Evolution of basic equations for nearshore wave field

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Abstract: In this paper, a systematic, overall view of theories for periodic waves of permanent form, such as Stokes and cnoidal waves, is described first with their validity ranges. To deal with random waves, a method for estimating directional spectra is given. Then, various wave equations are introduced according to the assumptions included in their derivations. The mild-slope equation is derived for combined refraction and diffraction of linear periodic waves. Various parabolic approximations and time-dependent forms are proposed to include randomness and nonlinearity of waves as well as to simplify numerical calculation. Boussinesq equations are the equations developed for calculating nonlinear wave transformations in shallow water. Nonlinear mild-slope equations are derived as a set of wave equations to predict transformation of nonlinear random waves in the nearshore region. Finally, wave equations are classified systematically for a clear theoretical understanding and appropriate selection for specific applications.

Keywords: Stokes waves, cnoidal waves, directional spectrum, mild-slope equation, diffraction, refraction

1. Introduction

The nearshore zone is the transition zone between deep ocean and land. The environmental gradient is steep in the zone, as found in changes in waves and currents, salinity and nutrient concentrations, and plants and animals, and thus the zone is called an ecotone. As a result, the nearshore zone is one of the zones highest in primary production on the earth’s surface; on the other hand, it experiences severe external forces such as storms and tsunamis, which result in serious natural disasters. Therefore, for conservation and utilization of the nearshore zone, it is indispensable to understand the various processes occurring there. Waves, especially, are the major external force. In addition, from a scientific viewpoint, waves provide us with interesting subjects to study in relation to the randomness and nonlinearity of oscillatory fluid motion. This paper describes the framework of theories for waves of permanent form and wave transformation in the nearshore zone. Theories for periodic waves which propagate without deformation are reviewed first. Then, a directional spectrum for describing random sea waves and theories for estimating directional spectra are introduced. Finally, governing equations for calculating wave transformation in the nearshore zone, from equations for linear periodic waves to those for nonlinear random waves, are discussed.

2. Validities of finite-amplitude wave theories

Fundamental sea waves are periodic waves which propagate at a uniform water depth without deformation. These waves are called waves of permanent form. No strict, analytical solution has been derived for these waves because of nonlinear boundary conditions along the water surface. The simplest analytical solution was derived by Airy (1841)1 for periodic waves of small amplitude, which allows boundary conditions to linearize. A finite-amplitude wave theory was developed by Stokes (1847)2 for deep water waves using the perturbation method, which is a technique to derive power series solutions for nonlinear equations. Small-amplitude wave theory (Airy theory) is interpreted as the first-order solution of the Stokes wave theory. Later, De
(1955) and Skjelbreia and Hendrickson (1960) extended the Stokes wave theory up to the fifth order in deep to shallow water.

For very shallow water, cnoidal wave theory was developed by Korteweg and de Vries (1895), who derived an equation for shallow water waves called the KdV equation and expressed the solution by using elliptic functions. Here, according to traditional usage, “deep water”, “shallow water” and “very shallow water” refer respectively to water depths which are large, comparable and small relative to wavelength. The meaning of “shallow water” is somewhat different from the general meaning. Systematic derivations of cnoidal wave theory were developed by Friedrichs (1948) and Keller (1948). Laitone (1960) derived a second-order solution, and a third-order solution was given by Chapepele (1962).

Since perturbation solutions need huge amounts of manipulations of long mathematical formulas, computer codes were developed for systematically deriving Stokes and cnoidal wave solutions of arbitrary orders. Schwartz (1974) derived higher-order solutions of Stokes waves by applying the method of conformal mapping. Fenton (1972) and Longuet-Higgins and Fenton (1974) studied higher-order solutions of a solitary wave, which is in the shallowest limit of the cnoidal waves. Nishimura et al. (1977) calculated the 24th-order cnoidal waves as well as the 51st-order Stokes waves.

Based on the above background, Isobe et al. (1982) showed mutual relationships among various perturbation solutions of finite-amplitude waves, which confirms that Stokes and cnoidal wave theories are the only two useful solutions among an infinite number of possible perturbation solutions. Then, they discussed the accuracies and validities of these theories of finite order. The essence will be introduced in the following.

Consider periodic waves which propagate at a uniform water depth without deformation. In a normal situation, the water is inviscid and incompressible, and hence the fluid motion is irrotational and solenoidal. This allows us to introduce velocity potential and stream function to formulate the problem. Here, the stream function, \( \psi(x, z, t) \), which is a function of the horizontal and vertical coordinates, \( x \) and \( z \), and the time, \( t \), is introduced. The stream function is related to the horizontal and vertical velocities, \( u \) and \( w \), as follows:

\[
\begin{align*}
  u &= \frac{\partial \psi}{\partial z}, \\
  w &= -\frac{\partial \psi}{\partial x}.
\end{align*}
\]

The water surface elevation is denoted by \( \eta(x, t) \).

For non-deforming waves, in the coordinate system moving with the same speed as the wave celerity, \( c \), the wave profile is standing still and the velocity field becomes steady. The horizontal coordinate, \( X \), in the moving coordinate system is defined as

\[
X = x - ct.
\]

Thus, the stream function, \( \Psi \), and the water surface elevation, \( N \), in the moving coordinate system are not functions of \( t \). These quantities are related to corresponding quantities in the fixed coordinate system as follows:

\[
\begin{align*}
Psi(X, z) &= \psi(x - ct, z) - cz \quad \text{[3]} \\
N(X) &= \eta(x - ct). \quad \text{[4]}
\end{align*}
\]

Now the governing equation for \( \Psi \) is the Laplace equation, which is obtained from the irrotationality condition:

\[
\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0. \quad \text{[5]}
\]

The boundary conditions are

\[
\begin{align*}
\Psi &= 0 \quad \text{on } z = -h \quad \text{[6]} \\
\Psi &= \text{const. } (= Q) \quad \text{on } z = N \\
\frac{1}{2} \left[ \left( \frac{\partial \Psi}{\partial z} \right)^2 + \left( \frac{\partial \Psi}{\partial X} \right)^2 \right] + gN &= \text{const. } (= P) \\
& \quad \text{on } z = N \quad \text{[8]}
\end{align*}
\]

where \( g \) is the gravitational acceleration. Equations [6] and [7] are called the kinematic boundary conditions at the bottom and surface, respectively. Since water flows along these boundaries, they must be flow lines and hence the values of the stream function must be constant along them. Equation [8] is the dynamic boundary condition on the surface. This means that the pressure on the water surface should be constant and equal to the atmospheric pressure. This equation can be obtained from Bernoulli’s equation, which is the integrated form of the equation of motion for an irrotational flow. In Eqs. [5] to [8], the unknown (dependent) variables are \( \Psi \) and \( N \). Among these equations, Eqs. [5] and [6] are linear in terms of the unknown variables, but Eqs. [7] and [8] are nonlinear. Although Eq. [7] seems to be linear, the unknown position of the boundary, \( N \), makes it nonlinear. Because of the nonlinearity, the strict and analytical solution has not been found. Various methods have been employed to derive approximate solutions. The following will give a
systematic view for the solution to the basic equation and boundary conditions.

