A primal representation of the Monge-Kantorovich norm

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Abstract

In this note, following [Chitescu et al., 2014], we show that the Monge-Kantorovich norm on the vector space of countably additive measures on a compact metric space has a primal representation analogous to the Hanin norm, meaning that similarly to the Hanin norm, the Monge-Kantorovich norm can be seen as an extension of the Kantorovich-Rubinstein norm from the vector subspace of zero-charge measures, implying a number of novel results, such as the equivalence of the Monge-Kantorovich and Hanin norms.

As presented in [Chitescu et al., 2014] and summarized in [Cobzaş et al., 2019], the vector space $cabv(X)$ of countably additive measures with bounded variation on a compact metric space $(X, d)$ can be normed in at least two distinct ways to induce the topology of weak convergence of measures (also called weak-$\ast$ convergence of measures) on subsets bounded with respect to the total variation norm. One $(cabv(X), \|\cdot\|_{MK})$ is by the Monge-Kantorovich norm (also called the Kantorovich-Rubinstein norm)

$$\|\mu\|_{MK} = \sup_{f \in Lip(X), \|f\|_{sum} \leq 1} \left\{ \int f \, d\mu \right\},$$

with $(Lip(X), \|\cdot\|_{sum})$ being the vector space of functions $f : X \to \mathbb{R}$ for which the Lipschitz seminorm

$$\|f\|_L = \sup_{x, y \in X, x \neq y} \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \right\}$$

is finite, normed by the sum norm, which is defined as

$$\|f\|_{sum} = \|f\|_L + \|f\|_\infty.$$
The other \((cabv(X), \| \cdot \|_H)\) is by the Hanin norm
\[
\| \mu \|_H = \inf_{\nu \in cabv(X, 0)} \left\{ \| \nu \|_{KR} + \| \mu - \nu \|_{TV} \right\},
\]
where \(\| \cdot \|_{TV}\) is the total variation norm, and \((cabv(X, 0), \| \cdot \|_{KR})\) is the vector subspace \(cabv(X, 0) = \{ \mu \in cabv(X) : \mu(X) = 0 \}\) of zero-charge measures normed by the Kantorovich-Rubinstein norm (also called the modified Kantorovich-Rubinstein norm)
\[
\| \mu \|_{KR} = \sup_{f \in Lip(X, x_0), \| f \|_{L} \leq 1} \left\{ \int f d\mu \right\},
\]
with \((Lip(X, x_0), \| \cdot \|_{L})\) being the vector subspace \(Lip(X, x_0) = \{ f \in Lip(X) : f(x_0) = 0 \}\) of Lipschitz continuous functions vanishing at an arbitrary fixed point \(x_0 \in X\), normed by the Lipschitz seminorm.

The topological dual of \((cabv(X, 0), \| \cdot \|_{KR})\) and \((Lip(X, x_0), \| \cdot \|_{L})\) are isometrically isomorphic. The topological dual of \((cabv(X), \| \cdot \|_H)\) and \((Lip(X), \| \cdot \|_{max})\) are isometrically isomorphic as well, with the max norm defined as
\[
\| f \|_{max} = \max \{ f, f, \| f \|_{\infty} \},
\]
leading to the dual representation
\[
\| \mu \|_H = \sup_{f \in Lip(X): \| f \|_{max} \leq 1} \left\{ \int f d\mu \right\}.
\]
Inspired by the similarity of the dual representation of the Hanin norm and the definition of the Monge-Kantorovich norm, we prove the following theorem, showing that analogously to the Hanin norm, the Monge-Kantorovich norm can be seen as an extension of the Kantorovich-Rubinstein norm from the subspace \(cabv(X, 0)\) to the whole space \(cabv(X)\), leading to a number of consequences.

