Osculating properties of decomposable scrolls

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Osculating spaces of decomposable scrolls (of any genus and not necessarily normal) are studied and their inflectional loci are related to those of their generating curves by using systematically an idea introduced by Piene and Sacchiero in the setting of rational normal scrolls. In this broader setting the extra components of the second discriminant locus –deriving from flexes– are investigated and a new class of uninflected surface scrolls is presented and characterized. Further properties related to osculation are discussed for (not necessarily decomposable) scrolls.

Introduction

The inflectional behavior of a projective variety belongs to its extrinsic geometry. In particular, flexes can appear on projective manifolds under (isomorphic) projections. Though this observation is obvious, it seems that several projective manifolds have been extensively investigated from the point of view of their osculatory behavior only in the linearly normal case. This is true e.g., for rational scrolls of any dimension [9] and also for elliptic surface scrolls [7]. In this paper we mainly consider decomposable scrolls, not necessarily linearly normally embedded, and we study their inflectional behavior.

Decomposable scrolls $X \subset \mathbb{P}^N$, whose construction generalizes that of rational normal scrolls, are generated by $n$ curves $C_i$ $(i = 1, \ldots, n)$ isomorphic each other, lying in linearly independent linear subspaces generating the whole $\mathbb{P}^N$ (see Section 1). They are very well suited to investigate their $k$-th inflectional loci $\Phi_k(X)$. We do that developing systematically the local description used in [9] and [7], and in Sections 1 and 2 we succeed to describe several properties of $\Phi_k(X)$, relating them to the inflectional loci of the generating sections $C_i$.

In particular, restricting to the case of rational non-normal scrolls our approach allows us to produce in Section 3 a new series of counterexamples to the even dimensional part of a conjecture of Piene and Tai [10]. While the odd dimensional part of this conjecture has been proved several years ago [10], [3], the even dimensional part is false for certain linearly normal scrolls, as shown by the first author [6]. However we want to stress that the new counterexamples exhibited here are rational scrolls, though, of course, not linearly normal. We also characterize these examples in the framework of decomposable scrolls (Theorem 3.4). This adds some information in order to correct the even dimensional part of the conjecture.

Let $X$ be a decomposable scroll. While describing $\Phi_k(X)$ for $k > 2$ involves inflectional loci of lower order, the description becomes very easy for $k = 2$. In particular, we show that for a decomposable scroll $X$, $\Phi_2(X)$ can have only two types of irreducible components. Let $G$ be any such a component. Then, either $G$ is a sub-fibre of a fibre of $X$, or $X$ is rational, some curve $C_i$ is a line, and $G$ is a sub-scroll of $X$ given by a Segre product (Proposition 4.2).

This precise description of $\Phi_2(X)$ allows us to study in Section 4 the second discriminant locus of a decomposable scroll $X \subset \mathbb{P}^N$. This is the Zariski closed subset $D$ of $\mathbb{P}^{N+1}$ parameterizing all hyperplane sections of $X$ admitting a triple point. The main component of $D$ is the second dual variety of $X$, which parameterizes...
osculating hyperplanes to \( X \) at general points and their limits. But when \( X \) has flexes, extra components \( \mathcal{D}_G \) of \( \mathcal{D} \) arise, coming from the irreducible components \( G \) of \( \Phi_2(X) \). Our study of \( \Phi_2(X) \) allows us to describe these components: either \( \mathcal{D}_G \) is a linear space or it is a 1-dimensional family of linear spaces. In particular, we show that \( \mathcal{D}_G \) is a scroll if and only if \( X \) is a rational normal scroll generated by some lines plus conics and/or twisted cubics (Example 4.3 and Proposition 4.4). Moreover, we characterize rational normal scrolls generated by some lines plus some conics as the decomposable scrolls admitting an irreducible component \( \mathcal{D}_G \) of \( \mathcal{D} \) which is a rational normal scroll (Theorem 4.7).

In Section 5, we come to surface scrolls, not necessarily linearly normal, regardless the fact they are decomposable. Here, we investigate those of invariant \(-1\), the triplet \((X,L,W)\), or the corresponding pair \((X,W)\), in place of the non-degenerate embedded variety \( \varphi_W(X) \subset \mathbb{P}^N \) and sometimes we do not distinguish between \( X \) and its image.

For any integer \( k \geq 0 \) let \( J_k L \) be the \( k \)-th jet bundle of \( L \). For every \( x \in X \) we denote by
\[
j_{k,x}^{(X,L)} : W \rightarrow (J_k L)_x
\]
the homomorphism associating to every section \( s \in W \) its \( k \)-th jet evaluated at \( x \). When the subspace \( W \) we are dealing with is clear from the context, or the discussion involves a single pair \((X,W)\), we simply write \( j_{k,x}^X \) or \( j_{k,x} \) respectively, instead of \( j_{k,x}^{(X,L)} \). Recall that \( j_{k,x}(\sigma) \) is represented in local coordinates by the Taylor expansion of \( \sigma \) at \( x \), truncated after the order \( k \). So, if \( |W| \) is very ample, the \( k \)-th osculating subspace to \( X \) at a point \( x \in X \) is defined as \( \text{Osc}^k_x(X) := \mathbb{P}(\text{Im} j_{k,x}^{(X,W)}) \). Identifying \( \mathbb{P}^N \) with \( \mathbb{P}(W) \) (the set of codimension 1 vector subspaces of \( W \)) we see that \( \text{Osc}^k_x(X) \) is a linear subspace of \( \mathbb{P}^N \). To avoid that it fills up the whole ambient space we assume that \( N \) is large enough. For instance, to discuss osculation for surfaces, i.e., \( k = n = 2 \), a reasonable assumption is that \( N \geq 6 \) or even 5, depending on the regularity of the surface we are dealing with. Recalling that \( \text{rk}(J_k L) = \binom{k+n}{n} - 1 \) we have \( \dim(\text{Osc}^k_x(X)) \leq \min\{N,\binom{k+n}{n} - 1\} \). Let \( \mathcal{U} \subseteq X \) be the Zariski dense open subset where the rank of the homomorphism \( j_{k,x}^{(X,W)} : W \rightarrow (J_k L)_x \) attains its maximum, say \( s(k) + 1 \). The \( k \)-th inflectional locus of \((X,W)\) is defined by \( \Phi_k(X) = X \setminus \mathcal{U} \). So \( x \in \Phi_k(X) \) if and only if \( \dim(\text{Osc}^k_x(X)) < s(k) \). By flex we simply mean a point in \( \Phi_2(X) \), while a higher flex is a point of \( \Phi_k(X) \) with \( k > 2 \). We say that \( X \) is uninflected to mean that \( \Phi_2(X) = \emptyset \). Of course \( \Phi_0(X) \subseteq \Phi_k(X) \) for \( h \leq k \). Let \( n = 1 \). If \( N < k \), then clearly \( \Phi_k(X) = X \). However, if \( N \geq k \) then \( \Phi_k(X) \not\subseteq X \) (e.g., see [1] p. 37, Ex C-2).

In particular, \( \Phi_N(X) = \emptyset \) if and only if \( X \) is a rational normal curve ([1] p. 39, Ex C-14).

Now let \( x \in \mathcal{U} \). A hyperplane \( H \in \mathbb{P}^{N \vee} \) is said to be \( k \)-th osculating to \( X \) at \( x \) if \( H \supseteq \text{Osc}^k_x(X) \). Then the \( k \)-th dual variety \( X^\vee_k \) of \((X,W)\) is defined as the closure in \( \mathbb{P}^{N \vee} \) of the locus parameterizing all \( k \)-th osculating hyperplanes to \( X \) at points of \( \mathcal{U} \).

By scroll we mean an embedded smooth projective variety \( Y \subset \mathbb{P}^N \) of dimension \( n \geq 1 \) endowed with a morphism \( \pi : Y \rightarrow C \) over a smooth curve \( C \) such that \( (f,\mathcal{O}_{\mathbb{P}^n}(1))f = (\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \) for every fibre \( f \) of \( \pi \), or the corresponding pair \((X,W)\) with \(|W|\) very ample, such that \( Y = \varphi_W(X) \). Of course \( X = C \) if \( n = 1 \). We need to fix some more notation.
Let \((X, W)\) be a scroll. As is known, for any \(k \geq 2\) we have a strict inequality \(\dim(\text{Osc}_k^i(X)) < \binom{k+n}{n} - 1\) at every point \(x \in X\). In fact, there are local coordinates \((u, v_2, \ldots, v_n)\) around every point \(x \in X\) such that the homogeneous coordinates \(x_i (i = 0, \ldots, N)\) of the points of the variety locally can be written as \(x_i = a_i(u) + \sum_{j=2}^{n} v_j b_{ij}(u)\), where \(a_i\) and \(b_{ij}\) are holomorphic functions of \(u\). Since every section \(\sigma \in W\) is a linear combination \(\sigma = \sum_{i=0}^{N} \lambda_i x_i\) we thus see that the second derivatives \(\sigma_{v_j v_n}\) vanish at every point. Then \(\dim(\text{Osc}_k^i(X)) \leq 2n\), and differentiating further up to the order \(k\) we see that

\[ \dim(\text{Osc}_k^i(X)) \leq nk \quad \text{for every} \ x \in X. \]

Finally, we set \(\mathbb{F}_e = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e))\) to denote the Segre–Hirzebruch surface of invariant \(e (e \geq 0)\). Then, as in \([5, \text{p. 372}]\), \(C_0\) stands for a section of minimal self-intersection and \(f\) for a fibre.

1 Decomposable scrolls and their flexes

The situation we consider for the most part of this paper is inspired by that in \([9\] and [7, Sec. 2]. Let \(C\) be a smooth curve of genus \(g\). For \(i = 1, \ldots, n\) let \(L_i\) be a very ample line bundle on \(C\) and let \(V_i \subseteq H^0(C, L_i)\) be a vector subspace such that \(|V_i|\) gives rise to an embedding

\[ \varphi_i : C \rightarrow \mathbb{P}^n = \mathbb{P}(V_i). \]

Set \(C_i = \varphi_i(C)\). Let \(V = \bigoplus_{i=1}^{n} V_i, \mathcal{E} = \bigoplus_{i=1}^{n} L_i\) and consider the projective bundle \(P = \mathbb{P}(\mathcal{E})\). By identifying \(V\) with a vector subspace of \(H^0(P, L)\), where \(L\) is the tautological line bundle on \(P\), we get an embedding

\[ \varphi : P \rightarrow \mathbb{P}^N = \mathbb{P}(V). \]

We set \(X = \varphi(P)\). According to \([7, \text{p. 151}]\) we say that \(X\) is the decomposable scroll generated by \(C_1, \ldots, C_n\). For a point \(p \in C\), let \(p_i = \varphi_i(p) \in C_i\). Geometrically, \(X\) is generated by the linear spaces \(f_p : = \langle p_1, \ldots, p_n \rangle \cong \mathbb{P}^{n-1}\) as the point \(p\) varies on \(C\); note that all the linear spans \(\langle C_i \rangle = \mathbb{P}^{r_i}\) of the \(C_i\)'s are skew each other and generate the whole ambient space \(\mathbb{P}^N\). Let \(t\) be a local parameter on \(C\) such that \(\varphi_i(p) = (x_0(0), \ldots, x_{r_i}(0))\) corresponds to \(t = 0\). Locally, around \(p\), the homomorphism \(j_k^{C_i} : V_i \rightarrow j_k L_i\) is represented by the matrix

\[ M_k^i(t) = \begin{pmatrix} x_0(t) & x_1(t) & \cdots & x_{r_i}(t) \\ x_0'(t) & x_1'(t) & \cdots & x_{r_i}'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{(k)}(t) & x_1^{(k)}(t) & \cdots & x_{r_i}^{(k)}(t) \end{pmatrix}. \]

The linear space spanned by the row vectors of the matrix \(M_k^i(t)\) defines the \(k\)-th osculating space to \(C_i\) at \(p_i\). Note that, if \(k > r_i\), then every \(k\)-th osculating space to \(C_i\) is the whole space \(\mathbb{P}^{r_i} = \mathbb{P}(V_i)\). Now let \(\lambda_1, \ldots, \lambda_n\) denote homogeneous coordinates corresponding to a local trivialization of \(\mathcal{E}\) around \(p_i\), and, for \(\lambda_n \neq 0\), set \(v_i = \lambda_i/\lambda_n\). Then \((t, v_1, \ldots, v_{n-1})\) provide local coordinates on \(X\) at a point \(x \in f_p \setminus \langle p_1, \ldots, p_{n-1} \rangle\). Writing down the parametric equations for \(X\) around \(f_p\) we can easily get the matrix \(M_k^X(t, v_1, \ldots, v_{n-1})\) representing \(j_k^X : V \rightarrow j_k L\) near \(x\). Set \(M_{k-1} = M_{\text{min}(k-1, r_i)}\).

**Lemma 1.1** [7, p. 152] We have

\[
M_k^X(t, v_1, \ldots, v_{n-1}) = \begin{pmatrix} v_1 M_k^1 \ v_2 M_k^2 & \ldots & v_{n-1} M_k^{n-1} & M_k^n \\ M_{k-1}^1 & 0 & \ldots & 0 & 0 \\ 0 & M_{k-1}^2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{k-1}^{n-1} & 0 \end{pmatrix} \begin{pmatrix} t \end{pmatrix}.
\]
Let $x \in f_p \setminus \langle p_1, \ldots, \widehat{p_i}, \ldots, p_n \rangle$, where $\widehat{}$ denotes suppression. Up to reordering the $C_i$'s, there is no restriction if we suppose that $s = n$, hence the matrix representing $j_k^X$ is that given by Lemma 1.1. Sometimes, however, it is convenient to order the $C_i$'s according to some criterion (e. g., in such a way that $r_1 \leq r_2 \leq \cdots \leq r_n$). In this case, we can write $x = u_1 p_1 + \cdots + u_{s-1} p_{s-1} + p_s + u_{s+1} p_{s+1} + \cdots + u_{n-1} p_n$, and then, with respect to the local coordinates $(t, u_1, \ldots, u_{n-1})$ the matrix representing $j_k^X$ near $x$ is the following

