IDENTITY FOR SCALAR-VALUED FUNCTIONS OF TENSORS
AND ITS APPLICATIONS IN CLASSICAL FIELD THEORIES AND GRAVITY

JÜRGEN STRUCKMEIER
Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Str. 1,
60438 Frankfurt am Main, Germany
Physics Department, Goethe-University, Max-von-Laue-Str. 1,
60438 Frankfurt am Main, Germany
struckmeier@fias.uni-frankfurt.de

ARMIN VAN DE VENN
Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Str. 1,
60438 Frankfurt am Main, Germany
Physics Department, Goethe-University, Max-von-Laue-Str. 1,
60438 Frankfurt am Main, Germany
venn@fias.uni-frankfurt.de

DAVID VASAK
Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Str. 1,
60438 Frankfurt am Main, Germany
vasak@fias.uni-frankfurt.de

Received September 7, 2022

We prove a theorem on scalar-valued functions of tensors, where “scalar” refers to absolute scalars as well as relative scalars of weight \( w \). The present work thereby generalizes an identity referred to earlier by Rosenfeld in his publication “On the energy-momentum tensor” \(^1\). The theorem provides a \((1,1)\)-tensor identity which can be regarded as the tensor analogue of the identity following from Euler’s theorem on homogeneous functions. The remarkably simple but nowadays rather unknown identity is independent of any internal symmetries of the constituent tensors, providing a powerful tool for deriving relations between field-theoretical expressions and physical quantities. We apply the identity especially for analyzing the metric and canonical energy-momentum tensors of matter and gravity and the relation between them. In Ref. \(^2\) we presented a generalized Einstein field equation for arbitrary version of vacuum space-time dynamics — including torsion and non-metricity. The identity allows to formulate an equivalent representation of this equation. Thereby the conjecture of a zero-energy universe is confirmed.

**Keywords:** Scalar-valued function of tensors; Semi-classical field theory of gravity and matter; Energy-momentum tensor of gravity; Generalized Einstein field equation; Zero-energy universe.

1. Introduction

In this paper we prove an identity for scalar-valued functions of tensors that constitutes the tensor analogue of Euler’s theorem on homogeneous functions of calculus. Its proof is rather straightforward and the identity itself appears almost trivial, but certainly not
obvious. The identity provides a valuable and effective means for establishing relations between tensor expressions. It remained rather unnoticed by the community.

A prominent example addressed here is the long-standing question of the permissibility of “improving” the canonical energy-momentum tensor for fields of spin higher than zero. This topic was addressed much earlier by Rosenfeld and by Belinfante in the context of symmetrizing non-symmetric canonical energy-momentum tensors. The common objective was to identify a form suitable for furnishing the source term in the classical Einstein field equation with its symmetric Einstein tensor. In later papers of Sciama, Kibble, and Hehl et al. non-symmetric canonical energy-momentum tensors were identified as the source of torsion of space-time in a generalized Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity.

The paper is organized as follows. In Section 2 we first prove the theorem for absolute scalars constituted by contracting \((n, n)\)-tensors that on their part may be tensor products of lower rank tensors. On this basis, the theorem is then generalized for tensors whose indices refer to different vector spaces, such as vierbeins (tetrads) and spinor-tensors (e.g. Dirac matrices), and to relative tensors of any weight \(w\). As an instructive example, we apply the theorem to the determinant of a \((0, 2)\)-tensor, which represents a relative scalar of weight \(w = 2\). We thus find immediately the derivative of this determinant with respect to a tensor component—without the need to refer to the definition of a determinant.

In Section 3 we apply the identity to Lagrangians of semi-classical field theories that describe the dynamics of scalar, vector, and spinor fields in an arbitrary geometry of space-time. These Lagrangians are relative scalars of weight \(w = 1\), commonly referred to as scalar densities. While the functional derivative of a Lagrangian density with respect to the metric gives the metric energy-momentum tensor, the canonical one is derived from Noether’s theorem. With help of the identity, a generic relation between the metric and the canonical (Noether) energy-momentum tensors is then established, and applied to the Klein-Gordon, Proca, and Dirac systems.

In Section 4 the identity is applied to a variety of gravity theories, some including torsion and even non-metricity, with the metric and canonical energy-momentum tensor derived in analogy to the treatment of matter systems. This analogy naturally leads to a conservation law of the total energy-momentum of matter and space-time (“Zero-Energy Universe”). The underlying energy-momentum balance equation is identified as the generalized consistency equation that in the specific case of the Einstein-Hilbert Lagrangian gives the well-known field equation of Einstein’s General Relativity. In the Riemann-Cartan geometry a Poisson-like equation for torsion emerges in addition to an extended field equation.

Section 5 finally gives the summary of the paper and the conclusions.

For curious readers, an Appendix is added displaying the derivation of the same results via the identity applied in the DeDonder-Weyl Hamiltonian formalism that is at the heart of the covariant canonical gauge theory of gravity.
2. Theorem for scalar-valued functions of tensors

2.1. Absolute tensors

In the following, we prove a theorem for a scalar that is formed from contractions of an arbitrary \((n, n)\) tensor \(T\) and for any \((n, n)\) tensor field \(T(x)\) at a point with the coordinate \(x\) in an arbitrary geometry of the underlying manifold. (A similar consideration can be found in Rosenfeld’s paper\(^\text{I}\) from 1940 that apparently got rather unnoticed so far.)

**Theorem 1.** Let \(S = S(T) \in \mathbb{R}\) be a scalar-valued function constructed from a complete contraction of a \((n, n)\)-tensor \(T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_n}\), with the ordered index set \(\{\eta_k\}\) a bijective permutation \(\{\xi_{\pi(k)}\}\) of the ordered index set \(\{\xi_j\}\) such that \(\eta_k \equiv \xi_{\pi(k)} \equiv \xi_j\). Then the following identity holds:

\[
\begin{align*}
-\frac{\partial S}{\partial T^{\mu}_{\nu, \xi_2\ldots\xi_n}_{\eta_1\ldots\eta_n}} T^{\mu}_{\nu, \xi_2\ldots\xi_n}_{\eta_1\ldots\eta_n} - \ldots - & \frac{\partial S}{\partial T^{\xi_1\ldots\xi_{n-1}, \nu}_{\eta_1\ldots\eta_n}} T^{\xi_1\ldots\xi_{n-1}, \nu}_{\eta_1\ldots\eta_n} \\
+ \frac{\partial S}{\partial T^{\xi_1\ldots\xi_n}_{\mu, \eta_2\ldots\eta_n}} T^{\xi_1\ldots\xi_n}_{\mu, \eta_2\ldots\eta_n} + \ldots + & \frac{\partial S}{\partial T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_{n-1}, \nu}} T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_{n-1}, \nu} \\
\equiv 0.
\end{align*}
\]

(1)

Thus, in each term, exactly one previously contracted index \(\xi_j\) resp. \(\eta_j\) of \(S\) is replaced by the now open indices \(\nu\) and \(\mu\).

**Proof.** The proof is based on the simple relation

\[
\frac{\partial S}{\partial T^{\alpha}_{\mu, \nu}} T^{\alpha}_{\mu, \nu} = \frac{\partial T^{\beta}}{\partial T^{\alpha}_{\mu}} T^{\alpha}_{\mu, \nu} = \delta^{\alpha}_{\beta} \delta^{\mu}_{\nu} T^{\alpha}_{\mu, \nu} = T^{\mu}_{\nu} = \delta^{\beta}_{\nu} \delta^{\alpha}_{\mu} T^{\mu}_{\nu} = \frac{\partial T^{\beta}}{\partial T^{\alpha}_{\mu}} T^{\mu}_{\nu} = \frac{\partial S}{\partial T^{\nu}_{\alpha}} T^{\mu}_{\nu}.
\]

It is straightforward to see that this relation applies for any pair of contracted indices. Then for any pair \(\xi_j\) and \(\eta_k\) of contracted indices \(\eta_k \equiv \xi_{\pi(k)} \equiv \xi_j\) of \(S = T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_n}\), we find for their respective replacement by \(\nu\) and \(\mu\) according to Eq. (1):

\[
\begin{align*}
-\frac{\partial S}{\partial T^{\xi_1\ldots\xi_{j-1}, \nu}_{\eta_1\ldots\eta_{k-1}, \eta_k, \eta_{k+1}\ldots\eta_n}} T^{\xi_1\ldots\xi_{j-1}, \nu}_{\eta_1\ldots\eta_{k-1}, \eta_k, \eta_{k+1}\ldots\eta_n} - \ldots - & \frac{\partial S}{\partial T^{\xi_1\ldots\xi_{n-1}, \mu, \nu}_{\eta_1\ldots\eta_k, \eta_{k+1}\ldots\eta_n}} T^{\xi_1\ldots\xi_{n-1}, \mu, \nu}_{\eta_1\ldots\eta_k, \eta_{k+1}\ldots\eta_n} \\
+ \frac{\partial S}{\partial T^{\xi_1\ldots\xi_n}_{\mu, \eta_1\ldots\eta_{k-1}, \eta_{k+1}\ldots\eta_n}} T^{\xi_1\ldots\xi_n}_{\mu, \eta_1\ldots\eta_{k-1}, \eta_{k+1}\ldots\eta_n} + \ldots + & \frac{\partial S}{\partial T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_{n-1}, \nu}} T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_{n-1}, \nu} \\
= -\delta^{\nu}_{\mu} \delta^{\xi_{j-1}}_{\xi_j} T^{\xi_1\ldots\xi_{j-1}, \nu}_{\eta_1\ldots\eta_{k-1}, \eta_k, \eta_{k+1}\ldots\eta_n} + \delta^{\mu}_{\eta_k} T^{\xi_1\ldots\xi_{j-1}, \mu}_{\eta_1\ldots\eta_{k-1}, \eta_k, \eta_{k+1}\ldots\eta_n} \\
- \delta^{\nu}_{\eta_{k+1}} T^{\xi_1\ldots\xi_{j-1}, \nu}_{\eta_1\ldots\eta_{k-1}, \eta_{k+1}, \eta_k, \eta_{k+2}\ldots\eta_n} + \delta^{\mu}_{\eta_{k+1}} T^{\xi_1\ldots\xi_{j-1}, \mu}_{\eta_1\ldots\eta_{k-1}, \eta_{k+2}, \eta_k, \eta_{k+2}\ldots\eta_n} \\
= -T^{\xi_1\ldots\xi_{j-1}, \mu}_{\eta_1\ldots\eta_{k+1}, \eta_k, \eta_{k+1}\ldots\eta_n} + T^{\xi_1\ldots\xi_{j-1}, \nu}_{\eta_1\ldots\eta_{k+1}, \eta_k, \eta_{k+1}\ldots\eta_n} + T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_{k-1}, \eta_{k+1}\ldots\eta_n} + T^{\xi_1\ldots\xi_n}_{\eta_1\ldots\eta_{k+1}, \eta_k, \eta_{k+1}\ldots\eta_n} \\
\equiv 0,
\end{align*}
\]

which proves the assertion.
The \((n, n)\)-tensor \(T_{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_n}\) may be in particular a tensor product of tensors of lower ranks. This is demonstrated in the following example:

**Example 1.** Let \(S\) be a scalar emerging from the contraction of arbitrary tensors \(A^\alpha_{\beta \xi}\) with \(B^\xi_{\alpha \eta}\) and \(C^\eta_{\beta \nu}\):

\[
S = A^\alpha_{\beta \xi} B^\xi_{\alpha \eta} C^\eta_{\beta \nu}.
\]  

(2)

Then

\[
- \frac{\partial S}{\partial A^\alpha_{\beta \xi}} A^\mu_{\beta \xi} - \frac{\partial S}{\partial A^\alpha_{\beta \xi}} A^\alpha_{\mu \xi} + \frac{\partial S}{\partial A^\alpha_{\beta \mu}} A^\alpha_{\beta \nu} \\
- \frac{\partial S}{\partial B^\alpha_{\nu \eta}} B^\mu_{\alpha \eta} - \frac{\partial S}{\partial B^\xi_{\alpha \eta}} B^\xi_{\nu \eta} + \frac{\partial S}{\partial B^\xi_{\alpha \mu}} B^\eta_{\nu \eta} + \frac{\partial S}{\partial C^\eta_{\nu \mu}} C^\nu_{\eta} \equiv 0.
\]  