**Theorem.** The Monge-Kantorovich norm has the primal representation
\[
\| \mu \|_{MK} = \inf_{\nu \in cabv(X, 0)} \left\{ \max \{ \| \nu \|_{KR}, \| \mu - \nu \|_{TV} \} \right\},
\]
the supremum in the dual representation is achieved as \(\exists f \in Lip(X) : \| f \|_{sum} = 1 \wedge \| \mu \|_{MK} = \int f d\mu\), the subset of measures in \(cabv(X)\) with finite support is dense in \((cabv(X), \| \cdot \|_{MK})\), the topological dual of \((cabv(X), \| \cdot \|_{MK})\) and \((Lip(X), \| \cdot \|_{sum})\) are isometrically isomorphic, and the norms \(\| \cdot \|_{MK}\) and \(\| \cdot \|_{H}\) are equivalent as \(\| \mu \|_{MK} \leq \| \mu \|_H \leq 2\| \mu \|_{MK}\) for all \(\forall \mu \in cabv(X)\).

To obtain the proof, we will apply techniques of convex analysis cited from [Zalinescu, 2002]. We start with a number of propositions.

**Proposition 1.** Given \(\lambda \in [0, 1]\), the mapping \(F_{\lambda} : (cabv(X, 0), \| \cdot \|_{KR}) \to \mathbb{R}\) defined as
\[
F_{\lambda}(\nu) = \lambda \| \nu \|_{KR}
\]
is proper, convex and continuous, and its convex conjugate \(F_{\lambda}^* : (Lip(X, x_0), \| \cdot \|_{L}) \to \mathbb{R}\) is the indicator
\[
F_{\lambda}^*(f) = 1_{\{ f \in Lip(X, x_0) : \| f \|_{L} \leq \lambda \}}(f).
\]
Proof. By [Zalinescu, 2002, Corollary 2.4.16],

\[ (\nu \rightarrow \|\nu\|_{KR})^* = (f \rightarrow i_{\{f \in Lip(X,x_0):\|f\|_{L} \leq 1\}}(f)). \]

By [Zalinescu, 2002, Theorem 2.3.1(v)],

\[ (\nu \rightarrow \lambda \|\nu\|_{KR})^* = (f \rightarrow \lambda i_{\{f \in Lip(X,x_0):\|f\|_{L} \leq 1\}}(\lambda^{-1}f)), \]

which is equivalent to the proposed conjugate relation. The mapping is clearly proper, convex and continuous by being the constant multiple of a norm with a positive multiplier.

Proposition 2. Given \( \lambda \in [0,1] \) and \( \mu \in cabv(X) \), the mapping \( G_{\lambda,\mu} : (cabv(X), \|\cdot\|_H) \rightarrow \mathbb{R} \) defined as

\[ G_{\lambda,\mu}(\nu) = (1-\lambda)\|\mu - \nu\|_{TV} \]

is proper, convex and lower semicontinuous, and its convex conjugate \( G_{\lambda,\mu}^* : (Lip(X), \|\cdot\|_{max}) \rightarrow \mathbb{R} \) is

\[ G_{\lambda,\mu}^*(f) = i_{\{f \in Lip(X):\|f\|_{\infty} \leq 1-\lambda\}}(f) - \int f d\mu. \]

Proof. By [Cobzaş et al., 2019, Theorem 8.4.10], the level sets of the mapping \( (\nu \rightarrow \|\nu\|_{TV}) \) are compact with respect to topology of the weak convergence of measures, hence compact with respect to the topology induced by \( \|\cdot\|_H \) as well by [Cobzaş et al., 2019, Theorem 8.5.7]. This implies that the level sets are closed in \( (cabv(X), \|\cdot\|_H) \), hence the mapping is lower semicontinuous, and clearly proper and convex as well. It is also sublinear, so that by [Zalinescu, 2002, Theorem 2.4.14(i)] one has the conjugate relation

\[ (\nu \rightarrow \|\nu\|_{TV})^* = i_{\partial(\nu \rightarrow \|\nu\|_{TV})(0)}, \]

where by definition the subdifferential at 0 is

\[ \partial(\nu \rightarrow \|\nu\|_{TV})(0) = \{ f \in Lip(X) \mid \forall \mu \in cabv(X) : \int f d\mu \leq \|\mu\|_{TV} \}. \]

