LAGRANGIAN EMBEDDINGS OF THE KLEIN BOTTLE
AND
COMBINATORIAL PROPERTIES OF MAPPING CLASS GROUPS

VSEVOLOD V. SHEVCHISHIN

ABSTRACT. In this paper we prove the non-existence of Lagrangian embeddings of the Klein bottle \( K \) in \( \mathbb{C}P^2 \). We exploit the existence of a special embedding of \( K \) in a symplectic Lefschetz pencil \( pr : X \rightarrow S^2 \) and study its monodromy. As the main technical tool, we develop the combinatorial theory of mapping class groups. The obtained results allow us to show that in the case when the class \([K] \in H_2(X, \mathbb{Z}_2)\) is trivial the monodromy of \( pr : X \rightarrow S^2 \) must have special form. Finally, we show that such a monodromy cannot be realised in the case \( \mathbb{C}P^2 \).

0. Introduction

In \([N-1]\), Stefan Nemirovski proposed a proof of the non-existence of a Lagrangian embedding of the Klein bottle \( K := \mathbb{R}P^2 \# \mathbb{R}P^2 \) into \( \mathbb{C}P^2 \). His main claim, that for any Lagrangian embedding of the Klein bottle in a complex algebraic surface the \( \mathbb{Z}_2 \)-homology class \([K]\) is non-trivial, is false: We construct a counterexample in \([1.6]\). Another attempt to prove this result was made by Klaus Mohnke in \([Mo]\). He used completely different techniques, but his proof also appears to be incomplete.

The main result of the present paper is the following:

\textbf{Theorem 1 (Main theorem).} Any Lagrangian embedded Klein bottle \( K \) in \( \mathbb{C}P^2 \) or in a compact ruled symplectic 4-manifold \((X, \omega)\) has non-trivial \( \mathbb{Z}_2 \)-homology class \([K] \in H_2(X, \mathbb{Z}_2)\).

By \([MD-Sa]\), a ruled symplectic manifold \( X \) is (a blow-up of) a compact complex ruled surface, and \( \omega \) is a Kähler form on \( X \). The characterisation of such \( X \) from \([MD-Sa]\) will be used in the proof.

According to \([AMP]\), every Lagrangian embedding of \( K \) in a symplectic 4-manifold \( X \) can be modified into a Morse-Lefschetz fibration \( pr : (\hat{X}, K) \rightarrow (Y, \Gamma) \), which is a Lefschetz fibration of a special form, see \([1.7]\). The technical core of our paper is a study of the monodromy of such Morse-Lefschetz fibrations. For this purpose, we obtain several new results on the combinatorial structure of the mapping class group \( \text{Map}_g \), see \([0.3]\).

As in \([N-1]\), we can deduce from \textbf{Theorem 1} the following result:

\textbf{Theorem 2.} There is no Lagrangian embedding of the Klein bottle \( K \) in \( \mathbb{C}P^2 \).

This research was carried out with the financial support of the Heisenberg program of the Deutsche Forschungsgemeinschaft.
Indeed, the $\mathbb{Z}_2$-self-intersection number of an embedded Lagrangian Klein bottle in any $(X, \omega)$ equals its $\mathbb{Z}_2$-Euler class and hence vanishes, whereas $h^2 \neq 0$ for the unique non-trivial $h \in H_2(\mathbb{C}P^2, \mathbb{Z}_2)$. Note that Lagrangian embeddings of the Klein bottle in blown-up $\mathbb{C}P^2$ do exist, as an easy example from [N-1] shows: One realises $K$ as the real algebraic blow-up of the real projective plane $\mathbb{R}P^2$ and complexifies this construction obtaining $\mathbb{C}P^2$ blown-up at one point.

The following application was pointed out by St. Nemirovski, see also [N-1], [Aud], and [Ga-La], p. 489.

**Theorem 3.** Let $X$ be $\mathbb{C}P^2$ or a compact ruled symplectic 4-manifold. Then two Lagrangian embeddings $\varphi_1, \varphi_2 : \mathbb{R}P^2 \to X$ with transversal images representing the same $\mathbb{Z}_2$-homology class have at least 3 intersection points.

Indeed, the $\mathbb{Z}_2$-self-intersection index of any Lagrangian embedding $\varphi : \mathbb{R}P^2 \to X$ coincides with the Euler characteristic of $\mathbb{R}P^2$. In particular, the class $[\varphi(\mathbb{R}P^2)]$ is non-trivial and two transversal Lagrangian embeddings must have an odd number of intersection points. In the case of a single intersection point, one could perform Lagrangian surgery on it producing a Lagrangian embedding of the Klein bottle $K$. Its $\mathbb{Z}_2$-homology class is the sum of the classes of the images, hence trivial which contradicts **Main Theorem**. Notice also that in $\mathbb{C}P^2$ the condition on the homology classes is fulfilled automatically since there is a unique non-trivial element in $H_2(\mathbb{C}P^2, \mathbb{Z}_2)$.

Audin [Aud] has established an additional restriction on the class $[K]$, namely, $[K]^2 \equiv 0 \mod 4$. Here $[K]^2$ is calculated as the Pontryagin square (of the Poincaré dual) of $[K]$. For example, in the case when $X = S^2 \times S^2$, the class $[K]$ can not be the sum of the vertical and the horizontal fibres, since the Pontryagin square of such a sum is $2 \mod 4$. On the other hand, it is not difficult to show that for any symplectic form of product type $\omega = p_1^*\omega_1 + p_2^*\omega_2$ there exists a Lagrangian embedding of the Klein bottle $K$ such that the projection $p_1 : S^2 \times S^2 \to S^2$ on the first factor, restricted to $K$, is an $S^1$-bundle over a circle $\Gamma$ in the first $S^2$. As we show in **Proposition 1.5** in this case $K$ is homologous to the fibre of the projection $p_1$.

0.1. **Scheme of the proof.** Let us recall Nemirovski’s original ideas. He showed in [N-1] that, given a Lagrangian submanifold $L$ in a projective manifold $X$ and an $S^1$-valued Morse function $f : L \to S^1$, there exist another submanifold $L'$ isotopic to $L$, a blow-up $\tilde{X} \to X$ of $X$ along a complex submanifold $B \subset X$ with $B \cap L' = \emptyset$, and a holomorphic Lefschetz fibration $\text{pr} : \tilde{X} \to \mathbb{C}P^1$ such that $(\tilde{X}, \text{pr}, L')$ form a Morse-Lefschetz fibration (see **Definition 1.1**). Auroux, Muñoz, and Presas [AMP] generalised this result to the case of an arbitrary ambient compact symplectic manifold $X$, see **Theorem 1.1**. For our purposes we need a slight refinement of this generalisation; it is established in **Lemma 1.2**.

In the case $\dim_{\mathbb{R}}X = 4$ the homology group $H_2(X, \mathbb{Z}_2)$ is naturally embedded in the group $H_2(\tilde{X}, \mathbb{Z}_2)$, and thus the blow-up procedure does not affect the (non-)triviality of the homology class of a surface avoiding the blow-up locus. Thus we may assume that the Lefschetz fibration $\text{pr} : X \to S^2$ is defined already on $X$. As the Morse function $f$ on $K$, Nemirovski chose a (topologically unique) non-singular fibration $\text{pr}_K : K \to S^1$ with
circle fibres. As a result, he obtained a rather simple geometric configuration, described in Lemma 1.2 below.

The key idea of Nemirovski was to exploit the arising structure of a Morse-Lefschetz fibration and construct a complex line bundle $L$ on $X$ such that $\langle c_1(L), [K] \rangle \equiv 1 \mod 2$, which would have implied the non-triviality of $[K]$. Instead, in our approach we use this structure to describe the necessary condition for the vanishing of $[K]$ in terms of the Lefschetz fibration and its monodromy. First of all, we find an expression for $[K]$ using the homology spectral sequence. This allows us to produce an example of a Lagrangian embedding of the Klein bottle in a projective surface with trivial homology class $[K]$, see §1.6. Next, we undertake a detailed study of the combinatorial structure of the mapping class group $\text{Map}_g$. This technique (see Subsec. 0.3) allows us to show that if the class $[K]$ is trivial, then the monodromy of our Lefschetz fibration must have very special form, see Proposition 3.1. Then we observe that “twisting” appropriately a part of this special monodromy, we can construct a new symplectic manifold $X'$ such that

$$\text{rank} H_1(X', \mathbb{Z}_2) = \text{rank} H_1(X, \mathbb{Z}_2) + 1.$$  

On the other hand, using the classification of ruled symplectic manifolds from [MD-Sa] we show that under the assumptions of the Main Theorem the manifold $X'$ must also be ruled, and hence the ranks of both groups $H_1(X, \mathbb{Z}_2)$ and $H_1(X', \mathbb{Z}_2)$ must be even. The obtained contradiction finishes the proof.

0.2. Luttinger surgery and totally real embeddings of the Klein bottle. After the preprint version of this paper was finished and distributed in May 2006, Nemirovski has found an alternative proof of Theorem 1, see [N-2]. He observes that the “twisting” of the manifold $X$ above is the Luttinger surgery along the Klein bottle $K$ (well-known in the case of (Lagrangian) embeddings of the torus $T^2$, see [Lutt, E-Po, ADK]). He shows also that equality (0.1) can be deduced using the Viro index of curves on the Klein bottle and holds for any totally real embedding of $K$ in an almost complex four-manifold $(X, J)$.

0.3. Braid combinatorial structure of mapping class groups. As we have mentioned above, for the proof of the Main Theorem we need effective tools for calculations in the mapping class group $\text{Map}_g$ of a surface $\Sigma$ of given genus $g \geq 1$. This requires a better understanding of the combinatorial structure of $\text{Map}_g$.

It was understood by Dehn [De] that $\text{Map}_g$ is generated by certain simple transformations called Dehn twists. Seeking for a simple presentation for $\text{Map}_g$ one first sees that the “basic” relation among Dehn twists is the braid relation. So it was expected that $\text{Map}_g$ should resemble a braid group, possibly generalised. And indeed, Wajnryb’s presentation [Waj] realises $\text{Map}_g$ as a quotient of a certain Artin–Brieskorn braid group $\text{Br}(S_g)$ with a simple system of generators $S_g$ and relations. Recently Matsumoto [Ma] gave a description of relations in Wajnryb’s presentation in terms of the so-called Garside elements $\Delta(S)$ corresponding to certain subsystems $S \subset S_g$. In Section 2 we obtain some combinatorial properties of the Wajnryb–Matsumoto presentation, which are used in a crucial way in the proof of the Main Theorem.

Recall that every Artin–Brieskorn braid group $\text{Br}(S)$ is constructed from a Coxeter system $(S, M_S)$ such that $S$ is the set of generators of $\text{Br}(S)$ and $M_S$ is a matrix on $S$ encoding the combinatorics of the defining set of relations. To every such system $(S, M_S)$
one can also associate the so-called Coxeter–Weyl group $W(S)$ by adding the relation $s^2 = 1$ for each generator $s \in S$. In particular, there exists a natural surjective homomorphism $Br(S) \to W(S)$ which is a bijection on generators. The kernel of this homomorphism is called the pure braid group $P(S)$.

Moreover, the abelianisation $P_{ab}(S)$ of the pure braid group $P(S)$ is a free abelian group with the basis $\{A_t : t \in T(S)\}$ indexed by the set $T(S)$ of quasireflections such that $A_t$ is the projection of $t^2$ to $P_{ab}(S)$.

Remark. This result is well-known for the usual braid group $Br_d$.

If the Coxeter–Weyl group $W(S)$ is finite, then it can be realised as a reflection group. The corresponding Coxeter systems $S$ are completely classified. For such systems Deligne [Del] and Brieskorn-Saito [Br-Sa] defined the so-called Garside element $\Delta(S) \in Br(S)$. Its square $\Delta^2(S)$ lies in the centre of $P(S)$, and we show in Lemma [2.10] that the projection of $\Delta^2(S)$ to $P_{ab}(S)$ is simply the sum $\sum_{t \in T(S)} A_t$ of all basis elements of $P_{ab}(S)$. This immediately gives a description of the projections of Matsumoto’s relation elements to the group $P_{ab}(S_g)$.

Our next important result is:

Theorem 5 (Theorem [2.8]). Every factorisation $t = t_1 \cdot t_2 \cdot \ldots \cdot t_l$ of the identity $1 \in W(S)$ into a product of quasireflections $t_i \in T(S)$ is Hurwitz equivalent to a factorisation into squares of quasireflections, i.e., of the form $t' = t_1' \cdot t_2' \cdot \ldots \cdot t_l'$ with $t_{2i-1} = t_{2i} \in T(S)$.

Remark. This theorem is a generalisation of a classical result of Hurwitz. Namely, the Coxeter–Weyl group corresponding to the usual braid group $Br_d$ is the symmetric group $\text{Sym}_d$, and the claim is equivalent in this case to the irreducibility of the space of branched coverings $f : C \to \mathbb{CP}^1$ of fixed degree and genus of the curve $C$. For the case of finite Weyl groups corresponding to simple complex Lie algebras, this result was obtained by Kanev [Kan].

In its turn, Theorem 5 is used to describe the elements in $P(S)$ that can be represented as products of quasireflections $t_i$ projecting into a given subset $T' \subset T(S)$. Clearly, one can replace $T'$ by the subgroup $G \subset W(S)$ generated by it. For such $G$, we denote by $P_{ab}(S)_G$ the group of coinvariants (see, e.g., [Bro]). Since $W(S)$ acts on $P_{ab}(S)$ permuting the basis $A_t$, $t \in T(S)$, it follows that $P_{ab}(S)_G$ has a natural basis formed by the set of $W(S)$-orbits in $T(S)$. We denote the elements of this basis by $A_{G,t}$. 
Theorem 6 (Theorem 2.9). Let $G$ be a subgroup of $\mathcal{W}(S)$ and $x$ an element of $\mathcal{P}(S)$ which can be represented as a product $x = \prod_i \hat{t}_i^{\epsilon_i} \prod_j [x_{2j-1}, x_{2j}]$ such that $\hat{t}_i$ are quasigenerators, $\epsilon_i = \pm 1$, and $[x_{2j-1}, x_{2j}]$ are commutators of some $x_j \in \mathcal{B}(S)$. Assume that the projections of $x_i$ and $\hat{t}_j$ to $\mathcal{W}(S)$ lie in $G$. Then the projection $[x]_G$ of $x$ to $\mathcal{P}_{ab}(S)_G$ lies in the free abelian group generated by basis elements $A_{G,t}$ with $t \in G \cap T(S)$.

Remark. The point is that basis elements $A_{G,t}$ with $t \in T(S) \setminus G$ do not appear in the expansion of $[x]_G \in \mathcal{P}_{ab}(S)_G$.

Next, we focus on the study of the structure of the braid group $\mathcal{B}(S_g)$ involved in the Wajnryb–Matsumoto presentation of the group $\mathcal{M}_{ag}$ of mapping classes for a surface $\Sigma$ of genus $g$.

In the case $g \geq 4$ the set $T(S_g)$ is infinite, and it is hard to describe it geometrically in terms of $\Sigma$. We show that there exists a $\mathcal{W}(S_g)$-equivariant map of this set onto the set of all non-zero homology classes $v \in H_1(\Sigma, \mathbb{Z}_2)$. The latter set is denoted by $\mathcal{H}_g$ and we obtain a $\mathcal{W}(S_g)$-equivariant homomorphism $\mathbb{Z}(T(S_g)) \to \mathbb{Z}(\mathcal{H}_g)$ of free abelian groups. Thinking of $\mathcal{M}_{ag}$ as a generalised braid group, the corresponding Coxeter–Weyl group is $\mathcal{S}p(2g, \mathbb{Z}_2)$ and the elements of $\mathcal{H}_g$ are the corresponding quasireflections. Note that in the cases $g = 1, 2, 3$ we have the equalities $T(S_g) = \mathcal{H}_g$ and $\mathcal{W}(S_g) = \mathcal{S}p(2g, \mathbb{Z}_2)$ (up to the centre $\mathbb{Z}_2$ in $\mathcal{W}(S_3)$ in the case $g = 3$).

Our next result describes generators of the kernel of the natural homomorphism $\mathcal{B}(S_g) \to \mathcal{S}p(2g, \mathbb{Z}_2)$. This gives us generators of the kernels of the homomorphisms $\mathcal{M}_{ag} \to \mathcal{S}p(2g, \mathbb{Z}_2)$ and $\mathcal{W}(S_g) \to \mathcal{S}p(2g, \mathbb{Z}_2)$. We call the latter kernel the Weyl–Torelli group of $\Sigma$ and denote it by $\mathcal{W}T_g$.

Theorem 7 (Corollary 2.14, Proposition 2.17).

i) The kernel of $\mathcal{M}_{ag} \to \mathcal{S}p(2g, \mathbb{Z}_2)$ is generated by squares $T_\delta^2$ of Dehn twists along non-separating curves $\delta \subset \Sigma$.

ii) The abelianisation $\mathcal{W}T_{g,ab}$ of $\mathcal{W}T_g$ is a $\mathbb{Z}_2$-vector space isomorphic to $\wedge^6 H_1(\Sigma, \mathbb{Z}_2)$.

iii) There exists a natural lattice extension

$$0 \to \mathbb{Z}(\mathcal{H}_g) \to \Lambda_g \to \mathcal{W}T_{g,ab} \to 0$$

such that $\Lambda_g$ can be realised as a sublattice in $\frac{1}{2}\mathbb{Z}(\mathcal{H}_g)$ and the image of the induced embedding $\mathcal{W}T_{g,ab} \subset \frac{1}{2}\mathbb{Z}(\mathcal{H}_g)/\mathbb{Z}(\mathcal{H}_g) \cong \mathbb{Z}_2(\mathcal{H}_g)$ is generated by the sums $L_v := \frac{1}{2} \sum_{v \neq 0 \in V} A_v$ where $V$ is a symplectic 6-dimensional subspace of $H_1(\Sigma, \mathbb{Z}_2)$.

iv) There exists a group extension

$$0 \to \Lambda_g \to \widehat{\mathcal{S}p}(2g, \mathbb{Z}_2) \to \mathcal{S}p(2g, \mathbb{Z}_2) \to 1$$

and a homomorphism of the extension $1 \to \mathcal{P}(S_g) \to \mathcal{B}(S_g) \to \mathcal{W}(S_g) \to 1$ onto this extension such that the homomorphism $\mathcal{W}(S_g) \to \mathcal{S}p(2g, \mathbb{Z}_2)$ has the usual meaning and $\mathcal{P}(S_g) \to \Lambda_g$ is the composition $\mathcal{P}(S_g) \to \mathcal{P}_{ab}(S_g) \to \mathbb{Z}(\mathcal{H}_g) \subset \Lambda_g$. 

As a result, we can perform our calculations in much smaller and geometrically transparent groups $\Lambda_g$ and $\text{Sp}(2g, \mathbb{Z}_2)$, rather than in $\text{Map}_g$ or in $\text{Br}(S_g)$. Basically, this means working in $\Lambda_g$ considered as an $\text{Sp}(2g, \mathbb{Z}_2)$-module.

Next we describe the image of the Wajnryb–Matsumoto relations in $\Lambda_g$. For this purpose, for every $\mu \in H_1(\Sigma, \mathbb{Z}_2)$ we denote by $\varphi_\mu : H_1(\Sigma, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ the homomorphism given by $v \mapsto v \cap \mu$ and define a homomorphism $\hat{\varphi}_\mu : \mathbb{Z}(\mathcal{H}_g) \rightarrow \mathbb{Z}$ by setting $\hat{\varphi}_\mu(A_v) := 1$ for every $v \in \mathcal{H}_g \subset H_1(\Sigma, \mathbb{Z}_2)$ with $\varphi_\mu(v) = 1$ and $\hat{\varphi}_\mu(A_v) := 0$ otherwise. Thus $\hat{\varphi}_\mu$ is the “counting function” for the set-theoretic map $\varphi_\mu : \mathcal{H}_g \rightarrow \mathbb{Z}_2$. Our result describes the mapping class group $\text{Map}_{g,1}$ of the surface $\Sigma$ with one point fixed. It is known that for all $g \geq 2$ the kernel of the natural homomorphism $\text{Map}_{g,1} \rightarrow \text{Map}_g$ is the fundamental group $\pi_1(\Sigma)$.

**Theorem 8 (Corollary 2.19).** The homomorphism $\hat{\varphi}_\mu : \mathbb{Z}(\mathcal{H}_g) \rightarrow \mathbb{Z}$ extends naturally to a homomorphism from the stabiliser group $\text{Br}(S_g)_\mu$ of the element $\mu$. The latter induces a homomorphism $\hat{\varphi}_\mu : \text{Map}_{g,1,\mu} \rightarrow \mathbb{Z}_4$. Moreover, for every $\gamma \in \pi_1(\Sigma)$ and the corresponding element $x_\gamma \in \text{Map}_{g,1}$ one has $\hat{\varphi}_\mu(x_\gamma) \equiv (2g - 2) \cdot \varphi_\mu(\gamma) \mod 4$.

This result is used to obtain

**Theorem 9.** Every topological Lefschetz fibration $\text{pr} : X \rightarrow Y$ with even genus $g = 2g'$ of the fibre $F := \text{pr}^{-1}(y)$ admits a $\mathbb{Z}_2$-section, i.e., a class $[\sigma] \in H_2(X, \mathbb{Z}_2)$ with $[\sigma] \cap [F] \equiv 1 \mod 2$.

**Remark.** This result is proved in **Theorem 2.21** in the form of a certain factorisation problem in $\text{Map}_{g,1}$.

Besides, we prove the following in **Corollary 2.16**

**Theorem 10.** For $g \geq 2$ there exists an embedding $H_1(\Sigma, \mathbb{Z}_2) \hookrightarrow T^3(H_1(\Sigma, \mathbb{Z}_2)) =: T^3$ of the $\mathbb{Z}_2$-homology space into its third tensor power and a homomorphism $\varphi$ from the kernel $\text{Ker}(\text{Map}_{g,1} \rightarrow \text{Sp}(2g, \mathbb{Z}_2))$ to $T^3$ such that the composition of $\varphi$ with the natural embedding $\pi_1(\Sigma) \hookrightarrow \text{Map}_{g,1}$ equals the composition of the projection $\pi_1(\Sigma) \rightarrow H_1(\Sigma, \mathbb{Z}_2)$ with the above embedding $H_1(\Sigma, \mathbb{Z}_2) \hookrightarrow T^3$.

Now let us give an explanation of how **Theorem 6** is involved in the proof of the main result. Recall that by Gervais (**Ger**, **Theorem C**) the group $\text{Map}_g$ with $g \geq 3$ has a universal central extension $\tilde{\text{Map}}_g$ whose kernel is $\mathbb{Z}$. There exists a natural lift of the monodromy of a topological Lefschetz pencil $\text{pr} : X \rightarrow Y$ to the extension $\tilde{\text{Map}}_g$ whose evaluation is an element $c$ in the kernel, i.e., an integer. This extension can be obtained as the pull-back of the universal covering

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{\text{Sp}}(2g, \mathbb{R}) \rightarrow \text{Sp}(2g, \mathbb{R}) \rightarrow 0$$

and one can show that the number $c$ is — roughly speaking — the Chern number $c_1$ of the symplectic vector sheaf $R^1\text{pr}_*\mathbb{R}^X_X$: the first derived image of the constant sheaf $\mathbb{R}^X_X$ over $X$ (compare with **ABKP**, **BKP**, **Bo-Tsch**). Recall that the stalk of $R^1\text{pr}_*\mathbb{R}^X_X$ over a regular value $y \in Y$ is simply the cohomology group $H^1(X_y, \mathbb{R})$ of the fibre $X_y := \text{pr}^{-1}(y) \cong \Sigma_g$ and the symplectic structure on $H^1(X_y, \mathbb{R})$ is given by the cup product. The description of the extension $\tilde{\text{Map}}_g$ given by Gervais shows that the natural generator of the kernel $\mathbb{Z}$ is the chain relation element (see 2.2) so that $c_1$ is the algebraic number of times that the
chain relation is involved in the evaluation of the monodromy. Moreover, the universality of the extension $\text{Map}_g$ implies that one cannot find any further invariants.

Now let $G$ be a subgroup of $\text{Map}_g$. Then there could exist a non-trivial central extension $0 \to A \to \tilde{G} \to G \to 0$ with a “larger” abelian group $A$, not induced by the extension $\text{Map}_g$. Then an appropriate homomorphism $\psi : A \to \mathbb{Z}$ or $\psi : A \to \mathbb{Z}_m$ would give a (new) “characteristic class” for Lefschetz fibrations with monodromy in $G$. Theorem 6 explains how one can construct such an extension $0 \to A \to \tilde{G} \to G \to 0$ in the case when $G$ is the pre-image of some subgroup $H < \text{Sp}(2g,\mathbb{Z}_2)$ with respect to the natural homomorphism $\text{Map}_g \to \text{Sp}(2g,\mathbb{Z}_2)$. Then the homomorphism $\hat{\varphi}_\mu$ from Theorem 8 is essentially an example of such a $\psi$.

0.4. Notation. Let $X$ be a 4-manifold and $Y$ an oriented surface. For a fibration $pr : X \to Y$ and a subset $U \subset Y$, we set $X_U := \text{pr}^{-1}(U)$. Thus $X_y = \text{pr}^{-1}(y)$ is the fibre over a point $y \in Y$.

Let $\Sigma$ be a CW-complex and $f : \Sigma \to \Sigma$ a continuous map. The **torus of the map** $f$ is the quotient space $Z$ of the cylinder $\Sigma \times [0,1]$ with respect to the equivalence relation $(z,1) \sim (f(z),0)$. In this paper $\Sigma$ will be a real surface and $f : \Sigma \to \Sigma$ a diffeomorphism. Observe that the map $\text{pr} : [z,t] \in (\Sigma \times [0,1]/\sim) \mapsto t \in S^1 = [0,1]/(0 \sim 1)$ is a well-defined projection $\text{pr} : Z \to S^1$ of a fibre bundle structure on $Z$ with fibre $\Sigma$. We consider it a part of the map torus structure on $Z$. Vice versa, for any fibre bundle structure $\text{pr} : Z \to S^1$ with fibre $\Sigma$ there exists a continuous map $f : \Sigma \to \Sigma$ such that $(Z,\text{pr})$ is the torus of $f : \Sigma \to \Sigma$. In this case $f : \Sigma \to \Sigma$ is called the **monodromy map of** $\text{pr} : Z \to S^1$ or simply the **monodromy of** $\text{pr} : Z \to S^1$. The monodromy is defined up to isotopy and two fibre bundles $\text{pr} : Z \to S^1$ and $\text{pr}' : Z' \to S^1$ are isomorphic iff their monodromies are isotopic. More generally, let $\text{pr} : Z \to Y$ be a fibre bundle with fibre $\Sigma$ and $\gamma : S^1 \to Y$ a loop. Then the monodromy of the pulled-back bundle $\gamma^* Z := Z \times_{Y,\gamma} S^1$ is called the **monodromy of** $\text{pr} : Z \to Y$ along $\gamma$. The monodromy defines the **monodromy homomorphism** between $\pi_1(Y,y_0)$ and the mapping class group of $\Sigma$.

Let $\text{pr} : Z \to S^1$ be a bundle with fibre $\Sigma$, $y_0 \in S^1$ a base point, and $\Gamma \subset Z$ a section of $\text{pr}$, so that $\text{pr} : \Gamma \to S^1$ is a homeomorphism. Set $\Sigma = \text{pr}^{-1}(y_0)$ and $z_0 := \Gamma \cap \Sigma$. Then $(Z,\text{pr})$ can be realised as the torus of a map $f \in \text{Diff}(\Sigma,z_0)$ such that $\Gamma \subset Z$ becomes the torus of the map $f : \{z_0\} \to \{z_0\}$. Moreover, such $f$ is unique up to isotopy in $\text{Diff}(\Sigma,z_0)$, and we call $f \in \text{Diff}(\Sigma,z_0)$ the **section monodromy of** $\text{pr} : Z \to S^1$. Similar notation applies to bundles $\text{pr} : Z \to Y$ with fibre $\Sigma$. In our case $\Sigma$ is oriented and $f$ is orientation preserving, $f \in \text{Diff}_+(\Sigma,z_0)$.

An embedded circle $\delta$ on a connected surface $\Sigma$ is **non-essential** if it bounds a disc in $\Sigma$ and **essential** otherwise. All the embedded circles used in the proofs will be essential. An embedded circle $\delta$ on $\Sigma$ is **separating** if $\Sigma \setminus \delta$ has two connected components and **non-separating** otherwise. If $\Sigma$ is closed and orientable, then $\delta$ is non-separating if and only if $[\delta] \neq 0 \in H_1(\Sigma,\mathbb{Z})$ or, equivalently, if and only if $[\delta] \neq 0 \in H_1(\Sigma,\mathbb{Z}_2)$.
0.5. Acknowledgments. The author would like to thank Stefan Nemirovski for numerous discussions. Dmitry Akhiezer, Denis Auroux, Kai Cieliebak, Yasha Eliashberg, Ursula Hamenstädt, Sergei Ivashkovich, Viatcheslav Kharlamov, Dieter Kotschick, Stepan Orevkov, Leonid Polterovich, and Wolfgang Sorgel have made many valuable comments on earlier drafts of the paper.

1. Topology of Lefschetz fibrations

1.1. Lagrangian submanifolds in Lefschetz fibrations. We refer to [Go-St] for the definition of topological Lefschetz fibrations \( \text{pr} : X \to Y \), topological Lefschetz pencils \( \text{pr} : X \to S^2 \), and (symplectic) blow-ups of such \( X \), and to [AMP] and [Do] for a general theory of symplectic Lefschetz fibrations and their Lagrangian submanifolds. The paper [N-1] contains a discussion of Lagrangian submanifolds in projective algebraic manifolds. In this paper only the special case of 4-dimensional ambient manifolds \( X \) will be considered.

We shall always assume that the base surface \( Y \) of a Lefschetz fibration is oriented.

Definition 1.1. Let \( X \) be a closed 4-manifold and \( L \subset X \) a closed embedded surface. A Morse–Lefschetz fibration of \((X,L)\) is a Lefschetz fibration \( \text{pr} : X \to Y \) with the following properties:

i) the image \( \text{pr}(L) \) is a smooth embedded curve \( \Gamma \) in \( Y \);

ii) all critical points of the restricted projection \( \text{pr}_L := \text{pr}|_L : L \to \Gamma \) are of Morse type;

iii) a point \( x \in \Gamma \) is a critical value of \( \text{pr}_L : L \to \Gamma \) if and only if it is a critical value of \( \text{pr} : X \to Y \).

Such a curve \( \Gamma \) must be an arc or a circle. It turns out that any given symplectic manifold \((X,\omega)\) and any given Lagrangian embedded closed surface \( L \subset X \) can be included in a SMLF over the two-sphere \( S^2 \). Namely, the result proved in [AMP] ensures the following.

Theorem 1.1. Let \((X,\omega)\) be a closed symplectic 4-manifold such that \([\omega] = c_1(L)\) for some line bundle \( L \) on \( X \), \( L \subset X \) a closed Lagrangian submanifold, and \( f : L \to S^1 \) a Morse function. Then there exists a symplectic blow-up \( \tilde{X} \to X \) at a finite locus \( B = \{b_1, \ldots, b_N\} \subset X \setminus L \), a Morse–Lefschetz fibration \( \text{pr} : \tilde{X} \to S^2 \) of \((X,\omega, L)\) and an embedding \( \gamma : S^1 \hookrightarrow S^2 \), such that \( \text{pr}|_L = \gamma \circ f \).

Observe that in the case of a usual Morse function \( f : L \to \mathbb{R} \) the image \( f(L) \) is a compact interval which can be embedded in the circle \( S^1 \). So this case reduces to the case of circle valued Morse functions.

Let us establish the version of Theorem 1.1 needed for our purpose.

