Turning Big Bang into Big Bounce: I. Classical Dynamics

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Abstract

The big bounce (BB) transition within a flat Friedmann-Robertson-Walker model is analyzed in the setting of loop geometry underlying the loop cosmology. We solve the constraint of the theory at the classical level to identify physical phase space and find the Lie algebra of the Dirac observables. We express energy density of matter and geometrical functions in terms of the observables. It is the modification of classical theory by the loop geometry that is responsible for BB. The classical energy scale specific to BB depends on a parameter that should be fixed either by cosmological data or determined theoretically at quantum level, otherwise the energy scale stays unknown.

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I. INTRODUCTION

It is commonly believed that the cosmological singularity problem \([1, 2, 3]\) may be resolved in a theory which unifies gravity and quantum physics. Recent analyses done within the loop quantum cosmology (LQC) concerning homogeneous isotropic universes of the Friedmann-Robertson-Walker (FRW) type, strongly suggest that the evolution of these universes does not suffer from the classical singularity. Strong quantum effects at the Planck scale cause that classical big bang is replaced by quantum big bounce (BB) \([4, 5, 6, 7, 8]\).

The resolution of the cosmic singularity problem offered by LQC requires the existence of a fundamental length, which effectively implies the discreteness of quantum geometry. However, the size of this length has not been determined satisfactory yet, i.e. derived within LQC. Presently, it is an ad-hoc assumption of standard LQC \([9, 10]\).

Our paper is an extended version of the classical part of \([11]\). Its goal is the demonstration that the resolution of the initial cosmological singularity is due to the modification of the classical theory by the loop geometry. Quantum effects seem to be of a secondary importance, but should be examined since the energy scale specific to BB has not been identified yet \([8, 11]\). The big bounce may occur deeply inside the Planck scale where it is commonly expected that quantum effects cannot be ignored.

The difference between standard LQC \([14, 15]\) and our nonstandard LQC is the following: (i) we determine an algebra of observables on the kinematical phase space; (ii) we solve the Hamiltonian constraints at the classical level to identify physical phase space (i.e. the space of Dirac’s observables); (iii) we express functions on the physical phase space, like the matter density and geometrical operators (length, area, volume), in terms of Dirac’s observables and an evolution parameter; and (iv) by quantization we mean: finding a self-adjoint representation of the algebra of the Dirac observables and solution to the eigenvalue problem for operators corresponding to functions specified in (iii). Roughly speaking, in standard LQC one does not identify an algebra of physical observables and one imposes the Hamiltonian constraints only at the quantum level.

The present paper concerns the classical level so we are mainly concern with items (i)-(iii). Our next paper \([16]\) will be an extended version of the quantum part of \([11]\), i.e. it will be devoted to the realization of item (iv).

For simplicity of exposition we restrict ourselves to the flat FRW model with massless scalar field. This model of the universe includes the initial cosmological singularity and has been intensively studied recently within LQC.

In order to have our paper self-contained, we recall in Sec. II the form of classical Hamiltonian in terms of holonomy-flux variables. In Sec. III we solve the constraint and analyze the relative dynamics in the physical phase space. Section IV is devoted to the algebra of observables. We consider the energy density of the scalar field and geometrical operators in Sec. V. Higher order holonomy corrections are shortly discussed in the appendix. We conclude in the last section.
II. HAMILTONIAN

The gravitational part of the classical Hamiltonian, $H_g$, in general relativity is a linear combination of the first-class constraints, and reads

$$H_g := \int_\Sigma d^3x (N^i C_i + N^a C_a + NC),$$

(1)

where $\Sigma$ is the spacelike part of spacetime $\mathbb{R} \times \Sigma$, $(N^i, N^a, N)$ denote Lagrange multipliers, $(C_i, C_a, C)$ are the Gauss, diffeomorphism and scalar constraint functions. In our notation $(a, b = 1, 2, 3)$ are spatial and $(i, j, k = 1, 2, 3)$ internal $SU(2)$ indices. The constraints must satisfy a specific algebra.

Having fixed local gauge and diffeomorphism freedom we can rewrite the gravitational part of the classical Hamiltonian (for the flat FRW model with massless scalar field) in the form (see, e.g. [7])

$$H_g = -\gamma^{-2} \int_\mathcal{V} d^3x \, N e^{-1} \varepsilon_{ijk} E^{\alpha j} E^{\beta k} F^i_{\alpha \beta},$$

(2)

where $\gamma$ is the Barbero-Immirzi parameter, $\mathcal{V} \subset \Sigma$ is an elementary cell, $\Sigma$ is spacelike hyper-surface, $N$ denotes the lapse function, $\varepsilon_{ijk}$ is the alternating tensor, $E_i^a$ is a densitized vector field, $e := \sqrt{\det E}$, and where $F^i_{\alpha \beta}$ is the curvature of an $SU(2)$ connection $A_i^a$.

