SOME REVERSE $l_p$-TYPE INEQUALITIES INVOLVING CERTAIN QUASI MONOTONE SEQUENCES

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ABSTRACT. In this paper, we give some $l_p$-type inequalities about sequences satisfying certain quasi monotone type properties. As special cases, reverse $l_p$-type inequalities for non-negative decreasing sequences are obtained. The inequalities are closely related to Copson’s and Leindler’s inequalities, but the sign of the inequalities is reversed.

1. Introduction

For non-negative number sequences the following, classical inequalities of Hardy and Littlewood, are well known [3, p. 255, Th 346].

Let $\{b_\nu\}_{\nu=1}^\infty$ be a sequence of non-negative numbers, $\alpha > 0$, $m$ and $n$ positive integers such that $n < m$. The following inequalities hold true:

\[
\sum_{\mu=n}^{m} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{m} b_{\nu} \right)^p \leq C \sum_{\mu=n}^{m} \mu^{\alpha-1} (\mu b_{\mu})^p,
\]

\[
\sum_{\mu=n}^{m} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{m} b_{\nu} \right)^p \leq C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (\mu b_{\mu})^p
\]

for $p \geq 1$; and

\[
\sum_{\mu=n}^{m} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{m} b_{\nu} \right)^p \geq C \sum_{\mu=n}^{m} \mu^{\alpha-1} (\mu b_{\mu})^p,
\]

\[
\sum_{\mu=n}^{m} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{m} b_{\nu} \right)^p \geq C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (\mu b_{\mu})^p
\]

for $0 < p \leq 1$, where positive constant $C$ depends only on numbers $\alpha$ and $p$, and does not depend on integers $m$, $n$, and the sequence $\{b_\nu\}_{\nu=1}^\infty$.

Closely related to these inequalities are classical Copson inequalities [1], Leindler’s inequalities [5, 6, 7], and those proved or used in [8, 9, 10, 2].

In the paper we prove some related inequalities which involve non-negative sequences satisfying certain monotone-type properties. As special cases, inequalities converse to (1.1), (1.2), (1.3) and (1.4) for the case of non-negative monotone decreasing number sequences are deduced.

In order to prove the inequalities we need the following

Theorem 1.1. Let $\{b_\nu\}_{\nu=1}^\infty$ be a sequence of non-negative numbers, $0 < \alpha < \beta$, $m$ and $n$ positive integers such that $n < m$. Then the following inequality holds

\[
\left( \sum_{\mu=n}^{m} b_{\mu}^{\beta} \right)^{1/\beta} \leq \left( \sum_{\mu=n}^{m} b_{\mu}^{\alpha} \right)^{1/\alpha}.
\]

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The proof of the theorem is due to Jensen [3, p. 28, Th. 19].

We call a number sequence \( \{a_\nu\}_{\nu=1}^\infty \) non-negative monotone decreasing (or increasing), and denote it by \( a_\nu \downarrow \) (or \( a_\nu \uparrow \)), if for each positive integer \( \nu \) the following conditions are satisfied
\[
\begin{align*}
(1) & \quad a_\nu \geq 0, \\
(2) & \quad a_{\nu+1} \leq a_\nu \quad {\rm (or} \ a_{\nu+1} \geq a_\nu {\rm, \ respectively)}.
\end{align*}
\]

We call a number sequence \( \{\lambda_\nu\}_{\nu=1}^\infty \) quasi lacunary monotone if it is a non-negative monotone sequence (i.e., \( \lambda_\nu \downarrow \) or \( \lambda_\nu \uparrow \)) and there are positive constants \( K_1 \) and \( K_2 \) such that for each positive integer \( \nu \) the following condition is satisfied
\[
K_1 \lambda_\nu \leq \lambda_{2\nu} \leq K_2 \lambda_\nu.
\]

Finally, we call a non-negative number sequence \( \{\lambda_\mu\}_{\mu=1}^\infty \) quasi geometrically increasing if there is a positive constant \( K \) such that for each positive integer \( m \) the following condition is satisfied
\[
\sum_{\mu=1}^m \lambda_\mu \leq K \lambda_m.
\]

2. Inequalities for quasi monotone sequences

In this section we give several reverse inequalities of \( l_\mu \)-type involving non-negative decreasing sequences, quasi lacunary monotone sequences or quasi geometrically increasing sequences.

