DISCRETE HAMILTONIAN STRUCTURE OF SCHLESINGER TRANSFORMATIONS

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Abstract. Schlesinger transformations are algebraic transformations of a Fuchsian system that preserve its monodromy representation and act on the characteristic indices of the system by integral shifts. One of the important reasons to study such transformations is the relationship between Schlesinger transformations and discrete Painlevé equations; this is also the main theme behind our work. In this paper we show how to write an elementary Schlesinger transformation as a discrete Hamiltonian system w.r.t. the standard symplectic structure on the space of Fuchsian systems. We then show how Schlesinger transformations reduce to discrete Painlevé equations by considering two explicit examples, d-P(D^{(1)}_{4}) (or difference Painlevé V) and d-P(A^{(1)}_{2})*.

1. Introduction

In the theory of ordinary linear differential equations on a complex domain, and in particular in the theory of Fuchsian systems, an important characteristic of the equation is its configuration of singularities and the characteristic indices at these singular points. Associated to this data is the notion of the monodromy representation of the equation. Roughly speaking, this representation describes how the fundamental solution matrix of the equation changes under analytic continuation along the closed paths around the singular points and it gives a significant insight into the global behavior of solutions of the equation.

The fruitful idea of deforming the equation by moving the location of critical points into different configurations, or by changing the characteristic indices, without changing its monodromy representation, goes back to B. Riemann, but the actual foundations of this theory of isomonodromic deformations in the Fuchsian case were laid down in the works of R. Fuchs [Fuc07], L. Schlesinger [Sch12], and R. Garnier [Gar26]. An extension of the theory to the non-Fuchsian case was done relatively recently in the series of papers by M. Jimbo, T. Miwa, and K. Ueno, [JMU81, JM82, JM81]. At present, the theory of isomonodromic deformations is a very active research field with deep connections to other areas such as the theory of integrable systems, classical theory of differential equations, and differential and algebraic geometry.

In this work we focus on the relationship between isomonodromic deformations and Painlevé equations. We need to distinguish between continuous and discrete isomonodromic deformations. In the continuous case we move the location of singular points and the resulting Schlesinger equations reduce to Painlevé-type nonlinear differential equations. On the other hand, if we deform the characteristic indices, the isomonodromy condition requires that the indices change by integral shifts, and so the resulting dynamic is discrete and is expressed in the form of difference equations called Schlesinger transformations. Similarly to the continuous case, these transformations reduce to discrete analogues of Painlevé-type equations and the study of this correspondence has been a major research topic in the field of discrete integrable systems over the last twenty years, [RGH91, PNGR92]. We need to remark here that Schlesinger transformations correspond to difference Painlevé equations, but the other two types of discrete Painlevé equations, q-difference and elliptic-difference Painlevé equations can also be considered in a modification of this approach.

Discrete Painlevé equations share many properties with the differential Painlevé equations, e.g., the existence of special solutions such as algebraic solutions or solutions that can be expressed in terms of special functions, affine Weyl group symmetries, and the geometric classification of equations in terms of rational surfaces. However, while it is well-known how to write differential Painlevé equations in the Hamiltonian form [Oka80], the discrete Hamiltonian structure for discrete Painlevé equations is at present missing.

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The aim of the present paper is to study such discrete Hamiltonian structure for Schlesinger transformations. In particular, we present an explicit formula for the discrete Hamiltonian function of an elementary Schlesinger transformation expressed in the same coordinates as the Hamiltonian function of the continuous Schlesinger equations considered by Jimbo, Miwa, Môri, and Sato in [JMMS80]. Note that this explicit formula gives a convenient tool for computing Schlesinger transformations and consequently for the derivation of discrete Painlevé equations, since the usual computation of a Schlesinger transformation from the compatibility condition between the Fuchsian equation and the deformation equation is often quite complicated.

We also present two explicit examples of discrete Painlevé equation computed in this framework. The first example that we consider is an isomonodromic deformation of a two-dimensional Fuchsian system with three finite singular points. A nice feature of this example is that it allows us to compare, in the same setting and using the same coordinates, continuous deformations that are described by the Painlevé-VI equation $P_{VI}$, and discrete deformations that are described by the difference Painlevé-V equation $d-P_{VI}$, which corresponds to the Bäcklund transformations of $P_{VI}$. Recall that in terms of the surface of initial conditions $d-P_{VI}$ is also denoted as $d-P(D_4^{(1)})$. In the second example we consider a discrete Painlevé equation of the type $d-P(A_2^{(1)*})$ that does not have a continuous Painlevé counterpart. In [Sak07] one of the authors posed a problem of representing this equation using Schlesinger transformations of some Fuchsian system. In [Boa09] P. Boalch described the Fuchsian system whose Schlesinger transformations are described by $d-P(A_2^{(1)*})$ without doing the explicit coordinate computation. Also note that D. Arinkin and A. Borodin clarified the correspondence between $d-P(A_2^{(1)*})$ and the difference Fuchsian equation in [AB06]. Here we compute the $d-P(A_2^{(1)*})$ example explicitly using the discrete Hamiltonian and use it to illustrate some interesting features, as well as some difficulties, that occur when we study discrete Painlevé equations using this approach.

The text is organized as follows. In the next section we briefly set up the context of the problem and describe in more details some of the main objects. In Section 3 we derive the equations of the elementary Schlesinger transformations and in Section 4 we show how to write them in the Hamiltonian form. Section 5 is dedicated to the $d-P(D_4^{(1)})$ and $d-P(A_2^{(1)*})$ examples, and the final section is conclusions and discussion of the result.

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2. Preliminaries

2.1. Fuchsian Equations. Schlesinger transformations that we study originate from the deformation theory of Fuchsian equations. Recall that a Fuchsian equation (or a Fuchsian system) is a matrix linear differential equation on the Riemann sphere $\mathbb{CP}^1$ such that all of its singular points are regular singular points. We consider a generic case when a Fuchsian equation can be written in the Schlesinger normal form, i.e., when the coefficient matrix is a rational function with at most simple poles at some (distinct) points $u_1, \ldots, u_n$ (and possibly at the point $u_0 = \infty$),

\[
\frac{dY}{dx} = A(x)Y = \left( \sum_{i=1}^{n} \frac{A_i}{x - u_i} \right) Y,
\]

Here $A_i = \text{res}_{u_i} A(x) dx$ are constant $m \times m$ matrices. We also put $A_{\infty} = \text{res}_{\infty} A(x) dx = -\sum_{i=1}^{n} A_i$ and so this equation is regular at infinity iff $A_{\infty} = 0$.

Assumption 2.1. From now on we make an additional semi-simplicity assumption that the coefficient matrices $A_i$ are diagonalizable.

2.2. Spectral Type and Accessory Parameters. Geometrically, Schlesinger (or isomonodromic) dynamic takes place on the space of coefficients $A(x)$ of the Fuchsian equation (2.1), considered modulo gauge transformations. Here we briefly outline the description of this space following [Sak10].
With these definitions, given \( \Theta \), we separate the data about locations of singular points \( u_i \) (that we think of as parameters of the dynamic) and the residue matrices \( A_i \) at those points. Thus, we define

\[
\tilde{F}(u) = \tilde{F}(u_1, \ldots, u_n) = \{(A_1, \ldots, A_n) \mid A_i \in \text{End}(\mathbb{C}^m)\}
\]

to be the set of all Fuchsian equations with possible singularities at the points \( u_i \). Further, since Schlesinger dynamic either preserves (in the continuous case) or shifts (in the discrete case) the eigenvalues of \( A_i \), we treat the eigenvalues as the parameters of the dynamic as well. Thus, the appropriate phase space is a quotient of the fiber of the eigenvalue map by gauge transformations. Local coordinates on the phase space are called accessory parameters. Note that the space \( A \) of accessory parameters is quite complicated and its dimension depends on the spectral type of \( A(x) \) that we define next.

Spectral type of \( A(x) \) encodes the degeneracy of eigenvalues of the coefficient matrices \( A_i \) via partitions \( m = m_1 + \cdots + m_l \), \( m_1 \geq \cdots \geq m_l \geq 1 \), where \( m \) is the matrix size and \( m_i \) denotes the multiplicities of the eigenvalues of \( A_i \). The spectral type \( m \) of \( A(x) \) is then defined to be the collection of these partitions for all indices (including \( \infty \)):

\[
m = m_1 \cdots m_1, m_2 \cdots m_2, \ldots, m_n \cdots m_n, m_1^\infty \cdots m_l^\infty.
\]

It can be used to classify Fuchsian systems up to isomorphisms and the operations of addition and middle convolution introduced by N. Katz [Kat96], as in the recent work by T. Oshima [Osh08].

**Assumption 2.2.** Since for a Fuchsian system we can use scalar local gauge transformation of the form \( \tilde{Y}(x) = z(x)^{-1}Y(x) \), where \( z(x) \) is a solution of the scalar equation

\[
\frac{dz}{dx} = \sum_{i=1}^{n} \frac{\theta_i}{x - u_i} z,
\]

to change the residue matrices by \( \tilde{A}_i = A_i - \theta_i 1 \), we can assume, without loss of generality, that one of the eigenvalues \( \theta_i = 0 \). Thus, we always assume that the eigenvalue \( \theta_i \) of the highest multiplicity \( m_i \) is zero and we denote the corresponding subset of \( \tilde{F}(u) \) of reduced Fuchsian equations by \( F(u) \). This amounts to choosing a representative in the quotient space of all Fuchsian equations by the group of local scalar gauge transformations.

In view of Assumptions 2.1 and 2.2, \( A_i \) is similar to a diagonal matrix \( \text{diag}\{\theta_1, \ldots, \theta^r_i, 0, \ldots, 0\} \), where \( r_i = \text{rank}(A_i) \). Omitting the zero eigenvalues, we put

\[
\Theta_i = \text{diag}\{\theta_1, \ldots, \theta^r_i\}.
\]

Denote by \( \tilde{T} \) the set of all possible diagonal matrices of the spectral type \( m \) with the highest degeneracy eigenvalues at finite points set to 0 and omitted, as above. Thus, \( \tilde{T} = \{\Theta = (\Theta_1, \ldots, \Theta_n; \Theta_{\infty})\} \) and \( r_i = m - m_1 \). Then we have an eigenvalue map \( \text{Sp} : F(u) \to T \) from the set of reduced Fuchsian equations to the set of characteristic indices, where \( T \subset \tilde{T} \) is the subset of \( \Theta \) satisfying the Fuchs relation (or the trace condition)

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \theta_i^j = 0.
\]

With these definitions, given \( \Theta \in T \), the fiber of the eigenvalue map is \( F(u)_{\Theta} = \text{Sp}^{-1}(\Theta) \).

We still need to take into account the global similarity transformations. It is convenient to do this in two steps. First, we use such transformations to normalize our equation at infinity by reducing \( A_\infty \) to a particular form, e.g., make it diagonal, and then take the further quotient by the stabilizer subgroup \( G_{A_\infty} \) of the group \( \text{GL}_m \) of global similarity transformations. Thus, for a given \( \Theta \in T \), we fix \( A_\infty \) such that \( \text{Sp}(A_\infty) = \Theta_{\infty} \) and denote by \( F(u)_{(\Theta, A_\infty)} \) the subset of all Fuchsian equations in \( F(u)_{\Theta} \) satisfying the condition \( A_1 + \cdots + A_n = -A_\infty \). Then, finally, our phase space is

\[
A = A(\Theta, A_\infty) = F(u)_{(\Theta, A_\infty)}/G_{A_\infty}.
\]
When $A_\infty$ is diagonal, which is the case that we are mostly interested in, the dimension of this space $A$ of accessory parameters of the spectral type $m$ is given by the formula

$$\dim A = (n - 1)m^2 - \sum_{i=1}^{n,\infty} \left( \sum_{j=1}^{l_i} (m_j)^2 \right) + 2.$$  

2.3. The Decomposition Space. To study the Hamiltonian structure of Schlesinger equations, and also of Schlesinger transformations, it is convenient to consider a larger decomposition space that is defined as follows.

In view of Assumption 2.1, there exist full sets of right eigenvectors $b_{i,j}$, $A_i b_{i,j} = \theta_{i,j}^i b_{i,j}$, and left eigenvectors $c^\dagger_i$, $c^\dagger_i A_i = \theta_{i,j}^i c^\dagger_i$, (here we use the $\dagger$ symbol to indicate a row-vector). In the matrix form, omitting vectors with indices $j > r_i$ that are in the kernel of $A_i$, we can write

$$B_i = \begin{bmatrix} b_{i,1} \cdots b_{i,r_i} \end{bmatrix}, \quad A_i B_i = B_i \Theta_i, \quad C^\dagger_i = \begin{bmatrix} c^\dagger_i \vphantom{c^\dagger_{r_i}} \vphantom{c^\dagger_{r_i}} \cdots \vphantom{c^\dagger_{r_i}} \vphantom{c^\dagger_{r_i}} \vphantom{c^\dagger_{r_i}} c^\dagger_{r_i} \end{bmatrix}, \quad C^\dagger_i A_i = \Theta_i C^\dagger_i,$$

with $\Theta_i$ defined by (2.2). But this means that we have a decomposition $A_i = B_i C^\dagger_i$ provided that $C^\dagger_i B_i = \Theta_i$, where the last condition is related to the normalization ambiguity of the eigenvectors.

Thus, given $A_i$, we can construct, in a non-unique fashion, a corresponding decomposition pair $(B_i, C^\dagger_i)$. The space of all such pairs for all finite indices $1 \leq i \leq n$, without any additional conditions, we call the decomposition space. We denote it as

$$B \times C = (C^{r_1} \times \cdots \times C^{r_n}) \times ((C^{r_1})^\dagger \times \cdots \times (C^{r_n})^\dagger) \simeq (C^{r_1} \times (C^{r_1})^\dagger) \times \cdots \times (C^{r_n} \times (C^{r_n})^\dagger),$$

since it is convenient to write an element $(B, C^\dagger)$ of this space as a list of $n$ pairs $(B_1, C^\dagger_1; \cdots; B_n, C^\dagger_n)$. This space can be equipped with the natural symplectic structure where we take the matrix elements of $B_i$ and $C^\dagger_i$ as canonical coordinates, i.e., we take our symplectic form to be

$$\omega = \sum_{i=1}^n \text{tr}(dC^\dagger_i \wedge dB_i).$$

Remark 2.3. To avoid symbol overcrowding we slightly abuse the notation and use the same symbols, e.g., $B_i$ or $b_{i,j}$, to denote both the canonical coordinate system in the decomposition space and an actual point in the space; the exact meaning of the symbol is always clear from the context.

The idea of using the decomposition space $B \times C$ to study the Hamiltonian structure of isomonodromic deformations goes back to the paper by M. Jimbo, T. Miwa, Y. Môri, and M. Sato [JMMS80]. Since then it has been used by many other researchers studying isomonodromic deformations, most notably by J. Harnad, see, e.g., [Har94].

Remark 2.4. There are two natural actions on the decomposition space $B \times C$. First, the group $G_{\mathbb{L}}$ of global gauge transformations of the Fuchsian system induces the following action. Given $P \in G_{\mathbb{L}}$, we have the action $A_i \mapsto PA_i P^{-1}$ which translates into the action $(B_i, C^\dagger_i) \mapsto (PB_i, C^\dagger_i P^{-1})$. We refer to such transformations as similarity transformations. It is often necessary to restrict this action to the subgroup $G_{A_\infty}$ preserving the form of $A_\infty$. Second, for any pair $(B_i, C^\dagger_i)$ the pair $(B_i Q_i, Q_i^{-1} C^\dagger_i)$ determines the same matrix $A_i$ for $Q_i \in G_{\mathbb{L}}$. The condition $Q_i^{-1} C^\dagger_i B_i Q_i = Q_i^{-1} \Theta_i Q_i = \Theta_i$ restricts $Q_i$ to the stabilizer subgroup $G_{\Theta_i}$ of $G_{\mathbb{L}}$. In particular, when all $\theta_{i,j}^i$ are distinct, $Q_i$ has to be a diagonal matrix. We refer to such transformations as trivial transformations. These two actions obviously commute with each other.