The resolution of the singularity, obtained within LQC, is based on rewriting the curvature $F^k_{ab}$ in terms of holonomies around loops. The curvature $F^k_{ab}$ may be determined [7] by making use of the formula (see the appendix)

$$F^k_{ab} = -2 \lim_{Ar \square_{ij} \to 0} Tr \left( \frac{h_{\square_{ij}}^{(\mu)} - 1}{\mu^2 V_0^{2/3}} \right) \tau^k_{\alpha} \omega^\alpha_{\mu} \omega^\beta_{\mu},$$

(3)

where

$$h_{\square_{ij}}^{(\mu)} = h_i^{(\mu)} h_j^{(\mu)} (h_i^{(\mu)})^{-1} (h_j^{(\mu)})^{-1},$$

(4)

is the holonomy of the gravitational connection around the square loop $\square_{ij}$, considered over a face of the elementary cell, each of whose sides has length $\mu V_0^{1/3}$ (where $\mu > 0$) with respect to the flat fiducial metric $^{o}q_{ab} := \delta_{ij} \omega_a^i \omega_a^j$; fiducial triad $^{o}e^a_k$ and cotriad $^{o}\omega_k^a$ satisfy $^{o}\omega_a^i ^{o}e^a_j = \delta^i_j$; the spatial part of the FRW metric is $q_{ab} = a^2(t) ^{o}q_{ab}$; $Ar \square_{ij}$ denotes the area of the square; $V_0 = \int_\mathcal{V} \sqrt{q} d^3x$ is the fiducial volume of $\mathcal{V}$. Figure 1 shows geometrical setup for determination of $h_{\square_{ij}}^{(\mu)}$.

The holonomy along straight edge $^{o}e^a_k \partial_a$ of length $\mu V_0^{1/3}$ reads

$$h_k^{(\mu)}(c) = \mathcal{P} \exp \left( \int_0^{\mu V_0^{1/3}} \tau(k) A_k^{(k)} dx^a \right) = \exp(\tau k \mu c) = \cos(\mu c/2) \mathbb{I} + 2 \sin(\mu c/2) \tau k,$$

(5)

where $\tau_k = -i \sigma_k / 2$ ($\sigma_k$ are the Pauli spin matrices) and $\mathcal{P}$ denotes the path ordering symbol. Equation (5) presents the holonomy calculated in the fundamental, $j = 1/2$, representation of SU(2). The connection $A_k^a$ and the density weighted triad $E_k^a$ which occurs in (5) is determined by the conjugate variables $c$ and $p$ as follows: $A_k^a = ^{o}e^a_k c V_0^{-1/3}$ and $E_k^a = ^{o}e^a_k \sqrt{q_0} p V_0^{-2/3}$, where $c = \gamma \dot{a} V_0^{1/3}$ and $|p| = a^2 V_0^{2/3}$. Equation (5) presents the holonomy calculated in the fundamental, $j = 1/2$, representation of SU(2).
FIG. 1: Holonomy of connection around the square loop. Suitable fiducial cotriads $\omega^k_a$ are chosen to diagonalise the connection $A^k_a$.

Making use of (2), (3) and the so-called Thiemann identity [23]

$$\varepsilon_{ijk} E^{ai} E^{bk} = \frac{sgn(p)}{2\pi G \gamma V_0^{1/3}} \sum_k \omega^{abc}_c T_r \left( h^{(\mu)}_k \{ h^{(\mu)}_k \}^{-1}, V \right) \tau_i$$

leads to $H_g$ in the form

$$H_g = \lim_{\mu \to 0} H^{(\mu)}_g,$$

where

$$H^{(\mu)}_g = -\frac{sgn(p)}{2\pi G \gamma^3 \mu^3} \sum_{ijk} N \varepsilon_{ijk} T_r \left( h^{(\mu)}_i h^{(\mu)}_j (h^{(\mu)}_i)^{-1}(h^{(\mu)}_j)^{-1} h^{(\mu)}_k \{ (h^{(\mu)}_k)^{-1}, V \} \right),$$

and where $V = |p|^{3/2} = a^3 V_0$ is the volume of the elementary cell $V$.

