Correct by construction

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Abstract

Matrix code allows one to discover algorithms and to render them in code that is both compilable and is correct by construction. In this way the difficulty of verifying existing code is avoided. The method is especially important for logically dense code and when precision programming is called for. The paper explains both these concepts. Logically dense code is explained by means of the partition stage of the Quicksort algorithm. Precision programming is explained by means of fast exponentiation.

1 Introduction

In 1969 E.W. Dijkstra introduced [4] Structured Programming. At the time the method was revolutionary: it advocated the elimination of the goto statement by the exclusive use of conditional, alternative, and repetitive clauses. It took only a few years for structured programming to evolve from heresy to orthodoxy. But by then Dijkstra had arrived at the conviction that the looming "software crisis" could only be averted by formally verifying code. At the time Hoare's method [13] was the only one available for structured programs. It was found difficult to use in practice. Dijkstra maintained his conviction that code needed to be verified. He defended it in his 1971 paper "Concern for correctness as guiding principle in program composition" [5]. For my purpose the crucial passage is

When correctness concerns come as an afterthought and correctness proofs have to be given once the program is already completed, the programmer can indeed expect severe troubles. If, however, he adheres to the discipline to produce the correctness proofs as he programs along, he will produce program and proof with less effort than programming alone would have taken.

Let us give the name “correctness-oriented programming” to this hypothetical method of producing the correctness proofs as he programs along.

Within imperative programming the only possibility of correctness-oriented programming is programming with verification conditions [1 14 17 18]. In [18] programming with verification conditions is furthest evolved and receives the name “Matrix Code”. Yet even then it needed further improvement to make it practically applicable. This was done in [19]. The present paper presents two situations where programming with verification conditions is especially useful: precision programming and logically dense code. These concepts are explained in Section 3. In Section 2 we give a self-contained exposition of Matrix Code.

2 Matrix Code

Before Dijkstra can make good on his promise, we will have to find a way out of a chicken-and-egg situation: a correctness proof has to be developed for a program that does not yet exist. Such a way
was first proposed, according to R-J. Back [1], by J. Reynolds [14] and by M.H. van Emden [17]. Van Emden’s method, “programming with verification conditions”, was developed into the Matrix Code of 2014 [18]. In this paper we follow the practical implementation of Matrix Code shown in [19].

2.1 Floyd’s verification method

In 1967 R.W. Floyd proposed a method for verifying programs [8]. Here verification consists of a mathematical proof that the final state has a certain property provided that the initial state satisfies certain conditions. The program is in the form of a flowchart consisting of tests and statements that read from and write to a set of shared variables that together comprise the state. Without loss of generality we can assume that there is one node, the start node, without incoming arc. Likewise there is one node, the halt node, without outgoing arc. Some of the nodes are provided with labels. This has to be done in such a way that there is no pair of labels between which there is an infinite path without a label. The start node gets label S; the halt node gets label H.

Floyd’s method associates each label with an assertion. An assertion asserts that a certain relation holds between some of the variables. Assertions are expressed in the form of a formula of first-order predicate logic in which the free variables have the same names as some of the flowchart’s variables. Floyd’s method assumes an interpretation for the function symbols and predicate symbols occurring in the assertions associated with the labels of the flowchart.

There is a “verification condition” associated with every pair \( \langle P, Q \rangle \) of labels connected by a path that does not contain any label. The verification condition has the form \( \{p\}S\{q\} \) where \( p \) and \( q \) are the assertions associated with \( P \) and \( Q \) and where \( S \) is the binary relation between states resulting from the execution of the code in the path from \( P \) to \( Q \). The verification condition \( \{p\}S\{q\} \) is true if and only if the truth of \( p \) in a state \( s \) and the execution of \( S \) starting in \( s \) results in state \( t \) implies that \( q \) is true in \( t \).

Floyd’s verification method is embodied in the following theorem.

**Theorem 1** Given a flowchart \( F \) with start label \( S \) and halt label \( H \). If the verification conditions of \( F \) are true, then \( \{s\}F\{h\} \) is true, where \( s \) and \( h \) are the assertions associated with \( S \) and \( H \).

A verification condition can be regarded as a fragment of executable code. Compare \( \{p\}S\{q\} \) with \( P: \text{execute } "S"; \text{ goto } Q \), where \( P \) labels assertion \( p \), where \( Q \) labels assertion \( q \), and \( S \) is the code connecting \( P \) to \( Q \). The value of this observation lies in the fact that \( S \) need not come from a conventional flowchart. It can be anything that has a binary relation among states as meaning. In particular, this relation can be a subset of the identity relation. For example \( \{p\} x>0 \{p \& x>0\} \) is defined and is true for any \( p \).