We are now ready to give an alternative description of the space of accessory parameters $A_{(\Theta, A_\infty)}$. Given the pair $(\Theta, A_\infty)$ as in the previous section, let

$$(B \times C)_{(\Theta, A_\infty)} = \{(B_1, C^\dagger_1; \cdots; B_n, C^\dagger_n) \in B \times C \mid C^\dagger_i B_i = \Theta_i, \sum_{i=1}^n B_i C^\dagger_i = -A_\infty\} \subset B \times C.$$
Then
\[ A = A(\Theta, A_{\infty}) = (B \times C)(\Theta, A_{\infty})/(G_{\Theta_1} \times \cdots \times G_{\Theta_n}) \times G_{A_{\infty}}. \]

The following diagram illustrates the relationship between different spaces defined in these two sections:

![Diagram]

**Figure 1.** The decomposition space and the space of accessory parameters

### 2.4. Schlesinger Equations and Schlesinger Transformations.

Consider now isomonodromic deformations of a Fuchsian system. For the continuous case, the most interesting situation is when we take the deformation parameters to be locations of critical points. Thus, we let \( A = A(x, u) \) (i.e., \( A_i = A_i(u) \)), and consider a family of Fuchsian equations on the function \( Y = Y(x; u) \),

\[
\frac{dY}{dx} = A(x; u)Y = \left( \sum_{i=1}^{n} \frac{A_i(u)}{x - u_i} \right) Y.
\]

In [Sch12] L. Schlesinger showed that the monodromy preserving condition is equivalent to the set of deformation equations

\[
\frac{\partial Y}{\partial u_i} = -\frac{A_i(u)}{x - u_i} Y, \quad i = 1, \ldots, n.
\]

The compatibility conditions for these deformation equations are called *Schlesinger equations*. These are partial differential equations on the coefficient matrices \( A_i(u) \) and they have the form

\[
\frac{\partial A_i}{\partial u_j} = \frac{[A_i, A_j]}{u_i - u_j} \quad (i \neq j), \quad \frac{\partial A_i}{\partial u_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}.
\]

In [JMMS80] Jimbo et al. showed that Schlesinger equations can be written as a Hamiltonian system on the decomposition space \( B \times C \),

\[
\frac{\partial B_i}{\partial u_j} = \frac{\partial H_j}{\partial C_i^\dagger}, \quad \frac{\partial C_i^\dagger}{\partial u_j} = -\frac{\partial H_j}{\partial B_i},
\]

with the Hamiltonian

\[
H_j = H_j(B, C^\dagger) = \sum_{i \neq j} \frac{\text{tr}(A_j A_i)}{u_j - u_i}, \quad A_i = B_i C_i^\dagger.
\]

Schlesinger equations admit reductions to Painlevé-type nonlinear differential equations, which remains valid on the level of the Hamiltonians as well; we review an example of \( P_{VI} \) in Section 5. The main question that we consider is what happens to this picture in the discrete case.

The discrete counterparts of Schlesinger equations are Schlesinger transformations. Those are rational transformations preserving the singularity structure and the monodromy data of the Fuchsian system (2.1), except for the integral shifts in the characteristic indices \( \theta^j_i \). The study of such transformations again goes back to Schlesinger [Sch12], and this and the more general case of irregular singular points was considered in great detail in [JM81]. Schlesinger transformations have the form \( Y(x) \rightarrow \bar{Y}(x) = R(x)Y(x) \), where \( R(x) \)
is a specially chosen rational matrix function called the multiplier of the transformation. The coefficient matrix \( A(x) \) of our Fuchsian system then transforms to \( \bar{A}(x) \) that is related to \( A(x) \) by the equation

\[
\bar{A}(x)R(x) = R(x)A(x) + \frac{dR(x)}{dx}.
\]

Similarly to the continuous case, Schlesinger transformations induce discrete Painlevé-type dynamic on the space of accessory parameters. This dynamic is very complicated, but it becomes much simpler when considered on the decomposition space, which allows us to represent it as a discrete Hamiltonian system using the same canonical coordinates as [JMMS80], as in Figure 2:

\[
\begin{align*}
\mathcal{B} \times \mathcal{C} \quad &\xrightarrow{\text{Discrete Hamiltonian Dynamic}} \quad \mathcal{B} \times \mathcal{C} \\
(\mathcal{B} \times \mathcal{C})(\Theta,A_\infty) \quad &\xrightarrow{\text{Painlevé Dynamic}} \quad (\bar{B} \times \bar{C})(\bar{\Theta},\bar{A}_\infty) \\
A(\Theta,A_\infty) \quad &\xrightarrow{\text{Schlesinger transformations}} \quad A(\bar{\Theta},\bar{A}_\infty) \\
\setminus \quad &\quad \setminus \\
\mathcal{T} \quad &\quad \mathcal{T}
\end{align*}
\]

Figure 2. Schlesinger Transformations

Unfortunately, at present we do not have a discrete version of the Hamiltonian reduction procedure from the decomposition space to the space of accessory parameters, this is an important problem for future research.

In this paper we focus on the elementary Schlesinger transformations that only change two of the characteristic indices by unit shifts, i.e., \( \bar{\theta}_\alpha = \theta_\alpha - 1 \) and \( \bar{\theta}_\beta = \theta_\beta + 1 \). For the examples section, we also assume \( \theta_\alpha \) and \( \theta_\beta \) have no multiplicities. In [JM81] such elementary transformation are denoted by \( \{ \alpha \beta \mu \nu \} \). In Section 4 we construct a function \( \mathcal{H}^+(\mathcal{B}, \mathcal{C}) \) of the decomposition space \( \mathcal{B} \times \mathcal{C} \) that is a discrete Hamiltonian function, in the sense explained in the next section, for such elementary Schlesinger transformation.

2.5. Discrete Lagrangian and Hamiltonian Formalism. In developing a discrete version of the Hamiltonian formalism, one approach is to discretize the continuous dynamic preserving its integrability properties, see the recent encyclopedic book by Y. Suris, [Sur03], but it requires underlying continuous dynamical system. An alternative and more direct procedure is to develop a discrete version based on the variational principles. In this case, the Lagrangian formalism is more natural, but it is possible to extend it to include the discrete version of the Hamiltonian formalism as well. This approach has its origin in the optimal control theory (see, e.g., B. Jordan and E. Polak [JP64], and J. Cadzow [Cad70]) and mechanics (see e.g., J. Logan [Log73] and S. Maeda [Mae82]). In the theory of integrable systems it was first used in the foundational works by A. Veselov [Ves91, Ves88] and A. Veselov and J. Moser [MV91]. A lot of recent work in this field is motivated by the development of very effective symplectic integrators, see an excellent review paper by J. Marsden and M. West [MW01] (and the references therein), as well as more recent works by S. Lall and M. West [LW06] and A. Bloch, M. Leok, and T. Oshawa [OBL11] that emphasize the Hamiltonian aspects of the theory. Below we give a very brief outline of this approach closely following [OBL11], we refer to this and other publications above for details.

Let \( Q \) be our configuration space. Then, in the discrete case, the Lagrangian \( L \in \mathcal{F}(Q \times Q) \) is a function on the state space \( Q \times Q \). For the discrete time parameter \( k \in \mathbb{Z} \), a trajectory of motion is a map \( q : \mathbb{Z} \to Q \),
$q(k) = q_k$, or, equivalently, a sequence $\{q_k \in Q\}$. The action functional on the space of such sequences is given by $\tilde{S}(\{q_k\}) = \sum_k L(q_k, q_{k+1})$. Using the variational principle $\delta \tilde{S} = 0$ we obtain discrete Euler-Lagrange equations that have the form

$$D_2 L(q_{k-1}, q_k) + D_1 L(q_k, q_{k+1}) = 0,$$

where $D_1$ (resp. $D_2$) denote vectors of partial derivatives w.r.t. the first (resp. second) sets of local coordinates on the state space $Q \times Q$. These equations then implicitly determine the map (or more precisely, the correspondence) $q_{k+1} = \phi(q_{k-1}, q_k)$, which then defines the discrete Lagrangian flow $F_L : Q \times Q \rightarrow Q \times Q$ on the state space via $F_L(q_{k-1}, q_k) = (q_k, q_{k+1})$. This flow is symplectic w.r.t. the discrete Lagrangian symplectic form

$$\Omega_L(q_k, q_{k+1}) := d\vartheta_L = D_1 D_2 L(q_k, q_{k+1}) dq_k \wedge dq_{k+1},$$

$F_L^* \Omega_L = \Omega_L$, where one-forms $\vartheta_L : Q \times Q \rightarrow T^* (Q \times Q)$ are defined by

$$\vartheta_L(q_k, q_{k+1}) = D_2 L(q_k, q_{k+1}) dq_{k+1}, \quad \vartheta_L(q_k, q_{k+1}) = - D_1 L(q_k, q_{k+1}) dq_k.$$

If we introduce right and left discrete Legendre transforms $FL^\pm : Q \times Q \rightarrow T^* Q$ and the momenta variables $p_k, p_{k+1}$ by

$$FL^+(q_k, q_{k+1}) = (q_{k+1}, p_{k+1}) = (q_{k+1}, D_2 L(q_k, q_{k+1})), \quad FL^-(q_k, q_{k+1}) = (q_k, p_k) = (q_k, -D_1 L(q_k, q_{k+1})),$$

we see that $\vartheta_L = (FL^\pm)^* \vartheta$ and $\Omega_L = (FL^\pm)^* \Omega$, where $\vartheta$ and $\Omega$ are the standard Liouville and symplectic forms on $T^* Q$ respectively.

We can then define the discrete Hamiltonian flow $\tilde{F}_L : T^* Q \rightarrow T^* Q$ by $\tilde{F}_L(q_k, p_k) = (q_{k+1}, p_{k+1})$, i.e., $\tilde{F}_L = FL^+ \circ (FL^-)^{-1}$. But then the equations

$$p_k = -D_1 L(q_k, q_{k+1}), \quad p_{k+1} = D_2 L(q_k, q_{k+1}),$$

mean that $L(q_k, q_{k+1})$ is just the generating function (of type one, see [GPS00] for the terminology) of the canonical transformation $\tilde{F}_L$. We can then define the right and left discrete Hamiltonian functions as generating functions of this canonical transformation of type two and three respectively. Namely, right discrete Hamiltonian is

$$H^+(q_k, p_{k+1}) := p_{k+1} q_{k+1} - L(q_k, q_{k+1}),$$

and then the map $\tilde{F}_L : T^* Q \rightarrow T^* Q$ is given (implicitly) by the right discrete Hamiltonian equations

$$q_{k+1} = D_2 H^+(q_k, p_{k+1}), \quad p_k = D_1 H^+(q_k, p_{k+1}).$$

Similarly, left discrete Hamiltonian

$$H^-(q_{k+1}, p_k) := -p_k q_k - L(q_k, q_{k+1})$$

gives left discrete Hamiltonian equations

$$q_k = -D_1 H^-(q_{k+1}, p_k), \quad p_{k+1} = -D_2 H^-(q_{k+1}, p_k).$$

For completeness, we mention that the generating function of type four,

$$R(p_k, p_{k+1}) := q_k p_k - q_{k+1} p_{k+1} + L(q_k, q_{k+1}) = q_k p_k - H^+(q_k, p_{k+1}) = -q_{k+1} p_{k+1} - H^-(q_{k+1}, p_k),$$

$q_k = D_1 R(p_k, p_{k+1}), \quad q_{k+1} = D_2 L(q_k, q_{k+1}),$

is nothing but the Lagrangian function on the space of momenta satisfying the discrete Euler-Lagrange equations

$$D_2 R(p_{k-1}, p_k) + D_1 R(p_k, p_{k+1}) = 0.$$
2.6. **Elementary Divisors.** For elementary Schlesinger transformations \( \{ \alpha \beta \}_{\mu \nu} \) the multiplier matrix \( R(x) \) has a very special simple form, and in this section we describe some useful properties of such matrices that we need for our computations. Consider a rational matrix function

\[
R(x) = I + \frac{G}{x - z}, \quad \text{where } G = fg^\dagger \text{ is a matrix of rank 1.}
\]

Because of their role in representing general matrix functions as a product of factors of this form, such matrices are sometimes called elementary divisors [Dzh09]. Assuming that \( \text{tr}(G) = g^\dagger f \neq 0 \), which is the case we need, we can consider instead of \( G \) a corresponding rank-one projector \( P = f(g^\dagger f)^{-1}g^\dagger, P^2 = P \).

**Lemma 2.5.** Let

\[
R(x) = I + \frac{z - \zeta}{x - z} \frac{fg^\dagger}{fg^\dagger f}.
\]

Then we have the following

(i) basic properties,

\[
\det R(x) = \frac{x - \zeta}{x - z}, \quad R(x)^{-1} = I + \frac{\zeta - z}{x - z} \frac{fg^\dagger}{fg^\dagger f}, \quad R(z)f = \left( \frac{x - \zeta}{x - z} \right) f, \quad g^\dagger R(z) = \left( \frac{x - \zeta}{x - z} \right) g^\dagger;
\]

(ii) the Vanishing Rule,

\[
\partial_t (w^\dagger R(x)v)f = 0, \quad g^\dagger \partial_{g^\dagger} (w^\dagger R(x)v) = 0;
\]

(iii) and the Exchange Rule,

\[
v \partial_v (w^\dagger R(x)v) - \partial_{w^\dagger} (w^\dagger R(x)v)w^\dagger = \partial_{g^\dagger} (w^\dagger R(x)v)g^\dagger - f \partial_t (w^\dagger R(x)v).
\]

**Proof.** Part (i) is a simple direct computation, see also [Dzh09].

Next, notice that for a rank-one projector \( P = f(g^\dagger f)^{-1}g^\dagger, \)

\[
f \partial_t (w^\dagger Pv) = Pv w^\dagger (I - P) \text{ and } \partial_{g^\dagger} (w^\dagger Pv) g^\dagger = (I - P)vw^\dagger P,
\]

and taking the trace proves the Vanishing Rule. Finally,

\[
v \partial_v (w^\dagger R(x)v) - \partial_{w^\dagger} (w^\dagger R(x)v)w^\dagger = vw^\dagger R(x) - R(x)vw^\dagger = \left( \frac{x - \zeta}{x - z} \right) (vw^\dagger P - Pvw^\dagger),
\]

and

\[
\partial_{g^\dagger} (w^\dagger R(x)v)g^\dagger - f \partial_t (w^\dagger R(x)v) = \left( \frac{x - \zeta}{x - z} \right) (vw^\dagger P - Pvw^\dagger), \]

which proves the Exchange Rule. \( \square \)

3. **Schlesinger Transformations**

In this section we derive equations of an elementary Schlesinger transformation \( \{ \alpha \beta \}_{\mu \nu} \) in terms of the coordinates on the decomposition space \( B \times C \). We now take \( R(x) \) to be of the form considered in Lemma 2.5. Then we have the following result.

