ALMOST SURE CONVERGENCE OF PRODUCTS OF 2 × 2 NONNEGATIVE MATRICES

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ABSTRACT. We study the almost sure convergence of the normalized columns in an infinite product of nonnegative matrices, and the almost sure rank one property of its limit points. Given a probability on the set of 2 × 2 nonnegative matrices, with finite support \( \mathcal{A} = \{ A(0), \ldots, A(s - 1) \} \), and assuming that at least one of the \( A(k) \) is not diagonal, the normalized columns of the product matrix \( P_n = A(\omega_1) \cdots A(\omega_n) \) converge almost surely (for the product probability) with an exponential rate of convergence if and only if the Lyapunov exponents are almost surely distinct. If this condition is satisfied, given a nonnegative column vector \( V \) the column vector \( \frac{P_n V}{\|P_n V\|} \) also converges almost surely with an exponential rate of convergence. On the other hand if we assume only that at least one of the \( A(k) \) do not have the form \( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \), \( ad \neq 0 \) or the form \( \begin{pmatrix} 0 & b \\ d & 0 \end{pmatrix} \), \( bc \neq 0 \), the limit-points of the normalized product matrix \( \frac{P_n V}{\|P_n V\|} \) have almost surely rank 1 – although the limits of the normalized columns can be distinct – and \( \frac{P_n V}{\|P_n V\|} \) converges almost surely with a rate of convergence that can be exponential or not exponential.

INTRODUCTION

Given a finite set of nonnegative matrices 
\[
A(0) = \begin{pmatrix} a(0) & b(0) \\ c(0) & d(0) \end{pmatrix}, \ldots, A(s - 1) = \begin{pmatrix} a(s - 1) & b(s - 1) \\ c(s - 1) & d(s - 1) \end{pmatrix}
\]
we consider the product matrix 
\[
P_n(\omega) = A(\omega_1) \cdots A(\omega_n) = \begin{pmatrix} \alpha_n(\omega) & \beta_n(\omega) \\ \gamma_n(\omega) & \delta_n(\omega) \end{pmatrix}, \quad \omega = (\omega_n) \in \{0, 1, \ldots, s\}^N
\]
and we are interested by the almost sure limit points of \( \frac{P_n V}{\|P_n V\|} \) and the almost sure convergence of the normalized columns of \( P_n \). The set \( \{0, \ldots, s - 1\}^N \) is endowed by the product probability, defined from \( p_0, \ldots, p_{s-1} > 0 \) with \( \sum_k p_k = 1 \). This product is easily computable when all the matrices are upper triangular, or when all the matrices are lower triangular, and also when they are stochastic [5, Proposition 1.2].

The book of Bougerol and Lacroix, since its purpose is different, do not give any indication in the particular case that we study in this paper: indeed the hypothesis of contraction they make, for the almost sure convergence of \( \frac{P_n V}{\|P_n V\|} \), is the existence – for any \( k \in \mathbb{N} \) – of a matrix \( M_k \), product of...
matrices belonging to the set \( \{ A(0), \ldots, A(s - 1) \} \), such that \( \frac{M_k}{\| M_k \|} \) converges to a rank 1 matrix when \( k \to \infty \) ([1, Part A III Definition 1.3]).

On the other hand, the weak ergodicity defined in [4, Definition 3.3] holds almost surely if and only if the product matrix \( P_n \) is almost surely positive for \( n \) large enough. However the strong ergodicity ([4, Definition 3.4]) do not necessary hold, even if all the \( A(k) \) are positive.

The outset of the present study is the following theorem. The norm we use is the norm-1, and we say that the normalized columns of \( P_n(\omega) \) converge if each column of \( P_n(\omega) \) divided by its norm-1 (if nonnull) converges in the usual sense. We say that the rate of convergence is exponential or geometric when the difference – between the entries of the normalized column and their limits – is less than \( Cr^n \) with \( C > 0 \) and \( 0 < r < 1 \).

**Theorem 1.** Let \( A = \{ A(0), \ldots, A(s - 1) \} \) be a finite set of nonnegative matrices such that at least one of the \( A(k) \) is not diagonal.

