N=4 Supersymmetric Gauge Theory in the Derivative Expansion

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Abstract

Maximally supersymmetric gauge theories have experienced renewed interest due to the AdS/CFT correspondence and its conjectured S-duality. These gauge theories possess a large amount of symmetry and have quasi-integrable properties. We derive the amplitudes in the derivative expansion of the spontaneously broken examples and perform all loop integrations. The S-matrix is found via an algebraic recursion and at each order is SL(2,Z) invariant.
1 Introduction

Supersymmetric gauge theories with four conserved supersymmetries is conjectured to have a non-perturbative self-equivalence \[1, 2\] and has been studied for many reasons. These theories have the most symmetry of gauge theories at arbitrary couplings, including conformal invariance. The well-known holographic description of IIB string theory via $N = 4$ gauge theory \[3, 4, 5, 6\], or vice versa, generates information in the latter at the non-perturbative corner and is a well-defined example of the gravity/gauge correspondence. The construction of the S-matrix in this work has manifest S-duality and unitarity at each order in the expansion.\[1\]

Spontaneously broken $N = 4$ supersymmetric gauge theories have a mass parameter (and a BPS self-dual set of states) and an expansion in this variable, $\Box/m^2$, is readily available. S-duality of $N = 4$ gauge theory requires that the S-matrix of gauge bosons and composite operators be invariant via,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad (1.1)$$

with the coupling,

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \quad (1.2)$$

This structure, together with the perturbative coupling dependence and instantons requires that the correlations be built from a classified set of automorphic functions, Eisenstien series and cusp form(s); for a review see \[9\]. One interesting feature of S-duality is that knowledge of the perturbative scattering is enough to construct the instanton series; together, these two series are redundant. S-duality also might lead to exact solutions for correlators. This approach has manifest S-duality in the expansion, and is non-perturbative.

Following along the lines of the recursion formulae in scalar field theory, we now consider amplitudes and correlators in in the broken gauge theory. The tensor manipulations and integrals have already been done in \[10\]. The primary difference is that $N = 4$ gauge theory contains a conjectured S-duality, and this symmetry of the amplitudes helps control the coupling structure. $N = 8$ was previously examined in the derivative expansion \[11\] and new insights regarding the finiteness was found; $N = 8$ contains an $SO(8)$ $N = 4$ subtheory.

\[1\]The S-duality invariance within the AdS/CFT correspondence was explored to all orders in \[7, 8\], and leads to the composite operator correlations in the gauge theory. It is possible to translate information from amplitudes to composite operator correlations.
2 Couplings and Modular Invariance

S-duality requires the quantum generating functional of the S-matrix (gauge bosons) and correlations to be invariant under $SL(2, z)$ transformations. The lagrangian is,

$$\mathcal{L} = \int d^4 x \, \text{Tr} \left( F^2 + \psi \nabla \psi + \phi \Box \phi + [\phi^i, \phi^j]^2 \right), \quad (2.1)$$

with gauge group G, e.g. SU(N). The toroidal modular parameter in general is a matrix,

$$\tau_{ij} = \frac{\theta}{2\pi} + \frac{4\pi i \theta_{ij}}{g^2}, \quad (2.2)$$

and for simplicity we take the gauge group to be $SU(2)$. The complications from enlarging $SU(2) \rightarrow U(1)$ to $SU(N) \rightarrow U(1)^{N-1}$ is in the modular functions and trace structures involved in the sewing.