(3)

For the proof of the identity (3) the respective terms of the sum are worked out explicitly:

\[
- \frac{\partial S}{\partial A^\alpha_{\beta \xi}} A^\mu_{\beta \xi} = -\delta^\alpha_\nu A^\mu_{\xi \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} = -A^\mu_{\xi \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \xi C^\eta_{\beta \eta}
\]

\[
- \frac{\partial S}{\partial A^\alpha_{\beta \xi}} A^\alpha_{\mu \xi} = -\delta^\beta_\nu A^\alpha_{\mu \xi} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} = -A^\alpha_{\mu \xi} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \xi C^\eta_{\beta \eta}
\]

\[
\frac{\partial S}{\partial A^\alpha_{\beta \mu}} A^\alpha_{\beta \nu} = \delta^\mu_\xi A^\alpha_{\beta \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} = A^\alpha_{\beta \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \xi C^\eta_{\beta \eta}
\]

\[
- \frac{\partial S}{\partial B^\alpha_{\nu \eta}} B^\mu_{\alpha \eta} = -A^\alpha_{\nu \eta} \delta^\mu_\xi B^\xi_{\alpha \eta} C^\eta_{\beta \nu} = -A^\alpha_{\nu \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \xi C^\eta_{\beta \eta}
\]

\[
- \frac{\partial S}{\partial B^\xi_{\alpha \eta}} B^\eta_{\nu \eta} = A^\alpha_{\nu \eta} \delta^\xi_\mu B^\xi_{\alpha \eta} C^\eta_{\beta \nu} = A^\alpha_{\nu \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \xi C^\eta_{\beta \eta}
\]

\[
\frac{\partial S}{\partial C^\eta_{\nu \mu}} C^\nu_{\eta} = A^\alpha_{\beta \eta} B^\xi_{\alpha \eta} \delta^\mu_\xi C^\eta_{\beta \nu} + A^\alpha_{\beta \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \delta^\mu_\xi C^\eta_{\beta \nu} = A^\alpha_{\beta \eta} B^\xi_{\alpha \eta} C^\eta_{\beta \nu} \xi C^\eta_{\beta \eta}.
\]

The terms on the right-hand sides obviously sum up to zero.

Of course, this example can be generalized to the contraction of any number of tensors of any rank. It is important to stress that the above identity is entirely unrelated to possible internal symmetries of the constituent tensors, and is valid in addition to any symmetry related identities like e.g. the Bianchi identities in field theories of gravity involving the Riemann tensor. Nonetheless, those internal tensor symmetries can be explored for simplifying the resulting identity. If, for instance, the tensor \(A^\alpha_{\beta \xi}\) in (2) is symmetric or skew-symmetric, then the first two terms on the left-hand side of the identity (3) can be merged into a single term.
2.2. Scalar functions involving the metric tensor

With $S = S(g, T) \in \mathbb{R}$ a scalar-valued function constructed from the metric tensor $g_{\mu \nu}$ and an $(n, m)$-tensor $T^{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_m}$ of the physical fields, the following identity then holds for $(n - m)/2 \in \mathbb{Z}$:

$$0 \equiv \frac{\partial S}{\partial g_{\mu \beta}} g^{\mu \beta} + \frac{\partial S}{\partial g^{\beta \mu}} g_{\beta \mu} - \frac{\partial S}{\partial T^{\nu \xi_2 \ldots \xi_n \eta_1 \ldots \eta_m}} T^{\mu \xi_2 \ldots \xi_n \eta_1 \ldots \eta_m}$$

$$- \ldots - \frac{\partial S}{\partial T^{\xi_1 \ldots \xi_{n-1} \nu \eta_1 \ldots \eta_m}} T^{\xi_1 \ldots \xi_{n-1} \mu \eta_1 \ldots \eta_m}$$

$$+ \frac{\partial S}{\partial T^{\xi_1 \ldots \xi_n \mu \eta_2 \ldots \eta_m}} T^{\xi_1 \ldots \xi_n \mu \eta_2 \ldots \eta_m}$$

$$+ \ldots + \frac{\partial S}{\partial T^{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_{m-1} \nu}} T^{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_{m-1} \nu}$$.

(4)

**Corollary 1.** The trace of Eq. (4) then yields the scalar identity:

$$\frac{\partial S}{\partial g^{\alpha \beta}} g_{\alpha \beta} \equiv \frac{n - m}{2} \frac{\partial S}{\partial T^{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_m}} T^{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_m}.$$

(5)

**Proof.** Contracting Eq. (4) immediately gives Eq. (5). □

2.3. Scalar functions involving tensors with multiple index classes

**Corollary 2.** The theorem (1) holds also for scalars $S$ constructed from generalized tensor objects—such as spinor-tensors—which are made of multiple index classes. Examples of such objects are Dirac matrices $\gamma^\mu$, which are $(1, 1)$-spinor-$(1, 0)$-tensors, as well as vierbeins (tetrads) with one Lorentz and one space-time index, respectively. One then encounters a specific identity for each particular index class, provided that all other indices are fully contracted.

**Proof.** For each fully contracted index group, Eq. (1) trivially vanishes. For the not fully contracted indices, Eq. (1) applies.

Examples are provided in Sects. 3.4 and 4.1.4. □

2.4. Relative scalar built from relative tensors

**Corollary 3.** Let a relative scalar of weight $w$—denoted by $\tilde{S} = S(\sqrt{-g})^w \in \mathbb{R}$—be given as a function of the metric $g_{\mu \nu}$ and a tensor $T^{\xi_1 \ldots \xi_n \eta_1 \ldots \eta_m}$ of rank $(n, m)$. Then the
following identity holds for \((n - m)/2 \in \mathbb{Z}\):

\[
\begin{align*}
&+ \frac{\partial S}{\partial g_{\mu \beta}} g_{\nu \beta} + \frac{\partial S}{\partial g_{\beta \mu}} g_{\beta \nu} \\
&- \frac{\partial \tilde{S}}{\partial T^\mu \xi_2 \ldots \xi_n} T^\mu \xi_2 \ldots \xi_n \eta_1 \ldots \eta_m - \ldots - \frac{\partial \tilde{S}}{\partial T^\mu \eta_1 \ldots \eta_m} T^\xi_1 \ldots \xi_{n-1} \mu \eta_1 \ldots \eta_m \\
&+ \frac{\partial \tilde{S}}{\partial T^\xi_1 \ldots \xi_n \mu \eta_2 \ldots \eta_m} T^\xi_1 \ldots \xi_n \eta_2 \ldots \eta_m - \ldots + \frac{\partial \tilde{S}}{\partial T^\xi_1 \ldots \xi_{n-1} \mu \eta_1 \ldots \eta_{m-1} \nu} T^\xi_1 \ldots \xi_n \eta_1 \ldots \eta_{m-1} \nu
\end{align*}
\]

\[\equiv \delta^\nu_\mu \tilde{S}.
\]  

(6)

**Proof.** Multiply Eq. (4) with \((\sqrt{-g})^w\), add \(\delta^\nu_\mu \tilde{S}\) on both sides of the identity, and combine the appropriate terms on the left-hand side. From the definition of the determinant, the general rule for the derivative of the determinant of the covariant metric with respect to a component of the metric is obtained as:

\[
\frac{\partial \sqrt{-g}}{\partial g_{\mu \beta}} = \frac{1}{2} g^\beta_\mu \sqrt{-g}.
\]  

(7)

One then gets:

\[
\left( \frac{\partial S}{\partial g_{\mu \beta}} g_{\nu \beta} + \frac{\partial S}{\partial g_{\beta \mu}} g_{\beta \nu} + \delta^\mu_\nu \tilde{S} \right) (\sqrt{-g})^w = \frac{\partial \tilde{S}}{\partial g_{\mu \beta}} g_{\nu \beta} + \frac{\partial \tilde{S}}{\partial g_{\beta \mu}} g_{\beta \nu}.
\]

Equation (6) is obviously the analogue of Euler’s theorem on homogeneous functions in the realm of tensor calculus. The relative scalar \(\tilde{S} = S(\sqrt{-g})^w\) of weight \(w\) may in particular be the tensor product of some relative tensors of lower ranks and weights. The weight \(w\) of \(\tilde{S}\) is then the sum of the weights of the relative tensors.

As a direct application, the above used relation for the derivative of the metric’s determinant may also be derived using (6).

**Example 2.** The components of the covariant metric tensor field \((g_{\mu \nu}(x))\) transform under the transition \(x \mapsto X\) of the space-time location as:

\[
g_{\mu \nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu} g_{\alpha \beta}(x) \frac{\partial x^\beta}{\partial X^\nu},
\]

and hence its determinant \(g = |g_{\alpha \beta}(x)|:\)

\[
|g_{\mu \nu}(X)| = |g_{\alpha \beta}(x)| \left| \frac{\partial x}{\partial X} \right|^2.
\]

The determinant \(g\) of the covariant metric tensor thus transforms as a *relative scalar* of weight \(w = 2\). According to the general form of the identity for relative scalars of weight \(w\) from Eq. (6), we get for the relative scalar \(g\) due to the symmetry \(g_{\beta \alpha} = g_{\alpha \beta}^\ast\):

\[
\frac{\partial g}{\partial g_{\beta \mu}} g_{\beta \nu} + \frac{\partial g}{\partial g_{\beta \mu}} g_{\nu \beta} \equiv 2 \delta^\nu_\mu g \quad \Rightarrow \quad \frac{\partial g}{\partial g_{\beta \mu}} g_{\beta \alpha} \equiv \delta^\nu_\mu g.
\]

(8)
Contracting (8) with the inverse metric \( g^{\alpha\nu} \) reproduces the derivative of the determinant \( g \) of the covariant metric with respect to the component \( g_{\nu\mu} \) of the metric from Eq. (7),

\[
\frac{\partial g}{\partial g_{\nu\mu}} = g^{\mu\nu} g,
\]

and thus for the negative square root of \( g \)

\[
\frac{\partial \sqrt{-g}}{\partial g_{\nu\mu}} = \frac{1}{2} g^{\mu\nu} \sqrt{-g}, \quad \frac{\partial \sqrt{-g}}{\partial g^{\nu\mu}} = -\frac{1}{2} g_{\mu\nu} \sqrt{-g}.
\] (9)

3. Physical applications

3.1. Lagrangian densities

In the following applications various relativistic field theories are considered where the underlying space-time is not restricted to the Minkowski geometry but takes general curvilinear space-times into account. In that case the invariance of the action integral requires that the Lagrangian is a scalar density rather than a density. This is accomplished by multiplying it with \( \sqrt{-g} \).

