Asymptotic Perron’s Method and Simple Markov Strategies in Stochastic Games and Control

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Abstract

We introduce a modification of Perron’s method, where semi-solutions are considered in a carefully defined asymptotic sense. With this definition, we can show, in a rather elementary way, that in a zero-sum game or a control problem (with or without model uncertainty), the value function over all strategies coincides with the value function over Markov strategies discretized in time. Therefore, there are always discretized Markov \( \varepsilon \)-optimal strategies, (uniform over the mesh of the time grid, and uniform with respect to the bounded initial condition). The asymptotic version of Perron’s method can be used either in a stochastic formulation of asymptotic solutions (as we actually do here), or in an analytic set-up (for example, in order to approximate value functions by discretization).

Keywords: stochastic games, asymptotic Perron’s method, Markov strategies, viscosity solutions

Mathematics Subject Classification (2010): 91A05, 91A15, 49L20, 49L25

1 Introduction

The aim of the paper is to introduce the Asymptotic Perron’s Method, i.e. constructing a solution of the HJB as the supremum of carefully defined asymptotic sub-solutions. Using this method we show, in a rather elementary way, that value functions of zero-sum games/control problems can be (uniformly) approximated by some simple Markov strategies for the weaker player. From this point of view, we can think of the method as an alternative to the shaken coefficients method of Krylov [Kry00] (in the case of only one player, under different technical assumptions), or to the related method of regularization of solutions of HJB’s by Świąch in [Swi96a] and [Swi96b] (for control problems or games in Elliott-Kalton formulation). The method of shaken coefficients has been recently used for games in Elliott-Kalton formulation in [BN].

To the best of our knowledge, this modification of Perron’s method does not appear in the literature. In addition, it seems to apply to more general situations than we consider here, and either using a stochastic formulation or an analytic one (see Remark 3.2). Compared to the method of shaken coefficients of Krylov, or to the regularization of solutions by Świąch, the analytic approximation of the value function/solution of HJB by smooth approximate solutions is replaced by the Perron construction. The careful definition of asymptotic sub-solutions then allows us to prove that such sub-solutions work well with Markov controls, basically obtaining approximately optimal strategies. The arguments display once again the robustness of the Perron construction, combined with viscosity comparison. There is basically a large amount of freedom in choosing the definition of sub-solutions, as long as they lie below the value function. Here, we consider such asymptotic sub-solutions.

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The formulation of the games/control problems we consider is similar to the formulation of discrete-game/control in the seminal work \[KS88b\]. What we do here is to propose a novel method to study such models in stochastic framework. Beyond the particular results obtained here, the (analytic version of the) method seems to also be useful in approximation of viscosity solutions by discretization, in the spirit of Barles-Souganidis \[BS91\]. However, we do not pursue this direction here, and leave it to future work.

Compared to the so called Stochastic Perron Method employed in \[S13b, BS13\], the method we introduce here is quite different. The idea in \[S13b, BS13\] was to use exact sub-solutions in the stochastic sense then do Perron. In order to do so, the flexibility of stopping rules or stopping times was needed, leading to the possibility to complete the analysis only over general feed-back strategies (elementary for the purpose of well posed-ness of the state equation) or general predictable controls. Here, we propose instead to use asymptotic sub-solutions in the Perron construction. The flexibility on this end, allows us to work with a simple deterministic time grid, resulting in approximation over Markov strategies. Obviously, the analytic part of the proof that the sup of sub-solutions is a viscosity super-solution is similar to \[S13b\] or \[BS13\], but those analytic proofs were already similar to the (purely analytic) arguments of Ishii for viscosity Perron \[Ish87\].

In relation to the short note \[S14\], not only that our main results here presents yet a different proof for the equality of value functions obtained there, but it actually proves a stronger statement. We have, however, decided to keep the content of \[S14\] separate, since the main message here is the introduction of Asymptotic Perron’s Method and its relation to simple Markov strategies. This appears to be a genuinely novel method. Combining it with the modeling issues in \[S14\] would obscure the message of the present paper.

2 Set-up and Main Results

We use the model and some definitions from \[S13b, S14\]. Therefore, in order to avoid unnecessary overlap, we present just the model and standing assumptions, an refer to the papers just mentioned for other details.

The stochastic state system:

\[
\begin{aligned}
&dX_t = b(t, X_t, u_t, v_t)dt + \sigma(t, X_t, u_t, v_t)dW_t, \\
&X_s = x \in \mathbb{R}^d.
\end{aligned}
\]

For each \(s\), the problem comes with a fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a fixed filtration \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) satisfying the usual conditions and a fixed Brownian motion \((W_t)_{0 \leq t \leq T}\), with respect to the filtration \(\mathbb{F}\). We suppress the dependence on \(s\) all over the paper. The coefficients \(b : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}^d\) and \(\sigma : [0, T] \times \mathbb{R}^d \times U \times V \to \mathcal{M}^{d \times d}\) satisfy the

Standing assumption:

1. **(C)** \(b, \sigma\) are jointly continuous on \([0, T] \times \mathbb{R}^d \times U \times V\)

2. **(L)** \(b, \sigma\) satisfy a uniform local Lipschitz condition in \(x\), i.e.

\[
|b(t, x, u, v) - b(t, y, u, v)| + |\sigma(t, x, u, v) - \sigma(t, y, u, v)| \leq L(K)|x - y|
\]

