ON THE BIFURCATION
OF PERIODIC ORBITS.

J.P. Françoise

Université de Paris VI, GSIB, UFR de mathématiques,
175 Rue de Chevaleret,
75013 Paris, France

In 1980, I visited IMPA and J. Sotomayor asked me to "explain" Dulac’s article
on limit cycles. Then Mauricio M. Peixoto explained me his interest in this question in
relation with his famous article on structural stability. This was for me the beginning
of a fantastic experience with the world of differential equations. The opportunity of
Mauricio’s 80th birthday allows me to express him all my gratitude for his influence and
to warmly thank all his followers and close friends in particular Yvan Kupka, Charles
C. Pugh, Jorge Sotomayor and René Thom.

INTRODUCTION

Given a polynomial vector field of the plane:

\[ X = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y, \]

where \( P \) and \( Q \) are both polynomials of degree less than \( d \), Hilbert’s 16th problem, part
B, calls for finding a bound (which depends only on \( d \)) to the number of limit cycles
(isolated periodic solutions) of \( X \). Despite the many contributions to this question, the
problem still remains open nowadays even in the case \( d = 2 \). One of the most spectacular
contribution to this field is due to N.N. Bautin ([1]). These notes aim at providing generalizations of Bautin’s theorem which involve several new ideas introduced in recent years. The creation and developments of the theory of nonlinear oscillations is due to the school of A.A. Andronov. The famous books ” Theory of oscillations ” by Andronov, Khaikin, Witt” followed by ”Qualitative theory of second order dynamical systems” and ”Theory of bifurcations of dynamical systems in the plane” run several editions in various languages. As one of the most prominent followers of Andronov, N.N. Bautin wrote in 1939 a masterpiece contribution to the classical center-focus problem posed by Poincaré. In his lecture given on the occasion of Arnol’d’s 60-th birthday at the Fields Institute in Toronto, June 1997, ([13], [14]) S. Smale listed 18 problems chosen with these criteria: simple statement, also preferably mathematically precise; personal acquaintance with the problem; a belief that the question, its solution, partial results, or even attempts at its solution are likely to have great importance for Mathematics and its developments in the next century. Besides the Riemann Hypothesis, one was on Hilbert’s list; the Hilbert’s 16-th problem on limit cycles. The author hopes that the specialists will be interested to have here collected techniques which appeared in a scattered way (from 1995 to 2000) and have been revisited in a more systematic manner. For non-specialists, this survey may contribute to introduce this vivid subject which involves so different methods such as projections of analytic sets, division by an ideal, complex foliations, Riemann surfaces, algebraic invariant theory and normal forms, and which belongs to the tradition of the qualitative theory of differential equations. Chapters I-II review results which have essentially already been published jointly with Yosi Yomdin ([8], [9]). Chapter III is perhaps the most original and develops an extension of Bautin’s approach to any dimension in the framework of the theory of perturbations of a Nambu dynamics ([12]).
I- CLASSICAL BAUTIN’s APPROACH

The first paragraph is devoted to the presentation of Bautin approach ([1]). We give the definitions of the center set, of the Bautin ideal and of the Bautin index. We show that the center set is a real algebraic variety. We select particular generators of the Bautin ideal which are adapted to the first return map in a sense to be precised. We give an estimate of the domain of convergence of the return mapping and we show accordingly to ([2], [3], [4]) that it is an $A_0$-Series. We conclude this part with Bautin’s argument based on the classical Rolle property to produce a bound to the number of real zeros of the return mapping. In this survey, we will mostly be concerned with homogeneous nonlinearities. In this paragraph, it is not necessary to assume that the perturbation is homogeneous: all definitions and theorems can be easily modified without any fundamental changes. Consideration of this particular case allows simplification of the notations and also it captures the essential properties of the general case.

I.1 Construction of the return mapping by Bautin’s method

We consider a polynomial plane vector field $X$ of type:

$$X = x \partial/\partial y - y \partial/\partial x + \sum_{i+j=d} [a_{i,j} x^i y^j \partial/\partial x + b_{i,j} x^i y^j \partial/\partial y] \tag{1.1}$$

and the associated flow, solution of the system:

$$\dot{x} = -y + \sum_{i,j/i+j=d} a_{i,j} x^i y^j = -y + P(x, y), \tag{1.2a}$$

$$\dot{y} = x + \sum_{i,j/i+j=d} b_{i,j} x^i y^j = -y + Q(x, y). \tag{1.2b}$$

The parameters of the vector field $(a, b)$ can take any real values and thus $(a, b)$ should be considered as a point in the vector space $R^{2(d+1)}$. We first recall Bautin’s approach to find the return mapping of (1.1) near the origin. Write (1.2) in polar coordinates $(r, \theta)$:
\[ x = r \cos(\theta),\ y = r \sin(\theta). \quad (1.3) \]

This leads to:

\[ 2r \dot{r} = 2(x \dot{x} + y \dot{y}),\ r \dot{r} = x P + y Q = r^{d+1} A(\theta), \quad (1.4) \]

\[ \dot{\theta} = (x \dot{y} - y \dot{x})/(x^2 + y^2) = 1 + r^{d-1} B(\theta), \quad (1.5) \]

where \( A(\theta) \) and \( B(\theta) \) are two trigonometric polynomials (in \( \cos(\theta), \sin(\theta) \)) linear in the parameters \((a, b)\).

This yields:

\[ \frac{dr}{d\theta} = r^d A(\theta)/[1 + r^{d-1} B(\theta)], \quad (1.6) \]

and thus

\[ \frac{dr}{d\theta} = \sum_{k=0}^{\infty} (-1)^k r^{k(d-1)+d} A(\theta) B(\theta)^k. \quad (1.7) \]

This equation (1.7) may be rewritten:

\[ \frac{dr}{d\theta} = \sum_{k \geq d} r^k R_k(\theta), \quad (1.8) \]

where the coefficients \( R_k(\theta) \) are trigonometric polynomials in \( \cos(\theta), \sin(\theta) \) and polynomials in the parameters \((a, b)\). To simplify the notations, the dependence on the parameters \((a, b)\) is not made explicit. Bautin looks for a solution of (1.8) \( r = r(\theta) \) so that \( r(0) = r_0 \), given as an expansion:

\[ r = r_0 + v_2(\theta)r_0^2 + ... + v_k(\theta)r_0^k + ... \quad (1.9) \]

Comparison between (1.8) and (1.9) yields:
\[ v_2(\theta) = \ldots = v_{d-1}(\theta) = 0, \quad (1.10) \]
\[ v'_d(\theta) = R_d(\theta), \quad (1.11) \]
\[ v'_k(\theta) = \sum_{i=2}^{k} B_{ik}[v_d(\theta), \ldots, v_{k-1}(\theta)]R_i(\theta), k \geq d + 1. \quad (1.12) \]

The polynomial \( B_{ik}[a_d, \ldots, a_{k-1}] \) displays integer coefficients and is the coefficient of \( X^{k-i} \) in \( (X + a_dX^d + \ldots + a_pX^p + \ldots)^i \). The relation (1.12) allows to determine inductively the functions \( v_k(\theta) \):

\[ v_k(\theta) = \int_{\theta}^{0} \left[ \sum_{i=2}^{k} B_{ik}[v_d(\phi), \ldots, v_{k-1}(\phi)]R_i(\phi) \right] d\phi. \quad (1.13) \]

Two facts can be easily derived from this construction:

i) \( v_k(\theta) \) is polynomial in \( \theta \) (of degree less than \( k \)) and in \( (\sin(\theta), \cos(\theta)) \).

ii) \( v_k(\theta) \) is polynomial in the parameters \( (a, b) \) of the vector field. Thus in particular the coefficients \( v_k(2\pi) \) of the return mapping are polynomials in \( (a, b) \).

**Definition I.1.1**

The vector field \( X \) displays a center at the origin \( (0 \in R^2) \) (or is said to be a center) if and only if \( r(2\pi) = r(0) \) for all values of \( r(0) \) close to 0.

**I.2 Convergence of the Taylor series, majorant series and A_0-series**

Let \( f_\lambda(x) = \sum a_k(\lambda)x^k \) be an analytic series in \( x \) with polynomial coefficients in the parameters \( \lambda = (\lambda_0, \ldots, \lambda_D) \). Denote \( |a_k| \) (norm of the polynomial \( a_k(\lambda) \)) as the sum of the absolute value of the coefficients.

