We present first results from the QCDSF collaboration for the kaon semileptonic decay form factors at zero momentum transfer, using two flavours of non-perturbatively $O(a)$-improved Wilson quarks. A lattice determination of these form factors is of particular interest to improve the accuracy on the CKM matrix element $|V_{us}|$. Calculations are performed on lattices with lattice spacing of about 0.08 fm with different values of light and strange quark masses, which allows us to extrapolate to chiral limit. Employing double ratio techniques, we are able to get small statistical errors.
1. Introduction

Recently high emphasis is placed on testing the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. In particular, the unitarity of the CKM matrix implies the unitarity constraint of the first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta.$$  \hspace{1cm} (1.1)

According to the PDG [1] $\delta$ is equal to 0.0008(11). The uncertainty is still quite substantial and has to be decreased. As $|V_{ub}|$ is much less than unity, about half of the error of $\delta$ comes from the uncertainty of $|V_{us}|$. Since the kaon semileptonic decay rate is proportional to $|V_{as}|^2 |f_+(0)|^2$, this matrix element can be determined by combining experimental results for this decay rate and theoretical calculations of the vector form factor at zero momentum transfer $f_+(0)$.

The kaon semileptonic decay form factors $f_+(q^2)$ and $f_-(q^2)$ are defined as

$$\langle \pi(p')|V_\mu|K(p)\rangle = f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu,$$  \hspace{1cm} (1.2)

where $q^2 = (p-p')^2$ is the momentum transfer and $V_\mu = \bar{s} \gamma_\mu u$ is the weak current. It is also convenient to introduce the scalar form factor defined as

$$f_0(q^2) = f_+(q^2) \left(1 + \frac{q^2}{M_K^2 - M_\pi^2} \xi(q^2)\right), \quad \text{and} \quad \xi(q^2) = \frac{f_-(q^2)}{f_+(q^2)}.$$  \hspace{1cm} (1.3)

The form factor $f_+(0)$ at zero momentum transfer was estimated by Leutwyler and Roos [2] in 1984. They obtained the value 0.961(8), which is still used as a reference value to extract $|V_{us}|$ from the experimental data. However, to estimate higher order terms in the chiral perturbation theory (ChPT) expansion, they used a model of the wave function of the pseudoscalar meson. Recent ChPT calculations [3] favour a slightly larger value of $f_+(0) = 0.984(12)$. Thus, it is desirable to calculate $f_+(0)$ non-perturbatively and lattice calculation provides such an opportunity.

Recently a lot of lattice groups reported calculations of $f_+(0)$ [4, 5, 6, 7, 8, 9, 10, 11]. Their results coincide with each other and are in agreement with that of Leutwyler and Roos. For a recent review see [12].

In this paper we present first preliminary results for these form factors from $N_f = 2$ flavours of light dynamical, non-perturbatively $O(a)$-improved Wilson fermions and Wilson glue.

2. Lattice setup

For our calculation we use a gauge ensemble at $\beta = 5.29$, which corresponds to a lattice spacing $a = 0.075$ fm, and $\kappa_{\text{sea}} = 0.13590$, which corresponds to a pion mass $M_\pi = 0.591(2)$ GeV. The lattice volume is $24^3 \times 48$ with spacial extent equal to 1.9 fm. The correlation functions have been calculated on about 800 gauge configurations using various source locations to reduce statistical noise. We identify the sea quark mass with the light quark mass and the valence quark mass with the strange quark mass using 3 values $\kappa_s = 0.13485, 0.13530$ and 0.13570. The corresponding kaon masses are $M_K = 0.780(11), 0.704(13), 0.629(14)$ GeV.

---

1For translation into physical units the Sommer parameter $r_0 = 0.467$ fm is used.
3. Correlators

On the lattice we calculate two and three point functions, which are defined by

\[ C_P(t, \vec{p}) = \sum_{x} \langle O_{P \text{sink}}(\bar{x}, t) O_{P \text{src}}^\dagger(\bar{0}, 0) \rangle e^{-i\vec{p} \cdot \bar{x}} e^{-E_P \bar{p} \cdot t}, \] (3.1)

\[ C_{\mu}^{Q}(t, t', \vec{p}, \vec{p}') = \sum_{x, x'} \langle O_{Q \text{sink}}(\bar{x}', t') V_{\mu}(\bar{x}, t) O_{P \text{src}}^\dagger(\bar{0}, 0) \rangle e^{-i\vec{p}' \cdot (\bar{x}' - \bar{x}) - i\vec{p} \cdot \bar{x}} \] (3.2)

