Efficiency of a thermal machine with final processing time

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Abstract. The Novikov and Curzon-Ahlborn approaches which lead to the improving of the classical Carnot effectiveness expression in the direction of its diminishing were studied in this paper. This effect is due to the account for finite time of non-isothermic processes of the heat exchange which lead to the non-zero values of the thermal machine output power. Obvious expressions for these values are found in both analytic and graphical forms.

Keywords: thermal machine, finite time action, effectiveness.

1. Introduction
1.1. General concept of a heat engine and its efficiency

The concept of a heat engine and its efficiency (often referred as efficiency coefficient) are the basis of the classical or equilibrium phenomenological thermodynamics created in the first half of the 19th century by Carnot, Clapeyron, Kelvin and Clausius [1]. As is known, the simplest prototype of a heat engine consists of three physical objects: two thermostats with fixed temperatures (a “heater” with a temperature \( T_1 \) and a “refrigerator” with a temperature \( T_2 \), and always \( T_1 > T_2 > 0 \), all inequalities are strict!), as well as of working fluid, capable of heat exchange with both thermostats (\( Q_1 \) and \( Q_2 \), respectively) and performing mechanical work \( W \) on the external environment.

All values \( Q_1 \), \( Q_2 \) and \( W \) are algebraic (with the condition that \( |Q_1| > |Q_2| \)), and by virtue of the first law of thermodynamics for any closed cycle of a heat engine, the equality \( W = Q_1 + Q_2 \) takes place. The signs of all these values depend on the direction of the process performed by the heat engine: direct (\( W > 0, Q_1 > 0, Q_2 < 0 \)) in the mode of a heat engine or reverse (\( W < 0, Q_1 < 0, Q_2 > 0 \)) in one of the modes: heat pump or refrigerator [2].

Accordingly, the efficiency of a heat engine in each of these sub-modes is determined by different dimensionless ratios of \( Q_1 \), \( Q_2 \) and \( W \), but from both physical and technical points of view, the efficiency \( \eta = W/Q_1 \), sometimes called the Carnot’s \( C \) function, is traditionally considered to be the most interesting which has an obvious lower bound (\( C > 0 \)). By proven by Carnot universality of the action of a heat engine (in the sense of independence of efficiency on the type of working body), function \( C \) depends only on temperatures \( T_1 \) and \( T_2 \), and so that \( C \) increases with increasing temperature difference between \( T_1 \) and \( T_2 \), which (and this turns out to be quite significant!) is measured either by the difference \( T_1 - T_2 > 0 \), or the ratio \( T_1/T_2 > 1 \).

Within the framework of Carnot’s physical representations (i.e., without using the first law of thermodynamics, according to which \( W = Q_1 + Q_2 \), the dimensionless function \( C(T_1, T_2) \) has the form \( (T_1 - T_2)/T' \), where \( T' \) is some finite (arbitrary, but fixed) temperature value. It is clear that the efficiency in the formulation by Carnot has no limit from above, which, of course, is not confirmed by real observations of heat engines. Later (1849), Clausius took into account the fact \( W = Q_1 + Q_2 \), so that \( \eta = 1 - |Q_2|/Q_1 \); since, by definition, \( 0 < |Q_2|/Q_1 < 1 \), the Carnot function is bounded both from below and above, so \( 0 < C(T_1, T_2) < 1 \). Clausius found the explicit form of this function by setting the ratio \( Q_2/Q_1 = T_2/T_1 \), therefore

\[
\eta_C = C(T_1, T_2) = 1 - T_2/T_1.
\]
1.2. The difficulties with physical meaning of Carnot–Clausius formulation

This Carnot–Clausius result for efficiency is given in all thermodynamic textbooks as a model of the logically consistent derivation of an important physical result. However, for almost all known heat engines, the indicated efficiency value turns out to be too high - the real efficiency is usually 0.25 to 0.40 of the Carnot’s efficiency. Therefore, in thermodynamics, there has long been a need to obtain a more realistic upper limit for the efficiency of heat engines. Of course, it was clear to Carnot himself that for any real heat engine with possible useless leaks of heat (besides its transformation into work) the inequality \( \eta < \eta_C \) holds; the same is true for friction losses during the performance of mechanical work by the working fluid. In addition, Kelvin [3] as far back as 1853 drew attention to the fact that the heat exchange process between thermostats and the working fluid cannot, strictly speaking, be purely isothermal, as is assumed in the ideal Carnot cycle, but quantitative estimates of the effect of this fact on the value of efficiency were received neither then nor long after that.

