ASYMPTOTICS FOR RUIN PROBABILITIES OF A
NON-STANDARD RENEWAL RISK MODEL WITH
DEPENDENCE STRUCTURES AND EXPONENTIAL LÉVY
PROCESS INVESTMENT RETURNS

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Abstract. Consider a non-standard renewal risk model with dependence structures, where claim sizes follow a one-sided linear process with independent and identically distributed step sizes, the step sizes and inter-arrival times respectively form a sequence of independent and identically distributed random pairs, with each pair obeying a dependence structure. An insurance company is allowed to make risk-free and risky investments, where the price process of the investment portfolio follows an exponential Lévy process. When the step-size distribution is dominatedly-varying-tailed, some asymptotic estimates for the finite-and infinite-time ruin probabilities are obtained.

1. Introduction. It is well known that the standard renewal risk model (proposed by Sparre Andersen in 1957) has played a fundamental role in risk theory. This standard framework is based on many independence assumptions; e.g., the claim sizes, \( X_n \), \( n \geq 1 \), and the inter-arrival times \( Y_n \), \( n \geq 1 \), respectively form a sequence of independent and identically distributed (i.i.d.) random variables, and the two sequences are mutually independent. It is worth pointing out that these independence assumptions are mainly for mathematical tractability rather than practical relevance. Moreover, these independence assumptions make the renewal risk model too restrictive for practical problems. Therefore, recently, more and more researchers have started to propose different non-standard extensions to the

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standard renewal risk model with various dependence structures. See [1, 2, 4, 7, 13, 16, 25, 28], among others.

In this paper, we improve the model through introducing suitable dependence structures among claim sizes, and between claim sizes and inter-arrival times. On the one hand, with the increasing complexity of insurance and reinsurance products, researchers have been paying an increasing amount of attentions to the modeling of dependent risks. Throughout this paper, we make an assumption on the dependence of claim sizes \( \{X_n, n \geq 1\} \) as follows.

**Assumption 1.1.** Let the claim sizes \( \{X_n, n \geq 1\} \) be a one-sided linear process defined as

\[
X_n = \sum_{j=1}^{n} \varphi_{n-j} \varepsilon_j + \varphi_n \varepsilon_0, \quad n = 1, 2, \ldots,
\]

where \( \{\varepsilon_n, n \geq 1\} \) is a sequence of i.i.d. nonnegative random variables (r.v.'s) with generic random variable (r.v.) \( \varepsilon \) and common distribution function (d.f.) \( F \) satisfying \( F(x) = 1 - F(x) > 0 \) for all \( x > 0 \), and \( \{\varphi_n, n \geq 0\} \) are nonnegative constants with \( \varphi_0 > 0 \).

As pointed out by a large amount of literature ([3], [27], [37], among others), linear processes including the ARMA model and the fractional ARIMA model, are often used in time series analysis, and are also a type of widely used dependence structures in insurance risk theory.

On the other hand, if the deductible retained to the insured is raised, then the inter-arrival time will increase and the claim sizes would decrease because small losses will be ruled out and retained by the insured. Hence, the independence between the claim size \( X_n \) and the inter-arrival time \( Y_n \) is unreasonable in many insurance applications. During the last few years, the non-standard renewal risk models with dependence structures between \( X_n \) and \( Y_n \) have attracted an increasing amount of attentions from researchers in risk theory. See [2], [3], [5], [7], [13], [16], [24] and [25].

Motivated by [39], we make another type of assumption on the dependence structure of \( (\varepsilon_n, Y_n) \) as follows.

**Assumption 1.2.** Let \( (\varepsilon_n, Y_n) \) be i.i.d. copies of a generic random pair \( (\varepsilon, Y) \) following a bivariate Sarmanov distribution of the form

\[
P(\varepsilon \in dx, Y \in dy) = (1 + \pi \phi_1(x) \phi_2(y)) P(\varepsilon \in dx) P(Y \in dy), x \geq 0, y \geq 0,
\]

where the kernels \( \phi_1(x) \) and \( \phi_2(y) \) are two functions and the parameter \( \pi \) is a real constant satisfying

\[
E\phi_1(\varepsilon) = E\phi_2(Y) = 0,
\]

And

\[
1 + \pi \phi_1(x) \phi_2(y) \geq 0, \quad \text{for all } x \in D_x, y \in D_y,
\]

where \( D_x = \{x \in \mathbb{R} : P(\varepsilon \in (x - \delta, x + \delta)) > 0 \text{ for all } \delta > 0\} \) and \( D_y = \{y \geq 0 : P(Y \in (y - \delta, y + \delta)) > 0 \text{ for all } \delta > 0\} \).

Clearly, \( \varepsilon \) and \( Y \) are independent if \( 1 + \pi \phi_1(x) \phi_2(y) = 1 \), for all \( x \in D_x, y \in D_y \). Following [39], we say that a random vector \( (\varepsilon, Y) \) follows a proper bivariate Sarmanov distribution if \( 1 + \pi \phi_1(x) \phi_2(y) \neq 1 \), for all \( x \in D_x, y \in D_y \).

It should be noted that a bivariate Sarmanov distribution is a more general dependence structure than the Farlie-Gumbel-Morgenstern (FGM) dependence structure.
RUIN PROBABILITIES WITH DEPENDENCE STRUCTURES

(see [35] for more details). As pointed out by [34], if \( \phi_1(x) = 1 - 2F(x) \), \( \phi_2(y) = 1 - 2G(x) \), then \((\varepsilon, Y)\) jointly follows a bivariate FGM distribution. For more detailed discussions on Sarmanov distribution, see [23], [34], among others.

Consider the non-standard renewal risk model with Assumptions 1.1 and 1.2, in which the inter-arrival times \( \{Y_n, n \geq 1\} \), not independent of the claim sizes \( \{X_n, n \geq 1\} \), form a sequence of i.i.d., nonnegative and non-degenerate at-zero r.v.’s with common distribution function \( G \). Then, the arrival times of the successive claims \( \tau_n = \sum_{i=1}^{n} Y_i, n \geq 1 \), constitute a renewal counting process \( N(t) = \sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}}, \ t \geq 0 \), where \( I_A \) denotes the indicator function of an event \( A \), and by convention, \( \tau_0 = 0 \).

Denote the renewal function of \( \{N(t), t \geq 0\} \) as \( \lambda_t = EN(t) = \sum_{n=1}^{\infty} P\{\tau_n \leq t\} \) for \( t \geq 0 \). (5)

The aggregate claims up to time \( t \) \((\geq 0)\) are given by the compound sum

\[
S_t = \sum_{n=1}^{N(t)} X_n
\]

with \( S_t = 0 \) when \( N(t) = 0 \).

Suppose that an insurer is allowed to make risk-free and risky investments. Based on a amount of empirical investigations of stock markets indicating that the price processes of many stocks have sudden downward or upward jumps which cannot be explained by a continuous geometric Brownian motion, we consider a more general exponential (also known as geometric) Lévy process with jumps to model the price process of the investment portfolio. The price processes of the risk-free and risky assets, respectively, satisfy

\[
Z_0(t) = e^{rt} \quad \text{and} \quad Z_1(t) = e^{L(t)}, \ t \geq 0,
\]

where \( r > 0 \) is the risk-free rate, and the process \( \{L(t), t \geq 0\} \) is a Lévy process. That is, \( L(0) = 0, \{L(t), t \geq 0\} \) has independent and stationary increments, is stochastically continuous, and is right continuous with left limit (see [31] for general theory of Lévy process). Let \( (\gamma, \sigma^2, \nu) \) be the characteristic triplet of a Lévy process \( \{L(t), t \geq 0\} \), where \( \gamma \in \mathbb{R}, \sigma \geq 0 \) are two constants and \( \nu \) is a Lévy measure satisfying \( \nu(\{0\}) = 0 \) and \( \int_{-\infty}^{\infty} \min(x^2, 1) \nu(dx) < \infty \).

Assume that an insurer continuously invests a constant fraction \( \theta \in [0, 1] \) of his or her reserve in the risky asset and the fraction \( 1 - \theta \) in the risk-free asset. The fraction \( \theta \) is so-called constant investment strategy. The assumptions on price processes of the investment portfolio are commonly used in mathematical finance and actuarial science. For more details, see [15], [20], [29], among others.

Following (5) in [17], we define the price process of the investment portfolio with the constant investment strategy \( \theta \in [0, 1] \), which satisfies the following stochastic differential equation (SDE):

\[
dZ_\theta(t) = Z_\theta(t-)dL_\theta(t), \ t > 0, \ Z_\theta(0) = 1,
\]
where \(d\bar{L}_\theta(t) = (1 - \theta)rdt + \theta d\bar{L}(t)\) and \(dZ_1(t) = Z_1(t-)d\bar{L}(t)\). By Proposition 8.22 in [12], the solution of the SDE [9] satisfies

\[
Z_\theta(t) = e^{L_\theta(t)} \quad \text{and} \quad Z_\theta(0) = 1, \quad t \geq 0,
\]

(9)

where

\[
L_\theta(t) = \bar{L}_\theta(t) - \frac{1}{2}[\bar{L}_\theta, \bar{L}_\theta]_t + \sum_{0 \leq \tau \leq t} \left( \log(1 + \Delta \bar{L}_\theta(s)) - \Delta \bar{L}_\theta(s) + \frac{1}{2}(\Delta \bar{L}_\theta(s))^2 \right)
\]

with \(\Delta \bar{L}_\theta(s) = \bar{L}_\theta(s) - \bar{L}_\theta(s-)\) and \([\bar{L}_\theta, \bar{L}_\theta]\) being the quadratic variation process of \(\bar{L}_\theta\).

By Lemma 2.5 in [15], the process \(\{L_\theta(t), t \geq 0\}\) is also a Lévy process with the characteristic triplet \((\gamma_\theta, \sigma_\theta^2, \nu_\theta)\) which is specified by the original Lévy process \(\{L(t), t \geq 0\}\) as follows.

\[
\gamma_\theta(t) = \gamma + (1 - \theta) \left( r + \frac{\sigma^2}{2} \right) + \int_{\mathbb{R}} \left[ \log(1 + \theta(e^x - 1))I_{|\log(1 + \theta(e^x - 1))| \leq 1} - \theta xI_{|x| \leq 1} \right] \nu(dx);
\]

\[
\sigma_\theta^2 = \theta^2 \sigma^2;
\]

\[
\nu_\theta(A) = \nu(\{x \in \mathbb{R} : \log(1 + \theta(e^x - 1)) \in A\}) \quad \text{for any Borel set } A \subset \mathbb{R}.
\]

(10)

For more details of the relation between \(\{L(t), t \geq 0\}\) and \(\{L_\theta(t), t \geq 0\}\), see [12], [20] and [22].

