Sharp tridiagonal pairs

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Abstract

Let $\mathbb{K}$ denote a field and let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider a pair of $\mathbb{K}$-linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfies the following conditions: (i) each of $A, A^*$ is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a tridiagonal pair on $V$. It is known that $d = \delta$ and for $0 \leq i \leq d$ the dimensions of $V_i, V_{d-i}, V_i^*, V_{d-i}^*$ coincide. We say the pair $A, A^*$ is sharp whenever $\dim V_0 = 1$. A conjecture of Tatsuro Ito and the second author states that if $\mathbb{K}$ is algebraically closed then $A, A^*$ is sharp. In order to better understand and eventually prove the conjecture, in this paper we begin a systematic study of the sharp tridiagonal pairs. Our results are summarized as follows. Assuming $A, A^*$ is sharp and using the data $\Phi = (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^\delta$) we define a finite sequence of scalars called the parameter array. We display some equations that show the geometric significance of the parameter array. We show how the parameter array is affected if $\Phi$ is replaced by $(A^*; \{V_i^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$ or $(A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^\delta$) or $(A; \{V_i\}_{i=0}^d; A^*; \{V_{d-i}\}_{i=0}^d)$. We prove that if the isomorphism class of $\Phi$ is determined by the parameter array then there exists a nondegenerate symmetric bilinear form $\langle \ , \rangle$ on $V$ such that $\langle Au, v \rangle = \langle u, A^*v \rangle$ and $\langle A^* u, v \rangle = \langle u, A^*v \rangle$ for all $u, v \in V$.

1 Tridiagonal pairs

Throughout the paper $\mathbb{K}$ denotes a field and $V$ denotes a vector space over $\mathbb{K}$ with finite positive dimension.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let $\text{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all $\mathbb{K}$-linear transformations from $V$ to $V$. For $A \in \text{End}(V)$ and for a subspace $W \subseteq V$, we call $W$ an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case $\theta$ is the eigenvalue of $A$ associated with $W$. We say $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$. We now recall the notion of a tridiagonal pair.

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Definition 1.1 [19] By a tridiagonal pair on $V$ we mean an ordered pair of elements $A, A^*$ taken from $\text{End}(V)$ that satisfy (i)–(iv) below:

(i) Each of $A, A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0, W \neq V$.

We say the pair $A, A^*$ is over $K$.

Note 1.2 It is a common notational convention to use $A^*$ to represent the conjugate-transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

We refer the reader to [1–3, 5, 19–21, 27, 35–37, 45, 66] for background on tridiagonal pairs. See [4, 6–11, 14, 15, 17, 18, 22–26, 29, 31–33, 47–49, 51–53, 55, 64, 69] for related topics.

Let $A, A^*$ denote a tridiagonal pair on $V$, as in Definition 1.1. By [19, Lemma 4.5] the integers $d$ and $\delta$ from (ii), (iii) are equal; we call this common value the diameter of the pair. By [19, Corollary 5.7], for $0 \leq i \leq d$ the spaces $V_i, V_i^*$ have the same dimension; we denote this common dimension by $\rho_i$. By the construction $\rho_i \neq 0$. By [19, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the shape of $A, A^*$. The following special case has received a lot of attention. By a Leonard pair we mean a tridiagonal pair with shape $(1, 1, \ldots, 1)$ [54, Definition 1.1]. There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the $q$-Racah polynomials and their relatives [62]. This family coincides with the terminating branch of the Askey scheme [30]. See [12, 13, 16, 34, 38–44, 46, 54, 55–53, 65, 67, 68] for more information about Leonard pairs. Our point of departure in this paper is the following conjecture.

Conjecture 1.3 [27] Let $A, A^*$ denote a tridiagonal pair over an algebraically closed field. Then $\rho_0 = 1$ where $\{\rho_i\}_{i=0}^d$ is the shape of $A, A^*$.

Note 1.4 Conjecture [1.3] has been proven for some special cases that are described as follows. Referring to Conjecture [1.3] there is a parameter $q$ associated with $A, A^*$ that is used to describe the eigenvalues; we discuss $q$ above Lemma [9.1]. In [28] Ito and the second author prove Conjecture [1.3] assuming $q$ is not a root of unity. There is a family of tridiagonal pairs with $q = 1$ that are said to have Krawtchouk type [27]. In [27] Ito and the second author prove Conjecture [1.3] assuming $A, A^*$ has Krawtchouk type.
In order to better understand and eventually prove Conjecture 1.3, in this paper we begin a systematic study of those tridiagonal pairs that satisfy its conclusion. We start with a definition.

**Definition 1.5** A tridiagonal pair \( A, A^* \) is said to be sharp whenever \( \rho_0 = 1 \), where \( \{ \rho_i \}_{i=0}^d \) is the shape of \( A, A^* \).

### 2 Tridiagonal systems

When working with a tridiagonal pair, it is often convenient to consider a closely related object called a tridiagonal system. To define a tridiagonal system, we recall a few concepts from linear algebra. Let \( A \) denote a diagonalizable element of \( \text{End}(V) \). Let \( \{ V_i \}_{i=0}^d \) denote an ordering of the eigenspaces of \( A \) and let \( \{ \theta_i \}_{i=0}^d \) denote the corresponding ordering of the eigenvalues of \( A \). For \( 0 \leq i \leq d \) let \( E_i : V \rightarrow V \) denote the linear transformation such that \((E_i - I)V_i = 0 \) and \( E_i V_j = 0 \) for \( j \neq i \) \((0 \leq j \leq d)\). Here \( I \) denotes the identity of \( \text{End}(V) \). We call \( E_i \) the primitive idempotent of \( A \) corresponding to \( V_i \) (or \( \theta_i \)). Observe that (i) \( \sum_{i=0}^d E_i = I \); (ii) \( E_i E_j = \delta_{i,j} E_i \) \((0 \leq i, j \leq d)\); (iii) \( V_i = E_i V \) \((0 \leq i \leq d)\); (iv) \( A = \sum_{i=0}^d \theta_i E_i \). Moreover

\[
E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j}.
\]

We note that \( \{ E_i \}_{i=0}^d \) is a basis for the \( \mathbb{K} \)-subalgebra of \( \text{End}(V) \) generated by \( A \).

Now let \( A, A^* \) denote a tridiagonal pair on \( V \). An ordering of the eigenspaces of \( A \) (resp. \( A^* \)) is said to be standard whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let \( \{ V_i \}_{i=0}^d \) denote a standard ordering of the eigenspaces of \( A \). Then the ordering \( \{ V_{d-i} \}_{i=0}^d \) is standard and no other ordering is standard. A similar result holds for the eigenspaces of \( A^* \). An ordering of the primitive idempotents of \( A \) (resp. \( A^* \)) is said to be standard whenever the corresponding ordering of the eigenspaces of \( A \) (resp. \( A^* \)) is standard. We now define a tridiagonal system.

**Definition 2.1** By a tridiagonal system on \( V \) we mean a sequence

\[
\Phi = (A; \{ E_i \}_{i=0}^d; A^*; \{ E_i^* \}_{i=0}^d)
\]

that satisfies (i)–(iii) below.

(i) \( A, A^* \) is a tridiagonal pair on \( V \).

(ii) \( \{ E_i \}_{i=0}^d \) is a standard ordering of the primitive idempotents of \( A \).

(iii) \( \{ E_i^* \}_{i=0}^d \) is a standard ordering of the primitive idempotents of \( A^* \).

We say \( \Phi \) is over \( \mathbb{K} \).

The following result is immediate from lines (1), (2) and Definition 2.1.
Lemma 2.2 Let \((A; \{E_i\}_{i=0}^d ; A^*; \{E_i^*\}_{i=0}^d)\) denote a tridiagonal system. Then for \(0 \leq i, j \leq d\) the following hold.

(i) \(E_i A^* E_j = 0\) if \(|i-j| > 1\).

(ii) \(E_i^* A E_j^* = 0\) if \(|i-j| > 1\).

Definition 2.3 Let \(\Phi = (A; \{E_i\}_{i=0}^d ; A^*; \{E_i^*\}_{i=0}^d)\) denote a tridiagonal system on \(V\). For \(0 \leq i \leq d\) let \(\theta_i\) (resp. \(\theta_i^*\)) denote the eigenvalue of \(A\) (resp. \(A^*\)) associated with the eigenspace \(E_i V\) (resp. \(E_i^* V\)). We call \(\{\theta_i\}_{i=0}^d\) (resp. \(\{\theta_i^*\}_{i=0}^d\)) the eigenvalue sequence (resp. dual eigenvalue sequence) of \(\Phi\). We observe \(\{\theta_i\}_{i=0}^d\) (resp. \(\{\theta_i^*\}_{i=0}^d\)) are mutually distinct and contained in \(K\). By the shape of \(\Phi\) we mean the shape of the tridiagonal pair \(A, A^*\). We say \(\Phi\) is sharp whenever the tridiagonal pair \(A, A^*\) is sharp.

