Partial synchronicity and the \((\max,+)\) semiring

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Abstract. In this paper we illustrate how non-stochastic \((\max,+)\) techniques can be used to describe partial synchronization in a Discrete Event Dynamical System. Our work uses results from the spectral theory of dioids and analyses \((\max,+)\) equations describing various synchronization rules in a simple network. The network in question is a transport network consisting of two routes joined at a single point, and our Discrete Events are the departure times of transport units along these routes. We calculate the maximum frequency of circulation of these units as a function of the synchronization parameter. These functions allow us further to determine the waiting times on various routes, and here we find critical parameters (dependent on the fixed travel times on each route) which dictate the overall behaviour. We give explicit equations for these parameters and state the rules which enable optimal performance in the network (corresponding to minimum waiting time).

1 Introduction

Considerable advances have been made in recent years in the algebraic description of Discrete Event Dynamical Systems (DEDS). Chief among these is the approach using the so-called \((\max,+)\) semiring \[1\], with corresponding development of algorithms for spectral calculations \[2\] and concrete applications \[3\].

In this paper we illustrate how non-stochastic \((\max,+)\) techniques can be used to describe partial synchronization (in a manner to be defined below) in a DEDS. Our work uses results from the spectral theory of dioids and analyses \((\max,+)\) equations describing various synchronization rules in a simple network.

The paper is structured as follows. In Section \[2\] we review necessary mathematical properties of the \((\max,+)\) semiring. Section \[3\] introduces the network on which our DEDS is based, and in Section \[4\] we describe the different \((\max,+)\) equations we will solve. The resulting solutions lead, in Section \[5\], to critical phenomena in the network, and we calculate which among the \((\max,+)\) equations give optimal results. A numerical example illustrates our results in Section \[6\]. We conclude in Section \[7\].
2 \hspace{1em} \textbf{(max,+)} semiring

The \textit{(max,\hspace{1em}+)\ semiring (or dioid) $R_{\text{max}}$\ is the set $R \cup \{-\infty\}$ with the two operations $\oplus, \otimes$ defined by $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. Note that both operations possess identity elements, $-\infty$ for $\oplus$ and 0 for $\otimes$, but while $\otimes$ is invertible, $\oplus$ is idempotent: $a \oplus a = a$. In the remainder of this paper we will often omit the explicit multiplication symbols $\times, \otimes$, as is the convention. Our convention will be that if an expression contains any explicit symbol from $R_{\text{max}}$ we will assume all such hidden symbols are also from $R_{\text{max}}$: $a + bc$ means $a + (b \otimes c)$, $a \oplus bc$ means $a \oplus (b \otimes c)$, and $(a \oplus b)(c + d)$ means $(a \oplus b) \otimes (c + d)$.

These operations can be extended to matrices $A, B \in R_{\text{max}}^{n \times n}$ by defining $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes B)_{ij} = \oplus_{k}(A_{ik} \otimes B_{kj})$. As this may seem unusual to the novice reader, a small example is in order.\[\begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} (3 \otimes 4) \oplus (-1 \otimes 0) \\ (0 \otimes 4) \oplus (5 \otimes 0) \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.\] (1)

We now look at solutions to the \textit{(max,+)} spectral problem $A \otimes X = \lambda \otimes X$.

\textbf{Theorem 1.} \cite{1, 4, 5} \hspace{1em} Let $A \in R_{\text{max}}^{n \times n}$. There exists a maximal eigenvalue \[\lambda = \bigoplus_{k=1}^{n} \bigoplus_{i_1, i_2, \ldots, i_k} (A_{i_1 i_2} \otimes A_{i_2 i_3} \otimes \ldots \otimes A_{i_k i_1}) / k.\] (2)

If we extend our notation to $a^n \overset{\text{def}}{=} a \otimes a \otimes \ldots \otimes a = n \times a$ and $\text{Tr}(A) = \bigoplus_{k=1}^{n} A_{kk}$ equation (3) reads \[\lambda = \bigoplus_{k=1}^{n} (\text{Tr}(A^k))^{1/k}.\] (3)

In the standard way, we associate with $A$ a weighted digraph (precedence graph) $G(A)$ with nodes $N = \{1, 2, \ldots, n\}$ and edges $E = \{(i, j) \mid A_{ij} \neq -\infty\}$. The weight of edge $(i, j)$ is simply $A_{ij}$. Denoting by $i \rightarrow j$ the existence of a path from $i$ to $j$ we define $A$ to be irreducible (and $G(A)$ to be strongly connected) iff $i \rightarrow j \hspace{1em} \forall \hspace{1em} 1 \leq i, j \leq n$.

