Geometry and beta-functions for $N = 2$ matter models in two dimensions

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ABSTRACT

We study renormalizable nonlinear $\sigma$-models in two dimensions with $N = 2$ supersymmetry described in superspace in terms of chiral and complex linear superfields. The geometrical structure of the underlying manifold is investigated and the one-loop divergent contribution to the effective action is computed. The condition of vanishing $\beta$-function allows to identify a class of models which satisfy this requirement and possess $N = 4$ supersymmetry.

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1 Introduction

Nonlinear $\sigma$-models with $N = 2$ supersymmetry have been extensively studied in the past. A complete off-shell description of these systems has been presented in terms of chiral, twisted chiral and semi-chiral superfields [1]. Besides these types of models there exists another class of theories that seems to be suitable for a description of supersymmetric extensions of the low-energy QCD effective action [2, 3, 4]. These models are generically constructed by means of two kinds of superfields, i.e. the chiral superfield $\Phi$ and the complex linear superfield $\Sigma$. Both formulations provide a description of the scalar multiplet, in its minimal and nonminimal version respectively. A duality transformation connects the two multiplets, so that a model defined in terms of e.g. chiral superfields is essentially equivalent to the dual one written in terms of linear superfields. This duality, implemented at the classical level by a Legendre transform on the action, has been studied at the one-loop level and proven to be maintained [5].

The lagrangian of a renormalizable nonlinear $\sigma$-model in $N = 2$, 2-dimensional superspace is given by a real, nonderivative function of the superfields, the potential. In the chiral description one can interpret the superfields as holomorphic coordinates on a Kähler manifold with metric simply given by the second, holomorphic-antiholomorphic derivatives of the potential. The geometrical quantities are then constructed in standard manner and the covariant structure of physical objects is easily implemented. The problem of finding the corresponding geometrical interpretation for the nonminimal formulation of the theory has been addressed in ref. 3, where the analysis has been extended to the generalized $\sigma$-model whose potential is a function of chiral and linear superfields. Here we try to learn more: for such a theory first we discuss the geometrical properties and find that, once the auxiliary $N = 1$ components of the complex linear superfields are set on-shell, the underlying manifold is Kähler. On one hand this is not surprising since, using the on-shell condition, we eliminate the extra auxiliary degrees of freedom that distinguish the nonminimal multiplet from the minimal one. On the other hand the result is not a priori expected since, as argued in 3, eliminating the auxiliary fields of the linear multiplet modifies the quadratic action of the physical components in a nontrivial way. Indeed we show that this nontrivial behavior of the auxiliary fields leads to a $N = 1$ or component action which is the Legendre transform of the chiral one. Thus the two models are connected by a duality transformation, at least classically and with the auxiliary fields on-shell. Actually the equivalence between the two formulations is more stringent than this: we have computed in $N = 2$ superspace the one-loop $\beta$-function for the mixed, chiral plus linear theory, and we have found that the quantum, off-shell result is consistent with the above mentioned duality between the chiral and the complex linear superfields.

Finally we have studied the condition of vanishing $\beta$-function and identified a class of models which satisfy this requirement and possess $N = 4$ supersymmetry. These issues
are presented in detail in the following sections.

2 The geometry

The chiral $N = 2$ nonlinear $\sigma$-model is described by the superspace action

$$S = \int d^2x \, d^4\theta \, K(\Phi, \bar{\Phi})$$

(2.1)

with $\Phi^\mu$, $\bar{\Phi}^{\bar{\mu}}$, $\mu, \bar{\mu} = 1, \ldots, n$, satisfying the chirality constraints $\bar{D}_\alpha \Phi^\mu = 0$, $D_\alpha \bar{\Phi}^{\bar{\mu}} = 0$. It is well known [7] that a simple geometrical interpretation emerges: $K$ represents the Kähler potential of an underlying manifold with complex coordinates $\Phi^\mu$, $\bar{\Phi}^{\bar{\mu}}$, whose metric is given by

$$g_{\mu\bar{\mu}} = \frac{\partial^2 K}{\partial \Phi^\mu \partial \bar{\Phi}^{\bar{\mu}}}$$

(2.2)

Here and in the following we introduce the notation

$$K_{\mu\ldots\bar{\mu}\ldots} = \frac{\partial}{\partial \Phi^\mu} \ldots \frac{\partial}{\partial \bar{\Phi}^{\bar{\mu}}} \ldots K(\Phi, \bar{\Phi})$$

(2.3)

so that

$$g_{\mu\bar{\mu}} = K_{\mu\bar{\mu}}$$

(2.4)

In this complex basis the standard quantities of riemannian geometry become very simple. The only non-vanishing components of the connection are

$$\Gamma^\mu_{\nu\rho} = g^{\mu\bar{\rho}} \frac{\partial}{\partial \Phi^\nu} g_{\rho\bar{\mu}} \quad \Gamma^{\bar{\mu}}_{\nu\bar{\rho}} = (\Gamma^\mu_{\nu\bar{\rho}})^*$$

(2.5)

The Riemann tensor can be written as

$$R_{\mu\bar{\rho}\nu\bar{\sigma}} = K_{\mu\bar{\rho}\nu\bar{\sigma}} - K_{\rho\bar{\sigma}} K_{\mu\bar{\nu}} K_{\rho\bar{\mu}}$$

(2.6)

where $K_{\rho\bar{\sigma}}$ is the inverse of the metric in (2.2). Finally the Ricci tensor takes the form

$$R_{\mu\bar{\mu}} = \frac{\partial}{\partial \Phi^\mu} \frac{\partial}{\partial \bar{\Phi}^{\bar{\mu}}} \ln \det g_{\nu\bar{\nu}}$$

(2.7)

The Kählerian nature of the geometry is most easily recognized once the theory is rewritten using the $N = 1$ formalism, so that the relevant complex structure is immediately identified. We briefly review the chiral example in order to compare later on with the corresponding results in the less familiar complex linear setting.
We expand the $N = 2$ chiral superfield $\Phi$ in $N = 1$ superfield components (we use the notations given in Appendix A)

$$\Phi(\theta_1, \theta_2) = \phi(\theta_1) + \frac{1}{2}\theta_2^\alpha \psi_\alpha(\theta_1) + \frac{1}{4}\theta_2^2 F(\theta_1)$$

(2.8)

where

$$\phi = \Phi|_{\theta_2=0} \quad \psi_\alpha = D_{2\alpha} \Phi|_{\theta_2=0} \quad F = -D_2^2 \Phi|_{\theta_2=0}$$

(2.9)

The constraint equation $\bar{D}_\alpha \Phi = 0$, which obviously implies $\bar{D}^2 \Phi = 0$, allows to solve for $\psi_\alpha$ and $F$ in terms of $\phi$. We have

$$D_{2\alpha} \Phi = D_{1\alpha} \Phi \quad D_2^2 \Phi = -D_1^2 \Phi + D_1^\alpha D_{2\alpha} \Phi = D_1^2 \Phi$$

(2.10)

so that in $N = 1$ language we obtain

$$\psi_\alpha = D_{1\alpha} \phi \quad F = -D_1^2 \phi$$

(2.11)

