Reflected Mean-Field Backward Stochastic Differential Equations. Approximation and Associated Nonlinear PDEs

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Abstract
Mathematical mean-field approaches have been used in many fields, not only in Physics and Chemistry, but also recently in Finance, Economics, and Game Theory. In this paper we will study a new special mean-field problem in a purely probabilistic method, to characterize its limit which is the solution of mean-field backward stochastic differential equations (BSDEs) with reflections. On the other hand, we will prove that this type of reflected mean-field BSDEs can also be obtained as the limit equation of the mean-field BSDEs by penalization method. Finally, we give the probabilistic interpretation of the nonlinear and nonlocal partial differential equations with the obstacles by the solutions of reflected mean-field BSDEs.

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1 Introduction

Mathematical mean-field approaches have been used in many fields. To work on a stochastic limit approach to a mean-field problem is inspired at the one hand by classical mean-field approaches in Statistical Mechanics and Physics, by similar methods in Quantum Mechanics and Quantum Chemistry, but also by a recent series of papers by Lasry and Lions (see [16] and the references inside cited) who studied mean-field games. And also it has been strongly inspired by the McKean-Vlasov partial differential equations (PDEs) which have found a great interest in the last years and have been studied with the help of stochastic methods by many authors. On the other hand, in the last years models of large stochastic particle systems with mean field interaction have been studied by many authors; they have described them by characterizing their asymptotic behavior when the size of the system becomes very large, and also have shown that probabilistic methods allow to study the solution of linear McKean-Vlasov PDE. The reader is referred, for example, to the works by Borkar and Kumar [4], Bossy [5], Bossy and Talay [6], Chan [11], Kotelenez [15], Mckean [19], Méleard [20], Overbeck [21], Pra and Hollander [24], Sznitman [26], [27], Talay and Vaillant [28], and all the references therein. More details may refer to Buckdahn, Djehiche, Li and Peng [7] and the references inside cited.

Buckdahn, Djehiche, Li and Peng [7] studied a special mean-field problem in a purely stochastic approach. They considered a stochastic differential equation that describes the dynamics of a particle $X^{(N)}$ influenced by the dynamics of $N$ other particles, which are supposed to be independent identically distributed and of the same law as $X^{(N)}$. This equation (of rank $N$) is then associated with a backward stochastic differential equation (BSDE). After having proven the existence and the uniqueness of a solution $(X^{(N)}, Y^{(N)}, Z^{(N)})$ for this couple of equations the authors of [7] investigated its limit behavior. With a new approach which uses the tightness of the laws of the above sequence of triplets in a suitable space, and combines it with BSDE methods and the Law of Large Numbers, it was shown that $(X^{(N)}, Y^{(N)}, Z^{(N)})$ converges in $L^2$ to the unique solution of a limit equation formed by a McKean-Vlasov stochastic differential equation and a Mean-Field backward stochastic differential equation. Furthermore, Buckdahn, Li and Peng [9] proved the existence and the uniqueness of the solution of mean-field BSDEs under the classical assumptions, the comparison theorem of mean-field BSDEs and gave a stochastic interpretation to McKean-Vlasov partial differential equations (PDEs) with the help of the solutions of mean-field BSDEs. Since then we want to work on another new special mean-field problem to get a new limit equation which is like reflected BSDE in some sense. On the other hand, since the works [9] and [7] on the mean-field BSDEs, there are many works on its generalizations, e.g., Wang [29] studied backward doubly SDEs of mean-field type and its applications; Shi, Wang and Yong [25] studied backward stochastic Volterra integral equations of mean-field type; Li and Luo [18] studied reflected BSDEs of mean-field type, they proved the existence and the uniqueness for reflected mean-field BSDEs; and also its applications, e.g., Andersson, Djehiche [11], Bensoussan, Sung, Yam and Yung [3], Buckdahn, Djehiche and Li [8], Li [17], Yong [31]. Reflected BSDEs were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez [13] in 1997. Later the theory of RBSDEs develops very quickly, because of its many applications, for example, in partial differential equations, finance and so on. More details may refer to Buckdahn and Li [10] and the references inside cited.

In this paper we will study another new special mean-field problem, and get its limit which
is a new type of reflected BSDEs, we call it reflected mean-field BSDEs. Our objective here is to characterize such an equation, at one hand, as the limit of classical BSDEs with reflection and, on the other hand, as the limit of mean-field BSDEs with a penalization approach. The approximating reflected BSDEs (N) are discussed, and with an example it is in particular shown that these reflected BSDEs (N) don’t obey the comparison principle. Furthermore, under an additional monotonicity assumption of the driving coefficient, the description of reflected mean-field BSDEs as monotonic limit of mean-field BSDEs without reflection is used to give them a stochastic interpretation of associated non-local PDEs with obstacles. We show that the solution of the reflected mean-field BSDE is the unique viscosity solution of the associated non-local PDE with obstacles.

More precisely, we consider the following mean-field BSDE with reflections:

\[(1.1)\]

(i) \( Y \in S_F^2([0,T]), Z \in L_F^2([0,T];\mathbb{R}^d) \) and \( K \in A_{F}^{2,c}([0,T]) \);
(ii) \( Y_t = E[\Phi(x, X_T)|x=X_T] + \int_t^T E[g(s, u, \Lambda_s)]|u=\Lambda_s \ ds + K_T - K_t - \int_t^T Z_s dW_s \);
(iii) \( Y_t \geq h(t, X_t), \text{ a.s., for all } t \in [0,T] \);
(iv) \( \int_0^T (Y_t - h(t, X_t)) dK_t = 0 \),

where we have used the notation \( \Lambda = (X, Y, Z) \); \( T > 0 \) is a given finite time horizon; \( W = (W_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion; \( X = (X_t)_{t \in [0,T]} \) is a driving \( n \)-dimensional adapted stochastic process.

Such type of mean-field BSDEs without reflections have been studied by Buckdahn, Li and Peng [9], they proved that such a mean-field BSDE gave a stochastic interpretation to the related nonlocal PDEs. In this paper we first prove that, under our standard assumptions the mean-field BSDE (1.1) with reflections will be the limit equation of the following reflected BSDE (N):

\[(1.2)\]

(i) \( Y^N \in S_F^2([0,T]), Z^N \in L_F^2([0,T];\mathbb{R}^d) \) and \( K^N \in A_{F}^{2,c}([0,T]) \);
(ii) \( Y^N_t = \xi^N + \int_t^T f^N(s, \Theta_N(Y_s, Z_s))ds + K^N_T - K^N_t - \int_t^T Z^N_s dW_s, \ t \in [0,T] \);
(iii) \( Y^N_t \geq L^N_t, \ \text{ a.s., for any } t \in [0,T] \);
(iv) \( \int_0^T (Y^N_t - L^N_t) dK^N_t = 0 \),

where for \( N \geq 1 \) and \( \omega \in \Omega \),

\[ \xi^N(\omega) := \frac{1}{N} \sum_{k=1}^N \Phi(\Theta^k(\omega), X^N_{T^k}(\omega), X^N_T(\Theta^k(\omega))) \]

\[ f^N(\omega, t, y, z) := \frac{1}{N} \sum_{k=1}^N g(\Theta^k(\omega), t, X^N_{T^k}(\omega), (y_0, z_0), X^N_T(\Theta^k(\omega)), (y_k, z_k)) \]

for \( t \in [0,T] \), \( y = (y_0, \cdots, y_N) \in \mathbb{R}^{N+1}, z = (z_0, \cdots, z_N) \in \mathbb{R}^{(N+1) \times d} \),

\[ L^N_t(\omega) := h(\omega, t, X^N_t(\omega)), \ t \in [0,T] \].

(More details refer to Theorem 4.1). Example 3.1 shows that such reflected BSDE (N) usually doesn’t have the comparison theorem.

Furthermore, more generally, for the obstacle process \( L = (L_s)_{0 \leq s \leq T} \in S_F^2([0,T]) \) and the terminal condition \( \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}) \) such that \( \xi \geq L_T, \text{ P-a.s.} \), we consider the following reflected
mean-field BSDE:

(i) \( Y \in S_2^F([0, T]), \ Z \in L^2_F([0, T]; \mathbb{R}^d) \) and \( K \in A_{2,c}^F([0, T]) \);

(ii) \( Y_t = \xi + \int_t^T E[g(s, y, z, Y_s)]_{|y = Y_s, z = Z_s} ds + K_T - K_t - \int_t^T Z_s dW_s \);

(iii) \( Y_t \geq L_t, \text{ a.s., for all } t \in [0, T] \);

(iv) \( \int_0^T (Y_t - L_t) dK_t = 0 \) \( \tag{1.3} \)

Under our assumptions the reflected mean-field BSDE (1.3) can be also got as the limit equation of the following penalized mean-field BSDEs:

\[ \tilde{Y}_t^n = \xi + \int_t^T E[g(s, y, z, \tilde{Y}_s^n)]_{|y = \tilde{Y}_s^n, z = \tilde{Z}_s^n} ds + n \int_t^T (\tilde{Y}_s^n - L_s)^- ds - \int_t^T \tilde{Z}_s^n dW_s. \] \( \tag{1.4} \)

(More details refer to Theorem 5.1).

Finally, this allows us to give a probabilistic representation for the solution of the following non-local PDEs with obstacles:

\[ \begin{aligned}
  \min\{ &u(t, x) - h(t, x), -\frac{\partial}{\partial t} u(t, x) - Au(t, x) \\
  &- E[f(t, x, X_t^{0,x_0}, u(t, X_t^{0,x_0}), u(t, x), Du(t, x), E[\sigma(t, x, X_t^{0,x_0})])] \} = 0,
  \\
  u(T, x) &= E[\Phi(x, X_T^{0,x_0})], \quad x \in \mathbb{R}^n,
\end{aligned} \] \( \tag{1.5} \)

with

\[ Au(t, x) := \frac{1}{2} \text{tr}(E[\sigma(t, x, X_t^{0,x_0})]E[\sigma(t, x, X_t^{0,x_0})]^T D^2 u(t, x)) + Du(t, x).E[b(t, x, X_t^{0,x_0})]. \]

Here the functions \( b, \sigma, f \) and \( \Phi \) are supposed to satisfy (H6.1), and (H6.2), respectively, and \( X_{0,-x_0} \) is the solution of the SDE (6.1). More details refer to Theorem 6.1.

Our paper is organized as follows: Section 2 recalls briefly some elements of the theory of backward SDEs and mean-field BSDEs which will be needed in what follows. In Section 3 we introduce the reflected BSDEs of rank \( N \), define the framework in which it is investigated and prove the existence and the uniqueness, and give an example to explain that this type of reflected BSDEs of rank \( N \) usually doesn’t have the comparison theorem anymore. In this section we also give an important inequality about RBSDE which is very useful—Lemma 3.5. In Section 4 we prove the convergence of the solution of the reflected BSDE of rank \( (N) \) to that of reflected mean-field BSDE (Theorem 4.1). In Section 5 we prove that the reflected mean-field BSDEs can also be obtained as the limit equations of the reflected BSDEs with the help of the penalization method (Theorem 5.1). In Section 6 we prove that the solution of the reflected mean-field BSDEs is the unique viscosity solution of the associated nonlinear and nonlocal partial differential equation with the obstacles (Theorem 6.1). We also prove that the value functions \( u_n(t, x) \) which are defined by the penalized mean-field BSDEs are Lipschitz with respect to \( x \), uniformly in \( t \), and \( n \in \mathbb{N} \) (Proposition 6.1).
2 Preliminaries

2.1 A Recall on BSDEs

In this section we will introduce some basic notations and results about BSDEs, which will be needed in the following sections. First we will extend slight the classical Wiener space \((\Omega, \mathcal{F}, P)\):

- For an arbitrarily given time horizon \(T > 0\) and a countable index set \(I\) (which will be clarified later), \(\Omega\) is the set of all families \((\omega^i)_{i \in I}\), where \(\omega^i : [0, T] \rightarrow \mathbb{R}^d\) is continuous with initial value 0 (i.e., \(\Omega = C_0([0, T]; \mathbb{R}^d)\)); we endow it with the product topology produced by the uniform convergence on its components \(C_0([0, T]; \mathbb{R}^d)\);
- Let \(\mathcal{B}(\Omega)\) denote the Borel \(\sigma\)-field over \(\Omega\) and \(B = (W^i)_{i \in I}\) be the coordinate process over \(\Omega : W^i_t(\omega) = \omega^i_t, t \in [0, T], \omega \in \Omega, i \in I\);
- Let \(P\) be the Wiener measure over \((\Omega, \mathcal{B}(\Omega))\), i.e., the coordinates \(W^i, i \in I\), are a family of independent \(d\)-dimensional Brownian motions with respect to \(P\). In the end,
- Let \(\mathcal{F}\) be the \(\sigma\)-field \(\mathcal{B}(\Omega)\) completed by the Wiener measure \(P\).

Define \(W := W^0\). The Probability space \((\Omega, \mathcal{F}, P)\) is endowed with the filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}\), where \(\mathcal{F}\) is generated by the Brownian motion \(W\), enlarged by the \(\sigma\)-field \(\mathcal{G} = \sigma\{W^i_t, t \in [0, T], i \in I \setminus \{0\}\}\) and completed by the collection \(\mathcal{N}_P\) of all \(P\)-null sets, that is

\[
\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{G}, t \in [0, T],
\]

where \(\mathcal{F}^W_t = \sigma\{W^i_r, r \leq t\} \vee \mathcal{N}_P\}_{t \in [0, T]}\). Notice that the Brownian motion \(W\) still has the martingale representation property with respect to the filtration \(\mathcal{F}\).

We also introduce the following spaces which will be used later:

\[
S^2_F([0, T]) = \{(Y_t)_{t \in [0, T]} \text{ continuous adapted process: } E[\sup_{t \in [0, T]} |Y_t|^2] < +\infty\};
\]

\[
L^2_F([0, T]; \mathbb{R}^d) = \{(Z_t)_{t \in [0, T]} \text{ } \mathbb{R}^d\text{-valued progressively measurable process: } E\left[\int_0^T |Z_t|^2 dt\right] < +\infty\}.
\]

(Notice that \(|z|\) denotes the Euclidean norm of \(z \in \mathbb{R}^d\). Now given a measurable function \(g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}\) which satisfies that \((g(t, y, z))_{t \in [0, T]}\) is \(\mathcal{F}\)-progressively measurable for all \((y, z)\) in \(\mathbb{R} \times \mathbb{R}^d\). We give the following standard assumptions:

(A1) There is some constant \(C \geq 0\) such that, \(P\)-a.s., for all \(t \in [0, T], y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d\),

\[
|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).
\]

(A2) \(g(\cdot, 0, 0) \in L^2_F([0, T]; \mathbb{R})\).

