FULL SPECTRUM OF LYAPUNOV EXPONENTS IN GAUGE FIELD THEORY
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Abstract. We analyze the Lyapunov exponents of U(1) gauge fields across the phase transition from the confinement to the Coulomb phase on the lattice which are initialized by quantum Monte Carlo simulations. We observe all features of a strange attractor with a tendency to regularity towards the continuum limit. Results are also displayed for the full spectrum of Lyapunov exponents of the SU(2) gauge system.

1. Classical chaotic dynamics from quantum Monte Carlo initial states. Chaotic dynamics in general is characterized by the spectrum of Lyapunov exponents. These exponents, if they are positive, reflect an exponential divergence of initially adjacent configurations. In case of symmetries inherent in the Hamiltonian of the system there are corresponding zero values of these exponents. Finally negative exponents belong to irrelevant directions in the phase space: perturbation components in these directions die out exponentially. Pure random gauge fields on the lattice show a characteristic Lyapunov spectrum consisting of one third of each kind of exponents [1].

The general definition of the Lyapunov exponent is based on a distance measure $d(t)$ in phase space,

$$L := \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)}.$$  

In case of conservative dynamics the sum of all Lyapunov exponents is zero according to Liouville’s theorem, $\sum L_i = 0$. For gauge field theories one utilizes the gauge invariant distance measure consisting of the local differences of energy densities between two three-dimensional field configurations on the lattice:

$$d := \frac{1}{N_P} \sum_P |\text{tr} U_P - \text{tr} U'_P|.$$  

Here the symbol $\sum_P$ stands for the sum over all $N_P$ plaquettes, so this distance is bound in the interval $(0, 2N)$ for the group SU(N). $U_P$ and $U'_P$ are the familiar plaquette variables, constructed from the basic link variables $U_{x,i}$,

$$U_{x,i} = \exp(a A_{x,i} T),$$  

located on lattice links pointing from the position $x = (x_1, x_2, x_3)$ to $x + ae_i$. The generator of the group U(1) is $T = -ig$ and $A_{x,i}$ is the vector potential. The elementary plaquette variable is constructed for a plaquette with a corner at $x$ and lying in the $ij$-plane as $U_{x,ij} = U_{x,i}U_{x+i,j}U^\dagger_{x+j,i}U^\dagger_{x,j}$. It is related to the magnetic field strength $B_{x,k}$:

$$U_{x,ij} = \exp(\varepsilon_{ijk}a^2B_{x,k} T).$$  

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The electric field strength $E_{x,i}$ is related to the canonically conjugate momentum $P_{x,i} = a\dot{U}_{x,i}$ via

$$E_{x,i} = \frac{1}{g^2a^2} \left(T\dot{U}_{x,i}U_{x,i}^\dagger\right).$$

The Hamiltonian of the lattice gauge field system can be casted into the form

$$H = \sum \left[ \frac{1}{2} \langle P, P \rangle + 1 - \frac{1}{4} \langle U, V \rangle \right].$$

Here the scalar product stands for $\langle A, B \rangle = \text{Re}(AB^\dagger)$. The staple variable $V$ is a sum of triple products of elementary link variables closing a plaquette with the chosen link $U$. This way the Hamiltonian is formally written as a sum over link contributions and $V$ plays the role of the classical force acting on the link variable $U$.

We prepare the initial field configurations from a standard four-dimensional Euclidean Monte Carlo program on an $N^3 \times 4$ lattice varying the inverse gauge coupling $\beta \propto g^{-2}$ [2]. We relate such four-dimensional Euclidean lattice field configurations to Minkowskian momenta and fields for the three-dimensional Hamiltonian simulation by selecting a fixed time slice of the four-dimensional lattice.

