Stability of the Drygas Functional Equation on Restricted Domain

Magdalena Piszczek and Joanna Szczawińska

Abstract. We study the stability of the Drygas functional equation on a restricted domain. The main tool used in the proofs is the fixed point theorem for functional spaces.

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1. Introduction

We say that a function $f: \mathbb{R} \to \mathbb{R}$ satisfies the Drygas equation iff

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \quad x, y \in \mathbb{R}. \quad (1)$$

The above equation was introduced in [3] in order to obtain a characterization of the quasi-inner-product spaces. Ebanks, Kannappan and Sahoo in [4] have obtained the general solution of the Eq. (1) as

$$f(x) = a(x) + q(x), \quad x \in \mathbb{R},$$

where $a: \mathbb{R} \to \mathbb{R}$ is an additive function and $q: \mathbb{R} \to \mathbb{R}$ is a quadratic function, i.e. $q$ satisfies the quadratic functional equation

$$q(x + y) + q(x - y) = 2q(x) + 2q(y), \quad x, y \in \mathbb{R}.$$ 

A set-valued version of Eq. (1) was considered by Smajdor in [8].

The stability in the Hyers–Ulam sense of the Drygas equation has been investigated by Jung and Sahoo in [6]. They have proved that if a function $f: X \to Y$, where $X$ is a real vector space and $Y$ is a Banach space satisfies the inequality

$$\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\| \leq \varepsilon, \quad x, y \in X \quad (2)$$

\[ \]
for same $\varepsilon > 0$, then there exists a unique additive function $a: X \to Y$ and a unique quadratic function $q: X \to Y$ such that
\[
\|f(x) - a(x) - q(x)\| \leq \frac{25}{3} \varepsilon, \quad x \in X.
\]
Their result was improved first by Yang in [9] and later by Sikorska in [7]. In the case when $X$ is an Abelian group they obtained sharper bounds: $\frac{3}{2} \varepsilon$ and $\varepsilon$ respectively instead of $\frac{25}{3} \varepsilon$ (cf. Proposition 1 in [9] and Theorem 3.2 in [7]). The stability and solution of the Drygas equation under some additional conditions was also studied by Forti and Sikorska in [5] in the case when $X$ and $Y$ are amenable groups.

In the paper we present the stability results for the Drygas equation on restricted domain. Let $X$ be a nonempty subset of a normed space and $Y$ be a normed space. We say that a function $f: X \to Y$ satisfies the Drygas functional equation on $X$ if
\[
f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in X, x+y, x-y \in X. \tag{3}
\]

One of the method of the proof is based on a fixed point result that can be derived from [1] (Theorem 1). To present it we need the following three hypothesis:

(H1) $X$ is a nonempty set, $Y$ is a Banach space, $f_1, \ldots, f_k: X \to X$ and $L_1, \ldots, L_k: X \to \mathbb{R}_+$ are given.

(H2) $T: Y^X \to Y^X$ is an operator satisfying the inequality
\[
\|T\xi(x) - T\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|
\]
for all $\xi, \mu \in Y^X, x \in X$.

(H3) $A: \mathbb{R}_+^X \to \mathbb{R}_+^X$ is defined by
\[
A\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \quad \delta \in \mathbb{R}_+^X, x \in X.
\]

Now we are in a position to present the above mentioned fixed point theorem.

**Theorem 1.** Let hypotheses (H1)–(H3) be valid and functions $\varepsilon: X \to \mathbb{R}_+$ and $\varphi: X \to Y$ fulfil the following two conditions
\[
\|T\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in X,
\]
\[
\varepsilon^*(x) := \sum_{n=0}^{\infty} A^n\varepsilon(x) < \infty, \quad x \in X.
\]
Then there exists a unique fixed point $\psi$ of $T$ with
\[
\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in X.
\]
Moreover,
\[
\psi(x) := \lim_{n \to \infty} T^n \phi(x), \quad x \in X.
\]

Throughout the paper \( \mathbb{N}_0 \) denotes the set of all non-negative integers.

