Neutrino wave packet propagation in gravitational fields

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Abstract

We discuss the propagation of neutrino wave packets in a Lense-Thirring space-time using a gravitational phase approach. We show that the neutrino oscillation length is altered by gravitational corrections and that neutrinos are subject to helicity flip induced by stellar rotation. For the case of a rapidly rotating neutron star, we show that absolute neutrino masses can be derived, in principle, from rotational contributions to the mass-induced energy shift, without recourse to mass generation models presently discussed in the literature.

Key words: neutrino wave packet, gravitational phase, helicity flip, absolute neutrino mass
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Recent experimental evidence from Superkamiokande \cite{1} and SNO \cite{2} has significantly endorsed the claim \cite{3} that neutrinos undergo flavour oscillations in vacuum due to the difference of their rest masses. This discovery, however, makes no pronouncements about the intrinsic physical properties of neutrinos, particularly whether they exist as plane waves or wave packets. In addition, the actual calculation of the oscillation length for solar neutrinos assumes a flat space-time background. While this assumption seems reasonable for neutrino propagation in our solar system, it may be unwarranted when the gravitational source is a neutron star or a black hole.

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The purpose of this work is to investigate the contribution of gravitation to the flavour oscillations of a two-neutrino system in a Lense-Thirring space-time background [4], assuming a wave packet description of the mass eigenstates in momentum space. This approach [5], so far untested in gravitational problems, offers an intuitively satisfying description of neutrinos. It relates directly the physical characteristics of the gravitational source, such as its stellar temperature and density, to the size of the wave packet. It also lends itself well to sensing distortions in space-time because of the common and intrinsically non-local nature of both curvature and wave packets. The choice of Lense-Thirring space-time is particularly relevant for our purpose, since rotational effects may induce a helicity transition while the neutrino is in transit.

A particularly interesting feature of our approach is the introduction of the gravitational phase that leads to a direct spin-gravity coupling interaction within the neutrino’s wavefunction. It was shown [6,7,8,9] that the gravitational phase

\[ \Phi_G \equiv \int_{x_0}^{x} dz^\lambda \gamma_{\alpha\lambda}(z)p^\alpha + \frac{1}{4} \int_{x_0}^{x} dz^\lambda \left[ \gamma_{\beta\lambda,\alpha}(z) - \gamma_{\alpha\lambda,\beta}(z) \right] L^{\alpha\beta}(z), \]  

enters the description of quantum particles in external gravitational fields in a way that is essential and independent of their spin. In (1), \( \gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \) is the metric deviation, and \( p^\mu \) and \( L^{\alpha\beta} \) are momentum and orbital angular momentum operators of the free particle. Application of (1) to closed space-time integration paths gives rise to a covariant Berry’s phase [10] with consequences for particle interferometry [11]. Equation (1) also yields the particle deflection predicted by general relativity in the geometrical optics approximation [12]. When applied to fermions, \( \Phi_G \) and the spin connection \( \Gamma_\mu \) reproduce all those gravitational-inertial effects that have been either observed [13,14] or predicted [15,16], and predicts, in particular, the non-conservation of helicity [17] for strictly massless fermions.

To show how \( \Phi_G \) acts on a neutrino propagating in vacuum, we first start with the covariant Dirac equation \[ i\gamma^\mu (x) \left( \nabla^\mu + i\Gamma^\mu \right) - m/\hbar \right) \psi(x) = 0, \]  

where \( m \) is the neutrino rest mass, and \( \nabla^\mu \) is the usual covariant derivative. We use geometric units of \( G = c = 1 \) [18], so that all physical quantities can be described in units of length. Then the corresponding Dirac Hamiltonian in Lense-Thirring space-time is

\begin{align*}
H_0 \approx & \left( 1 - \frac{2M}{r} \right) \alpha \cdot p + m \left( 1 - \frac{M}{r} \right) \beta + i\hbar \frac{M}{2r^3} (\alpha \cdot r) \\
& + \frac{4}{5} \frac{M\Omega R^2}{r^3} \hat{L}^z + \frac{1}{5} \frac{hM\Omega R^2}{r^3} \left[ \frac{3z}{r^2} (\sigma \cdot r) - \sigma^z \right],
\end{align*}  

