Remarks on an inequality involving the normal scalar curvature

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Abstract

We study a pointwise inequality for submanifolds in real space forms involving the scalar curvature, the normal scalar curvature and the mean curvature. We translate it into an algebraic problem, allowing us to prove a slightly weaker version of it. We also prove the conjecture for certain types of submanifolds of $\mathbb{C}^n$.

1 Introduction

In 1983, Guadelupe and Rodriguez proved the following:

Theorem 1.1 ([11]). Let $M^2$ be a surface in a real space form $\tilde{M}^{2+m}(c)$ of constant sectional curvature $c$. Denote by $K$ the Gaussian curvature of $M^2$, by $H$ the mean curvature vector and by $K^\perp$ the normal scalar curvature. Then

$$K \leq ||H||^2 - K^\perp + c$$

at every point $p$ of $M^2$, with equality if and only if the ellipse of curvature at $p$ is a circle.

Remark that this is an extension of the well-known inequality $K \leq ||H||^2$ for surfaces in $E^3$.

In [9] the following was conjectured as generalization of the previous Theorem.

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Conjecture 1.1 ([9]). Let $M^n$ be a submanifold of a real space form $\tilde{M}^{n+m}(c)$ of constant sectional curvature $c$. Denote by $\rho$ the normalized scalar curvature, by $H$ the mean curvature vector and by $\rho^\perp$ the normalized normal scalar curvature. Then

$$\rho \leq \|H\|^2 - \rho^\perp + c.$$  \hspace{1cm} (1)

The conjecture was proved for $m=2$ in [9], where also some classification results were obtained in case equality holds in (1) at every point.

Remark 1.1 (Added remark on recent developments). Nowadays, this conjecture is known as the DDVV-conjecture. Recently the conjecture was proved for $n=3$ in [6] and for $m=3$ in [13]. In a private communication [14], Z. Lu announced a proof for the general case. Also in the study of submanifolds attaining equality there is recently substantial progress: see [7] and [17]. All these results were obtained after the finishing of this paper.

For normally flat submanifolds, in particular for hypersurfaces, inequality (1) follows from a more general result of Chen ([2]). In particular, we have for any submanifold $M^n$ of a real space form $\tilde{M}^{n+m}(c)$:

$$\rho \leq \|H\|^2 + c.$$  \hspace{1cm} (2)

For immersions which are invariant with respect to the standard Kählerian and Sasakian structures on $E^{2k}$ and $S^{2k+1}(1)$ the conjecture was proved in [8] and for immersions which are totally real with respect to the nearly Kähler structure on $S^6(1)$ in [10].

In section 3 we will translate the conjecture to an algebraic problem involving symmetric matrices, followed by a proof of a weaker version. In section 4 we will prove the conjecture for $H$-umbilical Lagrangian submanifolds of $C^n \cong E^{2n}$, for minimal Lagrangian submanifolds of $C^3 \cong E^6$ and for ultra-minimal Lagrangian submanifolds of $C^4 \cong E^8$. We remark that some of these results have been generalized in the meantime by A. Mihai in [15], see [16] in the present volume. The reader should be warned however that the notations in [16] and in this paper are not always consistent.

## 2 Preliminaries

Let $M^n$ be a Riemannian manifold of dimension $n$ with Riemann-Christoffel curvature tensor $R$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T_pM$, then we define the normalized scalar curvature of $M^n$ at $p$ by

$$\rho = \frac{2}{n(n-1)} \sum_{i<j=1}^n \langle R(e_i,e_j)e_j, e_i \rangle.$$  \hspace{1cm} (3)

Now let $\tilde{M}^{n+m}$ be another Riemannian manifold with Riemann-Christoffel curvature tensor $\tilde{R}$ and let $f : M^n \rightarrow \tilde{M}^{n+m}$ be an isometric immersion. If $h$ is the second fundamental form, $A_U$ the shape-operator associated to a normal vector field $U$, and $R^\perp$ the curvature tensor of the normal connection, then the equations of Gauss and Ricci are given by

$$\langle R(X,Y)Z,T \rangle = \langle \tilde{R}(X,Y)Z,T \rangle + \langle h(X,T), h(Y,Z) \rangle - \langle h(X,Z), h(Y,T) \rangle.$$  \hspace{1cm} (4)
\[ \langle R^\perp(X,Y)U,V \rangle = \langle \bar{R}(X,Y)U,V \rangle + \langle [A_U,A_V]X,Y \rangle, \]  

(5)

for tangent vectors \( X, Y, Z \) and \( T \) and normal vectors \( U \) and \( V \).

