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Nonlocal symmetries of Plebański’s second heavenly equation

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We study nonlocal symmetries of Plebański’s second heavenly equation in an infinite-dimensional covering associated to a Lax pair with a non-removable spectral parameter. We show that all local symmetries of the equation admit lifts to full-fledged nonlocal symmetries in the infinite-dimensional covering. Also, we find two new infinite hierarchies of commuting nonlocal symmetries in this covering and describe the structure of the Lie algebra of the obtained nonlocal symmetries.

Keywords: Plebański’s second heavenly equation; Lax pair; differential covering; nonlocal symmetries
2000 Mathematics Subject Classification: 58H05, 58J70, 35A30

1. Introduction

In this paper we study nonlocal symmetries of Plebański’s second heavenly equation, [32],

\[ u_{xz} - u_{ty} - u_{xx} u_{yy} + u_{xy}^2 = 0. \]  \hspace{1cm} (1.1)

This equation attached considerable attention because of its importance in general relativity. In particular, the equation is a reduction of the Einstein equations that govern self-dual gravitation fields, see [12, 26, 30, 35] and references therein. Eq. (1.1) is an example of a nonlinear integrable equation in four independent variables. Here integrability means the existence of a Lax pair, or a differential covering,

\[
\begin{cases}
q_t = (u_{xy} + \lambda) q_x - u_{xx} q_y,
q_z = u_{yy} q_x - (u_{xy} - \lambda) q_y,
\end{cases}
\]  \hspace{1cm} (1.2)

with a non-removable parameter \( \lambda \). Expanding the pseudopotential \( q \) into a Taylor series \( q = \sum_{k=0}^{\infty} \lambda^k q_k \) yields a new covering (4.1) with pseudopotentials \( q_k \) over Eq. (1.1). The goal of this paper is to study nonlocal symmetries for Eq. (1.1) in this covering.

Infinite-dimensional Lie algebras of nonlocal symmetries are well known to play an important role in the theory of nonlinear integrable equations and provide a useful tool to study of the latter, see e.g. [5, 7, 16, 17, 20, 33, 34] and references therein. Eq. (1.1) has the infinite-dimensional Lie algebra \( \mathfrak{s} \) of local contact symmetries and, as it was shown in [24], is uniquely defined by this algebra, see also [11, 25] for discussion of geometric properties of Eq. (1.1) and related equations. The algebra \( \mathfrak{s} \) is the semi-direct product \( \mathfrak{s} = \mathfrak{s}_o \ltimes \mathfrak{s}_c \), where \( \mathfrak{s}_o = \left[ \left[ \mathfrak{s}, \mathfrak{s} \right], \left[ \mathfrak{s}, \mathfrak{s} \right] \right] \) is an infinite-dimensional ideal and \( \mathfrak{s}_c \) is a three-dimensional solvable Lie algebra. We show that all local symmetries of Eq. (1.1) have...
lifts to nonlocal symmetries, that is to symmetries of the system (1.1), (4.1). Note that not every integrable equation allows lifts of local symmetries to nonlocal symmetries, see e.g. [10]. Also, we find two infinite hierarchies a and b of nonlocal symmetries such that

\[ [a, a] = 0, \quad [b, b] = 0. \] (1.3)

Note that existence of an infinite hierarchy of commuting flows is one of the most important properties of integrability, see [1, 6, 14, 19, 31] and references therein. Also, we find the structure of the Lie algebra \( s \oplus a \oplus b \) (the sum as vector spaces). In particular, we show that

\[ [s_\infty, a] = [s_\infty, b] = 0, \quad [s_\infty, a] = a, \quad [s_\infty, b] = b, \quad [a, b] = a. \] (1.4)

There is a great many of works devoted to methods of studying nonlinear partial differential equations (PDEs) that admit nonlocal symmetries. For the case of potential symmetries, that is nonlocal symmetries corresponding to Abelian coverings, see [8, Ch. 7], [9], and references therein. In regard to applications of nonlocal symmetries in non-Abelian coverings to studying of exact solutions and other integrability properties of nonlinear PDEs, see [5, 16, 17, 34] and references therein.