The following results on BSDEs are classical; for the proof refer to, e.g., Pardoux and Peng [22], or El Karoui, Peng and Quenez [14].

**Lemma 2.1.** Suppose the generator \(g\) satisfies (A1) and (A2). Then, for any random variable \(\xi \in L^2(\Omega, \mathcal{F}_T, P)\), the BSDE associated with the data \((g, \xi)\)

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \hspace{0.5cm} 0 \leq t \leq T,
\]

has a unique solution \((Y, Z) \in S^2_F([0, T]) \times L^2_F([0, T]; \mathbb{R}^d)\).
Now we give the standard estimate for BSDEs.

**Lemma 2.2.** Suppose that \( g_k \) satisfies (A1) and (A2) and \( \xi_k \in L^2(\Omega, \mathcal{F}_T, P) \), \( k = 1, 2 \). Let \((Y^k, Z^k)\) denote the unique solution of the BSDE with the data \((g_k, \xi_k)\), \( k = 1, 2 \), respectively. For every \( \delta > 0 \), there exists some \( \gamma > 0, C > 0 \) only depending on \( \delta \) and the Lipschitz constants of \( g_k, k = 1, 2 \), such that,

\[
E\left[ \int_0^T e^{\gamma t}(|Y^1_t|^2 + |Z^1_t|^2)dt \right] \leq CE[e^{\gamma T}|\xi|^2] + \delta E[\int_0^T e^{\gamma t}|\overline{f}(t,Y^1_t, Z^1_t)|^2dt],
\]

where

\[
(Y, Z) = (Y^1 - Y^2, Z^1 - Z^2), \quad \overline{f} = g_1 - g_2, \quad \overline{\xi} = \xi_1 - \xi_2.
\]

Now we introduce one of the important results for BSDEs—the comparison theorem (see Proposition 2.4 in Peng [23] or Theorem 2.2 in El Karoui, Peng and Quenez [14]).

**Lemma 2.3.** (Comparison Theorem) Suppose two coefficients \( g_1 \) and \( g_2 \) satisfy (A1) and (A2) and two terminal values \( \xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P) \). \((Y^1, Z^1)\) and \((Y^2, Z^2)\) are the solutions of the BSDE with the data \((\xi_1, g_1)\) and \((\xi_2, g_2)\), respectively. Then we have:

(i) (Monotonicity) If \( \xi_1 \geq \xi_2 \) and \( g_1 \geq g_2 \), a.s., then \( Y^1_t \geq Y^2_t \), for all \( t \in [0, T] \), a.s.

(ii) (Strict Monotonicity) If, in addition to (i), also \( P\{\xi_1 > \xi_2\} > 0 \), then we have \( P\{Y^1_t > Y^2_t\} > 0 \), for all \( 0 \leq t \leq T \), and in particular, \( Y^1_0 > Y^2_0 \).

### 2.2 A Recall on Mean-field BSDEs

In this section we recall some basic results on a new type of BSDE, the so called Mean-Field BSDEs; more details refer to [7] and [9].

The driver of our mean-field BSDE is a function \( f = f(\omega, t, y, z, \tilde{y}, \tilde{z}) : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) which is \( \mathcal{F} \)-progressively measurable, for all \((y, z, \tilde{y}, \tilde{z})\), and satisfies the following assumptions:

(A3) There exists a constant \( C \geq 0 \) such that, \( P\)-a.s., for all \( t \in [0, T] \), \( y_1, y_2, \tilde{y}_1, \tilde{y}_2 \in \mathbb{R} \), \( z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^d \),

\[
|f(t, y_1, z_1, \tilde{y}_1, \tilde{z}_1) - f(t, y_2, z_2, \tilde{y}_2, \tilde{z}_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |\tilde{y}_1 - \tilde{y}_2| + |\tilde{z}_1 - \tilde{z}_2|).
\]

(A4) \( f(\cdot, 0, 0, 0, 0) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \).

The following results refer to [9].

**Lemma 2.4.** Under the assumptions (A3) and (A4), for any random variable \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), the mean-field BSDE

\[
Y_t = \xi + \int_t^T E[f(s, y, z, Y_s, Z_s)|_{y=Y_s,z=Z_s}] ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T,
\]

has a unique adapted solution

\[
(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^d).
\]
Lemma 2.5. (Comparison Theorem) Let \( f_1 = f_1(\omega, t, y, z, \tilde{y}, \tilde{z}) \), \( i = 1, 2 \), be two drivers satisfying the standard assumptions (A3) and (A4). Moreover, we suppose
(i) One of both coefficients is independent of \( \tilde{z} \);
(ii) One of both coefficients is nondecreasing in \( \tilde{y} \).

Let \( \xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P) \) and denote by \( (Y^1, Z^1) \) and \( (Y^2, Z^2) \) the solution of the mean-field BSDE (2.2) with data \( (\xi_1, f_1) \) and \( (\xi_2, f_2) \), respectively. Then if \( \xi_1 \leq \xi_2 \), \( P \)-a.s., and \( f_1 \leq f_2 \), \( P \)-a.s., it holds that also \( Y^1_t \leq Y^2_t \), \( t \in [0, T] \), \( P \)-a.s.

Remark 2.1. The conditions (i) and (ii) of Lemma 2.2 are, in particular, satisfied, if they hold for the same driver \( f_j \) but also if (i) is satisfied by one driver and (ii) by the other one.

3 Reflected BSDEs

After the short recall on BSDEs let us now consider reflected BSDEs (RBSDEs) and MFBSDEs with reflection. Let us first introduce the framework in which we want to study the limit approach to get reflected MFBSDEs. First we give the countable index set as follows:

\[ I := \{ \omega \mid \omega \in \{ 1, 2, 3, \ldots \}^k, k \geq 1 \} \cup \{ 0 \}. \]

We define \( \omega \oplus i = (i_1, \ldots, i_k, i_k', \ldots, i_k''') \in I \), for two elements \( \omega = (i_1, \ldots, i_k) \), \( i' = (i_k', \ldots, i_k''') \) of \( I \) (with the convention that \( i \oplus 0 = i \)); in particular, \( \omega(k) \oplus i = \omega(k, i_1, \ldots, i_k), k \geq 0 \). Notice that \( \omega(k) \oplus i \in I \), \( i \in I \).

Now we introduce a family of shift operators \( \Theta^k : \Omega \rightarrow \Omega \), \( k \geq 0 \), over \( \Omega \). We define \( \Theta^k(\omega) = (\omega(k) \oplus i)_i \in I \), \( \omega \in \Omega, k \geq 0 \), and notice that \( \Theta^k \) is an operator mapping \( \Omega \) into \( \Omega \) associating \( (\omega_i)_i \in I \) with \( (\omega(k) \oplus i)_i \in I \). Notice that all these operators \( \Theta^k : \Omega \rightarrow \Omega \) make the Wiener measure \( P \) invariant (i.e., \( P \circ [\Theta^k]^{-1} = P \)), which allows to regard \( \Theta^k \) as an operator defined over \( L^0(\Omega, \mathcal{F}, P) : \Theta^k(\xi)(\omega) := \xi(\Theta^k(\omega)), \xi \in \Omega \), for the random variables \( \xi(\omega) = f(\omega_1^{i_1}, \ldots, \omega_n^{i_n}), i_1, \ldots, i_n \in I, i_1, \ldots, i_n \in [0, T], f \in C(R^{d \times n}), n \geq 1 \), then we can extend this definition from this set of continuous Wiener functionals to the space \( L^0(\Omega, \mathcal{F}, P) \) with the help of the density of the set of smooth Wiener functionals in \( L^0(\Omega, \mathcal{F}, P) \). Notice that, for all \( \xi \in L^0(\Omega, \mathcal{F}, P) \), the random variables \( \Theta^k(\xi), k \geq 1 \), are independent and uniformly identically distributed (i.i.d.), with the same law as \( \xi \) and also independent of the Brownian motion \( W \).

The end, for simplicity of notation we introduce the \((N + 1)\)-dimensional shift operator \( \Theta_N = (\Theta^0, \Theta^1, \ldots, \Theta^N) \), which relates a random variable \( \xi \in L^0(\Omega, \mathcal{F}, P) \) with the \((N + 1)\)-dimensional random vector \( \Theta_N(\xi) = (\xi, \Theta^1(\xi), \ldots, \Theta^N(\xi)) \) (remark that \( \Theta^0 \) is the identical operator). If \( \xi \) is a random vector, \( \Theta^k(\xi) \) and \( \Theta_N(\xi) \) are introduced by a componentwise application of the corresponding operators.

For introducing the notion of a RBSDE we shall introduce still the following space of adapted increasing processes:

\[ A^2_{\mathcal{F}}([0, T]) = \{(K_t)_{t \in [0, T]} \in S^2_{\mathcal{F}}([0, T]) \mid \text{non-decreasing process: } K_0 = 0\}. \]

An RBSDE with one barrier is associated with a terminal condition \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), a generator \( g \) satisfying the assumptions (A1) and (A2), and an “obstacle” process \( L = (L_t)_{t \in [0, T]} \). We assume that \( L \in S^2_{\mathcal{F}}([0, T]) \) and \( L_T \leq \xi, P \)-a.s. A solution of an RBSDE with one barrier is a triplet
(Y, Z, K) of F-progressively measurable processes, taking its values in \( \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \) and satisfying the following properties

(i) \( Y \in S^2_F([0,T]), Z \in L^2_F([0,T]; \mathbb{R}^d) \) and \( K \in A^{2,c}_F([0,T]); \)

(ii) \( Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t \in [0,T]; \)

(iii) \( Y_t \geq L_t, \quad \text{a.s., for any } t \in [0,T]; \)

(iv) \( \int_0^T (Y_t - L_t)dK_t = 0. \)  (3.1)

For shortness, a given triplet \((\xi, g, L)\) is said to satisfy the Standard Assumptions (A) if the generator \(g\) satisfies (A1) and (A2), the terminal value \(\xi\) belongs to \(L^2(\Omega, \mathcal{F}_T, P)\), and the obstacle process \(L \in S^2_F([0,T])\) is such that \(L_T \leq \xi, P\text{-a.s.}\)

We begin by recalling two lemmata which are by now well-known results of the theory of reflected BSDEs and are borrowed from Theorem 5.2 and Theorem 4.1, respectively, of the paper by El Karoui, Kapoudjian, Pardoux, Peng, Quenez [13].

**Lemma 3.1.** Let \((\xi, g, L)\) be a triplet satisfying the Standard Assumptions (A). Then the above RBSDE admits a unique solution \((Y, Z, K)\).

**Lemma 3.2.** (Comparison Theorem) We suppose that two triplets \((\xi_1, g_1, L^1)\) and \((\xi_2, g_2, L^2)\) satisfy the Standard Assumptions (A) but we impose only for one of the both coefficients \(g_1\) and \(g_2\) to fulfill the Lipschitz condition (A1). Furthermore, we make the following assumptions:

(i) \( \xi_1 \leq \xi_2, \quad a.s.; \)

(ii) \( g_1(t, y, z) \leq g_2(t, y, z), \quad a.s., \quad \text{for all } (t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d; \)  (3.2)

(iii) \( L^1_t \leq L^2_t, \quad a.s., \quad \text{for all } t \in [0,T]. \)

Let \((Y^1, Z^1, K^1)\) and \((Y^2, Z^2, K^2)\) be adapted solutions of the RBSDEs with data \((\xi_1, g_1, L^1)\) and \((\xi_2, g_2, L^2)\), respectively. Then, \(Y^1_t \leq Y^2_t, \quad \text{for all } t \in [0,T], \ P\text{-a.s.}\)

We also shall recall the following both standard estimates of BSDEs with one reflecting barrier.

**Lemma 3.3.** Let \((Y, Z, K)\) be the solution of the above RBSDE with data \((\xi, g, L)\) satisfying the Standard Assumptions (A). Then there exists a constant \(C\) such that

\[
E[\sup_{t \leq s \leq T} |Y_s|^2 + \int_t^T |Z_s|^2 ds + |K_T - K_t|^2 |\mathcal{F}_t] \leq CE[\xi^2 + \left( \int_t^T g(s, 0, 0) ds \right)^2 + \sup_{t \leq s \leq T} L^2_s |\mathcal{F}_t], \ P\text{-a.s., } t \in [0,T]. \quad (3.3)
\]

The constant \(C\) depends only on the Lipschitz constant of \(g\).

Lemma 3.3 is based on Propositions 3.5 in [13] and its generalization by Proposition 2.1 in Wu and Yu [30]. The following statement refers to Proposition 3.6 in [13] or Proposition 2.2 in [30].
Lemma 3.4. Let $(\xi, g, L)$ and $(\xi', g', L')$ be two triplets satisfying the above Standard Assumptions (A). We suppose that $(Y, Z, K)$ and $(Y', Z', K')$ are the solutions of our RBSDE with the data $(\xi, g, L)$ and $(\xi', g', L')$, respectively. Then, for some constant $C$ which only depends on the Lipschitz constant of the coefficient $g'$, and with the notations

$$
\xi = \xi - \xi', \quad \gamma = g - g', \quad \overline{L} = L - L', \\
\overline{Y} = Y - Y', \quad \overline{Z} = Z - Z', \quad \overline{K} = K - K',
$$

it holds, for all $t \in [0, T]$, $P$-a.s.,

$$
E\left[ \sup_{s \in [t, T]} |\overline{Y}_s|^2 + \int_t^T |\overline{Z}_s|^2 ds + |\overline{K}_T - \overline{K}_t|^2 |\mathcal{F}_t\right] \\
\leq CE[|\xi|^2 + \left( \int_t^T |g(s, 0, 0)| ds \right)^2 + \sup_{s \in [t, T]} |L_s|^2] \\
+ |\xi'|^2 + \left( \int_t^T |g'(s, 0, 0)| ds \right)^2 + \sup_{s \in [t, T]} |L'_s|^2 |\mathcal{F}_t|, \tag{3.4}
$$

where

$$
\Psi_{t,T} = E[|\xi|^2 + \left( \int_t^T |g(s, 0, 0)| ds \right)^2 + \sup_{s \in [t, T]} |L_s|^2] \\
+ |\xi'|^2 + \left( \int_t^T |g'(s, 0, 0)| ds \right)^2 + \sup_{s \in [t, T]} |L'_s|^2 |\mathcal{F}_t|.
$$

However we will also need a slight version of the above standard estimate for RBSDEs, which is of the same nature as that given by Lemma 2.2 for BSDEs.

