A Note on the Trace Method for Random Regular Graphs

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Abstract

The main goal of this note is to illustrate the advantage of analyzing the non-backtracking spectrum of a regular graph rather than the ordinary spectrum. We show that by switching to non-backtracking spectrum, the method of proof used in [Pud15] yields a bound of

$$2\sqrt{d-1} + \frac{2}{\sqrt{d-1}}$$

instead of the original

$$2\sqrt{d-1} + 1$$

on the second largest eigenvalue of a random $d$-regular graph.

1 Introduction

Let $\Gamma$ be a $d$-regular graph on $N$ vertices. The adjacency matrix $A_\Gamma$ of $\Gamma$ has $N$ real eigenvalues

$$d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq -d.$$  

The first eigenvalue is the trivial eigenvalue $\lambda_1 = d$ corresponding to the constant eigenfunction. Denote by $\lambda(\Gamma) \overset{\text{def}}{=} \max(\lambda_2, -\lambda_N)$ the maximal absolute value of a non-trivial eigenvalue. It is well known that many properties of the graph can be measured by the value of $\lambda(\Gamma)$. In particular, $\Gamma$ has better expanding properties the smaller $\lambda(\Gamma)$ is (see e.g., [HLW06]). The Alon-Boppana bound states that $\lambda_2(\Gamma) \geq 2\sqrt{d-1} + o_N(1)$ [Alo86, Nil91], and Alon conjectured that for random $d$-regular graphs, $\lambda_2(\Gamma)$ is very close to $2\sqrt{d-1}$. This conjecture was proven by the first named author [Fri08]: for every fixed $d \geq 3$ and every $\varepsilon > 0$, if $\Gamma$ is a uniformly random $d$-regular graph on $N$ vertices, then $\lambda(\Gamma) \leq 2\sqrt{d-1} + \varepsilon$ asymptotically almost surely (a.a.s.), namely, this holds with probability tending to 1 as $N \to \infty$.

The proof in [Fri08] uses the trace method, where the non-trivial eigenvalues are bounded by counting closed cycles in the graph. (We elaborate more in Section 3 below). This method was also used in other works studying $\lambda(\Gamma)$ for random $\Gamma$: these include [BS87], [Fri91], as well as the more recent, shorter proof of Alon’s conjecture [Bor19]. Another paper in this line of works is [Pud15] by the second named author, which uses results from combinatorial group theory to establish the weaker result that $\lambda(\Gamma) \leq 2\sqrt{d-1} + 1$ a.a.s.

The Hashimoto non-backtracking matrix $B_{\Gamma}$ of the graph $\Gamma$ is a matrix depicting the adjacency of oriented edges in $\Gamma$. Famously, the Ihara-Bass formula relates the spectrum of $B_\Gamma$ with that of $A_\Gamma$ and shows that one determines the other: see Section 2 for details. Consequently, one can bound the largest non-trivial eigenvalue of the non-backtracking matrix $B_{\Gamma}$ and deduce a bound on $\lambda(\Gamma)$. This is indeed how the proof works in both proofs of Alon’s conjecture [Fri08, Bor19].

However, [Pud15] bounds $\lambda(\Gamma)$ directly. The main point of this note is that by passing to bounding the largest non-trivial eigenvalue of the Hashimoto matrix $B_{\Gamma}$, the method of proof in [Pud15] gives the following significantly better bound:

**Theorem 1.1.** Fix $d \geq 3$, and let $\Gamma$ be a random $d$-regular simple\footnote{A simple graph has no loops and parallel edges. In general, graphs in this note are not assumed to be necessarily simple.} graph on $n$ vertices chosen at uniform distribution. Then a.a.s.,

$$\lambda(\Gamma) \leq 2\sqrt{d-1} + \frac{2}{\sqrt{d-1}}.$$
This note should thus be thought of as an addendum to [Pud15]. The original result in [Pud15] and its improvement in Theorem 1.1 fall short of proving the full strength of Alon’s conjecture as in [Fri08, Bor19]. However, we find this addendum interesting for two main reasons. First, it illustrates what seems to be a fundamental advantage of analyzing the non-backtracking spectrum rather than the ordinary one. Second, the method of proof in [Pud15] and here is very different than the one in [Fri08, Bor19], and may be used in other directions. For example, this method, including the improvement suggested here, is also applied in [HP20], which gives bounds on the second eigenvalue of random Schreier graphs of the symmetric group. We remark that Section 1.2.3 of [FK14] also mentions a similar improvement – working with Hashimoto matrices as opposed to adjacency matrices – for the results of [BS87] and of [Fri91].