**Definition I.2.1**

The series \( f_\lambda \) is called an \( A_0 \)-series if the following two conditions are satisfied:

There are positive constants \( K_1, K_2, K_3, K_4 \) such that:
1- \( \text{deg}(a_k) \leq K_1 k + K_2 \),
2- \( |a_k| \leq K_3 K_4^k \).

\( A_0 \)-series form a subring of the ring of formal power series in \( x \) with polynomial coefficients in \( \lambda \). All the usual analytic operations, like substitution to a given analytic function, composition, inversion, etc... transform \( A_0 \)-series into themselves. (The proof is rather straightforward and is not provided here). The \( A_0 \)-series have been precisely introduced (in the subject) by M. Briskin and Y. Yomdin ([4]). The proof of the following easy lemma is also left to the reader:

**Lemma I.2.2**

An \( A_0 \)-series \( f_\lambda(x) \) converges in the disc \( D(0, R) \) of radius \( R = [1/(K_4(1+|\lambda|))^{K_1}] \).

In the following, we also denote by \( f_\lambda \) the complex analytic function defined for all \( \lambda \in C^D \) on the disc \( D(0, R) \) by the \( A_0 \)-series.

**Proposition I.2.3**

For all \( \theta \), the series \( \sum_{k \geq d} x^k R_k(\theta) \), is an \( A_0 \)-series with \( K_1 = 1/(d - 1) \), \( K_2 = -1/(d - 1) \), \( K_3 = 1/[2(d + 1)]^{1/d-1} \), \( K_4 = [2(d + 1)]^{1/d-1} \).

**Proof.**

This is indeed a simple consequence of Bautin’s method. For all \( \theta \), the norms of the polynomials \( A(\theta) \) and \( B(\theta) \) (seen as polynomials in \( (a, b) \)),

\[
A(\theta) = (xP + yQ)[\cos(\theta), \sin(\theta)], \quad (1.14a)
\]

\[
B(\theta) = (xQ - yP)[\cos(\theta), \sin(\theta)], \quad (1.14b)
\]

are bounded by:
\[ |A(\theta)| \leq 2(d+1), \quad |B(\theta)| \leq 2(d+1). \] (1.15)

Write:

\[ \frac{dr}{d\theta} = \sum_{k \geq d} r^k R_k(\theta) = \sum_{j \geq 0} (-1)^j r^{d+j(d-1)} A(\theta)B(\theta)^j. \] (1.16)

Denote \( k = d + j(d-1) \), then this yields:

\[ \deg[R_k(\theta)] \leq 1 + j \leq 1 + [(k-d)/(d-1)] \leq [(k-1)/(d-1)]. \] (1.17)

Furthermore, the norm of \( R_k(\theta) \) as polynomial in the parameters \((a, b)\) is estimated as follows:

\[ |R_k(\theta)| = |A(\theta)B(\theta)^j| \leq |A(\theta)||B(\theta)|^j \leq [2(d+1)]^{[(k-1)/(d-1)]}. \] (1.18)

**Theorem I.2.4**

For all \( \theta \), the series \( \sum_{k \geq d} x^k v_k(\theta) \), is an \( A_0 \)-series with \( K'_1 = 1/(d-1), \ K'_2 = 0, \ K'_3 = [2\pi K_3/4K_4^2][K_4 - 2C + ((K_4 - 2C)^2 - K_4^2)^2], \ 1/K'_4 = |K_4 - 2C + ((K_4 - 2C)^2 - K_4^2)|/[2K_4^2], \ C = K_4 + 2\pi K_3 K_4^2. \)

**Proof.**

First observe that \( \deg(v_d(\theta)) = \deg(R_d(\theta)) = 1 \) (The degree as polynomial in the parameters). Thus we have \( \deg(v_d(\theta)) \leq d/(d-1) \). Assume inductively that:

\[ \deg(v_j(\theta)) \leq j/(d-1) \text{ for } j = d, \ldots, k-1. \] (1.19)

The recurrence relation (1.13) displays:

\[ \deg[v_k(\theta)] \leq \max[\deg(B_{ik}[v_d(\phi)], \ldots, \deg(v_{k-1}(\phi))R_i(\phi))], \ i = 2, \ldots, k. \] (1.20)
The equation (1.17) and the induction assumption yield:

\[ \deg[B_{ik}[v_d(\phi), \ldots, v_{k-1}(\phi)]R_i(\phi)] \leq K_1(k - i) + K_1i + K_2 \leq K_1k. \] (1.21)

This shows the first part of the theorem on the bound of the degrees of the coefficients \( v_k(\theta) \). For the second part of the proof related to the bound on the norms of the coefficients, we use standard methods and notations of majorant series.

**Definition I.2.5**

The formal series \( \Psi(x) = \sum_{k \geq 1} \Psi_k x^k \) with positive coefficients dominates the formal series \( \Phi(x) = \sum_{k \geq 1} \Phi_k x^k \) with positive coefficients if and only if for all coefficients \( \Phi_k \leq \Psi_k, k \geq 1 \).

The series \( x + \sum_{k \geq 2} |v_k(\theta)| x^k \) is dominated by the series \( x + \sum_{k \geq 2} \Psi_k x^k \) so that:

\[ \psi_k = 2\pi \sum_{i=d}^{k} B_{ik}(\psi_d, \ldots, \psi_{k-1}) \ | \ R_i \ | . \] (1.22)

Denote

\[ \bar{R}(x) = \sum_{i \geq 2} K_3 K_4^i x^i = K_3(K_4 x)^2 /[1 - K_4 x], \] (1.23)

then the proposition I.2.3 yields:

\[ \sum_{i \geq d} | R_i | \ x^i \text{isdominatedby} \bar{R}(x). \] (1.24)

The series \( \Psi(x) = x + \sum_{k \geq 2} \Psi_k x^k \) is then dominated by the solution \( \bar{\Psi}(x) \) solution of the equation:

\[ \bar{\Psi}(x) - x = 2\pi \bar{R}[\bar{\Psi}(x)] = 2\pi K_3[K_4 \bar{\Psi}(x)]^2 /[1 - K_4 \bar{\Psi}(x)]. \] (1.25)

At this point we have obtained that \( \bar{\Psi}(x) \) is a solution of an algebraic equation of degree two. Estimates of the constant \( K_4' \) is then obtained by the distance to the first
zero of the discriminant and constant $K'_3$ is then adjusted from the first term of the development. Easy computations give then the proof of the theorem.

I.3 Definitions of the center set, of the Bautin ideal and of the Bautin index. Definition of $\phi$-equivalence of series.

Definition I.3.1

The center set $C$ is the set of values of parameters $(a, b)$ so that the corresponding vector field $X$ has a center at the origin.

Proposition I.3.2

The center set $C$ is a real algebraic manifold.

Proof

The center set $C$ is given as the zero set of the coefficients $v_k(2\pi)$ which are polynomials in the parameters $(a, b)$.

From now on, it is appropriated to change of notations and denote $L_k(a, b) = v_k(2\pi)$ the coefficients of the return mapping to emphasize their dependence in terms of the parameters $(a, b)$. We denote:

$$r \mapsto L(r) = r + L_d(a, b)r^d + ... + L_k(a, b)r^k + ...$$  \hfill (1.26)

the return mapping defined in (1.9) for $\theta = 2\pi$.

Definition I.3.3

The Bautin ideal is the ideal generated in the ring $R[a, b]$ by the coefficients $L_k(a, b)$.

Definition I.3.4
The Bautin index is the first integer \( k_0 \) so that the polynomials \( L_d(a, b), \ldots, L_{k_0}(a, b) \) generate the Bautin ideal.

The reader should be careful about the fact that the Bautin index does not depend only on the Bautin ideal. In its definition, we cannot substitute to the collection of the coefficients of the return mapping another system of generators of the Bautin ideal. Note that the existence of the Bautin index just follows from the fact that the ring \( R[a, b] \) is Noetherian.

The local Hilbert’s 16th problem is to find a bound depending only on \( d \) to the number of isolated periodic orbits of \( X \) in a neighborhood of the origin. Isolated periodic orbits of \( X \) defined in the domain of definition of the return mapping correspond exactly to the isolated solutions of the equation:

\[
L(r) - r = 0.
\] (1.27)

We gradually change into notations more pertinent to the general algebraic geometry setting. Let \( \Phi(x, \lambda) \) be an analytic series in \( x \) with polynomial coefficients in the parameter \( \lambda = (\lambda_1, \ldots, \lambda_D) \).