What is true for a single verification condition also holds true for sets of verification conditions. Such a set can start small. It can then point in the direction of a missing verification condition, which can then be added. This a way of following up on Dijkstra’s hint “produce the proof as he programs along”.

2.2 Matrix Code

A set of verification conditions with generic element \( \langle P\rangle C\{Q\} \) can be read as the specification of a matrix with columns and rows indexed by the labels. \( C \) is then the matrix element in column \( P \) and row \( Q \). Ergo, a set of verification conditions is a program in Matrix Code. Let us now investigate Matrix Code as a programming language.

**Syntax of Matrix Code** A program consists of a signature, a set of assertion declarations, and a set of triples of the form \( \langle P\rangle C\{Q\} \), where \( P \) and \( Q \) are labels and \( C \) is a command. The signature consists of a set of function symbols, a set of predicate symbols, and a set \( V \) of variables. The
assertion declarations associate an assertion with every label. An assertion is a formula. A command is a formula, an expression of the form \( v := \tau \), where \( v \in V \) and \( \tau \) is a term, or an expression of the form \( C; D \), where \( C \) and \( D \) are commands.

Formulas, terms, and variables are defined as in first-order predicate logic with respect to the signature of the program.

**Semantics of Matrix Code**  The meaning of a program is truth or falsity with respect to an interpretation \( I \) for the signature of the program. The meaning of an assertion is a set of states. The meaning of a command is a binary relation over the set of states.

The set of states is the set of tuples of type \( V \rightarrow D \), where \( D \) is the universe of discourse of \( I \).

The meaning of an assertion \( F \) in \( I \) is \( \{ s \in (V \rightarrow D) \mid F \text{ is true in } I \text{ in state } s \} \). If a command is a formula \( F \), then its meaning in \( I \) is \( \{ (s, s) \mid s \in (V \rightarrow D) \land F \text{ is true in } I \text{ in state } s \} \). If a command is \( v := \tau \), then its meaning is \( \{ (s, s_v^\tau) \mid s \in (V \rightarrow D) \} \) where \( s_v^\tau \) is defined by \( s_v^\tau(v) = s(v) \) if \( v \in V \) is not \( v \) and \( s_v^\tau(v) \) equals the value of the term \( \tau \) in \( I \) in the state \( s \). If a command is \( C; D \), then its meaning is the relational product of the meanings of \( C \) and of \( D \).

“\( F \) is true in \( I \) in state \( s \)” and “the value of the term \( \tau \) in \( I \) in the state \( s \)” are defined as in logic.

3  Motivation for programming in Matrix Code

3.1  Dense logic

Some code is harder to get right than other code. It would help to know more about how, and where errors occur. Documentation on this is hard to come by. It is in the nature of the situation: as long as the bug is still alive, one is frantically chasing it. As soon as it is found, it is seen to be an egregious failing of the most elementary intelligence, too embarrassing to confess.

This makes Jon Bentley’s confession [2] (chapter Sorting) valuable:

... I once spent the better part of two days chasing down a bug hiding in a short partitioning loop.

I suspect Bentley is not the only one, but I know of no similar published confessions. Here is my experience with Quicksort. In 1968 it was many times that I ran up and down the stairs between my desk on the fourth floor and the computer room on the ground floor in chaotic attempts at debugging a partition function. Accepting a version of CACM Algorithm 402 as correct was based on checking whether it had correctly sorted an array of length thirty-six, the longest that would fit on a single line of line-printer paper so that it could be checked at a glance.

In this paper we examine the Quicksort algorithm. In developing code for two versions of the partitioning function Dijkstra’s “he will produce program and proof with less effort than programming alone would have taken” will turn out to be not an empty promise.

3.2  Desire for precision programming

Structured programming replaces a versatile precision instrument for control by a few coarse-grained primitives. Initially three constructs were allowed that were borrowed from Algol 60. Later, Dijkstra replaced these by his language of guarded commands. In either form, structured programming is inimical to precision in control.

