Nonlinear Numerical Analysis of The Plate Based on Thermo-Magneto-Mechanical Coupling

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Abstract—The thermo-magneto-elastic problems of the thin rectangular plate under the interaction of mechanical field, temperature field, and electromagnetic field are studied. On the basis of the geometric equations, physical equations, kinetic equations, and electrodynamics equations of the plate, the thermo-magneto-elastic basic equations of a thin rectangular plate are developed. According to Joule’s heat effect in the electromagnetic field and thermal equilibrium equation, temperature field in the thin rectangular plate, integral eigenvalues are derived. Adopting the difference, the quasi-linearization methods, quasilinear differential equations are obtained. Change rules of the stresses, temperatures, and deformations in the thin rectangular plate with electromagnetic parameters are analysed. It is confirmed that the stresses, strains, and temperatures in the plates can be dominated by altering electromagnetic and mechanical parameters through an example calculation.

1. INTRODUCTION
In the nuclear industry, high-speed transmission, and electromechanical power equipment and other fields, there are more and more structures in the electromagnetic field. They will be coupled by the mechanical field, the temperature field, and the electromagnetic field. More complex mechanical behavior has been demonstrated and the operation of engineering systems is affected. Hence, researches on the thermo-magneto-elastic problem have both theoretically and practically important significance. Now, the accomplishments on the magnetoelastic theoretical analysis and numerical calculation of plates and shells are relatively complete [1-10]. These accomplishments laid a good foundation for researches of the electromagnetoelasticity and its applications. Nevertheless, the studies concentrate primarily on coupling problems of electromagnetic field and mechanical field. The studies about the thermo-magneto-elastic problem with exploring the temperature field, especially, researches about two-dimensional thermo-magneto-elastic problem for the current-carrying plates and shells have rarely been seen. Consequently, the studies about the thermo-magneto-elastic problems for the thin current-carrying plate have currently changed into one of the most major issues.

The thermo-magneto-elastic problems of the thin rectangular plate under the interaction of a mechanical field, a temperature field, and an electromagnetic field are studied in this paper. The basic equations and a numerical algorithm under the coupling field are developed. According to Joule’s heat effect in the electromagnetic field and thermal equilibrium equation, temperature field in the thin rectangular plate and integral eigenvalues are derived. The stresses, displacements, and temperatures in the thin rectangular plate under the coupling field are computed by using this method. The thermo-magneto-elastic effect of electric current and magnetic field on thin plate is studied.
2. THERMO-MAGNETO-ELASTIC BASIC EQUATIONS

Let's consider the rectangular plate in a magnetic field \( B = \{0, B_x, 0\} \). The plate, the even-distributed load, and the electric current distribution in a rectangular coordinate system \( Oxyz \) are shown in figure 1.

![Figure 1. Distribution diagram of electric current and magnetic field on a thin plate.](image)

When the temperature effect is considered, the relations between the internal forces and strains of the rectangular plate are [6]

\[
\begin{align*}
N_x &= D_y [\varepsilon_x + \nu \varepsilon_y - (1 + \nu) \varepsilon_x] \\
N_y &= D_x [\varepsilon_x + \nu \varepsilon_y - (1 + \nu) \varepsilon_x] \\
M_x &= D_u [\kappa_x + \nu \kappa_y - (1 + \nu) \kappa_x] \\
M_y &= D_u [\kappa_x + \nu \kappa_y - (1 + \nu) \kappa_x]
\end{align*}
\]

where \( N_x, N_y, M_x, \) and \( M_y \) are the internal forces and moments; \( D_y (= Eh/(1-\nu^2)) \) and \( D_u (= Eh^3/(12(1-\nu^3))) \) are the tensile and bending rigidities, respectively; \( E \) is elastic modulus; \( \nu \) is Poisson’s ratio; \( h \) is the thickness of the thin plate; \( \varepsilon_x \) and \( \varepsilon_y \) are the strains along the corresponding directions; \( \kappa_x \) and \( \kappa_y \) are the bending strains along the corresponding directions; \( \varepsilon_r \) and \( \kappa_r \) are the integral eigenvalues of the temperature field \( T \), the expression is as follows:

\[
\begin{align*}
\varepsilon_r &= \frac{1}{h} \int_{-h/2}^{h/2} \alpha_T T(x,y,z) \, dz \\
\kappa_r &= \frac{12}{h^2} \int_{-h/2}^{h/2} \alpha_T T(x,y,z) \, dz 
\end{align*}
\]