The paper is organized as follows. Section 2 recalls the Hashimoto non-backtracking matrix and the relation between the ordinary spectrum and the non-backtracking spectrum of a regular graph. In Section 3 we briefly describe the original proof from [Pud15], and in Section 4 give the adjustments needed to establish Theorem 1.1. Finally, Section 5 contains some remarks regarding the more general model of random coverings of a fixed base graph.

2 The Hashimoto non-backtracking matrix and the Ihara-Bass formula

Let \( \Gamma \) be an undirected \( d \)-regular graph, not necessarily simple, on \( N \) vertices and let \( A = A_\Gamma \) be its \( N \times N \) adjacency matrix. Denote by \( \overrightarrow{E} \) the set of oriented edges of \( \Gamma \), namely, each edge of \( \Gamma \) appears twice in this set, once with every possible orientation, so \(|\overrightarrow{E}| = Nd\). For \( e \in \overrightarrow{E} \), we denote by \( \overrightarrow{e} \) the same edge with the reverse orientation, and by \( h(e) \) and \( t(e) \) the head and tail of \( e \), respectively. The Hashimoto or non-backtracking matrix \( B = B_\Gamma \) is a \( |\overrightarrow{E}| \times |\overrightarrow{E}| \) 0-1 matrix with rows and columns indexed by the elements of \( \overrightarrow{E} \). The \( e,f \) entry is defined by

\[
B_{e,f} = \begin{cases} 
1 & \text{if } t(e) = h(f) \text{ and } f \neq \overrightarrow{e}, \\
0 & \text{otherwise}.
\end{cases}
\]

The Ihara-Bass formula states that

\[
\det \left( I_{\overrightarrow{E}} - Bx \right) = \left( 1 - x^2 \right)^{N(d/2-1)} \det \left( I_N - Ax + (d - 1) x^2 I_N \right).
\] (1)

In fact, a similar formula holds more generally for arbitrary finite graphs – see, e.g., [KS00, Ran18] for more details and proofs. The Ihara-Bass formula shows that the spectra of \( A \) and \( B \) completely determine each other:

\[
\text{Spec} \left( B \right) = \{ \pm 1 \} \cup \{ \mu \mid \mu^2 - \lambda \mu + (d - 1) = 0, \ \lambda \in \text{Spec} \left( A \right) \}.
\] (2)

In particular, every eigenvalue \( \lambda \in \text{Spec} \left( A \right) \) corresponds to two eigenvalues \( \frac{\lambda \pm \sqrt{\lambda^2 - 4(d - 1)}}{2} \in \text{Spec} \left( B \right) \). The trivial eigenvalue \( \lambda_1 = d \) corresponds to \( d - 1 \), \( 1 \in \text{Spec} \left( B \right) \). Every eigenvalue \( \lambda \in \text{Spec} \left( A \right) \) with \( |\lambda| \geq 2\sqrt{d - 1} \) gives rise to two real eigenvalues in \([- (d - 1), -1] \cup [1, d - 1]\), while every eigenvalue with \( |\lambda| < 2\sqrt{d - 1} \) corresponds to two non-real eigenvalues lying on the circle of radius \( \sqrt{d - 1} \) around \( 0 \) in \( \mathbb{C} \). For a nice treatment of this dictionary between \( \text{Spec} \left( A \right) \) and \( \text{Spec} \left( B \right) \) in the regular case, consult [LP16, Section 3].

