ROSENTHAL’S SPACE REVISITED

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Abstract. Let $E$ be a rearrangement invariant (r.i.) function space on $[0,1]$, and let $Z_E$ consist of all measurable functions $f$ on $(0,\infty)$ such that $f^*\chi_{[0,1]} \in E$ and $f^*\chi_{[1,\infty)} \in L^2$. We reveal close connections between properties of the generalized Rosenthal’s space, corresponding to the space $Z_E$, and the behaviour of independent symmetrically distributed random variables in $E$. The results obtained are applied to consider the problem of the existence of isomorphisms between r.i. spaces on $[0,1]$ and $(0,\infty)$. Exploiting particular properties of disjoint sequences, we identify a rather wide new class of r.i. spaces on $[0,1]$ “close” to $L^\infty$, which fail to be isomorphic to r.i. spaces on $(0,\infty)$. In particular, this property is shared by the Lorentz spaces $\Lambda_2(\log^{-\alpha}(e/u))$, with $0 < \alpha \leq 1$.

1. Introduction

Let $p > 2$. Given any sequence $w = (w_n)_{n=1}^\infty$ of positive scalars such that

(1) $\sum_{n=1}^\infty w_n^{2p/(p-2)} = \infty$ and $\lim_{n \to \infty} w_n = 0$,

we define $X_{p,w}$ to be the space of all sequences $(a_n)_{n=1}^\infty$ of scalars satisfying

$\sum_{n=1}^\infty |a_n|^p < \infty$ and $\sum_{n=1}^\infty |a_n|^2 w_n^2 < \infty$,

under the norm

$\| (a_n)_{n=1}^\infty \| := \max \left\{ \| (a_n)_{n=1}^\infty \|_p, \| (a_n w_n)_{n=1}^\infty \|_2 \right\}$,

where $\| (a_n)_{n=1}^\infty \|_r = \left( \sum_{n=1}^\infty |a_n|^r \right)^{1/r}, 1 \leq r < \infty$. Note that, up to isomorphism, the definition of the space $X_{p,w}$ does not depend on the sequence $w$, i.e., $X_{p,w} \approx X_{p,w'}$, as long as both $w$ and $w'$ satisfy (1); [26] Theorem 13]. Hence, we can denote the space $X_{p,w}$ simply by $X_p$.

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The space $X_p$, introduced by Rosenthal in 1970 (see [26]), turned out to be very useful when studying the geometric structure of $L^p$-spaces. Specifically, $X_p$ is isomorphic to the complemented subspace of $L^p$ spanned by a certain sequence of independent 3-valued symmetrically distributed random variables (r.v.’s) [26, p. 282–283]. Moreover, for each $p > 2$ and an arbitrary sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^p[0, 1]$ of mean zero independent r.v.’s, the mapping $T: X_p \to L^p$, defined by

$$T(a_n) := \sum_{n=1}^{\infty} a_n f_n,$$

is an isomorphic embedding; [26 Theorem 3 and p. 280].

Later on, Johnson, Maurey, Schechtman, and Tzafriri introduced, in the memoir [16] (see p. 218), the following generalized space of Rosenthal type. Let $Y$ be an arbitrary rearrangement invariant (r.i.) space on $(0, \infty)$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable subsets of $(0, \infty)$ of positive measure such that

$$m(A_n) \leq 1, \quad m(A_n) \to 0 \ (n \to \infty), \quad \sum_{n=1}^{\infty} m(A_n) = \infty$$

($m$ is the Lebesgue measure). Then, the space $\tilde{U}_Y$ is defined as a Banach space which is isomorphic to the closed linear span of the sequence $\{\chi_{A_n}\}_{n=1}^{\infty}$ in $Y$. It is worth to note that, up to isomorphism, the latter span does not depend on the particular choice of sequence $\{A_n\}_{n=1}^{\infty}$ satisfying conditions (2) [16, Lemma 8.7]. The sequence $\{\|\chi_{A_n}\|_Y^{-1} \chi_{A_n}\}_{n=1}^{\infty}$, clearly is equivalent to an unconditional basis in $\tilde{U}_Y$. Moreover, if the space $Y(0, 1)$ is not equal to $L^\infty(0, 1)$ up to an equivalent renorming, $\tilde{U}_Y$ is isomorphic to a complemented subspace of $Y$.

To establish a link between the concepts so far introduced, recall a further important definition from [16] (see also [22, §2f]). Given a r.i. space $E$ on $[0, 1]$, we define the r.i. space $Z_E$ on $(0, \infty)$ consisting of all measurable functions $f$ on $(0, \infty)$ such that

$$\|f\|_{Z_E} := \|f^* \chi_{[0,1]}\|_E + \|f^* \chi_{[1,\infty)}\|_{L^2} < \infty,$$

where $f^*$ is the non-increasing left-continuous rearrangement of $|f|$ (observe that $\|\cdot\|_{Z_E}$ is a quasinorm, which is equivalent to a norm; [22 Theorem 2.f.1]).

Then, denoting $U_E := \tilde{U}_{Z_E}$, it can be checked that Rosenthal's space $X_p$ coincides, up to equivalence of norms, with the space $U_{L^p[0,1]}$ (in particular, we choose $w_n = m(A_n)^{1/2-1/p}$, see details in [16 p. 221]).

The main aim of this paper is to reveal close connections between properties of the space $U_E$ and the behaviour of independent r.v.’s in the corresponding r.i. space $E$.

Let $E$ be a r.i. space on $[0, 1]$. According to [17 Theorem 1], if $L^q[0, 1] \subseteq E$ for some $q < \infty$, then there is a constant $C = C(q) > 0$ such that for every sequence
of independent symmetrically distributed r.v.’s from $E$ we have

\begin{equation}
\left\| \sum_{n=1}^{\infty} x_n \right\|_E \leq C \left\| \sum_{n=1}^{\infty} \tau_n \right\|_{Z_E},
\end{equation}

where the sequence $\{\tau_n\}_{n=1}^{\infty}$ consists of pairwise disjoint measurable functions defined on $(0, \infty)$ such that $\tau_n$ and $x_n$ are equimeasurable for each $n = 1, 2, \ldots$ (it is worth to mention that the opposite inequality holds in every r.i. space $E$). More recently, in [9] (for a simpler proof see [10, Theorem 25]), the latter result was sharpened; it was proved that inequality (4) holds in every r.i. space $E$ that has the so-called Kruglov property (for definitions see the next section). Observe, for instance, that the exponential Orlicz space $\text{Exp} L^p$, generated by an Orlicz function equivalent to the function $e^{u^p}$ for large $u > 0$, has the Kruglov property if and only if $0 < p \leq 1$ (clearly, $\text{Exp} L^p$ does not contain $L^q$ for any $q < \infty$).

In the first part of the paper we show that inequality (4) is fulfilled for the class of independent symmetrically distributed r.v.’s in a r.i. space $E$ with the Fatou property whenever a similar estimate holds for the subspace $U_E$ of $Z_E$. More precisely, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable subsets of $(0, \infty)$ satisfying (2), then inequality (4) is a consequence of the following much weaker condition: there is a constant $C > 0$ such that for every set $S \subseteq \mathbb{N}$, with $\sum_{n \in S} m(A_n) \leq 1$, and all $a_n \in \mathbb{R}$

\begin{equation}
\left\| \sum_{n \in S} a_n u_n \right\|_E \leq C \left\| \sum_{n \in S} a_n \chi_{A_n} \right\|_{Z_E},
\end{equation}

where $u_n$ are independent symmetrically distributed functions, equimeasurable with the characteristic functions $\chi_{A_n}$ (see Theorem 1). Moreover, we prove in Theorem 2 that estimate (5) combined with a certain geometrical property of the subspace $[u_n]$ of a r.i. space $E$ ensures that $E \approx Z_E$.

Next, we apply the results obtained to consider the problem of the existence of isomorphisms between r.i. spaces on $[0, 1]$ and $(0, \infty)$, which was first posed by Mityagin in [23]. This and other closely related questions were intensively studied in the memoir [16] (see also [22]), by using the approach based on a construction of the stochastic integral with respect to a symmetrized Poisson process. In particular, it was shown that a r.i. space $E$ is isomorphic to the space $Z_E$ whenever $0 < \alpha_E \leq \beta_E < 1$, where $\alpha_E$ and $\beta_E$ are the Boyd indices of $E$ (see [16, Theorem 8.6] or [22, Theorem 2.f.1]). Later on, in [5], this result was improved: it turned out that non-triviality of the Boyd indices of $E$ can be replaced with a weaker condition that both spaces $E$ and its Köthe dual $E'$ have the Kruglov property.

However, there exist r.i. spaces on $[0, 1]$ which are not isomorphic to r.i. spaces on $(0, \infty)$. Roughly speaking, this property is shared by some r.i. spaces, which are located “very close” to the extreme r.i. spaces on $[0, 1]$, $L^1$ and $L^\infty$. For instance, this holds for the Orlicz space $L_{F_\alpha}$, $0 < \alpha < 1/2$, where $F_\alpha(u)$ is an Orlicz function equivalent to the function $u \log^\alpha u$ for large $u > 0$ [16, p. 235]. Observe that the only r.i. space on $(0, \infty)$, which can be isomorphic to $L_{F_\alpha}$ is
the space $Z_{L_{F_{0}}}$ (see [10]) Corollary 8.15 and subsequent remarks)). In such a case the result follows easily from the fact that either the space $E$ itself or its dual $E^*$ does not contain sequences equivalent to the unit vector $\ell^2$-basis, because both spaces $U_E$ and $Z_E$, clearly, contain such sequences. Indeed, if we assume that $L_{F_{0}} \approx Z_{L_{F_{0}}}$, with $0 < \alpha < 1/2$, then it would imply by duality that $\text{Exp} L^{1/\alpha} \approx Z_{\text{Exp} L^{1/\alpha}}$ (see Lemma 1). But this is a contradiction because the exponential Orlicz space $\text{Exp} L^r$, for $r > 2$, contains no sequences equivalent to the unit vector $\ell^2$-basis (for instance, this is a consequence of Proposition 4 with its proof combined with the well-known fact that any disjoint sequence in $\text{Exp} L^r$ contains a subsequence equivalent to the unit vector $c_0$-basis; see e.g. [27]).

Here, we present more non-trivial examples of r.i. spaces $E$ of such a sort, showing that even the existence of complemented subspaces isomorphic to $\ell^2$ does not guarantee that $U_E$ is isomorphically embedded into $E$. Specifically, exploiting particular properties of disjoint sequences, we identify a rather wide new class of r.i. spaces on $[0, 1]$ “close” to $L^\infty$, which fail to be isomorphic to r.i. spaces on $(0, \infty)$ (see Theorems 3, 4 and 5). Furthermore, in Corollary 2 we provide examples of Lorentz spaces $\Lambda_2(\varphi)$ containing plenty of complemented subspaces isomorphic to $\ell^2$, but without subspaces isomorphic to the corresponding Rosenthal’s spaces and not isomorphic to r.i. spaces on $(0, \infty)$. In particular, these properties are shared by the Lorentz spaces $\Lambda_2(\log^{-\alpha}(e/u))$, with $0 < \alpha \leq 1$ (see Corollary 3).

In the concluding part of the paper, in Theorem 6 we prove a partial result related to the problem if the Kruglov property of a r.i. space $E$ is a necessary condition for the existence of an isomorphic embedding $T: U_E \to E$. We consider the case when $T$ sends the basis functions $\chi_{A_n}$, $n = 1, 2, \ldots$, of $Z_E$ to some independent symmetrically distributed r.v.’s in $E$.

2. Preliminaries

2.1. Rearrangement invariant spaces. For a detailed account of basic properties of rearrangement invariant spaces, we refer to the monographs [11, 20, 22].

