THE IMAGE OF THE LEPOWSKY HOMOMORPHISM FOR THE GROUP $F_4$

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ABSTRACT. Let $G_o$ be a semisimple Lie group, let $K_o$ be a maximal compact subgroup of $G_o$ and let $\mathfrak{k} \subset \mathfrak{g}$ denote the complexification of their Lie algebras. Let $G$ be the adjoint group of $\mathfrak{g}$ and let $K$ be the connected Lie subgroup of $G$ with Lie algebra $ad(\mathfrak{k})$. If $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ then $U(\mathfrak{g})^K$ will denote the centralizer of $K$ in $U(\mathfrak{g})$. Also let $P : U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$ be the projection map corresponding to the direct sum $U(\mathfrak{g}) = (U(\mathfrak{k}) \otimes U(\mathfrak{a})) \oplus U(\mathfrak{g})n$ associated to an Iwasawa decomposition of $G_o$ adapted to $K_o$. In this paper we give a characterization of the image of $U(\mathfrak{g})^K$ under the injective antihomorphism $P : U(\mathfrak{g})^K \longrightarrow U(\mathfrak{k})^M \otimes U(\mathfrak{a})$, considered by Lepowsky in [10], when $G_o$ is locally isomorphic to $F_4$.

1. Introduction

Let $G_o$ be a connected, noncompact, real semisimple Lie group with finite center, and let $K_o$ denote a maximal compact subgroup of $G_o$. We denote with $\mathfrak{g}_o$ and $\mathfrak{k}_o$ the Lie algebras of $G_o$ and $K_o$, and $\mathfrak{k} \subset \mathfrak{g}$ will denote the respective complexified Lie algebras. Let $G$ be the adjoint group of $\mathfrak{g}$ and let $K$ be the connected Lie subgroup of $G$ with Lie algebra $ad(\mathfrak{k})$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $U(\mathfrak{g})^K$ denote the centralizer of $K$ in $U(\mathfrak{g})$.

In order to contribute to the understanding of $U(\mathfrak{g})^K$ Kostant suggested to consider the projection map $P : U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$, corresponding to the direct sum $U(\mathfrak{g}) = (U(\mathfrak{k}) \otimes U(\mathfrak{a})) \oplus U(\mathfrak{g})n$ associated to an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ adapted to $\mathfrak{k}$. In [10] Lepowsky studied the restriction of $P$ to $U(\mathfrak{g})^K$ and proved, among other things, that one has the following exact sequence

$$0 \longrightarrow U(\mathfrak{g})^K \xrightarrow{P} U(\mathfrak{k})^M \otimes U(\mathfrak{a}),$$

where $U(\mathfrak{k})^M$ denotes the centralizer of $M$ in $U(\mathfrak{k})$, $M$ being the centralizer of $\mathfrak{a}$ in $K$. Moreover if $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ is given the tensor product algebra structure then $P$ becomes an antihomomorphism of algebras. Hence to go any further in this direction it is necessary to determine the image of $P$.

To determine the actual image $P(U(\mathfrak{g})^K)$ Tirao introduced in [12] a subalgebra $B$ of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ defined by a set of linear equations in $U(\mathfrak{k})$ derived from certain embeddings between Verma modules (see [11] below). Then he proved that $P(U(\mathfrak{g})^K)$ always lies in $B$, and furthermore that
\(P(U(g)^K) = B^W\rho\) when \(G_o = \text{SO}(n,1)\) or \(\text{SU}(n,1)\). Here \(W\) is the Weyl group of the pair \((g,a)\), \(\rho\) is half the sum of the positive roots of \(g\) and \(B^W\rho\) is the subalgebra of all elements in \(B\) that are invariant under the tensor product action of \(W\) on \(U(\mathfrak{t})^M\) and the translated action of \(W\) on \(U(\mathfrak{a})\) (see Corollary 3.3 of [9]).

In [3] we proved that \(P(U(g)^K) = B\) when \(G_0 = \text{Sp}(n,1)\), and more recently, we showed that \(B^W\rho = B\) when \(G_0 = \text{SO}(n,1)\) or \(\text{SU}(n,1)\) (see [4]). Hence these results established that \(P(U(g)^K) = B\) for every classical real rank one semisimple Lie group with finite center. This paper is devoted to proving that this result also holds for the exceptional Lie group \(F_4\). The main result of the present work is the following,

**Theorem 1.1.** If \(G_o\) is locally isomorphic to \(F_4\), then \(P(U(g)^K) = B\).

This result confirms our old belief that the following theorem holds,

**Theorem 1.2.** Let \(G_o\) be a real rank one semisimple Lie group. Then the image of the Lepowsky homomorphism \(P\) is the algebra \(B\).

We point out that our proof of Theorem 1.2 follows a general pattern in all cases, however at certain points in the argument there are some substantial differences. Certainly the cases of \(\text{Sp}(n,1)\) and \(F_4\) are the most difficult of the rank one groups.

The proof of Theorem 1.1 follows the same ideas used to prove the analogue theorem for the symplectic group \(\text{Sp}(n,1)\), however we had to overcome some difficulties to establish the transversality results needed (Section 6), and the a priori estimates of the Kostant degrees (Section 7). In Section 6 we give a new and simplified version of the corresponding transversality results obtained in the symplectic case (see Section 4 of [3]). This new version is sufficient because of the introduction of a simplifying hypothesis called the degree property, which is done in Section 7. In this section we use this property to obtain an a priori estimate of the Kostant degree of certain elements \(b \in B\). This allows us to reduce the proof of Theorem 1.1 to proving Theorem 7.12 (see Proposition 7.14). The proof of this last theorem is given in Section 8 following the ideas developed in the symplectic case. In fact, most of the results proved in Section 6 of [3] hold in this case with appropriate changes.

2. **The algebra \(B\) and the image of \(U(g)^K\)**

In this section we assume that \(G_o\) be a connected, noncompact, real semisimple Lie group with finite center and of split rank one, not locally isomorphic to \(\text{SL}(2,\mathbb{R})\). Let \(\mathfrak{t}_o\) be a Cartan subalgebra of the Lie algebra \(\mathfrak{m}_o\) of \(M_o\). Set \(\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o\) and let \(\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}\) be the corresponding complexification. Then \(\mathfrak{h}_o\) and \(\mathfrak{h}\) are Cartan subalgebras of \(\mathfrak{g}_o\) and \(\mathfrak{g}\), respectively. Choose a Borel subalgebra \(\mathfrak{t} \oplus \mathfrak{m}^+\) of the complexification \(\mathfrak{m}\) of \(\mathfrak{m}_o\) and take \(\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}_+ \oplus \mathfrak{n}\) as a Borel subalgebra of \(\mathfrak{g}\). Let \(\Delta\) and \(\Delta^+\) be, respectively, the corresponding sets of roots and positive roots of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\). If
\(\alpha \in \Delta\) let \(X_\alpha\) denote a nonzero root vector associated to \(\alpha\). Also, let \(\theta\) be the Cartan involution, and let \(g = \mathfrak{t} \oplus \mathfrak{p}\) be the Cartan decomposition of \(g\) corresponding to \((G_o, K_o)\).

If \((\; , \; )\) denotes the Killing form of \(g\), for each \(\alpha \in \Delta\) let \(H_\alpha \in \mathfrak{h}\) be the unique element such that \(\phi(H_\alpha) = 2\langle \phi, \alpha \rangle / \langle \alpha, \alpha \rangle\) for all \(\phi \in \mathfrak{h}^*\), and let \(\mathfrak{h}_\mathbb{R}\) be the real span of \(\{H_\alpha : \alpha \in \Delta\}\). Also set \(H_\alpha = Y_\alpha + Z_\alpha\) where \(Y_\alpha \in \mathfrak{t}\) and \(Z_\alpha \in \mathfrak{a}\), and let \(P_+ = \{\alpha \in \Delta^+ : Z_\alpha \neq 0\}\). If \(\alpha \in P_+\) is a simple root and \(\lambda = \alpha|_\mathfrak{a}\) we let \(g(\lambda)\) denote the real reductive rank one subalgebra of \(g_o\) associated to \(\lambda\). We point out that \(Y_\alpha \neq 0\) if and only if \([g(\lambda), g(\lambda)] \neq \mathfrak{sl}(2, \mathbb{R})\). Hence, by our assumptions on \(G_o\), for any simple root \(\alpha \in P_+\) we have \(Y_\alpha \neq 0\).

If \(\alpha \in P_+\) we have \(a = \mathbb{C}Z_\alpha\) and we can view the elements in \(U(\mathfrak{t}) \otimes U(\mathfrak{a})\) as polynomials in \(Z_\alpha\) with coefficients in \(U(\mathfrak{t})\). For any such a root \(\alpha\) we set \(E_\alpha = X_{-\alpha} + \theta X_{-\alpha}\). Now we introduce the subalgebra \(B\) of \(U(\mathfrak{t})^M \otimes U(\mathfrak{a})\) defined by Tirao in [12].

**Definition 2.1.** Let \(B\) be the algebra of all \(b \in U(\mathfrak{t})^M \otimes U(\mathfrak{a})\) that satisfy
\[
E_\alpha^m b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E_\alpha^m,
\]
for all simple roots \(\alpha \in P_+\) and all \(n \in \mathbb{N}\). Here the congruence is modulo the left ideal \(U(\mathfrak{t})\mathfrak{m}^+\) of \(U(\mathfrak{t})\). This definition is motivated by equations (2) (see Corollary 2.2).

In Theorem 5 of [12] Tirao proved that \(B\) is a subalgebra of \(U(\mathfrak{t})^M \otimes U(\mathfrak{a})\), and in Corollary 6 of the same paper he proved that \(P(U(g)^K) \subset B\). For further reference we state this Corollary.

**Corollary 2.2.** Let \(\alpha \in P_+\) be a simple root. Then for all \(n \in \mathbb{N}\) and \(u \in U(g)^K\) we have
\[
E_\alpha^m P(u)(n - Y_\alpha - 1) \equiv P(u)(-n - Y_\alpha - 1)E_\alpha^m,
\]
where the congruence is modulo \(U(\mathfrak{t})\mathfrak{m}^+\).

In order to prove Theorem 1.1 we will now introduce some notation and recall known results. Let \(\Gamma\) denote the set of all equivalence classes of irreducible holomorphic finite dimensional \(K\)-modules \(V_\gamma\) such that \(V_\gamma^M \neq 0\). Any \(\gamma \in \Gamma\) can be realized as a submodule of all harmonic polynomial functions on \(\mathfrak{p}\), homogeneous of degree \(d\), for a uniquely determined \(d = d(\gamma)\) (see [8]). We shall refer to the non negative integer \(d(\gamma)\) as the *Kostant degree* of \(\gamma\). If \(V\) is any \(K\)-module and \(\gamma \in \tilde{K}\) then \(V_\gamma\) will denote the isotypic component of \(V\) corresponding to \(\gamma\). Let \(V\) be a locally finite \(K\)-module such that \(V^M \neq 0\) and let \(v \in V^M\), \(v \neq 0\). Since \(V\) is locally finite, we can decompose \(v\) into \(K\)-isotypic \(M\)-invariants as follows
\[
v = \sum_{\gamma \in \Gamma} v_\gamma,
\]
where $v_\gamma \in V_\gamma$ denotes the $\gamma$-isotypic component of $v$. Then we define the *Kostant degree* of $v$ by,

\[(3) \quad d(v) = \max \{d(\gamma) : v_\gamma \neq 0\}.\]

Since we are mainly concerned with representations $\gamma \in \Gamma$ that occur as subrepresentations of $U(\mathfrak{t})$ we set,

\[(4) \quad \Gamma_1 = \{\gamma \in \Gamma : \gamma \text{ is a subrepresentation of } U(\mathfrak{t})\}.
\]

Remark: Theorem 2.4.

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\[\Gamma \in \mathfrak{g}\]

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hypothesis, there exists \( u \in U(\mathfrak{g})^K \) such that \( P(u) = b - P(v) \) and \( b = P(u + v) \in P(U(\mathfrak{g})^K) \). This completes the induction argument and we obtain that \( B \subset P(U(\mathfrak{g})^K) \), as we wanted to prove. \( \Box \)

In view of this last result the main objective of this paper is to prove Theorem 2.4 when \( G_0 \) is locally isomorphic to \( F_4 \).

3. The equations defining \( B \)

From now on we shall write \( u \equiv v \) instead of \( u \equiv v \mod (U(\mathfrak{g})m^+) \), for any \( u, v \in U(\mathfrak{g}) \). Next result was proved in Lemma 29 of [12] for \( G_0 \) of arbitrary rank.

**Lemma 3.1.** Let \( \alpha \in P_+ \) be a simple root. Set \( H_\alpha = Y_\alpha + Z_\alpha \) where \( Y_\alpha \in t \), \( Z_\alpha \in a \) and let \( c = \alpha(Y_\alpha) \). If \( \lambda = \alpha|a \) and \( m(\lambda) \) is the multiplicity of \( \lambda \), then \( c = 1 \) when \( 2\lambda \) is not a restricted root and \( m(\lambda) \) is even, or when \( m(\lambda) \) is odd, and \( c = \frac{3}{2} \) when \( 2\lambda \) is a restricted root and \( m(\lambda) \) is even.

In particular, if \( G_0 \) is locally isomorphic to \( F_4 \) we have \( c = \frac{3}{2} \). To simplify the notation set \( E = E_\alpha \), \( Y = Y_\alpha \) and \( Z = Z_\alpha \) for any simple root \( \alpha \in P_+ \). Notice that \( [E, Y] = cE \), where \( c \) is as in Lemma 3.1. Also, since \( E_\alpha = X_{-\alpha} + \theta X_{-\alpha} \) and \( \alpha \) is a simple root in \( P_+ \) it follows that \( E \) is \( m^+ \)-dominant.

We shall identify \( U(\mathfrak{g}) \otimes U(\mathfrak{a}) \) with the polynomial ring in one variable \( U(\mathfrak{g})[x] \), replacing \( Y \) by the indeterminate \( x \). To study equation (1) we change \( b(x) \in U(\mathfrak{g})[x] \) by \( c(x) \in U(\mathfrak{g})[x] \) defined by

\[
(5) \quad c(x) = b(x + H - 1),
\]

where \( H \) is an appropriate vector in \( t \) to be chosen later, depending on the simple root \( \alpha \in P_+ \) and such that \( [H, E] = \frac{1}{2}E \) (see (17)). Now, if \( \tilde{Y} = Y + H \) we have \( [E, \tilde{Y}] = E \). Then \( b(x) \in U(\mathfrak{g})[x] \) satisfies (1) if and only if \( c(x) \in U(\mathfrak{g})[x] \) satisfies

\[
(6) \quad E^n c(n - \tilde{Y}) \equiv c(-n - \tilde{Y}) E^n
\]

for all \( n \in \mathbb{N} \). Note that (5) is an equation in the noncommutative ring \( U(\mathfrak{g}) \).

Now, if \( p \) is a polynomial in one indeterminate \( x \) with coefficients in a ring let \( p^{(n)} \) denote the \( n \)-th discrete derivative of \( p \). That is, \( p^{(n)}(x) = \sum_{j=0}^n (-1)^j \binom{n}{j} p(x + \frac{n}{2} - j) \). In particular, if \( p = p_m x^m + \cdots + p_0 \) we have

\[
p^{(n)}(x) = \begin{cases} 0, & \text{if } n > m \\ m^ip_m, & \text{if } n = m. \end{cases}
\]

Also, if \( X \in \mathfrak{g} \) we shall denote with \( \dot{X} \) the derivation of \( U(\mathfrak{g}) \) induced by \( \text{ad}(X) \). Moreover if \( D \) is a derivation of \( U(\mathfrak{g}) \) we shall denote with the same symbol the unique derivation of \( U(\mathfrak{g})[x] \) which extends \( D \) and such that \( Dx = 0 \). Thus for \( b \in U(\mathfrak{g})[x] \) and \( b = b_m x^m + \cdots + b_0 \), we have \( Db = (Db_m) x^m + \cdots + (Db_0) \). Observe that these derivations commute with the operation of taking the discrete derivative in \( U(\mathfrak{g})[x] \).
Next theorem gives a triangularized version of the system ([1]), and in turn, of the system ([1]) that defines the algebra B. A proof of it is given in [1], where the system ([1]) is studied in a more abstract setting and in particular the LU-decomposition of its coefficient matrix is given.

**Theorem 3.2.** Let \( c \in U(\mathfrak{t})[x] \). Then the following systems of equations are equivalent:

(i) \( E^n c(n - \bar{Y}) \equiv c(n - \bar{Y})E^n, (n \in \mathbb{N}_0); \)

(ii) \( \bar{E}^{n+1}(c^{(n)})(\frac{n}{2} + 1 - \bar{Y}) + \bar{E}^n(c^{(n+1)})(\frac{n}{2} + \frac{1}{2} - \bar{Y})E \equiv 0, (n \in \mathbb{N}_0). \)

Moreover, if \( c \in U(\mathfrak{t})[x] \) is a solution of one of the above systems, then for all \( \ell, n \in \mathbb{N}_0 \) we have

(iii) \( (-1)^n \bar{E}^\ell(c^{(n)})(-\frac{n}{2} + \ell - \bar{Y})E^n - (-1)^\ell \bar{E}^n(c^{(\ell)})(-\frac{n}{2} + n - \bar{Y})E^\ell \equiv 0. \)

Observe that if \( c \in U(\mathfrak{t})[x] \) is of degree \( m \) and \( c = c_m x^m + \cdots + c_0 \), then all the equations of the system (ii) corresponding to \( n > m \) are trivial, because \( c^{(n)} = 0 \). Moreover the equation corresponding to \( n = m \) reduces to \( \bar{E}^{m+1}(c_m) \equiv 0 \), and more generally the equation associated to \( n = j \) only involves the coefficients \( c_m, \ldots, c_j \). In this sense the system (ii) is a triangular system of \( m + 1 \) linear equations in the \( m + 1 \) unknowns \( c_m, \ldots, c_0 \).

