EXTENSIONS OF THE REPRESENTATION
MODULES OF A PRIME ORDER GROUP

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Dedicated to Professor Kálmán Győry on his 65th birthday

Abstract. For the ring $R$ of integers of a ramified extension of the field of $p$-adic numbers and a cyclic group $G$ of prime order $p$ we study the extensions of the additive groups of $R$-representations modules of $G$ by the group $G$.

Let $\mathcal{F}$ be the field of fractions of a principal ideal domain $R$, $F$ a field which contains $R$, let $G$ be a finite group and $\Gamma$ a matrix $R$-representation of $G$. Let $M$ be an $RG$-module, which affords the $R$-representation $\Gamma$ of $G$, and $FM = F \otimes_R M$ the smallest linear space over $F$ which contains $M$ and $\hat{M} = FM^+/M$, the factor group of the additive group of the space $FM$ by the additive group of $M$. Clearly, the group $\hat{M}$ and the space $FM$ are $RG$-modules. Put $\hat{F} = F^+/R$.

Let $f : G \to \hat{M}$ be a 1-cocycle of $G$ with value in $\hat{M}$, i.e.

$$f(xy) = xf(y) + f(x), \quad (x, y \in G).$$

Define $[g, x]$ by $\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix}$ and set

$$\mathcal{Crys}(G, M, f) = \{ [g, x] \mid g \in G, \quad x \in f(g) \},$$

where $x$ runs over the cosets $f(g) \in \hat{M}$ for any $g \in G$.

Clearly, $\mathcal{Crys}(G, M, f)$ is a group, where the multiplication is the usual matrix multiplication. Of course $K_1 = \{ [e, x] \mid e$ is the unit element of $G, \quad x \in f(e) \}$ is a normal subgroup of $\mathcal{Crys}(G, M, f)$ such that $K_1 \cong M^+$ and $\mathcal{Crys}(G, M, f)/K_1 \cong G$.

The group $\mathcal{Crys}(G, M, f)$ is an extension of the additive group of the $RG$-module $M$ by $G$.

We are using the terminology of the theory of group representations [1].

A 1-cocycle $f : G \to \hat{M}$ is called coboundary, if there exists an $x \in FM$ such that $f(g) = (g - 1)x + M$ for every $g \in G$. The 1-cocycles $f_1 : G \to \hat{M}$ and

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$f_2 : G \to \hat{M}$ are called cohomologous if $f_1 - f_2$ is a coboundary. Let $H^1(G, \hat{M})$ be the first cohomology group. Clearly, each element of $H^1(G, \hat{M})$ defines a class of equivalence of groups.

If the 1-cocycles $f_1, f_2$ are cohomologous, then $\mathfrak{crs}(G, M, f_1)$ and $\mathfrak{crs}(G, M, f_2)$ are isomorphic. This isomorphism is called equivalence and these groups are called equivalent. In particular, the group $\mathfrak{crs}(G, M, f)$ is split (i.e. $\mathfrak{crs}(G, M, f) = M \rtimes G$) if and only if $f$ is coboundary.

The dimension of the group $\mathfrak{crs}(G, M, f)$ is called the $R$-rank of the $R$-module $M$. (Note that $M$ is a free $R$-module of finite rank.) The group $\mathfrak{crs}(G, M, f)$ is called irreducible (indecomposable), if $M$ is an irreducible (indecomposable) $RG$-module and the 1-cocycle $f$ is not cohomologous to zero.

The group $\mathfrak{crs}(G, M, f)$ is non-split, if the 1-cocycle $f$ defines a nonzero element of $H^1(G, \hat{M})$.

Note that the properties of the group $\mathfrak{crs}(G, M, f)$ were studied in [5,6,8], in the cases when $R$ is either the ring of rational integers $\mathbb{Z}$, or the $p$-adic integers $\mathbb{Z}_p$, or the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at $p$.

