Moiré surface states and “high-temperature” superconductivity in topological insulators

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Recently, moiré superlattices have been found on the surface of topological insulators (TI) due to the rotational misalignment of topmost layers. In this work, we study the effects of moiré superlattices on the topological surface states. We introduce a continuum model of Dirac electrons moving in a periodic potential to describe the moiré surface states and identify various (high-order) van Hove singularities (VHS), which explains the experimentally observed peaks in the density of states (DOS). We show that the power-law divergent DOS at high-order VHS significantly enhances electron-phonon superconductivity. By solving the gap equation, we derive an analytic formula for the transition temperature $T_c$, which exhibits a power-law dependence on the retarded electron-phonon interaction strength $\lambda^\ast$.

I. INTRODUCTION

In recent years, moiré superlattices have been studied extensively in various 2D van der Waals heterostructures exemplified by graphene and transition metal dichalcogenide (TMD) multilayers [1–3]. These moiré systems exhibit a variety of remarkable electronic properties due to strong correlation effects in flat minibands. Besides graphene and TMD, another large family of moiré superlattices can be found in topological insulators [4–14], in particular Bi$_2$Se$_3$ and Bi$_2$Te$_3$ bulk crystals. When these bulk crystals are grown by the molecular-beam epitaxy (MBE), it is common to find a small rotational misalignment of topmost quintuple layers, leading to a moiré superlattice on the surface. Interestingly, a scanning tunneling microscope (STM) measurement [6] has directly observed such moiré superlattice in Bi$_2$Te$_3$ and found multiple sharp peaks in the local density of states (LDOS). Despite the ubiquity of moiré superlattices in TI, their effects on topological surface states have not been studied theoretically.

In this letter, we study moiré surface states of TI. The topological nature of TI surface states prevents them from gap opening as long as time-reversal symmetry is preserved, hence the moiré surface states do not form isolated mini bands, unlike other moiré systems such as graphene and TMD. Instead, we find prominent van Hove singularities (VHS) in moiré surface states which give rise to divergent density of states (DOS). Under appropriate conditions, some of these VHS exhibit power-law divergent DOS, which are known as high-order VHS [15].

We further study superconductivity at high-order VHS, where the electron-phonon interaction effect is significantly enhanced due to the divergent DOS. We find a new analytic formula for the superconducting critical temperature $T_c$ (see Eq. (9)), which exhibits a power-law dependence of the retarded electron-phonon interaction $\lambda^\ast$ and is thus parametrically enhanced with respect to the exponentially small $T_c$ in ordinary metals and at ordinary VHS [16, 17]. Importantly, the absence of moiré band gaps in the surface state spectrum enables a drastic reduction of Coulomb repulsion through retardation effects between the bandwidth of the surface state (at order of eV) and the Debye frequency [19].

This work is organized as follows: we first introduce and study a model of moiré surface states as Dirac fermion in a periodic scalar potential in Sec. II. Within the model, we identify high-order VHS at the crosses of circular Fermi surfaces. Then, we solve the gap equation for the superconducting critical temperature $T_c$ in the presence of power-law divergent density of states, taking account of both electron-phonon interaction and Coulomb repulsion within the Anderson-Morel approximation [19] (Sec. III). In the end, we discuss several experimental platforms to search for moiré surface states and enhanced superconductivity.

II. DIRAC FERMION IN A PERIODIC SCALAR POTENTIAL

In this section, we introduce and study a model of moiré topological insulator surface states as Dirac fermion in a periodic scalar potential. A previous DFT study in Bi$_2$Se$_3$/MoS$_2$ [7] revealed folded Dirac cones within the bulk gap due to the moiré superlattice. Thus, we start with the massless Dirac fermion in two
dimensions (2D)

$$H_0(k) = v_F(k_x \sigma^y - k_y \sigma^x),$$

where $v_F$ is the Fermi velocity, $k = (k_x, k_y)$ is the two-dimensional momentum, $\sigma = (\sigma^x, \sigma^y)$ are the Pauli matrices. Now we allow the continuous translation symmetry be broken into discrete ones by the moiré superlattice, while leave the time-reversal symmetry intact. Then the lowest order perturbation can be described by a spin-independent periodic scalar potential $U(r)$:

