ON THE FAILURE OF THE POINCARÉ LEMMA FOR $\bar{\partial}_M$ II

C. Denson Hill and Mauro Nacinovich

§1 INTRODUCTION

The purpose of this paper is to repair some inaccuracies in the formulation of the main result of [HN1]. As were written, the main theorems 4.1 and 4.2 of [HN1] are in fact in contradiction to earlier results of one of the authors [N2]. In the process of writing an erratum, we actually discovered some new phenomena. As we found these quite interesting, it has lead us to incorporate the needed corrections into this self contained article.

An unfortunate misprint, in which $R^{-1}$ got replaced by $R$, led the authors to misinterpret what their proof in [HN1] actually demonstrated. Upon closer scrutiny, we realized that there are two distinct ways to proceed.

One is that we may still obtain the original conclusions of our main theorems, provided we slightly strengthen our original hypothesis (cf. Theorems 5.1 and 5.2). This however entails a much more complicated argument, involving the CR structure of the characteristic bundle, which is of considerable independent interest (cf. Theorems 4.3 and 4.6).

The other is that if we stick to our original hypothesis, then the conclusions we obtain are slightly weaker than originally claimed, but in our opinion still interesting. In fact a new invariant comes into play, which measures the rate of shrinking, even in the situation where the local Poincaré lemma is valid.

Recall that here we make an important distinction between the vanishing of the cohomology and the validity of the Poincaré lemma: Consider the inhomogeneous problem $\bar{\partial}_M u = f$, to be solved for $u$, with given data $f$ satisfying $\bar{\partial}_M f = 0$ in some domain $U$ containing a point $x_0$. The vanishing of the cohomology in $U$ refers to the situation in which, no matter how $f$ is prescribed in $U$, there is always a solution $u$ in $U$ (i.e., no shrinking). The validity of the Poincaré lemma at $x_0$ requires only that a solution $u$ exist in a smaller domain $V_f$ with $x_0 \in V_f \subset U$ (i.e., there is some shrinking which might, in principle, depend on $f$). Our new invariant measures the relative rate of shrinking of $V_f$ with respect to radius($U$), as $U$ shrinks to the point $x_0$.

Under our original hypothesis, we are able to show that the cohomology of small convex neighborhoods of $x_0$ is always infinite dimensional, with respect to any choice of the Riemannian metric (cf. Theorems 7.2 and 7.3). This means that special shapes are needed, if one is to have the vanishing of the cohomology for small sets.

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In §6 we have listed a number of natural examples satisfying the slightly strengthened hypothesis. They illustrate how common it is for the Poincaré lemma to fail.

At the end of §7 we give a very simple example satisfying our original hypothesis. It illustrates how, even if the Poincaré lemma were to be valid, a shrinking must occur which goes like $\text{radius}(V_j) \simeq C r^{3/2}$, where $\text{radius}(U) \simeq r$, as $r \to 0$.

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§2 A PRIORI ESTIMATES

In this section we rehearse the facts of functional analysis that we shall use later for the discussion of the local cohomology of the $\partial_M$-complex. For the proofs we refer to [AFN], [N1] and [HN1]. Let $M$ be a paracompact smooth differentiable manifold, of real dimension $N$. We consider a sequence of complex vector bundles $\{E^q \to M\}_{q=0,1,...}$ (with $E^q$ of rank $r_q$), and a complex of linear partial differential operators:

$$
(2.1) \quad C^\infty(M, E^0) \xrightarrow{A_0} C^\infty(M, E^1) \xrightarrow{A_1} C^\infty(M, E^2) \xrightarrow{} \cdots
$$

This means that, for all $i = 0, 1, \ldots$:

(i) $A_i : C^\infty(M, E^q) \to C^\infty(M, E^{q+1})$ are linear and $\text{supp} (A_q(u)) \subset \text{supp}(u)$ for all $u \in C^\infty(M, E^q)$ (Peetre’s theorem),

(ii) $A_{q+1} \circ A_q = 0$ for all $q = 0, 1, \ldots$.

We denote by $E^q$ the sheaf of germs of smooth sections of $E^q$ and, for every open $U \subset M$, set $\mathcal{E}^q(U) = C^\infty(U, E^q)$.

For every open $U \subset M$ and $x_0 \in M$, we obtain complexes

$$(U, \mathcal{E}^*, A_*) \quad \mathcal{E}^0(U) \xrightarrow{A_0} \mathcal{E}^1(U) \xrightarrow{A_1} \mathcal{E}^2(U) \xrightarrow{} \cdots$$

and

$$((x_0), \mathcal{E}^*, A_*) \quad \mathcal{E}^0_{x_0} \xrightarrow{A_0} \mathcal{E}^1_{x_0} \xrightarrow{A_1} \mathcal{E}^2_{x_0} \xrightarrow{} \cdots$$

We denote their cohomology groups by:

$$
(2.2) \quad H^q(U, \mathcal{E}^*, A_*) = \frac{\ker(A_q : \mathcal{E}^q(U) \to \mathcal{E}^{q+1}(U))}{\text{Image} (A_{q-1} : \mathcal{E}^{q-1}(U) \to \mathcal{E}^q(U))}
$$

and

$$
(2.3) \quad H^q((x_0), \mathcal{E}^*, A_*) = \frac{\ker(A_q : \mathcal{E}^q_{x_0} \to \mathcal{E}^{q+1}_{x_0})}{\text{Image} (A_{q-1} : \mathcal{E}^{q-1}_{x_0} \to \mathcal{E}^q_{x_0})} = \lim_{U \ni x_0} H^q(U, \mathcal{E}^*, A_*) ,
$$

respectively.

When $H^q((x_0), \mathcal{E}^*, A_*) = \{0\}$, we say that (2.1) admits the Poincaré lemma in degree $q$. 
By introducing a smooth Riemannian metric in $M$, and a smooth partition of unity $\{\chi_\ell, U_\ell\}$ subordinated to a coordinate trivialization atlas $(U_\ell, x_\ell)$ for $E^q$, we define for any compact subset $K$ of $M$ the seminorm $\|\cdot\|_{q,K,m}$ by:

$$
(2.4) \quad \|f\|_{q,K,m} = \sum_{\ell} \sup_{x \in K} \sum_{|\alpha| \leq m} \left| \frac{\partial^{|\alpha|}(\chi_\ell f_\ell)(x)}{\partial [x_\ell^1]^{\alpha_1} \cdots \partial [x_\ell^N]^{\alpha_N}} \right|_{r_q}
$$

where $f_\ell(x) \in \mathbb{C}^{r_q}$ is the trivialization of $f$ in $U_\ell$ and $|\cdot|_{r_q}$ is the standard Euclidean norm in $\mathbb{C}^{r_q}$.

Next we introduce smooth Hermitian scalar products $(\cdot|\cdot)_q$ on the fibers of the $E^q$’s. This allows us to define the $L^2$-scalar product of $f, g \in \mathcal{E}^q(U)$ by:

$$
(2.5) \quad \int_U (f|g)_q d\lambda,
$$

where $d\lambda$ is the measure associated to the Riemannian metric of $M$. It is well defined when $\text{supp}(f) \cap \text{supp}(g)$ is compact in $U$. Using these scalar products we obtain the (formal) adjoint complex

$$
(2.6) \quad \mathcal{C}^\infty(M, E^0) \overset{A_0^*}{\underset{A_1}{\leftarrow}} \mathcal{C}^\infty(M, E^1) \overset{A_1}{\leftarrow} \mathcal{C}^\infty(M, E^2) \leftarrow \cdots
$$

by defining the partial differential operator $A_q^*: \mathcal{C}^\infty(M, E^{q+1}) \to \mathcal{C}^\infty(M, E^q)$ by:

$$
(2.7) \quad \int_M (A_q^* f|g)_q d\lambda = \int_M (f|A_q g)_{q+1} d\lambda
$$

where $\mathcal{C}^\text{comp}_c(M, E^{q+1})$ is the subspace of $g \in \mathcal{C}^\infty(M, E^{q+1})$ with $\text{supp}(g)$ compact in $M$.

In [AFN] the following was proved:

**Theorem 2.1.** Assume that the complex $(2.1)$ admits the Poincaré lemma in degree $q$ at the point $x_0 \in M$, i.e. we assume that the sequence:

$$
(2.8) \quad \mathcal{E}^q_{x_0} \overset{A_{q-1}}{\underset{A_q}{\rightarrow}} \mathcal{E}^q_{x_0} \overset{A_q}{\rightarrow} \mathcal{E}^q_{x_0}
$$

is exact. Then, for every open neighborhood $\omega$ of $x_0$ in $M$ we can find an open neighborhood $\omega_1$ of $x_0$ in $\omega$ such that

$$
(2.9) \quad \forall f \in \mathcal{E}^q(\omega) \text{ with } A_q f = 0 \text{ in } \omega \quad \exists u \in \mathcal{E}^{q-1}(\omega_1) \text{ with } A_{q-1} u = f \text{ in } \omega_1.
$$

As a consequence of the open mapping theorem for Fréchet spaces we also have:

**Theorem 2.2.** Let $\omega_1 \subset \omega \subset M$ be open subsets such that the restriction map $\mathcal{E}^*(\omega) \to \mathcal{E}^*(\omega_1)$ induces the zero map in cohomology:

$$
(2.10) \quad \{H^q(\omega, \mathcal{E}^*, A_*) \to H^q(\omega_1, \mathcal{E}^*, A_*)\} = 0,
$$

i.e. such that $(2.9)$ holds true. Then, for every compact $K_1 \subset \omega_1$ and every integer $m_1 \geq 0$ there are a compact $K \subset \omega$, an integer $m \geq 0$ and a constant $c > 0$ such that: the function $u$ in $(2.9)$ can be chosen to satisfy:

$$
(2.11) \quad \|u\|_{q,K,m} \leq c\|f\|_{q,K,m}.
$$
From the preceding Theorem (see again [AFN]) one obtains:

**Theorem 2.3.** Assume that for (2.1) we have (2.9). Then for every compact subset \(K_1\) of \(\omega_1\) we can find a compact subset \(K\) of \(\omega\), an integer \(m \geq 0\), and a constant \(c > 0\) such that for every \(f \in \mathcal{E}^q(\omega)\) with \(A_q f = 0\) in \(\omega\) and for every \(v \in \mathcal{E}^q(\omega_1)\) with \(\text{supp}(v) \subset K_1\) we have:

\[
(2.12) \quad \left| \int_{\omega_1} (f|v)_q d\lambda \right| \leq c \|A^*_q v\|_{q-1,K_1,0} \|f\|_{q,K,m}.
\]

Using the Hermitian inner product on the fibers of \(E^q\) and the duality pairing associated to the \(L^2\)-product, we can define for every open \(U \subset M\) the space \(\mathcal{D}'^q(U)\) of \(E^q\)-valued distributions in \(U\). We note that to both (2.1) and (2.6) we can associate complexes of partial differential operators on distributions, which are compatible with the natural inclusion map \(\mathcal{E}^q(U) \hookrightarrow \mathcal{D}'^q(U)\) for every \(q = 0,1, \ldots\) and every open \(U \subset M\).

