Explicit expressions of spin wave functions

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We derive the explicit expressions of the canonical and helicity wave functions for massive particles with arbitrary spins. Properties of these wave functions are discussed.

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I. INTRODUCTION

To describe particles with high spins in amplitude analysis, one needs to construct the explicit expressions of wave functions. Detailed properties of the amplitudes are needed in tensor analysis \[1\, 2\] to give independent invariant amplitudes free of kinematics singularities and zeros \[3\, 4\]. We will give the explicit expressions of the canonical and helicity wave functions for massive particles with arbitrary spins in this paper. These wave functions satisfy Rarita-Schwinger conditions \[9\].

We will discuss quantum states in section II. Spin wave functions are given in section III and section IV.

II. QUANTUM STATES

Let \(L(p)\) be a Lorentz transformation that satisfies

\[
\hat{p}^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle. \tag{1}
\]

For massive particles one can choose the standard momentum to be \((k^\mu) = (w; \mathbf{0})\). \(w\) is the mass of the particle. The space-time metric is taken as \((g^{\mu\nu}) = \text{diag}\{1, -1, -1, -1\}\). Now we can define one particle states as \[10\]

\[
|p,\sigma\rangle = U(L(p))|k,\sigma\rangle, \tag{2}
\]

with \(U(L(p))\) the unitary representation of \(L(p)\) in Hilbert space. The one particle states satisfy

\[
\hat{p}^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle. \tag{3}
\]

We choose the othonormal condition to be

\[
\langle p',\sigma'|p,\sigma\rangle = (2\pi)^3 (2p^0) \delta(p' - \bar{p}) \delta_{\sigma'\sigma}. \tag{4}
\]

Under Lorentz transformations,

\[
U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'\rangle; \tag{5}
\]

where

\[
D_{\sigma'\sigma}(W(\Lambda,p)) = L^{-1}(\Lambda p)\Lambda L(p) \tag{6}
\]
is the Wigner rotation \[11\] and \(\{D_{\sigma'\sigma}\}\) furnishes a representation of \(SO(3)\). We also use the notation \(|\bar{p},j,m\rangle \equiv |p,j,m\rangle\).

There are infinite ways to define the Lorentz transformation that satisfies equation (1). Canonical state and helicity state are the two types mostly used.

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If one define the Lorentz transformation in equation (2) to be pure Lorentz boost

\[ L(p) = L(p\bar{p}) = R(\varphi, \theta, 0)L_z(|\vec{p}|)R^{-1}(\varphi, \theta, 0), \]  

(7)

the canonical state is obtained. Here \( R(\varphi, \theta, 0) \) is the rotation that takes \( z \)-axis to the direction of \( \vec{p} \) with spherical angles \( (\theta, \varphi) \), and the boost \( L_z(|\vec{p}|) \) takes the four-momentum \( (k^\mu) = (w; 0, 0, |\vec{p}|) \) to \( \left( \sqrt{w^2 + p_z^2}; 0, 0, |\vec{p}| \right) \). For a particle of spin-\( j, \sigma \sim (j, m) \). It can be shown that for the canonical states, equation (7) become

\[ U(\Lambda)|\vec{p}, j, m\rangle = \sum_{m'} D^{j}_{m', m}(L^{-1}(\Lambda p)AL(\vec{p}))|\Lambda p, j, m'\rangle. \]  

(8)

\( D^{j}_{m', m} \) is the ordinary D-function. Especially, under rotation \( R \),

\[ U(R)|\vec{p}, j, m\rangle = \sum_{m'} D^{j}_{m', m}(R)|\vec{R}p, j, m'\rangle. \]  

(9)

Defining the Lorentz transformation in another way will leads to helicity states (12):

\[ L(p) = L(p\bar{p})R^{-1}(\varphi, \theta, 0) = R(\varphi, \theta, 0)L_z(|\vec{p}|). \]  

(10)

We have

\[ U(\Lambda)|\vec{p}, j, \lambda\rangle = \sum_{\lambda'} D^{j}_{\lambda', \lambda}(L^{-1}(\Lambda p)AL(\vec{p}))|\Lambda p, j, \lambda'\rangle \]  

(11)

and

\[ U(R)|\vec{p}, j, \lambda\rangle = |\vec{R}p, j, \lambda\rangle. \]  

(12)

The two types of definitions are related to each other by

\[ |\vec{p}, j, \lambda\rangle_{\text{helicity}} = \sum_{m} D^{j}_{m, \lambda}(\varphi, \theta, 0)|\vec{p}, j, m\rangle_{\text{canonical}}. \]  

(13)

We see that the definition of state depends on the choice of Lorentz transformation in equation (2). There is a definition called spinor state (13), which is different from that of equation (2) and does not depend on the specific choice of Lorentz transformation; but it makes things more complex and is seldom used.