Since \( X \) is compact, any \( f \in Lip(X) \) achieves its minimum and maximum, hence

\[ \exists x_0 \in X : |f(x_0)| = \|f\|_{\infty}, \]

implying that

\[ \max \left\{ \int f d\mu : \mu \in cabv(X), \|\mu\|_{TV} = \xi \right\} = \int f d(\pm \xi \delta_{x_0}) = \xi \|f\|_{\infty} \]

with \( \delta_{x_0} \) being the Dirac measure at \( x_0 \), and the sign of \( \xi \) is opposite to that of \( f(x_0) \). It follows that

\[ \exists \mu \in cabv(X), \|\mu\|_{TV} = \xi : \int f d\mu > \|\mu\|_{TV} \iff \|f\|_{\infty} > 1, \]

\[ \square \]
which is true for any $\xi \geq 0$, implying that
\[
\partial(\nu \to \|\nu\|_{TV})(0) = \{f \in Lip(X) : \|f\|_\infty \leq 1\},
\]
leading to the conjugate relation
\[
(\nu \to \|\nu\|_{TV})^* = i_{\{f \in Lip(X) : \|f\|_\infty \leq 1\}}.
\]
By [Zalinescu, 2002] Theorem 2.3.1(v),
\[
(\nu \to \| - \nu\|_{TV})^* = (f \to i_{\{f \in Lip(X) : \|f\|_\infty \leq 1\}}(-f)),
\]
where $(-f)$ can clearly be replaced by $(f)$. By [Zalinescu, 2002] Theorem 2.3.1(vi),
\[
(\nu \to \|\mu - \nu\|_{TV})^* = \left( f \to i_{\{f \in Lip(X) : \|f\|_\infty \leq 1\}}((1 - \lambda)^{-1}f) \right.
\]
\[ - (1 - \lambda)^{-1} \int (1 - \lambda)^{-1}df\mu \right).
\]
By [Zalinescu, 2002] Theorem 2.3.1(v),
\[
(\nu \to (1 - \lambda)\|\mu - \nu\|_{TV})^* = \left( f \to (1 - \lambda)i_{\{f \in Lip(X) : \|f\|_\infty \leq 1\}}((1 - \lambda)^{-1}f) \right.
\]\[ - (1 - \lambda)^{-1} \int (1 - \lambda)^{-1}df\mu \right),
\]
which is clearly equivalent to the proposition. 

\textbf{Proposition 3.} Given $\mu \in cabv(X)$, the mapping $H_\mu : (cabv(X, 0), \|\cdot\|_{KR}) \times (cabv(X), \|\cdot\|_H) \to \mathbb{R}$ defined as
\[
H_\mu(\nu_1, \nu_2) = \max\{\|\nu_1\|_{KR}, \|\mu - \nu_2\|_{TV}\}
\]
is proper, convex and lower semicontinuous, and its convex conjugate $H_\mu^* : (Lip(X, x_0), \|\cdot\|_L) \times (Lip(X), \|\cdot\|_{max}) \to \mathbb{R}$ is
\[
H_\mu^*(f_1, f_2) = \min_{\lambda \in [0, 1]} \left\{ i_{\{f \in Lip(X, x_0) : \|f\|_L \leq \lambda\}}(f_1) + i_{\{f \in Lip(X) : \|f\|_\infty \leq 1 - \lambda\}}(f_2) - \int f_2 d\mu \right\}.
\]
Proof. By [Zalinescu, 2002] Corollary 2.8.12, the conjugate relation
\[
((\nu_1, \nu_2) \to \max\{\lambda^{-1}F_\mu(\nu_1), (1 - \lambda)^{-1}G_{\lambda, \mu}(\nu_2)\})^* = ((f_1, f_2) \to \min_{\lambda \in [0, 1]} \{F_\mu^*(f_1) + G_{\lambda, \mu}^*(f_2)\})
\]
holds, which together with the previous propositions gives the claimed conjugate relation. By [Zalinescu, 2002] Theorem 2.1.3(vii), $H_\mu$ is proper and convex. Clearly the mappings $((\nu_1, \nu_2) \to \lambda^{-1}F_\mu(\nu_1))$ and $(((\nu_1, \nu_2) \to (1 - \lambda)^{-1}G_{\lambda, \mu}(\nu_2))$ are lower semicontinuous, hence $H_\mu$ as well by being their pointwise maximum. 