The classical total Hamiltonian for FRW universe with a massless scalar field, $\phi$, reads

$$H = H_g + H_\phi \approx 0,$$

where $H_g$ is defined by (7). The Hamiltonian of the scalar field is known to be: $H_\phi = N p^2 |p|^{-3/2}$, where $\phi$ and $p_\phi$ are the elementary variables satisfying $\{ \phi, p_\phi \} = 1$. The relation $H \approx 0$ defines the physical phase space of considered gravitational system with constraints.

Making use of (5) we calculate (8) and get the modified total Hamiltonian $H^{(\lambda)}_g$ corresponding to (9) in the form

$$H^{(\lambda)} / N = -3 \frac{\sin^2(\lambda \beta)}{8\pi G \gamma^2} \frac{\lambda^2}{v} + \frac{P^2_\phi}{2 v},$$

where

$$\beta := \frac{c}{|p|^{1/2}}, \quad v := |p|^{3/2}$$

(11)
are the canonical variables proposed in [7]. The variable $\beta = \gamma \dot{a}/a$ so it corresponds to the Hubble parameter $\dot{a}/a$, whereas $v^{1/3} = aV^{1/3}_0$ is proportional to the scale factor $a$. The relationship between the coordinate length $\mu$ (which depends on $p$) and the physical length $\lambda$ (which is a constant) reads
\begin{equation}
\lambda = \mu |p|^{1/2} = \mu a V^{1/3}_0.
\end{equation}

The complete Poisson bracket for the canonical variables $(\beta, v, \phi, p_\phi)$ is defined to be
\begin{equation}
\{\cdot, \cdot\} := 4\pi G\gamma \left[ \frac{\partial}{\partial \beta} \frac{\partial}{\partial v} - \frac{\partial}{\partial v} \frac{\partial}{\partial \beta} \right] + \frac{\partial}{\partial \phi} \frac{\partial}{\partial p_\phi} - \frac{\partial}{\partial p_\phi} \frac{\partial}{\partial \phi}.
\end{equation}

The dynamics of a canonical variable $\xi$ is defined by
\begin{equation}
\dot{\xi} := \{\xi, H^{(\lambda)}\}, \quad \xi \in \{\beta, v, \phi, p_\phi\},
\end{equation}
where $\dot{\xi} := d\xi/d\tau$, and where $\tau$ is an evolution parameter. The dynamics in the physical phase space, $F^{(\lambda)}_{\text{phys}}$, is defined by solutions to (14) satisfying the condition $H^{(\lambda)} \approx 0$. The solutions of (14) ignoring the constraint $H^{(\lambda)} \approx 0$ are in the kinematical phase space, $F^{(\lambda)}_{\text{kin}}$.

In what follows we apply the Dirac method of dealing with Hamiltonian constraints [17]: the Poisson bracket is worked out before one makes use of the constraint equations.

### III. DYNAMICS

In the case a Hamiltonian is a constraint which may be rewritten in the form of a product of a simpler constraint and a function on $F^{(\lambda)}_{\text{kin}}$ which has no zeros, the original dynamics may be reduced (to some extent) to the dynamics with the simpler constraint.

Equation (14) can be rewritten as
\begin{equation}
H^{(\lambda)} = NH_0^{(\lambda)} \tilde{H}^{(\lambda)} \approx 0,
\end{equation}
where
\begin{equation}
H_0^{(\lambda)} := \frac{3}{8\pi G\gamma^2 v} \left( \kappa \gamma |p_\phi| + v \frac{|\sin(\lambda \beta)|}{\lambda} \right), \quad \tilde{H}^{(\lambda)} := \kappa \gamma |p_\phi| - v \frac{|\sin(\lambda \beta)|}{\lambda},
\end{equation}
\begin{equation}
\kappa^2 \equiv 4\pi G/3.
\end{equation}
It is clear that $H_0^{(\lambda)} = 0$ only in the case when $p_\phi = 0 = \sin(\lambda \beta)$. Such case, due to (21)-(25), implies no dynamics.