\[ \frac{Z_{P \text{src}}^* Z_{Q \text{sink}}}{4E_P \bar{p} E_Q \bar{p}' Z_V} \langle Q(p') | V_{\mu}^{(R)} | P(p) \rangle e^{-E_P \bar{p} t - E_Q \bar{p}' (t' - t)}, \] (3.3)

where \( P \) and \( Q \) denote either the \( K \) or \( \pi \) meson. The energy of meson \( P \) (\( Q \)) is denoted by \( E_P(\bar{p}) \) (\( E_Q(\bar{p}) \)). The renormalised vector current, including the renormalisation factor \( Z_V \), is denoted by \( V_{\mu}^{(R)} \). The overlap with the physical meson states is given by \( Z_{P \text{sink}} \) and \( Z_{P \text{src}} \).

In the following we will assume the point meson sources to be inserted at time \( t_{\text{src}} = 0 \) (thus \( t_{\text{src}} \) is omitted in Eq. (3.3)) and point sinks are inserted at \( t' = T/2 \). As \( t' \) is fixed the \( t' \) dependence of all quantities is ignored in the following. For this choice of \( t' \) the three point functions (and therefore also the double ratios) are symmetric with respect to \( T/2 \). We make use of this property and average over both time ranges to increase the precision of calculations.

Note that smearing of meson operators is not used. This is because it leads to a momentum dependence of the overlap \( Z_{P \text{sink}}(p) \) and \( Z_{P \text{src}}(p) \). See, for example, Appendix A in Ref. [13] and the discussion below.

4. Scalar form factor at \( q^2_{\text{max}} \)

The scalar form factor \( f_0(q^2) \) at \( q^2 = q^2_{\text{max}} = (M_K - M_\pi)^2 \) can be obtained from the double ratio of three point functions (which was originally proposed to calculate the \( B \rightarrow D \ell \nu \) form factor in Ref. [14]):

\[ R(t) = \frac{C_4^{K}(t, \bar{0}, 0) C_4^{\pi K}(t, \bar{0}, 0) C_4^{\pi}(t, 0, \bar{0})}{C_4^{K}(t, \bar{0}, 0) C_4^{\pi K}(t, \bar{0}, 0) C_4^{\pi}(t, 0, \bar{0})} \rightarrow \frac{(M_K + M_K)^2}{4M_K M_\pi} (f_0(q^2_{\text{max}}))^2 = \bar{R}. \] (4.1)

Our results for \( R(t) \) for three values of the strange quark masses are shown in the left panel of Fig. 1. Note that renormalisation constants and exponential factors exactly cancel in \( R(t) \). We can extract \( f_0(q^2_{\text{max}}) \) with a statistical uncertainty < 0.1%.

In the \( SU(3) \) symmetric limit \( \bar{R} \) is equal to unity. The deviation of \( \bar{R} \) from unity depends on the physical \( SU(3) \) breaking effects on \( f_0(q^2_{\text{max}}) \), which can be seen in the right panel of Fig. 1.

5. Interpolation to zero momentum transfer

To study the \( q^2 \) dependence of the form factor we calculate

\[ F(\bar{p}, \bar{p}') = \frac{f_+(q^2)}{f_0(q^2_{\text{max}})} \left( 1 + \frac{E_K(\bar{p}) - E_\pi(\bar{p}')}{{E_K(\bar{p}) + E_\pi(\bar{p}')}} \xi(q^2) \right). \] (5.1)
we show, as an example, the data for

\[ \kappa_s = 0.13485 \]

\[ \kappa_s = 0.13530 \]

\[ \kappa_s = 0.13570 \]

\[ R(t) \]

\[ a^2 \Delta M^2 = a^2 (M_K^2 - M_\pi^2) \]

\[ \xi(q^2) \]

\[ \bar{R}_k(t, \bar{p}, \bar{p}') = \frac{C_k^{K\pi}(t, \bar{p}, \bar{p}')}{{C}_k^4(t, \bar{p}, \bar{p})} \]

\[ \xi(q^2) = \frac{- (\bar{p} + \bar{p}')_k (E_K(\bar{p}) + E_K(\bar{p}')) + (\bar{p} + \bar{p}')_k (E_\pi(\bar{p}) + E_\pi(\bar{p}')) \bar{R}_k(\bar{p}, \bar{p}')} {(|\bar{p} - \bar{p}'|)_k (E_K(\bar{p}) + E_K(\bar{p}')) - (\bar{p} + \bar{p}')_k (E_\pi(\bar{p}) - E_\pi(\bar{p}')) \bar{R}_k(\bar{p}, \bar{p}')}} \]

\[ \bar{R}_k(\bar{p}, \bar{p}') = \lim_{t \to \infty} \bar{R}_k(t, \bar{p}, \bar{p}') \]