Indeed, Kelvin’s observation could open up a new stage in the study of a heat engine and its efficiency, but the real need for this actually arose only after a century (!) in the 40s–50s of the 20th century due to the needs of a completely new type of heat engines, namely, nuclear power plants (in the terminology of the time – nuclear reactors). The reason for the partial revision of the Carnot–Clausius theory was primarily that within the framework of this theory any heat engine has zero output power \( P = 0 \), since with nonzero work \( W > 0 \), the time \( t \) of its execution is very large (\( t \to \infty \)), so that \( P = W/t \to 0 \).

On the other hand, it is the very large (or even infinite) time of performing any thermodynamic process (except, possibly, adiabatic) in the cycle of a heat engine is a necessary condition for the quasi-equilibrium state of such a process. The rejection of this condition and, consequently, the transition to a finite process time leads to far-reaching consequences. First of all, it is necessary to abandon the assumption about the exact coincidence of the temperature of the working fluid and each of the two thermostats (on the corresponding isothermal sections of the Carnot cycle).

Moreover, taking into account the non-equilibrium and, therefore, irreversible process of thermal conductivity, the condition \( Q_1/T_1 + Q_2/T_2 = 0 \), which describes the preservation of the “external” entropy, established by Clausius when analyzing the Carnot equilibrium cycle, will be violated. With a small deviation from equilibrium, the term \( \Delta S > 0 \) should be added to the left-hand part of the Clausius equation, describing the production of “internal” entropy and always strictly positive due to the second law of thermodynamics. Thus, generally speaking, the entire entropy balance should be recalculated based on non-equilibrium thermodynamics (a linear approximation is sufficient).

In the context of the efficiency problem, it is significant that violation of the Clausius equality also simultaneously leads to a violation of the expression for the efficiency of the Carnot cycle based on this equality (see above). This raises the question of obtaining a modified expression for both efficiency \( \eta \) and nonzero power of heat engine \( P > 0 \). It is clear from the beginning that for a real heat engine operating over a Carnot-type cycle (but with a finite duration), the theoretical upper limit \( \eta \) for efficiency will be below the Carnot limit \( \eta_C \). An important question is whether the expression for \( \eta \) will preserve the valuable property of universality inherent for the expression for \( \eta_C \), which depends only on the bath temperatures \( T_1 \) and \( T_2 \), but not on the properties of the working fluid or its boundaries with thermostats. Moreover, in view of the refusal of the isothermal condition of the processes of production and release of heat by the working fluid to thermostats (or baths), the question arises of choosing the average temperature \( T_0 \) of the working fluid in the interval between the bath values \( T_1 \) and \( T_2 \). Thus, the value \( T_0 \) can play the role of a parameter optimizing the operation of a heat engine (in the sense of maximizing its average mechanical or electrical power \( P \)).

1.3. The short history of the search for more adequate physical meaning

As far as the authors know, for the first time this set of questions was raised by compatriots of Carnot, French physicists and engineers, in the late 40s - early 50s of the last century in connection with the problem of optimizing the design of atomic reactors. The first prototype of the Zoé reactor on uranium oxide with heavy water as a moderator was launched on December 15, 1948 and developed only a few
kilowatts of electrical power. The next generation reactor (Saclay, end of October 1952) already gave from 1,000 to 1,500 kW of power. A detailed description of all the issues related here is given in the report [4] of J. Yvon in August 1956.

One of the decisive factors of such a significant increase in power was precisely the selection of the optimal average temperature $T_0$ of the working fluid (heat-transfer fluid). In [4] it was stated that for this purpose it should be $T_0 = \sqrt{T_1 T_2}$, and the corresponding efficiency is $\eta = 1 - \sqrt{T_2/T_1}$, but no calculations were given (probably, for reasons of secrecy). An analysis of the literature indicates that another French physicist, a well-known thermodynamic expert P. Chambadal, who published a small book on atomic energy [5] in 1958 and later (in 1963) – rather a thorough analysis of the efficiency of a heat engine of any nature [6].

