MILNOR K-THEORY, F-ISOCRystals AND SYNTOMIC REGulators

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Abstract We introduce a category of filtered F-isocrystals and construct a symbol map from Milnor K-theory to the group of 1-extensions of filtered F-isocrystals. We show that our symbol map is compatible with the syntomic symbol map to the log syntomic cohomology by Kato and Tsuji. These are fundamental materials in our computations of syntomic regulators which we work in other papers.

1. Introduction

The purpose of this paper is to provide fundamental materials for computing the syntomic regulators on Milnor K-theory, which is based on the theory of F-isocrystals.

Let V be a complete discrete valuation ring such that the residue field k is a perfect field of characteristic p > 0 and the fractional field K is of characteristic 0. For a smooth affine scheme S over V, we introduce a category of filtered F-isocrystals, which is denoted by Fil-F-MIC(S) (see §2.1 for the details). Roughly speaking, an isocrystal is a crystalline sheaf which corresponds to a smooth Q_l-sheaf, and “F” means Frobenius action. We refer to the book [LS] for the general terminology of F-isocrystals. As is well-known, it is equivalent to a notion of an integrable connection with Frobenius action. According to this, we shall define the category Fil-F-MIC(S) without the terminology of F-isocrystals. Namely, we define it to be a category of coherent modules endowed with Hodge filtration, integrable connection and Frobenius action, so that the objects are described by familiar and elementary notation. However, the theory of F-isocrystals plays an essential role in verifying several functorial properties. The purpose of this paper is to introduce a symbol map on the Milnor K-group to the group of 1-extensions of filtered F-isocrystals. To be precise, let S be a smooth affine scheme over V and U → S a smooth V-morphism having a good compactification, which means that U → S extends to a projective smooth
morphism $X \to S$ such that $X \setminus U$ is a relative simple normal crossing divisor (abbreviated to NCD) over $S$. Suppose that the comparison isomorphism

$$\vartheta(S)_K^i \otimes_{\vartheta(S)_K} H^i_{\text{dR}}(U_K/S_K) \cong H^i_{\text{rig}}(U_k/S_k)$$

holds for each $i \geq 0$, where $U_K = U \times_Y \text{Spec} K$ and $U_k = U \times_Y \text{Spec} k$. Then, for an integer $n \geq 0$ such that $\text{Fil}^{n+1} H^i_{\text{dR}}(U_K/S_K) = 0$ where $\text{Fil}^i$ is the Hodge filtration, we construct a homomorphism

$$[-]_{U/S} : K^{n+1}_n(\vartheta(U)) \to \text{Ext}^i_{\text{Fil}^i \text{-MIC}(S)}(\vartheta_S, H^n(U/S)(n+1))$$

from the Milnor $K$-group of the affine ring $\vartheta(U)$ to the group of 1-extensions in the category of filtered $F$-isocrystals (Theorem 2.23). We call this the symbol map for $U/S$. We provide an explicit formula of our symbol map (Theorem 2.25). Moreover, we shall give the comparison of our symbol map with the syntomic symbol map $[-]_{\text{syn}} : K^{n+1}_n(\vartheta(U)) \to H^{n+1}_{\text{syn}}(U, \mathbb{Z}_p(n+1))$

to the syntomic cohomology of Fontaine-Messing or, more generally, the log syntomic cohomology (cf. [Ka2, Chapter I §3], [Ts1, §2.2]). See Theorem 3.7 for the details.

Thanks to the recent work by Nekovář–Niziol [N-N], there are the syntomic regulator maps $\text{reg}_{\text{syn}}^{i,j} : K^i(X) \otimes \mathbb{Q} \to H^{2j-i}_{\text{syn}}(X, \mathbb{Q}_p(j))$

in a very general setting, which includes the syntomic symbol maps (up to torsion) and the rigid syntomic regulator maps by Besser [Bes1]. They play the central role in the Bloch-Kato conjecture [B-K] and in the $p$-adic Beilinson conjecture by Perrin-Riou [P, 4.2.2] (see also [Co, Conjecture 2.7]). However, the authors do not know how to construct $\text{reg}_{\text{syn}}^{i,j}$ without “$\otimes \mathbb{Q}$”. We focus on the log syntomic cohomology with $\mathbb{Z}_p$-coefficients since the integral structure is important in our ongoing applications (e.g. [A-C]), namely a deformation method for computing syntomic regulators.

It is a notorious fact that it is never easy to compute the syntomic regulator maps. Indeed, it is nontrivial even for showing the nonvanishing of $\text{reg}_{\text{syn}}^{i,j}$ in a general situation. The deformation method is a method to employ differential equations, which is motivated by Lauder [Lau], who provided the method for computing the Frobenius eigenvalues of a smooth projective variety over a finite field. The overview is as follows. Suppose that a variety $X$ extends to a projective smooth family $f : Y \to S$ with $X = f^{-1}(a)$ and suppose that an element $\xi_X \in K_i(X)$ extends to an element $\xi \in K_i(Y)$. We deduce a differential equation such that a “function” $F(t) = \text{reg}_{\text{syn}}(\xi_{f^{-1}(t)})$ is a solution. Solve the differential equation. Then, we get the $\text{reg}_{\text{syn}}(\xi_X)$ by evaluating $F(t)$ at the point $a \in S$. Of course, this method works only in a good situation; for example, it is powerless if $f$ is a constant family. However, once it works, it has a big advantage in explicit computation of the syntomic regulators. We demonstrate it by a particular example, namely an elliptic curve with 3-torsion points.
Theorem 1.1 (Corollary 4.9). Let $p \geq 5$ be a prime. Let $W = W(\overline{\mathbb{F}}_p)$ be the Witt ring and $K := \text{Frac}(W)$. Let $a \in W$ satisfy $a \not\equiv 0, 1 \mod p$. Let $E_a$ be the elliptic curve over $W$ defined by a Weierstrass equation $y^2 = x^3 + (3x + 4 - 4a)^2$. Let

$$
\xi_a = \{h_1, h_2\} = \left\{ \begin{array}{l}
y - 3x - 4 + 4a \\
y + 3x + 4 - 4a
\end{array} \right\} \in K_2(E_a),
$$

where we note that the divisors $\text{div}(h_i)$ have supports in 3-torsion points. Then, there are overconvergent functions $\varepsilon_1(t), \varepsilon_2(t) \in K \otimes W[t, (1 - t)^{-1}]$ which are explicitly given as in Theorem 4.8 together with (4.17) and (4.18), and we have

$$
\text{reg}_{\text{syn}}(\xi_a) = \varepsilon_1(a) \frac{dx}{y} + \varepsilon_2(a) \frac{x dx}{y} \in H^2_{\text{syn}}(E_a, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(E_a/K).
$$

We note that the function $\varepsilon_i(t)$ is defined in terms of the hypergeometric series

$$
2F_1 \left( \begin{array}{c}
\frac{1}{3}, \frac{2}{3} \\
1
\end{array} ; t \right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n}{n!} \frac{(\frac{2}{3})_n}{n!} t^n, \quad (\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1).
$$

Concerning hypergeometric functions and regulators, the first author obtains more examples in [A]. There, he introduces certain convergent functions which satisfy Dwork type congruence relations [Dw] to describe the syntomic regulators. Also, in the joint paper [A-C], Chida and the first author discuss $K_2$ of elliptic curves in more general situations and obtain a number of numerical verifications of the $p$-adic Beilinson conjecture. In both works, our category Fil-$F$-MIC$(S)$ and symbol maps perform as fundamental materials.

Finally, we comment on the category of syntomic coefficients by Bannai [Ban1, 1.8]. His category is close to our Fil-$F$-MIC$(S)$, and symbol maps perform as fundamental materials. However, he did not work on the symbol maps or regulator maps. Our main interest is the syntomic regulators, especially the deformation method; for this, ours is sufficient.

Notation. For an integral domain $V$ and a $V$-algebra $R$ (resp. $V$-scheme $X$), let $R_K$ (resp. $X_K$) denote the tensoring $R \otimes_V K$ (resp. $X \times_V K$) with the fractional field $K$.

Suppose that $V$ is a complete valuation ring $V$ endowed with a non-archimedean valuation $|\cdot|$. For a $V$-algebra $B$ of finite type, let $B^\dagger$ denote the weak completion of $B$. Namely, if $B = V[T_1, \cdots, T_n]/I$, then $B^\dagger = V[T_1, \cdots, T_n]^{|I}$ is the ring of power series $\sum a_\alpha T^\alpha$ such that for some $r > 1$, $|a_\alpha r^{|\alpha|} \to 0$ as $|\alpha| \to \infty$. We simply write $B^\dagger_K = K \otimes_V B^\dagger$.

2. Filtered $F$-isocrystals and Milnor $K$-theory

In this section, we work over a complete discrete valuation ring $V$ of characteristic 0 such that the residue field $k$ is a perfect field of characteristic $p > 0$. We suppose that $V$ has a $p$-th Frobenius $F_V$, namely an endomorphism on $V$ such that $F_V(x) \equiv x^p \mod pV$, and fix it throughout this section. Let $K = \text{Frac}(V)$ be the fractional field. The extension of $F_V$ to $K$ is also denoted by $F_V$. 
A scheme means a separated scheme which is of finite type over $V$ unless otherwise specified. If $X$ is a $V$-scheme (separated and of finite type), then $\hat{X}_K$ will denote Raynaud’s generic fiber of the formal completion $\hat{X}$, $X^\text{an}_K$ will denote the analytification of the $K$-scheme $X_K$ and $j_X : \hat{X}_K \hookrightarrow X^\text{an}_K$ will denote the canonical immersion [Ber3, (0.3.5)].

2.1. The category of Filtered $F$-isocrystals

Let $S = \text{Spec}(B)$ be an affine smooth variety over $V$. Let $\sigma : B^\dagger \rightarrow B^\dagger$ be a $p$-th Frobenius compatible with $F_V$ on $V$, which means that $\sigma$ is $F_V$-linear and satisfies $\sigma(x) \equiv x^p \mod pB$. The induced endomorphism $\sigma \otimes \mathbb{Z} \mathbb{Q} : B^\dagger_K \rightarrow B^\dagger_K$ is also denoted by $\sigma$. We define the category $\text{Fil-}F$-$\text{MIC}(S, \sigma)$ (which we call the category of filtered $F$-isocrystals on $S$) as follows.

**Definition 2.1.** An object of $\text{Fil-}F$-$\text{MIC}(S, \sigma)$ is a datum $H = (H_{\text{dR}}, H_{\rig}, c, \Phi, \nabla, \text{Fil}^*)$, where

- $H_{\text{dR}}$ is a coherent $B_K$-module,
- $H_{\rig}$ is a coherent $B^\dagger_K$-module,
- $c : H_{\text{dR}} \otimes B_K B^\dagger_K \xrightarrow{\cong} H_{\rig}$ is a $B^\dagger_K$-linear isomorphism,
- $\Phi : \sigma^* H_{\rig} \xrightarrow{\cong} H_{\rig}$ is an isomorphism of $B^\dagger_K$-algebras with $\sigma^* H_{\rig} := B^\dagger_K \otimes_{\sigma, B^\dagger_K} H_{\rig}$,
- $\nabla : H_{\text{dR}} \rightarrow \Omega^1_{B_K} \otimes H_{\text{dR}}$ is an (algebraic) integrable connection and
- $\text{Fil}^*$ is a finite descending filtration on $H_{\text{dR}}$ of locally free $B_K$-module (i.e. each graded piece is locally free),

that satisfies $\nabla(\text{Fil}^i) \subset \Omega^1_{B_K} \otimes \text{Fil}^{i-1}$ and the compatibility of $\Phi$ and $\nabla$ in the following sense. Note first that $\nabla$ induces an integrable connection $\nabla_{\rig} : H_{\rig} \rightarrow \Omega^1_{B^\dagger_K} \otimes H_{\rig}$, where $\Omega^1_{B_K}$ denotes the sheaf of continuous differentials. In fact, firstly regard $H_{\text{dR}}$ as a coherent $\mathcal{O}_{S_K}$-module. Then, by (rigid) analytification, we get an integrable connection $\nabla^{\text{an}}$ on the coherent $\mathcal{O}_{S^\text{an}_K}$-module $(H_{\text{dR}})^{\text{an}}$. Then, apply the functor $j^\dagger_S$ to $\nabla^{\text{an}}$ to obtain an integrable connection on $\Gamma \left( S^\text{an}_K, j^\dagger_S((H_{\text{dR}})^{\text{an}}) \right) = H_{\text{dR}} \otimes_{B_K} B^\dagger_K$. This gives an integrable connection $\nabla_{\rig}$ on $H_{\rig}$ via the isomorphism $c$. Then, the compatibility of $\Phi$ and $\nabla$ means that $\Phi$ is horizontal with respect to $\nabla_{\rig}$, namely $(\sigma \otimes \Phi) \circ \sigma^* \nabla_{\rig} = \nabla_{\rig} \circ \Phi$. We usually write $\nabla_{\rig} = \nabla$ to simplify the notation.

A morphism $H' \rightarrow H$ in $\text{Fil-}F$-$\text{MIC}(S, \sigma)$ is a pair of homomorphisms $(h_{\text{dR}} : H'_{\text{dR}} \rightarrow H_{\text{dR}}, h_{\rig} : H'_{\rig} \rightarrow H_{\rig})$, such that $h_{\rig}$ is compatible with $\Phi$’s, $h_{\text{dR}}$ is compatible with $\nabla$’s and $\text{Fil}^*$’s and, moreover, they agree under the isomorphism $c$.

**Remark 2.2.** (1) The category $\text{Fil-}F$-$\text{MIC}(S, \sigma)$ can also be described by using simpler categories as follows. Let $\text{MIC}(S_K)$ denote the category of filtered $\mathcal{S}_K$-modules with integrable connection – that is, the category of data $(M_{\text{dR}}, \nabla, \text{Fil}^*)$ with $M_{\text{dR}}$ a coherent $B_K$-module, $\nabla$ an integrable connection on $M_{\text{dR}}$ and $\text{Fil}^*$ a finite descending filtration on $M_{\text{dR}}$ of locally free $B_K$-module that satisfies $\nabla(\text{Fil}^i) \subset \Omega^1_{B_K} \otimes \text{Fil}^{i-1}$. Let $\text{MIC}(B^\dagger_K)$ denote the category of coherent $B^\dagger_K$-modules with integrable connections $(M_{\rig}, \nabla)$ on $B^\dagger_K$, and let
$F$-$\text{MIC}(B^\dagger_{K},\sigma)$ denote the category of coherent $B^\dagger_{K}$-modules with integrable connections equipped with $(\sigma$-linear) Frobenius isomorphisms $(M_{\text{rig}},\nabla,\Phi)$. Then, Fil-$F$-$\text{MIC}(S,\sigma)$ is identified with the fiber product

$$F$-$\text{MIC}(B^\dagger_{K},\sigma) \times_{\text{MIC}(B^\dagger_{K})} \text{Fil-MIC}(S_{K}).$$

(2) Let $F$-$\text{Isoc}^\dagger(B_{k})$ denote the category of overconvergent $F$-isocrystals on $S_{k}$. Then, there is the equivalence of categories ([LS, Theorem 8.3.10])

$$F$-$\text{Isoc}^\dagger(B_{k}) \cong F$-$\text{MIC}(B^\dagger_{K},\sigma).$$

Therefore, by combining with the description in (1), we see that our category Fil-$F$-$\text{MIC}(S,\sigma)$ does not depend on $\sigma$, which means that there is the natural equivalence Fil-$F$-$\text{MIC}(S,\sigma) \cong Fil-$F$-$\text{MIC}(S,\sigma')$ (see also Lemma 5.3 in the Appendix). By virtue of this fact, we often drop “$\sigma$” in the notation.

For two objects $H$, $H'$ in the category Fil-$F$-$\text{MIC}(S,\sigma)$, we have a direct sum $H \oplus H'$ and the tensor product $H \otimes H'$ in a customary manner. The unit object for the tensor product, denoted by $B$ or $\mathcal{O}_{S}$, is $(B_{K},B^\dagger_{K},c,p^{-n}\sigma_{B},d,\text{Fil}^\bullet)$, where $c$ is the natural isomorphism, $d$ is the usual differential and $\text{Fil}^\bullet$ is defined by $\text{Fil}^{-n}B_{K} = B_{K}$ and $\text{Fil}^{-n+1}B_{K} = 0$. The category Fil-$F$-$\text{MIC}(S,\sigma)$ forms a tensor category with this tensor product and the unit object $B$ or $\mathcal{O}_{S}$.

The unit object can also be described as $B = B(0)$ or $\mathcal{O}_{S} = \mathcal{O}_{S}(0)$ by using the following notion of Tate object.

**Definition 2.3.** Let $n$ be an integer.

1. The Tate object in Fil-$F$-$\text{MIC}(S)$, which we denote by $B(n)$ or $\mathcal{O}_{S}(n)$, is defined to be $(B_{K},B^\dagger_{K},c,p^{-n}\sigma_{B},d,\text{Fil}^\bullet)$, where $c$ is the natural isomorphism, $d : B_{K} \rightarrow \Omega_{B_{K}}^1$ is the usual differential and $\text{Fil}^\bullet$ is defined by $\text{Fil}^{-n}B_{K} = B_{K}$ and $\text{Fil}^{-n+1}B_{K} = 0$.

2. For an object $H$ of Fil-$F$-$\text{MIC}(S)$, we write $H(n) := H \otimes B(n)$.

Now, we discuss the Yoneda extension groups in the category Fil-$F$-$\text{MIC}(S)$. A sequence

$$H_{1} \rightarrow H_{2} \rightarrow H_{3}$$

in Fil-$F$-$\text{MIC}(S)$ (or in Fil-$\text{MIC}(S_{K})$) is called exact if

$$\text{Fil}^{i}H_{1,\text{dR}} \rightarrow \text{Fil}^{i}H_{2,\text{dR}} \rightarrow \text{Fil}^{i}H_{3,\text{dR}}$$

are exact for all $i$. Then, the category Fil-$F$-$\text{MIC}(S)$ forms an exact category which has kernel and cokernel objects for any morphisms. Thus, the Yoneda extension groups

$$\text{Ext}^{\bullet}(H,H') = \text{Ext}^{\bullet}_{\text{Fil-F-MIC}(S)}(H,H')$$
in \(\text{Fil}-\text{MIC}(S)\) are defined in the canonical way (or one can further define the derived category of \(\text{Fil}-\text{MIC}(S)\) [BBDG, 1.1]). An element of \(\text{Ext}^j(H,H')\) is represented by an exact sequence

\[
0 \rightarrow H' \rightarrow M_1 \rightarrow \cdots \rightarrow M_j \rightarrow H \rightarrow 0
\]

and is subject to the equivalence relation generated by commutative diagrams

\[
\begin{array}{ccccccc}
0 & \rightarrow & H' & \rightarrow & M_1 & \rightarrow & \cdots & \rightarrow & M_j & \rightarrow & H & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H' & \rightarrow & N_1 & \rightarrow & \cdots & \rightarrow & N_j & \rightarrow & H & \rightarrow & 0.
\end{array}
\]

Note that \(\text{Ext}^\bullet(H,H')\) is uniquely divisible (i.e. a \(\mathbb{Q}\)-module) as the multiplication by \(m \in \mathbb{Z}_{>0}\) on \(H\) or \(H'\) is bijective.

Next, we discuss the functoriality of the category \(\text{Fil}-\text{MIC}(S)\) with respect to \(S\). Let \(S' = \text{Spec}(B')\) be another affine smooth variety with a \(p\)-th Frobenius \(\sigma' : B'^{\dagger} \rightarrow B'^{\dagger}\) compatible with \(F_V\) and \(i : S' \rightarrow S\) a morphism of \(V\)-schemes. Then, \(i\) induces a pullback functor

\[
i^* : \text{Fil}-\text{MIC}(S) \rightarrow \text{Fil}-\text{MIC}(S')
\]

in a natural way. In fact, if \(H = (H_{dR}, H_{\text{rig}}, c, \Phi, \nabla, \text{Fil}^\bullet)\) is an object of \(\text{Fil}-\text{MIC}(S)\), then we define \(i^*H = (H_{dR} \otimes_{B_K} B'^{\dagger}_K, H_{\text{rig}} \otimes_{B_K} B'^{\dagger}_K, c \otimes_{B_K} B'^{\dagger}_K, \Phi', \nabla', \text{Fil}^\bullet)\), where \(\nabla'\) and \(\text{Fil}^\bullet\) are natural pullbacks of \(\nabla\) and \(\text{Fil}^\bullet\), respectively, and where \(\Phi'\) is the natural Frobenius structure obtained as follows. We may regard \((H_{\text{rig}}, \nabla_{\text{rig}}, \Phi)\) as an overconvergent \(F\)-isocrystal on \(S\) via the equivalence \(\text{F-MIC}(B_K^{\dagger}) \simeq \text{F-Isoc}^i(B_K)\) [LS, Theorem 8.3.10]. Then, its pullback along \(i_k : S'_k \rightarrow S_k\) is an overconvergent \(F\)-isocrystal on \(S'_k\), which is again identified with an object of \(\text{F-MIC}(B'^{\dagger}_K)\). Thus, it is of the form \((H'_{\text{rig}}, \nabla'_{\text{rig}}, \Phi')\), and \(H'_{\text{rig}}\) is naturally isomorphic to \(H_{\text{rig}} \otimes_{B_K} B'^{\dagger}_K\) ([LS, Prop 8.1.15]). Now, \(\Phi'\) gives the desired Frobenius structure.

### 2.2. The complex \(\mathcal{J}(M)\)

In this subsection, we introduce a complex \(\mathcal{J}(M)\) for each object \(M\) in \(\text{Fil}-\text{MIC}(S,\sigma)\) which is, in the case where \(M = \mathcal{O}_S(r)\), close to the syntomic complex \(\mathcal{J}_n(R)_{S,\sigma}\) of Fontaine–Messing.

Before the definition, we prepare a morphism attached to each object of \(\text{Fil-MIC}(S_K)\). Let \(H = (H_{dR}, \nabla, \text{Fil}^\bullet)\) be an object of \(\text{Fil-MIC}(S_K)\). Let \(\Omega^{\bullet}_{B_K} \otimes \text{Fil}^{-\bullet}H_{dR}\) denote the de Rham complex

\[
\text{Fil}^jH_{dR} \rightarrow \Omega^1_{B_K} \otimes \text{Fil}^{j-1}H_{dR} \rightarrow \cdots \rightarrow \Omega^n_{B_K} \otimes \text{Fil}^{-n}H_{dR} \rightarrow \cdots,
\]

where \(\otimes\) denotes \(\otimes_{B_K}\) and the differentials are given by

\[
\omega \otimes x \mapsto d\omega \otimes x + (-1)^j \omega \wedge \nabla(x), \quad (\omega \otimes x \in \Omega^j_{B_K} \otimes H_{dR}).
\]
Now, we define a natural map
\[
\text{Ext}^i_{\text{Fil-MIC}(S_K)}(\mathcal{O}_{S_K}, H) \to H^i(\Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} H_{\text{dR}})
\] (2.3)
in the following way. Let
\[
0 \to H \to M_i \to M_{i-1} \to \cdots \to M_0 \to \mathcal{O}_{S_K} \to 0
\] (2.4)
be an exact sequence in Fil-MIC(S_K). This induces an exact sequence
\[
0 \to \Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} H \to \Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} M_i \to \cdots \to \Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} M_0 \to \Omega_{B_K}^\bullet \to 0
\]
of complexes, and hence a connecting homomorphism \( \delta : H^0(\Omega_{B_K}^\bullet) \to H^i(\Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} H) \).

By the forgetful functor Fil-F-MIC(S,σ) \to Fil-MIC(S_K), the morphism (2.3) clearly induces a canonical morphism
\[
\text{Ext}^i_{\text{Fil-F-MIC}(S,\sigma)}(\mathcal{O}_S, H) \to H^i(\Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} H_{\text{dR}}).
\] (2.5)

Let \( F\text{-mod} = F\text{-MIC}(\text{Spec}K) \), namely the category of finite-dimensional \( K \)-modules endowed with \( F_V \)-linear bijective homomorphisms. Then, we have a functor
\[
F\text{-MIC}(B_K^\dagger, \sigma) \to D^b(F\text{-mod}), \quad (M_{\text{rig}}, \Phi) \mapsto (\Omega_{B_K}^\bullet \otimes M_{\text{rig}}, \Phi)
\]
to the derived category of complexes in \( F\text{-mod} \), where \( \Phi \) in the target is defined to be \( \sigma \otimes \Phi \) (we use the same notation because we always extend the Frobenius action on the de Rham complex \( \Omega_{B_K}^\bullet \otimes M_{\text{rig}} \) by this rule). Here, we note that, by Beilinson’s lemma [Bei, Lemma 1.4] (as in [Ban3, Theorem 3.2], [DN, Theorem 2.17]), \( D^b(F\text{-mod}) \) is equivalent to the full subcategory of \( D^b(F\text{-mod}') \), where \( F\text{-mod}' \) denotes the category of (possibly infinite-dimensional) \( K \)-modules with \( F_V \)-linear (not necessarily bijective) endomorphism, consisting of complexes whose cohomology groups belong to \( F\text{-mod} \). Since \( \Omega_{B_K}^\bullet \otimes M_{\text{rig}} \) belongs to the latter category, we are regarding it as an object of the former.

We also note that the above functor does not depend on \( \sigma \). Indeed, the composition
\[
F\text{-Isoc}^\dagger(B_k) \to F\text{-MIC}(B_K^\dagger, \sigma) \to D^b(F\text{-mod})
\]
is the functor \((E, \Phi) \mapsto (R\Gamma_{\text{rig}}(S_k, E), \Phi_{\text{rig}})\), where \( \Phi_{\text{rig}} \) denotes the Frobenius action on the rigid cohomology ([LS, Proposition 8.3.12]), and this does not depend on \( \sigma \).

**Definition 2.4.** For an object \( M \in \text{Fil-F-MIC}(S,\sigma) \), we define \( \mathcal{S}(M) \) to be the mapping fiber of the morphism
\[
1 - \Phi : \Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} M_{\text{dR}} \to \Omega_{B_K}^\bullet \otimes M_{\text{rig}},
\]
where \( 1 \) denotes the inclusion \( \Omega_{B_K}^\bullet \otimes \text{Fil}^{-\bullet} M_{\text{dR}} \hookrightarrow \Omega_{B_K}^\bullet \otimes M_{\text{rig}} \) via the comparison and \( \Phi \) is the composition of it with \( \Phi \) on \( \Omega_{B_K}^\bullet \otimes M_{\text{rig}} \).

Note that, in a more down-to-earth manner, each term of \( \mathcal{S}(M) \) is given by
\[
\mathcal{S}(M)^i = \Omega_{B_K}^i \otimes \text{Fil}^{-i} M_{\text{dR}} \oplus \Omega_{B_K}^{i-1} \otimes M_{\text{rig}}
\] (2.6)
and the differential $\mathcal{I}(M)^i \to \mathcal{I}(M)^{i+1}$ is given by

$$(\omega, \xi) \mapsto (d_M \omega, (1 - \Phi) \omega - d_M \xi), \quad (\omega \in \Omega^i_{B_K} \otimes \text{Fil}^{-i} M_{\text{dR}}, \xi \in \Omega^i_{B_K} \otimes M_{\text{rig}}),$$

where $d_M$ is the differential (2.2).

An exact sequence

$$0 \to H \to M_i \to M_{i-1} \to \cdots \to M_0 \to \mathcal{O}_S \to 0$$

in Fil-$\text{MIC}(S, \sigma)$ gives rise to an exact sequence

$$0 \to \mathcal{I}(H) \to \mathcal{I}(M_i) \to \mathcal{I}(M_{i-1}) \to \cdots \to \mathcal{I}(M_0) \to \mathcal{I}(\mathcal{O}_S) \to 0$$

of complexes of $\mathbb{Q}_p$-modules. Let $\delta : \mathbb{Q}_p \cong H^0(\mathcal{I}(\mathcal{O}_S)) \to H^i(\mathcal{I}(H))$ be the connecting homomorphism, where the first isomorphism is given as follows:

$$H^0(\mathcal{I}(\mathcal{O}_S)) = \{ (f, 0) \in B_K \oplus \{ 0 \} = \mathcal{I}(\mathcal{O}_S)^0 \mid df = 0, \sigma(f) = f \} \cong \mathbb{Q}_p.$$  

We define a homomorphism

$$u : \text{Ext}^i_{\text{Fil-MIC}(S, \sigma)}(\mathcal{O}_S, H) \to H^i(\mathcal{I}(H))$$

by associating $\delta(1)$ to the above extension. The composition of $u$ with the natural map $H^i(\mathcal{I}(H)) \to H^i(\Omega^i_{B_K} \otimes \text{Fil}^{-i} H_{\text{dR}})$ agrees with (2.5).