This claim is of course vacuous until “precision programming” has been demonstrated by an example. The example I use is:

given a positive number \( X \) and a nonnegative integer \( Y \), set \( z \) equal to \( XY \) by means of repeated multiplication
In [7] (page 60) the authors arrive at the following program:
\[
\begin{align*}
\text{[ \text{x,y: int; z:=1; x:=X; y:=Y} } \\
\text{; do y mod 2 != 0 -> z := x*z; y := y-1} \\
\text{\quad y != 0 & y mod 2 = 0 -> x := x*x; y := y/2} \\
\text{od} \\
\end{align*}
\]

There can be two obstacles to understanding this code: lack of familiarity with guarded commands and its contorted structure. The latter is due to its derivation. Without the desire to derive it, one would write:
\[
\begin{align*}
\text{[ \text{x,y: int; z:=1; x:=X; y:=Y} } \\
\text{; do y != 0 ->} \\
\text{\quad \text{if y mod 2 != 0 -> z := x*z; y := y-1} \\
\text{\quad \text{if y mod 2 = 0 -> x := x*x; y := y/2} \\
\text{\quad fi} \\
\text{fi} \\
\text{od} \\
\end{align*}
\]

For those not familiar with guarded commands, this transcribes to C as follows.

```
int x,y; z = 1; x = X; y = Y;
while (y != 0) {
    if ((y%2) != 0) { z = x*z; y = y-1; continue; }
    if ((y%2) == 0) { x = x*x; y = y/2; continue; }
}
```

The authors note that the computation \(x*z\) is not necessary the first time because \(z\) is 1. Also, after \(y := y-1\) that variable is even, and this fact is not used. They continue with “All such attempts [at improvement] probably make the program text less clear, and in any case much longer.” This is apparently such a serious obstacle that the authors leave it the way it is. But rather than noting that structured programming comes with drawbacks, they continue with:

There was a time when constructing the fastest program, irrespective of cost, was considered profoundest wisdom. Now, fortunately, this has become obsolete.

Perhaps the authors have decided that attack is the best defence—if these flaws bother you, then there is something wrong with you, not with their program! What is demonstrated here is that for this problem the greater degree of precision needed is discouraged by the rigid corset imposed by structured programming.

### 4 Egyptian multiplication: an example of precision programming

We have illustrated precision programming by an example where it is lacking. We continue with an example where it is practised: Stepanov and Rose [16] address a similar problem without evading its complications. At the same time they work at a higher level of generality, noting that the trick that expedites the program just discussed has wider applicability. They express it thus: to compute \(na\), where \(n\) is a nonnegative integer and \(a\) is a number on which + is defined. They call it “Egyptian multiplication”. The trick is to exploit the associativity of +: \(na\) can be defined as

\[a + a + \cdots + a\]

This can be computed as

\[a + (a + (a + (a + (a + (a + a))))))\]
requiring seven operations or as

\[((a + a) + (a + a)) + ((a + a) + (a + a))\]

requiring three operations. More generally, in the order of \(n\) compared to \(\log n\) operations.

To get greater benefit of the general applicability of the idea, Stepanov and Rose use the facilities for generic programming in C++ to make their code applicable to monoids (algebras where + is associative). They go one step further and note that the case \(n = 0\) is not needed. By assuming that \(n > 0\) the code will not need the operation’s neutral element, so that the code is no longer restricted to monoids but becomes applicable to semigroups in general.

To lighten the load of language detail we refrain from the use of the generic facilities of C++ and compute \(na\) with positive integer \(n\) and a floating-point type for \(a\) as a sort of symbolic stand-in for the semigroup type. Its symbolic status is underlined by the fact that addition in the floating-point standard is not associative.

As Matrix Code, being a matrix, has no conveniently printed form, we write sets of verification conditions, which are only trivially different. Below we develop several successive versions.

Initially we only have the problem statement, so only the start label and the halt label with associated assertions.

**Label declarations**

---

\(S: n = n_0 \& a = a_0 \& n_0 > 0\)
\(H: n_0\cdot a_0\ \text{has been printed}\)

We cannot think of any sufficiently simple code \(C\) that ensures \(\{S\} C \{H\}\), so need to add assertions, which come with associated new labels.

In deciding what these are going to be we note that the problem would be simplest in the case where \(n\) is a power of 2: then we only need to double \(a\) a suitable number of times. And in the likely event that \(n\) is not initially a power of 2, it is still efficient to start by removing all factors of 2 from \(n\).

