Evolutions of hypergraphs and their embedded homology

Shiquan Ren, Chengyuan Wu, Jie Wu

Abstract. Hypergraphs are higher-dimensional generalizations of graphs. An (abstract) simplicial complex is a hypergraph such that all the faces of hyperedges are still hyperedges. The topology of a hypergraph can be investigated by its embedded homology, the homology of its associated complex, and the homology of its lower-associated complex. In this paper, we study the evolutions of hypergraphs as well as the induced homomorphisms of the embedded homology groups. The main result of this paper investigates the homomorphisms of the embedded homology induced from evolutions of hypergraphs and the relations with the lower-associated simplicial complexes.

1 Introduction

Hypergraphs are higher dimensional generalizations of graphs (cf. [4]). In a hypergraph, a hyperedge can join any number of vertices, while in a graph each edge joins two vertices. Let \( H \) be a hypergraph. An \( n \)-dimensional hyperedge \( \sigma \in H \), or simply an \( n \)-hyperedge, is a collection of \( n + 1 \) vertices in the vertex set of \( H \). The topology of hypergraphs has various applications in network study and computer science (e.g. [6, 16, 22]).

(Co)homology theory can be used to investigate the topology of graphs as well as hypergraphs. For example, in 2003 and 2007, E. Babson and D.N. Kozlov [1, 2] used the cohomology rings of the Hom-complexes of graphs as well as the Stiefel-Whitney classes to give topological obstructions to graph colorings. In 2002, G.M. Ziegler [21] used chain complexes, chain maps, and chain homotopy to investigate the coloring problem of hypergraphs. And in 2012, a homology theory for \( k \)-graphs was introduced by A. Kumjian, D. Pask and A. Sims [17].

So far, various (co)homology theories of hypergraphs have been studied. In 1991, A.D. Parks and S.L. Lipscomb [18] found that a hypergraph \( H \) can be regarded as an incomplete simplicial complex with missing faces. They defined the associated simplicial complex \( \Delta H \) of \( H \) by adding all the missing faces. They studied the homology groups of \( \Delta H \) to investigate the topology of \( H \). In 1992, by considering the \( p \)-boundary operators of \( n \)-hyperedges, F.R.K. Chung and R.L. Graham [7] constructed a cohomology with mod 2 coefficients of hypergraphs. In 2009, E. Entander [9] constructed the independence simplicial complexes for hypergraphs and studied the homology of these simplicial complexes.

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The topology of $\mathcal{H}$ cannot be fully detected by the homology groups of $\Delta \mathcal{H}$. In fact, even if two hypergraphs have the same associated complex, they may have different topological structures (cf. [5, Example 3.8]). In 2016, S. Bressan, J. Li, S. Ren and J. Wu [2] invented the embedded homology to investigate the topology of hypergraphs. Generally, the embedded homology and the homology of associated complexes are not isomorphic (cf. [5, Example 3.8]). The original idea of the embedded homology was given by A. Grigor’yan, Y. Lin, Y. Muranov and S.T. Yau [12]. Some related ideas were also discussed in [13, 14].

Besides the homology of associated complexes and the embedded homology, the homology of the lower-associated complexes of hypergraphs was considered by S. Ren, C. Wu and J. Wu [20] in 2017. The lower-associated simplicial complex $\delta \mathcal{H}$ was defined as the simplicial complex by deleting all the hyperedges in $\mathcal{H}$ with missing faces (cf. [20]).

A hypergraph $\mathcal{H}'$ is called an evolution\(^1\) of $\mathcal{H}$, if $\mathcal{H}'$ is obtained by adding an hyperedge to $\mathcal{H}$. In particular, suppose $\mathcal{H}$ is a simplicial complex and is denoted as $\mathcal{K}$. Then a simplicial complex $\mathcal{K}'$ is an evolution of $\mathcal{K}$ means that $\mathcal{K}'$ is obtained by adding a simplex $\sigma$ to $\mathcal{K}$ where $\partial \sigma \subseteq \mathcal{K}$. An evolution $\mathcal{H}'$ of $\mathcal{H}$ induces

(a). a homomorphism of the embedded homology from $H_\ast(\mathcal{H})$ to $H_\ast(\mathcal{H}')$;

(b). a homomorphism of the homology of associated complexes from $H_\ast(\Delta \mathcal{H})$ to $H_\ast(\Delta \mathcal{H}')$;

(c). a homomorphism of the homology of lower-associated complexes from $H_\ast(\delta \mathcal{H})$ to $H_\ast(\delta \mathcal{H}')$.

The evolution of hypergraphs is a mathematical model for the evolution of social networks (cf. [3, 8, 11, 19]), which involves important topics in sociology (cf. [19]), physics (cf. [3, 11]), trades (cf. [11]), and computer science (cf. [8]). For example, the evolution of the scientific collaboration network (cf. [3]) can be described as follows: each vertex represents a researcher and each hyperedge represents a collaboration. As the researchers collaborate, the number of hyperedges grows and the hypergraph evolves.

The aim of this paper is to investigate the topology of evolutions of hypergraphs. We consider the embedded homology $H_\ast(\mathcal{H})$ as well as the simplicial complexes $\Delta \mathcal{H}$ and $\delta \mathcal{H}$. We investigate the homomorphisms of $H_\ast(\mathcal{H})$ induced from evolutions of $\mathcal{H}$. We define the embedded homological dimension of $\mathcal{H}$ as

$$d^h(\mathcal{H}) = \sup\{k \mid H_k(\mathcal{H}) \neq 0\}$$

and prove the following theorem.

**Theorem 1.1** (Main Result). Let $\mathcal{H}$ be a hypergraph. Let $\mathcal{H}'$ be an evolution of $\mathcal{H}$ by adding a $d$-dimensional hyperedge $\sigma$, $d \geq 1$. Suppose

\(^1\)Evolutions of random graphs were originally considered by P. Erdős and A. Rényi in [10]. And evolutions of social networks were considered in [3, 8, 11, 19]. In this paper, we borrow the terminology "evolution" from [3, 8, 10, 11, 19] and consider the evolution of hypergraphs.
(i). for any hyperedges $\tau \in \mathcal{H}$ and $\tau' \in \mathcal{H}'$, $\tau \cap \tau'$ is either empty or a hyperedge of $\mathcal{H}$;
(ii). $\sigma$ is not a face of any $(d+1)$-hyperedge of $\mathcal{H}$;
(iii). $\delta H' \neq \delta H$.

Then

(a). $d^{bh}(\mathcal{H} \cap \partial \sigma)$ is either $d-1$ or $d-2$;
(b). If $d^{bh}(\mathcal{H} \cap \partial \sigma) = d-1$ and in addition, for any $d$-hyperedge $\tau \in \mathcal{H}$ with $\tau \cap \sigma$ nonempty, $\tau$ does not intersect with any other $d$-hyperedges of $\mathcal{H}$, then $H_{d-1}(\mathcal{H}) \cong H_{d-1}(\mathcal{H}') \oplus \mathbb{Z}$;
(c). If $d^{bh}(\mathcal{H} \cap \partial \sigma) = d-2$, then $H_i(\mathcal{H}') \cong H_i(\mathcal{H})$ for $i \neq d-1$ and $H_{d-1}(\mathcal{H}') \not\cong H_{d-1}(\mathcal{H})$.