**Lemma 3.1.** Consider transformation (2.2) with

\[
R(x) = I + \frac{z - \zeta}{x - z} \frac{fg^\dagger}{fg^\dagger f}, \quad z \neq \zeta.
\]

Then

(i) the poles \( \{ u_i \} \) of \( \tilde{A}(x) \) coincide with the poles \( \{ u_i \} \) of \( A(x) \) if and only if, for some choice of indices \( \alpha, \beta \neq \alpha, \mu, \) and \( \nu \), we have

\[
z = u_\alpha, \zeta = u_\beta, \quad f \sim \tilde{b}_{\alpha, \mu} \sim b_{\beta, \nu}, \quad g^\dagger \sim c_{\alpha}^{\mu} \sim \tilde{c}_{\beta}^{\nu}, \quad \tilde{\partial}_\alpha^\mu = \partial_\alpha^\mu - 1, \tilde{\partial}_\beta^\nu = \partial_\beta^\nu + 1,
\]

where \( \sim \) means proportional. Thus, such \( R(x) \) defines the elementary transformation \( \{ \alpha \beta \}_{\mu \nu} \).
(ii) For $R(x)$ satisfying (3.2), the residue matrices of $A(x)$ and $\tilde{A}(x)$ are connected by the following equations:

\begin{align*}
(3.3) & \quad \tilde{A}, R(u_i) = R(u_i)A_i \quad \text{or equivalently} \quad A, R(u_i)^{-1} = R(u_i)^{-1}\tilde{A}_i, \quad i \neq \alpha, \beta, \\
(3.4) & \quad \tilde{A}_\beta R(u_\beta) = R(u_\beta)A_\beta \quad \text{and} \quad A_\alpha R(u_\alpha)^{-1} = R(u_\alpha)^{-1}\tilde{A}_\alpha.
\end{align*}

Proof. From the Schlesinger transformation equations

\begin{align*}
(3.5) & \quad \tilde{A}(x)R(x) = R(x)A(x) + \frac{dR(x)}{dx} \quad \text{or} \quad R(x)^{-1}\tilde{A}(x) = A(x)R(x)^{-1} - \frac{dR(x)^{-1}}{dx},
\end{align*}

and

\[ \frac{dR(x)}{dx} R(x)^{-1} = -R(x)\frac{dR^{-1}}{dx} = \left( \frac{1}{x - \zeta} - \frac{1}{x - z} \right) fg^\dagger, \]

we see that

\[ \text{tr} \frac{dR}{dx} R(x)^{-1} = \left( \frac{1}{x - \zeta} - \frac{1}{x - z} \right) = \text{tr} \tilde{A}(x) - \text{tr} A(x) = \sum_{i=1}^n \frac{\text{tr}(\tilde{A}_i) - \text{tr}(A_i)}{x - u_i}, \]

we see that $z, \zeta \in \{ u_i \}_{i=1}^n$. Let $z = u_\alpha$ and $\zeta = u_\beta$. Then $\text{tr}(\tilde{A}_\alpha) - \text{tr}(A_\alpha) = -1$ and $\text{tr}(\tilde{A}_\beta) - \text{tr}(A_\beta) = 1$.

Since both $R(x)$ is regular and invertible at $u_i$ for $i \neq \alpha, \beta$, taking residues of (3.5) at $u_i$ gives (3.3). Since $R(x)$ is regular (but degenerate) at $u_\beta$, we can take a residue at $u_\beta$ of the first equation in (3.5) to get the second equation in (3.4). Similarly, taking the residue of the second equation in (3.5) at $u_\alpha$ we get the second equation in (3.4).

Further, at $u_\alpha$, the first equation in (3.5) becomes

\[ \frac{u_\alpha - u_\beta}{(x - u_\alpha)^2} \frac{fg^\dagger}{g^\dagger} = \left( \frac{A_\alpha}{x - u_\alpha} + \sum_{i \neq \alpha} \frac{\tilde{A}_i}{x - u_i} \right) \left( I + \frac{u_\beta - u_\alpha}{x - u_\alpha} \frac{fg^\dagger}{g^\dagger} \right) \]

\[ - \left( I + \frac{u_\beta - u_\alpha}{x - u_\alpha} \frac{fg^\dagger}{g^\dagger} \right) \left( \frac{A_\alpha}{x - u_\alpha} + \sum_{i \neq \alpha} \frac{\tilde{A}_i}{x - u_i} \right). \]

The $(x - u_\alpha)^{-2}$-terms give

\[ fg^\dagger = fg^\dagger A_\alpha - \tilde{A}_\alpha fg^\dagger, \]

and so we see that $g^\dagger$ has to be an eigenvector of $A_\alpha$ and $f$ has to be an eigenvector of $\tilde{A}_\alpha$. Choose an index $\mu$ and let $g^\dagger \sim c^\mu_\alpha$ and $f \sim b^\mu_\alpha, \nu$, then $\theta^\mu_\alpha = \theta^\mu_\alpha + 1$. Taking the residue at $u_\alpha$ gives

\[ \tilde{A}_\alpha + \sum_{i \neq \alpha} \frac{u_\alpha - u_\beta}{u_\alpha - u_i} \frac{\tilde{A}_i}{g^\dagger} = A_\alpha + \sum_{i \neq \alpha} \frac{u_\alpha - u_\beta}{u_\alpha - u_i} \frac{f}{g^\dagger} A_i, \]

but this equation follows from $-\tilde{A}_{\infty} = \sum_{i=1}^n \tilde{A}_i = \sum_{i=1}^n A_i = -A_{\infty}$ and

\[ \tilde{A}_i - A_i = \frac{u_\alpha - u_\beta}{u_\alpha - u_i} \frac{\tilde{A}_i}{g^\dagger} - \frac{f}{g^\dagger} A_i, \quad i \neq \alpha, \]

which is an immediate consequence of (3.3). Repeating this argument for $u_\beta$ and the second equation in (3.5) implies that $g^\dagger \sim c^\nu_\beta$, $f \sim b_{\beta,\nu}$, and $\theta^\nu_\beta = \theta^\nu_\beta + 1$, for some choice of the index $\nu$, which completes the proof.

We can extend elementary Schlesinger transformations to the decomposition space as follows.

**Corollary 3.2.** Let $A(x)$ and $\tilde{A}(x)$ be connected by an elementary Schlesinger transformation $\{ \alpha_\mu \beta \nu \}$ given by $R(x)$ of the form (3.1) satisfying (3.2). Then the corresponding points $(B, C^\dagger)$ and $(B, C^\dagger)$ in the decomposition space are related as follows:

(i) If $i \neq \alpha, \beta$, $R(u_i)$ is invertible and

\begin{align*}
(3.6) & \quad C^\dagger_i \sim C^\dagger_i R(u_i) \quad \text{(equivalently, } C^\dagger_i \sim C^\dagger_i R(u_i)^{-1} \text{) and} \quad B_i \sim R(u_i)B_i \quad (B_i \sim R(u_i)^{-1}B_i). \]
\end{align*}
(ii) For the special indices $\alpha$ and $\beta$ we have

$$C^\dagger_i \sim C^\dagger_i R(u_\beta), \quad B_\beta \sim R(u_\beta)B_\beta, \quad \text{and} \quad C^\dagger_\alpha \sim C^\dagger_\alpha R(u_\alpha)^{-1}, \quad B_\alpha \sim R(u_\alpha)^{-1}B_\alpha.$$  

Proof. Multiplying first equation in (3.3) on the right by $B_\beta$ gives $A_i R(u_\beta)B_i = R(u_\beta)A_i B_i = R(u_\beta)B_i \Theta_\beta$, and so the matrix $R(u_\beta)B_i$ is the matrix of eigenvectors of $A_i$, $R(u_i)B_i \sim B_i$. The remaining equations are proved similarly.

Remark 3.3. Recall that the matrices $R(u_\beta)$ and $R(u_\alpha)^{-1}$ are degenerate, $R(u_\beta)b_{\beta,\nu} = R(u_\alpha)^{-1}b_{\alpha,\mu} = 0$ and $c^\dagger_i R(u_\beta) = c^\dagger_i R(u_\alpha)^{-1} = 0$, see (i), so the proportionality coefficients in (3.7) at $b_{\alpha,\mu}$, $b_{\beta,\nu}$, $c^\dagger_i$, and $c^\dagger_i$ are zero.

4. The Generating Function

The main objective of this section is to obtain an explicit formula for the discrete Hamiltonian function $H^+(B, C^\dagger)$ that generates Schlesinger transformations dynamic on the decomposition space. In this case the equations (2.5) take the form (recall Remark 2.3)

$$\frac{\partial H^+}{\partial c^\dagger_i}(B, C^\dagger), \quad \text{and} \quad c^\dagger_i = \frac{\partial H^+}{\partial b_{i,j}}(B, C^\dagger),$$

where we use the identifications $T^*(\mathbb{C}^n) \simeq \mathbb{C}^n \times (\mathbb{C}^n)\dagger$.

First we introduce some notation. Let

$$\mathcal{I} = \{(i, j) | 1 \leq i \leq n, 1 \leq j \leq r_1\} \text{ and } \mathcal{I}' = \mathcal{I} \setminus \{(\alpha, \mu), (\beta, \nu)\},$$

and for a multi-index $I = (i, j) \in \mathcal{I}'$, we put

$$b_I = b_{i,j}, \quad c^\dagger_i = c^\dagger_i, \quad \theta_I = \theta^\dagger_i, \quad R_I = R(u_i) \text{ when } i \neq \alpha \text{ and } R_\alpha = I - \frac{b_{\alpha,\mu} c^\dagger_i}{c^\dagger_i b_{\alpha,\mu}} - \frac{b_{\beta,\nu} c^\dagger_i}{c^\dagger_i b_{\beta,\nu}}.$$

Theorem 4.1. Let

$$R(x) = I + \frac{u_{\alpha} - u_{\beta}}{x - u_{\alpha}} b_{\beta,\nu} c^\dagger_i,$$

$$S(B, C^\dagger; \Theta) = \sum_{I \in \mathcal{I}'} \theta_I \log(c^\dagger_i R_I b_I), \quad \text{and}$$

$$H^+(B, C^\dagger; \Theta) = (\theta^\dagger_i - \theta^\dagger_\alpha + 1) \log(c^\dagger_i b_{\beta,\nu}) + \theta^\dagger_\alpha \log(c^\dagger_i b_{\alpha,\mu}) + (\theta^\dagger_{\mu} - 1) \log(b_{\alpha,\mu} c^\dagger_i b_{\beta,\nu}) + S(B, C^\dagger; \Theta).$$

Then, if $R(x)$ is the matrix defining the elementary transformation $\{a_{\mu, \nu}\}$ from $A(x)$ to $A(x)$, $(B, C^\dagger) \in (\mathcal{B} \times \mathcal{C})(\Theta, A^\infty_i)$, $(B, C^\dagger) \in (B \times C)(\Theta, A^\infty_i)$ are the corresponding points in the decomposition space $\mathcal{B} \times \mathcal{C}$, and $\Theta_i = C^\dagger_i B_i$, we have

$$\frac{\partial H^+}{\partial c^\dagger_i}(B, C^\dagger; \Theta) \quad \text{and} \quad c^\dagger_i = \frac{\partial H^+}{\partial b_{i,j}}(B, C^\dagger; \Theta).$$

We prove this theorem using a sequence of Lemmas. The generic situation is very simple.

Lemma 4.2. Let $R(x)$ be any matrix of the form (3.1) with $u_i \neq z$, $z$, and let $A_i R(u_i) = R(u_i)A_i$, with $A_i = B_i C^\dagger_i$, $A_i = B_i C_i^\dagger$, and $C_i^\dagger B_i = C_i^\dagger B_i = \Theta_i$. Then

$$\frac{\partial c^\dagger_i}{\partial b_{i,j}} \left( \theta^\dagger_i \log \left( c^\dagger_i R(u_i) b_{i,j} \right) \right) \quad \text{and} \quad c^\dagger_i = \frac{\partial c^\dagger_i}{\partial b_{i,j}} \left( \theta^\dagger_i \log \left( c^\dagger_i R(u_i) b_{i,j} \right) \right).$$

Proof. Indeed, in view of Corollary 3.2, $C^\dagger_i \simeq C^\dagger_i R(u_i)$, and the normalization $C^\dagger_i B_i = \Theta_i$ gives

$$c^\dagger_i = \theta^\dagger_i c^\dagger_i R(u_i) b_{i,j} = \partial c^\dagger_i \left( \theta^\dagger_i \log \left( c^\dagger_i R(u_i) b_{i,j} \right) \right).$$

The other equation is proved in exactly the same way.

The following Corollary is then immediate.
Corollary 4.3. Equations (4.5) in Theorem 4.1 hold for all indices (i, j) with i \neq \alpha, \beta and for (\beta, j) for j \neq \nu. Moreover, for i \neq \alpha, \beta we have

$$A_i = \sum_{j=1}^{r_i} b_{i,j} \partial b_{i,j} \left( \mathcal{H}^+ (B, \hat{C}^\dagger; \Theta) \right), \quad \bar{A}_i = \sum_{j=1}^{r_i} \left( \partial_{\hat{e}_i} \mathcal{H}^+ (B, \hat{C}^\dagger; \Theta) \right) \hat{e}_i^\dagger.$$

For the index (\beta, \nu), since \( R(u^{_\beta})b_{\nu, \beta} = 0 \) and \( \hat{e}_\nu^\dagger R(u^{_\beta}) = 0 \), a different approach is needed to find expressions for \( \bar{B}_{\beta, \nu} \) and \( c^\nu_{\beta} \). We do it in Lemma 4.5.

Let us now consider the indices \( \alpha \) and \( \beta \) defining the transformation. In this case \( R(x) \) becomes either singular or degenerate and some regularization is necessary. We do it using the equations that cut out the locus \( \mathcal{A} \subset \mathcal{B} \times \mathcal{C} \).

Lemma 4.4. In the setting of Theorem 4.1 with \( R_\alpha \) given by (4.1), we have

\[
\begin{align*}
\bar{b}_{\alpha, \mu} &= \partial_{\hat{e}_\alpha} \left( (\theta^\mu_\alpha - 1) \log (\hat{e}_\alpha^\mu b_{\beta, \nu}) \right), \\
\bar{c}^\mu_{\alpha} &= \partial_{\beta} \left( \theta^\mu_\alpha \log (\bar{b}^\nu_{\beta} b_{\alpha, \mu}) \right),
\end{align*}
\]

and for \( j \neq \mu \)

\[
\begin{align*}
\bar{b}_{\alpha, j} &= \partial_{\hat{e}_\alpha} \left( \theta^j_\alpha \log (\hat{e}_\alpha^j b_{\alpha, j}) \right), \\
\bar{c}^j_{\alpha} &= \partial_{\beta} \left( \theta^j_\alpha \log (\bar{b}^\nu_{\beta} R_\alpha b_{\alpha, j}) \right).
\end{align*}
\]

Proof. From (3.2), \( \bar{b}_{\alpha, \mu} \sim b_{\beta, \nu} \). This and the normalization \( \bar{e}_\nu^\dagger b_{\alpha, \mu} = \bar{b}^\mu_{\alpha} = \theta^\mu_\alpha - 1 \) gives (4.6), (4.7) follows similarly from \( c^\mu_{\alpha} \sim \bar{e}^\nu_{\beta} \). This also gives an additional orthogonality conditions

\[
\begin{align*}
\bar{e}_\nu^\dagger b_{\beta, \nu} &= \bar{e}^\nu_{\beta} b_{\alpha, j} = 0 \quad \text{for } j \neq \mu, \text{ and also } \bar{e}_\nu^\dagger b_{\beta, \nu} \neq 0, \quad \bar{e}^\nu_{\beta} b_{\alpha, \mu} \neq 0,
\end{align*}
\]

so \( R_\alpha \) is well-defined.