Next we follow the way used by [22] to define the integrated risk process (IRP) as the result of the insurance business and the net gains of the investment through the stochastic differential equation below:

\[
dU_\theta(t) = cdt - dS_1 + U_\theta(t-)d\bar{L}_\theta(t), \quad t > 0, \quad U_\theta(0) = x.
\]

(11)

We assume that the \(\{L(t), t \geq 0\}\) is independent of \(\{\varepsilon_n, n \geq 1\}\) and \(\{Y_n, n \geq 1\}\), which implies the independence between the insurance process \(\{S_t, t \geq 0\}\) and the investment process \(\{L_\theta(t), t \geq 0\}\). Then, by Lemma 2.2 of [22], we can verify that the solution to the SDE (11), i.e., the insurer’s surplus process with stochastic return on investments, is

\[
U_\theta(0) = x, \quad U_\theta(t) = e^{L_\theta(t)} \left( x + \int_0^t e^{-L_{\theta}^{(v)}}(cp - dS_v) \right), \quad t > 0,
\]

(12)

where \(x \geq 0\) is the initial surplus of the insurance company and \(c > 0\) is the constant premium rate. Under the conditions of [11] and [2], we remark that the \(U_\theta(t)\) is a generalization of the surplus processes considered by [10], [17], [21], [30], [40] and [41].

Define the Laplace exponent of the process \(\{L_\theta(t), t \geq 0\}\) as

\[
\psi_\theta(s) = \log E[e^{-sL_\theta(1)}], \quad -\infty < s < +\infty.
\]

(13)

If \(\psi_\theta(s) < \infty\), then

\[
E[e^{-sL_\theta(t)}] = e^{t\psi_\theta(s)} < \infty, \quad s \geq 0.
\]

(14)

By the proof of Lemma 4.1 in [22], we can get that \(\psi_\theta(s) < \infty\) for all \(\theta \in (0, 1)\) and \(s \geq 0\), and if \(0 < EL(1) < \infty\) and either \(\sigma > 0\) or \(\nu((-\infty, 0)) > 0\), then there exists a unique positive \(\kappa_\theta > 0\) such that \(\psi_\theta(\kappa_\theta) = 0\). Thus, if \(0 < EL(1) < \infty\),
either \( \sigma > 0 \) or \( \nu((-\infty,0)) > 0 \), then the convexity of \( \psi(\cdot) \) with \( \psi(0) = 0 \) and \( \psi'(0) = -EL_0(1) < 0 \) implies that for any fixed \( \theta \in (0,1) \),

\[
\psi(\theta) < 0, \text{ for any } 0 < \theta < \kappa_\theta.
\]  

As usual, define the finite-time ruin probability and the infinite-time ruin probability of the non-standard renewal risk model (12), respectively, as

\[
\Psi(x, T) = P\left\{ \inf_{0 \leq t \leq T} U(t) < 0 \mid U(0) = x \right\}, \quad T \geq 0,
\]

and

\[
\Psi(x) = \Psi(x, \infty) = \lim_{T \to \infty} \Psi(x, T) = P\left\{ \inf_{0 \leq t < \infty} U(t) < 0 \mid U(0) = x \right\}.
\]  

The main goal of this paper is to investigate ruin probabilities of the non-standard renewal risk model (12) with stochastic investment returns under Assumptions (1) and (2), and examine how tail probabilities are influenced by the dependence structures. When the common distribution function of the step sizes in (1) is heavy tailed, we establish some asymptotic formulas that hold for finite or infinite time horizons.

There are a few recent articles that are related to our study on asymptotic estimates for ruin probabilities of non-standard renewal risk models with stochastic investment returns. [17], [18] and [38] considered the similar problem, but their works were concentrated on the one-sided linear, bivariate upper tail independent, and upper tail asymptotic independence (UTAI) claim-size models (without dependence structures between the claim sizes and the inter-arrival times), respectively. [16] used the dependence structure as proposed by [3] to characterize the relation between the claim sizes and the inter-arrival times (often termed as a time-dependent renewal risk model). [24] studied the uniformly asymptotic tail behavior for a time-dependent renewal risk model with stochastic return, under a dependence structure between the extended-regularly-varying tailed claim sizes and the inter-arrival times. But for the time-dependent renewal risk model, [16] and [24] considered the case of independent claim sizes. However, we not only use a different and meaningful bivariate Sarmanov dependent structure to characterize the dependence relation between claim sizes and inter-arrival times, but also adopt an interesting one-sided linear process to model the dependent claim sizes. Hence, our obtained results partly extend the results of the above papers. Until now, no paper has simultaneously addressed continuous-time renewal risk models with stochastic investment returns under dependent structures between claim sizes and inter-arrival times, and among claim sizes.

The rest of this paper consists of three sections. Section 2 introduces some notations and states the main results of the paper, Section 3 provides some important lemmas, and Section 4 proves the main results.

2. Notations and main results. Throughout this paper, \( C \) represents a generic positive constant, which may vary with the context. Hereafter, all limit relations are for \( x \to \infty \) unless stated otherwise. For two positive functions \( a(\cdot) \) and \( b(\cdot) \) satisfying

\[
0 \leq l_1 \leq \liminf_{x \to \infty} \frac{a(x)}{b(x)} \leq \limsup_{x \to \infty} \frac{a(x)}{b(x)} \leq l_2 < \infty,
\]
we write \( a(x) = O(b(x)) \) if \( l_2 < \infty \), \( a(x) = o(b(x)) \) if \( l_2 = 0 \); we also write \( a(x) \lesssim b(x) \) if \( l_2 = 1 \); \( a(x) \gtrsim b(x) \) if \( l_1 = 1 \), \( a(x) \sim b(x) \) if \( l_1 = l_2 = 1 \), and \( a(x) \asymp b(x) \) if \( 0 < l_1 \leq l_2 < \infty \).

In finance and insurance, heavy tailed random variables play an important role in modeling extremal events, since they can model jumps and jumbo claims realistically. Now we recall some related classes of heavy-tailed distribution functions. In this paper, we focus on the so-called dominated variation class, denoted by \( D \). By definition, a distribution \( F \) on \([0, \infty)\) belongs to the class \( D \), denoted by \( F \in D \), if

\[
\limsup_{x \to \infty} \frac{F(xy)}{F(x)} < \infty, \quad \text{for any } y > 0. \tag{18}
\]

We also recall some other classes of heavy-tailed distribution functions which are crucial for our purpose. We say that a d.f. \( F \) \([0, \infty)\) belongs to the class \( C \) (has a consistently varying tail), denoted by \( F \in C \), if

\[
\lim \liminf_{y \searrow 1, x \to \infty} \frac{F(xy)}{F(x)} = 1, \quad \text{or} \quad \lim \limsup_{y \nearrow 1, x \to \infty} \frac{F(xy)}{F(x)} = 1. \tag{19}
\]

We say that a d.f. \( F \) \([0, \infty)\) belongs to the extended regular variation class \((ERV)\), denoted by \( F \in ERV(-\alpha, -\beta) \), if there are some \( 0 < \alpha \leq \beta < \infty \) such that the relation

\[
y^{-\alpha} \leq \liminf_{x \to \infty} \frac{F(xy)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(xy)}{F(x)} \leq y^{-\beta}, \quad \text{for every } 0 < y < 1. \tag{20}
\]

We say a d.f. \( F \) has a regularly varying tail with tail index \( -\alpha < 0 \), denoted by \( F \in R_{-\alpha} \), if

\[
\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha}, \quad \text{for any } y > 0. \tag{21}
\]

We say that a d.f. \( F \) \([0, \infty)\) belongs to the class \( L \) (is long-tailed), denoted by \( F \in L \), if

\[
\lim_{x \to \infty} \frac{F(x-y)}{F(x)} = 1, \quad \text{for any (or, equivalently, for some) } y \neq 0. \tag{22}
\]

It is well known that

\[
R \subset ERV \subset C \subset D \cap L \subset D, \quad D \notin L \quad \text{and} \quad L \notin D.
\]

For more details of heavy-tailed distributions, see [14].

Now we need two significant indices of a general distribution function \( F \) following [33]. For any \( y > 0 \), we set

\[
F_*(y) = \liminf_{x \to \infty} \frac{F(xy)}{F(x)} \quad \text{and} \quad F^*(y) = \limsup_{x \to \infty} \frac{F(xy)}{F(x)} \tag{23}
\]

and then

\[
J_F^+ = -\lim_{y \to \infty} \frac{\log F_*(y)}{\log y} \quad \text{and} \quad J_F^- = -\lim_{y \to \infty} \frac{\log F^*(y)}{\log y}, \tag{24}
\]

where \( J_F^+ \) and \( J_F^- \) are called the upper and lower Matuszewska indices, respectively. Generally, \( 0 \leq J_F^- \leq J_F^+ \leq \infty \). Especially, if \( F \in D \), then \( J_F^+ < \infty \), if \( F \in ERV(-\alpha, -\beta) \), then \( \alpha \leq J_F^- \leq J_F^+ \leq \beta \), and if \( F \in R_{-\alpha} \) with \( \alpha > 0 \), then \( J_F^- = J_F^+ = \alpha \).
Next we introduce another important index for distribution $F$ as

$$L_F = \lim_{y \downarrow 1} F_*(y).$$

(25)

By \([23]\), we have

$$L_F = \lim_{y \downarrow 1} F_*(y) = 1 / \lim_{y \uparrow 1} F^*(y).$$

(26)

Assume that $\lim \phi_1(x) = d_1$. Relations \([3]\) and \([4]\) mean that

$$1 + \pi d_1 \phi_2(y) \geq 0$$

and

$$\int_0^\infty (1 + \pi d_1 \phi_2(y)) G(dy) = 1.$$

Hence, we may define a new r.v. $Y^*_n$ independent of $\{\varepsilon_n, n \geq 1\}$, $\{Y_n, n \geq 1\}$ and $\{L_\theta(t), t \geq 0\}$, with the distribution defined by

$$G_\pi(dy) = P(Y^*_n \in dy) = (1 + \pi d_1 \phi_2(y)) G(dy).$$

(27)

Hereafter, for the conciseness on expression, we assume that $(\varepsilon^*, Y^*)$ is an independent version of $(\varepsilon, Y)$, meaning that the former has the same marginal distributions as the latter but has independent components. Assume that $\hat{\varepsilon}^*$ and $\hat{Y}^*$ are two independent r.v.s, which are also independent of $\varepsilon^*$, $Y^*$, with distributions $F$ and $G$, respectively, defined by

$$\tilde{F}(dx) = \left(1 - \frac{\phi_1(x)}{b_1}\right) F(dx) \text{ and } \tilde{G}(dy) = \left(1 - \frac{\phi_2(x)}{b_2}\right) G(dy), \; x \in D_x, \; y \in D_y.$$ 

(28)

Let $\varepsilon^*$, $Y^*$, $\hat{\varepsilon}^*$, $\hat{Y}^*$ be independent of $Y^*_n$, $\{\varepsilon_n, n \geq 1\}$, $\{Y_n, n \geq 1\}$ and $\{L_\theta(t), t \geq 0\}$. Denote by $\{(\varepsilon^*_n, Y^*_n), n \geq 1\}$ and $\{(\hat{\varepsilon}^*_n, \hat{Y}^*_n), n \geq 1\}$ the i.i.d. copies of generic random pairs $(\varepsilon^*, Y^*)$ and $(\hat{\varepsilon}^*, \hat{Y}^*)$, respectively.

