Electromagnetic field energy density in artificial microwave materials with negative parameters

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Abstract

General relations for the stored reactive field energy density in passive linear artificial microwave materials are established. These relations account for dispersion and absorption effects in these materials, and they are valid also in the regions where the real parts of the material parameters are negative. These relations always give physically sound positive values for the energy density in passive metamaterials. The energy density and field solutions in active metamaterials with non-dispersive negative parameters are also considered. Basic physical limitations on the frequency dispersion of material parameters of artificial passive materials with negative real parts of the effective parameters are discussed. It is shown that field solutions in hypothetical materials with negative and non-dispersive parameters are unstable.

Keywords: Artificial materials, negative material parameters, energy density, equivalent circuit, instability.

1 Introduction

During a few recent years, artificial media with negative real parts of material parameters (called Veselago media, backward-wave media, double negative media) have attracted much attention in view of the first experimental realizations of such media and because of potential applications in sub-wavelength imaging. Some other applications have been proposed, including improvement of antenna performance. In paper [1], radiation from a small electric dipole inside a spherical shell of such material has been considered, with some conclusions regarding the antenna bandwidth. A two-dimensional model of an antenna coated by a dispersive material shell has been considered in [2]. These and many other issues involve considerations of the reactive energy stored in materials with negative parameters, and the goal of this paper is to develop a method for calculation of the stored energy density in complex materials and to discuss difficulties in the interpretation of the properties of exotic materials as well as the validity of the used models of materials.
with negative parameters. A general approach that we introduce in this paper allows to determine the time-averaged energy density of time-harmonic electromagnetic fields in dispersive and lossy materials with various dispersion laws. Also, the possibility to create artificial materials with dispersion-free and negative parameters using active inclusions [3] is revisited.

It is well known that the field energy density in materials can be uniquely defined in terms of the effective material parameters only in case of small (negligible) losses (e.g. [4]). This is because in the general case when absorption cannot be neglected, the terms

\[ \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \]

describe both the rate of changing the stored energy and the absorption rate. Only if the absorption is negligible, we can write

\[ \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial w_e}{\partial t} + \frac{\partial w_m}{\partial t}, \]

where \( w_e \) and \( w_m \) are the energy densities of the electric and magnetic fields, respectively.

For artificial materials based on metal or dielectric inclusions of various shapes absorption can be neglected when the operational frequency is far from the resonant frequencies of the inclusions and from the lattice resonances, if the material is periodical. For electromagnetic fields whose spectrum is concentrated near a certain frequency \( \omega_0 \), the time-averaged energy density in a material with scalar frequency-dispersive parameters \( \epsilon(\omega) \) and \( \mu(\omega) \) reads (e.g. [4, 5])

\[ w = \frac{1}{2} \left| \frac{d(\omega \epsilon(\omega))}{d\omega} \right|_{\omega=\omega_0} |E|^2 + \frac{1}{2} \left| \frac{d(\omega \mu(\omega))}{d\omega} \right|_{\omega=\omega_0} |H|^2. \]  

(3)

If in the vicinity of the operating frequency \( \omega_0 \) the frequency dispersion can be neglected and \( \epsilon \) and \( \mu \) can be assumed to be independent from the frequency, (3) simplifies to

\[ w = \frac{1}{2} \epsilon |E|^2 + \frac{1}{2} \mu |H|^2. \]  

(4)

The validity of this formula is restricted to positive values of \( \epsilon \) and \( \mu \) because no passive media in thermodynamic equilibrium can store negative reactive energy, as this is forbidden by the thermodynamics (the second principle)\(^1\) [4, 7]. Actually, this means that frequency dispersion cannot be neglected when estimating the stored energy in the frequency regions where the material parameters are negative.