Simultaneous the geometric equations, physical equations (1), kinetic equations, and electrodynamics equations of the plate [3], the following basic thermo-magneto-elastic equations of the rectangular plate are obtained

\[
\begin{align*}
\frac{\partial u}{\partial y} &= \frac{2S}{D_y(1-\nu)} - \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \theta_x \\
\frac{\partial v}{\partial y} &= \frac{N_y}{D_y} - \frac{1}{2} \theta_y - \frac{\nu}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \nu \frac{\partial u}{\partial x} + (1+\nu) \varepsilon_r \\
\frac{\partial w}{\partial y} &= -\theta_y \\
\frac{\partial \theta_x}{\partial y} &= \frac{1}{D_u} M_x + \nu \frac{\partial^2 w}{\partial x^2} + (1+\nu) \kappa_r
\end{align*}
\]
\[ \frac{\partial N_y}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial S}{\partial x} = (F_y + f_y) \]

\[ \frac{\partial \hat{Q}_y}{\partial y} = \frac{\rho h^+ h^-}{12 t^2} \left( \frac{12 w}{h^+ w} \right) - (F_y + f_y) - \nu \frac{\partial N_y}{\partial x} \frac{\partial w}{\partial x} \]

\[ - E h \frac{\partial w}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) - E h \frac{\partial^2 w}{\partial x^2} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \]

\[ - \nu \frac{\partial^2 M_y}{\partial x^2} + \frac{E h}{12} \frac{\partial^2 w}{\partial x^2} + \frac{E h}{12} \frac{\partial^2 \kappa_y}{\partial x^2} \]

\[ - \nu N_y \frac{\partial^2 w}{\partial x^2} + S \frac{\partial \theta}{\partial x} + \theta \frac{\partial S}{\partial x} \]

\[ \frac{\partial S}{\partial y} = \rho h \frac{\partial^2 u}{\partial t^2} - E h \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) - \nu \frac{\partial N_y}{\partial x} \]

\[ + E h \frac{\partial f_y}{\partial x} -(F_y + f_y) \]

\[ \frac{\partial M_y}{\partial y} = \hat{Q}_y + N_y \theta - S \frac{\partial w}{\partial x} + \frac{\rho h \frac{\partial^2 \theta}{\partial t^2}}{12} - 2 \frac{\partial}{\partial x} \left[ D_u (1-v) \frac{\partial \theta}{\partial x} \right] \]

\[ \frac{\partial E_y}{\partial y} = - \frac{1}{\sigma} \frac{\partial^2 B_y}{\partial x^2} + B_y \frac{\partial }{\partial t} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial u}{\partial t} \frac{\partial B_y}{\partial x} \]

\[ - \frac{(B_x + B_y)}{2} \frac{\partial}{\partial t} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial B_y}{\partial t} \]

\[ \frac{\partial B_y}{\partial y} = \sigma u \left[ E_y + \frac{\partial v}{\partial t} B_x - \frac{\partial w}{\partial t} (B_x + B_y) \right] + \frac{B_x - B_y}{h} \]

(3)

where \( \hat{Q}_y = Q_y - N_y \theta + S \frac{\partial w}{\partial x} + \frac{\partial M_y}{\partial x} \); \( F_y, f_y, \) and \( F_i \) are the mechanical loads; \( f_y, f_y, \) and \( f_y \) are the Lorentz force components [3]; \( u, v, \) and \( W \) are the displacement components; \( \theta \) is the rotation angle; \( E, \) and \( E \) are the electric field intensity components; \( B_x, B_y, \) and \( B_z \) are the magnetic induction intensity components; \( \rho \) is the mass density; \( \sigma \) is the electrical conductivity; \( \mu \) is the permeability; \( t \) is the time variable; \( B_x, B_y \) (i = x, y) are the values of \( B_i \) on the upper and lower surfaces of thin plate, respectively.

3. COMPUTATIONAL METHOD

Equations (3) can be expressed as the following boundary-value problems:

\[ \left\{ \begin{array}{l} \frac{\partial N_y}{\partial y} = F(x, y, N) \quad (y_1 \leq y \leq y_2) \\ C_i N(y_1) = c_i, \quad C_i N(y_2) = c_i \end{array} \right. \]
where $N = \{u, v, w, \theta, N, \dot{Q}, S, M, E, B\}^T$; $C_1$, $C_2$ are known rectangular matrices; $c_1$, $c_2$ are known vectors.