We have a comment.

Lemma 2.4 [19, Theorem 11.1] Let \(\Phi\) denote a tridiagonal system with eigenvalue sequence \(\{\theta_i\}_{i=0}^d\) and dual eigenvalue sequence \(\{\theta_i^*\}_{i=0}^d\). Then the expressions

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]

are equal and independent of \(i\) for \(2 \leq i \leq d - 1\).

When discussing tridiagonal systems we will use the following notational convention. Let \(\lambda\) denote an indeterminate and let \(K[\lambda]\) denote the \(K\)-algebra consisting of all polynomials in \(\lambda\) that have coefficients in \(K\). With reference to Definition 2.3 for \(0 \leq i \leq d\) we define the following polynomials in \(K[\lambda]\): \(\tau_i, \eta_i, \tau_i^*, \eta_i^*\) monic with degree \(i\). By [3], for \(0 \leq i \leq d\) we have

\[
E_i = \frac{\tau_i(A)\eta_{d-i}(A)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)}, \quad E_i^* = \frac{\tau_i^*(A^*)\eta_{d-i}^*(A^*)}{\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)}.
\]

In particular

\[
E_0 = \frac{\eta_d(A)}{\eta_d(\theta_0)}, \quad E_0^* = \frac{\eta_d^*(A^*)}{\eta_d^*(\theta_0^*)}, \quad E_d = \frac{\tau_d(A)}{\tau_d(\theta_d)}, \quad E_d^* = \frac{\tau_d^*(A^*)}{\tau_d^*(\theta_d^*)}.
\]

We mention a result for future use.

Lemma 2.5 [41, Proposition 5.5] With reference to Definition 2.3

\[
\eta_d = \sum_{i=0}^d \eta_{d-i}(\theta_0)\tau_i, \quad \eta_d^* = \sum_{i=0}^d \eta_{d-i}^*(\theta_0^*)\tau_i^*.
\]
3 Isomorphisms of tridiagonal systems

Throughout this section let $V'$ denote a vector space over $K$ such that $\dim V' = \dim V$.

By a $K$-algebra isomorphism from $\text{End}(V)$ to $\text{End}(V')$ we mean an isomorphism of $K$-vector spaces $\sigma : \text{End}(V) \to \text{End}(V')$ such that $(XY)^\sigma = X^\sigma Y^\sigma$ for all $X, Y \in \text{End}(V)$.

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a tridiagonal system on $V$. For a map $\sigma : \text{End}(V) \to \text{End}(V')$ we define

$$\Phi^\sigma := (A^\sigma; \{E_i^\sigma\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d).$$

**Definition 3.1** Let $\Phi$ denote a tridiagonal system on $V$ and let $\Phi'$ denote a tridiagonal system on $V'$. By an isomorphism of tridiagonal systems from $\Phi$ to $\Phi'$ we mean a $K$-algebra isomorphism $\sigma : \text{End}(V) \to \text{End}(V')$ such that $\Phi^\sigma = \Phi'$. We say $\Phi$ and $\Phi'$ are isomorphic whenever there exists an isomorphism of tridiagonal systems from $\Phi$ to $\Phi'$.

It is useful to interpret the concept of isomorphism as follows. Let $\gamma : V \to V'$ denote an isomorphism of $K$-vector spaces. Then there exists a $K$-algebra isomorphism $\sigma : \text{End}(V) \to \text{End}(V')$ such that $X^\sigma = \gamma X \gamma^{-1}$ for all $X \in \text{End}(V)$. Conversely let $\sigma : \text{End}(V) \to \text{End}(V')$ denote a $K$-algebra isomorphism. By the Skolem-Noether theorem [50, Corollary 9.1.2] there exists an isomorphism of $K$-vector spaces $\gamma : V \to V'$ such that $X^\sigma = \gamma X \gamma^{-1}$ for all $X \in \text{End}(V)$.

**Lemma 3.2** Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a tridiagonal system on $V$ and let $\Phi' = (B; \{F_i\}_{i=0}^d; B^*; \{F^*_i\}_{i=0}^d)$ denote a tridiagonal system on $V'$. Then the following (i), (ii) are equivalent.

(i) $\Phi$ and $\Phi'$ are isomorphic.

(ii) There exists an isomorphism of $K$-vector spaces $\gamma : V \to V'$ such that $\gamma A = B \gamma$, $\gamma A^* = B^* \gamma$, and $\gamma E_i = F_i \gamma$, $\gamma E^*_i = F^*_i \gamma$ for $0 \leq i \leq d$.

**Proof.** (i)$\Rightarrow$(ii): Let $\sigma$ denote an isomorphism of tridiagonal systems from $\Phi$ to $\Phi'$. By our comments below Definition 3.1 there exists an isomorphism of $K$-vector spaces $\gamma : V \to V'$ such that $X^\sigma = \gamma X \gamma^{-1}$ for all $X \in \text{End}(V)$. Observe that $\gamma$ satisfies the requirements of (ii).

(ii)$\Rightarrow$(i): Define $\sigma : \text{End}(V) \to \text{End}(V')$ such that $X^\sigma = \gamma X \gamma^{-1}$ for $X \in \text{End}(V)$. Observe that $\sigma$ is an isomorphism from $\Phi$ to $\Phi'$.

4 The $D_4$ action

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a tridiagonal system on $V$. Then each of the following is a tridiagonal system on $V$:

$$\begin{align*}
\Phi^* & := (A^*; \{E^*_i\}_{i=0}^d; A; \{E_i\}_{i=0}^d), \\
\Phi^1 & := (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d), \\
\Phi^2 & := (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d).
\end{align*}$$
Viewing $\ast$, $\downarrow$, $\downarrow\downarrow$ as permutations on the set of all tridiagonal systems,

\[ *^2 = \downarrow^2 = \downarrow\downarrow^2 = 1, \quad (8) \]

\[ \downarrow\ast = \ast\downarrow, \quad \downarrow\downarrow = \ast, \quad \downarrow\downarrow\downarrow = \downarrow\downarrow. \quad (9) \]

The group generated by symbols $\ast$, $\downarrow$, $\downarrow\downarrow$ subject to the relations (8), (9) is the dihedral group $D_4$. We recall that $D_4$ is the group of symmetries of a square, and has 8 elements. Apparently $\ast$, $\downarrow$, $\downarrow\downarrow$ induce an action of $D_4$ on the set of all tridiagonal systems. Two tridiagonal systems will be called relatives whenever they are in the same orbit of this $D_4$ action. The relatives of $\Phi$ are as follows:

| name | relative |
|------|----------|
| $\Phi$ | $(A; \{E_i\}_{i=0}^d; A\ast; \{E_i^*\}_{i=0}^d)$ |
| $\Phi\downarrow$ | $(A; \{E_i\}_{i=0}^d; A\ast; \{E_i^*\}_{i=0}^d)$ |
| $\Phi\downarrow\downarrow$ | $(A; \{E_{d-i}\}_{i=0}^d; A\ast; \{E_i^*\}_{i=0}^d)$ |
| $\Phi\downarrow\downarrow\downarrow$ | $(A; E_{d-i})_{i=0}^d; A\ast; \{E_i^*\}_{i=0}^d)$ |
| $\Phi\downarrow\ast$ | $(A\ast; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ |
| $\Phi\downarrow\downarrow\ast$ | $(A\ast; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ |
| $\Phi\ast$ | $(A\ast; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ |

From our comments below Note 1.2 we obtain the following.

**Lemma 4.1** Let $\Phi$ denote a tridiagonal system. Then the following (i), (ii) hold.

(i) The relatives of $\Phi$ all have the same shape.

(ii) Suppose $\Phi$ is sharp. Then each relative of $\Phi$ is sharp.

We will use the following notational convention.

**Definition 4.2** Let $\Phi$ denote a tridiagonal system on $V$. For $g \in D_4$ and for an object $f$ associated with $\Phi$ we let $f^g$ denote the corresponding object associated with $\Phi^g$.