2. Stability Results

**Theorem 2.** Let \( X \) be a subset with 0 of a normed space, \( Y \) be a Banach space and \( c \geq 0 \). Assume that \( p > 0 \) and a function \( f: X \to Y \) satisfies
\[
\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\| \leq c(\|x\|^p + \|y\|^p) \tag{4}
\]
for all \( x, y \in X \) such that \( x + y, x - y \in X \).

1. If \( p > 2 \) and \( -x, \frac{x}{2} \in X \) for all \( x \in X \), then there exists a function \( g: X \to Y \) satisfying the Drygas equation on \( X \) such that
\[
\|f(x) - g(x)\| \leq \frac{2c}{2p - 4} \|x\|^p, \quad x \in X.
\]

2. If \( 0 < p < 1 \) and \( -x, 2x \in X \) for all \( x \in X \), then there exists a function \( g: X \to Y \) satisfying the Drygas equation on \( X \) such that
\[
\|f(x) - g(x)\| \leq \frac{2c}{2 - 2p} \|x\|^p, \quad x \in X.
\]

3. If \( 1 < p < 2 \) and \( -x, \frac{1}{2}x, 2x \in X \) for all \( x \in X \), then there exists a function \( g: X \to Y \) satisfying the Drygas equation on \( X \) such that
\[
\|f(x) - g(x)\| \leq \left( \frac{2c}{4 - 2p} + \frac{2c}{2p - 2} \right) \|x\|^p, \quad x \in X.
\]

Moreover, \( g \) is the unique solution of the Eq. (3) such that \( \|f(x) - g(x)\| \leq M\|x\|^p \) for all \( x \in X \) and some \( M \geq 0 \).

**Proof.** First observe that the inequality (4) clearly forces \( f(0) = 0 \).

1. Replacing \( x \) and \( y \) by \( \frac{x}{2} \) in (4) we obtain
\[
\left\| f(x) - 3f\left( \frac{x}{2} \right) - f\left( -\frac{x}{2} \right) \right\| \leq \frac{2c}{2p} \|x\|^p, \quad x \in X. \tag{5}
\]

Consider functions \( T: Y^X \to Y^X \) and \( \varepsilon: X \to \mathbb{R}_+ \) given as follows
\[
T \xi(x) = 3\xi\left( \frac{x}{2} \right) + \xi\left( -\frac{x}{2} \right), \quad x \in X, \xi \in Y^X
\]
and
\[
\varepsilon(x) = \frac{2c}{2p} \|x\|^p, \quad x \in X.
\]

The inequality (5) now becomes
\[
\|Tf(x) - f(x)\| \leq \varepsilon(x), \quad x \in X.
\]
For every $\xi, \mu \in Y^X$ and $x \in X$

$$\|T\xi(x) - T\mu(x)\| \leq 3 \left\| \xi \left( \frac{x}{2} \right) - \mu \left( \frac{x}{2} \right) \right\| + \left\| \xi \left( -\frac{x}{2} \right) - \mu \left( -\frac{x}{2} \right) \right\|,$$

so $T$ satisfies the inequality (H2) with $f_1(x) = \frac{x}{2}, f_2(x) = -\frac{x}{2}, L_1(x) = 3, L_2(x) = 1, x \in X$. By (H3), the operator $\Lambda: \mathbb{R}_+^X \to \mathbb{R}_+^X$ is given by

$$\Lambda \eta(x) = 3 \eta \left( \frac{x}{2} \right) + \eta \left( -\frac{x}{2} \right), \quad x \in X, \eta \in \mathbb{R}_+^X.$$

In particular

$$\Lambda \varepsilon(x) = 4 \varepsilon \left( \frac{x}{2} \right) = \frac{4}{2^p} \varepsilon(x), \quad x \in X.$$

Since $\Lambda$ is linear, we can prove by induction

$$\Lambda^n \varepsilon(x) = \left( \frac{4}{2^p} \right)^n \varepsilon(x), \quad x \in X, n \in \mathbb{N}_0.$$