This ratio is the noisiest of all of them. Thus, \( \xi(q^2) \) could only be extracted with a large uncertainty. The relative error turns out to be 20% – 80% for our statistics, even after we make use of the symmetry:

\[ \bar{R}_k^{K\pi}(\bar{p}, \bar{p}') = \bar{R}_k^{\pi K}(\bar{p}', \bar{p}) = \bar{R}_k^{\pi K}(-\bar{p}', -\bar{p}) \].

**Figure 1:** Left: Time dependence of the double ratio \( R(t) \) (Eq. (4.1)) for three values of the strange quark masses. Right: Values of \( f_0(q_{\text{max}}^2) \) as a function of the \( SU(3) \) breaking parameter \( a^2 \Delta M^2 = a^2 (M_K^2 - M_\pi^2) \).
From $f_0(q^2_{\text{max}})$, $F(p, p')$ and $\xi(q^2)$ we can calculate the scalar form factor $f_0(q^2)$. The results are shown in the right panel of Fig. 2. To interpolate $f_0(q^2)$ to zero momentum transfer, we fit it with a monopole ansatz $f_0(q^2) = f_0(0)/(1 - q^2/M^2)$. The result of the fit is also shown in the figure. In the given range of momenta the data is well described by this ansatz.

6. Chiral extrapolation

In order to calculate the physical value of $f_+(0) = f_0(0)$ we have to extrapolate our results to the physical pion and kaon masses. We make use of the results of ChPT to guide our extrapolation. In ChPT $f_+(0)$ can be expanded in terms of light pseudoscalar meson masses giving

$$f_+(0) = 1 + f_2 + f_4 + \ldots , \quad f_n = \mathcal{O}(M_{\pi,K,\eta}^{2n}). \quad (6.1)$$

The leading correction $f_2$ receives only contributions from non-local operators and can be determined unambiguously in terms of $M_K, M_\pi$ and $f_\pi$ (see [15]). We compute $f_2$ at the actual pion and kaon masses and define

$$\Delta f = f_+(0) - (1 + f_2), \quad (6.2)$$

which receives only contributions from local operators.

The Ademollo-Gatto theorem [16] states that $\Delta f$ is proportional to $(M_K^2 - M_\pi^2)^2$. Hence, we may write

$$\Delta f = a + b(M_K^2 - M_\pi^2)^2, \quad (6.3)$$

to obtain $f_+(0)$ by extrapolating $f_+(0)$ to the physical point. The data points and the resulting fit are shown in the left panel of Fig. 3. We find $a \approx 0$, in agreement with the expected result $\Delta f = 0$ in the limit of flavour $SU(3)$. 

---

**Figure 2:** Left: Time dependence of double ratio $R_F(t, p, p')$ for $\kappa_s = 0.13485$. Right: Scalar form factor $f_0(q^2)$ for different values of the strange quark mass. The solid line is the result of a monopole fit as described in the text.
At the physical meson masses we find $\Delta f = -0.016(8)$ of Ref. [2]. Inserting the physical value of $f_2$, $f_2 = -0.0227$, into $\Delta f$, we obtain

$$f_+(0) = 0.9647(15)_{\text{stat}}.$$  

(6.4)

We extract the CKM matrix element $|V_{us}|$ from the experimental value of $|V_{us}| f_+(0)$ averaged over all decay modes [17], $|V_{us}| f_+(0) = 0.21673(46)$. This finally gives

$$|V_{us}| = 0.2247(5)_{\exp(4)}_{\text{stat}}.$$  

(6.5)

To compute $\xi(0)$ at the physical meson masses, we make the ansatz $\xi(0) = c(M_K^2 - M^2_\pi)$, in accord with $\xi(0) = 0$ in the $SU(3)$ limit. In the right panel of Fig. 3 we show the data points and the resulting fit. We obtain

$$\xi(0) = -0.10(2)_{\text{stat}}.$$  

This value is consistent with the experimental values $-0.01(6)$ from $K^0_{13}$ decay and $-0.125(23)$ from $K^+_{13}$ decay.