## 5 The split decomposition

In this section we recall the split decomposition associated with a tridiagonal system [19, Section 4]. With reference to Definition 2.23 for $0 \leq i \leq d$ we define

\[ U_i = (E_0^i V + E_1^i V + \cdots + E_d^i V) \cap (E_i V + E_{i+1} V + \cdots + E_d V). \quad (10) \]

By [19, Theorem 4.6]

\[ V = U_0 + U_1 + \cdots + U_d \]

(direct sum),
and for $0 \leq i \leq d$ both
\begin{align*}
U_0 + U_1 + \cdots + U_i &= E_0^i V + E_i^* V + \cdots + E_d^* V, \\
U_i + U_{i+1} + \cdots + U_d &= E_i V + E_{i+1} V + \cdots + E_d V.
\end{align*}

By [19, Corollary 5.7] $U_i$ has dimension $\rho_i$ for $0 \leq i \leq d$, where $\{\rho_i\}_{i=0}^d$ is the shape of $\Phi$.

By [19, Theorem 4.6] both
\begin{align*}
(A - \theta_i I)U_i &\subseteq U_{i+1}, \\
(A^* - \theta_i^* I)U_i &\subseteq U_{i-1}
\end{align*}

for $0 \leq i \leq d$, where $U_{-1} = 0$ and $U_{d+1} = 0$. The sequence $\{U_i\}_{i=0}^d$ is called the $\Phi$-split decomposition of $V$ [19, Section 4].

The following lemma will be useful.

**Lemma 5.1** With reference to Definition 2.3 each of the following maps is bijective.
\[
\begin{align*}
E_0^* V &\rightarrow E_0 V, \quad v \mapsto E_0 v, \\
E_d^* V &\rightarrow E_d V, \quad v \mapsto E_d v,
\end{align*}
\[
\begin{align*}
E_0^* V &\rightarrow E_0 V, \quad v \mapsto E_0 v, \\
E_d^* V &\rightarrow E_d V, \quad v \mapsto E_d v,
\end{align*}
\[
\begin{align*}
E_0^* V &\rightarrow E_0^* V, \quad v \mapsto E_0^* v, \\
E_d^* V &\rightarrow E_d^* V, \quad v \mapsto E_d^* v.
\end{align*}
\]

**Proof.** Let $\xi$ denote the map on the right in the top line. We show that $\xi$ is bijective. Let $\{U_i\}_{i=0}^d$ denote the $\Phi$-split decomposition of $V$. By [19, Lemmas 6.2, 6.5], the map $U_0 \rightarrow U_d$, $u \mapsto \tau_d(A)u$ is a bijection. By (10) we have $U_0 = E_0^* V$ and $U_d = E_d V$. By (5) we have $\tau_d(A) = \tau_d(\theta_d)E_d$. By these comments $\xi$ is bijective. Applying $D_d$ we find each of the remaining maps is bijective. \qed

## 6 Sharp tridiagonal systems and the parameter array

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a sharp tridiagonal system. A bit later in the paper we will associate with $\Phi$ some formulae that have the scalars
\begin{equation}
\text{tr}(E_0 E_0^*), \quad \text{tr}(E_0 E_d^*), \quad \text{tr}(E_d E_0^*), \quad \text{tr}(E_d E_d^*)
\end{equation}
in the denominator. So we take a moment to establish that these scalars are nonzero.

**Lemma 6.1** With reference to Definition 2.3 assume $\Phi$ is sharp. Then each of the traces (15) is nonzero.

**Proof.** We first show that $\text{tr}(E_0 E_0^*) \neq 0$. Composing the first and third maps in the first column of Lemma 5.1 we find that the map $E_0 V \rightarrow E_0 V, v \mapsto E_0 E_0^* v$ is bijective. By this and since $\dim E_0 V = 1$, we find $E_0 E_0^* E_0$ is a nonzero scalar multiple of $E_0$. Now take the trace and use $\text{tr}(E_0) = 1$ to find $\text{tr}(E_0 E_0^*) \neq 0$ as desired. Applying this to the relatives of $\Phi$ we find each of the remaining traces is nonzero. \qed

7
With reference to Definition 2.3 assume \( \Phi \) is sharp, and let \( \{ U_i \}_{i=0}^d \) denote the \( \Phi \)-split decomposition of \( V \). Note that \( U_0 \) has dimension 1. For \( 0 \leq i \leq d \) the space \( U_0 \) is invariant under

\[
(A^* - \theta_i^* I)(A^* - \theta_{i-1}^* I) \cdots (A^* - \theta_1^* I)(A - \theta_0 I); \tag{16}
\]

let \( \zeta_i \) denote the corresponding eigenvalue. Note that \( \zeta_0 = 1 \). We call the sequence \( \{ \zeta_i \}_{i=0}^d \) the split sequence of \( \Phi \).

**Definition 6.2** With reference to Definition 2.3 assume \( \Phi \) is sharp. By the parameter array of \( \Phi \) we mean the sequence \((\{ \theta_i \}_{i=0}^d; \{ \theta_i^* \}_{i=0}^d; \{ \zeta_i \}_{i=0}^d)\) where \( \{ \zeta_i \}_{i=0}^d \) is the split sequence of \( \Phi \). We remark that the parameter array of \( \Phi \) is defined only when \( \Phi \) is sharp.

We now state a conjecture which indicates the significance of the parameter array.

**Conjecture 6.3** [27] Let \( d \) denote a nonnegative integer and let

\[
(\{ \theta_i \}_{i=0}^d; \{ \theta_i^* \}_{i=0}^d; \{ \zeta_i \}_{i=0}^d) \tag{17}
\]

denote a sequence of scalars taken from \( \mathbb{K} \). Then there exists a sharp tridiagonal system \( \Phi \) over \( \mathbb{K} \) with parameter array (17) if and only if (i)–(iii) hold below.

(i) \( \theta_i \neq \theta_j, \theta_i^* \neq \theta_j^* \) if \( i \neq j \) \((0 \leq i, j \leq d)\).

(ii) \( \zeta_0 = 1, \zeta_d \neq 0 \), and

\[
\sum_{i=0}^d \eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*)\zeta_i \neq 0. \tag{18}
\]

(iii) The expressions

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]

are equal and independent of \( i \) for \( 2 \leq i \leq d - 1 \).

Suppose (i)–(iii) hold. Then \( \Phi \) is unique up to isomorphism of tridiagonal systems.

Later in the paper we will prove one direction of Conjecture 6.3.

### 7 Some formulae for the split sequence

In this section we obtain some formulae for the split sequence of a sharp tridiagonal system. To prepare for this we have a few lemmas which hold for a general tridiagonal system.
Lemma 7.1 With reference to Definition 2.3, for $0 \leq i, j \leq d$ both

$$E_0\tau^*_i(A^*)\tau_j(A)E_0^* = 0,$$  \hspace{1cm} (19)

$$E_0^*\tau_i(A)\tau^*_j(A^*)E_0 = 0$$ \hspace{1cm} (20)

provided $i \neq j$.

Proof. We first show (19) for $i < j$. In the left-hand side of (19) we insert a factor $I$ between $\tau^*_i(A^*)$ and $\tau_j(A)$. We expand using $I = \sum_{r=0}^{d} E_r$ and simplify the result using $E_r\tau_j(A) = \tau_j(\theta_r)E_r$ for $0 \leq r \leq d$. By these comments the left-hand side of (19) is equal to

$$\sum_{r=0}^{d} \tau_j(\theta_r)E_0\tau^*_i(A^*)E_rE_0^*.$$ \hspace{1cm} (21)

For $0 \leq r \leq d$ we examine term $r$ in (21). By the definition of $\tau_j$ we have $\tau_j(\theta_r) = 0$ for $0 \leq r \leq j - 1$. Using Lemma 2.2(i) we find $E_0A^sE_r = 0$ for $0 \leq s \leq r - 1$. By this and since $\tau^*_i(A^*)$ has degree $i$ we find $E_0\tau^*_i(A^*)E_r = 0$ for $i + 1 \leq r \leq d$. By these comments term $r$ in (21) vanishes for $0 \leq r \leq j - 1$ and $i + 1 \leq r \leq d$. Recall $i < j$ so term $r$ in (21) vanishes for $0 \leq r \leq d$. In other words (21) is 0. We have shown (19) for $i < j$. Next we show (19) for $i > j$. In the left-hand side of (19) we again insert a factor $I$ between $\tau^*_i(A^*)$ and $\tau_j(A)$. This time we expand using $I = \sum_{r=0}^{d} E_r^*$ and simplify the result using $\tau^*_i(A^*)E_r^* = \tau^*_i(\theta_r^*)E_r^*$ for $0 \leq r \leq d$. By these comments the left-hand side of (19) is equal to

$$\sum_{r=0}^{d} \tau^*_i(\theta_r^*)E_0E_r^*\tau_j(A)E_0^*.$$ \hspace{1cm} (22)

For $0 \leq r \leq d$ we examine term $r$ in (22). By the definition of $\tau^*_i$ we have $\tau^*_i(\theta_r^*) = 0$ for $0 \leq r \leq i - 1$. Using Lemma 2.2(ii) we find $E_i^*A^sE_0^* = 0$ for $0 \leq s \leq r - 1$. By this and since $\tau_j(A)$ has degree $j$ we find $E_i^*\tau_j(A)E_0^* = 0$ for $j + 1 \leq r \leq d$. By these comments term $r$ in (22) vanishes for $0 \leq r \leq i - 1$ and $j + 1 \leq r \leq d$. Recall $i > j$ so term $r$ in (22) vanishes for $0 \leq r \leq d$. In other words (22) is 0. We have shown (19) for $i > j$. To get (20) apply (19) to $\Phi^*$.

