Surfing on the Edge:
Chaos vs. Near-Integrability in the System of Jovian Planets

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ABSTRACT

We demonstrate that the system of Jovian planets (Sun+Jupiter+Saturn+Uranus+Neptune), integrated for 200 million years as an isolated 5-body system using many sets of initial conditions all within the uncertainty bounds of their currently known positions, can display both chaos and near-integrability. The conclusion is consistent across four different integrators, including several comparisons against integrations utilizing quadruple precision. We demonstrate that the Wisdom-Holman symplectic map using simple symplectic correctors as implemented in Mercury 6.2 (Chambers 1999) gives a reliable characterization of the existence of chaos for a particular initial condition only with timesteps less than about 10 days, corresponding to about 400 steps per orbit. We also integrate the canonical DE405 initial condition out to 5 Gy, and show that it has a Lyapunov Time of 200–400 My, opening the remote possibility of accurate prediction of the Jovian planetary positions for 5 Gy.

Subject headings: outer solar system, dynamics, chaos

1. Introduction

Both the man of science and the man of art live always at the edge of mystery, surrounded by it. Both, as a measure of their creation, have always had to do with the harmonization of what is new with what is familiar, with the balance between novelty and synthesis, with the struggle to make partial order in total chaos.... This cannot be an easy life.

— J. Robert Oppenheimer
When one speaks of the stability of our Solar System, one must carefully define the meaning of “stable”. We say that the Solar System is \textit{practically stable} if, barring interlopers, the known planets suffer no close encounters between themselves or the Sun, over the main-sequence lifetime of the Sun. In a practically stable Solar System, the orbital eccentricities, inclinations, and semi-major axes of all the planets remain within some bounded region, not too far from their present values. In this sense, work by many authors over the past 15 years has all but proven that the Solar System is practically stable (Laskar 1994; Laskar 1996; Laskar 1997; Ito and Tanikawa 2002). Good reviews exist (Lissauer 1999; Lecar et al. 2001), and we will not discuss it further in this paper. A second, more formal definition involves the question of whether the Solar System is \textit{chaotic} or not. In a chaotic system, nearby solutions tend to diverge from each other exponentially with time, although in a weakly chaotic system such as the Solar System, the exponential divergence can be preceded by an initial period of polynomial divergence. Let $d(t)$ be the distance between two solutions, with $d(0)$ being their initial separation. Then $d(t)$ increases approximately as $d(0)e^{\lambda t}$ in a chaotic system, where $\lambda$ is the \textit{Lyapunov exponent}. The inverse of the Lyapunov exponent, $1/\lambda$, is called the \textit{Lyapunov time}, and measures how long it takes two nearby solutions to diverge by a factor of $e$. A system that is not chaotic is called \textit{integrable} or \textit{regular}, and has a Lyapunov exponent of zero. A practical consequence of being chaotic is that small changes become exponentially magnified, so that uncertainties in the current positions of the planets are magnified exponentially with time. Even though the Solar System is practically stable, a positive Lyapunov exponent means that uncertainties in the current positions of the planets are magnified to the point that we cannot predict the precise positions of the planets in their orbits after a few (or at most a few tens of) Lyapunov times.

KAM theory tells us that essentially all Hamiltonian systems which are not integrable are chaotic. An initial condition (IC) not lying precisely on a KAM torus will eventually admit chaos, but with a time scale that depends critically on the IC. Symplectic integrators (Channell and Scovel 1990; Wisdom and Holman 1991; Sanz-Serna 1992) have many nice properties when used for long-term integrations of Hamiltonian systems, such as conservation of phase-space volume, and bounded energy error. However, the validity of symplectically-integrated numerical solutions also depends critically upon the integration time step $h$, with the longevity of the solution’s validity scaling as $e^{a/h}$ for some constant $a$ (Benettin and Giorgilli 1994; Reich 1999). For linear problems, the dependence is even stronger and manifests itself as a bifurcation in the Lyapunov exponent, going discontinuously from zero to a non-zero value (Lessnick 1996, Newman and Lee 2005 — but see Rauch and Holman 1999)). Since the solar system is not integrable, and experiences unpredictable small perturbations, it cannot lie permanently on a KAM torus, and is thus chaotic. The operative question is the time scale of the chaos. To compute the time scale accurately,
we must guarantee that the measured time scale is not an artifact of the integration method.

What is the Lyapunov time of the Solar System? \textcite{Sussman and Wisdom 1988} first demonstrated that the motion of Pluto is chaotic with a Lyapunov time of about 20 million years, corroborated over a longer integration later by \textcite{Kinoshita and Nakai 1996}. \textcite{Laskar 1989} performed an averaged integration of the 8 major planets (excluding Pluto) and found that the Lyapunov time was about 5 million years, with the divergence dominated by that of the inner planets. \textcite{Laskar 1990} believed that secular resonances are the cause of the chaos in the inner Solar System (but see Murray and Holman 2001), although he did not believe the system of Jovian planets was affected by the chaos displayed by the inner planets. \textcite{Sussman and Wisdom 1992} performed a full (non-averaged) integration of the entire Solar System and confirmed Laskar’s 5 million year Lyapunov time, and further found that the system of Jovian planets by itself had a Lyapunov time of between 7 and 20 million years, although their measurement of the Lyapunov time displayed a disturbing dependence on the timestep of the integration. This dependence was later discovered to be due to symplectic integration schemes effectively integrating a slightly different set of ICs; the effect can be corrected (Saha and Tremaine 1992; Wisdom et al. 1996), although it decreases with decreasing timestep.

Since there are no two-body resonances amongst the Jovian planets, the cause of the chaos between them was not understood until \textcite{Murray and Holman 1999} identified the cause as being the overlap of three-body resonances. \textcite{Murray and Holman 1999} also performed Lyapunov time measurements on a large set of Outer Solar Systems differing only in the initial semi-major axis of Uranus. They found that their three-body resonance theory correctly predicted which regions of ICs were chaotic, and which were not, at least over the 200-million-year integration timespan they used. For the “actual” Solar System, they found that the Lyapunov time was about 10 million years. \textcite{Guzzo 2005} went on to corroborate the three-body resonance theory by performing a large suite of integrations, numerically detecting a large web of three-body resonances in the outer Solar System.

\textcite{Murray and Holman 1999} noted that the widths $\Delta a/a$ of the individual resonant zones was of order $3 \times 10^{-6}$, so that changes in the ICs of that order can lead to regular motion. They note, however, that “the uncertainties in the ICs, and those introduced by our numerical model, are comfortably smaller than the width of the individual resonances, so [the outer] Solar System is almost certainly chaotic.” Given that \textcite{Guzzo 2005} has also detected many three-body resonances consistent with Murry + Holman’s theory, it would seem at first glance that chaos in the outer Solar System is a fact.

