Approximate Shannon Sampling in Importance Sampling: Nearly Consistent Finite Particle Estimates

Alec Koppel · Amrit Singh Bedi · Víctor Elvira · Brian M. Sadler

Received: date / Accepted: date

Abstract In Bayesian inference, we seek to compute information about random variables such as moments or quantiles on the basis of data and prior information. When the distribution of random variables is complicated, Monte Carlo (MC) sampling is usually required. Importance sampling is a standard MC tool for addressing this problem: one generates a collection of samples according to an importance distribution, computes their contribution to an unnormalized density, i.e., the importance weight, and then sums the result followed by normalization. This procedure is asymptotically consistent as the number of MC samples, and hence deltas (particles) that parameterize the density estimate, go to infinity. However, retaining infinitely many particles is intractable. Thus, we propose a scheme for only keeping a finite representative subset of particles and their augmented importance weights that is nearly consistent.

To do so in an online manner, we approximate importance sampling in two ways: we (1) replace the deltas by kernels, yielding kernel density estimates (KDEs); (2) and sequentially project KDEs onto nearby lower-dimensional subspaces. Theoretically, we characterize the asymptotic bias of this scheme as determined by a compression parameter and kernel bandwidth, which yields a tunable tradeoff between consistency and memory. In experiments, we observe a favorable tradeoff between memory and accuracy, providing for the first time near-consistent compressions of arbitrary posterior distributions.

Importance sampling (IS) is a Monte Carlo (MC) methodology used for the stochastic approximation of intractable integrals [40]. IS constitutes a natural alternative to Markov chain Monte Carlo (MCMC) methods due to its advantages [3], e.g., there is not burn-in period, the parallelization is simpler [11], there is a for-free approximation of the normalizing constant that turns out useful in many practical problems (e.g., model selection), and adaptive mechanisms can be proposed without compromising the convergence of the method [10].

The recent interest in IS techniques has run in parallel with the explosion of Bayesian inference modeling and inference in interrelated areas such as statistics [39], signal processing [11], machine learning [33], and countless applied sciences. In this paper, and without loss of generality, we focus on the Bayesian inference problem where one seeks to compute distributional information about random variables such as moments or quantiles on the basis of a prior [33]. Typically, these random variables describe an unknown state or parameter, which can be characterized by a measurement model and a prior distribution. This problem appears in a wide plethora of applications such as genet-
ics [6], communications [25], econometrics [23], robotics [58], among many other examples.

In few statistical models, exact Bayesian inference is possible. For instance, under linear dynamics with additive Gaussian noise, one may track the mean of the hidden state via Kalman filters [24]. When one hypothesizes a latent Gaussian structure to the unknown transition density, Gaussian processes may be used to track more generic quantities [38]. Unfortunately, when the statistical models include nonlinearities and/or the Gaussian (and additive) assumptions break down, the aforementioned approaches no longer apply. In such cases, one may use variational approaches which, while intractable in general, in special cases amount to global optimization [20]. Monte Carlo methods and more particularly IS-based techniques are an asymptotically exact and computationally feasible alternative.

More specifically, based upon independent samples simulated from a proposal distribution, IS methods approximately compute expectations of arbitrary functions of the unknown parameter via weighted samples generated from the proposal. Several efforts have been devoted to implementing adaptive schemes where the proposal is iteratively improved in order to improve the quality of the IS estimators [10]. Other efforts have been devoted to investigating the number of required samples in IS in order to ensure accurate approximations [11] or to develop statistics to diagnose the quality of the estimators [10,16]. However, in all cases, the IS approach yields costly representations of empirical quantities in optimization, which has been exploited recently to surmount memory challenges in the kernel and Gaussian process regression [27,28].

Consequently, we are able to establish the asymptotic bias of this method is a tunable constant depending on the kernel bandwidth parameter and a compression parameter. These results yield an approach to importance reweighting that mitigates particle degeneracy, i.e., retaining a large number of particles with small weights [30], by directly compressing the posterior approximation, rather than statistical tests that require sub-sampling [10,16]. The compression is performed online, without waiting until the total number of samples $N$ are available, which is typically impractical. Experiments then demonstrate that this approach yields an effective tradeoff of consistency and memory, in contrast to the classical curse of dimensionality of MC methods.

1 Elements of Importance Sampling

In Bayesian inference [13] (Ch. 7), we are interested in computing expectations

$$I(\phi) = \mathbb{E}_x[\phi(x) \mid y] = \int_{x \in \mathcal{X}} \phi(x)p(x \mid y)dx \quad (1)$$

on the basis of a set of available observations $\{y_k\}_{k \leq K}$, where $\phi : \mathbb{R}^p \rightarrow \mathbb{R}$ is an arbitrary function, $x$ is a random variable taking values in $\mathcal{X} \subset \mathbb{R}^p$ which is typically interpreted as the hidden parameter, and $y$ is some observation process whose realizations $y_k$ are assumed to be informative about parameter $x$. For example, $\phi(x) = x$ yields the computation of the posterior mean, and $\phi(x) = x^p$ denotes the $p$-th moment. In particular, define the posterior distribution

$$P(\{x \mid \{y_k\}_{k \leq K}\} = \frac{P(\{y_k\}_{k \leq K} \mid x) \cdot P(x)}{P(\{y_k\}_{k \leq K})}. \quad (2)$$

We seek to infer the posterior $\{\}$ with $K$ data points $\{y_k\}_{k \leq K}$ available at the outset. Even for this setting, estimating $\{\}$ has unbounded complexity [29,51] when the posterior is unknown. Thus, we prioritize efficient estimates of $\{\}$ from an online stream of samples from an importance distribution to be subsequently defined. Begin by defining posterior $q(x)$ and un-normalized posterior $\widetilde{q}(x)$:

$$q(x) = \tilde{q}(x) / Z,$$

and

$$\widetilde{q}(x) \equiv \tilde{q}(x \mid y) = \mathbb{P}(\{y_k\}_{k \leq K} \mid x) \cdot P(x)$$
is a non-negative function proportional to posterior distribution $q(x|y)\equiv q(x) = \mathbb{P}(x|y)$ that integrates to normalizing constant $Z \equiv \mathbb{P}\{\{y_k\}_{k=1}^K\}$.

In most applications, we only have access to a measurement model $\mathbb{P}\{\{y_k\}_{k=1}^K | x\}$ for a collection of observations $\{y_k\}$ drawn from a static conditional distribution $\mathbb{P}(y| x)$ and a prior for $\mathbb{P}(x)$. Therefore, the integral (1) cannot be evaluated, and hence one must resort to importance sampling.

In Monte Carlo, we approximate (1) by sampling. Hypothetically, we could draw $N$ (not necessarily equal to $K$) samples $x^{(n)} \sim q(x)$ and estimate the expectation in (1) by the sample average

$$E_q[x]\{\phi(x)\} \approx \frac{1}{N} \sum_{n=1}^{N} \phi(x^{(n)}),$$

but typically it is difficult to obtain samples $x^{(n)}$ from posterior $q(x)$ of the hidden state. To circumvent this issue, define the importance distribution $\pi(x)$ with the same (or larger) support as true density $q(x)$, and multiply and divide by $\pi(x)$ inside the integral (1):

$$\int_{x \in X} \phi(x)q(x)dx = \int_{x \in X} \phi(x)\frac{q(x)}{\pi(x)} \pi(x)dx,$$

where the ratio $q(x)/\pi(x)$ is the Radon-Nikodym derivative, or unnormalized density, of the target $q$ with respect to the proposal $\pi$. Then, rather than requiring samples from true posterior $x^{(n)} \sim q(x)$, one may sample from importance distribution $x^{(n)} \sim \pi(x)$, $n = 1, ..., N$, and approximate (1) as

$$\hat{I}_N(\phi) \equiv \frac{1}{N} \sum_{n=1}^{N} q(x^{(n)}) \phi(x^{(n)})$$

$$= \frac{1}{NZ} \sum_{n=1}^{N} g(x^{(n)})\phi(x^{(n)}),$$

where $g(x^{(n)}) \equiv \frac{q(x^{(n)})}{\pi(x^{(n)})}$ are the importance weights. We note that in practice, we cannot calculate $g(x^{(n)})$ since the target distribution $q(x)$ is unknown and hence we calculate it using Bayes rule as follows:

$$q(x^{(n)}) = \frac{\mathbb{P}\{\{y_k\}_{k=1}^K | x^{(n)}\} \mathbb{P}(x^{(n)})}{\int \mathbb{P}\{\{y_k\}_{k=1}^K | x\} \mathbb{P}(x) dx}.$$

1. Note that $q(x)$ and $\hat{q}(x)$ depend on the data $\{y_k\}_{k=1}^K$, although we drop the dependence to ease notation.