**Definition I.3.5**

The Bautin ideal of the series \( \Phi(x, \lambda) \) is the ideal generated in the ring \( R[\lambda] \) by the coefficients \( \Phi_k(x, \lambda) \). The center set of the series \( \Phi(x, \lambda) \) is the zero set of its Bautin ideal. The Bautin index \( d \) of the series \( \Phi(x, \lambda) \) is the minimal integer \( d \) such that the coefficients \( \Phi_1(x, \lambda), \ldots, \Phi_{d-1}(x, \lambda) \) generate the Bautin ideal of the series \( \Phi(x, \lambda) \).

**Definition I.3.6**

Two series \( \Phi(x, \lambda) \) and \( \Psi(x, \lambda) \) with polynomial coefficients in the parameters \( \lambda \) are said to be \( \phi \)-equivalent if for all integers \( k \geq 1 \), the polynomial \( \Phi_k(\lambda) - \Psi_k(\lambda) \) belongs to the ideal generated by \( (\Phi_1(\lambda), \ldots, \Phi_{k-1}(\lambda)) \).
It can be easily checked that this defines an equivalence relation. Also the definition yields the following:

**Lemma I.3.7**

Two series $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ with polynomial coefficients in the parameters $\lambda$ which are $\phi$-equivalent have the same Bautin index.

**I.4 The classical argument to show that the number of real zeros is bounded**

We recall here Bautin’s argument to produce a bound to the number of real zeros. Let $\Phi(x, \lambda)$ be an analytic series in $x$ with polynomial coefficients in the parameters $\lambda$ and with Bautin index $d$. Assume that $\Phi(0, \lambda) = 0$ and $\Phi(x, 0) = 0$.

**Theorem I.4.1**

There is a ball $B \in \mathbb{R}^D$, centered at 0, in the space $\mathbb{R}^D$ and an interval $I$ containing 0 such that for all $\lambda \in B$, the number of zeros of $\Phi(x, \lambda)$ contained in $I$ is less than or equal to $d$.

**Proof.**

Using the definition of the Bautin index $d$, we write:

$$
\Phi(x, \lambda) = \sum_{i=1}^{d} \Phi_i(\lambda)[1 + \Psi_i(x, \lambda)]x^i,
$$

(1.28)

with $\Psi_i(x, \lambda)$ analytic in $x$, with polynomial coefficients in $\lambda$ such that $\Psi_i(0, \lambda) = \Psi_i(x, 0) = 0$. Assume that $B$ and $I$ are small enough so that (for instance) $|\Psi_i(x, \lambda)| \leq 1/2$ on $I \times B$. Then divide $\Phi(x, \lambda)$ by $[1 + \Psi_d(x, \lambda)]$ and rewrite:

$$
[\Phi(x, \lambda)]/[1 + \Psi_d(x, \lambda)] = \Phi_1(\lambda) + \Phi_2(\lambda)[1 + \Psi'_i(x, \lambda)]x^2 + \ldots
$$

(1.29)

Then from Rolle’s lemma follows that the number of zeros of $\Phi(x, \lambda)$ is less than
1+ number of zeros of the derivative \([\Phi(x, \lambda)]/[1 + \Psi_d(x, \lambda)]\). Write then this derivative as

\[
\Phi_2(\lambda)[1 + \Psi^{(2)}_1(x, \lambda)]x + ... \Phi_{d-1}(\lambda)[1 + \Psi^{(2)}_1(x, \lambda)]x^{d-1}.
\] (1.30)

Then repeat the process (sometimes referred to as the division-derivation algorithm). We obtain the result by an easy induction.

It is clear that one cannot go far with this proof if we want an explicit control of the size of the domain (either of \(I\) or of \(B\)). For instance, we know that the coefficients \(\Phi_i(\lambda)\) are combinations of the \(d\) first coefficients:

\[
\Phi_i(\lambda) = P_{(i,1)}(\lambda)\Phi_1(\lambda) + ... + P_{(i,d)}(\lambda)\Phi_d(\lambda),
\] (1.31)

but how to control the size of the ”quotients” \(P_{(i,k)}(\lambda)\) which enter in the construction of the \(\Psi_i(x, \lambda)\)? We will see how to bypass these difficulties in the next chapter using complex analytic methods such as the polynomial Hironaka division theorem.

I.5 Formulation of the problem in terms of projection of analytic sets.

Let \(\Phi : R \times R^D \mapsto R\) be an analytic series with polynomial coefficients:

\[
\Phi(x, \lambda) = x + \Phi_2(\lambda)x^2 + ... + \Phi_k(\lambda)x^k + ...
\] (1.32)

We consider the subset \(\Sigma \subset R \times R^D\) defined as the zero-set of \(\Phi(x, \lambda) - x\):

\[
\Sigma = [(x, \lambda)/\Phi(x, \lambda) - x = 0].
\] (1.33)

Let \(\pi : \Sigma \mapsto R^D\) be the restriction to \(\Sigma\) of the natural projection \(\pi : R \times R^D \mapsto R^D\). The center set associated to the analytic series \(\Phi(x, \lambda) - x\) defined in I.3.5 is the set \(C \subset R^D\) of parameters \(\lambda\) such that the fiber of the projection \(\pi^{-1}(\lambda)\) is contained in the set \(\Sigma\). The theorem I.4.1 displays the following geometric interpretation:
Theorem I.5.1

There is a neighborhood \( I \times B \) of \((0,0)\) in \( R \times R^D \) such that for all points \( \lambda \) of \( B \) the number of isolated points of the fibers \( \pi^{-1}(\lambda) \) restricted to \( \Sigma \) is less than the Bautin index \( d \) of the analytic series \( \Phi(x, \lambda) - x \).

I.6 A \( S^1 \)-action on the space of parameters and a system of \( S^1 \)-invariant polynomials which generate the Bautin ideal.

We can change the coordinates in the plane \((x,y) \mapsto (x',y')\) under the action of the rotation group. This leaves invariant the linear part of the vector field \( X \) and thus it induces an action of the rotation group \( S^1 \) on the space of parameters \((a,b)\). This linear action induces an action on the ring of polynomials \( R[a,b] \) on the space of parameters. A polynomial is said to be invariant if it is fixed under this \( S^1 \)-action. The center set is obviously invariant under the \( S^1 \)-action because the fact, that the vector field \( X \) has all its orbits periodic in a neighborhood of the origin, does not depend on the choice of the system of coordinates on this neighborhood. It is natural to ask if the coefficients of the return mapping are \( S^1 \)-invariant polynomials. The answer is no, but it is possible to find another system of generators of the Bautin ideal which are \( S^1 \)-invariant and so that the associated analytic series is \( \phi \)-equivalent to the return mapping. This was proved by ([6]) using the approach of the successive derivatives. For \( d = 2,3 \) it was proved by H. Zoladek ([15]) using the classical Bautin approach. We recall here his proof and derive the result in its full generality for any \( d \), \((d \geq 2)\).

Let us consider the ring of polynomials in \((a,b),[\theta,(\sin(\theta),\cos(\theta))]\):

\[
R[a,b] \otimes R[\theta,(\sin(\theta),\cos(\theta))].
\]

The rotation group acts on the ring \( R[a,b] \) as displayed previously and so it acts on \( R[a,b] \otimes R[\theta,(\sin(\theta),\cos(\theta))] \) in an obvious way (just extend the action by the identity on the second factor). We denote \( p \mapsto \phi \ast p \) this action where \( \phi \) belongs to \( S^1 \) and \( p \)
belongs to $R[a, b] \otimes R[\theta, (\sin(\theta), \cos(\theta))]$. Now $S^1$ acts also on the ring

$$R[\theta, (\sin(\theta), \cos(\theta))]$$

in a natural way:

$$\phi \ast (\theta) = \theta + \phi, \phi \ast (\sin(\theta), \cos(\theta)) = (\sin(\theta + \phi), \cos(\theta + \phi)). \quad (1.34)$$

This action can be extended to an action $p \mapsto \phi \ast p$ on the ring

$$R[a, b] \otimes R[\theta, (\sin(\theta), \cos(\theta))]$$

by the identity on the first factor.

**Definition I.6.1**

An element $p$ of the ring $R[a, b] \otimes R[\theta, (\sin(\theta), \cos(\theta))]$ is said to be covariant if for all $\phi$: $\phi \ast p = \phi \ast p$.