However, much earlier (in 1957-58), the famous Soviet thermal physicist I.I. Novikov published in the open press two papers [7, 8], where he gave a mathematically not quite strict, but physically quite convincing justification of the formulas for $T_0$ and $\eta$. Only after nearly 20 years (1975), two American physicists Curzon and Ahlborn [6] offered a fairly consistent derivation of formulas for $T_0$ and $\eta$, which has since been simplified, generalized and improved many times [10]. We note, however, that in article [6] for some reason none of the above-mentioned sources is mentioned at all.

Later in the physical literature, a whole scientific field appeared, known as finite-time thermodynamics. A review of its state in the period of greatest development (the 80s) can be found in the review [11], and the current state (combined with the theory of optimization of heat engines) is fully reflected in the monograph [12].

The aim of this paper is twofold: first, to give a short review of the original attempts to account for the finite time corrections of the thermal machine efficiency and, also, to give mostly appropriate derivation of this efficiency on the grounds of variational methods.

2. Materials and method of research

2.1 Novikov's calculation for the efficiency of a nuclear power plant

In 1957, the Soviet thermal physicist I.I. Novikov published an article entitled “Efficiency of atomic energy installation” in which he gave a physically quite convincing justification of the formulas for the temperature of the working fluid and the efficiency of a heat engine. In brief, this rationale was as follows.

Coming to the expression for the actual useful work produced in the installation [7, 8]:

$$L_{\text{act}} = B \left( T_a - T_p - a \right) \left[ 1 - \frac{T_0}{T_p} \left( 1 + A \right) \right],$$

where $A$ and $B$ are constant values depending on the degree of irreversibility of the thermodynamic cycle, $T_0$ is the ambient temperature (i.e., of the atmosphere), $T_p$ is the average temperature of the working fluid in the cycle at the heat supply section, $T_a$ is the reactor temperature, $A$ is a positive value, representing the average value of the temperature difference between the primary refrigerant and the working substance (as a rule, some kind of fluid) in the process of heat supply to the latter.

From the maximum condition of the function $L_{\text{act}}$, we find the optimal temperature of the working fluid $T'$: $T' = \sqrt{(T_a - a)T_0(1 + A)}$. If the irreversibility of processes in the thermodynamic cycle can be neglected, i.e. put $A = 0$, then $T' = \sqrt{(T_a - a)T_0}$. The value of $A$, small in comparison with $T_a$, in some cases can also be neglected, then $T' = \sqrt{T_a/T_0}$. This is the approximate formula for calculating the optimal average temperature of heat supply to the working fluid in the cycle.

Using this formula, it is easy to determine the value of the effective efficiency of the nuclear power plant. The effective efficiency is the ratio of the produced useful work to the amount of heat released in the reactor $Q$, i.e.

$$\eta_e = \left( 1 - a \right) \left[ 1 - \left( \frac{T_0(1 + A)}{T_a - a} \right)^{1/2} \right].$$
If we ignore the heat leaks and the irreversibility of the thermodynamic cycle and neglect \( \alpha \), then the following simple formula is obtained for the coefficient of efficiency:

\[
\eta_c = 1 - \sqrt{\frac{T_0}{T_\alpha}}.
\]

Later in the articles [7,8], the author makes a very important explanation that these formulas do not depend on the type of primary coolant used, i.e. can be used for various types of nuclear power plants.

Thus, I.I. Novikov was the first to give a mathematically incompletely rigorous, but physically completely convincing justification of the formulas for the optimum temperature of the working fluid and efficiency.

2.2 Derivation by Curzon and Ahlborn

Only 20 years later, in 1975, two American physicists Curzon and Ahlborn offered a fairly consistent derivation of these formulas, which has since been repeatedly improved and simplified.

Considering the heat flows through the reservoir containing the working fluid proportional to the temperature difference between the reservoir walls, we can write for the isothermal expansion and contraction stage, respectively:

\[
F_1 = \alpha(T_1 - T_{1w}), \quad F_2 = \beta(T_{2w} - T_2),
\]

where \( F_1 \) and \( F_2 \) are heat flows, \( T_{1w} \) and \( T_{2w} \) are the temperature of the working fluid, \( T_1 \) is the heater temperature, \( T_2 \) is the refrigerator temperature, \( \alpha \) and \( \beta \) are heat transfer coefficients.