#### A. Relative dynamics

An equation of motion for a function $f$ defined on physical phase space, due to (14), reads
\begin{equation}
\dot{f} = \{f, NH_0^{(\lambda)} \tilde{H}^{(\lambda)}\} = \{f, NH_0^{(\lambda)}\} \tilde{H}^{(\lambda)} + NH_0^{(\lambda)} \{f, \tilde{H}^{(\lambda)}\} = NH_0^{(\lambda)} \{f, \tilde{H}^{(\lambda)}\},
\end{equation}

since $\tilde{H}^{(\lambda)} = 0$. By analogy, for other function $g$ we have
\begin{equation}
\dot{g} = \{g, NH_0^{(\lambda)} \tilde{H}^{(\lambda)}\} = NH_0^{(\lambda)} \{g, \tilde{H}^{(\lambda)}\}, \quad \text{for} \quad \tilde{H}^{(\lambda)} \approx 0.
\end{equation}
Therefore we have the relation
\[
\dot{f} \dot{g} = \frac{df}{dg} = \frac{NH_0^{(\lambda)}\{f, \tilde{H}^{(\lambda)}\}}{NH_0^{(\lambda)}\{g, \tilde{H}^{(\lambda)}\}} = \frac{\{f, \tilde{H}^{(\lambda)}\}}{\{g, \tilde{H}^{(\lambda)}\}}, \quad \text{as} \quad H_0^{(\lambda)} \neq 0, \tag{19}
\]
which we rewrite in the form
\[
\frac{df}{\{f, \tilde{H}^{(\lambda)}\}} = \frac{dg}{\{g, \tilde{H}^{(\lambda)}\}}. \tag{20}
\]
Equation (20) shows that in the case of the relative dynamics we have: (i) one phase space variable may be used as an ‘evolution parameter’ of all other variables, (ii) dynamics is gauge independent in the sense that there is no dependence on the specific choice of the lapse function \(N\), and (iii) suitable choice of \(N\) may lead to a simpler form of Hamiltonian.

### B. Solution of the relative dynamics

Since the relative dynamics is gauge independent, it is reasonable to choose the gauge which simplifies the calculations. Our choice is \(N := 1/H_0^{(\lambda)}\). In this gauge the equations of motion read
\[
p_\phi = 0, \tag{21}
\]
\[
\dot{\beta} = -4\pi G\gamma \frac{|\sin(\lambda \beta)|}{\lambda}, \tag{22}
\]
\[
\dot{\phi} = \kappa \gamma \text{sgn}(p_\phi), \tag{23}
\]
\[
\dot{v} = 4\pi G\gamma v \cos(\lambda \beta) \text{sgn}(\sin(\lambda \beta)), \tag{24}
\]
\[
\tilde{H}^{(\lambda)} = 0. \tag{25}
\]
Combining (23) with (24) gives
\[
\dot{v} \frac{\dot{\phi}}{\dot{\phi}} = 3\kappa v \cos(\lambda \beta) \text{sgn}(\sin(\lambda \beta)) \text{sgn}(p_\phi). \tag{26}
\]
Rewriting (26) (and using \(\dot{v}/\dot{\phi} = dv/d\phi\)) gives
\[
\frac{\text{sgn}(\sin(\lambda \beta))}{\cos(\lambda \beta)} \frac{dv}{v} = 3\kappa \text{sgn}(p_\phi) \, d\phi. \tag{27}
\]
Making use of the identity \(\sin^2(\lambda \beta) + \cos^2(\lambda \beta) = 1\) and (25) gives
\[
|\cos(\lambda \beta)| = \sqrt{1 - \left(\frac{\kappa \gamma \lambda p_\phi}{v}\right)^2}. \tag{28}
\]
Combining (27) with (28), for \(\beta \in [0, \pi/2\lambda]\), leads to
\[
\frac{dv}{\sqrt{v^2 - (\kappa \gamma \lambda p_\phi)^2}} = 3\kappa \text{sgn}(p_\phi) \, d\phi. \tag{29}
\]
Since \(p_\phi\) is just a constant (due to (21)) we can easily integrate (29) and get
\[
\ln \left| v + \sqrt{v^2 - (\kappa \gamma \lambda p_\phi)^2} \right| = 3\kappa \text{sgn}(p_\phi)(\phi - \phi_0). \tag{30}
\]
Rewriting (30) leads to
\[ 2v = \exp(3\kappa \text{sgn}(p_\phi) (\phi - \phi_0)) + (\kappa\gamma|p_\phi\lambda|^2 \cdot \exp(-3\kappa \text{sgn}(p_\phi) (\phi - \phi_0))). \] (31)

The solution for the variable \(\beta\) may be easily determined from (25) rewritten as
\[ \kappa\gamma|p_\phi| = v \frac{\sin(\lambda\beta)}{\lambda}, \] (32)

The final expression reads
\[ \sin(\lambda\beta) = \frac{2\kappa\lambda|p_\phi|}{\exp(3\kappa \text{sgn}(p_\phi) (\phi - \phi_0)) + (\kappa\gamma\lambda|p_\phi|^2 \cdot \exp(-3\kappa \text{sgn}(p_\phi) (\phi - \phi_0)))}. \] (33)

where the domain of the variable \(\beta\) has been extended to the interval \([0, \pi/\lambda]\).