As by-products, in Theorem 2.5, we investigate the homomorphisms of the homology of associated simplicial complexes as well as the homomorphisms of the homology of lower-associated simplicial complexes, induced from evolutions of hypergraphs. In Theorem 3.2, we study the homomorphisms of the embedded homology induced from evolutions of hypergraphs. And in Theorem 3.4, we give some relations between the embedded homology and the lower-associated simplicial complexes. The main result Theorem 1.1 will be proved with the helps of Theorem 3.2 and Theorem 3.4 in Section 4.

Throughout this paper, all the hyperedges are assumed to be nonempty without extra claim. Nevertheless, hypergraphs are allowed to be empty. We use the integer $k$ to denote the dimensions of hypergraphs, and write $k$ as a subscript. We use $H$ to denote homology groups and use $\tilde{H}$ to denote reduced homology groups.

2 Associated simplicial complexes and lower-associated simplicial complexes

In this section, we introduce the associated simplicial complexes and the lower-associated simplicial complexes of hypergraphs. In Subsection 2.1, we review the homology of the associated simplicial complexes, the homology of the lower-associated simplicial complexes and the embedded homology. In Subsection 2.2, we give some examples for these three homology groups discussed in Subsection 2.1. In Subsection 2.3, we prove Theorem 2.5.

2.1 Definitions and constructions

**Definition 1.** (cf. [4, 18]) Let $V_{\mathcal{H}}$ be a set. Let $2^V$ denote the powerset of $V$. A hypergraph is a pair $(V_{\mathcal{H}}, \mathcal{H})$ where $\mathcal{H}$ is a subset of $2^V \setminus \{\emptyset\}$. An element of $V_{\mathcal{H}}$ is called a vertex and an element of $\mathcal{H}$ is called a hyperedge. Let $k \geq 0$. We call a hyperedge consisting of $k+1$ vertices a $k$-hyperedge, or a hyperedge of dimension $k$. We denote $\mathcal{H}_k$ as the collection of all the $k$-hyperedges of $\mathcal{H}$. We define the dimension of $\mathcal{H}$, denoted
as \( \dim \mathcal{H} \), to be the maximal integer \( k \) such that \( \mathcal{H}_k \) is nonempty. Given a hyperedge \( \sigma \) in \( \mathcal{H} \), we define the boundary of \( \sigma \), denoted as \( \partial \sigma \), as the collection of all nonempty proper subsets of \( \sigma \).

Throughout this paper, we assume that \( V_\mathcal{H} \) is the union of the vertices of all the hyperedges in \( \mathcal{H} \). Hence we simply denote a hypergraph \((V_\mathcal{H}, \mathcal{H})\) as \( \mathcal{H} \). An (abstract) simplicial complex \( \mathcal{K} \) is a hypergraph satisfying the following two conditions (cf. [15, p. 107]):

(i). for any \( v \in V_\mathcal{K} \), the single-point set \( \{v\} \) is in \( \mathcal{K} \);

(ii). for any \( \sigma \in \mathcal{K} \) and any non-empty subset \( \tau \subseteq \sigma \), \( \tau \) is in \( \mathcal{K} \).

A hyperedge of a simplicial complex is called a simplex, and a \( k \)-hyperedge of a simplicial complex is called a \( k \)-simplex. We denote the boundary maps of simplicial complexes as \( \partial_* \).

Let \( \mathcal{H} \) be a hypergraph. Let the lower-associated simplicial complex \( \delta \mathcal{H} \) be the largest simplicial complex that can be embedded in \( \mathcal{H} \). Then

\[
\delta \mathcal{H} = \{ \eta \in \mathcal{H} \mid \text{for any } \tau \subseteq \eta, \tau \in \mathcal{H} \}. \tag{2.1}
\]

Let the associated simplicial complex \( \Delta \mathcal{H} \) be the smallest simplicial complex that \( \mathcal{H} \) can be embedded in (cf. [18]). Then

\[
\Delta \mathcal{H} = \{ \eta \subseteq \tau \mid \tau \in \mathcal{H} \}. \tag{2.2}
\]

We notice that

\[
\delta \mathcal{H} \subseteq \mathcal{H} \subseteq \Delta \mathcal{H}. \tag{2.3}
\]

In [28], one of the equalities holds iff. both of the equalities hold iff. \( \mathcal{H} \) is a simplicial complex.

Let \( k \) be a nonnegative integer. Let \( \mathcal{H}_k \) (resp. \( (\delta \mathcal{H})_k \) or \( (\Delta \mathcal{H})_k \)) be the collection of all the \( k \)-dimensional hyperedges of \( \mathcal{H} \) (resp. \( \delta \mathcal{H} \) or \( \Delta \mathcal{H} \)). Let \( Z(\mathcal{H}_k) \) (resp. \( Z((\delta \mathcal{H})_k) \) or \( Z((\Delta \mathcal{H})_k) \)) be the collection of all the formal linear combinations of the elements of \( \mathcal{H}_k \) (resp. \( (\delta \mathcal{H})_k \) or \( (\Delta \mathcal{H})_k \)) with coefficients in \( \mathbb{Z} \). Then \( Z((\Delta \mathcal{H})_*) \) is a chain complex with boundary map \( \partial_* \), and \( Z((\delta \mathcal{H})_* \) is a chain complex with boundary map the restriction of \( \partial_* \) to \( Z(\delta \mathcal{H}_*) \). Moreover, \( Z(\mathcal{H}_*) \) is a graded group such that as graded groups,

\[
Z((\delta \mathcal{H})_*) \subseteq Z((\Delta \mathcal{H})_*) \subseteq Z((\Delta \mathcal{H})_*).
\]

Let \( \text{Inf}_*(\mathcal{H}) \) be the largest chain complex with boundary map \( \partial_* \) that is contained in \( Z(\mathcal{H}_*) \). Let \( \text{Sup}_*(\mathcal{H}) \) be the smallest chain complex with boundary map \( \partial_* \) that contains \( Z(\mathcal{H}_*) \). Then

\[
\text{C}_*(\delta \mathcal{H}; \mathbb{Z}) \subseteq \text{Inf}_*(\mathcal{H}) \subseteq Z(\mathcal{H}_*) \subseteq \text{Sup}_*(\mathcal{H}) \subseteq \text{C}_*(\Delta \mathcal{H}; \mathbb{Z}). \tag{2.4}
\]
Here $C_{\ast}(\cdot;\mathbb{Z})$ denotes the chain complex of a simplicial complex with integral coefficients. In particular, if $\mathcal{H}$ is a simplicial complex, then all the equalities in (2.4) hold. By [5, Proposition 2.1 and Proposition 2.2],

$$\text{Inf}_k(\mathcal{H}) = \mathbb{Z}(\mathcal{H}_k) \cap \partial_k^{-1}(\mathbb{Z}(\mathcal{H}_{k-1})), \quad \text{Sup}_k(\mathcal{H}) = \mathbb{Z}(\mathcal{H}_k) + \partial_{k+1}(\mathbb{Z}(\mathcal{H}_{k+1})).$$

