Splitting algorithms and circuits analysis

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Abstract: The splitting algorithms of monotone operator theory find zeros of sums of relations. This corresponds to solving series or parallel one-port electrical circuits, or the negative feedback interconnection of two subsystems. One-port circuits with series and parallel interconnections, or block diagrams with multiple forward and return paths, give rise to current-voltage relations consisting of nested sums and inverses. In this extended abstract, we present new splitting algorithms specially suited to these structures, for interconnections of monotone and anti-monotone relations.

Keywords: Scaled Relative Graph, Nyquist, loop shaping, robustness

1. INTRODUCTION

The mathematical property of monotonicity originated in the study of networks of nonlinear resistors (Duffin, 1946; Zarantonello, 1960; Dolph, 1961; Minty, 1960, 1961a,b). Monotonicity generalizes the concept of passivity from linear circuit theory; loosely speaking, an element is monotone if it is passive with respect to any possible reference trajectory. Following the influential paper of Rockafellar (1976), monotone operator theory has grown to become a pillar of large scale optimization theory (Bauschke and Combettes, 2011; Ryu and Yin, 2022; Parikh and Boyd, 2013; Bertsekas, 2011).

Central to this theory are the family of splitting algorithms. These algorithms find zeros of sums of monotone operators, and allow computation to be performed separately for each operator. Recent work by the authors has revisited the study of electrical networks using modern splitting algorithms (Chaffey and Sepulchre, 2021). The main idea is that finding a zero of the sum of two operators is equivalent to solving the port behavior of their parallel (or series) interconnection. In turn, this is equivalent to solving the behavior of the negative feedback interconnection of two elements. This observation motivates the development of splitting algorithms which match more general circuit architectures. In Section 4, we describe an algorithm which solves the behavior of arbitrary series/parallel one-port circuits.

While splitting methods require each circuit element to be monotone, similar ideas can be applied to mixed monotone circuits, consisting of port interconnections of monotone and anti-monotone elements. This significantly expands the possible types of circuit behavior, allowing, for example, relaxation oscillations (van der Pol, 1926) and neuronal excitability (FitzHugh, 1961). In (Das et al., 2021), the authors have adapted Difference of Convex Programming (Lipp and Boyd, 2016; Yuille and Rangarajan, 2003) to solve such behaviors. In Section 5, we describe a new splitting algorithm which matches the mixed feedback structure of oscillators such as the van der Pol and FitzHugh-Nagumo models.

A Hilbert space $\mathcal{H}$ is a complete vector space equipped with an inner product, $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, and an induced norm $|x| = \sqrt{\langle x, x \rangle}$. In this abstract, we will treat general Hilbert spaces, although a common choice in practice is the space of square-summable, discrete time signals on $[0, T]$, denoted $l_2(T)$.

An operator on $\mathcal{H}$, is a possibly multi-valued map $R : \mathcal{H} \to \mathcal{H}$. The identity operator, which maps $u \in \mathcal{X}$ to itself, is denoted by $I$. The domain of an operator $R$ is denoted $\text{dom } R$. The graph, or relation, of an operator, is the set $\{ (u, y) \mid u \in \text{dom } R, y \in R(u) \} \subseteq \mathcal{H} \times \mathcal{H}$. We use the notions of an operator and its relation interchangeably, and denote them in the same way.

The standard operations on functions can be extended to relations. Let $R$ and $S$ be relations on an arbitrary Hilbert space $\mathcal{H}$. Then:

\begin{align*}
S^{-1} &= \{ (y, u) \mid y \in S(u) \} \\
S + R &= \{ (x, y + z) \mid (x, y) \in S, (x, z) \in R \} \\
SR &= \{ (x, z) \mid \exists y \text{ s.t. } (x, y) \in R, (y, z) \in S \}.
\end{align*}
Note that $S^{-1}$ always exists, but is not an inverse in the usual sense. In particular, in general $S^{-1} S \neq I$.

**Definition 1.** A relation $S \subseteq \mathcal{H} \times \mathcal{H}$ is called monotone if

$$\{(u_1-u_2,y_1-y_2) \geq 0$$

for any $(u_1,y_1),(u_2,y_2) \in S$. A monotone relation is called maximal if it is not properly contained in any other monotone relation.