Lemma 7.2 With reference to Definition 2.3 the following (i), (ii) hold.

(i) For $0 \leq i \leq d$ both

$$E_0\tau^*_i(A^*)\tau_i(A)E_0^* = (\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_i)E_0\tau^*_i(A^*)E_0E_0^*.$$ \hspace{1cm} (23)

$$E_0^*\tau_i(A)\tau^*_i(A^*)E_0 = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_i^*)E_0^*\tau_i(A)E_0^*.$$ \hspace{1cm} (24)

(ii) For $0 \leq i \leq d$ both

$$E_0^*\tau_i(A)\tau^*_i(A^*)E_0 = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_i^*)E_0^*\tau_i(A)E_0^*.$$ \hspace{1cm} (25)

$$E_0\tau^*_i(A^*)\tau_i(A)E_0^* = (\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_i)E_0E_0^*\tau^*_i(A^*)E_0.$$ \hspace{1cm} (26)

Proof. In the expression on the right in (23), eliminate the middle $E_0$ using the equation on the left in (25); evaluate the result using (7) and then (19) to get the expression on the left in (23). This gives (24). Lines (24)–(26) are similarly obtained. \hfill \Box
We now restrict our attention to sharp tridiagonal systems.

**Theorem 7.3** With reference to Definition 2.3, assume \( \Phi \) is sharp and let \( \{\zeta_i\}_{i=0}^d \) denote the split sequence of \( \Phi \). Then for \( 0 \leq i \leq d \) both

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)\tau_i(A)E_0^* = \zeta_i E_0^*, \tag{27}
\]

\[
(A - \theta_1 I)(A - \theta_2 I) \cdots (A - \theta_i I)\tau_i(A^*)E_0 = \zeta_i E_0. \tag{28}
\]

Moreover \( \zeta_i = \zeta_i^* \).

**Proof.** By the construction

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)\tau_i(A) - \zeta_i I
\]

vanishes on \( E_0^* V \), so (27) holds. Next we show \( \zeta_i = \zeta_i^* \). In (27) take the trace of both sides and use \( \text{tr}(MN) = \text{tr}(NM) \), \( \text{tr}(E_0^*) = 1 \), \( E_0^* A^* = \theta_0^* E_0^* \) to find

\[
(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)\text{tr}(\tau_i(A)E_0^*) = \zeta_i. \tag{29}
\]

Applying this to \( \Phi^* \),

\[
(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)\text{tr}(\tau_i(A^*)E_0) = \zeta_i. \tag{30}
\]

To get \( \zeta_i = \zeta_i^* \) we show the left-hand sides of (29) and (30) coincide. Observe that the left-hand sides of (29) and (30) coincide. Since \( E_0 \) has rank 1 we find \( E_0 E_0^* E_0 \) is a scalar multiple of \( E_0 \). Taking the trace we find \( E_0 E_0^* E_0 = \text{tr}(E_0 E_0^*)E_0 \). Using this and \( \text{tr}(MN) = \text{tr}(NM) \) we find that the trace of the right-hand side of (29) is equal to the left-hand side of (30) times \( \text{tr}(E_0 E_0^*) \). Similarly the trace of the right-hand side of (30) is equal to the left-hand side of (29) times \( \text{tr}(E_0 E_0^*) \). By these comments the left-hand sides of (29), (30) coincide so \( \zeta_i = \zeta_i^* \). To get (28) apply (27) to \( \Phi^* \) and use \( \zeta_i = \zeta_i^* \). \( \square \)

**Theorem 7.4** With reference to Definition 2.3, assume \( \Phi \) is sharp and let \( \{\zeta_i\}_{i=0}^d \) denote the split sequence of \( \Phi \). Then for \( 0 \leq i \leq d \) both

\[
E_0 \tau_i^*(A^*)\tau_i(A)E_0^* = \zeta_i E_0 E_0^*, \tag{31}
\]

\[
E_0^* \tau_i(A)\tau_i^*(A^*)E_0 = \zeta_i E_0 E_0. \tag{32}
\]

**Proof.** We first show (31). Multiplying both sides of (27) on the left by \( E_0 \),

\[
E_0(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)\tau_i(A)E_0^* = \zeta_i E_0 E_0^*. \tag{33}
\]

Observe that the expression

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I) - \tau_i^*(A^*)
\]

is a polynomial in \( A^* \) with degree less than \( i \), so it is a linear combination of \( \{\tau_r^*(A^*)\}_{r=0}^{i-1} \). By this and Lemma 7.3.1 we find that the left-hand side of (33) is equal to the left-hand side of (31). This gives (31). To get (32) apply (31) to \( \Phi^* \) and use \( \zeta_i = \zeta_i^* \). \( \square \)

In the proof of Theorem 7.3 we used some trace formula for \( \zeta_i \). Taking the trace in (31), (32) we obtain some more trace formulae for \( \zeta_i \). These formulae are summarized below.
Theorem 7.5 With reference to Definition 2.3, assume $\Phi$ is sharp and let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of $\Phi$. Then (i), (ii) hold below.

(i) For $0 \leq i \leq d$,
\[ \zeta_i = (\theta^*_0 - \theta^*_i)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_i) \text{tr}(\tau_1(A)E^*_0), \]  
\[ \zeta_i = (\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_i) \text{tr}(\tau_1^*(A^*)E_0). \]  

(ii) For $0 \leq i \leq d$,
\[ \zeta_i = \frac{\text{tr}(E_0\tau_1^*(A^*)\tau_i(A)E^*_0)}{\text{tr}(E_0^*E_0)}, \]  
\[ \zeta_i = \frac{\text{tr}(E^*_0\tau_i(A)\tau^*_1(A)E_0)}{\text{tr}(E^*_0E_0)}. \]  

8 The split sequence for the relatives of $\Phi$, part I

We now discuss the relationship between the split sequences for the relatives of $\Phi$. In this discussion we treat separately the last term in the split sequence, since its role is somewhat special as we shall see. In this section we treat the last term. In Section 9 we treat the remaining terms. In this section we also give a proof of Conjecture 6.3 in one direction.

Theorem 8.1 With reference to Definition 2.3, assume $\Phi$ is sharp and let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of $\Phi$. Then both

\[ \zeta_d = \eta^*_d(\theta^*_0)\tau_d(\theta_d)\text{tr}(E_dE^*_0), \]  
\[ \zeta_d = \eta_d(\theta_0)\tau^*_d(\theta^*_0)\text{tr}(E^*_dE_0). \]  

Proof. To get the equation on the left in (38), set $i = d$ in (34) and evaluate the result using the equation on the right in (5). The equation on the right in (38) is similarly obtained using (35). \[ \square \]

Theorem 8.2 With reference to Definition 2.3, assume $\Phi$ is sharp and let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of $\Phi$. Then (i), (ii) hold below.

(i) For the tridiagonal systems $\Phi$, $\Phi^*$, $\Phi\downarrow\updownarrow$, $\Phi\downarrow\downarrow^*$ the last term in the split sequence is equal to $\zeta_d$.

(ii) For the tridiagonal systems $\Phi\downarrow$, $\Phi\uparrow$, $\Phi^\dagger$, $\Phi^\dagger^*$ the last term in the split sequence is equal to
\[ \sum_{i=0}^d \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i. \]  

Proof. (i): We have $\zeta^*_d = \zeta_d$ by Theorem 7.3. Applying the equation on the left in (35) to $\Phi\downarrow\uparrow$ we find
\[ \zeta_d\downarrow\uparrow = \tau^*_d(\theta^*_d)\eta_{d}(\theta_0)\text{tr}(E_0E^*_d). \]
Comparing this with the equation on the right in (38) we get \( \zeta_d^{\downarrow \downarrow} = \zeta_d \). By these comments we get \( \zeta_d = \zeta_d^{\downarrow \downarrow} = \zeta_d^{\downarrow \downarrow} = \zeta_d^{\downarrow \downarrow} = \zeta_d \).