Let $G = \langle a \mid a^p = 1 \rangle$ be the cyclic group of prime order $p$, $R$ the ring of integers of the ramified finite extension $T$ of the field of $p$-adic numbers. We calculate the group $H^1(G, \hat{M})$ for some module $M$ of an indecomposable $R$-representation of $G$.

Let $\Phi_p(x) = x^{p-1} + \cdots + x + 1$ be a cyclotomic polynomial of degree $p$ and let $\eta(x)$ be a divisor of $\Phi_p(x)$ over the field $\mathbb{F}$ with $\deg(\eta(x)) < p - 1$ (provided that such nontrivial polynomial exists).

**Lemma 1.** Let $M_1$ and $M_2$ be $RG$-modules which afford an $R$-representation $\Gamma$ of $G = \langle a \mid a^p = 1 \rangle$.

(i) If $M_1 \cong M_2$ then $H^1(G, \hat{M}_1) \cong H^1(G, \hat{M}_2)$.

(ii) If the matrix $\Gamma(a)$ does not have 1 as eigenvalue, then $H^1(G, \hat{M}_1)$ is trivial.

**Proof.** See [1]. $\Box$

**Theorem 1.** Let $G = \langle a \mid a^p = 1 \rangle$ and $M_\eta = \eta(a)RG$. Then the $RG$-module $M_\eta$ is indecomposable and

$$H^1(G, \hat{M}_\eta) \cong R/(\eta(1)R),$$

where $R/(\eta(1)R)$ is the additive group of the factor ring of $R$ by the ideal $\eta(1)R$.

**Proof.** Let $t \in R$ be a prime element and $\overline{R} = R/(tR)$. Then in $\overline{R}$ we have that

$$x^p - 1 = (x - 1)^p; \quad \eta(x) = (x - 1)^n; \quad (x^p - 1)\eta^{-1}(x) = (x - 1)^{p-n},$$

where $n = \deg(\eta(x))$.

Put $\eta_1(x) = (x^p - 1)\eta^{-1}(x)$. Then $M_\eta$ and $RG/(\eta_1(a)RG)$ are isomorphic as $RG$-modules. If $\overline{M}_\eta = M_\eta/(tM_\eta)$, then by (1) follows that $\overline{M}_\eta$ is a root subspace of the linear operator $a$ over $\overline{R}$. It is easy to see that $\overline{M}_\eta$ is not decomposable into a direct sum of invariant subspaces. It follows that $M_\eta$ is an indecomposable
indecomposability of $X$ where $\lambda$ in each class of 1-cocycles there is a cocycle $f$ where $\alpha \hat{t}$ we get $\text{RG}$ This means that in the class of 1-cocycles there is a cocycle $f$ \begin{align*}
 F/R & \text{ is a subgroup of } \rho p \\
 a & \text{ rational number (see Theorem 3.5, [4] p.21) of the form } \frac{\lambda}{r} \text{.}
\end{align*}
Moreover, from $f(a^p) = 0$ (in $\widehat{M}_{\eta}$) it follows that if $\omega = a^{p-1} + \cdots + a + 1$, then $\omega \cdot f(a) = \lambda \cdot \eta(1) \omega \in M_{\eta}$ if and only if $\lambda \cdot \eta(1) \in R$. Therefore, $H^1(G, \widehat{M}_{\eta})$ is isomorphic to the subgroup \begin{align*}
 \{ \lambda + R \mid \lambda \in F, \lambda \cdot \eta(1) \in R \} \text{ of } F/R \text{ and }
\end{align*}
\begin{align*}
 \{ \lambda + R \in F/R \mid \lambda \cdot \eta(1) \in R \} & \cong R/\langle \eta(1)R \rangle.
\end{align*}
\begin{proof}
Suppose that $\eta(1) = t^s R$, where $t$ is a prime element of $R$. Put $K_{\eta}(G, M_{\eta}) = \left\{ \begin{pmatrix} e & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & \alpha t^{-s} \eta(a) \\ 0 & 1 \end{pmatrix} \mid m \in M_{\eta} \right\}$, where $\alpha$ runs over the representative elements of the cosets of $R/\langle t^s R \rangle$. Up to equivalence, the groups $K_{\eta}(G, M_{\eta})$ give all extensions of the additive group of the $RG$-module $M_{\eta}$ by the group $G$.

