IDENTITIES INVOLVING THE $(h, q)$-GENOCCHI POLYNOMIALS
AND $(h, q)$-ZETA-TYPE FUNCTION

A. BAGDASARYAN, E. ŞEN, Y. HE, S. ARACI, AND M. ACIKGOZ

Abstract. The fundamental objective of this paper is to obtain some interesting properties for $(h, q)$-Genocchi numbers and polynomials by using the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ and mentioned in the paper $q$-Bernstein polynomials. By considering the $q$-Euler zeta function defined by T. Kim, which can also be obtained by applying the Mellin transformation to the generating function of $(h, q)$-Genocchi polynomials, we study $(h, q)$-Zeta-type function. We derive symmetric properties of $(h, q)$-Zeta function and from these properties we give symmetric property of $(h, q)$-Genocchi polynomials.

2010 Mathematics Subject Classification. 11S80, 11B68.

Keywords and phrases. $q$-Genocchi numbers and polynomials, $(h, q)$ Genocchi numbers and polynomials, Kim’s $q$-Bernstein polynomials, Mellin transformation, $(h, q)$-Zeta function, fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$.

1. Preliminaries

The rapid development of $q$-calculus has led to the discoveries of new generalizations of the Bernstein polynomials and Genocchi polynomials involving $q$-integers. The $q$-calculus theory is a novel theory that is based on finite difference re-scaling. Remarkably, the $q$-calculus encompasses many results of eighteenth and nineteenth century mathematics: Euler’s identities for $q$-exponential functions, Gauss’s $q$-binomial formulae, and Heine’s formulae for $q$-hypergeometric functions. Kurt Hensel also invented $p$-adic numbers. In spite of their being already one hundred years old, the $p$-adic numbers are still today enveloped in an aura of mystery within the scientific community (see [1-44]).

In this paper, we also derive some interesting identities of $(h, q)$-Genocchi polynomial by using $p$-adic $q$-integral on $\mathbb{Z}_p$ and Kim’s $q$-Bernstein polynomials. So, first, we list the definition of the following notations that we use in this paper.

Let $p$ be a chosen odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ stand for the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The $p$-adic absolute value in $\mathbb{C}_p$ is defined by

$$|x|_p = p^{-r},$$

where $x = p^r \frac{m}{n} (r \in \mathbb{Q},$ and $m, n \in \mathbb{Z}$ with $(p, m) = (m, n) = (p, n) = 1)$. When one speaks of $q$-extension, $q$ is variously considered as an indeterminate, either a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1 - q|_p < p^{-\frac{1}{p-1}}$ so that $q^r = \exp(x \log q)$ for each $|x|_p \leq 1$.
The following distribution on \( \mathbb{Z}_p \) is defined by Kim as:
for \( q \in \mathbb{C}_p \) with \( |1 - q|_p < 1 \),
\[
\mu_q(x + p^n \mathbb{Z}_p) = (1 + q) \frac{(-q)^x}{(1 + q^p^n)}, \quad \text{(for details, see [30], [31], [32]).}
\]

We say that \( f \) is uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and denote this property by the symbol \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \), if the difference quotients
\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]
have a limit \( l = f'(a) \) as \( (x, y) \to (a, a) \). Thus, for \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \), the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim in [30], [31], [32] as follows:
\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} (-1)^x f(x) q^x.
\]

So that,
\[
\lim_{q \to 1} I_{-q}(f) = I_{-1}(f),
\]
where the notation of \( I_{-1}(f) \) is called the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) (see [3], [5], [6], [7], [9], [10], [14], [23], [29], [30], [31], [32], [33], [35], [36], [38], [39], [40], [41]).

In [33], for \( k, n \in \mathbb{N}^* \) and \( x \in [0,1] \), Kim’s \( q \)-Bernstein polynomials are defined by
\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q [1 - x]^{n-k}_q.
\]

It is obvious that \( \lim_{q \to 1} B_{k,n}(x, q) = B_{k,n}(x) \) which are called the classical Bernstein polynomials cf. [1], [2], [5], [7], [10], [27], [29], [33].

In the theory of \( q \)-calculus for a real parameter \( q \in (0,1) \), \( q \)-analogue of \( x \) is given by

\[
[x]_q = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \ldots + q^{x-1},
\]

\[
[x]_{-q} = \frac{1 - (-q)^x}{1 + q} = 1 + (-q) + q^2 + \ldots + (-1)^{x-1} q^{x-1}.
\]

We want to note that \( \lim_{q \to 1} [x]_q = x \) (see [1-44]). Let us now take \( f(x) = e^{tx} \) in (1.2), then we get
\[
t \int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}
\]
where \( G_n \) are Genocchi numbers. By using (1.4), we have
\[
\int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \sum_{n=0}^{\infty} \left( \frac{G_{n+1}}{n+1} \right) \frac{t^n}{n!}
\]
From the above, we readily see that
\[
\sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{G_{n+1}}{n+1} \right) \frac{t^n}{n!}.
\]
By comparing the coefficients of $\frac{x^n}{n!}$ in the both sides of the above equation, we procure the following:

$$
\frac{G_{n+1}}{n+1} = \int_{\mathbb{Z}_p} x^n \, d\mu_{-1} (x), \text{ cf. } [9, 6, 27, 14].
$$

In [20], the $q$-extension of Genocchi numbers are defined by

$$
G_{0,q} = 0, \quad q \left(qG_q + 1\right)^n + G_{n,q} = \begin{cases} [2]_q, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1, \end{cases}
$$

with the usual convention about replacing $(G_q)^n$ by $G_{n,q}$.