We now need to solve the equations \( \bar{C}_\alpha^\dagger \sim \bar{C}_\alpha^\dagger R(u_\alpha)^{-1} \) for \( \bar{C}_\alpha^\dagger \) and \( B_\alpha \sim R(u_\alpha)^{-1} B_\alpha \) for \( B_\alpha \). The matrix \( R(u_\alpha)^{-1} = I - b_{\beta, \nu} (\bar{e}^\nu_{\beta} b_{\beta, \nu})^{-1} \bar{e}^\nu_{\beta} \) has rank \( m - 1 \) with the null-spaces spanned by \( c^\mu_{\alpha} \) and \( b_{\alpha, \mu} \). We can invert it on the orthogonal complements of those subspaces using \( R_\alpha \). Indeed,

\[
R_\alpha R(u_\alpha)^{-1} = I - \frac{b_{\nu,\mu} c^\nu_{\beta}}{c^\nu_{\beta} b_{\nu, \mu}} \quad \text{and} \quad R(u_\alpha)^{-1} R_\alpha = I - \frac{b_{\nu,\mu} c^\nu_{\beta}}{c^\nu_{\beta} b_{\nu, \mu}}.
\]

Thus, for \( j \neq \alpha \), \( R_\alpha R(u_\alpha)^{-1} = \bar{b}_{\alpha, j} \sim R_\alpha b_{\alpha, j} \), which gives (4.8); (4.9) is similar. \( \square \)

Lemma 4.5. In the setting of Theorem 4.1 with \( \mathcal{H}^+ (B, \hat{C}^\dagger; \Theta) \) given by (4.4), we have

\[
\begin{align*}
\bar{b}_{\beta, \nu} &= \partial_{\hat{e}_\beta} \mathcal{H}^+ (B, \hat{C}^\dagger; \Theta), \\
\bar{c}^\nu_{\beta} &= \partial_{\beta} \mathcal{H}^+ (B, \hat{C}^\dagger; \Theta),
\end{align*}
\]

Proof. Using (4.3) and the residue at infinity condition \( -A_\infty = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{A}_i = -\bar{A}_\infty \), we get

\[
\begin{align*}
b_{\beta, \nu} c^\nu_{\beta} + \theta^\mu_\alpha \frac{b_{\nu, \mu} c^\nu_{\beta}}{c^\nu_{\beta} b_{\nu, \mu}} + \sum_{I \in I'} b_I (\partial b_{I}, S) = \bar{b}_{\beta, \nu} c^\nu_{\beta} + (\theta^\mu_\alpha - 1) \frac{b_{\beta, \nu} c^\nu_{\beta}}{c^\nu_{\beta} b_{\beta, \nu}} + \sum_{I \in I'} \left( \partial_{\hat{e}_I} S \right) \hat{e}_I.
\end{align*}
\]

Using the Exchange Rule (iii), whose adaptation for \( R_\alpha \) is straightforward, this becomes

\[
\begin{align*}
b_{\beta, \nu} c^\nu_{\beta} + \theta^\mu_\alpha \frac{b_{\nu, \mu} c^\nu_{\beta}}{c^\nu_{\beta} b_{\nu, \mu}} + \left( \partial_{\hat{e}_I} S \right) \hat{e}_I = \bar{b}_{\beta, \nu} c^\nu_{\beta} + (\theta^\mu_\alpha - 1) \frac{b_{\beta, \nu} c^\nu_{\beta}}{c^\nu_{\beta} b_{\beta, \nu}} + b_{\beta, \nu} (\partial b_{I}, S),
\end{align*}
\]

which in turn can be rewritten as

\[
\begin{align*}
b_{\beta, \nu} \left( c^\nu_{\beta} - (\theta^\mu_\alpha - 1) \frac{c^\nu_{\beta}}{c^\nu_{\beta} b_{\beta, \nu}} - \bar{b}_{\beta, \nu} S \right) = \left( \bar{b}_{\beta, \nu} - \theta^\mu_\alpha \frac{b_{\nu, \mu}}{c^\nu_{\beta} b_{\nu, \mu}} - \hat{e}_I S \right) \hat{e}_I.
\end{align*}
\]
Thus,
\[
\ddot{b}_{\beta,\nu} = \lambda b_{\beta,\nu} + \theta_\alpha^\mu \frac{b_{\alpha,\mu}}{c^\alpha_\beta b_{\alpha,\mu}} + \partial_{c^{\mu \dagger}_\beta} S, \\
\ddot{c}^{\mu \dagger}_\beta = \lambda^t c^{\mu \dagger}_\beta + (\theta_\alpha^\mu - 1) \frac{c^{\mu \dagger}_\beta}{c^\alpha_\beta b_{\beta,\nu}} + \partial_{b_{\beta,\nu}} S.
\]

Using the Vanishing Rule (ii) and the normalization, we get
\[
\ddot{b}_{\beta,\nu} b_{\beta,\nu} = \theta_\beta^\mu = \theta_\beta^\mu + 1 = \lambda b_{\beta,\nu} + \theta_\alpha^\mu \\
\ddot{c}^{\mu \dagger}_\beta b_{\beta,\nu} = \theta_\beta^\mu - 1 + \lambda c^{\mu \dagger}_\beta b_{\beta,\nu}.
\]

Thus,
\[
\lambda = \lambda^t = \frac{\theta_\beta^\mu - \theta_\alpha^\mu + 1}{\ddot{c}^{\mu \dagger}_\beta b_{\beta,\nu}} \\
\ddot{b}_{\beta,\nu} = (\theta_\beta^\mu - \theta_\alpha^\mu + 1) \frac{b_{\beta,\nu}}{\ddot{c}^{\mu \dagger}_\beta b_{\beta,\nu}} + \theta_\alpha^\mu \frac{b_{\alpha,\mu}}{c^\alpha_\beta b_{\alpha,\mu}} + \partial_{c^{\mu \dagger}_\beta} S = \partial_{c^{\mu \dagger}_\beta} \mathcal{H}^+, \\
\ddot{c}^{\mu \dagger}_\beta = (\theta_\beta^\mu - \theta_\alpha^\mu + 1) \frac{c^{\mu \dagger}_\beta}{\ddot{c}^{\mu \dagger}_\beta b_{\beta,\nu}} + (\theta_\alpha^\mu - 1) \frac{c^{\mu \dagger}_\alpha}{c^\alpha_\beta b_{\beta,\nu}} + \partial_{b_{\beta,\nu}} S = \partial_{b_{\beta,\nu}} \mathcal{H}^+,
\]

which completes the proof of the Lemma (and of Theorem 4.1), and also gives the expressions for \(\ddot{b}_{\beta,\nu}\) and \(\ddot{c}^{\mu \dagger}_\beta\).

\[\blacksquare\]

**Remark 4.6.** Even though the generating function \(\mathcal{H}^+(\mathbf{B}, \mathbf{C}^\dagger; \Theta)\) is defined on a dense open subspace of \((\mathbf{B}, \mathbf{C}^\dagger)\), in view of the orthogonality conditions (4.10), to get the well-defined map from \((\mathbf{B} \times \mathbf{C}^\dagger)_{(\Theta, \mathbf{A}_\infty)}\) to \((\mathbf{B} \times \mathbf{C}^\dagger)_{(\Theta, \mathbf{A}_\infty)}\) we need to restrict it to the subspace
\[
(\mathbf{B} \times \mathbf{C}^\dagger)^{\text{ort}} = \{(\mathbf{B}, \mathbf{C}^\dagger) \mid \ddot{c}^{\mu \dagger}_\alpha b_{\beta,\nu} = \ddot{c}^{\mu \dagger}_\beta b_{\alpha,\mu} = 0 \quad \text{for} \quad j \neq \mu\}.
\]

5. Examples

In this section we consider, using our discrete Hamiltonian framework, two explicit examples of discrete Schlesinger transformations that can be reduced to discrete Painlevé equations. We also try to illustrate in this section the role played by the geometry of Painlevé equations in not only determining the type of the equation, but also in studying the relationship between different explicit forms of equations of the same type.

Recall that, according to the geometric theory of Painlevé equations developed in [Sak01], both continuous and discrete Painlevé equations can be classified by the type of their space of initial conditions, introduced by Okamoto [Oka79] in the continuous case. This space can be obtained by successively blowing up the projective plane \(\mathbb{P}^2\) at nine (possibly infinitely-close) points that lie on a unique (possibly singular) cubic curve. The type of the resulting surface \(X\) is then given by the configurations of irreducible components \(D_i\) of its *unique* anti-canonical divisor \(D\) or, equivalently, by the root sub-lattice \(\mathcal{R}\) spanned by \(D_i\) in the orthogonal complement \(D^\perp\) of \(D\) in the Picard lattice \(\text{Pic}(X)\). The orthogonal complement \(R^\perp\) of \(R\) in \(D^\perp\) then corresponds to the symmetries of the equation — its Weyl group acts as automorphisms, called the *Cremona transformations*, of \(\text{Pic}(X)\) preserving \(R \subset D^\perp\) and the translational part of the Weyl group corresponds to discrete Painlevé equations.

We also want to emphasise the following point, see also [Sak07]. Given a discrete Painlevé equation, we can determine its type by constructing its space of initial conditions. However, there is no converse process, although there are some particularly nice forms (that we call *standard*) of equations of each type, see [Sak07, GRO03, Mur04]. Further, there can be many *non-equivalent equations* (i.e., equations that can not be transformed into each other by some change of coordinates) of the same type — such equations can correspond to different translation directions in the affine Weyl group of symmetries of \(X\). And even when the two different equations are equivalent, finding the actual change of coordinates that will transform one equation into the other can be a very challenging problem. In each case, understanding the geometry and
the blow-up structure of the space of initial conditions can be of great help in determining the relationship between two different equations of the same type, as we illustrate by the two explicit examples below.

For the first example, we consider the Fuchsian equation of matrix size $m = 2$ with $n = 3$ finite poles and rank-one condition $r_1 = r_2 = r_3 = 1$ for the residue matrices at those poles. It is well-known that continuous isomonodromic deformations of this system are described by the classical continuous Painlevé equation $P_{VI}$, the type of the space of initial conditions is $D_4^{(1)}$, and the discrete Painlevé V equation $d-P_V = d-P(D_4^{(1)})$ appears as Bäcklund transformations of $P_{VI}$. Thus, it is reasonable to expect that an elementary Schlesinger transformation of this system is described by $d-P_V$, and we explicitly show that this is indeed the case.

The second example is a more interesting one. In [Sak07] one of the authors (H.S.) posed a problem of writing down discrete Painlevé equations of type $A_0^{(1)*}$, $A_1^{(1)*}$ and $A_2^{(1)*}$ as Schlesinger transformations of some Fuchsian equations. One difficulty here is that these equations do not correspond to any continuous Painlevé differential equations. This problem was solved theoretically by Boalch in [Boa09], where it was shown that the moduli spaces of monodromy groups of some Fuchsian equations are analytically isomorphic to the spaces of initial conditions for these discrete Painlevé equations. In particular, Boalch showed that the Fuchsian system with matrix size $m = 3$ with $n = 2$ finite poles (and so there are no continuous deformation parameters) and rank-two residue matrices, $r_1 = r_2 = 2$, has as its symmetry the affine Weyl group of type $E_6^{(1)}$, and so the type of the corresponding surface is either $A_2^{(1)}$ (for the multiplicative or $q$-difference case) or $A_2^{(1)*}$ (for the additive or difference case). Since we consider the additive case, we expect that discrete Schlesinger transformations of this system should have type $d-P(A_2^{(1)*})$. However, an elementary discrete Schlesinger transformation, when written in coordinates, becomes very complicated and in fact it cannot be reduced to a standard form of $d-P(A_2^{(1)*})$, as written in [Sak07]. Using the geometry of the space of initial conditions, we see that this is an example of a situation when the standard form of the equation corresponds to a direction that is a combination of two different directions given by different elementary Schlesinger transformations, and then we explicitly show how to combine them into a single Schlesinger transformation that corresponds to the standard form of $d-P(A_2^{(1)*})$, this is a new result. The geometric approach that we use requires finding correspondences between the rational surfaces that appear as desingularizations of different coordinate forms of two difference equations of the same type. Such correspondence gives us a blow-down structure that can then be used to find a change of coordinates that relates the two equations that we consider, see the recent preprint [CT12] by A. Carstea and T. Takenawa for a more detailed introduction into this technique.

5.1. Discrete Painlevé–V equation (type $D_4^{(1)}$).

5.1.1. Elementary Schlesinger transformations. For this example we consider a Fuchsian system of the spectral type 11, 11, 11, 11, i.e., we have $n = 3$ (finite) poles and the matrix size is $m = 2$. In view of Assumption 2.2, it is possible to make the rank of the residue matrices at the finite poles equal to one. Further, using the Möbius transformations on $\mathbb{P}^1$, we can map $(u_0 = \infty, u_1, u_3)$ to $(\infty, 1, 0)$. We put $u_2 = t$ and from now on we use $(1, t, 0)$ instead of the indices $(1, 2, 3)$, so $\theta_2 = \theta_i$, etc. Then

$$A(x) = \frac{A_1}{x - 1} + \frac{A_i}{x - t} + \frac{A_0}{x}, \quad A_\infty = -(A_0 + A_1 + A_i) = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix},$$

and we have the following Riemann scheme and the corresponding Fuchs relation:

$$\begin{align*}
\begin{cases}
  x = 0 & x = 1 & x = t & x = \infty \\
  \theta_0 & \theta_1 & \theta_i & \kappa_1 \\
  0 & 0 & \kappa_2 & \kappa_2
\end{cases}, \quad \theta_0 + \theta_1 + \theta_i + \kappa_1 + \kappa_2 = 0.
\end{align*}$$

In the rank-one case there is no need for the second index, and so we put $b_i = b_{i,1}$, $c_i = c_i^1$. Then the spectral decomposition is simply $A_i = b_i c_i^1$ with $\text{tr}(A_i) = c_i^1 b_i = \theta_i$. 

