The role of the cotangent bundle in resolving ideals of fat points in the plane.

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Abstract

We study the connection between the generation of a fat point scheme supported at general points in \( \mathbb{P}^2 \) and the behaviour of the cotangent bundle with respect to some rational curves particularly relevant for the scheme. We put forward two conjectures, giving examples and partial results in support of them.

1 Introduction

In this paper we are concerned with minimal free graded resolutions of fat point ideals in \( \mathbb{P}^2 \). Given general points \( P_1, \ldots, P_n \in \mathbb{P}^2 \) (which, unless we say something explicit to the contrary, will always be assumed to be general), and nonnegative integers \( m_1, \ldots, m_n \), let \( I(Z) \) denote the ideal \( I(P_1)^{m_1} \cap \cdots \cap I(P_n)^{m_n} \) of \( R = K[\mathbb{P}^2] = K[x_0, x_1, x_2] \) (where \( K \) is any algebraically closed field and where \( I(P_i) \) is the ideal generated by all forms that vanish at \( P_i \)). We refer to \( I(Z) \) as a fat point ideal, and if \( Z \) is the subscheme defined by \( I(Z) \), we use \( m_1 P_1 + \cdots + m_n P_n \) or \( Z(m_1, \ldots, m_n) \) to denote the scheme \( Z \), and \( I_Z \) for its sheaf of ideals, so that, in particular, \( I(Z)_k = H^0(\mathbb{P}^2, I_Z(k)) \).

In order to understand better the geometry of \( Z \) as a subscheme of \( \mathbb{P}^2 \), the first thing that comes to mind is to see how many curves of given degree \( k \) contain \( Z \), that is, have singularities of multiplicity at least \( m_1, \ldots, m_n \) at the given points \( P_1, \ldots, P_n \); in other words, we want to determine the dimension, as a \( K \)-vector space, of the homogeneous component \( I(Z)_k \) of \( I(Z) \).

The Hilbert function \( h_Z \) of \( I(Z) \), \( h_Z(k) := \dim_K(I(Z)_k) \), is not known in general, even if it has been determined for many choices of \( Z \). For example, it is known for all \( Z \) with \( n \leq 9 \) (\( [N] \), or see \([Ha3]\)), for any \( n \) if \( m_1 = \cdots = m_n \leq 20 \) (\( [Hi1] \), \([CCMO]\)), and for any \( n \) if \( m_i \leq 7 \) (\( [Mi] \), \([Y]\)). (Some but not all of these and later citations assume \( K \) is the complex numbers.) It is also known for many additional cases. Let us say that the sequence of multiplicities \( m_i \) (and by extension \( Z \)) is uniform if \( m_1 = \cdots = m_n \geq 0 \) and if \( n \geq 9 \). Then \( [E] \) determines \( h_Z \) for all \( k \), as long as \( Z \) is uniform, if \( n \) is a square, extending results of \([HIF]\). The paper \([HR]\) determines \( h_Z \) in many other uniform cases.

All of these results are consistent with a well known conjecture by means of which one can explicitly write down the function \( h_Z \) given the multiplicities \( m_i \). Various equivalent versions of this conjecture have been given (see \([S], [Ha4], [G], [Hi2], [Ha2]\)). We will refer to them collectively as the SHGH Conjecture.
Let us say that a fat point subscheme Z is quasi-uniform if \( n \geq 9 \) and \( m_1 = \cdots = m_9 \geq m_{10} \geq \cdots \geq m_n \geq 0 \). Thus uniform implies quasi-uniform. As shown in [HHT], assuming the SHGH Conjecture, then \( h_Z(k) = \max(0, (k+2)/2 - \sum_i (m_i+1)) \) holds for all \( k \) for a quasi-uniform \( Z \). Since there are \( (k+2)/2 \) forms of degree \( k \) and since the requirement for a form to vanish to order \( m_i \) at a point \( P_i \) imposes \( (m_i+1) \) conditions, the SHGH Conjecture in this situation just says that the conditions imposed by the points are independent as long as \( h_Z(k) > 0 \).

To go deeper into the geometry of a fat point scheme, the next step consists in understanding the relations among the curves containing \( Z \), that is, determining the minimal free graded resolution \( 0 \rightarrow M_1 \rightarrow M_0 \rightarrow I(Z) \rightarrow 0 \) of \( I(Z) \). Here \( M_0 \) and \( M_1 \) are free \( R \)-modules of the form \( M_0 = \oplus_k R^k[-k] \) and \( M_1 = \oplus_k R^k[-k] \). If \( h_Z \) is known and if the graded Betti numbers \( t_k \) are known, then the values of \( s_k \) are easy to determine from the exact sequence above.

We are hence interested in the graded Betti numbers \( t_k \). It is not hard to see that \( t_k \) is the dimension of the cokernel of the map \( \mu_{k-1}(Z) : I(Z)_{k-1} \otimes R_1 \rightarrow I(Z)_k \), where \( R_1 \) denotes the \( K \)-vector space spanned in \( R \) by linear forms and \( \mu_{k-1} \) is the map induced by multiplication of elements of \( I(Z)_{k-1} \) by linear forms. This paper is a reflection about the geometric obstacles to the rank maximality of the maps \( \mu_k \). Let us denote by \( \Omega \) the cotangent bundle of \( P^2 \), and by \( p : X \rightarrow P^2 \) the blow up at the points \( P_i \). We first translate the problem of determining the rank of the maps \( \mu_k(Z) \) into two equivalent postulation problems for \( Z \), one in \( P^2 \) and the other in \( X \): determine, for each \( k \), the rank of the restriction map

\[(a) \; \rho_k = \rho_k(Z) : H^0(\Omega(k+1)) \rightarrow H^0(\Omega(k+1)|_Z); \quad \text{or} \]
\[(b) \; \eta_k = \eta_k(Z) : H^0(p^*\Omega(k+1)) \rightarrow H^0(p^*\Omega(k+1)|_{p^{-1}Z}).\]

We show that the point of view (a) gives some information about the failure of this rank maximality due to superfluous conditions imposed by \( Z \) to the restriction of \( \Omega \) to some curves; but in fact this is not enough, and the right point of view is (b), since it is then possible to take into account the splitting of \( p^*\Omega \) on the normalization of the appropriate rational curves, and this allows to count properly the superfluous conditions imposed by \( Z \) to the restriction of \( \Omega \) to each curve.

Hence, by studying several examples and proving certain results (e.g. [5,3]), we arrive at two conjectures about the failure of the rank maximality of \( \mu_k \), one when \( \mu_k \) is expected to be surjective and the other when injectivity is expected. The idea is the same in the two cases but the expected surjective case is much easier to formulate, and this is why we keep them distinct; in both cases the obstruction to rank maximality is described by the presence of particular rational curves whose intersection with \( Z \) is “too high”.

Notice that a similar line of thought leads to the SHGH Conjecture: in fact, determining \( h^0(P^2, \mathcal{I}_Z(k)) \) amounts to computing the rank of the restriction map \( r_k : H^0(P^2, \mathcal{O}_{P^2}(k)) \rightarrow H^0(Z, \mathcal{O}_Z) \). The SHGH Conjecture says that failure of \( r_k \) to have maximal rank is completely accounted for by the occurrence of curves \( C \subset P^2 \) whose strict transform \( \tilde{C} \subset X \) is an exceptional divisor (i.e., a smooth rational curve of self-intersection \(-1\)), such that the scheme-theoretic intersection \( C \cap Z \) is too big with respect to \( \mathcal{O}(k)|_C \) (or, expressing things on the blow up, such that the inverse image \( \tilde{Z} \) of \( Z \) meets \( \tilde{C} \) in too many points with respect to \( kL|_C \), which here just means that \( \tilde{C} \cdot F < -1 \), where \( F = kL - m_1E_1 - \cdots - m_mE_n \), \( L \) is the pullback to \( X \) of a general line in \( P^2 \) and \( E_i \) is the exceptional locus obtained by blowing up the point \( P_i \)).

Unfortunately, things are quite complicated when studying the postulation with respect to a rank 2 vector bundle; for example, as said above, we have to take into consideration the splitting of \( p^*\Omega \) on the normalization of a rational plane curve, which is not known in general (see [2,1], and is actually an interesting problem per se. In our examples we have made use, when necessary, of a Macaulay 2 script which allows us to compute splitting types (see Section A2.3 of [GHII]).
The use of the cotangent bundle in problems concerning the generation of homogeneous ideals of subschemes of a projective space was introduced by A.Hirschowitz, and used for the first time for curves in $\mathbb{P}^3$ (see [11]).

Our conjectures assume that the fat point scheme $Z$ postulates well in the degree $k$ we are considering, i.e. that $h^1(I_Z(k)) = 0$; but notice that, assuming the SHGH Conjecture, we can always reduce ourselves to considering fat points $Z$ with good postulation, and for these we need to study only the map $\mu_\alpha$, where $\alpha$ is the initial degree of $I(Z)$ (see [23]).