2. Preliminaries

All considerations in this paper are local. The presentation in this section closely follows [20, 21], see also [22, 23]. Let \( \pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n) \) be a trivial bundle, and \( J^\infty(\pi) \) be the bundle of its jets of the infinite order. The local coordinates on \( J^\infty(\pi) \) are \( (x^i, u^a, u_b^f) \), where \( I = (i_1, \ldots, i_n) \) is a multi-index, and for every local section \( f: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \) of \( \pi \) the corresponding infinite jet \( j_\infty(f) \) is a section \( j_\infty(f): \mathbb{R}^n \to J^\infty(\pi) \) such that \( u_b^f(j_\infty(f)) = \partial_{x^i} f^a = \frac{\partial f^a}{\partial x^i} \). We put \( u^a = u^a(0, \ldots, 0) \). Also, in the case of \( n = 4, m = 1 \) we denote \( x^1 = t, x^2 = x, x^3 = y, x^4 = z \) and \( u^i_{(i,j,k,l)} = u_{i,ij,k,l} \) with \( i \) times \( t \), \( j \) times \( x \), \( k \) times \( y \), and \( l \) times \( z \).

The vector fields

\[ D_k = \frac{\partial}{\partial x^k} + \sum_{\ell \geq 0} m \sum_{\alpha=1}^{m} u_{\ell+k}^\alpha \frac{\partial}{\partial u_{\ell}^\alpha}, \quad k \in \{1, \ldots, n\}, \]

\((i_1, \ldots, i_k, \ldots, i_n) + 1 = (i_1, \ldots, i_k + 1, \ldots, i_n),\) are called total derivatives. They commute everywhere on \( J^\infty(\pi): [D_k, D_\ell] = 0 \).

The evolutionary derivation associated to an arbitrary smooth function \( \varphi: J^\infty(\pi) \to \mathbb{R}^m \) is the vector field

\[ E_\varphi = \sum_{\ell \geq 0} m \sum_{\alpha=1}^{m} D_\ell(\varphi^\alpha) \frac{\partial}{\partial u_{\ell}^\alpha}, \] (2.1)

with \( D_\ell = D_{(i_1, \ldots, i_\ell)} = D_{i_1}^{i_1} \circ \cdots \circ D_{i_{\ell}}^{i_{\ell}}. \)

A system of PDEs \( F_r(x^i, u_r^f) = 0 \), \( \#I \leq s, r \in \{1, \ldots, R\}, \) of the order \( s \geq 1 \) with \( R \geq 1 \) defines the submanifold \( E = \{ (x^i, u_r^f) \in J^\infty(\pi) \mid D_K(F_r(x^i, u_r^f)) = 0, \#K \geq 0 \} \) in \( J^\infty(\pi) \).

\[ ^a\text{We identify here the Lie algebra } s \text{ of local symmetries with its lift to nonlocal symmetries in the covering (4.1)} \]

\[ ^b\text{See definition of an Abelian covering in Section 2} \]
A function $\varphi : J^m(\pi) \to \mathbb{R}^m$ is called a (generator of an infinitesimal) symmetry of $\mathcal{E}$ when $E_\varphi(F) = 0$ on $\mathcal{E}$. The symmetry $\varphi$ is a solution to the defining system

$$\ell_\mathcal{E}(\varphi) = 0,$$

where $\ell_\mathcal{E} = \ell_F|_\mathcal{E}$ with the matrix differential operator

$$\ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_{\alpha I}}{\partial u_{I}^\alpha} D_{I} \right).$$

Solutions to (2.2) constitute the Lie algebra $\mathfrak{Lie}(\mathcal{E})$ of (infinitesimal) symmetries of equation $\mathcal{E}$ with respect to the Jacobi bracket $\{ \varphi, \psi \} = E_{\varphi}(\psi) - E_{\psi}(\varphi)$. The subalgebra of contact symmetries of $\mathcal{E}$ is $\mathfrak{Lie}_0(\mathcal{E}) = \mathfrak{Lie}(\mathcal{E}) \cap C^\infty(\mathcal{J}^{1}(\pi), \mathbb{R}^m)$.

Denote $\mathcal{W} = \mathbb{R}^m$ with coordinates $w^s$, $s \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) differential covering of $\mathcal{E}$ is a trivial bundle $\tau : J^m(\pi) \times \mathcal{W} \to J^m(\pi)$ equipped with the extended total derivatives

$$\tilde{D}_k = D_k + \sum_{s=0}^{\infty} T^{(s)}_k (x^i, u^s_{I}, w^j) \frac{\partial}{\partial w^s}$$

such that $[\tilde{D}_k, \tilde{D}_{\ell}] = 0$ for all $i \neq j$ whenever $(x^i, u^0_{I}) \in \mathcal{E}$. We define the partial derivatives of $w^s$ by $w^s_{I} = \tilde{D}_k (w^s)$. This yields the system of covering equations

$$w^s_{I} = T^s_k (x^i, u^s_{I}, w^j).$$

This over-determined system of PDEs is compatible whenever $(x^i, u^0_{I}) \in \mathcal{E}$.