(2)
where \( r = \sqrt{x^2 + y^2 + z^2} \), \( M \) and \( R \) are the mass and radius of the gravitational source, \( \Omega \) is its angular velocity, and \( L^z = xp^y - yp^x \) is the orbital angular momentum operator in the \( z \)-direction. The gravitational phase can be introduced by means of the transformation \( \psi(x) \rightarrow \exp \left( i \Phi_G / \hbar \right) \psi(x) \) and the new Hamiltonian takes the form \( H = H_0 + H_{\Phi_G} \), where

\[
H_{\Phi_G} = \alpha \cdot (\nabla \Phi_G) + (\nabla_t \Phi_G)
\]

is a first-order correction. We then treat (3) as a perturbation for a two mass-species neutrino system. While the approach taken is based on standard quantum mechanics, in order to apply (1) it is necessary to assume that the average of all possible paths reduces to the integration path of \( \Phi_G \). Because some terms of \( H \) are non-diagonal with respect to spin, the wave packet approach should also make known whether the neutrino mass eigenstates are subject to a helicity transition in the gravitational field of a rotating source.

Adopting the Dirac representation, the matrix element is

\[ \langle \psi(r) | H_{\Phi_G} | \psi(r) \rangle, \]

and we assume for the wave packet description

\[
|\psi(r)\rangle = \frac{1}{(2\pi)^{3/2}} \int d^3k \xi(k) e^{ik \cdot r} |U(k)\rangle_\pm,
\]

where \( \xi(k) = \xi(k^x)\xi(k^y)\xi(k^z) \) is the normalized Gaussian wavefunction of width \( \sigma_p \) and centroid \( \langle k \rangle \), and

\[
\xi(k^j) = \frac{1}{(\sqrt{2\pi} \sigma_p)^{1/2}} \exp \left[ -\frac{(k^j - \langle k \rangle^j)^2}{4\sigma_p^2} \right].
\]

The normalized four-spinor \( |U(k)\rangle_\pm \) is

\[
|U(k)\rangle_\pm \propto \sqrt{E + m / 2} \left( \begin{array}{c} 1 \\ \frac{\hbar \alpha \cdot k}{E + m} \end{array} \right) \otimes |\pm\rangle,
\]

\[
|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

where \( E = \sqrt{p^2 + m^2} \) is the energy associated with the spinor. By symmetrizing over the exchange between \( k \) and \( k' \), the explicit construction of the matrix element shows contributions due to both a spin diagonal term and a helicity transition, in the form
\[
\langle \psi(r)|H_{\Phi_G}|\psi(r)\rangle = \frac{\hbar^{3/2}}{(2\pi)^3 V} \int d^3r \ d^3k \ d^3k' \ \xi(k) \xi(k') \sqrt{\frac{E + m}{2E}} \sqrt{\frac{E' + m}{2E'}}
\]

\[
\times \left\{ \cos \left[ (k - k') \cdot r \right] \left[ \hbar \left( \nabla \Phi_G \right)_S \left( \frac{k}{E + m} + \frac{k'}{E' + m} \right) + \left( \nabla_t \Phi_G \right)_S \left( 1 + \frac{\hbar^2 (k \cdot k')}{(E + m)(E' + m)} \right) \right] \langle \pm | \pm \rangle \\
- \sin \left[ (k - k') \cdot r \right] \left[ \hbar \left( \nabla \Phi_G \right)_S \times \left( \frac{k}{E + m} - \frac{k'}{E' + m} \right) - \frac{\hbar^2 \left( \nabla_t \Phi_G \right)_S (k \times k')}{(E + m)(E' + m)} \right] \cdot \langle \mp | \sigma | \pm \rangle \right\}, \tag{7}
\]

where \((\nabla \mu \Phi_G)_S = (\hbar/2) \left[ (\nabla \mu \Phi_G) (k) + (\nabla \mu \Phi_G) (k') \right]\) is the symmetrized form of the gravitational phase gradients, and \(V = (4\pi/3) (r^3 - R^3)\) is the volume of spatial integration from the star’s surface. It is clear from the last line of (7) that there would be no helicity transition contribution if we considered strictly plane waves, since a non-zero transition amplitude requires different momentum components within the wave packet to interact with the Pauli spin matrices to yield a non-zero result. Choosing the spin quantization axis to be along the neutrinos’ direction of propagation, the helicity transition element is

\[
\langle \mp | \sigma | \pm \rangle = [\cos \theta \cos \varphi \mp i \sin \varphi] \hat{x} + [\cos \theta \sin \varphi \pm i \cos \varphi] \hat{y} - \sin \theta \hat{z}, \tag{8}
\]

where the upper sign refers to the transition from negative to positive helicity.

