Quantum random walks and thermalisation

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Abstract

It is shown how to construct quantum random walks with particles in an arbitrary faithful normal state. A convergence theorem is obtained for such walks, which demonstrates a thermalisation effect: the limit cocycle obeys a quantum stochastic differential equation without gauge terms. Examples are presented which generalise that of Attal and Joye (2007, J. Funct. Anal. 247, 253–288).

Key words: quantum random walk; thermalisation; thermalization; quantum heat bath; repeated interactions; quantum Langevin equation; toy Fock space.

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1 Introduction

A quantum random walk [3] may be interpreted as the dynamics of a quantum system that interacts periodically with a stream of identical particles, each of which lies in the vacuum state. As observed in [1], a Gelfand–Naimark–Segal construction may be employed to consider particles in a more general state (see also Franz and Skalski [6]); for a particular choice of evolution, one generated by a Hamiltonian which describes dipole-type interaction, Attal and Joye demonstrated that the limit flow involves fewer quantum noises than might naïvely be expected, a so-called ‘thermalisation’ effect [1, Theorem 7].

Inspired by Attal and Joye’s work, techniques from [3] are used below to show that thermalisation occurs for a large class of quantum random walks. Let $\rho$ be a faithful normal state on the particle algebra $B(K)$, where the Hilbert space $K$ may be infinite dimensional, and let $d$ be a $\rho$-preserving conditional expectation on $B(K)$. Suppose the linear maps $\Phi(\tau) : B(h) \to B(h \otimes K)$ are such that, in a suitable sense,

$$\tau^{-1}\delta(\Phi(\tau)(a) - a \otimes I_K) + \tau^{-1/2}\delta^+(\Phi(\tau)(a)) \to \Psi(a) \quad \text{as } \tau \to 0^+$$

for all $a \in B(h)$, where $\delta := I_{B(h)} \otimes d$ and $\delta^+ := I_{B(h \otimes K)} - \delta$. Then the random walk with generator $\Phi(\tau)$ and particle state $\rho$ converges to a quantum stochastic cocycle $k$...
on $\mathfrak{h} \otimes \mathcal{F}$, where $\mathcal{F}$ is the Boson Fock space over $L^2(\mathbb{R}^+; k)$, which satisfies the following quantum Langevin equation:

\begin{equation}
  k_t(a) = a \otimes I_F + \sum_{\alpha, \beta} \int_0^t k_s(\psi_\beta^\alpha(a)) \, d\Lambda_\alpha^\beta(s) \quad \forall \, t \in \mathbb{R}^+, \, a \in B(\mathfrak{h}).
\end{equation}

The generator of this cocycle is a linear mapping $\psi : B(\mathfrak{h}) \to B(\mathfrak{h} \otimes \hat{k})$, where $\hat{k}$ is the Hilbert space in the GNS representation corresponding to $\rho$, the vector $\omega$ gives the associated state and $k := \hat{k} \otimes \mathbb{C}\omega$. The generator $\psi$, given explicitly in terms of $d$ and $\Psi$, is such that $\psi_\beta^0 = 0$ unless $\alpha = 0$ or $\beta = 0$: there is no contribution from the gauge integrals in (1). If $n := \dim K$ is finite then there are $n^2 - 1$ independent quantum noises when working with particles in the vacuum state and at most twice that in the situation described above, not $n^4 - 1$ as might first appear necessary: this is the thermalisation phenomenon.

The formulation adopted below uses matrix spaces over operator spaces, the Lindsay–Wills approach to quantum stochastics. This setting is both natural and fruitful, allowing consideration of walks on $C^*$ algebras \cite{3} and quantum groups \cite{7}, for example.

This note is structured as follows: Section 2 rapidly introduces quantum random walks and presents the main convergence theorem from \cite{3}; Section 3 contains the result outlined above, Theorem \cite{5,9} and its proof, which is remarkably simple; Section 4 presents two classes of examples, which include the Attal–Joye model (Examples 4.3 and 4.8).

### 1.1 Conventions and notation

All vector spaces have complex scalar field; all inner products are linear in the second argument. An empty sum or product equals the appropriate additive or multiplicative unit, respectively.

The symbol $:= \,$ is to be read as ‘is defined to equal’ (or similarly). The indicator function of a set $A$ is denoted by $1_A$. The sets of non-negative integers and non-negative real numbers are denoted by $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ := [0, \infty]$. The identity operator on a vector space $V$ is denoted by $I_V$. Algebraic, Hilbert-space and ultraweak tensor products are denoted by $\otimes$, $\otimes$, and $\otimes$, respectively, with the symbol $\otimes$ also denoting the spatial tensor product of operator spaces. The Dirac dyad $|u\rangle\langle v|$ denotes the linear transformation on an inner-product space $V$ such that $w \mapsto \langle v, w \rangle V u$. 

2
2 Preliminaries

2.1 Quantum random walks

Definition 2.1. Let $V$ be a concrete operator space acting on the Hilbert space $h$, i.e., a closed subspace of $B(h)$. Recall that the matrix space

$$V \otimes M B(H) := \{ T \in B(h \otimes H) : E^x T E^y \in V \ \forall \ x, y \in H \}$$

is an operator space for any Hilbert space $H$, where $E_x : u \mapsto u \otimes x$.

The inclusions $V \otimes B(H) \subseteq V \otimes_M B(H) \subseteq V \otimes B(H)$ hold, with the latter an equality if and only if $V$ is ultraweakly closed, and there is the natural identification $(V \otimes_M B(H_1)) \otimes_M B(H_2) = V \otimes_M B(H_1 \otimes H_2)$.

Definition 2.2. Suppose $H \neq \{0\}$ and let $W$ be another operator space. A linear map $\Phi : V \rightarrow W$ is $H$ bounded if it is completely bounded, or it is bounded and the space $H$ is finite dimensional. The Banach space of such $H$-bounded maps is denoted by $HB(V; W)$ and is equipped with the $H_b$ norm

$$\| \cdot \|_{Hb} : \Phi \mapsto \|\Phi\|_{Hb} := \begin{cases} (\dim H) \|\Phi\| & \text{if } \dim H < \infty, \\ \|\Phi\|_{cb} & \text{if } \dim H = \infty, \end{cases}$$

where $\| \cdot \|_{cb}$ denotes the completely bounded norm.