However, the conclusion that the isolated outer Solar System is chaotic cannot be taken for granted. For example, it is known that symplectic integration with too-large a timestep...
can inject chaos into an integrable system ([Herbst and Ablowitz 1989; Newman et al. 2000]). Although most authors verify their primary results by performing “checking” integrations with smaller timesteps, the checking integrations are not always performed for the full duration of the main integrations. This, combined with the fact that longer symplectic integrations require shorter timesteps ([Benettin and Giorgilli 1994; Reich 1999]) means that one cannot assume that a timestep good enough, for example, for a 100-million-year integration is also good enough for a 200-million-year integration. There is currently no known method for analytically choosing a short-enough timestep \textit{a priori}, and so the only method of verifying the reliability of an integration is to re-perform the entire integration with shorter-and-shorter timesteps until the results converge. Newman et al. (2000) used this method to demonstrate that, for a given set of ICs, the Wisdom and Holman (1991) symplectic mapping with a 400 day timestep (about 11 steps per orbit, a commonly-quoted timestep) admits chaos, but that the results converge to regularity for any timestep less than about 100 days. However, many authors who find chaos have also performed reasonable convergence tests, demonstrating that the chaos does not always disappear at convergence.

There exists compelling evidence for the absence of chaos in the outer Solar System. Laskar and others noted that when the entire solar system is integrated, the inner Solar System manifests chaos on a 5-million-year timescale, but the outer Solar System appears regular in these integrations. Although Laskar’s approximate theory can overlook some causes of chaos, there also exist full-scale integrations that indicate the absence of chaos. Grazier \textit{et al.} (1999) and Newman \textit{et al.} (2000) utilized a Störmer integrator whose per-step numerical errors were bounded by the double-precision machine epsilon (about \(2 \times 10^{-16}\)), as long as the orbital eccentricity is less than 0.5 and 1000 steps or more are taken per orbit. Furthermore, their integration method took care to ensure that the roundoff error was unbiased. Note that, except for the possibility of having the same property with a larger timestep, this is practically as good an integration as is possible using double-precision. Furthermore, such an integration is symplectic by default, since it is essentially \textit{exact} in double precision. Using this method, they performed an integration of the Jovian planets lasting over 800 million years, and found no chaos. Varadi \textit{et al.} (2003) performed a 207My integration of the entire Solar System, including even the effects of the Moon, and placed a lower bound of 30My on the Lyapunov time of the system of Jovian planets.

We are thus left with the disturbing fact that, utilizing “best practices” of numerical integration, some investigators integrate the system of Jovian planets and find chaos, while others do not.

In this paper, we demonstrate that this apparent dilemma has a simple solution. Namely, that the boundary, in phase space, between chaos and near-integrability is finer than pre-
viously recognized. In particular, the current observational uncertainty in the positions of the outer planets is a few parts in $10^7$ (Standish 1998; Morrison and Evans 1998). Within that observational uncertainty, we find that some ICs lead to chaos while others do not. So, for example, drawing 7-digit ICs from the same ephemeris at different times, one finds some solutions that are chaotic, and some that are not. Thus, different researchers who draw their initial conditions from the same ephemeris at different times can find vastly different Lyapunov timescales.

2. Methods

With the exception of the two sets of initial conditions (ICs) we have received from other authors (Murray and Holman 1999; Grazier et al. 2005) and the set included in Mercury 6.2 (Chambers 1999), all ICs used in this paper are drawn at various epochs from DE405 (Standish 1998), which is the latest planetary ephemeris publicly available from JPL. It has stated uncertainty for the positions and masses of the outer planets of a few parts in $10^7$.

To ensure that our integration agrees over the short term with DE405, we verified in several cases that we can integrate between different sets of DE405 ICs, separated by as much as 100 years, while maintaining at least 7 digits of agreement with DE405.

We integrate the system of Jovian planets using only Newtonian gravity. The inner planets are accounted for by adding their masses to the Sun and perturbing the Sun’s position and linear momentum to equal that of the Sun+Mercury+Venus+Earth+Mars system. This ensures that the resonances between the outer planets is shifted by an amount that is second order in this mass ratio, roughly $3 \times 10^{-11}$ (Murray and Holman 1999), which is far smaller than the uncertainty in the outer-planet positions. We assume constant masses for all objects and ignore many effects which are probably relevant over a 200 My timescale (see for example Laskar 1999). We account for solar mass loss at a rate of $\dot{m}/m \approx 10^{-7}$ per million years (Laskar 1999; Noerdlinger 2005), but note that we observe no noticeable difference if we keep the solar mass constant.

To reduce the possibility that our results are dependent on the integration scheme, we used three different numerical integration methods to verify many of our results in this paper. First, we used the Mercury 6.2 package (Chambers 1999), with the Wisdom-Holman (Wisdom and Holman 1991) symplectic mapping option (called MVS in the input files). We used stepsizes varying from 2 days to 400 days. Second, we used the NBI package, which contains a 14th-order Cowell-Störmer method with modifications by the UCLA research
group led by William Newman (Grazier et al. 1995, 1996; Varadi et al. 2003). NBI has been shown to have relative truncation errors below the double precision machine epsilon (about $2 \times 10^{-16}$) when more than 1000 steps per orbit are used and the orbital eccentricity is less than 0.5. More precisely, if the largest component of the phase-space vector of the solution at time $t$ has absolute value $M$, then the local errors per step of each of the components are all less than $2 \times 10^{-16}M$. Note that this means that the component-by-component relative error can be significantly greater than the machine precision for components of the solution that are significantly less than $M$, but all components have errors relative to $M$ which are smaller than the machine precision. Furthermore, the authors of NBI have gone through great pains to ensure that the roundoff error is unbiased. We used a 4-day timestep for all NBI integrations, which gives more than 1000 timesteps per Jupiter orbit. We have verified the above “exact to double precision” property by comparison against quadruple precision integrations described below. Note that for the above-defined timestep, NBI gives practically the best integration possible using only double precision, and that such an integration is symplectic by default since it essentially provides a solution which is exact in double precision. In our 200 million year integrations, NBI always had relative energy errors and angular momentum errors of less than $2 \times 10^{-11}$, with an average of about $2 \times 10^{-12}$.

