\mathcal{N}}_R, \ w \in T_p \bar{\mathcal{N}}_R.
\] (2.10)

By (2.10) and (2.11), \(\gamma_{\bar{\mathcal{N}}_R}\) is singular only if \(w = 0\) and \(p \in \mathcal{M}_R\). Thus \(\gamma_{\bar{\mathcal{N}}_R}\) is an elliptic symbol.
On the other hand, it is clear that \( \tilde{h}_R D_{\mathcal{F}_p+\mathcal{F}_1^-}^e \tilde{h}_R \) is well defined on \( \tilde{N}_R \) if we define it to equal to zero on \( \tilde{N}_R \setminus \mathcal{M}_R \).

Let \( A : L^2(\xi) \to L^2(\xi) \) be a second order positive elliptic differential operator on \( \tilde{N}_R \) preserving the \( \mathbb{Z}_2 \)-grading of \( \xi = \xi_+ \oplus \xi_- \), such that its symbol equals to \( |\eta|^2 \) at \( \eta \in T \tilde{N}_R \).

Let \( P_{\tilde{N}_R}^e : L^2(\xi) \to L^2(\xi) \) be the zeroth order pseudodifferential operator on \( \tilde{N}_R \) defined by

\[
P_{\tilde{N}_R}^e = A^{-\frac{1}{4}} \tilde{h}_R D_{\mathcal{F}_p+\mathcal{F}_1^-}^e \tilde{h}_R A^{-\frac{1}{4}} + \frac{V}{\beta}. \tag{2.12}
\]

Let \( P_{\tilde{N}_R}^e : L^2(\xi+) \to L^2(\xi+) \) be the obvious restriction. Moreover, the analogue of [8, (2.34)] now takes the form

\[
\text{ind} \left( P_{\tilde{N}_R}^e, (\xi) \right) = (\tilde{A}(TM) \text{ch}(E), [M]). \tag{2.13}
\]

For any \( 0 \leq t \leq 1 \), set

\[
P_{\tilde{N}_R}^e(t) = A^{-\frac{1}{4}} \tilde{h}_R D_{\mathcal{F}_p+\mathcal{F}_1^-}^e \tilde{h}_R A^{-\frac{1}{4}} + \frac{tv}{\beta} + A^{-\frac{1}{4}}(1-t)vA^{-\frac{1}{4}}. \tag{2.14}
\]

Then \( P_{\tilde{N}_R}^e(t) \) is a smooth family of zeroth order pseudodifferential operators such that the corresponding symbol \( \gamma(P_{\tilde{N}_R}^e(t)) \) is elliptic for \( 0 < t \leq 1 \). Thus \( P_{\tilde{N}_R}^e(t) \) is a continuous family of Fredholm operators for \( 0 < t \leq 1 \) with \( P_{\tilde{N}_R}^e(1) = P_{\tilde{N}_R}^e \).

By Lemma 2.1 and [8, Lemma 2.4(ii)], proceeding as the proof of [8, Proposition 2.5], one has the following proposition.

**Proposition 2.2** There exist \( R, \beta, \varepsilon > 0 \) such that the following identity holds:

\[
\dim \left( \ker \left( P_{\tilde{N}_R}^e, (0) \right) \right) = \dim \left( \ker \left( P_{\tilde{N}_R}^e, (0)^* \right) \right) = 0. \tag{2.15}
\]

By (2.13), Proposition 2.2 and [8, Theorem 0.1], one has

\[
0 = \left( \tilde{A}(TM) \text{ch}(E^+), [M] \right) = \text{rk}(E_0^+, \tilde{A}(M)) + \left( \tilde{A}(TM) f^* \left( \text{ch}(E_0^+) - \text{rk}(E_0^+) \right), [M] \right) = \text{deg}(f) \left[ \text{ch}(E_0^+), S^{2n} \right],
\]

which contradicts with \( \text{deg}(f) \left[ \text{ch}(E_0^+), S^{2n} \right] \neq 0 \).