Lemma 1.2. For any embedded Lagrangian Klein bottle \( K \) in a symplectic 4-manifold \((X,\omega)\) there exists a deformation \( \tilde{\omega} \) of the symplectic form \( \omega \), a symplectic blow-up \( \tilde{X} \) of \( X \) at several points \( x_1, \ldots, x_N \in X \) outside \( K \), and a symplectic Lefschetz fibration \( \text{pr} : \tilde{X} \to S^2 \) with the following properties:

(ML1) the projection \( \text{pr}(K) \) is an embedded circle \( \Gamma \subset S^2 \);

(ML2) no critical value \( y \in S^2 \) of \( \text{pr} : X \to S^2 \) lies on \( \Gamma \);

(ML3) the fibres of the restricted projection \( \text{pr}|_K : K \to \Gamma \) are circles;
there exists a section of the projection \( pr : X \to S^2 \) (i.e., a surface \( S \subset X \) whose projection \( pr|_S : S \to S^2 \) is a diffeomorphism) such that \( S \cap K = \emptyset \).

**Proof.** The deformation of \( \omega \) is needed to make its cohomology class \([\omega]\) rational. It is easy to show that for any fixed Riemannian metric \( g_X \) on \( X \) and any \( \epsilon > 0 \) there exists a closed 2-form \( \tilde{\omega} \) such that the class \([\tilde{\omega}]\) is rational and \( \| \tilde{\omega} - \omega \|_{C^0(X)} \leq \epsilon \). The latter condition ensures that \( \tilde{\omega} \) is also a symplectic form.

Since \( H^2(K, \mathbb{R}) = 0 \), the restriction \( \tilde{\omega}|_K \) is exact. Moreover, by Hodge theory, there exists a 1-form \( \alpha \) on \( K \) with \( d\alpha = \tilde{\omega}|_K \) and

\[
\|\alpha\|_{C^0(K)} \leq C \cdot \|\tilde{\omega}|_K\|_{C^0(K)} = C \cdot \|\tilde{\omega} - \omega\|_{C^0(K)} \leq C \cdot \epsilon
\]

with the constant \( C \) depending only on the embedding \( K \subset X \) and the metric \( g_X \). By the (generalised) Darboux theorem (see, e.g., [AG]), there exists a neighbourhood \( U \) of \( K \) in \( X \) which is symplectomorphic relative \( K \) to a neighbourhood \( U' \) of the zero section of the cotangent bundle \( T^*K \) equipped with the canonical form \( \omega_{T^*K} \). The condition \( \|\alpha\|_{C^0(K)} \leq C \cdot \epsilon \) ensures that the graph \( L_\alpha \subset T^*K \) of the form \( \alpha \) lies in \( U' \). Observe that \( \omega_{T^*K}|_{L_\alpha} = d\alpha = \tilde{\omega} \). It follows that there exists a diffeomorphism \( \Phi : U \to U \) such that \( \Phi^*\tilde{\omega}|_K = \omega|_K = 0 \) and such that \( \Phi \) is trivial near the boundary \( \partial U \). Replacing \( \tilde{\omega} \) by \( \Phi^*\tilde{\omega} \), we see that \( K \) is \( \tilde{\omega} \)-Lagrangian.

Now, the desired symplectic Lefschetz fibration is obtained by **Theorem 1.1** applied to the map \( \varphi : K \to S^1 \) which realises \( K \) as an \( S^1 \)-fibre bundle over \( S^1 \). The number of single blow-ups in \( \tilde{X} \to X \) must be large for the following reason. By the construction of the symplectic Lefschetz fibration \( pr : \tilde{X} \to S^2 \) in [AMP], the blow-up centres \( x_1, \ldots, x_N \) are the common zeroes of a pair of sections \( s_0, s_1 \) of a certain line bundle \( L \) on \( X \) with \( c_1(L) = k[\tilde{\omega}] \) and \( k \gg 0 \). Moreover, the intersection index of the zero loci \( s_0^{-1}(0) \) and \( s_1^{-1}(0) \) at each \( x_1, \ldots, x_N \) is equal to 1, so that \( N = k^2[\tilde{\omega}]^2 \gg 0 \). □

1.2. **Topology of Lefschetz fibrations at singular fibres.** Let \( \Sigma \) be an oriented surface and \( \delta \subset \Sigma \) an embedded circle. Fix an annular neighbourhood \( U \) of \( \delta \) and realise it as the annulus \( \{(\rho, \theta) : \frac{1}{2} < \rho < 2, 0 \leq \theta \leq 2\pi\} \) with the orientation given by \( d\rho \land d\theta \). Let \( \chi(\rho) \) be a non-decreasing function with \( \chi(\rho) \equiv 0 \) for \( \rho \leq \frac{1}{2} \) and \( \chi(\rho) \equiv 2\pi \) for \( \rho \geq 2 \).

**Definition 1.2.** The **positive Dehn twist** of \( \Sigma \) along \( \delta \) is the diffeomorphism \( T_\delta : \Sigma \to \Sigma \) which is identical outside the neighbourhood \( U \) above and is given by the formula

\[
T_\delta(\rho, \theta) = (\rho, \theta - \chi(\rho))
\]

inside \( U \). The **negative Dehn twist** is the inverse diffeomorphism, it is given by \( (\rho, \theta) \mapsto (\rho, \theta + \chi(\rho)) \).

The Dehn twist of a prescribed sign is unique up to isotopy. The sign of a Dehn twist is defined so that symplectic and usual algebraic Lefschetz fibrations have only positive Dehn twists in their monodromy.

The positivity of the Dehn twists in symplectic Lefschetz fibrations plays no rôle in the proof. This is the reason why the result can be generalised to topological Lefschetz fibrations.

The notation \( T_\delta \) will be used both for the Dehn twist along a prescribed embedded circle \( \delta \subset \Sigma \) with a given sign and some specified neighbourhood \( U \) and coordinates \( (\rho, \theta) \).
on it and for the isotopy class of such a twist inside the corresponding mapping class group, see §13 below.

Let Δ be a disc with the boundary ∂Δ =: γ, Z a 4-manifold, pr : Z → Δ a proper Lefschetz fibration with a unique critical point z* over the origin 0 ∈ Δ. Denote by Σ the generic fibre of pr : Z → Δ, say over y0 ∈ γ.

**Lemma 1.3.** (1) The manifold Z can be deformationally retracted on the singular fibre Σ*: = pr−1(0).

(2) The monodromy along the boundary circle ∂Δ acts on Σ as the Dehn twist along a certain embedded circle δ ⊂ Σ, and the singular fibre Σ* is obtained from Σ by contracting δ to the nodal point on Σ*.

(3) If δ is non-separating, then the homology group H2(Σ*,Z) is isomorphic to Z and generated by the image of the fundamental class [Σ] ∈ H2(Σ,Z); otherwise, the group H2(Σ*,Z) is Z⊕Z generated by the fundamental cycles of the two irreducible components of Σ*.

(4) The homology group H1(Σ*,Z) is the quotient of H1(Σ,Z) by the subgroup Z⟨[δ]⟩ generated by [δ], and the cohomology group H1(Σ*,Z) is the orthogonal submodule Z⟨[δ]⟩ ⊂ H1(Σ,Z) with respect to the natural pairing H1(Σ,Z) × H1(Σ,Z) → Z.

(5) The homology groups of the boundary ∂Z = Zγ can be included into exact sequences

\[ 0 \longrightarrow H_0(\gamma, \mathcal{H}_1(\Sigma,\mathbb{Z})) \longrightarrow H_1(\gamma,\mathbb{Z}) \longrightarrow H_1(\gamma,\mathcal{H}_0(\Sigma,\mathbb{Z})) \longrightarrow 0 \]

\[ 0 \longrightarrow H_0(\gamma, \mathcal{H}_2(\Sigma,\mathbb{Z})) \longrightarrow H_2(\gamma,\mathbb{Z}) \longrightarrow H_1(\gamma,\mathcal{H}_1(\Sigma,\mathbb{Z})) \longrightarrow 0 \]

in which \( \mathcal{H}_p(\Sigma,\mathbb{Z}) \) denotes the locally constant sheaf with the stalk \( H_p(\Sigma,\mathbb{Z}) \) over \( y \in \gamma \). In particular, we have natural isomorphisms

\[ H_1(\gamma,\mathbb{Z}) \cong H_1(\gamma,\mathcal{H}_0(\Sigma,\mathbb{Z})) \quad \text{and} \quad H_0(\gamma,\mathcal{H}_2(\Sigma,\mathbb{Z})) \cong H_2(\Sigma,\mathbb{Z}), \]

the group \( H_1(\gamma,\mathcal{H}_1(\Sigma,\mathbb{Z})) \) is the subgroup \( \langle [\delta] \rangle \) of \( H_1(\Sigma,\mathbb{Z}) \), i.e., it consists of \( \lambda \in H_1(\Sigma,\mathbb{Z}) \) with \( \lambda \cap [\delta] = 0 \); \( H_0(\gamma,\mathcal{H}_1(\Sigma,\mathbb{Z})) \) is the quotient group of \( H_1(\Sigma,\mathbb{Z}) \) with respect to the subgroup \( \mathbb{Z}\langle [\delta] \rangle \) generated by \( [\delta] \).

For the proof we refer to [AGV]. The exact sequences in (5) are obtained from the Leray spectral sequence of the projection \( pr : Y \rightarrow \gamma \). The circle δ is called the vanishing cycle of the monodromy at \( z^* \), and its homology class \( [\delta] \in H_1(\Sigma,\mathbb{Z}) \) is called the vanishing class.

### 1.3. Homotopy type of groups of surface diffeomorphisms.

Let \( \Sigma \) be a closed oriented surface of genus \( g = g(\Sigma) \) with the base point \( z_0 \in \Sigma \). Denote by \( \text{Diff}_+(\Sigma) \) the group of orientation preserving diffeomorphisms of \( \Sigma \), by \( \text{Diff}_+(\Sigma, z_0) \) the subgroup of diffeomorphisms preserving the base point \( z_0 \), and by \( \text{Diff}_+(\Sigma, \{z_0\}) \) the subgroup of diffeomorphisms preserving the base point \( z_0 \) and acting trivially on the tangent plane \( T_{z_0}\Sigma \). Denote by \( \text{Map}_g \), \( \text{Map}_{g,1} \), and \( \text{Map}_{g,[1]} \) the corresponding mapping class groups, i.e., the groups of connected components of \( \text{Diff}_+(\Sigma) \), \( \text{Diff}_+(\Sigma, z_0) \), and \( \text{Diff}_+(\Sigma, \{z_0\}) \). Observe that the natural action of \( \text{Diff}_+(\Sigma) \) on \( \Sigma \) induces the principle fibre bundle \( ev_{z_0} : \text{Diff}_+(\Sigma) \rightarrow \)
$\Sigma$ given by $f \in \text{Diff}_+(\Sigma) \mapsto f(z_0) \in \Sigma$ with the structure group $\text{Diff}_+(\Sigma, z_0)$. In this way we obtain the long exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_{k+1}(\Sigma) \xrightarrow{\partial} \pi_k(\text{Diff}_+(\Sigma, z_0)) \rightarrow \pi_k(\text{Diff}_+(\Sigma)) \rightarrow \pi_k(\Sigma) \xrightarrow{\partial} \cdots$$  \hspace{1cm} (1.1)

$$\cdots \rightarrow \pi_1(\Sigma) \xrightarrow{\partial} \pi_0(\text{Diff}_+(\Sigma, z_0)) \rightarrow \pi_0(\text{Diff}_+(\Sigma)) \rightarrow \pi_0(\Sigma) \rightarrow 1.$$  

Similarly, the natural action of $\text{Diff}_+(\Sigma, z_0)$ on $T_{z_0}\Sigma$ by means of the differential $D_{z_0}f : T_{z_0} \rightarrow T_{z_0}$ of a given $f \in \text{Diff}_+(\Sigma, z_0)$ induces the principle fibre bundle $D_{z_0} : \text{Diff}_+(\Sigma, z_0) \rightarrow GL_+(T_{z_0}\Sigma)$ with the structure group $\text{Diff}_+(\Sigma, [z_0])$. In this way we obtain a similar long exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_{k+1}(GL_+(2, \mathbb{R})) \xrightarrow{\partial} \pi_k(\text{Diff}_+(\Sigma, [z_0])) \rightarrow \pi_k(\text{Diff}_+(\Sigma, z_0)) \rightarrow \cdots$$  \hspace{1cm} (1.2)

$$\cdots \rightarrow \pi_1(GL_+(2, \mathbb{R})) \xrightarrow{\partial} \pi_0(\text{Diff}_+(\Sigma, [z_0])) \rightarrow \pi_0(\text{Diff}_+(\Sigma, z_0)) \rightarrow 1.$$

The following facts about mapping class groups are well-known, see [Bi-2] or [Iva].

**Proposition 1.4.** i) In the cases $g = 0$ and $g = 1$ the map $\pi_0(\text{Diff}_+(\Sigma, z_0)) \rightarrow \pi_0(\text{Diff}_+(\Sigma))$ is an isomorphism.

ii) In the case $g \geq 2$ the connected component $\text{Diff}_0(\Sigma)$ of $\text{Diff}_+(\Sigma)$ is contractible and the sequence (1.1) induces the exact sequence of groups

$$1 \rightarrow \pi_1(\Sigma) \xrightarrow{\partial} \text{Map}_{g,1} \rightarrow \text{Map}_g \rightarrow 1.$$  \hspace{1cm} (1.3)

Moreover, the group $\text{Map}_{g,1}$ is the subgroup of index 2 of the group $\text{Aut}(\pi_1(\Sigma))$ and consists of those automorphisms of $\pi_1(\Sigma)$ which act trivially on $H_2(\pi_1(\Sigma), \mathbb{Z})$, the map $\pi_1(\Sigma) \xrightarrow{\partial} \text{Map}_{g,1}$ associates with each $\gamma \in \pi_1(\Sigma)$ the inner homomorphism $i_\gamma : \alpha \in \pi_1(\Sigma) \mapsto \gamma \alpha \gamma^{-1} \in \pi_1(\Sigma)$, and the group $\text{Map}_g$ is the corresponding group of outer automorphisms of $\pi_1(\Sigma)$.

iii) For $g \geq 1$, the long exact sequence (1.2) induces the central extension

$$1 \rightarrow \mathbb{Z} = \pi_1(GL_+(2, \mathbb{R})) \xrightarrow{\partial} \text{Map}_{g,[1]} \rightarrow \text{Map}_{g,1} \rightarrow 1.$$  \hspace{1cm} (1.4)

$$\pi_1(\text{Diff}_+(S^2)) = \mathbb{Z}_2, \pi_1(\text{Diff}_+(T^2)) \cong \pi_1(T^2) \cong \mathbb{Z}^2.$$

The embedding $\pi_1(\Sigma, z_0) \xrightarrow{\partial} \text{Map}_{g,1}$ (for $g \geq 2$) is easy to describe geometrically. Namely, for any $\gamma \in \pi_1(\Sigma, z_0)$ we choose an isotopy $F_t : \Sigma \rightarrow \Sigma$ with $t \in [0,1]$ such that $F_0 \equiv \text{id}_\Sigma$ and such that the curve $t \in [0,1] \mapsto F_t(z_0)$ represents the class $\gamma \in \pi_1(\Sigma, z_0)$. It follows immediately from the definition that $F_1$ represents the image $\partial_\Sigma(\gamma) \in \text{Map}_{g,1}$.

In the special case when $\gamma \in \pi_1(\Sigma, z_0)$ is represented by a smooth embedded curve, still denoted by $\gamma$, one can give a more explicit description of $\partial_\Sigma(\gamma)$. Namely, let $\gamma_+$ and $\gamma_-$ be the right and left boundary components of a collar neighbourhood $U_\gamma$ of $\gamma$ in $\Sigma$. Then the Dehn twists $T_{\gamma_\pm}$ of $\Sigma$ along $\gamma_\pm$ are well-defined elements of the group $\text{Map}_{g,1}$ and, as it can be easily seen,

$$\partial_\Sigma(\gamma) = T_{\gamma_+} \circ T_{\gamma_-}^{-1}.$$  \hspace{1cm} (1.5)

It follows that $\text{Map}_{g,1}$ is generated by Dehn twists along smooth embedded curves $\gamma$ on $\Sigma$ which do not pass through $z_0$. 

---

**LAGRANGIAN KLEIN BOTTLE AND MAPPING CLASS GROUPS** 11
1.4. Homology of Lefschetz fibrations. In this subsection, we describe the necessary conditions for the homological triviality of a Morse–Lefschetz embedding of the Klein bottle in a Lefschetz fibration \( \text{pr}: X \to Y \). We shall consider a somewhat more general situation than the one obtained by applying Lemma 1.2.

Let \( \text{pr}: X \to Y \) be a topological Lefschetz fibration with generic fibre \( \Sigma \) and oriented base \( Y \). Assume that \( K \subset X \) is an embedded Klein bottle such that

(T1) the image \( \text{pr}(K) \) is an embedded circle \( \Gamma \subset Y \) containing no critical value of \( \text{pr} \);

(T2) the restricted projection \( \text{pr}|_K: K \to \Gamma \) is a bundle with the fibre \( S^1 \).

These conditions ensure that \((X, \text{pr}, K, \Gamma)\) is a Morse–Lefschetz fibration of a special form.

Identify \( \Sigma \) with the fibre \( \text{pr}^{-1}(y_0) \) over a fixed point \( y_0 \in \Gamma \). Let \( m \) be the fibre of \( K \) over \( y_0 \) so that \( m = \Sigma \cap K \). We call \( m \) the meridian circle of \( K \) but consider it mostly as a curve on \( \Sigma \). Let \( F_\Gamma: \Sigma \to \Sigma \) be the monodromy of the projection \( \text{pr}: X_\Gamma \to \Gamma \). Recall that \( F_\Gamma \) is defined up to isotopy. Consequently, we can choose \( F_\Gamma \) so that \( F_\Gamma(m) = m \).

Let us consider an easier special case first.

**Proposition 1.5.** Assume that \( [m] = 0 \in H_1(\Sigma, \mathbb{Z}_2) \) so that \( m \) separates \( \Sigma \). Then the genus \( g \) of \( \Sigma \) is even and \( K \) is \( \mathbb{Z}_2 \)-homologous to the fibre \( \Sigma = \text{pr}^{-1}(y_0) \).

**Remark.** In the situation of the **Main Theorem** (once we have applied Lemma 1.2), the fibre of the Lefschetz fibration is \( \mathbb{Z}_2 \)-homologically non-trivial because it has intersection index 1 with any exceptional section. Thus, **Proposition 1.5** allows one to prove the **Main Theorem** in the (easy) case when the meridian circle \( m \) is homologically trivial in the fibre of a Morse–Lefschetz fibration for \((X, K)\). In fact, it will be shown in §2.3 that the fibre is \( \mathbb{Z}_2 \)-homologically non-trivial for any topological Lefschetz fibration with even fibre genus (see **Theorem 2.21** and **Theorem 2**).

**Proof.** By assumption, the meridian circle \( m \) divides \( \Sigma \) into two pieces, \( \Sigma \setminus m = \Sigma' \sqcup \Sigma'' \). Since the monodromy \( F_\Gamma \) preserves the orientation on \( \Sigma \) and inverts the orientation of \( m \), \( F_\Gamma \) must interchange the pieces \( \Sigma' \) and \( \Sigma'' \). Realising \( X_{\Gamma} \) as the torus of the monodromy \( F_\Gamma: \Sigma \to \Sigma \), we see that the boundary of the \( \mathbb{Z}_2 \)-chain \( \Sigma' \times [0,1] \) in \( X_{\Gamma} \) is \([K] + [\Sigma] \). Further, the pieces \( \Sigma' \) and \( \Sigma'' \) have the same genus \( g' \), hence the genus of \( \Sigma \) is even, \( g = 2g' \). \( \square \)

Let us now tackle the more complicated case when the meridian is homologically non-trivial in the fibre of the Lefschetz fibration. Denote by \( \mu := [m] \in H_1(\Sigma, \mathbb{Z}_2) \) the homology class of the meridian and by \( y_i^* \), \( i = 1, \ldots, n \), the critical values of \( \text{pr}: X \to Y \) (there may be none). For every \( y_i^* \) fix a small disc \( D_i \) containing \( y_i^* \) and set \( \Gamma_i := \partial D_i \). The latter is an embedded curve surrounding \( y_i^* \). Denote \( Y^* := Y \setminus \bigcup_i D_i \). Clearly, \( Y^* \) is homotopy equivalent to the complement of the set of critical values. For any subset \( A \subset Y \), let \( X_A := \text{pr}^{-1}(A) \) be the part of \( X \) lying over \( A \). The set \( X_{\gamma_0} = \text{pr}^{-1}(Y^*) \) is denoted by \( X^* \). In the case of separating \( \Gamma \) we denote by \( Y_+, Y_- \) the resulting pieces of \( Y \) and by \( g^*_+, g^*_- \) their genera. Set \( Y^\pm := Y \pm \bigcap Y^* \) and \( X^\pm := X_{Y^\pm} = \text{pr}^{-1}(Y^\pm) \), the latter are the parts of \( X \) lying over \( Y^\pm \).

The homological spectral sequence for the fibre bundle \( \text{pr}: X^* \to Y^* \) degenerates at the term \( E^2_{p,q} \) and yields the exact sequence

\[
0 \to H_0(Y^*, \mathcal{H}_2(X_y, \mathbb{Z}_2)) \to H_2(X^*, \mathbb{Z}_2) \to H_1(Y^*, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \to 0,
\]
where \( \mathcal{H}_p(X, \mathbb{Z}_2) \) denotes the locally constant sheaf with the stalk \( H_p(X, \mathbb{Z}_2) \) at \( y \in Y^o \). Since the monodromy map \( F_1 \) inverts the orientation of \( m \) we cannot have the case \( Y^o = S^2 \). Consequently, the surface \( Y^o \) is an Eilenberg-MacLane \( \pi_1 \)-space and \( H_p(Y^o, \mathcal{H}_q(X, \mathbb{Z}_2)) = H_p(\pi_1(Y^o), \mathcal{H}_q(X, \mathbb{Z}_2)) \).

We use a special cell decomposition of \( Y^o \) and the induced presentation of \( \pi_1(Y^o) \) to calculate the homology groups \( H_p(\pi_1(Y^o), M) \) with coefficients in a given \( \pi_1(Y^o) \)-module \( M \). Let \( g_Y \) denotes the genus of \( Y \). The construction we use depends on whether the curve \( \Gamma \) separates \( Y \) or not.

In the non-separating case we consider a polygon \( C_Y \) with \( 2g_Y + n \) sides, \( n \) being the number of critical values of \( pr : X \to Y \) and thus the number of holes in \( Y^o \). Mark and orient the sides of \( C_Y \) according to the relation word

\[
R := [\xi_1, \eta_1] \cdots [\xi_{g_Y}, \eta_{g_Y}] \cdot \Gamma_1 \cdots \Gamma_n.
\]

Gluing the first \( 2g_Y \) sides of \( C_Y \) pairwise according to the marking and orientation we obtain a surface \( Y' \) of genus \( g_Y \) with one boundary circle divided in subsequent segments \( \Gamma_1, \ldots, \Gamma_n \). To obtain the surface \( Y^o \) we divide each \( \Gamma_i \) into three pieces, say \( \Gamma'_i, \Gamma''_i \), and \( \Gamma'''_i \), and glue \( \Gamma'_i \) to \( \Gamma'''_i \) reversing the orientation.

On the constructed surface \( Y^o \) all the end points of the sides \( \xi_i, \eta_i, \Gamma_j \) are identified into a single point. We take this point as the base point \( y_0 \) on \( Y^o \) and obtain the presentation

\[
\pi_1(Y^o, y_0) = \langle \xi_1, \eta_1, \ldots, \xi_{g_Y}, \eta_{g_Y} ; \Gamma_1, \ldots, \Gamma_n \mid [\xi_1, \eta_1] \cdot \cdots \cdot [\xi_{g_Y}, \eta_{g_Y}] \cdot \Gamma_1 \cdot \cdots \cdot \Gamma_n = 1 \rangle
\]

for the group \( \pi_1(Y^o, y_0) \). Here we identify the curves \( \xi_i, \eta_i, \Gamma_j \) with the corresponding elements in \( \pi_1(Y^o, y_0) \). Further, we identify the curve \( \Gamma \) with \( \xi_1 \).

In the case of separating \( \Gamma \) we modify the construction as follows. Order the critical values \( y_i^+ \) so that the first \( n^+ \) of them lie in \( Y_+ \) and the last \( n^- := n - n^+ \) in \( Y_- \). Set

\[
R^+ := [\xi_1, \eta_1] \cdots [\xi_{g_Y^+}, \eta_{g_Y^+}] \cdot \Gamma_1 \cdots \Gamma_{n^+} \quad R^- := [\xi_{g_Y^+ + 1}, \eta_{g_Y^+ + 1}] \cdots [\xi_{g_Y}, \eta_{g_Y}] \cdot \Gamma_{n^+ + 1} \cdots \Gamma_n.
\]

Take polygons \( C_Y^+ \) and \( C_Y^- \) with \( 2g_Y^+ + n^+ + 1 \) sides marked and oriented according to the relation words \( R^+ \cdot \Gamma^+ \) and \( R^- \cdot \Gamma^- \), respectively. Glue the polygons \( C_Y^+ \) and \( C_Y^- \) along the sides \( \Gamma^+ \) and \( \Gamma^- \) and denote by \( \Gamma \) the resulting interval. We obtain a polygon \( C_Y \) with \( 2g_Y + n \) sides marked and oriented according to the word \( R^+ \cdot R^- \). Construct \( Y^o \) from \( C_Y \) according to this word as it was done above. As the presentation of \( \pi_1(Y^o) \) we take

\[
\pi_1(Y^o, y_0) = \langle \xi_1, \eta_1, \ldots, \xi_{g_Y}, \eta_{g_Y} ; \Gamma_1, \ldots, \Gamma_n ; \Gamma \mid R^+ = \Gamma^{-1}, R^- = \Gamma \rangle.
\]

Observe that the images of the polygons \( C_Y^+, C_Y^- \) on \( Y^o \) are glued into pieces \( Y^o_+, Y^o_- \) according to the relation words \( R^+ \cdot \Gamma^+, R^- \cdot \Gamma^- \), respectively. One can see that the fundamental groups \( \pi_1(Y^o_+, Y^o_-) \) are embedded in \( \pi_1(Y^o) \) by identification of generators.

Now let \( \pi \) be any group, \( \pi = \langle \mathcal{X} \mid \mathcal{R} \rangle \) its presentation with the set of generators \( \mathcal{X} \) and the set of relations \( \mathcal{R} \), and \( M \) a \( \pi \)-module. Let \( Fr(\mathcal{X}) \) be the free group generated by \( \mathcal{X} \). Set \( \mathcal{X} \otimes M \) to be the direct sum of \( \mathcal{X} \) copies of \( M \), so that the elements of \( \mathcal{X} \otimes M \) are finite formal sums \( \sum_i x_i \otimes m_i \) with \( x_i \in \mathcal{X} \) and \( m_i \in M \) satisfying the distributive law for the first argument. Define \( \mathcal{R} \otimes M \) and \( Fr(\mathcal{X}) \otimes M \) in the same way. We consider \( \mathcal{X} \otimes M \) and \( \mathcal{R} \otimes M \) as subgroups of \( Fr(\mathcal{X}) \otimes M \), and to distinguish between two copies of \( Fr(\mathcal{X}) \otimes M \) we use the notation \( x \otimes_1 m \) and \( x \otimes_2 m \), respectively. Now set \( \mathcal{R}_0 := M \), \( \mathcal{R}_1 := \mathcal{X} \otimes M \), \( \mathcal{R}_2 := \mathcal{R} \otimes M \) and define the chain homomorphisms \( \partial_1 : \mathcal{R}_1 \to \mathcal{R}_0 \) and
\[ \partial_2 : R_2 \to R_1 \] by setting \( \partial_1(x \otimes_1 m) := xm - m \) and imposing the following derivation properties:

- \( \partial_2(x \otimes_2 m) = x \otimes_1 m \) if \( x \otimes_1 \)
- \( \partial_2(ab \otimes_2 m) = \partial_2(a \otimes_2 bm) + \partial_2(b \otimes_2 m) \) for arbitrary \( a, b \in \text{Fr}(X) \).

It follows that there exists a unique \( \partial_2 : R_2 \to R_1 \) with these properties. In particular, \( \partial_2(1 \otimes_2 1) = \partial_2(1 \cdot 1 \otimes_2 m) = 2\partial_2(1 \otimes_2 m) \) and hence \( \partial_2(1 \otimes_2 m) = 0 \) for the neutral element \( 1 \in \text{Fr}(X) \). Similarly one concludes that \( \partial_2(a^{-1} \otimes_2 m) = -\partial_2(a \otimes_2 a^{-1}m) \).

The geometric meaning of these homomorphisms is as follows. From the presentation \( \langle X | R \rangle \) \( R \) of \( \Sigma \), the cellular chain complex associated with the locally constant coefficient system on \( Z \) induced by \( M \). In particular \( \partial_1 \circ \partial_2 = 0 \). Moreover, \( Z \) is 2-equivalent to the Eilenberg-MacLane space \( B(\pi, 1) \) and hence for \( p = 0 \) and \( p = 1 \) the \( p \)-th homology group of the complex \( (R, \partial) \) are the desired homology groups \( H_p(\pi, M) \).

By the construction, \( Y^\circ \) is homotopy equivalent to the 2-dimensional cell complex \( Z \) associated with both presentations of \( \pi_1(Y^\circ) \) above. As the coefficient module \( M \) we use the homology group \( H_1(\Sigma, Z_2) \). The image of the homology class \( [K] \) under projection \( H_2(X^\circ, Z_2) \to H_1(Y^\circ, \mathcal{H}_1(X_y, Z_2)) \) is easy to describe: It is represented by the chain \( \Gamma \otimes_1 \lambda \in R_1 \).

Next, we describe the image of \( H_2(X^\circ, Z_2) \) in \( H_2(X, Z_2) \). For this purpose we use the Mayer-Vietoris exact sequence associated with the decomposition of \( X \) into \( X^\circ \) and \( \sqcup_i X_{D_i} \).

Recall that \( X_{D_i} = \pi^{-1}(D_i) \) and \( X^\circ \cap X_{D_i} = \pi^{-1}(\Gamma_i) \). The relevant segment of the exact sequence is

\[
\cdots \to H_2(\sqcup_i X_{\Gamma_i}, Z_2) \to H_2(X^\circ, Z_2) \oplus H_2(\sqcup_i X_{D_i}, Z_2) \to H_2(X, Z_2) \to \cdots.
\]

**Lemma 1.6.** The kernel of the homomorphism \( H_2(X^\circ, Z_2) \to H_2(X, Z_2) \) is generated by classes \( \Lambda_i \) whose projections to \( H_1(Y^\circ, \mathcal{H}_1(X_y, Z_2)) \) have the form \( \Gamma_i \otimes_1 \lambda_i \) where \( \lambda_i \in H_1(X_y, Z_2) \) are \( \cap \)-orthogonal to \( \delta_i \).