If the material has considerable losses near the frequency of interest, it is not possible to define the stored energy density in a general way (more precisely, it is not possible to express that in terms of the material permittivity and permeability functions) [4]. Knowledge about the material microstructure is necessary to find the energy density, and this problem is far from trivial. In the literature, the energy density expressions have been

\(^1\)In thermodynamically non-equilibrium states, e.g. in non-uniform magnetized plasmas, the field energy may take negative values [6] leading to power amplification and instabilities. We do not consider such situations in this paper.
derived for an absorbing classical dielectric (Lorentz dispersion) with a single resonant frequency [8] and for the case where also the permeability obeys the same dispersion law as the permittivity [9]. The known artificial materials with negative real parts of the material parameters have different and more general dispersion laws, which means that we need to develop a more general approach suitable for calculations of the stored energy density in general dispersive and lossy materials. Such method will be presented in this paper.

In the next section we start from a brief review of the general properties of the constitutive parameters of passive low-loss materials, when the energy density can be defined in terms of the effective material parameters by formula (3).

2 Limitations on material parameters and dispersion in low-loss passive linear media

In this section we consider passive linear materials in thermodynamic equilibrium in the frequency regions where absorption can be neglected and the field energy density can be found in terms of the effective permittivity and permeability functions. For simplicity of writing, we restrict the analysis to isotropic media.

2.1 Limitations on material parameters

Landau and Lifshitz in [4, §84] give a proof that for all linear passive materials in the frequency regions with weak absorption the value of $w$ in (3) is not only always positive\(^2\), but it is always larger than the energy density of the same fields $E$ and $H$ in vacuum. Indeed, the following inequalities can be derived from the causality requirement assuming negligible losses [4]:

$$\frac{d\varepsilon(\omega)}{d\omega} > 0$$ (5)

(Eq. (84.1) in [4]) and

$$\frac{d\varepsilon(\omega)}{d\omega} > \frac{2(\varepsilon_0 - \varepsilon)}{\omega}$$ (6)

(Eq. (84.2) in [4]). Summing these two inequalities one finds that [4, §84]

$$\frac{d(\omega\varepsilon(\omega))}{d\omega} > \varepsilon_0.$$ (7)

The same is true for the permeability as well:

$$\frac{d(\omega\mu(\omega))}{d\omega} > \mu_0.$$ (8)

This has a clear physical meaning: To create fields in a material, work must be done to polarize the medium, which means that in the absence of losses more energy will be stored in material than in vacuum. This result is very general and applies also to passive low-loss metamaterials with negative parameters.

\(^2\)Positiveness of the derivatives in (3) is equivalent to the Foster theorem in the circuit theory.
Inequality (6) can be cast in equivalent form
\[
\frac{d(\omega \epsilon(\omega))}{d\omega} > 2\epsilon_0 - \epsilon(\omega).
\] (9)

Depending on the value of \(\epsilon\), either (7) or (9) is stronger. As is seen from (9), when the permittivity is negative and large in the absolute value, the permittivity must be very dispersive.

Considering plane electromagnetic waves in transparent isotropic dispersive materials, Sivukhin [10] gave one more limitation on the relative material parameters:
\[
\frac{d(\omega \epsilon_r(\omega))}{d\omega} + \frac{\mu_r}{\epsilon_r} \frac{d(\omega \mu_r(\omega))}{d\omega} > 0.
\] (10)

This relation holds if both \(\epsilon_r\) and \(\mu_r\) are either positive or negative.

If in a certain model the material parameters are assumed to be completely lossless, the above inequalities can become equalities. For example, the lossless plasma permittivity function
\[
\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)
\] (11)
is just on the allowed limit, because in this case
\[
\frac{d(\omega \epsilon(\omega))}{d\omega} = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega^2}\right) = 2\epsilon_0 - \epsilon(\omega)
\] (12)

It is easy to check that the lossless Lorentz permittivity model
\[
\epsilon(\omega) = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)
\] (13)
satisfies all the above inequalities at all frequencies.