Our third integrator was the Taylor 1.4 package (Jorba and Zou 2005). Taylor 1.4 is a recent and impressive advance in integration technology. It is a general-purpose, off-the-shelf integrator which utilizes automatic differentiation to compute arbitrary order Taylor series expansions of the right-hand-side of the ODE. Taylor 1.4 automatically adjusts the order and stepsize at each integration step in an effort to minimize truncation error, and utilizes Horner’s rule in the evaluation of the Taylor series to minimize roundoff error. As the authors note, integration accuracy is gained more efficiently by increasing the order of the integration than by decreasing the timestep, since the accuracy increases exponentially with the order but only polynomially in the timestep. Although Taylor 1.4 allows the user to specify a constant order and timestep, we chose to allow it to use variable order and timestep while providing it with a requested relative error tolerance equal to $1/1000$ of the machine precision, in order to produce solutions which were exact to within roundoff error. We found that Taylor 1.4 typically used about 27th order with about a 220 day timestep. The fact that the solution is exact to machine precision over such a long timestep guarantees that accumulated roundoff is by far the smallest in the Taylor 1.4 integrations. Furthermore, Taylor integrators are extremely stable when applied to non-stiff problems, with the radius of convergence increasing linearly with integration order, and in our case the timestep is

\[1\] NBI is available at [http://astrobiology.ucla.edu/~varadi/NBI/NBI.html](http://astrobiology.ucla.edu/~varadi/NBI/NBI.html), or by searching the web for “NBI Varadi”.

well within the radius of convergence (Barrio et al. 2005). Finally, *Taylor 1.4* allows the user to specify the machine arithmetic to use, including software arithmetics. Out-of-the-box, *Taylor 1.4* supports the use of IEEE 754 double precision (64 bit representation with a 53 bit mantissa), Intel extended precision (80 bit representation with a 64-bit mantissa, giving a machine precision of about $10^{-19}$, accessible as `long double` when using GCC on an Intel machine), the `DoubleDouble` datatype (Briggs 1996) which provides software quadruple precision in C++, and the GNU Multiple Precision Library, which allows arbitrary precision floating point numbers in C++. Most of our integrations using *Taylor 1.4* used Intel extended precision, which is almost as fast as double precision and gives about 19 decimal digits of accuracy. Over our 200-million-year integrations using Intel extended precision, Taylor typically had relative energy errors of less than $8 \times 10^{-14}$; the worst relative energy error observed in any of our integrations was $2 \times 10^{-13}$. Integrations began with the Solar System’s barycentre at the origin with zero velocity. After 200 million years the barycentre drifted a maximum of $3 \times 10^{-10}$ AU, while the $z$ component of the angular momentum was always conserved to a relative accuracy better than $3 \times 10^{-14}$. We also performed a suite of quadruple precision integrations, in which energy and angular momentum were each conserved to at least 26 significant digits over 200 million years.
Fig. 1.— *Taylor 1.4* satisfies Brouwer’s Law: when local tolerance is set above machine precision of $10^{-19}$, phase-space error grows as $t^2$ due to biased truncation errors (upper three curves). But with tolerance set well below the machine precision, its result is exact (i.e., correctly rounded) to machine precision, so that phase-space error grows only as $t^{1.5}$ (lowest curve).
Now we analyze the error growth as a function of time for our non-symplectic integrators, applied to a non-chaotic IC. One consequence of having a solution whose numerical error is dominated by unbiased roundoff (ie., exact to machine precision) is that when integrating a non-chaotic system, the total phase-space error grows polynomially as $t^{1.5}$. If the local error is biased (either by biased roundoff, or due to truncation errors in the integration scheme), then the total phase-space error grows as $t^2$. (This is assuming that the integrator is not inherently symplectic, as is the case with both NBI and Taylor 1.4.) This is known as Brouwer’s Law \cite{Brouwer1937}. As noted above, \cite{Grazier2005} have demonstrated that the error in NBI’s integration is dominated by unbiased roundoff when using 1000 timesteps per orbit. We tested Taylor 1.4 in Intel extended precision for similar properties by integrating the system of Jovian planets for 200 million years using various integration tolerances up to and beyond the machine precision of $10^{-19}$, and compared these integrations to a Taylor 1.4 integration that used quadruple precision. In Figure 1, we plot the phase-space separation between the quadruple precision integration and several integrations using Intel extended precision. We see that when the relative integration tolerance is set above the machine precision, the error grows as $t^2$ and is therefore truncation dominated. But when the tolerance is set to $10^{-22}$ (about a factor of 1000 below the machine precision), the error grows as $t^{1.5}$, and is therefore dominated by unbiased roundoff. This is consistent with Taylor 1.4 producing results that are exact in Intel extended precision when given a local relative error tolerance of $10^{-22}$, just as NBI produces exact results in double precision when used with 1000 or more timesteps per orbit. We see that after 200 million years, the errors in the positions of the planets are of order $10^{-5}$ AU (in the case of non-chaotic ICs). This translates into a phase error, or equivalently an observational error, of substantially less than one arc-minute. A comparison of our Figure 1 with Figure 2 of \cite{Grazier1999} at their 200 million year mark also demonstrates that Taylor 1.4 using Intel extended precision provides about 3 extra digits of precision over NBI using double precision, as would be expected when comparing 19-digit and 16-digit integrations. As we show later (Figure 7), the differences in orbital elements are even smaller.

For comparison, we briefly mention the run-times of the integration algorithms. All timings are for a 2.8Ghz Pentium 4 processor. As we shall see later, the largest timestep for which the Mercury 6.2 integrations agreed with the others was 8 days; a 16-day timestep was almost as good. Thus we compare the Wisdom-Holman mapping with timesteps of 16 and 8 days to NBI with a 4-day timestep, and Taylor-1.4. In addition to the Taylor-1.4 extended precision (\texttt{long double}) timings used in this paper, we present its timings for standard IEEE 754 \texttt{double} precision, for comparison to the other \texttt{double} precision integrations. Table 1 presents the results. We found that the total runtime was linearly proportional to the inverse of the timestep, as would be expected. We note several observations. First, Taylor-1.4 is
not competitive in terms of efficiency. However, note that Taylor-1.4 is a “proof-of-concept” software package for general-purpose integration of any system of ODEs, and it currently generates code that can be quite inefficient. A carefully hand-coded Taylor series integrator for Solar System integrations is far more efficient, and can be competitive with the above codes (Carles Simo 2007, personal communication). Second, Wisdom-Holman with an 8-day timestep is the fastest case among the integrations we tested that showed complete convergence. Third, Wisdom-Holman with a 4-day timestep (51 hours, not shown in the table) is slower than NBI with a 4-day timestep. Finally, although we did not test NBI for convergence at timesteps less stringent than 4 days, it is possible that NBI maintains the lead in being more efficient than Wisdom-Holman, if it also shows convergence at larger timesteps.
Table 1: Run times for integrating the 5-body system for 200 million years on a 2.8Ghz pentium 4, for *Mercury 6.2* with timesteps of 16 and 8 days, for NBI, and for *Taylor-1.4* in *double* and *long double* precision.

| integrator       | WH dt=16d | WH dt=8d | NBI    | Taylor-1.4 double | Taylor-1.4 long double |
|------------------|-----------|----------|--------|-------------------|-----------------------|
| time (hours)     | 13        | 26       | 40     | 100               | 150                   |
We will not directly measure the Lyapunov time in this paper, since it is difficult to create an objective measure of the Lyapunov time over a finite time interval. Formally, the Lyapunov time is only defined over an infinite time interval. In practical terms, the divergence is almost always polynomial for some nontrivial duration before the exponential divergence emerges, and it is difficult to pinpoint the change-over objectively. However, when plotting the distance between two nearby trajectories as a function of time, it is usually evident by visual inspection whether or not exponential divergence has occurred by the end of the simulation. Thus we will plot the actual divergence between two numerical trajectories initially differing by perturbing the position of Uranus by $10^{-14}$ AU (about 1.5mm) in the $z$ direction. We will call these pairs of trajectories “siblings.” In the cases where we see only polynomial divergence between siblings over a 200 million year integration, we will abuse terminology and call these systems “regular”, “near-integrable”, or “non-chaotic”, although formally all we have shown is that the Lyapunov time is longer than can be detected in a 200-million-year integration.
Fig. 2.— Murray + Holman (1999) in quadruple precision.

