Abstract

Let $\mathfrak{F}$ be a class of group and $G$ a finite group. Then a set $\Sigma$ of subgroups of $G$ is called a $G$-covering subgroup system for the class $\mathfrak{F}$ if $G \in \mathfrak{F}$ whenever $\Sigma \subseteq \mathfrak{F}$.

We prove that: If a set of subgroups $\Sigma$ of $G$ contains at least one supplement to each maximal subgroup of every Sylow subgroup of $G$, then $\Sigma$ is a $G$-covering subgroup system for the classes of all $\sigma$-soluble and all $\sigma$-nilpotent groups, and for the class of all $\sigma$-soluble $P\sigma T$-groups.

This result gives positive answers to questions 19.87 and 19.88 from the Kourovka notebook.
The group $G$ is said to be: $\sigma$-primary if $G$ is a $\sigma_i$-group for some $i = i(G)$; $\sigma$-decomposable or $\sigma$-nilpotent if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$; $\sigma$-soluble if every chief factor of $G$ is $\sigma$-primary.

A set $\mathcal{H}$ of subgroups of $G$ is a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i$.

Recall that a subgroup $A$ of $G$ is said to be $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AHx = HxA$ for all $H \in \mathcal{H}$ and all $x \in G$.

We say that $G$ is a $P\sigma T$-group if $\sigma$-permutability is a transitive relation in $G$, that is, if $K$ is a $\sigma$-permutable subgroup of $H$ and $H$ is a $\sigma$-permutable subgroup of $G$, then $K$ is a $\sigma$-permutable subgroup of $G$.

Let $\mathfrak{F}$ be a class of group and $G$ a finite group. Then a set $\Sigma$ of subgroups of $G$ is called a $G$-covering subgroup system [5] for the class $\mathfrak{F}$ if $G \in \mathfrak{F}$ whenever $\Sigma \subseteq F$.

In this paper, we prove the following

**Theorem A.** Suppose that a set of subgroups $\Sigma$ of $G$ contains at least one supplement to each maximal subgroup of every Sylow subgroup of $G$. Then $\Sigma$ is a $G$-covering subgroup system for any class $\mathfrak{F}$ in the following list:

(i) $\mathfrak{F}$ is the class of all $\sigma$-soluble groups.

(ii) $\mathfrak{F}$ is the class of all $\sigma$-nilpotent groups.

(iii) $\mathfrak{F}$ is the class of all $\sigma$-soluble $P\sigma T$-groups.

The theory of $P\sigma T$-groups was built in the works [1, 2, 3]. Theorem A gives positive answers to questions 19.87 and 19.88 from the Kourovka notebook [6] and, also, allows us to give the following new characterization of $\sigma$-soluble $P\sigma T$-groups.

**Corollary 1.1.** $G$ is a $\sigma$-soluble $P\sigma T$-group if and only if each maximal subgroup of every Sylow subgroup of $G$ has a supplement $T$ in $G$ such that $T$ is a $\sigma$-soluble $P\sigma T$-group.

In the classical case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$: $G$ is $\sigma^1$-soluble (respectively $\sigma^1$-nilpotent) if and only if $G$ is soluble (respectively nilpotent); $\sigma^1$-permutable subgroups are also called $S$-permutable [7]; in this case a $P\sigma T$-group is also called a $PST$-group [7].

A significant place to the theory of $PST$-groups is given in the book [6]. From Theorem A we get also the following result in this line researches.

**Corollary 1.2.** $G$ is a soluble $PST$-group if and only if each maximal subgroup of every Sylow subgroup of $G$ has a supplement $T$ in $G$ such that $T$ is a soluble $PST$-group.
2 Basic lemmas

If \( n \) is an integer, the symbol \( \pi(n) \) denotes the set of all primes dividing \( n \); as usual, \( \pi(G) = \pi(|G|) \), the set of all primes dividing the order of \( G \). \( G \) is said to be a \( D_\pi \)-group if \( G \) possesses a Hall \( \pi \)-subgroup \( E \) and every \( \pi \)-subgroup of \( G \) is contained in some conjugate of \( E \).