In particular, we think of \( d - 1 \) as the trivial eigenvalue of \( B \), which, again, corresponds to the constant eigenfunction on oriented edges. By ordering the multiset of eigenvalues of \( B \) by their absolute value, we obtain

\[
d - 1 = |\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_{2N}| = |\mu_{2N+1}| = \ldots = |\mu_{dN}| = 1.
\] (3)
We let $\mu (\Gamma ) \overset{\text{def}}{=} |\mu_2|$ denote the largest absolute value of a non-trivial eigenvalue. If $N \geq 2$ then $\mu (\Gamma ) \in [\sqrt{d-1}, d-1]$. Notice that if $\mu (\Gamma ) > \sqrt{d-1}$, in which case $\mu_2$ is real, then

$$
\lambda (\Gamma ) = \mu (\Gamma ) + \frac{d-1}{\mu (\Gamma )}.
$$

\section{The proof in \cite{Pud15}}

As explained in the introduction to \cite{Pud15}, using well-known contiguity results for different models of random regular graphs, when $d = 2k$ is even, it is enough to prove Theorem 1.1 in the permutation model\footnote{To deal with odd values of $d$, one can show that the probabilistic bound holding for random $(d+1)$-regular graphs also holds for random $d$-regular graphs – see \cite[Claim 6.1]{Pud15}.}. In this model, a random $d$-regular graph on $N$ vertices is generated by sampling $k = \frac{d}{2}$ independent uniformly random permutations $\sigma_1, \ldots, \sigma_k$ in the symmetric group $S_N$ and constructing the corresponding Schreier graph depicting the action of $S_N$ on $[N] \overset{\text{def}}{=} \{1, \ldots, N\}$ with respect to $\sigma_1, \ldots, \sigma_k$. Namely, the $N$ vertices of the graph are labeled $1, \ldots, N$, and for every $1 \leq i \leq N$ and every $1 \leq j \leq k$, one adds an edge $(i, \sigma_j (i))$ to the graph. The resulting graph may contain loops and parallel edges.

In order to use the trace method, we bound the number of closed walks in the random graph $\Gamma$ generated in the permutation model. We may direct and label the edges of $\Gamma$ by labeling $(i, \sigma_j (i))$ by $j$ and directing it as follows:

$$
i \xrightarrow{j} \sigma_j (i).
$$

Then, every (closed) walk of length $t$ in $\Gamma$ corresponds to some word in $\{\sigma_1^\pm, \ldots, \sigma_k^\pm\}^t$. For example, the left-to-right walk

$$4 \xrightarrow{5} 3 \xrightarrow{4} 7 \xrightarrow{2} 10 \xrightarrow{5} 3 \xrightarrow{2} 4
$$

corresponds to the word $\sigma_5 \sigma_4^{-1} \sigma_2 \sigma_5^{-1} \sigma_2$. Moreover, the number of closed walks in $\Gamma$ corresponding to a given word is exactly the number of fixed points of the permutation in $S_N$ defined by the word\footnote{Here, for ease of description, we compose permutations from left to right, although this is completely immaterial in the analysis.} in $\sigma_1, \ldots, \sigma_k$.

The proof in \cite{Pud15} heavily depends on deep results in combinatorial group theory proven in \cite{Pud14, PP15}. These works consider random permutations sampled by fixed words in the free group $F_k$ with basis $X = \{x_1, \ldots, x_k\}$. Given a word $w \in F_k$, the corresponding random permutation is $w (\sigma_1, \ldots, \sigma_k)$ where, as before, $\sigma_1, \ldots, \sigma_k$ are independent, uniformly random permutations in $S_n$. For example, if $w = x_5 x_1^{-1} x_2 x_5^{-1} x_2 \in F_5$, the random permutation is $w (\sigma_1, \ldots, \sigma_5) = \sigma_5 \sigma_4^{-1} \sigma_2 \sigma_5^{-1} \sigma_2$.