(ii): Applying (i) to \( \Phi^{\downarrow} \) we get \( \zeta_d^{\downarrow} = \zeta_d^{\downarrow \downarrow} = \zeta_d^{\downarrow \downarrow} = \zeta_d^{\downarrow \downarrow} \), so it suffices to show that the expression (39) is equal to \( \zeta_d^{\downarrow} \). By (34) at \( i = d \),

\[
\zeta_d = \eta_d^*(\theta_0^*) \text{tr}(\tau_d(A) E_0^*).
\]

Applying this to \( \Phi^{\downarrow} \),

\[
\zeta_d^{\downarrow} = \eta_d^*(\theta_0^*) \text{tr}(\eta_d(A) E_0^*).
\]

Evaluating (7) at \( \lambda = A \),

\[
\eta_d(A) = \sum_{i=0}^{d} \eta_{d-i}(\theta_0) \tau_i(A).
\]

By these comments and (34),

\[
\zeta_d^{\downarrow} = \eta_d^*(\theta_0^*) \sum_{i=0}^{d} \eta_{d-i}(\theta_0) \text{tr}(\tau_i(A) E_0^*)
= \sum_{i=0}^{d} \eta_{d-i}(\theta_0) \eta_{d-i}(\theta_0^*) (\theta_0^* - \theta_1^*) \cdots (\theta_0^* - \theta_i^*) \text{tr}(\tau_i(A) E_0^*)
= \sum_{i=0}^{d} \eta_{d-i}(\theta_0) \eta_{d-i}(\theta_0^*) \zeta_i.
\]

So (ii) holds. \( \Box \)

Corollary 8.3 With reference to Definition 2.3 assume \( \Phi \) is sharp and let \( \{ \zeta_i \}_{i=0}^{d} \) denote the split sequence of \( \Phi \). Then \( \zeta_0 = 1 \), \( \zeta_d \neq 0 \), and (18) holds.

Proof. Obviously \( \zeta_0 = 1 \). By Lemma 6.1 and Theorem 8.1 we get \( \zeta_d \neq 0 \). Applying this to \( \Phi^{\downarrow} \) we find \( \zeta_d^{\downarrow} \neq 0 \). Now (18) follows from this and Theorem 8.2(ii). \( \Box \)

We can now easily prove Conjecture 6.3 in one direction.

Proof of Conjecture 6.3 (direction “only if”). Assume that there exists a sharp tridiagonal system \( \Phi \) over \( \mathbb{K} \) that has parameter array (17). We show that this parameter array satisfies (i)–(iii). Assertion (i) follows from Definition 2.3. Assertion (ii) is just Corollary 8.3. Assertion (iii) is just Lemma 2.4. \( \Box \)

9 The split sequence for the relatives of \( \Phi \), part II

In this section we continue to discuss the relationship between the split sequences for the relatives of \( \Phi \). We will need the following scalars. Given a tridiagonal system \( (A; \{ E_i \}_{i=0}^{d}; A^*; \{ E_i^* \}_{i=0}^{d}) \) over \( \mathbb{K} \) and given nonnegative integers \( r, s, t \) such that \( r + s + t \leq d \),
in [56, Definition 13.1] we defined a scalar \([r, s, t]_q \in \mathbb{K}\), where \(q + q^{-1} + 1\) is the common value of (4). For example, if \(q \neq 1\) and \(q \neq -1\) then

\[
[r, s, t]_q = \frac{(q; q)_{r+s}(q; q)_{r+t}(q; q)_{s+t}}{(q; q)_{r}(q; q)_{s}(q; q)_{t}(q; q)_{r+s+t}},
\]

where

\[(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).\]

**Lemma 9.1** [39, Theorem 5.5] With reference to Definition 2.3, for \(0 \leq i \leq d\) we have

\[
\eta_i = \sum_{h=0}^{i} [h, i-h, d-i]_q \eta_{i-h}(\theta_0) \tau_h. \tag{40}
\]

**Note 9.2** In [39] we gave a proof of (40) for the case of \(q \neq 1\), \(q \neq -1\). A similar proof establishes (40) for each of the following cases: (i) \(q \neq 1\), \(q \neq -1\); (ii) \(q = 1\), \(\text{Char}(\mathbb{K}) \neq 2\); (iii) \(q = -1\), \(\text{Char}(\mathbb{K}) \neq 2\); (iv) \(q = 1\), \(\text{Char}(\mathbb{K}) = 2\).

We are now ready to give the relationship between the split sequences for the relatives of \(\Phi\). These relationships will show that the split sequence for each relative of \(\Phi\) is determined by the parameter array of \(\Phi\). We start with a comment. By the last line of Theorem 7.3 in each column of the following array the relatives of \(\Phi\) have the same split sequence:

\[
\begin{array}{ccccc}
\Phi & \Phi^\perp & \Phi^\downarrow & \Phi^\perp \downarrow & \Phi^\downarrow \perp \\
\Phi^* & \Phi^* \perp & \Phi^* \downarrow & \Phi^* \perp \downarrow & \Phi^* \downarrow \perp
\end{array}
\]

Therefore we limit our attention to the split sequences for the relatives of \(\Phi\) in the first row.
Theorem 9.3 With reference to Definition 2.3, assume $\Phi$ is sharp and let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of $\Phi$. Then (i)–(iv) hold below.

(i) For $0 \leq i \leq d$ both
\[
\zeta_i^\downarrow \left( (\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_i) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right),
\]
\[
\zeta_i \left( (\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_i) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right).
\]

(ii) For $0 \leq i \leq d$ both
\[
\zeta_i^\uparrow \left( (\theta_0^*- \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right),
\]
\[
\zeta_i \left( (\theta_0^*- \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right).
\]

(iii) For $0 \leq i \leq d$ both
\[
\zeta_i^\uparrow \left( (\theta_d^*- \theta_{d-1}^*)(\theta_d^* - \theta_{d-2}^*) \cdots (\theta_d^* - \theta_{d-i}^*) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right),
\]
\[
\zeta_i \left( (\theta_d^*- \theta_{d-1}^*)(\theta_d^* - \theta_{d-2}^*) \cdots (\theta_d^* - \theta_{d-i}^*) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right).
\]

(iv) For $0 \leq i \leq d$ both
\[
\zeta_i^\uparrow \left( (\theta_d - \theta_{d-1})(\theta_d - \theta_{d-2}) \cdots (\theta_d - \theta_{d-i}) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right),
\]
\[
\zeta_i \left( (\theta_d - \theta_{d-1})(\theta_d - \theta_{d-2}) \cdots (\theta_d - \theta_{d-i}) \right) = \sum_{h=0}^i [h, i - h, d - i] q \zeta_h \left( \theta_0 \right) \zeta_h \left( \theta_0 \right).
\]

**Proof.** We start by obtaining the first equation in part (ii). Applying (34) to $\Phi^\uparrow$,
\[
\zeta_i^\uparrow = (\theta_0^* - \theta_1^*) (\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*) \text{tr}(\eta_0(A)E_0^*).
\]
(41)

Evaluate (40) at $\lambda = A$ and in the result multiply both sides on the right by $E_0^*$ to get
\[
\eta_0(A)E_0^* = \sum_{h=0}^i [h, i - h, d - i] q \eta_0(A) \eta_h(A)E_0^*.
\]

In this equation we take the trace of both sides and use (34), (41) to obtain the first equation in part (ii). Apply $D_d$ to this and use $\zeta_i = \zeta_i^*$ to obtain the remaining formulae. □
10  Bilinear forms, anti-automorphisms, and tridiagonal systems

With reference to Definition 2.3 assume for the moment that \( \Phi \) is sharp. Our next goal is to show that if Conjecture 6.3 is true then there exists a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( V \) that satisfies

\[
\langle Au, v \rangle = \langle u, Av \rangle, \quad \langle A^* u, v \rangle = \langle u, A^* v \rangle \quad \text{for all } u, v \in V.
\]

We will also obtain some related results involving anti-automorphisms. We start with some definitions. Throughout this section let \( V' \) denote a vector space over \( \mathbb{K} \) such that \( \dim V' = \dim V \).