\(\forall |x|, |y| \leq K, t \in [0, T], u \in U, v \in V\) for some \(L(K) < \infty\), and

3. **(GL)** \(b, \sigma\) satisfy a global linear growth condition in \(x\)

\[
|b(t, x, u, v)| + |\sigma(t, x, u, v)| \leq C(1 + |x|)
\]

\(\forall |x|, |y| \in \mathbb{R}^d, t \in [0, T], u \in U, v \in V\) for some \(C < \infty\).
Perron’s method is a local method (for differential operators), so we only need local assumptions, except for the global growth which ensures non-explosion of the state equation and comparison for the Isaacs equation (see \[S\]). We denote by \(A(s)\) and \(B(s)\) the collections of elementary feed-back strategies for the \(u\)-player and the \(v\)-player, respectively (see \[S\] [Definition 2.1]). We also denote by \(U(s)\) and \(V(s)\) the set of open-loop (predictable with respect to \(\mathbb{F}\))-controls for the \(u\)-player and the \(v\)-player, respectively. Feed-back strategies are denoted by greek letters \(\alpha, \beta\) and open-loop controls by roman letters \(u, v\). Fixed a strategy \(\alpha \in A(s)\) and either a strategy \(\beta \in B(s)\) or an open-loop control \(v \in V\), the state system has a unique strong solution. This is briefly proved in \[S\] and \[13b\], but the result is rather obvious anyway, and it will be used freely in the paper without further notice. We recall that, the only reason there to restrict feed-back strategies to be elementary, was to have well posed-ness of the state equation.

Now, for a bounded and continuous function \(g : \mathbb{R}^d \to \mathbb{R}\), following we define
\[
\begin{align*}
v(s, x) & \triangleq \sup_{\alpha \in A(s)} \left( \inf_{v \in V(s)} \mathbb{E}[g(X^x_T, x, \alpha, v)] \right) \leq \inf_{\alpha \in A(s)} \sup_{\beta \in B(s)} \left( \mathbb{E}[g(X^x_T, x, \alpha, \beta)] \right).
\end{align*}
\]

We assign the meaning for \(v^\ast\) as the value of a “robust-control problem” (\(v\) parametrizes ”Knightian uncertainty, see \[S\]) and \(V\) is the lower value of a genuine zero-sum game between two symmetric players (see \[S\]). Combining \[13b\] and \[14\] we (already) know that \(v = V\) is the unique bounded and continuous viscosity solution of the lower Isaacs equation
\[
\begin{align*}
\begin{cases}
-v_t - H^\ast(t, x, v_x, v_{xx}) = 0 \\
v(T, \cdot) = g(\cdot).
\end{cases}
\end{align*}
\]

with the notations
\[
H^\ast(t, x, p, M) \triangleq \sup_{u \in U} \inf_{v \in V} L(t, x, u, v, p, M),
\]
\[
L(t, x, u, v, p, M) \triangleq b(t, x, u, v) \cdot p + \frac{1}{2} Tr(\sigma(t, x, u, v)\sigma^T(t, x, u, v)M).
\]

We now define a very special class of elementary strategies, namely, simple Markov strategies. Actually, what we call simple Markov strategies below are called ”positional strategies ” in \[KS88b\], and are used extensively in deterministic games.

**Definition 2.1 (time grids and simple Markov strategies)** Fix \(0 \leq s \leq T\).

1. A time grid for \([s, T]\) is a finite sequence \(\pi\) of \(s = t_0 = t_1 < \cdots < t_n = T\).

2. Fix a time grid \(\pi\) as above. A strategy \(\alpha \in A(s)\) is called simple Markov strategy over \(\pi\) if there exist some functions \(\xi_k : \mathbb{R}^d \to \mathbb{U}, k = 1, \ldots, n\) measurable, such that
\[
\begin{align*}
\alpha(t, y(\cdot)) = \sum_{k=1}^n 1_{\{t_{k-1} < t \leq t_k\}} \xi_k(y(t_{k-1})).
\end{align*}
\]

The set of all simple Markov strategies over \(\pi\) is denoted by \(A^M(s, \pi)\).

3. We now define the set of all simple Markov strategies, over all possible time grids,
\[
A^M(s) \triangleq \bigcup_\pi A^M(s, \pi).
\]

In words, ”simple Markov” means that the player only changes actions over the time grid, and anytime he/she does so, the new control depends on the current position only. Consider the same optimization problems as above, but where the player \(u\) is restricted to using only simple Markov strategies, restricted to a fixed time grid, or not.
Denote by
\[ v^\pi(s, x) \triangleq \sup_{\alpha \in \mathcal{A}^b(s, \pi)} \left( \inf_{v \in \mathcal{V}(s)} \mathbb{E}[g(X_T^{s, x, \alpha, v})] \right) \leq V^\pi(s, x) \triangleq \sup_{\alpha \in \mathcal{A}^M(s, \pi)} \left( \inf_{\beta \in \mathcal{B}(s)} \mathbb{E}[g(X_T^{s, x, \alpha, \beta})] \right) \leq V(s, x), \]
and
\[ v^M(s, x) \triangleq \sup_{\alpha \in \mathcal{A}^M(s)} \left( \inf_{v \in \mathcal{V}(s)} \mathbb{E}[g(X_T^{s, x, \alpha, v})] \right) \leq V^M(s, x) \triangleq \sup_{\alpha \in \mathcal{A}^M(s)} \left( \inf_{\beta \in \mathcal{B}(s)} \mathbb{E}[g(X_T^{s, x, \alpha, \beta})] \right) \leq V(s, x). \]
The main result is that, as expected, the \( u \)-player (in either setting) cannot do better with general feed-back strategies than with simple Markov strategies.