For a word $w \in F_k$, denote by $F_w (N)$ the random variable counting the number of fixed points of the random permutation $w (\sigma_1, \ldots, \sigma_k) \in S_N$. Assume that $d = 2k$ is even and that $\Gamma$ is a random $d$-regular graph on $N$ vertices in the permutation model. The first step in the trace method for bounding $\lambda (\Gamma )$ is the observation that for any even $t \in \mathbb{Z}_{\geq 1}$,

$$
\mathbb{E} [\lambda (\Gamma)^t] \leq \mathbb{E} \left[ \sum_{i=2}^N \lambda_i (\Gamma)^t \right] = \mathbb{E} [\text{tr} \left( A^{t}\Gamma \right)] - d^t = \sum_{w \in (X \cup X^{-1})^t} (\mathbb{E} [F_w (N)] - 1). \quad (5)
$$

As shown in \cite{Pud14, PP15}, the asymptotic behaviour of the expectation $\mathbb{E} [F_w (N)]$ depends on an algebraic invariant of $w$, called the primitivity rank. This invariant, denoted $\pi (w)$, is equal to the smallest rank of a subgroup $H \leq F_k$ which contains $w$ as an imprimitive element, namely, such that $w$ does not belong to any free basis of $H$. There are no such subgroups if and only if $w$ is primitive in $F_k$, and in this case we define $\pi (w) = \infty$. The possible values of $\pi$ in $F_k$ are $0, 1, \ldots, k$ and $\infty$. 
The set of subgroups $H$ of rank $\pi (w)$ containing $w$ as an imprimitive element is denoted $\text{Crit} (w)$. It turns out that $|\text{Crit} (w)| < \infty$ for every $w$ [PP15, Section 4]. Theorem 1.8 in [PP15] states that

$$\mathbb{E} [\mathcal{F}_w (N)] = 1 + \frac{|\text{Crit} (w)|}{N^{\pi(w)-1}} + O \left( \frac{1}{N^{\pi(w)}} \right).$$

The big-O term was later effectivised in [Pud15, Proposition 5.1] to yield the following version of the theorem. Here $|w|$ denotes the length of $w$, namely, the number of letters when written in reduced form in the given basis $X$ of $F_k$.

**Theorem 3.1.** Let $w \in F_k$ and assume that $|w| = t$. Then for every $N > t^2$,

$$\mathbb{E} [\mathcal{F}_w (N)] \leq 1 + \frac{1}{N^{\pi(w)-1}} \left( |\text{Crit} (w)| + \frac{t^{2+2\pi(w)}}{N - t^2} \right) \leq 1 + \frac{|\text{Crit} (w)|}{N^{\pi(w)-1}} \left( 1 + \frac{t^{2+2\pi(w)}}{N - t^2} \right).$$

To count closed walks of length $t$ in $\Gamma$, we therefore care to know how many words of length $t$ there are in $F_k$ of every given primitivity rank. This, incorporated with the number of critical subgroups, is given by the following proposition:

**Theorem 3.2.** [Pud15, Proposition 4.3 and Theorem 8.2] Let $k \geq 2$ and $m \in \{1, \ldots, k\}$. Then

$$\limsup_{t \to \infty} \left[ \sum_{w \in F_k : |w| = t \& \pi (w) = m} |\text{Crit} (w)| \right]^{1/t} = \max \left( \sqrt{2k-1}, 2m-1 \right). \tag{6}$$

In fact, for $m \geq 2$, the limsup in the theorem can be replaced by ordinary lim, and for $m = 1$ it can be replaced by an ordinary lim on even values of $t$. The cases not covered by Theorem 3.2 are $\pi (w) \in \{0, \infty\}$. But $\pi (w) = 0$ if and only if $w = 1$, and $\pi (w) = \infty$ if and only if $w$ is primitive, if and only if $|\text{Crit} (w)| = 0$, if and only if $\mathbb{E} [\mathcal{F}_w (N)] = 1$ for all $N$. At any rate, words $w$ with $\pi (w) = \infty$ do not contribute to the summation (7) below.