A covering is said to be Abelian when system (2.4) can be mapped to the form

$$w^0_{I} = T^0_k (x^i, u^0_{I}), \quad w^j_{I} = T^j_k (x^i, u^0_{I}, w^0, \ldots, w^{j-1}), \quad j \in \mathbb{N}$$

by a change of variables $x^i$, $u^0_{I}$, $w^s$. Otherwise the covering is non-Abelian.

Denote by $\tilde{E}_\varphi$ the result of substitution of $\tilde{D}_k$ for $D_k$ in (2.1). A shadow of nonlocal symmetry of $\mathcal{E}$ corresponding to the covering $\tau$ with the extended total derivatives (2.3), or $\tau$-shadow, is a function $\varphi \in C^\infty(\mathcal{E} \times \mathcal{W}, \mathbb{R}^m)$, such that

$$\tilde{E}_\varphi(F) = 0$$

is a consequence of equations $D_k(F) = 0$ and (2.4). A nonlocal symmetry of $\mathcal{E}$ corresponding to the covering $\tau$ (or $\tau$-symmetry) is the vector field

$$\tilde{E}_{\varphi,A} = \tilde{E}_\varphi + \sum_{s=0}^{\infty} A^s \frac{\partial}{\partial w^s},$$

with $A^s \in C^\infty(\mathcal{E} \times \mathcal{W})$ such that $\varphi$ satisfies (2.5) and

$$\tilde{D}_k(A^s) = \tilde{E}_{\varphi,A}(T^s_k)$$

for $T^s_k$ from (2.3), see [10, Ch. 6, §3.2].

**Remark 2.1.** In general, not every $\tau$-shadow corresponds to a $\tau$-symmetry, since equations (2.7) provide an obstruction for existence of $A^s$ in (2.6). But for any $\tau$-shadow $\varphi$ there exists a covering $\tau_\varphi$ and a nonlocal $\tau_\varphi$-symmetry whose $\tau_\varphi$-shadow coincides with $\varphi$, see [10, Ch. 6, §5.8].
3. Local symmetries

The local contact symmetries of Eq. (1.1) are solutions \( \varphi = \varphi(t, x, y, z, u, u_t, u_x, u_y, u_z) \) to the equation

\[
\ell \varphi = D_tD_x \varphi - D_tD_y \varphi - u_x D_x^2 \varphi + 2u_vD_tD_y \varphi = 0.
\]

Direct computations\(^4\) show that the Lie algebra \( \mathfrak{s} = \text{Lie}_0(\mathcal{E}) \) is generated by the following functions

\[
\begin{align*}
\varphi_0(A) &= -A_t u_t - (xA_{zz} + yA_{zz}) u_x + (xA_{tt} + yA_{tt}) u_y + A_t u_z
\quad - \frac{1}{6} \left( x^3 A_{ttt} + 3 x^2 y A_{tzz} + 3 x y^2 A_{ttz} + y^3 A_{zzz} \right),
\varphi_1(A) &= -A_z u_t + A_t u_y - \frac{1}{2} \left( x^2 A_{tt} + 2 x y A_{tz} + y^2 A_{zz} \right),
\varphi_2(A) &= -A_t - yA_z,
\varphi_3(A) &= -A,
\psi_1 &= 3 u - x u_t - y u_y,
\psi_2 &= -3 t u_t - x u_x - y u_y - 3 z u_z,
\psi_3 &= -t u_x - z u_y,
\end{align*}
\]

where \( A = A(t, z) \) and \( B = B(t, z) \) below are arbitrary functions of \( t \) and \( z \). The structure of \( \mathfrak{s} \) is given by equations

\[
\begin{align*}
\{ \varphi_i(A), \varphi_j(B) \} &= \begin{cases} \varphi_{i+j}(A_i B_z - A_z B_i), & i + j \leq 4, \\ 0, & i + j > 4, \end{cases} & (3.1)
\{ \psi_i, \varphi_j(A) \} &= j \varphi_j(A), & (3.2)
\{ \psi_2, \varphi_1(A) \} &= \varphi_j (2(3 - j) A - 3(tA_t + zA_z)), & (3.3)
\{ \psi_3, \varphi_j(A) \} &= \begin{cases} \varphi_{j+1}(2 - j) A - t A_t - zA_z, & j \leq 2, \\ 0, & j \geq 3, \end{cases} & (3.4)
\{ \psi_1, \psi_2 \} &= 0, \quad \{ \psi_1, \psi_3 \} = \psi_3, \quad \{ \psi_2, \psi_3 \} = -2 \psi_3. & (3.5)
\end{align*}
\]