**Theorem 2.2** Under the standing assumptions, we have that \( v^M = v = V^M = V \), the unique continuous viscosity solution of the lower Isaacs equation. In addition
\[ v^\pi, V^\pi \to V, \text{ as } |\pi| \to 0 \]
uniformly on compacts in \([0, T] \times \mathbb{R}^d\).

### 3 Proof: the Asymptotic Perron’s Method

We introduce here the new version of Perron’s method, where sub-solutions of the (lower) Isaacs equations are replaced by asymptotic sub-solutions. In our particular framework, we use asymptotic stochastic sub-solutions (so the Perron Method here is asymptotic in the stochastic sense of \[S13a\] or \[BS13\]). However, we claim that, a similar Asymptotic Perron Method can be designed in the analytic framework (see Remark \[3.2\]), leading to either robust approximation of solutions of PDE’s, or results for games without Isaacs conditions in strong (unlike \[S13a\]) formulation, where strategies are exogenously restricted, similarly to \[BLQ13\]. We leave these lines of research to forthcoming work.

The definition of asymptotic (and stochastic) sub-solutions is different from the definition of stochastic sub-solutions in \[S14\] or \[S13b\], and, consequently, so are the proofs. The analytic part of the proof still resembles Ishii \[Ish87\] and the probabilistic part uses Itô along the smooth test function, but this is where similarities stop. As mentioned, the method we introduce here, since it comes in close relation to Markov strategies, can be viewed as a different alternative to the powerful method of shaken coefficients of Krylov, \[Kry00\] or to the work of Święch \[Swi96a\] and \[Swi96b\]. It only makes sense to perform the Asymptotic construction on one side of the optimization problem (to get nearly optimal Markov controls for the weaker player) as opposed to the two-sided analysis in \[S13b\].

**Definition 3.1 (Asymptotic Stochastic Sub-Solutions)** A function \( w : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is called an asymptotic (stochastic) sub-solution of the (lower) Isaacs equation, if it is bounded, continuous and satisfies \( w(T, \cdot) \leq g(\cdot) \). In addition, there exists a gauge function \( \varphi = \varphi_w : (0, \infty) \to (0, \infty) \), depending on \( w \) such that

1. \( \lim_{\varepsilon \searrow 0} \varphi(\varepsilon) = 0 \),
2. for each \( s \) (and the optimization problem coming with it), for each time \( s \leq r \leq T \), there exists a measurable function \( \xi : \mathbb{R}^d \to U \) such that, for each \( x \), each \( \alpha \in \mathcal{A}(s) \) and \( v \in \mathcal{V}(s) \), if we make the notation \( \alpha[r, \xi] \in \mathcal{A}(s) \), defined by
\[ \alpha[r, \xi](t, y(\cdot)) = 1_{\{s \leq t \leq r\}} \alpha(t, y(\cdot)) + 1_{\{r \leq t \leq T\}} \xi(y(r)), \]
then, for each \( r \leq t \leq T \) we have
\[ w(r, X_r^{s, x, \alpha, v}) = w(r, X_r^{s, x, \alpha[r, \xi], v}) \leq \mathbb{E}[w(t, X_t^{s, x, \alpha[r, \xi], v}) | \mathcal{F}_r] + (t - r) \varphi(t - r) \text{ a.s.} \] (2)

Denote by \( \mathcal{L} \) the set of asymptotic sub-solutions.
Remark 3.2 The definition of asymptotic solutions, tell us, that, for each \( r \) there exists a Markov control at that time, such that, if the control is held constant until later, the state equation plugged inside \( w \) will almost have the sub-martingale property in between \( r \) and any later time (reasonably close), for any choice of open loop controls \( v \). Since \( v \) can change wildly after \( r \), in this framework it is very convenient to consider asymptotic sub-solutions in the stochastic sense, resembling [S13b] or [BS13]. However, if the open loop controls are restricted to not change very soon after \( r \), one could consider asymptotic sub-solutions in analytic formulation. Without pursuing this direction here, in the definition of such a sub-solution, the inequality [2] could be replaced by

\[
 w(r, x_1) \leq \int_{\mathbb{R}^d} w(t, z) p(r, t, x_1, x_2; \xi(x_1), v) dx_2 + (t - r) \varphi(t - r), \forall x_1 \in \mathbb{R}^d, v \in V.
\]

Here, \( p(r, t, x_1, x_2, u, v) dx_2 \) is the transition law from time \( r \) to \( t \) of the process where \( u, v \) are held constant (which is, obviously a Markov process). Such definition, as mentioned above, would work well if controls \( v \) do not change in between \( r \) and \( t \), and would amount to "analytic asymptotic Perron's method", as opposed to the stochastic set-up we follow below.

Compared to [S13b], the next Proposition is not entirely trivial, but not hard either.