The explicit calculation of the integrals of (7) in spherical coordinates is prohibitively complicated by the coupling of the oscillatory functions to the amplitudes. However, an approximate, but analytic expression for (7) can be found by integrating over both position and momentum space after performing a 2nd-order Taylor series expansion with respect to \(m\) and an 11th-order Taylor expansion of the oscillatory functions. It can then be shown that (7) has the form

\[
\langle \psi(r)|H_{\Phi_G}|\psi(r)\rangle = \langle k | \left\{ \frac{M}{r} \left[ C_0(q,r) + C_1(q,r) \bar{m} + C_2(q,r) \bar{m}^2 \right] \\
+ \frac{M\Omega R^2}{r^2} \sin \theta \left[ D_0(q,r) + D_1(q,r) \bar{m} + D_2(q,r) \bar{m}^2 \right] \right\}, \tag{9}
\]

where \(\bar{m} = m/\langle p \rangle = m/(\hbar \langle k \rangle)\) and \(q = \langle k \rangle/\sigma_p\), with \(C_j(q,r)\) and \(D_j(q,r)\)
as dimensionless functions that can be tabulated. The first line of \( (9) \) is the contribution due to the diagonal components of \( \langle \psi(r) | H_{q,c} | \psi(r) \rangle \), while the second line is due to the off-diagonal components and refers to the helicity flip contribution of the perturbation.

For this paper, the gravitational source under consideration is a rapidly rotating \( 1.5M_\odot \) neutron star with \( R = 10 \text{ km} \) and \( \Omega = 1 \text{ kHz} \), at a distance of \( r = 10 \text{ kpc} \). Figure 1 contains a list of plots for the functions described in the matrix element \( (9) \). For this choice of parameters, it is clear from a comparison of Figures 1(a)-1(c) and Figures 1(d)-1(f) that \( |C_2(q,r)| \ll |C_1(q,r)| \ll |C_0(q,r)| \) and \( |D_2(q,r)| \ll |D_1(q,r)| \ll |D_0(q,r)| \) for all choices of \( q \). This suggests the trend towards convergence of the series due to expansion with respect to \( m \). To justify the truncation of the Taylor expansion of \( (7) \), we note from analyzing a one-dimensional analogue of the problem that the Gaussian functions which are present in the integrand have the effect of damping out the contribution of the higher-order expansion terms beyond the 11th order. A plot of this series expansion matches precisely with that due to the corresponding product function of the exact sinusoidal function with a one-dimensional Gaussian,

![Graphs](image-url)
except in the tail regions of the Gaussian envelope. In those isolated regions of parameter space, we estimate an error of 10-15% in the amplitude and are confident that this degree of error has no significant bearing on our results.

We stress some interesting features of (9). The most obvious one is the presence of terms linear in $m$, due to the Taylor expansion of $\langle \psi(r) | H_{\Phi_G} | \psi(r) \rangle$ for both the spin diagonal and off-diagonal terms. This fact has interesting implications, as shown below, in the calculation of the energy shifts. Given (8), we know that the helicity transition contribution to (9) is due entirely to the $z$-component of the transition amplitude, since the $x$- and $y$-components average out to zero over all spatial angles. Furthermore, the non-zero contribution is due entirely to the rotation of the source, which induces the helicity transition. Because of the $\sin \theta$ term, propagation of a neutrino along the $\pm z$-axis suggests that its initially prepared helicity state remains fixed throughout its motion. This is a sensible result that is consistent with our present understanding of spin-rotation coupling, since the rotational term in the Lense-Thirring metric tends to act like a uniform magnetic field that orients the particle spin either parallel or antiparallel to the field strength. This result also agrees with the fact that, barring quantum fluctuations about the classical integration path of (1), $\nabla_t \Phi_G = 0$ in the Lense-Thirring field, which implies helicity conservation [17].