2. In general, the importance distribution could be defined over any observation process $\pi(x \mid \{y_k\})$, not necessarily associated with time indices $k = 1, \ldots, K$. We define it this way for simplicity.

**Algorithm 1 IS: Importance Sampling with streaming samples**

**Require:** Observation model $\mathbb{P}(y \mid x)$ and prior $\mathbb{P}(x)$ or target distribution $q(x)$ (if known), importance distribution $\pi(x)$. Set of observations $\{y_k\}_{k=1}^K$ for $n = 0, 1, 2, \ldots$ do

Compute weight $g(x^{(n)})$ [cf. (4)] if $q(x)$ is known; else use (3).

Compute normalized weights $\tilde{w}^{(n)}$ by dividing by estimate for summand (6):

$$\tilde{w}^{(n)} = \frac{g(x^{(n)})}{\sum_{u=1}^{n} g(x^{(u)})}$$

for all $n$.

Estimate the expectation with the self-normalized IS as

$$\hat{I}_N(\phi) = \sum_{u=1}^{n} \tilde{w}^{(u)} \phi(x^{(u)})$$

The posterior distribution estimate is given by

$$\mu_n = \sum_{u=1}^{n} \tilde{w}^{(u)} \delta_{x^{(u)}}.$$

end for

Substituting (6) into (4), we obtain

$$g(x^{(n)}) \equiv \frac{\mathbb{P}\{\{y_k\}_{k=1}^K | x^{(n)}\} \mathbb{P}(x^{(n)})}{\pi(x^{(n)})}.$$

Note that (5) is unbiased, i.e., $E_{\pi(x)}[\hat{I}_N(\phi)] = E_{q(x)}[\phi(x)]$ and consistent with $N$. Moreover, its variance depends on how well the importance density $\pi(x)$ approximates the posterior [15].

Example priors and measurement models include Gaussian, Student’s t, and Uniform. Which one is appropriate depends on the context [13]. The normalizing constant $Z$ can be also estimated with IS as

$$Z \equiv \frac{1}{N} \sum_{n=1}^{N} g(x^{(n)}).$$

Note that in Eq. (5), the unknown $Z$ can be replaced by $\tilde{Z}$ in (3) Then, the new estimator is given by

$$\hat{I}_N(\phi) := \frac{1}{NZ} \sum_{n=1}^{N} g(x^{(n)})\phi(x^{(n)})$$

$$= \frac{1}{\sum_{j=1}^{N} g(x^{(j)})} \sum_{n=1}^{N} g(x^{(n)})\phi(x^{(n)}) = \sum_{n=1}^{N} w^{(n)}\phi(x^{(n)}),$$

where the “self-normalized” $w^{(n)}$ weights are defined

$$w^{(n)} \equiv \frac{g(x^{(n)})}{\sum_{u=1}^{n} g(x^{(u)})}.$$

The whole IS procedure is summarized in Algorithm 1. The estimator $\hat{I}_N(\phi)$ is the self-normalized importance
sampling (SNIS) estimator. It is important to note that the estimator \( J_N(\phi) \) can be viewed as integrating a function \( \phi \) with respect to distribution \( \mu_N \) defined as

\[
\mu_N(x) \equiv \sum_{n=1}^{N} w_n^{(n)} \delta_{x^{(n)}},
\]

(11)

which is called the particle approximation of \( q \) where \( \delta_{x^{(n)}} \) denotes the discrete Dirac delta (indicator) which is 1 if \( x = x^{(n)} \) and null otherwise. This delta expansion is one reason importance sampling is also referred to as histogram filters, as they quantify weighted counts of samples across the space.

As stated in [1], for consistent estimates of (1), we require that \( N \), the number of samples \( x^n \) generated from the importance distribution, and hence the parameterization of the importance distribution, to go to infinity \( N \to \infty \). Therefore, when we generate an infinite stream of particles, the parameterization of the importance density grows unbounded as it accumulates every particle previously generated. We are interested in allowing \( N \), the number of particles, to become large (possibly infinite), while the importance distribution’s complexity is moderate, thus overcoming an instance of the curse of dimensionality in Monte Carlo methods. In the next section, we propose a method to do so.

2 Compressing the Importance Distribution

The curse of dimensionality in importance sampling becomes apparent when we rewrite the last step of Algorithm 1 in vector notation. Specifically, define \( g_n \in \mathbb{R}^n \), \( w_n \in \mathbb{R}^n \) and \( X_n = [x^{(1)}; \ldots; x^{(n)}] \in \mathbb{R}^{p \times n} \). Then, after each new sample \( x^{(n)} \) is generated from the importance distribution, we incorporate it into the empirical measure (11) through the parameter updates

\[
g_n = [g_{n-1}; g(x^{(n)})],
\]

\[
w_n = z_n g_n, \quad X_n = [X_{n-1}; x^{(n)}]
\]

(12)

where we define \( z_n := 1/(1^T_n g_n) \) where \( 1_n \) is all one column vector with dimension \( n \). The unnormalized posterior distribution estimate parameterized by \( g_n \) and \( X_n \) is given by

\[
\bar{\mu}_n = \sum_{u=1}^{n} w_u^{(n)} \delta_{x^{(u)}}.
\]

(13)

The question is how to select a subset of columns of \( X_n \) and augment the weights \( g_n \) such that with an infinite stream of \( x^{(n)} \), we ensure the number of columns of \( X_n \) is finite and the empirical measure tends to its population counterpart, i.e., satisfies asymptotic posterior consistency [21]. Henceforth, we refer to the number of columns in matrix \( X \) which parameterizes (13) as the model order.

Greedy Subspace Projections

In this subsection, we design an algorithm to do so based Euclidean projections [42]. Suppose that instead of parameterizing the posterior distribution estimate \( \mu_{n-1} \) after \( n-1 \) samples by weights \( g_{n-1} \) and particle collection (or dictionary) \( X_{n-1} \) with \( n-1 \) columns as in (12), we instead parameterize it by some other dictionary \( D_{n-1} \) with \( M_{n-1} \) columns. Then, according to Algorithm 1 specifically (12), its weight and dictionary, are updated as

\[
\tilde{g}_n = [g_{n-1}; g(x^{(n)})], \quad \tilde{w}_n = z_n \tilde{g}_n, \quad \tilde{D}_n = [X_{n-1}; x^{(n)}].
\]

(14)

where we recall \( g(x^{(n)}) \) is the importance weight (5) and \( \tilde{\delta}_{x^{(n)}} \) is the Dirac delta function, both evaluated at sample \( x^{(n)} \). For convenience we define the delta function stacked across samples as \( \tilde{\delta}_{D} = \delta_{d_1}, \ldots, \delta_{d_{M}} \), and its outer product as the covariate matrix \( P_{D,D} := \delta_{D} \delta_{D}^T \), the matrix from this collection of particles. To select the subset of columns of \( \tilde{D}_n \) to formulate its compressed variant \( D_n \), we project the integral estimates onto subspaces \( H_D \) that consist only of densities that can be represented using dictionary \( D = [d_1, \ldots, d_M] \in \mathbb{R}^{p \times M} \), i.e., \( H_D = \{ y : y = \sum_{m=1}^{M} w_m \delta_{d_m} = w^T \delta_{D} \} \). We enforce efficiency by selecting dictionaries \( D \) such that \( M_n \ll n \).