**Proposition 3.3** Any \( w \in \mathcal{L} \) satisfies \( w \leq v^M \leq V^M \leq V \).

Proof: Fix \( \varepsilon \) and let \( \delta \) such that \( \varphi(\delta) \leq \varepsilon \). Choose a time partition such that \( t_k - t_{k-1} \leq \delta \). For this particular partition, we construct, recursively, going from time \( t_{k-1} \) to time \( t_k \), some measurable \( \xi_k : \mathbb{R}^d \to U \) satisfying the Definition 3.1. Now, we have, with \( \alpha \) formally defined as in the Definition 3.1 of simple Markov strategies, that, for the simple Markov strategy \( \alpha \) we have constructed,

\[
 w(t_{k-1}, X_{t_{k-1}}^{s,x;\alpha,v}) \leq \mathbb{E}[w(t_k, X_{t_k}^{s,x;\alpha,v}) | F_{t_{k-1}}] + (t_k - t_{k-1}) \varphi(t_k - t_{k-1}) \text{ a.s. } \forall k.
\]

This happens for any \( x \) and any open loop control \( v \). Taking expectations and summing up, we conclude that

\[
 w(s, x) \leq \mathbb{E}[w(T, X_T^{s,x;\alpha,v})] + \varepsilon \times (T - s), \forall v \in \mathcal{V}(s).
\]

Taking the infimum over \( v \), since \( w(T, \cdot) \leq g(\cdot) \), we conclude that, if \( |\pi| \leq \delta \) there exists \( \alpha \in \mathcal{A}^M(s, \pi) \) such that

\[
 w(s, x) \leq \inf_{v \in \mathcal{V}} \mathbb{E}[g(X_T^{s,x;\alpha,v})] + \varepsilon \times (T - s) \leq v^\pi(s, x) + \varepsilon \times (T - s) \forall x \in \mathbb{R}^d.
\]

Letting \( \varepsilon \searrow 0 \), we obtain the conclusion. ◊

The next lemma is rather obvious.

**Lemma 3.4** The set of asymptotic sub-solutions is directed upwards, i.e. \( w_1, w_2 \in \mathcal{L} \) implies \( w_1 \vee w_2 \in \mathcal{L} \).

Proof: the only important thing in the proof is to notice that one can choose the gauge function \( \varphi = \varphi_1 \vee \varphi_2 \) for \( w = w_1 \vee w_2 \). The choice of \( \xi \) is obvious. ◊

**Asymptotic Perron’s Method:** we define

\[
 w^- \triangleq \sup_{w \in \mathcal{L}} w \leq v^M \leq V^M \leq V.
\]

**Proposition 3.5 (Asymptotic Perron)** The function \( w^- \) is an LSC viscosity super-solution of the (lower) Isaacs equation.
Proof: from Proposition 4.1 in [BS12], there exist $\tilde{w}_n \in \mathcal{L}$ such that $w^- = \sup_n \tilde{w}_n$. We define the increasing sequence $w_n = \tilde{w}_1 \cup \cdots \cup \tilde{w}_n \in \mathcal{L} \cap w^-$. 

1. **Interior super-solution property** Let $\psi$ touch $w^-$ strictly below at some $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$. Let us assume, by contradiction, that the viscosity super-solution property fails at $(t_0, x_0)$. This means that

\[ \psi(t_0, x_0) + \sup_u \inf_v L(t_0, x_0, u, v; \psi_t(t_0, x_0), \psi_{xx}(t_0, x_0)) > 0. \]

We can choose a small neighborhood $B(t_0, x_0; \varepsilon) \subset [0, T) \times \mathbb{R}^d$ and some $\hat{u} \in U$ such that, over this neighborhood we have

\[ \psi(t, x) + \inf_u L(t, x, \hat{u}, v; \psi_t(t, x), \psi_{xx}(t, x)) > \varepsilon. \]

From here, we follow the usual Perron construction. The very different part will be to show that, after we “bump up” (an approximation of) $w^-$, it still stays an asymptotic sub-solution. More precisely, we know that, since $\psi$ touches $w^-$ below in a strict sense, there exists room of size $\eta > 0$ in between $w^-$ and $\psi$ over the compact (rectangular) torus $T \triangleq B(t_0, x_0, \varepsilon) - B(t_0, x_0, \varepsilon/2)$, i.e. $w^- \geq \psi + \eta$ on $T$. A Dini type argument (see, for example, [BS14]) shows that, one of the terms $w \triangleq w_n$ actually satisfies $w \geq \psi + \eta/2$. Define now, for $\gamma << \eta/2$ the function

\[ \hat{v} = \begin{cases} w \vee (\psi + \gamma), & \text{on } B(t_0, x_0; \varepsilon) \\ w, & \text{outside } B(t_0, x_0; \varepsilon) \end{cases} \]

Note that $\hat{v} = w$ on the overlapping $T$ (so, it is continuous) and $\hat{v}(t_0, x_0) = w^-(t_0, x_0) + \gamma > w^-(t_0, x_0)$. The proof would be finished if we can show that $\hat{v}$ is an asymptotic sub-solution.