\[4\]
Proposition 4. The mapping \((\nu \to \nu) : (cabv(X, 0), \|\cdot\|_{KR} \to (cabv(X), \|\cdot\|_H))\) is linear and continuous, and its adjoint \((\nu \to \nu)^*\) is \((f \to f - f(x_0)) : (Lip(X), \|\cdot\|_{\text{max}}) \to (Lip(X, x_0), \|\cdot\|_L)).\)

Proof. It is clear that \(\|\nu\|_H \leq \|\nu\|_{KR}\), so the linear operator \((\nu \to \nu)\) is bounded, hence continuous. For any \(\nu \in cabv(X, 0), f \in Lip(X)\) it holds that \(\int (f - f(x_0))d\nu = \int fd\nu - f(x_0)\nu(X) = \int fd\nu\), proving the adjoint relation. \(\square\)

We are now ready to prove the theorem.

Proof of Theorem. First we show that the condition \([Zalinescu, 2002, Theorem 2.8.1(iii)]\) holds for the mapping \(H_\mu\). Since \(H_\mu\) is finite everywhere, one has \(\text{dom} H_\mu = cabv(X, 0) \times cabv(X)\). For any \(\nu_1 \in cabv(X, 0)\), the restriction of the mapping \(H_\mu(\nu_1, \cdot)\) to any closed subset is lower semicontinuous. For any \(r > 0\), the closed and nonempty subset \(\{\nu \in cabv(X) : \|\nu\|_{TV} \leq r\}\) is complete with respect to any metric metrizing the topology of the weak convergence of measures by \([Cobza¸s et al., 2019, Theorem 8.4.10]\), such as the metric induced by \(\|\cdot\|_H\) by \([Cobza¸s et al., 2019, Theorem 8.5.7]\). Hence the restriction of the mapping \(H_\mu(\nu_1, \cdot)\) to this subset is lower semicontinuous, and therefore has points of continuity by \([Si and Zhang, 2020, Theorem 1.1]\), so that \(\exists \nu_2 \in cabv(X)\) such that \(H_\mu(\nu_1, \cdot)\) is continuous at \(\nu_2\).

Since the condition \([Zalinescu, 2002, Theorem 2.8.1(iii)]\) is satisfied, by \([Zalinescu, 2002, Corollary 2.8.2]\), one has

\[
\inf_{\nu \in \text{cab}(X, 0)} \{H_\mu(\nu, \nu)\} = \max_{f \in \text{Lip}(X)} \{-H_\mu^*((-f - f(x_0)), f)\},
\]

or equivalently

\[
\inf_{\nu \in \text{cab}(X, 0)} \{\max\{\|\nu\|_{KR}, \|\mu - \nu\|_{TV}\}\}
\]

\[
= \max_{f \in \text{Lip}(X)} \left\{ - \min_{\lambda \in [0, 1]} \left\{ i(f \in \text{Lip}(X, x_0) : \|f\|_L \leq \lambda) \{-f - f(x_0)\} + i(f \in \text{Lip}(X) : \|f\|_\infty \leq 1 - \lambda)(f) - \int f d\mu \right\} \right\},
\]

where, since \(\|f - f(x_0)\|_L = \|f\|_L\) and \(\min_x \{g(x)\} = -\max_x \{-g(x)\}\), the right side further simplifies to