(i) The normalized columns of \( P_n \) converge almost surely with an exponential rate of convergence if and only if the singular values \( \lambda_i(n) \) of \( P_n \) satisfy almost surely

\[
\lim_{n \to \infty} (\lambda_1(n))^\frac{1}{n} \neq \lim_{n \to \infty} (\lambda_2(n))^\frac{1}{n}.
\]

(ii) If (i) holds the limit-points of the normalized matrix \( \frac{P_n}{\| P_n \|} \) have almost surely rank 1 and, given a nonnegative column vector \( V \), the normalized column vector \( \frac{P_n V}{\| P_n V \|} \) converges almost surely with an exponential rate of convergence.

(iii) Nevertheless the normalized matrix \( \frac{P_n}{\| P_n \|} \) diverges almost surely, except in the case where the matrices \( A(0), \ldots, A(s - 1) \) have a common left eigenvector.

The different cases are detailed below:

- the case where at least one of the \( A(k) \) has rank one, in Remark 2 (i),
- the cases where all the \( A(k) \) have rank two and at least one of the \( A(k) \) has more than two nonnull entries in Sections 3 and 4,
- the cases where all the \( A(k) \) have rank two and two nonnull entries in Section 5.

**Remark 2.** (i) If one of the matrices \( A(k) \) has rank 1, the normalized columns of \( P_n \) are almost surely constant and equal for \( n \) large enough, as well as \( \frac{P_n V}{\| P_n V \|} \) for any nonnegative column vector \( V \) such that \( \forall n, P_n V \neq 0 \). So we can suppose in the sequel that all the \( A(k) \) have rank 2.

(ii) If both normalized columns of \( P_n \) converge (resp. converge exponentially) to the same limit, then for any nonnegative column vector \( V \) the normalized column \( \frac{P_n V}{\| P_n V \|} \) is a nonnegative linear combination of them, so it converges (resp. it converges exponentially) to the same limit. The limit points of \( \frac{P_n}{\| P_n \|} \) have rank 1.

(iii) If \( \frac{P_n}{\| P_n \|} \) converges (resp. converges exponentially), then for any nonnegative column vector \( V \) the normalized column \( \frac{P_n V}{\| P_n V \|} \) converges (resp. converges exponentially) because it is \( \frac{P_n V}{\| P_n V \|} \).

1. **Some triangular examples**

   - A case where the normalized columns converge almost surely to the same limit with a convergence rate in \( \frac{1}{n} \). Suppose that \( A(k) = \begin{pmatrix} 1 & b(k) \\ 0 & 1 \end{pmatrix} \) and that \( \exists k_0, b(k_0) \neq 0 \).
We have
\[ P_n = \begin{pmatrix} 1 & \sum_{i=1}^{n} b(\omega_i) \\ 0 & 1 \end{pmatrix}. \]

We obtain the normalized second column (resp. the normalized matrix) by dividing the second column of \( P_n \) (resp. \( P_n \) itself) by \( 1 + \sum_{i=1}^{n} b(\omega_i) \). So both normalized columns converge almost surely to \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), and the normalized matrix converges almost surely to \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Since the density of the set \( \{ i : \omega_i = k_0 \} \) is almost surely \( 1 \), \( \sum_{i=1}^{n} b(\omega_i) \) has the order of growth of \( n \). So the convergence rate of the second normalized column, and the one of the normalized matrix \( \| P_n \| \), have the order of \( 1/n \).

A case where the normalized columns converge almost surely to the same limit with an exponential convergence rate. Suppose that \( A(k) = \begin{pmatrix} 2 & b(k) \\ 0 & 1 \end{pmatrix} \) and that \( \exists k_0, b(k_0) \neq 0 \). We have
\[ P_n = \begin{pmatrix} 2^n & \sum_{i=1}^{n} 2^{i-1}b(\omega_i) \\ 0 & 2^n \end{pmatrix} \]
so the normalized columns of \( P_n \) converge almost surely to \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and the limit points of the normalized matrix \( \frac{P_n}{\| P_n \|} \) have the form \( \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \), \( \alpha \in [0, 1] \).