Now we write quantum states in terms of creation and annihilation operators:

\[ |\vec{p}, \sigma\rangle = \sqrt{(2\pi)^3 2p^0 a^1(\vec{p}, \sigma)}|0\rangle, \]  

(14)

with \( |0\rangle \) the vacuum state. Quantum fields are given by (14)

\[ \psi^{(+)}_l = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_{\sigma} U_l(\vec{p}, \sigma) a(\vec{p}, \sigma) e^{-ip \cdot x}, \]  

(15)

\[ \psi^{(-)}_l = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_{\sigma} V_l(\vec{p}, \sigma) a^1(\vec{p}, \sigma) e^{ip \cdot x}, \]  

(16)

\[ U(\Lambda, a)\psi^{(\pm)}_l U^{-1}(\Lambda, a) = \sum_{\Lambda'} G_{\Lambda' l}(\Lambda^{-1})\psi^{(\pm)}_{\Lambda'}(\Lambda x + a). \]  

(17)

The coefficient functions, \( U_l \) and \( V_l \), are wave functions in momentum space. \( a^\mu \) are parameters for translation. \( \{ G_{\Lambda' l} \} \) furnishes a representation of the Lorentz group. One finds that wave functions satisfy (18)

\[ \sum_{\Lambda'} G_{\Lambda' l}(\Lambda)U_{l'}(\vec{p}, \sigma) = \sum_{\sigma'} D_{l', \sigma'}(W(\Lambda, p))U_l(\vec{\Lambda} p, \sigma'), \]  

(18)

\[ \sum_{\Lambda'} G_{\Lambda' l}(\Lambda)V_{l'}(\vec{p}, \sigma) = \sum_{\sigma'} D^{*}_{l', \sigma'}(W(\Lambda, p))V_l(\vec{\Lambda} p, \sigma'); \]  

(19)
so we can define wave functions as

\[ U_l(\vec{p}, \sigma) = \sum_{l'} G_{l''} (L(\vec{p})) U_{l'} (\vec{k}, \sigma), \]  

\[ V_l(\vec{p}, \sigma) = \sum_{l'} G_{l''} (L(\vec{p})) V_{l'} (\vec{k}, \sigma). \]

For massive particles, \( \vec{k} = \hat{0} \).

III. WAVE FUNCTIONS FOR INTEGRAL SPIN PARTICLES

If the index \( l \) in previous section is chosen as Lorentz indexes, one arrived at vector fields:

\[
G(\Lambda)_{\mu}^{\nu} = \Lambda_{\nu}^{\mu},
U_{\mu}(\vec{p}, \sigma) = L(p)_{\mu}^{\nu} U_{\nu}(\vec{0}, \sigma),
V_{\mu}(\vec{p}, \sigma) = L(p)_{\mu}^{\nu} V_{\nu}(\vec{0}, \sigma).
\]

We use the following infinitesimal generators of the Lorentz group:

\[
(J_1^{\mu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},
(J_2^{\mu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
(J_3^{\mu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
(K_1^{\mu}) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
(K_2^{\mu}) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
(K_3^{\mu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};
\]

and get the explicit expressions of canonical wave functions (\( E \) is the energy of the particle)

\[
(e_\pm^\mu(\vec{p}, 0)) = \begin{pmatrix} \frac{\sqrt{2}}{w} \cos \theta \\ \frac{1}{w} \left( \frac{E}{w} - 1 \right) \sin 2\theta \cos \varphi \\ \frac{1}{w} \left( \frac{E}{w} - 1 \right) \sin 2\theta \sin \varphi \\ \frac{i}{w} \left( \frac{E}{w} - 1 \right) (1 + \cos 2\theta) + 1 \end{pmatrix},
\]

\[
(\pm e_\pm^\mu(\vec{p}, \pm 1)) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{w} \cos \theta \cos \varphi e^{\pm i\varphi} + 1 \\ \frac{1}{w} \left( \frac{E}{w} - 1 \right) \sin^2 \theta \cos \varphi e^{\pm i\varphi} + 1 \\ \frac{1}{w} \left( \frac{E}{w} - 1 \right) \sin^2 \theta \sin \varphi e^{\pm i\varphi} + 1 \\ \left( \frac{E}{w} - 1 \right) \cos \theta \sin \varphi e^{\pm i\varphi} \end{pmatrix};
\]