In the same way from the antichirality constraints, i.e. $D_\alpha \bar{\Phi} = 0$ and $D^2 \bar{\Phi} = 0$ one has

$$\bar{\psi}_\alpha = -D_{1\alpha} \bar{\phi} \quad \bar{F} = -D_1^2 \bar{\phi}$$

(2.12)

Correspondingly the action in (2.1) becomes

$$S = \frac{1}{4} \int d^2 x \ d^2 \theta_1 \ d^2 \theta_2 \ K(\Phi, \bar{\Phi}) = \frac{1}{2} \int d^2 x \ d^2 \theta_1 \ d^2 \theta_2 \ K_{\mu\bar{\nu}}(\phi, \bar{\phi}) D_{1\alpha} \phi^\mu D_{1}^\alpha \bar{\phi}$$

(2.13)

It is manifestly invariant under the first supersymmetry transformation $\delta \phi^\mu = i\epsilon^\alpha Q_{1\alpha} \phi^\mu$, $\delta \bar{\phi}^\bar{\mu} = i\epsilon^\alpha Q_{1\alpha} \bar{\phi}^\bar{\mu}$, while the invariance under the second supersymmetry

$$\delta \phi^\mu = i\eta^\alpha Q_{2\alpha} \Phi^\mu|_{\theta_2=0} = -\eta^\alpha \psi_\alpha$$

$$\delta \bar{\phi}^\bar{\mu} = i\eta^\alpha Q_{2\alpha} \bar{\Phi}^\bar{\mu}|_{\theta_2=0} = -\eta^\alpha \bar{\psi}_\alpha$$

(2.14)

leads to the determination of the complex structure $J^\mu$, $\bar{J}^\bar{\mu}$. Indeed from (2.14) and (2.11) it follows

$$\delta \phi^\mu = -\eta^\alpha D_{1\alpha} \phi^\mu \equiv i\eta^\alpha J^\mu$$

$$\delta \bar{\phi}^\bar{\mu} = \eta^\alpha D_{1\alpha} \bar{\phi}^\bar{\mu} \equiv i\eta^\alpha \bar{J}^\bar{\mu}$$

(2.15)

Thus we identify $J^\mu = i\delta^\mu$ and $\bar{J}^\bar{\mu} = -i\delta^\bar{\mu}$. Using the complex holomorphic basis $(\phi^\mu, \bar{\phi}^\bar{\mu})$ the complex structure can be written in standard form

$$J = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2.16)
All the defining properties of a Kähler geometry are automatically satisfied.

We wish to repeat these steps for the σ-model described in terms of complex linear superfields $\Sigma^a, \bar{\Sigma}^\alpha, a, \bar{a} = 1, \ldots, n$, satisfying the constraints $\bar{D}^2 \Sigma^a = 0, D^2 \bar{\Sigma}^\alpha = 0$,

$$S = \frac{1}{4} \int d^2 x \ d^4 \theta \ K(\Sigma, \bar{\Sigma}) = \frac{1}{4} \int d^2 x \ d^2 \theta_1 \ d^2 \theta_2 \ K(\Sigma, \bar{\Sigma})$$

Again we expand $\Sigma$ in terms of $N = 1$ superfields

$$\Sigma(\theta_1, \bar{\theta}_2) = f(\theta_1) + \frac{1}{2} \bar{\theta}_2^\alpha \lambda^\alpha(\theta_1) + \frac{1}{4} \bar{\theta}_2^\alpha G(\theta_1)$$

having defined the components as

$$f = \Sigma|_{\theta_2 = 0} \quad \lambda^\alpha = D_{2\alpha} \Sigma|_{\theta_2 = 0} \quad G = -D^2 \Sigma|_{\theta_2 = 0}$$

(The $N = 1$ component expansions are given explicitly in Appendix B). Now we want to use the constraint that $\Sigma$ satisfies i.e. $\bar{D}^2 \Sigma = 0$. It can be expressed as follows

$$\bar{D}^2 \Sigma = -D^2 \Sigma + D^2_1 \lambda$$

The situation here is somewhat different as compared to the one in the chiral example. In fact, from (2.19) and (2.20) we see that in the case of the linear superfield we can solve only for the highest component in terms of the lower ones

$$G = D^2_1 f - D^2_1 \lambda$$

while the $N = 1$ superfield $\lambda^\alpha$ is not determined. Indeed the nonminimal multiplet contains the same physical degrees of freedom as the minimal one, but it differs from it in the auxiliary field content. As we will show now the superfields $\lambda^\alpha$ are not dynamical and can be set on-shell algebraically.

Using the result in (2.21) the action (2.17) is reduced to $N = 1$ superspace

$$S = \frac{1}{8} \int d^2 x \ d^2 \theta_1 \ [D^2_1 \bar{\mathcal{F}}^T M D_{1\alpha} \mathcal{F} + \Lambda^\alpha \mathcal{M} \Lambda_\alpha$$

$$-D^2_1 \bar{\mathcal{F}}^T M P \Lambda_\alpha - \Lambda^\alpha P M D_{1\alpha} \mathcal{F}]$$

where we have defined

$$\mathcal{F} = \begin{pmatrix} f^a \\ \bar{f}^\bar{a} \end{pmatrix} \quad \Lambda_\alpha = \begin{pmatrix} \lambda^\alpha_a \\ \bar{\lambda}^\bar{\alpha}_a \end{pmatrix}$$

and

$$M = \begin{pmatrix} K_{ab} & K_{ab} \\ K_{\bar{a}b} & K_{\bar{a}b} \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
We are using the notation

\[ K_{a\ldots a} = \frac{\partial}{\partial f^{a}} \ldots \frac{\partial}{\partial f^{\bar{a}}} \ldots K(f, \bar{f}) \]  

(2.25)

As anticipated above, the superfields \( \Lambda_\alpha \) appear in (2.22) as auxiliary fields with equations of motion

\[ \Lambda_\alpha = M^{-1} P M D_{1\alpha} \mathcal{F} \]  

(2.26)

Substituting (2.26) in (2.22) we obtain

\[ S = \frac{1}{8} \int d^2 x \ d^2 \theta_1 \left[ D_{1\alpha}^a \mathcal{F}^T (M - M P M^{-1} P M) D_{1\alpha} \mathcal{F} \right] \]  

(2.27)

In addition to the manifest invariance under a first supersymmetry, the action in (2.27) possesses a second one inherited from the \( N = 2 \) reduction

\[ \delta \mathcal{F} = - \eta^\alpha M^{-1} P M D_{1\alpha} \mathcal{F} \]  

(2.28)

In general, from dimensional analysis and Lorentz and parity invariance, a second supersymmetry transformation, as in (2.15), is of the form

\[ \delta \mathcal{F} = i \eta^\alpha J D_{1\alpha} \mathcal{F} \]  

(2.29)

where \( J \) must be the complex structure of a complex manifold, as required by the closure of the algebra. Thus, from (2.28) we find the expression of the complex structure, while from the action in (2.27) we determine the metric

\[ J = i M^{-1} P M \quad g = M - M P M^{-1} P M \]  

(2.30)

It is easy to verify that the complex structure \( J \) satisfies

\[ J^2 = -1, \quad J^T g J = g, \quad N_{ij}^k \equiv J_i^l \partial_l J_{j}^k - J_j^l \partial_l J_{i}^k = 0, \quad \partial_{[k} J_{j]} = 0 \]  