**Corollary 4.** For a scalar density Lagrangian \( \tilde{L} \), i.e. for a relative scalar of weight \( w = 1 \), the identity (6) writes for \((n - m)/2 \in \mathbb{Z}\):

\[
\begin{align*}
&- \frac{\partial \tilde{L}}{\partial g^{\nu\mu}} g^{\mu\beta} - \frac{\partial \tilde{L}}{\partial g_{\nu\mu}} g^{\beta\mu} \\
&- \frac{\partial \tilde{L}}{\partial T^\mu \xi_2 \cdots \xi_n \eta_1 \cdots \eta_m} T^{\mu \xi_2 \cdots \xi_n \eta_1 \cdots \eta_m} - \cdots - \frac{\partial \tilde{L}}{\partial T^{\xi_1 \cdots \xi_{n-1}} \nu \eta_1 \cdots \eta_m} \eta_{n-1} \eta_n \cdots \eta_m \\
&+ \frac{\partial \tilde{L}}{\partial T^{\xi_1 \cdots \xi_n \nu} \eta_1 \cdots \eta_m} \eta_{n-1} \eta_n \cdots \eta_m + \cdots + \frac{\partial \tilde{L}}{\partial T^{\xi_1 \cdots \xi_n \eta_1 \cdots \eta_{m-1}} \mu} T^{\xi_1 \cdots \xi_n \eta_1 \cdots \eta_{m-1}} \mu \\
&\equiv \delta^\mu_\nu \tilde{L}.
\end{align*}
\] (10)

3.2. Klein-Gordon Lagrangian

The Klein-Gordon Lagrangian density \( \tilde{L}_{\text{KG}} = L_{\text{KG}} \sqrt{-g} \) for a massive complex scalar field \( \phi(x) \) is given by (see e.g. [9],

\[
\begin{align*}
\tilde{L}_{\text{KG}} (\phi, \bar{\phi}, \partial_\alpha \phi, \partial_\nu \bar{\phi}, g^{\mu\nu}) &= \left( \frac{\partial \bar{\phi}}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} g^{\alpha\beta} - m^2 \bar{\phi} \phi \right) \sqrt{-g} \\
&= \left[ \frac{1}{2} \left( \frac{\partial \bar{\phi}}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} + \frac{\partial \bar{\phi}}{\partial x^\beta} \frac{\partial \phi}{\partial x^\alpha} \right) g^{\alpha\beta} - m^2 \bar{\phi} \phi \right] \sqrt{-g}.
\end{align*}
\] (11)

In order to set up the pertaining identity (6), we set the required derivatives of \( \tilde{L}_{\text{KG}} \). This yields with the derivative of the determinant \( g \) of the covariant metric \( g_{\mu\nu} \) with respect to
In a dynamic space-time, the Proca Lagrangian is defined by (see e.g. 3.3. Proca Lagrangian)

\[
\mathcal{L}_P = \left(-\frac{1}{4} f_{\alpha\beta} f_{\xi\eta} g^{\alpha\xi} g^{\beta\eta} + \frac{1}{2} m^2 a_\alpha a_\beta g^{\alpha\beta}\right) \sqrt{-g}, \quad f_{\alpha\beta} \equiv \frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta}.
\]

All tensors forming \(\mathcal{L}_P\) are understood as tensor fields, taken at the same spacetime event \(x\).

We observe that the scalar density \(\mathcal{L}_P\) represents in this case a quadratic function of the field tensor \(f_{\alpha\beta}\) and the field vector \(a_\alpha\). Nonetheless, the identity (10) for the scalar density \(\mathcal{L}_P\) follows in the usual form as:

\[
- \frac{\partial \mathcal{L}_P}{\partial g^{\mu\nu}} g^{\mu\nu} - \frac{\partial \mathcal{L}_P}{\partial g^{\beta\mu}} g^{\beta\mu} + \frac{\partial \mathcal{L}_P}{\partial f_{\mu\beta}} f_{\mu\beta} + \frac{\partial \mathcal{L}_P}{\partial f_{\mu\beta}} f_{\beta\nu} + \frac{\partial \mathcal{L}_P}{\partial a_\mu} a_\nu \equiv \delta^\mu_\nu \mathcal{L}_P.
\]

Rewriting the identity (14) as a correlation of metric and canonical energy-momentum tensor densities, \(\tilde{T}^\mu_\nu\) and \(\tilde{\theta}^\mu_\nu\), we observe that these tensors do not coincide for the Proca system:

\[
\tilde{T}^\mu_\nu = 2 \frac{\partial \tilde{\mathcal{L}}_P}{\partial g^{\mu\nu}} g^{\mu\nu} - \frac{\partial \tilde{\mathcal{L}}_P}{\partial g^{\beta\mu}} g^{\beta\mu} + \frac{\partial \tilde{\mathcal{L}}_P}{\partial f_{\mu\beta}} f_{\mu\beta} + \frac{\partial \tilde{\mathcal{L}}_P}{\partial f_{\mu\beta}} f_{\beta\nu} + \frac{\partial \tilde{\mathcal{L}}_P}{\partial a_\mu} a_\nu
\]

\[
\equiv \tilde{\theta}^\mu_\nu + \frac{\partial \tilde{\mathcal{L}}_P}{\partial f_{\mu\beta}} f_{\nu\beta} + \frac{\partial \tilde{\mathcal{L}}_P}{\partial a_\mu} a_\nu.
\]
3.4. Regularized Dirac Lagrangian

3.4.1. Identity for the metric indices

The regularized Dirac Lagrangian $\mathcal{L}_D(\psi, \partial \psi, \bar{\psi}, \partial \bar{\psi}, \gamma^\mu)^{[10][11][18]}$, constructed upon the Dirac equation writes

$$\mathcal{L}_D = \frac{i}{2} \left( \bar{\psi} \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} - \bar{\psi} \gamma^\alpha \frac{\partial \bar{\psi}}{\partial x^\alpha} \gamma_\alpha \psi \right) - m \bar{\psi} \psi + \frac{i}{3M} \frac{\partial \bar{\psi}}{\partial x^\alpha} \sigma_\alpha^\beta \frac{\partial \psi}{\partial x^\beta}, \quad (16)$$

wherein the (1, 1)-spinor-(2, 0)-tensor field $\sigma_\alpha^\beta$ is defined as the commutator of the matrix product $\gamma^\alpha \gamma^\beta$:

$$\sigma_\alpha^\beta = \frac{i}{2} (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha) \Leftrightarrow \sigma_\alpha^\beta \equiv \frac{i}{2} (\gamma_\alpha^c \gamma_\beta^c - \gamma_\beta^c \gamma_\alpha^c). \quad (17)$$

With an explicit notation of the spinor indices as lower case Latin letters, the Dirac Lagrangian takes on the form:

$$\mathcal{L}_D = \frac{i}{2} \left( \bar{\psi}_a \gamma^a \frac{\partial \psi}{\partial x^\alpha} - \bar{\psi}_a \gamma^a \frac{\partial \bar{\psi}}{\partial x^\alpha} \gamma_\alpha \psi \right) - m \bar{\psi}_a \psi^a + \frac{i}{3M} \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \sigma_\alpha^\beta \frac{\partial \psi^a}{\partial x^\beta}. \quad (18)$$

With the spinor indices fully contracted, the identity $[1]$ with respect to the metric indices for the scalar quantity $\mathcal{L}_D$ is then given by:

$$\frac{\partial \mathcal{L}_D}{\partial \left( \frac{\partial \psi}{\partial x^\alpha} \right)} \frac{\partial \psi}{\partial x^\alpha} + \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \frac{\partial \mathcal{L}_D}{\partial \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} \right)} - \frac{\partial \mathcal{L}_D}{\partial \gamma_\alpha^\beta} \gamma_\alpha^\beta \equiv 0. \quad (19)$$

Equation (19) is verified by direct calculation:

$$\frac{\partial \mathcal{L}_D}{\partial \left( \frac{\partial \psi}{\partial x^\alpha} \right)} \frac{\partial \psi}{\partial x^\alpha} = \frac{i}{2} \bar{\psi}_a \gamma^a \frac{\partial \psi}{\partial x^\alpha} + \frac{1}{3M} \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \sigma_\alpha^\beta \frac{\partial \psi}{\partial x^\beta}, \quad \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \frac{\partial \mathcal{L}_D}{\partial \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} \right)} = - \frac{i}{2} \bar{\psi}_a \gamma^a \frac{\partial \psi}{\partial x^\alpha} + \frac{1}{3M} \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \sigma_\alpha^\beta \frac{\partial \psi}{\partial x^\beta},$$

$$- \frac{\partial \mathcal{L}_D}{\partial \gamma_\alpha^\beta} \gamma_\alpha^\beta = \frac{i}{2} \bar{\psi}_a \gamma^a \frac{\partial \psi}{\partial x^\alpha} - \frac{i}{2} \bar{\psi}_a \gamma^a \frac{\partial \psi}{\partial x^\alpha} \gamma_\alpha^\beta \mu \psi^c \gamma^c \mu \psi^c.$$

which obviously sums up to zero.

The identity $[10]$ for the covariant scalar density $\tilde{\mathcal{L}}_D$ is then

$$\frac{\partial \tilde{\mathcal{L}}_D}{\partial \left( \frac{\partial \psi}{\partial x^\alpha} \right)} \frac{\partial \psi}{\partial x^\alpha} + \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \frac{\partial \tilde{\mathcal{L}}_D}{\partial \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} \right)} - \frac{\partial \tilde{\mathcal{L}}_D}{\partial \gamma_\alpha^\beta} \gamma_\alpha^\beta \equiv \delta_\nu^\mu \tilde{\mathcal{L}}_D, \quad (20)$$

hence, rearranging the terms and skipping the spinor indices:

$$\tilde{\gamma}_\nu^\mu = \frac{\partial \tilde{\mathcal{L}}_D}{\partial \left( \frac{\partial \psi}{\partial x^\alpha} \right)} \frac{\partial \psi}{\partial x^\alpha} + \frac{\partial \bar{\psi}_a}{\partial x^\alpha} \frac{\partial \tilde{\mathcal{L}}_D}{\partial \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} \right)} - \delta_\nu^\mu \tilde{\mathcal{L}}_D \equiv \text{Tr} \left\{ \frac{\partial \tilde{\mathcal{L}}_D}{\partial \gamma_\alpha^\beta} \gamma_\alpha^\beta \right\} \equiv \tilde{T}_\nu^\mu. \quad (20)$$
The corresponding identity (1) for the scalar $L_{3.4.2.}$ indices and leave open the spinor indices: the Dirac Lagrangian with the metric
\[ T_{\mu\nu} = \theta_{\mu\nu} = \frac{i}{2} \left( \bar{\psi}_c \gamma^a \psi_a - \bar{\psi}_a \gamma^a \psi_a \right) - g_{\mu\nu} L_D \]
+ \frac{i}{3M} \theta_{\mu\nu} \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} \sigma^{\alpha\beta} \frac{\partial \psi}{\partial x^\beta} + \frac{\partial \bar{\psi}}{\partial x^\beta} \sigma^{\beta\alpha} \frac{\partial \psi}{\partial x^\alpha} \right). \quad (21)

3.4.2. Identity for the spinor indices

The corresponding identity (1) for the scalar $L_D$ is obtained if we contract the space-time indices and leave open the spinor indices:
\[ \frac{\partial L_D}{\partial (\psi^a)} \frac{\partial \psi^a}{\partial x^\alpha} + \frac{\partial L_D}{\partial \psi_b} \frac{\partial \psi_b}{\partial x^\alpha} + \frac{\partial L_D}{\partial \bar{\psi}_b} \frac{\partial \bar{\psi}_b}{\partial x^\alpha} + \frac{\partial L_D}{\partial \gamma^a_{b\alpha}} \gamma^a_{b\alpha} - \gamma^a_{b\alpha} \frac{\partial L_D}{\partial \gamma^c_{a\alpha}} = 0. \quad (22) \]

The identity (22) is also verified by direct calculation:
\[ \frac{\partial L_D}{\partial (\psi^a)} \frac{\partial \psi^a}{\partial x^\alpha} = \frac{i}{2} \bar{\psi}_c \gamma^c_{a\alpha} \psi_a + \frac{i}{3M} \bar{\psi}_c \sigma^{\alpha\beta} \psi_a \frac{\partial \psi^a}{\partial x^\beta} \]
- \[ \frac{\partial \bar{\psi}_b}{\partial x^\alpha} \left( \frac{\partial L_D}{\partial (\psi^a)} \frac{\partial \psi^a}{\partial x^\alpha} \right) = \frac{i}{2} \bar{\psi}_b \gamma^a_{c\beta} \psi_a - m \bar{\psi}_b \psi_a \]
- \[ \frac{\partial L_D}{\partial \psi_b} \frac{\partial \psi_b}{\partial x^\alpha} = \frac{i}{2} \bar{\psi}_b \gamma^c_{a\alpha} \psi_a + m \bar{\psi}_b \psi_a \]
- \[ \frac{\partial L_D}{\partial \gamma^b_{c\alpha}} \gamma^c_{a\alpha} = \frac{i}{2} \left( \bar{\psi}_b \gamma^c_{a\alpha} \frac{\partial \psi^c}{\partial x^\alpha} - \bar{\psi}_b \gamma^a_{c\beta} \frac{\partial \psi^c}{\partial x^\beta} \right) + \frac{i}{3M} \bar{\psi}_b \sigma^{\alpha\beta} \frac{\partial \psi^c}{\partial x^\beta} - \frac{1}{6M} \frac{\partial \bar{\psi}_b}{\partial x^\alpha} \left( \gamma^c_{a\alpha} \gamma^a_{d\beta} - \gamma^c_{b\beta} \gamma^a_{d\alpha} \right) \frac{\partial \psi^d}{\partial x^\beta} \]
- \[ \gamma^b_{c\alpha} \frac{\partial L_D}{\partial \gamma^c_{a\alpha}} = \frac{i}{2} \left( \bar{\psi}_b \gamma^a_{c\alpha} \frac{\partial \psi^a}{\partial x^\alpha} - \bar{\psi}_b \gamma^a_{c\beta} \frac{\partial \psi^a}{\partial x^\beta} \right) - \frac{i}{3M} \bar{\psi}_b \sigma^{\alpha\beta} \frac{\partial \psi^a}{\partial x^\beta} + \frac{1}{6M} \frac{\partial \bar{\psi}_b}{\partial x^\alpha} \left( \gamma^c_{b\beta} \gamma^a_{d\alpha} - \gamma^c_{b\beta} \gamma^a_{d\alpha} \right) \frac{\partial \psi^d}{\partial x^\beta} \]

Again, all terms on the right-hand side sum up to zero.