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Consider the elementary transformation \( \{ \frac{t+1}{t} \} \), i.e., \( \tilde{a}_1 = \theta_1 - 1 \), \( \tilde{b}_i = \theta_i + 1 \) and \( \tilde{a}_0 = \theta_0 \). Its multiplier 
matrix and the generating Hamiltonian function are

\[
R(x) = I + \frac{1 - t}{x - 1} \frac{b_t}{c_t^\dagger b_t}
\]

\[
\mathcal{H}^+(B, C; \Theta) = \log \left( \frac{c_t^\dagger b_t}{c_t^\dagger b_t} \right) + \theta_t \log \left( \frac{(c_t^\dagger b_t)(c_t^\dagger b_t)}{c_t^\dagger b_t} \right) + \theta_t \log \left( c_t^\dagger b_t \right) + \theta_0 \log \left( c_0^\dagger R(0)b_0 \right)
\]

\[
= (\theta_t + 1 - \theta_1 - \theta_0) \log \left( c_t^\dagger b_t \right) + (\theta_1 - 1) \log \left( c_t^\dagger b_t \right) + \theta_1 \log \left( c_t^\dagger b_t \right)
\]

\[
+ \theta_0 \log \left( c_t^\dagger b_t - (1 - t)c_t^\dagger b_0 c_t^\dagger b_t \right).
\]

This function generates the following dynamic equations:

\begin{align*}
(5.1) \quad \tilde{b}_1 &= \theta_t \frac{b_t}{c_t^\dagger b_t}, \\
(5.2) \quad \tilde{b}_0 &= \frac{b_t}{c_t^\dagger b_t} \frac{(c_t^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)b_t}{(c_t^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)(c_t^\dagger b_t)} = \theta_0 \frac{R(0)b_0}{c_0^\dagger R(0)b_0}, \\
(5.3) \quad \tilde{b}_1 &= \frac{b_t}{c_t^\dagger b_t} \frac{b_t}{c_t^\dagger b_t} + \theta_t \frac{b_t}{c_t^\dagger b_t} + \theta_0 \frac{(c_0^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)b_t}{(c_t^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)(c_0^\dagger b_0)}, \\
(5.4) \quad \tilde{c}_1^\dagger &= \frac{c_t^\dagger b_t}{c_t^\dagger b_t}, \\
(5.5) \quad \tilde{c}_0^\dagger &= \theta_t \frac{c_t^\dagger b_t}{c_t^\dagger b_t} \frac{(c_t^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)b_t}{(c_t^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)(c_t^\dagger b_t)} = \theta_0 \frac{c_0^\dagger R(0)}{c_0^\dagger R(0)b_0}, \\
(5.6) \quad \tilde{c}_1^\dagger &= \theta_t \frac{c_t^\dagger b_t}{c_t^\dagger b_t} \frac{c_t^\dagger b_t}{c_t^\dagger b_t} + \theta_t \frac{c_t^\dagger b_t}{c_t^\dagger b_t} + \theta_0 \frac{(c_0^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)b_t}{(c_t^\dagger b_t)(c_0^\dagger b_0) - (1 - t)(c_t^\dagger b_t)(c_0^\dagger b_0)}.
\end{align*}

Remark 5.1. To illustrate the general situation, we now show how to solve these equations to define a map
from \((B \times C)(\Theta, A)\) to \((B \times C)(\Theta, A)\) in this relatively simple example. This map is defined up to trivial 
transformations and becomes unique once we specify some normalization condition.

Indeed, equation (5.4) implies that \( c_t^\dagger = c_t c_t^\dagger \), where \( c_t \) is some proportionality constant. Thus, we can 
rewrite the multiplier matrix as \( R(x) = I + \frac{1 - t}{x - 1} \frac{b_t c_t^\dagger}{c_t^\dagger b_t} \). Equation (5.5) can be rewritten as \( c_0^\dagger \sim c_0^\dagger R(0), \) and 
so \( c_0^\dagger = c_0 c_0^\dagger R(0)^{-1} \), cf. (3.6). Finally, expression for \( c_t^\dagger \) can either be obtained by a direct computation or, 
simpler, from the equation \( c_t^\dagger A_\infty = c_t^\dagger \tilde{A}_\infty \). We then substitute these expressions into equations (5.1–5.3) 
and simplify. The final map is given by the following set of equations (with \( c_0, c_1, \) and \( c_t \) arbitrary constants)
corresponding to trivial transformations):
\[
\begin{align*}
\bar{c}_i^t &= c_t c_i^t, \\
c_0^t &= c_0 c_i^t \left( I + \frac{1 - t \beta_0 c_i^t}{t c_i^t b_t} \right), \\
c_1^t &= c_1 \left( c_2^t + (\theta_1 - \theta_t - 1) \frac{c_1^t}{c_i^t b_t} + \frac{(1 - t) b_0}{c_i^t b_t} \right) \left( I - \frac{b_0}{c_i^t b_t} \right), \\
\bar{b}_1 &= \frac{1}{c_1 b_t}, \\
\bar{b}_0 &= \frac{1}{c_0} \left( I - (1 - t) \frac{b_0 c_1^t}{c_i^t b_t} \right) c_0, \\
\bar{b}_t &= \frac{1}{c_t} \left( b_1 + (\theta_t - \theta_1 + 1) \frac{b_t}{c_i^t b_t} - \frac{(1 - t) b_1}{c_i^t b_t} \right) \left( I - \frac{b_0}{c_i^t b_t} \right) b_0.
\end{align*}
\]

Let us now look at the Schlesinger dynamic on the space of accessory parameters. Normalizing the eigenvectors $b_t$ and $c_i^t$, we can write

\[
A_t = a_t \begin{bmatrix} 1 \\ b_t \end{bmatrix} \begin{bmatrix} c_i \\ 1 \end{bmatrix} = a_t \begin{bmatrix} 1 \\ -\theta_1 - w \end{bmatrix} \begin{bmatrix} \gamma_t + w \\ 1 \end{bmatrix}, \quad \text{tr}(A_t) = \theta_t = a_t (c_t + b_t) = a_t (\beta_t + \gamma_t).
\]

Here $w = -b_0$, and so $\beta_0 = 0$ and $\gamma_0 = \theta_0/a_0$. The reason for introducing $w$ is to simplify the relations between parameters $a_t$, $b_t$, and $c_i$ that follow from the trace conditions and the equation $A_\infty = -\text{diag} \{\kappa_1, \kappa_2\}$. These relations, written in terms of $a_i$, $\beta_i$, and $\gamma_i$, are

\[
\begin{align*}
(5.7) & \quad a_0 + a_1 + a_t = 0 \\
(5.8) & \quad a_1 \beta_1 \gamma_1 + a_t \beta_t \gamma_t + w(\kappa_1 - \kappa_2) = 0 \\
(5.9) & \quad a_1 \beta_1 + a_t \beta_t + \kappa_2 = 0 \\
(5.10) & \quad \theta_0 + a_1 \gamma_1 + a_t \gamma_t + \kappa_1 = 0,
\end{align*}
\]

where the last equation also follows from (5.9) and the Fuchs relation. Thus, everything is parameterized by $a_1$, $\beta_1$, $a_t$, and $\beta_t$ subject to one additional constraint $a_1 \beta_1 + a_t \beta_t + \kappa_2 = 0$. One additional degree of freedom is related to the global gauge action by constant non-degenerate diagonal matrices that allows us to normalize one of the $a_i$s, but it is convenient to keep this freedom for computations. Finally, note that the normalization parameter $a_t$ should be considered as a part of either $b_t$ or $c_i^t$, where the actual choice depends on the context.

Same coordinates were used in [Sak10] to describe the isomonodromic dynamic in the continuous case and the reduction to $P_{Vf}$. With $a_t$ attached to $c_i^t$, the standard symplectic form on the decomposition space reduces as follows, see Proposition 4 in [Sak10] for a more general statement:

\[
\omega = \sum_{i=1}^{3} \text{tr} \left( dC_i^t \wedge dB_i^t \right) = da_0 \wedge d(-w) + da_1 \wedge d(\beta_1 - w) + da_t \wedge d(\beta_t - w)
\]
\[
= d(a_0 + a_1 + a_t) \wedge d(-w) + da_1 \wedge d\beta_1 + da_t \wedge d\beta_t + da_1 \wedge d\beta_1 + da_t \wedge d\beta_t
\]
\[
= \frac{da_1}{a_1} \wedge d(a_1 \beta_1) + \frac{da_t}{a_t} \wedge d(a_t \beta_t) = \left( \frac{da_1}{a_1} - \frac{da_1}{a_1} \right) \wedge d(a_1 \beta_1) = d \left( \frac{a_t}{a_1} \right) \wedge d(a_1 \beta_t),
\]

where we used (5.9) to eliminate $a_1 \beta_1$. It is convenient to introduce new symplectic coordinates $p = (a_1 \beta_1)/t$ and $q = -(ta_t)/a_1$. This gives, using (5.7–5.10),

\[
\begin{align*}
a_1 \beta_1 &= pq - \kappa_2, & a_1 \beta_t &= tp, & a_1 \gamma_1 &= \theta_1 + \kappa_2 - pq, \\
a_t \beta_t &= -pq, & a_t \beta_1 &= \frac{q}{t}(\kappa_2 - pq), & a_t \gamma_t &= \theta_t + pq.
\end{align*}
\]
For the continuous deformations, the only deformation parameter is \( t \) and the corresponding Hamiltonian, given by equation (2.4), under this reduction becomes

\[
\mathcal{H}_t = \frac{\text{tr}(A_t A_0)}{t} + \frac{\text{tr}(A_t A_1)}{t - 1} = \frac{(\theta_t + pq)(\theta_0 + p(q - t))}{t} + \frac{(\theta_1 + \kappa_2 - p(q - t))(t\theta_1 + \kappa_2 q - pq(q - t))}{t(t - 1)}.
\]

Since

\[
\omega = d \left( -\frac{q}{t} \right) \wedge d(tp) = dp \wedge dq - d \left( \frac{pq}{t} \right) \wedge dt,
\]

\[
\omega - d\mathcal{H}_t \wedge dt = dp \wedge dq - d \left( \mathcal{H}_t + \frac{pq}{t} \right) \wedge dt = dp \wedge dq - d\mathcal{H}_{V1} \wedge dt,
\]

where \( \mathcal{H}_{V1} = \mathcal{H}_t + pq/t \) is the Hamiltonian function for \( P_{V1} \).

Unfortunately, such a reduction procedure does not work in the discrete case and we need to work directly with the evolution equations. We give a brief outline of such computations and show the reduction to \( d-P_V \).

It is convenient to attach the normalization parameters \( \tilde{a}_i \) to \( \tilde{b}_i \) and \( a_i \) to \( c_i^\dagger \) (i.e., in what follows, \( b_i \) denotes the normalized vector, but in the formulas describing the dynamics we use \( \tilde{a}_i \tilde{b}_i \), and so on). It turns out that it is possible to separate the “normalization” dynamics of \( a_i \) and the “directional dynamics” of \( \beta_i \) and \( \gamma_i \), the latter one is given by the Painlevé equations.

Equation (5.1) becomes

\[
\tilde{a}_1 \tilde{b}_1 = \tilde{a}_1 \left[ \frac{1}{\beta_1 - \bar{w}} \right] - \frac{\theta_1 - 1}{c_i^\dagger b_i} \left[ \frac{1}{\beta_i - w} \right] = \frac{\theta_1 - 1}{c_i^\dagger b_i} \beta_t - \bar{w} = \beta_1 - \beta_t,
\]

Note that \( c_i^\dagger b_i = \gamma_i + \beta_i - w = \gamma_i + \beta_i = c_i^\dagger b_i \), and so \( \theta_1 - 1 = \tilde{a}_1 \tilde{c}_i^\dagger \tilde{b}_i = \tilde{a}_1 c_i^\dagger b_i = \tilde{\beta}_1 \), as it should be.

Equation (5.4) becomes

\[
\theta_1 \left[ \gamma_i + \beta_i - w \right] = \frac{\theta_1}{c_i^\dagger b_i} \left[ \gamma_i + \beta_i - w \right] = \frac{\theta_1}{c_i^\dagger b_i} (\gamma_i + \beta_i - w) - (\gamma_i + \beta_i - w) = \gamma_i (\beta_i - \beta_t) + \beta_i (\gamma_i - \gamma_t) = \beta_1 \gamma_i - \beta_t \gamma_i.
\]

Using this equation in the computation of the trace, \( c_i^\dagger b_i = \gamma_i + \beta_i = c_i^\dagger b_i \), we again see that the trace condition is satisfied, \( \tilde{a}_1 c_i^\dagger b_i = \theta_1 \). Important consequences of (5.11) and (5.12) are

\[
\tilde{c}_i^\dagger \tilde{b}_t = \gamma_t + \beta_t = \gamma_t + \beta_t = c_i^\dagger b_i,
\]

\[
(\tilde{c}_i^\dagger b_t)(\bar{w} - w) = (\gamma_t + \bar{w} + \beta_t - w)(\bar{w} - w) = \gamma_t (\beta_t - \beta_t) + \beta_t (\gamma_t - \gamma_t) = \beta_1 \gamma_t - \beta_t \gamma_t.
\]

We show shortly that the first equation generates the “normalization” dynamics.

Equation (5.2) becomes

\[
\tilde{a}_0 \left[ \frac{1}{\bar{w}} \right] = \frac{\theta_0}{(1 - t)(c_i^\dagger b_0)(c_i^\dagger b_1) - (c_i^\dagger b_1)(c_i^\dagger b_0)} \left( (1 - t)(c_i^\dagger b_0) \left[ \frac{1}{\beta_i - w} \right] - (c_i^\dagger b_1) \left[ \frac{1}{\bar{w}} \right] \right).
\]

The traces in this formula are, using (5.11) and (5.12),

\[
\tilde{c}_i^\dagger b_0 = \gamma_t + \bar{w} - w = \gamma_t,
\]

\[
\tilde{c}_i^\dagger b_1 = \gamma_0 + \bar{w} + \beta_t - w = \gamma_0 + \beta_1 = \frac{\theta_0 + \tilde{a}_0 \beta_1}{\tilde{a}_0},
\]

\[
\tilde{c}_i^\dagger b_t = \gamma_t + \beta_t = c_i^\dagger b_t = \gamma_t + \beta_t = \tilde{c}_i^\dagger b_1,
\]

\[
\tilde{c}_i^\dagger b_0 = \gamma_0 + \bar{w} - w = \frac{\theta_0 + \tilde{a}_0 (\bar{w} - w)}{\tilde{a}_0}.
\]

The equality of the normalized vectors then gives

\[
(1 - t)(\gamma_0)(\bar{w}) - (\gamma_0 + \beta_0)(\bar{w}) = (1 - t)(\gamma_0)(\beta_t - w) - (\gamma_0 + \beta_0)(\bar{w} - w),
\]

\[
(1 - t)(\gamma_t)(\bar{w}) = (\gamma_0 + \beta_0)(\bar{w} - w) = \beta_t \gamma_t - \gamma_1 \beta_t,
\]

which gives the following important relation

\[
\beta_t \gamma_t = t \beta_1 \gamma_1.
\]

As before, the normalization condition reduces to \( \tilde{\theta}_0 = \theta_0 \).
It is possible to show that the remaining evolution equations do not give any additional information. Let us now write everything in the symplectic coordinates $p$ and $q$. Equation (5.13) gives the equation for the normalization dynamic,
\[
\frac{\ddot{a}_t}{a_1} = \frac{\bar{a}_1(\gamma_1 + \beta_1)}{a_1(\gamma_1 + \beta_t)} = \frac{\ddot{\theta}_t + p\dot{q} + \frac{2}{t}(\kappa_2 - \bar{p}\bar{q})}{\theta_1 + \kappa_2 - p(q - \bar{t})}.
\]

Multiplying (5.14) by $a_1 \ddot{a}_t$, we get
\[
(\theta_1 + \kappa_2 - pq)\bar{q}(\kappa_2 - \bar{p}\bar{q}) = tp(\dot{\theta}_t + p\dot{\bar{q}}).
\]

Finally, combining (5.11) and (5.8) gives
\[
a_1\beta_1(\gamma_1 + \beta_t) + \beta_t(a_4\gamma_t - a_1\beta_t + \kappa_1 - \kappa_2) = \bar{a}_1\bar{\beta}_t(\bar{\gamma}_t + \bar{\beta}_1) + \bar{\beta}_1(\bar{a}_1\bar{\gamma}_t - \bar{a}_t\bar{\beta}_t + \kappa_1 - \kappa_2).
\]

Dividing by $\gamma_1 + \beta_t = \bar{\gamma}_t + \bar{\beta}_1$ and doing some re-arranging, we get
\[
pq + p\bar{q} - \kappa_2 = \frac{t(\theta_1 + \kappa_1)}{q - \bar{t} - \frac{a_1 + \kappa_2}{p}} + \frac{\dot{\theta}_1 + \kappa_1}{q - \bar{t} - \frac{a_1 + \kappa_2}{p}}.
\]

We now claim that after a certain change of variables equations (5.15), (5.16) transform into the standard form for $d$-$P_2$. Although this can be seen directly without too much difficulty, we prefer to use the geometry of the rational surface corresponding to the equation to explain how to find this coordinate transformation, illustrating some of the ideas that we need for the next, more complicated, example.