The complex $\mathcal{I}(\mathcal{O}_S(r))$ is close to the syntomic complex of Fontaine-Messing. More precisely, let $S_n := S \times W \text{Spec} W/p^n W$ and $B_n := B/p^n B$. The syntomic complex $\mathcal{I}_n(r)_{S, \sigma}$ is the mapping fiber of the morphism

$$1 - p^{-r} \sigma : \Omega^i_{B_n} \to \Omega^i_{B_n}$$

of complexes where we note that $p^{-r} \sigma$ is well-defined (see [Ka1, p. 410–411]). The $i$-th term of $\mathcal{I}_n(r)_{S, \sigma}$ is

$$\mathcal{I}_n(r)^i_{S, \sigma} = \begin{cases} \{ 0 \} \oplus \Omega^i_{B_n} & \text{if } i < r, \\ \Omega^i_{B_n} \oplus \Omega^i_{B_n} & \text{if } i \geq r, \end{cases}$$

and the differential is given by

$$(\omega, \xi) \mapsto (d\omega, (1 - p^{-r} \sigma) \omega - d\xi).$$

Hence, there is a natural map

$$H^i(\mathcal{I}(\mathcal{O}_S(r))) \to \mathbb{Q} \otimes \varprojlim_n H^i_{\text{zar}}(S_1, \mathcal{I}_n(r)_{S, \sigma}) =: H^i_{\text{syn}}(S, \mathbb{Q}_p(r)).$$  

Let

$$\text{Ext}^i_{\text{Fil-MIC}(S, \sigma)}(\mathcal{O}_S, \mathcal{O}_S(r)) \to H^i_{\text{syn}}(S, \mathbb{Q}_p(r))$$

be the composition morphism. Apparently, both sides of (2.10) depend on $\sigma$. However, if we replace $\sigma$ with $\sigma'$, there is the natural transformation between Fil-$\text{MIC}(S, \sigma)$ and Fil-$\text{MIC}(S, \sigma')$, thanks to the theory of $F$-isocrystals, and there is also a natural transformation on the syntomic cohomology. The map (2.10) is compatible under these transformations. In this sense, (2.10) does not depend on the Frobenius $\sigma$.  


Lemma 2.5. Suppose $\text{Fil}^0 H_{\text{dR}} = 0$. Then, the map $u$ in (2.7) is injective when $i = 1$. Moreover, the map (2.9) is injective when $i = 1$ and $r \geq 0$.

Proof. Let

$$0 \to H \to M \to \mathcal{O}_S \to 0$$

be an exact sequence in $\text{Fil}-\text{MIC}(S, \sigma)$. Since $\text{Fil}^0 H_{\text{dR}} = 0$, there is the unique lifting $e \in \text{Fil}^0 M_{\text{rig}}$ of $1 \in \mathcal{O}_S$. Then,

$$u(M) = (\nabla(e), (1 - \Phi)e) \in H^1(\mathscr{I}(H)) \subset \Omega^1_{B_K} \otimes \text{Fil}^{-1} H_{\text{rig}} \oplus H_{\text{rig}}$$

by definition of $u$. If $u(M) = 0$, then the datum $(B_K e, B_K^1 e, c, \Phi, \nabla, \text{Fil}^*)$ forms a subobject of $M$ which is isomorphic to the unit object $\mathcal{O}_S$. This gives a splitting of the above exact sequence. The latter assertion is immediate as

$$H^1_{\text{syn}}(S, \mathbb{Q}_p(1)) \subset \mathbb{Q} \otimes \lim_{\leftarrow n} (\Omega^1_{B_n} \oplus B_n), \quad H^1_{\text{syn}}(S, \mathbb{Q}_p(r)) \subset \mathbb{Q} \otimes \lim_{\leftarrow n} (\{0\} \oplus B_n), \quad (r \geq 2)$$

by (2.8) and

$$H^1(\mathscr{I}(\mathcal{O}_S(1))) \subset \Omega^1_{B_K} \oplus B_K^1, \quad H^1(\mathscr{I}(\mathcal{O}_S(r))) \subset \{0\} \oplus B_K^1, \quad (r \geq 2)$$

by (2.6).

2.3. Log objects

In this subsection, we introduce the “log object” in $\text{Fil}-\text{MIC}(S)$ concerning a $p$-adic logarithmic function. In the next subsection, it will be generalized to a notion of “polylog object”.

For $f \in B^\times$, let

$$\log^{(\sigma)}(f) := p^{-1} \log \left( \frac{f^p}{f^\sigma} \right) = -p^{-1} \sum_{n=1}^{\infty} \frac{(1 - f^p / f^\sigma)^n}{n} \in B^1.$$

An elementary computation yields $\log^{(\sigma)}(f) + \log^{(\sigma)}(g) = \log^{(\sigma)}(fg)$ for $f, g \in B^\times$.

Definition 2.6. For $f \in B^\times$, we define the log object $\mathcal{L} \log(f)$ in $\text{Fil}-\text{MIC}(S, \sigma)$ as follows.

- $\mathcal{L} \log(f)_{\text{dR}}$ is a free $B_K$-module of rank two; $\mathcal{L} \log(f)_{\text{dR}} = B_K e_{-2} \oplus B_K e_0$.
- $\mathcal{L} \log(f)_{\text{rig}} = B_K^1 e_{-2} \oplus B_K^1 e_0$.
- $c$ is the natural isomorphism.
- $\Phi$ is the $\sigma$-linear morphism defined by

$$\Phi(e_{-2}) = p^{-1} e_{-2}, \quad \Phi(e_0) = e_0 - \log^{(\sigma)}(f)e_{-2}.$$
\begin{itemize}
\item $\nabla$ is the connection defined by $\nabla(e_{-2}) = 0$ and $\nabla(e_0) = \frac{df}{f} e_{-2}$.
\item Fil* is defined by
\[
\text{Fil}^i \nabla \log(f)_{dR} = \begin{cases} 
\log(f)_{dR} & \text{if } i \leq -1, \\
B_K e_0 & \text{if } i = 0, \\
0 & \text{if } i > 0.
\end{cases}
\]
\end{itemize}

This is fit into the exact sequence
\begin{equation}
0 \rightarrow \mathcal{O}_S(1) \xrightarrow{\epsilon} \nabla \log(f) \xrightarrow{\pi} \mathcal{O}_S \xrightarrow{} 0 \tag{2.12}
\end{equation}
in Fil-F-MIC(S), where the two arrows are defined by $\epsilon(1) = e_{-2}$ and $\pi(e_0) = 1$. This defines a class in $\text{Ext}^1_{\text{Fil-F-MIC}(S,\sigma)}(\mathcal{O}_S, \mathcal{O}_S(1))$, which we write by $[f]_S$. It is easy to see that $f \mapsto [f]_S$ is additive. We call the group homomorphism
\begin{equation}
[-]_S : \mathcal{O}(S)^* \rightarrow \text{Ext}^1_{\text{Fil-F-MIC}(S,\sigma)}(\mathcal{O}_S, \mathcal{O}_S(1)) \tag{2.13}
\end{equation}
the symbol map.

**Lemma 2.7.** The composition
\[
\mathcal{O}(S)^{\times} \xrightarrow{[-]_S} \text{Ext}^1_{\text{Fil-F-MIC}(S,\sigma)}(\mathcal{O}_S, \mathcal{O}_S(1)) \rightarrow H^1_{\text{syn}}(S, \mathbb{Q}_p(1))
\]
agrees with the symbol map by Kato [Ka1] where the second arrow is the map (2.10). Namely, it is explicitly described as follows:
\[
f \mapsto \left(\frac{df}{f}, \log(\sigma)(f)\right) \in H^1_{\text{syn}}(S, \mathbb{Q}_p(1)) \subset \Omega^1_{\hat{B}_K} \oplus \hat{B}_K,
\]
where $\hat{B} := \varprojlim_n B/p^n B$ and $\hat{B}_K := \mathbb{Q} \otimes \hat{B}$.

**Proof.** By definition of $u$ in (2.7), one has
\[
u(\nabla \log(f)) = (\nabla(e_0), (1 - \Phi)e_0) = \left(\frac{df}{f}, \log(\sigma)(f)\right)
\]
as desired. \hfill $\square$

**Lemma 2.8.** The symbol map (2.13) is functorial with respect to the pullback. Namely, for a morphism $i : S' \rightarrow S$, the diagram
\[
\begin{array}{ccc}
\mathcal{O}(S)^{\times} & \xrightarrow{[-]_S} & \text{Ext}^1_{\text{Fil-F-MIC}(S,\sigma)}(\mathcal{O}_S, \mathcal{O}_S(1)) \\
i & \downarrow & \downarrow i^* \\
\mathcal{O}(S')^{\times} & \xrightarrow{[-]_{S'}} & \text{Ext}^1_{\text{Fil-F-MIC}(S',\sigma)}(\mathcal{O}_{S'}, \mathcal{O}_{S'}(1))
\end{array}
\]
is commutative, where $i^*$ denotes the map induced from the pullback functor (2.1).

**Proof.** Because of the injectivity of (2.10) (Lemma 2.5), the assertion can be reduced to the compatibility of Kato’s symbol maps by Lemma 2.7. \hfill $\square$
2.4. Polylog objects

In this subsection, we generalize the log object to polylog objects. To define the polylog objects, we need the \( p \)-adic polylog function.

For an integer \( r \), we denote the \( p \)-adic polylog function by

\[
\ln_r^{(p)}(z) := \sum_{n \geq 1, p \nmid n} \frac{z^n}{n^r} = \lim_{s \to \infty} \frac{1}{1 - z^{p^s}} \sum_{1 \leq n < p^s, p \nmid n} \frac{z^n}{n^r} \in \mathbb{Z}_p \left[ \frac{1}{1 - z} \right]^\wedge, \tag{2.14}
\]

where \( A^\wedge \) denotes the \( p \)-adic completion of a ring \( A \). As is easily seen, one has

\[
\ln_r^{(p)}(z) = (-1)^{r+1} \ln_{r+1}^{(p)}(z^{-1}), \quad z \frac{d}{dz} \ln_{r+1}^{(p)}(z) = \ln_r^{(p)}(z).
\]

If \( r \leq 0 \), this is a rational function. Indeed,

\[
\ln_0^{(p)}(z) = \frac{1}{1 - z} - \frac{1}{1 - z^p}, \quad \ln_{-r}^{(p)}(z) = \left(z \frac{d}{dz}\right)^r \ln_0^{(p)}(z).
\]

If \( r \geq 1 \), it is no longer a rational function but an overconvergent function.

**Proposition 2.9.** Let \( r \geq 1 \). Put \( x := (1 - z)^{-1} \). Then, \( \ln_r^{(p)}(z) \in (x - x^2)\mathbb{Q}_p[x]^\dagger \).

**Proof.** Since \( \ln_r^{(p)}(z) \) has \( \mathbb{Z}_p \)-coefficients, it is enough to show that \( \ln_r^{(p)}(z) \in (x - x^2)\mathbb{Q}_p[x]^\dagger \). We first note that

\[
(x^2 - x) \frac{d}{dz} \ln_{k+1}^{(p)}(z) = \ln_k^{(p)}(z). \tag{2.15}
\]

The limit in (2.14) can be rewritten as

\[
\lim_{s \to \infty} \frac{1}{x^{p^s} - (x - 1)^{p^s}} \sum_{1 \leq n < p^s, p \nmid n} \frac{x^{p^s} (1 - x^{-1})^n}{n^r}.
\]

This shows that \( \ln_r^{(p)}(z) \) vanishes at \( x = 0, 1 \). Therefore, it suffices to show that \( \ln_r^{(p)}(z) \in \mathbb{Q}_p[x]^\dagger \) because then it is divisible by \( x - x^2 \) in \( \mathbb{Q}_p[x]^\dagger \) by [Ma, Theorem 3.5].

Let \( w(x) \in \mathbb{Z}_p[x] \) be defined by

\[
\frac{1 - z^p}{(1 - z)^p} = x^p - (x - 1)^p = 1 - pw(x).
\]

Then,

\[
\ln_1^{(p)}(z) = p^{-1} \log \left( \frac{1 - z^p}{(1 - z)^p} \right) = -p^{-1} \sum_{n=1}^{\infty} \frac{p^n w(x)^n}{n} \in \mathbb{Z}_p[x]^\dagger.
\]

This shows that \( \ln_1^{(p)}(z) \in (x - x^2)\mathbb{Q}_p[x]^\dagger \), as required in case \( r = 1 \).

Now, let

\[
-(x - x^2)^{-1} \ln_1^{(p)}(z) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \in \mathbb{Z}_p[x]^\dagger.
\]
By (2.15), one has
\[
\ln_2^{(p)}(z) = c + a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1} + \cdots \in \mathbb{Q}_p[x],
\]
and hence, \( \ln_2^{(p)}(z) \in (x - x^2)\mathbb{Q}_p[x] \), as required in case \( r = 2 \). Continuing the same argument, we obtain \( \ln_r^{(p)}(z) \in (x - x^2)\mathbb{Q}_p[x] \) for every \( r \).

\[\square\]

Remark 2.10. The proof shows that \( \ln_r^{(p)}(z) \) converges on an open disk \( |x| < |1 - \zeta_p| \).

Definition 2.11. Let \( C = V[T,T^{-1},(1 - T)^{-1}] \) and \( \sigma_C \) a \( p \)-th Frobenius such that \( \sigma_C(T) = T^p \). Let \( n \geq 1 \) be an integer. We define the \( n \)-th polylog object \( \mathcal{P}ol_n(T) \) of Fil-\( p \)-\text{MIC}(Spec\( C \)) as follows.

- \( \mathcal{P}ol_n(T)_{\text{dR}} \) is a free \( C_K \)-module of rank \( n+1 \); \( \mathcal{P}ol_n(T)_{\text{dR}} = \bigoplus_{j=0}^n C_K e_{-2j} \).
- \( \mathcal{P}ol_n(T)_{\text{rig}} := \mathcal{P}ol_n(T)_{\text{dR}} \otimes_{C_K} C_K^\dagger \).
- \( c \) is the natural isomorphism.
- \( \Phi \) is the \( C_K^\dagger \)-linear morphism defined by
  \[
  \Phi(e_0) = e_0 - \sum_{j=1}^n (-1)^j \ln_j^{(p)}(T)e_{-2j}, \quad \Phi(e_{-2j}) = p^{-j}e_0, \quad (j \geq 1).
  \]
- \( \nabla \) is the connection defined by
  \[
  \nabla(e_0) = \frac{dT}{T-1} e_{-2}, \quad \nabla(e_{-2j}) = \frac{dT}{T} e_{-2j-2}, \quad (j \geq 1),
  \]
  where \( e_{-2n-2} := 0 \).
- \( \text{Fil}^* \) is defined by \( \text{Fil}^m \mathcal{P}ol_n(T)_{\text{dR}} = \bigoplus_{0 \leq j \leq -m} C_K e_{-2j} \). In particular, \( \text{Fil}^m \mathcal{P}ol_n(T)_{\text{dR}} = \mathcal{P}ol_n(T)_{\text{dR}} \) if \( m \leq -n \) and \( = 0 \) if \( m \geq 1 \).

When \( n = 2 \), we also write \( \text{dilog}(T) = \mathcal{P}ol_2(T) \) and call it the \text{dilog} object.

For a general \( S = \text{Spec}(B) \) and \( f \in B \) satisfying \( f, 1 - f \in B^\times \), we define the polylog object \( \mathcal{P}ol_n(f) \) to be the pullback \( u^* \mathcal{P}ol_n(T) \), where \( u : \text{Spec}(B) \to \text{Spec} V[T,T^{-1}, (1 - T)^{-1}] \) is given by \( u(T) = f \). When \( n = 1 \), \( \mathcal{P}ol_1(T) \) coincides with the log object \( \mathcal{L}og(1 - T) \) in Fil-\( F \)-\text{MIC}(C).

2.5. Relative cohomologies.

Let \( S = \text{Spec}(B) \) be a smooth affine \( V \)-scheme and let \( \sigma \) be a \( p \)-th Frobenius on \( B^\dagger \). In this subsection, we discuss objects in Fil-\( F \)-\text{MIC}(S) arising as a relative cohomology of smooth morphisms.

Let \( u : U \to S \) be a quasi-projective smooth morphism. We first describe a datum which we discuss in this subsection.

Definition 2.12. We define a datum
\[H^i(U/S) = (H_{\text{dR}}^i(U_K/S_K), H_{\text{rig}}^i(U_k/S_k), c, \nabla, \Phi, \text{Fil}^*) \]
as follows.
• $H^i_{\text{dR}}(U_K/S_K)$ is the $i$-th relative algebraic de Rham cohomology of $u_K$, namely the module of global sections of the $i$-th cohomology sheaf $R^i(u_K)_*\Omega^i_{U_K/S_K}$.

• $\nabla$ is the Gauss–Manin connection on $H^i_{\text{dR}}(U_K/S_K)$, and $\text{Fil}^i$ is the Hodge filtration defined from the theory of mixed Hodge modules by M. Saito [Sa1, Sa2] (however, we shall only be concerned with the de Rham cohomology under the setting 2.14 or 2.16, and then the Hodge filtration can be defined from the Hodge to de Rham spectral sequences).

• $(H^i_{\text{rig}}(U_k/S_k),\Phi)$ is the $B^1_{K^\text{rig}}$-module with $\sigma$-linear Frobenius structure obtained as the $i$-th relative rigid cohomology of $u_k$. In particular,

$$H^i_{\text{rig}}(U_k/S_k) = \Gamma(S^\text{an}_K, R^i u_{\text{rig}}^t j_{U_k}^t \mathcal{O}_{U_K^\text{rig}}),$$

where $R^i u_{\text{rig}}^t j_{U_k}^t \mathcal{O}_{U_K^\text{rig}} := R^i(u_K^\text{an})_* j_{U_k}^t \Omega^\text{rig}_{U_K^\text{an}}$ is the $i$-th relative rigid cohomology sheaf (we justify this notation and definition in Remark 2.13 below).

• $c: H^i_{\text{dR}}(U_K/S_K) \otimes_{B_K} B^1_K \to H^i_{\text{rig}}(U_k/S_k)$ is the natural morphism between the algebraic de Rham cohomology and the rigid cohomology.

Let us give a construction of the comparison morphism $c$ in this datum. We basically follow the construction in [So, 5.8.2]. Let $i: S^\text{an}_K \to S_K$ be the natural morphism of ringed topoi [Ber3, 0.3]. Then, by adjunction, we have a natural morphism $R^i u_{K^\text{rig}}^t \Omega^\text{rig}_{U_K/S_K} \to \tau_*(R^i u_K^\text{an})_* \Omega^\text{rig}_{U_K^\text{an}}$. Now, together with the natural morphisms

$$(R^i u_{K^\text{rig}}^t \Omega^\text{rig}_{U_K/S_K})^\text{an} \to R^i(u_K^\text{an})_* \Omega^\text{rig}_{U_K^\text{an}} \to R^i(u_K^\text{an})_* j_{U_K^\text{an}} \Omega^\text{rig}_{U_K^\text{an}} / S^\text{an}_K,$$

we get a morphism $R^i u_{K^\text{rig}}^t \Omega^\text{rig}_{U_K/S_K} \to \tau_* R^i(u_K^\text{an})_* j_{U_K^\text{an}} \Omega^\text{rig}_{U_K^\text{an}} / S^\text{an}_K$. By taking the module of global sections, we get a morphism $H^i_{\text{dR}}(U_K/S_K) \to H^i_{\text{rig}}(U_k/S_k)$, and therefore, the desired morphism $c: H^i_{\text{dR}}(U_K/S_K) \otimes_{B_K} B^1_K \to H^i_{\text{rig}}(U_k/S_k)$ because $H^i_{\text{rig}}(U_k/S_k)$ is a $B^1_K$-module.

**Remark 2.13.** (1) Let us justify our description of the rigid cohomology $H^i_{\text{rig}}(U_k/S_k)$ in Definition 2.12.

We begin by recalling a usual definition of the rigid cohomology of $u_k : U_k \to S_k$ over a frame $(S_k, S^\text{an}_K, \hat{S})$, where $S$ is a closure of $S$ in a projective space over $V$ and $\hat{S}$ is its completion. Let $\mathcal{B}$ be the closure of $U$ in a projective space over $\hat{S}$ and let $\mathcal{J} : \mathcal{B} \to \hat{S}$ be the extension of $u$ (we choose this notation because, in this article, $\mathcal{X}$ usually denotes $\mathcal{J}^{-1}(S)$ and $f$ denotes $f|_{\mathcal{X}}$). Then, by definition, the $i$-th relative rigid cohomology sheaf is $R^i(\mathcal{J}_K^\text{an})_* j_{U_K^\text{an}} \Omega^\text{rig}_{U_K^\text{an}} / \mathcal{S}_K^\text{an}$, where $\mathcal{J}_U : \hat{U}_K \to \mathcal{S}_K$ is the natural inclusion and $H^i_{\text{rig}}(U_k/S_k)$ is the module of global sections of this sheaf on $\mathcal{S}_K$.

However, by using the fact that $u: U \to S$ is a lift of $u_k: U_k \to S_k$ to a smooth morphism of smooth algebraic $V$-schemes, we may obtain the same module without referring to compactifications. In fact, first, since $S^\text{an}_K$ is a strict neighborhood of $\hat{S}_K$ [Ber3,

$^1$In [Sa1], [Sa2], the mixed Hodge modules are defined for complex algebraic varieties. However, the construction of the Hodge filtration and $D$-module structures work over an arbitrary field of characteristic zero.
(1.2.4)(ii), \( H^i_{\text{rig}}(U_k/S_k) \) is also the module of global sections of \( R^i\left( \overline{\mathcal{F}}^n_{\mathbb{K}} \right) \neq \Omega^n_{\overline{\mathcal{X}}_{\mathbb{K}}/\mathbb{S}^n_{\mathbb{K}}} \) on \( S^n_{\mathbb{K}} \) by overconvergence. Moreover, the restriction of this sheaf on \( S^n_{\mathbb{K}} \) is isomorphic to \( R^i\left( u^n_{\mathbb{K}} \right) \Omega^n_{U^n_{\mathbb{K}}/S^n_{\mathbb{K}}} \) by [LS, 6.2.2] because, again, \( U^n_{\mathbb{K}} \) is a strict neighborhood of \( \hat{U}_{\mathbb{K}} \).

Our definition of \( R^i u_{\text{rig}} j_!^! \Omega U^n_{\mathbb{K}} \) and the description of \( H^i_{\text{rig}}(U_k/S_k) \) in Definition 2.12 are thus justified.

(2) We will use the datum \( H^i_{\text{rig}}(U/S) \) only in the case where the rigid cohomology sheaf \( R^i u_{\text{rig}} j_!^! \Omega U^n_{\mathbb{K}} \) is known to be a coherent \( j_S^! \mathcal{O}_{S^n_{\mathbb{K}}} \)-module for all \( i \geq 0 \). In this case, we also have

\[
H^i_{\text{rig}}(U_k/S_k) = \Gamma(S^n_{\mathbb{K}}, R^i u_{\text{rig}} j_!^! \Omega U^n_{\mathbb{K}})
\]

by the vanishing of higher sheaf cohomologies for coherent \( j_S^! \mathcal{O}_{S^n_{\mathbb{K}}} \)-modules (this vanishing is perhaps well-known, but we included it as Lemma 2.19 at the end of this subsection because we could not find an appropriate reference).

Now, we have defined the datum \( H^i(U/S) \). This, however, does not immediately mean that it is an object of \( \text{Fil}-F\text{-MIC}(S) \). For this datum to be an object in \( \text{Fil}-F\text{-MIC}(S) \), we need the \( i \)-th relative cohomology \( H^i_{\text{rig}}(U_k/S_k) \) to be a coherent \( B^!_{\mathbb{K}} \)-module with Frobenius structure, and we need the morphism \( c \) to be an isomorphism. In the rest of this subsection, we discuss two settings under which these conditions hold. Briefly said, these two settings are: the case of proper smooth morphisms (Setting 2.14) and the case of smooth families of general dimension with “good compactification” of both the source and the target (Setting 2.16).

We start with the first setting.

**Setting 2.14.** \( u: U \to S \) is a projective smooth morphism of smooth \( V \)-schemes with \( S = \text{Spec}(B) \).

**Proposition 2.15.** Under Setting 2.14, the relative rigid cohomology sheaf \( R^i u_{\text{rig}} j_!^! \Omega U^n_{\mathbb{K}} \) is a coherent \( j_S^! \mathcal{O}_{S^n_{\mathbb{K}}} \)-module with Frobenius structure for each \( i \geq 0 \). Consequently, \( H^i_{\text{rig}}(U_k/S_k) \) is a coherent \( F B^!_{\mathbb{K}} \)-module for each \( i \geq 0 \). Moreover, the comparison morphism

\[
c: H^i_{\text{dR}}(U_K/S_K) \otimes_{B_K} B^!_{\mathbb{K}} \to H^i_{\text{rig}}(U_k/S_k)
\]

is bijective for each \( i \geq 0 \).

In particular, the datum

\[
H^i(U/S) = (H^i_{\text{dR}}(U_K/S_K), H^i_{\text{rig}}(U_k/S_k), c, \nabla, \Phi, \text{Fil}^*)
\]

is an object of \( \text{Fil}-F\text{-MIC}(S) \).

**Proof.** The first statement follows from Berthelot’s result [Laz, Theorem 4.1], [Ber, Théorème 5]. Now, since \( c \) is a morphism of coherent modules over the noetherian ring \( B^!_{\mathbb{K}} \), it suffices to prove that it is an isomorphism on the reduction by each maximal ideal of \( B^!_{\mathbb{K}} \), which is the extension of a maximal ideal of \( B_K \). Therefore, we may assume that \( S = \mathbb{S} = \text{Spec}(k) \) (after a possible extension of \( k \)), and then the claim follows from
comparison of the (absolute) algebraic de Rham cohomology and the rigid (or, in this case, crystalline) cohomology (e.g. [A-Bal, 4.2], [G, (7)]).

The second sufficient condition for $H^i(U/S)$ to be an object of Fil-$F$-MIC($S$) is, briefly said, that $u$ has a “good compactification”.

**Setting 2.16.** Let $S = \text{Spec}(B)$ be a smooth affine $V$-scheme and let $\overline{S}$ be a projective smooth $V$-scheme with an open immersion $S \hookrightarrow \overline{S}$ such that the complement $T$ is a relative simple NCD on $\overline{S}$ over $V$. Let $\overline{X}$ be a projective smooth $V$-scheme and let $\overline{f}: \overline{X} \rightarrow \overline{S}$ be a projective morphism. Let $\overline{D}$ be a relative NCD on $\overline{X}$ over $V$ and put $\overline{D}^v = \overline{f}^{-1}(T)$ and $\overline{D} = \overline{D}^h \cup \overline{D}^v$. We put $X = \overline{X} \setminus \overline{D}$, $f = \overline{f}|_X$ and $U = \overline{X} \setminus \overline{D}$. We then assume that the following conditions hold:

1. $\overline{D} = \overline{D}^h \cup \overline{D}^v$ is also a relative NCD over $V$.
2. $D := \overline{D}^h \cap X \hookrightarrow X$ is a relative NCD over $S$.
3. The morphism $\overline{f}: (\overline{X}, \overline{D}) \rightarrow (\overline{S}, T)$ is log smooth and integral, and $(\overline{S}, T)$ is of Zariski type.

The notation in Setting 2.16 can be summarized by the diagram

\[
\begin{array}{ccc}
X & \overset{\overline{f}}{\longrightarrow} & \overline{X} \\
\downarrow & & \downarrow \overline{f} \\
S & \overset{f}{\longrightarrow} & \overline{S}
\end{array}
\]

where the notation above $\hookrightarrow$ shows the complement of the subscheme.

**Proposition 2.17.** Under Setting 2.16, the relative rigid cohomology sheaf $R^i u_{\text{rig}}^! j_L^! \mathcal{O}_{U_K}$ is a coherent $j_S^! \mathcal{O}_{\overline{S}_{K_{\text{an}}}}$-module with Frobenius structure for each $i \geq 0$. Consequently, $H^i_{\text{rig}}(U_k/S_k)$ is a coherent $F\cdot B_K^!$-module for each $i \geq 0$. Moreover, the comparison morphism

\[
c: H^i_{dR}(U_K/S_K) \otimes_{B_K} B^!_K \rightarrow H^i_{\text{rig}}(U_k/S_k)
\]

is bijective for each $i \geq 0$.

In particular, the datum

\[
H^i(U/S) = (H^i_{dR}(U_K/S_K), H^i_{\text{rig}}(U_k/S_k), c, \nabla, \Phi, \text{Fil}^*)
\]

is an object of Fil-$F$-MIC($S$).

**Proof.** Our setting assures us that we are in the situation of [Sh3, Section 2], i.e. the assumptions before [Sh3, Theorem 2.1] are satisfied. Moreover, since $f$ is integral, the assumption $(\star)$ in [Sh3, Theorem 2.1] and $(\star)'$ in [Sh3, Theorem 2.3] are also satisfied [Sh2, Corollary 4.7]. Therefore, the first statement follows from [Sh3, Theorem 2.2] and [Sh3, Theorem 2.4].