(2.31)

where \( N_{ij}^k \) is the Nijenhuis tensor. Thus we conclude that \( J \) is the complex structure of a Kähler manifold described by a nonholomorphic set of coordinates \( (f^a, \bar{f}^{\bar{a}}) \). We can go to a canonical basis by performing the following field redefinition

\[ \mathcal{I} = \frac{\partial K}{\partial \mathcal{F}} \quad \text{with} \quad \mathcal{I} \equiv \left( \frac{\phi^a}{\phi^{\bar{a}}} \right) = \left( \frac{K^a}{K^{\bar{a}}} \right) \]  

(2.32)

In terms of the new variables the second supersymmetry transformation becomes

\[ \delta \mathcal{I} = M \delta \mathcal{F} = M (i \eta^\alpha J D_{1\alpha} \mathcal{F}) = i \eta^\alpha M J M^{-1} D_{1\alpha} \mathcal{I} \]  

(2.33)
so that the new complex structure takes the standard form

\[ J' \equiv MJM^{-1} = iP \]  

(2.34)

If we now express the action in (2.27) using the fields in (2.32) we obtain

\[ S = \frac{1}{8} \int d^2x \ d^2\theta_1 \left[ D_\alpha^I T(M^{-1} - PM^{-1}P) D_{1\alpha} T \right] \]  

(2.35)

where \( M \) has to be thought as a function of \( f(\phi, \bar{\phi}) \) and \( \bar{f}(\phi, \bar{\phi}) \) as given by inverting the relation in (2.32). The matrix \( [M^{-1} - PM^{-1}P] \) is block off-diagonal so that it consistently represents the Kähler metric (with only nonvanishing holomorphic-nonholomorphic components), in complete analogy with the result in (2.13) for the chiral formulation. The above equivalence, however, has been obtained after the elimination of the extra auxiliary fields \( \lambda_\alpha \) via their equations of motion.

We observe that the action (2.35) is nothing else than the \( N = 1 \) formulation we would have obtained performing a duality transformation on the original action in (2.17). Indeed, one can start from the first order action

\[ S = \int d^2x \ d^4\theta \left[ K(\Sigma, \bar{\Sigma}) - \Sigma \Phi - \bar{\Sigma} \bar{\Phi} \right] \]  

(2.36)

with \( \Phi, \bar{\Phi} \) satisfying the chirality constraints \( D_\alpha \Phi = 0, D_\alpha \Phi = 0 \). The \( \sigma \)-model in (2.17) is reobtained by functional integration over \( \Phi, \bar{\Phi} \) which imposes the linearity constraints on \( \Sigma, \bar{\Sigma} \). On the other hand, the equations of motion for \( \Sigma, \bar{\Sigma} \) give

\[ \Phi = \frac{\partial K}{\partial \Sigma}, \quad \bar{\Phi} = \frac{\partial K}{\partial \bar{\Sigma}} \]  

(2.37)

which evaluated at \( \theta_2 = 0 \) take the form

\[ \phi = \frac{\partial K}{\partial f}, \quad \bar{\phi} = \frac{\partial K}{\partial \bar{f}} \]  

(2.38)

i.e. (2.32). In this way one reconstructs a \( \sigma \)-model in terms of chiral superfields with a potential \( \tilde{K} \) given by the Legendre transform of \( K \)

\[ \tilde{K}(\Phi, \bar{\Phi}) = [K(\Sigma, \bar{\Sigma}) - \Sigma \Phi - \bar{\Sigma} \bar{\Phi}]|_{\Sigma=\Sigma(\Phi, \bar{\Phi}), \bar{\Sigma}=\bar{\Sigma}(\Phi, \bar{\Phi})} \]  

(2.39)

which is just the \( N = 2 \) formulation of the theory in (2.33).

The equations of motion (2.26) for the bosonic components of the \( N = 1 \) auxiliary superfields are explicitly given in Appendix B. There one can see that the auxiliary fields \( p_{\alpha\beta} \) are expressed in terms of the space-time derivative of the physical field \( B \) and therefore they acquire a nontrivial dynamics which modifies the quadratic action for the field \( B \) in
a substantial way. The same pattern is present in the fermionic sector once the equations (2.26) are used to eliminate the auxiliary $\bar{\beta}_\alpha$ field in terms of derivatives of the physical fermion $\zeta_\alpha$. Thus at the component level, as pointed out in ref. [6], the complex linear model seems to differ from the chiral one. However, since the equations of motion for the $\sigma$-model with linear multiplets are dual to the constraints of the chiral one, the two systems must become dual equivalent once the auxiliary fields of the linear multiplet are set on-shell and correspondingly the auxiliaries of the chiral multiplet are eliminated through the constraints. Indeed the elimination of the auxiliary fields from (2.22) leads to a physical action in $N = 1$ superspace (or in components) which is the Legendre transform of a chiral action.

Now we extend our analysis to the general $\sigma$-model whose potential is a function of chiral and complex linear superfields [2]

$$S = \int d^2x \ d^4\theta \ \Omega(\Phi^\mu, \bar{\Phi}^{\bar{\mu}}, \Sigma^a, \bar{\Sigma}^{\bar{a}})$$

with $\mu, \bar{\mu} = 1, \ldots, m$ and $a, \bar{a} = 1, \ldots, n$. In order to obtain the $N = 1$ reduction we expand the fields as in (2.8) and (2.18) and introduce the definitions

$$H = \begin{pmatrix} 0 & \Omega_{\mu \bar{\nu}} \\ \Omega_{\bar{\mu} \nu} & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \Omega_{ab} & \Omega_{\bar{a} \bar{b}} \\ \Omega_{\bar{a} b} & \Omega_{\bar{b} \bar{a}} \end{pmatrix}$$

$$N = \begin{pmatrix} \Omega_{a \mu} & \Omega_{a \bar{\mu}} \\ \Omega_{\bar{a} \mu} & \Omega_{\bar{a} \bar{\mu}} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2.41)

and

$$I = \begin{pmatrix} \phi^\mu \\ \bar{\phi}^{\bar{\mu}} \end{pmatrix}, \quad F = \begin{pmatrix} f^a \\ \bar{f}^{\bar{a}} \end{pmatrix}, \quad \Lambda_\alpha = \begin{pmatrix} \lambda^a_\alpha \\ \bar{\lambda}^{\bar{a}}_\alpha \end{pmatrix}$$

(2.42)

In terms of the above quantities the $N = 1$ action takes the form

$$S = \frac{1}{8} \int d^2x \ d^2\theta_1 \left[ -2D^\alpha_1 \bar{\mathcal{I}} \bar{D}_1 \mathcal{I} + D^\alpha_1 \mathcal{F}^T M D_1 \mathcal{F} + \Lambda^{\alpha T} M \Lambda_\alpha - D^\alpha_1 \mathcal{F}^T M P \Lambda_\alpha \\ -\Lambda^{\alpha T} P M D_1 \mathcal{F} + \Lambda^{\alpha T} [N, P] D_1 \mathcal{I} + D^\alpha_1 \mathcal{F}^T [P, N^T] \Lambda_\alpha \right]$$

(2.43)