Suppose $p = t^d \theta$, where $d > 1$ is the ramification index and $\theta$ is a unit in $R$. Set $\bar{\chi}_{ji} = t^j RG + (a - 1)^j RG$, \begin{align*}
 (1 \leq j < d, 1 \leq i < p).
\end{align*}
\begin{theorem}
The module $\bar{\chi}_{ji}$ is an $RG$-module affording an indecomposable $R$-representation of $G$ and $H^1(G, \bar{\chi}_{ji}) \cong R/t^{d-j} R$.
\end{theorem}
\begin{proof}
Suppose that the $RG$-module $\bar{\chi}_{ji}$ is decomposable into a direct sum of $RG$-submodules. Then $t^j = u_1 + u_2$, where $u_1, u_2$ are nonzero elements of $RG$ with $u_1 u_2 = 0$. Thus $e_1 = t^{-j} u_1$ is an idempotent. Since the trace $tr(e_1)$ of $e_1$ is a rational number (see Theorem 3.5, [4] p.21) of the form $rp^{-1}$ \begin{align*}
 (1 \leq r \leq p), \text{ we get } t^j r p^{-1} \in R, \text{ which is impossible for } j < d. \text{ This contradiction proves the indecomposability of } \bar{\chi}_{ji}. \text{ Clearly } F \bar{\chi}_{ji} = FG = F + (a - 1) FG. \text{ Therefore, in each class of 1-cocycles there is a cocycle } f : G \longrightarrow \bar{\chi}_{ji} \text{ such that } f(a) = \lambda + \bar{\chi}_{ji}, \text{ where } \lambda \in F \text{ with } \lambda \omega \in \bar{\chi}_{ji}. \text{ It follows that } \lambda p = t^j \alpha, \text{ where } \alpha \in R \text{ and }$ $H^1(G, \bar{\chi}_{ji}) \cong \{ \lambda + R \mid \lambda \in F, \lambda t^{d-j} \in R \}$ is a subgroup of $F/R$. \hfill \Box
\end{proof}
\begin{align*}
 K_{\eta}(G, \bar{\chi}_{ji}) = \left\{ \begin{pmatrix} e & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & \alpha t^{-d} \\ 0 & 1 \end{pmatrix} \mid m \in \bar{\chi}_{ji} \right\},
\end{align*}
where $\alpha$ runs over the representative elements of the cosets of $R/\langle t^{d-j} R \rangle$. 

Corollary 2. The groups $K_\alpha(G, \mathfrak{x}_{ji})$ give all extensions of the additive group of the $RG$-module $\mathfrak{x}_{ji}$ by $G$.

Lemma 2. The set $\{ \mathfrak{x}_{ji} \mid j = 1, \ldots, d - 1; i = 1, \ldots, \frac{p-1}{2} \}$ consists of pairwise non-isomorphic modules.

Proof. Let us consider an indecomposable $RG$-module $V_i = \overline{RG}/((a-1)^i\overline{RG})$, where $\overline{R} = R/(tR)$ and $1 \leq i \leq p$. It is easy to check that the elements

$$u_1 = t^i, \ldots, u_i = t^i(a-1)^{i-1}, \quad u_{i+1} = (a-1)^i, \ldots, u_p = (a-1)^{p-1}$$

form an $R$-basis in $\mathfrak{x}_{ji}$ and

$$\Phi_p(x) - (x-1)^{p-1} = p\theta(x),$$

where $\theta(x) \in \mathbb{Z}[x]$, $\deg(\theta(x)) \leq p-2$. Note that since $\theta(1) = 1$, it follows that $\theta(a)$ is a unit in the group ring $RG$. Using the identity

