Losing Treewidth by Separating Subsets

Anupam Gupta∗
CMU

Euiwoong Lee†
NYU

Jason Li∗
CMU

Pasin Manurangsi‡
UC Berkeley

Michał Włodarczyk§
University of Warsaw

Abstract

We study the problem of deleting the smallest set \( S \) of vertices (resp. edges) from a given graph \( G \) such that the induced subgraph (resp. subgraph) \( G \setminus S \) belongs to some class \( \mathcal{H} \). We consider the case where graphs in \( \mathcal{H} \) have treewidth bounded by \( t \), and give a general framework to obtain approximation algorithms for both vertex and edge-deletion settings from approximation algorithms for certain natural graph partitioning problems called \( k \)-\textsc{Subset Vertex Separator} and \( k \)-\textsc{Subset Edge Separator}, respectively.

For the vertex deletion setting, our framework combined with the current best result for \( k \)-\textsc{Subset Vertex Separator}, improves approximation ratios for basic problems such as \( k \)-\textsc{Treewidth Vertex Deletion} and \( \text{Planar-} \mathcal{F} \text{ Vertex Deletion} \). Our algorithms are simpler than previous works and give the first deterministic and uniform approximation algorithms under the natural parameterization.

For the edge deletion setting, we give improved approximation algorithms for \( k \)-\textsc{Subset Edge Separator} combining ideas from LP relaxations and important separators. We present their applications in bounded-degree graphs, and also give an APX-hardness result for the edge deletion problems.
Introduction

Let $H$ be a class of infinitely many graphs. In the $H$-Vertex Deletion (resp. $H$-Edge Deletion) problem, we are given a graph $G$ and we must find the smallest set of vertices (resp. edges) $X$ such that $G \setminus X$ belongs to class $H$. The simplest examples of such problems are Vertex Cover and Feedback Vertex Set problems, where $H$ is the set of all empty graphs (resp. forests) and hence $G \setminus X$ must exclude all edges (resp. cycles) of $G$. Indeed, the problem has often been studied in the context when $H$ is a class of graphs that exclude a fixed graph $F$ in some sense (e.g., minor, subgraph, or induced subgraph). In this case we let $F$ be a finite list of excluded graphs and define $H$ to be the class of graphs that exclude every graph from $F$. There has been a rich body of work studying parameterized complexity and kernelization of these problems parameterized by the size of the optimal solution [HVHJ11, FLMS12, CM15, DDdH16, KLP+16, EGK16, BBKM16, GLSS16, BCKP16, JP17, ALM+18].

In this paper, we focus on parameterized approximation algorithms parameterized by $F$ — our desired running time is of the form $f(F)\cdot \text{poly}(n)$. Note that the notion of approximation is inherent in this parameterization: even the simplest case of Vertex Cover, where the only graph in $F$ is a single edge and hence $f(F)$ is a constant, is NP-hard. Such an approximation algorithm could be used to obtain better kernels [ALM+18, ALM+17]. In addition to the 2-approximation algorithms for Feedback Vertex Set [BG96, BBF99, CGHW98], the systematic study of the parameterized approximability depending on $F$ has also been done in the context of both parameterized algorithms and approximation algorithms [FJP10, FLMS12, GL15, BCKP18, JP17, KS17, BRU17, ALM+18, Lee18, KK18]

Many of the above algorithmic successes are based on one of the following two techniques.

- Specifically designed linear programming (LP) relaxations for the problem, often inspired by other classical optimization problems: These approaches were used in [FJP10, GL15, Lee18] to solve Diamond Hitting Set, $k$-Star Transversal and $k$-Path Transversal, where the underlying classical problems were Feedback Vertex Set, Dominating Set, and Balanced Separator respectively. While these tools often give principled ways to find optimal approximation ratios, the previous connections were tailored to specific settings.

- Combinatorial algorithms that exploit graph-theoretic structures of $F$-free graphs, using the general algorithm to find a balanced separator [FHL08] as a subroutine: This route was taken by [JP17, KS17, BRU17, ALM+18] for Planar-$F$ Vertex Deletion, Minimum Planarization, Chordal Vertex Deletion, and Distance Hereditary Vertex Deletion. While seamlessly bridging between graph-theoretic properties and well-studied graph partitioning algorithms, this technique has the shortcoming that the best approximation factor for Balanced Separator is $\Omega(\sqrt{\log n})$, forcing approximation ratios to depend on $n$.  

The natural question is: can we apply both kinds of techniques to give stronger results? We attempt to give a positive answer to this question, by defining a new kind of graph partitioning problems that (1) can be approximated well using some variants of the LP-based graph partitioning techniques and (2) exploit fundamental graph-theoretic concepts (e.g., treewidth) more closely than

---

1 One notable exception that did not use any of the two techniques is the work of Fomin et al. [FLMS12], which uses the notion of lossless protrusion reduction that reduces the instance size while preserving an approximation ratio.

2 BRU17 did not use FHL08 as a black-box and analyzed a linear programming relaxation directly, but only has an approximation ratio of $\Omega(\log n)$. BRU17 and ALM+18 have an additional advantage of handling the weighted version, whereas our results cannot.
the traditional Balanced Separator problem. We demonstrate the power of these approaches for minor or treewidth deletion problems. We get simpler algorithms with better approximation ratios, and we hope that more of such intimate connections can be made between structural graph theory and graph partitioning algorithms through intermediate problems.

1.1 Our Results

A class $\mathcal{H}$ of graphs is called hereditary if $G \in \mathcal{H}$, all its induced subgraphs are in $\mathcal{H}$. Our main conceptual contribution is easy to state:

If $\mathcal{H}$ is hereditary, and if graphs in $\mathcal{H}$ have bounded treewidth, good approximations for $\mathcal{H}$-Vertex Deletion are implied by good approximations for a natural graph partitioning problem called $k$-Subset Vertex Separator.

(A similar result holds for $\mathcal{H}$-Edge Deletion and $k$-Subset Edge Separator.) What are these partitioning problems? Given graph $G = (V(G), E(G))$, a subset $R \subseteq V(G)$ of terminals, and an integer $k$, the $k$-Subset Vertex Separator (resp. $k$-Subset Edge Separator) problem asks to delete the minimum number of vertices (resp. edges) to partition $G$ so that each component has at most $k$ vertices from $R$. When $R = V(G)$, these problems are called $k$-Vertex Separator (resp. $k$-Edge Separator), and they generalize balanced separator problems studied in the context of cuts and metrics [ENRS00]. The case of general $R$ has close connections to problems such as Multiway Cut and Uniform Metric Labeling.

Let us now elaborate on how we design approximation algorithms for these separator problems, and how we develop the framework to use them to get new results for $\mathcal{H}$-Vertex Deletion and $\mathcal{H}$-Edge Deletion.

1.1.1 Vertex Deletion Problems

Our main result in this setting is a meta-theorem connecting $\mathcal{H}$-Vertex Deletion problems with the $k$-Subset Vertex Separator ($k$-SVS) problem. In this paper, unless specified by a subscript, all constants hidden in $O(\cdot)$ are absolute constants. Also throughout the paper, OPT denotes the cardinality of the optimal solution of an optimization problem. An algorithm for $k$-SVS is called an $(\alpha, \beta)$-bicriteria approximation algorithm if it returns a set $S \subseteq V(G)$ such that $|S| \leq \alpha \cdot \text{OPT}$ and each component of $G \setminus S$ has at most $\beta k$ vertices from $R$. This is a weaker requirement than a “true” approximation. Also, when we refer to an exact algorithm for any optimization problem, we mean an algorithm finding a solution of size OPT.

**Theorem 1.** Let $\mathcal{H}$ be a hereditary class of graphs with treewidth bounded by $t - 1$. Suppose that

(a) $\mathcal{H}$-Vertex Deletion admits an exact algorithm that runs in time $f(n, \text{OPT})$, and
(b) $k$-Subset Vertex Separator admits a $(\alpha(k), O(1))$-bicriteria approximation algorithm that runs in time $g(n, k)$ with $\alpha(k) = O(\log k)$.

Then there is a $2\alpha(t)$-approximation for $\mathcal{H}$-Vertex Deletion with running time $(f(n, O(\alpha(t))) + g(n, O(\alpha(t)))) \cdot \log n$.

E.g., an $(O(1), O(1))$-bicriteria approximation for $k$-SVS would give an $O(1)$-approximation for $\mathcal{H}$-Vertex Deletion running in $(g(n, O(t)) + f(n, O(t))) \log n$ time. The best approximation for $k$-SVS currently gives an $(O(\log k), 2)$-bicriteria approximation and runs in time $n^{O(1)}$ [Lee18], so Theorem 1 implies the following corollaries. All algorithms in this paper are deterministic.\footnote{The conference version of [Lee18] presents a randomized algorithm, but the journal version derandomized it.}
We first study minor deletion, where we want to exclude every graph in $\mathcal{F}$ as a minor. The celebrated result by Fomin et al. [FLMS12] studied the case when $\mathcal{F}$ has at least one planar graph (called Planar-$\mathcal{F}$ Vertex Deletion), and gave a randomized $c_\mathcal{F}$-approximation algorithm that runs in time $f(\mathcal{F}) \cdot O(nm)$, where $c_\mathcal{F}$ is a constant depending on $\mathcal{F}$. While requiring the excluded family $\mathcal{F}$ contain a planar graph seems restrictive, this case still captures fundamental optimization problems such as Vertex Cover and Feedback Vertex Set. The following corollary will be proved in Section 3.

Corollary 2. If $\mathcal{H}$ is minor-closed and graphs in $\mathcal{H}$ have treewidth at most $t$, $\mathcal{H}$-Vertex Deletion admits an $O((\log t) \cdot \alpha)$-approximation algorithm that runs in time $O(\mathcal{H}(n \log n) + n^{O(1)})$. In particular, Planar-$\mathcal{F}$ Vertex Deletion admits an $O((\log f) \cdot \alpha)$-approximation with running time $O(\mathcal{F}(n \log n) + n^{O(1)})$, where $\alpha$ denotes the number of vertices of any planar graph in $\mathcal{F}$.

The latter result improves the approximation ratio for Planar-$\mathcal{F}$ Vertex Deletion from $c_\mathcal{F}$ to $O(\log k)$. It is also deterministic, positively answering an open question in Kim et al. [KLP+16].

Theorem 1 also implies uniform algorithms with better running time and approximation ratio for many natural parameterized problems. A parameterized algorithm with parameter $k$ is called uniform if there is a single algorithm that takes an instance $I$ and a value of $k$ as input, and runs in time $f(k) \cdot |I|^{O(1)}$. Non-uniform algorithms for a parameterized problem indicate that there are different algorithms for each $k$, whose existence relies on non-constructive arguments.

If we have a sequence of families $\mathcal{H}_k$ where every graph in $\mathcal{H}_k$ has treewidth at most $k$, and we know a uniform exact fixed-parameter tractable (FPT) algorithm for $\mathcal{H}_k$-Vertex Deletion parameterized by both $k$ and OPT, we get a uniform approximation algorithm for $\mathcal{H}_k$-Vertex Deletion. Some examples include $k$-Treewidth Vertex Deletion, $k$-Pathwidth Vertex Deletion, and $k$-Treedepth Vertex Deletion, where $\mathcal{H}_k$ is the set of all graphs with treewidth, pathwidth, and treedepth at most $k$ respectively. Another example is $k$-Path Transversal where $\mathcal{H}_k$ contains all graphs with no simple path of length $k$ as a subgraph (or equivalently, as a minor). For these cases we get the following results that will be proved in Section 3.

Corollary 3. The following problems admit $O(\log k)$-approximations:

(a) $k$-Treewidth Vertex Deletion in time $2^{O(k^3 \log^2 k)} \cdot n \log n + n^{O(1)}$,
(b) $k$-Pathwidth Vertex Deletion in time $2^{O(k^2 \log k)} \cdot n \log n + n^{O(1)}$,
(c) $k$-Treedepth Vertex Deletion in time $2^{O(k^2 \log k)} \cdot n \log n + n^{O(1)}$,
(d) $k$-Path Transversal in time $2^{O(k \log^2 k)} \cdot n \log n + n^{O(1)}$.

The first result of Corollary 3 improves the best current approximation for $k$-Treewidth Vertex Deletion from [FLMS12], which did not explicitly state the dependency of the approximation ratio and running time on the treewidth $k$. Our algorithms also give the first uniform $O(k)$-approximation algorithms in this parameterization, while the protrusion-replacement technique of [FLMS12] makes their algorithms non-uniform. For $k$-Treedepth Vertex Deletion a $2^k$-approximation algorithm has been known [GHO+13]. The last result of Corollary 3 tightens the runtime of the $O(\log k)$-approximation algorithm for $k$-Path Transversal that runs in time $O(2^{k^2 \log k} n^{O(1)})$ [Lee18].

Another application of Theorem 1 arises in bidimensionality theory. For any family $\mathcal{H}$ with treewidth bounded by $k$, an efficient polynomial-time approximation scheme (EPTAS) for $\mathcal{H}$-Vertex Deletion on excluded-minor families was first given by Fomin et al. [FLRS11]. For $\mathcal{H}$-Vertex Deletion on $M$-minor free graphs $G$, the algorithm runs in time $n^{f(|\mathcal{H}, M|)}$ [FLRS11].
or becomes non-uniform and runs in time $2^{f(n,M)} \cdot n^{O(1)}$ \cite{FLMS12}. In Section 3.2 we prove the following meta-theorem that gives faster algorithms for many relevant problems, which are uniform as long as the promised exact algorithm is.

**Theorem 4.** Let $H$ be a hereditary class of graphs with treewidth bounded by $t - 1$, and $M$ be a fixed graph. Suppose that $H$-VERTEX DELETION admits an exact algorithm that runs in time $f(n, OPT)$. Then $H$-VERTEX DELETION admits an $(1 + \varepsilon)$-approximation algorithm on $M$-minor-free graphs with running time $f(n, O_M((t \log^3 t)/\varepsilon^2)) \cdot n \log n + n^{O(1)}$.