Recently, several mathematicians have studied on the concept of $(h, q)$-Genocchi polynomials and given some new properties about these polynomials cf. [6, 19, 22, 41]. By the same motivation, for $n \in \mathbb{N}^*$, we consider the following $(h, q)$-Genocchi polynomials

$$
G_{n+1,q}^{(h)} (x) = \int_{\mathbb{Z}_p} q^{(h-1)y} [x+y]^n_q \, d\mu_{-q} (y) .
$$

In the special case $x = 0$ in (1.5), we have $G_{n,q}^{(h)} (0) := G_{n,q}^{(h)}$ that are called the $(h, q)$-Genocchi numbers. By (1.5), we derive new relations by using aforementioned $q$-Bernstein polynomials and define their generating function. By applying Mellin transformation to this generating function, we obtain $(h, q)$-analogue of zeta function which interpolates $(h, q)$-Genocchi polynomials at negative integers. Next, we investigate symmetric properties of the $(h, q)$-zeta function. Further, from this investigation, we get symmetric property of $(h, q)$-Genocchi polynomials which we present in the next sections.

2. ON THE PROPERTIES OF THE $(h, q)$-GENOCCHI POLYNOMIALS

By (1.5), we easily get

$$
G_{n+1,q}^{(h)} (x) = \begin{aligned}
(n+1) \int_{\mathbb{Z}_p} q^{(h-1)y} [x+y]^n_q \, d\mu_{-q} (y)
&= (n+1) \frac{[2]_q}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{kx} \frac{(-1)^k}{1 + q^{h+k}} \\
&= [2]_q (n+1) \sum_{l=0}^{\infty} (-1)^l q^{hl} [x+l]^n_q .
\end{aligned}
$$
Further,
\[
\sum_{n=0}^{\infty} G_{n,q}^{(h)}(x) \frac{t^n}{n!} = [2]q t \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[x+n]_q} t^n
\]
\[
= [2]q t \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{([x]_q+q^*[n]_q) t}
\]
\[
= \left( \frac{e^{[x]_q t}}{q^x} \right) \left( [2]q q^x t \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[q^x t]_q} \right)
\]
\[
= \left( \sum_{n=0}^{\infty} [x]_q \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} q^{(n-1)x} G_{n,q}^{(b)} \frac{t^n}{n!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} q^{(k-1)x} [x]_q^{n-k} G_{k,q}^{(h)} \right) \frac{t^n}{n!}.
\]

Therefore, we obtain the following theorem.

**Theorem 1.** For \( n \in \mathbb{N}^* \), we have

\[
\frac{G_{n+1,q}^{(h)}(x)}{n+1} = [2]q \sum_{l=0}^{\infty} (-1)^l q^{hl} [x+l]_q^n.
\]

Moreover,

\[
G_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{(k-1)x} G_{k,q}^{(h)} [x]_q^{n-k} = q^{-x} \left( q^x G_{q}^{(h)} + [x]_q \right)^n
\]

with the usual convention about replacing \( (G_{q}^{(h)})^n \) by \( G_{n,q}^{(h)} \).

By Theorem 1, we attain the following:

\[
\sum_{n=0}^{\infty} G_{n,q}^{(h)}(x) \frac{t^n}{n!} = [2]q t \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[x+n]_q} t^n.
\]

Using (2.2), we are now ready to obtain symmetric property of \((h,q)\)-Genocchi polynomials, as follows:

\[
G_{n+1,q}^{(h)}(1-x) = (n+1) \int_{Z_\rho} q^{-(h-1)y} [1-x+y]_q^n d\mu_{-q-1}(y)
\]
\[
= (n+1) \frac{[2]q^{-1}}{(1-q^{-1})^x} \sum_{k=0}^{n} \binom{n}{k} q^{-k(1-x)} (-1)^k \frac{1}{1+q^{-(h+k)}}
\]
\[
= (-1)^n q^{h+n-1} \frac{(n+1)}{1-q^{-(h+1)}} \sum_{k=0}^{n} \binom{n}{k} q^{kx} (-1)^k \frac{1}{1+q^{h+k}}
\]
\[
= (-1)^n q^{h+n-1} G_{n+1,q}^{(h)}(x).
\]

So, we arrive at the following theorem.