Remark 2.3. Note that $T \mapsto E^x T E^y$ equals the slice map $I_{B(h)} \otimes \omega_{x,y}$, where $\omega_{x,y}$ is the ultraweakly continuous functional $X \mapsto \langle x, X y \rangle_H$. As $\{ \omega_{x,y} : x, y \in H \}$ is total in the predual $B(H)^*$, it follows that

$$(I_{B(h)} \otimes \omega)(T) \in V \ \forall \omega \in B(H)^*, \ T \in V \otimes_M B(H).$$

If $m : B(H_1) \rightarrow B(H_2)$ is linear, $h$ bounded and ultraweakly continuous then so is

$$I_{B(h)} \otimes m : B(h \otimes H_1) \rightarrow B(h \otimes H_2),$$

by [5, Lemma 1.5(b)], and the previous observation shows that such an ampliﬁcation respects the matrix-space structure: $(I_{B(h)} \otimes m)(V \otimes_M B(H_1)) \subseteq V \otimes_M B(H_2)$.

The following type of ampliﬁcation was introduced by Lindsay and Wills [9].
Proposition 2.4. If $\Phi \in HB(V; W)$ then the $H$ lifting of $\Phi$ is the unique map $\Phi \otimes_M I_{B(H)} : V \otimes_M B(H) \to W \otimes_M B(H)$ such that

$$E^x(\Phi \otimes_M I_{B(H)}(T))E_y = \Phi(E^x T E_y) \quad \forall x, y \in H, \, T \in V \otimes_M B(H).$$

The lifting is linear, $H$ bounded and is completely bounded if $\Phi$ is. It satisfies the inequalities $\|\Phi \otimes_M I_{B(H)}\| \leq \|\Phi\|_{HB}$ and $\|\Phi \otimes_M I_{B(H)}\|_{cb} \leq \|\Phi\|_{cb}$.

Proof. See [3, Theorem 2.5].

Proposition 2.5. For any $\Phi \in HB(V; V \otimes_M B(H))$ there exists a unique family of maps $\Phi^{(n)} : V \to V \otimes_M B(H^{\otimes n})$ indexed by $n \in \mathbb{Z}_+$, the quantum random walk with generator $\Phi$, such that $\Phi^{(0)} = I_V$ and

$$E^x \Phi^{(n+1)}(a)E_y = \Phi^{(n)}(E^x \Phi(a)E_y) \quad \forall x, y \in H, \, a \in V, \, n \in \mathbb{Z}_+.$$

These maps are linear, $H$ bounded and completely bounded if $\Phi$ is; if $n \geq 1$ then $\|\Phi^{(n)}\|_{HB} \leq \|\Phi\|^n_{HB}$ and $\|\Phi^{(n)}\|_{cb} \leq \|\Phi\|^n_{cb}$.

Proof. Given $\Phi^{(n)}$, use Proposition 2.4 to let $\Phi^{(n+1)} := (\Phi^{(n)} \otimes_M I_{B(H)}) \circ \Phi$. For the first inequality, see [3, Theorem 2.7]; the second is immediate.

2.2 Toy and Boson Fock space

Notation 2.6. Let $k$ be a Hilbert space containing the distinguished unit vector $\omega$ and let $\hat{k} := \hat{k} \ominus C\omega$, the orthogonal complement of $C\omega$ in $\hat{k}$. Define $\hat{x} := \omega + x \in \hat{k}$ for any $x \in k$.

Definition 2.7. The toy Fock space over $k$ is $\Gamma := \bigotimes_{n=0}^\infty \hat{k}(n)$, where $\hat{k}(n) := \hat{k}$ for all $n \in \mathbb{Z}_+$, with respect to the stabilising sequence $(\omega(n) := \omega)_{n=0}^\infty$; the suffix $(n)$ is used to indicate the relevant copy of $\hat{k}$. Note that $\Gamma = \Gamma_{n[} \otimes \Gamma_{n]}$, where $\Gamma_{n[} := \bigotimes_{m=0}^{n-1} \hat{k}(m)$ and $\Gamma_{n]} := \bigotimes_{m=n}^{\infty} \hat{k}(m)$, for all $n \in \mathbb{Z}_+$.

Definition 2.8. Let $\mathcal{F}$ be the Boson Fock space over $L^2(\mathbb{R}_+; k)$, the Hilbert space of square-integrable $k$-valued functions on the half line. Recall that $\mathcal{F}$ may be considered as the completion of $E$, the linear span of exponential vectors $\varepsilon(f)$ labelled by $f \in L^2(\mathbb{R}_+; k)$, with respect to the inner product

$$\langle \varepsilon(f), \varepsilon(g) \rangle_{\mathcal{F}} := \exp\left(\int_0^\infty \langle f(t), g(t) \rangle_k dt\right) \quad \forall f, g \in L^2(\mathbb{R}_+; k).$$
The following gives sense to the idea that the toy space $\Gamma$ approximates $F$.

**Proposition 2.9.** For all $\tau > 0$ there is a unique co-isometry $D_\tau : F \to \Gamma$ such that

$$D_\tau \varepsilon(f) = \bigotimes_{n=0}^{\infty} f(n; \tau), \quad \text{where} \quad f(n; \tau) := \tau^{-1/2} \int_{n\tau}^{(n+1)\tau} f(t) \, dt,$$

for all $f \in L^2(\mathbb{R}_+; k)$. Furthermore, $D_\tau^* D_\tau \to I_F$ strongly as $\tau \to 0^+$. (See [2].)

2.3 QS cocycles

**Definition 2.10.** An $h$ process $X$ is a family $\{X_t\}_{t \in \mathbb{R}_+}$ of linear operators in $h \otimes F$, such that the domain of each operator contains $h \otimes E$ and the map $t \mapsto X_t \varepsilon(f)$ is weakly measurable for all $u \in h$ and $f \in L^2(\mathbb{R}_+; k)$; this process is adapted if

$$\langle u \varepsilon(f), X_t v \varepsilon(g) \rangle_{h \otimes F} = \langle u \varepsilon(1_{[0,t]}f), X_t v \varepsilon(1_{[0,t]}g) \rangle_{h \otimes F} \langle \varepsilon(1_{[t,\infty]}f), \varepsilon(1_{[t,\infty]}g) \rangle_F$$

for all $u, v \in h$, $f, g \in L^2(\mathbb{R}_+; k)$ and $t \in \mathbb{R}_+$. (As is conventional, the tensor-product sign is omitted between elements of $h$ and exponential vectors.)