$$
\begin{pmatrix}
n_1 M^1_k & \cdots & u_{s-1} M^{s-1}_k & M^*_k & u_s M^{s+1}_k & \cdots & u_{n-1} M^n_k \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \overbrace{M^{s+1}_{k-1}}^{(t)} & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots \\
```

We say that two matrices $A$ and $B$ of type $m \times n$ are row equivalent if the vector subspace of $\mathbb{C}^n$ spanned by the rows of $A$ is the same as that spanned by the rows of $B$.

Here is an immediate application.

**Theorem 1.2** Let $X$ be a decomposable scroll generated by $C_1, \ldots, C_n$ and let $\Phi_2(X)$ be its inflectional locus.

(1) The following three conditions are equivalent:

(i) $\left( f_p \setminus \bigcup_{i=1}^{n} \langle p_1, \ldots, \widehat{p_i}, \ldots, p_n \rangle \right) \cap \Phi_2(X) \neq \emptyset$;

(ii) $p_i$ is a flex of $C_i$ for every $i = 1, \ldots, n$;

(iii) $f_p \subseteq \Phi_2(X)$.

(2) $p_i \in \Phi_2(X)$ if and only if it is a flex of $C_i$.

(3) Let $x \in \Phi_2(X)$: if $x \in f_p \setminus \langle p_1, \ldots, \widehat{p_i}, \ldots, p_n \rangle$ then $p_s$ is a flex of $C_s$.

**Proof.** To prove (1) it is enough to show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Let $x \in f_p \setminus \langle p_1, \ldots, p_{n-1} \rangle$, so that we can write $x = v_1 p_1 + \cdots + v_{n-1} p_{n-1} + p_n$. Then $x \in \Phi_2(X)$ if and only if $j^{X}_{2,x} : V \rightarrow (J_2 L_x)$ has rank $< 2n + 1$. Note that $\text{rk}(M_1^i(t)) = 2$ and 

$$\text{rk}(M_2^i(t)) \geq 2$$

for every $i$ and for every $t$. Then Lemma 1.1 shows that $\text{rk}(M_2^X(0, v_1, \ldots, v_{n-1})) < 2n + 1$ if and only if both $\text{rk}(M_2^0(0)) = 2$ and 

$$\text{rk}((v_1 M^1_n \ v_2 M^2_n \ \cdots \ v_{n-1} M^{n-1}_n)(0)) = 2.$$ 

The former condition says that $j_{2,p} : V_t \rightarrow (J_2 L_n)_p$ has rank 2, while by (i) the latter one is equivalent to saying that either $v_i = 0$ or $\text{rk}(M_2^i(0)) = 2$ for every $i = 1, \ldots, n - 1$. In conclusion we have that $j_{2,p} : V_t \rightarrow (J_2 L_i)_p$ has rank 2 for $i = n$ and for every $i$ such that $v_i \neq 0$. So, if $x \in \Phi_2(X)$ is a general point as in (i), we get (ii). On the other hand, if (ii) holds, then we see that $f_p \setminus \langle p_1, \ldots, p_{n-1} \rangle$, hence its closure $f_p$, lies in $\Phi_2(X)$. So (1) is proved. Moreover, the above argument proves the “only if” part of (2) when $x = p_n$, and (3) in the special case $s = n$. As to the “if” part of (2), note that if $x = p_n$ then the matrix $M_2^X(0, 0, \ldots, 0)$ of Lemma 1.1 has the
following special form:

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & M_2^n \\
M_1 & 0 & \ldots & 0 & 0 \\
0 & M_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & M_1^{n-1} & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

So, if \( p_n \) is a flex of \( C_n \) we get \( \text{rk}(M_2^X(0, 0, \ldots, 0)) = 2n \), since \( M_2^n(0) \) has rank 2. Now, let \( x \) be as in (3); so we can write \( x = u_1(p_1 + \ldots + u_{x-1}p_{x-1} + p_x + u_xp_{x+1} + \cdots + u_{n-1}p_n) \). Then one can easily see that the matrix representing \( j_2^X : V \to J_2L \) near \( x \) is

\[
\begin{pmatrix}
u_1M_2^1 & \ldots & u_{x-1}M_2^{x-1} & M_2^x & u_xM_2^{x+1} & \ldots & u_{n-1}M_2^n \\
M_1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & \ldots & M_1^n
\end{pmatrix}
\]

Thus the same argument as above works and shows that since \( x \in \Phi_2(X) \), \( p_n \) must be a flex of \( C_n \). This completes the proof of (3) and (2). \( \square \)

**Corollary 1.3** \( X \) is uninflected if and only if \( C_1, \ldots, C_n \) are uninflected.

The same argument proving Theorem 1.2 says more.

**Proposition 1.4** For any \( x \in \Phi_2(X) \) we have

\[
\text{Osc}_2^X(X) = \langle \text{Osc}_{p_1}^1(C_1), \ldots, \text{Osc}_{p_n}^1(C_n), \text{Osc}_{p_n}^1(C_n) \rangle
\]

for some \( s \), where, \( p_s \in \Phi_2(C_s) \). Moreover, \( \text{Osc}_2^X(X) \) is the same linear \( \mathbb{R}^{2n-1} \) for all \( x \in \Phi_2(X) \cap f_p \).

**Proof.** First, suppose that \( x \not\in \langle p_1, \ldots, p_{n-1} \rangle \). Then \( x = v_1p_1 + \cdots + v_{n-1}p_{n-1} + p_n \). As \( x \in \Phi_2(X) \), the first block of rows in the matrix \( M_2^X(0, v_1, \ldots, v_{n-1}) \) appearing in Lemma 1.1 for \( k = 2 \) has rank 2. In particular, \( p_n \in \Phi_2(C_n) \) by Theorem 1.2(3). Moreover, either \( v_1 = 0 \) or \( v_1M_2^1 \) is row equivalent to \( M_1^1 \). Hence \( M_2^X \) is row equivalent to the matrix

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & M_2^n \\
M_1 & 0 & \ldots & 0 & 0 \\
0 & M_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & M_1^{n-1} & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

This means exactly that

\[
\text{Osc}_2^X(X) = \langle \text{Osc}_{p_1}^1(C_1), \ldots, \text{Osc}_{p_{n-1}}^1(C_{n-1}), \text{Osc}_{p_n}^2(C_n) \rangle.
\]
Next, suppose that \( x \in \langle p_1, \ldots, p_{n-1} \rangle \setminus \langle p_1, \ldots, p_{n-2} \rangle \). Then, \( x = v_1p_1 + \cdots + v_{n-2}p_{n-2} + p_{n-1} \) can also be written as \( x = u_1p_1 + \cdots + u_{n-1}p_{n-1} + p_s + u_sp_{s+1} + \cdots + u_n p_n \), as done after Lemma 1.1, with \( s = n - 1 \) and \( u_{n-1} = 0 \). Then look at the matrix appearing after Lemma 1.1 in the present situation:

\[
\begin{pmatrix}
  u_1M_2^1 & \ldots & u_{n-2}M_2^{n-2} & M_2^{n-1} & 0 \\
  M_1^1 & \ldots & 0 & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & \ldots & M_1^{n-2} & 0 & 0 \\
  0 & \ldots & 0 & 0 & M_1^1
\end{pmatrix} \quad (0)
\]

If \( x \in \Phi_2(X) \), arguing as before we see that \( p_{n-1} \in \Phi_2(C_{n-1}) \) and this matrix is row equivalent to

\[
\begin{pmatrix}
  0 & \ldots & 0 & M_2^{n-1} & 0 \\
  M_1^1 & \ldots & 0 & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & \ldots & M_1^{n-2} & 0 & 0 \\
  0 & \ldots & 0 & 0 & M_1^1
\end{pmatrix} \quad (0)
\]

This means that

\[
\text{Osc}_x^2(X) = \langle \text{Osc}_{p_1}^1(C_1), \ldots, \text{Osc}_{p_{n-2}}^1(C_{n-2}), \text{Osc}_{p_{n-1}}^2(C_{n-1}), \text{Osc}_{p_n}^1(C_n) \rangle.
\]

Now, let \( s \leq n - 2 \). By repeating the argument for \( x \in \langle p_1, \ldots, p_s \rangle \setminus \langle p_1, \ldots, p_{s-1} \rangle \), we see that \( p_s \in \Phi_2(C_s) \) and \( \text{Osc}_x^2(X) \) is the linear span of \( \text{Osc}_{p_s}^1(C_s) \) and the spaces \( \text{Osc}_{p_i}^j(C_i) \) for \( i \neq s \). This proves the first assertion. Now, note that all \( \text{Osc}_{p_i}^1(C_i) \) are lines. Moreover, as we have shown, \( p_s \) is a flex for \( C_s \), hence \( \text{Osc}_{p_s}^2(C_s) \) is also a line. Thus, for any \( x \in X \), \( \text{Osc}_x^2(X) \) is the linear space generated by the \( n \) tangent lines to \( C_i \) at \( p_i \) for \( i = 1, \ldots, n \). Note that they generate a \( \mathbb{P}^{2n-1} \). It turns out that \( \text{Osc}_x^2(X) \) is the same \( \mathbb{P}^{2n-1} \) for all \( x \in \Phi_2(X) \cap f_p \). □

2 Higher flexes and fibres

Let \( X \) be a decomposable scroll over a smooth curve \( C \) generated by \( C_1, \ldots, C_n \) as in Section 1, and let \( f_p = \langle p_1, \ldots, p_n \rangle \) be the fibre over \( p \in C \). In this section we explore some connections between the higher inflectional loci \( \Phi_k(X) \) and the fibres of \( X \).

Remark 2.1 We have

\[
\text{Osc}_{p_s}^k(X) = \langle \text{Osc}_{p_1}^{k-1}(C_1), \ldots, \text{Osc}_{p_s}^k(C_s), \ldots, \text{Osc}_{p_n}^{k-1}(C_n) \rangle
\]

for any \( s = 1, \ldots, n \) (the only \( k \)-th osculating space on the right hand is that at \( p_s \)). In particular, if \( p_s \in \Phi_k(C_s) \), then \( p_s \in \Phi_k(X) \).

Proof. Up to reordering the curves we can suppose that \( s = n \). Then the matrix representing \( j_{k,p_s}^X \), is, according to Lemma 1.1,

\[
\begin{pmatrix}
  0 & \ldots & 0 & M_k^n \\
  M_{k-1}^1 & \ldots & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & M_{k-1}^{n-1} & 0 \\
  0 & \ldots & 0 & 0
\end{pmatrix}
\]

This proves the first assertion. Note that all linear spaces appearing on the right hand of (2) are skew each other.
Then the second assertion follows from the inequality:

\[
\dim \left( \text{Osc}^k_{p_1}(X) \right) \leq (n - 1)(k - 1) + \dim \left( \text{Osc}^k_{p_1}(C_s) \right) + (n - 1) \\
< (n - 1)k + k = nk.
\]

(4) □

As to the converse, if \( p_s \in \Phi_k(X) \), we cannot claim that \( p_s \in \Phi_k(C_s) \) if \( k > 2 \). However, we have

**Remark 2.2** If \( p_s \in \Phi_k(X) \), then either \( p_s \in \Phi_k(C_s) \), or \( p_j \in \Phi_{k-1}(C_j) \) for some \( j \neq s \).

**Proposition 2.3** Let \( p_i \in \Phi_k(C_i) \) for \( i = 1, \ldots, s, \ldots, n \), where \( \sim \) denotes suppression. Then for every point \( x \in f_p \setminus \langle p_1, \ldots, \hat{p}_s, \ldots, p_n \rangle \) we have

\[
\text{Osc}^k_x(X) = \langle \text{Osc}^{k-1}_{p_1}(C_1), \ldots, \text{Osc}^k_{p_s}(C_s), \ldots, \text{Osc}^{k-1}_{p_n}(C_n) \rangle
\]

(the only \( k \)-th osculating space on the right hand is that at \( p_s \)).

**Proof.** Up to reordering we can suppose that \( s = n \). Due to the assumption, we have \( \text{Osc}^k_{p_i}(C_i) = \text{Osc}^{k-1}_{p_i}(C_i) \) for \( i = 1, \ldots, n - 1 \). This means that the two matrices \( M^i_k \) and \( \hat{M}^{i-1}_{k-1} \) are row equivalent for \( i = 1, \ldots, n - 1 \). Now look at the matrix \( M \) of Lemma 1.1. By subtracting suitable linear combinations of the subsequent rows from the first block of rows we see that \( M \) is row equivalent to the matrix \( \hat{M}^0_k \). This proves the assertion. □

The same argument proves more. Actually, assume that \( p_i \in \Phi_k(C_i) \) for \( j = 1, \ldots, s \) and set \( \Lambda = \langle p_1, \ldots, p_s \rangle \). Up to reordering we can suppose that \( (i_1, \ldots, i_{s-1}, i_s) = (1, \ldots, s-1, n) \). Then for any \( x \in \Lambda \setminus \langle p_1, \ldots, p_s \rangle \) we can write \( x = v_1p_1 + \ldots + v_{s-1}p_{s-1} + p_n \). Arguing as in the proof of Proposition 2.3 we have that the matrices \( M^i_k \) and \( \hat{M}^{i-1}_{k-1} \) are row equivalent for \( i = 1, \ldots, s - 1 \). Now look at the matrix \( M \) of Lemma 1.1, representing \( j_{k,x} \). The first block of rows of \( M \) is

\[
\begin{pmatrix}
(v_1M^1_k & \ldots & v_{s-1}M^{s-1}_{k-1} & 0 & \ldots & 0 & M^n_k)
\end{pmatrix}.
\]

By subtracting suitable linear combinations of the subsequent rows of \( M \) from the first block we see that \( M \) is row equivalent to the matrix in \( \hat{M}^0_k \). Now, since also \( p_n \in \Phi_k(C_n) \) we have \( \text{rk}(M^n_k) < k + 1 \) and then the same computation done to prove Remark 2.1 holds at \( x \), giving \( \dim \left( \text{Osc}^k_x(X) \right) < nk \). Thus \( \Lambda \setminus \langle p_1, \ldots, p_{s-1} \rangle \subseteq \Phi_k(X) \). On the other hand \( \Phi_k(X) \cap f_p \) is a Zariski closed subset, hence \( \Lambda \subseteq \Phi_k(X) \).

Now suppose that \( (i_1, \ldots, i_s) = (1, \ldots, s) \), with \( s \leq n - 1 \) and \( p_n \notin \Phi_k(C_n) \). Then \( \text{rk}(M^n_k) = k + 1 \), and the same argument as above applied to any point \( x \in \langle p_1, \ldots, p_n \rangle \setminus \Lambda \) shows that

\[
\dim \left( \text{Osc}^k_x(X) \right) = \sum_{i=1}^{n-1} \text{rk}(\hat{M}^{i-1}_{k-1}) + (k + 1) - 1.
\]

In particular, if \( p_i \notin \Phi_{k-1}(C_i) \) for \( i = 1, \ldots, s \), then all the first \( n - 1 \) summands are equal to \( k \), hence \( \dim \left( \text{Osc}^k_x(X) \right) = nk \), and so \( x \notin \Phi_k(X) \). This proves the following

**Proposition 2.4** If \( p_{ij} \in \Phi_k(C_i) \) for \( j = 1, \ldots, s \leq n \), then \( \langle p_{i1}, \ldots, p_{is} \rangle \subseteq \Phi_k(X) \). Moreover, if \( p_{ij} \in \Phi_k(C_i) \setminus \Phi_{k-1}(C_i) \) for \( j = 1, \ldots, s \leq n - 1 \) and \( p_{ij} \notin \Phi_k(C_i) \) for \( j = s + 1, \ldots, n \), then \( \Phi_k(X) \cap f_p = \langle p_{i1}, \ldots, p_{is} \rangle \).

**Corollary 2.5** i) If \( p_i \in \Phi_k(C_i) \) for every \( i = 1, \ldots, n \), then \( f_p \subseteq \Phi_k(X) \).

ii) If \( f_p \subseteq \Phi_k(X) \), then \( p_i \in \Phi_k(C_i) \) for some \( i \).

**Proof.** i) is obvious; ii) follows from Remark 2.2, taking into account the inclusion \( \Phi_{k-1}(C_j) \subseteq \Phi_k(C_j) \). □

In particular, if \( f_p \subseteq \Phi_k(X) \) and \( k > 2 \), we see that not necessarily \( p_i \in \Phi_k(C_i) \) for all \( i \)'s. For \( n = 2 \) we can be more explicit.
Proposition 2.6 Let $n = 2$ and $k \geq 2$. Then $f_p \subseteq \Phi_k(X)$ if and only if either

a) $p_i \in \Phi_{k-1}(C_i)$ for some $i$, or
b) $p_i \in \Phi_k(C_i)$ for $i = 1, 2$.

Proof. Let $f_p \subseteq \Phi_k(X)$. By Corollary 2.5, ii), up to reordering, we can suppose that $p_1 \in \Phi_k(C_1)$. Then, by Proposition 2.3, for every $x \in f_p \setminus \{p_1\}$, we have

$$\text{Osc}_x^k(X) = (\text{Osc}_{p_1}^{k-1}(C_1), \text{Osc}_{p_2}^k(C_2)).$$  \hspace{1cm} (5)

Hence

$$\dim(\text{Osc}_x^k(X)) = \dim(\text{Osc}_{p_1}^{k-1}(C_1)) + \dim(\text{Osc}_{p_2}^k(C_2)) + 1.$$  \hspace{1cm} (6)

Since $x \in \Phi_k(X)$ this shows that either $p_1 \in \Phi_{k-1}(C_1)$, case a), or $p_2 \in \Phi_k(C_2)$, case b). To prove the converse, in both cases a) and b), up to renaming, we can assume that $p_1 \in \Phi_k(C_1)$. Then Proposition 2.3 gives again (5) for any $x \in f_p \setminus \{p_1\}$ and then (6) shows that

$$\dim(\text{Osc}_x^k(X)) \leq \begin{cases} 
  k - 2 + k + 1 & \text{in case a),} \\
  k - 1 + (k - 1) + 1 & \text{in case b).}
\end{cases}$$

Hence $f_p \setminus \{p_1\} \subseteq \Phi_k(X)$ in both cases, and then, taking the closure, we get $f_p \subseteq \Phi_k(X)$.

Theorem 2.7 Let $n = 2$. Suppose that $x \in \Phi_k(X)$ and let $f_p$ be the fibre of $X$ containing $x$.

i) If $x \neq p_1, p_2$, then $f_p \subseteq \Phi_k(X)$;

ii) if $x = p_i$, then either $f_p \subseteq \Phi_k(X)$ or $p_i \in \Phi_k(C_i)$.

Proof. If $x \neq p_1$, then we can write $x = vp_1 + p_2$. According to Lemma 1.1, $j_{k,x}^X$ is represented by the following matrix

$$M = \left( \begin{array}{cc}
  vM_1^k & M_2^k \\
  M_{k-1}^k & 0
\end{array} \right).$$

Since $x \in \Phi_k(X)$, $M$ has rank $\leq 2k$. This implies either

$\alpha)$ \ $\text{rk}(M_{k-1}^k) < k - 1$, i.e., $p_1 \in \Phi_{k-1}(C_1)$, or

$\beta)$ $\text{rk}(vM_k^1 M_k^2) < k$.

Condition $\beta)$ in turn implies both $\text{rk}(M_k^1) < k$ and $\text{rk}(M_k^2) < k$. The latter condition means that $p_2 \in \Phi_k(C_2)$, while the former is equivalent to saying that

either \ $v = 0$, or \ $\text{rk}(M_k^1) < k$.

In other words, either $x = p_2$ or $p_1 \in \Phi_k(C_1)$. In conclusion, if $x \neq p_1, p_2$ then either

$\alpha)$ $p_1 \in \Phi_{k-1}(C_1)$, or

$\beta)$ $p_i \in \Phi_k(C_i)$ for $i = 1, 2$.

In both cases $f_p \subseteq \Phi_k(X)$ by Proposition 2.6. This proves i). Now let $x = p_i$. By Remark 2.2 either $p_i \in \Phi_k(C_i)$ or $p_j \in \Phi_{k-1}(C_j)$ for $j \neq i$. But in the latter case Proposition 2.6 says that $f_p \subseteq \Phi_k(X)$ again. This proves ii).
Corollary 2.8 Let \( n = 2 \). Then \( \Phi_k(X) = \emptyset \) if and only if \( \Phi_k(C_i) = \emptyset \) for \( i = 1, 2 \).

Proof. If \( p_i \in \Phi_k(C_i) \) for some \( i \), we know that \( p_i \in \Phi_k(X) \) by Remark 2.2. This proves the “only if” part. To prove the “if” part suppose, by contradiction, that \( x \in \Phi_k(X) \), and let \( f_p \) be the fibre of \( X \) through \( x \). By Theorem 2.7 either \( f_p \subseteq \Phi_k(X) \) or \( p_i \in \Phi_k(C_i) \) for some \( i \). In both cases, taking into account Proposition 2.6, we see that \( \Phi_k(C_i) \neq \emptyset \) for some \( i \). But this is a contradiction. \( \square \)

Now suppose that \( r_1 \leq r_2 \), where \( \langle C_i \rangle = \mathbb{P}^r \). If \( r_1 < k - 1 \), then we have \( \dim(\text{Osc}_k^{k-1}(C_1)) \leq r_1 < k - 1 \) for every \( y \in C_1 \). In other words, \( \Phi_{k-1}(C_1) = C_1 \) and therefore \( \Phi_k(X) = X \) by Proposition 2.6. Let \( r_1 \geq k - 1 \). If \( r_2 < k \) (i.e., \( r_1 = r_2 = k - 1 \)), then \( \Phi_k(C_i) = C_i \) for \( i = 1, 2 \), hence \( \Phi_k(X) = X \) again, by Proposition 2.6. If \( r_1 = k - 1 \) but \( r_2 \geq k \) then \( \Phi_k(C_1) = C_1 \) but \( \Phi_k(C_2) \subseteq C_2 \) (e.g., see [1] p. 37, Ex. C-2); so \( \Phi_k(X) \) contains every fibre of \( X \) passing through a point of either \( \Phi_{k-1}(C_1) \) or \( \Phi_k(C_2) \). Taking into account also Remark 2.1, we thus get the following

Corollary 2.9 Let \( n = 2 \) and suppose that \( r_1 \leq r_2 \). If \( \Phi_{k-1}(C_1) = \emptyset \) but \( \Phi_k(C_1) = C_1 \), then \( \Phi_k(X) \) consists of \( C_1 \) plus the fibres containing a point of \( \Phi_k(C_2) \).

3 Surface scrolls: examples and applications

In this section we focus on the surface case \( (n = 2) \). Let \( X \subset \mathbb{P}^N \) be a decomposable surface scroll as in Section 1. We present some examples concerned with the dimension that \( \text{Osc}_k^X(X) \) can have at some point \( x \), and with the structure of \( \Phi_k(X) \), focusing in particular on the case of non-normal rational scrolls. First it is useful to recall the situation for normal rational scrolls.

Example 3.1 Notation as in Section 1; let \( C = \mathbb{P}^1, \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(r_1) \oplus \mathcal{O}_{\mathbb{P}^1}(r_2) \), with \( 1 \leq r_1 \leq r_2 \), and let \( X \subset \mathbb{P}^N \) be the image of \( \mathbb{P}(\mathcal{E}) \) in the embedding given by complete linear system associated with the tautological line bundle \( L \). Note that \( N = r_1 + r_2 + 1 \). Let \( p = (t_0 : t_1) \in \mathbb{P}^1 \) and set \( t = t_1 / t_0 \) (or \( t_0 / t_1 \)). At any point \( x \in X \setminus C_1 \) we can use local coordinates \((t, v)\) to write \( x = vp_1 + p_2 \) on the fibre \( f_p \); then, according to Lemma 1.1, the homomorphism \( j_k^X : H^0(X, L) \to J_kL \) is represented near \( x \) by the matrix

\[
M_k^X(t, v) = \begin{pmatrix} vM_k^1 & M_k^2 \\ M_{k-1} & 0 \\ 0 & 0 \end{pmatrix}(t).
\]

Note that

\[
\text{rk}(M_k^2(t)) = \min\{k + 1, r_2 + 1\}.
\]

Moreover

\[
\text{rk}(M_{k-1}^1(t)) = \begin{cases} \text{rk}(M_{k-1}^1) = k & \text{if } k - 1 \leq r_1, \\
\text{rk}(M_{r_1}^1) = r_1 + 1 & \text{otherwise}. \end{cases}
\]

It follows that

\[
\text{rk}(j_k^X) = \text{rk}(M_k^X(t, v)) = \text{rk}(M_k^2(t)) + \text{rk}(M_{k-1}^1(t)) = \min\{k + 1, r_2 + 1\} + \min\{k, r_1 + 1\}.
\]

Therefore

\[
\dim(\text{Osc}_k^X(X)) = \begin{cases} 2k & \text{if } k \leq r_1 + 1, \\
r_1 + r_2 + 1 & \text{if } r_1 + 1 \leq k \leq r_2, \\
r_1 + r_2 + 1 & \text{if } k \geq r_2. \end{cases}
\]