$$xy - 1 = (x-1)(y-1) + (x-1) + (y-1),$$

from (3) we obtain that

$$(a-1)^p = p(a-1) \cdot (\alpha_0 + \alpha_1(a-1) + \cdots + \alpha_{p-2}(a-1)^{p-2}),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{p-2} \in \mathbb{Z}$. Since $p = t^d\theta = t(t^{d-1}\theta)$, from (4) we get

$$(a-1)^p = (a-1)u_p = tm,$$

where $m \in \mathfrak{x}_{ji}$. According to (2) $(a-1)u_i = t^iu_{i+1}$, and from (5) we obtain that the $RG$-module $\overline{\mathfrak{x}}_{ji} = \mathfrak{x}_{ji}/(t\mathfrak{x}_{ji})$ is isomorphic to a direct sum $V_i \oplus V_{p-i}$ of indecomposable $RG$-modules, so by Theorem 2 and Lemma 1 the proof is complete. $\square$

Let $n > 1$ be the degree of a divisor of $\Phi_p(x)$, which is irreducible over $R$. We consider the following $RG$-modules:

$$\mathfrak{U}_{ji} = t^j(a-1)RG + (a-1)^{s+1}RG, \quad (1 \leq j < d, \quad 1 \leq s < n).$$

It is easy to check that the $RG$-module $\mathfrak{U}_{ji}$ satisfies the condition (ii) of Lemma 1, so $H^1(G, \mathfrak{U}_{ji}) = 0$.

Let $\mathfrak{I}_{js}$ be a submodule of the free module $\mathfrak{G}(2) = \{(x, y) \mid x, y \in RG\}$ of rank 2, which consists of the solutions $(x, y)$ of the equality

$$t^j(a-1)x + (a-1)^{s+1}y = 0.$$
Lemma 3. Let $\omega = \Phi_p(a)$ and set $u_1 = [0, \omega]$, $u_2 = [(a - 1)^s, -t^j]$ and $u_3 = [t^{-j}(\omega - (a - 1)^{p-1}), (a - 1)^{p-s-1}]$. Then $\mathfrak{Z}_{js}$ is an $RG$-module generated by $u_1, u_2, u_3$.

Proof. Clearly, $u_1, u_2, u_3 \in \mathfrak{Z}_{js}$. Let $u = [x, y]$ be an arbitrary element of $\mathfrak{Z}_{js}$. If $x = 0$ then $u = u_1$. Suppose $x \neq 0$. By substraction of the elements of $RGu_3$ from $u$ we obtain that $y = \gamma_0 + \gamma_1(a - 1) + \cdots + \gamma_{p-s-2}(a - 1)^{p-s-2}$ ($\gamma_r \in R$). By (6)

$$t^j(a - 1)x + (\gamma_0 + \gamma_1(a - 1) + \cdots + \gamma_{p-s-2}(a - 1)^{p-s-2}) \cdot (a - 1)^{s+1} = 0,$$

which is possible if and only if $\gamma_0 \equiv \cdots \equiv \gamma_{p-s-2} \equiv 0 \pmod{t^j}$. Now, since $u$ is an element of $RGu_2$, we obtain that $y = 0$. Then $t^j(a - 1)x = 0$ which implies $x = \alpha \omega$ ($\alpha \in R$) and $u = \alpha(t^ju_3 - (a - 1)^{p-s-1}u_2)$.

\[ \square \]

Theorem 3. The $RG$-module $\mathfrak{Z}_{js}$ is indecomposable. Moreover,

$$H^1(G, \mathfrak{Z}_{js}) \cong R/(t^dR) \oplus R/(t^{d-j}R)$$

and the $RG$-modules $\mathfrak{X}_{js}$ are pairwise non-isomorphic.

