Exactly self-similar blow-up of the generalized De Gregorio equation

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Abstract
We study exactly self-similar blow-up profiles for the generalized De Gregorio model for the three-dimensional Euler equation:

\[ u_t + au \cdot \nabla u = -p, \quad \nabla \cdot u = 0, \]

where \( u(x,t) \) is the velocity field and \( p(x,t) \) is the scalar pressure. The vanishing of \( \nabla \cdot u \) captures the incompressibility condition. The global wellposedness of the Euler equation in three dimensions for smooth and decaying initial data is wide open, attracting a great deal of research efforts. The interested reader is referred to the surveys [1, 5, 10, 11].

Let \( \omega = \nabla \times u \) be the vorticity, which then satisfy

\[ \omega_t + (u \cdot \nabla) \omega = \omega \cdot \nabla u. \]

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1. Introduction

The most fundamental equation concerning the motion of inviscid incompressible fluids in three-dimensional space is the Euler equation:

\[ u_t + u \cdot \nabla u = -p, \quad \nabla \cdot u = 0, \]

where \( u(x,t) \) is the velocity field and \( p(x,t) \) is the scalar pressure. The vanishing of \( \nabla \cdot u \) captures the incompressibility condition. The global wellposedness of the Euler equation in three dimensions for smooth and decaying initial data is wide open, attracting a great deal of research efforts. The interested reader is referred to the surveys [1, 5, 10, 11].

Let \( \omega = \nabla \times u \) be the vorticity, which then satisfy

\[ \omega_t + (u \cdot \nabla) \omega = \omega \cdot \nabla u. \]
The second term on the left-hand side, known as the advection term, has the effect of transporting the vorticity. Since \( u \) is divergence free, the advection term will not affect the \( L^p \) norms of the vorticity. The first term on the right-hand side, known as the vortex stretching term, is only present in the three-dimensional case. Schematically \( \nabla u \approx \omega \), so the vortex stretching term may be as bad as \( \omega^2 \) in the worst-case scenario, and may cause blow-up of the equation.

1.1. The generalized De Gregorio model

In [6, 7] De Gregorio proposed a one-dimensional equation to model the competition between advection and vortex stretching in the Euler equation. The equation belongs to the family

\[ w_t + auw_x = u_xw, \quad u_t = Hw \]

where \( H \) denotes the Hilbert transform, and \( a \) is a real parameter quantifying the relative strength of advection, modelled by \( uw_x \), and vortex stretching, modelled by \( u_xw \). Note that the relation \( u_t = Hw \) also mimics the Biot–Savart law relating the velocity and the vorticity. It turns out that this equation also models a variety of other equations, including the surface quasi-geostrophic equation, see [9]. Other similar 1D models of the Euler equation can be found in [13].

What De Gregorio studied in [6] is the case \( a = 1 \), which mirrors the Euler equation. The special case \( a = 0 \) had appeared in Constantin–Lax–Majda [4] and had been known to develop finite-time singularity for all non-trivial initial data. The generalization to arbitrary \( a \) was done by Okamoto–Sakajo–Wunsch [16]. When \( a < 0 \), advection and vortex stretching cooperate to cause a blow-up, as shown in [2]. When \( a > 0 \), advection can possibly tame the growth brought by vortex stretching and the picture is more interesting. Smooth blow-up solutions for small \( a \) were found by Elgindi–Jeong [9], Elgindi–Ghoul–Masmoudi [8] and Chen–Hou–Huang [3], and for \( a \leq 1 \) by Chen–Hou–Huang [3] and [12]. For \( C^\alpha \) solutions with \(|a\alpha| \) small enough, advection is not strong enough to inhibit vortex stretching, and blow-ups are also constructed in the above-cited papers. A numerical investigation of the behaviour of the solution with different values of \( a \) can be found in Lushnikov–Silantyev–Siegel [15].

1.2. The result

This note improves on the known results on the \( C^\alpha \) blow-ups. The ones constructed in [9] is exactly self-similar, i.e. of the form

\[ w(x,t) = \frac{1}{1-t} W \left( \frac{x}{(1-t)^{(1+\lambda)/\alpha}} \right) \]

where \( W \in C^\alpha \), but with the restriction that \( 1/\alpha \in \mathbb{Z} \). The construction in [3] works for all \( \alpha \in (0,1) \), as long as \(|a\alpha| \ll 1 \), but the solution is not exactly self-similar, but only asymptotically so. In this note we fill the gap by constructing exactly self-similar blow-up solutions for all \( \alpha \in (0,1) \), provided that \(|a\alpha| \ll 1 \). Specifically we show that

**Theorem 1.** There is \( c > 0 \) such that if \(|a\alpha| < c \), then there are \( W(\cdot; a) \in C^\alpha \) and \( \lambda(a) \in \mathbb{R} \) such that

\[ w(x,t) = \frac{1}{1-t} W \left( \frac{x}{(1-t)^{(1+\lambda(a))/\alpha}}; a \right) \]

is a self-similar solution. Moreover, \( W(\cdot; a) \) and \( \lambda(a) \) are analytic in \( a \).
1.3. The method

We mostly follow [9]. Plugging the ansatz in the equation

\[ w_t + uaw_x = u_x, \quad u_x = Hw = -|\nabla|^{-1}w_x \]

we get the steady-state equation for \( W \):

\[ F := \left( 1 + \frac{\lambda}{\alpha} \right) x + aU \right) W_x + (1 - U_x)W = 0, \quad U_x = HW. \]

