Study on total irregularity of graphs

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Abstract: The total irregularity of a graph \( G \) is defined by

\[
\text{irr}(G) = \frac{1}{2} \sum_{(u,v) \in E(G)} \left| d_G(u) - d_G(v) \right|
\]

For an ordered degree sequence of \( V(G) = \{v_1, v_2, \ldots, v_n\} \) with \( d(v_1) \leq d(v_2) \leq \ldots \leq d(v_n) \), \( irr_t(G) \) can be expressed in the form

\[
irr_t(G) = \sum_{i>j} (d(v_i) - d(v_j)) \text{irr}_t(G) = \sum_{i>j} (d(v_i) - d(v_j))
\]

where \( d_G(x) \) is the degree of \( x \in V(G) \). An edge \( e \in E(G) \) is said to be total irregular positive (negative, stable) inner edge if \( irr_t(G) \) \( \geq \) \( irr_t(G) \) \( \geq \) \( irr_t(G) \) respectively. Total irregular positive inner edge is denoted by TIPI edge. Similarly we use the notations TINI, TISI suitably. A graph \( G \) is called total irregular positive (negative, stable) inner graph if all the edges \( e \in E(G) \) are total irregular positive (negative, stable) inner edges; otherwise \( G \) is called a total irregular mixed inner graph. Total irregular positive inner graph is denoted by TIPI graph. Similarly we use the notations TINI, TISI suitably.

In this paper, we prove that the Complete graph \( K_{m,n} \) \( m \neq n \) and \( m, n \geq 2 \) is a TIPI graph and the Star graph \( S_n \) \( n \geq 3 \) is a TINI graph.

Keywords: Total irregularity, total irregular positive (negative, stable) inner edge, total irregular positive (negative, stable) inner graph.

1. Introduction

Let \( G \) be a simple undirected graph of order \( n = |V(G)| \) and size \( m = |E(G)| \). For \( v \in V(G) \), the degree of \( v \), denoted by \( d_G(v) \), is the number of edges incident to \( v \). In 1997, Michael O. Albertson [4] introduce the concept of “imbalance of an edge”, denoted by \( \text{imb}_G(e) \) and sum of all imbalances of the edges in a graph is called “irregularity” of a graph and it is denoted by \( \text{irr}(G) \).

Inspired by the structure and meaning of irregularity, in 2014, H. Abd, S. Brandt and D. Dimitrov [3] introduce a new irregularity measure, called the total irregularity. Furthermore, they showed that \( \text{irr}_t(T) \leq (n-2)\text{irr}(T) \) for any tree \( T \).

In 2015, Darko Dimitrov and RisteŠkrekovski [4] derived relation between irregularity and total irregularity for a connected graph \( G \) with \( n \) vertices that is \( \text{irr}_t(G) \leq \frac{n^2}{4} \text{irr}(G) \).

2 Preliminaries

Definition 2.1

A graph \( G \) is a bipartite graph if the vertex set \( V(G) \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that every edge of \( G \) join a vertex of \( V_1 \) and a vertex of \( V_2 \).

Example 2.1
For the above graph $G$, if $V_1(G) = \{v_2, v_4\}$ and $V_2(G) = \{v_3, v_5\}$, then $G$ is a bipartite graph.

**Definition 2.2**

A Complete bipartite graph $G$ is a simple bipartite graph with partition $(V_1(G), V_2(G))$ in which each vertex of $V_1(G)$ is joined to each vertex of $V_2(G)$. If $V_1(G) = r$ and $V_2(G) = s$, then such graph is denoted by $K_{r,s}$.

**Example 2.2**

\[ K_{2,3} \]

**Definition 2.3**

In a complete bipartite graph $K_{r,s}$, if either $r = 1$ or $s = 1$, then $K_{r,s}$ is called a star graph.

**Example 2.3**

[Star graph $K_{1,8}$]

**Definition 2.4**

The Friendship graph $F_n$ can be constructed by joining $n$ copies of the cycle graph $C_3$ with a common vertex.

**Example 2.4**

\[ F_2: \]

**3. Main Results**

**Lemma 3.1 (Edge Deletion Lemma)**

If $(u,v)$ is in $E(G)$, set $G'' = G - (u,v)$.

- a) If $\deg(u) = \deg(v)$, then $\text{irr}_c(G'') = \text{irr}_c(G) + 2[\Deg^c(u) + \Deg^c(v) - \Deg^> (v)]$.