\textbf{Theorem 2.} \cite{1} \hspace{1em} If $A \in R_{\text{max}}^{n \times n}$ is irreducible, then $\lambda$ as given by (3) is unique.

Note that from a graph-theoretic point of view, $\lambda$ is merely the maximum cycle mean of $G(A)$.

Our DEDS will be described by equations in $R_{\text{max}}$. Specifically, denoting the time of the $k$th occurrence of event $i$ by $x_i(k)$, we are interested in equations of the form.
where we have a finite family of \( 1 \leq i, j \leq n \) events and \( 1 \leq l \leq k - 1 \). For linear equations in \( R_{\text{max}} \), (4) becomes

\[
x_i(k) = \bigoplus_{j,l}(f_{ij}^l \otimes x_j(k - l))
\]

and we will show in Section 4 how this can be written as a matrix equation with unitary retard

\[
x_i(k) = \bigoplus_j (A_{ij} \otimes x_j(k - 1)).
\]

We are interested in the asymptotic properties of this system as \( k \to \infty \) so we study \( \lim_{k \to \infty} A^k \).

**Theorem 3.**\cite{6,7,8} If \( A \in R_{\text{max}}^{n \times n} \) is irreducible, there exists integers \( M, c(A) \) such that

\[
A^{k \otimes c(A)} = (\lambda)^{c(A)} \otimes A^k \quad \forall \ k \geq M.
\]

We call \( c(A) \) the cyclicity of \( A \) (see \cite{1} for details of how to calculate this integer).

## 3 Model

We consider an elementary model of two tourbuses with partial synchronization. Each tourbus goes on a fixed circular route \( R_1 \) and \( R_2 \), stopping every so often to drop off and pick up passengers. The tour buses meet at one location, a downtown station \( S \), where passengers can pass from one bus to the other one. The physical network consists of two loops meeting at \( S \), as shown in Figure 1.

We assume fixed travel times \( T_1 \) and \( T_2 \) for \( R_1 \) and \( R_2 \) respectively, and without loss of generality let \( T_1 \leq T_2 \). The simplest model is to describe this system using the \((\text{max,}+)\) equations \( X(k + 1) = A \otimes X(k) \), with

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad X(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}
\]

and \( x_i(k) \) is the departure time of bus \( i \) from station \( S \) along route \( R_i \). We consider the following extreme cases.
CASE M1 (no synchronization):
We have
\[ A = \begin{pmatrix} T_1 & -\infty \\ -\infty & T_2 \end{pmatrix} \]  
so that the graph \( G(A) \) is disconnected into two parts each having its own cycle mean \( \lambda_1 = T_1 \) and \( \lambda_2 = T_2 \) as shown in Figure 2 (the nodes here correspond to departure times of buses).

CASE M2 (complete synchronization):
We have
\[ A = \begin{pmatrix} T_1 & T_2 \\ T_1 & T_2 \end{pmatrix} \]  
so that the graph \( G(A) \) is strongly connected. There are 3 circuits in this graph with cycle means \( T_1, T_2, \) and \( T_1 + T_2 \), so the maximum cycle mean gives an eigenvalue of \( \lambda = T_2 \) (see Figure 3).

4 Partial Synchronicity
For the case of complete synchronization in section 3, it is clear that the large value of \( \lambda = T_2 \) slows down the overall system. We further suppose in this
section that we cannot alter the travel time $T_2$, but that we have a large amount of additional stock (buses) which we can add to the network.

