On the Complexity of a Charged Quantum Oscillator

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Abstract

In this paper, we study the effect of both electric and magnetic fields on the rate of complexity growth. Our system is a charged quantum oscillator and over a period of time, we study the maximum dynamic evolution of quantum states which might lead to a strong bound of the rate of computation. We show that by turning on the electric field, the rate of complexity decreases, whereas this rate has increasing behavior by magnetic field. In this regard we also find a critical value of magnetic field which, beyond that, the rate changes its behavior notably.

1 Introduction

In the field of computing and information processing, there are at least two fundamental questions: the speed of processing information and the amount of memory capacity of a computer. In other words the problem of the speed limit and the amount of memory limitations are challenging questions and notably they might be identified respectively, by the energy and the number of degrees of freedom that the system could achieve. In this way for a given quantum mechanical system, the speed limit can be translated to the maximum number of distinct states which this system passes per unit of time. Such states are supposed to be orthogonal and in this sense, the issue of speed limit can be identified by the quantum mechanics.

On the other hand the quantum mechanics teaches us to think that black holes are not `black' [1], but rather, if one considers a black hole as a data store or computational device, which compresses a given definite amount of energy, then it can indeed perform at a certain rate of operations per second [2]. There is also a theoretical upper limit for the information processing, more precisely, for a quantum system with a given average energy \(E\), in a second, one can set a theoretical upper limit on the number of operations which is given by \(\frac{2E}{\pi\hbar}\). This limitation known as Lloyd’s bound [3].

The quantum information theory is indeed one of the functional aspects of quantum mechanics which has a crucial role in the development of the computer science. Interestingly enough it is argued that there is a concrete realization between the black hole physics and the quantum information theory. Namely in order to study the physics of the black holes, quantum information theory may play an important role. For example, Bekenstein [4] argued that black holes set a theoretical maximum limit on information storage which means memory is bounded. Therefore, understanding quantum information from theoretical point of view would be an interesting issue or might even be crucial. In this way entanglement entropy and complexity are important quantities (for more details see [5,6]). It is also interesting to mention that the black hole can in principle set fundamental limits on density, entropy and computational complexity [7]. On the other hand in a seminal work, Brown et al. [8] discovered a surprising connection between
the action of the interior of the black hole and the above mentioned upper limit of processing. In fact what was argued goes as follows: the action of the interior of the black hole increases at a rate exactly equal to $\frac{2E}{\pi \hbar}$ and this led them to conclude that black holes produce complexity at the fastest possible rate. So it is a relevant question to ask that if we disturb the system is it possible that the system exceeds this bound.

On the other hand in computer science, there is a simple definition for computational complexity: complexity of an operation is the amount of resources required for running a problem; or equivalently, it is defined by the minimum of difficulties to be experienced in writing all possible algorithms for doing the problem [9]. There is also a standard definition of complexity in the quantum mechanics. In fact, complexity is a criterion that shows us how difficult the task is. In principle the complexity of quantum state which originates from the field of quantum computations is defined by the number of elementary unitary operations which is required to build up a desired state from a given reference state [10]. It was also argued that quantum complexity helps us to capture some certain features of the late time behavior of eternal black hole geometries as well [11].

In Ref. [12], the authors consider Lloyd’s bound in relation to holographic complexity by making the distinction between orthogonalizing and simple gates. In fact, the amount of information that a physical system can store or process directly relates to the number of available system states. Our main goal in this article is to examine Lloyd’s bound for a quantum system in different conditions, and then compare the results with a state in which quantum system states are not necessarily orthogonal. Given the fact that the quantum harmonic oscillator is one of the key issues in quantum mechanics, we intend to compute the variation of the complexity rate for this oscillator when the system is perturbed by electric as well as magnetic fields.

The layout of this paper is as follows. In section 2, we briefly consider the orthogonality time for two simple uncoupled harmonic oscillators in the presence of the magnetic and electric fields. In particular, we explore the minimum time needed for a given state to evolve into n orthogonal state. In Section 3, we investigate for any upper bound of complexification. Finally, concluding remarks are given in section 4.