Note that at any point \( x \in X \setminus C_1 \) the dimension of the \( k \)-th osculating space can be strictly smaller than \( 2k \). This is obvious when \( N < 2k \), but it can happen also for \( k \leq \left\lfloor \frac{N-1}{2} \right\rfloor \), e.g., for a very unbalanced rational normal scroll (i.e., with invariant \( e := r_2 - r_1 \) very large). In fact, for \( k \leq \left\lfloor \frac{N-1}{2} \right\rfloor \) we have \( \dim (\operatorname{Osc}^{k}_x(X)) < 2k \) if \( r_1 + 1 < k \) from (3). This means \( N + 1 - e < 2k \leq N - 1 \), hence \( e \geq 3 \) is enough. For \( k \geq 2 \), (3) also shows that

\[
 k + 2 \leq \dim (\operatorname{Osc}^{k}_x(X)) \leq 2k
\]

at any point \( x \in X \setminus C_1 \). In particular, letting \( k = 2 \) we see that \( \dim (\operatorname{Osc}^{2}_x(X)) = 4 \) for any \( x \in X \setminus C_1 \).

We want to stress that \( X \) was linearly normal in the example above. Here is an enlightening example showing how small the dimension of \( \operatorname{Osc}^{k}_x(X) \) can be at some point \( x \), for any \( k \), when we drop linear normality.

**Example 3.2** Fix integers \( k \geq 2 \) and \( r = r_2 \geq 3 \). Let \( C = \mathbb{P}^1 \), \( E = L_1 \oplus L_2 \), where \( L_1 = \mathcal{O}_{\mathbb{P}^1}(1), L_2 = \mathcal{O}_{\mathbb{P}^1}(k + r - 1) \), and let \( V_1 = H^0(\mathbb{P}^1, L_1) = \langle t_0, t_1 \rangle \), \( V_2 = \langle t_0^{k+r-1}, t_0^{k+r-2}, t_1, t_0^{r-2}t_1^{k+1}, \ldots, t_0^{k+r-2}, t_1^{k+r-1} \rangle \).

Note that \( \phi_2 : \mathbb{P}^1 \to \mathbb{P}^r \) defines an embedding, which is not linearly normal, since \( \dim V_2 = r + 1 < h^0(L_2) \). Then \( X \subset \mathbb{P}^{r+2} \), defined as in Section 1, is a rational non-normal scroll. Let \( L \) be the hyperplane bundle and let \( V \subset H^0(X, L) \) be the subspace giving rise to the embedding. Note that at the point \( p \in \mathbb{P}^1 \), corresponding to \( (t_0 : t_1) = (1 : 0) \) we have

\[
|V_2 - 2p| = \cdots = |V_2 - (k + 1)p|.
\]

This means that for every \( h \), \( 2 \leq h \leq k \) the homomorphism

\[
 j_{h, p}^{C_2} : V_2 \to (J_k L_2)_p
\]

has a 2-dimensional image (isomorphic to \( (J_1 L_2)_p \)), i.e., \( \operatorname{rk}(j_{h, p}^{C_2}) = 2 \). On the other hand, at every point \( q \in \mathbb{P}^1 \) it is obvious that \( \operatorname{rk}(j_{h, q}^{C_2}) = 2 \) for any \( h \geq 1 \). Now, let \( x \in f_p \). If \( x \in f_p \setminus \{p_2\} \), Proposition 2.3 shows that

\[
\operatorname{Osc}^{h}_x(X) = \langle \operatorname{Osc}^{h}_{p_1}(C_1), \operatorname{Osc}^{h-1}_{p_2}(C_2) \rangle
\]

for any \( h = 2, \ldots, k \). On the other hand, Remark 2.1 tells us that

\[
\operatorname{Osc}^{h}_{p_2}(X) = \langle \operatorname{Osc}^{h-1}(C_1), \operatorname{Osc}^{h}_{p_2}(C_2) \rangle
\]

for any \( h \geq 2 \). In both cases \( \operatorname{Osc}^{h}_x(X) \) is the linear span of two skew lines, namely \( C_1 \) and \( \operatorname{Osc}^{h}_{p_2}(C_2) \); hence

\[
\operatorname{Osc}^{k}_{x}(X) = \operatorname{Osc}^{k-1}_x(X) = \cdots = \operatorname{Osc}^1_x(X) \quad \text{for every } k \geq 3,
\]

at every point \( x \in f_p \). In particular,

\[
\dim (\operatorname{Osc}^{k}_x(S)) = 3 \quad \text{for all } k \geq 2 \text{ at any point } x \in f_p.
\]

We recall that if \( X \in \mathbb{P}^N \) is any scroll of dimension \( n \), then \( \dim (\operatorname{Osc}^{2}_x(X)) \geq n + 1 \) \( \textup{(2)} \) (see also \( \textup{[6]} \) for \( n = 2 \)).

**Example 3.3** Let \( C = \mathbb{P}^1 \), \( L_1 = \mathcal{O}_{\mathbb{P}^1}(m) \) with \( m \geq 2 \), \( V_1 = H^0(\mathbb{P}^1, L_1) \), and consider \( L_2 = \mathcal{O}_{\mathbb{P}^1}(d) \) with \( d \geq m + 2 \). The vector space \( H^0(\mathbb{P}^1, L_2) \) defines an embedding of \( C \) as a rational normal curve \( \Gamma \subset \mathbb{P}^d \). Projecting \( \Gamma \) from a general linear space \( T \) of dimension \( d - m - 2 \) to a \( \mathbb{P}^{m+1} \) we get an embedding. Let \( V_2 = V(T) \) be the vector subspace of \( H^0(\mathbb{P}^1, L_2) \) corresponding to this embedding. Let \( C_i \) be the image of \( C \) in the embedding defined by \( V_i \), \( i = 1, 2 \), and in the space \( \mathbb{P}^{2m+2} = \mathbb{P}(V_1 \oplus V_2) \) consider the decomposable rational scroll \( X \) generated by \( C_1 \) and \( C_2 \). Note that \( X = \mathbb{F}_{d-m} \). We claim that \( \Phi_m(X) = \emptyset \). Of course \( \Phi_m(C_1) = \emptyset \). Let \( O_T \) be the \( m \)-th osculating developable of \( \Gamma \) (i.e., the variety generated by the linear spaces \( \operatorname{Osc}^m_x(\Gamma) \), as \( x \) varies on \( \Gamma \)). Note that \( \dim (O_T) = m + 1 \), hence \( T \cap O_T = \emptyset \) for a general \( T \). Since no osculating space \( \operatorname{Osc}^m_x(\Gamma) \) meets the center of projection \( T \), we conclude that \( \Phi_m(C_2) = \emptyset \). Then the claim follows from Corollary 2.8.
Let us recall the following conjecture of Piene–Tai [10]. Let $S \subset \mathbb{P}^N (N \geq 5)$ be a non-degenerate smooth projective surface such that $\dim \left( \text{Osc}_S^i(S) \right) \leq 2k$ for all points $x \in S$ and for every $k$, with equality for $k = \left\lfloor \frac{N-1}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$ stands for the greatest integer function.

(i) If $N$ is odd, then $S$ is the balanced rational normal scroll of degree $N - 1$ (i. e., $S$ is $\mathbb{F}_0$ embedded by $|C_0 + [\frac{N-1}{2}]f|$).

(ii) If $N$ is even, then $S$ is the semibalanced rational normal scroll of degree $N - 1$ (i. e., $S$ is $\mathbb{F}_1$ embedded by $|C_0 + ([\frac{N-1}{2}] + 1)f|$).

Part (i) of this conjecture is true, as proved in [3], while part (ii) is not (see [6] Theorem A and comment after Corollary 2.3). Example 3.3 provides a new series of counterexamples to the even dimensional part of the conjecture. We want to stress that all these scrolls are decomposable, while those appearing in [6] Theorem A] are not, all being isomorphic to the elliptic $\mathbb{P}^1$-bundle of invariant $-1$. Moreover, we have the following characterization, which provides more information in order to correct the conjecture.

**Theorem 3.4** Let $X \subset \mathbb{P}^{2m+2} (m \geq 2)$ be a decomposable scroll with $n = 2$ such that $\Phi_m(X) = \emptyset$. Then either $X$ is the semibalanced rational normal scroll of degree $m + 1$, or $X$ is of the type described in Example 3.3.

**Proof.** By Corollary 2.8 it must be $\Phi_m(C_i) = \emptyset$, for $i = 1, 2$. In particular, $C_1$ cannot be a line, hence $r_1 = \dim \left( \left\langle C_1 \right\rangle \right) \geq 2$. We can assume that $r_1 \leq r_2$ and then from $r_1 + r_2 + 1 = 2m + 2$ we get that $r_1 \leq m$. As $\Phi_m(C_1) = \emptyset$, this implies that $r_1 = m$, and then $r_2 = m + 1$. So $C_1$ is a rational normal curve of degree $m$ in $\mathbb{P}^m$ while $C_2$ is either the rational normal curve of degree $m + 1$ in $\mathbb{P}^{m+1}$ or any other rational non-normal curve of some degree $d \geq m + 2$ in $\mathbb{P}^{m+1}$. In the former case $X$ is the semibalanced rational normal scroll. In the latter, $C_2$ is obtained by projecting a rational normal curve of degree $d$ in $\mathbb{P}^d$ to $\mathbb{P}^{m+1}$ from a general center as in Example 3.3.

The examples in the next part of this section are concerned with $\Phi_2(X)$. First we would like to stress that for the cubic scroll $X \subset \mathbb{P}^4$ the inflectional locus $\Phi_2(X)$ consists exactly of the generating line $C_1$. In fact this is the only semi-balanced rational normal scroll which is not uninflected. As to quartic rational normal scrolls in $\mathbb{P}^5$ the situation is also well known [11]. Let us note that the one isomorphic to $\mathbb{F}_0$ is uninflected according to Corollary 1.3, being generated by two conics $C_1, C_2$. On the other hand, the one isomorphic to $\mathbb{F}_2$ is generated by a line $C_1$ and a rational normal cubic $C_2$, which has no flexes. Hence, according to Theorem 1.2(1) its inflectional locus $\Phi_2$ consists exactly of $C_1$.

**Example 3.5** We consider quintic non-normal rational scrolls in $\mathbb{P}^6$. Let $X$ be as in Example 3.3, with $m = 1$ and $d = 4$. According to Theorem 1.2(1), the inflectional locus $\Phi_2(X)$ consists of the line $C_1$ and the fibres passing through the flexes of the non-normal quartic rational curve $C_2$. Now the center of projection $T$ is a point. If $T \not\in O_T$, then $C_2$ has no flexes, as we said, and then $\Phi_2(X) = C_1$. On the other hand, if $c \in O_T$, then $C_2$ has $\epsilon$ flexes, where $\epsilon$ is the number of osculating planes to $\Gamma$ passing through $T$. According to the enumerative formula counting the weighted number of 2-osculating lines and 3-osculating planes to $C_2$ [8] Theorem 3.2] we can see that $\epsilon = 1$ or 2. Depending on this, $\Phi_2(X)$ consists of $C_1$ plus one or two fibres.

**Example 3.6** In the same vein we can construct non-normal rational scrolls having a finite inflectional locus. Let $C, L_2, V_2$ be as in the previous example but now put $L_1 = O_{\mathbb{P}^1}(2)$ and $V_1 = H^0(\mathbb{P}^1, L_1)$. Again let $C_i$ be the image of $C$ in the embedding defined by $V_i$, $i = 1, 2$, and in the space $\mathbb{P}^6 = \mathbb{P}(V_1 \oplus V_2)$ consider the decomposable sextic rational scroll $X$ generated by $C_1$ and $C_2$. Now $C_1$ is a conic, hence it has no flexes. On the other hand $C_2$ has $\epsilon = 1$ or 2 flexes provided that the projection of $\Gamma$ giving rise to $C_2$ is made from a center $T \in O_T$. Therefore, according to Theorem 1.2 ((1) and (2)), $\Phi_2(X)$ consists of one or two points (the flexes of $C_2$).

**Example 3.7** Let $C$ be a smooth curve of genus 1, $L_1 = O_C(3p)$, for some point $p \in C$, $V_1 = H^0(C, L_1)$. Then $C_1 = \varphi_1(C)$ is a smooth plane cubic having exactly 9 flexes, one of which is $p_1 := \varphi_1(p)$. Now let $L_2$ be a line bundle of degree 5 on $C$. The vector space $H^0(C, L_2)$ defines an embedding of $C$ in $\mathbb{P}^4$ whose image, say $\Gamma$, is a quintic normal elliptic curve: then $\Gamma$ has no flexes (but 25 hyperflexes). Projecting $\Gamma$ from a point $c \in \mathbb{P}^4 \setminus \text{Sec}(\Gamma)$ to a $\mathbb{P}^3$ we get an embedding; let $V_2 = V(c)$ be the corresponding vector subspace of $H^0(C, L_2)$.
and let \( C_2 \) be the image of \( C \) in the embedding \( \varphi_2 : C \to \mathbb{P}^3 \) defined by \( V_2 \). In the space \( \mathbb{P}^6 = \mathbb{P}(V_1 \oplus V_2) \) consider the decomposable elliptic scroll \( X \) generated by \( C_1 \) and \( C_2 \). It can happen that \( p_2 := \varphi_2(p) \) is a flex of \( C_2 \) or not. According to Theorem 1.2(1), in the former case the whole fibre \( f_p \) is in the inflectional locus \( \Phi_2(X) \), while in the latter we have \( \Phi_2(X) \cap f_p = \{ p \} \). Note that if \( c \) is general enough, then \( C_2 \) has no flexes and therefore \( X \) has only 9 flexes: those of \( C_1 \).

4 The second discriminant locus of decomposable scrolls

Let \((X, L, W)\) be as at the beginning of Section 0, and let \( \mathcal{U} \subseteq X \) be the Zariski dense open subset of \( X \) where \( j_{k,x}^{(X,W)} : W \to (J_k L)_x \) attains the maximum rank \( s(k) + 1 \). If \( x \in \mathcal{U} \), the fact that \( H \in |W| \) is a \( k \)-th osculating hyperplane to \( X \) at \( x \) is equivalent to the fact that \( H = (\sigma)_0 \), where \( \sigma \in W \) and \( j_{k,x}(\sigma) = 0 \). Equivalently, this means that \( H \in |W - (k + 1)x| \), i.e., the hyperplane section cut out by \( H \) on \( X \) has a point of multiplicity \( \geq (k + 1) \) at \( x \). Note however that if \( x \notin \mathcal{U} \) and \( H \in |W - (k + 1)x| \), this does not necessarily mean that \( H \in X^\vee \). Actually \( H \in X^\vee \) if and only if \( H \) is a limit of \( k \)-th osculating hyperplanes to \( X \) at points of \( \mathcal{U} \). On the other hand we can consider the \( k \)-th discriminant locus \( \mathcal{D}_k(X,W) \) of \((X,W)\), which is defined as the image of

\[
\mathcal{J} := \{(x,H) \in X \times |W| \mid H \in |W - (k + 1)x|\}
\]

via the second projection of \( X \times |W| \). It parameterizes all hyperplane sections of \( X \subseteq \mathbb{P}^N \) admitting a singular point of multiplicity \( \geq k + 1 \); of course \( \mathcal{D}_k(X,W) \supseteq X^\vee \) with equality if and only if \( \mathcal{U} = X \), i.e., if and only if

\[
\dim (\text{Osc}_k^2(X)) = s(k) \quad \text{for every} \quad x \in X.
\]

In general \( \mathcal{D}_k(X,W) \) contains some extra components coming from the irreducible components of \( \Phi_k(X) \).

The discussion above says that \( \mathcal{D}_k(X,W) = X^\vee \) if and only if \( \Phi_k(X) = \emptyset \). From this point of view, the characterization of balanced rational normal scroll surfaces due to Ballico, Piene and Tai [3], mentioned after Example 3.3, can be rephrased as follows.

**Proposition 4.1** Let \( X \subseteq \mathbb{P}^N \) be any smooth surface, where \( N = 2m + 1 \geq 5 \). Then \( \mathcal{D}_m(X,W) = X^\vee \) if and only if \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( W = H^0(\mathbb{P}^1 \times \mathbb{P}^1, O_{\mathbb{P}^1 \times \mathbb{P}^1}(1, m)) \).

Now, let \( X \subseteq \mathbb{P}^N = \mathbb{P}(V) \) be a decomposable scroll as in Section 1. For simplicity we identify \( X \) with the corresponding abstract projective bundle \( P \). So, we denote by \( \mathcal{D}_2(X,V) \) the second discriminant locus of \((P,L,V)\). Its main component is the second dual variety \( X^\vee \) of \( X \). Note that if \( X \) is not linearly normal then \( X^\vee \) corresponds to a suitable linear section of the second dual variety of the linearly normal scroll giving rise to \( X \) via the projection to \( \mathbb{P}^N \). Here, relying on the results of Sections 1 and 2, we want to describe the extra components of \( \mathcal{D}_2(X,V) \). Of course we assume that \( \Phi_2(X) \neq \emptyset \). As a first thing we need to describe the irreducible components of \( \Phi_2(X) \).

**Proposition 4.2** Let \( X \subseteq \mathbb{P}^N \) be a decomposable scroll as in Section 1, generated by \( C_1, \ldots, C_n \), and assume that \( \Phi_2(X) \neq \emptyset \). Let \( G \) be an irreducible component of \( \Phi_2(X) \). Then, up to reordering the curves \( C_i \)'s, either

1. \( G = \langle p_1, \ldots, p_s \rangle \subseteq f_p = \langle p_1, \ldots, p_n \rangle \), or
2. \( X \) is rational and \( G = C_1 \times \langle p_1, \ldots, p_s \rangle \) is the image of \( \mathbb{P}^1 \times \mathbb{P}^{s-1} \) via the Segre embedding.

Moreover,

\[
\text{Osc}_k^2(X) = \langle \text{Osc}_{p_1}^2(C_1), \text{Osc}_{p_2}^1(C_2), \ldots, \text{Osc}_{p_n}^1(C_n) \rangle
\]

for all \( x \in G \) in case (1) and for all \( x \in G \cap f_p \) in case (2). In particular, \( \dim (\text{Osc}_k^2(X)) = 2n - 1 \) for any \( x \in G \) in both cases.

**Proof.** As \( \Phi_2(X) \neq \emptyset \) it follows from Theorem 1.2(3) that \( \Phi_2(C_i) \neq \emptyset \) for some \( i \). If \( \Phi_2(C_i) \neq C_i \) for every \( i = 1, \ldots, n \), then we get an irreducible component as in case (1). Actually, up to reordering the curves, we can assume that \( p_i \in \Phi_2(C_i) \) for \( i = 1, \ldots, s \). Then \( G := \langle p_1, \ldots, p_s \rangle \subseteq \Phi_2(X) \) by Proposition 2.4. Moreover,
since \( \Phi_2(C_i) \) is a finite set for every \( i \), we conclude that \( G \) is an irreducible component of \( \Phi_2(X) \). Now suppose that \( \Phi_2(C_i) = C_i \) for some \( i \). Then up to reordering the curves we can assume that \( \Phi_2(C_i) = C_i \) for \( i = 1, \ldots, s \) and \( \Phi_2(C_i) \neq C_i \) for \( i > s \). This implies that \( C_i \) is a line for \( i = 1, \ldots, s \) and a rational curve of higher degree for \( i > s \). In particular, \( C = \mathbb{P}^1 \), i.e., \( X \) is a rational scroll. Moreover \( G_p := \{ p_1, \ldots, p_s \} \subset \Phi_2(X) \) by Proposition 2.4, for every \( p \in \mathbb{P}^1 \). Let \( G := \bigcup_{p \in \mathbb{P}^1} G_p \). Then \( G \) is the sub-scroll of \( X \) generated by the lines \( C_1, \ldots, C_s \). In other words, \( G \) is \( \mathbb{P}(O_{\mathbb{P}^1}(1)^{\oplus s}) = \mathbb{P}^1 \times \mathbb{P}^{s-1} \) embedded in the linear span of \( C_1, \ldots, C_s \) via the Segre embedding. This gives case (2) and there are no further possibilities. The last assertions follow from Proposition 1.4, since \( C \subseteq \Phi_2(X) \).

Of course it may happen that a fibre \( G_p \) of an irreducible component of \( \Phi_2(X) \) of type (2) is contained in a larger component of \( \Phi_2(X) \) of type (1). This happens if \( C_j \) has a flex at the point \( p_j \) for some \( j > s \).

Now let us consider the second discriminant locus: for simplicity we set \( D = D_2(X, V) \) and denote by \( D_G \) the component of \( D \) arising from an irreducible component \( G \) of \( \Phi_2(X) \).

Let \( G \) be an irreducible component of \( \Phi_2(X) \). We say that \( G \) is of type (1) or (2) according to the cases of Proposition 4.2. Let \( G \) be of type (1). Then, recalling that \( \text{Osc}^2_p(X) \) is a fixed \( \mathbb{P}^{2n-1} \) for all \( p \in G \), we conclude that the component \( D_G \) is a linear \( \mathbb{P}^{N-2n} \). Now suppose that \( G \) is of type (2). By Proposition 4.2, \( \text{Osc}^2_p(X) \) is a fixed linear space \( T_p := \mathbb{P}^{2n-1} \) for \( p \in G \cap f_p \). So, letting \( p \) vary on \( \mathbb{P}^1 \) we see that

\[
D_G = \bigcup_{p \in \mathbb{P}^1} \{ H \in \mathbb{P}^{N^\vee} | H \supseteq T_p \}.
\]

We can think of \( T_p \) as \( \text{Osc}^2_{p_1}(X) \). Note that if \( \Phi_2(X) \neq X \), the tangent line to \( C_n \) varies as \( p \) varies on \( C \). Hence for points \( x, y \in G \), lying on general distinct fibres \( f_p, f_q \) of \( X \) we have

\[
\text{Osc}^2_{p_1}(X) = \text{Osc}^2_{p_2}(X) \neq \text{Osc}^2_{q_1}(X) = \text{Osc}^2_{q_2}(X).
\]

It follows that \( \dim(D_G) = N - 2n + 1 \), \( D_G \) being a family of \( \mathbb{P}^{N-2n} \) parameterized by \( \mathbb{P}^1 \). More precisely, we can describe the structure of \( D_G \) in this way. Consider the incidence correspondence

\[
P = \{(p_1, H) \in C_1 \times \mathbb{P}^{N^\vee} | H \supseteq \text{Osc}^2_{p_1}(X) \}.
\]

Note that \( P \) is a \( \mathbb{P}^{N-2n} \)-bundle over \( C_1 \times \mathbb{P}^1 \) via the first projection of \( C_1 \times \mathbb{P}^{N^\vee} \), since \( \text{Osc}^2_{p_1}(X) \) is a \( \mathbb{P}^{2n-1} \) for any \( p_1 \in C_1 \). Then \( D_G = \pi(P) \), where \( \pi \) is the second projection of \( C_1 \times \mathbb{P}^{N^\vee} \).