A mapping process $k$ is a family $\{k_\cdot(a)\}_{a \in V}$ of $h$ processes such that the map $a \mapsto k_\cdot(a)$ is linear for all $t \in \mathbb{R}_+$; this process is adapted if each $k_\cdot(a)$ is, it is strongly regular if

$$k_\cdot(\cdot) E_\varepsilon(f) \in B(V; B(h; h \otimes F)) \quad \forall f \in L^2(\mathbb{R}_+; k), \; t \in \mathbb{R}_+,$$

with norm locally uniformly bounded as a function of $t$, and it is CB regular if these two conditions hold with “bounded” replaced by “completely bounded”.

**Theorem 2.11.** For any $\phi \in \hat{k}B(V; V \otimes_{M} B(\hat{k}))$ there exists a unique strongly regular adapted mapping process $k^\phi$, the QS cocycle generated by $\phi$, such that

$$\langle u \varepsilon(f), (k^\phi_t(a) - a \otimes I_F) v \varepsilon(g) \rangle = \int_0^t \langle u \varepsilon(f), k^\phi_s(E_f^{(s)} \phi(a) E_{g^{(s)}}) v \varepsilon(g) \rangle \, ds$$

for all $u, v \in h$, $f, g \in L^2(\mathbb{R}_+; k)$, $a \in V$ and $t \in \mathbb{R}_+$. If $\phi$ is completely bounded then $k$ is CB regular.

*Proof.* See [9]; the proof contained there is valid for any operator space. \qed

QS cocycles are the correct limit objects for quantum random walks, as the next section makes clear.
2.4 Random-walk convergence

**Definition 2.12.** If \( \tau > 0 \) and \( \Phi \in \hat{k}B(V; V \otimes_M B(\hat{k})) \) then the embedded walk with generator \( \Phi \) and step size \( \tau \) is the mapping process \( K^{\Phi,\tau} \) such that
\[
K_t^{\Phi,\tau}(a) := (I_h \otimes D_\tau)^{*}(\Phi^{(n)}(a) \otimes I_m)(I_h \otimes D_\tau) \quad \text{if } t \in [n\tau, (n+1)\tau]
\]
for all \( a \in V \) and \( t \in \mathbb{R}_+ \).

**Definition 2.13.** If \( \tau > 0 \) and \( \Phi \in \hat{k}B(V; V \otimes_M B(\hat{k})) \) then the modification \( m(\Phi, \tau) \in \hat{k}B(V; V \otimes_M B(\hat{k})) \) is defined by setting
\[
m(\Phi, \tau)(a) := (\tau^{-1/2}\Delta + \Delta)(\Phi(a) - a \otimes I_\hat{k})(\tau^{-1/2}\Delta + \Delta) \quad \forall a \in V,
\]
where \( \Delta \) is the orthogonal projection from \( h \otimes k \) onto \( h \otimes k \) and \( \Delta := I_{h \otimes k} - \Delta \). Note that \( m(\Phi, \tau) \) is completely bounded if \( \Phi \) is. In block-matrix form,
\[
\begin{bmatrix}
\Phi_0\phi \\
\Phi^\phi x
\end{bmatrix}_{m(\Phi, \tau)}(a) = 
\begin{bmatrix}
\tau^{-1}(\Phi_0^\phi(a) - a) & \tau^{-1/2}\Phi^\phi_0(a) \\
\tau^{-1/2}\Phi_0^\phi(a) & \Phi^\phi_0(a) - a \otimes I_\hat{k}
\end{bmatrix}.
\]

**Remark 2.14.** For a sequence \( (\Phi_n) \) in \( HB(V; W) \), recall that
\[
\Phi_n \otimes_M I_{B(H)} \to 0 \text{ strongly}
\]
if and only if \( \begin{cases} 
\Phi_n \to 0 \text{ strongly} & \text{when } \dim H < \infty, \\
\Phi_n \to 0 \text{ in cb norm} & \text{when } \dim H = \infty,
\end{cases} \]
by [3] Proposition 2.11 and Lemma 2.13).

The following is a quantum analogue of Donsker’s invariance principle.

**Theorem 2.15.** Let \( \tau_n > 0 \) and \( \Phi_n, \phi \in \hat{k}B(V; V \otimes_M B(\hat{k})) \) be such that
\[
\tau_n \to 0+ \quad \text{and} \quad m(\Phi_n, \tau_n) \otimes_M I_{B(\hat{k})} \to \phi \otimes_M I_{B(\hat{k})} \text{ strongly}
\]
(i.e., pointwise in norm) as \( n \to \infty \). If \( f \in L^2(\mathbb{R}^+; k) \) and \( T \in \mathbb{R}_+ \) then
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \|K_t^{\Phi_n,\tau_n}(a)E_{\varepsilon(f)} - k_t^\phi(a)E_{\varepsilon(f)}\| = 0 \quad \forall a \in V.
\]
If, further, \( \|m(\Phi_n, \tau_n) - \phi\|_{\hat{k}b} \to 0 \) as \( n \to \infty \) then
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \|K_t^{\Phi_n,\tau_n}(\cdot)E_{\varepsilon(f)} - k_t^\phi(\cdot)E_{\varepsilon(f)}\|_{\hat{k}b} = 0;
\]
when \( \Phi_n \) and \( \phi \) are completely bounded, the same implication holds if \( \| \cdot \|_{\hat{k}b} \) is replaced by \( \| \cdot \|_{\text{cb}} \).
Proof. See [3, Theorem 7.6].

**Notation 2.16.** The conclusion (2) will be abbreviated to $K_{\Phi_n,\tau_n} \to k^\phi$; the stronger conclusion (3) will be denoted by $K_{\Phi_n,\tau_n} \to_{cb} k^\phi$, or by $K_{\Phi_n,\tau_n} \to_{cb} k^\phi$ if the completely bounded version holds.