**Lemma I.6.2**

Product and sum of two covariant polynomials are covariant polynomials.

The proof of this lemma is very easy and is omitted.

**Lemma I.6.3**

The coefficients $R_k(\theta)$ of the equation (1.8) are covariant polynomials.

**Proof.**

The trigonometric polynomials $A(\theta)$ and $B(\theta)$ are clearly covariant polynomials as seen from the definition of the actions. Now from Lemma I.6.2 it follows that the products $A(\theta)B(\theta)^k$ are covariant polynomials.
It is now necessary to be more careful with the inductive construction of the coefficients $v_k(\theta)$. We initiate the discussion with the term of lowest degree $v_d(\theta)$ defined by the condition:

$$v'_d(\theta) = R_d(\theta).$$  \hspace{1cm} (1.35)

The term $R_d(\theta)$ to be integrated is a trigonometric polynomial. Thus this yields:

$$v_d(\theta) = z_d(\theta) + s_d(\theta),$$  \hspace{1cm} (1.36)

where $z_d$ is a constant and $s_d(\theta)$ is a trigonometric polynomial with $s_d(0) = s_d(2\pi) = 0$. Note that

$$z_d = (1/2\pi) \int_0^{2\pi} R_d(\phi) d\phi,$$  \hspace{1cm} (1.37)

and the first coefficient of the return mapping is:

$$v_d(2\pi) = 2\pi z_d = \int_0^{2\pi} R_d(\phi) d\phi.$$  \hspace{1cm} (1.38)

Next step of the recurrence:

$$v'_{d+1}(\theta) = \sum_{i=d}^{d+1} B_{i,d+1} [v_d(\theta)] R_i(\theta),$$  \hspace{1cm} (1.39)

leads to:

$$v_{d+1}(\theta) = z_{d+1} \theta + r^{(2)}_{d+1} \theta^2 + s_{d+1}(\theta),$$  \hspace{1cm} (1.40)

where $s_{d+1}(\theta)$ is a trigonometric polynomial so that: $s_{d+1}(0) = s_{d+1}(2\pi) = 0$,

$$z_{d+1} = (1/2\pi) \int_0^{2\pi} \sum_{i=d}^{d+1} B_{i,d+1}[s_d(\phi)] R_i(\phi) d\phi,$$  \hspace{1cm} (1.41)

and where the coefficient $R_{d+1}$ is proportional to the coefficient $v_{d+1}$.

The general step $k$ of the recurrence displays:
\[ v_k(\theta) = z_k \theta + r_k^{(2)} \theta^2 + \ldots + r_k^{(k-d+1)} \theta^{k-d+1} + s_k(\theta), \]  

(1.42)

where \( s_k(\theta) \) is a trigonometric polynomial so that: \( s_k(0) = s_k(2\pi) = 0 \), such that:

\[ s'_k(\theta) = \sum_{i=2}^{k} B_{ik}[s_d(\theta), \ldots, s_{k-1}(\theta)] R_i(\theta), \]  

(1.43)

\[ z_k = (1/2\pi) \int_0^{2\pi} [\sum_{i=2}^{k} B_{ik}[s_d(\phi), \ldots, s_{k-1}(\phi)] R_i(\phi)] d\phi, \]  

(1.44)

and where the coefficients \( r_k^{(j)}, j = 2, \ldots, k - d + 1 \) belongs to the ideal generated by the preceding coefficients \( z_d, \ldots, z_{k-1} \). The construction of the coefficients \( z_k \) yields:

**Lemma I.6.4**

The two series \( L(x) - x = \sum_{k \geq d} v_k(2\pi)x^k \) and \( \Phi(x) - x = \sum_{k \geq d} z_k x^k \) are \( \phi \)-equivalent.

We prove now that the polynomials \( v_k \) are \( S^1 \)-invariant. The proof splits into several lemmas of independent interest.

**Lemma I.6.5**

Let \( T_k(\theta) \) be an element of the ring \( R[a, b] \otimes R[\theta, (\sin(\theta), \cos(\theta))] \) which is a covariant trigonometric polynomial, then the polynomial \( v_k \) defined by:

\[ v_k = \int_0^{2\pi} T_k(\phi) d\phi, \]  

(1.45)

is an invariant polynomial.

**Proof.**

Let \( \theta \) be an element of \( S^1 \), consider \( \theta * (v_k) \):
\[ \theta * (v_k) = \int_0^{2\pi} \theta * T_k(\phi) d\phi. \]  

(1.46)

The covariance property of \( T_k \) yields:

\[ \int_0^{2\pi} \theta * T_k(\phi) d\phi = \int_0^{2\pi} \theta^* T_k(\phi) d\phi = \int_0^{2\pi} T_k(\phi + \theta) d\phi = \int_0^{2\pi} T_k(\phi) d\phi. \]  

(1.47)

**Lemma I.6.6**

Let \( T_k(\theta) \) be a trigonometric polynomial without constant term which is covariant. The only trigonometric polynomial \( S_k(\theta) \) without constant term so that:

\[ S'_k(\theta) = T_k(\theta), \]  

(1.48)

is covariant.

**Proof.**

Write

\[ T_k(\theta) = \sum_l T_{kl}(a,b) e^{il\theta}. \]  

(1.49)

Let us consider the first action of \( S^1 \). Let \( \phi \) be an element of \( S^1 \). Under the action of this element, each term \( T_{kl}(a,b) \) gets multiplied by the factor \( e^{i\phi} \). Now the polynomial \( S_k(\theta) \) writes:

\[ S_k(\theta) = \sum_l [T_{kl}(a,b)/l] e^{il\theta}. \]  

(1.50)

Its covariance follows easily from this expression.

We conclude this paragraph with the theorem
Theorem I.6.7

The displacement function \( L(x) - x = \sum_{k \geq d} v_k (2\pi) x^k \) is \( \phi \)-equivalent to a series \( \Phi(x) - x = \sum_{k \geq d} z_k x^k \) with \( S^1 \)-invariant coefficients \( z_k \).

Proof.

The polynomial \( R_d(\theta) \) is covariant (cf. Lemma I.6.3) and thus the coefficient \( z_d \) is invariant as a consequence of formula (1.44) and of the Lemma I.6.5. The polynomial \( s_d \) is covariant as a result of the Lemma I.6.6. Assume inductively that the polynomials \( s_d \theta, ..., s_{k-1}(\theta) \) are covariant. Then Lemma I.6.3 and Lemma I.6.6 and the formula (1.43) yield the fact that \( s_k \) is covariant and the Lemma I.6.5 imply that the coefficient \( z_k \) is invariant.

II-A GLOBAL AND COMPLEX GENERALIZATION OF BAUTIN’S THEOREM.

Bautin’s method as displayed in the preceding chapter does not allow to produce a precise estimate on the size of the domain on which the number of zeros of an \( A_0 \)-series (such as the displacement function) is controlled. It is not surprising that to get such an estimate, complex analytic methods are more appropriated. We begin this chapter with general considerations on Bernstein classes.

II.1 Finiteness properties, Bernstein inequality.

In this part, the definitions of the Bernstein classes \( B_1 \) and \( B_2 \) are displayed and their equivalence is proved. Then it is shown that the number of zeros of functions of these classes is explicitly bounded. The closed disc centered at 0 \( \in C \) of radius \( R \) is denoted \( \bar{D}_R \). This paragraph relies entirely on ([8]) and ([9]).

Definition II.1.1
Let $R > 0$, $0 < \alpha < 1$ and $K > 0$ and $f$ holomorphic in a neighborhood of $\bar{D}_R$. The function $f$ belongs to the Bernstein class $B^{1}_{R,\alpha,K}$ if and only if:

$$\max[|f(z)|, z \in \bar{D}_R]/\max[|f(z)|, z \in \bar{D}_{\alpha R}] \leq K.$$  

(2.1)

**Definition II.1.2**

Let $N$ be an integer, $R > 0$ and $c > 0$ and $f(z) = \sum_{i \geq 0} a_i z^i$ be an analytic function on a neighborhood of $0 \in \mathbb{C}$. The function $f$ belongs to the Bernstein class $B^{2}_{N,R,c}$ if and only if for all $j \geq N$:

$$|a_j| R^j \leq c \max_{i=0,...,N}(|a_i| R^i).$$  

(2.2)

**Theorem II.1.3**

Let $f$ be an element of $B^{2}_{N,R,c}$. Then $f$ is analytic on the open disc $D_R$, and for all $R' < R$ and $\alpha < 1$, and $K = \alpha^{-N}[1 + \alpha(1 - \alpha^N)/(1 - \alpha) + c\beta/(1 - \beta)]$ where $\beta = R'/R$, $f$ belongs to $B^{1}_{R',\alpha,K}$. Conversely if $f$ belongs to $B^{1}_{R,\alpha,c}$, then $f \in B^{2}_{N,R,c}$ with $N = [(\log_2(K) - \log_2(1 - \alpha) + 1)/(\log_2(1/\alpha))]$, $c = [K(2K + 1)/(1 - \alpha)^2]$.  