**Remark.** The condition on \( \lambda_i \) is vacuous if \( [\delta_i] = 0 \in H_1(X_y, Z_2) \).

**Proof.** Clearly, \( H_2(\sqcup_i X_{\Gamma_i}) = \oplus_i H_2(X_{\Gamma_i}) \). As in the case of \( X^\circ \), the homological spectral sequence for each of these summands reduces to the exact sequence

\[
0 \to H_0(\Gamma_i, \mathcal{H}_2(X_y, Z_2)) \to H_2(X_{\Gamma_i}, Z_2) \to H_1(\Gamma_i, \mathcal{H}_1(X_y, Z_2)) \to 0.
\]

The group \( H_0(\Gamma_i, \mathcal{H}_2(X_y, Z_2)) \) is just \( Z_2 \) generated by \( [\Sigma] \). Since \( \pi_1(\Gamma_i) = \mathbb{Z} \), it follows that \( H_1(\Gamma_i, \mathcal{H}_1(X_y, Z_2)) \) equals \( H_1(\mathbb{Z}, M_i) \) where \( M_i \) is the group \( H_1(X_y, Z_2) \) with the action of \( \pi_1(\Gamma_i) = \mathbb{Z} \) given by the monodromy along \( \Gamma_i \), which is the Dehn twist \( T_{\delta_i} \) along the vanishing cycle \( \delta_i \). Thus \( H_1(\Gamma_i, \mathcal{H}_1(X_y, Z_2)) \) is simply the space \( [\delta_i]^{-1} \) of those \( \lambda \in H_1(X_y, Z_2) \) that are \( \cap \)-orthogonal to \( \delta_i \). On the other hand, the group \( H_0(\Gamma_i, \mathcal{H}_2(X_y, Z_2)) \) is \( Z_2 \) generated by the fibre class, and so the composition \( H_0(\Gamma_i, \mathcal{H}_2(X_y, Z_2)) \to H_2(X_{\Gamma_i}, Z_2) \to H_2(X_{D_i}, Z_2) \) is an embedding. Consequently, the sequence (1.3) splits and \( H_1(\Gamma_i, \mathcal{H}_1(X_y, Z_2)) \) can be considered as a subgroup of \( H_2(X_{\Gamma_i}, Z_2) \). Moreover, the image of \( H_1(\Gamma_i, \mathcal{H}_1(X_y, Z_2)) \) in \( H_2(X_{D_i}, Z_2) \) is trivial. The lemma follows. \( \square \)
Let us establish necessary conditions for the vanishing of $[K]$ in $H_2(X,\mathbb{Z}_2)$.

**Proposition 1.7.** Assume that $\mu = [m]$ is non-trivial but the projection of the class $[K]$ to $H_1(Y^o,\mathcal{H}_1(X_\ast,\mathbb{Z}_2))$ vanishes.

i) In the case of separating $G$ there exists a decomposition $\mu = \mu_+ + \mu_- \in H_1(\Sigma,\mathbb{Z}_2)$ such that for the pieces $Y^o_+$ and $Y^o_-$ the monodromy action of the fundamental group $\pi_1(Y^o_\pm)$ (resp., $\pi_1(Y^o_\mp)$) preserves $\mu_+$ (resp., $\mu_-$). In particular, both $\mu_+$ and $\mu_-$ are $F_\Gamma$-invariant.

ii) In the case of non-separating $G$ the monodromy group of $pr : X^o \to Y^o$ acts trivially on $\mu \in H_1(\Sigma,\mathbb{Z}_2)$. Moreover, there exists a class $\nu \in H_1(\Sigma,\mathbb{Z}_2)$ such that $\mu = (1 + F_{\eta_1}^{-1})\nu$ for the monodromy $F_{\eta_1}$ along $\eta_1$ and such that $\nu$ is invariant under the action of the remaining generators in the above presentation of $\pi_1(Y^o)$.

**Proof.** To simplify notation, we denote by $\xi_i, \eta_i, \Gamma_j$ the curves on $Y^o$, the corresponding elements in $\pi_1(Y^o)$, the generators of the associated free group, and the monodromy transformations along these curves.

i) It follows from Lemma 1.6 that the element $\Gamma \otimes_1 \mu$ can be represented in the form

$$\Gamma \otimes_1 \mu = \partial_2(R^+ \cdot \Gamma \otimes_2 \mu_+) + \partial_2(R^- \cdot \Gamma^{-1} \otimes_2 \mu_-) + \sum \Gamma_i \otimes_1 \lambda_i$$

for some $\mu_\pm, \lambda_i \in H_1(\Lambda,\mathbb{Z}_2)$ such that $\lambda_i$ are $\cap$-orthogonal to $\delta_i$. Let us expand $\partial_2(R^+ \cdot \Gamma \otimes_2 \mu_+)$ preserving the commutators $[\xi_i, \eta_i]$ and denoting by $w_i^\Gamma$ the final subword of $R^+ \cdot \Gamma$ after the letter $\Gamma_i$ and by $w_i^\Gamma$ the final subword of $R^+ \cdot \Gamma$ after the commutator $[\xi_i, \eta_i]$. In particular, $w_i^\Gamma, = \Gamma, w_i^{\Gamma_n+1} = \Gamma_n \cdot \Gamma, w_i^{\Gamma} = \Gamma_{\Gamma_n} \cdot \Gamma_{\Gamma_n} \cdot \Gamma_{\Gamma_n + \Gamma},$ and similarly for $w_i^\Gamma$. This gives

$$\partial_2(R^+ \cdot \Gamma \otimes_2 \mu_+) = \partial_2([\xi_1, \eta_1] \cdot \ldots \cdot \Gamma_n \cdot \Gamma \otimes_2 \mu_+)$$

and a similar formula for $\partial_2(R^- \cdot \Gamma^{-1} \otimes_2 \mu_-)$, with the only difference that the first summand will be $-\Gamma \otimes_2 \Gamma^{-1} \mu_-$. The expansion of the commutators gives

$$\partial_2([\xi_1, \eta_1] \otimes_2 \nu) = \partial_2(\xi_1 \eta_1^{-1} \eta_1^{-1} \otimes_2 \nu)$$

where

$$\xi \otimes_1 \eta \eta_1^{-1} \eta_1^{-1} \nu + \eta \otimes_1 \xi \eta_1^{-1} \eta_1^{-1} \nu - \xi \otimes_1 \xi \eta_1^{-1} \eta_1^{-1} \nu - \eta \otimes_1 \eta \eta_1^{-1} \nu$$

Collecting similar summands (and ignoring the signs since we are working with $\mathbb{Z}_2$-spaces) we obtain the desired decomposition $\mu = \mu_+ + \Gamma^{-1} \mu_-$ from the coefficient of $G$, the equality $\lambda_i = w_i^\Gamma \mu_\pm$ as the coefficient of $G_i$, and the equalities $(\eta_i - 1)\xi_i^{-1} \eta_1^{-1} w_i^\Gamma \mu_\pm = 0$, $(1 - \xi_i)\xi_i^{-1} \eta_1^{-1} w_i^\Gamma \mu_\pm = 0$ from the coefficients of $\xi_i$ and $\eta_i$. Now observe that the equality $\lambda_i = w_i^\Gamma \mu_\pm$ together with the $\cap$-orthogonality $\lambda_i \cap \delta_i \equiv 0 \mod 2$ implies that $\mu_\pm$ is invariant under the action of the Dehn twist $T_{\delta_i}$ which is the action of $G_i$. This yields the identity $w_i^\Gamma \mu_\pm = w_i^\Gamma \mu_\pm = \ldots = \Gamma \mu_\pm$ for all $i = 0; 1, \ldots, n^+$. Here we have set $w_0^\Gamma = \Gamma \cdot w_1^\Gamma$, this element coincides with $w_1^\Gamma$. 
To obtain (1.11), we use the Mayer-Vietoris sequence corresponding to the covering we see that each $H_i$ act trivially on $\xi_i^{-1}\eta_i^{-1}w_i^\mu$, so that $w_i^\mu$ also remains invariant, and then we conclude $w_i^\mu = w_i^\mu$. Summing up, we obtain the $\pi_1(Y_+^\circ)$-invariance of $\Gamma\mu_+$. Finally, since $\Gamma$ can be expressed in $\pi_1(Y_+^\circ)$ as a product of already treated elements, $\Gamma\mu_+ = \mu_+$. The argument for the case of $\mu_-$ is the same.

ii) This time we have $\Gamma = \xi_1$ in $\pi_1(Y^\circ)$ and the unexpanded relation reads

$$\xi_1 \otimes_1 \mu = \partial_2(R \otimes_2 \nu) + \sum_i \Gamma_i \otimes_1 \lambda_i.$$  

The expansion provides the same equalities as above, except for the coefficient of $\xi_1$ which now is $\mu + (\eta_i - 1)\xi_i^{-1}\eta_i^{-1}w_i^\mu$. As above, we can conclude the invariance of $\nu$ with respect to $\xi_i\eta_i$ for $i = 2, \ldots, n$ and with respect to all $\Gamma_i$’s. So the remaining equalities are $\xi_1^{-1}\eta_1^{-1}\nu = \eta_1^{-1}\nu$ and $(1 + \eta_1)\xi_1^{-1}\eta_1^{-1}\nu = \mu$. Since the commutator $[\xi_1, \eta_1]$ is equal in $\pi_1(Y^\circ)$ to an expression in the remaining generators, the actions of $\xi_1$ and $\eta_1$ on $\nu$ commute. Thus the first remaining equality is equivalent to the $\xi_1$-invariance of $\nu$, and the second yields $(1 + \eta_1^{-1})\nu = \mu$, as desired. $\square$

**Lemma 1.8.** The first homology group of $X$ can be included into the exact sequence

$$0 \to H_0(Y^\circ, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \to H_1(X, \mathbb{Z}_2) \to H_1(Y, \mathbb{Z}_2) \to 0.$$  

**Proof.** The claim is trivial in the special case when $Y = S^2$ and there are no critical points of $\text{pr}: X \to Y = S^2$. In the remaining cases we consider the homology of $X^\circ$. The Leray spectral sequence reduces to the exact sequence

$$0 \to H_0(Y^\circ, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \to H_1(X^\circ, \mathbb{Z}_2) \to H_1(Y^\circ, \mathbb{Z}_2) \to 0.$$  

To obtain (1.11), we use the Mayer-Vietoris sequence corresponding to the covering $X = X^\circ \cup (\bigcup_i X_{D_i})$ with $X^\circ \cap (\bigcup_i X_{D_i}) = \bigcup_i X_{\Gamma_i}$. This gives us the realisation of $H_1(X, \mathbb{Z}_2)$ as the cokernel of the homomorphism

$$\bigoplus_i H_1(X_{\Gamma_i}, \mathbb{Z}_2) \to H_1(X^\circ, \mathbb{Z}_2) \oplus \bigoplus_i H_1(X_{D_i}, \mathbb{Z}_2)$$

For each $X_{\Gamma_i}$ we have

$$0 \to H_0(\Gamma_i, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \to H_1(X_{\Gamma_i}, \mathbb{Z}_2) \to H_1(\Gamma_i, \mathbb{Z}_2) \to 0.$$  

Now observe that the embedding $X_{\Gamma_i} \subset X_{D_i}$ induces an isomorphism $H_0(\Gamma_i, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \cong H_1(X_{D_i}, \mathbb{Z}_2)$ and hence a natural splitting of (1.14). “Inserting” this splitting in (1.13) we see that each $H_1(\Gamma_i, \mathbb{Z}_2) \cong \mathbb{Z}_2$ “kills” the class in $H_1(Y^\circ, \mathbb{Z}_2)$ from (1.12) represented by the curve $\Gamma_i \subset Y$. $\square$
1.5. \( \mathbb{Z}_2 \)-\textit{sections of Lefschetz fibrations}. If \( K \) is \( \mathbb{Z}_2 \)-homologically non-trivial under the assumptions of \textit{Proposition 1.7}, then it must be \( \mathbb{Z}_2 \)-homologous to the generic fibre \( \text{pr}^{-1}(y_0) \). (The same situation occurred in \textit{Proposition 1.5} above.) In turn, the homological non-triviality of the generic fibre is equivalent to the existence of a class \( [\sigma] \in H_2(X,\mathbb{Z}_2) \) with \( [\sigma] \cap [K] \neq 0 \). In this paragraph we study the properties of such \( [\sigma] \). This will give us the possibility to construct examples of Lagrangian embeddings of the Klein bottle in symplectic manifolds \( (X,\omega) \) with trivial homology class.

We maintain the notation from the previous paragraph with a minor modification. In particular, \( Y \) is a compact surface with a smooth boundary \( \partial Y \), possibly empty or not connected, and \( \text{pr} : X \to Y \) a Lefschetz fibration with a generic fibre \( \Sigma = \text{pr}^{-1}(y_0) \) of genus \( g \) such that all critical values \( y_i^* \) do not lie on \( \partial Y \). Denote by \( Y^\circ \) the surface obtained from \( Y \) by cutting small pairwise disjoint discs \( D_i \) centred at \( y_i^* \). Then the restriction \( \text{pr} : \partial X \to \partial Y \) is a fibre bundle with fibre \( \Sigma \).

**Definition 1.3.** A \( \mathbb{Z}_2 \)-\textit{section} of \( \text{pr} : X \to Y \) is a relative homology class \( [\sigma] \in H_2(X,\partial X;\mathbb{Z}_2) \) which has non-trivial \( \mathbb{Z}_2 \)-intersection index with the class \( [\Sigma] \) of the generic fibre. We denote by \( \partial\sigma \in H_1(\partial X,\mathbb{Z}_2) \) the class of the boundary of \( [\sigma] \) and by \( \sigma^\vee \in H^2(X,\mathbb{Z}_2) \) the Poincaré dual class. For an open set \( U \subset Y \) with smooth boundary \( \partial U \) which avoids the critical values \( y_i^* \), we define the \textit{restriction of} \( \sigma \) on the piece \( X_U = \text{pr}^{-1}(U) \) by means of the restriction of \( \sigma^\vee \), so that \( (\sigma|_{X_U})^\vee := \sigma^\vee|_{X_U} \).

The latter definition can also be used to define \( \partial\sigma \), since \( (\partial\sigma)^\vee = \sigma^\vee|_{\partial X} \).

The cohomology group \( H^2(X,\mathbb{Z}_2) \) where \( \sigma^\vee \) “lives” admits a description dual to the one given above for the homology group \( H_2(X,\mathbb{Z}_2) \). In particular, we have the exact sequence

\[
0 \to H^1(Y^\circ,\mathcal{H}^1(X_y,\mathbb{Z}_2)) \to H^2(X,\mathbb{Z}_2) \to H^0(Y^\circ,\mathcal{H}^2(X_y,\mathbb{Z}_2)) \to 0,
\]

where \( \mathcal{H}^p(X_y,\mathbb{Z}_2) \) denotes the locally constant sheaf with the stalk \( H^p(X_y,\mathbb{Z}_2) \) at \( y \in Y^\circ \). \( H^0(Y^\circ,\mathcal{H}^2(X_y,\mathbb{Z}_2)) \) is naturally isomorphic to \( \mathbb{Z}_2([\Sigma]) \), and \( H^1(Y^\circ,\mathcal{H}^1(X_y,\mathbb{Z}_2)) = H^1(\pi_1(Y^\circ),\mathcal{H}^1(X_y,\mathbb{Z}_2)) \). Moreover, for any \( \pi_1(Y^\circ) \)-module \( M \) we can calculate the groups \( H^1(\pi_1(Y^\circ),M) \) using the resolvent \( 0 \to \mathcal{R}^0 \xrightarrow{d^{(1)}} \mathcal{R}^1 \xrightarrow{d^{(2)}} \mathcal{R}^2 \to 0 \), dual to the resolvent \( (\mathcal{R}_\bullet,\partial_\bullet) \) from the previous subsection. In the case of a finite presentation \( \langle \mathcal{X} | \mathcal{R} \rangle \) of \( \pi_1(Y^\circ) \) the cochain groups \( \mathcal{R}^* \) coincide with the corresponding chain groups, so that \( R^0 = M \), \( R^1 = \mathcal{X} \otimes M \), and \( R^2 = \mathcal{R} \otimes M \), and the index in \( \otimes^k \) indicates that the element lies in \( \mathcal{R}^k \). The differentials \( d^* \) are dual to \( \partial_\bullet \) and are given by the transposed matrices. One sees immediately the formula \( d^{(1)} : m \mapsto \sum_i x_i \otimes (x_i - 1) \cdot m \) for \( d^{(1)} : R^0 = M \to R^1 \). The formal expression for \( d^{(2)} : R^1 \to R^2 \) is \( d^{(2)} (\sum_i x_i \otimes m_i) = \sum_{ij} R_{ij} \otimes \partial R_{ij} \cdot m_i \) where the sums are taken over all \( x_i \in \mathcal{X} \) and \( R_{ij} \in \mathcal{R} \), and \( \partial R_{ij} \) denotes the \textit{Fox free differential} of the word \( R_{ij} \) in the free group \( \text{Fr}(\mathcal{X}) \) generated by \( \mathcal{X} \), see [Bi-2]. For a given \( R_{ij} \), the coefficient \( \sum_i R_{ij} \cdot m_i \) equals the value on \( R_{ij} \in \text{Fr}(\mathcal{X}) \) of the unique \textit{cross-homomorphism} \( \phi : \text{Fr}(\mathcal{X}) \to M \) with \( \phi(x_i) = m_i \). Recall that cross-homomorphisms are characterised by the property that \( \phi(a \cdot b) = \phi(a) + a \cdot \phi(b) \) for every \( a,b \in \text{Fr}(\mathcal{X}) \), which is simply the dual of the derivation properties of the differential \( \partial_2 \).
For \( g \geq 2 \), denote by \( \widetilde{\text{Map}}_g \) the group in the extension
\[
1 \to H_1(\Sigma, \mathbb{Z}_2) \to \widetilde{\text{Map}}_g \to \text{Map}_g \to 1,
\]
so that \( \widetilde{\text{Map}}_g \) is the quotient of \( \text{Map}_{g,1} \) by the image with respect to \( \partial_\pi \) of the kernel of the homomorphism \( \pi_1(\Sigma, z_0) \to H_1(\Sigma, \mathbb{Z}_2) \). A Dehn twist in \( \widetilde{\text{Map}}_g \) is a projection of a Dehn twist \( T_\delta \in \text{Map}_{g,1} \).

**Lemma 1.9.** Let \( \text{pr} : X \to Y \) be a Lefschetz fibration of a closed manifold \( X \) and \( \langle \xi, \eta, \Gamma \rangle \) the above presentation of the fundamental group \( \pi_1(Y^\circ) \) with a single relation word.

i) The image of the homomorphism \( H^2(X, \mathbb{Z}_2) \to H^2(X^\circ, \mathbb{Z}_2) \) is generated by the class of some \( \mathbb{Z}_2 \)-section (if such exists) and by classes from \( H^1(Y^\circ, \mathcal{H}^1(X, \mathbb{Z}_2)) \) represented by cocycles \( \lambda^\vee \) of the form
\[
\lambda^\vee = \sum_j (\xi_j \otimes \lambda_j^\vee + \eta_j \otimes \mu_j^\vee) + \sum_i \Gamma_i \otimes \delta_i^\vee \in H^1(Y^\circ, \mathcal{H}^1(X, \mathbb{Z}_2)),
\]
where \( n_i \in \mathbb{Z}_2 \), \( \lambda_j^\vee, \mu_j^\vee \in H^1(\Sigma, \mathbb{Z}_2) \), and \( \delta_i^\vee \) is the Poincaré dual of the class \([\delta_i]\).

ii) The obstruction to the existence of a \( \mathbb{Z}_2 \)-section of the fibration \( \text{pr} : X \to Y \) is a coset class \([\psi^\vee]\) of the quotient of the group \( \mathcal{R}^2 \) by the subgroup of all coboundaries \( d^2(\lambda^\vee) \) such that \( \lambda^\vee \) has the form \((1.16)\).

iii) In the case \( g \geq 2 \), the obstruction \([\psi^\vee]\) vanishes if and only if the monodromy \( F : \pi_1(Y^\circ) \to \text{Map}_g \) of \( \text{pr} : X^\circ \to Y^\circ \) can be lifted to a homomorphism \( F : \pi_1(Y^\circ) \to \widetilde{\text{Map}}_g \) such that \( F(\Gamma) \) is a Dehn twist in \( \widetilde{\text{Map}}_g \). The difference between such two lifts \( F_1, F_2 : \pi_1(Y^\circ) \to \widetilde{\text{Map}}_g \) is a cross-homomorphism \( \phi : \pi_1(Y^\circ) \to H_1(\Sigma) \) which is associated with the unique cochain \( \lambda^\vee_\phi \) of the form \((1.16)\) whose Poincaré dual \( \lambda^\vee_\phi \) has coefficients \( \lambda_j = \phi(\xi_j) \), \( \mu_j = \phi(\eta_j) \), and \( n_i\delta_i = \phi(\Gamma_i) \).

**Proof.** The first two assertions are dual to Lemma 1.6 so the proof proceeds by dualising the homomorphism and taking into account the duality between kernels and images. The last assertion follows from the fact that the natural embedding \( \partial_\pi : \pi_1(\Sigma, z_0) \to \text{Map}_g(\Sigma, z_0) \) induces an inclusion \( H_1(\Sigma, \mathbb{Z}_2) \to \text{Map}_g \) for any \( g \geq 2 \). Then one observes that the group-homological meaning of both constructions is the same.

It should be noticed that in the case \( g = 1 \) the obstruction \([\psi^\vee]\) to the existence of a \( \mathbb{Z}_2 \)-section is not determined by the monodromy in \( \text{Map}_1 = \text{Map}_{1,1} \). This reflects the fact that the homomorphism \( \partial_\pi : \pi_1(T^2) = \mathbb{Z}^2 \to \pi_0(\text{Diff}_+(T^2)) \) in \((1.1)\) is trivial.

1.6. **An example.** We construct an example of a Lagrangian embedding of the Klein bottle in a projective surface \( X \) equipped with a Kähler form \( \omega \) such that the homology class \([K]\) is trivial. In the example, \( K \) will be fibred over a non-separating circle in the two-torus \( T^2 \).

Let \( Y \) be a two-torus. Pick a flat metric on \( Y \). This gives us a complex structure and a Kähler form \( \omega_Y \) on \( Y \). We may assume that the \( \omega_Y \)-volume of \( Y \) is 1. Fix a geometric basis of \( Y \), represented by embedded curves \( \xi \) and \( \eta \) meeting transversally in a single point.

Let \( \Sigma \) be another two-torus, realised as the quotient of \( \mathbb{C} \) by the lattice \( \Lambda \) generated by vectors \( \alpha := 1 \) and \( \beta := e^{2\pi i/3} \). Let \( \omega_\Sigma \) be the flat Kähler form on \( \Sigma \) such that the
Consider the C-linear homomorphisms \( F_\xi := -\text{id} : \mathbb{C} \to \mathbb{C} \) and \( F_\eta := e^{\pi i/3}\text{id} : \mathbb{C} \to \mathbb{C} \). Clearly, they define holomorphic automorphisms of \( \Sigma \) preserving the base point \( z_0 := 0 \in \Sigma \), also denoted by \( F_\xi \) and \( F_\eta \).

Let \( pr : X \to Y \) be the fibration over the base \( Y \) with the fibre \( \Sigma \) and with monodromy \( F_\xi \) along \( \xi \) and \( F_\eta \) along \( \eta \). It follows that there exists a flat Kähler structure on \( X \) with the Kähler form \( \omega \), in which \( X \) is the product \( (Y, \omega_Y) \times (\Sigma, \omega_\Sigma) \) locally near each fibre. In particular, there exists a fibrewise Kähler form \( \omega_\Sigma \) on \( X \) such that \( \omega = pr^*\omega_Y + \omega_\Sigma \).

We claim that \( \omega \) is a polarisation of \( X \) corresponding to a certain holomorphic line bundle \( \mathcal{L} \). Indeed, since the monodromy preserves the base point \( x_0 \in \Sigma \), we obtain the horizontal section \( \sigma_0 \) which is constantly \( z_0 \). It follows that \( \sigma_0 \) is holomorphic. Now it is easy to see that \( \omega \) represents the Chern class \( c_1(\mathcal{L}) \) of the holomorphic line bundle \( \mathcal{L} := \mathcal{O}_X(\sigma_0 + X_y) \) of the divisor \( \sigma_0 + X_y \), \( X_y \) being the vertical fibre over some point \( y \in Y \). By Kodaira’s embedding theorem, see e.g., [Gr-Ha], \( \mathcal{L} \) is ample and hence \( X \) is projective algebraic.

Now let us compute \( H_2(X, \mathbb{Z}_2) \) using the homology spectral sequence. Its \( E^2_{p,q} \)-term consists of the groups \( H_p(Y, \mathcal{H}_q(X_y, \mathbb{Z}_2)) \). The groups \( H_0(Y, \mathcal{H}_2(X_y, \mathbb{Z}_2)) \) and \( H_2(Y, \mathcal{H}_0(X_y, \mathbb{Z}_2)) \) are both \( \mathbb{Z}_2 \)'s, generated by the class of the fibre \( X_y \) and the section \( \sigma_0 \), respectively. To compute \( H_1(Y, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \), observe first that the monodromy actions of \( F_\xi \) and \( F_\eta \) are given by the \( \mathbb{Z}_2 \)-matrices \( \text{id} \) and \( (1 \ 0) \), respectively. Using the notation of [14], we see that \( H_1(Y, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \) is represented by the cycles \( \lambda = \xi \otimes \lambda_\xi + \eta \otimes \lambda_\eta \) with \( \lambda_\xi, \lambda_\eta \in H_1(\Sigma, \mathbb{Z}_2) \) satisfying \( \partial_1(\lambda) = (F_\eta - \text{id})\lambda_\eta = 0 \). Since \( F_\eta - \text{id} \) is non-degenerate, \( \lambda \) must be of the form \( \xi \otimes \lambda_\xi \). The calculation from [111] shows that the boundaries can be written as
\[
\partial_2[(\xi, \eta) \otimes \nu] = \xi \otimes (F_\eta - \text{id})F_\xi^{-1}F_\eta^{-1}\nu + \eta \otimes ((id - F_\xi)F_\eta^{-1}F_\xi^{-1}\nu = \xi \otimes (F_\eta - \text{id})F_\xi^{-1}F_\eta^{-1}\nu
\]
and hence cover the entire group of chains. Thus \( H_1(Y, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \) is trivial. Clearly, the classes of the fibre and the section survive in \( H_2(X, \mathbb{Z}_2) \) (this means that the spectral sequence degenerates in the term \( E^2_{p,q} \), so that we obtain \( H_2(X, \mathbb{Z}_2) = \mathbb{Z}_2[(X_y), [\sigma_0]] \).

Now let us construct two Lagrangian embeddings of the Klein bottle in \( X \). Consider the following curves on \( \Sigma \):

\[
\beta_0 := \{t \cdot e^{2\pi i/3} \in \Sigma = \mathbb{C}/\Lambda : t \in \mathbb{R}\} \quad \beta_1 := \{\frac{1}{2} \cdot e^{\pi i/3} + t \cdot e^{2\pi i/3} \in \Sigma = \mathbb{C}/\Lambda : t \in \mathbb{R}\}
\]

Both curves are closed geodesics on \( \Sigma \) representing the homology class corresponding to \( \beta = e^{2\pi i/3} \in \Lambda \cong H_1(\Sigma, \mathbb{Z}) \). Moreover, both curves are invariant with respect to the transformation \( F_\xi = -\text{id} \). For the curve \( \beta_1 \), this follows from fact that \( 2 \cdot \frac{1}{2} e^{\pi i/3} = e^{\pi i/3} = e^{2\pi i/3} + 1 \) lies in the lattice \( \Lambda \).

Now define the surfaces \( K_0 \) and \( K_1 \) in \( X \) which consist of points \((y, z) \in X \) such that \( y \) lies on \( \xi \) and \( z \) on \( \beta_0 \) or \( \beta_1 \), respectively. In other words, we take a point \( y_0 \in \xi \), realise \( \beta_0 \) and \( \beta_1 \) on the fibre \( pr^{-1}(y_0) \), and then carry them around \( \xi \) using parallel transport. Since \( F_\xi(\beta_i) = \beta_i \), \( K_0 \) and \( K_1 \) are indeed closed surfaces. From the fact that \( F_\xi \) inverts the orientation on each \( \beta_i \) we conclude that \( K_i \) are Klein bottles. The local product structure of the symplectic form \( \omega \) ensures that \( K_i \) are \( \omega \)-Lagrangian.

To find the homology classes \([K_i] \in H_2(X, \mathbb{Z}_2)\) we observe the following. Since \( K_i \) are disjoint from the generic fibre of \( X \), each \( K_i \) must be \( \mathbb{Z}_2 \)-homologous to the fibre or homologically trivial. Since \( K_1 \) is disjoint from \( \sigma_0 \), \([K_1] = 0 \in H_2(X, \mathbb{Z}_2) \). On the other hand, the pair \( \beta_0, \beta_1 \) separates the torus \( \Sigma \), and applying the argument from the proof
of Proposition 1.5 to the union $\beta_0 \cup \beta_1$ instead of the meridian circle $m$ we conclude that $[K_0] + [K_1]$ is homologous to the fibre. Thus $[K_0] = [X_q] \neq 0 \in H_2(X, \mathbb{Z}_2)$.

2. Combinatorial structure of mapping class groups

2.1. Coxeter–Weyl and braid groups. Let us first recall standard definitions and facts concerning Coxeter–Weyl groups and Artin–Brieskorn braid groups.

Let $S = \{s_1, \ldots, s_r\}$ be any finite set. A \textit{Coxeter matrix} over $S$ is a symmetric $r \times r$-matrix $M = (m_{ij})$ with entries in $\mathbb{N} \cup \{\infty\}$, such that $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The pair $(S, M)$ is called the \textit{Coxeter system}. Often we shorten the notation for a Coxeter system to $S$ and understand $M$ as a structure on $S$, denoting it by $M_S$. If $s = s_i, s' = s_j \in S$, then $m_{ij}$ is also denoted by $m_{s_is'_i}$.

The \textit{Coxeter graph} $\Delta = \Delta(S, M)$ associated with a Coxeter system $(S, M)$ has $S$ as the set of vertices; $s_i$ and $s_j$ are connected by a wedge iff $m_{ij} \geq 3$; if $m_{ij} > 4$, this wedge is labelled by the entry $m_{ij} = m_{s_is_j}$. The labelled Coxeter graph $\Delta$ completely determines the associated Coxeter matrix.

For any subset $S' \subset S$ the induced Coxeter matrix $M' = M_{S'}$ onto $S'$. The associated graph $\Delta' := \Delta(S', M')$ is a full subgraph of $\Delta$, i.e., any two vertices $s_1, s_2 \in S'$ are connected and labelled in $\Delta'$ in the same way as in $\Delta$.

For any two letters $a, b$ and a non-negative integer $m$ we denote by $(ab)^m$ the word $abab\ldots$ of the length $m$ consisting of alternating letters $a$ and $b$. If $a$ and $b$ lie in a group $G$, then $(ab)^m$ denotes the corresponding product in $G$.

\textbf{Definition 2.1.} The \textit{Artin–Brieskorn braid group} $\text{Br}(S) = \text{Br}(S, M)$ of a Coxeter system $(S, M)$ is the group generated by $S$ with \textit{(generalised) braid relations}

$$\langle s_i s_j \rangle^{m_{ij}} = \langle s_j s_i \rangle^{m_{ij}} \quad \text{for each } i \neq j, \text{ such that } m_{ij} < \infty$$

as defining relations. The \textit{Coxeter–Weyl group} $W(S) = W(S, M)$ is obtained from the braid group $\text{Br}(S)$ by adding the \textit{reflection relation} $s^2 = 1$ for each $s \in S$.

The kernel of the natural projection $\pi : \text{Br}(S) \to W(S)$ is called the \textit{pure braid group} of the Coxeter system $(S, M)$ and is denoted by $P(S)$. Its \textit{abelianisation} is denoted by $P_{ab}(S)$.

To distinguish equalities in different groups, we use the notation $w \equiv w'$ for equality in $W(S)$ and $w = w'$ for equality in $\text{Br}(S)$. An \textit{expression} of an element $w$ from $W(S)$ or from $\text{Br}(S)$ is a word $s = s_i^{\epsilon_i} \ldots s_q^{\epsilon_q}$ in the alphabet $S$, $\epsilon_i = \pm 1$, whose evaluation in $W(S)$ or, respectively, in $\text{Br}(S)$ yields $w$. Such an expression $s$ is called \textit{positive} if all $\epsilon_i$ equal +1. In view of the reflection relation $s_i^{-1} = s_i$, we may consider only positive expressions of $w \in W(S)$. An element $w \in \text{Br}(S)$ is called positive if it admits a positive expression.