**Label declarations**

---

\(S: n = n_0 \& a = a_0 \& n_0 > 0\)
\(A: n_0\cdot a_0 = n\cdot a \& n > 0\)
\(B: A \& \text{odd}(n)\)
\(H: n_0\cdot a_0\ \text{has been printed}\)

**Triples**

---

\(\{S\} \ \text{skip} \ \{A\}\)
\(\{A\} \ \text{even}(n); \ a := a + a; \ n := n/2 \ \{A\}\)
\(\{A\} \ \text{odd}(n) \ \{B\}\)
\(\{B\} \ n = 1; \ \text{print} \ a \ \{H\}\)

This is already an executable program, although it only works for \(n\) a power of two. To handle the general case we need to add another triple emanating from \(B\) and to introduce a variable \(z\) to collect part of final result; see the additional assertion labeled with \(C\).

**Label declarations**

---

\(S: n = n_0 \& a = a_0 \& n_0 > 0\)
\(A: n_0\cdot a_0 = n\cdot a \& n > 0\)
\(B: A \& \text{odd}(n)\)
C: \( n_0 \cdot a_0 = z + n \cdot a \) & \( n > 0 \)
H: \( n_0 \cdot a_0 \) has been printed

Triples
--------
\{S\} skip \{A\}
\{A\} even(n); a := a+a; n := n/2 \{A\}
\{A\} odd(n) \{B\}
\{B\} n = 1; print a \{H\}
\{B\} n != 1; z := a; a := a+a; n := (n-1)/2 \{C\}

The algorithm proper can start with triples emanating from C, which gives rise to a new assertion D and triples emanating from it.

Label declarations
------------------
S: n = n0 \& a = a0 \& n0 > 0
A: n0\cdot a0 = n \cdot a \& n > 0
B: A \& odd(n)
C: n0\cdot a0 = z + n \cdot a \& n > 0
D: C \& odd(n)
H: n0\cdot a0 has been printed

Triples
--------
\{S\} skip \{A\}
\{A\} even(n); a := a+a; n := n/2 \{A\}
\{A\} odd(n) \{B\}
\{B\} n = 1; print a \{H\}
\{B\} n != 1; z := a; a := a+a; n := (n-1)/2 \{C\}
\{C\} odd(n) \{D\}
\{C\} even(n); a := a+a; n := n/2 \{C\}
\{D\} n = 1; print z+a \{H\}
\{D\} n > 1; z := z+a; a := a+a; n := (n-1)/2 \{C\}

No more labels are needed; no triples are missing. The items needed to reach this result became apparent as we started from the initial version that only contained the problem statement.

5 Liffig form of Matrix Code

The simplicity of the Matrix Code syntax causes it to allow some unfamiliar forms—forms that defy simple transcription to one’s favourite implemented language. For example \( x > 0; x < 11 \) is a legal command with a perfectly well-defined meaning. The same can be said of \( x := x + 1; x < 10 \), which has the same meaning as \( x < 9; x := x + 1 \) in the interpretation with the mathematical integers as universe of discourse. The latter form is preferred for ease of transcription.

“Liffig” is the name of a notation for Matrix Code programs that differs in encouraging ease of transcription. It imposes a helpful order on the set of triples. It integrates the label declarations with the set of triples. It gives priority to preferable forms of commands. Liffig is inspired by Dijkstra’s guarded commands. As a sample I present the Liffig form of the Egyptian multiplication example. See Figure 1.

The reader will recognize the if ..|.. fi of Dijkstra’s guarded commands. Each of these is preceded by a label. Each individual guarded command is terminated by goto L to indicate that
S: n = n0 & a = a0 & n0 > 0
  if true -> skip; goto A
  fi

A: n0*a0 = n*a & n > 0
  if even(n) -> a := a+a; n := n/2; goto A
  | odd(n) -> goto B
  fi

B: A & odd(n)
  if n = 1  -> print a; goto H;
  | n != 1  -> z := a; a := a+a; n := (n-1)/2; goto C
  fi

C: n0*a0 = z+n*a & n > 0
  if odd(n) -> goto D
  | even(n) -> a := a+a; n := n/2; goto C
  fi

D: C & odd(n)
  if n = 1  -> print z+a; goto H
  | n > 1  -> z := z+a; a := a+a; n := (n-1)/2; goto C
  fi

H: n0*a0 has been printed
  return

Figure 1: The program for Egyptian multiplication in Liffig form.

the assertion labelled by L holds at that point. Because of the added label before if ..].. fi
and the added goto's following I have named the language “Liffig”, to be parsed as “L/iffi/g”.

In the sequel I will use Liffig to develop two versions of the partition function of Quicksort. It
has the advantage that its verification conditions can be clearly recognized. It is easy to transcribe
to several implemented programming languages.