We denote by

\[
(2.13) \quad H^q(U, \mathcal{D}'^*, A_*) = \frac{\ker \left( A_q : \mathcal{D}'^q(U) \rightarrow \mathcal{D}'^{q+1}(U) \right)}{\text{Image} \left( A_{q-1} : \mathcal{D}'^{q-1}(U) \rightarrow \mathcal{D}'^q(U) \right)}
\]

and

\[
(2.14) \quad H^q((x_0), \mathcal{D}'^*, A_*) = \frac{\ker \left( A_q : \mathcal{D}'^q_{x_0} \rightarrow \mathcal{D}'^{q+1}_{x_0} \right)}{\text{Image} \left( A_{q-1} : \mathcal{D}'^{q-1}_{x_0} \rightarrow \mathcal{D}'^q_{x_0} \right)} = \lim_{U \ni x_0} H^q(U, \mathcal{D}'^*, A_*),
\]

the corresponding cohomology groups. The natural inclusion maps \(\mathcal{E}^q(U) \hookrightarrow \mathcal{D}'^q(U)\), induce natural maps in cohomology:

\[
(2.15) \quad H^q(U, \mathcal{E}^*, A_*) \rightarrow H^q(U, \mathcal{D}'^*, A_*)
\]

and

\[
(2.16) \quad H^q((x_0), \mathcal{E}^*, A_*) \rightarrow H^q((x_0), \mathcal{D}'^*, A_*)
\]

for every open \(U \subset M\) and \(x_0 \in M\).

Following the same arguments of [AFN] we can prove the following:

**Theorem 2.4.** If (2.16), for fixed \(q \geq 1\) and \(x_0 \in M\), is the zero map, then for every open neighborhood \(\omega\) of \(x_0\) in \(M\) there exists an open neighborhood \(\omega_1\) of \(x_0\) in \(\omega\) such that:

\[
(2.17) \quad \forall f \in \mathcal{E}^q(\omega) \text{ with } A_q f = 0 \text{ in } \omega \quad \exists u \in \mathcal{D}'^{q-1}(\omega_1) \text{ with } A_{q-1} u = f \text{ in } \omega_1.
\]

Moreover, we have the analogue of Theorem 2.3:

**Theorem 2.5.** Assume that for some \(q \geq 1\) and open \(\omega_1 \subset \omega \subset M\) the composition:

\[
(2.18) \quad H^q(\cdot, \mathcal{E}^*, A_*) \rightarrow H^q(\cdot, \mathcal{E}^*, A_*) \rightarrow H^q(\cdot, \mathcal{D}'^*, A_*)
\]
yields the zero map. Then for every compact subset $K_1 \subset \omega_1$ there exist integers $m, m_1 \geq 0$, a compact $K \subset \omega$ and a constant $c > 0$ such that for every $f \in \mathcal{E}^q(\omega)$ with $A_q f = 0$ in $\omega$ and for every $v \in \mathcal{E}^q(\omega_1)$ with $\text{supp}(v) \subset K_1$ we have:

\begin{equation}
\left| \int_{\omega_1} (f|v)_q d\lambda \right| \leq c \|A_q^* v\|_{q-1,K,m_1} \|f\|_{q,K,m}.
\end{equation}

§3 Preliminaries on CR manifolds and notation

In this paper $M$ will be a smooth ($C^\infty$) paracompact manifold, of real dimension $2n + k$, with a smooth CR structure of type $(n, k)$: $n$ is its complex CR dimension and $k$ its real CR codimension. As an abstract CR manifold $M$ is a triple $M = (M, HM, J)$, where $HM$ is a smooth real vector subbundle of rank $2n$ of the real tangent bundle $TM$, and where $J : HM \to HM$ is a smooth fiber preserving isomorphism such that $J^2 = -I$. It is also required that the formal integrability conditions $[C^\infty(M, T^{0,1}M), C^\infty(M, T^{0,1}M)] \subset C^\infty(M, T^{0,1}M)$ be satisfied. Here $T^{0,1}M = \{X + iJX \mid X \in HM\}$ is the complex subbundle of the complexification $CHM$ of $HM$ corresponding to the eigenvalue $-i$ of $J$; we have $T^{1,0}M \cap T^{0,1}M = 0$ and $T^{1,0}M \oplus T^{0,1}M = CHM$, where $T^{1,0}M = \overline{T^{0,1}M}$. When $k = 0$, we recover the abstract definition of a complex manifold, via the Newlander-Nirenberg theorem.

We denote by $H^0M = \{\xi \in T^*M \mid \langle X, \xi \rangle = 0, \forall X \in H_{\pi(\xi)}M\}$ the characteristic bundle of $M$. To each $\xi \in H^0M$, we associate the Levi form at $\xi$:

\begin{equation}
\mathcal{L}_\xi(X) = \xi([JX, \tilde{X}]) = d\tilde{\xi}(X, JX) \quad \text{for} \quad X \in H_xM
\end{equation}

which is Hermitian for the complex structure of $H_xM$ defined by $J$. Here $\tilde{\xi}$ is a section of $T^{0,1}M$ extending $\xi$ and $\tilde{X}$ a section of $HM$ extending $X$.

Let $\mathcal{E}^*(M) = \bigoplus_{h=0}^{2n+k} \mathcal{E}^{(h)}(M)$ denote the Grassmann algebra of smooth, complex valued differential forms on $M$. We denote by $\mathcal{J}$ the ideal of $\mathcal{E}^*(M)$ that annihilates $T^{0,1}M$:

\begin{equation}
\mathcal{J} = \left\{ \alpha \in \bigoplus_{h \geq 1} \mathcal{E}^{(h)}(M) \mid \alpha|_{T^{0,1}M} = 0 \right\}.
\end{equation}

By the formal integrability conditions we have $d \mathcal{J} \subset \mathcal{J}$. We also consider the powers $\mathcal{J}^p$ of the ideal $\mathcal{J}$, obtaining a decreasing sequence of $d$-closed ideals of $\mathcal{E}^*(M)$:

\begin{equation}
\mathcal{E}^*(M) = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \ldots \supset \mathcal{J}^{n+k-1} \supset \mathcal{J}^{n+k} \supset \mathcal{J}^{n+k+1} = \{0\}.
\end{equation}

Passing to the quotients, the exterior differential defines linear maps:

$\bar{\partial}_M : \mathcal{J}^p / \mathcal{J}^{p+1} \to \mathcal{J}^p / \mathcal{J}^{p+1}$. The grading of $\mathcal{E}^*(M)$ induces a grading of $\mathcal{J}^p / \mathcal{J}^{p+1}$:

\begin{equation}
\mathcal{J}^p / \mathcal{J}^{p+1} = \bigoplus_{j=0}^{n} \mathcal{Q}_{M,j}^{p,j}(M).
\end{equation}

As $\bar{\partial}_M \circ \bar{\partial}_M = 0$, we obtain the tangential Cauchy-Riemann complexes for $0 \leq p \leq n + k$:

\begin{equation}
(\mathcal{Q}_{M}^{p,*}, \bar{\partial}_M) = \left\{ 0 \to \mathcal{Q}_{M}^{p,0}(M) \overset{\bar{\partial}_M}{\to} \mathcal{Q}_{M}^{p,1}(M) \to \cdots \overset{\bar{\partial}_M}{\to} \mathcal{Q}_{M}^{p,n}(M) \to 0 \right\}.
\end{equation}
We also note that $Q^p_j(M) = C^\infty(M,Q^p_j)$ for complex vector bundles $Q^p_j$ on $M$ of rank $\binom{n+k}{p}$ and $(3.5)$ is a complex of partial differential operators of the first order. We denote by $\Omega^p_M$ the sheaf of smooth sections $f$ of $Q^p_M$ satisfying $\partial_M f = 0$. Note that $Q^0_M$ is the trivial complex line bundle over $M$; we set $\Omega^0_M = \mathcal{O}_M$ and call it the sheaf of germs of smooth CR functions on $M$.

If $(M,H,M,J_M)$ and $(N,H,N,J_N)$ are two CR manifolds, we say that a differentiable map $\phi : M \to N$ is CR iff: (i) $d\phi(HM) \subset HN$; (ii) $d\phi(J_M X) = J_N d\phi(X)$ for all $X \in HM$.

We say that a CR manifold $(M,H,M,J_M)$ of type $(n,k)$ is locally embeddable if for each point $x \in M$ we can find an open neighborhood $U$ of $x$ in $M$, an open subset $U$ of $\mathbb{C}^{n+k}$, and a smooth CR map $\phi : U \to \tilde{U}$ which is an embedding.

For a locally embeddable $M$ we shall give now a description in local coordinates of the Levi form and of the tangential Cauchy-Riemann complex.

Let $x \in M$ and $U$ open in $\mathbb{C}^{n+k}$ be as above. We can assume that $x$ is the origin $0$ of $\mathbb{C}^{n+k}$ and that

\[(3.6) \quad M \cap U = \{ x \in U \mid \rho_1(z) = 0, \ldots, \rho_k(z) = 0 \}\]

where $\rho_1, \ldots, \rho_k$ are real valued smooth functions on $U$ and

\[(3.7) \quad \partial \rho_1(z) \wedge \cdots \wedge \partial \rho_k(z) \neq 0 \quad \text{for} \quad z \in U.\]

The holomorphic tangent space to $M$ at a point $z \in M$, having chosen holomorphic coordinates on $T_z \mathbb{C}^{n+k}$, is identified with $T_z^{1,0}M$ and is described by:

\[(3.8) \quad H_zM \simeq T_z^{1,0}M = \left\{ u = (u^\alpha) \in \mathbb{C}^{n+k} \mid \sum_{\alpha=1}^{n+k} \frac{\partial \rho_j(z)}{\partial z^\alpha} u^\alpha = 0 \right\}.\]

We also have:

\[(3.9) \quad H^0_zM = \left\{ \xi = \left( \sum_{j=1}^k \lambda^j d^c \rho_j(z) \right) \mid \lambda^1, \ldots, \lambda^k \in \mathbb{R} \right\}
\]

and the Levi form at $\xi = \left( \sum_{j=1}^k \lambda^j d^c \rho_j(z) \right)$ is the complex Hessian

\[(3.10) \quad \sum_{j=1}^k \sum_{\alpha,\beta=1}^{n+k} \lambda^j \frac{\partial \rho_j(z)}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta \quad \text{for} \quad u = (u^\alpha) \in T_z^{1,0}M \simeq H_zM.\]

Let $z^{n+j} = t^j + is^j$ with $t^j, s^j \in \mathbb{R}$, for $j = 1, \ldots, k$. By a linear change of coordinates we can obtain that near $x = 0$:

\[(3.11) \quad \rho_j(z) = s^j - h_j(z^1, \ldots, z^n, t^1, \ldots, t^k), \quad \text{with} \quad h_j = 0(2) \quad \text{for} \quad j = 1, \ldots, k.\]

Near $x = 0$, the complex coordinates $z^1, \ldots, z^n$ and the real coordinates $t^1, \ldots, t^k$ define smooth coordinates on $M$. 