2. Spectrum of the stability matrix. Instead of the classical determination of the Lyapunov exponent by the rescaling method outlined in the preceding section, we now use the monodromy matrix approach [3]. The Lyapunov spectrum $L_i$ is expressed in terms of the eigenvalues $\Lambda_i$ of the monodromy matrix $M$:

$$L_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Lambda_i(t')dt', \quad i = 1, ..., f,$$

where $\Lambda_i(t)$ are the solutions of the characteristic equation for $f$ degrees of freedom

$$\det (\Lambda_i(t) \mathbb{1} - M(t)) = 0$$

at a given time $t$. Here $M$ is the linear stability matrix,

$$M = \begin{pmatrix} \frac{\partial S}{\partial U} & \frac{\partial \dot{U}}{\partial P} \\ \frac{\partial P}{\partial U} & \frac{\partial \dot{P}}{\partial P} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2 S}{\partial U^2} & 0 \end{pmatrix},$$

with the three-dimensional lattice action $S_3$.

Figure 1 displays the complex eigenvalues for selected U(1) gauge field configurations [4] prepared by a quantum Monte Carlo heat-bath algorithm. By investigating the time evolution of the monodromy matrix it turned out that the choice of the real time $t$ does not affect the eigenvalues appreciably. In the confinement phase ($\beta = 0.9$) the eigenvalues lie on either the real or on the imaginary axes. This is a nice illustration of a strange attractor of a chaotic system. Positive Lyapunov exponents eject the trajectories from oscillating orbits provided by the imaginary eigenvalues. Negative Lyapunov exponents attract the trajectories keeping them confined in the basin. In the Coulomb phase ($\beta = 1.1$) the real Lyapunov exponents become rare to eventually vanish in the continuum limit [5]. In all cases the spectrum is symmetric with respect to the real and imaginary axes: the former property is due to the fact that the equations of motion are real, the latter is due to the Hamiltonian being conservative (time independent). Also a number of zero-frequency modes occur which
is connected to symmetry transformations commuting (in the Poisson bracket sense) with the Hamiltonian, such as time independent gauge transformations.

Figure 2 depicts the complex eigenvalues for SU(2) gauge field configurations [3] taken from random starts with a high energy \( g^2aE = 0.8 \) and a low energy \( g^2aE = 0.1 \), respectively. Some caution is in order, since the gauge fields were not obtained from a Monte Carlo equilibrium configuration at definite coupling. Further, the entries of the monodromy matrix were constructed from derivatives with respect to components of the SU(2) matrix whereas the derivatives with respect to the phase were taken in the U(1) case. The SU(2) eigenvalues scatter in the complex plane but the symmetry of the spectrum with respect to the real and imaginary axes is still obvious. Again the prerequisites of a strange attractor are fulfilled and one observes some tendency to regularity at lower energy.

Figure 3 shows the real part of the squared Lyapunov spectrum of U(1) gauge
fields for several couplings. Since the U(1) spectrum is purely real or purely imaginary, its square allows for a one-dimensional representation of its distribution. It is symmetric for $\beta = 0$ and is shifted to negative values towards the Coulomb phase exhibiting an increasing number of zero modes with increasing $\beta$.

Figure 4 shows the real part of the Lyapunov spectrum of SU(2) gauge fields [3] extrapolated to $1/N \to 0$ from data taken at $N = 2, 3, 4, 5$ and 6 at high energy ($g^2aE = 0.8$). Since the SU(2) spectrum lies inside the complex plane, here only its real part is represented to gain a one-dimensional distribution. The overall pattern resembles that obtained earlier from smaller systems ($N = 2, 3$) with the rescaling method [6]. The structure of the ordered real part of the Lyapunov spectrum is similar at all energies considered, but the maximal point, $aL_{\text{max}}$, scales with the energy, $g^2aE$.

3. Summary. This contribution concentrated on the spectrum of the monodromy matrix for (classical) compact U(1) theory. It exhibited an interplay between
positive, imaginary and negative Lyapunov exponents in the confinement phase changing to a pure imaginary spectrum deep in the Coulomb phase leading to the regularity of the Maxwell theory. The spectrum for SU(2) theory was compared showing the characteristics of a strange attractor. Since the configurations were chosen from a random start corresponding to $\beta = 0$, it will be interesting to compute the spectrum of the stability matrix for finite $\beta$ and to investigate if the non-Abelian theory stays chaotic in the continuum limit.

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