We note, by selecting \( D = X_n \) at each iteration, the true density estimate sequence (12) can be viewed as
executing a sequence of subspace projections. In particular, for the unnormalized version of the posterior density \( \hat{\mu}_n \) we can write

\[
\hat{\mu}_n = \arg\min_{y \in \mathcal{H}} \left\| y - \left( \hat{\mu}_{n-1} + g(x^{(n)}) \delta_{X^{(n)}} \right) \right\|^2,
\]

where the equality comes from observing that \( \hat{\mu}_n \) can be represented using only the points \( X_n \), using (12). Notice that (15) expresses \( \hat{\mu}_n \) as the projection of \( \mathcal{D}_{\hat{\mu}_n} \) onto the subspace \( \mathcal{H}_{\hat{\mu}_n} \) defined by \( X_n \).

Rather than select dictionary \( \mathcal{D} = X_n \), we propose instead to select a different dictionary, \( \mathcal{D} = \mathcal{D}_n \), which is extracted from the particles observed thus far, at each iteration. The process by which we select \( \mathcal{D}_n \) will be discussed next, but is of dimension \( p \times M_n \), with \( M_n \ll n \). As a result, we generate an estimate sequence \( \tilde{I}_n \) that differs from the estimate sequence and is instead parameterized by dictionary \( \mathcal{D}_n \) and importance weights \( w_n \).

To be precise, we replace the update (15) in which the number of particles grows at each step by the subspace projection of the empirical unnormalized density estimate sequence onto subspace \( \mathcal{H}_{\mathcal{D}_n} \):

\[
\tilde{\mu}_n = \arg\min_{y \in \mathcal{H}_{\mathcal{D}_n}} \left\| y - \left( \tilde{\mu}_{n-1} + g(x^{(n)}) \delta_{X^{(n)}} \right) \right\|^2.
\]

In terms of the parameter space of coefficients (importance weights) only. In order to do so, we first define the density approximation update \( \tilde{\mu}_n \) without projection, given density estimate \( \mu_{n-1} \) as

\[
\tilde{\mu}_n = \mu_{n-1} + g(x^{(n)}) \delta_{X^{(n)}}.
\]

where the projection \( \mathcal{D}_n \) is defined in (14) as the data dictionary with the newest sample added to the previous one \( \mathcal{D}_{n-1} \), and \( g_n \) as its associated weights, which are used to construct \( \tilde{\mu}_n \). Note that \( \mathcal{D}_n \) has \( M_n = M_{n-1} + 1 \) columns, also the length of \( g_n \).

For a fixed dictionary \( \mathcal{D}_n \), the projection in (16) amounts to a least-squares problem on the importance weights. To see this, apply the equality in (9) to rewrite (16) in terms of the density expansions, and note that the coefficient vector is the only free parameter to write

\[
\arg\min_{\mathcal{D} \in \mathbb{R}^{p \times M_n}} \left\| \sum_{s=1}^{M_n} g_s \delta_{s} - \sum_{u=1}^{M_n} \bar{g}_u \delta_{u} \right\|^2
\]

where we expand the square and define the cross-delta matrix \( \mathcal{P}_{\mathcal{D}_n} \) whose \((s,u)\)th entry is given by \( \delta_{s} \delta_{u} \).

The other matrices \( \mathcal{P}_{\mathcal{D}_n}, \mathcal{D}_n \) and \( \mathcal{P}_{\mathcal{D}_n}, \mathcal{D}_n \) are similarly defined. Note that \( M_n \) is the number of columns in \( \mathcal{D}_n \), while \( M_n = M_{n-1} + 1 \) is the number of columns in \( \mathcal{D}_n \).

**Dictionary Update** The update procedure for the dictionary \( \mathcal{D}_{n+1} \) is based upon greedy sparse approximation, a topic studied in compressive sensing [34].

The function \( \tilde{I}_n = z_n I_{n-1} + z_n g(x^{(n)}) \delta_{X^{(n)}} \) defined by numerical integration of the latest Monte Carlo particles without projection is parameterized by dictionary \( \mathcal{D}_n \) [cf. (14)], whose model order is \( M_n = M_{n-1} + 1 \). We form \( \mathcal{D}_n \) by selecting a subset of \( M_n \) columns from \( \mathcal{D}_n \) that are best for approximating \( \tilde{I}_n \) in terms of integral approximation error. As previously noted, numerous approaches are possible for sparse representation. We make use of destructive orthogonal matching pursuit (OMP) [54] with allowed error tolerance \( \epsilon_n \) to find a dictionary matrix \( \mathcal{D}_n \) based on the one which adds the latest sample point \( \mathcal{D}_n \). This choice is due to the fact that we can tune its stopping criterion to guarantee the per-step behavior of the estimates is nearly unchanged, and hence in the limit is nearly consistent, while ensuring the model order of the learned density remains finite. The detailed steps for KOMP algorithm are summarized in Algorithm 3.

**Kernel Smoothing** The importance sampling forms an empirical distribution for the conditional distribution \( \mathcal{P}(x | y) \) based on observations \( \{y_n\} \). As the number of observations and particles jointly approach infinity, this expansion of deltas [cf. (11)] exactly approaches the population posterior. However, since deltas have no “volume,” there is no hope of finding a finite dimensional approximation based on samples that is nearly consistent.

Specifically, when \( \mathcal{X} \) is compact, a finite number of Euclidean balls may be used to cover the space, known as its covering number (its logarithm, a quantity studied in information theory, is called the metric entropy [12]). For the covering number of a compact set to be
Algorithm 3 Kernel Orthogonal Matching Pursuit (KOMP)

Require: distribution $\hat{\mu}$ defined by dict. $\mathbf{D} \in \mathbb{R}^{p \times M}$, coeffs. $\mathbf{\hat{g}} \in \mathbb{R}^{M}$, approx. budget $\epsilon > 0$
initialize $\mu = \hat{\mu}$, dictionary $\mathbf{D} = \mathbf{\hat{D}}$ with indices $\mathcal{I}$, model order $M = \hat{M}$, coeffs. $\mathbf{g} = \mathbf{\hat{g}}$
while candidate dictionary is non-empty $\mathcal{I} \neq \emptyset$ do
    for $j = 1, \ldots, \hat{M}$ do
        Find minimal approximation error with dictionary element $\mathbf{d}_j$ removed
        $\gamma_j = \min_{\mathbf{w} \in \mathbb{R}^{\mathcal{I}} \setminus \{j\}} \| \hat{\mu}(\cdot) - \sum_{k \in \mathcal{I} \setminus \{j\}} \mathbf{g}_k \mathbf{\kappa}(\mathbf{d}_k, \cdot) \|_H$
    end for
    Find dictionary index minimizing approximation error: $j^* = \arg \min_{j \in \mathcal{I}} \gamma_j$
    if minimal approximation error exceeds threshold $\gamma_{j^*} > \epsilon$
        stop
    else
        Prune dictionary $\mathbf{D} \leftarrow \mathbf{D} \setminus \{j^*\}$, remove both the columns associated with index $j^*$
        Revise set $\mathcal{I} \leftarrow \mathcal{I} \setminus \{j^*\}$ and model order $M \leftarrow M - 1$
    end if
Update weights $\mathbf{g}$ defined by current dictionary $\mathbf{D}$
$\mathbf{g} = \arg \min_{\mathbf{g} \in \mathbb{R}^{M}} \| \hat{\mu}(\cdot) - \mathbf{w}^T \mathbf{\kappa}(\cdot) \|_H$
end while
return $\mu, \mathbf{D}, \mathbf{g}$ of complexity $M \leq \hat{M}$ s.t. $\| \mu - \hat{\mu} \|_H \leq \epsilon$