6 Quicksort

The Quicksort algorithm is a common example in introductory programming textbooks; it is also
important in practice. The idea of the algorithm is that to sort an array, one begins by swapping
elements in such a way that the array ends up being partitioned. This means that the array is
decomposed into two parts in such a way that nothing in the left part is greater than anything in
the right part. In this way the problem of sorting the entire array is reduced to the subproblems of
sorting the left part and of sorting the right part. These two problems are smaller and can be solved
by the same method.

The problem reduction has a simple form in a suitable programming language. Hoare used Algol 60; a C version follows below.

```c
void Quicksort(int A[], int m, int n) {
  int i, j;
  if (m<n) {
    partition(A, &i, &j, m, n);
    Quicksort(A, m, j);
    Quicksort(A, i, n);
  }
}
```
Thus the only work that remains is the function partition. This can be found in dozens of introductory textbooks for programming. It so happens that the two main variants have been authored or co-authored by Hoare himself, which is my reason for restricting the treatment to these.

7 Partitioning tactics

An abstract statement of the goal of partitioning is

the array is decomposed into two parts in such a way nothing in the left part is greater than anything in the right part.

This implies that there is an item of the same type as the array elements (so that they can be compared) such that nothing in the left part exceeds the item nor anything in the right part is exceeded by the item. Such an item is called a “pivot”. So defined, the pivot may or may not itself be an array element.

For the choice of the pivot there are practical and theoretical considerations. Practical considerations include whether one wants to optimize for some special property that the array to be sorted might have. The properties that might arise in practice include that there may be few or many equal items and that the array may be random or nearly sorted. We next turn to theoretical considerations.

Partitioning results in an array of length \( n \) being subdivided into segments of lengths \( r \) and \( n - r \). Before partitioning the number of possible permutations is \( n! \)—afterwards this is \( r!(n-r)! \). Information theory measures the uncertainty in not knowing which of \( N \) equally probable possibilities is actually the case as \( \log N \) bits. Thus the reduction of uncertainty achieved by the partition is

\[
\log n! - \log r! - \log(n-r)! = \log \frac{n!}{r!(n-r)!}. \tag{1}
\]

This is maximal for \( r = n - r \); when the partition ends in the middle. To achieve this the partition algorithm needs a clairvoyant choice of pivot. An example of a realizable choice of pivot causes \( r \) to assume every value in \( 1, \ldots, n \) with equal probability. It can be shown that this makes (1) smaller than its maximum value by a factor of 1.386 [12].

One can get a larger yield out of a partition by choosing as pivot the median of a random sample of size \( m \) out of the \( n \) array elements. The resulting gain in Quicksort performance has, of course, to be balanced against the cost of computing the median. Strategies for choosing the pivot and adaptations of Quicksort to the several properties of the sequence to be sorted represent a vast area for research; one could devote a whole thesis to it.

8 Hoare’s 1961 partition

The first version of Quicksort was published in 1961 as Algorithms 63 and 64 [10, 11]. Curiously enough Google Scholar gives about a dozen references, to be compared with over a thousand to the journal article [12], which contains much ancillary information, but no code. It turns out that Algorithm 63 is quite interesting; see Figure 2. Multiple elements equal to the pivot \( X \) can end up in the middle and will not be touched by future recursive calls by Quicksort. It would be nice if this could be said for all elements equal to \( X \). The above listing does not say.

Perhaps Algorithm 63 does have this property. It seems a daunting task to prove this. See the listing in Appendix A. In fact, it seems a daunting task to even prove that the algorithm satisfies the weaker property claimed. This is only to be expected with code not constructed with a correctness proof in mind. Let us now embark on a derivation of a Matrix Code program.

---

1 In fact, somebody did; see “Quicksort” by Robert Sedgewick. PhD thesis, Stanford, 1975.
procedure partition (A,M,N,I,J); value M,N;
array A; integer M,N,I,J;
comment I and J are output variables, and A is the array (with
subscript bounds M:N) which is operated upon by this procedure.
Partition takes the value X of a random element of the array A,
and rearranges the values of the elements of the array in such a
way there exist integers I and J with the following properties:
M <= J < I <= N provided M < N
A[R] <= X for M <= R <= J
A[R] = X for J < R < I
A[R] >= X for I <= R <= N

Figure 2: Hoare's specification for the partition algorithm in Algorithm 63.