A case where the normalized columns converge almost surely to two different limit with an exponential convergence rate. Suppose that \( A(k) = \begin{pmatrix} 1 & b(k) \\ 0 & 2 \end{pmatrix} \), we have
\[ P_n = \begin{pmatrix} 1 & \sum_{i=1}^{n} 2^{n-i}b(\omega_i) \\ 0 & 2^n \end{pmatrix} \]
so the limit of the second normalized column is \( \begin{pmatrix} 1+s \\ 1+s \end{pmatrix} \) with \( s = \sum_{i=1}^{\infty} 2^{-i}b(\omega_i) \).

Another case where the normalized columns converge almost surely to the same limit, but the convergence rate is non-exponential. Suppose that the alphabet has two elements, \( A(0) = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \) and \( A(1) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \), and that \( p_0 = p_1 = \frac{1}{2} \). We have
\[ P_n = \begin{pmatrix} 2^{k_0(n)} & 2^{k_1(n)} \sum_{i=1}^{n} k_j(i-1) \\ 0 & 2^{k_1(n)} \end{pmatrix} \]
where \( k_j(i) = \# \{ i' \leq i : \omega_{i'} = j \} \).

By the well known recurrence property one has almost surely \( k_0(i) = k_1(i) \) for infinitely many \( i \), so both normalized columns converge to \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). The difference between this vector and the second column of \( P_n \) has entries \( \pm \frac{1}{1+\sum_{i=1}^{\infty} \frac{1}{2^{k_0(i)-k_1(i)}}} \).

With probability 1, this difference do not converge exponentially to 0 because \( \lim_{i \to \infty} \frac{k_0(i)-k_1(i)}{i} = 0 \).
2. The singular values of $P_n$

The singular values of $P_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$, let $\lambda_1(n)$ and $\lambda_2(n)$, are by definition the positive roots of the eigenvalues of $tP_nP_n$:

$$\lambda_i(n) := \sqrt{\frac{\alpha_n^2 + \beta_n^2 + \gamma_n^2 + \delta_n^2 \pm \sqrt{\alpha_n^2 + \beta_n^2 + \gamma_n^2 + \delta_n^2}^2 - 4(\alpha_n \delta_n - \beta_n \gamma_n)^2}{2}}.$$

Now the Lyapunov exponents $\lambda_i := \lim_{n \to \infty} \frac{1}{n} \log \lambda_i(n)$ exist almost surely by the subadditive ergodic theorem [3], and one has

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{2n} \log(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$$
$$\lambda_2 = \lambda_1 + \lim_{n \to \infty} \frac{1}{n} \log \frac{|\alpha_n \delta_n - \beta_n \gamma_n|}{\alpha_n^2 + \beta_n^2 + \gamma_n^2 + \delta_n^2}. \quad (2)$$

Notice that $\lambda_1$ is finite if $P_n$ is not eventually the null matrix: denoting by $\alpha$ and $\beta$ the smaller nonnull value and the greater value of the entries of the matrices $A(k)$ one has

$$\log \alpha \leq \lambda_1 \leq \log(2\beta).$$

As for $\lambda_2$, it belongs to $[-\infty, \lambda_1]$. For instance if $P_{n_0}$ has rank 1, $\lambda_2 = \lambda_2(n) = -\infty$ for $n \geq n_0$.

For any nonnegative matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we denote by $d_H(M)$ the Hilbert distance between the columns of $M$ and by $d_\infty(M)$ the norm-infinite distance between the normalized columns of $M$:

$$d_H(M) := \left| \log \frac{ad}{bc} \right| \quad \text{and} \quad d_\infty(M) = \left| \frac{a}{a+c} - \frac{b}{b+d} \right|.$$

Proposition 3. $d_H(M)$ and $d_\infty(M)$ are at least equal to $\frac{|ad-bc|}{a^2+b^2+c^2+d^2}$.