The explicit solution when \( a = 0 \) (see (4.2) and (4.3) of [9])

\[ (W, \lambda) = \left( -\frac{2\sin(\alpha\pi/2)|x|^{\alpha} \text{sgn} x}{|x|^{2\alpha} + 2\cos(\alpha\pi/2)|x|^{\alpha} + 1}, 0 \right), \]

\[ U_x = HW = \frac{2(\cos(\alpha\pi/2)|x|^{\alpha} + 1)}{|x|^{2\alpha} + 2\cos(\alpha\pi/2)|x|^{\alpha} + 1}. \]

\( W \) is odd and differentiable with respect to \( |x|^{\alpha} \). It suggests the change of variable \( f(x) = f(x^{1/\alpha}) \), and the need to study the Hilbert transform

\[ \tilde{Hf}(x) = Hf(x^{1/\alpha}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)dy}{x^{1/\alpha} - y} = \frac{1}{\pi} \int_{0}^{\infty} \frac{2yf(y)dy}{x^{2/\alpha} - y^2} \]

\[ = \frac{1}{\alpha\pi} \int_{0}^{\infty} \frac{2^{2/\alpha - 1}f(z)dz}{x^{2/\alpha} - z^{2/\alpha}} \quad \text{(with f odd and z = y}^{\alpha}\text{)} \]

in the new variable. In terms of the new variable we need to solve

\[ F = (1 + \lambda)x\tilde{W}_x + aU\tilde{W}_x + (1 - \tilde{HW})\tilde{W} = 0, \quad U_x = HW. \]

In section 3 we will generalize the estimates in [9] from the discrete range \( 1/\alpha \in \mathbb{Z} \) to the full range \( \alpha \in (0, 1) \), using some delicate analysis of the integral kernel. Armed with these estimates, in section 4 we will use the implicit function theorem to show that a perturbation of the explicit solution above exists as long as \( |\alpha| \) is small enough. The argument mostly follows [9], but it is easier if we let \( F \) map \( H^2 := \{ W \in H^1 : x W_x \in H^2 \} \) to \( H^2 \). This way all terms in \( F \) are automatically in \( H^2 \): for the most difficult term \( \tilde{U} W_x = (U/x)(x W_x) \) we will need the Hardy inequalities collected in section 2. Then we only need the \( H^2 \to H^2 \) bound of \( (dF)^{-1} \) for the implicit function theorem to work and give us the desired solution for small \( |\alpha| \).

2. Hardy inequalities

Here we record some useful Hardy-type inequalities in Sobolev spaces.

**Lemma 1** ([14], theorem 4.3). If \( p \in (1, \infty) \), \( \gamma < (p - 1)/p \), \( k \geq 1 \) and \( f(0) = \cdots = f^{(k-1)}(0) = 0 \), then

\[ \|x^{\gamma-k}f(x)\|_{L^p} \lesssim_{k,p,\gamma} \|x^{\gamma}f^{(k)}\|_{L^p}. \]

**Definition 1.** For any integer \( k \geq 0 \), any number \( p \in (1, \infty) \) and any weight \( w(x) \geq 0 \) define

\[ \|f\|_{W^k,p(w)} = \sum_{j=0}^{\infty} \|w^{(j)}f^{(j)}\|_{L^p}, \quad \|f\|_{W^k,p(w)} = \|f\|_{W^k,p(w)} + \|xf^{(k)}(x)\|_{W^{k-1,p}(w)}. \]

Let \( W^{k,p}(w) \) and \( W^{k,p}_o(w) \) denote the subspace of odd functions in \( W^{k,p}(w) \) and \( W^{k,p}(w) \), respectively.
Definition 2.

\[ I[f(x)] = \int_0^x \frac{f(y) - f(0) - yf'(0)}{y^2} \, dy. \]

Lemma 2. If \( k \geq 0 \) and \( p \in (1, \infty) \) then

\[ \| (f')_j \|_{W^{k,p}} \lesssim_{k,p,\gamma} \| f'' \|_{W^{k,p}} \lesssim \| f \|_{W^{k+2,p}}. \]

Proof. Without loss of generality assume \( \| f(0) \| = \| f'(0) \| = 0 \). Then the result follows from Lemma 1 because \( (x^{-2} f(x))^{(k)} \) is a linear combination of \( x^{-2} f^{(k-j)}(x) \), \( 0 \leq j \leq k \).

Lemma 3. If \( k \geq 0 \) and \( p \in (1, \infty) \) then

\[ \left\| g(x) \int_0^x f(y) \, dy \right\|_{W^{k,p}} \lesssim_k \sum_{l=0}^k \left\| g^{(l)}(x) \int_0^x f(y) \, dy \right\|_{L^p} + \sum_{n+l \leq k} \left\| g^{(n)} f^{(l-1)} \right\|_{L^p} \lesssim_{k,p,\gamma} \sum_{l=0}^k \sup_x |x g^{(l)}(x)| \| f \|_{L^p} + 1_k \| g \|_{C^{k-1}} \| f \|_{W^{k-1,p}}. \]

(by lemma 1)

Remark 1. If we discount the term where all the derivatives fall on the integral, we have \( 1 \leq n \leq k - 1 \) in the summation, so we only need the \( W^{k-2,p} \) norm of \( f \).

