Planar Graphs of Bounded Degree have Constant Queue Number*

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Abstract

A queue layout of a graph consists of a linear order of its vertices and a partition of its edges into queues, so that no two independent edges of the same queue are nested. The queue number of a graph is the minimum number of queues required by any of its queue layouts. A long-standing conjecture by Heath, Leighton and Rosenberg states that the queue number of planar graphs is constant. This conjecture has been partially settled in the positive for several subfamilies of planar graphs (most of which have bounded treewidth). In this paper, we make a further important step towards settling this conjecture. We prove that planar graphs of bounded degree (which may have unbounded treewidth) have constant queue number.

A notable implication of this result is that every planar graph of bounded degree admits a three-dimensional straight-line grid drawing in linear volume. Further implications are that every planar graph of bounded degree has constant track number, and that every $k$-planar graph (i.e., every graph that can be drawn in the plane with at most $k$ crossings per edge) of bounded degree has constant queue number.

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1 Introduction

Queue layouts of graphs form a well-known type of linear layouts and play an important role in various fields, e.g., in sorting [2], scheduling [20], VLSI circuit design [15], and graph drawing [13, 17]. A queue layout of a graph consists of a vertex ordering and a partition of its edges into queues, so that no two independent edges of the same queue are nested [14]; see Fig. 1a. The queue number $qn(G)$ of a graph $G$ is the smallest number of queues required by any of its queue layouts. Note that queue layouts form the “dual” concept of stack layouts [16, 24] (widely known as book embeddings), in which two edges of the same stack are allowed to nest but not to cross.

It is known that there exist non-planar graphs on $n$ vertices with $\Theta(n)$ queue number, for example, the queue number of the complete graph $K_n$ is $\lceil n/2 \rceil$ [14]. Moreover, there exist graphs of bounded degree that may require arbitrarily many queues [23]. Among the graphs having sublinear queue number are those with a subquadratic number of edges [13], and those that belong to any minor-closed graph family [7]. Constant queue number is achieved by all graphs of bounded treewidth. In particular, a graph with treewidth $w$ has queue number $O(2^w)$ [9, 21]. Improved bounds (linear in the parameter) are known for graphs of bounded pathwidth [7], bounded track number [9], or bounded bandwidth [13]; for a survey we refer the reader to [9].

A rich body of literature focuses on planar graphs. In fact, it is known that the graphs that admit queue layouts with only one queue are the arched-level planar graphs [14], which are planar graphs with at most $2n - 3$ edges over $n$ vertices (note that testing whether a graph is arched-level planar is NP-complete [13]). Trees are arched-level planar and therefore have queue number one [14]. Outerplanar graphs have queue number at most two [13], Halin graphs and series-parallel graphs have queue number at most three [12, 19], and planar 3-trees have queue number at most five [1]. However, it is still unknown whether the queue number of planar graphs is constant. In particular, back in 1992, Heath, Leighton and Rosenberg [13] conjectured that every planar graph has constant queue number. Notably, this conjecture has not been settled after almost three decades. The best-known upper bound is due to Dujmović [5], who showed that the queue number of planar graphs on $n$ vertices is $O(\log n)$ improving upon a previous polylogarithmic bound by Di Battista et al. [4]. On the other hand, the best-known lower bound is due to a family of planar 3-trees that require four queues [1].

It is worth noting that a positive answer to the conjecture by Heath, Leighton and Rosenberg [13] would have several remarkable implications. Here we name few of them. A first implication is that every planar graph would admit a three-dimensional grid drawing in linear volume [22], which is a major open problem in graph drawing first posed by Felsner et al. [10] back in 2003. A second one is that every planar graph would have constant track number [8]. Moreover, every bipartite planar graph would have a 2-layer drawing and a corresponding edge-coloring of constant size, in which no two edges of the same color cross [9]. Finally, it is known that if the queue number of planar graphs is constant, then the same holds for the broader family of $k$-planar graphs, i.e., of those graphs that can be drawn in the plane such that each edge is crossed at most $k$ times (for fixed values of $k$) [9].

**Our contribution.** In this paper, we make an important step towards settling the conjecture by Heath, Leighton and Rosenberg [13] that planar graphs have constant queue number. Namely, we prove that every planar graph of maximum degree $\Delta$ has queue number $O(\Delta^c)$, where $c$ is a small constant. This implies that the conjecture holds for planar graphs of bounded degree. More precisely, the main contribution of this paper is the following theorem.

**Theorem 1.** Every planar graph of maximum degree $\Delta$ has queue number at most $32(2\Delta - 1)^6 - 1$. 


Figure 1: Illustration of: (a) a queue layout of the complete graph $K_4$ on 4 vertices $\{u_1, \ldots, u_4\}$ with two queues (solid and dotted), (b) two nesting edges $(u_1, v_1)$ and $(u_2, v_2)$, (c) a 3-necklace, and (d) a 3-rainbow.

The proof of Theorem 1 is constructive and yields an algorithm that computes a queue layout of the input graph $G$ with at most $32(2\Delta - 1)^6 - 1$ queues in polynomial time. In particular, the algorithm works in $O(n)$ time, where $n$ is the number of vertices of $G$, under the unit-cost RAM model of computation with word size linear in $n \log \Delta$. We remark that this model of computation makes use of reasonable assumptions, which fit the reality of common programming languages [11]; for example, Chan [3] used it to efficiently solve fundamental problems in computational geometry.

Wood [22] proves that a graph on $n$ vertices that belongs to a proper minor-closed family of graphs, such as planar graphs, has a straight-line drawing on a $O(1) \times O(1) \times O(n)$ three-dimensional grid if and only if its queue number is constant. This result combined with Theorem 1 yields the following corollary. We remark that the best upper bound currently known for the volume of three-dimensional grid drawings of planar graphs is $O(n \log n)$ [5].