Now we can write the expressions for the transferred amount of heat for each stage, respectively:

\[
W_1 = F_1 t_1 = \alpha t_1(T_1 - T_{1w}), \quad W_2 = F_2 t_2 = \beta t_2(T_{2w} - T_2),
\]

where \( t_1 \) and \( t_2 \) are the corresponding intervals of the processes.

For the adiabatic stages, we take into account the absence of heat fluxes due to their reversibility:

\[
W_1/T_{1w} = W_2/T_{2w}.
\]

The power of the heat engine is determined by the expression:

\[
P = \frac{W_1 - W_2}{(t_1 + t_2)\gamma},
\]

where \((\gamma - 1)(t_1 + t_2)\) is the time required for performing adiabatic cycles.

Using the above ratios, we arrive at the following expression:

\[
P = \frac{\alpha \beta xy(T_1 - T_{2} - x - y)}{y[\beta T_1 y + \alpha T_2 x + xy(\alpha - \beta)]},
\]

where \( x = T_1 - T_{1w}, \ y = T_{2w} - T_2 \).

The power \( P \) is maximal at \( x \) and \( y \) values satisfying the conditions:

\[
\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0,
\]

i.e.

\[
\beta T_1 y(T_1 - T_2 - x - y) = x[\beta T_1 y + \alpha T_2 x + xy(\alpha - \beta)],
\]

\[
\alpha T_2 y(T_1 - T_2 - x - y) = y[\beta T_1 y + \alpha T_2 x + xy(\alpha - \beta)].
\]

It follows that:

\[
\gamma = \left(\frac{\alpha T_2}{\beta T_1}\right)^{1/2} x.
\]

We introduce the notation \( \mu = \frac{x}{T_1} \) and make a quadratic equation with respect to \( \mu \):

\[
(1 - \frac{\alpha}{\beta}) \mu^2 - 2 \left[\left(\frac{\alpha T_2}{\beta T_1}\right)^{1/2} + 1\right] \mu + \left(1 - \frac{T_2}{T_1}\right) = 0.
\]

We can find its solution that has a physical meaning:

\[
\mu = \frac{x}{T_1} = \frac{1 - \sqrt{T_2/T_1}}{1 + \sqrt{\alpha/\beta}}.
\]

Using the previous calculations, you can rewrite it as follows:
\[
\frac{y}{T_2} = \frac{\sqrt{T_1/T_2} - 1}{1 + \sqrt{\beta/\alpha}}
\]

Then:
\[
\eta' = \frac{W_1 - W_2}{W_1} = 1 - \frac{T_{2w}}{T_{1w}} = 1 - \frac{T_2 + y}{T_1 - x} = 1 - \sqrt{T_2/T_1}.
\]

Like Novikov, the authors of [9] make an important observation that the result obtained depends only on the temperature of the heat carriers.

2.3 Derivation of efficiency expression by means of Lagrange multipliers method

This derivation, being simpler and more general than the two above, also helps to find a compromise necessary to maximize the output power of the thermal installation.

During the first stage of the cycle between \( t_0 \) and \( t_1 = t_0 + \tau_1 \), the working fluid has a temperature \( \theta_1(t) \), receiving heat \( Q_1 \) from the heater at a temperature \( T_1 \). We assume that the heat transfer process occurs according to the Fourier law, therefore the expression for the heat flux \( \varphi_1(t) \) from the heater to the working body is \( \varphi_1(t) = C_1[T_1 - \theta_1(t)] \), where \( C_1 \) is assumed to be known. During the second stage between \( t_1 \) and \( t_2 = t_1 + \tau_a \), the temperature adiabatically drops from \( \theta_1(t) \) to \( \theta_2(t) \). During the third stage between \( t_2 \) and \( t_3 = t_2 + \tau_2 \), the working fluid has a temperature of \( \theta_2(t) \), giving off heat \( |Q_2| \) to the refrigerator, which has a temperature of \( T_2 \) and \( \varphi_2(t) = C_2[\theta_2(t) - T_2] \). Finally, between \( t_3 \) and \( t_4 = t_3 + \tau_a' \) there is a return to the initial state with the temperature increasing from \( \theta_2(t_3) \) to \( \theta_1(t_4) = \theta_1(t_0) \). The total cycle duration is \( \tau = t_4 + t_0 = \tau_1 + \tau_a + \tau_2 + \tau_a' \). The work done by the heat engine during the cycle is the difference between the amount of heat
\[
Q_1 = \int_{t_0}^{t_1} \varphi_1(t) dt
\]

obtained from the heater, and
\[
|Q_2| = \int_{t_2}^{t_3} \varphi_2(t) dt
\]
given to the refrigerator. Thus, the output power:
\[
P = \frac{1}{\tau} \left( C_1 \int_{t_0}^{t_1} [T_1 - \theta_1(t)] dt - C_2 \int_{t_2}^{t_3} [\theta_2(t) - T_2] dt \right).
\]