- b) If $\deg(u) > \deg(v)$ and $\deg(v) \neq 1$, then
\[ \text{irr}_t(G') = \text{irr}_t(G) + [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)] + [\text{Deg}^- (v) + \text{Deg}^-(v) - \text{Deg}^+ (v)] \]

c) If \( \text{deg}(u) > \text{deg}(v) \) and \( \text{deg}(v) = 1 \), then
\[ \text{irr}_t(G') = \text{irr}_t(G) + [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)] - \sum_{v \in V(G)} |\text{deg}(v) - \text{deg}(w)| \]

**Proof**

When we delete an edge \( e = uv \) in \( G \), \( \text{deg}(u) \) and \( \text{deg}(v) \) will be decreased by one and the remaining vertices having the same degree as before.

The total irregularity changes between \( G \) and \( G' \) are

The only changes between \( G \) and \( G' \) are

\[ \sum_{u \in V(G)} |\text{deg}(u) - \text{deg}(w)| \text{ and } \sum_{v \in V(G)} |\text{deg}(v) - \text{deg}(w)|. \]

a) Suppose \( \text{deg}(u) = \text{deg}(v) \)

The total contribution at \( u \) will be
\[ [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)], \text{ while the total contribution at } v \text{ will be } [\text{Deg}^- (v) + \text{Deg}^-(v) - \text{Deg}^+ (v)]. \]

Thus,
\[ \text{irr}_t(G') = \text{irr}_t(G) + [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)] + [\text{Deg}^- (v) + \text{Deg}^-(v) - \text{Deg}^+ (v)] \]

Since \( \text{deg}(u) = \text{deg}(v) \), which implies
\[ \text{Deg}^+ (u) = \text{Deg}^+ (v), \text{Deg}^- (u) = \text{Deg}^- (v) \text{ and } \text{Deg}^-(u) = \text{Deg}^-(v). \]

Therefore,
\[ \text{irr}_t(G') = \text{irr}_t(G) + 2[\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (v)]. \]

b) Suppose \( \text{deg}(u) > \text{deg}(v) \) and \( \text{deg}(v) \neq 1 \).

The total contribution at \( u \) will be
\[ [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)], \text{ while total contribution at } v \text{ will be } [\text{Deg}^- (v) + \text{Deg}^-(v) - \text{Deg}^+ (v)]. \]

Thus
\[ \text{irr}_t(G') = \text{irr}_t(G) + [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)] + [\text{Deg}^- (v) + \text{Deg}^-(v) - \text{Deg}^+ (v)]. \]

c) We delete an edge \( uv \) and \( \text{deg}(v) = 1 \) from \( G \), then \( G' \) contains two component.

That is \( G' = G - e (= uv) \setminus K_1 \)

Since the total irregularity of \( K_1 \) is zero, it is enough to find total irregularity of \( G - e (= uv) \).

The total contribution at \( u \) will be
\[ [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)], \text{ while the total contribution of } v \text{ will be } \sum_{v \in V(G)} |\text{deg}(v) - \text{deg}(w)| \]

Thus
\[ \text{irr}_t(G') = \text{irr}_t(G) - \sum_{v \in V(G)} |\text{deg}(v) - \text{deg}(w)| + [\text{Deg}^- (u) + \text{Deg}^-(u) - \text{Deg}^+ (u)]. \]

**Definition 3.2**

Let \( G = (V, E) \) be any graph with \( E \) is non-empty. An edge \( e \in E(G) \) is said to be a total irregular positive inner edge if \( \text{irr}_t(G + e) > \text{irr}_t(G) \). It is denoted by TIPI edge.

**Example 3.3**
Friendship graph $F_2$:

Consider $F_2$ with degree sequence $(2, 2, 2, 2, 4)$.

$\text{irr}_r(F_2) = 8$.

Let $e = v_1v_2$, $\text{irr}_r(F_2 - e) = 12$.

$\text{irr}_r(F_2 - e) = 12 > 8 = \text{irr}_r(F_2)$.

Therefore $e$ is a TIPI edge.

**Proposition 3.4**

$$\text{irr}_r(K_{m,n}) = mn(n - m), m \neq n \text{ and } m, n \geq 2.$$  

**Proof**

Let the bipartition of $K_{m,n}$ be $(X, Y)$, where $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$.

Any edge $e \in E(K_{m,n})$ is of the form $u_i v_j, 1 \leq i \leq m$ and $1 \leq j \leq n$.

Therefore $\text{deg}(u_i) = n, 1 \leq i \leq m$ and $\text{deg}(v_j) = m, 1 \leq j \leq n$.