**Proof**

If $f \in B^{2}_{N,R,c}$, the convergence of the series $f(z) = \sum_{i \geq 0} a_i z^i$ on the disc $D_R$ is consequence of the Cauchy-Hadamard formula and of the inequality:

$$|a_n| \leq \max_{i=0,...,N}(|a_i| R^i)(R^{-n}),$$  

(2.3)

for large $n$. Denote $m = \max(|f(z)|, z \in D_{\alpha R}$, Cauchy’s inequalities yield:

$$|a_i| \leq m/(\alpha R')^i.$$  

(2.4)

For $i = 0, ..., N$, this displays:

$$|a_i| R^i \leq m/(\alpha R')^i \leq [m/\alpha^N (R'/R)^N]$$  

(2.5)
and then for all \( j \geq N + 1 \):

\[
|a_j| R^j \leq c[m/\alpha^N (R'/R)^N].
\] (2.6)

This yields:

\[
\max(| f(z) |, z \in D_{R'}) \leq \sum_{i=0}^{N} |a_i|R'^i + \sum_{j \geq N+1} |a_j|R'^j
\] (2.7)

\[
\leq m \sum_{i=0}^{N} (1/\alpha R')^i R'^i + c[m/\alpha^N (R'/R)^N] \sum_{j \geq N+1} (R'/R)^j
\] (2.8)

, and thus:

\[
\max(| f(z) |, z \in D_{R'}) \leq [m/(\alpha^N)] \left[ 1 + \alpha(1 - \alpha^N)/(1 - \alpha) + c\beta/(1 - \beta) \right]
\] (2.9)

as expected. Conversely, assume that \( f \) belongs to \( B^1_{R,\alpha,K} \), \( f(z) = \sum_{i=0}^{\infty} a_i z^i \). Let \( N \) be an integer and \( \sigma = \max_{i=0,\ldots,N}(|a_i| R^i) \) and \( b = \max(|R(z)|, z \in D_{\alpha R}) \), where \( R(z) = \sum_{i \geq N+1} a_i z^i \). Consider \( P_N(z) = \sum_{i=0}^{N} a_i z^i \) and:

\[
\max(|P_N(z)|, z \in D_{\alpha R}) \leq \sum_{i=0}^{N} |a_i| (\alpha R)^i
\] (2.10)

\[
\max(|P_N(z)|, z \in D_{\alpha R}) \leq \sigma/(1 - \alpha).
\] (2.11)

This yields:

\[
\max(|f(z)|, z \in D_{\alpha R}) \leq K [b + (\sigma/(1 + \alpha)].
\] (2.12)

Cauchy inequalities yield:

\[
|a_j| R^j \leq K [b + (\sigma/(1 + \alpha)].
\] (2.13)

The following bound for \( b \) may be displayed:

\[
b \leq \sum_{j \geq N+1} [a_j] (\alpha R)^j \leq K [b + (\sigma/(1 + \alpha)] [\alpha^{N+1}/(1 - \alpha)],
\] (2.14)

\[
b [1 - (K \alpha^{N+1} / (1 - \alpha))] \leq K \sigma [\alpha^{N+1}/(1 - \alpha)^2].
\] (2.15)
Fix \( N \) so that:
\[
(K\alpha^{N+1}/(1 - \alpha) \leq 1/2; \quad (2.16)
\]
this is achieved for instance if: \( N = [(\log_2(K) - \log_2(1 - \alpha) + 1)/\log_2(1/\alpha)] \). Such a choice for \( N \) yields:
\[
b \leq 2K\sigma[\alpha^{N+1}/(1 - \alpha)^2] \leq 2K\sigma/(1 - \alpha)^2. \quad (2.17)
\]
Now this yields:
\[
|a_j|R^j \leq K[2\sigma K/(1 - \alpha)^2] + [\sigma/(1 - \alpha)], \quad (2.18)
\]
\[
|a_j|R^j \leq K(2K + 1)/(1 - \alpha)^2; \quad (2.19)
\]
the theorem now follows with: \( c = [K(2K + 1)/(1 - \alpha)^2] \).

The number of zeros of a function which belongs to a Bernstein class is bounded accordingly to what follows.

**Theorem II.1.4**

If \( f \in B^1_{R,\alpha,K} \), then the number of zeros of \( f \) in the disc \( \bar{D}_{\alpha R} \) is less than:
\[
\log_2(K)/[\log_2[(1 + \alpha^2)/2\alpha]] \quad (2.20)
\]

**Proof:**

Assume that \( f \) displays \( n \) zeros \( z_1, ..., z_n \) in the disc \( \bar{D}_{\alpha R} \). Denote \( g \) the holomorphic function on the disc \( D_R \) defined as:
\[
g(z) = f(z) \prod_{k=1}^{n} [(R^2 - z\bar{z}_k)/R(z - z_k)]. \quad (2.21)
\]
The maximum principle yields:
\[
\max(|g(z)|, z \in D_R) = \max(|f(Re^{i\theta})| \prod_{k=1}^{n} [(R^2 - Re^{i\theta} \bar{z}_k)/R(Re^{i\theta} - z_k)] |, \theta \in [0, 2\pi]), \quad (2.22)
\]
\[ = \max(|f(Re^{i\theta})| \prod_{k=1}^{n} [(R^2 - Re^{i\theta}z_k)/R(Re^{i\theta} - z_k)] \mid, \theta \in [0, 2\pi]), \quad (2.23) \]
\[ = \max(|f(z)|, z \in D_R). \quad (2.24) \]

Consider now:
\[ \max(|g(z)|, z \in D_{\alpha R}) = \max(|f(\alpha Re^{i\theta})| \prod_{k=1}^{n} [(R^2 - \alpha Re^{i\theta}z_k)/R(\alpha Re^{i\theta} - z_k)] |) \quad (2.25) \]
\[ \geq \max(|f(z)|, z \in D_{\alpha R}) \prod_{k=1}^{n} \min[|R^2 - \alpha Re^{i\theta}z_k)/R(\alpha Re^{i\theta} - z_k) |]. \quad (2.26) \]

Write now separately each quantities:
\[ | (R^2 - \alpha Re^{i\theta}z_k)/R(\alpha Re^{i\theta} - z_k) | \quad (2.27) \]
and \( z_k = R_ke^{i\theta_k} \). The minimum of the quantity when \( \theta - \theta_k \) varies in \([0, 2\pi]\) is
\[ (R + \alpha R_k)/(\alpha R + R_k). \quad (2.28) \]
The minimum of this quantity when \( 0 \leq R_k \leq \alpha R \) is
\[ (1 + \alpha^2)/2\alpha. \quad (2.29) \]
This yields the following inequality:
\[ \max(|g(z)|, z \in D_{\alpha R}) \geq \max(|f(z)|, z \in D_{\alpha R})[(1 + \alpha^2)/2\alpha]^n. \quad (2.30) \]
The inequalities:
\[ \max(|f(z)|, z \in D_R) = \max(|g(z)|, z \in D_R) \geq \max(|g(z)|, z \in D_{\alpha R}), \quad (2.31) \]
display:
\[ K \geq [(1 + \alpha^2)/2\alpha]^n, \quad (2.32) \]
and thus:

\[ n \leq \log_2(K)/\log_2[(1 + \alpha^2)/2\alpha]. \tag{2.33} \]

**Lemma II.1.5**

Let \( f \in B_{N,R,c}^2 \) and \( R'' = R/[2^{3N}\max(c, 2)] \), the number of zeros of \( f \) in the disc \( \bar{D}_{R''} \) is less than \( N \).

**Proof.**

Set \( R' = R/[\max(c, 2)] \) and \( \alpha = 2^{-3N} \). This displays:

\[
[(1 + \alpha^2)/2\alpha] = (2^{6N} + 1)/(2^{3N+1}) > 2^{3N-1},
\]

\[ K < 2^{3N,N+2}. \]

This yields that the number of zeros of \( f \) is less than:

\[ (3N^2 + 2)/(3N - 1) < N + 1, \] \(\tag{2.35}\)

as soon as \( N \geq 2 \).