**The idea of the proof** is quite simple, namely:

1. if, at time $r$, we have $w \geq \psi + \gamma$ (at that particular position $y(r)$), then, we follow from $r$ forward the nearly optimal strategy corresponding to the asymptotic sub-solution $w$ (which depends only on $y(r)$)

2. if, at time $r$, we have instead $w < \psi + \gamma$ (again, at that particular position $y(r)$) we follow from $r$ forward the strategy $\hat{u}$. In between $r$ and any later time $t$, the process $\psi + \gamma$ superposed to the state equation is not a true sub-martingale, but is an asymptotic one. The reason is that, it is a sub-martingale until the first time it exits $B(t_0, x_0; \varepsilon)$. However, the chance that this happens before $t$ can be estimated in terms of the size of the interval $t - r$, and bounded above by a gauge function.

We develop rigorously below the arguments described above. Fix $s \leq r \leq T$. Since $w$ is an asymptotic sub-solution, there exists a Markov strategy $\xi$ at time $r$ corresponding for the Definition of sub-solution 3.1 (for the initial time $s$). Now, we define

\[ \hat{\xi}(x) = 1_{\{r, x \notin B(t_0, x_0; \varepsilon/2) \vee w(r, x) \geq \psi(r, x) + \gamma/2\}} \xi(x) + 1_{\{r, x \in B(t_0, x_0; \varepsilon/2) \wedge w(r, x) < \psi(r, x) + \gamma/2\}} \hat{u}. \]

We want to show that $\hat{\xi}$ satisfies the property in the definition of the sub-solution $\hat{u}$ at $r$, with an appropriate choice of the gauge function $\varphi$ independent of $r$ or $s$. Let $\varphi_w$ be the gauge function of the sub-solution $w$. Consider any $\alpha \in A(s)$ and any $v \in V(s)$. By the definition of the subsolution $w$, and, taking into account that $w \leq \hat{v}$ we have that, on the event

\[ A \triangleq \{(r, X^s_{r, x^0, v}) \notin B(t_0, x_0; \varepsilon/2) \vee w(r, X^s_{r, x^0, v}) \geq \psi(r, X^s_{r, x^0, v}) + \gamma/2\} \in \mathcal{F}_r \]
we have that $X^{s,x;[r,\xi],v}_t = X^{s,x;[r,\xi],v}_t$, and, therefore

$$1_A \psi(r, X^{s,x;[r,\xi],v}_r) = 1_A \psi(r, X^{s,x;[r,\xi],v}_r) \leq E[1_A \psi(t, X^{s,x;[r,\xi],v}_t|F_r] + 1_A \times (t - r) \varphi(t - r) \leq E[1_A \psi(t, X^{s,x;[r,\xi],v}_t|F_r] + 1_A \times (t - r) \varphi_w(t - r).$$

(3)

On the complement of $A$, the process $\psi(t, X^{s,x;[r,\xi],v}_t)$ is a sub-martingale (by Itô) up to the first time $\tau$ where the process gets out of $B(t_0, x_0; \varepsilon)$, i.e.

$$\tau \triangleq \inf\{t \geq r|\{t, X^{s,x;[r,\xi],v}_t \notin B(t_0, x_0; \varepsilon)\}.\]

The sub-martingale property says that

$$1_A \psi(r + \gamma)(r, X^{s,x;[r,\xi],v}_r) \leq E[1_A \psi(r + \gamma)(\tau \wedge t, X^{s,x;[r,\xi],v}_\tau)|F_r] \leq E[1_A \psi(r \wedge t, X^{s,x;[r,\xi],v}_\tau)|F_r].$$

Fix $t \geq r$. Denote now the event

$$B \triangleq \{|X^{s,x;[r,\xi],v}_t| < \varepsilon, \forall r \leq t' \leq t\}.$$

We use here the norm $|(t, x)| \triangleq \max\{|t|, |x|\}$. Consequently, we have

$$E[1_A \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r] = E[1_A 1_B \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r] + E[1_A 1_B \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r].$$

Therefore,

$$E[1_A \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r] = E[1_A \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r] + E[1_A \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r].$$

Since $\psi$ is bounded by some constant $\|\psi\|_{\infty}$, we conclude that

$$1_A \psi(r, X^{s,x;[r,\xi],v}_r) = 1_A \psi(r, X^{s,x;[r,\xi],v}_r) \leq E[1_A \psi(r \wedge t, X^{s,x;[r,\xi],v}_r)|F_r] + 2\|\psi\|_{\infty} P[A \cap B|F_r], a.s.$$

(4)

We can now put together (3) and (4). If we can find a gauge function $\tilde{\varphi}$ such that

$$2\|\psi\|_{\infty} P[A \cap B|F_r] \leq 1_A \tilde{\varphi}(t - r) \times (t - r), a.s.$$

we are done, as one can choose the gauge function for $\tilde{\psi}$ as

$$\tilde{\psi} \triangleq \varphi_w \vee \tilde{\varphi}.$$

We do that in the Lemma 3.6 below, and finish the proof of the interior sub-solution property.