2 Orthogonality time in harmonic oscillator

Let us consider two simple uncoupled harmonic oscillators subjected to both uniform electric and magnetic fields. The corresponding Hamiltonian is then given by

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2} m (w^2 + \frac{q^2 B^2}{4m^2 c^2}) (x^2 + y^2) + \frac{qB}{2mc} (yp_x - xp_y) - \frac{q^2}{2m w^2} E^2,$$

(1)

where $m$ and $q$ stand for the mass and charge of system, respectively. The uniform electric and magnetic fields are supposed to be in $x$ and $z$ directions, respectively; Also in writing the above Hamiltonian, the identity $\vec{A} = B\hat{z} \times \vec{r}$, has been used. The eigenvalue of 2D charged harmonic oscillator is then given by

$$E = \sqrt{w^2 + \frac{w_e^2}{2} (n_1 + n_2 + 1) - \frac{w_c}{2} (n_1 - n_2) - 2m \frac{w_e^2}{w^2}},$$

(2)

where $n_1$ and $n_2$ are positive integers or zero also $w_e$ and $w_c$ are defined by

$$\frac{qB}{2m} \equiv w_c, \quad \frac{qE}{2m} \equiv w_e.$$  

(3)

\footnote{Note that through this paper we have used the natural units $\hbar = c = 1$.}
2.1 Minimum rate of orthogonality

Now let us compute the minimum time needed for any state of a given physical system to evolve into an orthogonal state. To do so we suppose an arbitrary quantum state as a superposition of energy eigenstates as follows

$$|\psi_0\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c_{n_1,n_2} |n_1,n_2\rangle,$$

where we assume that the system has a discrete spectrum. It is then straightforward to write the time evolution of $|\psi_0\rangle$ as follows

$$|\psi_{t\perp}\rangle = e^{-iHt\perp} |\psi_0\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c_{n_1,n_2} e^{-iE_{n_1,n_2}t\perp} |n_1,n_2\rangle.$$  

(5)

To find the minimum time for orthogonality, let us define

$$S(t\perp) \equiv \langle \psi_0 | \psi_{t\perp} \rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |c_{n_1,n_2}|^2 e^{-iE_{n_1,n_2}t\perp}.$$  

(6)

where solving $S(t\perp) = 0$ gives us the desired minimum time $t\perp$. After doing some algebra and making use of the following relation for $x \geq 0$

$$1 - \frac{2}{\pi} [x + \sin(x)] \leq \cos(x),$$  

(7)

one can deduce that

$$1 - \frac{2E}{\pi} t\perp + \frac{2}{\pi} Im(S) \leq Re(S),$$  

(8)

where $E$ is the average energy in the state $|\psi_0\rangle$. By imposing the orthogonality condition which leads to $ReS(t\perp) = ImS(t\perp) = 0$, one the minimum time $t\perp$, which it takes for $|\psi_0\rangle$ to evolve into an orthogonal state is given by

$$t\perp = \pi \frac{2E}{\pi}.$$

(9)

This means our quantum system of energy $E$ needs at least a time of $t\perp$ to go from one state to an orthogonal state. This is indeed the Margolus-Levitin theorem [13], which gives a fundamental limit on quantum computation.

Now the main aim is to compute the average energy in the state $|\psi_0\rangle$. In our case for a simple uncoupled harmonic oscillator, one has

$$E = \langle \psi_0 | H | \psi_0 \rangle = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \left| c_{n_1,n_2} \right|^2 \left[ \sqrt{w^2 + w_c^2(n_1 + n_2 + 1)} - w_c(n_1 - n_2) - 2m \frac{w^2}{w_c^2} \right].$$

(10)

