Extracting Chaos Control Parameters from Time Series Analysis

R B B Santos\textsuperscript{1} and J C Graves\textsuperscript{2}
\textsuperscript{1} Centro Universitário da FEI, Avenida Humberto de Alencar Castelo Branco 3972, 09850-901, São Bernardo do Campo, SP, Brazil
\textsuperscript{2} Instituto Tecnológico de Aeronáutica, Praça Marechal Eduardo Gomes 50, 12228-900, São José dos Campos, SP, Brazil
E-mail: rsantos@fei.edu.br

Abstract. We present a simple method to analyze time series, and estimate the parameters needed to control chaos in dynamical systems. Application of the method to a system described by the logistic map is also shown. Analyzing only two 100-point time series, we achieved results within 2\% of the analytical ones. With these estimates, we show that OGY control method successfully stabilized a period-1 unstable periodic orbit embedded in the chaotic attractor.

1. Introduction
Chaos control is a common feature found in a wide variety of complex systems of interest to science and engineering. Owing to incomplete knowledge of the initial conditions, and to roundoff errors due to finite precision calculations, time evolution of a chaotic system is effectively unpredictable even when the underlying dynamics is strictly deterministic. Chaotic oscillations in a dynamical system may be reduced or even suppressed by chaos control techniques that disturb the system slightly. In this paper, we present a simple way to estimate the control parameters of the OGY chaos control method\cite{OGY} through an analysis of the chaotic time series output by the system.

A wide class of physical and biological systems presents chaotic behavior in some range of operation. The generality of chaotic behavior is due to the fact that almost every real physical system is governed by nonlinear equations whose solutions are strongly dependent on the initial conditions at least in some part of the system parameter space\cite{Guckenheimer}. In traditional applications, dynamical systems are maintained in the neighborhood of one of its stable equilibrium points. Close to one of these points, the system dynamics becomes essentially linear and no sensitive dependency on the initial conditions is observed.

There are many routes for which a physical system may become chaotic, but all of them involve a set of bifurcations that disturbs the system stability as one or more parameters are changed. A complete description of routes to chaos is outside the scope of this paper. Perhaps the most well known route to chaos is the period doubling cascade pioneered by Feigenbaum\cite{Feigenbaum, Feigenbaum2}. The basic idea is that a system presents a cascade of subharmonic flip bifurcations until a point in which a strange attractor emerges and the dynamics becomes chaotic. Feigenbaum route to chaos is usually treated in the context of maps, particularly the logistic map.
The logistic map is a unimodal map with a single maximum point at $x = 1/2$ ruled by
\[ x_{n+1} = F(x_n, r) = rx_n(1 - x_n), \quad (1) \]
and the parameter $r$ controls the behavior of the map. It is widely known that a cascade of period doubling bifurcation beginning at $r = 3$ leads to the emergence of chaos at $r \approx 3.5699$. Apart from a trivial fixed point at $x = 0$, all other fixed points are given by $x^* = 1 - 1/r$ throughout the entire parameter range $0 < r \leq 4$.

Unidimensional maps as the logistic map are not so artificial as it may seem at first sight. Chaotic multidimensional systems in a strong dissipation regime may have just one relevant degree of freedom. In these systems, other degrees of freedom are often deemed irrelevant owing to the phase space volume contraction in those directions for which the Lyapunov exponents are negative. In the remaining relevant direction, sensitive dependence on the initial conditions holds[5].

Time series are typically used as a means to gain insight on the general dynamics of a chaotic system, including Lyapunov exponents, and some topological characteristics[6]. However, from a control engineering perspective, this kind of information is excessive. The novelty of the proposed method is to analyze the time series to extract just the information required to control chaos, drawing attention only to the dynamics on a small neighborhood of the unstable periodic orbits.

A recent aspect related to the understanding of chaotic systems is the possibility of chaos control[7]. Chaos control techniques allows us to keep a chaotic system in an unstable periodic orbit embedded within a strange attractor with the application of small perturbations to the underlying dynamics. Due to the sensitive dependence on the actual state of the system, a small perturbation may cause dramatic changes in the system evolution[8]. Owing precisely to the chaotic behavior of the system, its control turns out to be surprisingly efficient because every small region in the phase space of a strange attractor is crossed by orbits that visit every other region in the attractor.