### 1.1.2 Edge Deletion Problems

Our main result for the edge deletion problems is the following. Unlike the vertex version, the approximation ratio becomes an absolute constant, but the algorithm uses the maximum degree of $G$, $\deg(G)$, as an additional parameter.

**Theorem 5.** Let $H$ be a class of graphs closed under taking a subgraph. Suppose graphs in $H$ have treewidth bounded by $t - 1$, and $H$-EDGE DELETION admits an exact algorithm with running time $f(n, OPT)$. Then there is an $(3 + \varepsilon)$-approximation for $H$-EDGE DELETION with running time $f(n, n^{O_M((t \log^3 t)/\varepsilon^2)} + f(n, t \deg(G)/\varepsilon)) \log(n/\varepsilon)$.

The above theorem is based on our improved results on $k$-SUBSET EDGE SEPARATOR. The previous best approximation algorithm for $k$-SUBSET EDGE SEPARATOR was an $O(\log k)$-approximation that runs in time $n^{O(1)}$ \cite{Lee18}. While the existence of an $O(1)$-approximation algorithm that runs in time $n^{O(1)}$ would refute the Small Set Expansion Hypothesis \cite{RST12}, we show that one can get significantly better approximations factor using $k$ as a parameter.

**Theorem 6.** The following parameterized algorithms for $k$-SUBSET EDGE SEPARATOR exist:

(a) a $(2 + \varepsilon)$-approximation that runs in time $2^{O(t \deg(G)/\varepsilon)} n^{O(1)}$ for the case $R = V(G)$,

(b) a 2-approximation that runs in time $n^{k+O(1)}$, and

(c) a $(2 + \varepsilon)$-approximation that runs in time $2^{O(k^2 \deg(G)/\varepsilon)} n^{O(1)}$.

We now present corollaries of Theorem 5. Applying exact algorithms (parameterized by OPT) for well-known cases immediately follow the following corollary proved in Section 3.3

**Corollary 7.** If $H$ is minor-closed and with treewidth bounded by $t$, $H$-EDGE DELETION admits a $(3 + \varepsilon)$-approximation algorithm that runs in time $f(t, \deg(G), \varepsilon) \cdot n^{O(1)}$ for some function $f$. In particular, PLANAR-$F$ EDGE DELETION admits a $(3 + \varepsilon)$-approximation algorithm with running time $f(F, \deg(G), \varepsilon) \cdot n^{O(1)}$.

In Section 4.5 we also present implications of Theorem 5 to the NOISY PLANAR $k$-SAT($\delta$) problem studied by Bansal et al. \cite{BRU17}. For a fixed $k = O(1)$, an instance of NOISY PLANAR $k$-SAT($\delta$) is an instance of $\phi$ of $k$-SAT with $n$ variables and $m$ clauses where the factor graph of $\phi$ becomes planar after deleting $\delta m$ edges (See Section 4.5 for formal definitions). Bansal et al. \cite{BRU17} proved that for any $\varepsilon > 0$, there is an algorithm that achieves $(1 + O(\varepsilon + \delta \log m \log \log m))$-approximation in time $m^{O(\log \log m)^2/\varepsilon}$. We prove that if the degree of the factor graph is bounded, we can obtain an improved algorithm. Note that $k$-SAT with the maximum degree $O(1)$ has been actively studied and proved to be APX-hard (e.g., 3-SAT($5$)) for general factor graphs.

**Corollary 8.** For any $\varepsilon > 0$, there is an $(1 + O(\varepsilon + \delta))$-approximation algorithm for NOISY PLANAR $k$-SAT($\delta$) that runs in time $f(\varepsilon, \deg(\phi)) \cdot m^{O(1)}$ for some function $f$, where $\deg(\phi)$ indicates the maximum degree of the factor graph of $\phi$. 

4
For the edge deletion problems, the trivial reduction from VERTEX COVER, that gave \((2 - \varepsilon)\)-inapproximability for all vertex problems under the Unique Games Conjecture, does not work. When \(k = 1\), \(k\)-SUBSET EDGE SEPARATOR becomes the famous MULTIWAY CUT problem, which is hard to approximate within a factor \(\approx 1.2\) assuming the Unique Games Conjecture \cite{AMM17}. One can also speculate that the edge deletion problems may have an exact algorithm or PTAS when \(\text{deg}(G)\) is bounded. We prove the following hardness result that \(k\)-EDGE SEPARATOR with \(k = 3\) is APX-hard even when \(\text{deg}(G) = 4\). Taking \(F\) to be the set of all graphs with three vertices (which are all planar), this also proves that excluding \(F\) as a subgraph, minor, or immersion will not admit a PTAS even for bounded degree graphs.

**Theorem 9.** There exists a constant \(c > 1\) such that \(k\)-EDGE SEPARATOR is NP-hard to approximate within a factor of \(c\) even when \(k = 3\) and \(\text{deg}(G) = 4\). Consequently, when \(F\) is the set of all graphs with three vertices, deleting the minimum number of edges to exclude \(F\) as a subgraph, minor, or immersion is APX-hard for bounded degree graphs.

We note that the above hardness result only leaves open the case of \(\text{deg}(G) = 3\); when \(\text{deg}(G) = 2\), the graph is simply a disjoint union of paths and cycles, and hence \(k\)-EDGE SEPARATOR can be solved (exactly) in polynomial time.

### 1.2 Techniques

**Vertex Deletion.** We briefly sketch our proof techniques for Theorem 1 for general \(H\)-VERTEX DELETION using an algorithm for \(k\)-SUBSET VERTEX SEPARATOR as a black box. For simplicity, let us focus on \(k\)-TREEWIDTH VERTEX DELETION.

Let \(S^* \subseteq V\) be the optimal solution with \(|S^*| = \text{OPT}\). Our high-level approach is the following iterative algorithm that maintains a feasible solution \(R \subseteq V\) and refines it to a smaller solution in each iteration. (Initially we start from \(R = V\).) The simple but crucial lemma for us is Lemma 10 in Section \(3\) which states that if in the induced subgraph \(G[V \setminus S^*]\), which has treewidth at most \(k\), there are at most \(\varepsilon|R|\) vertices such that additionally deleting them from \(G[V \setminus S^*]\) ensures that each connected component has at most \(O(k/\varepsilon)\) vertices from \(R\). This type of argument guaranteeing the existence of a small separator that *finely separates* a subset (i.e., each component has \(O_{k,\varepsilon}(1)\) vertices from \(R\)) previously appeared in Fomin et al. \cite{FLRS11} for bidimensionality theory. Our lemma admits a simpler proof because we need less properties and do not need to be constructive.

Our main conceptual contribution that bridges treewidth deletion and \(k\)-SUBSET VERTEX SEPARATOR is to observe that the above lemma guarantees a feasible solution of \(O(k/\varepsilon)\)-SUBSET VERTEX SEPARATOR of size \(\text{OPT} + \varepsilon R\). Applying an \((\alpha, \beta)\)-bicriteria approximation algorithm for \(k\)-SUBSET VERTEX SEPARATOR will delete at most \(\alpha(\text{OPT} + \varepsilon R)\) vertices to make sure that each connected component has at most \(O(\beta k/\varepsilon)\) vertices from \(R\). Since \(R\) is a feasible solution, each connected component admits a solution of size \(O(\beta k/\varepsilon)\), which can optimally solved in \(f(\beta k/\varepsilon) \cdot n^{O(1)}\) time by deleting at most \(\text{OPT}\) vertices. In total, the size of our new solution is at most \(\alpha(\text{OPT} + \varepsilon R) + \text{OPT}\). By appropriately adjusting \(\varepsilon\), we can prove that unless \(R = O(\alpha \cdot \text{OPT})\), the size of the new solution is at most \(|R|/2\), which implies that we will achieve \(O(\alpha)\)-approximation in at most \(O(\log n)\) iterations. The current best \((O(\log k), 2)\)-bicriteria approximation algorithm for \(k\)-SUBSET VERTEX SEPARATOR immediately yields \(O(\log k)\)-approximation for \(k\)-TREEWIDTH VERTEX DELETION.

Recently, Bansal et al. \cite{BRU17} and Agarwal et al. \cite{ALM+18} used graph partitioning algorithms to solve treewidth deletion problems. Agarwal et al.’s approach was based on graphs with bounded treewidth admitting small *global separators* (i.e., whose deletion ensures each component has \(2n/3\) vertices), while Bansal et al. additionally used the fact that any subset \(R \subseteq V\) admits a small
**R-global separator** (i.e., whose deletion ensures each component has $2|R|/3$ vertices). Such small separators are found by a modification of traditional graph partitioning algorithms for global separators, which allows us to recurse into smaller components. Using such global partitioning algorithms gives an inherent loss of $\Omega(\log n)$. Our results indicate that computing a *finer-grained separator* of $R$, such as $k$-SUBSET VERTEX SEPARATOR, avoids the loss in terms of $n$. One downside of our approach is that it does not work for weighted settings. The idea of reducing to the subset version of a classical combinatorial optimization problem was also employed by Bonnet et al. [BBKM16] where they used SUBSET FEEDBACK VERTEX SET to solve BOUNDED $P$-BLOCK VERTEX DELETION.

It is an interesting open question to see whether $k$-SUBSET VERTEX SEPARATOR admits an $(\alpha, \beta)$-bicriteria approximation algorithm for absolute constants $\alpha, \beta$ since it will imply an $O(\alpha)$-approximation algorithm for $k$-TREEWIDTH VERTEX DELETION that does not depend on $k$ by Theorem [1]. The best inapproximability is $(2 - \varepsilon)$ coming from VERTEX COVER, and this is even for the case $R = V$.

**$k$-Subset Edge Separator.** Here we highlight our techniques for the edge deletion problems, which result in algorithms with better approximation factors than their vertex deletion counterparts. The gap in difficulty between vertex- and edge-deletion versions has been observed in other cut problems on undirected graphs, such as MULTIWAY CUT [GYY04] and MINIMUM $k$-WAY CUT [SV95]. Intuitively, the reason is that in edge deletion problems, we can charge the solution cost to the boundary size of each connected component in the remaining graph (without the deleted edges). Since every edge deleted belongs to the boundary of exactly two components, the sum of the boundary sizes of the components is exactly twice the solution cost. Charging the cost of an algorithm to the sizes of boundaries proves to be a more tractable strategy in many cases.

The $k$-Edge Separator problem is a special case of $k$-Subset Edge Separator where $R = V(G)$. Our two $(2 + \varepsilon)$-approximation algorithms for $k$-Edge Separator start by reducing the degree of the graph to $O(k)$, while sacrificing only an $(1 + \varepsilon)$ factor loss in approximation. This step relies on the observation that if a vertex has very high degree, then nearly all of its incident edges must be deleted in any feasible solution, so we might as well delete them all and sacrifice an $(1 + \varepsilon)$ factor loss.

After the degree of the graph is parameterized by $k$, our first algorithm begins with any feasible solution and iteratively improves it using local search. At each step, the algorithm examines connected components of at most $k$ vertices and looks for one which can improve the current solution. If the graph has degree $O(k)$, then there are only $k^{O(k)}n$ many connected components of size at most $k$, which leads to a running time FPT in $k$.

The second algorithm for $k$-Edge Separator relies on a direct reduction to an instance of UNIFORM METRIC LABELING by viewing each of the $k^{O(k)}n$ connected components of size at most $k$ as a color in UNIFORM METRIC LABELING. It then applies the 2-approximation algorithm of UNIFORM METRIC LABELING from [KT02].

For the more general $k$-Subset Edge Separator problem, the local search algorithm generalizes to one running in time $n^{k+O(1)}$. Instead of trying all connected components of size at most $k$, we try all subsets of $R$ of size at most $k$, not necessarily connected. Determining if a given subset improves the solution is more technical, requiring a gadget reduction to a minimum $s$-$t$ cut instance.

Finally, for the case when the graph degree is small, we can follow the UNIFORM METRIC LABELING

---

4 The hardness of node $k$-way cut follows from the observation that the instance is feasible if and there is an independent set of size $k$ in the graph, and independent set is hard to approximate [Zuc09].
reduction approach to obtain an algorithm FPT in both \(k\) and \(\deg(G)\). In particular, we prove that, modulo an \((1 + \varepsilon)\) loss in approximation, there are essentially \(2^{O(k^2 \deg(G))} n\) many relevant connected components to consider. For this, we use the idea of important cuts, a tool that has been used in FPT algorithms for other cut problems, such as \textsc{Multiway Cut} \cite{Mar06, CLL09}. Assigning each of these relevant components a color gives a \textsc{Uniform Metric Labeling} instance of size parameterized by both \(k\) and \(\deg(G)\), which is again approximated to factor 2.

### 1.3 Related Work

In this subsection, we briefly survey known parameterized approximation algorithms for \textsc{Planar-}\(\mathcal{F}\) \textsc{Vertex Deletion} and \textsc{Planar-}\(\mathcal{F}\) \textsc{Edge Deletion}, parameterized by \(\mathcal{F}\), leaving out a rich set of results on exact parameterized algorithms and kernelization (often parameterized by \(OPT\)). The edge-deletion version for \textsc{Planar-}\(\mathcal{F}\) \textsc{Vertex Deletion} that admit an \(O_\mathcal{F}(1)\)-approximation algorithm for the weighted case is when \(\mathcal{F}\) is a diamond \cite{FJP10} or \(\mathcal{F}\) is a simple path (where minor deletion and subgraph deletion become equivalent).

When we exclude a single graph \(\mathcal{F}\) as a subgraph, there is a simple \(k\)-approximation algorithm where \(k\) is the number of vertices in \(\mathcal{F}\). A nearly-matching hardness was proved by Guruswami and Lee \cite{GL15}, who showed that \(\mathcal{H}\)-\textsc{Vertex Deletion} is \(NP\)-hard to approximate within a factor of \(k - 1 - \varepsilon\) for any \(\varepsilon > 0\) \((k - \varepsilon\) assuming the Unique Games Conjecture) whenever \(\mathcal{F}\) is 2-vertex-connected. If \(\mathcal{F}\) is a star or a simple path with \(k\) vertices, \(O(\log k)\)-approximation algorithms are known \cite{GL15, Lee18}.