Once again the spinor superfields $\Lambda_\alpha$ are auxiliary and can be eliminated using the on-shell condition

$$\Lambda_\alpha = M^{-1} P M D_1 \mathcal{F} - M^{-1} [N, P] D_1 \mathcal{I}$$

(2.44)

By doing so we find an action in which only the $N = 1$ superfields $\phi^\mu, \bar{\phi}^{\bar{\mu}}, f^a$ and $\bar{f}^{\bar{a}}$ appear and interact in a nontrivial manner

$$S = \frac{1}{8} \int d^2x \ d^2\theta_1 \left[ -2D^\alpha_1 \bar{\mathcal{I}} \bar{D}_1 \mathcal{I} + D^\alpha_1 \mathcal{F}^T M D_1 \mathcal{F} \\ -\left(D^\alpha_1 \mathcal{F}^T M P - D^\alpha_1 \mathcal{F}^T [P, N^T]\right) M^{-1} (P M D_1 \mathcal{F} - [N, P] D_1 \mathcal{I}) \right]$$

(2.45)
The second supersymmetry transformations are easily computed

\[ \delta \mathcal{I} = -\eta^\alpha PD_{1\alpha} \mathcal{I} \]
\[ \delta \mathcal{F} = -\eta^\alpha M^{-1} P M D_{1\alpha} \mathcal{F} + \eta^\alpha M^{-1} [N, P] D_{1\alpha} \mathcal{I} \]  

(2.46)

They can be rewritten as

\[ \delta \mathcal{X} = i\eta^\alpha JD_{1\alpha} \mathcal{X} \]  

(2.47)

having defined the vector

\[ \mathcal{X} = \begin{pmatrix} \mathcal{I} \\ \mathcal{F} \end{pmatrix} = \begin{pmatrix} \phi^\mu \\ \bar{\phi}^\a \\ f^a \\ \bar{f}^\a \end{pmatrix} \]  

(2.48)

and the complex structure

\[ J = i \begin{pmatrix} P & 0 \\ -M^{-1} [N, P] & M^{-1} P M \end{pmatrix} \]  

(2.49)

It is straightforward to check that \( J \) satisfies the conditions in \( (2.31) \) so that the underlying manifold is Kähler. In complete analogy with what we had done before, we can go to a holomorphic basis

\[ \mathcal{I}' = \mathcal{I} \quad \mathcal{F}' = \begin{pmatrix} f^a \\ \bar{f}^\a \end{pmatrix} \]  

(2.50)

where \( f^a \equiv \partial \Omega / \partial f^a \) and \( \bar{f}^\a \equiv \partial \Omega / \partial \bar{f}^\a \). In the new coordinate system the second supersymmetry transformations become

\[ \delta \mathcal{X}' = i\eta^\alpha R J R^{-1} D_{1\alpha} \mathcal{X}' \]  

(2.51)

where

\[ R = \begin{pmatrix} 1 & 0 \\ N & M \end{pmatrix} \]  

(2.52)

The standard form of the complex structure is then obtained from \( (2.51) \)

\[ J' = R J R^{-1} = i \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \]  

(2.53)

As a final step one can write the action in terms of the new fields

\[ S = \frac{1}{8} \int d^2 x \ d^2 \theta_1 \ D_{1\alpha} \mathcal{X}'^T \mathcal{G} D_{1\alpha} \mathcal{X}' \]  

(2.54)

and check that the metric \( \mathcal{G} \) has only barred-unbarred components.
The above analysis leads to the conclusion that models described by chiral superfields, complex linear superfields or both of them, once reduced to $\mathcal{N}=1$ superspace and with appropriate choices of complex coordinates, i.e. appropriate field redefinitions, all behave in a very similar manner. In particular they all share the property of possessing a Kähler metric from which geometrical objects can be easily obtained. However we wish to emphasize that the reduction procedure for the chiral case is conceptually different as compared to the complex linear one. In fact the solution of the chirality constraint allows one to determine completely the higher $\mathcal{N}=1$ components in terms of the lowest one. On the contrary, for the linear superfield only the highest component is given by the constraint equation, and in order to obtain the standard form for the $\mathcal{N}=1$ action one has to use the on-shell conditions. Thus the relevant question is: are the two formulations, with or without auxiliary fields equivalent? The results obtained in our previous work \[5\] seem to confirm the equivalence: there we have performed a one-loop calculation using the explicit $\mathcal{N}=2$ formalism, and we have found that the classical duality transformations are maintained at the quantum level. We study further this issue in the next section where the $\beta$-function for the mixed model is computed at one loop.

3 The one-loop $\beta$-function

We compute in superspace and start with the $\mathcal{N}=2$ version of the model

$$S = \int d^2x \, d^4\theta \, \Omega(\Phi^\mu, \Phi^\bar{\mu}, \Sigma^a, \Sigma^\bar{a})$$ (3.1)

where $\mu, \bar{\mu} = 1, \ldots, m$ and $a, \bar{a} = 1, \ldots, n$. In order to perform perturbative calculations it is advantageous to use the background field method and split the fields

$$\Phi \to \Phi + \Phi_0 \quad \bar{\Phi} \to \bar{\Phi} + \bar{\Phi}_0 \quad \Sigma \to \Sigma + \Sigma_0 \quad \bar{\Sigma} \to \bar{\Sigma} + \bar{\Sigma}_0$$ (3.2)

The action is then expanded around the background $\Phi_0, \bar{\Phi}_0, \Sigma_0$ and $\bar{\Sigma}_0$. We separate the free kinetic action, which determines the quantum propagators, from the interactions. We consider vertices quadratic in the quantum fields since that is all we need for a one-loop calculation

$$S = \int d^2x \, d^4\theta \left[ \Phi^\mu \Phi^\nu \delta_{\mu\bar{\nu}} - \Sigma^a \Sigma^\bar{a} \delta_{a\bar{a}} + (\Omega_{\mu\bar{\mu}} - \delta_{\mu\bar{\mu}})\Phi^\mu \Phi^\bar{\mu} + (\Omega_{a\bar{a}} + \delta_{a\bar{a}})\Sigma^a \Sigma^\bar{a} 
+ \frac{1}{2} \Omega_{\mu\nu} \Phi^\mu \Phi^\nu + \frac{1}{2} \Omega_{\mu\bar{\mu}} \Phi^\mu \Phi^\bar{\mu} + \frac{1}{2} \Omega_{a\bar{a}} \Sigma^a \Sigma^\bar{a} + \frac{1}{2} \Omega_{a\bar{b}} \Sigma^a \Sigma^\bar{b} 
+ \Omega_{a\bar{a}} \Sigma^a \Phi^\bar{a} + \Omega_{a\bar{a}} \Sigma^\bar{a} \Phi^a + \Omega_{a\bar{a}} \Sigma^a \Phi^\bar{a} + \Omega_{a\bar{a}} \Sigma^\bar{a} \Phi^a + \ldots \right]$$ (3.3)

The quantum fields are explicit while the background is implicit in the vertices given by derivatives of the potential $\Omega$. Superspace Feynman diagrams and standard D-algebra
techniques are the ingredients for loop calculations. The quantization of the chiral superfield is common knowledge, whereas the quantum treatment of the complex linear superfield has been obtained recently [8], [5]. We refer the reader to the relevant references for details. Here we simply recall that the chiral superspace propagators are given by

\[ < \bar{\Phi}^\mu \Phi^\mu > = -\frac{\delta^{(4)}(\theta - \theta')}{\Box} \delta^{\mu\bar{\mu}} \] (3.4)

Correspondingly for the complex linear superfield one has an effective propagator (see [5])

\[ < \bar{\Sigma}^\alpha \Sigma^\alpha > = \delta^{\alpha\bar{\alpha}} \left( \frac{D^2 \bar{D}^2}{\Box} + \frac{D_\alpha \bar{D}^2 D^\alpha}{\Box} \right) \delta^{(4)}(\theta - \theta') \] (3.5)

Additional factors of $D^2$ and $\bar{D}^2$ come from each chiral, antichiral quantum line respectively, at the vertices. We have not mentioned the infinite tower of ghosts introduced by the Batalin-Vilkovisky gauge-fixing procedure of the complex linear superfield [8] since, as shown in [5], they essentially decouple from the external background and do not contribute at one loop.