**Theorem 2.** (Symmetric property of \(G_{n,q}^{(h)}(x)\)) Let \( n \in \mathbb{N}^* \), then we have

\[
G_{n+1,q}^{(h)}(1-x) = (-1)^n q^{h+n-1} G_{n+1,q}^{(h)}(x).
\]
Because of (2.2), we note that

\[
q^h \sum_{n=0}^{\infty} G_{n,q}^{(1)} \frac{t^n}{n!} + \sum_{n=0}^{\infty} G_{n,q}^{(h)} \frac{t^n}{n!} = [2]q \cdot t.
\]

By expression of (2.3), we derive the following recurrence formula:

\[
G_{0,q}^{(h)} = 0, \quad q^h G_{n,q}^{(1)} + G_{n,q}^{(h)} = \begin{cases} [2]q, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1. \end{cases}
\]

Thanks to (2.4) and Theorem 1, we have the following theorem.

**Theorem 3.** Let \( n \in \mathbb{N}^* \), then we have

\[
G_{0,q}^{(h)} = 0, \quad q^h \left(q G_{q}^{(h)} + 1\right)^n + G_{n,q}^{(h)} = \begin{cases} [2]q, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1. \end{cases}
\]

with the usual convention about replacing \( \left(G_{q}^{(h)}\right)^n \) by \( G_{n,q}^{(h)} \).

For \( n \in \mathbb{N} \), by Theorem 1 and Theorem 3, we can proceed as follows:

\[
q^2 G_{n,q}^{(h)} (2) = \left(q \left(q G_{q}^{(h)} + 1\right) + 1\right)^n
= \sum_{k=0}^{n} \binom{n}{k} q^k \left(q G_{q}^{(h)} + 1\right)^k
= nq \left(q G_{q}^{(h)} + 1\right)^1 + q^{1-h} \sum_{k=2}^{n} \binom{n}{k} q^k q^{h-1} \left(q G_{q}^{(h)} + 1\right)^k
= nq^{2-h} \left([2]q - \frac{[2]q}{1 + q^h}\right) - q^{1-h} \sum_{k=2}^{n} \binom{n}{k} q^k G_{k,q}^{(h)}
= nq^{2-h} [2]q + q^{2-2h} G_{n,q}^{(h)} , \text{ if } n > 1.
\]

Thus, we discover the following theorem.

**Theorem 4.** Let \( n \in \mathbb{N} \), then we have

\[
G_{n+1,q}^{(h)} (2) = (n + 1) q^{-h} [2]q + q^{-2h} G_{n+1,q}^{(h)}.
\]

With the help of Theorem 2, it is not difficult to see the following:

\[
(n + 1) q^{-h} \int_{\mathbb{Z}_p} q^{(h-1)x} \left[1 - x\right]_{q^{-1}}^n d\mu_{-q} (x)
= q^{n+h-1} (-1)^n (n + 1) \int_{\mathbb{Z}_p} q^{(h-1)x} \left[1 - x\right]_{q^{-1}}^n d\mu_{-q} (x)
= q^{n+h-1} (-1)^n G_{n+1,q}^{(h)} (-1) = C_{n+1,q^{-1}}^{(h)} (2).
\]

Consequently, we state the following theorem.

**Theorem 5.** The following equality holds true:

\[
(n + 1) q^{-h} \int_{\mathbb{Z}_p} q^{(h-1)x} \left[1 - x\right]_{q^{-1}}^n d\mu_{-q} (x) = G_{n+1,q^{-1}}^{(h)} (2).
\]
On account of Theorem 3 and Theorem 5, we derive the following formula:

\[ (n + 1) q^{h-1} \int_{\mathbb{Z}_p} q^{(h-1)x} [1 - x]_{q^{-1}}^n \, d\mu_q(x) = (n + 1) q^{h-1} [2]_q + q^{2h} G^{(h)}_{n+1,q^{-1}}. \]

From (2.6), we see that

\[ (n + 1) q^{h-1} \int_{\mathbb{Z}_p} q^{(h-1)x} [1 - x]_{q^{-1}}^n \, d\mu_q(x) = (n + 1) q^{h-1} [2]_q + q^{2h} G^{(h)}_{n+1,q^{-1}}. \]

So, we deduce the following corollary.

**Corollary 1.** The following equality holds true

\[ \int_{\mathbb{Z}_p} q^{(h-1)x} [1 - x]_{q^{-1}}^n \, d\mu_q(x) = [2]_q + q^{h+1} \frac{G^{(h)}_{n+1,q^{-1}}}{n + 1}. \]

3. **New properties on the \((h, q)\)-Genocchi numbers arising from the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) and \(q\)-Bernstein polynomials**

In this part, we give some interesting relations between the \((h, q)\)-Genocchi numbers and \(q\)-Bernstein polynomials arising from fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\). For \(x \in \mathbb{Z}_p\), we recall the definition of the aforementioned \(q\)-Bernstein polynomials as follows:

\[ B_{k,n}(x, q) = \binom{n}{k} x^k [1 - x]_{q^{-1}}^{n-k}, \text{ where } n, k \in \mathbb{N}^*. \]

By (3.1), Kim’s \(q\)-Bernstein polynomials have the following property:

\[ B_{k,n}(x, q) = B_{n-k,n}(1 - x, q^{-1}) \] (see [33]).