If \( 0 \neq b(x) \in U(\mathfrak{t})[x] \) and \( c(x) \in U(\mathfrak{t})[x] \) is defined by \( c(x) = b(x + H - 1), \)
where \( H \) is as in ([17]), we find it convenient to write, in a unique way,
\( b = \sum_{j=0}^m b_j x^j \) with \( b_j \in U(\mathfrak{t}), b_m \neq 0 \), and \( c = \sum_{j=0}^m c_j \varphi_j \) where \( c_j \in U(\mathfrak{t}) \) and \( \{\varphi_n\}_{n \geq 0} \) is the basis of \( \mathbb{C}[x] \) defined by,

(i) \( \varphi_0 = 1, \)

(ii) \( \varphi_n^{(1)} = \varphi_{n-1} \quad \text{if } n \geq 1, \)

(iii) \( \varphi_n(0) = 0 \quad \text{if } n \geq 1. \)

Moreover it is easy to prove that such a family is given by

\[ \varphi_n(x) = \frac{1}{n!} x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2) \cdots (x - \frac{n}{2} + 1), \quad n \geq 1. \]

Next lemma contains the results of Lemma 3.3 and Lemma 3.5 of [3]. Its proof is the same as that of the corresponding lemmas in [3].

**Lemma 3.3.** Let \( b = \sum_{j=0}^m b_j x^j \in U(\mathfrak{t})[x] \) and set \( c(x) = b(x + H - 1) \).
Then, if \( c = \sum_{j=0}^m c_j \varphi_j \) with \( c_j \in U(\mathfrak{t}) \) we have

\[ c_i = \sum_{j=1}^m b_j t_{ij} \quad 0 \leq i \leq m, \]

where
\[ t_{ij} = \sum_{k=0}^i (-1)^k \binom{i}{k} (H + \frac{i}{2} - 1 - k)^j. \]
Theorem 3.6. Also, we have

\[ \epsilon^{j-i}(t_{ij}) = \left( -\frac{1}{2} \right)^{j-i} j!E^{j-i}. \]

From these results and Theorem 3.2 we obtain the following theorem and its corollary in the same way as in [3].

Theorem 3.4. If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), then \( \hat{E}^{m+1}(c_j) \equiv 0 \) for all \( 0 \leq j \leq m \).

Corollary 3.5. If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), then \( \hat{E}^{2m+1-j}(b_j) \equiv 0 \) for all \( 0 \leq j \leq m \).

Next we rewrite equation (iii) of Theorem 3.2 for later reference. Given \( b = \sum_{j=0}^{m} b_j x^j \in B \) and \( c(x) = b(x+H-1) \) as above, it follows from Theorem 3.4 that equation (iii) of Theorem 3.2 is satisfied if \( \ell > m \) or \( n > m \), and it is trivial when \( \ell = n \). Also note that the equation corresponding to \( (n, \ell) \) is equivalent to that one corresponding to \( (\ell, n) \).

Theorem 3.6. Let \( b = \sum_{j=0}^{m} c_j x^j \in U(\mathfrak{t})[x] \) and \( c(x) = b(x+H-1) \). If \( c = \sum_{j=0}^{m} c_j \varphi_j \) with \( c_j \in U(\mathfrak{t}) \) and \( 0 \leq \ell, n \) we set

\[
\epsilon(\ell, n) = (-1)^n \sum_{n \leq i \leq m} \hat{E}^{\ell}(c_i) \varphi_{i-n} \left( \frac{-\alpha}{2} + \ell - \tilde{Y} \right) E^n \\
- (-1)^{\ell} \sum_{\ell \leq i \leq m} \hat{E}^{n}(c_i) \varphi_{i-\ell} \left( \frac{-\alpha}{2} + n - \tilde{Y} \right) E^{\ell}.
\]

Then, if \( b \in B \) we have \( \epsilon(\ell, n) \equiv 0 \mod (U(\mathfrak{t})m^+) \) for all \( 0 \leq \ell, n \).

Proof. The assertion follows from equation (iii) of Theorem 3.2 and the fact that \( c^{(k)} = \sum_{i=k}^{m} c_i \varphi_{i-k} \) for all \( 0 \leq k \leq m \).

4. The group \( \mathbf{F}_4 \)

Let \( G_0 \) be locally isomorphic to \( \mathbf{F}_4 \). The Dynkin-Satake diagram of \( \mathfrak{g} \), the complexification of the Lie algebra of \( G_0 \), is

![Diagram of the Dynkin-Satake diagram of \( \mathfrak{g} \)]

We can choose an orthonormal basis \( \{ \epsilon_i \}_{i=1}^{4} \) of \( \mathfrak{h}_R^* \) such that \( \alpha_4 = \epsilon_2 - \epsilon_3 \), \( \alpha_3 = \epsilon_3 - \epsilon_4 \), \( \alpha_2 = \epsilon_4 \), \( \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \). Moreover, if \( \sigma \) denotes the conjugation of \( \mathfrak{g} \) with respect to \( \mathfrak{g}_0 \), then \( \epsilon_1^\sigma = \epsilon_1 \) and \( \epsilon_i^\sigma = -\epsilon_i \) if \( 2 \leq i \leq 4 \).

Also, we have \( \epsilon_1^\theta = -\epsilon_1 \) and \( \epsilon_i^\theta = \epsilon_i \) for \( 2 \leq i \leq 4 \). From the diagram it follows that

\[
\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{ \epsilon_i : 1 \leq i \leq 4 \} \cup \{ \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 4 \} \\
\cup \{ \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \},
\]

\[
P_+ = \{ \epsilon_1, \epsilon_1 \pm \epsilon_2, \epsilon_1 \pm \epsilon_3, \epsilon_1 \pm \epsilon_4 \} \cup \{ \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \},
\]

\[
P_- = \{ \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_2 \pm \epsilon_3, \epsilon_2 \pm \epsilon_4, \epsilon_3 \pm \epsilon_4 \},
\]

...
where the signs may be chosen independently. Here $P_-$ denotes the set of roots in $\Delta^+(g, h)$ that vanish on $a$. Hence $P_-=\Delta^+(m, t)$ and from this it follows that $m \simeq so(7, C)$.

We have $t = \ker(e_1)$ and $e_1$ is the only root in $P_+$ that vanishes on $t$. If we set $\mu = e_1$, then $H_\mu = Z_\mu \in a$. Choose the root vector $X_\mu$ so that $\langle X_\mu, e_1 \rangle = 2$ and define $X_\mu = \theta X_\mu$. Then the ordered set $\{H_\mu, X_\mu, X_-\}$ is an s-triple. This choice characterizes $X_\mu$ up to a sign. On the other hand, it can be established that for any choice of nonzero root vectors $X_{\alpha_1}$ and $X_{\alpha_2}$ we have $[X_\mu, \theta X_{\alpha_1}] = t X_{\alpha_1}$ and $[X_\mu, X_{\alpha_2}] = -t \theta X_{\alpha_2}$ with $t^2 = 1$. Then normalize $X_\mu$ so that,

$$[X_\mu, \theta X_{\alpha_1}] = -X_{\alpha_1} \quad \text{and} \quad [X_\mu, X_{\alpha_2}] = \theta X_{\alpha_2}. \tag{8}$$

Now consider the Cayley transform $\chi$ of $g$ defined by

$$\chi = Ad(\exp \frac{\pi}{4}(\theta X_\mu - X_\mu)).$$

It is easy to see that

$$Ad(\exp t(\theta X_\mu - X_\mu))H_\mu = \cos(2t)H_\mu + \sin(2t)(X_\mu + \theta X_\mu).$$

Then $\chi(H_\mu) = X_\mu + \theta X_\mu$ and, since $\mu_0 = 0$, $\chi$ fixes all elements of $t$. Therefore $h_\mu = \chi(t \oplus a) = t \oplus C(X_\mu + \theta X_\mu) \subset k$ is a Cartan subalgebra of both $g$ and $\mathfrak{t}$.

For any $\phi \in \mathfrak{h}_t^* \subset \mathfrak{h}_t^*$ define $\tilde{\phi} \in \mathfrak{h}_t^*$ by $\tilde{\phi} = \phi \cdot \chi^{-1}$. Then $\Delta(g, h_\mu) = \{\tilde{\alpha} : \alpha \in \Delta(g, h)\}$ and $g_{\tilde{\alpha}} = \chi(g_\alpha)$. A root $\tilde{\alpha} \in \Delta(g_\mu, h_\mu)$ is said to be compact (respectively noncompact) if $g_{\tilde{\alpha}} \subset \mathfrak{t}$ (respectively $g_{\tilde{\alpha}} \subset p$). Let $\Delta(\mathfrak{t}, h_\mu)$ and $\Delta(\mathfrak{p}, h_\mu)$ denote, respectively, the sets of compact and noncompact roots.

Using Lemma 3.1 of [3] it follows that $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$ are compacts roots, and that $\tilde{\alpha}_2$ is a noncompact root. Also, since $X_\mu$ was chosen so that (8) holds, we obtain that $\tilde{\alpha}_1$ is a noncompact root. From this it follows that

$$\Delta(\mathfrak{t}, h_\mu) = \{\pm(\tilde{e}_i \pm \tilde{e}_j) : 1 \leq i < j \leq 4\}$$

$$\cup \{\frac{1}{2}(\pm \tilde{e}_1 \pm \tilde{e}_2 \pm \tilde{e}_3 \pm \tilde{e}_4) : \text{even number of minus signs}\},$$

$$\Delta(\mathfrak{p}, h_\mu) = \{\pm \tilde{e}_i : 1 \leq i \leq 4\}$$

$$\cup \{\frac{1}{2}(\pm \tilde{e}_1 \pm \tilde{e}_2 \pm \tilde{e}_3 \pm \tilde{e}_4) : \text{odd number of minus signs}\}.$$

Next we construct a particular Borel subalgebra $b_\mathfrak{t} = \mathfrak{h}_\mu \oplus \mathfrak{t}_+$ of $\mathfrak{t}$ that will be useful later on to describe the set $\Gamma$, as well as some of the properties of the elements of $\Gamma$ (see Proposition 5.1). For more details on the construction of the subalgebra $b_\mathfrak{t}$ and its relation with $\Gamma$ we refer the reader to [5].

Since $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ is the only simple root in $P_+$ set, as in the previous section, $E = X_{-\alpha_1} + \theta X_{-\alpha_1}$. Let $H_\mu \in t_\mathfrak{p}$ be such that $\alpha(H_\mu) > 0$ for all $\alpha \in \Delta^+(m, t)$. We say that $H_\mu$ is $\mathfrak{t}$-regular if in addition $\alpha(H_\mu) \neq 0$ for all $\alpha$ with $\tilde{\alpha} \in \Delta(\mathfrak{t}, h_\mu)$. Since $\mu$ is the only root in $\Delta^+(g, h)$ that vanishes on $t$ and since $\tilde{\mu}$ is a noncompact root, it follows that $\mathfrak{t}$-regular vectors exist. Given a $\mathfrak{t}$-regular vector $H_\mu$ consider the positive system

$$\Delta^+(\mathfrak{t}, h_\mu) = \{\tilde{\alpha} \in \Delta(\mathfrak{t}, h_\mu) : \alpha(H_\mu) > 0\}.$$
If \( \lambda_0 = \alpha_1|_a \) is the simple restricted root and \( H_+ \) is a \( \mathfrak{t} \)-regular vector we consider the following set,

\[
P_+(\lambda_0)^- = \{ \alpha \in P_+ : \alpha|_a = \lambda_0 \text{ and } \alpha(H_+) < 0 \}.
\]

**Definition 4.1.** A positive system \( \Delta^+(\mathfrak{t}, \mathfrak{h}_\mathfrak{t}) \) defined by a \( \mathfrak{t} \)-regular vector \( H_+ \) (see (3)) is said to be compatible with \( E \) if \( \alpha - \alpha_1 \) is a root for every \( \alpha \in P_+(\lambda_0)^- \) such that \( \alpha \neq \alpha_1 \).

The \( \mathfrak{t} \)-regular vectors in \( t_\mathbb{R} \), for \( g_0 \simeq f_4 \), are all of the form \( H_+ = (0, t_2, t_3, t_4) \) with \( t_2 > t_3 > t_4 > 0 \) and \( t_2 \neq t_3 + t_4 \). Different vectors \( H_+ \) define two different positive systems, they depend only on whether \( \pm(t_2 - t_3 - t_4) > 0 \), and they are both compatible with \( E \). From now on fix a \( \mathfrak{t} \)-regular vector \( H_+ = (0, t_2, t_3, t_4) \) with \( t_2 > t_3 > t_4 > 0 \) and \( t_2 > t_3 + t_4 \). The corresponding positive system in \( \Delta(\mathfrak{t}, \mathfrak{h}_\mathfrak{t}) \) is,

\[
\Delta^+(\mathfrak{t}, \mathfrak{h}_\mathfrak{t}) = \{ \bar{\epsilon}_i \pm \bar{\epsilon}_j : 2 \leq i < j \leq 4 \} \cup \{ \bar{\epsilon}_i \pm \bar{\epsilon}_1 : 2 \leq i \leq 4 \}
\]

\[
\cup \{ \frac{1}{2}(\bar{\epsilon}_1 \pm \bar{\epsilon}_3 \pm \bar{\epsilon}_4) : \text{even number of minus signs} \},
\]

and \( b_\mathfrak{t} = \mathfrak{h}_\mathfrak{t} \oplus \mathfrak{t}^+ \) is the associated Borel subalgebra. A simple system in \( \Delta^+(\mathfrak{t}, \mathfrak{h}_\mathfrak{t}) \) is given by,

\[
(9) \quad \Pi(\mathfrak{t}, \mathfrak{h}_\mathfrak{t}) = \{ \bar{\epsilon}_4 + \bar{\epsilon}_1, \bar{\epsilon}_3 - \bar{\epsilon}_4, \bar{\epsilon}_4 - \bar{\epsilon}_1, \frac{1}{2}(\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_3 - \bar{\epsilon}_4) \).
\]

Hence \( \mathfrak{t} \simeq \mathfrak{so}(9, \mathbb{C}) \).

Fix nonzero root vectors \( X_{\bar{\epsilon}_i + \bar{\epsilon}_1} \) \( (2 \leq i \leq 4) \), \( X_{\bar{\epsilon}_i \pm \bar{\epsilon}_j} \) \( (2 \leq i < j \leq 4) \) and define,

\[
(10) \quad X_{\bar{\epsilon}_i + \bar{\epsilon}_1} = \chi(X_{\epsilon_i + \epsilon_1}), \quad X_{\bar{\epsilon}_i - \bar{\epsilon}_1} = \chi(\theta X_{\epsilon_i + \epsilon_1}), \quad X_{\bar{\epsilon}_i \pm \bar{\epsilon}_j} = \chi(X_{\epsilon_i \pm \epsilon_j}).
\]

Then it follows from Proposition 2.4 of [5] that,

\[
(11) \quad X_{\bar{\epsilon}_i \pm \bar{\epsilon}_j} = X_{\epsilon_i \pm \epsilon_j},
\]

\[
X_{\bar{\epsilon}_i \pm \bar{\epsilon}_1} = \frac{1}{2}(X_{\epsilon_i + \epsilon_1} + [X_{\mu}, \theta X_{\epsilon_i + \epsilon_1}] + \theta X_{\epsilon_i + \epsilon_1})
\]

and

\[
X_{\bar{\epsilon}_i - \bar{\epsilon}_1} = \frac{1}{2}(X_{\epsilon_i + \epsilon_1} - [X_{\mu}, \theta X_{\epsilon_i + \epsilon_1}] + \theta X_{\epsilon_i + \epsilon_1})
\]

Hence,

\[
(12) \quad X_{\bar{\epsilon}_i + \bar{\epsilon}_1} - X_{\bar{\epsilon}_i - \bar{\epsilon}_1} = [X_{\mu}, \theta X_{\epsilon_i + \epsilon_1}] = X_{\epsilon_i} \in m^+,
\]

Then from (11) and (12) it follows that,

\[
(13) \quad m^+ = \langle \{ X_{\bar{\epsilon}_i \pm \bar{\epsilon}_j} : 2 \leq i < j \leq 4 \} \cup \{ X_{\bar{\epsilon}_i + \bar{\epsilon}_1} - X_{\bar{\epsilon}_i - \bar{\epsilon}_1} : 2 \leq i \leq 4 \} \rangle,
\]

where \( \langle S \rangle \) denotes the linear space spanned by the set \( S \).

Next we define, as in the case of \( \text{Sp}(n,1) \) (see Section 3 of [3]), a Lie subalgebra \( \tilde{\mathfrak{g}} \) of \( \mathfrak{g} \) that is both \( \sigma \) and \( \theta \) stable and its real form \( \tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \cap \tilde{\mathfrak{g}} \) is isomorphic to \( \mathfrak{sp}(2,1) \). Recall that \( \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \) is the only
simple root in $P_\pm$. Let $\tilde{\mathfrak{g}}$ be the complex Lie subalgebra of $\mathfrak{g}$ generated by the following nonzero root vectors,

$$\{X_{\pm\epsilon_2}, X_{\pm\epsilon_1}, X_{\pm(\epsilon_3+\epsilon_4)}\}.$$ 

Then $\tilde{\mathfrak{g}}$ is a simple Lie algebra stable under $\sigma$ and $\theta$. Therefore $\tilde{\mathfrak{g}}$ is the complexification of the real subalgebra $\mathfrak{g}_o = \mathfrak{g}_o \cap \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{p}}$ is a Cartan decomposition of $\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$ and $\mathfrak{p} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$. Moreover $\mathfrak{h} = (\mathfrak{t} \cap \tilde{\mathfrak{g}}) \oplus \mathfrak{a}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$ and $\tilde{m} = m \cap \tilde{\mathfrak{f}}$ is the centralizer of $\mathfrak{a}$ in $\tilde{\mathfrak{f}}$. That $\mathfrak{g}_o \simeq \mathfrak{sp}(2,1)$ follows from the Dynkin-Satake diagram of $\mathfrak{g}_o$.