**Corollary 1.** Every planar graph of bounded degree admits a straight-line drawing on a $O(1) \times O(1) \times O(n)$ three-dimensional grid.

Closely related to the queue layouts are the so-called track layouts. In a track layout, the vertices of a graph are partitioned into sequences, called tracks, such that the vertices in each track form an independent set and the edges between each pair of tracks do not cross. The track number of a graph is the minimum number of tracks required by any of its track layouts. Whether planar graphs have constant track number is still an open question, with the current best upper bound being $O(\log n)$ [3]. Dujmović, Pór and Wood [8] prove that a graph with queue number at most $q$ has track number $O(4^q)$, which together with Theorem 1 implies that planar graphs of bounded degree have constant track number. More precisely, we have the following.

**Corollary 2.** Every planar graph of maximum degree $\Delta$ has track number $O(4^{\Delta^6})$.

Another related problem is the 2-track thickness problem, in which one seeks for a 2-layer drawing of a bipartite graph and a corresponding coloring of its edges with as few colors as possible, such that no two edges of the same color cross; the minimum number of colors required in any such drawing is referred to as 2-track thickness. Again, it is not known whether the 2-track thickness of bipartite planar graphs is constant, with the best upper bound being $O(\log n)$ [9], which is due to a result by Dujmović and Wood [9] that relates the queue number of a bipartite planar graph and its 2-track thickness. Using the same result, together with Theorem 1, we obtain that every bipartite planar graph of bounded degree has constant 2-track thickness.

**Corollary 3.** Every bipartite planar graph of maximum degree $\Delta$ has 2-track thickness $O(\Delta^6)$.

A last corollary of Theorem 1 stems from another result by Dujmović and Wood [9], who prove that if the queue number of planar graphs is constant, then the same holds for $k$-planar graphs when $k$ is a fixed constant. Their proof relies on a planarization technique, in which each crossing is replaced by a degree-4 vertex. If the input is a $k$-planar graph of bounded degree, the obtained planarization is also of bounded degree, and we obtain the following corollary. Note that, Dujmović and Frati [6] recently proved that $k$-planar graphs have queue number $O(\log n)$. 
Corollary 4. For fixed \( k \geq 0 \), every \( k \)-planar graph of bounded degree has constant queue number.

Proof strategy. Our approach starts with a layering of the vertices of the input graph obtained from a breadth-first search (BFS) traversal of it. This layering only serves as a basis for computing the linear order of the vertices in the queue layout. In particular, the vertices that belong to earlier layers in the BFS-tree \( T \) will precede those belonging to subsequent layers of \( T \). As a consequence, all edges that belong to \( T \) do not nest in such a linear order. The main challenge with this approach is to deal with edges that are not part of \( T \). Such edges either connect vertices on the same layer or vertices lying on consecutive layers. While the second type of edges can be eliminated by subdividing them a constant number of times, the first type of edges may still result in arbitrarily large groups of pairwise nesting edges, commonly called rainbows. To cope with this issue, we change the order of the vertices on each layer so as to eliminate all nestings between edges connecting vertices of the same layer. On the other hand, this will unavoidably introduce rainbows formed by edges of \( T \). However, we reorder the vertices such that the maximum number of edges of \( T \) in the same rainbow is bounded by a polynomial in \( \Delta \). For ease of description, we first discuss our approach in a special case, namely when the edges that do not belong to \( T \) form a perfect matching on its leaves. Then we show how to use the solution of the special case for the general case.

Paper organization. In Section 2 we introduce notation and definitions that are used throughout this paper. We also present results from the literature that we exploit in our algorithm. In Section 3 we start by giving an overview of the proof technique and then continue with its detailed description. We conclude in Section 4 by posing a question which might shed some further light towards solving the conjecture by Heath, Leighton and Rosenberg.

2 Preliminaries

Queue Layouts. Let \( G = (V,E) \) be an \( n \)-vertex simple connected undirected graph, that is, a graph with neither self-loops nor multi-edges. We denote an edge between vertices \( u \) and \( v \) by \((u,v)\). Let \( \prec \) be a linear order of the vertices of \( G \). Consider two independent edges \((u_1,v_1)\) and \((u_2,v_2)\), that is, edges \((u_1,v_1)\) and \((u_2,v_2)\) do not share a common endvertex. Up to a renaming of their endvertices, we may assume that \( u_1 \prec v_1 \) and \( u_2 \prec v_2 \). We say that \((u_1,v_1)\) nests \((u_2,v_2)\) with respect to \( \prec \) if and only if \( u_1 \prec u_2 \prec v_2 \prec v_1 \); see Fig. 1b. Consider now \( k \) edges \((u_1,v_1),\ldots,(u_k,v_k)\) that are pairwise independent, such that \( u_i \prec v_i \) for each \( i = 1,\ldots,k \). If \( u_1 \prec v_1 \prec u_2 \prec v_2 \prec \ldots \prec u_k \prec v_k \), then we say that edges \((u_1,v_1),\ldots,(u_k,v_k)\) form a \( k \)-necklace; see Fig. 1c. On the other hand, if \( u_1 \prec \ldots \prec u_k \prec v_k \prec \ldots \prec v_1 \), then we say that edges \((u_1,v_1),\ldots,(u_k,v_k)\) form a \( k \)-rainbow; see Fig. 1d.

A preliminary result by Heath and Rosenberg [14] shows that a graph admits a queue layout with \( k \) queues if and only if there exist a linear order of its vertices in which no \((k+1)\)-rainbow is formed. Another useful tool for analyzing the queue number of graphs is the fact that the queue number of a graph is bounded by the queue number of any of its subdivisions. More precisely:

Lemma 1 (Dujmović and Wood [9]). Let \( D \) be a subdivision of a graph \( G \) obtained by subdividing each of the edges of \( G \) at most \( k \) times. If \( qn(D) \leq q \), then \( qn(G) \leq \frac{1}{2}(2q+2)^2k - 1 \).