As $p > 2$ we have $\frac{4}{2^p} < 1$. Consequently the series $\sum_{n=0}^{\infty} \Lambda^n \varepsilon(x)$ is convergent for every $x \in X$ and

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) = \sum_{n=0}^{\infty} \left( \frac{4}{2^p} \right)^n \varepsilon(x) = \frac{2^p}{2^p - 4} \varepsilon(x) = \frac{2c}{2^p - 4} \|x\|^p, \quad x \in X.$$

By Theorem 1, there exists a function $g: X \to Y$ such

$$g(x) = \lim_{n \to \infty} T^n f(x), \quad x \in X,$$

$$g(x) = 3g \left( \frac{x}{2} \right) + g \left( -\frac{x}{2} \right), \quad x \in X$$

and

$$\|f(x) - g(x)\| \leq \frac{2c}{2^p - 4} \|x\|^p, \quad x \in X.$$

Next we prove that $g$ satisfies the Drygas equation. Observe first that if a function $h: X \to Y$ satisfies the inequality

$$\|h(x + y) + h(x - y) - 2h(x) - h(y) - h(-y)\| \leq M(\|x\|^p + \|y\|^p) \quad (6)$$

for all $x, y \in X$ such that $x + y, x - y \in X$ and some $M > 0$, then

$$\|T h(x + y) + T h(x - y) - 2 T h(x) - T h(y) - T h(-y)\| \leq \frac{4M}{2^p}(\|x\|^p + \|y\|^p),$$
for \(x, y \in X\) satisfying \(x + y, x - y \in X\). Indeed, fix \(h: X \to Y\) and assume (6). Then

\[
Th(x + y) + Th(x - y) - 2Th(x) - Th(y) - Th(-y)
= 3\left(h\left(\frac{x + y}{2}\right) + h\left(\frac{x - y}{2}\right) - 2h\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right) - h\left(\frac{-y}{2}\right)\right)
+ \left(h\left(\frac{-x + y}{2}\right) + h\left(\frac{-x - y}{2}\right) - 2h\left(\frac{-x}{2}\right) - h\left(\frac{-y}{2}\right) - h\left(\frac{y}{2}\right)\right)
\]

for all \(x, y \in X, x + y, x - y \in X\). Hence

\[
\|Th(x + y) + Th(x - y) - 2Th(x) - Th(y) - Th(-y)\|
\leq 3\left\|h\left(\frac{x + y}{2}\right) + h\left(\frac{x - y}{2}\right) - 2h\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right) - h\left(\frac{-y}{2}\right)\right\|
+ \left\|h\left(\frac{-x + y}{2}\right) + h\left(\frac{-x - y}{2}\right) - 2h\left(\frac{-x}{2}\right) - h\left(\frac{-y}{2}\right) - h\left(\frac{y}{2}\right)\right\|
\leq 3M\left\|\frac{x}{2}\right\|^p + \left\|\frac{y}{2}\right\|^p\right\| + M\left\|\frac{1}{2}\right\|^p + \left\|\frac{-y}{2}\right\|^p\right\|
= \frac{4M}{2^p}(\|x\|^p + \|y\|^p).
\]

Consequently, proceeding by induction we get that if a function \(h: X \to Y\) satisfies the inequality (6), then

\[
\|T_n h(x + y) + T_n h(x - y) - 2T_n h(x) - T_n h(y) - T_n h(-y)\|
\leq M\left(\frac{4}{2^p}\right)^n (\|x\|^p + \|y\|^p)
\]

for all \(n \in \mathbb{N}_0\) and \(x, y \in X, x + y, x - y \in X\). On account of the above observation and (4)

\[
\|T_n f(x + y) + T_n f(x - y) - 2T_n f(x) - T_n f(y) - T_n f(-y)\|
\leq c\left(\frac{4}{2^p}\right)^n (\|x\|^p + \|y\|^p)
\]

for every \(n \in \mathbb{N}_0\) and \(x, y \in X\) such that \(x + y, x - y \in X\). Letting \(n \to \infty\) we get

\[
g(x + y) + g(x - y) = 2g(x) + g(y) + g(-y), \quad x, y \in X, x + y, x - y \in X.
\]