**Example 4.3** Let \( X \) be a decomposable scroll as in Section 1, generated by lines \( C_1, \ldots, C_{n-1} \) and by a non-degenerate rational curve \( C_n \subset \mathbb{P}^r \), \( r = r_n \geq 3 \), of degree \( d \). Then \( d \geq 3 \) and \( X \subset \mathbb{P}^N \), where \( N = 2n - 2 + r \). According to Proposition 4.2, \( G = C_1 \times \mathbb{P}^{n-2} \), Segre embedded in \( \mathbb{P}^{2n-3} = (C_1, \ldots, C_{n-1}) \). Let \( p, q \) be any two distinct points of \( C = \mathbb{P}^1 \). The tangent lines to \( C_n \) at \( p_n \) and \( q_n \) generate at most a \( \mathbb{P}^3 \). Therefore

\[
\dim \left( \langle C_1, \ldots, C_{n-1}, \text{Osc}^1_{p_n}(C_n), \text{Osc}^1_{q_n}(C_n) \rangle \right) \leq 2n + 1.
\]

Recall that the linear space above is just the linear span \( \langle \text{Osc}^2_{p_1}(X), \text{Osc}^2_{p_1}(X) \rangle \) by Proposition 4.2. So, if \( r \geq 4 \), for any two distinct points \( p, q \in C \) there exists a hyperplane \( H \) of \( \mathbb{P}^N \) containing both \( \text{Osc}^2_{p_1}(X) \) and \( \text{Osc}^2_{q_1}(X) \). Note that any such a hyperplane corresponds to a singular point of \( D_G \): actually, \( \pi_1^{-1}(H) = \{(p_1, H), (q_1, H)\} \). In particular, if \( r \geq 4 \), then \( \text{Sing}(D_G) \) contains the \( \mathbb{P}^2 \) parameterizing the double symmetric product of \( C_n \) with itself. Now let \( r = 3 \). If \( d \geq 4 \), then \( C_n \) is not normal. Hence it is the image of a rational normal curve \( \overline{C} \subset \mathbb{P}^d \) of degree \( d \) via a projection from a general linear space \( T \) of dimension \( d - 4 \). Any two tangents to \( \overline{C} \) span a \( \mathbb{P}^3 \), so, one sees by a dimension count that in the dual space \( \mathbb{P}^{d^\vee} \) there is a one dimensional family of hyperplanes of \( \mathbb{P}^{d^\vee} \) containing two tangent lines to \( \overline{C} \) and \( T \). Projecting to \( \mathbb{P}^3 \) they provide infinitely many pairs of coplanar tangent lines to \( C_n \) (see also [5] Remark 5.2)). This being a closed condition, implies that for any \( p_n \in C_n \) there exists some other point \( q_n \in C_n \) such that the two tangent lines \( \text{Osc}^1_{p_n}(C_n) \) and \( \text{Osc}^1_{q_n}(C_n) \) are coplanar. For such a pair of points,

\[
\dim \left( \langle C_1, \ldots, C_{n-1}, \text{Osc}^1_{p_n}(C_n), \text{Osc}^1_{q_n}(C_n) \rangle \right) = 2n.
\]
Hence $H := \langle \text{Osc}_{p_1}^2(X), \text{Osc}_{q_1}^2(X) \rangle$ is a hyperplane of $\mathbb{P}^{2n+1}$ giving rise to a singular point of $D_G$. Finally, let $r = 3 = d$. Then $C_n$ is a twisted cubic. Being a rational normal curve, we know that any two distinct tangent lines to $C_n$ do not meet. Thus, for any two distinct points $p, q \in C$ we have that

$$\langle \text{Osc}_{p_1}^2(X), \text{Osc}_{q_1}^2(X) \rangle = \langle C_1, \ldots, C_{n-1}, \text{Osc}_{p_n}^1(C_n), \text{Osc}_{q_n}^1(C_n) \rangle = \langle C_1, \ldots, C_n \rangle$$

is the whole $\mathbb{P}^{2n+1}$. In fact, in this case $D_G$ is a scroll over $C$ (see Proposition 4.4 below).

What we said in Example 4.3 when either $r \geq 4$ or $r = 3$ and $d \geq 4$ holds, “a fortiori”, if $C_1, \ldots, C_s$ are lines ($s \geq 1$) and for some $i = s + 1, \ldots, n$ either $r_i \geq 4$ or $r_i = 3$ and $\text{deg} \ C_i \geq 4$. Actually, also in this case there are hyperplanes $H$ of $\mathbb{P}^N$ containing both $\text{Osc}_{p_1}^2(X)$ and $\text{Osc}_{q_1}^2(X)$, for distinct points $p, q$ of $C$, and any such hyperplane gives rise to a singular point of $D_G$. From now on in this Section, we assume that

$$1 = \text{deg} \ C_1 = \cdots = \text{deg} \ C_s < \text{deg} \ C_{s+1} \leq \cdots \leq \text{deg} \ C_n.$$ 

Example 4.3 shows that $D_G$ is not a scroll if $\text{deg} \ C_n \geq 4$. On the other hand, we can prove the following

**Proposition 4.4** Let $X$ be a decomposable scroll generated by $C_1, \ldots, C_n$, where $C_i$ is a line for $i = 1, \ldots, s$ and

$$2 \leq \text{deg} \ C_{s+1} \leq \cdots \leq \text{deg} \ C_n \leq 3.$$ 

Let $G$ be the sub-scroll of $X$ generated by $C_1, \ldots, C_s$. Then $D_G$ is a rational scroll.

We need to point out some facts.

**Remark 4.5** Let $Y$ be the decomposable scroll generated by $C_{s+1}, \ldots, C_n$, and let $\mathbb{P}^M$ be its linear span in $\mathbb{P}^N$. We denote by $\Sigma$ the minimal sub-scroll of $Y$ generated by the curves $C_{s+1}, \ldots, C_{n-1}$ and by $F$ any hyperplane of $Y$. Note that $\Sigma \cap C_n = \emptyset$, while $\Sigma \cap F$ is a hyperplane of $F$. We have $\text{Pic}(Y) \cong \mathbb{Z}^2$ and we can choose as generators the classes of $\Sigma$ and $F$. Then:

(i) any hyperplane of $\mathbb{P}^M$ cuts $Y$ along a divisor $D$ linearly equivalent to $\Sigma + bF$, for some integer $b > 0$. In particular, since $\Sigma$ does not meet $C_n$ we see that

$$\text{deg} \ C_n = D C_n = (\Sigma + bF) C_n = b. \quad (9)$$

(ii) For any hyperplane $H$ of $\mathbb{P}^N$ not containing $\mathbb{P}^M$ set $h := H \cap \mathbb{P}^M$. If $h$ contains $\text{Osc}_{p_i}^1(C_i)$ for every $i = s + 1, \ldots, n$ then $h$ cuts $Y$ along a divisor of the form $D = 2F_p + R$ where $F_p = (p_{s+1}, \ldots, p_n)$ and $R$ is an effective divisor linearly equivalent to $\Sigma + \beta F$, with $\beta \geq 0$. Indeed, the tangent space to $Y$ at $p_i$ is

$$\text{Osc}_{p_i}^1(Y) = (p_{s+1}, \ldots, \text{Osc}_{p_i}^1(C_i), \ldots, p_n)$$

for $i = s + 1, \ldots, n$, by Remark 2.1 with $d = 1$. Hence $h$ is tangent to $Y$ at all points $p_{s+1}, \ldots, p_n$. Since they are linearly independent, this says that $h$ is tangent to $Y$ along the whole fibre $F_p$. Thus the divisor $D$ cut out by $h$ on $Y$ is singular at all points of $F_p$, hence the summand $2F_p$ appears in the expression of $D$ as positive linear combination of its irreducible components.

Now we can prove Proposition 4.4.

**Proof.** As the fibres of $\mathcal{P}$ are mapped linearly into $\mathbb{P}^{N'}$ by $\pi$, it is enough to show that the bundle projection of $\mathcal{P}$ induces a morphism $D_G \to C_1$. To do that we prove that $\pi$ is bijective, i.e., for any $H \in D_G$, the fibre $\pi^{-1}_N(H)$ consist of a single element. Equivalently, for any pair of distinct points $p, q \in C$, there is no hyperplane $H \subset \mathbb{P}^N$ containing both $\text{Osc}_{p_1}^2(X)$ and $\text{Osc}_{q_1}^2(X)$. Set

$$R_p = \langle \text{Osc}_{p_{s+1}}^1(C_{s+1}), \ldots, \text{Osc}_{p_n}^1(C_n) \rangle, \quad R_q = \langle \text{Osc}_{q_{s+1}}^1(C_{s+1}), \ldots, \text{Osc}_{q_n}^1(C_n) \rangle.$$ 

By Proposition 4.2 we know that

$$\text{Osc}_{p_1}^2(X) = \langle C_1, \ldots, C_s, R_p \rangle, \quad \text{Osc}_{q_1}^2(X) = \langle C_1, \ldots, C_s, R_q \rangle.$$
Thus the assertion follows once we show that $\langle R_p, R_q \rangle = \mathbb{P}^M$. By contradiction, suppose that there is a hyperplane $h$ of $\mathbb{P}^M$ containing both $R_p$ and $R_q$. Then, according to Remark 4.5 (ii), $h$ cuts $Y$ along a divisor $D = 2F_p + 2F_q + R$, with $R$ linearly equivalent to $\Sigma + \beta F$, for some integer $\beta \geq 0$. Then, dotting with $C_n$ and recalling (9), we get

$$\deg C_n = DC_n = 4 + \beta \geq 4,$$

a contradiction. □

A further property of $D_G$ is that it is degenerate in $\mathbb{P}^{N+s-1}$. In fact, for any $x \in G$, Osc$_x^2(X)$ contains the lines $C_1, \ldots, C_s$, and hence their linear span $\Lambda := \langle C_1, \ldots, C_s \rangle$ which is a $\mathbb{P}^{2s+1}$. By duality, this means that $D_G$ is contained in the linear subspace $\mathbb{P}^{N-2s} \subset \mathbb{P}^{N+s-1}$ parameterizing the hyperplanes containing $\Lambda$. Moreover, $\langle D_G \rangle = \mathbb{P}^{N-2s}$.

Next we want to determine the degree of $D_G$ when $G$ is of type (2). To do that, recall that $\langle C_i \rangle = \mathbb{P}^{r_i}$, and let $d_i = \deg C_i$.

**Proposition 4.6** Let $G$ be an irreducible component of $\Phi_2(X)$ of type (2). Then $\deg D_G = 2 \sum_{i=s+1}^n (d_i - 1)$.

**Proof.** Since $G$ is of type (2) we know that $d_1 = \cdots = d_s = 1$ and $d_i \geq 2$ for $i > s$. Also $r_1 = \cdots = r_s = 1$ and $r_i \geq 2$ for $i > s$. Recalling that $N = \sum_{i=1}^n r_i + n - 1$, we note that

$$\dim D_G = N - 2n + 1 = \sum_{i=1}^n r_i - n = \sum_{i=s+1}^n (r_i - 1).$$

So $\deg D_G$ is the number of elements of $D_G$ contained in a linear system $S \subset |\mathcal{V}|$ defined by $\sum_{i=s+1}^n (r_i - 1)$ linear conditions, general enough. Choose $r_i - 1$ general points in each $\mathbb{P}^{r_i}$ for $i = s + 1, \ldots, n$ and call $Z_i \subset \mathbb{P}^{r_i}$ the linear subspace they generate. Let $S$ be the linear system of hyperplanes of $\mathbb{P}^n$ defined by the condition of passing through all these points. Let $H \in D_G$ be a hyperplane of $\mathbb{P}^n$ not containing $\mathbb{P}^{r_i}$ for a given $i$, $s + 1 \leq i \leq n$. Then $h_i := H \cap \mathbb{P}^{r_i}$ is a hyperplane of $\mathbb{P}^{r_i}$ tangent to the curve $C_i$. More precisely, if $H \supset \text{Osc}_x^2(X)$ and $x \in f_p$, then Osc$_x^2(X) \supset \text{Osc}_x^1(C_i)$, and so $h_i$ is tangent to $C_i$ at $p_i$. On the other hand, if our $H$ is also in $S$, then, in particular, $h_i$ contains $Z_i$. Conversely, suppose that $h_i$ is a hyperplane of $\mathbb{P}^{r_i}$ containing $Z_i$ and tangent to $C_i$ at a point $p_i$, and set

$$H := \langle C_1, \ldots, C_{i-1}, h_i, C_{i+1}, \ldots, C_n \rangle.$$  

Clearly $H$ is a hyperplane of $\mathbb{P}^n$. Moreover, $H \in D_G$, because $H$ contains all $\langle C_j \rangle$ for $j \neq i$ and also Osc$_{p_i}^1(C_i)$. Furthermore, $H \in S$ since $H \supset Z_i$ for every $i = s + 1, \ldots, n$. It thus follows that

$$\deg D_G = \sum_{i=s+1}^n b_i,$$

where $b_i$ is the number of hyperplanes of $\mathbb{P}^{r_i}$ containing $Z_i$, that are tangent to $C_i$. To compute $b_i$ note that $\dim Z_i = r_i - 2$, so $Z_i$ is the axis of a pencil of hyperplanes of $\mathbb{P}^{r_i}$. The number of hyperplanes in this pencil that are tangent to $C_i$ is that of the ramification points of the morphism $C_i \to \mathbb{P}^1$ defined by the projection of $C_i$ from $Z_i$. Thus the Riemann–Hurwitz formula tells us that $b_i = 2(d_i - 1)$ and this concludes the proof. □

Relying on the above results we get the following characterization.

**Theorem 4.7** Let $X \subset \mathbb{P}^n$ be a decomposable scroll generated by $C_1, \ldots, C_n$, and let $d_i = \deg C_i$, for $i = 1, \ldots, n$. Suppose that $G$ is an irreducible component of type (2) of $\Phi_2(X)$. Then $D_G$ is a rational normal scroll if and only if, up to reordering the curves, $d_1 = \cdots = d_s = 1$ and $d_{s+1} = \cdots = d_n = 2$ for some $s \geq 1$.

**Proof.** As $G$ is of type (2), we can assume that $d_1 = \cdots = d_s = 1$ for some $s \geq 1$ and $d_i \geq 2$ for $i \geq s + 1$, by Proposition 4.2. As we noted, $D_G$ has dimension $N - 2n + 1$ and is non-degenerate in $\mathbb{P}^{N-2s}$. Thus, recalling Proposition 4.6, the inequality $\deg D_G \geq \text{codim} D_G + 1$ becomes

$$2 \sum_{i=s+1}^n (d_i - 1) \geq 2(n - s).$$  \hspace{1cm} (10)
Note that this is an equality if and only if
\[ d_i = 2 \quad \text{for} \quad i = s + 1 \ldots, n. \quad (11) \]

So, if \( D_C \) is a rational normal scroll, then (11) holds. On the other hand, if (11) holds then we know that \( X \) is a rational scroll, by Proposition 4.4, and then equality in (10) says that it is normal.  

5 A general lower bound

In [6, Theorem A] it is shown that the highest inflectional locus of an indecomposable linearly normal elliptic scroll of invariant \( e = -1 \) is empty. By adapting the argument used in [6] we can locate the highest inflectional locus of an elliptic indecomposable scroll of invariant \( e = 0 \), which is linearly normally embedded. Let \( C \) be a smooth curve of genus 1 and let \( S = \mathbb{P}(\mathcal{E}) \), where \( \mathcal{E} \) is the holomorphic rank-2 vector bundle on \( C \) defined by the non-split extension
\[
0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0. \quad (12)
\]

Let \( \pi : S \to C \) be the ruling projection and denote by \( C_0 \) the tautological section on \( S \). Let \( \delta \in \text{Div}(C) \) be a divisor of degree \( \deg \delta = m + 1 \geq 3 \) and set \( L := \mathcal{O}_S(C_0 + \pi^* \delta) \). Note that \( L \) is very ample, because \( \deg \delta \geq e + 3 \) [4, Ex. 2.12(b), p. 385] and the morphism given by \( |L| \) embeds \( S \) as a linearly normal scroll of degree \( 2m + 2 \in \mathbb{P}^N \), where \( N = 2m + 1 \) (note that \( m = \frac{N-1}{2} = \frac{2m+1-1}{2} \)). Let \( x \in S \). By [6, (1.0.m)] we have
\[
\dim(\text{Osc}^m_x(S)) = N - 1 - \dim(|L - (m + 1)x|) = 2m - \dim(|L - (m + 1)x|). \quad (13)
\]

On the other hand, by [6, Remark 1.2] we know that
\[
|L - (m + 1)x| = mf_x + |L - mf_x - x|, \quad (14)
\]

where \( f_x \) is the fibre through \( x \). Note that the line bundle \( L \otimes \mathcal{O}_S(-mf_x) = \mathcal{O}_S(C_0 + \pi^*(\delta - m\pi(x))) \) is not necessarily spanned, because \( \deg(\delta - m\pi(x)) = m + 1 - m = 1 < e + 2 \) [4, Ex. 2.12(a), p. 385]. We have that
\[
\dim(|L - mf_x - x|) = \dim(|L - mf_x|) - 1 \quad \text{if and only if} \quad L \otimes \mathcal{O}_S(-mf_x) \text{ is spanned at } x. \quad (15)
\]

Now, twisting (12) by \( \mathcal{O}_C(\delta - m\pi(x)) \) we immediately see that
\[
h^0(L - mf_x) = h^0(\mathcal{E}(\delta - m\pi(x)) = 2h^0(\mathcal{O}_C(\delta - m\pi(x)) = 2.
\]

Hence (15) gives
\[
\dim(|L - mf_x - x|) = 0 \quad \text{if and only if} \quad L \otimes \mathcal{O}_S(-mf_x) \text{ is spanned at } x
\]

and taking into account (14) and (13) we get
\[
\dim(\text{Osc}^m_x(S)) = 2m \quad \text{if and only if} \quad L \otimes \mathcal{O}_S(-mf_x) \text{ is spanned at } x.
\]

This proves the following

**Proposition 5.1** Let \( S \subset \mathbb{P}^{2m+1} \) be a linearly normal surface scroll over an elliptic curve \( C \), defined by an indecomposable vector bundle as in (12), and let \( L \) be the hyperplane bundle. Then \( x \in \Phi_m(S) \) if and only if the line bundle \( L \otimes \mathcal{O}_S(-mf_x) \) is not spanned at \( x \), where \( f_x \) is the fibre of \( S \) through \( x \).

Now let \( S \subset \mathbb{P}^N \) be any surface scroll. Though \( S \) can be not decomposable, according to Example 3.2 it seems natural to ask whether, under some assumption, we can get a global lower bound for the dimension of \( \text{Osc}^k_x(S) \), i.e., a lower bound holding at every point \( x \in S \), bigger than 3. We determine such a lower bound, depending on \( k \), under certain assumptions on the linear system (not necessarily complete) giving rise to the embedding.

In the following we use the same notation as in [6].
Theorem 5.2 Let $S \subset \mathbb{P}^N$ be a surface scroll embedded by $|V|$, where $V \subseteq H^0(S, L)$, $L = \mathcal{O}_{\mathbb{P}^N}(1)|_S$, and suppose that $N \geq 2k$. Let $x \in S$ and denote by $f_x$ the fibre of $S$ through $x$. If $|V - tf_x|$ is very ample for every non-negative integer $t \leq k - 2$, then

$$\dim \left( \text{Osc}^k_x(S) \right) \geq k + 2 \quad \text{for } k \geq 3.$$ 

Note that this global lower bound is the same holding for rational normal scrolls, as shown by Example 3.1.

Proof. First let us prove, by induction, that

$$\dim \left( \text{Osc}^k_x(S) \right) \geq k + 1 \quad \text{for any } k \geq 2. \quad (16)$$

For $k = 2$ this comes from [6] Theorem B (noting that $N \geq 4$ is enough in the proof). So let $k \geq 3$ and set $\mathcal{L} = L - (k - 2)f_x$ and $|W| = |V - (k - 2)f_x|$. Note that $|W|$ is very ample by assumption and that $S$ embedded by $|W|$ is also a scroll. Actually, for every fibre $f$ of $S$ we have

$$\mathcal{L}f = (L - (k - 2)f_x)f = Lf = 1.$$ 

Thus [6] Lemma (1.4) and Lemma (1.5) imply the following facts. The linear system $|W - f_x|$ is base-point free and if $\varphi_x : S \to \mathbb{P}^M$ denotes the associated morphism then one of the following conditions holds:

i) every fibre of $\varphi_x$ intersects any fibre $f$ of $S$ at a finite set,

ii) $(S, \mathcal{L}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$ and $W = H^0(S, \mathcal{L})$.

If ii) holds, then $L = \mathcal{O}(1, k - 1)$, hence $h^0(L) = 2k$, which implies that

$$N = \dim(|V|) \leq \dim(|L|) = 2k - 1,$$

but this contradicts our assumption that $N \geq 2k$. Therefore condition i) holds. Now suppose, by contradiction, that $(16)$ is not true, i.e., $\dim \left( \text{Osc}^k_x(S) \right) \leq k$. Because $|V - (k - 3)f_x|$ is very ample, by induction we know that $\dim \left( \text{Osc}^{k-1}_x(S) \right) \geq k$. So, due to the obvious inclusion $\text{Osc}^k_x(S) \supseteq \text{Osc}^{k-1}_x(S)$ we conclude that

$$\text{Osc}^k_x(S) = \text{Osc}^{k-1}_x(S).$$

Equivalently, this says that

$$|V - (k + 1)x| = |V - kx|.$$ 

This in turn, according to [6] Remark (1.2)], implies the equality

$$\dim(|V - kfx - x|) = \dim(|V - (k - 1)f_x - x|).$$

Hence

$$f_x \subseteq \text{Bs}(|V - (k - 1)f_x - x|) = \text{Bs}(|W - f_x - x|) = \varphi_x^{-1}(\varphi_x(x)).$$

But this contradicts condition i). To conclude the proof we show that equality cannot occur in $(16)$ for $k \geq 3$. First of all, since $|V - tf_x|$ is very ample for all $t \leq k - 2$, by applying [6] Remark (1.7)] inductively we see that

$$\dim(|W|) = \dim(|V|) - 2(k - 2) = N - 2(k - 2). \quad (17)$$

Note that

$$\dim(|V - (k + 1)x|) = \dim(|W - 3x|)$$ 

by [6] Remark (1.2)]. Due to $(16)$ and the assumption $N \geq 2k$, $S$ is embedded by $|W|$ as a scroll in a projective space of dimension $\geq 4$, hence $|W - 3x| \neq |W - 2x|$ by [6] Theorem B). Since $|W|$ is very ample this says that $\dim(|W - 3x|) < \dim(|W|) - 3$. Thus, recalling the equality

$$\dim(|V - (k + 1)x|) + \dim \left( \text{Osc}^k_x(S) \right) = N - 1,$$

we get

$$\dim \left( \text{Osc}^k_x(S) \right) > N + 2 - \dim(|W|). \quad (18)$$

Finally, combining $(17)$ with $(18)$ and assuming equality in $(16)$ gives $k \leq 2$. This completes the proof. \qed
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1 First section

1.1 First subsection

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This paper is a sample adapted from the \texttt{AMS-LATEX} document testmath.tex to illustrate the use of the document class \texttt{w-art} and publication-specific variants of that class for WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim.

1 Introduction

This paper contains examples of various features from \texttt{AMS-LATEX}.

2 Enumeration of Hamiltonian paths in a graph