Since the root vectors $X_\mu$ and $\theta X_\mu$ are in $\tilde{\mathfrak{g}}$, it follows that $\tilde{\mathfrak{g}}$ is stable under the Cayley transform $\chi$ of the pair $(\mathfrak{g}, \mathfrak{h})$. Hence the restriction of $\chi$ to $\tilde{\mathfrak{g}}$ is the Cayley transform associated to $(\tilde{\mathfrak{g}}, \mathfrak{h})$. Then $\mathfrak{h}_\tilde{\mathfrak{f}} = \chi(\mathfrak{h}) = \mathfrak{h}_\tilde{\mathfrak{f}} \cap \tilde{\mathfrak{f}}$ is a Cartan subalgebra of $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$. The positive system $\Delta^+(\tilde{\mathfrak{f}}, \mathfrak{h}_\tilde{\mathfrak{f}}) = \{\tilde{\alpha}|_{\tilde{\mathfrak{f}}} \in \Delta(\tilde{\mathfrak{f}}, \mathfrak{h}_{\tilde{\mathfrak{f}}}) : \tilde{\alpha} \in \Delta^+(\tilde{\mathfrak{f}}, \mathfrak{h}_{\tilde{\mathfrak{f}}})\}$ in $\Delta(\tilde{\mathfrak{f}}, \mathfrak{h}_{\tilde{\mathfrak{f}}})$. Moreover,

$$\Pi(\tilde{\mathfrak{f}}, \mathfrak{h}_{\tilde{\mathfrak{f}}}) = \{\delta = \bar{e}_2 - \bar{e}_1, \gamma_1 = \frac{1}{2}(\bar{e}_1 + \bar{e}_2 - \bar{e}_3 - \bar{e}_4), \gamma_2 = \bar{e}_3 + \bar{e}_4\}$$

is a simple system in $\Delta^+(\tilde{\mathfrak{f}}, \mathfrak{h}_{\tilde{\mathfrak{f}}})$ and the corresponding Dynkin diagram is

```
 o     o
 δ    γ₁    γ₂
```

Then $\Delta^+(\tilde{\mathfrak{f}}, \mathfrak{h}_{\tilde{\mathfrak{f}}}) = \{\delta, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where $\gamma_3 = \gamma_1 + \gamma_2 = \frac{1}{2}(\bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4)$ and $\gamma_4 = 2\gamma_1 + \gamma_2 = \bar{e}_1 + \bar{e}_2$. Hence $\tilde{\mathfrak{f}} \simeq \mathfrak{sp}(1,\mathbb{C}) \times \mathfrak{sp}(2,\mathbb{C})$.

A simple calculation shows that $\chi(\theta X_{-\alpha_i}) = \frac{\sqrt{2}}{2}E$, thus $E$ is a root vector in $\tilde{\mathfrak{f}}^+$ corresponding to $\gamma_3$. Then set $X_{\gamma_3} = E$. Now define $\varphi_1 = \bar{e}_3 + \bar{e}_1$, $\varphi_2 = \bar{e}_4 + \bar{e}_1$ and $\varphi_3 = \bar{e}_4 - \bar{e}_1$. Then in view of (10) we have,

$$X_{\gamma_4} = \chi(X_{\epsilon_2+\epsilon_1}), \quad X_{\delta} = \chi(\theta X_{\epsilon_2+\epsilon_1}), \quad X_{\varphi_1} = \chi(X_{\epsilon_3+\epsilon_1}),$$

and

$$X_{\varphi_2} = \chi(X_{\epsilon_4+\epsilon_1}), \quad X_{\varphi_2} = \chi(X_{\epsilon_4+\epsilon_1}).$$

It follows from (12) that $X_{\gamma_4} - X_\delta$ and $X_{\varphi_1} - X_{\delta_1}$ are in $m^+$ for $i = 1, 2$.

Normalize $X_{-\gamma_4}, X_{-\delta}, X_{-\varphi_1}$ and $X_{-\delta_1}$ so that $\langle X_{\gamma_4}, X_{-\gamma_4} \rangle = \langle X_\delta, X_{-\delta} \rangle = \langle X_{\varphi_1}, X_{-\varphi_1} \rangle = \langle X_{\delta_1}, X_{-\delta_1} \rangle = 1$, for $i = 1, 2$. Then it follows that,

$$\langle X_{\gamma_4} - X_\delta, X_{-\gamma_4} + X_{-\delta} \rangle = \langle X_{\varphi_1} - X_{\delta_1}, X_{-\varphi_1} + X_{-\delta_1} \rangle = 0.$$ 

Hence, $X_{-\gamma_4} + X_{-\delta}$ and $X_{-\varphi_1} + X_{-\delta_1}$ ($i = 1, 2$) are in $(m^+)\perp$, the orthogonal complement of $m^+$ in $\mathfrak{f}$ with respect to the Killing form of $\mathfrak{f}$.

To simplify the notation set, $X_{\pm 1} = X_{\pm \gamma_1}, X_{\pm 2} = X_{\pm \gamma_2}, X_{\pm 3} = X_{\pm \gamma_3}$, and $X_{\pm 4} = X_{\pm \gamma_4}$. Let $H_1 \in [\mathfrak{f}_{\gamma_1}, \mathfrak{f}_{-\gamma_1}]$ be such that $\gamma_1(H_1) = 2$, and
normalize $X_1$ and $X_{-1}$ so that $\{H_1, X_1, X_{-1}\}$ is an $s$-triple. Next normalize $X_2$ and $X_4$ (and accordingly $X_δ$) so that
\[ [X_1, X_2] = E \quad \text{and} \quad [X_1, E] = X_4. \]
From this, and the fact that $γ_2(H_1) = -2$, it follows that
\[ [X_{-1}, E] = 2X_2 \quad \text{and} \quad [X_{-1}, X_4] = 2E. \]

Now choose $H_2 \in [κ_2, t_2]$ such that $γ_2(H_2) = 2$ and normalize $X_{-2}$ so that $\{H_2, X_2, X_{-2}\}$ is an $s$-triple. Since $[κ_2, t_2] \subset t$ and $γ_1(H_2) = -1$, if we define
\[ H = \frac{1}{2}H_2, \]
we obtain a vector $H \in t$ such that $H(E) = \frac{1}{2}E$. This vector $H$ is the one used in (5). Also, since $δ(H_2) = 0$, we have $[X_δ, H] = 0$.

As in the previous sections set $Z = Z_{α_2}$, $Y = Y_{α_1}$ and $Y = Y + H$. From Lemma 3.1 it follows that $E(Y) = \frac{3}{2}E$, hence $E(Y) = E$. Now, since
\[ (e_1 + e_2)(H_4) = 0, \]
we have $(e_1 + e_2)(Y) = -(e_1 + e_2)(Z) = -1$ because $(e_1 + e_2)|_α = 2α_1|_α$ and $δ_1(Z) = \frac{1}{2}$ (see Lemma 3.1). Then $X_δ(Y) = X_δ$, and therefore $X_δ(Y) = X_δ$.

5. The $M$-spherical $K$-modules

In this section we describe the main properties of the $K$-modules in the classes $Γ$ and $Γ_1$ (see (4)). In the following proposition we collect several results that will be very useful later on, and in Proposition 5.3 we will prove some important properties of the Kostant degree $d(u)$ for $u \in U(κ)^M$ that make use of these results.

**Proposition 5.1.** Let $G_α$ be locally isomorphic to $F_4$, and let $κ = h_κ ⊕ κ^+$ be the Borel subalgebra of $κ$ defined before. Then $m^+ ⊂ κ^+$ and $E$ is a root vector in $κ^+$. Moreover:

(i) For any $γ ∈ K$ let $ξ_γ$ denote its highest weight. Then, $γ ∈ Γ$ if and only if $ξ_γ = \frac{1}{2}(γ_4 + δ) + \ell_2γ$ with $k, ℓ \in N_κ$. In this context we write $γ = γ_{k, ℓ}$, $ξ_γ = ξ_{k, ℓ}$ and $V_{k, ℓ}$ for the corresponding representation space. Also we shall refer to any $v ∈ V_{k, ℓ}^M$ as an $M$-invariant element of type $(k, ℓ)$.

(ii) For any $γ_{k, ℓ} ∈ Γ$ we have $d(γ_{k, ℓ}) = k + 2ℓ$.

(iii) If $γ ∈ Γ$ we have $γ ∈ Γ_1$ if and only if $ξ_γ = ξ_{k, ℓ}$ with $k$ even.

(iv) For any $γ_{k, ℓ} ∈ Γ$ we have $X_δ^k E^ℓ(V_{k, ℓ}^M) = V_{k, ℓ}^{k+ℓ}$ and $X_δ^p E^q(V_{k, ℓ}^M) = \{0\}$ if and only if $p > k$ or $p + q > k + ℓ$.

For a proof of this proposition we refer the reader to [5]. The construction of the Borel subalgebra $κ$ is contained in Section 3 of [5] and the statements in (i), (ii) and (iv) follow from Proposition 4.4, Theorem 4.5 and Theorem 5.3 of [5], respectively. On the other hand (iii) is a well known general fact. We point out that some of these results where first established in [7], others
where proved in [2] and they were generalized in [5] to any real rank one semisimple Lie group.

The following proposition is the analogue of part (ii) of Proposition 3.11 of [3]. We omit its proof since, up to minor changes, is the same as that of Proposition 3.11.

**Proposition 5.2.** Let $G_o$ be locally isomorphic to $F_4$. Let $\gamma_{k,\ell} \in \Gamma$ and let $V_{k,\ell}$ be a $K$-module in the class $\gamma_{k,\ell}$. Then if $0 \neq v \in V_{k,\ell}^M$ the set

$$\{ X^k_{\delta} E^{\ell+j}(v) : 0 \leq j \leq k \}$$

is a basis of the irreducible $\{ H_1, X_1, X_{-1} \}$-module of dimension $k+1$ generated by any non trivial highest weight vector of $V_{k,\ell}$. Moreover, $X^k_{\delta} E^{\ell+j}(v)$ is a weight vector of weight $\xi_{k,\ell} - j\gamma_1$ and the following identities hold,

\begin{align*}
\tag{18} X_1 X^k_{\delta} E^{\ell+j}(v) &= \frac{(j+\ell)}{2} X^k_{\delta} E^{\ell+j-1}(v) \quad 0 \leq j \leq k, \\
\tag{19} X_{-1} X^k_{\delta} E^{\ell+j}(v) &= \frac{2(j+1)(k-j)}{\ell+j+1} X^k_{\delta} E^{\ell+j+1}(v) \quad 0 \leq j \leq k, \\
\tag{20} X^j_{-1}(u_{k,\ell}) &= 2^j j! \binom{k}{j} \binom{\ell+j}{\ell}^{-1} X^k_{\delta} E^{\ell+j}(v) \quad 0 \leq j \leq k,
\end{align*}

where $u_{k,\ell}$ is the highest weight vector $X^k_{\delta} E^{\ell}(v)$.

In the following proposition we prove some important properties of the Kostant degree $d(u)$ for $u \in U(\mathfrak{t})^M$. Even though we give the proof for $F_4$, since our argument relies heavily on Proposition 5.1, the same proof hold for the other real rank one groups, SO$(n,1)$, SU$(n,1)$ and Sp$(n,1)$, with the appropriate changes. These result will be used in Section 8.

**Proposition 5.3.** Let $G_o$ be locally isomorphic to $F_4$. If $u, v \in U(\mathfrak{t})^M$ are nonzero vectors, then

(a) $d(u+v) \leq \max\{d(u), d(v)\}$,
(b) $d(uv) = d(u) + d(v)$,
(c) $d(u) = 0$ if and only if $u \in U(\mathfrak{t})^K$.

**Proof.** The assertions (a) and (c) follow directly from the definition of the Kostant degree. We start the proof of (b) by showing that $d(uv) \leq d(u) + d(v)$ for any $0 \neq u, v \in U(\mathfrak{t})^M$. Let us begin by considering $u \in V_{r,s} \subset U(\mathfrak{t})^M$ and $v \in V_{r',s'} \subset U(\mathfrak{t})^M$ where $V_{r,s}$ and $V_{r',s'}$ are, respectively, irreducible finite dimensional $K$-modules in the classes $\gamma_{r,s}$ and $\gamma_{r',s'}$ of $\Gamma$. Then $u \otimes v \in (V_{r,s} \otimes V_{r',s'})^M$ and we decompose it as follows,

\begin{equation}
\tag{21} u \otimes v = \sum_{i,j} w_{i,j},
\end{equation}

where $w_{i,j} \neq 0$ is the $\gamma_{i,j}$-isotypic component of $u \otimes v$. We recall that if $\gamma_{i,j} \in \Gamma$ then its highest weight is $\xi_{i,j} = \frac{1}{2}(\gamma_4 + \delta) + j\gamma_3$ and $d(\gamma_{i,j}) = i + 2j$,
see Proposition 5.1. We will show that \(d(w_{i,j}) \leq d(u) + d(v)\) for any \(w_{i,j}\) that occurs in \(\varphi\).

In view of (9) a simple system of roots in \(\Delta^+(\mathfrak{t}, \mathfrak{h})\) is given by,

\[
\Pi(\mathfrak{t}, \mathfrak{h}) = \{\bar{\epsilon}_4 + \bar{\epsilon}_1, \bar{\epsilon}_3 - \bar{\epsilon}_4, \bar{\epsilon}_4 - \bar{\epsilon}_1, \gamma_1 = \frac{1}{2}(\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_3 - \bar{\epsilon}_4)\}.
\]

Then it follows that

\[
\gamma_4 + \delta = (\bar{\epsilon}_4 + \bar{\epsilon}_1) + 2(\bar{\epsilon}_3 - \bar{\epsilon}_4) + 3(\bar{\epsilon}_4 - \bar{\epsilon}_1) + 4 \gamma_1
\]

and

\[
\gamma_3 = (\bar{\epsilon}_4 + \bar{\epsilon}_1) + (\bar{\epsilon}_3 - \bar{\epsilon}_4) + (\bar{\epsilon}_4 - \bar{\epsilon}_1) + \gamma_1.
\]

If \(V_{i,j} \subset U(\mathfrak{t})^M\) occurs in the decomposition of \(V_{r,s} \otimes V_{r',s'}\) it is known (see [9]) that its highest weight \(\xi_{i,j} = \frac{i}{2}(\gamma_4 + \delta) + j \gamma_3\) is given by,

\[
\xi_{i,j} = \xi_{r+r'+s+s'} - \left[c_1(\bar{\epsilon}_4 + \bar{\epsilon}_1) + c_2(\bar{\epsilon}_3 - \bar{\epsilon}_4) + c_3(\bar{\epsilon}_4 - \bar{\epsilon}_1) + c_4 \gamma_1\right],
\]

where \(c_i \in \mathbb{N}_0\) for \(1 \leq i \leq 4\). Hence comparing the coefficients of the simple root \(\bar{\epsilon}_4 + \bar{\epsilon}_1\) in the left hand side and the right hand side of (23) it follows that

\[
\frac{i}{2} + j = \frac{r + r'}{2} + s + s' - c_1.
\]

Then, since \(c_1 \geq 0\), we have

\[
d(w_{i,j}) = r + r' + 2(s + s') - 2c_1 = d(u) + d(v) - 2c_1 \leq d(u) + d(v).
\]

Therefore, using the definition (3) and (21) it follows that,

\[
d(u \otimes v) = \max\{d(w_{i,j})\} \leq d(u) + d(v).
\]

Now, using that the map \(u \otimes v \in U(\mathfrak{t})^M \otimes U(\mathfrak{t})^M \to uv \in U(\mathfrak{t})^M\) is a \(K\)-homomorphism it follows that \(d(uv) \leq d(u) + d(v)\).

Now let \(u \in V_{r,s} \oplus \cdots \oplus V_{r,s} (m\text{ summands})\) and \(v \in V_{r',s'} \oplus \cdots \oplus V_{r',s'} (n\text{ summands})\), where \(V_{r,s}\) and \(V_{r',s'}\) are irreducible finite dimensional \(K\)-submodules of \(U(\mathfrak{t})^M\) as above. Write \(u = u_1 + \cdots + u_m\) with \(u_k \in V_{r,s} (1 \leq k \leq m)\) and \(v = v_1 + \cdots + v_n\) with \(v_\ell \in V_{r',s'} (1 \leq \ell \leq n)\). Then using above calculation we obtain,

\[
d(uv) = d\left(\sum_{k,\ell} u_kv_{\ell}\right) \leq \max\{d(u_kv_{\ell}) : 1 \leq k \leq m, 1 \leq \ell \leq n\}
\]

\[
\leq \max\{d(u_k) + d(v_\ell) : 1 \leq k \leq m, 1 \leq \ell \leq n\} = d(u) + d(v).
\]

Consider now \(u, v \in U(\mathfrak{t})^M\) such that \(d(u) = p\) and \(d(v) = q\). It follows from (3) that,

\[
u = \sum_{d(\gamma) \leq p} u_\gamma \quad \text{and} \quad v = \sum_{d(\gamma) \leq q} v_\gamma.
\]
where \( u_\gamma \) and \( v_\tau \) denote, respectively, the \( K \)-isotypic components of \( u \) and \( v \) corresponding to the classes \( \gamma \) and \( \tau \) of \( \Gamma_1 \). Then using (24) we obtain,

\[
\begin{align*}
d(uv) &= d \left( \sum_{\gamma, \tau} u_\gamma v_\tau \right) \\
&\leq \max \{ d(u_\gamma v_\tau) : u_\gamma \neq 0, v_\tau \neq 0 \} \\
&\leq \max \{ d(u_\gamma) + d(v_\tau) : u_\gamma \neq 0, v_\tau \neq 0 \} \\
&= \max \{ d(\gamma) + d(\tau) : u_\gamma \neq 0, v_\tau \neq 0 \} \\
&\leq p + q = d(u) + d(v).
\end{align*}
\]

Our next goal is to show that \( d(uv) = d(u) + d(v) \) for any \( u, v \in U(\mathfrak{t})^M \).