For \(\mathcal{H}\)-\textsc{Edge Deletion}, the notion of immersion deletion has commonly been studied instead of minor deletion. Bansal et al. \cite{BRU17} gave an \(O(\log n \log \log n)\)-approximation for \(t\)-\textsc{Treewidth Edge Deletion}. The edge-deletion version for \textsc{induced subgraph deletion} was also studied \cite{BCKP18}.

There is also vast literature on general \(\mathcal{H}\)-\textsc{Vertex Deletion} or \(\mathcal{H}\)-\textsc{Edge Deletion} besides the aforementioned minor, immersion, subgraph, and induced subgraph deletions. Lund and Yannakakis \cite{LY93} considered the maximization version when we want to find the maximum \(S \subseteq V(G)\) such that the induced subgraph \(G[S] \in \mathcal{H}\), and showed that whenever \(\mathcal{H}\) is hereditary and nontrivial \((\mathcal{H}\) contains an infinite number of graphs and does not contain an infinite number of graphs), then the maximization version is hard to approximate within a factor \(2^{\log^{1/2-\varepsilon} n}\) for any \(\varepsilon > 0\). This inapproximability ratio was subsequently improved to \(n^{1-\varepsilon}\) for any \(\varepsilon > 0\) by Feige and Kogan \cite{FK05}.

\textsc{Chordal Vertex Deletion} \cite{ALM18, JP17, KK18} and \textsc{Odd Cycle Transversal} \cite{ACMM05} are other primary examples of \(\mathcal{H}\)-\textsc{Vertex Deletion}; they can be captured as a subgraph deletion when \(\mathcal{F}\) is the set of all chordless or odd cycles. The problem of reducing other width parameters (e.g., rankwidth, cliquewidth) have been studied \cite{ALM18}. Besides approximation algorithms, these problems also have been studied through the lens of their parameterized complexity (parameterized by \(OPT\)) and covering-packing duality (known as the Erdős-Pósa property). We refer the
reader to the introduction of [ALM+18] and [GL15] for more detailed survey.

2 Preliminaries

Unless otherwise specified by a subscript, all constants hidden in $O(\cdot)$ notations are absolute constants that do not depend on any parameter. For a graph $G = (V(G), E(G))$, let $n$ denote the number of vertices, and a subset $S$ of vertices or edges, let $G\setminus S$ be the graph after deleting $S$ from $G$. For disjoint subsets $C_1, \ldots, C_m \subseteq V(G)$, let $E(C_1, \ldots, C_m) := \{(u, v) \in E(G) : u \in C_i, v \in C_j, i \neq j\}$. For $C \subseteq V(G)$, let $\partial(C) := E(C, V \setminus C)$. For $v \in V$, let $\deg(v)$ denote the degree of $v$, and let $\deg(G)$ be the maximum degree of $G$.

Tree decomposition and treewidth. Given a graph $G = (V(G), E(G))$, a tree $T = (V(T), E(T))$ is called a tree decomposition of $G$ if every node (also called a bag) $t \in V(T)$ is a subset of $V(G)$, and the following conditions are met.

1. The union of all bags is $V(G)$.
2. For each $v \in V(G)$, the subtree of $T$ induced by $\{t \in V(T) : v \in t\}$ is connected.
3. For each $(u, v) \in E(G)$, there is a bag $t$ such that $u, v \in t$.

The width of $T$ is the cardinality of the largest bag minus 1, and the treewidth of $G$, denoted $tw(G)$, is the minimal width of a tree decomposition of $G$. If we restrict the tree $T$ to be a path, we obtain analogous notions of path decomposition and pathwidth of $G$, denoted $pw(G)$.

Treedepth. A treedepth decomposition of $G$ is a tree $T$ with an injective mapping $\phi : V(G) \to V(T)$, such that whenever $(u, v) \in E(G)$ then $\phi(u)$ and $\phi(v)$ are in ancestor-descendant relation. The treedepth of $G$, denoted $td(G)$, is the minimum height of a treedepth decomposition of $G$. We have $tw(G) \leq pw(G) \leq td(G) - 1$ [RRVST14].

Minors. We say that graph $M$ is a minor of graph $G$ if there exists a mapping $\phi$ from $V(M)$ to disjoint connected subgraphs of $G$, such that whenever $(u, v) \in E(M)$ then $E(\phi(u), \phi(v)) \neq \emptyset$. Otherwise we say that $G$ is $M$-minor-free. If $M$ is planar, then all $M$-minor-free graphs have treewidth bounded by $|V(M)|^{O(1)}$ [RSS06, CC16].

3 Vertex Deletion

In this section, we prove our results for the vertex deletion problems. We first prove Theorem \[1\] and then show its applications to Planar-$\mathcal{F}$ Vertex Deletion, uniform algorithms, and bidimensionality.

Our proof of Theorem \[1\] is based on the following simple lemma that reveals a natural connection between k-Subset Vertex Separator and $\mathcal{H}$-Vertex Deletion when graphs in $\mathcal{H}$ have bounded treewidth.

Lemma 10. Suppose graph $G$ has its treewidth bounded by $t - 1$ and let $R \subseteq V(G)$. Then for each natural number $\delta$ there exists a set $X \subseteq V(G)$ such that $|X| \leq \frac{t}{\delta} |R|$ and each connected component of $G \setminus X$ contains at most $\delta$ elements from $R$. What is more, if the tree decomposition is given, the set $X$ can be constructed in polynomial time.

Proof. Consider a tree decomposition of $G$ of width $t - 1$. For a bag $B$ let $r(B)$ denote the number of vertices from $R$ introduced in the subtree of the decomposition rooted at $B$. If $|R| \leq \delta$ then $X = \emptyset$ satisfies the claim and otherwise there is a bag $B$ with $r(B) > \delta$. Let $B_0$ be such a bag with all its descendant having $r(B) \leq \delta$. Vertices contained in $B_0$ form a cut with all connected components
formed by descendants of \( B_0 \) having at most \( \delta \) vertices from \( R \). We iterate this procedure and define \( X \) to be the union of all performed cuts. Each cut is formed by at most \( t \) vertices and there can be at most \( \frac{R}{\delta} \) iterations. The claim follows.

Now we recall and prove the main theorem for the vertex deletion problems.

**Theorem 1.** Let \( \mathcal{H} \) be a hereditary class of graphs with treewidth bounded by \( t - 1 \). Suppose that 
\[(a) \ \mathcal{H}\text{-VERTEX DELETION} \text{ admits an exact algorithm that runs in time } f(n, \text{OPT}), \text{ and} \]
\[(b) \ k\text{-SUBSET VERTEX SEPARATOR} \text{ admits a } (\alpha(k), O(1))\text{-bicriteria approximation algorithm}
\]
\[\text{that runs in time } g(n, k) \text{ with } \alpha(k) = O(\log k). \]

Then there is a \( 2\alpha(t) \)-approximation for \( \mathcal{H}\text{-VERTEX DELETION} \) with running time \( (f(n, O(\alpha(t))) + g(n, O(\alpha(t)))) \cdot \log n \).

**Proof.** Let \( \varepsilon > 0 \) be a constant determined later (depending on \( t \)). Our algorithm maintains a feasible solution \( R \subseteq V(G) \) (say we start from \( R = V(G) \)) and iteratively finds a smaller solution. Let \( S^* \subseteq V(G) \) be an optimal solution to \( \mathcal{H}\text{-VERTEX DELETION} \) and let \( R \subseteq V(G) \) be the current solution. The graph \( G \setminus S^* \) has its treewidth bounded by \( t - 1 \), therefore Lemma \( \ref{tvcw} \) with \( \delta = t/\varepsilon \) guarantees that there exists a set \( X \subseteq V(G) \setminus S^* \), \( |X| \leq \varepsilon|R| \) so that each connected component in \( G \setminus (S^* \cup X) \) has at most \( t/\varepsilon \) vertices from \( R \).

We launch the \( (\alpha(k), O(1))\)-bicriteria approximation for \( k\text{-SUBSET VERTEX SEPARATOR} \) on \( G \) with \( k = t/\varepsilon \) and \( \alpha(k) = O(\log k) \). It returns a set \( Y \subseteq V(G) \) of size at most
\[\alpha \cdot |S^* \cup X| \leq \alpha \cdot (\text{OPT} + \varepsilon|R|)\]
such that each connected component of \( G \setminus Y \) has at most \( O(t/\varepsilon) \) vertices from \( R \). Since \( \mathcal{H} \) is hereditary, \( R \cup Y \) is a valid solution and we have a bound \( O(t/\varepsilon) \) on the solution size for each connected component. We thus can solve \( \mathcal{H}\text{-VERTEX DELETION} \) on each component \( C \subseteq G \setminus Y \) in time \( f(n, O(t/\varepsilon)) \). We know that \( C \cap S^* \) is a feasible solution for each \( C \), so the sum of solution sizes is bounded by \( |S^*| = \text{OPT} \).

Let \( R' \) be the union of \( Y \) and all solutions obtained for the connected components in \( G \setminus Y \). It will be the new \( R \) in the next iteration. Since \( |R'| \leq |Y| + \text{OPT} \leq (\alpha + 1)\text{OPT} + \varepsilon|R| \), as long as
\[(\alpha(k) + 1) \cdot \text{OPT} + \alpha(k)\varepsilon|R| \leq (3/4)|R| \quad \Leftrightarrow \quad |R| \geq \frac{\alpha(k) + 1}{3/4 - \alpha(k)\varepsilon}\text{OPT},
\]
the size of the maintained solution is decreased by a factor of \( 3/4 \). Since \( \alpha(k) = O(\log k) \), if \( \varepsilon = c/\alpha(t) \) for small constant \( c > 0 \),
\[\frac{\alpha(k) + 1}{3/4 - \alpha(k)\varepsilon} = \frac{\alpha(t/\varepsilon) + 1}{3/4 - \alpha(t/\varepsilon)\varepsilon} = \frac{\alpha(t\alpha(t)/c) + 1}{3/4 - c\alpha(t\alpha(t)/c)/\alpha(t)} \leq 2\alpha(t)\]
The last inequality holds since \( \alpha(t\alpha(t)/c) \leq \alpha(t) + \alpha(t/c) \) gets multiplicatively closer to \( \alpha(t) \) as \( t \) grows, so that for small enough \( c > 0 \), we can ensure that the denominator is at least \( 3/5 \), and the numerator is at most \( (6/5)\alpha(t) \) for large enough \( t \). Therefore, if we begin with \( R = V(G) \) and iterate the procedure \( O(\log n) \) times, we have a \( 2\alpha(t) \)-approximation. The running time for each iteration is \( f(n, k) + g(n, k) = f(n, O(\alpha(t))) + g(n, O(\alpha(t))) \).

We combine this meta-theorem with a recent result for \( k\text{-SUBSET VERTEX SEPARATOR} \).

**Theorem 11 (Lee18).** There exists an \( (O(\log k), 2) \)-bicriteria approximation algorithm for \( k\text{-SUBSET VERTEX SEPARATOR} \) that runs in time \( n^{O(1)} \).
Corollary 12. Suppose $\mathcal{H}$ is a hereditary class of graphs with treewidth bounded by $t - 1$ and $\mathcal{H}$-Vertex Deletion admits an exact algorithm that runs in time $f(n, \text{OPT})$. Then $\mathcal{H}$-Vertex Deletion admits $O(\log t)$-approximation algorithm with running time $f(n, O(t \log t)) \log n + n^{O(1)}$.

We are ready to present the most general result improving upon [FLMS12] who gave a $c_{\mathcal{H}}$-approximation for $\mathcal{H}$-Vertex Deletion for some implicit constant $c_{\mathcal{H}}$. We emphasize that the constant hidden in term $O(\log t)$ is universal.

Corollary 2. If $\mathcal{H}$ is minor-closed and graphs in $\mathcal{H}$ have treewidth at most $t$, $\mathcal{H}$-Vertex Deletion admits an $O(\log t)$-approximation algorithm that runs in time $O_H(n \log n) + n^{O(1)}$. In particular, Planar-$\mathcal{F}$ Vertex Deletion admits an $O(\log f)$-approximation with running time $O_F(n \log n) + n^{O(1)}$, where $f$ denotes the number of vertices of any planar graph in $\mathcal{F}$.

Proof. For Planar-$\mathcal{F}$ Vertex Deletion, we use the Polynomial Grid Minor theorem [CC16] which says that if $G$ does not have a planar graph $F$ as a minor, the treewidth of $G$ is bounded by $|V(F)|^{O(1)}$. Planar-$\mathcal{F}$ Vertex Deletion admits a linear-time exact algorithm parameterized by the solution size [Bod97] so the assumptions of Corollary 12 are satisfied.

Due to the result of Robertson and Seymour [RS04], every minor-closed class can be represented as $\mathcal{F}$-minor-free graphs for some finite family $\mathcal{F}$. If the treewidth in $\mathcal{H}$ is additionally bounded, then at least one of graphs in $\mathcal{F}$ must be planar. Therefore $\mathcal{H}$-Vertex Deletion reduces to Planar-$\mathcal{F}$ Vertex Deletion.

3.1 Uniform Algorithms for Width Reduction

Another application of our approach emerges when we deal with a sequence of families $\mathcal{H}_k$. In contrary to the previously known techniques, Theorem 1 can produce uniform algorithms for $\mathcal{H}_k$-Vertex Deletion when provided with an exact uniform algorithm parameterized by both $k$ and OPT. We present such an exact algorithm for $k$-Treewidth Vertex Deletion, together with related problems, and combine it with our framework. Then we also cover $k$-Path Transversal problem where $\mathcal{H}_k$ consists all graphs with no simple path of length $k$.

Lemma 13. The problems of $k$-Treewidth / Pathwidth / Treedepth Vertex Deletion parameterized by $k$ and the solution size $p$ admit exact algorithms with running times $2^{O((k+p)^2k)}n$, $2^{O((k+p)(k+\log(k+p)))}n$, and $2^{O(k+p)k}n$ respectively.