By the analogy with the notation \( \pi(n) \), we write \( \sigma(n) \) to denote the set \( \{ \sigma_i | \sigma_i \cap \pi(n) \neq \emptyset \} \); \( \sigma(G) = \sigma(|G|) \). \( G \) is said to be: a \( \sigma \)-full group of Sylow type \([1]\) if every subgroup \( E \) of \( G \) is a \( D_{\sigma_i} \)-group for every \( \sigma_i \in \sigma(E) \).

Lemma 2.1 (See Theorem A \([3]\)). Every \( \sigma \)-soluble group is a \( \sigma \)-full group of Sylow type.

Lemma 2.2 (Theorem 1 in \([9]\)). \( G \) is \( \pi \)-separable if and only if
(i) \( G \) has a Hall \( \pi \)-subgroup and a Hall \( \pi' \)-subgroup;
(ii) \( G \) has a Hall \( \pi \cup \{ p \} \)-subgroup and a Hall \( \pi' \cup \{ q \} \)-subgroup for all \( p \in \pi' \) and \( q \in \pi \).

Lemma 2.3 (See Corollary 2.4 and Lemma 2.5 in \([1]\)). The class of all \( \sigma \)-nilpotent groups \( \mathcal{N}_\sigma \) is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if \( E \) is a normal subgroup of \( G \) and \( E/(E \cap \Phi(G)) \) is \( \sigma \)-nilpotent, then \( E \) is \( \sigma \)-nilpotent.

In view of Lemma 2.3, the class \( \mathcal{N}_\sigma \), of all \( \sigma \)-nilpotent groups, is a hereditary saturated formation and so from Proposition 2.2.8 in \([10]\) we get the following

Lemma 2.4 (See Proposition 2.2.8 in \([10]\)). If \( N \) is a normal subgroup of \( G \), then \((G/N)^{\mathcal{N}_\sigma} = G^{\mathcal{N}_\sigma} N/N\).

In this lemma, \( G^{\mathcal{N}_\sigma} \) denotes the \( \sigma \)-nilpotent residual of \( G \), that is, the intersection of all normal subgroups \( N \) of \( G \) with \( \sigma \)-nilpotent quotient \( G/N \).

Lemma 2.5 (See Theorem A in \([2]\)). If \( G \) is a \( \sigma \)-soluble \( P\sigma T \)-group and \( D = G^{\mathcal{N}_\sigma} \), then the following conditions hold:
(i) \( G = D \rtimes M \), where \( D \) is an abelian Hall subgroup of \( G \) of odd order, \( M \) is \( \sigma \)-nilpotent and every element of \( G \) induces a power automorphism in \( D \);
(ii) \( O_{\sigma_i}(D) \) has a normal complement in a Hall \( \sigma_i \)-subgroup of \( G \) for all \( i \).

Conversely, if Conditions (i) and (ii) hold for some subgroups \( D \) and \( M \) of \( G \), then \( G \) is a \( P\sigma T \)-group.

3 Proof of Theorem A

Proof of Theorem A. Assume that this theorem is false. We can assume without loss of generality that \( \sigma(G) = \{ \sigma_1, \sigma_2, \ldots, \sigma_t \} \).
(I) $G$ is not $\sigma$-nilpotent. Hence $t > 1$ and $D := G^{o_{\sigma}} \neq 1$.

Indeed, assume that $G$ is $\sigma$-nilpotent. Then $G$ is $\sigma$-soluble. Hence Statements (i) and (ii) hold for $G$. Moreover, in this case for every $i$ the product $H_i$, of all normal $\sigma_i$-subgroups of $G$, is the unique normal Hall $\sigma_i$-subgroup of $G$ and $G = H_1 \times H_2 \times \cdots \times H_t$. Hence every subgroup of $G$ is $\sigma$-permutable in $G$. Thus Statement (iii) also holds for $G$, contrary to our assumption on $G$. Hence (I) holds.

(i) Assume that this assertion is false and let $G$ be a counterexample of minimal order.

(*) $G$ has no non-identity normal $\sigma$-primary subgroups.