By [5, Proposition 2.4],

$$H_k(\text{Inf}_\ast(\mathcal{H})) = \text{Ker}(\partial_k |_{\mathbb{Z}(\mathcal{H}_k)})/(\mathbb{Z}(\mathcal{H}_k) \cap \partial_{k+1}\mathbb{Z}(\mathcal{H}_{k+1})), \quad (2.5)$$

$$H_k(\text{Sup}_\ast(\mathcal{H})) = (\partial_{k+1}\mathbb{Z}(\mathcal{H}_{k+1}) + \text{Ker}(\partial_k |_{\mathbb{Z}(\mathcal{H}_k)}))/\partial_{k+1}\mathbb{Z}(\mathcal{H}_{k+1}). \quad (2.6)$$

And the embedded homology of $\mathcal{H}$ is (cf. [5, Section 3.2])

$$H_\ast(\mathcal{H};\mathbb{Z}) = H_\ast(\text{Inf}_\ast(\mathcal{H})) \cong H_\ast(\text{Sup}_\ast(\mathcal{H})). \quad (2.7)$$

By (2.5), $d^h(\mathcal{H}) \leq \dim \mathcal{H}$.

On the other hand, the homology groups of $\delta \mathcal{H}$ and $\Delta \mathcal{H}$ are respectively

$$H_\ast(\delta \mathcal{H};\mathbb{Z}), \quad (2.8)$$

$$H_\ast(\Delta \mathcal{H};\mathbb{Z}). \quad (2.9)$$

In the remaining part of this paper, we omit the coefficients $\mathbb{Z}$ in (2.7), (2.8) and (2.9). We use $\tilde{H}$ to denote the reduced homology groups. In general, the embedded homology (2.5) and the homology groups (2.8), (2.9) are not isomorphic (cf. [5] and Subsection 2.2).

Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraphs. A morphism of hypergraphs from $\mathcal{H}$ to $\mathcal{H}'$ is a map $\varphi$ sending a vertex of $\mathcal{H}$ to a vertex of $\mathcal{H}'$ such that whenever $\sigma = \{v_0, \ldots, v_k\}$ is a hyperedge of $\mathcal{H}$, $\varphi(\sigma) = \{\varphi(v_0), \ldots, \varphi(v_k)\}$ is a hyperedge of $\mathcal{H}'$. By an argument similar to [5, Section 3.1], a morphism of hypergraphs $\varphi : \mathcal{H} \rightarrow \mathcal{H}'$ induces two simplicial maps

$$\delta \varphi : \delta \mathcal{H} \rightarrow \delta \mathcal{H}', \quad \Delta \varphi : \Delta \mathcal{H} \rightarrow \Delta \mathcal{H}'$$

By [5, Proposition 3.7] and applying an analogous argument of [5, Proposition 3.7] to $\delta \varphi$ and $\Delta \varphi$, we have the induced homomorphisms between the homology groups

$$(\delta \varphi)_* : H_\ast(\delta \mathcal{H}) \rightarrow H_\ast(\delta \mathcal{H}'), \quad (2.10)$$

$$(\Delta \varphi)_* : H_\ast(\Delta \mathcal{H}) \rightarrow H_\ast(\Delta \mathcal{H}'), \quad (2.11)$$

$$\varphi_* : H_\ast(\mathcal{H}) \rightarrow H_\ast(\mathcal{H}'). \quad (2.12)$$

**Definition 2.** Given a hypergraph $\mathcal{H}$, an **evolution** of $\mathcal{H}$ is a hypergraph $\mathcal{H}' = \mathcal{H} \cup \{\sigma\}$.

Here $\sigma$ is a hyperedge. Without loss of generality, we assume $\sigma \notin \mathcal{H}$ and $\dim \sigma \geq 1$. By considering the canonical inclusion of hypergraphs, evolutions of hypergraphs can be regarded as a particular family of morphisms of hypergraphs.
2.2 Some examples

Given two hypergraphs \( \mathcal{H} \) and \( \mathcal{H}' \), we consider the following conditions

1. \( \delta \mathcal{H} = \delta \mathcal{H}' \), which implies \( (1)' \). \( H_*(\delta \mathcal{H}) \cong H_*(\delta \mathcal{H}') \);

2. \( \Delta \mathcal{H} = \Delta \mathcal{H}' \), which implies \( (2)' \). \( H_*(\Delta \mathcal{H}) \cong H_*(\Delta \mathcal{H}') \).

3. \( H_*(\mathcal{H}) \cong H_*(\mathcal{H}') \).

The next example shows that (1) and (2) cannot imply (3), hence (1)' and (2)' cannot imply (3) as well.

Example 2.1. Let

\[
\mathcal{H} = \{\{v_0, v_1, v_2, v_3\}\},
\]

\[
\mathcal{H}' = \{\{v_0, v_1, v_2, v_3\}, \{v_0, v_1\}, \{v_0, v_3\}, \{v_0, v_2\}, \{v_1, v_3\}, \{v_1, v_2\}, \{v_2, v_3\}\}.
\]

Then \( \delta \mathcal{H} = \delta \mathcal{H}' = \emptyset \), and both \( \Delta \mathcal{H} \) and \( \Delta \mathcal{H}' \) are the tetrahedron. Hence

\[
H_*(\delta \mathcal{H}) = H_*(\delta \mathcal{H}') = 0, \quad \tilde{H}_*(\Delta \mathcal{H}) = \tilde{H}_*(\Delta \mathcal{H}') = 0.
\]

However, \( H_1(\mathcal{H}) = 0 \) and \( H_1(\mathcal{H}') = \mathbb{Z}^{\oplus 3} \).

The next example shows that (1) and (3) cannot imply (2)', hence cannot imply (2) as well.

Example 2.2. Let

\[
\mathcal{H} = \{\{v_0, v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_4, v_5\}\},
\]

\[
\mathcal{H}' = \{\{v_0, v_1, v_3\}, \{v_0, v_2\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4, v_5\}\}.
\]

Then \( \delta \mathcal{H} = \delta \mathcal{H}' = \emptyset \), and

\[
H_*(\delta \mathcal{H}) = H_*(\delta \mathcal{H}') = 0, \quad H_*(\mathcal{H}) = H_*(\mathcal{H}') = 0.
\]

However, \( H_1(\Delta \mathcal{H}) = \mathbb{Z} \), and \( H_1(\Delta \mathcal{H}') = 0 \).

The following example shows that \( \ref{2.10} \), \( \ref{2.11} \) and \( \ref{2.12} \) can be distinct.