In this context, OGY control method is a very simple one, relying on a variation of linear control with feedback proportional to the system output[1]. The technique has been applied successfully in a variety of chaotic systems since its first experimental demonstration[9]. One of the advantages of OGY technique is that it requires little knowledge of the equations governing the system. In a codimension-1 map
\[ x_{n+1} = F(x_n, r_n), \quad (2) \]
the controls needed to stabilize a period-1 unstable periodic orbit (fixed point) are small disturbances $\Delta r_n$ around the nominal value $r_0$ of the control parameter $r$:
\[ \Delta r_n = r_n - r_0 = -\gamma \hat{n} \cdot (x_n - x^*_0) \quad (3) \]
where $\hat{n}$ is a unit vector normal to the Poincaré section $\Sigma$ in the neighborhood of the unstable fixed point $x^*_0$, and $\gamma$ is a proportional gain.

Applied to the stabilization of a fixed point in the logistic map[3]
\[ x_{n+1} = F(x_n, r) = rx_n(1 - x_n), \quad (4) \]
OGY control method is based on the linearized map around the unstable fixed point $x^*_0 = 1 - 1/r_0$, and around the nominal control parameter $r_0$,
\[ x_{n+1} \approx x^*_0 + \alpha (x_n - x^*_0) + \beta (r_n - r_0) \quad (5) \]
that leads to a modified map

\[ x_{n+1} = (r_0 - \gamma (x_n - x_0^*) ) x_n (1 - x_n) \] (6)

when the system is close enough to the unstable fixed point \( x_0^* \). Optimal control is achieved by choosing \( \gamma = -\alpha / \beta \). Therefore, one must determine from a series of runs of the experiment the location of the fixed point \( x_0^* \), and the sensitivities

\[ \alpha = \left. \frac{\partial F}{\partial x} \right|_{x=x_0^*, r=r_0} \] (7)

and

\[ \beta = \left. \frac{\partial F}{\partial r} \right|_{x=x_0^*, r=r_0} \] (8)

in order to apply controls on the system. It should be pointed out that this conclusion is valid even in delayed feedback control\[10\]. Although the feedback, in this case, does not involve explicitly the unstable fixed point \( x_0^* \), knowledge of the location of the fixed point is still required to evaluate the sensitivities \( \alpha \) and \( \beta \).

Reconstruction of some aspects of the chaotic dynamics from time series is a subject explored through a variety of methods\[6, 11, 12\]. We introduce a very simple recurrence method to estimate the parameters required in order to apply the OGY method to control chaos in dynamical systems. For ease of presentation, we restrict this discussion to unidimensional maps as the logistic map, and to period one unstable periodic orbits. Therefore, it suffices to estimate the values of \( x_0^* \), \( \alpha \), and \( \beta \) from a time series.

It must be noted that the method proposed does not attempt to reconstruct the chaotic attractor. Instead, we focus on the local dynamics of the attractor around their unstable periodic orbits. In this way, we were able to reduce substantially the volume of information needed to be extracted from the experiment and processed in order to control a chaotic system.

1.1. Our Proposal

The dynamics of a chaotic system is typically slow around an unstable periodic orbit. Hence, we begin analyzing the behavior of a unidimensional map around one of its fixed points. A linearized version of the map around \( x = x_0^* \), and \( r = r_0 \) may be written as

\[ x_{n+1} \approx x_0^* + \alpha (x_n - x_0^*) + \beta (r_n - r_0) . \] (9)

Fixing the value of the control parameter to the nominal value \( r_0 \), and selecting only those situations in which there is a close encounter, that is, situations in which the variation

\[ \Delta_n = x_{n+1} - x_n \approx (\alpha - 1) (x_n - x_0^*) \] (10)

is small, less than some \( \varepsilon \), for instance. In these close encounter cases, the system is guaranteed to be in a small neighborhood of radius \( \varepsilon \) of the fixed point due to the slow dynamics of the map around \( x_0^* \). Then, it is straightforward to show that

\[ x_0^* \approx \left\langle \frac{x_n \Delta_m - x_m \Delta_n}{\Delta_m - \Delta_n} \right\rangle \] (11)

where \( \Delta_m \) is the corresponding variation for another close encounter in the neighborhood of the same fixed point, and the brackets indicate averaging over the several such close encounters provided by a long time series. Despite the simplicity of this recurrence method, fixed point
detection may be enhanced by techniques that manipulates the probability distribution on the attractor through fixed point transforms[11].