To determine the mass-induced energy shift for the neutrino oscillation length, we note that the off-diagonal elements of (9) contribute to a second-order effect in time-independent perturbation theory, where the unperturbed energy eigenvalues $E_0^\pm$ come from $H_0 \psi(x) = E_0^\pm \psi(x)$ for $E_0^\pm \approx \sqrt{\langle p \rangle^2 + m^2 - \frac{2M}{r} \langle p \rangle}$.
\[ \frac{4}{5} \frac{M \Omega R^2}{r^3} \left( L^\hat{z} \pm \frac{b}{2} \right), \] and

\[ E_0^+ - E_0^- = \frac{4}{5} \hbar M \Omega R^2. \] (10)

By virtue of (10), we can calculate the energy shift for a given neutrino, leading to a new energy of \[ E_\pm \approx E_0^\pm + (\Delta E)_m^\pm, \] where

\[ (\Delta E)_m^\pm = \langle \pm | H_{\Phi_G} | \pm \rangle \pm \frac{|\langle - | H_{\Phi_G} | + \rangle|^2}{E_0^+ - E_0^-}. \] (11)

This leads \[19,20\] to the final expression for the neutrino oscillation length

\[ L_{\text{osc.}} = \frac{2\pi}{\left( E_{m_2}^\pm - E_{m_1}^\pm \right)}, \] where

\[ E_{m_2}^\pm - E_{m_1}^\pm = \hbar \langle k \rangle \left\{ \frac{1}{2} \left( \bar{m}_2^2 - \bar{m}_1^2 \right) \right. \]

\[ + \left[ \frac{M}{r} C_1(q, r) \pm \frac{M \Omega R^2}{r^2} \sin^2 \theta F_1(q, r) \right] \left( \bar{m}_2 - \bar{m}_1 \right) \]

\[ + \left[ \frac{M}{r} C_2(q, r) \pm \frac{M \Omega R^2}{r^2} \sin^2 \theta F_2(q, r) \right] \left( \bar{m}_2^2 - \bar{m}_1^2 \right) \} \] (12)

and

\[ F_0(q, r) = \frac{5}{4} \langle k \rangle D_0^2(q, r) \] (13)

\[ F_1(q, r) = \frac{5}{2} \langle k \rangle D_0(q, r) D_1(q, r) \] (14)

\[ F_2(q, r) = \frac{5}{4} \langle k \rangle \left[ D_1^2(q, r) + 2D_0(q, r) D_2(q, r) \right]. \] (15)

The plots of (14) and (15) are listed in Figure 2.

It is clear from Figures 1 and 2 that the functions become extremely large for \( q \to 0 \), corresponding to very large momentum spread in the wave packet for a given neutrino energy, which then rapidly decay to zero for large \( q \). In particular, the plots show that the helicity transition terms in (12) will dominate as \( q \to 0 \), which is consistent with (7), since a large momentum spread in the wave packet is required to sense the differential rotational effects within a localized region of space-time near the star’s surface.

For solar neutrinos and those due to supernovae \[20\], the expected wave packet widths in momentum space are \( h \sigma_p \approx 10^{-5} \text{ MeV} \) and \( 2 \times 10^{-2} \text{ MeV} \), respectively. The corresponding values for \( q \approx 10^6 \) and \( q \approx 50 \) indicate that gravitational corrections have negligible effect on the neutrino oscillation lengths.
for these scenarios. However, the situation is quite different when applied to rapidly rotating neutron stars. For the neutron star parameters we consider, \( M/r \approx 7.175 \times 10^{-18} \) and \( M\Omega R^2/r^2 \approx 7.751 \times 10^{-36} \). In order to predict a 1\% correction to the value of \( \Delta m_{21}^2 = m_2^2 - m_1^2 \) as determined by solar neutrino experiments, we require \( |F_2(q, r)| \approx 1.3667 \times 10^{33} \), which suggests a choice of \( q \approx 2.1 \times 10^{-5} \) from Figure 2(b), and implies a wave packet width of \( \hbar \sigma_p \approx 4.762 \times 10^5 \) MeV. This prediction corresponds very well to the calculated width of \( \hbar \sigma_p \approx 3.260 \times 10^5 \) MeV for neutrinos emitted from neutron stars, as determined by a mean-free-path calculation [20], assuming a stellar temperature of \( 3 \times 10^6 \) K [21], and an effective stellar density of \( 10^{11} \) g/cm\(^3\) averaged over the neutron star’s expected core and surface densities. Our results therefore show that helicity flip likely plays a role in the case of neutrinos propagating in the field of rotating neutron stars.