Proof sketch. As these algorithms are variants of well-known previous algorithms, we briefly give a sketch of the proof here and give more detailed explanations in Section A. For any graph $H$ we have $\text{tw}(H) \leq \text{pw}(H) \leq \text{td}(H)$. Consider a solution $X \subseteq V(G)$ – it satisfies $|X| \leq p$ and $\text{tw}(G \setminus X) \leq k$. After adding $X$ to each bag of the tree decomposition for $G \setminus X$ we obtain a decomposition for $G$ with width at most $k + p$. We can thus use the linear-time constant approximation algorithm for treewidth [RGDS13] to find a tree decomposition of $G$ with width $O(k + p)$ in time $2^{O(k+p)}n$.

The problem of finding a tree (or path) decomposition of width $k$ parameterized by the width $t$ of the input tree decomposition has been studied by [BK91] who gave a $2^{O(tk+\log t)}n$-time algorithm for the pathwidth case and a $2^{O(t^2k)}n$-time algorithm for the treewidth case. A $2^{O(tk)}n$-time algorithm for finding a treedepth decomposition of width $k$ given a tree decomposition of width $t$ has been obtained by [RRVS14]. We slightly modify these procedures to handle vertex deletion and use them over the precomputed tree decomposition of width $O(k + p)$. A more detailed construction is presented in Section A.

Corollary 3. The following problems admit $O(\log k)$-approximations:
(a) $k$-Treewidth Vertex Deletion in time $2^{O(k^3 \log^2 k)} \cdot n \log n + n^{O(1)}$.
(b) $k$-Pathwidth Vertex Deletion in time $2^{O(k^2 \log k)} \cdot n \log n + n^{O(1)}$.
(c) $k$-Treedepth Vertex Deletion in time $2^{O(k^2 \log k)} \cdot n \log n + n^{O(1)}$.
(d) $k$-Path Transversal in time $2^{O(k^2 \log k)} \cdot n \log n + n^{O(1)}$.

Proof. For Treewidth / Pathwidth / Treedeptth Vertex Deletion, we inject the bounds from Lemma 13 into Corollary 12. To handle Path Transversal observe that the $k$-path-free graphs have treedepth bounded by $k$ and therefore also treewidth bounded by $k$ [RRVS14]. There is an exact algorithm for $k$-Path Transversal with running time $f_k(n, p) = O(k^p n)$, where $p$ is the bound on the solution size [Lee18]. The claim follows again from Corollary 12.

3.2 Applications in Bidimensionality

In this section we show how to obtain better guarantees over planar graphs or, more generally, over graphs with excluded minor. The main insight from bidimensionality we rely on is the following lemma allowing to truncate the solution candidate by increasing the working treewidth moderately.

Lemma 14 ([FLRS11], Corollary 1). Let $G$ be a $M$-minor-free graph, $X \subseteq V(G)$, and $tw(G \setminus X) \leq t$. Then for any $\varepsilon > 0$ there exists a set $X' \subseteq V(G)$ such that $|X'| \leq \varepsilon |X|$ and $tw(G \setminus X') = O_M(t/\varepsilon)$, where the hidden constant depends on the excluded minor $M$. Moreover, for given $G, X, \varepsilon$, the set $X'$ can be constructed in polynomial time, however with a slightly worse guarantee $tw(G \setminus X') = O_M(\frac{\log 1}{\varepsilon})$.

Proof. We retrace the proofs in [FLRS11] to give explicit dependence on $t$. Their Corollary 2 says that if $G$ is $M$-minor-free and $tw(G \setminus X) \leq t$, then $tw(G) = O_M(t \sqrt{|X|})$, i.e., $M$-minor-free graphs have “truly sublinear treewidth” with $\lambda = 1/2$. For such a family and with assumptions as above, Lemma 1 guarantees that there exists $\gamma_t(\varepsilon)$ and a set $X' \subseteq V(G)$, $|X'| \leq \varepsilon |X|$, such that every connected component $C$ of $G \setminus X'$ satisfies $|C \cap X| \leq \gamma_t(\varepsilon)$ and $|N(C)| \leq \gamma_t(\varepsilon)$. Moreover, the proof indicates that $\gamma_t(\varepsilon) = O_M \left( \left( \frac{t}{\varepsilon} \right)^2 \right)$ in the existential variant and $\gamma_t(\varepsilon) = O_M \left( \left( \frac{\log 1}{\varepsilon} \right)^2 \right)$ in the constructive variant. Injecting this bound into their Corollary 1 entails the claim. 

We now prove our meta-theorem for bidimensional problems. Roughly, a problem is bidimensional if the solution value for the problem on a $k \times k$ grid is $\Omega(k^2)$, which is true for $\mathcal{H}$-Vertex Deletion when $\mathcal{H}$ is a class of graphs with bounded treewidth. Introduced in Demaine et al. [DFHT05], it has been a unifying theory for many algorithms in minor-free graphs. Demaine and Hajiaghayi [DH05] and later Fomin et al. [FRS11] designed EPTASes for a large class of bidimensional problems. For $\mathcal{H}$-Vertex Deletion on $M$-minor free graphs $G$, there is an uniform algorithm runs in time $n^{f(\mathcal{H}, M)}$ [FRS11], and a non-uniform algorithm that runs in time $g(\mathcal{H}, M) \cdot n^{O(1)}$ [FLMS12].

As previously observed in [FLMS12], the main bottleneck of the running time was reducing treewidth, so our algorithm for $k$-Treewidth Vertex Deletion can be used to obtain improved running time for all bidimensional problem considered in [FRS11]. We formally present EPTASes for $\mathcal{H}$-Vertex Deletion with explicit running times that are uniform as long as the promised exact algorithm is. The only hidden factor we do not keep track of comes from the grid obstruction for excluded minor $M$.

Theorem 4. Let $\mathcal{H}$ be a hereditary class of graphs with treewidth bounded by $t - 1$, and $M$ be a fixed graph. Suppose that $\mathcal{H}$-Vertex Deletion admits an exact algorithm that runs in time $f(n, \text{OPT})$. Then $\mathcal{H}$-Vertex Deletion admits an $(1 + \varepsilon)$-approximation algorithm on $M$-minor-free graphs with running time $f(n, O_M((t \log^2 t)/\varepsilon^2)) \cdot n \log n + n^{O(1)}$. 

11
Proof. Let us start with finding an $O(\log t)$-approximate solution $X$ with Corollary 12 in time $f(n, O(\log t)) + n^{O(1)}$. Since $tw(G \setminus X) \leq t$, we can use the constructive variant of Lemma 11 with $f' = O(\varepsilon / \log t)$ to find $X'$ such that $|X'| \leq \frac{\varepsilon}{2} \cdot OPT$ and $tw(G \setminus X') \leq O_M\left(\frac{\log^2 t}{\varepsilon}\right)$.

Though tree composition of $G \setminus X'$ is not explicitly given, we can find a decomposition of width $O(tw(G \setminus X'))$ in time $2^{O(tw(G \setminus X'))} \cdot n$ [BGDSD+13].

We apply the constructive variant of Lemma 10 to graph $G \setminus X'$ with $R = X$ and $\delta = O_M\left(\frac{\log^2 t}{\varepsilon}\right)$. By choosing an appropriate constant, we obtain set $Y$ of size at most $O(tw(G \setminus X')) \cdot |X| \leq \frac{\varepsilon}{2} \cdot OPT$ such that each connected component $C$ of $G \setminus (X' \cup Y)$ satisfies $|C \cap X| \leq \delta$. We launch the exact algorithm for $H$-Vertex Deletion on each component with bound $OPT \leq \delta$ in total time $f(n, \delta)$ and return the sum of solutions together with $X' \cup Y$.

We illustrate some applications of the above theorem. For some problems, we additionally take advantage of the bidimensionality to show that the dependence on $k$ can be even subexponential.

**Corollary 15.** $k$-Path Transversal and $k$-Vertex Separator admit an EPTAS on $M$-minor-free graphs with running time $\exp\{O_M(\sqrt{\frac{k \log^3 k}{\varepsilon^2})}\} \cdot n \log n + n^{O(1)}$.

Proof. Graphs with excluded minor $M$ that are $k$-path free or have each component size bounded by $k$ have treewidth of order $O_M(\sqrt{k})$ [DH07]. There is an exact algorithm for $k$-Path Transversal with running time $f_k(n, p) = O(k^p n)$, where $p$ is the bound on the solution size [Lee18]. The same approach works for $k$-Vertex Separator: as long as the graph has a component of size at least $k+1$ we can find a connected subgraph of size $k+1$. At least one of its vertices must belong to the solution so we can perform branching with $k+1$ direct recursive calls and depth at most $p$. The claim follows from Theorem 1 with $f\left(n, O_M\left(\sqrt{\frac{k \log^3 k}{\varepsilon^2})}\right)\right) = \exp\{O_M(\sqrt{\frac{k \log^3 k}{\varepsilon^2})}\} \cdot n$.

**Corollary 16.** The $k$-Pathwidth Vertex Deletion and $k$-Treewidth Vertex Deletion problems admit EPTASes on $M$-minor-free graphs with running time $\exp\{O_M\left(\frac{k^2 \log^3 k}{\varepsilon^2}\right)\} \cdot n \log n + n^{O(1)}$. Also, $k$-Treewidth Vertex Deletion admits an analogous result with running time $\exp\{O_M\left(\frac{k^2 \log^3 k}{\varepsilon^2}\right)\} \cdot n \log n + n^{O(1)}$.

Proof. We apply Lemma 13 providing an exact algorithm for these problems, to Theorem 1. The respective running times of these routines are:

- $f_k^{pw}(n, p) = 2^{O((k+p) - (k+\log(k+p)))} \cdot n$,  
- $f_k^{td}(n, p) = 2^{O((k+p)k)} \cdot n$,  
- $f_k^{tw}(n, p) = 2^{O((k+p)^2k)} \cdot n$.

This completes the proof.

4 Edge Deletion

For $k$-Edge Separator and $k$-Subset Edge Separator the best previous approximation algorithm gave $O(\log k)$-approximation [Lee18]. We present an improved $(2 + \varepsilon)$-approximation algorithm for $k$-Edge Separator in Section 4.1 and give two extensions to $k$-Subset Edge Separator in Section 4.2 and 4.3 with (almost) the same approximation ratio. In Section 4.4 we apply these algorithms for $H$-Edge Deletion and study further applications in Section 4.5. Finally, in Section 4.6 we prove inapproximability results for $k$-Edge Separator for $k = 3$ which implies inapproximability for all edge deletion problems considered in this paper.
4.1 \(k\)-Edge Separator

We first give a \((2+\varepsilon)\)-approximation algorithm for \(k\)-Edge Separator that runs in time \(2^{O(k \log k)} n^{O(1)}\), proving the first part of Theorem 5. It can be proved in two ways. After showing that we can assume that the maximum degree is \(O(k)\) without loss of generality, the first proof is a reduction to Uniform Metric Labeling studied by Kleinberg and Tardos [KT02], where 2-approximation is achieved via the standard LP relaxation. The second proof is based on a direct local search algorithm, which creates a new component with at most \(k\) vertices whenever it improves the overall cost.

These two proofs lead to two approximation algorithms for \(k\)-Subset Edge Separator with similar approximation ratios that run in time \(f(k, \deg(G)) \cdot n^{O(1)}\) and \(nf(k)\) respectively. The first algorithm is based on the reduction to Uniform Metric Labeling extended by the technique of important separators [MR14], and the second algorithm extends the local search in the second proof by efficiently computing the best local move.

For \(k\)-Edge Separator, we present the local search based algorithm. We start by noting that considering only bounded degree graphs suffices, since in the optimal solution, large degree vertices will lose almost all incident edges.

Claim 17. For any \(\varepsilon > 0\), an \(\alpha\)-approximation algorithm for \(k\)-Edge Separator that runs in time \(f(k, \deg(G)) n^{O(1)}\) implies an \((\alpha + \varepsilon)\)-approximation algorithm for \(k\)-Edge Separator that runs in time \(f(k, 2k/\varepsilon) n^{O(1)}\).

Proof. Suppose that there exists an \(\alpha\)-approximation algorithm for \(k\)-Edge Separator that runs in time \(f(k, \deg(G)) n^{O(1)}\). To solve \(k\)-Edge Separator for a unbounded degree graph, given a graph \(G = (V(G), E(G))\), we remove all the edges incident on vertices whose degree is more than \(2k/\varepsilon\).

Let \(S^* \subseteq E(G)\) be the optimal solution. Every vertex \(v\) in \(G \setminus S^*\) has degree at most \(k - 1\), so for \(v\) with \(\deg(v) > 2k/\varepsilon\), the above operation deletes at most \(k - 1\) edges not in \(S^*\), which is at most \(\varepsilon/2\) fraction of edges in \(S^*\) incident to \(v\). Therefore, this operation deletes at most \(\varepsilon |E(S^*)| = \varepsilon \cdot \text{OPT}\) edges overall outside \(S^*\). Running the bounded degree algorithm on the resulting graph proves the claim.

Now we give an algorithm for \(k\)-Edge Separator.

Lemma 18 ((i) of Theorem 5). There is a \((2+\varepsilon)\)-approximation algorithm for \(k\)-Edge Separator that runs in time \(2^{O(k \log(k/\varepsilon))} n^{O(1)}\).

Proof. Our local search algorithm maintains the partition \((C_1, \ldots, C_m)\) of \(V(G)\) where \(|C_i| \leq k\) for each \(i\). This corresponds to deleting edges in \(E(C_1, \ldots, C_m)\). In each iteration, the algorithm considers every possible part \(C \subseteq V\) of size at most \(k\) such that the induced subgraph \(G[C]\) is connected. There are at most \(n \cdot \deg(G)^k\) such components to consider. For each \(C\), we consider the new partition where \(C\) is added to the partition, and each previous part \(C_i\) becomes \(C_i \leftarrow C_i \setminus C\). (Delete empty part from the partition.) If the new partition cuts fewer edges, implement this change and repeat until there is no possible improvement. Since each iteration strictly improves the current solution, the total running time is bounded by \(\deg(G)^k n^{O(1)}\).