Assume that \( d(u) = p \) and \( d(v) = q \). Then, using (25) and the fact that \( d(uv) \leq d(u) + d(v) \) for any \( u, v \in U(\mathfrak{t})^M \) it follows that,

\[
uv = \sum_{d(\gamma) = p, d(\tau) = q} u_\gamma v_\tau + w,
\]

where \( w \in U(\mathfrak{t})^M \) is such that \( d(w) < p + q \). Then, in view of (3) we may assume that

\[
(26) \quad u = \sum_{i + 2j = p} u_{i,j} \quad \text{and} \quad v = \sum_{r + 2s = q} v_{r,s},
\]

where \( u_{i,j} \) and \( v_{r,s} \) denote, respectively, the \( K \)-isotypic components of \( u \) and \( v \) corresponding to the classes \( \gamma_{i,j} \) and \( \gamma_{r,s} \) of \( \Gamma_1 \). Let \( k = \max \{ i \in \mathbb{N}_0 : u_{i,j} \neq 0 \text{ for some } j \} \) and \( \ell = \max \{ r \in \mathbb{N}_0 : v_{r,s} \neq 0 \text{ for some } s \} \). Then using (26), Leibnitz rule and part (iv) of Proposition 5.1 it follows that,

\[
\begin{align*}
&\hat{E}^{(p+q-k-\ell)/2} \hat{X}^{k+\ell}(uv) \\
&= \binom{k + \ell}{\ell} \left( \frac{p+q-k-\ell}{q-\ell} \right) \hat{E}^{(p-k)/2} \hat{X}^{k}(u_{k,\frac{p-k}{2}}) \hat{E}^{(q-\ell)/2} \hat{X}^{\ell}(v_{\ell,\frac{q-\ell}{2}}) \neq 0.
\end{align*}
\]

We point out that the right hand side of (27) is different from zero because, in view of (iv) of Proposition 5.1 it is a product of two dominant vectors. Also using Leibnitz rule, Proposition 5.1 (iv) and (27) it follows that,

\[
\hat{X}^{k+\ell}(uv) = \binom{k + \ell}{\ell} \hat{X}^{k}(u_{k,\frac{p-k}{2}}) \hat{X}^{\ell}(v_{\ell,\frac{q-\ell}{2}}) \neq 0,
\]

and

\[
\hat{X}^{k+\ell+1}(uv) = 0.
\]

To finish the proof write

\[
uv = \sum_{i,j} b_{i,j},
\]

where \( b_{i,j} \) denote the \( K \)-isotypic components of \( uv \) corresponding, respectively, to the classes \( \gamma_{i,j} \in \Gamma_1 \). Then from (27), (28) and (29) we obtain,

\[
\hat{X}^{k+\ell}(uv) = \sum_{j} \hat{X}^{k+\ell}(b_{k+\ell,j})
\]
and

\[ 0 \neq \sum_j E^{(p+q-k-\ell)/2} X_{\bar{d}}^{k+\ell}(b_{k+\ell,j}). \]

Therefore, from Proposition 5.1 (iv) it follows that there exists \( b_{k+\ell,j} \neq 0 \) such that \((p+q-k-\ell)/2 + k + \ell \leq k + \ell + j\). Thus

\[ d(uv) \leq d(u) + d(v) = p + q \leq k + \ell + 2j = d(b_{k+\ell,j}) \leq d(uv). \]

This completes the proof of the proposition. \( \square \)

6. Transversality results

In this section we prove several results that will allow us to deal with the congruence modulo \( U(\mathfrak{t})m^+ \) that occur in the equations that define the algebra \( B \) (see (1)). In particular, we reduce the congruence modulo \( U(\mathfrak{t})m^+ \) to a congruence modulo \( U(\mathfrak{t})\eta \), where \( \eta \subset m^+ \) is the abelian subalgebra defined as follows

\( \eta = \langle \{X_{\bar{c}_3+\bar{c}_4}, X_{\bar{c}_2+\bar{c}_3}, X_{\bar{c}_1+\bar{c}_2}\} \rangle. \)

Before stating the main results we introduce the following notation,

\( S_{23} = X_{\bar{c}_2+\bar{c}_3}, \quad S_{24} = X_{\bar{c}_2+\bar{c}_4}, \quad \text{and} \quad T_{ij} = X_{\bar{c}_i-\bar{c}_j} \quad (2 \leq i \neq j \leq 4). \)

Let \( q^+ \) be the linear span of \( \{X_\alpha : \alpha \in \Delta^+(\mathfrak{t}, \mathfrak{h}_\ell) \text{ and } \alpha \neq \gamma_1\} \). Since \( \gamma_1 \) is a simple root in \( \Delta^+(\mathfrak{t}, \mathfrak{h}_1) \) (see (22)) it follows that \( q^+ \) is a subalgebra of \( \mathfrak{t}^+ \). We are interested in considering weight vectors \( u \in U(\mathfrak{t})m^+ \) of weight \( \lambda = a(\gamma_4 + \delta) + b\gamma_3 \quad (a,b \in \mathbb{Z}) \), and such that \( \hat{X}(u) \equiv 0 \mod (U(\mathfrak{t})\eta) \) for every \( X \in q^+ \).

Consider the subalgebra \( q \subset \mathfrak{t} \) defined as follows

\[ q = q^+ \oplus \mathfrak{h}_\ell \oplus q^- , \]

where

\[ \mathfrak{h}_\ell = \ker(\gamma_4 + \delta) \cap \ker(\gamma_3) = \langle \{H_{\bar{c}_3-\bar{c}_4}, H_{\bar{c}_4-\bar{c}_3}\} \rangle \]

and

\[ q^- = \langle \{X_{-(\bar{c}_3-\bar{c}_4)}\} \rangle. \]

Then a simple calculation shows that,

\[ [q, \eta] \subset \eta. \]

Moreover, \( q = r \oplus u \) where \( r = \langle \mathfrak{h}_\ell \cup \{X_{\pm(\bar{c}_3-\bar{c}_4)}\} \rangle \simeq \mathfrak{gl}(2, \mathbb{C}), \mathfrak{h}_\ell \) is a Cartan subalgebra of \( r \) and \( u \) is the following nilpotent subalgebra,

\[ u = \langle \{X_{\bar{c}_3 \pm \bar{c}_j} : 3 \leq j \leq 4\} \cup \{X_{\bar{c}_1 \pm \bar{c}_i} : 2 \leq i \leq 4\} \cup \{X_{\gamma_2}, X_{\gamma_3}, X_{\psi_1}, X_{\psi_2}\} \rangle, \]

where

\[ \psi_1 = \frac{1}{2}(-\bar{c}_1 + \bar{c}_2 - \bar{c}_3 + \bar{c}_4), \quad \psi_2 = \frac{1}{2}(-\bar{c}_1 + \bar{c}_2 + \bar{c}_3 - \bar{c}_4). \]

The proof of the next two lemmas follow from a direct application of Poincaré-Birkhoff-Witt theorem. Let \( \mathfrak{g} \) be an arbitrary finite dimensional complex Lie algebra and let \( \mathfrak{l} \) be a subalgebra of \( \mathfrak{g} \). If \( \{X_1, \ldots, X_p\} \) is an
ordered basis of \( \mathfrak{l} \) complete it to an ordered basis \( \{Y_1, \ldots, Y_q, X_1, \ldots, X_p\} \) of \( \mathfrak{g} \). Now, if \( I = (i_1, \ldots, i_q) \in \mathbb{N}_0^q \) and \( J = (j_1, \ldots, j_p) \in \mathbb{N}_0^p \) define as usual \( Y^I X^J = Y_{i_1}^1 \cdots Y_{i_q}^q X_{j_1}^1 \cdots X_{j_p}^p \) in \( U(\mathfrak{g}) \). Then we have,

**Lemma 6.1.** Any \( u \in U(\mathfrak{g}) \) can be written in a unique way as \( u = a_1 X_1 + \cdots + a_p X_p \) where

\[
a_k = \sum a_{I,j_1,\ldots,j_k} Y^I X_1^{j_1} \cdots X_k^{j_k} \quad \text{for} \quad k = 1, \ldots, p,
\]

and the coefficients \( a_{I,j_1,\ldots,j_k} \) are complex numbers.

**Lemma 6.2.** Let \( \mathfrak{g} \) and \( \mathfrak{l} \) be as above. Let \( u \in U(\mathfrak{g}) \) and \( X \in \mathfrak{g} - \mathfrak{l} \) be such that \( X(\mathfrak{l}) \subset \mathfrak{l} \). If \( uX^n \equiv 0 \mod (U(\mathfrak{g}) \mathfrak{l}) \) for some \( n \in \mathbb{N} \), then \( u \equiv 0 \mod (U(\mathfrak{g}) \mathfrak{l}) \).

Let \( \mathfrak{h}^\perp \) be the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{k} \) with respect to the Killing form of \( \mathfrak{k} \). For any \( Z \in (\mathfrak{m}^+)\perp \) consider the linear map \( T_Z : \mathfrak{q} \times (\mathfrak{m}^+)\perp \to \mathfrak{h}^\perp \) given by

\[
T_Z(X,Y) = [X,Z] + Y, \quad X \in \mathfrak{q} \quad \text{and} \quad Y \in (\mathfrak{m}^+)\perp.
\]

Since \( [\mathfrak{q},\mathfrak{h}] \subset \mathfrak{h} \) and \( (\mathfrak{m}^+)\perp \subset \mathfrak{h}^\perp \) it follows that \( \text{Im}(T_Z) \subset \mathfrak{h}^\perp \), where \( \text{Im}(T_Z) \) denotes the image of the map \( T_Z \). The following proposition will be used in Theorem 6.4 to prove one of the main results of this section.

**Proposition 6.3.** There exists \( Z_o \in (\mathfrak{m}^+)\perp \) such that \( \text{Im}(T_{Z_o}) = \mathfrak{h}^\perp \).

**Proof.** Using 13 and the notation introduced in 14, 15 and 31 it is easy to check that,

\[
\mathfrak{h}^\perp = (\mathfrak{m}^+)\perp \oplus \langle \{X_{-\delta}, X_{-\delta_1}, X_{-\delta_2}, T_{32}, T_{42}, T_{43}\} \rangle.
\]

It is clear, from the definition of \( T_Z \), that \( (\mathfrak{m}^+)\perp \subset \text{Im}(T_Z) \) for every \( Z \in (\mathfrak{m}^+)\perp \). Now consider the vector,

\[
Z_o = X_{-\gamma_4} + X_{-\delta} + X_{-\phi_2} + X_{-\delta_2} + X_{-\gamma_3} + H_{\bar{\epsilon}_4 - \bar{\epsilon}_3},
\]

where \( H_{\bar{\epsilon}_4 - \bar{\epsilon}_3} \in \mathfrak{h} \) is such that \( (\bar{\epsilon}_4 - \bar{\epsilon}_3)(H_{\bar{\epsilon}_4 - \bar{\epsilon}_3}) = 2 \). Using 13 and 16 it follows that \( Z_o \in (\mathfrak{m}^+)\perp \). In view of 37, to prove that \( \text{Im}(T_{Z_o}) = \mathfrak{h}^\perp \) we need to show that \( \langle \{X_{-\delta}, X_{-\delta_1}, X_{-\delta_2}, T_{32}, T_{42}, T_{43}\} \rangle \) is contained in \( \text{Im}(T_{Z_o}) \). In fact, using that \( X_{\varphi_1}, X_{\varphi_2}, X_{\psi_1}, X_{\psi_2}, H_{\bar{\epsilon}_4 - \bar{\epsilon}_3} \) and \( T_{43} \) are in \( \mathfrak{h} \) (see 14, 15, 31 and 35 for the notation) a simple calculation shows that,

\[
T_{Z_o}(X_{\varphi_2},0) \equiv c_1 T_{42}, \quad T_{Z_o}(X_{\varphi_1},0) \equiv c_2 T_{32},
\]

\[
T_{Z_o}(X_{\psi_2},0) \equiv c_3 X_{-\delta_2}, \quad T_{Z_o}(X_{\psi_1},0) \equiv c_4 X_{-\delta_1},
\]

\[
T_{Z_o}(H_{\bar{\epsilon}_4 - \bar{\epsilon}_3},0) \equiv c_5 X_{-\delta}, \quad T_{Z_o}(T_{43},0) \equiv c_6 T_{43},
\]

where, in all cases, the congruence is modulo the subspace \((\mathfrak{m}^+)\perp \) and \( c_i \neq 0 \) for \( 1 \leq i \leq 6 \). This completes the proof of the proposition. \( \square \)
Theorem 6.4. Let \( u \in U(\mathfrak{k})m^+ \) be a vector of weight \( \lambda = a(\gamma_4 + \delta) + b\gamma_3 \), with \( a, b \in \mathbb{Z} \), and such that \( X(u) \equiv 0 \mod (U(\mathfrak{k})\eta) \) for every \( X \in q^+ \). Then \( u \equiv 0 \mod (U(\mathfrak{k})\eta) \).

Proof. Let \( U(\mathfrak{k}) = \bigcup_{j \geq 0} U_j(\mathfrak{k}) \) be the canonical ascending filtration of \( U(\mathfrak{k}) \). If \( v \in U(\mathfrak{k}) \) and \( v \neq 0 \) define,

\[
\deg(v) = \min \{ j : v \in U_j(\mathfrak{k}) \text{ and } v \notin U_{j-1}(\mathfrak{k}) \},
\]

where it is understood that \( U_{-1}(\mathfrak{k}) = \{0\} \). Let \( S \) be the set of all \( v \in U(\mathfrak{k})m^+ \) of weight \( \lambda = a(\gamma_4 + \delta) + b\gamma_3 \) (\( a, b \in \mathbb{Z} \)), so that \( X(v) \in U(\mathfrak{k})\eta \) for every \( X \in q^+ \) and \( v \notin U(\mathfrak{k})\eta \). The theorem will be proved if we show that \( S = \emptyset \). Assume on the contrary that \( S \neq \emptyset \) and choose \( u \in S \) such that \( \deg(u) = \min \{ \deg(v) : v \in S \} \). Set \( r = \deg(u) \) and let \( p_r : U_r(\mathfrak{k}) \to U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k}) \) denote the quotient map. The map \( p_r \) intertwines the representations of \( K \) on \( U_r(\mathfrak{k}) \) and on \( U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k}) \), and since \( u \notin U_{r-1}(\mathfrak{k}) \) we have \( p_r(u) \neq 0 \).

Let \( S(\mathfrak{t}) \) be the symmetric algebra of \( \mathfrak{t} \) and let \( S(\mathfrak{t}^*) \) denote the algebra of polynomial functions on \( \mathfrak{t} \). Let \( S_r(\mathfrak{t}) \) and \( S_r(\mathfrak{t}^*) \) denote the corresponding homogeneous subspaces of \( S(\mathfrak{t}) \) and \( S(\mathfrak{t}^*) \) of degree \( r \). There is an algebra isomorphism between \( S(\mathfrak{t}) \) and \( S(\mathfrak{t}^*) \) defined by the Killing form of \( \mathfrak{t} \); this isomorphism maps \( S_r(\mathfrak{t}) \) onto \( S_r(\mathfrak{t}^*) \) and intertwines the canonical representations of \( K \) on \( S_r(\mathfrak{t}) \) and on \( S_r(\mathfrak{t}^*) \). Composing this isomorphism with the natural \( K \)-isomorphism between \( U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k}) \) and \( S_r(\mathfrak{k}) \) we obtain a \( K \)-isomorphism,

\[
U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k}) \simeq S_r(\mathfrak{t}^*).
\]

Hence we can think of \( p_r(u) \) as a homogeneous polynomial function on \( \mathfrak{t} \) of degree \( r \), and regard \( p_r \) as a \( K \)-homomorphism from \( U_r(\mathfrak{k}) \) to \( S_r(\mathfrak{t}^*) \).

Let \( (m^+)^\perp \) be the orthogonal complement of \( m^+ \) in \( \mathfrak{t} \) with respect to the Killing form of \( \mathfrak{t} \). Since \( u \in U(\mathfrak{k})m^+ \) and the isomorphism given in (40) is defined by the Killing form of \( \mathfrak{t} \) it follows that,

\[
p_r(u)(Y) = 0 \quad \text{for every} \quad Y \in (m^+)^\perp.
\]

Now let \( X \in q^+ \). Since \([q^+, \eta] \subset \eta \) we have \( X^k(U(\mathfrak{k})\eta) \subset U(\mathfrak{k})\eta \) for every \( k \in \mathbb{N} \). Then, since by hypothesis \( \dot{X}(u) \in U(\mathfrak{k})\eta \), it follows that \( \dot{X}^k(u) \in U(\mathfrak{k})\eta \) for any \( k \in \mathbb{N} \). Therefore, using that \( (m^+)^\perp \subset \eta^\perp \) and that \( p_r \) is a \( K \)-homomorphism it follows by induction on \( k \) that

\[
X^k(p_r(u))(Y) = p_r(\dot{X}^k(u))(Y) = 0 \quad \text{for} \quad Y \in (m^+)^\perp \quad \text{and} \quad X \in q^+,
\]

where \( X(p_r(u)) \) denotes the action of \( X \) on the polynomial function \( p_r(u) \).