$$H(k; r) = H_0(k) + U(r)\sigma^0,$$

where $\sigma^0$ is the identity matrix, $U(r) = U(r + L_{1,2})$, and the $L_{1,2}$ are two primitive vectors of the moiré superlattice. A schematic diagram of this setup is shown in Fig. 1. This model can apply to bulk TI crystals with top layers twisted or the surface state in the interface between a topological insulator (TI) and a large-gap insulator.

The density of states (DOS) of the system described by Eq. (2) generally looks like the right panel in Fig. 1b, where positive and negative sides are qualitatively similar. Near zero energy, DOS grows linearly. As energy increases, VHS peaks emerge. At higher energy, new Dirac points are formed (known as satellite Dirac points $\Gamma_{36}$), so that the entire spectrum remains gapless. Among these VHS peaks, there are a few prominent ones that are, in fact, high-order VHS peaks, given appropriate parameters.

In comparison to ordinary VHS that are caused by saddle points in the energy dispersion, such high-order VHS peaks are caused by high-order critical points $k_c$ in momentum space, where the electron velocity and the Hessian matrix determinant both vanish, $\nabla E(k_c) = 0$, $\det D(k_c) = 0$, $(D_{ij} = \partial_i \partial_j E)$. Around these high-order VHS, the energy dispersion is characterized by high-order polynomials of momenta, and the DOS shows power-law divergence [21],

$$N(\xi) = \begin{cases} C_+ \xi^\nu, & \xi > 0 \\ C_- (-\xi)^\nu, & \xi < 0 \end{cases}.$$  

Here $-1 < \nu < 0$ is the power-law exponent and $C_\pm > 0$ are coefficients of the electron (hole) side. $E(k)$ denotes the energy dispersion and $\xi = E - E(k_c)$.

To be concrete, we first consider a periodic potential
as follows
\[ U(r) = 2U \sum_{j=1}^{3} \cos(G_j \cdot r), \]
where \( G_j = \frac{2\pi}{\sqrt{3}}L^{-1}(-\sin\frac{2\pi j}{3}, \cos\frac{2\pi j}{3}) \) are three reciprocal vectors, and \( U \) is the potential strength. Then there are two energy scales \( v_F/L \) and \( U \) in Eq. (2), and the low-energy physics is determined by a single dimensionless control parameter \( UL/v_F \).

As shown in the band structure (Fig. 1(b)), the first set of satellite Dirac points on positive side are found at \( M \) points, and there are generically six saddle points per moiré Brillouin zone (MBZ) between the main Dirac point \( \Gamma \) and satellite Dirac points \( M \). Remarkably, the Fermi surfaces passing through these saddle points are all perfect circles in a wide range of \( UL/v_F \) (Fig. 1b).

When \( U = 1.36v_F/L \), three ordinary saddle points and one local extremum merge into a high-order saddle point at the \( K \) point where all three Fermi surfaces intersect (Fig. 1c), and the dispersion around the \( K \) point becomes flattened (Fig. 1b). In the experiment, the potential strength \( U \) and the Fermi velocity \( v_F \) are mostly determined by the material, but we can tune this parameter \( UL/v_F \) by tuning the twisted angle \( \theta \) and the resulting superlattice constant \( L = a/\theta \). Thus, we can also define a magic angle \( \theta_m = 0.74Ua/v_F \) when our system hits high-order VHS, where \( a \) is the atomic lattice constant.

Among high symmetry points \( \Gamma, M \) and \( K \), the time-reversal invariant points \( \Gamma, M \) will always be at least doubly degenerate, while only \( K \) point can become spin singlet. We can thus expand the singlet dispersion \( E_k \) around the \( K \) point,
\[ E_{p+K} - E_K = \alpha p_x^2 + \beta(p_x^3 - 3p_x p_y^2) + \gamma p_y^4 + \cdots \] (5)
where \( p^2 = p_x^2 + p_y^2 \) with \( p_x \) (\( p_y \)) parallel (perpendicular) to the \( \Gamma K \) line. We then compute the Taylor coefficients \( \alpha, \beta \), and \( \gamma \) as functions of \( UL/v_F \) as shown in Fig. 2.

