ON SOME NEW APPLICATIONS OF NEWTON-RAPHSON METHOD

Treanungkur Mal
Independent Researcher, India
maltreanungkur@gmail.com

October 11, 2021

Abstract

I am going to provide a new technique of approximating area under the curve, using the Newton-Raphson Method. I am also going to provide a formula that would help us approximate any Definite Integral or help us find the area under the curve, under certain conditions. The relative error of this formula is very small, which makes it even more interesting.

Keywords: Newton-Raphson Method, Applications of Newton-Raphson Method, Approximating Area under the curve.

1 Introduction

In Numerical Analysis, we often use Newton-Raphson Method to approximate roots of a polynomial function because a polynomial with degree ≥ 5 is solvable iff it forms a solvable Galios Group [1]. The main formula of Newton-Raphson Method is :

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Proposition 1. Let us assume, a function \( f(x) \), which is an increasing continuous function on the interval \([α, β]\). Let, a root of \( f(x) \) lies in the interval \([α, β]\), for instance let it be \( a \), then

\[ \int_a^b f(x) \, dx \approx \frac{1}{2} \left\{ \frac{f(x_0)}{f'(x_0)} \left( f(x_0) + f(x_1) \right) + \ldots + \frac{f(x_n)}{f'(x_n)} \left( f(x_n) + f(x_{n+1}) \right) \right\} \]

where,

\[ x_0 = b, \quad x_{n+1} \approx a, \quad \text{and} \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]
Newton-Raphson Method

We have often encountered various polynomials in our life, most commonly we have often seen polynomials of degree 2, and we have also solved it using factorization, using Completing the square method, and using the Quadratic Formula. But it turns out to be for polynomials with degree $\geq 5$, it is not easily solvable using Radicals \[2\]. For instance, this equation:

$$f(x) = 5x^5 + 4x^4 - 3x^3 + 2x^2 + 4x + 1$$

One of the ways to solve higher degree polynomials like this, is by approximation.

The Newton-Raphson Method gives us the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

First, we have to choose a value $x_0$ around which the function $f(x)$ is increasing, then we have to draw a tangent passing through the point $(x_0, f(x_0))$. Then we have to find its x-intercept, and the above-mentioned formula gives the x-intercept of the tangent drawn through the point $(x_n, f(x_n))$. And by repeating the same process a couple of times we can get a value that is approximately equal to the root of $f(x)$.
3 Proof of Newton-Raphson Method

To understand my method of finding area under the curve we need to understand the proof of Newton-Raphson Method.

Let us assume, a function $f(x)$ around an initial value $x_0$ such that the function is increasing and $x_0$ is not a critical value, then it meets the curve of $f(x)$ at point $(x_0, f(x_0))$, now we have to draw a tangent from that specific point.

Now we have,

Slope of the tangent $(m) = f'(x_0)$ and,

$$y - y_{value} = m(x - x_{value})$$

Here, $(x_{value}, y_{value}) \equiv (x_0, f(x_0))$

$$\Rightarrow y - f(x_0) = m(x - x_0)$$

$$\Rightarrow y - f(x_0) = f'(x_0)(x - x_0)$$

This is the equation of the line, but we need to find its x-intercept, so let the coordinate of x-intercept be $(x_1, 0)$.

$$\Rightarrow 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow f(x_0) = f'(x_0)(x_0 - x_1)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

On generalizing this formula for 'n' we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton-Raphson Method is very useful in Numerical Analysis, and it is one of the widely used methods for approximating the roots of a function $[3]$. But it has some limitations like

(i) The initial value $(x_0)$ must not be a critical value.
(ii) It’s convergence is not guaranteed.
(iii) Division by zero problem can occur.
(iv) Inflection point issue might occur
(v) Symbolic derivative is required.