4. Theories of Gravity

In the following we consider a generalized, metric compatible space-time with Riemann-Cartan geometry, where the connection is not necessarily symmetric and torsion of space-time is thus admitted. The additional degrees of freedom carried by the torsion enforce
to treat connection and metric as independent fields (Palatini approach). The metric and torsion dependent curvature tensor is then called the Riemann-Cartan tensor. In some cases we will also, in addition, take non-metricity into account, or also restrict the geometry to the torsion-free Riemannian one with the Levi-Civita connection.

4.1. Scalars depending on the Riemann-Cartan tensor and the metric

4.1.1. General gravity Lagrangian for metric compatibility

Any relative scalar Lagrangian \( \tilde{L}_{Gr}(R, g) \) of weight \( w = 1 \) built from the Riemann-Cartan curvature tensor

\[
R^\eta_{\xi\beta\lambda} = \frac{\partial \Gamma^\eta_{\xi\lambda}}{\partial x^\beta} - \frac{\partial \Gamma^\eta_{\xi\beta}}{\partial x^\lambda} + \Gamma^\eta_{\tau\lambda} \Gamma^\tau_{\xi\beta} - \Gamma^\eta_{\tau\beta} \Gamma^\tau_{\xi\lambda}
\]  

(23)

and the metric satisfies the identity (10), whose particular form for \( \tilde{L}_{Gr} \) is given by:

\[
\frac{\partial \tilde{L}_{Gr}}{\partial g^\nu_\beta} g^{\mu_\beta} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\beta_\nu}} g^{\beta_\mu} = - \frac{\partial \tilde{L}_{Gr}}{\partial R^\eta_{\alpha_\beta\lambda}} R^\mu_{\alpha_\beta\lambda} + \frac{\partial \tilde{L}_{Gr}}{\partial R^\eta_{\mu_\beta\lambda}} R^\eta_{\nu_\beta\lambda} \\
+ \frac{\partial \tilde{L}_{Gr}}{\partial R^\eta_{\alpha_\mu\lambda}} R^\eta_{\alpha_\nu\lambda} + \frac{\partial \tilde{L}_{Gr}}{\partial R^\eta_{\alpha_\beta\nu}} R^\eta_{\alpha_\beta\nu} - \delta^\mu_\nu \tilde{L}_{Gr}.
\]  

(24)

As stated in Eq. (6) it is necessary and sufficient for the identity (24) to hold that \( \tilde{L}_{Gr}(R, g) \) is a scalar density function of the tensors \( R^\eta_{\alpha_\beta\lambda} \) and \( g^{\beta_\nu} \). Whether or not a tensor may on its part be represented by underlying quantities—such as the affine connection or the metric in the actual case—does not play a role for setting up the identity. The fact that the Riemann-Cartan tensor actually possesses internal symmetries also does not modify the form of the identity. Clearly, these symmetries will be reflected in a later analysis of the invariant. For instance, with the Riemann-Cartan tensor (23) being by its definition skew-symmetric in its last index pair, the last two derivative terms of the identity (24) are identical and can be merged into a single term.

The left-hand side of the identity (24) can again be interpreted as the metric energy-momentum tensor \( T^\mu_\nu \) of the Lagrangian system \( \tilde{L}_{Gr}(R, g) \), hence

\[
T^\mu_\nu = \frac{1}{\sqrt{-g}} \left( \frac{\partial \tilde{L}_{Gr}}{\partial g^{\beta_\nu}} g^{\mu_\beta} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\beta_\mu}} g^{\beta_\nu} \right),
\]  

(25)

whereas the corresponding canonical energy-momentum tensor \( \theta^\mu_\nu \) is derived for the corresponding absolute scalar \( L_{Gr} = \tilde{L}_{Gr}/\sqrt{-g} \) by means of Noether’s theorem as:

\[
\theta^\mu_\nu = 2 \frac{\partial L_{Gr}}{\partial R^\eta_{\alpha_\beta\mu}} R^\eta_{\alpha_\beta\nu} - \delta^\mu_\nu L_{Gr}.
\]  

(26)

Notice that we use a modified notation for the canonical energy-momentum tensor for gravity to distinguish it for later convenience from that of matter. This gives the relation between both tensors:

\[
T^\mu_\nu = \theta^\mu_\nu - R^\mu_{\alpha_\beta\lambda} \Gamma^\beta_\nu \frac{\partial \tilde{L}_{Gr}}{\partial R^\eta_{\alpha_\beta\lambda}} R^\eta_{\nu_\beta\lambda} + \frac{\partial \tilde{L}_{Gr}}{\partial R^\eta_{\mu_\beta\lambda}} R^\eta_{\nu_\beta\lambda}.
\]
We remark that the above tensors were formally set up in analogy to the energy-momentum
tensors of the Klein-Gordon and Proca systems. This is independent from interpreting any
of the tensors (25) and (26) as the physical energy-momentum tensors of the gravitational
field.

4.1.2. General gravity Lagrangian

For the case of a non-vanishing covariant derivative of the metric, hence for $g^{\alpha\beta}_{\mu} \neq 0$,
the gravity Lagrangian also depends on the covariant derivative of the metric $	ilde{L}_{Gr} = \mathcal{L}_{Gr}(R, g, \nabla g)$. The identity (24) is then amended by three additional terms that reflect
the three indices of $g^{\alpha\beta}_{\mu}$:

$$
\begin{align*}
- \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} - \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\alpha\beta}_{\mu}} g^{\alpha\beta}_{\nu} \\
- \frac{\partial \tilde{L}_{Gr}}{\partial R^{\alpha\beta}_{\mu\lambda}} R^{\mu\lambda}_{\alpha\beta} + \frac{\partial \tilde{L}_{Gr}}{\partial R^{\alpha\beta}_{\mu\lambda}} R^{\mu\lambda}_{\alpha\beta} + \frac{\partial \tilde{L}_{Gr}}{\partial R^{\alpha\beta}_{\mu\lambda}} R^{\mu\lambda}_{\alpha\beta} \equiv \delta^\mu_\nu \tilde{L}_{Gr},
\end{align*}
$$

(27)

The metric energy-momentum tensor is then generalized to:

$$
T^\mu_\nu = \frac{1}{\sqrt{-g}} \left( \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} \right),
$$

whereas the canonical energy-momentum tensor takes on the generalized form (see
Eq. (48) of Struckmeier et al.):

$$
\vartheta^\mu_\nu = \frac{\partial \tilde{L}_{Gr}}{\partial g^{\alpha\beta}_{\mu}} g^{\alpha\beta}_{\nu} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\alpha\beta}_{\mu}} g^{\alpha\beta}_{\nu} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\alpha\beta}_{\mu}} R^{\alpha\beta}_{\mu\lambda} R^{\alpha\beta}_{\mu\lambda} - \delta^\mu_\nu \tilde{L}_{Gr}.
$$

Due to the identity (10), the so-defined energy-momentum tensors are correlated by:

$$
T^\mu_\nu \equiv \vartheta^\mu_\nu - \frac{\partial \tilde{L}_{Gr}}{\partial g^{\alpha\beta}_{\mu}} R^{\mu\lambda}_{\alpha\beta} + \frac{\partial \tilde{L}_{Gr}}{\partial g^{\alpha\beta}_{\mu}} R^{\mu\lambda}_{\alpha\beta} \delta^\mu_\nu.
$$

4.1.3. Einstein-Hilbert-Cartan Lagrangian $\tilde{L}_E$

The Einstein-Hilbert-Cartan Lagrangian $\tilde{L}_E$ is defined by the Ricci scalar density $\tilde{R} = R \sqrt{-g}$. The latter emerges from the definition of the Riemann-Cartan curvature tensor from Eq. (23) via its non-trivial contraction $R_{\xi\nu} = R_{\xi\nu}^{\eta\lambda}$, followed by a contraction with the metric according to:

$$
16\pi G \tilde{L}_E = R^{\eta\nu\lambda}_{\xi\eta} g^{\xi\lambda} \sqrt{-g} = R_{\xi\lambda} g^{\xi\lambda} \sqrt{-g} = \tilde{R}.
$$

(28)

Note that the Ricci tensor $R_{\xi\lambda}$ is not symmetric for non-symmetric connection coefficients $\Gamma^\eta_{\lambda\xi} \neq \Gamma^\eta_{\xi\lambda}$. For the scalar density $\tilde{L}_E$, the left-hand side of the general equation (24), hence the derivatives of $\tilde{R}$ with respect to the metric are:

$$
\frac{\partial \tilde{R}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} + \frac{\partial \tilde{R}}{\partial g^{\nu\beta}_{\gamma}} g^{\mu\gamma}_{\alpha} = \tilde{R}^{\mu}_{\nu} + \tilde{R}^{\mu}_{\nu} - \delta^\mu_\nu \tilde{R}.
$$

(29a)
The derivatives of $\tilde{R}$ with respect to the Riemann tensor follow as:

\[
\frac{\partial \tilde{R}}{\partial R^\mu_{\xi\beta\lambda}} = \delta_\nu^{\mu} \delta_\xi^{\beta} g^{\xi\lambda} \tilde{R}^\nu_{\xi\lambda} = \tilde{R}^\nu_{\xi\lambda} = \tilde{R}^\mu_{\nu} \tag{29b}
\]

\[
\frac{\partial \tilde{R}}{\partial R^\eta_{\mu\beta\lambda}} = \delta^\eta_\xi \delta^\beta_\eta g^{\xi\lambda} \tilde{R}^\eta_{\nu\beta\lambda} = \tilde{R}^\mu_{\nu} = \tilde{R}^\mu_{\nu} \tag{29c}
\]

\[
\frac{\partial \tilde{R}}{\partial R^\eta_{\xi\beta\mu}} = \delta^\mu_\eta \delta^\beta_\eta g^{\xi\lambda} \tilde{R}^\eta_{\xi\nu\lambda} = \tilde{R}^\mu_{\nu} = \tilde{R}^\mu_{\nu} \tag{29d}
\]

\[
\frac{\partial \tilde{R}}{\partial R^\eta_{\xi\beta\mu}} = \delta^\mu_\eta \delta^\beta_\eta g^{\xi\lambda} \tilde{R}^\eta_{\xi\nu\lambda} = \tilde{R}^\mu_{\nu} = \tilde{R}^\mu_{\nu} \tag{29e}
\]

The identity (24) is obviously satisfied by Eqs. (29). The derivatives of the $\tilde{L}_E$ with respect to the metric define the metric energy-momentum tensor,

\[
T^\mu_\nu = \frac{1}{\sqrt{-g}} \left( \frac{\partial \tilde{L}_E}{\partial g^\mu\beta} g^{\mu\beta} + \frac{\partial \tilde{L}_E}{\partial g_{\beta\mu}} g_{\beta\mu} \right) = \frac{1}{16\pi G} \left( R^\mu_\nu + R^\mu_{\nu} - \delta^\mu_\nu R \right). \tag{30}
\]

By virtue of the identity (24), this tensor can be identically replaced by its derivatives with respect to the Riemann tensor. The corresponding canonical energy-momentum tensor $\vartheta^\mu_\nu$ is defined by:

\[
\vartheta^\mu_\nu = \frac{1}{8\pi G} \left( R^\mu_\nu - \frac{1}{2} g^{\mu\nu} R \right), \tag{31}
\]

Both tensors are thus related as:

\[
T^\mu_\nu = \vartheta^\mu_\nu - \frac{\partial \tilde{L}_E}{\partial R^\mu_{\xi\beta\lambda}} R^\eta_{\xi\nu\lambda} + \frac{\partial \tilde{L}_E}{\partial R^\eta_{\xi\beta\mu}} R^\eta_{\xi\nu\lambda} - \delta^\mu_\nu \tilde{L}_E = \frac{1}{16\pi G} \left( 2R^\mu_\nu - \delta^\mu_\nu R \right).
\]

and hence agree for a symmetric Ricci tensor $R^\mu_\nu \equiv R^\nu_\mu$, which is induced by symmetric connections $\Gamma^\eta_{\xi\lambda} \equiv \Gamma^\eta_{\lambda\xi}$ and hence a torsion-free space-time.