**Example 2.17.** Choose self-adjoint operators $H_{\text{sys}} \in B(h)$ and $H_{\text{par}} \in B(\hat{k})$, and let $V \in B(h; h \otimes k)$. Define $H_{\text{tot}}(\tau) := H_{\text{sys}} \otimes I_k + I_h \otimes H_{\text{par}} + H_{\text{int}}(\tau)$ for all $\tau > 0$, where

$$H_{\text{int}}(\tau) := \tau^{-1/2}(Q V E^\omega + E_\omega V^* Q^*) = \tau^{-1/2}
\begin{bmatrix}
0 & V^* \\
V & 0
\end{bmatrix}$$

and $Q : h \otimes k \hookrightarrow h \otimes \hat{k}$ is the natural embedding map. If $W(\tau) := \exp(-i\tau H_{\text{tot}}(\tau))$ then

$$(\tau^{-1/2} \Delta^+ + \Delta)(W(\tau) - I_{h \otimes \hat{k}})(\tau^{-1/2} \Delta^+ + \Delta)$$

$$= -i
\begin{bmatrix}
H_{\text{sys}} + \mu I_h & V^* \\
V & 0
\end{bmatrix} - \frac{1}{2}
\begin{bmatrix}
V^* V & 0 \\
0 & 0
\end{bmatrix} + O(\tau)$$

as $\tau \to 0^+$, where $\mu := \langle \omega, H_{\text{par}} \omega \rangle_{\hat{k}}$. Thus if $\tau_n \to 0^+$ and

$$\Phi(\tau) : B(h) \to B(h \otimes \hat{k}); \ a \mapsto (a \otimes I_k) W(\tau)$$

then, by Theorem 2.15, $K^\phi(\tau_n), \tau_n \to_{cb} k^\phi$, where $\phi(a) := (a \otimes I_k) F$ for all $a \in B(h)$ and

$$F :=
\begin{bmatrix}
-i(H_{\text{sys}} + \mu I_h) - \frac{1}{2} V^* V & -iV^* \\
-iV & 0
\end{bmatrix} \in B(h \otimes \hat{k}).$$

The cocycle $k^\phi$ obtained in the limit is an evolution of Hudson–Parthasarathy type; if $U_t := k^\phi_t(I_h)$ for all $t \in \mathbb{R}_+$ then $U_t$ is unitary, by [3] Proof of Theorem 7.1 and Theorem 7.5], and the adapted $h$ process $U$ satisfies the quantum Langevin equation

$$(4) \quad U_0 = I_{h \otimes F}, \quad dU_t = d\Lambda_F(t) U_t.$$  

(The cocycle $k^\phi$ may be recovered from $U$ by setting $k^\phi_t(a) := (a \otimes I_F) U_t.)$

**Example 2.18.** Let $\tau_n$ and $W(\tau)$ be as in Example 2.17 and let

$$\Phi(\tau) : B(h) \to B(h \otimes \hat{k}); \ a \mapsto W(\tau)^*(a \otimes I_k) W(\tau).$$
Then \( K^{\Phi(\tau_n), \tau_n} \to_{cb} k^\phi \), by Theorem 2.15, where

\[
\dot{\phi}(a) := \begin{bmatrix}
-\text{i}[a, H_{\text{sys}}] + V^*(a \otimes I_k)V - \frac{1}{2}\{a, V^*V\} - \text{i}aV^* + \text{i}V^*(a \otimes I_k) \\
-\text{i}(a \otimes I_k)V + \text{i}Va & 0
\end{bmatrix}
\]

for all \( a \in B(\mathfrak{h}) \), with the commutator \([x, y] := xy - yx\) and the anticommutator \(\{x, y\} := xy + yx\). Here, the limit cocycle is an inner Evans–Hudson flow: if \( U \) is the unitary adapted \( h \) process which satisfies (4) then, by \([8, \text{Theorem 7.4}]\),

\[
k^\phi_t(a) = U_t^*(a \otimes I_F)U_t \quad \forall a \in B(\mathfrak{h}), \ t \in \mathbb{R}_+.
\]

**Remark 2.19.** In Example 2.17, if \( k \) has orthonormal basis \( \{e_j\}_{j=1}^N \), where \( N \) may be infinite, then

\[
H_{\text{int}}(\tau) = \tau^{-1/2} \sum_{j=1}^N (V_j \otimes |e_j\rangle\langle\omega| + V^*_j \otimes |\omega\rangle\langle e_j|),
\]

where \( V_j := E^{e_j}V \) for all \( j \); the series converges strongly if \( N = \infty \). For \( N < \infty \), this is the dipole-interaction Hamiltonian used by Attal and Joye in \([1]\). With the convention that \( e_0 := \omega \), the quantum Langevin equation (4) takes the form

\[
U_0 = I_h \otimes I_F, \quad \text{d}U_t = \sum_{\alpha, \beta = 0}^N (F^\alpha_\beta \otimes I_F)U_t \text{d}\Lambda^\beta_\alpha(t),
\]

where \( F^\alpha_\beta := E^\alpha FE_\beta \) for all \( \alpha, \beta \).

### 3 Thermal walks

**Notation 3.1.** Let \( \varrho \) be a density matrix which acts on the Hilbert space \( K \), i.e., a positive operator with unit trace, and suppose that the corresponding normal state \( \rho : X \mapsto \text{tr}(\varrho X) \) on the von Neumann algebra \( B(K) \) is faithful (so \( K \) is separable). Fix an orthonormal basis \( \{e_j\}_{j=0}^N \) of \( K \), where \( N \in \mathbb{Z}_+ \) or \( N = \infty \), such that

\[
\varrho = \sum_{j=0}^N \lambda_j |e_j\rangle\langle e_j|.
\]

The eigenvalues \( \lambda_j \) are positive and sum to 1, so this series is norm convergent.
**Definition 3.2.** Let \((\hat{k}, \pi, \omega)\) be the GNS representation corresponding to \(\rho\) and let \(X \mapsto [X]\) denote the induced mapping from \(B(K)\) into \(\hat{k}\), so that \(\omega := [I_k]\). Note that the representation \(\pi : B(K) \to B(\hat{k})\) is ultraweakly continuous, injective and unital [4, Theorems 2.3.16 and 2.4.24], and \([\ker \rho]\) is dense in \(k := \hat{k} \otimes \mathbb{C} \omega\). 

**Lemma 3.3.** The slice map \(\tilde{\rho} := I_{B(h)} \otimes \rho : B(h \otimes K) \to B(h)\) is completely positive and such that

\[
\tilde{\rho}(T) = \sum_{j=0}^{N} \lambda_j E_{e_j} T E_{e_j} = \text{tr}_K((I_h \otimes \varrho)\ T) \quad \forall \ T \in B(h \otimes K),
\]

where \(\text{tr}_K\) is the partial trace over \(K\).