Here is what is currently known about resolution of fat point ideals in $\mathbb{P}^2$. For uniform ([H52] or quasi-uniform ([HHF]) $Z$, it is conjectured that the maps $\mu_k$ have maximal rank for all $k$. We refer to these as the Uniform Resolution and Quasi-Uniform Resolution Conjectures. The Uniform Resolution Conjecture has been proved for $m = 1$ ([CM]), $m = 2$ ([12]) and $m = 3$ ([C1]); more generally, if $m_i \leq 3$ for all $i$, and the length of $Z$ is sufficiently high, [B1] determines the graded Betti numbers in all degrees. Verifications of the Quasi-Uniform Resolution Conjecture in some cases were given in [HHF], under the assumption of the SHGH Conjecture. Some outright verifications were given by [HR]. By applying the results of [E] to results of [HHF], it also follows that the Uniform Resolution Conjecture holds for all $m$ not too small, as long as $n$ is an even square. Finally, the Betti numbers are known for all $Z$ with $n \leq 8$ ([Ca], [F3], [Ha6], [FFH]); the $n \leq 8$ results show that any general resolution conjecture will have to be more subtle than the SHGH Conjecture.

In Section 6 we prove that our Conjectures 6.1 and 6.7 together with the SHGH Conjecture imply the Uniform and Quasi-uniform Resolution Conjectures (see Proposition 6.8).

2 Preliminaries

We now establish some terminology and notations and recall some basic concepts.

By curve we will mean a 1-dimensional scheme without embedded components.

The surface obtained from $\mathbb{P}^2$ by blowing up general points $P_i$ is always denoted by $X$, $p : X \to \mathbb{P}^2$ is the morphism given by blowing up the points, $E_i$ is the exceptional curve obtained by blowing up the point $P_i$ and $L$ is the divisorial inverse image under $p$ of a line in $\mathbb{P}^2$. We will also use $L$ and $E_i$ to denote the linear equivalence class of the given divisor, in which case the divisor class group $\operatorname{Cl}(X)$ is the free abelian group on the basis $L, E_1, \ldots, E_n$. The intersection form on $X$ is such that the basis elements are orthogonal with $-L^2 = E_i^2 = -1$ for all $i$.

Given a divisor $F$ on $X$, we will use $F$ to denote its divisor class and sometimes even the sheaf $\mathcal{O}_X(F)$, and we will for convenience write $H^0(F)$ for $H^0(X, \mathcal{O}_X(F))$. For each $F$, there is a natural multiplication map $\mu_F : H^0(F) \otimes H^0(L) \to H^0(F + L)$.

If $Z = m_1P_1 + \cdots + m_nP_n$ is a fat point scheme, it is clear that, under the correspondence of $H^0(\mathbb{P}^2, I_Z(k))$ with $H^0(X, kL - \sum m_iE_i)$, the map

$$\mu_k(Z) : H^0(I_Z(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(I_Z(k + 1))$$

is just the map $\mu_{kL - \sum m_iE_i}$.

Given a curve $C \subset \mathbb{P}^2$, we denote the multiplicity of $C$ at $P_i$ by $m(C)_{P_i} = r_i$, and $\hat{C} = dL - \sum r_iE_i$ will denote its strict transform. Note that $d$ is just the degree of $C$. If $C \subset \mathbb{P}^2$ is an integral curve such that $\hat{C} \subset X$ is smooth and rational, we write $\mathcal{O}_{\hat{C}}(k)$ instead of $\mathcal{O}_{\mathbb{P}^2}(k)$. We recall that $\hat{C}$ is an exceptional divisor (of the first kind) in $X$ if $\hat{C} = dL - \sum r_iE_i$ is smooth and rational with $-1 = \hat{C}^2 = d^2 - \sum r_i^2$, which by the adjunction formula implies $-1 = K_X \cdot \hat{C} = -3d + \sum r_i$, since $K_X = -3L + E_1 + \cdots + E_n$. 


Let $Y$ be a smooth projective variety, $D$ a divisor and $A$ a subscheme of $Y$; the residual scheme $A' = \text{res}_DA$ is the subscheme of $Y$ whose sheaf of ideals $\mathcal{I}_{\text{res}DA}$ is given by the exact sequence: $0 \to \mathcal{I}_{\text{res}DA}(-D) \to \mathcal{I}_A \to \mathcal{I}_{A \cap D,D} \to 0$, where $\mathcal{I}_{A \cap D,D}$ is the sheaf of ideals on $D$ defining the scheme-theoretic intersection of $A$ and $D$ as a subscheme of $D$.

If $Z = m_1P_1 + \cdots + m_nP_n$ in $\mathbb{P}^2$ is a fat point scheme, and $C$ is a plane curve whose proper transform is $\tilde{C} = dL - \sum r_iE_i$, the residual sequence tensored by $\mathcal{O}_{\mathbb{P}^2}(k)$ becomes: $0 \to \mathcal{I}_Z(k-d) \to \mathcal{I}_Z(k) \to \mathcal{I}_{Z \cap C,C}(k) \to 0$, where $Z' = \text{res}_C Z$ has homogeneous ideal $(I(Z) : I(C))$.

Now if we set $F_k(Z) = kL - m_1E_1 - \cdots - m_nE_n$, we have $F_k(Z) - \tilde{C} = (k-d)L - \sum (m_i - r_i)E_i$ and its cohomology is the cohomology of a fat point scheme provided that $m_i - r_i \geq 0$ for all $i$; more precisely, $F_k(Z) - \tilde{C} = F_{k-d}(Z')$ if $r_i \leq m_i$. Thus divisors corresponding to residuals are easy to compute.

Setting $\Omega = \Omega_{\mathbb{P}^2}$, recall the Euler sequence on $\mathbb{P}^2$:

$$0 \to \Omega(1) \to \mathcal{O}_{\mathbb{P}^2} \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0.$$ 

Now let $C \subset \mathbb{P}^2$ be a degree $d$ integral curve and assume $\tilde{C} \subset X$ smooth and rational. Since the Euler sequence is a sequence of vector bundles, its pullback to $X$ restricted to $\tilde{C}$ is still exact, and gives

$$0 \to p^*\Omega(1)|_{\tilde{C}} \to \mathcal{O}_{\tilde{C}} \otimes H^0(\mathcal{O}_X(L)) \to \mathcal{O}_{\tilde{C}}(d) \to 0. \quad (*)$$

In the following we set

$$p^*\Omega(1)|_{\tilde{C}} \cong \mathcal{O}_{\tilde{C}}(-a_C) \oplus \mathcal{O}_{\tilde{C}}(-b_C),$$

where we always assume $a_C \leq b_C$; looking at the Chern classes in $(*)$ gives $a_C + b_C = d$. We will say that the splitting type of $C$ or $\tilde{C}$ is $(a_C,b_C)$ and the splitting gap is $b_C - a_C$.

In some cases we can immediately determine the splitting type. Suppose that $m$ is the maximum value of $m(C)_P$. See [As] or [F1], [F2] for the proof of the following lemma:

**Lemma 2.1** We have $\min(m,d-m) \leq a_C \leq d - m$, and $d = a_C + b_C$.

Note that the splitting type is completely determined if $d - m \leq m + 1$, and it is $(\min(m,d-m),\max(m,d-m))$. When $d - m > m + 1$ it is not known in general what the splitting type is, but it can be computed fairly efficiently; see Section A.2.3 in [GHI].

If $f : A \to B$ is a linear map between vector spaces, we say that $f$ is *exp-onto* (i.e., expected to be onto), resp. *exp-inj* (i.e., expected to be injective), if $\dim A \geq \dim B$, resp. $\dim A \leq \dim B$. The *expected dimension* for the cokernel of $f$ is defined to be $\exp\dim \text{cok}(f) := \max(0, \dim B - \dim A)$. So, for example,

$$\exp\dim \text{cok}(\mu_k(Z)) = \max(0, h^0(\mathcal{I}_Z(k+1)) - 3h^0(\mathcal{I}_Z(k))).$$

We say that a fat point scheme $Z$ has *good postulation in degree* $k$, if the map $r_k$ is of maximal rank, i.e. if $h^0(\mathcal{I}_Z(k))h^1(\mathcal{I}_Z(k)) = 0$. We say that $Z$ has *good postulation* if the maps $r_k$ have maximal rank for all $k$, and we say that $Z$ is *minimally generated* if the maps $\mu_k$ all have maximal rank (i.e., $Z$ is minimally generated if $\mu_k$ is onto when it is exp-onto and injective when it is exp-inj).

A few additional notions will be useful. Given a 0-dimensional scheme $Y$, we denote by $l(Y)$ the length of $Y$; hence $l(m_1P_1 + \cdots + m_nP_n) = \sum_i \binom{m_i+1}{2}$.

We define $\alpha = \alpha(Z)$ to be the least $k$ such that $h^0(\mathcal{I}_Z(k))$ is positive, and we define $\tau = \tau(Z)$ to be the least $k$ such that $h^1(\mathcal{I}_Z(k)) = 0$.

Recall that if $h^1(\mathcal{I}_Z(k)) = 0$ then $h^1(\mathcal{I}_Z(t)) = 0$ for $t \geq k$, and $\mu_t(Z)$ is surjective for $t \geq k + 1$, by the Castelnuovo-Mumford lemma [Mu2].
Remark 2.2 Let $Z$ be a fat points subscheme of $\mathbb{P}^2$ (supported at general points). Then $\alpha - 1 \leq \tau$. If $Z$ has good postulation, then $\alpha - 1 \leq \tau \leq \alpha$.

In fact, $\alpha - 1 \leq \tau$ follows by taking cohomology of $0 \to \mathcal{I}_Z(k) \to \mathcal{O}_\mathbb{P}^2(k) \to \mathcal{O}_Z \to 0$. Good postulation gives $h^0(\mathcal{I}_Z(k)) = h^1(\mathcal{I}_Z(k)) = 0$, which implies $\tau \leq \alpha$.