**Remark 3.1.** We have \( \mathfrak{s} = \mathfrak{s}_w \times \mathfrak{s}_o \), where \( \mathfrak{s}_w \) is generated by \( \varphi_i(A) \) and \( \mathfrak{s}_o \) is generated by \( \psi_j \). The algebra \( \mathfrak{s}_w \) admits the following description. Consider the (commutative associative) algebra of truncated polynomials \( \mathbb{R}_4[s] = \mathbb{R}[s]/(s^4) \) and the Lie algebra \( \mathfrak{h} \) of Hamiltonian vector fields on \( \mathbb{R}^2 \). [15]. Then \( \mathfrak{s}_w \) is isomorphic to the Lie algebra \( \mathbb{R}_4[s] \otimes \mathfrak{h} \) with the bracket \([f \otimes V, g \otimes W] = fg \otimes [V, W]\) for \( f, g \in \mathbb{R}_4[s] \) and \( V, W \in \mathfrak{h} \).

4. Infinite-dimensional covering, shadows, and nonlocal symmetries

Substituting

\[
q = \sum_{k=0}^{\infty} \lambda^k q_k.
\]

in the system (1.2) yields new (infinite-dimensional) covering

\[
\begin{align*}
q_{0,t} &= u_{xy} q_{0,x} - u_{xx} q_{0,y}, \\
q_{0,z} &= u_{yy} q_{0,x} - u_{xx} q_{0,y}, \\
q_{m,t} &= u_{xy} q_{m,x} - u_{xx} q_{m,y} + q_{m-1,x}, \\
q_{m,z} &= u_{yy} q_{m,x} - u_{xx} q_{m,y} + q_{m-1,y}, \quad m \geq 1.
\end{align*}
\]

\(^4\)We carried out all computations in the Jets software [4].
Theorem 4.2. The shadows $\Upsilon$ covering (4.1) defined as

$\langle \cdot, \Upsilon \cdot \rangle$

Remark 4.2. The bracket $R$ Hamiltonian vector fields on

The local symmetries

Theorem 4.1. Remark 4.1. To simplify notation, here and below we use $\phi$ for $\tilde{D}_xD_z(\phi)$, etc.

The nonlocal symmetries of Eq. (1.1) are described by the following theorems:

Theorem 4.1. The local symmetries $\Phi_0(A), \ldots, \Phi_3(A)$, $\Psi_1, \Psi_2, \Psi_3$ have the lifts $\Phi_0(A), \ldots, \Phi_3(A)$, $\Psi_1, \Psi_2, \Psi_3$ to the nonlocal symmetries in the covering (4.1) defined as $\Phi_i(A) = (\phi_i(A), \phi_i(A), \phi_i(A), \ldots, \phi_i(A), \ldots), \Psi_j(A) = (\psi_j(A), \psi_j(A), \psi_j(A), \ldots, \psi_j(A), \ldots)$, with

$\Phi_0(A) = -A_z q_k - (x A_{tz} + y A_{tz}) q_k + (x A_{tt} + y A_{tz}) q_k + A_t q_k,$

$\Phi_1(A) = -A_z q_k + A_t q_k,$

$\Phi_2(A) = \Phi_3(A) = 0,$

$\Psi_1(A) = -x q_k - y q_k,$

$\Psi_2(A) = -y q_k + y q_k,$

$\Psi_3(A) = -q_k + z q_k + (k + 1) q_k.$

The proof is similar to the proof of theorem 4.2 below and is therefore omitted.

Theorem 4.2. The shadows $\Upsilon_k$ $\hat{\Upsilon}_k$ to the nonlocal symmetries in the covering (4.1) defined as $\Upsilon_k = (q_k, \hat{\Upsilon}_k, \ldots, \hat{\Upsilon}_k, \ldots)$ with

$\Upsilon_k, m = \sum_{s=0}^m \langle s, q_k + m + 1 - x \rangle,$

where $\langle a, b \rangle = a_1 b_y - a_y b_x.$

Remark 4.2. The bracket $\langle \cdot, \cdot \rangle$ is a Lie bracket. It endows the space $C^\infty(\mathbb{R}^2)$ of smooth functions on $\mathbb{R}^2$ with the structure of a Lie algebra with the noncentral part isomorphic to the algebra of Hamiltonian vector fields on $\mathbb{R}^2$ and the center generated by the constant functions.
The structure of the algebra of nonlocal symmetries

Proof. We claim that

$$\mathcal{Y}_{k,m,T} = (\mathcal{Y}_{k,m}, u_s) + (q_m, q_{k,s}) + \mathcal{Y}_{k,m-1,s}. $$