The one-loop $\beta$-function is computed evaluating all the divergent contributions to the effective action: they are given by local expressions that by dimensional analysis do not contain any derivatives. Thus the spinor covariant derivatives always stay on the quantum lines of the Feynman diagrams and the D-algebra is straightforward (we make use repeatedly of the identities listed in Appendix A). Some care is required in collecting all the terms with their appropriate combinatoric factors. It is convenient to introduce the following notation

\[ W = (\Omega_{\mu\bar{\mu}} - \delta_{\mu\bar{\mu}}) \quad V = (\Omega_{a\bar{a}} + \delta_{a\bar{a}}) \quad U = \Omega_{ab} \quad Z = \Omega_{a\bar{\mu}} \] (3.6)

The divergent contributions are then grouped in various sets: those corresponding to graphs which contain only $\Phi W \bar{\Phi}$ interactions which give rise to the one-loop divergence

\[ \Omega_1^{(1)} \rightarrow \frac{1}{\epsilon} \text{tr} \ln (1 + W) \] (3.7)

We note that, if we were to set to zero the complex linear superfields, the above result would be just the standard one-loop divergence for the $N = 2$ chiral $\sigma$-model with a corresponding metric $\beta$-function proportional to the Ricci tensor of the Kähler manifold (cfr. eq. (2.7)).

Then there are the graphs with $\Sigma V \bar{\Sigma}$ interactions which contribute

\[ \Omega_2^{(1)} \rightarrow -\frac{1}{\epsilon} \text{tr} \ln (1 - V) \] (3.8)
It is also easy to show that the sum of diagrams containing any number of $\Sigma V\Sigma$ vertices and an equal number of $\Sigma U\Sigma$ and $\bar{U}\bar{V}\Sigma$ interactions give rise to a divergent contribution of the form

$$\Omega_3^{(1)} \rightarrow -\frac{1}{\epsilon} \text{tr} \ln \left(1 - U \frac{1}{1 - V} \bar{U} \frac{1}{1 - \bar{V}}\right)$$

(3.9)

The results in (3.8) and (3.9), with chiral superfields set equal to zero, have been obtained in ref. [5] and lead to the one-loop $\beta$-function for the complex linear $\sigma$-model.

Finally there are the remaining diagrams which contain both chiral and linear quantum lines. In order to account for this type of terms let us define an effective chiral propagator with the $W$-vertices resummed

$$<< \bar{\Phi}\Phi >> = \frac{1}{\Box} \frac{1}{1 + W} \delta^{(4)}(\theta - \theta') \equiv \Pi$$

(3.10)

and an effective linear propagator with the $V$-vertices and the $U$- and $\bar{U}$-vertices resummed

$$<< \Sigma\Sigma >> = \left[ \frac{D^2 \bar{D}^2}{\Box} + \frac{D_a \bar{D}^2 D^a}{\Box} \left(1 - U \frac{1}{1 - V} \bar{U} \frac{1}{1 - \bar{V}}\right)^{-1}\right] \frac{1}{1 - V} \delta^{(4)}(\theta - \theta') \equiv \hat{\Pi}$$

(3.11)

It is easy to show that the last class of diagrams can be written in terms of $\Pi$ and $\hat{\Pi}$ as

$$\text{tr} \left[ \mathcal{Z} \Pi \hat{\Pi} + \frac{1}{2} \left(\mathcal{Z} \Pi \hat{\Pi}\right)^2 + \frac{1}{3} \left(\mathcal{Z} \Pi \hat{\Pi}\right)^3 + \ldots \right] = \text{tr} \sum_{n=1}^{\infty} \frac{1}{n} \left(\mathcal{Z} \Pi \hat{\Pi}\right)^n$$

(3.12)

The corresponding divergence is given by

$$\Omega_4^{(1)} \rightarrow \frac{1}{\epsilon} \text{tr} \ln \left(1 + \mathcal{Z} \frac{1}{1 + \mathcal{W}} \hat{\mathcal{Z}} \frac{1}{1 - \bar{V}}\right)$$

(3.13)

Finally, adding all the various contributions and using (3.6), we obtain

$$\Omega^{(1)} \rightarrow \frac{1}{\epsilon} \left[ \text{tr} \ln \left(\Omega_{\mu\mu} - \Omega_{\mu a} \Omega_{a a}^{-1} \Omega_{a\mu}\right) - \text{tr} \ln \left(\Omega_{ab} \Omega_{b\bar{b}}^{-1} \Omega_{\bar{b}a} - \Omega_{aa}\right) \right]$$

(3.14)

Now we show that this result is in perfect agreement with what expected from the duality correspondence between the chiral and the complex linear formulations. Under duality trasformation on the linear superfields the action in (3.1) is mapped into a pure chiral $\sigma$-model

$$S_D = \int d^2x \; d^4\theta \; \hat{\Omega}(\Phi^\mu, \bar{\Phi}^{\bar{\mu}}, \Psi^a, \bar{\Psi}^{\bar{a}})$$

(3.15)

where

$$\Psi^a = \frac{\partial \Omega}{\partial \Sigma^a}, \quad \bar{\Psi}^{\bar{a}} = \frac{\partial \Omega}{\partial \Sigma^{\bar{a}}}$$

(3.16)
and $\tilde{\Omega}$ is the Legendre transform of the $\Omega$ potential

$$\tilde{\Omega}(\Phi^\mu, \tilde{\Phi}^\mu, \Psi^a, \tilde{\Psi}^a) = \left[ \Omega(\Phi, \tilde{\Phi}, \Sigma, \tilde{\Sigma}) - \Psi^a \Sigma^a - \tilde{\Psi}^a \tilde{\Sigma}^a \right] \bigg|_{\Sigma = \Sigma(\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}), \tilde{\Sigma} = \tilde{\Sigma}(\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi})}$$  \hfill (3.17)

Our claim is that under duality transformations $\Omega(1)$ in (3.14) is mapped into the one-loop divergent contribution that one would obtain from a chiral model described in terms of coordinates $\Phi^\mu, \Psi^a, \tilde{\Phi}^\mu, \tilde{\Psi}^a$ with $\mu, \tilde{\mu} = 1, \ldots, m$ and $a, \tilde{a} = 1, \ldots, n$

$$\Omega(1) \rightarrow \tilde{\Omega}(1) = \frac{1}{\epsilon} \mathrm{tr} \ln \tilde{\Omega}_{i\tilde{j}}$$  \hfill (3.18)

where now $i, \tilde{j} = 1, \ldots, m + n$. Let us check this result in the simple case $m = n = 1$.