Drawings and Representations. A drawing of a graph \( G \) is a mapping of the vertices of \( G \) to distinct points of the plane, and of the edges of \( G \) to Jordan arcs connecting their corresponding endvertices but not passing through any other vertex. A drawing is planar if no two edges intersect,
Figure 2: Illustration of: (a) the octahedron graph with a BFS-tree (drawn bold) rooted at a specific vertex (colored gray), and (b) its corresponding ordered concentric representation \[18\].

except possibly at a common endvertex. A graph is planar if it admits a planar drawing. A planar drawing subdivides the plane into topologically connected regions, called faces. The infinite region is called the outer face.

Central in our approach are also the so-called ordered concentric representations, which were recently studied by Pupyrev \[18\]. An ordered concentric representation of a planar graph \(G\) is a planar drawing of \(G\) where the vertices are located at concentric circles \(C_0, C_1, \ldots, C_{h-1}\) with decreasing radii centered at a point \(c\) of the plane, except for a single vertex \(r\), called the center of the representation, which is located at point \(c\). All vertices on circle \(C_i\) have graph-theoretic distance \(h - i\) from \(r\), for \(i = 0, 1, \ldots, h - 1\). It follows that each edge of \(G\) either has both its endvertices at the same circle (level edge) or at two consecutive circles (binding edge). Moreover, in an ordered concentric representation, the following three properties hold: (R.1) each level edge is drawn outside the circle on which its endvertices are located, (R.2) each binding edge consists of at most two segments, one required segment which is drawn between the two circles on which its endvertices are located and one optional segment which is drawn outside both of these circles, and (R.3) vertex \(r\) is incident to the outer face; for an illustration refer to Fig. 2. To compute an ordered concentric representation of a planar graph \(G\), Pupyrev \[18\] first computes a BFS-tree of \(G\). The root of this tree is the center of the representation, while its edges are binding edges drawn between the two corresponding circles (see R.2). His result is summarized as follows.

Lemma 2 (Pupyrev \[18\]). Given an \(n\)-vertex planar graph \(G = (V,E)\), it is possible to compute in \(O(n)\) time an ordered concentric representation \(\{C_0, C_1, \ldots, C_{h-1}\}\) of \(G\) with center \(r \in V\), such that \(h\) is the height of a breadth-first search tree of \(G\) rooted at \(r\).

Given a spanning tree \(T\) of a graph \(G\) rooted at a vertex \(r\), and a vertex \(v\) of \(G\), we denote by \(\text{dist}(v)\) the graph-theoretic distance of \(v\) from \(r\) in \(T\). Clearly, \(\text{dist}(r) = 0\), while for each leaf \(v\) of \(T\), it holds that \(\text{dist}(v) \leq h(T)\), where \(h(T)\) is the height of \(T\). For a vertex \(v\) of \(G\), we refer to the value \(h(T) - \text{dist}(v)\) as the layer of \(v\) in \(T\), which we denote by \(\ell(v)\). Note that given an ordered concentric representation \(R = \{C_0, C_1, \ldots, C_{h-1}\}\) of a graph \(G\) computed using a BFS-tree \(T\) of \(G\), for each vertex that belongs to circle \(C_i\) in \(R\), its layer in \(T\) is \(i\), for each \(i = 0, 1, \ldots, h - 1\).

3 Description of the Algorithm

In this section, we prove that every planar graph of bounded degree has constant queue number. To obtain this result, we employ a constructive approach. Namely, given a planar graph of maximum degree \(\Delta\), we describe an efficient algorithm that computes a queue layout of this graph with at most \(32(2\Delta - 1)^6 - 1\) queues. This section is organized as follows: First, we introduce in Section 3.1 a special subfamily of plane graphs with maximum degree \(\Delta\), for which it is possible to compute a
Figure 3: Illustration of a $\Delta$-matched graph $G$ with $\Delta = 3$, in which the edges of $M$ are drawn bold below the horizontal line $L : y = 0$. The label of each edge of $M$ corresponds to its nesting-value, while the label of each vertex of $G$ corresponds to its matching-value.

queue layout with at most $2\Delta - 2$ queues. Then, in Section 3.2, we reduce the problem of finding a queue layout of a general planar graph of degree $\Delta$ to the already discussed special case. This reduction increases the number of queues used to $32(2\Delta - 1)^6 - 1$. We conclude in Section 3.3 by discussing the time complexity of our algorithm.

3.1 The Special Case: $\Delta$-matched Graphs

In this section we consider a special subfamily of planar graphs of degree $\Delta$, which we call $\Delta$-matched graphs. Formally, a $\Delta$-matched graph $G = (V,E)$ consists of a $(\Delta - 1)$-ary tree $T$ with $|V|$ vertices, rooted at a vertex $r \in V$, and a perfect matching $M$ on the leaves of $T$, such that there exists a planar drawing $\Gamma$ of $G$ with the following properties (refer to Fig. 3 for an illustration):

(P.1) all leaves of $T$ lie on a horizontal line $L : y = 0$,
(P.2) all edges of $T$ (of $M$) are drawn above (below, respectively) $L$
(P.3) vertex $r$ lies on the outer face of $\Gamma$, and
(P.4) the vertices of layer $\ell > 0$ in $T$ are drawn on a horizontal line $L_\ell : y = \ell$.

Note that the root $r$ of $T$ has degree at most $(\Delta - 1)$, while the leaves of $T$ have degree 2 in $G$. All other vertices of $G$ have degree at most $\Delta$. Hence, graph $G$ has maximum degree at most $\Delta$. Our goal is to compute a queue layout of $G$ with at most $2\Delta - 2$ queues. Observe that drawing $\Gamma$ can be easily converted into an ordered concentric representation of $G$ with center $r$ in which all binding edges are part of $T$.