Here, we only consider the case $\theta \in [0, 1)$, since the insurer is not allowed to invest all his or her wealth into risky assets. Now we begin to state our main results.

**Theorem 2.1.** Consider the non-standard risk model \([12]\) with $\theta \in [0, 1)$. In addition to Assumptions 1.1 and 1.2, assume that in claim-size model \([7]\), the common distribution function $F$ of the step sizes $\{\varepsilon_n, n \geq 1\}$ belongs to the class $\mathcal{D}$ and the nonnegative coefficients $\{\varphi_n, n \geq 0\}$ satisfy $\sup_{n \geq 0} \varphi_n < \infty$ and $\varphi_0 > 0$. If $\lim_{x \to \infty} \phi_1(x)$ exists and $\phi_2(y)$ is bounded in \([2]\), then it holds for all $0 < T < \infty$ with $P\{\tau_1 \leq T\} > 0$,

$$L_F \int_0^T P \left\{ \varepsilon^* \sum_{i=0}^\infty \varphi_i e^{-L_\theta(s+\tau_i)} I_{(s+\tau_i, \leq T)} > x \right\} d\lambda^*_x$$

$$\leq \Psi(x, T) \leq L_F^{-1} \int_0^T P \left\{ \varepsilon^* \sum_{i=0}^\infty \varphi_i e^{-L_\theta(s+\tau_i)} I_{(s+\tau_i, \leq T)} > x \right\} d\lambda^*_x,$$

(29)

where $\lambda^*_x = \int_0^x (1 + \lambda_{t-u}) (1 + \pi d_1 \phi_2(u)) G(du)$.

Since $L_F = 1$ when $F \in \mathbb{C}$, we immediately obtain the following corollary from Theorem 2.1:

**Corollary 2.1.** Consider the non-standard risk model \([12]\) with Assumptions 1.1 and 1.2. If the common distribution function $F$ of the step sizes $\{\varepsilon_n, n \geq 1\}$ in
model \([\mathcal{I}]\) belongs to the class \(\mathcal{C}\), then under the remaining assumptions of Theorem 2.1, for all \(0 < T < \infty\) with \(P\{\tau_1 \leq T\} > 0\), we have
\[
\Psi(x, T) \sim \int_0^T P \left\{ \varepsilon^* \sum_{i=0}^{\infty} \varphi_i e^{-L_\theta(s + \tau_i)} I_{(s + \tau_i \leq T)} > x \right\} d\lambda^* \quad \text{for all } \varepsilon^* > 0.
\] (30)

**Theorem 2.2.** Consider the non-standard risk model \([\mathcal{I}]\) with Assumptions 1.1 and 1.2. Assume that \(0 < EL(1) < \infty\) and either \(\sigma > 0\) or \(\nu((\infty, 0)) > 0\), and for every fixed \(\theta \in (0, 1)\), let \(\kappa_\theta > 0\) be the unique value satisfying \(\psi_\theta(\kappa_\theta) = 0\), and for the case \(\theta = 0\), write \(\kappa_\theta = \infty\). If the common distribution function \(F\) of the step sizes \(\{\varepsilon_n, n \geq 0\}\) in model \([\mathcal{I}]\) belongs to the extended regular variation class \((E\mathcal{R}\mathcal{V})\) for some \(0 < \alpha < \beta < \infty\), then under the remaining assumptions of Theorem 2.1, relation (30) also holds for all \(0 < T \leq \infty\) with \(P\{\tau_1 \leq T\} > 0\).

Next we discuss some special cases of Theorem 2.2.

**Theorem 2.3.** Under the assumptions of Theorem 2.2, for the case \(\theta \in (0, 1)\), if \(F \in \mathcal{R}-\alpha\) for some \(0 < \alpha < \kappa_\theta\), then for all \(0 < T < \infty\) with \(P\{\tau_1 \leq T\} > 0\), we have
\[
\Psi(x, T) \sim \mathcal{F}(x) \int_0^T E \left\{ \sum_{i=0}^{\infty} \varphi_i e^{-L_\theta(s + \tau_i)} I_{(s + \tau_i \leq T)} \right\}^\alpha d\lambda^* \quad \text{for all } \varepsilon^* > 0,
\] (31)
for the case \(\theta = 0\), if \(F \in \mathcal{R}-\alpha\) for some \(0 < \alpha < \infty\), then relation (31) also holds for all \(0 < T \leq \infty\) with \(P\{\tau_1 \leq T\} > 0\).

**Corollary 2.2.** Under the conditions of Theorem 2.3, we have
\[
\Psi(x) \sim \mathcal{F}(x) \frac{E[(1 + \pi d_1 \varphi_2(\tau_1))e^{\tau_1 \psi_\theta(\alpha)}]}{1 - Ee^{\tau_1 \psi_\theta(\alpha)}} E \left\{ \sum_{i=0}^{\infty} \varphi_i e^{-L_\theta(\tau_i)} \right\}^\alpha.
\] (32)

**Remark 2.1.** By letting \(\varphi_0 = 1\), \(\varphi_n = 0\) for all \(n \geq 1\) in \([\mathcal{I}]\), and \(1 + \pi \varphi_1(x) \varphi_2(y) \equiv 1\), for all \(x \in D_\varepsilon\), \(y \in D_Y\) in \([\mathcal{I}]\), respectively, we can see that the non-standard renewal risk model \([\mathcal{I}]\) includes two special cases that the claim sizes are i.i.d. and \(\varepsilon\) is independent of \(Y\). Thus, Theorem 2.1 partly extends the results of \([20]\), in which the claim sizes are i.i.d. and the claim sizes, \(X_n, n = 1, 2, \ldots\), and the inter-arrival times \(Y_n, n = 1, 2, \ldots\), are mutually independent. Theorem 2.3 and Corollary 2.2 partly extend the results of \([17]\) and \([18]\), in which \(\varepsilon\) is considered to be independent of \(Y\).

**Remark 2.2.** Consider the case that step sizes of claims have regularly varying tails. Theorem 2.3 and Corollary 2.2 show that in finite and infinite times, the extreme of insurance risk always dominates the extreme of the financial risk because the tail probability of step sizes determines the exact decay rate of the ruin probabilities. However, for the case that the claim sizes are i.i.d., Theorem 4.4 in \([22]\) shows for the case of dangerous investment \((\psi_\theta(\alpha) > 0\) since \(\alpha > \kappa_\theta\)) that the extreme of the financial risk finally dominates the extreme of the insurance risk when the claim sizes have regularly varying tails.

3. **Lemmas.** In this section, we need a series of lemmas to prove our main results. First, we begin with some properties of the class \(\mathcal{D}\) in the lemma below, which is due to \([6]\), \([11]\) and \([33]\).
Lemma 3.1. If a distribution $F \in \mathcal{D}$, then for any $p > J_F^+$, we have

$$x^{-p} = o(F(x)), \quad (33)$$

and

$$F \in \mathcal{D} \text{ if and only if } L_F > 0.$$ 

Lemma 3.2. Let $X$ and $Y$ be two independent and nonnegative random variables, where $X$ is distributed by $F \in \mathcal{D}$ and $Y$ is nonnegative and nondegenerate at 0 satisfying $EY^p < \infty$ for some $p > J_F^+$. Then, the distribution of the product $XY$ belongs to the class $\mathcal{D}$ and $P\{XY > x\} \asymp F(x)$.

Proof. See Theorem 3.3 (iv) of [11] (also Lemma 3.8 of [33]).

Lemma 3.3. Let $\varepsilon$ and $\Theta$ be two independent and nonnegative random variables with $\varepsilon$ distributed by $F \in \mathcal{D}$. Then, we have the following two results.

(i) If $F \in \mathcal{D}$, then for arbitrarily fixed $\delta > 0$ and $J_F^+ < p_2 < \infty$, there exists a positive constant $C$ without relation to $\Theta$ and $\delta$ such that for all large $x$,

$$P\{\varepsilon\Theta > \delta x \mid \Theta\} \leq C \frac{F(x)(\delta^{-p_2}I_{\Theta > \delta} + I_{\Theta \leq \delta})}{x}. \quad (34)$$

Proof. See Lemma 3.2 in [20].

(ii) If $F \in \mathcal{E}RV(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, then for arbitrarily fixed $\delta > 0$ and $0 < p_1 < \alpha \leq J_F^+ \leq \beta < p_2 < \infty$, there exists a positive constant $C$ without relation to $\Theta$ and $\delta$ such that for all large $x$,

$$P\{\varepsilon\Theta > \delta x \mid \Theta\} \leq C \frac{F(x)(\delta^{-p_1}I_{\Theta > \delta} + \delta^{-p_2}I_{\Theta > \delta})}{x}. \quad (35)$$

Proof. See Lemma 4.1.5 in [35].

Lemma 3.4. A distribution $F$ on $[0, \infty)$ belongs to the class $\mathcal{D}$ if and only if for any distribution $W$ on $[0, \infty)$ satisfying $W(x) = o(F(x))$, there exists a positive function $w(\cdot)$ such that

$$w(x) \searrow 0, \quad xw(x) \nearrow \infty \text{ and } W(xw(x)) = o(F(x)).$$

Proof. See the proof of sufficiency in [42] and the proof of the necessity in [32].

Lemma 3.5. Define an exponential functional of a Lévy process $\{L(t), t \geq 0\}$ as

$$Z = \int_0^\infty e^{-L(s)} ds.$$ 

Then we have:

1. $Z < \infty$ almost surely if and only if $L(t) \to \infty$ almost surely as $t \to \infty$;
2. If $p > 0$ and $\psi(p) = \log E[e^{-pL(1)}] < 0$ with $EL(1) > 0$, then $EZ^p < \infty$.