\textbf{Definition 2.2.} The \textit{length} $\ell(w)$ of $w \in W(S)$ is the minimal possible length of an expression $s = s_1 \ldots s_l$ of $w$. The \textit{W-length} of $x \in \text{Br}(S)$ is the length $\ell_W(\bar{x})$ of its projection $\bar{x} := \pi(x)$ to $W(S)$. An expression $s$ is \textit{reduced} (or \textit{W-reduced}) if it is positive and its length equals its W-length.

A \textit{quasireflection} in a Coxeter–Weyl group $W(S)$ is any element $t$ conjugated to a generator $s \in S$. We denote by $T(S)$ the set of quasireflection in $W(S)$. 
Note that quasireflections are usual reflections in finite Coxeter–Weyl groups, and transpositions in the symmetric groups $\text{Sym}_n$. We will use the following standard properties of Coxeter–Weyl groups, see e.g. [Bou] or [Hum].

**Proposition 2.1.** i) (Strong Exchange Property.) Let $s = s_1 \cdots s_q \in W(S)$ be a (positive) expression of an element $w \in W(S)$, $s \in S$ a generator, and $t \in T(S,M)$ a quasireflection. Then $\ell(ws) = \ell(w) \pm 1$ and $\ell(wt) - \ell(w)$ is odd. Moreover, if $\ell(wt) < \ell(w)$, then there exists $j = 1, \ldots, q$ such that $w \equiv s_1 \cdots s_{j-1} s_{j+1} \cdots s_q t$. The analogous property holds for the left products $sw$ and $tw$.

ii) (Relations between generators.) The order of $s_i s_j$ in $W(S)$ is $m_{ij}$. Furthermore, for any subset $S' \subset S$, the natural homomorphism $W(S') \to W(S)$ is injective, i.e., there are no new relations in $W(S)$ among the generators of $W(S')$.

iii) (Uniqueness of the set of factors.) For any $w \in W(S)$ and any reduced expression $w = s_1 \cdots s_l$ the set of factors $S_w := \{s_1, \ldots, s_l\}$ is independent of the particular choice of the reduced expression; the set $S_w$ generates the unique minimal subsystem $S' \subset S$ such that $w$ lies in $W(S')$.

iv) (Conjugacy classes of generators.) Two generators $s', s'' \in S$ are conjugated if and only if they can be connected by a chain $s_0 = s', s_1, \ldots, s_{n-1}, s_n = s''$ such that each index $m_{s_i-1, s_i}$ is odd. In particular, for a connected graph $G(S)$ with no labelling (i.e. such that each $m_{ij}$ is 2 or 3), the set of quasireflections $T(S)$ consists of elements conjugated to any given $s \in S$.

**Lemma 2.2.** Suppose that $w \in W(S,M)$ admits two factorisations $w \equiv s_1 w_1 \equiv s_2 w_2$ with $s_1 \neq s_2 \in S$ and $w_1, w_2 \in W(S)$ such that $\ell(w_1) = \ell(w_2) = \ell(w) - 1$. Set $m := m_{s_1 s_2}$. Then $w$ admits a factorisation $w \equiv s_1^{n_1} w_1 \equiv s_2^{n_2} w_2$ with $\ell(w_1) = \ell(w) - m$.

A similar property holds for factorisation $w \equiv s_1^{n_1} w_1 \equiv s_2^{n_2} w_2$. 

**Proof.** Fix factorisations $w \equiv \langle s_1 s_2 \rangle^{n_1} v_1$ and $w \equiv \langle s_2 s_1 \rangle^{n_2} v_2$ such that $\ell(v_1) = \ell(w) - n_i$ and such that the lengths $n_i$ are the maximal possible. Then $1 \leq n_i \leq m$. Indeed, if $n_1 > m$, then $w$ admits a reduced expression which starts with $\langle s_1 s_2 \rangle^{m+1}$. But then $\langle s_1 s_2 \rangle^{m+1} = s_1 \langle s_2 s_1 \rangle^{m} = s_1 \langle s_1 s_2 \rangle^{m} = s_1 \langle s_2 s_1 \rangle^{m-1} \equiv \langle s_2 s_1 \rangle^{m-1}$ has smaller $W$-length, in contradiction with $\ell(v_1) = \ell(w) - n_1$.

There is nothing to prove if $n_1$ or $n_2$ equals $m$, so we assume that $n_1, n_2 < m$. Fix reduced expressions $v_i$ for $v_i$, so that $\langle s_1 s_2 \rangle^{n_1} v_1$ and $\langle s_2 s_1 \rangle^{n_2} v_2$ are reduced expressions for $w$. Then by the Exchange Property applied to the relation $\ell(s_2 w) = \ell(w) - 1$, $w$ admits an expression $s_2 w_2$ such that $w_2$ is obtained by removing a single generator $s^*$ from $\langle s_1 s_2 \rangle^{n_1} v_1$. It could not be the first letter $s_1$ since $s_1 \neq s_2$. It could not be the last letter in the subword $\langle s_1 s_2 \rangle^{n_1}$ since then $\langle s_1 s_2 \rangle^{n_1} \equiv \langle s_2 s_1 \rangle^{n_1}$ in contradiction with $n_1 < m = m_{s_1 s_2}$. It could not also be an inner letter in the subword $\langle s_1 s_2 \rangle^{n_1}$ since then we would obtain a squared generator as a subword of $s_2 w_2$ and conclude that $\ell(s_2 w_2) < \ell(w)$. Thus the letter $s^*$ is removed from $v_1$. Denoting by $v_2'$ the obtained word, we get a reduced expression $\langle s_2 s_1 \rangle^{n_1+1} v_2'$. This implies that $n_2 > n_1$. By symmetry, we must have $n_1 > n_2$. This contradiction shows that in fact $n_1 = n_2 = m$, as desired. □
Lemma 2.3. For any two reduced expressions \( s = s_1 \cdots s_l \) and \( s' = s'_1 \cdots s'_l \) of a given \( w \in W(S) \), their evaluations in \( Br(S) \) define the same element \( \hat{w} \in Br(S) \). In particular, the correspondence \( w \in W(S) \mapsto \hat{w} \in Br(S) \) is a set-theoretic section of the projection \( \pi : Br(S) \to W(S) \).

Proof. Using induction on the length of \( w \) we may assume that the claim holds for all elements \( v \in W(S) \) of length \( \ell(v) < \ell(w) \). This implies the lemma in the case \( s_1 = s'_1 \), so we may assume that \( s_1 \neq s'_1 \). Set \( m := m_{s_1,s'_1} \). By the previous lemma, \( w \) admits reduced factorisations \( \langle s_1s'_1 \rangle^m w_0 \equiv \langle s'_1s_1 \rangle^m w_0 \). By induction, the lift \( \hat{w}_0 \in Br(S) \) is well-defined. Since \( \langle s_1s'_1 \rangle^m = \langle s'_1s_1 \rangle^m \) in \( Br(S) \), the latter two factorisations define the same element \( \langle s_1s'_1 \rangle^m \hat{w}_0 = \langle s'_1s_1 \rangle^m \hat{w}_0 \in Br(S) \). On the other hand, \( \hat{s} := s_1 \cdots s_l \) equals \( \langle s_1s'_1 \rangle^m \hat{w}_0 \) in \( Br(S) \) since they have the same first letter \( s_1 \). The same argument for \( \hat{s}' := s'_1 \cdots s'_l \) finishes the proof. \( \square \)

Applying the Reidemeister–Schreier theorem (see e.g. [Co-Zi] or [Ly-Sch]) we obtain a presentation of \( P(S) \).

Proposition 2.4. i) The pure braid group \( P(S) \) is generated by the elements \( (s_1 \cdots s_l) \cdot s_0^2 \cdot (s_1 \cdots s_l)^{-1} \) such that the product \( s_1 \cdots s_l s_0 \) is \( W \)-reduced.

ii) The Reidemeister–Schreier defining set of relations in \( P(S) \) consists of products of the form \( \hat{w} \cdot R \cdot \hat{w}^{-1} \), where \( w \) varies over the Coxeter–Weyl group \( W(S) \), \( R \) over the defining set of braid relations in \( Br(S) \), and \( \hat{w} \cdot R \cdot \hat{w}^{-1} \) is written as a product of the above generators \( (s_1 \cdots) \cdot s_0^2 \cdot (s_1 \cdots)^{-1} \).

Definition 2.3. The elements produced by the proposition are called Reidemeister–Schreier generators and relations, or simply RS-elements. If we denote by \( a*b \) the conjugation of \( b \) by \( a \) so that \( a*b := aba^{-1} \), then all RS-generators have the form \( \hat{w} \ast s_0^2 \).

Remark. More precisely, the generators produced by the Reidemeister–Schreier theorem are elements of the free group \( Fr(S) \), and their explicit form depends on the concrete expression \( s_1 \cdots s_l \). However, by Lemma 2.3, we only need to know the corresponding element in \( W(S) \).

Proof. We only indicate the algorithm which realises any \( f \in P(S) \) as a product of generators. Let \( f \) be represented in the form \( f = f_1 \cdot \hat{w} \cdot s^\epsilon \cdot f_2 \) so that \( f_1 \) is a product of RS-generators, \( \hat{w} \) is a \( W \)-reduced element in \( Br(S) \), \( s^\epsilon = s^{\pm 1} \) a letter with \( s \in S \), and \( f_2 \) any element in \( Br(S) \). Find \( w_1 \in W(S) \) such that \( w_1 \equiv w \cdot s \). Then \( f = f_1 \cdot (\hat{w} \cdot s^\epsilon \cdot \hat{w}_1^{-1}) \cdot \hat{w}_1 \cdot f_2 \). Thus we only need to represent \( \hat{w} \cdot s^\epsilon \cdot \hat{w}_1^{-1} \) as a product of RS-generators. Here we must consider four cases according to the possible values of the \( W \)-length of \( w_1 \) and the sign \( \epsilon = \pm 1 \). Recall that \( \ell_W(w_1) = \ell_W(w) \pm 1 \). Then we obtain

- case \( \ell(w_1) = \ell(w) + 1 \) and \( \epsilon = +1 \): \( \hat{w}_1 = \hat{w} \cdot s \) and \( \hat{w} \cdot s \cdot \hat{w}_1^{-1} = 1 \);
- case \( \ell(w_1) = \ell(w) + 1 \) and \( \epsilon = -1 \): \( \hat{w}_1 = \hat{w} \cdot s \) and \( \hat{w} \cdot s^{-1} \cdot \hat{w}_1^{-1} = (\hat{w} \ast s^2)^{-1} \);
- case \( \ell(w_1) = \ell(w) - 1 \) and \( \epsilon = +1 \): \( \hat{w} = \hat{w}_1 \) and \( \hat{w} \cdot s \cdot \hat{w}_1^{-1} = \hat{w}_1 \ast s^2 \);
- case \( \ell(w_1) = \ell(w) - 1 \) and \( \epsilon = -1 \): \( \hat{w} = \hat{w}_1 \) and \( \hat{w} \cdot s^{-1} \cdot \hat{w}_1^{-1} = 1 \). \( \square \)
Proposition 2.5. i) The abelianisation $P_{ab}(S)$ of the pure braid group $P(S)$ is a free abelian group admitting a basis generated by RS-elements $\hat{w} \ast s^2 \in P(S)$.

ii) The projection of every element $x \ast s^2$ with $x \in Br(S)$ and $s \in S$ into $P_{ab}(S)$ is a generator $[\hat{w} \ast s^2]_a^b$ for some $w \in W(S)$ and $s \in S$, not unique in general. Moreover, $\hat{w} \ast \hat{s}$ can be obtained by a sequence of the following transformations:

(A0) replacing $x \ast s^2$ with $x \in Br(S)$ by $x' \ast s^2$ with some $x' \in Br(S)$ such that $x$ and $x'$ have equal projections to $W(S)$, $x \equiv x' \in W(S)$;

(A1) replacing $(x \cdot (s_1 s_2)^k) \ast s_3^2$ by $(x \cdot (s_2 s_1)^m \cdot s_1^k) \ast s_2^2$ where

- $-s_3 = s_1$ if $k$ is even and $s_3 = s_2$ if $k$ is odd,
- $s_3' = s_3$ if $m$ is even and $s_3' = s_2$ otherwise.

Proof. First let us show that relations (A0) and (A1) hold in $P_{ab}(S)$. The first one follows from the equality

$$(xs_2 y) \ast s_2^2 = (xs_2 y) s_2^2 (xs_2 y)^{-1} = xs_2 y^{-1} \cdot (xy) s_2^2 (xy)^{-1} \cdot (xs_2 y^{-1})^{-1}. $$

For the second one, we observe that it is sufficient to consider the case $x = 1$. Represent the braid relation $(s_1 s_2)^m = (s_2 s_1)^m$ in the form $(s_1 s_2)^m \cdot s_3 = s_2 (s_1 s_2)^m \cdot s_3$. An algebraic manipulation gives

$$(s_1 s_2)^{m-1} \cdot s_3 ((s_1 s_2)^{m-1})^{-1} = s_2,$$

so that squaring yields the desired relation for $k = m - 1$. The conjugation by $((s_1 s_2)^l)^{-1}$ and application of (A0) gives the rest. In the special case $s_1 = s_2 := s$ we must have $m = 1$ and $k = 0$, so that $m - k - 1 = 0$ and the only relation in (A1) is the trivial equality $(x \ast s) \ast s^2 = x \ast s^2$.

Fix $a \neq b \in S$ such that $m := m_{ab} \neq \infty$. Let $w_0 \in W(S)$ be any element. Set $a_{2i-1} := b_{2i} := a$, $a_{2i} := b_{2i-1} := b$, $w_{i} := w_i \cdot \langle ab \rangle^i$, and $w_{i}' := w_i \cdot \langle ba \rangle^i$. Observe that $w_i \ast w_i' \equiv w_i \ast w_i' \equiv 2m$-periodic, that is, $w_{i+2m} \equiv w_i$ and $w_{i+2m} \equiv w_i'$. Besides, $b_i = a_{2m-i}$ and $w_i' \equiv w_{2m-i}$. The induced RS-relation reads

$$\prod_{i=1}^m (\hat{w}_{i-1} \cdot a_i \cdot \hat{w}_i^{-1}) = \prod_{i=1}^m (\hat{w}_{i-1}' \cdot b_i \cdot \hat{w}_i'^{-1}).$$

Let $w_k$ be an element of maximal $W$-length. Then $\ell(w_{k-1}) = \ell(w_{k+1}) = \ell(w_k) - 1$. In this situation Lemma 22 implies that $v := w_{k \pm m}$ is the shortest element with $\ell(v) = \ell(w_k) - m$ and for every $i = 0, \ldots, 2m$ one has either $w_{k \pm m \pm i} \equiv v \langle ab \rangle^i$ or $w_{k \pm m \pm i} \equiv v \langle ba \rangle^i$. It follows that $w_i \equiv v \cdot v_i$ with $\ell(w_i) = \ell(v) + \ell(v_i)$ such that $v_0$ is either $\langle ab \rangle^l$ or $\langle ba \rangle^l$ for some $l = 0, \ldots, m$. This means that the relation we consider is obtained by conjugation by $v$ from the relation

$$(2.1) \prod_{i=1}^m (\hat{v}_{i-1} \cdot a_i \cdot \hat{v}_i^{-1}) = \prod_{i=1}^m (\hat{v}_{i-1}' \cdot b_i \cdot \hat{v}_i'^{-1}),$$

where $v_i$, $v_i'$ are defined in the same way as $w_i$, $w_i'$.

In view of the symmetry $a \leftrightarrow b$, we assume that $v_0 = \langle ab \rangle^l$ with $l \leq m$. If $l$ is even, we obtain the following sequences of words $v_i$ and $v_i'$:

$$v_i \equiv \begin{cases} \langle ab \rangle^{l+i} & i = 0, \ldots, m-l \\ \langle ab \rangle^m \equiv \langle ba \rangle^i & i = m-l \\ \langle ba \rangle^{m+l-i} & i = m-l, \ldots, m \end{cases}$$

$$v_i' \equiv \begin{cases} \langle ab \rangle^{l-i} & i = 0, \ldots, l \\ \langle ab \rangle^0 \equiv \langle ba \rangle^0 \equiv 1 & i = m-l \\ \langle ba \rangle^{i-l} & i = l, \ldots, m \end{cases}$$
If \( l \) is odd, these two sequences are interchanged. The element \( \hat{v}_{l-1} \cdot a_i \cdot \hat{v}_i^{-1} \) is non-trivial if and only if \( \ell(v_{l-1}) > \ell(v_i) \), and the same is true for \( \hat{v}_{l-1} \cdot b_i \cdot \hat{v}_i^{-1} \). The corresponding values of \( i \) are \( i = m - l + 1, \ldots, m \) for \( v_i \) and \( i = 1, \ldots, l \) for \( v'_i \). Comparing the corresponding factors, we see that the induced relation in \( P^i_{ab}(S) \) is the sum of relations (A1) over \( k = 1, \ldots, l \).

**Theorem 2.6.** Two quasireflections \( t = w * s \) and \( t' = w' * s' \) in \( W(S) \) with \( w, w' \in W(S) \) and \( s, s' \in S \) are equal if and only if the second can be obtained from the first by a sequence of the following transformations:

- \( T \) replacing \( (x \cdot (s_1 s_2)^k) * s_3 \) by \( (x \cdot (s_2 s_1)^{m-k}) * s'_3 \) where \( x \in W(S) \) and \( s_1, s_2, s_3, s'_3 \) have the same meaning as in part (A1) of Proposition 2.5.

In particular, the correspondence

\[
t \equiv w \ast s \in T(S) \iff A_i := \hat{w} \ast s^2 \in P_{ab}(S) \quad \text{for } w \in W(S) \text{ and } s \in S
\]

defines a bijection between the set of quasireflections in \( W(S) \) and the RS-basis of \( P_{ab}(S) \).

**Proof.** Denote by \( A(S) \) the RS-basis of \( P_{ab}(\Delta) \) constructed above and by \( [\hat{w} \ast s^2]_{ab} \) the image of \( \hat{w} \ast s^2 = \hat{w} \cdot s^2 \cdot \hat{w}^{-1} \) in \( A(S) \). Observe that the transformation (A1) from Proposition 2.5 is obtained by squaring the transformation (T) and that the analog of (A0) in \( T(S) \) is the trivial transformation. This implies the second claim of the theorem modulo the first claim.

In any case, we obtain a well-defined surjective map \( t : A(S) \rightarrow T(S) \) given by \( t : [\hat{w} \ast s^2]_{ab} \mapsto w \ast s \), and the first claim of the theorem is equivalent to the injectivity of \( t \). Since \( t \) is \( W(S) \)-invariant it is sufficient to show that for every \( s \in S \) and every \( w \in W(S) \) the condition \( w \ast s \equiv s \) implies the equality \( [\hat{w} \ast s^2]_{ab} = [s^2]_{ab} \). We proceed by induction on the length \( \ell(w) \). The case \( \ell(w) = 0 \) is trivial. Fix a pair \((w, s)\) with the property \( w \ast s \equiv s \). Assume that \( \ell(ws) = \ell(w) - 1 \). Then by the Exchange Property \( w = w's \) with \( \ell(w') = \ell(w) - 1 \). But then

\[
w \ast s \equiv (w' \ast s) \ast s \equiv (w' \ast s) \cdot s \cdot (sw')^{-1} \equiv w' \cdot s \cdot w'^{-1} \equiv w' \ast s,
\]

and hence \( [\hat{w} \ast s^2]_{ab} = [\hat{w'} \ast s^2]_{ab} = [s^2]_{ab} \) by the inductive assumption. In the case \( \ell(ws) = \ell(w) + 1 \) we conclude from Lemma 2.3 the equality \( \hat{w}s = \hat{w}s \) in \( Br(S) \). On the other hand, the equality \( w \ast s \equiv s \) means that \( w \) and \( s \) commute in \( W(S) \), and hence \( \ell(sw) = \ell(ws) = \ell(w) + 1 \). As above, we obtain \( \hat{s} \hat{w} = s \hat{w} \) and hence \( \hat{w}s = s \hat{w} \). This implies the desired identity \( [\hat{w} \ast s^2]_{ab} = [s^2]_{ab} \). The proof follows.

Summing up the results obtained so far, we have

**Theorem 2.7.** For any quasireflection \( t \in T(S) \) and any two representing expressions \( t \equiv w \ast s \) and \( t' \equiv w' \ast s' \) with \( w, w' \in W(S) \) and \( s, s' \in S \), the evaluations \( \hat{w} \ast s \) and \( \hat{w}' \ast s' \) define the same element \( t \in Br(S) \).

The set \( \hat{T}^2 := \{ \hat{t}^2 : t \in T(S) \} \) is a minimal generator set for the pure braid group \( P(S) \).
Proof. Let $s_1, s_2 \in S$ be generators and $k < m_{s_1 s_2}$ a non-negative integer. Then the relation $\langle s_1 s_2 \rangle^k s_3 = (s_2 s_1)^{m-k} \cdot s_3'$ with $s_3, s_3'$ as in the transformation (T) holds in $\text{Br}(S)$. Moreover, even if $s_1 \langle s_1 s_2 \rangle^k$ is not $W$-reduced, we obtain the equality $$(s_1 \langle s_1 s_2 \rangle^k)^{*} s_3 \stackrel{(T)}{=} \langle s_1 s_2 s_1 \rangle^{m-k-1} s_3' = \langle s_1 s_2 \rangle^{m-k} s_3' \stackrel{(T)}{=} \langle s_2 s_1 \rangle^{k-1} s_3$$ in $\text{Br}(S)$. Similarly,

$$(s_2 \langle s_1 s_2 \rangle^k)^{*} s_3 \stackrel{(T)}{=} \langle s_2 s_2 s_1 \rangle^{m-k-1} s_3' = \langle s_1 s_2 \rangle^{m-k-2} s_3'$$

for $k > 0$. This implies that the relation

$$(\hat{x} s \hat{s}^* s, s_3 \stackrel{(T)}{=} (x \cdot \langle s_1 s_2 \rangle^{m-k-1} \hat{s}^* s_3'),$$

where $x \in W(S)$ and $s_1, s_2, s_3, s_3'$ have the same meaning as in part (A1) of Proposition 2.5 holds in $\text{Br}(S)$ even if $x \cdot \langle s_1 s_2 \rangle^k$ or $x \cdot \langle s_2 s_1 \rangle^{m-k-1}$ is not $W$-reduced. It follows that $t = w * s \in \mathcal{T}(S) \mapsto \hat{t} := \hat{w} * s$ induces a well-defined bijection between $\mathcal{T}(S)$ and the sets $\mathcal{T}^2(S)$ and $\mathcal{T}(S) := \{ t : t \in \mathcal{T}(S) \}$ whose inverse can be obtained as the composition $\hat{t} \in \mathcal{T}(S) \mapsto \mathcal{T}^2(S) \mapsto A_t = [ \mathcal{T}_i ]_{ab} \in P_{ab}(S)$.

Now let $x \in \text{Br}(S)$ be an element which is either not $W$-reduced or non-positive. Then there exists a factorisation of the form $x = x' \cdot s' \cdot x''$ or respectively $x = x' \cdot s''^{-1} \cdot x''$ in which the sum of Br-lengths of $x'$ and $x''$ is smaller than that of $x$. Let $s \in S$ be any generator. Then $x * s^2$ can be factorised as

$$x * s^2 = (x' * (s^2 \cdot x'')) * s^2 = ((x' * s) \cdot (x' * s^2)) * (x' * s^{-2})$$

in the first case, and as

$$x * s^2 = (x' * s^{-1} \cdot x'') * s^2 = ((x' * s) \cdot (x' * s'^2)) * (x' * s^2)$$

in the second case. Using this procedure one can express any RS-generator $x * s^2$ as a word in the alphabet $\mathcal{T}^2(S)$. Consequently, $\mathcal{T}^2(S)$ generates $P(S)$. The minimality of the generator system $\mathcal{T}^2(S)$ follows from Theorem 2.6. Indeed, every proper subset of $\mathcal{T}^2(S)$ generates a proper subgroup of $P_{ab}(S)$.

Finally, let us observe that if $l(w \cdot s) = l(w) - 1$ for some $w \in \text{Br}(S)$ and $s \in S$, then $w = w' \cdot s$ with $w' \equiv w \cdot s$ and hence $\hat{w} * s = w' * (s \cdot s) = w' * s$. Thus every RS-generator $\hat{t}^2 \in \mathcal{T}^2(S)$ can be represented in the form $\hat{t}_2 = \hat{w} * s^2$ with a $W$-reduced word $w \cdot s$. □

Let us now present a procedure that will allow us to describe the possible factorisation of a given $a \in P(S)$ as a product of RS-generators $\hat{t}_2 = \hat{w}_i \cdot s_i^2 \in \mathcal{T}^2(S)$.

**Definition 2.4.** A factorisation of length $l$ of an element $x$ of a group $G$ is an expression $f = f_1 \cdot f_2 \cdots f_l$ whose evaluation in $G$ gives $x$. A Hurwitz move is a transformation of such an $f = f_1 \cdot f_2 \cdots f_l$ whereby a pair $f_i, f_{i+1}$ is replaced by $(f_{i+1})^{-1} f_{i+1} f_i f_{i+1}$ or by $(f_i f_{i+1})^{-1} f_i f_{i+1}$ and the remaining factors remain unchanged. We say either that $f_i$ is shifted to the right and $f_{i+1}$ is shifted with conjugation to the left or, respectively, that $f_i$ is shifted with conjugation.

A Hurwitz transformation is a sequence of Hurwitz moves. Two factorisations $f$ and $f'$ connected by a Hurwitz transformation are called Hurwitz equivalent.

Further details see e.g. in [Kl-Ku].
Theorem 2.8. (Hurwitz Problem in Coxeter–Weyl Groups). Let \( t = t_1 \cdot t_2 \cdots \cdot t_l \) be a factorisation of the identity \( 1 \in W(S) \) into quasirefections \( t_i \in T(S) \). Then the length \( l \) is even, \( l = 2l' \), and \( t \) is Hurwitz equivalent to a factorisation \( t' \) into squares of quasirefections, i.e., \( t' = t'_1 \cdot t'_2 \cdots \cdot t'_l \) with \( t_{2i-1} \equiv t'_{2i} \in T(S) \).

For the special case of finite Weyl groups \( W \) corresponding to simple complex Lie algebras, this result was obtained by Kanev [Kan], Proposition 2.3. His proof exploits another technique, namely, the geometry and combinatorics of the corresponding root systems.

Proof. Since the set of quasirefections \( T(S) \) is invariant under conjugations, any Hurwitz transformation of \( t \) changes \( t \) into a factorisation into quasirefections.

For a quasirefection \( t \in T(S) \) define its width as the smallest \( k \) such that \( t \) can be represented in the form \( t = w \cdot s \) with \( w \in W(S), \ s \in S \), and such that \( k = \ell(w) \).

Shifting the first factor \( t_1 \) with conjugation to the right end, we obtain a factorisation \( t_2 \cdot t_3 \cdots t_l \cdot \tilde{t}_1 \). Comparing it with the original factorisation \( t = t_1 \cdot t_2 \cdots \cdot t_l \equiv 1 \), we obtain the equality \( \tilde{t}_1 \equiv t_1 \). Consequently, a cyclic permutation of the factors of \( t \) gives a Hurwitz equivalent factorisation. Thus we may assume that the last factor \( t_l \) has the largest width among all \( t_i \).

Now let us represent each \( t_i, \ i = 1, \ldots, l - 1 \), in the form \( t_i \equiv w_i \cdot s_i \) with the smallest possible length \( \ell(w_i) \). Fix a reduced expression \( w_i \) for each \( w_i \). Observe that inverting an expression \( w \) of any element \( w \in W(S) \) produces an expression for the inverse element \( w^{-1} \). In view of this, we denote by \( w^{-1} \) the inversion of an expression \( w \).

Now consider the expression \( w := \prod_{i=1}^{l-1} w_i \cdot s_i \cdot w_i^{-1} \). Its evaluation in \( W(S) \) gives \( t_i^{-1} \equiv t_i \), and its product with \( t_i \) gives \( 1 \). By the Strong Exchange Property there exists a letter \( s \) in \( w \) whose removal from \( w \) gives an expression for \( 1 \).

Consider first the case when \( s \) is the middle letter \( s_j \) of the expression \( w_j \cdot s_j \cdot w_j^{-1} \) for some \( t_j \). Then the remaining pieces \( w_j \cdot w_j^{-1} \) cancel, and hence

\[
 t_1 \cdots t_{j-1} \cdot t_j \cdot t_{j+1} \cdots t_{l-1} \equiv t_1 \cdots t_{j-1} \cdot t_{j+1} \cdots t_{l-1} \cdot t_l
\]

On the other hand, shifting with conjugation the factor \( t_j \) in \( t_1 \cdots \cdot t_{j-1} \cdot t_j \cdot t_{j+1} \cdots \cdot t_{l-1} \) to the right end, we obtain a factorisation \( t_1 \cdots \cdot t_{j-1} \cdot t_{j+1} \cdots t_{l-1} \cdot \tilde{t}_j \). Consequently, \( \tilde{t}_j \equiv t_j \) and \( t_1 \cdots \cdot t_{j-1} \cdot t_j \cdot t_{j+1} \cdots t_{l-1} \cdot t_l \) is Hurwitz equivalent to the factorisation \( t_1 \cdots \cdot t_{j-1} \cdot t_{j+1} \cdots t_{l-1} \cdot t_l \). Since in this case \( t_1 \cdots \cdot t_{j-1} \cdot t_{j+1} \cdots t_{l-1} \equiv 1 \), we can complete the proof using induction on \( l \).

It remains to consider the case when the letter \( s \) appears, say, in the initial subword \( w_j \) in the expression \( w_j \cdot s_j \cdot w_j^{-1} \) for some \( t_j \). (The case when the letter \( s \) appears in the final subword \( w_j^{-1} \) can be treated similarly.) Write \( w_j \) in the form \( w' s w'' \). Then \( w' w'' \equiv (w' s w''^{-1}) \cdot w' s w'' \) and hence

\[
(2.2) \quad w' w'' \cdot s_j \cdot w_j^{-1} \equiv (w' s w''^{-1}) \cdot (w_j s_j w_j^{-1}).
\]

Now shift \( t_l \) with conjugation to the left in-between \( t_{j-1} \) and \( t_j \). This gives us a factorisation of the form \( t_1 \cdots \cdot t_{j-1} \cdot \tilde{t}_l \cdot t_j \cdot t_{j+1} \cdots \cdot t_{l-1} \). Comparing it with (2.2) we obtain the equality \( \tilde{t}_l \equiv w' s w''^{-1} \). Observe that the width of \( \tilde{t}_l \equiv w' s w''^{-1} \) is at most \( \ell(w') < \ell(w' s w'') \) and hence less than the width of \( t_l \). Now we can use induction on the sum of the widths of \( t_i \).  \( \square \)
Now consider the following situation. Let $G$ be a subgroup of $\mathcal{W}(S)$. Then $G$ acts on $\mathcal{P}_{ab}(S)$ by conjugation and we denote by $\mathcal{P}_{ab}(S)_G$ the group of \textit{coinvariants} (see [Bro]). Recall that $\mathcal{P}_{ab}(S)_G$ is the quotient of $\mathcal{P}_{ab}(S)$ by the subgroup generated by the elements of the form $w \cdot A - A$ with $w \in G$ and $A \in \mathcal{P}_{ab}(S)$. Since $\mathcal{W}(S)$ permutes basis elements in $\mathcal{P}_{ab}(S)$, the group $\mathcal{P}_{ab}(S)_G$ is a free abelian group with a basis given by the quotient set $\mathcal{T}(S)/G$. Another description of $\mathcal{P}_{ab}(S)_G$ is as the quotient of $\mathcal{P}(S)$ by the commutator group $[\widehat{G}, \mathcal{P}(S)]$ where $\widehat{G}$ is the pre-image of $G$ in $\text{Br}(S)$. We denote the elements of $\mathcal{T}(S)/G$ by $G \cdot t$ and the elements of the induced basis in $\mathcal{P}_{ab}(S)_G$ by $A_{G \cdot t}$ or $A_G$.