When \( U = 1.36v_F/L \), we find \( \alpha \) vanishes, while \( \beta \) remains finite, indicating a high-order saddle point described by a third-order polynomial \( E_{p+K} - E_K = \beta(p_x^3 - 3p_x p_y^2) \).

The density of states (DOS) of a \( C_3 \) saddle point diverges with power-law exponent \( \nu = -1/3 \) according to the scaling property of the dispersion [15, 21]. As shown in Fig. 2, the numerical power-law fitting of DOS gives \( \nu = -0.34 \), which agrees well with \( \nu = -1/3 \).

When \( U = 0.15v_F/L \), at energy much higher than the first set of satellite Dirac points, \( \Gamma \) point becomes a high-order Dirac point, where six circular Fermi surfaces intersect together as shown in Fig. 3a. The high-order Dirac point also exhibits power-law divergent DOS just like high-order VHS.

Next we consider a periodic potential with \( D_4 \) symmetry
\[ U(r) = 2U[\cos(2\pi_x/L) + \cos(2\pi y/L)], \] (6)
which corresponds to a square moiré superlattice. When \( U = 4.71v_F/L \), there are four high-order saddle points on four \( GM \) lines respectively (Fig. 3b), where the local dispersion becomes \( E_{q+V} - E_V = a q_x^2 + b q_y^2 + \cdots \) \((ab > 0)\). Here \( V \) denotes the momentum of high-order saddle point, and \( q_y \) (\( q_x \)) is parallel (perpendicular) to the \( GM \) line. Such kind of saddle point can split into at most two critical points: One ordinary saddle point and one ordinary extremum, and we call it a \( A_2 \) saddle point [21], where the energy contour is beak like (Fig. 3c). Details of high-order VHS \( A_2 \) and high-order Dirac points can be found in Appendix A.

In the \( D_4 \) potential given in Eq. (6), \( U(x + L/2, y + L/2) = -U(x, y) \), hence the system has an additional particle-hole symmetry \( E_k \rightarrow -E_k \), which does not exist for \( C_6 \) potential given in Eq. (1). This is consistent

FIG. 2. (a) Derivatives of the band dispersion at the \( K \) point under the potential given in Eq. (1). \( \alpha, \beta, \) and \( \gamma \) are 2nd, 3rd and 4th order derivatives with respective to momentum (see main text). When \( UL/v_F = 1.36, \alpha \) vanishes, making the \( K \) point a high-order saddle point. (b) DOS around the high-order VHS. The fitting is given by \( N(\xi) = c(\xi - \xi_0)^{\nu} + n \), where \( c = 0.20, \xi_0 = 2.59, \nu = -0.34 \), and \( n = -1.78 \). \( \nu \) = -0.34 agrees well with theoretical result \( \nu = -1/3 \) (see main text).

FIG. 3. (a) Fermi surface with a high-order Dirac point at the \( \Gamma \) point in \( C_6 \) potential Eq. (1). (b) Fermi surface with a high-order VHS of class \( A_2 \) on the high-symmetry line \( GM \) in \( D_4 \) potential Eq. (6). The ones passing through high-order VHS are plotted in thick black curves, among which those in (a) are perfect circles. The corresponding MBZ is plotted in dashed lines.
with the experiment result in bulk Bi$_2$Te$_3$ crystal, where particle-hole symmetry is broken [6]. In addition, high-symmetry points in the MBZ under the $D_4$ potential are all time-reversal invariant, thus cannot possess nondegenerate high-order VHS.

When the chemical potential is put at the energy of high-order saddle point, we expect interaction effect will be greatly enhanced due to the power-law divergent DOS. In the following we will discuss superconductivity near high-order VHS.