Indeed,

$$\mathcal{Y}_{k,m,T} = \sum_{s=0}^{m} (q_s, q_{k,m+1-s}) = \sum_{s=0}^{m} (q_s, q_{k,m+1-s}) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s,T}) =$$

$$= (q_0, q_{k,m+1}) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}) + q_{k,m+1-s} + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}, u_s) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}, u_s) =$$

$$= (q_0, q_{k,m+1}) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}, u_s) =$$

$$+ \sum_{s=1}^{m} (q_s, q_{k,m+1-s}, u_s) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}, u_s) + \sum_{s=1}^{m} (q_s, q_{k,m+1-s}, u_s) =$$

$$= (q_m, q_{k,s}) + \mathcal{Y}_{k,m-1,s} + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}, u_s) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}, u_s) =$$

$$= (q_m, q_{k,s}) + \mathcal{Y}_{k,m-1,s} + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}, u_s) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}, u_s) =$$

$$= (q_m, q_{k,s}) + \mathcal{Y}_{k,m-1,s} + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}, u_s) + \sum_{s=0}^{m} (q_s, q_{k,m+1-s}, u_s) =$$

Proof of the equality $\mathcal{Y}_{k,m,z} = (\mathcal{Y}_{k,m}, u_s) + (q_m, q_{k,s}) + \mathcal{Y}_{k,m-1,y}$ is analogous. $\square$

Remark 4.3. We note that the nonlocal symmetries $\mathcal{Y}_k$ are similar to the nonlocal symmetries for the four-dimensional Martínez Alonso–Shabat equation found in [29], but the Lie brackets on $C^\infty(\mathbb{R}^2)$ here and in the constructions of [29] are different.

Theorem 4.3. Eq. (1.1) has ‘invisible’ symmetries (symmetries with the zero shadow) in the covering (4.1) defined as

$$\Gamma_k = (0, \ldots, 0, q_0, q_1, q_2, \ldots, q_m, \ldots), \quad k \geq 1.$$

The proof is similar to the proof of theorem 4.2 and is therefore omitted.

5. The structure of the algebra of nonlocal symmetries

The structure of the algebra of nonlocal symmetries $\Phi_m(A), \Psi_k, \zeta_j, \Gamma_j$ of Eq. (1.1) in the covering (4.1) is described by the following theorem.

--

\[\text{see remark 4.1.}\]
Theorem 5.1. The Jacobi brackets of lifts of the local symmetries are lifts of the Jacobi brackets of the corresponding local symmetries, that is, the commutators for $\Phi_m(A)$, $\Psi_k$ satisfy equations (3.1)—(3.5) with $\varphi_m(A)$, $\psi_k$ replaced by $\Phi_m(A)$, $\Psi_k$, respectively. The other Jacobi brackets are

$$\{\Phi_m(A), Y_k\} = \{\Phi_m(A), \Gamma_i\} = \{Y_k, \Gamma_i\} = \{\Gamma_i, \Gamma_j\} = 0, \quad 0 \leq m \leq 3, \quad k, l \geq 0, \quad i, j \geq 1,$$

$$\{\Psi_1, Y_k\} = (k + 3) Y_k, \quad \{\Psi_2, Y_k\} = -2k Y_k, \quad \{\Psi_j, Y_k\} = -(k + 1) Y_{k+1},$$

$$\{\Psi_1, \Gamma_i\} = -(i - 1) \Gamma_i, \quad \{\Psi_2, \Gamma_i\} = 2(i - 1) \Gamma_k, \quad \{\Psi_3, \Gamma_i\} = (i - 1) \Gamma_{i-1},$$

$$\{\Gamma_i, Y_k\} = \begin{cases} -Y_{k-i+1}, & k \geq i - 1, \\ 0, & k < i - 1. \end{cases}$$

Proof. We will prove that $\{\Phi_0(A), Y_k\} = 0$. The proof for the other Jacobi brackets is similar and therefore is omitted.

We start with writing down explicitly the formula for computing the Jacobi bracket in case of some infinite-dimensional vectors $\Theta$ and $\Omega$. In order to facilitate applying this formula to vectors $\Phi_0(A)$ and $Y_k$ we enumerate coordinates of the vectors $\Theta$ and $\Omega$ in the following way: $\Theta = (\Theta_{-1}, \Theta_0, \ldots, \Theta_m, \ldots), \Omega = (\Omega_{-1}, \Omega_0, \ldots, \Omega_m, \ldots)$. Then the Jacobi bracket is of the form $\{\Theta, \Omega\} = \langle \Theta_{-1}, \{\Theta, \Omega\}_0, \{\Theta, \Omega\}_1, \ldots \rangle$, where

$$\{\Theta, \Omega\}_j = \sum_I \left( \tilde{D}_I(\Theta_{-1}) \frac{\partial}{\partial u_I} (\Omega_j) - \tilde{D}_I(\Omega_{-1}) \frac{\partial}{\partial u_I} (\Theta_j) + \sum_{m=0}^{\infty} \left( \tilde{D}_I(\Theta_m) \frac{\partial}{\partial q_{m,I}} (\Omega_j) - \tilde{D}_I(\Omega_m) \frac{\partial}{\partial q_{m,I}} (\Theta_j) \right) \right), \quad j \geq -1.$$