Lemma 3.5. As in Lemma 3.4 we suppose that $(\xi, g, L)$ and $(\xi', g', L')$ are two triplets satisfying the above Standard Assumptions (A), and $(Y, Z, K)$ and $(Y', Z', K')$ are the solutions of the associated RBSDEs, respectively. Then, for some $C > 0$ and all $\delta > 0$ we can find $\gamma > 0$ such that, with the notations of Lemma 3.4,

$$
E\left[ \int_0^T e^{\gamma t} (|\overline{Y}_t|^2 + |\overline{Z}_t|^2) dt \right] \\
\leq 2E \left[ e^{\gamma T} |\xi|^2 \right] + \delta E \left[ \int_0^T e^{\gamma t} |g(t, Y_t, Z_t)|^2 dt \right] \\
+ C \left( E \left[ \sup_{[0,T]} |e^{\gamma t} \overline{L}_t|^2 \right] \right)^{1/2} \Psi_{0,T}^{1/2} \tag{3.5}
$$

(Recall the definition of $\Psi_{t,T}$ given in Lemma 3.4.). The constant $C$ only depends on the bound and the Lipschitz constant of $g$ and $g'$, while $\gamma$ only depends on $\delta$ and on the Lipschitz constant of $g$ and $g'$.

Remark 3.1. The above estimate for $L = L'$ was established in the proof of Theorem 5.2 in [13]. Since the above lemma plays a crucial role in our approach we give its proof for the reader’s convenience.

Proof (of Lemma 3.5). Let $\delta > 0$ be sufficiently small and $\gamma > 0$. Then, by applying Itô’s formula to the process $(e^{\gamma t}\overline{Y}_t^2)_{t \in [0, T]}$ and by taking into account that $(\overline{Y}_t - \overline{L}_t) d\overline{K}_t \leq 0$, $t \in [0, T]$, we get

$$
E\left[ \int_0^T e^{\gamma t} (|\overline{Y}_t|^2 + |\overline{Z}_t|^2) dt \right] \\
\leq E \left[ e^{\gamma T} |\xi|^2 \right] + 2E \left[ \int_0^T e^{\gamma t} (g(t, Y_t, Z_t) - g'(t, Y'_t, Z'_t)) dt \right] \\
+ 2E \left[ \int_0^T e^{\gamma t} (|\overline{L}_t|^2 + |\overline{K}_t|^2) dt \right].
$$
Thus, since \( g'(\omega, t, \ldots) \) is Lipschitz, with a Lipschitz constant \( L \) which does not depend on \((\omega, t)\), we can conclude from the latter relation that, for some constant \( C_{L, \delta} \) only depending on \( \delta \) and \( L \),

\[
E \left[ \int_0^T e^{\gamma t} (|Y_t|^2 + |Z_t|^2) dt \right] \\
\leq E \left[ e^{\gamma T} |\xi|^2 \right] + E \left[ \int_0^T e^{\gamma t} (C_{L, \delta} |Y_t|^2 + \frac{1}{2} |Z_t|^2) dt \right] \\
+ \frac{1}{2} \delta E \left[ \int_0^T e^{\gamma t} |g(t, Y_t, Z_t)|^2 dt \right] + 2E \left[ \sup_{t \in [0, T]} |e^{\gamma T} |T_t| (K_T + K'_T) \right].
\]

Consequently, for \( \gamma := C_{L, \delta} + \frac{1}{2} \),

\[
E \left[ \int_0^T e^{\gamma t} (|Y_t|^2 + |Z_t|^2) dt \right] \leq 2E \left[ e^{\gamma T} |\xi|^2 \right] + \delta E \left[ \int_0^T e^{\gamma t} |g(t, Y_t, Z_t)|^2 dt \right] \\
+ 4 \left( E \left[ \sup_{t \in [0, T]} |e^{\gamma T} |T_t| \right] \right)^{1/2} \left( E \left[ (K_T + K'_T)^2 \right] \right)^{1/2}.
\]

Then the result announced in the lemma follows from Lemma 3.3.

For an arbitrarily given natural number \( N \geq 0 \) we consider a measurable function \( f : \Omega \times [0, T] \times \mathbb{R}^{N+1} \times \mathbb{R}^{(N+1) \times d} \rightarrow \mathbb{R} \), \((f(t, y, z))_{t \in [0, T]}\) is \( \mathbf{F} \)-progressively measurable for all \((y, z)\) in \( \mathbb{R}^{N+1} \times \mathbb{R}^{(N+1) \times d} \). We make the following standard assumptions, which extend naturally (A1) and (A2):

(B1) There is some constant \( C \geq 0 \) such that, P-a.s., for all \( t \in [0, T] \), \( y_1, y_2 \in \mathbb{R}^{N+1} \), \( z_1, z_2 \in \mathbb{R}^{(N+1) \times d} \),

\[
|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).
\]

(B2) \( f(\cdot, 0, 0) \in L^2(0, T; \mathbb{R}) \).

Let now \( f : \Omega \times [0, T] \times \mathbb{R}^{N+1} \times \mathbb{R}^{(N+1) \times d} \rightarrow \mathbb{R} \) be a measurable function satisfying the assumptions (B1) and (B2). As above we suppose that \( L \in S^2(\Omega, [0, T]), \xi \in L^2(\Omega, \mathcal{F}_T, P) \) and \( L_T \leq \xi \), P-a.s. For a triplet \((\xi, f, L)\) with these properties we say that it satisfies the Standard Assumptions (B).

The above statements allow to extend the existence and uniqueness result to RBSDEs whose data triplet \((\xi, f, L)\) satisfies the Standard Assumptions (B).

**Proposition 3.1.** For every data triplet \((\xi, f, L)\) satisfying the Standard Assumptions (B) the RBSDE (N)

\[
(Y_t - L_t) = \xi + \int_t^T f(s, \Theta N(Y_s, Z_s)) ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t \in [0, T];
\]

admits a unique solution \((Y, Z, K)\).
Proof. Given an arbitrary couple \((U,V) \in H^2 = L^2_F([0,T]) \times L^2_F([0,T];\mathbb{R}^d)\) we put \(g(t) = f(t,\Theta_N(U_t,V_t)), t \in [0,T]\), and we denote by \((Y,Z,K)\) the unique solution of the RBSDE with data triplet \((\xi,g,L)\). For this we observe that the process \(g\) is in \(L^2_F([0,T])\) (and so it satisfies (A1) and (A2)) and we recall that Lemma 3.1 guarantees the existence and the uniqueness of the triplet \((Y,Z,K)\). We denote the mapping \((U,V) \rightarrow (Y,Z)\) by \(\Phi\). For proving that the above RBSDE admits a unique solution it suffices to show that, for a suitable equivalent norm in \(H^2\), the mapping \(\Phi : H^2 \rightarrow H^2\) is a contraction. Indeed, if \(\Phi\) is a contraction mapping on \(H^2\) then there exists a unique couple \((Y,Z) \in H^2\) such that \(\Phi(Y,Z) = (Y,Z)\). Due to the definition of \(\Phi\), there is some \(K \in A^2_F([0,T])\) such that \((Y,Z,K)\) is a solution of the RBSDE with data triplet \((\xi,f(.,\Theta_N(Y,Z)),L)\). Consequently, \((Y,Z,K)\) is a solution of the above RBSDE. The uniqueness of the solution of our RBSDE follows immediately from the fact that whenever \((Y,Z,K)\) is a solution the couple \((Y,Z)\) is the unique fixed point of \(\Phi\) in \(H^2\).

For proving that the mapping \(\Phi\) is a contraction with respect to an appropriate equivalent norm on \(H^2\), we consider arbitrary \((U,V), (U',V') \in H^2\) and apply Lemma 3.5 to \((Y,Z) = \Phi(U,V), (Y',Z') = \Phi(U',V')\). For \((\overline{Y},\overline{Z}) = (Y-Y',Z-Z'), (\overline{U},\overline{V}) = (U-U',V-V'), \overline{\xi} = 0, \overline{L} = 0, \overline{g}(t) = f(t,\Theta_N(U_t,V_t)) - f(t,\Theta_N(U'_t,V'_t)), t \in [0,T],\) we thus get

\[
E \left[ \int_0^T e^{\gamma t}(|\overline{Y}_t|^2 + |\overline{Z}_t|^2)dt \right] \leq \delta E \left[ \int_0^T e^{\gamma t}(|\overline{g}(t)|^2)dt \right],
\]

for any \(\delta > 0\); the constant \(\gamma > 0\) depends only on \(\delta\) and on the Lipschitz constant \(L\) of \(f(\omega,t,..)\). On the other hand,

\[
E \left[ |\overline{g}(t)|^2 \right] \leq L^2(N+1) \sum_{k=0}^N E \left[ |\Theta^k(U_t,\overline{V}_t)|^2 \right] = L^2(N+1)^2 E \left[ |\overline{U}_t|^2 + |\overline{V}_t|^2 \right], \quad t \in [0,T]
\]

(Recall that the random vectors \(\Theta^k(U_t,\overline{V}_t), k \geq 0\), obey the same probability law). Consequently, for \(\delta := \frac{1}{2}(L^2(N+1)^2)^{-1}\),

\[
E \left[ \int_0^T e^{\gamma t}(|\overline{Y}_t|^2 + |\overline{Z}_t|^2)dt \right] \leq \frac{1}{2} E \left[ \int_0^T e^{\gamma t}(|\overline{U}_t|^2 + |\overline{V}_t|^2)dt \right].
\]

This shows that the mapping \(\Phi : H^2 \rightarrow H^2\) is contractive with respect to the norm

\[
\|(U,V)\|_{H^2} = \left( E \left[ \int_0^T e^{\gamma t}(|U_t|^2 + |V_t|^2)dt \right] \right)^{1/2}, \quad (U,V) \in H^2.
\]

The proof is complete.

Remark 3.2. Let us remark that for the type of RBSDE which we have studied in Proposition 3.1 the comparison principle does, in general, not hold. A consequence is that the penalization method can't be used for the proof of the existence for such a RBSDE.

Let us give an example:
Example 3.1. (1) We consider the BSDE without reflection

\[ Y_t = \xi + \int_t^T f(s, \Theta_N(Y_s, Z_s))ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{3.8} \]

with \( \xi \in L^\infty(\Omega, \mathcal{F}_T^W, P), f(s, y, z) = -y_1, \ y = (y_0, y_1, \cdots, y_N), \ z = (z_0, z_1, \cdots, z_N). \)

Then, the equation \((3.8)\) takes the form

\[ Y_t = \xi - \int_t^T \Theta^1(Y_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \]

Since \( \Theta^1(Y_s), s \in [0, T], \) is independent of \( \omega, \) \( Z \) is obtained from the martingale representation property of \( \xi \in L^\infty(\Omega, \mathcal{F}_T^W, P), \)

\[ \xi = \mathbb{E}[\xi] + \int_0^T Z_s dW_s, \ \text{where} \ Z = (Z_s) \in L^2_{\mathcal{F}_T}(0, T); \tag{3.9} \]

and \( Y \in S^2_T(0, T) \) is the unique solution of the following equation:

\[ Y_t = \mathbb{E}[\xi|\mathcal{F}_t^W] - \int_t^T \Theta^1(Y_s)ds, \quad t \in [0, T]. \tag{3.10} \]

Using the notation \( I_i = (1, \cdots, 1) \in \mathbb{R}^i, i \geq 1, \) and the fact that \( \xi \) as \( \mathcal{F}_t^W \)-measurable random variable coincides \( \mathbb{P} \)-a.s. with some Borel measurable functional \( \Phi : C_0([0, T]) \rightarrow \mathbb{R} \) combined with \( W, \xi = \Phi(W), \mathbb{P} \)-a.s., we see that the unique solution of \((3.10)\) is of the form

\[ Y_t = \mathbb{E}[\Phi(W)|\mathcal{F}_t^W] + \sum_{i=1}^\infty (-1)^i \int_t^T \int_{t_i}^T \cdots \int_{t_{i-1}}^T \mathbb{E}[\Phi(W_{t+1})|\mathcal{F}_{t_i}^{W_{t+1}}]dt_i \cdots dt_2 dt_1 \]

\[ = \mathbb{E}[\Phi(W)|\mathcal{F}_t^W] + \sum_{i=1}^\infty (-1)^i \int_t^T \frac{(s-i-1)!}{(i-1)!} \mathbb{E}[\Phi(W_{t+1})|\mathcal{F}_{t_i}^{W_{t+1}}]ds, \quad t \in [0, T]. \tag{3.11} \]

Indeed, due to the definition of \( \Theta^k, k \geq 1, \)

\[ \Theta^1(Y_s) = \mathbb{E}[\Phi(W^{(1)}(s))|\mathcal{F}_{s}^{W^{(1)}}(s)] + \sum_{i=1}^\infty (-1)^i \int_s^T \int_{t_i}^T \cdots \int_{t_{i-1}}^T \mathbb{E}[\Phi(W_{t+1})|\mathcal{F}_{t_i}^{W_{t+1}}]dt_i \cdots dt_2 dt_1, \quad s \in [0, T], \]

and it can be easily checked that \((Y, \Theta^1(Y))\) satisfies \((3.10), (3.11)\) with \((3.9)\) yields \((3.8).\)

Consequently, \( Y \) given by \((3.11)\) and \( Z \) by \((3.9)\) is the unique solution of \((3.8).\) We also observe that, if \( |\xi| \leq C, \mathbb{P} \)-a.s., then

\[ |Y_t| \leq C \sum_{i=0}^\infty \frac{(T-t)^i}{i!} = Ce^{T-t}, \quad t \in [0, T]. \]

Consequently, \((3.8)\) can be regarded also as an RBSDE with reflection barrier \( L_t = -Ce^T, t \in [0, T], \)

and its unique solution \((Y, Z, K)\) is given by \((3.11), (3.10)\) and \( K_t = 0, t \in [0, T].\)

(2) Let us now consider the RBSDE introduced above with \( T = 2, \) and \( \xi = |W_1|^2 \wedge 1, \)

\[ C = 1, \ L_t = -e^T, \ t \in [0, T]. \) Then, again \( K_t = 0, t \in [0, T], \) and from \((3.8), \)

\[ E[Y_t] = E[|W_1|^2 \wedge 1] - \int_t^T E[\Theta^1(Y_s)]ds \]

\[ = E[|W_1|^2 \wedge 1] - \int_t^T E[Y_s]ds, \quad t \in [0, 2], \]

\[ 12 \]
i.e., $E[Y_t] = E[W_1^2 \wedge 1]e^{-(2-t)}$, $t \in [0, 2]$. On the other hand, since $\Theta^1(Y_s)$, $s \in [0, 2]$, is independent of $W$, 

$$E[Y_t|\mathcal{F}_t^W] = |W_1|^2 \wedge 1 - \int_t^2 E[Y_s]ds, \quad t \in [0, 2],$$

and thus,

$$E[Y_t|\mathcal{F}_t^W] = |W_1|^2 \wedge 1 - \int_t^2 E[Y_s]ds \leq |W_1|^2 \wedge 1 - E[|W_1|^2 \wedge 1](1 - e^{-1}) < 0, \quad \text{on } \{ |W_1|^2 \wedge 1 < E[|W_1|^2 \wedge 1](1 - e^{-1}) \}. \quad (3.12)$$

Consequently, $P\{E[Y_t|\mathcal{F}_t^W] < 0\} > 0$, and, hence, also $P\{Y_1 < 0\} > 0$.