Proof. By the classical inequality $|\log t| \geq \frac{2|t-1|}{t+1}$ one has $d_H(M) \geq \frac{2|ad-bc|}{ad+bc}$. Now $ad \leq a^2 + d^2$ and $bc \leq b^2 + c^2$, so $d_H(M) \geq \frac{|ad-bc|}{a^2+b^2+c^2+d^2}$. On the other side $d_\infty(M) = \frac{|ad-bc|}{ab+ad+bc+cd} \geq \frac{|ad-bc|}{a^2+b^2+c^2+d^2}$ because $2(a^2+b^2+c^2+d^2) = (a^2+b^2) + (a^2+d^2) + (b^2+c^2) + (c^2+d^2) \geq 2ab + 2ad + 2bc + 2cd$. \hfill \Box

Corollary 4. If $d_H(P_n)$ or $d_\infty(P_n)$ converges exponentially to 0, the normalized columns of $P_n$ converge exponentially to the same limit and $\lambda_2 < \lambda_1$.

Proof. Let $p, q \geq n$. The Hilbert distance between the $i^{th}$ column of $P_p$ and the $j^{th}$ column of $P_q$ is $d_H(P_pP_n, P_nP_q)$, where $P_{n,p,q}$ is the matrix whose columns are the $i^{th}$ column of $A(\omega_{n+1}) \ldots A(\omega_p)$ and the $j^{th}$ column of $A(\omega_{n+1}) \ldots A(\omega_q)$. By the well known property of the Birkhoff coefficient $\gamma_B$ ([4, Section 3]) this distance is at most $d_H(P_n)\gamma_B(P_{n,p,q}) \leq d_H(P_n)$, so it converge exponentially to 0. By the obvious inequality $d_\infty(M) \leq d_H(M)$ the norm-infinite distance between the $i^{th}$ normalized column of $P_p$ and the $j^{th}$ normalized column of $P_q$ ($i = j$ or $i \neq j$) converges exponentially to 0, so both normalized columns are Cauchy and converge exponentially to the same limit when $n$ tends to infinity, and $\lambda_2 < \lambda_1$ by (2) and Proposition 3. \hfill \Box
3. The case where $P_n$ is almost surely positive for $n$ large enough

**Theorem 5.** Suppose that $A^* = \sum_k A(k)$ is not triangular and that at least one of the $A(k)$ has more than two nonnull entries. Then, with probability 1, the normalized columns of $P_n$ converge exponentially to the same limit, as well as $\frac{P_n}{\|P_n\|}$ for any nonnegative column vector $V$, and $\lambda_2 < \lambda_1$; the limit points of $\frac{P_n}{\|P_n\|}$ have rank 1; the weak ergodicity, in the sense of [4, Definition 3.3], holds.

**Proof.** By the hypotheses either one of the $A(k)$ is positive, or one of the $A(k)$ is 
\[
\begin{pmatrix}
a(k) & b(k) \\
0 & c(k)
\end{pmatrix}
\] (its square is positive), or one of the $A(k)$ is triangular with three nonnull entries and another $A(k')$ is such that $A(k) + A(k') > 0$. In this last case $A(k)A(k')A(k)$ is positive. So in all cases there exist $k, k'$ such that $A(k)A(k')A(k)$ is positive and, denoting by $k(n)$ the number of occurrences of the word $kk'k$ in $\omega_1 \ldots \omega_n$, the limit of $k(n)$ is almost surely $s^{-3}$.

Clearly the Hilbert distance $d_H$ and the Birkhoff coefficient $\tau_B$ ([4, Section 3]) have the following property:
\[
d_H(M_1 \ldots M_t) \leq d_H(M_1)\tau_B(M_2) \ldots \tau_B(M_t)
\]
where $d_H$ means the distance between the rows (or the columns) of $M$.

Now we split the product matrix $P_n$ in the following way:

\[
P_n = M_1 \ldots M_k \text{ where } M_i = A(\omega_{n-i+1}) \ldots A(\omega_n), \ n_0 = 0 < n_1 < \ldots < n_k,
\]
the indexes $n_1, n_3, \ldots$ corresponding to the disjoint occurrences of $A(k)A(k')A(k)$:

\[
M_1 = A(\omega_1) \ldots A(\omega_{n_1-1})A(k)A(k')A(k) \text{ and } M_3 = M_5 = \ldots = A(k)A(k')A(k).
\]