**Theorem 2.1.** Let a sequence \( \{a_\nu\}_{\nu=1}^\infty \) be such that \( a_\nu \downarrow \), \( \{\lambda_\mu\}_{\mu=1}^\infty \) and \( \{\gamma_\nu\}_{\nu=1}^\infty \) be lacunary monotone sequences, \( m \) and \( n \) positive integers such that \( m \geq 16n \). If \( p \geq 1 \), then the following inequality holds
\[
\sum_{\mu=m}^n \lambda_\mu \left( \sum_{\nu=n}^m a_\nu \gamma_\nu \right)^p \geq C \sum_{\mu=4n}^m \lambda_\mu (ma_\mu \gamma_\mu)^p.
\] (2.1)

If, in addition, \( \{2^{\mu} \lambda_\nu\}_{\nu=1}^\infty \) is a quasi geometrically increasing sequence, then the following inequality holds
\[
\sum_{\mu=m}^n \lambda_\mu \left( \sum_{\nu=\mu}^m a_\nu \gamma_\nu \right)^p \geq C \sum_{\mu=8n}^m \lambda_\mu (ma_\mu \gamma_\mu)^p.
\] (2.2)

Here and later on \( C \) and \( C_i \) will denote positive constants depending only on \( p \) and the sequences \( \{\lambda_\mu\}_{\mu=1}^\infty \) and \( \{\gamma_\nu\}_{\nu=1}^\infty \), and not depending on \( m, n \) and the sequence \( \{a_\nu\}_{\nu=1}^\infty \).

**Theorem 2.2.** Let a sequence \( \{a_\nu\}_{\nu=1}^\infty \) be such that \( a_\nu \downarrow \), \( \{\lambda_\mu\}_{\mu=1}^\infty \) and \( \{\gamma_\nu\}_{\nu=1}^\infty \) be lacunary monotone sequences, \( m \) and \( n \) positive integers such that \( m \geq 4n \). If \( 0 < p \leq 1 \), then the following inequality holds
\[
\sum_{\mu=4n}^m \lambda_\mu \left( \sum_{\nu=4n}^\mu a_\nu \gamma_\nu \right)^p \leq C \sum_{\mu=n}^m \lambda_\mu (ma_\mu \gamma_\mu)^p.
\] (2.3)

If, in addition, \( \{2^{\mu} \lambda_\nu\}_{\nu=1}^\infty \) is a quasi geometrically increasing sequence, then the following inequality holds
\[
\sum_{\mu=4n}^m \lambda_\mu \left( \sum_{\nu=\mu}^m a_\nu \gamma_\nu \right)^p \leq C \sum_{\mu=n}^m \lambda_\mu (ma_\mu \gamma_\mu)^p.
\] (2.4)
3. Proof of Theorem 2.1

We prove inequality (2.2). For given $n$ and $m$ we choose positive integers $N$ and $M$ such that $2^{N-1} < n \leq 2^N$ and $2^M \leq m < 2^{M+1}$. Then the following inequality holds

$$I = \sum_{\mu=n}^{m} \lambda_{\mu} \left( \sum_{\nu=\mu}^{m} a_{\nu} \gamma_{\nu} \right)^p \geq \sum_{\mu=2^N}^{2^M} \lambda_{\mu} \left( \sum_{\nu=\mu}^{2^M} a_{\nu} \gamma_{\nu} \right)^p.$$ 

By splitting the first sum into blocks of length $2^i$, we obtain

$$I \geq \sum_{i=N+1}^{M} \sum_{\mu=2^{i-1}+1}^{2^i} \lambda_{\mu} \left( \sum_{\nu=\mu}^{2^i} a_{\nu} \gamma_{\nu} \right)^p.$$ 

By bounding the third sum from below, taking into account that $\{\lambda_{\mu}\}_{\mu=1}^{\infty}$ is a quasi lacunary monotone sequence, we have

$$I \geq \sum_{i=N+1}^{M} \left( \sum_{\mu=2^i}^{2^i} \lambda_{\mu} \right) \left( \sum_{\nu=2^i}^{2^{i+1}} a_{\nu} \gamma_{\nu} \right)^p \geq \sum_{i=N+1}^{M} \lambda_{2^i} \left( \sum_{\nu=2^i}^{2^{i+1}} a_{\nu} \gamma_{\nu} \right)^p,$$

where $C_1$ depends only on the sequence $\{\lambda_{\mu}\}_{\mu=1}^{\infty}$. Now, we split the second sum into blocks of length $2^{i-1}$, remove the terms with index $i = M$, and taking into consideration that $a_{\nu} \downarrow$ and $\{\lambda_{\mu}\}_{\mu=1}^{\infty}, \{\gamma_{\nu}\}_{\nu=1}^{\infty}$ are lacunary monotone sequences, we get

$$I \geq C_2 \sum_{i=N+1}^{M-1} 2^i \lambda_{2^i} \left( \sum_{j=1}^{M-1} a_{2^j+1} \sum_{\nu=2^j+1}^{2^{j+1}} \gamma_{\nu} \right)^p \geq C_3 \sum_{i=N+1}^{M-1} 2^j \lambda_{2^i} \left( \sum_{j=1}^{M-1} a_{2^j+1} 2^{j+1} \gamma_{2^j} \right)^p.$$ 