We add to route $R_2$ a large number $m$ of buses such that the interval between departure times on this route becomes a very small number $\delta$ (which we can think of as $T_2/m$ if required). Denote by $y_1(k), y_2(k), \ldots, y_m(k)$ the departure times of the $m$ new buses, which operate according to the rules

$$
y_i(k) = \delta \otimes y_{i-1}(k-1), \quad y_0(k) = \delta \otimes y_m(k-1).
$$

For this route we do not produce a timetable, as we assume to a first approximation there is virtually no waiting for a bus.

For this system, it would seem there should be no interdependence between $x_1(k)$ and $x_2(k)$. However, what of a passenger who wants to make a non-stop round trip $R_1 \rightarrow R_2 \rightarrow R_1$—would some synchronization rule benefit such a passenger? To investigate further we set $T_2 = nT_1 + r$ and look at the $R_{\text{max}}$ equation $x_1(k) = f(x_1(k-l))$ for different values of $l$ and linear $f$.

**CASE P1:**

$$x_1(k) = T_1 x_1(k-1) \oplus T_1 T_2 x_1(k-(l+1)) \quad (12)$$

Using $n$ artificial states (i.e. states that do not in fact correspond to the departure of any physical bus) $z_1, z_2, \ldots, z_l$, we can rewrite (12) as

$$
x_1(k) = T_1 x_1(k-1) \oplus r T_1 z_l(k-1)
$$

$$z_i(k) = T_1^{n/l} \otimes z_{i-1}(k-1), \quad i = 1, \ldots, l \quad (13)$$

$$z_0(k) \overset{\text{def}}{=} x_1(k).$$

At the expense of increasing the dimension, our system of equations now has
unitary retard as in (3) with column vector
\[
X(k) = \begin{pmatrix}
x_1(k) \\
z_i(k) \\
z_{l-1}(k) \\
\vdots \\
z_1(k)
\end{pmatrix}
\] (14)
and matrix
\[
A = \begin{pmatrix}
T_1 & T_1 + r & -\infty & -\infty & \cdots & -\infty \\
-\infty & -\infty & nT_1/l & -\infty & \cdots & -\infty \\
-\infty & -\infty & -\infty & nT_1/l & \cdots & -\infty \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
nT_1/l & -\infty & -\infty & -\infty & \cdots & -\infty
\end{pmatrix}.
\]

The full precedence graph \(G(A)\) for the system \(x_1, z_1, \ldots, z_l, x_2, y_1, \ldots, y_m\) is shown in Figure 4. The disconnected parts of the graph \(G(A)\) have maximum cycle means \(\lambda_2 = \delta\) and
\[
\lambda_1 = \left\{ \begin{array}{ll}
\frac{T_1}{(n+1)T_1 + r} & \text{if } l > n \\
\text{otherwise}
\end{array} \right.
\] (15)
since \(T_1 \oplus r = T_1\) (by definition of \(r\)) and, for \(l > n\) (equivalently \(l = l \oplus (n+1)\)),
\[
\frac{(n+1)T_1 + r}{l+1} < \frac{(n+1)T_1 + T_n}{l+1} = \frac{(n+2)T_n}{l+1} \leq T_1.
\] (16)

Note that \(l > n\) effectively corresponds to no synchronization, while for the choice \(l \leq n\), \((n-l+1)\) out of every \((n+1)\) buses are synchronized. The minimum possible \(\lambda_1\) is \(\lambda_1 = T_1\) given by \(l > n\).

**CASE P2:**
\[
x_1(k) = T_1 x_1(k - 1) \oplus rT_1^{l+1} x_1(k - (l + 1))
\] (17)
As in the previous case P1, we can rewrite (17) as

\[
x_1(k) = T_1 x_1(k - 1) \oplus r T_1 z_i(k - 1)
\]

\[
z_i(k) = T_1 \otimes z_{i-1}(k - 1), i = 1, \ldots, l
\]

(18)

\[
z_0(k) \overset{\text{def}}{=} x_1(k).
\]

and the system is described by \(X(k + 1) = A \otimes X(k)\) with \(X(k)\) as in (14) and

\[
A = \begin{pmatrix}
T_1 & T_1 + r & -\infty & -\infty & \cdots & -\infty \\
-\infty & -\infty & T_1 & -\infty & \cdots & -\infty \\
-\infty & -\infty & -\infty & T_1 & \cdots & -\infty \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T_1 & -\infty & -\infty & -\infty & \cdots & -\infty
\end{pmatrix}
\]