Having an estimate for $x^*_0$, it is easy to show that

$$\alpha \approx 1 + \left\langle \frac{\Delta_n}{x_n - x^*_0} \right\rangle.$$  \hfill (12)

Finally, in order to estimate $\beta$, we vary the control parameter to a value $r_1 = r_0 + \delta r$ slightly different from $r_0$, and estimate the new fixed point $x^*_1$ using equation (11). Hence, averaging over many different values of $r_1$ close to $r_0$, we discover that

$$\beta \approx (1 - \alpha) \left\langle \frac{\delta x^*}{\delta r} \right\rangle = (1 - \alpha) \left\langle \frac{x^*_1 - x^*_0}{r_1 - r_0} \right\rangle.$$  \hfill (13)

The procedure outlined is very easy to perform, and we should expect to find a fair estimate of the control parameters after analyzing relatively small number of long enough time series of the output of the chaotic system. In the next section, we apply this procedure to control the very simple logistic map.

2. Results and Discussion
To demonstrate the power of this simple method, we estimate $x^*_0$, $\alpha$, $\beta$, and $\gamma$ for the logistic map with only two runs of 100 points each. In these simulations, we set $r_0 = 3.90$ and $r_1 = 3.91$, right in the middle of a heavily chaotic region, and we also fixed the radius for a close encounter to be detected in 10% of the total variation of the logistic map. Figure 1 shows a typical trajectory obtained in these very short simulations.

![Figure 1](image.png)

**Figure 1.** A 100-point chaotic trajectory in the logistic map with $r_0 = 3.90$. The circles indicate close encounters along the trajectory.

We performed the analysis proposed in section 1.1 and the results are shown in table 1. Even these short few runs were enough for a fair estimation of the parameters needed to control the chaos in the system.
Table 1. Theoretical and simulated values for the fixed point $x^*$, sensitivities $\alpha$ and $\beta$, and proportional gain $\gamma$.

| Parameter | $x^*$       | $\alpha$ | $\beta$ | $\gamma$ |
|-----------|-------------|----------|---------|----------|
| Theory    | 0.74359     | -1.90    | 0.1907  | 9.97     |
| Simulation| 0.74347     | -1.86    | 0.1892  | 9.85     |
| Error     | 0.016%      | 2.1%     | 0.79%   | 1.2%     |

Using the estimated parameters, we were able to stabilize a period-1 unstable periodic orbit embedded in the heavily chaotic region of the logistic map as shown in figure 2. In order to keep the orbit stabilized, additional controls amounting to only $\Delta r/r_0 = 0.074\%$ had to be continuously applied.

Figure 2. Stabilized trajectory in the logistic map ($r_0 = 3.90$) using OGY chaos control method with parameters estimated from time series. The arrow marks the first time that controls were applied.

3. Conclusion
We presented a simple procedure to estimate the set of parameters needed in order to control chaos. Analysis of only two very short time series of the logistic map was enough to estimate the control parameters within 2% of their analytical values. With these estimates, OGY method could successfully stabilize an unstable periodic orbit. The method applied here to stabilize a fixed point in the logistic map may be extended to stabilize a period-$k$ unstable periodic orbit in higher dimensional systems where $\alpha$ is replaced by the jacobian matrix $[\alpha_{ij}] = [\partial F_i/\partial x_j]$, and $\beta$ is replaced by a matrix of sensitivities $[\beta_{ij}] = [\partial F_i/\partial r_j]$. 
Acknowledgments
RBBS acknowledges partial financial support from Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP). JCG wishes to thank Fundação Educacional Inaciana Pe. Sabóia de Medeiros and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) for financial support. We are grateful to an anonymous referee for suggestions that made the exposition more precise.

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