By applying Theorem 1.1 to this inequality taking into account that $1 \leq p$, then changing the order of summation, we have

$$I \geq C_4 \sum_{j=N+1}^{M-1} 2^{j+1} \sum_{\mu=2^j}^{2^{j+1}} a_{\mu}^p \gamma_{\mu}^p \lambda_{\mu} \geq C_3 \sum_{j=N+1}^{M-1} a_{2^j+1} 2^{j+1} \gamma_{2^j}^p \sum_{j=N+1}^{M-1} 2^i \lambda_{2^i} \geq C_3 \sum_{j=N+1}^{M-1} a_{2^j+1} 2^{j+1} \gamma_{2^j}^p \lambda_{2^j}.$$ 

Since $a_{\nu} \downarrow$, taking into consideration that $\{\gamma_{\nu}\}_{\nu=1}^{\infty}$ and $\{\lambda_{\mu}\}_{\mu=1}^{\infty}$ are quasi lacunary monotone and quasi geometrically increasing sequences, respectively, we obtain

$$I \geq C_4 \sum_{j=N+1}^{M-1} 2^{j+1} \sum_{\mu=2^j+1}^{2^{j+1}} a_{\mu}^p \gamma_{\mu}^p \lambda_{\mu}.$$ 

We rewrite the above inequality in the form

$$I \geq C_4 \sum_{\mu=2^{N+2}+1}^{2^{M+1}} \lambda_{\mu} (\mu a_{\mu} \gamma_{\mu})^p,$$

and since $2^{N+2} < 8n$, we obtain

$$I \geq C_4 \sum_{\mu=8n}^{n} \lambda_{\mu} (\mu a_{\mu} \gamma_{\mu})^p.$$ 

Thus, we have proved inequality (2.2) assuming that $N+1 \leq M-1$. In fact, for $m \geq 16n$ we get $2^{N-1} \leq n \leq \frac{n}{16} \leq 2^{M-3}$, yielding that the condition $N+1 \leq M-1$ is satisfied.
In order to prove inequality (2.1), put

\[ J = \sum_{\mu=n}^{m} \lambda_{\mu} \left( \sum_{\nu=n}^{\mu} a_{\nu} \gamma_{\nu} \right)^{p}, \]

in a similar manner, but by making use of the fact that \( \{\lambda_{\mu}\}_{\mu=1}^{\infty} \) is solely a quasi lacunary monotone sequence (i.e. without a quasi geometrically increasing sequence assumption), we obtain

\[ J \geq C_{5} \sum_{\mu=2^{N+1}}^{2^{M-1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p}. \]

Thus

\[ \sum_{\mu=2^{N+1}}^{2^{M-1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p} \leq C_{6}J. \]  
(3.1)

Since \( a_{\nu} \downarrow \) and \( \{\lambda_{\mu}\}_{\mu=1}^{\infty}, \{\gamma_{\nu}\}_{\nu=1}^{\infty} \) are lacunary monotone sequences, we have

\[ J_{1} = \sum_{\mu=2^{M}}^{2^{M+1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p} \leq \sum_{\mu=2^{M}}^{2^{M+1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p} \leq C_{7} \sum_{\mu=2^{M-1}}^{2^{M-1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p}. \]

Hence, for \( N+1 \leq M-1 \) we obtain

\[ J_{1} \leq \sum_{\mu=2^{N+1}}^{2^{M-1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p}. \]

Thus, inequality (3.1) yields

\[ J_{1} \leq C_{8}J; \]

or

\[ \sum_{\mu=2^{M}}^{2^{M+1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p} \leq C_{8}J. \]  
(3.2)

Adding inequalities (3.1) and (3.2) together, we obtain

\[ \sum_{\mu=2^{N+1}}^{2^{M+1}} \lambda_{\mu}(\mu a_{\mu} \gamma_{\mu})^{p} \leq C_{9} \sum_{\mu=n}^{m} \lambda_{\mu} \left( \sum_{\nu=n}^{\mu} a_{\nu} \gamma_{\nu} \right)^{p}. \]