**Lemma 3.6** There exist constants $C, C'$ (depending only on the function $v$) such that, for any $s$ and any $r \geq s$, if $t \geq r$ is close enough to $r$ we have

$$P[A \cap B|F_r] = 1_A P[A \cap B|F_r] \leq 1_A C P\left(\frac{1}{C t - r} \leq \epsilon\right), a.s.,$$

independently over all strategies $\alpha \in A(s)$ and controls $v \in \mathcal{V}$. The function

$$\tilde{\varphi}(t) \triangleq \frac{2\|v\|_{\infty} C'}{t} P\left(\frac{1}{C t - r} \leq \epsilon\right)$$

satisfies $\lim_{t \to 0} \tilde{\varphi}(t) = 0$ and therefore is a gauge function.
Proof: To begin with, we emphasize that we do not need such a precise bound on conditional probabilities, to finish the proof of Theorem 2.2 (both the interior super-solution part or the terminal condition). The simple idea of the proof is to see that, conditioned on \( A^c \), the event we care about amounts to a continuous semi-martingale with bounded volatility and bounded drift to exit from a fixed box in the interval of time \([r, t]\). If the size of \( t - r \) is small enough, that amounts to just the martingale part exiting of a smaller fixed box in between \( t \) and \( r \). This can be rephrased, through a time change, in terms of a Brownian motion, and estimated very precisely to be of the order \( \mathbb{P} \left( N(0, 1) \geq \frac{1}{c \sqrt{t - r}} \right) = o(t - r) \), where \( N(0, a^2) \) is a normal with mean zero and standard deviation \( a \). This is basically the whole proof, in words. The precise mathematics below follows exactly these lines. We first notice that

\[
A^c \cap B^c \subset \{ (r, X_{r}^{s,x;\alpha,v}) \in B(t_0, x_0; \varepsilon/2) \text{ and } X_{t'}^{s,x;\alpha,v} \notin B(t_0, x_0; \varepsilon), \text{ for some } r \leq t' \leq t \}.
\]

If \( t - r < \varepsilon/2 \) we have

\[
A^c \cap B^c \subset \{ (r, X_{r}^{s,x;\alpha,v}) \in B(t_0, x_0; \varepsilon/2) \text{ and } |X_{t'}^{s,x;\alpha,v} - X_{r}^{s,x;\alpha,v}| \geq \varepsilon/2, \text{ for some } r \leq t' \leq t \}.
\]

Over \( B(t_0, x_0; \varepsilon) \) both the drift and the volatility of the state system are uniformly bounded by some constant \( C \). Therefore, if we choose

\[
t - r \leq \frac{\varepsilon}{4C}
\]

the integral of the drift part cannot exceed \( \varepsilon/4 \) in size. We, therefore, conclude that, with the notation

\[
D \triangleq \{ (r, X_{r}^{s,x;\alpha,v}) \in B(t_0, x_0; \varepsilon/2) \},
\]

if \( t - r \leq \frac{\varepsilon}{2} \land \frac{\varepsilon}{4C} \), then

\[
A^c \cap B^c \subset D \cap \left\{ \left| \int_{r}^{t'} \sigma(q, X_{q}^{s,x;\alpha,v}, \alpha[r, \hat{\xi}](q, X_{t'}^{s,x;\alpha,v}), v_{q})dW_{q} \right| \geq \varepsilon/4 \text{ some } r \leq t' \leq t \right\}.
\]

Denote by

\[
M_{t'} \triangleq \int_{s}^{t'} \sigma(q, X_{q}^{s,x;\alpha,v}, \alpha[r, \hat{\xi}](q, X_{t'}^{s,x;\alpha,v}), v_{q})dW_{q}, \forall \ s \leq t' \leq T.
\]

We study separately the coordinates. More precisely we consider on \( \mathbb{R}^{d} \) the max norm as well, and, for \( M_{t'} = (M_{t'}^{1}, \ldots, M_{t'}^{d}) \), we also have

\[
A^c \cap B^c \subset \bigcup_{l=1}^{d} \left( D \cap \{ \rho_{l} \leq t \} \right),
\]

where

\[
\rho_{l} \triangleq \inf \{ r \leq t' \leq \tau \mid |M_{t'}^{l} - M_{r}^{l}| \geq \varepsilon/4 \}.
\]

We estimate the probabilities above (conditioned on \( F_{r} \)) individually. We want to use Dambis-Dubins-Schwarz. In order to do so rigorously, and without enlarging the probability space to accommodate an additional Brownian motion, we first need to make the volatility explode at \( T \). Choose any function

\[
f \rightarrow [s, T) \rightarrow (0, \infty), \quad \int_{s}^{t} f^{2}(q)dq < \infty \quad \forall q < T, \quad \int_{s}^{T} f^{2}(q)dq = \infty.
\]

Fix some \( \varepsilon' > 0 \) and define

\[
M_{t'}^{x} \triangleq \int_{0}^{t'} \left( 1_{\{0 \leq q \leq T - \varepsilon'\}} \sigma(q, X_{q}^{s,x;\alpha,v}, \alpha[r, \hat{\xi}](q, X_{t'}^{s,x;\alpha,v}), v_{q}) + 1_{\{T - \varepsilon' < q \leq T\}} f(q) \right) dW_{q}, \forall s \leq t' < T.
\]
If \( t_0 + \varepsilon < T - \varepsilon' \), then \( \tau \leq T - \varepsilon' \). In this case, we have

\[
\rho_l = \inf \{ r \leq t' \leq \tau \| M_{t'} - M_r \| \geq \varepsilon/4 \}.
\]