Allowing for an asymmetric connection, Eq. (31) yields an asymmetric canonical energy-momentum tensor. In their covariant representations both energy-momentum tensors are:

\[
T^\mu_\nu = \vartheta^\mu_\nu - \frac{\partial \tilde{L}_E}{\partial R^\mu_{\xi\beta\lambda}} R^\eta_{\xi\nu\lambda} + \frac{\partial \tilde{L}_E}{\partial R^\eta_{\mu\beta\lambda}} R^\eta_{\nu\beta\lambda} = \vartheta^\mu_\nu + \frac{1}{16\pi G} \left( 2R^\mu_\nu - \delta^\mu_\nu R \right),
\]

which gives the relations

\[
\vartheta^\nu_{\nu} = T^\nu_{\nu}, \quad \vartheta_{\nu\nu} = \frac{1}{8\pi G} R_{\nu\nu}.
\]

Note that these derivations hold for both, the metric and the Palatini formulation as either one is based on a covariant scalar density Lagrangian. We remark, though, that in the case of the metric formalism, the addition of the Gibbons-York-Hawking boundary term to the Einstein-Hilbert Lagrangian leads to a non-covariant Lagrangian density—and hence to a scalar density that is not invariant under space-time diffeomorphisms. As the covariance
of the scalar density $\hat{S}$ constitutes a necessary and sufficient condition for the validity of Eq. (6), it is not possible to apply it to the boundary term in curved geometries.

4.1.4. Vierbein formulation of the Einstein-Hilbert-Cartan Lagrangian

For including fermions to the variety of matter fields the description of space-time geometry must be modified by considering inertial frames at every point of the underlying Riemann-Cartan geometry. These frames are provided by a “vierbein” (aka tetrad) field of four orthonormal basis vectors $e_i(x)$ with the coordinates $e_i^\mu(x)$. The Lorentz (or nonholonomic) indices $i$ refer to the local inertial (Lorentz) frame while $\mu$ are the metric (holonomic) indices. Dual vierbeins, $e^i(x)$, satisfy the orthonormality relation $e^i e_j = \eta_{ij}$ with the Minkowski metric $\eta_{ij}$. The Lagrangian (28) may now be expressed equivalently in terms of the vierbeins as

$$16\pi G \hat{\mathcal{L}}_E = R^k_{i\ell\lambda} e_k^\xi e_j^\lambda \eta^{ij} \varepsilon = R \varepsilon = \hat{R}, \quad (35)$$

where $\varepsilon \equiv \sqrt{-g}$ is the determinant of the dual vierbein field $e^i$. The identity (24) then separates into the two index classes, namely the metric index class (holonomic) indices

$$- \frac{\partial \hat{\mathcal{L}}_E}{\partial e_m^\nu} \varepsilon_m^\mu + \frac{\partial \hat{\mathcal{L}}_E}{\partial R^n_{m\mu\lambda}} R^n_{m\nu\lambda} + \frac{\partial \hat{\mathcal{L}}_E}{\partial R^n_{m\beta\mu}} R^n_{m\beta\lambda} \equiv \delta^\nu_{\nu} \hat{\mathcal{L}}_E, \quad (36)$$

and the Lorentz index class (nonholonomic indices) of the vierbeins:

$$- \frac{\partial \hat{\mathcal{L}}_E}{\partial e_n^\beta} \varepsilon_n^\mu + \frac{\partial \hat{\mathcal{L}}_E}{\partial \eta^{mk}} \eta^{kn} + \frac{\partial \hat{\mathcal{L}}_E}{\partial R^n_{k\beta\lambda}} R^n_{k\beta\lambda} - \frac{\partial \hat{\mathcal{L}}_E}{\partial R^n_{m\beta\lambda}} R^n_{m\beta\nu} \equiv \delta^\nu_{m} \hat{\mathcal{L}}_E. \quad (37)$$

To confirm Eqs. (36) and (37), we use

$$\frac{\partial \varepsilon}{\partial e^\nu_\nu} = - e^\nu_\nu \varepsilon,$$

to find for the terms with open metric indices and contracted vierbein indices:

$$\frac{\partial \hat{R}}{\partial R^n_{m\nu\lambda}} R^n_{m\nu\lambda} = \delta^k_{\nu} \delta^m_{\nu} \varepsilon_m^\mu e_j^\lambda e^j_\lambda \eta^{ij} \varepsilon = R^{k\nu_{\lambda}} e_k^\lambda e^j_\lambda \eta^{ij} \varepsilon = R^{k\nu_{\nu}} \varepsilon = R^{\mu}_{\nu} \varepsilon,$$

$$\frac{\partial \hat{R}}{\partial R^n_{m\beta\nu}} R^n_{m\beta\nu} = \delta^k_{\beta} \delta^m_{\beta} \varepsilon_m^\mu e_j^\lambda e^j_\lambda \eta^{ij} \varepsilon = R^{k\beta_{\nu}} e_k^\lambda e^j_\lambda \eta^{ij} \varepsilon = R^{\mu_{\beta}} \varepsilon = R^{\mu}_{\mu} \varepsilon,$$

which sums up to yield the right-hand side of (36). With the canonical energy-momentum tensor density corresponding to Eq. (26), which is obtained here from Eq. (36) as

$$\hat{\mathcal{J}}^\mu = \frac{\partial \hat{\mathcal{L}}_E}{\partial R^n_{m\nu\lambda}} R^n_{m\nu\lambda} + \frac{\partial \hat{\mathcal{L}}_E}{\partial R^n_{m\beta\nu}} R^n_{m\beta\nu} - \delta^\mu_{\nu} \hat{\mathcal{L}}_E,$$
the “vierbein” energy-momentum tensor $\tilde{T}^\mu_\nu$ is not symmetric and coincides with the canonical one:

$$\tilde{T}^\mu_\nu = \frac{1}{\varepsilon} \frac{\partial \tilde{L}_E}{\partial e^\mu_m} = \eta^\mu_\nu = \frac{1}{8\pi G} \left(R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R\right).$$

In the same way, we set up the terms with contracted metric indices while leaving the vierbein indices open:

$$-\frac{\partial \tilde{R}}{\partial e^\eta_m} e^\eta_\tau = -R^k_{\xi \lambda} \left(\delta^\xi_\tau \delta^\eta_\tau e^\lambda_j + e^\xi_\tau \delta^\lambda_\eta e^\eta_j - e^\xi_\tau e^\lambda_\eta e^\eta_\tau\right) e^\eta_\tau \eta^{ij} \varepsilon = (\delta^m_\tau R - 2R^m_\tau) \varepsilon$$

$$\frac{\partial \tilde{R}}{\partial \eta^{nk}} \eta^{nk} + \frac{\partial \tilde{R}}{\partial \eta^{km}} \eta^{km} = (R^m_\tau + R^n_\tau) \varepsilon$$

$$-\frac{\partial \tilde{R}}{\partial R_{\tau}^{n_{k\beta\lambda}}} R^{n_{k\beta\lambda}} = \delta^l_\tau \delta^k_\eta \delta^\beta_\xi e^\xi_l e^\lambda_j R^{n_{k\beta\lambda}} \eta^{ij} \varepsilon = R^m_\tau \varepsilon$$

$$-\frac{\partial \tilde{R}}{\partial R_{\tau}^{m_{n\beta\lambda}}} R^{m_{n\beta\lambda}} = -\delta^l_\tau \delta^m_\beta \delta^\beta_\xi e^\xi_l e^\lambda_j R^{m_{n\beta\lambda}} \eta^{ij} \varepsilon = -R^m_\tau \varepsilon,$$

which sums up to $\delta^m_\tau \tilde{R}$ and thus verifies the identity (37).

4.1.5. Quadratic gravity Riemann tensor squared

The gravity Lagrangian $\tilde{L}_{\text{Ric}^2}$ constructed form the complete contraction of two Riemann-Cartan tensors, also referred to as the Kretschmann scalar, was already proposed by Einstein in a private letter to H. Weyl[14]. It is defined by:

$$\tilde{L}_{\text{Ric}^2} = -\frac{1}{4} R^\xi_{\eta\rho\sigma} R^\eta_{\xi\sigma\lambda} g^{\rho\sigma} g^{\alpha\lambda} \sqrt{-g}.$$

It is again directly verified that the left-hand side of the identity (24) gives

$$\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^{\mu\beta}} g^{\mu\beta} + \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^{\nu\sigma}} g^{\nu\sigma} = R^\mu_{\eta\rho\tau} R^{\eta\rho\sigma} \nu \sqrt{-g} - \delta^\mu_\nu \tilde{L}_{\text{Ric}^2}, \quad (38a)$$

while the terms on the right-hand side evaluate to

$$-\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\alpha_{\beta\lambda}} R^{\mu}_{\alpha\beta\lambda} = -\frac{1}{2} R^\mu_{\alpha\beta\lambda} R^{\alpha\beta\lambda} \sqrt{-g} \quad (38b)$$

$$\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\rho_{\nu\beta\lambda}} R^{\rho}_{\nu\beta\lambda} = \frac{1}{2} R^\rho_{\nu\beta\lambda} R^{\alpha\beta\lambda} \sqrt{-g} \quad (38c)$$

$$\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\eta_{\alpha\beta}} R^{\rho}_{\alpha\beta} \eta \rho \nu = \frac{1}{2} R^\rho_{\alpha\beta} R^{\alpha\beta} \nu \sqrt{-g} \quad (38d)$$

$$\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\alpha_{\beta\mu}} R^{\alpha\beta}_{\mu} \nu = \frac{1}{2} R^{\alpha\beta}_{\mu} R^{\alpha\beta} \nu \sqrt{-g}. \quad (38e)$$
The terms (38) indeed satisfy the general form of the identity (24). As the terms (38b) and (38c) cancel, the remaining terms (38d) and (38e) form the canonical energy-momentum tensor, which is symmetric as it agrees with the metric one:

\[
T_{\nu\mu} = 2 \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^\nu\mu} = R^\rho\tau\mu R^\rho\tau\nu - g_{\nu\mu} \mathcal{L}_{\text{Ric}^2}
\]

\[
\vartheta_{\nu\mu} = 2 \frac{\partial \tilde{\vartheta}_{\text{Ric}^2}}{\partial R^\eta_{\alpha\beta}} R^\eta_{\alpha\beta\nu} - g_{\nu\mu} \mathcal{L}_{\text{Ric}^2} = T_{\nu\mu}.
\]

Consequently, a pure \(L_{\text{Ric}^2}\) model for the dynamics of the free gravitational field leads to a symmetric canonical energy-momentum tensor and is thus not eligible to describe torsion dynamics of spacetime, which emerges in the case of a non-symmetric energy-momentum tensor of the matter fields.