3 Balancing Consistency and Memory

Supervised Learning In supervised learning, we are given a bunch of feature vectors and learn a statistical model that maps features to targets, e.g., binary labels. This task can be cast as a stochastic optimization problem whose objective quantifies model fit averaged over training examples. Since the data distribution is unknown, numerical optimization schemes must interact with statistical sampling procedures in order to attain good performance. Recently, use of importance distributions to weight updates, e.g., coordinate [2,13] or stochastic gradient [9] descent, have been developed. Doing so yields faster deep network training by weighting mini-batches. While efforts to have the importance distribution co-evolve with learning update rules have been proposed [47], they limit its representation to the current mini-batch, which ensures low-complexity representations at the cost of discarding prior information. In contrast, our method would allow weighting based on any particle history, not just a mini-batch.

Reinforcement Learning (RL). In reinforcement learning, [43], an autonomous agent moving through a space, selects actions, after which a reward is revealed and a random transition to a new state occurs. The agent’s goal is to learn a mapping from states to actions, i.e., a policy, that accumulates the most long-term rewards. Typically, an agent chooses actions according to a policy and then updates the policy via rewards observed [57]; however, this theoretically requires an inordinate amount of random actions to be chosen before reasonable performance is learned [33,49], an issue known as the explore-exploit tradeoff. To lessen its deleterious effect, exploratory actions may be chosen via an importance distribution [44] or policy updates may be chosen from previous experience known to be safe [37]. Use of importance sampling to overcome explore-exploit is an active area of research [50,32]. The methods developed here would permit off-policy policy evaluation or prioritized reply using generic posteriors, rather than Gaussian policies or uniform weightings.
where \( N \) is the number of particles. To establish required consistency, we first details the technical conditions required as follows.

3.1 Assumptions and Technicalities

**Assumption 1** Recall the definition of the target distribution \( q \) from Sec. 2 following (2). Denote the integral of test function \( \phi: \mathcal{X} \rightarrow \mathbb{R} \) as \( q(\phi) \).

(i) Assume that \( \phi \) is absolutely integrable, i.e., \( \phi q(|\phi|) < \infty \), and has absolute value at most unit \( |\phi| \leq 1 \).

(ii) The test function has absolutely continuous second derivative, and \( \int_{\mathbb{X}} \phi''(x) dx < \infty \).

**Assumption 2** The kernel function in (20) is chosen such that \( \int_{\mathbb{X}} k_{\phi}(x) = 1 \), \( \int_{\mathbb{X}} x k_{\phi}(x) = 0 \), and \( \sigma^2_k = \int_{\mathbb{X}} x^2 k_{\phi}(x) > 0 \).

**Assumption 3** Denote as \( \hat{m} \) and \( \bar{m} \) the mean embeddings of distributions \( \hat{\rho}, \bar{\rho} \in \mathcal{H} \) in an RKHS (20)

\[
\hat{m} = \mathbb{E}_x[k_{\phi}(x)] \quad \text{and} \quad \bar{m} = \mathbb{E}_x[k_{\phi}(x)].
\]

The RKHS norm between full distributions lower-bounds the distance between their mean embeddings: \( ||\hat{\rho} - \bar{\rho}||_{\mathcal{H}} \leq ||\hat{m} - \bar{m}||_{\mathcal{H}} \), which are related by a multiplicative factor \( ||\hat{\rho} - \bar{\rho}||_{\mathcal{H}} = K ||\hat{m} - \bar{m}||_{\mathcal{H}} \).

Assumption [1] is a textbook condition in the analysis of Monte Carlo methods, and appears in [35]. Assumptions [2] and [3] are required conditions for establishing consistency of kernel density estimates and are standard — see [58] [Theorem 6.28]. Assumption [3] is reasonable considering that the distribution is completely characterized by its moments [14]. Hence if the full distribution is close according to some metric, then the moments are close — see [45] [eqn. (14)]. This assumption formalizes this statement for means and metrics in RKHSs [46].

We begin by noting that under Assumption [1] we have classical statistical consistency of importance sampling as the number of particles becomes large as stated in Lemma [1] in Appendix A. This result permits characterizing the bias of Algorithm 2 given next.

**Theorem 1** Define the second moment of the true unnormalized density \( \rho \) as in Lemma [7]. Then, under Assumptions [1-3] (in Appendix [7.7]), the estimator of Alg. 3 exhibits posterior contraction:

\[
\sup_{|\phi| \leq 1} \left( \mathbb{E}[\hat{I}_N(\phi) - I(\phi)] \right) \leq \mathcal{O} \left( \frac{\epsilon + \sigma^2_k h^2 + \frac{1}{\sqrt{Nh}} + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) + h^3}{h} \right) + \frac{24}{N} \rho.
\]

and hence, as \( N \rightarrow \infty \), is consistent when compression budget and bandwidth go to null \( \epsilon, h \rightarrow 0 \).

**Proof:** Inspired by [1], begin by denoting \( \hat{I}_N(\phi) \) as the integral estimate given by Algorithm 2 when we use kernel smoothing with bandwidth parameter \( h \). Consider the bias of the integral estimate \( \hat{I}_N(\phi) - I(\phi) \), and add and subtract \( \hat{I}_N(\phi) \), the uncompressed kernelized self-normalized importance estimator, as well as \( I_N(\phi) \) the result of Algorithm 1, i.e., the SNIS estimator with no smoothing.

\[
\hat{I}_N(\phi) - I(\phi) = \hat{I}_N(\phi) - I_N(\phi) + \hat{I}_N(\phi) - I_N(\phi) + I_N(\phi) - I(\phi).
\]

Take the expectation on both sides with respect to the population posterior (2) to obtain

\[
\mathbb{E}[\hat{I}_N(\phi) - I(\phi)] = \mathbb{E}[\hat{I}_N(\phi) - I_N(\phi)] + \mathbb{E}[I_N(\phi) - I(\phi)] + 2\mathbb{E}[I_N(\phi) - I(\phi)]
\]

(22)

where we add and subtract \( \mathbb{E}[I(\phi)] \) inside the second-to-last term and group like terms.