Since \( 2^{N+1} \leq 4n, \ 2^{M+1} > m \), the above inequality implies the inequality (2.1). This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

We prove the inequality (2.3). Let positive integers \( N \) and \( M \) be defined by the inequalities \( 2^{N-1} < n \leq 2^{N} \) and \( 2^{M} < m < 2^{M+1} \). This yields

\[ I = \sum_{\mu=4n}^{m} \lambda_{\mu} \left( \sum_{\nu=\mu}^{m} a_{\nu} \gamma_{\nu} \right)^{p} \leq \sum_{\mu=2^{N+1}+1}^{2^{M+1}} \lambda_{\mu} \left( \sum_{\nu=\mu}^{2^{M+1}} a_{\nu} \gamma_{\nu} \right)^{p} \]

\[ = \sum_{i=N+1}^{M} \sum_{\mu=2^{i}+1}^{2^{i}+1} \lambda_{\mu} \left( \sum_{\nu=\mu}^{2^{i}+1} a_{\nu} \gamma_{\nu} \right)^{p}. \]
By bounding the third sum from above, taking into account that \( \{\lambda_\mu\}_{\mu=1}^{\infty} \) is a quasi lacunary monotone sequence, we get

\[
I \leq \sum_{i=N+1}^{M} \left( \sum_{\nu=2i+1}^{2^{i+1}} a_{\nu} \gamma_{\nu} \right) p \sum_{\mu=2i+1}^{2^{i+1}} \lambda_{\mu} \leq C_1 \sum_{i=N+1}^{M} 2^i \lambda_{2i} \left( \sum_{\nu=2i+1}^{2^{i+1}} a_{\nu} \gamma_{\nu} \right) p,
\]

where positive constant \( C_1 \) depends only on the sequence \( \{\lambda_\mu\}_{\mu=1}^{\infty} \). Now, we split the second sum into blocks of length \( 2^j \) and taking into consideration the fact that \( a_\nu \downarrow \), and \( \{\gamma_\nu\}_{\nu=1}^{\infty} \) is a lacunary monotone sequences, we have

\[
I \leq C_1 \sum_{i=N+1}^{M} 2^i \lambda_{2i} \left( \sum_{j=1}^{M} \sum_{\nu=2i+1}^{2^{i+1}} \gamma_{\nu} \right) p \leq C_2 \sum_{i=N+1}^{M} 2^i \lambda_{2i} \left( \sum_{\nu=2i+1}^{2^{i+1}} a_{\nu} 2^i \gamma_{\nu} \right) p.
\]

By applying Theorem 2.1 and then changing the order of summation, we obtain

\[
I \leq C_2 \sum_{i=N+1}^{M} 2^i \lambda_{2i} \left( \sum_{j=1}^{M} \sum_{\nu=2i+1}^{2^{i+1}} a_{\nu} 2^j \gamma_{2j} \right) \leq C_2 \sum_{i=N+1}^{M} 2^i \lambda_{2i} \left( \sum_{j=1}^{M} \sum_{i=N+1}^{j} 2^i \lambda_{2i} \right).
\]

Further, the fact that \( \{\lambda_\mu\}_{\mu=1}^{\infty} \) is a quasi geometrically increasing sequence yields

\[
I \leq C_3 \sum_{j=N+1}^{M} a_{2^j} 2^{j(p-1)} \gamma_{2j} \lambda_{2j}.
\]

Since \( a_\nu \downarrow \), taking into consideration that \( \{\gamma_\nu\}_{\nu=1}^{\infty} \) and \( \{\lambda_\mu\}_{\mu=1}^{\infty} \) are quasi lacunary monotone sequences, we get

\[
a_{2^j} 2^{j(p-1)} \gamma_{2j} \lambda_{2j} \leq C_4 \sum_{\mu=2^{j-1}+1}^{2^j} a_{\mu} \gamma_{\mu} \lambda_{\mu}.
\]

Therefore,

\[
I \leq C_5 \sum_{j=N+1}^{M} \sum_{\mu=2^{j-1}+1}^{2^j} \lambda_{\mu} (\mu a_\mu \gamma_\mu)^p = C_5 \sum_{\mu=2^{N+1}}^{2^M} \lambda_{\mu} (\mu a_\mu \gamma_\mu)^p.
\]

where positive constant \( C_5 \) does not depend on \( N \) and \( M \). Since \( 2^M \leq m \) and \( 2^N + 1 > m \), we obtain

\[
I \leq C_6 \sum_{\mu=m}^{m} \lambda_{\mu} (\mu a_\mu \gamma_\mu)^p,
\]

which proves inequality (2.4).