Thus, from Corollary 1, (3.1) and (3.2) we see that

\[ \int_{\mathbb{Z}_p} B_{k,n}(x, q) q^{(h-1)x} \, d\mu_q(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1 - x, q^{-1}) q^{(h-1)x} \, d\mu_q(x) \]

\[ = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} q^{(h-1)x} [1 - x]_{q^{-1}}^{n-l} \, d\mu_q(x) \]

\[ = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_q + q^{h+1} \frac{G^{(h)}_{n-l,q^{-1}}}{n - l + 1} \right). \]

For \(n, k \in \mathbb{N}^* \) with \(n > k\), we compute

\[ \int_{\mathbb{Z}_p} B_{k,n}(x, q) q^{(h-1)x} \, d\mu_q(x) \]

\[ = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_q + q^{h+1} \frac{G^{(h)}_{n+l,q^{-1}}}{n + l + 1} \right) \]

\[ = \left\{ \begin{array}{ll}
[2]_q + q^{h+1} G^{(h)}_{n+1,q^{-1}} & \text{if } k = 0, \\
\binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_q + q^{h+1} \frac{G^{(h)}_{n+l,q^{-1}}}{n + l + 1} \right) & \text{if } k > 0.
\end{array} \right. \]
Let us take the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ for the $q$-Bernstein polynomials of degree $n$ as follows:

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q) q^{(h-1)x} d\mu_q(x) = \binom{n}{k} \int_{\mathbb{Z}_p} [x]^k q^{n-1} q^{(h-1)x} d\mu_q(x)$$

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G^{(h)}_{l+k+1,q}}{l+k+1}.$$

Therefore, by (3.3) and (3.4), we attain the following theorem.

**Theorem 6.** Let $n, k \in \mathbb{N}^*$ with $n > k$. Then we have

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G^{(h)}_{l+k+1,q}}{l+k+1}$$

$$= \begin{cases} [2]_q + \frac{q^{h+1}}{n+1} G^{(h)}_{n+1,q^{-1}} & \text{if } k = 0, \\ \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} [2]_q + \frac{q^{h+1}}{n+1} G^{(h)}_{n+1,q^{-1}} & \text{if } k > 0. \end{cases}$$

Putting $k = 0$ in the above theorem, we procure the following corollary.

**Corollary 2.** The following holds true:

$$\sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{G^{(h)}_{l+1,q}}{l+1} = [2]_q + \frac{q^{h+1}}{n+1} G^{(h)}_{n+1,q^{-1}}.$$

Let $n_1, n_2, k \in \mathbb{N}^*$ with $n_1 + n_2 > 2k$. Then we get

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} d\mu_q(x)$$

$$= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}_p} [1-x]^{n_1+n_2-l} q^{(h-1)x} d\mu_q(x)$$

$$= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} [2]_q + \frac{q^{h+1}}{n_1+n_2-l+1} G^{(h)}_{n_1+n_2-l+1,q^{-1}}$$

$$= \begin{cases} [2]_q + \frac{q^{h+1}}{n_1+n_2+1} G^{(h)}_{n_1+n_2+1,q^{-1}} & \text{if } k = 0, \\ \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} [2]_q + \frac{q^{h+1}}{n_1+n_2-l+1} G^{(h)}_{n_1+n_2-l+1,q^{-1}} & \text{if } k \neq 0. \end{cases}$$

As a result, we state the following theorem.

**Theorem 7.** Let $n_1, n_2, k \in \mathbb{N}^*$ with $n_1 + n_2 > 2k$, then we get

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} d\mu_q(x)$$

$$= \begin{cases} [2]_q + \frac{q^{h+1}}{n_1+n_2+1} G^{(h)}_{n_1+n_2+1,q^{-1}} & \text{if } k = 0, \\ \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} [2]_q + \frac{q^{h+1}}{n_1+n_2-l+1} G^{(h)}_{n_1+n_2-l+1,q^{-1}} & \text{if } k \neq 0. \end{cases}$$
From the binomial theorem, we can derive the following equation.

\[
(3.5) \quad \int_{\mathbb{Z}_p} B_{k,n_1} (x, q) B_{k,n_2} (x, q) q^{(h-1)x} d\mu_q (x)
\]

\[
= \prod_{i=1}^{2} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{2k+l} q^{(h-1)x} d\mu_q (x)
\]

\[
= \prod_{i=1}^{2} \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{G_{l+2k+1,q}^{(h)}}{l+2k+1}.
\]

Thus, by (3.5), we obtain the following theorem.

**Theorem 8.** Let \( n_1, n_2, k \in \mathbb{N}^* \) with \( n_1 + n_2 > 2k \), then we have

\[
\sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{G_{l+2k+1,q}^{(h)}}{l+2k+1} = [2]_q + \frac{q^{h+1}}{n_1+n_2+1,q^{-1}} \text{ if } k = 0,
\]

\[
= \sum_{l=0}^{2k} (-1)^{2k+1} \binom{2k}{l} \left( [2]_q + \frac{q^{h+1}}{n_1+n_2-l+1,q^{-1}} \right) \text{ if } k \neq 0.
\]

Now also, by the same method, substituting \( k = 0 \) in the above theorem, we discover the following corollary.