**Definition 2.** A relation $S : \mathcal{H} \to \mathcal{H}$ is anti-monotone if $-S$ is monotone.

3. **SPLITTING TWO-ELEMENT CIRCUITS**

There is a large body of literature on splitting algorithms, which solve problems of the form $0 \in M_1(u) + M_2(u)$, where $M_1$ and $M_2$ are maximal monotone relations. There is a direct analogy with electrical circuits: if $M_1$ and $M_2$ are resistances, their series interconnection is given by the relation $v = M_1(i) + M_2(i)$; if $M_1$ and $M_2$ are conductances, their parallel interconnection is given by $i = M_1(v) + M_2(v)$. Given a current, the corresponding voltage across a parallel interconnection can be found using a splitting algorithm, by solving $0 = M_1(v) + M_2(v) - i$. Here, we briefly describe two splitting algorithms – the forward/backward splitting, and the Douglas-Rachford splitting. For the convergence properties of these algorithms, we refer the reader to (Giselsson and Moursi, 2019; Bauschke and Combettes, 2011; Ryu and Yin, 2022). Given an operator $S$ and a scaling factor $\alpha$, the $\alpha$-resolvent of $S$ is defined to be the operator

$$\text{res}_{\alpha S} = (I + \alpha S)^{-1}.$$

If $S$ is maximal monotone, $\text{res}_S$ is single-valued (Minty, 1961).

3.1 **Forward/backward splitting**

The simplest splitting algorithm is the forward/backward splitting (Passty, 1979; Gabay, 1983; Tseng, 1988). Suppose $M_1$ and $\text{res}_{\alpha M_2}$ are single-valued. Then:

$$0 \in M_1(x) + M_2(x)$$

if and only if

$$0 \in x - \alpha M_1(x) - (x + \alpha M_2(x))$$

and

$$x = \text{res}_{\alpha M_2}(I - \alpha M_1)x.$$

The fixed point iteration $x^{j+1} = \text{res}_{\alpha M_2}(x^j - \alpha M_1(x^j))$ is the forward/backward splitting algorithm.

3.2 **Douglas-Rachford splitting**

The reflected resolvent, or Cayley operator, is the operator

$$R_{\alpha S} := 2\text{res}_S - I.$$

Given two operators $M_1$ and $M_2$, and a scaling factor $\alpha$, the Douglas-Rachford algorithm (Douglas and Rachford, 1956; Lions and Mercier, 1979) is the iteration

$$z^{k+1} = T(z^k),$$

$$x^k = \text{res}_{\alpha M_2}(z^k),$$

where $T$ is given by

$$T = \frac{1}{2}(I + R_{\alpha M_1} R_{\alpha M_2}).$$

4. **SPLITTING N-ELEMENT CIRCUITS**

If our circuit is composed of three elements, with one series interconnection and one parallel interconnection (Figure 1), it has the form $M = M_1 + (M_2 + M_3)^{-1}$. A naive approach to solving the behavior of this circuit is to use a splitting algorithm such as the forward/backward algorithm, with the resolvent step applied for $M_1$ and the forward step applied for $(M_2 + M_3)^{-1}$. Applying this forward step amounts to solving $v = (M_2 + M_3)^{-1}(i)$ for some $u$, which may be rewritten as $0 \in (M_2 + M_3)(v) - i$. This can be solved by again applying the forward/backward algorithm.

![Fig. 1. Three elements with one series interconnection and one parallel interconnection.](image)

This naive procedure has poor complexity: for every forward/backward step for $M_1 + (M_2 + M_3)^{-1}$, an entire fixed point iteration has to be computed for (an offset version of) $M_2 + M_3$. In (Chaffey and Sepulchre, 2021), we propose an alternative procedure for $n$-element circuits. Here, we sketch this procedure on the circuit of Figure 1. Rather than apply a forward step for the relation $(M_2 + M_3)^{-1}$, we simply apply a single step of the fixed point iteration needed to compute this forward step, using the forward/backward algorithm. Given $v^*$, we want to solve $0 \in (M_1 + (M_2 + M_3)^{-1})(i) - v^*$. Assume that $M_3$, $\text{res}_{\alpha M_2}$, and $\text{res}_{\alpha_2 M_1}$ are single-valued. We then have:

$$v^* \in v + M_1(i)$$

$$v \in (M_2 + M_3)^{-1}(i),$$

where $v$ is the voltage over $M_2$, illustrated in Figure 1. Equation (2) gives

$$i + \alpha_2 M_1(i) \ni i - \alpha_2 v + \alpha_2 v^*$$

$$i = (I + \alpha_2 M_1)^{-1}(i - \alpha_2 v + \alpha_2 v^*)$$

$$i = \text{res}_{\alpha_2 M_1}(i - \alpha_2 v + \alpha_2 v^*).$$

Equation (3) gives

$$v + \alpha_1 M_2(v) \ni v - \alpha_1 M_2(v) + \alpha_1 i$$

$$v = (I + \alpha_1 M_2)^{-1}(v - \alpha_1 M_2(v) + \alpha_1 i)$$

$$v = \text{res}_{\alpha_1 M_2}(v - \alpha_1 M_2(v) + \alpha_1 i).$$

This shows that a fixed point of the iteration

$$i^{k+1} = \text{res}_{\alpha_1 M_2}(i^k - \alpha_1 M_2(i^k) + \alpha_1 i^k)$$

$$i^k = \text{res}_{\alpha_2 M_1}(i^k - \alpha_2 i^k + \alpha_2 v^*)$$

is a solution to our original problem $0 \in (M_1 + (M_2 + M_3)^{-1})(i) - v^*$.