According to Dumbis-Dubins-Schwarz (see [KSS88a], page 174) applied with the obvious shift of time origin from \( r \) to zero, the process \((Z_{s'})_{0 \leq s' < \infty} \) defined as

\[
Z_{s'} = M_{s'} - M_r, \quad 0 \leq s' < \infty, \quad \text{for} \ A_{s'} = \inf \{ t' \geq r \mid \langle M_{s'} \rangle_{t'} - \langle M_r \rangle_r > s' \}.
\]

is a Brownian motion with respect to the filtration \( \mathcal{G}_{s'} \triangleq \mathcal{F}_{s'}, \quad 0 \leq s' < \infty \). Therefore, since \( \mathcal{F}_r \subset \mathcal{G}_0, Z \) is independent of \( \mathcal{F}_r \) and

\[
M_{t'} - M_r = Z_{\langle M_{s'} \rangle_{t'} - \langle M_r \rangle_r}, \quad r \leq t' < T.
\]

On the other hand, on \( D \) we have that since \( \sigma \) is uniformly bounded (independently on the strategy and the control) on \( D \) and before \( \tau \) (the exit time from \( B(t_0, x_0; \varepsilon) \)) so

\[
\langle M_{s'} \rangle_{t'} - \langle M_r \rangle_r \leq C^2(t' - r), \quad r \leq t' \leq \tau.
\]

We then conclude that

\[
D \cap \{ \rho_l \leq t \} \subset \left\{ \max_{0 \leq s' \leq C^2(t-r)} |Z_{s'}| \geq \varepsilon/4 \right\}.
\]

Let \( M^+_{\tau} \) and \( M^-_{\tau} \) the distributions of the running max and running min of a standard BM starting at time zero. Since the Brownian motion \( Z \) is independent of \( \mathcal{F}_r \), we can obtain that

\[
\mathbb{P}(D \cap \{ \rho_l \leq t \} | \mathcal{F}_r) \leq \mathbb{P}(M^+_{\mathcal{C}^2(t-r)} \geq \varepsilon/4) + \mathbb{P}(M^-_{\mathcal{C}^2(t-r)} \geq \varepsilon/4) = 2 \mathbb{P}(M^+_{\mathcal{C}^2(t-r)} \geq \varepsilon/4).
\]

It is well known, from the reflection principle, (see, for example, Karatzas and Shreve [KSS88a], page 80) that this is the same as

\[
\mathbb{P}(D \cap \{ \rho_l \leq t \} | \mathcal{F}_r) \leq 4 \mathbb{P}(N(0, C^2(t-r)) \geq \varepsilon/4) = 4 \mathbb{P} \left( N(0, 1) \geq \frac{1}{C \sqrt{t-r}} \epsilon \right).
\]

We emphasize that all estimates are independent of: the initial time \( s \), the later time \( r \) and the even later time \( t \) as long as \( t - r \leq \frac{\varepsilon}{2} \wedge \frac{\varepsilon}{4C} \), independent of \( \alpha \in \mathcal{A}(s) \) and of \( v \in \mathcal{V}(s) \). Summing all the terms, we obtain

\[
\mathbb{P}[A^c \cap B^c | \mathcal{F}_r] \leq 4d \mathbb{P} \left( N(0, 1) \geq \frac{1}{C \sqrt{t-r}} \varepsilon \right).
\]

We can multiply now with \( 1_{A^c} \) which is measurable with respect to \( \mathcal{F}_r \). It is well known that

\[
\mathbb{P} \left( N(0, 1) \geq \frac{1}{C \sqrt{t-r}} \varepsilon \right) \mathcal{F}_r \rightarrow 0, \quad \text{as} \quad t \searrow 0,
\]

and this finishes the proof of the lemma. Instead of appealing to Dambis-Dubins-Schwarz theorem, we could also use the (smooth solution) characterizing the exit probably of a Brownian motion from a box, and super-POSE it to the state process, resulting in a super-martingale (just using Itô formula). This would still bound the probability we are interested in by the exit probability of a standard Brownian motion from a box.

Proof of Proposition 3.3 continued:

2. **The terminal condition** \( w^-(T, \cdot) \geq g(\cdot) \) The proof of this is done again, by contradiction. The "bump-up" analytic construction we use is similar to [BS13], and the rest is based on similar arguments to the interior super-solution property and a very similar estimate to Lemma 3.6 above.
The only difference, if the Dambis-Dubins-Schwarz route is followed, is to see that, with all notations as above, if \( r \leq t < T \), then

\[
\{ \rho_t \leq t \} = \lim_{\varepsilon \searrow 0} \{ \rho_t \leq t \wedge (T - \varepsilon) \} = \lim_{\varepsilon \searrow 0} \{ |M^{\varepsilon}_t - M^r_t| \geq \varepsilon/4 \text{ some } r \leq t' \leq \tau \wedge t \wedge (T - \varepsilon') \}.
\]

We can first let \( \varepsilon' \searrow 0 \) to obtain the conclusion, that (with the same notation \( \hat{v} \) for the "bump-up" function), with the gauge function we just chose in the proof of the Lemma above we have

\[
\hat{v}(r, X^{s,x;\alpha,v}_t) \leq \mathbb{E}[\hat{v}(t, X^{s,x;\alpha[r,\hat{\xi}],v}_t)|\mathcal{F}_t] + (t - r)\varphi(t - r) \ a.s.
\]

for all \( t \in [r, T) \) close enough to \( r \), i.e. \( t - r \leq \frac{\varepsilon}{4C} \). Obviously, letting \( t \to T \) we obtain the same for \( t = T \). The proof is now complete. \( \diamond \)