Modeling of artificial magnetic materials requires more care because the very notion of permeability loses its meaning at high frequencies. Thus, the model permeability expressions obtained from quasi-static considerations do not necessarily satisfy the basic physical requirements at high frequencies. An important example is the effective permeability of a mixture of chiral or omega particles [11] or split-ring resonators [12,13], or of arrays of “swiss rolls” [13]:
\[
\mu = \mu_0 \left(1 + \frac{A\omega_p^2}{\omega_0^2 - \omega^2}\right).
\] (14)

This function has a physically sound behavior at low frequencies \([\mu(\omega) = O(\omega^2)]\) and near the resonance, but in the limit \(\omega \to \infty\) it does not tend to \(\mu_0\). However, in the limit of extremely high frequencies materials cannot be polarized at all because of inertia of electrons, so the parameters must tend to \(\epsilon_0\) and \(\mu_0\). As a result, this expression becomes non-physical [due to instantaneous response of the material, condition (8) is not satisfied]

\[^3\text{For this reason, some authors use the Lorentz dispersion law (36) to model the effective permeability of dense arrays of split-ring resonators, e.g. [14,15]. That model is physically sound at high frequencies, but fails in the low-frequency limit.}\]
at frequencies larger than $\sqrt{3}\omega_0$. As explained in [4, §82], the integrals in the Kramers-Krönig relations should be truncated at a high enough frequency where the permeability becomes nearly real and constant (formula (82.17) in [4]). Note, however, that inequality

$$\frac{d\mu(\omega)}{d\omega} > 0$$

(15)

for the permeability function (14) is still satisfied at all frequencies.

2.2 Instability of field solutions in media with non-dispersive negative parameters

In this section we will introduce a circuit model for complex passive metamaterials needed to study the energy density in complex media and use it first to show that although field solutions of the Maxwell equations for materials with non-dispersive negative material parameters exist, they are unstable. This fact appears to be quite obvious, but for some reason it appears to be neglected in the literature. Indeed, let us consider a material model with constant and negative $\epsilon$ and $\mu$, as e.g. in [1]. Within the frame of this model, the energy density is given by (4) and it is negative [1]. If due to a fluctuation the field at a certain moment of time increases, the stored energy will decrease. In this situation the field will increase even more, so that the total energy of the system be minimized.

As a first simple example, let us consider a parallel-plate capacitor filled by a hypothetical material with a negative permittivity $\epsilon < 0$ which does not depend on the frequency. Let us assume that the capacitor is charged and the voltage between its terminals is $u(0)$. At $t = 0$ it is connected to a resistor (resistance $R$). The voltage satisfies the differential equation

$$u(t) + CR\frac{du(t)}{dt} = 0,$$

(16)

whose solution is

$$u(t) = u(0)e^{-\frac{t}{RC}}.$$  

(17)

Under our assumption, capacitance $C$ is negative, which means that the capacitor does not discharge but, on the contrary, charges itself, collecting free charges from the conductor and resistor. This is an expected result, because in this process the total field energy in the system decreases.

The same conclusion follows if the transient regimes in these systems are analyzed. Figure 1 shows a capacitor that is connected to a time harmonic voltage source $V = \ldots$
Figure 2: Connection of a capacitor filled by a lossless plasma to a time-harmonic voltage source (left) and the equivalent circuit (right).

\[ V_0 \sin(\omega t) \text{ at time } t = 0 \text{ (there is no charge in the capacitor at } t < 0). \text{ The current in the circuit reads} \]

\[ i(t) = V_0 \omega C \frac{\cos(\omega t) + \omega RC \sin(\omega t) - e^{-\frac{t}{RC}}}{1 + \omega^2 R^2 C^2}. \quad (18) \]

If the capacitance is negative, the current exponentially grows. This result shows that the time-harmonic solutions in media with negative non-dispersive parameters are unstable, because a small increase in the source amplitude due to noise will exponentially grow. This is the reason for the known fact [16] that finite-difference time-domain schemes for media with negative and non-dispersive material parameters are unstable.

In reality, these instabilities do not exist because all passive materials with negative parameters are frequency dispersive. To illustrate this, let us consider a more realistic case where the negative permittivity corresponds to a lossless plasma:

\[ \epsilon(\omega) = \epsilon_0 \epsilon_r = \epsilon_0 \left(1 - \frac{\omega^2}{\omega_p^2}\right). \quad (19) \]

At the frequencies \( \omega < \omega_p \) the permittivity is real and negative. The real parameter \( \omega_p \) is called plasma frequency. At microwaves, dense arrays of ideally conducting wires can be described (although only for specific excitations [17]) by this model (e.g. [18–20]).