On the other hand, for the terminal condition $\xi' = 0$, $(Y', Z', K') = (0, 0, 0)$ is the unique solution of our RBSDE. This shows that, although $P\{\xi > \xi'\} = 1$, we have $P\{Y_1 < Y_1'\} > 0$, i.e., in general, our RBSDE doesn’t satisfy a comparison principle.

4 A Limit Approach for Mean-Field BSDEs with Reflection

The objective of this section is to study the limit of RBSDE(N) as $N$ tends to infinity. For this we choose the framework we have already introduced for the study of the approximation of the reflected mean-field BSDE by RBSDEs. Let $(\Phi, g, \mathcal{X})$ be a data triplet satisfying the assumptions (C1)-(C3):

(C1) $g : \Omega \times [0, T] \times (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d)^2 \rightarrow \mathbb{R}$ is a bounded measurable function, and $g$ is Lipschitz with respect to $(u, v)$, i.e., $P$-a.s., for all $t \in [0, T]$ and $(u, v), (u', v') \in (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d)^2$,

$$|g(t, (u, v)) - g(t, (u', v'))| \leq C \left( |u - u'| + |v - v'| \right);$$

(C2) $\Phi : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a bounded measurable function, and $\Phi(\omega, ., .)$ is Lipschitz, i.e., $P$-a.s., for all $(x, \hat{x}), (x', \hat{x}') \in \mathbb{R}^m$,

$$|\Phi(x, \hat{x}) - \Phi(x', \hat{x}')| \leq C \left( |x - x'| + |\hat{x} - \hat{x}'| \right).$$

(C3) $\mathcal{X} = (X^N)_{N \geq 1}$ is a Cauchy sequence in $\mathcal{S}_F^2([0, T]; \mathbb{R}^m)$, i.e., there is a (unique) process $X \in \mathcal{S}_F^2([0, T]; \mathbb{R}^m)$ such that

$$E\left[ \sup_{t \in [0, T]} |X_t^N - X_t|^2 \right] \rightarrow 0 \quad \text{as } N \rightarrow +\infty.$$ 

Moreover, let $h : \Omega \times [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$ be a function with the following properties:

(C4) $h : \Omega \times [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a bounded measurable function which is

- $\mathcal{F}$-progressively measurable, for every fixed $x \in \mathbb{R}^m$;
- Lipschitz continuous, for every fixed $(\omega, t) \in \Omega \times [0, T]$, with a Lipschitz constant that doesn’t depend on $(\omega, t)$;
- continuous in $t$, for every fixed $(\omega, x) \in \Omega \times \mathbb{R}^m$. 

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Given such a quadruplet \((\Phi, g, h, \mathcal{X})\) such that \((\Phi, g, \mathcal{X})\) fulfills the assumptions (C1)-(C3), \(h\) satisfies the assumption (C4) and \(h(T, x) \leq \Phi(x, x')\), \(P\text{-a.s.}\), for all \((x, x') \in \mathbb{R}^m \times \mathbb{R}^m\), we say that \((\Phi, g, h, \mathcal{X})\) satisfies the Standard Assumptions (C) and we put, for \(N \geq 1\) and \(\omega \in \Omega\),

\[
\xi^N(\omega) := \frac{1}{N} \sum_{k=1}^{N} \Phi(\Theta^k(\omega), X^N_T(\omega), X^N_T(\Theta^k(\omega))),
\]

\[
f^N(\omega, t, y, z) := \frac{1}{N} \sum_{k=1}^{N} g(\Theta^k(\omega), t, X^N_t(\omega), (y_0, z_0), X^N_t(\Theta^k(\omega)), (y_k, z_k)),
\]

for \(t \in [0, T]\), \(y = (y_0, \cdots, y_N) \in \mathbb{R}^{N+1}\), \(z = (z_0, \cdots, z_N) \in \mathbb{R}^{(N+1) \times d}\),

\[
L^N_t(\omega) := h(\omega, t, X^N_t(\omega)), \quad t \in [0, T].
\]

We notice that, for each \(N \geq 1\), the triplet \((\xi^N, f^N, L^N)\) satisfies the Standard Assumptions for an RBSDE: \((\xi^N, f^N)\) satisfies (B1)-(B2) and \(L^N \in S^2_F([0, T])\) is such that \(L^N_T \leq \xi^N\). Thus, due to Proposition 3.1, we have for all \(N \geq 1\) a unique solution \((Y^N, Z^N, K^N)\) of the RBSDE (N)

\[
\begin{align*}
(i) \quad & Y^N \in S^2_F([0, T]), \quad Z^N \in L^2_F([0, T]; \mathbb{R}^d) \quad \text{and} \quad K^N \in A^{2,c}_F([0, T]); \\
(ii) \quad & Y^N_t = \xi^N + \int_t^T f^N(s, \Theta_N(Y_s, Z_s)) ds + K^N_T - K^N_t - \int_t^T Z^N_s dW_s, \quad t \in [0, T]; \\
(iii) \quad & Y^N_t \geq L^N_t, \quad \text{a.s., for any} \ t \in [0, T]; \\
(iv) \quad & \int_0^T (Y^N_t - L^N_t) dK^N_t = 0. 
\end{align*}
\]

We remark that, the driving coefficient of the above RBSDE(N) can be written as follows:

\[
f^N(s, \Theta_N(Y^N_s, Z^N_s)) = \frac{1}{N} \sum_{k=1}^{N} (\Theta^k g)(s, (X^N_s, Y^N_s, Z^N_s), \Theta^k(X^N_s, Y^N_s, Z^N_s)),
\]

\(s \in [0, T]\). Our objective is to show that the unique solution of RBSDE(N) converges to the unique solution \((Y, Z, K)\) of the Reflected Mean-Field BSDE

\[
\begin{align*}
(i) \quad & Y \in S^2_F([0, T]), \quad Z \in L^2_F([0, T]; \mathbb{R}^d) \quad \text{and} \quad K \in A^{2,c}_F([0, T]); \\
(ii) \quad & Y_t = \mathbb{E}[\Phi(x, X_T)]_{x=X_T} + \int_t^T \mathbb{E}[g(s, u, A_s)]_{u=A_s} ds + K_T - K_t - \int_t^T Z_s dW_s; \\
(iii) \quad & Y_t \geq h(t, X_t), \quad \text{a.s., for all} \ t \in [0, T]; \\
(iv) \quad & \int_0^T (Y_t - h(t, X_t)) dK_t = 0,
\end{align*}
\]

where we have used the notation \(\Lambda = (X, Y, Z)\).

**Lemma 4.1.** Under the Standard Assumptions (C) on the data quadruplet \((\Phi, g, h, \mathcal{X})\) the above Reflected Mean-Field BSDE possesses a unique solution \((Y, Z, K) \in S^2_F([0, T]) \times L^2_F([0, T]; \mathbb{R}^d) \times A^{2,c}_F([0, T])\).

The proof is standard. For the convenience we give the proof here.

**Proof.** Let \(H^2 := L^2_F([0, T]; \mathbb{R}) \times L^2_F([0, T]; \mathbb{R}^d)\). Similar to the discussion in the beginning of the proof of Proposition 3.1 it is sufficient to prove the existence and the uniqueness for the above
BSDE in $H^2$. Indeed, if $(Y, Z)$ is a solution of our BSDE in $H^2$, an easy standard argument shows that it is also in $B^2 := S^2_F([0, T]; \mathbb{R}) \times L^2_F([0, T]; \mathbb{R}^d)$. On the other hand, the uniqueness in $H^2$ implies obviously that in its subspace $B^2$.

For proving the existence and uniqueness in $H^2$ we consider for an arbitrarily given couple of processes $(U, V) \in H^2$ the coefficient $g_s^{U,V} = E[g(s, \lambda, \Lambda_s)]_{\lambda = \Lambda_s}, s \in [0, T]$, for $\Lambda_s = (X_s, U_s, V_s)$. Since $g$ is an element of $L^2_F([0, T]; \mathbb{R})$ it follows from Lemma 3.1 that there is a unique solution $\Phi(U, V) := (Y, Z) \in H^2$ of the reflected BSDE:

\begin{align*}
(i) & \quad Y \in S^2_F([0, T]), Z \in L^2_F([0, T]; \mathbb{R}^d) \text{ and } K \in A^{2, \infty}([0, T]); \\
(ii) & \quad Y_t = E[\Phi(x, X_T)]_{x = X_T} + \int_t^T g_s^{U,V} ds + K_T - K_t - \int_t^T Z_s dW_s; \\
(iii) & \quad Y_t \geq h(t, X_t), \text{ a.s., for all } t \in [0, T]; \\
(iv) & \quad \int_0^T (Y_t - h(t, X_t)) dK_t = 0,
\end{align*}

For such defined mapping $\Phi : H^2 \to H^2$ it suffices to prove that it is a contraction with respect to an appropriate equivalent norm on $H^2$, in order to complete the proof. For this end, we consider two couples $(U^1, V^1), (U^2, V^2) \in H$ and $(Y^k, Z^k) = \Phi(U^k, V^k), k = 1, 2$. Then, due to Lemma 3.5, for all $\delta > 0$ there is some constant $\gamma > 0$ (only depending on $\delta$) such that, with the notation $(\overline{Y}, \overline{Z}) = (Y^1 - Y^2, Z^1 - Z^2),$

\begin{align*}
E & \left[ \int_0^T e^{\gamma t} |(\overline{Y}_t)|^2 + |(\overline{Z}_t)|^2 dt \right] \\
& \leq \delta E \left[ \int_0^T e^{\gamma t} |u^{1, t} - u^{2, t}|^2 + |v^{1, t} - v^{2, t}|^2 dt \right].
\end{align*}

Let $(\overline{U}, \overline{V}) = (U^1 - U^2, V^1 - V^2)$. Then, from the Lipschitz continuity (C1) of $g$ (with Lipschitz constant $C$ which doesn’t depend on $(\omega, t)$)

\begin{align*}
E & \left[ |u^{1, t} - u^{2, t}|^2 \right] + E \left[ |v^{1, t} - v^{2, t}|^2 \right] \\
& = E \left[ |E[g(t, \lambda, \Lambda_t)]_{\lambda = \Lambda_t^1} - E[g(t, \lambda, \Lambda_t^2)]_{\lambda = \Lambda_t^2}|^2 \right] \\
& \leq CE \left[ |\Lambda_t^1 - \Lambda_t^2|^2 \right] \\
& = CE \left[ |U_t^1 - U_t^2|^2 + |V_t^1 - V_t^2|^2 \right],
\end{align*}

where $\Lambda_t^1 := (X_t, U_t^1, V_t^1), \ \Lambda_t^2 := (X_t, U_t^2, V_t^2)$. Consequently, we have

\begin{align*}
E & \left[ \int_0^T e^{\gamma t} |(\overline{Y}_t)|^2 + |(\overline{Z}_t)|^2 dt \right] \leq \delta C \int_0^T e^{\gamma t} E \left[ |(\overline{U}_t)|^2 + |(\overline{V}_t)|^2 \right] dt \\
& = \frac{1}{2} E \left[ \int_0^T e^{\gamma t} |(\overline{U}_t)|^2 + |(\overline{V}_t)|^2 dt \right],
\end{align*}

for $\delta := \frac{1}{2C}$. This shows that if we endow the space $H^2$ with the norm

$$\|(U, V)\|_{H^2} = \left( E \left[ \int_0^T e^{\gamma t} |(U_t^1 + V_t^1)|^2 dt \right] \right)^{1/2}$$

the mapping $\Phi : H^2 \to H^2$ becomes a contraction. Thus, the proof is complete.

We now can formulate the following theorem:
Theorem 4.1. Under the Standard Assumptions (C) on the data quadruplet \((\Phi, g, h, X)\), the unique solution \((Y^N, Z^N, K^N)\) of RBSDE\((N)\) (4.1) converges to the unique solution \((Y, Z, K)\) of the above MFBSEDE (4.2) with reflection:

\[
E \left[ \sup_{t \in [0,T]} |Y^N_t - Y_t|^2 + \int_0^T |Z^N_t - Z_t|^2 dt + \sup_{t \in [0,T]} |K^N_t - K_t|^2 \right] \to 0, 
\]

as \(N \to +\infty\).

**Proof.** First we want to prove that

**Step 1.** For all \(p \geq 2\),

\[
E \left[ \frac{1}{N} \sum_{k=1}^{N} (\Theta^k g)(t, \Lambda_t, \Theta^k(\Lambda_t)) - E [g(t, u, \Lambda_t)] \big|_{u=\Lambda_t} \right]^p \to 0,
\]

and

\[
E \left[ \frac{1}{N} \sum_{k=1}^{N} (\Theta^k \Phi)(X_T, \Theta^k(X_T)) - E [\Phi(x, X_T)] \big|_{x=X_T} \right]^p \to 0,
\]

as \(N \to +\infty\) (notice that \((\Theta^k \Phi)(\omega, X_T, \Theta^k(X_T))(\omega) := \Phi(\Theta^k(\omega), X_T(\omega), X_T(\Theta^k(\omega)))\)).