We examine the gauge boson scattering through the quantum generating functional of the S-matrix. This functional is built from the operators,

$$\text{Tr} F^n, \; \text{Tr} F^n \text{Tr} F^m, \; \ldots \quad (2.3)$$

together with covariant derivatives. Label the operators by the dimension, $\mathcal{O}^i_{(d_i)}$; the dimensionality is labeled with the expectation value of a scalar field giving the correct dimensions,

$$\langle \phi \rangle^{4-d_i} \mathcal{O}^i_{(d_i)} \quad (2.4)$$

Because of S-duality the terms in the quantum generating functional must be modular invariant functions. These have been classified \[12\], and separable into normalizable and non-normalizable ones, and convergent versus non-convergent. The former condition is not relevant because it is tantamount to integrating over the coupling constants over the inequivalent vacua. The nonconvergent modular forms could be zeta function regularized; however, this results in a modular anomaly (an example is produced below). Any automorphic function may be represented as an infinite sum of the independent ones for complex values of $s$. 
\[ E_{s}(q,-q) = \sum_{p,q} \frac{\tau_{2}^{s}}{(p\tau + q)^{s-q}(p\bar{\tau} + q)^{s+q}} \]  

They are divergent for \( \text{Res} < 1 \). The scattering of gravitons and the bounds on the planar limit of \( N = 4 \) strong coupling correlators indicate that only half-integral or integral values of \( s \) enter into the description of the scattering. For non-vanishing \( q \) these functions transform with weight \( q \). They satisfy the \( SL(2, \mathbb{Z}) \) invariant Laplacian, and potentially leads to differential relations in the terms of the S-matrix expansion.

The set of functions we consider consist of the ring,

\[ \prod E_{s}(q_{i},-q_{i}) \]  

with \( \sum s_{i} = s = n/2, \; n \) an integer, and \( \sum q_{i} = 0 \) with \( |q_{i}| \leq s_{i} \). The amplitudes containing the fermions have non-zero \( q \) \( (N_{\psi} = q/2) \).

In \( k \)-space, the amplitude for four gauge bosons is an expansion,

\[ \sum \text{Tr} \nabla^{2n} F^{4} h_{nn}(\tau, \bar{\tau}) \]  

with the derivatives distributed among the four field strengths (the color structures are implied, and multi-trace structures are found by taking a product of lower trace ones). The general term is,

\[ \prod_{i=1}^{\mu} \prod_{j=1}^{n} \partial_{\mu_{\nu}(i,j)} \prod_{i=1}^{m_{i}} \frac{n_{i}}{\phi_{\nu_{\mu}(i,j)}} \prod_{i=1}^{p} \frac{n_{i}}{\psi_{\nu_{\mu}(i,j)}} \]  

and may be grouped into gauge invariant operators, in the functional,

\[ \prod_{i=1}^{\mu} \prod_{j=1}^{n} \nabla_{\mu_{\nu}(i,j)} \prod_{i=1}^{m_{i}} \frac{n_{i}}{\phi_{\nu_{\mu}(i,j)}{\psi_{\nu_{\mu}(i,j)}}} \]
\[ \prod_i \phi_{a(i,j)} \prod_{j=1}^{n'} \nabla_{\rho_{a(i,j)}} \prod_i \psi_{a(i,j)} \]

with the color indices implied. Unitarity is built into the sewing, and may also be seen via expanding the usual Feynman diagrams at small \( k^2/m^2 \). In order to generate the unitarity cuts of the massive modes, the gauge multiplet and the dyons, one has to resum the terms in the derivative expansion. Alternatively, from the expected cuts in the amplitudes, we may impose more conditions on coefficients beyond the recursion.

The prefactors \( h \) are invariant, and model the coupling dependence,

\[ h \left( \frac{a \tau + b}{c \tau + d} \right) = h(\tau, \bar{\tau}) . \]  

Our task is to find the ring of functions that enter into the \( h \) functions and then to utilize the sewing to build the S-matrix. In this manner, unlike the \( \phi^n \) theory, the coupling structure in \( \tau \) is simplified and also allows for a determination of the complete set of instanton corrections to the amplitudes! One of the two series, perturbative and non-perturbative, is redundant. The perturbative derivative series is invariant under S-duality. In agreement with the scattering functions, we examine functions \( h_k \) in the ring with \( s = m/2 \), an integer or half-integer (this will be elaborated further). The Eisenstein functions admit the representation,

\[ E_{s}^{(q,-q)} = \sum_{p,q} \frac{\tau_{2}^{s}}{(p \tau + q)^{s-q}(p \bar{\tau} + q)^{s+q}} \]

and expansion, found via a Possion resummation,

\[ E_{s}^{(q,-q)} = 2\zeta(s)\tau_{2}^{s} + \alpha_{s}\tau_{2}^{1-s} + \mathcal{O}(e^{-\tau_{2}}) . \]