Let \( \{e_1, \ldots, e_n\} \) be as above and suppose that \( \{u_1, \ldots, u_m\} \) is an orthonormal basis for \( T_p^\perp M \). Then we define the normalized normal scalar curvature of \( M^n \) at \( p \) by

\[
\rho^\perp = \frac{2}{n(n-1)} \sqrt{\sum_{i<j}^{n} \sum_{\alpha<\beta}^{m} \langle R^\perp(e_i,e_j)u_\alpha,u_\beta \rangle^2},
\]

(6)

which corresponds to the definition proposed in [9]. Another extrinsic curvature invariant that we will use is the mean curvature vector of the submanifold at \( p \):

\[
H = \frac{1}{n} \text{tr}(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i,e_i).
\]

(7)

3 A translation of the problem

From now on, we use the following convention: if \( A \) and \( B \) are \((n \times n)\)-matrices, we define \( \langle A, B \rangle = \text{tr}(A^t \cdot B) \). The associated norm is then given by \( \|A\|_2 = \text{tr}(A^t \cdot A) = \sum_{i,j} (A_{ij})^2 \).

The scalar product, and hence the norm are preserved by orthogonal transformations.

3.1 The translation

The following theorem reduces the conjecture to an inequality involving symmetric \((n \times n)\)-matrices.

**Theorem 3.1.** Conjecture 1.1 is true for submanifolds of dimension \( n \) and codimension \( m \) if for every set \( \{B_1, \ldots, B_m\} \) of symmetric \((n \times n)\)-matrices with trace zero the following inequality holds:

\[
\sum_{\alpha,\beta=1}^{m} \|[B_\alpha,B_\beta]\|^2 \leq \left( \sum_{\alpha=1}^{m} \|B_\alpha\|^2 \right)^2. 
\]

(8)

**Proof.** Let \( M^n \) be a submanifold of \( \tilde{M}^{n+m}(c) \). Take \( p \in M^n \) and suppose that \( \{e_1, \ldots, e_n\} \) is an orthonormal basis for \( T_p M \) and that \( \{u_1, \ldots, u_m\} \) is an orthonormal basis for \( T_p^\perp M \). In summations, Latin indices will always range from 1 to \( n \), whereas Greek indices range from 1 to \( m \). Further, we use the notations introduced in the previous section. We define a symmetric \((1,2)\)-tensor \( b \), taking normal values, by

\[
b(X,Y) = h(X,Y) - \langle X,Y \rangle H
\]

for all \( X,Y \in T_p M \). Remark that

\[
\|b\|^2 = \sum_{i,j} \|b(e_i,e_j)\|^2 = \sum_{i,j} \|h(e_i,e_j)\|^2 - 2 \sum_i \langle h(e_i,e_i), H \rangle + n \|H\|^2 = \|h\|^2 - n \|H\|^2.
\]

(9)

Now we define a set \( \{B_1, \ldots, B_m\} \) of symmetric operators on \( T_p M \) by

\[
\langle B_\alpha X,Y \rangle = \langle b(X,Y), u_\alpha \rangle
\]
for all $X, Y \in T_pM$. It is clear that $B_\alpha = A_{u_\alpha} - \langle H, u_\alpha \rangle \text{id}$, and thus
\[ [B_\alpha, B_\beta] = [A_{u_\alpha}, A_{u_\beta}]. \quad (10) \]