Without loss of generality we assume that $m < n$.

Now $\text{irr}_r(K_{m,n}) = \sum_{i=1}^{m} \sum_{j=1}^{n} |\text{deg}(u_i) - \text{deg}(v_j)|$

$$= mn(n - m).$$

Therefore, $\text{irr}_r(K_{m,n}) = mn(n - m)$.

**Proposition 3.5**

$$\text{irr}_r(K_{m,n} - e) = mn(n - m) + 2m - 2, \quad m \neq n \text{ and } m, n \geq 2.$$  

**Proof**

Any edge $e \in E(K_{m,n})$ is of the form $u_i v_j, 1 \leq i \leq m$ and $1 \leq j \leq n$.

Now we delete an edge $u_i v_j$ from $K_{m,n}$.

By using Edge deletion Lemma for $u_i v_j$

Since $\text{deg}(u_i) = n, 1 \leq i \leq m$ and $\text{deg}(v_j) = m, 1 \leq j \leq n$.

Without loss of generality we assume that $m < n$.

$\text{irr}_r(K_{m,n} - u_i v_j) = \text{irr}_r(K_{m,n})$

$$+ [\text{Deg}^- (u_i) + \text{Deg}^- (u_i) - \text{Deg}^+(v_j)]$$

$$+ [\text{Deg}^- (v_j) + \text{Deg}^- (v_j) - \text{Deg}^+(u_i)]$$

$\text{Deg}^- (u_i) = 0, \text{Deg}^- (u_i) = m - 1$ and $\text{Deg}^+(u_i) = n - 1$

$\text{Deg}^- (v_j) = n - 1, \text{Deg}^- (v_j) = m - 1$ and $\text{Deg}^+(v_j) = 0$.

$\text{irr}_r(K_{m,n} - u_i v_j) = mn(n - m) + (0 + (m - 1) - (n - 1)) + ((m - 1) + (n - 1) - 0)$

$$= mn(n - m) + 2m - 2.$$  

Similar result holds for any edge $e = u_i v_j, 1 \leq i \leq m$ and $1 \leq j \leq n$.

Hence, $\text{irr}_r(K_{m,n} - e) = mn(n - m) + 2m - 2, m \neq n \text{ and } m, n \geq 2$.  

Proposition 3.6

In $K_{m,n}$ every edge $e \in E(K_{m,n})$ is a TIPI edge.

Proof

Any edge $e \in E(K_{m,n})$ is of the form $u_i v_j$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Now we delete any edge $u_i v_j$ from $K_{m,n}$, then the resulting graph will be isomorphic to the following graph.

But $\text{irr}_i(K_{m,n} - e) = mn(n-m) + 2m - 2$ which is strictly greater than $mn(n-m) = \text{irr}_i(K_{m,n})$.

Since $e$ is arbitrary, $\text{irr}_i(K_{m,n} - e) > \text{irr}_i(K_{m,n})$ for all $e \in E(K_{m,n})$.

Therefore every edge $e \in E(K_{m,n})$ is a TIPI edge.

Definition 3.7

Let $G = (V, E)$ be any graph with $n$ vertices and $E$ is non-empty. If all the edges $e \in E(G)$ are TIPI edges of $G$, then $G$ is called a total irregular positive inner graph. It is denoted by TIPI graph.

Theorem 3.8

The Complete graph $K_{m,n}$, $m \neq n$ and $m, n \geq 2$ is a TIPI graph.

Definition 3.9

Let $G = (V, E)$ be any graph with $E$ is non-empty. An edge $e \in E(G)$ is said to be a total irregular negative inner edge if $\text{irr}_i(G - e) < \text{irr}_i(G)$. It is denoted by TINI edge.

Example 3.10

The Graph $G$ with degree sequence (1, 2, 2, 3).

$\text{irr}_i(G) = 14$. Let $e = v_2 v_4$

The graph $G - e$ with degree sequence (1, 2, 2, 3).

$\text{irr}_i(G - e) = 8 < 14 = \text{irr}_i(G)$

Therefore, $e$ is a TINI edge.
Example 3.11

\[ \text{irr}_t(S_n) = n(n-1), \ n \geq 1. \]

**Proof**

The vertices of \( S_n \) be \( u, v_1, v_2, \ldots, v_n \).

In \( S_n \), \( \text{deg}(u) = n \) and \( \text{deg}(v_1) = \text{deg}(v_2) = \ldots = \text{deg}(v_n) = 2 \).