Now that the coherence is proved for all $i$, the proof reduces to the absolute case as in the proof of Proposition 2.15. Then, since the algebraic de Rham cohomology (resp.
the rigid cohomology) is isomorphic to the algebraic log de Rham cohomology (resp. log rigid cohomology by e.g. [Tz1, Theorem 3.5.1]), the claim follows from the comparison theorem between algebraic log de Rham cohomology and log rigid cohomology [Bal-Ch, Corollary 2.6]. □

We also have a Gysin exact sequence in Fil-$F$-MIC$(S)$ for curves under this setting.

**Proposition 2.18** (Gysin exact sequence). Let $U = \text{Spec}(A)$ and $S = \text{Spec}(B)$ be smooth affine $V$-schemes and let $u: U \rightarrow S$ be a smooth morphism of relative dimension one with connected fibers. Assume that there exists a projective smooth curve $f: X \rightarrow S$ with an open immersion $U \hookrightarrow X$ such that $f|_U = u$ and that the complementary divisor $D := X \setminus U$ is finite étale over $S$. Moreover, assume that $u$ satisfies the conclusions of Proposition 2.17 (namely, the coherence of the rigid cohomology and the bijectivity of the comparison morphism), e.g. that we are in Setting 2.16.

Then, we have an exact sequence

$$0 \rightarrow H^1(X/S) \rightarrow H^1(U/S) \rightarrow H^0(D/S)(-1) \rightarrow H^2(X/S) \rightarrow 0.$$ 

in Fil-$F$-MIC$(S)$.

**Proof.** First, $H^1(X/S)$ and $H^1(D/S)$ are objects of Fil-$F$-MIC$(S)$ by Proposition 2.15, and so is $H^1(U/S)$ by assumption. Next, it is a standard fact about de Rham cohomology that we have an exact sequence

$$0 \rightarrow H^1_{\text{dR}}(X_K/S_K) \rightarrow H^1_{\text{dR}}(U_K/S_K) \rightarrow H^0_{\text{dR}}(D_K/S_K)(-1) \rightarrow H^2_{\text{dR}}(X_K/S_K) \rightarrow 0$$

whose morphisms are horizontal and compatible with respect to Fil. Therefore, by the comparison isomorphism on each term and by the flatness of $B_K^\dagger$ over $B_K$, we get a corresponding exact sequence

$$0 \rightarrow H^1_{\text{rig}}(X_k/s_k) \rightarrow H^1_{\text{rig}}(U_k/s_k) \rightarrow H^0_{\text{rig}}(D_k/s_k)(-1) \rightarrow H^2_{\text{rig}}(X_k/s_k) \rightarrow 0$$

for rigid cohomologies. The compatibility of this sequence with Frobenius structures on each term can be checked on each closed point of $\hat{S}_K$ and therefore reduced to the absolute case [Ban2, Theorem 2.19]. □

The following lemma is the promised statement in Remark 2.13 (2).

**Lemma 2.19.** Let $X$ be a smooth affine $V$-scheme and let $\mathcal{M}$ be a coherent $j^\dagger_X \mathcal{O}_{X_K^an}$-module. Then, for any $j \geq 1$, we have $H^j(X_K^an, \mathcal{M}) = 0$.

**Proof.** This lemma is essentially given in the proof of [LS, 6.2.12]. We recall the argument for the convenience of the reader. Choose a closed immersion $X \hookrightarrow \mathbb{A}^N_V$ to an affine space and let $Y$ be the closure of $X$ in $X \hookrightarrow \mathbb{A}^N_V \hookrightarrow \mathbb{P}^N_V$. Then, $V_\rho := X_K^an \cap B^N(0, \rho^+)$ for $\rho > 1$ form a cofinal family of strict neighborhoods of $X_K$.

By the coherence of $\mathcal{M}$, we can take a coherent $\mathcal{O}_{V_{\rho_0}}$-module $M$ for some $\rho_0 > 1$ such that $\mathcal{M}|_{V_{\rho_0}} = j_X^\dagger M$ [LS, 5.4.4]. Then, if $j_\rho: V_\rho \hookrightarrow V_{\rho_0}$ denotes the inclusion for $1 < \rho < \rho_0$, we have isomorphisms

$$\text{RG}^j(X_K^an, \mathcal{M}) = \text{RG}^j(V_{\rho_0}, \mathcal{M}|_{V_{\rho_0}}) = \lim_{\rho} \text{RG}^j(V_{\rho_0}, (j_\rho)_*M) = \lim_{\rho} \text{RG}^j(V_\rho, M|_{V_\rho}).$$
In fact, the first identification holds because $\mathcal{M}$ is a $j^+_U \mathcal{O}_{X_K^an}$-module by a standard argument (as in [Bal-Ber, 1.1]), the second one follows from quasi-compactness and separatedness of $V_{\rho_0}$ and the third one holds because $j_\rho$ is affinoid. Now, the claim follows because each $V_\rho$ is affinoid and $M|_{V_\rho}$ is coherent.

\section{Extensions associated to Milnor symbols}

In this subsection, we discuss how we associate an extension in Fil-$F$-MIC$(U)$ to a Milnor symbol.

Let $\rho: U = \text{Spec}(A) \to S = \text{Spec}(B)$ be a smooth morphism of smooth affine $V$-scheme.

We assume that $\rho$ satisfies the consequences of Proposition 2.17.

\begin{assumption}
For each $i \geq 0$, the $i$-th relative rigid cohomology sheaf $R^i u_{\text{rig}} j^+_U \mathcal{O}_{U_K^an}$ is a coherent $j^+_S \mathcal{O}_{S_K^an}$-module with Frobenius structure and the natural morphism
\[ H^i_{\text{dR}}(U_K/S_K) \otimes_{B_K} B^+_K \to H^i_{\text{rig}}(U_k/S_k) \]
\end{assumption}

(2.17) is bijective.

\begin{remark}

(1) As we have discussed in Proposition 2.17, Assumption 2.20 is satisfied if we are in Setting 2.16.

(2) Assume that there is a projective smooth morphism $f: X \to S$ with an open immersion $U \to X$ such that $X \setminus U$ is a relative simple normal crossing divisor over $S$. Then, the coherence of the $i$-th relative rigid cohomology sheaf for each $i \geq 0$ assures the rest of Assumption 2.20. In fact, we may prove that (2.17) is an isomorphism as in the proof of Proposition 2.17 and that $\Phi$ is an isomorphism as in the proof of [Sh3, Theorem 2.4].

(3) Note that (a part of) Assumption 2.20 allows us to interpret the relative rigid cohomology as a cohomology of Monsky–Washnitzer type.

More precisely, for a smooth morphism $\rho: U = \text{Spec}(A) \to S = \text{Spec}(B)$ of affine smooth $V$-schemes, assume that, for each $i \geq 0$, the $i$-th rigid cohomology sheaf $R^i u_{\text{rig}} j^+_U \mathcal{O}_{U_K^an}$ satisfies $H^j(S^an_K, R^i u_{\text{rig}} j^+_U \mathcal{O}_{U_K^an}) = 0$ for all $j \geq 1$ (e.g. if all $R^i u_{\text{rig}} j^+_U \mathcal{O}_{U_K^an}$ are coherent). Then, the $B^+_K$-module $H^i_{\text{rig}}(U_k/S_k) = \Gamma(S^an_K, R^i u_{\text{rig}} j^+_U \mathcal{O}_{S_K^an})$ is isomorphic to the cohomology $H^i_{\text{MW}}(U_k/A_k) = H^i(\Omega^\bullet_{A_K/B_K}^{\text{MW}})$ of the complex of continuous differentials $\Omega^\bullet_{A_K/B_K}^{\text{MW}} = \Gamma(U^an_K, j^+_U \Omega^\bullet_{U_K^an}^{\text{MW}}/S^an_K)$. This follows from our assumption and the vanishing of the cohomologies $H^j(U^an_K, j^+_U \Omega^k_{U_K^an}^{\text{MW}}/S^an_K) = 0$ for all $j \geq 1$ and $k \geq 0$ (which also follows from Lemma 2.19).

Recall that, for a commutative ring $R$, the $r$-th Milnor $K$-group $K^M_r(R)$ is defined to be the quotient of $(R^\times)^{\otimes r}$ by the subgroup generated by
\[ a_1 \otimes \cdots \otimes b \otimes \cdots \otimes (-b) \otimes \cdots \otimes a_r, \quad a_1 \otimes \cdots \otimes b \otimes \cdots \otimes (1-b) \otimes \cdots \otimes a_r. \]

Recall from §2.3 the log object $\mathcal{L}og(f)$ in Fil-$F$-MIC$(U)$ for $f \in \mathcal{O}(U)^\times$ and the extension (2.12) which represents the class
\[ [f]_U \in \text{Ext}^1_{\text{Fil-F-MIC}(U)}(\mathcal{O}_U, \mathcal{O}_U(1)). \]
For \( h_0, h_1, \ldots, h_n \in \mathcal{O}(U)^\times \), we associate an \((n + 1)\)-extension
\[
0 \rightarrow \mathcal{O}_U(n + 1) \rightarrow \log(h_n)(n) \rightarrow \cdots \rightarrow \log(h_1)(1) \rightarrow \log(h_0) \rightarrow \mathcal{O}_U \rightarrow 0
\]
which represents the class
\[
[h_0]_U \cup \cdots \cup [h_n]_U \in \text{Ext}_{\text{Fil}-\text{MIC}(U)}^{n+1}(\mathcal{O}_U, \mathcal{O}_U(n+1)).
\]

It is a standard argument to show that the above cup-product is additive with respect to each \( h_i \), so that we have an additive map
\[
(\mathcal{O}(U)^\times)^{\otimes n+1} \rightarrow \text{Ext}_{\text{Fil}-\text{MIC}(U)}^{n+1}(\mathcal{O}_U, \mathcal{O}_U(n+1)).
\]

**Proposition 2.22.** \([f]_U \cup [f]_U = 0\) for \( f \in \mathcal{O}(U)^\times \) and \([f]_U \cup [1 - f]_U = 0\) for \( f \in \mathcal{O}(U)^\times \) such that \( 1 - f \in \mathcal{O}(U)^\times \). Hence, the homomorphism
\[
K_{n+1}^M(\mathcal{O}(U)) \rightarrow \text{Ext}_{\text{Fil}-\text{MIC}(U)}^{n+1}(\mathcal{O}_U, \mathcal{O}_U(n+1)), \quad \{h_0, \ldots, h_n\} \mapsto [h_0]_U \cup \cdots \cup [h_n]_U
\]
is well-defined (note \( \log(-f) = \log(f) \) by definition).

**Proof.** To prove this, it follows from Lemma 2.8 that we may assume \( U = \text{Spec} V[T, T^{-1}] \) and \( f = T \) for the vanishing \([f]_U \cup [f]_U = 0\) and \( U = \text{Spec} V[T, (T - T^2)^{-1}] \) and \( f = T \) for the vanishing \([f]_U \cup [1 - f]_U = 0\).

Here, we show the latter vanishing. Let \( C = V[T, (T - T^2)^{-1}] \) and \( U = \text{Spec}(C) \). Recall the dilog object \( D := \text{Dilog}(T) \) which has a unique increasing filtration \( W_* \) (as an object of Fil-F-MIC(U)) that satisfies
\[
W_j D_{\text{adR}} = \begin{cases} 
0 & \text{if } j \leq -5, \\
C_K e_{-4} & \text{if } j = -4, -3, \\
C_K e_{-4} \oplus C_K e_{-2} & \text{if } j = -2, -1, \\
\text{Dilog}(T)_{\text{adR}} & \text{if } j \geq 0
\end{cases}
\]
and the filtration Fil* on \( W_j D_{\text{adR}} \) is given to be Fil* \( W_j D_{\text{adR}} = W_j D_{\text{adR}} \cap \text{Fil}^i D_{\text{adR}} \). Then, it is straightforward to check that \( W_{-4} \cong \mathcal{O}_U(2), W_{-2} \cong \text{log}(T)(1), W_0/W_{-4} \cong \text{log}(1 - T) \) and \( W_0/W_{-2} \cong \mathcal{O}_U \). Thus, \( [T]_U \cup [1 - T]_U \) is realized by the extension
\[
0 \rightarrow W_{-4} \rightarrow W_{-2} \rightarrow W_0/W_{-4} \rightarrow W_0/W_{-2} \rightarrow 0.
\]

Consider a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_U(2) & \rightarrow & \mathcal{O}_U(2) & \rightarrow & \mathcal{O}_U(0) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_U(2) & \rightarrow & \mathcal{O}_U(2) & \oplus & W_{-2} & \rightarrow & \mathcal{O}_U(0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & W_{-4} & \rightarrow & W_{-2} & \rightarrow & W_0/W_{-4} & \rightarrow & 0
\end{array}
\]
where \( \iota \) is the first inclusion, \( \pi_2 \) is the first projection, \( \pi_2 \) is the composition with the second projection and the inclusion \( W_{-2} \hookrightarrow W_0 \), \( \pi_3 \) is the quotient \( W_0 \to W_0/W_{-2} \cong \mathcal{O}_U \), and add: \((x,y) \mapsto x+y\). The above diagram shows the vanishing \([T]_U \cup [1-T]_U = 0\).

Let \( n \) be a nonnegative integer, let \( h_0, \ldots, h_n \in \mathcal{O}(U)^x \) and suppose that \( \text{Fil}^{n+1} H^{n+1}(U/S) = 0 \). Under this setting, we define an object

\[
M(U/S)_{h_0, \ldots, h_n}
\]

in \( \text{Fil}-\text{MIC}(U) \) in the following way.

Let \( \mathcal{M}_{h_0, \ldots, h_n} \) be the complex

\[
\mathcal{L}og(h_n)(n) \to \mathcal{L}og(h_1)(1) \to \mathcal{L}og(h_0)
\]

in \( \text{Fil}-\text{MIC}(U) \), where the first term is placed in degree 0, and the differentials are the compositions of the projection \( \mathcal{L}og(h_i)(i) \to \mathcal{O}_U e_0 \) and the injection \( \mathcal{O}_U e_{-2} \to \mathcal{L}og(h_{i-1})(i-1) \) defined by \( e_0 \mapsto e_{-2} \). This fits into a distinguished triangle

\[
0 \to \mathcal{O}_U(n+1) \to \mathcal{M}_{h_0, \ldots, h_n} \to \mathcal{O}_U[-n] \to 0
\] (2.19)

in the derived category of \( \text{Fil}-\text{MIC}(U) \).

First, we define the de Rham part of \( M(U/S)_{h_0, \ldots, h_n} \). Let \( \mathcal{M}_{h_0, \ldots, h_n, \text{dR}} \) denote the de Rham realization of \( \mathcal{M}_{h_0, \ldots, h_n} \). This can be seen as the de Rham realization of a complex of mixed Hodge modules by M. Saito \cite{Sa1}, \cite{Sa2}. Set

\[
M(U/S)_{h_0, \ldots, h_n, \text{dR}} := H^n_{\text{dR}}(U_K/S_K, \mathcal{M}_{h_0, \ldots, h_n, \text{dR}}) = H^n(U_K, \Omega^*_{U_K/S_K} \otimes \mathcal{M}_{h_0, \ldots, h_n, \text{dR}}),
\]

which fits into an exact sequence

\[
0 \to H^n_{\text{dR}}(U_K/S_K) \to M(U/S)_{h_0, \ldots, h_n, \text{dR}} \to \mathcal{O}_S \xrightarrow{\delta} H^n_{\text{dR}}(U_K/S_K).
\]

Note that this implies that \( M(U/S)_{h_0, \ldots, h_n, \text{dR}} \) is locally free of finite rank. Again, by the theory of mixed Hodge modules, all the terms underly variations of mixed Hodge structures. Hence, they carry the Hodge filtration, which we write by \( \text{Fil}^i \), and the integrable connection \( \nabla \) that satisfies the Griffiths transversality. All the arrows are compatible with respect to \( \nabla \) and strictly compatible with respect to \( \text{Fil}^i \). Moreover, since the image of \( \delta \) is contained in \( \text{Fil}^{n+1} H^n_{\text{dR}}(U_K/S_K) \) and \( \text{Fil}^{n+1} H^n_{\text{dR}}(U_K/S_K) = 0 \) by the assumption, we have an exact sequence

\[
0 \to H^n_{\text{dR}}(U_K/S_K) \to M(U/S)_{h_0, \ldots, h_n, \text{dR}} \to \mathcal{O}_S \to 0
\]

in the category \( \text{Fil-MIC}(S_K) \).

Let \( \mathcal{M}_{h_0, \ldots, h_n, \text{rig}} \) be the corresponding complex in \( F\text{-MIC}(A^1_K) \) to \( \mathcal{M}_{h_0, \ldots, h_n} \) which can be seen as a complex of overconvergent \( F \)-isocrystals. In particular, this can also be seen as a complex of coherent \( J^1_U \mathcal{O}_{U^\text{an}} \)-modules with Frobenius structure. We define

\[
M(U/S)_{h_0, \ldots, h_n, \text{rig}} := H^n_{\text{rig}}(U_K/S_K, \mathcal{M}_{h_0, \ldots, h_n, \text{rig}}) := \Gamma(S^\text{an}, R^n u_{\text{rig}}^* \mathcal{M}_{h_0, \ldots, h_n, \text{rig}}),
\]

where \( R^n u_{\text{rig}}^* \mathcal{M}_{h_0, \ldots, h_n, \text{rig}} = R^u(a^\text{an})_* \left( J^1_U \Omega^*_{U^\text{an}} \otimes \mathcal{M}_{h_0, \ldots, h_n, \text{rig}} \right) \).
Now, we explain how we get a comparison isomorphism
\[ c: M(U/S)_{h_0,\ldots,h_n,dR} \otimes_{B_K} B_K^\dagger \to M(U/S)_{h_0,\ldots,h_n,\text{rig}}. \]

First, we have a canonical homomorphism
\[ H^i_{dR}(U_K/S_K, M_{h_0,\ldots,h_n,dR}) \otimes_{B_K} B_K^\dagger \to H^i_{rig}(U_k/S_k, M_{h_0,\ldots,h_n,\text{rig}}) \]
for each \( i \) (the construction is the same as in the case of trivial coefficients in the beginning of the previous subsection). To prove that this is an isomorphism, as in the de Rham part, note that we have an exact sequence
\[ 0 \to H^n_{rig}(U_k/S_k) \to M(U/S)_{h_0,\ldots,h_n,\text{rig}} \to B_K^\dagger \to H^{n+1}_{rig}(U_k/S_k) \]
because \( H^1(S, R^n_{U,S} K_n^\dagger U^\prime_{O,S}) = 0 \) by Assumption 2.20. Thus, the comparison homomorphism is an isomorphism by the flatness of \( B_K^\dagger \) over \( B_K \), by Assumption 2.20 and by five lemma.

Now, we have constructed an object \( M_{h_0,\ldots,h_n}(U/S) \) in Fil-F-MIC(\( S \)) which fits into the exact sequence
\[ 0 \to H^n(U/S)(n+1) \to M_{h_0,\ldots,h_n}(U/S) \to \mathcal{O}_S \to 0. \tag{2.20} \]
The extension class (2.20) is additive with respect to each \( h_i \) (one can show this in the same way as the proof of bi-additivity of \([h_0]_U \cup \cdots \cup [h_n]_U \), but based on the theory of mixed Hodge modules concerning the strictness of the filtrations; for the rigid part, we use the functoriality of the relative rigid cohomology and the comparison to algebraic de Rham cohomology), so that we have a homomorphism
\[ (\mathcal{O}(U)^X)^{\otimes n+1} \to \text{Ext}^1_{\text{Fil-F-MIC}(S)}(\mathcal{O}_S, H^n(U/S)(n+1)). \tag{2.21} \]

**Theorem 2.23.** Suppose Fil\(^{n+1}\) \( H^{n+1}_{dR}(U_K/S_K) = 0 \). Then, under the assumption 2.20, the homomorphism (2.21) factors through the Milnor K-group, so that we have a map
\[ [-]_{U/S} : K_{n+1}^M(\mathcal{O}(U)) \to \text{Ext}^1_{\text{Fil-F-MIC}(S)}(\mathcal{O}_S, H^n(U/S)(n+1)) \tag{2.22} \]
which we call the symbol map for \( U/S \).

**Remark 2.24.** If it were possible to define a natural projection
\[ \text{Ext}^{n+1}(\mathcal{O}_U, \mathcal{O}_U(n+1)) \to \text{Ext}^1(\mathcal{O}_S, H^n(U/S)(n+1)) \]
under the assumption \( H^{n+1}(U/S) = 0 \), then the object \( M_{h_0,\ldots,h_n}(U/S) \) should correspond to the class \([h_0]_U \cup \cdots \cup [h_n]_U \), and hence, Theorem 2.23 would be immediate from Proposition 2.22. However, this is impossible since we do not take into consideration the boundary conditions, such as admissibility, for constructing our category. We need to prove Theorem 2.23 independently while almost the same argument works as well.
**Proof.** Write the homomorphism (2.21) by $\tilde{\rho}$. It is enough to show the following.

$$\tilde{\rho}(h_0 \otimes \cdots \otimes f \otimes f \otimes \cdots \otimes h_n) = 0, \quad \tilde{\rho}(h_0 \otimes \cdots \otimes f \otimes (1 - f) \otimes \cdots \otimes h_n) = 0.$$ 

We show the latter. Let $f \in \mathcal{O}(U)^\times$ such that $1 - f \in \mathcal{O}(U)^\times$. Let $u : U \to \text{Spec} \mathbb{V}[T, (T - T^2)^{-1}]$ be the morphism given by $u^*T = f$. Recall the diagram (2.18), and take the pullback by $u$. It follows that

$$0 \to \mathcal{O}_U(n + 1) \to \text{Log}(h_n) \to \cdots \to \text{Log}(f(i + 1)) \to \text{Log}(1 - f)(i) \to \cdots \to \text{Log}(h_0) \to \mathcal{O}_U \to 0$$

is equivalent to

$$0 \to \mathcal{O}_U(n + 1) \to \text{Log}(h_n) \to \cdots \to \mathcal{O}_U(i + 2) \to \mathcal{O}_U(i) \to \cdots \to \text{Log}(h_0) \to \mathcal{O}_U \to 0$$

as an $(n + 1)$-extension in $\text{Fil}^-\text{MIC}(U)$. This implies that $\mathcal{M}$ is quasi-isomorphic to $\mathcal{M}'$ as mixed Hodge modules or $\mathcal{F}$-isocrystals. Hence, $\mathcal{N}$ gives rise to a splitting of

$$0 \to \text{H}^n(U/S)(n + 1) \to M_{h_0, \ldots, f, 1 - f, \ldots, h_n}(U/S) \to \mathcal{O}_S \to 0.$$ 

This completes the proof of $\tilde{\rho}(h_0 \otimes \cdots \otimes f \otimes (1 - f) \otimes \cdots \otimes h_n) = 0$.

To see $\tilde{\rho}(h_0 \otimes \cdots \otimes f \otimes f \otimes \cdots \otimes h_n) = 0$, we consider the similar diagram to (2.18) obtained from $\text{Sym}^2 \text{Log}(T)$. Then, the rest is the same. \qed

### 2.7. Explicit formula

In this subsection, we continue to use the setting of the previous subsection. In particular, $U = \text{Spec}(A) \to S = \text{Spec}(B)$ is a smooth morphism of smooth $V$-schemes that satisfies Assumption 2.20. We fix a $p$-th Frobenius endomorphism $\varphi$ (resp. $\sigma$) on $A^\dagger$ (resp. $B^\dagger$).

Let $n \geq 0$ be an integer. Suppose that $\text{Fil}^{n+1} \text{H}^{n+1}_{dR}(U_K/S_K) = 0$. Recall the maps

$$\text{K}^M_{n+1}(A) \xrightarrow{[-]_{U/S}} \text{Ext}^1_{\text{Fil}^-\text{MIC}(S)}(\mathcal{O}_S, \text{H}^n(U/S)(n + 1)) \xrightarrow{(2.7)} \text{H}^1(\mathcal{F}(\text{H}^n(U/S)(n + 1))).$$

Note that the last cohomology group is

$$\left\{ (\omega, \xi) \in (\Omega^1_{B_K} \otimes \text{Fil}^n \text{H}^{n}_{dR}(U_K/S_K)) \oplus \left( B^1_K \otimes \text{H}^{n}_{dR}(U_K/S_K) \right) \Big| (1 - \varphi_{n+1})\omega = d\xi, d(\omega) = 0 \right\},$$

where $\varphi_i := p^{-i}\varphi$ and $d$ is the differential map induced from the Gauss-Manin connection as in (2.2). Let

$$\begin{array}{ccc}
\Omega^1_{B_K} \otimes \text{Fil}^n \text{H}^{n}_{dR}(U_K/S_K) & \xrightarrow{\text{Ext}^1_{\text{Fil}^-\text{MIC}(S)}(\mathcal{O}_S, \text{H}^n(U/S)(n + 1))} & B^1_K \otimes \text{H}^{n}_{dR}(U_K/S_K) \\
\delta & & R_{\sigma} \\
\end{array}$$

(2.23)
be the compositions of (2.7) and the projections \((\omega, \xi) \mapsto \omega\) and \((\omega, \xi) \mapsto \xi\), respectively. The map \(\delta\) agrees with the map (2.5), and hence, it does not depend on \(\sigma\), while \(R_\sigma\) does. We put
\[
D_{U/S} := \delta \circ [-]_{U/S}, \quad \text{reg}^{(\sigma)}_{U/S} := R_\sigma \circ [-]_{U/S}.
\]

The purpose of this section is to give an explicit description of these maps.

**Theorem 2.25.** Suppose \(\text{Fil}^{n+1}H_{\text{dR}}^{n+1}(U_K/S_K) = 0\). Let \(\xi = \{h_0, \ldots, h_n\} \in K_{n+1}^M(A)\). Then,
\[
D_{U/S}(\xi) = (-1)^n \frac{dh_0}{h_0} \land \frac{dh_1}{h_1} \land \cdots \land \frac{dh_n}{h_n}, \quad \text{reg}^{(\sigma)}_{U/S}(\xi) = (-1)^n \sum_{i=0}^n (-1)^i p^{-1} \log\left(\frac{\widetilde{h_i^p}}{h_i}\right) \left(\frac{dh_0}{h_0}\right)^{\varphi_1} \land \cdots \land \left(\frac{dh_{i-1}}{h_{i-1}}\right)^{\varphi_1} \land \frac{dh_{i+1}}{h_{i+1}} \land \cdots \land \frac{dh_n}{h_n}.
\]

Here, one can think of (2.25) as an element of
\[
\Omega^1_{B_K} \otimes \text{Fil}^n H_{\text{dR}}^n(U_K/S_K) = \Omega^1_{B_K} \otimes \Gamma(X_K, \Omega^2_{X_K/S_K}(\log D_K))
\]
in the following way. Let \(\widetilde{\Omega}^i_{X_K}(\log D_K) := \Omega^i_{X_K}(\log D_K)/\text{Im}(\Omega^i_{S_K} \otimes \Omega^{i-1}_{X_K}(\log D_K))\) which fits into the exact sequence
\[
0 \longrightarrow \Omega^1_{S_K} \otimes \Omega^{i-1}_{X_K/S_K}(\log D_K) \longrightarrow \widetilde{\Omega}^i_{X_K}(\log D_K) \longrightarrow \Omega^i_{X_K/S_K}(\log D_K) \longrightarrow 0.
\]

We think (2.25) of being an element of \(\Gamma(X_K, \widetilde{\Omega}^{n+1}_{X_K}(\log D_K))\). However, since \(\text{Fil}^{n+1}H_{\text{dR}}^{n+1}(U_K/S_K) = 0\) by the assumption, it turns out to be an element of
\[
\Gamma(X_K, \Omega^1_{S_K} \otimes \Omega^2_{X_K/S_K}(\log D_K)) = \Omega^1_{B_K} \otimes \Gamma(X_K, \Omega^2_{X_K/S_K}(\log D_K)).
\]

**Proof.** In this proof, we omit to write the symbol “\(\land\)”.

First, we describe the extension \([\xi]_{U/S}\). Let \(0 \longrightarrow H_{\text{dR}}^n(U/S)(n+1) \longrightarrow M_{\xi}(U/S) \longrightarrow \mathcal{O}_S \longrightarrow 0\) be the extension \([\xi]_{U/S}\) in Fil-F-MIC\((S, \sigma)\). Let \(e_\xi \in \text{Fil}^0 M_{\xi}(U/S)_{\text{dR}}\) be the unique lifting of \(1 \in \mathcal{O}(S_K)\). Then,
\[
\iota(D_{U/S}(\xi)) = \nabla(\xi), \quad \iota(\text{reg}^{(\sigma)}_{U/S}(\xi)) = (1 - \Phi)e_\xi
\]
by definition (see also (2.11)), where \(\nabla\) and \(\Phi\) are the data in \(M_{\xi}(U/S) \in \text{Fil}-\text{MIC}(S, \sigma)\).