Proof. See Proposition 2.1 and Lemma 2.1 of [26], respectively.
\[ \vartheta_i(T) = \sum_{n=1}^{\infty} \varphi_n e^{-L_0(\tau_n)} I_{(\tau_n \leq T)}, \quad \vartheta_0(T) = \sum_{n=1}^{\infty} \varphi_n e^{-L_0(\tau_n)}; \]

\[ \tau^{\pi}_1 = Y^*_1, \quad \tau^{\pi}_n = Y^*_1 + \sum_{i=2}^{n} Y_i, \quad n = 2, \ldots; \]

\[ \tilde{\tau}^{\pi}_1 = \tilde{Y}^*_1, \quad \tilde{\tau}^{\pi}_n = \tilde{Y}^*_1 + \sum_{i=2}^{n} Y_i, \quad n = 2, \ldots. \]

The following inequalities will play crucial roles in the proof of Theorem 2.1. For \( 0 < p \leq 1 \), by Hölder’s inequality and \( C_r \) inequality, we have for any \( 0 \leq s \leq T \) and \( i \geq 1 \),

\[
\begin{align*}
E \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0 \left( \sum_{k=i+1}^{n} Y_k + s \right)} I_{\left( \sum_{k=i+1}^{n} Y_k + s \leq T \right)} \right)^p & \\
& \leq \left( \sup_{n \geq 0} \varphi_n \right)^p \sum_{n=i}^{\infty} E \left( e^{-L_0 \left( \tau_{n-i} + s \right)} I_{(\tau_{n-i} + s \leq T)} \right)^p \\
& \leq \left( \sup_{n \geq 0} \varphi_n \right)^p \sum_{n=i}^{\infty} E \int_{0}^{T} e^{-pL_0(t+s)} I_{(t+s \leq T)} P\{\tau_{n-i} \in dt\} \\
& = \left( \sup_{n \geq 0} \varphi_n \right)^p \sum_{n=i}^{\infty} \int_{0}^{T} e^{(t+s) \psi_p(p)} I_{(t+s \leq T)} P\{\tau_{n-i} \in dt\} \\
& \leq \left( \sup_{n \geq 0} \varphi_n \right)^p \max \left( 1, e^{2T \psi_p(p)} \right) \sum_{n=i}^{\infty} P\{\tau_{n-i} \leq T\} \\
& = \left( \sup_{n \geq 0} \varphi_n \right)^p \max \left( 1, e^{2T \psi_p(p)} \right) [1 + EN(T)].
\end{align*}
\]

For \( p > 1 \), by Hölder’s inequality and \( C_r \) inequality, we have for any \( 0 \leq s \leq T \) and \( i \geq 1 \),
Lemma 4.1 in [22] has shown that $\psi(s) < \infty$ for all $\theta \in (0, 1)$ and $s \geq 0$. Especially, (7) and (13) imply $\psi_0(p) = -rp < 0$. Since there exists some $h > 0$ such that $E[e^{hN(T)}] < \infty$ by Lemma 3.2 in [19], we can obtain that,

$$E[N(T)^z] < \infty \text{ for any } z > 0.$$ 

Then, we can get for any $p > 0$, $0 \leq s \leq T$, and $i \geq 1$,

$$E \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_p} \left( \sum_{k=i+1}^{n} y_{k+s} \right) \mathbb{I} \left( \sum_{k=i+1}^{n} y_{k+s} \leq T \right) \right)^p \leq C \max \left( 1, e^{2T\psi(p)} \right) 2^p \left\{ 1 + E[N(T)^{2p-1}] + i^{2p-2}(1 + EN(T)) \right\} < \infty.$$ 

Hence, we have for any $\delta > 0$, any $p > 0$ and any $q \geq 0$,
The following lemma will play a crucial role in the proof of the main results.

**Lemma 3.6.** Under the conditions of Theorem 2.1, for every $i \geq 1$, the relations

$$P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n e^{-L_{\varphi}(\tau_n)} I(\tau_n \leq T) > x \right\}$$

$$\sim P\left\{ \frac{\varepsilon_i}{C} \sum_{n=i}^{\infty} \varphi_n e^{-L_{\varphi}(\tau_n)} I(\tau_n \leq T) > x \right\}$$

$$= P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n e^{-L_{\varphi}(\tau_n)} I(\tau_n \leq T) > x \right\}$$

and

$$P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n e^{-L_{\varphi}(\tau_n)} I(\tau_n \leq T) > x \right\} \asymp F(x)$$

hold for all $0 < T < \infty$ with $P\{\tau_1 \leq T\} > 0$, and the distribution function of $\varepsilon_i \varphi_1(T)$ belongs to the class $D$.

**Proof.** By the conditions of Theorem 2.1, let $\lim_{T \to \infty} \varphi_1(x) = d_1$, and there exist two constants $b_1 > 1$ and $b_2 > 1$ such that $|\varphi_1(x)| \leq b_1 - 1$ and $|\varphi_2(y)| \leq b_2 - 1$ for all $x \in D_x$ and $y \in D_y$. Obviously, $d_1 < b_1$. According to the definitions of bivariate Sarmanov distribution and the distributions $F$ and $\tilde{G}$ in (28), we have for every
i \geq 1,
\begin{align*}
P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} > x \right\} \\
= \int_0^\infty \int_0^\infty P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\sum_{k \neq i}^{n} Y_k + v)} I_{(\sum_{k \neq i}^{n} Y_k + v \leq T)} > x \right\} \\
\cdot \left[ 1 + \pi b_1 b_2 - \pi b_1 b_2 \left( 1 - \frac{\phi_1(u)}{b_1} \right) - \pi b_1 b_2 \left( 1 - \frac{\phi_2(v)}{b_2} \right) \\
+ \pi b_1 b_2 \left( 1 - \frac{\phi_1(u)}{b_1} \right) \left( 1 - \frac{\phi_2(v)}{b_2} \right) \right] F(du) G(dv) \\
= (1 + \pi b_1 b_2) P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\sum_{k \neq i}^{n} Y_k + Y_i^*)} I_{(\sum_{k \neq i}^{n} Y_k + Y_i^* \leq T)} > x \right\} \\
- \pi b_1 b_2 P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\sum_{k \neq i}^{n} Y_k + Y_i^*)} I_{(\sum_{k \neq i}^{n} Y_k + Y_i^* \leq T)} > x \right\} \\
- \pi b_1 b_2 P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^*)} I_{(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^* \leq T)} > x \right\} \\
+ \pi b_1 b_2 P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^*)} I_{(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^* \leq T)} > x \right\} \\
= (1 + \pi b_1 b_2) P \{ \varepsilon_i^* \vartheta_i(Y_i^*, T) > x \} - \pi b_1 b_2 P \{ \varepsilon_i^* \vartheta_i(Y_i^*, T) > x \} \\
- \pi b_1 b_2 P \{ \varepsilon_i^* \vartheta_i(\bar{Y}_i^*, T) > x \} + \pi b_1 b_2 P \{ \varepsilon_i^* \vartheta_i(\bar{Y}_i^*, T) > x \},
\end{align*}
\begin{align*}
\text{where } \{ \varepsilon_i^*, i \geq 1 \}, \{ Y_i^*, i \geq 1 \}, \{ \tilde{\varepsilon}_i^*, i \geq 1 \}, \{ \bar{Y}_i^*, i \geq 1 \} \text{ and } \{ L_{\theta}(t), t \geq 0 \} \text{ are mutually independent, and are also independent of } \{ \varepsilon_i, i \geq 1 \} \text{ and } \{ Y_i, i \geq 1 \}.
\end{align*}
Since \( \lim_{x \to \infty} \phi_1(x) = d_1 \), and by relation \text{[28]}, we have
\begin{align*}
\overline{F}(x) &= \int_x^\infty \left( 1 - \frac{\phi_1(u)}{b_1} \right) F(du) \sim \left( 1 - \frac{d_1}{b_1} \right) \overline{F}(x).
\end{align*}

Hence, since \( F(x) \in \mathcal{D} \), it follows that
\begin{align*}
\overline{F}(x) \in \mathcal{D} \text{ and } \overline{F}(x) \sim F(x).
\end{align*}

As for the first term on the right-hand side of \text{[39]}, since \( Y_i^* \) is identically distributed to \( Y \) and independent of \( \{ L_{\theta}(t), t \geq 0 \} \) and \( \{ Y_k, k \geq 1 \} \), relation \text{[36]} implies that for some \( p > J_p^* \) and every \( i \geq 1 \),
\begin{align*}
E[\vartheta_i(Y_i^*, T)]^p &= E \left( \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} \right)^p < \infty.
\end{align*}

By relation \text{[41]} and independence among \( \varepsilon_i^*, Y_i^*, \{ Y_n, n \geq 1 \} \) and \( \{ L_{\theta}(t), t \geq 0 \} \), Lemma 3.2 implies that the distribution of \( \varepsilon_i^* \vartheta_i(Y_i^*, T) \) belongs to the class \( \mathcal{D} \) and
\begin{align*}
P \{ \varepsilon_i^* \vartheta_i(Y_i^*, T) > x \} \sim \overline{F}(x).
\end{align*}
Because of independence among \( \tilde{Y}_i^* \), \( \{Y_n, n \geq 1\} \) and \( \{L_\theta(t), t \geq 0\} \), relations \( |\varphi_2(y)| \leq b_2 - 1 \) and [28], we get

\[
E \left[ e^{-L_\theta(\sum_{k \neq i} Y_k + \tilde{Y}_i^*)} I(\sum_{k \neq i} Y_k + \tilde{Y}_i^* \leq T) \right]^p = E \int_0^T \left[ e^{-L_\theta(s)} I(s \leq T) \right]^p P(\tilde{z}_n^* \in ds)
\]

\[
= E \int_0^T \int_0^{T-v} [e^{-L_\theta(u+v)}] P(\tau_{n-1} \leq du) P(\tilde{Y}^* \in dv)
\]

\[
\leq E \int_0^T \int_0^{T-v} [e^{-L_\theta(u+v)}] P(\tau_{n-1} \leq du) P(Y \in dv)
\]

\[
= E[e^{-L_\theta(\tau_n)} I(\tau_n \leq T)]^p.
\]

Similar to the proof of relation [36], the aforementioned inequality implies that for any \( \delta > 0 \), any \( p > 0 \) and any \( q \geq 0 \),

\[
\sum_{i=1}^\infty q \left( E \left[ \sum_{n=1}^\infty \varphi_{n-i} e^{-L_\theta(\sum_{k \neq i} Y_k + \tilde{Y}_i^*)} I(\sum_{k \neq i} Y_k + \tilde{Y}_i^* \leq T) \right]^p
\]

\[
+ EI \left\{ \sum_{n=1}^\infty \varphi_{n-i} e^{-L_\theta(\sum_{k \neq i} Y_k + \tilde{Y}_i^*)} I(\sum_{k \neq i} Y_k + \tilde{Y}_i^* \leq T) < \delta \right\} \right) < \infty.
\]

(43)

For each \( i \geq 1 \), similar to the proof of the distribution of \( \varphi_i^* \theta_i(Y_i^*, T) \), we can get that distributions of \( \varphi_i^* \theta_i(Y_i^*, T), \varphi_i^* \theta_i(\tilde{Y}_i^*, T), \varphi_i^* \theta_i(\hat{Y}_i^*, T) \) also belong to the class \( D \) and

\[
P(\varphi_i^* \theta_i(Y_i^*, T) > x) \approx F(x), \quad P(\varphi_i^* \theta_i(\tilde{Y}_i^*, T) > x) \approx F(x),
\]

\[
P(\varphi_i^* \theta_i(\hat{Y}_i^*, T) > x) \approx F(x).
\]

(44)