If we fill a capacitor with such material, its impedance becomes

\[ Z = \frac{1}{j\omega C} = \frac{1}{j\omega C_0 \left(1 - \frac{\omega^2}{\omega_p^2}\right)}, \quad (20) \]

where \( C_0 \) is the capacitance of the same capacitor filled by vacuum. Obviously, this corresponds to a parallel connection of a capacitor with capacitance \( C_0 \) and an inductor with \( L = 1/(C_0 \omega_p^2) \), as illustrated in Figure 2. Connection of a capacitor filled by a lossless plasma to a source is equivalent to connection of a usual parallel resonant circuit to the same source. Although the filling material has a negative permittivity at \( \omega < \omega_p \), both the capacitance and inductance in the equivalent circuit are positive. Naturally, the solution for the current in this circuit (that can be readily found using the Laplace transform, for example) does not contain any growing exponents.

Similarly to the case of capacitors filled with hypothetical non-dispersive materials with negative permittivities, current through an inductor filled with a hypothetical non-dispersive material with a negative permeability exponentially grows in time, if such inductor is connected to a voltage source via a resistor. For a more realistic consideration, let us assume that the magnetic material that fills the inductor’s coil can be modeled by the permeability function (14), where the magnitude factor \( A \) does not depend on the
frequency. In a certain frequency region (higher than the resonant frequency $\omega_0$) the permeability is negative.

Let us consider an inductor filled by this material and connected via a resistor to a time-harmonic voltage source (Figure 3, left). The impedance of this inductor is

$$Z(\omega) = j\omega L_0 \mu_r(\omega) = j\omega L_0 + \frac{j\omega^3 L_0 A}{\omega_0^2 - \omega^2},$$

where $L_0$ is the inductance of the same coil without any magnetic material inside. This is the same as the impedance of an inductor magnetically coupled to a resonant circuit (Figure 3, right) [2]:

$$Z(\omega) = j\omega L_0 + \frac{j\omega^3 M^2/L}{\omega_0^2 - \omega^2}.$$  

Thus, the circuit shown on the right of Figure 3 which contains only usual constant and positive inductances and a capacitance is equivalent to the circuit containing a magnetic core with permittivity (14) (Figure 3, left) if we choose $M^2/L = L_0 A$. As in the case of materials with negative permittivity, it is obvious that no instabilities occur in this simple passive circuit.

Next we will determine the stored energy density in particular realizations of Veselago media at microwaves, taking into account dispersion and dissipation.

### 3 Passive dispersive and lossy materials with negative parameters

For media without magnetoelectric interactions that can be adequately characterized by two materials parameters: the permittivity and permeability, it is possible to consider energies stored in the electric and magnetic fields separately. Indeed, the properties of linear media do not depend on what particular external field we apply. Having the full freedom to choose the external sources, we can always realize a situation where in a certain (small) volume only electric or magnetic field is non-zero. Because we deal with effective materials, the period of the microstructure or the average distance between inclusions is considerably smaller than the wavelength, otherwise one cannot introduce effective permittivity and permeability. Thus, we can take a representative sample of the material that contains many inclusions but whose size is still much smaller than the wavelength, and probe its properties in (nearly) uniform electric and magnetic fields.
3.1 Field energy density in wire media

Negative effective permittivity is most often realized by dense arrays of parallel thin metal wires. For plane electromagnetic waves whose wave vector is orthogonal to the wires, the effective permittivity for electric fields directed along the wires can be modeled by the plasma permittivity function

$$\epsilon = \epsilon_0 \left[ 1 - \frac{\omega_p^2}{\omega (\omega - j\Gamma)} \right].$$

(23)

There exist several models for the equivalent plasma frequency $\omega_p$, and here we will use the quasi-static model [20] that is not limited to the case of small wire radius and allows to estimate the loss factor. For example, if the skin effect in the wires can be neglected (uniform current distribution over the wire cross section), the effective parameters read [20]

$$\omega_p^2 = \frac{2\pi}{a^2 \epsilon_0 \mu_0 \log \frac{a^2}{4r_0 (a-r_0)}},$$