Using the equation of Gauss (11) and (9), we find
\[ \rho = \frac{2}{n(n-1)} \sum_{i<j} (R(e_i, e_j)e_j, e_i) \]
\[ = \frac{2}{n(n-1)} \sum_{i<j} \left( c + \langle h(e_i, e_i), h(e_j, e_j) \rangle - \|h(e_i, e_j)\|^2 \right) \]
\[ = c + \frac{2}{n(n-1)} \left( \frac{n^2}{2} \|H\|^2 - \frac{1}{2} \|b\|^2 \right) \]
\[ = c + \frac{n}{n-1} \|H\|^2 - \frac{1}{n(n-1)} \|b\|^2, \]
and thus
\[ \|H\|^2 - \rho + c = \frac{1}{n(n-1)} \|b\|^2 = \frac{1}{n(n-1)} \sum_{\alpha} \|B_\alpha\|^2 \geq 0. \]

From the equation of Ricci (15) and (10), we get
\[ \rho^\perp = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha, \beta} \|[A_{u_\alpha}, A_{u_\beta}]\|^2} = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha, \beta} \|[B_\alpha, B_\beta]\|^2}. \]

We conclude that
\[ \rho \leq \|H\|^2 - \rho^\perp + c \iff (\rho^\perp)^2 \leq (\|H\|^2 - \rho + c)^2 \]
\[ \iff \sum_{\alpha, \beta} \|[B_\alpha, B_\beta]\|^2 \leq \left( \sum_{\alpha} \|B_\alpha\|^2 \right)^2. \]

\[ \Box \]

**Remark 3.1.** By proving Theorem 2 for $m = 2$, we obtain a simple proof of the conjecture for codimension 2 submanifolds:
\[ (\|B_1\|^2 + \|B_2\|^2)^2 \geq 4\|B_1\|^2\|B_2\|^2 \geq 2\|[B_1, B_2]\|^2, \]
where the second inequality is due to Chern, do Carmo and Kobayashi [5], see Lemma 3.1 below.

### 3.2 Proof of a weaker version of the inequality

First, we recall two inequalities.
Lemma 3.1 ([5]). If $B_1$ and $B_2$ are symmetric $(n \times n)$-matrices, then

$$||[B_1, B_2]||^2 \leq 2||B_1||^2||B_2||^2,$$

with equality if and only if $B_1 = B_2 = 0$ or, after a suitable orthogonal transformation,

$$B_1 = \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ \mu_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \mu_2 & 0 & 0 & \cdots & 0 \\ 0 & -\mu_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$  \hspace{1cm} (11)

Theorem 3.2 ([12]). Let \(\{B_1, \ldots, B_m\}\) be a set of symmetric \((n \times n)\)-matrices. Then

$$\sum_{\alpha,\beta=1}^m ||[B_\alpha, B_\beta]||^2 + \sum_{\alpha,\beta=1}^m \langle B_\alpha, B_\beta \rangle^2 \leq \frac{3}{2} \left( \sum_{\alpha=1}^n ||B_\alpha||^2 \right)^2.$$ 

We will use these inequalities to proof the following, weaker version of conjecture 1.1:

Theorem 3.3. Let \(M^n\) be a submanifold of a real space form \(\tilde{M}^{n+m}\). Then

(i) \(\rho \leq ||H||^2 - \frac{2m-1}{3m-3} \rho^2 + c,\)

(ii) \(\rho \leq ||H||^2 - \frac{2n^2+n-3}{3n^2+6n-12} \rho^2 + c.\)

Proof. Define the matrices \(B_\alpha\) as in the proof of theorem 3.1. After a suitable orthogonal transformation, we may assume that \(\langle B_\alpha, B_\beta \rangle = ||B_\alpha||^2 \delta_{\alpha\beta}.\) The inequality of Cauchy-Schwarz yields \((\sum_{\alpha=1}^m ||B_\alpha||^2)^2 \leq m \sum_{\alpha=1}^m ||B_\alpha||^4,\) and thus

\[
\sum_{\alpha \neq \beta=1}^m ||B_\alpha||^2||B_\beta||^2 \leq (m-1) \sum_{\alpha=1}^m ||B_\alpha||^4.
\]