Example 2.3. Let \( \mathcal{H} = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}\} \). Let \( \sigma = \{v_0, v_1, v_2\} \). Let \( \mathcal{H}' = \mathcal{H} \cup \{\sigma\} \). Let \( \varphi \) be the embedding of \( \mathcal{H} \) into \( \mathcal{H}' \). Then

(a). \( H_0(\mathcal{H}) = H_0(\mathcal{H}') = 0 \), \( H_1(\mathcal{H}) = \mathbb{Z} \), and \( H_1(\mathcal{H}') = 0 \). Moreover, \( \varphi_* \) is the identity map from zero to zero in dimension 0, and the zero map on \( \mathbb{Z} \) in dimension 1;

(b). \( \delta \mathcal{H} = \delta \mathcal{H}' = \emptyset \). Moreover, \( (\delta \varphi)_* \) is the identity map from zero to zero in all dimensions;

(c). \( \Delta \mathcal{H} \cong S^1 \) and \( \Delta \mathcal{H}' \cong \ast \). Moreover, \( (\Delta \varphi)_* \) is the identity map on \( \mathbb{Z} \) in dimension 0, and the zero map on \( \mathbb{Z} \) in dimension 1.

Remark 1: If both \( \mathcal{H} \) and \( \mathcal{H}' \) are simplicial complexes, then \( \varphi \) is a simplicial map. In this case, \( (\delta \varphi)_* \), \( (\Delta \varphi)_* \) and \( \varphi_* \) are same.
2.3 Homology of associated simplicial complexes and lower-associated simplicial complexes

Before giving Theorem 2.5, we first prove the next lemma.

Lemma 2.4. (a). $\delta H' = \delta H$ iff. there exists $\tau \subset \sigma$ such that $\tau \notin H$;
(b). $\Delta H' = \Delta H$ iff. there exists $\tau \in H$ such that $\sigma \subseteq \tau$;
(c). Suppose for any $\tau \subset \sigma$, $\tau \in H$. Then $\delta H'$ is the union of $\delta H$ and $\sigma$. Moreover, $\delta H' \simeq \delta H/\partial \sigma$;
(d). Suppose $\sigma$ is a maximal face of $H'$. Then $\Delta H'$ is the union of $\Delta H$ and $\Delta \sigma$. Moreover, $\Delta H' \simeq \Delta H/(\Delta H \cap \partial \sigma)$.

Proof. (a). The assertion follows from (2.1).
(b). The assertion follows from (2.2).
(c). It follows from (2.1) that $\delta H'$ is the union of $\delta H$ and $\sigma$. Now we prove the second assertion. It follows from the assumption of (c) that $\partial \sigma \subseteq H$. Since $\partial \sigma$ is a simplicial complex, we have $\partial \sigma \subseteq \delta H$. Since $\delta H'$ is the union of $\delta H$ and $\sigma$, $\delta H'$ is homotopy equivalent to the quotient space $\delta H/\partial \sigma$.
(d). Since $\sigma$ is a maximal face of $H'$, we have that for any $\tau \in H$, $\sigma$ is not a subset of $\tau$. It follows from (2.2) that $\Delta H'$ is the union of $\Delta H$ and $\Delta \sigma$. Now we prove the second assertion.

Case 1. $\Delta H \cap \partial \sigma$ is empty. Then
$$\Delta H' = \Delta H \cup \Delta \sigma \simeq \Delta H \cup \{\ast\} = \Delta H/\emptyset.$$ Hence we have (d).

Case 2. $\Delta H \cap \partial \sigma$ is nonempty. Since $\Delta \sigma$ is contractible, by taking the union of $\Delta H$ and $\Delta \sigma$, $\Delta H'$ is homotopy equivalent to the quotient space $\Delta H/(\Delta H \cap \partial \sigma)$.
Summarizing both Case 1 and Case 2, we obtain (d).

With the help of Lemma 2.4 (c) and (d), we obtain the next theorem.

**Theorem 2.5.** (a) Suppose for any \( \tau \subseteq \sigma, \tau \in \mathcal{H}. \) Then for any \( k \neq d - 1 \) and \( d, \)

\[
\tilde{H}_k(\delta \mathcal{H}') \cong \tilde{H}_k(\delta \mathcal{H}); \tag{2.13}
\]

(b) Suppose \( \Delta \mathcal{H} \cap \partial \sigma \) is nonempty, and \( \sigma \) is a maximal face of \( \mathcal{H}'. \) Then we have a long exact sequence of reduced homology groups

\[
\cdots \xrightarrow{\partial^*} \tilde{H}_k(\Delta \mathcal{H} \cap \partial \sigma) \xrightarrow{i^*} \tilde{H}_k(\delta \mathcal{H}) \xrightarrow{j^*} \tilde{H}_k(\Delta \mathcal{H}') \xrightarrow{\partial^*} \tilde{H}_{k-1}(\Delta \mathcal{H} \cap \partial \sigma) \xrightarrow{i^*} \cdots \xrightarrow{j^*} \tilde{H}_0(\Delta \mathcal{H}') \longrightarrow 0. \tag{2.14}
\]

Here \( i' \) is the inclusion \( \Delta \mathcal{H} \cap \partial \sigma \hookrightarrow \Delta \mathcal{H} \) and \( j' \) is the inclusion \( \Delta \mathcal{H} \hookrightarrow \Delta \mathcal{H}'. \) In particular, if \( \tilde{H}_{k-1}(\Delta \mathcal{H} \cap \partial \sigma) = \tilde{H}_k(\Delta \mathcal{H} \cap \partial \sigma) = 0, \) then

\[
\tilde{H}_k(\Delta \mathcal{H}') \cong \tilde{H}_k(\Delta \mathcal{H}). \tag{2.15}
\]

**Proof.** (a) Let \( i \) be the inclusion \( \partial \sigma \hookrightarrow \delta \mathcal{H}. \) Let \( q \) be the quotient map \( \delta \mathcal{H} \rightarrow \delta \mathcal{H}/\partial \sigma. \) By [15, Theorem 2.13], we have a long exact sequence

\[
\cdots \xrightarrow{\partial^*} \tilde{H}_k(\partial \sigma) \xrightarrow{i^*} \tilde{H}_k(\delta \mathcal{H}) \xrightarrow{q^*} \tilde{H}_k(\delta \mathcal{H}/\partial \sigma) \xrightarrow{\partial^*} \tilde{H}_{k-1}(\partial \sigma) \xrightarrow{i^*} \cdots \xrightarrow{q^*} \tilde{H}_0(\delta \mathcal{H}/\partial \sigma) \longrightarrow 0. \tag{2.16}
\]

By Lemma 2.4 (c), there is a homotopy equivalence

\[ h : \delta \mathcal{H}/\partial \sigma \longrightarrow \delta \mathcal{H}'. \]