**II.2 The Hironaka polynomial division theorem**

The precise statement of the following theorem was provided by P. Milman. The terminology "Hironaka polynomial division theorem" is also due to him. This technical result allows to clarify the presentation of the "Quantitative Bautin’s theorem" first proved in ([8]).

**Theorem II.2.1**

Let \( I \) be an ideal of \( C[\lambda_1, ..., \lambda_D] \). There is a system of generators \( g_1, ..., g_s \) of the ideal \( I \) and constants \( C \) and \( C_1 \) such that for all elements \( f \) of degree \( k \) of \( I \), there is a decomposition:
\[
f(\lambda) = \sum_{i=1}^{s} \phi_i(\lambda) g_i(\lambda),
\]

with

\[
deg(\phi_i) \leq deg(f) = k,
\]

and

\[
|\phi_i| \leq CC_1^k |f|.
\]

We include here a full proof of this theorem. The non classical part of this division theorem is displayed in the control of the norms (equation 2.37b) of the quotients. The classical proof uses the inversion of an operator defined on Banach spaces of analytic functions but we do not follow these lines here.

**Definition II.2.2**

In the following, a total ordering \( \leq \) on \( N^D \) is said to be compatible with the addition if:

i) For all indices \( \alpha \in N^D, \beta \in N^D \), then \( \alpha \leq \alpha + \beta \), ii) For all indices \( \alpha^1, \alpha^2, \beta \), \( \alpha^1 + \beta \leq \alpha^2 + \beta \) if and only if \( \alpha^1 \leq \alpha^2 \).

Here, we choose (for instance) the total ordering defined in such manner:

\( \alpha \leq \beta \) if and only if \( \alpha_1 + \ldots + \alpha_D = |\alpha| \leq \beta_1 + \ldots + \beta_D = |\beta| \) or if \( |\alpha| = |\beta| \), then \( C(\alpha) \leq C(\beta) \) where \( C(\alpha) = \sum_{i=0}^{D} \alpha_i \epsilon^i \), \( \epsilon \) is a transcendent number, \( 0 \leq \epsilon \leq 1 \). Note firstly that \( \leq \) defines a total ordering compatible with the addition. Indeed, the transcendent nature of \( \epsilon \) yields

\[
C(\alpha) = C(\beta) \iff \alpha = \beta
\]

**Definition II.2.3**

24
Let $f \in C[\lambda]$, $f \neq 0$, $f = \sum f_\alpha \lambda^\alpha$. The largest exponent $\alpha$ so that $f_\alpha \neq 0$ is called the privileged exponent of $f$ and is denoted $exp(f)$. The monomial $[f_\alpha \lambda^\alpha, \alpha = exp(f)]$ is called the initial monomial and denoted $In(f)$.

**Definition II.2.4**

Given $s$ polynomials $g_1, ..., g_s$, the associated partition of $N^D$ is defined as follows:

\[
\Delta_1 = exp(g_1) + N^D, ..., \Delta_i = exp(g_i) + N^D - \bigcup_{j<i} \Delta_j, ..., \bar{\Delta} = N^D - \bigcup_{i=1}^s \Delta_i.
\]

**Definition II.2.5**

Let $I$ be an ideal of $C[\lambda_1, ..., \lambda_D]$. Consider the set:

\[exp(I) = (exp(f), f \in I).\]  \hspace{1cm} (2.39)

It can be shown that this set has finitely many extremal points:

\[\alpha^1, ..., \alpha^s.\]  \hspace{1cm} (2.40)

Choose $g_1, ..., g_s$ in the ideal $I$ so that $exp(g_i) = \alpha^i, i = 1, ..., s$. Such a set of polynomials is called a standard basis, Hironaka basis or Grobner basis (of the ideal $I$ relatively to the ordering $\leq$).

**Proposition II.2.6**

Let $f \in C[\lambda]$, $deg(f) = k$, and $g_1, ..., g_s \in C[\lambda]$, there is a constant $C$ which depends only on the polynomials $g_i$ and there are unique $h_1, ..., h_s, h \in C[\lambda]$ such that:

\[f = h_1 g_1 + ... + h_s g_s + h,\]  \hspace{1cm} (2.41a)

\[h_i = \sum h_{i,\alpha} \lambda^\alpha \Rightarrow \alpha + exp(g_i) \in \Delta_i,\]  \hspace{1cm} (2.41b)

\[h = \sum h_\alpha \lambda^\alpha \Rightarrow \alpha \in \bar{\Delta},\]  \hspace{1cm} (2.41c)

\[Max(|h_i|, |h|) \leq CC_1^k |f|,\]  \hspace{1cm} (2.41d)
\[ Max[\deg(h_i), \deg(h)] \leq k. \] (2.41e)

**Proof:**

Given \( f \), denote \( \text{In}(f) = f_{\alpha_0} \lambda^{\alpha_0} \). If \( \alpha_0 \in \Delta \) set \( h_1 = f_{\alpha_0} \lambda^{\alpha_0} \) and \( h_i^{(1)} = 0, i = 1, \ldots, s, f^{(1)} = f - h^{(1)} \). This yields \( |f^{(1)}| \leq |f|, \deg(f^{(1)}) \leq \deg(f) \) and \( |h^{(1)}| \leq |f| \), \( \deg(h^{(1)}) \leq \deg(f) \). If \( \alpha_0 \in \Delta_i \) (it may belong to several \( \Delta_j \), choose one). Then set:

\[ h_j^{(1)} = 0 \] if \( j \neq i \) and \( h^{(1)} = 0 \). Write \( \alpha_0 = \exp(g_i) + \beta_0, \text{In}(g_i) = \gamma_i \lambda^{\exp(g_i)} \), then denote \( h_i^{(1)} = f_{\alpha_0} \lambda^{\beta_0} / \gamma_i \) and \( f^{(1)} = f - h_i^{(1)} g_i \). This yields:

\[ |h_i^{(1)}| \leq C |f| \] (2.42)

where:

\[ C = \max(1/\gamma_j), j = 1, \ldots, s \] (2.43)

\[ \deg(h_i^{(1)}) = |\beta_0| \leq k - \min(|\exp(g_j)|) \] (2.44)

and thus this displays:

\[ |f^{(1)}| \leq |f| + CG |f| \leq (1 + CG) |f| \] (2.45)

where:

\[ G = \max(|g_i|), i = 1, \ldots, s. \]

Note furthermore that:

\[ \deg(f^{(1)}) \leq \max[\deg(f), \deg(h_i^{(1)} g_i)] \] (2.46)

and

\[ \deg(h_i^{(1)} g_i) = |\beta_0| + |\deg(g_i)| \leq |\beta_0| + |\exp(g_i)| \leq |\alpha_0| \leq k \] (2.47)

and thus

\[ \deg(f^{(1)}) \leq k \] (2.48).

Then, repeat the whole process with \( f^{(1)} \) in place of \( f \). Note that the privileged exponent of \( f^{(1)} \) is strictly less than the privileged exponent of \( f \). The process stops ultimately
when the privileged exponent becomes zero. The choice of the ordering yields that the number of steps involved is less than $k(1 + \epsilon^{1-D})$. This yields the result of the proposition with:

$$C_1 = (1 + CG)^{1+\epsilon^{1-D}}$$  \hspace{1cm} (2.49)

The theorem follows of the preceding proposition and of the property that if a polynomial $f$ belongs to an ideal $I$ and if $g_1,\ldots,g_s$ is a Grobner basis of the ideal $I$, then the division of $f$ yields $h = 0$. (See for instance Lejeune-Jalabert ([10])).

II.3 The main theorem

**Theorem II.3.1**

Let $f_\lambda(x) = \sum_{k\geq 1} f_k(\lambda)x^k$ be an $A_0$-series. For any $\lambda \in C^D$, the function $f_\lambda$ belongs to the Bernstein class $B_{d-1,R,c}$ where $d$ is the Bautin index of the series $f_\lambda(x)$, $R = [(C_1\bar{\lambda})^{K_1}K_4]^{-1}$, $c = [MCK_3(C_1\bar{\lambda})^{K_2}]/R$, if $R \leq 1$, $c = [MCK_3(C_1\bar{\lambda})^{K_2}]/R$, if $R \geq 1$, $\bar{\lambda} = max(1,|\lambda|)$. The constants $K_1,\ldots,K_4$ are those which appear in the definition (I.2.1) of an $A_0$-series. The constants $C$ and $C_1$ appears in the ”Hironaka polynomial division theorem” and $M$ is the norm of the matrix of the change of basis $(f_k)$ to a Grobner basis $(g_l)$ of the Bautin ideal of the series $f_\lambda(x)$.