5.1.2. Geometry of $d$-$P(D_4^{\text{I}})$. Let us briefly review the geometric description of the difference Painlevé V equation from the point of view of the theory of rational surfaces [Sak01]. We think about it as a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with parameters $a_0, \ldots, a_4, s$ or, equivalently, as an automorphism of a field $\mathbb{C}(a_0, \ldots, a_4; s; f, g)$,
\[
\begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & s; \bar{f}, \bar{g} \\ \bar{a}_4 & \bar{a}_0 & \bar{a}_3 & \bar{f}, \bar{g} \end{pmatrix} \mapsto \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4; \bar{s}; \bar{f}, \bar{g} \\ \bar{a}_3 & \bar{a}_0 & \bar{a}_4 & \bar{s} \end{pmatrix},
\]
where $\delta = a_1 + 2a_2 + a_3 + a_4 + a_0$ and where $\bar{f}$ and $\bar{g}$ are given by the equation that is usually called the discrete Painlevé-V equation
\[
\begin{aligned}
f + \bar{f} &= a_3 + a_0 \left( \frac{a_1}{g + 1} + \frac{a_0}{sg + 1} \right), \\
g\bar{g} &= \frac{(f + \bar{a}_2)(\bar{f} + \bar{a}_2 + \bar{a}_4)}{sf(\bar{f} + \bar{a}_3)}.
\end{aligned}
\]

We refer to it as the standard form of $d$-$P_2$. We lift this map to an isomorphism $\varphi$ between rational surfaces $X_a = X(a_0, \ldots, a_4)$ and $X_{\bar{a}}$ obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ via a sequence of blowups that resolve the indeterminacies of the map $\varphi$,
\[
\begin{array}{ccc}
X_a & \xrightarrow{\varphi} & X_{\bar{a}} \\
\mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1 \times \mathbb{P}^1.
\end{array}
\]

Decomposing $\varphi$ as a sequence of two maps, $(f, g) \mapsto (\bar{f}, \bar{g})$ it is easy to see that the indeterminate points of $\varphi_1$ are $(\infty, -1)$ and $(\infty, -1/s)$ and the indeterminate points of $\varphi_2$ are $(-\bar{a}_2, 0), (-\bar{a}_2 - \bar{a}_4, 0), (0, \infty)$, and $(\bar{a}_3, \infty)$. Applying $\varphi^{-1}$ we obtain that the indeterminate points of $\varphi$ are $p_1(-\bar{a}_2 - \bar{a}_4, 0), p_2(-\bar{a}_2, 0), p_3(0, \infty), p_4(a_3, \infty), p_5(\infty, -1), \text{ and } p_6(\infty, -1/s)$. In exactly the same way we find that the indeterminate points of $\bar{\varphi}^{-1}$ are $\bar{p}_1(-\bar{a}_2 - \bar{a}_4, 0), \ldots, \bar{p}_6(\infty, -1/s)$.

As usual, we denote a local coordinate description of a blowup at a point $(x_0, y_0)$ by $(x, y) \leftrightarrow (u, v) \cup (U, V)$ where $x - x_0 = u = UV$ and $y - y_0 = uv = V$, then the exceptional divisor is given by $u = V = 0$. For example, extending the map $\varphi$ at the blowup at $p_1(-\bar{a}_2 - \bar{a}_4, 0)$, we use $u_1 = f + a_2 + a_4$ and $u_1v_1 = g$. Then $\bar{f} = \bar{f}(u_1, v_1)$ and $\bar{g} = g(u_1, v_1)$ become well-defined and we see that the exceptional divisor $E_1$ is mapped to the line $\bar{f} = \bar{a}_2$. On the level of divisor classes on Pic($X$) we get the map $E_1 \mapsto H_f - E_2$, where $H_f$ is the class of the total transform of the line $f = \text{const}$. 17
Proceeding along the same lines and noticing that we need to blow-up at two additional points \( p_7(0,a_1) \) on \( E_5 \) and \( p_8(0,a_0/s) \) on \( E_7 \) (here we give the coordinates of the points in the \((u_i,v_i)\) charts), we get the following blowup diagram describing the construction of the surface \( X_\bullet \) corresponding to our equation:

The Picard lattice of \( X \) is generated by the total transforms \( H_f \) and \( H_g \) of the coordinate lines and the classes of the exceptional divisors \( E_i \),

\[
\text{Pic}(X) = \mathbb{Z}H_f \oplus \mathbb{Z}H_g \oplus \bigoplus_{i=1}^{8} \mathbb{Z}E_i.
\]

The anti-canonical divisor \(-K_X\) uniquely decomposes as a positive linear combination of \(-2\)-curves \( D_i \),

\[
-K_X = 2H_f + 2H_g - \sum_{i=1}^{8} E_i = D_0 + D_1 + 2D_2 + D_3 + D_4,
\]

where

\[
D_0 = H_g - E_1 - E_2, \quad D_1 = H_g - E_3 - E_4, \quad D_2 = H_f - E_5 - E_6, \quad D_3 = E_5 - E_7, \quad D_4 = E_6 - E_8.
\]

The configuration of these \(-2\)-curves (on which the blow-up points are located) is described by the affine Dynkin diagram of type \( D_4^{(1)} \),

which is why we say that the difference Painlevé V equation has the type \( d\text{-}P(D_4^{(1)}) \). The corresponding divisors \( D_i \) generate a sub-lattice \( R = \text{Span}_\mathbb{Z}\{D_0,\ldots, D_4\} \subset (-K_X)^+ \simeq E_8^{(1)} \) in \( \text{Pic}(X) \). Symmetries of \( X \) are given by the affine Weyl group of \textit{Cremona transformations} generated by the complementary root sub-lattice \( R^+ \) in \((-K_X)^+ \) acting as automorphisms of \( \text{Pic}(X) \). In this case, \( R^+ = \text{Span}_\mathbb{Z}\{\alpha_0,\ldots, \alpha_4\} \), where

\[
\begin{align*}
\alpha_0 &= E_1 - E_2 \\
\alpha_1 &= E_3 - E_4 \\
\alpha_2 &= H_f - E_1 - E_3 \\
\alpha_3 &= H_g - E_5 - E_7 \\
\alpha_4 &= H_g - E_6 - E_8
\end{align*}
\]
which is again of the type $D^{(1)}_4$. Finally, we can compute the action of $\varphi_*$ on $\text{Pic}(X)$ to be

$$
\begin{align*}
H_f \mapsto 5H_f + 2H_g - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - E_6 - E_7 - E_8, & \quad E_1 \mapsto H_f - E_2, \\
H_g \mapsto 2H_f + H_g - E_1 - E_2 - E_3 - E_4, & \quad E_2 \mapsto H_f - E_1, \\
E_5 \mapsto 2H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_8, & \quad E_3 \mapsto H_f - E_1, \\
E_6 \mapsto 2H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_7, & \quad E_4 \mapsto H_f - E_3, \\
E_7 \mapsto 2H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_6, & \quad E_8 \mapsto H_f - E_3,
\end{align*}
$$

and so the induced action $\varphi_*$ on the sub-lattice $R^\perp$ is given by translation:

$$
\begin{align*}
\alpha_0 & \mapsto \alpha_0 & \alpha_2 & \mapsto \alpha_2 + \delta & \alpha_4 & \mapsto \alpha_4 - \delta \\
\alpha_1 & \mapsto \alpha_1 & \alpha_3 & \mapsto \alpha_3 - \delta & \delta & = -K_X.
\end{align*}
$$

5.1.3. **Reduction to the standard form.** We now demonstrate how to reduce equations (5.15, 5.16) to equations (5.17). First, we resolve the indeterminacies of $\psi : (p, q) \mapsto (\bar{p}, \bar{q})$ to get an isomorphism $\psi : \bar{X}_a \to \bar{X}_a$. The surface $\bar{X} = \bar{X}_a$, with $a$ denoting the collection of parameters of the map, $a = \{\theta_0, \theta_1, \theta_1', \kappa_1, \kappa_2\}$, is constructed by the following blow-up diagram:

Here the coordinates of the blow-up points are given w.r.t. the following local coordinate systems:

$$
\begin{align*}
p_1 \left( \frac{1}{p} = 0, q = 0 \right) & \leftrightarrow p_2 \left( \frac{1}{p} = 0, pq = -\theta_1 \right) \quad \text{and} \quad p_2' \left( \frac{1}{pq} = 0, q = 0 \right), \\
p_3 \left( p = 0, \frac{1}{q} = 0 \right) & \leftrightarrow p_4 \left( p = 0, \frac{1}{pq} = \frac{1}{\kappa_2} \right) \quad \text{and} \quad p_4' \left( p = 0, \frac{1}{pq} = \frac{1}{\theta_1 + \kappa_2} \right), \\
p_5 \left( \frac{1}{p} = 0, q = t \right) & \leftrightarrow p_7 \left( \frac{1}{p} = 0, p(q - t) = -\theta_0 \right), \\
p_6 \left( \frac{1}{p} = 0, q = 1 \right) & \leftrightarrow p_8 \left( \frac{1}{p} = 0, p(q - 1) = 1 - \kappa_1 + \kappa_2 \right).
\end{align*}
$$

Note that the resulting configuration of exceptional divisors looks quite different from the previous case and moreover, rank(Pic($\bar{X}$)) = 12. The reason for this is that the surface $\bar{X}$ is not relatively minimal for $\psi$, see also [CT12] — there are two pairs of $-1$-curves that are exchanged by $\psi$: $F_2 \leftrightarrow F_2'$ and $F_2' \leftrightarrow H_2 - F_3$, and so blowing down any of these pairs will still result in an isomorphism between the corresponding surfaces. We blow down $F_2$ and $H_2 - F_3$, which eliminates $-4$ and $-3$-curves and gives us the diagram on the left. This diagram now looks very similar to the configuration we had for the standard form of $d$-$P_V$ (on the right).
and in particular we immediately see that $\bar{X}$ is again of type $D_4^{(1)}$. We now need to find the blowing-down structure for $\bar{X}$ to match the blowing-down structure for $X$. Thus, in Pic($\bar{X}$) we put $E_9 = F'_2$, $E_{10} = H_q - F_3$ and we are looking for rational 0-curves $H_f$, $H_g$, and $-1$-curves $E_i$, $i = 1, \ldots, 8$ such that

\[ H_f \cdot H_g = 1, \quad E_i^2 = -1, \quad H_f^2 = H_g^2 = H_f \cdot E_i = H_f \cdot E_i = E_i \cdot E_j = 0, \quad 1 \leq i \neq j \leq 10. \]

Using the above diagrams, the genus formula $2g(C) - 2 = C^2 + C \cdot K_{\bar{X}}$, and the distinguished $-2$-curve $D_4$, we see that it makes sense to take $H_f = H_p + H_q - F_1 - F_3$ and $E_i = F_i$, $i = 5, \ldots, 8$. Then the possible choices for $H_g$ are $H_q, H_p + H_q - F_2 - F_3, H_p + H_q - F_1 - F_3$, and $H_p + H_q - F'_4 - F_3$. Attempting to represent $F_1 - F_2 = H_g - E_1 - E_2$ eliminates the first two possibilities, and the remaining two are equivalent. To summarize, we have the following blowing-down structure:

\[ H_f = H_p + H_q - F_1 - F_3, \quad E_4 = H_p - F_3, \quad E_i = F_i, \quad i = 2, 4, \ldots, 8, \]
\[ H_g = H_p + H_q - F_3 - F'_4, \quad E_9 = F'_2, \quad E_1 = H_p + H_q - F_1 - F_3 - F'_4, \quad E_{10} = H_q - F_3. \]

To find the expressions for coordinates $f$ and $g$ we proceed as follows. The linear system $|H_f|$ consists of curves of bi-degree $(1, 1)$ passing through $p_1$ an $p_3$, and so $|H_f| = \{c_1pq + c_2 = 0\}$. Thus, $[pq : 1]$ is a projective coordinate on $|H_f|$, and so we can take $f = pq$. Similarly, $[pq - (\theta_1 + \kappa_2) : p]$ is a projective coordinate on $|H_g|$. We take $g = -\frac{1}{t} \left( q - \frac{\theta_1 + \kappa_2}{p} \right)$, where the additional factor $-1/t$ is needed to ensure that the $(f, g)$-coordinates of $p_3$ are $(\infty, -1)$, as in the standard case.

We can now match the parameters of our elementary Schlesinger transformations with the parameters $s$ and $a_i$ in the standard form (5.17) of $d-P_V$. For example, $E_1$ blows down to $(f, g) = (\theta_1 + \kappa_2, 0) = (-a_2 - a_4, 0)$. Doing this for other divisors $E_i$ results in the following

\[ a_0 = \theta_1 - 1 + k_1, \quad a_1 = -\theta_1 - k_1, \quad a_2 = \theta_4, \quad a_3 = \kappa_2, \quad a_4 = -\theta_1 - \theta_4 - \kappa_2, \quad \delta = -1. \]

Thus, not only can we now immediately see that equations (5.15, 5.16) transform into the standard form (5.17) of $d-P_V$, but also that under this identification of parameters,

\[ \bar{\theta}_1 = -\bar{a}_2 - \bar{a}_3 - \bar{a}_4 = -a_2 + \delta - a_3 - a_4 = \theta_1 - 1, \quad \bar{\theta}_4 = -\bar{a}_0 - \bar{a}_1 - \bar{a}_2 - \bar{a}_3 - \bar{a}_4 - 1 = \theta_4 + 1, \]

and the other parameters remain unchanged. Thus, we can identify the action of $d-P_V$ on the Riemann Scheme of the Fuchsian system under this correspondence. We use the same observation in the next example.

5.2. Discrete Painlevé equation $d-P(A_2^{(1)})$.

5.2.1. Schlesinger Transformations. We now consider Fuchsian systems of the spectral type 111,111,111, which corresponds to $n = 2$ poles, matrix size $m = 3$, and, rank($A_i$) = 2. Thus,

\[ A_i = B_i C_i = \begin{bmatrix} b_{i,1} \quad b_{i,2} \end{bmatrix} \begin{bmatrix} c_{i,1}^1 \\ c_{i,2}^1 \end{bmatrix}. \]
Using a Möbius transformation preserving \( u = \infty \), we can map the poles to \( u_1 = 0 \) and \( u_2 = 1 \), then the Riemann scheme and the Fuchs relation are

\[
\begin{align*}
\left\{ x &= 0 \quad x = 1 \quad x = \infty \\
\theta_1^1 &\quad \theta_2^1 &\quad \kappa_1 \\
\theta_1^2 &\quad \theta_2^2 &\quad \kappa_2 \\
0 &\quad 0 &\quad \kappa_3
\right\}, \quad \theta_1^1 + \theta_2^1 + \theta_1^2 + \theta_2^2 + \sum_{j=1}^{3} \kappa_j = 0.
\end{align*}
\]

This example does not have any continuous deformation parameters but it admits non-trivial Schlesinger transformation. As before, consider an elementary Schlesinger transformation \( \{ \frac{1}{x} \} \), \( \tilde{\theta}_1^1 = \theta_1^1 - 1, \tilde{\theta}_2^1 = \theta_2^1 + 1 \), and \( \tilde{\theta}_j^1 = \theta_j^1 \) otherwise.