Let’s compute the sup of both sides of (23) over range \( |\phi| \leq 1 \) followed by fact that a sup of a sum is upper-bounded by the sum of individual terms:

\[
\sup_{|\phi| \leq 1} \left( \mathbb{E}[\hat{I}_N(\phi) - I(\phi)] \right) \leq \sup_{|\phi| \leq 1} \left( \mathbb{E}[\hat{I}_N(\phi) - \hat{I}_N(\phi)] \right) + \sup_{|\phi| \leq 1} \left( \mathbb{E}[\hat{I}_N(\phi) - I(\phi)] \right) + 2\sup_{|\phi| \leq 1} \left( \mathbb{E}[I_N(\phi) - I(\phi)] \right).
\]

(24)

Now, compute the absolute value of both sides of (24), and to the last two terms, pull the absolute value inside the supremum, i.e., Then, (21) (Lemma 1) applied to the last term yields:

\[
\sup_{|\phi| \leq 1} \left( \mathbb{E}[\hat{I}_N(\phi) - I(\phi)] \right) \leq \sup_{|\phi| \leq N} \left( \mathbb{E}[\hat{I}_N(\phi) - I_N(\phi)] \right) + \sup_{|\phi| \leq 1} \left( \mathbb{E}[I_N(\phi) - I(\phi)] \right) + \frac{24}{N} \rho
\]

(25)

Thus, it remains to characterize the first two terms on the right-hand side of (25). We first focus on the second term \( \mathbb{E}[\hat{I}_N(\phi) - I_N(\phi)] \), which is the bias due to kernel smoothing. [36] [Theorem 6.28] establishes this bias as

\[
\mathbb{E}[\hat{I}_N(\phi) - I(\phi)]^2 = \frac{1}{4} \sigma^4_k h^4 \phi''(x) + \frac{1}{hN} + \mathcal{O} \left( \frac{1}{N} \right) + \mathcal{O}(h^6).
\]

(26)

Apply Jensen’s inequality to pull the square outside the expectation, and compute the square root of both sides
to obtain an upper-estimate of the bias of $|\mathbb{E}[\hat{I}_N(\phi) - I(\phi)]|$: 

$$|\mathbb{E}[\hat{I}_N(\phi) - I(\phi)]| \lesssim O \left( \sigma^2 h^2 + \frac{1}{\sqrt{Nh}} + O\left( \frac{1}{\sqrt{N}} \right) + h^3 \right).$$  

(27)

Now, we return to the first term on the right-hand side of (25), which is the integral approximation error associated using the compressed kernel density rather than the full one. This expression is exactly the absolute value of the Maximum Mean Discrepancy (MMD) [19, 31].

$$\left| \mathbb{E}[\hat{I}_N(\phi) - I(\phi)] \right| = \| \text{MMD}(\hat{\pi}_N, \hat{\pi}_N^\epsilon) \|$$  

(28)

where $\hat{\pi}_N^\epsilon$ is the occupancy measure (similarly defined to (34)) associated with Algorithm 2, i.e., with compression, whereas $\hat{\pi}_N$ is its uncompressed counterpart. Notably, [22, Lemma 4] establishes the equivalence of the MMD and the RKHS norm between the mean embeddings $\hat{\mu}_N$ and $\hat{\mu}_N$, respectively, which are simply the averages of their current dictionaries:

$$\hat{\mu}_N = \frac{1}{M} \sum_{u=1}^{M} \kappa_{d(u)}(\cdot), \quad \hat{\mu}_N = \frac{1}{N} \sum_{n=1}^{N} \kappa_{x(n)}(\cdot)$$

Substituting the preceding expression into (28), and applying [22, Lemma 4] yields

$$\text{MMD}(\hat{\pi}_N^\epsilon, \hat{\pi}_N) = \left\| \frac{1}{M} \sum_{u=1}^{M} \kappa_{d(u)}(\cdot) - \frac{1}{N} \sum_{n=1}^{N} \kappa_{x(n)}(\cdot) \right\|_H$$  

(29)

which, given the full distributions $\hat{\mu}_N$ and $\hat{\mu}_N$, are $e$-close with respect to the RKHS norm, by Assumption 3, their mean embeddings are $e$-close with respect to the RKHS norm as well (up to a multiplicative factor $K$). Specifically, taking (25), substituting the bias due to kernel smoothing, and the error caused by compression from (28)-(29), we obtain

$$\left| \mathbb{E}[\hat{I}_N(\phi) - I(\phi)] \right| \leq O \left( \epsilon + \sigma^2 h^2 + \frac{1}{\sqrt{Nh}} + O\left( \frac{1}{\sqrt{N}} \right) + h^3 \right) + \frac{24}{N^\rho}$$

as stated in Theorem 1.

Theorem 1 establishes that the compressed kernelized importance sampling scheme proposed in Section 2 is nearly asymptotically consistent, and can be made arbitrarily close to exact consistency by sending the bandwidth $h$ and compression budget $\epsilon$ to null. However, when these parameters are fixed positive constants, they provide a tunable tradeoff between bias and memory. That is, when the compression budget is a positive constant, then the memory of the posterior distribution representation is finite, as we formalize next.

Theorem 2 Denote as $\hat{\mu}_N$ the empirical distribution (cf. [19]) defined by Alg. 2 whose model order is $M_n$ after $n$ particles generated from importance density $\pi(x)$.

Under Assumptions 2-4 (in Appendix A), for compact feature space $\mathcal{X}$ and bounded importance weights $g(x^{(n)})$, $M_n < \infty$ for all $n$.

Theorem 2 (proof in Appendix B) contrasts the classical bottleneck in the number of particles required to represent an arbitrary posterior, which grows unbounded [1]. While this is a pessimistic estimate, experimentally we observe orders of magnitude reduction in complexity compared to exact importance sampling, which is the focus of the subsequent section.

4 Experiments

4.1 Direct Importance Sampling

In this section, we conduct a simple numerical experiment to demonstrate the efficacy of the proposed algorithm in terms of balancing model parsimony and statistical consistency. We consider the problem of estimating the expected value of function $\phi(x)$ with the target $q(x)$ and the proposal $\pi(x)$ given by

$$\phi(x) = 2\sin \left( \frac{\pi}{1.5x} \right), \quad q(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(x - 1)^2}{2} \right),$$

$$\pi(x) = \frac{1}{\sqrt{4\pi}} \exp \left( -\frac{(x - 1)^2}{4} \right).$$  

(31)

to demonstrate that generic Monte Carlo integration allows one to tract generic quantities of random variables that are difficult to compute under more usual the probabilistic hypotheses. For [31], since $q(x)$ is known, this is referred to as “direct importance sampling”. We run Algorithm 1 i.e., classic importance sampling, and Algorithm 2 for the aforementioned problem. For Alg. 2 we select compression budget $\epsilon = 3$, Gaussian kernel with bandwidth $h = 0.01$. We track both the self-normalized integral estimate [9] and the un-normalized integral estimate (where one uses [9] but skips the division by normalizing constant Z). The reason for plotting the un-normalized integral estimate is that this is the quantity that enters into the compression routine [10]. We further track the absolute integral approximation error, and the number of particles that parameterize the empirical measure. We first represent...
the histogram of the particle generated in Fig. 1. These results are displayed in Fig. 3. In Fig. 2a, we plot the unnormalized integral approximation error for Algorithms 1–2, which are close, and the magnitude of the difference depends on the choice of compression budget. This trend is corroborated in the evolution of (normalized) integral estimates in Fig. 2b, with very little error incurred for both algorithms. From the equality in (6), we need the target distribution \( q_t \) and the likelihood \( \eta_t = \eta = \mathcal{N}(0, \sigma^2) \). From the equality in (9), we need the likelihood and a prior distribution to calculate \( q(x^{(n)}) \) using Bayes Rule [cf. (7)]. Here we fix the likelihood (measurement model) and prior as

\[
\mathbb{P}\left( \{y_k \}_{k \leq K} \mid x^{(n)} \right) = \frac{1}{(2\pi \sigma_1^2)^{K/2}} \exp \left( -\frac{\|y - x^{(n)}\|^2}{2\sigma_1^2} \right),
\]

\[
\mathbb{P}\left( x^{(n)} \right) = \frac{1}{(2\pi \sigma_2^2)^{3}} \exp \left( -\frac{(x^{(n)})^2}{2\sigma_2^2} \right). \tag{32}
\]

For the simulations we considered \( K = 10 \), \( b = 5 \), \( \sigma = 0.1 \), \( \sigma_1 = 0.4 \), \( \sigma_2 = 1.6 \), and compression budget \( \epsilon = 10^{-3} \). A uniform distribution \( U[3, 7] \) is used as the importance distribution.