4.1.6. Quadratic gravity with Ricci tensor squared

The scalar density made of the square of the (not necessarily symmetric) Ricci tensor \(R_{\eta\alpha} = R^\rho_{\eta\rho\alpha}\) is defined by the following contraction with the metric

\[
\mathcal{L}_{\text{Ric}^2} = \frac{1}{2} R_{\eta\lambda} R^\eta_{\alpha\lambda} g^{\alpha\lambda} \sqrt{-g}.
\]

For the Lagrangian (39), the general identity (24) reduces to:

\[
- \partial \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^\nu\beta} g^\mu\beta + \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^\beta\nu} g^\beta\mu = \left( R_{\nu\beta} R^\mu_{\beta\alpha} + R^\beta_{\mu\alpha} R_{\nu\beta} \right) \sqrt{-g} - \delta^\mu_{\nu} \tilde{L}_{\text{Ric}^2}.
\]

Explicitly, the derivative terms with respect to the metric are

\[
\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^\nu\beta} g^\mu\beta = \left( R_{\nu\beta} R^\mu_{\beta\alpha} + R^\beta_{\mu\alpha} R_{\nu\beta} \right) \sqrt{-g} - \delta^\mu_{\nu} \tilde{L}_{\text{Ric}^2},
\]

whereas the derivative terms with respect to the Ricci tensor follow as

\[
\frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\eta_{\alpha\beta}} R_{\eta\mu\nu} = R^\mu_{\beta\alpha} R^\beta_{\nu\mu} \sqrt{-g}.
\]

Clearly, the right-hand sides of Eqs. (41) satisfy Eq. (40). The metric energy-momentum tensor is thus:

\[
T^\nu_{\mu} = 2 \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial g^\nu\beta} g^\mu\beta = R^\mu_{\beta\alpha} R_{\nu\beta} + R^\beta_{\mu\alpha} R_{\nu\beta} - \delta^\mu_{\nu} \mathcal{L}_{\text{Ric}^2},
\]

whereas the canonical energy-momentum tensor emerges as:

\[
\vartheta^\mu_{\nu} = \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\mu_{\beta\alpha}} R_{\nu\beta} + \frac{\partial \tilde{L}_{\text{Ric}^2}}{\partial R^\beta_{\mu\alpha}} R_{\beta\nu} - \delta^\mu_{\nu} \mathcal{L}_{\text{Ric}^2} = R^\mu_{\beta\alpha} R_{\nu\beta} + R^\beta_{\mu\alpha} R_{\beta\nu} - \delta^\mu_{\nu} \mathcal{L}_{\text{Ric}^2}.
\]

The canonical energy-momentum tensor coincides with the metric one. \(\vartheta^\mu_{\nu}\) is thus symmetric in this case even if the Ricci tensor itself is non-symmetric. As a consequence, it cannot complement a non-symmetric energy-momentum tensor of a source field in an Einstein-type equation and hence does not describe the dynamics of torsion of spacetime.
4.1.7. Quadratic gravity with Ricci scalar squared

The scalar density made of the square of the Ricci scalar \( R = R_{\rho\alpha}g^{\rho\alpha} = R^\rho_{\quad \rho\alpha}g^{\rho\alpha} \) is defined by the following contraction with the metric

\[
\hat{\mathcal{L}}_{\text{RIS}^2} = \frac{1}{2} R_{\rho\lambda} R^\rho_{\quad \rho\alpha} g^{\rho\alpha} g^{\epsilon \lambda} \sqrt{-g}.
\]

(42)

The identity (40) consisting of the derivative terms with respect to the metric

\[
\frac{\partial \hat{\mathcal{L}}_{\text{RIS}^2}}{\partial g^{\beta\rho}} g^{\beta\mu} + \frac{\partial \hat{\mathcal{L}}_{\text{RIS}^2}}{\partial g^{\beta\sigma}} g^{\sigma\mu} = R^\beta_{\quad \beta} (R^\mu_{\quad \nu} + R^\mu_{\quad \nu}) \sqrt{-g} - \delta^\mu_{\quad \nu} \hat{\mathcal{L}}_{\text{RIS}^2},
\]

and the derivative terms with respect to the Ricci tensor

\[
\frac{\partial \hat{\mathcal{L}}_{\text{RIS}^2}}{\partial R^\mu_{\quad \beta}} R^\nu_{\quad \beta} = R^\beta_{\quad \beta} R^\mu_{\quad \nu} \sqrt{-g}
\]

(43a)

\[
\frac{\partial \hat{\mathcal{L}}_{\text{RIS}^2}}{\partial R^\beta_{\quad \mu}} R^\nu_{\quad \beta} = R^\beta_{\quad \beta} R^\mu_{\quad \nu} \sqrt{-g}
\]

(43b)

is obviously satisfied. The metric energy-momentum tensor is given by:

\[
T^\mu_{\quad \nu} = \frac{2}{\sqrt{-g}} \frac{\partial \hat{\mathcal{L}}_{\text{RIS}^2}}{\partial g^{\beta\rho}} g^{\beta\mu} = R^\beta_{\quad \beta} (R^\mu_{\quad \nu} + R^\mu_{\quad \nu}) - \delta^\mu_{\quad \nu} \hat{\mathcal{L}}_{\text{RIS}^2},
\]

whereas the canonical energy-momentum tensor emerges as:

\[
\partial^\mu_{\quad \nu} = \frac{\partial \hat{\mathcal{L}}_{\text{RIS}^2}}{\partial R^\mu_{\quad \beta}} R^\nu_{\quad \beta} - \delta^\mu_{\quad \nu} \hat{\mathcal{L}}_{\text{RIS}^2} = R^\beta_{\quad \beta} (R^\mu_{\quad \nu} + R^\mu_{\quad \nu}) - \delta^\mu_{\quad \nu} \hat{\mathcal{L}}_{\text{RIS}^2}.
\]

Also in this case, the canonical energy-momentum tensor coincides with the metric one.

4.2. Consistency equation of the gauge theory of gravity

From the covariant canonical Hamiltonian formulation of the gauge theory of gravity (CCGG) one derives the second rank tensor “consistency equation”:

\[
-2 \frac{\partial \hat{H}_{\text{Gr}}}{\partial g^{\alpha\nu}} g^{\alpha\mu} + 2 \bar{k}^{\alpha\mu\beta} \frac{\partial \hat{H}_{\text{Gr}}}{\partial k^{\alpha\beta}} - \hat{q}^{\alpha\beta} \frac{\partial \hat{H}_{\text{Gr}}}{\partial q^{\alpha\beta}} + \hat{q}^{\alpha\beta} \frac{\partial \hat{H}_{\text{Gr}}}{\partial \bar{q}^{\alpha\beta}} = \frac{\partial \hat{H}_{0}}{\partial a^{\alpha\mu}} g^{\alpha\nu} + \frac{\partial \hat{L}_{\text{Gr}}}{\partial g^{\alpha\nu}} g^{\alpha\mu},
\]

\[
= \frac{\partial \hat{L}_{0}}{\partial a^{\alpha\mu}} g^{\alpha\nu} - \frac{\partial \hat{L}_{0}}{\partial a^{\alpha\mu}} a^{\alpha\nu} + 2 \frac{\partial \hat{L}_{0}}{\partial g^{\alpha\mu}} g^{\alpha\nu},
\]

(44)

which has the Lagrangian representation:

\[
-2 \frac{\partial \hat{L}_{\text{Gr}}}{\partial g^{\alpha\nu}} g^{\alpha\mu} + 2 \frac{\partial \hat{L}_{\text{Gr}}}{\partial g^{\alpha\mu\beta}} g^{\alpha\nu} = \frac{\partial \hat{L}_{\text{Gr}}}{\partial R^{\mu}_{\quad \nu\alpha\beta}} R^{\nu}_{\quad \eta\alpha\beta} + \frac{\partial \hat{L}_{\text{Gr}}}{\partial R^{\mu}_{\quad \alpha\beta}} R^{\nu}_{\quad \eta\alpha\beta}
\]

\[
= \frac{\partial \hat{L}_{0}}{\partial a^{\alpha\mu}} g^{\alpha\nu} - \frac{\partial \hat{L}_{0}}{\partial a^{\alpha\mu}} a^{\alpha\nu} + 2 \frac{\partial \hat{L}_{0}}{\partial g^{\alpha\mu}} g^{\alpha\nu}.
\]

(45)
\( \mathcal{L}_{Gr} \) represents a generic Lagrangian density of vacuum gravity, and \( \mathcal{L}_0 \) can be seen as representing scalar and massive vector fields, i.e., \( \mathcal{L}_0 = \mathcal{L}_{KG} + \mathcal{L}_P \). By virtue of the identities for scalar density valued functions of arbitrary tensors and the metric, given here by

\[
\delta^\nu_\mu \mathcal{L}_0 = \frac{\partial \mathcal{L}_0}{\partial \phi} \frac{\partial \phi}{\partial x^\nu} + \frac{\partial \mathcal{L}_0}{\partial a_{\alpha \mu}} a_{\nu ; \beta} + \frac{\partial \mathcal{L}_0}{\partial a_{\beta ; \mu}} a_{\nu ; \beta} - 2 \frac{\partial \mathcal{L}_0}{\partial g^\alpha_\nu} g^{\alpha \mu}
\]

\[
\delta^\nu_\mu \mathcal{L}_Gr = -2 \frac{\partial \mathcal{L}_Gr}{\partial g^\alpha_\nu} g^{\alpha \mu} + 2 \frac{\partial \mathcal{L}_Gr}{\partial a_{\alpha \beta ; \mu}} g_{\alpha \nu ; \beta} + \frac{\partial \mathcal{L}_Gr}{\partial g_{\alpha \beta ; \mu}} g_{\alpha \beta ; \nu} - \frac{\partial \mathcal{L}_Gr}{\partial R^\alpha_{\mu \alpha \beta}} R^\beta_{\nu \alpha \mu} + 2 \frac{\partial \mathcal{L}_Gr}{\partial R^\beta_{\mu \alpha \beta}} R^\alpha_{\nu \beta}
\]

the consistency equation (45) has the equivalent representation:

\[
\frac{\partial \mathcal{L}_Gr}{\partial g_{\alpha \beta ; \mu}} g_{\alpha \beta ; \nu} - \delta^\nu_\mu \mathcal{L}_Gr = \left(- \frac{\partial \mathcal{L}_0}{\partial \phi} \frac{\partial \phi}{\partial x^\nu} + \frac{\partial \mathcal{L}_0}{\partial a_{\beta ; \mu}} a_{\beta ; \nu} - \delta^\nu_\mu \mathcal{L}_0 \right).
\]

(46)

The right-hand side of Eq. (46) is exactly minus the covariant representation of the non-symmetric canonical energy-momentum tensor density \( \bar{\theta}^\nu_\mu \), of the system \( \mathcal{L}_0 \):

\[
\bar{\theta}^\nu_\mu = \frac{\partial \mathcal{L}_0}{\partial \phi} \frac{\partial \phi}{\partial x^\nu} + \frac{\partial \mathcal{L}_0}{\partial a_{\beta ; \mu}} a_{\beta ; \nu} - \delta^\nu_\mu \mathcal{L}_0.
\]

We note that the covariant derivatives of the vector field term causes the connection besides the metric to act as an additional coupling to the left-hand side of Eq. (46). The latter can be interpreted as the canonical energy-momentum tensor density associated with the “free” gravitational field Lagrangian \( \mathcal{L}_{Gr} \). Hence, Eq. (46) is equivalently written as

\[
\frac{\partial \mathcal{L}_{Gr}}{\partial g_{\alpha \beta ; \mu}} g_{\alpha \beta ; \nu} + 2 \frac{\partial \mathcal{L}_{Gr}}{\partial R^\beta_{\mu \alpha \beta}} R^\alpha_{\nu \beta} - \delta^\nu_\mu \mathcal{L}_{Gr} = - \bar{\theta}^\nu_\mu.
\]

(47)

This is the general Einstein-type equation that applies for any postulated “free gravity” Lagrangian \( \mathcal{L}_{Gr} \) describing the dynamics of the gravitational field in source-free regions of space-time including torsion and non-metricity. Moreover, written as

\[
\bar{\theta}^\mu_\nu + \bar{\theta}^\nu_\mu = 0
\]

(48)

this represents a balance equation of energy and momentum of matter and space-time, often expressed as the zero-energy universe [13][10][17][18].