The power terms are associated with perturbation theory, and the exponential series the instanton corrections. The ring \( S_{n/2}^{q} \) of modular invariant functions is the set,

\[ \left\{ \prod_{j=1}^{p} E_{s_{j}}^{(q_{j},-q_{j})} \right\} \]  

with \( \sum_j s_j \leq n \) and \( \sum q_j = q \). These functions converge individually for \( s > 1 \). The regularized versions could be utilized, for example, \( \bar{E}_{1} = \ln \tau_{2} \eta(\tau, \bar{\tau}) \), but they possess
a modular anomaly as a result. The basis was already commented on, and follows from examining compatibility with the perturbative coupling series.

A n-gauge boson scattering amplitude has the coupling series,

\[(g^2)^{n/2-1}, (g^2)^{n/2}, \ldots, (g^2)^{n/2-2+n_{\text{max}}/2}.\] (2.16)

We have labeled the maximum coupling with the integer \(n_{\text{max}}\), as the ring of functions related to this expansion always truncate. It is impossible to generate an infinite series in \(g^2\) with \(s \leq n/2\). The series of terms may be regrouped into,

\[g^{n-1} (g^2)^{n_{\text{max}}/2} \left[ \left( \frac{1}{g^2} \right)^{n_{\text{max}}/2}, \ldots, \left( \frac{1}{g^2} \right)^{-n_{\text{max}}/2+1} \right] = g^{n-1} (g^2)^{n_{\text{max}}/2} h_{\eta}(\tau, \bar{\tau}) \] (2.17)

with \(k = n_{\text{max}}\), with \(n_{\text{max}} = n + 2\); there are no inverse powers of \(g\) in perturbation theory. The anomalous, from the modular point of view, front factor may be removed with a field redefinition,

\[A \rightarrow g^{-2}A \quad x \rightarrow gx.\] (2.18)

Then the Lagrangian is,

\[\int d^4x \frac{1}{g^2} \text{Tr} \left( \partial A + \frac{1}{2} A^2 \right)^2\] (2.19)

and the \(g^2\) factor in front is absorbed by the implied metric \(\sqrt{g}\) in the coordinate redefinition. The coordinate redefinition is a slight improvement on the statement of S-duality of \(N = 4\) super gauge theory, similar to Einstein frame in IIB superstring theory.

The amplitudes have the expanded form by varying the quantum generating functional,

\[\langle A(k_1) \ldots A(k_n) \rangle = \sum_q h_q^{(n)}(\tau, \bar{\tau}) f_q(k_1, \ldots, k_n)\] (2.20)

with \(h_q\) elements in the finite ring \(S_{n/2+1}^{(0)}\). The form is similar for the fermions, and \(n + 2\) is changed in general to \(n_A + n_\phi + n_{\psi}/2 + 2\). Via the expansion of the modular functionis, the maximum loop is \(n_{\text{max}} - 1\) for the \(n\)-point function.
The four-point function in massless $N = 4$ has the property that the $A^4$ term receives corrections at tree-level and one-loop only. In the spontaneously broken theory the modular ring has $n_{max} = 6$, $n/2 + 1 \rightarrow 4$. This corresponds to a truncation of the $A^4$ after five loops. The relevant ring contains,

\[ E_3, E_{3/2}^2, |E_{3/2}^{(1, -1)}|^2, E_2 \]

\[ |E_{3/2}^{-1/2, 1/2}|^2, |E_{3/2}^{-3/2, 1/2}|^2, |E_{3/2}^{+3/2, -3/2}|^2 \]

where only the convergent ones are included. These separate functions leaves unknown parameters, which are determined by the derivative expansion, up to five loops if all functions are included. This is simpler than an infinite series, as the four-point with no derivatives has only six functions. The structure is similar at higher order in derivatives and number of gauge bosons, and the instanton corrections are determined by the perturbative terms.