Let $I = [0, 1]$ or $(0, \infty)$. A Banach lattice $E$ on $I$ is said to be a rearrangement invariant (in brief, r.i.) (or symmetric) space if from the conditions: functions $x(t)$ and $y(t)$ are equimeasurable, i.e.,

$$m\{t \in I : |x(t)| > \tau\} = m\{t \in I : |y(t)| > \tau\} \text{ for all } \tau > 0$$

and $y \in E$ it follows $x \in E$ and $\|x\|_E = \|y\|_E$ (throughout, $m$ denotes the Lebesgue measure).

In particular, every measurable on $I$ function $x(t)$ is equimeasurable with the non-increasing, right-continuous rearrangement of $|x(t)|$ given by

$$x^*(t) := \inf\{ \tau > 0 : m\{s \in I : |x(s)| > \tau\} \leq t \} , \quad t > 0.$$ 

We note that for any r.i. space $E$ on $[0, 1]$ we have: $L^\infty[0, 1] \subset E \subset L^1[0, 1]$. Denote by $E_0$ the closure of $L^\infty[0, 1]$ in the r.i. space $E$ on $[0, 1]$ (the separable part of $E$). The space $E_0$ is r.i., and it is separable if $E \neq L^\infty$. The fundamental function $\phi_E$ of a symmetric space $E$ is defined by $\phi_E(t) := \|\chi_{[0,t]}\|_E$, $t > 0$. In what
follows, $\chi_A$ is the characteristic function of a set $A$. The function $\phi_E$ is quasi-concave, that is, it is nonnegative and increases, $\phi_X(0) = 0$, and the function $\phi_E(t)/t$ decreases. Without loss of generality, we will assume that $\|\chi_{[0,1]}\|_E = 1$ for every r.i. space $E$.

It is well known that the dilation operator $\sigma_\tau x(t) := x(t/\tau)\chi_{[0,\min(1,\tau)]}(t)$, $0 \leq t \leq 1$, is bounded on every r.i. space $E$ on $[0,1]$ and $\|\sigma_\tau\|_{E\to E} \leq \max(1,\tau)$ (see e.g. [20 Ch. II, §4.3]). The numbers $\alpha_E$ and $\beta_E$ given by

$$\alpha_E := \lim_{\tau \to 0} \frac{\ln \|\sigma_\tau\|_E}{\ln \tau}, \quad \beta_E := \lim_{\tau \to \infty} \frac{\ln \|\sigma_\tau\|_E}{\ln \tau}$$

satisfy the inequalities $0 \leq \alpha_E \leq \beta_E \leq 1$ and are called the Boyd indices of $E$.

The Köthe dual $E'$ of a r.i. space $E$ on $I$ consists of all measurable functions $y$ such that

$$\|y\|_{E'} := \sup \left\{ \int_I |x(t)y(t)| \, dt : x \in E, \|x\|_E \leq 1 \right\} < \infty.$$ 

If $E^*$ denotes the Banach dual of $E$, then $E' \subset E^*$ and $E' = E^*$ if and only if $E$ is separable. A r.i. space $E$ on $I$ is said to have the Fatou property if whenever $\{x_n\}_{n=1}^\infty \subseteq E$ and $x$ measurable on $[0,1]$ satisfy $x_n \to x$ a.e. on $I$ and $\sup_{n=1,2,...} \|x_n\|_E < \infty$, it follows that $x \in E$ and $\|x\|_E \leq \lim \inf_{n \to \infty} \|x_n\|_E$. It is well-known that a r.i. space $E$ has the Fatou property if and only if the natural embedding of $E$ into its Köthe bidual $E''$ is a surjective isometry.

An important example of r.i. spaces are the Orlicz spaces. Let $\Phi$ be an Orlicz function, i.e., increasing convex function on $[0,\infty)$ such that $\Phi(0) = 0$. Then, the Orlicz space $L_\Phi := L_\Phi(I)$ consists of all measurable on $I$ functions $x$ such that the Luxemburg–Nakano norm

$$\|x\|_{L_\Phi} := \inf \{ \lambda > 0 : \int_I \Phi(|x(t)|/\lambda) \, dt \leq 1 \}$$

is finite (see e.g. [19]). In particular, if $\Phi(u) = u^p$, $1 \leq p < \infty$, then $L_\Phi = L^p$. If $\Phi(u)$ is equivalent for large $u > 0$ to the function $e^{\lambda u^p}$, $p > 0$, we obtain the exponential Orlicz space $\text{Exp} L^p[0,1]$.

Every increasing concave function on $[0,1]$, with $\varphi(0) = 0$, and $1 \leq q < \infty$ generate the Lorentz space $\Lambda_q(\varphi)$ endowed with the norm

$$\|x\|_{\Lambda_q(\varphi)} := \left( \int_0^1 x^*(t)^q \, d\varphi(t) \right)^{1/q}.$$ 

2.2. The Kruglov property and comparison of sums of independent functions and their disjoint copies in r.i. spaces. Let $f$ be a measurable function on $[0,1]$. Denote by $\pi(f)$ the random variable (in brief, r.v.) $\sum_{i=1}^N f_i$, where $f_i$ are independent copies of $f$ (that is, independent r.v.’s equidistributed with $f$) and $N$ is a r.v. independent of the sequence $\{f_i\}$ and having the Poisson distribution with parameter 1. The following property has its origin in Kruglov’s paper [21].
and was actively studied and used by Braverman [12]. We say that a r.i. space \( E \) on \([0, 1]\) has the Kruglov property if the relation \( f \in E \) implies that \( \pi(f) \in E \).

Roughly speaking, a r.i. space \( E \) has the Kruglov property if it is located sufficiently “far away” from the space \( L^\infty \). In particular, if \( E \) contains \( L^p \) with some \( p < \infty \), then \( E \) has the Kruglov property. However, the latter condition is not necessary; for instance, the exponential Orlicz space \( \text{Exp} L^p \) has the Kruglov property if and only if \( 0 < p \leq 1 \) (see [12, § 2.4], [8]), but clearly \( \text{Exp} L^p \) does not contain \( L^q \) with any \( p > 0 \) and \( 1 \leq q < \infty \).

The Kruglov property is closely related to the famous Rosenthal inequality [26] and more generally to the problem of the comparison of sums of independent functions and their disjoint copies in r.i. spaces.

Let \( E \) be a r.i. space on \([0, 1]\). As was already mentioned in Section 1 by [17] Theorem 1], if \( L^q[0, 1] \subseteq E \) for some \( q < \infty \), then the inequality (4) holds for some constant \( C = C(q) > 0 \) and for each sequence of independent symmetrically distributed functions \( \{x_n\}_{n=1}^\infty \subset E \). Here, \( \bar{x}_n \) are disjoint copies of \( x_n \) defined on the semi-axis \([0, \infty)\) (for instance, we may take \( \bar{x}_n(t) = x_n(t \cdot n + 1) \chi_{[n-1, n)}(t), n = 1, 2, \ldots \)). We will refer such a sequence \( \{\bar{x}_n\} \) as a disjointification of the sequence \( \{x_n\} \). Using an operator approach initiated in [8] (see also [10]), Astashkin and Sukochev have showed that inequality (4) holds for a wider class of r.i. spaces with the above-defined Kruglov property.

It is easy to check that the above r.v. \( \pi(f) \) is equidistributed with the sum

\[
Kf(t) := \sum_{n=1}^\infty \sum_{i=1}^n f_{n,i}(t)\chi_{E_n}(t), \quad 0 \leq t \leq 1,
\]

where \( E_n \) are disjoint subsets of \([0, 1] \), \( m(E_n) = 1/(en!) \), \( n = 1, 2, \ldots \), and \( f_{n,i} \) are functions identically distributed with \( f \), \( i = 1, \ldots, n, n = 1, 2, \ldots \) such that \( f_{n,1}, \ldots, f_{n,n}, \chi_{E_n} \) are independent for each positive integer \( n \). It turns out that the above mapping \( K \) can be treated as a linear operator defined on suitable r.i. spaces (see [10] p. 1029). Moreover, given a r.i. space \( E \) on \([0, 1]\), the space \( E \) has the Kruglov property if and only if the operator \( K \) is bounded in \( E \). For this reason, \( K \) is called the Kruglov operator.

We will say that subsets \( F_n \) of \([0, 1] \), \( n = 1, 2, \ldots \), are independent if the characteristic functions \( \chi_{F_n} \), \( n = 1, 2, \ldots \), are independent on \([0, 1]\).

Standard Banach space notation is used throughout. In particular, \( X \approx Y \), where \( X \) and \( Y \) are Banach spaces, means that \( X \) and \( Y \) are isomorphic. We will write \( Y \preceq X \) if there is an isomorphic embedding \( T : Y \to X \). The notation \( f \asymp g \) will mean that there exists a constant \( C > 0 \) not depending on the arguments of the quantities (norms) \( f \) and \( g \) such that \( C^{-1} f \leq g \leq C f \). Finally, in what follows, \( C, c \) etc. denote constants whose value may change from line to line.

3. Rosenthal’s space \( \mathcal{U}_E \) and comparison of sums of independent functions and their disjoint copies in r.i. spaces.

Let \( \{A_n\}_{n=1}^\infty \) be an arbitrary (fixed) sequence of disjoint measurable subsets of \((0, \infty)\) satisfying conditions (2). Denote by \( u_n \) independent symmetrically
distributed r.v.’s supported on \([0,1]\) and equimeasurable with the characteristic functions \(\chi_{A_n}, n = 1,2,\ldots\). As it was mentioned in Section 1 if a r.i. space \(E\) has the Kruglov property (see Section 2.2), then there is a constant \(C > 0\) such that for any sequence \(\{x_n\}_{n=1}^{\infty}\) of independent symmetrically distributed r.v.’s from \(E\) inequality (4) holds. Clearly, then the above r.v.’s \(u_n, n = 1,2,\ldots\), satisfy condition (5). In this section, assuming that a r.i. space \(E\) has the Fatou property, we prove the converse non-trivial implication: from (5) it follows (4). Moreover, starting with this result we will show that estimate (5) combined with a geometrical property of the closed linear span \([u_n]\) in \(E\) implies that \(E \approx Z_E\).

First, we consider independent r.v.’s \(v_n, n = 1,2,\ldots\), which are identically distributed with the characteristic functions \(\chi_{A_n}, n = 1,2,\ldots\).

**Proposition 1.** Let \(E\) be a r.i. space on \([0,1]\). Suppose that there exists \(C > 0\) such that for every set \(S \subseteq \mathbb{N}\) such that \(\sum_{n \in S} m(A_n) \leq 1\) and all \(a_n \in \mathbb{R}, n \in S\), we have

\[
\left\| \sum_{n \in S} a_n v_n \right\|_E \leq C \left\| \sum_{n \in S} a_n \chi_{A_n} \right\|_{Z_E}.
\]

Then, the Kruglov operator \(K\) is bounded from \(E\) into \(E''\).