Note that in our analysis only statistical errors are estimated. Analysis of systematic errors which include extrapolation errors of $f_0(q^2)$ to zero momentum transfer and chiral extrapolation errors is underway.

7. Conclusions and outlook

In this work we have presented preliminary results for the kaon semileptonic decay form factors and $|V_{us}|$ from $N_f = 2$ non-perturbatively $O(a)$-improved Wilson fermions. We found

$$f_+(0) = 0.9647(15)_{\text{stat}} \land |V_{us}| = 0.2247(5)_{\exp(4)}_{\text{stat}}.$$  

(7.1)

---

2Here and in the following only the statistical error is quoted.
in agreement with the results of other lattice groups [4, 5, 6, 7, 8, 9, 10] as well as the estimate of
Leutwyler and Roos [2], within the error bars.

The QCDSF collaboration is going to improve the accuracy of the results by performing calcula-
tions at lighter quark masses down to pion masses of about 300 MeV and using partially twisted
boundary conditions [9]. In addition, we are going to study discretisation effects using gauge en-
sembles at different lattice spacings in the range of 0.115 fm to 0.070 fm. Furthermore, we plan to
study finite size effects.

Acknowledgments

The numerical calculations have been performed on the APE1000 and apeNEXT at NIC/DESY
(Zeuthen), the BlueGene/L at NIC/FZJ (Jülich) and EPCC (Edinburgh). Some of the configurations
have been generated on the BlueGene/L at KEK by the Kanazawa group as part of the DIK research
programme. This work was supported in part by the DFG, by the EU Integrated Infrastructure Ini-
tiative Hadron Physics (I3HP) under contract number RII3-CT-2004-506078. SMM is partially
supported by the RFBR grants 05-02-16306a, 07-02-00237a, 06-02-04010 and INTAS YS Fellow-
ship 05-109-4821. SMM also acknowledges support from the Lattice 2007 organizers.

References

[1] W. M. Yao et al. [Particle Data Group], J. Phys. G 33 (2006) 1.
[2] H. Leutwyler and M. Roos, Z. Phys. C 25 (1984) 91.
[3] J. Bijnens and P. Talavera, Nucl. Phys. B 669 (2003) 341 [arXiv:hep-ph/0303103]. M. Jamin,
   J. A. Oller and A. Pich, JHEP 0402 (2004) 047 [arXiv:hep-ph/0401080]. V. Cirigliano, G. Ecker,
   M. Eidemuller, R. Kaiser, A. Pich and J. Portoles, JHEP 0504 (2005) 006 [arXiv:hep-ph/0503108].
[4] D. Becirevic et al., Nucl. Phys. B 705 (2005) 339 [arXiv:hep-ph/0403217].
[5] D. Becirevic et al., Eur. Phys. J. A 24S1 (2005) 69 [arXiv:hep-lat/0411016].
[6] M. Okamoto [Fermilab Lattice Collaboration], arXiv:hep-lat/0412044.
[7] N. Tsutsui et al. [JLQCD Collaboration], PoS LAT2005 (2006) 357 [arXiv:hep-lat/0510068].
[8] C. Dawson, T. Izubuchi, T. Kaneko, S. Sasaki and A. Soni, Phys. Rev. D 74 (2006) 114502
   [arXiv:hep-ph/0607162].
[9] P. A. Boyle, J. M. Flynn, A. Jüttner, C. T. Sachrajda and J. M. Zanotti, JHEP 0705 (2007) 016
   [arXiv:hep-lat/0703005].
[10] D. J. Antonio et al., arXiv:hep-lat/0702026.
[11] J. Zanotti, $K \rightarrow \pi$ form factor with 2 + 1 dynamical domain wall fermions, in these proceedings.
[12] A. Jüttner, Status of Kaon physics on the lattice, in these proceedings.
[13] K. C. Bowler et al. [UKQCD Collaboration], Phys. Rev. D 54 (1996) 3619 [arXiv:hep-lat/9601022].
[14] S. Hashimoto, A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, Phys.
   Rev. D 61 (2000) 014502 [arXiv:hep-ph/9906376].
[15] D. Becirevic, G. Martinelli and G. Villadoro, Phys. Lett. B 633 (2006) 84 [arXiv:hep-lat/0508013].
[16] M. Ademollo and R. Gatto, Phys. Rev. Lett. 13 (1964) 264.
[17] M. Moulson [FlaviaNet Working Group on Kaon Decays], arXiv:hep-ex/0703013.