Interaction dependence of composite fermion effective masses

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We estimate the composite fermion effective mass for a general two particle potential \( r^{-\alpha} \) using exact diagonalization for polarized electrons in the lowest Landau level on a sphere. Our data for the ground state energy at filling fraction \( \nu = 1/2 \) as well as estimates of the excitation gap at \( \nu = 1/3, 2/5 \) and 3/7 show that \( m_{\text{eff}} \sim \alpha^{-1} \).

73.40.Hm, 71.10.Pm

The dynamics of interacting planar electrons in the lowest Landau level of a strong magnetic field show many interesting features at filling fractions \( \nu < 1 \) experimentally observed as the fractional quantum Hall effect (FQHE). It emerged that the picture of composite fermions (CF) \([1,2]\) moving in a reduced magnetic field is central to the understanding of the FQHE. The field theoretic formulation of this idea has received much attention (see e.g. \([3,4]\)), in particular after Halperin, Lee and Read \([5]\) described the polarized \( \nu = 1/2 \) state as a fermi liquid state of composite fermions. Since the CF picture explains gaps due to the electron–electron interaction as Landau level gaps of composite fermions, their effective mass has to be understood as a result of this interaction.

Numerical diagonalization of the interaction Hamiltonian for electrons on a sphere has a long history \([6]\) as a testing ground for the understanding of the FQHE. Rezayi and Read \([7]\) have shown that even for the small number of electrons \( N \approx 10 \) accessible to exact diagonalization the pattern of the angular momenta of \( \nu = 1/2 \) ground states follows Hund’s rule applied to composite fermions in a zero magnetic field. Later on, Morf and d’Ambrumenil \([8]\) demonstrated that the ground state follows Hund’s rule applied to composite fermions (CF) \([1,2]\) moving in a reduced magnetic field.

Interaction dependence of composite fermion effective mass is also the relevant parameter \([8]\) for the excitation gap of \( \nu = \pi/(2p + 1) \) FQHE states \([8]\). The basic features of the FQHE as seen in finite size studies are to a high degree independent of the exact form of the two particle interaction potential \( V(r) \). Most studies therefore used a simple \( 1/r \) potential. With the advent of a qualitative theory of the FQHE it now seems appropriate to study the dependence of the numerically obtained effective masses on the chosen potential.

The single particle wave functions on a sphere of radius \( R \) pierced by \( \Phi = 2\pi \) flux quanta are monopole harmonics of angular momenta \( j = S, S + 1, \ldots \) with energy

\[
E = \frac{\hbar^2}{2mR^2} (j(j+1) - S^2].
\]

We will use the ion disc radius \( a = (\pi \text{ density})^{-1/2} \) as basic length unit: \( R = a\sqrt{S}/N \). It is related to the magnetic length by \( a = 2\alpha\sqrt{S}/N \).

The quasipotential coefficients \([6]\) for an interaction potential \( V(r) = \frac{\alpha^2}{2} \left( \frac{r}{a} \right)^\alpha \) with chord distance \( r \) are

\[
V_J = (-1)^{2S+J} \frac{(2S+1)\Gamma(1-\frac{\alpha}{2})}{N^{\alpha/2} \Gamma(\frac{\alpha}{2})} \times \sum_{k} \frac{\Gamma(\frac{\alpha}{2}+k)}{\Gamma(2+k-\frac{\alpha}{2})} \left\{ \begin{array}{ccc} S & S & J \end{array} \right\} \left\{ \begin{array}{ccc} S & S & k \end{array} \right\} \left\{ \begin{array}{ccc} S & S & 0 \end{array} \right\}^2.
\]

Fig. \([6]\) gives an impression of the \( \alpha \) dependence of the \( V_J \).

We have calculated the ground state energy and angular momenta for \( \nu = 1/2 \) systems with up to \( N = 13 \) electrons. The composite fermions feel no magnetic field at this filling fraction and form a “CF-atom” with shells \( j = 0, 1, 2, \ldots \) of degeneracy \( 2j+1 \). The ground state angular momentum follows Hund’s rule applied to the CF-atom (e.g. \( J = 0 \) for \( N = n^2 \) indicating \( n \) closed shells) for \( 0.2 \leq \alpha \leq 1.99 \). At \( \alpha = 0.1 \) we find small derivations in two cases \( (L = 1 \text{ instead of } 3 \text{ for } N = 6 \text{ and } L = 4 \text{ instead of } 6 \text{ for } N = 12) \) but these ground states are almost degenerate with states with the “right” angular momentum.

Morf and d’Ambrumenil \([8]\) found that the ground state energy per particle of the \( \nu = 1/2 \) system with Coulomb interaction can be interpreted, up to a correction linear in \( 1/N \), as kinetic energy \( T(N,m^*) \) of the CF-atom with effective CF mass \( m^* \). This energy is calculated by summing up the contributions of the individual particles given by eq. \([6]\) with \( S = 0 \) and \( m = m^* \). It can be written as sum of a constant linear in \( 1/N \) (in units of
We find the same pattern of deviation of the energy of partially filled shells from linear behaviour is proportional to the effective mass parameter $C$ introduced in (2):

$$\frac{\hbar^2}{m^* a} = C \frac{e^2}{2 \varepsilon}. \quad (3)$$

We find the same pattern of $N$-dependence of the ground state energies in the range $0.1 \leq \alpha \leq 1.99$. Fig. 3(a) shows that for a long-range potential $\alpha = 0.1$ the ground state energy comes very close to the prediction of free composite fermions.