Let \( j \) be the inclusion \( \delta \mathcal{H} \hookrightarrow \delta \mathcal{H}'. \) By the proof of Lemma 2.4 (c), the diagram

\[
\begin{array}{ccc}
\delta \mathcal{H} & \xrightarrow{q} & \delta \mathcal{H}/\partial \sigma \\
\downarrow j & & \downarrow h \\
\delta \mathcal{H}' & & \end{array}
\]

commutes up to homotopy. Hence \( h \circ q \simeq j. \) Hence by substituting \( \delta \mathcal{H}/\partial \sigma \) with \( \delta \mathcal{H}' \) and substituting \( q \) with \( j \) in (2.16), we obtain a long exact sequence of reduced homology groups

\[
\cdots \xrightarrow{\partial^*} \tilde{H}_k(\partial \sigma) \xrightarrow{i^*} \tilde{H}_k(\delta \mathcal{H}) \xrightarrow{j^*} \tilde{H}_k(\delta \mathcal{H}') \xrightarrow{\partial^*} \tilde{H}_{k-1}(\partial \sigma) \xrightarrow{i^*} \cdots \xrightarrow{j^*} \tilde{H}_0(\delta \mathcal{H}') \longrightarrow 0. \tag{2.17}
\]

Suppose \( k \neq d - 1 \) and \( d. \) Then both \( \tilde{H}_k(\partial \sigma) \) and \( \tilde{H}_{k-1}(\partial \sigma) \) are zero. Hence by the exactness of (2.17), we have (2.13).
(b). Let $q'$ be the quotient map $\Delta \mathcal{H} \to \Delta \mathcal{H}/(\Delta \mathcal{H} \cap \partial \sigma)$. By Lemma 2.4 (d), there is a homotopy equivalence

$$h' : \Delta \mathcal{H}/(\Delta \mathcal{H} \cap \partial \sigma) \longrightarrow \Delta \mathcal{H}' .$$

By the proof of Lemma 2.4 (d), the diagram

$$\begin{array}{ccc}
\Delta \mathcal{H} & \xrightarrow{q'} & \Delta \mathcal{H}/(\Delta \mathcal{H} \cap \partial \sigma) \\
\downarrow{j'} & & \downarrow{h'} \\
\Delta \mathcal{H}' & \leftarrow & \Delta \mathcal{H}'
\end{array}$$

commutes up to homotopy. Hence $h' \circ q' \simeq j'$. Since $\Delta \mathcal{H} \cap \partial \sigma$ is nonempty, by applying [15, Theorem 2.13], we obtain (2.14). In addition, suppose $\tilde{H}_{k-1}(\Delta \mathcal{H} \cap \partial \sigma) = \tilde{H}_k(\Delta \mathcal{H} \cap \partial \sigma) = 0$. Then (2.15) follows immediately from the exactness of (2.14).

**Remark 2:** Supplementary to Theorem 2.5 (b), suppose $\Delta \mathcal{H} \cap \partial \sigma$ is empty, and for any $\sigma \subseteq \tau$, $\tau / \in \mathcal{H}$. Then $H_*(\Delta \mathcal{H}') \cong H_*(\Delta \mathcal{H}) \oplus H_*(\Delta \sigma)$.

### 3 Evolutions of hypergraphs, lower-associated simplicial complexes, and the embedded homology

In Definition 2, an evolution of a hypergraph $\mathcal{H}$ is defined as a hypergraph obtained by adding a new hyperedge to $\mathcal{H}$. In this section, we study evolutions of hypergraphs as well as their induced homomorphisms on the embedded homology groups.

In Subsection 3.1, we study the induced homomorphisms of evolutions of hypergraphs on the embedded homology. We prove Theorem 3.2. In Subsection 3.2, we study the relations between the embedded homology and the lower-associated simplicial complexes. We prove Theorem 3.4.

In the remaining part of this paper, we let $\mathcal{H}$ be a hypergraph. We let $d$ be a positive integer and let $\mathcal{H}'$ be an evolution of $\mathcal{H}$ by adding a hyperedge $\sigma$ of dimension $d$.

#### 3.1 Evolutions of hypergraphs and their embedded homology

The next lemma is a consequence of [3, Theorem 3.10].

**Lemma 3.1.** Suppose for any $\tau \in \mathcal{H}$ and any $\tau' \in \mathcal{H}'$, $\tau \cap \tau'$ is either empty or a hyperedge of $\mathcal{H}$. Then for any $k \geq d^h(\mathcal{H} \cap \partial \sigma) + 2$,

$$H_k(\mathcal{H}') \cong H_k(\mathcal{H}) \oplus H_k((\mathcal{H} \cap \partial \sigma) \cup \sigma).$$

**Proof.** We consider the two hypergraphs $\mathcal{H}$ and $(\mathcal{H} \cap \partial \sigma) \cup \sigma$. Their union is $\mathcal{H}'$, and their intersection is $\mathcal{H} \cap \partial \sigma$. With the help of our assumptions on $\mathcal{H}$, $\mathcal{H}'$ and $\sigma$, it follows that
for any \( \tau \in \mathcal{H} \) and any \( \eta \in (\mathcal{H} \cap \partial \sigma) \cup \sigma, \tau \cap \eta \) is either empty or a hyperedge of \( \mathcal{H} \cap \partial \sigma \). Thus by \([\mathbf{5}, \text{Theorem 3.10}]\), we have a long exact sequence of homology groups

\[
\cdots \to H_k(\mathcal{H} \cap \partial \sigma) \to H_k(\mathcal{H}) \oplus H_k((\mathcal{H} \cap \partial \sigma) \cup \sigma) \to H_k(\mathcal{H}') \to H_{k-1}(\mathcal{H} \cap \partial \sigma) \to \cdots \to H_0(\mathcal{H}') \to 0.
\]

(3.2)

On the other hand, for any \( k \geq d^b(\mathcal{H} \cap \partial \sigma) + 1 \), it follows from (2.5) or (2.6) that

\[
H_k(\mathcal{H} \cap \partial \sigma) = 0.
\]

(3.3)

By substituting (3.3) into the long exact sequence (3.2), we obtain (3.1) for any \( k \geq d^b(\mathcal{H} \cap \partial \sigma) + 2 \).

Given a hypergraph \( \mathcal{H} \) and a nonnegative integer \( k \), let

\[
X_k(\mathcal{H}) = \mathbb{Z}(\mathcal{H}_{k-1}) \cap \partial_k \mathbb{Z}(\mathcal{H}_k),
\]

\[
Y_k(\mathcal{H}) = \partial_{k+1} \mathbb{Z}(\mathcal{H}_{k+1}) + \ker(\partial_k|_{\mathbb{Z}(\mathcal{H}_k)}).
\]

By applying Lemma 3.1, the next theorem follows.

**Theorem 3.2.** (a) (i). \( H_k(\mathcal{H}') \cong H_k(\mathcal{H}) \) for any \( k \neq d - 1 \) and \( d \); (ii). \( H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H}) \) iff. \( X_d(\mathcal{H}') = X_d(\mathcal{H}) \); (iii). \( H_d(\mathcal{H}') \cong H_d(\mathcal{H}) \) iff. \( Y_d(\mathcal{H}') = Y_d(\mathcal{H}) \).

(b). Suppose for any \( d \)-hyperedge \( \tau \in \mathcal{H} \) with \( \tau \cap \sigma \) nonempty, \( \tau \) does not intersect with any other \( d \)-hyperedges of \( \mathcal{H} \). If \( d = d^b(\mathcal{H} \cap \partial \sigma) + 1 \), then \( H_{d-1}(\mathcal{H}) \cong H_{d-1}(\mathcal{H}') \oplus \mathbb{Z} \).