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Denote by $C = (C_{\alpha,j})$ the matrix

$$C = -2i \left( I + i \left( \frac{\partial h_\ell(z)}{\partial t^j} \right) \right)^{-1} \left( \frac{\partial h_j(z)}{\partial \bar{z}^\alpha} \right)$$

and consider the partial differential operators

$$\bar{L}_\alpha = \frac{\partial}{\partial \bar{z}^\alpha} + \sum_{j=1}^k C_{\alpha,j}^j \frac{\partial}{\partial t^j}.$$ 

On a neighborhood $V$ of $0$ in $M$ the natural pull-back composed with the projection onto the quotient define an identification:

$$Q_{M}^{p,q}(V) \simeq \left\{ \sum_{1 \leq \alpha_1 < \ldots < \alpha_p \leq n+k} \sum_{1 \leq \beta_1 \leq \ldots \leq \beta_q \leq n} a_{\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q} \, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q} \right\}$$

where $a_{\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q} = a_{\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q}(z^1, \ldots, z^n, t^1, \ldots, t^k)$ are smooth complex valued functions on $V$.

With this identification, and observing that $[\bar{L}_\alpha, \bar{L}_\beta] = 0$ for $1 \leq \alpha, \beta \leq n$, we obtain:

$$\bar{\partial}_M \left( \sum_{1 \leq \alpha_1 < \ldots < \alpha_p \leq n+k} \sum_{1 \leq \beta_1 \leq \ldots \leq \beta_q \leq n} a_{\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q} \, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q} \right)$$

$$= \sum_{1 \leq \alpha_1 < \ldots < \alpha_p \leq n+k} \sum_{1 \leq \beta_1 \leq \ldots \leq \beta_q \leq n} (\bar{L}_\beta a_{\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q}) \, d\bar{z}^{\beta} \wedge dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}.$$ 

§4 THE CR STRUCTURE OF THE CHARACTERISTIC BUNDLE

In this section we define regular points of $H^0M$ and prove a result that was formulated in [Tu, Theorem 2] for a generically embedded CR manifold. In that paper we could not find a proof of the stated Levi flatness of the CR structure of the conormal bundle. This fact is interesting for our discussion, because it entails the regularity of the characteristic codirections $\xi \in H^0M$ at which the rank of the Levi form is maximal. This regularity is requested in Theorems 5.1 and 5.2 to discuss the non validity of the Poincaré Lemma.

We have preferred to consider the characteristic bundle, which is more intrinsically related to the differential geometry of (abstract) CR manifolds, rather than the conormal bundle of [Tu]. However, our results imply those of [Tu, Theorem 2], since, when $M$ is a generic CR submanifold of a complex manifold $\tilde{M}$,
the characteristic bundle $H^0M$ is the image, by the dual map $J^*$ of the complex structure $J : T\tilde{M} \to T\tilde{M}$, of the conormal bundle $\tilde{M}^*(M)$ of $M$ in $\tilde{M} : H^0M = J^* \left( \tilde{M}^*(M) \right)$, and $J^* : T^*\tilde{M} \to T^*\tilde{M}$ is biholomorphic.

Let $M$ be smooth abstract CR submanifold, of type $(n, k)$. Denote by $\vartheta$ the tautological 1-form on $H^0M$. If $\pi : H^0M \to M$ is the natural projection, we have:

\[(4.1) \quad \vartheta(X) = \xi(\pi_*(X)) \quad \text{if} \quad \xi \in H_x^0M \quad \text{and} \quad X \in T_\xi H^0M.\]

For each point $\xi \in H^0M$, we set:

\[(4.2) \quad \begin{cases} N_\xi H^0M = \{ X \in T_\xi H^0M | \vartheta(X) = 0 \}, \\ H_\xi H^0M = \{ X \in N_\xi H^0M | d\vartheta(X, Y) = 0 \quad \forall Y \in T_\xi H^0M \}. \end{cases} \]

Let $\hat{H}^0M$ be the open subset of the nonzero cotangent vectors in $H^0M$. Then $\xi \to N_\xi H^0M$ is a smooth distribution of hyperplanes in $T\hat{H}^0M$. We note however, as the discussion below will clarify, that in general the dimension of $H_\xi H^0M$ is not constant and therefore $\xi \to H_\xi H^0M$ may fail to be a smooth subbundle of $T\hat{H}^0M$.

If $U$ is an open subset of $H^0M$, we define:

\[(4.3) \quad \begin{cases} \mathfrak{N}(U) = \{ X \in C^\infty(U, T\hat{H}^0M) | X(\xi) \in N_\xi H^0M \quad \forall \xi \in U \}, \\ \mathfrak{H}(U) = \{ X \in C^\infty(U, T\hat{H}^0M) | X(\xi) \in H_\xi H^0M \quad \forall \xi \in U \}. \end{cases} \]

**Lemma 4.1** Let $U$ be an open subset of $\hat{H}^0M$. Then:

\[(4.4) \quad \begin{cases} [\mathfrak{H}(U), \mathfrak{N}(U)] \subset \mathfrak{N}(U), \\ [\mathfrak{H}(U), \mathfrak{H}(U)] \subset \mathfrak{H}(U). \end{cases} \]

**Proof** For $X \in \mathfrak{H}(U)$ and $Y \in \mathfrak{N}(U)$, we have:

\[\vartheta([X, Y]) = -d\vartheta(X, Y) + X\vartheta(Y) - Y\vartheta(X) = 0.\]

Indeed $d\vartheta(X, Y) = 0$ because $X(\xi) \in H_\xi H^0M$, and $\vartheta(Y) = 0$, $\vartheta(X) = 0$ because $Y(\xi), X(\xi) \in N_\xi H^0M$, for all $\xi \in U$. This shows that $[\mathfrak{H}(U), \mathfrak{N}(U)] \subset \mathfrak{N}(U)$.

Let now $X, Y \in \mathfrak{H}(U)$. Then $[X, Y] \in \mathfrak{N}(U)$ and, to complete the proof of the Lemma, we need only to verify that $d\vartheta([X, Y], Z) = 0$ for all $Z \in C^\infty(U, T\hat{H}^0M)$. Since $\vartheta \neq 0$ at each point $\xi \in U$, we can as well assume that $\vartheta(Z)$ is constant in $U$. We have:

\[0 = d\vartheta(X, Z) = X\vartheta(Z) - Z\vartheta(X) - \vartheta([X, Z]) = -\vartheta([X, Z])\]

and likewise $\vartheta([Y, Z]) = 0$. Therefore

\[\vartheta([X, [Y, Z]]) = -d\vartheta(X, [Y, Z]) + X\vartheta([Y, Z]) - [Y, Z]\vartheta(X) = 0\]

and likewise $\vartheta([Y, [X, Z]]) = 0$. Hence we have:

\[d\vartheta([X, Y], Z) = [X, Y]\vartheta(Z) - Z\vartheta([X, Y]) - \vartheta([[[X, Y], Z]])\]

and

\[\vartheta([X, [Y, Z]]) + \vartheta([Y, [X, Z]]) = 0.\]
This shows that \([\mathcal{I}(U), \mathcal{I}(U)] \subset \mathcal{I}(U)\). □

Assume now that we are given a generic CR embedding \(M \hookrightarrow \tilde{M}\) of \(M\) into an \((n + k)\)-dimensional complex manifold \(\tilde{M}\). This yields an embedding \(H^0M \hookrightarrow T^*\tilde{M}\). The cotangent bundle \(T^*\tilde{M}\) has a complex structure, that we denote by \(\mathfrak{J} : TT^*\tilde{M} \to TT^*\tilde{M}\). We shall still denote by \(\pi : T^*\tilde{M} \to \tilde{M}\) the natural projection, extending \(H^0M \xrightarrow{\pi} M\).

**Lemma 4.2** Let \(x \in M\) and \(\xi \in H^0_x M \setminus \{0\}\). Then

\[
H_\xi H^0M = T_\xi H^0M \cap \mathfrak{J}T_\xi H^0M.
\]

**Proof** In the proof we can as well assume that \(\tilde{M}\) is an open subset of \(\mathbb{C}^{n+k}\). In particular we shall utilize the standard trivialization of the cotangent bundle and identify \(H^0M\) with a submanifold of \(\tilde{M} \times (\mathbb{R}^{2n+2k})^*\):

\[
H^0M = \left\{ (x, x^*) \in M \times (\mathbb{R}^{2n+2k})^* \mid \langle x^*, H_x M \rangle = \{0\} \right\}.
\]

We can assume that \(M\) is defined in \(\tilde{M}\) by

\[
M = \{ x \in \tilde{M} \mid \rho_1(x) = 0, \ldots, \rho_k(x) = 0 \},
\]

with \(\rho_1, \ldots, \rho_k \in C^\infty(\tilde{M}, \mathbb{R})\) and \(\partial \rho_1(x) \wedge \cdots \wedge \partial \rho_k(x) \neq 0\) for all \(x \in \tilde{M}\).