5. **SPLITTING THE DIFFERENCE**

A mixture of positive and negative feedback is a ubiquitous mechanism, in both biology and engineering, for the generation of switches and oscillations (Sepulchre et al., 2019; Sepulchre and Stan, 2005; Stan et al., 2007; Stan and Sepulchre, 2007; Chua et al., 1987). Again adopting the
For further details of the implementation, the reader is referred to (Chaffey, 2022, Example 4.3).

Given the mixed monotone structure of Figure 2, we can find the steady state behavior of the system by solving a zero-finding problem: 0 ∈ Α1(x) + Α2(x) − B(x) − i.

The authors have explored methods to solve these problems using an adaptation of Difference of Convex Programming in (Das et al., 2021). The method involves iterating the operator (Α1 + Α2)⁻¹B. Computing (Α1 + Α2)⁻¹ at every iteration is an expensive operation; in this section, we propose the mixed monotone Douglas-Rachford algorithm (Algorithm 7), which replaces (Α1 + Α2)⁻¹ with a single step of the Douglas-Rachford iteration needed to invert it.

For operators Α1 and Α2 and step size α, we define Τα(Α1, Α2) to be the Douglas-Rachford operator:

\[ Τα(Α1, Α2) = \frac{1}{2}(I + Rα_Α1Rα_Α2). \]

Recall that RαS denotes the Cayley operator 2αS − I.

Algorithm 1 Mixed-Monotone Douglas-Rachford

1. **Data:** Maximal monotone Α1, Α2. Monotone, single-valued B. Initial value x₀. Convergence tolerance ε > 0.
2. Define Α’₁ by x ↦ Α₁(x) − yj for all j.
3. j = 1
4. do
5. Solve
   \[ x_{j+1} = \text{res}_α_Α2(z_j), \]
   \[ y_{j+1} = B(x_{j+1}), \]
   \[ z_{j+1} = Τα(Α’₁, Α₂)(z_j). \]
6. j = j + 1.
7. while |x_{j+1} − x_j| > ε

Note that a fixed point of this algorithm is a solution to 0 ∈ Α₁(x) + Α₂(x) − B(x): we know, by convergence of the Douglas-Rachford algorithm, that x is a solution to 0 ∈ Α’₁(x) + Α₂(x), which is equal to Α₁(x) + Α₂(x) − B(x) at a fixed point. (Chaffey, 2022, Thm. 4.1) gives a convergence condition for this algorithm. Figure 3 shows steady-state solutions to the van der Pol oscillator computed with Algorithm 7. The system is treated as an interconnection of operators on l₂, the space of length T periodic signals. For further details of the implementation, the reader is referred to (Chaffey, 2022, Example 4.3).

Fig. 2. A parallel mixed monotone circuit, which is a prototype structure for systems such as the van der Pol and FitzHugh-Nagumo oscillators.

Fig. 3. Steady-state solutions to the van der Pol oscillator for μ = 0.0002 (blue), 1.5 (orange) and 10 (red). Algorithmic parameters are a step size of α = 0.05, convergence tolerance of ε = 0.01 and 5000 time steps.

REFERENCES

Bauschke, H.H. and Combettes, P.L. (2011). *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* CMS Books in Mathematics. Springer New York, New York, NY. doi:10.1007/978-1-4419-9467-7.

Bertsekas, D.P. (2011). Incremental proximal methods for large scale convex optimization. *Mathematical Programming,* 129(2), 163–195. doi:10.1007/s10107-011-0472-0.

Chaffey, T. (2022). *Input/Output Analysis: Graphical and Algorithmic Methods.* Ph.D. thesis, University of Cambridge.

Chaffey, T. and Sepulchre, R. (2021). Monoctone Circuits. In *Proceedings of the European Control Conference.*

Chua, L.O., Desoer, C.A., and Kuh, E.S. (1987). *Linear and Nonlinear Circuits.* McGraw-Hill Series in Electrical Engineering. McGraw-Hill, New York.