As for coordinates $q_{m,I}, \Phi_{0,j}(A)$ depends at most on $q_{j,i}, q_{j,x}, q_{j,y}, q_{j,z}, q_j$ or $q_{j+1}$ and $Y_{k,j}$ depends at most on $q_{m,x}, q_{k+j+1-m,x}, q_{m,y}, q_{k+j+1-m,y}$ for $m = 0, \ldots, j$. The following observations will be used frequently:

$$\sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m) \frac{\partial}{\partial q_{m,I}} (\Phi_{0,j}) = \sum_{I=x,y,z} \tilde{D}_I(a_j) \frac{\partial}{\partial q_{j,I}} (\Phi_{0,j}), \quad j \geq -1, \quad i = 0, 1, 2, 3,$$

$$\sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m) \frac{\partial}{\partial q_{m,I}} (Y_{k,j}) = \sum_{m=0}^{j} (\tilde{D}_x(a_m) q_{k+j+1-m,y} - \tilde{D}_z(a_m) q_{k+j+1-m,y}) q_{m,y} - \tilde{D}_y(a_m) q_{k+j+1-m,x} \tilde{D}_y(a_m) q_{k+j+1-m,x} q_{m,x} \sum_{m=0}^{j} (a_m, q_{k+j+1-m}) - \langle a_{k+j+1-m}, q_m \rangle.$$

We have

$$\{\Phi_i(A), Y_k\}_1 = \sum_I \tilde{D}_I(\phi_i(A)) \frac{\partial}{\partial u_I} (q_k) - \sum_I \tilde{D}_I(q_k) \frac{\partial}{\partial u_I} (\phi_i(A)) +$$

$$+ \sum_I \sum_{m=0}^{\infty} \tilde{D}_I(\Phi_i(m,A)) \frac{\partial}{\partial q_{m,I}} (q_k) - \sum_I \sum_{m=0}^{\infty} \tilde{D}_I(Y_{k,m}) \frac{\partial}{\partial q_{m,I}} (\phi_i(A)), \quad 0 \leq i \leq 3.$$
It is easy to see, that \( \sum_l D_l (q_k) \frac{\partial}{\partial u_l} (\Phi_i(A)) = \Phi_i^k(A) \) for \( i = 0, 1, 2, 3 \), hence \( \{ \Phi_i(A), \Upsilon_k \} - 1 = 0 \). Then for \( j \geq 0 \) we get

\[
\{ \Phi_j(A), \Upsilon_k \}_j = \sum_l \hat{D}_l (\Phi_j(A)) \frac{\partial}{\partial u_l} (\Upsilon_k, \{ ) - \sum_l \hat{D}_l (q_k) \frac{\partial}{\partial u_l} (\Phi_j(A)) + \\
+ \sum_l \sum_{m=0}^{\infty} \left( D_l (\Phi_{j,m}(A)) \frac{\partial}{\partial q_{m,l}} (\Upsilon_k, \{ ) - \sum_l \sum_{m=0}^{\infty} \left( D_l (\Upsilon_{k,m}) \frac{\partial}{\partial q_{m,l}} (\Phi_j(A)) \right) .
\]