Now calculate the total irregularity of \( F_n \).

Since the degree difference between any two vertices \( v_i, v_j, i \neq j \) is zero.

So we have \( \text{irr}_t(S_n) = \sum_{i=1}^{n} \text{deg}(u) - \text{deg}(v_i) = n(n-1) \).

Therefore, \( \text{irr}_t(S_n) = n(n-1), n \geq 2. \)

**Proposition 3.12**

\[ \text{irr}_t(S_n - e) = (n-1)(n-2), n \geq 3. \]

**Proof**

Let the vertices of \( S_n \) be \( u, v_1, v_2, \ldots, v_n \).

Any edge \( e \in E(S_n) \) is of the form \( u v_i \), \( 1 \leq i \leq n \).

Now we delete an edge between \( u \) and \( v_i \), \( \text{deg}(u) = n \) and \( \text{deg}(v_i) = 1 \).

By using Lemma 2.5

\[ \text{irr}_t(S_n - u v_i) = \text{irr}_t(S_n) + [\text{Deg}^<(u) + \text{Deg}^>(u) - \text{Deg}^x(u)] - \sum_{j=1}^{n} [\text{deg}(v_i) - \text{deg}(v_j)]. \]

\[ = n(n-1) + (n-1) - (n-1) = n(n-1) - 2(n-1) = (n-1)(n-2). \]

Therefore, \( \text{irr}_t(S_n - e) = (n-1)(n-2). \)

**Proposition 3.13**

In \( S_n \), every edge \( e \in E(S_n) \) is a TINI edge.

**Proof**

Let the vertices of \( S_n \) be \( u, v_1, v_2, \ldots, v_n \).
We delete any edge $e \in E(S_n)$ from $S_n$ the resulting graph will be isomorphic to the above graph. But $irr_T(S_n - e) = (n - 1)(n - 2)$ which is strictly less than $nn - 1 = irr_T(S_n)$. Since $e$ is an arbitrary $irr_T(S_n - e) < irr_T(S_n)$ for all edges $e \in E(S_n)$. Therefore every edge $e \in E(S_n)$ is a TINI edge.

**Definition 3.14**

Let $G = (V, E)$ be any graph with $E$ is non-empty. If all the edges $e \in E(G)$ are TINI edges of $G$, then $G$ is called a total irregular negative inner graph. It is denoted by TINI graph.

**Theorem 3.15**

The Star graph $S_n, n \geq 3$ is a TINI graph.

**Definition 3.16**

Let $G = (V, E)$ be any graph with $E$ is non-empty. An edge $e \in E(G)$ is said to be a total irregular stable inner edge if it is denoted by TISI edge.

**Example 3.17**

Consider the graph $G$ with degree sequence $(2, 2, 3, 3, 3, 5)$

\[ irr_{irr}(G) = 18. \]

Let $e = v_4v_5$

The graph $G - e$ with degree sequence $(2, 2, 2, 3, 3, 5)$

\[ irr_{irr}(G - e) = 18. \]

Therefore $e$ is a TISI edge.

**Definition 3.18**

Let $G = (V, E)$ be any graph with $E$ is non-empty. If all the edges $e \in E(G)$ are TISI edges of $G$, then $G$ is called a total irregular stable inner graph. It is denoted by TISI graph.

**Open problem**

Existence or non-existence of a TISI graph is an open problem.

**Definition 3.19**
Let $G = (V, E)$ be any graph with $n$ vertices and the edge set $E$ is non-empty. If $G$ is none of total irregular positive inner, total irregular negative inner and total irregular stable inner graph then, $G$ is called as total irregular mixed inner graph. It is denoted by TIMI graph.

Example 3.20

The graph $G$ with degree sequence $(2, 2, 3, 3, 4)$.

Let $e_1 = v_3v_4$ and $e_2 = v_4v_5$.

$G - e_1$ with degree sequence $(1, 2, 2, 3, 4)$

$irr_G(G - e_1) = 14$.

$G - e_2$ with degree sequence $(2, 2, 2, 4, 4)$

$irr_G(G - e_2) = 8$.

Therefore $G$ is TIMI graph.

4. Conclusion:

In this paper, we consider a new measure on total irregularity of simple undirected graph. We present edge deletion lemma. Using this lemma we prove that the Complete graph $K_{m,n}$, $m \neq n$ and $m, n \geq 2$ is a TIPI graph, and the Star graph $S_n$, $n \geq 3$ is a TINI graph and also give an open problem for TISI graph.

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