Next we use the approach in [39], but there are many changes in this proof because the coefficients \( \{\varphi_n, n \geq 1\} \) bring much trouble. By Chebyshevs inequality, [33], [41] and [42], we can obtain for some \( p > J_F^+ \),

\[
P(\varphi_i(Y_i^*, T) > x) \leq \frac{E[\varphi_i(Y_i^*, T)]^p}{x^p} = o(F(x)) = o(P(\varphi_i(Y_i^*, T) > x)).
\]

According to the fact that the distribution of \( \varphi_i^* \theta_i(Y_i^*, T) \) belongs to the class \( D \), Lemma 3.4 implies that there exists a positive function \( \tilde{g}(\cdot) \) such that

\[
\tilde{g}(x) \downarrow 0, \quad x \tilde{g}(x) \not\rightarrow \infty \quad \text{and} \quad P(\varphi_i(Y_i^*, T) > x \tilde{g}(x)) = o(P(\varphi_i(Y_i^*, T) > x)).
\]
This and (40) imply that
\[
P\{\xi_i^* \vartheta_i(Y_i^*, T) > x\} = \int_0^{x \bar{g}(x)} \bar{F}(x/z) P\{\vartheta_i(Y_i^*, T) \in dz\} + o(P\{\xi_i^* \vartheta_i(Y_i^*, T) > x\})
\]
\[
= (1 - \frac{d_1}{b_1} + o(1)) \int_0^{x \bar{g}(x)} \bar{F}(x/z) P\{\vartheta_i(Y_i^*, T) \in dz\} + o(P\{\xi_i^* \vartheta_i(Y_i^*, T) > x\})
\]
\[
= (1 - \frac{d_1}{b_1} + o(1)) P\{\xi_i^* \vartheta_i(Y_i^*, T) > x\}.
\]  
(45)

By the boundness of \(\phi_2(y)\), we have
\[
P\{\vartheta_i(\tilde{Y}_i^*, T) > x \bar{g}(x)\}
\]
\[
= \int_0^{\int_0^{x \bar{g}(x)}} P\left\{\sum_{n=1}^{\infty} \varphi_{n-i} \xi_i^{* -L_a(\sum_{k \neq i} Y_k + v)} I(\sum_{k \neq i} Y_k + v \leq T) > x \bar{g}(x)\right\} \left(1 - \frac{\phi_2(v)}{b_2}\right) G(dv)
\]
\[
= O(P\{\vartheta_i(\tilde{Y}_i^*, T) > x \bar{g}(x)\}) = o(P\{\xi_i^* \vartheta_i(Y_i^*, T) > x\}),
\]

which implies
\[
P\{\xi_i^* \vartheta_i(\tilde{Y}_i^*, T) > x\}
\]
\[
= \int_0^{\int_0^{x \bar{g}(x)}} \bar{F}(x/z) P\{\vartheta_i(\tilde{Y}_i^*, T) \in dz\} + O(P\{\vartheta_i(\tilde{Y}_i^*, T) > x \bar{g}(x)\})
\]
\[
= (1 - \frac{d_1}{b_1} + o(1)) P\{\xi_i^* \vartheta_i(\tilde{Y}_i^*, T) > x\} + o(P\{\xi_i^* \vartheta_i(\tilde{Y}_i^*, T) > x\}).
\]  
(46)

From relations (39), (45), (46) and (27), we obtain that
\[
P\{\xi_i \vartheta_i(T) > x\}
\]
\[
= (1 + \pi d_1 b_2 + o(1)) P\{\xi_i^* \vartheta_i(Y_i^*, T) > x\} - (\pi d_1 b_2 + o(1)) P\{\xi_i^* \vartheta_i(\tilde{Y}_i^*, T) > x\}
\]
\[
= (1 + \pi d_1 b_2 + o(1)) \int_0^{\int_0^{x \bar{g}(x)}} P\{\xi_i^* \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\sum_{k \neq i} Y_k + v)} I(\sum_{k \neq i} Y_k + v \leq T) > x\} G(dv)
\]
\[
- (\pi d_1 b_2 + o(1)) \int_0^{\int_0^{x \bar{g}(x)}} P\{\xi_i^* \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\sum_{k \neq i} Y_k + v)} I(\sum_{k \neq i} Y_k + v \leq T) > x\}
\]
\[
\times (1 - \frac{\phi_2(v)}{b_2}) G(dv)
\]
\[
= (1 + o(1)) \int_0^{\int_0^{x \bar{g}(x)}} P\{\xi_i^* \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\sum_{k \neq i} Y_k + v)} I(\sum_{k \neq i} Y_k + v \leq T) > x\}
\]
\[
\times (1 + \pi d_1 \phi_2(v)) G(dv)
\]
\[
= (1 + o(1)) P\{\xi_i^* \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n^T + Y_n^*)} I(\tau_n^T \leq T) > x\}
\]
\[
= (1 + o(1)) P\{\xi_i^* \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n^T)} I(\tau_n^T \leq T) > x\},
\]
which implies that relation (37) holds for all large \( x \).

Similar to the proof of relation (36), by relation (27) and independence among \( Y^*_n \), \( \{Y_n, n \geq 1\} \) and \( \{L_\alpha(t), t \geq 0\} \), we have for any \( \delta > 0 \), any \( p > 0 \) and any \( q \geq 0 \),

\[
\sum_{i=1}^{\infty} i^q \left( E \left[ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\alpha(\sum_{k=1}^{n} Y_k + Y^*_n)} I(\sum_{k=1}^{n} Y_k + Y^*_n \leq T) \right] \right)^p
\]

\[
+ E \left\{ \sum_{i=1}^{\infty} \varphi_{n-i} e^{-L_\alpha(\sum_{k=1}^{n} Y_k + Y^*_n)} I(\sum_{k=1}^{n} Y_k + Y^*_n \leq T) \right\} < \delta \)
\]

\[
= \sum_{i=1}^{\infty} i^q \left( E \left[ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau^*_n)} I(\tau^*_n \leq T) \right] \right)^p
\]

\[
+ E \left\{ \sum_{i=1}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau^*_n)} I(\tau^*_n \leq T) \right\} \right) \]

\[
\leq C[EN^*_\tau(T)^{2p+q+1}] < \infty,
\]

(47)

where \( \{N^*_\tau(t), t \geq 0\} \) is regarded as a delayed renewal counting process with claim-arrival times \( \tau^*_1 = Y^*_1, \tau^*_n = Y^*_n + \sum_{i=2}^{n} Y_i, \ n = 2, 3, \ldots \).

Further, lemma 3.2 together with (47) implies that the distribution of

\[
\varepsilon_i^* \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau^*_n)} I(\tau^*_n \leq T)
\]

belongs to the class \( \mathcal{D} \) and \( P(\varepsilon_i^* \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau^*_n)} I(\tau^*_n \leq T) > x) \) \( \sim \mathcal{F}(x) \). Hence, the obtained (37) implies that relation (38) holds and the distribution function of \( \varepsilon_i \vartheta_i(T) \) also belongs to the class \( \mathcal{D} \). This ends the proof of Lemma 3.6.

\[ \square \]

**Lemma 3.7.** Under the conditions of Theorem 2.1, there exists some positive function \( g(x) \) satisfying \( g(x) \to \infty \) and \( g(x) = o(x) \) such that, for every \( i \geq 1 \) and all \( 0 < T < \infty \) with \( P(\tau_i \leq T) > 0 \)

\[
P \left\{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau_n)} I(\tau_n \leq T) > g(x) \right\} = o(\mathcal{F}(x))
\]

\[
= o \left( P \left( \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau_n)} I(\tau_n \leq T) > x \right) \right) \}
\]

and

\[
P \left\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau_n^*)} I(\tau_n^* \leq T) > g(x) \right\} = o(\mathcal{F}(x))
\]

\[
= o \left( P \left( \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau_n^*)} I(\tau_n^* \leq T) > x \right) \right) \}.
\]

**Proof.** Obviously, let \( g(x) = x / \ln x \) which satisfies the assumption condition. Take \( p^* > J^+_P + \varepsilon \) for some \( \varepsilon > 0 \). By Chebyshevs inequality, (33) and (36), we can obtain for every \( i \geq 1 \),

\[
P \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\alpha(\tau_n)} I(\tau_n \leq T) > \frac{x}{\ln x} \right) \leq \left( \frac{x}{\ln x} \right)^{-p^*} E[\vartheta_i(T)]^{p^*}
\]

\[
= x^{-p^* - \varepsilon} \left( \ln x \right)^{p^* - \varepsilon} E[\vartheta_i(T)]^{p^*} = o(\mathcal{F}(x)).
\]

By the same method as the proof of the first relation, we can prove the second relation together with relation (47). This ends the proof of Lemma 3.7.

\[ \square \]
Lemma 3.8. Under the conditions of Theorem 2.1, for every \( 1 \leq i < j \), it holds for all \( 0 < T < \infty \) with \( P\{\tau_1 \leq T\} > 0 \) that

\[
P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\tau_n)} I_{(\tau_n \leq T)} > x, \xi_j \sum_{n=j}^{\infty} \varphi_{n-j} e^{-L_0(\tau_n)} I_{(\tau_n \leq T)} > x \}
= o\left( F(x) \right) = o\left( P\{ \xi_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\tau_n)} I_{(\tau_n \leq T)} > x \} \right).
\]  

(48)

Proof. Let \( g(x) \) be the function defined in Lemma 3.7. According to Lemma 3.7, we have for some \( p > J_1^+ \) and \( 1 \leq i < j \),

\[
P\{ \xi, \theta_i(T) > x, \xi_j \theta_j(T) > x \}
\leq P\{ \xi, \theta_i(T) > x, \xi_j \theta_j(T) \leq g(x) \} + P\{ \theta_j(T) > g(x) \}
\leq P\{ \xi, \theta_i(T) > x, \xi_j > x/g(x) \} + o\left( F(x) \right).
\]

As in the proof of (39), we split the probability \( P\{ \xi, \theta_i(T) > x, \xi_j > x/g(x) \} \) on the right-hand side of the aforementioned inequality into four parts as

\[
P\{ \xi, \theta_i(T) > x, \xi_j > x/g(x) \}
= (1 + \pi b_1 b_2) P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq j} n_{k} + Y_k^*)} I (\sum_{k \neq j} n_{k} + Y_k^* \leq T) > x, \xi_j > x/g(x) \}
- \pi b_1 b_2 P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq j} n_{k} + Y_k^*)} I (\sum_{k \neq j} n_{k} + Y_k^* \leq T) > x, \xi_j > x/g(x) \}
- \pi b_1 b_2 P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq j} n_{k} + Y_k^*)} I (\sum_{k \neq j} n_{k} + Y_k^* \leq T) > x, \xi_j > x/g(x) \}
+ \pi b_1 b_2 P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq j} n_{k} + Y_k^*)} I (\sum_{k \neq j} n_{k} + Y_k^* \leq T) > x, \xi_j > x/g(x) \}
= I_1 + I_2 + I_3 + I_4.
\]

As for \( I_1 \), by the fact that \( \{ \xi^*_{i}, i \geq 1 \}, \{ Y^*_i, i \geq 1 \}, \{ \xi^*_i, i \geq 1 \}, \{ Y^*_i, i \geq 1 \} \) and \( L_0(t), t > 0 \) are mutually independent, and are also independent of \( \{ \xi, i \geq 1 \} \) and \( \{ Y_i, i \geq 1 \} \), Lemma 3.6 implies that

\[
P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq j} n_{k} + Y_k^*)} I (\sum_{k \neq j} n_{k} + Y_k^* \leq T) > x, \xi_j > x/g(x) \}
\lesssim P\{ \xi \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq j} n_{k} + Y_k^*) + Y^*_i} I (\sum_{k \neq j} n_{k} + Y_k^* + Y^*_i \leq T) > x \} P\{ \xi_j > x/g(x) \}
\lesssim C F(x) F(x/g(x)) = o\left( F(x) \right),
\]

where the last inequality follows from (47) and lemma 3.2.