Inequality (2.3) can be proved in an analogous way, but without a quasi geometrically increasing assumption for the sequence \( \{2^\mu \lambda_{2^\mu}\}_{\mu=1}^{\infty} \).

5. Inequalities for non-negative decreasing sequences

For \( \alpha, \lambda \in \mathbb{R} \) put

\[
\lambda_{\mu} = \mu^{\alpha-1} \quad (\mu = 1, 2, \ldots),
\gamma_{\nu} = \nu^{\lambda} \quad (\nu = 1, 2, \ldots).
\]

Obviously, the obtained sequences \( \{\mu^{\alpha-1}\}_{\mu=1}^{\infty} \) and \( \{\nu^{\lambda}\}_{\nu=1}^{\infty} \) are both quasi lacunary monotone sequences.

If, in addition, \( \alpha > 0 \), then \( \{2^\mu \mu^{\alpha-1}\}_{\mu=1}^{\infty} \) is a quasi geometrically increasing sequence.

By applying Theorems 2.1 and 2.2 for such sequences \( \{\lambda_\mu\}_{\mu=1}^{\infty} \) and \( \{\gamma_\nu\}_{\nu=1}^{\infty} \), we deduce the following \( l_p \)-type inequalities for non-negative decreasing sequences, which are converse to inequalities (1.1), (1.2), (1.3) and (1.4).
Theorem 5.1. Let a sequence \( \{a_\nu\}_{\nu=1}^{\infty} \) be such that \( a_\nu \downarrow, \) and \( \alpha > 0, \lambda \in \mathbb{R}, \) \( m \) and \( n \) positive integers such that \( m \geq 16n. \) If \( p \geq 1, \) then the following inequalities hold
\[
\sum_{\mu=n}^{m} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{m} a_\nu \nu^{\lambda} \right)^p \geq C \sum_{\mu=n}^{m} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\
\sum_{\mu=n}^{m} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{m} a_\nu \nu^{\lambda} \right)^p \geq C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.
\]

Hereafter \( C \) denotes positive constant depending only on \( \alpha, \lambda \) and \( p, \) and not depending on \( m, n \) and the sequence \( \{a_\nu\}_{\nu=1}^{\infty}. \)

Theorem 5.2. Let a sequence \( \{a_\nu\}_{\nu=1}^{\infty} \) be such that \( a_\nu \downarrow, \) and \( \alpha > 0, \lambda \in \mathbb{R}, \) \( m \) and \( n \) positive integers such that \( m \geq 4n. \) If \( 0 < p \leq 1, \) then the following inequalities hold
\[
\sum_{\mu=4n}^{m} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{m} a_\nu \nu^{\lambda} \right)^p \leq C \sum_{\mu=n}^{m} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\
\sum_{\mu=4n}^{m} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{m} a_\nu \nu^{\lambda} \right)^p \leq C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.
\]

Note that Theorems 5.1 and 5.2 given above imply several inequalities proved earlier [3,4,11].

Namely the following Corollaries are simple consequences of these theorems and the inequalities (1.1), (1.2), (1.3) and (1.4).

Corollary 5.1. Let a sequence \( \{a_\nu\}_{\nu=1}^{\infty} \) be such that \( a_\nu \downarrow, \alpha > 0, \lambda \in \mathbb{R}, \) and \( n \) a positive integer. If \( p > 0, \) then the following inequalities hold
\[
\sum_{\mu=1}^{n} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{n} a_\nu \nu^{\lambda} \right)^p \geq C \sum_{\mu=1}^{n} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\
\sum_{\mu=1}^{n} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{n} a_\nu \nu^{\lambda} \right)^p \geq C \sum_{\mu=1}^{n} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.
\]

Corollary 5.2. Let a sequence \( \{a_\nu\}_{\nu=1}^{\infty} \) be such that \( a_\nu \downarrow, \alpha > 0, \lambda \in \mathbb{R}, \) and \( n \) a positive integer. If \( p \geq 1, \) then the following asymptotic equivalences hold
\[
\sum_{\mu=1}^{n} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{n} a_\nu \nu^{\lambda} \right)^p \asymp \sum_{\mu=1}^{n} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\
\sum_{\mu=1}^{n} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{n} a_\nu \nu^{\lambda} \right)^p \asymp \sum_{\mu=1}^{n} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.
\]

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SOME REVERSE $l_p$-TYPE INEQUALITIES...

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