Assume that $G$ has a minimal normal $\sigma$-primary subgroup, $R$ say.

Let $P/R$ be any non-identity Sylow subgroup of $G/R$. Then for some prime $p$ and for a Sylow $p$-subgroup $G_p$ of $G$ we have $G_pR/R = P/R$, so $G_p$ is non-identity.

Now let $V/R$ be any maximal subgroup of $P/R$, that is, $|P:V| = |P/R : V/R| = p$. Then $V = R(G_p \cap V)$, so

$$p = |G_p R : R(G_p \cap V)| = (|G_p| |R| : |G_p \cap R|) : (|R| (G_p \cap V) : |R \cap G_p \cap V|) = |G_p : G_p \cap V|,$$

so $G_p \cap V$ is a maximal subgroup of $G_p$. Hence $G$ has a $\sigma$-soluble subgroup $T$ such that $(G_p \cap V)T = G$.

But then $RT/R \cong T/(T \cap R)$ is a $\sigma$-soluble subgroup of $G/R$ such that

$$(V/R)(RT/R) = (R(G_p \cap V)/R)(TR/R) = G/R.$$

Therefore the hypothesis holds for $G/R$, so $G/R$ is $\sigma$-soluble by the choice of $G$. But then $G$ is $\sigma$-soluble, a contradiction. Hence we have (*).

(**) $t = 2$, that is, $\sigma(G) = \{\sigma_1, \sigma_2\}$.

Assume that $t > 2$ and let $P_i$ be a Sylow $p_i$-subgroup of $G$ for some $p_1 \in \sigma_1 \cap \pi(G)$, $p_2 \in \sigma_2 \cap \pi(G)$ and $p_3 \in \sigma_3 \cap \pi(G)$. Let $V_i$ be a maximal subgroup of $P_i$. Then, by hypothesis, $G$ has $\sigma$-soluble subgroups $T_1$, $T_2$ and $T_3$ such that $G = V_iT_i$ for $i = 1, 2, 3$.

Let $R$ be a minimal normal subgroup of $T_1$. Then $R$ is $\sigma$-primary, $R$ is a $\sigma_k$-group say. Since $|G : T_2| = |T_1T_2 : T_2| = |T_1 : T_1 \cap T_2|$ is a $p_2$-number and $|T_1 : T_1 \cap T_3|$ is a $p_3$-number, where $p_2 \in \sigma_2$ and $p_3 \in \sigma_3$, we have either $R \leq T_1 \cap T_2$ or $R \leq T_1 \cap T_3$, $R \leq T_1 \cap T_2$ say. Hence $R^G = R^{T_1T_2} = R^{T_2} \leq T_2$, so $G$ has a non-identity normal $\sigma$-primary subgroup, contrary to Claim (*). Thus (**') holds.

The final contradiction for (i). Let $\pi = \sigma_1 \cap \pi(G)$. Since $T_1$ is $\sigma$-soluble, $T_1$ has a Hall $\sigma_k$-subgroup for all $k$ by Lemma 2.1. Then a Hall $\sigma_2$-subgroup of $T_1$ is a Hall $\pi'$-subgroup of $G$ and a Hall $\sigma_1$-subgroup of $T_2$ is a Hall $\pi-$subgroup of $G$.

Now we show that $G$ has a Hall $\pi \cup \{p\}$-subgroup for every $p \in \sigma_2 \cap \pi(G)$. If $|\sigma_2 \cap \pi(G)| = 1$ it is evident. Now assume that $|\sigma_2 \cap \pi(G)| > 1$ and let $q \in (\sigma_2 \cap \pi(G)) \setminus \{p\}$. Let $V$ be a maximal
subgroup of a Sylow $q$-subgroup $Q$ of $G$. And let $T$ be a $\sigma$-soluble supplement to $V$ in $G$. Then $T$ is $\pi$-separable by Claim (**). Hence $T$ has a Hall $\pi \cup \{p\}$-subgroup $H$ by Lemma 2.2. But $|G : T|$ is a $\{q\}$-number, where $p \neq q \notin \sigma_1$, so $H$ is a Hall $\pi \cup \{p\}$-subgroup of $G$.