We observe a comparable tradeoff to that which may be gleaned from Section 4 in particular, we are able to obtain complexity reduction by orders of magnitude reduction with extremely little integral estimation error. This suggests the empirical validity of our compression rule based on un-normalized importance weights operating in tandem with kernel smoothing.

### 5 Conclusions

We focused on a Bayesian inference where one streams simulated Monte Carlo samples to approximate an unknown posterior via importance sampling. Doing so may consistently approximate any function of the posterior at the cost of infinite memory. Thus, we proposed Algorithm 2 (CKIS) to approximate the posterior by a kernel density estimate (KDE) projected onto nearby lower-dimensional subspace, which allows online compression as particles arrive in perpetuity. We established that the bias of CKIS depends on kernel bandwidth and compression budget, providing a tradeoff between statistical accuracy and memory. Experiments demonstrated that we attain memory-reduction by orders of magnitude with very little estimation error, which motivates future application to memory-efficient versions of Monte Carlo localization, which has been eschewed due to its computational burden, when the total number of samples goes to infinity (Theorem 2).

### Appendix

#### A Proof of Consistency of Importance Sampling

Here we state a result on the sample complexity and asymptotic consistency of importance sampling estimators in terms of integral error. We increase the granularity of the proof found in the literature so that the modifications required for our results on compressed importance sampling estimates are laid bare.

**Lemma 1** [Theorem 2.1] Suppose \( \pi \), the proposal distribution is absolutely continuous w.r.t. \( \eta \), the population posterior, and both are defined over \( X \). Then define their Radon-Nikodym derivative:

\[
\frac{d\eta}{d\pi}(x) = \frac{g(x)}{\int_X g(x) \pi(dx)}, \quad \rho = \frac{\pi(g^2)}{\eta(g^2)},
\]

where \( g \) is the unnormalized density of \( \eta \) with respect to \( \pi \). Moreover, \( \rho \) is its second moment ("variance" of unnormalized density). Under Assumption 1, Alg. 1 contracts to true posterior as

\[
\sup_{|\phi| \leq 1} \mathbb{E}[|I_N(\phi) - I(\phi)|] \leq \frac{12}{N} \rho, \quad \mathbb{E}[[I_N(\phi) - I(\phi))^2] \leq \frac{4}{N} \rho, \tag{33}
\]

and hence approaches exact consistency as \( N \to \infty \).
Let us denote the empirical random measure by $\pi$ which we rewrite in greater the completeness and coherence.

**Proof:** This proof is a more detailed in [1][Theorem 2.1] which we rewrite in greater the completeness and coherence. Let us denote the empirical random measure by $\pi^N$ as

$$\pi^N := \frac{1}{N} \sum_{n=1}^{N} \delta_{\mathbf{x}^{(n)}}, \quad \mathbf{x}^{(n)} \sim \pi.$$  \hspace{1cm} (34)

where $\pi^N$ is the occupancy measure, which when weighted, yields the importance sampling empirical measure [11]. Note that the integral approximation at $N$ is denote by $I_N(\phi)$. With the above notation is hand, it holds that

$$\pi^N(g) = \int \frac{1}{N} \sum_{n=1}^{N} g(\mathbf{x}) \delta_{\mathbf{x}^{(n)}}(\mathbf{x}) d\mathbf{x} = \frac{1}{N} \sum_{n=1}^{N} g(\mathbf{x}^{(n)}),$$ \hspace{1cm} (35)

and similarly

$$\pi^N(\phi g) = \int \frac{1}{N} \sum_{n=1}^{N} \delta_{\mathbf{x}^{(n)}}(\mathbf{x}) \phi(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \frac{1}{N} \sum_{n=1}^{N} \phi(\mathbf{x}^{(n)}) g(\mathbf{x}^{(n)}).$$ \hspace{1cm} (36)

From the above equalities, for the bias in the estimator, we can write

$$I_N(\phi) - I(\phi) = \frac{\pi^N(\phi g)}{\pi^N(g)} - I(\phi)$$ \hspace{1cm} (37)

$$= \frac{\pi^N(\phi g)}{\pi^N(g)} - I(\phi) \frac{\pi^N(g)}{\pi^N(g)}$$ \hspace{1cm} (38)

$$= \frac{1}{\pi^N(g)} \left[ \pi^N(\phi g) - I(\phi) \pi^N(g) \right]$$ \hspace{1cm} (39)

$$= \frac{1}{\pi^N(g)} \pi^N \left( (\phi - I(\phi)) g \right).$$ \hspace{1cm} (40)

Let us define $\hat{\phi} := \phi - I(\phi)$ and note that $\pi(\hat{\phi} g) = 0$. \hspace{1cm} (41)

Rewriting the bias, we get

$$I_N(\phi) - I(\phi) = \frac{1}{\pi^N(g)} \pi^N(\hat{\phi} g)$$

$$= \frac{1}{\pi^N(g)} \left[ \pi^N(\hat{\phi} g) - \pi(\hat{\phi} g) \right]$$ \hspace{1cm} (42)

where the second equality holds from (41). The first term in the bracket is an unbiased estimator for the second one which means that

$$E \left[ \pi^N(\hat{\phi} g) - \pi(\hat{\phi} g) \right] = 0.$$ \hspace{1cm} (43)

Take the expectation on both sides of (42), we get

$$E[I_N(\phi) - I(\phi)] = E \left[ \frac{1}{\pi^N(g)} \left[ \pi^N(\hat{\phi} g) - \pi(\hat{\phi} g) \right] \right].$$ \hspace{1cm} (44)

We can add the expression in (43) to the right hand side of (44) since it is zero, we get

$$E[I_N(\phi) - I(\phi)] = E \left[ \frac{1}{\pi^N(g)} \left[ \pi^N(\hat{\phi} g) - \pi(\hat{\phi} g) \right] \right]$$

$$+ E \left[ \frac{1}{\pi(g)} \left[ \pi^N(\hat{\phi} g) - \pi(\hat{\phi} g) \right] \right].$$ \hspace{1cm} (45)

Taking the expectation operator outside, we get

$$E[I_N(\phi) - I(\phi)] = E \left[ \left( \frac{1}{\pi^N(g)} - \frac{1}{\pi(g)} \right) \left[ \pi^N(\hat{\phi} g) - \pi(\hat{\phi} g) \right] \right].$$ \hspace{1cm} (46)

The memory-reduction scheme nearly preserves statistical consistency, while yielding reasonable complexity, whereas Alg. 1 attains exact consistency as its memory grows unbounded with index $n$. The memory-reduction scheme nearly preserves statistical consistency, while yielding reasonable complexity, whereas Alg. 1 attains exact consistency as its memory grows unbounded with index $n$. The memory-reduction scheme nearly preserves statistical consistency, while yielding reasonable complexity, whereas Alg. 1 attains exact consistency as its memory grows unbounded with index $n$.
Next, we split the set of integration to $A = \{2\pi\mu_{MC}^{N}(g) > \pi(g)\}$ and its complement using the property

$$E[f(X)] = E[f(X)1_{A}(X)] + E[f(X)1_{A}^{c}(X)],$$

where $1_{A}$ the indicator function of the set $A$ with selecting $A = \{2\pi\mu_{MC}^{N}(g) > \pi(g)\}$, which takes value 1 if $x \in A$ and 0 if $x \notin A$. We get

$$|E[I_{N}(\phi) - I(\phi)]| \leq |E[I_{N}(\phi) - I(\phi)1_{2\pi\mu_{MC}^{N}(g) > \pi(g)}]| + |E[I_{N}(\phi) - I(\phi)1_{2\pi\mu_{MC}^{N}(g) \leq \pi(g)}]|.\quad (48)$$