while helicity wave functions are

\[
(e_h^\mu(\vec{p}, 0)) = \begin{pmatrix} \frac{\sqrt{2}}{w} \sin \theta \cos \varphi \\ \frac{\sqrt{2}}{w} \sin \theta \sin \varphi \\ \frac{w}{E} \cos \theta \\ 0 \end{pmatrix},
(\pm e_h^\mu(\vec{p}, \pm 1)) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp \cos \theta \cos \varphi + i \sin \varphi \\ \mp \cos \theta \sin \varphi - i \cos \varphi \\ \pm \sin \theta \cos \varphi \pm i \sin \varphi \end{pmatrix};
\]

We have

\[
U(\vec{p}, \sigma) = V^*(\vec{p}, \sigma) = e(\vec{p}, \sigma).
\]

Wave functions for higher integral spins can be defined recurrently by
\[ e_{\mu_1 \mu_2 \cdots \mu_j}(\vec{p}, j, \sigma) = \sum_{\sigma'_{j-1}, \sigma_j} (j - 1, \sigma'_{j-1}; 1, \sigma_j) e_{\mu_1 \mu_2 \cdots \mu_{j-1}}(\vec{p}, j - 1, \sigma'_{j-1}) e_{\mu_j}(\vec{p}, \sigma_j). \]  

Using the C-G coefficient relation

\[
\sum_{\sigma'_{j-1} \cdots \sigma_n} (j_1, \sigma_1; j_2, \sigma_2; j_1 + j_2, \sigma_3)(j_1 + j_2, \sigma_3; j_3, \sigma_3; j_1 + j_2 + j_3, \sigma_4) \cdots
\times (j_1 + j_2 + \cdots + j_{n-1}, \sigma_{j_n}; j_n, \sigma_n;j_1 + j_2 + \cdots + j_n, \sigma_{n-1} + \sigma_n)
\]

\[
= \left[ \prod_{i=1}^{n} \frac{(2j_i)!}{(j_i + \sigma_i)!} \right]^{1/2} \left\{ \frac{\sqrt{\sum_{i=1}^{n} (j_i + \sigma_i)! \left[ \sum_{i=1}^{n} (j_i - \sigma_i)! \right]^2}}{2^n \sum_{i=1}^{n} j_i!} \right\}^{1/2},
\]

we find

\[
e_{\mu_1 \mu_2 \cdots \mu_j}(\vec{p}, j, \sigma)
= \sum_{\sigma_1, \sigma_2, \cdots, \sigma_j} \left\{ \frac{2^j (j + \sigma)! (j - \sigma)!}{(2j)! \prod_{i=1}^{j} [(1 + \sigma_i)! (1 - \sigma_i)!]} \right\}^{1/2} \delta_{\sigma_1 + \sigma_2 + \cdots + \sigma_j, \sigma} e_{\mu_1}(\vec{p}, \sigma_1) e_{\mu_2}(\vec{p}, \sigma_2) \cdots e_{\mu_j}(\vec{p}, \sigma_j).
\]

It is easy to show

\[
\Lambda^{\mu_1 \nu_1} \Lambda^{\mu_2 \nu_2} \cdots \Lambda^{\mu_j \nu_j} e^{\nu_1 \nu_2 \cdots \nu_j}(\vec{p}, j, \sigma) = \sum_{\sigma} D^{j}_{\sigma, \sigma} (W(\Lambda, \vec{p})) e^{\mu_1 \mu_2 \cdots \mu_j}(\vec{p}, j, \sigma').
\]

\[ e_{\mu_1 \mu_2 \cdots \mu_j}(\vec{p}, j, \sigma) \] satisfies all of the Rarita-Schwinger conditions: space-like, symmetric and traceless.

**IV. WAVE FUNCTIONS FOR HALF-INTEGRAL SPIN PARTICLES**

The convention of \( \gamma \)-matrices used here follows that of Bjorken and Drell [4], so the generators of the Lorentz group are

\[
\vec{J} = \frac{1}{2} \begin{pmatrix} \vec{\tau} & 0 \\ 0 & \vec{\tau} \end{pmatrix}, \quad \vec{K} = \frac{1}{2} \begin{pmatrix} 0 & \vec{\tau} \\ \vec{\tau} & 0 \end{pmatrix};
\]

with \( \vec{\tau} \) Pauli matrices.