(24)

$$\Gamma = \frac{2}{\sigma \mu_0 r_0^2 \log \frac{a^2}{4r_0 (a-r_0)}}.$$  

(25)

Here $a$ is the array period, $r_0$ is the wire radius, $\sigma$ is the conductivity of the wire material, and $\epsilon_0$ and $\mu_0$ are the parameters of the matrix. The matrix is assumed to be a lossless magnetodielectric, so $\epsilon_0$ and $\mu_0$ are real numbers.

To determine the stored field energy density in this material, we position a small (in terms of the wavelength or the decay factor in the effective medium) piece of this material in a parallel-plate capacitor. Generalizing formula (20) using (23), we have for the admittance

$$Y = j\omega C_0 + \frac{1}{j\omega L + R},$$

(26)

where $C_0$ is now the capacitance of the capacitor filled with the matrix material (permittivity $\epsilon_0$), and

$$L = 1/(\omega_p^2 C_0), \quad R = \Gamma/(\omega_p^2 C_0).$$

(27)

Obviously, the equivalent circuit is a parallel connection of a capacitor and an inductor with a loss resistor, Figure 4. This circuit has the same input impedance as the actual capacitor filled by a material sample.
However, before using this circuit in order to calculate the stored reactive energy in the medium, we must ensure that the circuit structure indeed corresponds to the microstructure of the material under study. It is well known that from the input impedance of a circuit it is impossible to uniquely determine the circuit structure. In other words, different circuits can have the same input impedance at all frequencies (e.g., [7]). In the context of this study this means that our circuit model correctly describes the input impedance of the material-loaded capacitor, but it may fail to properly model the material microstructure.

In our particular case the material is realized as an array of wires along the electric field direction (that is, running from one plate of the capacitor to the other). Apparently, this array of wires possesses some inductance and resistance connected in series, so we see that our model indeed corresponds to the microstructure of the medium, and we can use it.

In the time-harmonic regime the time-averaged stored reactive energy is

$$W = \frac{1}{2} \left( C_0 |V_C|^2 + L |I_L|^2 \right), \quad (28)$$

where $V_C$ is the voltage amplitude on the capacitor and $I_L$ is the amplitude of the current through the inductor (see the equivalent circuit in Figure 4). This can be written as

$$W = \frac{1}{2} C_0 |V_C|^2 \left[ 1 + \frac{L}{C_0 (\omega^2 L^2 + R^2)} \right]. \quad (29)$$

For a parallel-plate capacitor (the plate area $S$, the distance between the plates $d$), we have $C_0 = \epsilon_0 S / d$ and $V_C = Ed$. The total energy is the energy density $w_e$ multiplied by the capacitor volume $Sd$:

$$W = w_e S d = \frac{1}{2} \frac{\epsilon_0 S}{d} |E|^2 d^2 \left[ 1 + \frac{L}{C_0 (\omega^2 L^2 + R^2)} \right]. \quad (30)$$

Thus, the energy density reads

$$w_e = \frac{\epsilon_0}{2} \left( 1 + \frac{\omega_p^2}{\omega^2 + \Gamma^2} \right) |E|^2, \quad (31)$$

where we have substituted the values of the circuit parameters from (27). Parameters $\omega_p$ and $\Gamma$ are given by (24) and (25), respectively.