(c). Suppose for any \( \tau \in \mathcal{H} \) and any \( \tau' \in \mathcal{H}' \), \( \tau \cap \tau' \) is either empty or a hyperedge of \( \mathcal{H} \).

(i). If \( d \geq d^b(\mathcal{H} \cap \partial \sigma) + 2 \), then \( H_d(\mathcal{H}') \cong H_d(\mathcal{H}) \); (ii). If \( d \geq d^b(\mathcal{H} \cap \partial \sigma) + 3 \), then \( H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H}) \).

**Proof.** (a)-(i). Let \( \partial_\delta ' \) be the boundary map of \( \Delta \mathcal{H}' \). By [\mathbf{5}, Proposition 3.2], the restriction of \( \partial_\delta ' \to \Delta \mathcal{H} \) is \( \partial_\delta \). Since \( \mathcal{H}' \) is an evolution of \( \mathcal{H} \) by adding \( \sigma \), for any \( k \neq d \), \( H_k(\mathcal{H}') = H_k(\mathcal{H}) \). Thus by (2.5) or (2.6), for any \( k \neq d - 1 \) and \( d \), \( H_k(\mathcal{H}') \cong H_k(\mathcal{H}) \).

(a)-(ii). Suppose \( X_d(\mathcal{H}') = X_d(\mathcal{H}) \). Then by (2.5), \( H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H}) \). Conversely, suppose \( H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H}) \). If \( X_d(\mathcal{H}') \neq X_d(\mathcal{H}) \), then \( X_d(\mathcal{H}) \) is a nontrivial subgroup of \( X_d(\mathcal{H}') \). Since both \( X_d(\mathcal{H}') \) and \( X_d(\mathcal{H}) \) are subgroups of \( \ker(\partial_{d-1}|_{\mathbb{Z}(\mathcal{H}_d)} \mathcal{H}_d) \), it follows from (2.5) that \( H_{d-1}(\mathcal{H}') \) is a nontrivial subgroup of \( H_{d-1}(\mathcal{H}) \). This contradicts our assumption. Hence \( X_d(\mathcal{H}') = X_d(\mathcal{H}) \). Therefore, we obtain the equivalence in (a)-(ii).

(a)-(iii). Suppose \( Y_d(\mathcal{H}') = Y_d(\mathcal{H}) \). Then by (2.6), \( H_d(\mathcal{H}') \cong H_d(\mathcal{H}) \). Conversely, suppose \( H_d(\mathcal{H}') \cong H_d(\mathcal{H}) \). If \( Y_d(\mathcal{H}') \neq Y_d(\mathcal{H}) \), then \( Y_d(\mathcal{H}) \) is a nontrivial subgroup of \( Y_d(\mathcal{H}') \). By (2.6), \( H_d(\mathcal{H}) \) is a nontrivial subgroup of \( H_d(\mathcal{H}') \) as well. This contradicts our assumption. Hence \( Y_d(\mathcal{H}') = Y_d(\mathcal{H}) \). Therefore, we obtain the equivalence in (a)-(iii).

Before proving (b) and (c), we notice that

\[
d \geq \dim(\mathcal{H} \cap \partial \sigma) + 1 \geq d^b(\mathcal{H} \cap \partial \sigma) + 1.
\]

(3.4)
Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ be the $(d - 1)$-faces of $\sigma$.

(b). Suppose for any $d$-hyperedge $\tau \in \mathcal{H}$ with $\tau \cap \sigma$ nonempty, $\tau$ does not intersect with any other $d$-hyperedges of $\mathcal{H}$. Suppose $d = d^h(\mathcal{H} \cap \partial \sigma) + 1$. Then

$$H_{d-1}(\mathcal{H} \cap \sigma) \neq 0. \quad (3.5)$$

We claim that

$$\partial_d \sigma = \sum_{i=0}^{d} (-1)^i \sigma_i \notin \partial_d(\mathcal{Z}(\mathcal{H}_d)). \quad (3.6)$$

If (3.6) is not true, then there exist a positive integer $m$, $d$-hyperedges $\tau_1, \tau_2, \ldots, \tau_m$ of $\mathcal{H}$, and integers $a_1, a_2, \ldots, a_m$ such that

$$\partial_d \left( \sum_{i=1}^{m} a_i \tau_i \right) = \sum_{i=0}^{d} (-1)^i \sigma_i. \quad (3.7)$$

By (3.6), it is clear that there exists some nonzero $a_i$, $1 \leq i \leq m$. Without loss of generality, we assume that all the $a_i$'s are nonzero, $1 \leq i \leq m$. Otherwise we delete the summand $a_i \tau_i$. Hence there exists $\tau_\lambda$, $1 \leq \lambda \leq m$, such that $\tau_\lambda \cap \sigma \neq \emptyset$. By our assumption, $\tau_\lambda$ does not intersect with any other $d$-hyperedges of $\mathcal{H}$ except $\sigma$. Hence if we use $\eta_0, \eta_1, \ldots, \eta_d$ to denote the $(d - 1)$-faces of $\tau_\lambda$, then each $\eta_i$, $0 \leq i \leq d$, is a summand of (3.7) up to a coefficient $\pm 1$. Consequently, by (3.7), we have

$$\{\eta_0, \ldots, \eta_d\} = \{\sigma_0, \ldots, \sigma_d\}.$$

It follows that $\tau_\lambda = \sigma$. This contradicts that $\sigma \notin \mathcal{H}$. Therefore, (3.6) holds.

Consequently, with the help of (3.6),

$$\partial_d(\mathcal{Z}(\mathcal{H}'_d)) = \partial_d(\mathcal{Z}(\mathcal{H}_d) \oplus \mathcal{Z}(\sigma))$$

$$= \partial_d(\mathcal{Z}(\mathcal{H}_d)) \oplus \partial_d(\mathcal{Z}(\sigma))$$

$$= \partial_d(\mathcal{Z}(\mathcal{H}_d)) \oplus \mathcal{Z}(\sum_{i=0}^{d} (-1)^i \sigma_i). \quad (3.8)$$

On the other hand, it follows from (3.5) that for each $0 \leq i \leq d$, $\sigma_i \in \mathcal{H}_{d-1}$. Hence

$$\mathcal{Z}(\sum_{i=0}^{d} (-1)^i \sigma_i) \subseteq \mathcal{Z}(\mathcal{H}_{d-1}). \quad (3.9)$$

With the help of (2.5), (3.8) and (3.9), it follows that

$$H_{d-1}(\mathcal{H}')$$

$$= \text{Ker}(\partial_{d-1} |_{\mathcal{Z}(\mathcal{H}_{d-1})}/(\mathcal{Z}(\mathcal{H}_{d-1}) \cap (\partial_d(\mathcal{Z}(\mathcal{H}_d)) \oplus \mathcal{Z}(\sum_{i=0}^{d} (-1)^i \sigma_i))))$$

$$= \text{Ker}(\partial_{d-1} |_{\mathcal{Z}(\mathcal{H}_{d-1})}/((\mathcal{Z}(\mathcal{H}_{d-1}) \cap \partial_d(\mathcal{Z}(\mathcal{H}_d))) \oplus \mathcal{Z}(\sum_{i=0}^{d} (-1)^i \sigma_i))). \quad (3.10)$$
Moreover,

\[ \mathbb{Z}(\sum_{i=0}^{d} (-1)^i \sigma_i) \subseteq \text{Ker}(\partial_{d-1} \mid_{\mathbb{Z}(\mathcal{H}_{d-1})}). \]

Therefore, by comparing (3.10) with the embedded homology of \( \mathcal{H} \) given in (2.5), we obtain (b).