(2) The idea of the proof is the same as before so we only give a sketch. Replacing \(y\) by \(x\) in (4) we obtain

\[
\left\|\frac{1}{3}f(2x) - \frac{1}{3}f(-x) - f(x)\right\| \leq \frac{2c}{3}\|x\|^p, \quad x \in X.
\]
Let functions $T : Y^X \to Y^X$ and $\varepsilon : X \to \mathbb{R}_+$ be define by formulas

$$T \xi(x) = \frac{1}{3} \xi(2x) - \frac{1}{3} \xi(-x), \quad x \in X, \; \xi \in Y^X$$

and

$$\varepsilon(x) = \frac{2c}{3} \|x\|^p, \quad x \in X.$$

The inequality (7) takes now the form

$$\|T f(x) - f(x)\| \leq \varepsilon(x), \quad x \in X.$$

Obviously $T$ satisfies the inequality (H2) with $f_1(x) = 2x, f_2(x) = -x, L_1(x) = L_2(x) = \frac{1}{3}, \; x \in X$. The operator $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ is given by

$$\Lambda \eta(x) = \frac{1}{3} \eta(2x) + \frac{1}{3} \eta(-x), \quad x \in X, \; \eta \in \mathbb{R}_+^X.$$

In particular

$$\Lambda \varepsilon(x) = \frac{1}{3} \varepsilon(2x) + \frac{1}{3} \varepsilon(-x) = \frac{2p + 1}{3} \varepsilon(x), \quad x \in X.$$

Proceeding by induction, we obtain

$$\Lambda^n \varepsilon(x) = \left(\frac{2p + 1}{3}\right)^n \varepsilon(x), \quad x \in X, \; n \in \mathbb{N}_0.$$

Since $p < 1, \frac{2p + 1}{3} < 1$, so the series $\sum_{n=0}^{\infty} \Lambda^n \varepsilon(x)$ is convergent for every $x \in X$ and

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) = \frac{3}{2 - 2p} \varepsilon(x) = \frac{2c}{2 - 2p} \|x\|^p, \quad x \in X.$$

By Theorem 1, there exists a function $g : X \to Y$ such

$$g(x) = \lim_{n \to \infty} T^n f(x), \quad x \in X,$$

$$g(x) = \frac{1}{3} g(2x) - \frac{1}{3} g(-x), \quad x \in X$$

and

$$\|f(x) - g(x)\| \leq \frac{2c}{2 - 2p} \|x\|^p, \quad x \in X.$$

A trivial verification shows that

$$\|T^n f(x + y) + T^n f(x - y) - 2T^n f(x) - T^n f(y) - T^n f(-y)\|$$

$$\leq \left(\frac{2p + 1}{3}\right)^n c(\|x\|^p + \|y\|^p),$$

for every $n \in \mathbb{N}_0$ and $x, y \in X$ satisfying $x + y, x - y \in X$. Hence, letting $n \to \infty$ we obtain

$$g(x + y) + g(x - y) = 2g(x) + g(y) + g(-y), \quad x, y \in X, \; x + y, x - y \in X.$$
(3) In this case let $f_e : X \to Y$ and $f_o : X \to Y$ be the even and the odd part of the function $f$, respectively. That means $f_e(x) = \frac{f(x) + f(-x)}{2}, f_o(x) = \frac{f(x) - f(-x)}{2}$ for $x \in X$ and $f = f_e + f_o$. It is easy to see that $f(0) = f_e(0) = f_o(0) = 0$. It follows that

$$
\|f_e(x + y) + f_e(x - y) - 2f_e(x) - f_e(y) - f_e(-y)\|
= \frac{1}{2}\|f(x + y) + f(-x - y) + f(x - y) + f(-x + y)
- 2f(x) - 2f(-x) - f(y) - f(-y) - f(-y) - f(y)\|
\leq \frac{1}{2}(\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\|
+ \|f(-x - y) + f(-x + y) - 2f(-x) - f(-y) - f(y)\|)
\leq c(\|x\|^p + \|y\|^p)
$$

and analogously

$$
\|f_o(x + y) + f_o(x - y) - 2f_o(x) - f_o(y) - f_o(-y)\|
\leq c(\|x\|^p + \|y\|^p)
$$

for every $x, y \in X$ such that $x + y, x - y \in X$. Hence $f_e, f_o$ satisfy the inequality (4).