First, we assign an integer value, called nesting-value, to all edges of $M$. Recall that each edge of $M$ connects two vertices of $G$ that are leaves in $T$, and by Property P.1 are along the horizontal line $L : y = 0$ in the drawing $\Gamma$ of $G$. We denote the order in which the endvertices of the edges of $M$ appear along $L$ by $\prec_L$. Consider now an edge $e$ of $M$. Edge $e$ has nesting-value zero if $e$ is not nested by any other edge in $\prec_L$. Edge $e$ has nesting-value $i > 0$ if the maximum nesting-value of all edges nesting $e$ is equal to $i - 1$; refer to Fig. 3 for an illustration.

Based on the nesting-values of the edges of $M$, we compute in a bottom-up traversal of $T$ an integer value for each vertex $v$ of $G$, the so-called matching-value of $v$, $mv(v)$. For a vertex $v$ on layer 0 of $T$, the matching-value of $v$ is the nesting-value of the unique edge of $M$ incident to it. For a vertex $v$ on layer $\ell \leq 1$ of $T$, the matching-value of $v$ equals the minimum matching-value of its neighbors in layer $\ell - 1$ in $T$; refer to Fig. 3 for an illustration. In other words, the matching-value
of vertex \( v \) equals the minimum nesting-value of the edges of \( M \) incident to the leaves of the subtree of \( T \) rooted at \( v \). In addition, the matching-values of any two consecutive leaves of \( T \) along \( L \) differ by at most one. Hence, if the leaves of a subtree rooted at a vertex \( v \) have minimum matching-value \( \alpha \) and maximum matching-value \( \beta \), then for every value \( m \) in \([\alpha, \beta]\) there exists at least one leaf of this subtree with matching-value \( m \).

Finally, based on the matching-values, we partition the vertices of \( G \) that belong to a certain layer of \( T \) to layer-groups. Formally, the layer-group \( g(v) \) of a vertex \( v \) of \( G \) is defined as follows:

\[
g(v) = g \quad \text{if} \ v \ \text{belongs to layer} \ \ell \ \text{and} \ \text{mv}(v) \in \left[ g \cdot (\Delta - 1)^\ell, (g + 1) \cdot (\Delta - 1)^\ell \right]. \tag{1}
\]

Observe that \( g(v) = \text{mv}(v) \) if \( v \) belongs to layer 0. We further denote by \( V^{g}_{\ell} \) the set of vertices of \( G \) that belong to layer \( \ell \) in \( T \), and that are contained in layer-group \( g \). Remark that \( \{V^{g}_{\ell}\}_{\ell, g} \) is a partition of the vertices of \( G \). We are now ready to present the main result of this section.

**Lemma 3.** Every \( \Delta \)-matched graph has queue number at most \( 2\Delta - 2 \).

**Proof.** Let \( G \) be a \( \Delta \)-matched graph, and let \( \{V^{g}_{\ell}\}_{\ell, g} \) be a partition of the vertex-set of \( G \) as described above. Recall that \( \Gamma \) is a planar drawing of \( G \) satisfying Properties P[1−4] We construct a linear order \( \prec \) of the vertices of \( G \) as follows. For every two distinct vertices \( u \) and \( v \) of \( G \), we have that \( u \prec v \) if and only if one of the following conditions holds:

\begin{enumerate}[C.1]
  \item \( \ell(u) < \ell(v) \), or
  \item \( \ell(u) = \ell(v) \) and \( g(u) < g(v) \), or
  \item \( \ell(u) = \ell(v) = \ell, \ g(u) = g(v) \) and \( u \) is to the left of \( v \) along \( L_\ell \) in \( \Gamma \).
\end{enumerate}

Observe that all edges of \( M \) form a necklace. Indeed, if \( e \) is an edge of \( M \), then the endvertices of \( e \) belong to the same layer-group and they are consecutive in \( \prec \). Therefore, we can assign all edges of \( M \) in one queue, say \( Q_0 \).

The remaining edges of \( G \) belong to \( T \). As a result, their endvertices belong to consecutive layers of \( T \). Let \( u \) be a vertex of \( G \) that belongs to layer \( \ell + 1 \) in \( T \), and assume that \( u \) is contained in layer-group \( g \), i.e., \( u \in V^{g}_{\ell+1} \). Let also \( v_1, v_2, \ldots, v_d \) be the neighbors of \( u \) in layer \( \ell \) (not necessarily in this left-to-right order), where \( d \leq \Delta - 1 \). Without loss of generality, we assume that \( \text{mv}(v_1) \leq \text{mv}(v_2) \leq \cdots \leq \text{mv}(v_d) \). This implies that \( \text{mv}(u) = \min\{\text{mv}(v_i) \mid i = 1, \ldots, d\} = \text{mv}(v_1) \).

Since \( u \) is contained in \( V^{g}_{\ell+1} \), it follows by Eq. (1) that

\[
\text{mv}(v_1) = \text{mv}(u) \in \left[ g \cdot (\Delta - 1)^{\ell+1}, (g + 1) \cdot (\Delta - 1)^{\ell+1} \right]. \tag{2}
\]

Hence

\[
g(\Delta - 1) \cdot (\Delta - 1)^{\ell} = g \cdot (\Delta - 1)^{\ell+1} \leq \text{mv}(v_1) < (g + 1) \cdot (\Delta - 1)^{\ell+1} = (g + 1)(\Delta - 1) \cdot (\Delta - 1)^{\ell}. \tag{3}
\]

It follows that for vertex \( v_1 \) of layer \( \ell \) we have

\[
g(\Delta - 1) \leq g(v_1) < (g + 1)(\Delta - 1).
\]

Alternatively,

\[
v_1 \in V^{g(\Delta - 1)}_{\ell} \cup \cdots \cup V^{(g+1)(\Delta - 1)-1}_{\ell} = \bigcup_{k=0}^{\Delta-2} V^{g(\Delta - 1)+k}_{\ell}.
\]
Figure 4: The queue layout for the $\Delta$-matched graph $G$ of Fig. 3 produced by our algorithm. This layout consists of four queues drawn black-bold, black-solid and gray (denoted by $Q_0$, $Q_1$ and $Q_2$ in the algorithm). The vertices of $G$ that belong to the same layer in $T$ are drawn within the same dotted rectangle.