To make calculations easier, one can start by considering the set of evolutions that pass through an exact cycle of $N_t$ which are supposed to be mutually orthogonal states at a constant...
rate in time $\tau$; In this case one has $N_t = \sum_{n=0}^{N} (n+1)$ where we have set $n = n_1 + n_2$ and $c_{n_1,n_2} = \sqrt{\frac{1}{N_t}}$, therefore, one obtains

$$E = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \left( \frac{1}{N_t} \right) \sqrt{w^2 + w^2_c(n_1 + n_2 + 1)} - \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \left( \frac{1}{N_t} \right) w_c(n_1 - n_2) - 2m \frac{w^2}{w^2}$$

$$= \sum_{n=0}^{N} \left( \frac{n+1}{N_t} \right) \sqrt{w^2 + w^2_c(n + 1)} - 2m \frac{w^2}{w^2}$$

$$= \left( 1 + \frac{2N}{3} \right) \sqrt{w^2 + w^2_c} - 2m \frac{w^2}{w^2}. \quad (11)$$

On the other hand, making use of the inequality (8), one can show that

$$\tau_\perp \geq \frac{\pi}{2(1 + \frac{2N}{3}) \sqrt{w^2 + w^2_c} - 4m \frac{w^2}{w^2}}. \quad (12)$$

This is indeed sets a bound for a time which takes for a given quantum system (charged harmonic oscillator subjected to both electric and magnetic fields) of energy $E$ to go from one state to an orthogonal state. In what follows it is shown that this bound could potentially introduces an upper bound of the rate of complexity.

### 3 Upper Bound of Complexification

In principle as mentioned, the complexity of quantum state might be characterized by the number of operations (elementary unitary operations) which is needed to build up a desired state from a given reference state. In particular, Lloyd in his seminal work [3] showed that the speed of processing information which a physical device can do, is limited by its energy and also it is interesting to mention that the amount of information that a system can process is indeed limited by the number of degrees of freedom that it possesses. This limitation which set a fundamental upper limit in computation speed for a classical computer is named as Lloyd’s bound. Basically the information process done by a computer is supposed to take a given initial state to a final state via successive application of logic gates. In principle, a logical gate refers to the actual physical device that performs a logical operation. On the other hand any gate or operation takes some time $t$ to perform its task. If one implements a Hamiltonian action
to a task which is done by given gate, then the corresponding operation might be given by $U(t) = e^{iH_g t}$. Therefore by definition, unitary evolution takes the initial state $|0\rangle$ to desired final state $U(t)|0\rangle$ after time $t$. Now let us apply a sequential application of $n$ gates, namely we have

$$|0\rangle \rightarrow U(t)|0\rangle. \quad (13)$$

In reference [12], it was shown that for a series computation in which

$$U(t) = T \prod_i U(t_{i+1}, t_i), \quad (14)$$

where $U(t_{i+1}, t_i)$ stand for an orthogonalizing gate and $T$ is the time ordered operator. In the present case the rate of complexification gets a strong bound as follows

$$\dot{C} \leq \frac{1}{\tau_\perp}. \quad (15)$$

where $\tau_\perp$ is the orthogonalization time of the system. Thus in our case the rate of complexification is given by

$$\dot{C} \leq \left( E_m \sqrt{1 + \frac{w_c^2}{w^2}} - \frac{4m w_c^2}{\pi w^2} \right), \quad (16)$$

noting that we have defined $E_m = \frac{2}{\pi} (1 + \frac{2N}{3}) w$. In figure.(2) we have plotted the rate of complexity for different values of $w_c$ and $w_e$. As it is shown from figure.(1) and (2), there is a critical value for magnetic field which beyond that the rate of complexity changes its behavior drastically which is given by

$$w_c^{cri} = \frac{2\sqrt{3}}{(3 + 2N)\omega} \sqrt{m\omega_c^2[(3 + 2N)\omega^3 + 3m\omega_c^2]} \quad (17)$$