From the precedence graph shown in Figure 5 we have \(\lambda_2 = \delta\) and

\[
\lambda_1 = \frac{(l + 1)T_1 + r}{l + 1}
\]

since obviously

\[
\frac{(l + 1)T_1 + r}{l + 1} \leq T_1
\]

for all values of \(l\). This system corresponds to one out of every \((l + 1)\) buses

Figure 5: Precedence Graph for CASE P2, equation (17)

being synchronized. The limiting case is

\[
\lim_{l \to \infty} \frac{(l + 1)T_1 + r}{l + 1} = T_1
\]

so the minimum value of \(\lambda_1\) occurs at \(l = \infty\).
5 Optimal Solutions

In both cases P1 and P2 of Section 4 the minimum value of \( \lambda_1 \) is \( T_1 \): In P1 it is given by any choice \( l > n \) while in P2 it is given by the extreme value \( l = \infty \). These values both correspond to no synchronization, and this is the result we would expect, that the system runs faster if the buses do not wait for each other.

If our goal is to choose the synchronization rule which maximizes the speed of the system, the problem is solved. In this Section we look at the separate problem of minimizing the waiting time of passengers that make a circular journey over all or part of the network (by circular we mean a journey that begins and ends at the same point). We itemize such journeys \( J_i \) as follows:

\[
\begin{align*}
J_1 & : R_1 \rightarrow R_1 \\
J_2 & : R_1 \rightarrow R_2 \rightarrow R_1 \\
J_3 & : R_2 \rightarrow R_1 \rightarrow R_2 \\
J_4 & : R_2 \rightarrow R_2
\end{align*}
\]

For cases P1, P2 we look at regions \( l \leq n \) and \( l > n \) to give synchronization rules \( S_i \) as follows:

\[
\begin{align*}
S_1 & : \text{Equation (12), } l \leq n \\
S_2 & : \text{Equation (12), } l > n \\
S_3 & : \text{Equation (17), } l \leq n \\
S_4 & : \text{Equation (17), } l > n
\end{align*}
\]

For \( i, j \in \{1, 2, 3, 4\} \) we pair each journey \( J_i \) with rules \( S_j \) to give 16 different models. Let \( w_{ij} \) be the total waiting time on journey \( J_i \) with rule \( S_j \). These waiting times can be broken into two components, the waiting time before boarding the bus \( w_{ij}^b \) and the waiting time in transit \( w_{ij}^t \), which can only occur for journeys \( J_2 \) and \( J_3 \). From our calculation of \( \lambda_1 \) and \( \lambda_2 \) in Section 4 we determine \( w_{ij}^b, w_{ij}^t \) and \( w_{ij} = w_{ij}^b + w_{ij}^t \) to be the values given by equations (29)–(31) in the Appendix.

We now make some observations on the solutions given by equation (31). Our goal is to minimize the waiting times, or more specifically the average waiting time per passenger as a function of the synchronization parameter \( l \): If \( n_i \) people take journey \( J_i \) and \( N = \sum_i n_i \) then we want to find the minimum of

\[
W_i \overset{\text{def}}{=} \frac{\sum_j n_j w_{ji}}{N}
\]

Note that

- \( w_{i1} \) is minimized by choosing the largest possible value of \( l \), i.e. \( l = n \).
- \( w_{i2} = \lim_{l \to \infty} w_{i4} \)
- similarly \( w_{i3} \) is minimized by choosing \( l = n \).
Finally examine \( w_{44} \). \( w_{14} \) and \( w_{34} \) are both clearly minimized by \( l = \infty \). To minimize \( w_{24} \) we view it as a continuous function of \( l \) and solve

\[
\frac{dw_{24}}{dl} = \frac{d}{dl} \frac{(3l - 2n + 1)T_1 + (2n - 2l + 3)r}{2(l + 1)} = 0.
\tag{23}
\]