Let $A = (a_{ij})$ be the adjacency matrix of graph $G$. The corresponding Kirchhoff matrix $K = (k_{ij})$ is obtained from $A$ by replacing in $-A$ each diagonal entry by the degree of its corresponding vertex; i.e., the $i$th diagonal entry is identified with the degree of the $i$th vertex. It is well known that

$$\det K(i|i) = \text{ the number of spanning trees of } G, \quad i = 1, \ldots, n \quad (1)$$

where $K(i|i)$ is the $i$th principal submatrix of $K$.

Let $C_{ij}$ be the set of graphs obtained from $G$ by attaching edge $(v_i v_j)$ to each spanning tree of $G$. Denote by $C_i = \bigcup_j C_{ij}$. It is obvious that the collection of Hamiltonian cycles is a subset of $C_i$. Note that the cardinality of $C_i$ is $k_{ii} \det K(i|i)$. Let $\tilde{X} = \{\hat{x}_1, \ldots, \hat{x}_n\}$. Let $x = \{\hat{x}_1, \ldots, \hat{x}_n\}$.

Define multiplication for the elements of $\tilde{X}$ by

$$\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \ldots, n. \quad (2)$$

Let $\hat{k}_{ij} = k_{ij} \hat{x}_j$ and $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$. Then the number of Hamiltonian cycles $H_c$ is given by the relation [8]

$$\left( \prod_{j=1}^{n} \hat{x}_j \right) H_c = \frac{1}{2} \hat{k}_{ij} \det \hat{K}(i|i), \quad i = 1, \ldots, n. \quad (3)$$

The task here is to express (3) in a form free of any $\hat{x}_i, i = 1, \ldots, n$. The result also leads to the resolution of enumeration of Hamiltonian paths in a graph.

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It is well known that the enumeration of Hamiltonian cycles and paths in a complete graph $K_n$ and in a complete bipartite graph $K_{n_1,n_2}$ can only be found from first combinatorial principles [4]. One wonders if there exists a formula which can be used very efficiently to produce $K_n$ and $K_{n_1,n_2}$. Recently, using Lagrangian methods, Goulden and Jackson have shown that $H_c$ can be expressed in terms of the determinant and permanent of the adjacency matrix [3]. However, the formula of Goulden and Jackson determines neither $K_n$ nor $K_{n_1,n_2}$ effectively. In this paper, using an algebraic method, we parametrize the adjacency matrix. The resulting formula also involves the determinant and permanent, but it can easily be applied to $K_n$ and $K_{n_1,n_2}$. In addition, we eliminate the permanent from $H_c$ and show that $H_c$ can be represented by a determinantal function of multivariables, each variable with domain $\{0, 1\}$. Furthermore, we show that $H_c$ can be written by number of spanning trees of subgraphs. Finally, we apply the formulas to a complete multigraph $K_{n_1 \ldots n_p}$.

The conditions $a_{ij} = a_{ji}, i, j = 1, \ldots, n,$ are not required in this paper. All formulas can be extended to a digraph simply by multiplying $H_c$ by 2.

### 3 Main Theorem

**Notation** For $p, q \in P$ and $n \in \omega$ we write $(q, n) \leq (p, n)$ if $q \leq p$ and $A_{q,n} = A_{p,n}$.

\begin{notation}
For $p, q \in P$ and $n \in \omega$
\end{notation}

Let $B = (b_{ij})$ be an $n \times n$ matrix. Let $n = \{1, \ldots, n\}$. Using the properties of [2], it is readily seen that

**Lemma 3.1**

$$\prod_{i \in n} \left( \sum_{j \in n} b_{ij} \hat{x}_i \right) = \left( \prod_{i \in n} \hat{x}_i \right) \text{per } B$$

(4)

where $\text{per } B$ is the permanent of $B$.

Let $\hat{Y} = \{\hat{y}_1, \ldots, \hat{y}_n\}$. Define multiplication for the elements of $\hat{Y}$ by

$$\hat{y}_i \hat{y}_j + \hat{y}_j \hat{y}_i = 0, \quad i, j = 1, \ldots, n.$$

(5)

Then, it follows that

**Lemma 3.2**

$$\prod_{i \in n} \left( \sum_{j \in n} b_{ij} \hat{y}_j \right) = \left( \prod_{i \in n} \hat{y}_i \right) \det B.$$  

(6)

Note that all basic properties of determinants are direct consequences of Lemma 3.2. Write

$$\sum_{j \in n} b_{ij} \hat{y}_j = \sum_{j \in n} b^{(\lambda)}_{ij} \hat{y}_j + (b_{ii} - \lambda_i)\hat{y}_i \hat{y}$$

(7)

where

$$b^{(\lambda)}_{ii} = \lambda_i, \quad b^{(\lambda)}_{ij} = b_{ij}, \quad i \neq j.$$  

(8)

Let $B^{(\lambda)} = (b^{(\lambda)}_{ij})$. By (6) and (7), it is straightforward to show the following result:

**Theorem 3.3**

$$\det B = \sum_{l=0}^{n} \sum_{I_l \subseteq n} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det B^{(\lambda)}(I_l | I_l),$$

(9)

where $I_l = \{i_1, \ldots, i_l\}$ and $B^{(\lambda)}(I_l | I_l)$ is the principal submatrix obtained from $B^{(\lambda)}$ by deleting its $i_1, \ldots, i_l$ rows and columns.
**Remark 3.4** Let $M$ be an $n \times n$ matrix. The convention $M(n|n) = 1$ has been used in (9) and hereafter.

Before proceeding with our discussion, we pause to note that Theorem 3.3 yields immediately a fundamental formula which can be used to compute the coefficients of a characteristic polynomial [9]:

**Corollary 3.5** Write

$$\det(B - xI) = \sum_{l=0}^{n} (-1)^{l} b_{l} x^{l}.$$ Then

$$b_{l} = \sum_{I \subseteq \mathbf{n}} \det(B_{I \setminus I}). \quad (10)$$

Let

$$K(t, t_{1}, \ldots, t_{n}) = \begin{pmatrix}
D_{1}t & -a_{12}t_{2} & \cdots & -a_{1n}t_{n} \\
-a_{21}t_{1} & D_{2}t & \cdots & -a_{2n}t_{n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1}t_{1} & -a_{n2}t_{2} & \cdots & D_{nt}
\end{pmatrix}. \quad (11)$$

Then

$$D(t_{1}, \ldots, t_{n}) = \frac{\partial}{\partial t} \det K(t, t_{1}, \ldots, t_{n})|_{t=1}.$$ Then

$$D(t_{1}, \ldots, t_{n}) = \sum_{i \in \mathbf{n}} D_{i} \det K(t = 1, t_{1}, \ldots, t_{n}; i|i), \quad (13)$$

where $K(t = 1, t_{1}, \ldots, t_{n}; i|i)$ is the $i$th principal submatrix of $K(t = 1, t_{1}, \ldots, t_{n})$.

Theorem 3.3 leads to

$$\det K(t_{1}, \ldots, t_{n}) = \sum_{l \in \mathbf{n}} (-1)^{|l|} \prod_{i \in l} t_{i} \prod_{j \in l} (D_{j} + \lambda_{j} t_{j}) \det A(\lambda_{l}) |T|T|. \quad (14)$$

Note that

$$\det K(t = 1, t_{1}, \ldots, t_{n}) = \sum_{l \in \mathbf{n}} (-1)^{|l|} \prod_{i \in l} t_{i} \prod_{j \in l} (D_{j} + \lambda_{j} t_{j}) \det A(\lambda_{l}) |T|T| = 0. \quad (15)$$

Let $t_{i} = \hat{x}_{i}, i = 1, \ldots, n$. Lemma 3.1 yields

$$\left( \sum_{i \in \mathbf{n}} a_{l _i} x_i \right) \det K(t = 1, x_{1}, \ldots, x_{n}; l|l)$$

$$= \left( \prod_{i \in \mathbf{n}} \hat{x}_{i} \right) \sum_{l \subseteq \mathbf{n} - \{i\}} (-1)^{|l|} \text{per } A(\lambda_{l}) |I|I \det A(\lambda_{l}) (I \cup \{i\}|T \cup \{i\}). \quad (16)$$

\begin{pmatrix}
D_{1} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & D_{2} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & D_{n}
\end{pmatrix}
6 Sh. First Author, Sh. Second Author, and Sh. Third Author: Short Title

\[
\begin{align*}
&= \prod_{i\in n} \hat{x}_i \\
&= \sum_{I \subseteq n-\{l\}} (-1)^{|I|} \mathbf{A}^{(\lambda)}(I|I) \\
&\det \mathbf{A}^{(\lambda)}(\overline{I}\cup\{l\}|\overline{I}\cup\{l\}) \\
\label{sum-ali}
\end{align*}
\]

By (3), (6), and (7), we have

**Proposition 3.6**

\[ H_c = \frac{1}{2n} \sum_{l=0}^{n} (-1)^{l} D_l, \] (17)

where

\[ D_l = \sum_{i \leq n} D(t_1, \ldots, t_n)\mathbb{1}_{i=\begin{cases} 0, & i \in t_l \\ 1, & \text{otherwise} \end{cases}} \text{ for } i=1,\ldots,n. \] (18)

**4 Application**

We consider here the applications of Theorems 5.2 and 5.3 to a complete multipartite graph \( K_{n_1 \cdots n_p} \). It can be shown that the number of spanning trees of \( K_{n_1 \cdots n_p} \) may be written

\[ T = n^{p-2} \prod_{i=1}^{p} (n-n_i)^{n_i-1} \] (19)

where

\[ n = n_1 + \cdots + n_p. \] (20)

It follows from Theorems 5.2 and 5.3 that

\[ H_c = \frac{1}{2n} \sum_{l=0}^{n} (-1)^{l}(n-l)^{p-2} \sum_{l_1,\ldots,l_p=0}^{p} \prod_{i=1}^{p} \left( \frac{n_i}{l_i} \right) \cdot \left[ (n-l)-(n_i-l_i) \right]^{n_i-l_i} \cdot \left[ (n-l)^2 - \sum_{j=1}^{p} (n_i-l_i)^2 \right]. \] (21)

\[ \ldots \binom{n_i}{l_i} \{ l \ldots \} \]

and

\[ H_c = \frac{1}{2} \sum_{l=0}^{n-1} (-1)^{l}(n-l)^{p-2} \sum_{l_1,\ldots,l_p=0}^{p} \prod_{i=1}^{p} \left( \frac{n_i}{l_i} \right) \cdot \left[ (n-l)-(n_i-l_i) \right]^{n_i-l_i} \left( 1 - \frac{l_p}{n_p} \right) \left[(n-l)-(n_p-l_p)\right]. \] (22)

The enumeration of \( H_c \) in a \( K_{n_1 \cdots n_p} \) graph can also be carried out by Theorem 7.5 or 7.6 together with the algebraic method of (2). Some elegant representations may be obtained. For example, \( H_c \) in a \( K_{n_1 n_2 n_3} \) graph may be written

\[ H_c = \frac{n_1! n_2! n_3!}{n_1+n_2+n_3} \sum_{i} \left[ \binom{n_1}{i} \binom{n_2}{n_3-i} \binom{n_3}{n_3-n_2+i} \right. \]
\[ + \binom{n_1-1}{i} \binom{n_2-1}{n_3-n_2+i} \binom{n_3-1}{n_3-n_2+i} \] (23)
5 Secret Key Exchanges

Modern cryptography is fundamentally concerned with the problem of secure private communication. A Secret Key Exchange is a protocol where Alice and Bob, having no secret information in common to start, are able to agree on a common secret key, conversing over a public channel. The notion of a Secret Key Exchange protocol was first introduced in the seminal paper of Diffie and Hellman [1]. presented a concrete implementation of a Secret Key Exchange protocol, dependent on a specific assumption (a variant on the discrete log), specially tailored to yield Secret Key Exchange. Secret Key Exchange is of course trivial if trapdoor permutations exist. However, there is no known implementation based on a weaker general assumption.

The concept of an informationally one-way function was introduced in [5]. We give only an informal definition here:

**Definition 5.1** A polynomial time computable function \( f = \{ f_k \} \) is informationally one-way if there is no probabilistic polynomial time algorithm which (with probability of the form \( 1 - k^{-e} \) for some \( e > 0 \)) returns on input \( y \in \{0, 1\}^k \) a random element of \( f^{-1}(y) \).

In the non-uniform setting [5] show that these are not weaker than one-way functions:

**Theorem 5.2** (non-uniform) The existence of informationally one-way functions implies the existence of one-way functions.

We will stick to the convention introduced above of saying “non-uniform” before the theorem statement when the theorem makes use of non-uniformity. It should be understood that if nothing is said then the result holds for both the uniform and the non-uniform models.

It now follows from Theorem 5.2 that

**Theorem 5.3** (non-uniform) Weak SKE implies the existence of a one-way function.

More recently, the polynomial-time, interior point algorithms for linear programming have been extended to the case of convex quadratic programs [11, 13], certain linear complementarity problems [7, 10], and the nonlinear complementarity problem [6]. The connection between these algorithms and the classical Newton method for nonlinear equations is well explained in [7].

6 Review

We begin our discussion with the following definition:

**Definition 6.1** A function \( H : \mathbb{R}^n \to \mathbb{R}^n \) is said to be \( B \)-differentiable at the point \( z \) if (i) \( H \) is Lipschitz continuous in a neighborhood of \( z \), and (ii) there exists a positive homogeneous function \( BH(z) : \mathbb{R}^n \to \mathbb{R}^n \), called the \( B \)-derivative of \( H \) at \( z \), such that

\[
\lim_{v \to 0} \frac{H(z + v) - H(z) - BH(z)v}{\|v\|} = 0.
\]

The function \( H \) is \( B \)-differentiable in set \( S \) if it is \( B \)-differentiable at every point in \( S \). The \( B \)-derivative \( BH(z) \) is said to be strong if

\[
\lim_{(v,v') \to (0,0)} \frac{H(z + v) - H(z + v') - BH(z)(v - v')}{\|v - v'\|} = 0.
\]

**Lemma 6.2** There exists a smooth function \( \psi_0(z) \) defined for \( |z| > 1 - 2a \) satisfying the following properties:

(i) \( \psi_0(z) \) is bounded above and below by positive constants \( c_1 \leq \psi_0(z) \leq c_2 \).

(ii) If \( |z| > 1 \), then \( \psi_0(z) = 1 \).

(iii) For all \( z \) in the domain of \( \psi_0 \), \( \Delta_0 \ln \psi_0 \geq 0 \).

(iv) If \( 1 - 2a < |z| < 1 - a \), then \( \Delta_0 \ln \psi_0 \geq c_3 > 0 \).
Thus there is a bijection from $\phi$ to $\lambda$. Let $h(r) \geq 0$ be a suitable smooth function satisfying $h(r) \geq c_3$ for $1 - 2\alpha < |z| < 1 - \alpha$, and $h(r) = 0$ for $|z| > 1 - \frac{\alpha}{2}$. The radial Laplacian

$$\Delta_0 \ln \psi_0(r) = \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right) \ln \psi_0(r)$$

has smooth coefficients for $r > 1 - 2\alpha$. Therefore, we may apply the existence and uniqueness theory for ordinary differential equations. Simply let $\ln \psi_0(r)$ be the solution of the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right) \ln \psi_0(r) = h(r)$$

with initial conditions given by $\ln \psi_0(1) = 0$ and $\ln \psi'_0(1) = 0$.

Next, let $D_\nu$ be a finite collection of pairwise disjoint disks, all of which are contained in the unit disk centered at the origin in $C$. We assume that $D_\nu = \{ z \mid |z - \nu| < \delta\}$. Suppose that $D_\nu(a)$ denotes the smaller concentric disk $D_\nu(a) = \{ z \mid |z - \nu| \leq (1 - 2\alpha)\delta\}$. We define a smooth weight function $\Phi_0(z)$ for $z \in C - \bigcup_\nu D_\nu(a)$ by setting $\Phi_0(z) = 1$ when $z \notin \bigcup_\nu D_\nu$ and $\Phi_0(z) = \psi_0((z - \nu)/\delta)$ when $z$ is an element of $D_\nu$. It follows from Lemma 6.2 that $\Phi_0$ satisfies the properties:

(i) $\Phi_0(z)$ is bounded above and below by positive constants $c_1 \leq \Phi_0(z) \leq c_2$.

(ii) $\Delta_0 \ln \Phi_0 \geq 0$ for all $z \in C - \bigcup_\nu D_\nu(a)$, the domain where the function $\Phi_0$ is defined.

(iii) $\Delta_0 \ln \Phi_0 \geq c_3\delta^{-2}$ when $(1 - 2\alpha)\delta < |z - \nu| < (1 - \alpha)\delta$.

Let $A_\nu$ denote the annulus $A_\nu = \{ (1 - 2\alpha)\delta < |z - \nu| < (1 - \alpha)\delta\}$, and set $A = \bigcup_\nu A_\nu$. The properties (i) and (ii) of $\Phi_0$ may be summarized as $\Delta_0 \ln \Phi_0 \geq c_3\delta^{-2}\chi_A$, where $\chi_A$ is the characteristic function of $A$.

Suppose that $\alpha$ is a nonnegative real constant. We apply Proposition 3.6 with $\Phi(z) = \Phi_0(z)e^{\alpha|z|^2}$. If $u \in C_0^\infty(R^2 - \bigcup_\nu D_\nu(a))$, assume that $D$ is a bounded domain containing the support of $u$ and $A \subset D \subset R^2 - \bigcup_\nu D_\nu(a)$. A calculation gives

$$\int_D \left(\sum_{\nu} |u|^2 \Phi_0(z) e^{\alpha|z|^2} + c_3\delta^{-2}\int_A |u|^2 \Phi_0 e^{\alpha|z|^2}\right).$$

The boundedness, property (i) of $\Phi_0$, then yields

$$\int_D \left(\sum_{\nu} |u|^2 e^{\alpha|z|^2} + c_3\delta^{-2}\int_A |u|^2 e^{\alpha|z|^2}\right).$$

Let $B(X)$ be the set of blocks of $\Lambda_X$ and let $b(X) = |B(X)|$. If $\phi \in Q_X$ then $\phi$ is constant on the blocks of $\Lambda_X$.

$$P_X = \{ \phi \in M \mid \Lambda_\phi = \Lambda_X \}, \quad Q_X = \{ \phi \in M \mid \Lambda_\phi \geq \Lambda_X \}.$$  \hfill (24)

If $\Lambda_\phi \geq \Lambda_X$ then $\Lambda_\phi = \Lambda_Y$ for some $Y \geq X$ so that

$$Q_X = \bigcup_{Y \geq X} P_Y.$$  

Thus by Möbius inversion

$$|P_Y| = \sum_{X \geq Y} \mu(Y, X) |Q_X|.$$  

Thus there is a bijection from $Q_X$ to $W^{B(X)}$. In particular $|Q_X| = w^{B(X)}$.

Next note that $b(X) = \dim X$. We see this by choosing a basis for $X$ consisting of vectors $v^k_i$ defined by

$$v^k_i = \begin{cases} 1 & \text{if } i \in \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$
\[v^\ast(k)\_\{(i)\}=
\begin{cases} 1 & \text{if } i \in \Lambda(k), \\
0 & \text{otherwise.}
\end{cases}\]

\textbf{Lemma 6.3} Let \( A \) be an arrangement. Then

\[\chi(A, t) = \sum_{B \subseteq A} (-1)^{|B|} t^{\dim(B)}.\]

In order to compute \( R'' \) recall the definition of \( S(X, Y) \) from Lemma 3.1. Since \( H \in B, A_H \subseteq B \). Thus if \( T(B) = Y \) then \( B \in S(H, Y) \). Let \( L'' = L(A'') \). Then

\[R'' = \sum_{H \in B \subseteq A} (-1)^{|B|} t^{\dim(B)} = \sum_{Y \in L''} \sum_{B \in S(H, Y)} (-1)^{|B|} t^{\dim(B)} = -\sum_{Y \in L''} \sum_{B \in S(H, Y)} (-1)^{|B-H_H|} t^{\dim(B)} = -\sum_{Y \in L''} \mu(H, Y) t^{\dim(Y)} = -\chi(A'', t).\]  \( (25) \)

\textbf{Corollary 6.4} Let \( (A, A', A'') \) be a triple of arrangements. Then

\[\pi(A, t) = \pi(A', t) + t\pi(A'', t).\]

\textbf{Definition 6.5} Let \( (A, A', A'') \) be a triple with respect to the hyperplane \( H \in A \). Call \( H \) a separator if \( T(A) \notin L(A') \).

\textbf{Corollary 6.6} Let \( (A, A', A'') \) be a triple with respect to \( H \in A \).

(i) If \( H \) is a separator then

\[\mu(A) = -\mu(A'')\]

and hence

\[|
\mu(A)\mid = |\mu(A'')|\]

(ii) If \( H \) is not a separator then

\[\mu(A) = \mu(A') - \mu(A'')\]

and

\[|\mu(A)| = |\mu(A')| + |\mu(A'')|\]

\textbf{Proof.} It follows from Theorem 5.2 that \( \pi(A, t) \) has leading term

\[(-1)^{r(A)} \mu(A) t^{r(A)}.\]

The conclusion follows by comparing coefficients of the leading terms on both sides of the equation in Corollary 6.4. If \( H \) is a separator then \( r(A') < r(A) \) and there is no contribution from \( \pi(A', t) \).
The Poincaré polynomial of an arrangement will appear repeatedly in these notes. It will be shown to equal the Poincaré polynomial of the graded algebras which we are going to associate with $\mathcal{A}$. It is also the Poincaré polynomial of the complement $M(\mathcal{A})$ for a complex arrangement. Here we prove that the Poincaré polynomial is the chamber counting function for a real arrangement. The complement $M(\mathcal{A})$ is a disjoint union of chambers

$$M(\mathcal{A}) = \bigcup_{C \in \text{Cham}(\mathcal{A})} C.$$ 

The number of chambers is determined by the Poincaré polynomial as follows.

**Theorem 6.7** Let $\mathcal{A}_R$ be a real arrangement. Then

$$|\text{Cham}(\mathcal{A}_R)| = \pi(\mathcal{A}_R, 1).$$

**Proof.** We check the properties required in Corollary 6.6: (i) follows from $\pi(\Phi_l, t) = 1$, and (ii) is a consequence of Corollary 3.5.

**Theorem 6.8** Let $\phi$ be a protocol for a random pair $(X, Y)$. If one of $\sigma_\phi(x', y)$ and $\sigma_\phi(x, y')$ is a prefix of the other and $(x, y) \in S_{X, Y}$, then

$$\langle \sigma_j(x', y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y') \rangle_{j=1}^\infty.$$

**Proof.** We show by induction on $i$ that

$$\langle \sigma_j(x', y) \rangle_{j=1}^i = \langle \sigma_j(x, y) \rangle_{j=1}^i = \langle \sigma_j(x, y') \rangle_{j=1}^i.$$
The induction hypothesis holds vacuously for \( i = 0 \). Assume it holds for \( i - 1 \), in particular \([\sigma_j(x', y')]_{j=1}^{i-1} = [\sigma_j(x, y')]_{j=1}^{i-1}\). Then one of \([\sigma_j(x', y')]_{j=1}^{i-1}\) and \([\sigma_j(x, y')]_{j=1}^{i-1}\) is a prefix of the other which implies that one of \( \sigma_i(x', y') \) and \( \sigma_i(x, y') \) is a prefix of the other. If the \( i\)th message is transmitted by \( P_X \) then, by the separate-transmissions property and the induction hypothesis, \( \sigma_i(x, y) = \sigma_i(x, y') \), hence one of \( \sigma_i(x, y) \) and \( \sigma_i(x', y) \) is a prefix of the other. By the implicit-termination property, neither \( \sigma_i(x, y) \) nor \( \sigma_i(x', y) \) can be a proper prefix of the other, hence they must be the same and \( \sigma_i(x', y) = \sigma_i(x, y) = \sigma_i(x, y') \). If the \( i\)th message is transmitted by \( P_Y \) then, symmetrically, \( \sigma_i(x, y) = \sigma_i(x', y') \) by the induction hypothesis and the separate-transmissions property, and, then, \( \sigma_i(x, y) = \sigma_i(x, y') \) by the implicit-termination property, proving the induction step. \( \square \)