(c). Suppose for any \( \tau \in \mathcal{H} \) and any \( \tau' \in \mathcal{H}', \tau \cap \tau' \) is either empty or a hyperedge of \( \mathcal{H} \).

(c)-(i). Suppose \( d \geq d^h(\mathcal{H} \cap \partial \sigma) + 2 \).

Then by Lemma 3.1 holds for \( k = d \). On the other hand, since \( \partial_d \sigma \neq 0 \), we have

\[ \text{Ker}(\partial_d \mid_{\mathbb{Z}((\mathcal{H} \cap \partial \sigma) \cup \sigma)_{d-1}}) = 0. \]

Hence it follows from (2.5) that \( H_d((\mathcal{H} \cap \partial \sigma) \cup \sigma) = 0 \). Therefore, with the help of (3.1) for \( k = d \), \( H_d(\mathcal{H}') \cong H_d(\mathcal{H}) \).

(c)-(ii). Suppose \( d \geq d^h(\mathcal{H} \cap \partial \sigma) + 3 \).

Then by Lemma 3.1 holds for \( k = d - 1 \).

**Case 1.** For each \( 0 \leq i \leq d \), \( \sigma_i \in \mathcal{H} \). Then both

\[ \text{Ker}(\partial_{d-1} \mid_{\mathbb{Z}((\mathcal{H} \cap \partial \sigma) \cup \sigma)_{d-1}}) \]

and

\[ \mathbb{Z}((\mathcal{H} \cap \partial \sigma) \cup \sigma)_{d-1}) \cap \partial_d \mathbb{Z}(\sigma) \]

are

\[ \mathbb{Z}(\sum_{i=0}^{d} (-1)^i \sigma_i). \]

Hence by (2.5), \( H_{d-1}((\mathcal{H} \cap \partial \sigma) \cup \sigma) = 0 \). With the help of (3.1) for \( k = d - 1 \), it follows that \( H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H}) \).

**Case 2.** There exists \( 0 \leq i \leq d \) such that \( \sigma_i \notin \mathcal{H} \). Then both (3.11) and (3.12) are zero. Hence by (3.1) for \( k = d - 1 \) and (2.5), \( H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H}) \).

Summarizing both Case 1 and Case 2, we obtain (c)-(ii). \( \square \)

The next corollary is a particular case of Theorem 3.2 (c).

**Corollary 3.3.** Suppose for any \( \tau, \tau' \in \mathcal{H} \), \( \tau \cap \tau' \) is either empty or a hyperedge of \( \mathcal{H} \). If \( d \geq 3 \) and \( \mathcal{H} \cap \partial \sigma \) is a contractible simplicial complex, then \( H_*(\mathcal{H}') \cong H_*(\mathcal{H}) \).

**Proof.** Suppose \( \mathcal{H} \cap \partial \sigma \) is a contractible simplicial complex. It follows from the definition of simplicial complexes that for any \( \tau \in \mathcal{H} \), \( \tau \cap \sigma \) is either empty or a hyperedge of \( \mathcal{H} \). Hence with the help of our assumption, for any \( \tau \in \mathcal{H} \) and any \( \tau'' \in \mathcal{H}', \tau \cap \tau'' \) is either empty or a hyperedge of \( \mathcal{H} \). Since \( \mathcal{H} \cap \partial \sigma \) is contractible, \( d^h(\mathcal{H} \cap \partial \sigma) = 0 \). Hence \( d \geq d^h(\mathcal{H} \cap \partial \sigma) + 3 \).

By Theorem 3.2 (c), the corollary follows. \( \square \)
3.2 The embedded homology and the lower-associated simplicial complexes

We recall that $d$ is a positive integer and $\mathcal{H}'$ is an evolution of $\mathcal{H}$ by adding a hyperedge $\sigma$ of dimension $d$. By Example 2.1

- $\delta \mathcal{H}' = \delta \mathcal{H}$ and $\Delta \mathcal{H}' = \Delta \mathcal{H}$ cannot imply $H_*(\mathcal{H}') = H_*(\mathcal{H})$.

By Example 2.2

- $\delta \mathcal{H}' = \delta \mathcal{H}$ and $H_*(\mathcal{H}') = H_*(\mathcal{H})$ cannot imply $H_*(\Delta \mathcal{H}') = H_*(\Delta \mathcal{H})$.

The next theorem investigates further relations among the homology groups.

**Theorem 3.4.** The followings are equivalent

(a). $H_*(\delta \mathcal{H}') = H_*(\delta \mathcal{H})$;

(b). $\delta \mathcal{H}' = \delta \mathcal{H}$.

Moreover, suppose $H_*(\mathcal{H}') \cong H_*(\mathcal{H})$, and $\sigma$ is not a face of any $(d+1)$-hyperedge of $\mathcal{H}$. Then both (a) and (b) hold.

**Proof.** We prove the first assertion. (b) $\implies$ (a): trivial. (a) $\implies$ (b): suppose to the contrary, (b) does not hold. Then $\delta \mathcal{H}' = \delta \mathcal{H} \cup \{\sigma\}$, and

$$\text{Ker}\partial_{d-1} |_{Z(\delta \mathcal{H}')_{d-1}} = \text{Ker}\partial_{d-1} |_{Z(\delta \mathcal{H})_{d-1}}.$$  

Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ be the $(d-1)$-faces of $\sigma$. Then for each $0 \leq i \leq d$, $\sigma_i \in (\delta \mathcal{H})_{d-1}$; and

$$\partial_d \sigma = \sum_{i=0}^{d} (-1)^i \sigma_i.$$  \hspace{1cm} (3.13)

Since $\sigma \in (\delta \mathcal{H}')_d$, (3.13) is an element of $\partial_d(Z(\delta \mathcal{H}')_d)$. We use $m$ to denote a positive integer.

**CASE 1.** (3.13) is not an element of $\partial_d(Z(\delta \mathcal{H})_d)$.

**SUBCASE 1.1.** for any $m$, $m(\partial_d \sigma) \notin \partial_d(Z(\delta \mathcal{H})_d)$.

Then

$$\partial'_d(Z(\delta \mathcal{H}')_d) = \partial_d(Z(\delta \mathcal{H})_d) \oplus Z(\partial_d \sigma).$$  \hspace{1cm} (3.14)

By the definition of the homology groups of simplicial complexes, it follows from (3.14) that $H_{d-1}(\delta \mathcal{H}) \not\cong H_{d-1}(\delta \mathcal{H}')$.