3 Sewing and Amplitudes

The sewing relation, as presented in $\phi^n$ theory, may be used to generate all of the coefficients in the derivative expansion. The approach is the same as in the scalar theory, with the added complication of many tensors. However, all loop integrals may be performed and the amplitudes are generated via algebraic computations. The parameter $L$ counts $L - 1$ loops in these diagrams, and for simplicity we keep the parameter as $L$. We have to sum over all the vertices, with $n_A$, $n_\phi$, and $n_\psi$ lines, together with multi-derivatives acting on them. The general vertex is listed in the previous section.

The recursion is illustrated in figure (1) of [10]. The left hand side takes the form, with $m_l + m_r = n$-point (numbers of lines to the left and right of the quantum vertex),

\[ \sum_L \left[ \prod_{i=1}^{m_l} \left( \prod_{j=1}^{m_\phi} \partial_{\mu_{\phi(i,j)}} A_{\mu_i}(k_i) \right) \right] \left[ \prod_{i=m_l+1}^{m_r} \left( \prod_{j=1}^{m_\psi} \partial_{\mu_{\psi(i,j)}} A_{\mu_i}(k_i) \right) \right] \]

\[ \left( \prod_{i=1}^{L} \left( \prod_{j=1}^{m_\phi} \partial_{\mu'_{\phi(i,j)}} \prod_{j=1}^{m_\psi} A_{\mu'_\psi} \right) \right) \left[ \prod_{i=1}^{L} \left( \prod_{j=1}^{m_\psi} \partial_{\mu'_{\psi(i,j)}} \prod_{j=1}^{m_\phi} A_{\mu'_\phi} \right) \right] \times t^{\nu_{\phi}, \mu_i, m_\phi, \nu_{\psi}, \mu'_i, m_\psi} \]

(3.2)
with $\sum m_i^A = \sum \tilde{m}_i^A = L$; the $\tau$ is within the tensor $t$, as well as the coefficients of the derivative structure. We have to sum over all possible combinations, as iterated from lower to higher orders in derivatives and numbers of external states. Internal fermions and scalars must be included in the sum.

The right-hand side of the equation contains the quantum vertices,

$$t_{\mu_0, \nu_0, m_0} \prod_{i=1}^n \prod_{j=1}^n \partial_{\mu_0(i,j)} \prod \tilde{m}^A_{i} \prod \tilde{A}_{\mu_0(i,j)} \prod \tilde{A}_{\tilde{\mu}(i,j)}$$ (3.4)

The integrals have to be performed in the former. Contractions of the internal fields generate the propagators, and due to the tensor structure the integrals are more complicated. However, all internal momenta or derivatives may be extracted from within the integral with the use of the identity,

$$\partial^\mu \Delta^L = L \left[ (2 - d) + m^2 \partial^2 \right] \times \left( \frac{k^\mu}{k^2} \right) \Delta^L.$$ (3.5)

The massive propagator is

$$\Delta_{\mu \nu} = (x^2)^{-d/2+1} K_{d/2} (m x) P_{\mu \nu},$$ (3.6)

in terms of the modified Bessel function and the tensor structure (gauge dependent) of the propagator. This identity is easily proved in x-space, and its application to the gauge field correlations is, upon iteration,

$$\int e^{ixk} \prod_{j=1}^n \partial_{\mu_j} \Delta^L \sum_{\sigma,\sigma'} \prod \eta_{\mu_0,\nu_0} \prod k_{\mu_0}(\frac{1}{k^{2n_1+2n_2}})(-2)^{n_2-1} \times \left[ (2 - d) + m^2 \partial^2 \right]^n \Delta^L.$$ (3.7)