This inequality, together with lemma 3.1 and theorem 3.2 gives

\[
\frac{3}{2} \left( \sum_{\alpha=1}^n ||B_\alpha||^2 \right)^2 \geq \sum_{\alpha,\beta=1}^m ||[B_\alpha, B_\beta]||^2 + \sum_{\alpha=1}^m ||B_\alpha||^4
\]

\[
\geq \sum_{\alpha,\beta=1}^m ||[B_\alpha, B_\beta]||^2 + \frac{1}{m-1} \left( \sum_{\alpha \neq \beta=1}^m ||B_\alpha||^2||B_\beta||^2 \right)
\]

\[
\geq \sum_{\alpha,\beta=1}^m ||[B_\alpha, B_\beta]||^2 + \frac{1}{2(m-1)} \sum_{\alpha,\beta=1}^m ||[B_\alpha, B_\beta]||^2
\]

\[
\geq \frac{2m-1}{2m-2} \sum_{\alpha,\beta=1}^m ||[B_\alpha, B_\beta]||^2. \hspace{1cm} (12)
\]

Inequality (12) implies

\[
\frac{3}{2} (||H||^2 - \rho + c)^2 \geq \frac{2m-1}{2m-2} (\rho^2)^2,
\]

which yields the first inequality stated in the theorem. To prove the second one, remark that we may replace \(m\) by the dimension of the image of \(b.\) The result follows from the observation \(\dim(\text{im}(b)) \leq \frac{n(n+1)}{2} - 1.\) \hspace{1cm} \(\square\)
4 Lagrangian submanifolds

In this section, we prove the conjecture for three families of Lagrangian submanifolds, namely for $H$-umbilical Lagrangian submanifolds of $\mathbb{C}^n \cong \mathbb{E}^{2n}$, for minimal Lagrangian submanifolds of $\mathbb{C}^3 \cong \mathbb{E}^6$ and for ultraminimal Lagrangian submanifolds of $\mathbb{C}^4 \cong \mathbb{E}^8$.

Recall that a submanifold $M$ of a Kählerian manifold $\tilde{M}^{2n}$ is called Lagrangian if at every point the almost complex structure $J$ of $\tilde{M}^{2n}$ induces an isomorphism between $T_p M$ and $T_{-p} M$. In particular $\dim(M) = n$. The second fundamental form satisfies the following symmetry property:

$$\langle h(X, Y), JZ \rangle = \langle h(X, Z), JY \rangle,$$

(13)

for $X, Y, Z \in T_p M$.

4.1 $H$-umbilical Lagrangian immersions in $\mathbb{C}^n$

It was proven in [4] that there are no totally umbilical Lagrangian submanifolds in complex space forms, except totally geodesic ones. $H$-umbilical Lagrangian submanifolds are introduced in [3] as the ‘simplest’ Lagrangian submanifolds next to totally geodesic ones. Their second fundamental form satisfies

$$h(E_1, E_1) = \lambda J E_1, \quad h(E_2, E_2) = \ldots = h(E_n, E_n) = \mu J E_1, \quad h(E_1, E_j) = \mu J E_j, \quad h(E_j, E_k) = 0 \text{ for } j, k \in \{2, \ldots, n\}, j \neq k.$$

(14)

for some suitable functions $\lambda$ and $\mu$ and a suitable orthonormal local frame field $\{E_1, \ldots, E_n\}$ on $M^n$.

We prove the following:

**Theorem 4.1.** Let $M^n$ be a $H$-umbilical Lagrangian immersion in $\mathbb{C}^n \cong \mathbb{E}^{2n}$. Then

$$\rho \leq \|H\|^2 - \rho^\perp,$$

with equality at every point if and only if $M^n$ is totally geodesic.

**Proof.** From (14) the form of the shape-operators is easily deduced. We now use theorem 3.1. Defining the matrices $B_\alpha$ as in the proof of that theorem, we easily see that

$$\sum_{\alpha, \beta=1}^n \| [B_\alpha, B_\beta] \|^2 = 2(n - 1)\mu^2 \left( (n - 2)\mu^2 + 2(\lambda - \mu)^2 \right),$$

$$\left( \sum_{\alpha=1}^n \| B_\alpha \|^2 \right)^2 = (n - 1)^2 \left( \frac{1}{n}(\lambda - \mu)^2 + 2\mu^2 \right)^2,$$

such that

$$\sum_{\alpha, \beta=1}^n \| [B_\alpha, B_\beta] \| \leq \left( \sum_{\alpha=1}^n \| B_\alpha \|^2 \right)^2 \Leftrightarrow 2n\mu^4 - 4n\mu^2(\lambda - \mu)^2 + \frac{n-1}{n^2}(\lambda - \mu)^4 \geq 0.$$