Similarly it can be proved that $G$ has a Hall $\pi' \cup \{p\}$-subgroup for all $p \in \pi$. Therefore $G$ is $\pi$-separable by Lemma 2.2 and so $G$ is $\sigma$-soluble by Claim (**), contrary to the choice of $G$. Hence Statement (i) holds.

(iii) Assume that this assertion is false and let $G$ be a counterexample of minimal order. Then $G$ is $\sigma$-soluble by Part (i).

(1) If $R$ is a non-identity normal subgroup of $G$, then the hypothesis holds for $G/R$. Hence $G/R$ is a $\sigma$-soluble $P\sigma T$-group (See the proof of Claim (*)).

(2) If $R$ is an abelian minimal normal subgroup of $G$, then $R$ is not a Sylow subgroup of $G$.

Indeed, assume that $R$ is Sylow subgroup of $G$ and let $V$ be an y maximal subgroup of $R$. Then for every supplement $T$ to $V$ in $G$ we have that $T \cap R$ is normal in $G$, the minimality of $R$ implies that $T = G$. Hence $G$ is a $\sigma$-soluble $P\sigma T$-group, a contradiction. Hence (2) holds.

(3) $D$ is $\sigma$-nilpotent.

Assume that this is falls. Then $D$ is not $\sigma$-primary. Let $R$ be a minimal normal subgroup of $G$, so $R$ is a $\sigma_i$-group for some $i$ since $G$ is $\sigma$-soluble. Moreover, from Lemmas 2.3 and 2.4 we get that

$$(G/R)^{\Omega_i} = G^{\Omega_i}R/R = DR/R \simeq D/(D \cap R)$$

is a Hall $\sigma$-nilpotent subgroup of $G/R$ by Claim (1). Hence $R$ is the unique minimal normal subgroup of $G$, $R < D$ and $R \notin \Phi(G)$ since $D$ is not $\sigma$-nilpotent. Therefore $C_G(R) \leq R$ and $D/R$ is a Hall subgroup of $G/R$. Moreover, $D/R$ is not a $\sigma_i$-group since $D$ is not $\sigma$-primary. Let $p$ be a prime dividing $|D/R|$ such that $p \notin \sigma_i$. And let $P$ be a Sylow $p$-subgroup of $D$. Then $P \cap R = 1$ and $P$ is a Sylow $p$-subgroup of $G$ since $D/R$ is a Hall subgroup of $G/R$.

Let $V$ be a maximal subgroup of $P$ and $T$ a supplement to $V$ in $G$ such that $T$ is a $P\sigma T$-group. Then $T^{\Omega_i} \leq D$ and $T^{\Omega_i}$ is a Hall abelian subgroup of $T$ such that every subgroup of $T^{\Omega_i}$ is normal in $T$ by Lemma 2.5. Moreover, $R \leq T$ since $|G : T|$ is a $\{p\}$-number. Hence $T^{\Omega_i} \cap R$ is a normal abelian Hall subgroup of $R$. Hence either $T^{\Omega_i} \cap R = 1$ or $T^{\Omega_i} \cap R = R$ and so $R \leq T^{\Omega_i}$.

First assume that $T^{\Omega_i} \cap R = 1$. Then $T^{\Omega_i} \leq C_G(R)$, so $T^{\Omega_i} = 1$ and hence $T$ is $\sigma$-nilpotent. From $P = P \cap VT = V(P \cap T)$ it follow that $T$ is not a $\sigma_i$-group. Hence for a Hall $\sigma_i$-subgroup $E$ of $T$ we have $E \neq 1$ and $E \leq C_G(R) \leq R$, a contradiction. Therefore $R \leq T^{\Omega_i}$, so $R = T^{\Omega_i}$ is a $q$-group for some prime $q \neq p$ since $C_G(R) \leq R$ and $T^{\Omega_i}$ is abelian. Let $Q$ be a Sylow $q$-subgroup of $T$. Then $R = Q$ since $R = T^{\Omega_i}$ is a Hall subgroup of $T$. Moreover, $R$ is a Sylow $q$-subgroup of $G$ since $p \neq q$ and $|G : T|$ is a $\{p\}$-number, contrary to Claim (2). Hence we have (3).