Remark 2.3 Since $\mu_k(Z)$ (being the 0-map) is trivially injective for all $k < \alpha$ and it is surjective for $k \geq \tau + 1$, we need only consider $\mu_k$ in degrees $k$ (if any) with $\alpha \leq k \leq \tau$.

If $Z$ has good postulation, then either $\tau = \alpha - 1$, and the Betti numbers for $I(Z)$ are completely determined, or $\tau = \alpha$, in which case we need only consider $\mu_\alpha$; if $\mu_\alpha$ is exp-onto, then $Z$ is minimally generated if and only if $\mu_\alpha$ is surjective, while if $\mu_\alpha$ is exp-inj, $Z$ is minimally generated if and only if $\mu_\alpha$ is injective.

Now drop the good postulation assumption, and take any $Z$; if $k \geq \alpha$, assuming the SHGH Conjecture it is always possible (and easy to do explicitly, by factoring out the fixed part of $H^0(\mathcal{I}_Z(k))$; see [GH]) to replace $k$ and $Z$ by a $k'$ and $Z'$ (supported at the same points) such that the kernels of $\mu_k(Z)$ and $\mu_{k'}(Z')$ have the same dimension, but such that $Z'$ has good postulation in degree $k'$. Thus (assuming the SHGH Conjecture) we can reduce to considering only fat points $Z$ with good postulation and with $\alpha = \tau$, and for these we need to study only the map $\mu_\alpha$.

The forthcoming Remark 2.4 and Lemma 2.5 will be useful in the next section:

Remark 2.4 Let $C$ be a curve of degree $d$ in $\mathbb{P}^2$; then the exact sequence $0 \to \mathcal{O}_\mathbb{P}^2(t - d) \to \mathcal{O}_\mathbb{P}^2(t) \to \mathcal{O}_\mathbb{P}^2(t)|_C \to 0$ gives

$$h^0(\mathcal{O}_\mathbb{P}^2(t)|_C) = \left(\frac{t+2}{2}\right) \text{ for } 0 \leq t \leq d - 1, \quad h^0(\mathcal{O}_\mathbb{P}^2(t)|_C) = \frac{1}{2}(2td + 3d - d^2) \text{ for } t \geq d.$$ 

The same exact sequence twisted by $\Omega$ and the cohomology of the cotangent bundle (see for example [OSS]):

$$h^0(\mathbb{P}^2, \Omega(k)) = h^2(\mathbb{P}^2, \Omega(-k)) = \begin{cases} k^2 - 1 & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0 \end{cases}, \quad h^1(\mathbb{P}^2, \Omega(k)) = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$$

together give:

$$h^0(\Omega(t)|_C) = \begin{cases} (t-1)(t+1) & \text{if } 1 \leq t \leq d - 2 \\ d(2t-d) & \text{if } t \geq d - 1 \end{cases}, \quad h^1(\Omega(t)|_C) = 0 \text{ for } t \geq \max(1, d - 1).$$

Lemma 2.5 Let $C$ be a plane curve having a singularity of multiplicity $r$ at a point $P$, and let $Z$ be the $m$-fat point supported at $P$; then $l(Z \cap C) = \binom{m+1}{2} - \binom{m-r+1}{2}$.

Proof. Let $x, y$ be local coordinates at $P$, $A_m := K[x, y]/(x, y)^{m}$ the coordinate ring of $Z$, $f = 0$ a local equation for $C$, where $f$ has initial degree $r$, and $(\bar{f}) := (f)A_m$; then, $l(Z \cap C)$ is the dimension of the $K$-vector space $A_m/(\bar{f})$. If $r \geq m$, $\bar{f} = 0$, so $\dim_K A_m/(\bar{f}) = \binom{m+1}{2}$ (which was already obvious since $Z \subset C$). If $r < m$, it is easy to prove that the vector space $(\bar{f})$ has dimension $\binom{m+1-r}{2}$ using an appropriate induction.

$\blacksquare$
3 Various equivalent postulation problems

In this and in the following sections \( k \) will always denote a positive integer, and \( Z \), as usual, a fat point subscheme supported at general points of \( \mathbf{P}^2 \).

In this section we are going to translate the problem of determining the rank of the maps \( \mu_k(Z) : I(Z)_k \otimes R_i \to I(Z)_{k+1} \) into three different, but closely related, postulation problems. By postulation problem we mean the computation of the rank of a restriction map \( H^0(F) \to H^0(F|_Y) \) with \( F \) a vector bundle and \( Y \) a subscheme of a given scheme. One of these approaches, i.e. the translation into a postulation problem in the 3-fold \( \mathbf{P}(\Omega) \) with respect to a rank 1 bundle, is here because we find it intrinsically interesting, although we’ll use it only to understand the geometry of certain examples. The other two approaches will lead to conjectures [6.1] and [6.7].

We now define the three restriction maps in which we are interested: \( \rho_k = \rho_k(Z) \), \( \psi_k = \psi_k(Z) \), \( \eta_k = \eta_k(Z) \).

The multiplication map \( \mu_k = \mu_k(Z) \) comes from considering the Euler sequence twisted by \( \mathcal{I}_Z(k) \) and taking cohomology:

\[
\begin{aligned}
(1*) \quad 0 \to & H^0(\Omega(k+1) \otimes \mathcal{I}_Z) \to H^0(\mathcal{I}_Z(k)) \otimes H^0(\mathcal{O}_{\mathbf{P}^2}(1)) \xrightarrow{\mu_k} H^0(\mathcal{I}_Z(k+1)) \to H^1(\Omega(k+1) \otimes \mathcal{I}_Z) \to H^1(\Omega(k+1)) \to \ldots
\end{aligned}
\]

In the forthcoming Lemma [3.1] we compare this to the cohomology sequence obtained by restricting \( \Omega \) to \( Z \):

\[
\begin{aligned}
(2*) \quad 0 \to & H^0(\Omega(k+1) \otimes \mathcal{I}_Z) \to H^0(\Omega(k+1)) \xrightarrow{\rho_k} H^0(\Omega(k+1)|_Z) \to H^1(\Omega(k+1) \otimes \mathcal{I}_Z) \to H^1(\Omega(k+1)) = 0
\end{aligned}
\]

Now consider the projective bundle \( \pi : \mathbf{P}(\Omega) \to \mathbf{P}^2 \) with the invertible sheaf

\[
\mathcal{E}_t = \mathcal{O}_{\mathbf{P}(\Omega)}(1) \otimes \pi^* \mathcal{O}_{\mathbf{P}^2}(t).
\]

We set

\[
T = \pi^{-1}(Z) \subset \mathbf{P}(\Omega).
\]

By [Hi] Ex. III.8.1, III.8.3 and III.8.4, \( R^i \pi_* \mathcal{O}_{\mathbf{P}(\Omega)}(1) = 0 \) for \( i > 0 \), hence \( R^i \pi_* \mathcal{E}_t \cong R^i \pi_* \mathcal{O}_{\mathbf{P}(\Omega)}(1) \otimes \mathcal{O}_{\mathbf{P}^2}(t) = 0 \) for \( i > 0 \), so that \( H^i(\Omega(t)) \cong H^i(\mathcal{E}_t) \) for all \( i \geq 0 \); in particular, \( H^1(\mathcal{E}_k+1) = 0 \) for any \( k \geq 0 \). Taking \( \psi_k \) to be the canonical restriction map, we have the exact sequence:

\[
(3*) \quad 0 \to H^0(\mathcal{E}_{k+1} \otimes \mathcal{T}) \to H^0(\mathcal{E}_{k+1}) \xrightarrow{\psi_k} H^0(\mathcal{E}_{k+1}|_T) \to H^1(\mathcal{E}_{k+1} \otimes \mathcal{T}) \to 0.
\]

We will also work in the blow up \( p : X \to \mathbf{P}^2 \). Set

\[
\tilde{Z} = \sum_{i \geq 1} m_i E_i \subset X
\]

and consider the exact sequence:

\[
(4*) \quad 0 \to H^0(p^*(\Omega(k+1) \otimes \mathcal{I}_Z)) \to H^0(p^*(\Omega(k+1))) \xrightarrow{\eta_k} H^0(p^*(\Omega(k+1)|_Z)) \to H^1(p^*(\Omega(k+1) \otimes \mathcal{I}_Z)) \to 0
\]

where \( H^1(p^*(\Omega(k+1))) = 0 \) for the following reason: \( R^i p_* \mathcal{O}_X = 0 \) for \( i > 0 \) and \( p_* \mathcal{O}_X \cong \mathcal{O}_{\mathbf{P}^2} \) hence, by [Hi] III.8.3, \( R^i p_* p^* \Omega(k+1) \cong R^i p_* (\mathcal{O}_X \otimes p^* \Omega(k+1)) \cong R^i p_* \mathcal{O}_X \otimes p^* \Omega(k+1) = 0 \) for \( i > 0 \) and \( p_* p^* \Omega(k+1) \cong \Omega(k+1) \), and so by [Hi] ex. III.8.1 \( H^i(p^*(\Omega(k+1))) \cong H^i(p_* p^* \Omega(k+1)) = H^i(\Omega(k+1)) \) for all \( i \geq 0 \).
Lemma 3.1 If $Z$ has good postulation in degrees $k$ and $k+1$, then $\mu_k$ is injective, resp. surjective, if and only if $\rho_k$ is injective, resp. surjective. Moreover, if $h^1(I_Z(k)) = 0$, then
\[
\exp\dim cok \mu_k = \exp\dim cok \rho_k = \max(0, 2l(Z) - (k+2)), \quad \text{and}
\]
\[cok \mu_k = cok \rho_k = H^1(\Omega(k+1) \otimes I_Z).\]

Proof. If $h^0(I_Z(k)) = 0$, $\mu_k$ is injective, that is, $H^0(\Omega(k+1) \otimes I_Z) = 0$, so also $\rho_k$ is injective. If $h^1(I_Z(k)) = 0$, we have that $cok \mu_k = H^1(\Omega(k+1) \otimes I_Z) = cok \rho_k$, and $\ker \mu_k = H^0(\Omega(k+1) \otimes I_Z) = \ker \rho_k$, so that the difference between the dimension of the domain and the dimension of the codomain is the same: $h^0(I_Z(k))h^0(\mathcal{O}_{P^2}(1)) - h^0(I_Z(k+1)) = 3((k+2) - l(Z)) - ((k+3) - l(Z)) = k(k+2) - 2l(Z) = h^0(\Omega(k+1)) - h^0(\Omega(k+1)|_Z)$.