The unital \(*\)-homomorphism \(\tilde{\pi} := I_{B(h)} \otimes \pi : B(h \otimes K) \to B(h \otimes \hat{k})\) is injective and such that

\[
E^{[X]}(I_h \otimes X)^* T (I_h \otimes Y) = \tilde{\rho}((I_h \otimes X)^* T (I_h \otimes Y)) \quad \forall \ X, Y \in B(K), \ T \in B(h \otimes K).
\]

**Proof.** The existence of \(\tilde{\rho}\) and \(\tilde{\pi}\) as claimed follows from [10, Proposition IV.5.13 and Theorem IV.5.2]. The identities are immediate if \(T \in B(h) \otimes B(K)\), so hold everywhere by ultraweak continuity. \(\square\)

**Definition 3.4.** If \(\Phi \in KB(V; V \otimes M B(K))\) then \(\tilde{\pi} \circ \Phi \in \hat{k}B(V; V \otimes M B(\hat{k}))\) is the GNS **generator** of the quantum random walk with **generator** \(\Phi\) and **particle state** \(\rho\). (The vector state on \(B(\hat{k})\) given by \(\omega\) corresponds to the state \(\rho\) on \(B(K)\).)

**Definition 3.5.** Fix a conditional expectation \(d\) on \(B(K)\), i.e., a linear idempotent onto a \(C^*\) subalgebra \(D\) such that

\[
d(X^*X) \geq 0 \quad \text{and} \quad d(d(X)Y) = d(X)d(Y) = d(Xd(Y)) \quad \forall \ X, Y \in B(K).
\]

Recall that \(d\) is completely positive, by [10, Corollary IV.3.4]. Suppose further that \(d\) preserves the state \(\rho\), i.e., \(\rho \circ d = \rho\). Then \(d\) is ultraweakly continuous, so \(D\) is a von Neumann algebra, and \(d(I_K) = I_K\), since \(I_K - d(I_K)\) is an orthogonal projection with \(\rho(I_K - d(I_K)) = 0\).

**Notation 3.6.** Letting \(\delta := I_{B(h)} \otimes d\), where \(d\) is as above, the lifted map \(\delta\) is a \(\tilde{\rho}\)-preserving conditional expectation onto \(B(h) \otimes D\) which leaves \(V \otimes M B(K)\) invariant. Furthermore,

\[
\delta(a \otimes I_K) = a \otimes I_K \quad \text{and} \quad \delta(T_1 \delta(T_2)) = \delta(T_1)\delta(T_2) = \delta(\delta(T_1)T_2)
\]

\(9\)
for all $a \in B(h)$ and $T_1, T_2 \in B(h \otimes K)$. These, together with the identity $\tilde{\rho} \circ \delta = \tilde{\rho}$, imply that

$$\tilde{\rho}(\delta(T_1)T_2) = \tilde{\rho}(T_1 \delta(T_2)) \quad \forall T_1, T_2 \in B(h \otimes K) \quad \text{and}$$

$$\tilde{\rho}((a \otimes I_K)T) = a\tilde{\rho}(T) \quad \forall a \in B(h), \ T \in B(h \otimes K).$$

The conditional expectation $\delta$ plays a vital rôle in scaling quantum random walks in order to obtain convergence. The following example is a natural choice for this map, that given by the eigenbasis $\{e_j\}_{j=0}^N$.

**Example 3.7.** Define the diagonal map

$$\delta_e : B(h \otimes K) \to B(h \otimes K); \ T \mapsto (I_h \otimes S)^*(T \otimes I_K)(I_h \otimes S),$$

where the Schur isometry $S \in B(K; K \otimes K)$ is such that $Se_j = e_j \otimes e_j$ for all $j$. If $D_e$ is the maximal Abelian subalgebra of $B(K)$ generated by $\{e_j\}_{j=0}^N$, then $\delta_e$ is the lifting of the unique $\rho$-preserving conditional expectation $d_e$ onto $D_e$.

**Definition 3.8.** If $\tau > 0$ and $\Phi \in KB(V; V \otimes_M B(K))$ then the modification $m_d(\Phi, \tau) \in KB(V; V \otimes_M B(K))$ is defined by setting

$$m_d(\Phi, \tau) := \tau^{-1/2} \delta \circ \Phi + \tau^{-1/2} \delta^\perp \circ \Phi = (\tau^{-1/2} \delta^\perp \circ \Phi' \circ \Phi',$$

where $\Phi' \in KB(V; V \otimes_M B(K))$ is such that $\Phi'(a) := \Phi(a) - a \otimes I_K$ for all $a \in V$ and $\delta^\perp := I_{B(h \otimes K)} - \delta$. If $\Phi$ is completely bounded then so is $m_d(\Phi, \tau)$.

**Theorem 3.9.** Let $\tau_n > 0$ and $\Phi_n, \Psi \in KB(V; V \otimes_M B(K))$ be such that

$$\tau_n \to 0^+ \quad \text{and} \quad m_d(\Phi_n, \tau_n) \otimes_M I_{B(K)} \rightarrow \Psi \otimes_M I_{B(K)}$$

as $n \to \infty$. If $\psi \in \widehat{kB(V; V \otimes_M B(\widehat{k}))}$ is defined by setting

$$\psi(a) := \Delta^\perp(\tilde{\pi} \circ \Psi)(a)\Delta^\perp + \Delta(\tilde{\pi} \circ \delta^\perp \circ \Psi)(a)\Delta^\perp + \Delta^\perp(\tilde{\pi} \circ \delta^\perp \circ \Psi)(a)\Delta$$

then $\psi$ is completely bounded whenever $\Psi$ is and $K_{\tilde{\pi} \circ \Phi_n, \tau_n} \rightarrow k_{\psi}$. Furthermore,

if $\|m_d(\Phi_n, \tau_n) - \Psi\|_{kb} \rightarrow 0$ then $K_{\tilde{\pi} \circ \Phi_n, \tau_n} \rightarrow_{kb} k_{\psi}$

and, when $\Phi_n$ and $\Psi$ are completely bounded,

if $\|m_d(\Phi_n, \tau_n) - \Psi\|_{cb} \rightarrow 0$ then $K_{\tilde{\pi} \circ \Phi_n, \tau_n} \rightarrow_{cb} k_{\psi}$.
Proof. Note first that $\Phi' = (\tau \delta + \tau^{1/2} \delta^\perp) \circ m_d(\Phi, \tau)$ for all $\tau > 0$. If $X, Y \in \ker \rho$ and $a \in V$ then (5) and the identity $\tilde{\rho} \circ \delta = \tilde{\rho}$ imply that

$$E^\omega m(\tilde{\pi} \circ \Phi, \tau)(a) E_\omega = \tau^{-1} E^\omega \tilde{\pi}(\Phi'(a)) E_\omega = \tau^{-1} \tilde{\rho}(\Phi'(a)) = \tilde{\rho}(m_d(\Phi, \tau))(a) E_\omega,$$

$$E^{|X|} m(\tilde{\pi} \circ \Phi, \tau)(a) E_{|X|} = \tau^{-1/2} E^{|X|} \tilde{\pi}(\Phi'(a)) E_\omega = E^{|X|}(\tilde{\pi} \circ (\tau^{1/2} \delta + \delta^\perp) \circ m_d(\Phi, \tau))(a) E_{|X|},$$

and

$$E^\omega m(\tilde{\pi} \circ \Phi, \tau)(a) E_{|Y|} = E^\omega(\tilde{\pi} \circ (\tau^{1/2} \delta + \delta^\perp) \circ m_d(\Phi, \tau))(a) E_{|Y|}.$$