Replace $y$ by $x$ in (4). By the evenness of $f_e$,

$$
\|f_e(2x) - 4f_e(x)\| \leq 2c\|x\|^p, \quad x \in X
$$

which gives

$$
\left\|f_e(x) - \frac{f_e(2x)}{4}\right\| \leq \frac{1}{2}c\|x\|^p, \quad x \in X.
$$

Let

$$
T\xi(x) = \frac{\xi(2x)}{4}, \quad \xi \in Y^X, \quad x \in X,
\Lambda\delta(x) = \frac{\delta(2x)}{4}, \quad \delta \in \mathbb{R}^+_X, \quad x \in X
$$

and $\varepsilon(x) = \frac{1}{2}c\|x\|^p, x \in X$. By Theorem 1, there exists a function $g_e : X \to Y$ such that

$$
g_e(x) = \lim_{n \to \infty} T^nf_e(x), \quad x \in X,
\quad g_e(x) = \frac{g_e(2x)}{4}, \quad x \in X
$$

and

$$
\|f_e(x) - g_e(x)\| \leq \frac{2c}{4 - 2^p}\|x\|^p, \quad x \in X.
$$
Moreover,
\[
\|T^n f_e(x + y) + T^n f_e(x - y) - 2T^n f_e(x) - T^n f_e(y) - T^n f_e(-y)\| \\
\leq \left(\frac{2^p}{4}\right)^n c(\|x\|^p + \|y\|^p),
\]
for every \( n \in \mathbb{N}_0 \) and \( x, y \in X \) satisfying \( x + y, x - y \in X \). Hence \( g_e \) satisfies the Drygas equation.

In the same way, replacing \( y \) by \( x \) in (4) and using the oddness of \( f_o \), we obtain
\[
\|f_o(2x) - 2f_o(x)\| \leq 2c\|x\|^p, \quad x \in X
\]
which with \( x \) replacing by \( \frac{x}{2} \) yields
\[
\left\| f_o(x) - 2f_o\left(\frac{x}{2}\right) \right\| \leq \frac{2}{2^p} c\|x\|^p, \quad x \in X.
\]
Define now
\[
T \xi(x) = 2\xi\left(\frac{x}{2}\right), \quad \xi \in Y^X, \quad x \in X,
\]
\[
\Lambda \delta(x) = 2\delta\left(\frac{x}{2}\right), \quad \delta \in \mathbb{R}_+^X, \quad x \in X
\]
and \( \varepsilon(x) = \frac{2}{2^p} c\|x\|^p, \quad x \in X \). By Theorem 1, there exists a function \( g_o : X \to Y \) such that
\[
g_o(x) = \lim_{n \to \infty} T^n f_o(x), \quad x \in X,
\]
\[
g_o(x) = 2g_o\left(\frac{x}{2}\right), \quad x \in X
\]
and
\[
\|f_o(x) - g_o(x)\| \leq \frac{2c}{2^p - 2}\|x\|^p, \quad x \in X.
\]
By
\[
\|T^n f_o(x + y) + T^n f_o(x - y) - 2T^n f_o(x) - T^n f_o(y) - T^n f_o(-y)\| \\
\leq \left(\frac{2^p}{2}\right)^n c(\|x\|^p + \|y\|^p), \quad n \in \mathbb{N}_0, \quad x, y \in X, \quad x + y, x - y \in X,
\]
g_0 satisfies the Drygas equation. Thus \( g = g_e + g_o \) also satisfies the Drygas equation and
\[
\|f(x) - g(x)\| \leq \frac{2c}{4 - 2^p}\|x\|^p + \frac{2c}{2^p - 2}\|x\|^p, \quad x \in X.
\]
It remains to prove the uniqueness of the function $g$. We show the case $p > 2$ in details. Let us assume that functions $g_1, g_2 : X \to Y$ fulfill the Drygas equation on $X$ and
\[
\|f(x) - g_i(x)\| \leq M_i \|x\|^p, \quad x \in X
\]
for some $M_i \geq 0$, $i = 1, 2$. Hence $\|g_1(x) - g_2(x)\| \leq (M_1 + M_2)\|x\|^p, x \in X$. Since $g_1, g_2$ satisfy the Drygas equation,
\[
g_i(x) = 3g_i\left(\frac{x}{2}\right) + g_i\left(-\frac{x}{2}\right), \quad x \in X, \ i = 1, 2.
\]
Thus
\[
\|g_1(x) - g_2(x)\| \leq 3\|g_i\left(\frac{x}{2}\right) - g_2\left(\frac{x}{2}\right)\| + \|g_1\left(-\frac{x}{2}\right) - g_2\left(-\frac{x}{2}\right)\| \\
\leq 4(2p) (M_1 + M_2)\|x\|^p
\]
for all $x \in X$. It is easy to check that,
\[
\|g_1(x) - g_2(x)\| \leq \left(\frac{4}{2p}\right)^n (M_1 + M_2)\|x\|^p, \quad x \in X, \ n \in \mathbb{N}_0.
\]
Letting $n$ to $\infty$ we obtain
\[
g_1(x) = g_2(x), \quad x \in X.
\]
The proofs of the other cases runs as before. □