**Corollary 3.** The following identity

\[
\sum_{l=0}^{n_1+n_2} \binom{n_1+n_2}{l} (-1)^l \frac{G_{l+1,q}^{(h)}}{l+1} = [2]_q + \frac{q^{h+1}}{n_1+n_2+1,q^{-1}}
\]

holds true.

For \( x \in \mathbb{Z}_p \) and \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, ..., n_s, k \in \mathbb{N}^* \) with \( \sum_{l=1}^{s} n_l > sk \). Then we take the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) for the \( q \)-Bernstein polynomials of degree \( n \) as follows:

\[
\int_{\mathbb{Z}_p} B_{k,n_1} (x, q) B_{k,n_2} (x, q) ... B_{k,n_s} (x, q) q^{(h-1)x} d\mu_q (x)
\]

\[
= \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} [x]_q^{sk} [1 - x]_{q^{-1}}^{n_1+n_2+...+n_s-sk} q^{(h-1)x} d\mu_q (x)
\]

\[
= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{s+k} \int_{\mathbb{Z}_p} [1 - x]_{q^{-1}}^{n_1+n_2+...+n_s-sk} q^{(h-1)x} d\mu_q (x)
\]

\[
= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{s+k+l} \times [2]_q + \frac{q^{h+1}}{n_1+n_2+...+n_s-l+1,q^{-1}} \text{ if } k = 0,
\]

\[
= \prod_{i=1}^{s} \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{s+k+l} \times [2]_q + \frac{q^{h+1}}{n_1+n_2+...+n_s-l+1,q^{-1}} \text{ if } k \neq 0.
\]

Consequently, we obtain the following theorem.
Theorem 9. Let \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, \ldots, n_s, k \in \mathbb{N}^* \) with \( \sum_{l=1}^s n_l > sk \). Then we have

\[
\int_{z_p} \left( \prod_{i=1}^s B_{k,n_i}(x) \right) q^{(h-1)x} d\mu_q(x)
\]

\[
= \left\{ \begin{array}{ll}
[2]_q + \frac{q^{k+1}}{n_1+n_2+\ldots+n_s+1} G_{n_1+n_2+\ldots+n_s+1,q}^{(h)} & \text{if } k = 0, \\
\prod_{i=1}^s \left( \sum_{l=0}^{sk} \binom{n_i}{k} \binom{sk}{l} (-1)^{sk+l} \right) \times \left( [2]_q + \frac{q^{k+1}}{n_1+n_2+\ldots+n_s-l+1} G_{n_1+n_2+\ldots+n_s-l+1,q}^{(h)} \right) & \text{if } k \neq 0.
\end{array} \right.
\]

From the definition of \( q \)-Bernstein polynomials and the binomial theorem, we easily see that

\[
\int_{z_p} B_{k,n_1}(x) B_{k,n_2}(x, q) B_{k,n_3}(x, q) \ldots B_{k,n_s}(x, q) q^{(h-1)x} d\mu_q(x)
\]

\[
= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_i+n_s-sk} \left( \sum_{d=1}^s (n_d-k) \right) (-1)^l \int_{z_p} [x]_q^{sk+l} q^{(h-1)x} d\mu_q(x)
\]

\[
= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=1}^{n_i+n_s-sk} \left( \sum_{d=1}^s (n_d-k) \right) (-1)^l \frac{G_{l+sk+1,q}^{(h)}}{l+sk+1}.
\]

Thus, from (3.6), we discover the following theorem.

Theorem 10. Let \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, \ldots, n_s, k \in \mathbb{N}^* \) with \( \sum_{l=1}^s n_l > sk \), then the following identity holds true:

\[
\sum_{l=0}^{n_i+n_s-sk} \binom{n_i+n_s-sk}{l} \left( \sum_{d=1}^s (n_d-k) \right) (-1)^l \frac{G_{l+sk+1,q}^{(h)}}{l+sk+1}
\]

\[
= \left\{ \begin{array}{ll}
[2]_q + \frac{q^{k+1}}{n_1+n_2+\ldots+n_s+1} G_{n_1+n_2+\ldots+n_s+1,q}^{(h)} & \text{if } k = 0, \\
\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \times \left( [2]_q + \frac{q^{k+1}}{n_1+n_2+\ldots+n_s-l+1} G_{n_1+n_2+\ldots+n_s-l+1,q}^{(h)} \right) & \text{if } k \neq 0.
\end{array} \right.
\]

Taking \( k = 0 \) in the above theorem, we deduce the following.