ON SOME NEW APPLICATIONS OF NEWTON-RAPHSON METHOD

4 Extending Newton-Raphson Method to Approximate Area

I have discovered an interesting method to find area under the curve by extending the concept of Newton-Raphson Method.

\[
\int_{a}^{b} f(x) \, dx \approx \frac{1}{2} \left\{ \frac{f(x_0)}{f'(x_0)} (f(x_0) + f(x_1)) + \ldots + \frac{f(x_n)}{f'(x_n)} (f(x_n) + f(x_{n+1})) \right\}
\]

where,

\[ x_0 = b, \quad x_{n+1} \approx a, \quad and \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

So to find area of a function \( f(x) \) we use Integration, or to be precise we use Definite Integration. Already we have many methods of approximating Definite Integrals, like Midpoint Rule, Trapezoidal Rule, Simpson’s Rule, Riemann Sum, etc.[4]

Now we will see how Newton Raphson Method helps us also to find Area under the curve. Now, I will provide a proof of the Rule given above.

Figure 2: This is the graphical representation of the length of the sides of trapezium at \( x_0 \) and \( x_1 \)
5 Proof of Approximating Area using Newton Raphson’s Method

The idea here is to form many trapeziums and then find their area, in the above picture, the black stippled lines are the trapeziums formed by using the Newton-Raphson Method \[5\]. Eventually, as we progress the last figure formed will be a triangle, i.e. when \(f(x_{n+1}) \approx f(a)\). Let the area of the first trapezium be \(A_1\) then,

\[
\text{area}(A_1) = \frac{1}{2} (|x_0 - x_1|)(f(x_0) + f(x_1))
\]

Note,

\[
(|x_0 - x_1|) = \left( |x_0 - x_0 + \frac{f(x_0)}{f'(x_0)}| \right) = \left( \frac{f(x_0)}{f'(x_0)} \right)
\]

\[
\Rightarrow \text{area}(A_1) = \frac{1}{2} \left( \frac{f(x_0)}{f'(x_0)} \right) (f(x_0) + f(x_1))
\]

where, \(x_0 = b\), upper limit of the integral. Similarly,

\[
\text{area}(A_2) = \frac{1}{2} \left( \frac{f(x_1)}{f'(x_1)} \right) (f(x_1) + f(x_2))
\]

\[
\text{area}(A_3) = \frac{1}{2} \left( \frac{f(x_2)}{f'(x_2)} \right) (f(x_2) + f(x_3))
\]

\[
\ldots
\]

\[
\text{area}(A_{n+1}) = \frac{1}{2} \left( \frac{f(x_n)}{f'(x_n)} \right) (f(x_n) + f(x_{n+1}))
\]

where,

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}
\]

Since,

\[
f(x_{n+1}) \approx f(a)
\]

So,

\[
f(x_{n+1}) \approx 0 \text{ i.e. } \text{area}(A_{n+1}) = \text{area (One and only triangle)}.
\]

Therefore total Area :

\[
(area(A_1) + area(A_2) + \ldots + area(A_{n+1})) \approx \int_a^b f(x) \, dx
\]
\[
\Rightarrow \frac{1}{2} \left\{ \frac{f(x_0)}{f'(x_0)} (f(x_0) + f(x_1)) + \ldots + \frac{f(x_n)}{f'(x_n)} (f(x_n) + f(x_{n+1})) \right\} \approx \int_{a}^{b} f(x) \, dx
\]

Therefore our final answer,
\[
\int_{a}^{b} f(x) \, dx \approx \frac{1}{2} \left\{ \frac{f(x_0)}{f'(x_0)} (f(x_0) + f(x_1)) + \ldots + \frac{f(x_n)}{f'(x_n)} (f(x_n) + f(x_{n+1})) \right\}
\]
where,
\[x_0 = b, \, x_{n+1} \approx a, \text{ and } x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\]