The following examples show that the assumptions putting on the set $X$ can not be omitted.

**Example 3.** Let $p > 2, X = (-\infty, -1] \cup \{0\} \cup [1, \infty)$ and $f(x) = |x|, x \in X$. Then
\[
|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq 2(|x|^p + |y|^p)
\]
for all $x, y \in X$ such that $x + y, x - y \in X$. Consider functions $g_a : X \to \mathbb{R}$ given by $g_a(x) = ax^2, x \in X$, where $a$ is any real constant. The functions $g_a$ obviously satisfy the Drygas equation and
\[
|f(x) - g_a(x)| \leq |x|^p, \quad x \in X
\]
for all $a \in [0, 1]$.

**Example 4.** Let $p \in (0, 1), X = [-1, 1]$ and $f(x) = x^3, x \in X$. Then for all $x, y \in X$ such that $x + y, x - y \in X$
\[
|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq 6(|x|^p + |y|^p).
\]
Every function $g_a : X \to \mathbb{R}$ given by $g_a(x) = ax, x \in X$ with $a \in \mathbb{R}$ satisfies the Drygas equation and
\[
|f(x) - g_a(x)| \leq |x|^p, \quad x \in X
\]
for all $a \in [0, 1]$. 
Example 5. Let $1 < p < 2$, $X = (-\infty, -1] \cup \{0\} \cup [1, \infty)$ and $f(x) = \frac{1}{2}(|x| + x)$, $x \in X$. Then

$$|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq |x|^p + |y|^p$$

for all $x, y \in X$ such that $x + y, x - y \in X$. Functions $g_a : X \to \mathbb{R}$ given by $g_a(x) = ax, x \in X$ are solutions of the Drygas equation and

$$|f(x) - g_a(x)| \leq |x|^p, \quad x \in X$$

for all $a \in [0, 1]$.

By the same method, we can also obtain the stability result for $p = 0$, but in order to obtain the best known bound we have to make more technical substitutions. The idea is adapted from [7].

**Theorem 6.** Let $X$ be such a subset of an Abelian group that $0, -x, 2x, 3x \in X$ for all $x \in X, Y$ a Banach space and $c \geq 0$. If a function $f : X \to Y$ satisfies

$$\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\| \leq c$$

(8)

for all $x, y \in X$ with $x + y, x - y \in X$, then there exists a function $g : X \to Y$ satisfying the Drygas equation on $X$ such that

$$\|f(x) - g(x)\| \leq c, \quad x \in X.$$

Moreover, $g$ is the unique function satisfying equation (3), such that $\|f(x) - g(x)\| \leq M, x \in X$ for some $M \geq 0$.