Equations (31) and (33) present the dependence of the canonical variables \(v\) and \(\beta\) on the evolution parameter \(\phi\), which is a monotonic function due to (23).

**IV. ALGEBRA OF OBSERVABLES**

A function, \(O\), defined on phase space is a Dirac observable if
\[ \{O, H^{(\lambda)}\} \approx 0. \] (34)

Since we have
\[ \{O, H^{(\lambda)}\} = \{O, NH_0^{(\lambda)} \tilde{H}^{(\lambda)}\} = NH_0^{(\lambda)}\{O, \tilde{H}^{(\lambda)}\} + \{O, NH_0^{(\lambda)}\} \tilde{H}^{(\lambda)}, \] (35)

it is clear that on the constraint surface, \(\tilde{H}^{(\lambda)} = 0\), the Dirac observable satisfies (independently on the choice of \(N\)) a much simpler equation
\[ \{O, \tilde{H}^{(\lambda)}\} \approx 0. \] (36)

Thus, we put \(N := 1/H_0^{(\lambda)}\) and solve (34) in the whole phase space, i.e. we solve the equation
\[ \frac{\sin(\lambda\beta)}{\lambda} \frac{\partial O}{\partial \beta} - v \cos(\lambda\beta) \frac{\partial O}{\partial v} - \frac{\kappa \text{sgn}(p_\phi)}{4\pi G} \frac{\partial O}{\partial \phi} = 0. \] (37)

A function \(O = O(O_1, \ldots, O_k)\) satisfies (37) if
\[ \{O_1, \tilde{H}^{(\lambda)}\} = 0 = \{O_2, \tilde{H}^{(\lambda)}\} = \ldots = \{O_k, \tilde{H}^{(\lambda)}\}, \] (38)

where \(k + 1\) is the dimension of the kinematical phase space. It is so because one has
\[ \{O, \tilde{H}^{(\lambda)}\} = \frac{\partial O}{\partial O_1} \{O_1, \tilde{H}^{(\lambda)}\} + \ldots + \frac{\partial O}{\partial O_k} \{O_k, \tilde{H}^{(\lambda)}\}. \] (39)

In what follows we consider only elementary observables. The set of such observables, \(E\), is defined by the requirements: (i) each element of \(E\) is a solution to (37), (ii) elements of \(E\) are functionally independent on the constraint surface, \(\tilde{H}^{(\lambda)} = 0\), (iii) elements of \(E\) satisfy...
a Lie algebra, and (iv) two sets of observables satisfying two algebras are considered to be the same if these algebras are isomorphic.

In our case \( k = 3 \) and solutions to (37) are found to be
\[
O_1 := p_\phi, \quad O_2 := \phi - \frac{s}{3\kappa} \arth(\cos(\lambda\beta)), \quad O_3 := s v \frac{\sin(\lambda\beta)}{\lambda},
\]  
(40)
where \( s := \text{sgn}(p_\phi) \). One may verify that the observables satisfy the Lie algebra
\[
\{O_2, O_1\} = 1, \quad \{O_1, O_3\} = 0, \quad \{O_2, O_3\} = \gamma \kappa.
\]  
(41)

Because of the constraint \( \tilde{H}(\lambda) = 0 \) (see (32)), we have
\[
O_3 = \gamma \kappa O_1.
\]  
(42)

Thus, we have only two elementary Dirac observables which may be used to parameterize the physical phase space \( \mathcal{F}_{\text{phys}}^{(\lambda)} \). To identify the Poisson bracket in \( \mathcal{F}_{\text{phys}}^{(\lambda)} \) consistent with the Poisson bracket (13) defined in \( \mathcal{F}_{\text{kin}}^{(\lambda)} \), we find a symplectic twoform corresponding to (13). It reads
\[
\omega = \frac{1}{4\pi G\gamma} d\beta \wedge dv + d\phi \wedge dp_\phi.
\]  
(43)