Proof. It is easy to see that

$$u_1 = t^j(a - 1), \ldots, u_{i-1} = t^j(a - 1)^{i-1}, \quad u_i = (a - 1)^i, \ldots, u_{p-1} = (a - 1)^{p-1}$$

form an $R$-basis in the $RG$-module $\Omega_{js}$ and

$$\overline{\Omega}_{js} = \Omega_{js}/(t\Omega_{js}) \cong V_s \oplus V_{p-s-1}.$$

Since $s < n$, it follows that the $RG$-module $\Omega_{js}$ is indecomposable. Moreover, it follows that the $RG$-modules $\Omega_{js}$ are pairwise non-isomorphic and $RG$-modules $\mathfrak{Z}_{js}$, $\Omega_{js}$ and $RG^2$ form an exact sequence

$$0 \rightarrow \mathfrak{Z}_{js} \rightarrow RG^{(2)} \rightarrow \Omega_{js} \rightarrow 0.$$ 

Therefore, $\mathfrak{Z}_{js}$ is the kernel of a minimal projective covering of the indecomposable $RG$-module $\Omega_{js}$, so $\mathfrak{Z}_{js}$ is also indecomposable.

Lemma 4. Let $\mathfrak{Z}_{js} = (F\mathfrak{Z}_{js})^+ / \mathfrak{Z}_{js}$, $\hat{F} = F^+/R$ and $M = (a - 1)\mathfrak{Z}_{js}$. Then

$$\mathfrak{Z}_{js}/M = \hat{F} \nu_1 + \hat{F} \nu_2,$$

where $\nu_1 = [0, \omega] + M$, $\nu_2 = [\omega, 0] + M$ and $a \nu_1 = \nu_1$, $a \nu_2 = \nu_2$.

Proof. Clearly, $ax = x$ ($x \in \mathfrak{Z}_{js}/M$) and $\hat{F} \nu_1 = \hat{F}[0, \omega] + M \in \mathfrak{Z}_{js}/M$. Moreover,

$$\omega \hat{F} u_3 + M = \hat{F}[t^{-1}p\omega, 0] + M = \hat{F}(tp^{-1})[t^{-1}p\omega, 0] + M$$

$$= \hat{F}[\omega, 0] + M.$$
By analogy
\[ \omega \hat{F} u_2 + M = \hat{F}[0, -t^j \omega] + M = \hat{F}[0, \omega] = \hat{F} \nu_1. \]
Therefore \( \hat{3}_{js}/M = \hat{F} \nu_1 + \hat{F} \nu_2. \)
\[ \square \]

From Lemma 4 it follows that each class of 1-cocycles of the group \( G \) with values in the group \( \hat{3}_{js} = (F \hat{3}_{js})^+/\hat{3}_{js} \) contains a 1-cocycle \( f \) such that
\[ f(a) = \alpha[0, \omega] + \beta[\omega, 0] + Z_{js}, \]
where \( \alpha, \beta \in F \) and \( \omega(\alpha[0, \omega] + \beta[\omega, 0]) \in 3_{js}. \) This condition holds if and only if \( \alpha p, \beta p \in R. \) Moreover
\[ \alpha[0, \omega] + \beta[\omega, 0] \in 3_{js} + (a - 1)\hat{3}_{js} \]
if and only if \( \alpha \in R \) and \( \beta \in t^{-j}R. \) Using properties of the 1-cocycle \( f \) it is easy to show that the two 1-cocycles \( f_j \) \( (j = 1, 2): \)
\[ f_1(a) = \alpha_1[0, \omega] + \beta_1[\omega, 0] + 3_{js}, \quad f_2(a) = \alpha_2[0, \omega] + \beta_2[\omega, 0] + 3_{js} \]
are cohomologous if and only if
\[ p\alpha_1 \equiv p\alpha_2 \pmod{t^d} \quad \text{and} \quad p\beta_1 \equiv p\beta_2 \pmod{t^{d-j}}, \]
where \( \alpha_j, \beta_j \in F, \ p\alpha_j, p\beta_j \in R. \) Note that \( p = t^d \theta. \)

It follows that the map \( f \mapsto (p\alpha + t^d R, p\beta + t^{d-j} R) \) gives the isomorphism
\[ H^1(G, \hat{3}_{js}) \cong R/(t^d R) \oplus R/(t^{d-j} R). \]
Therefore, according to (ii) of Lemma 1, the \( RG \)-modules \( 3_{js} \) \( (1 \leq j < d) \) are pairwise non-isomorphic.
\[ \square \]