If \( \phi \) is a protocol for \((X, Y)\), and \((x, y), (x', y)\) are distinct inputs in \( S_{X,Y} \), then, by the correct-decision property, \( (\sigma_j(x, y))_{j=1}^{\infty} \neq (\sigma_j(x', y'))_{j=1}^{\infty} \).

Equation (25) defined \( P_Y \)'s ambiguity set \( S_{X\mid Y}(y) \) to be the set of possible \( X \) values when \( Y = y \). The last corollary implies that for all \( y \in S_Y \), the multiset \[^{1}\]{\sigma_{\phi}(x, y) : x \in S_{X\mid Y}(y)} \) is prefix free.

### 7 One-Way Complexity

\( \hat{C}_1(X\mid Y) \), the one-way complexity of a random pair \((X, Y)\), is the number of bits \( P_X \) must transmit in the worst case when \( P_Y \) is not permitted to transmit any feedback messages. Starting with \( S_{X,Y} \), the support set of \((X, Y)\), we define \( G(X\mid Y) \), the characteristic hypergraph of \((X, Y)\), and show that

\[
\hat{C}_1(X\mid Y) = \lceil \log \chi(G(X\mid Y)) \rceil.
\]

Let \((X, Y)\) be a random pair. For each \( y \) in \( S_Y \), the support set of \( Y \), Equation (25) defined \( S_{X\mid Y}(y) \) to be the set of possible \( x \) values when \( Y = y \). The characteristic hypergraph \( G(X\mid Y) \) of \((X, Y)\) has \( S_X \) as its vertex set and the hyperedge \( S_{X\mid Y}(y) \) for each \( y \) in \( S_Y \).

We can now prove a continuity theorem.

**Theorem 7.1** Let \( \Omega \subset \mathbb{R}^n \) be an open set, let \( u \in BV(\Omega; \mathbb{R}^m) \), and let

\[
T^u_x = \left\{ y \in \mathbb{R}^m : y = \bar{u}(x) + \left\langle \frac{D\bar{u}}{|D\bar{u}|}(x), z \right\rangle \text{ for some } z \in \mathbb{R}^n \right\}
\]

for every \( x \in \Omega \setminus S_u \). Let \( f : \mathbb{R}^m \to \mathbb{R}^k \) be a Lipschitz continuous function such that \( f(0) = 0 \), and let \( v = f(u) : \Omega \to \mathbb{R}^k \). Then \( v \in BV(\Omega; \mathbb{R}^k) \) and

\[
Jv = (f(u^+) - f(u^-)) \odot \nu_u \cdot \mathcal{H}_{n-1}|_{S_u}.
\]

In addition, for \( |D\bar{u}|\)-almost every \( x \in \Omega \) the restriction of the function \( f \) to \( T^u_x \) is differentiable at \( \bar{u}(x) \) and

\[
\bar{D}v = \nabla(f|_{T^u_x})(\bar{u}) \frac{D\bar{u}}{|D\bar{u}|} \cdot |D\bar{u}|.
\]

Before proving the theorem, we state without proof three elementary remarks which will be useful in the sequel.

**Remark 7.2** Let \( \omega : [0, +\infty[ \to [0, +\infty[ \) be a continuous function such that \( \omega(t) \to 0 \) as \( t \to 0 \). Then

\[
\lim_{h \to 0^+} g(\omega(h)) = L \iff \lim_{h \to 0^+} g(h) = L
\]

for any function \( g : [0, +\infty[ \to \mathbb{R} \).

[^1]: A multiset allows multiplicity of elements. Hence, \( \{0, 0, 1, 0\} \) is prefix free as a set, but not as a multiset.
Remark 7.3 Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function and assume that
\[
L(z) = \lim_{h \to 0^+} \frac{g(hz) - g(0)}{h}
\]
exists for every $z \in \mathbb{Q}^n$ and that $L$ is a linear function of $z$. Then $g$ is differentiable at 0.

Remark 7.4 Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, and let $f : \mathbb{R}^m \to \mathbb{R}$ be a function. Then the restriction of $f$ to the range of $A$ is differentiable at 0 if and only if $f(A) : \mathbb{R}^n \to \mathbb{R}$ is differentiable at 0 and
\[
\nabla(f\vert_{\text{range}(A)})(0)A = \nabla(f(A))(0).
\]

Proof. We begin by showing that $v \in BV(\Omega; \mathbb{R}^k)$ and
\[
|Dv| (B) \leq K |Du| (B) \quad \forall B \in \mathcal{B}(\Omega),
\]
where $K > 0$ is the Lipschitz constant of $f$. By (13) and by the approximation result quoted in (13) it is possible to find a sequence $(u_n) \subset C^1(\Omega; \mathbb{R}^m)$ converging to $u$ in $L^1(\Omega; \mathbb{R}^m)$ and such that
\[
\lim_{n \to +\infty} \int_{\Omega} |\nabla u_n| \, dx = |Du| (\Omega).
\]
The functions $v_n = f(u_n)$ are locally Lipschitz continuous in $\Omega$, and the definition of differential implies that $|\nabla v_n| \leq K |\nabla u_n|$ almost everywhere in $\Omega$. The lower semicontinuity of the total variation and (13) yield
\[
|Dv| (\Omega) \leq \liminf_{n \to +\infty} |Dv_n| (\Omega) = \liminf_{n \to +\infty} \int_{\Omega} |\nabla v_n| \, dx
\]
\[
\leq K \liminf_{n \to +\infty} \int_{\Omega} |\nabla u_n| \, dx = K |Du| (\Omega).
\]
Since $f(0) = 0$, we have also
\[
\int_{\Omega} |v| \, dx \leq K \int_{\Omega} |u| \, dx;
\]
therefore $u \in BV(\Omega; \mathbb{R}^k)$. Repeating the same argument for every open set $A \subset \Omega$, we get (29) for every $B \in \mathcal{B}(\Omega)$, because $|Dv|$, $|Du|$ are Radon measures. To prove Lemma 6.2, first we observe that
\[
S_u \subset S_v, \quad \tilde{v}(x) = f(\tilde{u}(x)) \quad \forall x \in \Omega \setminus S_u.
\]
In fact, for every $\varepsilon > 0$ we have
\[
\{y \in B_\rho(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\} \subset \{y \in B_\rho(x) : |u(y) - \tilde{u}(x)| > \varepsilon/K\},
\]

\[
\lim_{\rho \to 0^+} \frac{|\{y \in B_\rho(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\}|}{\rho^n} = 0
\]
whenever $x \in \Omega \setminus S_u$. By a similar argument, if $x \in S_u$ is a point such that there exists a triplet $(u^+, u^-, \nu_u)$ satisfying (14), (15), then
\[
(v^+(x) - v^-(x)) \otimes \nu_u = (f(u^+(x)) - f(u^-(x))) \otimes \nu_u \quad \text{if } x \in S_v
\]
and $f(u^-) = f(u^+) \neq f(u^-)$ if $x \in S_u \setminus S_v$. Hence, by (1.8) we get
\[
Jv(B) = \int_{B \cap S_v} (v^+ - v^-) \otimes \nu_v \, d\mathcal{H}_{n-1} = \int_{B \cap S_v} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1}
\]
\[
= \int_{B \cap S_v} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1}
\]
and Lemma 6.2 is proved. \hfill \square

To prove (31), it is not restrictive to assume that $k = 1$. Moreover, to simplify our notation, from now on we shall assume that $\Omega = \mathbb{R}^n$. The proof of (31) is divided into two steps. In the first step we prove the statement in the one-dimensional case ($n = 1$), using Theorem 5.3. In the second step we achieve the general result using Theorem 7.1.
Step 1
Assume that \( n = 1 \). Since \( S_n \) is at most countable, (7) yields that \( \bar{D}v \) of \( (S_u \setminus S_n) = 0 \), so that (19) and (21) imply that \( Dv = Dv + \bar{J}v \) is the Radon-Nikodým decomposition of \( Dv \) in absolutely continuous and singular part with respect to \( \bar{D}u \). By Theorem 5.3, we have

\[
\frac{\bar{D}v}{\bar{D}u}(t) = \lim_{s \to t^+} \frac{Dv([t, s])}{\bar{D}u([t, s])}, \quad \frac{\bar{D}u}{\bar{D}u}(t) = \lim_{s \to t^+} \frac{Du([t, s])}{\bar{D}u([t, s])}
\]

\( \bar{D}u \)-almost everywhere in \( \mathbb{R} \). It is well known (see, for instance, [12, 2.5.16]) that every one-dimensional function of bounded variation \( w \) has a unique left continuous representative, i.e., a function \( \hat{w} \) such that \( \hat{w} = w \) almost everywhere and \( \lim_{s \to t^-} \hat{w}(s) = \hat{w}(t) \) for every \( t \in \mathbb{R} \). These conditions imply

\[
\hat{u}(t) = Du([-\infty, t]), \quad \hat{t}(t) = Dv([-\infty, t]) \quad \forall t \in \mathbb{R}
\]

and

\[
\hat{v}(t) = f(\hat{u}(t)) \quad \forall t \in \mathbb{R}.
\]

Let \( t \in \mathbb{R} \) be such that \( |\bar{D}u|(t, s) > 0 \) for every \( s > t \) and assume that the limits in (22) exist. By (23) and (24) we get

\[
\frac{\hat{v}(s) - \hat{v}(t)}{|\bar{D}u|(t, s)} = \frac{f(\hat{u}(s)) - f(\hat{u}(t))}{|\bar{D}u|(t, s)}
\]

\[
= \frac{f(\hat{u}(s)) - f(\hat{u}(t)) + \frac{\bar{D}u(t)}{|\bar{D}u|(t, s)} |\bar{D}u|(t, s) - f(\hat{u}(t))}{|\bar{D}u|(t, s)}
\]

for every \( s > t \). Using the Lipschitz condition on \( f \) we find

\[
\frac{\hat{v}(s) - \hat{v}(t)}{|\bar{D}u|(t, s)} \leq K \frac{|\hat{u}(s) - \hat{u}(t)|}{|\bar{D}u|(t, s)}.
\]

By (29), the function \( s \to |\bar{D}u|(t, s) \) is continuous and converges to 0 as \( s \downarrow t \). Therefore Remark 7.2 and the previous inequality imply

\[
\frac{\bar{D}v}{\bar{D}u}(t) = \lim_{h \to 0^+} \frac{f(\hat{u}(t) + h \bar{D}u(t)) - f(\hat{u}(t))}{h} \quad |\bar{D}u| \text{-a.e. in } \mathbb{R}.
\]
By (22), \( \dot{u}(x) = \ddot{u}(x) \) for every \( x \in \mathbb{R} \setminus S_u \); moreover, applying the same argument to the functions \( u'(t) = u(-t), v'(t) = f(u'(t)) = v(-t) \), we get

\[
\frac{\bar{D}v}{\bar{D}u}(t) = \lim_{h \to 0} \frac{f(\bar{u}(t) + h \frac{\bar{D}u}{\bar{D}u}(t)) - f(\bar{u}(t))}{h} \quad \text{for} \quad \bar{D}u \text{-a.e. in } \mathbb{R}
\]

and our statement is proved.

**Step 2**

Let us consider now the general case \( n > 1 \). Let \( \nu \in \mathbb{R}^n \) be such that \( |\nu| = 1 \), and let \( \pi_\nu = \{ y \in \mathbb{R}^n : (y, \nu) = 0 \} \). In the following, we shall identify \( \mathbb{R}^n \) with \( \pi_\nu \times \mathbb{R} \), and we shall denote by \( y \) the variable ranging in \( \pi_\nu \) and by \( t \) the variable ranging in \( \mathbb{R} \). By the just proven one-dimensional result, and by Theorem 3.3 we get

\[
\lim_{h \to 0} \frac{f(\bar{u}(y + t\nu) + h \frac{\bar{D}u}{\bar{D}u}(t)) - f(\bar{u}(y + t\nu))}{h} = \frac{\bar{D}v}{\bar{D}u}(t) \quad \text{for } \bar{D}u \text{-a.e. in } \mathbb{R}
\]

for \( \mathcal{H}_{n-1} \)-almost every \( y \in \pi_\nu \). We claim that

\[
\frac{\langle \bar{D}u, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(y + t\nu) = \frac{\bar{D}u_y}{\bar{D}u_y}(t) \quad \text{for } \bar{D}u \text{-a.e. in } \mathbb{R}
\]

(34) for \( \mathcal{H}_{n-1} \)-almost every \( y \in \pi_\nu \). In fact, by (16) and (18) we get

\[
\int_{\pi_\nu} \frac{\bar{D}u_y}{\bar{D}u_y} \cdot |\bar{D}u_y| \, d\mathcal{H}_{n-1}(y) = \int_{\pi_\nu} \bar{D}u_y \, d\mathcal{H}_{n-1}(y)
\]

\[
= \langle \bar{D}u, \nu \rangle = \frac{\langle \bar{D}u, \nu \rangle}{\langle \bar{D}u, \nu \rangle} \cdot |\bar{D}u_y| = \int_{\pi_\nu} \frac{\langle \bar{D}u, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(y + t\nu) \cdot |\bar{D}u_y| \, d\mathcal{H}_{n-1}(y)
\]

and (24) follows from (13). By the same argument it is possible to prove that

\[
\frac{\langle \bar{D}v, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(y + t\nu) = \frac{\bar{D}v_y}{\bar{D}u_y}(t) \quad \text{for } \bar{D}u \text{-a.e. in } \mathbb{R}
\]

(35) for \( \mathcal{H}_{n-1} \)-almost every \( y \in \pi_\nu \). By (24) and (25) we get

\[
\lim_{h \to 0} \frac{f(\bar{u}(y + t\nu) + h \frac{\langle \bar{D}u, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(y + t\nu)) - f(\bar{u}(y + t\nu))}{h} = \frac{\langle \bar{D}v, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(y + t\nu)
\]

for \( \mathcal{H}_{n-1} \)-almost every \( y \in \pi_\nu \), and using again (14), (15) we get

\[
\lim_{h \to 0} \frac{f(\bar{u}(x) + h \frac{\langle \bar{D}u, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(x)) - f(\bar{u}(x))}{h} = \frac{\langle \bar{D}v, \nu \rangle}{\langle \bar{D}u, \nu \rangle}(x)
\]
\[ |\langle \tilde{D}u, \nu \rangle| \text{-a.e. in } \mathbb{R}^n. \]

Since the function \( |\langle \tilde{D}u, \nu \rangle| / |\tilde{D}u| \) is strictly positive \( |\langle \tilde{D}u, \nu \rangle| \text{-almost everywhere} \), we obtain also

\[
\lim_{h \to 0} \frac{f(\tilde{u}(x) + h \langle \tilde{D}u, \nu \rangle(x)) - f(\tilde{u}(x))}{h} = \frac{\langle \tilde{D}u, \nu \rangle(x) \langle \tilde{D}v, \nu \rangle(x)}{|\tilde{D}u|}.
\]

\[ |\langle \tilde{D}u, \nu \rangle| \text{-almost everywhere in } \mathbb{R}^n. \]

Finally, since

\[
\frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} = \langle \tilde{D}u, \nu \rangle = \langle \tilde{D}u, \nu \rangle \quad \text{\( \tilde{D}u \)-a.e. in } \mathbb{R}^n,
\]

\[
\frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} = \langle \tilde{D}v, \nu \rangle = \langle \tilde{D}v, \nu \rangle \quad \text{\( \tilde{D}u \)-a.e. in } \mathbb{R}^n,
\]

and since both sides of (33) are zero \( |\tilde{D}u| \)-almost everywhere on \( |\langle \tilde{D}u, \nu \rangle| \)-negligible sets, we conclude that

\[
\lim_{h \to 0} \frac{f(\tilde{u}(x) + h \langle \tilde{D}u, \nu \rangle(x)) - f(\tilde{u}(x))}{h} = \langle \tilde{D}v, \nu \rangle,
\]

\( \tilde{D}u \)-a.e. in \( \mathbb{R}^n \). Since \( \nu \) is arbitrary, by Remarks 7.3 and 7.4 the restriction of \( f \) to the affine space \( T_x^u \) is differentiable at \( \tilde{u}(x) \) for \( |\tilde{D}u| \)-almost every \( x \in \mathbb{R}^n \) and (26) holds.

It follows from (13), (14), and (15) that

\[
D(t_1, \ldots, t_n) = \sum_{I \in \mathcal{I}} (-1)^{|I| - 1} |I| \prod_{i \in I} \prod_{j \in I} (D_{ij} + \lambda_j t_j) \det A^{(\lambda)}(T_I). \tag{36}
\]

Let \( t_i = \hat{x}_i, i = 1, \ldots, n \). Lemma 1 leads to

\[
D(\hat{x}_1, \ldots, \hat{x}_n) = \prod_{i \in \mathcal{I}} \hat{x}_i \sum_{I \in \mathcal{I}} (-1)^{|I| - 1} |I| \det A^{(\lambda)}(I_I) \det A^{(\lambda)}(T_I). \tag{37}
\]

By (3), (13), and (37), we have the following result:

**Theorem 7.5**

\[
H_{\text{c}} = \frac{1}{2n} \sum_{l=1}^{n} l(-1)^{l-1} A_l^{(\lambda)}, \tag{38}
\]

where

\[
A_l^{(\lambda)} = \sum_{I_l \subseteq \mathcal{I}} \det A^{(\lambda)}(I_I) \det A^{(\lambda)}(T_I), |I| = l. \tag{39}
\]
It is worth noting that $A_l^{(\lambda)}$ of (39) is similar to the coefficients $b_l$ of the characteristic polynomial of (10). It is well known in graph theory that the coefficients $b_l$ can be expressed as a sum over certain subgraphs. It is interesting to see whether $A_l^{(\lambda)}$, $\lambda = 0$, structural properties of a graph.

We may call (38) a parametric representation of $H_c$. In computation, the parameter $\lambda_i$ plays very important roles. The choice of the parameter usually depends on the properties of the given graph. For a complete graph $K_n$, let $\lambda_i = 1$, $i = 1, \ldots, n$. It follows from (39) that

$$A_l^{(1)} = \begin{cases} n!, & \text{if } l = 1 \\ 0, & \text{otherwise} \end{cases}.$$ (40)

By (38)

$$H_c = \frac{1}{2}(n - 1)!.$$ (41)

For a complete bipartite graph $K_{n_1n_2}$, let $\lambda_i = 0$, $i = 1, \ldots, n$. By (39),

$$A_l = \begin{cases} -n_1!n_2!\delta_{n_1n_2}, & \text{if } l = 2 \\ 0, & \text{otherwise} \end{cases}.$$ (42)

Theorem 7.5 leads to

$$H_c = \frac{1}{n_1 + n_2}n_1!n_2!\delta_{n_1n_2}.$$ (43)

Now, we consider an asymmetrical approach. Theorem 3.3 leads to

$$\det K(t = 1, t_1, \ldots, t_n; l) = \sum_{I \subseteq n - \{l\}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I}(D_j + \lambda_j t_j) \det A^{(\lambda)}(\mathcal{T} \cup \{l\})$$

which reduces to Goulden–Jackson’s formula when $\lambda_i = 0$, $i = 1, \ldots, n$ [9].

8 Various font features of the amsmath package

8.1 Bold versions of special symbols

In the amsmath package \texttt{\textbackslash boldsymbol} is used for getting individual bold math symbols and bold Greek letters—everything in math except for letters of the Latin alphabet, where you’d use \texttt{\textbackslash mathbf}. For example,

\begin{verbatim}
A_\infty + \pi A_0 \sim \mathbf{A}_\infty + \pi \mathbf{A}_0
\end{verbatim}

looks like this:

$$A_\infty + \pi A_0 \sim A_\infty + \pi A_0.$$
8.2 “Poor man’s bold”

If a bold version of a particular symbol doesn’t exist in the available fonts, then \boldsymbol can’t be used to make that symbol bold. At the present time, this means that \boldsymbol can’t be used with symbols from the msam and msbm fonts, among others. In some cases, poor man’s bold (\pmb) can be used instead of \boldsymbol:

\[ \frac{\partial x}{\partial y} \right| \frac{\partial y}{\partial z} \]

\[ \frac{\partial x}{\partial y} \left( \frac{\partial y}{\partial z} \right) \]

So-called “large operator” symbols such as \sum and \prod require an additional command, \mathop, to produce proper spacing and limits when \pmb is used. For further details see The \TeXbook.

\[ \sum_{\substack{i<B\	ext{i odd}}} \prod_{\kappa} \kappa F(r_i) \]

9 Compound symbols and other features

9.1 Multiple integral signs

\iint, \iiint, and \iiiint give multiple integral signs with the spacing between them nicely adjusted, in both text and display style. \idotsint gives two integral signs with dots between them.

\[ \iiint_{A} f(x,y,z) \, dx \, dy \, dz \]

\begin{align*}
\iiint_{A} f(x,y,z) \, dx \, dy \, dz &= \dotsb \int_{A} f(x_1, \ldots, x_k) \\
\end{align*}

9.2 Over and under arrows

Some extra over and under arrow operations are provided in the amsmath package. (Basic \LaTeX provides \overrightarrow and \overleftarrow).

\[ \overrightarrow{\psi(t) E_i h} = \overleftarrow{\psi(t) E_i h} \]

\begin{align*}
\overrightarrow{\psi(t) E_i h} &= \overleftarrow{\psi(t) E_i h} \\
\end{align*}

\begin{align*}
\begin{align*}
\end{align*}
\end{align*}
These all scale properly in subscript sizes:

\[ \int_{\overrightarrow{AB}} ax\,dx \]

9.3 Dots

Normally you need only type \dots for ellipsis dots in a math formula. The main exception is when the dots fall at the end of the formula; then you need to specify one of \dotsc (series dots, after a comma), \dotsb (binary dots, for binary relations or operators), \dotsm (multiplication dots), or \dotsi (dots after an integral). For example, the input

Then we have the series $A_1,A_2,\dotsc$, the regional sum $A_1+A_2+\dotsb$, the orthogonal product $A_1A_2\dotsm$, and the infinite integral

\[ \int_{A_1}\int_{A_2}\dotsi \]

produces

Then we have the series $A_1, A_2, \ldots$, the regional sum $A_1 + A_2 + \cdots$, the orthogonal product $A_1A_2\cdots$, and the infinite integral

\[ \int_{A_1} \int_{A_2} \ldots \]

9.4 Accents in math

Double accents:

\[ \hat{H} \quad \check{C} \quad \tilde{T} \quad \acute{A} \quad \grave{G} \quad \acute{D} \quad \ddot{D} \quad \breve{B} \quad \bar{B} \quad \vec{V} \]

This double accent operation is complicated and tends to slow down the processing of a \LaTeX file.

9.5 Dot accents

\dddot and \dddddot are available to produce triple and quadruple dot accents in addition to the \dot and \ddot accents already available in \LaTeX:

\[ \dddot{Q} \quad \ddddot{R} \]

9.6 Roots

In the \texttt{amsmath} package \texttt{\leftroot} and \texttt{\uproot} allow you to adjust the position of the root index of a radical:

\[ \sqrt{\leftroot{-2}\uproot{2}\beta\{k} \]

gives good positioning of the $\beta$: \[ \sqrt[k]{\beta} \]
9.7 Boxed formulas
The command \boxed puts a box around its argument, like \fbox except that the contents are in math mode:

\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}

9.8 Extensible arrows
\xleftarrow and \xrightarrow produce arrows that extend automatically to accommodate unusually wide subscripts or superscripts. The text of the subscript or superscript are given as an optionalresp. mandatory argument: Example:

0 \xleftarrow{\zeta} F \times [n-1] \xrightarrow{\partial \alpha \beta} E\hat{\partial} b

9.9 \overset, \underset, and \sideset
Examples:

\overset{*}{X}\qquad\underset{*}{X}\qquad\overset{a}{\underset{b}{X}}

The command \sideset is for a rather special purpose: putting symbols at the subscript and superscript corners of a large operator symbol such as \sum or \prod, without affecting the placement of limits. Examples:

\sideset{\alpha \beta \gamma}{\ast \ast \ast} \prod_k \sum_{0 \le i \le m} E_i \beta x

9.10 The \text command
The main use of the command \text is for words or phrases in a display:

\text{y = y' if and only if} y'_k = \delta_k y_{\tau(k)}

9.11 Operator names
The more common math functions such as log, sin, and lim have predefined control sequences: \log, \sin, \lim. The amsmath package provides \DeclareMathOperator and \DeclareMathOperator* for producing new function names that will have the same typographical treatment. Examples:

\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|
\[
\meas_1\left\{ u \in R_+^1 : f^*(u) > \alpha \right\} = \meas_n\left\{ x \in R^n : |f(x)| \geq \alpha \right\} \quad \forall \alpha > 0.
\]

\[\esssup\text{ and } \meas\text{ would be defined in the document preamble as}\]
\[
\DeclareMathOperator*{\esssup}{ess\,sup} \\
\DeclareMathOperator{\meas}{meas}
\]

The following special operator names are predefined in the \texttt{amsmath} package: \texttt{\varlimsup}, \texttt{\varliminf}, \texttt{\varinjlim}, and \texttt{\varprojlim}. Here’s what they look like in use:

\[
\begin{align*}
\varlimsup_{n \to \infty} Q(u_n, u_n - u^\#) &\leq 0 \\
\varliminf_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} &\to 0 \\
\varinjlim (m_i^\lambda \cdot)^* &\leq 0 \\
\varprojlim_{p \in S(A)} A_p &\leq 0
\end{align*}
\]

\begin{align*}
\text{(48)} \\
\text{(49)} \\
\text{(50)} \\
\text{(51)}
\end{align*}

9.12 \texttt{\mod} and its relatives

The commands \texttt{\mod} and \texttt{\pod} are variants of \texttt{\pmod} preferred by some authors; \texttt{\mod} omits the parentheses, whereas \texttt{\pod} omits the “mod” and retains the parentheses. Examples:

\[
x \equiv y + 1 \pmod{m^2}
\]

\[
x \equiv y + 1 \mod{m^2}
\]

\[
x \equiv y + 1 \mod{m^2}
\]

\[
\begin{align*}
x &\equiv y + 1 \mod{m^2} \\
x &\equiv y + 1 \pmod{m^2}
\end{align*}
\]

9.13 Fractions and related constructions

The usual notation for binomials is similar to the fraction concept, so it has a similar command \texttt{\binom} with two arguments. Example:

\[
\sum_{\gamma \in \Gamma_C} I_{\gamma} = 2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} + \cdots + (-1)^i \binom{k}{i} 2^{k-i} + \cdots + (-1)^k
\]

\[
= (2 - 1)^k = 1
\]

\[
\begin{align*}
(55)
\end{align*}
\]
\[ \sum_{\gamma \in \Gamma_C} I_\gamma = 2^k - \binom{k}{1}2^{k-1} + \binom{k}{2}2^{k-2} + \cdots + (-1)^l \binom{k}{l}2^{k-l} + \cdots + (-1)^k = (2-1)^k = 1 \]

There are also abbreviations
\[ \frac \binom \tfrac \dbinom \tbinom \]
for the commonly needed constructions
\[ \displaystyle \frac \binom \tfrac \dbinom \tbinom \]
\[ \textstyle \frac \binom \tfrac \dbinom \tbinom \]

The generalized fraction command \texttt{\textbackslash genfrac} provides full access to the six \LaTeX{} fraction primitives:
\[ \over: \frac{n+1}{2} \quad \overwithdelims: \left( \frac{n+1}{2} \right) \quad \text{(56)} \]
\[ \atop: \frac{n+1}{2} \quad \atopwithdelims: \left( \frac{n+1}{2} \right) \quad \text{(57)} \]
\[ \above: \frac{n+1}{2} \quad \abovewithdelims: \left[ \frac{n+1}{2} \right] \quad \text{(58)} \]

9.14 Continued fractions

The continued fraction
\[ \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \cdots}}}} \quad \text{(59)} \]
can be obtained by typing
\[ \texttt{\textbackslash cfrac{1}{\textbackslash sqrt{2} + \textbackslash cfrac{1}{\textbackslash sqrt{2} + \textbackslash cfrac{1}{\textbackslash sqrt{2} + \textbackslash cfrac{1}{\textbackslash sqrt{2} + \cdots}}}}} \]
\[ \cfrac{1}{\sqrt{2} + \cfrac{1}{\sqrt{2} + \ldotsb} } \]

Left or right placement of any of the numerators is accomplished by using \( \cfrac[l]{} \) or \( \cfrac[r]{} \) instead of \( \cfrac \).

9.15 Smash

In amsmath there are optional arguments \( t \) and \( b \) for the plain \TeX\ command \( \texttt{\textbackslash smash} \), because sometimes it is advantageous to be able to ‘smash’ only the top or only the bottom of something while retaining the natural depth or height. In the formula \( X_j = (1/\sqrt{\lambda_j})X'_j \texttt{\textbackslash smash}[b] \) has been used to limit the size of the radical symbol.

\[ X_{-j} = (1/\sqrt{\smash[b]{\lambda_j}})X'_j \]

Without the use of \texttt{\textbackslash smash}[b] the formula would have appeared thus: \( X_j = (1/\sqrt{\lambda_j})X'_j \), with the radical extending to encompass the depth of the subscript \( j \).

9.16 The ‘cases’ environment

‘Cases’ constructions like the following can be produced using the \texttt{cases} environment.

\[
P_{r-j} = \begin{cases} 
0 & \text{if } r-j \text{ is odd}, \\
r!( -1 ) ^ { ( r-j ) / 2 } & \text{if } r-j \text{ is even}.
\end{cases}
\]

\begin{equation}
P_{r-j} = \begin{cases} 
0 & \text{if } r-j \text{ is odd}, \\
r!( -1 ) ^ { ( r-j ) / 2 } & \text{if } r-j \text{ is even}.
\end{cases}
\end{equation}

Notice the use of \texttt{\textbackslash text} and the embedded math.

9.17 Matrix

Here are samples of the matrix environments, \texttt{\textbackslash matrix}, \texttt{\textbackslash pmatrix}, \texttt{\textbackslash bmatrix}, \texttt{\textbackslash Bmatrix}, \texttt{\textbackslash vmatrix} and \texttt{\textbackslash Vmatrix}:

\[
\begin{vmatrix}
\vartheta & \varepsilon \\
\varepsilon & \vartheta
\end{vmatrix}
\]

\begin{equation}
\begin{vmatrix}
\vartheta & \varepsilon \\
\varepsilon & \vartheta
\end{vmatrix}
\end{equation}

\begin{equation}
\begin{vmatrix}
\vartheta & \varepsilon \\
\varepsilon & \vartheta
\end{vmatrix}
\end{equation}
To produce a small matrix suitable for use in text, use the `smallmatrix` environment.

\begin{math}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\end{math}

To show the effect of the matrix on the surrounding lines of a paragraph, we put it here: \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and follow it with enough text to ensure that there will be at least one full line below the matrix.

\begin{align*}
\text{\textbackslash hdotsfor\{number\}} \text{ produces a row of dots in a matrix spanning the given number of columns:}
\end{align*}

\[ W(\Phi) = \begin{vmatrix}
\varphi & 0 & \ldots & 0 \\
\varphi_{k_2} & \varphi & \ldots & 0 \\
\varphi_{k_1} & \varphi_{k_2} & \ddots & \varphi_{k_{n-1}} \\
\varphi_{k_{n-1}} & \varphi_{k_{n-2}} & \ldots & \varphi_{k_n}
\end{vmatrix} \]

\begin{align*}
\text{\textbackslash hdotsfor\{1.5\}\{3\}. The number in square brackets will be used as a multiplier; the normal value is 1.}
\end{align*}

9.18 The `\substack` command

The `\substack` command can be used to produce a multiline subscript or superscript: for example

\begin{align*}
\sum_{\substack{0 \le i \le m \\ 0 < j < n}} P(i,j)
\end{align*}

produces a two-line subscript underneath the sum:

\( \sum_{0 \le i \le m \atop 0 < j < n} P(i,j) \) (62)

A slightly more generalized form is the `subarray` environment which allows you to specify that each line should be left-aligned instead of centered, as here:

\begin{align*}
\sum_{\begin{subarray}{l}
0 \le i \le m \\
0 < j < n
\end{subarray}} P(i,j)
\end{align*}

\( \sum_{\begin{subarray}{l}
0 \le i \le m \\
0 < j < n
\end{subarray}} P(i,j) \) (63)
9.19 Big-g-g delimiters

Here are some big delimiters, first in \texttt{\smallsize}:

\[
\left( E_y \int_0^{t_\varepsilon} L_{x,y^{2x}(s)} \varphi(x) \, ds \right)
\]

and now in \texttt{\Large} size:

\[
\left( E_y \int_0^{t_\varepsilon} L_{x,y^{2x}(s)} \varphi(x) \, ds \right)
\]
A Examples of multiple-line equation structures

Note: Starting on this page, vertical rules are added at the margins so that the positioning of various display elements with respect to the margins can be seen more clearly.

A.1 Split

The `split` environment is not an independent environment but should be used inside something else such as `equation` or `align`.

If there is not enough room for it, the equation number for a `split` will be shifted to the previous line, when equation numbers are on the left; the number shifts down to the next line when numbers are on the right.

\[ f_{h,\varepsilon}(x,y) = c \mathbf{E}_{x,y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, du + h \int L_{x,z} \varphi(x) \rho_{x}(dz) + h \left[ \frac{1}{t_{\varepsilon}} \left( \mathbf{E}_{y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, ds - t_{\varepsilon} \int L_{x,z} \varphi(x) \rho_{x}(dz) \right) \right. \]

\[ \left. + \frac{1}{t_{\varepsilon}} \left( \mathbf{E}_{y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, ds - \mathbf{E}_{x,y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, ds \right) \right] \]

\[ = h \hat{L}_{x} \varphi(x) + h \theta_{\varepsilon}(x,y), \]

Some text after to test the below-display spacing.

\[
\begin{equation}
\begin{split}
 f_{h,\varepsilon}(x,y) = c \mathbf{E}_{x,y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, du + h \int L_{x,z} \varphi(x) \rho_{x}(dz) + h \left[ \frac{1}{t_{\varepsilon}} \left( \mathbf{E}_{y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, ds - t_{\varepsilon} \int L_{x,z} \varphi(x) \rho_{x}(dz) \right) \right. \\
\left. + \frac{1}{t_{\varepsilon}} \left( \mathbf{E}_{y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, ds - \mathbf{E}_{x,y} \int_{0}^{t_{\varepsilon}} L_{x,y_{\varepsilon}(s)} \varphi(x) \, ds \right) \right] \\
= h \hat{L}_{x} \varphi(x) + h \theta_{\varepsilon}(x,y),
\end{split}
\end{equation}
\]
Unnumbered version:

\[
\begin{split}
 f_{h, \varepsilon}(x, y) &= \varepsilon \mathbf{E}_{x,y} \int_0^{t_\varepsilon} L_{x,y,\varepsilon(\varepsilon u)} \varphi(x) \, du \\
&= h \int L_{x,z} \varphi(x) \rho_x(dz) \\
&\quad + h \left[ \frac{1}{t_\varepsilon} \left( \mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^\varepsilon(s)} \varphi(x) \, ds - t_\varepsilon \int L_{x,z} \varphi(x) \rho_x(dz) \right) \\
&\quad + \frac{1}{t_\varepsilon} \left( \mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^\varepsilon(s)} \varphi(x) \, ds - \mathbf{E}_{x,y} \int_0^{t_\varepsilon} L_{x,y,\varepsilon(\varepsilon s)} \varphi(x) \, ds \right) \right] \\
&= h L_{x} \varphi(x) + h \theta_\varepsilon(x, y),
\end{split}
\]

Some text after to test the below-display spacing.
If the option `centertags` is included in the options list of the `amsmath` package, the equation numbers for `split` environments will be centered vertically on the height of the `split`:

\[
|I_2| = \left| \int_0^T \psi(t) \left\{ u(a, t) - \int_{\gamma(t)}^a \frac{d\theta}{k(\theta, t)} \int_0^\theta c(\xi) u_\xi(\xi, t) \, d\xi \right\} \, dt \right|
\leq C_{6} \left| f \int_\Omega |\bar{S}_{a,-1,0} W_2(\Omega, \Gamma_l)| \right| \left| \|u\|_{\omega_{-2}} W_2^\Omega(\Omega; \Gamma_l, T) \right|.
\] (65)

Some text after to test the below-display spacing.
Use of split within align:

\begin{align}
|I_1| &= \left| \int_{\Omega} gRu \, d\Omega \right| \\
&\leq C_3 \left[ \int_{\Omega} \left( \int_a^x g(\xi,t) \, d\xi \right)^2 \, d\Omega \right]^{1/2} \\
&\quad \times \left[ \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_a^x c u_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \\
&\leq C_4 \left| \int_{\Omega} S_{a,-}^{-1,0} W_2(\Omega,\Gamma_l) \right| \left| u \overset{\circ}{\to} W_2^3(\Omega; \Gamma_r, T) \right|.
\end{align}

(66)

\begin{align}
|I_2| &= \left| \int_0^T \psi(t) \left\{ u(a,t) - \int_{\gamma(t)}^a \frac{d\theta}{k(\theta,t)} \int_a^\theta c(\xi) u_t(\xi,t) \, d\xi \right\} dt \right| \\
&\leq C_6 \left[ \int_{\Omega} \left| S_{a,-}^{-1,0} W_2(\Omega,\Gamma_l) \right| \left| u \overset{\circ}{\to} W_2^3(\Omega; \Gamma_r, T) \right| \right].
\end{align}

(67)

Some text after to test the below-display spacing.

\begin{align}
|I_1| &= \left| \int_{\Omega} gRu \, d\Omega \right| \\
&\leq C_3 \left[ \int_{\Omega} \left( \int_a^x g(\xi,t) \, d\xi \right)^2 \, d\Omega \right]^{1/2} \\
&\quad \times \left[ \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_a^x c u_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \\
&\leq C_4 \left| \int_{\Omega} S_{a,-}^{-1,0} W_2(\Omega,\Gamma_l) \right| \left| u \overset{\circ}{\to} W_2^3(\Omega; \Gamma_r, T) \right|.
\end{align}

(66)

\begin{align}
|I_2| &= \left| \int_0^T \psi(t) \left\{ u(a,t) - \int_{\gamma(t)}^a \frac{d\theta}{k(\theta,t)} \int_a^\theta c(\xi) u_t(\xi,t) \, d\xi \right\} dt \right| \\
&\leq C_6 \left| \int_{\Omega} S_{a,-}^{-1,0} W_2(\Omega,\Gamma_l) \right| \left| u \overset{\circ}{\to} W_2^3(\Omega; \Gamma_r, T) \right|.
\end{align}

(67)
Unnumbered align, with a number on the second split:

\[ I_1 = \left| \int_{\Omega} gR u \, d\Omega \right| \]
\[ \leq C_3 \left[ \int_{\Omega} \left( \int_{a}^{x} g(\xi, t) \, d\xi \right)^2 \, d\Omega \right]^{1/2} \]
\[ \times \left[ \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_{a}^{x} cu_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \]
\[ \leq C_4 \| f \| \left\| \tilde{S}_{a,-}^{-1,0} W_2(\Omega, \Gamma) \right\| \| u \|^{-1}_{(0; \Gamma_r, T)} \] \[ \leq C_4 \left[ \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_{a}^{x} cu_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \]
\[ \leq C_4 \left[ \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_{a}^{x} cu_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \]
\[ \leq C_6 \left[ f \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_{a}^{x} cu_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \] \text{(67')}

\begin{align*}
I_2 &= \left| \int_{0}^{T} \psi(t) \left\{ u(a, t) - \int_{\gamma(t)}^{a} \frac{d\theta}{k(\theta, t)} \int_{a}^{\theta} c(\xi) u_t(\xi, t) \, d\xi \right\} \, dt \right| \\
&\leq C_6 \left[ f \int_{\Omega} \left\{ u_x^2 + \frac{1}{k} \left( \int_{a}^{x} cu_t \, d\xi \right)^2 \right\} \, d\Omega \right]^{1/2} \]
\end{align*}
A.2 Multline

Numbered version:

\[
\int_a^b \left\{ \int_a^b \left[ f(x)^2 g(y)^2 + f(y)^2 g(x)^2 \right] - 2f(x)g(x)f(y)g(y) \right\} \, dx \right\} \, dy \\
= \int_a^b \left\{ g(y)^2 \int_a^b f^2 + f(y)^2 \int_a^b g^2 - 2f(y)g(y) \int_a^b fg \right\} \, dy \quad (68)
\]

To test the use of \texttt{\label} and \texttt{\ref}, we refer to the number of this equation here: \texttt{(68)}.

\begin{multline}
\int_a^b \left\{ \int_a^b \left[ f(x)^2 g(y)^2 + f(y)^2 g(x)^2 \right] - 2f(x)g(x)f(y)g(y) \right\} \, dx \right\} \, dy \\
= \int_a^b \left\{ g(y)^2 \int_a^b f^2 + f(y)^2 \int_a^b g^2 - 2f(y)g(y) \int_a^b fg \right\} \, dy
\end{multline}

Unnumbered version:

\[
\int_a^b \left\{ \int_a^b \left[ f(x)^2 g(y)^2 + f(y)^2 g(x)^2 \right] - 2f(x)g(x)f(y)g(y) \right\} \, dx \right\} \, dy \\
= \int_a^b \left\{ g(y)^2 \int_a^b f^2 + f(y)^2 \int_a^b g^2 - 2f(y)g(y) \int_a^b fg \right\} \, dy
\]

Some text after to test the below-display spacing.
A.3 Gather

Numbered version with \notag on the second line:

\begin{align}
D(a, r) &\equiv \{ z \in \mathbb{C} : |z - a| < r \}, \\
\text{seg}(a, r) &\equiv \{ z \in \mathbb{C} : \Re z = \Re a, |z - a| < r \}, \\
c(e, \theta, r) &\equiv \{ (x, y) \in \mathbb{C} : |x - e| < y \tan \theta, 0 < y < r \}, \\
C(E, \theta, r) &\equiv \bigcup_{e \in E} c(e, \theta, r).
\end{align}

Unnumbered version.

\begin{align*}
D(a, r) &\equiv \{ z \in \mathbb{C} : |z - a| < r \}, \\
\text{seg}(a, r) &\equiv \{ z \in \mathbb{C} : \Re z = \Re a, |z - a| < r \}, \\
c(e, \theta, r) &\equiv \{ (x, y) \in \mathbb{C} : |x - e| < y \tan \theta, 0 < y < r \}, \\
C(E, \theta, r) &\equiv \bigcup_{e \in E} c(e, \theta, r).
\end{align*}
A.4 Align

Numbered version:

\[ \gamma_x(t) = (\cos tu + \sin tx, v), \quad (72) \]
\[ \gamma_y(t) = (u, \cos tv + \sin ty), \quad (73) \]
\[ \gamma_z(t) = \left( \cos tu + \frac{\alpha}{\beta} \sin tv, -\frac{\beta}{\alpha} \sin tu + \cos tv \right). \quad (74) \]

Unnumbered version:

\[ \gamma_x(t) = (\cos tu + \sin tx, v), \]
\[ \gamma_y(t) = (u, \cos tv + \sin ty), \]
\[ \gamma_z(t) = \left( \cos tu + \frac{\alpha}{\beta} \sin tv, -\frac{\beta}{\alpha} \sin tu + \cos tv \right). \]

A variation:

\[ x = y \quad \text{by (84)} \]  \hspace{1cm} (75)
\[ x' = y' \quad \text{by (85)} \]  \hspace{1cm} (76)
\[ x + x' = y + y' \quad \text{by Axiom 1.} \]  \hspace{1cm} (77)
A.5 Align and split within gather

When using the \texttt{align} environment within the \texttt{gather} environment, one or the other, or both, should be unnumbered (using the * form); numbering both the outer and inner environment would cause a conflict.

Automatically numbered \texttt{gather} with \texttt{split} and \texttt{align*}:

\begin{align*}
\phi(x,z) &= z - \gamma_{10} x - \gamma_{mn} x^m z^n \\
&= z - M^{r-1} x - M^{-(m+n)} x^m z^n
\end{align*}

Here the \texttt{split} environment gets a number from the outer \texttt{gather} environment; numbers for individual lines of the \texttt{align*} are suppressed because of the star.

\begin{align*}
\zeta^0 &= (\xi^0)^2, \\
\zeta^1 &= \xi^0 \xi^1, \\
\zeta^2 &= (\xi^1)^2,
\end{align*}

Some text after to test the below-display spacing.

The *ed form of \texttt{gather} with the non-*ed form of \texttt{align}.

\begin{align*}
\varphi(x,z) &= z - \gamma_{10} x - \gamma_{mn} x^m z^n \\
&= z - M^{r-1} x - M^{-(m+n)} x^m z^n
\end{align*}

\begin{align}
\zeta^0 &= (\xi^0)^2, \\
\zeta^1 &= \xi^0 \xi^1, \\
\zeta^2 &= (\xi^1)^2,
\end{align}
A.6 Alignat

Numbered version:
\begin{alignat}{3}
V_i &= v_i - q_i v_j, & \quad X_i &= x_i - q_i x_j, & \quad U_i &= u_i, \quad \text{for } i \neq j; \quad (82) \\
V_j &= v_j, & \quad X_j &= x_j, & \quad U_j u_j + \sum_{i \neq j} q_i u_i. \quad (83)
\end{alignat}

Some text after to test the below-display spacing.

\begin{alignat*}{3}
V_i &= v_i - q_i v_j, & \quad X_i &= x_i - q_i x_j, & \quad U_i &= u_i, \quad \text{for } i \neq j; \\
V_j &= v_j, & \quad X_j &= x_j, & \quad U_j u_j + \sum_{i \neq j} q_i u_i.
\end{alignat*}

Some text after to test the below-display spacing.
The most common use for \texttt{alignat} is for things like

\begin{alignat}{2}
  x &= y \quad \text{by (66)} \label{eq:C} \\
  x' &= y' \quad \text{by (82)} \label{eq:D} \\
  x + x' &= y + y' \quad \text{by Axiom 1.} \label{eq:E}
\end{alignat}

Some text after to test the below-display spacing.
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**MSC (2000)** 04A25

This is an example input file. Comparing it with the output it generates can show you how to produce a simple document of your own.

1 Introduction

The class file w-art.cls represents an adaptation of the L\TeX\-standard class file article.cls and the \AMS\ class file amsart.cls with the size option 10pt to the specific requirements of journal production at WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim. It can be used through the L\TeX\-command