**Proof.** Replace $(x, y)$ in (8) by $(2x, x)$, next by $(x, 2x), (-x, -2x)$ and $(x, x)$ (cf. the proof of Theorem 3.2 in [7]). Then

$$\|f(3x) - 2f(2x) - f(-x)\| \leq c,$$

$$\|f(3x) + f(-x) - 2f(2x) - f(-2x)\| \leq c,$$

$$\| - f(-3x) - f(x) + 2f(-x) + f(2x) + f(-2x)\| \leq c,$$

$$\|f(2x) + f(0) - 3f(x) - f(-x)\| \leq c,$$

for $x \in X$. Which with $\|f(0)\| \leq \frac{c}{2}$ give

$$\|2f(3x) - f(-3x) - 9f(x)\| \leq 6c, \quad x \in X,$$

whence

$$\left\|f(x) - \frac{2}{9}f(3x) + \frac{1}{9}f(-3x)\right\| \leq \frac{2}{3}c, \quad x \in X. \quad (9)$$

Let functions $T : Y^X \to Y^X$ and $\varepsilon : X \to \mathbb{R}_+$ be defined as follows

$$T\xi(x) = \frac{2}{9}\xi(3x) - \frac{1}{9}\xi(-3x), \quad x \in X, \ \xi \in Y^X$$

and

$$\varepsilon(x) = \frac{2}{3}c, \quad x \in X.$$
The inequality (9) now takes the form
\[ \|Tf(x) - f(x)\| \leq \varepsilon(x), \quad x \in X. \]
As before, using Theorem 1, there exists a function \( g: X \to Y \) such that
\[
g(x) = \lim_{n \to \infty} T^n(x), \quad x \in X,
\]
and
\[
g(x) = \frac{2}{9}g(3x) - \frac{1}{9}g(-3x), \quad x \in X
\]
and
\[ \|f(x) - g(x)\| \leq c, \quad x \in X. \]
In the same manner as in the proofs of Theorem 2 we show that \( g \) satisfies the Drygas equation and \( g \) is unique. \( \square \)

3. Nonstability Results

In this section we show that for \( p \in \{1, 2\} \) the Drygas equation is not stable. The idea of the construction of the examples comes from the paper [2].

Example 7. Let \( \phi: \mathbb{R} \to \mathbb{R} \) be defined as
\[
\phi(x) = \begin{cases} 
-\alpha, & x \leq -1, \\
\alpha x, & -1 < x < 1, \\
\alpha, & 1 \leq x,
\end{cases}
\]
where \( \alpha > 0 \). The function \( f: \mathbb{R} \to \mathbb{R} \) given by
\[ f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \quad x \in \mathbb{R} \]
satisfies
\[
|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq 8\alpha(|x| + |y|), \quad (10)
\]
but there exist no pair \((g, k)\) of a function \( g: \mathbb{R} \to \mathbb{R} \) satisfying the Drygas equation and a constant \( k \geq 0 \) such that
\[ |f(x) - g(x)| \leq k|x|, \quad x \in \mathbb{R}. \]