We have to sum over all pairs $(n_1, n_2)$ such that $n_1 + n_2 = n$ and all partitions of $n$ to the $\eta$ and $k$'s. The gauge boson scattering is,

$$\left[ \prod_{i=1}^L \left( \prod_{j=1}^{m_0^A} \partial_{\mu_0(i,j)} \prod A_{\mu_0(i,j)} \right) \right] \left[ \prod_{i=1}^L \left( \prod_{j=1}^{\tilde{m}_0^A} \tilde{\partial}_{\tilde{\mu}(i,j)} \prod \tilde{A}_{\tilde{\mu}(i,j)} \right) \right]$$ (3.8)
\begin{equation}
\sum_{\sigma, \sigma'} \prod_{\nu} \eta_{\mu_{\sigma} \nu_{\sigma'}} \prod_{\mu} k_{\mu_{\sigma}} \left( \frac{1}{k^{2n_{1}+2n_{2}}} \right) (-2)^{n_{2}-1} \times \left[ (2 - d) + m^2 \partial_{m}^2 \right] \Delta_{L} \times \prod_{\rho(i), \tilde{\rho}(j)} \eta_{\mu_{\rho(i)} \nu_{\rho(j)}} \tag{3.9}
\end{equation}

The tensor structure follows from the derivative diagram, and the last term comes from the Pfaffian,

\begin{equation}
\langle \prod_{i}^{L} A_{\mu_{i}} \prod_{j}^{L} A_{\tilde{\rho}_{j}} \rangle \tag{3.10}
\end{equation}

The expression may seem complicated, but all loop integrals have been performed. The algebraic complexity is also less than typical loop diagram calculations, which at intermediate steps may contain millions of terms.

The integrals are evaluated to,

\begin{equation}
\int d^d x \ e^{i x \cdot k} \Delta_{L} = (k^2)^{-d/2-L(d/2-1)} \sum_{n} (k^2/m^2)^n \alpha_n^{(L)} \tag{3.11}
\end{equation}

in dimensional regularization, and

\begin{equation}
(k^2)^{-d/2-L(d/2-1)} \sum_{m,n} \left( \frac{k^2}{\Lambda^2} \right)^m \left( \frac{k^2}{m^2} \right)^n \alpha_{m,n}^{(L)} \tag{3.12}
\end{equation}

in a momentum cutoff scheme. \(N = 4\) is finite, and divergences at intermediate steps in the calculation should drop out in the final equations.

Equating the former with the quantum vertices generates a recursion in the couplings and tensor \(tt \sim t\), with the coupling structure found from the ring of Eisenstein functions up to number coefficients. The products of Eisenstein function iterate from \(n\) and \(m\)-point to the higher derivative terms in \(t\) (together with the enlarged basis in the latter). The iteration formulae allow for a computation of the scattering amplitude via polynomial equations, with no integrals. Similar results are available for the correlations of the composite operators.

\section{Discussion}

The scattering amplitudes of spontaneously broken \(N = 4\) supersymmetric gauge theory have been examined in the derivative expansion. The gauge theory may be iteratively constructed via sewing, and is equivalent to the sewing in the Feynman
diagram expansion. Unitarity is obvious in this construction. Amplitudes may be constructed without any integrals; the integrals involved are of the free-field type and may all be integrated. Unitarity is obvious in this construction.

Scalar and gauge field theory has also been examined in \cite{10, 13}; S-duality has been implemented in the $N = 4$ gauge theory and order by order the gauge boson expansion is invariant under $SL(2, \mathbb{Z})$ duality. S-duality in the amplitudes has the advantage that from the perturbative structure, all of the instanton corrections are computable and follow from the former. The instanton expansion via the terms, $e^{-2n\tau_2}$, are connected to the perturbative terms and are redundant; the perturbative terms generate an instanton calculus via an $SL(2, \mathbb{Z})$ completion.

The derivative expansion, and the concise expressions which follow from it, should enable further progress several areas. The intermediate coupling regime is accessible as well as some possible information regarding the verification of duality.

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