Lemma 4. If \( k \geq 0 \) and \( p \in (1, \infty) \) then

\[ \left\| \frac{x f(x)}{1 + x^2} \right\|_{W^{k,p}} \lesssim_{k,p,\gamma} \| f'' \|_{W^{\max(k-1,1),p}} \lesssim \| f \|_{W^{\max(k,2),p}}. \]

Proof. By Lemma 3 (note that we have subtracted the term with all \( k \) derivatives hitting \( f' \)) and Lemma 2,

\[ \left\| \frac{d^k x f(x)}{dx^k 1 + x^2} - \frac{x f^{(k)}(x)}{1 + x^2} \right\|_{L^p} \lesssim_{k,p,\gamma} \| (f')_j \|_{W^{\max(k-2,0),p}} \lesssim_{k,p,\gamma} \| f'' \|_{W^{\max(k-1,1),p}}. \]

If \( k = 0 \) then there is nothing more to prove. If \( k \geq 1 \) then similarly

\[ \left\| \frac{d^{k-1} x f^{(k)}(x)}{dx^{k-1} 1 + x^2} - \frac{x f^{(k)}(x)}{1 + x^2} \right\|_{L^p} \lesssim_{k,p,\gamma} \| f'' \|_{W^{\max(k-1,1),p}}. \]

Also,

\[ \left\| \frac{x f'(x)}{1 + x^2} \right\|_{W^{k-1,p}} \lesssim_k \| f(x) - f(0) \|_{W^{k-1,p}} - \frac{f'(0)}{1 + x^2} \| x f'(0) \|_{W^{k-1,p}} \lesssim_k \| 1/(1 + x^2) \|_{C^{k-1}} \| (f(x) - f(0))/x \|_{W^{k-1,p}} + \| 1/(1 + x^2) \|_{W^{k-1,p}} \| f'(0) \|_{W^{k-1,p}} \lesssim_{k,p,\gamma} \| f'' \|_{W^{\max(k-1,1),p}}. \]

(by lemma 1 and W1,p \( \subset C^0 \))
3. Hilbert transform on Hölder functions

In this section we generalize the bounds for the Hilbert transform in [9].

**Definition 3.** For a function \( f \) let \( \tilde{f}(x) = f(x^{1/\alpha}) \) (\( x \geq 0 \)).

For example,

\[
\tilde{W}(x) = -\frac{2\sin(\alpha\pi/2)x}{x^2 + 2\cos(\alpha\pi/2)x + 1}, \quad \tilde{H}W(x) = \frac{2(\cos(\alpha\pi/2)x + 1)}{x^2 + 2\cos(\alpha\pi/2)x + 1}.
\]

**Remark 2.** In this section we only consider functions defined on \( \{x : x \geq 0\} \), unless stated otherwise.

**Definition 4.** For \( r \geq 1 \) (not necessarily an integer) define

\[
H^r(f)(x) = \frac{1}{\pi} \int_0^\infty \frac{2y^{2r-1}}{x^r - y^r} f(y) dy
\]

and

\[
\tilde{H}f = H^{(1/\alpha)}f
\]

so that

\[
\tilde{H}f(x) = (HEf(\cdot, x))(x^{1/\alpha})
\]

where \( Ef(x) = f(|x|) \, \text{sgn} \, x \) is the odd extension of \( f \). Then for odd \( V \),

\[
\tilde{H}V(x) = HEV(x^{1/\alpha}) = HV(x^{1/\alpha}) = \tilde{H}V(x).
\]

The bound on the operator \( H^r \) on Sobolev spaces is the key linear estimate underlying the proof of [9], which only treats the case \( r \in \mathbb{N}^+ \), using partial fraction decomposition of the integrand. Now we treat the general case by bounding the integrand (and its derivatives) directly. First we need some elementary inequalities.

**Lemma 5.** If \( r \geq 1 \) and \( t \geq 0 \) then

\[
\frac{2t^{2r-1}}{1 - t^2} \leq \frac{2t}{1 - t^2} \leq \frac{2rt^{2r-1}}{1 - t^2} \leq \frac{2r^{2r-1}}{1 - t^2} + 2r.
\]

**Proof.** First we show the first two inequalities. Clearing the denominator and cancelling the factor \( 2t \) give \( r^{2r-2} - rt^2 \leq 1 - t^2 \leq r - rt^2 \). The first inequality is equivalent to \( (r - 1)t^2 + 1 \geq r^{2r-2} \), and the second equivalent to \( t^2 + r - 1 \geq rt^2 \). Both follow from Young’s inequality. The last inequality is nothing but \( |r - t^{2r-1}| \leq |1 - t^2| \), which holds because \( t \) and \( t^{2r-1} \) are always between 1 and \( t^2 \). \( \square \)

**Lemma 6.** For \( r \geq 1 \) we have \( \|H^r\|_{L^2 \rightarrow L^2} \leq Cr \) for some constant \( C \).

**Proof.** We have

\[
H^r(f)(x) = \frac{1}{\pi} \int_0^\infty \frac{2yf(y)dy}{x^r - y^r} = \frac{1}{\pi} \int_0^\infty \left( \frac{1}{x-y} - \frac{1}{x+y} \right) f(y)dy = HEf(x)
\]

where \( Ef(x) = f(|x|) \, \text{sgn} \, x \), so \( H^{(1)} \) is an isometry on \( L^2(\mathbb{R}^+) \).
For general $r$ we have
\[ \pi H^{(r)}f(x) = \int_0^\infty \frac{2y^{2r-1}}{x^{2r} - y^{2r}} f(y) \, dy = \int_0^\infty \frac{2t^{2r-1}}{1 - t^2} f(tx) \, dt \quad (y = tx) \]
so
\[ \pi (H^{(r)} - H^{(1)})f(x) = \int_0^\infty \left( \frac{2t^{2r-1}}{1 - t^2} - \frac{2t}{1 - t^2} \right) f(tx) \, dt. \]