**Theorem 2.9.** Let $G$ be a subgroup of $\mathcal{W}(S)$ and $x$ an element of $\mathcal{P}(S)$ which can be represented as the product $x = \prod_i \hat{t}_i \cdot \prod_j [x_{2j-1}, x_{2j}]$ such that $\hat{t}_i$ are quasigenerators, $\epsilon_i = \pm 1$, and $[x_{2j-1}, x_{2j}]$ are commutators of some $x \in \text{Br}(S)$. Assume that the projections of $x_i$ and $\hat{t}_j$ to $\mathcal{W}(S)$ lie in $G$. Then the projection $[x]_G$ of $x$ to $\mathcal{P}_{ab}(S)_G$ lies in the free abelian group generated by basis elements $A_{G \cdot t}$ with $t \in G \cap \mathcal{T}(S)$.

**Proof.** First, we reduce the general case to the special one with no commutators. For this purpose we represent each $x_i$ in the form $x_i = x_i' \cdot x_i''$ so that $x_i'$ is a product of quasigenerators projecting to $G$, and $x_i''$ is a product of squares of quasigenerators. Using the commutation relations $[x, y \cdot z] = [x, y] \cdot (y \cdot [x, z])$ and $[x \cdot y, z] = (x \cdot [y, z]) \cdot [x, z]$ we expand each commutator $[x_{2j-1}, x_{2j}]$ as a product of quasigenerators projecting to $G$ and commutators $[y, \hat{t}^2]$ such that $y$ projects to $G$ and $\hat{t}^2$ is a squared quasigenerator. Since such $[y, \hat{t}^2]$ lie in $\mathcal{P}(S)$ and project to zero in $\mathcal{P}_{ab}(S)_G$, we obtain the desired reduction.

Observe that Hurwitz moves do not destroy the properties of factors $\hat{t}_i$ listed in the hypothesis. Thus applying **Theorem 2.8**, we transform the original factorisation into a new one, still denoted in the same way, in which $t_{2i-1} \equiv t_{2i} \in \mathcal{W}(S)$. This means that it is enough to consider the case when $x$ is the product of two factors, $x = \hat{t}_1 \cdot \hat{t}_2$ with $t_1 \equiv t_2$. In the case when the signs $\epsilon_i$ are equal, say, $\epsilon_1 = \epsilon_2 = +1$, we can rewrite this product in the form $\hat{t}_1 \hat{t}_2^{-1} \hat{t}_2$. Since $t_2 \in G \cap \mathcal{T}(S)$ by our assumptions, the square $\hat{t}_2^2$ has the desired form. Thus we may additionally assume that $x = \hat{t}_1 \cdot \hat{t}_2^{-1}$.

We claim that the condition $t_1 \equiv t_2 \in \mathcal{W}(S)$ implies that $\hat{t}_2$ can be obtained from $\hat{t}_1$ by conjugation with some $z \in \mathcal{P}(S)$. For this purpose we write $\hat{t}_1 = y_1 \cdot s_1$ with $s_1 \in S$ and $y_1 \in \text{Br}(S)$ and conjugate both $\hat{t}_i$ by $y_1^{-1}$. This reduces the situation to the special case when $\hat{t}_1 = s_1$ is a usual generator of $\text{Br}(S)$. Adjusting the notation, we still have $\hat{t}_2 = y_2 \cdot s_2$. Let $w$ be the image of $y_2$ in $\mathcal{W}(S)$. Our claim would follow from the equality $\hat{w} \cdot s_2 = s_1$ in the braid group. Let us prove it. Assume first that $\ell(ws_2) < \ell(w)$. Then by the Exchange Property $w = w' s_2$ with $\ell(w') = \ell(w) - 1$. In this case $\ell(w' s_2) > \ell(w')$, $\hat{w} \cdot s_2 = (\hat{w}' s_2) \cdot s_2 = \hat{w}' \cdot s_2$, and still $\hat{w} \cdot s_2 \equiv s_1$ in $\mathcal{W}(S)$. Thus we may assume that $\ell(ws_2) > \ell(w)$. Observe that the equality $w \cdot s_2 \equiv s_1$ is equivalent to $ws_2 \equiv s_1 w$. As $\ell(ws_2) > \ell(w)$, the expressions $ws_2$ and $s_1 w$ are reduced for any reduced expression $w$ of $w$. Consequently, $\hat{w} \cdot s_2 = s_1 \cdot \hat{w}$ in the braid group. This is the desired identity $\hat{w} \cdot s_2 = s_1$, which proves the claim.

Summing up, it remains to consider a commutator $\hat{t} z \hat{t}^{-1} z^{-1}$ where $z \in \mathcal{P}(S)$ and $\hat{t} \in G$. By definition, such elements project to 0 in $\mathcal{P}_{ab}(S)_G$. This finishes the proof. \(\square\)
**Definition 2.5.** A Coxeter system \((S, M)\) is **irreducible** if its Coxeter graph is connected, and has **finite type** if the group \(W(S)\) is finite. It is known that every irreducible Coxeter system \((S, M)\) of finite type has the unique longest element \(w_o = w_o(S) \in W(S)\). The canonical lift \(\hat{w}_o \in \text{Br}(S)\) is called the **Garside element** of \(\text{Br}(S)\) and is denoted by \(\Delta(S)\).

The classification theory of irreducible Coxeter groups of finite type (see [Bou] or [Hum]) says that such a group is either the Weyl group of a simple complex Lie group, or the dihedral group \(D_{2m}\) with the Coxeter system on \(S = \{s_1, s_2\}\) given by relation \(m_{s_1, s_2} = m, m = 5, 7, 8, 9, \ldots\), or else one of the groups \(H_3, H_4\). It is known that the longest element \(w_o \in W(S)\) has the property \(\ell(w_o) = \ell(w_o w) + \ell(w)\) for any \(w \in W(S)\). For the properties of the Garside element we refer the reader to [Br-Sa] and [Del].

**Lemma 2.10.** The square \(\Delta^2(S)\) of the Garside element lies in \(P(S)\) and is equal in \(P_{ab}(S)\) to the sum \(\sum_{t \in T(S)} A_t\).

**Proof.** The property \(\ell(w_o w) = \ell(w_o) - \ell(w)\) applied to \(w = w_o\) implies that \(w_o\) is an idempotent. Thus \(\Delta^2 = \hat{w}_o^2\) lies in \(P(S)\).

Take any generator \(s \in S\) and any reduced expression \(w\) of \(w_o\). Then \(\ell(sw_o) = \ell(w_o) - \ell(s) < \ell(w_o)\). By the Strong Exchange Property, removing an appropriate letter \(s'\) from \(w\) we obtain a word \(w_1\) which is an expression for \(sw_o\). Comparing the lengths we see that \(w_1\) is reduced. Writing \(w_1\) as \(s_2 \cdots s_l\) with \(l := \ell(w_o)\), we see that \(ss_2 \cdots s_l\) is a reduced expression for \(w_o\). Since \(w_o\) is an idempotent, \(s_1 \cdots s_2 s\) is also a reduced expression for \(w_o\). Consequently, \(\Delta^2 = s_1 s_2 \cdots s_l s_1 s_2 \cdots s_l s\). Denoting \(s_1 := s\), transform the latter expression into the product

\[
\Delta^2 = \prod_{i=1}^l (\hat{s}_1^{-1} \hat{s}_2^{-1} \cdots \hat{s}_{i-1}^{-1} * \hat{s}_i^2).
\]

It shows that \(\Delta^2\) is equal in \(P_{ab}(S)\) to a sum \(\sum_{t \in T(S)} n_t A_t\) with non-negative integers \(n_t\) such that \(n_s > 0\) for any \(s \in S\). Since \(\Delta^2\) lies in the centre of \(\text{Br}(S)\), the sum \(\sum_{t \in T(S)} n_t A_t\) is invariant under the action of \(W(S)\). Since \(W(S)\) permutes the basis elements \(A_t\) of \(P_{ab}(S)\) and any quasireflection \(t \in T(S)\) is conjugated to some \(s \in S\), we obtain the inequality \(n_t \geq 1\) for all \(t \in T(S)\).

On the other hand, it is shown in [Br-Sa] and [Del] that \(\Delta^2 = \hat{\Pi}^h\) where \(\Pi\) is the so-called **Coxeter element** in \(W(S)\) and \(h\) is the **Coxeter number** of the system \(S\), see e.g. [Bou] or [Hum] for definitions. The formula for the Coxeter number can be reformulated as the equality of the length \(l = \ell(w_o)\) and the number of quasireflection. This shows that \(\sum_{t \in T(S)} n_t = l\). Consequently, all \(n_t\) are equal to 1, and the lemma follows.

**2.2. Combinatorial structure of Map.** In this paragraph we solve certain factorisation problems in the mapping class group, which is the key ingredient in our proof of the **Main Theorem**.

We use special finite presentations of \(\text{Map}_g\) and \(\text{Map}_{g,[1]}\) due to Wajnryb [Waj] and Matsumoto [Ma] which realise these groups as quotients of braid groups corresponding to certain Coxeter systems \(S_g\) with additional relations given in terms of Garside elements of appropriate subsystems \(S'_g\) of \(S_g\). Let us give a geometric description of these relations. For further details see e.g. [Bi-1] and [Ger].
Definition 2.6. (Chain and lantern relations.) Consider a surface $C$ which is a torus with 2 holes. Denote its boundary circles by $\eta', \eta''$. Consider the curves $\alpha, \beta, \beta'$ on the surface as shown on Figure 1. Note that if $\alpha, \beta, \beta'$ are embedded circles on a surface $\Sigma$ such that $\beta$ and $\beta'$ are disjoint and each meets $\alpha$ transversally at a single point, then a collar neighbourhood $U$ of the graph $\alpha \cup \beta \cup \beta'$ is a torus with 2 holes and the whole configuration is diffeomorphic to the one on Figure 1. We call such a surface $C \subset \Sigma$ and the whole configuration $(C, \alpha, \beta, \beta')$ a chain in $\Sigma$ defined by $\alpha, \beta, \beta'$. For any chain configuration, the chain relation element is defined by the formula

$$C(\alpha, \beta, \beta') := (T_{\beta}T_{\alpha}T_{\beta'})^4(T_{\eta'}T_{\eta''})^{-1}.$$  

Consider a surface $L$ which is a sphere with 4 holes. Denote its boundary circles by $\alpha_1, \ldots, \alpha_4$. Realise it as a disc with 3 holes and consider the curves $\beta_1, \beta_2, \beta_3$ on it as shown on Figure 2. Observe that if $\beta_1, \beta_2$ are embedded circles on a surface $\Sigma$ which meet at two points, then a collar neighbourhood $U$ of the graph $\beta_1 \cup \beta_2$ is a disc with 3 holes and the whole configuration is diffeomorphic to the one on Figure 2. We call such a surface $L \subset \Sigma$ and the whole configuration $(L, \alpha_i, \beta_j)$ a lantern in $\Sigma$ defined by $\beta_1, \beta_2$. For any lantern configuration, the lantern relation element is defined by the formula

$$L(\beta_1, \beta_2) := (T_{\alpha_1}T_{\alpha_2}T_{\alpha_4})(T_{\beta_1}T_{\beta_2}T_{\beta_3})^{-1}.$$  

A chain or, respectively, lantern relation is the equality $C(\alpha, \beta, \beta') = 1$ or, respectively, $L(\beta_1, \beta_2) = 1$ for the corresponding relation elements. A configuration (or relation) is called non-separating if all the circles involved in it are non-separating.

As suggested by the terminology, the relations above hold in the mapping class group $\text{Map}_{g,k,l}$ of the surface of genus $g$ with $k$ marked points and $l$ boundary circles. (Our notation uses the fact that $\text{Map}_{g,k,l}$ can be defined as the mapping class group of the surface of genus $g$ with $k+l$ marked points such that $l$ of them, say, $z_1, \ldots, z_l$, are framed, i.e., equipped with a trivialisation of the tangent plane $T_{z_i} \Sigma$.) We will only consider the case with at most one marked point, i.e., $k+l \leq 1$, and shorten our notation to $\text{Map}_{g,1}$ or $\text{Map}_{g,[1]}$, dropping the vanishing index $k$ or $l$. 

---

**Figure 1.** Curves in a chain configuration.
Definition 2.7. For $g = 1$, we set $S_1 := A_2$ so that the graph of $S_1$ consists of two vertices $s_1$ and $s_2$ connected by an edge. For $g \geq 2$, the graph of $S_g$ is defined as the extension of the Dynkin linear graph $A_{2g}$ by a single vertex $s_0$ connected to $s_4$. Thus, the graph of $S_g$ looks like

$$s_0 \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad \cdots \quad s_{2g-1} \quad s_{2g}$$

Each element $\hat{t} \in \text{Br}(S_g)$ conjugated to a generator $s_i$ is called a quasigenerator. The set of quasigenerators is denoted by $\hat{T}(S_g)$.

Theorem 2.11. The group $\text{Map}_{g,[1]}$ is isomorphic to the quotient of $\text{Br}(S_g)$ obtained by adding the following relations:

- In the case $g = 1$: No additional relations.
- In the case $g = 2$: A single non-separating chain relation $C(\alpha, \beta, \beta') = 1$.
- In the case $g \geq 3$: A single non-separating chain relation $C(\alpha, \beta, \beta') = 1$ and a single non-separating lantern relation $L(\beta', \beta'') = 1$.

The braid generators $s_i$, $i = 0, 1, \ldots, 2g$, can be realised as Dehn twists along the curves $a_i$ shown on Figure 3. In particular, $a_0 = \beta_2$, $a_1 = \beta_1$, $a_2i = \alpha_i$ for $i = 1, \ldots, g$, and each $a_{2i-1}$ is $\mathbb{Z}$-homologous to $[\beta_{i-1}]+[\beta_i]$ for $i = 2, \ldots, g$. 

![Figure 2. Curves in a lantern configuration.](image1)

![Figure 3. Geometric generators of $\text{Map}_{g,[1]}$.](image2)
• The kernel of the homomorphism $\text{Map}_{1,1} \to \text{Map}_1$ is the central free abelian group generated by $(T_{a_1}T_{a_1})^6 = (s_1s_2)^6$. In the case $g \geq 2$, the kernel of the homomorphism $\text{Map}_{g,1} \to \text{Map}_g$ is normally generated by the product $T_{\beta_g}T_{\beta'_g}^{-1}$ where $T_{\beta_g}, T_{\beta'_g}$ are the Dehn twists along the curves $\beta_g, \beta'_g$. □

This presentation was found by Wajnryb [Wa], with an error corrected in [Bi-Wa]. In [Ma], Matsumoto has found simple explicit words expressing the relations in terms of Garside elements of certain Coxeter subsystems of $S_g$.

**Theorem 2.12.** i) The chain relation can be given by the element $C_0 := \Delta^{-4}(A_4)\Delta^2(A_5)$ where $A_4$ and $A_5$ are the Coxeter subsystems $\{s_2, s_3, s_4, s_0\}$ and $\{s_1, s_2, s_3, s_4, s_0\}$, respectively.

ii) The lantern relation can be given by the element $C_0 \cdot \Delta^{-2}(E_6)\Delta(E_7)$ where $C_0$ is the chain relation element above and $E_6, E_7$ are the Coxeter subsystems $\{s_0, s_1, s_2, \ldots, s_6\}$ and $\{s_0, s_1, s_2, \ldots, s_6\}$, respectively.

iii) The relation $T_{\beta_g} = T_{\beta'_g}$ can be replaced by the commutator relation $[T_{\beta_g}, \Delta^2(A_{2g})] = 1$ where $A_{2g}$ is the Coxeter subsystem $\{s_1, s_2, \ldots, s_{2g}\}$. □

**Remark.** It is known that any two non-separating chain (or lantern) configurations are conjugate by some $f \in \text{Diff}_+(\Sigma, z_0)$. This implies the fact, used implicitly in the above theorem, that the subgroup normally generated by a single non-separating chain relation $C(\alpha, \beta, \beta')$ is independent of a specific choice of the chain configuration, and the same holds for lantern configurations. For the proof of this and further details, we refer to [Ger].

We call $\Delta^2(A_{2g})$ the **hyperelliptic element**, the commutator $[T_{\beta_g}, \Delta^2(A_{2g})]$ the (basic) **hyperelliptic relation element**, and $[T_{\beta_g}, \Delta^2(A_{2g})] = 1$ the **hyperelliptic relation**. The reason for this terminology is that the element $\Delta^2(A_{2g}) \in \text{Br}(S_g)$ represents the hyperelliptic involution of $\Sigma$ which can be realised on Figure 3 as the rotation by 180° about the horizontal axis going along the chain of curves $a_1, a_2, \ldots, a_{2g}$. Besides, we call the relation $\Delta^2(E_6) = \Delta(E_7)$ the **modified lantern relation** and the product $\Delta^{-2}(E_6)\Delta(E_7)$ the (basic) **modified lantern relation element**. The same names will be used for any conjugates of these elements.

Observe that all $s_i$ are conjugate in $\text{Br}(S_g)$, so that quasi-generators $\hat{t} \in \hat{T}(S_g)$ are preferred lifts of Dehn twists along non-separating curves $\delta$. The $\mathbb{Z}_2$-homology class $[\delta] \in H_1(\Sigma, \mathbb{Z}_2)$ is determined by the image of $\hat{t}$ in $\text{W}(S_g)$. Hence, we obtain a well-defined set-theoretic $\text{W}(S_g)$-equivariant map $T(S_g) \to \mathcal{H}_g := H_1(\Sigma, \mathbb{Z}_2)\backslash\{0\}$. (This map is bijective if $g = 1, 2, 3$, but if $g \geq 4$, then the set $T(S_g)$ is infinite, and the map is only surjective.) Furthermore, the projection of a quasi-generator $\hat{t}$ to $\text{Sp}(2g, \mathbb{Z}_2)$ has order 2. Therefore there exists a well-defined homomorphism $\text{W}(S_g) \to \text{Sp}(2g, \mathbb{Z}_2)$.

**Definition 2.8.** The kernel $W_{Ig} := \text{Ker}(\text{W}(S_g) \to \text{Sp}(2g, \mathbb{Z}_2))$ is called the **Weyl–Torelli group** of the surface $\Sigma$. 
It follows that $\mathcal{H}_g$ is the quotient set $T(S_g)/W\mathcal{I}_g$. Consequently, the group $\mathbb{Z}(\mathcal{H}_g)$ is naturally isomorphic to the coinvariant group $P_{ab}(S_g)W\mathcal{I}_g$. This allows us to apply Theorem 2.9 with the group $\mathbb{Z}(\mathcal{H}_g)$ instead of $\mathbb{Z}(T(S_g))$. Moreover, we can reduce the extension $1 \to P(S_g) \to Br(S_g) \to W(S_g) \to 1$ to an extension $1 \to \mathbb{Z}(\mathcal{H}_g) \to E \to W(S_g) \to 1$. However, the group $W(S_g)$ is bigger than $Sp(2g,\mathbb{Z}_2)$ for all $g \geq 3$, and their difference — the Weyl–Torelli group $W\mathcal{I}_g$ — is generated by lantern relation elements and is therefore invisible in $Map_g$. Worse still, $W\mathcal{I}_g$ is infinite for all $g \geq 4$, which makes effective calculation in $W\mathcal{I}_g$ difficult. In order to see which part of $W\mathcal{I}_g$ can be factored out, we need a description of the kernel $\text{Ker}(Br(S_g) \to Sp(2g,\mathbb{Z}_2))$.

**Proposition 2.13.** The kernel of the natural homomorphism $Br(S_g) \to Sp(2g,\mathbb{Z}_2)$ is generated by squared quasigenerators $\ell^2$ and by modified lantern relation elements $x \ast (\Delta^{-2}(E_0)\Delta(E_7))$, $x \in Br(S_g)$.

**Proof.** By Lemma 2.10 every element of the kernel of the homomorphism $Br(S_g) \to Map_g$ can be represented as a product of squared quasigenerators $\ell^2$ and lantern relation elements. Consequently, it is sufficient to find the corresponding presentation in the mapping class group $Map_g$.

Denote by $G$ the kernel $\text{Ker}(Map_g \to Sp(2g,\mathbb{Z}_2))$ and by $G_0$ the subgroup generated by products $T_{\delta_1}^{\epsilon_1}T_{\delta_2}^{\epsilon_2}$ with $\epsilon_i = \pm 1$ and $[\delta_1] = [\delta_2] \neq 0 \in H_1(\Sigma,\mathbb{Z}_2)$. We use $\equiv$ to denote equality modulo 2, i.e., the equality in $\mathbb{Z}_2$, or in $H_1(\Sigma,\mathbb{Z}_2)$, or in $Sp(2g,\mathbb{Z}_2)$. Let $\alpha_1,\ldots,\alpha_g; \beta_1,\ldots,\beta_g$ be the geometric basis of $\Sigma$ fixed above.

Fix some $f \in G$ and write $f$ as a product of Dehn twists along non-separating curves, $f = \prod_{i=1}^n T_{\delta_i}^{\epsilon_i}$. The braid relations $T_\gamma T_\delta = T_\delta T_\gamma$ with $\epsilon = \pm 1$ and $\delta' = T_\gamma(\delta)$ allow us to use Hurwitz moves to re-organise the product by moving any single Dehn twist $T_\gamma$ in both directions without changing it. Note that the generators $T_{\delta_1}^{\epsilon_1}T_{\delta_2}^{\epsilon_2}$ of $G_0$ are stable under conjugation.

**Step 1.** Assume that there exist factors $T_{\delta_i}$ with $\delta_i \cap \beta_1 \equiv 0$. Collect all such factors $T_{\delta_i}$ on the right using the braid relation. Henceforth, we assume that $\delta_i \cap \beta_1 \equiv 1$ for $i = 1,\ldots,n_1$ and $\delta_i \cap \beta_1 \equiv 0$ for $i = n_1 + 1,\ldots,n$.

**Step 2.** Assume that $\delta_i \cap \delta_j \equiv 1$ for some distinct $i,j \leq n_1$. Bring them together, so that $j = i + 1$, and then apply the braid relation. It transforms $T_{\delta_i}T_{\delta_{i+1}}$ into $T_\gamma T_{\delta_i}$ with $\gamma \equiv \delta_i + \delta_{i+1}$. Consequently, $\gamma \cap \beta_1 \equiv 0$, and we can shift $T_\gamma$ to the right. This allows us to diminish the number $n_1$ of $\delta_i$ with $\delta_i \cap \beta_1 \equiv 1$. Repeating this procedure, we come to the case when $\delta_i \cap \delta_j \equiv 0$ for all $i,j \leq n_1$.

**Step 3.** Assume that $n_1 > 0$, i.e., there exist factors $T_{\delta_i}$ with $\delta_i \cap \beta_1 \equiv 1$. Take the initial subproduct $\prod_{i=1}^{n_1} T_{\delta_i}$ and multiply it by $T_{\beta_1}^{-1}T_{\beta_1}$ on the left. Apply the braid relation twice to the initial subword $T_{\beta_1}^{-1}T_{\beta_1}T_{\delta_i}$:

$$T_{\beta_1}^{-1}T_{\beta_1}T_{\delta_1} \Rightarrow T_{\beta_1}^{-1}T_{\delta_1}T_{\beta_1} \Rightarrow T_{\beta_1}^{-1}T_{\gamma}T_{\delta},$$

where $\delta := T_{\gamma}(\delta_1)$ and $\gamma := T_{\delta}(\beta_1)$. Then $\delta \equiv \delta_1 + \beta_1$ and $\delta \cap \delta_1 \equiv 1$ for all $i = 2,\ldots,n_1$. Then shift $T_{\delta}$ to the right behind all $T_{\delta_i}$, $i = 2,\ldots,n_1$:

$$T_{\beta_1}^{-1}T_{\gamma}T_{\delta} \prod_{i=2}^{n_1} T_{\delta_i} \Rightarrow T_{\beta_1}^{-1}T_{\gamma} \prod_{i=2}^{n_1} T_{\delta_i}T_{\delta}.$$
where \( \delta'_i := T_{\delta_i} \). Then \( \delta'_i \equiv \delta_i + \delta \) and \( \delta'_i \cap \beta_i \equiv 0 \). Finally, shift \( T_{\beta_i}^{-1} \) and the newly obtained \( T_{\delta'_i} \), \( i = 2, \ldots, n \), to the right. We obtain a new decomposition \( f = \prod_{i=1}^{n-2} T_{\delta''_i} \) of the original \( f \) in which \( \delta''_i \cap \beta_i \equiv 0 \) for \( i = 3, \ldots, n + 2 \) and \( \delta''_i \cap \beta_i \equiv 1 \).

**Step 4.** If \( \delta''_i \cap \beta_i \equiv 0 \), then \( \prod_{i=1}^{n-2} T_{\delta''_i}(\beta_i) \equiv \beta_1 + \delta''_i \neq \beta_i \) which is in contradiction with the assumption that \( f_* \equiv \text{id} \) in \( \mathbb{Z}_2 \)-homology. If \( \delta''_i \cap \beta_i \equiv 1 \), transform \( T_{\delta''_i} T_{\delta''_j} \) into \( T_{\delta''_i} T_{\delta''_j} \) with \( \delta''_i \cap \beta_i \equiv 0 \). This leads to the same contradiction. Thus \( \delta''_i \cap \beta_i \equiv 0 \) and \( \delta''_i \cap \beta_i \equiv \delta''_i \cap \beta_i \equiv 1 \). But then \( \beta_i \equiv \prod_{i=1}^{n-2} T_{\delta''_i}(\beta_i) \equiv \beta_1 + \delta''_i + \delta''_j \) and hence \( \delta''_i \equiv \delta''_j \). This means that \( T_{\delta''_j} T_{\delta''_i} \) lies in \( G_0 \). Shift it to the right. In this way we have represented \( f \) in the form \( \prod_i T_{\delta_i} \cdot f_1 \) with \( f_1 \in G_0 \) and \( \delta_i \cap \beta_i \equiv 0 \).

**Step 5.** Repeat Steps 4-7 subsequently for \( \beta_2, \ldots, \beta_g \) instead of \( \beta_1 \). For each \( \beta_k \), the previously reached relations \( \delta_i \cap \beta_j \equiv 0 \) with \( i = 1, \ldots, n \) and \( j = 1, \ldots, k - 1 \) remain undestroyed. As a result, we represent \( f \) in the form \( \prod_i T_{\delta_i} \cdot f_2 \) with \( f_2 \in G_0 \) and \( \delta_i \cap \beta_j \equiv 0 \) for all \( j = 1, \ldots, g \). It follows that after this step each \( \mathbb{Z}_2 \)-class \( [\delta_i] \) is a linear combination of \( \mathbb{Z}_2 \)-classes \( [\beta_1], \ldots, [\beta_g] \), and hence \( \delta_i \cap \delta_j \equiv 0 \) for each \( i, j = 1, \ldots, n \). Consequently, the application of the braid relation does not change the \( \mathbb{Z}_2 \)-homology class of the remaining \( \delta_i \), for if \( T_{\delta_j} T_{\delta_i} T_{\delta_i}^{-1} = T_{\delta_j} \), then \( \delta_j \equiv \delta_j \).

**Step 6.** Consider \( \Delta_{\alpha_g} \) be the set of those \( \delta_i \) for which \( \alpha_g \cap \delta_i \equiv 1 \) and let \( V_{\alpha_g} \) be the \( \mathbb{Z}_2 \)-vector space spanned by \( \Delta_{\alpha_g} \). Assume that \( \dim_{\mathbb{Z}_2} V_{\alpha_g} \geq 3 \). Find three circles in \( \Delta_{\alpha_g} \), say, \( \delta_1, \delta_2, \delta_3 \), which are \( \mathbb{Z}_2 \)-linearly independent. Then \( [\delta_2] \equiv [\delta_1] + [\gamma_2] \) and \( [\delta_3] \equiv [\delta_1] + [\gamma_3] \) for some classes \( [\gamma_2], [\gamma_3] \in H_1(\Sigma, \mathbb{Z}_2) \) such that \( [\delta_1], [\gamma_2], [\gamma_3] \) are \( \mathbb{Z}_2 \)-linearly independent. Note that \( \gamma_1 \cap \alpha_g \equiv (\delta_1 - \delta_1) \cap \alpha_g \equiv 0 \). Realise the classes \( [\gamma_2], [\gamma_3] \in H_1(\Sigma, \mathbb{Z}_2) \) by embedded curves \( \gamma_2, \gamma_3 \subset \Sigma \) disjoint from \( \delta_1 \) and from each other. Pick a point \( z_0 \in \Sigma \) disjoint from \( \delta_1, \gamma_2, \gamma_3 \) and choose embedded arcs \( a_1, a_2, a_3 \) connecting \( z_0 \) with \( \delta_1, \gamma_2, \gamma_3 \), respectively, and disjoint (except for the end points) from each other and from \( \delta_1, \gamma_2, \gamma_3 \). Then a collar neighbourhood \( U \) of the graph formed by \( \delta_1, \gamma_2, \gamma_3 \) and \( a_1, a_2, a_3 \) is a disc with three holes bounded by \( \delta_1, \gamma_2, \gamma_3 \), see Figure 2. Multiply the product \( \prod_{i=1}^n T_{\delta_i} \) by the corresponding lantern relation

\[
T_{\delta_1}^{-1} T_{\delta_1}^{-1} T_{\delta_1}^{-1} T_{\delta_1} T_{\beta_2} T_{\beta_3} T_{\delta_1} T_{\beta_2} T_{\beta_3} = \text{id},
\]

see Figure 2. The factors \( T_{\delta_1}^{-1} T_{\delta_2} T_{\delta_3} \) and \( T_{\delta_1}^{-1} T_{\delta_3} T_{\delta_3} \) cancel out. Shift the factor \( T_{\delta_1}^2 \in G_0 \) to the right. Now we obtain the product \( \prod_{i=1}^n T_{\delta_i} \) which contains less factors \( T_{\delta_i} \) with \( \delta_i \cap \alpha_g \equiv 1 \). Repeating this procedure, we come to the case when \( \dim_{\mathbb{Z}_2} V_{\alpha_g} \leq 2 \).

**Step 7.** We maintain the notation \( \Delta_{\alpha_g} \) and \( V_{\alpha_g} \) of Step 6. In the case \( \dim_{\mathbb{Z}_2} V_{\alpha_g} = 0 \) we have \( \delta_i \cap \alpha_g \equiv 0 \) for every factor \( T_{\delta_i} \) and proceed to the next step. Assume that \( \dim_{\mathbb{Z}_2} V_{\alpha_g} = 1 \). Then \( \delta_i \equiv \delta_j \) for every \( \delta_i, \delta_j \in \Delta_{\alpha_g} \). Consequently, \( \prod_{i=1}^n T_{\delta_i}(\alpha_g) \equiv \alpha_g + n_{\alpha_g} \delta_1 \) where \( \delta_1 \) is any element of \( \Delta_{\alpha_g} \) and \( n_{\alpha_g} \) is the cardinality of \( \Delta_{\alpha_g} \). It follows that \( n_{\alpha_g} \) is even and we can collect the factors \( T_{\delta_i} \) in pairs \( T_{\delta_{2i-1}} T_{\delta_{2i}} \) lying in \( G_0 \).