Now we reach a step in [Pud15] which was required because the original proof directly analyzed the ordinary spectrum and counted closed walks with possible backtracking in $\Gamma$. Note that the word corresponding to a closed walk in $\Gamma$ is reduced if and only if the walk has no backtracking. So for arbitrary walks of length $t$, we need to consider not only reduced words of length $t$ but any words in $(X \cup X^{-1})^t$. Using (6) and the “extended cogrowth formula” – [Pud15, Theorem 4.4] – we obtain:

**Theorem 3.3.** [Pud15, Cor. 4.5] Let $k \geq 2$ and $m \in \{0, 1, \ldots, k\}$. Then

$$\limsup_{t \to \infty} \left[ \sum_{w \in (X \cup X^{-1})^t : \pi (w) = m} |\text{Crit} (w)| \right]^{1/t} = \begin{cases} 
\sqrt{2k-1} - 1 & \text{if } 2m - 1 \leq \sqrt{2k-1} \\
\frac{2k-1}{2m-1} + 2m - 1 & \text{if } 2m - 1 \geq \sqrt{2k-1}.
\end{cases}$$

**Remark 3.4.** The proof of the extended cogrowth formula never appeared explicitly in print. The writing of “Notes on the cogrowth formula: the regular, biregular and irregular cases”, which is mentioned in the reference list in [Pud15], was never completed. One reason is that the discussion leading to the current note took place already in the beginning of 2016. This discussion led to the realization that the extended cogrowth formula was immaterial for the current method of proof. The second named author still stands behinds the statement of Theorem 4.4 in [Pud15]. The ideas in that proof are not too far from some of the existing proofs to Grigorchuk’s cogrowth formula.

\footnote{For completeness, let us mention that for $k \geq 3$, the number of primitive words of length $t$ behaves like $(2k-3)^t$ – see [PW14].}
The final computation in [Pud15] goes as follows. With assumptions as above, it follows from (5) that for all \( t < \sqrt{N} \) even,

\[
\mathbb{E} \left[ \lambda (\Gamma)^t \right] = \sum_{w \in (X \cup X^{-1})^t} (\mathbb{E} \left[ \mathcal{F}_w (N) \right] - 1) = \sum_{m=0}^{k} \sum_{\pi(w)=m} \mathbb{E} \left[ \mathcal{F}_w (N) \right] - 1
\]

\[
\frac{1}{N^{m-1}} \sum_{w \in (X \cup X^{-1})^t : \pi(w)=m} |\text{Crit}(w)| \left( 1 + \frac{t^2 + 2\pi(w)}{N - t^2} \right).
\]

If \( t \approx c \log N \) for some constant \( c = c(d) \), taking the \( t \)-th root of both side and then taking the limit as \( N \to \infty \), we may use Theorem 3.3 to estimate the right hand side and deduce that \( \mathbb{E} \left[ \lambda (\Gamma)^t \right]^{1/t} \leq 2 \sqrt{d-1} + 0.835 \) – for details see Section 6.1 in [Pud15]. Using Markov’s inequality gives that a.a.s. \( \lambda (\Gamma) \leq 2 \sqrt{d-1} + 0.84 \).

Finally, as explained in Footnote 2, this can be used to get a slightly weaker bound when \( d \) is odd, and all together, for every \( d \geq 3 \), a random \( d \)-regular graph on \( N \) vertices satisfies a.a.s. \( \lambda (\Gamma) \leq 2 \sqrt{d-1} + 1 \).

### 4 Proof of Theorem 1.1

To establish the bound in Theorem 1.1, we adapt the proof from Section 3 to the non-backtracking spectrum. The trace of the \( t \)-th power \( B^t \) of the Hashimoto matrix \( B = B_{T} \) is equal to the number of **cyclically non–backtracking closed walks** of length \( t \) in \( \Gamma \). Each such walk consists of a sequence of \( t \) edges \( e_1, \ldots, e_t \) so that \( t \cdot (e_i) = h \cdot (e_{(i+1)} \mod t) \) and \( e_{(i+1)} \mod t \neq \overline{e_i} \) for \( i = 1, \ldots, t \). When \( d = 2k \) is even we may use the permutation model as in Section 3 to sample a random \( d \)-regular graph on \( N \) vertices. Every cyclically non-backtracking closed walk corresponds to a fixed point of a cyclically reduced word in the permutations \( \sigma_1, \ldots, \sigma_k \). The total number of cyclically non-backtracking closed walks in \( \Gamma \) is, therefore,

\[
\sum_{w \in \mathcal{CR}_t (F_k)} \mathcal{F}_w (N),
\]

where \( \mathcal{CR}_t (F_k) \) denotes the set of cyclically reduced words of length \( t \) in \( F_k \). There is an exact formula for the number of such words:

**Proposition 4.1.** [Man11, Prop. 17.2] The number of cyclically reduced words of length \( t \) in \( F_k \) is

\[
|\mathcal{CR}_t (F_k)| = (2k - 1)^t + k + (-1)^t (k - 1).
\]

The trace method in the non-backtracking case is based on the following equality:

\[
\sum_{\mu \in \text{Spec}(B)} \mu^t = \text{tr} (B^t) = \# \{ \text{cyclically non–backtracking walks of length } t \}.
\]

If \( t \) is even, for every real eigenvalue \( \mu \), the summand \( \mu^t \) is positive. Since every non-real eigenvalue \( \mu \) lies on \( \{ z \in \mathbb{C} : |z| = \sqrt{d-1} \} \), the summand \( \mu^t \) in this case has real part at least \(-\sqrt{d-1}^t \). Recall also that the trivial eigenvalue is \( d - 1 \) and that (at least) \( N (d - 2) + 1 \) out of the \( Nd \) eigenvalues are \( \pm 1 \). Hence, recalling the notation (3), for \( t \) even we have

\[
\text{tr} (B^t) = (d - 1)^t + \sum_{i=2}^{2N-1} \mu_i (\Gamma)^t + N (d - 2) + 1,
\]
\[
\text{Re} \left[ \mu_2 (\Gamma)^t \right] = \text{tr} \left( B^t \right) - (d - 1)^t - \sum_{i=3}^{2N-1} \text{Re} \left[ \mu_i (\Gamma)^t \right] - N (d - 2) - 1
\]
\[
\leq \left[ \sum_{w \in \mathcal{E}_N(k) : \pi(w) = m} \mathcal{F}_w (N) \right] - (d - 1)^t + 2N \sqrt{d - 1} - N (d - 2) - 1
\]

Prop. 4.1
\[
\leq \sum_{m=1}^{k} \sum_{w \in \mathcal{E}_N(k) : \pi(w) = m} \left( \mathcal{F}_w (N) - 1 \right) + d - 1 + 2N \sqrt{d - 1} - N (d - 2) - 1
\]

Note that the last summation starts with \( m = 1 \) and not with \( m = 0 \) because only the trivial word \( w = 1 \) has primitivity rank \( \pi(w) = 0 \), and the empty word is the only (cyclically) reduced word giving 1.

Theorem 3.2 implies that for any \( \varepsilon > 0 \)
\[
\sum_{w \in \mathcal{E}_N(k) : \pi(w) = m} |\text{Crit} (w)| \leq \sum_{w \in \mathcal{F}_k : |w| = t \& \pi(w) = m} |\text{Crit} (w)| \leq \left[ \max \left( \sqrt{2k-1}, 2m-1 \right) + \varepsilon \right]^t 
\]

for every large enough \( t \). Taking expectations on both sides of (8), we obtain that for every \( \varepsilon > 0 \) and large enough \( t \),
\[
\mathbb{E} \left[ \text{Re} \left[ \mu_2 (\Gamma)^t \right] \right] \leq 2N \sqrt{d - 1}^t + \sum_{m=1}^{k} \frac{1}{N^{m-1}} \left( 1 + \frac{t^2 + 2m}{N - t^2} \right) \sum_{w \in \mathcal{E}_N(k) : \pi(w) = m} |\text{Crit} (w)| \leq 2N \sqrt{d - 1}^t + \left( 1 + \frac{t^2 + 2k}{N - t^2} \right) \sum_{m=1}^{k} \frac{1}{N^{m-1}} \left[ \max \left( \sqrt{2k-1}, 2m-1 \right) + \varepsilon \right]^t 
\]