Furthermore, we have

\[
\{ \Phi_0(A), \Upsilon_k \}_j = \\
= \sum_{l=x,y} ( D_l (\Phi_{0,m}(A)) \frac{\partial}{\partial q_{m,l}} (\Upsilon_k, \{ ) - \sum_l \sum_{m=0}^{\infty} ( D_l (\Upsilon_{k,m}) \frac{\partial}{\partial q_{m,l}} (\Phi_0(A)) ) = \\
= \sum_{m=0}^{\infty} ( \hat{D}_x (\Phi_{0,m}(A)) - \hat{D}_x (\Phi_{0,k+1-m}(A)) ) q_{k+1-m,y} - \hat{D}_x (\Phi_{0,k+1-m}(A)) q_{m,y} - \hat{D}_y (\Phi_{0,m}(A)) - \hat{D}_y (\Phi_{0,k+1-m}(A)) q_{k+1-m,x} + \\
+ \hat{D}_y (\Phi_{0,k+1-m}(A)) q_{m,x} - \sum_{l=x,y,z} \hat{D}_l (\Upsilon_{k,l}) \frac{\partial}{\partial q_{m,l}} (\Phi_0(A)) = \\
= \sum_{m=0}^{\infty} ( \langle \Phi_{0,m}(A), q_{k+1-m} \rangle - \langle \Phi_{0,k+1-m}(A), q_m \rangle - \sum_{l=x,y,z} \langle \Phi_{0,m}(A), q_{k+1-m} \rangle - \langle \Phi_{0,k+1-m}(A), q_m \rangle \\
+ \sum_{l=x,y,z} ( \langle q_{m,l}, q_{k+1-m} \rangle + \langle q_{m,l}, q_{k+1-m} \rangle + \langle q_{m,l}, q_{k+1-m} \rangle ) \frac{\partial}{\partial q_{m,l}} (\Phi_0(A)) ) = \\
= \sum_{m=0}^{\infty} ( \langle \Phi_{0,m}(A), q_{k+1-m} \rangle - \langle \Phi_{0,k+1-m}(A), q_m \rangle + \\
+ \langle \Phi_{0,k+1-m}(A), q_m \rangle - \langle \Phi_{0,k+1-m}(A), q_m \rangle ) = 0.
\]

\[\square\]

**Remark 5.1.** Denote by \( a \) the ideal of the Lie algebra \( \mathfrak{s} \) generated by \( \Upsilon_j \) and by \( b \) the subalgebra generated by \( \Gamma_j \). Identify \( s_{\infty} \) and \( s_0 \) with the ideal generated by \( \Phi_m(A) \) and the subalgebra generated by \( \Psi_k \), respectively. Then we obtain

\[
\mathfrak{s} = (s_{\infty} \oplus (a \times b)) \ltimes s_0
\]

(here \( \oplus \) denotes a direct sum of commuting Lie algebras), in particular we have equations (1.3) and (1.4) together with \( [s_{\infty}, s_{\infty}] = s_m, [s_0, s_{\infty}] = s_{\infty}, \) and \( [s_0, s_0] = s_0. \)
6. Conclusion

In this paper we have found two infinite hierarchies of commuting nonlocal symmetries for Plebański's second heavenly equation. One of these hierarchies can be employed to construct new solutions using the techniques presented e.g. in [10, 19, 31], see also [5, 16, 17, 34]. Furthermore, we find the lifts of local symmetries and the structure of the Lie algebra of the nonlocal symmetries. We emphasize that finding an explicit form of nonlocal symmetries and the commutator relations for an infinite-dimensional symmetry algebra of nonlocal symmetries (rather than just shadows) is quite rare. We know just a number of similar results in the literature, [1–3, 5, 13, 16–18, 29]. We hope that it would be very interesting to study how the infinite-dimensional Lie algebra of nonlocal symmetries reflects the algebraic structure behind the integrability properties of the considered equation, cf. [27, 28].

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References

[1] M. Ablowitz, S. Chakravarty, and L. Takhtajan, A self-dual Yang–Mills hierarchy and its reductions to integrable systems in 1 + 1 and 2 + 1 dimensions, Comm. Math. Phys. 158 (1993) 289–314.
[2] H. Baran, I.S. Krasil’shchik, O.I. Morozov, and P. Vojčák, Coverings over Lax integrable equations and their nonlocal symmetries, Theor. Math. Phys. 188 (2016) no. 3, 1273–1295.
[3] H. Baran, I.S. Krasil’shchik, O.I. Morozov, and P. Vojčák, Nonlocal symmetries of Lax integrable equations: a comparative study, arXiv:1611.04938
[4] H. Baran and M. Marvan, Jets: A software for differential calculus on jet spaces and diffieties, http://jets.math.slu.cz (2016).
[5] P.M. Bies, P. Görka, E.G. Reyes, The dual modified Korteweg–de Vries–Fokas–Qiao equation: Geometry and local analysis, J. Math. Phys. 53 No 7 (2012) 073710 (19 pages)
[6] M. Blaszak, Multi-Hamiltonian Theory of Dynamical Systems (Springer, Berlin, 1998).
[7] M. Blaszak, Classical R-matrices on Poisson algebras and related dispersionless systems, Phys. Lett. A 297 (2002), 191–195.
[8] G.W. Bluman, S. Kumei, Symmetries and Differential Equations. Applied Mathematical Sciences 81 (Springer, Berlin, 1989)
[9] G.W. Bluman, A.F. Cheviakov, S.C. Anco, Applications of Symmetry Methods to Partial Differential Equations. Applied Mathematical Sciences 168 (Springer, Berlin, 2010)
[10] A.V. Bocharov et al., Symmetries of Differential Equations in Mathematical Physics and Natural Sciences, (edited by A.M. Vinogradov and I.S. Krasil’shchik). (Factorial Publ. House, Moscow, 1997 (in Russian), English translation: Amer. Math. Soc., 1999)
[11] B. Doubrov and E.V. Ferapontov, On the integrability of symplectic Monge–Ampère equations, J. Geom. Phys. 60 (2010) 1604–1616.
[12] M. Dunajski, Solitons, Instantons and Twistors (Oxford University Press, Oxford, 2009)
[13] J.D. Finley III and J.D. McIver, Non-Abelian infinite algebra of generalized symmetries for the SDiff(2) Toda equation, J. Phys. A 37 (2004) 5825–5847.
[14] B. Fuchssteiner and A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D 4 (1981/82) 47–66.
[15] D.B. Fuks, Cohomology of infinite-dimensional Lie algebras, (Consultants Bureau, New York, 1986).
[16] R. Hernández Heredero, E.G. Reyes, Geometric Integrability of the Camassa–Holm Equation. II, *Int. Math. Res. Notices* 2012 No 13 (2012) 3089–3125.