First, the dependence of $\Psi$ on $z$ can be solved easily by utilizing power series expansion. The following expression can be verified to satisfy the governing equation [5] only by substituting it into Eq. [5]:

$$\Psi(X, z) = \sum_{i=0}^{\infty} (-1)^i \left(\frac{z + h}{2i+1}\right)^{(2i+1)l!} F^{(2i)}(X)$$  \[9\]

where the superscript, $(2l)$, represents the $2l$-th derivative. In addition, the kinematic bottom boundary condition [6] is automatically satisfied since $\Psi = 0$ on $z = -h$. Now the problem is to find the solutions of $F(X)$ and $N(X)$ which satisfy Eqs. [7] and [8]. However, since these equations are nonlinear, the strict analytical solution cannot be found.

As depicted in Fig. 1, the independent parameters of wave dimensions are the water depth, $h$, wavelength, $L$, and wave height, $H$. From these three parameters, we can define various pairs of two independent non-dimensional parameters. Isobe et al. (1982) proved that a regular double power series solution can be obtained using the following pair of non-dimensional parameters:

$$\varepsilon = H L^2 / h^3$$  \[10\]

$$\delta = (h/L)^2.$$  \[11\]

The parameter $\varepsilon$ is called Ursell’s parameter and $\delta$ the square of the relative water depth. The double power series solution is expressed as

$$F(X; \varepsilon, \delta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon^i \delta^j F_{ij}(X)$$  \[12\]

$$N(X; \varepsilon, \delta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon^i \delta^j N_{ij}(X).$$  \[13\]

Availability of a solution of this type means that the problem was solved in a theoretical sense, but the accuracy of the double power series solution of finite order is very low and therefore the solution is of little practical use. However, a major advantage of the solution exists in giving an overall view of the perturbation solutions of finite-amplitude waves, as discussed below.

Instead of expanding into the double power series, we first try to find a single power series solution in $\varepsilon$:

$$F(X; \varepsilon, \delta) = \sum_{i=0}^{\infty} \varepsilon^i F_{\varepsilon i}(X; \delta)$$  \[14\]

$$N(X; \varepsilon, \delta) = \sum_{i=0}^{\infty} \varepsilon^i N_{\varepsilon i}(X; \delta).$$  \[15\]

A physical interpretation of this expansion is that $\delta$ and hence the relative water depth, $h/L$, may not necessarily be small, being zero-order quantity (in the order of unity), but $\varepsilon$ and hence the wave steepness, $H/L$, must be a small quantity. By expanding Eqs. [7] and [8] in terms of $\varepsilon$, then substituting Eqs. [14] and [15] into them, and rearranging the terms in ascending order of $\varepsilon$, we obtain a pair of differential equations for $F_{\varepsilon i}$ and $N_{\varepsilon i}$ for each order of $i$. When we solve these equations in ascending order from zero, each pair of differential equations becomes linear, and thereby a solution can be found without difficulty, although the mathematical manipulation is very long and tedious. The result is summarized as follows:

$$\psi = \left[ c + \sum_{i=0}^{\infty} \varepsilon^i b_{ij} \right] (z + h) + \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \varepsilon^{n+2} a_{ni} \cos nk(x - ct)$$  \[16\]

$$\eta = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \varepsilon^{n+2} a_{nl} \cos nk(x - ct).$$  \[17\]

where $k = 2\pi/L$ is the wave number, and $a_{ni}$ and $b_{ni}$ are constants which can be determined by the relative water depth $\delta$. The theory using this expansion is called Stokes wave theory, which was first derived by Stokes (1847). Stokes wave theory is valid in deep and shallow water. However, in very shallow water, convergence of the series is very slow or the series is divergent even for slightly large values of $\varepsilon$. This occurs in cases of small relative water depth, $h/L$, and finite wave steepness, $H/L$. In these cases, the alternative power series solution should be used.
The single power series solution in terms of $\delta$ is expressed as follows:

$$F(X; \varepsilon, \delta) = \sum_{j=0}^{\infty} \delta^j F_{ij}(X; \varepsilon)$$  \[18\]

$$N(X; \varepsilon, \delta) = \sum_{j=0}^{\infty} \delta^j N_{ij}(X; \varepsilon).$$  \[19\]

A physical interpretation of the above is that Ursell’s parameter, $\varepsilon$, may not necessarily be small, being zero-order quantity, but the square, $\delta = (h/L)^2$, of the relative water depth and hence the relative wave height, $H/h$, must be a small quantity. Similarly to the Stokes wave expansion, Eqs. [7] and [8] are expanded into the power series in terms of $\delta$, Eqs. [18] and [19] are substituted into these equations, and terms are rearranged in ascending order of $\delta$. When solving pairs of differential equations for $F_{ij}$ and $N_{ij}$ in ascending order of $\delta$, the second-order equations are nonlinear, but the solutions are available by using elliptic functions. Since the remaining equations are linear, the solution can be found without difficulty but with huge efforts in calculation. It is summarized as follows:

$$\psi = \left[ c + \sum_{j=1}^{\infty} \delta^j b_{0j} \right] (z + h) + \sum_{j=1}^{\infty} \left\{ (-1)^{j+1} \frac{(z + h)^{2j+1}}{(2j+1)!} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \delta^n j b_{nj} \right\} \times cn^{2n} \left( \frac{2K(x - ct)}{L}, \kappa \right)^{(2j)} \right] \right]$$  \[20\]

$$\eta = \sum_{j=1}^{\infty} \delta^j a_{0j} + \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \delta^n j a_{nj} \times \left[ \frac{2K(x - ct)}{L}, \kappa \right] \right]$$  \[21\]

where $cn$ is the Jacobian elliptic function, $K$ the elliptic integral of the first kind, and $\kappa$ the modulus. The parameters $a_{nj}$ and $b_{nj}$ are constants to be determined by $\varepsilon$. This theory is known as cnoidal wave theory and is valid for very shallow water, as will be understood from the assumption. However, because of the irregularity found in the coefficients after the 10th order, the convergence radius becomes very small even in very shallow water. This is because of singular points in the imaginary number. Care should be taken when we use solutions of the 10th or higher order.