On the representation of Eqs. (4), the difference scheme is constructed in the $x$-direction [11], simultaneously, we find the derivatives of basic unknowns with respect to time in Eqs. (3) according to the time step by using Newmark’s stable finite equidifferent formulas [6].

Thus, Eqs. (3) can be expressed in the following form:

$$\begin{align*}
\frac{dN}{dy} &= F(y, N) \quad (y_1 \leq y \leq y_2) \\
C_1N(y_1) &= c_1, \quad C_2N(y_2) = c_2
\end{align*}$$

(5)

The problems represented in Eqs. (5) are nonlinear. Usually, applying the method of linearization, nonlinear problems can be converted into a series of linear problems.

$$\begin{align*}
\frac{dN^{(k+1)}}{dy} &= F(y, N^{(k)}) + \Gamma(y, N^{(k)})(N^{(k+1)} - N^{(k)}) \\
C_1N^{(k+1)}(y_1) &= c_1, \quad C_2N^{(k+1)}(y_2) = c_2
\end{align*}$$

(6)

where $\Gamma(y, N^{(k)})$ is Jacobi’s matrix. So we gain a set of linear ordinary differential equations.

4. ELECTROMAGNETIC TEMPERATURE EFFECT

Based on electrodynamics equations, generalized Ohm’s law, at the same time, noticing applied current density $J_{x(z), y}$, we have [12]:

$$\begin{align*}
J_x &= J_{x(z), y} + \sigma E_y + \sigma \left( \frac{\partial w}{\partial t} B_x - \frac{\partial w}{\partial t} B_y \right) \\
J_y &= J_{x(z), y} + \sigma E_x + \sigma \left( \frac{\partial w}{\partial t} B_x - \frac{\partial w}{\partial t} B_y \right)
\end{align*}$$

(7)

Due to Joule’s heat effect in electromagnetic field, heat source is bound to be generated in the thin plate. Taking into account the thin plate discussed and the low frequency electromagnetic field, the skin effect is not evident. We can think that the current distribution is uniform. So the power density of the heat source is [13]

$$Q = 0.86 \frac{J_z^2}{\sigma} = 0.86 \frac{J_x^2 + J_y^2}{\sigma}$$

(8)

We consider that the power distribution of heat source in the plate thickness direction is symmetrical. So the temperature curve equation along the $z$-direction is

$$T = T_x + \frac{Qh^2}{8\lambda} \left[ 1 + \frac{4\lambda}{h\alpha_p} - 4 \left( \frac{z}{h} \right)^2 \right] \frac{\rho h c}{2\alpha_p} T_x$$

(9)

By using Eqs. (2), (7), (8), and (9), we have:

$$e_x^{(k+1)} = e_x^0 T_x + \frac{\alpha_p h^2}{8\sigma \lambda} \left[ J_{x(z), y}^2 + \sigma \left( 2E_x^{(k+1)} E_y^{(k+1)} - (E_x^{(k+1)})^2 \right) + 2J_{x(z), y} \sigma E_y^{(k+1)} \right]$$
\[ \sigma \left[ B_z \left( 2 \frac{\partial \nu^{(i)}}{\partial t} \frac{\partial \nu^{(i)}}{\partial t} - \left( \frac{\partial \nu^{(i)}}{\partial t} \right)^2 \right) - 3 \left( \frac{\partial \nu^{(i)}}{\partial t} \right)^2 \right] \]

\[ + 2 \frac{\partial \nu^{(i)}}{\partial t} \frac{\partial \nu^{(i)}}{\partial t} (B_z^{(i)})^2 + 2 \left( \frac{\partial \nu^{(i)}}{\partial t} \right)^2 B_z^{(i)} B_z^{(i)} \]

\[ - 2B_z \left( \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} + \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} \right) \]

\[ - 2 \frac{\partial \nu^{(i)}}{\partial t} \frac{\partial \nu^{(i)}}{\partial t} (B_z^{(i)})^2 + 2 \frac{\partial \nu^{(i)}}{\partial t} \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} \]

\[ - 2J_w \sigma \left( \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} - \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} - \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} + \frac{\partial \nu^{(i)}}{\partial t} B_z^{(i)} \right) \]

\[ - 2\sigma \left[ B_z \left( E_r^{(i+1)} \frac{\partial \nu^{(i)}}{\partial t} + E_r^{(i)} \frac{\partial \nu^{(i)}}{\partial t} - E_r^{(i)} \frac{\partial \nu^{(i)}}{\partial t} \right) \right] \]

\[ - \left( E_r^{(i)} B_z^{(i)} \frac{\partial \nu^{(i)}}{\partial t} + E_r^{(i)} B_z^{(i)} \frac{\partial \nu^{(i)}}{\partial t} + E_r^{(i)} B_z^{(i)} \frac{\partial \nu^{(i)}}{\partial t} \right) \]

\[ - 2E_r^{(i)} B_z^{(i)} \frac{\partial \nu^{(i)}}{\partial t} \] \hspace{1cm} (10)

\[ \kappa_r^{(i+1)} = 0 \] \hspace{1cm} (11)