Now, using the description of 1-cocycles it is easy to prove the following

**Corollary 3.** Put
\[ K_{\alpha, \beta}(G, 3_{js}) = \left\langle \left( \begin{array}{cc} 
\alpha t^{-d}[0, \omega] + \beta t^{-d}[\omega, 0] \\
1 
\end{array} \right) \left| m \in Z_{js} \right. \right\rangle, \]
where \( \alpha \) and \( \beta \) independently run over the representative elements of the cosets \( R/(t^d R) \) and \( R/(t^{d-j} R), \) respectively. Up to equivalence, the groups \( K_{\alpha, \beta}(G, 3_{js}) \) give all extensions of the additive group of the \( RG \)-module \( 3_{js} \) by the group \( G. \)

If \( R \) is the quadratic extension of the ring of \( p \)-adic integers, then the \( R \)-representations of \( G \) were described by P.M. Gudivok (see [7]).

Finally, we have the following result
Theorem 4. Let $Φ_p(x)$ be decomposable into the product of at least two irreducible polynomials over $R$. Then the dimensions of the non-split indecomposable groups $Crys(G,M,̂)$ are unbounded.

Proof. Let $Φ_p(x) = η_1(x) · · · η_k(x)$ ($k > 2$) be a decomposition into a product of polynomials irreducible over $R$ and suppose that

$$η_1(x) = x^n - α_{n-1}x^{n-1} - · · · - α_1x - α_0 ∈ R[x].$$

Note that $deg(η_1(x)) = deg(η_2(x)) = · · · = deg(η_k(x)) = n$ and $kn = p - 1$.

We will use the technic of the integral representation of finite groups, which was developed by S.D. Berman and P.M. Gudivok in [2, 3, 7]. Let $ε$ be a primitive $p$th root of unity such that $η_1(ε) = 0$ and let $r_j$ be a natural number, such that $ε_j = ε^{r_j}$ is a root of the polynomial $η_j(x)$, where $r_1 = 1$ and $j = 1, . . . , k$. Let $˜ε = (0 0 α_0 1 . . . 0 α_1 . . . 0 )$ be the comparing matrix of $η_1(x)$.

The following $R$-representations of $G = \langle a \mid ap = 1 \rangle$ are irreducible:

$$δ_0 : a ↦ 1; \quad δ_1 : a ↦ ˜ε; \quad δ_j : a ↦ ˜ε_j = ˜ε_j^{r_j}, \quad (j = 2, . . . , k).$$

Note that the module which affords representation $δ_1$ is $R[ε]$ with $R$-basis $1, ε, . . . , ε^{n-1}$.

Let $m ∈ N$. Define the following $R$-representation of $G = \langle a \rangle$ of degree $(3n+1)m$:

$$Γ_m : a ↦ \begin{pmatrix} Δ_{1m}(a) & U_m(a) \\ 0 & Δ_{2m}(a) \end{pmatrix},$$

where

$$Δ_{1m}(a) = δ_0^{(m)}(a) + δ_1^{(m)}(a) = \begin{pmatrix} E_m ⊗ δ_0(a) & 0 \\ 0 & E_m ⊗ δ_1(a) \end{pmatrix};$$

$$Δ_{2m}(a) = δ_2^{(m)}(a) + δ_3^{(m)}(a) = \begin{pmatrix} E_m ⊗ δ_2(a) & 0 \\ 0 & E_m ⊗ δ_3(a) \end{pmatrix};$$

$$U_m(a) = \begin{pmatrix} E_m ⊗ υ & J_m(1) ⊗ υ \\ E_m ⊗ \overline{υ} & E_m ⊗ \overline{υ} \end{pmatrix};$$

$u = (0, 0, . . . , 0, 1)$ defines a nonzero element of $Ext(δ_0, δ_j)$;

$J_m(λ)$ is a Jordan block of degree $m$ with $λ$ in the main diagonal;

$\overline{υ}$ is a matrix in which the first row is $(0, . . . , 0, 1)$ and all other rows are zero. The matrix $\overline{υ}$ defines a nonzero element of the group $Ext(δ_1, δ_j)$, where $j = 2, 3$;

$E_m$ is the unity matrix of degree $m$. 