The relationship between the double power series solution, $F_{ij}$ and $N_{ij}$, and the solution, $F_a$ and $N_a$, of the Stokes wave theory is expressed as follows:

$$F_a(X; \delta) = \sum_{j=0}^{\infty} \delta^j F_{ij}(X)$$  \[22\]

$$N_a(X; \delta) = \sum_{j=0}^{\infty} \delta^j N_{ij}(X).$$  \[23\]

As depicted in Fig. 2, each order solution, $F_a$ and $N_a$, includes all elements in the corresponding column of the double power series. On the other hand, the cnoidal wave solution is related to the double power series solution as

$$F_{cj}(X; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j F_{ij}(X)$$  \[24\]

$$N_{cj}(X; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j N_{ij}(X)$$  \[25\]

which means that the solutions, $F_{cj}$ and $N_{cj}$, include all elements in the corresponding row.

Since the order of summation is arbitrary, we can take summation in the oblique direction first, and then in the direction perpendicular to it. However, the first summation in the oblique direction includes only a small number of elements in the double power series. For example, if we add the elements in the 45-degree right-down direction first, only one element is included in the zero-order solution, two in the first-order solution, three in the second-order solution, and
so on. On the other hand, each order solution of Stokes or cnoidal wave theory includes an infinite number of elements in the vertical or horizontal direction. Therefore, although theories other than Stokes and cnoidal wave theories exist and can be derived, their accuracies are lower than the two wave theories and they are less useful.

Finally, Nishimura et al. (1977) examined the accuracy of finite-order Stokes and cnoidal wave theories by calculating errors for the nonlinear boundary conditions, which are satisfied only approximately by finite-order solutions. As an example, Fig. 3 shows relative error curves for the dynamic boundary condition, for various orders of Stokes and cnoidal waves. In the horizontal axis, is the deep-water wavelength. Also drawn in the figure is the curve for the highest waves of permanent form obtained by Yamada and Shiotani (1968) by numerical calculation. The curves of breaking wave height on sloping bottoms proposed by Goda (1970) are added by chain lines. These are higher than those of the highest waves of permanent form. As can clearly be seen, Stokes wave theory is valid in deep to shallow water and cnoidal wave theory in shallow to very shallow water. There is an overlap region. From this and other similar figures, an appropriate wave theory can be selected for practical use as shown in Fig. 4. Roughly speaking, Stokes wave theory should be used if the shallow-water Ursell’s parameter is smaller than 25, and for larger values we should switch to cnoidal wave theory. Here, for the sake of convenience in practical use, Ursell’s parameter is defined by considering shallow water approximation as , in which wave period is used instead of wavelength in the original definition. Figure 5 shows examples of wave profiles calculated by Stokes and cnoidal wave theories as well as by 19th-order Stream Function Method (SFM 19; Dean 1965), which is an almost exact numerical solution. As can be expected, Stokes wave theory is accurate for small Ursell’s parameters, whereas cnoidal wave theory is accurate for large Ursell’s parameters. With this, the relationships and background of various finite-amplitude wave theories have been clarified.

3. Estimation of directional spectrum

Although it is essential to study periodic waves to understand fundamental characteristics of sea waves, as done in the previous section, real sea waves are random since they are generated by wind blowing over the sea surface. Goda (2008) reviewed fundamental concepts for analyzing random waves, i.e., spectral analysis and individual wave analysis, and their applications to engineering practices. Here, as an essential element for analyzing random wave transformation in the subsequent sections, a method to estimate directional spectra is introduced.

Frequency spectrum is normally calculated from the time series of water surface fluctuation during a certain period. The total number of data is normally in the order of hundreds to thousands. When we need a directional spectrum, several kinds of time series of oscillating quantities such as water surface fluctuation and water particle velocity at various locations are used. However, the number of these oscillating quantities is very small compared to the number of
To calculate frequency spectra. Therefore, a theory is needed to estimate directional spectra with very high accuracy. The following is an example to improve accuracy.

Directional spreading of the energy of random sea waves is indispensable to predict wave transformation in the nearshore region, as shown by Goda (2008). A component of random waves has a certain wave period, $T$, wavelength, $L$, and wave direction, $\theta$. By changing the variables into the wave number vector, $k = (k \cos \theta, k \sin \theta)$ ($k = 2\pi L$: wavenumber), and the angular frequency, $\omega = 2\pi/T$, the amplitude of the component waves representing the energy in the interval between $k$ and $k + dk$, and $\omega$ and $\omega + d\omega$ is expressed as $Z(dk, d\omega)$. Then the random water surface displacement, $\eta$, is obtained by integrating all components:

$$\eta(x, t) = \int \int e^{i(kx - \omega t)} Z(dk, d\omega).$$  \[26\]

The wave number–frequency spectrum is defined as the energy density function in terms of the wave number vector and frequency:

$$S(k, \omega) dk d\omega = \langle Z(dk, d\omega) Z^*(dk, d\omega) \rangle$$  \[27\]

where $\langle \rangle$ denotes the ensemble mean and $^*$ the complex conjugate. Thus, the right-hand side represents the expected value of squared amplitude, which is proportional to the energy of component waves; so $S(k, \omega)$ represents the energy density. Since the wave number, $k$, and angular frequency, $\omega$, are uniquely related to each other through an equation called the dispersion relation, $S$ becomes a function of $\omega$ and $\theta$ as $S(\theta, \omega)$, which is called the directional spectrum.