Proof of Theorem \[2.2 \] According to \[S13b\], \( V \) is the unique bounded continuous viscosity solution of the Isaacs equation. Since \( w^- \) is a LSC viscosity super-solution, the comparison result in \[S13b\] ensures that, actually, \( w^- = V^M = V \). In order to prove the second part of Theorem \[2.2 \] we note that we have constructed \( L \supseteq w^m \not
 V \). By continuity and the Dini’s criterion, the above convergence is uniform on compacts. This means that, for each \( \varepsilon \) there exists \( w \) (one of the terms of the sequence) such that

\[
V - \varepsilon \leq w, \text{ on } C = [0, T] \times \{|x| \leq N\}.
\]

Let \( \varphi \) be the gauge function of this particular \( w \), and let \( \delta \) such that \( \varphi(\delta) \leq \varepsilon \). According to the proof of Proposition \[3.3 \] if \( |\pi| \leq \delta \), there exists \( \hat{\alpha} \in \mathcal{A}^M(s, \pi) \) such that

\[
w(s, x) \leq \inf_{v \in \mathcal{V}} \mathbb{E}[\varrho(X^{s,x;\hat{\alpha},v}_T)] + \varepsilon \times (T - s) \ \forall x.
\]

This implies that

\[
V(s, x) - \varepsilon \times (1 + (T - s)) \leq \inf_{v \in \mathcal{V}} \mathbb{E}[\varrho(X^{s,x;\hat{\alpha},v}_T)], \ \forall |x| \leq N.
\]

Not only that the approximation is uniform on \( C \), but, for fixed time \( s \), the uniform approximation can be realized over the same simple Markov strategy \( \hat{\alpha} \in \mathcal{A}^M(s, \pi) \), for \( |\pi| \leq \delta \). \( \diamond \)

In case the state system actually only depends on \( u \) and not on \( v \) (i.e. we have a control problem rather than a game), then, with the obvious observation that

\[
v^M(s, x) \leq V(s, x) \leq \sup_{u \in \mathcal{U}(s)} \mathbb{E}[\varrho(X^{s,x;u}_T)],
\]

one can use our result about games to conclude that, in a control problem (one-player) like in \[BS13 \] (but under our stronger standing assumptions here), the value functions over open-loop controls, elementary feed-back strategies and simple Markov strategies coincide. In addition, the approximation with simple Markov strategies is uniform over the mesh of the grid, uniform on compacts. We remind the reader that, in \[BS13 \], the value function studied was defined over open-loop controls, i.e.

\[
V_{\delta}(s, x) \triangleq \sup_{u \in \mathcal{U}(s)} \mathbb{E}[\varrho(X^{s,x;u}_T)].
\]

In the case of games, \textbf{if the Isaacs condition is satisfied}, we know from \[S13b \] that the game has a value. Applying the new asymptotic Perron method to both players (on both sides) we obtain that, for each \( \varepsilon \), there exist \( \varepsilon \)-saddle point \textbf{within the class of simple Markov strategies, uniformly in bounded} \( x \), which means

\[
(\alpha(\varepsilon), \beta(\varepsilon)) \in \mathcal{U}^M(s) \times \mathcal{V}^M(s)
\]

such that

\[
\mathbb{E}[\varrho(X^{s,x;\alpha(\varepsilon),\beta(\varepsilon)}_T)] - \varepsilon \leq \mathbb{E}[\varrho(X^{s,x;\alpha(\varepsilon),\beta(\varepsilon)}_T)] \leq \mathbb{E}[\varrho(X^{s,x;\alpha(\varepsilon),\beta}_T)] + \varepsilon \ (\forall) \ (\alpha, \beta) \in \mathcal{U}(s) \times \mathcal{V}(s), |x| \leq N.
\]
If the Isaacs condition fails, we can still model the game, in a martingale formulation, as in [S13a], and a value over feed-back mixed strategies does exist. Using again the Asymptotic Perron’s method, for both players, we can obtain the existence of $\varepsilon$-saddle point within the class of mixed strategies of simple Markov type, uniformly in bounded $x$. A mixed strategy $\mu$ of simple Markov type (for the player $u$) is defined by a time grid $\pi$ and some functions $\xi_k : \mathbb{R}^d \to \mathcal{P}(U), k = 1, \ldots, n$ measurable, such that

$$
\mu(t, y(\cdot)) = \sum_{k=1}^n 1_{\{t_{k-1} < t \leq t_k\}} \xi_k(y(t_{k-1})) \in \mathcal{P}(U).
$$

In other words, player $u$ decides at time $t_{k-1}$ based only on the position at that time, what distribution he/she will be sampling continuously from until $t_k$. Obviously, one can define similarly mixed strategies of Markov type for the $v$-player. In order to do the analysis and obtain the approximate mixed Markov saddle strategies, one would have to go inside the short proofs in [S13a] and apply Asymptotic Perron’s Method for the auxiliary (and strongly defined) games in the proofs there. In other words, the above paragraph for games over pure strategies satisfying Isaacs condition applies to the auxiliary game in [S13a], leading to $\varepsilon$-saddle points in the class of mixed strategies of Markov type for the original game.

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