III. SUPERCONDUCTIVITY NEAR HIGH-ORDER VHS

In this section we show that due to power-law divergent DOS, superconducting critical temperature is greatly enhanced near high-order VHS. It is especially relevant for moiré surface states of topological insulators because the metallic nature of the system ensures an drastic downward renormalization of the Coulomb repulsion, whereas the absence of retardation effects in systems with isolated moiré bands presents a challenge to electron-phonon superconductivity.

To find an analytic formula for the critical temperature $T_c$, we employ the Anderson-Morel approximation to solve the gap equation. We assume the dimensionless interaction takes a simple form: piece wise constant attractive phonon interaction $\lambda > 0$ and repulsive interaction $\mu > 0$,

\[
g(\xi, \xi') = \begin{cases} 
\mu, & \xi D < |\xi|, |\xi'| < W \\
\mu - \lambda, & 0 < |\xi|, |\xi'| < \xi D
\end{cases},
\]

(7)

which is normalized with the constant DOS away from the van Hove singularity. Here $\xi = E - E_F$ is the electron energy measured from Fermi energy $E_F$. Notice that $g > 0$ means attraction and $g < 0$ repulsion. When we set the chemical potential $E_F$ to be exactly at the high-order VHS, the normalized DOS can be described by the piece wise function

\[
n(\xi) = \begin{cases} 
|\Lambda|^{-\nu} |\xi|^{-\nu}, & |\xi| < \Lambda \\
\frac{1}{\nu} & \Lambda < |\xi| < W
\end{cases},
\]

(8)

where $-1 < \nu < 0$ is the power-law exponent of the DOS. Four energy scales are involved in this problem: the superconducting critical temperature $T_c$, the high-order VHS peak cutoff $\Lambda$, the Debye frequency $\epsilon_D$, and the bandwidth $W$, which satisfy $T_c \ll \Lambda < \epsilon_D < W$.

Before we get into formal calculations, we first consider several limits with attractive interaction $\lambda$ and simple expression of DOS. The critical temperature $T_c$ is determined by the condition $\lambda \chi = 1$ with the pair susceptibility $\chi = \int_T^{\infty} n(\xi) \xi^{-1} d\xi$. When DOS is constant $n(\xi) = 1$, pair susceptibility is logarithmically divergent in temperature $\chi \sim \log(\epsilon_D/T)$, which leads to the BCS formula $T_c \sim \epsilon_D \exp(-1/\lambda)$. When DOS has an ordinary VHS with cutoff $\Lambda$, $n(\xi) = \log(\Lambda/|\xi|)$, we have $\chi \sim \log^2(\Lambda/T)$ and hence Labbé-Bok formula $T_c \sim A \exp(-1/\sqrt{\lambda})$ [10, 17]. When DOS has a high-order VHS with cutoff $\Lambda$ and power-law exponent $\nu$, $\chi \sim (T/\Lambda)^{\nu}$ and hence we have the power-law formula $T_c \sim \Lambda(1/\lambda)^{1/\nu}$, where $-1 < \nu < 0$.

With interaction Eq. (7) and DOS Eq. (8), we find an exact analytic formula of the critical temperature $T_c$ which generalizes the well known BCS formula,

\[
T_c = \frac{\Lambda}{I(\nu)\nu} \left[ \frac{1}{\lambda - \mu^*} - \log \left( \frac{\epsilon_D}{\Lambda} \right) + \frac{1}{|\nu|} \right]^{-1/|\nu|},
\]

(9)

where $\mu^* = \mu/[1 + \mu \ln(W/\epsilon_D)]$ is the screened repulsion, $I(\nu) = 2(2^{1/\nu} - 1)\Gamma(\nu)\zeta(\nu)$, $\Gamma(\nu)$ and $\zeta(\nu)$ are the gamma function and the zeta function respectively, and we put $\nu = -|\nu|$ to remind the reader $-1 < \nu < 0$. We direct readers to Appendix B for detailed derivation.