The twoform \( \omega \) is degenerate on \( \mathcal{F}_{\text{phys}}^{(\lambda)} \) due to the constraint \( \tilde{H}(\lambda) = 0 \). Making use of the explicit form of this constraint (32) and the functional form of \( O_1 \) and \( O_2 \), leads to the symplectic form \( \Omega \) on \( \mathcal{F}_{\text{phys}}^{(\lambda)} \). Direct calculations give (see App. B)
\[
\Omega := \omega|_{\tilde{H}(\lambda)=0} = dO_2 \wedge dO_1,
\]  
(44)

where \( \omega|_{\tilde{H}(\lambda)=0} \) denotes the reduction of \( \omega \) to the constraint surface. The Poisson bracket corresponding to (44) reads
\[
\{\cdot, \cdot\} := \frac{\partial \cdot}{\partial O_2} \frac{\partial}{\partial O_1} - \frac{\partial \cdot}{\partial O_1} \frac{\partial}{\partial O_2}
\]  
(45)
so the algebra satisfied by \( O_1 \) and \( O_2 \) has a simple form given by
\[
\{O_2, O_1\} = 1.
\]  
(46)

Our kinematical phase space, \( \mathcal{F}_{\text{kin}}^{(\lambda)} \), is four dimensional. In relative dynamics one variable is used to parametrize three others. Since the constraint relates the variables, we have only two independent variables. This is the reason we have only two elementary physical observables parametrizing \( \mathcal{F}_{\text{phys}}^{(\lambda)} \).

V. FUNCTIONS ON PHASE SPACE

In this section we discuss the functions on the constraint surface that may describe singularity aspects of our cosmological model. Considered functions are not observables, but they can be expressed in terms of observables and an evolution parameter \( \phi \). They do become observables for each fixed value of \( \phi \), since in such case they are only functions of observables.
A. Energy density

An expression for the energy density \( \rho \) of the scalar field \( \phi \) reads

\[
\rho(\lambda, \phi) = \frac{1}{2} \frac{p_\phi^2}{v^2}.
\]

(47)

In terms of elementary observables we have

\[
p_\phi = \mathcal{O}_1, \quad v = \kappa \gamma \lambda |\mathcal{O}_1| \cosh(3\kappa(\phi - \mathcal{O}_2)).
\]

(48)

For fixed \( p_\phi \) the density \( \rho \) takes its maximum value at the minimum value of \( v \). Rewriting (31) in the form

\[
\frac{v}{\Delta} = \cosh(3\kappa s(\phi - \phi_0) - \ln \Delta), \quad \text{where} \quad \Delta := \kappa \gamma \lambda |p_\phi|,
\]

(49)

we can see that \( \cosh(\cdot) \) takes minimum value equal to one at \( 3\kappa s(\phi - \phi_0) = \ln \Delta \). Thus, the maximum value of the density, \( \rho_{\text{max}} \), corresponds to \( v = \Delta \) and reads

\[
\rho_{\text{max}} = \frac{1}{2\kappa^2 \gamma^2} \frac{1}{\lambda^2}.
\]

(50)

We can determine \( \rho_{\text{max}} \) if we know \( \lambda \). However, \( \lambda \) is a free parameter of the formalism. Thus, finding the critical energy density of matter corresponding to the big bounce is an open problem.

It is tempting to apply (50) to the Planck scale. To make use of Planck’s length \( l_{\text{Pl}} := \sqrt{\hbar G/c^3} \) and Planck’s energy density \( \rho_{\text{Pl}} := c^5/\hbar G^2 \), we multiply (50) by \( c^2 \) and recall that \( \kappa^2 \equiv 4\pi G/3 \). Thus (50) reads

\[
\rho_{\text{max}} = \frac{3}{8\pi G \gamma^2} \frac{1}{\lambda^2}.
\]

(51)

Substituting \( \lambda = l_{\text{Pl}} \) into (51) gives \( \rho_{\text{max}} = 3/8\pi \gamma^2 \rho_{\text{Pl}} \simeq 2.07 \rho_{\text{Pl}} \). Resolving (51) in terms of \( \lambda \) makes possible finding \( \lambda \) corresponding to \( \rho_{\text{Pl}} \). We get \( \lambda = \sqrt{3/8\pi \gamma^2} l_{\text{Pl}} \simeq 1.44 l_{\text{Pl}} \). (We have used \( \gamma \simeq 0.24 \) determined in black hole entropy calculations [26, 27].) Surprisingly, the classical expression (51) fits the Planck scale.

A natural next step is an examination of the energy scale of the big bounce at the quantum level. Preliminary calculations, [11], suggest that the energy scale would be described by the classical expression (51).