**Proof**

Write first the condition for $f_\lambda(x)$ to be an $A_0$-series as follows:

$$deg[f_k(\lambda)] \leq K_1k + K_2,$$

$$|f_k| \leq K_3K_4^k.$$  

Denote $I$ the ideal generated by the $d$ first coefficients $f_1(\lambda),\ldots,f_d(\lambda)$ and write:

$$f_k(\lambda) = \sum_{j=1}^{s}[\phi_j^k(\lambda)g_j(\lambda)].$$  \hspace{1cm} (2.50)

27
The Hironaka polynomial division theorem yields:

\[ | \phi^k_j(\lambda) | \leq CC_1^{(K_1k+K_2)} | f_k | \bar{\lambda}(K_1k+K_2) \]  \hspace{1cm} (2.51a)

\[ \leq [CK_3C_1^{K_2}][C_1^{K_1} K_4]^{k} \bar{\lambda}(K_1k+K_2). \]  \hspace{1cm} (2.51b)

The relation:

\[ g_j(\lambda) = \sum_{l=1}^{d} \Phi_{jl}(\lambda) f_l(\lambda), \]  \hspace{1cm} (2.52)

yields:

\[ | f_k(\lambda) | \leq [CK_3C_1^{K_2} \bar{\lambda}^{K_2}][C_1^{K_1} K_4 \bar{\lambda}^{K_1}]^{k} \sum_{j=1}^{s} | g_j(\lambda) |, \]  \hspace{1cm} (2.52)

\[ \leq M[CK_3C_1^{K_2} \bar{\lambda}^{K_2}][C_1^{K_1} K_4 \bar{\lambda}^{K_1}]^{k} \max[| f_1(\lambda) |, ..., | f_d(\lambda) |]. \]  \hspace{1cm} (2.53)

Write now:

\[ R = [(C_1 \bar{\lambda})^{K_1} K_4]^{-1}, \]  \hspace{1cm} (2.54)

the equation (2.53) yields:

\[ | f_k(\lambda) | R^k \leq [MCK_3(C_1 \bar{\lambda})^{K_2} / R^d] \max[| f_1(\lambda) |, ..., | f_d(\lambda) |] R^d. \]  \hspace{1cm} (2.55)

If \( R \leq 1 \), this displays:

\[ | f_k(\lambda) | R^k \leq [MCK_3(C_1 \bar{\lambda})^{K_2} / R^d] \max[| f_1(\lambda) |, R, ..., | f_d(\lambda) |] R^d. \]  \hspace{1cm} (2.56)

If \( R \geq 1 \), the equation (2.53) yields:

\[ | f_k(\lambda) | R^k \leq [MCK_3(C_1 \bar{\lambda})^{K_2} / R] \max[| f_1(\lambda) |, R, ..., | f_d(\lambda) |] R, \]  \hspace{1cm} (2.57)

and thus:

\[ | f_k(\lambda) | R^k \leq [MCK_3(C_1 \bar{\lambda})^{K_2} / R] \max[| f_1(\lambda) |, R, ..., | f_d(\lambda) |] R^d. \]  \hspace{1cm} (2.58)

This concludes the proof.

The theorem II.3.1 and the lemma II.1.5 now implies the following:
Theorem II.3.2

The number of zeros of $f\lambda(x)$ in the disc $\bar{D}_{R''}$ is less than $d - 1$ with

$$R'' = R/[2^{3(d-1)}\max(c, 2)].$$

(2.59)

III. Extension to any dimension of Bautin’s theory

These last years, the dynamics of plane systems was extensively studied and several new techniques were developed. Some are specific to 2-dimensional systems but often these methods can be appropriately extended to multi-dimensional systems. The algorithm of the successive derivatives was derived some years ago ([5]) to find the first non-vanishing derivative (relatively to the parameter $\epsilon$) of the return mapping (near the origin) of a plane vector field $X_0 + \epsilon X_1$ of type:

$$X_0 + \epsilon X_1 = x\partial/\partial y - y\partial/\partial x + \epsilon \sum_{i,j/i+j=2}^{d} [a_{i,j}x^i y^j \partial/\partial x + b_{i,j}x^i y^j \partial/\partial y].$$

(3.1)

The algorithm was then used in the center-focus problem (cf. [6]), which directly relates to Hopf bifurcations of higher order and to several other problems on limit cycles of plane vector fields. How to extend appropriately this situation in any dimension? We have to perturb a dynamics which is integrable and displays only periodic orbits. Assuming that the perturbation depends of finitely many parameters (say for instance it is polynomial), we expect also that the perturbed system displays a first return-mapping which is analytic with a Taylor expansion with coefficients which depend polynomially of the parameters. This return-mapping should label all the periodic orbits (at first return) of the perturbed system by its fixed points. The principal aim of this paragraph is to present a framework where such demands are realized. In this framework, a generalization of the algorithm of the successive derivatives is provided.

III-1 Controlled Nambu dynamics and (*)-property.
Let $f = (f_1, \ldots, f_{n-1}) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a generic submersion (meaning that $f$ is a submersion outside a critical set $f^{-1}(C)$, where $C$ is a set of isolated points). Let $\Omega = dx_1 \wedge dx_2 \wedge \ldots dx_n$ be a volume form on $\mathbb{R}^n$. Consider the vector field $X_0$ such that:

$$\iota_{X_0} \Omega = df_1 \wedge \ldots \wedge df_{n-1}.$$  \hspace{1cm} (3.2)

The functions $f_i, (i = 1, \ldots, n-1)$ are first integrals of the vector field $X_0$:

$$df_i \wedge \iota_{X_0} \Omega = (X_0, f_i) dx_1 \wedge dx_2 \wedge \ldots dx_n = df_i \wedge df_1 \wedge df_2 \wedge \ldots df_{n-1} = 0.$$ \hspace{1cm} (3.3)

This type of dynamics is well-known in Physics and named Nambu’s dynamics ([12]).

For $c$ varying in a neighborhood of 0, assume that the curves $f^{-1}(c)$ have a compact connected component $\gamma_c$. Let $\Sigma$ be a small neighborhood of the zero-section of the normal bundle to $\gamma_0$. For $c$ small enough, the curves $\gamma_c$ are closed periodic orbits of $X_0$ and they cut transversely $\Sigma$. Choose $c$ as a coordinate on the transverse section $\Sigma$ to the flow of $X_0$. Lastly, assume that there are 2-forms $\omega_i$ such that:

$$\iota_{X_0} \omega_i = df_i; \quad i = 1, \ldots, n - 1.$$ \hspace{1cm} (3.4)

There are of course different possible choice of the forms $\omega_i$ and accordingly different possible perturbations have to be considered. If the condition (3.4) is fulfilled, we will say that the singularity of the Nambu dynamics (3.3) is controlled (or alternatively that the Nambu dynamics itself is controlled).

The appropriated extension of the ($\ast$)-property first discussed in ([5]) is presented in the following.

**Definition III.1.1**

Let $f = (f_1, \ldots, f_{n-1}) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a generic submersion. Assume that $f^{-1}(c)$ contains a compact curve $\gamma_c$. The application displays the ($\ast$)-property if for all polynomial 1-forms $\omega$ such that
\[ \int_{\gamma_c} \omega = 0, \]  
\( (3.5) \)

for all \( c \); there exist polynomial \( g_i, R \) such that:

\[ \omega = g_1 df_1 + ... + g_{n-1} df_{n-1} + dR. \]  
\( (3.6) \)

It was proved in ([5]) that the function \( f_1 : R^2 \to R^1, f_1 : (x_1, x_2) \to (x_1^2 + x_2^2) \) displays the \((\ast)\)-property. Several generalizations were proposed after but the core of the argument in the computation of the successive derivatives is captured in this notion. The generalization proposed in this article provides a new presentation of the \((\ast)\)-property which seems interesting as well for the 2-dimensional case. Indeed, the definition of the vector field \( X_0 \) given in the preceding introduction yields the:

**Proposition III.1.2**

Let \( \omega \) be a 1-form such that \( \omega(X_0) = 0 \), then there are functions \( g_1, ..., g_{n-1} \) so that:

\[ \omega = g_1 df_1 + ... + g_{n-1} df_{n-1}. \]  
\( (3.7) \)

Note that the condition \( \omega(X_0) = 0 \), equivalent to \( \omega \wedge df_1 \wedge ... \wedge df_{n-1} = 0 \), yields \( \omega = g_1 df_1 + ... + g_{n-1} df_{n-1} \) where the coefficients \( g_k \) are obtained as ratio of minors of the Jacobian matrix of the \( f_j \).