To prove the first convergence we need to consider arbitrary \(t \in [0, T]\) and \(u \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d\). Notice that the sequence of random variables \((\Theta^k g)(t, u, \Lambda_t), \ k \geq 1\), is i.i.d. and has the same law as \(g(t, u, \Lambda_t)\), from the Strong Law of Large Numbers we get that

\[
\frac{1}{N} \sum_{k=1}^{N} (\Theta^k g)(t, u, \Lambda_t) \longrightarrow E [g(t, u, \Lambda_t)],
\]

\(P\text{-a.s.}, \text{ as } N \to +\infty\). For an arbitrarily small \(\varepsilon > 0\), let \(\Lambda^\varepsilon_t : \Omega \to \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d\) be a random vector which has only a countable number of values, and also satisfies \(|\Lambda_t - \Lambda^\varepsilon_t| \leq \varepsilon\), everywhere on \(\Omega\). Then, obviously,

\[
\frac{1}{N} \sum_{k=1}^{N} (\Theta^k g)(t, \Lambda^\varepsilon_t, \Theta^k(\Lambda^\varepsilon_t)) \longrightarrow E [g(t, u, \Lambda^\varepsilon_t)] \big|_{u=\Lambda^\varepsilon_t},
\]

\(P\text{-a.s.}, \text{ as } N \text{ tends to } +\infty\). On the other hand, from the Lipschitz continuity of \(g(\omega, t, \cdot, \cdot)\), uniformly in \((\omega, t, \cdot)\), we know that \(\Lambda_t\) also have the convergence:

\[
\frac{1}{N} \sum_{k=1}^{N} (\Theta^k g)(t, \Lambda_t, \Theta^k(\Lambda_t)) \longrightarrow E [g(t, u, \Lambda_t)] \big|_{u=\Lambda_t},
\]

\(P\text{-a.s.}, \text{ as } N \to +\infty\). Finally, from the boundedness of \(g\) and, thus, of that of the convergence, we get the wished result. Similarly, we also obtain the \(L^p\)-convergence for the terminal conditions, for all \(p \geq 2\).

**Step 2.** Recalling the argument given in Step 1 we see that, for all \(p \geq 2\), as \(N \to +\infty\),

\[
R^{N,p}_1 := E \left[ \frac{1}{N} \sum_{k=1}^{N} (\Theta^k \Phi)(X_T, \Theta^k(X_T)) - E [\Phi(x, X_T)] \big|_{x=X_T} \right]^p \to 0,
\]

and

\[
R^{N,p}_2 := E \left[ \int_0^T \frac{1}{N} \sum_{k=1}^{N} (\Theta^k g)(t, \Lambda_t, \Theta^k(\Lambda_t)) - E [g(t, u, \Lambda_t)] \big|_{u=\Lambda_t} \right]^p \to 0.
\]
For estimating the distance between \((Y^N, Z^N, K^N)\) and \((Y, Z, K)\) we apply Lemma 3.5 and get, for \(\delta \in (0, 1)\) which will be specified later, and for some \(\gamma > 0\) (depending on \(\delta\) and on the Lipschitz constant of \(g\)),

\[
E \left[ \int_0^T e^{\gamma t}(|Y^N_t - Y_t|^2 + |Z^N_t - Z_t|^2) dt \right] \leq 2E[e^{\gamma T} |\tilde{\xi}^N|^2] + \\
+ \delta E \left[ \int_0^T e^{\gamma t} |\mathcal{T}^N(t, Y_t, Z_t)|^2 dt \right] + \left( CE \left[ \sup_{t \in [0, T]} e^{\gamma t} |\mathcal{T}^N_t|^2 \right] \right)^{1/2} (\Psi_{0, T}^N)^{1/2},
\]

where

\[
\tilde{\xi}^N := \frac{1}{N} \sum_{k=1}^N (\Theta^k \Phi)(X^N_T, \Theta^k(X^N_T)) - E [\Phi(x, X_T)]_{x = X_T},
\]

\[
\mathcal{T}^N(t, Y_t, Z_t) := \frac{1}{N} \sum_{k=1}^N (\Theta^k g)(t, \Lambda^N_t, \Theta^k(\Lambda^N_t)) - E [g(t, u, \Lambda_t)]_{u = \Lambda_t},
\]

with \(\Lambda^N_t = (X^N_t, Y^N_t, Z^N_t)\), \(\Lambda_t = (X_t, Y_t, Z_t)\),

and

\[
\Psi_{t, T}^N := E \left[ |\xi^N|^2 + \left( \int_t^T |f^N(s, \Theta_N(Y^N_s, Z^N_s))| ds \right)^2 \right] + \left( \int_t^T \left| E [g(s, \mathcal{T}_t^N)]_{\mathcal{T}_t^N} ds \right|^2 + \sup_{s \in [t, T]} \left| h(s, X^N_s) \right|^2 + \sup_{s \in [t, T]} \left| h(s, X^N_s) \right|^2 |\mathcal{T}_t| \right],
\]

for \(t \in [0, T]\). In virtue of the boundedness of the coefficients \(\Phi, g\) and \(h\) it follows that, for some constant \(C\), \(\Psi_{t, T}^N \leq C\), \(t \in [0, T]\), \(P\)-a.s. On the other hand, recalling that the coefficients \(\Phi(\omega,.)\), \(g(\omega, t, .)\) and \(h(\omega, t, .)\) are Lipschitz, with some Lipschitz constant \(L\) which is independent of \((\omega, t)\), and using the the fact that, for any random variable \(\xi\), the variables \(\Theta^k(\xi)\), \(k \geq 0\), obey the same probability law, we see that

\[
E \left[ |\tilde{\xi}^N|^2 \right] \leq 2R_{1, T}^{N, 2} + 8L^2 E \left[ |X^N_T - X_T|^2 \right],
\]

\[
E \left[ \int_0^T e^{\gamma t} |\mathcal{T}^N(t, Y_t, Z_t)|^2 dt \right] \leq 2e^{\gamma T} R_{2, T}^{N, 2} + 8L^2 E \left[ \int_0^T e^{\gamma t} |\Lambda^N_t - \Lambda_t|^2 dt \right],
\]

\[
E \left[ \sup_{t \in [0, T]} |\mathcal{T}^N_t|^2 \right] \leq L^2 E \left[ \sup_{t \in [0, T]} |X^N_t - X_t|^2 \right].
\]

Consequently, with the notation

\[
R_N := 4e^{\gamma T} \left( R_{1, T}^{N, 2} + R_{2, T}^{N, 2} + 4L^2 E[|X^N_T - X_T|^2] \right) + 2L^2 E \left[ \int_0^T |X^N_t - X_t|^2 dt \right] + CL \left( E \left[ \sup_{t \in [0, T]} |X^N_t - X_t|^2 \right] \right)^{1/2}
\]

we have

\[
E \left[ \int_0^T e^{\gamma t} (|Y^N_t - Y_t|^2 + |Z^N_t - Z_t|^2) dt \right] \leq R_N + 8L^2 \delta E \left[ \int_0^T e^{\gamma t} (|Y^N_t - Y_t|^2 + |Z^N_t - Z_t|^2) dt \right],
\]

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and choosing \( \delta := \frac{1}{10}L^{-2} \) we obtain
\[
E \left[ \int_0^T e^{\gamma t} (|Y_i^N - Y_t|^2 + |Z_i^N - Z_t|^2) dt \right] \leq 2R_N, \quad N \geq 1.
\]
Hence, since \( R_N \) converges to zero as \( N \) tends to \( +\infty \), we also have
\[
E \left[ \int_0^T e^{\gamma t} (|Y_i^N - Y_t|^2 + |Z_i^N - Z_t|^2) dt \right] \rightarrow 0, \quad \text{as} \ N \rightarrow +\infty.
\]
Applying now Lemma 3.4 we obtain the following estimate, for all \( t \in [0, T] \), \( P \)-a.s.,
\[
E[\max_{s \in [t, T]} |Y^N_s - Y_s|^2 + \int_t^T |Z^N_s - Z_s|^2 ds + |(K^N_t - K_T) - (K^N_t - K_t)|^2]
\leq CE[\xi^N_t]^2 + (\int_t^T \mathcal{T}_N(s, Y_s, Z_s) ds)^2] + CE[\max_{s \in [t, T]} |\mathcal{T}_s|^2]^{1/2} E[\Psi^N_{t,T}]^{1/2},
\]
which right-hand side converges to zero according to our preceding convergence result. Consequently, \( E[\max_{s \in [t, T]} |Y^N_s - Y_s|^2] \rightarrow 0 \) and, for all \( t \in [0, T] \), \( E[|K^N_t - K_t|^2] \rightarrow 0 \), as \( N \) tends towards \( +\infty \). In order to conclude, it suffices to observe that the fact that \( K \) is a square integrable, increasing continuous process implies that we even have
\[
E[\max_{t \in [0, T]} |K^N_t - K_t|^2] \rightarrow 0, \quad \text{as} \ N \rightarrow 0.
\]
Indeed, given an arbitrary \( \varepsilon > 0 \) we can find some finite partition \( 0 = t_0 < t_1 < \cdots < t_M = T \) such that \( E[\max_{1 \leq i \leq M} (K_{t_i} - K_{t_{i-1}})^2] \leq \varepsilon^2 \). Then, since the processes \( K^N \) and \( K \) are increasing,
\[
\max_{t \in [0, T]} |K^N_t - K_t| = \max_{1 \leq i \leq M} \left( \max_{t \in [t_{i-1}, t_i]} |K^N_t - K_t| \right)
\leq \max_{1 \leq i \leq M} (|K^N_{t_{i-1}} - K_{t_{i-1}}| + |K^N_{t_i} - K_{t_i}|),
\]
and, consequently,
\[
E\left[\max_{t \in [0, T]} |K^N_t - K_t|^2\right] \leq 2 \sum_{1 \leq i \leq M} \left( E[|K^N_{t_{i-1}} - K_{t_{i-1}}|^2] + E[|K^N_{t_i} - K_{t_{i-1}}|^2] \right)
\rightarrow 4 \sum_{1 \leq i \leq M} E[|K_{t_i} - K_{t_{i-1}}|^2] \leq 4E[|K_T - \max_{1 \leq i \leq M} (K_{t_i} - K_{t_{i-1}})|]
\leq 4 \left( E[|K_T|^2] \right)^{1/2} \varepsilon, \quad \text{as} \ N \rightarrow +\infty.
\]
The proof is complete.

5 Existence of a solution of the Reflected MFBSDE: approximation via penalization

In [13] the penalization method for BSDEs is used to prove the existence for the reflected BSDE. Can we use also here this method adapted to mean-field BSDEs, in order to study reflected MFBSDEs? In this section we will give a positive answer to this question. We will see that we can get the
reflected mean-field BSDEs by using the penalization method to the mean-field BSDEs. The result of this section will be very useful in Section 6.

For the given coefficient \( g(\omega, t, y, z, \bar{y}) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) which satisfies (A3), and \( g \) is nondecreasing with respect to \( \bar{y} \), the obstacle process \( L \in S^2_{\mathcal{F}}([0, T]) \), and the terminal condition \( \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}) \) such that \( \xi \geq L_T \), P-a.s., we consider the following reflected mean-field BSDE:

\[
\begin{align*}
(i) & \ Y_t \in S^2_{\mathcal{F}}([0, T]), \ Z_t \in L^2_{\mathcal{F}}([0, T]; \mathbb{R}^d) \text{ and } K_t \in A^2_{\mathcal{F}}([0, T]); \\
(ii) & \ Y_t = \xi + \int_t^T E[g(s, y, z, Y_s)]|_{y=Y_s, z=Z_s} ds + K_T - K_t - \int_t^T Z_s dW_s; \\
(iii) & \ Y_t \geq L_t, \text{ a.s., for all } t \in [0, T]; \\
(iv) & \int_0^T (Y_t - L_t) dK_t = 0.
\end{align*}
\]

(5.1)

Similarly to the proof of Lemma 4.1 we know that the above equation (5.1) has a unique solution \((Y, Z, K)\).

We define \( f(s, y, z) := E[g(s, y, z, Y_s)] \), where \((Y, Z, K)\) is the solution of (5.1), \( s \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d \).

For each \( n \in \mathbb{N} \), let \((Y^n, Z^n) \in S^2_{\mathcal{F}}([0, T]) \times L^2_{\mathcal{F}}([0, T]; \mathbb{R}^d)\) denote the solution of the following BSDE which in fact, also can be seen as a special mean-field BSDE:

\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s) ds + n \int_t^T (Y^n_s - L_s)^- ds - \int_t^T Z^n_s dW_s, \quad (5.2)
\]

Then from the comparison theorem–Lemma 2.3 we know

\[
Y^n_t \leq Y^{n+1}_t, \quad 0 \leq t \leq T, \text{ a.s., } \forall n \in \mathbb{N}.
\]

(5.3)

We define \( K^n_t := n \int_0^T (Y^n_s - L_s)^- ds, \quad 0 \leq t \leq T \).

Therefore, from the proof on Pages 719-723 in [13], we know:

\[
\begin{align*}
(i) & \ Y^n_t \uparrow \bar{Y}_t, \quad 0 \leq t \leq T, \text{ a.s.;} \\
(ii) & \ E\left( \sup_{0 \leq t \leq T} |Y^n_t - \bar{Y}_t|^2 + \int_0^T |Z^n_s - \bar{Z}_s|^2 ds + \sup_{0 \leq t \leq T} |K^n_t - \bar{K}_t|^2 \right) \to 0,
\end{align*}
\]

(5.4)

where \((\bar{Y}, \bar{Z}, \bar{K})\) is the solution of the following reflected BSDE:

\[
\begin{align*}
(i) & \ \bar{Y} \in S^2_{\mathcal{F}}([0, T]), \ \bar{Z} \in L^2_{\mathcal{F}}([0, T]; \mathbb{R}^d) \text{ and } \bar{K} \in A^2_{\mathcal{F}}([0, T]); \\
(ii) & \ \bar{Y}_t = \xi + \int_t^T f(s, \bar{Y}_s, \bar{Z}_s) ds + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s; \\
(iii) & \ \bar{Y}_t \geq L_t, \text{ a.s., for all } t \in [0, T]; \\
(iv) & \int_0^T (\bar{Y}_t - L_t) d\bar{K}_t = 0.
\end{align*}
\]

(5.5)

By comparing the equations (5.1) and (5.5) we get from the uniqueness of the solution of RBSDE (5.5) that

\[
Y_t = \bar{Y}_t, \ K_t = \bar{K}_t, \quad 0 \leq t \leq T, \text{ a.s., and } Z_t = \bar{Z}_t, \text{ a.e.a.s.}
\]

(5.6)
Therefore, we know:

\begin{align}
(\text{i}) \quad & Y^n_t \uparrow Y_t, 0 \leq t \leq T, \text{a.s.;} \\
(\text{ii}) \quad & E\left( \sup_{0 \leq t \leq T} |Y^n_t - Y_t|^2 + \int_0^T |Z^n_s - Z_s|^2 ds + \sup_{0 \leq t \leq T} |K^n_t - K_t|^2 \right) \to 0, \\
\end{align}

(5.7)

where \((Y, Z, K)\) is the solution of reflected MFBSDE \((5.1)\).