Proof. We observe that \( f \) is odd and bounded by \( 2\alpha \). Now, we show that (10) holds. For \( x = y = 0 \) and \( x, y \in \mathbb{R} \) such that \(|x| + |y| \geq 1 \) it is obvious. Consider the case \( 0 < |x| + |y| < 1 \). There exists \( N \in \mathbb{N} \) such that
\[ \frac{1}{2^N} \leq |x| + |y| < \frac{1}{2^{N-1}}. \]
Then \(|2^{N-1}x| < 1, |2^{N-1}y| < 1, |2^{N-1}(x + y)| < 1, |2^{N-1}(x - y)| < 1 \). Hence
\[ 2^n x, 2^n y, 2^n (x + y), 2^n (x - y) \in (-1, 1) \quad \text{for } n = 0, 1, \ldots, N - 1. \]
By the definition of $f$,
\[
|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)| = \left| \sum_{n=0}^{\infty} \frac{\phi(2^n(x+y))}{2^n} + \sum_{n=0}^{\infty} \frac{\phi(2^n(x-y))}{2^n} - 2\sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \right|
\leq 4\alpha \sum_{n=N}^{\infty} \frac{1}{2^n} = 8\alpha \frac{1}{2^N}
\leq 8\alpha(|x| + |y|).
\]

Assume that there exist a function $g: \mathbb{R} \to \mathbb{R}$ satisfying the Drygas equation and a constant $k \geq 0$ such that
\[
|f(x) - g(x)| \leq k|x|, \quad x \in \mathbb{R}.
\]
Since $g$ fulfills the Drygas equation, there exist an additive function $h: \mathbb{R} \to \mathbb{R}$ and a quadratic function $q: \mathbb{R} \to \mathbb{R}$ such that $g(x) = h(x) + q(x), x \in \mathbb{R}$. Whence, as $f$ is bounded by $2\alpha$, we have
\[
|h(x) + q(x)| \leq k|x| + 2\alpha, \quad x \in \mathbb{R}.
\]
In particular,
\[
|h(nx) + q(nx)| \leq k|nx| + 2\alpha, \quad x \in \mathbb{R}, \ n \in \mathbb{N}.
\]

The function $q$ satisfies the quadratic functional equation, which implies
\[
|h(x) + nq(x)| \leq k|x| + \frac{1}{n}2\alpha, \quad x \in \mathbb{R}, \ n \in \mathbb{N},
\]
Hence $q(x) = 0, x \in \mathbb{R}$ and
\[
|h(x)| \leq k|x| + 2\alpha, \quad x \in \mathbb{R}.
\]
It follows that, the additive function $h$ is bounded in the neighborhood of 0, and consequently $h(x) = ax, x \in \mathbb{R}$ for some constant $a \in \mathbb{R}$. Thus
\[
|f(x) - ax| \leq k|x|, \quad x \in \mathbb{R}
\]
which gives
\[
\left| \frac{f(x)}{x} \right| \leq k + |a|, \quad x \in \mathbb{R} \setminus \{0\}.
\] (11)

Let $N$ be such that $N\alpha > k + |a|$ and take an $x \in (0, \frac{1}{2N-1})$. Then $2^n x \in (0,1)$ for $n = 0, 1, \ldots, N-1$ and
\[
f(x) = \sum_{n=0}^{N-1} \frac{\phi(2^n x)}{2^n} + \sum_{n=N}^{\infty} \frac{\phi(2^n x)}{2^n} > Nx\alpha
\]
so
\[
\frac{f(x)}{x} > N\alpha > k + |a|,
\]
which is contrary to (11). \qed
For \( p = 2 \) we have the same example like in the case of the quadratic equation (see [2]).

**Example 8.** Let \( \phi: \mathbb{R} \to \mathbb{R} \) be defined as

\[
\phi(x) = \begin{cases} 
\alpha, & x \in (-\infty, -1] \cup [1, +\infty), \\
\alpha x^2, & x \in (-1, 1),
\end{cases}
\]

where \( \alpha > 0 \). Put

\[
f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{4^n}, \quad x \in \mathbb{R}.
\]

Then \( f \) satisfies

\[
|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq 32\alpha(|x|^2 + |y|^2)
\]

and there do not exist a function \( g: \mathbb{R} \to \mathbb{R} \) satisfying the Drygas equation and a constant \( k \geq 0 \) such that

\[
|f(x) - g(x)| \leq k|x|^2, \quad x \in \mathbb{R}.
\]

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Magdalena Piszczek and Joanna Szczawińska
Institute of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków, Poland
e-mail: magdap@up.krakow.pl;
        jszczaw@up.krakow.pl

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