Letting $\Theta := m_d(\Phi, \tau) - \Psi$, this working shows that

$$(m(\tilde{\pi} \circ \Phi, \tau) - \psi)(a) = \Delta^\perp (\tilde{\pi} \circ \Theta)(a) \Delta^\perp + \Delta (\tilde{\pi} \circ \delta^\perp \circ \Theta)(a) \Delta^\perp + \Delta (\tilde{\pi} \circ \delta \circ \Theta)(a) \Delta + \tau^{1/2} R(a),$$

where

$$R(a) := \Delta (\tilde{\pi} \circ \delta \circ m_d(\Phi, \tau))(a) \Delta^\perp + \Delta (\tilde{\pi} \circ \delta^\perp \circ m_d(\Phi, \tau))(a) \Delta + \Delta (\tilde{\pi} \circ (\tau^{1/2} \delta + \delta^\perp) \circ m_d(\Phi, \tau))(a) \Delta.$$

The result follows, by Theorem 2.15.

Remark 3.10. If $\Psi \in KB(V; V \otimes M B(K))$ and $\psi$ is defined by (10) then the identities (5) and (8) imply that

$$E^\omega \psi(a) E_\omega = \tilde{\rho}(\Psi(a)) = E^\omega \tilde{\pi}(\Psi(a)) E_\omega,$$

$$E^{|X|} \psi(a) E_{|X|} = \tilde{\rho}((I_h \otimes X)^*(\delta^\perp \circ \Psi)(a)) = E^{|d^\perp(\Psi, X)} \tilde{\pi}(\Psi(a)) E_{|X|},$$

and

$$E^\omega \psi(a) E_{|Y|} = 0$$

for all $X, Y \in \ker \rho$ and $a \in V$, where $d^\perp := I_B(K) - d$. If $n := \dim K < \infty$ then at most

$$2 \dim \left\{ d^\perp(X) : X \in \ker \rho \right\} \leq 2(n^2 - 1)$$

independent noises appear in the quantum Langevin equation (1) satisfied by $k^\psi$, with equality in the above if $d(X) = \rho(X) I_K$ for all $X \in B(K)$. 

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**Remark 3.11.** Although the construction in Example 3.7 may appear to depend on the choice of orthonormal basis which diagonalises $\varrho$, this is not really so. To see this, suppose $\{f_j\}_{j=0}^N$ is another eigenbasis for $\varrho$, labelled so that $e_j$ and $f_j$ have the same eigenvalue for all $j$, let $d_\varrho$ and $d_\tau$ be the $\rho$-preserving conditional expectations from $B(K)$ onto the subalgebras generated by $\{|e_j\rangle\langle e_j|\}_{j=0}^N$ and $\{|f_j\rangle\langle f_j|\}_{j=0}^N$, and let $\delta_\varrho$ and $\delta_\tau$ be their lifts to $B(h \otimes K)$.

If $U \in B(K)$ is the unique unitary operator such that $Ue_j = f_j$ for all $j$ then
\[ \rho(U^*XU) = \rho(X) \quad \text{and} \quad U^*d_\varrho(X)U = d_\varrho(U^*XU) \quad \forall X \in B(K). \]
Hence, if $\Phi, \Psi \in KB(V; V \otimes_M B(K))$ and $\tilde{U} := I_h \otimes U$,
\[ (m_{d_\varrho}(\Phi, \tau) - \Psi)(a) = \tilde{U}^*(m_{d_\tau}(\Phi, \tau) - \tilde{\Psi})(a)\tilde{U} \quad \forall a \in V, \]
where $\tilde{\Phi} : a \mapsto \tilde{U}\Phi(a)\tilde{U}^*$ et cetera. Let $\psi_\delta$ and $\psi_\tau$ be defined by (10), but with $\delta$ equal to $\delta_\varrho$ and $\delta_\tau$, respectively, and $\Psi$ replaced by $\tilde{\Psi}$ in the latter case; if $a \in V$ and $X, Y \in \ker \rho$ then
\[ E^\omega \psi_\delta(a)E_\omega = \tilde{\rho}(\tilde{\Psi}(a)) = \tilde{\rho}(\Psi(a)) = E^\omega \psi_\tau(a)E_\omega, \]
\[ E^{[X]} \psi_\delta(a)E_{[X]} = \tilde{\rho}((I_h \otimes X)^*(\delta_\tau^\perp \circ \tilde{\Psi})(a)) \]
\[ = \tilde{\rho}(\tilde{U}^*(I_h \otimes X^*)\tilde{U}^*\delta_\tau^\perp(\tilde{\Psi}(a))\tilde{U}) \]
\[ = \tilde{\rho}((I_h \otimes U^*XU)^*\delta_\varrho^\perp(\tilde{\Psi}(a))) = E^{[U^*XU]} \psi_\delta(a)E_\omega \]
\[ E^\omega \psi_\tau(a)E_{[Y]} = E^\omega \psi_\tau(a)E_{[U^*YU]} \]
and
\[ E^{[X]} \psi_\tau(a)E_{[Y]} = 0 = E^{[U^*XU]} \psi_\tau(a)E_{[U^*YU]}. \]
Thus if $W \in B(\hat{k})$ is the unique unitary operator such that $W[X] = [U^*XU]$ for all $X \in B(K)$ then
\[ \psi_\tau(a) = (I_h \otimes W)^*\psi_\delta(a)(I_h \otimes W) \quad \forall a \in V; \]
the change of orthonormal basis used to define the diagonal map is manifest as a change of coordinates (an isometric isomorphism of $\hat{k}$ which preserves $\omega$) and unitary conjugation of the map $\Psi$.

### 4 Examples

Henceforth $V$ will be a von Neumann algebra $\mathcal{M}$ and $\mathcal{M} \otimes_M B(K) = \mathcal{M} \otimes B(K)$. As above, $d$ is a conditional expectation on $B(K)$ which preserves the faithful normal state $\rho$ and $\delta = I_{B(h)} \otimes d$. 