Lemma 3.2 The following conditions are equivalent:

(i) $\rho_k$ is injective, resp. surjective;
(ii) $\psi_k$ is injective, resp. surjective;
(iii) $\eta_k$ is injective, resp. surjective.

Proof. $i) \Leftrightarrow ii)$: one has $\pi_*(\mathcal{E}_t) \cong \Omega(t)$, and (see for example [II], 2.1) $\pi_*(\mathcal{E}_t|_T) \cong \Omega(t)|_Z$, $\pi_*(\mathcal{E}_t \otimes \mathcal{I}_T) \cong \Omega(t) \otimes I_Z$; hence $H^0(\mathcal{E}_t) \cong H^0(\Omega(t))$, $H^0(\mathcal{E}_t \otimes \mathcal{I}_T) \cong H^0(\Omega(t) \otimes I_Z)$, $H^0(\mathcal{E}_t|_T) \cong H^0(\Omega(t)|_Z)$.

$i) \Leftrightarrow iii)$: one has $p_*\mathcal{O}_X \cong \mathcal{O}_{P^2}$; by Prop. 2.3 of [AH], one has also: $p_*\mathcal{O}_Z \cong \mathcal{O}_Z$; hence it follows (see for example the proof of Lemma 2.3 in [II], taking into account that $p^{-1}(Z) = Z$) that $p_*(\mathcal{I}_Z) \cong \mathcal{I}_Z$. By the projection formula we get $p_*(p^*\Omega(k+1)) \cong \Omega(k+1)$, so $p_*(p^*\Omega(k+1)|_Z) \cong \Omega(k+1)|_Z$ and $p_*(p^*\Omega(k+1) \otimes \mathcal{I}_Z) \cong \Omega(k+1) \otimes I_Z$. Hence the dimensions of the first three vector spaces in (4*) and in (2*) are the same, so we conclude that $\rho_k$ is of maximal rank if and only if $\eta_k$ is.

4 Superfluous conditions for the cotangent bundle

Now we are interested in studying the behaviour of the restriction of $\Omega(k+1)$ to a curve in $P^2$. This will help us in the study of $\rho_k$ and hence (see Section [II]) of $\mu_k$. In what follows $C$ will be a curve of degree $d$ in $P^2$.

Definition 4.1 We denote by
\[\beta = \beta_{C,Z,k} : H^0(\Omega(k+1)|_C) \rightarrow H^0(\Omega(k+1)|_{C \cap Z})\]
the restriction map. We also set
\[\gamma(C, Z, k) := \exp\dim cok \beta_{C,Z,k} = \max\{0, 2l(Z \cap C) - h^0(\Omega(k+1)|_C)\}.\]

If $m(C)_{P^2} = r_i \leq m_i + 1$, by Lemma 2.5 $l(Z \cap C) = \sum(r_im_i - \binom{r_i}{2})$. So by Remark 2.4 we find for $k + 2 \geq d$ and $r_i \leq m_i + 1$
\[\gamma(C, Z, k) = \max\{0, 2\sum(r_im_i - \binom{r_i}{2}) - d(2k + 2 - d)\}.\]
Proposition 4.2 Assume $h^1(I_Z(k)) = 0$. If $C \subset P^2$ is a curve of degree $d \leq k + 2$, then
\[ \dim \operatorname{cok} \mu_k \geq \dim \operatorname{cok} \beta_{C,Z,k}. \]
In particular, if there exists a (not necessarily integral) curve $C$ of degree $d \leq k + 2$ such that
\[ \dim \operatorname{cok} \beta_{C,Z,k} > \exp \dim \operatorname{cok} \mu_k, \]
then $\mu_k$ is not of maximal rank.

Proof. Since $h^1(I_Z(k)) = 0$, $Z$ has good postulation in degree $k$ and $k + 1$, so, by Lemma 3.1, $\dim \operatorname{cok} \mu_k = \dim \operatorname{cok} \rho_k = h^1(I_Z \otimes \Omega (k + 1))$.

Now set $t := k + 1$ and consider the commutative diagram:
\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & I_{\text{res} C,Z} \otimes \Omega (t - d) & I_Z \otimes \Omega (t) & I_{Z \cap C,C} \otimes \Omega (t) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Omega (t - d) & \Omega (t) & \mathcal{O}_C \otimes \Omega (t) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_{\text{res} C,Z} \otimes \Omega (t - d) & \mathcal{O}_Z \otimes \Omega (t) & \mathcal{O}_{Z \cap C,C} \otimes \Omega (t) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O} \tag{1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\end{array}
\]

Taking cohomology we get:
\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^0(I_{\text{res} C,Z} \otimes \Omega (t - d)) & H^0(I_Z \otimes \Omega (t)) & H^0(I_{Z \cap C,C} \otimes \Omega (t)) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^0(\Omega (t - d)) & H^0(\Omega (t)) & H^0(\Omega (t)|_C) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^0(\mathcal{O}_{\text{res} C,Z}) & H^0(\mathcal{O}^2 Z) & H^0(\mathcal{O}^2_{Z \cap C,C}) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\gamma \cdot H^1(I_{\text{res} C,Z} \otimes \Omega (t - d)) & H^1(I_Z \otimes \Omega (t)) & H^1(I_{Z \cap C,C} \otimes \Omega (t)) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where $h^1(\Omega (t)|_C) = 0$ and $h^2(I_{\text{res} C,Z} \otimes \Omega (t - d)) = 0$ since $h^2(\Omega (t - d)) = 0$ (this because $d \leq t + 1$). One has that $\dim \operatorname{cok} \mu_k = \dim \operatorname{cok} \rho_k = h^1(I_Z \otimes \Omega (t)) \geq h^1(I_{Z \cap C,C} \otimes \Omega (t)) = \dim \operatorname{cok} \beta_{C,Z,k}$.  

As a consequence we have a first criterion to find schemes $Z$ for which $\mu_k$ fails to have maximal rank:

Corollary 4.3 Assume $h^1(I_Z(k)) = 0$. If there exists a (not necessarily integral) curve $C$ of degree $d \leq k + 2$ such that
\[ \gamma(C, Z, k) > 0 \text{ if } \mu_k \text{ is exp-onto, or} \]
\[ \gamma(C, Z, k) > 2l(Z) - k(k + 2) \text{ if } \mu_k \text{ is exp-inj,} \]
then $\mu_k$ is not of maximal rank.

Proof. This follows directly by Proposition 4.2 since $\dim \operatorname{cok} \beta_{C,Z,k} \geq \gamma(C, Z, k)$.  

Example 4.4 Let $Z = Z(3, 2, 1, 1)$; then $\mu_4(Z)$ does not have maximal rank. To see this directly, let $C$ be the line through $P_1$ and $P_2$. Then $Z$ has good postulation (see [H55] or [H33] for calculating the Hilbert function), $l(Z) = 11$, $h^0(I_Z(3)) = 0$, $h^0(I_Z(4)) = 4$, $h^0(I_Z(5)) = 10$, and $\alpha = \tau = 4$. Thus $I(Z)$ is generated in degrees at most 5, but since $C$ is in the base locus of $H^0(I_Z(4))$ but the zero locus of the whole ideal is just $Z$, there must be a generator of degree 5, so the map $\mu_4(Z)$ is not surjective, and since it is exp onto it is hence not of maximal rank (see [2.3]).