In the limit $\nu \rightarrow 0$, the power-law dependence of the effective interaction strength $\lambda^* = \lambda - \mu^*$ disappears because the $1/|\nu|$ term in the bracket dominates, then Eq. (9) reduces to the BCS formula of exponential dependence, $\lim_{\nu \rightarrow 0} T_c = 1.13\epsilon_D \exp(-1/\lambda^*)$ (see Appendix B). Reproducing the correct numerical prefactor indicates that our new formula is exact.

The new formula Eq. (9) is surprisingly an analytic function of the retarded attractive interaction strength $\lambda^* = \lambda - \mu^*$ with power-law dependence. The analytic nature of the formula suggests a dramatic enhancement of superconductivity compared to ordinary metals and at ordinary VHS when $\lambda^*$ is small. We compare our new formula with previous studies on various DOS in Appendix B.

![FIG. 4](image-url)

(a) Transition temperature $T_c$ at different effective interaction $\lambda^*$ with $\epsilon_D = 80$ K and $\nu = -1/3$ (the chemical potential is at the high-order VHS). The Anderson-Morel plot refers to the original Anderson-Morel formula $T_c = 1.13\epsilon_D \exp(-1/\lambda^*)$ [19], and the high-order VHS plot refers to Eq. (9). The solid blue line corresponds to $\Lambda = \epsilon_D$ and the dashed one corresponds to $\Lambda = \epsilon_D/2$, in either case $T_c$ with a high-order VHS is much higher than the one without, especially when the effective interaction $\lambda^*$ is small. (b) Transition temperature $T_c$ when the chemical potential $E_F$ is not exactly at the high-order VHS energy with $\Lambda = \epsilon_D$. $T_c$ starts to drop when $E_F$ is comparable to $\Lambda$.

We plot the transition temperature $T_c$ as a function
of effective interaction $\lambda^*$ in Fig. 4a with parameters relevant to topological insulators. We find that the transition temperature $T_c$ is enhanced enormously by the high-order VHS compared to the original Anderson-Morel result due to the power-law nature of the expression. This enhancement is robust as long as $\Lambda$ and $\epsilon_D^*$ are at the same order. When $\lambda^* = 0.2$ and $\Lambda = \epsilon_D^*/2$, the enhancement can still achieve $\gtrsim 200$ times.

Numerical evidence also shows that the high-order VHS has a robust effect on superconductivity even when the chemical potential is not exactly at the VHS energy $E_F \neq 0$ (Fig. 4b). We find that $T_c$ starts to drop when $E_F$ is comparable to $\Lambda$, and how quickly $T_c$ drops depends critically on $\lambda^*$. Nevertheless, when $\lambda^* > 0.2$, $T_c$ remains half of $T_{c,0} = T_c(E_F = 0)$ even when $E_F$ is significantly away from zero. This plot can be compared to future experimental data of $T_c$ when varying the filling $E_F$.

Now we discuss several possible systems to realize our model of Dirac fermion in a periodic potential and the phonon-induced superconductivity within. A prototypical system is the moiré topological insulator surface states, where the effective potential can be comparable to the kinetic energy at scale $v_F/L$. In real TIs, the Dirac velocity $v_F$ can be a few eV A’s [22–24], and the moiré supercell constant $L$ can be several or even tens of nm’s depending on the lattice mismatch [5–9], then the energy scale $v_F/L$ is at order of tens of meVs, which is comparable to the effective potential at moiré scale [25–27]. Furthermore, Bi$_2$Se$_3$ is believed to have strong electron-phonon interaction [28, 29]. Great efforts have been put into extract the electron-phonon coupling strength $\lambda$ of Bi$_2$Se$_3$ both theoretically and experimentally, and most studies fall into the range from $\lambda = 0.2$ to 0.5 (Fig. 4) [30–33]. Bi$_2$Se$_3$ also has a relatively large dielectric constant $\epsilon > 50$ [28, 29], and the bandwidth of topological surface bands reaches at least 800 meV [18], so the renormalized Coulomb repulsion $\mu^*$ is negligible, i.e. $\lambda^* \approx \lambda$. With the numbers given, we anticipate that superconductivity can occur on moiré surface states of topological insulators with transition temperature up to $T_c \sim 10$ K.