**SUBCASE 1.2.** There exists a smallest $m \geq 2$ such that $m(\partial_d \sigma) \in \partial_d(Z(\delta \mathcal{H})_d)$.

Then

$$\partial'_d(Z(\delta \mathcal{H}')_d) = \partial_d(Z(\delta \mathcal{H})_d) \oplus Z(\partial_d \sigma)/ \sim$$  \hspace{1cm} (3.15)
where \( \sim \) is the equivalent relation by identifying \( m(\partial_d \sigma) \) with an element in \( \partial_d (\mathbb{Z}(\delta \mathcal{H})_d) \). By the definition of homology groups of simplicial complexes, it follows from (3.15) that 
\[ H_{d-1}(\delta \mathcal{H}) \not\cong H_{d-1}(\delta \mathcal{H}') \]

Summarizing both Subcase 1.1 and Subcase 1.2, we have that in Case 1, (a) does not hold.

**Case 2.** (3.13) is an element of \( \partial_d (\mathbb{Z}(\delta \mathcal{H})_d) \).

Then since \( \sigma \not\in (\delta \mathcal{H})_d \), there exists a positive integer \( l, a_1, a_2, \ldots, a_l \in \mathbb{Z} \) and \( \tau_1, \tau_2, \ldots, \tau_l \in (\delta \mathcal{H})_d, \tau_i \neq \sigma \) for each \( 1 \leq i \leq l \), such that
\[ \partial_d (\sum_{i=1}^{l} a_i \tau_i) = \partial_d \sigma. \]

Hence
\[ \sum_{i=1}^{l} a_i \tau_i - \sigma \in \text{Ker} \partial_d \mid_{\mathbb{Z}(\delta \mathcal{H}')_d}. \] (3.16)

Moreover, for any \( m \), we have
\[ m (\sum_{i=1}^{l} a_i \tau_i - \sigma) \notin \text{Ker} \partial_d \mid_{\mathbb{Z}(\delta \mathcal{H})_d}. \] (3.17)

On the other hand,
\[ \partial_{d+1} \mathbb{Z}((\delta \mathcal{H}')_{d+1}) = \partial_{d+1} \mathbb{Z}((\delta \mathcal{H})_{d+1}). \] (3.18)

By the definition of the homology groups of simplicial complexes, it follows from (3.16), (3.17) and (3.18) that
\[ H_d(\delta \mathcal{H}') = H_d(\delta \mathcal{H}) \oplus \mathbb{Z}. \]

Summarizing both Case 1 and Case 2, we get a contradiction with (a). Therefore, our assumption is not true. That is, (a) implies (b). The first assertion follows.

We turn to prove the second assertion. Suppose \( H_*(\mathcal{H}') = H_*(\mathcal{H}) \) and \( \sigma \) is not a face of any \( (d+1) \)-hyperedge of \( \mathcal{H} \). To prove the second assertion, we suppose to the contrary, \( \delta \mathcal{H}' \neq \delta \mathcal{H} \). We want to get a contradiction. We notice that \( \partial_d \sigma \in \mathbb{Z}(\mathcal{H}_{d-1}) \).

**Case 1’.** \( \partial_d \sigma \notin \partial_d (\mathbb{Z}(\mathcal{H}_d)) \).

Then
\[ \partial_d (\mathbb{Z}(\mathcal{H}_d')) \cap \mathbb{Z}(\mathcal{H}_{d-1}') = (\partial_d (\mathbb{Z}(\mathcal{H}_d)) \cap \mathbb{Z}(\mathcal{H}_{d-1})) \oplus \mathbb{Z}(\partial_d \sigma). \] (3.19)

It follows from (2.5) and (3.19) that \( H_{d-1}(\mathcal{H}) \not\cong H_{d-1}(\mathcal{H}') \).

**Case 2’.** \( \partial_d \sigma \in \partial_d (\mathbb{Z}(\mathcal{H}_d)) \).

Then since \( \sigma \notin \mathcal{H}_d \), there exists a positive integer \( l, a_1, a_2, \ldots, a_l \in \mathbb{Z} \) and \( \tau_1, \tau_2, \ldots, \tau_l \in \mathcal{H}_d, \tau_i \neq \sigma \) for each \( 1 \leq i \leq l \), such that
\[ \sum_{i=1}^{l} a_i \tau_i - \sigma \in \text{Ker} \partial_d \mid_{\mathbb{Z}(\mathcal{H}_d')} \] (3.20)
Moreover, for any $m$, we have
\[ m\left(\sum_{i=1}^{l} a_i \tau_i - \sigma\right) \notin \operatorname{Ker} \partial_d |_{\mathcal{Z}(\mathcal{H}_d)}. \quad (3.21) \]

On the other hand, it follows from our assumption that $\sigma$ is not a face of any $(d+1)$-hyperedge of $\mathcal{H}$ that
\[ \mathcal{Z}(\mathcal{H}'_d) \cap \partial_{d+1} \mathcal{Z}(\mathcal{H}'_{d+1}) = \mathcal{Z}(\mathcal{H}_d) \cap \partial_{d+1} \mathcal{Z}(\mathcal{H}_{d+1}). \quad (3.22) \]

By (2.5), (3.20), (3.21) and (3.22), we obtain
\[ H_d(\mathcal{H}') = H_d(\mathcal{H}) \oplus \mathbb{Z}. \]

Summarizing both Case 1’ and Case 2’, we get a contradiction with our assumption $H_*\mathcal{H}' \cong H_*\mathcal{H}$. Therefore, (a) and (b) hold. \qed

4 Proof of Theorem 1.1

We prove Theorem 1.1.

Proof. (a). Suppose to the contrary, $d^b(\mathcal{H} \cap \partial \sigma)$ is neither $d-1$ nor $d-2$. Then by (3.4), we have $d \geq d^b(\mathcal{H} \cap \partial \sigma) + 3$. By Theorem 3.2 (c), it follows that $H_*\mathcal{H}' \cong H_*\mathcal{H}$. Hence by the second assertion of Theorem 3.4 we have $\delta \mathcal{H}' = \delta \mathcal{H}$. This contradicts our assumption (iii) of Theorem 1.1. Therefore, $d^b(\mathcal{H} \cap \partial \sigma)$ is either $d-1$ or $d-2$. The first assertion follows.

(b). The assertion follows from Theorem 3.2 (b).

(c). By Theorem 3.2 (a) (i) and Theorem 3.2 (c) (i), we have $H_i(\mathcal{H}') \cong H_i(\mathcal{H})$ for $i \neq d-1$. To prove (c), we suppose to the contrary that $H_{d-1}(\mathcal{H}') \cong H_{d-1}(\mathcal{H})$. Then $H_*\mathcal{H}' \cong H_*\mathcal{H}$. By the second assertion of Theorem 3.4 $\delta \mathcal{H}' = \delta \mathcal{H}$. This contradicts our assumption (iii) of Theorem 1.1. Hence $H_{d-1}(\mathcal{H}') \neq H_{d-1}(\mathcal{H})$ and (c) follows. \qed

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