B. Geometrical operators

In the case the volume \( V = a^3V_0 \) of an elementary cell \( V \) is a cube, geometrical operators have simple dependence on canonical variables. Since the volume operator, \( V \), is given by

\[
V(\phi) = v,
\]

(52)

the area operator, \( A \), reads

\[
A(\phi) = v^{2/3},
\]

(53)
and the length operator, \( L \), is found to be
\[
L(\phi) = v^{1/3}. 
\] (54)

As far as we know, the geometrical operators have been considered so far only in the kinematical Hilbert space of the loop quantum gravity [28, 29, 30, 31, 32]. We propose the examination of these operators in the physical Hilbert space of the loop quantum cosmology.

It results from (32) that we have
\[
v_{\min} = \kappa \gamma \lambda |p_\phi| 
\] (55)
so the geometrical operators are bounded from below by zero (as \( \lambda |p_\phi| \to 0 \)).

An examination of the spectra of the geometrical operators at the quantum level is the next natural step. An interesting question is: Do these operators have the nonzero minimum and discrete eigenvalues? It is expected that the story will turn out to be similar to the case of the linear harmonic oscillator, where we have the nonzero ground-state energy and discrete energy levels. We present an answer to this intriguing question in our forthcoming paper [16].

VI. CONCLUSIONS

We have shown that the resolution of the initial singularity of the flat FRW model with massless scalar field is due to the modification of the model at the classical level by making use of the loop geometry. The modification is parametrized by the continuous parameter \( \lambda \). Each value of \( \lambda \) specifies the critical energy density of the scalar field corresponding to the big bounce. As there is no specific choice of \( \lambda \), the BB may occur at any low and high densities. The former case (big \( \lambda \)) contradicts the data of observational cosmology (there was no BB in the near past!) and leads to weakly controlled modification of the expression for the curvature \( F_k^{ab} \), i.e. gravitational part of the Hamiltonian (see the appendix). The latter case (small \( \lambda \)) gives much better approximation for the classical Hamiltonian (see the appendix), but may easily lead to densities much higher than the Planck scale density, where the classical formalism is believed to be inadequate. Finding specific value of the parameter \( \lambda \), i.e. the energy scale specific to BB is an open problem.

Our approach is quite different from the so-called effective or polymerization method (see, e.g. [33]), where the replacement \( \beta \to \sin(\lambda \beta)/\lambda \) in the Hamiltonian finishes the procedure of quantization. In our method this replacement has been done entirely at the classical level: Eq. (10) results from using an explicit form of the holonomy (5) in (8). Quantization consists in finding a self-adjoint representation of observables on the physical phase space and an examination of the spectra of these observables [11, 12].

The elementary observables \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) constitute a complete set of constants of motion in the constraint surface. They are used to parametrize the physical phase space and are “building blocks” for the compound observables like the energy density of the scalar field and the geometrical operators. So they have deep physical meaning. The role of \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) becomes even more important at the quantum level as they enable finding quantum operators corresponding to the classical compound observables [13].

Our theoretical framework may be used for examination of the discreteness aspects of geometrical quantum operators, which may help in the selection of \( \lambda \). An extension of our formalism to the quantum level is straightforward. The algebra of observables is defined
in the physical phase space (hyper-surface in the kinematical phase space determined by the constraint equation). The carrier space of the self-adjoint representation of the algebra defines the physical Hilbert space. Examination of the eigenvalue problem of the length, area and volume operators (for fixed value of an evolution parameter they are observables) may lead to the specification of an unique $\lambda$ (or an interval of allowed values). Our next paper \cite{16} is devoted to examination of these problems.

It may happen, however, that the value of the parameter $\lambda$ cannot be determined, for some reason, theoretically. The story may turn out to be similar to the case of the short-range repulsive part of the potential of the nucleon-nucleon interaction introduced to explain the scattering data \cite{34} and the nuclear matter saturation of energy \cite{35}. In such a case $\lambda$ will become a phenomenological variable parameterizing our ignorance of microscopic properties of the universe.

An independent source of information on discreteness aspects of geometry is the observational cosmology. The cosmic projects for the detection of gamma ray bursts may reveal that the velocity of cosmic photons depend on their wave lengths, which may be ascribed to the foamy nature of spacetime \cite{36, 37, 38}. The detection of the primordial gravitational waves created at the big bounce may bring valuable information on the geometry of this phase \cite{39, 40, 41, 42}. The observational cosmology data may help to determine the phenomenological value of the parameter $\lambda$.