**Step 8.** Assume that \( \dim_{\mathbb{Z}_2} V_{\alpha_g} = 2 \). Take two elements from \( \Delta_{\alpha_g} \), say \( \delta_1 \) and \( \delta_2 \), which are linearly independent. Then every \( \delta_i \in \Delta_{\alpha_g} \) is \( \mathbb{Z}_2 \)-homologous to \( \delta_1 \), or to \( \delta_2 \), or to \( \delta_1 + \delta_2 \). Find \( \gamma_1, \gamma_2 \in H_1(\Sigma, \mathbb{Z}_2) \) such that \( \delta_1 \cap \gamma_1 \equiv \delta_2 \cap \gamma_2 \equiv 1 \), \( \delta_1 \cap \gamma_2 \equiv \delta_2 \cap \gamma_1 \equiv \gamma_1 \cap \gamma_2 \equiv 0 \),
and $\delta_i \cap \gamma_j \equiv 0$ for every $\delta_i$ not lying in $\Delta_{\alpha_g}$. (This can be done by choosing an appropriate $\mathbb{Z}_2$-Darboux basis in $H_1(\Sigma, \mathbb{Z}_2)$.) Then
\[ \prod_{i=1}^n T_{\delta_i}(\gamma_1) - \gamma_1 \equiv \sum_{i=1}^n (\gamma_1 \cap \delta_i) \delta_i \]
and hence
\[ \gamma_2 \cap \left( \prod_{i=1}^n T_{\delta_i}(\gamma_1) - \gamma_1 \right) \equiv n_{12} \]
where $n_{12}$ is the number of $\delta_i \in \Delta_{\alpha_g}$ with $\delta_i \equiv \delta_1 + \delta_2$. Consequently, $n_{12}$ is even and we can collect the factors $T_{\delta_i}$ with $\delta_i \equiv \delta_1 + \delta_2$ in pairs as at Step 8. Similarly one shows that the numbers $n_1$ and $n_2$ of $\delta_i \in \Delta_{\alpha_g}$ with $\delta_i \equiv \delta_1$ or, respectively, $\delta_i \equiv \delta_2$ are also even. Thus all factors $T_{\delta_i}$ with $\delta_i \in \Delta_{\alpha_g}$ can be collected in pairs $T_{\delta_{2i-1}} T_{\delta_{2i}}$ lying in $G_0$.  

**Step 9.** After Step 8 the remaining factors $T_{\delta_i}$ fulfill the relation $\delta_i \cap \alpha_g \equiv 0$. We repeat Steps 8-9 for $\alpha_{g-1}, \alpha_{g-2}$ and so on. Notice that the previously achieved $\mathbb{Z}_2$-orthogonality with $\beta_1, \ldots, \beta_i; \alpha_g, \ldots$ remains unaffected. Therefore, we conclude that $G_0 = G$.

Thus we have shown that the kernel $\text{Ker} (\text{Map}_g \to \text{Sp}(2g, \mathbb{Z}_2))$ is generated by products $T_{\delta_1}^{\epsilon_1} T_{\delta_2}^{\epsilon_2}$ with $\epsilon_i = \pm 1$ and $[\delta_1] = [\delta_2] \neq 0 \in H_1(\Sigma, \mathbb{Z}_2)$.

**Step 10.** Multiplying a generator $T_{\delta_1}^{\epsilon_1} T_{\delta_2}^{\epsilon_2}$ by $T_{\delta_1}^{-1}$ and $T_{\delta_2}^{-2}$ if necessary, we take it to the form $T_{\delta_1} T_{\delta_2}^{-1}$. Let $\delta_1 =: \alpha, \delta_2 =: \alpha'$, so that our generator is $T_{\alpha} T_{\alpha'}^{-1}$.

Write this product in the form $T_{\alpha} T_{\gamma}^{-2} T_{\gamma}^{-1}$. Shift $T_{\gamma}^{2}$ to the right conjugating $T_{\alpha}^{-1}$ and then shift $T_{\gamma}^{-2}$ with conjugation to the right. This transforms $T_{\alpha} T_{\gamma}^{-2} T_{\gamma}^{-1}$ into $T_{\alpha} T_{\alpha'}^{-1} T_{\gamma}^{-2} T_{\gamma}^{2}$, with $\alpha'' = T_{\gamma}^{2}(\alpha')$. This allows us to change the homology class $[\alpha']$ to $[\alpha''] = (\alpha' + 2(\alpha' \cap \gamma) \cdot [\gamma]$. We claim that there exists a sequence of such “moves” which transforms the integer homology class $[\alpha']$ into the class $[\alpha]$. Clearly, it is sufficient to find an inverse transformation of $[\alpha]$ into $[\alpha']$. Fix a curve $\beta$ such that $\alpha \cap \beta = 1$. Then $[\alpha'] = (2k + 1)[\alpha] + 2l[\beta] + 2m[\gamma]$ for some non-separating curve $\gamma$ with $\gamma \cap \alpha = \gamma \cap \beta = 0$, where both $l$ and $m$ could be zero. Applying $T_{\beta}^{\pm 2}$ to $\alpha$, we can change $2l$ into $2l \pm 2(2k + 1)$. Iterating this we can transform $l$ into $l'$ with $|l'| \leq |2k + 1|$. On the other hand, we can replace $\beta$ by a new curve $\beta'$ in the homology class $[\alpha] \pm [\beta]$. Then $l$ remains unchanged and $2k + 1$ changes to $2k + 1 \pm 2l$. Consequently, these operations allow us to cancel $l$ out.

A similar procedure is applied to eliminate $m$. Indeed, for a curve $\gamma'$ in the homology class $[\gamma] + [\beta]$ the map $T_{\gamma'}^{\pm 2} T_{\gamma}^{\pm 2}$ transforms $m$ into $m \pm 2(2k + 1)$. Thus we can replace $m$ by $m'$ with $|m'| \leq |2k + 1|$. However the equality $m' = \pm (2k + 1)$ is impossible since $[\alpha']$ is a primitive cohomology class. To change $k$, we fix an embedded curve $\delta$ such that $\delta \cap \gamma = 1$ and $\delta \cap \alpha = \delta \cap \beta = 0$ and choose an embedded curve $\delta'$ in the class $[\delta] + [\alpha]$. Then $T_{\delta'}^{\pm 2}$ transforms the class $(2k + 1)[\alpha] + 2m[\gamma]$ into $(2k + 1 \pm 4m)[\alpha] + 2m([\gamma] \mp 2[\delta])$. So we could also make $|2k + 1|$ smaller than $2m$, possibly changing the class $[\gamma]$. The procedure terminates at $l = m = 0$ and $2k + 1 = \pm 1$. Since a Dehn twist $T_{\delta}$ is independent of the choice of the orientation on $\delta$, we obtain the equality $[\alpha] = [\alpha']$ of integral homology classes.
**Step 11.** Let $\alpha, \alpha'$ be the simple curves with $[\alpha] = [\alpha'] \in H_1(\Sigma, \mathbb{Z})$ obtained above. It follows that $\alpha' = F_1(\alpha)$ for some $F_1 \in \text{Map}_g$. As above, let $\beta$ be a curve such that $\alpha \cap \beta = 1$. Set $\beta' := F_1(\beta)$. Then $\beta' \cap \alpha = 1$ and hence $[\beta'] = [\beta] + l[\alpha] + m[\gamma]$ in $H_1(\Sigma, \mathbb{Z})$ with some primitive $[\gamma] \in H_1(\Sigma, \mathbb{Z})$ such that $\gamma \cap \alpha = \gamma \cap \beta = 0$. Then $T_{\alpha'}^{-1}$ preserves the curve $\alpha'$ (up to isotopy) and $[T_{\alpha'}^{-1}(\beta')] = [\beta] + m[\gamma]$. Further, find a curve $\gamma'$ in the homology class $[\alpha] + [\gamma] = [\alpha'] + [\gamma]$ which is disjoint from $\alpha'$. Then $T_{\alpha'}^l T_{\gamma'}^{-l}$ preserves the curve $\alpha'$ (up to isotopy) and transforms the class $[\beta] + m[\gamma]$ into $[\beta]$.

In this way we have constructed $F_2 \in \text{Map}_g$ which takes $\alpha$ to $\alpha'$ and preserves the integral class $[\beta]$. Observe that the action of $F_2$ on the $\gamma$-orthogonal complement to $\mathbb{Z}([\alpha],[\beta])$ can be realised as a product of Dehn twists along curves disjoint from $\alpha' = F_2(\alpha)$ and from $\beta' := F_2(\beta)$. After an appropriate correction of $F_2$ we obtain an $F$ lying in the Torelli group $\mathcal{I}_g := \text{Ker}(\text{Map}_g \to \text{Sp}(2g,\mathbb{Z}))$ such that $\alpha' = F(\alpha)$.

**Step 12.** Now let us apply the explicit description of the Torelli group. It is known that $\mathcal{I}_2$ is generated by Dehn twists $T_\delta$ along separating curves ([Pa], see also [Jo2]). Every such curve cuts $\Sigma$ into two pieces, say $\Sigma'$ and $\Sigma''$, each of them being a surface of genus 1 with one hole. The Dehn twist along $T_\delta$ is given by $(T_\alpha T_\beta)^6$ for any geometric basis $\alpha$ and $\beta$ of $\Sigma'$, i.e., two curves on $\Sigma'$ meeting transversally at a single point. But then $T_\alpha$ and $T_\beta$ are conjugated to the Coxeter subsystem $A_2 := \{s_1, s_2\}$, and hence $(T_\alpha T_\beta)^6$ is conjugated to $\Delta^4(A_2)$ and is a product of squared quasigenerators $t^2$.

By [Jo1] (see also [Jo2]), the Torelli group $\mathcal{I}_g$, $g \geq 3$, is generated by products $T_\eta T_{\eta'}^{-1}$ where $\eta$ and $\eta'$ are disjoint non-separating curves such that $\eta \sqcup \eta'$ cuts $\Sigma$ into two pieces. Denote these pieces by $\Sigma'$ and $\Sigma''$. Each of them is a surface with two boundary circles, and their genera $g'$ and $g''$ are related by $g' + g'' = g - 1$. If $g' = 0$ or $g'' = 0$, the curves $\eta, \eta'$ are isotopic and the product $T_\eta T_{\eta'}^{-1}$ is trivial. It follows that such products $T_\eta T_{\eta'}^{-1}$ with additional condition $g' = 1$ also generate the Torelli group $\mathcal{I}_g$. However, in the case $g' = 1$ the piece $\Sigma'$ is a chain surface, see *Figure 1*. Using the definition of the chain relation and the curves on *Figure 1* we obtain

$$T_\eta T_{\eta'}^{-1} = T_\eta^2 T_{\eta'}^{-1} T_{\eta'}^{-1} = T_\eta^2 (T_\beta T_\alpha T_{\beta'})^4.$$ 

Now observe that the chain configuration $\{\beta, \alpha, \beta'\}$ is conjugated to the configuration $\{a_1, a_2, a_3\} =: A_3$ on *Figure 3*. Consequently, $(T_\beta T_\alpha T_{\beta'})^4$ is conjugated to the squared Garside element $\Delta^2(A_3)$ and can be represented as a product of squared quasigenerators $t^2$. \qed

**Corollary 2.14.** i) The kernel $\text{Ker}(\text{Map}_g \to \text{Sp}(2g,\mathbb{Z}))$ is generated by the squares $T_\delta^2$ of Dehn twists along non-separating curves $\delta \subset \Sigma \setminus \{z_0\}$.

ii) The Weyl--Torelli group $W\mathcal{I}_g = \text{Ker}(\text{W}(S_g) \to \text{Sp}(2g,\mathbb{Z}))$ is generated by elements conjugated to $w_o(E_7)$. \label{cor:2.14}

Recall that *Proposition 1.7* reduces the topological statement about the (non-)existence of special embeddings of the Klein bottle into topological Lefschetz fibrations to certain algebraic relations in the mapping class group $\text{Map}_g$. It turns out that these relations hold in $\text{Sp}(2g,\mathbb{Z})$. Lifting the corresponding elements to the braid group $\text{Br}(S_g)$, we can project them to $\mathbb{Z}\langle \mathcal{H}_g \rangle$. So we need tools to distinguish the combinatorial structure of $\mathcal{H}_g$ and $\mathbb{Z}\langle \mathcal{H}_g \rangle$. 


Let us describe the first such tool. Let $R$ be any commutative ring and $H$ a free $R$-module of finite rank. Denote by $T^*H$ the tensor algebra of $H$ over $R$. For each degree $d$, let $T^d_{\text{Sym}} H \subset T^d H$ be the submodule consisting of tensors invariant with respect to the natural action of the symmetric group $\text{Sym}_d$ permuting the tensor factors $H \otimes \cdots \otimes H$. Define the \textit{shuffle product} in $T^*H$ as follows. For $A \subset T^k H$ and $B \subset T^l H$ set $A \bullet B := \sum_{\sigma \in \text{Sym}_{k+l}} \sigma (A \otimes B)$ where the sum $\sum'$ is taken over all permutations $\sigma \in \text{Sym}_{k+l}$ of tensor factors which preserve the order of the first $k$ and of the last $l$ factors. In other words, $\sigma \in \text{Sym}_{k+l}$ must satisfy the condition $\sigma(i) < \sigma(j)$ if $i < j \leq k$ and if $k < i < j$. In particular, $v \bullet w = v \otimes w + w \otimes v$ and $u \bullet (v \otimes w) = u \otimes v \otimes w + v \otimes u \otimes w + v \otimes w \otimes u$ for $u, v, w \in H$. One verifies immediately the following properties of the introduced structures:

- $v^\otimes d \in T^d_{\text{Sym}} H$ for any $v \in H$; $A \bullet B \in T^{k+l}_{\text{Sym}} H$ if $A \subset T^k H$ and $B \subset T^l H$;
- the $\bullet$-product satisfies the associativity, commutativity, and distributivity laws;
- for $v, w \in H$ one has the \textit{binomial formula}:

$$
(2.5) \quad (v + u)^\otimes d = v^\otimes d + v^{\otimes (d-1)} \bullet w + v^{\otimes (d-2)} \bullet w^\otimes 2 + \cdots + w^\otimes d;
$$

- $v^\otimes k \bullet v^\otimes l = \binom{k+l}{k} v^{\otimes (k+l)}$ for $v \in H$.

The last property shows that in the case when $H \cong R^\oplus r$ is the free module of rank $r$, the algebra $(T^*_{\text{Sym}} H, \bullet)$ is isomorphic to the $r$-th tensor power $A^* \otimes \cdots \otimes A^*$ of the so-called \textit{algebra of divided powers} $A^*$.

In our case, we set $R := \mathbb{Z}_2$ and $H := H_1(\Sigma, \mathbb{Z}_2)$. Let us define homomorphisms $\varphi^d : \mathbb{Z}_2(\mathcal{H}_g) \to T^d H$ by setting $\varphi^d(A_v) := v^\otimes d$ for each $v \in \mathcal{H}_g$ and use the same notation $\varphi^d : \text{P}(S_g) \to T^d H$ for the composition. Clearly, each $\varphi^d$ takes values in $T^d_{\text{Sym}} H \cong \text{Sym}^d(H)$. The meaning of $\varphi^1$ is obvious: it maps each $A_v$ to the vector $v \in H_1(\Sigma, \mathbb{Z}_2)$. To describe $\varphi^2$, we observe that the space $\text{Sym}^2(H)$ is naturally isomorphic to the $\mathbb{Z}_2$-Lie algebra $\text{sp}(2g, \mathbb{Z}_2)$. Explicitly, the isomorphism is given by $M \in \text{Sym}^d(H) \leftrightarrow J \cdot M \in \text{sp}(2g, \mathbb{Z}_2)$, where $J$ is the “symplectic” matrix $J := (\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix})$ with square $g \times g$ blocks, and $\text{sp}(2g, \mathbb{Z}_2)$ is realised as a matrix Lie subalgebra of $\text{Mat}(2g, \mathbb{Z}_2)$ with respect to a symplectic basis $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$ of $H$. Further, $\text{sp}(2g, \mathbb{Z}_2)$ admits a natural group extension

$$
1 \to \text{sp}(2g, \mathbb{Z}_2) \to \text{Sp}(2g, \mathbb{Z}_4) \to \text{Sp}(2g, \mathbb{Z}_2) \to 1,
$$

and an explicit calculation with any $T^2_s$ shows that the homomorphism $J \cdot \varphi^2 : \text{P}(S_g) \to \text{sp}(2g, \mathbb{Z}_2) \hookrightarrow \text{Sp}(2g, \mathbb{Z}_4)$ coincides with the composition $\text{P}(S_g) \hookrightarrow \text{Br}(S_g) \to \text{Map}_g \to \text{Sp}(2g, \mathbb{Z}_4)$. Observe also that the matrix $J$ can be given by $\sum_{i=1}^g \alpha_i \cdot \beta_i$ for any symplectic basis $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$ of $H$. Besides, recall the definition of the hyperelliptic relation element $[T_{\beta_g}, \Delta^2(A_{2g})]$ given in \textbf{Theorem 2.12}.

\textbf{Lemma 2.15.} \hspace{1em} i) $\varphi^3(\Delta^2(A_5)) = \varphi^3(\Delta^2(E_7)) = 0$.

\hspace{1em} ii) $\varphi^3([T_{\beta_g}, \Delta^2(A_{2g})]) = \beta_g \bullet (\sum_{i=1}^g \alpha_i \cdot \beta_i)$.

\textbf{Proof.} \hspace{1em} i) Note that $T(A_5) = \mathcal{H}_2$ and $T(E_7) = \mathcal{H}_3$. So in both cases $\Delta^2(\ldots)$ is the sum $\sum_{v \neq 0} A_v$ where the subspace $V \subset H_1(\Sigma, \mathbb{Z}_2)$ is of dimension 4 and 6, respectively. Consequently, $\varphi^3(\Delta^2(\ldots))$ is the sum $\sum_{v \in V} v^{\otimes 3}$. Fixing a basis $e_1, \ldots, e_r$ of $V$ ($r = 4$ or 6 according to the case) and expanding the sum $\sum_{v \in V} v^{\otimes 3}$ using the binomial formula \textbf{2.5} we obtain the sum of monomials $e^n := e_1^{\otimes n_1} \cdots e_r^{\otimes n_r}$ with $n_1 + \cdots + n_r = 3$. Since at
least one $n_i$ vanishes, each monomial $e^n$ appears an even number of times, and the sum vanishes.

ii) For simplicity, we use the same notation $a_i, α_j, β_k ∈ ℤ_2$ for the $ℤ_2$-homology classes of the corresponding curves. Set $v_{ij} := \sum_{j=i}^{j-1} a_k$ and $v_i := v_{i,2g}$. It follows that the projection of the squared Garside element $Δ^2(A_{2g})$ to $ℤ_2(ℤ_g)$ is the sum $\sum_{1≤i,j≤g} A_{v_{ij}}$. Since $β_g$ is disjoint from each $a_j$ with $j < 2g$, the hyperelliptic relation element $[T_β_3, Δ^2(A_{2g})]$ projects to the sum $\sum_{i=1}^{2g} (A_{v_i} + A_{v_i+β_g})$. Hence

$$ϕ^3([T_β_3, Δ^2(A_{2g})]) = \sum_{i=1}^{2g} (v_i ⊗^3 + (v_i + β_g) ⊗^3).$$

Expanding and using the identity $2 = 0 ∈ ℤ_2$ we obtain

$$ϕ^3([T_β_3, Δ^2(A_{2g})]) = \sum_{i=1}^{2g} v_i \cdot β_g ⊗^2 + v_i ⊗^2 \cdot β_g).$$

Plugging $v_i = a_i + a_{i+1} + ⋯ + a_{2g}$ into the sum $\sum_{i=1}^{2g} v_i$ we see that every $a_j$ appears $j$ times. Thus

$$\sum_{i=1}^{2g} v_i = a_1 + a_3 + ⋯ + a_{2g-1} = β_1 + (β_1 + β_2) + ⋯ + (β_{g−1} + β_g) = β_g.$$

Similarly, plugging $v_i ⊗^2 = \sum_{j=i}^{2g} a_j ⊗^2 + \sum_{i<j<k≤2g} a_j \cdot a_k$ into the sum $\sum_{i=1}^{2g} v_i ⊗^2$ we obtain

$$\sum_{i=1}^{2g} v_i ⊗^2 = \sum_{j=1}^{2g} j \cdot a_j ⊗^2 + \sum_{1<j<k≤2g} j \cdot a_j \cdot a_k.$$

The first sum gives

$$a_1 ⊗^2 + a_3 ⊗^2 + ⋯ + a_{2g-1} = β_1 ⊗^2 + (β_1 + β_2) ⊗^2 + ⋯ + (β_{g−1} + β_g) ⊗^2 = β_1 + β_2 + β_3 + ⋯ + β_{g−1} + β_g.$$

The second sum can be rewritten as $\sum_{1<j<k≤2g} j \cdot a_j \cdot a_k = \sum_{j=1}^{2g} [2g-2j+1] \cdot a_k$. The interior summation yields $\sum_{k=2j}^{2g} a_k = (α_j + ⋯ + α_g) + (β_j + β_g)$. So, setting $β_0 := 0$, we can expand the second sum as

$$\sum_{1≤j<k≤2g} j \cdot a_j \cdot a_k = \sum_{j=1}^{2g} (β_{2j-1} + β_{2j}) \cdot ((α_j + ⋯ + α_g) + (β_j + β_g)).$$

Let us analyse the terms in the last sum. Firstly, $\sum_{j=1}^{2g} (β_{2j−1} + β_{2j}) \cdot β_g = β_g \cdot β_g = 2β_g ⊗^2 = 0$. Similarly, $β_j \cdot β_j = 0$ and $\sum_{j=1}^{2g} (β_{2j−1} + β_{2j}) \cdot β_j = β_1 + β_2 + β_3 + ⋯ + β_{g−1} + β_g$. The terms containing a given $α_k$ are

$$\sum_{j=1}^{k} (β_{2j−1} + β_{2j}) \cdot α_k = (β_0 + β_k) \cdot α_k = β_k \cdot α_k.$$

So finally we can compute

$$ϕ^3([T_β_3, Δ^2(A_{2g})]) = β_g \cdot β_g ⊗^2 + (β_1 ⊗^2 + \sum_{j=1}^{2g} β_j \cdot α_j) \cdot β_g = β_0 \cdot β_0 \cdot \sum_{j=1}^{2g} α_j \cdot β_j,$$

as desired.

\[ \square \]

**Corollary 2.16.** For $g ≥ 2$ the homomorphism $ϕ^3$ extends to the group $\text{Ker}(Map_{g,1} → Sp(2g, ℤ_2))$ so that the composition $ϕ^3 ◦ ϕ: π_1(Σ) → T^3$ can be realised as the composition of the projection $π_1(Σ) → H_1(Σ, ℤ_2)$ with the embedding $H_1(Σ, ℤ_2) ⊂ T^3$ given by $γ → γ \cdot \sum_{i=1}^{2g} α_i \cdot β_i$.

**Proof.** Apply Corollary 2.14 i).
Proposition 2.17. i) The abelianisation $WI_{g,ab}$ of the Weyl–Torelli group $WI_g$ is a $\mathbb{Z}_2$-vector space naturally isomorphic to $\wedge^6 H_1(\Sigma, \mathbb{Z}_2)$.

ii) There exists a natural lattice extension

$$0 \to \mathbb{Z}(\langle H_g \rangle) \to \Lambda_g \to WI_{g,ab} \to 0$$

such that $\Lambda_g$ can be realised as a sublattice in $\frac{1}{2}\mathbb{Z}(\langle H_g \rangle)$ and the image of the induced embedding $WI_{g,ab} \subset \frac{1}{2}\mathbb{Z}(\langle H_g \rangle)/\mathbb{Z}(\langle H_g \rangle) \cong \mathbb{Z}_2(\langle H_g \rangle)$ is generated by the sums $L_v := \sum_{v \neq 0 \in V} A_v$ in which $V$ is a symplectic 6-dimensional subspace of $H_1(\Sigma, \mathbb{Z}_2)$.

iii) There exists a group extension

$$(2.6) \quad 0 \to \Lambda_g \to \tilde{Sp}(2g, \mathbb{Z}_2) \to Sp(2g, \mathbb{Z}_2) \to 1$$

and a homomorphism of the extension $1 \to P(S_g) \to Br(S_g) \to W(S_g) \to 1$ onto this extension such that the homomorphism $W(S_g) \to Sp(2g, \mathbb{Z}_2)$ has the usual meaning and $P(S_g) \to \Lambda_g$ is the composition $P(S_g) \to P_{ab}(S_g) \to \mathbb{Z}(H_g) \subset \Lambda_g$.

A subspace $V \subset H_1(\Sigma, \mathbb{Z}_2)$ is called symplectic if the restriction of the intersection form to $V$ is non-degenerate.

Proof. Let $L_0 := C \cdot \Delta^{-2}(E_6) \Delta(E_7)$ and $L'_0 := \Delta^{-2}(E_6) \Delta(E_7)$ be the “basic” non-modified and modified lantern relation elements. Then their projections to $W(S_g)$ are equal and give the longest element $w_o(E_7)$ of the Coxeter subgroup $W(E_7) \subset W(S_g)$. Since $w_o(E_7)$ has order 2 and its conjugates generate $WI_g$, the abelianisation $WI_{g,ab}$ is a $\mathbb{Z}_2$-vector space.

Now consider the natural extension

$$0 \to \mathbb{Z}(\langle H_g \rangle) \to Q \to WI_g \to 1$$

so that $Q$ is the quotient of the kernel $\text{Ker}(Br(S_g) \to Sp(2g, \mathbb{Z}_2))$ by the kernel $\text{Ker}(P(S_g) \to \mathbb{Z}(\langle H_g \rangle))$. Every element in $Q$ is the product of conjugates $x \ast L_0$ with $x \in Br(S_g)$ and elements of $\mathbb{Z}(\langle H_g \rangle)$. Since $\mathbb{Z}(\langle H_g \rangle)$ lies in the centre of $Q$ and the chain relation $C = L_0(L'_0)^{-1}$ lies in $\mathbb{Z}(\langle H_g \rangle)$, the commutators $[x \ast L_0, y \ast L_0]$ and $[x \ast L'_0, y \ast L_0]$ (and so on) are equal and generate the commutator group $Q' := [Q, Q]$.

We claim that the intersection $Q' \cap \mathbb{Z}(\langle H_g \rangle)$ is trivial. Suppose that $z \in Q' \cap \mathbb{Z}(\langle H_g \rangle)$. The condition $z \in Q'$ means that $z$ is represented by an element $\hat{z} = \prod_j [x_j \ast L_0, y_j \ast L_0]$ from $Br(S_g)$. Since $x_j \ast L_0$ and $y_j \ast L_0$ project to id $\in Sp(2g, \mathbb{Z}_2)$, we can apply to them the algorithm from Proposition 2.13. However, we use only the steps that do not involve the lantern relation, i.e. omit Steps 0-3. As a result, each $x_j \ast L_0$ and $y_j \ast L_0$ is represented as a product of elements from $\mathbb{Z}(\langle H_g \rangle)$ and Dehn twists $T_{\delta_i}$ such that the $\mathbb{Z}_2$-homology classes $[\delta_i]$ are linear combinations of $[\beta_1], \ldots, [\beta_g]$. Since $\mathbb{Z}(\langle H_g \rangle)$ lies in the centre, it follows that the commutators $[x_j \ast L_0, y_j \ast L_0]$ are products of Dehn twists $T_{\delta_i}$ with $[\delta_i] \in \mathbb{Z}_2([\beta_1], \ldots, [\beta_g])$.

Now consider $z$ as an element of $P(S_g)$. Write the projection of $z$ to $\mathbb{Z}(\langle H \rangle)$ in the form $z = \sum_{v \in H_g} n_v A_v$ with $n_v \in \mathbb{Z}$. Let $G$ be the subgroup of $W(S_g)$ obtained from $WI_g$ by adding quasireflections $t \in T(S_g)$ whose projections to $Sp(2g, \mathbb{Z}_2)$ are Dehn twists $T_{\delta_i}$ with $[\delta_i] \in \mathbb{Z}_2([\beta_1], \ldots, [\beta_g])$. Since $G$ contains $WI_g$, there exists a $G$-equivariant projection $G_g = T(S_g)/NW_I_g \to T(S_g)/G$ such that coset classes in $T(S_g)/G$ can be described as $G$-orbits in $G_g$. Clearly, each $[\delta] \in \mathbb{Z}_2([\beta_1], \ldots, [\beta_g])$ is fixed by $G$. The remaining orbits are as follows. Set $V := \mathbb{Z}_2([\beta_1], \ldots, [\beta_g])$ and $U := \mathbb{Z}_2([\alpha_1], \ldots, [\alpha_g])$. 
Then for every $u \in \mathcal{H}_g$ with non-trivial projection to $U$ parallel to $V$ the $G$-orbit $G \cdot u$ consists of all vectors $u + v$ with $v$ varying over $V$. In particular, $\sum_{v \in V} n_{u+v} = 0$ by \textbf{Theorem 2.9}. Set $V_1 := \mathbb{Z}_2([\beta_2], \ldots, [\beta_g])$. Then for the special case $u = [\alpha_1]$ we obtain $\sum_{v \in V_1} (n_{[\alpha_1]+v} + n_{[\alpha_1+\beta_1]+v}) = 0$. The possibility to choose an arbitrary $\mathbb{Z}_2$-symplectic basis provides two further relations:

$$\sum_{v \in V_1} (n_{[\beta_1]+v} + n_{[\alpha_1+\beta_1]+v}) = 0 \quad \text{and} \quad \sum_{v \in V_1} (n_{[\alpha_1]+v} + n_{[\beta_1]+v}) = 0.$$  

Comparing them, we obtain the relation $\sum_{v \in V_1} n_{w+v} = 0$ first for $w = [\alpha_1], [\beta_1], [\alpha_1 + \beta_1]$, and then for all $w \in \mathcal{H}_g$ which are $\mathbb{Z}_2$-orthogonal to $\beta_2, \ldots, \beta_g$. Now set $V_k := \mathbb{Z}_2([\beta_{k+1}], \ldots, [\beta_g])$ and write the latter equality for $w = [\beta_1]$ in the form $\sum_{v \in V_k} (n_{[\beta_1]+v} + n_{[\beta_1+\beta_2]+v}) = 0$.

The same argument as above provides the relation $\sum_{v \in V_2} n_{w+v} = 0$, again first for $w = [\beta_1]$ and then for all $w \in \mathcal{H}_g$ that are $\mathbb{Z}_2$-orthogonal to $V_2$. Repeating this argument, we obtain the relation $n_{w} = 0$ for all $w \in \mathcal{H}_g$. This means the desired triviality of $z$ in $\mathbb{Z}\langle \mathcal{H}_g \rangle$.

The property $Q' \cap \mathbb{Z}\langle \mathcal{H}_g \rangle = 1$ implies that the abelianisation $\Lambda := Q_{ab} = Q/ Q'$ includes into the extension (2.6). Since the chain relation element $C$ and $\Delta^2(E_6)$ lie in $\mathbb{Z}\langle \mathcal{H}_g \rangle$, in the description of the extension (2.6) we can replace $L_0$ by $\Delta(E_7)$. The square $\Delta^2(E_7)$ is the sum $\sum_{t \in \mathcal{T}(E_7)} A_t$. The quasireflection set $\mathcal{T}(E_7)$ coincides with the set $\mathcal{H}_3$ of non-zero vectors in $\mathbb{Z}_2([\alpha_1], \ldots, [\beta_3])$. Conjugating by $\textbf{Sp}(2g, \mathbb{Z}_2)$ we obtain the description of generators of $\text{WI}_{g,ab} = \Lambda_g/\mathbb{Z}\langle \mathcal{H}_g \rangle$ claimed in the lemma.