**Remark 1.** Clearly, from the condition \(\sum_{n \in S} m(A_n) \leq 1\) and definition of the norm in \(Z_E\) (see (3)) it follows that (6) can be equivalently rewritten as

\[
(6') \quad \left\| \sum_{n \in S} a_n v_n \right\|_E \leq C' \left\| \sum_{n \in S} a_n \chi_{A'_n} \right\|_E,
\]

where sets \(A'_n \subseteq [0,1]\) are pairwise disjoint and \(m(A'_n) = m(A_n), n = 1,2,\ldots\)

**Proof.** According to [10, Theorem 22(i)], it suffices to prove that there is a constant \(C' > 0\) such that for every sequence \(\{x_n\}_{n=1}^l \subseteq E\) of independent functions, with \(\sum_{n=1}^l m(\{t : x_n(t) \neq 0\}) \leq 1\), we have

\[
(7) \quad \left\| \sum_{n=1}^l x_n \right\|_E \leq C' \left\| \sum_{n=1}^l \mathbb{1}_{A'_n} \right\|_E,
\]

where \(\{\mathbb{1}_{A'_n}\}_{n=1}^l\) is a disjointification of the sequence \(\{x_n\}_{n=1}^l\) (we may and will assume that all the functions \(\mathbb{1}_{A'_n}\) are supported on \([0,1]\)). Moreover, without loss of generality, we suppose that \(x_n \geq 0, n = 1,\ldots,l\). For arbitrary \(\varepsilon > 0\) and \(k \in \mathbb{N}\) we set

\[
G_n^k := \{t : \varepsilon(k - 1) < x_n(t) \leq \varepsilon k\}, \quad F_n^k := \{t : \varepsilon(k - 1) < \mathbb{1}_{A'_n}(t) \leq \varepsilon k\}.
\]

Observe that, for every \(n = 1,2,\ldots,l\), the sets \(G_n^k, k = 1,2,\ldots\) (resp. \(F_n^k, k = 1,2,\ldots,n = 1,2,\ldots,l\)) are pairwise disjoint. Due to properties (2), for each \(n = 1,\ldots,l\) and all \(k \in \mathbb{N}\), we can find pairwise disjoint sets \(S_n^k \subseteq \mathbb{N}\) such that

\[
m(G_n^k) = m(F_n^k) = \sum_{i \in S_n^k} m(A_i).
\]
Define now the step-functions

\[ y_n := \sum_{k=1}^{\infty} \varepsilon_k \cdot \chi_{G^k_n} \quad \text{and} \quad z_n := \sum_{k=1}^{\infty} \varepsilon_k \cdot \chi_{F^k_n}, \quad n = 1, \ldots, l. \]

Clearly, the functions \( y_n, n = 1, \ldots, l \), are independent and

\[ x_n \leq y_n, \quad n = 1, \ldots, l. \]

Fix \( n = 1, 2, \ldots, l \). Then, the sets \( G^k_n, k \in \mathbb{N} \), are pairwise disjoint. Therefore, thanks to (8), we can represent the set \( G^k_n \) in the form

\[ G^k_n = \bigcup_{i \in S^k_n} G^{k,i}_n, \quad k \in \mathbb{N}, \]

where \( G^{k,i}_n \subseteq [0,1] \) are pairwise disjoint for all \( i \in S^k_n, k \in \mathbb{N}, \) and \( m(G^{k,i}_n) = m(A_i), i \in S^k_n \). Furthermore, we see that

\[ y_n = \sum_{k=1}^{\infty} \varepsilon_k \sum_{i \in S^k_n} \chi_{G^{k,i}_n}, \quad n = 1, \ldots, l. \]

Next, denote by \( v^{k,i}_n \) independent copies of the characteristic functions \( \chi_{G^{k,i}_n}, i \in S^k_n, k \in \mathbb{N}, n = 1, 2, \ldots, l \). Then, for each \( n = 1, 2, \ldots, l \), the sequence \( \{\varepsilon_k \cdot v^{k,i}_n\}_{i \in S^k_n, k \in \mathbb{N}} \) is a disjointification of the sequence \( \{\varepsilon_k \cdot v^{k,i}_n\}_{i \in S^k_n, k \in \mathbb{N}} \) (see Section 2.2). Therefore, if

\[ f_n := \sum_{k=1}^{\infty} \varepsilon_k \sum_{i \in S^k_n} v^{k,i}_n, \quad n = 1, 2, \ldots, l, \]

then, by [15, Proposition 1] (see also [10, Proposition 7]), we have

\[ m(\{t : y_n(t) > \tau\}) \leq 2m(\{t : \sup_{k \in \mathbb{N}, i \in S^k_n} \varepsilon_k \cdot v^{k,i}_n(t) > \tau\}) \]

\[ \leq 2m(\{t : f_n(t) > \tau\}). \]

Since \( y_n, n = 1, \ldots, l \) (respectively, \( f_n, n = 1, \ldots, l \)) are nonnegative independent r.v.’s, the sequence \( \{y_n\}_{n=1}^l \) (resp. \( \{f_n\}_{n=1}^l \)) has the same distribution as the sequence \( \{y^*_n(t_n)\}_{n=1}^l \) (resp. \( \{f^*_n(t_n)\}_{n=1}^l \)), which is defined on the probability space \([0,1], \prod_{n=1}^l m_n \) (for each \( n = 1, \ldots, l \), \( m_n \) is the Lebesgue measure on \([0,1])\). Furthermore, from \([10]\) and definition of the rearrangement of a measurable function it follows that

\[ \sigma_{1/2}(y^*_n)(t_n) = y^*_n(2t_n) \leq f^*_n(t_n), \quad 0 \leq t_n \leq 1/2. \]
It can easily be checked that the functions \( \sigma_1/2 y_n, n = 1, 2, \ldots, l, \) are independent on the interval \([0, 1/2]\). Indeed, for arbitrary intervals \( I_1, \ldots, I_l \) of \( \mathbb{R} \) we have

\[
m(\{t \in [0, 1/2] : (\sigma_1/2 y_j)(t) \in I_j, j = 1, \ldots, l\})
\]

\[
= m(\{t \in [0, 1/2] : y_j(2t) \in I_j, j = 1, \ldots, l\})
\]

\[
= \frac{1}{2} m(\{t \in [0, 1] : y_j(t) \in I_j, j = 1, \ldots, l\})
\]

\[
= \frac{1}{2} \prod_{j=1}^{l} m(\{t \in [0, 1] : y_j(t) \in I_j\})
\]

\[
= \frac{1}{2^{l+1}} \prod_{j=1}^{l} m(\{t \in [0, 1/2] : y_j(2t) \in I_j\})
\]

\[
= \frac{1}{2^{l+1}} \prod_{j=1}^{l} m(\{t \in [0, 1/2] : (\sigma_1/2 y_j)(t) \in I_j\})
\]

Hence, from (11), we have

\[
\left\| \sigma_{1/2} \left( \sum_{n=1}^{l} y_n \right) \right\|_E = \left\| \sum_{n=1}^{l} \sigma_{1/2}(y_n) \right\|_E
\]

\[
= \left\| \sum_{n=1}^{l} (\sigma_{1/2} y_n)^*(t_n) \right\|_{E([0,1]^l)}
\]

\[
\leq \left\| \sum_{n=1}^{l} f_n(t_n) \right\|_{E([0,1]^l)}
\]

\[
= \left\| \sum_{n=1}^{l} f_n \right\|_E.
\]

Since \( \|\sigma_r\|_{E \to E} \leq \max(1, \tau) \) (see Section 2.1 or [20, Ch.II, §4.3]), from this inequality it follows

\[
\left\| \sum_{n=1}^{l} y_n \right\|_E = \left\| \sigma_2 \left( \sum_{n=1}^{l} y_n \right) \right\|_E
\]

\[
\leq 2 \left\| \sigma_{1/2} \left( \sum_{n=1}^{l} y_n \right) \right\|_E \leq 2 \left\| \sum_{n=1}^{l} f_n \right\|_E.
\]

Therefore, combining this together with (9), we have

(12) \[
\left\| \sum_{n=1}^{l} x_n \right\|_E \leq 2 \left\| \sum_{n=1}^{l} f_n \right\|_E.
\]
On the other hand, from (8) it follows that there are pairwise disjoint sets
\( F_{k,i}^n \subseteq [0,1] \) such that
\[
m(F_{k,i}^n) = m(A_i), \quad i \in S_k^n, \quad k \in \mathbb{N}, \quad n = 1, \ldots, l, \quad \text{and}
\]
\[
F_k^n = \bigcup_{i \in S_k^n} F_{k,i}^n, \quad k \in \mathbb{N}, \quad n = 1, \ldots, l.
\]

Moreover, by the above definitions, \( v_{k,i}^n \) are being independent copies of the characteristic functions \( \chi_{A_i}, \quad i \in S_k^n, \quad k \in \mathbb{N}, \quad n = 1, 2, \ldots, l \). Since the sets \( A_i \) (resp. \( F_{k,i}^n \), \( i \in S_k^n, \quad k \in \mathbb{N}, \quad n = 1, 2, \ldots, l \), are pairwise disjoint and
\[
\sum_{n=1}^{l} \sum_{k=1}^{\infty} \sum_{i \in S_k^n} m(A_i) \leq \sum_{n=1}^{l} m\{|t : x_n(t) \neq 0\}) \leq 1,
\]
by the hypothesis of the proposition (see also Remark 1), we have
\[
\left\| \sum_{n=1}^{l} f_n \right\|_E \leq C \left\| \sum_{n=1}^{l} \sum_{k=1}^{\infty} \varepsilon_k \sum_{i \in S_k^n} \chi_{A_i} \right\|_E z_E
\]
\[
= C \left\| \sum_{n=1}^{l} \sum_{k=1}^{\infty} \varepsilon_k \sum_{i \in S_k^n} \chi_{F_{k,i}^n} \right\|_E
\]
\[
= C \left\| \sum_{n=1}^{l} z_n \right\|_E, \quad \text{(13)}
\]
where \( z_n := \sum_{k=1}^{\infty} \varepsilon_k \cdot \chi_{F_k^n}, \quad n = 1, 2, \ldots, l \).

Further, for every \( n = 1, \ldots, l, \quad k = 2, 3, \ldots \) and all \( t \in F_k^n \) we have
\[
\overline{x}_n(t) > \varepsilon (k - 1) \geq \frac{1}{2} \varepsilon k = \frac{1}{2} z_n(t).
\]

Hence, taking into account the disjointness of the sets \( F_k^n, \quad k \in \mathbb{N}, \quad n = 1, \ldots, l \),
we obtain
\[
\left\| \sum_{n=1}^{l} \overline{x}_n \right\|_E \geq \frac{1}{2} \left\| \sum_{n=1}^{l} z_n \sum_{k=2}^{\infty} \chi_{F_k^n} \right\|_E.
\]
Additionally, since the sets \( F_1^n \subseteq [0,1], \quad n = 1, 2, \ldots, l \), are pairwise disjoint, then
\[
\left\| \sum_{n=1}^{l} z_n \chi_{F_1^n} \right\|_E \leq \varepsilon \| \chi_{[0,1]} \|_E = \varepsilon.
\]
As a result, from inequalities (12) and (13) we get
\[
\left\| \sum_{n=1}^{l} x_n \right\|_E \leq 2C \left\| \sum_{n=1}^{l} z_n \right\|_E \\
\leq 2C \left( \left\| \sum_{n=1}^{l} z_n \sum_{k=2}^{\infty} \chi_{F_n^k} \right\|_E + \left\| \sum_{n=1}^{l} z_n \chi_{F_n^1} \right\|_E \right) \\
\leq 4C \left( \varepsilon + \left\| \sum_{n=1}^{l} x_n \right\|_E \right)
\]
Letting \( \varepsilon \to 0 \), we obtain (7) with \( C' = 4C \). \( \square \)

Next, we proceed with comparing the sequence \( \{v_i\} \) with the sequence \( \{u_i\} \) of independent symmetrically distributed r.v.’s equimeasurable with the characteristic functions \( \chi_{A_i}, i = 1, 2, \ldots \).