IV. CONCLUSION

In this work, we study the moiré topological insulator surface states using a continuum model of Dirac electrons moving in periodic potentials at moiré scale. Within the continuum model, we identify various types of high-order VHS. We further compute the superconducting transition temperature $T_c$ when the chemical potential is close to the high-order VHS. When exactly at the high-order VHS, we give an analytic formula of $T_c$, showing a power-law instead of exponential dependence of the retarded electron-phonon interaction strength. This result suggests a significantly enhanced superconductivity at high-order VHS, especially when the electron-phonon interaction is weak. In the end, we discuss several real materials that can demonstrate the enhancement of superconductivity due to high-order VHS.

Note added.– After this work is completed, we became aware of a related and independent work [34].

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Appendix A: High-order Dirac point and high-order VHS of type $A_2$
In this appendix, we solve the BCS gap equation at high-order VHS within Anderson-Morel approximation [35, 36],

$$\Delta_k = - \sum_{k'} V_{k,k'} \frac{\Delta_{k'}}{2E_{k'}} \tanh \frac{E_{k'}}{2T}$$  \quad (B1)$$

where $\Delta_k$ is the superconducting gap, and $E_k = \sqrt{v^2 + \Delta^2}$ is the quasiparticle energy. Due to the complexity of the gap equation, now we proceed within the Anderson-Morel approximation [19], in which both the interaction strength and the gap are assumed to be piece wise constant. We consider four energy scales: the interaction strength and the gap are assumed to be piece wise constant attractive phonon interaction $\lambda$ and a repulsive Coloumb interaction $\mu$,

$$g(\xi, \xi') = \begin{cases} 
\mu, & \epsilon_D < |\xi|, |\xi'| < W \\
\mu - \lambda, & 0 < |\xi|, |\xi'| < \epsilon_D.
\end{cases} \quad (B2)$$

which is normalized with the constant DOS $N_0$ away from the van Hove singularity, $V_k \xi_k = N_0 g(\xi, \xi')$. The normalized density of state $n(\xi) = N(\xi)/N_0$, however, is modified from the original Anderson-Morel model to account for the high-order VHS. For now we set the chemical potential $E_F$ to be exactly at the high-order van Hove singularity,

$$n(\xi) = \begin{cases} 
C|\xi|^{-\nu}, & |\xi| < \Lambda \\
1, & \Lambda < |\xi| < W
\end{cases} \quad (B3)$$

where $-1 < \nu < 0$ is the power-law exponent of the DOS. The continuity condition enforced that $C = |\Lambda|^{-\nu}$. Within the Anderson-Morel approximation, the gap also takes a simple form,

$$\Delta(\xi) = \begin{cases} 
\Delta_1, & |\xi| < \omega_D \\
\Delta_2, & \omega_D < |\xi| < W
\end{cases} \quad (B4)$$

Plugging in $N(\xi')$ and $V(\xi, \xi')$, the gap equation becomes

$$\left( \frac{\Delta_1}{\Delta_2} \right) = \begin{pmatrix} 
(\lambda - \mu) \left\{ I(\nu) \left( \frac{\xi}{\Lambda} \right)^{\nu} + \log \left( \frac{\xi}{\Lambda} \right) + \frac{1}{\nu} \right\} \\
-\mu \left\{ I(\nu) \left( \frac{\xi}{\Lambda} \right)^{\nu} + \log \left( \frac{\xi}{\Lambda} \right) + \frac{1}{\nu} \right\}
\end{pmatrix}
\begin{pmatrix} 
\Delta_1 \\
\Delta_2
\end{pmatrix}$$  \quad (B5)$$

where $I(\nu) = 2(2^{1-\nu} - 1)\Gamma(\nu)\zeta(\nu)$, $\Gamma(\nu)$ and $\zeta(\nu)$ are the gamma function and the zeta function respectively. Here we use an important integral,