Consider the second term of (48), and use the fact that $|\phi| \leq 1$, and so $[\mu_{n}^{N}(\phi)|, I(\phi)| \leq 1$ since they are mean values w.r.t. probability measures $\mu_{n}^{N}, q$ respectively. Then we use that $E[I_{A}] = P(A)$ and obtain

$$|E[I_{N}(\phi) - I(\phi)]| \leq |E[I_{N}(\phi) - I(\phi)1_{2\pi\mu_{MC}^{N}(g) > \pi(g)}]| + 2P(2\pi\mu_{MC}^{N}(g) \leq \pi(g)).\quad (49)$$

So the constant 2 comes from the fact that $|I_{N}(\phi) - I(\phi)| \leq |I_{N}(\phi)| + |I(\phi)| \leq 2$. For the first term on the right hand side of (49), from the set condition i.e., the definition of $A$, it holds that

$$\frac{1}{2\pi\mu_{MC}^{N}(g)\pi(g)} < \frac{2}{\pi(g)}\quad (50)$$

which implies that

$$|E[I_{N}(\phi) - I(\phi)]| \leq \frac{2}{\pi(g)^{2}}E[|\pi(g) - \pi_{N}(g)||2\pi\mu_{MC}^{N}(g) - \pi(g)|] + 2P(2\pi\mu_{MC}^{N}(g) \leq \pi(g)).\quad (51)$$

Finally, to upper bound the first term on the right hand side of (51), we bound the expectation first using Cauchy-Schwarz

$$E[|\pi(g) - \pi_{N}(g)||2\pi\mu_{MC}^{N}(g) - \pi(g)|] \leq E[(\pi(g) - \pi_{N}(g))^{2}]^{\frac{1}{2}}E[(2\pi\mu_{MC}^{N}(g) - \pi(g))^{2}]^{\frac{1}{2}}\quad (52)$$

The first expectation on the right hand side of (52) is bounded as follows: by definition of $\pi_{n}^{N}$ we have for $x_{n} \sim \pi$ independent that

$$E[(\pi(g) - \pi_{N}(g))^{2}] = E[(\pi(g) - N\sum_{n=1}^{N}g(x_{n}))^{2}]\quad (53)$$

which since $E[g(x_{n})] = \pi(g)$ and by independence of the $x_{n}(n)$ is equal to

$$= \frac{1}{N^{2}}\sum_{n=1}^{N}Var(g(x_{n})) = \frac{1}{N^{2}}\sum_{n=1}^{N}Var(g(x_{n}))\quad (53)$$

and since $x_{n}(n)$ identically distributed, $x_{n}(n) \sim \pi$, this is equal to

$$\frac{N^{2}}{N^{2}}Var_{n \sim \pi}(g) = \frac{1}{N}(\pi(g^{2}) - \pi(g)^{2}) \leq \frac{1}{N}\pi(g^{2}).$$

The second expectation on the right hand side of (52) is bounded in a similar way along with the fact that $|\phi| \leq 1$ so $|\phi| \leq 2$. Then we utilize different upper bounds on the right hand side of (52) to obtain

$$|E[I_{N}(\phi) - I(\phi)]| \leq \frac{2}{\pi(g)^{2}}E[|\pi(g) - \pi_{N}(g)||2\pi\mu_{MC}^{N}(g) - \pi(g)|] + 2P(2\pi\mu_{MC}^{N}(g) \leq \pi(g))\quad (54)$$

$$\leq \frac{2}{\pi(g)^{2}}\frac{1}{\sqrt{N}}\pi(g^{2})^{1/2}2\frac{1}{\sqrt{N}}\pi(g^{2})^{1/2}$$

$$+ 2P(2\pi\mu_{MC}^{N}(g) \leq \pi(g))\quad (55)$$
where the inequalities follow from the fact that the test function is bounded by $|\phi|$. Next, note that
\[
P \left( 2n^N (g) \leq \pi(g) \right) = P \left( 2 \left( \pi^N (g) - \pi(g) \right) \leq -\pi(g) \right)
\leq P \left( 2 \left( \pi^N (g) - \pi(g) \right) \geq \pi(g) \right).
\]
(56)
where the first equality if obtained by subtracting $-2\pi(g)$ from both sides inside the bracket. Next, we need to use Markov inequality, which is given by
\[
P(X \geq a) \leq \frac{E(X)}{a}.
\]
(57)
Utilizing this, we can write
\[
P \left( 2 \left| \pi^N (g) - \pi(g) \right| \geq \pi(g) \right) \leq \frac{2E \left| \pi^N (g) - \pi(g) \right|}{\pi(g)}
\leq \frac{4 \pi^2(g^2)}{N \pi(g)^2}.
\]
This implies that
\[
P \left( 2n^N (g) \leq \pi(g) \right) \leq \frac{4 \pi^2(g^2)}{N \pi(g)^2}.
\]
(59)
Utilize the upper bound in (59) into (66), we obtain
\[
\sup_{|\phi| \leq 1} |E[I_{\mathcal{N}}(\phi) - I(\phi)]| \leq \frac{12 \pi^2(g^2)}{N \pi(g)^2}
\]
which proves the result for the bias.

\section*{B Proof of Theorem 2}

We begin by presenting a lemma which allows us to relate the stopping criterion of our sparsification procedure to a Hilbert subspace distance.

\textbf{Lemma 2} Define the distance of an arbitrary feature vector $x$ evaluated by the feature transformation $\phi(x)$ to $\mathcal{H}_D = \text{span}(\phi(D_n))^M$ of the subspace of the real spanned by a dictionary $D$ of size $M$, as
\[
\text{dist}(\phi(x), \mathcal{H}_D) = \min_{y \in \mathcal{H}_D} |\phi(x) - \phi(y)|.
\]
(61)
This set distance simplifies to following least-squares projection when $D \in \mathbb{R}^{p \times M}$ is fixed
\[
\text{dist}(\phi(x), \mathcal{H}_D) = \left| \phi(x) - \phi(x)^T D P_{D,D}^{-1} \phi_D \right|.
\]
(62)
\textbf{Proof:} The distance to the subspace $\mathcal{H}_D$ is defined as
\[
\text{dist}(\phi(x), \mathcal{H}_D) = \min_{y \in \mathcal{H}_D} |\phi(x) - \phi(y)|
\leq \min_{\phi \in \mathbb{R}^M} |\phi(x) - \phi(x)^T D P_{D,D}^{-1} \phi_D |
\]
where the first equality comes from the fact that the dictionary $D$ is fixed, so $\phi \in \mathbb{R}^M$ is the only free parameter. Now plug in the minimizing weight vector $\nu^* = \phi(x_0) P_{D_D, D_D}^{-1} \phi_D$ into (63) which is obtained in an analogous manner to the logic which yields (18) - (19). Doing so simplifies (63) to the following
\[
\text{dist}(\phi(x_n), \mathcal{H}_D) = \left| \phi(x_n) - \phi(x_n)^T D P_{D,D}^{-1} \phi_D \right|.
\]
(64)
Next, we establish the finiteness of the model order.