Fig. 3.— Murray + Holman (1999) in long double.
As a prelude to our main results, we first crudely reproduce the results of Murray and Holman (1999). Murray + Holman performed a survey in which the semi-major axis of Uranus $a_U$ was varied around its actual value, plotting the measured Lyapunov time as a function of $a_U$. Broadly, they found that below $a_U \approx 19.18$, the Lyapunov time was substantially less than $10^7$ years, sometimes as small as $10^5$ years; in the range $a_U \approx 19.20 - 19.22$, the Lyapunov time fluctuated between about 1–10 million years, interspersed with cases in which the motion was regular; near $a_U = 19.24$ it was uniformly regular; near $a_U = 19.26$ there was a tightly-packed region of both chaotic and regular orbits; and near $a_U = 19.28$ the motion was again uniformly regular. We have reproduced this survey using quadruple precision integrations, using a very coarse grid in $a_U$ due to computing constraints. (On a 2.8GHz Intel Pentium 4, a quadruple precision integration using Taylor 1.4 proceeds roughly at 100My per month of CPU time.) Figure 2 displays the distance between siblings, for various values of $a_U$. Although we were not able to perform the survey using a finer grid in $a_U$ due to the computational expense, we see that in broad outline we obtain results similar to Murray + Holman. In Figure 3 we repeat the same survey using Intel extended precision. We see that the results are qualitatively identical, demonstrating that 19 digits of precision (which requires about 1/20 of the CPU time of quadruple precision) gives qualitatively identical, and quantitatively very similar, results. Henceforth in this paper, all Taylor 1.4 integrations are performed using Intel extended precision, with the relative local error tolerance set to $10^{-22}$.

3. Results

3.1. Corroborating previous results

As an early step, the author obtained from Murray + Holman their initial conditions (ICs) used in Murray and Holman (1999), and verified using accurate integrations that their system was chaotic. The author then obtained from P. Sharp the ICs used by Grazier et al. (2005), and verified that their system was regular, at least over a 200My timespan. The author then spent significant time eliminating several possible reasons for the discrepancy, such as incorrect ICs, incorrect deletion of the inner planets, etc. The author also integrated from one set of ICs to the other (having along the way to account for the fact that they were in different co-ordinate systems), and finally found that the systems agreed with each other to a few parts in $10^7$ when integrated to the same epoch. After some time it became apparent that neither group of authors had made any obvious errors. The conclusion seemed to be that both systems nominally represented the outer Solar System to within observational error, but one was chaotic and one was not.
3.2. Ephemeris Initial Conditions Drawn at Different Times