```latex
\documentclass[<abbr>,fleqn, other options]{w-art}
```

where \texttt{<abbr>} is an abbreviation of the journal name.

| Abbreviation | Journal Name | Description |
|--------------|--------------|-------------|
| adp          | Ann. Phys. (Leipzig) | Annalen der Physik |
| cpp          | Contrib. Plasma Phys. | Contributions to Plasma Physics |
| mn           | Math. Nachr. | Mathematische Nachrichten |
| mlq          | Math. Log. Quart. | Mathematical Logic Quarterly |
| pop          | Fortschr. Phys. | Fortschritte der Physik |
| pss          | phys. stat. sol. | physica status solidi |
| pssa         | phys. stat. sol. (a) | physica status solidi (a) |
| pssb         | phys. stat. sol. (b) | physica status solidi (b) |
| zamm         | ZAMM · Z. Angew. Math. Mech. | Zeitschrift für Angewandte Mathematik und Mechanik |

One difference to the standard layout is the indentation by 3 cc or 4 cc of floats (figures and tables) and mathematical environments (\texttt{\[ \ldots \]}, \texttt{equation}, \texttt{multline}, \ldots). To achieve this effect the new floats vchtable and vchfigure were added which are to be used in combination with \texttt{vchcaption}. The standard \texttt{table}, \texttt{figure}, and \texttt{caption} commands are nevertheless still working. So if there is the need to place a table or figure over the full width of the page these floats may still be used. In order to get short captions flushed left in contrast to the standard centered form the class loads internally the \texttt{caption2.sty} package by Harald Axel Sommerfeldt with the options \texttt{nooneline}, \texttt{small}, \texttt{bf}.

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\** Second author footnote.
\*** Third author footnote.
2 Required packages

This class requires the standard \LaTeX{} packages \texttt{calc}, \texttt{sidecap}, and \texttt{caption2} and the \texttt{AMS-\LaTeX} packages\footnote{If these packages are not part of your installation you may download them from the nearest CTAN server.}.

2.1 New documentclass options

\texttt{separatedheads} (default): This gives the normal section, subsection and subsubsection headings with white space above and below the heading.

\texttt{embeddedheads}: With the \texttt{embeddedheads} option all section headings on all levels will be typeset as run-in headings without numbering like the standard \LaTeX{} paragraph. If the numbering shall remain one has to set \texttt{\setcounter{secnumdepth}{3}} explicitly in the preamble of the document.

\texttt{autolastpage} (default): The pagenumber of the last page will automatically be determined by the classfile. In the \texttt{\pagespan{}}{} command only the first entry should be entered. If the second entry is also given it will be ignored. The pagenumber of the lastpage will be used in the running head of the first page. In order to get correct results the document has to be run through \LaTeX{} at least twice.

\texttt{noautolastpage}: The value of the second argument of the \texttt{\pagespan{}}{} command will be printed as the last pagenumber.

\texttt{referee}: Prints the document with a larger amount of interline whitespace.

2.2 Floating objects – figures and tables

We have two different table environments: \texttt{table} and \texttt{vchtable}. The same holds true for figure: \texttt{figure} and \texttt{vchfigure}. The vch-types including their captions (vchcaption) are typically left indented by an amount equal to the indentation of mathematical formulas.

For the caption layout the \texttt{caption2.sty} package is preloaded.

Additionally the \texttt{sidecap} package of the \LaTeX{}-distribution will be loaded with the option “rightcaption” by the \texttt{w-art} class. This package defines the \texttt{SCfigure} and \texttt{SCTable} environment for figures and tables with captions on one side.

2.2.1 Tables

The \LaTeX{} code for Table 2 is
\begin{verbatim}
\begin{table}
\caption{The caption inside a table environment.}
\label{tab:2}\renewcommand{\arraystretch}{1.5}
\begin{tabular}{lll} \hline
\end{tabular}
\caption{The caption inside a vchtable environment.}
\label{tab:3}\renewcommand{\arraystretch}{1.5}
\begin{tabular}{lll} \hline
\end{tabular}
\begin{verbatim}
\caption{The caption inside a table environment.}
\label{tab:2}\renewcommand{\arraystretch}{1.5}
\begin{tabular}{lll} \hline
\caption{The caption inside a vchtable environment.}
\label{tab:3}\renewcommand{\arraystretch}{1.5}
\begin{tabular}{lll} \hline
\end{verbatim}
\end{verbatim}
Table 2  The caption inside a table environment.

| Description 1 | Description 2 | Description |
|---------------|---------------|-------------|
| Row 1, Col 1  | Row 1, Col 2  | Row 1, Col 3|
| Row 2, Col 1  | Row 2, Col 2  | Row 2, Col 3|

\begin{tabular}{l|c|c}
Row 1, Col 1 & Row 1, Col 2 & Row 1, Col 3 \\
Row 2, Col 1 & Row 2, Col 2 & Row 2, Col 3 \hline
\end{tabular}
\end{vchtable}

Table 3  The caption inside a vchtale environment.

| Description 1 | Description 2 | Description |
|---------------|---------------|-------------|
| Row 1, Col 1  | Row 1, Col 2  | Row 1, Col 3|
| Row 2, Col 1  | Row 2, Col 2  | Row 2, Col 3|

2.2.2  Figures

The \LaTeX{} code for Fig. 1 is

\begin{figure}[htb]
\includegraphics[width=.5\textwidth]{empty.eps}
\caption{The usual figure environment. It ...}
\label{fig:1}
\end{figure}

Fig. 1  The usual figure environment. It may be used for figures spanning the whole page width.

The \LaTeX{} code for Fig. 2 (a vchfigure) is

\begin{vchfigure}[htb]
\includegraphics[width=.5\textwidth]{empty.eps}
\vchcaption{A vchfigure environment with a vchcaption. Figure and caption are leftindented.}
\label{fig:2}
\end{vchfigure}

The \LaTeX{} code for Fig. 3 and 4 is

\begin{figure}[htb]
\begin{minipage}[t]{.45\textwidth}
\includegraphics[width=\textwidth]{empty.eps}
\caption{Two figures side by side with different numbers.}
\label{fig:3}
\end{minipage}
\end{figure}
Fig. 2 A `vchfigure` environment with a `vchcaption`. Figure and caption are leftindented.

\end{minipage}
\hfil
\begin{minipage}[t]{.45\textwidth}
\includegraphics[width=\textwidth]{empty.eps}
\caption{This is the second picture.}
\label{fig:4}
\end{minipage}
\end{figure}

Fig. 3 Two figures side by side with different numbers.

Fig. 4 This is the second picture.

The \LaTeX{} code for Fig. 5a and b is

\begin{figure}[htb]
\begin{minipage}[t]{.45\textwidth}
\includegraphics[width=\textwidth]{empty.eps} a)
\hfil
\includegraphics[width=\textwidth]{empty.eps} b)
\caption{Two figures with one number. The figures are referred to as a) and b).}
\label{fig:5}
\end{minipage}
\end{figure}
2.2.3 SCfigure and SCTable environments

The SCfigure and SCTable environment may be used as provided and described in the documentation of the sidecap package.

So a typical SCfigure environment would look as follows:

\begin{SCfigure}[<relwidth>][<float>]
\includegraphics[<options>]{filename.eps} \\
\caption{Caption of a SCfigure.}
\label{fig:x1} % Give a unique label
\end{SCfigure}

where \texttt{relwidth} (optional) is the caption width relative to the width of the figure or table. A large value (e.g., 50) reserves the maximum width that is possible.

And \texttt{float} (optional) is like the floating position parameter of the original table/figure environments. Default is \texttt{[tbp]}.

The alignment rules are:

- Figures and tables on top of a page should be top aligned with the caption.
- Figures and tables on bottom of a page should be bottom aligned with the caption.

The \LaTeX\ code for Fig. 6 is

\begin{SCfigure}[4][htb]
\includegraphics[width=.3\textwidth]{empty.eps} \\
\caption{Caption of a SCfigure figure. These captions are always bottom aligned.}
\label{fig:6}
\end{SCfigure}

In order to print a figure and a table side by side the \texttt{\setfloattype} command is introduced. The \LaTeX\ code for Fig. 7 and Table 4 is

\begin{figure}[htb]
\begin{minipage}{.45\textwidth}
\includegraphics[width=\textwidth]{empty.eps} \\
\caption{Figure and table side by side. This is the picture.}
\label{fig:8}
\end{minipage}
\end{figure}
Fig. 6 Caption of a SCfigure figure. These captions are always bottom aligned.

\hfil
\begin{minipage}{.45\textwidth}
\setfloattype{table}
\caption{This is the table. \ldots}
\label{tab:4}
\renewcommand{\arraystretch}{1.5}
\begin{tabular}{lll}
\ldots
\end{tabular}
\end{minipage}
\end{figure}

Fig. 7 Figure and table side by side. This is the picture.

Table 4 This is the table. Picture and table are both numbered independently.

| Description 1 | Description 2 | Description 3 |
|---------------|---------------|---------------|
| Row 1, Col 1  | Row 1, Col 2  | Row 1, Col 3  |
| Row 2, Col 1  | Row 2, Col 2  | Row 2, Col 3  |

3 Test of math environments

Equations are always left-aligned. Therefore the option fleqn is used for the documentclass command by default. Note that fleqn does not work with unnumbered displayed equations written as $$ Ax =b $$, so please use \[ Ax=b \] or an equation* or gather* environment instead.

By default the equations are consecutively numbered. This may be changed by putting the following command inside the preamble

\numberwithin{equation}{section}

The latex math display environment \[ \ldots \]

$$ \sum_{i=1}^{\infty} \frac{1}{i^2} $$
An equation environment:

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$ (1)

For more mathematical commands and environments please refer to the document *-tma.tex and the documentation of the \textit{AMSTeX} classes.

### 3.1 Some predefined theorem like environments

Some predefined theorem like environments may be used by loading the package \textit{w-thm.sty}. This package will load by itself the package \textit{amsthm.sty}. So it will be easy to define new theorem- and definition-like environments. For further details refer to the documentation of the \textit{amsthm.sty} package.

| environment     | caption   | theoremstyle |
|-----------------|----------|--------------|
| thm, theorem    | Theorem  | theorem      |
| prop, proposition | Proposition | theorem   |
| lem, lemma      | Lemma    | theorem      |
| cor, corollary  | Corollary| theorem      |
| axiom           | Axiom    | theorem      |
|defs, defn, definition | Definition | definition |
|example          | Example  | definition   |
|rem, remark      | Remark   | definition   |
|notation         | Notation | definition   |

**Theorem 3.1** This is a theorem.

**Theorem 3.2** Another theorem.

**Proof.** This is a proof.

**Definition 3.3** This is a definition.

**Proposition 3.4** This is a proposition.

**Lemma 3.5** This is a lemma.

**Corollary 3.6** This is a corollary.

**Example 3.7** This is an example.

**Remark 3.8** This is a remark.

### 3.2 Definition of new theorem like environments

Because \textit{w-thm.sty} uses \textit{amsthm.sty} the definition of new theorem like environments will be done in the same manner as in the \textit{amsthm} package. The definition of

\begin{verbatim}
\theoremstyle{plain}
\newtheorem{criterion}{Criterion}
\theoremstyle{definition}
\newtheorem{condition}{Condition}
\end{verbatim}

inside the preamble of the document will give the following environments.

**Criterion 1** This is a Criterion.

**Condition 3.9** This is a Condition.

If the name of a predefined environment has to be changed it can be done by e.g. typing

\begin{verbatim}
\renewcommand{\definitionname}{Definitions}
\end{verbatim}

after the \texttt{\begin{document}} command.
Acknowledgements An acknowledgement may be placed at the end of the article.

The style of the following references should be used in all documents.

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Greek symbols – \texttt{w-greek.sty}

\begin{table}[h]
\centering
\begin{tabular}{llllll}
\hline
$\alpha$ & $\beta$ & $\gamma$ & $\delta$ & $\epsilon$ & $\zeta$ \\
$\alpha$ & $\beta$ & $\gamma$ & $\delta$ & $\epsilon$ & $\zeta$ \\
$\theta$ & $\vartheta$ & $\iota$ & $\kappa$ & $\lambda$ & $\upsilon$ \\
$\theta$ & $\vartheta$ & $\iota$ & $\kappa$ & $\lambda$ & $\upsilon$ \\
\hline
\end{tabular}
\caption{Slanted greek letters}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{llllll}
\hline
$\Gamma$ & $\Delta$ & $\Theta$ \\
$\Gamma$ & $\Delta$ & $\Theta$ \\
$\Lambda$ & $\Xi$ & $\Pi$ \\
$\Lambda$ & $\Xi$ & $\Pi$ \\
$\Sigma$ & $\Upsilon$ & $\Omega$ \\
$\Sigma$ & $\Upsilon$ & $\Omega$ \\
$\Psi$ & $\Phi$ & $\Psi$ \\
$\Psi$ & $\Phi$ & $\Psi$ \\
\hline
\end{tabular}
\caption{Upright greek letters}
\end{table}
Table 3: Boldface variants of slanted greek letters

| Slanted Greek Letter | Boldface Variants |
|---------------------|-------------------|
| \( \alpha \)        | \textbf{\( \alpha \)} |
| \( \beta \)         | \textbf{\( \beta \)} |
| \( \gamma \)        | \textbf{\( \gamma \)} |
| \( \delta \)        | \textbf{\( \delta \)} |
| \( \epsilon \)      | \textbf{\( \epsilon \)} |
| \( \zeta \)         | \textbf{\( \zeta \)} |
| \( \eta \)          | \textbf{\( \eta \)} |
| \( \Gamma \)        | \textbf{\( \Gamma \)} |
| \( \Delta \)        | \textbf{\( \Delta \)} |
| \( \Theta \)        | \textbf{\( \Theta \)} |

Table 4: Boldface variants of upright greek letters

| Upright Greek Letter | Boldface Variants |
|---------------------|-------------------|
| \( \alpha \)        | \textbf{\( \alpha \)} |
| \( \beta \)         | \textbf{\( \beta \)} |
| \( \gamma \)        | \textbf{\( \gamma \)} |
| \( \delta \)        | \textbf{\( \delta \)} |
| \( \epsilon \)      | \textbf{\( \epsilon \)} |
| \( \zeta \)         | \textbf{\( \zeta \)} |
| \( \eta \)          | \textbf{\( \eta \)} |
| \( \Gamma \)        | \textbf{\( \Gamma \)} |
| \( \Delta \)        | \textbf{\( \Delta \)} |
| \( \Theta \)        | \textbf{\( \Theta \)} |