Here I have the benefit of a sequel to Algorithm 63 in the form of a paper [15] by Stepanov in
which he shows that the well-known problem of the Dutch National Flag [6] is a disguise for Hoare's
Algorithm 63 as it should be done. The “Flag” problem is to arrange an array of which the elements
have at most three different elements. We will distinguish them as red, white, and blue. The task is
to permute them so that there is no red to the right of any white and no white to the right of any
blue. Thus, if all three occur, then in the desired state all the reds are at the left (lower indexes),
all the blues are at the right (higher indexes), and all the whites in between.

Stepanov noticed that red, white, and blue can be read as encodings for less than, equal to,
and greater than the pivot in Algorithm 63. In addition to this encoding, Dijkstra goes to great
lengths to hide the connection; he speaks in terms of buckets, patriotically coloured pebbles, and a
mini-computer to inspect contents of buckets.

Now that the secret is out, a good way to improve Algorithm 63 is to write a program for the
Dutch National Flag problem for array segments containing no items other than those I shall call
red, white, and blue. I follow the idea of Stepanov’s code [15] (page 209), but will write the program
in Liffig. In the course of transcription to C the array will become an array of integers, a pivot
X will appear, and red, white, and blue will be translated to less than, equal to, and greater than X.

Initially the program is only a problem statement.

S: a[m..n] is allocated, with m <= n &
a[m..n] contains no elements other than red, white, blue
...
H: a[] is a permutation of a0[] &
there exist f, s, and t such that m<=f<=s<=t<=n+1 &
a[m..f-1]=red & a[f..s-1]=white & a[t..n]=blue

The problem is to find suitable text to replace the dots. The main item is an assertion A that is easy
to reach from S and that is easy to adjust in the direction of H.

S: a[m..n] is allocated, with m <= n &
a[m..n] contains no elements other than red, white, blue &
a[] = a0[]
if true -> f := m; s := f; t := n+1; goto A
fi
H: a[] is a permutation of a0[] & m<=f<=t<=n+1 &
a[m..f-1]=red & a[f..t-1]=white & a[t..n]=blue
if true -> *j = f-1; *i = t; return
fi
Assertion $A$ can be illustrated with an array of 12 elements, as shown below.

| array content | R R R W W W U U U B B B |
|---------------|--------------------------|
| index         | m f s t n                |

The array contents, red, white, unknown, and blue, are shown as R, W, U, and B respectively. The indexes $m$, $f$, $s$, $t$, and $n$ have the names of the corresponding variables in the code.

Code has been added at $S$ to establish $A$. The code at $H$ is added to pass the result of the computation to the call of the function of which we are writing the body.

At $A$ the task is to make the gap between $s$ and $t$ smaller. We first pick off the easy case $s=t$, where there is no need to make the gap smaller. The case $s<t$ is almost equally simple.

See Figure 3.

---

$S$: $a[m..n]$ is allocated, with $m \leq n$ & $a[m..n]$ contains no elements other than red, white, blue & $a[] = a0[]$
if true -> $f := m$; $s := f$; $t := n+1$; goto $A$
fi

$H$: $a[]$ is a permutation of $a0[]$ & $m \leq f \leq s \leq t \leq n+1$ & $a[m..f-1]=$red & $a[f..t-1]=$white & $a[t..n]=$blue
if true -> *$j = f-1$; *$i = t$; return
fi

$A$: $s \leq t$ & $a[]$ is a permutation of $a0[]$ & $m \leq f \leq s \leq t \leq n+1$ & $a[m..f-1]=$red & $a[f..s-1]=$white & $a[t..n]=$blue
if $s = t$ -> *$j := f-1$; *$i := t$; return
| $s < t$ -> goto $B$
if

$B$: $A$ & $s < t$
if $a[s]=$red -> swap($a,f,s$); $f := f+1$; $s := s+1$; goto $A$
| $a[s]=$white -> $s := s+1$; goto $A$
| $a[s]=$blue -> $t := t-1$; swap($a,s,t$); goto $A$
fi

---

Figure 3: Liffig code for the Dutch National Flag problem. See Appendix 3 for the transcription to C.

## 9 Foley and Hoare partition

I have written at length about how the first Hoare partition, the one in Algorithm 63 [10], can be improved by identifying it with the problem of the Dutch National Flag. As a result the main idea of Quicksort was eclipsed by the Dutch National Flag. Many implementations of Quicksort do not follow Algorithm 63, but are more like the version published by Foley and Hoare [9]. It can be described as follows.
After choosing the pivot, an upward sweep is performed until an obstacle is encountered in the form of an element that is greater than the pivot. A downward sweep is performed until an obstacle is encountered in the form of an element that is smaller than the pivot. With both sweeps stopped, the obstacles are swapped. The two sweeps and subsequent swaps are repeated until the unpartitioned segment vanishes.