The multiplier matrix \( R(x) \) and other related matrices in this case are

\[
R(x) = I - \frac{1}{x} \frac{b_{2,1} c_{2}^{1\dagger}}{c_{1}^{1\dagger} b_{2,1}}, \quad R(x)^{-1} = I + \frac{1}{x - 1} \frac{b_{2,1} c_{2}^{1\dagger}}{c_{1}^{1\dagger} b_{2,1}}
\]

\[
R_1 = I - \frac{b_{1,1} c_{1}^{1\dagger}}{c_{1}^{1\dagger} b_{1,1}} - \frac{b_{2,1} c_{1}^{1\dagger}}{c_{1}^{1\dagger} b_{2,1}}, \quad R_2 = R(1) = I - \frac{b_{2,1} c_{2}^{1\dagger}}{c_{2}^{1\dagger} b_{2,1}} = R(0)^{-1},
\]

and we immediately see that

\[
R_1 R_2 = I - \frac{b_{2,1} c_{1}^{1\dagger}}{c_{1}^{1\dagger} b_{2,1}}, \quad R_2 R_1 = I - \frac{b_{1,1} c_{1}^{1\dagger}}{c_{1}^{1\dagger} b_{1,1}}.
\]

The generating Hamiltonian function can be written as

\[
\mathcal{H}^+(B, C^\dagger; \Theta) = (\theta_2^1 - \theta_1^1 - \theta_2^2 + 1) \log(c_{2}^{1\dagger} b_{2,1}) + (\theta_1^1 - \theta_2^1 - 1) \log(c_{1}^{1\dagger} b_{2,1}) + \theta_2^2 \log D_1 + \theta_2^2 \log D_2,
\]

where

\[
D_1 = (c_{2}^{1\dagger} b_{2,1})(c_{2}^{1\dagger} b_{1,2})(c_{2}^{1\dagger} b_{1,1}) - (c_{1}^{1\dagger} b_{2,1})(c_{2}^{1\dagger} b_{1,1})(c_{2}^{1\dagger} b_{1,2}) - (c_{1}^{1\dagger} b_{1,2})(c_{1}^{1\dagger} b_{1,1})(c_{2}^{1\dagger} b_{1,2}),
\]

\[
D_2 = (c_{2}^{1\dagger} b_{2,1})(c_{2}^{1\dagger} b_{2,2}) - (c_{2}^{1\dagger} b_{2,2})(c_{2}^{1\dagger} b_{2,1}).
\]

We restrict this function to the subspace \( \mathcal{B} \times \mathcal{C}^\dagger \) ort given by the equations \( c_{2}^{1\dagger} b_{2,1} = c_{2}^{1\dagger} b_{1,2} = 0 \). It then generates the map from \( \mathcal{B} \times \mathcal{C} \)\( (\Theta, A_\infty) \) to \( \mathcal{B} \times \mathcal{C} \)\( (\tilde{\Theta}, \tilde{A}_\infty) \), where \( A_\infty = \tilde{A}_\infty = \text{diag} \{ \kappa_1, \kappa_2, \kappa_3 \} \). The map is
implicitly given by the equations

\[ b_{1,1} = (\theta^1_1 - 1) \frac{b_{2,1}}{c_1^{\dagger} b_{2,1}} \]

\[ \bar{b}_{1,2} = \theta^2_1 \left( I - \frac{b_{2,1} c_1^{\dagger}}{c_1^{\dagger} b_{2,1}} \right) \frac{b_{1,2}}{c_1^{\dagger} b_{1,2}} = \theta^2_1 R_1 \frac{b_{1,2}}{c_1^{\dagger} b_{1,2}}, \]

\[ b_{2,1} = (\theta^2_1 - \theta^1_1 - \theta^3_2 + 1) \frac{b_{2,1}}{c_2^{\dagger} b_{2,1}} + \theta^1_1 \frac{b_{1,1}}{c_2^{\dagger} b_{1,1}} - \theta^2_1 \frac{c_1^{\dagger} b_{1,1}}{c_2^{\dagger} b_{1,1}} \frac{b_{1,2}}{c_1^{\dagger} b_{1,2}} \]

\[ + \theta^2_2 \left( \frac{c_2^{\dagger} b_{2,2}}{D_2} \right) b_{2,1} - \left( \frac{c_2^{\dagger} b_{2,1}}{D_2} \right) b_{2,2}, \]

\[ \bar{b}_{2,2} = \theta^2_2 \left( \frac{c_1^{\dagger} b_{2,1}}{D_2} \right) b_{2,2} - \left( \frac{c_1^{\dagger} b_{2,2}}{D_2} \right) b_{2,1} = \theta^2_2 \frac{R_2 b_{2,2}}{c_2^{\dagger} R_2 b_{2,2}}, \]

\[ c_1^{\dagger} = \theta^1_1 \frac{c_2^{\dagger}}{c_1^{\dagger} b_{1,1}}, \]

\[ c_2^{\dagger} = \theta^2_1 \frac{c_1^{\dagger}}{c_2^{\dagger} b_{1,2}} \left( I - \frac{b_{1,1} c_2^{\dagger}}{c_2^{\dagger} b_{1,1}} \right) = \theta^2_1 \frac{c_1^{\dagger}}{c_2^{\dagger} b_{1,2}} R_1, \]

\[ c_2^{\dagger} = (\theta^2_1 - \theta^1_1 - \theta^3_2 + 1) \frac{c_2^{\dagger}}{c_2^{\dagger} b_{2,1}} + (\theta^1_1 - 1) \frac{c_1^{\dagger}}{c_1^{\dagger} b_{2,1}} - \theta^2_1 \frac{c_1^{\dagger} b_{1,1}}{c_1^{\dagger} b_{2,1}} \frac{c_2^{\dagger}}{c_1^{\dagger} b_{1,2}}, \]

\[ + \theta^2_2 \left( \frac{c_2^{\dagger} b_{2,2}}{D_2} \right) c_2^{\dagger} - \left( \frac{c_2^{\dagger} b_{2,2}}{D_2} \right) c_2^{\dagger} \]

\[ c_2^{\dagger} = \theta^2_2 \left( \frac{c_1^{\dagger} b_{2,1}}{D_2} \right) c_2^{\dagger} - \left( \frac{c_1^{\dagger} b_{2,1}}{D_2} \right) c_2^{\dagger} = \theta^2_2 \frac{c_2^{\dagger} R_2}{c_2^{\dagger} b_{2,2}}, \]

\[ \text{cf. (3.6–3.7). These equations can be solved to give the map from} \ (\mathcal{B} \times \mathcal{C})(\theta, \mathcal{A}_\infty) \ \text{to} \ (\mathcal{B} \times \mathcal{C})(\theta, \mathcal{A}_\infty) \ \text{explicitly,} \]

\[ c_1^{\dagger} = c_1^{\dagger} \left( c_2^{\dagger} + (\theta^1_1 - \theta^2_1 - 1) \frac{c_1^{\dagger} b_{2,2}}{c_1^{\dagger} b_{1,2}} + \frac{c_1^{\dagger} b_{2,2}}{c_1^{\dagger} b_{1,2}} c_2^{\dagger} + \frac{c_1^{\dagger} (b_{2,1} c_1^{\dagger} + b_{1,2} c_1^{\dagger} b_{1,1}) c_1^{\dagger} c_1^{\dagger} R_2}{(\theta^1_1 - \theta^2_1 - 1) c_1^{\dagger} b_{1,2}} \right) \]

\[ c_1^{\dagger} = \frac{c_2^{\dagger} c_1^{\dagger}}{c_1^{\dagger} b_{2,1}} \left( I - \frac{b_{2,1} c_1^{\dagger}}{c_1^{\dagger} b_{2,1}} \right) = c_2^{\dagger} c_1^{\dagger}, \]

\[ c_2^{\dagger} = c_2^{\dagger}, \]

\[ c_2^{\dagger} = c_2^{\dagger} \left( c_2^{\dagger} + \frac{c_2^{\dagger} (b_{1,1} c_1^{\dagger} + b_{1,2} c_1^{\dagger} b_{1,1}) b_{2,1} c_1^{\dagger}}{(\theta^1_1 - \theta^2_1 - 1) c_1^{\dagger} b_{1,2}} \right). \]
where we still need to substitute some coordinates on the phase space. Unfortunately, using symplectic coordinates similar to the previous example results in very complicated equations, so we use the following parameterization instead.

Assuming the generic situation, we can use the similarity and trivial transformations, see Remark 2.4, to first map the vectors $b_{1,1}$, $b_{1,2}$, and $b_{2,1}$ to the standard basis vectors using the global similarity transformations, and then use the trivial transformations to adjust the scale on the coordinate axes so that all components of $b_{2,2}$ are equal to 1. Then, using the condition $C_iB_i = \Theta_i$, we get the following parameterization:

\[
B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \theta_1^1 & 0 & \alpha \\ 0 & \theta_1^2 & \beta \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} x - \theta_2^1 & -x \\ -y & y + \theta_2^2 \\ \theta_2^1 & 0 \end{bmatrix},
\]

where the discrepancy in the notation of the elements of $C_1^j$ and $C_2^j$ is due to the fact that we can express $\alpha$ and $\beta$ in terms of $x$, $y$, and the parameters $\theta_i^j$ and $\kappa_i$ by comparing the coefficients of the characteristic polynomial of $A_{\infty} = -B_1C_1^j - B_2C_2^j$. The resulting expressions, although easy to obtain, are quite large and we omit them. Using equations (5.18–5.25) we get the following dynamic in the coordinates $(x, y)$:

\[
\begin{align*}
\bar{x} &= \frac{(\alpha - \beta)(\theta_1^1(y + \theta_2^2) - \alpha(x - y - \theta_2^2))}{\alpha(\theta_1^2 - \theta_2^1 - 1)} \\
\bar{y} &= \frac{(\alpha - \beta)y - \beta\theta_2^1}{(\theta_1^1 - \theta_2^1 - 1)} \left( 1 + \frac{\theta_2^1}{\alpha} + \frac{(\beta - \alpha)x + (\theta_2^1 + 1 + \alpha)(\theta_2^1 + \alpha)}{(\alpha - \beta)(y + \theta_2^2) - \alpha(\theta_2^1 + 1)} \right),
\end{align*}
\]

where we still need to substitute $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$.

With the help of Mathematica we can find the indeterminate points of the map $\psi : (x, y) \rightarrow (\bar{x}, \bar{y})$ to get the following blowup diagram (note that in this case it is slightly more convenient to compactify $\mathbb{C}^2$ to $\mathbb{P}^2$ rather than $\mathbb{P}^1 \times \mathbb{P}^1$),
where the projective coordinates of the blowup points are

\begin{align*}
  p_1 &= (1 : 1 : 0), \quad p_2 = (\theta_1^2 : 0 : 1), \\
  p_3 &= \left( \frac{\theta_i^2 + \kappa_1}{\theta_i^2 - \theta_i^1} : \frac{\theta_i^2 + \theta_i^2 + \kappa_1}{\theta_i^2 - \theta_i^1} : 1 \right), \\
  p_6 &= (0 : -\theta_2^2 : 1), \quad p_7 = (0 : 1 : 0), \\
  p_8 &= (1 : 0 : 0), \\
  p_9 &= \left( \frac{\theta_i^2 - \theta_i^1}{\theta_1^2 - \theta_1^1} : \frac{\theta_1^2 + \theta_1^2 + \kappa_3}{\theta_1^2 - \theta_1^1} : 1 \right), \\
  p_5 &= \left( \frac{\theta_i^2 + \theta_i^2 + \kappa_3}{\theta_i^2 - \theta_i^1} : 1 \right),
\end{align*}

and

\begin{align*}
  p_5 \text{ is given in the coordinates } (u, v) = \left( \frac{Y-X}{X}, \frac{Z-Y}{Z} \right), \quad \text{and the quadric } Q \text{ passing through the points } p_1, \ldots, p_6 \\
  \text{is given in the homogeneous coordinates } [X : Y : Z] \text{ by}
\end{align*}

\begin{align*}
  (\theta_1^2 - \theta_2^2)(X - Y - \theta_2^2 Z)(X - Y - \theta_2^2 Z) + (\theta_1^2 - \theta_2^2 Z)(\theta_2^2 Z - X) + \theta_2^2 Y = 0.
\end{align*}

The anti-canonical divisor $-K_X$ uniquely decomposes as a positive linear combination of $-2$-curves $D_i$,

\begin{align*}
  -K_X = 3E - \sum_{i=1}^{9} E_i = D_0 + D_1 + D_2,
\end{align*}

where

\begin{align*}
  D_0 &= 2E - \sum_{i=1}^{6} E_i, \quad D_1 = E - E_1 - E_2 - E_8, \quad D_2 = E_1 - E_9.
\end{align*}

Thus, we see that the configuration of components $D_i$ is indeed described by the Dynkin diagram of type $A_2^{(1)}$. To this diagram correspond two different types of surfaces, the generic one corresponds to the multiplicative system of type $A_2^{(1)}$, and the degenerate configuration where all three components $D_i$ intersect at one point corresponds to the additive system denoted by $A_2^{(1)*}$, this is our case:

\[\text{Dynkin diagram } A_2^{(1)} \quad \text{A}_2^{(1)\text{-surface}} \quad \text{A}_2^{(1)*\text{-surface}}.\]

5.2.2. **Geometry of d-P(A_2^{(1)*}).** Following Murata [Mur04] and Sakai [Sak07], we take the standard form of the d-P(A_2^{(1)*}) to be the following. We consider d-P(A_2^{(1)*}) to be a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with parameters $b_1, \ldots, b_8$

\begin{align*}
  (b_1, b_2, b_4, f, g) \rightarrow (\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{f}, \bar{g}), \quad \bar{b}_1 = b_1, \quad \bar{b}_3 = b_3, \quad \bar{b}_5 = b_5 + \delta, \quad \bar{b}_7 = b_7 - \delta, \\
  \bar{b}_2 = b_2, \quad \bar{b}_4 = b_4, \quad \bar{b}_6 = b_6 + \delta, \quad \bar{b}_8 = b_8 - \delta,
\end{align*}

$\delta = b_1 + \cdots + b_8$, and $\bar{f}$ and $\bar{g}$ are given by the equation

\begin{align*}
  (f + g)(\bar{f} + g) &= \frac{(g + b_1)(g + b_2)(g + b_3)(g + b_4)}{(g - b_5)(g - b_6)} \\
  (\bar{f} + g)(\bar{f} + \bar{g}) &= \frac{((\bar{f} - b_1)(\bar{f} - b_2)(\bar{f} - b_3)(\bar{f} - b_4))}{(\bar{f} + b_7 - \delta)(\bar{f} + b_8 - \delta)}.
\end{align*}
Note that equation (5.29) is the asymmetric version of equation (1.3) in [GRO03]. We get the following blowup diagram describing the construction of the surface $X_b$ corresponding to our equation:

\[
\begin{array}{c}
\text{Pic}(X_b) = \mathbb{Z}H_f \oplus \mathbb{Z}H_g \oplus \bigoplus_{i=1}^{8} \mathbb{Z}E_i,
\end{array}
\]

As in Section 5.1.2, $\text{Pic}(X_b) = \mathbb{Z}H_f \oplus \mathbb{Z}H_g \oplus \bigoplus_{i=1}^{8} \mathbb{Z}E_i$, The anti-canonical divisor $-K_X = 2H_f + 2H_g - \sum_{i=1}^{8} E_i$ decomposes as $-K_X = D_0 + D_1 + D_2$, where

\[
D_0 = H_f + H_g - E_1 - E_2 - E_3 - E_4, \quad D_1 = H_f - E_5 - E_6, \quad D_2 = H_g - E_7 - E_8.
\]