Now, a method to estimate the wave number–frequency spectrum from measured time series of oscillating quantities is introduced. Based on the linear (small-amplitude) wave theory, any oscillating quantity, $\xi(x, t)$, is expressed by using the transfer function, $H(k, \omega)$, from the water surface elevation:

$$\xi(x, t) = \int \int H(k, \omega) e^{i(kx - \omega t)} S(k, \omega) dk.$$  \[28\]

Some representative transfer functions are shown in Table 1. Suppose $M$ kinds of time series of various quantities at various locations are obtained. From Eq. [28], the cross-power spectra matrix, $\Phi_{mn}(\omega)$ ($m, n = 1$ to $M$), is expressed as

$$\Phi_{mn}(\omega) = \int H_m(k, \omega) H_n^*(k, \omega) e^{i(kx_n - x_m)} S(k, \omega) dk.$$  \[29\]

The symbol $\Phi_{mn}$ represents power spectra for $m = n$, and cross spectra for $m \neq n$. The problem now is how to estimate the wave number–frequency spectrum as a function of $\Phi_{mn}$. Many methods have been...
proposed to improve accuracy. Isobe et al. (1984)\(^{19}\) applied the maximum likelihood method to derive an explicit formula. The result shows that the estimated spectrum, \(S(k, \omega)\), is expressed as

\[
\hat{S}(k, \omega) = a / \left[ \sum_{m=1}^{M} \sum_{n=1}^{M} \Phi_{m,n}^{-1}(\omega)H_m(k, \omega)H_n(k, \omega)e^{ik(x_n - x_m)} \right]^{30}
\]

where \(a\) is a constant for adjusting the value of total energy. The above equation has a high resolution and can detect peaks of discrete spectrum. The method is called Extended Maximum Likelihood Method (EMLM).

In the practical ocean wave observation, the water surface fluctuation (or pressure fluctuation) and two-component horizontal velocities are often measured, since these can be measured by installing a wave gage (or pressure gage) and a two-component current meter at the same location. For this case, Eq. [30] is simplified to yield the following equation:

\[
\hat{S}(\theta, \omega) = a / \left[ M_0M_2(\gamma^2 \cos^2 \hat{\theta} + \sin^2 \hat{\theta}) - M_1^2 \sin^2(\hat{\theta} - \hat{\theta}_m) \right]^{31}
\]

where

\[
M_0 = m_{00}
\]
\[
M_1 = \sqrt{m_{10}^2 + m_{01}^2}
\]
\[
M_2 = \frac{m_{20} + m_{02}}{2} + \sqrt{\left(\frac{m_{20} - m_{02}}{2}\right)^2 + m_{11}^2}
\]
\[
\hat{\theta} = \theta - \theta_p
\]

In the above definitions, \(\hat{\theta}_m\) is called the average direction, which is the gravitational center of the directional spectrum; \(\theta_p\) the principal direction in which the mean square value of the velocity becomes maximum; and \(\gamma\) the long-crestedness parameter, which represents the peakedness of the directional spectrum. For this case, the same equation was derived by Oltman-Shay and Guza (1984).\(^{20}\) Figure 6 shows an example of a comparison between true and estimated directional spectra from numerical simulation. The EMLM gives estimation close to the true spectrum only from the three components, whereas the method derived by Longuet-Higgins et al. (1963)\(^{21}\) has less resolution.

Other methods have been proposed by Hashimoto and Kobune (1988)\(^{22}\) based on the Bayesian approach, and by Hashimoto et al. (1994)\(^{23}\) based on the maximum entropy principle. For a combined incident and reflected wave field, the phase difference between a component of incident waves and the corresponding component of reflected waves is not random but deterministic depending on the location of the reflective boundary. Isobe and Kondo (1984)\(^{24}\) and Hashimoto et al. (1997)\(^{25}\) proposed methods to separate incident and reflected waves by modifying the maximum likelihood method and maximum entropy method.
Unlike the methods introduced above, directional spectra can be estimated from the simultaneous two-dimensional water surface profile as measured by remote sensing technology. An example is given by Hashimoto and Tokuda (1999).26

Based on directional spectra estimated from field data, Mitsuyasu et al. (1975)27 proposed a standard form of directional spectra by introducing the Mitsuyasu-type directional function. Once an incident directional spectrum is given, the time series of the water surface elevation can be generated by superposing component periodic waves. In the next section, equations for predicting the transformation of periodic waves as well as random waves in the nearshore zone are introduced.

4. Equations for wave transformation in the nearshore zone

When waves enter the nearshore region, water motion is restricted by the sea bottom, shoreline and coastal structures, and hence the waves transform. There are many elements of wave transformation, such as shoaling, refraction, diffraction, reflection and breaking. To understand nearshore processes such as wave-induced current, sediment transport, wave run-up and overtopping on coastal structures, and wave forces on breakwaters, we need to predict the wave field in the nearshore region. Historically, each element of wave transformation has been treated separately, such as by refraction diagrams or diffraction diagrams. Refraction means wave transformation involving a change in wave direction and height, due to a change in water depth. This includes wave shoaling, which is the transformation of waves incident perpendicular to the coast with parallel bottom contours. Diffraction is caused mainly by barriers which interrupt wave propagation and create a sheltered region behind them. Figure 7 depicts the various components of nearshore wave transformation.

These elements occur simultaneously in the nearshore zone. The mild-slope equation was derived to analyze combined refraction and diffraction of periodic waves of small amplitude. Since then, various wave equations have been proposed to take more elements into consideration. In the following, the mild-slope equation is first introduced and then equations extended for randomness and nonlinearity of waves are explained.

4.1 Mild-slope equation. The mild-slope equation was derived by Berkhoff (1972)28 to analyze combined refraction and diffraction of small-amplitude periodic waves.

From the periodic small-amplitude wave theory at a uniform water depth, the velocity potential, \( \phi \), is expressed as follows:

\[
\phi = \frac{g}{i\omega} \frac{\cosh k(z + h)}{\cosh kh} \eta e^{-i\omega t} \tag{41}
\]

where \( i \) is the unit of imaginary number. The water particle velocity is obtained by taking the gradient of the velocity potential, \( \phi \). The above expression is equivalent to the first-order (i.e., small amplitude) solution of the velocity potential, \( \phi \). The above expression is equivalent to the first-order (i.e., small amplitude) solution of the Stokes wave theory expressed by Eq. [16]. The symbol, \( \eta \), defined by the following equation, denotes the complex amplitude of the water surface elevation, \( \eta \):

\[
\eta = \eta e^{-i\omega t}. \tag{42}
\]

Since we deal with refraction-diffraction problems, the complex amplitude, \( \eta \), is not constant but is a gradually varying function of the horizontal coordinates, \( (x, y) \).