(The generalization is straightforward.) We rewrite (3.14) as

$$\Omega(1) = \frac{1}{\epsilon} \ln \frac{\Omega_{\Phi\Phi} \Omega_{\Sigma\Sigma} - \Omega_{\Phi\Sigma} \Omega_{\Sigma\Phi}}{\Omega_{\Sigma\Sigma} \Omega_{\Sigma\Sigma} - \Omega_{\Sigma\Sigma}^2}$$  \hfill (3.19)

Using the Legendre transform in (3.17) which defines $\tilde{\Omega}$, we express the derivatives of $\Omega$ with respect to $\Phi, \tilde{\Phi}, \Sigma, \tilde{\Sigma}$ in terms of derivatives of $\tilde{\Omega}$ with respect to $\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}$. With the definition

$$\Delta = \tilde{\Omega}_{\Psi\psi}^2 - \tilde{\Omega}_{\psi\Psi} \tilde{\Omega}_{\phi\phi}$$  \hfill (3.20)

we obtain

$$\Omega_{\Sigma\Sigma} = \frac{\tilde{\Omega}_{\Phi\Phi}}{\Delta} \quad \Omega_{\Sigma\Sigma} = -\frac{\tilde{\Omega}_{\Psi\Psi}}{\Delta}$$

$$\Omega_{\Phi\Sigma} = \frac{\tilde{\Omega}_{\Psi\Psi} \tilde{\Omega}_{\phi\phi} - \tilde{\Omega}_{\phi\phi} \tilde{\Omega}_{\Phi\psi}}{\Delta}$$

$$\Omega_{\phi\phi} = \tilde{\Omega}_{\phi\phi} + \frac{1}{\Delta} (\tilde{\Omega}_{\phi\phi} \tilde{\Omega}_{\phi\phi} - \tilde{\Omega}_{\phi\phi} \tilde{\Omega}_{\phi\phi}) \tilde{\Omega}_{\phi\phi}$$

$$+ \frac{1}{\Delta} (\tilde{\Omega}_{\Psi\Psi} \tilde{\Omega}_{\phi\phi} - \tilde{\Omega}_{\phi\phi} \tilde{\Omega}_{\Psi\Psi}) \tilde{\Omega}_{\phi\phi}$$  \hfill (3.21)

Substituting in (3.13) we obtain

$$\Omega(1) = \frac{1}{\epsilon} \ln (\tilde{\Omega}_{\phi\phi} \tilde{\Omega}_{\phi\phi} - \tilde{\Omega}_{\phi\phi} \tilde{\Omega}_{\phi\phi})$$  \hfill (3.22)

This is indeed the one-loop divergent correction to the effective potential of a chiral $\sigma$-model described by coordinates $(\Phi, \Psi, \tilde{\Phi}, \tilde{\Psi})$ and Kähler potential $\tilde{\Omega}$

$$\tilde{\Omega}(1) = \frac{1}{\epsilon} \mathrm{tr} \ln \tilde{\Omega}_{i\tilde{j}} = \frac{1}{\epsilon} \ln \det \tilde{\Omega}_{i\tilde{j}}$$  \hfill (3.23)
where

\[ \hat{\Omega}_{ij} = \left( \begin{array}{cc} \hat{\Omega}_{\Phi\Phi} & \hat{\Omega}_{\Phi\Psi} \\ \hat{\Omega}_{\Psi\Phi} & \hat{\Omega}_{\Psi\Psi} \end{array} \right) \] (3.24)

We note that there is a one-to-one correspondence between mixed models with vanishing one-loop \( \beta \)-function and chiral models with \( \beta^{(1)} = 0 \).

In the next section we study the condition of vanishing \( \beta \)-function to see whether a theory with \( N = 4 \) supersymmetry can be constructed.

4 \( N = 4 \) supersymmetry

We consider the action in (3.1) in its simplest version, i.e. \( m = n = 1 \). In this case the target manifold is 4-dimensional. With a duality transformation we can map the coordinates \((\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})\) into a set of chiral complex coordinates \((\Phi, \bar{\Phi}, \Psi, \bar{\Psi})\) and correspondingly for the potential

\[ \Omega(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \rightarrow \hat{\Omega}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \] (4.1)

with \( \hat{\Omega} \) obtained as Legendre transform of \( \Omega \) as in (3.17).

It is well known that the chiral model described by the potential \( \hat{\Omega} \) has \( N = 4 \) extended supersymmetry if the corresponding complex manifold is hyperkähler. In this case, if we express the supersymmetry transformations in terms of \( N = 1 \) superfields

\[ \delta \phi^i = i J^{(A)i}_j \epsilon^{(A)\alpha} D_\alpha \phi^j \quad A = 1, 2, 3 \] (4.2)

the three complex structures must satisfy the standard conditions (see eq. (2.31))

\[ J^{(A)2} = -1 \quad J^{(A)T} g J^{(A)} = g \quad N^{(A)k}_{ij} = 0 \quad d J^{(A)} = 0 \] (4.3)

and, in addition, the quaternionic relation

\[ J^{(A)i}_k J^{(B)k}_j = -\delta^{AB} \delta^i_j + \epsilon^{ABC} J^{(C)i}_j \] (4.4)

We know also that all hyperkähler manifolds are Ricci flat. Thus \( N = 4 \) chiral models have vanishing one-loop \( \beta \)-function, which from eqs. (3.22) and (2.7) means

\[ \hat{\Omega}_{\Phi\Phi} \hat{\Omega}_{\Psi\Psi} - \hat{\Omega}_{\Phi\Psi} \hat{\Omega}_{\Psi\Phi} = c - \text{number} \] (4.5)

Since we are considering a 4-dimensional target manifold, the converse is also true \[ \frac{1}{4} \], i.e. requiring (4.5) to be satisfied we have Ricci flatness and consequently the manifold is hyperkähler and the corresponding chiral \( \sigma \)-model is \( N = 4 \) supersymmetric.

Now we want to implement these results directly on the mixed chiral-linear model. Since duality is maintained at one loop, a vanishing \( \beta^{(1)} \)-function for the chiral model
leads to a vanishing $\beta^{(1)}$-function for the mixed model. In other words the condition in (4.5) is dual equivalent to

$$\frac{\Omega_{\Phi\Phi} \Omega_{\Sigma\Sigma} - \Omega_{\Phi\Sigma} \Omega_{\Sigma\Phi}}{\Omega_{\Sigma\Sigma} \Omega_{\Sigma\Sigma} - \Omega_{\Sigma \Sigma} \Omega_{\Sigma \Sigma}} = c - \text{number}$$

which obviously implies $\beta^{(1)}_{\text{mixed}} = 0$, cfr. (3.19). At this point if we were sure that $N = 4$ supersymmetry is not broken by duality transformations, we could conclude that a model which satisfies (4.6) is $N = 4$ supersymmetric. Motivated by this expectation, we proceed by explicit construction and find a class of mixed models which exhibit $N = 4$ supersymmetry.