To obtain the isomorphism $\text{WI}_{g,ab} \cong \wedge^6 H_1(\Sigma, \mathbb{Z}_2)$, we compute $\varphi^6(\Delta^2(E_7))$. An explicit calculation using the binomial formula (2.5) shows that $\varphi^6(\Delta^2(E_7)) = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3$. Further, note that over the coefficient ring $\mathbb{Z}_2$ we have the identity $v \cdot v = 0$. It follows that the algebra generated by $H_1(\Sigma, \mathbb{Z}_2)$ and by the shuffle product is simply the $\mathbb{Z}_2$-Grassmann algebra over $H_1(\Sigma, \mathbb{Z}_2)$. Consequently, the shuffle product $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3$ can be identified with the wedge product $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \in \wedge^6 H_1(\Sigma, \mathbb{Z}_2)$. Vice versa, any symplectic vector subspace $V \subset H_1(\Sigma, \mathbb{Z}_2)$ of dimension 6 is of the form $V = x \cdot \mathbb{Z}_2([\alpha_1], \ldots, [\beta_3])$ for some $x \in \textbf{Sp}(2g, \mathbb{Z}_2)$. For such $V$, the conjugate $x \star \Delta^2(E_7)$ equals the sum $\sum_{v \neq 0 \in V} A_v := L_V \in \mathbb{Z}_2(\mathcal{H}_g)$, and $\varphi^6(L_V)$ is the wedge product of the vectors in any basis of $V$. So the isomorphism $\text{WI}_{g,ab} \cong \wedge^6 H_1(\Sigma, \mathbb{Z}_2)$ follows from the fact that the elements $\textbf{Sp}(2g, \mathbb{Z}_2)$-conjugate to $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3$ span the space $\wedge^6 H_1(\Sigma, \mathbb{Z}_2)$. \hfill $\square$

Now we introduce the second tool used to explore the structure of the group $\mathbb{Z}\langle \mathcal{H}_g \rangle$. To any $\varphi_\mu = (\mu \cap \cdot) : H_1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}_2$ we associate a homomorphism $\hat{\varphi}_\mu : \mathbb{Z}\langle \mathcal{H}_g \rangle \to \mathbb{Z}$ by setting $\hat{\varphi}_\mu(A_v) := 1 \in \mathbb{Z}$ if $\varphi_\mu(v) \equiv 1 \in \mathbb{Z}_2$ and $\hat{\varphi}_\mu(A_v) := 0$ if $\varphi_\mu(v) \equiv 0$ for any element $v \in \mathcal{H}_g \subset H_1(\Sigma, \mathbb{Z}_2)$. Thus $\hat{\varphi}_\mu$ counts the algebraic number of generators $A_v$ with $\varphi_\mu(v) \equiv 1$. Extend this homomorphism to a homomorphism $\hat{\varphi}_\mu : \textbf{P}(S_g) \to \mathbb{Z}$ in the obvious way.
Lemma 2.18. Let \( \hat{\varphi}_\mu : \mathbb{Z}(\mathcal{H}_g) \to \mathbb{Z} \) be the homomorphism induced by \( \varphi_\mu : H_1(\Sigma, \mathbb{Z}^2) \to \mathbb{Z}^2 \).

i) For any chain relation element \( C \) the value \( \hat{\varphi}_\mu(C) \) is 0 or \(-4\).

ii) For any squared modified lantern relation element \( L^2 \) the value \( \hat{\varphi}_\mu(L^2) \) is 0 or \(-8\).

iii) For any element \( x \) representing a lantern relation \( L \) the value \( \hat{\varphi}_\mu(L) \) is 0 or \(-8\).

iv) \( \hat{\varphi}_\mu([T_\beta, \Delta^2(A_{2g})]) \equiv (2g - 2)\varphi_\mu(\beta_g) \mod 4 \).

Corollary 2.19. The homomorphism \( \hat{\varphi}_\mu : \mathbb{Z}(\mathcal{H}_g) \to \mathbb{Z} \) extends naturally to a homomorphism between \( \mathbb{Z} \) and the stabiliser group \( Br(S_g)_\mu \) of the element \( \mu \). The latter induces a homomorphism \( \hat{\varphi}_\mu : \text{Map}_{g,1,\mu} \to \mathbb{Z}_4 \).

Proof. First we show the claims of the corollary. It follow immediately from part iii) of the lemma that there exists a natural extension \( \hat{\varphi}_\mu : \Lambda_g \to \mathbb{Z} \), such that \( \hat{\varphi}_\mu(L) = 0 \) or \(-4\) for any modified lantern relation element \( L \). To extend \( \hat{\varphi}_\mu \) onto \( Br(S_g)_\mu \), we observe that the group \( Sp(2g, \mathbb{Z}_2)_\mu \) is generated by (the images of) Dehn twists \( T_\delta \) with \( \delta \cap \mu \equiv 0 \mod 2 \). Consequently, \( Br(S_g)_\mu \) is generated by such Dehn twists, \( P(S_g) \), and lantern relation elements. We set \( \hat{\varphi}_\mu(T_\delta) = 0 \) for such Dehn twists. To see that this definition makes sense, consider an element \( x \) which is a product of \( T_\delta \) with \( \delta \cap \mu \equiv 0 \mod 2 \), and which lies in the subgroup generated by \( P(S_g) \) and lantern relation elements. Then \( x \) can be projected to \( \Lambda_g \), and hence its square \( x^2 \) lies in \( P(S_g) \). Now using Theorem 2.9, we obtain \( \hat{\varphi}_\mu(x^2) = 0 \). This implies that \( \hat{\varphi}_\mu(x) = 0 \) and shows that the homomorphism \( \hat{\varphi}_\mu : Br(S_g)_\mu \to \mathbb{Z} \) is indeed well-defined. The existence of the induced homomorphism \( \hat{\varphi}_\mu : \text{Map}_{g,1,\mu} \to \mathbb{Z}_4 \) is trivial (modulo the lemma, of course).

Now we prove the assertions of the lemma.

i) Obviously, we have the following conjugation property: \( \hat{\varphi}_{x\ast \mu}(x \ast A) = \hat{\varphi}_\mu(A) \) for any \( x \in Sp(2g, \mathbb{Z}_2) \) and any \( A \in \mathbb{Z}(\mathcal{H}_g) \). Hence we may assume that \( C \) is the basic chain relation element \( C_0 := \Delta^{-4}(A_4)\Delta^2(A_5) \). Further, restricting \( \varphi_\mu \) to \( \mathbb{Z}_2(\alpha_1, \alpha_2, \beta_1, \beta_2) \) we can also suppose that \( g = 2 \) and that this restriction is a non-trivial homomorphism.

We make use of the standard facts on root systems, see [Bou]. In particular, we have the natural identifications \( S_2 = A_5 \) of the Coxeter systems and \( Sp(4, \mathbb{Z}_2) = W(A_5) \) of the groups. Besides, the set \( \mathcal{H}_2 \) is naturally identified with the set \( \Phi^+(A_5) \) of positive roots of the system \( A_5 \).

We claim that there are exactly two orbits of the action of \( W(A_4) \) on \( \Phi^+(A_5) \). The first one is clearly the root system \( \Phi^+(A_4) \). The explicit description of the root systems gives the five remaining elements: \( \{v_1, v_1 + v_2, v_1 + v_2 + v_3, \ldots, v_1 + \cdots + v_4 + v_0\} \). Here we are using the notation from Definition 2.7 and denote by \( v_i \) the simple root corresponding to \( s_i, \; i = 1, \ldots, 4, 0 \) We see explicitly that \( v_1 + v_2 \) is obtained from \( v_1 \) with the help of the reflection \( s_2 = T_{a_2}, \; v_1 + v_2 + v_3 \) from \( v_1 + v_2 \) with the help of \( s_3 = T_{a_3} \) and so on. This gives us the desired description of the orbits.

It follows that it is sufficient to calculate \( \hat{\varphi}_\mu(\Delta^{-4}(A_4)\Delta^2(A_5)) \) for the cases in which \( \mu = \beta_1 = a_1 \) and \( \mu = \beta_2 = a_0 \). By definition, \( \hat{\varphi}_\mu(\Delta^2(A_5)) \) is the number of vectors \( v \in H_1(\Sigma, \mathbb{Z}^2) \) satisfying \( \varphi_\mu(v) = 1 \). So there are \( 2^3 = 8 \) such vectors, and \( \hat{\varphi}_\mu(\Delta^2(A_5)) = 8 \).

To simplify notation, we replace the system \( \{s_2, s_3, s_4, s_0\} \) by \( \{s_1, s_2, s_3, s_4\} \). This does not change anything in the case \( g = 2 \) because of the existence of an appropriate isomorphism. Then \( \Delta^2(A_4) = \sum_{1 \leq j \leq k \leq 4} A_{v_{jk}} \) with \( v_{jk} := \sum_{i=j}^k a_i \in H_1(\Sigma, \mathbb{Z}^2) \). Then \( \hat{\varphi}_{\beta_1}(\Delta^2(A_4)) \)
is the number of vectors $v_{jk}$ "containing" the summand $\alpha_1 = a_2$. For such vectors we must have $1 \leq j \leq 2 \leq k \leq 4$, so that $\hat{\varphi}_\beta(A_4) = 2 \times 3 = 6$ and $\hat{\varphi}_\beta(A_4) = 12$. A similar consideration for $\mu = \beta_2$ leads to the vectors $v_{jk}$ with $1 \leq j \leq k = 4$, and hence $\hat{\varphi}_\beta(A_4) = 4 \times 2 = 8$. So $\hat{\varphi}_\mu(C)$ equals either $8 - 8 = 0$ or $8 - 12 = -4$.

ii) Now we consider the squared modified lantern relations $L^2$. We claim that after replacing $A_4$ by $E_6$ and $A_5$ by $E_7$, the argument remains essentially the same. In particular, we have the natural identifications $S_3 = E_7$ of the Coxeter systems and $\Phi^+(E_7) = H_3$ of positive roots, and the natural isomorphism $W(E_7) = Sp(6,\mathbb{Z}_2) \times \mathbb{Z}_2$ of the groups. After conjugation we come to the case of the basic lantern relation $\Delta^{-4}(E_6) \Delta^2(E_7)$. Similar to the previous situation, we may assume that $g = 3$ and that the restricted $\varphi_\mu$ is non-trivial. As above, $\hat{\varphi}_\mu(\Delta^2(E_7))$ is the number of vectors $v \in H_1(\Sigma,\mathbb{Z}_2)$ satisfying $\varphi_\mu(v) = 1$. This time this number is $2^5 = 32 = \hat{\varphi}_\mu(\Delta^2(E_7))$.

Now let us describe the $W(E_6)$-orbits in $T(E_7)$. Since the quasi-reflections are in bijection with positive roots of the corresponding Lie algebra, we have 63 quasi-reflections in $T(E_7)$ and 36 in $T(E_6)$. Next, the embedding $E_6 \subset E_7$ of the Coxeter systems induces an embedding of the set of quasi-reflections. Thus $T(E_6)$ is a $W(E_6)$-orbit in $T(E_0)$, and there are $63 - 36 = 27$ remaining quasi-reflections.

One of these remaining quasi-reflections is the element $s_1$. (Here we are using the notation from Definition [27].) To determine its $W(E_6)$-orbit let us consider the complex Lie algebra $g$ of type $E_7$. Fix a Cartan subalgebra $h$ in $g$ and a compatible system of simple roots, the latter being in natural bijection with the system $E_7 = \{s_0; s_1, \ldots, s_6\}$. For each $t \in T(E_7)$ denote by $\alpha_t \in h^*$ the corresponding positive root, and by $g_t^\pm$ the root subspace corresponding to $\pm \alpha_t$. Further, let $g' \subset g$ be the Lie subalgebra of type $E_6$ defined by the embedding $E_6 \subset E_7$ and $h' \subset h$ the compatible embedding of the Cartan subalgebra. Denote by $B : h^* \times h^* \rightarrow \mathbb{C}$ the canonical bilinear form of $h^*$, normalised by the condition $B(\alpha_1, \alpha_1) = +2$.

Denote by $V^+$ (respectively, by $V^-$) the sum of root spaces $g_t^+$ (respectively, $g_t^-$) over quasi-reflections $t \in T(E_7) \setminus T(E_6)$. Since for every root $\alpha_t$ with $t \in T(E_7) \setminus T(E_6)$ the coefficient of $\alpha_t$ is positive, $V^+$ is invariant with respect to the adjoint action of $g^\prime$. This means that the entire orbit $W(E_6) \cdot \alpha_{s_1}$ lies in $\Phi^+(E_7)$ (this is different from the action of the full Weyl group $W(E_7)$ which inverts every root).

From the Dynkin diagram we see that $B(\alpha_{s_1}, \alpha_{s_2}) = -1$ and $\alpha_{s_1}$ is orthogonal to the remaining simple roots $\alpha_{s_0}; \alpha_{s_3}, \ldots, \alpha_{s_6}$ in $E_6$. This means that for every $v \in h'$ we have $\alpha_{s_1}(v) = -\omega_{s_2}(v)$, where $\omega_{s_2}$ is the fundamental weight of the system $E_6$ dual to $\alpha_{s_2}$. Consequently, $V^+$ contains an irreducible $g'$-submodule with the minimal weight $-\omega_{s_2}$. It is known that the dimension of this submodule is 27. Thus $\dim V^+ \geq 27$. By the same argument, $\dim V^- \geq 27$. Comparing dimensions, we conclude that the $g'$-irreducible decomposition of $g$ is $g = g' \oplus V^+ \oplus V^- \oplus C(\alpha_{s_1}^\vee)$, where $\alpha_{s_1}^\vee$ is the coroot dual to $\alpha_{s_1}$. Now observe that by [Bou, Ch.VI, §1, Éxercice 23], the weight $-\omega_{s_2}$ is minuscule. This property is equivalent to the assertion that the $g'$-weights of $V^+$ form a single $W(E_6)$-orbit.

Let us turn back to the calculation of the possible values of $\hat{\varphi}_\mu(\Delta^2(E_6))$. For this purpose we use the following explicit construction of the root system of type $E_6$. Let

\footnote{I would like to thank W. Sörgel for this reference.}
\( \mathfrak{h}_R^* \) (the real form of the dual Cartan subalgebra \( \mathfrak{h}^* \)) be the space spanned by vectors \( \varepsilon_1, \ldots, \varepsilon_6; \varepsilon \) satisfying the linear relation \( \varepsilon_1 + \cdots + \varepsilon_6 = 0 \). Define the bilinear form on \( \mathfrak{h}_R^* \) by setting \( B(\varepsilon_i, \varepsilon_i) = -\frac{2}{3}, B(\varepsilon_i, \varepsilon_j) = -\frac{1}{6}, B(\varepsilon, \varepsilon) = \frac{1}{2} \), and \( B(\varepsilon_i, \varepsilon) = 0 \). Then the set
\[
\Phi := \{ \pm 2\varepsilon, \varepsilon_i + \varepsilon_j + \varepsilon_k \pm \varepsilon (i > j > k); \varepsilon_i - \varepsilon_j (i \neq j) \}
\]
is a root system of type \( E_6 \), and the set
\[
\pi := \{ \alpha_s := \varepsilon_i - \varepsilon_{i-1} (i = 2, \ldots, 6); \alpha_{s_0} := \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon \}
\]
is the system of simple roots with respect to the appropriate Weyl chamber. The corresponding positive roots are
\[
\Phi^+ := \{ 2\varepsilon, \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon (i > j > k), \varepsilon_i - \varepsilon_j (i > j) \}.
\]
Denote by \( \Lambda \) the integer lattice generated by \( \pi \). Since the curves \( a_0, a_2, \ldots, a_6 \) form a basis of the integer homology group of \( \Sigma \), we can identify \( \Lambda \) with \( \mathbb{H}_1(\Sigma, \mathbb{Z}) \). As we have shown, every homomorphism \( \varphi_\mu : \mathbb{H}_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}_2 \) is obtained by \( \mathbb{Z}_2 \)-reduction from either a homomorphism \( \lambda \in \Lambda \mapsto B(\alpha_t, \lambda) \in \mathbb{Z} \) with some \( \alpha_t \in \Phi \), or a homomorphism \( \lambda \in \Lambda \mapsto B(\gamma, \lambda) \in \mathbb{Z} \) with some weight \( \gamma \) of the \( \mathfrak{g}^* \)-module \( V^+ \). Moreover, all homomorphisms \( \varphi_\mu \) of the same type yield the same value of \( \hat{\varphi}_\mu(\Delta^2(\mathbb{E}_6)) \).

As a representative of the first \( \mathbb{W}(\mathbb{E}_6) \)-orbit we take the root \( 2\varepsilon \). Then \( \hat{\varphi}_\mu(\Delta^2(\mathbb{E}_6)) \) is the number of roots \( \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon \) with \( i > j > k \). Thus \( \hat{\varphi}_\mu(\Delta^2(\mathbb{E}_6)) = (\frac{6}{3}) = 20 \) in this case. This gives \( \hat{\varphi}_\mu(L^2) = 32 - 2 \times 20 = -8 \).

As a representative of the second \( \mathbb{W}(\mathbb{E}_6) \)-orbit we take the weight \( \varepsilon_1 + \varepsilon_2 \). To count the roots in question we use the following observations: First, since \( \varepsilon \) is orthogonal to \( \varepsilon_1 \) and \( \varepsilon_2 \), we can replace \( \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon \) by \( \varepsilon_i + \varepsilon_j + \varepsilon_k \). Second, since \( \varepsilon_1 + \cdots + \varepsilon_6 = 0 \) and we are interested only in the parity of \( B(\varepsilon_1 + \varepsilon_2, \alpha_t) \), we can replace each \( \varepsilon_i + \varepsilon_j + \varepsilon_k \) by \( \varepsilon_1 + \varepsilon_2 - \varepsilon_i - \varepsilon_j + \varepsilon_k \). This yields twice each \( \varepsilon_i + \varepsilon_j + \varepsilon_k \) with \( i > j > k \geq 2 \). The explicit combinatorics is:

- \( \varepsilon_2 - \varepsilon_1 \) is orthogonal to \( \varepsilon_1 + \varepsilon_2 \);
- \( B(\varepsilon_1 + \varepsilon_2, \varepsilon_i - \varepsilon_1) = -1 \) for \( i \geq 3 \), and there are 4 such roots;
- \( B(\varepsilon_i + \varepsilon_2, \varepsilon_i - \varepsilon_2) = -1 \) for \( i \geq 3 \), and there are 4 such roots;
- \( B(\varepsilon_i + \varepsilon_2, \varepsilon_i + \varepsilon_j + \varepsilon_k) = -1 \) for \( i > j > k \geq 3 \), and there are \( 2 \times 4 = 8 \) such roots;
- \( B(\varepsilon_i + \varepsilon_2, \varepsilon_i + \varepsilon_j + \varepsilon_j) = 0 \) for \( i > j \geq 3 \).

Thus \( \hat{\varphi}_\mu(\Delta^2(\mathbb{E}_6)) = 16 \) and \( \hat{\varphi}_\mu(L^2) = 32 - 2 \times 16 = 0 \).

\[ \text{iii) Every separating curve } \gamma \subset \Sigma \setminus \{z_0\} \text{ divides } \Sigma \text{ into two pieces } \Sigma' \text{ and } \Sigma'' \text{ one of which, say } \Sigma'', \text{ contains } z_0. \text{ Let } p \geq 1 \text{ be the genus of the other piece } \Sigma'. \text{ Then the whole configuration is conjugated to the one in which } \Sigma' \text{ contains the curves } a_1, \ldots, a_{2p} \text{ and } \Sigma'' \text{ the curves } a_{2p+2}, \ldots, a_{2g} \text{ (see the curve } \gamma_3 \text{ on Figure 3). In particular, in the case } p = g \text{ the curve } \gamma \text{ surrounds the base point } z_0 \text{ and corresponds to the boundary curve } \partial \text{ on Figure 3. By the chain relation (this time separating), } T_{\gamma_p} T_{\gamma_{p-1}} = (T_{\beta_p} T_{\alpha_p} T_{\beta_p})^4, \text{ where the curve } \beta''_p \text{ is like the one shown on Figure 1} \text{ and where we set } T_{\gamma_0} = 1 \in \text{Map}_{g,1} \text{ in the special case } p = 1. \text{ Thus } T_{\gamma} \text{ is conjugated to the alternating product of the elements } (T_{\beta_p} T_{\alpha_p} T_{\beta_p})^4 \text{. The incidence relations in the configuration } \beta_p, \alpha_p, \beta_p \text{ correspond to the Coxeter system } A_3, \text{ whereas the product } (T_{\beta_p} T_{\alpha_p} T_{\beta_p})^4 \text{ is the squared Garside element } \Delta^2(A_3). \text{ Thus } (T_{\beta_p} T_{\alpha_p} T_{\beta_p})^4 \text{ lies in } \mathbb{P}(S_g) \text{ and equals in } \mathbb{Z}(H_g) \text{ to the sum } \sum_{1 \leq i \leq j \leq 3} A_{\nu_{ij}} \text{ in} \]
which \( v_{11} = [\beta_p] \in H_1(\Sigma, \mathbb{Z}_2), \) \( v_{22} = [\alpha_p], \) \( v_{33} = [\beta_p] = [\beta_p], \) and \( v_{ij} = \sum_{k=i}^{j} v_{kk}. \) An explicit calculation shows that \( \sum_{1 \leq i \leq j \leq 3} A_{v_{ij}} = 2A_{\beta_p} + 2A_{\alpha_p} + 2A_{\alpha_p + \beta_p}. \) It follows that \( \hat{\varphi}_\mu \) takes value 0 or 4 on every such product \( (T_{\beta_p}T_{\alpha_p}T_{\beta_p})^4. \) This implies assertion \( \mathfrak{v}. \)

iv) The subgroup \( \partial_\pi(\pi_1(\Sigma, z_0)) \subset \text{Map}_{g,1} \) is normally generated by the basic hyperelliptic relation element \( [T_{\beta_g}, \Delta^2(A_{2g})]. \) Its geometric realisation is \( \partial_\pi(\beta'_g) \) where the element \( \beta'_g \in \pi_1(\Sigma, z_0) \) is represented by an embedded curve, still denoted by \( \beta'_g, \) which is isotopic to \( \beta_g \) and passes through the base point \( z_0. \) On \( \text{Figure 3} \) this curve corresponds to the arc \( \beta'_g, \) and the correct picture on the \textit{closed} surface is obtained by contracting the boundary circle \( \partial \) to the base point \( z_0. \) In particular, after this contraction the curves \( \beta_g \) and \( \beta'_g \) will cut a regular neighbourhood of the curve \( \beta'_g. \) Note that the element \( \Delta^2(A_{2g}) \) is represented by a certain hyperelliptic involution of \( \Sigma \) which maps \( \beta_g \) to \( \beta'_g. \)

In view of this algebraic description, it is sufficient to find the possible values of \( \hat{\varphi}_\mu([T_{\beta_g}, \Delta^2(A_{2g})]). \) First, we observe that in the group \( \mathbb{Z} \langle H_g \rangle \) we have the equality \( \Delta^2(A_{2g}) = \sum_{1 \leq i < j \leq 2g} A_{v_{ij}} \) where \( v_{ij} := \sum_{k=i}^{j} [a_k] \in H_1(\Sigma, \mathbb{Z}_2). \) Then \( T_{\beta_g}(v_{ij}) = v_{ij} \) for \( i < j < 2g, \) so that in the group \( \mathbb{Z} \langle H_g \rangle \) we obtain

\[
[T_{\beta_g}, \Delta^2(A_{2g})] = \sum_{i=1}^{2g} \bigg( A_{v_{i,2g} + [\beta_g]} - A_{v_{i,2g}} \bigg).
\]

It is easy to see that if \( \varphi_\mu([\beta_g]) = 0, \) then \( \hat{\varphi}_\mu(A_{v_{i,2g} + [\beta_g]} - A_{v_{i,2g}}) = 0 \) for all \( i = 1, \ldots, 2g. \) So it remains to consider the case \( \varphi_\mu([\beta_g]) = 1. \)

Here we observe that \( [a_1], \ldots, [a_{2g}] \) form a basis of \( H_1(\Sigma, \mathbb{Z}_2) \) in which \( [\beta_g] = [a_1] + [a_3] + \cdots + [a_{2g-1}]. \) Thus the homomorphism \( \varphi_\mu : H_1(\Sigma, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \) is completely determined by its values \( \varphi_\mu(a_1), \ldots, \varphi_\mu(a_{2g}). \) In other words, we can obtain all homomorphisms \( \varphi_\mu \) by varying these values. Clearly, the group of such transformations of \( \varphi_\mu \) preserving the value \( \varphi_\mu([\beta_g]) = 1 \in \mathbb{Z}_2 \) is generated by the following two types of \textit{simple transformations}: either we “switch” a single value \( \varphi_\mu(a_{2k}) \) on an even curve \( a_{2k}, \) or the values \( \varphi_\mu(a_{2k-1}) \) and \( \varphi_\mu(a_{2k+1}) \) on two consecutive odd curves \( a_{2k-1} \) and \( a_{2k+1}. \) Since \( \varphi_\mu([\beta_g]) = 1, \) every value \( \hat{\varphi}_\mu(A_{v_{i,2g} + [\beta_g]} - A_{v_{i,2g}}) \) is either +1 or −1. The crucial observation is that for any simple transformation we get an \textit{even} number of sign changes for the values \( \hat{\varphi}_\mu(A_{v_{i,2g} + [\beta_g]} - A_{v_{i,2g}}). \) Namely, 2\( k \) signs change when we “switch” the value \( \varphi_\mu(a_{2k}), \) and 2 signs change when we “switch” the values \( \varphi_\mu(a_{2k-1}) \) and \( \varphi_\mu(a_{2k+1}). \) It follows that the value \( \hat{\varphi}_\mu([T_{\beta_g}, \Delta^2(A_{2g})]) \mod 4 \) remains unchanged, so that it depends only on \( \varphi_\mu([\beta_g]). \)

An explicit calculation in the case when \( \varphi_\mu \) vanishes on all \( [\alpha_1], \ldots, [\alpha_g]; [\beta_1], \ldots, [\beta_{g-1}] \) shows that \( \hat{\varphi}_\mu([T_{\beta_g}, \Delta^2(A_{2g})]) = 2g - 2, \) as claimed. \( \square \)

2.3. Factorisation problems in mapping class groups. In this paragraph we consider several factorisation problems in the group \( \text{Map}_{g,1} \) and in its subgroup \( \text{Map}_{g,1,\mu} \) stabilising a given non-zero homology class \( \mu \in H_1(\Sigma, \mathbb{Z}_2). \) The topological meaning of these problems is the existence or non-triviality of certain special homology classes in Lefschetz fibrations.

Recall that \( \text{Map}_g \) denotes the group in the extension

\[
1 \rightarrow H_1(\Sigma, \mathbb{Z}_2) \rightarrow \widetilde{\text{Map}}_g \rightarrow \text{Map}_g \rightarrow 1,
\]
so that $\widetilde{\text{Map}}_g$ is the quotient of $\text{Map}_{g,1}$ by the image with respect to $\partial_\pi$ of the kernel of the homomorphism $\pi_1(\Sigma, z_0) \to H_1(\Sigma, \mathbb{Z}_2)$. For our purposes, only the projections of $F \in \text{Map}_{g,1}$ to $\text{Map}_g$ will be relevant. Let $\text{Map}_{g,\mu}$ be the stabiliser of $\mu$ in $\text{Map}_g$.

**Proposition 2.20.** There exists a homomorphism $\varphi_\mu : \widetilde{\text{Map}}_{g,\mu} \to \mathbb{Z}_4$ with the following properties:

i) $\varphi_\mu$ vanishes on the subgroup $\widetilde{\text{Map}}'_{g,\mu} \subset \widetilde{\text{Map}}_{g,\mu}$ generated by Dehn twists $T_\delta$ with $\delta \cap \mu \equiv 0 \mod 2$ and commutators $[F, F']$ with $F, F' \in \text{Map}_{g,\mu}$.

ii) The restriction of $\varphi_\mu$ to the kernel $\text{Ker}(\text{Map}_{g,\mu} \to \text{Sp}(2g, \mathbb{Z}_2))$ is the mod 4 reduction of the homomorphism $\varphi_\mu$.

iii) In particular, $\varphi_\mu(\partial_\pi(\gamma)) \equiv (2g - 2)\varphi_\mu(\gamma) \mod 4$ for any $\gamma \in \pi_1(\Sigma)$.

**Proof.** Combine Theorem 2.9 for the group $G := W(S)_\mu$ with Lemma 2.13 and Corollary 2.19.

**Theorem 2.21.** Let $g = 2g' \geq 2$ be even and let $\text{id} = \prod_i [F_{2i-1}, F_{2i}] \circ \prod_j T_{\delta_j}$ be a factorisation in $\text{Map}_g$ in which $T_{\delta_j}$ are Dehn twists along embedded circles $\delta_1, \ldots, \delta_n \subset \Sigma \setminus \{z_0\}$ and $[F_{2i-1}, F_{2i}]$ denotes the commutator of $F_{2i-1}, F_{2i} \in \text{Map}_{g'}$. Then there exist lifts $\tilde{F}_{2i-1}, \tilde{F}_{2i} \in \text{Map}_{g,1}$ and curves $\delta_j'$ on $\Sigma \setminus \{z_0\}$ such that each $\delta_j'$ is isotopic to $\delta_j$ on $\Sigma$ and $\prod_i [\tilde{F}_{2i-1}, \tilde{F}_{2i}] \cdot \prod_j T_{\delta_j'} = \partial_\gamma \epsilon \in \text{Map}_{g,1}$ for some $\gamma \in \pi_2(\Sigma, z_0)$ with trivial $\mathbb{Z}_2$-homology class $[\gamma] \equiv 0 \in H_1(\Sigma, \mathbb{Z}_2)$.

**Proof.** Fix some lifts $\tilde{F}_i \in \text{Map}_{g,1}$ of $F_i \in \text{Map}_g$ and lift $T_{\delta_j}$ to $\text{Map}_{g,1}$ in the natural way. For any $v \in H_1(\Sigma, \mathbb{Z}_2)$ we denote by $\vartheta_v$ the corresponding element in $\text{Map}_g$. Then $\prod_i [\tilde{F}_{2i-1}, \tilde{F}_{2i}] \cdot \prod_j T_{\delta_j} = \vartheta_{w_0} \in \text{Map}_g$ for some $w_0 \in H_1(\Sigma, \mathbb{Z}_2)$.

Denote by $W$ the set of all $w \in H_1(\Sigma, \mathbb{Z}_2)$ for which $\vartheta_w$ can be represented as the product $\prod_i [\tilde{F}_{2i-1}, \tilde{F}_{2i}] \cdot \prod_j T_{\delta_j'}$ for some possible lifts $\tilde{F}_i$ and $T_{\delta_j'}$. Then $W = w_0 + V$ for an appropriate $\mathbb{Z}_2$-subspace $V \subset H_1(\Sigma, \mathbb{Z}_2)$, and the assertion of the theorem is equivalent to the claim that $w_0 \in V$. Assuming the contrary, $V$ must be a proper subspace of $H_1(\Sigma, \mathbb{Z}_2)$.

Let $\mu \in H_1(\Sigma, \mathbb{Z}_2)$ be an element such that the functional $\varphi_\mu : v \in H_1(\Sigma, \mathbb{Z}_2) \mapsto \mu \cap v \in \mathbb{Z}_2$ vanishes on $V$ but is non-zero on $w_0$. We claim that $\mu \in H_1(\Sigma, \mathbb{Z}_2)$ is invariant under the action of all factors $F_i, T_{\delta_j}$. It follows from Lemma 2.13 that $V$ consists of (the Poincaré duals of) coboundaries $d(\lambda) \lambda'$ with $\lambda'$ having the form (1.16). The equivalent dual condition is that, considering the factorisation $\prod_i [F_{2i-1}, F_{2i}] \cdot \prod_j T_{\delta_j}$ as a single relation word $R$, the boundary $\partial_2(R \otimes_2 \mu)$ cancels all such $\lambda'$. The calculation done in the proof of Proposition 2.17 shows that

$$\partial_2(R \otimes_2 \mu) = \sum_j T_{\delta_j} \otimes w_j^T \mu + \sum_i (F_{2i-1} \otimes (F_{2i} - \text{id}) w_i^F \mu + F_{2i} \otimes (\text{id} - F_{2i-1}) w_i^F \mu)$$

in which $w_j^T, w_i^F$ are certain final subwords of $R$. More precisely, $w_j^T$ consists of all letters of $R$ after $T_{\delta_j}$, and $w_i^F$ of all letters of $R$ after $F_{2i}$. Now, consider the conditions $w_j^T \mu \cap \delta_j \equiv 0 \mod 2$ starting from the last letter $T_{\delta_j}$ and going backwards. Recursively, we obtain the desired congruence $\mu \cap \delta_j \equiv 0 \mod 2$. A similar argument for the conditions $(F_{2i} - \text{id}) w_i^F \mu \cap \lambda_i \equiv 0$ and $(F_{2i-1} - \text{id}) w_i^F \mu \cap \nu_i \equiv 0$ gives the desired result for $F_i$. 