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APPENDIX A: HOLONOMY CORRECTIONS

The curvature of $SU(2)$ connection $F_{ab}^k = \partial_a A_b^k - \partial_b A_a^k + \epsilon_{ij}^k A_i^a A_j^b$, entering the expression \cite{2} for the gravitational part of the Hamiltonian, can be expressed in terms of holonomies. Using the mean-value and Stokes’ theorems we have

$$\tau_k F_{ab}^k(\vec{x}) \approx \frac{1}{s_{ab}} \int_\sigma \tau_k F_{cd}^k dx^c \wedge dx^d \approx \frac{1}{s_{ab}} \left( \mathcal{P} \exp \left( \int_{\partial \sigma} \tau_k A_c^k dx^c \right) - 1 \right),$$  \hspace{1cm} (A1)

where $\partial \sigma$ is the boundary of a small surface $\sigma$ with center at $\vec{x}$, and where $s_{ab} := \int_\sigma dx^a \wedge dx^b$. The expression for $F_{ab}^k$ is exact but in the limit when we shrink the area enclosed by the loop $\partial \sigma$ to zero. If we choose $\partial \sigma$ in the form of the square $\Box_{ij}$ with sides length $\mu$, the expression for a small value of $\mu = \mu_0$ has the form \cite{18}

$$F_{ab}^k(\mu_0) = \lim_{\mu \to \mu_0} \left\{ - 2 \text{Tr} \left( \frac{h^{(\mu)}_{\Box_{ij}} - 1}{\mu^2 V^2/3} \right) \tau^k \omega_a^i \omega_b^j + O(\mu^4) \right\},$$  \hspace{1cm} (A2)

and we have

$$F_{ab}^k = \lim_{\mu_0 \to 0} F_{ab}^k(\mu_0).$$  \hspace{1cm} (A3)
In the standard LQC the $\mathcal{O}(\mu^4)$ holonomy corrections are ignored (see, e.g. [7, 8]). It was found in [18, 19] that including higher order corrections leads to new curvature singularities different from the initial singularity and increases an ambiguity problem of loop cosmology. However, the holonomy corrections do not change the result that the big bounce is a consequence of the loopy nature of geometry [20].

Taking only the first term of (A2) leads to the simplest modification of gravity, but may be insufficient for the description of the inflationary phase. The choice of $\mu_0$ based on the expectation that Big Bounce should occur at the Planck scale [7] has little justification [10]. The significance of Planck’s scale for quantum gravity seems to be rather a belief than proved result (see, e.g. [21]). Heuristic reasoning playing game at the same time with Heisenberg’s uncertainty principle, Schwarzschild’s radius and process of measurement cannot replace a proof (see, e.g. [22]).

APPENDIX B: SYMPLECTIC FORM

The symplectic form on the physical phase space $\Omega$ may be obtained from the symplectic form on the kinematical phase space $\omega$ by taking into account the constraint (32).

The symplectic form corresponding to (13) reads

$$\omega = d\phi \wedge dp_{\phi} + \frac{1}{4\pi G\gamma} d\beta \wedge dv = d\phi \wedge dp_{\phi} + \frac{1}{3\kappa^2\gamma} d\beta \wedge dv.$$  \hspace{1cm} (B1)

Making use of (32) we get

$$dv = \frac{\kappa \gamma \lambda}{\sin(\lambda \beta)} dp_{\phi} - \lambda \cotg(\lambda \beta) d\beta.$$  \hspace{1cm} (B2)

Insertion of (B2) into (B1) gives

$$\Omega = d\phi \wedge dp_{\phi} + \frac{\text{sgn}(p_{\phi})}{3\kappa} \frac{\lambda}{\sin(\lambda \beta)} d\beta \wedge dp_{\phi}.$$  \hspace{1cm} (B3)

Since

$$\frac{\lambda}{\sin(\lambda \beta)} = -\frac{d \text{arcth}(\cos(\lambda \beta))}{d\beta},$$  \hspace{1cm} (B4)

we have

$$\Omega = \left( d\phi - \frac{\text{sgn}(p_{\phi})}{3\kappa} \frac{d \text{arcth}(\cos(\lambda \beta))}{d\beta} d\beta \right) \wedge dp_{\phi}. $$  \hspace{1cm} (B5)

On the other hand, due to (40), we have

$$\mathcal{O}_1 = p_{\phi}, \hspace{1cm} \mathcal{O}_2 = \phi - \frac{\text{sgn}(p_{\phi})}{3\kappa} \text{arcth}(\cos(\lambda \beta)).$$  \hspace{1cm} (B6)

Therefore,

$$\Omega = d\mathcal{O}_2 \wedge d\mathcal{O}_1.$$  \hspace{1cm} (B7)
Thus, the physical phase space may be parametrized by the variables $\mathcal{O}_1$ and $\mathcal{O}_2$, and the corresponding Poisson bracket is given by (45).

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