Thus we start with the chiral-linear model in $N = 2$ superspace and look for two extra supersymmetries, which we demand to mix chiral and linear multiplets, to be linear in the fields and compatible with the chiral and linear constraints. These requirements lead to the following transformations

$$\begin{align*}
\delta \Phi &= \epsilon_{\alpha} \bar{\partial}^\alpha \Sigma \\
\delta \Sigma &= \epsilon_{\alpha} D^\alpha \Phi + \bar{\epsilon}_{\alpha} \bar{D}^\alpha \bar{\Sigma} \\
\delta \bar{\Phi} &= \bar{\epsilon}_{\alpha} \bar{\partial}^\alpha \bar{\Sigma} \\
\delta \bar{\Sigma} &= \bar{\epsilon}_{\alpha} \bar{D}^\alpha \bar{\Phi} + \epsilon_{\alpha} D^\alpha \Sigma
\end{align*}$$

(4.7)

where $\epsilon_{\alpha} = \xi_{\alpha} + i\zeta_{\alpha}$, $\bar{\epsilon}_{\alpha} = \xi_{\alpha} - i\zeta_{\alpha}$ are the complex parameters. It is easy to show that the corresponding algebra closes off-shell

$$[\delta_{(1)}, \delta_{(2)}] = [\delta_{\epsilon_{(1)}}, \delta_{\epsilon_{(2)}}] = 0 \quad [\delta_\epsilon, \delta_{\bar{\epsilon}}] = i\epsilon_\alpha \bar{\epsilon}_\beta \partial^{\alpha\beta}$$

(4.8)

Moreover the $\sigma$-model action is left invariant by the above transformations if and only if there exist two functions $G, \bar{G}$ such that

$$\begin{align*}
\Omega_{\Phi} D^\alpha \bar{\Sigma} + \Omega_{\Sigma} D^\alpha \Phi + \Omega_{\Sigma} D^\alpha \Sigma &= D^\alpha G \\
\Omega_{\bar{\Phi}} \bar{D}^\alpha \bar{\Sigma} + \Omega_{\bar{\Sigma}} \bar{D}^\alpha \bar{\Phi} + \Omega_{\Sigma} \bar{D}^\alpha \Sigma &= \bar{D}^\alpha \bar{G}
\end{align*}$$

(4.9)

It follows that the second derivatives of the potential must satisfy

$$\begin{align*}
\Omega_{\Phi\Phi} &= \Omega_{\Sigma\Sigma} \\
\Omega_{\Phi\Sigma} &= \Omega_{\Sigma\Sigma} \\
\Omega_{\Sigma\Phi} &= \Omega_{\Sigma\Sigma}
\end{align*}$$

(4.10)

The relations in (4.10) imply (4.6) and therefore $\beta^{(1)} = 0$. However, since the last conditions are stronger than (4.6) the class of models selected by (4.10) does not exhaust the whole class of $N = 4$ systems in four dimensions. A more general approach would require the definition of nonlinear supersymmetry transformations, with (4.6) as integrability conditions for the invariance of the action.

In order to study the geometry underlying the $N = 4$ model we consider its reduction to $N = 1$ superspace. We expand the chiral and the linear superfields into their $N = 1$ components as in (2.8), (2.18). Moreover we eliminate the components $F, G, \psi_\alpha$
as in (2.11), solving the constraints, and $\lambda_\alpha$ as in (2.44) using the on-shell condition. Finally for the $N = 1$ lowest components $\chi = (\phi, \bar{\phi}, f, \bar{f})$, we obtain three extra supersymmetries which, as in (4.2), can be expressed in the form

$$\delta \chi = i\eta^\alpha J(1) D_{1\alpha} \chi + i\xi^\alpha J(2) D_{1\alpha} \chi + i\zeta^\alpha J(3) D_{1\alpha} \chi$$

(4.11)

where $J(1)$ is given in (2.49) and

$$J(2) = \frac{i}{2} \begin{pmatrix} PM^{-1}[N, P] & 1 - PM^{-1}PM \\ 2 + PSM^{-1}[N, P] & S + SPM^{-1}PM \end{pmatrix}$$

(4.12)

$$J(3) = \frac{1}{2} \begin{pmatrix} -M^{-1}[N, P] & -P + M^{-1}PM \\ 2P - SM^{-1}[N, P] & SP + SM^{-1}PM \end{pmatrix}$$

The matrices $M$, $N$ and $P$ are given in (2.41) and $S$ is defined as

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(4.13)

Through direct computation, one verifies that $J^{(A)}$, $A = 1, 2, 3$ satisfy the conditions in (4.3) and (4.4), so that the complex manifold is hyperkahler.

5 Conclusions

We have studied two-dimensional supersymmetric $\sigma$-models described in terms of both chiral and complex linear superfields. Classically the linear superfield is dual equivalent to the chiral one, therefore on the basis of the correspondence complex linear $\rightarrow$ chiral, one is naturally lead to borrow the well known results obtained for $N = 2$ chiral non linear $\sigma$-models. The main properties of these models are the associated Kähler geometry, a one-loop metric $\beta$-function proportional to the Ricci tensor, a vanishing $\beta$-function at two and three loops and a nonvanishing correction at four loops [11]. However some caution must be used in a straightforward implementation of this program: one must be aware of the fact that the minimal and the nonminimal multiplets do differ in their auxiliary field content. As we have seen in detail a complete equivalence is obtained only if some $N = 1$ components of the nonminimal multiplet are set on-shell. In order to maintain a complete off-shell formulation of the theory one has to work in $N = 2$ superspace. Using this formalism we have computed the one-loop $\beta$-function for the mixed $\sigma$-model and shown that it is in perfect agreement with the expectations from duality correspondence.

At this point it might be interesting to push the calculation at higher perturbative orders. In addition since, both chiral and linear superfields can interact with supersymmetric Yang-Mills it would be worth to continue and extend the work started in ref. [12].
Having at our disposal two $N = 2$ multiplets that can be consistently coupled and quantized, it is natural to look for additional supersymmetries. Here we have constructed the simplest class of models which realizes an $N = 4$ invariance with an underlying hyperkahler geometry. This has been achieved starting from the condition of vanishing $\beta$-function and considering supersymmetry transformations linear in the fields. We have restricted our attention to the case of one minimal scalar coupled to one nonminimal scalar field, but the generalization to the case with $2n + 2n$ fields is actually straightforward and can be easily implemented. It would be interesting to consider the equation $\beta^{(1)} = 0$ in full generality, allowing for additional supersymmetry transformations nonlinear in the fields. This would lead to the identification of the most general class of $N = 4$ invariant models.