Since $\|f(tx)\|_{L^2} = \|f\|_{L^2}/\sqrt{t}$, we have
\[ \|H^{(r)}\|_{L^2 \to L^2} \leq 1 + \frac{1}{\pi} \int_0^\infty \left( \frac{2t}{1 - t^2} - \frac{2r^{2r-1}}{r^{2r} - 1} \right) \frac{dt}{\sqrt{t}}. \]

Note that the integrand is nonnegative by lemma 5. By the same lemma,
\[ \int_0^2 \left( \frac{2t}{1 - t^2} - \frac{2r^{2r-1}}{r^{2r} - 1} \right) \frac{dt}{\sqrt{t}} \leq 2r \int_0^2 \frac{dt}{\sqrt{t}} = 4\sqrt{2}r. \]

For $t \geq 2$,
\[ \frac{2t}{1 - t^2} - \frac{2r^{2r-1}}{r^{2r} - 1} \leq \frac{8r}{3r} \]
so
\[ \int_2^\infty \left( \frac{2t}{1 - t^2} - \frac{2r^{2r-1}}{r^{2r} - 1} \right) \frac{dt}{\sqrt{t}} \leq \int_2^\infty \frac{8rdt}{3r\sqrt{t}} = \frac{8\sqrt{2}}{3} r \]
and then
\[ \|H^{(r)}\|_{L^2 \to L^2} \leq 1 + \frac{20\sqrt{2}}{3\pi} r \leq \left( 1 + \frac{20\sqrt{2}}{3\pi} \right) r. \]

\[ \square \]

**Lemma 7.** For $r \geq 1$, if $f(0) = 0$ then for $k = 1, 2$ and $x \neq 0$,
\[ (H^{(r)}f)^{(k)}(x) = \frac{1}{\pi} \int_0^\infty \frac{2rx^{2r-k} y^{k-1}}{x^{2r} - y^{2r}} f^{(k)}(y) \, dy. \]

**Proof.** Using $H^{(1)}f = HEf$, where $Ef(x) = f(|x|) \text{ sgn} x$, we see that the identity holds for $r = 1$.
For $r > 1$ we have (note the singularity of the kernel has been subtracted)
\[ \pi ((H^{(r)} - H^{(1)})f)'(x) = \int_0^\infty \partial_x \left( \frac{2y^{2r-1}}{x^{2r} - y^{2r}} - \frac{2y^{2r}}{x^{2r} - y^{2r}} \right) f(y) \, dy. \]
By Euler’s theorem on homogeneous functions ($xF_x + yF_y = 0$ for $F$ homogeneous of degree 0) applied to the starred equality,
\[ \partial_x \frac{y^{2r}}{x^{2r} - y^{2r}} = -\partial_x \frac{x^{2r}}{x^{2r} - y^{2r}} = \frac{x^{2r}}{x^{2r} - y^{2r}} \]
so
\[ \pi ((H^{(r)} - H^{(1)})f)'(x) = -\int_0^\infty \partial_x \left( \frac{2y^{2r-1}}{x^{2r} - y^{2r}} - \frac{2x}{x^3 - y^3} \right) f(y) \, dy. \]
Since \( f(0) = 0 \) we can integrate by parts to get the identity for \( k = 1 \).

Similarly,
\[
\frac{\partial_x}{x^\nu - y^\nu} x^{2r-1} = \frac{1}{y} \frac{\partial_y}{x^\nu - y^\nu} x^{2r-1} = -\frac{1}{x} \frac{\partial_x}{x^\nu - y^\nu} x^{2r-2} = \frac{\partial_y}{x^\nu - y^\nu} x^{2r-2}
\]
so
\[
\pi((H^{(r)} - H^{(1)})f)''(x) = -\int_0^\infty \partial_y \left( \frac{2xy^{2r-2}}{x^\nu - y^\nu} - \frac{2y}{x^2 - y^2} \right) f'(y)dy.
\]
Integrating by parts we get the identity for \( k = 2 \). This time we do not need \( f'(0) = 0 \) because the parenthesis vanishes at \( y = 0 \).

\[
\square
\]

**Lemma 8.** For \( r \geq 1 \), if \( f(0) = 0 \) then \( \| (H^{(r)}f)' \|_{L^2} \leq Cr'' \| f'' \|_{L^2} \) for some constant \( C \).

**Proof.** Since \( f(0) = 0 \), taking the second derivative commutes with \( E \) and \( H \). Then \((H^{(1)}f)'' = (HEf)'' = HEf'' = H^{(1)}(f'')', \) so \( \| (H^{(1)}f)'' \|_{L^2} = \| H^{(1)}(f'') \|_{L^2} = \| f'' \|_{L^2} \) because \( H^{(1)} \) is an isometry on \( L^2(\mathbb{R}^+) \).