It is reasonable to make the assumption that either our Solar System is chaotic, or it is not. It cannot be both. This appears to be the assumption that most practitioners make when measuring “the” Lyapunov time of our Solar System. Although this is probably a reasonable assumption, it does not follow that all initial conditions (ICs) drawn from an ephemeris are equivalent. The most recent ephemeris published by JPL is DE405 (Standish 1998). It is based upon hundreds of thousands of observations, all of which have finite error. Thus, the ephemeris does not represent the exact Solar System. For the outer planets, the best match of DE405 to the observations yields residual errors of a few parts in $10^7$ (Standish 1998; Morrison and Evans 1998). The masses of the planets are also known to only a few parts in $10^7$. It is possible that within these error bounds there exist different solutions with different Lyapunov times. In particular, it may be possible that some solutions display chaos on a 200-million-year timescale, while others do not.
Fig. 4.— Distance in AU between “sibling” trajectories as a function of time for 21 sets of initial conditions drawn from DE405 at 30-day intervals. Each plot represents one initial condition. Siblings are integrated with the Wisdom-Holman symplectic map with timesteps: 50 (solid line), 100 (long dash), 200 (short dash), and 400 days (dotted). For any given plot, note disagreement across timesteps, and that the switch between chaos and regularity is not always monotonic with timestep.
To test this hypothesis, we drew 21 sets of ICs from DE405, at 30-day intervals starting at Julian date 2448235 (9.5 Dec, 1990). We represent each of our 21 sets of ICs using a 3-digit numeral, 000, 030, 060, ..., 570, 600, representing the number of days after JD2448235 at which the initial condition is drawn. We chose a 2-year total interval across which to draw our ICs in order to ensure a reasonable sample of inner-planet positions before deleting the inner planets (a Martian year is about 2 Earth years). We drew the initial conditions by taking the output of the program `testeph.f`, which is included with the DE405 ephemeris. The output of `testeph.f` is rounded to 7 digits, since no more digits are justifiable.\footnote{Although this is probably not the best way to uniformly sample initial conditions from within the error ball representing the error in the observations, it \textit{does} represent a reasonable way to reproduce how users of DE405, taking initial conditions from DE405 to 7 digits, get their samples.} As described above, we augment the mass of the Sun with that of the inner planets and augment the Sun’s position and linear momentum to match that of the inner Solar System. For each of the 21 ICs, we generate a “sibling” IC which is offset by $10^{-14}$ AU (1.5mm). We then integrate both of them for 200 million years, and measure the distance between them at 1-million-year intervals. To ensure that our results are not integrator- or timestep-dependent, we perform each integration in several ways: first, with the Wisdom-Holman symplectic mapping included with \textit{Mercury 6.2} \cite{Chambers1999} with timesteps of 400, 200, 100, 50, 32, 16, 8, 4, and 2 days; second, with \textsc{NBI} with a timestep of 4 days; and third, with \textit{Taylor 1.4} in Intel extended precision with a relative error tolerance of $10^{-22}$. Results for the Wisdom-Holman integrations with larger timesteps are displayed in Figure 4. After staring at these figures for awhile, several key observations become apparent. First, when comparing across the 21 sets of initial conditions, there is remarkable disagreement about whether or not the outer Solar System displays exponential divergence. This will be discussed further below, using more accurate integrations. Looking at an individual graph, but across timesteps, we note that there are very few cases (e.g. $t = 150, 330, 420$) in which there is universal agreement between all four timesteps that divergence is polynomial. There are also only a few cases that universally agree that the divergence is exponential. In most cases, there is substantial disagreement across timesteps whether a particular initial condition admits chaos. This is consistent with the observation of Newman \textit{et al.} (2000). However, in contrast to Newman \textit{et al.} (2000), we note that the “switch” from chaos to non-chaos is not always monotonic in timestep. For example, for system 000, timesteps of 400 and 50 days admit chaos, while the “in-between” timesteps of 200 and 100 days do not. For system 180, timesteps of 400 and 200 days display non-chaos, while for 100 and 50 day timesteps, chaos is apparent; this is precisely opposite to what one would expect if large timesteps were injecting chaos into the system. As we shall see later in the paper (Table 2), there appears also to be no observable correlation between the timestep and the percentage of ICs that admit chaos. Thus we
hypothesize that the discrepancy across timesteps is due more to perturbations in the ICs and our use of only low-order symplectic correctors (Wisdom et al. 1996; Chambers 1999), than it is due to unreliable integration at large timesteps. Corroborating this hypothesis would require us to re-perform these experiments using higher-order symplectic correctors or “warmup” (Saha and Tremaine 1992), which is a possible direction for future research.
Fig. 5.— Similar to previous Figure, but more accurate integrations. NBI: solid lines; Wisdom-Holman: dashed lines with $dt = 8$ days (long dash), 4 days (short dash); Dotted lines ($Taylor-1.4$) are shifted below others because $Taylor-1.4$ is more accurate. For any given IC, there is good agreement between the shapes of the curves, indicating agreement on the existence (or lack) of chaos. (Divergence saturates at $\approx 100$ AU.)
Fig. 6.— Curves identical to the previous figure, but plotted on a log-log scale, which depicts polynomial divergence as a straight line.
Figures 5 and 6 plot the divergence between sibling trajectories for all 21 ICs, as integrated by more accurate integrations showing convergence: NBI, the Wisdom-Holman mappings with timesteps of 8 and 4 days, and Taylor 1.4. Wisdom-Holman with 2 day timesteps agreed with these curves, but are omitted to reduce clutter; Wisdom-Holman with a timestep of 16 days also showed good agreement in all but two cases. Both figures are identical except that Figure 5 uses a log-linear scale, while Figure 6 uses a log-log scale; different features are visible using the different scales. After staring at these graphs for awhile, at least two observations present themselves. First, if one looks at the system corresponding to any single IC, there is usually good agreement between the integrations as to the future divergence between the sibling trajectories of that particular case. This demonstrates that convergence has occurred and makes it unlikely that the results are integrator dependent. Second, looking across cases, it appears that the future of the outer Solar System over the next 200 million years is quite uncertain, varying from nearly integrable, to chaotic with a Lyapunov time of order 10 million years or less. This is quite a startlingly diverse array of possible outcomes, considering that the ICs for these systems are all drawn from the same ephemeris, all less than 2 years apart, and presumably differing from each other by only a few parts in $10^7$. The author has, in fact, verified that several of the ICs, when all integrated up to the same epoch in the vicinity of 1991, agree with each other to a few parts in $10^7$. 
Fig. 7.— The final difference at the 200My mark between siblings, in semi-major axis and eccentricity, for the 21 cases depicted in the previous Figures. The integrator was NBI. Eccentricity difference is the Euclidean distance between siblings in the 4-dimensional space consisting of the four orbital eccentricities. Semi-major axis difference is the Euclidean distance between siblings in the 4-dimensional space consisting of the four \( \Delta a/a \) values. Some values have been moved slightly for textual clarity, but in no case by more than the width or height of a character.
Figures 5 and 6 demonstrate that in the chaotic cases, the siblings can have their respective planets on opposite sides of the Solar System after 200My. However, Figure 7 demonstrates that changes in the orbital elements are much less drastic, demonstrating that the Solar System is practically stable over a 200My timescale even when it is chaotic.

3.3. Explicitly Perturbed Initial Conditions

Following Murray and Holman (1999), we performed several surveys in which we perturbed the semi-major axis $a_U$ of Uranus from its current value, but kept all other initial conditions (ICs) constant. We used all three previously mentioned integrators; Wisdom-Holman with timesteps of 4 and 8 days; NBI; and Taylor-1.4. The IC was the default one from the file big.in included with Mercury 6.2 (Chambers 1999), which according to the documentation is from JD2451000.5. The inner planets were deleted, with their mass and momentum augmenting the Sun's as described elsewhere in this paper. We completed surveys in which $a_U$ was changed in steps of $2 \times 10^{-k}$ for $k = 6, 7, 8$, which corresponds to $\Delta a_U/a_U$ in steps of $10^{-(k+1)}$. We went 10 steps in each direction for each value of $k$. For each step, we generated a “sibling” IC by randomly perturbing the positions of all planets by an amount bounded by $10^{-14}$ AU. We then integrated both for 200My, and plotted the distance between them as a function of time. For $k = 7, 8$, there was no significant difference between any of the integrations. That is, all siblings at all steps had virtually identical divergences when changing $a_U$ in 10 steps of $2 \times 10^{-7}$ AU, for a total change of $2 \times 10^{-6}$. This corresponds to $\Delta a_U/a_U$ stepped by $10^{-(8,9)}$, for a total change in $\Delta a_U/a_U$ of $10^{-7}$ in each direction. However, for $k = 6$, some of the steps showed chaos while others did not. The change was not monotonic: over the 21 steps (10 in either direction plus the “baseline” case), there were three “switches” between chaos and stability.

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3 Note that, for no good reason, this is different from the perturbations used to generate siblings in the rest of the paper.
Two systems with $\Delta a_U/a_U = 1e-7$, one chaotic, one not

Fig. 8.— Two systems, both using ICs within the error bounds of the best known positions for the outer planets. Both have identical initial conditions except for the semi-major axis $a_U$ of Uranus, which differs between the two systems by $10^{-7}$ in $\Delta a_U/a_U$. One admits chaos, while the other does not. The “upper” six curves (all starting with sibling distances near $10^{-5}$) are all double-precision integrations, two using Wisdom-Holman with timesteps of 4 and 8 days, and one using NBI. Of the six, the three plotted with points are the chaotic trajectory and the three plotted with lines are the non-chaotic trajectory. The “lower” two curves (starting with sibling distances near $10^{-8}$) are integrated in extended precision with Taylor 1.4. The chaotic one fits an exponential curve with a Lyapunov time of about 12 million years, while the non-chaotic one has the two trajectories separating approximately as $t^{1.5}$. 
Figure 8 plots two of these 21 systems. The value of $\Delta a_U/a_U$ differs between the two systems by one part in $10^7$. One of the systems appears chaotic, and the other does not, over a 200My timespan. The non-chaotic one has a semi-major axis of $a_U + 2 \times 10^{-6}$, while the chaotic one has semi-major axis $a_U + 4 \times 10^{-6}$. All other ICs in the two systems are identical. To ensure that the result is not integrator dependent, we have repeated the integrations with the Wisdom-Holman mapping included in Mercury 6.2 with timesteps of 8 and 4 days; and with NBI with a timestep of 4 days. As can be seen, all the integrations agree quite well with one another. Note that the Taylor 1.4 integrations provide 3 extra digits of precision, and so the curves for Taylor 1.4 are displaced about 3 orders of magnitude below the curves computed in double precision. Otherwise the shapes of the curves are virtually identical. We note that the chaotic one has a Lyapunov time of about 12 million years, while the regular one has the sibling trajectories separating from each other polynomially in time as $t^{1.5}$.