The \( \epsilon_r^{(i+1)} \) and \( \kappa_r^{(i+1)} \) are substituted in the set of the differential equations, the thermo-magneto-elastic coupling equations of a thin rectangular plate can be obtained. All unknowns can be solved through discrete-orthogonalization method.

5. EXAMPLES ANALYSIS

A thin rectangular plate made from aluminium alloy with four edges fixed shown in figure 1 is placed in a magnetic field \( B = \{ 0, B, 0 \} \). The electric current density in the plate is \( J_a = \{ J_{a}, 0, 0 \} \). The mechanical load is \( F = \{ 0, 0, F \} \). We know \( E = 71 \text{ GPa} \), \( \nu = 0.34 \), \( \rho = 2670 \text{ kg/m}^3 \), \( \sigma = 3.63 \times 10^3 \text{ (Ω·m)}^{-1} \), \( \mu = 1.256 \times 10^{-8} \text{ H/m} \), \( \lambda = 237 \text{ W/(m·C)} \), \( \alpha_r = 235 \text{ W/(m}^2 \cdot \text{C)} \), \( \alpha_c = 2.35 \times 10^{-4} \text{ C}^{-1} \), \( J_{a} = J_{a} \sin \alpha A/m^2 \), \( \omega = \pi \times 10^8 \text{ s}^{-1} \), \( a = 2 \text{ m} \), \( b = 0.5 \text{ m} \), \( h = 1 \times 10^{-3} \text{ m} \).

The initial conditions are \( N(x, y, t)|_{t=0} = 0 \), \( \dot{u}(x, y, t)|_{t=0} = 0 \), \( \dot{v}(x, y, t)|_{t=0} = 0 \), \( \dot{w}(x, y, t)|_{t=0} = 0 \), \( \dot{\theta}(x, y, t)|_{t=0} = 0 \).

The boundary conditions are \( x = 0 : u = 0, v = 0, w = 0, \theta_r = 0 \).
\[ x = a : u = 0, v = 0, w = 0, \theta_x = 0 \]
\[ y = 0 : u = 0, v = 0, w = 0, \theta_y = 0, B_x = 0.1 \sin \omega t \]
\[ y = b : u = 0, v = 0, w = 0, \theta_y = 0, B_x = 0 \]

Figure 2 gives the deflection distribution at \( x = 1 \text{ m} \) in the \( y \)-direction for \( F_y = 300 \text{ N/m}^2 \), \( J_y = 5 \text{ MA/m}^2 \), \( B_y = 0.1 \text{ T} \), and \( t = 6 \text{ ms} \). Figure 3 gives the stress distribution at \( x = 1 \text{ m} \) for \( F_y = 100 \text{ N/m}^2 \), \( J_y = 3 \text{ MA/m}^2 \), \( B_y = 0.2 \text{ T} \), and \( t = 5 \text{ ms} \). Figure 4 gives the deflection distribution at \( x = 1 \text{ m} \) in the \( y \)-direction for similar matters and different moment. Figure 5 gives the variation of the temperature at \( x = 1 \text{ m}, y = 0.25 \text{ m} \) with time for \( F_y = 20 \text{ N/m}^2 \), \( B_y = 0.01 \text{ T} \), and different values of \( J_y \).
Figure 5. The temperature versus time for different $J_x$.

6. CONCLUSIONS
The thermo-magneto-elastic problems of the thin rectangular plate have been studied in this paper. Adopting the difference and quasi-linearization methods, we have converted nonlinear partial differential equations including 10 essential unknowns into the quasilinear differential equations, which can be solved through discrete-orthogonalization method. Numerical solutions of the thermo-magnetoelastic stress and deformation in the thin rectangular plate have been given. The relations between the stresses, temperatures, deformations in the thin rectangular plate and the electromagnetic parameters have been analysed. The calculation results show that the stress, strain, and temperature in plates can be controlled by changing the electromagnetic and mechanical parameters.

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