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A Conventions

In $N = 2$ superspace we define the supersymmetry generators\[\]

\[\begin{align*}
Q_\alpha &= i \left( \frac{\partial}{\partial \theta^\alpha} - \frac{1}{2} \bar{\theta}^\beta i \partial_\alpha \beta \right), \\
\bar{Q}_\beta &= i \left( \frac{\partial}{\partial \bar{\theta}^\beta} - \frac{1}{2} \theta^\alpha i \partial_\alpha \beta \right)
\end{align*}\]  

(A.1)

which satisfy the algebra

\[\{Q_\alpha, \bar{Q}_\beta\} = i \partial_\alpha \beta\]  

(A.2)

The spinor covariant derivatives are

\[\begin{align*}
D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} \bar{\theta}^\beta i \partial_\alpha \beta, \\
\bar{D}_\beta &= \frac{\partial}{\partial \bar{\theta}^\beta} + \frac{1}{2} \theta^\alpha i \partial_\alpha \beta
\end{align*}\]  

(A.3)

with

\[\{D_\alpha, \bar{D}_\beta\} = i \partial_\alpha \beta\]  

(A.4)

We list here some other relations that we have repeatedly used in the calculation of the $\beta$-function:

\[\begin{align*}
[D_\alpha, \bar{D}_\beta] &= -i \partial_\alpha \beta \bar{D}_\beta \\
D^2 \bar{D}_\beta D^2 &= \Box D^2 \\
D_\alpha i \partial_\alpha \beta \bar{D}_\beta &= -2 D^2 \bar{D}^2 - D_\alpha \bar{D}^2 \bar{D}_\alpha 
\end{align*}\]  

(A.5) \quad (A.6) \quad (A.7)

The $D$-algebra in the loop is completed using the identities

\[\begin{align*}
\delta^{(4)}(\theta - \theta') D^2 \bar{D}^2 \delta^{(4)}(\theta - \theta') &= \frac{1}{2} \delta^{(4)}(\theta - \theta') D^2 \bar{D}^2 D_\alpha \delta^{(4)}(\theta - \theta') = \delta^{(4)}(\theta - \theta') 
\end{align*}\]  

(A.8)

The reduction to $N = 1$ superspace is performed in terms of new coordinates

\[\begin{align*}
\theta_1^\alpha &= \theta^\alpha + \bar{\theta}^\alpha, \\
\theta_2^\alpha &= \theta^\alpha - \bar{\theta}^\alpha
\end{align*}\]  

(A.9)

and corresponding covariant derivatives

\[\begin{align*}
D_{1\alpha} &= D_\alpha + \bar{D}_\alpha = 2 \frac{\partial}{\partial \theta_1^\alpha} + \frac{1}{2} \bar{\theta}_1^\beta i \partial_\alpha \beta, \\
D_{2\alpha} &= D_\alpha - \bar{D}_\alpha = 2 \frac{\partial}{\partial \theta_2^\alpha} - \frac{1}{2} \bar{\theta}_2^\beta i \partial_\alpha \beta
\end{align*}\]  

(A.10)

which satisfy $\{D_{1\alpha}, D_{1\beta}\} = 2i \partial_\alpha \beta$. The supersymmetry generators are

\[\begin{align*}
Q_{1\alpha} &= Q_\alpha + \bar{Q}_\alpha, \\
Q_{2\alpha} &= Q_\alpha - \bar{Q}_\alpha
\end{align*}\]  

(A.11)
The nonminimal scalar multiplet in components

The most general solution to the constraint $D^2 \Sigma = 0$ has the following form

$$\Sigma(\theta, \bar{\theta}) = B + \theta^\alpha \rho_\alpha + \bar{\theta}^\alpha \bar{\zeta}_\alpha + \theta^\alpha \bar{\theta}^\beta (p_{\alpha \beta} - \frac{i}{2} \partial_{\alpha \beta} B) - \theta^2 H + \theta^2 \bar{\theta}^\beta \bar{\beta}_\alpha$$

+ $\bar{\theta}^2 \theta^\alpha (-\frac{i}{2} \partial_{\alpha \beta} \bar{\zeta}^\beta) - \frac{1}{4} \theta^2 \bar{\theta}^2 (3 \Box B + 2i \partial^{\alpha \beta} p_{\alpha \beta})$ (B.1)

where the fields appearing in the expansion are given by (we use the conventions of refs. [9, 6])

$$B = \Sigma| \quad \rho_\alpha = D_\alpha \Sigma| \quad \bar{\zeta}_\alpha = \bar{D}_\alpha \Sigma|$$
$$p_{\alpha \beta} = \bar{D}_\beta D_\alpha \Sigma| \quad H = D^2 \Sigma| \quad \bar{\beta}_\alpha = \frac{1}{2} D^\beta \bar{D}_\alpha D_\beta \Sigma|$$ (B.2)

In order to perform the reduction to $N = 1$ superspace we rewrite the previous multiplet as an expansion in $\theta_1, \theta_2$ using the definitions (A.9). In terms of $N = 1$ superfields the $\Sigma$ multiplet is then given by (see eq. (2.18))

$$\Sigma(\theta_1, \theta_2) = f(\theta_1) + \frac{1}{2} \theta_1^\alpha \lambda_\alpha(\theta_1) + \frac{1}{4} \theta_2^2 G(\theta_1)$$ (B.3)

where

$$f(\theta_1) = B + \frac{1}{2} \theta_1^\alpha (\rho_\alpha + \bar{\zeta}_\alpha) + \frac{1}{4} \theta_1^2 (p^\alpha_\alpha - H)$$
$$\lambda_\alpha(\theta_1) = (\rho_\alpha - \bar{\zeta}_\alpha) + \theta_1^\beta \left( p_{(\alpha \beta)} - \frac{i}{2} \partial_{\alpha \beta} B - \frac{1}{2} C_{\beta \alpha} H \right) - \frac{1}{2} \theta_1^2 \left( \bar{\beta}_\alpha + \frac{i}{2} \partial_{\alpha \beta} \bar{\zeta}^\beta \right)$$
$$G(\theta_1) = -(p^\alpha_\alpha + H) + \theta_1^\alpha \left( \bar{\beta}_\alpha + \frac{i}{2} \partial_{\alpha \beta} \bar{\zeta}^\beta \right) - \frac{1}{4} \theta_1^2 (3 \Box B + 2i \partial^{\alpha \beta} p_{\alpha \beta})$$ (B.4)

Corresponding expansions can be written for the $\bar{\Sigma}$–multiplet which satisfies the constraint $D^2 \bar{\Sigma} = 0$. Its components are obtained from the previous ones by simply interchanging barred and unbarred quantities and $\lambda_\alpha \to -\bar{\lambda}_\alpha$.

Using the previous expressions for the $N = 1$ superfields we can write the equations of motion (2.26) for the auxiliary fields $\Lambda_\alpha$ in components. For instance, setting the fermions to zero, the equations of motion for the auxiliary bosonic fields are

$$\mathcal{P}_{(\alpha \beta)} - \frac{1}{2} C_{\beta \alpha} P M^{-1} P M \mathcal{P}^\gamma_{\gamma} = \frac{1}{2} (1 + P M^{-1} P M) i \partial_{\alpha \beta} B + \frac{1}{2} C_{\beta \alpha} (1 - P M^{-1} P M) \mathcal{H}$$ (B.5)

where we have defined

$$\mathcal{B} = \begin{pmatrix} B^\alpha \\ B^a \end{pmatrix} \quad \mathcal{P}_{\alpha \beta} = \begin{pmatrix} p^\alpha_{\beta} \\ \bar{p}^a_{\alpha \beta} \end{pmatrix} \quad \mathcal{H} = \begin{pmatrix} H^a \\ H^a \end{pmatrix}$$ (B.6)
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