\( l \) drops out of equation (23), meaning the existence of an extremum does not depend on \( l \) but rather on the solution of \((2n + 2)T_1 = (2n + 5)r \). Thus we have a critical point

\[
r_c = \left( \frac{2n + 2}{2n + 5} \right) T_1.
\tag{24}
\]

The slope of the function \( w_{24}(l) \) can now be written in terms of the critical remainder \( r_c \) as

\[
\frac{dw_{24}}{dl} = \frac{(2n + 5)(r_c - r)}{(l + 1)^2}.
\tag{25}
\]

Hence \( w_{24}(l) \) is

- strictly increasing for \( r < r_c \): minimum is obtained by choosing lowest value of \( l \), i.e. \( l = n + 1 \)
- strictly decreasing for \( r > r_c \): minimum is obtained by choosing highest value of \( l \), i.e. \( l = \infty \)
- constant for \( r = r_c \): choose any value of \( l \)

In equation (23) we rewrite \( W \) with the optimal values of \( l \) chosen as indicated above: for \( w_{14} \) we insert in the matrix \( W^m (\equiv W^{\text{min}}) \) the values at \( l = n + 1 \), since the values at \( l = \infty \) correspond to column 2, \( w_{12} \).

Now that we have chosen \( l \) we should choose the synchronization rule \( S_j \) that minimizes \( W_i \). In rows 1, 3 and 4 of \( W^m \) it is clear that the minimum values are in column 2 corresponding to \( S_2 \). Let us look at row 2, \( w_{22}^m \). In Figure 6 we plot \( w_{21}^m, w_{22}^m \) and \( w_{24}^m \) as functions of \( r \).

\[
r_s = \left( \frac{2n + 2}{2n + 3} \right) T_1
\tag{26}
\]

along with equation (24) defines three different regimes as follows:

- \( r > r_s : w_{21}^m < w_{22}^m < w_{24}^m \)
- \( r_c < r < r_s : w_{21}^m < w_{22}^m < w_{24}^m \)
- \( r < r_c : w_{21}^m < w_{24}^m < w_{22}^m \)

Further we note the importance of the critical value \( r_s \) is that for \( r > r_s \) rule \( S_2 \) gives shortest waiting times for all journeys \( J_i \), and in this regime \( \overline{W}_2 \) is
Figure 6: $w_{21}^m$, $w_{22}^m$, and $w_{24}^m$ plotted against $0 \leq r \leq T_1$ (see equation (12)).

The critical points are $r_c = (2n + 2)T_1/(2n + 5)$ and $r_s = (2n + 2)T_1/(2n + 3)$.

The critical points are $r_c = (2n + 2)T_1/(2n + 5)$ and $r_s = (2n + 2)T_1/(2n + 3)$. The minimal irrespective of the values $n_i$ (see equation (22)). For $r < r_s$ the values of $n_i$ dictate which $W_i$ is minimal: In particular, if

$$n_2 > \frac{r}{(2n + 3)(r_s - r)}(3n_1 + n_3)$$

then $W_1$ is minimal. We now have a rule that governs optimum performance in our network: If $r > r_s$ implement rule $S_2$, otherwise monitor passenger numbers $n_i$ and if at any stage equation (27) holds, implement rule $S_1$.

Observe further from equation (26) that $4T_1/5 < r_s < T_1$, since $1 < n < \infty$. This means that even for a random choice of $T_1$ and $T_2$, the chances that $r$ is less than $r_s$ are greater than 80%:

$$P(r < r_s \mid \text{arbitrary } T_1, T_2) \geq 0.8$$

6 Example

Let $T_1 = 3$ units of time and $n = 2$ so that $r_s = 18/7$ as given by equation (26).