For \( r > 1 \), we change variable as before to get
\[
\| (H^{(r)}f)'' \|_{L^2} \leq \left( 1 + \frac{1}{\pi} \int_0^\infty \left( \frac{2rt}{1 - t^2} - \frac{2t}{1 - t^2} \right) \frac{dt}{\sqrt{t}} \right) \| f'' \|_{H^r}.
\]
Note that the integrand is nonnegative by lemma 5. By the same lemma,
\[
\int_0^2 \left( \frac{2rt}{1 - t^2} - \frac{2t}{1 - t^2} \right) \frac{dt}{\sqrt{t}} \leq 2r \int_0^2 \frac{dt}{\sqrt{t}} = 4\sqrt{2}r.
\]
For \( t \geq 2 \),
\[
\frac{2rt}{1 - t^2} - \frac{2t}{1 - t^2} \leq \frac{2t}{t^2 - 1} \leq \frac{8}{3t}
\]
so
\[
\int_2^\infty \left( \frac{2rt}{1 - t^2} - \frac{2t}{1 - t^2} \right) \frac{dt}{\sqrt{t}} \leq \int_2^\infty \frac{8dt}{3\sqrt{t}} = \frac{8\sqrt{2}}{3}
\]
and then
\[
\| (H^{(r)}f)'' \|_{L^2} \leq \left( 1 + \frac{1}{\pi} \left( 4\sqrt{2}r + \frac{8\sqrt{2}}{3} \right) \right) \| f'' \|_{L^2} \leq \left( 1 + \frac{20\sqrt{2}}{3\pi} \right) r'' \| f'' \|_{H^r}.
\]
\[
\square
\]

By lemmas 6 and 8 we get

**Lemma 9.** For \( r \geq 1 \) we have \( \| H^{(r)} \|_{H^r_{0} \rightarrow H^r} \leq Cr \) for some constant \( C \), where \( H^r_0 \) denotes the space of \( H^r \) functions vanishing at \( 0 \).

The estimates in this section provide a substitute for lemma A.5 of [9]. They work for all Hölder exponents \( 0 < \alpha \leq 1 \), but we only need up to two derivatives, as opposed three in [9]. Two derivatives suffice to close the estimate, as can be seen in lemma 4 above (take \( k = 2 \)) and lemma 11 below (\( \| f''(0) \| \leq \| f'' \|_{H^r} \)). The rest of the proof morally follows the steps in [9], but here we work out the estimates in detail for the sake of completeness, as well as a slight different choice of function spaces.
4. Hölder steady states for nonzero a

We first define the spaces which we are going to work with.

**Definition 5.** Let \( \dot{H}^2 = \{ f \in H^2 : x f, \dot{f} \in H^2 \} \) with \( \|f\|_{\dot{H}^2} = \|f\|_{H^2} + \|\dot{f}\|_{H^2} \). We subscript a space by 0 to indicate the subspace of functions that vanish at 0. For example, \( H^2_0 = \{ f \in H^2 : f(0) = 0 \} \).

By lemma 2.2 of [9], for \( 0 < \alpha \leq 1 \) and \( f \in H^2_0 \) we have \( (\dot{H}f)_x(0) = f_x(0) \cot(\alpha \pi/2) \).

**Definition 6.** Let \( X = \{ V \in H^2_0 : V_x(0) = 0 \} \) and \( Y = \{ V \in H^2_0 : V_x(0) + 2 \sin(\alpha \pi/2) \dot{H}V(0) = 0 \} \).

We will solve the steady-state equation \( F = 0 \) (see section 1.3) using the implicit function theorem, so we need to find its differential.

\[
\frac{dF}{d(\bar{W}, 0, 0)}(V, \mu, 0) = LV + \mu x \bar{W}
\]

where

\[
LV = V + xV_x - \bar{W}HV - VH\bar{W}.
\]

**Lemma 10.** If \( 0 < \alpha \leq 1 \), then \( L \) is an isomorphism from \( X \) to \( Y \) and \( L^{-1}f(x) = \frac{x(1 - x^2) \sin(\alpha \pi/2)}{(1 + 2x \cos(\alpha \pi/2) + x^2)^2} \)

\[
\times \int_0^x \frac{1 - y^2}{y} \sin \frac{\alpha \pi}{2} g(y) dy + \left( \frac{1 + y^2}{y} \cos \frac{\alpha \pi}{2} + \frac{2}{\sin \frac{\alpha \pi}{2}} \right) h(y) dy
\]

\[
- \frac{x((1 + x^2) \cos(\alpha \pi/2) + 2x)}{(1 + 2x \cos(\alpha \pi/2) + x^2)^2}
\]

\[
\times \int_0^x \left( \frac{1 + y^2}{y} \cos \frac{\alpha \pi}{2} + \frac{2}{\sin \frac{\alpha \pi}{2}} \right) g(y) + \frac{1 - y^2}{y} \sin \frac{\alpha \pi}{2} h(y) dy
\]

where

\[
g(x) = \frac{f(x)}{x} - \frac{f'(0)}{1 + 2x \cos(\alpha \pi/2) + x^2},
\]

and

\[
h(x) = \frac{\dot{H}f(x)}{x} - \frac{(1 - x^2) \dot{H}f(0)}{x(1 + 2x \cos(\alpha \pi/2) + x^2)}
\]

\[
= \frac{\dot{H}f(x) - \dot{H}f(0)}{x} + \frac{2(\cos(\alpha \pi/2) + x)}{1 + 2x \cos(\alpha \pi/2) + x^2} \dot{H}f(0)
\]

\[
= \frac{\dot{H}f(x) - \dot{H}f(0) - x(\dot{H}f)'(0)}{x} - \frac{2x(\cos \alpha \pi + x \cos(\alpha \pi/2))}{1 + 2x \cos(\alpha \pi/2) + x^2} \dot{H}f(0).
\]

The operator norm of \( L^{-1} \) is bounded, uniformly in \( \alpha \).