Now consider vertex $v_i$, such that $1 < i \leq d$. We claim that

$$g(\Delta - 1) \leq g(v_i) < (i - 1) + (g + 1)(\Delta - 1)$$

Recall that $mv(v_i)$ is the minimum nesting-value of the edges of $M$ incident to the leaves of the subtree of $T$ rooted at $v_i$. Since $mv(v_{i-1}) \leq mv(v_i)$, for every integer value $m$ such that $mv(v_{i-1}) \leq m < mv(v_i)$, there exists a leaf in the subtree of $T$ rooted at $v_{i-1}$ with matching-value $m$. Since vertex $v_{i-1}$ belongs to layer $\ell$ in $T$, the subtree of $T$ rooted at $v_{i-1}$ has at most $(\Delta - 1)^\ell$ leaves. Therefore

$$mv(v_i) \leq (\Delta - 1)^\ell + mv(v_{i-1})$$

holds, which applied $i - 1$ times, gives

$$mv(v_i) \leq (i - 1)(\Delta - 1)^\ell + mv(v_1).$$

Eq. (3) and Eq. (5) give

$$mv(v_i) < (i - 1)(\Delta - 1)^\ell + (g + 1)(\Delta - 1) \cdot (\Delta - 1)^\ell = ((i - 1) + (g + 1)(\Delta - 1)) \cdot (\Delta - 1)^\ell.$$ 

Since $mv(v_1) \leq mv(v_i)$, we conclude that

$$g(\Delta - 1) \cdot (\Delta - 1)^\ell \leq mv(v_i) < ((i - 1) + (g + 1)(\Delta - 1)) \cdot (\Delta - 1)^\ell$$

holds, which implies our initial claim in Eq. (4), as desired. Using Eq. (4) and the fact that $i \leq d \leq \Delta - 1$, we get for $i = 1, \ldots, d$

$$v_i \in V_\ell^g(\Delta - 1) \cup \ldots \cup V_\ell^{g(\Delta - 1) - 2} \cup \bigcup_{k=0}^{2\Delta - 4} V_\ell^{g(\Delta - 1) + k}.$$ (6)

Recall that we have already assigned all edges of $G$ that belong to $M$ in a single queue denoted by $Q_0$. We are now ready to describe how to assign each edge of $G$ that is in $T$ to the remaining $2\Delta - 3$ queues. For each layer $\ell > 0$ and for each layer-group $g$ of layer $\ell$ in $T$, we assign the edges between $V_\ell^g$ and $V_\ell^{g(\Delta - 1) + k}$ to queue $Q_{k+1}$, for $k = 0, \ldots, 2\Delta - 4$; for an illustration refer to Fig. 4. By Eq. (6), all edges of $G$ have been assigned to one of the $2\Delta - 2$ queues $Q_0, \ldots, Q_{2\Delta - 3}$.

To complete the proof of the lemma, it remains to show that no two independent edges of the same queue are nested. As already mentioned, all edges of $Q_0$ form a $|Q_0|$-necklace and therefore
Figure 5: Illustration of different operations performed by our algorithm. The starting configuration is the ordered concentric representation of a planar graph of degree 3 illustrated in (a), in which \((u_1, v_1)\) is a binding edge, while \((u_2, v_2)\) and \((u_3, v_3)\) are level edges.

For some \(0 < k \leq 2\Delta - 3\), consider two independent edges \((x_1, y_1)\) and \((x_2, y_2)\) assigned to the same queue \(Q_k\), such that \(x_1 \prec y_1\) and \(x_2 \prec y_2\). Without loss of generality, we may further assume that \(x_1 \prec x_2\). We will prove that \(y_1 \prec y_2\), which implies that edges \((x_1, y_1)\) and \((x_2, y_2)\) are not nested. Since \(x_1 \prec x_2\), by Conditions \(C.2\) and \(C.3\) we have that \(\ell(x_1) \leq \ell(x_2)\). Since the endvertices of \((x_1, y_1)\) and \((x_2, y_2)\) belong to consecutive layers in \(T\), it follows that \(\ell(y_1) = \ell(x_1) + 1\) and \(\ell(y_2) = \ell(x_2) + 1\). For the sake of contradiction, assume that \(y_2 \prec y_1\), which implies that \(\ell(y_2) \leq \ell(y_1)\). By combining the above inequalities, we may conclude that \(\ell(x_1) = \ell(x_2) = \ell\) and \(\ell(y_1) = \ell(y_2) = \ell + 1\), for some \(\ell \geq 0\).

Let \(g(y_1) = g_1\) and \(g(y_2) = g_2\). Since both \((x_1, y_1)\) and \((x_2, y_2)\) are assigned to queue \(Q_k\), it follows that \(g(x_1) = (\Delta - 1)g_1 + (k - 1)\) and \(g(x_2) = (\Delta - 1)g_2 + (k - 1)\). Since \(x_1 \prec x_2\), by Conditions \(C.2\) and \(C.3\) we conclude that \((\Delta - 1)g_1 + (k - 1) = g(x_1) \leq g(x_2) = (\Delta - 1)g_2 + (k - 1)\), which implies that \(g_1 \leq g_2\). On the other hand, since \(y_2 \prec y_1\), by Conditions \(C.2\) and \(C.3\) we similarly get \(g_2 \leq g_1\). Therefore, \(g_1 = g_2\) and consequently \(g(x_1) = g(x_2)\).