We first write down the term \(M_{\xi}(U/S)\) explicitly. Write \(l_i := p^{-1} \log(h_i^p/h_i)\). Let \(\{e_{i,0}, e_{i,-2}\}\) be the basis of \(\mathcal{L}(\log(h_i))(i)\) such that \(\text{Fil}^{-i} \mathcal{L}(\log(h_i))(i)_{\text{dR}} = A_K e_{i,0}\) and
\[
\nabla(e_{i,0}) = \frac{dh_i}{h_i} e_{i,-2}, \quad \Phi(e_{i,0}) = p^{-i} e_{i,0} - p^{-i} l_i e_{i,-2}, \quad \Phi(e_{i,-2}) = p^{-i-1} e_{i,-2}.
\]

Recall the \((n+1)\)-extension
\[
0 \longrightarrow \mathcal{O}_U(n+1) \longrightarrow \mathcal{L}(\log(h_n))(n) \longrightarrow \cdots \longrightarrow \mathcal{L}(\log(h_1))(1) \longrightarrow \mathcal{L}(\log(h_0)) \longrightarrow \mathcal{O}_U \longrightarrow 0.
\]
Let \( (T^•_{A_K/B_K}, D) \) be the total complex of the double complex \( \Omega^•_{A_K/B_K} \otimes \mathcal{L} \log(h_*) \text{dR} \). In a more down-to-earth manner, we have

\[
T^q_{A_K/B_K} := \bigoplus_{i=0}^{d} \Omega^i_{A_K/B_K} \otimes \mathcal{L} \log(h_{i-q})(i-q) \text{dR}, \quad q \in \mathbb{Z},
\]

where we denote \( \mathcal{L} \log(h_j)(j) := 0 \) if \( j < 0 \) or \( j > n \), and the differential \( D : T^q \to T^{q+1} \) is defined by

\[
D(\omega^i \otimes x_j) = d\omega^i \otimes x_j + (-1)^i \omega^i \wedge \nabla(x_j) + (-1)^i \omega^i \otimes \pi(x_j)
\]

for \( \omega^i \otimes x_j \in \Omega^i_{A_K/B_K} \otimes \mathcal{L} \log(h_j)(j) \text{dR} \), where \( \pi : \mathcal{L} \log(h_{i})(i) \to \mathcal{L} \log(h_{i-1})(i-1) \) is the composite of the projection \( \mathcal{L} \log(h_{i})(i) \to \mathcal{L} \log(h_{i-1})(i-1) \) and the injection \( A_K e_{i,0} \cong A_K e_{i-1,-2} \hookrightarrow \mathcal{L} \log(h_{i-1})(i-1) \) defined by \( e_{i,0} \mapsto e_{i-1,-2} \). We have the exact sequence

\[
0 \to \Omega^{\bullet + n}_{A_K/B_K} \to T^•_{A_K/B_K} \to \Omega^•_{A_K/B_K} \to 0, \quad (2.28)
\]

where the first arrow is induced from \( \mathcal{O}_U \cong \mathcal{O}_U e_{d-2} \hookrightarrow \mathcal{L} \log(h_d) \text{dR} \), the second arrow is induced from the projection \( \mathcal{L} \log(h_0) \text{dR} \to \mathcal{O}_U e_{0,0} \cong \mathcal{O}_U \) and the differential on \( \Omega^{\bullet + n}_{A_K/B_K} \) is the usual differential operator \( d \) (not \((-1)^n d\)). For the rigid part, let \( (T^•_{A_K/B_K}, D) \) be defined in the same way by replacing \( \Omega^•_{A_K/B_K} \) with \( \Omega^•_{A_K/B_K} \) (see Remark 2.21 (3)). Then, we also have an exact sequence corresponding to (2.28). Now, we have a description

\[
M_ξ(U/S) \text{dR} = H^0(T^•_{A_K/B_K}), \quad M_ξ(U/S) \text{rig} = H^0(T^•_{A_K/B_K}^{\text{rig}})
\]

(the description of the rigid part follows from the exact sequence (2.28) on two sides with Remark 2.21 (3)), and we have an exact sequence

\[
0 \to H^n(U/S)(n+1) \to M_ξ(U/S) \to \mathcal{O}_S \to 0
\]

in \( \text{Fil-F-MIC}(S, σ) \).

Before going to the proof of Theorem 2.25, we give an explicit description of \( e_ξ \) in (2.27). Put

\[
\omega^0 := 1, \quad \omega^i := \frac{dh_0}{h_0} \frac{dh_1}{h_1} \ldots \frac{dh_{i-1}}{h_{i-1}} \in \Omega^i_{A_K/B_K} \quad (1 \leq i \leq n+1).
\]

However, we note that \( \omega^{n+1} = \omega^n \wedge \frac{dh_n}{h_n} = 0 \) as \( \omega^n \in \text{Fil}^{n+1} H^{n+1}_{\text{dR}}(U_K/S_K) = 0 \). Put

\[
e_ξ' = \sum_{i=0}^{n} \omega^i e_{i,0} \in T^0_{A_K/B_K}.
\]

It is a direct computation to show \( D(e_ξ') = 0 \), so that one has \( e_ξ' \in M_ξ(U/S) \text{dR} \). This is obviously a lifting of \( 1 \in \mathcal{O}_S(U_K) \). We claim

\[
e_ξ = e_ξ'.
\]

(2.29)

To do this, it is enough to show that \( e_ξ' \in \text{Fil}^0 M_ξ(U/S) \text{dR} \). Let \( j : U_K \hookrightarrow X_K \). We think \( T^•_{A_K/B_K} \) of being a complex of \( \mathcal{O}_U \)-modules. Let \( \text{Fil}^0 \mathcal{L} \log(h_i) X_K = \mathcal{O}_X e_{i,0}, \)
Fil^k \log(h_i)X_K = \mathcal{O}_X e_{i,0} + \mathcal{O}_X e_{i,-2}$ for $k < 0$, and $\operatorname{Fil}^k \log(h_i)X_K = 0$ for $k > 0$. Let $\mathcal{F}_U^\bullet \subset j_*T_{A_K/B_K}^q$ be the subcomplex of locally free $\mathcal{O}_X$-modules such that

$$\mathcal{F}_U^q = \bigoplus_{i=0}^d \mathcal{O}_{X_K}^i (\log D_K) \otimes \log(h_{i-q})X_K \subset j_*T_{A_K/B_K}^q,$$

and put

$$\operatorname{Fil}^k \mathcal{F}_U^q \mathcal{F}_U^q = \bigoplus_{i=0}^d \mathcal{O}_{X_K}^i (\log D_K) \otimes \operatorname{Fil}^{k-q} \log(h_{i-q})X_K \subset \mathcal{F}_U^q.$$

Then,

$$\operatorname{Fil}^k M_\xi(U/S)_{\text{dR}} = \text{Im}[H^0(X_K, \operatorname{Fil}^k \mathcal{F}_U^\bullet /S) \to H^0(X_K, \mathcal{F}_U^\bullet /S) \cong M_\xi(U/S)].$$

Notice that $\operatorname{Fil}^0 \mathcal{F}_U^q = 0$ for $q < 0$ as $\operatorname{Fil}^k \log(h_i)X_K = 0$ for $k > 0$. Hence, there is a natural map

$$\Gamma(X_K, \ker[\operatorname{Fil}^0 \mathcal{F}_U^0 /S \xrightarrow{D} \operatorname{Fil}^0 \mathcal{F}_U^1 /S]) \to H^0(X_K, \operatorname{Fil}^k \mathcal{F}_U^\bullet /S),$$

and the element $e'_\xi$ lies in the image of the left-hand side by definition. This shows $e'_\xi \in \operatorname{Fil}^0 M_\xi(U/S)_{\text{dR}}$ and hence completes the proof of (2.29).

We prove (2.26). Applying $1 - \Phi$ on (2.29). We have

$$(1 - \Phi)e_\xi = \sum_{i=0}^n \omega^i e_{i,0} - \varphi_i(\omega^i)(e_{i,0} - l_ie_{i,-2}) = \sum_{i=0}^n (1 - \varphi_i(\omega^i)) e_{i,0} + l_i\varphi_i(\omega^i)e_{i,-2}. \quad (2.30)$$

Put

$$Q_0 := l_0, \quad Q_k := \sum_{i=0}^k (1)^i l_i \left( \frac{dh_0}{h_0} \right) \cdots \left( \frac{dh_{i-1}}{h_{i-1}} \right) \varphi_1 \frac{dh_{i+1}}{h_{i+1}} \cdots \frac{dh_k}{h_k} \quad (1 \leq k \leq n).$$

A direct calculation yields

$$dQ_k = (1 - \varphi_{k+1} \left( \frac{dh_0}{h_0} \cdots \frac{dh_k}{h_k} \right)) = (1 - \varphi_{k+1}) (\omega^{k+1}).$$

It follows

$$D(Q_k e_{k+1,0}^{k+1})$$

$$= (1 - \varphi_{k+1}) (\omega^{k+1}) e_{k+1,0}^{k+1} + (1)^k Q_k e_{k+1,0}^{k+1} - l_{k+1} e_{k+1,2} + (1)^k Q_k e_{k,-2}$$

$$= (1 - \varphi_{k+1}) (\omega^{k+1}) e_{k+1,0}^{k+1} + (1)^k Q_k e_{k+1,0}^{k+1} - l_{k+1} e_{k+1,2} + (1)^k Q_k e_{k,-2}$$

$$= (1 - \varphi_{k+1}) (\omega^{k+1}) e_{k+1,0}^{k+1} + l_{k+1} e_{k+1,0}^{k+1} e_{k+1,2} + (1)^k Q_k e_{k,-2} - (1)^{k+1} Q_k e_{k+1,0}^{k+1}$$

for $0 \leq k \leq n - 1$. Hence,

$$D \left( \sum_{k=0}^{n-1} Q_k e_{k+1,0}^{k+1} \right) = (1 - \Phi)(e_\xi) - l_0 e_{0,-2} + (1)^n (1 - \varphi_{n+1})(\eta)e_{n,-2} + Q_0 e_{0,-2} - (1)^n Q_n e_{n,-2}$$

$$= (1 - \Phi)(e_\xi) - (1)^{n} Q_n e_{n,-2}$$
We work over a fine log scheme (3.1. Log syntomic cohomology of sheaves on In this section, we compare the symbol map (3.26 with the Syntomic Symbol maps and this shows (cf. (2.27)), by (2.30). This shows (cf. (2.27))
\[
\text{reg}_{U/S}^{(\sigma)}(\xi) = (-1)^n Q_n = (-1)^n \sum_{i=0}^{n} (-1)^i l_i \left( \frac{dh_0}{h_0} \right)^{\varphi_1} \cdots \left( \frac{dh_{i-1}}{h_{i-1}} \right)^{\varphi_1} \frac{dh_i}{h_i} \cdots \frac{dh_n}{h_n},
\]
as required.

We prove (2.25). Note that \( D_{U/S}(\xi) \) is characterized by \( D_{U/S}(\xi) e_n, -2 = \nabla(e_{\xi}) \), where \( \nabla \) is the Gauss-Manin connection on \( M_{\xi}(U/S)_{\text{dR}} \) (cf. (2.27)). Let \( T^*_A K \) be the complex defined in the same way as \( T^*_A K / B_K \) by replacing \( \Omega^*_{A_K / B_K} \) with \( \Omega^*_{A_K} \). Let
\[
\tilde{\omega}^0 := 1, \quad \tilde{\omega}^i := \frac{dh_0}{h_0} \frac{dh_1}{h_1} \cdots \frac{dh_{i-1}}{h_{i-1}} \in \Omega^i_{A_K} \quad (1 \leq i \leq n + 1),
\]
and \( \tilde{e}_{\xi} := \sum_{i=0}^{n} \tilde{\omega}^i e_{i,0} \in T_{A_K}^0 \). Then,
\[
D(\tilde{e}_{\xi}) = \sum_{i=0}^{n} (-1)^i \left( \frac{dh_0}{h_0} \cdots \frac{dh_i}{h_i} e_{i,2} + \frac{dh_0}{h_0} \cdots \frac{dh_{i-1}}{h_{i-1}} e_{i-1,-2} \right) = (-1)^n \frac{dh_0}{h_0} \cdots \frac{dh_n}{h_n} e_{n,-2}
\]
and this shows (cf. (2.25)),
\[
\nabla(e_{\xi}) = (-1)^n \frac{dh_0}{h_0} \cdots \frac{dh_n}{h_n} e_{n,-2} \in \Omega^1_{B_K} \otimes \Gamma(X_K, \Omega^n_{X_K / S_K}(\log D_K)) e_{n,-2} \quad (2.31)
\]
as required. \( \square \)

3. Comparison with the Syntomic Symbol maps

In this section, we compare the symbol map \([-]_{U/S}\) introduced in §2.6 with the symbol maps to the log syntomic cohomology. We refer to Kato’s article [Ka3] for the formulation and terminology of log schemes. Throughout this section, we write \((X_n, M_n) := (X, M) \otimes \mathbb{Z}/p^n \mathbb{Z}\) for a log scheme \((X, M)\).

3.1. Log syntomic cohomology

We work over a fine log scheme \((S, L)\) flat over \( \mathbb{Z}_p \). We endow the DP-structure \( \gamma \) on \( I = p \mathcal{O}_S \) compatible with the canonical DP-structure on \( p \mathcal{O}_S \). The log de Rham complex for a morphism \( f : (X, M_X) \to (S, L) \) of log schemes is denoted by \( \omega^*_{X/S} \) ([Ka3, (1.7)]).

**Proposition 3.1** [Ts2, Corollary 1.11]. Let \((Y_n, M_n)\) be a fine log scheme over \((S_n, L_n)\). Let \((Y_n, M_n) \rightarrow (Z_n, N_n)\) be a \((S_n, L_n)\)-closed immersion. Assume that \((Z_n, N_n)\) has \( p \)-bases over \((S_n, L_n)\) locally in the sense of [Ts2, Definition 1.4]. Let \((D_n, M_{D_n})\) be the DP-envelope of \((Y_n, M_n)\) in \((Z_n, N_n)\). Let \( J^{[r]}_{D_n} \subset \mathcal{O}_{Z_n} \) be the \( r \)-th DP-ideal of \( D_n \). Then, the complex
\[
J^{[r-1]}_{D_n} \otimes \omega^*_{Z_n / S_n}
\]
of sheaves on \((D_n)_{\text{et}} = (Y_n)_{\text{et}}\) does not depend on the embedding \((Y_n, M_n) \rightarrow (Z_n, N_n)\). In particular, if \((Y_n, M_n)\) has \( p \)-bases over \((S_n, L_n)\) locally, then the natural morphism
is a quasi-isomorphism.

Concerning $p$-bases of log schemes, the following result is sufficient in most cases.

**Lemma 3.2** [Ts2, Lemma 1.5]. If $f : (X, M_X) \to (Y, M_Y)$ is a smooth morphism of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$, then $f$ has $p$-bases locally.

Following [Ts2, §2], we say a collection $\{(X_n, M_{X,n}, i_n)\}_n$ (abbreviated to $\{(X_n, M_{X,n})\}_n$) is an adic inductive system of fine log schemes, where $(X_n, M_{X,n})$ are fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$, and $i_n : (X_n, M_{X,n}) \to (X_{n+1}, M_{X,n+1}) \otimes \mathbb{Z}/p^n\mathbb{Z}$ are isomorphisms.

Let $\{(Y_n, M_n)\}_n \hookrightarrow \{(Z_n, N_n)\}_n$ be an exact closed immersion of adic inductive systems of fine log schemes over $\{(S_n, L_n)\}_n$. We consider the following condition.

**Condition 3.3.**

1. For each $n \geq 1$, $(Z_n, N_n)$ has $p$-bases over $(S_n, L_n)$ locally,
2. Let $(D_n, M_{D,n}) \to (Z_n, N_n)$ be the DP-envelope of $(Y_n, M_n)$ compatible with the DP-structure on $(S_n, L_n)$, and let $J^{[r]}_{D_n} \subset \mathcal{O}_{D_n}$ be the $r$-th DP-ideal. Then, $J^{[r]}_{D_n}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $J^{[r]}_{D_{n+1}} \otimes \mathbb{Z}/p^n\mathbb{Z} \cong J^{[r]}_{D_n}$.

Suppose that there is a compatible system $\sigma = \{\sigma_n : (S_n, L_n) \to (S_n, L_n)\}_n$ of $p$-th Frobenius endomorphisms.

**Condition 3.4.** There is a hypercovering $\{Y_n\}_n \to \{Y_n\}_n$ in the étale topology which admits an embedding $\{(Y_n^\nu, M_n^\nu)\}_n \hookrightarrow \{(Z_n^\nu, N_n^\nu)\}_n$ into an adic inductive system of simplicial log schemes over $\{(S_n, L_n)\}_n$, such that each $\{(Y_n^\nu, M_n^\nu)\}_n \to \{(Z_n^\nu, N_n^\nu)\}_n$ satisfies Condition 3.3, and there is a $p$-th Frobenius $\{\varphi_n^\nu : (Z_n^\nu, N_n^\nu) \to (Z_n^\nu, N_n^\nu)\}_n$ compatible with $\sigma$.

Let $(Y, M) = \{(Y_n, M_n)\}_n \to \{(S_n, L_n)\}_n$ together with the $p$-th Frobenius $\sigma$ on $(S, L)$ satisfy Condition 3.4. We define the log syntomic complexes according to the construction in [Ts2, p. 539–541]². Let $r \geq 0$ be an integer. Let $(D_{n^r}, M_{D,n^r}) \to (Z_{n^r}, N_{n^r})$ be the DP-envelopes of $(Y_{n^r}, M_{n^r})$ compatible with the DP-structure on $(S_n^r, L_n)$ and $J^{[r]}_{D_n^r} \subset \mathcal{O}_{D_n^r}$ be the $r$-th DP-ideal. Let

$$J^{[r]}_{n, (Y^\nu, M^\nu), (Z^\nu, N^\nu)/(S, L)} := J^{[r-]}_{D_n^r} \otimes \mathcal{O}_{Z_n^r/S_n}$$

be the complex of sheaves on $(D_{1^r})_{\text{ét}} = (Y_{1^r})_{\text{ét}}$. We also write

$$O^\bullet_{n, (Y^\nu, M^\nu), (Z^\nu, N^\nu)/(S, L)} = J^{[0]}_{n, (Y^\nu, M^\nu), (Z^\nu, N^\nu)/(S, L)}.
$$

If $r < p$, then there is the well-defined morphism

$$\varphi^\nu_r : J^{[r]}_{n, (Y^\nu, M^\nu), (Z^\nu, N^\nu)/(S, L)} \longrightarrow O^\bullet_{n, (Y^\nu, M^\nu), (Z^\nu, N^\nu)/(S, L)}$$

²Tsuji [Ts2] defined the log syntomic complexes only in case $S = \text{Spec} \mathbb{Z}_p$ with trivial log structure. However, the same construction works in general as long as $\sigma$ is fixed.
satisfying $p^r \varphi_r^n = (\varphi_{Z^n})^*$ (cf. [Ts2, p.540]). It follows from Proposition 3.1 that one can “glue” the complexes $J_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L)$ so that we have a complex

$$J_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L)$$

in the derived category of sheaves on $(Y_1)_n$ ([Ts2, p.541] (see also [Ka2, Remark(1.8)]). Moreover, the Frobenius $\varphi_r^n$ are also glued as $\sigma$ is fixed (this can be shown by the same argument as in [Ka2, p.212]); we have

$$\varphi_r^n : J_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L) \rightarrow O_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L).$$

Then, we define the log syntomic complex $\mathcal{S}_n(r)(Y,M),(Z^*,\rho^n)/(S,L,\sigma)$ to be the mapping fiber of

$$1 - \varphi_r^n : J_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L) \rightarrow O_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L).$$

In a more down-to-earth manner, the degree $q$-component of $\mathcal{S}_n(r)(Y,M),(Z^*,\rho^n)/(S,L,\sigma)$ is

$$J_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L)^{\oplus q-1}O_{n,(Y^n,M^n),Z^n,\rho^n}/(S,L)$$

and the differential maps are given by $(\alpha,\beta) \mapsto (d_\alpha (1 - \varphi_{d+1})\alpha - d\beta)$. We define the log syntomic cohomology groups by

$$H^i_{\text{syn}}((Y,M)/(S,L,\sigma),Z/p^n(r)) := H^i_{\text{et}}(Y_1,\mathcal{S}_n(r)(Y,M),(Z^*,\rho^n)/(S,L,\sigma)), $$

$$H^i_{\text{syn}}((Y,M)/(S,L,\sigma),Z_p(r)) := \lim_{n} H^i_{\text{syn}}((Y,M)/(S,L,\sigma),Z/p^n(r))$$

and $H^i_{\text{syn}}((Y,M)/(S,L,\sigma),\mathbb{Q}_p(r)) := H^i_{\text{syn}}((Y,M)/(S,L,\sigma),Z_p(r)) \otimes \mathbb{Q}$. Note that they depend on the choice of $\sigma$.

Let $(Y',M') \rightarrow (S',L',\sigma')$ satisfy Condition 3.4 and let

$$(Y',M') \xrightarrow{f} (Y,M)$$

be a commutative diagram of fine log schemes in which $\sigma$ and $\sigma'$ are compatible. Then, there is the pullback

$$f^* : H^i_{\text{syn}}((Y,M)/(S,L,\sigma),Z/p^n(r)) \rightarrow H^i_{\text{syn}}((Y',M')/(S',L',\sigma'),Z/p^n(r)).$$

**Proposition 3.5.** Let $0 \leq r < p$. Suppose that $(Y,M) \rightarrow (S,L)$ is smooth. Let $\Phi_r : H^i_{\text{zar}}(Y_n,\omega^r_{Y_n/S_n}) \rightarrow H^i_{\text{zar}}(Y_n,\omega^r_{Y_n/S_n})$ be the $\sigma$-linear map induced from $\varphi_r^n$. Then, there is an exact sequence

$$\cdots \rightarrow H^{i-1}_{\text{zar}}(Y_n,\omega^r_{Y_n/S_n}) \xrightarrow{1-\Phi_r^1} H^i_{\text{zar}}(Y_n,\omega^r_{Y_n/S_n}) \rightarrow H^i_{\text{syn}}((Y,M)/(S,L,\sigma),Z/p^n(r)) \rightarrow H^i_{\text{zar}}(Y_n,\omega^r_{Y_n/S_n}) \xrightarrow{1-\Phi_r^1} H^i_{\text{zar}}(Y_n,\omega^r_{Y_n/S_n}) \rightarrow \cdots$$
Proof. Since \((Y, M) \to (S, L)\) is smooth, it has \(p\)-bases locally by Lemma 3.2. Therefore, the natural morphism \((3.1)\) is a quasi-isomorphism, and the exact sequence follows. \(\square\)

If \(S = \text{Spec}V\) with \(V\) a \(p\)-adically complete discrete valuation ring and \(Y \to S\) is proper, the exact sequence in Proposition 3.5 remains true after taking the projective limit with respect to \(n\). Indeed, each term is a \(V/p^nV\)-module of finite length, so that the Mittag-Leffler condition holds. The author does not know whether it is true in general \((S, L)\).

We attach the following lemma which we shall often use.

Lemma 3.6. For a ring \(A\), let \(A\hat{\cdot} := \lim \leftarrow A/p^nA\) denote the \(p\)-adic completion. Let \((S, L)\) be a fine log scheme flat over \(\mathbb{Z}(p)\) such that \(S\) is affine and noetherian. Let \((Y, M) \to (S, L)\) be a smooth morphism of fine log schemes such that \(Y \to S\) is proper. Let \((Y_n, M_n) = (Y, M)\otimes \mathbb{Z}/p^n\mathbb{Z}\) and \((S_n, L_n) = (S, L)\otimes \mathbb{Z}/p^n\mathbb{Z}\). Then,

\[
\mathcal{O}(S)\hat{\cdot} \otimes \mathcal{O}(S)H^i_{\text{zar}}(Y, \omega_{\cdot \geq k} \geq Y/S) \cong \lim_n H^i_{\text{zar}}(Y_n, \omega_{Y_n/S_n}^\geq k)
\]

for any \(k\). Note that \(\mathcal{O}(S)\hat{\cdot}\) is a noetherian ring ([A-M, Theorem 10.26]).

Proof. For an abelian group \(M\) and an integer \(n\), we denote by \(M[\cdot n]\) the kernel of the multiplication by \(n\). An exact sequence

\[
0 \to \omega_{Y/S}^\geq k/p^n \to \omega_{Y/S}^\geq k \to \omega_{Y_n/S_n}^\geq k \to 0
\]

gives rise to an exact sequence

\[
0 \to H^i_{\text{zar}}(Y, \omega_{\cdot \geq k}^\geq Y/S)/p^n \to H^i_{\text{zar}}(Y_n, \omega_{Y_n/S_n}^\geq k) \to H^{i+1}_{\text{zar}}(Y, \omega_{\cdot \geq k}^\geq Y/S)[p^n] \to 0
\]

of finitely generated \(\mathcal{O}(S)\)-modules as \(Y \to S\) is proper. Therefore, the assertion is reduced to show that, for any \(p\)-torsion free noetherian ring \(A\) and any finitely generated \(A\)-module \(M\),

\[
A\hat{\cdot} \otimes_A M \cong \lim_n M/p^nM,
\]

(3.3)

where the transition map \(M/p^{n+1} \to M/p^n\) is the natural surjection, and

\[
\lim_n M[p^n] = 0,
\]

(3.4)

where the transition map \(M[p^{n+1}] \to M[p^n]\) is multiplication by \(p\). The isomorphism (3.3) is well-known ([A-M, Prop. 10.13]). We show (3.4). Let

\[
0 \to M_1 \to M_2 \to M_3 \to 0
\]

be any exact sequence of finitely generated \(A\)-modules. Then,

\[
0 \to M_1[p^n] \to M_2[p^n] \to M_3[p^n] \to M_1/p^n \to M_2/p^n \to M_3/p^n \to 0
\]
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is exact. Suppose that $M_2[p^n] = 0$ for all $n$. Then, by taking the projective limit, we have

$$0 \rightarrow \lim_{\leftarrow n} M_3[p^n] \rightarrow \lim_{\leftarrow n} (M_1/p^n) \rightarrow \lim_{\leftarrow n} (M_2/p^n) \rightarrow 0$$

Since $A \rightarrow A^\wedge$ is flat ([A-M, Prop. 10.14]), the bottom arrow is injective, and hence, the vanishing $\lim_{\leftarrow n} M_3[p^n] = 0$ follows. For the proof of (3.4), apply this to $M_3 = M$ and $M_2$ a free $B$-module of finite rank. 

3.2. Syntomic symbol maps

We review syntomic symbol maps ([F-M, p.205], [Ka2, Chapter I §3], [Ts1, §2.2], [Ts2, p.542]). Let $(Y,M)$ be a fine log scheme which is flat over $\mathbb{Z}_p$. We write $(Y_n,M_n) = (Y,M) \otimes \mathbb{Z}/p^n \mathbb{Z}$, as before. Suppose that $\{ (Y_n,M_n) \} \rightarrow \operatorname{Spec} \mathbb{Z}/p^n \mathbb{Z}$ satisfies Condition 3.4, where $\operatorname{Spec} \mathbb{Z}/p^n \mathbb{Z}$ is endowed with the trivial log structure, the identity as the Frobenius and the canonical DP-structure on $p\mathbb{Z}/p^n \mathbb{Z}$. For a sheaf $M$ of monoid, we denote by $M^{gp}$ the associated sheaf of abelian group.