The upward and downward sweeps do two comparisons per array element in Algorithm 63: one comparison with the pivot and one comparison to ensure that the next index does not exceed the array bound. This work is rewarded when the array is long, but has relatively few different values. When multiple values are rare, it pays to avoid checking for exceeding the array bound by means of a “sentinel”. In the outline sketched above only an array element greater than the pivot is counted as an obstacle in the upward sweep. Suppose that this criterion is changed to “not smaller”, then the comparison with the array bound can be dispensed with. With the analogous modification applied to the downward sweep we get a partition in which only one comparison is needed per array element. The cost of this speed-up is that the exchange of obstacles is also done when these are equal (to each other because equal to the pivot). Using the pivot as sentinel has two advantages. The first is that often the array to be sorted is such that equal elements are rare. This raises the question why to choose as pivot an array element in the first place. This is easy to avoid, which is another reason for choosing the sentinel version of partitioning.

In Section 8 a Liffig program was developed from a specification informally obtained from the published code in [10, 11]. I showed some intermediate steps in the development of the Liffig program. Now is the time to do the same on the basis of the code published in [9], except that no intermediate steps will be shown. See Figure 3.

10 Concluding remarks

Scope of Matrix Code Commands are limited to assertions and to assignments in which the right-hand side is a term of logic. This makes Matrix Code a marvel of simplicity, syntactically and semantically. It can only be recommended for writing bodies of function definitions in another language. This other language is needed to make the Matrix Code program into a procedure body.

Relation of Matrix Code to logic An intriguing aspect of Matrix Code, the purest form of imperative programming, is that so much of it is defined by first-order predicate logic, a formalism that antedates programming languages by decades. Syntactically, signatures, formulas (assertions and commands), variables, terms are adopted from this logic. Semantically, the value of a term and the truth of a formula are defined as in Tarski’s semantics for logic. Added to logic are :=, ;, as well as the notion of program on the syntactic side. On the semantic side, states and binary relations over them are foreign to logic.

Relation of Matrix Code to logic programming An important advantage of Matrix Code is that it can grow from small beginnings by accretion of verification conditions. Even such small beginnings have computations. These cannot give incorrect results as long as the verification conditions are true. But they can fail to give desired answers as long as not all requisite verification conditions have been added.

Logic programming has the same advantage. The programs grow by accretion of facts or parts of a procedure definition which have meaning by themselves as a Horn clauses of logic and as such are true or false. The similarity between flowchart programs and a certain style of logic programs was noted in [3].

Relation to matrix theory So far in this paper it was only noted that verification conditions, being of the form \( \{P\} S \{Q\} \), can be read as a listing of items specifying a matrix with element \( S \) in column \( P \) and row \( Q \). In [18] it is shown that there is a deeper justification of viewing collections
Figure 4: Liffig code for the partition function of Foley and Hoare [9].
of verification conditions as matrices. For example, if $M_1$ and $M_2$ are matrix code programs on the same vocabulary, then their matrix product is defined and denotes a matrix code program. The assertions associated with the labels can be regarded as a column vector $A$. The equation $MA = A$ can be interpreted as a correctness proof of program $M$ with respect to the assertions $A$.

**Termination**  Truth of the verification conditions in a matrix program only guarantees that no incorrect results can be produced. It leaves open the possibility that no results appear due to non-termination. In my experience the development process is such that one is aware of the reason it cannot give rise to an infinite computation when a loop is closed.

**Summary for the practitioner**  Hoare’s verification method was widely seen as more important than Floyd’s. After all, the former is for structured programs, while Floyd’s flowcharts reek of goto statements. Hoare acknowledged the inspiration from Floyd, while the public saw Hoare playing Jesus to Floyd in the role of John the Baptist. It is in the nature of things who gets all the publicity. But once one admits concern for correctness as the overriding one, program structure may not offer any advantage.

But merely going back to flowcharts does not give anything in exchange for sacrificing structure. One would first face the difficulty of writing a correct flowchart only to be left with the difficulty of finding assertions that yield true verification conditions. It is only when one sees that a shortcut can be taken that another path ahead is revealed. The shortcut is to *skip the flowchart* and to start writing assertions on a blank slate, not encumbered by any flowchart, with the specification of the programming problem as only constraint. The first two assertions are the precondition and the postcondition for the program as a whole. Either a verification condition is found that connects the two by feasible code or an intermediate assertion is found that can be so connected. Thus the solution to the programming problem is found by discovering missing assertions or adding missing verification conditions. All of this works because sets of verification are already a programming language of sorts. For the practitioner the technicalities of Section 2 are dispensable: only this paragraph is needed, to be followed by some of the examples.