A map \( \langle \cdot, \cdot \rangle : V \times V' \to \mathbb{K} \) is called a bilinear form whenever the following conditions hold for \( u, v \in V \), for \( u', v' \in V' \), and for \( \alpha \in \mathbb{K} \): (i) \( \langle u + v, u' \rangle = \langle u, u' \rangle + \langle v, u' \rangle \); (ii) \( \langle \alpha u, u' \rangle = \alpha \langle u, u' \rangle \); (iii) \( \langle u + v', u' \rangle = \langle u, u' \rangle + \langle v, u' \rangle \); (iv) \( \langle \alpha u, v' \rangle = \alpha \langle u, v' \rangle \). We observe that a scalar multiple of a bilinear form is a bilinear form. Let \( \langle \cdot, \cdot \rangle : V \times V' \to \mathbb{K} \) denote a bilinear form. Then the following are equivalent: (i) there exists a nonzero \( v \in V \) such that \( \langle v, v' \rangle = 0 \) for all \( v' \in V' \); (ii) there exists a nonzero \( v' \in V' \) such that \( \langle v, v' \rangle = 0 \) for all \( v \in V \). The form \( \langle \cdot, \cdot \rangle \) is said to be degenerate whenever (i), (ii) hold and nondegenerate otherwise. By a bilinear form on \( V \) we mean a bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{K} \). This form is said to be symmetric whenever \( \langle u, v \rangle = \langle v, u \rangle \) for all \( u, v \in V \).

**Lemma 10.1** With reference to Definition 2.3 let \( \langle \cdot, \cdot \rangle \) denote a nonzero bilinear form on \( V \) that satisfies (42). Then \( \langle \cdot, \cdot \rangle \) is nondegenerate.

**Proof.** It suffices to show that the space \( W = \{ w \in V \mid \langle w, V \rangle = 0 \} \) is zero. Using (42) we routinely find \( A W \subseteq W \) and \( A^* W \subseteq W \), so either \( W = 0 \) or \( W = V \) by Definition 1.1(iv). But \( W \neq V \) since \( \langle \cdot, \cdot \rangle \) is nonzero so \( W = 0 \) as desired. \( \square \)

**Lemma 10.2** With reference to Definition 2.3 let \( \langle \cdot, \cdot \rangle \) denote a nonzero bilinear form on \( V \) that satisfies (42). Then (i), (ii) hold below.

(i) \( \langle E_i V, E_j V \rangle = 0 \) and \( \langle E_i^* V, E_j^* V \rangle = 0 \) if \( i \neq j \) \( 0 \leq i, j \leq d \).

(ii) For \( 0 \leq i \leq d \) the restriction of \( \langle \cdot, \cdot \rangle \) to each of \( E_i V, E_i^* V \) is nondegenerate.

**Proof.** (i): Let \( i, j \) be given with \( i \neq j \). We show \( \langle E_i u, E_j v \rangle = 0 \) for \( u, v \in V \). Recall that \( E_i \) is contained in the subalgebra of \( \text{End}(V) \) generated by \( A \), so using (42) we have \( \langle E_i u, E_j v \rangle = \langle u, E_i E_j v \rangle = \langle u, 0 \rangle = 0 \). The proof of \( \langle E_i^* V, E_j^* V \rangle = 0 \) is similar.

(ii): Combine (i) above with Lemma 10.1 \( \square \)

**Lemma 10.3** With reference to Definition 2.3 assume \( \Phi \) is sharp. Then up to scalar multiple there exists at most one bilinear form \( \langle \cdot, \cdot \rangle \) on \( V \) that satisfies (42).

**Proof.** The dimension of \( E_0 V \) is 1 since \( \Phi \) is sharp; pick a nonzero \( \eta \in E_0 V \). Let \( \langle \cdot, \cdot \rangle \) denote a nonzero bilinear form on \( V \) that satisfies (42), and note that \( \langle \eta, \eta \rangle \neq 0 \) by Lemma
Suppose another bilinear form \( \langle , \rangle' \) on \( V \) satisfies (12). We show that \( \langle , \rangle' \) is a scalar multiple of \( \langle , \rangle \). Since \( \langle \eta, \eta \rangle \neq 0 \) there exists \( \alpha \in \mathbb{K} \) such that \( \langle \eta, \eta \rangle' = \alpha \langle \eta, \eta \rangle \).

Define a map \( (,): V \times V \to \mathbb{K} \) by
\[
(u,v) = \langle u, v \rangle' - \alpha \langle u, v \rangle \quad (u,v \in V).
\]

The map \( (,): V \times V \to \mathbb{K} \) is a bilinear form on \( V \) that satisfies (12) and \( \langle \eta, \eta \rangle = 0 \) so \( (,): V \times V \to \mathbb{K} \) is zero by our preliminary comment. Now \( \langle , \rangle' = \alpha \langle , \rangle \) and the result follows.

**Lemma 10.4** With reference to Definition 2.3 assume \( \Phi \) is sharp. Let \( \langle , \rangle \) denote a bilinear form on \( V \) that satisfies (12). Then \( \langle , \rangle \) is symmetric.

**Proof.** We assume \( \langle , \rangle \) is nonzero; otherwise we are done. Pick a nonzero \( \eta \in E_0V \) and note that \( \langle \eta, \eta \rangle \neq 0 \) by Lemma 10.2(ii). Define a map \( (,): V \times V \to \mathbb{K} \) by \( (u,v) = \langle v, u \rangle \) for \( u, v \in V \). Then \( (,): V \times V \to \mathbb{K} \) is a bilinear form on \( V \) that satisfies (12), so by Lemma 10.3 there exists \( \alpha \in \mathbb{K} \) such that \( (,): V \times V \to \mathbb{K} \) is a bilinear form on \( V \). Now \( \langle \eta, \eta \rangle = \alpha \langle \eta, \eta \rangle \) and \( \langle \eta, \eta \rangle \neq 0 \) so \( \alpha = 1 \). Therefore \( \langle u, v \rangle = \langle v, u \rangle \) for all \( u, v \in V \), so \( \langle , \rangle \) is symmetric.

By a \( \mathbb{K} \)-algebra anti-isomorphism from \( \text{End}(V) \) to \( \text{End}(V') \) we mean an isomorphism of \( \mathbb{K} \)-vector spaces \( \sigma: \text{End}(V) \to \text{End}(V') \) such that \( (XY)^\sigma = Y^\sigma X^\sigma \) for all \( X,Y \in \text{End}(V) \). By an anti-automorphism of \( \text{End}(V) \) we mean a \( \mathbb{K} \)-algebra anti-isomorphism from \( \text{End}(V) \) to \( \text{End}(V) \). Bilinear forms and anti-isomorphisms are related as follows. Let \( \langle , \rangle : V \times V' \to \mathbb{K} \) denote a nondegenerate bilinear form. Then there exists a unique anti-isomorphism \( \sigma: \text{End}(V) \to \text{End}(V') \) such that \( \langle Xv, v' \rangle = \langle v, X^\sigma v' \rangle \) for all \( v \in V, v' \in V' \), \( X \in \text{End}(V) \). Conversely, given an anti-isomorphism \( \sigma: \text{End}(V) \to \text{End}(V') \) there exists a bilinear form \( \langle , \rangle : V \times V' \to \mathbb{K} \) such that \( \langle Xv, v' \rangle = \langle v, X^\sigma v' \rangle \) for all \( v \in V, v' \in V' \), \( X \in \text{End}(V) \). This bilinear form is nondegenerate, and uniquely determined by \( \sigma \) up to multiplication by a nonzero scalar in \( \mathbb{K} \). We say the form \( \langle , \rangle \) is associated with \( \sigma \).

**Lemma 10.5** Let \( \langle , \rangle \) denote a nondegenerate bilinear form on \( V \) and let \( \sigma \) denote the anti-automorphism of \( \text{End}(V) \) associated with \( \langle , \rangle \). Assume \( \langle , \rangle \) is symmetric. Then \( X^\sigma = X \) for all \( X \in \text{End}(V) \).

**Proof.** We fix \( u \in V \) and show \( (X^\sigma - X)u = 0 \). For all \( v \in V \) we have \( \langle Xu, v \rangle = \langle X^\sigma v, u \rangle = \langle v, (X^\sigma)^\sigma u \rangle = \langle X^\sigma u, v \rangle \), so \( \langle (X - X^\sigma)u, v \rangle = 0 \). Therefore \( (X^\sigma - X)u = 0 \) since \( \langle , \rangle \) is nondegenerate.

**Lemma 10.6** Let \( \sigma \) denote an anti-isomorphism from \( \text{End}(V) \) to \( \text{End}(V') \). Then \( \text{tr}(X) = \text{tr}(X^\sigma) \) for all \( X \in \text{End}(V) \).