The above cycle is closed, so the entropy change in the time interval between \( t_0 \) and \( t_4 \) is zero:
\[
\Delta S = C_1 \int_{t_0}^{t_1} \frac{T_1 - \theta_1(t)}{\theta_1(t)} dt - C_2 \int_{t_2}^{t_3} \frac{\theta_2(t) - T_2}{\theta_2(t)} dt = 0.
\]

We need to maximize \( P \) as a function of temperatures \( \theta_1(t) \), \( \theta_2(t) \) and durations \( \tau_1 \), \( \tau_a \), \( \tau_2 \), \( \tau_a' \) for given values of \( T_1 \), \( T_2 \), \( C_1 \), \( C_2 \), \( \tau \) with \( \Delta S = 0 \). Introducing the Lagrange multiplier \( \lambda/\tau \) and writing down that \( P - \lambda \Delta S/\tau \) is constant with respect to \( \theta_1(t) \) and \( \theta_2(t) \), we get:
\[
\theta_1(t) = \sqrt{\lambda T_1}, \quad \theta_2(t) = \sqrt{\lambda T_2}.
\]

The optimal cycle for a thermal installation is a Carnot cycle with constant temperatures \( \theta_1(t) \equiv \theta_1 \) and \( \theta_2(t) \equiv \theta_2 \). Maximizing \( P \) with respect to \( \tau_a \) and \( \tau_a' \), we obtain \( \tau_a / \tau \approx 0 \), \( \tau_a' / \tau \approx 0 \): the adiabatic steps should be as short as possible. This allows you to write that \( P - \lambda \Delta S/\tau \) is invariable relative to \( \tau_1 / \tau = 1 - \tau_2 / \tau \), which gives:
\[
C_1(T_1 - \theta_1)(1 - \lambda/\theta_1) + C_2(\theta_2 - T_2)(1 - \lambda/\theta_2).
\]

Or using (1):
\[
\sqrt{\lambda} = \frac{\sqrt{C_1 T_1} + \sqrt{C_2 T_2}}{\sqrt{C_1} + \sqrt{C_2}}.
\]

From the condition \( \Delta S = 0 \), we get the optimal duration of isothermal steps of the cycle:
\[ \tau_1 = \frac{\sqrt{C_2}}{\sqrt{C_1} + \sqrt{C_2}} , \quad \tau_2 = \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}} . \]

Maximum power is
\[ P_{\text{max}} = \frac{C_1 C_2}{(\sqrt{C_1} + \sqrt{C_2})^2} (T_1 - T_2). \]

This power is equivalent at best to \( \frac{1}{4} \) (attained with \( C_1 = C_2 \)) heat flux \( C(T_1 - T_2) \), which could be transferred from the heater to the refrigerator with a coefficient \( C \) equivalent to \( \sqrt{C_1 C_2} \). Finally, the efficiency value obtained from the equality of the total entropy change to zero during the entire cycle:
\[ Q_1 \theta_1 - |Q_2| \theta_2 = 0. \]

Consequently,
\[ \eta = 1 - \frac{T_2}{T_1}. \]

As you can see, the formula obtained using the Lagrange multipliers method is identical to that obtained earlier by other methods, and still remains independent of the characteristics of the working body.

At the isothermal expansion section, the gas receives from the heater the amount of heat \( Q_1 \), and at the isothermal compression section it gives the amount of heat \( |Q_2| \), therefore, a fraction of the heat which is converted into useful work is equal to \( W = Q_1 - |Q_2| \).