The spin-\( \frac{1}{2} \) canonical wave functions are ( here \( \alpha = \ln ((E + |\vec{p}|)/w) \)

\[
U_c(\vec{p}, \frac{1}{2}) = \begin{pmatrix} \cosh \frac{\alpha}{2} \\ \cos \theta \sinh \frac{\alpha}{2} \\ \sin \theta \cosh \frac{\alpha}{2} \end{pmatrix}, \quad U_c(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} 0 \\ \cosh \frac{\alpha}{2} \\ \sin \theta e^{i\phi} \sinh \frac{\alpha}{2} \end{pmatrix};
\]

\[
V_c(\vec{p}, \frac{1}{2}) = \begin{pmatrix} -\cos \theta \sinh \frac{\alpha}{2} \\ 0 \\ \cosh \frac{\alpha}{2} \end{pmatrix}, \quad V_c(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} 0 \\ \sin \theta e^{i\phi} \sinh \frac{\alpha}{2} \\ -\cos \theta \cosh \frac{\alpha}{2} \end{pmatrix};
\]

and the helicity wave functions are

\[
U_h(\vec{p}, \frac{1}{2}) = \begin{pmatrix} \cos \frac{\alpha}{2} e^{-i\phi} \\ \sin \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} \cosh \frac{\alpha}{2} \end{pmatrix}, \quad U_h(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} -\sin \frac{\alpha}{2} e^{-i\phi} \\ \cos \frac{\alpha}{2} \cosh \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} \sinh \frac{\alpha}{2} \end{pmatrix};
\]

\[
V_h(\vec{p}, \frac{1}{2}) = \begin{pmatrix} \sin \frac{\alpha}{2} \\ \cos \frac{\alpha}{2} e^{-i\phi} \\ -\sin \frac{\alpha}{2} \cosh \frac{\alpha}{2} \end{pmatrix}, \quad V_h(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} -\cos \frac{\alpha}{2} e^{-i\phi} \\ \sin \frac{\alpha}{2} \sinh \frac{\alpha}{2} \\ \cos \frac{\alpha}{2} \cosh \frac{\alpha}{2} \end{pmatrix};
\]
Spin-\(n + \frac{1}{2}\) wave functions read

\[
U_{\mu_1 \mu_2 \cdots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) = \sum_{\sigma_1, \sigma_2, \cdots, \sigma_{n+1}} \left\{ \begin{array}{l} 2^n \frac{(n + \frac{1}{2} + \sigma)! (n + \frac{1}{2} - \sigma)!}{(n+1)!} \\ \prod_{i=1}^{n+1} \frac{((1 + \sigma_i)!(1 - \sigma_i)!)}{i!} \end{array} \right\} \frac{1}{2} \delta_{\sigma_1 + \sigma_2 + \cdots + \sigma_{n+1}} \delta_{\sigma} \\
\times e_{\mu_1}(\vec{p}, \sigma_1) e_{\mu_2}(\vec{p}, \sigma_2) \cdots e_{\mu_n}(\vec{p}, \sigma_n) U(\vec{p}, \sigma_{n+1});
\]

\[
V_{\mu_1 \mu_2 \cdots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) = \sum_{\sigma_1, \sigma_2, \cdots, \sigma_{n+1}} \left\{ \begin{array}{l} 2^n \frac{(n + \frac{1}{2} + \sigma)! (n + \frac{1}{2} - \sigma)!}{(n+1)!} \\ \prod_{i=1}^{n+1} \frac{((1 + \sigma_i)!(1 - \sigma_i)!)}{i!} \end{array} \right\} \frac{1}{2} \delta_{\sigma_1 + \sigma_2 + \cdots + \sigma_{n+1}} \delta_{\sigma} \\
\times e^{*}_{\mu_1}(\vec{p}, \sigma_1) e^{*}_{\mu_2}(\vec{p}, \sigma_2) \cdots e^{*}_{\mu_n}(\vec{p}, \sigma_n) V(\vec{p}, \sigma_{n+1}).
\]

They satisfy Dirac equations and Rarita-Schwinger conditions [9]; especially

\[
\gamma^{\mu k} U_{\mu_1 \mu_2 \cdots \mu_k \cdots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) = 0,
\]

\[
\gamma^{\mu k} V_{\mu_1 \mu_2 \cdots \mu_k \cdots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) = 0.
\]