**Example 5.1** Evaluate the following Integral using Newton Raphson’s Rule :
\[
\int_{-0.5}^{1} 2x^2 + 3x + 1 \, dx
\]

**Solution.** For solving the given Definite Integral we need to find some values for 'x' :
\[x_0 = 1, \, f(x_0) = 6, \, f'(x_0) = 7\]
\[x_1 = 0.142, \, f(x_1) = 1.466, \, f'(x_1) = 3.568\]
\[x_2 = -0.26, \, f(x_2) = 0.3552, \, f'(x_2) = 1.96\]
\[x_3 = -0.44, \, f(x_3) = 0.0672, \, f'(x_3) = 1.24\]
\[x_4 = -0.49 \approx -0.5, \]

So, we can stop here as \(x_4 \approx -0.5\)

Let,
\[
I_{\text{approx}} = \int_{-0.5}^{1} 2x^2 + 3x + 1 \, dx \approx \frac{1}{2} \left\{ \frac{f(x_0)}{f'(x_0)} (f(x_0) + f(x_1)) + \ldots + \frac{f(x_n)}{f'(x_n)} (f(x_n) + f(x_{n+1})) \right\}
\]
\[
\Rightarrow I_{\text{approx}} \approx \frac{1}{2} \left\{ \frac{6}{7} (7.466) + \frac{1.466}{3.568} (1.8212) + \frac{0.3552}{1.96} (0.4224) + \frac{0.0672}{1.24} (0.0774) \right\}
\]
\[
\Rightarrow I_{\text{approx}} \approx 4.0609
\]

By solving it using Normal Integration we get,
\[
I_{\text{original}} = 3.375
\]
Therefore, Relative Error = \(|I_{original} - I_{approx}| = 0.6859\)

and Relative Error in % = \(\frac{0.6859}{3.376} \cdot 100\% = 0.2032 \cdot 100\% = 2.302\%\)

Finally, On comparing with all the widely used methods of approximating Definite Integrals, we get the Relative Errors for the given sum in % as follows:

Relative Error for the given sum using Midpoint Rule = 1.8518 %
Relative Error for the given sum using Trapezoidal Rule = 3.7037 %
Relative Error for the given sum using Left Riemann Sum = 40.7407 %
Relative Error for the given sum using Right Riemann Sum = 48.1481 %
Relative Error for the given sum using Newton Raphson’s Rule = 2.302%

By the above examples, I have tried to show that Newton Raphson Rule for solving Indefinite Integrals is very efficient as well as it is very correct, compared to other methods or rules which are widely used today [6].

6 Acknowledgements

I would like to thank My Mother Shukla Mal, who always motivated me in life.

I would also like to thank My Father Ashis Kumar Mal, who always encouraged me to know more about Mathematics.

Finally, I thank My Dear Brother Subhash Baur, who is the prime reason for me, being in love with Mathematics, along with that I also thank My Dear Sisters Sarmistha Mal and Susmita Mal.

References

[1] M. Crisfield, “Accelerating and damping the modified newton-raphson method,” Computers & structures, vol. 18, no. 3, pp. 395–407, 1984.

[2] S. Akram and Q. U. Ann, “Newton raphson method,” International Journal of Scientific & Engineering Research, vol. 6, no. 7, pp. 1748–1752, 2015.

[3] L. Wedepohl, H. Nguyen, and G. Irwin, “Frequency-dependent transformation matrices for untransposed transmission lines using newton-raphson method,” IEEE Transactions on Power Systems, vol. 11, no. 3, pp. 1538–1546, 1996.
[4] T. J. Ypma, “Historical development of the newton–raphson method,” SIAM review, vol. 37, no. 4, pp. 531–551, 1995.

[5] S. Abbasbandy, “Improving newton–raphson method for nonlinear equations by modified adomian decomposition method,” Applied mathematics and computation, vol. 145, no. 2-3, pp. 887–893, 2003.

[6] A. Ben-Israel, “A newton-raphson method for the solution of systems of equations,” Journal of Mathematical analysis and applications, vol. 15, no. 2, pp. 243–252, 1966.