For $0 \leq r < p$, there is the natural map

$$\Gamma(Y_n,M^{gp}_{n+1}) \rightarrow H^1_{\text{syn}}((Y,M),\mathbb{Z}/p^n(1))$$

(3.5)

where we omit to write “/(Spec $\mathbb{Z}/p^n \mathbb{Z}$, $\mathbb{Z}/p^n \mathbb{Z}$, $\mathbb{Z}/p^n \mathbb{Z}$)” in the notation of the syntomic cohomology or syntomic complexes below. This is defined in the following way. We define the complex $C_n$ as

$$C_n := (1 + J_{D_n} \rightarrow M^{gp}_{D_n}) \quad (1 + J_{D_n} \text{ is placed in degree 0}),$$

where $M^{gp}_{D_n}$ denotes the associated sheaf of abelian groups. We define the map of complexes

$$s : C_{n+1} \rightarrow \mathcal{H}_n(1)(Y,M),(\mathbb{Z}^*,N^*)$$

as the map

$$s^0 : 1 + J^{[1]}_{D_{n+1}} \rightarrow J^{[1]}_{D_n}, \quad a \mapsto \log(a)$$

in degree 0, and the map

$$s^1 : M^{gp}_{D_{n+1}} \rightarrow \mathcal{O}_{D_n} \otimes \mathcal{O}_{Z_n} \omega_{Z_n}^1 \oplus \mathcal{O}_{D_n}$$

$$b \mapsto (d\log(b), p^{-1} \log(b^p \varphi_{n+1}(b)^{-1}))$$

in degree 1. Here, $\varphi_{n+1}(b)b^{-p}$ belongs to $1 + p\mathcal{O}_{D_{n+1}}$ and the logarithm $p^{-1} \log(b^p \varphi_{n+1}(b)^{-1}) \in \mathcal{O}_{D_n}$ is well-defined. One easily verifies that the maps $s^0$ and $s^1$ yield a map of complexes. Since there is a natural quasi-isomorphism $C_{n+1} \cong M^{gp}_{n+1}[-1]$, the map $s$ induces a morphism ([Ts1, (2.2.3)])

$$M^{gp}_{n+1}[1] \rightarrow \mathcal{H}_n(1)(Y,M),(\mathbb{Z}^*,N^*)$$

in the derived category, and hence, (3.5) is obtained.
Now, suppose that $M$ is defined by a divisor $D \subset Y$. Let $U := Y \setminus D$. Then, we have $M_{\text{sp}} = j_\ast \mathcal{O}_U^\times$ and obtain a map

$$\mathcal{O}(U_{n+1})^\times \longrightarrow H^1_{\text{syn}}((Y,M),\mathbb{Z}/p^n(1)). \quad (3.7)$$

This map and the product structure of syntomic cohomology give rise to a map

$$[-]_{\text{syn}} : K^M_r(\mathcal{O}(U_{n+1})) \longrightarrow H^r_{\text{syn}}((Y,M),\mathbb{Z}/p^n(r)) \quad (3.8)$$

for $0 \leq r \leq p-1$ (cf. [Ka1, Proposition 3.2]), which we call the syntomic symbol map.

### 3.3. Comparison of Symbol maps for $U/S$ with Syntomic symbol maps

Let $W = W(k)$ be the Witt ring of a perfect field $k$ of characteristic $p > 0$ and $K$ the fractional field. Let $F_W$ be the $p$-th Frobenius on $W$ or $K$. We omit to write “/(Spec$W$,W$^\times$,F$W$)” in the notation of syntomic complexes or syntomic cohomology, as long as there is no fear of confusion, e.g.

$$H^i_{\text{syn}}((Y,M),\mathbb{Z}/p^n(r)) = H^i_{\text{syn}}((Y,M)/(\text{Spec} W,W^\times,F_W),\mathbb{Z}/p^n(r)).$$

For a flat (log) $W$-scheme $X$, we write $X_K := X \otimes W K$ and $X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$ as before.

Let $Q$ be a smooth affine scheme over $W$ and $T$ a reduced relative simple NCD over $W$. Let $Y$ be a smooth scheme over $W$ and let

$$f : Y \longrightarrow Q$$

be a projective morphism that is smooth outside $f^{-1}(T)$. Let $D \subset Y$ be a reduced relative simple NCD over $W$. Put $S := Q \setminus T$ and $U := Y \setminus (D \cup f^{-1}(T))$. Let $L$ be the log structure on $Q$ defined by $T$ and $M$ the log structure on $Y$ defined by the reduced part of $D \cup f^{-1}(T)$. We then suppose that the following conditions hold.

- $U$ and $S$ are affine.
- The divisor $D \cup f^{-1}(T)$ is a relative simple NCD over $W$, and $D \cap f^{-1}(S)$ is a relative simple NCD over $S$.
- $f : (Y,M) \to (Q,L)$ is smooth.
- There is a system $\sigma = \{\sigma_m : (Q_m,L_m) \to (Q_m,L_m)\}_m$ of $p$-th Frobenius endomorphisms compatible with $F_W$.
- Assumption 2.20 holds for $U/S$; namely, the $i$-th relative rigid cohomology sheaf $R^i f_{\text{rig}}^! j_U^! \mathcal{O}_{U_{\text{an}}}^\times$ is a coherent $j_S^! \mathcal{O}_{S_{\text{an}}}^\times$-module for each $i \geq 0$ (this implies the comparison isomorphism (2.17) by Remark 2.21 (2)).

**Theorem 3.7.** Under the above setting, let $\omega^{\bullet}_{Y/Q}$ (resp. $\omega^{\bullet}_{Y_m/Q_m}$) denote the de Rham complex of $(Y,M)/(Q,L)$ (resp. $(Y_m,M_m)/(Q_m,L_m)$). Let $0 \leq n < p-1$ be an integer which satisfies $H^n(\omega^{2n+1}_{Y_m/Q_m}) = H^{n+1}(\omega^{2n+1}_{Y_m/Q_m}) = 0$ for all $m > 0$. Then, the following diagram is $(-1)^n$-commutative:
Here, the extension group is taken in the category $\text{Fil}^-\text{F}^-\text{MIC}(S,\sigma)$. See (2.22) and (3.8) for the definitions of the symbol maps $[-]_{U/S}$ and $[-]_{\text{syn}}$, and see (2.23) for $R_\sigma$.

The isomorphism $i$ follows from Proposition 3.5 and Lemma 3.6 under the assumption $H^{n+1}(\omega_{Y_m/Q_m}) = 0$.

To prove Theorem 3.7, we prepare for three lemmas.

**Lemma 3.8.** Let $V = \text{Spec} C$ be a smooth affine $W$-scheme. Let $\sigma = \{\sigma_n : C_n \to C_n\}_n$ be a compatible system of $p$-th Frobenius. Let $V(R) = \text{Hom}_W(\text{Spec} C, R)$ denote the set of $R$-valued points for a $W$-algebra $R$. Put $W_n := W/p^n W$. Then, for any $a_1 \in V(W_1)$, there is a unique $W$-valued point $a = (a_n) \in V(W) = \lim_{\longrightarrow n} V(W_n)$ which makes the diagram

\[
\begin{array}{ccc}
\text{Spec} W_n & \xrightarrow{a_n} & V_n \\
F_W & \downarrow & \sigma_n \\
\text{Spec} W_n & \xrightarrow{a_n} & V_n.
\end{array}
\]

commutative for all $n$.

**Proof.** For $a_n \in V(W_n)$, we define $\phi(a_n)$ to be the morphism which makes the following diagram commutative:

\[
\begin{array}{ccc}
\text{Spec} W_n & \xrightarrow{a_n} & V_n \\
F_W \cong & \downarrow & \sigma_n \\
\text{Spec} W_n & \phi(a_n) & V_n.
\end{array}
\]

One easily verifies that $\phi(a_n)$ is a $W$-morphism (i.e. $\phi(a_n) \in V(W_n)$) and commutes with the reduction map $\rho_n : V(W_n) \to V(W_{n-1})$. Put

\[V(W_n)_{b_{n-1}} := \{a_n \in V(W_n) \mid \rho_n(a_n) = b_{n-1}\}\]
for \(b_{n-1} \in V(W_{n-1})\). This is a nonempty set as the reduction map \(\rho_n\) is surjective (formal smoothness property). We claim that a map

\[
\phi : V(W_n)_{b_{n-1}} \to V(W_n)_{\phi(b_{n-1})}
\]

is a constant map. Indeed, let \(C_n = W_n[t_1, \ldots, t_m]/I\). One can write \(\sigma_n(t_i) = t_i^p + pf_i(t)\). Let \(a_n\) be given by \(t_i \mapsto \alpha_i\). Then, \(\phi(a_n)\) is given by the ring homomorphism

\[
t_i \mapsto F_W^{-1}(\alpha_i^p + pf(\alpha_1, \ldots, \alpha_m)) \mod p^n W.
\]

which depends only on \(\{\alpha_i \mod p^{n-1} W\}_i\), namely on \(\rho_n(a_n) = b_{n-1}\). This means that \(\phi(a_n)\) is constant on the set \(V(W_n)_{b_{n-1}}\).

We prove Lemma 3.8. We have shown that if \(b_{n-1} \in V(W_{n-1})\) satisfies \(\phi(b_{n-1}) = b_{n-1}\), then \(\phi^n(a_n) = \phi(a_n)\) for all \(a_n \in V(W_n)_{b_{n-1}}\) and \(m \geq 1\). Let \(a_1 \in V(W_1)\) be an arbitrary element and take an arbitrary sequence

\[
(a_1, c_2, \ldots, c_n, \ldots) \in \lim_{\to} V(W_n) = V(W).
\]

Since \(\phi\) is the identity on \(V(W_1)\), one has \(\phi^2(c_2) = \phi(c_2)\) as \(c_2 \in V(W_2)_{a_1}\). Then, \(\phi^3(c_3) = \phi^2(c_3) = \phi(c_3)\) as \(c_3 \in V(W_3)_{\phi(c_2)}\). Continuing this, \(a_n := \phi^{n-1}(c_n)\) satisfies that \(\phi(a_n) = a_n\) in \(V(W_n)\) and \(\rho_n(a_n) = \phi^{n-1}(c_n-1) = \phi^{n-2}(c_n-2) = a_{n-1} \in V(W_{n-1})\). Hence, the sequence \(\{a_n\}_n\) defines a \(W\)-valued point \(a = (a_n) \in V(W)\) which makes a diagram

\[
\begin{array}{ccc}
\text{Spec} W_n & \xrightarrow{a_n} & V_n \\
\downarrow F_W & & \downarrow \sigma_n \\
\text{Spec} W_n & \xrightarrow{a_n} & V_n
\end{array}
\]

commutative for all \(n \geq 1\). We show that such \(a\) is unique. Suppose that another \(W\)-valued point \(a' = (a_1', a_2', \ldots, a_n', \ldots)\) makes the above diagram commutative. This means \(\phi(a'_n) = a'_n\) for all \(n \geq 1\). Since \(\phi\) is a constant map on \(V(W_n)_{a'_n-1}\), the point \(a'_n = \phi(a'_n)\) is uniquely determined by \(a'_n-1\), and hence by \(a_1\). Therefore, \(a'\) is uniquely determined by \(a_1\), which means \(a = a'\).

\[\square\]

**Lemma 3.9.** Suppose that \(k = W/pW\) is an algebraically closed field. Let \(V = \text{Spec} C\) be a smooth affine scheme over \(W\). Let \(\hat{C}\) be the \(p\)-adic completion and \(\hat{C}_K := \hat{C} \otimes_W K\). Let \(H_K\) be a locally free \(\hat{C}_K\)-module of finite rank. Let \(\{P_a\}_{a \in V(k)}\) be a set of \(W\)-valued points of \(V\) such that each \(P_a \in V(W)\) is a lifting of \(a \in V(k)\). For an element \(x \in H_K\), let \(x|_{P_a} \in \kappa(P_a) \otimes_C K\) denote the reduction at \(P_a\), where \(\kappa(P_a) \cong K\) is the residue field. If \(x\) satisfies \(x|_{P_a} = 0\) for all \(a \in V(k)\), then \(x = 0\).

**Proof.** Since there is an inclusion \(H_K \hookrightarrow E_K\) into a free \(\hat{C}_K\)-module of finite rank which has a splitting, we may replace \(H_K\) with \(E_K\), and hence, we may assume that \(H_K\) is a free \(\hat{C}_K\)-module. Then, one can reduce the proof to the case of rank one, i.e. \(H_K = \hat{C}_K\). Let \(x \in \hat{C}\) satisfy \(x|_{P_a} = 0\) for all \(P_a\). Suppose that \(x \neq 0\). There is an integer \(n\) such that \(x \in p^n \hat{C} \setminus p^{n+1} \hat{C}\). Write \(x = p^n y\) with \(y \in \hat{C} \setminus p\hat{C}\) that satisfies \(y|_{P_a} = 0\) for all \(a \in V(k)\). Then, \(y\) is zero in \(\hat{C}/p\hat{C} \cong C/pC\) by Hilbert nullstellensatz, namely \(y \in p\hat{C}\). This is a contradiction. \[\square\]
Lemma 3.10. Let $Y$ be a projective smooth scheme over $W$ and $D_Y$ a relative NCD in $Y$ over $W$. Let $M$ be the log structure on $Y$ defined by $D_Y$ and put $U = Y \setminus D_Y$. Then, there is a canonical isomorphism
\[ c : H^i_{\text{syn}}((Y, M), \mathbb{Q}_p(j)) \longrightarrow H^i_{\text{rig-syn}}(U, \mathbb{Q}_p(j)). \] (3.10)
Moreover, a diagram
\[
\begin{array}{ccc}
K_i^M(\mathcal{O}(U)) & \longrightarrow & H^i_{\text{syn}}((Y, M), \mathbb{Q}_p(i)) \\
\downarrow^{[-]_{\text{syn}}} & & \downarrow^c \\
H^i_{\text{rig-syn}}(U, \mathbb{Q}_p(i)) & \leftarrow & H^i_{\text{syn}}((Y, M), \mathbb{Q}_p(i))
\end{array}
\] (3.11)
is commutative.

Proof. Write $U_K = U \times_W K$ and $(Y_K, M_K) = (Y, M) \times_W K$. There is a canonical isomorphism
\[ R\Gamma_{\log\text{-syn}}(U, \mathbb{Q}_p(j)) \cong \text{Cone}[\text{Fil}^j R\Gamma_{\log\text{-dR}}((Y_K, M_K)) \xrightarrow{1-p^{-j} \phi_{\text{crys}}} R\Gamma_{\log\text{-crys}}((Y_1, M_1)/K)[{-1}]] \]
arising from (3.1) (see also Proposition 3.5), where $\phi_{\text{crys}}$ is the $p$-th Frobenius which is defined thanks to the comparison of log de Rham and log crystalline cohomology. There is also a canonical isomorphism ([Bes1, Remark 8.7, 3])
\[ R\Gamma_{\text{rig-syn}}(U, \mathbb{Q}_p(j)) \cong \text{Cone}[\text{Fil}^j R\Gamma_{\text{rig-dR}}(U_K/K) \xrightarrow{1-p^{-j} \phi_{\text{rig}}} R\Gamma_{\text{rig}}(U_1/k)][{-1}], \] (3.12)
where $\phi_{\text{rig}}$ is the $p$-th Frobenius which is defined thanks to the comparison of the algebraic de Rham cohomology and the rigid cohomology. Now, the isomorphism $c$ is induced by the comparison morphism between the rigid and the log crystalline cohomology [Sh1] and the one between the de Rham cohomology and the log de Rham cohomology.

By the construction, $c$ is compatible with respect to the cup-product and the period map to the etale cohomology,
\[
\begin{array}{ccc}
H^i_{\text{syn}}(U, \mathbb{Q}_p(i)) & \xrightarrow{c} & H^i_{\text{rig-syn}}(U, \mathbb{Q}_p(i)) \\
\rho_{\text{syn}} & & \rho_{\text{rig-syn}} \\
\downarrow & & \\
H^i_{\text{ét}}(U_K, \mathbb{Q}_p(i))
\end{array}
\]
where $\rho_{\text{syn}}$ is as in [Ts1, §3.1], and $\rho_{\text{rig-syn}}$ is as in [Bes1, Corollary 9.10]. Moreover, both $\rho_{\text{syn}} \circ [-]_{\text{syn}}$ and $\rho_{\text{rig-syn}} \circ \text{reg}_{\text{rig-syn}}$ agree with the etale symbol map ([Ts1, Proposition 3.2.4], [Bes1, Corollary 9.10]). We show the commutativity of the diagram (3.11). Thanks to the compatibility with respect to the cup-product, one can reduce the assertion to the case $i = 1$; namely, it is enough to show that for $f \in \mathcal{O}(U)^{\times}$ an element $u := c[f]_{\text{syn}} - \text{reg}_{\text{rig-syn}}(f)$ is zero. There is an exact sequence
\[
0 \longrightarrow H^0_{\text{dR}}(U_K/K) \longrightarrow H^1_{\text{rig-syn}}(U, \mathbb{Q}_p(1)) \xrightarrow{h} \text{Fil}^1 H^1_{\text{dR}}(U_K/K)
\]
from (3.12). By the construction of $c$ and the definition of $[-]_{\text{syn}}$ in (3.8) and [Bes1, Def.6.5 and Prop.10.3], one has $h(u) = df/f - df/f = 0$. Therefore, $u$ lies in the image of $H^0_{\text{dR}}(U_K/K)$. We claim that the composition of the maps

$$H^0_{\text{dR}}(U_K/K) \xrightarrow{\rho_{\text{rig-syn}}} H^1_{\text{rig-syn}}(U,\mathbb{Q}_p(1)) \xrightarrow{\rho_{\text{rig-syn}}} H^1_{\text{et}}(U_K,\mathbb{Q}_p(1))$$

is injective. We may assume that $U$ is connected. Moreover, we may replace $W$ with $W(\overline{k})$, so that we may further assume that $U$ has a $W$-valued point. Then, $H^0_{\text{dR}}(U_K/K) \cong K$ and the injectivity can be reduced to the case $U = \text{Spec}W$, which can be easily verified. We turn to the proof of $u = 0$. It is enough to show $\rho_{\text{rig-syn}}(u) = 0$. However,

$$\rho_{\text{rig-syn}}(u) = \rho_{\text{rig-syn}}(\text{reg}_{\text{rig-syn}}(f)) - \rho_{\text{rig-syn}}(c[f]_{\text{syn}}) = \rho_{\text{rig-syn}}(\text{reg}_{\text{rig-syn}}(f)) - \rho_{\text{syn}}([f]_{\text{syn}}) = [f]_{\text{et}} - [f]_{\text{et}} = 0,$$

where $[-]_{\text{et}}$ is the etale symbol map, so we are done. \hfill \Box

**Proof of Theorem 3.7.** By replacing $W$ with $W(\overline{k})$, we may assume that $k$ is an algebraically closed field. We fix a $p$-th Frobenius on $\mathcal{O}(U)^\dagger$ compatible with $\sigma$. Recall the diagram (3.9). Let $\xi \in K_{n+1}^M(\mathcal{O}(U))$ and let

$$\langle \xi \rangle_{U/S}, \langle \xi \rangle_{\text{syn}} \in \mathcal{O}(S)^\dagger \otimes \mathcal{O}(S_K) H^n_{\text{dR}}(U_K/S_K)$$

be the elements sent along the diagram in a counterclockwise direction and a clockwise direction, respectively. We want to show $\langle \xi \rangle_{U/S} = (-1)^n \langle \xi \rangle_{\text{syn}}$. For a closed point $a \in S_1(k)$, we take the unique lifting $P_a \in S(W)$ as in Lemma 3.8. By Lemma 3.9, it is enough to show

$$\langle \xi \rangle_{U/S}|_{P_a} = \langle \xi \rangle_{\text{syn}}|_{P_a} \in \kappa(P_a) \otimes_{k} H^n_{\text{dR}}(U_K/S_K)$$

for every $a$, where $\kappa(P_a) \cong K$ denotes the residue field of $P_a$. Therefore, to show the $(-1)^n$-commutativity of the diagram (3.9), we may specialize the diagram at $P_a$ so that the proof is reduced to the case $(Q,L,\sigma) = (\text{Spec}W,W^X,F_W)$. Summing up the above, the proof of Theorem 3.7 is reduced to showing $(-1)^n$-commutativity of the following diagram:

$$\begin{array}{ccc}
K^M_{n+1}(\mathcal{O}(U)) & \xrightarrow{[-]_{U/W}} & K^M_{n+1}(\mathcal{O}(U)) \\
\otimes & & \otimes \\
\text{Ext}^1(W,H^n(U/W)(n+1)) & \xrightarrow{\phi_{F_W}} & H^n_{\text{syn}}((Y,M)_{\mathcal{O}_Y},\mathbb{Z}_p(n+1)) \\
\cong & & \cong \\
H^n_{\text{dR}}(U_K/K) & \xrightarrow{\phi_{\text{F}_W}} & H^n_{\text{zar}}(Y,\Omega^\bullet_{Y/W}(\text{log}D)).
\end{array}$$

Here, $Y$ is a projective smooth scheme over $W$, $D$ a relative NCD over $W$, $U = Y \setminus D$ and $M$ is the log structure on $Y$ defined by $D$. The extension group is taken in the category of Fil-$F$-MIC($\text{Spec}W,F_W$).
To compare $[-]_{U/W}$ and $[-]_{\text{syn}}$, we use the explicit formula for $[-]_{U/W}$ (Theorem 2.25) and the theory of rigid syntomic regulators by Besser [Bes1]. Consider a diagram

$$
\begin{array}{ccc}
K_{n+1}^{M}(\mathcal{O}(U)) & \stackrel{\text{reg}_{U/W}}{\longrightarrow} & H_{\text{rig-syn}}^{n+1}(U, \mathbb{Q}_{p}(n+1)) \\
\downarrow & & \downarrow \\
H_{\text{rig-syn}}^{n}(U, \mathbb{Q}_{p}(n+1)) & \stackrel{c}{\cong} & H_{\text{syn}}^{n}(Y, M, \mathbb{Z}_{p}(n+1)) \\
\end{array}
$$

(3.14)

where $\text{reg}_{U/W} := \phi_{U/W} \circ [-]_{U/W}$. Here, the isomorphism $i$ follows from [Bes1, (8.5)] and $c$ is the canonical isomorphism (3.10) in Lemma 3.10. The commutativity of the right upper triangle is proven in Lemma 3.10, and that of the right lower square is immediate from the construction of $c$. We show the commutativity of the left. Let $\xi = \{h_0, \ldots, h_n\} \in K_{M}^{n+1}(\mathcal{O}(U))$. Then, using [Bes1, Def.6.5 and Prop.10.3], one can show that

$$
\text{reg}_{\text{rig-syn}}(\xi) = \sum_{i=0}^{n} (-1)^{i} p^{-1} \log \left( \frac{h_{i}^{p}}{h_{i}} \right) \left( \frac{dh_{0}}{h_{0}} \right)^{\varphi_{1}} \wedge \cdots \wedge \left( \frac{dh_{i-1}}{h_{i-1}} \right)^{\varphi_{1}} \wedge \frac{dh_{i+1}}{h_{i+1}} \wedge \cdots \wedge \frac{dh_{n}}{h_{n}}
$$

(3.15)

under the inclusion $H_{\text{dR}}^{n}(U_{K}/K) \hookrightarrow \Omega_{A_{K}/K}^{n}/d\Omega_{A_{K}/K}^{n-1} \cong \Omega_{A_{K}/K}^{n}/d\Omega_{A_{K}/K}^{n-1}$ in a similar way to the proof of [Ka1, Cor. 2.9]. This agrees with $(-1)^{n}\text{reg}_{U/W}(\xi)$ by Theorem 2.25, and hence, the commutativity of the left square follows. This completes the proof of $(-1)^{n}$-commutativity of the diagram (3.13), and hence Theorem 3.7.

4. $p$-adic regulators of $K_{2}$ of curves

For a regular scheme $X$ and a divisor $D$, let $(X, D)$ denote the log scheme whose log structure is defined by the reduced part of $D$.

4.1. Symbol map on $K_{2}$ of a projective smooth family of curves

Let $p > 2$ be a prime and let $W$ be the Witt ring of a prefect field $k$ of characteristic $p$. Put $K = \text{Frac}(W)$, the fractional field. Let $F_{W}$ be the $p$-th Frobenius on $W$. Let $Q$ be a smooth affine curve over $W$ and $T \subset Q$ a closed set that is finite etale over $W$. Put $S := Q \setminus T$. Let $Y$ be a smooth quasi-projective surface over $W$ and let

$$
f : Y \longrightarrow Q
$$

be a projective surjective $W$-morphism such that $f$ is smooth outside $F := f^{-1}(T)$, and the fibers are connected. Let $D \subset Y$ be a reduced relative simple NCD over $W$. Put $X := f^{-1}(S) = Y \setminus F$, $D_X := D \cap X$ and $U := X \setminus D_X$. Suppose that the following conditions hold.
i) There is a system $\sigma = \{\sigma_n : (Q_n, T_n) \to (Q_n, T_n)\}_n$ of $p$-th Frobenius endomorphisms compatible with $F_W$, where $(X_n, M_n) := (X, M) \otimes \mathbb{Z}/p^n\mathbb{Z}$ as before.

ii) The divisor $D + F$ is a relative simple NCD over $W$, and $D_X$ is finite étale over $S$.

iii) The multiplicity of each component of $F$ is prime to $p$.

iv) Assumption 2.20 holds for $U/S$; the $i$-th relative rigid cohomology sheaf $R^i u_{\text{rig}} j_U^! \mathcal{O}_{U^K}$ is a coherent $j_S^! \mathcal{O}_{S^K}$-module for each $i \geq 0$, where $u : U \to S$ (this implies the comparison isomorphism (2.17) by Remark 2.21 (2)).

Here is the summary of notation.

$$
\begin{array}{c}
X \xleftarrow{F} Y \xleftarrow{D + F} U \\
\downarrow \quad \downarrow f \\
S \xleftarrow{T} Q
\end{array}
$$

where the notation above $\hookrightarrow$ shows the complement of the subscheme.

The following lemma provides a sufficient condition for that the condition iv) holds.

**Lemma 4.1.** Let $C$ be a smooth affine curve over $W$ and $v : V \to C$ a smooth $W$-morphism. Suppose that there is a commutative square

$$
\begin{array}{c}
V \xrightarrow{v} Z \\
\downarrow \quad \downarrow g \\
C \xleftarrow{} \mathcal{C}
\end{array}
$$

where $\hookrightarrow$ are open immersions that satisfies the following.

- $\mathcal{C}$ (resp. $\overline{Z}$) is a smooth projective curve (resp. a smooth projective scheme) over $W$.
- Put $Z := g^{-1}(C)$. Then $Z \to C$ is projective smooth.
- Put $T' := \mathcal{C} \setminus C$. Then $T'$ is finite étale over $W$, $\overline{Z} \setminus V$ is a relative simple NCD over $W$, and $Z \setminus V$ is a relative simple NCD over $C$.
- The multiplicity of an arbitrary component of $g^{-1}(T')$ is prime to $p$.

Then, $R^i u_{\text{rig}} j_V^! \mathcal{O}_{V^K}$ is a coherent $j_C^! \mathcal{O}_{C^K}$-module for each $i \geq 0$.

**Proof.** The conditions imply that the morphism

$$
g : (\overline{Z}, \overline{Z} \setminus V) \to (\mathcal{C}, T')
$$

of log schemes is smooth so that $V/C$ is adapted to Setting 2.16. Therefore, $R^i u_{\text{rig}} j_V^! \mathcal{O}_{V^K}$ is a coherent $j_C^! \mathcal{O}_{C^K}$-module for each $i \geq 0$ by Proposition 2.17.

We turn to the setting (4.1). We have the symbol map

$$
[-]_{U/S} : K^M_2(\mathcal{O}(U)) \to \text{Ext}^1_{\text{Fil-F-MIC}(S)}(\mathcal{O}_S, H^1(U/S)(2))
$$

by Theorem 2.23. We omit to write the subscript “Fil-F-MIC(S)” in the extension groups, as long as there is no fear of confusion.
Lemma 4.2. The following diagram is commutative:

\[
\begin{array}{ccc}
K^M_2(\mathcal{O}(U)) & \xrightarrow{\partial} & \mathcal{O}(D_X)^\times \\
[-]_{U/S} & \downarrow & [-]_{D_X/S} \\
\text{Ext}^1(\mathcal{O}_S, H^1(U/S)(2)) & \xrightarrow{\text{Res}} & \text{Ext}^1(\mathcal{O}_S, H^0(D_X/S)(1)).
\end{array}
\]

Here, the right vertical arrow is the symbol map defined in Theorem 2.23, and \(\partial\) is the tame symbol which is defined by

\[
\partial : \{f, g\} \mapsto (-1)^{\text{ord}_{D_X}(f)}\text{ord}_{D_X}(g) \frac{f^{\text{ord}_{D_X}(g)}}{g^{\text{ord}_{D_X}(f)}}_{D_X}.
\]  

(4.2)

Proof. We first note that \(K^M_2(\mathcal{O}(U))\) is generated by symbols \(\{f, g\}\) with \(\text{ord}_{D_X}(f) = 0\). Suppose \(\text{ord}_{D_X}(f) = 0\), namely \(f \in \mathcal{O}(U \cup D_X)^\times\). The natural morphism

\[
p : \text{Cone}[\mathcal{L}\log f(1) \to \mathcal{L}\log(g)][-1] \to \mathcal{L}\log f(1)
\]

and the residue map induce the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \to & H^1(U/S)(2) & \to & M_{f,g}(U/S) & \to & \mathcal{O}_S & \xrightarrow{\delta_{f,g}} & H^2(U/S)(2) \\
\downarrow & & \downarrow & & \downarrow p & & \downarrow \text{Res} & \downarrow & \downarrow \text{Res} \\
\mathcal{O}_S & \xrightarrow{\delta_f} & H^1(U/S)(2) & \xrightarrow{\text{Res}} & H^1(\mathcal{L}\log f(1)) & \to & H^1(U/S)(1) & \to & H^2(U/S)(2) \\
\downarrow & & & & \downarrow & & \downarrow \text{Res} & \downarrow & \downarrow \text{Res} \\
0 & \to & H^0(D_X/S)(1) & \to & H^0(\mathcal{L}\log f|D_X) & \to & H^0(D_X/S) & \xrightarrow{\delta_{f|D_X}} & H^1(D_X/S)(1)
\end{array}
\]

(4.3)

with exact rows where \(\delta_{f,g}(1) = \frac{df}{f} - g\), \(\delta_f(1) = \frac{df}{f}\) and \(\delta_{f|D_X}(1) = \frac{df}{f}|_{D_X}\). The maps “Res”s are residue maps appearing in the Gysin exact sequence; as for the middle one, we are considering the Gysin exact sequence with coefficient in \(\mathcal{L}\log f(1)\) which is constructed in the same manner as Proposition 2.18. We show that \(p'\) agrees with the map \(\delta_g : 1 \mapsto \frac{dg}{g}\).