If the losses can be neglected ($\Gamma \ll \omega$), the same result follows from (3), since

$$\frac{d(\omega \epsilon(\omega))}{d\omega} = \epsilon_0 \left( 1 + \frac{\omega_p^2}{\omega^2} \right). \quad (32)$$

Let us next consider a wire medium where the matrix is a lossy dielectric, with the permittivity $\epsilon = \epsilon_0 - j\sigma_d / \omega$, where $\sigma_d$ is the conductivity of the matrix material. Following the approach of [20], we find that the effective permittivity is

$$\epsilon = \epsilon_0 \left[ 1 - \frac{\omega_p^2}{\omega (\omega - j\Gamma)} \right] - j \frac{\sigma_d}{\omega}, \quad (33)$$
where the plasma parameters $\omega_p$ and $\Gamma$ remain the same as for wires in a lossless matrix (the physical reason for this is that the negative permittivity appears due to the inductance of the wire array, and that inductance does not depend on the dielectric loss). In the equivalent circuit shown in Figure 4, the matrix loss will be reflected by an additional loss resistance $R_d = d/ (\sigma d S)$ connected in parallel with the capacitor $C_0$. This means that the expression for the stored energy density (31) does not change if the matrix has some non-zero conductivity: dielectric losses in the matrix have no effect on the stored energy density function, while losses in the wires have a strong effect.

Let us assume next that the matrix has no conductivity, but there are some magnetic losses: the matrix parameters are $\epsilon_0$ and $\mu = \mu_0 - j \mu''$, where $\epsilon_0$ and $\mu_0$ are real. The effective permittivity of the wire medium in this matrix can be found using formulas of [20] with substitution $L \rightarrow L(1 - j \mu''/\mu_0)$ where $L$ is the inductance per unit length of the wire array in the matrix with the permeability $\mu_0$. The result is the same as (23) where the plasma frequency does not depend on $\mu''$ and is given by (24), but the loss factor $\Gamma$ is different:

$$\Gamma = \frac{2}{\sigma \mu_0 r_0^2} \log \frac{a^2}{4a(a-r_0)} + \omega \frac{\mu''}{\mu_0}.$$ (34)

In addition to the loss factor due to resistive wires, there is a factor measuring magnetic losses in the background medium. The structure of the equivalent circuit in Figure 4 does not change, but the resistance

$$R = \frac{\Gamma}{\omega^2 \sigma C_0}$$ (35)

is now frequency-dependent. However, the stored energy density can be still calculated using formula (31), because the additional impedance is real.

### 3.2 Field energy density in artificial Lorentzian dielectrics

Negative permittivity can be alternatively realized using artificial dielectrics with resonant inclusions. Frequency dispersion in such materials is described by the Lorentz formula

$$\epsilon = \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega^2 - \omega_0^2 + j \omega \Gamma}\right),$$ (36)

which is widely used as a model of natural materials in solid state physics. We will consider microwave materials designed as a collection of short metal needles oriented along the electric field direction and distributed in a lossless matrix. If the length of the needles is much smaller than the wavelength and the distance between the needles is much larger than the needle length but much smaller than the wavelength, formula (36) is a good estimate for the effective permittivity. Parameters $\omega_p$, $\omega_0$ and $\Gamma$ can be estimated in terms of the particle dimensions and the inclusion concentration using the antenna model of an individual inclusion [21] and an appropriate mixing rule (e.g., [21, 22]).

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4The inclusion shape is actually not critical for the validity of the model relation (36). They can be metal ellipsoids or spheres, for example.
Making use of the same approach as above, we consider a capacitor filled with a material having this dispersion law. Its admittance is

\[ Y = j\omega C_0 \varepsilon = j\omega C_0 + \frac{j\omega C_0 \omega_p^2}{\omega_0^2 - \omega^2 + j\omega \Gamma}. \] (37)

Apparently, this is the admittance of a parallel connection of a capacitor \( C_0 \) and a series resonant circuit with the elements

\[ C = \frac{C_0 \omega_p^2}{\omega_0^2}, \quad L = \frac{1}{(\omega_p^2 C_0)}, \quad R = \frac{\Gamma}{(\omega_p^2 C_0)}. \] (38)

It differs from the equivalent circuit for wire media (Figure 4) by the additional capacitance \( C \) in series with \( L \) and \( R \). This equivalent circuit is a valid model for the microstructure of this material because currents along needles are modeled by inductance \( L \) and charges at the ends of the needles by capacitance \( C \). The loss is due to non-ideally conducting material of the needles (the matrix material is assumed to be lossless), so it is appropriately modeled by resistor \( R \) in series with the inductance.