Entering the auxiliary designation \( q = |Q_2|/Q_1 \), we write the formula of efficiency and power:
\[ \eta = 1 - \frac{|Q_2|}{Q_1} = 1 - q, \quad q > 0. \]
\[ P = \frac{W}{\tau} = \frac{Q_1 - |Q_2|}{\tau} = \left( \frac{Q_1 - |Q_2|}{|Q_2|} \right) \frac{Q_1}{\tau} = \eta Q_1 / \tau, \]
where \( \tau \) is the cycle time. From here
\[ W = \eta Q_1. \]

Using the second law of thermodynamics, we write the condition for the equality of the total entropy change to zero during the entire cycle:
\[ \frac{Q_1}{T_1} - \frac{|Q_2|}{T_2} = 0. \]

Consequently,
\[ \eta = 1 - \frac{T_2}{T_1}. \]

The resulting formula of Carnot’s efficiency does not have a root that would appear if we took into account the process time.

Now we use the second law of thermodynamics for working fluid temperatures \( \theta_1 \) and \( \theta_2 \):
\[ \frac{Q_1}{\theta_1} - \frac{|Q_2|}{\theta_2} = 0. \]

From here
\[ \frac{\theta_1}{\theta_2} = \frac{Q_1}{|Q_2|} = \frac{1}{q}, \]
and the efficiency formula is
\[ \eta = 1 - \frac{\theta_2}{\theta_1} = 1 - q. \]

Denoting the Carnot efficiency (4) as \( \eta_c \) and subtracting (6) from (4), we get:
\[ \eta_c - \eta = q - \frac{T_2}{T_1}. \]

Similarly, we suppose (as in the conclusion with the help of additional Lagrange multipliers) that the heat transfer process during the first stage of the cycle occurs according to the Fourier law, therefore the heat flux \( \varphi_1(t) \) from the heater to the working fluid has the form \( \varphi_1(t) = C_1[T_1 - \]

\[ \theta_1(t) \] where \( C_1 \) is assumed to be known. During the third stage, the working fluid has a temperature \( \theta_2(t) \), giving off heat \( |Q_2| \) to a refrigerator having a temperature \( T_2 \) and \( \varphi_2(t) = C_2[\theta_2(t) - T_2] \).

Then, since \( Q_1 = \int_{t_0}^{t_1} \varphi_1(t)dt \), and \( |Q_2| = \int_{t_2}^{t_3} \varphi_2(t)dt \):

\[
Q_1 = C_1 \int_{t_0}^{t_1} (T_1 - \theta_1)dt = C_1 \tau_1(T_1 - \theta_1) = C_1 \tau_1 T_1 \left(1 - \frac{\theta_1}{T_1}\right), \quad (8)
\]

\[
|Q_2| = C_2 \int_{t_2}^{t_3} (\theta_2 - T_2)dt = C_2 \tau_2(\theta_2 - T_2) = C_2 \tau_2 T_2 \left(\frac{\theta_2}{T_2} - 1\right). \quad (9)
\]

Expressing from (5) \( |Q_2| = qQ_1 \) and \( \theta_2 = q\theta_1 \), and also using (8) and (9), we get:

\[
\theta_1 = \frac{qC_1 \tau_1 T_1 + C_2 \tau_2 T_2}{q(C_1 \tau_1 + C_2 \tau_2)}. \quad (10)
\]

Substituting this expression into (8), we get:

\[
Q_1 = \frac{C_1 C_2 \tau_1 \tau_2 T_1(q - T_2/T_1)}{q(C_1 \tau_1 + C_2 \tau_2)}. \quad (11)
\]

Using (7) and (2) and denoting \( C_1 C_2 \tau_1 \tau_2/(C_1 \tau_1 + C_2 \tau_2) \) as \( C' \), we write the expression for \( Q_1 \) in the form:

\[
Q_1 = C'T_1 \frac{\eta_c - \eta}{1 - \eta}. \quad (10)
\]

Substituting (10) into (3), we get:

\[
P = \frac{W}{\tau} = \frac{\eta Q_1}{\tau} = \frac{C'}{T_1 \eta} \frac{\eta_c - \eta}{1 - \eta}. \quad (11)
\]

Combining \( C' \) and \( \tau = \tau_1 + \tau_2 \) into one constant \( C \), we finally write:

\[
P = CT_1 \eta \frac{\eta_c - \eta}{1 - \eta}. \quad (12)
\]

**Figure 1.** Dependence graph (12) for \( \eta_c \), equal to 0.4, 0.6, 0.8.

Let’s consider the function (12). Differentiation gives the following expression for the first derivative:

\[
P' = CT_1 \frac{\eta^2 - 2\eta + \eta_c}{(1 - \eta)^2}.
\]

Equating it to zero, we get the following expression for the critical point:

\[
\eta = 1 - \sqrt{1 - \eta_c}.
\]

We next obtain the second derivative of the function (12):

\[
P'' = 2CT_1 \frac{(\eta_c - 1)}{(1 - \eta)^3}. \quad (13)
\]
Since \( \eta \) and \( \eta_c \) are less than 1, expression (13) is negative, which allows us to say that this critical point is the maximum of function (12), which was to be proved.