From Figure 2(a), it also follows that \( |F_1(q, r)| \approx 1.2150 \times 10^{38} \) for the same choice of \( q \) as when applied to Figure 2(b), and so the contribution in (12) due to linear mass difference \( \Delta m_{21}^2 = m_2^2 - m_1^2 \) is not negligible. This result suggests the possibility of performing, in principle, a parameter fit of (12) for suitable choices of \( q, \Delta m_{21}^2 \), and \( \Delta m_{21}^2 \) so that we can infer the absolute neutrino masses entirely from observations, and without any reference to the mass generation mechanisms presently under consideration in the literature [19]. In practice, however, such an undertaking would require an enormously large neutrino flux and large counting rates to obtain statistically significant measurements. To our knowledge, there are no experimental facilities available that can meet these requirements. Nonetheless, the theoretical possibility of determining absolute neutrino masses by this technique makes for an interesting consideration in the development of future neutrino observatories.

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\section{References}

[1] Y. Fukuda et al. (Superkamiokande Collaboration), \textit{Phys. Rev. Lett.} \textbf{86}, 5656-5660 (2001); \textbf{86}, 5651-5655 (2001); \textbf{82}, 2644 (1999); \textit{Phys. Lett.} \textbf{B433}, 9 (1998); \textbf{436}, 33 (1998); \textbf{467}, 185 (1999).

[2] Q. R. Ahmad et al. (SNO Collaboration) \textit{Phys. Rev. Lett.} \textbf{89}, 011302 (2002); \textbf{89}, 011301 (2002); \textbf{87}, 071301 (2001).
[3] V. Gribov and B. Pontecorvo, *Phys. Lett.* **B28**, 493 (1969).

[4] J. Lense and H. Thirring, *Z. Phys.* **19**, 156 (1918); (English translation: B. Mashhoon, F.W. Hehl and D.S. Theiss, *Gen. Rel. Grav.* **16**, 711(1984)).

[5] C. Giunti, *Found. Phys. Lett.* **17**, 103 (2004); [arXiv:hep-ph/0302026](http://arxiv.org/abs/hep-ph/0302026).

[6] Y.Q. Cai and G. Papini, *Phys. Rev. Lett.* **66**, 1259 (1991); **68**, 3811 (1992).

[7] D. Singh and G. Papini, *Nuovo Cimento* **B115**, 223 (2000).

[8] G. Papini, in *Advances in the Interplay between Quantum and Gravity Physics*, edited by Peter G. Bergmann and V. de Sabbata, 317 (Kluwer Academic, Dordrecht 2002); [arXiv:gr-qc/0110056](http://arxiv.org/abs/gr-qc/0110056).

[9] G. Papini in *Relativity in Rotating Frames*, edited by G. Rizzi and M.L. Ruggiero (Kluwer Academic, Dordrecht 2004) Ch.16 and references therein; [arXiv:gr-qc/0304082](http://arxiv.org/abs/gr-qc/0304082).

[10] Y. Q. Cai and G. Papini, *Mod. Phys. Lett.* **A4**, 1143 (1989); *Class. Quantum Grav.* **7**, 269 (1990); *Gen. Rel. Grav.* **22**, 259 (1990).

[11] Y. Q. Cai and G. Papini, *Class. Quantum Grav.* **6**, 407 (1989).

[12] G. Lambiase, G. Papini, R. Punzi and G. Scarpetta, *Phys. Rev D* **71**, 073011 (2005).

[13] S. A. Werner, J.-L. Staudenmann and R. Colella, *Phys. Rev. Lett.* **42**, 1103 (1979). L. A. Page, *Phys. Rev. Lett.* **35**, 543 (1975).

[14] U. Bonse and T. Wroblewski, *Phys. Rev. Lett.* **51**, 1401 (1983).

[15] F.W. Hehl and W.-T. Ni, *Phys. Rev. D* **42**, 2045 (1990).

[16] B. Mashhoon, *Phys. Rev. Lett.* **61**, 2639 (1988).

[17] D. Singh, N. Mobed, and G. Papini, *J. Phys. A: Math. Gen.* **37**, 8329 (2004).

[18] C. S. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, (W. H. Freeman & Co. 1973).

[19] M. Fukugita and T. Yanagida, *Physics of Neutrinos and Applications to Astrophysics*, (Springer-Verlag Press 2003).

[20] C. W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood Academic Press 1993).

[21] S. L. Shapiro and S. A. Teukolsky, *Black holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects*, (John Wiley & Sons 1983).