The last inequality is satisfied for every $\lambda$ and $\mu$ since the bilinear form $2nx^2 - \frac{4}{n}xy + \frac{n-1}{n^2}y^2$ is positive definite. \[\square\]
4.2 Minimal Lagrangian submanifolds of \( \mathbb{C}^3 \)

**Theorem 4.2.** Let \( M^3 \) be a minimal Lagrangian submanifold of \( \mathbb{C}^3 \). Then

\[
\rho \leq -\rho^\perp
\]

and equality holds at a point \( p \) if and only if there exists an orthonormal basis \( \{e_1, e_2, e_3\} \) of \( T_p M \) such that

\[
A_{Je_1} = \begin{pmatrix}
a & 0 & 0 \\
0 & -a & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix}
0 & -a & 0 \\
-a & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_{Je_3} = 0,
\]

with respect to this basis. If equality holds at every point of a minimal Lagrangian submanifold of \( \mathbb{C}^3 \), then \( M^3 \) is either a cylinder on complex curve in \( \mathbb{C}^2 \) (with respect to a different complex structure) or a “twisted special Lagrangian cone”, both in the sense of [17].

**Proof.** Let \( M^3 \) be a minimal Lagrangian submanifold of \( \mathbb{C}^3 \). Take \( p \in M^3 \) and consider the function

\[
f : \{ X \in T_p M \mid \|X\| = 1 \} \to \mathbb{R} : X \mapsto \langle h(X, X), JX \rangle.
\]

Take \( e_1 \in T_p M \) such that \( f \) attains its maximum value in \( e_1 \). Then \( \langle h(e_1, e_1), JY \rangle = 0 \) for every \( Y \perp e_1 \). Using (13), this implies that \( e_1 \) is an eigenvector of \( A_{Je_1} \). Choosing \( e_2 \) and \( e_3 \) such that \( \{e_1, e_2, e_3\} \) is an orthonormal basis for \( T_p M \) which diagonalizes \( A_{Je_1} \), we have that the shape-operators take the following form:

\[
A_{Je_1} = \begin{pmatrix}
a + b & 0 & 0 \\
0 & -a & 0 \\
0 & 0 & -b
\end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix}
0 & -a & 0 \\
-a & c & -d \\
0 & -d & -c
\end{pmatrix}, \quad A_{Je_3} = \begin{pmatrix}
0 & 0 & -b \\
0 & -d & -c \\
-b & -c & d
\end{pmatrix}.
\]

We now compute \( \rho \), using Gauss’ equation:

\[
3\rho = \sum_{i<j=1}^3 \langle R(e_i, e_j)e_j, e_i \rangle
\]

\[
= \sum_{i<j=1}^3 (\langle h(e_i, e_i), h(e_j, e_j) \rangle - \langle h(e_i, e_j), h(e_i, e_j) \rangle)
\]

\[
= (-2a^2 - ab) + (-2b^2 - ab) + (ab - 2c^2 - 2d^2)
\]

\[
= -2(a^2 + b^2 + c^2 + d^2) - ab.
\]

The computation of \( \rho^\perp \) using Ricci’s equation is completely analogous to that in [10], yielding

\[
9(\rho^\perp)^2 = \sum_{\alpha<\beta=1}^3 \sum_{i<j=1}^3 \langle R^\perp(e_i, e_j)e_\alpha, e_\beta \rangle^2
\]

\[
= \frac{1}{2} \sum_{\alpha<\beta=1}^3 \|[A_{Je_\alpha}, A_{Je_\beta}]\|^2
\]

\[
= 4(a^4 + b^4 + c^4 + d^4) + 4a^3b + 4ab^3
\]

\[+3a^2b^2 + 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 8c^2d^2 - 8abc^2 - 8abd^2.
\]
Using the same argument as in [10], we obtain that $9(\rho^\perp)^2 \leq (3\rho)^2$, which implies the inequality stated in the theorem, since $\rho \leq 0$ from [2]. Equality holds if and only if $c = d = 0$ and $ab = 0$. By, if necessary, changing the role of $e_2$ and $e_3$, we obtain the result.