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4.1 Hudson–Parthasarathy evolutions

**Remark 4.1.** Suppose \( F \in \mathcal{M} \otimes B(\mathcal{K}) \) and let \( \Psi : \mathcal{M} \rightarrow \mathcal{M} \otimes B(\mathcal{K}) \) be such that \( \Psi(a) = (a \otimes I_\mathcal{K})F \) for all \( a \in \mathcal{M} \). If \( \psi \) is defined by (10) then \( \psi(a) = (a \otimes I_\mathcal{K})G \) for all \( a \in \mathcal{M} \), by (7), where

\[
G = \Delta^{\frac{1}{2}}(\pi)\Delta^{\frac{1}{2}} + \Delta^{\frac{1}{2}}(\delta^\perp(F))\Delta^{\frac{1}{2}} + \Delta^{\frac{1}{2}}(\delta^\perp(F))\Delta.
\]

Furthermore, the cocycle \( k^\psi \) is such that \( k^\psi_t(a) = (a \otimes I_\mathcal{F})X_t \) for all \( a \in \mathcal{M} \) and \( t \in \mathbb{R}_+ \), where the adapted \( \mathfrak{h} \) process \( X = \{X_t := k^\psi_t(I_\mathcal{K})\}_{t \in \mathbb{R}_+} \) satisfies the Hudson–Parthasarathy equation

\[
X_0 = I_{\mathcal{H} \otimes \mathcal{F}}, \quad dX_t = d\Lambda_G(t)X_t.
\]

(See [8] Proof of Theorem 7.1.)

**Theorem 4.2.** Let \( H_\mathcal{H} \) and \( H_\mathcal{O} \) be self-adjoint elements of \( \mathcal{M} \otimes B(\mathcal{K}) \) such that \( H_\mathcal{H} = \delta(H_\mathcal{H}) \) and \( H_\mathcal{O} = \delta^\perp(H_\mathcal{O}) \). If \( \tau > 0 \) and \( \Lambda_G(\tau) := H_\mathcal{H} + \tau^{-\frac{1}{2}}H_\mathcal{O} \) then the completely isometric map

\[
\Phi(\tau) : \mathcal{M} \rightarrow \mathcal{M} \otimes B(\mathcal{K}); \quad a \mapsto (a \otimes I_\mathcal{K})\exp(-i\tau H_\Lambda(\tau))
\]

is such that \( ||m_d(\Phi(\tau), \tau) - \Psi||_{cb} \rightarrow 0 \) as \( \tau \rightarrow 0^+ \), where

\[
\Psi : \mathcal{M} \rightarrow \mathcal{M} \otimes B(\mathcal{K}); \quad a \mapsto (a \otimes I_\mathcal{K})(-i(H_\mathcal{H} + H_\mathcal{O}) - \frac{1}{2}\delta(H_\mathcal{O}^2)).
\]

Thus \( K^\Phi(\tau_n) \rightarrow_{cb} k^\psi \) if \( \tau_n \rightarrow 0^+ \), where \( \psi : \mathcal{M} \rightarrow \mathcal{M} \otimes B(\mathcal{K}) \) is such that

\[
\begin{align*}
E^a\psi(a)E_{\omega} &= -ia\bar{\rho}(H_\mathcal{H}) - \frac{1}{2}a\bar{\rho}(H_\mathcal{O}^2), \\
E^{[X]}\psi(a)E_{\omega} &= -ia\bar{\rho}((I_\mathcal{H} \otimes X)^*H_\mathcal{O}), \\
E^a\psi(a)E_{[Y]} &= -ia\bar{\rho}(H_\mathcal{O}(I_\mathcal{H} \otimes Y))
\end{align*}
\]

and

\[
E^{[X]}\psi(a)E_{[Y]} = 0
\]

for all \( X, Y \in \ker \rho \) and \( a \in \mathcal{M} \), and \( U_t := k^\psi_t(I_\mathcal{H}) \) is unitary for all \( t \in \mathbb{R}_+ \).

**Proof.** If \( a \in \mathcal{M} \) then \( \delta(\Phi(\tau)'(a)) = (a \otimes I_\mathcal{K})\delta(\exp(-i\tau H_\Lambda(\tau)) - I_{\mathcal{H} \otimes \mathcal{K}}) \), by (7), and the same holds with \( \delta \) replaced by \( \delta^\perp \). As \( \tau \rightarrow 0^+ \),

\[
\tau^{-1}\delta(\exp(-i\tau H_\Lambda(\tau)) - I_{\mathcal{H} \otimes \mathcal{K}}) = -iH_\mathcal{H} - \frac{1}{2}\delta(H_\mathcal{O}^2) + O(\tau^{1/2})
\]

and

\[
\tau^{-1/2}\delta^\perp(\exp(-i\tau H_\Lambda(\tau)) - I_{\mathcal{H} \otimes \mathcal{K}}) = -iH_\mathcal{O} + O(\tau^{1/2}).
\]
which gives the first claim. Theorem 3.9 and Remark 3.10, simplified using the identities (7), (9) and \( \tilde{\rho} = \tilde{\rho} \circ \delta \), complete the result; unitarity holds for the adapted process \( U = \{U_t\}_{t \in \mathbb{R}_+} \) by [8, Theorem 7.5].

**Example 4.3.** Let \( \delta = \delta_e \) be the diagonal map of Example 3.7 and suppose \( H_d := H_{\text{sys}} \otimes I_K + I_h \otimes H_{\text{par}} \), where the self-adjoint operators \( H_{\text{sys}} \in \mathcal{M} \) and

\[
H_{\text{par}} = \sum_{j=0}^{N} \mu_j |e_j\rangle \langle e_j| \in B(K),
\]

with this series strongly convergent when \( N = \infty \). Let \( K_x := K \otimes \mathbb{C} e_0 \), choose \( V \in \mathcal{M} \otimes B(\mathbb{C}; K_x) \) and define

\[
H_o := QVE_\omega + E_\omega V^* Q^* = \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix},
\]

where \( Q : h \otimes K_x \hookrightarrow h \otimes K \) is the natural embedding. If \( \tau_n \rightarrow 0^+ \) and \( \Phi(\tau) \) is defined by (13) then Theorem 4.2 implies that \( K_e \pi \circ \Phi(\tau_n), \tau_n \rightarrow cb \), where

\[
E^\omega \psi(a) E_\omega = -ia(H_{\text{sys}} + \rho(H_{\text{par}}) I_h) - \frac{1}{2} \lambda(a H^2_o),
\]

\[
E^{[X]} \psi(a) E_\omega = -ia(E^{X} \rho e_0 QV + V^* Q^* E_{X^* e_0}),
\]

\[
E^{Y} \psi(a) E_Y = -ia(E^{Y^* e_0 QV + V^* Q^* E_{Y^* e_0}})
\]

and \( E^{[X]} \psi(a) E_Y = 0 \)

for all \( X, Y \in \ker \rho \) and \( a \in \mathcal{M} \). If \( \dim K < \infty \) then this agrees with [1, Theorem 7]. Letting \( \lambda_0 = 1 \) and \( \lambda_j = 0 \) for all \( j > 0 \), the above is also in formal agreement with Example 2.17; note that the GNS space is spanned by \( \{|e_j\rangle \langle e_0|\}_{j=0}^{N} \) in this case.