Let \((C_1, \ldots, C_m)\) be the resulting partition output by the local search and \((C_1^*, \ldots, C_m^*)\) be the optimal partition. For each \(C_i^*\), either \(C_i^*\) is a part in \((C_1, \ldots, C_m)\), or the local move with \(C_i^*\) does not improve \((C_1, \ldots, C_m)\). In the latter case, as the local improvement with \(C_i^*\) newly deletes edges
in $\partial C^*_i \setminus E(C_1, \ldots, C_m)$ and restores currently deleted edges in $E(G[C^*_i]) \cap E(C_1, \ldots, C_m)$, we can conclude that
\[
|\partial C^*_i \setminus E(C_1, \ldots, C_m)| \geq |E(G[C^*_i]) \cap E(C_1, \ldots, C_m)|.
\]
Note that in the former case, the above is trivially satisfied. If we add the above for every $i = 1, \ldots, m^*$, the left-hand side is two times the number of edges in $E(C^*_1, \ldots, C^*_m) \setminus E(C_1, \ldots, C_m)$, and the right-hand side is the number of edges in $E(C_1, \ldots, C_m) \setminus E(C^*_1, \ldots, C^*_m)$. Therefore,
\[
2|E(C^*_1, \ldots, C^*_m) \setminus E(C_1, \ldots, C_m)| \geq |E(C_1, \ldots, C_m) \setminus E(C^*_1, \ldots, C^*_m)| \Rightarrow 2\text{OPT} \geq |E(C_1, \ldots, C_m)|.
\]
Therefore, this algorithm runs in time $2^{O(k \log \deg(G))} n^{O(1)}$ and gives a 2-approximation. Applying Claim \cite{17} for any $\varepsilon > 0$, we have a $(2 + \varepsilon)$-approximation algorithm that runs in time $2^{O(k \log(k/\varepsilon))} n^{O(1)}$ for general graphs.

\section*{4.2 $k$-Subset Edge Separator in Time $n^{k+O(1)}$}

Note that the above local search algorithm, without the degree reduction step, also implies a 2-approximation algorithm in time $n^{k+O(1)}$, since there are at most $n^k$ subsets of $V$ of size at most $k$. For $k$-\textsc{Subset Edge Separator} where each part can contain much more than $k$ vertices as long as it has at most $k$ vertices from $R$, even the degree bound does not yield a polynomial bound on the number of choices we need to consider in the local search algorithm. For example, given a subset $R' \subseteq R$ with $|R'| \leq k$, there can be exponentially many $C \subseteq V$ such that $C \cap R = R'$ and $G[C]$ is connected.

The modified local search algorithm for $k$-\textsc{Subset Edge Separator}, in each iteration, finds the best local improvement over all possible subsets. The following lemma shows that it can be done in polynomial time. It immediately proves (ii) of Theorem \ref{6} which gives a 2-approximation algorithm for $k$-\textsc{Subset Edge Separator} in time $n^{k+O(1)}$.

\textbf{Lemma 19.} Let $(C_1, \ldots, C_m)$ be a partition of $V$ and $\emptyset \neq R' \subsetneq R$. There is a polynomial time algorithm to find $C^* \subseteq V$ that minimizes the cost $|E(C_1 \setminus C, \ldots, C_m \setminus C, C)|$ over every set $C$ that satisfies $R \cap C = R'$.

\textbf{Proof.} From $G$, merge all vertices in $R'$ to a vertex $s$, and merge all vertices in $R \setminus R'$ to a vertex $t$, while creating parallel edges if needed. Let $G_1$ be the resulting graph. Let $B \subseteq E(G_1)$ be the edges cut by the current solution. Call them blue edges. Finding the best $C^*$ in $G$ is equivalent to finding the best $s$-$t$ cut $(S, V(G_1) \setminus S)$ in $G_1$ ($s \in S$) that minimizes
\[
|\partial G_1[S] \setminus B| + |B \setminus G_1[S]|,
\]
which is exactly the total cost of the new partition. The first term is the number of edges that are newly deleted by adding $S$ to the partition, and the second term is the number of the previously deleted edges minus the number of the undeleted edges in $S$.

We find the minimum $S$ by reducing to the classic Min $s$-$t$ Cut problem. Starting from $G_1$, we do the following operations to obtain $G_2$.

- For each non-blue edge, do not change anything.
- For each blue edge $e = (u, v)$ with $u, v \in V \setminus \{s, t\}$ we introduce a new vertex $t_e$ and replace $(u, v)$ by three edges $(s, t_e), (u, t_e), (v, t_e)$.
  - If $u, v \in S$, we can put $t_e$ to $S$ and do not cut any edge.
– If $u \notin S$ and $v \notin S$, we cut one edge by putting $t_e$ to $V \setminus S$.
– If $|S \cap \{u, v\}| = 1$, we cut one edge by putting $t_e$ to $S$.

• For each blue edge $e = (s, v)$ with $v \neq t$, do not change anything.
  – If $v \in S$, we do not cut any edge.
  – If $v \notin S$, we cut one edge.

• For each blue edge $e = (u, t)$ with $u \neq s$, keep this edge and add one more edge $(s, u)$.
  – We cut one edge whether $u \in S$ or not.

Note that for each $(u, v) \in E(G_1)$, we cut exactly one edge in $G_2$ if (1) it is non-blue and cut by $S$, or (2) it is blue and not completely contained in $S$. This is exactly the objective function that we want to minimize in $S_1$. Therefore, the minimum $s$-$t$ cut in $G_2$ gives the optimal solution in $G_1$, which in turn gives the optimal local improvement $C^*$ in $G$. \qed

4.3 $k$-Subset Edge Separator Parameterized by Degree

In this section, we provide a $(2 + \varepsilon)$-approximation algorithm for $k$-SUBSET EDGE SEPARATOR parameterized by $k$ and the maximum degree of the graph, proving (iii) of Theorem [6]. Throughout this section, we will fix $\varepsilon > 0$ and the maximum degree $d$ of the graph. Our algorithm has three main steps. First, we will reduce our search space of solutions to $k$-SUBSET EDGE SEPARATOR to a smaller set of canonical solutions, which behave more nicely. In particular, in each canonical solution $S \subseteq E(G)$, every connected component of $G \setminus S$ containing a vertex in $R$ has a small number of edges leaving the component. We will show that there always exists a canonical solution of size $\leq (1 + \varepsilon)OPT$. Then, we will find a 2-approximation to the best canonical solution by reducing to the UNIFORM METRIC LABELING problem, which we will define later. Formulating the UNIFORM METRIC LABELING instance requires another ingredient, the concept of important cuts, a tool popular in FPT algorithm design.

We begin with canonical solutions.

**Definition 20.** A solution $S \subseteq E(G)$ to $k$-SUBSET EDGE SEPARATOR is called $\varepsilon$-canonical if, for each connected component $C \subseteq V(G)$ of $G \setminus S$ satisfying $C \cap R \neq \emptyset$, we have $|\partial C| \leq 2k \deg(G)/\varepsilon$.

**Observation 21.** There exists a $\varepsilon$-canonical solution with size at most $(1 + \varepsilon)OPT$.

**Proof.** Consider the optimal solution $S^*$, which we modify as follows. For each component $C \subseteq V(G)$ with $|\partial C| > 2k \deg(G)/\varepsilon$, further delete all edges incident to each vertex in $R \cap C$. We delete $\leq k \deg(G)$ edges, which can be charged evenly to the boundary edges of $C$, so that each edge gets charged $\leq \varepsilon/2$. Every edge gets charged twice, so the total number of additional edges deleted is $\leq \varepsilon \cdot OPT$. It is clear that $S^*$ with these additional edges deleted is $\varepsilon$-canonical. \qed

At this point, we are looking for a solution that separates the graph into pieces with a small number of vertices in $R$ and small boundary. Our next step is to provide a “cover” for all possible such pieces, which we will use in our UNIFORM METRIC LABELING reduction. In particular, we look for a set $\mathcal{C} \subseteq 2^V$ of subsets of vertices of small size such that every piece $C \subseteq V$ that we might possibly look for satisfies $C \subseteq C'$ for some $C' \in \mathcal{C}$.

**Lemma 22.** There exists a set $\mathcal{C} \subseteq 2^V$ of subsets of vertices of size $4^{O(kM)} \cdot n$ such that (1) every subset $C \in \mathcal{C}$ satisfies $1 \leq |C \cap R| \leq k$, and (2) for every connected component $C \subseteq V$ satisfying $1 \leq |C \cap R| \leq k$ and $|\partial C| \leq M$, we have $C \subseteq C'$ for some $C' \in \mathcal{C}$.
Note that we will set \( M := 2k \text{deg}(G)/\varepsilon \) when applying Lemma \([22]\) later on. The main ingredient in the proof of Lemma \([22]\) is the concept of important cuts in a graph, along with a result bounding the number of important cuts of a bounded size.

**Definition 23** (Important cut). For vertices \( s, t \in V(G) \), an \( s \)-\( t \) cut is a subset \( X \subseteq V(G) \) of vertices such that \( s \in X \) and \( t \notin X \). An important \( s \)-\( t \) cut is an \( s \)-\( t \) cut \( X \subseteq V(G) \) with the following two additional properties:

1. The induced graph \( G[X] \) is connected.
2. There is no \( s \)-\( t \) cut \( X' \subseteq V(G) \) such that \( |\partial X'| \leq |\partial X| \) and \( X \subseteq X' \).

**Theorem 24** ([CFK+15], Theorem 8.11). For fixed vertices \( s, t \in V(G) \) and integer \( p \geq 0 \), there are at most \( 4^p \) important \( s \)-\( t \) cuts of size at most \( p \). Moreover, all of these can be enumerated in time \( O(4^p \cdot n^{O(1)}) \).

Using this theorem, we now prove Lemma \([22]\).

**Proof (Lemma \([22]\)).** Fix a vertex \( s \in R \). Our goal is to establish a set \( C_s \subseteq 2^V \) of size \( 4^{O(kM)} \) such that (1) every subset \( C \in C_s \) satisfies \( 1 \leq |C \cap R| \leq k \), and (2) for every connected component \( C \subseteq V \) satisfying \( s \in C \), \( 1 \leq |C \cap R| \leq k \), and \( |\partial C| \leq M \), we have \( C \subseteq C' \) for some \( C' \in C_s \). Then, we can take \( C := \bigcup_{s \in R} C_s \) of size \( 4^{O(kM)} \cdot n \), which satisfies the lemma.

Consider the following construction. Take the graph \( G \), add a new vertex \( t \), and for each vertex \( v \in R \setminus \{s\} \), add \( M+1 \) parallel edges connecting \( v \) and \( t \); call the new graph \( H \). Apply Theorem \([24]\) on \( H, s, t \) with \( p := (k-1)(M+1) + M \), giving \( 4^{O(kM)} \) important \( s \)-\( t \) cuts. Let \( C_s \) be these cuts; we now show that this set works. For every connected component \( C \subseteq V \), we have

\[
|\partial_H C| = |\partial_G C| + |C \cap (R \setminus \{s\})| \cdot (M+1).
\]

If \( C \) contains \( s \) and satisfies \( |C \cap R| \leq k \) and \( |\partial_G C| \leq M \), then \( |\partial_H C| \leq M + (k-1)(M+1) \leq p \). Therefore, either \( C \in C_s \), or there is an important cut \( X \in C_s \) such that \( |\partial_H X| \leq |\partial_H C| \) and \( C \subseteq X \). In the latter case, \( X \) cannot contain \( \geq k \) vertices in \( R \setminus \{s\} \), since that would mean \( |\partial_H X| \geq |X \cap (R \setminus \{s\})| \cdot (M+1) \geq k(M+1) > p \geq |\partial_H C| \), contradicting the assumption that \( |\partial_H X| \leq |\partial_H C| \). Therefore, \( X \) contains \( \leq k \) vertices in \( R \), including \( s \). Finally, we have

\[
|\partial_G X| = |\partial_H X| - |X \cap (R \setminus \{s\})| \cdot (M+1) \leq |\partial_H C| - |C \cap (R \setminus \{s\})| \cdot (M+1) = |\partial_G C|.
\]

Hence, the subset \( X \in C_s \) satisfies the conditions of the lemma for \( C \).

We invoke Lemma \([22]\) with \( M := 2k \text{deg}(G)/\varepsilon \) and compute the corresponding set \( C \). The last step in the algorithm is to reduce the problem to an instance of Uniform Metric Labeling.

**Definition 25** (Uniform Metric Labeling). Given a graph \( G = (V,E) \), a set of labels \( L \), and cost matrix \( A \in \mathbb{R}^{V \times L} \) where entry \( A_{v,\ell} \) is the cost of labeling vertex \( v \in V(G) \) with label \( \ell \in L \), the **Uniform Metric Labeling** problem is to label each vertex in \( V(G) \) with exactly one label in \( L \) that minimizes

\[
\sum_{v \in V} A_{v,l(v)} + \sum_{(u,v) \in E} 1_{l(u) \neq l(v)}
\]

where \( l(v) \) is the label of vertex \( v \) and \( 1_{l(u) \neq l(v)} \) equals 1 if the labels of \( u \) and \( v \) are different, and 0 otherwise.
**Theorem 26 ([KT02]).** There is a polynomial-time 2-approximation algorithm for Uniform Metric Labeling.

We reduce to Uniform Metric Labeling as follows: the labels are the subsets in $C$ along with a dummy label, called $\bot$. For each vertex $v \in V(G)$ and label $C \in C$, the cost $A_{v,C}$ is 0 if $v \in C$, and $\infty$ otherwise. That is, we do not allow a vertex to be labeled by a subset that does contain that vertex. For label $\bot$, the cost is $A_{v,\bot} = 0$ if $v \notin R$, and $\infty$ otherwise. That is, we do not allow a vertex in $R$ to be labeled $\bot$. Observe that this Uniform Metric Labeling instance has size $4^{O(\log n)} = 2^{O(k^2 \deg(G)/\varepsilon) n^{O(1)}}$.