The mild-slope equation is obtained by multiplying a vertical distribution function, \( \cosh k(z + h) \), with the governing Laplace equation and integrating from the bottom to the mean water surface in the vertical direction:

\[
\nabla (c c_0 \nabla \eta) + k^2 c c_0 \eta = 0 \tag{43}
\]

where \( \nabla = (\partial/\partial x, \partial/\partial y) \) is the two-dimensional horizontal gradient operator, \( c \) the wave celerity and \( c_0 \) the group velocity (i.e., energy transfer velocity). For a given set of boundary conditions at all boundaries of the computational domain, the above elliptic partial differential equation can be solved uniquely.

Berkhoff et al. (1982)29 proved that the mild-slope equation includes the eikonal equation with diffraction effect and the energy conservation equa-
tion between two wave rays. This implies that the mild-slope equation includes a refraction process with modification due to diffraction. Hence, wave fields such as those in a harbor of varying water depth and in a nearshore zone with coastal structures can be predicted.

4.2 Parabolic approximation. The mild-slope equation requires boundary conditions on all boundaries of the computational domain, along both the offshore boundary and shoreline. All boundaries, including the shoreline, have more or less effects on the wave field. This means that each point in the wave field interacts with all other points and the solution must be calculated simultaneously for the whole wave field. However, from the physical viewpoint, waves entering a nearshore region transform due to refraction and diffraction, lose energy due to breaking, and finally disappear near the shoreline. This implies that the shoreline boundary has little effect on the major wave field as long as reflection is negligible. This understanding reminds us to calculate the wave field step by step from the offshore boundary to the onshore ward direction. From a mathematical viewpoint, a parabolic equation has such characteristics. Radder (1979)\(^3\) approximated the mild-slope equation by a parabolic equation using Cartesian coordinates. To extend applicability, Isobe (1986)\(^3\) used curvilinear coordinates which approximate wave rays and fronts.

A curvilinear coordinate system, \((\lambda, \mu)\), of which the directions of \(\lambda\) and \(\mu\) correspond to those of wave rays and fronts, respectively, is introduced as shown in Fig. 8. On splitting the velocity potential into incident and reflected wave components and assuming the latter is much smaller than the former, we can eliminate the second derivative in terms of \(\lambda\) and obtain the following parabolic equation:

\[
2ik \frac{1}{h_\lambda} \frac{\partial \tilde{\eta}}{\partial \lambda} + \frac{1}{cc^2 h_\lambda} \frac{\partial}{\partial \mu} \left( cc^2 \frac{h_\lambda}{h_\mu} \frac{\partial \tilde{\eta}}{\partial \mu} \right) + \left[ \frac{i}{cc^2 h_\mu} \frac{\partial}{\partial \lambda} (kcc^2 h_\mu) + 2k(k-K) \right] \tilde{\eta} = 0 \quad [44]
\]

where

\[
\tilde{\eta} = \eta \exp \left( -i \int K h_\lambda d\lambda \right). \quad [45]
\]

The quantities \(h_\lambda\) and \(h_\mu\) are the scale factors of the curvilinear coordinate system and taken to be \(1/K\), in which \(K\) is the approximated value of the wave number at each location. Since the spatial change of phase in \(\tilde{\eta}\) is approximately removed in \(\tilde{\eta}\) by definition [45], the change in \(\tilde{\eta}\) is very gradual and hence a coarse grid system can be used in the numerical calculation with sufficient accuracy. Equation [44] includes the second-order derivative of \(\tilde{\eta}\) with respect to \(\mu\), but the first order with respect to \(\lambda\). Therefore, the elliptic-type mild-slope equation is approximated by the parabolic-type partial differential equation. The second derivative with respect to \(\mu\) expresses a diffraction effect in the wave front direction during wave propagation. Because of the parabolic nature of the partial differential equation, it can be solved from the offshore incoming wave boundary to the onshore direction step by step. This remarkably reduces the required computer storage and computational time.

Figure 9 compares analytical and numerical solutions of wave height distribution along the sheltered side of a semi-infinite breakwater. Due to diffraction, incoming wave energy penetrates into the sheltered region, although the diffraction coefficient, \(K_d\), which is the ratio of wave height in the sheltered region to the incident wave height, is smaller than unity. Good agreement is found especially for a small angle of incidence \(\theta_i\). It should be noted that wave energy is proportional to the square of the wave height, and hence the diffraction in terms of energy is not very large. This means diffraction occurs to a certain degree, but diffracted wave energy may often be neglected.

Figure 10 is an example to compare calculated and (laboratory) experimental results on the distributions of the relative wave height, \(H/H_I\) (the ratio of the local wave height to the incident wave height). Since wave reflection is negligible due to the sloping beach in the harbor, the assumption of the parabolic
approximation is satisfied and hence good agreement can be found.

Parabolic equation models are extended to weakly nonlinear waves (Kirby and Dalrymple, 1983; Liu and Tsay, 1984) and breaking waves (Dalrymple et al., 1984). Diffraction-refraction of random waves can be treated by superimposing results for component waves with various frequencies and directions. A refraction, diffraction and breaking model of random waves was proposed by Isobe (1987) by introducing energy dissipation due to wave breaking. The model can be used in general situations of wave transformation in the nearshore zone as long as wave nonlinearity and multi-reflection are not critical factors. Figure 11 shows a comparison of calculated and measured significant wave height, $H_{1/3}$, and mean direction, $\theta_m$.

4.3 Time-dependent mild slope equations for random waves. When multi-reflection is important, we need to solve for the mild-slope equation of the elliptic type. Time-dependent equations are proposed as a numerical solution technique. Since the wave profile is calculated time by time, a dissipation term due to wave breaking can be incorporated easily. Watanabe and Maruyama (1986) proposed time-dependent mild-slope equations for calculating the nearshore wave field due to refraction, diffraction and breaking. For random waves, Isobe (1994) derived a set of time-dependent equations as introduced in the following.