**Proposition 2.** Let \( E \) be a r.i. space on \([0, 1]\). Then, for every \( S \subseteq \mathbb{N} \) such that \( \sum_{i \in S} m(A_i) \leq 1 \) and all \( a_i \in \mathbb{R}, i \in S \), we have
\[
\left\| \sum_{i \in S} a_i v_i \right\|_E \leq 16 e \cdot \left\| \sum_{i \in S} a_i u_i \right\|_E.
\]

**Proof.** First, since \( u_i, i = 1, 2, \ldots \), are independent symmetrically distributed r.v.’s, the sequence \( \{u_n\}_{n=1}^{\infty} \) is 1-unconditional in \( E \) (see, e.g., [12, Proposition 1.14]). Therefore, we may (and will) assume that coefficients \( a_i, i \in S \), are nonnegative.

For each \( i \in S \), recalling that \( m(A_i) > 0 \), we denote by \( \alpha_i \) the least root of the equation
\[
2t(1 - t) = \frac{1}{4} m(A_i).
\]
Straightforward calculations show that
\[
\frac{1}{8} m(A_i) < \alpha_i < \frac{1}{2} m(A_i), \quad i \in S.
\]

Let \( \{G_i, H_i\}_{i \in S} \) be a family of independent subsets of \([0, 1]\) such that \( m(G_i) = m(H_i) = \alpha_i, i \in S \). Then, clearly, \( h_i := \chi_{H_i} - \chi_{G_i}, i \in S \), are independent symmetrically distributed r.v.’s. Moreover, since \( m(\{t : |u_i(t)| = 1\}) = m(A_i) \) for each \( i \in S \), and, due to independence,
\[
m(\{t : |h_i(t)| = 1\}) = 2\alpha_i(1 - \alpha_i) = \frac{1}{4} m(A_i), \quad i \in S,
\]
we have
\[
m(\{t : |h_i(t)| > \tau\}) \leq m(\{t : |u_i(t)| > \tau\}), \quad \tau > 0.
\]
Hence, by the well-known Kwapien-Rychlik inequality (see e.g. [28, Ch. V, Theorem 4.4]), for all \( a_i \geq 0 \) and \( \tau > 0 \), we get
\[
m\left( \left\{ t : \left| \sum_{i \in S} a_i h_i(t) \right| > \tau \right\} \right) \leq 2m\left( \left\{ t : \left| \sum_{i \in S} a_i u_i(t) \right| > \tau \right\} \right).
\]
Next, denoting $h := \sum_{i \in S} a_i h_i$, we represent $h = h' - h''$, where

$$h' := \sum_{i \in S} a_i \chi_{H_i}, \quad h'' := \sum_{i \in S} a_i \chi_{G_i}.$$  

Since $h'$ and $h''$ are independent, for each $\tau > 0$ it follows

$$m(\{t : |h(t)| > \tau\}) \geq m(\{t : |h'(t)| > \tau\} \cap \{t : h''(t) = 0\})$$
(17)

$$= m(\{t : |h'(t)| > \tau\}) \cdot m(\{t : h''(t) = 0\}).$$

Further, since $G_i$ are independent, by (15), we have

$$m(\{t : h''(t) = 0\}) \geq m(\bigcap_{i \in S} ([0, 1] \setminus G_i)) = \prod_{i \in S} (1 - m(G_i))$$
(18)

$$= \prod_{i \in S} (1 - \alpha_i) \geq \prod_{i \in S} \left(1 - \frac{1}{2} m(A_i)\right).$$

Finally, from the elementary inequality

$$\log(1 - x) \geq -\frac{x}{1 - x}, \quad 0 \leq x < 1,$$

and the assumption that $\sum_{i \in S} m(A_i) \leq 1$ it follows

$$\log \left(\prod_{i \in S} \left(1 - \frac{1}{2} m(A_i)\right)\right) = \sum_{i \in S} \log \left(1 - \frac{1}{2} m(A_i)\right)$$

$$\geq -\frac{1}{2} \sum_{i \in S} \frac{m(A_i)}{1 - \frac{1}{2} m(A_i)}$$

$$\geq -\sum_{i \in S} m(A_i) \geq -1.$$  

Combining the latter inequality with (17) and (18), we obtain

$$m(\left\{t : \left|\sum_{i \in S} a_i h_i(t)\right| > \tau\right\}) \geq \frac{1}{e} m(\left\{t : \left|\sum_{i \in S} a_i \chi_{H_i}(t)\right| > \tau\right\}).$$
(19)

On the other hand, one can easy see that, by (15), for all $i \in S$

$$m(\{t : v_i(t) > \tau\}) \leq 8m(\{t : \chi_{H_i}(t)(t) > \tau\}), \quad \tau > 0.$$  

Therefore, by passing to the rearrangements of r.v.’s $v_i$ and $\chi_{H_i}$, $i \in S$, in the same way as in the proof of Proposition 11 we deduce that for all $\tau > 0$ and $a_i \geq 0$

$$m(\left\{t : \left|\sum_{i \in S} a_i \chi_{H_i}(t)\right| > \tau\right\}) \geq \frac{1}{8} m(\left\{t : \left|\sum_{i \in S} a_i v_i(t)\right| > \tau\right\}).$$

Summing up this inequality, (16) and (19), we arrive at the estimate

$$m(\left\{t : \left|\sum_{i \in S} a_i v_i(t)\right| > \tau\right\}) \leq 16 e \cdot m(\left\{t : \left|\sum_{i \in S} a_i u_i(t)\right| > \tau\right\}), \quad \tau > 0.$$  

As a result, applying 20, Ch.II, §4.3, Corollary 2, we obtain (14).  

□
Now, from Propositions [12][2][9] (or [10] Theorem 25), and [3] Theorem 2.4 it follows the first main result of the paper.

**Theorem 1.** Let $E$ be a r.i. space on $[0,1]$. Suppose there is a constant $C > 0$ such that for every set $S \subseteq \mathbb{N}$, with $\sum_{n \in S} m(A_n) \leq 1$, and all $a_n \in \mathbb{R}$, $n \in S$, we have (5), that is,

$$\left\| \sum_{n \in S} a_n u_n \right\|_E \leq C \left\| \sum_{n \in S} a_n \chi_{A_n} \right\|_{Z_E},$$

where $u_n$ are independent symmetrically distributed functions, equimeasurable with $\chi_{A_n}$. Then, the Kruglov operator $K$ is bounded from $E$ into $E''$.

Therefore, if $E$ has the Fatou property, then it possesses the Kruglov property and hence there is a constant $C > 0$, depending only on $E$, such that for every sequence $\{x_n\}_{n=1}^{\infty}$ of independent symmetrically distributed r.v.'s from $E$ inequality (1) holds, that is,

$$\left\| \sum_{n=1}^{\infty} x_n \right\|_E \leq C \left\| \sum_{n=1}^{\infty} \beta_n \right\|_{Z_E},$$

where $\{\beta_n\}_{n=1}^{\infty}$ is a disjointification of $\{x_n\}_{n=1}^{\infty}$.

If we assume that, additionally, for some constant $C > 0$ and every $S \subseteq \mathbb{N}$ such that $\sum_{n \in S} m(A_n) \leq 1$ and all $a_n \in \mathbb{R}$, $n \in S$,

$$\left\| \sum_{n \in S} a_n u_n \right\|_{E'} \leq C \left\| \sum_{n \in S} a_n \chi_{A_n} \right\|_{Z_{E'}},$$

then the spaces $E$ and $Z_E$ are isomorphic.

Theorem [11] asserts that $E \approx Z_E$ under some conditions related to both spaces $E$ and $E'$. Next, we prove a statement, showing that the same result holds provided that, along with inequality (5), the subspace $\{u_n\}$ of $E$ has a certain geometrical property.

We will repeatedly use the following auxiliary result.

**Lemma 1.** For every r.i. space $E$ on $[0,1]$, we have $(Z_E)' = Z_{E'}$. Moreover, if $E$ has the Fatou property (resp. is separable), then so has (resp. is) $Z_E$.

**Proof.** Since $Z_E$ is a r.i. space on $[0,\infty)$, then

$$\|y\|_{(Z_E)'} = \sup_{\|x\|_{Z_E} \leq 1} \int_0^\infty x^*(t)y^*(t) \, dt$$

(see, for instance, [20] Ch.II, §2.2, property 140)). Hence, by definition of the norm in $Z_E$, we have

$$\|y\|_{(Z_E)'} \leq \sup_{\|x\|_{E} \leq 1} \int_0^1 x^*(t)y^*(t) \, dt + \sup_{\|(x^*(k))\|_{E'} \leq 1} \sum_{k=1}^{\infty} x^*(k)y^*(k)$$

$$= \|y\chi_{[0,1]}\|_{E'} + \left( \sum_{k=1}^{\infty} y^*(k)^2 \right)^{1/2}$$

$$\leq \|y\|_{Z_{E'}},$$
and the first assertion of the lemma follows.

Next, suppose that $E$ has the Fatou property. Let a sequence $\{x_n\}_{n=1}^{\infty} \subseteq Z_E$ satisfy the conditions $0 \leq x_n \uparrow x$ and $\sup_n \|x_n\|_{Z_E} < \infty$. Observe that then $x_n \uparrow x$ a.e. on $[0,1]$ (see e.g. [20 Ch.II, §2.2, property 11]). Therefore, by the hypothesis and the inequality

$$\max \left\{ \sup_n \|x_n^*\chi_{[0,1]}\|_{E}, \sup_n \|x_n^*\chi_{[1,\infty)}\|_{L^2} \right\} \leq \sup_n \|x_n\|_{Z_E} < \infty,$$

we have $x_n^*\chi_{[0,1]} \in E$ and $x_n^*\chi_{[1,\infty)} \in L^2(0,\infty)$. As a result, $x \in Z_E$ and $\|x\|_{Z_E} = \lim_{n \to \infty} \|x_n\|_{Z_E}$. This means that $Z_E$ has the Fatou property.

It remains to prove that $Z_E$ is separable provided if $E$ is. To this end, in view of [20 Ch.II, §4.5, Theorem 4.8], it suffices to show that each nonnegative function $x \in Z_E$ can be approximated in $Z_E$ by its truncations, i.e., we need to deduce that $\|x - x_n\|_{Z_E} \to 0$ and $\|x - x^n\|_{Z_E} \to 0$ as $n \to \infty$, where $x_n := x\chi_{[0,n]}$ and $x^n := \min(x,n)$, $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Since $E$ and $L^2(0,\infty)$ are separable r.i. spaces, there is $\delta > 0$ such that

$$\max \left\{ \|x^*\chi_{[0,\delta]}\|_{E}, \|x^*\chi_{[1,1+\delta]}\|_{L^2} \right\} < \varepsilon. \quad (20)$$

On the other hand, taking into account that $m\{t > 0 : x(t) > \varepsilon\} < \infty$ and $\|x^*\chi_{[n,\infty)}\|_{L^2(0,\infty)} \to 0$ as $n \to \infty$, we can find a positive integer $N$ satisfying the conditions:

$$m(\{t > N : x(t) > \varepsilon\}) < \delta \quad (21)$$

and

$$\|x^*\chi_{[N,\infty)}\|_{L^2(0,\infty)} < \varepsilon. \quad (22)$$

From definition of the rearrangement of a measurable function and inequality (21) it follows that, for all $n \geq N$,

$$m(\{t > 0 : (x\chi_{[n,\infty)})^*(t) > \varepsilon\}) = m(\{t > n : x(t) > \varepsilon\}) < \delta.$$

Combining this inequality with (20), we have

$$\|(x\chi_{[n,\infty)})^*\chi_{[0,1]}\|_E \leq \|x^*\chi_{[0,\delta]}\|_E + \|(x\chi_{[n,\infty)})^*\chi_{[\delta,1]}\|_E \leq \varepsilon (1 + \|\chi_{[0,1]}\|_E) = 2\varepsilon \quad (23)$$