With \(d^c = (\partial - \bar{\partial})/i\), we obtain:

\[
H^0M = \left\{ \left( x, \sum_{j=1}^k a^j d^c \rho_j(x) \right) \mid x \in M, a^1, \ldots, a^k \in \mathbb{R} \right\}.
\]

In this way a tangent vector to \(H^0M\) at \(\xi_0 = (x_0, x_0^*) = \left(x_0, \sum_{j=1}^k a_0^j d^c \rho_j(x_0)\right)\) is identified to a vector in \(\mathbb{R}^{2n+2k} \times (\mathbb{R}^{2n+2k})^*\) of the form:

\[
X = \left( v_0, \sum_{j=1}^k \lambda^j \langle d^c \rho_j(x_0), \cdot \rangle + \sum_{j=1}^k a_0^j dd^c \rho_j(x_0)(v_0, \cdot) \right),
\]

with \(v_0 \in T_{x_0}M\) and \(\lambda^1, \ldots, \lambda^k \in \mathbb{R}\). We recall the notation \(\mathcal{J}\) for the complex structure of \(\tilde{M}\), and hence here of \(\mathbb{C}^{n+k}\), and \(\mathfrak{J}\) for the complex structure of \(T^*\tilde{M}\), and hence here of \(T^*\mathbb{C}^{n+k}\). We have:

\[
\mathfrak{J}X = \left( \mathcal{J}v_0, \sum_{j=1}^k \lambda^j \langle d^c \rho_j(x_0), \mathcal{J} \cdot \rangle + \sum_{j=1}^k a_0^j dd^c \rho_j(x_0)(v_0, \mathcal{J} \cdot) \right).
\]

Therefore \(X\) and \(\mathfrak{J}X\) both belong to \(T_\xi H^0M\) if, and only if,

(i) \(v_0 \in H^0_x M\).
and there exist real numbers $\mu^1, \ldots, \mu^k$ such that:

\[
\sum_{j=1}^{k} \lambda^j \langle d^c \rho_j(x_0), \mathcal{J}w \rangle + \sum_{j=1}^{k} a_0^j dd^c \rho_j(x_0)(v_0, \mathcal{J}w) = \sum_{j=1}^{k} \mu^j \langle d^c \rho_j(x_0), w \rangle + \sum_{j=1}^{k} a_0^j dd^c \rho_j(x_0)(\mathcal{J}v_0, w) \\
\forall w \in \mathbb{R}^{2n+2k}
\]

From this we deduce that:

\[
\sum_{j=1}^{k} a_0^j (dd^c \rho_j(x_0)(v_0, \mathcal{J}w) - dd^c \rho_j(x_0)(\mathcal{J}v_0, w)) = \sum_{j=1}^{k} \mu^j \langle d^c \rho_j(x_0), w \rangle \quad \forall w \in T_{x_0}M
\]

and

\[
\sum_{j=1}^{k} a_0^j (dd^c \rho_j(x_0)(v_0, \mathcal{J}w) - dd^c \rho_j(x_0)(\mathcal{J}v_0, w)) = - \sum_{j=1}^{k} \lambda^j \langle d^c \rho_j(x_0), \mathcal{J}w \rangle = - \sum_{j=1}^{k} \lambda^j \langle d\rho_j(x_0), w \rangle \forall w \in (T_{x_0}M)^\perp
\]

Here the perpendicular is taken with respect to the standard metric of $\mathbb{R}^{2n+2k}$.

We can find the real numbers $\mu^j$ to satisfy (4.11) if and only if

\[
\sum_{j=1}^{k} a_0^j (dd^c \rho_j(x_0)(v_0, \mathcal{J}w) - dd^c \rho_j(x_0)(\mathcal{J}v_0, w)) = 0 \quad \forall w \in H_{x_0}M.
\]

When $v_0, w \in H_{x_0}M$ we have

\[
dd^c \rho_j(x_0)(v_0, \mathcal{J}w) = -dd^c \rho_j(x_0)(\mathcal{J}v_0, w) \quad \forall j = 1, \ldots, k
\]

and therefore (4.11) is equivalent to

\[
v_0 \in \ker \mathcal{L}_{x_0}.
\]

Finally we note that (4.12) uniquely determines the real coefficients $\lambda^j$ for any given $v_0 \in \ker \mathcal{L}_{x_0}$.

In this way we characterized a vector $X \in T_{x_0}H^0M \cap \Im T_{x_0}H^0M$: it has the form (4.9) with

\[
2 \sum_{j=1}^{k} a_0^j dd^c \rho_j(v_0, \mathcal{J}w) + \sum_{j=1}^{k} \lambda^j \langle d\rho_j(x_0), w \rangle = 0 \quad \forall w \in (T_{x_0}M)^\perp.
\]

Next we compute $H_{x_0}H^0M$. To this aim we observe that, denoting by $x_\alpha^*$ the dual real variables of $(\mathbb{R}^{2n+2k})^*$ we can write a tangent vector in the form:

\[
X = \sum_{\alpha} v^\alpha \frac{\partial}{\partial x^\alpha} + \sum_{j=1}^{k} \lambda^j \langle d^c \rho_j(x), e_\alpha \rangle \frac{\partial}{\partial x^\alpha} + \sum_{j=1}^{k} a^j dd^c \rho_j(x)(v, e_\alpha) \frac{\partial}{\partial x^\alpha}
\]
where \( e_\alpha (\alpha = 1, \ldots, 2n + 2k) \) is the canonical basis of \( \mathbb{R}^{2n+2k} \simeq \mathbb{C}^{n+k} \) and the vector \( v = \sum_{\alpha=1}^{2n+2k} v^\alpha (x, x^*) \frac{\partial}{\partial x^\alpha} \) belongs to \( T_x M \). It is convenient to consider the coefficients \( v^\alpha, \lambda^j \) to be smooth functions of \( x, x^* \), defined everywhere in \( \tilde{T}^* \tilde{M} \). A vector field of this form belongs to \( \mathfrak{N}(U) \) (for a neighborhood \( U \) of \( \xi_0 \) in \( H^0 M \)) iff

\[
\sum_{j=1}^{k} a^j \langle d^c \rho_j (x), v \rangle = 0 \quad \text{in} \quad U.
\]

By differentiation we obtain:

\[
\sum_{\alpha=1}^{2n+2k} \left( \sum_{j=1}^{k} \frac{\partial a^j}{\partial x^\alpha} \langle d^c \rho_j, v \rangle + \sum_{j=1}^{k} a^j \left\langle d^c \rho_j, \frac{\partial v}{\partial x^\alpha} \right\rangle \right) dx^\alpha_* = 0 \quad \text{in} \quad U.
\]

We note that

\[
x^\alpha = \sum_{j=1}^{k} a^j \langle d^c \rho_j (x), e_\alpha \rangle \quad \text{in} \quad U
\]

and therefore

\[
dx^\alpha_* = \sum_{\beta=1}^{2n+2k} \sum_{j=1}^{k} \frac{\partial a^j}{\partial x^\beta} \langle d^c \rho_j, e_\alpha \rangle dx^\beta_* \quad \text{in} \quad U.
\]

Let

\[
Y = \sum_{\alpha=1}^{2n+2k} w^\alpha \frac{\partial}{\partial x^\alpha} + \sum_{j=1}^{k} \mu^j \langle d^c \rho_j(x), e_\alpha \rangle \frac{\partial}{\partial x^\alpha_*} + \sum_{j=1}^{k} a^j \langle d^e \rho_j(x), (w, e_\alpha) \rangle \frac{\partial}{\partial x^\alpha_*}
\]

be another vector field in \( U \). Let \([X, Y] = \sum_{\alpha=1}^{2n+2k} \left( A^\alpha \frac{\partial}{\partial x^\alpha} + B^\alpha \frac{\partial}{\partial x^\alpha_*} \right)\). Then:

\[
A^\alpha = v^\beta \frac{\partial w^\alpha}{\partial x^\beta} - w^\beta \frac{\partial v^\alpha}{\partial x^\beta} + \sum_{j=1}^{k} \lambda^j \langle d^c \rho_j, e_\beta \rangle \frac{\partial w^\alpha}{\partial x^\beta_*} - \sum_{j=1}^{k} \mu^j \langle d^c \rho_j, e_\beta \rangle \frac{\partial v^\alpha}{\partial x^\beta_*} + \sum_{j=1}^{k} a^j \langle d^e \rho_j, (v, e_\beta) \rangle \frac{\partial w^\alpha}{\partial x^\beta_*} - \sum_{j=1}^{k} a^j \langle d^e \rho_j, (w, e_\beta) \rangle \frac{\partial v^\alpha}{\partial x^\beta_*}.
\]

Assume that \( X \in \mathfrak{N}(U) \) and impose the condition that \( d\vartheta(X, Y) = 0 \) for all \( Y \in C^\infty(U, T^*H^0 M) \). We can choose \( Y \) is such a way that \( \vartheta(Y) = \text{constant} \) on a neighborhood of \( U \) in \( T^* \tilde{M} \), so that:

\[
\sum_{\alpha=1}^{2n+2k} \left( \sum_{j=1}^{k} \frac{\partial a^j}{\partial x^\alpha_*} \langle d^c \rho_j, w \rangle + \sum_{j=1}^{k} a^j \left\langle d^c \rho_j, \frac{\partial w}{\partial x^\alpha_*} \right\rangle \right) dx^\alpha_* = 0 \quad \text{in} \quad U.
\]
Then $d\vartheta(X,Y) = -\vartheta([X,Y])$ and we are lead to the condition:

$$
\sum_{j=1}^{k} a^j \langle d^c \rho_j, [v, w] \rangle
- \sum_{j=1}^{k} \lambda_j \langle d^c \rho_j, w' \rangle + \sum_{j=1}^{k} \mu^j \langle d^c \rho_j, v' \rangle
- \sum_{j=1}^{k} a^j dd^c \rho_j(v, w') + \sum_{j=1}^{k} a^j dd^c \rho_j(w, v')
= - \sum_{j=1}^{k} a^j dd^c \rho_j(v, w)
- \sum_{j=1}^{k} \lambda_j \langle d^c \rho_j, w' \rangle + \sum_{j=1}^{k} \mu^j \langle d^c \rho_j, v' \rangle
- \sum_{j=1}^{k} a^j dd^c \rho_j(v, w') + \sum_{j=1}^{k} a^j dd^c \rho_j(w, v') = 0,
$$

(4.22)

where

$$
v' = \sum_{j=1}^{k} \sum_{\alpha=1}^{2n+2k} \frac{\partial a^j}{\partial x^\alpha} \langle d^c \rho_j, v \rangle e_\alpha,
\quad w' = \sum_{j=1}^{k} \sum_{\alpha=1}^{2n+2k} \frac{\partial a^j}{\partial x^\alpha} \langle d^c \rho_j, w \rangle e_\alpha
$$

are the components of $v$ and $w$ orthogonal to $HM$. The vector $X_{\xi_0}$ belongs to $H_{\xi_0}H^0M$ if and only if (4.22) holds at the point $\xi_0 = (x_0, x^0)$ for all choices of $\mu^0_0, \ldots, \mu^0_k \in \mathbb{R}$ and for every $w_0 \in T_{x_0}M$. Taking first $w_0 = 0$ and arbitrary $\mu^0_0$, we obtain:

$$
\langle d^c \rho_j(x_0), v'(\xi_0) \rangle = 0 \quad \text{for} \quad j = 1, \ldots, k,
$$

i.e. $v(\xi_0) \in H_{x_0}M$. Next, letting $w_0$ vary in $H_{x_0}M$, we obtain that:

$$
\sum_{j=1}^{k} a^j_0 dd^c \rho_j(v(\xi_0), w_0) = 0 \quad \forall w_0 \in H_{x_0}M
$$

and hence $v(\xi_0) \in \ker L_{\xi_0}$. Finally, we have:

$$
2 \sum_{j=1}^{k} a^j_0 dd^c \rho_j(v(\xi_0), w'_0) + \sum_{j=1}^{k} \lambda_j^0 \langle d^c \rho_j(x_0), w'_0 \rangle = 0 \quad \forall w'_0 \in T_{x_0}M \cap (H_{x_0}M)^\perp.
$$

Since $\langle d^c \rho_j, Jw \rangle = \langle d\rho_j, w \rangle$, and $J ((T_{x_0}M)^\perp) = T_{x_0}M \cap (H_{x_0}M)^\perp$, by (4.14) the proof of the lemma is complete. □

**Theorem 4.3.** Let $M$ be a generic CR submanifold, of type $(n,k)$. Assume that the Levi form $L_\xi$ has constant rank $m$ for all $\xi$ in an open subset $U$ of $\mathcal{H}^0M$. Then $U$ is a CR submanifold of $T^*\tilde{M}$, of type $(n-m, 2k + 2m)$, which is Levi flat and foliated by complex submanifolds of dimension $(n-m)$. Note that the embedding $U \hookrightarrow T^*\tilde{M}$ is not generic when $m < n$.