![Fig. 9. Comparison of diffraction coefficients behind a semi-infinite breakwater between numerical solution calculated by the parabolic equation and analytical solution.](image1.png)

![Fig. 10. Comparison of calculated and measured relative wave heights in a harbor with a sloping bottom.](image2.png)
In the mild-slope equation [43], the complex amplitude, $\eta$, is defined as Eq. [42] by using the angular frequency, $\omega$, of the periodic waves. However, for random waves, angular frequencies are different per component waves. In addition, the values of the coefficients, $ccg$ and $k^2$, are dependent on the angular frequency. Therefore, we cannot use the mild-slope equation directly to calculate random wave transformation. Here, we consider narrow-banded frequency spectra with a representative angular frequency, $\omega_0$, around which random wave energy is concentrated, and define $\tilde{\eta}$ by

$$\eta = \tilde{\eta}e^{-i\omega t}. \quad [46]$$

Comparison with Eq. [42] yields

$$\tilde{\eta} = \tilde{\eta}e^{-i\omega' t} \quad [47]$$

where $\omega' = \omega - \omega_0$, which is small compared to $\omega$ by assumption. So, $\tilde{\eta}$ is defined independently of the angular frequency of each component of random waves and a slowly varying function of time, $t$.

On considering Eq. [47], the mild-slope equation must also be satisfied by $\tilde{\eta}$:

$$\nabla(ceg \nabla \tilde{\eta}) + k^2 cceg \tilde{\eta} = 0. \quad [48]$$

Now for the sake of mathematical simplicity, $\tilde{\eta}$ is transformed to $\bar{f} = \tilde{\eta}/\sqrt{ccg}$, which simplifies the mild-slope equation into the Helmholtz equation:

$$\nabla^2 \bar{f} + k^2 \bar{f} = 0. \quad [49]$$

To include energy dissipation due to breaking, the energy dissipation coefficient, $D$, is introduced in the equation:

$$\nabla^2 \bar{f} + k^2(1 + iD)\bar{f} = 0. \quad [50]$$

Since the values of the coefficients change with the angular frequency, we cannot deal with random waves by the above equation. If we introduce time derivatives of $\bar{f}$ into a similar equation:

$$\nabla^2 \bar{f} - ia_1 \nabla^2 \left( \frac{\partial \bar{f}}{\partial t} \right) + (b_0 + ic_0) \bar{f}$$

$$+ i(b_1 + ic_1) \frac{\partial \bar{f}}{\partial t} - b_2 \frac{\partial^2 \bar{f}}{\partial t^2} = 0 \quad [51]$$

and compare with Eq. [50], the following constraints are obtained for the constants, $a_1$, $b_0$, $b_1$, $b_2$, $c_0$ and $c_1$:

$$k^2 = (b_0 + b_1 \omega' + b_2 \omega'^2)/(1 - a_1 \omega') \quad [52]$$

$$D = c_0 + c_1 \omega'. \quad [53]$$

By determining the values which approximate the above two equations, all the coefficients in Eq. [51] become independent of frequency and thus the
equation can be used for component waves with any frequency. Thus by inputting $f$ along the offshore boundary defined from incident random water surface fluctuation, which is obtained by superimposing component waves for a given wave spectrum, Eq. [51] can be used to analyze refraction, diffraction and breaking of random waves. Since temporal evolution is obtained in the numerical calculation, the equation is called time-dependent mild-slope equation for random waves.

Figure 12 shows an example to compare calculated and measured water surface fluctuation due to shoaling random waves. The incident wave profile which is shown in the middle figure was measured on a horizontal bottom. As seen in the bottom figure, good agreement can be found as long as the small-amplitude assumption is satisfied. Moreover, although discrepancies especially in the steepness of wave crests become prominent just prior to the breaking point, integral properties such as wave energy and radiation stress can be accurately calculated. However, to predict properties which are strongly affected by nonlinearity, nonlinear theory must be used.

4.4 Boussinesq equations. For nonlinear waves, Boussinesq equations were derived by Peregrine (1967).\(^{38}\) The basic assumption is the same as in the case of cnoidal waves, explained in section 2. Ursell’s parameter, $\varepsilon$, is assumed to be in the order of unity, whereas the square, $\delta = (h/L)^2$, of the relative water depth and hence the relative wave height, $H/h$, are small. By taking the terms up to the second order of $\delta$, the following Boussinesq equations were derived:

$$\frac{\partial \eta}{\partial t} + \nabla [(h + \eta)\bar{u}] = 0$$  \[54\]

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \nabla)\bar{u} + g \nabla \eta = \frac{k}{2} \nabla \left[ \nabla \left( h \frac{\partial \bar{u}}{\partial t} \right) \right] - \frac{h^3}{6} \nabla \left[ \nabla \frac{\partial \bar{u}}{\partial t} \right]$$  \[55\]

where $\bar{u} = (\bar{u}, \bar{v})$ is the horizontal velocity averaged over the water depth.

Boussinesq equations include terms up to the second order of the relative wave height and square of the relative water depth, which are indicators of wave nonlinearity and dispersivity, respectively. Therefore, Boussinesq equations are weakly nonlinear and weakly dispersive wave equations, which implies the equations are valid for moderately high waves in shallow water.

The right-hand side of Eq. [55] represents the effect of wave dispersion. For very shallow water waves of small amplitude, the wave celerity is independent of the wave period, and hence all component waves propagate by the same wave celerity, $\sqrt{gh}$, which means waves propagate by keeping the same profile. However, since the two terms on the right-hand side of Eq. [55] result in a slight difference in celerity depending on the wave period of each component wave, each component wave propagates with slightly different celerity, the total wave profile changes gradually, and finally the component waves separate from each other from front to back due to the difference in wave celerity. This phenomenon is called wave dispersion, which is the reason why Boussinesq equations are dispersive equations. If these terms are neglected, the resultant equations are called nonlinear shallow water equations, which do not include dispersion caused by the effect of finite water depth.

To extend the applicable range of Boussinesq equations, modified Boussinesq equations were proposed. As an example, Madsen and Sorensen (1992)\(^{39}\) introduced additional terms to improve the dispersion relation in deep water. In addition, apart from the theoretical assumption and derivation, the nonlinear term in the expression of Boussinesq equations is the same as the fully nonlinear term. Therefore, the modified version of Boussinesq equations is valid for most of the practical situations to calculate wave field, such as wave height distribution.
However, because of the approximation included in the theory, detailed wave properties such as vertical distribution of velocity cannot be described accurately.