Let $C = d_H(M_1) < \infty$, $r = \tau_B(A(k)A(k')A(k)) < 1$ and $r' \in [r^{s^{-3/3}}, 1]$. We have for $n$ large enough

\[
d_H(P_n) \leq C\tau(A(\omega_{n_1+1}) \ldots A(\omega_n)) \leq Cr^{k(n)/3-1} \leq Cr^n
\]
and we conclude with Corollary 4 and [4, Lemma 3.3]. \qed

4. The case where $A^* = \sum_k A(k)$ is triangular not diagonal

Assuming for instance that $A^*$ is upper triangular not diagonal, we have

\[
P_n(\omega) = \begin{pmatrix}
\alpha_n(\omega) & 0 \\
\delta_n(\omega)s_n(\omega) & \delta_n(\omega)
\end{pmatrix}
\]
with $s_n(\omega) = \sum_{i=1}^{\infty} \alpha_{i-1}(\omega) b(\omega_i)/\delta_{i-1}(\omega)$.

To know if $\lim_{n \to \infty} s_n(\omega)$ is finite or infinite, and to know the rate of convergence, we use the exponentials of the expected values of $\log a(\cdot)$ and $\log d(\cdot)$:

\[
p := a(0)^p_0 \ldots a(s-1)^p_{s-1} \text{ and } q := d(0)^q_0 \ldots d(s-1)^q_{s-1}.
\]

By the law of large numbers, for any $\varepsilon > 0$ we have almost surely for any integer $n \geq 0$

\[
k \frac{p^n}{q^n} (1 - \varepsilon)^n \leq \frac{\alpha_n(\omega)}{\delta_n(\omega)} \leq K \frac{p^n}{q^n} (1 + \varepsilon)^n \quad (\kappa, K \text{ constants}).
\]

On the other side, since $A^*$ is not diagonal, the set of the integers $n$ such that $b(\omega_n) \neq 0$ has almost surely the positive density $\frac{1}{s} \# \{k \colon b(k) \neq 0\}$. Given $\varepsilon > 0$, for $n$ large enough this set has a nonempty intersection with $[n(1 - \varepsilon), n]$. So we
We assume that the $A$ and $P$ limit points of the normalized matrix $P$ differ by a constant, as well as the normalized matrix $\lambda$. Theorem 6. Suppose that $A^*$ is upper triangular not diagonal, and that all the $A(k)$ have rank 2. Then $P_n$ has almost surely the following properties:

(i) If $p > q$, $P_n V$ converges to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for any nonnegative column vector $V$, and the limit points of $P_n V$ have rank 1. The rate of convergence of $P_n V$ is exponential if and only if $p > q$, and one has $\lambda_2 < \lambda_1$ if and only if $p > q$.

(ii) If $p < q$, setting $s(\omega) = \lim_{n \to \infty} s_n(\omega)$, $P_n V$ converges exponentially to $\begin{pmatrix} 0 \\ \frac{s(\omega)}{s(\omega)+1} \end{pmatrix}$ if and only if the second entry of $V$ is nonnull. The normalized product $P_n \frac{P_n V}{\|P_n V\|}$ converges exponentially to $\begin{pmatrix} 0 \\ \frac{s(\omega)}{s(\omega)+1} \end{pmatrix}$ and $\lambda_2 < \lambda_1$.

Proof. (i) If $p > q$ one has almost surely $\lim_{n \to \infty} s_n(\omega) = \infty$ because, among the indexes $i$ such that $b_i(\omega) \neq 0$, one has $\frac{\alpha_{i-1}(\omega)}{s_{i-1}(\omega)} \geq 1$ infinitely many times. The difference between $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second normalized column of $P_n$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so it tends almost surely to 0 by (5), with an exponential rate of convergence if and only if $p > q$. We conclude about $P_n \frac{P_n V}{\|P_n V\|}$ and $P_n \frac{P_n V}{\|P_n V\|}$ by using Remark 2 (ii).

If $p > q$ one has almost surely, from (2) and (4), $\lambda_2 \leq \lambda_1 + \lim_{n \to \infty} \frac{1}{n} \log \frac{\delta_{i-1}(\omega)}{s_{i-1}(\omega)} < \lambda_1.$

If $p = q$, (2), (4) and (5) imply almost surely $\lambda_1 = \lambda_2$.