Alternatively, note that Corollary 4.3 applies: $\mu_4$ is exp onto but $\gamma(C, Z, 4) > 0$ since, using the fact $\Omega(5)|_C \cong OC(3) \oplus OC(4)$, we see $h^0(\Omega(5)|_C) = 9 < 10 = 2l(Z \cap C)$. In other words, $Z \cap C$ imposes one superfluous condition to the sections of $\Omega(5)|_C$. But a point of $P^2$ imposes 2 condition to a rank 2 bundle; so if we wish to understand what’s going on geometrically we have to move to $P(\Omega)$. Here (cf. Lemma 3.2) we have to check the dimension of the space of global sections of $E_5 = Op_{p_{(\Omega)}}(1) \otimes \pi^*p_{25}(5)$ vanishing on the 1-dimensional scheme $T = \pi^{-1}(Z)$, or, equivalently, on the 0-dimensional scheme $T' := (T \cap P(E)) \cup (T \cap P(G))$, where $E \oplus G$ is a local trivialization of $\Omega$. (In fact, since $E_5$ is $Op_1(1)$ on the fibers, the inverse image of a point $\pi^{-1}(P)$ can be replaced by two generic points in the fiber. For the non reduced case, and for further details, see [H1], [I3], [G1].) Hence we are looking at the postulation with respect to the invertible sheaf $E_5$ of the 0-dimensional scheme $T'$ in $P(\Omega)$; since $\Omega(5)|_C \cong OC(3) \oplus OC(4)$, we have a curve $D := P(OC(3)) \subset P(\Omega)|_C \subset P(\Omega)$ and $T' \cap D$ has length 5, while $E_5|_D \cong Op_D(3)$ (see [H1] V.2.6). It is now clear that it is possible to find a subscheme of $T'$ of length $l(T') - 1$ imposing the same conditions as $T'$ to $E_5$, that is, $T'$ does not postulate well with respect to $E_5$.

Example 4.5 Another similar example is given by $Z = Z(4, 3, 3, 3, 2)$; here $\mu_7(Z)$ is again exp onto and fails to have maximal rank. To see this, let $C$ be the conic through the 5 points $P_i$. Again $Z$ has good postulation ([H55], [H33]), and we have $l(Z) = 31$, $h^0(I_Z(6)) = 0$, $h^0(I_Z(7)) = 5$, $h^0(I_Z(8)) = 14$, $\alpha = \tau = 7$. As before, $C$ is in the base locus of $I(Z)_7$, so while the map $\mu_7$ is exp onto it is not surjective, hence does not have maximal rank. Alternatively, again Corollary 4.3 applies: $\gamma(C, Z, 7) > 0$. In more detail, $Z$ does not postulate well with respect to $\Omega(8)$ (i.e., the number of sections of $\Omega(8)$ vanishing on $Z$ is greater than the length of $Z$ would lead us to expect), since $h^0(\Omega(8)|_C) = 28 < 2 \cdot 15 = 2l(Z \cap C)$. In other words, $Z \cap C$ imposes 2 superfluous conditions on the sections of $\Omega(8)|_C \cong Op_1(13)^{\oplus 2}$ (here we have used the fact that $\Omega|_C \cong Op_1(13)^{\oplus 2}$). If we work in $P(\Omega)$ (cf. Lemma 3.2), we have to consider the postulation with respect to $E_8$ of the 1-dimensional scheme $T = \pi^{-1}(Z)$, or, as in the previous example, of the 0-dimensional scheme $T' := (T \cap P(E)) \cup (T \cap P(G))$, $E \oplus G$ again being a local trivialization of $\Omega$. Since $\Omega(8)|_C \cong Op_1(13)^{\oplus 2}$, we have two curves, $D_1$ and $D_2$, both contained in $P(\Omega)|_C$, where $T' \cap D_i$ has length 15, while $E_8|_{D_i} \cong Op_1(13)$. It is hence possible to find a subscheme of $T'$ of length $l(T') - 2$ imposing the same conditions as $T'$ to $E_8$; i.e., $T'$ does not postulate well with respect to $E_8$. These two superfluous conditions give a contribution of 2 to the cokernel.

Example 4.6 The map $\mu_5$ for $Z = 3P_1 + 3P_2 + 3P_3$ fails to have maximal rank; $Z$ postulates well (H11), $l(Z) = 18$, $h^0(I_Z(4)) = 0$, $h^0(I_Z(5)) = 3$, $h^0(I_Z(6)) = 10$, $\alpha = \tau = 5$, and $\mu_5$ is exp-inj. But $\mu_5$ is not injective; actually, if $L_{ij}$ is the line through $P_i$ and $P_j$, and $C$ is the union of $L_{12}$, $L_{13}$ and $L_{23}$, the cubic $C$ is a fixed component for $|I(Z)_5|$, hence the three generators of $I(Z)_5$ are of the form $CF_i$, $i = 1, 2, 3$, where $F_1$, $F_2$ and $F_3$ are the three conics which generate $I(P_1 + P_2 + P_3)$. Since $h^0(I_{P_1 + P_2 + P_3}(3)) = 7$, the dimension of the image of $\mu_5$ is also 7, i.e. $I(Z)$ needs 3 generators in degree 6, not just one.

Alternatively, note that Corollary 4.3 applies: $\gamma(C, Z, 5) = 3 > 2l(Z) - 5(5 + 2) = 1$; here $C$ is a triangle, hence reducible with arithmetic genus 1. What happens here is that $\Omega(6)|_{L_{ij}} \cong Op_{L_{ij}}(4) \oplus Op_{L_{ij}}(5)$, so that $Z \cap L_{ij}$ imposes one superfluous condition to the sections of $\Omega(5)|_{L_{ij}}$.
for each one of the three lines $L_{ij}$; one of these superfluous conditions wouldn’t bother the rank maximality of $\mu_5$, which is expected to be injective with a 1-dimensional cokernel; the other two conditions give a contribution of 2 to the cokernel. If we reinterpret the situation in $\mathbb{P}(\Omega)$, we have to consider the postulation with respect to $\mathcal{E}_6$ of a certain 0-dimensional scheme $T'$, analogously to what happens in the previous examples; here there are three curves $D_{ij} := \mathbb{P}(\mathcal{O}_{L_{ij}}(4))$ such that $\mathcal{E}_6|_{D_{ij}} \cong \mathcal{O}_{D_{ij}}(4)$, while $T' \cap D_{ij}$ has length 6; $T'$ does not postulate well with respect to $\mathcal{E}_6$. Notice anyway that the reducible curve $D_{12} \cup D_{13} \cup D_{23}$ causing troubles is now the union of three disjoint smooth rational curves, since a point $P$ in $D_{ij}$ is the point in $\mathbb{P}(\Omega|_{L_{ij}})$ representing the tangent direction of $L_{ij}$ at $P$.

These first three examples are easy to treat by taking into account the occurrence of fixed components. The next example (as well as example 5.4) shows that this is not always the case.

**Example 4.7** If $Z = 9P_1 + \cdots + 9P_7$, the map $\mu_{24}$ fails to have maximal rank. Again $Z$ postulates well ([Ha2], [Ha5], [Ha3]), $l(Z) = 315$, $h^0(\mathcal{I}_Z(23)) = 0$, $h^0(\mathcal{I}_Z(24)) = 10$, $h^0(\mathcal{I}_Z(25)) = 36$, $\alpha = \tau = 24$, and $\mu_{24}$ is exp-inj. But $|I(Z)|_{24}$ is fixed component free and $\mu_{24}$ is not injective; if it were, dim $\text{Im} \mu_{24}$ would be 30, but in fact it is 29 ([Ha2]). Once more, Corollary 4.3 applies: $\gamma(C, Z, 24) > 2l(Z) - 24(24 + 2)$, with $C := \sum C_i$, where $C_i$ is a cubic with $m(C_i) P_i = 1$ for $i \neq j$, $2$ for $i = j$. Again, $C$ is not irreducible. What happens here is that the superfluous conditions imposed by $Z \cap C$ on $\Omega(25)|_C$ are more than the expected dimension for the cokernel of $\rho_{24}$, since $2l(Z \cap C) - h^0(\Omega(25)|_C) = 616 - 609 = 7 > 2l(Z) - 24(24 + 2) = 6$. Thus $\rho_{24}$, and hence $\mu_{24}$, are not injective by Corollary 4.3 (Notice that taking into account just one of the curves $C_i$ is not enough: in fact, $h^0(\Omega(25)|_{C_i}) = 141 < 2l(Z \cap C_i) = 142$; but this only says that dim cok$\rho_{24} \geq 1$.)

## 5 Superfluous conditions for the pullback of the cotangent bundle

In examples 4.4, 4.5, 4.6 and 4.7 failure of $\mu_k(Z)$ to have maximal rank was related to $Z$ imposing too many conditions on the global sections of $\Omega(k+1)|_C$, and we checked it just by a dimension count, i.e. the expected dimension $\gamma(C, Z, k)$ of the cokernel of $\beta_{C, Z, k}$ was too big. But $\Omega(k+1)|_C$ is a rank two vector bundle, so it can happen that the dimension of the cokernel is bigger than its expected dimension. This of course cannot occur with a rank one bundle on $\mathbb{P}^1$, since if $A$ is a 0-dimensional scheme on $\mathbb{P}^1$, the cokernel of the restriction map $H^0(\mathcal{O}_A(t)) \rightarrow H^0(\mathcal{O}_A)$ always has the expected dimension.

Instead if we consider for example the restriction map $H^0(\mathcal{O}_A \oplus \mathcal{O}_A(2)) \rightarrow H^0(\mathcal{O}_A)$ where $A$ is the union of two points, then the expected dimension of the cokernel is 0 but the actual dimension is 1; $A$ imposes 1 condition too many on $H^0(\mathcal{O}_A)$. This is possible because the splitting gap of $\mathcal{O}_A \oplus \mathcal{O}_A(2)$ is 2. In the previous examples this behaviour did not arise: in examples 4.4 and 4.5 $C$ is a line or a smooth conic with splitting gap 1, respectively 0; in example 4.7 $C_i$ is a singular cubic, so we don’t look at $\Omega(k+1)|_{C_i}$, but we can look at the splitting of the pull-back of $\Omega(k+1)$ on $\tilde{C}_i$, and we find that the splitting gap is 1.