4.5 Time-dependent nonlinear mild slope equations for random waves. A strongly nonlinear wave equation is derived in a somewhat different way. It was shown by Luke (1967) that the velocity potential, $\phi$, and water surface fluctuation, $\eta$, which terminate the following Lagrangian, satisfy the governing equation and boundary conditions for waves on a varying water depth:

$$ L[\phi, \eta] = \int_{t_1}^{t_2} \int_A \int_{-h}^{0} \left[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 \right] dz \, dA \, dt $$

where $t_1$ and $t_2$ are the limits of time, $t$, and $A$ denotes the area of concern in the horizontal plane.

In general, wave equations are two-dimensional equations which are obtained by integrating the governing three-dimensional equations in the vertical direction. For the vertical integration, vertical distribution functions are given theoretically or empirically. Masel (1993) derived an extended mild slope equation by introducing a vertical distribution function of the hyperbolic cosine type and integrating the governing equation. A clear and generalized concept of this procedure was proposed by Nadaoka et al. (1994) in deriving a strongly nonlinear, strongly dispersive wave equation by applying the Galerkin method to the Euler equations of motion. Isobe (1994) derived a general form of nonlinear, dispersive wave equations by using the Lagrangian defined by Eq. [56] as follows.

The velocity potential, $\phi$, which is a function in three dimensions, including the vertical direction, is expanded into a series in terms of a certain set of vertical distribution functions, $Z_\alpha(z) (\alpha = 1 \text{ to } M)$:

$$ \phi(x, z, t) = \sum_{\alpha=1}^{M} Z_\alpha(z; h(x)) f_\alpha(x, t) \equiv Z_\alpha f_\alpha $$

where $x = (x, y)$ is the two-component horizontal coordinate and the summation convention is applied when the same subscript appears twice in a term. The function, $Z_\alpha$, may be dependent on the local water depth, $h(x)$, and $f_\alpha$ is the weighting function for $Z_\alpha$ and a function of $x$ and $t$ but not of $z$.

On substituting Eq. [57] into Eq. [56] and applying the variational principle with respect to $f_\alpha$ and $\eta$, we obtain the following set of time-evolutional partial differential equations:

$$ \begin{align*}
Z_\alpha \frac{\partial \eta}{\partial t} + \nabla (A_{\alpha \beta} \nabla f_\beta) - B_{\alpha \beta} f_\beta + (C_{\alpha \beta} - C_{\alpha \beta}) \nabla f_\beta \nabla h \\
+ \frac{\partial Z_\alpha}{\partial h} Z_\alpha f_\beta \nabla \eta \nabla h &= 0 \\
&\quad \text{[58]}
\end{align*} $$

where

$$ \begin{align*}
A_{\alpha \beta} &= \int_{-h}^{0} Z_\alpha Z_\beta dz \\
B_{\alpha \beta} &= \int_{-h}^{0} \frac{\partial Z_\alpha}{\partial z} \frac{\partial Z_\beta}{\partial z} dz \\
C_{\alpha \beta} &= \int_{-h}^{0} \frac{\partial Z_\alpha}{\partial h} Z_\beta dz \\
D_{\alpha \beta} &= \int_{-h}^{0} \frac{\partial Z_\alpha}{\partial h} \frac{\partial Z_\beta}{\partial h} dz \\
&\quad \text{[60, 61, 62, 63]}
\end{align*} $$

and the superscript, $\eta$, denotes the value at the water surface, $z = \eta$.

Since Eq. [58] is a vector equation consisting of $M$ components and Eq. [59] is a scalar equation, $(M + 1)$ equations are included in these equations. On the other hand, the number of unknown functions is $(M + 1): M$ for $f_\alpha$ and $1$ for $\eta$. Hence we can solve these simultaneous equations step by step if the appropriate initial and boundary conditions are given.

All wave equations such as the mild-slope equation and Boussinesq equations are obtained by integrating three-dimensional basic equations in the vertical direction for a corresponding vertical distribution function. In this sense, the nonlinear time-dependent mild-slope equations explained above are a general form of wave equations, since the set of vertical distribution functions can be given arbitrarily. For example, if we give only one component of a hyperbolic cosine-type vertical distribution function and neglect nonlinear terms, we obtain the same result as the mild-slope equation. If we give a parabolic vertical distribution function, the result is equivalent to Boussinesq equations. Finally, if we give a sufficient number of vertical distribution functions which can accurately approximate the strict solution of the vertical profile of the velocity potential, we can calculate an accurate numerical solution for nonlinear wave transformation in the nearshore area.

The accuracy of the dispersion relation, i.e., the relation between the wave frequency and wavelength or wave celerity, is often used to examine the validity
of the approximation to linear wave theory. The linear dispersion relation is strictly satisfied by assuming a hyperbolic cosine-type vertical distribution function. As an alternative, a set of polynomial functions can satisfy the dispersion relation with sufficient accuracy. This set has an advantage in that it can give an accurate vertical distribution to analyze nonlinear wave transformation in very shallow water. Figure 13 compares the linear dispersion relations of the strict solution and series expressions in terms of polynomial functions. The agreement is good in very shallow water even for a small number of terms, such as $M = 2$, and also good even in deep water if we use a sufficient number of terms, such as $M = 4$. This implies that if we use a sufficient number of polynomial functions, transformation of nonlinear waves can accurately be predicted from deep water to very shallow water.

Figure 14 compares calculated and measured water surface fluctuation, $\eta$, and horizontal water particle velocity, $u_h$, at the bottom. Although the profile has steep crests resulting from strong nonlinearity, the agreement is good. Figure 15 is for waves travelling over a submerged breakwater. Although both nonlinearity and dispersion are strong, a good agreement can be obtained. Such a high accuracy cannot be found if Boussinesq equations are used because they are weakly nonlinear and weakly dispersive wave equations. These imply that the time-dependent nonlinear mild-slope equations are valid if an appropriate set of vertical distribution functions are selected.