Since \( u \) is a vector of weight \( \lambda = a(\gamma_4 + \delta) + b\gamma_3 \), it follows from the definition of \( \mathfrak{h}_t \) that \( \dot{H}(u) = 0 \) for every \( H \in \mathfrak{h}_t \). Then,

\[
H^k(p_r(u))(Y) = 0 \quad \text{for} \quad Y \in \mathfrak{t}, \ H \in \mathfrak{h}_t \quad \text{and} \quad k \in \mathbb{N}.
\]

Let \( 0 \neq \overline{u} \in U(\mathfrak{k})/U(\mathfrak{k})\eta \) be the image of \( u \) under the quotient map. Normalize \( X_{\overline{e}_3-\overline{e}_4} \) and \( X_{-(\overline{e}_3-\overline{e}_4)} \) so that \( \{X_{\overline{e}_3-\overline{e}_4}, H_{\overline{e}_3-\overline{e}_4}, X_{-(\overline{e}_3-\overline{e}_4)} \} \) is an
\( s \)-triple. Since \( X_{\tilde{c}_1 - \tilde{c}_4} \in q^+ \) and \( H_{\tilde{c}_1 - \tilde{c}_4} \in \mathfrak{h}_t \), and by hypothesis \( \pi \) is a dominant vector of weight zero with respect to above \( s \)-triple, we obtain that \( \dot{X}_{-(\tilde{c}_1 - \tilde{c}_4)}(\pi) = 0 \). Hence, from (34) we obtain that \( \dot{X}(u) \in U(\mathfrak{t})\eta \) for \( X \in q^- \). Then, since \( [q^-, \eta] \subset \eta \), it follows that

\[
(44) \quad X^k(p_r(u))(Y) = 0 \quad \text{for} \quad Y \in (m^+)_{\perp}, \quad X \in q^- \quad \text{and} \quad k \in \mathbb{N}.
\]

Now recall that for \( k \in K \) and \( f \in S_r(\mathfrak{t}^*) \) the action of \( k \) on \( f \) is given by \( (kf)(Y) = f(Ad(k^{-1})Y) \) for every \( Y \in \mathfrak{t} \). Then, from (41), (42), (43) and (44) it follows that

\[
(45) \quad p_r(u)(Ad(exp X)Y) = 0 \quad \text{for} \quad X \in q^+ \cup \mathfrak{h}_t \cup q^- \quad \text{and} \quad Y \in (m^+)_{\perp}.
\]

Let \( Q \) be the connected Lie subgroup of \( K \) with Lie algebra \( q \) (see (32)). Since the set \( \exp q^+ \cdot \exp \mathfrak{h}_t \cdot \exp q^- \) generates \( Q \) we obtain that,

\[
(46) \quad p_r(u)(Ad(g)Y) = 0 \quad \text{for} \quad g \in Q \quad \text{and} \quad Y \in (m^+)_{\perp}.
\]

Now consider the map \( \Phi : Q \times (m^+)_{\perp} \to \eta^\perp \) defined by \( \Phi(g, Y) = Ad(g)Y \). The fact that the image of \( \Phi \) is contained in \( \eta^\perp \) follows from a simple calculation using that \( [q, \eta] \subset \eta \) and that \( \eta \subset m^+ \). Let \( e \in Q \) be the identity element and \( Z \in (m^+)\) then \( (d\Phi)(e, Z) \) is the map \( T_Z : q \times (m^+)_{\perp} \to \eta^\perp \) defined in (30). It follows from Proposition (43) that \( (d\Phi)(e, Z_o) \) is surjective. This implies that the image of \( \Phi \) contains an open set of \( \eta^\perp \), then in view of (46) we obtain that,

\[
(47) \quad p_r(u)(Y) = 0 \quad \text{for every} \quad Y \in \eta^\perp.
\]

Recall that \( \eta = \langle \{X_2, S_{23}, S_{24}\} \rangle \) (see (30)). Extend the basis of \( \eta \) to a basis \( B = \{Z_1, \ldots, Z_q, X_2, S_{23}, S_{24}\} \) of \( \mathfrak{t} \), where \( q = \dim \mathfrak{t} - 3 \). If \( I = (i_1, \ldots, i_q) \in \mathbb{N}^q_0 \) and \( J = (j_1, j_2, j_3) \in \mathbb{N}^3_0 \), set \( |I| = i_1 + \cdots + i_q \), \( |J| = j_1 + j_2 + j_3 \) and \( Z^I = Z_1^{i_1} \cdots Z_q^{i_q} \) in \( S(\mathfrak{t}) \). If we regard \( p_r(u) \) as an element in \( S_r(\mathfrak{t}) \) we can write

\[
p_r(u) = \sum_{I, J} b_{I, J} Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3},
\]

where \( b_{I, J} \in \mathbb{C} \) and the sum extends over all \( I \) and \( J \) such that \( |I| + |J| = r \). Now, identifying \( \mathfrak{t}^* \) with \( \mathfrak{t} \) via the Killing form of \( \mathfrak{t} \) and considering a basis \( \mathcal{B} \) of \( \mathfrak{t} \) dual to \( \mathcal{B} \) it follows from (47) that \( b_{I, 0} = 0 \), for all \( I \) such that \( |I| = r \). Therefore

\[
(48) \quad p_r(u) = \sum_{|J| > 0} b_{I, J} Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3},
\]

where the sum extends over all \( I \) and \( J \) such that \( |I| + |J| = r \). On the other hand, since \( p_r \) is a \( K \)-homomorphism from \( U_r(\mathfrak{t}) \) to \( S_r(\mathfrak{t}) \) it follows that \( p_r(u) \) has weight \( \lambda = a(\gamma_4 + \delta) + b_\gamma \) with respect to \( \mathfrak{h}_t \). Then, (48) implies that

\[
(49) \quad u = \sum_{|J| > 0} b_{I, J} Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3} + u',
\]
where the monomials \( Z^{I}X_{2}^{J}S_{2}^{k}b_{2}^{j} \) are in \( U(\mathfrak{t}) \), the sum extends over all \( I \) and \( J \) such that \(|I| + |J| = r \) and \( u' \) is a vector of weight \( \lambda \) in \( U_{r-1}(\mathfrak{t}) \). Moreover, since the sum in the first term of (49) is a vector in \( U(\mathfrak{t})\eta \) and \( X(U(\mathfrak{t})\eta) \subset U(\mathfrak{t})\eta \) for \( X \in \mathfrak{q}^{+} \), it follows by hypothesis that \( X(u') \in U(\mathfrak{t})\eta \) for every \( X \in \mathfrak{q}^{+} \). Also, since \( u \in U(\mathfrak{t})\mathfrak{m}^{+} \) and \( u \notin U(\mathfrak{t})\eta \) the same facts hold for \( u' \), therefore \( u' \in \mathcal{S} \). This is a contradiction since \( \deg(u') < \deg(u) \). Then \( \mathcal{S} = \emptyset \) and the proof of the theorem is completed. \( \square \)

**Corollary 6.5.** Let \( u \in U(\mathfrak{t})\mathfrak{m}^{+} \) be a \( \mathfrak{q}^{+} \)-dominant vector of weight \( \lambda = a(\gamma_4 + \delta) + b\gamma_3 \) with \( a, b \in \mathbb{Z} \). Then \( u \in U(\mathfrak{t})\eta \).

Next theorem will be used in an important way in Section 8. Its proof is similar to that of Theorem 6.4. Consider the following subalgebra of \( \mathfrak{g} \):

\[
\tilde{\mathfrak{q}} = \{ X \in \mathfrak{t} : \dot{X}(V^{+}_\gamma) = 0 \text{ for every } \gamma \in \Gamma_1 \}.
\]

It is easy to see that,

\[
\tilde{\mathfrak{q}} = \mathfrak{t}^{+} \oplus \mathfrak{h}_t \oplus \langle \{ X_{-\tilde{\gamma}_1 + \tilde{\gamma}_4}, X_{-\tilde{\gamma}_4 + \tilde{\gamma}_1}, X_{-\tilde{\gamma}_3 + \tilde{\gamma}_1} \} \rangle,
\]

where \( \mathfrak{h}_t \) is as in (33). Let \( \tilde{Q} \) denote the connected Lie subgroup of \( K \) with Lie algebra \( \tilde{\mathfrak{q}} \).

If \( Z \in (\mathfrak{m}^{+})^\perp \) consider the linear map \( \tilde{T}_Z : \tilde{\mathfrak{q}} \times (\mathfrak{m}^{+})^\perp \to \mathfrak{t} \) given by

\[
\tilde{T}_Z(X,Y) = [X,Z] + Y, \quad X \in \tilde{\mathfrak{q}} \text{ and } Y \in (\mathfrak{m}^{+})^\perp.
\]

Next proposition is the analogue of Proposition 6.3 and will be used in the proof of Theorem 6.7.

**Proposition 6.6.** If \( Z_0 \in (\mathfrak{m}^{+})^\perp \) is as in (38) it follows that \( \text{Im}(\tilde{T}_{Z_0}) = \mathfrak{t} \).

**Proof.** Using the definition of \( \mathfrak{h} \) (see (33)) it is easy to see that,

\[
\mathfrak{t} = \mathfrak{h}^\perp \oplus \langle \{ X_{-\tilde{\gamma}_2 - \tilde{\gamma}_3}, X_{-\tilde{\gamma}_2 - \tilde{\gamma}_1}, X_{-\tilde{\gamma}_3 - \tilde{\gamma}_1} \} \rangle.
\]

Now, since \( \mathfrak{q} \subset \tilde{\mathfrak{q}} \) it follows from Proposition 6.3 that,

\[
\tilde{T}_{Z_0}(q \times (\mathfrak{m}^{+})^\perp) = Z_0(q \times (\mathfrak{m}^{+})^\perp) = \mathfrak{h}^\perp.
\]

Hence, it follows from (52) that to complete the proof we need to show that \( X_{-\tilde{\gamma}_2 - \tilde{\gamma}_3}, X_{-\tilde{\gamma}_2 - \tilde{\gamma}_1}, \) and \( X_{-\tilde{\gamma}_3 - \tilde{\gamma}_1} \) are in the image of \( \tilde{T}_{Z_0} \). In fact, a simple calculation shows that,

\[
\tilde{T}_{Z_0}(X_{-\tilde{\gamma}_4 + \tilde{\gamma}_1}, 0) \equiv a_1 X_{-\tilde{\gamma}_2 - \tilde{\gamma}_3}, \quad \tilde{T}_{Z_0}(X_{\gamma_1}, 0) \equiv a_2 X_{-\tilde{\gamma}_3 - \tilde{\gamma}_1},
\]

and

\[
\tilde{T}_{Z_0}(X_{-\tilde{\gamma}_3 + \tilde{\gamma}_1}, 0) \equiv a_3 X_{-\tilde{\gamma}_2 - \tilde{\gamma}_3} + a_4 X_{-\tilde{\gamma}_3 - \tilde{\gamma}_1},
\]

where, in all cases, the congruence is module the subspace \( \mathfrak{h}^\perp \) and the constants \( a_i \) are nonzero for \( 1 \leq i \leq 4 \). This completes the proof. \( \square \)

**Theorem 6.7.** Let \( u \in U(\mathfrak{t})\mathfrak{m}^{+} \) be a \( \mathfrak{t}^{+} \)-dominant vector of weight \( \lambda = a(\gamma_4 + \delta) + b\gamma_3 \) with \( a, b \in \mathbb{N}_0 \). Then \( u = 0 \).
Proof. Let $U(\mathfrak{g}) = \bigcup_{j \geq 0} U_j(\mathfrak{g})$ and let $u \in U(\mathfrak{g})\mathfrak{m}^+$ be a $\mathfrak{m}^+$-dominant vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ with $a, b \in \mathbb{N}_0$. Assume that $u \neq 0$ and set $r = \deg(u)$ (see (32)). Let $p_r : U_r(\mathfrak{g}) \to S_r(\mathfrak{g}^*)$ be the $K$-homomorphism defined in the proof of Theorem 6.4. Observe that $p_r(u) \neq 0$ because $u \notin U_{r-1}(\mathfrak{g})$.

Since $u \in U(\mathfrak{g})\mathfrak{m}^+$, and the $K$-homomorphism $p_r : U_r(\mathfrak{g}) \to S_r(\mathfrak{g}^*)$ is defined via the Killing form of $\mathfrak{g}$, it follows that

$$p_r(u)(Y) = 0 \quad \text{for every} \quad Y \in (\mathfrak{m}^+)^{\perp}.$$  

Also, since $u$ is a $\mathfrak{m}^+$-dominant vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$, it follows from Proposition 5.1 that $u \in \mathfrak{V}_\lambda^{\mathfrak{m}^+}$ for $\gamma \in \Gamma_1$ with highest weight $\lambda$. Hence $X(u) = 0$ for every $X \in \mathfrak{q}$. Then since $p_r$ is a $K$-homomorphism we have

$$X^k(p_r(u))(Y) = p_r(X^k)(Y) = 0 \quad \text{for} \quad X \in \mathfrak{q}, \quad Y \in \mathfrak{g} \quad \text{and} \quad k \in \mathbb{N}.$$  

Now, since $\{\exp X : X \in \mathfrak{q}\}$ generates $\mathfrak{q}$, it follows from (55) and (56) that

$$p_r(u)(Ad(g)Y) = 0 \quad \text{for} \quad g \in \tilde{Q} \quad \text{and} \quad Y \in (\mathfrak{m}^+)^{\perp}.$$  

That is, $p_r(u)$ vanishes on the image of the map $\tilde{\Phi} : \tilde{Q} \times (\mathfrak{m}^+)^{\perp} \to \mathfrak{g}$ defined by $\tilde{\Phi}(g, Y) = Ad(g)Y$. Now, if $e \in \tilde{Q}$ is the identity element and $Z \in (\mathfrak{m}^+)^{\perp}$, then $(d\tilde{\Phi})(e, Z) = \tilde{T} : \mathfrak{q} \times (\mathfrak{m}^+)^{\perp} \to \mathfrak{g}$. Then it follows from Proposition 6.6 that $(d\tilde{\Phi})(e, Z_0)$ is surjective. This implies that the image of $\tilde{\Phi}$ contains an open set of $\mathfrak{g}$, hence $p_r(u) = 0$ as a polynomial function on $\mathfrak{g}$, which is a contradiction. Therefore $u = 0$ as we wanted to prove.  

Before stating the next results we define the following subalgebra of $\mathfrak{g}$,

$$\mathfrak{s} = \mathfrak{t}^- \oplus \mathfrak{h}_\mathfrak{g} \oplus \langle \{X_1^{\epsilon_3 + \epsilon_4}, X_1^{\epsilon_4 + \epsilon_5}, T_{34}, T_1, X_1\} \rangle.$$  

The following result is the analogue of Proposition 4.9 of [3]. Although its proof uses the same idea as that of Proposition 4.9 we include it here because of some technical differences.

**Proposition 6.8.** Let $u_0, u_1 \in U(\mathfrak{g})$ be such that $X_1(u_0) = X_1(u_1) = 0$. If $u_0 + u_1 E \equiv 0 \mod (U(\mathfrak{g})\eta)$ then $u_0 \equiv u_1 \equiv 0 \mod (U(\mathfrak{g})\eta)$.

**Proof.** Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{g}$ defined in (57). If $\{S_1, \ldots, S_t\}$ is an ordered basis of $\mathfrak{s}$, the following is an ordered basis for $\mathfrak{g}$

$$\{S_1, \ldots, S_t, T_{23}, T_{24}, X_\delta, X_{\psi_2}, X_{\delta_1}, X_{\psi_1}, X_{\delta_2}, X_4, X_3, X_2, S_{23}, S_{24}\},$$  

we refer the reader to (14), (15), (31) and (35) for the notation.

Let $U_1$ (respectively $U_2$) be the subspace of $U(\mathfrak{g})$ spanned by those monomials that, when written in the Poincaré-Birkhoff-Witt bases of $U(\mathfrak{g})$ associated to (58), end with powers of $X_2$ (respectively $S_{23}$) or before. Using that $X_1(\mathfrak{s}) \subset \mathfrak{s}$ and taking a close look at the action of $X_1$ on the other elements of the basis (58) it follows that $X_1(U_1) \subset U_1$ and $X_1(U_2) \subset U_2$.