This will be proved before the proof of lemma 13.
Lemma 11. For $0 < \alpha \leq 1$ and $n = 0, \pm 1$,
\[
T_{n,\alpha f} = \frac{x}{1 + x^2} \int_0^x y^n \left( \frac{f(y)}{y} - \frac{f'(0)}{1 + 2y\cos(\alpha \pi / 2) + y^2} \right) dy,
\]
\[
S_{n,\alpha f} = \frac{x^{1-\max(n,0)}}{1 + x^2} \int_0^x y^n \left( \frac{Hf(y) - Hf(0)}{y} - \frac{2(\cos(\alpha \pi / 2) + y)H(0)}{1 + 2y\cos(\alpha \pi / 2) + y^2} \right) dy
\]
are bounded from $Y$ to $H^2_\alpha$, with $\|T_{n,\alpha}\|_{Y \to H^2} \leq C$ and $\|S_{n,\alpha}\|_{Y \to H^2} \leq C/\alpha$.

Proof. Clearly $T_{n,\alpha f}(0) = S_{n,\alpha f}(0) = 0$, so it remains to bound the $H^2$ norm.

For $T_{n,1}$, when $n = -1$ we have
\[
T_{-1,f} = \frac{xf(x)}{1 + x^2} + \frac{x f'(0)}{1 + x^2} \int_0^x y dy = \frac{xf(x)}{1 + x^2} + \frac{x \ln(1 + x^2) f'(0)}{2(1 + x^2)}.
\]
The first term is bounded by lemma 4, while the second has the desired bound because $x\ln(1 + x^2)/(1 + x^2) \in H^2$ and $|f'(0)| \lesssim \|f\|_{H^2}$. For $n = 0, 1$, we use the old integrand. By lemma 3, Hardy’s inequality and lemma 9, the contribution of the first term in $T_{0,1}$ is $\lesssim \|f(x)/x\|_{H^2} \lesssim \|f\|_{H^2}$, and the contribution of the first term in $T_{1,1}$ is $\lesssim \|f\|_{H^2}$. For the second term in $T_{n,1}$ we have
\[
\frac{x}{1 + x^2} \int_0^x y^n dy = \frac{x}{1 + x^2} \left( \arctan x \text{ resp. } \frac{1}{2} \ln(1 + x^2) \right) \in H^2
\]
so it has the desired bound because $|f'(0)| \lesssim \|f\|_{H^2}$.

For $T_{n,\alpha}$, we have
\[
(T_{n,1} - T_{n,\alpha})f(x) = \frac{x f'(0)}{1 + x^2} \int_0^x \frac{2(1 - \cos(\alpha \pi / 2))y^{n+1}}{(1 + y^2)(1 + 2y\cos(\alpha \pi / 2) + y^2)} dy
\]
where all higher derivatives of $g_{n,\alpha}$ are bounded, uniformly in $\alpha \in [0, 1]$, so $T_{n,\alpha f}$ has the desired bound, because $x/(1 + x^2) \in H^\infty$ and $|f'(0)| \lesssim \|f\|_{H^2}$.

For $S_{-1,\alpha}$, we rewrite the integrand as
\[
IHf(y) - \frac{2(\cos \alpha \pi + y\cos(\alpha \pi / 2))}{1 + 2y\cos(\alpha \pi / 2) + y^2} Hf(0).
\]
For the first term, by lemma 4 and 9,
\[
\left\| \frac{x I\!f(x)}{1 + x^2} \right\|_{H^2} \lesssim \|I\!f\|_{H^2} = \|f^{(1/\alpha)}\|_{H^2} \lesssim \|f\|_{H^2}/\alpha.
\]
For the second term we have
\[
\frac{x}{1 + x^2} \int_0^x \frac{2(\cos \alpha \pi + y\cos(\alpha \pi / 2))}{1 + 2y\cos(\alpha \pi / 2) + y^2} dy
\]
whose $H^2$ norm is bounded, uniformly in $\alpha$, so the second term has the desired bound because $|Hf(0)| \lesssim \|Hf\|_{H^2} \lesssim \|f\|_{H^2}/\alpha$. 

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For $S_{n,\alpha}$ ($n = 0, 1$) we use the old integrand. By lemma 3, Hardy’s inequality and lemma 9, the contribution of the first term in $S_{0,\alpha}$ is
\[
\lesssim \left\| \frac{\tilde{H}f(x) - \tilde{H}f(0)}{x} \right\|_{H^1} \lesssim \|\tilde{H}f\|_{H^1} \lesssim \|f\|_{H^1}/\alpha
\]
and the contribution of the first term in $S_{1,\alpha}$ is
\[
\lesssim \left\| \frac{1}{x} \int_0^x \tilde{H}f(y)dy \right\|_{H^1} \lesssim \|\tilde{H}f\|_{H^1} \lesssim \|f\|_{H^1}/\alpha.
\]
For the second term in $S_{0,\alpha}$ we have
\[
\frac{x}{1 + x^2} \int_0^x \frac{2(\cos(\alpha \pi/2) + y)}{1 + 2\cos(\alpha \pi/2) + y^2} dy = \frac{x \ln(1 + 2\cos(\alpha \pi/2) + x^2)}{1 + x^2}
\]
and for the remaining terms in $S_{1,\alpha}$ we have
\[
\frac{1}{1 + x^2} \int_0^x \left( 1 + \frac{2y(\cos(\alpha \pi/2) + y)}{1 + 2\cos(\alpha \pi/2) + y^2} \right) dy = \frac{1}{1 + x^2} \left( 3x - \int_0^x \frac{2\cos(\alpha \pi/2)}{1 + 2\cos(\alpha \pi/2) + y^2} dy \right)
\]
\[
= \frac{1}{1 + x^2} \left( 3x - \cos \frac{\alpha \pi}{2} \ln \left( 1 + 2\cos \frac{\alpha \pi}{2} + x^2 \right) \right)
\]
\[
+ \frac{2}{1 + x^2} \frac{\cos^2 \frac{\alpha \pi}{2}}{\sin \frac{\alpha \pi}{2}} \arctan \frac{x \sin \frac{\alpha \pi}{2}}{1 + x \cos \frac{\alpha \pi}{2}}
\]
whose $H^2$ norms are bounded, uniformly in $\alpha$, so they also have the desired bound as before.