4.6 Classification of wave equations. Fundamental wave theories are for waves which propagate at a uniform water depth without deformation. Stokes and cnoidal wave theories have been derived for deep to shallow, and shallow to very shallow waters, respectively, according to the assumption on water depth. To deal with wave transformation in the nearshore region, the effect of changing water depth, i.e., the bottom slope, must be taken into account. The mild-slope equation includes the effect of the bottom slope on the basis of the first-order Stokes wave theory, i.e., small-amplitude wave theory. Parabolic equations have been derived from the mild-slope equation for simpler numerical calculation. Time-dependent mild-slope equations have been proposed mainly to incorporate with a wave breaking model. Furthermore, time-dependent mild-slope equations for random waves can deal with refraction, diffraction and breaking of random waves.

Nonlinear shallow water equations have been derived for nonlinear waves in very shallow water. The equations do not include the effect of wave dispersion. Boussinesq equations, which in the case of a horizontal bottom reduce to the first-order cnoidal wave equations, include weakly dispersive terms in addition to the terms in nonlinear shallow water waves. Modified versions can represent strong dispersivity in relatively deep water.

Time-dependent nonlinear mild-slope equations for random waves can represent both strong nonlinearity and strong dispersivity and deal with nonlinear random waves in a time-series numerical calculation.
Table 2 summarizes the classification of various wave equations according to the assumptions in deriving the equations. Since the more strict and general equations need more computer storage and computational time, appropriate selection of wave equations is essential in practice.

In this section, we systematically reviewed various wave equations which can be obtained by integrating the three-dimensional governing equations in the vertical direction. If vertical irregularity is essential, such as in wave transformation due to three-dimensional structures, direct numerical calculation of three-dimensional governing equations is indispensable. The progress in this field is also remarkable with the aid of improved computer performance (JSCE, 2012).

5. Concluding remarks

In this paper, an overall view of wave equations to analyze the nearshore wave field has been introduced as systematically as possible. Fundamental wave theories are for periodic waves which propagate at a uniform water depth without deformation, i.e., waves of permanent form. As analytical solutions of this type of waves, Stokes and cnoidal wave theories were developed for deep to shallow water and for shallow to very shallow water, respectively. Independent parameters for waves of permanent form are the water depth, \( h \), wavelength, \( L \), and wave height, \( H \), from which two independent non-dimensional parameters can be defined arbitrarily. If we select the square, \( \delta = (h/L)^2 \), of relative water depth and Ursell’s parameter, \( \varepsilon = HL^2/h^3 \), a regular double power series solution can be derived. Based on this solution, a systematic understanding of various single power series solutions, i.e., perturba-

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### Table 2. Assumptions behind various wave equations

| Wave equations         | Bottom slope | Multi reflection | Relative water depth (h/L)^2 | Nonlinearity | Randomness |
|------------------------|--------------|-----------------|-----------------------------|--------------|------------|
| MSE^1)                 | small        | yes             | arbitrary                   | small (finite^5) | no (yes^8) |
| Parabolic eq.          | small        | no^5)           | arbitrary                   | small (finite^5) | no (yes^8) |
| T-D^2) MSE             | small        | yes             | arbitrary                   | small        | no (yes^8) |
| T-D^2) MSE for random waves | small      | yes             | arbitrary                   | small        | yes        |
| NL^3) shallow water eqs. | arbitrary^6 | yes             | very small                  | arbitrary    | yes        |
| Boussinesq eqs.        | arbitrary^6 | yes             | small (arbitrary^7)         | finite^6)    | yes        |
| NL^3) MSE              | arbitrary^6 | yes             | arbitrary                   | arbitrary    | yes        |

1) MSE: mild slope equation, 2) T-D: time-dependent, 3) NL: nonlinear, 4) within restriction of shallow water assumption, 5) dependent on vertical distribution functions, 6) improved version includes incident and reflected waves, 7) improved version, 8) improved version, 9) in spite of theoretical assumption, resultant equations includes a fully nonlinear term, 10) by linear superposition of component waves.
tion solutions, can be achieved. Stokes and cnoidal wave theories have been proved to be the only two practically effective solutions among an infinite number of possible perturbation solutions. Furthermore, validity ranges for these wave theories of various orders were determined by calculating the relative error included in the solutions. In practice, we have only to use Stokes and cnoidal waves for Ursell’s parameters of less than and larger than 25, respectively.

To deal with random waves, the concept of a directional spectrum and methods to estimate directional spectra were introduced. The maximum likelihood method, Bayesian approach and maximum entropy method were among the latter. By applying these methods to field data, we can accumulate samples of directional spectra, from which standard forms have been proposed. From the standard directional spectra, we can give incident waves to analyze transformation of random waves in the nearshore zone.

Various wave equations to calculate wave transformation in the nearshore zone were introduced based on the assumption in deriving the equations. The mild-slope equation initiates numerical calculation of combined refraction and diffraction. Parabolic approximations were proposed to save computer storage and computational time, and time-dependent mild-slope equations were proposed to simplify numerical calculation and introduce a dissipation term due to wave breaking. Time-dependent mild-slope equations were further extended for random waves. As a set of nonlinear wave equations, Boussinesq equations were derived for weakly nonlinear and weakly dispersive waves. Finally, nonlinear mild-slope equations were derived to analyze shoaling, refraction, diffraction, reflection and breaking of nonlinear random waves.

As introduced above, wave transformation in the nearshore zone is of theoretical and practical interest. More accurate and detailed analysis methods will be developed with the help of advanced computer performance.

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Profile

Masahiko Isobe was born in 1952 in Tokyo. He graduated from the Faculty of Engineering at the University of Tokyo, specializing in civil engineering. He continued his study at the Graduate School of Engineering and obtained master’s and doctoral degrees in the field of coastal engineering under the supervision of Prof. Kiyoshi Horikawa. He began his professional career as Research Associate in 1978 at the University of Tokyo. Since then he has experienced Assistant Professor (1981 to 1983) and Associate Professor (1983 to 1987) at the Yokohama National University, and Associate Professor (1987 to 1992) and Professor (1992 to date) at the University of Tokyo. He was one of the core founders of the Graduate School of Frontier Sciences which was established at Kashiwa Campus, the newly developed and third major campus of the University of Tokyo, in 1998. He served as the Dean (2005 to 2007) of the graduate school and the Vice President (2009 to 2011) of the university. His contribution in research is mainly on the wave hydrodynamics and coastal environment. He has been working on the nonlinear wave dynamics in the nearshore area and water quality problems in the enclosed sea. As a result of his achievements, he was awarded annual incentive prize and annual research award of the Japan Society of Civil Engineers in 1986 and 1997, respectively.