3.4. Accurate integrations over the age of the Solar System

The author has reported related results for integrations lasting $10^9$ years (Hayes, Wayne B. 2007). The essential conclusion is the same, in that even after $10^9$ years, there remain some ICs (about 10%) that show no evidence of chaos, although some of the ICs appearing as regular over 200My develop exponential divergence later.

Figure 9 displays the sibling divergence over $5 \times 10^9$ years of the “canonical” IC used by DE405 (JED 2440400.5, June 28, 1969). As we can see, this IC shows little evidence of chaos for about the first 1.5Gy, and then develops slow exponential divergence with a Lyapunov time between about 200 and 400 million years. The individual planets each show similarly-shaped divergence curves (not shown), with the magnitude of divergence increasing with orbital radius. After 5 Gy, the uncertainty in Jupiter’s position for this IC is less than 1 AU, while the uncertainty in Neptune’s position is about 9 AU. Thus, there is a non-negligible chance that, if the Solar System lies close enough to the “canonical” IC of DE405, that we can know within about 10–15 degrees where each outer planet will be in its orbit when the Sun ends its main-sequence lifetime and becomes a red giant. Note that the levelling-off that starts at about the 4 Gy mark is not saturation (which occurs closer to 100 AU separation, while the separation at 5 Gy here is less than 10); the outer Solar System instead seems to be entering again into a period of polynomial (non-chaotic) divergence.
Fig. 9.— Sibling divergence over 5 Gy of the canonical IC of DE405, integrated using Taylor-1.4.

3.5. Percentage of initial conditions displaying chaos

Observing the distance between sibling trajectories in Figures 5 and 6 at the 200My mark, we can reasonably simplify the distinction between chaotic and regular trajectories. By choosing a cutoff distance $C$ and restricting our view to the various double-precision integrations, we can claim that siblings differing by less than $C$ after 200My are regular (in the sense of having no observable Lyapunov exponent), while those differing by more than $C$ are chaotic with a measurable positive Lyapunov exponent. Table 2 lists the number of systems that are chaotic by the above definition, as a function of timestep and cutoff. In addition to the 21-sample group of ICs displayed in Figures 5 and 6, we also drew a second sample group of 10 samples, spaced at 10-year intervals from 1900 to 1990. Since the testeph.f program included in DE405 provides only 7 digits of precision (corresponding to the accuracy to which the positions are known), taking ICs from DE405 at different times effectively takes ICs from different exact orbits, differing from each other by as much as one part in $10^7$. (As noted in a previous footnote, this method of sampling may not ideally represent an unbiased
sample from the observational error ball; instead, it is an unbiased sample from the set of 7-digit-rounded initial conditions drawn from DE405.) We make several observations. First, the fraction of sampled systems that are chaotic by this simple definition is roughly about (70±10)%, and is relatively independent of both the timestep and the two sample groups (30-day vs. 10-year samples), although it of course increases as we decrease the cutoff. Recall that for a given initial condition, different stepsizes can give different results, so that which systems are chaotic changes as the timestep changes. However, here we measure only the number of chaotic systems as a function of timestep. If chaos were being “injected” into the system by the integrator, we would expect that the number of systems displaying chaos should increase with increasing timestep. However, this is not observed. A possible interpretation of this is that even a 400 day timestep reliably determines whether some system is chaotic, but the system is slightly different for each timestep due to the IC perturbation introduced by the symplectic integration (Saha and Tremaine 1992; Wisdom et al. 1996). Verifying this would require us to implement “warmup” (Saha and Tremaine 1992) or higher-order symplectic correctors than are included with Mercury 6.2 (Chambers 1999), which we have not done. However, the fact that the fraction of systems displaying chaos is independent of timestep argues against the “chaos is injected by the integrator” hypothesis, at least for the timesteps used in this paper.
| dt  | C=1 /10 | C=1 /21 | C=1 total | C=1 % | C=0.1 /10 | C=0.1 /21 | C=0.1 total | C=0.1 % |
|-----|---------|---------|-----------|-------|-----------|-----------|-------------|--------|
| 400 | 7       | 14      | 21        | 67.7  | 9         | 16        | 25          | 80.6   |
| 200 | 7       | 11      | 18        | 58.1  | 8         | 15        | 23          | 74.2   |
| 100 | 6       | 11      | 17        | 54.8  | 8         | 13        | 21          | 67.7   |
| 050 | 6       | 14      | 20        | 64.5  | 8         | 15        | 23          | 74.2   |
| 032 | 7       | 10      | 17        | 54.8  | 9         | 15        | 24          | 77.4   |
| 016 | 7       | 14      | 21        | 67.7  | 8         | 16        | 24          | 77.4   |
| 008 | 8       | 13      | 21        | 67.7  | 8         | 14        | 22          | 71.0   |
| 004 | 8       | 13      | 21        | 67.7  | 8         | 18        | 26          | 83.9   |
| 002 | 8       | 13      | 21        | 67.7  | 8         | 15        | 23          | 74.2   |

Table 2: The percentage of initial conditions drawn from DE405 that admit chaos. Column labels: $C$ is the cutoff described in the text; $dt$ is the timestep; /10 means “out of the 10 initial conditions drawn at 10-year intervals from 1900 to 1990”; /21 means “out of the 21 initial conditions drawn at 30-day intervals starting at 1990”; total is the sum of the previous two columns; % is the percentage of systems out of the 31 that display chaos according to the cutoff.
3.6. Chaos in the inner solar system is robust

Varadi et al. (2003) performed a 207My integration of the entire Solar System, including some non-Newtonian effects and a highly-tuned approximation to the effects of the Moon, and placed a lower bound of 30My on the Lyapunov time of the outer Solar System. However, they still saw chaos in the inner solar system. To test the robustness of chaos in the inner solar system, we performed several integrations of 8 planets (Mercury through Neptune) using the Wisdom-Holman mapping with timesteps of 8, 4, 2, 1, and 0.5 days. We treated the Earth-Moon system as a single body. To ensure that chaos in the outer solar system did not “infect” the inner solar system, we used only those DE405 ICs from Figures 5 and 6 for which the outer solar system was regular over 200My. We then integrated the system until chaos appeared. In all cases, even though the outer solar system was regular, the inner solar system displayed chaos over a short timescale such that information about the inner planetary positions was lost within about 20–50My. Thus, unlike the outer Solar System, we observe that chaos in the inner solar system is robust.