Let \([A \to B]\) denote the complex

\[
\cdots \to 0 \to A \to B \to 0 \to \cdots
\]

with \(A\) placed in degree zero. Consider a commutative diagram

\[
\begin{array}{ccc}
[\mathcal{L}\log(f)_{\text{dR}} \to \mathcal{L}\log(g)_{\text{dR}}] & \xrightarrow{\text{Res}} & [\mathcal{L}\log(f)_{\text{dR}}/\mathcal{O}_{U_K} e_{-2,f} \to \mathcal{L}\log(g)_{\text{dR}}] \\
\downarrow & & \downarrow \cong \\
[\mathcal{L}\log(f)_{\text{dR}} \to 0] & \xrightarrow{p} & [\mathcal{L}\log(f)_{\text{dR}}/\mathcal{O}_{U_K} e_{-2,f} \to 0] \\
\downarrow & & \downarrow \cong \\
[\mathcal{L}\log(f)_{\text{dR}} \to \mathcal{O}_{U_K}] & \xrightarrow{p'} & [\mathcal{O}_{U_K} \to 0]
\end{array}
\]

of complexes in Fil-F-MIC(\(U\)) where the two horizontal morphisms on the left-hand side are the canonical surjections, and the vertical ones are the projections. Under the identification \(\mathcal{L}\log(f)_{\text{dR}}/\mathcal{O}_{U_K} e_{-2,f} \cong \mathcal{O}_{U_K}\) (which sends \(e_{0,f}\) to 1), the homomorphism \(p'\) is
nothing but the connecting morphism arising from the extension $0 \to \mathcal{O}_U \to \mathcal{L}og(g)_{\text{dR}} \to \mathcal{O}_{U_1} \to 0$. This means $p' = \delta_g$.

Since the composition $\text{Res} \circ p'$ is multiplication by $\text{ord}_D(g)$, the diagram (4.3) induces

$$0 \to H^1(U/S)(2) \to M_{f,g}(U/S) \to \mathcal{O}_{S_K} \xrightarrow{\delta_{f,g}} H^2(U/S)(2) \quad (4.4)$$

Now, we show the lemma. Let $\xi \in K^M_2(\mathcal{O}(U)) \cap \text{Ker}(d\log)$ be arbitrary. Fix $t \in \mathcal{O}(U)_{\times}$ such that $\text{ord}_{D_X}(t) > 0$. Replacing $\xi$ with $m\xi$ for some $m > 0$, one can express

$$\xi = \{f,t\} + \sum_j \{u_j,v_j\}$$

with $\text{ord}_{D_X}(f) = \text{ord}_{D_X}(u_j) = \text{ord}_{D_X}(v_j) = 0$. Then, (4.4) induces a commutative diagram

$$0 \to H^1(U/S)(2) \to M_{\xi}(U/S) \to \mathcal{O}_{S_K} \to 0 \quad (4.5)$$

Since the tame symbol of $\xi$ is $f|_{D_X}$, the assertion follows.

**Proposition 4.3.** Let $K^M_2(\mathcal{O}(U))_{\partial=0}$ be the kernel of the tame symbol $\partial : K^M_2(\mathcal{O}(U)) \to \mathcal{O}(D_X)_{\times}$. Then, the symbol map $[-]_{U/S}$ uniquely extends to

$$[-]_{X/S} : K^M_2(\mathcal{O}(U))_{\partial=0} \to \text{Ext}^1(\mathcal{O}_S,H^1(X/S)(2)).$$

**Proof.** Let $N$ be the kernel of the connecting homomorphism $H^0(D_X/S)(-1) \to H^2(X/S)$ in the Gysin exact sequence. Since $H^0_{\text{rig}}(D_X/S)$ is an overconvergent $F$-isocrystal on $S$ of weight 0, the weight of $N_{\text{rig}}$ is $-2$, and therefore, we have $\text{Hom}_{\text{Fil-F-MIC}(S)}(\mathcal{O}_S,N(2)) = 0$. This shows that the Gysin exact sequence induces an exact sequence

$$0 \to \text{Ext}^1(\mathcal{O}_S,H^1(X/S)(2)) \to \text{Ext}^1(\mathcal{O}_S,H^1(U/S)(2)) \to \text{Ext}^1(\mathcal{O}_S,N(2)),$$

where the extension group is taken in the category of Fil-F-MIC(S). Now, the construction of $[-]_{X/S}$ is immediate from Lemma 4.2. □

**Lemma 4.4.** Let

$$H^\bullet_{\text{syn}}((Y,F),\mathbb{Z}_p(r)) = \lim_{n} H^\bullet_{\text{syn}}((Y,F),\mathbb{Z}/p^n(r))$$

denote the syntomic cohomology with coefficients in $\mathbb{Z}_p$, where we omit to write “$(W,W^\times,F_W)$”. Then, the syntomic symbol map $[-]_{\text{syn}}$ induces a map
Milnor K-theory, F-isocrystals and syntomic regulators

\[ K_2^M (\mathcal{O}(U)) \mathcal{O} \to H^2_{\text{syn}}((Y, F), \mathbb{Z}_p(2)), \]

which we write by the same notation. When \( Q = S = \text{Spec} W \) (\( F = \emptyset \)), this is compatible with the regulator map by Besser [Bes1] (or equivalently by Nekovář-Niziol [N-N]), which means that a diagram

\[ K_2^M (\mathcal{O}(U)) \mathcal{O} \to H^2_{\text{syn}}((Y, F), \mathbb{Z}_p(2)), \]

is commutative, where \( K_i(-) \subset K_i(-) \otimes \mathbb{Q} \) denotes the Adams weight piece.

**Proof.** To show the former, it is enough to show that a diagram

\[ K_2^M (\mathcal{O}(U)) \mathcal{O} \to \mathcal{O}(D^*) \to \mathcal{O}(D^*) \]

is commutative. Then, the required symbol map is induced from an exact sequence

\[ 0 \to H^2_{\text{syn}}((Y, F), \mathbb{Z}/p^n(2)) \to H^2_{\text{syn}}((Y, D + F), \mathbb{Z}/p^n(2)) \to H^1_{\text{syn}}((D, D \cap F), \mathbb{Z}/p^n(1)). \]

To show the commutativity of (4.6), it is enough to check \([\partial(\xi)]_{\text{syn}} = \text{Res}([\xi]_{\text{syn}})\) for an element \( \xi = \{f, g\} \in K_2^M (\mathcal{O}(U)) \) such that \( \text{ord}_{D_X} (f) = 0 \). One has \([\partial(\xi)]_{\text{syn}} = [(f|_{D_X})^{\text{ord}_{D_X}} (g)]_{\text{syn}} = \text{ord}_{D_X} (g) \text{Res}([f|_{D_X}]_{\text{syn}}). \) However, one has

\[ \text{Res}([f]_{\text{syn}} \cup [g]_{\text{syn}}) = [f]_{\text{syn}}|_{D_X} \cup \text{Res}'([g]_{\text{syn}}) = \text{Res}'([g]_{\text{syn}})[f|_{D_X}]_{\text{syn}}, \]

where

\[ \text{Res}' : H^1_{\text{syn}}((Y, D + F), \mathbb{Z}/p^n(1)) \to H^0_{\text{syn}}((D, D \cap F), \mathbb{Z}/p^n(0)) = \mathbb{Z}/p^n \mathbb{Z}. \]

By the construction of \([-]_{\text{syn}}\) in §3.2, one can verify \( \text{Res}'([g]_{\text{syn}}) = \text{ord}_{D_X} (g) \). Hence, \([\partial(\xi)]_{\text{syn}} = \text{Res}([\xi]_{\text{syn}}), \) as required.

When \( Q = S = \text{Spec} W \), the latter assertion can be derived from Lemma 3.10 as \( H^2_{\text{syn}}((Y, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(Y_K/K) \to H^2_{\text{syn}}((Y, D), \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(U_K/K) \) is injective. \( \square \)
Theorem 4.5.

\[
\begin{array}{c}
\text{Ext}^1(\mathcal{O}_S, H^1(X/S)(2)) \\
\downarrow \scriptstyle R_\sigma \quad (2.23) \\
\mathcal{O}(S)_K \otimes \mathcal{O}(S) H^1_{\text{dR}}(X/S) \\
\cap \\
\mathcal{O}(S)_K \otimes \mathcal{O}(S) H^1_{\text{dR}}(X/S) \\
\leftarrow \\
\mathcal{O}(Q) \otimes \mathcal{O}(Q) H^1_{\text{zar}}(Y, \omega^*_{Y/Q})
\end{array}
\]

\[K_2^M(\Theta(U)) \otimes_0 \] is \((-1)\)-commutative where the extension group is taken in the category Fil-F-MIC(S,σ), and \(\omega^*_{Y/Q}\) is the log de Rham complex of \((Y,F)/(Q,T)\).

**Proof.** Noticing that \(H^1_{\text{dR}}(X_K/S_K) \to H^1_{\text{dR}}(U_K/S_K)\) is injective, one can derive the \((-1)\)-commutativity from Theorem 3.7. \(
\square
\)

4.2. Syntomic regulators of \(K_2\) of Elliptic Curves with 3-torsion points

Let

\[f_Q : X_Q \to S_Q = \text{Spec} \mathbb{Q}[t, (t - t^2)^{-1}]\]

be a family of elliptic curves given by a Weierstrass equation

\[y^2 = x^3 + (3x + 4(1 - t))^2.\]

This is the universal elliptic curve over the modular curve \(X_1(3) \cong \mathbb{P}^1_\mathbb{Q}\) with 3-torsion points \((x,y) = (0, \pm 4(1-t))\) and \(x = \infty\). The \(j\)-invariant of the generic fiber \(X_t\) is

\[j(X_t) = \frac{27(1 + 8t)^3}{t(1-t)^3}.\]

Let \(p \geq 5\) be a prime. Let \(W = W(\mathbb{F}_p)\) be the Witt ring of the algebraic closure \(\mathbb{F}_p\) and \(K = \text{Frac} W\) the fractional field. We define an elliptic fibration

\[\tilde{f} : \tilde{Y} \to \mathbb{P}^1_W(t)\] (4.7)

over the projective line in the following way. For \(\alpha \in W \cup \{\infty\}\), let \(P_\alpha\) denote the \(W\)-valued point of \(\mathbb{P}^1_W(t)\) given by \(t = \alpha\). Let \(S_0 = \text{Spec} W[t] \subset \mathbb{P}^1_W(t)\), and \(Y_0\) the subscheme in \(\mathbb{P}^2_W \times_W S_0\) defined by a homogeneous equation

\[X_2^2 X_0 = X_1^3 + X_0 (3X_1 + 4(1-t)X_0)^2,\]

where \((X_0, X_1, X_2)\) denote the homogeneous coordinates of \(\mathbb{P}^2_W\). We set \(x = X_1/X_0\) and \(y = X_2/X_0\). Let \(f_0 : Y_0 \to S_0\) be the natural morphism. The singular fibers are \(f_0^{-1}(P_0)\) and \(f_0^{-1}(P_1)\), and their singular points are \(R_0 = \{(x,y,t) = (-4,0,0)\}\) and \(R_1 = \{(x,y,t) = \)}
We define the fibration \( (4.7) \) to be the blowing-up at \( \{ R_0, R_1 \} \), and then we have an elliptic fibration

\[ \tilde{f}_0 : \tilde{Y}_0 \rightarrow S_0 \]

that satisfies the following.

- \( \tilde{Y}_0 \) is smooth over \( W \).
- The singular fiber \( \tilde{f}_0^{-1}(P_0) \) is a nonreduced curve with two components whose multiplicities are 1 and 2, and the reduced part \( \tilde{f}_0^{-1}(P_0)_{\text{red}} \) is the Neron 2-gon over \( W \), say \( \tilde{f}_0^{-1}(P_0) = Z + 2E \).
- The singular fiber \( \tilde{f}_0^{-1}(P_1) \) is the Neron 3-gon over \( W \), say \( \tilde{f}_0^{-1}(P_1) = F_1 + F_2 + F_3 \).

Next, let \( s = 1/t, z = s^2x \) and \( w = s^3y \). Then, the Weierstrass equation of the generic fiber of \( \tilde{f}_0 \) turns out to be \( w^2 = z^3 + s^2(3z + 4s(s - 1))^2 \). Let \( S_\infty := \text{Spec} W[s, (1 - s)^{-1}] \). The homogeneous equation

\[ X_2^2X_0 = X_1^3 + s^2X_0(3X_1 + 4s(s - 1)X_0)^2, \quad z = X_1/X_0, \quad w = X_2/X_0 \]

defines a subscheme \( Y_\infty \subset \mathbb{P}^2_W \times W S_\infty \) and a projective flat morphism \( f_\infty : Y_\infty \rightarrow S_\infty \). The singular fiber is \( f_\infty^{-1}(P_\infty) \) with one singular point \( R_\infty = \{(z, w, s) = (0, 0, 0)\} \), and the scheme \( Y_\infty \) is nonregular only at \( R_\infty \). It is a standard exercise to resolve it. Namely, 4-time blowing-ups \( Y_\infty \rightarrow \cdots \rightarrow Y_\infty \) provide an elliptic fibration

\[ \tilde{f}_\infty : \tilde{Y}_\infty \rightarrow S_\infty, \]

and it satisfies the following.

- \( \tilde{Y}_\infty \) is smooth over \( W \).
- The singular fiber \( \tilde{f}_\infty^{-1}(P_\infty) \) is the (nonreduced) curve of Kodaira type IV* over \( W \), say \( \tilde{f}_\infty^{-1}(P_\infty) = C_1 + C_2 + C_3 + 2E_1 + 2E_2 + 2E_3 + 3F \) (e.g. [Si, IV, Fig. 4.4]). This is a relative simple NCD over \( W \).

We define the fibration \((4.7)\) to be

\[ \tilde{f} : \tilde{Y} = \tilde{Y}_0 \cup \tilde{Y}_\infty \rightarrow \mathbb{P}^1_W(t). \]

Let \( Q := \mathbb{P}^1_W \setminus \{ P_1, P_\infty \} \supset S := \mathbb{P}^1_W \setminus \{ P_0, P_1, P_\infty \} \) and put \( Y := \tilde{f}^{-1}(Q), \ X := \tilde{f}^{-1}(S) \). Let \( c \in 1 + pW \) and let \( \sigma = \{ \sigma_n : (Q_n, P_{0, n}) \rightarrow (Q_n, P_{0, n}) \}_{n=1} \) be the system of \( p \)-th Frobenius endomorphisms given by \( \sigma_n(t) = ct^p \). Let \( \overline{D}_\pm \subset \overline{Y} \) be the closure of sections \( (x, y) = (0, \pm 4(1 - t)) \) and \( \overline{D}_\infty \) the closure of the infinity section \( x = \infty \). Put \( \overline{D} = \overline{D}_+ \cup \overline{D}_- \cup \overline{D}_\infty \), \( D = Y \cap \overline{D}, \ D_X = D \cap X \) and \( U = X \setminus D_X \). By the construction of \( \overline{f} \), one sees that the following holds.

- Each component of \( \overline{D} \) defines a section of \( \overline{f} \) (and hence is isomorphic to \( \mathbb{P}^1_W \)).
- In the fiber \( f^{-1}(P_0) = Z + 2E \), \( \overline{D} \) meets only with \( Z \), and each intersection is transversal.
• In the fiber $f^{-1}(P_1)$, $\overline{D}_+$ meets only with $F_1$, $\overline{D}_-$ meets only with $F_2$ and $\overline{D}_\infty$ meets only with $F_3$. Each intersection is transversal.

• In the fiber $f^{-1}(P_\infty)$, $\overline{D}_+$ meets only with $C_1$, $\overline{D}_-$ meets only with $C_2$ and $\overline{D}_\infty$ meets only with $C_3$. Each intersection is transversal.

Now the data $(Y/Q,S,D,\sigma)$ satisfy the conditions i), . . . , iv) in the beginning of §4.1 (note iv) follows from Lemma 4.1).

Let $\hat{B}$ (resp. $B^\dagger$) denote the $p$-adic completion (resp. the weak completion) and write $\hat{B}_K := \hat{B} \otimes W K$, $B^\dagger_K := B^\dagger \otimes W K$ as before.

We consider a Milnor symbol

$$\xi := \left\{ \frac{y - 3x - 4(1-t)}{-8(1-t)}, \frac{y + 3x + 4(1-t)}{8(1-t)} \right\} \in K_2^M(\mathcal{O}(U)). \quad (4.8)$$

It is a simple exercise to show $\partial(\xi) = 0$, where $\partial$ is the tame symbol (4.2). Hence, we have a 1-extension

$$[\xi]_{X/S} \in \text{Ext}_F^{1,1}(\mathcal{O}_S, H^1(X/S)(2))$$

by Proposition 4.3. Put $h = 3x + 4(1-t)$. One has

$$d\log(\xi) = d\log\left(\frac{-y-h}{-8(1-t)}\right) \wedge d\log\left(\frac{y+h}{8(1-t)}\right)$$

$$= \frac{dy - dh}{y-h} \wedge \frac{dy + dh}{y+h} + \frac{dt}{t-1} \wedge \left( \frac{dy - dh}{y-h} - \frac{dy + dh}{y+h} \right)$$

$$= \frac{6}{x^3} \left( dy \wedge dx - \frac{x}{t-1} dy \wedge dt + \frac{y}{t-1} dx \wedge dt \right) \quad \text{(by } y^2 - h^2 = x^3\text{)}$$

$$= \frac{3dt}{t-1} \wedge \frac{dx}{y} \in \Gamma(X_K, \Omega^2_{X_K/K}).$$

Hence,

$$D_{X/S}(\xi) = -3 \frac{dt}{t-1} \otimes \frac{dx}{y} \in \Omega^1_{B_K} \otimes H^1_{\text{dR}}(X_K/S_K)$$

by Theorem 2.25 (2.25). The purpose of this section is to describe

$$\text{reg}_{X/S}(\xi) \in B^1_{K} \otimes H^1_{\text{dR}}(X_K/S_K)$$

in terms of the hypergeometric functions

$$2F_1\left(\begin{array}{c} a, b \\ 1 \end{array}; t \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! n!} t^n,$$

where $(a)_n := a(a+1) \cdots (a+n-1)$ denotes the Pochhammer symbol. Put

$$\omega := \frac{dx}{y}, \quad \eta := \frac{xdx}{y} \quad (4.9)$$
a $B_K$-basis of $H^1_{\text{dR}}(X_K/S_K)$. Let
\[
F(t) = \frac{1}{2\sqrt{3}} 2 F_1 \left( \frac{1}{3}, \frac{2}{3}; t \right)
\]
and put
\[
\tilde{\omega} := \frac{1}{F(t)} \frac{dx}{y}, \quad \hat{\eta} := 4(1 - t)(F(t) + 3tF'(t)) \frac{dx}{y} + F(t) \frac{x dx}{y}
\]  
(4.10)
a $K((t))$-basis of $K((t)) \otimes H^1_{\text{dR}}(X_K/S_K)$. Let $\Delta := \text{Spec} W[[t]] \to \mathbb{P}^1_W(t)$ and $O := \text{Spec} W[[t]]/(t) \to \Delta$. The fibration $\mathcal{E} := f^{-1}(\Delta)$ over $\Delta$ is a Tate elliptic curve with central fiber $f^{-1}(O) = F = Z + 2E$. Let $q \in tW[[t]]$ be the Tate period of $\mathcal{E}/\Delta$ which is characterized by
\[
j(X_t) = \frac{27(1 + 8t)^3}{t(1 - t)^3} = \frac{1}{q} + 744 + 196884q + \cdots
\]
\[
\implies q = \frac{1}{27} t + \frac{250289}{243} t^2 - \frac{5507717}{243} t^3 + \frac{25287001}{81} t^4 + \cdots
\]
Let $\omega_{\mathcal{E}/\Delta}$ be the de Rham complex of $(\mathcal{E},F)/(\Delta,O)$. Thanks to the fundamental theorem on log crystalline cohomology by Kato, we have the canonical isomorphism
\[
H^1_{\text{zar}}(\mathcal{E},\omega^*_{\mathcal{E}/\Delta}) \cong H^1_{\text{crys}}((\mathcal{E}_1,F_1)/(\Delta,O)),
\]
([Ka3, Theorem 6.4], see also Proposition 3.1) where $(\mathcal{E}_n,Z_n) = (\mathcal{E},Z) \otimes \mathbb{Z}/p^n\mathbb{Z}$, as before.

**Proposition 4.6.** Let $\nabla : K((t)) \otimes H^1_{\text{dR}}(X_K/S_K) \to K((t)) dt \otimes H^1_{\text{dR}}(X_K/S_K)$ be the Gauss-Manin connection. Then,
\[
\nabla(\omega) = \frac{dt}{3t} \omega + \frac{dt}{12(t^2 - t)} \eta, \quad \nabla(\eta) = \frac{4dt}{3t} \omega + \frac{dt}{3t} \eta,
\]
\[
\nabla(\tilde{\omega}) = \frac{dq}{q} \otimes \hat{\eta}, \quad \nabla(\hat{\eta}) = 0.
\]
(4.11)  
(4.12)

Let $L := \text{Frac}(W[[t]])$. Then, $\left\{ \frac{1}{2} \tilde{\omega}, \frac{1}{2} \hat{\eta} \right\}$ forms the de Rham symplectic basis of $L \otimes W[[t]] H^1_{\text{zar}}(\mathcal{E},\omega^*_{\mathcal{E}/\Delta})$ in the sense of §5.1. Moreover, $\tilde{\omega}$ and $\hat{\eta}$ lie in $H^1_{\text{zar}}(\mathcal{E},\omega^*_{\mathcal{E}/\Delta})$.

**Proof.** One can derive (4.11) from the well-known formula on the Gauss-Manin connection for an elliptic fibration (e.g. [Sas, p.304], [A-C, Theorem 7.1]). We show that $\left\{ \frac{1}{2} \tilde{\omega}, \frac{1}{2} \hat{\eta} \right\}$ forms the de Rham symplectic basis. Then, (4.12) follows from Proposition 5.1. First, thanks to (4.11), one can directly show
\[
\nabla(\tilde{\omega}) = \frac{dt}{12(t^2 - t) F(t)^2} \otimes \hat{\eta}, \quad \nabla(\hat{\eta}) = 0,
\]
(4.13)
where we note that the hypergeometric function $F(t)$ satisfies $t(1 - t)F''(t) + (1 - 2t)F'(t) - \frac{2}{3} F(t) = 0$ (e.g. [NIST, 16.8.5]). Let $\left\{ \tilde{\omega}_{\text{dR}}, \hat{\eta}_{\text{dR}} \right\}$ be the de Rham symplectic basis. Then,
\( \nabla(\hat{\omega}_{\text{dr}}) = \frac{dq}{q} \otimes \hat{\eta}_{\text{dr}}, \quad \nabla(\hat{\eta}_{\text{dr}}) = 0 \) (4.14)

by Proposition 5.1. Since \( \ker(\nabla) \cong K \), we have \( \hat{\eta} = a\hat{\eta}_{\text{dr}} \) for some \( a \in K^\times \). Since \( \hat{\omega} \) and \( \hat{\omega}_{\text{dr}} \) are regular 1-forms, there is a \( h \in L^\times \) such that \( \hat{\omega} = h\hat{\omega}_{\text{dr}} \). We then have

\[
\nabla(\hat{\omega}) = \nabla(h\hat{\omega}_{\text{dr}}) = dh \otimes \hat{\omega}_{\text{dr}} + h\nabla(\hat{\omega}_{\text{dr}}) = dh \otimes \hat{\omega}_{\text{dr}} + h \frac{dq}{q} \otimes \hat{\eta}_{\text{dr}}.
\]

By (4.13), \( h = b \) is a constant. Let \( G_m = \text{Spec} W[t] \) be the reduced regular locus of \( F \) and \( i : G_m \rightarrow E \) the inclusion. Put \( u_0 = y/(x+4) \) and \( u = (u_0 - \sqrt{-3})/(u_0 + \sqrt{-3}) \). Then, \( G_m = \text{Spec} W[u,u^{-1}] \) and

\[
i^* \hat{\omega} = 2\sqrt{-3}i^* \left( \frac{dx}{y} \right) = 2 \frac{du}{u} = 2i^* \hat{\omega}_{\text{dr}}.
\]

This shows \( \hat{\omega} = 2\hat{\omega}_{\text{dr}} \). By (4.13) and (4.14), we have

\[
\frac{dt}{12(t^2-t)}F(t)^2 \otimes \hat{\eta} = 2 \frac{dq}{q} \otimes \hat{\eta}_{\text{dr}} = 2a^{-1} \frac{dq}{q} \otimes \hat{\eta}.
\]

Take the residue at \( t = 0 \), and then we see \( a = 2 \).

There remains to show \( \hat{\omega},\hat{\eta} \in H^1_{\text{zar}}(\mathcal{E},\omega_{\mathcal{E}/\Delta}) \). It is straightforward to see that \( dx/y \in \Gamma(\mathcal{E},\omega_{\mathcal{E}/\Delta}) \), so that one has \( \hat{\omega} \in \Gamma(\mathcal{E},\omega_{\mathcal{E}/\Delta}) \). The Gauss-Manin connection induces a connection

\[
\nabla : H^1_{\text{zar}}(\mathcal{E},\omega_{\mathcal{E}/\Delta}) \rightarrow \frac{dt}{t} \otimes H^1_{\text{zar}}(\mathcal{E},\omega_{\mathcal{E}/\Delta}),
\]

and one from (4.12)

\[
\hat{\eta} = q \frac{d}{dq} \hat{\omega} = u(t) \cdot t \frac{d}{dt} \hat{\omega}, \quad \text{where } u(t) := t^{-1}q \frac{dt}{dq} \in W[[t]]^\times
\]

(note that \( W[[q]] = W[[t]] \)). This shows \( \hat{\eta} \in H^1_{\text{zar}}(\mathcal{E},\omega_{\mathcal{E}/\Delta}) \). This completes the proof. \( \square \)

**Proposition 4.7.** Let \( c \in 1+pW \) and let \( \sigma \) be the \( p \)-th Frobenius on \( \hat{B}_K \) defined by \( \sigma(t) = ct^p \). Let \( \Phi \) be the \( \sigma \)-linear Frobenius on \( H^1_{\text{zar}}(\mathcal{E},\omega_{\mathcal{E}/\Delta}) \). Put

\[
\tau^\sigma(t) = p^{-1} \log \left( \frac{q^p}{q^\sigma} \right) = -p^{-1} \log(27p^{-1}c) + p^{-1} \log(q_0^p/q_0^\sigma), \quad \left( q_0 := \frac{27q}{t} \right),
\]

where \( \log : 1+pW[[t]] \rightarrow W[[t]] \) is defined by the customary Taylor expansion. Then,

\[
(\Phi(\hat{\omega}) \quad \Phi(\hat{\eta})) = (\hat{\omega} \quad \hat{\eta}) \left( \begin{array}{cc} p & 0 \\ -p\tau^\sigma(t) & 1 \end{array} \right).
\]

**Proof.** Note \( W[[t]] = W[[q]] \). This is a special case of Theorem 5.6 in the Appendix. \( \square \)

Let \( \xi \) be the symbol (4.8). Let \( \sigma : B^\dagger \rightarrow B^\dagger \) be the \( p \)-th Frobenius defined by \( \sigma(t) = ct^p \). Let

\[
0 \rightarrow H^1_{\text{dr}}(X/S)(2) \rightarrow M_\xi(X/S) \rightarrow \mathcal{O}_S \rightarrow 0
\]
be the 1-extension in $\Fil F$-MIC$(S, \sigma)$, associated to $[\xi]_{X/S}$. Let $e_\xi \in \Fil^0 M_\xi(X_K/S_K)_{\dR}$ be the unique lifting of $1 \in \mathcal{O}(S_K)$. Define $\varepsilon_i(t), E_i(t)$ by

$$\reg^{(\sigma)}_{X/S}(\xi) = e_\xi - \Phi(e_\xi) = \varepsilon_1(t)\omega + \varepsilon_2(t)\eta$$

$$= E_1(t)\tilde{\omega} + E_2(t)\tilde{\eta}. \quad (4.16)$$

It follows that $\varepsilon_i(t) \in B^+_K$. By (4.10), one immediately has

$$\varepsilon_1(t) = \frac{E_1(t)}{F(t)} + 4(1-t)(F(t) + 3tF'(t))E_2(t), \quad (4.17)$$

$$\varepsilon_2(t) = F(t)E_2(t). \quad (4.18)$$

The power series $E_i(t)$ are explicitly described as follows.