For proving the statement on the equality case, it suffices to remark that when the shape operator has the form (15), then the cubic form $\langle h(X, Y), JZ \rangle$ has $S_3$-symmetry in the sense of [1] and therefore the classification following from the classification in [1].

We can extend the previous theorem to 3-dimensional Lagrangian submanifolds of complex space forms. For a complex space form of constant holomorphic sectional curvature $4c$, the curvature tensor takes the form

$$\tilde{R}(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY - 2\langle JX, Y \rangle JZ).$$

This implies that for a Lagrangian immersion in such a space, the equations of Gauss and Ricci read respectively:

$$\langle R(X, Y)Z, T \rangle = c(\langle Y, Z \rangle \langle X, T \rangle - \langle X, Z \rangle \langle Y, T \rangle) - \langle h(X, Z), h(Y, T) \rangle,$$

$$\langle R^\perp(X, Y)U, V \rangle = c(\langle JY, U \rangle \langle JX, V \rangle - \langle JX, U \rangle \langle JY, V \rangle) + \langle [A_U, A_V]X, Y \rangle.$$

An analogous computation as in the proof of the previous theorem now yields the following:

**Theorem 4.3.** Let $M^3$ be a minimal Lagrangian submanifold of a complex space form of constant holomorphic sectional curvature $4c$. Then

$$(\rho^\perp)^2 \leq (\rho - c)^2 - 2c(\rho - c) + \frac{c^2}{3}$$

and equality holds at a point $p$ if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM$ such that

$$A_{Je_1} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{Je_3} = 0,$$

with respect to this basis.

In [5] an analogous inequality relating $\rho$ and $\rho^\perp$ is obtained for complex submanifolds of complex space forms.

### 4.3 Ultra-minimal Lagrangian submanifolds of $\mathbb{C}^4$

A submanifold $M^n$ of a Riemannian manifold $\bar{M}^{n+m}$ is called ultra-minimal if around each point $p \in M^n$ there exist a local orthonormal tangent frame and a local orthonormal normal frame, such that the shape operators take the form

$$A_{U_\alpha} = \begin{pmatrix} A_1^\alpha & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & A_k^\alpha & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix},$$
where $A_j^a$ is an symmetric $(n_j \times n_j)$-matrix, with tr$(A_j^a) = 0$, and $n_1 \neq n$.

**Theorem 4.4.** Let $M^4$ be an ultra-minimal submanifold of $\mathbb{C}^4$. Then
\[ \rho \leq -\rho^\perp, \]
and equality holds at a point $p$ if and only if there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_p M$ such that
\[ A_{Je_1} = \begin{pmatrix} a & b & 0 & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} b & -a & 0 & 0 \\ -a & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{Je_3} = 0, \quad A_{Je_4} = 0, \]
with respect to this basis.

**Proof.** Since $M^4$ is ultra-minimal, there are two cases to consider, namely $n_1 = n_2 = 2$ and $n_1 = 3, n_2 = 1$.
In the first case, using the symmetry conditions for Lagrangian immersions, we obtain that
\[ A_{Je_1} = \begin{pmatrix} a & b & 0 & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} b & -a & 0 & 0 \\ -a & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{Je_3} = 0, \quad A_{Je_4} = 0, \]
(17)
Using Ricci’s equation, one can verify that
\[ 36(\rho^\perp)^2 = \frac{1}{2} \sum_{\alpha < \beta = 1} \| [A_{Je_\alpha}, A_{Je_\beta}] \|^2 = 4 (a^2 + b^2)^2 + (c^2 + d^2)^2 \]
and from Gauss’ equation
\[ 6 \rho = -2(a^2 + b^2 + c^2 + d^2), \]
thus we have $\rho \leq -\rho^\perp$, with equality if and only if $a = b = 0$ or $c = d = 0$.
In the second case, the ultra-minimality condition yields that $A_\alpha^a = 0$ for $\alpha = 1, 2, 3, 4$ and hence the problem reduces to the one solved in Theorem 4.2. We obtain $\rho \leq -\rho^\perp$, with equality if and only if the shape operators take the form (17), with $b = 0$. $\square$

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