Now since \(x_1 \prec x_2\), by Condition \(C.3\) it follows that \(x_1\) is to the left of \(x_2\) along \(L_{\ell}\) in the drawing \(\Gamma\) of \(G\). Similarly, since \(g(y_1) = g(y_2)\) and \(y_2 \prec y_1\), it follows by Condition \(C.3\) again that \(y_2\) is to the left of \(y_1\) along \(L_{\ell+1}\) in the drawing \(\Gamma\) of \(G\). However, this implies that edges \((x_1, y_1)\) and \((x_2, y_2)\) cross in \(\Gamma\), which is a contradiction to the fact that \(\Gamma\) is a planar drawing of \(G\). Therefore, \(y_1 \prec y_2\), and edges \((x_1, y_1)\) and \((x_2, y_2)\) are not nested, as desired, which concludes the proof.

3.2 The Generalization: Planar Graphs of Maximum Degree \(\Delta\)

In this section, we use the approach presented in the previous section to general planar graphs of maximum degree \(\Delta\). In a high-level description, our approach consists of three main steps. Given a planar graph \(G\) of maximum degree \(\Delta\), we first compute an auxiliary planar graph \(G_1\) of maximum degree \(\Delta\) by subdividing some of the edges of \(G\) a constant number of times. Then, we exploit structural properties of graph \(G_1\) to obtain a \(\Delta\)-matched graph \(G_2\) of maximum degree \(\Delta\) by replacing some of the vertices of \(G_1\) with appropriately-defined \(\Delta\)-matched instances. It follows, by Lemma \(3\) that the queue number of \(G_2\) is at most \(2\Delta - 2\). In a third step, we show that a queue layout of \(G\) can be obtained from a queue layout of \(G_2\) by introducing a number of additional queues that is polynomial in \(\Delta\), thus confirming Theorem \(1\).

First, let us argue that we may assume without loss of generality that \(G\) has minimum degree at least two. Note that Theorem \(1\) clearly holds for \(\Delta \leq 2\), as all graphs of maximum degree at most two have queue number one \(14\). So, assume for the remainder that \(\Delta \geq 3\). Suppose that \(v\) is a vertex of degree one in \(G\). We introduce two new vertices \(v_1\) and \(v_2\), and three new edges \((v, v_1)\), \((v, v_2)\) and \((v_1, v_2)\) in \(G\). It follows that vertex \(v\) has degree three and each of the two introduced vertices \(v_1\) and \(v_2\) has degree two. By applying this procedure to all degree-1 vertices in \(G\), we
obtain a planar supergraph of $G$ with minimum degree at least two and maximum degree $\Delta$. It is not difficult to see that the queue number of $G$ is at most the queue number of this supergraph. So, in the following we will assume that every vertex in $G$ has degree at least two.

We start by describing how to construct graph $G_1$ from graph $G$. If $G$ has no degree-2 vertex, then we subdivide an edge of $G$ once, in order to introduce such a degree-2 vertex. As a consequence, in the following we will assume that $G$ contains at least one degree-2 vertex, which we denote by $r$. Let $R = \{C_0, C_1, \ldots, C_{h-1}\}$ be an ordered concentric representation of $G$ centered at $r$. Let $T$ be the BFS-tree of $G$ that was used in order to compute $R$; refer to Section 2. Recall that an edge of $G$ that is not in $T$, is either a level edge (if it connects two vertices of the same level in $T$) or a binding edge (if it connects two vertices on consecutive levels in $T$).

We proceed by subdividing each binding edge of $G$ that does not belong to $T$ (if any) once. By this simple operation, each binding edge of $G$ is split into a level edge and an edge that can be assigned to tree $T$. For an illustration of this operation refer to the binding edge $(u_1, v_1)$ in Fig. 5a, which is subdivided once by introducing vertex $v'_1$ in Fig. 5b, as a result, the edge $(v_1, v'_1)$ is binding and part of $T$, while the edge $(u_1, v'_1)$ is level. Additionally, we subdivide each level edge $(u, v)$ twice, if $u$ or $v$ has degree greater than two. This yields three edges $(u, u', (u', v')$ and $(v', v)$, the first and last of which we assign to the tree $T$, while the middle edge $(u', v')$ becomes a level edge. This guarantees that the endvertices of all level edges have degree two. For an illustration refer to the level edge $(u_2, v_2)$ in Fig. 5b whose endvertices have degree three; by subdividing this edge twice with vertices $u'_2$ and $v'_2$ in Fig. 5c, we guarantee that two of the newly formed edges, that is, $(u_2, u'_2)$ and $(v_2, v'_2)$, become binding and part of $T$, while the middle edge $(u'_2, v'_2)$ becomes a level edge whose endvertices have degree two. The resulting subdivision of $G$ is graph $G_1$. Note that each edge of $G$ is subdivided at most three times in order to obtain $G_1$.

