1. Motivation

At asymptotically high baryon chemical potential ($\mu_B$) and low temperature ($T$) – where QCD can be treated in a perturbative manner – the ground-state of quark matter is found to be that of a colour-superconductor (for a recent review see e.g. [1]). Unfortunately, the sign problem prevents us from using lattice QCD to determine the ground-state at the more moderate densities typical in the cores of compact (neutron) stars.

One way to proceed is to study model field theories such as the Nambu – Jona-Lasinio (NJL) model. This purely fermionic field theory, in which colour-neutral quarks interact via a four-point contact term, not only contains the same global symmetries as two flavour QCD, but can be simulated on the lattice even with $\mu_B \neq 0$. In [2] we show that the ground-state of the lattice model with $\mu_u = \mu_d = \mu_B$, i.e. with “up” and “down” quarks sharing a common Fermi surface, exhibits s-wave superfluidity via a standard BCS pairing between quarks of different flavours; i.e.

$$\langle ud \rangle \neq 0; \quad \Delta_{BCS} \neq 0.$$  \hfill (1)

Within the cores of compact stars, however, the Fermi momenta $k_F^u$ and $k_F^d$ are expected to be separate. A simple argument based on that of [3] suggests that for a two flavour Fermi liquid of massless quarks and electrons with $\mu_B = 400\text{MeV}$ and both weak equilibrium ($\mu_d = \mu_u + \mu_e$) and charge neutrality ($2n_u/3 - n_d/3 - n_e = 0$) enforced, all the Fermi momenta of the system are determined:

$$k_F^u = \mu_u = \mu_B - \mu_e/2 = 355.5\text{MeV},$$  
$$k_F^d = \mu_d = \mu_B + \mu_e/2 = 444.5\text{MeV},$$  
$$k_F^e = \mu_e = 89\text{MeV}.$$  \hfill (2)

The effect of separating the Fermi surfaces of pairing quarks in QCD should be to make the colour-superconducting phase less energetically favourable. Introducing $\mu_I \propto (\mu_u - \mu_d) \neq 0$ could prove a good method, therefore, to investigate the stability of the superfluid phase.

2. The Model

The action of the lattice NJL model (with $a \to 1$) is given by

$$S_{NJL} = \sum_{xy} \bar{\Psi}_x x M[\Phi, \mu_B, \mu_I]_{xy} \Psi_y + \frac{g}{2} \sum_x \text{Tr} \Phi_1^1 \Phi_1^1,$$  \hfill (3)

where $\Psi \equiv (u, d)^T$ is the $SU(2)$ doublet of staggered up and down quarks defined on lattice sites $x$ and $\Phi \equiv \sigma + i \cdot \vec{\tau}$ is a matrix of bosonic auxiliary fields defined on dual sites $\bar{x}$. The fermion
kinetic matrix \( M_{xy} \) is defined in \(^2\) and we choose the same bare parameters used therein.

One can separate the Fermi surface of up and down quarks by simultaneously setting baryon chemical potential \( \mu_B \equiv (\mu_u + \mu_d)/2 \neq 0 \) and isospin chemical potential \( \mu_I \equiv (\mu_u - \mu_d)/2 \neq 0 \). With \( \mu_I = 0 \), \( \tau_2 M \tau_2 = M^* \), which is a sufficient condition to show that \( \det M \) is both real and positive \(^4\). With \( \mu_I \neq 0 \) however, this is no longer true such that once again we are faced with the sign problem.

The fact that physically the two scales are ordered \( \mu_I \lesssim \mu_B \) suggests that one may be able to apply techniques recently developed to study QCD with \( \mu_B \ll T \) \(^5\). First, however, we present the results of a partially quenched calculation.

3. Partially Quenched \( \mu_I \)

Whilst the primary motivation for investigating \( \mu_I \neq 0 \) is to study the superfluid phase which sets in at large \( \mu_B \), this requires one to introduce an explicit symmetry breaking parameter \((j)\); it is currently not clear how to study \( \mu_I \neq 0 \) in the \( j \to 0 \) limit \(^2\). Instead, we choose to study the chiral symmetry restoring phase transition with the aim of controlling the systematics of introducing \( \mu_I \neq 0 \).

The first step we take is to perform a “partially quenched” calculation in which \( \mu_I = 0 \) when generating the background fields and is made non-zero only during the measurement of fermion observables. In particular, we measure the up and down quark condensates

\[
\langle \bar{u}u \rangle, \langle \bar{d}d \rangle \equiv \frac{1}{V} \frac{\partial \ln \mathcal{Z}}{\partial m_{u,d}} = \frac{1}{2} \langle \text{tr}(1 \pm \tau_3)M^{-1} \rangle \tag{4}
\]

as functions of \( \mu_B \) for various \( \mu_I \) on a \( 12^4 \) lattice. Some results are presented in Fig. 1.

The results agree qualitatively with those of mean-field studies of the model in which the introduction of a small but non-zero \( \mu_I \) is seen to suppress the up quark condensate and enhance the down quark condensates for various \( \mu_B \) on a \( 12^4 \) lattice.

\[
n_{B,I} = \frac{1}{2V} \frac{\partial \ln \mathcal{Z}}{\partial \mu_{B,I}} = \frac{1}{4} \langle \bar{u}\gamma_0 u \pm \bar{d}\gamma_0 d \rangle \tag{5}
\]

on a \( 12^4 \) lattice at \( \mu_B = 0.6 \), which from Fig. 1 can be seen to be where the effect of having \( \mu_I \neq 0 \) is largest. In QCD, one can show that e.g. \( \langle \bar{u}\psi \psi \rangle \) expanded about \( \mu_B = 0 \) is analytic in \( \mu^2_B \), such that for small imaginary \( \mu_B \) the quantity remains real \(^6\). For our simulations, however, this is not the case, and measured quantities are, in general, complex. Therefore, we fit the data by the Taylor series

\[
\left( \frac{\langle \bar{u}u \rangle}{\langle \bar{d}d \rangle} \right) = \sum_{n=0}^{\infty} \left( \frac{A_n}{B_n} \right) \left( \frac{\tilde{\mu}_I}{\mu_B} \right)^n \tag{6}
\]
then analytically continue to real 

and

each truncated at some suitable point. We can then analytically continue to real \( \mu_I \) using e.g.

\[
\left( \begin{array}{c} n_B \\ n_I \end{array} \right) = \sum_{n=0}^{\infty} \left( \begin{array}{c} C_n \\ D_n \end{array} \right) \left( \frac{\mu_I}{\mu_B} \right)^n,
\]

Figure 2 shows the real and imaginary parts of the condensates as functions of \( \mu_I \) with \( \mu_B = 0.6 \) on a 12\( ^4 \) lattice.

![Figure 2](image-url)

Figure 3 shows the real and imaginary parts of \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) as functions of imaginary \( \mu_I \) with \( \mu_B = 0.6 \) on a 12\( ^4 \) lattice.

![Figure 3](image-url)

Whilst these results are only preliminary, we have shown that we can calculate the coefficients in (6) and (7) as functions of \( \mu_I \) and in principle, reproduce reliable forms of the curves in Fig. 1. With this aim, we plan to repeat this exercise for various values of \( \mu_B \) in both the the chirally broken and restored phases on various lattice volumes. Also, whilst it is difficult to study the diquark condensate in the \( j \to 0 \) limit, it would be interesting to compare the response of \( \langle ud \rangle \) to \( \mu_I \) at fixed \( j \) to that of \( \langle \bar{\psi}\psi \rangle \) at fixed mass.

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