**Theorem 4.8.** Let $\xi$ be as in (4.8). Put $\nu = (-1 + \sqrt{-3})/2$. Then, $E_1(t), E_2(t) \in K[[t]]$, and they are characterized by

$$\frac{d}{dt} E_1(t) = -3 \left( F(t) \frac{dt}{t-1} - F(t)^\sigma \frac{p^{-1}dt^\sigma}{t^\sigma - 1} \right), \quad E_1(0) = 0, \quad (4.19)$$

$$\frac{d}{dt} E_2(t) = -E_1(t) \frac{q'}{q} - 3F(t)^\sigma \tau^{(\sigma)}(t) \frac{p^{-1}dt^\sigma}{t^\sigma - 1}, \quad E_2(0) = -9\ln_2^{(p)}(-\nu). \quad (4.20)$$

**Proof.** Apply $\nabla$ on (4.16). By Proposition 4.6 together with the fact that $\Phi \nabla = \nabla \Phi$, one has

$$\nabla(\varepsilon_1 - \Phi(\nabla(\varepsilon_1))) = dE_1(t) \otimes \tilde{\omega} + \left( E_1(t) \frac{dq}{q} + dE_2(t) \right) \otimes \tilde{\eta}. \quad (4.21)$$

Since

$$\nabla(\varepsilon_1) = -d\log(\xi) = -3 \frac{dt}{t-1} \otimes \frac{dx}{y} = -3F(t) \frac{dt}{t-1} \otimes \tilde{\omega}$$

(cf. (2.31)), the left-hand side of (4.21) is

$$-3F(t) \frac{dt}{t-1} \otimes \tilde{\omega} + 3p^{-1}\sigma \left( F(t) \frac{dt}{t-1} \right) \otimes p^{-1}\Phi(\tilde{\omega})$$

$$= -3F(t) \frac{dt}{t-1} \otimes \tilde{\omega} + 3F(t)^\sigma \frac{p^{-1}dt^\sigma}{t^\sigma - 1} \otimes (\tilde{\omega} - \tau^{(\sigma)}(t)\tilde{\eta}) \quad \text{(Proposition 4.7)}$$

$$= -3 \left( F(t) \frac{dt}{t-1} - F(t)^\sigma \frac{p^{-1}dt^\sigma}{t^\sigma - 1} \right) \otimes \tilde{\omega} - 3F(t)^\sigma \tau^{(\sigma)}(t) \frac{p^{-1}dt^\sigma}{t^\sigma - 1} \otimes \tilde{\eta}.$$

Therefore, one has

$$\frac{d}{dt} E_1(t) = -3 \left( F(t) \frac{dt}{t-1} - F(t)^\sigma \frac{p^{-1}dt^\sigma}{t^\sigma - 1} \right), \quad (4.22)$$
and
\[
\frac{d}{dt} E_2(t) = -E_1(t) \frac{q'}{q} - 3F(t)^{\sigma} \tau^{(\sigma)}(t) \frac{p^{-1} dt^{\sigma}}{t^{\sigma} - 1}. \tag{4.23}
\]

The differential equation (4.22) implies that $E_1(t) \in K[[t]] \cap \hat{B}_K$. It determines all coefficients of $E_1(t)$ except the constant term. Note $\tau^{(\sigma)}(t) \in W[[t]]$. Taking the residue at $t = 0$ of the both sides of (4.23), one concludes $E_1(0) = 0$. We thus have the full description of $E_1(t)$. Since $E_1(t) \in tK[[t]]$, the differential equation (4.23) implies that $E_2(t) \in K[[t]] \cap \hat{B}_K$, and it determines all coefficients of $E_2(t)$ except the constant term.

The rest of the proof is to show $E_2(0) = -9\ln^2(-\nu)$. To do this, we look at the syntomic cohomology of the singular fiber $F$. Recall that $F$ has two components $Z, E$ and the multiplicity of $Z$ (resp. $E$) is 1 (resp. 2). The reduced part $F_{\text{red}} = Z \cup E$ is a Neron 2-gon, and the divisor $D$ intersects only with $Z$. Let $F_*$ be the simplicial nerve of the normalization of $F_{\text{red}}$. Let $i_{F_\text{red}} : F_{\text{red}} \rightarrow \mathcal{E}$ and let $i_{F_*} : F_* \rightarrow \mathcal{E}$. There is a commutative diagram

\[
\begin{array}{ccc}
H^2_{\text{syn}}(\mathcal{E}, \mathbb{Z}_p(2)) & \xrightarrow{i^*_{F_*}} & H^2_{\text{syn}}(F_*, \mathbb{Z}_p(2)) \\
\rho_* & & \cong \quad \cong \\
H^2_{\text{syn}}((\mathcal{E}, F)/(\Delta, O, \sigma), \mathbb{Z}_p(2)) & & H^1_{\text{zar}}(F_*, \mathcal{O}_{F_*}) \\
\cong & & \quad u' \\
H^1_{\text{zar}}(\mathcal{E}, \Omega^*_{\mathcal{E}/\Delta}) & \xrightarrow{i^*_{F_{\text{red}}}} & H^1_{\text{zar}}(F_{\text{red}}, \Omega^*_{\mathcal{E}/\Delta} | F_{\text{red}}),
\end{array}
\]

where $u$ and $u'$ are the maps induced from the canonical maps $\Omega^*_{\mathcal{E}/\Delta} | F_{\text{red}} \rightarrow \mathcal{O}_{F_*}$ and $\Omega^*_{F_*} \rightarrow \mathcal{O}_{F_*}$. By Theorem 4.5, $\rho_{\sigma}(\xi)_{\text{syn}}$ is related to $[\xi]_{\text{syn}} \in H^2_{\text{syn}}(\mathcal{E}, \mathbb{Z}_p(2))$ as follows:

\[
\rho_{\sigma}([\xi]_{\text{syn}}) = -E_1(t) \tilde{\omega} - E_2(t) \tilde{\eta} \in H^1_{\text{zar}}(\mathcal{E}, \Omega^*_{\mathcal{E}/\Delta}). \tag{4.24}
\]

To compute $E_2(0)$, we shall compute the both sides of

\[
(u' \circ i^*_{F_*})([\xi]_{\text{syn}}) = -E_2(0)(u \circ i^*_{F_{\text{red}}})(\tilde{\eta}) \in H^1_{\text{zar}}(F_*, \mathcal{O}_{F_*}). \tag{4.25}
\]

Let $z_0 = y/(x + 4)|_Z$ be a coordinate of $Z \cong \mathbb{P}^1_W$. Let $R_1, R_2$ be the intersection points of $Z$ and $E$ given by $z_0 = \sqrt{-3}, -\sqrt{-3}$, respectively. We use another coordinate $z := \nu^2(1 - z_0)/2$ of $Z$, so that the rational functions

\[
h_1 := \frac{y - 3x - 4(1 - t)}{-8(1 - t)}, \quad h_2 := \frac{y + 3x + 4(1 - t)}{8(1 - t)} \in \mathcal{O}(U)^\times
\]

in the Milnor symbol $\xi$ in (4.8) satisfy

\[
h_1|_Z = z^3, \quad h_2|_Z = (1 - \nu z)^3, \quad h_1|_E = h_2|_E - 1. \tag{4.26}
\]
The points $R_1, R_2$ are given by $z = -1, -\nu$, respectively. Let $\mathcal{J}_n(r)V = \mathcal{J}_n(r)V_{(W,F_W)}$ denote the syntomic complex for $V = Z, E$ or $Z \cap E$. We fix isomorphisms

\[ H^2_{\text{syn}}(F_\ast, Z/p^n(r)) \cong H^2_{\text{fl}}(F_{\text{red}}, \mathcal{J}_n(r)Z \oplus \mathcal{J}_n(r)E \rightarrow \mathcal{J}_n(r)Z \cap E) \]

\[ H^2_{\text{DR}}(F_\ast/W) \cong H^2_{\text{zar}}(F_{\text{red}}, \Omega^Z \oplus \Omega^E \rightarrow \mathcal{O}Z \cap E) \]

\[ H^2_{\text{zar}}(F_\ast, \mathcal{O}_F) \cong H^2_{\text{zar}}(F_{\text{red}}, \mathcal{O}Z \oplus \mathcal{O}E \rightarrow \mathcal{O}Z \cap E), \]

where $i : (f,g) \mapsto f|_{Z \cap E} - g|_{Z \cap E}$. We also fix an isomorphism

\[ \alpha : H^1_{\text{zar}}(F_\ast, \mathcal{O}_F) \cong \text{Coker}[H^0(\mathcal{O}_Z) \oplus H^0(\mathcal{O}_E) \rightarrow H^0(\mathcal{O}_{R_1}) \oplus H^0(\mathcal{O}_{R_2})] \cong W, \]

where the last isomorphism is given by $(c_1, c_2) \mapsto c_1 - c_2$.

We compute the right-hand side of (4.25). The map $u \circ i_{F_{\text{red}}}^\ast$ agrees with the composition of the maps,

\[ H^1_{\text{zar}}(\mathcal{E}, \mathcal{O}_F) \rightarrow H^1_{\text{zar}}(\mathcal{E}, \mathcal{O}_E) \rightarrow H^1_{\text{zar}}(F_\ast, \mathcal{O}_F). \]

Let $y_1 = 2y$ and $x_1 = x + 3$ so that we have $y^2 = x^3 + (3x + 4 - 4t)^2 \iff y_1^2 = 4x_1^3 - g_2x_1 - g_3$. Let $\mathcal{E} = U_0 \cup U_\infty$ be an affine open covering given by $U_0 = \{y_1^2 = 4x_1^3 - g_2x_1 - g_3\}$ and $U_\infty = \{w^2 = 4v - g_2v^3 - g_3v^4\}$ with $v = 1/x_1$, $w = y_1/x_1$. Then, one can compute the cohomology $H^i_{\text{zar}}(\mathcal{E}, \mathcal{F}^\ast)$ for a bounded complex $\mathcal{F}^\ast$ of quasi-coherent sheaves by the total complex of Cech complex

\[ \Gamma(U_0, \mathcal{F}^\ast) \oplus \Gamma(U_\infty, \mathcal{F}^\ast) \rightarrow \Gamma(U_0 \cap U_\infty, \mathcal{F}^\ast), \quad (f_0, f_\infty) \mapsto f_0|_{U_0 \cap U_\infty} - f_\infty|_{U_0 \cap U_\infty}. \]

In particular, $H^1_{\text{zar}}(\mathcal{E}, \mathcal{F}^\ast)$ is the kernel of the map

\[ \Gamma(U_0 \cap U_\infty, \mathcal{F}^0) \times (\Gamma(U_\infty, \mathcal{F}^1) \oplus \Gamma(U_\infty, \mathcal{F}^1)) \rightarrow \Gamma(U_0 \cap U_\infty, \mathcal{F}^1) \times (\Gamma(U_\infty, \mathcal{F}^2) \oplus \Gamma(U_\infty, \mathcal{F}^2)) \]

\[ (f) \times (g_0, g_\infty) \mapsto (df - (g_0|_{U_0 \cap U_\infty} - g_\infty|_{U_0 \cap U_\infty})) \times (dg_0, dg_\infty) \]

modulo the image of the map

\[ \Gamma(U_\infty, \mathcal{F}^0) \oplus \Gamma(U_\infty, \mathcal{F}^0) \rightarrow \Gamma(U_0 \cap U_\infty, \mathcal{F}^0) \times (\Gamma(U_\infty, \mathcal{F}^1) \oplus \Gamma(U_\infty, \mathcal{F}^1)) \]

\[ (g_0, g_\infty) \mapsto (g_0|_{U_0 \cap U_\infty} - g_\infty|_{U_0 \cap U_\infty}) \times (dg_0, dg_\infty), \]

where $d : \mathcal{F}^i \rightarrow \mathcal{F}^{i+1}$ is the differential map. We compute

\[ u \circ i_{F_{\text{red}}}^\ast(\hat{\eta}) = i_F^\ast(u''(\hat{\eta})) = i_F^\ast(u''\left(\frac{1}{\sqrt{-3}}\hat{\eta}\right)), \quad \hat{\eta}_1 := \frac{x_1dx_1}{y_1}, \]  

(4.27)

where the last equality follows from the fact that $u''(dx/y) = 0$. The cohomology class $\hat{\eta}_1 \in H^1_{\text{zar}}(\mathcal{O}_E)$ is represented by a Cech cocycle

\[ \left(\begin{array}{c} y_1 \\ 2x_1 \end{array}\right) \times \left(\frac{x_1dx_1}{y_1}, \frac{(g_2v + 2g_3v^2)dv}{4w}\right) \in \Gamma(U_0 \cap U_\infty, \mathcal{O}_E) \times (\Gamma(U_0, \mathcal{O}_E) \oplus \Gamma(U_\infty, \mathcal{O}_E)). \]
Then, \( u''(\tilde{\eta}) \) is represented by a Čech cocycle \( (y_1/2x_1) = (y/(x+3)) \in \Gamma(U_0 \cap U_\infty, \mathcal{O}_\mathcal{E}) \), and hence, (4.27) is represented by a Čech cocycle

\[
\left( \frac{1}{\sqrt{-3}} \frac{y}{x+3} \right)_Z \times (0,0) = \left( \frac{1}{\sqrt{-3}} \frac{z_0(z_0^2+3)}{z_0^2+2},0 \right) \times (0,0)
\]

\[
= \left( \frac{1}{\sqrt{-3}} z_0,0 \right) \times (0,0) - \left( \frac{1}{\sqrt{-3}} \frac{-z_0}{z_0^2+2},0 \right) \times (0,0)
\]

\[
\in \Gamma(U_0 \cap U_\infty, \mathcal{O}_Z \oplus \mathcal{O}_E) \times (\Gamma(U_0, \mathcal{O}_{Z\cap E}) \oplus \Gamma(U_\infty, \mathcal{O}_{Z\cap E})).
\]

Since \( z_0 \in \Gamma(U_0, \mathcal{O}_Z) \) and \( z_0/(z_0^2+2) \in \Gamma(U_\infty, \mathcal{O}_Z) \), this is equivalent to

\[
(0,0) \times \left( \left( \frac{1}{\sqrt{-3}} \cdot \sqrt{-3}, \frac{1}{\sqrt{-3}} \cdot (-\sqrt{-3}) \right), \left( \frac{1}{\sqrt{-3}} \cdot -\sqrt{-3}, \frac{1}{\sqrt{-3}} \cdot \sqrt{-3} \right) \right)
\]

\[
= (0,0) \times ((1,-1),(1,-1)).
\]

This shows

\[
\alpha \left( (h \circ i_{F_{\text{red}}}^*)(\tilde{\eta}) \right) = 1 - (-1) = 2
\] (4.28)

giving the right-hand side of (4.25).

We write down the left-hand side of (4.25) explicitly. Put \( D_F := D \cap F = D \cap Z \) (= three points \( z = 0, \nu^2, \infty \)). Let \( F_U := F_{\text{red}} \cap U = (Z \setminus D_F) \cup E \) and its simplicial nerve is denoted by \( F_{U,*} \). One has \( h_i|_{F_U} \in \mathcal{O}(F_U)^\times \) and

\[
\xi|_{F_U} = \{ h_1|_{F_U}, h_2|_{F_U} \} \in K_2^M(\mathcal{O}(F_U)).
\]

We think \( h_i|_{F_U} \) to be an element of \( K_1(F_{U,*}) \) and \( \xi_{F_U} \) to be an element of \( K_2(F_{U,*}) \) via the canonical maps \( K^M(\mathcal{O}(F_U)) \rightarrow K_i(F_{U,*}) \). To compute the left-hand side of (4.25), it is enough to compute

\[
[\xi|_{F_U}]_{\text{syn}} = [h_1|_{F_U}]_{\text{syn}} \cup [h_2|_{F_U}]_{\text{syn}} \in H^2_{\text{syn}}((F_*,D_F),\mathbb{Z}_p(2))
\]

as there is a commutative diagram

\[
H^2_{\text{syn}}(F_*,\mathbb{Z}_p(2)) \xleftarrow{\cong} H^1_{\text{dR}}(F_*/W) \xrightarrow{u'} H^1_{\text{zar}}(F_*,\mathcal{O}_{F_*})
\]

\[
H^2_{\text{syn}}((F_*,D_F),\mathbb{Z}_p(2)) \xleftarrow{\cong} H^1_{\text{dR}}((F_*,D_F)/W) \xrightarrow{\text{ }} H^1_{\text{zar}}(F_*,\mathcal{O}_{F_*}).
\]

We further replace the log syntomic cohomology with the rigid syntomic cohomology. Take an (arbitrary) affine open set \( E' \subset E \) such that \( E' \supset Z \cap E \). Put \( Z' := Z \setminus D_F \) and \( F_U' := Z' \cup E' \) and let \( F_{U,*}' \) be the simplicial nerve. Then, there is a commutative diagram

\[
H^2_{\text{syn}}((F_*,D_F),\mathbb{Z}_p(2)) \xleftarrow{\cong} H^1_{\text{dR}}((F_*,D_F)/W) \xrightarrow{\text{ }} H^1_{\text{zar}}(F_*,\mathcal{O}_{F_*})
\]

\[
H^2_{\text{rig-syn}}(F_{U,*}',\mathbb{Q}_p(2)) \xleftarrow{\cong} H^1_{\text{dR}}(F_{U,*}'/K) \xrightarrow{\text{ }} H^1_{\text{zar}}(F_*,\mathcal{O}_{F_*}) \otimes \mathbb{Q}.
\]
Thanks to the compatibility with the rigid regulator maps (Lemma 3.10), it is enough to compute
\[
\text{reg}_{\text{rig-syn}}(h_1|_{F'_U}) \cup \text{reg}_{\text{rig-syn}}(h_2|_{F'_U}) \in H_{\text{rig-syn}}^2(F'_U, \mathbb{Q}_p(2)).
\] (4.29)
We take a \(p\)-th Frobenius \(\varphi_{Z'}\) on \(\mathcal{O}(Z')^\dagger\) given by \(\varphi_{Z'}(z) = z^p\). Note \(\varphi : Z' \to Z'\) fixes \(R_1\) and \(R_2\). We also take a \(p\)-th Frobenius \(\varphi_{E'}\) on \(\mathcal{O}(E')^\dagger\) that fixes \(R_1\) and \(R_2\). Then, for \((V, V) = (Z', Z)\) or \((E', E)\), the rigid syntomic complex \(S_{\text{rig-syn}}(r)_V := R\Gamma_{\text{rig-syn}}(V, \mathbb{Q}_p(r))\) is given as follows,
\[
S_{\text{rig-syn}}(r)_V = \text{Cone}[\Gamma(V_K, \Omega^{2\nu}_{V_K} (\log \partial V_K))^{1-p^r\varphi_{V}} \Omega^{1}_{\mathcal{O}(V)_{K}^\dagger}[−1],
\]
where \(\partial V = V \setminus V\) and \(\Omega^1_{\mathcal{O}(V)_{K}^\dagger} = \Omega^1_{\mathcal{O}(V)_{K}^\dagger / K}\) denotes the module of continuous differentials. Moreover, the rigid syntomic cohomology of \(F'_U,\) is described as follows:
\[
H^j_{\text{rig-syn}}(F'_U, \mathbb{Q}_p(r)) \cong H^j(S_{\text{rig-syn}}(r)_{Z'} \oplus S_{\text{rig-syn}}(r)_{E'} \to S_{\text{rig-syn}}(r)_{Z \cap E}),
\]
where \(i : (f, g) \to f|_{Z \cap E} - g|_{Z \cap E}\). Under this identification, it follows from [Bes1, Prop.10.3] that we have cocycles
\[
\text{reg}_{\text{rig-syn}}(h_1|_{F'_U}) = \left(3 \frac{dz}{z^0}, 0\right) \times (0, 0) \times (0, 0)
\]
\[
\text{reg}_{\text{rig-syn}}(h_2|_{F'_U}) = \left(\frac{-3\nu dz}{1 - \nu z}, 3p^{-1} \log \left(\frac{1 - \nu z^p}{1 - \nu^p z^p}\right)\right) \times (0, 0) \times (0, 0)
\]
\[\in (\Omega^{1}_{\mathcal{O}(Z')_{K}^\dagger} \oplus \mathcal{O}(Z')_{K}^\dagger) \times (\Omega^{1}_{\mathcal{O}(E')_{K}^\dagger} \oplus \mathcal{O}(E')_{K}^\dagger) \times (\mathcal{O}(R_1) \oplus \mathcal{O}(R_2))\]
for \(h_i\) in (4.26), and then
\[
(4.29) = \left(-9p^{-1} \log \left(\frac{1 - \nu z^p}{1 - \nu^p z^p}\right) \frac{dz}{z^0}, 0\right) \times (0, 0) \times (0, 0)
\]
\[= \left(9d(\ln_2(p)(\nu z)), 0\right) \times (0, 0) \times (0, 0)
\]
\[\equiv (0, 0) \times (0, 0) \times (9\ln_2(p)(-\nu), 9\ln_2(p)(-\nu^2))\]
in \(H_{\text{rig-syn}}^2(F'_U, \mathbb{Q}_p(2))\). Here, we use the fact that \(\ln_2(p)(z)\) is an overconvergent function on \(\mathbb{P}^1 \setminus \{1, \infty\}\) (Proposition 2.9). This shows
\[
\alpha((u' \circ i_{F'_U}^*)([\xi]_{\text{syn}})) = 9\ln_2(p)(-\nu) - 9\ln_2(p)(-\nu^2) = 18\ln_2(p)(-\nu),
\]
which provides the left-hand side of (4.25). Combining this with (4.28), one finally has
\[
18\ln_2(p)(-\nu) = -2E_2(0).
\]
This completes the proof.

**Corollary 4.9.** Let \(a \in W\) satisfy \(a \not\equiv 0, 1 \mod p\). Let \(c = F_W(a) a^{-p}\) so that \(\sigma(t) = ct^p\) satisfy \(\sigma(t)|_{t=a} = F_W(a)\). Let \(E_a\) be the fiber of \(f\) at \(t = a\) and put \(U_a := E_a \cap U\). Let \(\xi|_{E_a} \in K^M_2(\mathcal{O}(U_a))_{0=0}\) be the restriction of the Milnor symbol \(\xi\) in (4.8), and we think it to be an
element of the Adams weight piece $K_2(E_a)^{(2)}$ under the natural map $K_2^M(O(U_a))_{\vartheta=0} \to K_2(E_a)^{(2)}$. Let

$$\text{reg}_{\text{rig-syn}} : K_2(E_a)^{(2)} \to H^2_{\text{rig-syn}}(E_a, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(E_a/K)$$

be the regulator map by Besser [Bes1] or, equivalently, by Nekovář-Niziol [N-N]. Then,

$$\text{reg}_{\text{rig-syn}}(\xi|_{E_a}) = -\varepsilon_1(a) \frac{dx}{y} - \varepsilon_2(a) \frac{x dx}{y} \in H^1_{\text{dR}}(E_a/K).$$

Proof. Recall the diagram in Theorem 4.5 and take the pullback of it at the point $t = a$ of $S^*$. Then, $[-]_{\text{syn}}$ turns out to be the regulator map $\text{reg}_{\text{rig-syn}}$ (Lemma 4.4). Now, the assertion is immediate from Theorem 4.8.

5. Appendix: Frobenius on totally degenerating abelian schemes

5.1. De Rham symplectic basis for totally degenerating abelian varieties

Let $R$ be a regular noetherian domain and $I$ a reduced ideal of $R$. Let $L := \text{Frac}(R)$ be the fractional field. Let $J/R$ be a $g$-dimensional commutative group scheme such that the generic fiber $J_\eta$ is a principally polarized abelian variety over $L$. If the fiber $T$ over $\text{Spec} R/I$ is an algebraic torus, we call $J$ a totally degenerating abelian scheme over $(R, I)$ (cf. [F-C] Chapter II, 4). Assume that the algebraic torus $T$ is split. Assume further that $R$ is complete with respect to $I$. Then, there is the uniformization $\rho : \mathbb{G}_m^g \to J$ in the rigid analytic sense. We fix $\rho$ and the coordinates $(u_1, \ldots, u_g)$ of $\mathbb{G}_m^g$. Then a matrix

$$q = \begin{pmatrix} q_{11} & \cdots & q_{1g} \\ \vdots & \ddots & \vdots \\ q_{g1} & \cdots & q_{gg} \end{pmatrix}, \quad q_{ij} = q_{ji} \in L$$

of multiplicative periods is determined up to $\text{GL}_g(\mathbb{Z})$, and this yields an isomorphism

$$J \cong \mathbb{G}_m^g / q^\mathbb{Z}$$

of abelian schemes over $R$, where $\mathbb{G}_m^g / q^\mathbb{Z}$ denotes Mumford’s construction of the quotient scheme ([F-C] Chapter III, 4.4).

In what follows, we suppose that the characteristic of $L$ is zero. The morphism $\rho$ induces

$$\rho^* : \Omega^1_{J/R} \to \bigoplus_{i=1}^g \hat{\Omega}^1_{J_{\mathbb{Z}}, i}, \quad \hat{\Omega}^1_{J_{\mathbb{Z}}, i} := \lim_{\leftarrow n} \Omega^1_{J/R/I^n[u_i, u_i^{-1}]/R}. $$

Let

$$\text{Res}_i : \hat{\Omega}^1_{J_{\mathbb{Z}}, i} \to R, \quad \text{Res}_i \left( \sum_{m \in \mathbb{Z}} a_m u_i^m \frac{du_i}{u_i} \right) = a_0$$

be the residue map. The composition of $\rho^*$ and the residue map induces a morphism

$$\Omega^\bullet_{J/R} \to R^g[-1]$$

of complexes, and hence a map

$$\tau : H^1_{\text{dR}}(J/R) := H^1_{\text{zar}}(J, \Omega^\bullet_{J/R}) \to R^g.$$  \hfill (5.2)
Let $U$ be defined by

$$0 \rightarrow U \rightarrow H^1_{\text{dR}}(J_\eta/L) \xrightarrow{\tau} L^g \rightarrow 0.$$ 

Note that the composition $\Gamma(J_\eta, \Omega^1_{J_\eta/L}) \xrightarrow{\sim} H^1_{\text{dR}}(J_\eta/L) \xrightarrow{\tau} L^g$ is bijective. Let $\langle x, y \rangle$ denote the symplectic pairing on $H^1_{\text{dR}}(J_\eta/L)$ with respect to the principal polarization on $J_\eta$. We call an $L$-basis $\hat{\omega}_i, \hat{\eta}_j \in H^1_{\text{dR}}(J_\eta/L)$, $1 \leq i, j \leq g$, a de Rham symplectic basis if the following conditions are satisfied.

1. $\hat{\omega}_i \in \Gamma(J_\eta, \Omega^1_{J_\eta/L})$ and $\tau(\hat{\omega}_i) \in (0, \ldots, 1, \ldots, 0)$, where “1” is placed in the $i$-th component. Equivalently, $\rho^*(\hat{\omega}_i) = \frac{du_i}{u_i}$.

2. $\hat{\eta}_j \in U$ and $\langle \hat{\omega}_i, \hat{\eta}_j \rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta.

If we fix the coordinates $(u_1, \ldots, u_g)$ of $G_m$, then $\hat{\omega}_i$ are uniquely determined by (DS1). Since the symplectic pairing $\langle x, y \rangle$ is annihilated on $U \otimes U$ and $\Gamma(J_\eta, \Omega^1_{J_\eta/L}) \otimes \Gamma(J_\eta, \Omega^1_{J_\eta/L})$, the basis $\hat{\eta}_j$ are uniquely determined as well by (DS2).

**Proposition 5.1.** Let $V$ be a subring of $R$. Suppose that $(R, I)$ and $V$ satisfy the following.

1. There is a regular integral noetherian $\mathbb{C}$-algebra $\tilde{R}$ complete with respect to a reduced ideal $\tilde{I}$ and an injective homomorphism $i : R \rightarrow \tilde{R}$ such that $i(V) \subset \mathbb{C}$ and $i(I) \subset \tilde{I}$ and

   $\hat{\Omega}^1_{L/V} \rightarrow \hat{\Omega}^1_{\mathbb{C}^g}$

   is injective where we put $\tilde{L} := \text{Frac}(\tilde{R})$ and

   $\hat{\Omega}^1_{L/V} := L \otimes_R \left( \varprojlim_n \Omega^1_{R_n/V} \right)$, $\hat{\Omega}^1_{\mathbb{C}^g} := \tilde{L} \otimes_R \left( \varprojlim_n \Omega^1_{\tilde{R}_n/\mathbb{C}} \right)$, $R_n := R/I^n$, $\tilde{R}_n := \tilde{R}/\tilde{I}^n$.