(ii) If $p < q$, the almost sure exponential convergence of $P_n V$ and $P_n \frac{P_n V}{\|P_n V\|}$ is due to the fact that $s(\omega) - s_n(\omega)$ tends exponentially to 0 from (4). The almost sure inequality $\lambda_2 < \lambda_1$ is due to $\lambda_2 \leq \lambda_1 + \lim_{n \to \infty} \frac{1}{n} \log \frac{\alpha_n(\omega)}{s(\omega)}$.

5. The case where all the $A(k)$ have only two nonnull entries

We assume that the $A(k)$ have rank 2, so the normalized columns of $P_n$ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Suppose first that all the $A(k)$ are diagonal with two nonnull entries. If the constants $p$ and $q$ defined in (3) are distinct, then with probability 1 one has $\lambda_2 < \lambda_1$, the normalized column vector $\frac{P_n V}{\|P_n V\|}$ converges exponentially for any nonnegative vector $V$, as well as the normalized matrix $\frac{P_n}{\|P_n\|}$. If $p = q$, then with probability 1 we have $\lambda_1 = \lambda_2$, the normalized column matrix $\frac{P_n V}{\|P_n V\|}$ do not necessarily converge, and the limit points of the normalized matrix $\frac{P_n}{\|P_n\|}$ have the form $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha \in [0, 1]$.

Suppose now that at least one of the $A(k)$ has the form $\begin{pmatrix} 0 & b(k) \\ c(k) & 0 \end{pmatrix}$, $b(k)c(k) \neq 0$. Let $i_1, i_2, \ldots$ be the indexes such that $A(\omega_i)$ has this form. For any $n \in \mathbb{N}$ the
product matrix $P_n$ or $P_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has the diagonal form $\begin{pmatrix} \alpha_n & 0 \\ 0 & \delta_n \end{pmatrix}$ with

$\alpha_n = a(\omega_1) \ldots a(\omega_{i-1}) b(\omega_{i+1}) d(\omega_{i+1}) \ldots a(\omega_{i-1}) b(\omega_{i+1}) \ldots$

$\delta_n = d(\omega_1) \ldots d(\omega_{i-1}) c(\omega_{i+1}) a(\omega_{i+1}) \ldots a(\omega_{i-1}) b(\omega_{i+1}) \ldots$

Clearly, the first as well as the second normalized column diverge. With probability 1 the limit points of the normalized matrix $P_n ||P_n||$ have the form $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha \in [0,1]$, and $\lambda_1 = \lambda_2$.

6. THE ALMOST SURE DIVERGENCE OF $\frac{P_n}{||P_n||}$

It remains to prove the last item of Theorem 1, and more generally the following

**Proposition 7.** (i) Let $(A_n)$ be a sequence of complex-valued $d \times d$ matrices. If the sequence of normalized matrices $\frac{A_n}{||A_n||}$ converges, the matrices $A$ such that $A_n = A$ for infinitely many $n$ have a common left-eigenvector, which is a row of the limit matrix.

(ii) Given a set $\mathcal{A} = \{A(0), \ldots, A(s-1)\}$ of complex-valued $d \times d$ matrices without common left-eigenvector, a positive probability vector $(p_0, \ldots, p_{s-1})$ and the product probability on $\{0, \ldots, s-1\}^N$, the sequence of the normalized matrices $\frac{A(\omega_1) \ldots A(\omega_n)}{||A(\omega_1) \ldots A(\omega_n)||}$ diverges for almost all sequence $(\omega_n)_{n \in \mathbb{N}}$ in $\{0, \ldots, s-1\}^N$.

