Order Parameter as an Additional State Variable of Unstable Traffic Flow

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Abstract. We discuss a phenomenological approach to the description of unstable vehicle motion on multilane highways that could explain in a simple way such observed self-organizing phenomena as the sequence of the phase transitions “free flow → synchronized motion → jam” and the hysteresis in them.

We introduce a new variable called order parameter that accounts for possible correlations in the vehicle motion at different lanes. So, it is principally due to “many-body” effects in the car interaction in contrast to such variables as the mean car density and velocity being actually the zeroth and first moments of the “one-particle” distribution function. Therefore, we regard the order parameter as an additional independent state variable of traffic flow and formulate the corresponding evolution equation governing the lane changing rate.

In this context we analyze the instability of homogeneous traffic flow manifesting itself in both of these phase transitions and endowing them with the hysteresis. Besides, the jam state is characterized by the vehicle flows at different lanes being independent of one another.

1 Introduction

The existence of a new basic phase in vehicle flow on multilane highways called the synchronized motion was recently discovered by Kerner and Rehborn [1], impacting significantly the physics of traffics as a whole. In particular, it turns out that the spontaneous formation of moving jams on highways proceeds mainly through a sequence of two transitions: “free flow → synchronized motion → stop-and-go pattern” [2]. Besides, all these transitions exhibit the hysteresis [2,3,4]. As follows from the experimental data [1,3,4] the synchronized mode is essentially a multilane effect. Recently Kerner [5,6] assumed that the transition “free flow → synchronized mode” is caused by “Z”-like form of the overtaking probability depending on the car density.

There have been proposed several macroscopic models dealing with multilane traffic flow [7,8,9,10,11,12,13]. Both these models specify the traffic dynamics completely in terms of the car density $\rho$, mean velocity $v$, and, may be, the velocity variance $\theta$ or ascribe these quantities to the vehicle flow at each lane individually. Nevertheless, a quantitative description of the synchronized mode is far from being developed well because of its complex structure [3]. In particular, it can form the totally homogeneous (i) and
homogeneous-in-speed (ii) flows [1]. Especially in the latter case there is no explicit relationship between the mean car velocity $v$ and density $\rho$, with the value of $v$ being actually constant and less than that of free flow. The other important feature is the key role of some cars bunched together and traveling much faster than the typical ones, which enables to regard them as a special car group [1]. Therefore, in the synchronized mode the function of car distribution in the velocity space should have two maxima and we will call such fast car groups platoons in speed. These features of the synchronized mode have been substantiated also in [15] using single-car-data. In particular, it has been demonstrated that the synchronized mode exhibits small correlations between fluctuations in the car flow, velocity and density. There is only a strong correlation between the velocities at different lanes taken at the same time and decreasing sufficiently fast as the time difference increases. By contrast, there are strong long-time correlations between the flow and density in the free flow state as well as the stop-and-go mode.

Keeping in mind a certain analogy with aggregation processes in physical systems Mahnke et al. [16,17] proposed a kinetic model for the formation of the synchronized mode treated as the motion of a large car cluster. In the present paper following practically the spirit of the Landau theory of phase transitions we develop a phenomenological approach to the description of this process. We ascribe to the vehicle flow an additional internal parameter will be called below the order parameter $h \in (0, 1)$ characterizing the possible correlations in the vehicle motion at different lanes and write for it a governing equation. For the car motion where drivers do not change lane at all we set $h = 0$, in the opposite limit $h = 1$.

2 Order Parameter and the Individual Driver Behavior

For fixed values of $\rho$ and $v$ the order parameter $h$ is assumed to be uniquely determined, thus, for a uniform vehicle flow we write:

$$\tau \frac{dh}{dt} = -\Phi(h, \rho, v),$$

(1)

where $\tau$ is the delay time and the function $\Phi(h, \rho, v)$ fulfills the inequality:

$$\frac{\partial \Phi}{\partial h} > 0.$$  

(2)

We note that the time $\tau$ characterizes the delay in the driver decision of changing lanes but not in the control over the headway, so, this delay can be prolonged. The particular value $h(v, \rho)$ of the order parameter results from the compromise between the danger of an accident during changing lanes and the will of driver to move as fast as possible. Obviously, the lower is the mean vehicle velocity $v$ for a fixed value of $\rho$, the weaker is the lane-changing danger and the stronger is the will to move faster. Besides, the higher is the
vehicle density $\rho$ for a fixed value of $v$, the stronger is this danger (here the will has no effect). Thus, the dependence $h(v, \rho)$ is an decreasing function of $v$ and $\rho$, so, due to (3):

$$\frac{\partial \Phi}{\partial v} > 0, \quad \frac{\partial \Phi}{\partial \rho} > 0,$$

with the latter inequality being caused by the danger effect only. Equation (3) describes actually the behavior of the drivers that prefer to move faster than the statistically mean vehicle and whose readiness for risk is greatest. Exactly this group of drivers (platoons in speed) govern the value of $h$.