### 2.2 The dimension of \( M \) is odd

Let \( M \) be a compact spin manifold of dimension \( 2n-1 \), with Riemannian metric \( g \). Let \( S^{2n-1}_r \) be \( (2n-1) \)-sphere of radius \( r \) with the standard metric \( g_0 \). Let \( F \) be a foliation on \( M \).

Let \( f : M \to S^{2n-1} \) be a map of non-zero degree and \( f_*|_{F} \) is 1-contracting.

Consider

\[
M \times S^1 \xrightarrow{f \times \frac{1}{r} \text{id}} S^{2n-1} \times S^1 \xrightarrow{h} S^{2n-1} \times S^1 \cong S^{2n}, \tag{2.17}
\]

where \( S^1_r \) is the one-dimensional sphere of radius \( r \), \( f \times \frac{1}{r} \text{id} \) is defined as \( (f \times \frac{1}{r} \text{id})(p, t) = (f(p), \frac{1}{r} t) \), \( (p, t) \in M \times S^1 \), and \( h \) is map of non-zero degree. \( h|_{F \times S^1} \) is 1-contracting.

Consider now the following metric. On \( M \times S^1 \), \( g + ds^2 \) where \( ds^2 \) is the standard metric on \( S^1 \), on \( S^{2n-1} \times S^1 \), \( g_0 + ds^2 \) where \( ds^2 \) is the standard metric on \( S^1 \), and on \( S^{2n} \), \( \tilde{g} \) is the standard metric on the unit sphere.

\( \tilde{g} \) Springer
The composed map \( \tilde{f} = h \circ (f \times \frac{1}{r} id) : M^{2n-1} \times S^1_r \to S^{2n} \) is of non-zero degree. \( \tilde{f}|_{F \times S^1} \) is also 1-contracting, for \( v \in F \),

\[
\| \tilde{f}_* (v, t) \| = \| h_* (f_* v, \frac{t}{r}) \| \leq \| f_* v \| + \| \frac{t}{r} \| \leq \| v \| + \frac{1}{r} \| t \| \leq \| v \| + \| t \|.
\] (2.18)

We assume \( r > 1 \).

We can now apply the same method used for the even-dimensional case. Construct complex spinor bundles \( S \) over \( M^{2n-1} \times S^1_r \) and \( E_0 \) over \( S^{2n} \), respectively; and consider the bundle \( S \otimes E \) over \( M^{2n-1} \times S^1_r \), where \( E = f_* E_0 \).

Fix \( x \in M^{2n-1} \times S^1_r \), such that \( \tilde{f}(x) \) is a regular value. As before, we can choose the frames near \( x \) and \( \tilde{f}(x) \) as follows. Let \( \{ f_1, \ldots, f_{2n-1}, f_{2n} \} \) be a \((g + ds^2)\)-orthonormal tangent frame near \( x \in M^{2n-1} \times S^1_r \) such that \((\nabla f_k)_x = 0\) for each \( k, f_1, \ldots, f_{2n-1} \) are tangent to \( M^{2n-1} \), \( f_{2n} \) is tangent to \( S^1_r \) and \( \{ f_1, \ldots, f_{rkF} \} \) is a basis of \( F \). Let \( \{ e_1, \ldots, e_{2n} \} \) be a \( g_0 + ds^2 \)-orthonormal tangent frame near \( \tilde{f}(x) \in S^{2n} \) such that \((\nabla e_k) \tilde{f}(x) = 0\) for each \( k \). Moreover, the bases \( \{ f_1, \ldots, f_{2n} \} \) and \( \{ e_1, \ldots, e_{2n} \} \) can be chosen so that \( e_j = \lambda_j f_* f_j \), \( 1 \leq j \leq rkF \), \( j = 2n \) for appropriate \( \lambda_j \) and \( \lambda_{2n} \).