The forthcoming example 5.4 instead, illustrates a situation such that $\mu_k(Z)$ is exp-onto, but there exists a curve $C$ with splitting gap 2, and dim cok $\beta_{C, Z, k} > 0$, so $\mu_k(Z)$ is not onto although exp-dim cok $\beta_{C, Z, k} = \gamma(C, Z, k) = 0$. So it seems evident that, if we want to formulate a conjecture about the rank maximality of $\mu_k(Z)$, it is necessary to take into consideration the splitting type, and to consider the real cokernel of the maps $\beta_{C, Z, k}$; this is what we are going to do next.

**Definition 5.1** Let $C$ be a curve of degree $d$ in $\mathbb{P}^2$, such that its strict transform $\tilde{C} = dL - \sum r_iE_i$ is smooth and rational in the surface $X$ obtained by blowing up the points $P_i$. Given a positive
integer $k$ and taking cohomology of the exact sequence $0 \to p^*\Omega(k+1)|_{\hat{C}} \otimes \mathcal{I}_{\hat{C} \cap \tilde{Z}} \to p^*\Omega(k+1)|_{\hat{C}} \to p^*\Omega(k+1)|_{\hat{C} \cap \tilde{Z}} \to 0$, where $\tilde{Z} = \sum m_i E_i$, we get the restriction map

$$\theta = \theta_{C,Z,k} : H^0(p^*\Omega(k+1)|_{\hat{C}}) \to H^0(p^*\Omega(k+1)|_{\hat{C} \cap \tilde{Z}}).$$

In order to measure the superabundance of conditions imposed by $\hat{C} \cap \tilde{Z}$ on the sections of $p^*\Omega(k+1)|_{\hat{C}}$ we also set

$$\delta_0(C, Z, k) = \dim \cok \theta_{C,Z,k}.$$

Writing $a$ and $b$ for $a_C$ and $b_C$, we have $p^*\Omega(k+1)|_{\hat{C}} \cong \mathcal{O}_C(-a + dk) \oplus \mathcal{O}_C(-b + dk)$. Moreover, $\hat{C} \cdot \tilde{Z} = \sum r_i m_i$.

Since $b \leq d$ and by assumption $k \geq 1$, we have $dk - b \geq 0$ so that $h^1(p^*\Omega(k+1)|_{\hat{C}}) = 0$. Hence

$$\delta_0(C, Z, k) = h^1(p^*\Omega(k+1)|_{\hat{C}}) = h^1(\mathcal{O}_C(-a + dk - \sum r_i m_i) \oplus \mathcal{O}_C(-b + dk - \sum r_i m_i)) = \max(0, l(\tilde{Z} \cap \hat{C}) - h^0(\mathcal{O}_C(-a + dk)) + \max(0, l(\tilde{Z} \cap \hat{C}) - h^0(\mathcal{O}_C(-b + dk)),$

so that finally

$$\delta_0(C, Z, k) = \max(0, \sum r_i m_i - dk + a - 1) + \max(0, \sum r_i m_i - db + b - 1).$$

In certain cases, $\delta_0$ is nothing more than $\gamma$;

**Theorem 5.2** Let $C \subset \mathbb{P}^2$ be a curve whose strict transform $\hat{C} = dL - \sum r_i E_i$ is smooth and rational in $X$, and assume $d \leq k + 2$ and $r_i - 1 \leq m_i$ for all $i$. Then

$$\cok \beta_{C,Z,k} \cong \cok \theta_{C,Z,k}$$

hence $\delta_0(C, Z, k) \geq \gamma(C, Z, k)$, with equality if and only if $\cok \beta_{C,Z,k}$ has its expected dimension (in this occurs, for example, if $\sum r_i m_i - dk + a - 1 \geq 0$).

**Proof.** The maps $\theta = \theta_{C,Z,k}$ and $\bar{\theta} = \bar{\theta}_{C,Z,k} : H^0(p^*\Omega(k+1)|_{\hat{C}}) \to H^0(p^*\Omega(k+1)|_{\hat{C} \cap \tilde{Z}})$ are the maps on cohomology coming from the exact sequences $0 \to p^*\Omega(k+1)|_{\hat{C}} \otimes \mathcal{I}_{\hat{C} \cap \tilde{Z}} \to p^*\Omega(k+1)|_{\hat{C}} \to p^*\Omega(k+1)|_{\hat{C} \cap \tilde{Z}} \to 0$ and its pushforward by $p_*$, and it is clear that $\cok \bar{\theta} \cong \cok \theta$.

We also have an exact sequence $0 \to \mathcal{O}_C \to p_*\mathcal{O}_{\hat{C}} \to S \to 0$, where $S = \oplus_{P \in \text{Sing}(C)} \mathcal{O}_P/O_P$ and $\mathcal{O}_P$ denotes the integral closure of $\mathcal{O}_P$. Letting $\bar{\delta}_P$ be the length $l(\mathcal{O}_P/O_P)$, one has (by [Ht], Ex. IV.1.8 and Cor. V.3.7) $p_0(C) = p_0(\hat{C}) + \sum_{P \in \text{Sing}(C)} \bar{\delta}_P$. But $0 = p_0(\hat{C}) = \binom{d-1}{2} - \sum \binom{i}{2}$ and $p_0(C) = \binom{d-1}{2}$, so $\sum_{P \in \text{Sing}(C)} \bar{\delta}_P = \sum \binom{i}{2}$, hence $l(S) = \sum \binom{i}{2}$.

There is a natural map $0 \to \mathcal{O}_{\hat{C} \cap \tilde{Z}} \to p_*\mathcal{O}_{\hat{C} \cap \tilde{Z}}$; let us denote the cokernel by $S'$. Since $r_i \leq m_i + 1$ for all $i$, by lemma [25] we have $l(S') = l(\tilde{C} \cap \tilde{Z}) - l(C \cap Z) = \sum r_i m_i - \sum (r_i m_i - \binom{i}{2}) = \sum \binom{i}{2}$. Now consider the diagram

$$
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \\
\downarrow & & \downarrow \\
0 & \to & p_*\mathcal{O}_{\hat{C}} \\
\downarrow & & \downarrow \\
0 \oplus R^1 p_*\mathcal{I}_{\hat{C} \cap \tilde{Z}, \hat{C}} & \to & S' \to 0
\end{array}
$$

There is a map $S \to S'$ making the diagram commute, and it has to be surjective since $R^1 p_*\mathcal{I}_{\hat{C} \cap \tilde{Z}, \hat{C}} = 0$ by [Ht] III.11.2. Hence it is a surjective map between sheaves supported at points and of the same length, so we conclude $S' \cong S$, which gives us the exact sequence $0 \to \mathcal{O}_{\hat{C} \cap \tilde{Z}} \to p_*\mathcal{O}_{\hat{C} \cap \tilde{Z}} \to S' \to 0$.

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Tensoring this and the exact sequence at the beginning of the proof by \( \Omega(k+1) \), taking into account that \( p_!\mathcal{O}_{\tilde{C}} \otimes \Omega(k+1) = p_!\mathcal{O}_{\tilde{C}}^* \Omega(k+1)|_{\tilde{C}} \), and finally writing the projection formula, \( \mathbb{H} \), III.8.3), recalling that \( H^1(\Omega(k+1)|_{\tilde{C}}) = 0 \) for \( k+2 \geq d \) (see Remark 2.4), and finally writing \( \beta = \beta_{\tilde{C},Z,k} \), we get
\[
0 \rightarrow H^0(\Omega(k+1)|_{\tilde{C}}) \rightarrow H^0(p_!\mathcal{O}(k+1)|_{\tilde{C}}) \rightarrow H^0(S^2) \rightarrow 0
\]
\[
0 \rightarrow H^0(\Omega(k+1)|_{\tilde{C} \cap Z}) \rightarrow H^0(p_!\mathcal{O}(k+1)|_{\tilde{C} \cap Z}) \rightarrow H^0(S^2) \rightarrow 0
\]
The snake lemma now gives \( \text{cok } \overline{\beta} \cong \text{cok } \beta \).

The inequality \( \delta_0(C, Z, k) \geq \gamma(C, Z, k) \) is now clear, since \( \delta_0 \) is the dimension of \( \text{cok } \theta_{\tilde{C},Z,k} \), while \( \gamma \) is merely the expected dimension of \( \text{cok } \beta_{\tilde{C},Z,k} \). For the rest, assuming \( \sum r_i m_i - dk + b - 1 \geq 0 \) and using \( h^0(p^*\mathcal{O}(k+1)|_{\tilde{C}}) = h^0(p_!\mathcal{O}(k+1)|_{\tilde{C}}) = h^0(\Omega(k+1)|_{\tilde{C}}) \), we have \( \delta_0(C, Z, k) = 2l(\tilde{Z} \cap C) - h^0(p^*\mathcal{O}(k+1)|_{\tilde{C}}) = 2l(\tilde{Z} \cap C) - \sum (\delta_i') - h^0(\Omega(k+1)|_{\tilde{C}}) = 2l(\tilde{Z} \cap C) - h^0(\Omega(k+1)|_{\tilde{C}}) = \gamma(C, Z, k). \)

**Corollary 5.3** Assume \( h^1(I_Z(k)) = 0 \) and moreover that there exists an integral curve \( C \subset \mathbb{P}^2 \) such that \( \tilde{C} = dL - \sum r_i E_i \) is smooth and rational in \( X \) with \( d \leq k+2 \), \( r_i - 1 \leq m_i \) for all \( i \) and \( \delta_0(C, Z, k) > \exp \text{-dim cok } \mu_k(Z) \). Then \( \mu_k(Z) \) is not of maximal rank.