Now we consider the following penalized mean-field BSDEs:
\[
\tilde{Y}^n_t = \xi + \int_t^T E[g(s, y, z, \tilde{Y}^n_s)]|_{y=\tilde{Y}^n_s, z=\tilde{Z}_s} ds + n \int_t^T (\tilde{Y}^n_s - L_s)^- ds - \int_t^T \tilde{Z}^n_s dW_s.
\]

(5.8)

From Lemmas 2.4 and 2.5 we know that it has a unique solution \((\tilde{Y}^n, \tilde{Z}^n)\), and
\[
\tilde{Y}^n_t \leq \tilde{Y}^{n+1}_t, 0 \leq t \leq T, \text{a.s.,} \forall n \in \mathbb{N}.
\]

(5.9)

We define \(\tilde{K}^n_t := n \int_0^t (\tilde{Y}^n_s - L_s)^- ds, 0 \leq t \leq T\). Then we can prove that

**Theorem 5.1.** Under our assumptions, we have

\begin{align}
(\text{i}) \quad & \tilde{Y}^n_t \uparrow Y_t, 0 \leq t \leq T, \text{a.s.;} \\
(\text{ii}) \quad & E\left( \sup_{0 \leq t \leq T} |\tilde{Y}^n_t - Y_t|^2 + \int_0^T |\tilde{Z}^n_s - Z_s|^2 ds + \sup_{0 \leq t \leq T} |\tilde{K}^n_t - K_t|^2 \right) \to 0, \\
\end{align}

(5.10)

where \((Y, Z, K)\) is the solution of reflected MFBSDE \((5.1)\).

**Proof.** We define \(f_n(s, y, z) = E[g(s, y, z, \tilde{Y}^n_s)], n \in \mathbb{N}\). Then MFBSDE \((5.8)\) becomes a classical BSDE with the generator \(f_n(s, y, z) + n(y - L_s)^-\) and the terminal condition \(\xi\). Here in order to be clear, we denote \(C_0\) the Lipschitz constant of \(g\). Applying Lemma 2.2 to the BSDEs \((5.2)\) and \((5.8)\), for \(\delta = \frac{1}{1+4C_0^2}\), there exists a constant \(\gamma\) such that

\[
E\left[ \int_0^T e^{\gamma t}(|Y^n_t - \tilde{Y}^n_t|^2 + |Z^n_t - \tilde{Z}^n_t|^2) dt \right] \leq \delta E\left[ \int_0^T e^{\gamma t}|f(t, Y^n_t, Z^n_t) - f_n(t, Y^n_t, Z^n_t)|^2 dt \right] \\
= \delta E\left[ \int_0^T e^{\gamma t}|E[g(t, y, z, Y_t)]|_{y=Y^n_t, z=Z^n_t} - E[g(t, y, z, \tilde{Y}^n_t)]|_{y=Y^n_t, z=Z^n_t}|^2 dt \right] \\
\leq 2\delta C_0^2 \int_0^T e^{\gamma t}E|Y^n_t - \tilde{Y}^n_t|^2 dt \\
\leq 4\delta C_0^2 \int_0^T e^{\gamma t}E|Y_t - Y^n_t|^2 dt + 4\delta C_0^2 \int_0^T e^{\gamma t}E|Y^n_t - \tilde{Y}^n_t|^2 dt.
\]

(5.11)

Therefore, we get
\[
E\left[ \int_0^T e^{\gamma t}(|Y^n_t - \tilde{Y}^n_t|^2 + |Z^n_t - \tilde{Z}^n_t|^2) dt \right] \leq 4C_0^2 E\left[ \int_0^T e^{\gamma t}|Y_t - Y^n_t|^2 dt \right],
\]

(5.12)

Furthermore, from \((5.7)\) and \((5.9)\) the proof is complete. \(\blacksquare\)
6 Relation between a Reflected MFBSDE and an obstacle problem for a nonlinear parabolic nonlocal PDE

In this section we will show that reflected MFBSDEs studied before allow to give a probabilistic representation for the solutions of non-local PDEs with obstacles.

We consider measurable functions \( b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times d} \) which are assumed to satisfy the following conditions:

(i) \( b(\cdot, 0, 0) \) and \( \sigma(\cdot, 0, 0) \) are continuous and there exists some constant \( C > 0 \) such that 
\[
|b(t, x, x)| + |\sigma(t, x, x)| \leq C(1 + |x|), \quad \text{for all } 0 \leq t \leq T, \ x, x \in \mathbb{R}^n;
\]

(ii) \( b \) and \( \sigma \) are Lipschitz in \( x, \bar{x} \), i.e., there exists some constant \( C > 0 \) such that
\[
|b(t, x_1, \bar{x}_1) - b(t, x_2, \bar{x}_2)| + |\sigma(t, x_1, \bar{x}_1) - \sigma(t, x_2, \bar{x}_2)| \leq C(|x_1 - x_2| + |\bar{x}_1 - \bar{x}_2|),
\]
for all \( 0 \leq t \leq T, \ x_1, \bar{x}_1, x_2, \bar{x}_2 \in \mathbb{R}^n \).

We now study the following SDE with the initial condition \((t, \zeta) \in [0, T] \times L^2(\Omega, F_t, P; \mathbb{R}^n)\):

\[
\begin{align*}
&\frac{dX^t_s}{dt} = E[b(s, x, X^0_s)]|_{x=\hat{X}^t_s}ds + E[\sigma(s, x, X^0_s)]|_{x=\hat{X}^t_s}dB_s, \ s \in [t, T], \\
&X^t_0 = \zeta.
\end{align*}
\]

Under the assumption (H6.1), SDE (6.1) has a unique strong solution. Indeed, we first get the existence and uniqueness of the solution \( X^{0,x_0} \in \mathcal{S}_p(0, T; \mathbb{R}^n) \) to the McKean-Vlasov SDE (6.1). Once knowing \( X^{0,x_0} \), SDE (6.1) becomes a classical equation with the coefficients \( \tilde{b}(s, x) = E[b(s, x, X^{0,x_0}_s)] \) and \( \tilde{\sigma}(s, x) = E[\sigma(s, x, X^{0,x_0}_s)] \). From standard arguments we also can have, for any \( p \geq 2 \), there exists \( C_p \in \mathbb{R} \) which only depends on the Lipschitz and the growth constants of \( b \) and \( \sigma \) such that, for all \( t \in [0, T] \) and \( \zeta, \zeta' \in L^p(\Omega, F_t, P; \mathbb{R}^n) \),

\[
E[\sup_{0 \leq s \leq T} |X_{s,t}^{t,\zeta} - X_{s,t}^{t,\zeta'}|^p |F_t] \leq C_p |\zeta - \zeta'|^p, \ \text{P-a.s.},
\]

\[
E[\sup_{0 \leq s \leq T} |X_{s,t}^{t,\zeta}|^p |F_t] \leq C_p (1 + |\zeta|^p), \ \text{P-a.s.}
\]

These standard estimates are well-known in the classical case. More details may refer to, e.g., [7].

Let now be given two real-valued mappings \( f(t, x, \bar{x}, \bar{y}, y, z) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) and \( \Phi(x, \bar{x}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) which satisfy the following conditions:

(i) There exists a constant \( C > 0 \) such that
\[
|f(t, x_1, \bar{x}_1, y_1, z_1) - f(t, x_2, \bar{x}_2, y_2, z_2)| + |\Phi(x_1, \bar{x}_1) - \Phi(x_2, \bar{x}_2)| \leq C(|x_1 - x_2| + |\bar{x}_1 - \bar{x}_2| + |y_1 - y_2| + |z_1 - z_2|),
\]
for all \( 0 \leq t \leq T, \ x_1, \bar{x}_1, x_2, \bar{x}_2 \in \mathbb{R}^n, \ y_1, \bar{y}_1, y_2, \bar{y}_2 \in \mathbb{R} \) and \( z_1, z_2 \in \mathbb{R}^d \);

(ii) \( f \) and \( \Phi \) satisfy a linear growth condition, i.e., there exists some \( C > 0 \) such that, for all \( \bar{x}, x \in \mathbb{R}^n \),
\[
|f(t, x, \bar{x}, 0, 0, 0)| + |\Phi(x, \bar{x})| \leq C(1 + |x| + |\bar{x}|);
\]

(iii) \( f(t, x, \bar{x}, \bar{y}, y, z) \) is continuous in \( t \) for all \( (x, \bar{x}, \bar{y}, y, z) \);

(iv) \( f(t, x, \bar{x}, \bar{y}, y, z) \) is nondecreasing with respect to \( \bar{y} \);

(v) \( E[\Phi(x, X^{0,x_0}_T)] \geq h(T, x) \), for all \( x \in \mathbb{R}^n \).
We consider the following reflected BSDE:

(i) \( Y^{t,\zeta} \in S_\mathcal{F}^2([t,T]), \) \( Z^{t,\zeta} \in L_\mathcal{F}^2([t,T];\mathbb{R}^d) \) and \( K^{t,\zeta} \in A_{\mathcal{F}}^{2,c}([0,T]); \)

(ii) \( Y^{t,\zeta}_s = E[\Phi(x, X^{t,\zeta}_T)]|_{x=X^{t,\zeta}_s} + \int_s^T E[f(r, x, X^{t,\zeta}_r, Y^{t,\zeta}_r, y, z)]|_{x=X^{t,\zeta}_s, y=Y^{t,\zeta}_s, z=Z^{t,\zeta}_s} dr + K^{t,\zeta}_T - K^{t,\zeta}_s - \int_s^T Z^{t,\zeta}_r dW_r; \)

(iii) \( Y^{t,\zeta}_s \geq h(s, X^{t,\zeta}_s), \) a.s., for all \( s \in [0,T]; \)

(iv) \( \int_t^T (Y^{t,\zeta}_s - h(s, X^{t,\zeta}_s))dK^{t,\zeta}_s = 0, \) \( (6.3) \)

We first consider the equation \( (6.3) \) when \((t, \zeta) = (0, x_0): \) We know that there exists a unique solution \((Y^{0,x_0}, Z^{0,x_0}, K^{0,x_0}) \in S_\mathcal{F}^2(0,T;\mathbb{R}) \times L_\mathcal{F}^2(0,T;\mathbb{R}^d) \times A_{\mathcal{F}}^{2,c}(0,T);\) to the Reflected Mean-Field BSDE \( (6.3). \) Once we get \((Y^{0,x_0}, Z^{0,x_0}, K^{0,x_0}), \) equation \( (6.3) \) becomes a classical reflected BSDE whose coefficients \( \tilde{f}(s, X^{t,\zeta}_s, y, z) = E[f(s, x, X^{0,x_0}_s, Y^{0,x_0}_s, y, z)]|_{x=X^{t,\zeta}_s} \) satisfies the assumptions (A1) and (A2), and \( \tilde{\Phi}(X^{t,\zeta}_T) = E[\Phi(x, X^{0,x_0}_T)]|_{x=X^{t,\zeta}_T} \in L^2(\Omega, \mathcal{F}_T, P). \) Thus, from Lemma 3.1 we know that there exists a unique solution \((Y^{t,\zeta}, Z^{t,\zeta}, K^{t,\zeta}) \in S_\mathcal{F}^2(0,T;\mathbb{R}) \times H_\mathcal{F}^2(0,T;\mathbb{R}^d) \times A_{\mathcal{F}}^{2,c}(0,T);\) to equation \( (6.3). \)

Now we introduce the random field:

\[
u(t, x) = Y^{t,x}_s|_{s=t}, \quad (t, x) \in [0,T] \times \mathbb{R}^n,
\]

where \( Y^{t,x} \) is the solution of RBSDE \( (6.3) \) with \( x \in \mathbb{R}^n \) at the place of \( \zeta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n). \)

Notice that, it is obvious that \( \nu \) is a deterministic function, for all \( t \in [0,T], x \in \mathbb{R}^n, \) and as we told above: once we get \((Y^{0,x_0}, Z^{0,x_0}, K^{0,x_0}), \) equation \( (6.3) \) becomes a classical reflected BSDE whose coefficients \( \tilde{f}(s, X^{t,\zeta}_s, y, z) = E[f(s, x, X^{0,x_0}_s, Y^{0,x_0}_s, y, z)]|_{x=X^{t,\zeta}_s} \) satisfies the assumptions (A1) and (A2), and \( \tilde{\Phi}(X^{t,\zeta}_T) = E[\Phi(x, X^{0,x_0}_T)]|_{x=X^{t,\zeta}_T} \in L^2(\Omega, \mathcal{F}_T, P). \) Therefore, from Proposition 6.1 and Theorem 3.2 in [10], we immediately get that

(i) \(|\nu(t, x) - \nu(t, y)| \leq C|x - y|, \) for all \( x, y \in \mathbb{R}^n; \)

(ii) \(|\nu(t, x)| \leq C(1 + |x|), \) for all \( x \in \mathbb{R}^n; \)

(iii) \( \nu \) is continuous in \( t. \) \( (6.5) \)

In this section we want to consider the following non-local PDE with an obstacle

\[
\begin{align*}
&\min\{\nu(t, x) - h(t, x), -\partial_t \nu(t, x) - Au(t, x) \} \\
&\quad - E[f(t, x, X^{t,\zeta}_t, \nu(t, x), X^{t,\zeta}_t, y, z)] = 0, \\
&\quad (t, x) \in [0,T] \times \mathbb{R}^n, \\
&\quad \nu(T, x) = E[\Phi(x, X^{0,x_0}_T)], \quad x \in \mathbb{R}^n,
\end{align*}
\]

with

\[
Au(t, x) := \frac{1}{2} tr[E[\sigma(t, x, X^{t,\zeta}_t)\sigma(t, x, X^{t,\zeta}_t)]D^2\nu(t, x)] + D\nu(t, x).E[b(t, x, X^{0,x_0}_t)].
\]

Here the functions \( b, \sigma, f \) and \( \Phi \) are supposed to satisfy (H6.1), and (H6.2), respectively, and \( X^{0,x_0} \) is the solution of the SDE \( (6.1). \) We want to prove that the value function \( \nu(t, x) \) introduced by \( (6.4) \) is the unique viscosity solution of equation \( (6.6). \) Now we have to do with nonlocal PDEs with
obstacles. Furthermore, unlike [2] here the nonlocal term is not produced by a diffusion process with jumps. We first recall the definition of a viscosity solution of equation (6.6). The reader more interested in viscosity solutions is referred to Crandall, Ishii and Lions [12].