Similarly, according to (28) and (40), we have

\[
I_2 = o\left( F(x) \right), \quad I_3 = o\left( F(x) \right), \quad I_4 = o\left( F(x) \right).
\]

Together with (38), we complete the proof of the lemma. \qed
Lemma 3.9. Under the conditions of Theorem 2.1, for every \( N \geq 1 \), it holds for all \( 0 < T < \infty \) with \( P\{\tau_1 \leq T\} > 0 \) that

\[
\sum_{i=1}^{N} P\{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\} \lesssim \sum_{i=1}^{N} P\{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\} \lesssim \sum_{i=1}^{N} P\{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\}. \tag{49}
\]

Proof. Firstly, we give the asymptotic lower bound of (49). From (48), we easily obtain that for every \( N \geq 1 \),

\[
P\{\sum_{i=0}^{N} \varepsilon_i \vartheta_i(T) > x\} \geq P\{\bigcup_{i=1}^{N} \{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\} \}
\geq \sum_{i=1}^{N} P\{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\}
- \sum_{1 \leq i < j \leq N} P\{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x, \varepsilon_j \sum_{n=j}^{\infty} \varphi_{n-j}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\}
\gtrsim (1 - o(1)) \sum_{i=1}^{N} P\{\sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\tau_n)} I_{(\tau_n \leq T)}> x\}, \tag{50}
\]

which gives the asymptotic lower bound of (49).

Then we turn to the asymptotic upper bound of (49). For any \( 0 < \omega < 1 \), we have

\[
P\{\sum_{i=0}^{N} \varepsilon_i \vartheta_i(T) > x\} \leq P\{\bigcup_{i=0}^{N} (\varepsilon_i \vartheta_i(T) > \omega x)\}
+ P\{\sum_{i=0}^{N} \varepsilon_i \vartheta_i(T) > x, \bigcap_{i=0}^{N} (\varepsilon_i \vartheta_i(T) \leq \omega x)\}
= P_1(x) + P_2(x).
\]

For the conciseness on expression, denote

\[
\vartheta_i(Y_\tau^*, T) = \sum_{n=i}^{\infty} \varphi_{n-i}e^{-L_\delta(\sum_{k=1}^{n} Y_k + Y_\tau^*)} I_{(\sum_{k=1}^{n} Y_k + Y_\tau^* \leq T)}, \quad i = 1, 2, \ldots
\]

From Chebyshev inequality, (33), (36), (37), (38) and Lemma 3.7, it follows that for any \( \delta > 0 \), there exists a sufficiently large \( x_0 > 0 \) such that for all \( x \geq x_0 \) and
some \( p > J_F^+ \),

\[
P_1(x) \leq \sum_{i=1}^N P\{\varepsilon_i \sum_{n=1}^\infty \varphi_n e^{-\delta_n} I_{(\tau_n \leq T)} > \omega x\} + P\{\varepsilon_0 \sum_{n=1}^\infty \varphi_n e^{-\delta_n} I_{(\tau_n \leq T)} > \omega x\}
\]

\[
\leq \sum_{i=1}^N P\{\varepsilon_i \partial_i(T) > \omega x\} + (\omega x)^{-p} \varepsilon_0^p E[\theta_0(T)]^p
\]

\[
\leq (1 + \delta) \sum_{i=1}^N P\{\varepsilon_i^* \partial_i(Y^*_n, T) > \omega x, \partial_i(Y^*_n, T)\}
\]

\[
\leq g(x) + (1 + \delta) \sum_{i=1}^N P\{\partial_i(Y^*_n, T) > g(x)\} + \delta \bar{F}(x)
\]

\[
\lesssim (1 + \delta) \int_0^{g(x)} P\{\varepsilon_i^* > x/y\} P\{\partial_i(Y^*_n, T) \in dy\} + (1 + \delta) \delta \bar{F}(x) + \delta \bar{F}(x)
\]

\[
\lesssim \frac{(1 + \delta)}{F^*(1/\omega)} \int_0^{g(x)} P\{\varepsilon_i^* > x/y\} P\{\partial_i(Y^*_n, T) \in dy\} + \delta(2 + \delta) \bar{F}(x)
\]

\[
\lesssim \frac{(1 + \delta)}{F^*(1/\omega)} \sum_{i=1}^N P\{\varepsilon_i \sum_{n=1}^\infty \varphi_n e^{-\delta_n} I_{(\tau_n \leq T)} > x\}
\]

\[
+ \delta(2 + \delta) O\left(P\{\varepsilon_i \sum_{n=1}^\infty \varphi_n e^{-\delta_n} I_{(\tau_n \leq T)} > x\}\right),
\]

where \( g(x) \) is the function defined in Lemma 3.7. Hence, by letting \( \omega \to 1 \), and the arbitrariness of \( \delta \), (23) implies that

\[
P_1(x) \lesssim L_F^{-1} \sum_{i=1}^N P\{\varepsilon_i \sum_{n=1}^\infty \varphi_n e^{-\delta_n} I_{(\tau_n \leq T)} > x\}.
\]

As for \( P_2(x) \), we have

\[
P_2(x) = P\{\varepsilon_i \partial_i(T) > x, \bigcap_{i=0}^N (\varepsilon_i \partial_i(T) \leq \omega x), \bigcup_{i=0}^N (\varepsilon_i \partial_i(T) > x/(N + 1))\}
\]

\[
\leq \sum_{i=1}^N P\{\varepsilon_i \partial_i(T) > x/(N + 1), \sum_{j=1, j\neq i}^N \varepsilon_j \partial_j(T) + \varepsilon_0 \partial_0(T) > (1 - \omega)x\}
\]

\[
+ P\{\varepsilon_0 \partial_0(T) > x/(N + 1), \sum_{j=1}^N \varepsilon_j \partial_j(T) > (1 - \omega)x\}
\]

\[
\leq \sum_{i=1}^N \sum_{j=1, j\neq i}^N P\{\varepsilon_i \partial_i(T) > x/(N + 1), \varepsilon_j \partial_j(T) > (1 - \omega)x/N\}
\]
and the definition of the class $D$. This completes the proof.

As for $P_{21}(x)$, since $(1-\omega)/(N+1) < 1/(N+1)$ with $\omega < 1$, then it follows from (48) and the definition of the class $D$ that

$$P_{21}(x) \leq \sum_{i=1}^{N} \frac{P\{\varepsilon_i \theta_i(T) > (1-\omega)x/(N+1)\}}{P\{\varepsilon_i \theta_i(T) > (1-\omega)x/(N+1)\}} \leq o(1).$$

As for $P_{22}(x)$, it follows from Chebyshev inequality, (33), (36) and (38) that for some $p > J^+_F$,

$$P_{22}(x) \leq \sum_{i=1}^{N} P\{\varepsilon_0 \theta_0(T) > (1-\omega)x/N\} \leq N[(1-\omega)x/N]^{-p} E[\theta_0(T)]^p \leq o(F(x)) = o\left(\left\{ \varepsilon_i \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} \right\} \right).$$

Similar to the proof of (54), we have

$$P_{23}(x) = o(F(x)) = o\left(\left\{ \varepsilon_i \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} \right\} \right).$$

Hence, combining (54)-(55), we immediately get that

$$P\left\{ \sum_{i=0}^{N} \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} > x \right\} \leq L^{-1}_F \sum_{i=1}^{N} \left\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} > x \right\}.$$

This completes the proof.

Lemma 3.10. Under the conditions of Theorem 2.1, it holds for all $0 < T < \infty$ with $P\{\tau_1 \leq T\} > 0$ that

$$\sum_{i=1}^{\infty} P\{\varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} > x \} \leq P\left\{ \sum_{i=1}^{\infty} \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} > x \right\} \leq L^{-1}_F \sum_{i=1}^{\infty} P\{\varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_{\theta}(\tau_n)} I_{(\tau_n \leq T)} > x \}. \quad (56)$$
Proof. Firstly, we give the asymptotic lower bound of (36). For every integer $N$ such that $\sum_{i=N+1}^{\infty} \frac{1}{i^2} < 1$, we have

$$P\left\{ \sum_{i=N+1}^{\infty} \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\tau_n)} I(\tau_n \leq T) > x \right\}$$

$$\leq P\left\{ \sum_{i=N+1}^{\infty} \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\tau_n)} I(\tau_n \leq T) > \sum_{i=N+1}^{\infty} \frac{x}{i^2} \right\}$$

$$\leq \sum_{i=N+1}^{\infty} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\tau_n)} I(\tau_n \leq T) > \frac{x}{i^2} \right\}$$

$$= \sum_{i=N+1}^{\infty} \left[ (1 + \pi b_1 b_2) P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq i}^{n} Y_k + Y_i^*)} I(\sum_{k \neq i}^{n} Y_k + Y_i^* \leq T) > \frac{x}{i^2} \right\} + \pi b_1 b_2 P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^*)} I(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^* \leq T) > \frac{x}{i^2} \right\} + \pi b_1 b_2 P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^*)} I(\sum_{k \neq i}^{n} Y_k + \bar{Y}_i^* \leq T) > \frac{x}{i^2} \right\} \right]$$

$$= J_1(x, N) + J_2(x, N) + J_3(x, N) + J_4(x, N).$$

As for $J_1(x, N)$, by (34) in Lemma 3.3 and (36), it can be shown that for some $p > J_0^+$, all large $x$ and sufficiently large $N$,

$$J_1(x, N) \leq C \overline{F}(x)(1 + \pi b_1 b_2) \sum_{i=N+1}^{\infty} \left( 2^p E \left[ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq i}^{n} Y_k + Y_i^*)} I(\sum_{k \neq i}^{n} Y_k + Y_i^* \leq T) \right]^p \right)$$

$$+ EI \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\sum_{k \neq i}^{n} Y_k + Y_i^*)} I(\sum_{k \neq i}^{n} Y_k + Y_i^* \leq T) < \frac{x}{i^2} \right) \leq C \overline{F}(x)(1 + \pi b_1 b_2)E[N(T)^{4p+1}I(N(T)\geq N+1)] = o(\overline{F}(x)). \quad (57)$$