There is, however, another characteristics of the driver behavior, it is the mean velocity $v = \nu(h, \rho)$ chosen by the statistically mean driver taking into account also the danger resulting from the frequent lane changes by the “fast” drivers. Following typical assumptions the velocity $\nu(h, \rho)$ as a function of $\rho$ is considered to be decreasing:

$$\frac{\partial \nu}{\partial \rho} < 0 \quad \text{and} \quad \rho \nu(\rho) \to 0 \quad \text{as} \quad \rho \to \rho_0,$$

where $\rho_0$ is the upper limit vehicle density on road. In general, the dependence of $\nu(h, \rho)$ on $h$ should be increasing for small values of the vehicle density, $\rho \ll \rho_0$, because in this case the lane-changing makes no substantial danger to traffic and practically all the drives can pass by vehicles moving at lower speed without risk. By contrast, when the vehicle density is sufficiently high, $\rho \sim \rho_0$, the lane-changing is due to the car motion of the most “impatient” drivers whose behavior makes an additional danger to the main part of other drivers and the velocity $\nu(h, \rho)$ has to decrease as the order parameter $h$ increases. For certain intermediate values of the vehicle density, $\rho \approx \rho_c$, this dependence is to be weak as well as near the boundary points, so:

$$\frac{\partial \nu}{\partial h} > 0 \quad \text{for} \quad \rho < \rho_c, \quad \frac{\partial \nu}{\partial h} < 0 \quad \text{for} \quad \rho > \rho_c, \quad \frac{\partial \nu}{\partial h} = 0 \quad \text{at} \quad h = 0, 1.$$

Then the governing equation (1) takes the form:

$$\tau \frac{dh}{dt} = -\phi(h, \rho), \quad \text{where} \quad \phi(h, \rho) \overset{\text{def}}{=} \Phi[h, \rho, \nu(h, \rho)]$$

and the condition $\phi(h, \rho) = 0$ specifies the steady state dependence $h(\rho)$ of the order parameter on the vehicle density.

Let us, now, study properties and stability of this steady state solution. From Eq. (1) we get

$$\frac{\partial \phi}{\partial h} = \frac{\partial \Phi}{\partial h} + \frac{\partial \Phi}{\partial v} \frac{\partial \nu}{\partial h}, \quad \frac{\partial \phi}{\partial \rho} = \frac{\partial \Phi}{\partial \rho} + \frac{\partial \Phi}{\partial v} \frac{\partial \nu}{\partial \rho}.$$

As mentioned above, the value of $\partial \Phi/\partial \rho$ is solely due to the danger during changing lanes, so this term can be ignored until the vehicle density $\rho$ becomes sufficiently high. Thus, in a certain region $\rho < \rho_h < \rho_0$ the derivative
The region of the traffic flow instability in the $h\rho$-plane and the form of the curve $h(\rho)$ displaying the dependence of the order parameter on the vehicle density.

$$\frac{\partial \phi}{\partial \rho} \sim (\frac{\partial \Phi}{\partial v})(\frac{\partial \theta}{\partial \rho}) < 0$$

by virtue of (3) and (4) and the function $h(\rho)$ is increasing or decreasing for $\frac{\partial \phi}{\partial h} > 0$ or $\frac{\partial \phi}{\partial h} < 0$, respectively. This statement follows directly from the relation $\frac{dh}{d\rho} = -\frac{\partial \phi}{\partial \rho} \frac{\partial \phi}{\partial h}^{-1}$.

For long-wave perturbations of the vehicle distribution on a highway the density $\rho$ can be treated as a constant. So, according to the governing equation (6), the steady-state traffic flow is unstable if $\frac{\partial \phi}{\partial h} < 0$. Due to (2) and (5) the first term in the expression for $\frac{\partial \phi}{\partial h}$ in (7) is dominant in the vicinity of the lines $h = 0$ and $h = 1$, thus, in these regions the curve $h(\rho)$ is increasing and the steady state traffic flow is stable. For $\rho < \rho_c$ the value $\frac{\partial \theta}{\partial h} > 0$, inequality (8), and, thereby, the region $\{0 < h < 1, 0 < \rho < \rho_c\}$ corresponds to the stable vehicle motion. However, for $\rho > \rho_c$ there can be an interval of the order parameter $h$ where the derivative $\frac{\partial \phi}{\partial h}$ changes the sign and the vehicle motion becomes unstable. Therefore, as the car density $\rho$ grows causing the increase of the order parameter $h$ it can go into the instability region wherein $\frac{dh}{d\rho} < 0$. Under these conditions the curve $h(\rho)$ is to look like “S” (Fig. 1a) and its decreasing branch corresponds to the unstable vehicle flow. The lower increasing branch matches the free-flow state, whereas the upper one should be related to the synchronized phase because it is characterized by the order parameter coming to unity.