We will soon take \( t \) to be a function of \( N \) so that as \( N \to \infty \), \( N^{1/t} \to c \) for a constant \( c \) specified below. Then for every \( \varepsilon > 0 \) and every large enough \( N \),
\[
\left( 1 + \frac{t^2 + 2k}{N - t^2} \right) \cdot 2 (k + 1) \leq (1 + \varepsilon)^t.
\]

Because the right hand side of (10) is at most \((k + 1)\) times the maximal summand (among the \( k + 1 \) summands), we get that for every \( \varepsilon > 0 \) and large enough \( N \),
\[
\mathbb{E} \left[ \text{Re} \left[ \mu_2 (\Gamma)^t \right] \right] \leq \left[ (1 + \varepsilon) \cdot \max \left( \left\{ N^{1/t} \sqrt{d - 1} \right\} \cup \left\{ \frac{2m - 1}{N^{(m-1)/t}} \mid 2m - 1 \in \left[ \sqrt{d - 1}, d - 1 \right] \right\} \right) \right]^t,
\]

where we used the observation that if \( 2m - 1 < \sqrt{d - 1} \) then the term corresponding to \( m \) in (10) is \( \frac{N^{1/t}}{N^{(m-1)/t}} \), and is thus strictly smaller than the first term \( 2N \sqrt{d - 1}^t \). A simple analysis yields that, at least for large values of \( d \), the optimal value of \( t = t(N) \) is such that
\[
N^{1/t} \to e^{-\sqrt{d-1}}.
\]
as $N \to \infty$. With this value, whenever $2m - 1 \in [\sqrt{d-1}, d-1]$, write $m = \beta \sqrt{d-1}$ with $\beta > \frac{1}{2}$. Then
\[
\frac{2m - 1}{N^{(m-1)/t}} = \frac{2\beta \sqrt{d-1} - 1}{(N^{1/t})^{3\sqrt{d-1}}} < N^{1/t} \frac{2\beta}{(N^{1/t})^{3\sqrt{d-1}}} \leq N^{1/t} \frac{2\beta}{e^{2\beta/\epsilon}} \leq N^{1/t} \frac{2\beta}{e^{\epsilon} \cdot \sqrt{d-1}},
\]
where the last inequality follows as $\frac{2\beta}{e^{2\beta/\epsilon}} \leq 1$ with equality if and only if $\beta = \epsilon/2$. Therefore, with this value of $t$, we obtain from (11) that for every $\epsilon > 0$,
\[
\mathbb{E} \left[ \Re \left[ \mu_2 (\Gamma)^t \right] \right] \leq \left( 1 + \epsilon \right) \cdot \sqrt{d-1} \cdot \frac{2}{e^{\epsilon} \cdot \sqrt{d-1}}
\]
for every large enough $N$. Recall that if $\mu_2 (\Gamma)$ is non-real, then it has absolute value $\sqrt{d-1}$, and so we always have $\Re \left[ \mu_2 (\Gamma)^t \right] \geq -\sqrt{d-1}$ for $t$ even. Therefore, for $x = \frac{2\beta}{e^{2\beta/\epsilon}}$,
\[
\Pr \left\{ \mu (\Gamma) > (1 + 2\epsilon) e^{x} \sqrt{d-1} \right\} \cdot \left( 1 + 2\epsilon \right) e^{x} \sqrt{d-1} \leq \mathbb{E} \left[ \Re \left[ \mu_2 (\Gamma)^t \right] \right] \leq \left( 1 + \epsilon \right) e^{x} \sqrt{d-1},
\]
which yields that for every $\epsilon > 0$
\[
\Pr \left\{ \mu (\Gamma) > (1 + 2\epsilon) \cdot \sqrt{d-1} \cdot e^{x} \sqrt{d-1} \right\} \to 0.
\]
Finally, by (4), when $\mu (\Gamma) > \sqrt{d-1}$, we have that $\lambda (\Gamma) = \mu (\Gamma) + \frac{d-1}{\mu (\Gamma)}$, and as $e^{x} + e^{-x} < 2e^{x^2/2}$ for $x > 0$, we conclude that
\[
\Pr \left\{ \lambda (\Gamma) > 2\sqrt{d-1} \cdot e^{x} \left( \frac{d-1}{\sqrt{d-1}} \right) \right\} \to 0.
\]
As $d$ is even so far, we have $d \geq 4$ and $\frac{2}{e^{x} \left( \frac{d-1}{\sqrt{d-1}} \right)} \leq 0.1$, and for $y < 0.1$, $e^{y} < 1 + 1.1y$, and
\[
2\sqrt{d-1} \cdot e^{x} \left( \frac{d-1}{\sqrt{d-1}} \right) \leq 2\sqrt{d-1} \left( 1 + \frac{2}{e^{x} \left( \frac{d-1}{\sqrt{d-1}} \right)} \right) < 2\sqrt{d-1} + \frac{0.6}{\sqrt{d-1}}.
\]
Hence,