This displays an alternative to the \((\ast)\)-property now presented as follows:

**Proposition III.1.3**

A generic submersion \( f : R^n \to R^{n-1} \) displays the \((\ast)\)-property if for any polynomial 1-form \( \omega \) such that

\[ \int_{\gamma_c} \omega = 0, \]  
\( (3.5) \)

31
for all $c$; then there exists a polynomial $R$ such that:

$$\omega(X_0) = X_0.R.$$  \hfill (3.8)

Such a function $R$ can be (in principle) constructed with the following pattern. Choose $R$ arbitrarily on the transverse section $\Sigma$, then extend $R$ to the whole tubular neighborhood of $\gamma_0$ saturated by the orbits $\gamma_c$ by integration of the 1-form $\omega$ along the orbits of $X_0$.

**III-2 The successive derivatives of the first return mapping of the perturbed system.**

Now perturb $X_0$ into $X_\epsilon = X_0 + \epsilon X_1$. Let $M$ be a point of $\Sigma$ close to 0 and let $\gamma_\epsilon$ be the trajectory of $X_\epsilon$ passing by the point $M$. The next first intersection point of $\gamma_\epsilon$ with $\Sigma$ defines the so-called first return mapping of $X_\epsilon$ relatively to the transverse section $\Sigma$: $c \mapsto L(c, \epsilon)$. The mapping $L$ is analytic and it displays a Taylor development (in $\epsilon$):

$$L(c, \epsilon) = c + \epsilon L_1(c) + \ldots + \epsilon^k L_k(c) + O(\epsilon)^{k+1}. \hfill (3.9)$$

The expression of the first coefficient $L_1(c)$ is classical and belongs to the lore of bifurcation theory. With the vector field $X_\epsilon$ and the 1-forms $\omega_i$ (cf. [F]), introduce the 1-forms:

$$\iota_{X_\epsilon} \omega_i = \iota_{X_0} \omega_i + \epsilon \iota_{X_1} \omega_i = df_i + \epsilon \iota_{X_1} \omega_i. \hfill (3.10)$$

**Definition III.2.1**

The perturbation $X_\epsilon$ of the controlled Nambu dynamics is said to be admissible if for all the 2-forms $\omega_i$, the 1-forms $\iota_{X_1} \omega_i$ have polynomial coefficients. Note that has said above this displays different admissible perturbations depending of the choice of the forms $\omega_i$. 

32
Recall that the parameter $c$ chosen as coordinates on the transverse section $\Sigma$ is the restriction of the functions $f = (f_1, ..., f_{n-1})$ to the section.

Then the $i^{th}$-component of $L_1(c)$ is equal to:

$$L_{1,i}(c) = \int_{\gamma_0} \iota_{X_1} \omega_i. \quad (3.11)$$

Assume now that the first derivative $L_1(c)$ vanishes identically and that the submersion $f$ displays the (*)-property then there exist $g_{ij}$ and $R_i$ such that:

$$\iota_{X_1} \omega_i = \sum_j g_{ij} \iota df_j + dR_i. \quad (3.12)$$

Following the lines of the algorithm of the successive derivatives, the expression $3.12$ yields:

$$L_{2,i}(c) = -\int_{\gamma_0} \sum_j g_{ij} \iota_{X_1} \omega_j. \quad (3.13)$$

This is indeed the second step of a general recursive scheme which displays as follows:

Assume that all the $k^{th}$-first derivatives of the return mapping of the perturbed vector field vanish identically. This yields:

$$L_{k,i}(c) = \int_{\gamma_0} \sum_j g_{ij}^{k-1} \iota_{X_1} \omega_j = 0. \quad (3.14)$$

The (*)-property yields new functions $g_{ij}^k$, $R^k$ such that:

$$\sum_j g_{ij}^{k-1} \iota_{X_1} \omega_j = \sum_j g_{ij}^k df_j + dR^k. \quad (3.15)$$

This yields the following expression of the $(k+1)^{th}$-derivative of the return mapping of the perturbation:

$$L_{k+1,i}(c) = \int_{\gamma_0} \sum_j g_{ij}^k \iota_{X_1} \omega_j. \quad (3.16)$$
The algorithm implies of course the first

**Theorem III.2.2**

Let $X_0$ be a controlled Nambu dynamics which displays the (*)-property and let $X_1$ be an admissible perturbation. Then the perturbed dynamics $X_\varepsilon$ has an analytic first return map. The coefficients of the Taylor expansion of this return mapping depend polynomially on the coefficients of the perturbation.

From the general theory of projections of analytic sets ([11]), it now follows:

**Theorem III.2.3**

There exists a uniform bound to the number of isolated periodic orbits, which correspond to fixed points of the first return mapping of $X_0 + \varepsilon X_1$ which intersect the transverse section $\Sigma$ in the neighborhood of $0$.

The general framework presented here should of course be illustrated with specific examples (of dimension larger than 2). Some have been worked out recently by Seok Hur (Paris VI) and will be matter to further publications.
REFERENCES

[1] N.N. Bautin: On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. Amer. Math. Soc. Transl. 1, 5, 336-413 (1962). Translated from Mat. Sbornik, 30, 181-196, (1952).

[2] M. Briskin, J.-P. Francoise, Y. Yomdin: The Bautin ideal of the Abel Equation. Nonlinearity 11, 431-443 (1998).

[3] M. Briskin, J.-P. Francoise, Y. Yomdin: Center conditions, composition of polynomials and moments on algebraic curves. Ergod. Th. Dynam. Sys. 19, 1201-1220 (1999).

[4] M. Briskin, Y. Yomdin: Algebraic Families of Analytic Functions. J. Diff. Equations. 136 (2), 248-267 (1997).

[5] J.-P. Francoise: Successive derivatives of a first-return map, application to quadratic vector fields. Erg. Th. Dynam. Sys., 16, 87-96 (1996).

[6] J.-P. Francoise, R. Pons: Computer Algebra Methods and The stability of differential systems. Random and computational dynamics, vol. 3, n4, 265-287 (1995).

[7] J.-P. Francoise, C.C. Pugh: Keeping track of limit cycles. J. Differential Equations 65, 139-157 (1986).

[8] J.-P. Francoise, Y. Yomdin: Bernstein inequality and applications to analytic geometry and differential equations. J. Funct. Analysis 146, 185-205 (1997).

[9] J.-P. Francoise, Y. Yomdin: Projection of analytic sets and Bernstein inequalities. Singularities Symposium-Lojasiewicz 70, Edts B. Jacubczyk, W. Pawlucki, Y. Stasica, Banach Center Publications, Warszawa 44, 103-108 (1998).

[10] M. Lejeune-Jalabert: Effectivité de calculs polynomiaux. Publications de l’Institut Fourier, Université de Grenoble, (1984)

[11] S. Lojasiewicz, R. Tougeron, M. Zurro.: Eclatement des coefficients des series entiéres et deux théorèmes de Gabrielov. Manuscripta Matematica, 92, 325-337, (1997).

[12] Nambu Broken Symmetry. Selected papers of Y. Nambu edited and selected with a foreword by T. Eguchi and K. Nishijima. World scientific series in 20th century physics, 13. World scientific publishing co. inc. (1995)
[13] S. Smale: Dynamics retrospective: great problems, attempts that failed. Physica D, 51, 267-273, (1991).

[14] S. Smale: Mathematical Problems for the Next Century. The Mathematical Intelligencer, 20, 7-15, (1998).

[15] Y. Yomdin: Global finiteness properties of analytic families and algebra of their Taylor coefficients. The Arnol’d Fest (Toronto, Ontario, 1997), 527-555, Fields Inst. Commun. 24, Amer. Math. Soc., Providence, RI, (1999).

[16] H. Zoladek: Quadratic systems with center and their perturbations. J. Differential Equations, 109, 223-273, (1994).