Based on graph $G_1$, we next describe how to construct graph $G_2$. In this step, we need to guarantee one additional property of $\Delta$-matched graphs, that is, all level edges of a $\Delta$-matched graph belong to layer 0. Suppose that in the concentric representation of $G_1$ there exists a level edge $(u, v)$ at layer $0 < \ell < h$. We create two complete $(\Delta - 1)$-ary trees of height $\ell$, say $T_u$ and $T_v$, and we identify their roots with vertices $u$ and $v$, respectively. Since the height of each of $T_u$ and $T_v$ is $\ell$, it follows that all leaves of $T_u$ and all leaves of $T_v$ can be placed along $C_0$ consecutively while preserving planarity. We replace edge $(u, v)$ by $(\Delta - 1)^\ell$ edges forming a matching $M_{u,v}$, such that the $i$-th leaf of $T_u$ from the left along $C_0$ is connected to the $i$-th leaf of $T_v$ from the right along $C_0$. Observe that the edges of $M_{u,v}$ form a $(\Delta - 1)^\ell$-rainbow. For an illustration of this operation, refer to edges $(u_1, v'_1)$ and $(u_3, v_3)$ in Fig. 5c, which are two level edges that do not have their endvertices along the outermost circle $C_0$. Edge $(u_1, v'_1)$ is replaced by the two gray-colored binary trees $T_{u_1}$ and $T_{v'_1}$ of height two in Fig. 5d, while the edge $(u_3, v_3)$ is replaced by the two black-colored binary trees $T_{u_3}$ and $T_{v_3}$ of height one. Also, the four edges of $M_{u_1,v'_1}$ and the two edges of $M_{u_3,v_3}$ are drawn dotted with mostly rectilinear segments in Fig. 5d. Once we apply the aforementioned procedure to all level edges of $G_1$, we obtain graph $G_2$ together with an ordered concentric representation $R_2$, in which all level edges have their endvertices along the outermost circle $C_0$. Since we assumed that $G$ has minimum degree at least two, all leaves of $T$ lie on $C_0$.

We now claim that $R_2$ can be converted into a drawing $\Gamma$ of $G_2$ that satisfies Properties P1, P4. Since $r$ belongs to the outer face of $R_2$, there is a curve $C$ that starts at $r$, cuts all circles of $R_2$ once, and crosses no edge of $G_2$. Hence, we can use $C$ to cut all the circles of $R_2$ and stretch $R_2$ so that each circle becomes a line segment by preserving the planarity of the drawing. In other words, we can obtain the drawing $\Gamma$ of $G_2$ through a suitable homeomorphic transformation of $R_2$. Properties P1 and P4 hold by construction in $\Gamma$, while Properties P2 and P3 follow from Properties R1 and R3 of the ordered concentric representation $R_2$, respectively. It follows that graph $G_2$ is a $\Delta$-matched graph. Thus, by Lemma 3 graph $G_2$ admits a queue layout $L_2$ with $2\Delta - 2$ queues.
Next we derive a queue layout $L_1$ for $G_1$ from the queue layout $L_2$ for $G_2$. Recall that $G_2$ was obtained from $G_1$ by replacing each level edge $(u, v)$ of $G_1$ by two complete $(\Delta - 1)$-ary trees $T_u$ and $T_v$, and by a set $M_{u,v}$ of matching edges. For the desired queue layout $L_1$ of $G_1$ we order the vertices of $G_1$ according to their ordering in $L_2$. For every edge of $G_1$ we assign the same queue as in $L_2$, provided that this edge is also an edge of $G_2$. Otherwise, such an edge is a level edge of $G_1$ and we assign it to the queue $Q_0$. Thus, in queue layout $L_1$, all level edges are assigned to queue $Q_0$, while queues $Q_1, \ldots, Q_{2\Delta - 3}$ contain all edges of the BFS-tree $T$.

It remains to show that no two (level) edges in $Q_0$ nest. Recall that the ordering of vertices is inherited from queue layout $L_2$ and hence satisfies Conditions C1, C2, C3. For any level edge $(u, v)$ in $G_1$ we have $\ell(u) = \ell(v)$. Moreover, the set of matching edges in $G_2$ incident to the leaves of $T_u$ is given by $M_{u,v}$, and the same holds for $T_v$. This implies that $mv(u) = mv(v) = \min\{\text{nesting-value of } e | e \in E\}$ and consequently $g(u) = g(v)$ by Eq. (1). Now suppose for the sake of contradiction that level edge $(u_1, v_1)$ nests level edge $(u_2, v_2)$ with $u_1 \prec u_2 \prec v_2 \prec v_1$. By Condition C1 it follows that $\ell(u_1) \leq \ell(u_2) \leq \ell(v_2) \leq \ell(v_1)$ and as $\ell(u_1) = \ell(v_1)$, all four vertices have the same level $\ell$. Secondly, in $G_2$ all edges of $M_{u_1,v_1}$ nest all edges of $M_{u_2,v_2}$. Hence, $mv(v_2) - mv(v_1) = |M_{u_2,v_2}| = (\Delta - 1)^\ell$, i.e., the difference between the minimum nesting-value of edges in $M_{u_2,v_2}$ and the minimum nesting-value of edges in $M_{u_1,v_1}$ is $(\Delta - 1)^\ell$. This however, by Eq. (1) gives $g(v_2) = g(v_1) + 1$, which contradicts the fact that $v_2 \prec v_1$ according to Condition C2.

So far, we constructed a queue layout $L_1$ of graph $G_1$ with $2\Delta - 2$ queues. As $G_1$ is obtained from $G$ by subdividing each edge at most three times, we can apply Lemma 1 with $k = 3$ and $q = 2\Delta - 2$, and conclude that there is a queue layout $L$ of $G$ with at most $32(2\Delta - 1)^6 - 1$ queues. Hence, $qn(G) \leq 32(2\Delta - 1)^6 - 1$, which concludes the proof of Theorem 1.