For the common case of metricity, hence for a covariantly conserved metric \( (g_{\alpha \beta ; \nu} \equiv 0) \), Eq. (47) reduces to:

\[
2 \frac{\partial \mathcal{L}_{Gr}}{\partial R^\beta_{\mu \alpha \beta}} R^\alpha_{\nu \beta} - g^{\mu \nu} \mathcal{L}_{Gr} = - \bar{\theta}^\mu_\nu.
\]

(49)

For the Einstein-Hilbert-Cartan Lagrangian \( \mathcal{L}_{Gr,E} = R/16\pi G \), discussed in Sect. 4.1.3 which constitutes a particular model Lagrangian to describe the dynamics of the free gravitational field as discussed in Sect. 4.1.3, Eq. (49) immediately yields the Einstein equation,
which also holds for non-zero torsion in the case of a non-symmetric matter field tensor $\theta^{\mu \nu}$, which requires the Ricci tensor $R^{\mu \nu}$ to be non-symmetric as well:

$$R^{(\mu \nu)} - \frac{1}{2} g^{\mu \nu} R = -8\pi G \theta^{(\mu \nu)}$$

$$R^{[\mu \nu]} = -8\pi G \theta^{[\mu \nu]}.$$

The first equation is the classical Einstein equation. The second equation follows from Eq. (34) and can be expressed in terms of the skew-symmetric portion of the affine connection which is the torsion $S^{\lambda \mu \nu}$ of space-time:

$$[(\nabla_\lambda - 2S_{\lambda}) S^{\lambda \mu \nu} - (\nabla_\mu S_\nu - \nabla_\nu S_\mu)] = -8\pi G \theta^{[\mu \nu]}.$$

In the last equation we used the Bianchi identity relating torsion and the antisymmetric Ricci tensor. Obviously, a non-symmetric canonical energy-momentum tensor emerges as a source of torsion of space-time.

5. Conclusions

The theorem on (relative) scalar-valued functions of tensors reflects the linear structure of tensor spaces and thereby constitutes an analogue to Euler’s theorem on homogeneous functions. When applied to Lagrangian densities $\tilde{\mathcal{L}}$ of classical field theories, the resulting identities provide relations between the metric and canonical versions of the energy-momentum tensors of matter fields. By the same mathematical reasoning, analogous tensors for generic theories of gravity are derived. The identity thereby sheds new light on the long-standing ambiguity of the proper definition of the energy-momentum of gravity. Moreover, as a further application of the identity to the gauge theory of gravity, a relation of the energy-momentum tensors of both, gravity and matter, emerges. In combination with the identities that hold separately for both tensors, that relation may be cast into a simple but nonetheless most general form of an energy-momentum balance equation that confirms the conjecture of “zero-energy universe”. The symmetric portion of that balance equation reproduces thereby a generalized version of Einstein’s field equation, while the skew-symmetric portion gives a Poisson-type equation for the torsion of space-time.

Appendix A. The DeDonder-Weyl-Hamilton application

Speaking of Hamiltonians, we refer to the DeDonder-Weyl (DW) Hamiltonian formulation. These Hamiltonians emerge from a covariant Legendre transformation of the pertaining Lagrangian. That means both, the temporal and the spatial derivatives of the fields, are converted into their conjugates, namely the canonical momenta. This contrasts with the conventional Hamiltonians of continuum mechanics and field theories, where merely the temporal derivative of the respective field is transformed into a canonical momentum while the spatial derivatives are retained. For various DeDonder-Weyl Hamiltonians, we set up the pertaining identities and finally show that the general Einstein-type equation emerging from DeDonder-Weyl Hamiltonians of the gauge theory of gravity may be cast via the identity into a simpler form while retaining its scope for non-vanishing
torsion of space-time and non-metricity. This way the results in the main text of this paper are reproduced.

For the scalar density DW Hamiltonian $\tilde{H}$, the momentum fields $\tilde{p}$ representing tensor densities must be expressed as absolute tensors $p = \tilde{p}/\sqrt{-g}$ prior to setting up the identity according to Eq. (10). An example is worked out in Sect. 3.3.

For Klein-Gordon field the pertinent DW Hamiltonian density $\tilde{H}_{KG}(\phi, \tilde{\phi}, \tilde{\pi}^{\alpha}, g_{\mu\nu})$ for a system of complex fields in a dynamic space-time is given by:

$$\tilde{H}_{KG} = \tilde{\pi}^{\alpha} \frac{\partial \phi}{\partial x^{\alpha}} + \frac{\partial \tilde{\phi}}{\partial x^{\alpha}} \tilde{\pi}^{\alpha} - \tilde{L}_{KG}, \quad \tilde{\rho}^{\alpha} = \frac{\partial \tilde{L}_{KG}}{\partial (\frac{\partial \phi}{\partial x^{\alpha}})}, \quad \tilde{\pi}^{\alpha} = \frac{\partial \tilde{L}_{KG}}{\partial (\frac{\partial \tilde{\phi}}{\partial x^{\alpha}})}.$$  \hspace{1cm} (A.1)

Here, the scalar density $\tilde{H}_{KG}$ is defined as a function of the tensor densities $\tilde{\rho}^{\alpha}(x)$ and $\tilde{\pi}^{\beta}(x)$ rather than of absolute tensor fields. To set up the pertaining identity, $\tilde{H}_{KG}$ can be rewritten as:

$$\tilde{H}'_{KG} = \frac{1}{2} \left( \tilde{\rho}^{\alpha} \tilde{\rho}^{\beta} + \tilde{\pi}^{\beta} \tilde{\rho}^{\alpha} \right) g_{\alpha\beta} + m^2 \tilde{\phi} \tilde{\phi} \sqrt{-g}. \hspace{1cm} (A.2)$$

As can be directly verified, the identity now takes on the form:

$$\frac{\partial \tilde{H}'_{KG}}{\partial g_{\beta\mu}} g_{\beta\mu} = \frac{\partial \tilde{H}'_{KG}}{\partial g_{\alpha\mu}} g_{\alpha\mu} - \tilde{\rho}^{\mu} \frac{\partial \tilde{H}'_{KG}}{\partial \tilde{\rho}^{\nu}} + \tilde{\pi}^{\mu} \frac{\partial \tilde{H}'_{KG}}{\partial \pi_{\nu}} g_{\beta\nu} = \delta_{\mu}^{\nu} \tilde{H}'_{KG}. \hspace{1cm} (A.3)$$

The correlation of the derivatives with respect to the metric of $\tilde{H}'_{KG}$ and $\tilde{H}_{KG}$ follows as:

$$2 \frac{\partial \tilde{H}'_{KG}}{\partial g_{\beta\mu}} g_{\beta\mu} = 2 \frac{\partial \tilde{H}_{KG}}{\partial g_{\beta\mu}} g_{\beta\mu} + \delta_{\mu}^{\nu} \left( \tilde{H}'_{KG} + \tilde{L}_{KG} \right) = 2 \frac{\partial \tilde{L}_{KG}}{\partial g_{\beta\nu}} g_{\beta\mu} + \delta_{\nu}^{\mu} \left( \tilde{H}'_{KG} + \tilde{L}_{KG} \right),$$

hence

$$\frac{\partial \tilde{H}_{KG}}{\partial g_{\beta\mu}} g_{\beta\mu} = \frac{\partial \tilde{L}_{KG}}{\partial g_{\beta\nu}} g_{\beta\mu}. \hspace{1cm}$$

In terms of the proper DW Hamiltonian (A.1), the identity writes:

$$\frac{\partial \tilde{H}_{KG}}{\partial g_{\beta\mu}} g_{\beta\mu} + \frac{\partial \tilde{H}_{KG}}{\partial g_{\alpha\mu}} g_{\alpha\mu} - \tilde{\rho}^{\mu} \frac{\partial \tilde{L}_{KG}}{\partial \tilde{\rho}^{\nu}} - \tilde{\pi}^{\mu} \frac{\partial \tilde{L}_{KG}}{\partial \pi_{\nu}} \tilde{\pi}^{\nu} \equiv - \delta_{\mu}^{\nu} \tilde{L}_{KG}. \hspace{1cm} (A.4)$$

representing the Hamiltonian version of the equivalence of metric and canonical energy-momentum tensor densities of the Klein-Gordon system:

$$\tilde{T}_{\nu}^{\mu} = 2 \frac{\partial \tilde{H}_{KG}}{\partial g_{\mu\beta}} g_{\nu\beta} \equiv \tilde{\rho}^{\mu} \frac{\partial \tilde{H}_{KG}}{\partial \tilde{\rho}^{\nu}} + \frac{\partial \tilde{H}_{KG}}{\partial \pi_{\nu}} \tilde{\pi}^{\nu} - \delta_{\nu}^{\mu} \tilde{L}_{KG} = \tilde{\rho}_{\nu}^{\mu}. \hspace{1cm} (A.5)$$

The equivalent covariant Proca Hamiltonian density $\tilde{H}_{P}(\tilde{p}^{\mu}, a_{\nu}, g_{\mu\nu})$ is obtained by the covariant Legendre transformation as:

$$\tilde{H}_{P} = \tilde{p}^{\alpha} \frac{\partial a_{\alpha}}{\partial x^{\beta}} - \tilde{L}_{P} = - \frac{1}{4} \tilde{p}^{\alpha} \tilde{p}^{\beta} \tilde{\pi}^{\alpha} g_{\alpha\xi} g_{\beta\eta} \frac{1}{\sqrt{-g}} - \frac{1}{2} m^2 a_{\alpha} a_{\xi} g^{\alpha\xi} \sqrt{-g}.$$  \hspace{1cm} (A.6a)

$$\tilde{p}^{\alpha\beta} = \frac{\partial \tilde{L}_{P}}{\partial f_{\alpha\beta}} = f^{\alpha\beta} \sqrt{-g} \hspace{1cm} (A.6b)$$
In order to apply the theorem and hence to set up the related identity, the Hamiltonian $\tilde{H}_P$ must first be expressed as the equivalent function of absolute tensors:

$$\tilde{H}_P = -\frac{1}{2}p^{\alpha\beta} p^{\epsilon\eta} g_{\alpha\xi} g_{\beta\eta} \sqrt{-g} - \frac{1}{2} m^2 a_{\alpha} a_{\xi} g^{\alpha\xi} \sqrt{-g}. \tag{A.7}$$

The identity (10) then follows as

$$\frac{\partial \tilde{H}_P}{\partial g_{\mu\beta}} g_{\nu\beta} + \frac{\partial \tilde{H}_P}{\partial g_{\beta\mu}} = \frac{\partial \tilde{H}_P}{\partial p^{\mu\beta}} p_{\mu\beta} - \frac{\partial \tilde{H}_P}{\partial p^{\beta\mu}} p_{\beta\mu} + \frac{\partial \tilde{H}_P}{\partial a_{\mu}} a_{\nu} \equiv \delta_{\nu}^\mu \tilde{H}_P', \tag{A.8}$$

which is again verified by direct calculation:

$$\frac{\partial \tilde{H}_P}{\partial p^{\mu\beta}} p_{\mu\beta} = \frac{1}{2} p^\mu_{\beta} p_{\nu\beta} g_{\xi\eta} \sqrt{-g} = \frac{1}{2} p^\mu_{\beta} p_{\nu\beta} \sqrt{-g}$$

$$\frac{\partial \tilde{H}_P}{\partial p^{\beta\mu}} p_{\beta\mu} = \frac{1}{2} p^\beta_{\mu} p_{\beta\nu} g_{\xi\eta} \sqrt{-g} = \frac{1}{2} p^\beta_{\mu} p_{\beta\nu} \sqrt{-g}$$

$$\frac{\partial \tilde{H}_P}{\partial a_{\nu}} a_{\nu} = -m^2 a_{\nu} a_{\xi} g^{\mu\xi} \sqrt{-g} = -m^2 a_{\nu} a_{\mu} \sqrt{-g}$$

$$2 \frac{\partial \tilde{H}_P}{\partial g_{\beta\mu}} g_{\beta\nu} = \left( \frac{1}{2} p^\beta_{\lambda} p^{\alpha\mu} g_{\beta\nu} - \frac{1}{2} p^\alpha_{\beta} p^\mu_{\nu} g_{\alpha\xi} g_{\beta\eta} + m^2 a_{\alpha} a_{\xi} g^{\alpha\xi} \sqrt{-g} \right) \sqrt{-g}$$

$$+ g^{\mu\alpha} g_{\beta\nu} \tilde{H}_P'$$

$$= \frac{1}{2} p^\beta_{\mu} p_{\beta\nu} \sqrt{-g} - \frac{1}{2} p_{\beta\nu} p^\mu_{\beta} \sqrt{-g} + m^2 a_{\nu} a_{\mu} \sqrt{-g} + \delta_{\nu}^\mu \tilde{H}_P'. \tag{A.8}$$

Summing up all terms on the right-hand side we see that all cancel except $\delta_{\nu}^\mu \tilde{H}_P'$, which then gives Eq. (A.8).