4. Discussion

We conclude that perturbations in the initial conditions (ICs) of the outer planets as small as one part in $10^7$ can change the behaviour from regular to chaotic, and back, when measured over a timespan of 200 million years. We believe this is the first demonstration of the “switch” from chaos to near-integrability with such a small perturbation of the ICs. Since our knowledge of the orbital positions of the outer planets is comparable to one part in $10^7$, it follows that, even if our simplistic physical model accounting only for Newtonian gravity were the correct model, it would be impossible at present to determine the Lyapunov time of the system of Jovian planets. Furthermore, it implies that an IC with 7 digits of precision (which is all an ephemeris can justifiably provide) can randomly lie on a chaotic or non-chaotic trajectory. Since our results converge in the limit of small timestep for the Wisdom-Holman mapping, and the converged results also agree with two very different high-accuracy integrations, and finally since the high-accuracy integrations in turn agree very well with quadruple-precision integrations, we believe that the results in this paper are substantially free of significant numerical artifacts.

Guzzo (2005) corroborates the existence of a large web of 3-body resonances in the outer Solar System, and finds that their placement is consistent with Murray and Holman’s (1999) theory. Guzzo used his own fourth-order symplectic integrator (Guzzo 2001), and performed what appear to be reasonable convergence tests to verify the robustness of his main results. Thus, Murray and Holman’s theory appears to explain the existence and
placement of 3-planet resonances. Furthermore, chaotic regions in Hamiltonian systems are usually densely packed with both chaotic and regular orbits (Lichtenberg and Lieberman 1992). This paper corroborates the observation of densely packed regular and chaotic orbits, at a scale previously unexplored for the system of Jovian planets.

As discussed in the text relating to Figure 8, we performed surveys across the semi-major axis $a_U$ of Uranus in steps of $2 \times 10^{-k}$ AU for $k = 6, 7, 8$. We found that, around the current best-estimate value of $a_U$, perturbations smaller than $2 \times 10^{-6}$ AU had no effect on the existence of chaos. However, this does not imply that perturbations this small cannot have an effect; it simply means that the “border” between chaos and regularity is not within $2 \times 10^{-6}$ AU of the current best estimate of $a_U$. However, it is clear that the border between chaos and regularity is between the two systems depicted in Figure 8, which differ from each other in $a_U$ by $2 \times 10^{-6}$ AU. Although a survey across $a_U$ for values between those two systems may not be very relevant from a physical standpoint, it might be very interesting from a dynamical systems and chaos perspective to probe the structure of the border between chaos and regularity. In particular, it may be interesting to see if the border itself has some sort of fractal structure (Mandelbrot 1982). Such chaotic structure has already been observed in the circular restricted three-body problem (Murison 1989).

Newman et al. (2000) gave a compelling demonstration that the Wisdom + Holman symplectic mapping with too-large a timestep could introduce chaos into a near-integrable system by first showing that they could reproduce the chaos with a 400-day timestep, and then showing that the integration converged to being regular with a timestep of about 50 days or less. Our results appear to hint that an even smaller timestep may be required: our curves depicting divergence-of-nearby-orbits did not fully converge until the timestep was 8 days or less. On the other hand, the symplectic integrators produce solutions that effectively integrate a system with slightly perturbed ICs, although these perturbations decrease with decreasing timestep (Saha and Tremaine 1992, Wisdom et al. 1996). Our Figure 4 demonstrates that the behaviour does not always “switch” monotonically in timestep from chaotic to non-chaotic as observed by Newman et al. (2000); Table 2 also hints that the “amount” of chaos does not appear to increase with increasing timestep. An alternate interpretation is that even a 400-day timestep accurately integrates an orbit, but not the orbit that we chose. In particular, it may accurately integrate an orbit whose IC is perturbed slightly from the one we chose, and an appropriate correction may allow us to recover the correct orbit using an integration with much larger timestep (Saha and Tremaine 1992, Wisdom et al. 1996). However, the symplectic correctors in Mercury 6.2 are clearly not good enough to perform this recovery at a 400-day timestep; better correctors (Wisdom 2006) or warmup (Saha and Tremaine 1992) may be able to achieve this.
The convergence of our results at timesteps of 8 days or less, as well as the agreement with two different non-symplectic integrators (including the one used by Newman et al.), indicate that the IC perturbations described by Saha and Tremaine (1992) and Wisdom et al. (1996) are negligible in our smaller timestep cases. Thus, using “warmup” (Saha and Tremaine 1992) or higher-order symplectic correctors will not substantially alter our conclusions, although they might allow the same conclusions to be drawn using larger timesteps.

With the exception of the results plotted in Figure [9], all of our simulations had a duration of 200 million years (My). All of them display an initial period of polynomial divergence, before the appearance of exponential divergence (if any). However, the duration of initial polynomial divergence differs greatly across systems, and has been observed by others (Lecar et al. 2001) to last significantly longer that 200My; Grazier et al. (2005) observed it to last the entire duration of their 800My simulation. It would be interesting to create a table like Table [2] but including a “simulation duration” dimension as well. Certainly the evidence hints that more systems make the “switch” from polynomial to exponential divergence as the duration of the simulation increases.

Our physical model is very simplistic, accounting only for Newtonian gravity between the Sun and Jovian planets. Although we ignore many physical effects which are known to effect the detailed motion of the planets (Laskar 1999; Varadi et al. 2003), it is unclear if such effects would substantially alter the chaotic nature of solutions. We at first believed that the largest such effect ignored was solar mass loss. Our first simulations did not account for solar mass loss, which amounts to about one part in $10^7$ per million years (Laskar 1999; Noerdlinger 2005). Since we find that perturbations in position of that order can shift the system in-and-out of chaos, a naive analysis might lead one to suspect that solar mass loss might shift the planetary orbits in-and-out of resonance on a timescale that is fast compared to the Lyapunov time, thus smoothing out the sibling divergence. We thus modified our model to include solar mass loss, but surprisingly it made absolutely no observable difference to any of the figures presented in this paper. To ensure that we did not make an error, we simulated systems with ever increasing mass loss until the Sun was losing 10% of its mass per 100My. We noted that the planetary orbital semi-major axes expanded significantly, as would be expected, but that the sibling divergences did not change until mass loss was at a rate of about 1% per 100My (1000 times greater than in reality). Thus, we conclude that solar mass loss also makes no difference to our results.
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REFERENCES

Barrio, R., F. Blesa, and M. Lara (2005). VSVO formulation of the Taylor method for the numerical solution of ODEs. Computers and Mathematics with Applications 50, 93–111.

Benettin, G. and A. Giorgilli (1994). On the Hamiltonian interpolation of near-to-the-identity symplectic mappings with application to symplectic integration algorithms. Journal of Statistical Physics 74(5/6), 1117–1143.

Briggs, K. (1996). The doubledouble package: Software quadruple precision in C++. Unpublished.