**Theorem 4.2.** For $d$ even, $d \geq 4$, a random $d$-regular graph $\Gamma$ on $N$ vertices satisfies
\[
\lambda (\Gamma) \leq 2\sqrt{d-1} + \frac{0.6}{\sqrt{d-1}}
\]
a.a.s. as $N \to \infty$.

By [Pud15, Claim 6.1], an a.a.s. bound on $\lambda (\Gamma)$ for a $(d+1)$-regular random graph also holds a.a.s. for a $d$-regular graph. We conclude that if $d \geq 3$ is odd, then a random $d$-regular graph $\Gamma$ on $N$ vertices satisfies
\[
\lambda (\Gamma) \leq 2\sqrt{d} + \frac{0.6}{\sqrt{d}} \leq 2\sqrt{d-1} + \frac{1}{\sqrt{d-1}} + \frac{0.6}{\sqrt{d}} < 2\sqrt{d-1} + \frac{2}{\sqrt{d-1}}.
\]
This completes the proof of Theorem 1.1. \(\square\)

**Remark 4.3.** As explained in [Pud15, Section 6.2], for small values of $d$, the optimal value of $t = t(N)$ in the proof above is different and leads to better bounds.
5 Random coverings of a fixed graph

A random $2k$-regular graph on $N$ vertices in the permutation model can be thought of as a random $N$-degree covering space of the bouquet with one vertex and $k$ loops – see [Pud15, Section 1]. More generally, [Pud15] deals with random coverings of an arbitrary finite connected graph. In this case, extending Alon’s conjecture, the first named author conjectured in [Fri03] that for every $\varepsilon > 0$, a random $N$-degree covering of a fixed graph $\Delta$ satisfies that a.a.s. all new eigenvalues of the covering are at most $\rho + \varepsilon$ in absolute value, where $\rho$ is the spectral radius of the universal cover of $\Delta$. In this more general case, [Pud15] provided the best bounds at the time it was written (and see the references therein for earlier bounds). Slightly later, this conjecture was proven when the base graph is regular in [FK14] (later split into a series of papers starting with [FK19]) and in [Bor19]. More recently, the conjecture was proven in full, namely, for arbitrary finite base graph, in [BC19].

We remark that the improvement suggested in the current note applies more generally to random covers of a fixed regular graph. In this case, [Pud15, Thm 1.5] states that the largest new eigenvalue of a random $N$-cover of a fixed $d$-regular graph is a.a.s. less than $2\sqrt{d-1} + 0.84$. This can be improved to the same statement as in Theorem 4.2, namely, to $2\sqrt{d-1} + \frac{0.6}{\sqrt{d-1}}$.

In the irregular case, there is no direct dictionary between the spectrum of the ordinary spectrum and that of the non-backtracking spectrum. However, [BC19] uses a variety of non-backtracking operators to prove the conjecture about random coverings of arbitrary graphs. In fact, they manage to prove something much stronger than the original conjecture in [Fri03], and show the new spectrum of a random covering of $\Delta$ is contained a.a.s. in an $\varepsilon$-neighborhood of the spectrum of the covering tree.

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