Let

$$\nabla : H^1_{\text{dR}}(J_\eta/L) \rightarrow \hat{\Omega}^1_{L/V} \otimes_L H^1_{\text{dR}}(J_\eta/L)$$

be the Gauss-Manin connection. Then,

$$\nabla(\hat{\omega}_i) = \sum_{j=1}^g \frac{dq_{ij}}{q_{ij}} \otimes \hat{\eta}_j, \quad \nabla(\hat{\eta}_j) = 0. \quad (5.3)$$

**Proof.** By the assumption (C), we may replace $J_\eta/L$ with $(J_\eta \otimes_L \tilde{L})/\tilde{L}$. We may assume $R = \tilde{R}$, $V = \mathbb{C}$ and $J = J_\eta \otimes_R \tilde{R}$. There is a smooth scheme $J_S/S$ over a connected smooth affine variety $S = \text{Spec} A$ over $\mathbb{C}$ with a Cartesian square
such that \( \text{Spec}R \to S \) is dominant. Let \( D \subset S \) be a closed subset such that \( J_S \) is proper over \( U := S \setminus D \). Then, the image of \( \text{Spec}R/I \) is contained in \( D \) since \( J \) has a total degeneration over \( \text{Spec}R/I \). Thus, we may replace \( R \) with the completion \( \hat{A}_D \) of \( A \) by the ideal of \( D \). Let \( L_D = \text{Frac} \hat{A}_D \) then \( \hat{\Omega}_{L_D/C}^1 \cong L_D \otimes_A \Omega_A^1/C \). Let \( m \subset A \) be a maximal ideal containing \( I \) and \( \hat{A}_m \) the completion by \( m \). Then, \( \hat{A}_D \subset \hat{A}_m \) and \( \hat{\Omega}_{A_D/C} \subset \hat{\Omega}_{A_m/C} \). Therefore, we may further replace \( \hat{A}_D \) with \( \hat{A}_m \). Summing up the above, it is enough work in the following situation.

\[
R = \hat{A}_m \cong \mathbb{C}[[x_1, \ldots, x_n]] \supset I = (x_1, \ldots, x_n), \quad V = \mathbb{C}, \quad L = \text{Frac}R
\]

\[
J = \text{Spec}R \times_A J_S \longrightarrow \text{Spec}R.
\]

Note \( \hat{\Omega}_{L/C}^1 \cong L \otimes_A \Omega_A^1/C \). Let \( h : J \to S^{an} \) be the analytic fibration associated to \( J_S/S \). Write \( J_\lambda = h^{-1}(\lambda) \) a smooth fiber over \( \lambda \in U^{an} := (S \setminus D)^{an} \). Let \( \nabla : \mathcal{O}_{U^{an}} \otimes R^1 h_* \mathbb{C} \longrightarrow \Omega^1_{U^{an}} \otimes R^1 h_* \mathbb{C} \) be the flat connection compatible with the Gauss-Manin connection on \( H^1_{\text{dr}}(J_S/S)|_U \) under the comparison

\[
\mathcal{O}_{U^{an}} \otimes R^1 h_* \mathbb{C} \cong \mathcal{O}_{U^{an}} \otimes_A H^1_{\text{dr}}(J_S/S).
\]

We describe a de Rham symplectic basis in \( \hat{\omega}^{an}_i, \hat{\omega}^{an}_j \in \mathcal{O}_{U^{an}} \otimes_A H^1_{\text{dr}}(J_S/S) \) and prove (5.3) for the above flat connection. Write \( J_\lambda = (\mathbb{C}^\times)^g/\lambda^{2n} \) for \( \lambda \in U^{an} \). Let \( (u_1, \ldots, u_g) \) denote the coordinates of \( (\mathbb{C}^\times)^g \). Let \( \delta_i \in H_1(J_\lambda, \mathbb{Z}) \) be the homology cycle defined by the circle \( |u_i| = \varepsilon \) with \( 0 < \varepsilon \ll 1 \). Let \( \gamma_j \in H_1(J_\lambda, \mathbb{Z}) \) be the homology cycle defined by the path from \((1, \ldots, 1)\) to \((q_{j1}, \ldots, q_{jg})\). As is well-known, the dual basis \( \delta_i^*, \gamma_j^* \in H^1(J_\lambda, \mathbb{Z}) \) is a symplectic basis, namely

\[
\langle \delta_i^*, \delta_i^* \rangle = \langle \gamma_j^*, \gamma_j^* \rangle = 0, \quad \langle \delta_i^*, \gamma_j^* \rangle = \frac{1}{2\pi \sqrt{-1}} \delta_{ij},
\]

where \( \delta_{ij} \) denotes the Kronecker delta. We have \( \hat{\omega}^{an}_i = du_i/u_i \) by (DS1), and then

\[
\hat{\omega}^{an}_i = 2\pi \sqrt{-1} \delta_i^* + \sum_{j=1}^g \log q_{ij} \gamma_j^*.
\]

(5.4)

Let \( \tau^B : R^1 h_* \mathbb{Z} \rightarrow \mathbb{Z}(-1)^g \) be the associated map to \( \tau \). An alternative description of \( \tau^B \) is

\[
\tau^B(x) = \frac{1}{2\pi \sqrt{-1}}((x, \delta_1), \ldots, (x, \delta_g)),
\]
where \((x,\delta)\) denotes the natural pairing on \(H^1(J_\lambda,\mathbb{Z})\otimes H_1(J_\lambda,\mathbb{Z}).\) Obviously, \(\tau^B(\gamma_j^*) = 0.\) This implies that \(\hat{\eta}_j\) is a linear combination of \(\gamma_1^*,\ldots,\gamma_d^*\) by (DS2). Since \(\langle \hat{\omega}_i^\an, \hat{\eta}_j \rangle = \delta_{ij} = \langle \hat{\omega}_i^\an, \gamma_j^* \rangle,\) one concludes

\[
\hat{\eta}_j^\an = \gamma_j^*.
\]

Now, (5.3) is immediate from this and (5.4).

Let \(\hat{\omega}_i, \hat{\eta}_j \in H^1_{\text{dr}}(J_\eta/L)\) be the de Rham symplectic basis. Let \(x \in D^\an\) be the point associated to \(m.\) Let \(V^\an\) be a small neighborhood of \(x\) and \(j : V^\an \to S^\an\) an open immersion. Obviously, \(\hat{\omega}_i^\an = du_i/u_i \in \Gamma(V^\an, j_*\mathcal{O}^\an) \otimes A H^1_{\text{dr}}(J_S/S)\). Thanks to the uniqueness property, this implies \(\hat{\eta}_j^\an = \gamma_j^* \in \Gamma(V^\an, j_*\mathcal{O}^\an) \otimes A H^1_{\text{dr}}(J_S/S);\) in other words, \(\gamma_j^* \in \Gamma(V^\an \setminus D^\an, j^{-1}R^1h_*\mathbb{Q}).\) Let \(\hat{S}_m\) be the ring of power series over \(\mathbb{C}\) containing \(\hat{A}_m\) and \(\Gamma(V^\an, j_*\mathcal{O}^\an).\) There is a commutative diagram

\[
\begin{array}{ccc}
\hat{A}_m \otimes A \Omega^1_{A/C} \otimes A H^1_{\text{dr}}(J_S/S) & \to & \hat{S}_m \otimes A \Omega^1_{A/C} \otimes A H^1_{\text{dr}}(J_S/S) \\
\Omega^1_{A/C} \otimes A H^1_{\text{dr}}(J_S/S) & \to & \Gamma(V^\an, j_*\mathcal{O}^\an) \otimes \Omega^1_{A/C} \otimes A H^1_{\text{dr}}(J_S/S)
\end{array}
\]

with all arrows injective. Hence, the desired assertion for \(\hat{\omega}_i, \hat{\eta}_j\) can be reduced to that of \(\hat{\omega}_i^\an, \hat{\eta}_j^\an\). This completes the proof. \qed

5.2. Frobenius on De Rham symplectic basis

Let \(V\) be a complete discrete valuation ring such that the residue field \(k\) is perfect and of characteristic \(p,\) and the fractional field \(K := \text{Frac}V\) is of characteristic zero. Let \(F_V\) be a \(p\)-th Frobenius endomorphism on \(V.\)

Let \(A\) be an integral flat \(V\)-algebra equipped with a \(p\)-th Frobenius endomorphism \(\sigma\) on \(A\) which is compatible with \(F_V.\) Assume that \(A\) is \(p\)-adically complete and separated and that there is a family \((t_i)_{i \in I}\) of finitely many elements of \(A\) such that it forms a \(p\)-basis of \(A_n := A/p^nA\) over \(V_n := V/p^nV\) for all \(n \geq 1\) in the sense of [Ka1, Definition 1.3]. The latter assumption is equivalent to that \((t_i)_{i \in I}\) forms a \(p\)-basis of \(A_1\) over \(V_1\) since \(A\) is flat over \(V\) (loc. cit. Lemma 1.6). Then \(\Omega^1_{A_n/V_n}\) is a free \(A_n\)-module with basis \((dt_i)_{i \in I}\) (loc. cit. Lemma 1.8). Write \(A_K := A \otimes V K.\)

**Definition 5.2.** We define the category \(F\text{-MIC}(A_K, \sigma)\) as follows. An object is a triplet \((M, \nabla, \Phi)\) where

- \(M\) is a locally free \(A_K\)-module of finite rank,
- \(\nabla : M \to \hat{\Omega}^1_{A/V} \otimes A M\) is an integrable connection, where \(\hat{\Omega}^1_{A/V} := \lim \downarrow \Omega^1_{A_n/V_n}\) a free \(A\)-module with basis \((dt_i)_{i \in I},\)
- \(\Phi : \sigma^*M \to M\) is a horizontal \(A_K\)-linear map.

A morphism in \(F\text{-MIC}(A_K, \sigma)\) is an \(A\)-linear map of the underlying \(A\)-modules which is commutative with \(\nabla\) and \(\Phi.\) Let \(L := \text{Frac}(A)\) be the fractional field. The category \(F\text{-MIC}(L, \sigma)\) is defined in the same way by replacing \(A\) with \(L\) and \(\hat{\Omega}^1_{A/V}\) with \(L \otimes A \hat{\Omega}^1_{A/V}.\)
Lemma 5.3 [E-K, 6.1]. Let $\sigma'$ be another $F_V$-linear $p$-th Frobenius on $A$. Then, there is the natural equivalence

$$F\text{-MIC}(A, \sigma) \xrightarrow{\cong} F\text{-MIC}(A, \sigma'), \quad (M, \nabla, \Phi) \mapsto (M, \nabla, \Phi')$$

of categories, where $\Phi'$ is defined in the following way. Let $(\partial_i)_{i \in I}$ be the basis of $\hat{T}_{A/V} := \lim_{\longleftarrow n} (\Omega^1_{A_n/V_n})^*$ which is the dual basis of $(dt_i)_{i \in I}$. Then,

$$\Phi' = \sum_{n_i \geq 0} \left( \prod_{i \in I} (\sigma'(t_i) - \sigma(t_i))^{n_i} \right) \Phi \prod_{i \in I} \partial_i^{n_i}.$$ 

Let $f : X \to \text{Spec}A$ be a projective smooth morphism. Write $X_n := X \times_V V/p^{n+1} V$. Then, one has an object

$$H^i(X/A) := (H^i_{\text{dR}}(X/A) \otimes_V K, \nabla, \Phi) \in F\text{-MIC}(A_K),$$

where $\Phi$ is induced from the Frobenius on crystalline cohomology via the comparison ([B-O] 7.4)

$$H_{\text{cris}}^i(X_0/A) \cong \varprojlim_n H^i_{\text{dR}}(X_n/A_n) \cong H^i_{\text{dR}}(X/A).$$

Assume that there is a smooth $V$-algebra $A^a$ and a smooth projective morphism $f^a : X^a \to \text{Spec}A^a$ with a Cartesian diagram

$$\begin{array}{ccc}
X & \longrightarrow & \text{Spec}A \\
\downarrow & & \downarrow \\
X^a & \longrightarrow & \text{Spec}A^a
\end{array}$$

such that $\text{Spec}A \to \text{Spec}A^a$ is flat. One has the overconvergent $F$-isocrystal $R^i f^a_{\text{rig},*} \mathcal{O}_{X^a}$ on $\text{Spec}A^a$ ([Et] 3.4.8.2). Let

$$H^i(X^a/A^a) := (H^i_{\text{rig}}(X_0^a/A_0^a), \nabla, \Phi) \in F\text{-MIC}^\dagger(A_K^a)$$

denote the associated object via the natural equivalence $F\text{-Isoc}^\dagger(A_0^a) \cong F\text{-MIC}^\dagger(A_K^a)$ ([LS] 8.3.10). The comparison $H^i_{\text{rig}}(X_0^a/A_0^a) \cong H^i_{\text{dR}}(X^a/A^a) \otimes_{A^a} (A^a)^\dagger_K$ in (2.16) induces an isomorphism $H^i(X/A) \cong A_K \otimes_{(A^a)^\dagger_K} H^i(X^a/A^a)$ in $F\text{-MIC}(A_K)$.

Definition 5.4 (Tate objects). For an integer $r$, a Tate object $A_K(r)$ is defined to be the triplet $(A_K, \nabla, \Phi)$ such that $\nabla = d$ is the usual differential operator, and $\Phi$ is a multiplication by $p^{-r}$.

We define for $f \in A \setminus \{0\}$

$$\log^\sigma(f) := p^{-1} \log \left( \frac{f^p}{f^\sigma} \right) = \sum_{n=1}^\infty \frac{p^{n-1} g^n}{n}, \quad \frac{f^p}{f^\sigma} = 1 - pg$$

which belongs to the $p$-adic completion of the subring $A[g] \subset L$. In particular, if $f \in A^\times$, then $\log^\sigma(f) \in A$. 

Definition 5.5 (Log objects). Let \( q = (q_{ij}) \) be a \( g \times g \)-symmetric matrix with \( q_{ij} \in A^\times \). We define a log object \( \mathcal{L}og(q) = (M, \nabla, \Phi) \) in \( F\text{-MIC}(A, \sigma) \) to be the following. Let

\[
M = \bigoplus_{i=1}^{g} A_K e_i \oplus \bigoplus_{i=1}^{g} A_K f_i
\]

be a free \( A_K \)-module with a basis \( e_i, f_j \). The connection is defined by

\[
\nabla(e_i) = \sum_{j=1}^{g} dq_{ij} \otimes f_j, \quad \nabla(f_j) = 0
\]

and the Frobenius \( \Phi \) is defined by

\[
\Phi(e_i) = e_i - \sum_{j=1}^{g} \log(\sigma)(q_{ij}) f_j, \quad \Phi(f_j) = p^{-1} f_j.
\]

It is immediate to check that the log objects are compatible under the natural equivalence in Lemma 5.3. In this sense, our \( \mathcal{L}og(q) \) does not depend on \( \sigma \). By definition, there is an exact sequence

\[
0 \to \bigoplus_{j=1}^{g} A_K(1) f_j \to \mathcal{L}og(q) \to \bigoplus_{i=1}^{g} A_K(0) e_i \to 0.
\]

Theorem 5.6. Let \( R \) be a flat \( V \)-algebra which is a regular noetherian domain complete with respect to a reduced ideal \( I \). Suppose that \( R \) has a \( p \)-th Frobenius \( \sigma \). Let \( J \) be a totally degenerating abelian scheme with a principal polarization over \( (R, I) \) in the sense of §5.1. Let \( \text{Spec} R[h^{-1}] \to \text{Spec} R \) be an affine open set such that \( J \) is proper over \( \text{Spec} R[h^{-1}] \) and \( q_{ij} \in R[h^{-1}]^\times \), where \( q = (q_{ij}) \) is the multiplicative period as in (5.1). Suppose that \( R/pR[h^{-1}] \) has a \( p \)-basis over \( \text{Spec} V/pV \). Let \( A = R[h^{-1}]^\wedge \) be the \( p \)-adic completion of \( R[h^{-1}] \). Put \( L := \text{Frac}(A) \) and \( J_A := J \otimes_R A \). Let \( J_\eta \) be the generic fiber of \( J_A \). Then, there is an isomorphism

\[
(H^1_{\text{dR}}(J_\eta/L), \nabla, \Phi) \otimes_{A_K} A_K(1) \cong \mathcal{L}og(q) \in F\text{-MIC}(L)
\]

which sends the de Rham symplectic basis \( \widehat{\omega_i}, \widehat{\eta_j} \in H^1_{\text{dR}}(J_\eta/L) \) to \( e_i, f_j \), respectively.

Proof. Let \( q_{ij} \) be indeterminates with \( q_{ij} = q_{ji} \), and \( t_1, \ldots, t_r \) \((r = g(g+1)/2)\) are products \( \prod q_{i,j}^{n_{i,j}} \) such that \( \sum n_{i,j} x_i x_j \) is positive semi-definite and they give a \( \mathbb{Z} \)-basis of the group of the symmetric pairings. Let \( J_\eta = \mathbb{G}_m/q^\mathbb{Z} \) be Mumford’s construction of the quotient group scheme over a ring \( \mathbb{Z}_p[[t_1, \ldots, t_r]] \) ([F-C] Chapter III, 4.4). Then, there is a Cartesian square

\[
\begin{array}{ccc}
J & \to & J_\eta \\
\downarrow & & \downarrow \\
\text{Spec} R & \to & \text{Spec} \mathbb{Z}_p[[t_1, \ldots, t_r]]
\end{array}
\]
such that the bottom arrow sends \( t_i \) to an element of \( I \) by the functoriality of Mumford’s construction ([F-C] Chapter III, 5.5). Thus, we may reduce the assertion to the case of \( J = J_q, R = \mathbb{Z}_p[[t_1, \ldots, t_r]] \), \( I = (t_1, \ldots, t_r) \) and \( h = \prod q_{ij} \). Since \( \log(q) \) and \( H^1_{\text{DR}}(J_\eta/L) \) are compatible under the natural equivalence in Lemma 5.3, we may replace the Frobenius \( \sigma \) on \( R \) with a suitable one. Thus, we may assume that it is given as \( \sigma(q_{ij}) = q_{ij}^p \) and \( \sigma(a) = a \) for \( a \in \mathbb{Z}_p \). Under this assumption, \( \log(\sigma(q_{ij})) = 0 \) by definition. Therefore, our goal is to show

\[
\nabla(\hat{\omega}_i) = \sum_{j=1}^{g} dq_{ij} \otimes \hat{\eta}_j, \quad \nabla(\hat{\eta}_j) = 0,
\]

(5.6)

\[
\Phi(\hat{\omega}_i) = p\hat{\omega}_i, \quad \Phi(\hat{\eta}_j) = \hat{\eta}_j.
\]

(5.7)

Since the condition (C) in Proposition 5.1 is satisfied, (5.6) is nothing other than (5.3). We show (5.7). Let \( q^{(p)} :=(q_{ij}^p) \) and \( J_{q^p} := \mathbb{G}_m^g/(q^{(p)})^\mathbb{Z} \). Then, there is the natural morphism \( \sigma_J : J_{q^p} \to J_q \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{G}_m^g & \to & \mathbb{G}_m^g \\
\downarrow & & \downarrow \\
J_{q^p} & \xrightarrow{\sigma_J} & J_q \\
\downarrow & & \downarrow \\
\text{Spec}R & \xrightarrow{\sigma} & \text{Spec}R
\end{array}
\]

Let \( [p] : J_{q^p} \to J_{q^p} \) denote the multiplication by \( p \) with respect to the commutative group scheme structure of \( J_q \). It factors through the canonical surjective morphism \( J_{q^p} \to J_q \) so that we have \( [p]' : J_q \to J_{q^p} \). Define \( \varphi := \sigma_J \circ [p]' \). Under the uniformization \( \rho : \mathbb{G}_m^g \to J_q \), this is compatible with a morphism \( \mathbb{G}_m^g \to \mathbb{G}_m^g \) given by \( u_i \to u_i^p \) and \( a \to \sigma(a) \) for \( a \in R \), which we also write \( \Phi \). Therefore,

\[
\Phi = \varphi^* : H^1_{\text{DR}}(J_\eta/L) \to H^1_{\text{DR}}(J_\eta/L).
\]

In particular, \( \Phi \) preserves the Hodge filtration so that \( \Phi(\hat{\omega}_i) \) is again a linear combination of \( \hat{\omega}_i \)’s. Since

\[
\rho^* \Phi(\hat{\omega}_i) = \Phi(\hat{\omega}_i),
\]

one concludes \( \Phi(\hat{\omega}_i) = p\hat{\omega}_i \). However, since \( \Phi(\ker \nabla) \subset \ker \nabla \) and \( \ker \nabla \) is generated by \( \hat{\eta}_j \)’s by (5.6), \( \Phi(\hat{\eta}_j) \) is again a linear combination of \( \hat{\eta}_j \)’s. Note

\[
\langle \Phi(x), \Phi(y) \rangle = p \langle x, y \rangle.
\]

Therefore,

\[
\langle \Phi(\hat{\eta}_j), p\hat{\omega}_j \rangle = \langle \Phi(\hat{\eta}_j), \Phi(\hat{\omega}_j) \rangle = p \langle \hat{\eta}_j, \hat{\omega}_j \rangle.
\]

This implies \( \Phi(\hat{\eta}_j) = \hat{\eta}_j \), so we are done.
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References

[A-Bal] Y. André and F. Baldassarri, De Rham cohomology of differential modules over algebraic varieties, in Progress in Mathematics vol. 189 (Birkhäuser, 2000).

[A] M. Asakura, New $p$-adic hypergeometric functions and syntomic regulators, Journal de Théorie des Nombres de Bordeaux, To appear.

[A-C] M. Asakura and M. Chida, A numerical approach toward the $p$-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$, Res. Math. Sci. 10 (2023), Article 11.

[A-M] M. Atiyah and I. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Series in Math (Westview Press, Boulder, CO, 2016).

[Bal-Ber] F. Baldassarri and P. Berthelot, On Dwork cohomology for singular hypersurfaces, in Geometric Aspects of Dwork Theory (de Gruyter, 2004), 177–244.

[Bal-Ch] F. Baldassarri and B. Chiarellotto, Algebraic versus rigid cohomology with logarithmic coefficients, in Barsotti Symposium in Algebraic Geometry, Perspectives in Mathematics vol. 15 (Elsevier, 1994), 11–50.

[Ban1] K. Bannai, Rigid syntomic cohomology and $p$-adic polylogarithms, J. Reine Angew. Math. 529 (2000), 205–237.

[Ban2] K. Bannai, On the $p$-adic realization of elliptic polylogarithms for CM-elliptic curves, Duke Math. J. 113 (2) (2002), 193–236.

[Ban3] K. Bannai, Syntomic cohomology as a $p$-adic absolute Hodge cohomology, Math. Z. 242 (2002), 443–480.

[Bei] A. Beilinson, On the derived category of perverse sheaves, in K-Theory, Arithmetic and Geometry, Lecture Notes in Mathematics vol. 1289 (Springer, 1987), 27–41.

[BBDG] A. Beilinson, B. Bernstein and P. Deligne, Faisceaux pervers, in Analyse et topologie sur les espaces singuliers I, Astérisque vol. 100 (1982).

[Ber1] P. Berthelot, Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$, in Groupe de travail d’analyse ultramétrique, vol. 9, no. 3 (1981).

[Ber2] P. Berthelot, Finitude et pureté cohomologique en cohomologie rigide avec un appendice par Aise Johan de Jong, Invent. Math. 128 (1997), 329–377.

[Ber3] P. Berthelot, Cohomologie rigide et cohomologie rigide à supports propres, Première partie, Preprint, https://perso.univ-rennes1.fr/pierre.berthelot/.

[B-O] P. Berthelot and A. Ogus, Notes on Crystalline Cohomology (Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978), vi+243.

[Bes1] A. Besser, Syntomic regulators and $p$-adic integration I: Rigid syntomic regulators, Israel J. Math. 120 (2000), part B, 291–334.

[Bes2] A. Besser, Syntomic regulators and $p$-adic integration II: $K_2$ of curves, Israel J. Math. 120 (2000), part B, 335–359.

[B-K] S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, in P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Y. I. Manin and K. A. Ribet (eds.) The Grothendieck Festschrift I, Progress in Mathematics vol. 86 (Boston, Birkhäuser, 1990), 333–400.
[Co] P. Colmez, Fonctions $L$ $p$-adiques, in Séminaire Bourbaki, vol. 41 (1998/99), Astérisque No. 266 (2000), Exp. No. 851, 3, 21–58.

[DN] F. Déglose and W. Niziol, On $p$-adic absolute Hodge cohomology and syntomic coefficients, I, Comment. Math. Helv. 93(1) (2018), 291–334.

[Dw] B. Dwork, $p$-adic cyles, Publ. Math. IHES vol. 37 (1969), 27–115.

[E-K] M. Emerton and M. Kisin, An introduction to the Riemann-Hilbert correspondence for unit $F$-crystals, in Geometric Aspects of Dwork Theory vol. 1-2 (Walter de Gruyter, Berlin, 2004), 677–700.

[Et] J.-Y. Étesse, Images directes I: Espaces rigides analytiques et images directes, J. Théor. Nombres Bordeaux 24(1) (2012), 101–151.

[F-C] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Springer-Verlag, Berlin, 1990).

[F-M] J.-M. Fontaine and W. Messing, $p$-adic periods and $p$-adic etale cohomology, in K. A. Ribet (ed.) Current Trends in Arithmetical Algebraic Geometry, Contemp. Math. vol. 67 (Amer. Math. Soc., Providence, 1987), 179–207.

[G] R. Gerkmann, Relative rigid cohomology and point counting on families of elliptic curves, J. Ramanujan Math. Soc. 23(1) (2008), 1–31.

[Ka1] K. Kato, The explicit reciprocity law and the cohomology of Fontaine-Messing, Bull. Soc. Math. France 119(4) (1991), 397–441.

[Ka2] K. Kato, On $p$-adic vanishing cycles (application of ideas of Fontaine-Messing), in Algebraic Geometry (Sendai, 1985), Adv. Stud. Pure Math. 10 (Amsterdam, North-Holland, 1987), 207–251.

[Ka3] K. Kato, Logarithmic structures of Fontaine-Illusie, in Algebraic Analysis, Geometry, and Number Theory (Johns Hopkins University Press, Baltimore 1989), 191–224.

[Lau] A. Lauder, Rigid cohomology and $p$-adic point counting, J. Théor. Nombres Bordeaux 17(1) (2005), 169–180.

[Laz] C. Lazda, Incarnations of Berthelot’s conjecture, J. Number Theory 166 (2016), 135–157.

[Ma] L. N. Macarro, Division theorem over the Dwork–Monsky–Washnitzer completion of polynomial rings and Weyl algebras, in Rings, Hopf Algebras, and Brauer Groups (CRC Press, 2020), 175–191.

[N-N] J. Nekovář and W. Niziol, Syntomic cohomology and $p$-adic regulators for varieties over $p$-adic fields, Algebra Number Theory 10(8) (2016), 1695–1790. With appendices by Laurent Berger and Frédéric Déglose.

[NIST] NIST Handbook of Mathematical Functions, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Cambridge Univ. Press, 2010).

[P] B. Perrin-Riou, Fonctions $L_p$-adiques des representations $p$-adique, Astérisque 229 (1995).

[Sa1] M. Saito, Modules de Hodge polarisables, Publ. Res. Inst. Math. Sci. 24(6) (1988), 849–995.

[Sa2] M. Saito, Mixed Hodge modules, Publ. Res. Inst. Math. Sci. 26(2) (1990), 221–333.

[Sas] T. Sasai, Monodromy representations of homology of certain elliptic surfaces, J. Math. Soc. Japan 26(2) (1974), 296–305.

[Sh1] A. Shiho, Crystalline fundamental groups II – log convergent cohomology and rigid cohomology, J. Math. Sci. Univ. Tokyo 9(1) (2002), 1–163.

[Sh2] A. Shiho, Relative log convergent cohomology and relative rigid cohomology I, arXiv:0707.1742.

[Sh3] A. Shiho, Relative log convergent cohomology and relative rigid cohomology III, arXiv:0805.3229.
[Si] J. Silverman, Advanced topics in the arithmetic of elliptic curves, in *Graduate Texts in Mathematics* vol. 151 (Springer-Verlag, New York, 1994).

[So] N. Solomon, $p$-adic elliptic polylogarithms and arithmetic applications, thesis, 2008.

[LS] B. L. Stum, *Rigid Cohomology*, Cambridge Tracts in Mathematics vol. 172 (Cambridge University Press, Cambridge, 2007), xvi+319.

[Ts1] T. Tsuji, $p$-adic étale cohomology and crystalline cohomology in the semi-stable reduction case, *Invent. Math.* **137** (1999), 233–411.

[Ts2] T. Tsuji, On $p$-adic nearby cycles of log smooth families, *Bull. Soc. Math. France.* **128** (2000), 529–575.

[Tz1] N. Tsuzuki, On the Gysin isomorphism of rigid cohomology, *Hiroshima Math J.* **29**(3) (1999), 479–527.

[Tz2] N. Tsuzuki, On base change theorem and coherence in rigid cohomology, *Doc. Math.* (2003), 891–918.