It is clear that any solution to this Uniform Metric Labeling instance is a valid solution to $k$-Subset Edge Separator: the components containing vertices in $R$ are precisely the maximal connected components of the same label, and each such component must have $\leq k$ vertices in $R$. Moreover, the best $\varepsilon$-canonical solution $S$ for $k$-Subset Edge Separator can be transformed into a solution for Uniform Metric Labeling with the same solution value as follows: for each connected component $C \subseteq V$ in $G \setminus S$ with a vertex in $R$, take a set $C' \subseteq C$ with $C \subseteq C'$ and color all vertices in $C$ with label $C'$; for connected components without a vertex in $R$, label all their vertices $\bot$. Thus, we can compute a 2-approximation to Uniform Metric Labeling and obtain a solution within factor 2 of the best $\varepsilon$-canonical solution, or within factor $2(1 + \varepsilon)$ of the optimum. Of course, we can make the approximation factor $2 + \varepsilon$ by resetting $\varepsilon \leftarrow \varepsilon/2$.

### 4.4 $\mathcal{H}$-Edge Deletion

Our main theorem for $\mathcal{H}$-Edge Deletion is the following. Unlike the vertex deletion, our algorithm uses $\deg(G)$ as an extra parameter, and it is an interesting open problem whether we can remove this dependence.

**Theorem 5.** Let $\mathcal{H}$ be a class of graphs closed under taking a subgraph. Suppose graphs in $\mathcal{H}$ have treewidth bounded by $t - 1$, and $\mathcal{H}$-Edge Deletion admits an exact algorithm with running time $f(n, \text{OPT})$. Then there is an $(3 + \varepsilon)$-approximation for $\mathcal{H}$-Edge Deletion with running time

$$
\left(\min\left(2^{O(\log n/k \deg(G)/\varepsilon)} n^{O(t \deg(G)/\varepsilon)}\right) + f(n, \deg(G)/\varepsilon)\right) \log(n/\varepsilon).
$$

**Proof.** As for the vertex deletion version, our algorithm maintains a feasible solution and iteratively tries to improve it. Let $S^* \subseteq E(G)$ be an optimal solution to $\mathcal{H}$-Edge Deletion and let $R_E \subseteq E(G)$ be the solution. From $G$ and $R_E$, construct a graph $G'$ where we subdivide each edge $e = (u, v) \in R_E$; formally, create a new vertex $r_e$ and replace $(u, v)$ by $(u, r_e)$ and $(v, r_e)$. Let $R'_V := \{r_e : e \in R_E\} \subseteq V(G')$. From $S^*$, let $S' \subseteq E(G')$ be such that for each edge $e = (u, v) \in S^*$,

- If $e \notin R_E$, put $e$ to $S'$.
- If $e \in R_E$, arbitrarily choose one endpoint (say $u$) and put $(u, r_e)$ to $S'$.

By construction, $|S'| = |S^*|$ and $|R'_V| = |R_E|$.

Note that $G' \setminus S'$ can be obtained from $G \setminus S^*$ by subdividing edges (when $e \in R_E \setminus S^*$) and add degree one vertices (when $e = (u, v) \in R_E \cap S^*$, $G' \setminus S'$ additionally has $(v, r_e)$ compared to $G \setminus S^*$). Both operations do not increase the treewidth, so the fact that graph $G \setminus S^*$ has its treewidth bounded by $t$ implies that $G' \setminus S'$ has its treewidth bounded by $t$. By Lemma 10 with $\delta = t \cdot \beta$ ($\beta$ to be chosen later) guarantees that there exists a set $X_V' \subseteq V(G')$, $|X_V'| \leq \frac{|R'_V|}{\beta}$, so that each connected component in $(G' \setminus S') \setminus X_V'$ has at most $t \cdot \beta$ vertices from $R'_V$. Since subdividing edges does not
increase the maximum degree, \( \deg(G') \leq \deg(G) \). Let \( X'_E \subseteq E(G') \) be the set of edges incident on \( X'_V \). Then \(|X'_E| \leq \frac{\deg(G)|R'_E|}{\beta} \leq \frac{\deg(G)|R_E|}{\beta} \) so that each connected component in \( G' \setminus (S' \cup X'_E) \) has at most \( t \cdot \beta \) vertices from \( R'_V \).

We launch the \((2 + \varepsilon)\) approximation for \( k\)-\textsc{Subset Edge Separator} with \( k = t \cdot \beta \) on \( G' \) and \( R'_V \). It returns a set \( Y' \subseteq E(G') \) of size at most

\[
(2 + \varepsilon) \cdot |S' \cup X'_E| \leq (2 + \varepsilon)(\OPT + \frac{\deg(G)|R_E|}{\beta})
\]

with each connected component of \( G' \setminus Y' \) having at most \( t \cdot \beta \) vertices from \( R'_V \). Let \( Y \) be \( Y' \) projected back to \( G \); formally, \( Y := \{e = (u, v) \in E(G) : e \in Y' \text{ or } (u, r_e) \in Y' \text{ or } (v, r_e) \in Y'\} \). Since \( G' \setminus Y' \) has at most \( t \cdot \beta \) vertices from \( R'_V \), \( G \setminus Y \) has at most \( t \cdot \beta \) edges from \( R_E \).

Since \( \mathcal{H} \) is hereditary, \( R_E \cup Y \) is a valid solution and we have a bound \( t \cdot \beta \) on the solution size for each connected component. We thus can solve \( \mathcal{H}\)-\textsc{Edge Deletion} on each component \( C \subseteq G \setminus Y \) in time \( f(n, t \cdot \beta) \). We know that \( C \cap S^* \) is a feasible solution for each \( C \), so the sum of solution sizes is bounded by \(|S^*| = \OPT \). Therefore, we obtain a feasible solution of size

\[
(2 + \varepsilon) \cdot \left( \OPT + \frac{\deg(G)|R_E|}{\beta} \right) + \OPT.
\]

Let \( \beta = \deg(G)/\varepsilon \), so that the above quantity becomes

\[
(3 + \varepsilon)\OPT + \varepsilon(2 + \varepsilon)|R_E|.
\]

This becomes better than \((1 - \varepsilon)|R_E|\) when

\[
|R_E|(1 - \varepsilon(3 + \varepsilon)) \geq (3 + \varepsilon)\OPT \iff |R_E| \geq \frac{(3 + \varepsilon)}{(1 - \varepsilon(3 + \varepsilon))}\OPT = (3 + O(\varepsilon))\OPT.
\]

Therefore, if we repeat this iteration until there is no improvement by a factor of \((1 - \varepsilon)\), the final solution is guaranteed to be within \((3 + O(\varepsilon))\OPT \). The running time is

\[
\min(2^{O(k^2 \deg(G)/\varepsilon)} n^{O(1)}, n^{k+O(1)}) = \min(2^{O(k^2 \deg(G)/\varepsilon)} n^{O(1)}, n^{O(t \deg(G)/\varepsilon)})
\]

for \( k\)-\textsc{Subset Edge Separator} with \( k = t \beta = t \deg(G)/\varepsilon \) plus \( f(t \deg(G)/\varepsilon)n^{O(1)} \) for the final step. There can be at most \( \log(n/\varepsilon) \) iterations. \( \square \)

### 4.5 Applications for Bounded Degree Graphs

We prove the corollaries of Theorem [5] introduced in Section [1]

**Corollary 7.** If \( \mathcal{H} \) is minor-closed and with treewidth bounded by \( t \), \( \mathcal{H}\)-\textsc{Edge Deletion} admits a \((3 + \varepsilon)\)-approximation algorithm that runs in time \( f(t \deg(G), \varepsilon) \cdot n^{O(1)} \) for some function \( f \). In particular, \( \textsc{Planar-} \mathcal{F} \) \textsc{Edge Deletion} admits a \((3 + \varepsilon)\)-approximation algorithm with running time \( f(\mathcal{F}, \deg(G), \varepsilon) \cdot n^{O(1)} \).

**Proof.** For \( \textsc{Planar-} \mathcal{F} \) \textsc{Edge Deletion}, we use the Polynomial Grid Minor theorem [CC16] which says that if \( G \) does not have a planar graph \( F \) as a minor, the treewidth of \( G \) is bounded by \(|V(F)|^{O(1)} \). \( \textsc{Planar-} \mathcal{F} \) \textsc{Edge Deletion} admits a linear-time exact algorithm parameterized by the solution size [Con90] so the assumptions of Theorem [5] are satisfied.
Due to the result of Robertson and Seymour [RS04], every minor-closed class can be represented as $F$-minor-free graphs for some finite family $F$. If the treewidth in $\mathcal{H}$ is additionally bounded, then at least one of graphs in $F$ must be planar. Therefore $\mathcal{H}$-Edge Deletion reduces to Planar-$F$ Edge Deletion.

We also present implications of Theorem 5 to the Noisy Planar $k$-SAT($\delta$) problem studied by Bansal et al. [BRU17]. For a fixed $k = O(1)$, an instance of Noisy Planar $k$-SAT($\delta$) is an instance of $\phi$ of $k$-SAT where the factor graph $H$ of $\phi$ is almost planar.

Formally, given a $k$-CNF formula $\phi$ with $n$ variables and $m$ clauses, the factor graph of $\phi$ is a bipartite graph $H = (A, B)$ where $A$ contains a vertex for every variable appearing in $\phi$, $B$ contains a vertex for every clause appearing in $\phi$, and a clause-variable $x$ is connected to a variable-vertex $x$ if and only if $x$ belongs to $C$. As an instance of Noisy Planar $k$-SAT($\delta$), the factor graph of $\phi$ is promised to be a planar graph with $\delta m$ additional edges for some $\delta > 0$.

Bansal et al. [BRU17] proved that for any $\varepsilon > 0$, there is an algorithm that achieves $(1 + O(\varepsilon + \delta \log m \log \log m))$-approximation in time $m^{O((\log \log m)/\varepsilon)}$. We prove that if the degree of the factor graph is bounded, we can obtain an improved algorithm. Note that $k$-SAT with the maximum degree $O(1)$ has been actively studied and proved to be APX-hard (e.g., 3-SAT(5)) for general factor graphs.

**Corollary 8.** For any $\varepsilon > 0$, there is an $(1 + O(\varepsilon + \delta))$-approximation algorithm for Noisy Planar $k$-SAT($\delta$) that runs in time $f(\varepsilon, \deg(\phi)) \cdot m^{O(1)}$ for some function $f$, where $\deg(\phi)$ indicates the maximum degree of the factor graph of $\phi$.

**Proof.** Recall that we treat $k$ as an absolute constant. Given an instance $\phi$ of Noisy Planar $k$-SAT($\delta$), the factor graph $H$ of $\phi$ has $\Theta(m)$ vertices. Deleting $O(\delta m)$ edges from $H$ will make $H$ planar, and additionally deleting $O(\varepsilon m)$ edges will make its treewidth bounded by $O(1/\varepsilon)$.

We apply Corollary 7 to delete $O((\varepsilon + \delta)m)$ edges of $H$ to reduce its treewidth to $O(1/\varepsilon)$. Its running time is $f(\varepsilon, \deg(H)) \cdot m^{O(1)}$. Delete all clauses that lost at least one of the incident edges. We deleted $O((\varepsilon + \delta)m)$ clauses. Now that the treewidth is bounded by $O(1/\varepsilon)$, apply an exact algorithm for $k$-SAT that runs in time $2^{O(1/\varepsilon)} \cdot m^{O(1)}$ [KM96].

### 4.6 Inapproximability of $k$-Edge Separator

We end this section by proving the inapproximability of $k$-EdgeSeparator, as stated below.

**Theorem 9.** There exists a constant $c > 1$ such that $k$-EdgeSeparator is NP-hard to approximate within a factor of $c$ even when $k = 3$ and $\deg(G) = 4$. Consequently, when $\mathcal{F}$ is the set of all graphs with three vertices, deleting the minimum number of edges to exclude $\mathcal{F}$ as a subgraph, minor, or immersion is APX-hard for bounded degree graphs.

In fact, we will prove an NP-hardness of approximation for a slightly different partitioning problem, which is sometimes referred to as Partitioning into Triangles (PIT). In PIT, we are given a graph $G = (V, E)$ and the goal is to find the largest collection of disjoint triangles, i.e., disjoint subsets of vertices $S_1, \ldots, S_k \subseteq V$ such that each $S_i$ is of size three and induces a 3-clique. We will show the following hardness of approximation for PIT:

**Lemma 27.** There exists $\varepsilon > 0$ such that it is NP-hard, given a graph $G = (V, E)$ with $\deg(G) = 4$, to distinguish between the following two cases, where $n$ denotes $|V|$:

- (Completeness) The vertex set $V$ can be partitioned into $n/3$ disjoint triangles.
• (Soundness) Every collection of disjoint triangles has size less than \((1 - \varepsilon)n/3\).

PIT is a classic NP-complete problem (see [GJ79]), and it should be remarked that Kann [Kan91] already showed that the problem is Max SNP-hard when \(\deg(G) = 6\); indeed, this already suffices for proving Theorem 9 if we relax the degree requirement to \(\deg(G) = 6\). Nevertheless, even if we want \(\deg(G) = 4\), the proof is still simple, and the reduction is in fact exactly the same as that of van Rooij et al. [vRvKNB13], who showed the NP-hardness of (the exact version of) PIT when \(\deg(G) = 4\). The authors of [vRvKNB13] also showed that PIT becomes polynomial time solvable when \(\deg(G) \leq 3\) and, hence, the degree requirement cannot be improved in Lemma 27. Nevertheless, we are not aware of either an efficient algorithm or hardness result for the \(\deg(G) = 3\) case for \(k\)-Edge Separator, and we leave that as an open question.

Before we prove Lemma 27, let us first state how it implies Theorem 9.

Proof of Theorem 9. The reduction is trivial: we keep the input \(G\) to PIT as it is, and set \(k = 3\). Moreover, let \(c = 1 + \varepsilon/4\) where \(\varepsilon\) is the constant from Lemma 27.

(Completeness) Suppose that there exists a partition of \(V\) into \(n/3\) triangles \(S_1, \ldots, S_{n/3}\). There are \(n\) uncut edges with respect to this partition, and hence it cuts exactly \(|E| - n\) edges.

(Soundness) Suppose that every collection of disjoint triangles has size less than \((1 - \varepsilon)n/3\). Consider any partition of \(V\) into disjoint subsets \(T_1, \ldots, T_k\), each of size at most three. Our assumption implies that less than \((1 - \varepsilon)n\) vertices are adjacent to two uncut edges. Hence, the total number of uncut edges is less than \((1 - \varepsilon)n + \varepsilon n/2 = (1 - \varepsilon/2)n\), and the number of cut edges is more than \(|E| - (1 - \varepsilon/2)n\).