Similarly, according to (28), (40) and the boundness of $\phi_2(y)$, we obtain that as first $x \to \infty$ and then $N \to \infty$,

$$J_2(x, N) = J_3(x, N) = J_4(x, N) = o(\overline{F}(x)).$$

Hence, (38) implies that as first $x \to \infty$ and then $N \to \infty$,

$$P\left\{ \sum_{i=N+1}^{\infty} \varepsilon_i \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_0(\tau_n)} I(\tau_n \leq T) > x \right\} = o(\overline{F}(x))$$

$$= o \left( P\left\{ \varepsilon_1 \sum_{n=1}^{\infty} \varphi_{n-1} e^{-L_0(\tau_n)} I(\tau_n \leq T) > x \right\} \right)$$
and

\[
\sum_{i=N+1}^{\infty} P\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \} 
\]

\[
\leq \sum_{i=N+1}^{\infty} P\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > \frac{x}{i^2} \}
\]

\[
= o(\mathcal{F}(x)) = o \left( P\{ \sum_{n=1}^{\infty} \varphi_{n-1} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \} \right).
\]

This means that for any $0 < \delta < 1$, there exist sufficiently large $N_0$ and $x_1$ such that for all $x \geq x_1$,

\[
P\{ \sum_{i=N_0+1}^{\infty} \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \}
\]

\[
\leq \delta \mathcal{F}(x) \leq \delta P\{ \sum_{n=1}^{\infty} \varphi_{n-1} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \},
\]

(58)

and

\[
\sum_{i=N_0+1}^{\infty} P\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \}
\]

\[
\leq \delta \mathcal{F}(x) \leq \delta P\{ \sum_{n=1}^{\infty} \varphi_{n-1} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \},
\]

(59)

By applying Lemma 3.9, it holds that for the aforementioned large $N_0$,

\[
P\{ \sum_{i=1}^{N_0} \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \}
\]

\[
\geq \sum_{i=1}^{N_0} P\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \}.
\]

This, together with (59), implies that,

\[
P\{ \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \}
\]

\[
\geq P\{ \sum_{i=1}^{N_0} \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \}
\]

\[
\geq (1 - \delta) \sum_{i=1}^{\infty} P\{ \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \},
\]

which, by the arbitrariness of $\delta$, gives the asymptotic lower bound.
Then we proceed to the asymptotic upper bound. For any $0 < l < 1$ and $N_1 \geq 1$, we have
\[
P\left\{ \sum_{i=1}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\}
\leq P\left\{ \sum_{i=1}^{N_1} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > lx \right\}.
\]
\[
+ P\left\{ \sum_{i=N_1+1}^{\infty} \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > (1-l)x \right\} = K_1(x, N_1) + K_2(x, N_1).
\]
For any $0 < \rho < 1$, we have
\[
K_1(x, N_1) = P\left\{ \sum_{i=1}^{N_1} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > lx \right\}
\leq P\left\{ \bigcup_{i=0}^{N_1} (\varepsilon_i \partial(T) > \rho lx) \right\} + P\left\{ \sum_{i=0}^{N_1} \varepsilon_i \partial(T) > lx, \bigcap_{i=0}^{N_1} (\varepsilon_i \partial(T) \leq \rho lx) \right\}.
\]
Similar to the proof of the asymptotic upper bound for any fixed $N$ in Lemma 3.9, by letting $l \rho \to 1 (l \to 1)$, we can obtain that
\[
K_1(x, N_1) = (1 + o(1)) L_F^{-1} \sum_{i=1}^{N_1} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\}
\leq (1 + o(1)) L_F^{-1} \sum_{i=1}^{\infty} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\}.
\]
Also, similar to [58], first letting $x \to \infty$ and then $N_1 \to \infty$,
\[
K_2(x, N_1) = o \left( F(x) \right) = o \left( \sum_{i=1}^{\infty} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\} \right).
\]
This completes the proof of Lemma 3.10.

**Lemma 3.11.** Under the conditions of Theorem 2.1, it holds for all $0 < T < \infty$ with $P\{\tau_1 \leq T\} > 0$ that
\[
\lim_{N \to \infty} \limsup_{x \to \infty} \frac{\sum_{i=N+1}^{\infty} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\}}{F(x)} = 0. \tag{60}
\]
and
\[
\sum_{i=1}^{\infty} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\}
\sim \sum_{i=1}^{\infty} P\left\{ \varepsilon_i \sum_{n=i}^{\infty} \varphi_n i e^{-L_\theta(\tau_n)} I_{(\tau_n \leq T)} > x \right\}. \tag{61}
\]
Thus, we can express $\Psi(x,T)$ in (17), respectively, as

$$\Psi(x,T) = \sum_{i=1}^{N} P\{ \varepsilon_i \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n)} I(\tau_n \leq t) > x \} > x \}. \quad (62)$$

Similar to the proof of (57), it follows from (34) in Lemma 3.3 and (47) that for some $p > J_P^+$,

$$\sum_{i=N+1}^{\infty} P\{ \varepsilon_i \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n)} I(\tau_n \leq t) > x \}$$

$$\leq C \bar{F}(x) \sum_{i=N+1}^{\infty} \{ 2^p E \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n)} I(\tau_n \leq t) \} + E I \{ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n)} I(\tau_n \leq t) < \frac{1}{1} \}$$

$$\leq C \bar{F}(x) E[N_{N_1}(T)]^{4p+1} I(N_{N_1}(T) > N+1).$$

Then, (60) holds that as first $x \to \infty$ and then $N \to \infty$.

Hence, this, together with (59) and (62), implies that (61) holds for large $x$. \( \square \)

Now we turn to the proof of Theorem 2.1.

4. Proofs. Firstly, let us denote the discounted net loss process by

$$V_\theta(t) = x - e^{-L_a(t)} U_\theta(t) = \int_0^t e^{-L_a(v)} (dS_v - cdv), \; t \geq 0. \quad (63)$$

By substituting (1) and (6) into (63), we have for any $t \geq 0$,

$$V_\theta(t) = \sum_{n=1}^{\infty} X_n e^{-L_a(\tau_n)} I(\tau_n \leq t) - c \int_0^t e^{-L_a(v)} dv$$

$$= \sum_{n=1}^{\infty} e^{-L_a(\tau_n)} I(\tau_n \leq t) \sum_{i=1}^{n} \varphi_{n-i} \varepsilon_i + \sum_{n=1}^{\infty} e^{-L_a(\tau_n)} I(\tau_n \leq t) \varphi_n \varepsilon_0 - c \int_0^t \int_0^t e^{-L_a(v)} dv$$

$$= \sum_{i=1}^{\infty} \varepsilon_i \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_a(\tau_n)} I(\tau_n \leq t) + \varepsilon_0 \sum_{n=1}^{\infty} \varphi_n e^{-L_a(\tau_n)} I(\tau_n \leq t) - c \int_0^t e^{-L_a(v)} dv$$

$$= \sum_{i=0}^{\infty} \varepsilon_i \bar{\vartheta}_i(t) - cZ_t, \quad (64)$$

where $Z_t = \int_0^t e^{-L_a(s)} ds$.

Thus, we can express $\Psi(x,T)$ in (16) and $\Psi(x)$ in (17), respectively, as

$$\Psi(x,T) = \sum_{i=0}^{\infty} \varepsilon_i \bar{\vartheta}_i(s) - cZ_t, \quad (65)$$

and

$$\Psi(x) = \sum_{i=0}^{\infty} \varepsilon_i \bar{\vartheta}_i(s) - cZ_t.$$
4.1. Proof of Theorem 2.1.

Proof. Relations (64) and (65) imply
\[ P\{ \sum_{i=0}^{\infty} \varepsilon_i \vartheta_i(T) > x + cZ_\infty \} \leq \Psi(x, T) \leq P\{ \sum_{i=0}^{\infty} \varepsilon_i \vartheta_i(T) > x \}, \]
where \( Z_\infty = \int_{0}^{\infty} e^{-L_\theta(s)} ds \).

We first prove the asymptotic upper bound of (29). Note that by (56) and (61),
\[ \sum_{i=0}^{\infty} \vartheta_i(T) \leq \{ l < \}
\]
In the sequel, we derive the asymptotic lower bound of (29). For \( 0 < l < 1 \), it follows that
\[ \Psi(x, T) \geq P\{ \sum_{i=0}^{\infty} \varepsilon_i \vartheta_i(T) > x + cZ_\infty \} \]
\[ \geq P\{ \sum_{i=0}^{\infty} \varepsilon_i \vartheta_i(T) > x + cZ_\infty, cZ_\infty \leq lx \} \]
\[ \geq P\{ \sum_{i=0}^{\infty} \varepsilon_i \vartheta_i(T) > (1 + l)x \} - P\{ \sum_{i=0}^{\infty} \varepsilon_i \vartheta_i(T) > (1 + l)x, cZ_\infty > lx \}
\]
\[ = q_1(x, T) - q_2(x, T). \]
By Lemma 3.9, Lemma 3.6, (23) and Lemma 3.7, we have for sufficiently large \( N \),

\[
q_1(x, T) \geq P\left\{ \sum_{i=0}^{N} \varepsilon_i \vartheta_i(T) > (1 + l)x \right\} \geq \sum_{i=1}^{N} P\{ \varepsilon_i \vartheta_i(T) > (1 + l)x \}
\]

\[
\sim \sum_{i=1}^{N} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > (1 + l)x \}
\]

\[
= \sum_{i=1}^{N} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > (1 + l)x \}
\]

\[
\geq \sum_{i=1}^{N} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > (1 + l)x, \vartheta_i(Y_{\pi}^*, T) \leq g(x) \}
\]

\[
= \sum_{i=1}^{N} \int_{0}^{g(x)} P\{ \varepsilon_i^* > (1 + l)x/y \} P\{ \vartheta_i(Y_{\pi}^*, T) \in dy \}
\]

\[
\geq F_*(1 + l) \sum_{i=1}^{N} \int_{0}^{g(x)} P\{ \varepsilon_i^* > x/y \} P\{ \vartheta_i(Y_{\pi}^*, T) \in dy \}
\]

\[
\geq F_*(1 + l) \sum_{i=1}^{N} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > x \} - F_*(1 + l) \sum_{i=1}^{N} P\{ \vartheta_i(Y_{\pi}^*, T) > g(x) \}
\]

\[
= F_*(1 + l) \left( \sum_{i=1}^{\infty} - \sum_{i=N+1}^{\infty} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > x \} \right) - o(F(x))
\]

\[
= F_*(1 + l) \sum_{i=1}^{\infty} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > x \} - o \left( \sum_{i=1}^{\infty} P\{ \varepsilon_i^* \vartheta_i(Y_{\pi}^*, T) > x \} \right),
\]

where \( g(x) \) be the function defined in Lemma 3.7 and the last equation follows from (60).