Brouwer, D. (1937). On the accumulation of errors in numerical integration. Astron. J. 46, 149–153.

Chambers, J. E. (1999, April). A hybrid symplectic integrator that permits close encounters between massive bodies. Monthly Notices of the Royal Astronomical Society 304, 793–799.

Channell, P. J. and C. Scovel (1990). Symplectic integration of Hamiltonian systems. Nonlinearity 3, 231–259.

Grazier, K. R., W. I. Newman, J. M. Hyman, and P. W. Sharp (2005). Long simulations of the solar system: Brouwer’s law and chaos. Preprint.

Grazier, K. R., W. I. Newman, W. M. Kaula, and J. M. Hyman (1999, August). Dynamical Evolution of Planetesimals in the Outer Solar System. I. The Jupiter/Saturn Zone. Icarus 140, 341–352.

Grazier, K. R., W. I. Newman, W. M. Kaula, F. Varadi, and J. M. Hyman (1995, May). An Exhaustive Search for Stable Orbits between the Outer Planets. Bulletin of the American Astronomical Society 27, 829.
Grazier, K. R., W. I. Newman, F. Varadi, D. J. Goldstein, and W. M. Kaula (1996, June). Integrators for Long-Term Solar System Dynamical Simulations. *Bulletin of the American Astronomical Society* 28, 1181.

Guzzo, M. (2001). Improved leap-frog symplectic integrators for orbits of small eccentricity in the perturbed Kepler problem. *Celestial Mechanics and Dynamical Astronomy* 80, 63–80.

Guzzo, M. (2005, March). The web of three-planet resonances in the outer Solar System. *Icarus* 174, 273–284.

**Hayes, Wayne B.** (2007). Is the Outer Solar System Chaotic? *Nature Physics*. Accepted.

Herbst, B. M. and M. J. Ablowitz (1989). Numerically induced chaos in the nonlinear Schroedinger equation. *Physical Review Letters* 62, 2065–2068.

Ito, T. and K. Tanikawa (2002, October). Long-term integrations and stability of planetary orbits in our Solar system. *Monthly Notices of the Royal Astronomical Society* 336, 483–500.

Jorba, A. and M. Zou (2005). A software package for the numerical integration of ODEs by means of high-order Taylor methods. *Experimental Mathematics* 14, 99–117.

Kinoshita, H. and H. Nakai (1996). Long-Term Behavior of the Motion of Pluto over 5.5 Billion Years. *Earth Moon and Planets* 72, 165–173.

Laskar, J. (1989, March). A numerical experiment on the chaotic behaviour of the solar system. *Nature* 338, 237–238.

Laskar, J. (1990, December). The chaotic motion of the solar system - A numerical estimate of the size of the chaotic zones. *Icarus* 88, 266–291.

Laskar, J. (1994). Large-scale chaos in the Solar System. *Astronomy and Astrophysics* 287L, 9–12.

Laskar, J. (1996). Large Scale Chaos and Marginal Stability in the Solar System. *Celestial Mechanics and Dynamical Astronomy* 64, 115–162.

Laskar, J. (1997). Large scale chaos and the spacing of the inner planets. *Astronomy and Astrophysics* 317, L75–L78.

Laskar, J. (1999, August). The limits of Earth orbital calculations for geological time-scale use. *Royal Society of London Philosophical Transactions Series A* 357, 1735–1759.
Lecar, M., F. A. Franklin, M. J. Holman, and N. W. Murray (2001). Chaos in the Solar System. *Annual Review of Astronomy and Astrophysics* 39, 581–631.

Lessnick, M. K. (1996). Ph. D. thesis, UCLA.

Lichtenberg, A. J. and M. A. Lieberman (1992). *Regular and Chaotic Dynamics*. Springer.

Lissauer, J. J. (1999). Chaotic motion in the solar system. *Reviews of Modern Physics* 71(3), 835–845.

Mandelbrot, B. (1982). *The Fractal Geometry of Nature*. W. H. Freeman & Company.

Morrison, L. V. and D. W. Evans (1998, November). Check on JPL DE405 using modern optical observations. *A&AS* 132, 381–386.

Murison, M. A. (1989, December). The fractal dynamics of satellite capture in the circular restricted three-body problem. *AJ* 98, 2346–2359.

Murray, N. and M. Holman (1999, March). The Origin of Chaos in the Outer Solar System. *Science* 283, 1877–1881.

Murray, N. and M. Holman (2001, April). The role of chaotic resonances in the Solar System. *Nature* 410, 773–779.

Newman, W. I. and A. Y. Lee (2005, May). Symplectic Integration Methods and Chaos: Timestep Selection and Lyapunov Time. *AAS/Division of Dynamical Astronomy Meeting 36*.

Newman, W. I., F. Varadi, A. Y. Lee, W. M. Kaula, K. R. Grazier, and J. M. Hyman (2000, May). Numerical Integration, Lyapunov Exponents and the Outer Solar System. *Bulletin of the American Astronomical Society* 32, 859.

Noerdlinger, P. D. (2005). Solar mass loss, the astronomical unit, and the scale of the solar system. Preprint available at http://home.comcast.net/ pdnoerd/SMassLoss.html.

Rauch, K. P. and M. Holman (1999, February). Dynamical Chaos in the Wisdom-Holman Integrator: Origins and Solutions. *AJ* 117, 1087–1102.

Reich, S. (1999). Backward error analysis for numerical integrators. *SIAM J. Numer. Anal* 36(5), 1549–1570.

Saha, P. and S. Tremaine (1992, October). Symplectic integrators for solar system dynamics. *AJ* 104, 1633–1640.
Sanz-Serna, J. M. (1992). Symplectic integrators for Hamiltonian problems: an overview. In A. Iserles (Ed.), *Acta Numerica 1992*, pp. 243–286. Cambridge University Press.

Standish, E. M. (1998). JPL planetary and lunar ephemerides, DE405/LE405. *JPL IOM 312.F–98–048*. August 26, 1998. Available at http://ssd.jpl.nasa.gov/iau-comm4/relateds.html.

Sussman, G. J. and J. Wisdom (1988, July). Numerical evidence that the motion of Pluto is chaotic. *Science* 241, 433–437.

Sussman, G. J. and J. Wisdom (1992). Chaotic evolution of the solar-system. *Science* 257, 56–62.

Varadi, F., B. Runnegar, and M. Ghil (2003, July). Successive Refinements in Long-Term Integrations of Planetary Orbits. *The Astrophysical Journal* 592, 620–630.

Wisdom, J. (2006, April). Symplectic Correctors for Canonical Heliocentric n-Body Maps. *AJ* 131, 2294–2298.

Wisdom, J. and M. Holman (1991, October). Symplectic maps for the n-body problem. *The Astronomical Journal* 102, 1528–1538.

Wisdom, J., M. Holman, and J. Touma (1996). Symplectic Correctors. *Fields Institute Communications, Vol. 10*, p. 217 10, 217.