Then we upgrade the $H^2$ norm to the $\tilde{H}^2$ norm.

**Lemma 12.** For $0 < \alpha \leq 1$, $n = 0, \pm 1$, $T_{n,\alpha}$ and $S_{n,\alpha}$ are bounded from $Y$ to $\tilde{H}^1_0$, with $
\|T_{n,\alpha}\|_{Y \rightarrow \tilde{H}^1_0} \leq C$ and \n\|S_{n,\alpha}\|_{Y \rightarrow \tilde{H}^1_0} \leq C/\alpha.

**Proof.** Since
\[
(T_{n,1} - T_{n,\alpha})f = \frac{xf'(0)}{1 + x^2} g_{n,\alpha}(x)
\]
where all higher derivatives of $xg_{n,\alpha}'$ are bounded, $T_{n,\alpha}$ can be controlled as before.

For $S_{n,\alpha}$, as before we can fall the derivative on the integral and it remains to bound
\[
\frac{x^{\min(n,0) + 1}}{1 + x^2} \left( \tilde{H}f(x) - \tilde{H}f(0) - \frac{2x(\cos(\alpha \pi/2) + x)\tilde{H}f(0)}{1 + 2x \cos(\alpha \pi/2) + x^2} \right)
\]
whose $H^2$ norm is \n\[ \lesssim \|\tilde{H}f\|_{H^1} \lesssim \|f\|_{H^1}/\alpha \]
by lemma 9.

**Proof of lemma 10.** The formal expression has been derived in section 4.2 of [9]. The $\tilde{H}^2$ bound of $L^{-1}f$ comes from those of $T_{n,\alpha}f$ and $S_{n,\alpha}f$. Note that the only terms in $L^{-1}$ not covered by them are
\[
\frac{-x^3 \sin(\alpha \pi/2)}{(1 + 2x \cos(\alpha \pi/2) + x^2)^2} \int_0^\infty y \cos(\alpha \pi/2) h(y)dy
\]
and
\[- \frac{x^3 \cos(\alpha \pi/2)}{(1 + 2x \cos(\alpha \pi/2) + x^2)^2} \int_0^\infty -y \sin(\alpha \pi/2) h(y) dy\]

which cancel each other. To show the bound is uniform in \(\alpha\), it suffices to note that each appearance of \(S_{\alpha, \alpha}\) (via \(h\)) in \(L^{-1}\) is accompanied by a factor of \(\sin(\alpha \pi/2) < 2\alpha\), which offsets the worse bounds of \(S_{\alpha, \alpha}\). Finally, \(L^{-1}f(0) = (L^{-1}f)'(0) = 0\) because the integrals vanish at 0, and they are multiplied by a factor of \(x\).

Recall that in terms of the new variable we need to solve
\[F = (1 + \lambda) x \tilde{W}_x + a U \tilde{W}_x + (1 - H \tilde{W}) \tilde{W} = 0, \quad U_x = H \tilde{W}.\]

We restrict ourselves to solutions with \(\tilde{W} \in \tilde{W} + X\), i.e. \(\tilde{W} - \tilde{W} \in X = \{V \in H^1_0 : V(0) = 0\}\).

**Lemma 13.** \(F : (\tilde{W} + X) \times \mathbb{R} \times \mathbb{R} \to H^1_0\) is analytic, i.e. it can be written as a convergent power series (in fact it is a polynomial) in terms of \(\tilde{W}, \lambda\) and \(a\), each term of which being a bounded multilinear operator from \(H^1_0 \times \mathbb{R} \times \mathbb{R}\) to \(H^1_0\).

**Proof.** Multiplicative closedness of \(H^2\) takes care of every term but \(U \tilde{W}_x\), which we deal with now. Since \(U \tilde{W}_x = (U/x)(x \tilde{W}_x)\), in terms of the variable \(y = x^{\alpha}\) we have \(U \tilde{W}_x(y) = (\tilde{U}(y)/y^{1/\alpha})(y \lambda \tilde{W}_x)\). For the second factor we have \(\|y \tilde{W}_x\|_{H^1} \leq \|\tilde{W}\|_{H^2}\). To bound the first factor we start from the identity \(x U_x = x H \tilde{W}\). In terms of \(y\) it is \(\alpha y \tilde{U}_y = y^{1/\alpha} H \tilde{W}\), so

\[\frac{\tilde{U}(y)}{y^{1/\alpha}} = \frac{1}{\alpha y^{1/\alpha}} \int_0^y H \tilde{W}(z) \alpha x^{\alpha - 1/\alpha} dz.\]