Furthermore, (25) and (67) imply that

\[
\lim_{l_{\alpha} \to x} q_1(x, T) \geq L_F \int_{l_{\alpha}}^{T} P\{ \varepsilon^* \sum_{l=0}^{\infty} \varphi_l e^{-L_\beta (\tau_l + s)} I(\tau_l + s \leq T) > x \} d\lambda^*_s.
\]

Finally, we deal with \( q_2(x, T) \). It follows from Lemma 3.1, Lemma 3.5, (60), (61), and (67) that for some \( p > J_F^+ \),

\[
q_2(x, T) \leq P\{ c Z_{\infty} > l x \} \leq (l x)^{-p} E(Z_{\infty}^p) = O(x^{-p}) = o(F(x))
\]

\[
= o \left( \int_{0}^{T} P\{ \varepsilon^* \sum_{l=0}^{\infty} \varphi_l e^{-L_\beta (\tau_l + s)} I(\tau_l + s \leq T) > x \} d\lambda^*_s \right).
\]

Hence, the proof of Theorem 2.1 is completed. \( \square \)

4.2. Proof of Theorem 2.2.

**Proof.** If \( F \in \mathcal{ERV}(-\alpha, -\beta) \) for some \( 0 < \alpha \leq \beta < \infty \), by Theorem 2.1, then we can obtain relation (30) for all \( 0 < T < \infty \) with \( P\{ \tau_l \leq T \} > 0 \).

Now we consider the case that the results in Lemmas 3.6-3.11 also hold for \( T = \infty \) under the conditions of Theorem 2.2.

Firstly, we deal with the following inequalities, which are crucial to the proof of Theorem 2.2.
For $0 < p \leq 1$ and $q \geq 0$, by $C_r$ inequality and $\sup_{n \geq 0} \varphi_n < \infty$, we have
\[
\sum_{i=1}^{\infty} i^q \mathbb{E} \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta (\tau_n)} \right)^p \leq (\sup_{n \geq 0} \varphi_n)^p \sum_{i=1}^{\infty} i^q \sum_{n=i}^{\infty} \mathbb{E} (e^{-L_\theta (\tau_n)})^p = (\sup_{n \geq 0} \varphi_n)^p \sum_{n=1}^{\infty} \mathbb{E} (e^{-L_\theta (\tau_n)})^p \sum_{i=1}^{n} i^q \leq C \sum_{n=1}^{\infty} n^{q+1} \mathbb{E} (e^{-L_\theta (\tau_n)})^p.
\]

For $p > 1$, and $q \geq 0$, by Hölder’s inequality, we have
\[
\sum_{i=1}^{\infty} i^q \mathbb{E} \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta (\tau_n)} \right)^p \leq (\sup_{n \geq 0} \varphi_n)^p \sum_{i=1}^{\infty} i^q \sum_{n=i}^{\infty} n^{2p-2} \sum_{n=1}^{\infty} n^{2p-2} \mathbb{E} (e^{-L_\theta (\tau_n)})^p \leq C \sum_{i=1}^{\infty} i^q \sum_{n=i}^{\infty} n^{2p-2} \mathbb{E} (e^{-L_\theta (\tau_n)})^p \leq C \sum_{n=1}^{\infty} n^{2p+q-1} \mathbb{E} (e^{-L_\theta (\tau_n)})^p.
\]

For any fixed $\theta \in [0, 1)$, by (14), (15) and the definition of $\kappa_\theta$, we obtain $\psi_\theta(p) < 0$ for any $0 < p < \kappa_\theta$. From (15), we have
\[
\mathbb{E} e^{-pL_\theta (\tau_1)} = \mathbb{E} e^{\tau_1 \psi_\theta(p)} < 1.
\]

Then, by independent and stationary increments of Lévy process $\{L_\theta(t), t \geq 0\}$, we can get for any $0 < p < \kappa_\theta$ and any $q \geq 0$,
\[
\sum_{i=1}^{\infty} i^q \mathbb{E} \left( \sum_{n=i}^{\infty} \varphi_{n-i} e^{-L_\theta (\tau_n)} \right)^p \leq C \sum_{n=1}^{\infty} n^{2p+q+1} \mathbb{E} (e^{-L_\theta (\tau_n)})^p = C \sum_{n=1}^{\infty} n^{2p+q+1} (\mathbb{E} e^{-pL_\theta (\tau_1)})^n < \infty. \tag{68}
\]

Take $J^+_{p} < p < \kappa_\theta$. As for the first term on the right-hand side of (39) in Lemma 3.6, by relation (68) and independence among $\xi^*_i$, $Y^*_i$, $\{Y_n, n \geq 1\}$ and $\{L_\theta(t), t \geq 0\}$, lemma 3.2 implies that the distribution of $\xi^*_i \vartheta_i(Y^*_i, T)$ belongs to the class $D$ and
\[
P\{\xi^*_i \vartheta_i(Y^*_i, T) > x\} \asymp \mathcal{F}(x).
\]

Because of relations $|\varphi_2(y)| \leq b_2 - 1$ and (28), we get $\mathbb{E}[e^{-pL_\theta (\hat{Y}_i^*)}] \leq \mathbb{E}[e^{-pL_\theta (\tau_1)}] < 1$.

Similarly, we can get that distributions of $\hat{\xi}^*_i \vartheta_i(Y^*_i, T), \xi^*_i \vartheta_i(\hat{Y}^*_i, T), \hat{\xi}^*_i \vartheta_i(\hat{Y}^*_i, T)$ also belong to the class $D$ and
\[
P\{\hat{\xi}^*_i \vartheta_i(Y^*_i, T) > x\} \asymp \mathcal{F}(x), \ P\{\xi^*_i \vartheta_i(Y^*_i, T) > x\} \asymp \mathcal{F}(x), \ P\{\hat{\xi}^*_i \vartheta_i(\hat{Y}^*_i, T) > x\} \asymp \mathcal{F}(x).
\]
Similar to the proof of relation (68), relation (27) and the boundness of $\phi_2(y)$ imply for any $0 < p < \kappa_\theta$ and any $q \geq 0$,

$$\sum_{i=1}^{\infty} i^q E \left( \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\theta (\sum_{k \neq i} Y_k + Y_*^i)} \right)^p \leq C \sum_{n=1}^{\infty} n^{2p+q+1} E \left( e^{-p L_\theta (\sum_{k \neq i} Y_k + Y_*^i)} \right)$$

$$\leq C \sum_{n=1}^{\infty} n^{2p+q+1} E (e^{-p L_\theta(Y_*^i)})^n < C \sum_{n=1}^{\infty} n^{p+q+1} \sum_{i=1}^{\infty} \varphi_{n-i} e^{-p L_\theta(Y_*^i)} < \infty.$$  

(69)

The remaining results in Lemma 3.6 also hold for $T = \infty$. Hence, Lemma 3.6 also holds for $T = \infty$ under the conditions of Theorem 2.2.

Take $J^T_F < p < \kappa_\theta$. By (68) and (69), the results in Lemmas 3.7-3.9 also hold for $T = \infty$ under the conditions of Theorem 2.2.

Next we deal with Lemmas 3.10 and 3.11 for $T = \infty$. As for $J_1(x, N)$ in Lemma 3.10, by (35) in Lemma 3.3 and (68), it can be shown that for any fixed $p_1 > 0$ and $p_2 > 0$ such that $0 < p_1 < \alpha \leq J_F^T \leq \beta < p_2 < \kappa_\theta \leq \infty$, all large $x$ and sufficiently large $N$,

$$J_1(x, N) \leq C \mathcal{F}(x)(1 + \pi b_1 b_2) \sum_{i=N+1}^{\infty} \left\{ i^{2p_1} E \left[ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\theta(\sum_{k \neq i} Y_k + Y_*^i)} \right] \right\}^{p_1} \nonumber$$

$$+ i^{2p_2} E \left[ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\theta(\sum_{k \neq i} Y_k + Y_*^i)} \right]^{p_2} \nonumber$$

$$= C \mathcal{F}(x)(1 + \pi b_1 b_2) \sum_{i=N+1}^{\infty} \left\{ i^{2p_1} E \left[ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} \right] \right\}^{p_1} \nonumber$$

$$+ i^{2p_2} E \left[ \sum_{n=1}^{\infty} \varphi_{n-i} e^{-L_\theta(\tau_n)} \right]^{p_2} \nonumber$$

$$= o(\mathcal{F}(x)).$$

Similarly, we obtain that as first $x \to \infty$ and then $N \to \infty$,

$$J_2(x, N) = J_3(x, N) = J_4(x, N) = o(\mathcal{F}(x)).$$

The remaining results in Lemma 3.10 also hold for $T = \infty$. Hence, relation (68) also holds for $T = \infty$ under the conditions of Theorem 2.2. Finally, we consider that (60) and (61) in Lemma 3.11 also hold for $T = \infty$ under the conditions of Theorem 2.2. As for (60), it follows from (35) in Lemma 3.3 and (68) that for any fixed $p_1 > 0$ and $p_2 > 0$ such that $0 < p_1 < \alpha \leq J_F^T \leq \beta < p_2 < \kappa_\theta \leq \infty$, all large $x$ and sufficiently large $N$,
The remaining results in Lemma 3.11 also hold for $T = \infty$. Hence, relation (61) also holds for $T = \infty$ under the conditions of Theorem 2.2. By the above results, we immediately obtain relation (30) also holds for $T = \infty$ under the conditions of Theorem 2.2.

4.3. Proof of Theorem 2.3.

Proof. Since $F \in \mathcal{R}_{\alpha}$, by Theorem 2.2 and Breiman’s theorem (see Breiman (1965)), we immediately obtain (31).

4.4. Proof of Corollary 2.2.

Proof. From (31), and independent and stationary increments of Lévy process \( \{L_\theta(t), t \geq 0\} \), we can derive that for $T = \infty$,

\[
\Psi(x) \sim F(x) \int_0^\infty E \left( \sum_{i=0}^\infty \varphi_i e^{-L_\theta(s+\tau_i)} \right)^\alpha d\lambda_s^*
\]

\[
= F(x) \int_0^\infty e^{-\alpha L_\theta(s)} E \left( \sum_{i=0}^\infty \varphi_i e^{-[L_\theta(s+\tau_i) - L_\theta(s)]} \right)^\alpha d\lambda_s^*
\]

\[
= F(x) E \left( \sum_{i=0}^\infty \varphi_i e^{-L_\theta(\tau_i)} \right)^\alpha \int_0^\infty e^{\psi_\theta(\alpha)} d\lambda_s^*
\]

\[
= F(x) E \left( \sum_{i=0}^\infty \varphi_i e^{-L_\theta(\tau_i)} \right)^\alpha \frac{E \left[ (1 + \pi d_1 \phi_2(\tau_1) e^{\tau_1 \psi_\theta(\alpha)}) \right]}{1 - E e^{\tau_1 \psi_\theta(\alpha)}}.
\]

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