**Proof.** We have \( \exp \text{-dim cok } \mu_k < \delta_0(C, Z, k) = \dim \text{cok } \theta_{\tilde{C},Z,k} = \dim \text{cok } \beta_{\tilde{C},Z,k} \) and we conclude by Proposition 4.2.

We now show how to use this last result. A significant difference here with the three previous examples is that the splitting gap for (any irreducible component of) \( C \) was 0 or 1 previously; in Example 5.4 it is 2.

**Example 5.4** Let \( Z = 4P_1 + \cdots + 4P_7 + P_8 \); then \( \mu_{11}(Z) \) fails to have maximal rank (see [FIII]). Note that \( Z \) has good postulation and \( I(Z)_{11} \) is fixed component free (apply [Ha5] or [Ha3]). We have \( l(Z) = 71 \), \( h^0(I_Z(10)) = 0 \), \( h^0(I_Z(11)) = 7 \), \( h^0(I_Z(12)) = 20 \), \( \alpha = \tau = 11 \), hence \( \mu_a \) is exp-onto. The map \( \mu_{11}(Z) \) is not surjective. This can be attributed to the existence of a rational curve \( C \) of degree 8 with \( r_i := m(C) P_i = 3 \) for \( 0 \leq i \leq 7 \), and \( r_8 = 1 \); \( \tilde{C} \subset X \) is a smooth rational curve of self-intersection \( \tilde{C}^2 = 0 \). This time we cannot read failure of maximal rank on the sections of \( \Omega \), since \( h^0(\Omega(12)|_{\tilde{C}}) = 128 = 2l(Z \cap C) \); i.e., \( \gamma(C, Z, 11) = 0 \). Instead, the splitting gap for \( C \) is 2, since (see the proof of Lemma 12 of [FIII]) \( p^*(\Omega(1)|_{\tilde{C}}) \cong \mathcal{O}_{\tilde{C}}(-3) \oplus \mathcal{O}_{\tilde{C}}(-5) \), hence \( p^*(\Omega(12)|_{\tilde{C}}) \cong \mathcal{O}_{\tilde{C}}(85) \oplus \mathcal{O}_{\tilde{C}}(83) \). The scheme \( \tilde{Z} := \sum m_i E_i \) intersects \( \tilde{C} \) in a 0-dimensional scheme of length \( \sum r_i m_i = 85 \), so \( \tilde{Z} \cap \tilde{C} \) is too much for \( \mathcal{O}_{\tilde{C}}(83) \) (and not enough for \( \mathcal{O}_{\tilde{C}}(85) \)); that is, the cohomology of the exact sequence \( 0 \rightarrow p^*(\Omega(12)|_{\tilde{C}} \otimes I_{\tilde{C} \cap Z} \rightarrow p^*(\Omega(12)|_{\tilde{C}} \rightarrow p^*(\Omega(12)|_{\tilde{C} \cap Z} \rightarrow 0 \)

\[
0 \rightarrow H^0(\mathcal{O}_{\tilde{C}} \oplus \mathcal{O}_{\tilde{C}}(-2)) \rightarrow H^0(p^*\mathcal{O}(12)|_{\tilde{C}}) \rightarrow H^0(p^*\mathcal{O}(12)|_{\tilde{C} \cap Z}) \rightarrow H^1(\mathcal{O}_{\tilde{C}} \oplus \mathcal{O}_{\tilde{C}}(-2)) \rightarrow 0
\]
so \( \theta \) is not of maximal rank. (This cannot happen if the splitting gap is 0 or 1.) Since \( \delta_0(C, Z, 11) = 1 > 0 \), we see by Corollary 5.3 that \( \mu_{11}(Z) \) fails to have maximal rank.

The previous examples might lead one to think that the curves \( C \) that need to be taken into consideration are the ones with \( C^2 \leq 0 \). The following example shows that this is not the case.
Example 5.5 Let $Z = 15(P_1 + \ldots + P_4) + 13(P_3 + P_5) + 9P_7 + 2(P_8 + \ldots + P_{11})$; then $l(Z) = 719$, $h^0(I_Z(37)) = 22$, $h^0(I_Z(38)) = 61$, so that $\mu_{37}(Z)$ is exp-onto but in fact it does not have maximal rank; precisely, dim cok($\mu_{37}$) = 1; this has been computed with Macaulay 2 ([GS]).

Now consider a curve $C$ whose strict transform $\tilde{C}$ is an irreducible curve in the linear system $|34L - 14(E_1 + \ldots + E_4) - 12(E_5 + E_6) - 8E_7 - 2(E_8 + \ldots + E_{11})|$ (such a $C$ exists, since $\tilde{C} = 2D$ with $D$ a Cremona transform of a line, hence the linear system above contains Cremona transforms of conics). One has $\tilde{C}^2 = 4$. The splitting type for $\tilde{C}$ is $(14,20)$ (it is possible to compute it with the script in [GHI]); then (5.1) $\delta_0(C,Z,37) = 1$. Here too we cannot work in $\mathbb{P}^2$; in fact, $\gamma(C,Z,37) = 0$.

6 Two conjectures

In each of our examples above, failure of $\mu_k(Z)$ to be surjective is accompanied by $\delta_0(C,Z,k) > 0$. This seems to be fairly general behavior, which leads us to advance the following conjecture.

Conjecture 6.1 Let $Z = \sum m_iP_i$ be a fat point scheme in $\mathbb{P}^2$ (for general points $P_i$), with $h^1(I_Z(k)) = 0$. Say $\mu_k(Z)$ is exp-onto. Then $\mu_k(Z)$ fails to be surjective if and only if there exists an integral curve $C \subset \mathbb{P}^2$ whose strict transform $\tilde{C} = dL - \sum r_iE_i$ is smooth and rational in $X$, with $d \leq k + 2$, $r_i \leq m_i + 1$ and $\delta_0(C,Z,k) > 0$.

Remark 6.2 In fact, in every example we have found for which $\mu_k(Z)$ fails to have maximal rank, we have $h^0(dL - \sum m_iE_i - \tilde{C}) > 0$, and hence $d \leq k$.

The “if” part of Conjecture 6.1 is true, and is Corollary 5.3. Here are some counterexamples to the “if” part of conjecture 6.1 with $d > k + 2$ and $r_i > m_i + 1$ for some $i$:

- $Z = P_1$, $k = 1$, $\tilde{C} = 4L - 3E_1 - E_2 - \cdots - E_8$;
- $Z = P_1 + \cdots + P_4$, $k = 2$, $\tilde{C} = 5L - 3E_1 - 2(E_2 + E_3 + E_4) - (E_4 + \cdots + E_8)$;
- $Z = P_1 + \cdots + P_7$, $k = 3$, $\tilde{C} = 8L - 3(E_1 + \cdots + E_7) - E_8$;
- $Z = 2P_1 + 2P_2 + P_3 + \cdots + P_7$, $k = 4$, $\tilde{C} = 7L - 4E_1 - 3E_2 - 2(E_3 + \cdots + E_8)$.

The problem in each case is, in some sense, that $C$ is too big.

In the case when $\mu_k(Z)$ is exp-inj the situation is more complicated. We have already seen in Examples 4.6, 4.7 that the curve $C$ needs not be irreducible; the following example shows that it can also be nonreduced.

Example 6.3 Let $Z = 60(P_1 + \ldots + P_8)$; then $l(Z) = 14640$, $h^0(I_Z(169)) = 0$, $h^0(I_Z(170)) = 66$, $I_Z(171)) = 238$, $\alpha = \tau = 170$, so that $\mu_{170}(Z)$ is exp-inj but in fact it does not have maximal rank (see 1a2); precisely, exp-dim cok($\mu_{170}$) = 40, while the actual dimension is 48.

Let $C_j$ be a sextic with $r_{j,i} = m(C_j)P_i = 2$ for $i \neq j$ and $r_{j,j} = m(C_j)P_j = 3$, $j = 1, \ldots, 8$. The splitting type for $C_i$ is $(3,3)$ by 2.1; then (5.1) $\delta_0(C_j,Z,170) = 2 \geq 0$, $60 \sum_i r_{j,i} - 6170 + 3 - 1 = 4$. In order to take into account the contribution of each $C_j$, we do as in Example 4.7 and we consider $C = \sum C_j$, but this is still not enough since $8 \cdot 4 < \exp-dim \text{cok}(\mu_{170})$.

So we go on: since $\text{res}_{C_j}Z = 57P_j + 58 \sum_{i \neq j} P_i$, we find $\delta_0(C_j, \text{res}_{C_j}Z,170 - 6) = 2$. If we add up the contribution not only of $Z$ but also of $\text{res}_{C_j}Z$ for all the $C_j$’s, we then find dim cok($\mu_{170}$) $\geq 8(4 + 2) = 48$. It is useless to go on, since $\delta_0(C_j, \text{res}_{C_j}Z,170 - 12) = 0$.

Notice that since the splitting type for $C_j$ is balanced, we can work directly in $\mathbb{P}^2$; it is easy to check that $\gamma(2C,Z,170) = 48$ so it is enough to apply Proposition 1.2.
In this last example we have seen that it is enough to consider $\gamma$, but this is not always the case for injectivity too. In fact, in the following example bijectivity is expected, and $\gamma = 0$, while $\delta_0 = 1$.