$$\int_0^{\pi} d\xi n(\xi) \frac{\tanh (\beta \xi/2)}{\xi} = \left( \frac{T}{\Lambda} \right)^{\nu} \left( I(\nu) - \int_{\beta \Lambda}^{\infty} dx x^{\nu-1} \right) + \int_{\beta \Lambda}^{\epsilon_D} \frac{dx}{x}, \quad I(\nu) = \int_0^{\infty} dx x^{\nu-1} \tanh \left( \frac{x}{2} \right), \quad (B6)$$

where we used $T_c \ll \Lambda$. Solving the consistency equation gives the critical temperature

$$T_c = \frac{\Lambda}{I(\nu)^{1/\nu}} \left[ \frac{1}{\lambda - \mu} - \log \left( \frac{\epsilon_D}{\Lambda} \right) + \frac{1}{|\nu|} \right]^{-1/|\nu|}. \quad (B7)$$

Now we compare our result with previous studies
Tang et al. generalized Anderson and Morel’s result to the partially flat band scenario in the context of topological crystalline insulator interface superconductivity, where $n(\xi) = \alpha N_0$ within the flat band width $\Lambda$ \[4, 37\].

$$T_c = 1.13\Lambda \left( \frac{\epsilon_D}{\Lambda} \right)^{1/\alpha} \exp\left( -\frac{1}{\alpha(\lambda - \mu^*)} \right)$$  \hspace{0.5cm} (B11)

where in Fig. 7 we choose $\alpha = 1/(\nu + 1)$ such that the total density of states within $\Lambda$ coincide with our model.

The VHS plot refers to Labbé and Bok’s work in the context of cuprate superconductors in the limit $\Lambda \gg \epsilon_D$, where $n(\xi) = n_1 \log(\Lambda/\xi) + n_0$ within the VHS peak cutoff $\Lambda$ \[10, 17\].

$$T_c = \frac{\Lambda}{2} \exp\left( 0.819 + \frac{1}{n_1} - \sqrt{F} \right)$$  \hspace{0.5cm} (B12)

where $F \equiv \left( \frac{1}{n_1} + 0.819 \right)^2 + \left( \log(\frac{\lambda}{\lambda - \mu^*})^2 - 2 - \frac{2}{n_1} \left( \log\left( \frac{2\pi \nu}{\Lambda} \right) - \frac{1}{\lambda - \mu^*} \right) \right)$. Again we choose $n = -1 + 1/(\nu + 1)$ to normalized the total density of states within $\Lambda$. Here $\mu^*$ is renormalized slightly differently from the usual Anderson-Morel screening. We refer the reader to Ref. \[17\] for a more detailed discussion.

Hélikkiä et al. also considered a scenario with power-law divergent DOS $N(\xi) = \xi^\nu/\Lambda^\nu$ in the context of the multiple Dirac point \[38\]. Unlike us, they work in the limit $\Lambda \gg \epsilon_D$ and neglect the Coulomb repulsion. In our language, their formula can be rewritten as

$$T_c = \frac{2(\nu + 1)^{1/\nu}}{J(\nu)^{1/\nu}} \Lambda \left( \frac{\epsilon_D}{\Lambda} \right)^{1/|\nu|-1} \left( \frac{1}{\lambda} \right)^{-1/|\nu|}$$  \hspace{0.5cm} (B13)

where $J(\nu) = \Gamma(-\nu/2)\Gamma(3 + \nu/2)/\sqrt{\pi}$. The measured $T_c$ for twisted bilayer graphene (tBLG) is taken from Ref. \[39\] and the one for cuprates is taken to be $100\,K$ as a typical value. We use parameters relevant for tBLG in Fig. 7 (a), and those relevant for cuprates in Fig. 7 (b). We pick $\nu = -1/3$ for tBLG and $\nu = -1/4$ for cuprates, which corresponds to the leading order high-order VHS for systems with $C_3$ symmetry and $D_4$ symmetry respectively. The tBLG plot is not compatible with the VHS result since Labbé and Bok assumes that the flat band range $\Delta_{FB}$ is much larger compared to the Debye frequency $\epsilon_D$, which does not hold in tBLG.

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