\textbf{Proof:} Consider the model order of the unnormalized density estimate $\tilde{\mu}_{n-1}$ and $\tilde{\mu}_n$ generated by Algorithm 2 denoted by $M_{n-1}$ and $M_n$, respectively, at two arbitrary subsequent instances $n - 1$ and $n$. Suppose the model order of the density estimate $\tilde{\mu}_{n-1}$ is less than or equal to that of $\tilde{\mu}_n$, i.e. $M_n \leq M_{n-1}$. This relation holds when the stopping criterion of KOMP (defined in Algorithm 2, stated as $\min_{j=1 \ldots, M_{n-1}} \gamma_j > \epsilon$, is not satisfied for the updated dictionary matrix with the newest sample point $x^{(n)}$ appended: $D_n = [D_{n-1}; x^{(n)}]$) [cf. (14)], which is of size $M_{n-1} + 1$. Thus, the negation of the termination condition of KOMP in Algorithm 2 must hold for this case, stated as
\[
\min_{j=1 \ldots, M_{n-1}} \gamma_j \leq \epsilon.
\]
(65)
Observe that the left-hand side of (65) lower bounds the approximation error $\gamma_{M_{n-1}+1}$ of removing the most recent feature vector $x^{(n)}$ due to the minimization over $j$, that is, $\min_{j=1 \ldots, M_{n-1}} \gamma_j \leq \gamma_{M_{n-1}+1}$. Consequently, if $\gamma_{M_{n-1}+1} \leq \epsilon$, then (65) holds and the model order does not grow. Thus it suffices to consider $\gamma_{M_{n-1}+1}$. The definition of $\gamma_{M_{n-1}+1}$ with the substitution of $I_n$ in (17) allows us to write
\[
\gamma_{M_{n-1}+1} = \min_{u \in \mathbb{R}^{M_{n-1}}} \sum_{k=1, \ldots, M_{n-1}+1} |\tilde{\mu}_n - 1 + g(x^{(n)})\kappa_{x^{(n)}}(x) - \sum u_k \kappa_{D_k}(x)|
\]
\[
= \min_{u \in \mathbb{R}^{M_{n-1}}} \left[ \sum_{k \in \mathcal{X} \setminus \{M_{n-1}+1\}} g(x^{(n)})\kappa_{D_k}(x) + g(x^{(n)})\kappa_{x^{(n)}}(x) - \sum u_k \kappa_{D_k}(x) \right],
\]
(66)
where we denote the $k^{th}$ column of $D_n$ as $d_k$. The minimal error is achieved by considering the square of the expression inside the minimization and expanding terms to obtain
\[
\left[ \sum_{k \in \mathcal{X} \setminus \{M_{n-1}+1\}} g(x^{(n)})\kappa_{D_k}(x) + g(x^{(n)})\kappa_{x^{(n)}}(x) - \sum u_k \kappa_{D_k}(x) \right]^2
\]
\[
= g^T T D_n D_n^T g + g(x^{(n)})^2 - 2 g(x^{(n)}) u^T T D_n D_n^T u + 2 g(x^{(n)}) u^T T D_n D_n^T g(x^{(n)})
\]
\[
- 2 w^T T D_n D_n^T u,
\]
(67)
To obtain the minimum, we compute the stationary solution of (67) with respect to $u \in \mathbb{R}^{M_{n-1}}$ and solve for the minimizing $u^*$, which in a manner similar to the logic in (18) - (19), is given as
\[
u^* = [g(x^{(n)}) P_{D_n D_n}^{-1} \kappa_{D_n}(x^{(n)})] + g(x^{(n)})
\]
(68)
Plug $u^*$ in (68) into the expression in (66) and using the short-hand notation $\sum u_k \kappa_{D_k}(x) = u^T \kappa_{D_n}(x)$, doing so simplifies (66) to
\[
\left[ g^T T D_n D_n^T g + g(x^{(n)}) \kappa_{x^{(n)}}(x) - u^T \kappa_{D_n}(x) \right]^2
\]
\[
= [g^T \kappa_{D_n}(x) + g(x^{(n)}) \kappa_{x^{(n)}}(x) - u^T \kappa_{D_n}(x)]^2
\]
\[
- [g(x^{(n)}) P_{D_n D_n}^{-1} \kappa_{D_n}(x^{(n)}) + g(x^{(n)})] \kappa_{x^{(n)}}(x) \]
\[
= [g(x^{(n)}) \kappa_{x^{(n)}}(x) - g(x^{(n)}) P_{D_n D_n}^{-1} \kappa_{D_n}(x^{(n)})]^2 \kappa_{D_n}(x)
\]
\[
= [g(x^{(n)}) \kappa_{x^{(n)}}(x) - \kappa_{D_n}(x^{(n)})]^2 \kappa_{D_n}(x)
\]
(69)
Notice that the right-hand side of (69) may be identified as the distance to the subspace $\mathcal{H}_M$ in (64) defined in Lemma 2 scaled by a factor of $g(x^{[n]})$. We may upper-bound the right-hand side of (69) as

$$g(x^{[n]})|\kappa_{\mathcal{H}_M}(x) - \kappa_{\mathcal{H}_M}(x^{[n]})|^2 P_{\mathcal{H}_M}^{-1} \kappa_{\mathcal{H}_M}(x) = g(x^{[n]}) \text{dist} (\kappa_{\mathcal{H}_M}(x), \mathcal{H}_M)$$

(70)

where we have applied (62) regarding the definition of the subspace distance on the right-hand side of (70) to replace the absolute value term. Now, when the KOMP stopping criterion is violated, i.e., (65) holds, which implies $\gamma_{M-1+1} \leq \epsilon$. Therefore, the right-hand side of (70) is upper-bounded by $\epsilon$, we can write

$$g(x^{[n]}) \text{dist} (\kappa_{\mathcal{H}_M}(x), \mathcal{H}_M) \leq \epsilon.$$  

(71)

After rearranging the terms in (71), we write

$$\text{dist} (\kappa_{\mathcal{H}_M}(x), \mathcal{H}_M) \leq \frac{\epsilon}{g(x^{[n]})}.$$  

(72)

where we have divided both sides by $g(x^{[n]})$. Observe that if (72) holds, then $\gamma_M \leq \epsilon$ holds, but since $\gamma_M \geq \min_j \gamma_j$, we may conclude that (66) is satisfied. Consequently the model order at the subsequent step does not grow $M_n \leq M_{n-1}$ whenever (72) is valid.

Now, let’s take the contrapositive of the preceding expressions to observe that growth in the model order ($M_n = M_{n-1} + 1$) implies that the condition

$$\text{dist} (\kappa_{\mathcal{H}_M}(x), \mathcal{H}_M) > \frac{\epsilon}{g(x^{[n]})}$$

(73)

holds. Therefore, each time a new point is added to the model, the corresponding map $\kappa_{\mathcal{H}_M}(x^{[n]})$ is guaranteed to be at least a distance of $\frac{\epsilon}{g(x^{[n]})}$ from every other feature map in the current model. In canonical works such as [7,8], the largest self-normalized importance weight is shown to be bounded by a constant. Under the additional hypothesis that the un-normalized importance weight is bounded by some constant $g$, then we have via (73):

$$\text{dist} (\kappa_{\mathcal{H}_M}(x), \mathcal{H}_M) > \frac{\epsilon}{g}.$$  

(74)

Therefore, For a fixed compression budget $\epsilon$ and step size $\eta$, the KOMP stopping criterion is violated for the newest point whenever distinct dictionary points $d_k$ and $d_j$ for $j, k \in \{1, \ldots, M_{n-1}\}$, satisfy the condition $\text{dist} (\kappa_{\mathcal{H}_M}(d_k), \kappa_{\mathcal{H}_M}(x)) > \frac{\epsilon}{g}$. Next, we follow the similar argument as provided in the proof of Theorem 3.1 in [15]. Since $X$ is compact and $\kappa_x$ is continuous, the range $\kappa_x(X)$ of the feature space $X$ is compact. Therefore, the number minimum of balls (covering number) of radius $\kappa$ (here, $\kappa = \frac{\epsilon}{g}$) needed to cover the set $\kappa_x(X)$ is finite (see, e.g., [3]) for a finite compression budget $\epsilon$. The finiteness of the covering number states that the number of elements in the dictionary $M_X$ will be finite as $N \rightarrow \infty$ denoted by $M_X$.