Now, \( \tilde{\varphi}_\mu(\theta_{w_0}) \equiv 0 \mod 4 \) by Proposition \(^{2.20}\) \( i \). On the other hand, \( \hat{\varphi}_\mu(\theta_{w_0}) \equiv (2g - 2) \varphi_\mu(w_0) \equiv 2(\mu \cap w_0) \equiv 2 \) by Proposition \(^{2.20}\) \( iii \). This contradiction shows that the desired lifts exist.

\[ \square \]

**Remark.** From this theorem and Lemma \(^{1.9}\) we obtain Theorem \(^{3}\) and thence the non-triviality of the \( \mathbb{Z}_2 \)-homology class of the fibre for any topological Lefschetz fibration with even fibre genus stated in the remark following Proposition \(^{1.5}\)

From the proof of Theorem \(^{2.22}\) we can conclude the following:

**Corollary 2.22.** Let \( g \geq 3 \) be odd and let \( i = \prod_i [F_{2i-1}, F_{2i}] \cap \prod_j T_{\delta_j} \) be a factorisation in \( \text{Map}_g \) as in Theorem \(^{2.22}\). Assume that for every lift \( i, \beta \) \( \prod_j [F_{2j-1}, F_{2j}] \cap \prod_j T_{\delta_j} = \partial_\pi \gamma \in \text{Map}_g \) with \( \gamma \in \pi_1(\Sigma, z_0) \) the \( \mathbb{Z}_2 \)-homology class \( [\gamma]_{\mathbb{Z}_2} \) is non-trivial. Then there exists \( \mu \in H_1(\Sigma, \mathbb{Z}_2) \) stabilised by all \( F_i \) and all \( T_{\delta_j} \) such that \( \mu \cap [\gamma] \equiv 1 \mod 2 \) for every \( \gamma \in \pi_1(\Sigma, z_0) \) as above.

**Theorem 2.23.** \( i \) Let \( g \geq 1 \) be odd. Then the element \( (T_{\alpha_1} T_{\beta_1})^3 \) cannot be represented in \( \text{Map}_g \) as a product of Dehn twists and commutators \( \prod_i T_{\delta_i} \cdot \prod_j [F_{2j-1}, F_{2j}] \) such that \( \delta_i \cap \beta_j \equiv 0 \mod 2 \) and such that every \( F_j \) fixes the homology class \( [\beta_j]_{\mathbb{Z}_2} \in H_1(\Sigma, \mathbb{Z}_2) \).

\( ii \) Let \( g \geq 2 \) be even, let \( \prod_i T_{\delta_i} \cdot \prod_j [F_{2j-1}, F_{2j}] \) be a factorisation in \( \text{Map}_g \) of the element \( (T_{\alpha_1} T_{\beta_1})^3 \) satisfying the properties above, and let \( \hat{F}_j \) be some lifts of the involved factors to \( \text{Map}_{g_1} \). Then \( (T_{\alpha_1} T_{\beta_1})^{-3} \prod_i T_{\delta_i} \cdot \prod_j [F_{2j-1}, \hat{F}_{2j}] = \partial_\pi (\gamma) \) for some \( \gamma \in \pi_1(\Sigma, z_0) \) satisfying \( [\gamma] \cap [\beta_1] \equiv 1 \mod 2 \).

**Proof.** \( i \) We consider the special case \( g = 1 \) first, abbreviating the notation \( \alpha_1, \beta_1 \) to \( \alpha, \beta \). Here \( S_1 = \{ s_1, s_2 \} = A_2 \), and hence \( W(S_1) = \text{Sym}_3 \) and \( \text{Br}(S_1) = \text{Br}_3 \). Moreover, we can identify \( \text{Map}_{1,(-)} \) with \( \text{Br}_3 \) setting \( T_\alpha, T_\beta \) to be the standard generators of \( \text{Br}_3 \). Besides we have \( T(S_1) = H_1 = \{ [\alpha], [\beta], [\alpha + \beta] \} \), all three are \( \mathbb{Z}_2 \)-homology classes. Let us apply Theorem \(^{2.29}\) to the subgroup \( G := \mathbb{Z}_2(T_\beta) \subset \text{Sym}_3 \). It has two orbits in \( T(S_1) \), namely, \( \{ [\beta] \} \) and \( \{ [\alpha], [\alpha + \beta] \} \). Let \( A_{G,[\beta]} \), \( A_{G,[\alpha]} \) be the corresponding basis of \( \mathbb{P}_{ab}(S_1)_G \). As in the proof above, the projection from \( \mathbb{P}_{ab}(S_1) = \mathbb{Z}(A_{[\alpha]}, A_{[\beta]}, A_{[\alpha + \beta]}) \) onto the component \( \mathbb{Z}(A_{G,[\alpha]}) \) is given by the homomorphism \( \tilde{\varphi}_{[\beta]} : \mathbb{P}_{ab}(S_1) \to \mathbb{Z} \). Then Theorem \(^{2.29}\) shows that \( \tilde{\varphi}_{[\beta]}(F) = 0 \) for every factorisation \( F \) as in the theorem, whereas \( \tilde{\varphi}_{[\beta]}((T_\alpha T_\beta)^3) = \tilde{\varphi}_{[\beta]}(A_{[\alpha]} + A_{[\beta]} + A_{[\alpha + \beta]}) = 2 \). This contradiction excludes the existence of such a factorisation.

Now consider the general case of an odd \( g \geq 3 \) and assume that a factorisation \( \prod_i T_{\delta_i} \cdot \prod_j [F_{2j-1}, F_{2j}] \) as in the hypotheses exists. Lifting it to \( \text{Br}(S_g) \) we obtain an element \( \tilde{F} \) lying in the kernel \( \text{Ker}(\text{Br}(S_g) \to \text{Sp}(2g, \mathbb{Z}_2)) \). Repeating the argument from the proof of Theorem \(^{2.22}\) we obtain that \( \tilde{\varphi}_{[\beta]}(\tilde{F}) \equiv 0 \mod 4 \). Note that such a lift \( \tilde{F} \) differs from \( (T_{\alpha_1} T_{\beta_1})^3 \) (considered now as an element from \( \text{Br}(S_g) \)) by a product of chain, lantern, and hyperelliptic relation elements. Since \( g \) is odd, \( \tilde{\varphi}_{[\beta]}(x) \equiv 0 \mod 4 \) for every relation element \( x \) (including the hyperelliptic one), so the value \( \tilde{\varphi}_{[\beta]}((T_{\alpha_1} T_{\beta_1})^3) \) must be a multiple of 4. On the other hand, the calculation done in the case \( g = 1 \) shows that \( \tilde{\varphi}_{[\beta]}((T_{\alpha_1} T_{\beta_1})^3) = 2 \), a contradiction.
\textbf{ii)} As above, we apply the argument used in the proof of \textbf{Theorem 2.21}. This yields \( \hat{\phi}_{[\beta_1]}((T_{\alpha_1}T_{\beta_1})^3, \partial_3(\gamma)) \equiv 0 \mod 4 \). Since \( \hat{\phi}_{[\beta_1]}((T_{\alpha_1}T_{\beta_1})^3) = 2 \), the assertion follows from \textbf{Proposition 2.20 ii).}

\section{3. Proof of the main result}

In this section we give the proof of the \textbf{Main Theorem}. We maintain the notation of \textbf{Proposition 1.7}. In particular, \( m \) denotes the meridian circle of \( K \), considered as a curve on \( \Sigma \), and \( \mu = [m]_{\mathbb{Z}_2} \) its homology class on \( \Sigma \).

In the case when \( m \) separates \( \Sigma \), the theorem follows from the remark immediately after \textbf{Proposition 1.7}. So we assume henceforth that \( m \) is non-separating on \( \Sigma \). Furthermore, we may assume that the hypothesis of \textbf{Proposition 1.7} is fulfilled and the projection of \([K]\) to \( H_1(Y^\circ, \mathcal{H}_1(X_y, \mathbb{Z}_2)) \) vanishes, since otherwise there is nothing to prove.

Let us make the following observation about the monodromy \( F_1 \) along \( \Gamma \). Realise the meridian \( m \) as the curve \( \beta_1 \) in some geometric basis \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \) of \( \Sigma \). Then we can isotopically deform \( F_1 \) so that \( \beta_1 = m \) is \( F_1 \)-stable. The induced map \( F_1 : \beta_1 \rightarrow \beta_1 \) inverts the orientation. Observe that the map \((T_{\alpha_1}T_{\beta_1})^3 \) has the same property in the homotopy, namely, it maps the free homotopy class of \( \beta_1 \) onto itself inverting the orientation. It follows that the homotopy class \( F_1 := F_1(T_{\alpha_1}T_{\beta_1})^{-3} \) admits a representative (still denoted by \( F_1 \)) which fixes \( \beta_1 \) pointwise. In particular, \( F_1 \) is a diffeomorphism of \( \Sigma \setminus \beta_1 \). The classification of diffeomorphisms of surfaces implies that \( F_1 \) can be represented as a product of Dehn twists \( T_\delta \) along curves disjoint from \( \beta_1 \). Fix such a decomposition of \( F_1 \) and lift it to the group \( \text{Map}_{g,1} \). The obtained elements of \( \text{Map}_{g,1} \) are denoted by \( \hat{F}_1 \) and \( \hat{F}_1 := \hat{F}_1 \cdot (T_{\alpha_1}T_{\beta_1})^3 \).

### 3.1. Fine structure of the monodromy

In this paragraph we refine the result of \textbf{Proposition 1.7}.

\textbf{Proposition 3.1.} Under the hypotheses of \textbf{Proposition 1.7 i)}, assume in addition that \([K]\) vanishes in \( H_2(X, \mathbb{Z}_2) \). Then the decomposition \( \mu = \mu_+ + \mu_- \in H_1(\Sigma, \mathbb{Z}_2) \) constructed in \textbf{Proposition 1.7} satisfies \( \mu \cap \mu_+ = \mu \cap \mu_- = 1 \mod 2 \).

Moreover, the class \( \mu \) is \textbf{not} invariant either with respect to the monodromy action of \( \pi_1(Y^\circ) \) or with respect to the action of \( \pi_1(Y^\circ) \).

\textbf{Proof.} We must exclude the following two possibilities:

- One of the classes \( \mu_\pm \) vanishes, say, \( \mu_- = 0 \) and so \( \mu_+ = \mu \).
- Both \( \mu_+, \mu_- \) are non-trivial, but \( \mu \cap \mu_+ = \mu \cap \mu_- \equiv 0 \mod 2 \).

Consider first the case \( \mu_- = 0 \). Then the monodromy \( F_1 \) admits a representation as a product \( F_+ := \prod_i [F_{2i-1}, F_{2i}] \cdot \prod_j T_{\delta_j} \) such that \( \mu \) is invariant with respect to all \( F_i \) and all \( T_{\delta_j} \). This gives the equality \((T_{\alpha_1}T_{\beta_1})^3 = F_+ \cdot (F_1)^{-1} \) with the same property for the right hand side. \textbf{Theorem 2.23} excludes such a possibility for odd \( g \) and ensures the following in the case of even \( g \geq 2 \). For any lift \( \hat{F}_+ \) of \( F_+ \) to the group \( \text{Map}_{g,1} \) we obtain the relation \((T_{\alpha_1}T_{\beta_1})^3 = \partial_x(\gamma) \cdot \hat{F}_+ \cdot (\hat{F}_1)^{-1} \) with some \( \gamma \in \pi_1(\Sigma) \) whose homology class \([\gamma] \in H_1(\Sigma, \mathbb{Z}_2) \) satisfies \([\gamma] \cap \mu \equiv 1 \mod 2 \). Now choose such a lift \( \hat{F}_+ \) of \( F_+ \) as a part of a lift of the whole monodromy using \textbf{Theorem 2.21}. The lift \( \hat{F}_1 \in \text{Map}_{g,1} \) gives us a section \( \sigma_K : \Gamma \rightarrow X \) of
the fibration \( \text{pr}: X \to Y \) over \( \Gamma \) which is disjoint from \( K \). On the other hand, the lift \( \hat{F}_+ \)
defines a \( \mathbb{Z}_2 \)-section \( \sigma|_{\Gamma} \) of \( \text{pr}: X \to Y \) over \( \Gamma \) which is the restriction of some \( \mathbb{Z}_2 \)-section \( \sigma \).

The condition \( \{\gamma\} \cap \mu \equiv 1 \mod 2 \) means that the \( \mathbb{Z}_2 \)-intersection index \( \sigma \cap [K] \) is non-zero, hence \([K]\) is non-trivial. This rules out the case \( \mu_-=0 \).

This shows also that \( \mu \) cannot be invariant with respect to the monodromy action of \( \pi_1(Y_+^0) \) or \( \pi_1(Y_0^0) \).

Let us assume that both \( \mu_+ \) and \( \mu_- \) are non-trivial but \( \mu_+ \cap \mu \equiv \mu_- \cap \mu \equiv 0 \). Then \( g \geq 2 \). Set \( l := \hat{\varphi}_{\mu_+} (\hat{F}_-) \). In the factorisation of \( \hat{F}_- = \hat{F}_- \circ (T_{\alpha_1} T_{\beta_1})^{-1} \) considered above there are no Dehn twist \( T_{\delta_j} \) with \( \delta_j \cap \mu \equiv 1 \). So using the definition of \( \hat{\varphi}_{\mu}, \hat{\varphi}_{\mu_{\pm}} \) we conclude that \( \hat{\varphi}_{\mu} (\hat{F}_- \circ (T_{\alpha_1} T_{\beta_1})^{-1}) = 2 \) and \( \hat{\varphi}_{\mu_{\pm}} (\hat{F}_-) = l+2 \).

Let \( F_+ \) and \( F_- \) be the factorisations of \( F_g \) of the form \( F_+ = \prod [F_{2i-1}, F_{2i}] \cdot \prod T_{\delta_j} \) arising from the monodromy homomorphisms \( \mathcal{F}_\pm : \pi_1(Y_0^\circ) \to \text{Map}_g \). Lifting the monodromy homomorphisms \( \mathcal{F}_\pm \) to \( \text{Map}_{g,1} \) we obtain lifts \( \hat{F}_\pm \in \text{Map}_{g,1} \) of \( F_g \), which differ from \( \hat{F}_g \) by some elements lying in the image of the homomorphism \( \partial_\pi : \pi_1(\Sigma) \to \text{Map}_{g,1} \).

Assume that \( g \) is odd. Then \( \hat{\varphi}_{\mu}, \hat{\varphi}_{\mu_{\pm}} \) vanish \( \mod 4 \) on the image of \( \partial_{\pi} \). Applying Theorem 2.9 we conclude that \( \hat{\varphi}_{\mu_+} (\hat{F}_+) \equiv \hat{\varphi}_{\mu_-} (\hat{F}_-) \equiv 0 \mod 4 \). This contradicts the relation \( \hat{\varphi}_{\mu_+} (\hat{F}_-) - \hat{\varphi}_{\mu_-} (\hat{F}_+) = 2 \).

Assume that \( g \) is even. Define \( \gamma_{\pm} \in \pi_1(\Sigma) \) from the relations \( \hat{F}_\pm = \hat{F}_g \circ \partial_\pi(\gamma_{\pm}) \). By Theorem 2.27 there exists a lift \( \hat{\mathcal{F}} : \pi_1(Y_0^\circ) \to \text{Map}_g \) of the monodromy homomorphism \( \mathcal{F} : \pi_1(Y_0^\circ) \to \text{Map}_g \), see Section 2.3. By Lemma 1.9 every such lift \( \hat{\mathcal{F}} \) corresponds to a \( \mathbb{Z}_2 \)-section \( \sigma \). We can suppose that the lifts \( \hat{F}_\pm \in \text{Map}_{g,1} \) of \( F_g \) are compatible with such a section \( \sigma \). This means that the projections of \( \hat{F}_\pm \) to \( \text{Map}_g \) are equal. This shows that \( \{\gamma_+\} = \{\gamma_-\} \in H_1(\Sigma, \mathbb{Z}_2) \). Comparing the values of \( \hat{\varphi}_{\mu}, \hat{\varphi}_{\mu_{\pm}} \) and using \( \hat{\varphi}_{\mu_{\pm}} (\hat{F}_+) = 0 \), we obtain \( \hat{\varphi}_{\mu_{\pm}} (\partial_\pi(\gamma_{\pm})) = -l \) and \( \hat{\varphi}_{\mu_{\pm}} (\partial_\pi(\gamma_{-})) = -l - 2 \). Now, using Proposition 2.20, we obtain \( \{\gamma_+\} \cap \mu \equiv 1 \mod 2 \). This means that \([K] \cap [\sigma] \neq 0\), which contradicts the hypotheses of the proposition.

\[ \square \]

### 3.2. Proof of Main Theorem

Let us change the notation slightly and denote by \( X_0 \) the original symplectic manifold (\( \mathbb{CP}_2 \) or ruled one) and by \( Y \) the blow-up of \( X_0 \) constructed in Lemma 1.2.

Let us assume that the claim of the Main Theorem is false and \( K \subset X \) is homologically trivial. First, we sum up the properties of \( \text{pr}: X \to Y \) obtained so far. By the construction of Lemma 1.2 \( Y \) is the sphere \( S^2 \) and \( \Gamma \) separates \( Y \) into discs \( Y_\pm \). By Proposition 1.3 the meridian circle \( m \) does not separate \( \Sigma \), and by Proposition 3.1 the monodromy of \( \text{pr}: X \to Y \) has the following structure: there exist \( \mu_+, \mu_- \in H_1(\Sigma, \mathbb{Z}_2) \) such that \( \mu_+ \cap \mu_- \equiv 1 \mod 2 \) and such that the monodromy \( \mathcal{F}_\pm : \pi_1(Y_\pm^0) \to \text{Map}_g \) of each piece preserves the class \( \mu_\pm \) but does not preserve \( \mu \). Finally, the monodromy \( F_g \) along \( \Gamma \) can be realised by a map, still denoted by \( F_g \in \text{Diff}_+(\Sigma) \), which maps \( m \) onto itself reversing the orientation on \( m \).

Fix local “polar” coordinates \((r, \varphi)\) in a neighbourhood \( U_m \) of \( m \) on \( \Sigma \) so that \( m \) is defined by the equation \( r = 1 \), \( \varphi \) is an angle coordinate on \( m \), and the map \( F_g \) is given by \((r, \varphi) \mapsto (r^{-1}, -\varphi)\). It follows that there exists a geometric realisation \( T_m \in \text{Diff}_+(\Sigma) \) of the Dehn twist along \( m \) which commutes with \( F_g \). Using this map \( T_m \) and a local
trivialisation of the bundle \( \mathfrak{pr} : X_\Gamma \rightarrow \Gamma \) used in the definition of the monodromy map \( F_\Gamma \). We obtain a map \( \Psi : X_\Gamma \rightarrow X_\Gamma \) which preserves the fibres and acts on each fibre \( X_y \) \((y \in \Gamma)\) by the map \( T_m \).

Construct a new manifold \( X' \) by gluing together the pieces \( X_{Y_+} \) and \( X_{Y_-} \) along \( X_\Gamma = \partial X_{Y_+} = \partial X_{Y_-} \) via the map \( \Psi : X_\Gamma \rightarrow X_\Gamma \). Obviously, \( X' \) admits a Lefschetz fibration \( \mathfrak{pr}' : X' \rightarrow Y \) over the same base \( Y = S^2 \). Moreover, the monodromy of \( \mathfrak{pr}' : X' \rightarrow Y \) remains unchanged in \( Y_+ \) and gets conjugated by \( T_m \) in \( Y_- \). Since \([m]_{Z_2} = \mu \) and \( \mu_+ = \mu + \mu_- \), the whole monodromy of \( \mathfrak{pr}' : X' \rightarrow Y \) preserves the class \( \mu_+ \).

**Lemma 3.2.** \( \text{rank } H_1(X', Z_2) = \text{rank } H_1(X, Z_2) + 1 \).

**Proof.** Let \( V_+ \) and \( V_- \) be the \( Z_2 \)-subspaces of \( H_1(\Sigma, Z_2) \) generated by the vanishing classes \([\delta_i]\) of the projections \( \mathfrak{pr} : X_{Y_+} \rightarrow Y_+ \) and \( \mathfrak{pr} : X_{Y_-} \rightarrow Y_- \), respectively. Then by **Lemma 1.8**

\[
H_1(X, Z_2) = H_1(\Sigma, Z_2)/(V_+ + V_-) \quad \text{and} \quad H_1(X', Z_2) = H_1(\Sigma, Z_2)/(V_+ + T_\mu(V_-))
\]

Set \( W_0 := Z_2\langle \mu_+, \mu_- \rangle \subseteq H_1(\Sigma, Z_2) \) so that \( H_1(\Sigma, Z_2) = Z_2\langle \mu_+, \mu_- \rangle \oplus W_0 \) is an orthogonal decomposition. It follows from the second assertion of **Proposition 3.1** and the relation \( \mu = \mu_+ + \mu_- \) that \( V_+/(V_+ \cap W_0) \) has rank 1 and is generated by the coset class of \( \mu_+ \). Similarly, \( V_-/(V_- \cap W_0) = Z_2\langle \mu_- \rangle \). Since \( T_\mu(\mu_-) = \mu_+ \), we obtain

\[
(V_+ + V_-)/((V_+ + V_-) \cap W_0) = Z_2\langle \mu_+, \mu_- \rangle \quad \text{and} \quad (V_+ + T_\mu(V_-))/((V_+ + T_\mu(V_-)) \cap W_0) = Z_2\langle \mu_+ \rangle.
\]

The lemma follows. \( \square \)

**Remark.** I am grateful to Stefan Nemirovski for the following observation. The twisting construction of the manifold \( X' \) above is the **Luttinger surgery** of \( X \) along the Klein bottle \( K \), see \([0.2] \) and \([N-2] \). In its turn, the description of the construction shows that the Luttinger surgery along the Klein bottle \( K \) is compatible with the projection \( \mathfrak{pr} \) of the Morse-Lefschetz fibration \( \mathfrak{pr} : (X, K) \rightarrow (Y, \Gamma) \) provided that the restricted projection \( \mathfrak{pr} : K \rightarrow \Gamma \cong S^1 \) is an \( S^1 \)-bundle without critical points. Note that this compatibility property holds also if \( K \) is replaced by the torus \( T^2 \) (see \([ADK] \)).

**Lemma 3.3.** Both \( \text{rank } H_1(X, Z_2) \) and \( \text{rank } H_1(X', Z_2) \) must be even.

**Proof.** The classification of ruled symplectic manifolds \([MD-Sa] \) implies that a finite sequence of blow-ups and blow-downs transforms \( X \) into a product \( S^2 \times Y \) where \( Y \) is a closed oriented surface. Since blow-ups do not change \( \pi_1(X) \), we conclude the first part of the lemma.

The second part is obtained in the same way once we show that \( X' \) is also a ruled symplectic manifold. First, we observe that the gluing map \( \Psi : X_\Gamma \rightarrow X_\Gamma \) can be extended to a symplectomorphism of some neighbourhood of \( X_\Gamma \). It follows that \( X' \) carries a symplectic form \( \omega' \) that coincides with the original form \( \omega \) on \( X \) on the pieces \( X_{Y_\pm} \).

At this point we use the specific structure of \( \omega \) and the monodromy of \( \mathfrak{pr} : X \rightarrow Y \). Recall that \( X \) was constructed as a symplectic blow-up of the original \( X_0 \). It follows that there exist symplectic sections \( E_1, \ldots, E_N \subset X \) of the projection \( \mathfrak{pr} : X \rightarrow S^2 \) such that for the class \([D] := [\Sigma] + \sum_i [E_i] \in H_2(X, Z) \) one has \( c_1(X) \cdot [D] > 0 \). The sections \( E_i \) are simply the exceptional spheres resulting from the blow-up construction. Since \( E_i \) are disjoint from \( K \), they survive in \( X' \), and we obtain symplectic sections \( E'_1, \ldots, E'_N \) in \( X' \).
Moreover, for the class \( [D'] := [\Sigma] + \sum_i [E'_i] \in H_2(X, \mathbb{Z}) \) we again have \( \chi(X') \cdot [D'] > 0 \). In this situation the characterisation theorem of McDuff and Salamon, see Corollary 1.5 in [MD-Sa], says that \( X' \) must indeed be a ruled symplectic manifold.

The obtained contradiction shows that under the hypothesis of the Main Theorem the monodromy of \( pr : X \to Y \) can not have the structure described in Proposition 3.1. This implies the assertion of the Main Theorem.

REFERENCES

[ABKP] Amorós, J.; Bogomolov, F.; Katzarkov, L.; Pantev, T.: Symplectic Lefschetz fibrations with arbitrary fundamental groups; with an appendix by Ivan Smith, J. Diff. Geom., 54 (2000), 489–545, Math. Rev.: 1823313 (2002g:57051)

[A-G] Arnold, V. I.; Givental, A. B.: Symplectic geometry, Dynamical systems, IV, 1–138, Encyclopaedia Math. Sci., 4, Springer, Berlin, 2001; Math. Rev.: 0842908 (88b:58044).

[AGV] Arnold, V.; Gusein-Sade, S.; Varchenko, A.: Singularities of Differential maps, vol.II: Monodromy and Asymptotic Integrals, viii+492 p., Birkhäuser, 1988.

[Aud] Audin, M.: Quelques remarques sur les surfaces lagrangiennes de Givental’, J. Geom. Phys., 7 (1990), 583–598, Math. Rev.: 1131914 (92i:57022)

[Aur] Auroux, D.: A stable classification of Lefschetz fibrations, Geom. Topol., 9 (2005), 203–217, Math. Rev.: 2115673 (2005i:57032)

[ADK] Auroux, D.; Donaldson, S. K.; Katzarkov, L.: Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves, Math. Ann., 326 (2003), 185-203.

[AMP] Auroux, D.; Muñoz, V.; Presas, F.: Lagrangian submanifolds and Lefschetz pencils, J. Symplectic Geom., 3 (2005), 171–219, Math. Rev.: 2199539

[Bi-1] Birman, J.: Mapping class groups of surfaces, In Proceedings of Conference on Braids (Santa Cruz, CA, 1986), 13–43, Contemp. Math., 78, Amer. Math. Soc., Providence, RI, 1988, Math. Rev.: 0975076 (90g:57013).

[Bi-2] Birman, J.: Braids, links, and mapping class groups, Annals of Math. Studies., 82 (1975), Princeton Univ. Press and Univ. of Tokyo Press., 229 p., (1975).

[Bi-Waj] Birman, J.; Wajnryb, B.: Errata: Presentations of the mapping class group, Isr. J. Math., 88 (1994), 425-427.

[BKP] Bogomolov, F.; Katzarkov, L.; Pantev, T.: Hyperelliptic Szpiro inequality, J. Diff. Geom., 61 (2002), 51–80, Math. Rev.: 1949784 (2003k:57030)

[Bo-Tsch] Bogomolov, F.; Tschinkel, Yu.: Simple examples of symplectic four-manifolds with exotic properties, Acta Appl. Math. 75 (2003), 25-28.

[Bou] Bourbaki, N.: Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968, 288 pp.

[Br-Sa] Brieskorn, E.; Saito, K.: Artin-Gruppen und Coxeter-Gruppen, Invent. Math., 17(1972), 245–271, Math. Rev.: 0323910 (54#2263)

[Bro] Brown, K. S.: Cohomology of groups, Springer-Verlag, Berlin, 1982, x+306 pp., Math. Rev.: 0672956 (83k:20002)

[Co-Zi] Collins, D. J.; Zieschang, H.: Combinatorial group theory and fundamental groups, Algebra VII, Encyclopaedia Math. Sci., 58, pp.1–166; Springer, Berlin, 1993, Math. Rev.: 1265269 (95g:57004).

[De] Dehn, M.: Die Gruppe der Abbildungsklassen, Acta Math., 69(1938), 135-206.

[Del] Deligne, P.: Les immeubles des groupes de tresses généralisés, Invent. Math., 17(1972), 273–302, Math. Rev.: 0422673 (54 #10659).
Donaldson, S.: *Lefschetz fibrations in symplectic geometry*, Doc. Math. J. DMV, Extra Volume ICMIII (1998), 309–314.

Eliashberg, Ya.; Polterovich, L.: *New applications of Luttinger’s surgery*, Comment. Math. Helv., 69 (1994), 512-522.

Gatien, D.; Lalonde, F.: *Holomorphic cylinders with Lagrangian boundaries and Hamiltonian dynamics*, Duke Math. J. 102 (2000), no. 3, 485–511. *Math. Rev.*: MR1756107 (2002h:53146)

Gervais, S.: *Presentation and central extensions of mapping class groups*, Trans. AMS, 348 (1996), 3097–3132, *Math. Rev.*: 1327256 (96j:57016)

Gompf, R. E.; Stipsicz, A. I.: *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20, AMS, Providence, RI, 1999, xvi+558 pp, *Math. Rev.*: MR1707327 (2000h:57038).

Gompf, R.: *Toward a topological characterization of symplectic manifolds*, J. Symplectic Geom., 2 (2004), 177–206. *Math. Rev.*: MR2108373 (2005j:53100).

Griffiths, P., Harris, J.: *Principles of algebraic geometry*, John Wiley & Sons, N.-Y., (1978).

Humphreys, J. E.: *Reflection groups and Coxeter groups*, Cambridge Univ. Press, 1990, 204+x pp.

Ivanov, N.: *Mapping class groups*, Daverman, R. J. (ed.) et al., Handbook of geometric topology. Amsterdam: Elsevier, 523-632 (2002).

Johnson, D.: *The structure of the Torelli group I: A finite set of generators for Τ*, Ann. of Math., 118 (1983), 423-442, *Math. Rev.*: 85a:57005.

Johnson, D.: *A survey of the Torelli group*, Contemp. Math., 20 (1983), 165-179, *Math. Rev.*: 85d:57009.

Kanev, V.: *Irreducibility of Hurwitz spaces*, 34 pages, Preprint N. 241, Dipartimento di Matematica, Università di Palermo, ArXiv: math.AG/0509154.

Kharlamov, V. M.; Kulikov, V. S.: *On braid monodromy factorizations*, Izvestiya: Mathematics, 67:1 (2003), 151-164.

Lyndon, R. C.; Schupp, P. E.: *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin–New-York, 1977, xiv+339 pp.

Matsumoto, M.: *A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities*, Math. Ann., 316 (2000), 401–418, *Math. Rev.*: 1752777 (2001e:57002)

McDuff, D., Salamon, D.: *A survey of symplectic 4-manifolds with b_+ = 1*, Turk. J. Math. 20, 47–60, (1996).

Mohnke, K.: *How to (symplectically) thread the eye of a (Lagrangian) needle*, Preprint, 15 pages, ArXiv: math.SG/0106139.

Nemirovskii, S.: *Lefschetz pencils, Morse functions, and Lagrangian embeddings of the Klein bottle*, Izvetsiya Mathematics, 66:1 (2002), 151-164.

Nemirovski, S.: *Homology class of a Lagrangian Klein bottle*, Preprint (Version 3), 6 pp., ArXiv: math.SG/0106122 v3.

Powell, J.: *Two theorems on the mapping class group of a surface*, Proc. AMS, 68 (1978), 347-350, *Math. Rev.*: 58:13045.

Wajnryb, B.: *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math., 45 (1983), 157–174, *Math. Rev.*: 0719117 (85g:57007)

**Mathematisches Institut der Universität Bonn, Beringstrasse 1, D-53115 Bonn, Germany**

*E-mail address: sewa@math.uni-bonn.de*