**Definition 6.1.** A real-valued continuous function \( u \in C_p([0, T] \times \mathbb{R}^n) \) is called

(i) a viscosity subsolution of equation (6.6) if, firstly, \( u(T, x) \leq E[\Phi(x, X^0_{t,0})] \), for all \( x \in \mathbb{R}^n \), and if, secondly, for all functions \( \varphi \in C^3_{l,b}([0, T] \times \mathbb{R}^n) \) and \((t, x) \in [0, T) \times \mathbb{R}^n \) such that \( u - \varphi \) attains its local maximum at \((t, x)\),

\[
\begin{align*}
\min \{ u(t, x) - h(t, x), -\frac{\partial}{\partial t} \varphi(t,x) - D \varphi(t,x) \cdot E[b(t, x, X^0_{t,0})] \\
- \frac{1}{2} \text{tr} \{ E[\sigma(t, x, X^0_{t,0})] \} E[\sigma(t, x, X^0_{t,0})]^T D^2 \varphi(t,x) \\
- E[ f(t, x, X^0_{t,0}, u(t, x, X^0_{t,0}), u(t, x), D \varphi(t,x) \cdot E[\sigma(t, x, X^0_{t,0})]]] \} \leq 0;
\end{align*}
\]

(ii) a viscosity supersolution of equation (6.6) if, firstly, \( u(T, x) \geq E[\Phi(x, X^0_{t,0})] \), for all \( x \in \mathbb{R}^n \), and if, secondly, for all functions \( \varphi \in C^3_{l,b}([0, T] \times \mathbb{R}^n) \) and \((t, x) \in [0, T) \times \mathbb{R}^n \) such that \( u - \varphi \) attains its local minimum at \((t, x)\),

\[
\begin{align*}
\min \{ u(t, x) - h(t, x), -\frac{\partial}{\partial t} \varphi(t,x) - D \varphi(t,x) \cdot E[b(t, x, X^0_{t,0})] \\
- \frac{1}{2} \text{tr} \{ E[\sigma(t, x, X^0_{t,0})] \} E[\sigma(t, x, X^0_{t,0})]^T D^2 \varphi(t,x) \\
- E[ f(t, x, X^0_{t,0}, u(t, x, X^0_{t,0}), u(t, x), D \varphi(t,x) \cdot E[\sigma(t, x, X^0_{t,0})]]] \} \geq 0;
\end{align*}
\]

(iii) a viscosity solution of equation (6.6) if it is both a viscosity sub- and a supersolution of equation (6.1).

**Remark 6.1.** (i) We recall that \( C_p([0, T] \times \mathbb{R}^n) = \{ u \in C([0, T] \times \mathbb{R}^n) : \text{There exists some constant } p > 0 \text{ such that } \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \frac{|u(t,x)|}{1+|t|^p} < +\infty \} \).

(ii) The space \( C^3_{l,b}([0, T] \times \mathbb{R}^n) \) denotes the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded. Therefore, that function in \( C^3_{l,b}([0, T] \times \mathbb{R}^n) \) is of at most linear growth.

We now can give the main statement of this section.

**Theorem 6.1.** Under the assumptions (H6.1) and (H6.2) the function \( u(t, x) \) defined by (6.4) is the unique viscosity solution of equation (6.6).

For each \((t, x) \in [0, T) \times \mathbb{R}^n\), and \( n \in \mathbb{N} \), let \( \{(nY_{s,t}^{t,x}, nZ_{s,t}^{t,x}), t \leq s \leq T\} \) denote the solution of the MFBSDE

\[
\begin{align*}
nY_{s,t}^{t,x} &= E[\Phi(x, X^0_{t,0})]|_{x=X^t_t} + \int_s^T E[f(r, x, X^0_{r,0}, nY_{r,t}^{r,t}, y, z)]|_{x=X^t_t, y=nY_{r,t}^{r,t}, z=nZ_{r,t}^{r,t}} \, dr \\
&+ n \int_s^T (nY_{r,t}^{r,t} - h(r, X^t_t)) \, dr - \int_s^T nZ_{r,t}^{t,x} \, dW_r, \quad t \leq s \leq T.
\end{align*}
\]

(6.7)

We define

\[
u_n(t, x) := nY_{t,t}^{t,x}, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^n.
\]

(6.8)

It is known from [3] that \( u_n(t, x) \) defined by (6.8) is in \( C([0, T] \times \mathbb{R}^n) \), has linear growth in \( x \), and is the unique continuous viscosity solution of the following equation:

\[
\begin{cases}
-\frac{\partial}{\partial t}u_n(t, x) - Au_n(t, x) - \{ E[f(t, x, X^0_{t,0}, u_n(t, x, X^0_{t,0}), u_n(t, x), D u_n(t, x), E[\sigma(t, x, X^0_{t,0})]]] \\
+ n(u_n(t, x) - h(t, x))^- \} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\
u_n(T, x) = E[\Phi(x, X^0_{T,0})], \quad x \in \mathbb{R}^n.
\end{cases}
\]

(6.9)
with

\[ Au(t, x) := \frac{1}{2} \text{tr}(E[\sigma(t, x, X_{t}^{0,x})]E[\sigma(t, x, X_{t}^{0,x})]^T D^2 u(t, x)) + D u(t, x) E[b(t, x, X_{t}^{0,x})]). \]

We have the uniqueness of viscosity solution \( u_n \) only in the space \( C_p([0, T] \times \mathbb{R}^n) \) (in \([9]\) the authors gave an example to explain why the uniqueness is only in \( C_p([0, T] \times \mathbb{R}^n) \)). More details refer to \([9]\).

**Lemma 6.1.**

\[
u_n(t, x) \uparrow u(t, x), \text{ for all } (t, x) \in [0, T] \times \mathbb{R}^n. \quad (6.10)
\]

**Proof.** When \((t, x) = (0, x_0)\), the equation (6.7) is the penalized mean-field BSDE, from Section 5, we know \( \eta_r^{0,x_0} \uparrow Y_r^{0,x_0} \), \( 0 \leq r \leq T \), P-a.s., in particular, \( u_n(0, x_0) \uparrow u(0, x_0) \).

When \((t, x) \neq (0, x_0)\), recall that SDE (6.1) becomes the classical equation with the coefficients \( \tilde{b}(r, x) = E[b(r, x, X_{r}^{0,x})] \) and \( \tilde{\sigma}(r, x) = E[\sigma(r, x, X_{r}^{0,x})] \), the equation (6.7) becomes the classical penalized BSDE with the coefficient \( \tilde{f}_n(r, x, y, z) := E[f(r, x, X_{r}^{0,x}, \eta_r^{0,x_0}, y, z)] + n(y - h(r, x))^- \), and terminal condition \( \tilde{\Phi}(x) := E[\Phi(x, X_{T}^{0,x})] \). Notice that now still \( \tilde{f}_n(r, x, y, z) \leq \tilde{f}_{n+1}(r, x, y, z) \), following the proof on Pages 719-723 in \([13]\), we can get \( \eta_s^{\tilde{x},x} \uparrow Y_s^{\tilde{x},x} \), \( t \leq s \leq T \), P-a.s., therefore, \( u_n(t, x) \uparrow u(t, x) \), for all \((t, x) \in [0, T] \times \mathbb{R}^n. \)

On the other hand, notice that because \( u_n \) and \( u \) are continuous, from Dini’s theorem it follows that the above convergence is uniform on compacts. We can also prove that, \( u_n \) has linear growth in \( x \) and is Lipschitz in \( x \), uniformly with respect to \( n \in \mathbb{N} \).

**Proposition 6.1.** There exists a constant \( C \) independent of \( n \), such that, for every \( n \in \mathbb{N} \),

\[
(i) \quad |u_n(t, x)| \leq C(1 + |x|), \text{ for all } x \in \mathbb{R}^n, t \in [0, T];
(ii) \quad |u_n(t, x) - u_n(t, y)| \leq C|x - y|, \text{ for all } x, y \in \mathbb{R}^n, t \in [0, T]. \quad (6.11)
\]

The proof is given in the appendix for convenience.

**Proof of Theorem 6.1. Step 1:** We first prove that \( u(t, x) \) is a viscosity supersolution of (6.6). Indeed, let \((t, x) \in [0, T] \times \mathbb{R}^n \) and let \( \varphi \in C_{0}^{2}(\mathbb{R}^n) \) be such that \( u - \varphi > u(t, x) - \varphi(t, x) = 0 \) everywhere on \((0, T] \times \mathbb{R}^n) - \{(t, x)\}. \) Then, because \( u \) is continuous and \( u_n(t, x) \uparrow u(t, x), 0 \leq t \leq T \), \( x \in \mathbb{R}^n \), there exists some sequence \((t_n, x_n)\), \( n \geq 1 \), at least along a subsequence, such that

i) \( (t_n, x_n) \to (t, x), \) as \( n \to +\infty; \)

ii) \( u_n - \varphi \geq u_n(t_n, x_n) - \varphi(t_n, x_n) \) in a neighborhood of \((t_n, x_n)\), for all \( n \geq 1; \)

iii) \( u_n(t_n, x_n) \to u(t, x), \) as \( n \to +\infty. \)

Consequently, because \( u_n \) is a viscosity solution and hence a supersolution of equation (6.9), we have, for all \( n \geq 1,
\[
\frac{\partial}{\partial t} \varphi(t_n, x_n) + \frac{1}{2} \text{tr}(E[\sigma(t, x_n, X_{t_n}^{0,x_n})]E[\sigma(t, x_n, X_{t_n}^{0,x_n})]^T D^2 \varphi(t_n, x_n)) + D \varphi(t_n, x_n) E[b(t_n, x_n, X_{t_n}^{0,x_n})] + \{E[f(t_n, x_n, X_{t_n}^{0,x_n}, u_n(t_n, X_{t_n}^{0,x_n}), u_n(t_n, x_n), D \varphi(t_n, x_n) E[\sigma(t_n, x_n, X_{t_n}^{0,x_n})])]
\]
\[+ n(u_n(t_n, x_n) - h(t_n, x_n))^- \} \leq 0, \quad (6.12)
\]

Therefore,
\[
\frac{\partial}{\partial t} \varphi(t_n, x_n) + \frac{1}{2} \text{tr}(E[\sigma(t, x_n, X_{t_n}^{0,x_n})]E[\sigma(t, x_n, X_{t_n}^{0,x_n})]^T D^2 \varphi(t_n, x_n)) + D \varphi(t_n, x_n) E[b(t_n, x_n, X_{t_n}^{0,x_n})] + E[f(t_n, x_n, X_{t_n}^{0,x_n}, u_n(t_n, X_{t_n}^{0,x_n}), u_n(t_n, x_n), D \varphi(t_n, x_n) E[\sigma(t_n, x_n, X_{t_n}^{0,x_n})])] \leq 0, \quad (6.13)
\]

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Taking the limit, from (H6.1), (H6.2) and (6.11) it follows from Lebesgue dominated convergence theorem we get that

\[
\frac{\partial}{\partial t} \varphi(t, x) + \frac{1}{2} \text{tr}(E[\sigma(t, x, X_{t}^{0, x_0})]E[\sigma(t, x, X_{t}^{0, x_0})^T D^2 \varphi(t, x)]) + D \varphi(t, x).E[b(t, x, X_{t}^{0, x_0})] \\
+ E[f(t, x, X_{t}^{0, x_0}, u(t, X_{t}^{0, x_0}), u(t, x), D \varphi(t, x).E[\sigma(t, x, X_{t}^{0, x_0})])] \leq 0, 
\]

(6.14)

Because our \( u(t, x) \geq h(t, x) \), we prove that \( u(t, x) \) is the viscosity supersolution.

**Step 2:** The function \( W(t, x) \) is a viscosity subsolution of equations (6.6).

Indeed, let \((t, x) \in [0, T) \times \mathbb{R}^n\) be a point at which \( u(t, x) > h(t, x) \), let \( \varphi \in C_{l, b}^3([0, T] \times \mathbb{R}^n) \) be such that \( u - \varphi < h(t, x) \) everywhere on \((0, T] \times \mathbb{R}^n) - \{(t, x)\}. Then, because \( u \) is continuous and \( u_n(t, x) \uparrow u(t, x) \), \( 0 \leq t \leq T, x \in \mathbb{R}^n \), there exists some sequence \((t_n, x_n)\), \( n \geq 1 \), at least along a subsequence, such that,

\[ i) \quad (t_n, x_n) \to (t, x), \quad \text{as} \quad n \to +\infty; \]
\[ ii) \quad u_n - \varphi \leq u_n(t_n, x_n) - \varphi(t_n, x_n) \quad \text{in a neighborhood of} \quad (t_n, x_n), \quad \text{for all} \quad n \geq 1; \]
\[ iii) \quad u_n(t_n, x_n) \to u(t, x), \quad \text{as} \quad n \to +\infty. \]

Consequently, because \( u_n \) is a viscosity solution and hence a subsolution of equation (6.9), we have, for all \( n \geq 1, \)

\[
\frac{\partial}{\partial t} \varphi(t_n, x_n) + \frac{1}{2} \text{tr}(E[\sigma(t_n, x_n, X_{t_n}^{0, x_0})]E[\sigma(t_n, x_n, X_{t_n}^{0, x_0})^T D^2 \varphi(t_n, x_n)]) + D \varphi(t_n, x_n).E[b(t_n, x_n, X_{t_n}^{0, x_0})] \\
+ \{E[f(t_n, x_n, X_{t_n}^{0, x_0}, u_n(t_n, x_n), X_{t_n}^{0, x_0}), u_n(t_n, x_n), D \varphi(t_n, x_n).E[\sigma(t_n, x_n, X_{t_n}^{0, x_0})]] \\
+n(u_n(t_n, x_n) - h(t_n, x_n)) \} \geq 0, 
\]

(6.15)

From the assumption that \( u(t, x) > h(t, x) \) and the uniform convergence of \( u_n \), we can get that for \( n \) large enough \( u_n(t_n, x_n) > h(t_n, x_n) \), hence, taking the limit as \( n \to \infty \) in the above inequality from (H6.1), (H6.2) and (6.11) it follows from Lebesgue dominated convergence theorem we get:

\[
\frac{\partial}{\partial t} \varphi(t, x) + \frac{1}{2} \text{tr}(E[\sigma(t, x, X_{t}^{0, x_0})]E[\sigma(t, x, X_{t}^{0, x_0})^T D^2 \varphi(t, x)]) + D \varphi(t, x).E[b(t, x, X_{t}^{0, x_0})] \\
+ E[f(t, x, X_{t}^{0, x_0}, u(t, X_{t}^{0, x_0}), u(t, x), D \varphi(t, x).E[\sigma(t, x, X_{t}^{0, x_0})])] \geq 0, 
\]

(6.16)

we prove that \( u(t, x) \) is the viscosity subsolution. \( \square \)

**Remark 6.2.** For the uniqueness of the viscosity solution, from [9] we know that we have the uniqueness of viscosity solution \( u \) in the space \( C_p([0, T] \times \mathbb{R}^n) \). Similar to the proof of Theorem 5.1 in [10] and Theorem 7.1 in [9] it is not hard to prove the uniqueness.