The stored reactive energy is the sum of the energies stored in all reactive elements:

\[ W = w_e S d = \frac{1}{2} \left( C_0 |V_{C_0}|^2 + L|I_L|^2 + C|V_C|^2 \right), \] (39)

where \( V_{C_0} \) and \( V_C \) are the voltages at the respective elements. Solving for \( I_L \) and \( V_C \) and substituting the equivalent circuit parameters (38) we find

\[ w_e = \frac{\varepsilon_0}{2} \left[ 1 + \frac{(\omega^2 + \omega_0^2)\omega_p^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \right] |E|^2. \] (40)

This result coincides with that obtained earlier in [9], where the motion equation for the electric polarization was directly solved. For the case of negligible losses (\( \Gamma \to 0 \)), the same result follows from (3).

The present method extends to the case of many resonant frequencies (multi-phase mixtures of inclusions of several different sizes) by simply adding more parallel \( LCR \) branches to the equivalent circuit.

### 3.3 Field energy density in dense arrays of split rings

Dense arrays of split rings and other similar structures can be modeled in the quasi-static regime by the following effective permeability (e.g. [12, 13]):

\[ \mu = \mu_0 \left( 1 + \frac{A\omega^2}{\omega_0^2 - \omega^2 + j\omega \Gamma} \right), \] (41)

where the magnitude factor \( A \) and the loss factor \( \Gamma \) do not depend on the frequency. Similarly to the approach introduced above for artificial dielectrics, we position a small (in terms of the wavelength or the decay length in the effective medium) sample in the magnetic field of a solenoid with inductance \( L_0 \). The inductance becomes

\[ Z(\omega) = j\omega L_0 \mu_r(\omega) = j\omega L_0 + \frac{j\omega^3 L_0 A}{\omega_0^2 - \omega^2 + j\omega \Gamma}. \] (42)
Figure 5: Magnetic material sample in the probe magnetic field of a solenoid (left) and the equivalent circuit (right).

An equivalent circuit with the same impedance\(^5\) is shown in Figure 5. Indeed, the input impedance seen by the source is

\[
Z = j\omega L_0 + \frac{j\omega^3 M^2 / L}{1/LC - \omega^2 + j\omega R/L}. \tag{43}
\]

This is the same as (42) if

\[
\frac{M^2}{L} = L_0 A, \quad \frac{1}{LC} = \omega_0^2, \quad \frac{R}{L} = \Gamma. \tag{44}
\]

This is a correct equivalent representation from the microscopic point of view, because the material which we model is a collection of capacitively loaded loops magnetically coupled with the incident magnetic field.

The total stored reactive energy is the sum of the energies stored in all reactive elements:

\[
W = \frac{1}{2} (L_0 |I|^2 + L |I_L|^2 + C|V_C|^2). \tag{45}
\]

Expressing \(I_L\) and \(V_C\) in terms of \(I\), we get, similarly to the derivations in [2],

\[
W = \frac{1}{2} L_0 |I|^2 \left[ L_0 + \frac{\omega^2 M^2 C (1 + \omega^2 LC)}{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2} \right] |I|^2. \tag{46}
\]

Rearranging terms and substituting the equivalent parameters (44), this can be written as

\[
W = \frac{1}{2} L_0 |I|^2 \left[ 1 + \frac{A\omega^2 (\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \right]. \tag{47}
\]

Considering the stored energy in one unit-length section of the solenoid, we have

\[
W = w_m S = \frac{1}{2} \mu_0 n^2 S \frac{|H|^2}{n^2} \left[ 1 + \frac{A\omega^2 (\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \right], \tag{48}
\]

where \(S\) is the solenoid cross section area and \(n\) is the number of turns per unit length. We have substituted the solenoid inductance per unit length (a tightly wound long solenoid) \(L_0 = \mu_0 n^2 S\) and used the relation \(I = H/n\) between the current \(I\) and the magnetic field inside the solenoid \(H\). Finally, the stored field energy density is found to be

\[
w_m = \frac{\mu_0}{2} \left[ 1 + \frac{A\omega^2 (\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \right] |H|^2. \tag{49}
\]

\(^5\)A similar approach was used in [2] to determine the equivalent quality factor of a simple radiator loaded by an artificial magnetic.