The ratio between the two cases is more than

\[
\frac{|E| - n}{|E| - (1 - \varepsilon/2)n} = 1 + \frac{\varepsilon n/2}{|E| - (1 - \varepsilon/2)n} \geq 1 + \frac{\varepsilon n/2}{2n} = c,
\]

where the inequality comes from \(\deg(G) \leq 4\). This concludes our proof.

We now turn our attention back to the proof of Lemma 27. As stated earlier, we exactly follow the reduction of van Rooij et al. [vRvKNB13]. They reduce from \(\text{Max 1-in-3SAT}\) problem, in which we are given a 3CNF formula and the goal is to find an assignment that assigns exactly one literal to be true in each clause. Since we want to prove hardness of approximation, we will need hardness of approximation of \(\text{Max 1-in-3SAT}\), which is well-known and can be stated as follows.

Lemma 28. There exists \(\delta > 0\) such that it is NP-hard, given a 3CNF formula such that each variable appears in at most \(d = O(1)\) clauses, to distinguish between the following two cases:

• (Completeness) There is an assignment such that exactly one literal in each clause is true.

• (Soundness) Every assignment satisfies less than \((1 - \delta)\) fraction of clauses.

We will also need the gadgets from [vRvKNB13], which can be summarized as follows. Since this is exactly the same as those used in [vRvKNB13], we do not provide full constructions of them.

\(^5\)The result stated in Lemma 28 is folklore, although we are not aware of it being stated in this form before. However, it is quite easy to see that it is true, as follows. First, recall that Max-3SAT is NP-hard to approximate even on bounded degree instances [Has00, Tre01]. Then, observe that we can use the reduction of Schaefer [Sch78] from 3SAT to 1-in-3SAT, which is approximation-preserving and also preserves boundedness of the degrees.

\(^6\)For the purpose of our proof, each clause in a 3CNF formula contains exactly three literals.
Figure 4.1: Illustration of a fan and a cloud. A fan is depicted in Figure 4.1a. A (4, 1)-cloud is depicted in Figure 4.1b; here the true vertices are marked by “T”, the false vertex by “F” and the inner vertices by “I”. The dashed triangle corresponds to the true collection (as defined in Definition 30), whereas the remaining two triangles correspond to the false collection.

here; we refer the readers to Lemma 8 of [vRvKNB13] for more details. Illustrations of the gadgets can be founded in Figure 4.1, which is reconstructed (with slight modifications) from Figure 5 of [vRvKNB13].

Definition 29. A fan is a graph of five vertices $O_1, O_2, O_3, I_1, I_2$ and seven edges: $\{I_1, I_2\}$ and $\{I_i, O_j\}$ for all $i \in [2]$ and $j \in [3]$. In other words, it is a union of three triangles having one edge $\{I_1, I_2\}$ in common. We call $O_1, O_2, O_3$ outer vertices and $I_1, I_2$ inner vertices of the fan.

Definition 30. For $a, b \in \mathbb{N}$, an $(a, b)$-cloud is a graph of $2(a + b) - 3$ vertices that satisfies the following properties:

- The vertices can be divided into three groups: $a$ true vertices, $b$ false vertices and $(a + b) - 3$ inner vertices.
- Each true vertex and each false vertex has degree two.
- There are only two different collections of disjoint triangles that contain all inner vertices. In one collection, every false vertex is included but none of the true vertices are included; we call this collection the true collection. In the other collection, every true vertex is included but none of the false vertices are included; we call this collection the false collection.

Lemma 31 ([vRvKNB13]). For every $a, b \in \mathbb{N}$ such that $a \equiv b \mod 3$, an $(a, b)$-cloud exists.

We are now ready to prove Lemma 27.

Proof of Lemma 27. As stated earlier, we follow the reduction of van Rooij et al. [vRvKNB13] from Max 1-in-3SAT to PIT, although we will have to be slightly more careful in the analysis, as we want to not only prove hardness for the exact version but also the approximate version of the problem.

Van Rooij et al.’s reduction can be described as follows:

- First, notice that we can assume without loss of generality that the number of occurrences of each literal is divisible by three; this can be easily ensure by duplicating all the clauses twice.
- For each variable $x$, let $a(x)$ be the number of occurrences of the literal $x$ and $b(x)$ be the number of occurrences of the literal $\neg x$. We create an $(a(x), b(x))$-cloud for each variable $x$. 

21
• We create a fan for each clause $C$. For each literal in the clause, we identify one outer vertex of the fan to a vertex corresponding to that literal in the cloud of the variable. Note that, for each variable $x$, since there are $a(x)$ and $b(x)$ vertices corresponding to $x$ and $\neg x$ respectively, the identification can be done in such a way that each literal vertex is identified with exactly one vertex from a clause cloud, which also ensures that the graph has maximum degree four.

Finally, let $\varepsilon = \delta/(8d + 8)$. Moreover, let us use $N$ and $M$ to denote the number of variables and the number of clauses of the 3CNF formula respectively, and $n$ to denote the number of vertices of the resulting graph. Notice that, from the bounded degree assumption, we have $M \leq dN/3$. Moreover, from the sizes of each gadgets, we have $n \leq 2M + \sum_x 2(a(x) + b(x)) = 8M$.

(Completeness) Suppose that there exists an assignment $\phi$ such that each clause contains exactly one true literal. Then, we can define our partition as follows. For each variable $x$, we pick the true or false collection for the $x$-cloud based on the value $\phi(x)$. For each clause, we pick the triangle with two inner vertices and the outer vertex corresponding to the true literal. It is clear that this is indeed a partition of vertices into disjoint triangle as desired.

(Soundness) We will show the contrapositive; suppose that there exists $k \geq (1 - \varepsilon)n/3$ disjoint triangles $S_1, \ldots, S_k$. We call a variable $x$ good if the triangles restricted to only those entirely contained in the $x$-cloud is either the true collection or the false collection. Notice that, if each inner vertex of $x$-cloud is in at least one of the selected triangles, then $x$-cloud must be good, since the inner vertices of the $x$-cloud are not adjacent to any vertices outside of the cloud. However, there are at most $\varepsilon n$ vertices outside of the disjoint triangles, meaning that at most $\varepsilon n \leq 8\varepsilon M$ variables are not good.

Next, we call a clause $C$ good if (1) the three variables whose literals are in $C$ are good and (2) the inner vertices of $C$-fan are in at least one of the selected triangles. Notice that there are at most $d(8\varepsilon M) = 8\varepsilon dM$ clauses that violate (1). More, again, since there are at most $\varepsilon n \leq 8\varepsilon M$ vertices outside of the union of the triangles, at most $8\varepsilon M$ clauses violate (2). Hence, all but at most $8\varepsilon (d + 1)M \leq \delta M$ clauses are good.

We will define an assignment $\phi$ as follows. For each good $x$, we define $\phi(x)$ to be true if the triangles correspond to the true collection of the $x$-cloud, and we let $\phi(x)$ be false otherwise. For the remaining $x$'s, we assign $\phi(x)$ arbitrarily. It is easy to see that $\phi$ satisfies all the good clauses; this is simply because exactly one literal in each good clause $C$, the one whose triangle with the two inner vertices in the $C$-fan is selected, is set to true. Hence, $\phi$ satisfies all but $\delta M$ clauses, which concludes our proof.

References

[ACMM05] Amit Agarwal, Moses Charikar, Konstantin Makarychev, and Yury Makarychev. $O(\sqrt{\log n})$ approximation algorithms for min uncut, min 2cnf deletion, and directed cut problems. In Proceedings of the thirty-seventh annual ACM symposium on Theory of computing, pages 573–581. ACM, 2005.

[ALM+17] Akanksha Agrawal, Daniel Lokshtanov, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. Feedback vertex set inspired kernel for chordal vertex deletion. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1383–1398. Society for Industrial and Applied Mathematics, 2017.

[ALM+18] Akanksha Agrawal, Daniel Lokshtanov, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. Polylogarithmic approximation algorithms for weighted-\textit{F}-deletion problems. APPROX ’18, 2018. To appear.
Haris Angelidakis, Yury Makarychev, and Pasin Manurangsi. An improved integrality gap for the călinescu-karloff-rabani relaxation for multiway cut. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 39–50. Springer, 2017.

Vineet Bafna, Piotr Berman, and Toshihiro Fujito. A 2-approximation algorithm for the undirected feedback vertex set problem. *SIAM Journal on Discrete Mathematics*, 12(3):289–297, 1999.

Édouard Bonnet, Nick Brettell, O-joung Kwon, and Dániel Marx. Parameterized vertex deletion problems for hereditary graph classes with a block property. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 233–244. Springer, 2016.

Anudhyan Boral, Marek Cygan, Tomasz Kociumaka, and Marcin Pilipczuk. A fast branching algorithm for cluster vertex deletion. *Theory of Computing Systems*, 58(2):357–376, 2016.

Ivan Bliznets, Marek Cygan, Paweł Komosa, and Michal Pilipczuk. Hardness of approximation for ℋ-free edge modification problems. *ACM Transactions on Computation Theory (TOCT)*, 10(2):9, 2018.

Ann Becker and Dan Geiger. Optimization of pearl’s method of conditioning and greedy-like approximation algorithms for the vertex feedback set problem. *Artificial Intelligence*, 83(1):167–188, 1996.

Hans Bodlaender, Pal Gronas Drange, Markus S. Dregi, Fedor V. Fomin, Daniel Lokshtanov, and Michal Pilipczuk. A $O(c^k n)$ 5-approximation algorithm for treewidth. 45, 04 2013.

Hans L. Bodlaender and Ton Kloks. Better algorithms for the pathwidth and treewidth of graphs. In Javier Leach Albert, Burkhard Monien, and Mario Rodríguez Artalejo, editors, *Automata, Languages and Programming*, pages 544–555, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.

Hans L. Bodlaender. Treewidth: Algorithmic techniques and results. In Igor Prívara and Peter Ružička, editors, *Mathematical Foundations of Computer Science 1997*, pages 19–36, Berlin, Heidelberg, 1997. Springer Berlin Heidelberg.

Nikhil Bansal, Daniel Reichman, and Seeun William Umboh. Lp-based robust algorithms for noisy minor-free and bounded treewidth graphs. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1964–1979. Society for Industrial and Applied Mathematics, 2017.

Niv Buchbinder, Roy Schwartz, and Baruch Weizman. Simplex transformations and the multiway cut problem. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2400–2410. Society for Industrial and Applied Mathematics, 2017.

Chandra Chekuri and Julia Chuzhoy. Polynomial bounds for the grid-minor theorem. *Journal of the ACM (JACM)*, 63(5):40, 2016.

Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized algorithms*, volume 3. Springer, 2015.

Fabián A Chudak, Michel X Goemans, Dorit S Hochbaum, and David P Williamson. A primal–dual interpretation of two 2-approximation algorithms for the feedback vertex set problem in undirected graphs. *Operations Research Letters*, 22(4-5):111–118, 1998.

Jianer Chen, Yang Liu, and Songjian Lu. An improved parameterized algorithm for the minimum node multiway cut problem. *Algorithmica*, 55(1):1–13, 2009.

Yixin Cao and Dániel Marx. Interval deletion is fixed-parameter tractable. *ACM Transactions on Algorithms (TALG)*, 11(3):21, 2015.
Bruno Courcelle. The monadic second-order logic of graphs. i. recognizable sets of finite graphs. Information and computation, 85(1):12–75, 1990.

Pål Grønås Drange, Markus Dregi, and Pim van't Hof. On the computational complexity of vertex integrity and component order connectivity. Algorithmica, 76(4):1181–1202, 2016.

Erik D Demaine, Fedor V Fomin, MohammadTaghi Hajiaghayi, and Dimitrios M Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and h-minor-free graphs. Journal of the ACM (JACM), 52(6):866–893, 2005.

Erik D Demaine and MohammadTaghi Hajiaghayi. Bidimensionality: new connections between fpt algorithms and ptass. In Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, pages 590–601. Society for Industrial and Applied Mathematics, 2005.

Erik D Demaine and MohammadTaghi Hajiaghayi. The bidimensionality theory and its algorithmic applications. The Computer Journal, 51(3):292–302, 2007.

Eduard Eiben, Robert Ganian, and O-joung Kwon. A Single-Exponential Fixed-Parameter Algorithm for Distance-Hereditary Vertex Deletion. In Piotr Faliszewski, Anca Muscholl, and Rolf Niedermeier, editors, 41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016), volume 58 of Leibniz International Proceedings in Informatics (LIPIcs), pages 34:1–34:14, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

G. Even, J. Naor, S. Rao, and B. Schieber. Divide-and-conquer approximation algorithms via spreading metrics. Journal of the ACM, 47(4):585–616, 2000.

Uriel Feige, MohammadTaghi Hajiaghayi, and James R Lee. Improved approximation algorithms for minimum weight vertex separators. SIAM Journal on Computing, 38(2):629–657, 2008.

Samuel Fiorini, Gwenaël Joret, and Ugo Pietropaoli. Hitting diamonds and growing cacti. In International Conference on Integer Programming and Combinatorial Optimization, pages 191–204. Springer, 2010.

Uriel Feige and Shimon Kogan. The hardness of approximating hereditary properties. Technical Report, 2005.

Fedor V. Fomin, Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. Planar f-deletion: Approximation, kernelization and optimal fpt algorithms. In Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science, FOCS ’12, pages 470–479, Washington, DC, USA, 2012. IEEE Computer Society.

Fedor V. Fomin, Daniel Lokshtanov, Venkatesh Raman, and Saket Saurabh. Bidimensionality and eptas. In Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’11, pages 748–759, Philadelphia, PA, USA, 2011. Society for Industrial and Applied Mathematics.

Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Sebastian Ordyniak, Felix Reidl, Peter Rossmanith, Fernando Sánchez Villaamil, and Somnath Sikdar. Kernelization using structural parameters on sparse graph classes. In Hans L. Bodlaender and Giuseppe F. Italiano, editors, Algorithms – ESA 2013, pages 529–540, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.