(because $\|\chi_{[0,1]}\|_E = 1$; see Section 2.1). Moreover, since

$$m(\{t > 0 : x(t)\chi_{[n,\infty)}(t) > x^*(N)\}) \to 0 \quad \text{as} \quad n \to \infty,$$

there exists a positive integer $M > N$ such that for all $n \geq M$

$$m(\{t > 0 : (x\chi_{[n,\infty)})^*(t) > x^*(N)\}) = m(\{t > 0 : x(t)\chi_{[n,\infty)}(t) > x^*(N)\}) < \delta.$$

Hence, from (20) it follows that

$$\|(x\chi_{[n,\infty)})^*\chi_{[1,\infty)}(x\chi_{[n,\infty)})^* > x^*(N)\|_{L^2} \leq \|x^*\chi_{[1,1+\delta]}\|_{L^2} < \varepsilon, \quad n \geq M.$$
On the other hand, in view of (22),
\[ \| (x\chi_{[n,\infty)})^* \chi_{[x\chi_{[n,\infty)})^* \leq x^*(N)} \|_{L^2} \leq \| x^* \chi_{[x^*(N)} \|_{L^2} \leq \| x^* \chi_{[N,\infty)} \|_{L^2} < \varepsilon, \quad n \geq M. \]
Summing up the last inequalities, we have that for all \( n \geq M \)
\[ \| (x\chi_{[n,\infty)})^* \chi_{[1,\infty)} \|_{L^2} \leq \| (x\chi_{[n,\infty)})^* \chi_{[1,\infty)} \chi_{[x\chi_{[n,\infty)})^* > x^*(N)} \|_{L^2} \]
\[ + \| (x\chi_{[n,\infty)})^* \chi_{[x\chi_{[n,\infty)})^* \leq x^*(N)} \|_{L^2} \leq 2\varepsilon. \]
This inequality and (23) imply that \( \| x\chi_{[n,\infty)} \|_{Z_E} \leq 4\varepsilon \) for all \( n \geq M \). Since \( \varepsilon > 0 \) is arbitrary, this yields \( \| x - x_n \|_{Z_E} \to 0 \) as \( n \to \infty \).

Finally, we prove a similar assertion for the upper truncations \( x^n, \quad n \in \mathbb{N} \). Suppose that, as above, \( \delta > 0 \) satisfies condition (20). Then, if a positive integer \( N' \) is sufficiently large, we have \( m(\{ t > 0 : x(t) > N' \}) < \delta \). Combining this inequality with (20), for all \( n \geq N' \) we get
\[ \| x - x^n \|_{Z_E} = \| x\chi_{[x^n]} \|_{Z_E} \leq \| x^* \chi_{[0,\delta]} \|_E < \varepsilon, \]
whence \( \| x - x^n \|_{Z_E} \to 0 \) as \( n \to \infty \). \( \square \)

Let \( \{ A_n \}_{n=1}^\infty \) be a sequence of pairwise disjoint measurable subsets of \( (0, \infty) \) satisfying conditions (2). Moreover, let \( E \) be a r.i. space on \([0,1]\) and \( \phi_E \) its fundamental function. Denoting by \( u_n, \quad n = 1, 2, \ldots \), supported on \([0,1]\) independent symmetrically distributed r.v.’s, which are equimeasurable with the characteristic functions \( \chi_{A_n}, \quad n = 1, 2, \ldots \), we set
\[ f_n := \frac{\chi_{A_n}}{\phi_E(m(A_n))}, \quad g_n := \frac{\chi_{A_n}}{\phi_E'(m(A_n))}, \]
\[ \tilde{f}_n := \frac{u_n}{\phi_E(m(A_n))}, \quad \tilde{g}_n := \frac{u_n}{\phi_E'(m(A_n))}, \quad n = 1, 2, \ldots \]
Since \( \phi_{E'}(t) = t/\phi_E(t), \quad 0 < t \leq 1 \) [20 Ch.II, §4.6], then \( \{ f_n, g_n \} \) and \( \{ \tilde{f}_n, \tilde{g}_n \} \) are biorthogonal systems in \( E \). Also, we denote
\[ \langle f, g \rangle := \int_0^1 f(t)g(t) \, dt, \quad f \in E, g \in E'. \]

**Proposition 3.** Let \( E \) be a r.i. space on \([0,1]\), and let \( S \subseteq \mathbb{N} \) be such that \( \sum_{i \in S} m(A_i) \leq 1 \). Suppose that the mapping
\[ Pf := \sum_{n \in S} \langle f, g_n \rangle \tilde{f}_n \]
is a bounded projection on \( E \). Then, there is a constant \( C > 0 \), which depends only on \( E \) and \( \| P \| \), such that for all \( a_n \in \mathbb{R} \)
\[ \left\| \sum_{n \in S} a_n u_n \right\|_{E'} \leq C \left\| \sum_{n \in S} a_n \chi_{A_n} \right\|_{Z_{E'}}, \]

(26)
Proof. First, we estimate
\[
\left\| \sum_{n \in S} a_n \tilde{g}_n \right\|_{E'} = \sup \left\{ \left\langle \sum_{n \in S} a_n \tilde{g}_n, f \right\rangle : \|f\|_E \leq 1 \right\}
= \sup \left\{ \left\langle \sum_{n \in S} a_n \tilde{g}_n, Pf \right\rangle : \|P\| \leq 1 \right\}
\leq \sup \left\{ \left\langle \sum_{n \in S} a_n \tilde{g}_n, Pf \right\rangle : \|Pf\|_E \leq 1 \right\}.
\]
Moreover,
\[
\left\langle \sum_{n \in S} a_n \tilde{g}_n, Pf \right\rangle = \sum_{n \in S} a_n \left\langle f, \tilde{g}_n \right\rangle = \int_0^\infty \left( \sum_{n \in S} a_n g_n \right) \cdot \left( \sum_{m \in S} \left\langle f, \tilde{g}_m \right\rangle f_m \right) dt,
\]
and since \(f_m\) are disjoint copies of the functions \(\tilde{f}_m, m \in S\), by [17, Theorem 1], there is \(C' > 0\), depending only on \(E\), such that
\[
\left\| \sum_{n \in S} \left\langle f, \tilde{g}_m \right\rangle f_m \right\|_{Z_E} \leq C' \left\| \sum_{m \in S} \left\langle f, \tilde{g}_m \right\rangle \tilde{f}_m \right\|_E = C' \|Pf\|_E.
\]
Hence,
\[
\left\| \sum_{n \in S} a_n \tilde{g}_n \right\|_{E'} \leq \sup \left\{ \int_0^\infty \left( \sum_{n \in S} a_n g_n \right) \cdot \left( \sum_{m \in S} \left\langle f, \tilde{g}_m \right\rangle f_m \right) dt : \left\| \sum_{n \in S} \left\langle f, \tilde{g}_m \right\rangle f_m \right\|_{Z_E} \leq C' \|P\| \right\}.
\]
Since \((Z_E)' = Z_{E'}\), by Lemma 11 the latter inequality yields that for all \(a_n \in \mathbb{R}\) we obtain the inequality
\[
\left\| \sum_{n \in S} a_n \tilde{g}_n \right\|_{E'} \leq C' \|P\| \left\| \sum_{n \in S} a_n g_n \right\|_{Z_{E'}},
\]
which is equivalent to desired estimate (26). □

From Theorem 11 and Proposition 3 it follows

**Theorem 2.** Let \(E\) be a r.i. space on \([0, 1]\) with the Fatou property. Suppose that there exists \(C > 0\) such that for every set \(S \subseteq \mathbb{N}\), with \(\sum_{n \in S} m(A_n) \leq 1\), and all \(a_n \in \mathbb{R}, n \in S\), inequality (3) holds and the projection \(P\) corresponding to such a set \(S \subseteq \mathbb{N}\) (see (24) and (25)) is bounded on \(E\). Then, \(E \approx Z_E\).

4. **Existence of an isomorphic embedding** \(T : U_E \to E\): the case when \(T(\chi_{A_n}), n = 1, 2, \ldots\), are ”almost” disjoint.

As was said in Section 11 if a r.i. space \(E\) and its Köthe dual \(E'\) possess the Kruglov property, then the spaces \(E\) and \(Z_E\) are isomorphic (see [11]). In turn, according to Theorem 11 a r.i. space \(E\) with the Fatou property has the Kruglov property whenever there is an isomorphic embedding of Rosenthal’s space \(U_E\) into \(E\). Moreover, in the proof of the latter result the functions \(T(\chi_{A_n})(= u_n)\),
\[ n = 1, 2, \ldots, \text{ were independent, symmetrically distributed and equimeasurable with the characteristic functions } \chi_{A_n}, n = 1, 2, \ldots. \text{ A natural question appears: Let } T \text{ be an isomorphic embedding of Rosenthal’s space } U_E \text{ into } E. \text{ What we can say about the functions } T(\chi_{A_n}), n = 1, 2, \ldots? \text{ Further, we consider two different cases, when these functions are ”almost” disjoint and independent. As a consequence, we will obtain new examples of r.i. spaces } E \text{ such that } E \not\cong Z_E. \]

We begin with an auxiliary result, which was proved earlier in the separable case by Raynaud (see [25, Proposition 1]). However, for the reader’s convenience we provide here a simple alternative proof of this fact. Let \( G \) denote the separable part of the exponential Orlicz space \( \text{Exp} L^2 \) (i.e., the closure of \( L^\infty \) in \( \text{Exp} L^2 \)).

**Proposition 4.** Let \( E \) be a r.i. space on \([0, 1]\). Suppose that there exists a sequence \( \{x_n\}_{n=1}^\infty \subseteq E \) with \( \|x_n\|_E \simeq \|x_n\|_{L^1}, n = 1, 2, \ldots, \) which is equivalent in \( E \) to the unit vector \( \ell^2 \)-basis. Then, \( E \supseteq G. \)

**Proof.** Clearly, it can be assumed that \( E \not\cong L^1 \). Since \( \{x_n\}_{n=1}^\infty \) is equivalent in \( E \) to the unit \( \ell^2 \)-basis, we have \( x_n \to 0 \) weakly in \( E \) and so \( x_n \to 0 \) weakly in \( L^1 \). Hence, \( \{x_n\}_{n=1}^\infty \) has no convergent subsequences in \( L^1 \). Applying then the well-known result by Aldous and Fremlin [2], we select a subsequence \( \{x_{n_k}\} \subseteq \{x_n\} \) such that for some \( c > 0 \) and all \( a_k \in \mathbb{R} \)

\[
\left\| \sum_{k=1}^\infty a_k x_{n_k} \right\|_{L^1} \geq c \left\| (a_k) \right\|_2.
\]

Combining this inequality with the assumptions and with the embedding \( E \subseteq L^1 \), we conclude that the norms of \( E \) and \( L^1 \) are equivalent on the infinite-dimensional subspace \( \{x_{n_k}\} \) in \( E \).

In other words, the canonical embedding \( I: E \to L^1 \) is not strictly singular. Assuming that \( E \not\supseteq G \), by [7, Theorem 2], we obtain that this embedding is not disjointly strictly singular. This means that there is a sequence of pairwise disjoint functions \( \{h_i\}_{i=1}^\infty \) from \( E \) such that the norms of \( E \) and \( L^1 \) are equivalent on the closed linear span \([h_i]\). But this is a contradiction. Indeed, if the norms of \( E \) and \( L^1 \) were equivalent on the span \([h_i]\) of pairwise disjoint functions \( h_i, i = 1, 2, \ldots \), one can easily check that there exists \( \delta > 0 \) such that for every \( i = 1, 2, \ldots \)

\[
m(\{t \in [0, 1] : |h_i(t)| > \delta \|h_i\|_E\}) > \delta
\]

(see also [18, Theorem 1]). Clearly, the sets

\[
U_i(\delta) := \{t \in [0, 1] : |h_i(t)| > \delta \|h_i\|_E\}, \quad i = 1, 2, \ldots,
\]

are pairwise disjoint and \( m(U_i(\delta)) > \delta \). Hence,

\[
m\left( \bigcup_{i=1}^\infty U_i(\delta) \right) = \sum_{i=1}^\infty m(U_i(\delta)) = \infty,
\]

which is not possible because the union \( \bigcup_{i=1}^\infty U_i(\delta) \) is contained in \([0, 1]\) (other proofs of this and some close results see in [24] and [4, Corollary 3]). \( \square \)
Corollary 1. Suppose $E$ is a separable r.i. space on $[0,1]$ such that $E \not\approx G$. Then, if $E$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ equivalent in $E$ to the unit vector $\ell^2$-basis, there is a disjoint sequence $\{x_n\}_{n=1}^{\infty} \subset E$ with the same property.