**Example 6.4** Let $Z = 11(P_1 + \ldots + P_7) + 5P_8 + 2P_9$; then $l(Z) = 480$, $h^0(I_Z(30)) = 16$, $h^1(I_Z(30)) = 0$, $h^0(I_Z(31)) = 48$, so that $\mu_{30}(Z)$ is exp-bijective but in fact it does not have maximal rank. To see this, it is enough to apply Corollary 5.3 with $C = 19L - 7E_1 + \ldots + E_7 - 4E_8 - E_9$. The splitting type for $C$ is $(8, 11)$ (to compute it, use [GHI]); then 5.1 gives $\delta_0(C, Z, 30) = \max(0, -9 + 8 - 1) + \max(0, -9 + 11 - 1) = 1$.

On the other hand, it is easy to check (4.1) that $\gamma(C, Z, 30) = 0$, while Corollary 4.3 is useless here.

These examples motivate the following definition:

**Definition 6.5** Let $C \subset \mathbb{P}^2$ be a degree $d$ curve, with $m(C)_{P_1} = r_1$. The $h$-iterated residual scheme of $Z = \sum m_i P_i$ with respect to $C$ is defined inductively as follows:

$$res_{C, 0}Z := Z, \quad res_{C, h}Z := res_{C}(res_{C, h-1}Z).$$

Notice that $res_{C, h}Z = (\sum (m_i - hr_i))P_i$ if $m_i \geq hr_i$.

Assume now that the strict transform $\tilde{C} = dL - \sum r_i E_i$ is smooth rational with $a := a_C$, $b := b_C$. Let $t - hd \geq 1$; we define inductively the $h$-superabundance of $C$:

$$\delta_h(C, Z, t) := \delta_0(C, res_{C, h}Z, t - hd).$$

We finally set

$$\delta(C, Z, t) := \sum_{h=0, \ldots, \left[\frac{d}{h}\right]} \delta_h(C, Z, t).$$

Now let $F$ be as usual $F = tL - \sum m_i E_i$, and set $A_h(C, Z, t) = -F \cdot \tilde{C} + a - 1 + h\tilde{C}^2$, $B_h(C, Z, t) = -F \cdot \tilde{C} + b - 1 + h\tilde{C}^2$. Then $B_h(C, Z, t) \geq A_h(C, Z, t)$, and if $m_i \geq hr_i$, $t - hd \geq 1$, we have:

$$\delta_h(C, Z, t) = \max(0, \sum r_i(m_i - hr_i) - d(t - hd) + a - 1) + \max(0, \sum r_i(m_i - hr_i) - d(t - hd) + b - 1) = \max(0, A_h(C, Z, t)) + \max(0, B_h(C, Z, t)).$$

To understand better the connection between $\delta$ and $\gamma$, the following proposition is helpful:

**Proposition 6.6** Let $C \subset \mathbb{P}^2$ be a curve with $\tilde{C} = dL - \sum r_i E_i$ smooth rational. Assume that $(p + 1)r_i - 1 \leq m_i$, $t + 2 \geq d(p + 1)$, and assume also that $A_h(C, Z, t) \geq 0$ for $h = 0, \ldots, p$. Then, denoting by $(p + 1)C$ the $p^{th}$ infinitesimal neighborhood of $C$ in $\mathbb{P}^2$, one has:

$$\sum_{h=0, \ldots, p} \delta_h(C, Z, t) = \gamma((p + 1)C, Z, t).$$

**Proof.** First notice that, if $A_0(C, Z, k) \geq 0$, then using adjunction formula $\delta_0(C, Z, k) = A_0(C, Z, k) + B_0(C, Z, k) = 2\sum (r_i m_i - \binom{k}{2}) - d(2k + 2 - d) \geq 0$. Hence, if $k + 2 \geq d$ and $r_i \leq m_i + 1$, then $\gamma(C, k) = 2\sum (r_i m_i - \binom{k}{2}) - d(2k + 2 - d) = 2l(Z \cap C) - h^0(\Omega(k + 1)|_C) = \delta_0(C, Z, k)$ (see 4.1 5.2).

We have $res_{C, h}Z = \sum (m_i - hr_i)P_i$, since by assumption $hr_i \leq m_i$, for $0 \leq h \leq p$.

So, since by assumption $r_i - 1 \leq m_i - hr_i$, $t - hd + 2 \geq d$ and $A_h(C, Z, t) \geq 0$ for $h = 0, \ldots, p$, we have $\delta_h(C, Z, t) = \delta_0(C, \sum (m_i - hr_i)P_i, t - hd) = \gamma(C, \sum (m_i - hr_i)P_i, t - hd) = 2l((\sum (m_i - hr_i)P_i) \cap C) - h^0(\Omega(t - hd + 1)|_C)$ for $h = 0, \ldots, p$. 
It is easy to check (see [2,4,2,6]) and use $t - dh + 2 \geq d$ and $r_i - 1 \leq m_i - hr_i$ for $0 \leq h \leq p$, and $m_i((p + 1)C)_{P_i} = (p + 1)r_i$ that
\[
\sum_{h=0}^{p} h^{0}(\Omega(t - hd + 1)_{(C)}) = h^{0}(\Omega(t + 1)_{((p + 1)C)}), \text{ and}
\sum_{h=0}^{p} l((\sum m_i - hr_i)_{P_i} \cap C) = l((\sum m_i)_{P_i} \cap (p + 1)C). \text{ So conclusion follows adding up.}
\]

We are now ready to formulate a conjecture for the case where injectivity is expected.

**Conjecture 6.7** Let $Z = \sum_i m_i P_i$ be a fat point scheme in $P^2$ (for general points $P_i$), with $h^{1}(I_Z(k)) = 0$. Say $\mu_k(Z)$ is exp-inj. Then $\mu_k(Z)$ fails to be injective if and only if there exists a curve $C \subset P^2$ such that: $\tilde{C} = dL - \sum r_i E_i$ has $r_i \leq m_i + 1$ and $d \leq k + 2$; $C = \sum_j C_j$, where each $C_j$ is integral with $\tilde{C}_j$ smooth and rational in $X$ and $\tilde{C}_j \cdot \tilde{C}_{j_2} = 0$; and $\delta(C_j, Z, k) > 2l(Z) - k(k + 2)$. The “if” part of conjecture [6.7] is true if for example $j = 1$ and $A_h(C, Z, t) \geq 0$ for $h = 0, \ldots, n_1$ by Proposition [6.6] and Corollary [4.3].

Notice that all the results on the generation for fat point schemes (see the introduction for a list of them) are consistent with Conjectures [6.1] and [6.7].

We end by proving that the SHGH Conjecture together with Conjectures [6.1] and [6.7] imply the Uniform and Quasi-uniform Resolution Conjectures (for the statement of these conjectures see the Introduction).

**Proposition 6.8** The SHGH Conjecture together with Conjectures [6.1] and [6.7] imply the Uniform and Quasi-uniform Resolution Conjectures.

**Proof.** Since uniform implies quasi-uniform, let $Z$ be a quasi-uniform point scheme, i.e. $Z = m \sum_{i=1,\ldots,9} P_i - \sum_{i=10,\ldots,n} m_i P_i$, $n \geq 9$, $m \geq m_1 \geq \ldots m_n \geq 0$. We want to prove that, assuming the SHGH Conjecture, Conjecture [6.1] and Conjecture [6.7], the map $\mu_k(Z)$, or equivalently the map $\mu_F$ with $F = kL - m \sum_{i=1,\ldots,9} E_i - \sum_{i=10,\ldots,n} m_i E_i$, is of maximal rank.

We can write $F = (k - 3m)L - mK_X + \sum_{i \geq 10} (m - m_i) E_i$. We can assume that $h_Z(k) > 0$, otherwise $\mu_k(Z)$ is the zero map, hence trivially injective; since $Z$ is quasi-uniform, the SHGH conjecture then says (see introduction) that $h_{Z}(k) = (k^{2} + 2 - 9(m^{2} + 1)) - \sum_{i} (\frac{m_i^{2} + 1}{2})$. In particular $(k^{2} + 2 - 9(m^{2} + 1)) > 0$, which gives $k \geq 3m$. If $k = 3m$, then $n = 9$, in which case $F = m(3L - E_i - \ldots - E_9)$, so $h^0(F) = 1$ and $\mu_F$ has maximal rank.

Now let $k > 3m$. In order to prove that $\mu_F$ has maximal rank, by [6.1] and [6.7], it is enough to prove that $\delta_0(C, Z, k) = 0$ for each $C = dL - \sum r_i E_i$ smooth rational in $X$; since $\delta_{0}(C, Z, k) = \max(0, -F \cdot \tilde{C} + a) + \max(0, -F \cdot \tilde{C} + b - 1)$ with $a \leq b \leq d$, we’ll just prove that $-F \cdot \tilde{C} + b - 1 \leq 0$.

By the SHGH Conjecture, $\tilde{C}^2 \geq -1$, so by adjunction formula $K_X \cdot \tilde{C} = -\tilde{C}^2 - 2 \leq -1$. We hence find: $-F \cdot \tilde{C} + b - 1 = (-k + 3m) L + mK_X - \sum_{i \geq 10} (m - m_i) E_i \cdot \tilde{C} + b - 1 \leq -(k - 3m)d - m - \sum_{i \geq 10} r_i (m - m_i) + d - 1 < 0$.

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