### 4.2 Evans–Hudson evolutions

**Definition 4.4.** An Evans–Hudson flow is a solution \( k \) of the quantum stochastic differential equation (1) which is \(*\)-homomorphic, i.e., each \( k_t(a) \) extends to a bounded operator on \( h \otimes F \) and the mapping \( a \mapsto k_t(a) \) is a \(*\)-homomorphism from \( \mathcal{M} \) to \( B(h \otimes F) \) for all \( t \in \mathbb{R}_+ \).

**Remark 4.5.** If \( F \in \mathcal{M} \otimes B(K) \) and \( \Psi : \mathcal{M} \rightarrow \mathcal{M} \otimes B(K) \) such that

\[
\Psi(a) = (a \otimes I_K)F + F^*(a \otimes I_K) + \delta(\delta^+(F)^*(a \otimes I_K)\delta^+(F)) \quad \forall a \in \mathcal{M}
\]
then a short calculation shows that $\psi$ defined by (10) is such that

$$\psi(a) = (a \otimes I_\mathbb{k})G + G^*(a \otimes I_\mathbb{k}) + G^*\Delta(a \otimes I_\mathbb{k})\Delta G \quad \forall a \in \mathcal{M},$$

with $G$ as in (11). It follows [8, Theorem 7.4] that $k^\psi(a) = X_t^*(a \otimes I_\mathbb{k})X_t$ for all $a \in \mathcal{M}$ and $t \in \mathbb{R}_+$, where $X$ is the solution of (12); if $X$ is co-isometric then $k^\psi$ is an inner Evans–Hudson flow.

Recall that $[x,y] := xy - yx$ and $\{x,y\} := xy + yx$ are the commutator and the anticommutator, respectively.

**Theorem 4.6.** If $H_d$, $H_o$ and $H_{\text{tot}}$ are as in Theorem 4.2 then the ultraweakly continuous unital $\ast$-homomorphism

$$(14) \quad \Phi(\tau) : \mathcal{M} \to \mathcal{M} \overline{\otimes} B(\mathbb{K}); \quad a \mapsto \exp(i\tau H_{\text{tot}})(a \otimes I_\mathbb{k}) \exp(-i\tau H_{\text{tot}})$$

is such that $\|m_d(\Phi(\tau), \tau) - \Psi\|_{\text{cb}} \to 0$ as $\tau \to 0^+$, where

$$\Psi : \mathcal{M} \to \mathcal{M} \overline{\otimes} B(\mathbb{K}); \quad a \mapsto -i[a \otimes I_\mathbb{k}, H_d + H_o] + \delta(H_o(a \otimes I_\mathbb{k})H_o) - \frac{1}{2}\{a \otimes I_\mathbb{k}, \delta(H_o^2)\}.$$  

Hence $K_{\pi \circ \Phi(\tau_n), \tau_n} \to_{\text{cb}} k^\psi$ if $\tau_n \to 0^+$, where $\psi : \mathcal{M} \to \mathcal{M} \overline{\otimes} B(\mathbb{K})$ is completely bounded and

$$E^x\psi(a)E_\omega = -i[a, \tilde{\rho}(H_d)] + \tilde{\rho}(H_o(a \otimes I_\mathbb{k})H_o) - \frac{1}{2}\{a, \tilde{\rho}(H_o^2)\};$$

$$E^{[X]}\psi(a)E_\omega = -i[a, \tilde{\rho}((I_\mathbb{h} \otimes X)^*H_o)],$$

$$E^\omega\psi(a)E_{[Y]} = -i[a, \tilde{\rho}(H_o(I_\mathbb{h} \otimes Y))],$$

and $E^{[X]}\psi(a)E_{[Y]} = 0$ for all $X, Y \in \ker \rho$ and $a \in \mathcal{M}$, and $k^\psi$ is an inner Evans–Hudson flow.

**Proof.** This follows in the same manner as Theorem 4.2.

**Remark 4.7.** Since $\Phi(\tau)$ in (14) is an ultraweakly continuous $\ast$-homomorphism, so are $\Phi(\tau)^{(n)}$, for all $n \in \mathbb{Z}_+$, and the embedded walk $K_{\pi \circ \Phi_n(\tau_n), \tau_n}$. It follows, by strong convergence, that the limit cocycle $k^\psi$ of Theorem 4.6 is $\ast$-homomorphic; this shows directly that $k^\psi$ is an Evans–Hudson flow.
Example 4.8. If $\delta, H_d, H_o$ and $Q$ are as in Example 4.3 and the generator $\Phi(\tau)$ is defined by (14) then $K^{\pi\Phi(\tau_n)\tau_n} \to_{cb} k^\psi$ if $\tau_n \to 0^+$, where $\psi$ is such that

$$E^\omega \psi(a) E_\omega = -i[a, H_{sys}] + \bar{\rho}(H_o(a \otimes I_K)H_o) - \frac{1}{2}\{a, \bar{\rho}(H_o^2)\},$$

$$E^{[X]} \psi(a) E_\omega = -i[a, E^X \bar{\rho}(QV + V^*Q^*E_Y e_0)],$$

$$E^\omega \psi(a) E_{[Y]} = -i[a, E^Y \bar{\rho}(QV + V^*Q^*E_Y e_0)],$$

and $E^{[X]} \psi(a) E_{[Y]} = 0$

for all $X, Y \in \ker \rho$ and $a \in \mathcal{M}$. When $\dim K < \infty$, the map $a \mapsto E^\omega \psi(a) E_\omega$ is the Lindblad generator of [1, Corollary 13]. Formally, the above agrees with Example 2.18 when $\lambda_0 = 1$ and $\lambda_j = 0$ for all $j > 0$.

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