Since $u_0 + u_1 E \in U(\mathfrak{g})\eta$ in view of Lemma 6.1 we can write

$$u_0 + u_1 E = aX_2 + bS_{23} + cS_{24},$$  

Proposition 6.9. Let \( u \in U_1 \), \( b \in U_2 \) and \( c \in U(\mathfrak{k}) \). Then applying \( \hat{X}_1 \) we obtain that,
\[
\tag{60} u_1 X_4 = \hat{X}_1(a) X_2 + a E + \hat{X}_1(b) S_{23} + \hat{X}_1(c) S_{24},
\]
and for every \( k \geq 2 \) we get,
\[
\tag{61} 0 = \hat{X}_1^k(a) X_2 + k \hat{X}_1^{k-1}(a) E + \binom{k}{2} \hat{X}_1^{k-2}(a) X_4 + \hat{X}_1(b) S_{23} + \hat{X}_1(c) S_{24}.
\]

Now set \( \mathfrak{n} = \langle \{ S_{23}, S_{24} \} \rangle \). If \( n \) is sufficiently large so that \( \hat{X}_1^n(a) = 0 \), using equation (61) and decreasing induction on \( j \) it follows that \( \hat{X}_1^j(a) = 0 \) for every \( 0 \leq j \leq n \). In particular, \( a = 0 \). Hence, from (59) and (60) we obtain that \( u_1 X_4 \in U(\mathfrak{k}) \mathfrak{n} \) and \( u_0 + u_1 E \in U(\mathfrak{k}) \mathfrak{n} \). Now, using Lemma 6.2 and the fact that \( E(\mathfrak{n}) = 0 \) it follows that \( u_0 \equiv u_1 \equiv 0 \) mod \( U(\mathfrak{k}) \mathfrak{n} \), therefore \( u_0 \equiv u_1 \equiv 0 \) mod \( U(\mathfrak{k}) \mathfrak{n} \) as we wanted to prove.

Next proposition will be used in Theorem 8.6 of Section 8.

**Proposition 6.9.** Let \( \{ \eta_j : j \in \mathbb{N}_0 \} \) be a sequence in \( U(\mathfrak{k}) \) such that \( \eta_j \neq 0 \) for a finite number of \( j \)'s, \( \hat{X}_1(\eta_j) = 0 \) for every \( j \in \mathbb{N}_0 \) and \( \sum_{j \geq 0} \eta_j E^j \equiv 0 \) mod \( U(\mathfrak{k}) \eta \). Then
\[
\sum_{i \geq 0} \eta_{2i} E^{2i} \equiv 0 \quad \text{and} \quad \sum_{i \geq 0} \eta_{2i+1} E^{2i+1} \equiv 0,
\]
where the congruence is mod \( U(\mathfrak{k}) \eta \).

**Proof.** Let \( \Delta = 2 X_4 X_2 - E^2 \). Since \( X_2, X_4 \) and \( E \) commute with each other it follows that \( (-1)^j \Delta^j \equiv E^{2j} \) mod \( U(\mathfrak{k}) \eta \) for every \( j \in \mathbb{N}_0 \). Also observe that \( \hat{X}_1(\Delta) = 0 \). From now on the proof follows in the same way as that of Proposition 4.11 of [3], simply changing the congruence mod \( U(\mathfrak{k}) X_2 \) for a congruence mod \( U(\mathfrak{k}) \eta \).

\[
\text{\bf 7. An estimate on the Kostant degree}
\]

In this section we introduce the degree property and show that every \( b \in P(U(\mathfrak{g})^K) \) has the degree property. This result is used in Proposition 7.11 below. We also show that to prove Theorem 2.4 and therefore our main result Theorem 1.11 it is enough to prove Theorem 7.12 below.

**Definition 7.1.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{k})^M \otimes U(\mathfrak{a}) \) with \( b_m \neq 0 \). We say that \( b \) has the degree property if \( d(b_{m-j}) \leq m + 2j \) for every \( 0 \leq j \leq m \).

We begin by recalling a few facts about \( \mathfrak{s} \)-triples in \( \mathfrak{g} \). Recall that an \( \mathfrak{s} \)-triple is a set of three linearly independent elements \( \{ x, e, f \} \) in \( \mathfrak{g} \) such that \([x, e] = 2e, \ [x, f] = -2f \) and \([e, f] = x \). The \( \mathfrak{s} \)-triple \( \{ x, e, f \} \) is called normal if \( e, f \in \mathfrak{p} \) and \( x \in \mathfrak{k} \). A normal \( \mathfrak{s} \)-triple \( \{ x, e, f \} \) is called principal if \( e \) (and hence \( f \)) is a regular element in \( \mathfrak{p} \). Theorem 3 of [8] guarantee that principal normal \( \mathfrak{s} \)-triples exist, and in Theorem 6 of the same paper it is proved that any two principal normal \( \mathfrak{s} \)-triples are \( K_\theta \)-conjugate, where \( K_\theta \) is the subgroup of all elements in \( G \) that commute with \( \theta \).
Fix a principal normal \( \mathfrak{s}\)-triple \( \{x, e, f\} \) in \( \mathfrak{g} \) and set \( z = x/2 \). In Proposition 1 of \([11]\) it is proved that the map \( \text{ad}(z) : \mathfrak{p} \rightarrow \mathfrak{p} \) is diagonalizable with eigenvalues 1, \(-1\) and possibly 0. Since in our case \( \mathfrak{g} \cong \mathfrak{f}_4 \), the eigenvalues of \( \text{ad}(z) \) in \( \mathfrak{g} \) are \(-2\), \(-1\), 0, 1 and 2 (see the proof of Proposition 1 of \([11]\)), then the next result follows.

**Lemma 7.2.** The map \( \text{ad}(z) : \mathfrak{k} \rightarrow \mathfrak{k} \) is diagonalizable and its highest eigenvalue is 2.

In Corollary 9 of \([11]\) it is shown that if \( \mathfrak{g}_o \) is a semisimple Lie algebra over \( \mathbb{R} \), different from \( \mathfrak{sl}(2, \mathbb{R}) \), and \( V_{\gamma} \) is an irreducible \( K \)-module of type \( \gamma \in \Gamma \) then \( d(\gamma) \) is the highest eigenvalue of \( z \) in \( V_{\gamma} \). From this result the following lemma follows.

**Lemma 7.3.** Let \( V \) be a finite dimensional \( K \)-module and let \( n \) be the highest eigenvalue of \( z \) in \( V \). If \( u \in V^M \) and \( u \neq 0 \), then \( d(u) \leq n \).

As an application of Lemma 7.2 and Lemma 7.3 we obtain the following result that will be useful in what follows.

**Lemma 7.4.** If \( u \in U_m(\mathfrak{t})^M \) and \( u \neq 0 \), then \( d(u) \leq 2m \).

Recall that \( P : U(\mathfrak{g}) \rightarrow U(\mathfrak{t}) \otimes U(\mathfrak{a}) \) is the projection on the first summand of the direct sum \( U(\mathfrak{g}) = (U(\mathfrak{t}) \otimes U(\mathfrak{a})) \oplus U(\mathfrak{g})n \), associated to an Iwasawa decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n} \) adapted to \( \mathfrak{k} \). The proof of the following result follows easily by choosing an appropriate Poincaré-Birkhoff-Witt bases of \( U(\mathfrak{g}) \).

**Lemma 7.5.** \( P(U_m(\mathfrak{g})) = \sum_{0 \leq \ell \leq m} U_{m-\ell}(\mathfrak{t}) \otimes Z^\ell \) for every \( m \geq 0 \).

Let \( \sigma : S(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \) be the symmetrization mapping. It is known that \( \sigma \) is a \( K \)-linear isomorphism. Let \( \varphi : U(\mathfrak{t}) \otimes S(\mathfrak{p}) \rightarrow U(\mathfrak{g}) \) be the \( K \)-linear isomorphism defined by \( \varphi(u \otimes p) = u \sigma(p) \). Then we have,

\[
U(\mathfrak{g})^K = \sum_{m \geq 0} \left( U(\mathfrak{t}) \sigma(S_m(\mathfrak{p})) \right)^K.
\]

**Theorem 7.6.** Let \( u \in (U(\mathfrak{t}) \sigma(S_m(\mathfrak{p})))^K \) where \( m \) is the smallest possible. Then \( P(u) = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{t})^M \otimes U(\mathfrak{a}) \), \( b_m \neq 0 \) and \( d(b_{m-j}) \leq m + 2j \) for \( 0 \leq j \leq m \).

**Proof.** Let \( \overline{u} \in (U(\mathfrak{t}) \otimes S_m(\mathfrak{p}))^K \) be such that \( \varphi(\overline{u}) = u \). Write \( S_m(\mathfrak{p}) = \sum W_\tau \) where the sum runs over a finite set \( J \subset \Gamma \). Then by Schur’s Lemma we have,

\[
(62) \quad (U(\mathfrak{t}) \otimes S_m(\mathfrak{p}))^K = \sum_{\tau \in J} (U(\mathfrak{t})_{\tau^*} \otimes W_\tau)^K,
\]

where \( \tau^* \) is the contragredient representation of \( \tau \), and \( U(\mathfrak{t})_{\tau^*} \) denotes the \( \tau^* \)-isotypic component of \( U(\mathfrak{t}) \).
Let \( q \) be a subspace of \( p \) such that \( p = a \oplus q \) and let \( \{X_1, \ldots, X_r\} \) be an ordered bases of \( q \). If \( a = (a_1, \ldots, a_r) \) with \( a_i \in N_0 \), and \( X^a = X_1^{a_1} \cdots X_r^{a_r} \) in \( S(p) \), it follows that \( \{Z^\ell X^a : 0 \leq \ell + |a| \leq m\} \) is a bases of \( S_m(p) \), where \( |a| = a_1 + \cdots + a_r \). Then, in view of (62), we can write

\[
\tilde{u} = \sum_{0 \leq \ell + |a| \leq m} u_{\ell,a} \otimes Z^\ell X^a,
\]

where \( u_{\ell,a} \) belongs to the \( K \)-module \( V = \sum_{\tau \in J} U(\mathfrak{g})^M \) for every pair \((\ell, a)\). Then,

\[
P(u) = \sum_{0 \leq \ell + |a| \leq m} P(u_{\ell,a} \sigma(Z^\ell X^a)) = \sum_{0 \leq \ell + |a| \leq m} u_{\ell,a} P(\sigma(Z^\ell X^a)).
\]

Now, since \( \sigma(Z^\ell X^a) \in U_{\ell+|a|}(\mathfrak{g}) \), it follows from Lemma (7.5) that

\[
P(\sigma(Z^\ell X^a)) = \sum_{0 \leq j \leq \ell + |a|} v_{\ell,a,j} \otimes Z^j,
\]

with \( v_{\ell,a,j} \in U_{\ell+|a|-j}(\mathfrak{g}) \). Hence from (63) we have,

\[
P(u) = \sum_{0 \leq j \leq m} \left( \sum_{j \leq \ell + |a| \leq m} u_{\ell,a} v_{\ell,a,j} \right) \otimes Z^j.
\]

Then from the uniqueness of the coefficients \( b_j \) it follows that,

\[
b_j = \sum_{j \leq \ell + |a| \leq m} u_{\ell,a} v_{\ell,a,j} \quad \text{for} \quad 0 \leq j \leq m,
\]

where \( v_{\ell,a,j} \in U_{\ell+|a|-j}(\mathfrak{g}) \subset U_{m-j}(\mathfrak{g}) \) for every pair \((\ell, a)\). Hence from (64) we obtain that,

\[
b_j \in \langle V \cdot U_{m-j}(\mathfrak{g}) \rangle^M \subset U(\mathfrak{g})^M \quad \text{for} \quad 0 \leq j \leq m.
\]

Recall that \( \langle S \rangle \) denotes the linear space spanned by the set \( S \). Observe that in this case \( \langle V \cdot U_{m-j}(\mathfrak{g}) \rangle \) is a \( K \)-module.

Now, since the highest eigenvalue of \( z \) in \( p \) is 1, it follows that the highest eigenvalue of \( z \) in \( S_m(p) \) is \( m \). Then \( d(\tau) \leq m \) for every \( \tau \in J \), and therefore \( d(\tau^*) \leq m \) for every \( \tau \in J \). This implies that the highest eigenvalue of \( z \) in \( V \) is less or equal to \( m \). On the other hand, we know that the highest eigenvalue of \( z \) in \( U_{m-j}(\mathfrak{g}) \) is less or equal to \( 2(m-j) \), hence the highest eigenvalue of \( z \) in \( \langle V \cdot U_{m-j}(\mathfrak{g}) \rangle \) is less or equal to \( m + 2(m-j) \). Then, from Lemma (7.5) and (65) it follows that \( d(b_j) \leq m + 2(m-j) \) for \( 0 \leq j \leq m \), and therefore \( d(b_{m-j}) \leq m + 2j \) for \( 0 \leq j \leq m \), as we wanted to prove. \( \square \)

**Theorem 7.7.** Let \( b \in P(U(\mathfrak{g})^K) \) be such that \( b = b_m \otimes Z^m + \cdots + b_0 \) with \( b_m \neq 0 \), then \( d(b_{m-j}) \leq m + 2j \) for every \( 0 \leq j \leq m \).

**Proof.** Let \( u \in U(\mathfrak{g})^K \) be such that \( P(u) = b \). Since \( b_m \neq 0 \), it follows from Corollary 7.3 of [9] that \( u \in (U(\mathfrak{g}) \sigma(S_m(p)))^K \) and \( m \) is the smallest possible. Hence the result follows from Theorem (7.6) \( \square \)
Our next goal is to show that Theorem 2.4 follows from Theorem 7.12 bellow. In the next lemma we single out a particular element $\omega \in B$. This element is a scalar multiple of $P(\Omega)$, where $\Omega$ is the Casimir of $g$.

**Lemma 7.8.** There exist $\omega = \omega_2 \otimes Z^2 + \omega_1 \otimes Z + \omega_0 \in P(U(g)^K) \subset B$ such that $\omega_2 = 1$, $\omega_1$ is a nonzero scalar, $\omega_0$ is a scalar multiple of the Casimir element of $m$ and $d(\omega_0) \leq 4$.

**Proposition 7.9.** For any $b \in U(\mathfrak{t})^M \otimes U(\mathfrak{a})$ there exist $n \in \mathbb{N}_0$ such that $b\omega^n$ has the degree property.

**Proof.** Let $b = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{t})^M \otimes U(\mathfrak{a})$. Fix $n \in \mathbb{N}_0$ sufficiently large so that $d(b_{m-j}) \leq m + 2n + 2j$ for every $0 \leq j \leq m$. A simple calculation shows that

$$\omega^n = \sum_{k=0}^{[k/2]} \omega_{k,n} \otimes Z^{2n-k}, \quad (66)$$

where $\omega_{k,n} = \sum_{i=0}^{[k/2]} (n_{k-i})(k-i) \omega_1^{k-2i} \omega_0^i$ for $0 \leq k \leq 2n$, and that

$$b\omega^n = \sum_{j=0}^{m+2n} \left( \sum_{k=0}^{\min\{j, 2n\}} b_{m+k-j} \omega_{k,n} \right) \otimes Z^{m+2n-j}. \quad (67)$$

Then if $(b\omega^n)_\ell$ denotes the coefficient of $Z^\ell$ in $b\omega^n$, we have

$$d((b\omega^n)_{m+2n-j}) \leq \max\{d(b_{m+k-j} \omega_{k,n}) : 0 \leq k \leq j\}$$

$$= \max\{d(b_{m+k-j}) + d(\omega_{k,n}) : 0 \leq k \leq j\}$$

$$\leq \max\{m + 2n + 2(j - k) + 2k : 0 \leq k \leq j\}$$

$$= m + 2n + 2j,$$

for every $0 \leq j \leq m + 2n$. Hence $b\omega^n$ has the degree property. \hfill \Box

It is now convenient to introduce the following notation, for any $m \in \mathbb{N}_0$ and $0 \leq r \leq m$ define $d_r$ as follows,

$$d_r = \left[\frac{3m - 2r + 2}{2}\right]. \quad (68)$$

In the next lemma we obtain an upper bound on the Kostant degree of the coefficients $b_r$ of certain $b \in U(\mathfrak{t})^M \otimes U(\mathfrak{a})$.

**Lemma 7.10.** Let $b = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{t})^M \otimes U(\mathfrak{a})$ with $b_m \neq 0$. If $b\omega$ has the degree property then $d(b_r) \leq 2d_r$ for every $0 \leq r \leq m$.

**Proof.** Let $(b\omega)_\ell$ denote the coefficient of $Z^\ell$ in $b\omega$. It follows from (66) and (67), or directly by computing $b\omega$, that

$$b_{m-j} = (b\omega)_{m+2-j} - b_{m-j+1} \omega_1 - b_{m-j+2} \omega_0 \quad (69)$$
by induction on $j$. That is, 
\[ d \text{ of Kostant degree } 2 \]

\[ \text{equal to } p \]

\[ \text{the coefficients of } b \]

for $0 \leq j \leq m$, with the understanding that \( b_{m+1} = b_{m+2} = 0 \). Then, since \( \omega_1 \) is a scalar and \( d(\omega_0) \leq 4 \), from (69) we obtain that
\[ d(b_{m-j}) \leq \max\{d((b\omega)_{m+2-j}), d(b_{m-j+1}), d(b_{m-j+2}) + 4\}. \]

Hence, using (70) and the fact that \( b\omega \) has the degree property, it follows by induction on \( j \) that \( d(b_{m-j}) \leq m + 2 + 2j \) for every \( 0 \leq j \leq m \). Now, since the Kostant degree of any element of \( U(\mathfrak{g})^M \) is even (see (ii) and (iii) of Proposition 5.1), it follows that \( d(b_r) \leq 2d_r \) for every \( 0 \leq r \leq m \). □

Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \), where \( d_r \) is as in (68). Using Proposition 5.1 and the above bound on \( d(b_r) \) we can decompose the coefficients \( b_r \) of \( b \) as follows,
\[ b_r = \sum_{t=0}^{2d_r} b^r_{2i,t-2i} \] \[ \text{for } 0 \leq r \leq m, \]
where \( b^r_{2i,t-2i} \) is the component of \( b_r \) in the isotypic component of \( U(\mathfrak{t})^M \) of type \( (2i,t-2i) \). Consider now the following linear subspace of \( B \),
\[ \widetilde{B} = \{ b \in B : b^r_{2i,j} = 0 \text{ if } i + j \leq k \text{ and } 0 \leq 2k \leq \deg(b) \}. \]

That is, \( \widetilde{B} \) consists of the elements \( b \in B \) such that the \( K \)-types \( b^r_{2i,j} \) that occur in the coefficient \( b_{2k} \) of \( b \), have Kostant degree greater than \( 2k \) for all \( k \) such that \( 0 \leq 2k \leq \deg(b) \).