For the proper DW Hamiltonian $\text{(A.6a)}$, we have

$$\frac{\partial \tilde{H}_P}{\partial p^{\mu\beta}} p_{\mu\beta} = -\frac{1}{2} p^\beta_{\mu} p_{\nu\beta} \frac{1}{\sqrt{-g}} = -\frac{1}{2} p^\beta_{\mu} p_{\nu\beta} \sqrt{-g} = \frac{\partial \tilde{H}_P}{\partial p^{\mu\beta}} p_{\nu\beta}$$

$$\frac{\partial \tilde{H}_P}{\partial a_{\mu}} a_{\nu} = -m^2 a_{\nu} a_{\mu} \sqrt{-g} = \frac{\partial \tilde{H}_P}{\partial a_{\mu}} a_{\nu}$$

$$2 \frac{\partial \tilde{H}_P}{\partial g_{\beta\mu}} g_{\beta\nu} = -\frac{1}{2} p_{\beta\nu} p^{\mu\beta} \sqrt{-g} - \frac{1}{2} p_{\beta\nu} p^\mu_{\beta} \sqrt{-g} + m^2 a_{\nu} a_{\mu} \sqrt{-g} - \delta_{\nu}^\mu \tilde{L}_P$$

$$= 2 \frac{\partial \tilde{H}_P}{\partial g_{\beta\mu}} g_{\beta\nu} - \delta_{\nu}^\mu \tilde{H}_P' - \delta_{\nu}^\mu \tilde{L}_P,$$

where $\tilde{L}_P$ is expressed in canonical variables according to $f_{\alpha\beta} = \tilde{p}_{\alpha\beta}$. The identity (A.8) can now be written in terms of the proper DW Proca Hamiltonian $\tilde{H}_P$ as:

$$\frac{\partial \tilde{H}_P}{\partial g_{\mu\beta}} g_{\nu\beta} + \frac{\partial \tilde{H}_P}{\partial g_{\beta\mu}} g_{\beta\nu} - \tilde{p}^{\mu\beta} \frac{\partial \tilde{H}_P}{\partial \tilde{p}^{\beta\mu}} - \tilde{p}^{\beta\mu} \frac{\partial \tilde{H}_P}{\partial \tilde{p}^{\beta\nu}} + \frac{\partial \tilde{H}_P}{\partial a_{\nu}} a_{\nu} \equiv -\delta_{\nu}^\mu \tilde{L}_P.$$

With the canonical energy-momentum tensor density of the Proca system,

$$\tilde{\theta}_{\nu}^\mu = \frac{\partial \tilde{L}_P}{\partial f_{\beta\mu}} f_{\beta\nu} - \delta_{\nu}^\mu \tilde{L}_P = \tilde{p}^{\beta\mu} \frac{\partial \tilde{H}_P}{\partial \tilde{p}^{\beta\nu}} - \delta_{\nu}^\mu \tilde{L}_P,$$
this yields the correlation of metric and canonical energy-momentum tensors in the Hamiltonian representation for the Proca system:

\[ \hat{\theta}^\mu_\nu = \tilde{T}^\mu_\nu - \tilde{p}^\mu_\beta a_\beta \, a_\nu. \]

Inserting the general form of the canonical field equations in a dynamic space-time,

\[ a_\nu;\beta = \frac{\partial \tilde{\mathcal{H}}_P}{\partial \tilde{p}^\nu_\beta}, \quad \tilde{p}^\mu_\beta ;\beta = - \frac{\partial \tilde{\mathcal{H}}_P}{\partial a_\mu} \]

gives finally:

\[ \tilde{T}^\mu_\nu = \hat{\theta}^\mu_\nu + \tilde{p}^\mu_\beta a_\nu;\beta + \tilde{p}^\mu_\beta ;\beta a_\nu = \hat{\theta}^\mu_\nu + (a_\nu \tilde{p}^\mu_\beta) ;\beta. \]  (A.9)

The divergence term on the right-hand side of Eq. (A.9) coincides with that discussed by Landau & Lifshitz in §33. Here, we encounter the term for the case of a dynamic space-time.

In order to define the covariant DW Hamiltonian for the Dirac field, we first define the Dirac momentum density fields, \( \tilde{\pi}^\mu \) and \( \tilde{\bar{\pi}}^\mu \), representing the conjugates of the spinor fields \( \psi \) and \( \bar{\psi} \), respectively, and hence the conjugate fields of their partial derivatives, \( \partial \psi / \partial x^\mu \) and \( \partial \bar{\psi} / \partial x^\mu \). They are derived from the Lagrangian density \( \tilde{\mathcal{L}}_D \) via

\[ \tilde{\pi}^\mu = \frac{\partial \tilde{\mathcal{L}}_D}{\partial (\partial \psi / \partial x^\mu)}, \quad \tilde{\bar{\pi}}^\mu = \frac{\partial \tilde{\mathcal{L}}_D}{\partial (\partial \bar{\psi} / \partial x^\mu)}. \]  (A.10)

Due to the quadratic “velocity” dependence of (16), the corresponding covariant Hamiltonian is obtained via the Legendre transformation

\[ \tilde{\mathcal{H}}_D (\psi, \tilde{\pi}^\alpha, \bar{\psi}, \tilde{\bar{\pi}}^\alpha, \gamma^\alpha) = \tilde{\pi}^\alpha \frac{\partial \psi}{\partial x^\alpha} + \frac{\partial \bar{\psi}}{\partial x^\alpha} \tilde{\bar{\pi}}^\alpha - \tilde{\mathcal{L}}_D (\psi, \partial_\alpha \psi, \bar{\psi}, \partial_\alpha \bar{\psi}, \gamma^\alpha) \]

as:

\[ \tilde{\mathcal{H}}_D = \frac{i M}{2} \left( \bar{\psi} \gamma_\alpha \tilde{\pi}^\alpha - \tilde{\pi}^\alpha \gamma_\alpha \bar{\psi} \right) \sqrt{-g} - \tilde{\mathcal{L}}_D + (m - M) \bar{\psi} \psi \sqrt{-g}, \]  (A.11)

with \( \tau_{\alpha\beta} \) being the inverse of the matrix \( \sigma^{\beta\alpha} \), the latter defined in Eq. (17).

\[ \tau_{\alpha\beta} = \frac{i}{6} (\gamma_\alpha \gamma_\beta + 3 \gamma_\beta \gamma_\alpha), \quad \tau_{\nu\alpha} \sigma^{\alpha\mu} = \delta^\mu_\nu \mathbf{1}. \]

The DW Hamiltonian representation of the identity (20) is then:

\[ \text{Tr} \left\{ \frac{\partial \tilde{\mathcal{H}}_D}{\partial \tilde{\pi}^\alpha} \gamma_\nu \right\} = \frac{\partial \tilde{\mathcal{H}}_D}{\partial \tilde{\pi}^\alpha} \tilde{\pi}^\alpha + \tilde{\pi}^\mu \frac{\partial \tilde{\mathcal{H}}_D}{\partial \tilde{\pi}^\mu} - \delta_\nu^\mu \tilde{\mathcal{L}}_D, \]

which display the identity of metric and canonical energy-momentum tensor densities in the Hamiltonian representation. Note that these tensors are not symmetric in their covariant or contravariant representations.
In closing this Appendix, we display, for the sake of completeness, the DW Hamiltonian representation of the consistency equation as derived from the covariant canonical transformation framework in the Hamiltonian picture:

\[
-2 \frac{\partial \tilde{\mathcal{H}}_R}{\partial g_{\alpha \mu}} g_{\alpha \nu} + 2 \tilde{k}_{\alpha \mu \beta} \frac{\partial \tilde{\mathcal{H}}_R}{\partial k_{\alpha \nu \beta}} - \tilde{q}_{\nu \alpha \beta} \frac{\partial \tilde{\mathcal{H}}_R}{\partial q_{\alpha \nu \beta}} + \tilde{q}_{\nu \mu \beta} \frac{\partial \tilde{\mathcal{H}}_R}{\partial q_{\alpha \mu \beta}} - \partial \tilde{\mathcal{H}}_0 \partial a_{\mu} a_{\nu} - \tilde{p}_{\mu \beta} \frac{\partial \tilde{\mathcal{H}}_0}{\partial p_{\beta \nu \beta}} + 2 \frac{\partial \tilde{\mathcal{H}}_0}{\partial g_{\alpha \mu}} g_{\alpha \nu},
\]  
(A.12)

By covariant Legendre transformation it gives Eq. (45).

References

1. L. Rosenfeld, *Mem. Acad. Roy. Belgique, cl. sc.* 18 (1940) 1.
2. J. Struckmeier, J. Muench, D. Vasak, J. Kirsch, M. Hanauske and H. Stoecker, *Phys. Rev. D* 95 (6 2017) 124048, http://arxiv.org/abs/1704.07246 arXiv:1704.07246.
3. F. J. Belinfante, *Physica* 6 (1939) 887.
4. D. W. Sciama, The analogy between charge and spin in general relativity, in *Recent Developments in General Relativity*, (Pergamon Press, Oxford; PWN, Warsaw, 1962), pp. 415–439. Festschrift for Infeld.
5. T. W. B. Kibble, *J. Math. Phys.* 2 (3 1961) 212.
6. F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, *Rev. Mod. Phys.* 48 (1976) 393.
7. J. Struckmeier, D. Vasak and J. Kirsch, Generic Theory of Geometrodynamics from Noether’s Theorem for the Diff(M) Symmetry Group, in *Discoveries at the Frontiers of Science: From Nuclear Astrophysics to Relativistic Heavy Ion Collisions*, eds. J. Kirsch, J. Steinheimer-Froschauer, S. Schramm and H. Stöcker (Springer Nature Switzerland AG, 2020) pp. 143–181.
8. J. Struckmeier and D. Vasak, *Astron. Nachr.* (2021).
9. W. Greiner and J. Reinhardt, *Field Quantization* (Springer, Berlin, Heidelberg, 1996).
10. S. Gasiorowicz, *Elementary particle physics* (Wiley, New York, 1966).
11. J. Struckmeier and A. Redelbach, *Int. J. Mod. Phys. E* 17 (2008) 435, http://arxiv.org/abs/0811.0508 arXiv:0811.0508.
12. J. York, *Physical Review Letters* 28 (April 1972) 1082.
13. G. Gibbons and S. Hawking, *Physical Review D* 15 (May 1977) 2752.
14. A. Einstein, Private letter to Hermann Weyl ETH Zürich Library, Archives and Estates (3, 1918).
15. P. Jordan, *Annalen der Physik* 428 (1939) 64.
16. D. Sciama, *Monthly Notices of the Royal Astronomical Society* 113 (1953) 34.
17. R. Feynman, W. Morinigo and W. Wagner, *Feynman Lectures On Gravitation* (Frontiers in Physics) (Westview Press, Boulder, Colorado, 1995).
18. S. Hawking, *The Theory of Everything* (New Millenium Press, 2003).
19. T. De Donder, *Théorie Invariantive Du Calcul des Variations* (Gauthier-Villars & Cie., Paris, 1930).
20. H. Weyl, *Ann. Math.* 36 (1935) 607.
21. L. Landau and E. Lifschitz, *Klassische Feldtheorie* (Akademie Verlag Berlin, 1989).
22. J. Struckmeier and H. Reichau, General U(N) gauge transformations in the realm of covariant Hamiltonian field theory, in *Exciting Interdisciplinary Physics*, ed. W. Greiner, *Proceedings of the “Symposium on Exciting Physics: Quarks and gluons/atomic nuclei/biological systems/networks”, Makutsi Safari Farm, South Africa, 13–20 November 2011*, (Springer International Publishing Switzerland, 2013), p. 367. http://arxiv.org/abs/1205.5754 arXiv:1205.5754.