**Proof.** We have: (i) $[\mathcal{M}(U), \mathcal{N}(U)] \subset \mathcal{M}(U)$ by Lemma 4.1; (ii) $\mathcal{J}(\mathcal{N}(U)) = \mathcal{N}(U)$, by Lemma 4.2; (iii) $\mathcal{N}(U)$ is a distribution of constant rank $(2n - 2m)$ by Lemma 4.2. Hence the real Frobenious theorem provides the foliation, and by the classical theorem of Levi-Civita, each leaf is a complex submanifold of dimension $(n-m)$. □
We say that a point $\xi_0 \in \bar{H}^0 M$ is regular if there exists an open neighborhood $U$ of $\xi_0$ in $\bar{H}^0 M$ and a smooth submanifold $V$ of $U$ with

\[(4.23) \quad \xi_0 \in V \subset U, \quad T_\xi V \subset H_\xi \bar{H}^0 M \forall \xi \in V, \quad \text{and} \quad T_\xi V = H_{\xi_0} \bar{H}^0 M.\]

Since the rank of the Levi form $L_\xi$ is a lower semicontinuous function of $\xi \in \bar{H}^0 M$, Theorem 4.3 yields:

**Corollary 4.4** Let $M$ be a generic CR submanifold, of type $(n, k)$, and $\xi_0 \in \bar{H}^0 M$ with $\text{rank} L_\xi \limsup \xi \rightarrow \xi_0 \text{rank} L_\xi$. Then $\xi_0$ is regular and there is an open neighborhood $U$ of $\xi_0$ in $\bar{H}^0 M$ consisting of regular points $\xi$ with $\text{rank} L_\xi = \text{rank} L_{\xi_0}$.

We have:

**Lemma 4.5** Let $M$ be a generic CR submanifold, of type $(n, k)$, of a complex manifold $\bar{M}$. Let $\xi_0 \in \bar{H}^0 M$ be regular. If the Levi form $L_{\xi_0}$ has rank $m$, then there exists an $(n - m)$-dimensional smooth complex submanifold $V^*$ of a neighborhood of $\xi_0$ in $T^* \bar{M}$, with:

\[(4.24) \quad \xi_0 \in V^* \subset H^0 M.\]

The projection $W = \pi(V^*)$ is a smooth complex submanifold of a neighborhood of $\pi(\xi_0)$ in $\bar{M}$, contained in $M$.

**Proof** The dimension of $H_\xi H^0 M$ is an upper semicontinuous function of $\xi \in \bar{H}^0 M$. Therefore it remains constant and equal to $2(n - m)$ on an open neighborhood of $\xi_0$ in $V$. Let $V^*$ be such a neighborhood. We have $T_\xi V^* = H_\xi H^0 M$ for all $\xi \in V^*$. Hence $T_\xi V^* = T_\xi V^*$ for all $\xi \in V^*$ and then, by the theorem of Levi-Civita, $V^*$ is a complex submanifold of dimension $(n - m)$ of a neighborhood of $\xi_0$ in $T^* \bar{M}$. Since the fibers of $H^0 M \rightarrow M$ are totally real, $V^*$ is transversal to the fibers and therefore the map $V^* \ni \xi \rightarrow \pi(\xi) \in M$ is a local diffeomorphism. It becomes a diffeomorphism after substituting to $V^*$ its intersection with a suitable small neighborhood of $\xi_0$ in $T^* \bar{M}$. Finally, $\pi(V^*)$ is a complex submanifold of an open neighborhood of $\pi(\xi_0)$, contained in $M$, because the projection $T^* \bar{M} \rightarrow \bar{M}$ is holomorphic. $\square$

We have:

**Theorem 4.6** Let $M$ be a generic CR submanifold, of type $(n, k)$, of a complex manifold $\bar{M}$. Let $x_0$ be a point of $M$ and $\xi_0 \in \bar{H}^0 M$ a regular point of $H^0 M$. Then:

Then there exists an open neighborhood $\bar{W}$ of $x_0$ in $\bar{M}$, an $(n - m)$-dimensional complex submanifold $W$ of $\bar{W}$ with

$$x_0 \in W \subset M \cap \bar{W}$$

and a real valued smooth function $\rho : \bar{W} \rightarrow \mathbb{R}$ with $\rho(x) = 0$ for $x \in M \cap \bar{W}$, $d^c \rho(x_0) = \xi_0$ and $\frac{\partial \rho}{\partial z^\alpha} \bigg|_W$ holomorphic in $W$ for $\alpha = 1, \ldots, n + k$ (here $z^1, \ldots, z^{n+k}$ are holomorphic coordinates in $\bar{W}$).

**Proof** It suffices to consider the situation where $\tilde{M} = \bar{W}$ is a neighborhood of $x_0$ in $\mathbb{C}^{n+k}$. Then $T^* \tilde{M}$ can be identified to the product manifold $\tilde{W} \times (\mathbb{C}^{n+k})^*$, where...
is the space of \( \mathbb{C} \)-linear forms in \( \mathbb{C}^{n+k} \). If \( M \) is described by (4.7), then \( H^0M \) gets identified to the real submanifold of \( \tilde{M} \times (\mathbb{C}^{n+k})^* \):

\[
\left\{ \left( z, (1/i) \sum_{j=1}^{k} a^j \partial \rho_j(z) \right) \mid z \in M, a^0, \ldots, a^k \in \mathbb{R} \right\}.
\]

By Lemma 4.5, there is a complex submanifold \( V^* \) of dimension \( (n-m) \) of \( T^*\tilde{M} \) that is contained in \( H^0M \), contains the point \( \xi_0 \), and whose projection \( W = \pi(V^*) \) in \( M \) is an \( (n-m) \) smooth complex submanifold of \( \tilde{M} \). Define \( \rho = \sum_{j=1}^{k} a^j(z) \rho_j(z) \) with real valued smooth functions \( a^j \) such that \( \left( z, (1/i) \sum_{j=1}^{k} a^j(z) \partial \rho_j(z) \right) \) belongs to \( V^* \) when \( z \in W \). \( \square \)

§5 The main theorems (corrected version)

**Theorem 5.1.** Let \( M \) be a locally embeddable CR manifold of type \( (n,k) \). Let \( x_0 \in M \) and assume that \( H^0_{x_0}M \) contains a regular point \( \xi_0 \) of \( H^0M \) such that the Levi form \( L_{\xi_0} \) has \( q \) positive eigenvalues and \( (n-q) \) eigenvalues which are \( \leq 0 \). Then the local cohomology groups \( H^q_{\bar{\partial}M}((x_0), Q^{p,*}_M) = \lim_{U \ni x_0} H^q_{\bar{\partial}M}(U, Q^{p,*}_M) \) are infinite dimensional for all \( 0 \leq p \leq n+k \).

In fact a more general statement is valid. To formulate it, we consider the \( \bar{\partial}M \)-complex on currents (see [HN1], [NV]) and consider, for an open subset \( \omega \) of \( M \), and \( 0 \leq p \leq n+k \), its cohomology groups, that we denote by \( H^q_{\bar{\partial}M}(\omega, \mathcal{D}' \otimes Q^{p,*}_M) \) (for \( 0 \leq q \leq n \)). We define the local cohomology on currents by:

\[
H^q_{\bar{\partial}M}((x_0), \mathcal{D}' \otimes Q^{p,*}_M) = \lim_{\omega \ni x_0} H^q_{\bar{\partial}M}(\omega, \mathcal{D}' \otimes Q^{p,*}_M)
\]

Note that we have a natural map

\begin{equation}
H^q_{\bar{\partial}M}((x_0), Q^{p,*}_M) \rightarrow H^q_{\bar{\partial}M}((x_0), \mathcal{D}' \otimes Q^{p,*}_M). 
\end{equation}

We have:

**Theorem 5.2.** With the same assumptions of Theorem 5.1, the map (5.1) has an infinite dimensional image.

We give first the proof of Theorem 5.1, then indicate the small changes needed to prove Theorem 5.2.