On the other side, composite fermions are not free. The interaction energy of particles in shells describes a similar pattern with relative minima for closed shells. In order to test the influence of CF interactions on this method of obtaining $m^*$ we assume that the composite fermions interact via the same potential $r^{-\alpha}$ as the electrons. The energy of closed shells as well as the inter-shell energy can be calculated analytically using the shell model formalism of nuclear theory (1). The ground state energy of the outer partially filled shell is calculated numerically by exact diagonalization. The sum of these contributions is $V(N, \alpha)$, the ground state energy of $N$ interacting particles of infinite mass on a sphere without magnetic field. We then checked that the electron ground state energies can be fitted by $a_0 + a_1/N + V(N, \alpha) + T(N, \hat{m}^*)$. This provides another value for the effective mass $\hat{m}^*$, calculated for interacting composite fermions. For $\alpha < 1$ (see Fig. 3(a)) this appears to be less convincing than the free CF ansatz suggesting that in this case the CF interaction is even weaker. For $\alpha > 1$, however, Fig. 3(b) shows that the data can be interpreted by assuming a larger effective mass of the interacting composite fermions.

The resulting values for $C$ and $\hat{C}$, as shown in Fig. 3, are (for $\alpha$ not too big) linear in $\alpha$: $C = 0.164\alpha$, $\hat{C} = 0.195\alpha$.

The Coulomb system at filling fractions $\nu = p/(2p+1)$ has been studied extensively in the past, cf. e.g. (2). Apart from the ground state at $L = 0$ one finds a band of low-lying excitations with $L = 1, 2, \ldots, \Phi^* + 1$, the exciton or magnetoroton. $(\Phi^* = \text{the reduced magnetic field of the CF picture.})$ In the CF picture the ground state corresponds to $p$ filled Landau Levels and the excitation gap in the limit $L \to \infty$ measures the distance to the $(p + 1)$th level. Fig. 4 shows the $\nu = 1/3$ exciton mode, getting flatter for long-range potentials but with a visible magnetoroton minimum at $L_{\nu}/R \approx 1.4$.

At $\nu = 3/7$ the case $N = 12$ is the only one accessible to numerical diagonalization. ($N = 9, \Phi = 16$) for example is also a $\nu = 1/2$ state (an effect called “aliasing” in (2)) and since the reduced magnetic field is zero, this seems to be the preferable interpretation. This makes a systematic study of finite size effects impossible. Therefore we take the gap $\Delta E$ at $L = \Phi^* + 1$ for the highest available electron number ($N = 10$ for $\nu = 1/3, 2/5$; $N = 12$ for $\nu = 3/7$) as an estimate for the CF gap between the $p$ and $(p + 1)$th Landau level. The gap energies for different $\nu$ fit to one curve if divided by $\Delta \nu = 1/2 - \nu$ (Fig. 5). The CF picture expects a large $p$ behaviour

$$\Delta E = \frac{C \varepsilon^2}{p \varepsilon a}. \quad (4)$$

With $\Delta \nu \to 1/2p$ as $p \to \infty$ our fit gives $C = 0.35\alpha$. We therefore find an enhancement of $m^*$ for all potentials by roughly a factor of 2 as $\nu \to 1/2$. d’Ambrumenil and Morf report (3) that for Coulomb interaction a correction of the gap energy by the potential energy of two charges $q = 1/(2p + 1)$ at distance $2R$ allows a consistent extrapolation in $1/N$. Unfortunately, this does no longer work for systems with $\alpha \neq 1$. Nevertheless, our data are consistent with the value $C \approx 0.31$ for $\alpha = 1$ extracted from their data in (2). However, the dependence of $E$ on $\Delta \nu$ appears to be linear for all values of $\alpha$ contrary to the analytical result $\Delta E \sim |\Delta \nu|^{2/3}$ of (2).

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FIG. 1. The first three quasipotential coefficients $V_i$ for $N = 12, \nu = 1/2$ as functions of $\alpha$.

FIG. 2. The ground state energy per particle plotted as function of $1/N$. The part of the energy linear in $1/N$ is subtracted. For the data shown by stars $V(N, \alpha)$ was first subtracted. (a) Data for $\alpha = 0.1$; (b) data for $\alpha = 1.5$.

FIG. 3. The mass parameters $C$ and $\bar{C}$ as functions of $\alpha$. The fits shown are $C = 0.195\alpha$ and $\bar{C} = 0.164\alpha$.

FIG. 4. Exciton mode at $\nu = 1/3$ with data from $N = 8, 9, 10$ electron systems as function of the wave vector $Ll_0'/R$. The energy is in units of $e^2/\varepsilon l_0'$ with $l_0'$ the “corrected magnetic length” of Ref. [9], $l_0' = a\sqrt{\nu/2}$. The lines are a guide to the eye only.

FIG. 5. Exciton gap energies in units of $e^2/(\varepsilon a\Delta\nu)$ for $\nu = 1/3, N = 10; \nu = 2/5, N = 10; \nu = 3/7, N = 12$. The shown fit is $E = 1.40\Delta\nu\alpha$. 
free CF
interacting CF
\[ \nu = 1/3 \quad + \]
\[ \nu = 2/5 \quad \times \]
\[ \nu = 3/7 \quad \ast \]