Corollary 4. The identity

\[
\sum_{l=0}^{n_1+\ldots+n_s} \binom{n_1+\ldots+n_s}{l} (-1)^l \frac{G_{l+1,q}^{(h)}}{l+1} = [2]_q + \frac{q^{k+1}}{n_1+n_2+\ldots+n_s+1} G_{n_1+n_2+\ldots+n_s+1,q}^{(h)}
\]

holds true.
4. Further Remarks

In this final part, we consider the $q$-Euler Zeta function in $\mathbb{C}$, which is defined by Kim \[30\]

\begin{equation}
\zeta_q^{(h)}(s, x) = \left[2\right]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mh}}{[m + x]_q^s}
\end{equation}

where $q \in \mathbb{C}$, $h \in \mathbb{N}$ and $\Re(s) > 1$. It is clear that for the special case $h = 0$ and $q \to 1$ in (4.1), it reduces to the ordinary Hurwitz-Euler zeta function. Now, we consider (4.1) in the form

\begin{equation}
\zeta_q^{(h)} \left( s, bx + \frac{bj}{a} \right) = \left[2\right]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mh}}{m + bx + \frac{bj}{a}}[q^s].
\end{equation}

By using some operations to the above identity, that is, for any positive integers $m$ and $b$, there exist unique non-negative integers $k$ and $i$ such that $m = bk + i$ with $0 \leq i \leq b - 1$ and thus for $a \equiv 1(\text{mod } 2)$ and $b \equiv 1(\text{mod } 2)$, we can compute as follows:

\begin{equation}
\zeta_q^{(h)} \left( s, bx + \frac{bj}{a} \right) = \left[a\right]_q^s \left[2\right]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mh}}{ma + abx + bj}[q^s]
= \left[a\right]_q^s \left[2\right]_q \sum_{m=0}^{\infty} \sum_{i=0}^{b-1} \frac{(-1)^{i+mb} q^{i+mb} sh}{[i + mb] a + abx + bj}[q^s]
= \left[a\right]_q^s \left[2\right]_q \sum_{i=0}^{b-1} (-1)^i q^{iah} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{ab (m + x) + ai + bj}[q^s].
\end{equation}

From this, we see that

\begin{equation}
\sum_{j=0}^{a-1} (-1)^j q^{jib} \zeta_q^{(h)} \left( s, bx + \frac{bj}{a} \right) = \left[a\right]_q^s \left[2\right]_q \sum_{j=0}^{a-1} (-1)^j q^{jib} \sum_{i=0}^{b-1} (-1)^i q^{iah} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{ab (m + x) + ai + bj}[q^s].
\end{equation}

Replacing $a$ by $b$ and $j$ by $i$ in (4.2), we derive the following

\begin{equation}
\zeta_q^{(h)} \left( s, ax + \frac{ai}{b} \right) = \left[b\right]_q^s \left[2\right]_q \sum_{j=0}^{a-1} (-1)^j q^{jib} \sum_{i=0}^{b-1} (-1)^i q^{iah} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{ab (m + x) + ai + bj}[q^s].
\end{equation}

By considering the above identity in (4.3), we can easily state the following theorem.

**Theorem 11.** The following holds true

\begin{equation}
\frac{\left[2\right]_q}{\left[a\right]_q} \sum_{i=0}^{a-1} (-1)^i q^{ibh} \zeta_q^{(h)} \left( s, bx + \frac{bj}{a} \right) = \frac{\left[2\right]_q}{\left[b\right]_q} \sum_{i=0}^{b-1} (-1)^i q^{iah} \zeta_q^{(h)} \left( s, ax + \frac{ai}{b} \right).
\end{equation}
Now, setting $b = 1$ in the above theorem, we easily procure the following distribution formula

\[(4.4) \quad \zeta_q^h(s, ax) = \frac{[2]_q}{[2]_q^a} \sum_{i=0}^{a-1} (-1)^i q^{ih} \zeta_q^a(s, x + \frac{i}{a}).\]

Putting $a = 2$ in (4.4), it leads to the following corollary.

**Corollary 5.** The following identity holds true:

\[
\zeta_q^h(s, 2x) = \frac{[2]_q}{[2]_q^a} \left( \zeta_q^{(h)}(s, x) - q^h \zeta_q^{(h)}(s, x + \frac{1}{2}) \right).
\]

By (2.1) and (4.1), we have

\[(4.5) \quad G_n^{(h), q}(x + y) = \frac{[2]_q}{[2]_q^a} \frac{[b]_q}{[b]_q^a} \sum_{i=0}^{b-1} (-1)^i q^{ih} G_{n, q^a}(x + \frac{ai}{b}).\]

On account of (2.2), we develop as follows:

\[(4.6) \quad \sum_{n=0}^{\infty} G_n^{(h), q}(x + y) \frac{t^n}{n!} = \left[2]_q t \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{[x+y+m]q} \right] \frac{t^n}{n!} = \left[2]_q t \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{(q^n t)[x+y+m]q} \right] \frac{t^n}{n!}.
\]

(4.7)

By using Cauchy product in (4.7), we see that (4.7) can be written as

\[\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} q^{(j-1)h} G_{j, q^a}^{(h)}(x) [y]_{q^{a^n}}^{n-j} \right) \frac{t^n}{n!}.
\]

Thus, by comparing the coefficients of $\frac{t^n}{n!}$ in (4.6) and (4.7), we state the following corollary.
Corollary 6. The following equality holds true:

\[ G_{n,q}^{(h)}(x + y) = \sum_{j=0}^{n} \binom{n}{j} q^{j-1} y G_{j,q}^{(h)}(x) [y]_q^{n-j}. \]  

By using (4.8), after some computations, we readily derive the following symmetric relation:

Theorem 13. The following equality holds:

\[ [2]_{q^h} \sum_{i=0}^{m} \binom{m}{i} [a]_{q}^{i-1} [b]_{q}^{m-i} G_{i,q^h}^{(h)}(bx) S_{m-i,q^h}^{(h+i-1)}(a) \]

\[ = [2]_{q^s} \sum_{i=0}^{m} \binom{m}{i} [b]_{q}^{i-1} [a]_{q}^{m-i} G_{i,q^s}^{(h)}(ax) S_{m-i,q^s}^{(h+i-1)}(b) \]

where \( S_{m,q}^{(i)}(a) = \sum_{j=0}^{a-1} (-1)^j q^{ji} [j]_q^m. \)

REFERENCES

[1] M. Acikgoz, S. Araci and I. N. Cangul, A note on the modified q-Bernstein polynomials for functions of several variables and their reflections on q-Volkenborn integral, *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 707–712, 2011.
[2] M. Acikgoz, S. Araci, A study on the integral of the product of several type Bernstein polynomials, *IST Transaction of Applied Mathematics-Modelling and Simulation*, vol.1, no. 1, pp. 10-2010.
[3] S. Araci, M. Acikgoz and E. Şen, On the extended Kim’s p-adic q-deformed fermionic p-adic integrals in the p-adic integer ring, *Journal of Number Theory* 133 (2013) 3348-3361.
[4] S. Araci, E. Şen and M. Acikgoz, Theorems on Genocchi polynomials of higher order arising from Genocchi basis, *Taiwanese Journal of Mathematics* (in press).
[5] S. Araci, D. Erdal and J. J. Seo, A study on the fermionic p-adic q-integral representation on \( \mathbb{Z}_p \) associated with weighted q-Bernstein and q-Genocchi polynomials, *Abstract and Applied Analysis*, Volume 2011, Article ID 649248, 10 pages.
[6] S. Araci, J. J. Seo and D. Erdal, New construction weighted \((h, q)\)-Genocchi numbers and polynomials related to Zeta type function, *Discrete Dynamics in Nature and Society*, Volume 2011, Article ID 487490, 7 pages.
[7] S. Araci, M. Acikgoz and F. Qi, On the q-Genocchi numbers and polynomials with weight zero and their applications, *Nonlinear Functional Analysis and Applications* Vol. 18, No. 2 (2013), pp. 193-203.
[8] S. Araci, M. Acikgoz, F. Qi and H. Jolany, A note on the modified \( q \)-Genocchi numbers and polynomials with weight \((\alpha, \beta)\) and their interpolation function at negative integers, *Fasc. Math Journal* (in press).
[9] S. Araci, M. Acikgoz, K. H. Park and H. Jolany, On the unification of two families of multiple twisted type polynomials by using p-adic q-integral on \( \mathbb{Z}_p \) at \( q = -1 \), *Bulletin of the Malaysian Mathematical Sciences Society* (Article in press).
[10] S. Araci and M. Acikgoz, A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, *Advanced Studies in Contemporary Mathematics* (Kyungshang), vol. 22, no. 3, pp. 399–406, 2012.
[11] A. Bayad and T. Kim, Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials, *Russian Journal of Mathematical Physics*, vol. 18, no. 2, pp. 133–143, 2011.
[12] A. Bagdasaryan, An elementary and real approach to values of the Riemann zeta function, *Phys. Atom. Nucl.* 73, 251–254 (2010).
[13] A. Bagdasaryan, Elementary evaluation of the zeta and related functions: an approach from a new perspective, *AIP, Numerical Analysis and Applied Mathematics*, 1281, 1094–1097 (2010).
[15] E. Cetin, M. Acikgoz, I. N. Cangul, and S. Araci, A note on the \((h, q)\)-Zeta-type function with weight \(\alpha\), *Journal of Inequalities and Applications* 2013, 2013:100.

[16] J. Choi, P. J. Anderson, and H. M. Srivastava, Some \(q\)-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order \(n\), and the multiple Hurwitz zeta function, *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 723–737, 2008.

[17] Y. He and C. Wang, Some formulae of products of the Apostol-Bernoulli and Apostol-Euler Polynomials, *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 927953, 11 pages, 2012.

[18] Y. He, Symmetric identities for Carlitz’s \(q\)-Bernoulli numbers and polynomials, Advances in Difference Equations 2013 (2013), 246. doi:10.1186/1687-1847-2013-246.

[19] L.-C. Jang, K.-W. Hang, Y.-H. Kim, A note on \((h, q)\)-Genocchi polynomials of higher order, *Advances in Difference Equations*, Volume 2010, Article ID 309480, 6 pages.

[20] H. Jolany and H. Sharifi, Some results for the Apostol-Genocchi polynomials of higher order, *Bulletin of Malaysian Mathematical Sciences Society* (in press).

[21] H. Jolany, R. E. Alikelaye and S. S. Mohamad, Some results on the generalization of Bernoulli, Euler and Genocchi polynomials, *Acta Universitatis Apulensis*, No. 27, 2011, pp. 299-306.

[22] N.-S. Jung, H.-Y. Lee, J.-Y. Kang, C.-S. Ryoo, The twisted \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\) and \(q\)-Bernstein polynomials with weight \(\alpha\), *Journal of Inequalities and Applications* 2012, 2012:67.