**Proposition 7.11.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), \( b_m \neq 0 \), and \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \). Then there exist \( \widetilde{b} \in \widetilde{B} \) such that \( d(\widetilde{b}_r) \leq 2d_r \) for \( 0 \leq r \leq m \), \( \widetilde{b}_m = b_m \) if \( m \) is odd, and \( d(b_m - \widetilde{b}_m) \leq m \) if \( m \) is even. Moreover \( b^r_{2i,j} = b^r_{2i,j} \)

if \( i + j = d_r \) for every \( 0 \leq r \leq m \).

**Proof.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( b_m \neq 0 \) and \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \). Set \( p = 2[m/2] \) and using (71) define,
\[ c_p = \sum_{t=0}^{p} \sum_{\max(0,t-\frac{p}{2}) \leq i \leq \lfloor t/2 \rfloor} b^p_{2i,t-2i}. \]

That is, \( c_p \) contains all the \( K \)-types of \( b_p \) of Kostant degree smaller or equal to \( p \). Hence, \( c_p \in U(\mathfrak{t})^M \) and \( d(c_p) \leq p \). Since \( p \) is even \( c_p \otimes Z^p \in (U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W \). Then from Proposition 2.3 it follows that \( c_p \otimes Z^p \) is the leading term of an element \( c^{(p)} = c_p \otimes Z^p + \cdots \in P(U(\mathfrak{g})^K) \). Now define \( b^{(p)} = b - c^{(p)} \in B \). All the \( K \)-types that occur in the \( p \)-coefficient of \( b^{(p)} \) have Kostant degree greater than \( p \) and, since \( c^{(p)} \in P(U(\mathfrak{g})^K) \), it follows from Theorem 7.7 that \( d(b^{(p)}_r) \leq 2d_r \) for \( 0 \leq r \leq m \). Moreover, the \( K \)-types of Kostant degree \( 2d_r \) of \( b^{(p)}_r \) and \( b_r \) are the same for \( 0 \leq r \leq m \).

Considering now the \((p-2)\)-coefficient of \( b^{(p)} \) we construct, in a similar way, elements \( c^{(p-2)} \in P(U(\mathfrak{g})^K) \) and \( b^{(p-2)} = b^{(p)} - c^{(p-2)} \in B \), such that the coefficients of \( b^{(p-2)} \) corresponding to degrees greater than \( p - 2 \) are...
the same as those of \( b^{(p)} \), and all the \( K \)-types that occur in the \((p - 2)\) coefficient of \( b^{(p-2)} \) have Kostant degree greater than \( p - 2 \). Moreover, since \( c^{(p-2)} \in P(U(\mathfrak{g})^K) \), Theorem 7.7 implies that \( d(b^{(p-2)}) \leq 2d_r \) for \( 0 \leq r \leq m \), and that the \( K \)-types of Kostant degree \( 2d_r \) of \( b^{(p-2)} \) and \( b_r \) are the same for every \( 0 \leq r \leq m \).

Continuing in this way we obtain a sequence \( b^{(p)}, b^{(p-2)}, \ldots, b^{(0)} \) of elements in \( B \) of degree at most \( m \). If we set \( \tilde{b} = b^{(0)} \) it is clear that \( \tilde{b} \in \tilde{B} \) and that \( \tilde{b} \) has all the required properties.

Finally, in Proposition 7.14 below, we show that the next theorem implies Theorem 2.4 (and therefore Theorem 1.1). The proof of Theorem 7.12 will be done in the next section.

**Theorem 7.12.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in \tilde{B} \) be such that \( d(b_r) \leq 2d_r \) for every \( 0 \leq r \leq m \). Then \( b = 0 \).

If we assume that Theorem 7.12 holds we obtain the following corollary.

**Corollary 7.13.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( b_m \neq 0 \) and \( b_0 \) has the degree property. Then \( m \) is even and \( b \) has the degree property.

**Proof.** Since \( b_0 \) has the degree property, it follows from Lemma 7.10 that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \). Then Proposition 7.11 implies that there exist \( \tilde{b} \in \tilde{B} \) such that \( d(\tilde{b}_r) \leq 2d_r \) for \( 0 \leq r \leq m \), \( \tilde{b}_m = b_m \) if \( m \) is odd and \( \tilde{b}_m = b_m \) if \( m \) is even and \( \tilde{b}_m = b_m \). Then, \( \tilde{b} \) has the degree property for every \( 0 \leq r \leq m \).

**Proposition 7.14.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) with \( b_m \neq 0 \). Then, \( m \) is even and \( b \) has the degree property. In particular \( d(b_m) \leq m \), and therefore Theorem 2.4 holds.

**Proof.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be as in the statement of the theorem. It follows from Proposition 7.9 that there exist \( n \in \mathbb{N}_0 \) such that \( b_0^n \) has the degree property. Now, since \( b_0^{n-1} = b_m \otimes Z^{m+2(n-1)} + \cdots + b_0 \in B \) and \( b_m \neq 0 \), it follows from Corollary 7.13 that \( m + 2(n-1) \) is even and \( b_0^n \) has the degree property. Hence \( m \) is even, and from Corollary 7.13 and induction on \( k \) it follows that \( b_0^{n-k} \) has the degree property for every \( 0 \leq k \leq n \). In particular \( b \) has the degree property, as we wanted to prove.

8. **Proof of Theorem 7.12**

Our goal in this section is to prove Theorem 7.12. To do this, given any \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \), we will construct a linear system of equations in \( U(\mathfrak{k}) \) where the unknowns are \( \mathfrak{k}^+ \)-dominant vectors associated to certain \( K \)-types of the coefficients of \( b \) (see Theorem 8.6). This system will allow us to carry out a decreasing induction process that, when applied to \( b \in \tilde{B} \), will lead to the proof of Theorem 7.12.
Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \).

As indicated in (71), we can decompose the coefficient \( b_r \) of \( b \) as follows,

\[
(73) \quad b_r = \sum_{t=0}^{2d_r} \sum_{\max\{0, t-d_r\} \leq i \leq [t/2]} b_{r, t-2i}^r \quad \text{for} \quad 0 \leq r \leq m.
\]

We find it very convenient to keep in mind the following array of the \( K \)-types that occur in \( b_r \).

\[
(74) \quad b_r = b_{r, 0}^r + b_{r, 2d_r-2, 1}^r + b_{r, 2d_r-4, 1}^r + \cdots + b_{0, d_r}^r
\]

Observe that the parameter \( t \) used in (73) may be regarded as a label for the skew diagonals of the array (74). In fact, for \( 0 \leq t \leq 2d_r \) we shall refer to the set \( \{ b_{r, t-2i}^r : \max\{0, t-d_r\} \leq i \leq [t/2] \} \) as the skew diagonal associated to \( t \). Also observe that the Kostant degree is constant along the rows of the array (74), it takes the values \( 2d_r, 2d_r - 2, \ldots, 0 \) from the top to the bottom row of the array corresponding to \( b_r \).

Let \( T \in \mathbb{N}_o \) denote the label of the skew diagonals in the array corresponding to \( b_0 \). We will use \( T \) as a parameter for a decreasing induction.

For \( m \leq T \leq 2d_0 \) if \( m \) is even, and \( m - 1 \leq T \leq 2d_0 \) if \( m \) is odd, consider the following propositional function associated to \( b_r \),

\[
(75) \quad P(T) : b_r = \sum_{t=0}^{\min\{T-r, 2d_r\}} \sum_{\max\{0, t-d_r\} \leq i \leq [t/2]} b_{r, t-2i}^r, \quad 0 \leq r \leq m.
\]

Observe that \( P(T) \) holds if and only if \( b_{r, 2t-2i}^r = 0 \) for \( t > \min\{T-r, 2d_r\} \) for every \( 0 \leq r \leq m \). Also, in view of (71), it follows that \( P(2d_0) \) holds. This will be the starting point of our inductive argument.

Let \( E, X_\delta, H, Y \) and \( \bar{Y} \) be as in Section I. Recall that \( \hat{E}(H) = -\frac{1}{2}E \), \( \hat{X}_\delta(\bar{Y}) = X_\delta \) and \( \hat{X}_\delta(H) = \hat{E}(X_\delta) = 0 \). In the following lemma we state some properties of the derivations \( \hat{E} \) and \( \hat{X}_\delta \), we refer to Lemma 6.1 of \( \#3 \) for their proof.

**Lemma 8.1.** (i) \( \hat{E}^k(H^k) = k!(\frac{1}{2}E)^k \) and \( \hat{E}^k(H) = 0 \) if \( k > j \).

(ii) \( \hat{E}^k \varphi_k(H) = (-\frac{1}{2}E)^k \), where \( \varphi_k \) is as in (7).

(iii) \( \hat{X}_\delta^k((-Y)^k) = k!(\bar{X}_\delta)^k \) and \( \hat{X}_\delta^k((-\bar{Y})^k) = 0 \) if \( k > j \).

(iv) \( \hat{X}_\delta^k \varphi_k(a - \bar{Y}) = (-X_\delta)^k \) for any \( a \in \mathbb{C} \).
The following proposition is the analogue of Proposition 6.2 of [3]. Its proof is the same as that of Proposition 6.2 and it is obtained by applying \( X^{T-n-\ell} \) to \( 2 \) of Theorem 3.3 and using Lemma 3.3 and Lemma 5.1. Also observe that the derivation \( X_{\delta} \) preserves the ideal \( U(t)^{m+} \).

**Proposition 8.2.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \), and assume that \( P(T) \) holds for \( m \leq T \leq 2d_0 \). Then for every \((\ell, n)\) such that \( 0 \leq \ell, n \) and \( \ell + n \leq T \) we have

\begin{equation}
(76) \quad (-1)^n \Sigma_1 E^n - (-1)^\ell \Sigma_2 E^\ell \equiv 0 \mod (U(t)^{m+}),
\end{equation}

where

\begin{equation}
(77) \quad \Sigma_1 = \sum_{(i, r) \in I_1} A_{i, r}(T, n, \ell) X_{\delta}^{T-\ell-i} \hat{E}^{\ell-i-r}(b_r) E^{r-i} X_{\delta}^{i-n},
\end{equation}

and

\begin{equation}
A_{i, r}(T, n, \ell) = \left( \frac{-1}{2} \right)^{r-i}(1)^{r-i-n} r!(T-n-\ell) \left( \begin{array}{c} r \\ i \\ n \end{array} \right),
\end{equation}

\( I_1 = \{(i, r) \in N^2_0 : n \leq i \leq \min\{m, T-\ell\}, i \leq r \leq \min\{m, i+\ell\}\} \),

\( I_2 = \{(i, r) \in N^2_0 : \ell \leq i \leq \min\{m, T-n\}, i \leq r \leq \min\{m, i+n\}\} \).

Next proposition is the analogue of Proposition 6.3 of [3] and its proof is the same as that of Proposition 6.3. It is obtained by replacing \( b_r \) in (77) by its decomposition in \( K \)-types given in (75), then one uses Proposition 5.1(iv) to simplify the sums \( \Sigma_1 \) and \( \Sigma_2 \), and finally one multiply both sums on the right by \( X_{\delta}^{T-\ell} \) and then change in each term a certain number of \( X_{\delta} \)'s by the same number of \( X_4 \)'s so that \( \Sigma_1 \) and \( \Sigma_2 \) become weight vectors with respect to \( h_t \). Here we use that \( X_{\delta} \equiv X_4 \mod (U(t)^{m+}) \) and that the derivation \( X_{\delta} \) preserves the ideal \( U(t)^{m+} \).

**Proposition 8.3.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \), and assume that \( P(T) \) holds for \( m \leq T \leq 2d_0 \). Then for every \((\ell, n)\) such that \( 0 \leq \ell, n \) and \( \ell + n \leq T \) we have

\begin{equation}
(78) \quad (-1)^n \Sigma_1 E^n - (-1)^\ell \Sigma_2 E^\ell \equiv 0 \mod (U(t)^{m+}),
\end{equation}

where

\begin{equation}
\Sigma_1 = \sum_{(i, r) \in I_1} A_{i, r}(T, n, \ell) X_{\delta}^{T-\ell-i} \hat{E}^{\ell-i-r}(b'_r) E^{r-i} X_{\delta}^{i-n},
\end{equation}

\begin{equation}
\Sigma_2 = \sum_{(i, r) \in I_2} A_{i, r}(T, n, \ell) X_{\delta}^{T-n-i} \hat{E}^{n-i-r}(b'_r) E^{r-i} X_{\delta}^{\ell-n},
\end{equation}

\( I_1 = \{(i, r) \in N^2_0 : n \leq i \leq \min\{m, T-\ell\}, i \leq r \leq \min\{m, i+\ell\}\} \),

\( I_2 = \{(i, r) \in N^2_0 : \ell \leq i \leq \min\{m, T-n\}, i \leq r \leq \min\{m, i+n\}\} \).
with the understanding that the $K$-types $b^r_{2k,T-r-2k}$ that do not occur in $b_r$ 
are assumed to be zero. Moreover, in equation \( (78) \) all the terms of the left
hand side are weight vectors of weight \((2T - \ell - n)\gamma_1 + T(\gamma_2 + \delta)\).

The equations \( (78) \) may be regarded as a system of linear equations where
the unknowns, $X^T_{\delta - l} \hat{E}^{j+i-r}(b^r_{2k,T-r-2k})$, are derivatives of the $K$-types 
that occur in the $T - r$ skew diagonal of the coefficient $b_r$ of $b$ (see \( (74) \)).
Since the unknowns in this system are, in general, not \( \mathfrak{k}^+ \)-dominant we are
going to replace the system by an equivalent one where all the unknowns
become \( \mathfrak{k}^+ \)-dominant vectors associated to the $K$-types $b^r_{2k,T-r-2k}$.

Let $\tilde{c}(\ell, n)$ be the left hand side of equation \( (78) \). For \( 0 \leq n \leq \min\{2m, T\} \)
and \( 0 \leq L \leq \min\{2m, T\} - n \) consider the following linear combination,

\[
E_L(n) = \sum_{\ell=0}^{L} (-2)^{\ell} \binom{L}{\ell} \tilde{c}(\ell, n) E^{L-\ell} X^{\ell+n}_4.
\]

Under the hypothesis of Proposition \( 8.3 \) we have $E_L(n) \equiv 0 \mod (U(\mathfrak{k})\mathfrak{m}^+)$. 
Also set,

\[
E^1_L(n) = \sum_{\ell=0}^{L} (-2)^{\ell} \binom{L}{\ell} \Sigma_1 E^{L-\ell} X^{\ell+n}_4 \quad \text{and} \quad E^2_L(n) = \sum_{\ell=0}^{L} 2^{\ell} \binom{L}{\ell} \Sigma_2 X^{\ell+n}_4.
\]

Then it follows that

\[
E_L(n) = (-1)^n E^1_L(n) E^n - E^2_L(n) E^L.
\]

The following lemma is the analogue of Lemma 6.5 of \cite{3}. For the sym-
plectic group \( \text{Sp}(n,1) \) the vectors $D_k(b_{2i,j})$ are \( \mathfrak{k}^+ \)-dominant, however in \( F_4 \)
this property does not hold.

**Lemma 8.4.** Let $b_{2i,j} \in U(\mathfrak{k})^M$ be an $M$-invariant element of type \((2i, j)\). 
For $0 \leq k \leq 2i$ define,

\[
D_k(b_{2i,j}) = \sum_{\ell=0}^{\ell} (-2)^{\ell} \binom{k}{\ell} \binom{\ell}{i} (\ell + k)^{-1} X^{2i-\ell} \hat{E}^{j+i}(b_{2i,j}) X^{\ell}_4.
\]

Then $D_k(b_{2i,j})$ is a vector of weight $i(\gamma_4 + \delta) + (j + k)\gamma_3$ with respect to $\mathfrak{h}_\mathfrak{f}$, 
\( \hat{X}(D_k(b_{2i,j})) \equiv 0 \mod (U(\mathfrak{k})\mathfrak{h}) \) for every $X \in \mathfrak{q}^+$ and \( \hat{X}(D_k(b_{2i,j})) = 0$. 

**Proof.** Recall that $\mathfrak{q}^+$ is the linear span of \{ $X_\alpha : \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{f}) - \{\gamma_1\}$ \}. 
Since $\gamma_1$ is a simple root in $\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{f})$, if $\alpha$ is a positive root it follows that $\alpha - \gamma_1$ is either a positive root different from $\gamma_1$ or it is not a root. Hence if $u \in U(\mathfrak{k})$ is a $\mathfrak{k}^+$-dominant vector we have,

\[
\hat{X}_\alpha(\hat{X}_{-\ell}(u)) = 0 \quad \text{for every} \quad \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{f}) - \{\gamma_1\} \quad \text{and} \quad \ell \in \mathbb{N}_0.
\]

Then, in view of \( (20) \), it follows that

\[
\hat{X}(\hat{X}^{2i-\ell} \hat{E}^{j+i}(b_{2i,j})) = 0 \quad \text{for every} \quad X \in \mathfrak{q}^+.
\]

On the other hand, since $\hat{E}(\mathfrak{h}) = \hat{X}_4(\mathfrak{h}) = 0$ and $[\mathfrak{q}^+, \mathfrak{h}] \subset \mathfrak{h}$ it follows that,

\[
\hat{X}(E^n) \equiv 0 \quad \text{and} \quad \hat{X}(X^n_{\alpha}) \equiv 0 \mod (U(\mathfrak{k})\mathfrak{h}) \quad \text{for every} \quad X \in \mathfrak{q}^+.
\]
Hence from (81), (82) and (83) we obtain that,

\[ (84) \quad X(D_k(b_{2i,j})) \equiv 0 \mod (U(t)\eta) \quad \text{for every} \quad X \in q^+. \]

Now, since \( \dot{X}_1(E) = X_4 \) and \( \dot{X}_1(X_4) = 0 \), using (18) it follows that \( \dot{X}_1(D_k(b_{2i,j})) = 0 \). The details of this calculation can be found in the proof of Lemma 6.5 of [3]. Finally, it is easy to check that each term of \( D_k(b_{2i,j}) \) is a vector of weight \( i(\gamma_4 + \delta) + (j + k)\gamma_3 \) with respect to \( \eta_t \).