### 3 Phase Transitions and the Fundamental Diagram

The obtained dependence $h(\rho)$ actually describes the first order phase transition in the vehicle motion. Indeed, when increasing the car density exceeds the value $\rho_1$ the free flow becomes absolutely unstable and the synchronized mode forms through a sharp jump in the order parameter. If, however, after that the car density decreases the synchronized mode will persist until the car density attains the value $\rho_2 < \rho_1$. It is a typical hysteresis and the region $(\rho_2, \rho_1)$ corresponds to the metastable phases of traffic flow. It should
be noted that the stated approach to the description of the phase transition “free flow → synchronized mode” is rather similar to the hypothesis by Kerner [5,6] about “Z”-like dependence of the overtaking probability on the car density that can cause this phase transition.

Let us, now, discuss a possible form of the fundamental diagram showing $j = \rho \vartheta(\rho)$ where, by definition, $\vartheta(\rho) = \vartheta[h(\rho), \rho]$. Fig. 2 displays the dependence $\vartheta(h, \rho)$ of the mean vehicle velocity on the density $\rho$ for the fixed limit values of the order parameter $h = 0$ or 1. For small values of $\rho$ these curves practically coincide with each other. As the vehicle density $\rho$ grows and until it comes close to the critical value $\rho_c$ where the lane change danger becomes substantial, the velocity $\vartheta(1, \rho)$ practically does not depend on $\rho$. So at the point $\rho_c$ at which the curves $\vartheta(1, \rho)$ and $\vartheta(0, \rho)$ meet each other the former curve, $\vartheta(1, \rho)$, is to exhibit sufficiently sharp decrease in comparison with the latter one. Therefore, on one hand, the function $j_1(\rho) = \rho \vartheta(1, \rho)$ has to be decreasing for $\rho > \rho_c$. On the other hand, at the point $\rho_c$ for $h \ll 1$ the effect of the lane change danger is not extremely strong, it only makes the lane change ineffective, $\partial \vartheta/\partial h \approx 0$ (compare (5)). So it is reasonable to assume the function $j_0(\rho) = \rho \vartheta(0, \rho)$ increasing near the point $\rho_c$. Under the adopted assumptions the relative arrangement of the curves $j_0(\rho), j_1(\rho)$ is demonstrated in Fig. 2b, and Fig. 2c shows the fundamental diagram of traffic flow resulting from Fig. 1 and Fig. 2b.

The developed model predicts also the same type phase transition for large values of the order parameter. In fact, in an extremely dense traffic flow changing lanes is sufficiently dangerous and the function $\Phi(h, v, \rho)$ describing the driver behavior is to depend strongly on the vehicle density as $\rho \rightarrow \rho_0$. In addition, the vehicle motion becomes slow. Under such conditions the former term in the expression for $\partial \vartheta / \partial \rho$ in (3) should be dominant and, so, $\partial \vartheta / \partial \rho > 0$ and the stable vehicle motion corresponding to $\partial \vartheta / \partial h > 0$ is characterized by the decreasing dependence of the order parameter $h(\rho)$ on...
the vehicle density $\rho$ for $\rho > \rho_h$. Therefore, as the vehicle density $\rho$ increases the curve $h(\rho)$ can again go into the instability region (in the $h\rho$-plane), which has to give rise to a jump from the synchronized mode to a jam. The latter matches small values of the order parameter $h$ (Fig. 1b), so, it should comprise the vehicle flows along different lane where lane changing is depressed, making them practically independent of one another.

4 Conclusion

We have introduced an additional state variable of the traffic flow, the order parameter $h$, that accounts for internal correlations in the vehicle motion caused by the lane changing. Since such correlations are due to the “many-body” effects in the car interaction the order parameter is regarded as an independent state variable. Keeping in mind general properties of the driver behavior we have written the governing equation for this variable.

It turns out that in this way such characteristic properties of the traffic flow instability as the sequence of the phase transitions “free flow $\rightarrow$ synchronized motion $\rightarrow$ jam” can be described without additional assumptions. Moreover, in this model both the phase transitions are of the first order and exhibits hysteresis. Besides, the synchronized mode corresponds to highly correlated vehicle flows along different lanes, $h \approx 1$, whereas in the free flow and the jam these correlations are depressed, $h \ll 1$. So, the jam phase actually comprises mutually independent car flows along different lanes.

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