The symmetry sub-lattice $R^\perp = \text{Span}_\mathbb{Z}\{\alpha_0, \ldots, \alpha_6\}$, where

\[
\begin{align*}
\alpha_0 &= E_1 - E_4, & \alpha_1 &= E_2 - E_3, \\
\alpha_2 &= E_1 - E_2, & \alpha_3 &= H_f - E_1 - E_7, \\
\alpha_4 &= E_7 - E_8, & \alpha_5 &= H_g - E_1 - E_5, \\
\alpha_6 &= E_5 - E_6
\end{align*}
\]

which is of the type $E_6^{(1)}$. We compute the action of $\varphi_*$ on $\text{Pic}(X)$ to be

\[
\begin{align*}
(5.31) \quad & H_f \mapsto 6H_f + 3H_g - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - 3E_7 - 3E_8, \\
(5.32) \quad & H_g \mapsto 3H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_7 - E_8, \\
(5.33) \quad & E_1 \mapsto 2H_f + H_g - E_2 - E_3 - E_4 - E_7 - E_8, \\
(5.34) \quad & E_2 \mapsto 2H_f + H_g - E_1 - E_4 - E_7 - E_8, \\
(5.35) \quad & E_3 \mapsto 2H_f + H_g - E_1 - E_2 - E_4 - E_7 - E_8, \\
(5.36) \quad & E_4 \mapsto 2H_f + H_g - E_1 - E_2 - E_3 - E_7 - E_8, \\
(5.37) \quad & E_5 \mapsto 3H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_6 - E_7 - E_8, \\
(5.38) \quad & E_6 \mapsto 3H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_5 - E_7 - E_8, \\
(5.39) \quad & E_7 \mapsto H_f - E_8, \\
(5.40) \quad & E_8 \mapsto H_f - E_7,
\end{align*}
\]

and so the induced action $\varphi_*$ on the sub-lattice $R^\perp$ is given by the following translation:

\[
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, 1, 0, -1, 0)\delta, \quad \text{where } \delta = -K_X.
\]

5.2.3. Reduction to the standard form. To match the surface $\tilde{X}$ described by the blow-up diagram (5.26) with the surface $X_b$ described by diagram (5.30), we look for the blow-down structure in $\text{Pic}(\tilde{X})$, i.e., we look for rational classes $\mathcal{H}_f, \mathcal{H}_g, \mathcal{E}_1, \ldots, \mathcal{E}_8$ in $\text{Pic}(\tilde{X})$ such that

\[
\mathcal{H}_f \cdot \mathcal{H}_g = 1, \quad \mathcal{E}_i \cdot \mathcal{E}_i = -1, \quad \mathcal{H}_f^2 = \mathcal{H}_g^2 = \mathcal{H}_f \cdot \mathcal{E}_i = \mathcal{H}_g \cdot \mathcal{E}_i = \mathcal{E}_i \cdot \mathcal{E}_j = 0, \quad 1 \leq i, j \leq 8,
\]

where $\mathcal{H}_f$ is the class of the exceptional divisor $H_f$, $\mathcal{H}_g$ is the class of the exceptional divisor $H_g$, $\mathcal{E}_i$ are the classes of the curves $E_i$, and $\mathcal{E}_i \cdot \mathcal{E}_j = 0$.
and the resulting configuration matches diagram (5.30). Comparing the \(-2\)-curves on both diagrams,

\[
\begin{align*}
D_0 &= 2E - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4, \\
D_1 &= E - E_1 - E_7 - E_8 = \mathcal{H}_f - \mathcal{E}_5 - \mathcal{E}_6, \\
D_2 &= E_1 - E_9 = \mathcal{H}_g - \mathcal{E}_7 - \mathcal{E}_8,
\end{align*}
\]

we get the following identification (it is easy to see that it satisfies the genus and the rationality conditions):

\[
\begin{align*}
\mathcal{H}_f &= E - E_1, \quad \mathcal{E}_1 = E_3, \quad \mathcal{E}_3 = E_5, \quad \mathcal{E}_5 = E_7, \quad \mathcal{E}_7 = E - E_1 - E_2, \\
\mathcal{H}_g &= E - E_2, \quad \mathcal{E}_2 = E_4, \quad \mathcal{E}_4 = E_6, \quad \mathcal{E}_6 = E_8, \quad \mathcal{E}_8 = E_9.
\end{align*}
\]

To complete the correspondence it remains to define the base coordinates \(f\) and \(g\) of the linear systems \(|\mathcal{H}_f|\) and \(|\mathcal{H}_g|\) that will map the exceptional fibers of the divisors \(\mathcal{E}_i\) to the points \(\pi_i\) such that \(\pi_5\) and \(\pi_6\) are on the line \(f = \infty\), \(\pi_7\) and \(\pi_8\) are on \(g = \infty\), and \(\pi_1, \ldots, \pi_4\) are on the line \(f + g = 0\). Since the pencil \(|\mathcal{H}_f|\) consists of all curves on \(\mathbb{P}^2\) passing through \(p_1(1 : 1 : 0)\),

\[
|\mathcal{H}_f| = |E - E_1| = \{ aX + bY + cZ = 0 \mid a + b = 0 \} = \{ a(X - Y + \beta Z) + cZ = 0 \},
\]

we can define the projective base coordinate as \(f_1 = [X - Y + \beta Z : Z]\), with \(\beta\) being some parameter. Similarly,

\[
|\mathcal{H}_g| = |E - E_2| = \{ aX + bY + cZ = 0 \mid \theta_2^2 a + c = 0 \} = \{ a(X - \theta_2^2 Z + \alpha Y) + bY = 0 \},
\]

and \(g_1 = [X - \theta_2^2 Z + \alpha Y : Y]\). Then

\[
\begin{align*}
f_1(\pi_5) &= f_1(p_7) = [1 : 0], & g_1(\pi_7) &= g_1(p_1) = [1 + \alpha : 1], \\
f_1(\pi_6) &= f_1(p_8) = [1 : 0], & g_1(\pi_8) &= g_1(p_1) = [1 + \alpha : 1].
\end{align*}
\]

Thus, we take \(\alpha = -1\) (and looking at the other points we notice that it is also convenient to choose \(\beta = -\theta_2^2\)), and so we provisionally put

\[
f_2 = \frac{X - Y - \theta_2^2 Z}{Z}, \quad g_2 = \frac{Y}{X - Y - \theta_2^2 Z}.
\]

In these coordinates equation (5.27) of the quadric \(Q\) becomes

\[
Z^2(f_2 + \theta_2^2 - \theta_1^2)(\theta_1^2 - \theta_2^2) f_2 + (\theta_2^2 - \theta_2^2)(\theta_1^2 - \theta_2^2) g_2 - \theta_2^2 = 0,
\]

and so the points \(\pi_1, \ldots, \pi_4\) lie on the line \((\theta_1^2 - \theta_1^2) f_2 + (\theta_2^2 - \theta_2^2)((\theta_1^2 - \theta_2^2) g_2 - \theta_2^2) = 0\). Hence, if we make the final choice of the base coordinates to be

\[
f = \frac{\theta_1^2 - \theta_1^2(X - Y - \theta_2^2 Z)}{\theta_2^2 - \theta_2^2 Z}, \quad g = \frac{\theta_2^2(X - \theta_2^2 Z - \theta_2^2 Y)}{X - Y - \theta_2^2 Z},
\]

the points \(\pi_1, \ldots, \pi_4\) will be on the line \(f + g = 0\). Specifically, we have:

\[
\begin{align*}
\pi_1(\theta_1^2 + \kappa_1, -\theta_1^2 - \kappa_1), & \quad \pi_3(\theta_1^2 + \kappa_3, -\theta_1^2 - \kappa_3), & \quad \pi_5(\infty, \theta_2^2), & \quad \pi_7(\theta_1^2 - \theta_1^2, \infty), \\
\pi_2(\theta_1^2 + \kappa_2, -\theta_1^2 - \kappa_2), & \quad \pi_4(0,0), & \quad \pi_6(\infty, \theta_2^2), & \quad \pi_8(\theta_1^2 + 1, \infty).
\end{align*}
\]

Thus, we immediately get the identification between the parameters in the Riemann scheme of our Fuchsian system and the parameters \(b_i\) in the standard form of the Painlevé equation:

\[
\begin{align*}
b_1 &= \theta_1^2 + \kappa_1, & b_2 &= \theta_2^2 + \kappa_2, & b_3 &= \theta_1^2 + \kappa_3, & b_4 &= 0, & b_5 &= \theta_2^2, & b_6 &= \theta_2^2, & b_7 &= \theta_1^2 - \theta_1^2, & b_8 &= -\theta_2^2 - 1.
\end{align*}
\]

Then \(\delta = b_1 + \cdots + b_8 = -1\) and the induced action of the standard Painlevé dynamic on the Riemann scheme is

\[
\begin{align*}
\left\{\begin{array}{llll}
x = 0 & x = 1 & x = \infty \\
\theta_1^2 & \theta_2^2 & \kappa_1 \\
\theta_1^2 & \theta_2^2 & \kappa_2 \\
0 & 0 & \kappa_3
\end{array}\right\} \xrightarrow{\mathcal{A}(\lambda)^{+}} \left\{\begin{array}{llll}
x = 0 & x = 1 & x = \infty \\
\theta_1^2 & \theta_2^2 - 1 & \kappa_1 + 1 \\
\theta_1^2 & \theta_2^2 - 1 & \kappa_2 + 1 \\
0 & 0 & \kappa_3 + 1
\end{array}\right\}
\end{align*}
\]
whereas our elementary Schlesinger transformation acts as

$$
\begin{bmatrix}
    u_1 & u_2 & \infty \\
    \theta_1^1 & \theta_2^1 & \kappa_1 \\
    \theta_1^2 & \theta_2^2 & \kappa_2 \\
    0 & 0 & \kappa_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
    u_1 & u_2 & \infty \\
    \theta_1^1 - 1 & \theta_2^1 + 1 & \kappa_1 \\
    \theta_1^2 & \theta_2^2 & \kappa_2 \\
    0 & 0 & \kappa_3
\end{bmatrix}.
$$

Thus, these two transformations correspond to the different translation directions in the symmetry root sub-lattice of the surface $X$ and so are not equivalent. Indeed, we compute the action of $\psi_*$ of an elementary Schlesinger transformation $\{ \frac{1}{2} \}$ on the classes $H_f$, $H_g$, and $E_i$ to be

$$
\begin{align*}
H_f & \mapsto 2H_f + 3H_g - E_1 - E_2 - E_3 - E_4 - 2E_5 - 2E_8, \\
H_g & \mapsto 3H_f + 5H_g - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 3E_5 - E_6 - 2E_8, \\
E_1 & \mapsto H_f + 2H_g - E_2 - E_3 - E_4 - E_5 - E_8, \\
E_2 & \mapsto H_f + 2H_g - E_1 - E_3 - E_4 - E_5 - E_8, \\
E_3 & \mapsto H_f + 2H_g - E_1 - E_2 - E_4 - E_5 - E_8, \\
E_4 & \mapsto H_f + 2H_g - E_1 - E_2 - E_3 - E_5 - E_8, \\
E_5 & \mapsto E_7, \\
E_6 & \mapsto 2H_f + 2H_g - E_1 - E_2 - E_4 - E_5 - E_8, \\
E_7 & \mapsto 2H_f + 3H_g - E_1 - E_2 - E_3 - E_4 - 2E_5 - E_6 - 2E_8, \\
E_8 & \mapsto H_g - E_5,
\end{align*}
$$

and compare with the standard dynamic $\varphi_*$ given by (5.31–5.40) to see this explicitly:

$$
\begin{align*}
\psi_* : (a_0, a_1, a_2, a_3, a_4, a_5, a_6) & \mapsto (a_0, a_1, a_2, a_3, a_4, a_5, a_6) + (0, 0, 0, -1, 1, 1, -1)\delta, \\
\varphi_* : (a_0, a_1, a_2, a_3, a_4, a_5, a_6) & \mapsto (a_0, a_1, a_2, a_3, a_4, a_5, a_6) + (0, 0, 0, 1, 0, -1, 0)\delta,
\end{align*}
$$

where in both cases $\delta = -K_X$.

In fact, we can represent the standard Painlevé dynamic as a composition of two different Schlesinger transformations, combined with automorphisms of our Fuchsian system, as follows.

Let $\sigma_i(j, k)$ be the map that exchanges the $j$-th and the $k$-th eigenvectors of $A_i$. If the eigenvalues $\theta_i^j$ and $\theta_i^k$ are non-zero, this map is just a permutation on the decomposition space $B \times C$. For this example we need $\sigma_1(1, 3)$, but it also defines a mapping on $B \times C$, since $b_i^1 \in \text{Ker}(C_i^1)$ and $C_i^1 \in \text{Ker}(B_1)$. Combining $\sigma_1(1, 3)$ with the scalar gauge transformation $\rho_1(-\theta_i^j)$, where $\rho_i(s) : A(x) \mapsto (x-u_i)A(x)$, we get the action $\Sigma_1(1, 3) = \rho_1(-\theta_i^j) \circ \sigma_1(1, 3)$ on the spectral data (as encoded in the Riemann scheme) and the decomposition space:

$$
\Sigma_1(1, 3) : \left\{ \begin{bmatrix}
    u_1 & u_2 & \infty \\
    \theta_1^1 & \theta_2^1 & \kappa_1 \\
    \theta_1^2 & \theta_2^2 & \kappa_2 \\
    0 & 0 & \kappa_3
\end{bmatrix} \mapsto \begin{bmatrix}
    u_1 & u_2 & \infty \\
    \theta_1^1 - 1 & \theta_2^1 + 1 & \kappa_1 \\
    \theta_1^2 & \theta_2^2 & \kappa_2 \\
    0 & 0 & \kappa_3
\end{bmatrix} \right\}
\rightarrow
\left\{ \begin{bmatrix}
    u_1 & u_2 & \infty \\
    \theta_1^1 & \theta_2^1 & \kappa_1 + \theta_i^j \\
    \theta_1^2 & \theta_2^2 & \kappa_2 + \theta_i^j \\
    0 & 0 & \kappa_3 + \theta_i^j
\end{bmatrix} \right\}.
$$

Thus, comparing this decomposition with the standard Painlevé dynamic

$$
\begin{align*}
d^P(A_2(1, 3)^r) = \Sigma_1(1, 3) \circ \{ \frac{1}{2} \} & \circ \Sigma_1(1, 3) \circ \{ \frac{1}{2} \}.
\end{align*}
$$

Using Mathematica, we can verify directly that this composition dynamic satisfies equation (5.29).

6. Conclusions

To summarize, we gave an explicit expression for the generating function for elementary Schlesinger transformations in terms of the coordinates on the decomposition space for the coefficients of the underlying Fuchsian system. We also argued that this function should be thought of as a discrete Hamiltonian of
the dynamic generated by Schlesinger transformations, thus continuing to establish a parallel between the theories of continuous and the discrete isomonodromic transformations. We used this discrete Hamiltonian to compute some explicit examples, including the derivation of the discrete Painlevé equation of type d-\( P(\lambda_2^{(1)}) \) in this framework, thus answering an earlier question of Sakai about the representability of this equation in terms of Schlesinger transformations. We also emphasized the role played by the geometry of rational surfaces corresponding to the equations in comparing different equations of the same type.

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