Extensions of the representations modules
**Lemma 5.** (see [2,3]) $\Gamma_m$ is an indecomposable $R$-representation of $G$.

Let $\mathfrak{W}_m = R^l$ be a module of $l$-dimension vectors over $R$ affording the $R$-representation $\Gamma_m$. Put $\hat{F} = F^+/R$, $\hat{\mathfrak{W}}_m = F\mathfrak{W}_m^+/\mathfrak{W}_m$. Clearly $\hat{F}^l \cong \hat{\mathfrak{W}}_m$.

Define $\tau : F \to F^n$ by

$$\tau(w) = w(\alpha_0, \alpha_0 + \alpha_1, \alpha_0 + \alpha_1 + \alpha_2, \ldots, \alpha_0 + \cdots + \alpha_{n-2}, 1), \quad (7)$$

where the $\alpha_j$ are coefficients of $\eta_1(x)$ and $w \in F$.

**Lemma 6.** (i) Each 1-cocycle of $G = \langle a \mid a^p = 1 \rangle$ at $\hat{\mathfrak{W}}_m$ is cohomologous to a cocycle $\hat{f}$, such that

$$\hat{f}(a) = (X, 0, \ldots, 0) + \mathfrak{W}_m,$$

where $X \in F^n$ and $pX = 0$ in $\hat{F}^m$ (i.e. $pX \in R^m$).

(ii) Let $z \in F^n$ such that $(\bar{\varepsilon} - E_n)z = 0$ in $\hat{F}^n$. Then $z = \tau(w) \pmod{R^n}$, with $w \in F$ such that $\eta_1(1)w = 0$ in $\hat{F}$.

(iii) If $V = R/(p\eta(1)R)$ is the residual of ring $R$ by the ideal $(p\eta(1)R)$, then $H^1(G, \hat{\mathfrak{W}}_m) \cong V^m$.

**Proof.** (i) follows from (ii) of Lemma 1. (ii) is easy to check.

(iii) by (i) we can put $\hat{f}(a) = (X, 0, 0, 0)$ and $\hat{g}(a) = (Y, 0, 0, 0)$, where $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$ and $pX = pY = 0$. Note that all the equalities considered here are understood modulo the group $R$. Suppose that these 1-cocycles are cohomologous and $Z \in F^l$ is such that

$$\Gamma_m(a) - E_i)Z + \hat{f}(a) = \hat{g}(a). \quad (8)$$

Put $Z = (Z_1, Z_2, Z_3, Z_4)$, where $Z_1 \in F^m$ and $Z_2, Z_3, Z_4$ are $m$-dimensional vectors, with $i$-components belong to $F^n$, and denoted by $Z_2^i, Z_3^i$ and $Z_4^i$, respectively. By (8) we get

$$(E_m \otimes u)Z_3 + (J_m \otimes u)Z_4 + X = Y; \quad (9)$$

$$(E_m \otimes (\bar{\varepsilon} - E_n))Z_2 + (E_m \otimes \bar{\mathfrak{W}})(Z_3 + Z_4) = 0; \quad (10)$$

$$(E_m \otimes (\bar{\varepsilon} - E_n))Z_3 = 0, \quad (E_m \otimes (\bar{\varepsilon} - E_n))Z_4 = 0. \quad (11)$$