**Proof.** Fix a basis \( \{v_i\}_{i=1}^n \) for \( V \) and a basis \( \{v'_i\}_{i=1}^n \) for \( V' \). For the purpose of this proof we identify each element of \( \text{End}(V) \) (resp. \( \text{End}(V') \)) with the matrix in Mat\(_n(\mathbb{K})\) that represents it with respect to \( \{v_i\}_{i=1}^n \) (resp. \( \{v'_i\}_{i=1}^n \)). By the Skolem-Noether theorem [50, Corollary 9.122] there exists an invertible \( R \in \text{Mat}_n(\mathbb{K}) \) such that \( X^\sigma = R^{-1}X^tR \) for all \( X \in \text{End}(V) \). Now \( \text{tr}(X^\sigma) = \text{tr}(R^{-1}X^tR) = \text{tr}(X^t) = \text{tr}(X) \) as desired.

The following is a mild generalization of [3, Theorem 1.2].
Proposition 10.7  [3, Theorem 1.2] With reference to Definition 2.3, let $V'$ denote a vector space such that $\dim V' = \dim V$, and let $\sigma$ denote an anti-isomorphism from $\text{End}(V)$ to $\text{End}(V')$. Then (i)–(iii) hold below.

(i) $\Phi^\sigma$ is a tridiagonal system on $V'$.

(ii) $\Phi$ and $\Phi^\sigma$ have the same eigenvalue sequence and dual eigenvalue sequence.

(iii) Assume $\Phi$ is sharp. Then $\Phi^\sigma$ is sharp. Moreover $\Phi$ and $\Phi^\sigma$ have the same split sequence.

Proof. (i): We first show that the pair $A^\sigma, A^{\sigma^*}$ is a tridiagonal pair on $V'$. To do this we verify that $A^\sigma, A^{\sigma^*}$ satisfy conditions (i)–(iv) of Definition 1.1. Concerning Definition 1.1(i), we claim that $A^\sigma$ is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^d$. For $f \in \mathbb{K}[\lambda]$ we have $f(A) = 0$ if and only if $f(A^\sigma) = 0$. Therefore $A$ and $A^\sigma$ have the same minimal polynomial. The minimal polynomial of $A$ is $\prod_{i=0}^d(\lambda - \theta_i)$ so the minimal polynomial of $A^\sigma$ is $\prod_{i=0}^d(\lambda - \theta_i)$. By this and since $\{\theta_i\}_{i=0}^d$ are mutually distinct, $A^\sigma$ is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^d$. Similarly $A^{\sigma^*}$ is diagonalizable with eigenvalues $\{\theta_i^*\}_{i=0}^d$. Concerning Definition 1.1(ii), for $0 \leq i \leq d$ we apply $\sigma$ to (3) and find $E_i^\sigma$ is the primitive idempotent of $A^\sigma$ associated with $\theta_i$. Define $V_i = E_i^\sigma V'$ and note that $\{V_i\}_{i=0}^d$ is an ordering of the eigenspaces of $A^\sigma$. Applying $\sigma$ to the equation in Lemma 2.2(i) we find $E_i^\sigma A^\sigma E_i^\sigma = 0$ if $|i-j| > 1$ (0 $\leq i, j \leq d$). Therefore $A^\sigma V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$. The verification of Definition 1.1(iii) is similar, using $V_i^* = E_i^{\sigma^*} V'$ for $0 \leq i \leq d$. Concerning Definition 1.1(iv), let $W$ denote a subspace of $V'$ such that $A^\sigma W \subseteq W$ and $A^{\sigma^*} W \subseteq W$. We show $W = 0$ or $W = V'$. Let $\langle \ , \ \rangle : V \times V' \to \mathbb{K}$ denote the bilinear form associated with $\sigma$. Define $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$. Since $\langle \ , \ \rangle$ is nondegenerate $\dim W + \dim W^\perp$ is equal to the common dimension of $V, V'$. Observe $AW^\perp \subseteq W^\perp$; indeed for all $w \in W$ and all $v \in W^\perp$, $\langle Av, w \rangle = \langle v, A^\sigma w \rangle = 0$ since $A^\sigma w \in W$. We similarly obtain $A^* W^\perp \subseteq W^\perp$. Now $W^\perp = 0$ or $W^\perp = V$ since $A, A^*$ is a tridiagonal pair on $V$, and therefore $W = V'$ or $W = 0$. We have shown the pair $A^\sigma, A^{\sigma^*}$ satisfies conditions (i)–(iv) of Definition 1.1, so $A^\sigma, A^{\sigma^*}$ is a tridiagonal pair on $V'$. By the construction $\{E_i^\sigma\}_{i=0}^d$ (resp. $\{E_i^{\sigma^*}\}_{i=0}^d$) is a standard ordering of the primitive idempotents of $A^\sigma$ (resp. $A^{\sigma^*}$). Now $(A^\sigma; \{E_i^\sigma\}_{i=0}^d; A^*; \{E_i^{\sigma^*}\})$ is a tridiagonal system on $V'$ by Definition 2.3.

(ii): We mentioned in the proof of (i) above that for $0 \leq i \leq d$ the scalar $\theta_i$ is the eigenvalue of $A^\sigma$ associated with the primitive idempotent $E_i^\sigma$. Similarly $\theta_i^*$ is the eigenvalue of $A^{\sigma^*}$ associated with the primitive idempotent $E_i^{\sigma^*}$. The result follows.

(iii): The primitive idempotent $E_0$ has rank 1 since $\Phi$ is sharp, so $\text{tr}(E_0) = 1$. By this and Lemma 10.6 we have $\text{tr}(E_0^\sigma) = 1$. Now the primitive idempotent $E_0^\sigma$ has rank 1 so $\Phi^\sigma$ is sharp. By (31), Lemma 10.6 and (ii) above, we find $\Phi$ and $\Phi^\sigma$ have the same split sequence.

Let $\tilde{V}$ denote the dual space of $V$; consisting of all $\mathbb{K}$-linear transformations from $V$ to $\mathbb{K}$. Define $\langle \ , \ \rangle : V \times \tilde{V} \to \mathbb{K}$ by $\langle v, f \rangle = f(v)$ for $v \in V, f \in \tilde{V}$. Then $\langle \ , \ \rangle$ is a nondegenerate bilinear form. We call this form the canonical bilinear form between $V$ and $\tilde{V}$. Let $\sigma : \text{End}(V) \to \text{End}(\tilde{V})$ denote the anti-isomorphism associated with $\langle \ , \ \rangle$,
so that \((X^\sigma f)v = f(Xv)\) for all \(v \in V\), \(f \in \tilde{V}\), \(X \in \text{End}(V)\). We call \(\sigma\) the canonical anti-isomorphism from \(\text{End}(V)\) to \(\text{End}(\tilde{V})\).

**Definition 10.8** [3] Let \(\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a tridiagonal system on \(V\). Let \(\tilde{V}\) denote the dual space of \(V\) and let \(\sigma : \text{End}(V) \rightarrow \text{End}(\tilde{V})\) denote the canonical anti-isomorphism. By Proposition 10.7 \(\Phi^\sigma\) is a tridiagonal system on \(\tilde{V}\); we call this tridiagonal system the dual of \(\Phi\).

**Corollary 10.9** With reference to Definition 2.3 assume \(\Phi\) is sharp. Then the dual of \(\Phi\) is sharp and has the same parameter array as \(\Phi\).

**Proof.** Immediate from Proposition 10.7 and Definition 10.8.

**Proposition 10.10** With reference to Definition 2.3 let \(\langle , \rangle\) denote a nondegenerate bilinear form on \(V\) and let \(\dagger\) denote the associated anti-automorphism of \(\text{End}(V)\). Then the following (i), (ii) are equivalent.

(i) The bilinear form \(\langle , \rangle\) satisfies (42).

(ii) \(A^\dagger = A\) and \(A^{*\dagger} = A^*\).

**Proof.** (i)\(\Rightarrow\)(ii): Concerning \(A\), we fix \(v \in V\) and show \((A - A^\dagger)v = 0\). For all \(u \in V\) we have \(\langle Au, v \rangle = \langle u, A^\dagger v \rangle\) by construction and \(\langle Au, v \rangle = \langle u, Av \rangle\) by (42). Therefore \(\langle u, (A - A^\dagger)v \rangle = 0\) and this gives \((A - A^\dagger)v = 0\) since \(\langle , \rangle\) is nondegenerate. We have shown \(A^\dagger = A\) and one similarly shows \(A^{*\dagger} = A^*\).

(ii)\(\Rightarrow\)(i): For \(u, v \in V\), by construction \(\langle Xu, v \rangle = \langle u, X^\dagger v \rangle\) for all \(X \in \text{End}(V)\). In this equation we set \(X = A\) and use \(A^\dagger = A\) to get \(\langle Au, v \rangle = \langle u, Av \rangle\). Similarly we obtain \(\langle A^*u, v \rangle = \langle u, A^*v \rangle\). Therefore \(\langle , \rangle\) satisfies (42).