[23] H.-M. Kim, Some identities on the Bernstein and \(q\)-Genocchi polynomials, *Bull. Korean Math. Soc.* 50 (2013), No. 4, pp. 1289-1296.

[24] D. S. Kim, T. Kim, S.-H. Lee, J.-J. Seo, Symmetric identities for the \(q\)-Euler polynomials *Adv. Studies Theor. Phys.* Vol. 7, 2013, no. 24, 1149-1155.

[25] T. Kim, S. H. Rim, D.V. Dolgy, and S. H. Lee, Some identities of Genocchi polynomials arising from Genocchi basis, *Journal of Inequalities and Applications* 2013, 2013:13.

[26] T. Kim, On the \(q\)-extension of Euler and Genocchi numbers, *J. Math. Anal. Appl.* 326 (2007) 1458-1465.

[27] T. Kim, A note \(q\)-Bernstein polynomials, *Russ. J. Math. Phys.* 18(2011), 41-50.

[28] T. Kim, \(q\)-Volkenborn integration, *Russ. J. Math. Phys.* 9(2002), 288-299.

[29] T. Kim, J. Choi, Y. H. Kim, and L. C. Jang, On \(p\)-adic analogue of \(q\)-Bernstein polynomials and related integrals, *Discrete Dynamics in Nature and Society*, Article ID 179430, 9 pages, doi:10.1155/2010/179430.

[30] T. Kim, \(q\)-Euler numbers and polynomials associated with \(p\)-adic \(q\)-integrals, *J. Nonlinear Math. Phys.*, 14 (2007), no. 1, 15–27.

[31] T. Kim, New approach to \(q\)-Euler polynomials of higher order, *Russ. J. Math. Phys.*, 17 (2010), no. 2, 218–225.

[32] T. Kim, Some identities on the \(q\)-Euler polynomials of higher order and \(q\)-Stirling numbers by the fermionic \(p\)-adic integral on \(\mathbb{Z}_p\), *Russ. J. Math. Phys.*, 16 (2009), no.4, 484–491.

[33] T. Kim, A. Bayad, Y. H. Kim, A Study on the \(p\)-adic \(q\)-integrals representation on \(\mathbb{Z}_p\) associated with the weighted \(h\)-Bernstein and \(q\)-Bernoulli polynomials, *Journal of Inequalities and Applications*, Article ID 513821, 8 pages.

[34] T. Kim, Euler numbers and polynomials associated with zeta functions, *Abstract and Applied Analysis*, Article ID 581582, 11 pages.

[35] T. Kim, A note on \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) associated with \(q\)-Euler numbers, *Advances Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 133–137, 2007.

[36] T. Kim, On the analogs of Euler numbers and polynomials associated with \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) at \(q = -1\), *J. Math. Anal. Appl.* 331 (2) (2007) 779–792.

[37] T. Kim, A Note on the \(q\)-Genocchi Numbers and Polynomials, *Journal of Inequalities and Applications* 2007 (2007) doi:10.1155/2007/71452. Article ID 71452, 8 pages.

[38] T. Kim, A note on \(p\)-adic invariant integral in the rings of \(p\)-adic integers, *Adv. Stud. Contemp. Math.* 13 (1) (2006) 95–99.

[39] T. Kim, Symmetry of power sum polynomials and multivariate fermionic \(p\)-adic invariant integral on \(\mathbb{Z}_p\), *Russ. J. Math. Phys.* 16 (2009), no. 1, 93-96.

[40] T. Kim, Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, *Adv. Stud. Contemp. Math.* 20, 23–28 (2010).

[41] C. S. Ryoo, On the structure of the zeros of \((h, q)\)-Genocchi polynomials, *International Journal of Pure and Applied Mathematics*, Volume 78, No. 2 2012, 263-271.
[42] H. M. Srivastava and A. Pinter, Remarks on some relationships between the Bernoulli and Euler polynomials, *Applied Math. Letter*, 17(2004), 375-380.

[43] H. M. Srivastava, B. Kurt and Y. Simsek, Some families of Genocchi type polynomials and their interpolation functions, *Integral Transforms Special Functions* 23 (2012), 919-938; see also Corrigendum, *Integral Transforms Special Functions* 23 (2012), 939-940.

[44] H. M. Srivastava, Some generalizations and basic (or $q$-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inform. Sci. 5 (2011), 390-444.

Russian Academy of Sciences, Institute for Control Sciences, 65 Profsoyuznaya, 117997 Moscow, RUSSIA

E-mail address: abagdasari@hotmail.com

Department of Mathematics, Faculty of Science and Letters, Namık Kemal University, 59030 Tekirdağ, TURKEY

E-mail address: erdogan.math@gmail.com

Department of Mathematics, Kunming University of Science and Technology, Kunming, Yuannan 650500, People’s Republic of China

E-mail address: hyyhe@yahoo.com.cn

Atatürk Street, 31290 Hatay, TURKEY

E-mail address: mtsrk@hotmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY

E-mail address: acikgoz@gantep.edu.tr