From (11) and by (ii) we have

$$Z_3 = (\tau(v_1), \ldots, \tau(v_m)), \quad Z_4 = (\tau(u_1), \ldots, \tau(u_m)), \quad \tau \text{ is from (7)}$$

where $u_j, v_j \in F$, $\tau$ is from (7) and

$$\eta_1(1)u_j = \eta_1(1)v_j = 0. \quad (13)$$

Clearly, the equality (10) consists of $m$ matrix equalities of the form

$$(\bar{\varepsilon} - E_n)Z_2^i + \bar{\mathfrak{W}}\tau(w) = 0, \quad (14)$$
where $Z^i_2 \in F^n$ is the $i$th component of $Z_2$, $i = 1, \ldots, m$ and $w \in F$. Since $u\tau(w) = w$ and $\bar{u}\tau(w) = (w, 0, \ldots, 0)$, when all the rows of (14) are added together we obtain

$$-\eta(1)Z^i_2 + w = 0,$$

where $Z^m_2$ is the last component of the vector $Z_2$. According to (12) and (15), (10) gives the equalities

$$-\eta(1)z_j + v_j + u_j = 0, \quad (j = 1, \ldots, m)$$

where $z_j$ are some components of $Z_2$. From (9)

$$v_j + u_j + u_{j+1} + x_j = y_j, \quad (j = 1, \ldots, m-1)$$

$$v_m + u_m + x_m = y_m,$$

where $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$ and $pX = pY = 0$. Multiplying (17) by $\eta_1(1)$ and using (16) we obtain for the components of $X$ and $Y$

$$\eta_1(1)x_j = \eta_1(1)y_j \quad (j = 1, \ldots, m).$$

Therefore, if the 1-cocycles $\mathfrak{f}$ and $\mathfrak{g}$ are cohomologous then (18) holds.

Conversely, suppose that (18) holds. Then it is not difficult to construct vectors $Z_2, Z_3, Z_4$ that satisfy (9) and (10), which is equivalent to (8), i.e. the 1-cocycles $\mathfrak{f}$ and $\mathfrak{g}$ are cohomologous. It follows that by going from a cocycle to an element of the cohomology group, we need to change each component in $X$ by $\beta = \alpha \cdot p^{-1}$ modulo the group $R$, where $\alpha \in R$. Moreover, if $\eta_1 \cdot \beta \in R$, then must change $\beta$ to 0.

**Theorem 5.** Let $\varepsilon \in R$, where $\varepsilon^p = 1$ and $p > 2$. Then the description of the non-split indecomposable groups $\text{Crys}(G, M, \mathfrak{f})$ is a wild type problem.

**Proof.** For arbitrary matrices $A, B \in M(m, R)$ the map

$$\Gamma_{A, B} : a \mapsto \begin{pmatrix} E & 0 & E & A & E \\ \varepsilon E & E & E & B \\ \varepsilon^2 E & 0 & 0 \\ \varepsilon^3 E & 0 \\ \varepsilon^4 E \end{pmatrix}$$

is an $R$-representation of $G$ of degree $l = 5m$. The $R$-representations $\Gamma_{A, B}$ and $\Gamma_{A_1, B_1}$ are $R$-equivalent if and only if

$$C^{-1}AC \equiv A_1 \pmod{(1 - \varepsilon)}, \quad C^{-1}BC \equiv B_1 \pmod{(1 - \varepsilon)}$$

for some invertible matrix $C$. It follows that the description of the $R$-representations $\Gamma_{A, B}$ of $G$ is a wild type problem.

For the module affording the representation $\Gamma_{A, B}$ of $G$ we put $R^l$. Let $X$ be an $m$-dimensional vector over $F$ with $pX \in R^m$. Then there is a 1-cocycle $\mathfrak{f}_X : G \rightarrow \hat{R}^l$, such that $\mathfrak{f}_X(a) = (X, 0, \ldots, 0) + R^l$. The 1-cocycles $\mathfrak{f}_X$ and $\mathfrak{f}_Y$ are cohomologous if and only if

$$(1 - \varepsilon)(X - Y) \in R^m.$$  

Putting $X = (p^{-1}, 0, \ldots, 0)$ we obtain that $H^1(G, \hat{R}^l) \neq 0$. □
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