M. R. Garey and David S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

Archontia C Giannopoulou, Bart MP Jansen, Daniel Lokshtanov, and Saket Saurabh. Uniform kernelization complexity of hitting forbidden minors. ACM Transactions on Algorithms (TALG), 13(3):35, 2017.

Venkatesan Guruswami and Euiwoong Lee. Inapproximability of H-transversal/packing. Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, page 284, 2015.
Archontia C. Giannopoulou, Daniel Lokshtanov, Saket Saurabh, and Ondrej Suchý. Tree deletion set has a polynomial kernel but no opt\(^{O(1)}\) approximation. *SIAM Journal on Discrete Mathematics*, 30(3):1371–1384, 2016.

Archontia C. Giannopoulou, Michal Pilipczuk, Jean-Florent Raymond, Dimitrios M. Thilikos, and Marcin Wrochna. Linear Kernels for Edge Deletion Problems to Immersion-Closed Graph Classes. In Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl, editors, *44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)*, volume 80 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 57:1–57:15, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Multiway cuts in node weighted graphs. *Journal of Algorithms*, 50(1):49–61, 2004.

Johan Håstad. On bounded occurrence constraint satisfaction. *Inf. Process. Lett.*, 74(1-2):1–6, 2000.

Pinar Heggernes, Pim Vant Hof, Bart MP Jansen, Stefan Kratsch, and Yngve Villanger. Parameterized complexity of vertex deletion into perfect graph classes. In *International Symposium on Fundamentals of Computation Theory*, pages 240–251. Springer, 2011.

Bart MP Jansen and Marcin Pilipczuk. Approximation and kernelization for chordal vertex deletion. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1399–1418. Society for Industrial and Applied Mathematics, 2017.

Viggo Kann. Maximum bounded 3-dimensional matching is MAX SNP-complete. *Inf. Process. Lett.*, 37(1):27–35, 1991.

Eun Jung Kim and O-joung Kwon. Erdős-Pósa property of chordless cycles and its applications. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1665–1684. SIAM, 2018.

Eun Jung Kim, Alexander Langer, Christophe Paul, Felix Reidl, Peter Rossmanith, Ignasi Sau, and Somnath Sikdar. Linear kernels and single-exponential algorithms via protrusion decompositions. *ACM Transactions on Algorithms (TALG)*, 12(2):21, 2016.

Sanjeev Khanna and Rajeev Motwani. Towards a syntactic characterization of PTAS. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 329–337. ACM, 1996.

Ken-ichi Kawarabayashi and Anastasios Sidiropoulos. Polylogarithmic approximation for minimum planarization (almost). In *Proceedings of the 2017 IEEE 53rd Annual Symposium on Foundations of Computer Science*, FOCS ’17, pages 779–788, 2017.

Jon Kleinberg and Eva Tardos. Approximation algorithms for classification problems with pairwise relationships: Metric labeling and markov random fields. *Journal of the ACM (JACM)*, 49(5):616–639, 2002.

Euiwoong Lee. Partitioning a graph into small pieces with applications to path transversal. *Mathematical Programming*, 2018. Preliminary version in SODA 2017.

Carsten Lund and Mihalis Yannakakis. The approximation of maximum subgraph problems. In *International Colloquium on Automata, Languages, and Programming*, pages 40–51. Springer, 1993.

Dániel Marx. Parameterized graph separation problems. *Theoretical Computer Science*, 351(3):394–406, 2006.

Dániel Marx and Igor Razgon. Fixed-parameter tractability of multicut parameterized by the size of the cutset. *SIAM Journal on Computing*, 43(2):355–388, 2014.
A. Details of parameterized width-reduction algorithms

In this section we give a more detailed proof sketch of Lemma 13. Indeed, we show a more general claim that the algorithms from [BK91] and [RRVS14] can be extended to finding a tree-decomposition (or path / treewidth-decomposition) of small width $k$ given a decomposition of a larger width $t$ can be extended to handle vertex deletion ($t = O(k + p)$ in Lemma 13). The reasoning presented here is a proof sketch and we focus on explaining why the arguments from previous works remain valid in the extended versions of the algorithms. Since the main claim of [BK91] is just a linear running time for fixed $k$ and $t$, we need to explicitly bound the number of states in the dynamic programming routines.

A.1 Original routines for finding small tree or path decompositions

Definition 32. For an integer sequence $a$, its typical sequence $\tau(a)$ is obtained by iterating the following operations until none is possible anymore:

1. removal of repetitions of consecutive elements,
2. removal of a subsequence $a_{i+1}, \ldots, a_j$ satisfying $\forall i < k < j . a_i \leq a_k \leq a_j$ or $\forall i < k < j . a_j \leq a_k \leq a_i$.

The sequence $\tau(a)$ is uniquely defined [BK91, Lemma 3.2].

Lemma 33 (Lemma 3.3 and 3.5 in [BK91]). There are $O(4^k)$ typical sequences of integers in $[0,k]$. The length of each one is at most $2k + 1$. 

26
Definition 34. For integer sequences $a, b$ we write $a < b$ if one can extend them to sequences $a', b'$ of equal length by adding consecutive repetitions, such that $a' \leq b'$ on each index.

The relation $<$ is transitive, and $a < b$ holds if and only if $\tau(a) < \tau(b)$ [BK91, Lemma 3.7 and 3.10].

All algorithms in question work on a given tree decomposition of width at most $t$. We can assume that this is a nice tree decomposition, i.e., a binary tree $T$, in which every node $x$ is assigned a bag $B_x \subseteq V(G)$ and belongs to one of the following types:

1. **start:** the node is a leaf of $T$ with only one vertex in $B_x$,
2. **join:** the node has two children $y, z$ satisfying $B_x = B_y = B_z$,
3. **introduce:** the node has one child $y$ and $B_x$ is formed by adding one vertex to $B_y$,
4. **forget:** the node has one child $y$ and $B_x$ is formed by removing one vertex from $B_y$.

We define $T_x$ be the subtree of $T$ rooted at $x$, and $G_x$ to be a subgraph of $G$ induced by vertices introduced in $T_x$. A partial tree (path) decomposition is a decomposition of a subgraph $H$ of $G_x$ of width at most $k$.

Definition 35. For a partial path decomposition $Y = (Y_1, Y_2, \ldots, Y_m)$ we define its restriction with respect to the set $B_x$ to be $Z = (Y_1 \cap B_x, Y_2 \cap B_x, \ldots, Y_m \cap B_x)$. Let $1 = t_1 < t_2 < \cdots < t_q$ be all indices for which $Z_{t_{i-1}} \neq Z_{t_i}$ and let sequence $a^i$ indicate $([Y_{t_i}], [Y_{t_{i+1}}], \ldots, [Y_{t_{i+1}-1}])$. The characteristic of $Y$ is given by the sequence $(Z_t^q)_{i=1}^q$ called the interval model, and the list of sequences $(\tau(a^i))_{i=1}^q$.

The number of possible interval models in a node is $2^{O(t \log t)}$ and the maximal length of such a model is $2t + 3$ (Lemma 3.1 in [BK91]). Therefore the number of all possible characteristics in a node is bounded by $2^{O(t \log t) \cdot O(4^k)} = 2^{O(kt + t \log t)}$ (Lemma 4.1 in [BK91]).

Definition 36. For a partial tree decomposition $Y$ we define its restriction $Z$ with respect to the set $B_x$ again by intersecting each bag with $B_x$. A leaf of a restriction is called maximal if its bag is not contained in a bag of any other node. The trunk of such a decomposition is obtained by iteratively removing leaves that are not maximal and then replacing all nodes of degree 2 with edges. Each edge $e$ in a trunk induces a partial path decomposition of some subgraph of $G_x$—let $Z^e$ denote its interval model and $a^e$ the associated list of typical sequences. The characteristic of $Y$ is given by the trunk of $Z$, a family of interval models for each edge in the trunk, and a family of lists of typical sequences for each edge in the trunk.

Since the number of leaves in a trunk is at most $t$, the number of its nodes is $O(t)$ and the number of such trees is $2^{O(t \log t)}$. Since we need to store as many as $t$ typical sequences for each edge of the trunk, the number of all possible characteristics in a node $x$ is $2^{O(t^2k)}$.

For two characteristics of partial path decomposition we say the one majorizes another, written as $((Z_i), (a^i)) \prec ((Z'_i), (b^i))$, if $Z_i = Z'_i$ for all $i$ and $a^i \prec b^i$ for all $i$. The same notion applies to partial tree decomposition when the trunks are the same and majorization occurs on each edge of the trunk.

In the original algorithm one maintains a full set of characteristics for each node $x$ describing the minimal interface between $G_x$ and the rest of the graph. We can only store characteristics that admit a respective partial tree (path) decomposition. Moreover if there is a partial decomposition of $G_x$ with characteristic $C$, then the full set must contain a characteristic that is being majorized by $C$. This ensures that if there is a partial decomposition that can be extended to a full decomposition, then the interface contains its characteristic or a characteristic of another extendable partial decomposition.
A.2 Extension to vertex deletion

Consider the given nice tree decomposition $T$ as defined in the previous subsection. We can assume (by adding extra forget nodes) that the bag in the root is empty. We build a directed acyclic graph $T'$ with nodes given by triples $(x, X, \ell)$, where $x$ is a node from $T$, $X$ is a subset of $B_x$, and integer $\ell$ indicates how many vertices we have deleted in $G_x$.

Since $T$ is nice, each graph vertex $v \in V$ has one tree node that forgets it and possibly many nodes that introduce it. Note that the forget node for $v$ is an ancestor of all its introduction nodes, and $v$ only appears in the subtree of $T$ rooted at its forget node. Formally, $(x, X, \ell)$ stores the dynamic programming state (i.e., a set of characteristics) to compute a tree decomposition of width $k$ as in the previous subsection, where the bag containing $x$ becomes $X$ instead of $B_x$ while at most $\ell$ vertices are deleted among vertices in $G_x$ whose forget node is in the subtree rooted at $x$. Deleted vertices will be counted at its forget node.

For a start node $x$ we just add $(x, B_x, 0)$ to $T'$. If $x$ is a join node with children $y, z$ we add an edge from $(x, X, \ell)$ to $(y, X, \ell)$ and $(z, X, \ell)$ for all $X \subseteq B_x$, $0 \leq \ell \leq n$. For a node $x$ that introduces a vertex $v$ into its child $y$’s bag, we create nodes $(x, X, \ell)$ connected to $(y, X \setminus \{v\}, \ell)$ for all $v \in X \subseteq B_x$, $0 \leq \ell \leq n$, as well as dummy nodes $(x, X \setminus \{v\}, \ell)$ connected to $(y, X \setminus \{v\}, \ell)$. Dummy nodes represent an introduction operation that has been canceled. Finally for a forget node that removes a vertex $v$ from its child $y$, we have an edge from $(x, X, \ell)$ to $(y, X \cup \{v\}, \ell)$ for all $X \subseteq B_x$, $0 \leq \ell \leq n$, and to $(y, X, \ell - 1)$ for all $X \subseteq B_x$, $1 \leq \ell \leq n$. This represents branching into scenarios where $v$ will or will not be deleted.

The number of nodes in $T'$ is bounded by $V(T) \cdot 2^k \cdot n$. The following claim can be easily verified from the construction.

**Claim 37.** Fix a node $x \in T$ and consider a directed subtree $T'_x$ of our DAG such that:

1. for each $y \in T_x$, there is exactly one $y^' := (y, B^y_x, \ell^y_0) \in T'_x$,
2. for each $y$ and its child $z$ in $T_x$, there is an edge from $y^'$ to $z^'$ in $T'_x$.

Then $T'_x$ is a nice tree decomposition of $G_x \setminus S$, where $S$ is the union of $B_x \setminus B^y_x$ and some subset of size $\leq \ell$ in $G_x$. For each vertex $v \in V(G)$, $v$ is in $T'$ if and only if all $y$ with $v \in B^y_x$ satisfies $v \in B^y_x$.

Given this observation, we can fill out the dynamic programming tables for the DAG as we did for $T$. We can run the original routine on $T'$ that handles introduce and join nodes as before, only with set $B_x$ replaced by $X$. In a dummy node we just copy results from the child. For a forget node $(x, X, \ell)$, we compute the characteristics coming from branch $(y, X \cup \{v\}, \ell)$ as in the original routine and then add all characteristics copied from node $(y, X, \ell - 1)$.

**Theorem 38.** $k$-Treewidth Vertex Deletion and $k$-Pathwidth Vertex Deletion parameterized by $k$ and the width $t$ of the given tree decomposition admit exact algorithm with running times respectively $2^{O(t^2k)}n$ and $2^{O((t + t \log t))}n$.

**Proof sketch.** Sections 4.3, 4.4, and 4.5 in [BK91] describe how to compute the full set for partial path decompositions in nodes of type join, forget, introduce in time polynomial with respect to the size of characteristics’ space. Sections 5.3, 5.4, and 5.5 deal with the same cases for partial tree decompositions.

We can run these routines on $T'$ and compute the sum of characteristics’ sets when branching in forget nodes. The invariant of the full set for node $(x, X, \ell)$ becomes: each characteristic is induced
a partial decomposition of subgraph $H$ of $G_x$, such that $V(H) \cap B_x = X$ and $|V(H)| + \ell = |V(G_x)|$
and for each such partial decomposition the full set contains one being majorized by it.

The size of $T'$ is quadratic in $n$, so an explicit implementation of the algorithm would be burdened with a quadratic time. However we can observe that for each characteristic for fixed $x, X$ we only need to remember the smallest $\ell$ for which it is feasible. Thus, we can work with only $V(T) \cdot 2^t$

nodes and store the smallest feasible $\ell$ for each characteristic in a full set.

**Theorem 39.** $k$-TreeDepth Vertex Deletion parameterized by $k$ and the width $t$ of the given


tree decomposition admits an exact algorithm with running time $2^{O(tk)} n$. 

**Proof sketch.** We apply the same reasoning as for treewidth deletion. The number of states necessary to remember in a single node is bounded explicitly by $2^{O(tk)}$ (Lemma 15 in [RRVS14]) and the running time is analyzed in Lemma 17.