11 **Conjecture [6.3] and bilinear forms**

In this section we show how the truth of Conjecture 6.3 implies the existence of a nonzero bilinear form that satisfies (42). Actually, to obtain this bilinear form we do not need the full strength of Conjecture 6.3; just the following weaker version.

**Conjecture 11.1** Two sharp tridiagonal systems over \(K\) are isomorphic if and only if they have the same parameter array.

**Note 11.2** In [28] Conjecture 11.1 is proved assuming the parameter \(q\) from above Lemma 9.1 is not a root of unity, and that \(K\) is algebraically closed. In [27] Conjecture 11.1 is proved assuming the tridiagonal pairs in question have Krawtchouk type, and that \(K\) is algebraically closed.

**Corollary 11.3** Assume Conjecture 11.1 is true. Then every sharp tridiagonal system over \(K\) is isomorphic to its dual.

**Proof.** Follows from Corollary 10.9.
Theorem 11.4 Assume Conjecture 11.1 is true. With reference to Definition 2.3 assume \( \Phi \) is sharp. Then there exists a nonzero bilinear form on \( V \) that satisfies (42). This form is unique up to multiplication by a nonzero scalar in \( \mathbb{K} \). This form is nondegenerate and symmetric.

**Proof.** Let \( \tilde{V} \) denote the dual space of \( V \). Let \( (\ , \) : \( V \times \tilde{V} \to \mathbb{K} \) denote the canonical bilinear form and let \( \sigma : \text{End}(V) \to \text{End}(\tilde{V}) \) denote the canonical anti-isomorphism. By Lemma 3.2 and Corollary 11.3 there exists an isomorphism of vector spaces \( \gamma : V \to \tilde{V} \) such that \( \gamma A = A^\sigma \gamma \) and \( \gamma A^* = A^{*\sigma} \gamma \). Define a map \( (\ , \) : \( V \times V \to \mathbb{K} \) by

\[
(u, v) = (u, \gamma v)
\]

and observe that \( (\ , \) is a bilinear form on \( V \). The form \( (\ , \) is nondegenerate since \( (\ , \) is nondegenerate. We show \( (\ , \) satisfies (42). For \( u, v \in V \),

\[
(Au, v) = (Au, \gamma v) = (u, A^\sigma \gamma v) = (u, \gamma Av) = (u, Av)
\]

and similarly \( (A^*u, v) = (u, A^*v) \). Therefore \( (\ , \) satisfies (42). By Lemma 10.3 \( (\ , \) is unique up to multiplication by a nonzero scalar in \( \mathbb{K} \). By Lemma 10.4 \( (\ , \) is symmetric. \( \square \)

Theorem 11.5 Assume Conjecture 11.1 is true. With reference to Definition 2.3 assume \( \Phi \) is sharp. Then there exists a unique anti-automorphism \( \dagger \) of \( \text{End}(V) \) that fixes each of \( A, A^* \). Moreover \( X^\dagger\dagger = X \) for all \( X \in \text{End}(V) \).

**Proof.** Let \( (\ , \) denote the bilinear form on \( V \) from Theorem 11.4 and let \( \dagger \) denote the associated anti-automorphism of \( \text{End}(V) \). Then \( A^\dagger = A \) and \( A^{*\dagger} = A^* \) by Proposition 10.10. The anti-automorphism \( \dagger \) is unique by Lemma 10.3 and Proposition 10.10. For \( X \in \text{End}(V) \) we have \( X^\dagger\dagger = X \) by Lemma 10.5 and since \( (\ , \) is symmetric by Lemma 10.4. \( \square \)

12 Directions for further research

In this section we give some suggestions for further research. We start with a definition.

**Definition 12.1** With reference to Definition 2.3 let \( D \) (resp. \( D^* \)) denote the \( \mathbb{K} \)-subalgebra of \( \text{End}(V) \) generated by \( A \) (resp. \( A^* \)). Let \( T \) denote the \( \mathbb{K} \)-subalgebra of \( \text{End}(V) \) generated by \( A \) and \( A^* \).

We use the following convention. For \( \mathbb{K} \)-subspaces \( R, S \) of \( \text{End}(V) \) let \( RS \) denote the \( \mathbb{K} \)-subspace spanned by \( \{rs \mid r \in R, s \in S \} \).
Conjecture 12.2 With reference to Definition 2.3 and Definition 12.1 the following (i)–(iv) hold.

(i) The elements of $E_0^*D E_0^*$ mutually commute.

(ii) Each of the following holds:

\begin{align*}
E_0^* D D^* E_0 &= E_0^* D E_0^* E_0, \\
E_0^* D D^* D E_0^* &= E_0^* D E_0^* D E_0^*, \\
E_0^* D D^* D D^* E_0 &= E_0^* D E_0^* D E_0^* E_0, \\
E_0^* D D^* D D^* D E_0^* &= E_0^* D E_0^* D E_0^* D E_0^* E_0, \\
& \quad \cdots \\
E_0^* D D^* D D^* D D^* E_0 &= E_0^* D E_0^* D E_0^* D D^* E_0, \\
& \quad \cdots \\
E_0^* D D^* D D^* D D^* D E_0^* &= E_0^* D E_0^* D E_0^* D D^* D E_0^* E_0, \\
& \quad \cdots \\
E_0^* D D^* D D^* D D^* D D^* D E_0^* &= E_0^* D E_0^* D E_0^* D D^* D D^* E_0, \\
& \quad \cdots \\
E_0^* D D^* D D^* D D^* D D^* D D^* D E_0^* &= E_0^* D E_0^* D E_0^* D D^* D D^* D D^* E_0, \\
& \quad \cdots \\
E_0^* D D^* D D^* D D^* D D^* D D^* D D^* D E_0^* &= E_0^* D E_0^* D E_0^* D D^* D D^* D D^* D D^* E_0.
\end{align*}

(iii) The $\mathbb{K}$-algebra $E_0^* T E_0^*$ is generated by $E_0^* D E_0^*$.

(iv) The $\mathbb{K}$-algebra $E_0^* T E_0^*$ is commutative.

Note 12.3 In [28] Conjecture 12.2 is proven assuming the parameter $q$ from above Lemma 9.1 is not a root of unity, and that $\mathbb{K}$ is algebraically closed.

Conjecture 12.4 With reference to Definition 2.3 and Definition 12.1 the following (i), (ii) hold.

(i) $E_0^* T E_0^*$ is a field with identity $E_0^*$.

(ii) Viewing $\mathbb{K} E_0^*$ as a field with identity $E_0^*$, the field $E_0^* T E_0^*$ is an $r$-dimensional field extension of $\mathbb{K} E_0^*$, where $r = \dim E_0^* V$.

Note 12.5 Conjecture 12.4 implies Conjecture 12.3 since if $\mathbb{K}$ is algebraically closed then the field $\mathbb{K} E_0^*$ has no finite-dimensional field extensions other than itself.

Problem 12.6 With reference to Definition 2.3, let $\{U_i\}_{i=0}^d$ denote the $\Phi$-split decomposition of $V$ from Section 5. By (13) and (14), for $0 \leq i \leq d/2$ the space $U_i$ is invariant under

\begin{equation}
(A^* - \theta_{i+1}^*) (A^* - \theta_{i+2}^*) \cdots (A^* - \theta_{d-i}^*) (A - \theta_{d-i} I) \cdots (A - \theta_{i+1} I) (A - \theta_I). \tag{43}
\end{equation}

The restriction of (43) to $U_i$ is invertible by [19, Lemma 6.5]. Assuming $\Phi$ is sharp, find the eigenvalues for the action of (43) on $U_i$ in terms of the parameter array of $\Phi$.

Problem 12.7 With reference to Definition 2.3, let $\{U_i\}_{i=0}^d$ denote the $\Phi$-split decomposition of $V$ from Section 5. For $0 \leq i \leq d$ define a linear transformation $F_i : V \to V$ such that $(F_i - I) U_i = 0$ and $F_i U_j = 0$ if $i \neq j$ ($0 \leq j \leq d$). In other words $F_i$ is the projection from $V$ onto $U_i$. Assuming $A, A^*$ is sharp, find $F_i$ as a polynomial in $A, A^*$ and express the coefficients of this polynomial in terms of the parameter array.

Problem 12.8 With reference to Definition 2.3 assume $\Phi$ is sharp. For $0 \leq i \leq d$ find each of

\begin{align*}
\text{tr}(E_i E_0^*), \quad \text{tr}(E_i E_d^*), \quad \text{tr}(E_i^* E_0), \quad \text{tr}(E_i^* E_d)
\end{align*}

in terms of the parameter array of $\Phi$. 

20
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