Therefore, we can find positive scalars \( \{ \lambda_i \}_{i=1}^{rkF} \) and \( \lambda_{2n} \) such that \( e_i = \lambda_i \tilde{f}_* f_i \). Then we have that

\[
1 = \tilde{g} (e_i, e_j) = \tilde{g} (\lambda_i \tilde{f}_* f_i, \lambda_j \tilde{f}_* f_j) = \lambda^2 \tilde{g} (f_* f_i, f_* f_j).
\]

Thus for \( 1 \leq i \leq rkF \),

\[
1 = \lambda^2 \tilde{g} (f_* f_i, f_* f_j) \leq \lambda^2 g_0 (f_* f_i, f_* f_j) \leq \lambda^2 g (e_i, e_i) = \lambda^2
\] (2.19)

and \( 1 \leq \lambda^2 \). For \( i = 2n \),

\[
1 = \lambda^2 \tilde{g} (f_* f_{2n}, f_* f_{2n}) \leq \lambda^2 g_0 (f_* f_{2n}, f_* f_{2n}) \leq \lambda^2 g (e_i, e_i) = \lambda^2
\]

Then

\[
r^2 \leq \lambda^2
\]

In this case, (2.7) is replaced by

\[
\left( D_{F \oplus F_1^+, \beta, \varepsilon}^\beta \phi, \phi \right)^2 (2.20)
\]

\[
\geq \frac{1}{4} \frac{rkF}{rkF - 1} \left( \tilde{k}_g^F \left( -1 - \frac{2}{(rkF - 1)r} \right) \right) \| \phi \|^2 + O_{r,e} \left( \frac{1}{\beta + \varepsilon^2} \right) \| \phi \|^2.
\]

If \( \tilde{k}_g^F > 1 \), since (2.20) is valid for all \( r > 1 \), one also gets \( \text{ind} \left( P_{R, \beta, \varepsilon, +}^{\varepsilon, +} \right) = 0 \). But the Atiyah–Singer index theorem gives (see (2.13), (2.16))

\[
\text{ind} \left( P_{R, \beta, \varepsilon, +}^{\varepsilon, +} \right) \neq 0.
\]

As in [8, Sect. 2.5], the same proof applies for the case where \( F \) is spin, with an obvious modification of the (twisted) sub-Dirac operators (cf. [8, (2.58)]), using [1, Theorem 0.2] in (2.16).
Remark 2.3 (cf. [4,6]) Recall that a map $f : M \to N$ between Riemannian manifold is $(\varepsilon, \Lambda^k)$-contracting if
\[
\| f^* \alpha \| \leq \varepsilon \| \alpha \|, \quad \alpha \in \Lambda^k(N).
\] (2.21)
Note that “1-contracting” means $(1, \Lambda^1)$-contracting.

We have the following immediate consequence.

Theorem 2.4 Let $M$ be a closed Riemannian manifold of dimension $n$ and $F$ be a foliation on $M$. Suppose $TM$ or $F$ is spin and there exists a smooth map $f : (M, g) \to (S^n, g_0)$ of non-zero degree such that for any $v, w \in F$, $\| f_* v \wedge f_* w \| \leq \| v \wedge w \|$. Then there exits $x \in M$ with $k_F g_0(x) \leq 1$.

Proof It follows from the proof of Theorem 1.3. We only need to point out that \( \{ \lambda_i \}_{i=1}^{rkF} \) satisfy
\[
1 = \| e_i \wedge e_j \|_{g_0} = \| \lambda_i f_* f_i \wedge \lambda_j f_* f_j \|_{g_0} = \lambda_i \lambda_j \| f_* (f_i \wedge f_j) \|_{g_0} \leq \lambda_i \lambda_j \| f_i \wedge f_j \|_{g} = \lambda_i \lambda_j.
\] (2.22)
Thus $\lambda_i \lambda_j \geq 1$. Then proceeding as the proof of Theorem 1.3, we get Theorem 2.4. \( \square \)

Remark 2.5 In Theorem 1.2, Llarull also studied the case $k_g(x) \equiv 1$. In our foliation case, if $k_F^g(x) \equiv 1$, maybe it can be treated by combining the methods in [6] and [7]. We will study it in a future paper.

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