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**Appendixes**

**A  The original Quicksort**

Here follows the transcription to C of Algorithms 63 and 64 [10, 11].

```c
void partition(int A[], int M, int N, int* i, int* j){
    /* Makes X a random element of A[M..N] and permutes this
    segment in such a way that
        M <= J < I <= N provided M < N
        A[R] <= X for M <= R <= J
        A[R] = X for J < R < I
        A[R] >= X for I <= R <= N
    */
    int I = *i; int J = *j; // using Hoare’s I and J from now on
    int X; int F;
```
F = rndm(M,N); X = A[F];
I = M; J = N;
up: for(; I <= N; ++I)
    if (X < A[I]) goto down;
    I = N;
down: for(; J >= M; --J)
    if (X > A[J]) goto change;
    J = M;
change: if (I<J) { exchange(A, I, J);
        ++I; --J;
        goto up;
    }
else if (I < F) {exchange(A, I, F); ++I;}
else if (F < J) {exchange(A, F, J); --J;}
*i = I; *j = J;
}

void quicksort(int A[], int M, int N) { // sorts A[M..N]
int I, J;
if (M<N) { partition(A, M, N, &I, &J);
quicksort(A, M, J);
quicksort(A, I, N);
}
}

B The original Quicksort verified

This is the transcription to C of the Liffig code in Figure 3 in Section 8. Correctness has top priority. Hence close similarity to the Liffig version is more important than absence of code optimization opportunities.

void partition(int a[], int m, int n, int* i, int* j){
    int X; int F;
    int f=m; int s=m, t=n+1; goto A;
A: /* m<=k<f => a[k]=red &
    f<=k<s => a[k]=white
    s<=k<t => a[k] unknown
    t<=k<n+1 => a[k]=blue
    m<=f & f<=s & s<=t+1 & t<=n+1
    */
    if (s == t) { *j = f-1; *i = t; return; }
    if (s <= t) goto B;
    assert(0);
B: // A & s<t
    if (a[s] < X) {swap(a,f,s); ++f; ++s; goto A; }
    if (a[s] == X) { ++s; goto A; }
    if (a[s] > X) { --t; swap(a,s,t); goto A; }
    assert(0);
}
C The Quicksort of Foley and Hoare

Here follows the transcription to C of the Quicksort of Foley and Hoare [9].

void partition(int A[], int* ip, int* jp, int m, int n){
    int r, f; int i, j;
    f = (m+n)/2; r = A[f]; i = m; j = n;
    while (i <= j) {
        while (A[i]<r) ++i;
        while (A[j]>r) --j;
        if (i <= j) { swap(A, i, j); ++i; --j; }
    }
    *ip = i; *jp = j;
}
void Quicksort(int A[], int m, int n) {
    int i, j;
    if (m<n) {
        partition(A, &i, &j, m, n);
        Quicksort(A, m, j);
        Quicksort(A, i, n);
    }
}

D The Quicksort of Foley and Hoare: transcription to C of the Liffig code

Here follows the transcription to C of the Liffig code for the Quicksort of Foley and Hoare developed in Section [9].

void partition(int a[], int* Ip, int* Jp, int m, int n){
    int r; int i,j;
    S: r = (a[m]+a[n])/2 ; i = m; j = n; goto A;
    H: *Ip = i; *Jp = j; return;
    A: if (i == j) { goto F; }
        if (i < j) { goto B; }
        assert(0);
    B: if (a[i] < r) { ++i; goto A; }
        if (a[i] >= r) { goto C; }
        assert(0);
    C: if (a[j] > r) { --j; goto D; }
        if (a[j] <= r) { swap(a,i,j); ++i; --j; goto E; }
        assert(0);
    D: if (i == j) { goto F; }
        if (i < j) { goto C; }
        assert(0);
    E: if (i > j) { goto H; }
        if (i <= j) { goto A; }
        assert(0);
    F: if (a[i] <= r) { ++i; goto H; }
        if (a[j] >= r) { --j; goto H; }
        assert(0);
}
void Quicksort(int A[], int m, int n) {
    int i, j;
    if (m<n) {
        partition(A, &i, &j, m, n);
        Quicksort(A, m, j);
        Quicksort(A, i, n);
    }
}

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