Proof. By Proposition 11, we may assume that $\|x_n\|_E/\|x_n\|_{L^1} \to \infty$ as $n \to \infty$. Then, by the Kadec-Pełczyński alternative [18], there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that for some disjoint sequence $\{z_j\} \subset E$ we have

$$\|x_{n_j} - z_j\|_E \to 0 \quad \text{as} \quad j \to \infty.$$ 

Since $\{x_n\}$ is equivalent in $E$ to the unit vector $\ell^2$-basis, applying now the principle of small perturbations (see e.g. [11, Theorem 1.3.9]), we can assume that $\{z_j\}_{j=1}^{\infty}$ is equivalent in $E$ to the $\ell^2$-basis as well. \qed

It is clear that for every r.i. space $E$ on $[0,1]$ Rosenthal’s space $U_E$ (as a subspace of $Z_E$) contains a subspace isomorphic to $\ell^2$. Hence, if $U_E \subsetneq E$, the space $E$ must share the above property. So, if a r.i. space $E$ does not contain a subspace isomorphic to $\ell^2$, $U_E$ cannot be embedded isomorphically into $E$, which implies that $E \not\approx Z_E$. So, if $E$ is a separable r.i. space such that $E \not\approx G$ and it does not contain disjoint sequences equivalent to the unit vector basis of $\ell^2$, then $U_E \subsetneq E$ (see Corollary 1). In particular, if $p > 2$, the separable part $(\exp L^p)_0$ of the exponential Orlicz space $\exp L^p$ has the latter properties since each disjoint sequence of this space contains a subsequence equivalent to the unit vector basis of $c_0$ (see, e.g., [27]). As a result, we obtain the simplest examples of r.i. spaces $E$ such that $E \not\approx Z_E$.

Further, it is known that, if a r.i. space $E$ is not equal to $L^\infty(0,1)$ up to an equivalent renorming, Rosenthal’s space $U_E$ contains a complemented subspace of $Z_E$ isomorphic to $\ell^2$ [16, Lemma 8.7 and subsequent Remark]. Therefore, if we know that $E \approx Z_E$, then $E$ must contain a complemented subspace, which is isomorphic to $\ell^2$ as well. According to [16, Proposition 8.17], there are some Orlicz spaces, “close” to $L^1$, that fail to contain such subspaces and hence that are not isomorphic to $Z_E$ (in fact, they are not isomorphic to any r.i. space on $(0,\infty)$; see [16, Corollary 8.15]). The simplest example of such a space is the Orlicz space $L_{F_\alpha}$, where $F_\alpha(u)$ is an Orlicz function equivalent to the function $u \log^\alpha u$ for large $u > 0$, where $0 < \alpha < 1/2$ (see also a discussion in the concluding part of Section 1).

Here, we prove results showing that the existence of complemented subspaces isomorphic to $\ell^2$ does not guarantee that $U_E$ is isomorphically embedded into $E$ and, a fortiori, that $E \approx Z_E$. Specifically, we will provide examples of Lorentz spaces containing plenty of complemented subspaces isomorphic to $\ell^2$, but without subspaces isomorphic to the corresponding Rosenthal’s spaces.

First, we introduce a lattice version of a notion from [26, see p. 293]. We say that a Banach lattice $E$ has the disjoint $Q_2$-property (in brief, $E \in DQ_2$) whenever there is a constant $C_E > 0$ (depending only on $E$) such that given a disjoint sequence $\{h_n\}$ in $E$ with $\|h_n\|_E = 1$, which is equivalent to the unit vector $\ell^2$-basis, there exists a subsequence $\{h_n\} \subset \{h_n\}$ that is $C_E$-equivalent to the unit vector $\ell^2$-basis.
Let a Banach lattice $E$ have the $\mathcal{D}Q_2$-property (with the constant $C_E$). Suppose that $\{x_n\}_{n=1}^{\infty} \subset E$ is a disjoint sequence, which is equivalent to the unit $\ell^2$-basis and semi-normalized (i.e., $C^{-1} \leq \|x_n\|_E \leq C$ for some $C > 0$ and all $n = 1, 2, \ldots$). Then, it is obvious that $\{x_n\}_{n=1}^{\infty}$ contains a subsequence, which is $C'_E$-equivalent to the unit vector $\ell^2$-basis, where $C'_E := C_F \cdot C$.

**Theorem 3.** Let $E$ be a separable r.i. space, $E \in \mathcal{D}Q_2$. If $U_E \subset \subset E$, then $E \supseteq G$.

**Proof.** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of disjoint subsets of $(0, \infty)$ satisfying conditions (2). Then, for every $l \in \mathbb{N}$, there are pairwise disjoint sets $S_i^l \subset \mathbb{N}$, $i = 1, 2, \ldots$, such that

$$\sum_{n \in S_i^l} m(A_n) = \frac{1}{l}.$$  

Denote $B_i^l := \bigcup_{n \in S_i^l} A_n$, $i = 1, 2, \ldots$. Consider the block-basis $\{\chi_{B_i^l}\}_{i=1}^{\infty}$ of $\{\chi_{A_n}\}_{n=1}^{\infty}$. According to definition of the norm in $Z_E$ (see (3)), each set consisting of $l$ distinct functions $\chi_{B_i^l}$ is isometrically equivalent in $Z_E$ to the set $\{\chi((i-1)/l, i/l)\}_{i=1}^{l}$ in $E$, i.e., for all distinct $i_1, \ldots, i_l \in \mathbb{N}$ and $a_j \in \mathbb{R}$

$$\sum_{j=1}^{l} a_j \chi_{B_j^l} \bigg\|_{Z_E} = \left\| \sum_{i=1}^{l} a_i \chi((i-1)/l, i/l) \right\|_E$$  

(cf. [26] Corollary 8).

On the other hand, the sequence $\{\chi_{B_i^l}\}_{i=1}^{\infty}$ is $C_l$-equivalent in $Z_E$ to the unit vector $\ell^2$-basis. Indeed, for arbitrary $a_i \in \mathbb{R}$ there is a set $S'_i \subset \mathbb{N}$ with card $S'_i = l$, such that, with constants depending of $l$, we have

$$\sum_{i=1}^{\infty} a_i \chi_{B_i^l} \bigg\|_{Z_E} = \sum_{i \in S'_i} a_i \chi_{B_i^l} \bigg\|_E + \sum_{i \not\in S'_i} a_i \chi_{B_i^l} \bigg\|_{L^2} < C_l \left( \sum_{i \in S'_i} a_i \chi_{B_i^l} \bigg\|_{L^2} + \sum_{i \not\in S'_i} a_i \chi_{B_i^l} \bigg\|_{L^2} \right)$$

$$2C_l \sum_{i=1}^{\infty} a_i \chi((i-1)/l, i/l) \bigg\|_{L^2} = \frac{1}{\sqrt{l}} \| (a_i) \|_2.$$  

From the hypothesis, there exists an isomorphism $T: U_E \to E$. Then, if $y_i^l := T(\chi_{B_i^l})$, $i = 1, 2, \ldots$, by (28), we have

$$\sum_{i=1}^{\infty} a_i y_i^l \bigg\|_E \|T\| \sum_{i=1}^{\infty} a_i \chi_{B_i^l} \bigg\|_{Z_E} \simeq \frac{1}{\sqrt{l}} \| (a_i) \|_2,$$

with constants depending on $l$ and $\|T\|$.

In the case when $\|y_i^l\|_E \simeq \|y_i^l\|_{L_1}$, $i = 1, 2, \ldots$, for some $l \in \mathbb{N}$, all the conditions of Proposition 4 are satisfied, and so the desired result follows.
Assume, conversely, that for each \( l \in \mathbb{N} \) we have
\[
\lim \inf_{i \to \infty} \frac{\|y_i^l\|_{L^1}}{\|y_i^l\|_E} = 0.
\]
Denoting \( u_i^l := (1/\phi_E(1/l))y_i^l \), \( i, l = 1, 2, \ldots \), where \( \phi_E \) is the fundamental function of the space \( E \), we get
\[
\|T\|^{-1} \leq \|u_i^l\|_E \leq \|T\|, \quad i, l = 1, 2, \ldots,
\]
and clearly for every \( l = 1, 2, \ldots \)
\[
\lim \inf_{i \to \infty} \frac{\|u_i^l\|_{L^1}}{\|u_i^l\|_E} = 0.
\]
Then again, by the Kadec-Pelczyński alternative \[18\], for each \( l = 1, 2, \ldots \) there is subsequence \( \{u_i^l\} \subseteq \{u_i^l\} \), where a sequence \( \{i_j\} \) depends on \( l \in \mathbb{N} \), such that for some disjoint sequence \( \{z_j^l\} \subseteq E \) it holds
\[
\|u_{i_j}^l - z_j^l\|_E \to 0 \quad \text{as} \quad j \to \infty.
\]
Applying the principle of small perturbations (see e.g. \[1\] Theorem 1.3.9), we can assume that \( \{z_j^l\}_{j=1}^\infty \) is 2-equivalent in \( E \) to the sequence \( \{u_{i_j}^l\}_{j=1}^\infty \), and so, by \[30\],
\[
(2\|T\|)^{-1} \leq \|z_j^l\|_E \leq 2\|T\|, \quad j, l = 1, 2, \ldots,
\]
which means that for every \( l = 1, 2, \ldots \) the sequence \( \{z_j^l\}_{j=1}^\infty \) is semi-normalized with a constant independent of \( l \). Moreover, taking into account \[29\], we see that \( \{z_j^l\}_{j=1}^\infty \) is equivalent in \( E \) to the unit vector \( \ell^2 \)-basis (with constants depending on \( l = 1, 2, \ldots \)). Since \( E \in \mathcal{D}Q_2 \), for each \( l \in \mathbb{N} \) the sequence \( \{z_j^l\}_{j=1}^\infty \) contains a further subsequence \( \{z_{j_k}^l\}_{k=1}^\infty \) (where \( \{j_k\} \) also depends on \( l \in \mathbb{N} \)) that is \( C'_E \)-equivalent to the unit vector \( \ell^2 \)-basis. Clearly, then the sequence \( \{u_{i_{j_k}}^l\}_{k=1}^\infty \) is \( 2C'_E \)-equivalent to the same basis, i.e.,
\[
\sum_{k=1}^\infty a_k u_{i_{j_k}}^l \bigg|_E \lesssim 2C'_E \|a_k\|_2.
\]
Moreover, from \[27\] and the above notation it follows that
\[
\sum_{k=1}^l a_k u_{i_{j_k}}^l \bigg|_E \lesssim \frac{\|T\|}{\phi_E(1/l)} \sum_{k=1}^l a_k \chi_{B_{i_{j_k}}^l} \bigg|_{Z_E} = \frac{1}{\phi_E(1/l)} \sum_{j=1}^l a_j \chi_{((j-1)/l, j/l)} \bigg|_E
\]
for all \( a_j \in \mathbb{R} \). Combining this with \[31\], we obtain
\[
\sum_{j=1}^l a_j \chi_{((j-1)/l, j/l)} \bigg|_E \asymp \phi_E(1/l) \left( \sum_{j=1}^l a_j^2 \right)^{1/2}, \quad l \in \mathbb{N},
\]
with constants independent of \( l \in \mathbb{N} \) and \( a_j \in \mathbb{R} \).
Next, one can easily check that equivalence (32) implies that \( \phi_E(t) \approx t^{1/2}, 0 < t \leq 1 \). Indeed, for every \( l \in \mathbb{N} \) we have

\[
\chi(0,1) = \sum_{i=1}^{l} \chi((i-1)/l, i/l],
\]

whence, by (32),

\[
(33) \quad 1 = \left\| \chi(0,1) \right\|_E \approx \sqrt{l} \phi_E(1/l).
\]