(4) $D$ is nilpotent.

Assume that this false and let $R$ be a minimal normal subgroup of $G$. Then $R \leq D$ and
C_G(R) \leq R and D/R is a Hall subgroup of G/R (see the proof of Claim (3)). Hence D \leq O_{\pi_i}(G) for some i by Claim (3). Let P be a Sylow p-subgroup of G, where p \in \pi(G) \setminus \pi_i. Let V be a maximal subgroup of P and T a supplement to V in G such that T is a P_\sigma T-group. Then R \leq D \leq T, so \tau^{\pi_i} \cap R is a normal abelian Hall subgroup of R. Hence R \leq T^{\pi_i} (see the proof of Claim (\ast)). On the other hand, T^{\pi_i} \leq D. Therefore R = T^{\pi_i} is a Sylow q-subgroup of G for some q \neq p, contrary to Claim (2).

(5) D is a Hall subgroup of G.

Suppose that this is false and let P be a Sylow p-subgroup of D such that 1 < P < G_p, where G_p \in \text{Syl}_p(G). We can assume without loss of generality that G_p \leq H_1.

(a) D = P is a minimal normal subgroup of G.

Let R be a minimal normal subgroup of G contained in D. Since D is nilpotent by Claim (4), R is a q-group for some prime q. Moreover, D/R = (G/R)^{\pi_i} is a Hall subgroup of G/R by Claim (1) and Lemma 2.3. Suppose that PR/R \neq 1. Then PR/R \in \text{Syl}_p(G/R). If q \neq p, then P \in \text{Syl}_p(G). This contradicts the fact that P < G_p. Hence q = p, so R \leq P and therefore P/R \in \text{Syl}_p(G/R) and we again get that P \in \text{Syl}_p(G). This contradiction shows that PR/R = 1, which implies that R = P is the unique minimal normal subgroup of G contained in D. Since D is nilpotent by Claim (4), a p'-complement E of D is characteristic in D and so it is normal in G. Hence E = 1, which implies that R = D = P.

(b) D \notin \Phi(G). Hence for some maximal subgroup M of G we have G = D \rtimes M.

(c) If G has a minimal normal subgroup L \neq D, then G_p = D \times (L \cap G_p). Hence O_{\pi'}(G) = 1.

Indeed, DL/L \simeq D is a Hall subgroup of G/L by Claim (1). Hence G_pL/L = RL/L, so G_p = D \times (L \cap G_p). Thus O_{\pi'}(G) = 1 since D < G_p by Claim (a).

(d) V = C_G(D) \cap M is a normal subgroup of G and C_G(D) = D \times V \leq H_1.

In view of Claim (b), C_G(D) = D \times V, where V = C_G(D) \cap M is a normal subgroup of G. By Claim (a), V \cap D = 1 and hence V \simeq DV/D is \sigma-nilpotent by Lemma 2.2. Let W be a \sigma_1-complement of V. Then W is characteristic in V and so it is normal in G. Therefore we have (d) by Claim (c).

The final contradiction for (5). Let Q be a Sylow q-subgroup of G, where q \in \pi(G) \setminus \pi(H_1). Let V be a maximal subgroup of P and T a supplement to V in G such that T is a P_\sigma T-group. Then T^{\pi_i} \leq D and T^{\pi_i} is a Hall abelian subgroup of T. Then D is not a Sylow q-subgroup of T and so T^{\pi_i} = 1, which implies that T is \sigma-nilpotent. But then for a Sylow q-subgroup T_q of T we have 1 < T_q \leq C_G(D) \leq H_1, a contradiction.

(6) Every subgroup H of D is normal in G. Hence every element of G induces a power automorphism in D.