Das, A., Chaffey, T., and Sepulchre, R. (2021). Oscillations in Mixed-Feedback Systems. *arXiv:2103.16379 [cs, eess].*

Dolphins, C.L. (1961). Recent developments in some non-self-adjoint problems of mathematical physics. *Bulletin of the American Mathematical Society,* 67(1), 1–70. doi: 10.1090/S0002-9904-1961-10419-X.

Douglas, J. and Rachford, H.H. (1956). On the Numerical Solution of Heat Conduction Problems in Two and Three Space Variables. *Transactions of the American Mathematical Society,* 82(2), 421–439. doi: 10.2307/1993056.

Duffin, R.J. (1946). Nonlinear networks. I. *Bulletin of the American Mathematical Society,* 52(10), 833–839. doi: 10.1090/S0002-9904-1946-08650-4.

FitzHugh, R. (1961). Impulses and Physiological States in Theoretical Models of Nerve Membrane. *Biophysical Journal,* 1(6), 445–466. doi:10.1016/s0006-3495(61)86902-6.

Gabay, D. (1983). Chapter IX Applications of the Method of Multipliers to Variational Inequalities. In M. Fortin and R. Glowinski (eds.), *Studies in Mathematics and Numerical Analysis.*
Its Applications, volume 15 of Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-Value Problems. Elsevier. doi:10.1016/S0168-2024(08)70034-1.

Giselsson, P. and Moursi, W.M. (2019). On compositions of special cases of Lipschitz continuous operators. arXiv:1912.13165 [math].

Lions, P.L. and Mercier, B. (1979). Splitting Algorithms for the Sum of Two Nonlinear Operators. SIAM Journal on Numerical Analysis, 16(6), 964–979. doi: 10.1137/0716071.

Lipp, T. and Boyd, S. (2016). Variations and extension of the Convex–Concave procedure. Optimization and Engineering, 17(2), 263–287. doi:10.1007/s11081-015-9294-x.

Minty, G.J. (1960). Monotone networks. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 257(1289), 194–212. doi: 10.1098/rspa.1960.0144.

Minty, G.J. (1961a). On the maximal domain of a “monotone” function. The Michigan Mathematical Journal, 8(2), 135–137. doi:10.1307/mmij/102899564.

Minty, G.J. (1961b). Solving Steady-State Nonlinear Networks of ‘Monotone’ Elements. IRE Transactions on Circuit Theory, 8(2), 99–104. doi: 10.1109/TCT.1961.1086765.

Parikh, N. and Boyd, S. (2013). Proximal Algorithms. Foundations and Trends in Optimization, 1(3), 123–231.

Passty, G.B. (1979). Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. Journal of Mathematical Analysis and Applications, 72(2), 383–390. doi:10.1016/0022-247X(79)90234-8.

Rocksellare, R.T. (1976). Monotone operators and the proximal point algorithm. SIAM Journal on Control and Optimization, 14(5), 877–898. doi:10.1137/0314056.

Ryu, E.K. and Yin, W. (2022). Large-Scale Convex Optimization via Monotone Operators. Draft edition.

Sepulchre, R., Drion, G., and Franci, A. (2019). Control Across Scales by Positive and Negative Feedback. Annual Review of Control, Robotics, and Autonomous Systems, 2(1), 89–113. doi:10.1146/annurev-control-053018-023708.

Sepulchre, R. and Stan, G.B. (2005). Feedback mechanisms for global oscillations in Lure systems. Systems & Control Letters, 54(8), 809–818. doi: 10.1016/j.sysconle.2004.12.004.

Stan, G.B., Hamadeh, A., Sepulchre, R., and Goncalves, J. (2007). Output synchronization in networks of cyclic biochemical oscillators. In IEEE American Control Conference, 3973–3978. doi:10.1109/ACC.2007.4282673.

Stan, G.B. and Sepulchre, R. (2007). Analysis of Interconnected Oscillators by Dissipativity Theory. IEEE Transactions on Automatic Control, 52(2), 256–270. doi: 10.1109/tac.2006.890471.

Tseng, P. (1988). Applications of a Splitting Algorithm to Decomposition in Convex Programming and Variational Inequalities. SIAM Journal on Control and Optimization, 29(1), 119–138. doi:10.1137/0329006.

van der Pol, B. (1926). On “Relaxation-Oscillations”. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 2(11), 978–992. doi: 10.1080/14786442608564127.

Yuille, A.L. and Rangarajan, A. (2003). The Concave-Convex Procedure. Neural Computation, 15(4), 915–936. doi:10.1162/08997660360581958.

Zarantonello, E.H. (1960). Solving Functional Equations by Contractive Averaging. Technical Report PB166988, Mathematics Research Center, Univ. of Wisconsin, Madison.