To examine firstly the regime $r > r_s$, let $r = 8/3$, given by $T_2 = 26/3$ units of time. Under rule $S_2$ we calculate $w_{22}^m = 33/18$ with actual departure times $x_1(k) = 0, 3, 6, 9, 12, 15, \ldots$ Under rule $S_3$ we have $w_{23}^m = 35/18$ with departure times $x_1(k) = 0, 3, 6, 35/3, 44/3, 53/3, \ldots$ Thus rule $S_2$ gives optimal performance on $J_2$. Secondly, let $r = 7/3$ given by $T_2 = 25/3$, so that we are in the regime $r < r_s$. Rule $S_2$ gives $w_{22}^m = 39/18$ with departure times $x_1(k) = 0, 3, 6, 9, 12, 15, \ldots$ Under rule $S_3$ we have $w_{23}^m = 34/18$ with departure times $x_1(k) = 0, 3, 6, 34/3, 43/3, 52/3, \ldots$ Here optimal performance on $J_2$ is given by rule $S_3$.

Note in both regimes of this example the departure times $x_1(k)$ under $S_2$ are identical, the reason for the differences in $w_{22}^m$ lying in the transfer waiting time.
7 Conclusion

In this paper we have solved the (max,+) equations for a specific model with two routes joining at one point. We have shown how partial synchronicity can be described by these equations by introducing a synchronization parameter \( l \). Using results from (max,+) spectral theory we have calculated the maximum frequency (minimum cycle mean) of buses on such routes. These numbers allow us to calculate waiting times on various routes: We observe new critical behaviour which depends on the relation between \( T_1 \) and \( T_2 \), the (fixed) travel times on each route. Finally we state the rules for optimal performance in this network.

Further work is envisaged in looking at more general (max,+) equations, in particular a system with \( m_1 \) buses on \( R_1 \) and \( m_2 \) on \( R_2 \). Another direction in which this work could possibly be extended is to have further routes \( R_3, R_4, \ldots \)

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Appendix

\[
W^b = \begin{pmatrix}
\frac{(n+1)T_1 + r}{2(l+1)} & \frac{T_1}{2} & \frac{(l+1)T_1 + r}{2(l+1)} & \frac{(l+1)T_1 + r}{2(l+1)} \\
\frac{(n+1)T_1 + r}{2(l+1)} & \frac{T_1}{2} & \frac{(l+1)T_1 + r}{2(l+1)} & \frac{(l+1)T_1 + r}{2(l+1)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(29)

\[
W^t = \begin{pmatrix}
\frac{(n-l)T_1 + r}{l+1} & 0 & \frac{r}{l+1} & \frac{r}{l+1} \\
0 & T_1 - r & \frac{(n-l)r}{l+1} & \frac{(T_1 - r)(l-(l+1))}{l+1} \\
\frac{(n+1)T_1 + r}{2(l+1)} & \frac{r}{2} & \frac{(l+1)T_1 + r}{2(l+1)} & \frac{(l+1)T_1 + r}{2(l+1)} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(30)

\[
W = W^b + W^t = \begin{pmatrix}
\frac{3n-2l+3r}{2(l+1)} & \frac{T_1}{2} & \frac{(l+1)T_1 + 3r}{2(l+1)} & \frac{T_1}{2} \\
\frac{(n+1)T_1 + r}{2(l+1)} & \frac{3T_1}{2} - r & \frac{(l+1)T_1 - 2l+3r}{2(l+1)} & \frac{(l+1)T_1 + r}{2(l+1)} \\
\frac{(n+1)T_1 + r}{2(l+1)} & \frac{T_1}{2} & \frac{(l+1)T_1 + r}{2(l+1)} & \frac{(l+1)T_1 + r}{2(l+1)} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(31)

\[
W^m \equiv W^{min} = \begin{pmatrix}
\frac{T_1}{2} + \frac{3r}{2(l+1)} & \frac{T_1}{2} & \frac{T_1}{2} + \frac{3r}{2(l+1)} & \frac{T_1}{2} + \frac{3r}{2(l+1)} \\
\frac{T_1}{2} + \frac{r}{2(l+1)} & \frac{3T_1}{2} - r & \frac{T_1}{2} + \frac{r}{2(l+1)} & \frac{T_1}{2} + \frac{r}{2(l+1)} \\
\frac{T_1}{2} + \frac{r}{2(l+1)} & \frac{T_1}{2} & \frac{T_1}{2} + \frac{r}{2(l+1)} & \frac{T_1}{2} + \frac{r}{2(l+1)} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(32)

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