**Proof of Theorem 5.1**

By [BHN1], it suffices to show that \( H^q_{\bar{\partial}M}((x_0), Q^{p,*}_M) \neq 0 \). We can assume that \( M \) is a generic CR submanifold of an open subset \( \tilde{M} \) of \( \mathbb{C}^{n+k} \), and \( M \) is defined by (4.7). If \( L_{\xi_0} \) has rank \( n \) our statement reduces to [AFN]. If \( L_{\xi_0} \) has rank \( (n-d) \), we utilize Theorem 4.6. By shrinking \( \tilde{M} \), and changing the holomorphic coordinates, we put ourselves in the situation where \( x_0 = 0 \), there is a \( d \)-dimensional complex linear space \( W = \{ z^{d+1} = 0, \ldots, z^{n+k} = 0 \} \) such that

\begin{equation}
x_0 = 0 \in W \cap \tilde{M} \subset M \subset \tilde{M}.
\end{equation}
and

\[(5.3) \quad \partial \rho_1(z) = i dz^{n+1} \quad \text{in} \quad W \cap \tilde{M}, \quad \partial \rho_j(0) = i dz^{n+j}|_0 \quad \text{for} \quad j = 1, \ldots, k.\]

Set $\zeta = (z^1, \ldots, z^n)$ and $t = (t^1, \ldots, t^k)$. By the implicit function theorem, after shrinking $\tilde{M}$, we can take the defining functions in the form:

\[(5.4) \quad \rho_j(z) = s^j - h_j(z), \quad \text{with} \quad h_j(z) = h_j(\zeta, t) = 0(2),\]

and by \((5.3)\) we obtain moreover that

\[(5.5) \quad h_1(z) = 0 \quad \text{and} \quad \frac{\partial h_1(z)}{\partial z^\alpha} = 0 \quad \text{for} \quad z \in W \cap \tilde{M} \quad \text{and} \quad \alpha = d + 1, \ldots, n.\]

By substituting the complex variable $z^{n+1}$ by

\[(5.6) \quad z^{n+1} - i \sum_{\alpha, \beta = d+1}^{n+k} \frac{\partial^2 h_1(0)}{\partial z^\alpha \partial \bar{z}^\beta} z^\alpha \bar{z}^\beta,\]

and making a linear change of the variables $z^{d+1}, \ldots, z^n$, we obtain that:

\[(5.7) \quad h_1(\zeta, t) = \sum_{\alpha = d+1}^{d+q} z^\alpha z^\alpha - \sum_{\alpha = d+q+1}^{n} z^\alpha \bar{z}^{\alpha} + 0(3) \quad \text{at} \quad 0.\]

By Taylor’s formula we get:

\[(5.8) \quad h_1(\zeta, t) = \sum_{\alpha, \beta = d+1}^{n+k} \frac{\partial^2 h_1(z^1, \ldots, z^d, 0, \ldots, 0)}{\partial z^\alpha \partial \bar{z}^\beta} z^\alpha \bar{z}^\beta
\]

\[+ \Re \sum_{\alpha, \beta = d+1}^{n+k} \frac{\partial^2 h_1(z^1, \ldots, z^d, 0, \ldots, 0)}{\partial z^\alpha \partial \bar{z}^\beta} z^\alpha \bar{z}^\beta
\]

\[+ o \left( \sum_{\alpha, \beta = d+1}^{n} z^\alpha \bar{z}^{\alpha} \right).\]

By \((5.7)\) the second summand in the right hand side vanishes at 0. Thus we can find $r_0 > 0$ so that $B(r_0) = \{|z| \leq r_0\} \subseteq \tilde{M}$ and

\[(5.9) \quad h_1(z) = h_1(\zeta, t) \leq 2 \sum_{\alpha = d+1}^{d+q} z^\alpha \bar{z}^\alpha - (1/2) \sum_{\alpha = d+q+1}^{n} z^\alpha \bar{z}^\alpha
\]

\[+ \sum_{j=1}^{k} \left(t^j\right)^2 \quad \text{for} \quad z \in B(r_0) \cap M.\]

Let us fix a real number $\nu > 2$ and set:

\[(5.10) \quad \phi(z) = \phi(\zeta, t) = -it^1 + h_1(\zeta, t) - \nu \sum_{\alpha = d+1}^{d+q} z^\alpha \bar{z}^\alpha - \nu \sum_{j=1}^{k} (t^j + ih_j(\zeta, t))^2.\]
Since \( h_j(z) = \Re z^{n+j} = 0 \) in \( W \cap \tilde{M} \), we can find a real number \( r_1 \) with \( 0 < r_1 \leq r_0 \) such that

\[(5.11) \quad \Re \phi(z) \leq 0 \quad \text{for} \quad z \in B(r_1) \cap M.\]

Define, for every real \( \tau > 0 \)

\[(5.12) \quad f_\tau = e^{\frac{1}{2} \phi(z)} d z^1 \wedge \cdots \wedge d z^p \wedge d \bar{z}^{d+1} \wedge \cdots \wedge d \bar{z}^{d+q}.\]

This is a smooth \((p, q)\)-form, that defines a form in \( Q^{p, q}(B(r_1) \cap M) \) satisfying

\[(5.13) \quad \bar{\partial} M f_\tau = 0 \quad \text{in} \quad B(r_1) \cap M.\]

We next define:

\[(5.14) \quad \psi(z) = it^1 - h_1(z) - \nu \sum_{\alpha=1}^{d} z^\alpha \bar{z}^\alpha - \nu \sum_{\alpha=d+q+1}^{n} z^\alpha \bar{z}^\alpha - \nu \sum_{j=1}^{k} (t^j + i h_j(\zeta, t))^2.\]

Then we can find a positive \( r \) with \( 0 < r \leq r_1 \leq r_0 \) such that

\[(5.15) \quad \Re \psi(z) \leq -\frac{1}{2} \sum_{\alpha=1}^{n+k} z^\alpha \bar{z}^\alpha \quad \text{for} \quad z \in B(r) \cap M.\]

Now \( \nu \) and \( r \) are fixed and we set:

\[(5.16) \quad g_\tau = e^{\frac{1}{2} \psi(z)} d z^{p+1} \wedge \cdots \wedge d z^n \wedge d \bar{z}^1 \wedge \cdots d \bar{z}^d \wedge d \bar{z}^{d+q+1} \wedge \cdots d \bar{z}^n.\]

For each \( \tau > 0 \) the form \( g_\tau \) defines an element of \( Q^{n+k-p, n-q}(B(r) \cap M) \) and we have:

\[(5.17) \quad \bar{\partial} M f_\tau = 0, \quad \bar{\partial} M g_\tau = 0 \quad \text{in} \quad B(r) \cap M.\]

Let \( \chi = \chi(\zeta, t) \) denote a smooth real valued function defined in \( \mathbb{R}^{2n+k} \) such that \( \chi = 1 \) for \( |\zeta|^2 + |t|^2 < \frac{1}{2} \) and \( \chi = 0 \) for \( |\zeta|^2 + |t|^2 > \frac{2}{3} \). If the Poincaré lemma is valid, we have, for all \( R > 0 \) sufficiently large, an a priori estimate:

\[(5.18) \quad \left| \int \chi(R\zeta, Rt) f_\tau \wedge g_\tau \right| \leq C \left( \sup_{z \in B(r) \cap M} |D_z a D_{\bar{z}} b f_\tau| \right) \cdot \left( \sup_{R} |\bar{\partial} M \chi(R\zeta, Rt) \wedge g_\tau| \right)\]

with constants \( C = C(R) \) and \( m = m(R) \) which depend on \( R \) but are independent of \( \tau \).

Next we note that

\[\phi(\zeta, t) + \psi(\zeta, t) = -\nu \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha - 2\nu \sum_{j=1}^{k} \left( t^j + i h_j(\zeta, t) \right)^2.\]
Upon replacing $z$ by $z/\sqrt{\tau}$, we have:

$$ \int \chi(R\zeta, R\tau) f_\tau \wedge g_\tau = (-1)^q(n+k+d-p)\tau^{n+\frac{d}{2}} $$

$$ \times \int \chi(R\sqrt{\tau}(\zeta, t)) \exp \left( -\nu \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha - 2\nu \sum_{j=1}^{k} t^j + O(\sqrt{\tau}) \right)^2 $$

$$ \times dz^1 \wedge \cdots \wedge dz^{n+k} \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n $$

Therefore we obtain that

$$ (5.19) \quad \tau^{-(n+\frac{k}{2})} \int \chi(R\zeta, R\tau) f_\tau \wedge g_\tau \longrightarrow \text{constant} > 0 \quad \text{for } \tau \to 0. $$

Next we observe that we have an estimate of the form:

$$ (5.20) \quad \sup_{z \in B(r) \cap M} \left| D_z^a \bar{D}_{\bar{z}}^b f_\tau \right| \leq c_1 \tau^{-m} $$

with a positive constant $c_1$ which is independent $\tau$, because $\Re \phi \leq 0$ in $B(r) \cap M$.

On the other hand we have, for a positive constant $c_2$:

$$ (5.21) \quad \sup |R(\bar{\partial}_M \chi)(Rz) \wedge g_\tau| \leq c_2 \cdot R \cdot \exp \left( -R^{-2}/(4\tau) \right), $$

by (5.15).

By letting $\tau$ approach 0, with any fixed large $R > 0$, we see that (5.19) cannot possibly hold true. This proves the theorem.

**Proof of Theorem 5.2**

The only changes needed to obtain Theorem 5.2 are in formulas (5.18) and (5.21). In (5.18) instead of the sup norm of $\bar{\partial}_M g_\tau$ we need to introduce the sup norm of its derivatives up to some finite order $m_1 \geq 0$. This modifies (5.21) by a factor $\tau^{-m_1}$ in the right hand side, but again the right hand side of (5.18) tends to 0 when $\tau \to 0$, yielding a contradiction that, in view of Theorem 2.5, proves the statement of Theorem 5.2.

§6 Examples

1. Let $X$ be the complex manifold consisting of the pairs $(L_1, L_3)$ where $L_i$ is a complex linear subspace of $\mathbb{C}^4$ of dimension $i$ and $L_1 \subset L_3$. This is a compact complex manifold of complex dimension 5. Fix the canonical basis $e_1, e_2, e_3, e_4$ of $\mathbb{C}^4$. A local chart of $X$ in an open neighborhood $U$ of $(\langle e_1 \rangle, \langle e_1, e_2, e_3 \rangle)$ in $X$ is given by the coordinates $z^1, \ldots, z^5$ where $L_1$ and $L_3$ are generated, respectively, by the first column and by the three columns of the matrix:

$$ \begin{pmatrix} 1 & 0 & 0 \\ z^1 & 1 & 0 \\ z^2 & 0 & 1 \\ z^3 & 4 & 5 \end{pmatrix}. $$
Consider on $\mathbb{C}^2$ the structure of a right module over the division ring $\mathbb{H}$ of the quaternions and let $M$ be the subset of $X$ consisting of the pairs $(L_1, L_3)$ with $L_1 \cdot \mathbb{H} \subset L_3$. In $U$ the equation of $M$ is:

\[
\begin{vmatrix}
1 & 0 & 0 & -z^1 \\
z^1 & 1 & 0 & 1 \\
z^2 & 0 & 1 & -z^3 \\
z^3 & z^4 & z^5 & -\bar{z}^2
\end{vmatrix} = 0
\]

and therefore it is easy to verify that $M$ is a $CR$ manifold of type $(3, 2)$ and that the Levi form of $M$ has in every non-zero characteristic codirection exactly one positive, one negative and one zero eigenvalue. Thus at each point of $M$ the Poincaré lemma fails in degree $1$.