Since D is nilpotent by Claim (4), it is enough to consider the case when H \leq O_p(D) for some p\pi(D).
Let $R$ be any Sylow $r$-subgroup of $G$, where $r \not\in \pi(D)$. Let $V_1, V_2, \ldots, V_i$ be the set of all maximal subgroups of $R$. Let $T_i$ be a supplement to $V_i$ in $G$ such that $T_i$ is a $P\sigma T$-group with $D_i = T^\sigma i$.

Since $G = V_i T_i$, $R = V_i (T_i \cap R)$. Hence for some $a_i \in T_i \cap R$ we have we have $a_i \not\in V_i$. We show that $a_i \in N_G(H)$.

First observe that $D \leq T_i$ since $|G:T_i|$ is a $q$-number, where $r \not\in \pi(D)$. Moreover, $D_i \leq D$. But $D_i$ is a Hall subgroup of $T_i$ and every subgroup of $D_i$ is normal in $T_i$, so $D_i$ is a Hall subgroup of $D$. So either $H \leq O_p(D) \leq D_i$ or $O_p(D) \cap D_i = 1$. In the former case we have $a_i \in N_G(H)$ since every subgroup of $D_i$ is normal in $T_i$. Now assume that $O_p(D) \cap D_i = 1$, so $D_i \cap O_p(D) < a_i = 1$ since $D_i \leq D$ and $r \not\in \pi(D)$, so $O_p(D)(a_i) \simeq D_i O_p(D)(a_i) / D_i$ is $\sigma$-nilpotent. Hence $[O_p(D), a_i] = 1$, so $a_i \in N_G(H)$.

Let $V = \langle a_1, a_2, \ldots, a_i \rangle$. Then $V \leq N_G(H)$. Moreover, if $V < R$, then for some $i$ we have $V \leq V_i$. But then $a_i \not\in V_i$ and $a_i \in V \leq V_i \leq V_i$, a contradiction. Therefore $V = R \leq N_G(H)$. Hence $R^G \leq N_G(H)$. Therefore $E^G \leq N_G(H)$, where $E$ is a Hall $\pi(D)'$-subgroup of $G$. But then $E^G D/E^G \simeq D/(D \cap E^G)$ is nilpotent, so $D \leq E^G$ and hence $G = GE = E^G$. Hence we have (6).

(7) If $p$ is a prime such that $(p - 1, |G|) = 1$, then $p$ does not divide $|D|$. In particular, $|D|$ is odd.

Assume that this is false. Then, by Claim (4), $D$ has a maximal subgroup $E$ such that $|D : E| = p$ and $E$ is normal in $G$. It follows that $C_G(D/E) = G$ since $(p - 1, |G|) = 1$. Since $D$ is a Hall subgroup of $G$, it has a complement $M$ in $G$. Hence $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is $\sigma$-nilpotent. Therefore $G/E$ is $\sigma$-nilpotent. But then $D \leq E$, a contradiction. Hence $p$ does not divide $|D|$. In particular, $|D|$ is odd.

(8) $D$ is abelian.

In view of Claim (5), $D$ is a Dedekind group. Hence $D$ is abelian since $|D|$ is odd by Claim (7).

From Claims (5)–(8) we get that $G$ is $\sigma$-soluble $P\sigma T$-group, contrary to the choice of $G$. Hence Statement (iii) holds.

(ii) Assume that this assertion is false and let $G$ be a counterexample of minimal order. Then $G$ is a $\sigma$-soluble $P\sigma T$-group by Part (iii) since every $\sigma$-nilpotent group is a $\sigma$-soluble $P\sigma T$-group. Then $G^{ab}$ is a Hall subgroup of $G$ of odd order and every subgroup of $G^{ab}$ is normal in $G$ by lemma 2.5. Moreover, the hypothesis holds on $G/R$ for every minimal normal subgroup $R$ of $G$ and hence $G/R$ is $\sigma$-nilpotent by the choice of $G$, so $R = G^{ab}$ is a group of prime order $p$ for some prime $p$ and $R$ is a Sylow $p$-subgroup of $G$. But then the maximal $V$ subgroup of $R$ is identity and so $G$ is the unique supplement to $V$ in $G$, so $H$ is $\sigma$-nilpotent, a contradiction. Therefore Statement (ii) holds.

The theorem is proved.
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