As indicated at the beginning of the section we are interested in proving that \( P(T) \) implies \( P(T - 1) \) for \( m \leq T \leq 2d_0 \). To do this we need to show that the \( K \)-types \( b_{2i,T-r-2i}^r \) that occur in the \( T - r \) skew diagonal of \( b_r \) are equal to zero for \( 0 \leq r \leq m \). That is,

\[ b_{2i,T-r-2i}^r = 0 \quad \text{if} \quad 0 \leq T - r - 2i \leq \min\{T, 2d_0 - T\} - r, \]

for \( 0 \leq r \leq m \). For this purpose we introduce another propositional function \( Q(n) \) defined for \( 0 \leq n \leq \min\{T, 2d_0 - T\} + 1 \) as follows,

\[ (85) \quad Q(n) : \ b_{2i,T-r-2i}^r = 0 \quad \text{if} \quad 0 \leq T - r - 2i < n \quad \text{for} \quad 0 \leq r \leq m. \]

Clearly \( Q(0) \) is true. Also, since we have that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \), we obtain that (85) holds if \( T - r - 2i > \min\{T, 2d_0 - T\} - r \).

Next theorem is the analogue of Theorem 6.6 of [3] and its proof is the same as that of Theorem 6.6, it consist in rewriting the sum \( \mathcal{E}_{L}^1(n) \) in terms of the vectors \( D_k(b_{2i,j}) \) defined in Lemma 6.4 and the sum \( \mathcal{E}_{L}^2(n) \) in terms of \( \frak{t}^+ \)-dominant vectors. We refer the reader to Section 6 of [3] for the details.

**Theorem 8.5.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) be such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \), let \( m \leq T \leq 2d_0 \) and \( 0 \leq n \leq \min\{T, 2d_0 - T\} \). Then if \( P(T) \) and \( Q(n) \) are true we have,

\[ (86) \quad \sum_{T - L \leq 2k + r \leq T - n} B_{r,k}(T, n, L) D_{L + 2k + r - T} \left( \begin{array}{cc} b_{2k,T-r-2k}^r & X_4 \end{array} \right)^{T - k} E^n \]

\[ \quad - \sum_{T - L \leq 2k + r \leq T - n} (-2)^e \binom{L}{e} \binom{T - n - e}{T - r - e} u_{T - r - n, n} (X_4 X_4)^{(T + r + n)/2} E^L \equiv 0, \]

for all \( L \) such that \( 0 \leq L \leq \min\{2m, T\} - n \). Here the congruence is module the left ideal \( U(t)m^+ \), \( u_{T - r - n, n} = r!(-1)^r \dot{X}_T^{T - n - r} E^n (b_{T - r - n, n}) \) and

\[ B_{r,k}(T, n, L) = r!(-1)^T 2^{T - r - 2k} \binom{L}{r} \left( \begin{array}{c} T - L \end{array} \right) \left( \begin{array}{c} T - L \end{array} \right) \]

Moreover, the left hand side of equation (86) is a weight vector of weight \( T(\gamma_4 + \delta) + (n + L)\gamma_3 \).

We are now in a good position to obtain the system of equations that we are looking for. Using the notation introduced in (31) define,

\[ (87) \quad U = X_4 X_4 - T_{23} S_{23} + T_{24} S_{24}. \]
Then $U$ is a $\mathfrak{f}^+$-dominant vector of weight $\gamma_4 + \delta$ with respect to $\mathfrak{h}_\mathfrak{f}$ and $U \equiv X_\delta X_4 \pmod{(U(\mathfrak{f})\mathfrak{p})}$. For any $T$ and $n$ such that $m \leq T \leq 2d_0$ and $0 \leq n \leq \min\{T, 2d_0 - T\}$ consider the following sets,

$$L(T, n) = \{L \in \mathbb{N}_0 : 0 \leq L \leq \min\{2m, T\} - n, \ L \neq n\},$$

$$R_F(T, n) = \{r \in \mathbb{N}_0 : 0 \leq r \leq \min\{m, \min\{T, 2d_0 - T\} - n\}, \ r \equiv T - n\},$$

the congruence is mod (2) and the subindex $F$ stands for $F_4$. Let $|L(T, n)|$ and $|R_F(T, n)|$ denote the cardinality of these sets. The set $L(T, n)$ was also considered for the symplectic group $\text{Sp}(n,1)$ while $R_F(T, n)$ is the analogue of the set $R(T, n)$ defined in Section 6 of [3].

Next theorem gives a system of linear equations where the unknowns, $u_{T-r-n,n}$, are $\mathfrak{f}^+$-dominant vectors associated to the $K$-types that occur in the $T - r$ skew diagonal of the coefficient $b_r$ of $b$ for $0 \leq r \leq m$ (see (83)).

**Theorem 8.6.** Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$, and let $m \leq T \leq 2d_0$ and $0 \leq n \leq \min\{T, 2d_0 - T\}$. Then if $P(T)$ and $Q(n)$ are true we have,

$$\sum_{r \in R_F(T, n)} \left( \sum_{\ell} (-1)^{\ell} \binom{T}{T-\ell}(T-n-\ell) \right) u_{T-r-n,n}^* U(T+r+n)/2 = 0,$$

for every $L \in L(T, n)$. Here $u_{T-r-n,n}^* = r!(1)^r X^{T-n-r} (b_{T-r-n,n})$.

**Proof.** Let $u$ denote the left hand side of equation (86). Then, in view of Theorem 8.5 $u$ is a vector in $U(\mathfrak{f})^+$ of weight $\lambda = T(\gamma_4 + \delta) + (n + L)\gamma_3$ with respect to $\mathfrak{h}_\mathfrak{f}$.

On the other hand, using that $X_i X_\delta \equiv 0 \pmod{(U(\mathfrak{f})\mathfrak{p})}$ for every $X_i \in q^+$, together with (83), (84) and the fact that $E_\mathfrak{f}, X_\delta$ and $X_4$ commute with $\eta$ and that $\{q^+, \eta\} \subset \mathfrak{h}_\mathfrak{f}$, it follows that $X_i X_\delta \equiv 0 \pmod{(U(\mathfrak{f})\mathfrak{p})}$ for every $X_i \in q^+$. Then applying Theorem 6.3 we obtain that $u \equiv 0 \pmod{(U(\mathfrak{f})\mathfrak{p})}$, that is,

$$\sum_{r \in R_F(T, n)} B_{r,k}(T, n, L) D_{L+2k+r-T}(b_{T-r-n,n}^* (X_\delta X_4)^{T-k} E^n - \sum_{r \notin R_F(T, n)} (-2)^{\ell} \binom{T}{T-\ell}(T-n-\ell) u_{T-r-n,n}^* (X_\delta X_4)^{(T+r+n)/2} E^n \equiv 0.$$

Since $U \equiv X_\delta X_4 \pmod{(U(\mathfrak{f})\mathfrak{p})}$ (see (87)), we replace $X_\delta X_4$ by $U$ in (89). Also, recall that $X_1 X_\delta = X_1 X_4 = 0$ and $X_1(D_k(b_{2i,j})) = 0$ for $b_{2i,j} \in U(\mathfrak{f})^+$ of type $(2i, j)$ and $0 \leq k \leq 2i$ (see Lemma 8.4). Hence, since $L \neq n \pmod{2}$, it follows from Proposition 6.9 and Lemma 6.2 that

$$\sum_{r \in R_F(T, n)} \left( \sum_{\ell} (-1)^{\ell} \binom{T}{T-\ell}(T-n-\ell) \right) u_{T-r-n,n}^* U(T+r+n)/2 \equiv 0,$$

modulo the left ideal $U(\mathfrak{f})\mathfrak{p}$. Now, since the left hand side of equation (90) is a $\mathfrak{f}^+$-dominant vector of weight $T(\gamma_4 + \delta) + n\gamma_3$, applying Theorem 6.4 we
can replace the congruence mod(\(U(\mathfrak{f})\)) by an equality. This completes the proof of the theorem. \(\square\)

For \(T\) and \(n\) fixed, Theorem 8.6 gives a system of \(|L(T, n)|\) linear equations in the \(|R_F(T, n)|\) unknowns \(u^r_{T-r-n, n}\). This system is the analogue of the one given in Theorem 6.7 of [3]. The main advantage of this system is that the unknowns are all \(\mathfrak{f}^+\)-dominant vectors. Let \(A(T, n)\) denote the coefficient matrix of this system. In Section 6 of [3] a very thorough study of this matrix is carried out (see Subsection 6.2). This is done by considering a \((k + 1) \times (k + 1)\) matrix \(A(s)\) with polynomial entries \(A_{ij}(s) \in \mathbb{C}[s]\) that generalizes \(A(T, n)\). This matrix is defined as follows,

\[
A_{ij}(s) = \sum_{0 \leq \ell \leq \min\{L_i, 2j + \delta\}} (-2)^\ell \binom{L_i}{\ell} \binom{s - \ell}{2j + \delta - \ell},
\]

where \(0 \leq L_0 < \cdots < L_k\) is a sequence of integers and \(\delta \in \{0, 1\}\). In Theorem 6.15 of [3] it is obtained an explicit formula for \(\det A(s)\) as a product of polynomials of degree one in \(s\). Hence, we know the exact values of \(s\) for which \(A(s)\) is singular. Moreover, from the proof of Theorem 6.15 it follows that whenever \(A(s)\) is singular the reason is that it has several pairs of equal rows. In this case the strategy consist in replacing one equation in each one of these pairs by a new equation obtained from Theorem 8.5. We refer the reader to Subsection 6.3 of [3] for the details.

Since our goal in this section is to prove Theorem 7.12 we need to restate Theorem 8.6 for elements \(b \in \widetilde{B}\). If \(b = \sum_{r=0}^m b_r \otimes Z^r \in \widetilde{B}\), it follows from (72) that for \(r\) even we have \(b^r_{2i, j} = 0\) if \(d(b^r_{2i, j}) = 2(i + j) \leq r\). Hence, when \(T - n \equiv 0\) and \(r \in R_F(T, n)\) is such that \(d(b^r_{T-r-n, n}) = T - r + n \leq r\), we have \(u^r_{T-r-n, n} = 0\) in equation (88). Then we may consider a new index set defined as follows,

\[
\widetilde{R}_F(T, n) = \begin{cases} 
\{r \in R_F(T, n) : r < \frac{T+n}{2}\}, & \text{if } T - n \equiv 0 \\
\{T, 2d_0 - T\}, & \text{if } T - n \equiv 1,
\end{cases}
\]

where the congruence is mod \((2)\). For \(b \in \widetilde{B}\) we restate Theorem 8.6 as follows. This theorem is the analogue of Theorem 6.19 of [3] and it will be our main tool in the proof of Theorem 7.12.

**Theorem 8.7.** Let \(b = b_m \otimes Z^m + \cdots + b_0 \in \widetilde{B}\) be such that \(d(b_r) \leq 2d_r\) for \(0 \leq r \leq m\), and let \(m \leq T \leq 2d_0\) and \(0 \leq n \leq \min\{T, 2d_0 - T\}\). Then if \(P(T)\) and \(Q(n)\) are true we have,

\[
\sum_{r \in \widetilde{R}_F(T, n)} \left( \sum_{\ell} (-2)^\ell \binom{L_i}{\ell} \binom{T-n-\ell}{r-\ell} u^r_{T-r-n, n} U^{(T+r+n)/2} \right) = 0,
\]

for every \(L \in L(T, n)\). Here \(u^r_{T-r-n, n} = r ! (-1)^r \hat{X}^T_{-n-r} \hat{E}^n(b^r_{T-r-n, n})\).

Now we recall the definition of the sets \(R(T, n)\) and \(\widetilde{R}(T, n)\) used in the case of the group \(\text{Sp}(n, 1)\) (see Section 6 of [3]). Let \(b = b_m \otimes Z^m + \cdots + b_0 \in \widetilde{B}\)
with \( b_m \neq 0 \). For positive integers \( T \) and \( n \) such that \( m \leq T \leq 4m \) and \( 0 \leq n \leq \min\{T, 4m - T\} \) consider the following set

\[
R(T, n) = \{ r \in \mathbb{N}_0 : 0 \leq r \leq \min\{m, \min\{T, 4m - T\} - n\}, \ r \equiv T - n \},
\]

where the congruence is mod (2). The set \( \tilde{R}(T, n) \) is defined as in (91) replacing \( R_F(T, n) \) by \( R(T, n) \) (see (116) in [3]). Next we will show that Theorem 7.12 follows from Proposition 6.21 and Proposition 6.22 of [3].

**Proof of Theorem 7.12.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in \tilde{B} \) be such that \( d(b_r) \leq 2d_r \) for \( 0 \leq r \leq m \). We need to show that \( b = 0 \). Assume on the contrary that \( b \neq 0 \) and that \( m = \deg(b) \), that is \( b_m \neq 0 \). We will obtain a contradiction by showing that \( b_m = 0 \). In view of the definition of \( \tilde{B} \) (see (72)) to do this it is enough to show that \( P \left( \frac{2m}{4} \right) \) holds if \( m \) is even and that \( P(m - 1) \) is true if \( m \) is odd. Since \( P(2d_0) \) holds (see (111) and (115)) this will follow from the fact that \( P(T) \) implies \( P(T - 1) \) for any \( m \leq T \leq 2d_0 \).

Consider first \( m \geq 1 \). Let \( m \leq T \leq 2d_0 \) and \( 0 \leq n \leq \min\{T, 2d_0 - T\} \), and assume that \( P(T) \) and \( Q(n) \) hold. Since \( 2d_0 \leq 4m \), it follows that \( \min\{T, 2d_0 - T\} \leq \min\{T, 4m - T\} \) and a simple calculation shows that

\[
\min\{m, \min\{T, 2d_0 - T\} - n\} \leq \min\{m, \min\{T, 4m - T\} - n\}.
\]

Hence \( R_F(T, n) \subset R(T, n) \) and therefore \( \tilde{R}_F(T, n) \subset \tilde{R}(T, n) \).

Now set, \( w_{T-r-n,n}^r = 0 \) if \( r \in \tilde{R}(T, n) \) and \( r \not\in \tilde{R}_F(T, n) \) and \( u_{T-r-n,n}^r = r!(-1)^rX_\delta^{T-n-r}E^n(b_{T-r-n,r}^r) \) if \( r \in \tilde{R}_F(T, n) \). Then from Theorem 8.7 we obtain for every \( L \in L(T, n) \) that,

\[
(92) \quad \sum_{r \in \tilde{R}(T, n)} \left( \sum_{\ell} (-2)^\ell \left( \frac{L}{\ell} \right) \binom{T-n-\ell}{r-\ell} \right) u_{T-r-n,n}^r U^{(T+r+n)/2} = 0,
\]

Observe that, except for the fact that the vector \( X_\delta X_4 \) is replaced by \( U \), the system of equations given by (92) is the same as that of Theorem 6.19 of [3], in particular, their coefficient matrices are exactly the same. Then that \( P(T) \) implies \( P(T - 1) \) for any \( m \leq T \leq 2d_0 \) follows from Proposition 6.21 and Proposition 6.22 of [3]. We point out that the proof of these propositions are based on a very thorough study of the coefficient matrix of these system. We refer the reader to Theorem 6.15, Corollary 6.16 and Proposition 6.20 of [3] for the details.

Consider now \( m = 0 \). Assume that \( b = b_0 \in \tilde{B}, b \neq 0 \), and that \( d(b) = d(b_0) \leq 2d_0 = 2 \). From the definition of \( \tilde{B} \) (see (72)) we have \( b = b_0 = b_{0,0} + b_{0,1}^0 \), therefore \( b_{0,0}^0 \neq 0 \) or \( b_{0,1}^0 \neq 0 \), in particular \( d(b) = 2 \). Consider the element \( b^2 \omega = b^2 \otimes Z^2 + \omega b^2 \otimes Z + b^2 \omega_0 \in B \), where \( \omega = 1 \otimes Z^2 + \omega_1 \otimes Z + \omega_0 \) is the element in \( P(U(g))K \) defined in Lemma 7.3.

From Proposition 5.3 we have \( d(b^2) = 4 \), hence the component of Kostant degree four of \( b^2 \) is nonzero. Now, as in Proposition 7.11 we can remove the components of Kostant degree less or equal to two from \( b^2 \) and the components of Kostant degree less or equal to zero from \( b^2 \omega_0 \). This procedure
defines an element $\tilde{b} = \tilde{b}_2 \otimes Z^2 + \tilde{b}_1 \otimes Z + \tilde{b}_0 \in \tilde{B}$ with $d(\tilde{b}_r) \leq 2d_r$ for $0 \leq r \leq 2$, and such that the component of Kostant degree four of $\tilde{b}_2$ is the same as that of $b^2$. Then $\tilde{b} \neq 0$, which contradicts the first part of the proof. Therefore $b = 0$, as we wanted to prove. \qed

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