7 Appendix

**Proof of Proposition 6.1.** (i) From (6.10) we know \( u_1(t, x) \leq u_n(t, x) \leq u(t, x) \), for all \( n \in \mathbb{N}, x \in \mathbb{R}^n, t \in [0, T] \). Furthermore, since \( u_1 \) and \( u \) are linear growth in \( x \), the proof of (i) is complete.

(ii) To simplify the notations, we define \( \tilde{b}(s, x) = E[b(s, x, X_{s}^{0, x_0})] \) and \( \tilde{\sigma}(s, x) = E[\sigma(s, x, X_{s}^{0, x_0})] \), \( \tilde{\Phi}(x) = E[\Phi(x, X_{T}^{0, x_0})], \tilde{f}(s, x, y, z) = E[f(s, x, X_{s}^{0, x_0}, y, z)] \). Then, from the equation (6.7)
we know \((nY^{t,x},nZ^{t,x})\) is the solution of the following equation:

\[
\begin{align*}
nY_s^{t,x} &= \widetilde{\Phi}(X_T^{t,x}) + \int_s^T \tilde{f}(r, X_r^{t,x}, nY_r^{t,x}, nZ_r^{t,x})dr \\
&\quad + n\int_s^T (nY_r^{t,x} - h(r, X_r^{t,x})) - dr - \int_s^T nZ_r^{t,x}dW_r, \quad t \leq s \leq T,
\end{align*}
\] (7.1)

and \((nY^{t,y},nZ^{t,y})\) is the solution of the following equation:

\[
\begin{align*}
nY_s^{t,y} &= \widetilde{\Phi}(X_T^{t,y}) + \int_s^T \tilde{f}(r, X_r^{t,y}, nY_r^{t,y}, nZ_r^{t,y})dr \\
&\quad + n\int_s^T (nY_r^{t,y} - h(r, X_r^{t,y})) - dr - \int_s^T nZ_r^{t,y}dW_r, \quad t \leq s \leq T.
\end{align*}
\] (7.2)

For an arbitrarily given \(\varepsilon > 0\), we consider the function \(\psi_\varepsilon(x) = (|x|^2 + \varepsilon)\frac{1}{2}, \ x \in \mathbb{R}^n\). Obviously, \(|x| \leq \psi_\varepsilon(x) \leq |x| + \varepsilon\frac{1}{2}, \ x \in \mathbb{R}^n\). Moreover, for all \(x \in \mathbb{R}^n\),

\[
D\psi_\varepsilon(x) = \frac{x}{(|x|^2 + \varepsilon)\frac{1}{2}}, \quad D^2\psi_\varepsilon(x) = \frac{I}{(|x|^2 + \varepsilon)\frac{1}{2}} - \frac{x \otimes x}{(|x|^2 + \varepsilon)\frac{3}{2}}.
\]

Therefore, we have

\[
|D\psi_\varepsilon(x)| \leq 1, \quad |D^2\psi_\varepsilon(x)||x| \leq \frac{C}{(|x|^2 + \varepsilon)\frac{3}{2}}|x| \leq C, \ x \in \mathbb{R}^n,
\] (7.3)

where the constant \(C\) is independent of \(\varepsilon\). On the other hand, in order to be clear we denote \(\mu\) is the Lipschitz constant of \(h, \Phi, \) and \(f\). We consider the following two BSDEs:

\[
\begin{align*}
\bar{Y}_s &= \widetilde{\Phi}(X_T^{t,x}) + \mu\psi_\varepsilon(X_T^{t,x} - X_T^{t,y}) + \int_s^T (\tilde{f}(r, X_r^{t,x}, \bar{Y}_r, \bar{Z}_r) + \mu|X_r^{t,x} - X_r^{t,y}|)dr \\
&\quad + \int_s^T n\left(\bar{Y}_r - \left(h(r, X_r^{t,x}) + \mu\psi_\varepsilon(X_r^{t,x} - X_r^{t,y})\right)\right) - dr - \int_s^T \bar{Z}_r dW_r, \quad s \in [t, T);
\end{align*}
\] (7.4)

and

\[
\begin{align*}
\tilde{Y}_s &= \widetilde{\Phi}(X_T^{t,x}) - \mu\psi_\varepsilon(X_T^{t,x} - X_T^{t,y}) + \int_s^T (\tilde{f}(r, X_r^{t,x}, \tilde{Y}_r, \tilde{Z}_r) - \mu|X_r^{t,x} - X_r^{t,y}|)dr \\
&\quad + \int_s^T n\left(\tilde{Y}_r - \left(h(r, X_r^{t,x}) - \mu\psi_\varepsilon(X_r^{t,x} - X_r^{t,y})\right)\right) - dr - \int_s^T \tilde{Z}_r dW_r, \quad s \in [t, T].
\end{align*}
\] (7.5)

Obviously, the coefficients satisfy the assumptions (A1) and (A2), therefore from Lemma 2.1 they have unique solutions \((\bar{Y}, \bar{Z})\) and \((\tilde{Y}, \tilde{Z})\), respectively. Notice that the solutions of (7.4) and (7.5) depend on \(n\), for simplifying notations and causing no confusion we still denote the solutions by \((\bar{Y}, \bar{Z})\) and \((\tilde{Y}, \tilde{Z})\), respectively. Furthermore, from the comparison theorem for BSDEs (Lemma 2.3)

\[
\bar{Y}_s \leq nY_s^{t,x} \leq \bar{Y}_s, \quad \tilde{Y}_s \leq nY_s^{t,y} \leq \tilde{Y}_s, \quad \text{P-a.s., for all } s \in [t, T].
\] (7.6)

Now we introduce two other BSDEs:

\[
\begin{align*}
\bar{Y}_s' &= \widetilde{\Phi}(X_T^{t,x}) + \int_s^T [\tilde{f}(r, X_r^{t,x}, \bar{Y}_r) + \mu\psi_\varepsilon(X_r^{t,x} - X_r^{t,y}), \bar{Z}_r + \mu D\psi_\varepsilon(X_r^{t,x} - X_r^{t,y})(\sigma(r, X_r^{t,x}) - \sigma(r, X_r^{t,y}))]dr \\
&\quad + n\int_s^T (\bar{Y}_r - h(r, X_r^{t,x})) - dr - \int_s^T \bar{Z}_r dW_r, \quad s \in [t, T);
\end{align*}
\] (7.7)
where the constant $\bar{n}$ has unique solutions $\tilde{\beta} > \bar{s}$, $\tilde{\gamma} = \bar{s} = \tilde{\gamma}(\Phi(\mu f - n(\tilde{\beta} - \bar{s}))$ and hence, from Lemma 2.1 (7.7) and (7.8) have unique solutions $(\tilde{Y}', \tilde{Z}')$ and $(\breve{Y}', \breve{Z}')$, respectively.

On the other hand, from the uniqueness of the solution of BSDE we know that

$$\tilde{Y}'_s = \tilde{Y}_s - \mu \psi \varepsilon(X^{t,x}_s - X^{t,y}_s), \text{ for all } s \in [t, T], \text{ P-a.s.,}$$

$$\tilde{Z}'_s = \tilde{Z}_s - \mu D \psi \varepsilon(X^{t,x}_s - X^{t,y}_s)(\tilde{\sigma}(s, X^{t,x}_s) - \tilde{\sigma}(s, X^{t,y}_s)), \text{ dsdP-a.e. on } [t, T] \times \Omega;$$

and

$$\breve{Y}'_s = \breve{Y}_s + \mu \psi \varepsilon(X^{t,x}_s - X^{t,y}_s), \text{ for all } s \in [t, T], \text{ P-a.s.,}$$

$$\breve{Z}'_s = \breve{Z}_s + \mu D \psi \varepsilon(X^{t,x}_s - X^{t,y}_s)(\breve{\sigma}(s, X^{t,x}_s) - \breve{\sigma}(s, X^{t,y}_s)), \text{ dsdP-a.e. on } [t, T] \times \Omega.$$

For any $\beta > 0$ applying Itô’s formula to $e^{\beta s}|\tilde{Y}'_s - \tilde{Y}'_s|^2$ we get

$$e^{\beta r} |\tilde{Y}'_t - \tilde{Y}'_t|^2 + E[\int_t^T e^{\beta s}|\tilde{Y}'_s - \tilde{Y}'_s|^2 ds | \mathcal{F}_r] + E[\int_t^T e^{\beta s}|\tilde{Z}'_s - \tilde{Z}'_s|^2 ds | \mathcal{F}_r] =$$

$$E[\int_t^T 2e^{\beta s}|\tilde{Y}'_s - \tilde{Y}'_s|(f(s, X^{t,x}_s, \tilde{Y}'_s) + \mu \psi \varepsilon(X^{t,x}_s - X^{t,y}_s), \tilde{Z}'_s - \mu D \psi \varepsilon(X^{t,x}_s - X^{t,y}_s)(\tilde{\sigma}(s, X^{t,x}_s) - \tilde{\sigma}(s, X^{t,y}_s)))]ds | \mathcal{F}_r]$$

$$-E[\int_t^T 2e^{\beta s}n(\tilde{Y}'_s - \tilde{Y}'_s)(\tilde{Y}'_s - h(s, X^{t,x}_s)) - (\tilde{Y}'_s - h(s, X^{t,x}_s))^-)ds | \mathcal{F}_r]$$

$$+E[\int_t^T 2e^{\beta s}(\breve{Y}'_s - \tilde{Y}'_s)\Delta g(s)ds | \mathcal{F}_r],$$

(7.11)

where

$$\Delta g(s) = 2\mu |X^{t,x}_s - X^{t,y}_s| + 2\mu D \psi \varepsilon(X^{t,x}_s - X^{t,y}_s)(\tilde{\sigma}(s, X^{t,x}_s) - \tilde{\sigma}(s, X^{t,y}_s))$$

$$+\mu D^2 \psi \varepsilon(X^{t,x}_s - X^{t,y}_s)(\tilde{\sigma}(s, X^{t,x}_s) - \tilde{\sigma}(s, X^{t,y}_s), \tilde{\sigma}(s, X^{t,x}_s) - \tilde{\sigma}(s, X^{t,y}_s)).$$

(7.12)

Notice that $(\tilde{Y}'_s - \tilde{Y}'_s)((\tilde{Y}'_s - h(s, X^{t,x}_s))^- - (\tilde{Y}'_s - h(s, X^{t,x}_s))^-) \leq 0.$

From (7.3) and the Lipschitz continuity of $\tilde{b}$ and $\tilde{\sigma}$ we get $|\Delta g(s)| \leq C |X^{t,x}_s - X^{t,y}_s|$, P-a.s.,

where the constant $C$ is independent of $\varepsilon$ and $n$. Therefore, from (7.11) we have

$$e^{\beta r} |\tilde{Y}'_t - \tilde{Y}'_t|^2 + E[\int_t^T e^{\beta s}|\tilde{Y}'_s - \tilde{Y}'_s|^2 ds | \mathcal{F}_r] + E[\int_t^T e^{\beta s}|\tilde{Z}'_s - \tilde{Z}'_s|^2 ds | \mathcal{F}_r]$$

$$\leq CE[\int_t^T e^{\beta s}|\tilde{Y}'_s - \tilde{Y}'_s|^2 ds | \mathcal{F}_r] + \frac{1}{2} E[\int_t^T e^{\beta s}|\tilde{Z}'_s - \tilde{Z}'_s|^2 ds | \mathcal{F}_r]$$

$$+CE[\int_t^T e^{\beta s}|X^{t,x}_s - X^{t,y}_s|^2 ds | \mathcal{F}_r] + C \varepsilon, \text{ P-a.s., } r \in [t, T],$$

(7.13)

where the constant $C$ is independent of $\varepsilon$ and $n$. Then, take $\beta = C + 1$ with the help of (6.2) we get

$$|\tilde{Y}'_t - \tilde{Y}'_t|^2 \leq CE[\int_t^T |X^{t,x}_s - X^{t,y}_s|^2 ds | \mathcal{F}_t] + C \varepsilon$$

$$\leq CE[\sup_{t \leq s \leq T} |X^{t,x}_s - X^{t,y}_s|^2 | \mathcal{F}_t] + C \varepsilon$$

$$\leq C|x - y|^2 + C \varepsilon, \text{ P-a.s.}$$

(14.1)
Furthermore, from (6.8), (7.6), (7.9), (7.10) and (7.14) we have

\[ |u_n(t,x) - u_n(t,y)|^2 = |\tilde{Y}_t^{t,x} - \tilde{Y}_t^{t,y}|^2 \]
\[ \leq |\tilde{Y}_t - \tilde{Y}_t'|^2 \leq 2|\tilde{Y}_t' - \tilde{Y}_t''|^2 + 16\mu^2(|X^t,x - X^t,y|^2 + \epsilon) \]
\[ \leq C|x - y|^2 + C\epsilon. \]

Then, let \( \varepsilon \) tend to 0 the proof is complete.

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