### 3.3 Time Complexity

In this section, we analyze the runtime of our algorithm to construct a queue layout with at most $32(2\Delta - 1)^6 - 1$ queues for a given planar graph $G$ of maximum degree $\Delta$. First assume that the input graph $G = (V, E)$ is a $\Delta$-matched graph. In this case, the runtime of our algorithm is dominated by the computation of the nesting-values of the edges of $M$ and of the matching-values of the vertices of $G$. The former can be easily accomplished in $O(|M|)$ time by the drawing $\Gamma$ of $G$, while the latter in $O(|V|)$ by a bottom-up traversal of $T$. Having computed these values, the calculation of the layer-group of each vertex can be done in $O(|V|)$ time in total. The linear order can be determined in $O(|V|)$ time by a single iteration through the vertices of each layer $\ell$ of $T$ in the order that they appear along $L_\ell$; recall Conditions C1, C2, C3. Finally, the assignment of the edges of $G$ into queues can be performed in $O(|E|)$ time, since for each edge the corresponding queue is determined based on the layer-groups of its endvertices. Hence, the algorithm supporting Lemma 3 runs in $O(|V| + |E|)$ time, which is in $O(|V|)$, since $G$ is planar and hence $|E| = O(|V|)$.

For the case of a general planar graph $G = (V, E)$ of maximum degree $\Delta$, however, the construction of the graph $G_2$ may increase the runtime dramatically. In particular, note that graph $G_2$ may have size exponential in $|V|$, when $G$ is sparse. To see this, consider a graph obtained from a path on $n$ vertices $(v_1, v_2, \ldots, v_n)$ by adding an edge connecting $v_1$ to $v_3$. If $v_2$ is chosen as the center of the ordered concentric representation $R$, then edge $(v_1, v_3)$ becomes a level edge; see Fig. 6a. Since the endvertex $v_3$ of $(v_1, v_3)$ has degree greater than two, the edge $(v_1, v_3)$ will be subdivided twice. Let $v'_1$ and $v'_3$ be the subdivision vertices; refer to Fig. 6b for an illustration. Observe that the edge $(v'_1, v'_3)$ is a new level edge, whose endvertices are not in the outermost circle of the representation. Then each of the trees $T_{v'_1}$ and $T_{v'_3}$ replacing $(v'_1, v'_3)$ has $2^{n-3} - 1$ vertices; see Fig. 6c. So, in order to keep the runtime of our algorithm polynomial in the size of $G$, we avoid
introducing trees $T_u$ and $T_v$ explicitly for each level edge $(u, v)$. In fact, the introduction of these trees was convenient for proving the correctness of our approach. But, as we will argue below, to determine the correct queue layout we only need to know the size of the set $M_{u,v}$.

By Lemma 2 the ordered concentric representation $R$ of $G$ can be computed in linear time. Based on $R$, the computation of graph $G_1$ needs an additional $O(|E|)$ time. To avoid introducing trees $T_u$ and $T_v$ for each level edge $(u, v)$ of $T$ (as required for the computation of graph $G_2$), we first reroute each level edge in $R$, so that one part of it lies outside the outermost circle $C_0$ of $R$. This can be done without introducing crossings in $O(|E|)$ time in total, by a single traversal over all level edges in $R$ starting from those level edges of the outermost circle and moving inwards in the representation $R$. This edge rerouting guarantees that for any two level edges $(u, v)$ and $(u', v')$, all edges of $M_{u,v}$ nest all edges of $M_{u',v'}$ if and only if the part of $(u, v)$ that lies outside $C_0$ of $R$ nests the corresponding part of $(u', v')$ in $R$. Therefore, instead of introducing two trees $T_u$ and $T_v$ for each level edge $(u, v)$ of $T$, we assign a weight $w(u, v)$ to the edge $(u, v)$ equal to $|M_{u,v}|$ and compute the nesting-values of the edges of $M$ and the matching-values of the vertices of $G_2$ based on the weights of these edges as follows; e.g., the weight of the level edge $(v'_1, v'_3)$ of Fig. 6b is four. Consider an edge $(u, v)$ such that $(u, v)$ either belongs to $M$ or $(u, v)$ is a level edge of some layer $\ell > 0$; note that in the former case we assume that $w(u, v) = 1$. Observe that the order in which the endvertices of the edges of $M$ and the edge segments of the level edges lying outside $C_0$ appear along $C_0$ defines a linear order on their endvertices. Edge $(u, v)$ has nesting-value zero if $(u, v)$ is not nested by any other edge in this order. Otherwise, let $(u', v')$ be the edge with maximum nesting-value that nests edge $(u, v)$. Then, edge $(u, v)$ has nesting-value equal to the nesting-value of $(u', v')$ plus $w(u', v')$. Note that once all nesting-values of edges are computed, the computation of the matching-values of vertices can be done as in the unweighted case.

Up to this point, the time complexity of the algorithm is in $O(|V| + |E|) = O(|V|)$, assuming a RAM model of computation that supports standard operations on $w$-bit words with unit cost such that $w \geq \log |V|$, so that each value (in particular, each weight, nesting-value, and matching-value) associated with a vertex or with an edge fits in a word. Since each of these values does not exceed the maximum number of leaves of a $(\Delta - 1)$-ary tree of height at most $|V|$, that is, $O(\Delta^{|V|})$ or equivalently $O(2^{|V|\log \Delta})$, words of size linear in $|V| \log \Delta$ suffice. Thus, the running time of our algorithm is $O(|V|)$, as the computation of the weights of all level edges can be done in $O(|V|)$ time based on the drawing $\Gamma$ of $G$.

4 Open Problems

Clearly, the major question that remains open is to settle the conjecture by Heath, Leighton and Rosenberg for planar graphs without any restriction in the maximum degree (either in the positive
or in the negative). As a further intermediate question, we ask the following: Given a planar graph, is it possible to compute a linear order of its vertices and a partition of its edges into one stack and a constant number of queues? These layouts are known as mixed layouts and have been introduced in the paper by Heath, Leighton and Rosenberg, who conjectured that every planar graph admits a mixed layout with one stack and one queue. Pupyrev [18] recently disproved this conjecture by demonstrating a planar graph for which one stack and one queue do not suffice, and conjectured that for bipartite planar graphs one stack and one queue are always sufficient.

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