[17] R. Hernández Heredero, E.G. Reyes, Nonlocal symmetries, compacton equations, and integrability, *International Journal of Geometric Methods in Modern Physics* 10 No 9 (2013) 1350046 (24 pages).

[18] S. Igonin and M. Marvan, On construction of symmetries and recursion operators from zero-curvature representations and the Darboux–Egoroff system, *J. Geom. Phys.* 85 (2014) 106–123.

[19] I.S. Krasil’chik, V.V. Lychagin, and A.M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, (Gordon and Breach, NY, 1986).

[20] J. Krasil’schik and A.M. Verbovetsky, Geometry of jet spaces and integrable systems, *J. Geom. Phys.* 61 (2011) 16331674.

[21] I.S. Krasil’schik, A.M. Verbovetsky and R. Vitolo, A unified approach to computation of integrable structures, *Acta Appl. Math.* 120 (2012) 199–218.

[22] I.S. Krasil’schik and A.M. Vinogradov, Nonlocal symmetries and the theory of coverings, *Acta Appl. Math.* 2 (1984) 79–86.

[23] I.S. Krasil’schik and A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations, *Acta Appl. Math.* 15 (1989) 161–209.

[24] B. Kruglikov and O. Morozov, SDiff(2) and uniqueness of the Plebański equation, *JMP* 53 (2012) 083506 (11 p.).

[25] B. Kruglikov and O. Morozov, Integrable dispersionless PDEs in 4D, their symmetry pseudogroups and deformations, *Lett. Math. Phys.* 105 (2015) 1703–1723.

[26] A.A. Malykh, Y. Nutku, and M.B. Sheftel, Partner symmetries and non-invariant solutions of four-dimensional heavenly equations, *J. Phys. A: Math. Gen.* 37 (2004), 7527–7545.

[27] O.I. Morozov, Deformed cohomologies of symmetry pseudo-groups and coverings of differential equations, *J. Geom. Phys.* 113 (2017) 215–225.

[28] O.I. Morozov, Deformations of infinite-dimensional Lie algebras, exotic cohomology and integrable nonlinear partial differential equations, arXiv:1706.01090.

[29] O.I. Morozov and A. Sergyeyev, The four-dimensional Martínez Alonso–Shabat equation: Reductions and nonlocal symmetries, *J. Geom. Phys.* 85 (2014) 40–45.

[30] F. Neyzi, Y. Nutku, and M.B. Sheftel, Multi-Hamiltonian Structure of Plebanski’s Second Heavenly Equation, *J. Phys. A: Math. Gen.* 38 (2005), 8473–8485.

[31] P.J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed. (Springer, NY, 1993).

[32] J.F. Plebański, Some solutions of complex Einstein equations, *J. Math. Phys.* 16 (1975) 2395–2402.

[33] A. Sergyeyev, Infinite hierarchies of nonlocal symmetries of the Chen–Kontsevich–Schwarz type for the oriented associativity equations, *J. Phys. A* 42 (2009) 404017. (p. 15)

[34] E.G. Reyes, Pseudo-potentials, nonlocal symmetries and integrability of some shallow water equations, *Selecta Math. New Ser.* 12 (2006) 241–270.

[35] D. Yazıcı and M.B. Sheftel, Symmetry Reductions of Second Heavenly Equation and (2 + 1)-Dimensional Hamiltonian Integrable Systems. *JNMP* 15 Suppl. 3 (2008), 417–425.