2. Let $m_1, \ldots, m_\ell$ be integers $\geq 1$ and fix complex coordinates $w^1, w^2, z_j^1, \ldots, z_j^{m_j+1}, \xi_j^1, \ldots, \xi_j^{m_j}$, for $j = 1, \ldots, \ell$ in $\mathbb{C}^{2(m_1+\cdots+m_\ell)+\ell+2}$. We define a $CR$ submanifold of type $(2[m_1+\cdots+m_\ell]+\ell, 2)$ of $\mathbb{C}^{2(m_1+\cdots+m_\ell)+\ell+2}$ by:

\[
M := \left\{ \begin{array}{l}
\Im w^1 = \Im \sum_{j=1}^{\ell} \sum_{h=1}^{m_j} z_j^h \bar{\xi}_j^h \\
\Im w^2 = \Im \sum_{j=1}^{\ell} \sum_{h=1}^{m_j} z_j^{h+1} \bar{\xi}_j^h.
\end{array} \right.
\]

For every $\xi \neq 0$ in $H^0 M$, the Levi form $L_\xi$ has $m_1+\cdots+m_\ell$ positive, $m_1+\cdots+m_\ell$ negative, and $\ell$ zero eigenvalues. Thus the Poincaré lemma for $M$ is not valid at any point of $M$ in dimension $q = m_1 + \cdots + m_\ell$.

3. Let $N \subset \mathbb{C}^5$ be defined by

\[
N = \left\{ (z, w) \in \mathbb{C}^2 \times \mathbb{C}^3 \left| \begin{array}{c}
\Im w^1 = z^1 \bar{z}^2 + z^2 \bar{z}^1 \\
\Im w^2 = z^1 \bar{z}^3 + z^3 \bar{z}^1
\end{array} \right. \right\}
\]

At each $\xi \in H^0 M \setminus \{0\}$ the Levi form $L_\xi$ has one positive, one negative and one zero eigenvalue. We set $t^j = \Re w^j$, $j = 1, 2$ and use $t^1, t^2, z^1, z^2, z^3$ as global coordinates on $N$. The complex vector fields

\[
\begin{cases}
\bar{L}_1 = \frac{\partial}{\partial \bar{z}^1} - iz^2 \frac{\partial}{\partial t^1} - iz^3 \frac{\partial}{\partial t^2} \\
\bar{L}_2 = \frac{\partial}{\partial \bar{z}^2} - iz^1 \frac{\partial}{\partial t^1} + \omega_1 \frac{\partial}{\partial \bar{\xi}_1} \\
\bar{L}_3 = \frac{\partial}{\partial \bar{z}^3} - iz^1 \frac{\partial}{\partial t^2} + \omega_2 \frac{\partial}{\partial \bar{\xi}_2} \\
\bar{L}_4 = \frac{\partial}{\partial \bar{z}^4} - iz^2 \frac{\partial}{\partial t^2} + \omega_3 \frac{\partial}{\partial \bar{\xi}_3}
\end{cases}
\]

give a basis of $T^{0,1} N$ at each point of $N$. According to our result, there exist smooth complex valued functions $\omega_1, \omega_2, \omega_3$, defined in a neighborhood $U$ of $0$ in $\mathbb{R}_1^2 \times \mathbb{C}^3$ such that $\bar{L}_k \omega_j = \bar{L}_j \omega_k$ for all $1 \leq j, k \leq 3$, but the system $\bar{L}_j u = \omega_j$ for $j = 1, 2, 3$ has no solution in a neighborhood of $0$. We define on $M = U \times C_\xi$ a $CR$ structure by requiring that the vector fields

\[
\begin{cases}
\bar{L}_1 = \frac{\partial}{\partial \bar{z}^1} - iz^2 \frac{\partial}{\partial t^1} - iz^3 \frac{\partial}{\partial t^2} + \omega_1 \frac{\partial}{\partial \bar{\xi}_1} \\
\bar{L}_2 = \frac{\partial}{\partial \bar{z}^2} - iz^1 \frac{\partial}{\partial t^1} + \omega_2 \frac{\partial}{\partial \bar{\xi}_2} \\
\bar{L}_3 = \frac{\partial}{\partial \bar{z}^3} - iz^1 \frac{\partial}{\partial t^2} + \omega_3 \frac{\partial}{\partial \bar{\xi}_3} \\
\bar{L}_4 = \frac{\partial}{\partial \bar{z}^4} - iz^2 \frac{\partial}{\partial t^2} + \omega_3 \frac{\partial}{\partial \bar{\xi}_3}
\end{cases}
\]
form a basis of $T^{0,1}M$ at each point of $M$. This $M$ gives an example of a $CR$
manifold, of type $(4,2)$, such that the Levi form has for each $\xi \in H^0M\setminus \{0\}$ at least
3 eigenvalues $\geq 0$ and 3 eigenvalues $\leq 0$ (actually one positive, one negative and
two zero eigenvalues), which according to [H1], [H2] is not locally $CR$ embeddable
near 0. For similar examples see also [HN2]

4. Let $S^5$ be the unit sphere in $\mathbb{C}^3$ and consider the manifold $M$ consisting of the
complex lines that are tangent to $S$. This is a $CR$ submanifold, of hypersurface
type, generically embedded into the complex 4-dimensional manifold $\tilde{M}$ consisting
of all complex lines of the three dimensional complex projective space $\mathbb{CP}^3$. The
$CR$ dimension of $M$ is 3 and its Levi form $L_\xi$ has at each nonzero $\xi \in H^0M$ one
positive, one negative and one zero eigenvalue. Thus for all $x_0 \in M$ the Poincaré
lemma is not valid at $x_0$ in degree 1.

5. Consider the real hypersurface $M$ in $\mathbb{C}^3$ described by:

$$M = \{3z^3 = z^1z^11 + (z^1 + z^1)^m z^2z^2 \},$$

where $m$ is an integer $\geq 2$. Then $\xi_0 = (d\Re z^3)_{(0,0,0)}$ is a regular point. Indeed,
with $\rho = 3z_3 - z^1z^11 - (z^1 + z^1)^m z^2z^2$, we have $\partial \rho = (i/2)dz^3$ constant, and hence
holomorphic, along the holomorphic curve $W = \{z_1 = 0, z_3 = 0\} \subset M$. Therefore
$V = \{(z,\xi) | z \in W, \xi = d\Re z^3/2 \} \subset H^0M$ is a holomorphic curve in $T^*\mathbb{C}^3$.
Since the Levi form $L_{\xi_0}$ has rank 1, we have in fact $T_{\xi_0}V = H_{\xi_0}H^0M$ because
$T_{\xi_0}V \subset H_{\xi_0}H^0M$ and they have the same dimension. The Levi form $L_{\xi_0}$ has
one positive and one zero eigenvalue. Then the Poincaré lemma fails in degree 1 at
$x_0 = (0,0,0)$. Note that in this case the rank of the Levi form is not maximal at
the regular point $\xi_0$, and that the rank of the Levi form $L_\xi$ is not constant in any
neighborhood of $\xi_0$ in $H^0M$.

6. Let $\chi : \mathbb{R} \to \mathbb{R}$ be equal to 0 for $t \leq 1$ and equal to $t \exp(1/(1-t))$ when $t > 1$.
Then $\chi$ is smooth and convex non decreasing. Let $q_1, q_2$ be positive integers, with
$q_1 \geq 2, q_2 \geq 1$, and consider the hypersurface in $\mathbb{C}^{q_1+q_2}$:

$$M = \{(z, w) \in \mathbb{C}^{q_1} \times \mathbb{C}^{q_2} | \|z\|^2 + \chi(\|w\|^2) = 1 \}.$$

Then $M$ is the boundary of a smooth bounded convex set and therefore all global
cohomology groups of the tangential Cauchy-Riemann complex $\mathcal{H}^p(M, \mathcal{Q}^{p,*}_M)$ are
zero for all $p = 0,\ldots, q_1 + q_2$ and $1 \leq j \leq q_1 + q_2 - 2$ (see e.g. [N2]). However,
for all points $x_0 = (z_0, w_0) \in M$ with $\|w_0\| < 1$, if $\xi_0 \in H^0_{x_0}\setminus \{0\}$ the Levi form
$L_{\xi_0}$ has $q_1 - 1$ eigenvalues of the same sign and the others equal to zero. Therefore
the local cohomology groups $\mathcal{H}^{q_1-1}(x_0, \mathcal{Q}^{p,*}_M)$ are infinite dimensional for all $p =
0,\ldots, q_1 + q_2$.

§7 Further remarks on the Poincaré Lemma

We get back to the general situation considered in §2. Assume that the complex
(2.1) admits the Poincaré lemma in dimension $q$ at some point $x_0 \in M$. Fix any
Riemannian metric $g$ on $M$ and denote by $B(x_0, r)$ the ball of center $x_0$ and radius
$r$ for the distance defined by $g$. By Theorem 2.1, for every $r > 0$ there exists some
$r' > 0$ such that (2.9) is valid with $\omega = B(x_0, r)$ and $\omega_1 = B(x_0, r')$. For each $r > 0$
denote by $\omega_1 (x)$ the supremum of these $\omega' > 0$. In case the Poincaré lemma is not
valid in dimension \( q \) at \( x_0 \), and (2.9) does not hold for \( \omega = B(x_0, r) \) and any open \( \emptyset \neq \omega_1 \subset \omega \), we set \( \kappa_q(r) = 0 \). We set:

\[
(7.1) \quad \nu_q^-(x_0) = \liminf_{r \downarrow 0} \frac{\log \kappa_q(r)}{\log r} \quad \text{and} \quad \nu_q^+(x_0) = \limsup_{r \downarrow 0} \frac{\log \kappa_q(r)}{\log r}.
\]

The values \( \nu_q^-(x_0) \leq \nu_q^+(x_0) \) can be either real numbers \( \geq 1 \) or \( +\infty \). [We are making the convention that \( \log 0 = -\infty \).] If \( d \) is the distance associated to the Riemannian metric \( g \) and \( d_1 \) the distance associated to another Riemannian metric \( g_1 \) on \( M \), then there exist constants \( C_1 > 0 \), \( C_2 > 0 \), \( r_0 > 0 \) such that:

\[
(7.2) \quad d(x_0, x) \leq C_1 d_1(x_0, x) \leq C_2 d(x_0, x) \quad \forall x \in M \quad \text{with} \quad d(x_0, x) < r_0.
\]

Therefore we obtain:

**Lemma 7.1** The numbers \( \nu_q^\pm(x_0) \) are independent of the Riemannian metric \( g \).

If there exist a Riemannian metric \( g \) and a real \( r_0 > 0 \) such that

\[
H^q(B(x_0, r), \mathcal{E}^*, A^*) = 0 \quad \text{for all} \quad 0 < r < r_0,
\]

then \( \nu_q^+(x_0) = \nu_q^-(x_0) = 1 \).

The small balls of a Riemannian metric can be considered as convex sets. Thus the condition that \( \nu_q^+(x_0) > 1 \) has the meaning that convexity is not sufficient for the vanishing of the cohomology, but a small open subset (if there is any) on which the cohomology vanishes in degree \( q \) needs to have a special shape.