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It is important to note that in this particular case formula (3) leads to an incorrect expression even if the losses are negligible ($\Gamma \rightarrow 0$). For $\omega > \sqrt{3}\omega_0$ the stored energy density obtained from (3) is less than the energy stored in vacuum, and it becomes even negative at still higher frequencies. This is a manifestation of the failure of the quasistatic permeability model discussed above in Section 2.1. Formula (49) should be used even in the case of small losses.

4 Artificial negative-permittivity materials containing active inclusions

In paper [3] it was shown that in principle it is possible to realize a material with non-dispersive negative material parameters using artificial “molecules” that contain electronic circuits (impedance inverters). In these active materials, the basic physical limitation (7) does not apply. In the light of the previous consideration, it is of interest to study if in this case the stored energy density can be negative and clarify the physical meaning of this effect.

Let us consider an artificial molecule in form of a short dipole antenna loaded by a bulk impedance (Figure 6, left). To realize a wide-band non-dispersive negative permittivity, the load must be a negative capacitance [3]:

$$Z_{\text{load}} = -\frac{1}{j\omega C_{\text{inp}}} - \frac{Nl_{\text{eff}}^2}{j\omega\varepsilon_0(1 + |\epsilon|)},$$

where $C_{\text{inp}}$ is the input capacitance of the dipole antenna, $N$ is the number of particles per unit volume, $l_{\text{eff}}$ is the effective antenna length, and $\epsilon$ is the effective negative permittivity that we want to realize [the low-density limit of the Clausius-Mossotti formula was used in [3] to derive (50)]. On the right of Figure 6, the equivalent circuit of the whole molecule is shown. The negative capacitance is realized by an impedance inverter, which is connected to an equivalent circuit of a short dipole antenna ($C_{\text{inp}}$ is its input capacitance). $R_{\text{loss}}$ is the loss resistance of the antenna (the radiation resistance is compensated by the interaction field, as we suppose that the array of particles is regular or enough dense [21]).
The total impedance connected to the voltage source (external electric field) reads

\[ Z = R_{\text{loss}} - \frac{N_{\text{eff}}^2}{j\omega\varepsilon_0(1 + |\varepsilon|)}, \]  

that is, effectively we have a capacitor filled with a non-dispersive negative-permittivity material excited by a voltage source. As we know from section 2.2, this system is unstable, which is an expected result. This is in agreement with experiments [23].

5 Conclusions

A general approach that allows to determine the stored energy density in complex composite microwave materials has been presented. The method is based on an equivalent circuit representation of small material samples excited by electric and magnetic fields. Introduction of equivalent circuit parameters for specific microstructures of media is physically equivalent to an appropriate averaging procedure, needed to determine the properties of the effective medium. Particular examples of wire media (negative-epsilon material) and arrays of split-rings (negative-mu material) have been considered, as well as the usual Lorentzian dielectrics with losses. The last case has been considered in the literature using a different approach, and the present result agrees with the known formula.

The above derivations show how the energy density can be found for any passive and lossy composite, if its microstructure is known. The energy density is determined in terms of the energy stored in the reactive elements of the equivalent circuits. Naturally, in all cases the stored energy is positive, as it should be in all passive materials.

This conclusion appears to be very natural if one remembers that passive metamaterials exhibiting negative material parameters are anyway made from usual materials like metals or dielectrics. On the microscopic level, the stored energy is the electromagnetic field energy in the matrix material (normally a dielectric) and in the inclusions (normally a metal of another dielectric). This energy is a strictly non-negative definite function. The energy stored in a sample of the effective medium is the average of the corresponding microscopic quantity, and there is no reason to expect that for some specific shapes of metal inclusions the effective material will store negative energy. For example, the use of passive metamaterials in the design of antennas basically means adding some extra metal or dielectric elements like metal wires or split-ring resonators to a simpler antenna. On the fundamental level, this means only changing the antenna shape.

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