\[
\max_{f \in \text{Lip}(X)} \left\{ \max_{\lambda \in [0, 1]} \left\{ \int f d\mu - i(f \in \text{Lip}(X, x_0) : \|f\|_L \leq \lambda)(f)
\right.\right. \\
\left.\left. - i(f \in \text{Lip}(X) : \|f\|_\infty \leq 1 - \lambda)(f) \right\} \right\}. \quad (30)
\]
Given $f \in Lip(X)$, it is clear that

$$\max_{\lambda \in [0, 1]} \left\{ \int f d\mu - i_{\{f \in Lip(X, x_0) : \|f\|_L \leq \lambda\}}(f) - i_{\{f \in Lip(X) : \|f\|_\infty \leq 1 - \lambda\}}(f) \right\} = \begin{cases} \int f d\mu & \text{if } \exists \lambda \in [0, 1] : \|f\|_\infty \leq 1 - \lambda \land \|f\|_L \leq \lambda, \\ -\infty & \text{otherwise.} \end{cases}$$

(31)

All we need to show now is that

$$\exists \lambda \in [0, 1] : \|f\|_\infty \leq 1 - \lambda \land \|f\|_L \leq \lambda \iff \|f\|_\text{sum} \leq 1. \quad (32)$$

If $\|f\|_\infty \leq 1 - \lambda \land \|f\|_L \leq \lambda$, then $\|f\|_\infty + \|f\|_L \leq \lambda + (1 - \lambda) = 1$. If $\|f\|_\text{sum} \leq 1$, then $\lambda = \|f\|_L$ suffices. Hence one has

$$\inf_{\nu \in cabv(X, 0)} \{ \max_{f \in Lip(X) : \|f\|_\text{sum} \leq 1} \left\{ \int f d\mu \right\} \} = \max_{f \in Lip(X), \|f\|_\text{sum} \leq 1} \left\{ \int f d\mu \right\}, \quad (33)$$

which is exactly the primal formula we set out to prove, together with the variant of the dual formula with a maximum instead of a supremum, implying that $\exists f_* \in Lip(X) : \|f_*\|_\text{sum} \leq 1 \land \|\mu\|_M = \int f_* d\mu$. If $\|f_*\|_\text{sum} < 1$ for such an $f_*$, then we get the contradiction $\int \|f_*\|_\text{sum} f_* d\mu > \|\mu\|_M$, so that one actually has $\|f_*\|_\text{sum} = 1$.

For the density claim, notice that the missing ingredient for the proof of [Cobzaş et al., 2019, Proposition 8.5.3] to work with $\|\cdot\|_M$ instead of $\|\cdot\|_H$ is the formula $\forall \nu \in cabv(X, 0), \mu \in cabv(X) : \|\mu\|_M \leq \|\nu\|_KR + \|\mu - \nu\|_{TV}$, which in light of the primal representation clearly holds.

For the duality claim, notice again that the missing ingredient for the proof of [Cobzaş et al., 2019, Proposition 8.5.5] to work with $\|\cdot\|_M$ instead of $\|\cdot\|_H$ is exactly the density claim we just proved, hence $(cabv(X), \|\cdot\|_M) \cong (Lip(X), \|\cdot\|_\text{sum})$ holds as well.

For the equivalence claim, notice that on one hand, one has $\|\nu\|_KR + \|\mu - \nu\|_{TV} \geq \max\{\|\nu\|_KR, \|\mu - \nu\|_{TV}\}$ for $\forall \nu \in cabv(X, 0)$, hence $\|\mu\|_H \geq \|\mu\|_M$ for $\forall \mu \in cabv(X)$. On the other hand, one has $2\max\{\|\nu\|_KR, \|\mu - \nu\|_{TV}\} \geq \|\nu\|_KR + \|\mu - \nu\|_{TV}$ for $\forall \nu \in cabv(X, 0)$, hence $2\|\mu\|_M \geq \|\mu\|_H$ for $\forall \mu \in cabv(X)$. Therefore the norms $\|\cdot\|_M$ and $\|\cdot\|_H$ are equivalent. \hfill \Box

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