If one interests in a case of weak magnetic field and also near orthogonality of states, then the condition (6) can be written as $S(\tau) \approx \epsilon e^{i\alpha}$, where $\alpha$ is a free parameter and $\epsilon$ is supposed to be infinitesimal. Therefore in this case one obtains

$$\dot{C} \lesssim \left( E_m (1 + \frac{w_c^2}{2w^2}) - \frac{4m w_c^2}{\pi w^2} \right) [1 - \epsilon (\frac{2}{\pi} \sin \alpha - \cos \alpha] \quad (18)$$

on the other hand the maximum value of $\frac{2}{\pi} \sin \alpha - \cos \alpha$ is $\sqrt{1 + \left(\frac{2}{\pi}\right)^2}$ which leads to

$$\dot{C} \lesssim \left( E_m (1 + \frac{w_c^2}{2w^2}) - \frac{4m w_c^2}{\pi w^2} \right) \left(1 - \epsilon \sqrt{1 + \left(\frac{2}{\pi}\right)^2} \right) \quad (19)$$

Figure.(3) shows compares the rate rate of complexity in orthogonal and near orthogonal cases where we have assumed that the magnetic field is small.

### 4 Remarks and Conclusions

Basically, it is known that the speed limit to information processing can be addressed by studying the bounds on the growth rate of the complexity, and in this paper, we discussed this issue in the charged harmonic quantum oscillator. We used the minimal time of orthogonalization as one is used in the Margolus-Levitin bound where $\langle E \rangle$, the expectation value of the energy of the system, plays the crucial role. On the hand in Lloyd’s bound (the bound on the rate of
$\omega_c = 0, \omega_e = 0$
$\omega_c = 0.05, \omega_e = 0.2$
$\omega_c = 0.05, \omega_e = 0.3$
$\omega_c = 0.01, \omega_e = 0.2$
$\omega_c = 0.01, \omega_e = 0.3$

$0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1.0$

0.990
0.995
1.000
1.005
1.010
1.015

Figure 2: The rate of normalized complexification of the quantum harmonic oscillator. When the magnetic field induction reaches a definite critical value given by (17), the behavior of this rate changes. We have set $N = 200$ and $m = 1$.

$\omega_e = 0, \epsilon = 0$
$\omega_e = 0.8, \epsilon = 0$
$\omega_e = 0.8, \epsilon = 0.1$
$\omega_e = 1.2, \epsilon = 0$
$\omega_e = 1.2, \epsilon = 0.1$

$0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4$

0.85
0.90
0.95
1.00

Figure 3: The rate of variation in the complexity of the quantum harmonic oscillator for orthogonal (solid line) and near orthogonal (dashed line) states for $N = 200$.

computation), the minimal time to perform a task is controlled by the energy $E = \langle H \rangle$ where defines a limitation on the computations. Actually the computation speed might be given by the number of operations per unit time step or the time rate of change of the complexity. In this way the Lloyd’s bound defines the upper bound as $\frac{d}{dt}C \leq \frac{2E}{\pi \hbar}$. The computational speed is bounded from above by the instantaneous energy in unit time step. This is potentially gives us the bound of the growth rate of the holographic complexity as it is discussed in the context of AdS/CFT. Noting that in this notation all operations (or equivalently all gates) can be implemented quantum mechanically, however, there are input and output at any steps which are classical states. These states are orthogonal and have no superpositions with each other.

The main goal of this paper has been to investigate the complexity of a charged harmonic oscillator by making use the orthogonalizing states. This achieved by studying the maximum speed of dynamical evolution which this followed by finding the maximum number of distinct states that the system can pass through, per unit of time. For a quantum system the distinct states can be understood by orthogonality of states. We considered the harmonic oscillator in the presence of both magnetic and electric fields and found the minimum time needed for
any state of our system to evolve into an orthogonal state. We observed that the time of orthogonality for small and large values of magnetic field behaves differently, and also, the rate of complexity increases/decreases by turning on the magnetic/electric field.

Recently the study of complexity of charged black holes from the holographic point of view and investigating this issue at the quantum field theory side, received attractive attention (see for example [8, 14–16]). In this way considering the rate of change of the complexity of a charged quantum harmonic oscillator in the presence of external fields seems to be important. Motivated by this fact, as a future work we will implement this model in computation the rate of complexity for a charged thermofield double state of free real scalar quantum field theory in the presence of background electric field, to investigate the effect of such fields on complexification in this context.

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