Therefore, \( \phi_E(1/l) \approx 1/\sqrt{l}, l \in \mathbb{N} \). Combining this together with the quasi-concavity of \( \phi_E \), we obtain that \( \phi_E(t) \approx \sqrt{t}, 0 < t \leq 1 \). As a consequence, from (32) it follows that

\[
\left\| \sum_{j=1}^{l} a_j \chi((j-1)/l, j/l) \right\|_E \approx \frac{1}{\sqrt{l}} \left( \sum_{j=1}^{l} a_j^2 \right)^{1/2}
\]

\[= \left\| \sum_{j=1}^{l} a_j \chi((j-1)/l, j/l) \right\|_{L^2}, \quad l \in \mathbb{N},\]

with constants independent of \( l \in \mathbb{N} \) and \( a_j \in \mathbb{R} \). Clearly, this implies that \( E \approx L^2 \), and the desired result follows. \( \Box \)

**Theorem 4.** Let \( E \) be a separable r.i. space on \([0,1]\) such that both \( E \) and \( E' \) have the \( DQ_2 \)-property. If \( E \approx Z_E \), then \( G \subseteq E \subseteq G' \).

**Proof.** It follows from Theorem 3 that we need only to prove that \( E \subseteq G' \).

Suppose that \( T \) is an isomorphism from \( Z_E \) onto \( E \). Clearly, then \( T^* \) is an isomorphism from \( E^* \) onto \( (Z_E)^* \). Since \( E \) is separable, we have \( E^* = E' \) and, by Lemma 1, \( Z_E \) is a separable space with \( (Z_E)^* = (Z_E)' = Z_{E'} \). Thus, \( E' \approx Z_{E'} \), and hence, by Theorem 3, \( E' \supseteq G \), which implies \( E \subseteq E'' \subseteq G' \). \( \Box \)

Let \( 1 \leq p \leq \infty \). Recall that a Banach lattice \( E \) is said to be \( p \)-disjointly homogeneous (\( p \text{-DH} \)) if every disjoint normalized sequence contains a subsequence equivalent to the unit vector \( \ell^p \)-basis (\( c_0 \)-basis if \( p = \infty \)). Moreover, \( E \) is called uniformly \( p \text{-DH} \) if there is a constant \( C_E \), which depends only on \( E \), such that from every disjoint normalized sequence \( \{x_n\} \) we can select a subsequence \( \{x_{n_k}\} \subseteq \{x_n\} \), which is \( C_E \)-equivalent to the \( \ell^p \)-basis (for a detailed account of these properties see the survey \([13]\) and references therein).

Every \( p \text{-DH} \) Banach lattice for \( 1 < p < \infty \) is reflexive \([3]\). Also, it is obvious that each uniformly \( 2 \text{-DH} \) lattice has the \( DQ_2 \)-property.

**Theorem 5.** Let \( E \) be a uniformly \( 2 \text{-DH} \) r.i. space on \([0,1]\). Suppose that at least one of the following conditions holds:

(i) Rosenthal’s space \( \mathcal{U}_E \) is isomorphically embedded into the space \( E \);
(ii) \( E \) is isomorphic to a r.i. space on \((0, \infty)\).

Then, \( E \supseteq G \).

Moreover, if additionally the Köthe dual \( E' \) is uniformly \( 2 \text{-DH} \) and \( E' \) satisfies at least one of the conditions (i) and (ii), then \( G \subseteq E \subseteq G' \).
Proof. Since $E$ is a uniformly 2-$\mathcal{D}\mathcal{H}$, then the condition (i) implies the embedding $E \supseteq G$ by Theorem 3.

Let now $E$ be isomorphic to a r.i. space $Y$ on $(0, \infty)$. Denote $x_{n,i} := \chi_{[(i-1)/n,i/n)}$, $n, i \in \mathbb{N}$, and assume first that, for every $n \in \mathbb{N}$, the sequence $\{x_{n,i}\}_{i=1}^{\infty}$ is equivalent in $Y$ to the unit vector $\ell^2$-basis. Then, if $T$ is an isomorphism of $Y$ onto $E$, each sequence $\{y_{n,i}\}_{i=1}^{\infty}$, $n \in \mathbb{N}$, where $y_{n,i} := T(x_{n,i})$, $n, i \in \mathbb{N}$, is equivalent in $E$ to the unit vector $\ell^2$-basis as well. In the case when $\|y_{n,i}\|_E \approx \|y_{n,i}\|_{L^1}$, $i = 1, 2, \ldots$, for some $n \in \mathbb{N}$, the desired result follows, as above, by Proposition 4. Hence, it remains to consider the case when for each $n \in \mathbb{N}$ we have

$$\liminf_{i \to \infty} \frac{\|y_{n,i}\|_{L^1}}{\|y_{n,i}\|_E} = 0.$$ 

Then, denoting $u_{n,i} := (1/\phi_E(1/n))y_{n,i}$, $i, n = 1, 2, \ldots$ and reasoning as in the proof of Theorem 4 we can find, for every $n \in \mathbb{N}$, a subsequence $\{u_{n,i}\}_{i=1}^{\infty}$, which is 2-equivalent in $E$ to some disjoint semi-normalized (with a constant independent of $n$) sequence $\{z_{n,j}\}_{j=1}^{\infty}$. Thanks to the uniform 2-$\mathcal{D}\mathcal{H}$ property of $E$, passing if it necessary to a further subsequence, we can assume that there is a constant $D' > 0$ such that for every $n \in \mathbb{N}$ the sequence $\{u_{n,i}\}_{i=1}^{\infty}$ is $D'$-equivalent in $Y$ to the unit vector $\ell^2$-basis. On the other hand, for every $n \in \mathbb{N}$ the sequence $\{y_{n,i}\}_{i=1}^{\infty}$ (together with $\{x_{n,i}\}_{i=1}^{\infty}$ in $Y$) is $B$-symmetric in $E$ for some $B > 0$. Hence, for every $n \in \mathbb{N}$ the sequence $\{u_{n,i}\}_{i=1}^{\infty}$ and hence the sequence $\{(1/\phi_E(1/n))x_{n,i}\}_{i=1}^{\infty}$ is $D$-equivalent in $Y$ to the unit vector $\ell^2$-basis for some $D > 0$, i.e.,

$$D^{-1}\phi_E(1/n)\|(a_i)\|_2 \leq \left\| \sum_{i=1}^{\infty} a_ix_{n,i} \right\|_Y \leq D\phi_E(1/n)\|(a_i)\|_2$$

for all $n \in \mathbb{N}$ and $(a_i) \in \ell^2$. Clearly, this implies that $Y = L^2(0, \infty)$ (see the concluding part of the proof of Theorem 4). Since $E \approx Y$ by condition, we infer that $E = L^2[0, 1]$ (with equivalence of norms), and so in this case everything is done.

Conversely, suppose that the sequence $\{y_{1,i}\}_{i=1}^{\infty}$ is not equivalent in $Y$ to the unit vector $\ell^2$-basis; then, the same is true also for all sequences $\{y_{n,i}\}_{i=1}^{\infty}$, $n \in \mathbb{N}$. As was said above, for every $n \in \mathbb{N}$ the sequence $\{y_{n,i}\}_{i=1}^{\infty}$ is $B$-symmetric in $E$ for some $B > 0$. Moreover, since $\{x_{n,i}\}_{i=1}^{\infty}$, $n \in \mathbb{N}$, spans an 1-complemented subspace in $Y$ (see e.g. [20] Ch. II, §3.2]), we can assume that, for every $n \in \mathbb{N}$, the span $[y_{n,i}, i \in \mathbb{N}]$ is a $B$-complemented subspace in $E$. Then, according to [16] Lemma 8.10, there is a constant $A' > 0$ such that for every $n \in \mathbb{N}$ the sequence $\{y_{n,i}\}_{i=1}^{\infty}$ is $A'$-equivalent in $E$ to a disjoint sequence in $E$. Since the latter space is uniformly 2-$\mathcal{D}\mathcal{H}$ and $\{x_{n,i}\}_{i=1}^{\infty}$ is a $B$-symmetric sequence in $E$, we conclude that there is a constant $A > 0$ such that for every $n \in \mathbb{N}$ the sequence $\{(1/\phi_E(1/n))x_{n,i}\}_{i=1}^{\infty}$ is $A$-equivalent in $Y$ to the unit vector $\ell^2$-basis. As above, this yields that $Y = L^2(0, \infty)$ and hence $E = L^2[0, 1]$ (with equivalence of norms), which completes the proof. \qed
It is well known that every Lorentz space \( \Lambda_2(\varphi) \) has the uniform 2-\( \mathcal{D} \mathcal{H} \) property (see e.g. [13, Theorem 5.1]). Therefore, since the embedding \( \Lambda_2(\varphi) \supseteq G \) is equivalent to the condition \( \sum_{k=1}^{\infty} \varphi(e^{-k}) < \infty \) (see e.g. [6, Lemma 3]), we get the following consequence of Theorem 5.

**Corollary 2.** Let \( \varphi \) be an increasing concave function on \([0,1]\) with \( \varphi(0) = 0 \). Suppose that at least one of the following conditions holds:

(i) Rosenthal’s space \( U_{\Lambda_2(\varphi)} \) is isomorphically embedded into the space \( \Lambda_2(\varphi) \);

(ii) the space \( \Lambda_2(\varphi) \) isomorphic to a r.i. space on \((0, \infty)\).

Then, \( \sum_{k=1}^{\infty} \varphi(e^{-k}) < \infty \).

In particular, we get the following new examples of r.i. spaces on \([0,1]\) that are not equivalent to any r.i. spaces on \((0, \infty)\).

**Corollary 3.** Let \( 0 < \alpha \leq 1 \). Then, the Lorentz space \( \Lambda_2(\log^{-\alpha}(e/u)) \) has the following properties:

(a) any disjoint sequence in \( \Lambda_2(\log^{-\alpha}(e/u)) \) contains a subsequence 2-equivalent to the unit vector basis of \( l^2 \), which spans a 2-complemented subspace in \( \Lambda_2(\log^{-\alpha}(e/u)) \);

(b) Rosenthal’s space \( U_{\Lambda_2(\log^{-\alpha}(e/u))} \) fails to be isomorphically embedded into \( \Lambda_2(\log^{-\alpha}(e/u)) \) and \( \Lambda_2(\log^{-\alpha}(e/u)) \) is not isomorphic to any r.i. space on \((0, \infty)\).

5. **Existence of an isomorphic embedding** \( T: U_E \to E \): the case when \( T(\chi_{A_n}), n = 1, 2, \ldots, \) are independent.

In the final section, we treat the special case when there is an isomorphic embedding \( T: U_E \to E \) such that the functions \( T(\chi_{A_n}), n = 1, 2, \ldots, \) are independent symmetrically distributed r.v.’s.

Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of disjoint measurable subsets of \((0, \infty)\) satisfying conditions (2). In the same way as in the beginning of the proof of Theorem 3 for every \( m \in \mathbb{N} \), we find pairwise disjoint sets \( S^i \subseteq \mathbb{N}, i = 1, 2, \ldots, \) such that \( \sum_{m \in S^i} m(A_n) = 1/l \) and denote \( B^i = \bigcup_{m \in S^i} A_n, i = 1, 2, \ldots \).

Next, suppose that \( E \) is a r.i. space such that \( U_E \) is isomorphically embedded into \( E, T: U_E \to E \) is an isomorphism, \( y^i_l := T(\chi_{B^i}) \), \( i, l \in \mathbb{N} \). In contrast to the preceding section, we assume that sequences \( \{y^i_l\}_{l=1}^{\infty}, l \in \mathbb{N} \), do not contain “almost” disjoint subsequences, which means (see the proof of Theorem 3) that \( \| y^i_l \|_E \asymp \| y^i_l \|_{L^1}, i = 1, 2, \ldots, \) for each \( l \in \mathbb{N} \). Then, it is easy to check (see also [15]) that for every \( l \in \mathbb{N} \) there exists a constant \( \varepsilon_l > 0 \) such that

\[
m\left( \{ t : |y^i_l(t)| > \varepsilon_l \| y^i_l \|_E \} \right) \geq \varepsilon_l.
\]

However, we will need the following stronger condition: there are \( \alpha, \beta, \gamma > 0 \), an infinite sequence \( \{l_k\}_{k=1}^{\infty} \subset \mathbb{N} \), and a sequence of sets \( F_k \subseteq \mathbb{N}, k = 1, 2, \ldots, \) such that \( \gamma l_k \leq \text{card} F_k \leq l_k \) and for each \( i \in F_k \)

\[
m\left( \{ t : |y^i_{l_k}(t)| > \alpha \} \right) \geq \frac{\beta}{l_k}.
\]

Furthermore, let us consider the family \( \{B^i_{l_k}, i \in F_k, k \in \mathbb{N} \} \). One can readily check now that definition of the sets \( B^i_l, i, l \in \mathbb{N} \), and the conditions imposed on
the sets $F_k$, $k \in \mathbb{N}$, assure that the latter family satisfies requirements (2). Since Rosenthal’s space $\mathcal{U}_E$ is invariant (up to isomorphism) on the particular choice of a sequence of sets satisfying (2)\cite{16} Lemma 8.7, without loss of generality, we can replace the initial sequence $\{A_n\}_{n=1}^\infty$ with the family $\{B_{i_k}^k, i \in F_k, k \in \mathbb{N}\}$.

**Theorem 6.** Let $E$ be a r.i. space on $[0, 1]$ such that there exists an isomorphic embedding $T: \mathcal{U}_E \to E$. Suppose that there is a sequence $\{l_k\}_{k=1}^\infty \subset \mathbb{N}$ such that the functions $y_{i_k}^k := T(\chi_{B_{i_k}^k})$, $k, i \in \mathbb{N}$, are independent symmetrically distributed r.v.’s satisfying the above conditions (34). Then, the Kruglov operator $K$ is bounded from $E$ into $E''$.

Moreover, there is a constant $C > 0$ such that

$$(35) \quad \varphi_E \left( \frac{\beta}{2l_k} \right) \leq \frac{C}{l_k}, \quad k = 1, 2, \ldots,$$

where $\varphi_E$ is the fundamental function of the space $E$.

**Proof.** First, for each $k = 1, 2, \ldots$, we compare the finite sequences $\{y_{i_k}^k\}_{i \in F_k}$ and $\{u_{i_k}^k\}_{i \in F_k}$, where $u_{i_k}^k$ are, as above, independent symmetrically distributed r.v.’s equimeasurable with the characteristic functions $\chi_{B_{i_k}^k}$, $k, i = 1, 2, \ldots$. From (34) it follows that for all $\tau > 0$

$$m \left( \{ t : |y_{i_k}^k(t)| > \tau \} \right) \geq \beta m \left( \{ t : |u_{i_k}^k(t)| > \tau \} \right), \quad i \in F_k, k = 1, 2, \ldots$$

Hence, applying the result of Kwapien-Rychlik,\cite{28} Ch.V, Theorem 4.4.], for all $\tau > 0$ and $a_{i_k}^k \in \mathbb{R}$, we get

$$m \left( \left\{ t : \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k u_{i_k}^k(t) > \tau \right\} \right) \leq \frac{2}{\beta} m \left( \left\{ t : \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k y_{i_k}^k(t) > \beta \alpha \tau \right\} \right).$$

So, by\cite{20} Ch.II, §4.3, Corollary 2],

$$\left\| \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k u_{i_k}^k \right\|_E \leq \frac{2}{\beta^2 \alpha} \left\| \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k l_{i_k} \right\|_E.$$

On the other hand, since $T$ is an isomorphism, we have

$$(36) \quad \left\| \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k y_{i_k}^k \right\|_E \leq \frac{\|T\|}{\beta} \left\| \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k \chi_{B_{i_k}^k} \right\|_{Z_E}.$$

Combining the last inequalities, we infer that

$$\left\| \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k u_{i_k}^k \right\|_E \leq \frac{2\|T\|}{\beta^2 \alpha} \left\| \sum_{k=1}^\infty \sum_{i \in F_k} a_{i_k}^k \chi_{B_{i_k}^k} \right\|_{Z_E}.$$

Applying now Theorem I (to the family $\{B_{i_k}^k, i \in F_k, k \in \mathbb{N}\}$), we complete the proof of the first assertion.
Further, since card $F_k \leq l_k$ and $m(B^k_i) = 1/l_k$, from (36) it follows that
\[
\left\| \sum_{i \in F_k} y^k_i \right\|_E \leq C' \left\| \sum_{i \in F_k} \chi_{B^k_i} \right\|_E \leq C', \quad k = 1, 2, \ldots
\]
Moreover, taking into account the fact that $y^k_i$, $i \in F_k$, are independent symmetrically distributed r.v.’s, the inequality card $F_k \geq \gamma l_k$ and (34), we obtain
\[
\left\| \sum_{i \in F_k} y^m_k \right\|_E \geq \alpha m_k \cdot \left\| \chi_{\bigcap_{i \in F_k} \{y^m_k \geq \alpha\}} \right\|_E
\]
\[
= \alpha \gamma m_k \cdot \varphi_E \left( \prod_{i \in F_k} m(\{y^m_k \geq \alpha\}) \right)
\]
\[
\geq \alpha \gamma m_k \cdot \varphi_E \left( \left( \frac{\beta}{2m_k} \right)^{\gamma m_k} \right).
\]
Combining these inequalities, we obtain (35).

\[ \square \]

Corollary 4. Let $E$ be the exponential Orlicz space $\text{Exp}L^p$, $p > 0$. Then, there exists an isomorphic embedding $T: U_E \to E$, satisfying the conditions of Theorem 6 if and only if $0 < p \leq 1$.

Proof. One can easily check that, for $E = \text{Exp}L^p$, we have $\varphi_E(u) \asymp \log^{-1/p}(e/u)$, $0 < u \leq 1$. Therefore, a direct calculation shows that (35) is fulfilled in this case if and only if $0 < p \leq 1$. Moreover, if $0 < p \leq 1$, the space $\text{Exp}L^p$ has the Kruglov property (see [12, the beginning of §2.4] and [10, 4.3.1]), which implies that there exists an isomorphic embedding $T: U_E \to E$, satisfying the conditions of Theorem 6 (indeed, we take $u^m_i$ for $y^m_i$, an arbitrary sequence of positive integers $\{m_k\}_{k=1}^\infty$ and any set of cardinality $m_k$ for $F_k$, $k = 1, 2, \ldots$). Thus, the desired result follows.

\[ \square \]

References

[1] F. Albiac, N. J. Kalton, Topics in Banach space theory (Springer, New York, 2006).
[2] D. Aldous and D. Fremlin, Colacunary sequences in $L$-spaces, Studia Math. 71 (1982), 297–304.
[3] C. D. Aliprantis and O. Burkinshaw, Positive operators, Springer, 2006.
[4] S. V. Astashkin, Disjointly strictly singular inclusions of symmetric spaces, Math. Notes 65 (1999), 3–12.
[5] S. V. Astashkin, Rademacher series and isomorphisms of rearrangement invariant spaces on the semi-axis, J. Funct. Anal. 260 (2010), 195–207.
[6] S. V. Astashkin, Compact and strictly singular operators in rearrangement invariant spaces and Rademacher functions, Positivity (to appear).
[7] S. V. Astashkin, F. L. Hernández, and E. M. Semenov, Strictly singular inclusions of rearrangement invariant spaces and Rademacher spaces, Studia Math. 193 (2009), 269–283.
[8] S. V. Astashkin and F. A. Sukochev, Sums of independent random variables in rearrangement invariant spaces: an operator approach, Isr. J. Math., 145 (2005), 125–156.
[9] S. V. Astashkin and F. A. Sukochev, Series of independent, mean zero random variables in rearrangement-invariant spaces having the Kruglov property, J. Math. Sci. (N.Y.) 148 (2008), 795–809.
[10] S. V. Astashkin and F. A. Sukochev, *Independent functions and the geometry of Banach spaces*, Russian Math. Surveys 65 (2010), 1003–1081.

[11] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[12] M. Sh. Braverman, *Independent random variables and rearrangement invariant spaces*, London Math. Soc. Lecture Note Ser., 194, Cambridge Univ. Press, Cambridge 1994.

[13] T. Fiegel, W. B. Johnson, and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure, with applications to Lorentz function spaces*, J. Approx. Theory 13 (1975), 395–412.

[14] J. Flores, F. L. Hernández, and P. Tradacete, *Disjointly homogeneous Banach lattices and applications*. Ordered Structures and Applications: Positivity VII. Trends in Mathematics, Springer, 179–201 (2016).

[15] P. Hitczenko and S. Montgomery-Smith, *Measuring the magnitude of sums of independent random variables*, Ann. Probab. 29 (2001), 447–466.

[16] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc., 19 (1979) 298 pp.

[17] W. B. Johnson and G. Schechtman, *Sums of independent random variables in rearrangement invariant function spaces*, Ann. Probab. 17 (1989), 789–808.

[18] M. I. Kadec and A. Pelczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1961/1962), 161–176.

[19] M. A. Krasnoselskii and Ya. B. Rutikii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

[20] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators* (Amer. Math. Soc., Providence R. I., 1982).

[21] V. M. Kruglov, *A remark on the theory of infinitely divisible laws*, Teor. Veroyatn. i Primezn. 15 (1970), 331–336 (in Russian).

[22] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces vol. II* (Springer-Verlag, Berlin, 1979).

[23] B. S. Mityagin, *The homotopy structure of the linear group of a Banach space*, Russian Math. Surveys 25 (1970), 59–103.

[24] S. Ya. Novikov, *A characteristic of subspaces of a symmetric space*, in: Studies in the Theory of Functions of Several Variables (in Russian), Yaroslavl State Univ. (1980), 140–148.

[25] Y. Raynaud, *Complemented Hilbertian subspaces in rearrangement invariant function spaces*, Illinois J. Math. 39 (1995), 212–250.

[26] H. P. Rosenthal, *On the subspaces of L_p (p > 2) spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273–303.

[27] E. V. Tokarev, *On subspaces of some symmetric spaces*, Teor. Funkcii, Functional. Anal. i Prilozhen. 24(1975), 156–161 (in Russian).

[28] N. N. Vakhania, V. I. Tarieladze and S. A. Chobanyan, *Probability distributions in Banach spaces*, Kluwer Academic Publ. (1991).