Suppose that \( \nu_q^+(x_0) = 1 \). Then one could say that the cohomology vanishes asymptotically at \( x_0 \), in dimension \( q \). This occurs for example if \( \kappa_q(r) \approx cr \), with \( 0 < c < 1 \).

Suppose that the limit in \( \nu_q^-(x_0) \) is infinite. Then one could say that the Poincaré lemma fails asymptotically at \( x_0 \), in dimension \( q \). This occurs for example if \( \kappa_q(r) \approx C \exp(-a/r) \), for positive constants \( C \) and \( a \).

**Theorem 7.2.** Let \( M \) be a locally embeddable CR manifold of type \( (n, k) \). Let \( x_0 \in M \) and assume that there exists \( \xi \in H^n_{x_0} \) such that the Levi form \( \mathcal{L}_\xi \) has \( q \) positive eigenvalues and \( (n-q) \) eigenvalues which are \( \leq 0 \). Let \( g \) be any Riemannian metric on \( M \). Then there are constants \( r_0 > 0 \) and \( C > 0 \) such that, for every \( p = 0, 1, \ldots, n+k \), and \( 0 < r' \leq r \leq r_0 \), the maps:

\[
(7.3) \quad \mathcal{H}_{\overline{\partial}_M}^q(B(x_0, r), \mathcal{Q}^{p*}_M) \to \mathcal{H}_{\overline{\partial}_M}^q(B(x_0, r'), \mathcal{Q}^{p*}_M)
\]

induced by the restriction have infinite dimensional image if \( r' > Cr^{3/2} \). In particular \( \nu_q^-(x_0) \geq 3/2 \).

In fact a more general statement is valid, considering the \( \overline{\partial}_M \)-complex on currents. We have:

**Theorem 7.3.** Let \( M \) be a locally embeddable CR manifold of type \( (n, k) \). Let \( x_0 \in M \) and assume that there exists \( \xi \in H^n_{x_0} \) such that the Levi form \( \mathcal{L}_\xi \) has \( q \) positive eigenvalues and \( (n-q) \) eigenvalues which are \( \leq 0 \). Then there are constants \( r_0 > 0 \) and \( C > 0 \) such that, for every \( p = 0, 1, \ldots, n+k \), and \( 0 < r' \leq r \leq r_0 \), the maps:

\[
(7.4) \quad \mathcal{H}^q(B(x_0, r), \mathcal{O}^{p*}) \to \mathcal{H}^q(B(x_0, r'), \mathcal{O}^{p*})
\]
has an infinite dimensional image if \( r' > C r^{3/2} \).

We give first the proof of Theorem 7.2, then indicate the small changes needed to prove Theorem 7.3.

**Proof of Theorem 7.2**

Again by \([BHN1]\), assuming as we can that \( M \) is a generic CR submanifold of an open subset \( M \) of \( \mathbb{C}^{n+k} \), we know that (7.3) has an infinite dimensional image whenever it has a non zero image. Thus it will suffice to find conditions on \( 0 < r' \leq r \) that are necessary in order that (7.3) has zero image.

Therefore, we assume that \( x_0 = 0 \) and \( M \) is a closed generic CR submanifold of an open neighborhood \( M \) of 0 in \( \mathbb{C}^{n+k} \), where it is given by the equations:

(7.5) \[ s^j = h_j(\zeta, t) = 0(2) \quad \text{for} \quad j = 1, \ldots, k. \]

Here \( z^{n+j} = t^j + is^j \) with \( t^j, s^j \) real for \( j = 1, \ldots, k \) and \( \zeta = (z^1, \ldots, z^n) \).

We can also assume that

\[
 h_1(\zeta, t) = \sum_{\alpha=1}^{n} \epsilon_\alpha z^\alpha \bar{z}^\alpha + 3 \sum_{\alpha=1}^{n} \sum_{j=1}^{k} a_{\alpha,j} z^\alpha t^j + \sum_{j,\ell=1}^{k} b_{j,\ell} t^j t^\ell + 0(3),
\]

with \( \epsilon_\alpha = 1 \) for \( 1 \leq \alpha \leq q \) and \( \epsilon_\alpha \leq 0 \) if \( q < \alpha \leq n \). By substituting the complex variable \( z^{n+1} \) by \( z^{n+1} - \sum_{\alpha=1}^{n} \sum_{j=1}^{k} a_{\alpha,j} z^\alpha z^{n+j} - i \sum_{j,\ell=1}^{k} b_{j,\ell} z^{n+j} z^{n+\ell} \) we can then reduce to the case where

(7.6) \[ h_1(\zeta, t) = \sum_{\alpha=1}^{q} z^\alpha \bar{z}^\alpha + \sum_{\alpha=q+1}^{n} \epsilon_\alpha z^\alpha \bar{z}^\alpha + 0(3). \]

Let us fix a real number \( \nu > 1 + \max_{q+1 \leq \alpha \leq n} |\epsilon_\alpha| \) and set:

(7.7) \[ \phi(z) = \phi(\zeta, t) = -it^1 + h_1(\zeta, t) - \nu \sum_{\alpha=1}^{q} z^\alpha \bar{z}^\alpha - \nu \sum_{j=1}^{k} (t^j + ih_j(\zeta, t))^2 \]

(7.8) \[ f_\tau = e^{\frac{1}{\tau} \phi(z)} dz_1 \wedge \cdots \wedge dz^p \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q \]

and

(7.9) \[ \psi(z) = \psi(\zeta, t) = it^1 - h_1(\zeta, t) - \nu \sum_{\alpha=q+1}^{n} z^\alpha \bar{z}^\alpha - \nu \sum_{j=1}^{k} (t^j + ih_j(\zeta, t))^2 \]

(7.10) \[ g_\tau = e^{\frac{1}{\tau} \psi(z)} dz^{p+1} \wedge \cdots \wedge dz^{n+k} \wedge d\bar{z}^{q+1} \wedge \cdots \wedge d\bar{z}^n. \]

Clearly we have

(7.11) \[ \bar{\partial} f_\tau = 0, \quad \bar{\partial} g_\tau = 0. \]
We use the same cutoff function $\chi$ as in §5. If (7.3) has zero image, for all $R > 0$ with $R^{-1} < r'$, we have, by Theorem 2.3, an a priori estimate:

\[
\left| \int \chi(R\zeta, Rt) f_\tau \wedge g_\tau \right| 
\leq C \left( \sup_{|\zeta|^2 + |t|^2 \leq r^2} |D^\alpha f_\tau| \right) \cdot \left( \sup_{|\alpha| \leq m} |R(\bar{\partial}_M \chi)(R\zeta, Rt) \wedge g_\tau| \right)
\]

(7.12)

with constants $C$ and $m$ which depends on $r, r'$ but are independent of $\tau$.

Next we note that

\[
\phi(\zeta, t) + \psi(\zeta, t) = -\nu \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha - 2\nu \sum_{j=1}^{k} (t^j + ih_j(\zeta, t))^2.
\]

We have:

\[
\int \chi(R\zeta, Rt) f_\tau \wedge g_\tau
= (-1)^{(n+k-p)q} \tau^{n+\frac{k}{2}} \int \chi \left( R\sqrt{\tau}(\zeta, t) \right) \exp \left( -\nu \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha - 2\nu \sum_{j=1}^{k} (t^j + 0(\sqrt{\tau}))^2 \right) 
\times dz^1 \wedge \cdots \wedge dz^{n+k} \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n
\]

Therefore we obtain that

(7.13) \quad \tau^{-(n+\frac{k}{2})} \left| \int \chi(R\zeta, Rt) f_\tau \wedge g_\tau \right| \rightarrow \text{constant} > 0

for $\tau \to 0$.

Next we observe that we have an estimate of the form:

(7.14) \quad \sup_{|\zeta|^2 + |t|^2 \leq r^2} |D^\alpha f_\tau| \leq \exp \left( c_1 r^3 / \tau \right)

with a positive constant $c_1$ which is independent of $\tau$. Indeed the Taylor series of $\phi$ at 0 has a purely imaginary first degree term and a real second degree term which is $\leq 0$, and the factor involving $\tau^{-m}$ is absorbed by the constant $c_1$.

On the other hand we have, for positive constants $c_2, c_3$:

(7.15) \quad \sup_{|\zeta|^2 + |t|^2 \leq R^2} |R(\bar{\partial}_M \chi)(R\zeta, Rt) \wedge g_\tau| \leq c_3 \cdot R \cdot \exp \left( -c_2 R^{-2} / \tau \right).

Indeed the Taylor series of $\psi$ at 0 has a purely imaginary first order term and a real second order term which is negative definite, while the form in the left hand side is different from zero only for $|\zeta|^2 + |t|^2 > R^{-2}/2$.

Therefore

\[
\sup_{R > 0} \left| g_\tau \right| \leq c_2 r^3 \quad \forall R > 0 \quad \text{with} \quad Rr' > 1.
\]
Hence \( r' \leq \sqrt{c_1/c_2} \cdot r^{3/2} \). This completes the proof of the theorem.

**Proof of Theorem 7.3**

The only changes needed to obtain Theorem 7.3 are in formulas (7.12) and (7.15). Both remain valid when instead of the sup norm of \( \bar{\partial} M \) we need to introduce the sup norm of its derivatives up to some finite order \( m_1 \geq 0 \). In view of Theorem 2.5, we obtain the statement of Theorem 7.3.

**Example** Let us consider, for an integer \( m \geq 2 \), the \( CR \) manifold \( M \) of type \((2,1)\):

\[
M = \{ (z^1, z^2, z^3) \in \mathbb{C}^3 \mid |z^1|^2 + |z^2|^2 + |z^3|^{2m} = 1 \}.
\]

The Levi form is definite and nondegenerate at all points of \( M \) where \( z^3 \neq 0 \). But at the points of \( S = \{ |z^1|^2 + |z^2|^2 = 1, z^3 = 0 \} \subset M \) it is degenerate with one zero and one non zero eigenvalue. The Poincaré lemma in degree 1 is valid at all points \( x \in M \setminus S \) (see [N2] or [AH1],[AH2]), but we have \( \nu_1^- (x) \geq 3/2 \) at all points \( x \) of \( S \). In particular the cohomology in degree 1 of the intersection of \( M \) with a small Euclidean ball centered at \( x \in S \) is always infinite dimensional. And the same is true if, instead of small Euclidean balls, one uses small balls in any Euclidean metric. Also, like in Example 6, all of the global cohomology groups in degree 1 are zero, because \( M \) is the smooth boundary of a weakly pseudoconvex domain (see [BHN2]).

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