Taking the derivative and integrating by parts we get

\[\frac{d}{dy} \frac{\tilde{U}(y)}{y^{1/\alpha}} = \frac{H \tilde{W}(y)}{\alpha y} - \frac{1}{\alpha y^{1+1/\alpha}} \int_0^y \frac{H \tilde{W}(z)}{\alpha x^{\alpha - 1/\alpha}} dz = \frac{1}{\alpha y^{1+1/\alpha}} \int_0^y H \tilde{W}(z) z^{1/\alpha} dz.\]

and similarly,

\[\frac{d^2}{dy^2} \frac{\tilde{U}(y)}{y^{1/\alpha}} = \frac{1}{\alpha y^{2+1/\alpha}} \int_0^y H \tilde{W}(z) z^{1+1/\alpha} dz.\]

By lemma 1,

\[\|\tilde{U}(y)/y^{1/\alpha}\|_{L^2} \leq C_{\alpha}\|H \tilde{W}\|_{L^2}/\alpha, \quad \|((\tilde{U}(y)/y^{1/\alpha})')\|_{L^2} \leq C_{\alpha}'\|H \tilde{W}'\|_{L^2}/\alpha.\]

The order of \(C_{\alpha}\) and \(C_{\alpha}'\) can be found in chapter 0 of [14]: Since the weights are \(\chi_{x^{2-2/\alpha}}\), it corresponds to \(\epsilon = \pm 2 - 2/\alpha\) (and \(p = 2\)) in (0.4), so \(C_{\alpha} \lesssim 1/(1 - \epsilon) \leq 1/(2/\alpha - 1) \leq \alpha\) for \(\alpha \in (0, 1)\). Hence

\[\|\tilde{U}(y)/y^{1/\alpha}\|_{H^1} \lesssim H \tilde{W}\|_{H^1}\|\tilde{W}\|_{H^2}/\alpha\]

by lemma 9, so

\[\|U \tilde{W}_x\|_{H^1} \lesssim \|\tilde{W}\|_{H^1}/\alpha, \quad \|\tilde{W}\|_{H^1}/\alpha \lesssim \|\tilde{W}\|_{H^1}.\]

In particular,

\[dF(\tilde{W}, 0, 0) (V, \mu, 0) = \mu x \tilde{W}_x + x \tilde{W}_x - \tilde{W} HV - V \tilde{H} \tilde{W} + V = LV + \mu x \tilde{W}_x.\]
**Lemma 14.** \( dF_{(\tilde{\omega},0,0)}(\cdot,\cdot,0) \) is an isomorphism from \( X \times \mathbb{R} \) to \( H_0^2 \).

**Proof.** Recall from lemma 10 that
\[
dF_{(\tilde{\omega},0,0)}(V,\mu,0) = LV + \mu x \tilde{W}_x
\]
where
\[
LV = V + xV_x - \tilde{W}\tilde{H}V - \tilde{H}\tilde{W}
\]
is an isomorphism from \( X = \{ V \in H_0^2 : V_x(0) = 0 \} \) to \( Y = \{ V \in H_0^2 : V_x(0) + 2\sin(\alpha \pi/2)\tilde{H}V(0) = 0 \} \), so it suffices to show \( (x\tilde{W}_x)_x + 2\sin(\alpha \pi/2)\tilde{H}(x\tilde{W}_x) \) does not vanish at 0. This is because \( (x\tilde{W}_x)_x(0) = \tilde{W}_x(0) = -2\sin(\alpha \pi/2) \) and \( \tilde{H}(x\tilde{W}_x)(0) = H(x\tilde{W}_x)/\alpha = -\int \tilde{W}_x dx/\alpha \).

**Proof of theorem 1.** Let \( W = \alpha \Omega, U = \alpha \Upsilon, b = \alpha a \) and \( F = \alpha \Phi \). Then
\[
\Phi = (1 + \lambda)\alpha \Omega_x + b\tilde{T} \tilde{\Omega}_x + (1 - \alpha \tilde{H}\tilde{\Omega})\Omega.
\]
Note that \( \Phi \) is analytic (in the sense of lemma 13) from \( \bar{H}_0^2 \) to \( H_0^2 \), and that derivatives of all orders in terms of \( \Omega, \lambda \) and \( b \) are bounded uniformly in \( \alpha \) (the bound for \( \alpha \lambda \tilde{H}\tilde{\Omega} \) comes from lemma 9). Also
\[
d\Phi_{(\tilde{\Omega},0,0)}(V,\mu,0) = LV + \mu x \tilde{W}_x/\alpha
\]
where
\[
\frac{(x\tilde{W}_x)_x + 2\tilde{H}(x\tilde{W}_x)}{\alpha} = -2\sin(\alpha \pi/2)/\alpha
\]
is bounded and bounded away from 0, uniformly in \( \alpha \). Hence \( d\Phi(\cdot,\cdot,0) \) is invertible at \( (\Omega,0,0) \), and on a neighbourhood (whose size is independent of \( \alpha \)), its inverse is analytic, with derivatives of all orders bounded, uniformly in \( \alpha \). By the implicit function theorem, there is \( c > 0 \) such that if \( c > |b| = |\alpha a| \), then there are \( W(\cdot;a) \in \bar{H}_0^2 \) and \( \lambda(a) \in \mathbb{R} \) such that \( \Phi(W(\cdot;a),\lambda(a),a) = 0 \). Then
\[
w(x,t) = \frac{\text{sgn} x}{1-t} W\left(\frac{x}{(1-t)(1+\lambda(a))/\alpha};a\right)
\]
is a self-similar solution. Since \( W(\cdot;a) \in \bar{H}_0^2 \subset C^1 \), \( w(\cdot,t) \in C^\alpha \). Finally, the analyticity of \( W(\cdot;a) \) and \( \lambda(a) \) in \( a \) follows from lemma 13.

**Data availability statement**

No new data were created or analysed in this study.

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