**Proof.** (i) Let $P_n = A_1 \ldots A_n$, we suppose that the limit $P = \lim_{n \to \infty} \frac{P_n}{||P_n||}$ exists and we denote $\lambda_n = \frac{||P_n||}{||P_{n-1}||}$. If $A_n = A$ for $n = n_1, n_2, \ldots$ with $n_1 < n_2 < \ldots$ one has

$$PA = \lim_{k \to \infty} \frac{P_{n_1-1}}{||P_{n_1-1}||} A$$

$$= \lim_{k \to \infty} \lambda_{n_k} \frac{P_{n_k-1}}{||P_{n_k-1}||} A$$

(6)

One deduce $||PA|| = \lim_{k \to \infty} \left| \lambda_{n_k} \frac{P_{n_k-1}}{||P_{n_k-1}||} \right| = \lim_{k \to \infty} \lambda_{n_k}$, and the equality (6) becomes $PA = ||PA|| P$. Since the matrix $P$, of norm 1, has at least one row with a nonnull entry, this row is a left-eigenvector of $A$ related to the eigenvalue $||PA||$.

(ii) Given $A \in \mathcal{A}$ and $n_0 \in \mathbb{N}$, the set of the sequences $(\omega_n)$ such that $A(\omega_n) \in \mathcal{A} \setminus \{A\}$ for any $n \geq n_0$ has probability 0. Hence, with probability 1, the sequence $(A(\omega_n))$ has infinitely many occurrences of each of the matrices of $\mathcal{A}$ and, by (i), $\frac{A(\omega_1) \ldots A(\omega_n)}{||A(\omega_1) \ldots A(\omega_n)||}$ diverges almost surely. \hfill \Box

7. THE RANK ONE PROPERTY OF THE INFINITE PRODUCTS OF MATRICES

Here is a general result, deduced from [2].

**Theorem 8.** Let $(A_n)$ be a sequence of complex-valued $d \times d$ matrices, we denote by $P_n$ their normalized product with respect to the euclidean norm:

$$P_n := \frac{A_1 \ldots A_n}{||A_1 \ldots A_n||}$$

There exists a sequence $(Q_n)$ of matrices of rank 1 such that

$$\lim_{n \to \infty} ||P_n - Q_n|| = 0$$
Proof. Denoting by $C_i$ the columns of $U$, by $R_i$ the rows of $V$, and denoting by $s_j(n)$ the sum $i_1(n) + \cdots + i_j(n)$ we have

$$P_n = \sum_{0<i\leq s_1(n)} C_i(n)R_i(n) + \frac{\lambda_2(n)}{\lambda_1(n)} \sum_{s_1(n)<i\leq s_2(n)} C_i(n)R_i(n) + \ldots \quad (7)$$

so the converse implication of the theorem holds with $Q_n := C_1(n)R_1(n)$. To prove the direct implication we need the following lemma:

**Lemma 9.** Any matrix $A$ of the form $A = \sum_{i=1}^r C_i R_i$, where the nonnull columns $C_i$ are orthogonal as well as the nonnull rows $R_i$, has rank $r$.

**Proof.** We complete $\{R_1, \ldots, R_r\}$ to a orthogonal base. The rank of $A$ is the rank of the family $\{AR_i^*\}$, but $AR_i^* = \begin{cases} C_i & (i \leq r) \\ 0 & (i > r) \end{cases}$ and consequently this family has rank $r$. \hfill \square

Now using the compacity of the set of vectors of norm 1, there exists at least one increasing sequence of integers $(n_k)$ such that the columns $C_{i_k}(n_k)$, the rows $R_{i_k}(n_k)$, the reals $\frac{\lambda_2(n_k)}{\lambda_1(n_k)}$ and the integers $s_j(n_k)$ converge. Let $C_i, R_i, \alpha_i$ and $s_j$ be their respective limits, one deduce from (7) that $(P_{n_k})$ converge and

$$\lim_{k \to \infty} P_{n_k} = \sum_{0<i\leq s_1(n)} C_i R_i + \alpha_2 \sum_{s_1<n<s_2} C_i R_i + \ldots$$

If $i_1(n) \neq 1$ for infinitely many $n$, or if $\frac{\lambda_2(n_k)}{\lambda_1(n_k)}$ do not converge to 0, we can choose the sequence $(n_k)$ such that $i_1 \geq 2$ or $\alpha_2 \neq 0$. Consequently – from Lemma 9 – $\lim_{k \to \infty} P_{n_k}$ has rank at least 2, so it is not possible that $\lim_{n \to \infty} \|P_n - Q_n\|_2 = 0$ with $Q_n$ of rank 1. \hfill \square

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