3. Results and discussion
The new formula for the real efficiency of a heat engine

\[
\eta = 1 - \frac{1}{\sqrt{1 - \eta_c}} = 1 - \frac{\sqrt{T_2/\sqrt{T_1}}}{T_1},
\]

meeting the maximum power condition (for given heat transfer parameters during thermal contact of the working fluid and thermostats), fully corresponds to the result obtained for the first time by Novikov (see Section 2). It is clear that for any values of \( T_1 \) and \( T_2 \) the value of \( \eta \) is obviously less than \( \eta_c \). (e.g., in lowest approximations \( \eta \approx (1/2)\eta_c+(1/8)\eta_c^2 \)) which, in turn, is obviously less than one (see figure), and at which the heat engine has zero power.

Simple calculations show that at a given temperature difference, the values of \( \eta_c \) correspond much better to the actually observed efficiency values than the theoretically correct, but almost strongly overestimated value of \( \eta_c \). This is clearly shown by the table below from [9].

| Power Plant                  | \( T_1 (^\circ C) \) | \( T_2 (^\circ C) \) | \( \eta \) (Carnot) | \( \eta \) (Endoreversible) | \( \eta \) (Observed) |
|-----------------------------|----------------------|----------------------|---------------------|----------------------------|--------------------|
| West Thurrock (UK) coal-fired power plant | 25                   | 565                  | 0.64                | 0.40                       | 0.36               |
| CANDU (Canada) nuclear power plant            | 25                   | 300                  | 0.48                | 0.28                       | 0.30               |
| Larderello (Italy) geothermal power plant        | 80                   | 250                  | 0.33                | 0.178                      | 0.16               |

4. Conclusions
Derivation of the formula of efficiency by means of the method of Lagrange multipliers is simpler and more general than the well-known approaches of Novikov and Curzon & Alborne and also allows to select the average temperature of the working fluid that maximizes the mechanical power of the power plant. The next step is to see how this conclusion may be extended on more complex types of thermal machines – for example those with thermal radiation as the working body.

5. References
[1] Gelfer Ya M 1969 The History and Methodology of Thermodynamics and Statistical Physics (Moscow: Publishing House Vysshaya Shkola) p 475
[2] Rudoy Yu G 2011 The effectiveness of the thermal machine in the regimes of heat engine, heat pump and refrigerator Physics in Higher Education 17(4) pp 126–33 URL: http://pinhe.lebedev.ru/tom17n4.en.htm#The_Effectiveness_of_the_Thermal_Machine
[3] Thomson W 1853 Philosophical Magazine 5(30) https://doi.org/10.1080/14786445308562743
[4] Yvon J 1955 La pile de Saclay. Experience acquise en deux ans sur le transfert de chaleur par gaze comprime Communication du C.K.A. à la Conférence de Genève p 36
[5] Chambadal P 1957 Les Centrales Nucleaires (Paris: Armand Colin) p 58
[6] Chambadal P 1963 Evolution et Applications du Concept d’Entropie (Paris: Dunod) p 280
[7] Novikov I I 1957 Efficiency of atomic energy installation Atomnaya Energiya 3(11) pp 409-412
[8] Novikov I I 1958 Efficiency of atomic energy installation (a review) J. of Nuclear Energy II 7 pp 125–128
[9] Curzon F L and Ahlborn B 1975 Efficiency of a Carnot engine at maximum power output American J. of Phys. 43 pp 22–24
[10] Van den Broeck C 2005 Thermodynamic efficiency at maximum power Phys. Rev. Letters 95 190602 (1-3)
[11] Andresen B, Salamon P and Berry R S 1984 Thermodynamics in finite time Phys. Today 37(9) pp 62–70
[12] Mironova V A, Amelkin S A, Tsirlin A M 2000 Mathematical Methods of Thermodynamics at Finite Time (Moscow: Publishing House Khimia) p 384